Gribov horizon in Noncommutative QED

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Abstract

It is known that Noncommutative QED (NCQED) exhibits Gribov ambiguities in the Landau gauge. These ambiguities are related to zero modes of the Faddeev-Popov operator and arise in the ghost propagator when it has a pole. In this work, we establish a positive Faddeev-Popov operator for NCQED and the condition for the ghost propagator not to have poles, the so-called Gribov no-pole condition. This condition is implemented in the path integral and allows for the calculation of the photon propagator in momentum space, which is dependent on the squared non-commutativity parameter. In the commutative limit, the standard QED is recovered.
1 Introduction

The occurrence of so-called Gribov ambiguities for Yang-Mills theories was first discussed in [1], where Gribov showed that for non-Abelian gauge theories on flat topologically trivial space-times, gauge fixing is problematic. Namely, it is not possible to choose for the gauge potential one representative on each gauge orbit, that is, by considering the quotient of the space of gauge connections with respect to the gauge group. Gribov ambiguity amounts to the fact that there are in general different field configurations which obey the same gauge-fixing condition, but which are related by a gauge transformation, i.e., they are on the same gauge orbit. As first shown by Singer [2] and independently by Narasimhan and Ramadas [3], this occurrence can be given a precise mathematical characterisation in terms of topological non-triviality of $G$, the pertinent gauge group involved. On this basis, QED is singled out, with the relevant homotopy group $\Pi_1(\hat{U}(1))$ being trivial. For Quantum Chromodynamics (QCD), however, Gribov copies pose a real threat to the accurate description of the low-energy regime of the theory, the gauge group of $SU(N)$ Yang-Mills theories being nontrivial for $N \geq 2$. Over the last few decades, several solutions have been proposed in order to restrict the functional integral to the region in the space of connections where only one representative for each gauge orbit is present, the so-called first Gribov region. The most notable efforts are the Gribov-Zwanziger approach [5] and its refined version [7].

In [8, 9] the problem has been addressed in the context of gauge theories on noncommutative space-time. It has been shown that, differently from the commutative case, noncommutative QED (NCQED) exhibits Gribov ambiguity and gauge configurations have been found, with an infinite number of copies. The goal of the present work is to investigate the characterisation of the first Gribov region. Following the same approach as in non-Abelian gauge theories [7], the ensuing constraint on the physical space of connections is implemented by requiring that the ghost propagator does not develop poles. This implies a modification of the photon propagator.

The paper is organised as follows. In section 2, we shortly review the problem in the framework of standard gauge theory. We then recall in section 3 the formulation of QED in the noncommutative setting with Moyal type noncommutativity and review the derivation of the equation for Gribov copies introduced in [8, 9]. In section 4, we apply the techniques developed in [7] to characterise the first Gribov region for NCQED. In section 5, we evaluate the no-pole condition in the Landau gauge and find the Gribov region function $V(\Omega)$. Finally, we calculate the photon propagator in noncommutative momentum space. We conclude with some remarks and future perspectives.

2 Gribov ambiguity in Yang-Mills

Let $M = \mathbb{R}^4$ be the space-time manifold and let us define the group of gauge transformations, $G = \hat{U}(N)$, as the smooth maps $g : M \to U(N)$, with boundary condition $g(x) \to 1$ as $|x| \to \infty$.

A pure gauge theory of fundamental interactions, without matter fields, is a theory where the dynamical fields are the gauge connections, locally represented by Lie algebra valued one-forms $A$, with $F = dA + A \wedge A$ the local curvature two-form. When $M$ is the Euclidean space-time,
the classical action describing the dynamics is

\[ S = \frac{1}{4} \int F_{\mu\nu}^a F^{\mu\nu a} d^4x \quad (2.1) \]

where \( F = F_{\mu\nu}^a \tau_a dx^\mu \wedge dx^\nu \) and \( \tau_a \) are the generators of the Lie algebra. The quadratic, free, sector of the theory, upon integrating by parts with suitable boundary conditions, can be written as

\[ S = \frac{1}{2} \int d^n x \int d^n y \ A^\mu_{\nu}(x) \Delta_{ab}^{\mu\nu}(x,y) A^\nu_{\mu}(y), \quad (2.2) \]

with

\[ \Delta_{ab}^{\mu\nu}(x,y) = (-\partial^2 \delta^{\mu\nu} + \partial^\mu \partial^\nu) \delta^{(4)}(x-y) \delta_{ab} \quad (2.3) \]

and

\[ Z[J] = \int_{\mathcal{A}} [d\mu(A)] \exp \left( -\frac{1}{2}(S[A] + S_I[A,J]) \right), \quad (2.4) \]

the generating functional and \( \mathcal{A} \) the space of gauge connections. If the operator \( \Delta_{ab}^{\mu\nu} \) were invertible, as it is the case for scalar theories, the Gaussian integral in (2.4) could be formally performed to give:

\[ Z[J] = (\det \Delta)^{-\frac{1}{2}} \exp \left( \frac{1}{2} \int J \Delta^{-1} J \right) \quad (2.5) \]

with \( \Delta^{-1} \) the Euclidean propagator, but this is not the case for gauge theories unless we manage to integrate over equivalence classes of gauge connections. Indeed, because of gauge invariance of the free action under \( A \to A^g = gA g^{-1} + dg g^{-1} \), field configurations of the form \( dg g^{-1} \) (so-called pure gauge terms) are in the kernel of \( \Delta_{ab}^{\mu\nu} \), such that

\[ \Delta_{ab}^{\mu\nu} \partial^\nu \delta_{gg^{-1}} = 0 \quad (2.6) \]

showing that the operator (2.3) has eigenvectors with zero eigenvalues (so-called zero modes). To restrict the functional integration to gauge inequivalent potentials, one has to limit the functional integral (2.4) to the quotient space \( \mathcal{B} = \mathcal{A}/\mathcal{G} \). Mathematically, this amounts to choosing a surface \( \Sigma_f \subset \mathcal{A} \) which possibly intersects the gauge orbits only once. The choice of \( \Sigma_f \) is physically rephrased as a gauge fixing, \( f(A) = h \), for some chosen functions \( f, h \). Within the standard quantization procedure of gauge theories, this is achieved through the introduction of the Faddeev-Popov determinant, which is only unambiguous when no topological obstructions arise. But precisely the latter is at the origin of Gribov ambiguities. Therefore, we shortly summarise the standard procedure and the related topological issues.

The kinematical configuration space \( \mathcal{A} \) is usually assumed to be globally equivalent to the product \( \mathcal{B} \times \mathcal{G} \). In such a case, one has for the integration measure

\[ [d\mu(\mathcal{A})] = [d\mu(\mathcal{B})] [d\mu(\mathcal{G})] = [d\mu(\mathcal{B})] [d\alpha], \quad (2.7) \]

for gauge transformations close to the identity, \( g(x) \simeq 1 + \alpha^a(x) \tau_a \). By performing a change of variables \( [d\alpha] \to [df(A)] \), with the insertion of the Jacobian

\[ \text{Det} \frac{\delta f^a(x)}{\delta \alpha^b(y)} \equiv \text{Det} \Delta(x, y), \quad (2.8) \]

one arrives at

\[ [d\mu(\mathcal{A})] \text{Det} \Delta = [[d\mu(\mathcal{B})][d\alpha] \text{Det} \Delta = [d\mu(\mathcal{B})] [df]]. \quad (2.9) \]
And, finally, by integrating over $[df]$ with the insertion of a delta function $\delta(f(A) - h(x))$ which implements the gauge choice, one obtains the measure on the quotient space:

$$[d\mu(A)] \ Det \delta(f(A) - h(x)) = [d\mu(B)].$$

The Jacobian in (2.8) is the known Faddeev-Popov determinant.

However, the gauge fixing described above is not enough to remove unphysical degrees of freedom if the theory is non-Abelian. Indeed, let us consider the gauge orbit

$$A^g = gA g^{-1} + dg g^{-1} \simeq A + D\alpha,$$

with here $-(\partial_\mu D^\mu)\delta^{(4)}(x - y)\delta^{ab}$ is the FP operator for this choice of gauge fixing and $D\alpha = d\alpha + \alpha^a \wedge A^b[\tau_a, \tau_b]$. The gauge fixing condition $\partial_\mu A^\mu = 0$ yields the so-called equation of copies [1]

$$\partial_\mu D^\mu \alpha = 0,$$

which may have nontrivial solutions, whenever the gauge group is non-Abelian, yielding to Gribov ambiguities.\(^1\) The problem can be traced back to the topological non-equivalence of the kinematical space of connections $A$ and the physical space $B$, \([2], [3]\). Indeed (2.10) is only valid if $A = B \times G$ globally, which, for $G = \hat{U}(N)$ is only true for $N = 1$ namely for QED (see for example \([10]\) for a clear pedagogical discussion).

### 3 The Gribov copies equation for NC QED

For each two functions $f, g$ defined on Euclidean space-time $\mathbb{R}^4$, the noncommutative Moyal star product $f \star g$ \([11]\) can be given the following asymptotic expansion

$$(f \star g)(x) = f(x) \exp\left\{\frac{i}{2} \Theta^{\rho\sigma} \hat{\partial}_\rho \hat{\partial}_\sigma\right\} g(x),$$

with $\rho, \sigma = 1, \ldots, 4$. The antisymmetric matrix $\Theta$, has the following nonzero components

$$\Theta_{1,2} = -\Theta_{2,1} = \Theta_1, \quad \Theta_{3,4} = -\Theta_{4,3} = \Theta_2,$$

where $\Theta_i$ are real deformation parameters, in principle all different from each other, characterizing noncommutativity. For simplicity, we perform a rescaling in order to make all parameters equal. When $\Theta_i \rightarrow 0$, the star product tends to the standard commutative point-wise product of $f$ and $g$.

#### 3.1 Gauge transformations

Gauge theories with gauge group $G = \text{Maps}(\mathbb{R}^4, U(N))$ are modified in the noncommutative setting by replacing the point-wise product with the non-local product (3.13). The elements of the gauge group, $U_*(x)$, are defined as star exponentials

$$U_*(x) = \exp_*(i\alpha(x)^iT_i) = 1 + i\alpha^i(x)T_i - \frac{1}{2}(\alpha^i \star \alpha^j)(x)T_iT_j + \ldots$$

\(^1\) In the Abelian case we only have trivial solutions, if we further assume that $\lim_{x \rightarrow \infty} \alpha(x) = 0$. 

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with $T_i$ the Lie algebra generators of the structure group, $T_i \in \mathfrak{u}(N)$. The associated matter fields transform under deformed gauge transformations according to

$$\phi(x) \longrightarrow U_*(x) \triangleright \phi(x) = \exp_*(i\alpha^i(x)T_i) \triangleright \phi(x).$$

(3.16)

with $\triangleright$ indicating simultaneously the appropriate representation of the Lie algebra generators and the $\triangleright$ multiplication in space-time. At the infinitesimal level, we have then

$$\phi(x) \longrightarrow \phi(x) + i(\alpha \triangleright \phi)(x),$$

(3.17)

with

$$(\alpha \triangleright \phi)(x) = i(\alpha^i(x) \triangleright (T_j \triangleright \phi))(x).$$

(3.18)

The gauge potential transforms as

$$A_\mu \rightarrow A'_\mu = U_\ast \ast A_\mu \ast U_\ast^\dagger + iU_\ast \ast \partial_\mu U_\ast^\dagger.$$  

(3.19)

Specialising to NCQED, the infinitesimal transformation reads therefore

$$A_\mu \rightarrow A'_\mu[\alpha] = A_\mu + D_\mu[\alpha] + \mathcal{O}(\alpha),$$

(3.20)

where the covariant derivative $D_\mu$, now only due to space-time noncommutativity, is given by

$$D_\mu[\alpha] = \partial_\mu[\alpha] + i(\alpha \ast A_\mu - A_\mu \ast \alpha)$$

(3.21)

and reduces to the standard Abelian form in the commutative limit $\Theta \rightarrow 0$. The field strength $F$ is given by

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu.$$  

(3.22)

and transforms covariantly under star-gauge transformations.

It is precisely the occurrence of the covariant derivative which makes NCQED similar to non-Abelian gauge theories in commutative space-time, and, independently from the specific form of the gauge action $S[A]$ one deals with,\(^2\) it is a meaningful question and a relevant issue for the quantization of the theory to investigate the existence of nontrivial solutions for the equation of Gribov copies. The latter is readily obtained by considering $A_\mu$ and $A'_\mu$ satisfying the same gauge condition, e.g. the Landau gauge,

$$\partial^\mu A_\mu = \partial^\mu A'_\mu = 0.$$  

(3.24)

It follows that $\alpha$ yields a Gribov copy if

$$-\partial^\mu D_\mu \alpha(x) = 0.$$  

(3.25)

The problem has been addressed in [8,9] where the equation has been solved for a class of particularly simple potentials, and an infinite number of copies, which cannot be discarded by boundary conditions, i.e. they are non-trivial, has been found.

It is therefore clear that the standard quantization procedure applied to NCQED does not provide a well-defined measure of integration over gauge fields. In order to overcome the problem, we propose to apply the same techniques adopted for non-Abelian gauge theories, which have led to restricting the functional integral to an integration region free of copies: the Gribov region.

\(^2\) Besides the natural candidate, which is obtained by replacing the point-wise product with the noncommutative one, i.e.

$$S[A] = \frac{1}{4} \int F_{\mu\nu} \ast F^{\mu\nu}$$

(3.23)

other proposals have been considered in the literature. See for example [12] and refs. therein.
4 Establishing the Gribov Region

Once the existence of copies has been established in terms of zero modes of the Faddeev-Popov operator, we need to characterise the region \( \Omega \) in the space of connections where there are no ambiguities. The definition which we adopt here for NCQED was proposed by Gribov in 1978 for non-Abelian theories [1]. For the present case it reads:

\[
\Omega \equiv \{ A_\mu, \partial_\mu A_\mu = 0, \Delta(x,y) > 0 \}, \tag{4.26}
\]

where the Faddeev-Popov operator is explicitly given by

\[
\Delta(x,y) = -\partial_\mu D^\mu \delta(x-y) = -\partial^2 \delta(x-y) + i\lambda [A_\mu, \partial_\mu \delta(x-y)]_*, \tag{4.27}
\]

and \([f,g]_* = f \star g - g \star f\).

The boundary \( \partial \Omega \) represents the first Gribov horizon. For non-Abelian gauge theories it was shown [13, 14] that \( \Omega \) is convex, bounded in all directions in the space of gauge connections and that each gauge orbit passes at least once in \( \Omega \). The inverse of the FP operator, or equivalently the ghost propagator with external gauge field, \( G(k,A) = \Delta^{-1}(k,A) \), with \( k \) the momentum variable, can be used to implement the restriction to \( \Omega \) [1]. In the following we make the non-trivial assumption that the results established in [13, 14] may be extended to NCQED, namely that

- The Gribov region is bounded in every direction (in the functional space of transverse gauge potentials).
- The Faddeev-Popov determinant changes sign at the Gribov horizon.
- Every gauge orbit passes inside the Gribov horizon.

Under these assumptions, we will adapt to NCQED the procedure introduced in [15] to restrict the functional integration to the Gribov region. In the next subsection we provide some arguments to support these assumptions.

4.1 Properties of the NC Faddeev-Popov operator

The FP operator is given by

\[
\Delta(x,y,A) = -\partial_\mu D^\mu \delta(x-y) = -\partial^2 \delta(x-y) + i\lambda [A_\mu, \partial_\mu \delta(x-y)]_*. \tag{4.28}
\]

- **The Gribov region is limited in every direction**

  First of all, note that \( \Delta(x,y,A) \) is hermitian in Euclidean space so that for real functions \( f(x) \) and \( g(x) \) one has

  \[
  \int d^Dx \int d^Dy f(x) \star \Delta(x,y,A) \star g(y) \in \mathbb{R} \tag{4.29}
  \]

  Now, suppose that \( \Delta(x,y,A) \) develops a zero mode for some value of \( A \). This must come from some negative contribution from the term \( \mathcal{M}(x,y,A) = i\lambda [A_\mu, \partial_\mu \delta(x-y)]_* \) because...
the term $\partial^2 \delta(x - y)$ is positive. Therefore one must have for some value of $A$ and some function $\psi(x)$ that
\[ \int d^Dx \int d^Dy \psi(x) \ast M(x, y, A) \ast \psi(y) = a < 0 \] (4.30)

Now introduce a real parameter $t$ multiplying $A$ so as to parameterize its amplitude isotropically and consider the expression
\[ \int d^Dx \int d^Dy \psi(x) \ast \Delta(x, y, tA) \ast \psi(y) = \int d^Dx \int d^Dy \psi(x)(-\partial^2)\psi(x) + ta \] (4.31)
where we used that $M(x, y, tA) = tM(x, y, A)$. The first term is positive and since $a$ is negative, it is clear that this expression will become negative for some sufficiently high value of $t$. This means that the Gribov region, defined as the region in $A$ space such that $\Delta(x, y, A)$ is positive, will be eventually crossed for some finite value of $t$ and thus it must be limited in every direction.

- **The Faddeev-Popov determinant changes sign at the Gribov horizon.** This follows from the definition of the Gribov horizon and can be seen from the above remarks. As the amplitude of $A$ becomes large the $FP$ reaches a zero mode (the Gribov horizon) and afterward it becomes negative as we saw by raising the parameter $t$ in the proof above.

- **Every gauge orbit passes inside the Gribov horizon.** In order to prove this, one would need a functional $f(A)$ such that under an infinitesimal gauge transformation $\delta A_\mu = D_\mu \alpha$ its extrema are determined by the gauge condition
\[ \delta f(A) = 0 \Rightarrow \partial_\mu A_\mu = 0 \] (4.32)
and its second variation is positive, thus making these extrema in fact the minima
\[ \delta^2 f(A) > 0 \Rightarrow \partial_\mu D_\mu > 0 \] (4.33)

The argument then go on to show that for every gauge orbit, as one moves along the orbit, the functional $f(A)$ will eventually attain its absolute minimum. But since the condition for the minimum is exactly the condition that defines the Gribov region, $\partial_\mu D_\mu > 0$, it would follow that every gauge orbit will pass through the Gribov region. For the usual non-abelian case, the functional is $f(A) = \frac{1}{2} \int d^Dx A_\mu^a A_\mu^a$.

We have verified that the same functional works for the NC case, by using the cyclicity of the product. Given
\[ f(A) = \int d^Dx A_\mu \ast A_\mu = \int d^Dx A_\mu A_\mu. \] (4.34)

The first variation is
\[ 0 = \delta f(A) = \int d^Dx A_\mu \ast \delta A_\mu = \int d^Dx A_\mu \ast (\partial_\mu \alpha + i[\alpha, A_\mu]) = \int d^Dx A_\mu \ast \partial_\mu \alpha = \int d^Dx \partial_\mu A_\mu \ast \alpha = \int d^Dx \partial_\mu A_\mu \ast \alpha, \] (4.35)
which implies $\partial_\mu A_\mu = 0$. Here we have used the cyclicity of Moyal product under the integral ($\int f \star b \star c$ is cyclic) and $\int a \star b = \int b \star a$. Notice that the last property is in general not true for other star products. Then the second variation gives

$$0 < \delta^2 f(A) = \int dx \delta \partial_\mu A_\mu \star \alpha = \int \partial_\mu D_\mu \alpha \star \alpha = \int \partial_\mu D_\mu \alpha \cdot \alpha$$

which, for arbitrary $\alpha$ implies the positivity of $\partial_\mu D_\mu$, namely the condition defining the Gribov region. As mentioned above, to provide a complete proof one would need the show that the functional $f(A)$ attains its absolute minimum along every orbit in the NC case as well. This is a reasonable expectation but to pursue such a formal proof is beyond the scope of this work.

## 5 NCQED no-pole condition

The requirement that the FP operator has no zero eigenvalues is implemented in the standard approach [1] in terms of the request that the inverse of the operator doesn’t have poles: this is the so-called no pole condition. The latter is seen to restrict the region of integration to the Gribov region by introducing a factor $V(\Omega)$ in the generating functional,

$$Z(A,c,\bar{c}) = \int_{\Omega} [dA][d\bar{c}][dc] V(\Omega) \delta(\partial \cdot A) \exp \left[ -S[A] - \int dx \, dy \, \bar{c}(x) \star \Delta(x,y) \star c(y) \right]$$

with $\Delta(x,y)$ given in (4.27). We recall that we are in the Euclidean setting. We work in the Landau gauge, $\delta(\partial \cdot A)$. To determine $V(\Omega)$ we use the relationship between the ghost sector of the theory and the FP operator, which emerges from calculating the exact ghost propagator

$$\langle \bar{c}(p) c(-p) \rangle = \frac{\delta}{\delta J_\epsilon(x)} \frac{\delta}{\delta J_\epsilon(y)} Z[J]$$

with

$$Z[J] = \int_{\Omega} [dA][d\bar{c}][dc] V(\Omega) \delta(\partial_\mu A_\mu)$$

$$\times \exp \left[ -S[A] - \int dx \, dy \, \bar{c}(x) \star \Delta(x,y) \star c(y) + \int dx \, (J_\epsilon(x) \star c(x) + \bar{c}(x) \star J(x)) \right].$$

Thanks to the cyclicity and closure properties of the Moyal product\(^3\) we may perform the Gaussian integral over ghost fields in standard fashion, obtaining

$$Z(J) = C \det(\Delta) \exp \left[ - \int d^d x d^d y J_\epsilon(x) \star \Delta^{-1}(x,y) \star J_\epsilon(y) \right]$$

so that

$$\langle \bar{c}(p) c(-p) \rangle = \int [dA] V(\Omega) \delta(\partial_\mu A_\mu) \det(-\partial^\mu D_\mu) \Delta^{-1}(x,y) \exp(-S[A])$$

with $\Delta^{-1}(x,y)$ the ghost propagator at first order in $\hbar$. According to [15] it is convenient to interpret $\Delta(x,y)$ as the space representation of an abstract operator $\Delta$

$$\Delta(x,y) = \langle x | \Delta | y \rangle.$$ 

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\(^3\)They amount respectively to the fact that $\int f_1 \star \ldots \star f_n = \int f_n \star f_1 \star \ldots$ and $\int f \star g = f \cdot g$. 

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Inserting completeness relations $\int d^d x \langle x | x \rangle = 1$, we may obtain $\Delta(x, y)$ in momentum space
\[
\Delta(p, q) = \int d^d x \int d^d y \langle x | \Delta | y \rangle \ast \langle y | q \rangle = \int d^d x \int d^d y e^{-ipx} \ast \Delta(x, y) \ast e^{iqy}. \tag{5.43}
\]
The explicit calculation of star products, performed in App. A yields
\[
\Delta(p, q) = q^2 \delta(p - q) + 2i\lambda \sin \left(\frac{1}{2} \Theta_{\rho \sigma} p^\rho q^\sigma \right) \tilde{A}_\mu(p - q) q_\mu. \tag{5.44}
\]
where
\[
\langle x | p \rangle = \frac{e^{ipx}}{(2\pi)^{d/2}} \tilde{A}_\mu(p - q) q_\mu = \frac{e^{ipx}}{(2\pi)^{d/2}} e^{-i(p - q)x} A_\mu(x) \tag{5.45}
\]
Upon introducing the matrix notation
\[
\mathbb{1}_{pq} = \langle p | \mathbb{1} | q \rangle = \delta(p - q), \tag{5.46}
\]
\[
A_{pq} = \langle p | A | q \rangle = -2i \sin \left(\frac{1}{2} \Theta_{\rho \sigma} p^\rho q^\sigma \right) \tilde{A}_\mu(p - q) q_\mu, \tag{5.47}
\]
we may rewrite Eq. (5.44) as
\[
\Delta_{pq} = q^2 (\mathbb{1}_{pq} - \lambda A_{pq}). \tag{5.48}
\]
The no-pole condition works as a bound on the allowed amplitudes of the gauge field configurations and implies that inverse of this expression makes sense. Then, the inverse of the operator $\Delta$ can be written as
\[
\Delta_{pq}^{-1} = \frac{1}{q^2 (\mathbb{1}_{pq} - \lambda A_{pq})} = \frac{1}{q^2} \sum_{n=0}^{\infty} [(\lambda A)^n]_{pq}. \tag{5.49}
\]
We are interested in the poles of this expression, it is then convenient to study the corresponding normalized trace
\[
\frac{1}{V} \Delta_{pq}^{-1} \big|_{p = q = k} = \frac{1}{k^2} \left( 1 - \frac{\sigma(k, A)}{1} \right). \tag{5.50}
\]
with $V = \int \frac{d^d x}{(2\pi)^d}$ the infinite volume factor. This equation defines the so-called form factor $\sigma(k, A)$, which encapsulates all non-trivial information about the pole structure of the full ghost propagator, in fact, if it vanishes we recover the free propagator expression. In order to implement the no-pole condition one has to integrate over all gauge field configurations leading to
\[
\Delta_{kk}^{-1}(k) = \langle \Delta_{kk}^{-1}(k, A) \rangle_{\text{conn}} = \frac{V}{k^2} \left( 1 + \langle \sigma(k, A) \rangle_{\text{conn}} \right) = \frac{V}{k^2} \left( 1 - \frac{1}{1 - \langle \sigma(k, A) \rangle_{\text{1PI}}} \right) \tag{5.51}
\]
where $\langle \cdots \rangle_{\text{conn}}$ is the sum of all connected diagrams and $\langle \cdots \rangle_{\text{1PI}}$ is the sum of all one-particle-irreducible diagrams (see [15] for details). The no-pole condition is the requirement that the ghost propagator does not develop a pole for any value of $k \neq 0$. The evaluation of the exact
no-pole condition thus amounts to the computation of the exact value $\langle \sigma(k, A) \rangle_{1PT}$. This is of course a very difficult problem (see the discussion in Appendix (D)) and we will study here an approximate solution by considering only the first non-trivial contribution. From expression (5.49), we get
\[ k^2 \Delta^{-1}_{kk}(k, A) = 1_{kk} + gA_{kk} + \lambda^2 A^2_{kk} + \mathcal{O}(\lambda^3). \] (5.52)

Then, we can write $\sigma(k, A)$ as
\[ \sigma(k, A) = \frac{1}{V} (1_{kk} + \lambda A_{kk} + \lambda^2 A^2_{kk} + \mathcal{O}(\lambda^3)) - 1. \] (5.53)
\[ = \frac{\lambda^2}{V} \int \frac{d^dq}{(2\pi)^d} \int \frac{d^dp}{(2\pi)^d} A_{kp} (q^2 \Delta^{-1}_{pq}) A_{qk} \] (5.54)

where we used (5.49) and the relations
\[ 1_{kk} = V, \] (5.55)
\[ A_{kk} = -2 \sin(0) A_0 (0) \frac{i k_\mu}{k^2} = 0, \] (5.56)

The first nontrivial contribution to $\sigma(k, A)$ is thus obtained approximating the ghost propagator inside the integral (5.49) by its tree-level expression, that is $q^2 \Delta^{-1}_{pq} \approx 1_{pq}$. Therefore, noting that
\[ A^2_{kk} = \int \frac{d^dq}{(2\pi)^d} \langle q | A | q \rangle \langle q | A | k \rangle = -4 \int \frac{d^dq}{(2\pi)^d} \sin \left( \frac{1}{2} \Theta_{\rho,\sigma} k_\rho q_\sigma \right) \tilde{A}_\mu(k - q) \frac{q_\mu}{q^2} \sin \left( \frac{1}{2} \Theta_{\rho,\sigma} q_\rho k_\sigma \right) \tilde{A}_\nu(q - k) \frac{k_\nu}{k^2} \] (5.57)
equation (5.54) becomes
\[ \sigma(k, A, \Theta) = -4 \frac{\lambda^2}{V} \int \frac{d^dq}{(2\pi)^d} \sin \left( \frac{1}{2} \Theta_{\rho,\sigma} k_\rho q_\sigma \right) \tilde{A}_\mu(k - q) \frac{q_\mu}{q^2} \sin \left( \frac{1}{2} \Theta_{\rho,\sigma} q_\rho k_\sigma \right) \tilde{A}_\nu(q - k) \frac{k_\nu}{k^2} \] (5.58)

where the $\Theta$ dependence was highlighted in the argument of $\sigma$. Considering the Landau gauge $q_\mu A_\mu(k - q) = k_\mu A_\mu(k - q)$ and changing $q \rightarrow q + k$, we obtain,
\[ \sigma(k, A, \Theta) = \frac{4 \lambda^2 k_\mu k_\nu}{V \mu} \int \frac{d^dq}{(2\pi)^d} \sin^2 \left( \frac{1}{2} \Theta_{\rho,\sigma} q_\rho k_\sigma \right) \tilde{A}_\mu(-q) \frac{1}{(k + q)^2} \tilde{A}_\nu(q). \] (5.59)

The latter may be further simplified by observing that, in the Landau gauge, $\tilde{A}_\mu(-q) \tilde{A}_\nu(q)$ is transversal,
\[ \tilde{A}_\mu(-q) \tilde{A}_\nu(q) = \omega(A) \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \] (5.60)

Multiplying by $\delta_{\mu\nu}$ the factor $\omega(A)$ is found to be $\omega(A) = \frac{1}{\sigma - 1} |\tilde{A}|^2$. Thus we obtain
\[ \sigma(k, A, \Theta) = \frac{4 \lambda^2}{V} \left( \frac{2}{d - 1} - \frac{k_\mu k_\nu}{k^2} \right) \int \frac{d^dq}{(2\pi)^d} \sin^2 \left( \frac{1}{2} \Theta_{\rho,\sigma} q_\rho k_\sigma \right) \frac{|\Lambda(q)|^2}{(k + q)^2} \left( 1 - \frac{q_\mu q_\nu}{q^2} \right). \] (5.61)

Observing that $\frac{\sin^2(x)}{x^2} \leq 1$ we have $\sigma(k, \Theta) \leq I(k, \Theta)$. The function $I(k, \Theta)$ is defined as
\[ I(k, A, \Theta) = \frac{\lambda^2 \Theta_{\rho,\sigma} \Theta_{\omega,\lambda} k_\mu k_\nu k_\sigma k_\lambda}{V \mu (d - 1)} \int \frac{d^dq}{(2\pi)^d} \left| \tilde{A}_\omega(q) \right|^2 \frac{k^2 q^2}{(k + q)^2} \left| \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right| \frac{q_\mu q_\nu}{q^2} \] (5.62)
\[ = \frac{k_\mu k_\nu k_\sigma k_\lambda}{k^4} I_{\mu\nu\lambda\sigma}(k, \Theta), \] (5.63)
with
\[
\mathcal{I}_{\mu\nu\lambda\sigma}(k, A, \Theta) = \frac{\lambda^2}{V} \Theta_{\rho\sigma} \Theta_{\omega\lambda} \int \frac{d^d q}{(2\pi)^d} \tilde{A}_\lambda(q)^2 \frac{k^2 q^2}{(k + q)^2} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{q_\rho q_\omega}{q^2}. \tag{5.64}
\]

We are interested in finding an upper-bound for the latter, so to have a refined no-pole condition \(I(k, \Theta) < 1\). In Appendix (B) and (C) we perform an explicit analysis in \(d = 2\) and \(d = 4\) and show that \(I(k, A, \Theta)\) is an increasing function of \(k\), with the maximum value attained at \(k \to \infty\).

In what follows we shall therefore assume \(d = 2, 4\), although the analysis could be probably extended to other dimensions. Because of that, the no-pole condition is certainly satisfied if we impose
\[
I_{\text{max}}(A, \Theta) < 1, \tag{5.65}
\]

with
\[
I_{\text{max}}(A, \Theta) = \frac{k_\mu k_\nu k_\lambda k_\sigma}{k^4} \mathcal{I}_{\mu\nu\lambda\sigma}(\infty, A, \Theta). \tag{5.66}
\]

Taking the limit \(k^2 \to \infty\) in (5.64), we obtain
\[
\mathcal{I}_{\mu\nu\lambda\sigma}(\infty, A, \Theta) = \frac{\lambda^2}{V} \Theta_{\rho\sigma} \Theta_{\omega\lambda} \int \frac{d^d q}{(2\pi)^d} |\tilde{A}_\lambda(q)|^2 \left( \delta_{\mu\nu} q_\rho q_\omega - \frac{q_\mu q_\nu q_\rho q_\omega}{q^2} \right). \tag{5.67}
\]

Using the following properties of regularized momentum integrals [16]
\[
\int \frac{d^d q}{(2\pi)^d} f(q^2) q_\rho q_\omega = \frac{\delta_{\rho\omega}}{d} \int \frac{d^d q}{(2\pi)^d} q^2 f(q^2), \tag{5.68}
\]
\[
\int \frac{d^d q}{(2\pi)^d} f(q^2) q_\omega q_\rho q_\mu q_\nu = \frac{\delta_{\mu\nu} \delta_{\omega\rho} + \delta_{\mu\rho} \delta_{\nu\omega} + \delta_{\mu\omega} \delta_{\nu\rho}}{d(d + 2)} \int \frac{d^d q}{(2\pi)^d} q^4 f(q^2), \tag{5.69}
\]
we rewrite (5.67) as
\[
\mathcal{I}_{\mu\nu\lambda\sigma}(\infty, A, \Theta) = \frac{\lambda^2}{V} \Theta_{\rho\sigma} \Theta_{\omega\lambda} \left( \frac{(d + 1)\delta_{\mu\nu} \delta_{\rho\omega} - \delta_{\mu\rho} \delta_{\nu\omega} - \delta_{\mu\omega} \delta_{\nu\rho}}{d(d + 2)(d - 1)} \right) \int \frac{d^d q}{(2\pi)^d} q^2 |\tilde{A}_\lambda(q)|^2
\]
\[
= \frac{\lambda^2}{V} \left( \frac{(d + 1)\delta_{\mu\nu} \Theta^2_{\rho\lambda} - \Theta_{\mu\sigma} \Theta_{\nu\lambda} - \Theta_{\mu\lambda} \Theta_{\nu\sigma}}{d(d + 2)(d - 1)} \right) \int \frac{d^d q}{(2\pi)^d} q^2 |\tilde{A}_\lambda(q)|^2. \tag{5.70}
\]

Substituting (5.70) in (5.66), we conclude that
\[
I_{\text{max}}(A, \Theta) = \frac{\lambda^2}{V} \frac{k_\mu k_\nu k_\lambda k_\sigma}{k^4} \left( \frac{(d + 1)\delta_{\mu\nu} \Theta^2_{\rho\lambda} - \Theta_{\mu\sigma} \Theta_{\nu\lambda} - \Theta_{\mu\lambda} \Theta_{\nu\sigma}}{d(d + 2)(d - 1)} \right) \int \frac{d^d q}{(2\pi)^d} q^2 |\tilde{A}_\lambda(q)|^2
\]
\[
= \frac{\lambda^2}{V} \left( \frac{(d + 1)\Theta^2}{d(d + 2)(d - 1)} \right) \int \frac{d^d q}{(2\pi)^d} q^2 |\tilde{A}_\lambda(q)|^2, \tag{5.71}
\]
where we have used \(k_\sigma k_\lambda \Theta^2_{\sigma\lambda} = k^2 \Theta^2\) and \(k_\mu k_\sigma \Theta_{\mu\sigma} = 0\).

Thanks to the latter, we may estimate the first Gribov region and exhibit its dependence on \(\Theta\). Indeed, since \(\sigma(k, A, \Theta)\) is always smaller or equal than \(I(k, A, \Theta)\), which we have shown to be an increasing function of \(k\), we may choose in Eq. (5.37)
\[
V(\Omega) = \vartheta(1 - I_{\text{max}}(A, \Theta)). \tag{5.72}
\]
or using the Heaviside function integral form,

$$V(\Omega) = \int_{-\infty}^{+\infty+\varepsilon} \frac{d\tau}{2\pi i \tau} e^{\tau(1-I_{max}(\Theta))}. \quad (5.73)$$

We can insert this into the path integral (2.2) which takes the form

$$Z(J) = C \int \frac{d\tau}{2\pi i \tau} e^{\tau I_{max}(A, \Theta)} \int [dA] e^{-\tau I_{max}(A, \Theta)} \frac{1}{2} F_{\mu \nu} F_{\mu \nu} + \frac{1}{4 \pi} \partial_\mu A_\nu + A_\mu \ast J_\mu. \quad (5.74)$$

The tree-level photon propagator can be read from the quadratic part and is given by

$$\Delta_{\mu \nu}(k^2) = \left[ (1 + \gamma)k^2 \delta_{\mu \nu} + \left( \frac{1}{\alpha - 1} \right) k_\mu k_\nu \right], \quad \gamma \equiv \frac{2(d + 1)}{d(d + 2)(d - 1)} \frac{\tau \Theta^2 \lambda^2}{V}. \quad (5.75)$$

Therefore, the transverse photon propagator is given by

$$\langle \tilde{A}_\mu(-k) \tilde{A}_\nu(k) \rangle = \frac{1}{(1 + \gamma)k^2} \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (5.76)$$

We see that, at least within Moyal type non-commutativity\(^4\) there is no qualitative modification of the photon propagator, which is changed only by a scale factor that can be absorbed by a \(\Theta\) dependent renormalization of the gauge field. So the effect of Gribov copies is not felt and, in particular, there is no confining phase due to Gribov effects. It is a situation very similar to the case of the \(N = 4\) Super Yang-Mills, where Gribov effects are suppressed and no scale is generated \([17]\), thus maintaining the conformal invariance of the theory even when Gribov copies are taken into account. In the present case, the scale associated with the Gribov region is the noncommutative parameter \(\Theta\), which is not dynamically generated but is already present as a fundamental scale of the theory from the start.

This is to be contrasted with the original (commutative space) non-abelian Gribov analysis \([1]\), where the modification of the quadratic part of the action coming from the form factor is \(~|\tilde{A}_\lambda(q)|^2\) and it leads to an effective gluon propagator of the form \(~q^2/k^2\), where \(\kappa\) is the Gribov parameter determining the scale associated with the Gribov horizon. This scale is dynamically generated and can be computed by a gap equation,

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4 + \kappa^4} = 1 \quad (5.77)$$

whereas in the present case, the modification of the quadratic part induced by \(I_{max}(A, \Theta)\), (5.71), is of the form \(~\gamma q^2 |\tilde{A}_\lambda(q)|^2\), with \(\gamma\) proportional to \(\Theta^2\). This produces an ill-defined gap equation. In fact, the gap equation is obtained by extremising the vacuum energy functional, \(E(\tau) = -\ln Z\), with respect to \(\tau\). So, after integrating over the gauge fields in (5.74) considering only the quadratic part, we would need to find the extremum of

$$f(\tau) = -\ln \tau + \tau + \int \frac{d^4 q}{(2\pi)^d} \ln \left( (1 + \gamma)q^2 \right) \quad (5.78)$$

The integral in this expression is ill-defined and in fact it is zero under dimensional regularization. Considering further that we eventually want to take the thermodynamic limit \(V \rightarrow \infty\), keeping \(\gamma\)

\(^4\)Indeed, if the star product is not closed, already the quadratic part of the action, consisting of \(\int (\partial_\mu A_\nu - \partial_\nu A_\mu) \ast (\partial_\sigma A_\rho - \partial_\rho A_\sigma)\) is deformed. Namely, the \(\ast\) product cannot be removed, resulting in a modification of \(\Delta_{\mu \nu}(k^2)\).
finite (which means take $\tau \to \infty$), we see that this equation provides no consistent solution in this limit. Therefore, within the approximations considered, there is no dynamically generated scale associated with the Gribov region in NCQED. We conclude that the scale associated with the Gribov parameter in the non-abelian setting is replaced by the nondynamical noncommutative scale here. We can observe that in the commutative limit, when $\Theta \to 0$, all Gribov features are removed and the theory returns to standard QED.

As a final, important, remark, notice that our analysis is not exact. Besides the approximation of working only to leading nontrivial order in the form factor $\sigma(k, A, \Theta)$, we also consider a stronger no-pole condition than the needed one, since $I(k, A, \Theta)$ is an upper bound for $\sigma(k, A, \Theta)$. The result could be made more precise by maximising directly (5.61). There are however technical problems which we haven’t succeeded to solve for the moment.

6 Conclusion

We have analysed the no-pole condition for NCQED with Moyal type non-commutativity. By exploiting the formal analogy of the latter with non-Abelian gauge theories, we have restricted the effective action to the Gribov region by introducing a constraining factor, $V(\Omega)$, which is obtained through the ghost propagator in a fashion similar to SU($N$) gauge theories. As already stressed, our result is to be intended as a first estimate of the Gribov region, not only because we work to leading nontrivial order in the form factor $\sigma(k, A, \Theta)$, but also because the request that $I(k, \Theta) < 1$ is a stronger no-pole condition than the needed one, $I(k, \Theta)$ representing an upper bound for $\sigma(k, A, \Theta)$. This is due to technical difficulties in maximising $\sigma(k, \Theta)$ in a $k$-invariant way. Still, this upper bound guarantees the absence of poles for the ghost propagation. Also, it permits the calculation of a non-commutative transversal photon propagator proportional to $\frac{1}{(1 + \gamma)k^2}$, which results to be dependent on $\Theta^2$, the squared non-commutative parameter.

We therefore conclude that, in the approximation chosen, there is no qualitative modification of the photon propagator, it being only changed by a scale factor that can be absorbed by a $\Theta$ dependent renormalisation of the gauge field. Let us stress, however, that this result has only been proven for the Moyal star product. Already mild modifications of the latter, such as the Wick-Voros product, would introduce a momentum dependent weight which could modify the behaviour of the propagator (see for example [18] for an application to scalar field theory). It would be interesting to analyse this aspect more in detail. Some immediate research questions can be investigated following these results. First, naturally suggested by the present analysis we would like to find a more precise solution of $\sigma(k, \Theta) < 1$ dependent on the non-commutative parameter. Another possible path is to extend the analysis beyond the original Gribov semiclassical treatment and construct the analogous of the full Gribov-Zwanziger action [4, 5, 6] for the NCQED. Finally, for the reasons outlined above, it would be interesting to repeat the analysis for a different kind of non-commutativity.

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### A Faddeev-Popov operator in momentum space

We start writing $\Delta(x, y)$ in the momentum space as

$$\Delta(p, q) = \int d^d x \int d^d y \, e^{-ipx} \star \Delta(x, y) \star e^{iqy}. \quad (A.79)$$

We can write that as

$$\Delta(p, q) = \int d^d x \int d^d y \, e^{-ipx} \star (-\partial^2 \delta(x - y) + i\lambda A_{\mu}(x) \partial_{(x - y)} \star \delta(x - y)) \star e^{iqy}.$$  

By substituting the delta function in momentum space

$$\delta(x - y) = \int \frac{dk}{2\pi^d} e^{ik(x - y)} \quad (A.81)$$

we arrive at

$$\Delta(p, q) = \int d^d x \int d^d y \int \frac{dk}{(2\pi)^d} e^{-ipx} \star \left( k^2 e^{ik(x - y)} - \lambda A_{\mu}(x) \star k_{\mu} e^{ik(x - y)} + \lambda k_{\mu} e^{ik(x - y)} \star A_{\mu}(x) \right) \star e^{iqy}.$$  

By substituting the delta function in momentum space

$$\delta(x - y) = \int \frac{dk}{2\pi^d} e^{ik(x - y)} \quad (A.81)$$

we arrive at

$$\Delta(p, q) = \int d^d x \int d^d y \int \frac{dk}{(2\pi)^d} e^{-ipx} \star \left( k^2 e^{ik(x - y)} + \lambda A_{\mu}(x) \star k_{\mu} e^{ik(x - y)} \right) \star e^{iqy}.$$  

On using Eq. (3.13) to perform the star product in the $x$ and $y$ variables we get as an intermediate step

$$\Delta(p, q) = \int d^d x \int d^d y \int \frac{dk}{(2\pi)^d} e^{-ipx} \star \left( k^2 e^{ik(x - y)} + \lambda A_{\mu}(x) \star k_{\mu} e^{ik(x - y)} \right) \star e^{iqy}.$$  

By computing the first term in (A.83) and using the closure of the star product (see footnote 5), we obtain

$$\int d^d x \int d^d y \int \frac{dk}{(2\pi)^d} e^{-ipx} \star k^2 e^{ik(x - y)} \star e^{iqy}.$$  

By computing the first term in (A.83) and using the closure of the star product (see footnote 5), we obtain

$$\int d^d x \int d^d y \int \frac{dk}{(2\pi)^d} e^{-ipx} \star k^2 e^{ik(x - y)} \star e^{iqy} = q^2 \delta(q - p). \quad (A.84)$$
From the second term in (A.83), using the cyclicity and closure of the product

\[- \lambda \int d^4x \int d^4y \int \frac{dk}{(2\pi)^d} k_\mu e^{\frac{i}{2} \Theta_{\rho\sigma} k_{\rho q\sigma}} e^{-ipx} * A_\mu(x) * e^{ik(x-y)+iqy} \]

\[
= -\lambda \int d^4x \int d^4y \int \frac{dk}{(2\pi)^d} k_\mu \delta(q-k) e^{\frac{i}{2} \Theta_{\rho\sigma} k_{\rho q\sigma}} e^{-ipx} * A_\mu(x) * e^{ikx} 
= -\lambda \int d^4x q_\mu e^{-ipx} * A_\mu(x) * e^{iqx} 
= -\lambda \int d^4x q_\mu e^{iqx} * e^{-ipx} * A_\mu(x) 
= -\lambda \int d^4x q_\mu e^{\frac{i}{2} \Theta_{\rho\sigma} q_\rho q_\sigma} e^{i(q-p)x} A_\mu(x) 
= -\lambda q_\mu A_\mu(p-q) e^{\frac{i}{2} \Theta_{\rho\sigma} q_\rho q_\sigma}. \tag{A.85}
\]

Following the same procedure in the third term in (A.83),

\[
\lambda \int d^4x \int d^4y \int \frac{dk}{(2\pi)^d} k_\mu e^{\frac{i}{2} \Theta_{\rho\sigma} k_{\rho p\sigma}} e^{ik(x-y)-ipx} * A_\mu(x) * e^{iqy} 
= \lambda q_\mu A_\mu(p-q) e^{\frac{i}{2} \Theta_{\rho\sigma} p_\rho q_\sigma}. \tag{A.86}
\]

Inserting (A.84), (A.85) and (A.86) in (A.83), we prove that

\[
\Delta(p, q) = q^2 \delta(p-q) + \lambda q_\mu A_\mu(p-q) \left( e^{\frac{i}{2} \Theta_{\rho\sigma} p_\rho q_\sigma} - e^{-\frac{i}{2} \Theta_{\rho\sigma} p_\rho q_\sigma} \right) 
= q^2 \delta(p-q) + 2i \lambda q_\mu A_\mu(p-q) \sin \left( \frac{1}{2} \Theta_{\rho\sigma} p_\rho q_\sigma \right). \tag{A.87}
\]

### B Analysis of $\sigma(k, \Theta)$ in two dimensions

In this Section we consider the function $\sigma(k, \Theta)$ in $d = 2$ dimensions. In polar coordinates of the plane, $(q, \alpha)$, $q^2 = q^\mu q_\mu$, $\alpha \in [0, 2\pi]$, Eq. (5.61) becomes

\[
\sigma(k, \Theta) = \frac{4\lambda^2}{V} \int_0^\infty dq \frac{4\pi^2}{V} q|A(q)|^2 \int_0^{2\pi} d\alpha \frac{\sin^2 \left( \frac{1}{2} \Theta q k \sin \alpha \right) \sin^2 \alpha}{(k-q)^2}, \tag{B.88}
\]

where we have chosen $k_\mu = (0, k)$. Observing that $\frac{\sin^2 \alpha}{\pi} \leq 1$, we have

\[
\sigma(k, \Theta) \leq I(k, \Theta) = \frac{4\lambda^2}{V} \int_0^\infty dq \frac{4\pi^2}{V} q|A(q)|^2 \int_0^{2\pi} d\alpha \frac{\left( \frac{1}{2} \Theta q k \sin \alpha \right)^2 \sin^2 \alpha}{(k-q)^2} \tag{B.89}
\]

\[
= \frac{\lambda^2}{4\pi V} \int_0^\infty dq \frac{4\pi^2}{V} q|A(q)|^2 \left( \frac{3k^2 - q^2}{4k^4} \vartheta(k-q) - \frac{(k^2 - 3q^2)}{4q^4} \vartheta(q-k) \right)
\]

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where $\theta(x)$ is the Heaviside step function. We are interested in finding an upper-bound for the latter. Therefore we compute

$$\frac{\partial I(k, \Theta)}{\partial k} = \frac{\lambda^2}{V} \Theta^2 \int_0^\infty dq q^3 |A(q)|^2 \left( \frac{k^2 (k^2 - 3q^2) \delta(q - k)}{16q^2} + \frac{q^2 (3k^2 - q^2) \delta(k - q)}{16k^2} \right)$$

$$+ \frac{k^4 (3q^2 - 2k^2) \theta(q - k) + q^6 \theta(k - q)}{8k^3 q^2} \right) = \frac{\lambda^2}{V} \Theta^2 \int_0^\infty dq q^3 |A(q)|^2 \left( \frac{k^4 (3q^2 - 2k^2) \theta(q - k) + q^6 \theta(k - q)}{8k^3 q^2} \right) > 0 \text{ unless } |A(q)| = 0 \text{ (B.90)}$$

This is always positive, therefore $I(k, \Theta)$ is an increasing function of $k$ and it reaches its maximum at $k \to \infty$. We have

$$I_{\text{max}}(\Theta) = \lim_{k \to \infty} I(k, \Theta) = \frac{\lambda^2}{V} \Theta^2 \frac{3}{16\pi} \int_0^\infty dq q^3 |A(q)|^2 \text{ (B.91)}$$

and the no-pole condition is certainly verified if we impose

$$I_{\text{max}}(\Theta) < 1 \text{ (B.92)}$$

Thanks to the latter, we may estimate the first Gribov region and exhibit its dependence on $\Theta$.

C Analysis of $\sigma(k, A, \Theta)$ in Four Dimensions

Let us now address the problem in $d = 4$ dimensions, in the Euclidean setting as previously done. We work in spherical coordinates $(q, \psi, \alpha, \phi)$ with $q^2 = q^\mu q_\mu$, $\psi, \alpha \in [0, \pi]$, $\phi \in [0, 2\pi]$ and choose $k_\mu = (k, 0, 0, 0)$. Since the computation is Euclidean we are free to choose the reference frame as we like, so we choose the easiest one and the result will not depend on the choice. As for the noncommutativity matrix $\Theta$, according to the choice we have made with (3.14), the only contributing factor will be $\Theta^{12}$, it being

$$\Theta^{\mu\nu} q_\mu k_\nu = -\Theta^{12} q_2 k_1 = \Theta k q \sin \psi \cos \alpha \text{ (C.93)}$$

with $\Theta$ a real parameter. Eq. (5.61) becomes then

$$\sigma(k, A, \Theta) = \frac{4\lambda^2}{3V} \int_0^\infty dq \frac{q^3 |A(q)|^2}{16\pi^4} \int_0^{2\pi} d\phi \int_0^\pi \sin \alpha d\alpha \int_0^\pi \sin^4 \psi d\psi \frac{\sin^2 \left( \frac{k}{2} k q \sin \psi \cos \alpha \right)}{(k^2 + q^2 - 2kq \cos \psi)}$$

$$= \frac{8\pi \lambda^2}{3V} \int_0^\infty dq \frac{q^3 |A(q)|^2}{16\pi^4} \int_0^\pi \sin \alpha d\alpha \int_0^\pi \sin^4 \psi d\psi \frac{\sin^2 \left( \frac{k}{2} k q \sin \psi \cos \alpha \right)}{(k^2 + q^2 - 2kq \cos \psi)} \text{ (C.94)}$$

As for the two-dimensional case, since $\frac{\sin^2(x)}{x^2} \leq 1$ we have

$$\sigma(k, A, \Theta) \leq I(k, A, \Theta) = \frac{8\pi \lambda^2}{3V} \int_0^\infty dq \frac{q^3 |A(q)|^2}{16\pi^4} \int_0^\pi \sin \alpha d\alpha \int_0^\pi \sin^4 \psi d\psi \frac{\left( \frac{k}{2} k q^2 \sin^2 \psi \cos^2 \alpha \right)}{(k^2 + q^2 - 2kq \cos \psi)}$$

$$= \frac{2\pi \lambda^2}{V} \int_0^\infty dq \frac{q^3 |A(q)|^2}{16\pi^4} \int_0^\pi \cos^2 \alpha \sin \alpha d\alpha \int_0^\pi \psi d\psi \frac{k^2 \sin^6 \psi}{(k^2 + q^2 - 2kq \cos \psi)}$$

$$= \frac{4\pi \lambda^2}{9V} \int_0^\infty dq \frac{q^3 |A(q)|^2}{16\pi^4} \int_0^\pi \psi d\psi \frac{k^2 \sin^6 \psi}{(k^2 + q^2 - 2kq \cos \psi)} \text{ (C.94)}$$

$$= \frac{\lambda^2}{36\pi^2 V} \int_0^\infty dq q^3 |A(q)|^2 \left( \frac{q^4 - 5k^2 q^2 + 10k^4}{16k^4} \theta(q - k) + k^2 \theta(10q^4 - 5k^2 q^2 + k^4) \theta(q - k) \right)$$
where $\vartheta(x)$ is the Heaviside step function. We now proceed as in the previous section, by analysing the behaviour of the $I(k, A, \Theta)$ as a function of $k$. The derivative of $I(k, A, \Theta)$ with respect to $k$ is given by:

$$\frac{\partial I}{\partial k} = \frac{4\pi^2 \lambda^2 \Theta^2}{9V} \int_0^\infty dq q^5 |A(q)|^2 \left( \frac{10k^4 + q^4 - 5k^2 q^2}{16k^4} \delta(k - q) - \frac{10k^2 q^4 + k^6 - 5k^4 q^2}{32q^6} \delta(q - k) \right)$$

$$+ \frac{4\pi^2 \lambda^2 \Theta^2}{9V} \int_0^\infty dq q^5 |A(q)|^2 \left( \frac{2q^8 (5k^2 - 2q^2) \vartheta(k - q) + k^6 (-10k^2 q^2 + 3k^4 + 10q^4) \vartheta(q - k)}{16k^3 q^6} \right)$$

$$+ \frac{\pi \lambda^2 \Theta^2}{12V} k^5 \vartheta(k)|A(k)|^2.$$  

(C.95)

The latter is always strictly positive for $|A(q)| \neq 0$, therefore $I(k; A; \Theta)$ is an increasing function of $k$ and its maximum is obtained by performing the limit $k \to \infty$:

$$I_{\text{max}}(A, \Theta) = \frac{5\lambda^2 \Theta^2}{288\pi^2 V} \int_0^\infty dq q^5 |A(q)|^2.$$  

(C.96)

Therefore the no-pole condition is attained by imposing

$$I_{\text{max}}(A, \Theta) < 1$$  

(C.97)

and the first Gribov region is taken into account in the functional integral (5.37) by posing

$$V(\Omega) = \vartheta(1 - I_{\text{max}}(A, \Theta)),$$  

(C.98)

with $\vartheta$ the Heaviside function. The same conclusions as for the two dimensional case apply: the horizon depends on the square of the non-commutativity parameter and is removed in the commutative limit. The estimate can be probably made more precise by maximising directly the function $\sigma(k; A, \Theta)$, provided one is able to overcome the even more challenging technical difficulties in dealing with the higher-dimensional integral (C.94).

D What to expect of an all order computation?

Here we discuss perspectives on the extension of analysis to include higher order terms. We start with equation (5.57)

$$\hat{k}_{ik}^2 = \int \frac{d^dq}{(2\pi)^d} \langle k|A|q\rangle \langle q|A|k\rangle$$

$$= -4 \int \frac{d^dq}{(2\pi)^d} \sin\left(\frac{1}{2} \Theta_{\rho\sigma} k^\rho q^\sigma\right) \tilde{A}_\mu(k - q) \frac{q_\mu}{q^2} \sin\left(\frac{1}{2} \Theta_{\rho\sigma} q^\rho k^\sigma\right) \tilde{A}_\nu(q - k) \frac{k_\nu}{k^2}.$$  

(D.99)
One can easily proceed and write a general formal expression for the term of order \( n \)

\[
\mathcal{A}_{kk}^n = \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} \langle k | \mathcal{A} | q_1 \rangle \langle q_1 | \mathcal{A} | q_2 \rangle \cdots \langle q_{n-2} | \mathcal{A} | q_{n-1} \rangle \langle q_{n-1} | \mathcal{A} | k \rangle
\]

\[
= (-2i)^n \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} \sin \left( \frac{1}{2} \Theta_{\rho \sigma} k^\rho q_1^\sigma \right) \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_1^\rho q_2^\sigma \right) \cdots \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_{n-1}^\rho q_n^\sigma \right)
\]

\[
\hat{\mathcal{A}}_\mu(k - q_1) \hat{\mathcal{A}}_\sigma(q_1 - q_2) \cdots \hat{\mathcal{A}}_\nu(q_{n-1} - k) \frac{q_1^\rho q_2^\sigma \cdots q_{n-1}^\sigma}{q_1^2 q_2^2 \cdots q_{n-1}^2}.
\]

where the Landau gauge property \( q_\mu A_\mu(k - q) = k_\mu A_\mu(k - q) \) was used. It then follows that the exact ghost form factor is given by

\[
\sigma(k, A) = \sum_n \sigma^{(n)}(k, A),
\]

where

\[
\sigma^{(n)}(k, A) = \frac{(-2i)^n}{V} \frac{k_\mu k_\nu}{k^2} \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} \sin \left( \frac{1}{2} \Theta_{\rho \sigma} k^\rho q_1^\sigma \right) \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_1^\rho q_2^\sigma \right) \cdots \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_{n-1}^\rho q_n^\sigma \right)
\]

\[
\hat{\mathcal{A}}_\mu(k - q_1) \hat{\mathcal{A}}_\sigma(q_1 - q_2) \cdots \hat{\mathcal{A}}_\nu(q_{n-1} - k) \frac{q_1^\rho q_2^\sigma \cdots q_{n-1}^\sigma}{q_1^2 q_2^2 \cdots q_{n-1}^2}.
\]

Now, we worked semiclassically because we stopped at the lowest nontrivial order, which is the quadratic order \( \sigma^{(2)}(k, A) \). But in order to go to higher orders one cannot ignore interactions and the problem must be addressed as discussed around (5.51), that is, one must deal with \( \sigma(k) = \langle \sigma(k, A) \rangle_{1PI} \). This amounts to compute at each order \( n \) the following expression

\[
\sigma^{(n)}(k) = \frac{(-2i)^n}{V} \frac{k_\mu k_\nu}{k^2} \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} \sin \left( \frac{1}{2} \Theta_{\rho \sigma} k^\rho q_1^\sigma \right) \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_1^\rho q_2^\sigma \right) \cdots \sin \left( \frac{1}{2} \Theta_{\rho \sigma} q_{n-1}^\rho q_n^\sigma \right)
\]

\[
\hat{\mathcal{A}}_\mu(k - q_1) \hat{\mathcal{A}}_\sigma(q_1 - q_2) \cdots \hat{\mathcal{A}}_\nu(q_{n-1} - k) \frac{q_1^\rho q_2^\sigma \cdots q_{n-1}^\sigma}{q_1^2 q_2^2 \cdots q_{n-1}^2}.
\]

Therefore, one must be able to compute the \( n \)-point function \( \langle \hat{\mathcal{A}}_\mu(k - q_1) \hat{\mathcal{A}}_\sigma(q_1 - q_2) \cdots \hat{\mathcal{A}}_\nu(q_{n-1} - k) \rangle_{1PI} \), a not so easy task indeed.

The desired results rests on a proper understanding of the behavior of \( \sigma(k) \) as a function of \( k \). In the semiclassical quadratic case, analyzed in the paper, the result indicating no qualitative change in the propagator can be traced to the presence of sines in the expression for \( \sigma^{(2)}(k) \), that leads to a stronger bound of the integral suppressing the Gribov effects. The expression for \( \sigma^{(n)}(k) \) also displays sines and these should similarly provide stronger bounds in comparison to the non-abelian commutative case. So, one is led to speculate that Gribov effects could be suppressed at higher order as well. But, on the other hand, this will depend on the behavior of the gauge field \( n \)-point function in a complicated self-consistent way, thus making it difficult to settle for a definite answer as to what happens at higher orders.
References

[1] V. N. Gribov, “Quantization of Nonabelian Gauge Theories,” Nucl. Phys. B 139 (1978), 1
[2] I. M. Singer, “Some Remarks on the Gribov Ambiguity,” Commun. Math. Phys. 60 (1978), 7-12
[3] M. S. Narasimhan and T. R. Ramadas, “Geometry of SU(2) gauge fields,” Commun. Math. Phys. 67 (1979), 121-136
[4] D. Zwanziger, Action from the Gribov horizon Nucl. Phys. B, 321 (1989) 591-604.
[5] D. Zwanziger, Local and Renormalizable Action From the Gribov Horizon. Nucl. Phys. B, 323 (1989) 513-544.
[6] D. Zwanziger, Renormalizability of the critical limit of lattice gauge theory by BRS invariance. Nucl. Phys. B, 399 (1993) 477-513.
[7] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, A Refinement of the Gribov-Zwanziger approach in the Landau gauge: Infrared propagators in harmony with the lattice results. Phys. Rev. D, 78 (2008) 065047.
[8] F. Canfora, M. A. Kurkov, L. Rosa, and P. Vitale, “The Gribov problem in Noncommutative QED,” JHEP 01 (2016), 014
[9] M. Kurkov and P. Vitale, “The Gribov problem in Noncommutative gauge theory,” Int. J. Geom. Meth. Mod. Phys. 15 (2018) no.07, 1850119.
[10] V. P. Nair. Quantum field theory a modern perspective. New York, USA: Springer (2005)
[11] J. E. Moyal. Quantum mechanics as a statistical theory. Proc. Cambridge Phil. Soc., 45:99–124, (1949).
[12] P. Martinetti, P. Vitale and J. C. Wallet, “Noncommutative gauge theories on $\mathbb{R}^2_\theta$ as matrix models,” JHEP 09 (2013), 051 [arXiv:1303.7185 [hep-th]].
[13] G. Dell’Antonio and D. Zwanziger, “Ellipsoidal Bound on the Gribov Horizon Contradicts the Perturbative Renormalization Group”, Nucl. Phys. B, 326:333–350, (1989).
[14] G. Dell’Antonio and D. Zwanziger, ”Every gauge orbit passes inside the Gribov horizon”, Commun. Math. Phys., 138:291–299, (1991).
[15] M. A. L. Capri, D. Dudal, M. S. Guimaraes, L. F. Palhares, and S. P. Sorella, ”An all-order proof of the equivalence between Gribov’s no-pole and Zwanziger’s horizon conditions”, Phys. Lett. B, 719:448–453, (2013).
[16] M. A. Anacleto, F. A. Brito, O. Holanda, and E. Passos. ”Induction of the Lorentz-violating effective actions in quantum electrodynamics”. Int. J. Mod. Phys. A, 32(21):1750128, (2017).
[17] M. A. L. Capri, M. S. Guimaraes, I. F. Justo, L. F. Palhares, and S. P. Sorella, ”On the irrelevance of the Gribov issue in $\mathcal{N} = 4$ Super Yang-Mills in the Landau gauge”, Phys. Lett. B, 735:277–281, (2014).
[18] S. Galluccio, F. Lizzi and P. Vitale, “Twisted Noncommutative Field Theory with the Wick-Voros and Moyal Products,” Phys. Rev. D 78 (2008), 085007 [arXiv:0810.2095 [hep-th]].