Validity of Borodin and Kostochka Conjecture for 4K₁-free Graphs

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Abstract: Problem of finding an optimal upper bound for the chromatic no. of even 3K₁-free graphs is still open and pretty hard. Here we prove Borodin & Kostochka Conjecture for 4K₁-free graphs G i.e. if Δ(G) ≥ 9 and G is 4K₁-free, then χ(G) ≤ max{ω, Δ-1}.

Introduction:

In [1], [2], [3], [4] chromatic bounds for graphs are considered especially in relation with ω and Δ. Gyárfás [5] and Kim [6] show that the optimal χ-binding function for the class of 4K₁-free graphs has order ω²/log(ω). If we forbid additional induced subgraphs, the order of the optimal χ-binding function drops below ω²/log(ω). In 1941, Brooks' theorem stated that for any connected undirected graph G with maximum degree Δ, the chromatic number of G is at most Δ unless G is a complete graph or an odd cycle, in which case the chromatic number is Δ+1 [5]. In 1977, Borodin & Kostochka [6] conjectured that if Δ(G) ≥ 9, then χ(G) ≤ max{ω, Δ-1}. In 1999, Reed proved the conjecture for Δ ≥ 10¹⁴ [7]. Also D. W. Cranston and L. Rabern [8] proved it for claw-free graphs. Here we prove Borodin & Kostochka conjecture for 4K₁-free graphs.

Notation: For a graph G, V(G), E(G), Δ, ω, χ denote the vertex set, edge set, maximum degree, size of a maximum clique, chromatic number of G respectively. For u ∈ V(G), N(u) = {v ∈ V(G) / uv ∈ E(G)}, and N(u) = N(u) ∪ {u}. If S ⊆ V, then <S> denotes subgraph of G induced by S. If C is some coloring of G and if u ∈ V(G) is colored m in C, then u is called a m-vertex, if N(u) has a unique r-vertex, then r is called a unique color of u and if N(u) has more than one r-vertex, then r is called a repeat color of u. Also if P is a path in G s.t. vertices on P are alternately colored say i and j, then P is called an i-j path. All graphs considered henceforth are simple. We consider here simple and undirected graphs. For terms which are not defined here refer to Bondy and Murty [9].

Main Result: Let G be 4K₁-free and Δ ≥ 9, then χ ≤ max{Δ-1, ω}. Proof: Let if possible G be a smallest, connected, 4K₁-free graph with Δ ≥ 9 and χ > max{Δ-1, ω}. Then clearly as G ≠ C₂m₁ or Kₙ(G), χ = Δ > ω. Let u ∈ V(G). Then G-u ≠ Kₙ(G)-₁ (else χ = ω). If Δ(G-u) ≥ 9, then by minimality χ(G-u) ≤ max{ω(G-u), Δ(G-u)-1}. Clearly if ω(G-u) ≤ Δ(G-u)-1, then χ(G-u) = Δ(G-u)-1 ≤ Δ-1 and otherwise χ(G-u) = ω(G-u) ≤ ω < Δ. In any case χ(G-u) ≤ Δ-1. Also if Δ(G-u) < 9, then as G-u ≠ C₂m₁, by Brook's Theorem χ(G-u) ≤ Δ (G-u) < 9 ≤ Δ. Thus always χ(G-u) ≤ Δ-1 and in fact, χ(G-u) = Δ-1 and deg v ≥ Δ-1 ∀ v ∈ V(G).

Let u ∈ V(G) be s.t. deg u = Δ. Let S = {1,..., Δ-1, Δ} be a Δ-coloring of G with only u colored Δ. Then N(u) has Δ-2 vertices Aᵢ with unique colors i (1≤i≤Δ-2) and a pair of vertices say X, Y with the same color Δ-1. Clearly Aᵢ has a j-vertex for 1≤i≤j≤Δ-2 (else color Aᵢ by j, u by i).

Case 1: ∃ a (Δ-1)-coloring of G-u s.t. AᵢAⱼ ∉ E(G) for some i, j ∈ {1,..., Δ-2}.

(A) For no m, Aᵢ is the only m-vertex of both Aᵢ and Aⱼ for 1≤i, j, m≤Δ-2.
Let if possible Aᵢ, Aⱼ both have Aₘ as the only m-vertex. Then as Aₘ has at the most one repeat color, w.l.g. Aᵢ be the only j-vertex of Aₘ. Then color Aᵢ, Aⱼ by m, Aₘ by j, u by i, a contradiction.
(B) Aᵢ, Aⱼ do not have more than two common adjacent Aₘ in N(u).
Let Aᵢ, Aⱼ be both adjacent to say Aₖ, Aₙ, 1≤i, j, k, l ≤Δ-2. As each of Aᵢ, Aⱼ has at the most one repeat color, w.l.g. let Aₘ be the only m-vertex of both Aᵢ and Aⱼ, a contradiction to (A).
(C) Aᵢ is non-adjacent to at the most three Aₖ s ⇒ As Δ-2 ≥ 7, Aᵢ is adjacent to at least three Aₘ, i.e. Aᵢ, Aₖ, Aₙ, k, m≤Δ-2.
Let if possible AᵢAⱼ ∉ E(G) for 2≤k≤5. As G is 4K₁-free, ∃ at most two more 1-vertices a₁₁, a₁₂ and as ∃ 1-k path from A₁ to Aₙ, either a₁₁ or a₁₂ is adjacent to Aₖ with two k-vertices for 2≤k≤5. Again a₁₁ cannot have three repeat colors (else N(a₁₁) has a color say r missing. Color a₁₁ by r. Then either (i)
some $A_k$ ($2 \leq k \leq 5$) has no 1-vertex, hence color $A_k$ by $1$, $u$ by $k$ or (iii) $a_{12} A_k \in E(G)$ ($2 \leq k \leq 5$), $a_{12}$ has four repeat colors and $N(a_{12})$ has color $t$ missing. Color $a_{12}$ by $t$, $A_k$ by $1$, $u$ by $k$). Thus w.l.o.g. let $A_{a_{11}}$, $A_{a_{12}} \in E(G)$ for $i = 2$, 3 and $j = 4$, 5 s.t. $a_{11}$ has two repeat colors 2, 3 and $a_{12}$ has two repeat colors 4, 5. Clearly $A_1 A_j \in E(G)$ $\forall$ $6 \leq i \leq A - 2$ (else either $a_{11}$ or $a_{12}$ has three repeat colors).

**Claim 1:** Whenever $A_1$ has a unique i-vertex say $B$ for $6 \leq i \leq A - 1$, $A_1$ is the only 1-vertex of $B$.

Let if possible $B a_{11} \in E(G)$. Then $B$ has a unique m-vertex for $2 \leq m < A - 1$ (else $N(B)$ has some color $r$ missing. Color $B$ by $r$, $A_j$ by $1, u$ by 1). As $a_{11}$ has two repeat colors 2, 3, $B$ is its only i-vertex. Then $G$ has at the most one more i-vertex say $b$ (else $a_{11}, a_{11}, b_{12} > 4K_i$). Again by $(A)$, $B$ is not the only i-vertex of any $A_k$, for $2 \leq k \leq 5$. Hence $A_k b \in E(G)$ for $2 \leq k \leq 5$. Now $A_k B \notin E(G)$ for $k = 2, 3$ (else color $A_k$ by 1, $a_{11}$ by $i$, $B$ by $2/3$, $A_i$ by $i$, $u$ by 1) and $b$ has two k-vertices for $k = 2, 3$ (else color $A_k$ by $i$, $b$ by $k$, $u$ by $k$) $\Rightarrow A_m$ is the only m-vertex of $b$ for $m = 4, 5$ (else $b$ has color $r$ missing in $N(b)$. Color $b$ by $r$, $A_i$ by $i$, $u$ by 2). Now $A_m$ has two i-vertices (else $b$ is the only i-vertex of $A_m$. Color $b$ by $m$, $A_m$ by $i$, $b$ by $m$), $m \in \{4, 5\}$ $\Rightarrow a_{12}$ is the only 1-vertex of $A_m$, $m \in \{4, 5\}$. Again $B a_{12} \notin E(G)$ (else $B$ has three i-vertices and color say $r$ missing in $N(B)$. Color $B$ by $r$, $A_j$ by $1, u$ by 1) $\Rightarrow a_{12} b \in E(G)$. Then color $b$ by 4, $A_i$ by 1, $a_{12}$ by $i$, $u$ by 4, a contradiction. This proves Claim 1.

Now $a_{11}$ has an i-vertex for $k = 1, 2$ (else color $a_{1k}$ by $i$. If $a_{1k}$ is the only 1-vertex of $A_m$ ($2 \leq m \leq 5$), then color $A_m$ by 1, $u$ by $m$ and if every $A_m$ has two i-vertices, then if $k = 1 (2)$, color $A_2$ ($A_4$) by 1, $a_{12}(a_{11})$ by 2 (4), $u$ by 2 (4)).

Let $a_{11} b_{11} \in E(G)$. As $a_{11}$ has two repeat colors 2, 3, $b_{11}$ is the only i-vertex of $a_{11}$.

**Claim 2:** $a_{11}$ is the only 1-vertex of $b_{11}$.

Let if possible $a_{12} b_{11} \in E(G)$. As $a_{11}$ has two repeat colors 4, 5, $b_{11}$ is the only i-vertex of $a_{12}$. Then $G$ doesn't have an i-vertex say $b_{12} \notin \{B, b_{11}\}$ (else $a_{11}, a_{12}, b_{12} > 4K_i$). Again by $(A)$, $B$ cannot be the only i-vertex of any $A_m$ for $2 \leq m \leq 5$. Hence $a_{12} b_{11} \in E(G)$ for $2 \leq m \leq 5$. If $A_k$ is the only k-vertex of $b_{11}$ for some $k$, $2 \leq k \leq 5$, then if $b_{11}$ is the only i-vertex of $A_k$ color $A_k$ by $i$, $b_{11}$ by $k$, $u$ by $k$ and if $A_k$ has two i-vertices, then $a_{11}$ being the only 1-vertex of $A_k$ color $A_k$ by 1, $a_{11}$ by $i$, $b_{11}$ by $k$, $u$ by $k$, contradictions in both the cases. Hence let $b_{11}$ have repeat colors $k \forall k, 2 \leq k \leq 5$. But then $b_{11}$ has color $r$ missing in $N(b_{11})$. Color $b_{11}$ by $r$ and $a_{11}$ by $i$. Then $A_2 a_{12} \in E(G)$ (else color $A_2$ by 1, $u$ by 2). Again as $a_{12}$ has two repeat colors 4, 5, $A_2$ is its only 2-vertex and hence color $A_2$ by 1, $a_{12}$ by 2, $u$ by 2, a contradiction. This proves Claim 2.

Similarly if $b_{12}$ is an i-vertex of $a_{12}$, then $a_{12}$ ($b_{12}$) is the only 1-vertex (i-vertex) of $b_{12}$ ($a_{12}$). Now $A_{a_{11}}, A_b b_{12} \in E(G)$ for $m = 2, 3$ and $n = 4, 5$ (else let $A_2 b_{11} \notin E(G)$. If $a_{11}$ is the only 1-vertex of $A_2$, then color $a_{11}$ by $i$, $b_{11}$ by 1, $A_2$ by 2, $u$ by 2 and if $A_2 a_{12} \in E(G)$, then color $a_{11}$ by $i$, $b_{11}$ by 1, $a_{12}$ by 2, $A_2$ by 1, $u$ by 2).

As $A_1$ has at the most one repeat color, w.l.o.g. let $A_1$ have unique 2, 3, 4 vertices. Let $P(R)$ be a 2-1 (4-1) path from $A_2$ ($A_4$) to $A_1$. As $a_{12}$ ($a_{11}$) has a unique 2-vertex (4-vertex), clearly $P = \{A_2, a_{11}, a_{21}, A_1\}$ and $R = \{A_4, a_{12}, a_{41}, A_1\}$.

**Claim 3:** $a_{21} a_{12}, A_2 a_{12} \notin E(G)$. Similarly $a_{41} a_{11}, A_4 a_{11} \notin E(G)$.

Let if possible $a_{32} a_{12} \in E(G)$. Then $G$ has no other 2-vertex $a_{22} \notin \{A_2, a_{31}\}$ (else $a_{22}, a_{12}, A_2 > 4K_i$). Also $a_{22} b_{11} \in E(G)$ (else $A_2$ is the only 2-vertex of $b_{11}$. If $b_{11}$ is the only i-vertex of $A_2$, then color $b_{11}$ by 2, $A_2$ by 2, $A_2$ by 2, $A_2$ by 2, $u$ by 2 and if $A_2 a_{12} \in E(G)$, then color $b_{11}$ by 2, $A_2$ by 2, $A_2$ by 2, $u$ by 4, contradictions in both the cases. Hence $a_{22} a_{12} \notin E(G)$.

$\Rightarrow a_{32} b_{11} \in E(G)$ (else color $b_{11}$ by 1, $a_{11}$ by 1, $a_{21}$ by 1, $A_1$ by 2, $u$ by 1).

Next let if possible $A_2 a_{12} \notin E(G)$. Then $b_{12}$ is the only i-vertex of $A_2$ and $A_2 B \notin E(G)$. Also $A_2$ is the only 2-vertex of $a_{12}$ and hence $G$ has no other 2-vertex say $a_{22}$ (else $a_{22}, a_{12}, A_1, a_{11} > 4K_i$) $\Rightarrow B a_{21} \in$
E(G) (else color B by 2, A1 by i, u by 1). As a21 has two 1-vertices and i-vertices, a23\(b_2 \notin E(G)\). Also as A2 has two 1-vertices A2\(b_2 \notin E(G)\). Color b2 by 2, a12 by i. If a12 is the only 1-vertex of A4, then color A4 by 1, u by 4 and if A\(a_{41} \in E(G)\), then color a11 by 4, A4 by 1, u by 4, contradictions in both the cases. Hence A2\(a_{12} \notin E(G)\). This proves Claim 3.

Claim 4: Whenever A1 has a unique i-vertex B for 6\(\leq \Delta -1\), either A2 or A3 has two i-vertices.

Let \(b_{11}\) be the only i-vertex A2. Now \(b_{11}\) is not the only i-vertex of a21 (else \(<a_{21}, A_2, b_{21}, B > = 4K_1\)). Thus a21 has two i-vertices. This proves Claim 4.

Now as \(\Delta \geq 9\), and A1 has at the most one repeat color, A1 has at least two unique k-vertices for \(k \in \{6, 7, ..., \Delta -1\}\). Let B, C be the unique i-vertex, k-vertex of A1 resp by i, k \{6, 7, ..., \Delta -1\}. Again as a31 has two 1-vertices, each of A2 and a31 has at the most one other repeat color. By Claim 4, w.l.g. let A2, a21 have two i-vertices, k-vertices resp. \(\Rightarrow\) A2, a21 has a unique 4-vertex each. Similarly A3, a31 has a unique 2-vertex each. Now A3\(a_{11} \notin E(G)\) (else color a11 by 2, A2 by 4, u by 2). Also A2\(A_4 \notin E(G)\) (else color A4 by 2, A2 by 4, a11 by 2, a21 by 1, A2 by 2, u by 1) \(\Rightarrow a_{21}a_{11} \notin E(G)\) (else \(<a_{21}, a_{11}, A_2, A_4 > = 4K_1\)). As a11 is the unique 1-vertex of A2, color a11 by 2, a21 by 4, a11 by 2, A2 by 1, u by 2, a contradiction.

This proves (C).

If A\(A_j \notin E(G)\) (1\(\leq i, j \leq \Delta -2\)), then as \(\Delta -2 \geq 7\), by (C), \(\exists m (1 \leq m \leq \Delta -2)\) s.t. A\(A_m A_j A_m \in E(G)\). Also by (B), \(\exists m\) maximum two m’s (1\(\leq m \leq \Delta -2\)).

Case 1.1: \(\exists i, j\) s.t. A\(A_i \notin E(G)\) and A\(A_i A_m A_j A_k A_m \in E(G)\), 1\(\leq i, j, k, m \leq \Delta -2\).

W.l.g. let i = 1, j = 2, k = 5, j = 6. Also by (C), let A\(A_i A_4, A_2 A_7 \in E(G)\). Then by (B), A\(A_i A_7, A_3 A_i \notin E(G)\). By (A), w.l.g. let A1, A2 have two 5-vertices, 6-vertices resp. Clearly A\(A_i (A_3)\) is the unique 4-vertex (7-vertex) of A1 (A2). Also by (C), A\(A_i\) is adjacent to at least one of A, i \{3, 4, 6\} and if A\(A_i A_4 \in E(G), i \{3, 4, 6\}\), then A\(A_i\) has two i-vertices (else A\(A_i A_4 \in E(G)\)), i \{3, 4, 6\}, then A\(A_i\) has two i-vertices (else A\(A_i A_4 \in E(G)\)), A\(A_i\) is adjacent to at least three of A4, A5, A6, A7 and either A3, A4 or A3, A4 have a common adjacent A, s.t. A1 is their only i-vertex, a contradiction to (A). W.l.g. let A\(A_3 A_1 \in E(G)\). Again A\(A_3\) is the unique 3-vertex of A1. Now \(\exists 2\) i paths from A3 to A1 (i = 1, 3, 4). Also as G is 4K1-free, \(\exists\) at most two more 2-vertices a21, a22 and at least one of them say a21 has two repeat colors from {1, 3, 4}.

Case 1.1.1: a21A3, a21A4 \(\notin E(G)\) and a21 has two repeat colors 3, 4.

Then a22\(A_1 \in E(G)\) and a22 has two 1-vertices and a22 is the only 2-vertex of A1. W.l.g. let a22 have a unique 3-vertex (else a22 has a color r missing in N(a22)). Color a22 by r, A1 by 1, u by 1). Then a22\(A_3 \notin E(G)\) (else color a22 by 3, A1 by 1, A1 by 2, u by 3). Consider a 3-2 path T from A3 to A2 with a31 being the 3-vertex of A2 on T. As a22 has a unique 3-vertex, clearly a22\(A_31 \in E(G)\). Now a22\(A_31 \in E(G)\) (else alter colors along \{A2, a31, a21, A1\}, color A1 by 3, u by 1). Then G does not have a 3-vertex a22 \(\notin \{A_1, a_{31}\}\) (else \(<A_2, a_{22}, a_{31}, A_1 > = 4K_1\)). Now a22\(A_{31} \in E(G)\) (else A3 is the only 3-vertex of both A1 and A7, contrary to (A)). But as a31 has three 2-vertices, A7 is its only 7-vertex. Also by (C), A7 is adjacent to at least one A\(j \in \{3, 4, 6\}\) and has two j-vertices (else A\(j\) is the only j-vertex of A1 and A7, contrary to (A)). Hence A7 is the only 2-vertex of A7. Then color a31 by 7, A7 by 2, A2 by 3, u by 7, a contradiction. This proves Case 1.1.1.

Case 1.1.2: A3, A4 do not have a common adjacent 2-vertex.

W.l.g. let a21\(A_1, a_{21}A_3 \in E(G)\) and a22\(A_4 \in E(G)\). Then a21\(A_4, a_{22}A_3 \notin E(G)\). Clearly a21 has two 1-vertices and 3-vertices and hence a unique 4-vertex. Let a41 be the unique 4-vertex of A2. Then as \(\exists a\) 2-4 path S from A2 to A4, clearly a23a41 \(\notin E(G)\). Now a23a41 \(\notin E(G)\) (else G does not have a 4-vertex a23 \(\notin \{a_{11}, A_1\}\), as otherwise \(<A_2, a_{21}, a_{22}, A_1 > = 4K_1 \Rightarrow A_1a_{41} \in E(G)\) as otherwise A7 and A1 have a common unique 4-vertex A1, a contradiction to (A). But then color A1 by 2, A2 by 4, A1 by 7, u by 7). Let a42 be the unique 4-vertex of a21. Then a42a22 \(\in E(G)\) (else alter colors along \{A2, a32, a41, A2\}, color A1 by 4, u by 1). Thus a22 has three 4-vertices and hence a unique i-vertex for 1\(\leq i < \Delta -1\), i \{2, 4\}. Now A3 has a unique j-vertex for j = 1 or 3. Consider a 2-j path T from A2 to A4 and let a4j be the
unique j-vertex of A₂. Clearly a₂₁a₁₁ ∈ E(G). Again a₂₂a₁₁ /∈ E(G) (else G does not have a j-vertex a₂
∈ {a₁, A₁}, as otherwise <a₂, a₂₂, a₁₁, A₁> = 4K₁ ⇒ a₁₁a₁₁ ∈ E(G). Color A₁ by 2, A₂ by j, a₁₁ by 7, u by 7). Hence ∃ a₂₁ s.t. a₂₂a₂₁ ∈ E(G) (else color a₂₁ by j, A₁ by 2, u by 4). Now clearly a₂₂ is the unique 2-
vertex of a₂ and vice versa ⇒ A₁a₂₂ ∈ E(G) (else color a₂₁ by j, a₁₁ by 2, A₁ by 2, u by 4). Clearly A₁ has two 1-vertices (else A₁ is its unique 1-vertex. Alter colors along {A₂, a₂₂, a₁₁, A₁}, color A₁ by 4, u by 1) ⇒ j = 3 and a₁₁ is the unique 3-vertex of A₁ ⇒ A₁a₁₁ /∈ E(G). Then by (C), A₁ is adjacent to at least two Aₖ for k ∈ {5, 6, 7}. Let A₄Aₕ ∈ E(G), for m = 5 or 7. Then Aₕ is the unique m-vertex of A₁ and A₂, a contradiction to (A).

**Case 1.2:** ∀ i, j s.t. AᵢAᵢ /∈ E(G), Aᵢ, Aᵢ have only one common adjacent Aᵢ in N(u), 1 ≤ i, j, k ≤ Δ-2.
W.l.g. let i = 1, j = 2 and k = 3. By (C), let A₁Aₚ ∈ E(G) for m = 4, 5, A₂Aₚ ∈ E(G) for l = 6, 7. **Let if possible** AᵢAᵢ /∈ E(G). Now Aᵢ is adjacent to at the most one of Aₖ, Aₙ (else we get **Case 1.1** with Aₖ and Aₙ) and hence by (C), AₐAₖ ∈ E(G). Also by (C), w.l.g. let A₂ₖ ∈ E(G). Again Aₐₖ ∈ E(G) (else we get **Case 1.1** with Aₖ and Aₙ) and hence by (C), AₐₖAₖ ∈ E(G) ⇒ Aₐₖ /∈ E(G) (else we get **Case 1.1** with Aₖ and Aₙ) and Aₐₖ /∈ E(G). But then we get **Case 1.1** with A₁ and Aₖ, a contradiction. Hence AᵢAᵢ ∈ E(G) for 4 ≤ i ≤ 7. Again AᵢAₖ, AₖAₖ ∈ E(G) (else we get **Case 1.1** with Aₖ, Aₖ or Aₖ, Aₖ). Also either both A₁ have two 3-vertices for 1 ≤ i ≤ 3 ≤ 7 or say A₁ has a unique 3-vertex. Again if A₁ has a unique 3-vertex, then A₂, A₄, A₅ all have two 3-vertices (else a contradiction to (A)). Hence w.l.g. let A₁, A₄, A₅ have two 3-vertices. As G is 4K₁-free, G has at the most two 2-vertices say a₂₁ (i = 1, 2). W.l.g. let A₁a₂₁ ∈ E(G). Now a₂₁ has at the most two repeat colors (else a color say r is missing in N(a₂₁). Color a₂₁ by r, A₁ by 2, u by 1). Also as ∃ i-2 paths from Aᵢ to A₂ for i = 1, 4, 5, either a₂₁ or a₂₂ has two j-vertices for j = 1, 2, 6, 7. W.l.g. let a₂₁ have two repeat colors 1, 4 with A₁a₂₁, Aₐ₅a₅ ∈ E(G) ⇒ aₕa₂₂ ∈ E(G) and a₂₂ has two 5-vertices. Again at least two of {1, 4, 5} are unique colors of A₂.

**Case 1.2.1.** A₂ has a unique 1-vertex and 5-vertex.
Let A₁a₂₁, A₂a₂₁ ∈ E(G). As a₂₁ has two repeat colors 1, 4, it has a unique 5-vertex and clearly as ∃ 2-5 path from Aₚ to Aₚ, a₂₀aₚ ∈ E(G). Now a₂₀aₚ ∈ E(G) (else G doesn’t have a 5-vertex a₂₀ /∈ {Aₚ, aₚ}) as otherwise <Aₚ, a₂₀, Aₚ, a₂₁> = 4K₁. As a₂₁ has three 2-vertices, Aₚ is its only 6-vertex. Also aₚ has the only 5-vertex of Aₚ. Color aₚ by 6, Aₚ by 5, u by 6) ⇒ aₐₚaₐₚ ∈ E(G). Also a₌ₕaₚ ∈ E(G) (else color Aₚ by 5, aₚ by 2, a₂₁ by 6, Aₚ by 2, u by 1). But then a₂₂₁a₂₁ ∈ E(G) (else G doesn’t have a 1-
vertex a₂₁ /∈ {A₁, aₐ₁} as otherwise <A₁, a₁₂, A₁, a₂₁> = 4K₁ and a₁₁ has three repeat colors 2, 6, 7 with color say r is missing in N(a₁₁). Color a₁₁ by r, A₁ by 1, u by 2) ⇒ a₁₁a₂₁ ∈ E(G). Let a₂₁a₂₁ ∈ E(G). Then a₂₂ (a₁₂) is the only 2-vertex (1-vertex) of a₁₂ (a₂₂). Color a₂₁ by 1, a₁₂ by 2, Aₕ by 2, u by 5, a contradiction.

**Case 1.2.2.** A₂ has a unique 1-vertex and 4-vertex.
Let A₁a₁₁, A₂a₁₁ ∈ E(G). As A₂ has two 5-vertices, w.l.g. let a₂₂ have a unique 1-vertex. Then a₂₂₁a₁₁ /∈ E(G) (else if ∃ a₁₂, then <A₁, a₁₂, A₂, a₂₂> = 4K₁ and if a₁₂ doesn’t exist, then a₁₁ has three repeat colors 2, 6, 7 and color say r is missing in N(a₁₁). Color a₁₁ by r, A₁ by 2, u by 2 ⇒ a₂₁a₂₁ ∈ E(G) and a₂₁a₁₁ ∈ E(G). Then a₂₂ (a₁₂) is the only 2-vertex (1-vertex ) of a₁₂ (a₂₂). Color a₂₁ by 1, a₁₂ by 2, Aₕ by 2, u by 5, a contradiction.

This proves **Case 1.**

**Case 2:** In every (Δ-1)-coloring of G-u, all vertices with unique colors in N(u) are adjacent.

Clearly Δ-1 ≤ ω and hence Δ-1 = ω ≥ 8 ⇒ \( \bigcup_{i=1}^{\Delta-2} A_i \) is a maximum clique in G-u and \( \{X, Y\} = N(u) \)
\( \bigcup_{i=1}^{\Delta-2} A_i \).

I. At most two vertices in \( \bigcup_{i=1}^{\Delta-2} A_i \) are non-adjacent to both X and Y.
Let if possible \( A_1, A_2, A_3 \) be non-adjacent to both \( X \) and \( Y \). Then clearly \( \exists \) a \((\Delta-1)\)-vertex say \( Z \) in \( V(G) \) s.t. \( ZA_i \in E(G) \) for \( i = 1, 2, 3 \). Moreover, as \( G \) is \( 4K_4 \)-free, \( Z \) is their only \((\Delta-1)\)-vertex. If \( A_i \) is the only \( i \)-vertex of \( Z \) for some \( i \) (\( 1 \leq i \leq 3 \)), then color \( A_i \) by \((\Delta-1)\), \( Z \) by \( i \), \( u \) by \( i \), a contradiction. Hence \( Z \) has at least two \( i \)-vertices for \( i = 1, 2, 3 \). But then \( Z \) has some color \( r \) missing in \( N(Z) \). Color \( Z \) by \( r \), \( A_i \) by \((\Delta-1)\), \( u \) by \( i \), a contradiction.

II. Every vertex \( A_i \) of \( N(u) \) has at least one \( j \)-vertex \( j \neq i \) (else color \( A_i \) by \( j \) and \( u \) by \( j \)). \( 1 \leq i, j \leq \Delta-2 \).

III. \( X \) has a \( k \)-vertex for every \( k = 1, ..., \Delta-2 \).

Let if possible \( X \) not have a \( k \)-vertex. Also as \( \cup_{i=1}^{\Delta-2} A_i \) is a maximum clique in \( G \), \( \exists i \) \((1 \leq i \leq \Delta-2)\) s.t. \( YA_i \notin E(G) \). Then color \( X \) by \( k \). Now \( i = k \) (else we get Case 1 as \( Y \) and \( A_i \) are unique vertices in \( N(u) \)). As \( \Delta \geq 9 \) and each of \( Y \) and \( A_i \) has at the most one repeat color, clearly \( \exists j \) \((1 \leq j \leq \Delta-2)\) s.t. \( A_j \) is the only \( j \)-vertex of both \( Y \) and \( A_i \). Also \( A_j \) has either a unique \( i \)-vertex \( A_i \) or \((\Delta-1)\)-vertex \( Y \). Color \( Y \) and \( A_i \) by \( j \), \( A_j \) by \((\Delta-1)\), \( u \) by \((\Delta-1) \) \((j)\), a contradiction.

IV. \( X \) is adjacent to at least \( \omega-5 \) vertices in \( \bigcup_{i=1}^{\Delta-2} A_i \).

Let if possible \( X \) be non-adjacent to \( A_i \), \( i = 1, ..., 5 \). By I, w.l.g. let \( YA_i \in E(G) \) for \( i = 1, 2, 3 \). Also let \( YA_k \notin E(G) \) for some \( k \geq 4 \). By II and III, \( Y \) and \( A_k \) each has at the most one repeat color and hence w.l.g. let \( A_1 \) be the unique \( 1 \)-vertex of \( Y \) and \( A_k \). Now \( A_1 \) has two \((\Delta-1)\)-vertices (else color \( Y \) and \( A_k \) by \( 1 \), \( A_1 \) by \((\Delta-1)\), \( u \) by \( k \)) \( \Rightarrow A_k \) is the unique \( k \)-vertex of \( A_i \). Then color \( Y \) and \( A_k \) by \( 1 \), \( A_1 \) by \( k \) and we get Case 1 with two non-adjacent, unique vertices \( X \), \( A_i \), a contradiction.

V. \( X \) is not the only \((\Delta-1)\)-vertex of any \( A_i \).

Let if possible \( X \) be the only \((\Delta-1)\)-vertex of some \( A_i \). By IV, \( \exists j, k \) s.t. \( XA_k \), \( XA_j \in E(G) \). Also let \( XA_m \notin E(G) \) for some \( m \). If \( A_i \) is the only \( i \)-vertex of \( X \) and \( A_m \), then color \( X \), \( A_m \) by \( i \), \( A_i \) by \((\Delta-1)\), \( u \) by \( m \), a contradiction. Hence let \( A_i \) be not the only \( i \)-vertex of either \( X \) or \( A_m \). As \( X \) and \( A_m \) have at the most one repeat color, w.l.g. let \( A_i \) be the only \( k \)-vertex of \( X \) and \( A_m \).Again if \( X \) is the only \((\Delta-1)\)-vertex of \( A_i \), then as before we get a contradiction. Hence let \( A_i \) have two \((\Delta-1)\)-vertices. But then color \( A_i \) by \( i \), \( A_i \) by \((\Delta-1)\), \( X \) by \( k \), \( A_m \) by \( k \), \( u \) by \( m \), a contradiction.

By IV, w.l.g. let \( XA_k \in E(G) \) for \( k = 1, 2, 3 \) and \( XA_4 \notin E(G) \). Also w.l.g. let \( A_1 \) be the only \( 1 \)-vertex of \( X \) and \( A_4 \). By V, \( A_1 \) has two \((\Delta-1)\)-vertices. If any \( A_i \) \((1 \leq i \leq \Delta-2, i \neq 4) \) is non-adjacent to \( Y \), then as before by coloring \( X \), \( A_4 \) by \( 1 \) and \( A_i \) by \( 4 \), we get Case 1 and hence \( YA_k \in E(G) \) for every \( k \neq 4 \). Similarly \( XA_4 \in E(G) \) for every \( k \neq 4 \). As \( \Delta \geq 9 \), \( \exists i \) s.t. \( A_i \) is the only \( i \)-vertex of \( X \), \( Y \) and \( A_i \). Color \( X \), \( Y \), \( A_i \) by \( i \), \( A_i \) by \((\Delta-1)\), \( u \) by \( 4 \), a contradiction.

This proves Case 2 and completes the proof of the Main Result.

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