Instantaneous field singularities in electromagnetic waves

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Abstract

Singularities, i.e. places of discontinuity of physical parameters are extremely general objects appearing in both stationary and wave processes. They commonly occur in purely coherent, highly directional electromagnetic waves, such as laser beams, determining additional spatial or 'topological' properties of the beams independently of their propagational dynamics. For instance, a helicoidal structure of the wavefront curved around a line of undefined phase, so-called phase singularity, adds an orbital degree of freedom to electromagnetic waves. The phase singularities are typical to all types of scalar and one-component vectorial waves where the wave field can be defined only by intensity and phase distributions in space and time. The situation becomes more complex when the electromagnetic wave, as a vectorial object, depends in a different way on coordinates in its field components. This leads to a nonuniform field pattern containing singular points of an undefined instantaneous orientation of the electromagnetic field. In contrast to the phase singularities in a scalar wave, the points of instantaneous field (IF) singularities can be completely isolated by surrounding fields in 3D space at any fixed moment of time. Here we present a systematic description of the IF singularities of the electromagnetic waves in a paraxial approximation. Based on the IF distributions around the singularities, we provide a general qualitative classification of the wave beams. We also detail some of the common types of singularities where the transverse components of electric and magnetic fields have a form of the field sources and show the compensation mechanism of such vectorial field distributions by the longitudinal field components.

1. Introduction

The approach to describe propagating electromagnetic radiation in free space is based on the differential form of Maxwell’s electromagnetic theory and the corresponding wave equation [1]. This gives a detailed account of electromagnetic waves as synchronized transverse oscillations of electric and magnetic fields satisfying the superposition principle at a point. As result, all fundamental wave parameters, such as amplitude, wave frequency, direction in space (wave vector), an orientation of fields (polarization) and phase refer to the behaviour of the wave at a particular point in space and time. However, there is no doubt that the local wave parameters can be different at different points of the wave [2–10] even in purely coherent and highly directional electromagnetic radiations, such as paraxial laser beams [11–13]. It is also true that as the state of the wave field changes, the parameters may not always be defined throughout space. Typically, there are some particular points in the wave field where the local parameters, such as the phase, state, and handedness of the polarization or orientation of electric (magnetic) vectors are undefined [4]. These points are known as singularities in the wave field [14]. As all certain values of a singular parameter in the wave occur in the neighbourhood of a singular point (but not in the point itself), singularities define spatial properties of the surrounding wave field [9]. This requires one to deal with additional ‘integral’ characteristics describing spatial topological properties of the wave field [4, 5]. They are related to the fundamental geometrical aspects of electromagnetic waves and are associated with the inhomogeneous distributions of local parameters which constitute the wave singularities. These singularities are typically classified as phase and polarization singularities, depending on which parameter is undefined (or
singular) at some points in the wave. Phase singularities, commonly known in light fields as optical vortices [15], are features of scalar components of vectorial electromagnetic waves, leading to a circulation of the electromagnetic energy around the point in two dimensions [16–19]. They occur naturally for all types of scalar waves, both classical and quantum [20, 21], and have been studied in detail in a number of works in the different branches of wave physics (see, for instance, vortices in electron beams [22–24]).

Other types of singularities may occur in electromagnetic waves when the vectorial properties of the latter cannot be ignored. Well-known are the polarization singularities that naturally appear in completely polarized electromagnetic waves where the state of polarization varies with position, first described in detail by Nye [25, 26]. They refer to the points where were the direction of the polarization or the handedness of the polarization ellipse is not defined [9, 25]. As the state of polarization is actually determined by electromagnetic vectors averaged over the period of wave oscillations, the polarization pattern alone cannot properly expose the deep nature of singularities in these fields. The polarization provides only rough and, sometimes, distorted picture of instantaneous orientation of electric vectors in the vicinity of a singular point in the vector wave. The simplest example is a vector field of a uniformly circularly (generally elliptically) polarized optical vortex which cannot be properly explained in terms of either pure phase or polarization singularities, as we will show in section 2. On the other hand, the instantaneous orientations of electromagnetic vectors in the wave field provide a natural and efficient way to describe the most fundamental features of physical discontinuities in paraxial vector waves. Corresponding instantaneous vector field singularities (IF singularities) are not stable, but exhibit periodic evolution in propagation in space as the field vectors oscillate in time. There are some special IF singularities with unique source- and sink-like structures of the surrounding field in transverse components of the paraxial wave. We will consider them in detail in section 3. Then, in section 4, we will discuss wave fields with mixed topology and polarization singularities.

To clarify the concept of IF singularities, we will start from the general principles of electromagnetic wave topology and its trivial cases of homogeneously (generally elliptically) polarized fields and then we will move on to more complex scenarios involving scalar and vector singularities in the waves. Moreover, we consider here only electromagnetic waves which carry finite energy flux and are self-similar in space. The former leads to spatially localized waves in the form of light beams. The latter means that the intensity of the beam is similar in each plane $S_\perp$ orthogonal to the propagation direction.

2. Paraxial model of electromagnetic beams

In general, the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields in a free propagating electromagnetic wave are complicated functions of both space and time. The situation becomes simpler for monochromatic harmonic waves with a constant frequency $\omega$ where one can naturally separate spatial and time dependencies. This allows one to describe a harmonic wave as a product of complex-valued spatial mode and a complex exponential of time, $\mathbf{F} = \mathbf{F}(x, y, z) e^{-i\omega t}$, which is a wave number, $c$ is speed of light, and $\mathbf{F}$ represents electric or magnetic field, respectively. When the electromagnetic radiation is a highly directional along a certain spatial axis (for example $z$), a spatial mode can be expressed as $\mathbf{F}(x, y, z) = \mathbf{F}(x, y) e^{i k \ell z}$ with a complex amplitude $\mathbf{F}(x, y, z)$ satisfying the spatially asymmetric equation

$$\left(\nabla^2 + k^2\right) \mathbf{F} = 0,$$

where $\nabla = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y + \mathbf{e}_z \partial_z$ and $\partial_u = \partial/\partial u, u = x, y, z$, and $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the unit vectors in Cartesian coordinates, $k = \omega/c$ is a wave number, $c$ is speed of light, and $\mathbf{F}$ represents electric or magnetic field, respectively. When the electromagnetic radiation is a highly directional along a certain spatial axis (for example $z$), a spatial mode can be expressed as $\mathbf{F}(x, y, z) = \mathbf{F}(x, y) e^{i k \ell z}$ with a complex amplitude $\mathbf{F}(x, y, z)$ satisfying the spatially asymmetric equation

$$\left(\nabla^2 + 2i k \partial_z\right) \mathbf{F} = 0.$$

As was shown by Lax et al. [27], in the zero–order paraxial approximation, the electromagnetic fields are purely transverse, $\mathbf{F} = \mathbf{F}_\perp$, and the second $z$-derivative in the equation (2) could be ignored with respect to a slowly-varying first $z$-derivative of the amplitude function $\mathbf{F}_\perp$. Consequently, the equation for the complex amplitude takes a form of parabolic equation:

$$\left(\nabla_\perp^2 + 2i k \partial_z\right) \mathbf{F}_\perp = 0,$$

where $\nabla_\perp = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y$.

Note that the transversality of the field breaks down in the next order correction to the paraxial approximation [27]. By analysing the higher-order terms in the series expansion of Maxwell’s equations, Lax found that the first–order terms of expansion contain a small longitudinal field component ($F_z$) which is obtained from the transverse components through the relation $k F_z = i \nabla_\perp F_\perp$. The role of longitudinal components in the next order of the paraxial approximation will be discussed in sections 4 and 5. Here, our main
interest is the zero-order paraxial approximation as the basis to clarify the topology and field singularities phenomena in the coherent electromagnetic waves.

A scalar form of the equation (3) is represented by a parabolic equation of the Schrödinger type that describes the complex amplitude of a Cartesian component $F_{\perp} = F_{\perp}^0$ of the electromagnetic wave

$$\nabla^2 F_{\perp} = -2ik\partial_z F_{\perp}.$$ (4)

This equation admits solutions in a form of self-similar beams preserving their structure in propagation [28]. Many of such solutions with both, infinite and finite energies, have been obtained in the last decades [13, 29–35]. Main attention has been paid to the spatially-localized solutions with finite energy flux [29–34] as the only ones having physical sense. They include, among others, the well-known Hermite–Gaussian and Laguerre–Gaussian beams, elegant and generalized Laguerre–Gaussian beams, Bessel–Gauss beams, hypergeometric-Gauss beams, fractional-order elegant Laguerre–Gaussian beams, Bessel–Gauss beams with quadratic radial dependence, Mathieu and Ince–Gaussian beams, and optical vortex beams (see [35] and references therein).

As indicated by their names, all spatially localized characteristics of such beams are predominantly associated with the Gaussian envelope

$$G(x, y, z) = \frac{1}{\xi(z)} \exp\left(-\frac{\rho^2}{w_0^2 \xi(z)}\right),$$ (5)

where $\rho = \sqrt{x^2 + y^2}$ is the radial coordinate of an arbitrary point, $\xi(z) = 1 + iz/z_0$, $z_0 = kw_0^2/2$ and $w_0$ is a beam waist in the plane $z = 0$.

The scalar function $G$ is crucial in understanding paraxial electromagnetic wave behaviour because it serves as a background envelope for more general wave solutions, ensuring finiteness of their energy as $\int_0^{\infty} (2/\pi w_0^2) |G|^2 dS_z = 1$ (the star denotes complex conjugation).

The general, rigorous solutions to the parabolic equation (2) can be represented as a product [36]

$$F_{\perp} = U(x, y, z)G(x, y, z),$$ (6)

where $U(x, y, z)$ is a complex vector modulating function and $G(x, y, z)$ is of the form of equation (5). While the Gaussian amplitude describes evolution of the localized envelope of scalar waves or beams in space, the modulation function $U$ determines their all topological and vector properties. Despite the infinite number of possible modulation functions $U$, their general structure can be defined by just a few basic topological properties. Therefore, in the following we will concentrate on the functions $U$ having the most typical topological properties. However, before going any further, let us clarify the relevant terminology.

As the functions $U$ are vector fields defining the orientation of electric (magnetic) fields in space, they are directly related to the state of polarization of the beams. Typically, only beams with spatially variant polarization in their transverse cross-sections are deemed vector beams. In contrast, beams with a constant polarization are generally considered as scalar.

At this point, it is worth clarifying the relationship between the state of polarization and field orientation as these two terms are often, yet incorrectly, used interchangeably. The state of polarization is a time-averaged parameter of a beam and, so far, can be purely homogeneous while the instantaneous orientation of electric (magnetic) vectors provides a complex pattern in the transverse cross-section of the beam. In such cases, the electromagnetic fields of the beam are solutions to the full vector wave equation, and cannot be defined by one scalar function. According to superposition principle, it is always possible to represent the transverse vector function $U$ (corresponding to $E$ or $B$ fields) in terms of their Cartesian components:

$$U = A_{x} U_{x}(x, y, z)e_{x} + A_{y} U_{y}(x, y, z)e_{y},$$ (7)

where $A_{x}$ and $A_{y}$ are arbitrary, generally complex constants.

We consider the beam to be a ‘scalar’ when its electromagnetic field can be described by the real part of a single scalar modulating function:

$$\frac{\text{Re}\{A_{x} U_{x}\}}{\text{Re}\{A_{x}\}} = \frac{\text{Re}\{A_{y} U_{y}\}}{\text{Re}\{A_{y}\}} = \text{Re}\{U\}.$$ (8)

In case of a ‘vector beam’ the transverse Cartesian components of the modulating function (7) $U_{x} = A_{x} U_{x}(x, y, z)e_{x}$ and $U_{y} = A_{y} U_{y}(x, y, z)e_{y}$, depend in different ways on coordinates:

$$\frac{\text{Re}\{A_{x} U_{x}\}}{\text{Re}\{A_{x}\}} \neq \frac{\text{Re}\{A_{y} U_{y}\}}{\text{Re}\{A_{y}\}}.$$ (9)

The simplest example of the scalar beam is an elliptically polarized Gaussian beam. This trivial case occurs if the modulating function is constant: $U = A_{x} e_{x} + A_{y} e_{y}$, where $A_{x} = a_{x} e^{i\phi}$ and $A_{y} = a_{y} e^{i\phi}$; $a_{x}$ and $a_{y}$ are real numbers and $\phi$ is the phase difference between Cartesian components of $U$. Then one immediately obtains the...
simplest solution of equation (2) in form of complex amplitude \( F_x = (a_x e_x + a_y e_y)G \) of the elliptically polarized beam \( F_x e^{i(k_x - ut)} \), which certainly is a scalar wave. This paraxial, fundamental Gaussian beam \([11–13]\) is only one of the infinite numbers of possible variations of electromagnetic beams described by equation (6). However, homogeneous distributions of instantaneous electric (magnetic) field vectors with the same orientation at each point of the transverse plane of beams are sufficiently rare. Generally, beams have complex distributions of phase, polarization or the instantaneous directions of the electromagnetic fields changing from point to point in the transverse plane.

In the next section we will provide examples of scalar and vector beams with nontrivial topology of modulating functions.

3. Scalar and vector beams with topological charges

It appears that singularities described by modulating function \( U \) essentially define a spatial and vectorial topology of the surrounding wave field but not the evolution of wave packet in space and hence these two aspects can be analysed separately. The envelope function \( G \) generates scalar ‘wave space’ for the existence of field singularities, ensuring the finite energy flux of electromagnetic field over the infinite cross-section. The representation of the singularities by a single abstract modulating function greatly simplifies their formal description, consequently allowing better understanding of their fundamental nature.

In general, the function \( U(\rho, \varphi, z) \) depends on all three coordinates. By changing variables as \( R_t = \sqrt{X^2 + Y^2} = \rho/(w_0 \xi) \), where \( X = x/(w_0 \xi) \); \( Y = y/(w_0 \xi) \), following standard procedure shown in works \([30, 37]\), we insert expression (6) into wave equation (2) and obtain differential equation for the modulating function \( U(R_t, \varphi, \xi) \) in the form

\[
\nabla_\xi^2 U - 4\xi^2\partial_\xi U = 0
\]

(10)

where \( \nabla_\xi = e_x \partial_x + e_y \partial_y \).

This equation admits the separation of variables

\[
U(R_t, \varphi, \xi) = u(R_t, \varphi)Z(\xi),
\]

(11)

where the function \( Z(\xi) = \exp(K^2(1 - \xi)/4\xi) \), and \( K \) is a constant. Then the function \( u(R_t, \varphi) \) satisfies the two-dimensional vector Helmholtz equation

\[
(\nabla_\xi^2 + K^2)u = 0.
\]

(12)

In order to analyse possible solutions of equation (12) let us start with an important particular case when the constant \( K \) is equal to zero and, hence \( Z = 1 \). Then the modulating function \( U \) is independent of the longitudinal coordinate \( \xi \), and \( U = u \). In this case, the vector Helmholtz equation (12) reduces to the vectorial Laplace equation for the function \( U(R_t, \varphi) \)

\[
\nabla_\xi^2 U(R_t, \varphi) = 0.
\]

(13)

Components \( U_x \) and \( U_y \) of the modulating function \( U \) are, in general, independent scalar functions, interpreted as complex amplitudes of electromagnetic waves along Cartesian coordinates \( x \) and \( y \), respectively. When the polarization state of the light field is homogenous (generally elliptical), the modulating function \( U \) can be represented by a single scalar functions \( U_j(x, y, z) = U_j(x, y, z) = U(x, y, z) \) as

\[
U = (a_x e_x + a_y e_y)U,
\]

(14)

where \( U \) satisfies to the two-dimensional Laplace equation

\[
\nabla^2 U = 0.
\]

(15)

From the spectrum of possible solutions of equation (15), well-known as harmonic functions \([38]\), we will consider the simplest but critically important solutions which play a key role in classification of light singularities in scalar light beams

\[
U(R_t, \varphi) = R_t^n \exp(is\varphi),
\]

(16)

where \( n = 0; \pm 1; \pm 2... \), and \( s = \pm 1 \).

The function (16) has singularity on the line \( R_t = 0 \) when \( n = 0 \), and its modulus on this line is either 0 for \( n > 0 \) or \( \infty \) for \( n < 0 \). Finite energy solution (6) with modulating function (16) can be realized for \( n \geq 0 \), except for some specific superpositions in which the self-similarity of the resulting light beam is lost \([39]\). Then the product \( sn \) in an exponent of equation (16) can be represented by a new integer \( l = 0; \pm 1; \pm 2... \), and order \( n \) in \( R_t \) becomes \( n = |l| \), when \( n \geq 0 \). The physical solutions now read
While the harmonic function (17) is the simplest solution of equation (15), it gives full information about the topological structure in the field’s components. The azimuthal index \( l \) defines the topological structure of the wave surface. In optics this index is named a topological charge of an optical vortex. The term ‘optical vortex’ refers to the general solution (6) with scalar modulating function containing addend \( l\varphi \) in its phase, \( \vartheta(\rho, \varphi, z) \). Then the wave front of the optical vortex has a form of a helicoid-like surface twisted around the singular line \( R_t = 0 \). Moreover, the phase circulating around the singularity line in the function (17) results in phase discontinuity that is multiple of the topological charge

\[
\oint d\theta = \oint \nabla \theta dl = 2\pi l,
\]

where \( dl \) is the infinitesimal vector element of the loop. Consequently, the directional electromagnetic wave with the modulating function (17) carries discrete angular momentum [5] along the propagation axis \( z \)

\[
M_z = \langle U^* U \rangle / \langle U U^* \rangle = l\hbar,
\]

where \( L_z = -i\hbar \partial / \partial \varphi \) is the generator of rotations around \( z \).

The above discussed features apply to any scalar modulating functions with the singular phase factor \( l\varphi \).

In general, the scalar Cartesian components of the function \( u(R_t, \varphi) \) in equation (11) satisfy the two-dimensional Helmholtz equation (12) which takes a scalar form

\[
(\nabla_i^2 + K^2) u = 0.
\]

Among an infinite number of possible solutions of equation (20) there are axially symmetric solutions with the phase topology similar to (17)

\[
u(R_t, \varphi) = j_{2l}(\rho R_t) \exp(il\varphi),
\]

where \( j_{2l}(x) \) is the \( l \)-order Bessel function of the first kind. The corresponding scalar components of modulating function (11) are

\[
U(R_t, \varphi, \xi) = j_{2l}(\rho R_t) Z(\xi) \exp(il\varphi),
\]

These functions constitute basis of scalar Bessel–Gauss beams with a phase singularity of a type of optical vortex located along the optical axis (when \( l = 0 \)).

According to equations (6) and (14), the electric fields of arbitrarily homogeneously polarized Gaussian and Bessel–Gauss types of vortex beams with modulating functions equations (17) and (22) are

\[
E_x = \text{Re} \left\{ (E_{0x} e_x + E_{0y} e_y) \times R_{1\text{i}} l \exp(i\rho \varphi) \right\},
\]

\[
E_y = \text{Re} \left\{ (E_{0x} e_x + E_{0y} e_y) \times j_{2l}(\rho R_t) Z(\xi) \exp(i\rho \varphi) \right\},
\]

where \( E_{0x} \) and \( E_{0y} \) are real constant amplitudes of the \( x \)- and \( y \)-components of the electric field, respectively. By introducing a new constant \( k = k \sin \alpha \) in equation (22) as the angular parameter of the wave vector \( \mathbf{k} \), one can define \( K = \beta_0 k \) and \( Z(\xi) = \exp \left\{ -i\beta^2 z / (2k\xi) \right\} \). When \( l = 0 \) the equation (24) represents the well-known zero order Bessel–Gauss beam [36, 40]. The magnetic field component corresponding to field (24) can be uniquely determined by \( cB_y = e_z \times E_y \).

Following our earlier discussion (see equations (8) and (9)), in general, the two-component fields (23) and (24) describe vector waves because of different phase factors, \( \cos(l\varphi) \) and \( \cos(l\varphi + \phi) \), in their Cartesian components, thus \( \text{Re} \{ A_x U_x / \text{Re} \{ A_x \} \} = \text{Re} \{ A_y U_y / \text{Re} \{ A_y \} \} \). Exception are the beams with zero topological charge \( l = 0 \) and the beams with purely homogeneous linear states of polarization (when \( \phi = m\pi, m = 0; 1; 2 \ldots \) ). They include elliptically polarized fundamental Gaussian and Bessel–Gauss beams (described at the end of the previous section) which are both typical scalar beams. The latter are scalar optical vortex beams as their electric field oscillates in a single direction and they can be described by real part of a single scalar function.

Figures 1(a) and (b) show examples of the electromagnetic field distribution with function (23) corresponding to the linearly (\( y \)-polarized) Gaussian beams with topological charges \( l = 1 \) and \( l = -1 \), respectively. The wavefronts of these scalar beams continuously twist such that the field pattern rotates around the beam axis. The pattern contains the zero-field twisted surface which separates the out-of-phase fields. In general, the number of such surfaces corresponds to the topological charge carried by the beam.

As we mentioned above, the fields (23) and (24) represent, in general, elliptically polarized vector beams. Their modulating vector functions (14) admits separation of variables.
where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors and $\varphi_0$ is an arbitrary initial phase. In the more explicit form, taking in account that $\mathbf{n} = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ is a unit vector and $\sigma = \pm 1$ with positive (negative) sign corresponding to right (left) hand polarization, and $\varphi_0$ is an arbitrary initial phase. In the more explicit form, taking in account that $\mathbf{e}_x = (\mathbf{e}_r + i\mathbf{e}_\theta)e^{i\varphi_0}/\sqrt{2}$, the equation (27) may be represented in the polar basis of radial and azimuthal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as:

$$g(\varphi) = e^{i[l+\sigma\varphi\hat{\mathbf{e}}]}\mathbf{e}_x^{\pm},$$

where $\mathbf{e}_x^{\pm} = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ is a unit vector and $\sigma = \pm 1$ with positive (negative) sign corresponding to right (left) hand polarization, and $\varphi_0$ is an arbitrary initial phase. In the more explicit form, taking in account that $\mathbf{e}_x^{\pm} = (\mathbf{e}_r + i\mathbf{e}_\theta)e^{i\varphi_0}/\sqrt{2}$, the equation (27) may be represented in the polar basis of radial and azimuthal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as:

$$g(\varphi) = (\mathbf{e}_r + i\sigma\mathbf{e}_\theta) \exp\{i[(l + \sigma)\varphi + \varphi_0]\}.$$

Distributions of electric and magnetic fields in vector vortex beams corresponding to the modulating function (28) with $\sigma l = 1$ and $\sigma l = -1$ are shown in figure 2. One can see that, in contrast to scalar case, these vector fields are spatially inhomogeneous and evolve over the period of the fields oscillations. As this spatial evolution depends on both topological charge $l$ and handedness $\sigma$, the vector vortex beams carry topology- and polarization-dependent discrete angular momentum along their propagation [41]

$$M_z = (l + \sigma)\hbar.$$

It follows from equation (29) that the field rotation around the phase singularity can be decreased or increased by varying circulation of the field vectors at a point.

To conclude this paragraph, let us reiterate the most relevant points. As it is clearly seen in figures 1 and 2, the instantaneous spatial distribution of electromagnetic fields is dramatically different in scalar and vector vortex beams. The transverse field pattern of scalar beams consists of lines of discontinuities at each instant of time. These lines rotate with wave’s frequency leading to the time averaged null intensity along the axis (line of singularity). In contrast, the field pattern of vector beam always possesses a point singularity (null of the field) in its transverse cross-section. This type of discontinuity, which we call a vector singularity, will be discussed in detail in the next section. As for now, let us note that modulating functions of both vector and scalar vortex beams may not be rigidly coupled in space with the envelope Gaussian wave (5) which serves as a background for topological charges. For clarity we have chosen the specific case of topological charges and the singularity lines located along at the maximum of the Gaussian function (5) (along the beam axis). Generally, the wave field can have many separated topological charges with sufficiently arbitrary trajectories. The singularity lines may extend over three dimensions, and be embedded in the volume filled by the background Gaussian function forming closed loops or infinite, unbounded lines [42–45].

The singularities essentially change a spatial topology of a wave field and, as consequence, the overall intensity pattern. They are the cause of redistributions of the energy flux in the beam [19]. However, as the singularities exist only in the host Gaussian envelope (5), they do not influence the general hyperbolic character.

\[ U(R, \varphi, \xi) = f(R, \xi)g(\varphi), \]

with function $g(\varphi)$

$$g(\varphi) = (a_x e_x + a_y e_y) e^{i(l\varphi + \varphi_0)},$$

and $f = R^{|l|}$ for the Laplace’s type, and $f = J_0(|l|R) \cos(\xi)$ for the Helmholtz type of solutions (23), respectively. Let us look in detail at the particular but interesting case of circularly polarized vortex beams which can be obtained directly from the vector function (26) for $a_x = a_y = \sqrt{2}$ and $\varphi = \pm \pi/2$. Then the function (26) becomes

$$g(\varphi) = e^{i(l\varphi + \varphi_0)}\mathbf{e}_x^{\pm}.$$
of the beam self-diffraction. The topological modulating function itself has a physical sense only in combination with the background function which is responsible for the finite energy flux through the cross-section of the beam. In other words, the function (5) generates ‘space’ for existence of topological objects in a wave field and it ensures that the spatially unlimited paraxial beam carries a finite power.

4. Vector topologies in paraxial electromagnetic waves

In an elliptically polarized vortex beam the phase topology of scalar components of the field ‘competes’ with the polarization induced orientation of the field as the latter already cannot be ignored [46]. This ‘competition’ leads to rich topological patterns of the instantaneous electromagnetic field. The field rotates at every fixed point in space (except the point of singularity) with the wave frequency, while the twisted wave front of the beam rotates with the same frequency around a point of the phase singularity (see figure 2). The result of the both local (at point) and global (in space) rotations is an inhomogeneous distribution of the electromagnetic field in any

Figure 2. The instantaneous near-axis distribution of the electromagnetic field in the vector circularly polarized optical vortex with topological charge and handedness (a) $l = +1, \sigma = +1$; and (b) $l = +1, \sigma = -1$. The evolution of a fixed arbitrary point on the wave surface is shown with a green circle.
transverse plane of the vortex beam. As the state of polarization is a time-averaged effect, the instantaneous distribution of the vector field has generally much richer topology. Despite the homogeneous state of polarization, the instantaneous field (IF) pattern of the beam contains vector singularities as points of an undefined orientation of the fields. In this section, we will discuss these vectorial topological objects which we named IF singularities.

Unlike elliptically polarized vortex beam, the time-averaged pattern of an inhomogeneously distributed field may lead to nontrivial polarization distribution which may vary in space exhibiting the so-called polarization singularities. These singularities have been considered extensively in the literature \cite{9,25,26,47–63}. Here, we will focus on a specific situation when a beam is linearly polarized at any point, but the polarization direction varies azimuthally. In such beams the local rotation of the electric (magnetic) field at a point cancels the rotation of total field caused by the phase singularity. This mutual cancellation can be achieved through a superposition of waves with opposite topological charges and orthogonal states of polarizations.

The idea to use superposition of uniformly polarized beams to create nonuniformly polarized waves was first proposed by Hajnal \cite{48,49}. This fruitful concept was further developed to construct many specific polarization structures in optical beams \cite{50,64–70}. Particularly, Gori [50] showed that the field of a superposed vortices with the opposite topological charges can be linearly polarized at any point, but the polarization direction changes with the angular coordinate. Following this approach, we consider the circularly polarized field with modulating functions \cite{27} and \cite{28}. Let us introduce the superposition of these fields having opposite signs of topological charge $l$ and handedness $\sigma$, as

$$g(\varphi) = \sigma e^{i(l \varphi + \varphi_0)} + \sigma^* e^{-i(l \varphi + \varphi_0)}. \quad (30)$$

The corresponding electric and magnetic fields of vector Gaussian beams are

$$E_\parallel = E_0 \text{Re} \left\{ R_{31} G(\rho, z) e^{i(kz - \omega t)} \right\} \times (\cos[(l + \sigma) \varphi + \varphi_0]) e_\rho - \sigma \sin[(l + \sigma) \varphi + \varphi_0] e_\varphi, \quad (31)$$

$$B_\perp = B_0 \text{Re} \left\{ R_{31} G(\rho, z) e^{i(kz - \omega t)} \right\} \times (\sigma \sin[(l + \sigma) \varphi + \varphi_0]) e_\rho + \cos[(l + \sigma) \varphi + \varphi_0] e_\varphi,$$

where $B_0 = E_0/c$.

The field of vector Bessel–Gauss beams can be found in the similar way

$$E'_\parallel = E_0 \text{Re} \left\{ J_{l} (KR_0) Z(\xi) G(\rho, z) e^{i(kz - \omega t)} \right\} \times (\cos[(l + \sigma) \varphi + \varphi_0]) e_\rho - \sigma \sin[(l + \sigma) \varphi + \varphi_0] e_\varphi, \quad (32)$$

$$B'_\perp = B_0 \text{Re} \left\{ J_{l} (KR_0) Z(\xi) G(\rho, z) e^{i(kz - \omega t)} \right\} \times (\sigma \sin[(l + \sigma) \varphi + \varphi_0]) e_\rho + \cos[(l + \sigma) \varphi + \varphi_0] e_\varphi.$$

Both (equations \cite{31} and \cite{32}) types of beams possess similar topological properties. Hence, without loss of generality, we will concentrate below on the Gaussian beams expressed by equation \cite{31}. Figure 3 depicts few examples of electric field distribution in such beams. These patterns correspond to pure linear states of polarization at any point of a transverse plane, but the direction of the polarization depends on the azimuthal position. The numbers $l$ and $\sigma$ in equations \cite{31} and \cite{32} refer to different states within the symmetry group of the beams. The group with the same index $l$ splits into two subgroups according to $\sigma$. These subgroups $l\sigma$ have own symmetry with unique distribution of the electric (magnetic) field and cannot be transformed from one to another without symmetry breaking. Each subgroup $l\sigma > 0$ or $l\sigma < 0$ with a fixed $l$ permits local rotations of fields by the same angle at every point of the wave front by changing initial phase $\varphi_0$. The local rotations by the angle $\varphi_0$ are equal to a total rotation of the electric (magnetic) field pattern by the angle $\varphi_0/2$ around the beam centre preserving field symmetry in every subgroup except $l\sigma = -1$ (see figure 3).

Despite the absence of topological charges in the vector beams depicted by equations \cite{31} and \cite{32}, the field configuration still carries IF singularities originating from the undefined direction of the IF vectors. They may have a source, spiral, sink, saddle or circulation morphology as shown in figure 3. In particular case described in this section, the IF singularities are isolate points at the axis, $\rho = 0$, in a beam transverse plane. Their position and time-averaged morphology coincide with vector point singularities (V-points) proposed by Freund as points at which the orientation the electric (magnetic) vectors of a linearly polarized field becomes undefined \cite{52}. However, there are significant differences between them. The V-points are singularities in the state of polarization rather than fields and, thus, as it was embraced by Freund, they always belong to linearly polarized
waves. In contrast, the IF singularities may exist in any state of polarization. The relevant example are the singularities in circularly polarized optical vortices shown in section 2 (see figure 2 and the text below). Their electric (magnetic) fields contain IF singularities at every moment of time (see figure 3). Another difference is generally nonstationary behaviour of the IF singularities, as we will show in the next section. Thus, the vector field singularities are pure property of IF configuration. From the equation (32), one can see that in vicinity of IF singularities

\[ \oint E \mathbf{n} \, dl = c \oint B \mathbf{e} \, dl \]
\[ c \oint B \mathbf{n} 
\]

where \( \mathbf{n} \) is the unit vector normal to the infinitesimal element of the loop \( dl \). The line integral of the fields is calculated over the closed equiphasic loop coinciding with a circle of equal intensity in the beam transverse cross-section. These integrals are equal to zero for any subgroups except for \( l \sigma = -1 \) because \( g_\rho \) and \( g_\phi \) components of vector function

\[ g(\varphi) = \cos[(l + \sigma)\varphi + \varphi_0]e_\rho - \sigma \sin[(l + \sigma)\varphi + \varphi_0]e_\phi \]

change sign \( 2(l + \sigma) \) times along the closed loop. Nominaly, such field configuration is similar to two-dimensional vector fields of \( N = 2l + \sigma \) charge 'sources' (or currents) symmetrically localized in the vicinity of the axis \( \rho = 0 \) for the \( l \sigma < 0 \) and at infinity \( \rho \rightarrow \infty \) for \( l \sigma > 0 \). The parity of these sources leads to cancellation of the loop integrals equation (33) in the vicinity of the IF singularity.

The exceptional subgroup \( l \sigma = -1 \) is unique as the vector field \( g \) retains fixed orientation along the loop. This corresponds to a spirally oriented vector field \( g \) with two extreme cases \( \varphi_0 = 0 \) and \( \varphi_0 = \pi/2 \) of pure radial and azimuthal orientations when the function (34) is independent of the coordinate \( \varphi \)

\[ g = e_\rho, \quad \text{when } \varphi_0 = 0, \]

and

\[ g = -\sigma e_\phi, \quad \text{when } \varphi_0 = \pi/2. \]

Electric fields associated with these modulating functions correspond to axially symmetric, radially and azimuthally polarized beams, respectively, known in optics as TM and TE beams (see [63] and references

\[ \begin{array}{ccc}
\sigma = -1 & \sigma = +1 & \text{Intensity} \\
\hline
l = +1 & \varphi_0 = 0 & \text{figure 2} \\
l = +1 & \varphi_2 = \pi/2 & \text{figure 2} \\
l = +2 & \varphi_0 = 0 & \text{figure 2} \\
l = +3 & \varphi_0 = 0 & \text{figure 2} \\
\end{array} \]
The electromagnetic fields of the TM and TE beams are

\[
\begin{align*}
\mathbf{E}^{\text{TM}}_r &= \mathbf{E}_0 \Re \{ f (R, \xi) G e^{i(kz - \omega t)} \} \mathbf{e}_\rho, \\
\mathbf{B}^{\text{TM}}_r &= \mathbf{B}_0 \Re \{ f (R, \xi) G e^{i(kz - \omega t)} \} \mathbf{e}_\phi,
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{E}^{\text{TE}}_r &= \mathbf{E}_0 \Re \{ f (R, \xi) G e^{i(kz - \omega t)} \} \mathbf{e}_\phi, \\
\mathbf{B}^{\text{TE}}_r &= -\mathbf{B}_0 \Re \{ f (R, \xi) G e^{i(kz - \omega t)} \} \mathbf{e}_\rho.
\end{align*}
\]

As it is clearly seen, the field pattern has uncompensated instantaneous radial component of electric (for the TM) and magnetic (for the TE) field on any closed circular loop in the beam transverse section. Such pattern corresponds to intriguing field configurations (see figure 4) as the integrals in equation (33) are not equal to zero on the loop. Formally, this means the uncompensated the field’s sources and their currents in the vicinity of the IF singularity. To demonstrate this, let us consider the fields (37) and (38) in the plane \( z = 0 \) near the point \( R_0 = 0 \). Taking into account that \( f = R \) and using the first-order Taylor series approximation of the field functions near the axis \( \rho = 0 \), both (37) and (38) fields can be approximated as...
Consider a closed circular loop of equal field amplitudes in the \( z = 0 \) plane with the centre at a point \( \rho = 0 \) and a radius \( R \) as shown in figure 5. Let \( I = e_\varphi \) be a unit vector along the loop and \( n = e_\rho \) be a unit vector normal to the infinitesimal element \( dl \) of the loop. Then the electric and magnetic field fluxes in radial direction through the loop, for \( R \to 0 \), are

\[
\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} E_{TM} \cdot dl}{\Delta S} \approx (2E_0/w_0) \cos \omega t,
\]

\[
\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} B_{TM} \cdot dl}{\Delta S} \approx -(2B_0/w_0) \cos \omega t,
\]

where we used the fact that the loop length \( \oint dl = 2\pi R \) and area, \( \Delta S = \pi R^2 \). On the other hand, as the electric (magnetic) field flows through the closed loop, the dual to it magnetic (electric) field circulates around the loop:

\[
\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} B_{TE} \cdot dl}{\Delta S} \approx (2B_0/w_0) \cos \omega t,
\]

\[
\lim_{\Delta S \to 0} \frac{\oint_{\Delta S} E_{TE} \cdot dl}{\Delta S} \approx (2E_0/w_0) \cos \omega t.
\]

The expressions (41) can be converted into the field fluxes through the closed infinitesimal cylindrical surface \( \Omega \) with the length \( c\Delta t \) along the \( z \) direction and the volume \( \Delta V = c\Delta t \Delta S \) in the time interval \( t + \Delta t \), as:

\[
\oint_{\Delta S} E_{TM} \cdot nd\Omega = c \int_t^{t+\Delta t} \oint_{\Delta S} E_{TM} \cdot nd\Omega dt
\]

\[
\approx 2\pi R^2 (E_0/w_0) \int_t^{t+\Delta t} \cos \omega t dt
\]

\[
= 2\pi R^2 (E_0/w_0) [\sin(\omega (t + \Delta t)) - \sin(\omega t)]
\]

\[
\oint_{\Delta S} B_{TM} \cdot nd\Omega = c \int_t^{t+\Delta t} \oint_{\Delta S} B_{TM} \cdot nd\Omega dt
\]

\[
\approx -2\pi R^2 (B_0/w_0) \int_t^{t+\Delta t} \cos \omega t dt
\]

\[
= 2\pi R^2 (B_0/w_0) [\sin(\omega t) - \sin(\omega (t + \Delta t))].
\]
Formally, according to equations (42) and (44), the IF singularities equations (35) and (36), are sources of non-compensated fluxes and circulations of the transverse fields through the side area of the cylindrical surface. The field oscillations periodically change signs of the fluxes (44) and circulations (42) in any fixed plane section \( z = \text{const} \), leading to zero average values over the wave period \( T \), as it shown in the figure 4. These fluxes and circulations remain finite with the same signs between two zero points on a half period \( T/2 \) of oscillations (or on a half wavelength distance along \( z \)-axis).

Note, that this flux anomaly appearing in the transverse components of the wave field is compensated by including the longitudinal field components as their fluxes through the closed surface (here through bases of the cylindrical surface) exactly balance the fluxes of the transverse fields. Although in the zero-order paraxial approximation the longitudinal \( z \)-components of electric and magnetic fields are ignored, they appear in the next, first-order paraxial approximation as \( \omega \mathbf{F}_z = i\mathbf{\nabla} \mathbf{F}_z \) [27]. As an example, let us consider the longitudinal field flux through the plane \( z = 0 \) in the radially polarized beam of equation (39). In such a beam the electric field with the complex amplitude \( \mathbf{F}_z = 2i\mathbf{E}_0/\omega w_0 \) is the only nonzero longitudinal field, and is defined by equations (37) and (39) as:

\[
E_z = 2\mathbf{E}_0 \sin(\omega t) / kw_0.
\]  

The longitudinal component equation (45) is out of phase with respect to the transverse components equation (39) and its orientation is shown in figure 4. The resulting electric field flux of this longitudinal field through the cylindrical bases \( \Delta S \) is

\[
\oint_{\Delta S} \mathbf{E}_z \cdot d\mathbf{\Omega} = -2\pi cR^2(E_0/\omega w_0)[\sin(\omega(t+\Delta t)) - \sin(\omega t)],
\]

which exactly cancels the flux of the transverse components of the electromagnetic field depicted by equation (43).

Figure 5 schematically shows fluxes and circulations of electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) fields on the cylindrical surface around the beams axes with the length of side equal to a fraction of wavelength \( \Delta \lambda \).

The aforementioned IF singularities of electromagnetic waves are caused solely by the spatial configuration of the surrounding vector fields. As the points of undefined orientation of the fields are the zero-field points, these fields acquire physical values everywhere in the waves.

5. Polarization topology via moving IF singularities

All considerations on instantaneous vector singularities in the previous section refer to vector beams with a uniform state of polarization inhomogeneously distributed in space. In general, both orientation of the field vectors and ellipticity of the polarization can vary from point to point. In three dimensions, when the state of elliptic polarization changes with position, points of circular polarization appear along the so-called C-lines, while the states of linear polarization with undetermined handedness define L-surfaces [25, 26, 52, 56]. In the transverse cross-section, a L-surface is represented by the line. This line is swept out by at least one moving isolated singularity where instantaneous electric and magnetic fields are zero, and their directions are therefore undetermined. These singularities were first described by Nye and were originally named ‘wave disclinations’ [25]. Later, Dennis called them ‘relative singularities’ because their instantaneous position is phase-dependent [56].

The difference between the IF singularities in uniformly polarized beams, described above, and wave disclinations in beams with variant state of polarization is their dynamics. The IF singularities may have a stable spatial position and exist even in a wave with homogeneous state of polarization, while wave disclinations change their location in space over time sweeping out L-surface. Although the instantaneous value of the electric (magnetic) field is equal to zero at points of wave disclinations, the field averaged over a period of wave oscillations has a nonzero value. This time-averaged field imprints a nonzero intensity on the L-surface on which the polarization becomes linear. As their only difference is in the time-averaged picture, the wave disclinations are a particular case of IF singularities, in our terminology. The morphology of a moving wave disclinations belongs to the group \( k\tau = \pm 1 \) of IF singularities and may be of source, spiral, sink, circulation or saddle type in the transverse components of the fields (see figure 5).
The dynamic behaviour of the IF singularities affects the time-averaged polarization states leading to polarization singularities—places related to C-lines and L-surfaces in beams with variation of state of polarization. Because of their period-averaged nature, polarization singularities are not equal to the IF singularities and may even appear in points different from that of IF singularities. The dynamics of vector singularities can be simply illustrated through superposition of scalar and vector beams shown in previous section, as it leads to beams with typical polarization singularities. More precisely, we combine \( m \) vector modulating functions (25) and (28) with arbitrary topological charges \( l \) and handedness \( \sigma \)

\[
U = \sum_{n=1}^{m} A_n (e_p + i\sigma_n e_e) f_n \exp(i(l_n + \sigma_n) \varphi),
\]

where \( m \) is a number of the functions in the superposition and \( A_n \) is an arbitrary constant. For simplicity, we ignore initial phase in each function.

The simplest superposition of the type (47) with nontrivial vector topology is sum of just two terms. The first one is a non-singular modulation function with the topological charge \( l = 0 \) and handedness \( \sigma = -1, +1 \). The second term is a function with topological charge \( l = -1, +1 \) and handedness \( \sigma = -1, +1 \), which corresponds to a circularly polarized single charge vortex beam. The superposition of functions with the same handedness \( \sigma_1 \times \sigma_2 = 1 \) leads to the case of homogeneously circularly polarized vortex beams considered in the previous section. The difference is only the position of a singularity, which is now located off the optical axis. The other pair with opposite signs of handedness \( \sigma_1 \times \sigma_2 = -1 \) gives us vector beams with the following modulation functions

\[
U_{11} = A_1 (e_p + i\varphi) \exp(i \varphi) f_1
+ A_2 (e_p - i\varphi) f_2,
\]

\[
U_{12} = A_1 (e_p - i\varphi) \exp(-i \varphi) f_1
+ A_2 (e_p + i\varphi) f_2 \exp(2i \varphi),
\]

and

\[
U_{21} = A_1 (e_p - i\varphi) \exp(-i \varphi) f_1
+ A_2 (e_p + i\varphi) f_2,
\]

\[
U_{22} = A_1 (e_p + i\varphi) \exp(i \varphi) f_1
+ A_2 (e_p - i\varphi) f_2 \exp(-2i \varphi).
\]

The pairs \((U_{11}, U_{12})\) and \((U_{21}, U_{22})\) represent functions with opposite chiralities and can be transformed into each other via mirror reflection: \( \sigma \to -\sigma \); \( l \to -l \). Hence, it is sufficiently to consider only the field (48).

Vector beams represented by this superposition are named Poincaré beams because their polarization states cover entire surface of the Poincaré sphere [71–76]. If we choose the functions \( f_n = R^{|l|} \), then the electric fields of both beams are

\[
E_1 = \text{Re} \left[ E_{01} (e_p + i\varphi) \exp(i \varphi) \right]
+ E_{02} R_l (e_p - i\varphi) G(\rho, z) e^{i(kz - \omega t)}
\]

\[
E_2 = \text{Re} \left[ E_{01} (e_p - i\varphi) \exp(-i \varphi) \right]
+ E_{02} R_l (e_p + i\varphi) G(\rho, z) e^{i(kz - \omega t)},
\]

where \( E_{0n} \) are arbitrary constants. The Bessel–Gauss type of Poincaré beams can be obtained in a similar way by taking \( f_n = R^{|l|} (KR)^Z (\xi) \).

The electric field evolution of the beams given by equation (50) is depicted in figures 6(a) and (b). At any moment of time, the isolated point of the IF singularity lies on a L-line in the transverse plane of the beams. This point rotates completing a full circle along the L-line during a half period of field oscillations. At the same time the IF singularity experiences a periodic transformation of its structure (see the bottom panels in figure 6) similar to that of stationary IF singularities in circularly polarized vortex beams (see figure 2). In three dimensions the distribution of the IF of Poincaré beams (50) with a single topological charge contains a single line of IF singularity where the field is discontinuous. This line sweeps out the L-surface with the field oscillation as the IF singularity rotates with double wave frequency around the beam axis while the axis becomes a C-line with the circular state of polarization. As topological charge of Poincaré beams (50) is uncompensated, these beams carry orbital angular momentum and the whole polarization pattern rotates in propagation.

It is worth to note that unlike L-surfaces, C-lines are free from physical discontinuities of electromagnetic fields and have no IF singularities. The C-lines result from dynamic behaviour of the IF singularities on the L-surface. The topology of the polarization patterns in the vicinity of the C-point (transverse section of the C line) is related to average field orientations rather than to the field discontinuous. Those patterns are typically
named as 'lemon', 'star' and 'monstar' [56, 77]. Classification of C-points constitutes an extended topic discussed extensively in the literature (see for instance [9, 47, 48, 52, 56, 77–79]). As they do not possess the IF singularities, which are our main interest here, we only remark that Poincaré beams given by equation (50) and demonstrated in figure 6 (a) and (b), contain a 'star' and a 'lemon' C-points, respectively.

In closing of this section let us note that we have chosen one of the simplest possible superposition of modulating functions (28) to clarify the effect of IF singularities on polarization singularities. The selected superposition of the scalar function (corresponding to circularly polarized Gaussian beam) and the vector function (corresponding to circularly polarized vortex beam) is not the only one covering all states of polarization and leading to polarization singularities. The other simplest type of Poincaré beams can be constructed by the following superposition [80]

$$U = A_1 \varepsilon_x + A_2 R \varepsilon_y \exp(\pm i \phi).$$  \hspace{1cm} (51)

Each of the constituent functions is a scalar itself as it represents a single Cartesian component, but their superposition is a fully vectorial function (see equation (9)). In addition, superposition of non-coaxial beams, as well as tightly focusing beams, may produce complex singular structures in the longitudinal field components [58, 81–83]. However, the detailed analysis of these intriguing possibilities is out of a frame of this paper.
6. Conclusions

Although the singular electromagnetic waves, as exact solutions of the paraxial wave equation appear to be well-known, their instantaneous structure has been rarely investigated. This is due to the fact that their intensive research is usually carried out in the optical range of the electromagnetic spectrum, where the IF structure changes too fast and mostly can be averaged in conventional experimental observations. In general, however, the temporal structure of the singular electromagnetic field is experimentally resolved for longer wavelengths [49], and all the calculated details are potentially directly observable. In optics, the recently grown up field of ultrashort laser pulses allows the direct utilization of the instantaneous light field to drive, direct and switch electric currents [84]. The ability to use laser pulses to manipulate and steer electrons on ultrashort, typically femto- and attosecond timescales [85–87], requires a knowledge of the wave field distributions in space and time, and the complicated temporal regime of the field structure cannot be ignored. Despite the fact that attosecond science is still in its infancy [87], the use of the instantaneous structure of singular light fields may lead to novel results. The impact can be the determination of how a complicated temporal regime of the singular field can affect the dynamics of light–matter interaction on ultrashort timescales. Thus, the traditional analysis of the singular electromagnetic beams in terms of the averaged over the period of wave oscillation parameters, such as polarization, phase difference, and intensity distribution may not be sufficient for practical applications and may result in difficulties in understanding their rich properties when the instantaneous singularities cannot be ignored. For example, the uniform polarization of a singular beam carrying a phase singularity may not guarantee the uniform orientation of the instantaneous electromagnetic vectors over the transverse wavefront, as it is in the case of circularly polarized vortex beams. On the other hand, the vast majority of complexly polarized singular beams have fixed in space and time orientation of the electromagnetic fields carrying neither spin nor orbital angular momentum but having intensity distributions similar to those of vortex beams. To be consistent, in this work, we consider most general cases of the paraxial electromagnetic waves carrying singularities, provide their general qualitative classification, and emphasize the basic physical principles underlying the role of instantaneous singularities in the behaviour of the waves.

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