Unitless Frobenius Quantales

Cédric de Lacroix1 · Luigi Santocanale1

Received: 3 June 2022 / Accepted: 5 December 2022 / Published online: 27 December 2022 © The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
It is often stated that Frobenius quantales are necessarily unital. By taking negation as a primitive operation, we can define Frobenius quantales that may not have a unit. We develop the elementary theory of these structures and show, in particular, how to define nuclei whose quotients are Frobenius quantales. This yields a phase semantics and a representation theorem via phase quantales. Important examples of these structures arise from Raney’s notion of tight Galois connection: tight endomaps of a complete lattice always form a Girard quantale which is unital if and only if the lattice is completely distributive. We give a characterisation and an enumeration of tight endomaps of the diamond lattices $M_n$ and exemplify the Frobenius structure on these maps. By means of phase semantics, we exhibit analogous examples built up from trace class operators on an infinite dimensional Hilbert space. Finally, we argue that units cannot be properly added to Frobenius quantales: every possible extension to a unital quantale fails to preserve negations.

Keywords Quantale · Frobenius quantale · Girard quantale · Residuated lattice · Unit · Dualizing element · Serre duality · Tight map · Trace class operator · Nuclear map

Mathematics Subject Classification 06F07 · 18F75 · 06F05

Introduction

It is often stated, see for example [9, 18, 19], that a Frobenius quantale has a unit. Indeed, as far as these quantales are defined via a dualizing element (the linear falsity), this element necessarily is the unit of the dual multiplication. Then, by duality, the standard multiplication...
of the quantale has a unit. Negations are here taken as defined operators, by means of false and the implications.

It is possible, however, to consider negations as primitive operators and axiomatize them so to be coherent with the implications. We follow here this approach, thus exploring an axiomatization which might be considered folklore: while the axiomatization is explicitly considered in [10, §3.3], it is also closely related to the notions of Girard couple [8] and of Serre duality [20]. Models of the given axioms are quantales coming with a notion of negation, yet they might lack a unit. When they have a unit, these structures coincide with the standard Frobenius and Girard quantales. We call these structures unitless Frobenius quantales. It is the goal of this paper to explore in depth such an axiomatization. We get to the conclusion that unitless Frobenius and Girard quantales are structures of interest and worth further research.

In support of this conclusion we present several examples of these structures and characterize when they have units. Also, we show that the standard theory of quantic nuclei and phase quantales can be lifted to unitless Girard quantales and even to unitless Frobenius quantales. Slightly surprisingly, the new axiomatization immediately clarifies how to generalise to Frobenius quantales the usual double negation nucleus. As far as we are aware of, this nucleus is always described in the literature as arising from a cyclic element, yielding therefore a Girard quantale and, overall, a representation theorem for Girard quantales. We prove therefore that unitless Frobenius quantales can be represented as phase quantales, i.e. as quotients of free quantales over a semigroup by a double negation nucleus. As a consequence, many of our examples (and in principle all of them) arise as phase quantales.

We present and study the example that prompted us to develop this research. It is known [8, 9, 13, 21, 23] that the quantale of sup-preserving endomaps of $L$ has the structure of a Frobenius quantale if and only if $L$ is a completely distributive lattice. If $L$ is completely distributive, then every sup-preserving endomap of $L$ is tight, in the sense of [17], or nuclear, in the sense of [11]. If $L$ is not completely distributive, then tight maps still form a Girard quantale, yet a unitless one as we define. A statement finer than the one above then asserts that the Girard quantale of tight endomaps of a complete lattice $L$ is unital if and only if $L$ is completely distributive. We illustrate the theory so far developed on this example by showing that the Girard quantale of tight endomaps can also be constructed via a double negation nucleus. We further illustrate this example by characterizing tight endomaps of $M_n$ (the modular lattice of height 3 with $n$ atoms), by enumerating these maps, by characterising the operations on tight maps arising from the quantale structure.

In this setting, a natural question is whether it is possible to embed unitless Frobenius quantales into unital ones. This question has always a trivial positive answer. Yet, given the choice of negations as primitive operations, a finer question is whether it is possible to find such an embedding that also preserves the negations. A simple but surprising argument shows that this is never possible, unless the given quantale has already a unit. Studying further this phenomenon, we emphasize that, contrary to the unital case, this is due to the fact that positive elements are not closed under infima.

Let us mention other motivations that prompted us to develop this research. The literature on quantales often emphasizes that these structures might not have units. We could not identify a work whose main topic are units in quantales, yet the role of units has been discussed from slightly different perspectives. For example, completeness and complexity results for the Lambek calculus—which we consider as a fragment of non-commutative linear logic whose connections to quantale theory are well-known [25]—have shown to importantly differ depending on the presence of a unit in the calculus [2, 14].
The need of considering unitless semigroups analogous to Frobenius quantales has shown up when devising categorical frameworks for quantum computation [1]. While the Frobenius algebras considered in that work are not quantales, the Frobenius quantales that we construct in Sect. 3 are built up from the same Frobenius algebras of trace class operators considered in [1]; the strict analogy between the two situations can be formalised using the language of autonomous categories.

Finally, coming to the past research of one of the authors, the constructions carried out in [24] rely on Girard quantales; however, to perform them, it is unclear whether the units play any role, units even appear to be problematic. Fundamental results in that paper depend on considering an infinite family of Girard quantale embeddings that preserve all the structure except for the units.

1 Background

Let us recall a few notions that we need in the course of the paper. We address the reader to standard monographies, e.g. [7, 9, 10, 13, 18], for an in-depth presentation of these notions. A complete lattice \(L\) is a poset such that supremum \(\bigvee_{i \in I} x_i\) exists, for each family \(\{x_i \in L \mid i \in I\}\). A function \(f : L_0 \longrightarrow L_1\) between posets is sup-preserving if \(f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)\), for each family \(\{x_i \in L_0 \mid i \in I\}\) and whenever the supremum \(\bigvee_{i \in I} x_i\) exists. Whenever it exists, the supremum \(\bigvee\{y \mid y \leq x_i, \text{ for all } i \in I\}\) is the infimum of the family \(\{x_i \in L \mid i \in I\}\). Therefore, a complete lattice has also infima of arbitrary families. If \(L\) is a complete lattice, then the poset \(L^{\text{op}}\) (with the opposite ordering, \(x \leq_{\text{op}} y\) if and only if \(y \leq x\)) is also a complete lattice, where the suprema in \(L^{\text{op}}\) are infima in \(L\). If \(g : L_0 \longrightarrow L_1\) is sup-preserving, then there exists a unique sup-preserving map \(g : L_0^{\text{op}} \longrightarrow L_1^{\text{op}}\) satisfying \(f(x) \leq y\) if and only if \(g(y) \leq_{\text{op}} x\)—that is, \(x \leq g(y)\)—for each \(x, y \in L\). We use the notation \(\rho(f)\) for this map and say that \(\rho(f)\) is the right adjoint of \(f\). In a similar way, if \(g = \rho(f)\), then we let \(f = \ell(g)\) and say that \(f\) is the left adjoint of \(g\). We also write \(f \dashv g\) if \(g = \rho(f)\) and \(f = \ell(g)\).

Complete lattices and sup-preserving maps form a category, denoted by SLatt. The operation \((-)^{\text{op}} : \text{SLatt} \longrightarrow \text{SLatt}\), that we can extend to sup-preserving maps by setting \(f^{\text{op}} := \rho(f)\), is a contravariant functor from \text{SLatt} to itself. More than that, it is actually a category isomorphism between SLatt and its opposite category SLatt^{op}. A map \(g : L_0 \longrightarrow L_1\) is inf-preserving if it can be considered as a map \(L_0^{\text{op}} \longrightarrow L_1^{\text{op}}\) in SLatt.

A Galois connection on complete lattices \(L_0, L_1\) is a pair of maps \(f : L_0 \longrightarrow L_1\) and \(g : L_1 \longrightarrow L_0\) such that, for each \(x \in L_0\) and \(y \in L_1\), we have \(y \leq f(x)\) if and only if \(x \leq g(y)\). It is a consequence of this definition that we can see \(f\) as a sup-preserving map from \(L_0\) to \(L_1^{\text{op}}\) and that \(g = \rho(f) : L_1 = (L_1^{\text{op}})^{\text{op}} \longrightarrow L_0^{\text{op}}\). Every Galois connection is uniquely determined in this way. We say that \(f : L_0 \longrightarrow L_1\) is self-adjoint if the pair \(f, f\) is a Galois connection.

A closure operator on a complete lattice \(L\) is a map \(j : L \longrightarrow L\) which is isotone (i.e. order preserving), increasing \((x \leq j(x),\) for each \(x \in L\)), and idempotent \(j(j(x)) = j(x),\) for each \(x \in L\). If \(j : L \longrightarrow L\) is a closure operator, then \(L_j := \{x \in L \mid j(x) = x\}\) is, with the ordering induced from \(L\), a complete lattice, where suprema (in \(L_j\), denoted by \(\bigvee_j\)) are computed as follows:

\[
\bigvee_j\{x_i \in L_j \mid i \in I\} = j(\bigvee\{x_i \in L_j \mid i \in I\}).
\]
We shall say that \( x \in L \) is \( j \)-closed if \( x \in L_j \). The map \( j : L \rightarrow L_j \) is then a surjective sup-preserving map. If \( f : L_0 \rightarrow L_1 \) is sup-preserving, then \( \rho(f) \circ f \) is a closure operator in \( L_0 \), and every closure operator can be obtained in this way.

**Definition 1** A *quantale* is a pair \((Q, *)\) where \( Q \) is a complete lattice and \(*\) is a semigroup operation that distributes with arbitrary suprema in each place:

\[
( \bigvee_{i \in I} x_i ) * ( \bigvee_{j \in J} y_j ) = \bigvee_{i \in I, j \in J} x_i * y_j ,
\]

for each pair of families \( \{ x_i \mid i \in I \} \) and \( \{ y_j \mid j \in J \} \). If the semigroup operation \(*\) has a unit, then we say that the quantale is *unital*. If \((Q_i, *_i), i = 0, 1\), are quantales, then a sup-preserving map \( f : Q_0 \rightarrow Q_1 \) is a quantale homomorphism if \( f(x *_0 y) = f(x) *_1 f(y) \), for each \( x, y \in Q_0 \).

The category \( \text{SLatt} \) is actually a \(*\)-autonomous category, see e.g. [3, 11]. As such, it comes with a tensor product and a (unital) quantale is exactly a (monoid) semigroup object in the monoidal category \( \text{SLatt} \). For a quantale \((Q, *)\) and fixed \( x, y \in Q \), equation (1) exhibits the maps \( x * (-) : Q \rightarrow Q \) and \((-) * y : Q \rightarrow Q \) as being sup-preserving. Thus these two maps have right adjoints \( x \backslash (-) : Q^{\text{op}} \rightarrow Q^{\text{op}} \) and \( (-) / y : Q^{\text{op}} \rightarrow Q^{\text{op}} \). We have therefore

\[ x * y \leq z \text{ iff } y \leq x \backslash z \text{ iff } x \leq z / y , \]

for each \( x, y, z \in Q \). These operations, that we call the *implications*, have also the following explicit expressions:

\[ x \backslash z = \bigvee \{ y \mid x * y \leq z \} , \quad z / y = \bigvee \{ x \mid x * y \leq z \} . \]

Notice that if \( f : Q_0 \rightarrow Q_1 \) is a quantale homomorphism, then \( f(x \backslash z) \leq f(x) \backslash f(z) \) but in general this inequality might be strict (and a similar remark applies to \( f(z / y) \)).

### 2 Frobenius and Girard Quantales

The usual definition of Frobenius and Girard quantales requires a dualizing possibly cyclic element. It goes as follows:

**Definition 2** (Definition A) Let \((Q, *)\) be a quantale and let \( 0 \in Q \). The element 0 is dualizing if, for every \( x \in Q \), we have

\[ 0 / (x \backslash 0) = (0 / x) \backslash 0 = x . \]

The element 0 is cyclic if, for every \( x \in Q \), we have

\[ x \backslash 0 = 0 / x . \]

A *Frobenius quantale* is a tuple \((Q, *, 0)\) where \((Q, *)\) is a quantale and \( 0 \in Q \) is dualizing. If moreover 0 is cyclic then \((Q, *, 0)\) is a *Girard quantale*.

It is well known that a Frobenius quantale, if so defined, is unital, see [19, Corollary to Proposition 1] and [9, Proposition 2.6.3]. Indeed, \( 0 \backslash 0 = 0 / 0 \) turns out to be the unit of the quantale. We propose next a slightly different definition of Frobenius and Girard quantales, which does not rely on dualizing elements. The definition is more general in that, as we shall see, it allows to consider unitless quantales.
**Definition 3** (Definition B) A *Frobenius quantale* is a tuple \((Q, \ast, \perp)\) where \((Q, \ast)\) is a quantale and \(\perp(-), (-)^\perp : Q \to Q\) are inverse antitone maps satisfying

\[
x \perp y = x^\perp / y, \quad \text{for every } x, y \in Q.
\]

The map \((-)^\perp\) is called the right negation while the map \((-) \perp\) the left negation. A *Girard quantale* is a Frobenius quantale for which right and left negations coincide.

We informally call a structure as defined in Definition 3 a *unitless Frobenius* (or *Girard*) quantale. This naming, however, is slight imprecise, as these quantales might well have a unit.

Let us make straight the relation between Definition 2 and Definition 3.

**Lemma 4** (cf. [10, Lemma 3.18] and Lemma 15) If \((Q, \ast, \perp(-), (-)^\perp)\) is a Frobenius quantale, as defined in Definition 3, with unit \(1\), then \(\perp 1 = 1^\perp\) is a dualizing element of \(Q\). This establishes a bijection between the Frobenius quantale structures defined in Definition 2 and those defined in Definition 3 that, moreover, are unital.

The identity (2) is considered in [10], where it is called contraposition. Let us consider a few consequences of (2). This identity is (under the assumption that negations are inverse to each other) equivalent to the following ones:

\[
x \perp y = x^\perp / y^\perp, \quad x / y = \perp x \perp y, \quad \perp x \perp y = x / y^\perp.
\]

Invertible maps satisfying the above kind of conditions were called Serre dualities in [20]. The last of the three identities above was considered in [8].

**Definition 5** We say that a pair of antitone maps \(\perp(-), (-)^\perp : Q \to Q\) is Serre if they satisfy (2). If they are inverse to each other, then we say that they are form a *Serre duality*.

The following Lemma immediately follows from the previous definition.

**Lemma 6** A pair of antitone maps is Serre if and only if the equivalence below holds (for each \(x, y, z \in Q\)):

\[
x \ast z \leq \perp y \iff z \ast y \leq x^\perp.
\]

Notice that the equivalence in (6) amounts to what we called elsewhere (see e.g. [22]) the *shift relations*:

\[
x \ast y \leq z \iff \perp z \ast x \leq \perp y \iff y \ast z^\perp \leq x^\perp,
\]

which also appear in [10, §3.2.2] under the naming of (IGP).

If \(Q\) is unital, then a Serre pair \(\perp(-), (-)^\perp\) necessarily forms a Galois connection. If \(Q\) is not unital, then a Serre pair need not to be a Galois connection. For example, if \(x \ast y = 0\) for each \(x, y \in Q\), then the constant map \(x^\perp = 0, x \in Q\), is Serre with itself, but it is not self-adjoint if \(Q\) has at least two elements. We shall restrict ourself, notably in the next section, to consider Serre pairs forming a Galois connection. Of course, if \(\perp(-), (-)^\perp\) are inverse to each other, then they are also adjoint.

Given a quantale \((Q, \ast)\) and pair of antitone maps \(\perp(-), (-)^\perp : Q \to Q\) inverse to each other, we can define two distinct dual operators:

\[
x \oplus \perp y := \perp(y^\perp \ast x^\perp), \quad x \perp \oplus y := (\perp y \ast \perp x)^\perp.
\]
Proposition 7 (cf. [10, Lemma 3.17]) If \( \bot(-), (-)\bot \) is a Serre duality, then the two dual multiplications coincide and they are determined by the implication and the negations as follows

\[
x \oplus_{\bot} y = \bot x \setminus y = x / y^{\bot} = x \bot \oplus y.
\]

Our next aim is that of providing simple examples of unitless Frobenius quantales that indeed lack units.

Example 8 (Chu construction) We give a first example of unitless Girard quantale by observing that the usual Chu construction (or Twist product, see e.g. [3, 12, 19]), does not require a quantale to be unital and yields a Girard quantale as defined in Definition 3. For a quantale \((Q, \ast)\), the quantale \(C(Q) = (Q \times Q^{\text{op}}, \star)\) is defined by

\[
(x_1, x_2) \star (y_1, y_2) := (x_1 \ast y_1, y_1 \setminus x_2 \wedge y_2 / x_1).
\]

One can directly check the associativity of \(\star\) and that it distributes in both arguments over joins of \(Q \times Q^{\text{op}}.\) Recall that

\[
(x_1, x_2) \setminus (z_1, z_2) = (x_1 \setminus z_1 \wedge x_2 / z_2, z_2 \ast x_1),
\]

\[
(z_1, z_2) / (y_1, y_2) = (z_1 / y_1 \wedge z_2 \setminus y_2, y_1 \ast z_2).
\]

The duality being given by

\[
(x_1, x_2)^{\bot} := (x_2, x_1),
\]

the reader will have no difficulties proving that \((-)^{\bot}\) is Serre self-dual.

Proposition 9 The quantale \(C(Q)\) is unital if and only if \(Q\) is unital.

Indeed, the first projection is a surjective semigroup homomorphism and so, if \(C(Q)\) has a unit \((u_1, u_2)\), then \(u_1\) is a unit of \(Q.\) On the other hand, it is well known that if \(u_1\) is a unit of \(Q,\) then \((u_1, \top)\) is a unit of \(C(Q).\)

Example 10 (Couples of quantales) In [8] the authors define a couple of quantales as a pair of quantales \(C, Q\) that are related by a sup-preserving map \(\phi: C \longrightarrow Q;\) moreover, \(C\) is asked to be a \(Q\)-bimodule and the following equations

\[
\phi(c_1) \cdot c_2 = c_1 \cdot \phi(c_2) = c_1 \ast c_2
\]

are to be satisfied. If \(Q\) is a quantale, then \(C = Q^{\text{op}}\) is a canonical \(Q\)-bimodule, with, for \(x, y \in Q\) and \(q \in Q^{\text{op}},\) \(y \cdot q = q / y\) and \(q \cdot x = x \setminus q.\) Thinking of \(\phi: Q^{\text{op}} \longrightarrow Q\) as a sort of negation, the first equation in (5) yields \(x / \phi(y) = \phi(x) \setminus y,\) that is, the last equation in (3). Notice that, by taking the above equality as definition of a binary operator on \(Q^{\text{op}},\) then this operator is necessarily associative. If, moreover, \(\phi\) is an involution, then \(Q\) becomes a unitless Girard quantale. Viceversa, unitless Girard quantale structures on \(Q\) give rise to couples of quantales \(\phi: C \longrightarrow Q\) with \(C = Q^{\text{op}}\) and \(\phi\) an antitone involution. Let us notice, however, that Girard couples of quantales, in contrast to couples of quantales, turn out to have some unit [8, Proposition 8].

\(\square\) Springer
### 3 Nuclei and Phase Quantales

In this section we show how to generalise the elementary theory of nuclei from unital Girard quantales to unitless Girard quantales and also unitless Frobenius quantales.

Let us recall that a quantic nucleus (or simply a nucleus) on a quantale \((Q, \ast)\) is a closure operator \(j\) on \(Q\) such that, for all \(x, y \in Q\),

\[
j(x) \ast j(y) \leq j(x \ast y).
\]

It is easily seen that a closure operator \(j\) is a nucleus if and only if the two conditions below hold:

\[
x \ast j(y) \leq j(x \ast y), \quad j(x) \ast y \leq j(x \ast y), \quad \text{for all } x, y \in Q.
\]

Given a nucleus \(j\) on \((Q, \ast)\), let \(Q_j\) be the set of fixed points of \(j\). From the elementary theory of closure operators, \(Q_j\) is a complete lattice and \(j : Q \longrightarrow Q_j\) is a surjective sup-preserving map. \(Q_j\) has a canonical structure of a quantale \((Q_j, \ast_j)\) as well, where the multiplication is given by

\[
x \ast_j y := j(x \ast y).
\]

Moreover, \(j : (Q, \ast) \longrightarrow (Q_j, \ast_j)\) is a surjective quantale morphism.

**Definition 11** For a quantale \((Q, \ast)\), a Serre Galois connection is a Galois connection \(l, r : Q \longrightarrow Q\) such that \(l \circ r = r \circ l\) and, for all \(x, y, z \in Q\),

\[
x \ast y \leq l(z) \iff z \ast x \leq l(y).
\]

That is, \((l, r)\) is Serre if \(l \circ r = r \circ l\) and the following identity holds:

\[
l(z)/y = z\backslash l(y).
\]

We shall refer to (6) as the shift relation. Recall that if \((l, r)\) is Serre and moreover \((l, r)\) are inverse to each other, then we say that \((l, r)\) is a Serre duality.

We say that \(r : Q \longrightarrow Q\) is a Serre map if \((r, r)\) is a Serre Galois connection.

**Proposition 12** If \((l, r)\) is a Serre Galois connection, then \(j := l \circ r = r \circ l\) is a nucleus, and the restriction of \((l, r)\) to \(Q_j\) yields a Frobenius quantale structure on \((Q_j, \ast_j)\).

**Proof** We firstly prove that \(x \ast j(y) \leq j(x \ast y)\), that is, \(x \ast r l(y) \leq r l(x \ast y)\). Using the shift relation, this inclusion is equivalent to \(l(x \ast y) \ast x \leq l r l(y)\). Since \(l r l(y) = l(y)\), the latter inclusion is \(l(x \ast y) \ast x \leq l(y)\) and equivalent, by the shift relation, to \(x \ast y \leq r l(x \ast y)\), where the last inclusion holds since \(r \circ l\) is a closure operator. The inclusion \(j(x) \ast y \leq j(x \ast y)\) is proved similarly, using now \(j = l \circ r\).

Next, the restrictions of \((l, r)\) to \(Q_j\) have \(Q_j\) as codomain: if \(x \in Q_j\), then \(j(r(x)) = r(l(r(x))) = r(x)\), so \(r(x) \in Q_j\). Similarly, \(l(x) \in Q_j\). These restrictions are also inverse to each other: for \(x \in Q_j\), \(l(r(x)) = r(l(x)) = j(x) = x\). Let us argue that these restrictions are Serre: for \(x, y, z \in Q_j\), \(x \ast j(y) \leq j(x \ast y)\) \iff \(x \ast y \leq j(z \ast x)\leq l(y)\), so the shift relation holds in \(Q_j\).

**Proposition 13** Let \(j\) be a nucleus on \((Q, \ast)\) and suppose that \(l, r : Q_j \longrightarrow Q_j\) is a Serre duality on \((Q_j, \ast_j)\). Then \((l \circ j, r \circ j)\) is a Serre Galois connection on \((Q, \ast)\) and the Frobenius quantale structure \((Q_j, \ast_j, l, r)\) is induced from \((l \circ j, r \circ j)\) as described in Proposition 12.
Proof Let $j, l, r$ be as stated. For $x, y, z \in Q$, we have $x \leq r(j(y))$ iff $j(x) \leq r(j(y))$ iff $j(y) \leq l(j(x))$ iff $y \leq l(j(x))$.

Moreover
\[
x * y \leq r(j(z)) \iff j(x * y) = j(x) * j(y) \leq r(j(z))
\]
\[
\text{iff } j(z * x) = j(z) * j(x) \leq l(j(y)) \text{ iff } z * x \leq l(j(y)).
\]

Notice now that $j \circ r = r$ and $j \circ l = l$, since by assumption the codomain of $r$ and $l$ is $Q_j$.

Then
\[
(l \circ j) \circ (r \circ j) = l \circ r \circ j = j
\]
and, similarly, $(r \circ j) \circ (l \circ j) = j$. Finally, the restriction of $r \circ j$ (resp., $l \circ j$) to $Q_j$ is $r$ (resp., $l$), for example, for $x \in Q_j$, we have $(r \circ j)(x) = r(j(x)) = r(x)$.

We add next some remarks.

Definition 14 If $0$ is an element of a quantale $(Q, *)$, then we say that $0$ is weakly cyclic if
\[
0/(x \setminus 0) = (0/x) \setminus 0, \quad \text{for all } x \in Q.
\]

We say that a Serre Galois connection $(l, r)$ on $(Q, *)$ is representable by $0$ if
\[
r(x) = x \setminus 0, \quad l(x) = 0/x, \quad \text{for all } x \in Q.
\]

Observe that if a Serre Galois connection $(l, r)$ is representable by $0 \in Q$, then $0$ is a weakly cyclic element.

Lemma 15 If $(Q, *)$ is unital, then every Serre Galois connection is representable.

Proof If $1$ is the unit of $(Q, *)$, then
\[
r(y) = r(y)/1 = y\{l(1) \text{ and } l(y) = 1\{y = r(1)/y.
\]
Moreover
\[
r(1) = r(1)/1 = 1\{l(1) = l(1),
\]
and therefore $(l, r)$ is representable by $0 = r(1) = l(1) \in Q$.

Lemma 16 Let $0$ be a weakly cyclic element of a quantale $(Q, *)$ and set $r(x) := x \setminus 0$ and $l(x) := 0/x$. Then $(l, r)$ is a representable Serre Galois connection. If $0$ is $j$-closed (with $j = r \circ l = l \circ r$), then $(Q_j, *)$ is unital. If $(Q, *)$ is unital, then $0$ is $j$-closed.

Proof The first statement is obvious. If $0$ is $j$-closed, then the relations $r(x) = x \setminus 0$ and $l(x) = 0/x$ hold within $Q_j$, whence $0$ is a dualizing element of $Q_j$ which therefore is unital.

Finally, if $1$ is the unit of $(Q, *)$, then $0 = r(1) = rlr(1) = l(1) = lrl(1)$, showing that $0$ is $j$-closed.

Lemma 17 For $(l, r)$ a Galois connection on $Q$, the condition $l \circ r = r \circ l$ holds if and only if the images of $l$ and $r$ in $Q$ coincide.

Proof For a map $f$ let $\text{Img}(f)$ denote its image. Notice that $\text{Img}(r) = Q_{rl}$, since $r(x) = r(l(r(x)))$ and similarly $\text{Img}(l) = Q_{lor}$. Therefore, if $r \circ l = l \circ r$, then $\text{Img}(r) = \text{Img}(l)$. Conversely, if $\text{Img}(r) = \text{Img}(l)$, then $Q_{rl} = Q_{lor}$ and since a closure operator is determined by the set of its fixed points, then $l \circ r = r \circ l$. 

Springer
Example 18 (Trivial quantales)

As we do not require units, each complete lattice \( Q \) can be endowed with the trivial quantale structure, defined by:

\[ x * y := \bot, \quad \text{for each } x, y \in Q. \]

The trivial quantale structure clearly is not unital, unless \( Q \) is a singleton. Notice that, for the trivial structure,

\[ x \setminus y = y / x = \top, \quad \text{for each } x, y \in Q. \]

If \( Q \) is autodual, say with \( l, r : Q \to Q \) inverse to each other, then \( (Q, *, l, r) \) is a unitless Frobenius quantale.

The trivial structure is a source of counter-examples to quick conjectures. Consider \( (2, *) \), the two-element Boolean algebra equipped with the trivial quantale structure. Then we have

\[ x * y \leq r(z) \quad \text{if and only if} \quad y * z \leq l(x) \quad \text{no matter how we choose} \quad (l, r). \]

Thus, we might choose \( (l, r) \) not antitone and yet satisfying the shift relation (take \( r = l \) be the identity of \( 2 \)), or choose \( (l, r) \) antitone but not a Galois connection (take \( r = l \) be the constant function taking \( \bot \) as its unique value).

Also, consider the diamond lattice with 3 atoms (see Fig. 2) and endow it with the trivial quantale structure. Take for \( r \) a duality that cyclically permutes the 3 atoms and let \( l \) be its inverse, so to have \( l \neq r \). Then \( (l, r) \) is a Serre duality on the commutative quantale over \( M_3 \). On the other hand, every representable Serre duality on a commutative quantale is necessarily self-adjoint (i.e. we have \( r = l \)).

It might be asked whether \( (l, r) \) representable on \( Q \) implies that \( (l, r) \) is representable on \( Q_j \). This is not the case. On the chain \( 0 < 1 < 2 \), consider the following commutative quantale structure:

\[
\begin{array}{c|ccc}
& 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 \\
\end{array}
\]

By commutativity, 0 is weakly cyclic and, letting \( r(x) = x \setminus 0 \), we have \( r(0) = r(1) = 2 \) and \( r(2) = 1 \). Therefore \( Q_j = \{1, 2\} \) and the \( *_j \) is the trivial quantale structure on \( Q_j \). As the implication of the trivial structure is constant, it cannot yield the (unique) duality of \( \{1, 2\} \). Considering Lemma 16, notice here that 0 is not \( j \)-closed.

It is not difficult to reproduce similar counter-examples with the multiplication being non-trivial (i.e. \( x * y \neq \bot \), for some \( x, y \in Q \)).

Phase quantales. We explore next the construction described in Propositions 12 and 13 when \((Q, \ast)\) is the free quantale \((P(S), \bullet)\) over a semigroup \((S, \cdot)\). Recall (see e.g. [9]) that

\[
X \bullet Y := \{ x \cdot y \mid x \in X, \ y \in Y \},
\]

\[
X \setminus Y = \{ s \in S \mid s \cdot x \in Y, \ \forall x \in X \},
\]

\[
Y / X = \{ s \in S \mid x \cdot s \in Y, \ \forall x \in X \}.
\]

It is well-known (see e.g. [15]) that Galois connections \((l, r)\) on \( P(S) \) bijectively correspond to binary relations \( R \) via the correspondence

\[ x R y \iff x \in l(\{ y \}) \iff y \in r(\{ x \}). \]
so

\[ r(Z) = \{ u \in S \mid zRu, \text{ for all } z \in Z \}, \quad l(Y) = \{ u \in S \mid uRy, \text{ for all } y \in Y \}. \]

We aim to characterize Serre Galois connections via properties of their corresponding relation.

**Proposition 19** A Galois connection \((l, r)\) on \((P(S), \bullet)\) satisfies the shift relation (6) if and only if its corresponding binary relation \(R\) satisfies

\[ x \cdot y R z \iff x R y \cdot z, \text{ for all } x, y, z \in S. \]  

**(Proof)** If \((l, r)\) satisfies (6), then the following is a chain of equivalent statements (where \(X, Y, Z\) are subsets of \(S\)):

\[
\forall x \in X, y \in Y, z \in Z, zRx \cdot y \iff X \bullet Y \subseteq r(Z) \\
\quad \iff Z \bullet X \subseteq l(Y) \\
\quad \iff \forall x \in X, y \in Y, z \in Z, z \cdot xRy.
\]

Considering singletons we obtain that \(R\) satisfies

\[ \forall x, y, z \in S, zRx \cdot y \iff z \cdot xRy, \]

and clearly the latter condition suffices for (6) to hold. \(\square\)

We call **associative** a relation satisfying (7). To complete the characterisation of Serre Galois connections, we need to characterise the condition \(l \circ r = r \circ l\) in terms of the corresponding relation \(R\). Notice first that the condition trivially holds if \(R\) is symmetric, in which case \(r = l\).

By Lemma 17, \(r \circ l = l \circ r\) if and only if \(Im(r) = Im(l)\), that is

\[ \forall X \exists Y \ s.t. \ r(X) = l(Y), \text{ and } \forall Y \exists X \ s.t. \ r(X) = l(Y). \]

Clearly, these two conditions hold if and only if they hold with \(X\) (in the first case, or \(Y\) in the second) restricted to singleton, yielding

\[
\forall x \exists Y_x \forall z \ (xRz \iff zRy, \forall y \in Y_x), \quad (8) \\
\forall y \exists X_y \forall z \ (zRy \iff xRz, \forall x \in X_y). \quad (9)
\]

We call **weakly-symmetric** a relation satisfying (8) and (9).

We exemplify our previous observations, in particular the use of Proposition 12, in different ways.

**Example 20** (Unital Frobenius quantales from pregroups) Here we illustrate how the same tools can be used to build Frobenius quantales that are not Girard quantales.

Recall that a pregroup, see e.g. [5], is an ordered monoid \((M, 1, \cdot, \leq)\) coming with functions \(l, r : M \longrightarrow M\) satisfying

\[ x \cdot x' \leq 1, \quad x^l \cdot x \leq 1, \quad 1 \leq x^r \cdot x, \quad 1 \leq x \cdot x^l, \]

for all \(x \in M\). That is, a pregroup is a posetal rigid category. Consider the relation \(R\) defined by \(xRy\) if and only if \(x \cdot y \leq 1\). Clearly, \(R\) is associative. Observe now that \(x \cdot z \leq 1\) if and only if \(z \cdot x'^r \leq 1\). That is, the condition in (8) is satisfied by letting \(Y_x = \{ x'^r \}\). Similarly, (9) is satisfied by letting \(X_y = \{ y^l \}\). Since \(M\) has a unit, both \(P(M)\) and \(P(M)_j\) are unital. \(\diamond\)
Example 21 (Unitless Girard quantales from \(\mathbb{C}^*\)-algebras) Let us consider an algebra \(A\) coming with an associative symmetric pairing (i.e. a bilinear form) \(\langle - , - \rangle\) into the base field \(\mathbb{K}\). We do not assume that \(A\) has a unit. We recall that a pairing is said to be associative if it satisfies \((x \cdot y, z) = \langle x, y \cdot z \rangle\), for each \(x, y, z \in A\) (in which \(A\) is called a Frobenius algebra).

The binary relation \(R\), defined by \(x R y\) if and only if \(\langle x, y \rangle = 0\), is then an associative relation on the semigroup reduct \((A, \cdot)\) of \(A\), so we can consider the powerset quantale \((P(A), \bullet)\) and the Serre Galois connection \((l, r)\) the relation \(R\) gives rise to. Since \(R\) is symmetric, we denote \(l(X) = r(X) = X^\perp\), as usual from standard algebra, and so \(j(X) = X^\perp\). Clearly, if \(A\) has a unit, then \(P(A)\) and its quotient \(P(A)_j\) have a unit as well.

We argue next that the converse holds, if we can transform the pairing into a sort of inner product (as for example with \(\mathbb{C}^*\)-algebras). Namely, suppose that \(A\) comes with an involution \((-)^* : A \rightarrow A\) such that, for each \(f \in A\),

\[
(f, f^*) = 0 \implies f = 0. \quad (10)
\]

In particular, assuming \((10)\), we have \(A^\perp = \{0\}\). Under these assumptions, the following statement holds:

Proposition 22 If \(P(A)_j\) has a unit, then \(A\) has a unit.

Proof Obviously, every set of the form \(X^\perp\) is a subspace of \(A\). Let us say that a subspace is closed if it is of the form \(j(Y)\) or, equivalently, \(X^\perp\) (for some \(X\) or some \(Y\)).

We firstly observe that every one dimensional subspace of \(A\) is closed. That is, we have \(j(f) = \mathbb{K}f\), for each \(f \in A \setminus \{0\}\). Indeed, for \(g \in A\) and \(k = \frac{\langle g, f^* \rangle}{\langle f, f^* \rangle}\), we can write \(g = (g - kf) + kf\) with \(g - kf \in \{f^*\}^\perp\). Because of \((10)\), we also have \(f^* \notin \{f^*\}^\perp\). That is, we have and \(\mathbb{K}f \lor \{f^*\}^\perp = A\) and \(\mathbb{K}f \land \{f^*\}^\perp = \{0\}\) in the lattice of subspaces of \(A\). The relation \(\mathbb{K}f \lor \{f^*\}^\perp = A\) a fortiori implies the relation \(j(\mathbb{K}f) \lor j\{f^*\}^\perp = A\), taken in the lattice of closed subspaces. Using the duality, we derive \(\{0\} = A^\perp = (j(\mathbb{K}f) \lor j\{f^*\}^\perp) = j(\mathbb{K}f) \land j\{f^*\}^\perp = \{f\}^\perp \land \{f^*\}^\perp\), where we have used that both meets \(\land\) (the meet in the lattice of subspaces of \(A\)) and \(\land\) (the meet in the lattice of closed subspaces of \(A\)) are computed as intersection. Then we also have \(\{0\} = \{f^*\}^\perp \land \{f^*\}^\perp = \{f^*\}^\perp \land \{f\}^\perp\). It follows that both \(\mathbb{K}f\) and \(j(\{f\}) = \{f\}^\perp\) are complements of \(\{f^*\}^\perp\) in the lattice of subspaces of \(A\). Since \(\mathbb{K}f \subseteq j(\{f\})\), we derive \(\mathbb{K}f = j(\{f\})\) using modularity of this lattice.

Let now \(U\) be the unit of \(P(A)_j\). Then \(U \bullet \{f\} = j(U) \bullet \{f\} \subseteq j(U \bullet \{f\}) = j(U \bullet j(\{f\})) = j(\{f\}) = \mathbb{K}f\) and, similarly, \(\{f\} \bullet U \subseteq \mathbb{K}f\).

Thus, for each \(u \in U\) and \(f \in A\), \(u \cdot f = kf\) and \(f \cdot u = k f\) for some \(k, k' \in \mathbb{K}\). In particular, for \(u, v \in U\), then \(ku = u \cdot v = k'v\), showing that \(U\) has dimension 1 (the unit \(U\) cannot be \(\{0\}\) unless \(A = \{0\}\)). We claim now that, given \(u \in U\), there exists \(k \in \mathbb{K}\) such that, for all \(f \in A\), \(u \cdot f = kf\). Indeed, let \(f, g \in A \setminus \{0\}\) and write \(u \cdot f = kf\) and \(u \cdot g = kg\). If \(f = kg\), then \(k f f = u \cdot f = u \cdot kg = k(u \cdot g) = kkg = kgf\), thus \(k f = kg\). Suppose now that \(f, g\) are linearly independent. Then, for some \(k \in \mathbb{K}\), \(kf + kg = k(f + g) = u \cdot (f + g) = u \cdot f + u \cdot g = kf + kg\) which yields the relation \((k - k f) f = (kg - k) g\) and \(k f = k = kg\). We have proved the claim from which it readily follows that \(k' u := 1_k u\) is a left unit of \(A\). Similarly, we can find a right unit \(u'\) and, as usual, \(u = u'\). Thus \(A\) is unital. \(\square\)

The \(\mathbb{C}^*\)-algebra of \(n \times n\) matrices over the complex numbers \(\mathbb{C}\) comes with the pairing \(\langle A, B \rangle = tr(A \cdot B)\), where \(A^*\) is the conjugate transpose of \(A\). This algebra is unital and gives
rise to a well known Girard quantale, see e.g. [9, §2.6.15]. We can adapt this construction to consider classes of linear operators on an infinite dimensional Hilbert space $H$. A continuous linear mapping $f : H \rightarrow H$ is trace class, see e.g. [6, §18], if

$$\sum_{e \in \mathcal{E}} \langle|f|e, e\rangle_H < \infty,$$

where $\langle - , - \rangle_H$ above is the inner product of the Hilbert space $H$, $\mathcal{E}$ is an orthonormal basis, and $|f|$ is the unique positive operator such that $f^* \circ f = |f|^2$. For such an operator, we can define

$$tr(f) := \sum_{e \in \mathcal{E}} \langle f(e), e \rangle_H,$$

yielding an associative symmetric pairing. Now, $f$ is trace class if and only if its adjoint $f^*$ is trace class. With respect to the involution given by the adjoint, this pairing satisfies (10). The algebra of trace class operators does not have a unit. First of all, it is easily seen that the identity is not trace class. Indeed, this algebra is a well-known proper (when $H$ is infinite dimensional) ideal of the algebra of bounded linear operators on $H$. Most importantly, this algebra cannot have a unit. In fact, for each $h \in H$ there exists a trace class operator $c_h$ and $p_h \in H$ such that $c_h(p_h) = h$. For example, if $\mathcal{E}$ is a basis for $H$, we can let $e_0 \in \mathcal{E}$, and define $c_h(e_0) = h$, and $c_h(e) = 0$, for $e \in \mathcal{E} \setminus \{e_0\}$. Then a unit $u$ is forcedly the identity:

$$u(h) = u(c_h(p_h)) = (u \circ c_h)(p_h) = c_h(p_h) = h,$$

for all $h \in H$.

We collect these observations into a formal statement.

**Theorem 23** The collection of closed subspaces of the algebra of trace class operators is a unitless Girard quantale.

Similar considerations can be developed for Hilbert-Schmidt operators, see [6]. Indeed, the composition of two such operators is trace class, showing that the formula for the pairing in (11) also applies to these operators.

Our next goal is to give a representation theorem, analogous to the representation theorem for Girard quantales, see e.g. [19, Theorem 2]. The representation theorem is here extended to Frobenius and unitless quantales.

**Proposition 24** Given a Frobenius quantale $(Q, \ast, \dashv (-), (-)\bot)$, define $xRy$ iff $x \leq^\bot y$. Then $R$ is an associative weakly symmetric relation yielding a Serre Galois connection $(l, r)$ on $(\mathcal{P}(Q), \bullet)$, whose quotient $(\mathcal{P}(Q)_j, \bullet_j, l, r)$ is isomorphic to $(Q, \ast, \dashv (-), (-)\bot)$.

**Proof** Let us verify that $R$ is associative:

$$zRx \ast y \text{ iff } z \leq^\bot (x \ast y) \text{ iff } x \ast y \leq z^\bot \text{ iff } z \ast x \leq^\bot y \text{ iff } z \ast x Ry.$$
Finally, \( j(X) = \downarrow \bigvee X \), since
\[
j(X) = l(r(X)) = \downarrow (\downarrow \bigvee (\downarrow \bigvee X)) = \downarrow (\downarrow (\bigvee X)) = \downarrow (\bigvee X) .
\]

It is then easily seen that the mapping \( \downarrow : Q \rightarrow P(Q)_j \) is inverted by \( \bigvee : P(Q)_j \rightarrow Q \). These maps are quantale homomorphisms, indeed we have, for all \( x \) and \( y \) in \( Q \),
\[
\downarrow x \ast_j \downarrow y = \downarrow (\bigvee (\downarrow x \downarrow y)) = \{ z \in Q | z \leq x \ast y \} = \downarrow (x \ast y).
\]

\[\square\]

## 4 The Girard Quantale of Tight Maps

We present in this section the main example of a unitless Girard quantale, the one that prompted this research.

We denote by \( L^L \) the lattice of all function from \( L \) to \( L \), by \( [L, L]_\lor \) the lattice of all join-preserving endomaps of \( L \), and by \( [L, L]_\land \) the lattice of all meet-preserving endomaps of \( L \). For a function \( f : L \rightarrow L \), its Raney’s transforms \( f^\lor \) and \( f^\land \) are defined by
\[
f^\lor(x) := \bigvee_{t \not\leq x} f(t) , \quad f^\land(x) := \bigwedge_{t \not\geq x} f(t) .
\]

Let us remark that, in the definition above, we do not require that \( f \) has any property, such as being monotone.

**Definition 25** An endomap \( f : L \rightarrow L \) is **tight** if \( f = f^\land \lor \). We write \( [L, L]_\lor \) for the set of tight endomaps of \( L \). We say that \( f : L \rightarrow L \) is **cotight** if \( f = f^\lor \land \) and write \( [L, L]_\land \) for the set of cotight maps from \( L \) to \( L \).

We aim at demonstrating the following theorem.

**Theorem 26** For any complete lattice \( L \), the tuple \( ([L, L]_\lor, \circ, (\cdot)^* \) is a Girard quantale (as defined in Definition 3), where \( \circ \) is function composition and the duality \((\cdot)^* \) is defined by \( f^* := \rho(f)^\lor \).

In order to prove the theorem, let us recall some elementary properties of Raney’s transforms, most of which are stated and proved in [11]. In the following, \( L \) shall be an arbitrary but fixed complete lattice.

**Lemma 27** For any function \( f : L \rightarrow L \), \( f^\lor \) has a right-adjoint, and, in particular \( f^\lor \) is sup-preserving and monotone. If \( f \) is inf-preserving, then the following relation holds:
\[
\rho(f^\lor) = (\ell(f))^\land . \tag{12}
\]

**Proof** For each \( x, y \in L \), we have
\[
f^\lor(x) = \bigvee_{t \not\leq x} f(t) \leq y \text{ iff for all } t, \ x \not\leq t \text{ implies } f(t) \leq y , \]
\[
\text{ iff for all } t, \ f(t) \not\leq y \text{ implies } x \leq t , \]
\[
\text{ iff } x \leq \bigwedge_{t \not\leq y} t .
\]
Therefore, if we define \( g(y) := \bigwedge_{f(t) \not\leq y} t \), then \( g \) is right-adjoint to \( f^\vee \). Let us argue that equality (12) holds if \( f \) is inf-preserving. We shall show that, for all \( x \) and \( y \) in \( L \), we have
\[
f^\vee(x) \leq y \text{ iff } x \leq (\ell(f))^\wedge(y).
\]

On the one hand we have
\[
\bigvee_{x \not\leq t} f(t) \leq y \text{ iff for all } t \in L \text{ s.t. } x \not\leq t, f(t) \leq y.
\]

On the other hand we have
\[
x \leq \bigwedge_{t \not\leq y} (\ell(f)(t)) \text{ iff for all } t \in L \text{ s.t. } t \not\leq y, x \leq (\ell(f))(t).
\]

For the implication (a) \( \Rightarrow \) (b), let \( t \in L \) be such that \( t \not\leq y \). Suppose that \( x \not\leq (\ell(f))(t) \). Then, by (a), \( f(\ell(f)(t)) \leq y \) and, by the adjunction, \( t \leq f(\ell(f)(t)) \), whence we deduce \( t \leq y \), a contradiction. Therefore \( x \leq (\ell(f))(t) \). The implication (b) \( \Rightarrow \) (a) is proved in a similar way.

By duality, it follows that \( f^\wedge \) is inf-preserving and that \( \ell(f^\wedge) = (\rho(f))^\vee \) when \( f \) is sup-preserving.

Recall that the set \( L^L \) is pointwise ordered, i.e. we have \( f \leq g \) iff \( f(t) \leq g(t) \), for all \( t \in L \). The reader will have no difficulties verifying the following statement.

**Proposition 28** (cf. [11, Proposition 4.6]) The operation \((-)^\wedge : L^L \rightarrow L^L\) is right-adjoint to \((-)^\vee : L^L \rightarrow L^L\).

Let us briefly analyse the typing of the Raney’s transforms. Lemma 27 exhibits the Raney’s transforms as overloaded operators: for example, the codomain of \((-)^\vee\) can be taken as any of the complete lattices among \( L^L, [L, L]^\vee, [L, L]^t \), as suggested in the left diagram in Fig. 1.

The adjunction \((-)^\vee \dashv (-)^\wedge\) of Lemma 28 restricts from \( L^L \) as suggested in the diagram on the left of Fig. 1. This is further illustrated in the diagram on the right of the figure where

![Fig. 1 Typing of the Raney’s transforms](image-url)
the vertical inclusions on the left are inf-preserving and the vertical inclusions on the right are sup-preserving. The two Raney’s transforms in the bottom row are inverse isomorphisms, as a standard consequence of the characterisation of factorizarion systems in the category of sup-lattices and sup-preserving maps, see [9].

We recall next some usual consequences of the adjunction \((-)^\vee \dashv (-)^\wedge\) stated in Proposition 28.

**Lemma 29** A map \(f : L \rightarrow L\) is tight if and only if it lies in the image of the Raney’s transform \((-)^\vee\). For each map \(f : L \rightarrow L\), \(f^{\wedge\wedge}\) is the greatest tight map below \(f\), and \(f^{\vee\wedge}\) is the least cotight map above \(f\).

**Corollary 30** For every complete lattice \(L\), the set \([L, L]_\vee\) is closed under arbitrary suprema.

**Proof** Proposition 28 implies that the operation \((-)^{\wedge\wedge}\) is an interior operator, that is, the dual of a closure operator. As the set of fixed-points of a closure operator is closed under arbitrary infima, the set of fixed-points of an interior operator is closed under arbitrary suprema. \(\Box\)

**Lemma 31** If \(f\) is sup-preserving and \(g\) is any function, then

\[(f \circ g)^\vee = f \circ (g)^\vee.\] (13)

**Proof** Let \(f\) and \(g\) be as stated, and compute as follows:

\[(f \circ g^\vee)(x) = f(g^\vee(x)) = f(\bigvee_{x \not\leq t} g(t)) = \bigvee_{x \not\leq t} (f \circ g)(t) = (f \circ g)^\vee(x).\]

**Corollary 32** Tight maps are closed under composition.

**Proof** If \(f\) and \(g\) are tight maps, then \(f\) is sup-preserving, and so equation (13) ensures that \(f \circ g = f \circ g^{\wedge\wedge} = (f \circ g^\wedge)^\vee\), so \(f \circ g\) belongs to the image of \((-)^\vee\). As we have seen in Lemma 29, this is enough to ensure that this map is tight. \(\Box\)

Corollaries 30 and 32 ensure that \(([L, L]_\vee, \circ)\) is a subquantale of \(([L, L], \circ)\). The following proposition suffices to prove that \(([L, L]_\vee, \circ, (-)^*)\) is a Girard quantale, as stated in Theorem 26, where we recall that \(f^* := \rho(f)^\vee\).

**Proposition 33** The map \((-)^*\) is an involutive Serre duality on \(([L, L]_\vee, \circ)\).

**Proof** First the map \((-)^*\) is an involution. Indeed using the equality (12), we have

\[f^{**} = \rho(\rho(f)^\vee)^\vee = \ell(\rho(f))^{\wedge\wedge} = f^{\wedge\wedge} = f.\]

Since this map is the composition of a monotone map with an antitone one, it is antitone. Let us now show that the shift relation (6) holds, so \((-)^*\) is a Serre duality.

\[f \circ g \leq h^* \text{ iff } g \leq \rho(f) \circ h^*\] (*)&

\[f \circ g \leq (\rho(f) \circ h^*)^{\wedge\wedge},\]

since \(k^{\wedge\wedge}\) is the greatest tight map below \(k\), see Lemma 29,

\[= (\rho(f) \circ (h^*)^{\wedge\wedge})^{\vee} = (\rho(f) \circ \rho(h)^\vee) = \rho(h \circ f)^\vee = (h \circ f)^*,\]

\[\text{ iff } h \circ f \leq g^*, \text{ since } (-)^* \text{ is an antitone involution,}\]

where in ((*) we have used the dual of equation (13), the fact that \(g^{\wedge\wedge} = g\) if \(g\) is cotight, the fact that \(\rho(h)\) is cotight when \(h\) is tight (since \(\rho(h) = \rho(h^{\wedge\wedge}) = \ell(h^{\wedge})^\wedge\) and usual properties of adjoints). \(\Box\)
Let us recall at this point Raney’s theorem:

**Theorem 34** (Raney [17]) A complete lattice is completely distributive if and only if \( id_L \) is tight.

The following statement appears in [8, 13] and, with few subtle differences, in [23]:

**Theorem 35** The quantale \([L, L]_\lor, \circ\) is a Girard quantale if and only if \( L \) is a completely distributive lattice.

We refine this result by arguing that the Girard quantale \([L, L]_\lor, \circ, (-)^\lor\) has a unit if and only if \( L \) is a completely distributive lattice, see Theorem 38. If the quantale of tight maps is unital, then its unit necessarily is the identity, and then every map \( f \in [L, L]_\lor \) is tight, \( f = f \circ id_L = f \circ (id_L)^\lor = (f \circ (id_L)^\lor)^\lor \). We obtain therefore Theorem 35 as a corollary.

To achieve our goals, we recall some elementary properties of the quantale \([L, L]_\lor, \circ\). In particular, a characterisation of tight maps shall be given using the fundamental morphisms \( c_y \) and \( a_x \). For given \( x, y \in L \), these maps are defined by

\[
\begin{align*}
    c_y(t) &:= \begin{cases} y, & t \neq \bot, \\ \bot, & t = \bot, \end{cases} \\
    a_x(t) &:= \begin{cases} \top, & t \not\leq x, \\ \bot, & t \leq x. \end{cases}
\end{align*}
\]

These two maps are tight. Indeed, \( c_y = (K_y)^\lor \), where \( K_y : L \rightarrow L \) is the constant function with value \( y \in L \), and \( a_x = (\delta_x)^\lor \) where \( \delta_x(t) = \top \) if \( t = x \) and, otherwise, \( \delta_x(t) = \bot \). Therefore \( c_y \) and \( a_x \) are tight and then, using Corollary 32, \( c_y \circ a_x \) is tight as well. Let us remind the following fact:

**Lemma 36** The set of tight maps is generated under (arbitrary) joins by the maps \( c_y \circ a_x \), for \( x \) and \( y \) in \( L \).

**Proof** Recall that \( f \) is a tight map if and only if there exists \( g \in L^L \) such that \( g^\lor = f \).

Now for such a \( g \), we have

\[
g^\lor(x) = \bigvee_{x \not\leq t} g(t) = \bigvee \{ (c_{g(t)} \circ a_t)(x) \mid t \in L \}, \tag{14}
\]

since

\[
(c_{g(t)} \circ a_t)(x) = \begin{cases} c_{g(t)}(\top) = g(t), & x \not\leq t, \\ \bot, & x \leq t. \end{cases}
\]

\[\square\]

**Proposition 37** If \( u \) is a left or right unit of the quantale \([L, L]_\lor, \circ\), then \( u \) is the identity of \( L \).

**Proof** Let \( u \in [L, L]_\lor \) be a left unit, then, for each \( y \in L \),

\[
u(y) = u(c_y(\top)) = (u \circ c_y)(\top) = c_y(\top) = y,
\]

so \( u \) is the identity of \( L \). In a similar way, if \( u \in [L, L]_\lor \) is a right unit, then, for each \( x, t \in L \),

\[
t \leq x \iff \bot = a_x(t) = (a_x \circ u)(t) \iff u(t) \leq x.
\]

Thus \( u(t) = t \), for each \( t \in L \), and we conclude again that \( u \) is the identity of \( L \). \[\square\]
Theorem 38  The Girard quantale \([L, L]_\vee, \circ, (-)^*\) is unital if and only if \(L\) is a completely distributive lattice.

Proof  If \(L\) is a completely distributive lattice, then \([L, L]_\vee = L^L\) and \(L^L\) is a Girard quantale [9, 13, 23]. Conversely, If \([L, L]_\vee\) has a unit, then by the previous Proposition this unit is the identity. Thus \(\text{id}_L \in [L, L]_\vee\) and, by Raney’s theorem, \(L\) is a completely distributive lattice.  

5 Tight Maps from Serre Galois Connections

In this section we further illustrate the use of Serre Galois connections by arguing that the Girard quantale of tight maps arises from a Serre self-adjoint Galois connection on the set of inf-preserving functions \([L, L]_\wedge\).

If \(f\) is a monotone map, then we let \(\overline{f}\) be the least \(g \in [L, L]_\wedge\) such that \(f \leq g\).

Lemma 39  If \(g\) is sup-preserving, then

\[
\overline{g \circ f} = \overline{g \circ f}. \tag{15}
\]

Proof  Since \(f \leq \overline{f}\), then \(g \circ \overline{f} \leq g \circ \overline{f}\) by monotonicity.

To derive the converse inclusion, it will be enough to show that \(g \circ \overline{f} \leq g \circ f\). From \(g \circ f \leq g \circ \overline{f}\) it follows \(f \leq \rho(g) \circ g \circ \overline{f}\). Since \(\rho(g) \circ g \circ \overline{f} \in [L, L]_\wedge\), we have \(\overline{f} \leq \rho(g) \circ g \circ \overline{f}\) and \(g \circ \overline{f} \leq g \circ f\).  

Lemma 40  We have

\[
\overline{g \circ f} = \overline{g \circ f}. \tag{16}
\]

Proof  By monotonicity, we have \(g \circ f = (g \circ f)^\wedge \leq g \circ \overline{f}\). Let us show that the converse inclusion holds:

\[
\overline{g \circ f} \leq g \circ f \iff \overline{g \circ f} \leq (g \circ f)^\wedge = (g \circ f)^\wedge \wedge
\]

\[
\iff g \circ f \leq (g \circ f)^\wedge,
\]

where the latter relation holds, since \(k^\wedge\) is the least cotight map above \(k\).  

Consider the following operation on the set \([L, L]_\wedge\):

\[
g \bullet f := \overline{g \circ f}.
\]

Lemma 41  \(\bullet\) is a semigroup operations on \([L, L]_\wedge\).

Proof  Using the previous two lemmas, compute as follows:

\[
h \bullet (g \bullet f) = \overline{h \circ g \circ f} = \overline{h \circ g \circ \overline{f}}, \quad \text{by (15)}
\]

\[
(h \bullet g) \bullet f = \overline{h \circ g \circ f} = \overline{h \circ g \circ \overline{f}}, \quad \text{by (16)}.
\]
For $f \in [L, L]_\land$, we define

$$f^\perp := \ell(f)^\wedge = \rho(f^\vee).$$

**Proposition 42** $([L, L]_\land, \bullet)$ is a quantale and $(-)^\perp$ is a self-adjoint Serre Galois connection on $[L, L]_\land$.

**Proof** We firstly argue that $(-)^\perp$ is a self-adjoint Galois connection on $[L, L]_\land$ which, moreover, satisfies the shift relations (6) w.r.t. the operation $\bullet$. We have

$$f \leq g^\perp = \rho(g^\vee) \text{ iff } g^\vee \leq \ell(f) \text{ iff } g \leq \ell(f)^\wedge = \rho(f^\vee) = f^\perp$$

and

$$g \cdot f = g^\vee \circ f \leq \rho(h^\vee) = h^\perp \text{ iff } g^\vee \circ f \leq \rho(h^\vee)$$

$$\text{iff } f \leq \rho(g^\vee) \circ \rho(h^\vee) = \rho(h^\vee \circ g^\vee)$$

$$\text{iff } (h^\vee \circ g)^\vee = h^\vee \circ g^\vee \leq \ell(f)$$

$$\text{iff } h^\vee \circ g \leq \ell(f)^\wedge$$

$$\text{iff } h \cdot g = \overline{h^\vee \circ g} \leq \ell(f)^\wedge = f^\perp.$$

Let now

$$g \backslash h := \rho(g^\vee) \circ h,$$

$$h / f := h^\perp \backslash f^\perp.$$ 

It is immediate to see that $g \cdot f \leq h$ if and only if $f \leq g \backslash h$. To see that these two relations are equivalent to $g \leq h / f$, use the shift relations. □

**Proposition 43** The nucleus induced by the self-adjoint Serre Galois connection $(-)^\perp$ is $(-)^{\vee \wedge}$. The structure $([L, L]_\land^\vee, \cdot \vee \wedge, (-)^\perp)$ is that of a Girard quantale on the set of cotight maps of $L$ and $(-)^\vee : [L, L]_\land^\vee \rightarrow [L, L]_\vee^\vee$ yields a Girard quantale isomorphism from $([L, L]_\land^\vee, \cdot \vee \wedge, (-)^\perp)$ to $([L, L]_\vee^\vee, \circ, (-)^\star)$.

**Proof** First we compute the nucleus induced by $(-)^\perp$:

$$f^{\perp \perp} = \rho(\rho(f^\vee)^\vee) = \rho(\ell(f^{\vee \wedge})) = f^{\vee \wedge}.$$

Secondly, we observe that

$$g \cdot \cdot \vee \wedge f = g^\vee \circ f^{\vee \wedge} = (g^\vee \circ f^\vee)^\wedge,$$

using equation (16). Then, for $f, g$ cotight, we have

$$(f^\perp)^\vee = \rho(f^\vee)^\vee = (f^\vee)^\star,$$

and

$$(g \cdot \cdot \vee \wedge f)^\vee = (g^\vee \circ f^\vee)^{\vee \wedge} = g^\vee \circ f^\vee.$$ 

□
Remark 1 Recall that $[L, L]_\wedge$ is isomorphic in the category of SLatt to the tensor product $L^{op} \otimes L$. The binary operation $\bullet$ described here is, up to isomorphism, the structure described in [13, Theorem 3.3.5]. Indeed, up to isomorphism, the elementary tensors are, for $x, y \in L$,

$$(y \otimes x)(t) := \begin{cases} \top, & t = \top, \\ y, & x \leq t < \top, \\ \bot, & \text{otherwise}. \end{cases}$$

Observing now that $(y \otimes x)^\vee = c_y \circ a_x$, it is then immediate to verify that the formula used in [13] to define the semigroup structure holds:

$$(v \otimes u) \bullet (y \otimes x) = c_v \circ a_u \circ (y \otimes x) = \begin{cases} \bot, & y \leq u, \\ v \otimes x, & \text{otherwise}. \end{cases}$$

6 No Unital Extensions

Let $L$ be a complete lattice which is not completely distributive. In view of Theorem 38, implying that the Girard quantale $[L, L]_\vee$ has no unit, it might be asked whether it is possible to embed $[L, L]_\vee$ into a unital Girard quantale. Without further constraints, this is always possible, by firstly adding a unit, see e.g. [13, Lemma 1.1.11], and then by embedding the resulting quantale into its Chu construction. Such an embedding, however, does not preserve the duality.

We defined unitless Frobenius quantales as certain semigroups coming with a notion of duality. Given this choice of elementary operations, a more appropriate demand is to find an embedding that preserve sups, the binary multiplication, and the dualities. Given that

$$x \setminus y = (\perp y \ast x)^\perp, \quad x / y = (y \ast x^\perp), \quad \bigwedge_{i \in I} x_i = (\bigvee_{i \in I} x_i^\perp)^\perp$$

such an embedding necessarily preserves the implications and the infima.

Definition 44 We say that a quantale embedding $\iota : (Q_0, \ast_0) \rightarrow (Q_1, \ast_1)$ is strongly continuous if it preserves infima and the two implications.

Theorem 45 proves that such an embedding into a unital quantale never exists, neither for $[L, L]_\vee$, nor for any Frobenius quantale with no unit.

Theorem 45 If a Frobenius quantale has a strongly continuous embedding into a unital Frobenius quantale, then it has a unit.

Proof Let $(Q, \ast, (\perp (-), (-)^\perp))$ be a Frobenius quantale and let $u := \bigwedge \{ x \setminus x \mid x \in Q \}$. Then, for each $x \in Q$, $u \leq x \setminus x$ and so $x \ast u \leq x$. Also, $u \leq \perp x \ast x^\perp x = x / x$, so $u \ast x \leq x$.

Let us suppose next that this quantale has a strongly continuous embedding into an unit quantale $(Q', 1, \ast)$. Let us regard $Q$ as a subset of $Q'$ closed under suprema, infima, the binary multiplication, and the two implications. Then, for each $x \in Q$, $1 \leq x \setminus x$ and so $1 \leq x$. It follows that, for each $x \in Q$, $x \setminus x \ast 1 \leq x \ast u$, thus $x = x \ast u$. Similarly, $x \leq u \ast x$ and $x = u \ast x$.

Notice in the previous proof that we might have $1 < u$. That is, strongly continuous embeddings need not to preserve units. Examples of strongly continuous embeddings among unital Girard quantales arise from chain refinements. Let $C_0, C_1$ be two complete chains

\[ \text{Springer} \]
with $C_0 \subseteq C_1$ and $C_0$ being closed under meets and suprema of $C_1$. If $C_1$ is finite, $C_0$ being closed under meets and suprema simply means that the endpoints of $C_1$ belongs to $C_0$. Under these assumptions, there is a strongly continuous embedding of $[C_0, C_0]_\vee$ into $[C_1, C_1]_\vee$ which does not preserve the unit, while it preserves composition and the duality, see [24, Proposition 81].

### 7 Tight Endomaps of $M_n$

As a last step, we investigate tight maps for some simple non-distributive finite lattices.

**Lemma 46** Let $L$ be a finite lattice. Suppose $f : L \longrightarrow L$ is a sup-preserving map such that $f(L)$ is distributive. Then $f$ is tight.

**Proof** We use Lemma 36 and show that $f = \bigvee_i c_{y_i} \circ a_{x_i}$ for some family $\{ (y_i, x_i) \mid i \in I \}$.

That is, we are going to show that, given $x \in L$, there exists a family $\{ (y_i, x_i) \mid i \in I \}$ such that $c_{y_i} \circ a_{x_i} \leq f$, for each $i \in I$, and $\bigvee_{i \in I} c_{y_i} \circ a_{x_i}(z) = f(x)$.

Write $f(x) = \bigvee_i f(y_i)$ with $f(y_i)$ join-prime in $f(L)$. For each $i \in I$, let $x_i \in L$ be maximal in the set $\{ z \mid f(y_i) \not\geq f(z) \}$. Write $f(x) = \bigvee_i f(y_i)$ with $f(y_i)$ join-prime in $f(L)$. For each $i \in I$, let $x_i \in L$ be maximal in the set $\{ z \mid f(y_i) \not\geq f(z) \}$. This shows that $c_{f(y_i)} \circ a_{x_i} \leq f$.

Observe that, for each $i \in I$, $x \not\leq x_i$, since otherwise $f(y_i) \leq f(x) \leq f(x_i)$. Consequently

$$(\bigvee_{i \in I} c_{f(y_i)} \cdot a_{x_i})(x) = \bigvee_{i \in I} c_{f(y_i)}(a_{x_i}(x)) = \bigvee_{i \in I} f(y_i) = f(x).$$

$\Box$

**Proposition 47** If $L$ is either the pentagon $N_5$ or the diamond lattice $M_3$ (see Fig. 2), then $[L, L]_\vee = [L, L]_\vee \setminus [L, L]_\times$, where $[L, L]_\times$ is the set of order isomorphisms of $L$.

**Proof** If $f \in [L, L]_\times \cap [L, L]_\vee$, then $id_L = f^{-1} \circ f \in [L, L]_\vee$, thus $L$ is distributive by Raney theorem. Therefore, since neither $N_5$ nor $M_3$ are distributive, if $L$ is any of these two lattices, then $[L, L]_\times \cap [L, L]_\vee = \emptyset$. On the other hand, if $L \in \{ N_5, M_3 \}$ and $f \notin [L, L]_\times$, then $f$ is not injective and its image has cardinality at most 4. Since every lattice of cardinality at most 4 is distributive, then $f$ is tight, using Lemma 46.

Notice that, for $L = N_5$, $[L, L]_\vee = \{ \pi \circ L \}$, while for $L = M_3$, $[L, L]_\vee$ is isomorphic to the permutation group on 3 elements.

The relation $[L, L]_\vee = [L, L]_\vee \setminus [L, L]_\times$ does not hold in general. We exemplify this by characterizing $[L, L]_\vee$ for $L = M_n$, the modular lattice of height 3 $n$ atoms. The cadinalities

Fig. 2 The pentagon $N_5$ (left) and the diamond $M_3$ (right)
of \([M_n, M_n]_\lor\) have been described in [16]. To describe \([M_n, M_n]_\lor\), let \(\text{At}(M_n)\) denote the set of atoms of \(M_n\).

**Proposition 48** A sup-preserving map \(f : M_n \rightarrow M_n\) is tight if and only if its image \(f(M_n)\) is distributive, if and only if the cardinality of \(f(M_n) \cap \text{At}(M_n)\) is at most 2.

**Proof** If \(|f(M_n) \cap \text{At}(M_n)| \leq 2\), then \(|f(M_n)| \leq 4\) and \(f(M_n)\) is a distributive lattice. If \(|f(M_n) \cap \text{At}(M_n)| \geq 3\), then \(f(M_n)\) is isomorphic to \(M_k\) with \(3 \leq k \leq n\), thus it is not distributive.

By Lemma 46, if \(f(M)\) is distributive, then \(f\) is tight. Conversely, suppose that \(|f(M_n) \cap \text{At}(M_n)| \geq 3\). Let \(a, b, c\) be such that \(f(a), f(b), f(c) \in \text{At}(M_n)\) and observe that \(f(a) \neq \bot\).

Then

\[
f^\lor \lor (a) = \bigvee_{a \neq y} \bigwedge_{t \neq y} f(t) = \bigvee_{y \in \text{At}(M_n), y \neq a} \bigwedge_{t \in \text{At}(M_n), t \neq y} f(t)
\]

so \(f\) is not tight. \(\square\)

**Corollary 49** We have

\[
|M_n, M_n]_\lor | = 2 + 2n + 2n^2 + \binom{n}{2}n(n - 1) = \frac{1}{2}n^4 - n^3 + \frac{5}{2}n^2 + 2n + 2.
\]

**Proof** If a tight function is not of the form \(c_\bot, c_\top, c_j, a_j\) (for \(j \in \text{At}(M_n)\)), \(c_j \circ a_m, c_j \lor a_m\) (\(j, m \in \text{At}(M_n)\)), then it takes two atoms and sends it to distinct atoms, while all other atoms are sent to \(\top\). \(\square\)

Observe that if \(f\) is tight and \(|f(M_n)| \leq 3\), then \(f\) has one of this form:

\[
c_y \circ a_x, \quad c_y \lor a_x,
\]

where \(x, y \in M_n\). Notice that \(c_y \lor a_x\) is the map which sends \(t\) such that \(\bot < t \leq x\) to \(y\) and \(t\) such that \(t \leq x\) to \(\top\). If \(|f(M_n)| = 4\), then \(f\) is of the form

\[
f_{x_1, y_1, x_2, y_2} := c_{y_2} \circ a_{x_1} \lor c_{y_1} \circ a_{x_2},
\]

where \((x_1, x_2)\) and \((y_1, y_2)\) are pairs of distinct atoms of \(M_n\). Observe that

\[
f_{x_1, y_1, x_2, y_2}(t) = \begin{cases} 
\bot, & t = \bot, \\
y_1, & t = x_1, \\
y_2, & t = x_2, \\
\top, & \text{otherwise}.
\end{cases}
\]

The following proposition achieves the goal of giving a more immediate formula for the negation in the quantale \((|M_n, M_n]_\lor, \circ\).

**Proposition 50** We have

\[
(c_y \circ a_x)^* = c_x \lor a_y \quad \text{and} \quad (f_{x_1, y_1, x_2, y_2})^* = f_{y_1, x_2, y_2, x_1}
\]

where above \(x_1 \neq x_2\) and \(y_1 \neq y_2\).
Let us call a quantale positive. The restriction of \( f \) to the set of fixed points of \( h \) yields an order isomorphism with the image of \( f \). Therefore, there can be at most two atoms that are fixed by \( h \). The map that is equal to \( h \) except that it sends \( \perp \) to \( \perp \) is then join-preserving and tight, by Proposition \( 48 \). Then \( f \setminus f = h^\vee \) is this map, and clearly this map is above the identity.

\[ (c_y \circ a_x)^* = (c_y \wedge a_x)^* = (c_y)^* \vee (a_x)^* = a_y \vee c_x. \]

Using this and assuming that \( x_1, x_2 \) and \( y_1, y_2 \) are pairs of distinct atoms, we also compute as follows:

\[ (f_{x_1, y_1, x_2, y_2})^* = ((c_{y_2} \circ a_{x_1}) \vee (c_{y_1} \circ a_{x_2}))^* = (c_{y_2} \circ a_{x_1})^* \wedge (c_{y_1} \circ a_{x_2})^* = (a_{y_2} \vee c_{x_1}) \wedge (a_{y_1} \vee c_{x_2}) = f_{y_1, x_2, y_2, x_1}, \]

where the last equality is derived by observing that the pointwise meet of \( c_{x_1} \vee a_{y_2} \) and \( c_{x_2} \vee a_{y_1} \) satisfies the pattern for \( f_{y_1, x_2, y_2, x_1} \) given in equation (17). For example, if \( t = y_1 \), then \( (a_{y_2} \vee c_{x_1})(t) = \top \) and \( (a_{y_1} \vee c_{x_2})(t) = x_2 \). A similar argument is used when \( t = y_2 \). Finally, if \( t \nleq y_i, i = 1, 2 \), then \( (a_{y_2} \vee c_{x_1})(t) = (a_{y_1} \vee c_{x_2})(t) = \top. \]

We might further study quantales of the form \( [L, L]' \), by trying to identify similarities and differences with unital quantales. Let us say that an element \( p \in Q \) is positive if, for all \( x \in Q \),

\[ x \leq x \circ p \wedge p \circ x. \]

Let us call a quantale positive if each element of the form \( x \setminus x \) or \( x/x \) is positive. The proof Theorem 45 immediately yields the following statement.

**Lemma 51** A Frobenius quantale is unital if and only if it is positive and positive elements are closed under infima.

Similarly, a residuated partially-ordered semigroup (see [10, §3.2]) is positive if each element of the form \( x \setminus x \) or \( x/x \) is positive. It was argued in [4, Theorem 3.27] that a residuated partially-ordered semigroup embeds into a residuated partially-ordered monoid if and only if it is positive.

The following statement shows that positivity is no longer sufficient when we step from residuated semigroups to Frobenius quantales and residuated lattices—where for residuated lattices, a straightforward modification of Theorem 45 shows that the finite unitless quantales \( [M_n, M_n]' \) cannot be embedded into unital ones.

**Proposition 52** The quantales \( [M_n, M_n]' \) are positive.

**Proof** Let us show that \( f \setminus f \) is above the identity whenever \( f \in [M_n, M_n]' \). It is easily seen that, within \( [M_n, M_n]' \), \( f \setminus g = (\rho(f) \circ g)^{\wedge \vee} \).

Let \( h = \rho(f) \circ f \) and recall that \( id_{M_n} \leq h \). It follows that, for each \( a \in At(M_n) \),

\[ h(a) = \begin{cases} \top, & f(a) = f(\top), \\ a, & f(a) < f(\top). \end{cases} \]

The restriction of \( f \) to the set of fixed points of \( h \) yields an order isomorphism with the image of \( f \). Therefore, there can be at most two atoms that are fixed by \( h \). The map that is equal to \( h \) except that it sends \( \perp \) to \( \perp \) is then join-preserving and tight, by Proposition 48. Then \( f \setminus f = h^{\wedge \vee} \) is this map, and clearly this map is above the identity.
The reader will recognize then that a main difference with unital quantales is that positive elements are not closed under infima in $[M_n, M_n]_\lor$.

**Acknowledgements** The authors are thankful to Nick Galatos, the anonymous referee, and the editor for precious pointers, remarks, and guidance for improving a first version of this paper.

**Funding** This work was supported by the Agence Nationale de la Recherche, Project LAMBDACOMB ANR-21-CE48-0017.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Competing interest** The authors declare they have no competing interests.

**References**

1. Abramsky, S., Heunen, C.: H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics. In: Mathematical Foundations of Information Flow. Proc. Symp. Appl. Math., 71, 1–24. American Mathematical Society, Providence, RI (2012). https://doi.org/10.1090/psapm/071/599
2. Andréka, H., Mikulás, S.: Lambek calculus and its relational semantics: completeness and incompleteness. J. Logic Lang. Inf. 3(1), 1–37 (1994)
3. Barr, M.: *-autonomous categories. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg 752, 140 (1979). https://doi.org/10.1007/BFb0064582
4. Blount, K.: On the structure of residuated lattices. Ph.D. thesis, Vanderbilt University (1999)
5. Buszkowski, W.: Lambek grammars based on pregroups. In: de Groote, P., Morrill, G., Retoré, C. (eds.) Logical Aspects of Computational Linguistics, 4th International Conference, LACL 2001, Le Croisic, France, June 27–29, 2001, Proceedings. Lecture Notes in Computer Science, 2099, 95–109. Springer (2001). https://doi.org/10.1007/3-540-48199-0_6
6. Conway, J.B.: A Course in Operator Theory. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI 21, 372 (2000). https://doi.org/10.1090/gsm/021
7. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, 2nd edn., p. 298. Cambridge University Press, New York (2002). https://doi.org/10.1017/CBO9780511809088
8. Egger, J.M., Kruml, D.: Girard couples of quantales. Appl. Categ. Struct. 18(2), 123–133 (2010). https://doi.org/10.1007/s10485-008-9138-3
9. Eklund, P., Gutiérrez García, J., Höhle, U., Kortelainen, J.: Semigroups in Complete Lattices. Dev. Math. 54, 326 (2018). https://doi.org/10.1007/978-3-319-78948-4
10. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated lattices: an algebraic glimpse at substructural logics. Stud. Logic Found. Math. 151, (2007). https://doi.org/10.1016/S0049-227X(07)80005-X
11. Higgs, D.A., Rowe, K.A.: Nuclearity in the category of complete semilattices. J. Pure Appl. Algebra 57(1), 67–78 (1989). https://doi.org/10.1016/0022-4049(89)90028-5
12. Kalman, J.A.: Lattices with involution. Trans. Am. Math. Soc. 87, 485–491 (1958). https://doi.org/10.2307/1939112
13. Kruml, D., Paseka, J.: Algebraic and categorical aspects of quantales. Handbook of Algebra, North-Holland 5, 323–362 (2008). https://doi.org/10.1016/S1570-7954(07)05006-1
14. Kuznetsov, S.L.: Relational models for the Lambek calculus with intersection and unit. In: Fahrenberg, U., Gehrke, M., Santocanale, L., Winter, M. (eds.) Relational and Algebraic Methods in Computer Science—19th International Conference, RAMiCS 2021, Marseille, France, November 2–5, 2021, Proceedings. Lecture Notes in Computer Science, 13027, 258–274. Springer (2021). https://doi.org/10.1007/978-3-030-88701-8_16
15. Ore, O.: Galois connexions. Trans. Am. Math. Soc. 55(3), 493–513 (1944)
16. Quintero, S., Ramírez, S., Rueda, C., Valencia, F.: Counting and computing join-endomorphisms in lattices. In: Fahrenberg, U., Jipsen, P., Winter, M. (eds.) Relational and Algebraic Methods in Computer Science—18th International Conference, RAMiCS 2020, Palaiseau, France, April 8–11, 2020, Proceed-
ings [postponed]. Lecture Notes in Computer Science, Springer **12062**, 253–269 (2020). https://doi.org/10.1007/978-3-030-43520-2_16
17. Raney, G.N.: Tight Galois connections and complete distributivity. Trans. Am. Math. Soc. **97**, 418–426 (1960). https://doi.org/10.2307/1993380
18. Rosenthal, K.I.: Quantales and Their Applications. Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow **234**, 165 (1990)
19. Rosenthal, K.I.: A note on Girard quantales. Cahiers de Topologie et Géométrie Différentielle Catégoriques **31**(1), 3–11 (1990)
20. Rump, W.: Frobenius quantales, Serre quantales and the Riemann–Roch theorem. Studia Logica (2021). https://doi.org/10.1007/s11225-021-09970-1
21. Santocanale, L.: Dualizing sup-preserving endomaps of a complete lattice. In: Spivak, D.I., Vicary, J. (eds.) Proceedings of the 3rd Annual International Applied Category Theory Conference 2020, ACT 2020, Cambridge, USA, 6–10th July 2020. EPTCS, **333**, 335–346 (2020). https://doi.org/10.4204/EPTCS.333.23
22. Santocanale, L.: Skew metrics valued in Sugihara semigroups. In: Fahrenberg, U., Gehrke, M., Santocanale, L., Winter, M. (eds.) Relational and Algebraic Methods in Computer Science—19th International Conference, RAMiCS 2021, Marseille, France, November 2–5, 2021, Proceedings. Lecture Notes in Computer Science, **13027**, 396–412. Springer (2021). https://doi.org/10.1007/978-3-030-88701-8_24
23. Santocanale, L.: The involutive quantaloid of completely distributive lattices. In: Fahrenberg, U., Jipsen, P., Winter, M. (eds.) Relational and Algebraic Methods in Computer Science—18th International Conference, RAMiCS 2020, Palaiseau, France, April 8–11, 2020, Proceedings [postponed]. Lecture Notes in Computer Science, vol. 12062, pp. 286–301. Springer (2020). https://doi.org/10.1007/978-3-030-43520-2_18
24. Santocanale, L., Gouveia, M.J.: The continuous weak order. J. Pure Appl. Algebra (2021). https://doi.org/10.1016/j.jpaa.2020.106472
25. Yetter, D.N.: Quantales and (noncommutative) linear logic. J. Symb. Logic **55**(1), 41–64 (1990). https://doi.org/10.2307/2274953

**Publisher’s Note**  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.