A Theoretical Study of Second Lyapunov Method on Stability of Zero Solution for a Delay Differential System

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Abstract. This study examined the asymptotic stability of delay differential systems using Lyapunov’s second method. It showed that the zero solutions are asymptotically stable when satisfying some necessary and sufficient conditions. The solution origin of the given system is asymptotically stable for small perturbations, which has no effect on equilibrium system if some conditions are held and it is asymptotically stable for large perturbations that moving system of its equilibrium then its comeback to it if other conditions are held. The proof of theorems in this article results by using and applying Gronwell-Bellman inequality which leads us to main result.

Keywords: Lyapunov function, Delay differential equations, Asymptotic, Stability.

1. Introduction

In the past ten years, the control community has intensively studied time delay systems TDS analysis and control in response to evolving theories of adapted control and an essential requirement of applications in information technology [1, 2, 3, 4]. In addition, they are known as systems with after effect that belong to the category of functional differential equations, which are infinite-dimensional contrasted with ordinary differential equations. Time-delays occur in various engineering systems including aircraft, medicine, chemical control, laser models and Internet biology [5, 6]. As the case with systems that are free of delay, the Lyapunov method is efficient for analyzing stability of TDS. For TDS, two main methods of Lyapunov exist: the Lyapunov-Krasovskii [7, 8, 9] and Lyapunov-Razumikhin. There are numerous authors dealt with stability of TDS as being studied in [10, 11, 12, 13, 14, 15]. In this article, the second Lyapunov method was employed to determine the asymptotic stability of a null solution of proposed system.

2. Preliminaries.

Definition (2.1): If $F$ is a function from $T \times R^n$ to $T \times R^n$, where $T = [0, \infty)$ then $F$ satisfies Lipschitz condition, if it satisfies the inequality in [2],

$$\|F(t,x_1,x_2,...,x_n) - F(t,y_1,y_2,...,y_n)\| \leq \sum_{k=1}^{n} l_k \|x_k - y_k\|,$$

Where $t \in [0, \infty)$.

Definition (2.2): Assume that $x(t) = \theta(t)$ is a solution for the system [5],

$$\dot{x}(t) = ax(t)$$

(1)
is defined for \( t_0 \leq t \). Let \( x(t) = \vartheta(t) \) be another solution for (1) with initial condition \( x_0 = \vartheta(t_0) \), we say that the solution \( x(t) = \vartheta(t) \) is stable with respect to Lyapunov’s concept if there exist for any number \( \epsilon > 0 \) and \( \delta > 0 \) such that

\[
\|\vartheta(t_0) - \vartheta(t_0)\| \leq \delta \implies |\vartheta(t) - \vartheta(t)| \leq \epsilon
\]

This means that if the solution \( \vartheta(t) \) was closed from \( \vartheta(t) \) at the initial condition, so it will be too for \( t_0 \leq t \).

**Definition (2.3):** We say that the solution \( x(t) = \theta(t) \) is asymptotically stable, if it is stable w.r.t Lyapunov’s concept and

\[
\lim_{t \to \infty} \|\theta(t) - \vartheta(t)\| = 0 \quad [6]
\]

Now, we review some basics of Lyapunov’s theorem of stability. If \( v(x) : D \to \mathbb{R} \) is a function defined on \( D \subseteq \mathbb{R}^n \), then the cases are possible.

**Case A.**

The solution origin of system (1) is asymptotically stable for small perturbations, which has no effect on equilibrium system if these conditions are held:

C1: \( v(x) > 0 \), for \( x > 0 \).

C2: \( v(0) = 0 \).

C3: \( v(x) \) has partial derivatives.

C4: \( \dot{v}(x) < 0 \), for \( x \neq 0 \) and \( x \) lies at neighborhood of the origin.

**Case B.**

The solution origin of system (1) is asymptotically stable for large perturbations that moving system of its equilibrium then it comeback to it if these conditions are held:

C1: \( v(x) > 0 \), for \( x > 0 \).

C2: \( v(0) = 0 \).

C3: \( v(x) \) has a continuous partial derivative.

C4: \( \dot{v}(x) < 0 \), for \( x \neq 0 \) and \( x \) lies at neighborhood of the origin and satisfying the condition \( v(x) \to 0 \), when \( \|x\| \to \infty \).

**Remark (2.4):** The function \( v(x) \) is called Lyapunov’s function, this function exist mean absolutely that the system is stable [10].

**Definition (2.5):** (Gronwall-Bellman inequality) Assuming \( x(t) \) is a continuous function and [10]:

\[
h(r) \geq 0, \text{ if } x(t) \leq \omega(t) + \int_{t_0}^{t} h(r)x(r)dr, t \geq t_0, \text{ where } \omega(t) \text{ is decreasing and integrable function then}
\]

\[
x(t) \leq \omega(t)e^{\int_{t_0}^{t} h(r)dr}
\]
3. **Goal of Research**

We will study the stability of this system of differential equation:

\[ \dot{x}(t) = a(t) x(t) + b(t) x(t - \varphi). \]  \hspace{1cm} (2)

In our work, we shall discuss the following two problems.

**Problem (3.1):**

Studying stability of system (2) without delay that is \( \varphi = 0 \). So the system will become

\[ \dot{x}(t) = G(t)x(t), \]

where \( G(t) = a(t) + b(t) \).

**Problem (3.2):**

We are going to study the stability of system (2) as is.

4. **Results and Discussion:**

Suppose that the following system exists:

\[ \dot{x}(t) = G(t) x(t). \]  \hspace{1cm} (3)

We seeking the Lyapunov function whose quadratic formula is

\[ v(x, t) = x^T H(t)x. \]  \hspace{1cm} (4)

Differentiating (4) with respect to \( t \) we find

\[ \dot{v}(x, t) = x^T H(t)x + x^T \dot{H}(t)x + \dot{x}^T H(t)x. \]

\[ \implies \dot{v}(x, t) = \dot{x}^T G^T H(t)x + \dot{x}^T H(t)x + \dot{x}^T H(t)x. \]  \hspace{1cm} (5)

Substituting \( \dot{x} \) into (5),

\[ \implies \dot{v}(x, t) = \dot{x}^T G^T H(t)x + \dot{x}^T H(t)x + \dot{x}^T G(t)x. \]  \hspace{1cm} (6)

Taking out the common factor \( x^T \dot{x} \),

\[ \implies \dot{v}(x, t) = x^T \left( G^T H(t) + \dot{H}(t) + H(t) \dot{G}(t) \right)x. \]  \hspace{1cm} (7)

Assuming that

\[ -w(x, t), \]

\[ v(x, t) = \]

where \( J(t) \) is a square matrix of rank \( n \times n \) and \( w(t) = x^T J(t)x(t) \) may be determined from \( G(t) \).

From (7) and (8) we find that

\[ x^T \left( G^T H(t) + \dot{H}(t) + H(t) \dot{G}(t) \right)x = x^T J(t)x(t) \]
\[ \Rightarrow G^T H(t) + \dot{H}(t) + H(t) G(t) = -J(t). \] (9)

Multiplying each term of (9) by \( e^{\int_0^t G_T(r)dr} \) and \( e^{\int_0^t G(r)dr} \) from left and right respectively for \( H(t) \)

\[
e^{\int_0^t G_T(r)dr} G^T H(t)e^{\int_0^t G(r)dr} + e^{\int_0^t G_T(r)dr} \dot{H}(t)e^{\int_0^t G(r)dr} + e^{\int_0^t G_T(r)dr} H(t) G(t)e^{\int_0^t G(r)dr} = -e^{\int_0^t G_T(r)dr} J(t)e^{\int_0^t G(r)dr}
\]

\[
\Rightarrow e^{\int_0^t G_T(r)dr} \dot{H}(t)e^{\int_0^t G(r)dr} = -e^{\int_0^t G_T(r)dr} J(t)e^{\int_0^t G(r)dr}
\]

\[
\Rightarrow \frac{d}{dt} \left( e^{\int_0^t G_T(r)dr} H(t)e^{\int_0^t G(r)dr} \right) = -e^{\int_0^t G_T(r)dr} J(t)e^{\int_0^t G(r)dr}.
\]

Integrating for \( t \),

\[
e^{\int_0^t G_T(r)dr} H(t)e^{\int_0^t G(r)dr} = -\int_0^t e^{\int_0^\omega G_T(r)dr} J(\omega)e^{\int_0^\omega G(r)dr} d\omega
\]

\[
\Rightarrow H(t) = -e^{-\int_0^t G_T(r)dr} \left[ \int_0^t e^{\int_0^\omega G_T(r)dr} J(\omega)e^{\int_0^\omega G(r)dr} d\omega \right] e^{-\int_0^t G(r)dr}
\]

\[
\Rightarrow H(t) = -\int_0^t e^{\int_0^\omega G_T(r)dr} J(\omega)e^{\int_0^\omega G(r)dr} d\omega.
\]

Since each of \( v(x, t) \) and \( w(x, t) \) is strictly positive, there is a \( \rho(t_0) \) so that

\[
\rho(t_0) \cdot v(x, t) \leq w(x, t) \Rightarrow \rho(t_0) \cdot v(x, t) \leq -\dot{v}(x, t)
\]

\[
-\rho(t_0) \cdot v(x, t) \geq \dot{v}(x, t) \Rightarrow \dot{v}(x, t) \leq -\rho(t_0)
\]

\[
\ln v(x,y) \leq -\rho(t_0) + \ln v(0) \Rightarrow v(x,y) \leq e^{-\rho(t_0)} v(x_0,y_0).
\]

But

\[
v(x,t) \geq \alpha(t) \|x(t)\|^2
\]

\[
\Rightarrow \|x(t)\|^2 \leq \frac{1}{\alpha(t)} v(x,t)
\]

\[
\Rightarrow \|x(t)\|^2 \leq \frac{1}{\alpha(t)} e^{-t\rho(t_0)} v(x_0,y_0), \forall t \in (0, \infty)
\]

\[
\Rightarrow \|x(t)\|^2 \leq \frac{1}{\alpha(t_0)} e^{-t\rho(t_0)} v(x_0,y_0).
\]

And we have

\[
v(x,t) \leq \beta(t) \|x(t)\|^2
\]
\[ v(x_0, t_0) \leq \beta(t_0)\|x(0)\|^2 \quad (15) \]

\[ \|x(t)\|^2 \leq \frac{1}{\alpha(t_0)} e^{-\rho(t_0)\|y_0\|^2} \leq \frac{\beta(t_0)}{\alpha(t_0)} e^{-\rho(t_0)\|x(0)\|^2} \quad (16) \]

\[ \|x(t)\|^2 \leq \frac{\beta(t_0)}{\alpha(t_0)} e^{-\rho(t_0)\|x(0)\|^2}. \quad (17) \]

So, the last inequality gives us the necessary condition for system’s stability

\[ \lim_{t \to \infty} \|x(t)\|^2 = 0. \]

Hence, (3) is asymptotically stable.

We now begin our study of the stability of system (2)

\[ \dot{x}(t) = a(t) \cdot x(t) + b(t) \cdot x(t - \varphi). \quad (18) \]

We will take a special case of Gronwall-Bellman inequality

\[ x(t) = e^{\int_0^t a(r)dr} u(t). \quad (19) \]

Substituting (19) into (18), we see that

\[ \dot{x}(t) = a(t) \cdot e^{\int_0^t a(r)dr} u(t) + b(t) \cdot \int_0^t e^{\int_0^r a(\sigma)\,d\sigma} \, d\sigma u(t) \]

\[ \Rightarrow a(t) \cdot x(t) + b(t) \cdot x(t - \varphi) = a(t) \cdot e^{\int_0^t a(r)dr} u(t) + b(t) \cdot \int_0^t e^{\int_0^r a(\sigma)\,d\sigma} \, d\sigma u(t) \]

\[ \Rightarrow a(t) \cdot e^{\int_0^t a(r)dr} u(t) + b(t) \cdot e^{\int_0^t a(r)dr} u(t - \omega) = a(t) \cdot e^{\int_0^t a(r)dr} u(t) + b(t) \cdot \int_0^t e^{\int_0^r a(\sigma)\,d\sigma} \, d\sigma u(t - \omega) \]

\[ \Rightarrow \dot{u}(t) = b(t) \cdot e^{\int_0^t a(r)dr} u(t - \omega). \]

Moreover, if taking a quadratic Lyapunov function \( v = u^T H(t) u \), notice that, its hold the inequality

\[ \alpha \|u\|^2 \leq v(x) \leq \beta \|u\|^2. \]

Differentiating, we see that

\[ \dot{v} = u^T H(t) u + u^T \dot{H}(t) u + u^T H(t) \dot{u} \]

\[ \Rightarrow \dot{v} = u^T (t - \omega) e^{\int_{t-\omega}^t a(r)dr} b^T H(t) u(t) + u^T \dot{H}(t) u(t) + u^T H(t) b(t) e^{\int_0^t a(r)dr} u(t - \omega). \]

Integrating from zero to \( t \),
\[ v(x) = v(x(0)) + \int_0^t u^T(r) e^{T(r)} a(z) dz b^T(r) H(r) u(r) + u^T(r) \dot{H}(r) u(r) \]

\[ + u^T(r) H(r) b(r) e^{T(r)} a(z) dz u(r - \omega) dr. \]

Using the inequality

\[ \alpha \|u\|^2 \leq v(t) \leq \beta \|u\|^2. \]

We see that

\[ \implies v(x) \leq v(x(0)) + \frac{1}{\alpha} \int_0^t \left[ \sqrt{v^T(r - \omega)} e^{T(r)} a(z) dz b^T(r) H(r) \sqrt{v(r)} + \sqrt{v^T(r)} \dot{H}(r) \sqrt{v(r)} \\ + \sqrt{v(r)} H(r) b(r) e^{T(r)} a(z) dz \sqrt{r - \omega} \right] dr. \]

By taking the non-decreasing norm \( \|v(t)\|_I = \sup \|v(r)\|, \) where \(-\omega \leq r \leq \omega,\) it is observed that

\[ \implies v(x) \leq v(x(0)) + M + \frac{1}{\alpha} \int_0^t \left( 2 \|b(r)\| \|H(r)\| + e^{T(r)} a(z) dz \right) dr. \]

Applying Gronwell-Bellman inequality, we see that

\[ \implies v(x) \leq (v(x(0)) + M) e^{\frac{1}{\alpha} \int_0^t \left( 2 \|b(r)\| \|H(r)\| + e^{T(r)} a(z) dz \right) dr}. \]

By inequality, we have that

\[ \alpha \|u\|^2 \leq v(t) \leq \beta \|u\|^2 \]

\[ \implies \alpha \|x\|^2 \leq (\beta \|x\|^2 + M) e^{\frac{1}{\alpha} \int_0^t \left( 2 \|b(r)\| \|H(r)\| + e^{T(r)} a(z) dz \right) dr}. \]

\[ \implies \|x\|^2 \leq \frac{1}{\alpha} (\beta \|x\|^2 + M) e^{\frac{1}{\alpha} \int_0^t \left( 2 \|b(r)\| \|H(r)\| + e^{T(r)} a(z) dz \right) dr}. \]

Hence, this is a condition for the system stability. Finally, it is observed that the criterion of stability is satisfied when the square of the norm does not exceed
\[ \frac{1}{\alpha} \left( \beta \|x\|^2 + M \right) e^{\frac{1}{\alpha} \int_0^t \left( 2\|b(r)\|\|H(r)\| + e^{e^{r-a}\|a(r)\|} + \|H'(r)\| \right) dr} \]

Moreover, the criterion is satisfied when

\[ \lim_{t \to \infty} \frac{1}{\alpha} \int_0^t \left( 2\|b(r)\|\|H(r)\| + e^{e^{r-a}\|a(r)\|} + \|H'(r)\| \right) dt < 0. \]

We note that condition of stability depends on delay \( \omega \) and \( a(t), b(t) \).

5. **Conclusions:**

We proved in this article the asymptotic stability of the null solution of proposed system (2), that is if the trivial solution of the system, \( x(t) = (a(t) + b(t))x(t) \), is asymptotically stable and formula,

\[ \|x\|^2 \leq \frac{1}{\alpha} \left( \beta \|x\|^2 + M \right) e^{\frac{1}{\alpha} \int_0^t \left( 2\|b(r)\|\|H(r)\| + e^{e^{r-a}\|a(r)\|} + \|H'(r)\| \right) dr} \]

holds, then the zero solution for (2) is asymptotically stable too, when \( M = \sup_{-\omega \leq r \leq 0} v(x(t)) \) and \( v(x(t)) \) is the Lyapunov function and \( \alpha, \beta \) are eigenvalues for the matrix \( G \). For the future works, we suggest to authors to study the asymptotic stability of solutions of stochastic differential delay equations by using the technique of Lyapunov-Krasovskii functional. So, may someone can study Lyapunov’s second method in problems of the stability of solutions of systems with impulse effect.

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