New integral formulas and identities involving special numbers and functions derived from certain class of special combinatorial sums

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Abstract
By applying $p$-adic integral, in Simsek (Montes Taurus J Pure Appl Math 3(1):38–61, 2021), we constructed generating function for the special numbers and polynomials involving novel combinatorial sums and numbers. The aim of this paper is to use these combinatorial sums and numbers to derive various new formulas and relations associated with the Bernstein basis functions, the Fibonacci numbers, the Harmonic numbers, the alternating Harmonic numbers, the Bernoulli polynomials of higher order, binomial coefficients and new integral formulas for the Riemann integral. We also investigate and study on open problems involving these numbers. Moreover, we give relation among these numbers, the Digamma function, and the Euler constant. Moreover, by applying special values of these combinatorial sums, we give decomposition of the multiple Hurwitz zeta function which interpolates the Bernoulli polynomials of higher order. Finally, we give conclusions for the results of this paper with some comments and observations.

Keywords Generating function · Special numbers and polynomials · Bernoulli-type numbers and polynomials · Fibonacci numbers · Harmonic numbers · Stirling numbers · Daehee numbers · Digamma function · Hurwitz zeta function.

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1 Introduction
Due to the works in [1,24], we see that Daniel Bernoulli and Goldbach studied not only on interpolating a sequence, but also on the partial sums of the harmonic series. But these studies were not without much success. Bernoulli and Goldbach were working the following partial sums of the harmonic series
\[ f(v) = \sum_{j=1}^{v} \frac{1}{j}, \]

where \( v \) is a positive integer. In one of his studies, Euler found many formulas on these topics involving the Digamma function, the Euler constant, interpolating a sequence and partial sums of the harmonic series. We can find these clues in Euler’s letter to Christian Goldbach dated October 13, 1729. In this letter, Euler mentioned the formulas he found and sent the following formula, one of them, to Goldbach without any proof:

\[ f \left( \frac{1}{2} \right) = 2 - 2 \log(2) \]

(cf. [24, p.136], [1, pp. 283–284]). In [24, p.136] and [1, pp. 283–284], the following formula was given:

\[ f \left( \frac{1}{2} \right) = \int_{0}^{1} \frac{1 - \sqrt{x}}{1 - x} \, dx. \]

The motivation of this paper is of come from the spirit of historical mathematical works of Bernoulli, Euler and Goldbach with the help of applications of the following combinatorial numbers which was defined in [35]:

\[ y(n, \lambda) = \sum_{j=0}^{n} \frac{(-1)^n}{(j + 1)\lambda^{j+1}} (\lambda - 1)^{n+1-j}. \]  

(1)

It would be very appropriate to give a brief information about how the numbers \( y(n, \lambda) \) can be found. By applying \( p \)-adic integral on the set of \( p \)-adic integers in [35], we constructed generating function for the special numbers and polynomials involving combinatorial sums and many special numbers and polynomials. When we investigated and studied on interpolation functions for these new classes special numbers and polynomials with series algebraic operations, we performed on one of these series multiplications with its inspiration, the numbers \( y(n, \lambda) \) given in Eq. (1) were defined. In [35], some of their properties for these numbers were discussed and given.

In [35], we have put forward the following open problems for the numbers \( y(n, \lambda) \):

1. One of the first questions that comes to mind what is generating function for the numbers \( y(n, -1) \) and the numbers \( y(n, \lambda) \).
2. Some of the other questions are what are the special families of numbers the numbers \( y(n, -1) \) are related to.
3. What are the combinatorial applications of the numbers \( y(n, -1) \).
4. Can we find a special arithmetic function representing this family of numbers?

Partial solutions to some of the above open problems can be explored and studied. By using (1), we can prove the following theorems including formulas and identities related to the numbers \( y(n, \lambda) \), the harmonic numbers, the Fibonacci numbers, the Bernstein basis functions, and the Riemann integral:

**Theorem 1** Let \( n \in \mathbb{N}_0 \). Then we have

\[ y(n, \lambda) = \frac{-1}{\lambda} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n+1}{j} B_k S_1(j, k) \frac{B_{n+1}^k}{(\lambda)}, \]  

(2)
where $B_{j}^{n}(\lambda)$, $B_j$, and $S_1(j, k)$ denote the Bernstein basis function and the Bernoulli numbers, and the Stirling numbers of the first kind, respectively.

**Theorem 2** Let $n \in \mathbb{N}_0$. Then we have

$$y(n, \lambda) = \frac{(-1)^n}{(\lambda - 1)^{n+2}} \int_0^{\frac{\lambda - 1}{\lambda}} \frac{1 - x^{n+1}}{1 - x} \, dx.$$  

**Theorem 3** Let $n \in \mathbb{N}_0$. Then we have

$$y \left( n, \frac{1}{2} \right) = 2^{n+2} \left( H_{\frac{n}{2}} - H_n + \frac{(-1)^{n+1}}{n+1} \right),$$  

(3)

where $H_n$ denotes the harmonic numbers and $H_0 = 0$, and $[x]$ denotes the integer part of $x$.

**Theorem 4** Let $n \in \mathbb{N}_0$. Then we have

$$y \left( n, \frac{1}{2} \right) = \frac{1}{2^{n+2}} \int_0^{1} \frac{1 + (-1)^n x^{n+1}}{1 + x} \, dx.$$  

(4)

**Theorem 5** Let $n \in \mathbb{N}_0$. Then we have

$$\int_0^{1} \frac{1 - x^{n}}{1 - x} \, dx - \int_0^{1} \frac{1 - x^{n+1}}{1 - x} \, dx = \frac{y \left( n, \frac{1}{2} \right) + (-1)^n}{2^{n+2}}.$$  

(5)

**Theorem 6** Let $n \in \mathbb{N}_0$. Then we have

$$y \left( n, \frac{1 + \sqrt{5}}{2} \right) = \sum_{j=0}^{n} \frac{(-1)^{j+1}}{(j+1) \left( \frac{1 + \sqrt{5}}{2} F_{j+1} + F_j \right) \left( \frac{1 + \sqrt{5}}{2} F_{n-j+1} + F_{n-j} \right)},$$  

(6)

where $F_n$ denotes the Fibonacci numbers.

Combinatorial sums and numbers, special numbers and polynomials are the main topics that can be used in almost all areas of mathematics. In addition, they are also one of the most frequently used topics in applied sciences involving mathematical modeling and algorithmic solutions for real-world problems.

We give the following some basic standard notations, formulas, and definitions:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers, the set of integer numbers, the set of real numbers and the set of complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Supposing that for $z \in \mathbb{C}$ with $z = x + iy$ ($x, y \in \mathbb{R}$); $\text{Re}(z) = x$ and $\text{Im}(z) = y$ and also $\log z$ denotes the principal branch of the many-valued function $\text{Im}(\log z)$ with the imaginary part of $\log z$ constrained by

$$-\pi < \text{Im}(\log z) \leq \pi,$$

and $\log e = 1$.

$$\mathbb{0}^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N} \end{cases}.$$
The Bernoulli polynomials of order $k$, $B_n^{(k)}(x)$, are defined by the following generating function:

$$
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},
$$

where $|t| < 2\pi$ (cf. [1,3,5–8,10–17,19–35,38,39]; and references therein).

When $x = 0$ in (7), we have the Bernoulli numbers of order $k$:

$$
B_n^{(k)} = B_n^{(k)}(0),
$$

and putting $k = 1$, we have the Bernoulli numbers, $B_n$:

$$
B_n = B_n^{(1)},
$$

(cf. [1,3,5–8,10–17,19–35,38,39]; and references therein).

Harmonic numbers, denoted by $f(n)$ in Euler’s works and also in generally represented with the symbol $H_n$ and $H(n)$, are defined by means of the following generating function:

$$
\sum_{n=1}^{\infty} H_n z^n = \log \left( \frac{1}{1-z} \right) \frac{z}{z-1},
$$

where $|z| < 1$ (cf. [1,5,7,21,24,38,39]).

In [1,24,38], the Harmonic numbers are given by the following integral representation:

$$
H_n = \frac{1}{0} \frac{1 - x^n}{1-x} dx,
$$

and

$$
H_n = -n \int_0^1 x^{n-1} \log(1-x) dx = -n \int_0^1 (1-x)^{n-1} \log(x) dx,
$$

where $H_0 = 0$.

The Bernstein basis function $B_n^j(\lambda)$ is defined by

$$
B_n^j(\lambda) = \binom{n}{j} \lambda^j (1-\lambda)^{n-j},
$$

where $j \in \{0, 1, 2, \ldots, n\}$ (cf. [19]).

The Stirling numbers of the first kind, $S_1(n,k)$, are defined by the following generating function:

$$
\frac{(\log(1+z))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k) \frac{z^n}{n!},
$$

with $S_1(n,k) = 0$ if $k > n$, and $k \in \mathbb{N}_0$ (cf. [1,3,5–8,10–17,19–35,38,39]; and references therein).

In [11,12], with the aid of the Volkenborn integral on the set of $p$-adic integers, Kim defined the Daehee numbers $D_n$ by the following generating function:

$$
\frac{\log(1+z)}{z} = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!},
$$
where \( z \neq 0 \) and \( |z| < 1 \) (cf. [11,12]).

By using the above equation, the following explicit formula for the Daehee numbers \( D_n \) is found:

\[
D_n = (-1)^n \frac{n!}{n+1}
\]

(12)

(cf. [11,12]).

Substituting \( t = \log(1 + z) \), \( x = 0 \), and \( z \neq 0 \) into (7), by using Riordan’s method given in [21, p. 45, Exercise 19 (b)], after some elementary calculations, we have

\[
\log(1 + z) = z \sum_{n=0}^{\infty} \frac{B_n}{n!} (\log(1 + z))^n
\]

(13)

Combining the above equation with (11), after some elementary calculations, assuming that \( |z| < 1 \), we get

\[
\sum_{m=1}^{\infty} (-1)^{m+1} \frac{z^{m-1}}{m} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} B_n S_1(m, n) \frac{z^m}{m!}
\]

such that we here use the fact that \( S_1(m, n) = 0 \) if \( n > m \). After some elementary calculations, equating the coefficients of \( z^m \) on both sides of the previous equation, one has the following well-known novel formula, which has been proven by other different methods:

\[
\sum_{n=0}^{m} \frac{B_n S_1(m, n)}{m+1} = (-1)^m m!
\]

(14)

(cf. [3,12, p. 117], [21, p. 45, Exercise 19 (b)]).

There are many other proofs of (14). One of them was given by Kim [12] with the \( p \)-adic invariant integral on the set of \( p \)-adic integers. On the other hand, the relation is given by Eq. (14), Riordan [21, p. 45, Exercise 19 (b)] represented it by the notation \((b)_n\).

By using (12) and (14), in the next section, we give relations among the numbers \((b)_n\), the Daehee numbers \( D_n \), and the numbers \( y(n, \lambda) \).

1.1 Further remarks and observations on the numbers \( y(n, \lambda) \)

Substituting \( \lambda = 2 \) into (1), in [35] we defined the following combinatorial numbers:

\[
y(n) := y(n, 2) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)2^{j+1}}.
\]

(15)

In [35], we also give a relationship between the numbers \( y(n) \) and the \( \lambda \)-Apostol–Daehee numbers.

Putting \( \lambda = -1 \) in (1), we have

\[
y(n, -1) = \frac{1}{2(n + 1)} \sum_{j=0}^{n} \frac{1}{(j)}
\]

(16)

(cf. [35]).
Remark 1  Recently many authors have studied on binomial coefficients involving the following well-known sums and their applications:

\[ \sum_{j=0}^{n} \binom{n}{j}, \]
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j}, \]
\[ \sum_{j=0}^{n} \binom{n}{j} j^k \]

and so on (cf. [5,7,15–17,21,23,25–32,34,35,41]).

Remark 2  Most recently, in [32, Equations (3), (13), and (14)], we constructed the following generating function for the numbers \( y_6(m, n; \lambda, p) \) involving finite sums involving higher powers of binomial coefficients

\[ pF_{p-1}\left[ \begin{array}{c} -n, -n, \ldots, -n \\ 1, 1, \ldots, 1 \end{array} \right] (-1)^p \lambda e^t = n! \sum_{m=0}^\infty y_6(m, n; \lambda, p) \frac{t^m}{m!}, \]

where

\[ n!y_6(m, n; \lambda, p) = \sum_{k=0}^{n} \binom{n}{k} p^k \lambda^k, \]

and \( pF_q \) denotes the well-known generalized hypergeometric function which is defined by

\[ pF_q\left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} ; z \right] = \sum_{m=0}^\infty \left( \prod_{j=1}^{p} \binom{\alpha_j}{m} \prod_{j=1}^{q} \binom{\beta_j}{m} \right) \frac{z^m}{m!}, \]

where the above series converges for all \( z \) if \( p < q + 1 \), and for \( |z| < 1 \) if \( p = q + 1 \). Assuming that all parameters have general values, real or complex, except for the \( \beta_j, j = 1, 2, \ldots, q \) none of which is equal to zero or a negative integer and also

\[ (\lambda)^v = \prod_{j=0}^{v-1} (\lambda + j), \]

and \((\lambda)^0 = 1 \) for \( \lambda \neq 1 \), where \( v \in \mathbb{N}, \lambda \in \mathbb{C} \). For the generalized hypergeometric function and their applications see (cf. [13,32,42]; and references therein).

Here, we noting that (in future studies) the relationships between the numbers \( y(n, \lambda) \) and the numbers \( y_6(0, n; \lambda, p) \), which have potential to be used in not only in mathematics but also in other areas, may also be investigated.

Let’s briefly summarize the next sections of the article.

In Sect. 2, proofs of the theorems given in the first section are given.

In Sect. 3, with the help of the numbers \( y(n, \lambda) \), we give some formulas involving the Bernoulli numbers, the Stirling numbers, the Psi (or Digamma) function and the Euler constant.
In Sect. 4, with the aid of the numbers $y(n, \lambda)$, we give decomposition of the multiple Hurwitz zeta functions involving the Bernoulli polynomials of higher order. The article is completed with a conclusion section.

## 2 Proofs of main Theorems

In this section, the proofs of the theorems from Theorems 1 to 6, as well as the new novel formulas and relations derived in the light of the results of these theorems are given.

**Proof of Theorem 1** From (1), we get

$$y(n, \lambda) = \frac{1}{\lambda} \sum_{j=0}^{n} (-1)^{j+1} \frac{(n+1)}{j} \frac{1}{(j+1)} B_{j+1}^{n+1}(\lambda).$$

(17)

Combining (17) with (14) and (12), after some elementary calculations, we arrive at the desired result. $\square$

**Proof of Theorem 2** By using (1), we give integral representation for the numbers $y(n, \lambda)$. We set

$$y(n, \lambda) = \frac{(-1)^n}{(\lambda - 1)^{n+2}} \sum_{j=0}^{n} \int_{0}^{\frac{\lambda-1}{j}} x^j dx.$$  

(18)

By using the above equation, we also get

$$y(n, \lambda) = \frac{(-1)^n}{(\lambda - 1)^{n+2}} \int_{0}^{\frac{\lambda-1}{j}} \frac{1 - x^{n+1}}{1 - x} dx.$$  

(19)

Combining (18) and (19), proof is completed. $\square$

**Proof of Theorem 3.** Putting $\lambda = \frac{1}{2}$ in (1), we have

$$y\left(n, \frac{1}{2}\right) = 2^{n+2} \sum_{j=0}^{n} \frac{(-1)^{j+1}}{j+1}.$$  

(20)

We know from Sofo’s work [38, Eq. (1.5)] that the right-hand side of the previous equation gives us a formula of the following well-known alternating harmonic numbers, which are related to the Riemann zeta function, the Polylogarithm (or de Jonquière’s) function, the Psi (or Digamma) function and the other special functions:

$$\sum_{j=1}^{n} \frac{(-1)^j}{j} = H_{\left[n + \frac{1}{2}\right]} - H_n.$$  

(21)

Combining (20) with (21), proof of Theorem 3 is completed. $\square$

**Proof of Theorem 4** Using (20), we get

$$y\left(n, \frac{1}{2}\right) = -2^{n+2} \sum_{j=0}^{n} (-1)^{j} \int_{0}^{1} x^j dx.$$  

(22)
Combining (22) with the following well-known formula
\[\sum_{j=0}^{n} (-x)^j = \frac{1 + (-1)^n x^{n+1}}{1 + x},\]
we obtain
\[y\left(n, \frac{1}{2}\right) = \frac{1}{-2^{n+2}} \int_{0}^{1} 1 + \frac{(-1)^n x^{n+1}}{1 + x} dx.\]

This last equation shows us that the proof of the theorem completed. \(\square\)

**Proof of Theorem 5** Combining (3) with (9), we get
\[y\left(n, \frac{1}{2}\right) = 2^{n+2} \left( \int_{0}^{1} \frac{1 - x^{[\frac{3}{2}]} - 1}{1 - x} dx - \int_{0}^{1} \frac{1 - x^n - 1}{1 - x} dx + \frac{(-1)^{n+1}}{n+1} \right).\]

After some elementary calculations in the above equation, we arrive at the desired result. \(\square\)

**Proof of Theorem 6** Putting \(\lambda = \frac{1+\sqrt{5}}{2}\) into (1), we get
\[y\left(n, \frac{1 + \sqrt{5}}{2}\right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j + 1) \left(\frac{1+\sqrt{5}}{2}\right)^{j+1} \left(\frac{1+\sqrt{5}}{2} - 1\right)^{n+1-j}}.\]

Combining the previous equation with the following well-known identities involving the Fibonacci numbers:
\[
\left(\frac{1 + \sqrt{5}}{2}\right)^{j+1} = \frac{1 + \sqrt{5}}{2} F_{j+1} + F_j
\]
(cf. [14]), after some elementary calculations, we arrive at the desired result. \(\square\)

### 3 Formulas and relations derived from the numbers \(y\left(n, \lambda\right)\)

In this section, we give some formulas involving the Bernoulli numbers, the Stirling numbers, the Psi (or Digamma) function and the Euler constant, which are derived from main theorems.

Combining (3) with (10), we arrive at the following theorem:

**Theorem 7** Let \(n \in \mathbb{N}_0\). Then we have
\[y\left(n, \frac{1}{2}\right) = 2^{n+2} \left( n \int_{0}^{1} (1 - x)^{n-1} \log(x) dx - \left[\frac{n}{2}\right] \int_{0}^{1} (1 - x)^{\left[\frac{3}{2}\right]-1} \log(x) dx \right) + \frac{(-1)^{n+1} 2^{n+2}}{n+1}.\]

We now give relation among the numbers \(y\left(n, \frac{1}{2}\right)\), the Digamma function and the Euler constant.
The Euler’s constant (or Euler–Mascheroni) constant is given by

$$\gamma = \lim_{m \to \infty} \left( -\log(m) + \sum_{j=1}^{m} \frac{1}{j} \right)$$

and the Psi (or Digamma) function

$$\psi(z) = \frac{d}{dz} \log(\Gamma(z)),$$

where $\Gamma(z)$ denotes the Euler gamma function, which is defined by

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt,$$

where $z = x + iy$ with $x > 0$. For $z = n \in \mathbb{N}$,

$$\Gamma(n + 1) = n!$$

(cf. [1,24,39]).

Sofo [38] gave the following formula:

$$H_n = \gamma + \psi(n + 1), \quad (23)$$

where $H_0 = 0$.

Combining (3) with (23), we arrive at the following theorem:

**Theorem 8** Let $n \in \mathbb{N}_0$. Then we have

$$\sum_{v=0}^{n} \frac{S_1(v, 1)}{v!} - \sum_{v=0}^{[\frac{n}{2}]} \frac{S_1(v, 1)}{v!} + \sum_{j=0}^{n} B_j S_1(n, j) = 2^{-n-2} y\left(n, \frac{1}{2}\right). \quad (26)$$

**Proof** Combining (8) with (11), we obtain

$$\sum_{n=1}^{\infty} H_n z^n = - \sum_{n=0}^{\infty} S_1(n, 1) \frac{z^n}{n!} \sum_{n=0}^{\infty} z^n.$$
Therefore
\[ \sum_{n=1}^{\infty} H_n z^n = - \sum_{n=0}^{\infty} \sum_{v=0}^{n} S_1(v, 1) \frac{z^n}{v!}. \]

We have the following well-known formula:
\[ H_n = - \sum_{v=0}^{n} \frac{S_1(v, 1)}{v!}. \] (27)

Combining (27) with (3) and (14) yield the desired result. \( \square \)

Since
\[ S_1(v, 1) = (v - 1)!, \]
after some elementary calculations, we see that (26) reduces to the following corollary:

**Corollary 2**
\[ \sum_{j=0}^{n} B_j S_1(n, j) = 2^{-n-2} y\left(n, \frac{1}{2}\right) + H_{\left[n\frac{1}{2}\right]} - H_n. \]

### 4 Decomposition of the multiple Hurwitz zeta functions with the help of the numbers \( y(n, \lambda) \)

In this section, by the aid of the numbers \( y(n, \lambda) \), we give so-called decomposition of the multiple Hurwitz zeta functions in terms of the Bernoulli polynomials of higher order.

Substituting \( \lambda = e^{-t} \) into (1), we get
\[ y\left(n, \frac{1}{e^t}\right) = \sum_{m=0}^{\infty} \sum_{j=0}^{n} (-1)^{j-1} \frac{B_{m+n+1-j}^{n+1-j}}{(j+1)(m+n+1-j)(n+1-j)!} t^m. \] (29)

Using (28), we also obtain
\[ y\left(n, \frac{1}{e^t}\right) = \sum_{j=0}^{n} (-1)^{j-1} \frac{B_{m+n+1-j}^{n+1-j}}{(j+1)} \sum_{v=0}^{\infty} \binom{v+n+j}{v} e^{t(v+n+2)}, \]
where it is tacitly assumed that \(|e^t| < 1\). Substituting Taylor series of \( e^{tx} \) into the above equation yields
\[ y\left(n, \frac{1}{e^t}\right) = \sum_{j=0}^{n} (-1)^{j-1} \frac{B_{m+n+1-j}^{n+1-j}}{(j+1)} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \binom{v+n+j}{v} (v+n+2)^m \frac{t^m}{m!}. \] (30)

After completing the necessary calculations in (29) and (30), the coefficient of \( \frac{t^m}{m!} \) are equalized, the following result, which includes Bernoulli polynomials of higher order and their interpolation function, is found:
Theorem 10  Let $m, n \in N_0$. Then we have
\[
\sum_{j=0}^{n} \frac{1}{j+1} \left( (-1)^n \sum_{v=0}^{\infty} \binom{v+n-j}{v} (v+n+2)^m + \frac{(-1)^j B_{m+n+1-j} (n+2)}{B_{n+1-j} (n+1-j)!} \right) = 0.
\] (31)

We now give a few special values of Eqs. (29), (30), and (31) as follows:
\[
y \left( n, \frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)} \zeta_{n+1-j} (-m, n+2) \frac{t^m}{m!},
\] (32)

where $\zeta_d (s, x)$ denotes the $d$-ple (multiple) Hurwitz zeta functions or the Hurwitz zeta function of order $d \in \mathbb{N}$, which defined by
\[
\zeta_{d} (s, x) = \sum_{v=0}^{\infty} \binom{v+d-1}{d-1} \frac{1}{(x+v)^s}
= \sum_{v=0}^{\infty} \frac{(v+d-1)}{(d-1)} \frac{1}{(x+v)^s}
= \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \ldots \sum_{v_d=0}^{\infty} \frac{1}{(x+v_1+v_2+\ldots+v_d)^s}
\]
where $\text{Re}(s) > d$, when $d = 1$, we have the Hurwitz zeta function
\[
\zeta (s, x) = \zeta_1 (s, x) = \sum_{v=0}^{\infty} \frac{1}{(x+v)^s},
\] (cf. [2,4,9,18,36,37,39,40]).

In [4], Choi gave the following formula for the function $\zeta_d (s, x)$:
\[
\zeta_d (s, x) = \frac{1}{(d-1)!} \sum_{v=0}^{\infty} \sum_{c=0}^{d-1} |S_1 (d, c+1)| v^c \frac{1}{(x+v)^s}. \quad (33)
\]

Combining (32) with (33), we obtain
\[
y \left( n, \frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)(n-j+1)!} \sum_{v=0}^{\infty} \binom{v+n+2}{v} \sum_{c=0}^{n-j} |S_1 (n-j, c+1)| v^c \frac{t^m}{m!}
\] (34)

where $n \in \mathbb{N}$.

The above formula was also studied by Bayad and Simsek [2], and Kim [9]. It is well-known that the function $\zeta_d (s, x)$ interpolates the Bernoulli polynomials of order $d$ at negative integers. That is
\[
\zeta_d (-m, x) = \frac{(-1)^d m! B_d (m+x)}{(d+m)!} \quad \text{and} \quad \zeta (-m, x) = \frac{B_{m+1} (x)}{m+1}. \quad (35)
\]
where \( m \in \mathbb{N}_0 \) (cf. \([2,4,9,18,36,39,40]\)).

Joining Eqs. (34) with (28) and (32), after some elementary calculations, we arrive at the following theorem:

**Theorem 11** Let \( m, n \in \mathbb{N} \). Then we have

\[
\sum_{j=0}^{n} \frac{1}{j+1} \left( (-1)^n \zeta_{n+1-j}(-m, n+2) + \frac{(-1)^j B_{m+n+1-j}^{(n+1-j)} \binom{n+1-j}{n+1-j} (n+2)}{m+n+1-j} (n+1-j)! \right) = 0. \tag{37}
\]

It is worth noting here that Eqs. (31) and (37) give the decomposition of the function \( \zeta_d(s, x) \). That is, Putting \( n = 0 \) in (37), we get

\[
-\frac{B_{m+1}(2)}{m+1} = \zeta(-m, 2).
\]

Substituting \( n = 1 \) into (37), we obtain

\[
2\zeta_2(-m, 3) - \zeta(-m, 3) = \frac{1}{(m+2)(m+1)} \left( 2B_{m+2}^{(2)}(3) + \frac{d}{dx} \{B_{m+2}(x)\} \big|_{x=3} \right).
\]

Substituting \( n = 2 \) into (37), we also obtain

\[
6\zeta_3(-m, 4) + 3\zeta_2(-m, 4) + 2\zeta(-m, 4) = -\frac{B_{m+3}^{(3)}(4)}{(m+3)(m+2)(m+1)} + \frac{B_{m+2}^{(2)}(4)}{2(m+2)(m+1)} - \frac{B_{m+1}(4)}{3(m+1)}.
\]

**Remark 3** Many other decompositions are obtained by continuing as above. It is known that the decomposition of the multiple Hurwitz zeta function is given by different techniques and methods in the literature. In this paper, we do not focus on the other kinds decompositions.

### 5 Conclusion

In this study, some properties of the numbers \( y(n, \lambda) \) with special finite sums containing these numbers have been found. Many new novel formulas and relations blended with well-known special functions including the Euler gamma function, the Euler–Mascheroni constant, the Psi function, special polynomials including Bernstein basis functions, and special numbers including the Fibonacci numbers, the Bernoulli numbers, the Euler numbers, the Stirling numbers, and the Daehee numbers, those of known by the names of well-known mathematicians. On the other hand, new formulas and integral representations of the numbers \( y(n, \lambda) \) have been proved by using integral representations of the harmonic numbers and the alternating harmonic numbers, which are based on very old studies and have rich properties and relationships, and formulas related to other special functions, involving the Digamma function, the Euler gamma function, and the Euler constant. In this study, by the aid of the numbers \( y(n, \lambda) \), we give so-called decomposition of the multiple Hurwitz zeta functions which interpolated the Bernoulli polynomials of the higher order. In this study, we also make some efforts to find partial solutions to some of the open problems for the numbers \( y(n, \lambda) \) given in \([35]\). In our next studies, it is planned to investigate the solutions of the above open questions, including the numbers \( y(n, \lambda) \).
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