Constrained energy problems
with external fields

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Abstract. Given a positive definite kernel in a locally compact space \( X \), a closed set \( \Sigma \), a measure \( \sigma \geq 0 \), and a positive continuous \( g \), we study the minimal energy problem in the presence of an external field \( f \) over the class of all measures \( \nu \geq 0 \) supported by \( \Sigma \) and such that \( \int g \, d\nu = 1 \), \( \sigma - \nu \geq 0 \). Under general assumptions, we establish the existence of a minimizing measure \( \lambda_\sigma^\Sigma \) and analyze its continuity properties in the weak* and strong topologies when \( \sigma \) and \( \Sigma \) are varied. We also give a description of the \( f \)-weighted potential of \( \lambda_\sigma^\Sigma \) and single out its characteristic properties. Such results are mostly new even for classical kernels in \( \mathbb{R}^n \), which is important in applications.

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1. Introduction and statement of the problem

Let \( X \) be a locally compact Hausdorff space and \( \mathfrak{M} = \mathfrak{M}(X) \) the linear space of all real-valued Radon measures \( \nu \) on \( X \) equipped with the vague (=weak*) topology, i.e., the topology of pointwise convergence on the class of all real-valued continuous functions on \( X \) with compact support.

A kernel \( \kappa \) on \( X \) is meant to be an element from \( \Phi(X \times X) \), where \( \Phi(Y) \) consists of all lower semicontinuous functions \( \psi : Y \to (-\infty, \infty] \) such that \( \psi \geq 0 \) unless \( Y \) is compact. Given \( \nu, \mu \in \mathfrak{M} \), the mutual energy and the potential with respect to the kernel \( \kappa \) are defined by

\[
\kappa(\nu, \mu) := \int \kappa(x, y) \, d(\nu \otimes \mu)(x, y) \quad \text{and} \quad \kappa_\nu(\cdot) := \int \kappa(\cdot, y) \, d\nu(y),
\]

respectively. (Here and in the sequel, when introducing notation, we shall always tacitly assume the corresponding object on the right to be well defined.) For \( \nu = \mu \) we get the energy \( \kappa(\nu, \nu) \) of \( \nu \). Let \( \mathcal{E} \) consist of all \( \nu \in \mathfrak{M} \) with \( -\infty < \kappa(\nu, \nu) < \infty \).
In this work we consider a positive definite kernel $\kappa$, which means that it is symmetric (i.e., $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in X$) and the energy $\kappa(\nu, \nu)$, $\nu \in \mathfrak{M}$, is nonnegative whenever defined. Then $\mathcal{E}$ forms a pre-Hilbert space with the scalar product $\kappa(\nu, \mu)$ and the seminorm $\|\nu\| := \sqrt{\kappa(\nu, \nu)}$ (see [3]). The topology on $\mathcal{E}$ defined by means of this seminorm is called strong.

For an arbitrary closed set $E \subset X$, let $\mathfrak{M}^+(E)$ consist of all nonnegative $\nu \in \mathfrak{M}$ with the support $S_\nu \subset E$, and let $\mathcal{E}^+(E) := \mathfrak{M}^+(E) \cap \mathcal{E}$. Given a measure $\nu$ and a function $\psi$, for the sake of brevity we shall write $\langle \psi, \nu \rangle := \int \psi d\nu$.

Fix an external field $f$. We assume that either $f \in \Phi(X)$ (Case I), or $f = \kappa_\zeta$, where $\zeta \in \mathcal{E}$ is a signed measure (Case II). Then the $f$-weighted potential $W^f$ and the $f$-weighted energy $G^f(\nu)$ of $\nu \in \mathcal{E}$ are respectively given by the formulas

$$W^f(x) := \kappa_\nu(x) + f(x), \quad G^f(\nu) := \|\nu\|^2 + 2\langle f, \nu \rangle = (W^f + f, \nu).$$

Note that $W^f, \nu \in \mathcal{E}$, is defined and $\neq -\infty$ at least nearly everywhere (n.e.) in $X$, that is, except at most for some set $N \subset X$ with the interior capacity $C(N) = 0$.

Having fixed also a nonempty closed set $\Sigma \subset X$, we consider a function $g > 0$, defined and continuous at least in some open neighborhood $U_\Sigma$ of $\Sigma$, and a measure $\sigma \in \mathfrak{M}^+(\Sigma)$ which will serve as a constraint.

We are interested in the constrained minimal $f$-weighted energy problem

$$G^f_\sigma(\Sigma, g) := \inf_{\nu \in \mathcal{E}^\sigma(\Sigma, g)} G^f(\nu),$$

(1.1)

where

$$\mathcal{E}^\sigma(\Sigma, g) := \{\nu \in \mathcal{E}^+(\Sigma) : \langle g, \nu \rangle = 1, \nu \leq \sigma\}$$

and $\nu \leq \sigma$ means that $\sigma - \nu \geq 0$. (In (1.1), as usual, the infimum over the empty set is taken to be $+\infty$.) Along with its electrostatic interpretation, this problem has also found applications in approximation theory (see [1] [2] [7]). If

$$G^f_\sigma(\Sigma, g) < \infty$$

(1.2)

(or, which is equivalent, if the class

$$\mathcal{E}^\sigma(\Sigma, g) := \{\nu \in \mathcal{E}^\sigma(\Sigma, g) : G^f(\nu) < \infty\}$$

is nonempty), then we shall consider the problem on the existence of $\lambda^f_\Sigma \in \mathcal{E}^\sigma(\Sigma, g)$ with minimal $f$-weighted energy $G^f(\lambda^f_\Sigma) = G^f_\sigma(\Sigma, g)$. Such a $\lambda^f_\Sigma$ (if exists) will be called an equilibrium measure corresponding to the data $\kappa, \Sigma, \sigma, g,$ and $f$.

If $X = \mathbb{R}^2$, $\kappa(x, y) = -\log |x - y|$, and $g = 1$, the constrained energy problem has been analyzed by P. Dragnev, E. Saff and E. Rakhmanov: see [1] [2], where $f \in \Phi(\mathbb{R}^2)$ is fast growing at infinity, and [7], where $\Sigma = [-1, 1]$ and $f = 0$.

However, the methods applied in this note and the results obtained differ essentially from those in [1] [2] [7]. Namely, our approach is mainly based on the use of both the strong and vague topologies, which enables us in both Cases I and II to establish the existence of an equilibrium measure $\lambda^f_\Sigma$ for noncompact $\Sigma$ and to study continuity properties of $\lambda^f_\Sigma$ as a function of $(\Sigma, \sigma)$. We also obtain variational inequalities for the $f$-weighted equilibrium potential $W^f_\lambda$ and single out its characteristic properties, modifying properly the arguments from [1] [2] [7].
For the sake of simplicity we shall restrict ourselves to the case where either $X$ is a countable union of compact sets or $\inf_{x \in X} g(x) > 0$. Then the concept of local $\nu$-negligibility and that of $\nu$-negligibility coincide for any $\nu \geq 0$ with $\langle g, \nu \rangle < \infty$; hence, every $N$ with $C(N) = 0$ is $\nu$-negligible if, moreover, $\nu \in E$.

Before formulating the results obtained, we observe the following lemma.

**Lemma 1.1.** $G_f^\sigma(\Sigma, g) > -\infty$.

**Proof.** Indeed, in Case II it is an immediate consequence of the representation

$$G_f(\nu) = \|\nu\|^2 + 2\kappa(\nu, \zeta) = \|\nu + \zeta\|^2 - \|\zeta\|^2, \quad \nu \in E.$$  \hspace{1cm} (1.4)

Let Case I take place. If $X$ is compact, then $f \in \Phi(X)$ is bounded from below by $-c$, where $c > 0$, while $\nu(X) \leq \left[\min_{x \in X} g(x)\right]^{-1} < \infty$ for all $\nu \in E^\sigma(\Sigma, g)$, and the required inequality follows. Otherwise, $f$ has to be $\geq 0$; hence, $G_f^\sigma(\Sigma, g) \geq 0$. □

### 2. Main results

Following [5], we call a (positive definite) kernel $\kappa$ **perfect** if $E^+ := E^+(X)$, treated as a topological subspace of $E$, is strongly complete and the strong topology on $E^+$ is finer than the induced vague topology. It follows that a perfect kernel has to be **strictly positive definite**, and the seminorm $\| \cdot \|$ is then actually a norm.

**Remark 2.1.** It is well known (see, e.g., [4, 5, 6]) that the class of perfect kernels includes the Riesz kernels $|x - y|^{\alpha - n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ (in particular, the Newtonian kernel $|x - y|^{2 - n}$ in $\mathbb{R}^n$, $n \geq 3$), the restriction of the logarithmic kernel $-\log |x - y|$ in $\mathbb{R}^2$ to the open unit disk, and the Green kernel $g_D$, where $D$ is an open set in $\mathbb{R}^n$, $n \geq 2$, and $g_D$ is its generalized Green function.

Let $\nu_E$ denote the trace of $\nu \in M$ upon a $\nu$-measurable set $E$.

**Theorem 2.2.** Assume (1.2) to hold, $\kappa$ to be perfect, and let $C(\Sigma)$ be finite

If, moreover, either $g|_{U\Sigma}$ is bounded or there exist $r \in (1, \infty)$ and $\omega \in E$ such that $g^r(x) \leq \kappa_\omega(x) \text{ n.e. in } U\Sigma,$ \hspace{1cm} (2.1)

then the following assertions hold true:

(a) There exists a unique equilibrium measure $\lambda_\Sigma^\sigma$.

(b) Let $\Sigma_s \subset U\Sigma$, $s \in S$, be a decreasing ordered family of closed sets such that $C(\Sigma_s) < \infty$ and $\bigcap_{s \in S} \Sigma_s = \Sigma$. Let $\sigma_s \in M^+(\Sigma_s)$, $s \in S$, decrease and converge vaguely to $\sigma$. Then

$$G_f^\sigma(\Sigma, g) = \lim_{s \in S} G_f^{\sigma_s}(\Sigma_s, g)$$  \hspace{1cm} (2.2)

and $\lambda_{\Sigma_s}^\sigma \to \lambda_\Sigma^\sigma$ strongly (hence, also vaguely).  

\footnote{Even for the Newtonian kernel, sets of finite capacity might be noncompact (see [5]).}
(c) Let \( \{K\} \) be the increasing ordered family of all compact subsets of \( \Sigma \). Then there exists a net \((\beta^*_K)_{K \in \{K\}} \subset (1, \infty)\) that decreases to 1 and such that, for all \( \beta_K \in [1, \beta^*_K] \),

\[
G^*_f(\Sigma, g) = \lim_{K \uparrow \Sigma} G^{\beta_K \sigma_K}_f(K, g).
\]

Furthermore, \( \lambda^{\beta_K \sigma_K}_K \to \lambda^*_\Sigma \) strongly (and, hence, vaguely).

Given a closed set \( E \subset X \) with \( C(E) > 0 \) and a universally measurable function \( \psi \) bounded from below nearly everywhere in \( E \), we write

\[
\inf_{x \in E} \psi(x) := \sup \left\{ q : \psi(x) \geq q \text{ n. e. in } E \right\}.
\]

Then

\[
\psi(x) \geq \inf_{x \in E} \psi(x) \text{ n. e. in } E,
\]

which follows from the countable subadditivity of \( C(\cdot) \) over universally measurable sets with interior capacity zero [5]. If \( \psi \) is bounded from above n. e. in \( E \), write

\[
\sup_{x \in E} \psi(x) := \inf_{x \in E} - \psi(x).
\]

In the next theorem we assume that \( \sigma_K \in \mathcal{E} \) for every compact \( K \subset \Sigma \) and \( \langle g, \sigma_{\Sigma_0} \rangle > 1 \), where \( \Sigma_0 := \{x \in \Sigma : f(x) < \infty\} \). Then (1.2) necessarily holds, since one can choose a compact \( K_0 \subset \Sigma_0 \) so that \( \sigma_{K_0} / \langle g, \sigma_{K_0} \rangle \in \mathcal{E}(\Sigma, g) \); cf. [2, 8].

**Theorem 2.3.** Given \( \lambda \in \mathcal{E}^*(\Sigma, g) \), the following assertions are equivalent:

(i) \( \lambda \) is an equilibrium measure \( \lambda^*_\Sigma \).

(ii) There exists \( w_\lambda \in \mathbb{R} \) such that

\[
W^f_\lambda(x) \geq w_\lambda g(x) \text{ n. e. in } S_{\sigma-\lambda},
\]

\[
W^f_\lambda(x) \leq w_\lambda g(x) \text{ n. e. in } S_\lambda.
\]

(iii) \(-\infty < \ell \leq L < \infty\), where

\[
\ell := \sup_{x \in S_\lambda} \frac{W^f_\lambda(x)}{g(x)}, \quad L := \inf_{x \in S_{\sigma-\lambda}} \frac{W^f_\lambda(x)}{g(x)}.
\]

**Remark 2.4.** It follows that, if \( \lambda \) is an equilibrium measure, then the collection of all \( w_\lambda \) for whom both (2.4) and (2.5) hold forms the finite closed interval \([\ell, L] \).

Of course, if \( g = 1 \), \( f = 0 \) and \( \kappa \) satisfies the maximum principle, then \( \ell = L \) and \([\ell, L] \) consists of just one point. However, this is not the case in general (see Sec. 5).

**Remark 2.5.** Relation (2.5) actually holds for every \( x \in S_\lambda \) if, moreover, \( f \in \Phi(X) \).

The rest of the article is organized as follows. Theorem 2.3 will be proved in Sec. 5. The proof of Theorem 2.2 to be given in Sec. 4 is based on a theorem on the strong completeness of \( \mathcal{E}^*(\Sigma, g) \), which is the main subject of the next section.

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2 Observe that, under the assumptions made, both \( C(S_{\sigma-\lambda}) \) and \( C(S_\lambda) \) are nonzero.
3. Auxiliary assertions

**Theorem 3.1.** Let \( \kappa \) be perfect, \( E \subset U_\Sigma \) be a closed set with \( C(E) < \infty \), and let \( g \) be as in Theorem 2.2. Then

\[
\mathcal{E}^+(E,g) := \{ \nu \in \mathcal{E}^+(E) : \langle g, \nu \rangle = 1 \},
\]

is strongly complete. In more detail, every strongly fundamental net \((\nu_s)_{s \in S} \subset \mathcal{E}^+(E,g)\) converges strongly (and, hence, vaguely) to a unique \( \nu_0 \in \mathcal{E}^+(E,g) \). If, moreover, \( \sigma_0 \in \mathcal{M}^+(E) \) is given, then the same holds true for \( \mathcal{E}^\sigma_0(E,g) \) instead of \( \mathcal{E}^+(E,g) \).

**Proof.** For every \( B \subset E \) there exists a uniquely determined measure \( \theta_B \in \mathcal{E}^+(\overline{B}) \), called the interior capacitary distribution associated with \( B \), with the properties

\[
\theta_B(X) = \| \theta_B \|^2 = C(B), \quad (3.1)
\]

\[
\kappa \theta_B(x) \geq 1 \quad \text{n. e. in } B. \quad (3.2)
\]

Indeed, this follows from \( C(E) < \infty \) and the perfectness of \( \kappa \) due to [5, Th. 4.1].

One can certainly assume that \( C(E) > 0 \), since otherwise \( \mathcal{E}^+(E,g) \) is empty. Also observe that there is no loss of generality in assuming \( g \) to satisfy (2.1) for \( E \) instead of \( U_\Sigma \), since otherwise \( g|_E \) is bounded from above (say by \( M \)), which combined with (3.2) again gives (2.1) for \( \omega := M^r \theta_E, r \in (1, \infty) \) being arbitrary.

Fix a strongly fundamental net \((\nu_s)_{s \in S} \subset \mathcal{E}^+(E,g)\); then one can assume it to be strongly bounded. Due to the perfectness of the kernel, such a net converges to some \( \nu_0 \in \mathcal{E}^+ \) strongly and, therefore, vaguely. The latter yields \( S_{\nu_0} \subset E \) and \( \langle g, \nu_0 \rangle \leq 1 \). To prove that \( \mathcal{E}^+(E,g) \) is strongly complete, it is enough to show that

\[
\langle g, \nu_0 \rangle = 1. \quad (3.3)
\]

To this end, we shall treat \( E \) as a locally compact space with the topology induced from \( X \). Given a set \( B \subset E \), let \( \chi_B \) denote its characteristic function and let \( CB := E \setminus B \). Further, let \( \{K\} \) be the increasing family of all compact subsets \( K \) of \( E \). Since \( g\chi_K \) is upper semicontinuous on \( E \) while \( (\nu_s)_{s \in S} \) converges to \( \nu_0 \) vaguely, for every \( K \in \{K\} \) we have

\[
\langle g\chi_K, \nu_0 \rangle \geq \limsup_{s \in S} \langle g\chi_K, \nu_s \rangle.
\]

On the other hand, Lemma 1.2.2 from [5] gives

\[
\langle g, \nu_0 \rangle = \lim_{K \in \{K\}} \langle g\chi_K, \nu_0 \rangle.
\]

Combining the last two relations, we obtain

\[
1 \geq \langle g, \nu_0 \rangle \geq \limsup_{(s, K) \in S \times \{K\}} \langle g\chi_K, \nu_s \rangle = 1 - \liminf_{(s, K) \in S \times \{K\}} \langle g\chi_K, \nu_s \rangle,
\]

\( S \times \{K\} \) being the directed product of the directed sets \( S \) and \( \{K\} \). Hence, if we prove

\[
\liminf_{(s, K) \in S \times \{K\}} \langle g\chi_K, \nu_s \rangle = 0, \quad (3.4)
\]

the desired relation (3.3) follows.
To obtain (3.3), consider the interior capacitary distribution \( \theta_{CK}, K \in \{ K \} \) being given. Then application of Lemma 4.1.1 and Theorem 4.1 from [5] yields
\[
\| \theta_{CK} - \theta_{\tilde{K}} \|^2 \leq \| \theta_{CK} \|^2 - \| \theta_{\tilde{K}} \|^2 \quad \text{provided } K \subset \tilde{K}.
\]
Furthermore, it is clear from (3.1) that the net \( \| \theta_{CK} \|, K \in \{ K \} \), is bounded and nonincreasing, and hence fundamental in \( \mathbb{R} \). The preceding inequality thus implies that \( (\theta_{CK})_{K \in \{ K \}} \) is strongly fundamental in \( \mathcal{E}^+ \). Since it converges vaguely to zero, zero is also its strong limit due to the perfectness of the kernel; hence,
\[
\lim_{K \in \{ K \}} \| \theta_{CK} \| = 0.
\]
Write \( q := r(r - 1)^{-1} \), where \( r \in (1, \infty) \) is the number involved in condition (2.1). Combining (2.1) with (3.2) shows that the inequality
\[
g(x) \chi_{CK}(x) \leq \kappa_\omega(x)^{1/r} \kappa_{\theta_{CK}}(x)^{1/q}
\]
subsists n. e. in \( E \), and hence \( \nu_\omega \)-almost everywhere in \( X \). Having integrated this relation with respect to \( \nu_\omega \), we then apply the Hölder and, subsequently, the Cauchy-Schwarz inequalities to the integrals on the right. This gives
\[
\langle g \chi_{CK}, \nu_\omega \rangle \leq (\kappa_\omega, \nu_\omega)^{1/r} (\kappa_{\theta_{CK}}, \nu_\omega)^{1/q} \leq \| \omega \|^{1/r} \| \theta_{CK} \|^{1/q} \| \nu_\omega \|.
\]
Taking limits here along \( S \times \{ K \} \), we obtain (3.3) and, hence, (3.3).

**Lemma 3.2.** Assume \( \kappa \) to be perfect. In both Cases I and II, the \( f \)-weighted energy \( G_f \) is lower semicontinuous on \( \mathcal{E}^+ \) in the strong topology.

**Proof.** Actually, in Case II \( G_f \) is continuous on \( \mathcal{E}^+ \) in the strong topology, which is seen from (1.4). Let Case I take place; then \( f \in \Phi(X) \), and hence, \( (f, \nu) \) is vaguely lower semicontinuous on \( \mathcal{E}^+ \) (see [5]). Since so is \( \kappa(\nu, \nu) \), the desired conclusion follows in view of the fact that the strong topology is finer than the vague one. \( \square \)

**Lemma 3.3.** Assume that (1.2) holds. For \( \lambda \in \mathcal{E}_f^+(\Sigma, g) \) to be an equilibrium measure, it is necessary and sufficient that
\[
\langle W_f^\lambda, \nu - \lambda \rangle \geq 0 \quad \text{for all } \nu \in \mathcal{E}_f^+(\Sigma, g).
\]

**Proof.** Since \( \mathcal{E}_f^+(\Sigma, g) \) is convex, for any its elements \( \nu, \mu \), and \( h \in (0, 1] \) we get
\[
G_f(h \nu + (1 - h) \mu) - G_f(\mu) = 2h(W_f^\mu, \nu - \mu) + h^2 \| \nu - \mu \|^2.
\]
(It has been used here that \( G_f \) is finite on \( \mathcal{E}_f^+(\Sigma, g) \); see (1.3) and Lemma 1.1.) If \( \mu = \lambda \) is an equilibrium measure, then the left (hence, the right) side of (3.6) is \( \geq 0 \), which leads to (3.5) by letting \( h \to 0 \). Conversely, if (3.5) holds, then with \( \mu = \lambda \) and \( h = 1 \) gives \( G_f(\nu) \geq G_f(\lambda) \) for all \( \nu \in \mathcal{E}_f^+(\Sigma, g) \), as required. \( \square \)
4. Proof of Theorem 2.2

(a) Fix \((\nu_s)_{s \in S} \subset E_f^\sigma(\Sigma, g)\) with the property that \(\lim_{s \in S} G_f(\nu_s) = G_f^\sigma(\Sigma, g)\); such a net will be called minimizing. Then identity \((4.6)\) with \(h = 1/2\) implies

\[\|\nu_s - \nu_d\|^2 \leq 2G_f(\nu_s) + 2G_f(\nu_d) - 4G_f^\sigma(\Sigma, g)\]

for all \(s, d \in S\), which establishes the strong fundamentality of \((\nu_s)_{s \in S}\) when combined with the above definition and Lemma 1.1. Therefore, by Theorem 3.1 it converges strongly and vaguely to a unique \(\nu_0 \in E^\sigma(\Sigma, g)\). On account of Lemma 3.2 we thus get

\[G_f^\sigma(\Sigma, g) \leq G_f(\nu_0) \leq \liminf_{s \in S} G_f(\nu_s) = G_f^\sigma(\Sigma, g)\]

consequently, \(\nu_0\) is an equilibrium measure \(\lambda_S^\sigma\).

The uniqueness of \(\lambda_S^\sigma\) follows in a standard way. Indeed, if \(\lambda, \hat{\lambda} \in E_f^\sigma(\Sigma, g)\) are two equilibrium measures, then the sequence \((\mu_n)_{n \in \mathbb{N}}\) with \(\mu_{2n} = \lambda\) and \(\mu_{2n+1} = \hat{\lambda}\) is minimizing; therefore, what has just been proved yields \(\lambda = \hat{\lambda}\) as required.

(b) Under the assumptions of (b), \(E^\sigma(\Sigma, g) \subset E^{\sigma_i}(\Sigma_d, g) \subset E^{\sigma_i}(\Sigma_s, g)\) for all \(s, d \in S\) whenever \(s \leq d\). Hence, \(G_f^\sigma(\Sigma_s, g)\) increases as \(s\) ranges through \(S\) and

\[G_f^\sigma(\Sigma, g) = \lim_{s \in S} G_f^\sigma(\Sigma_s, g).\]  \(\quad (4.1)\)

By reason of \((4.1)\), this yields \(G_f^\sigma(\Sigma_s, g) < \infty\) for every \(s \in S\). Therefore, by (a), there exists a unique equilibrium measure \(\lambda_s := \lambda_{S_s}^\sigma\). Since \(\lambda_d \in E_f^{\sigma_*}(\Sigma_s, g)\) for all \(d \geq s\), we conclude from Lemma 5.3 that \((W_f^{\lambda_0^*}, \lambda_d - \lambda_s) \geq 0\) and, consequently,

\[\|\lambda_d - \lambda_s\|^2 \leq G_f^\sigma(\Sigma_d, g) - G_f^\sigma(\Sigma_s, g)\]  \(\quad (4.2)\)

However, as follows from \((4.1)\), the net \(G_f^\sigma(\Sigma_s, g)\), \(s \in S\), is fundamental in \(\mathbb{R}\). When combined with \((4.2)\), this implies that \((\lambda_s)_{s \in S}\) is strongly fundamental in \(E^{\sigma_*}(\Sigma, g)\) for every \(\Sigma \in S\). Therefore, by Theorem 3.1 \((\lambda_s)_{s \in S}\) converges strongly and vaguely to a unique measure \(\nu_0\) and \(\nu_0 \in E^{\sigma_0}(\Sigma, g)\) for every \(\Sigma \in S\). Since \(\nu_0 \leq \sigma_0\) and \(\sigma_0 - \nu_0 \to \sigma - \nu_0\) vaguely as \(\ell\) ranges over \(S\), we get \(\nu_0 \leq \sigma\). Thus, actually \(\nu_0 \in E^\sigma(\Sigma, g)\) and, by Lemma 5.2

\[G_f(\Sigma, g) \leq G_f(\nu_0) \leq \liminf_{s \in S} G_f(\lambda_s).\]

Together with \((4.1)\), this gives \((2.2)\) and \(\nu_0 = \lambda_S^\sigma\), and the proof of (b) is complete.

(c) To prove (c), we start by establishing the relation

\[G_f^\sigma(\Sigma, g) = \lim_{K \uparrow \Sigma} G_f^{\sigma \cap}(K, g).\]  \(\quad (4.3)\)

For every \(\nu \in E_f^\sigma(\Sigma, g)\), write \(\hat{\nu}_K := \nu_K / \langle g, \nu_K \rangle\). Since, by \([5\text{ Lemma 1.2.2}]\),

\[1 = \lim_{K \uparrow \Sigma} \langle g, \nu_K \rangle, \quad \langle f, \nu \rangle = \lim_{K \uparrow \Sigma} \langle f, \nu_K \rangle, \quad \|\nu\|^2 = \lim_{K \uparrow \Sigma} \|\nu_K\|^2,\]

we obtain

\[G_f(\nu) = \lim_{K \uparrow \Sigma} G_f(\hat{\nu}_K).\]  \(\quad (4.4)\)
Having fixed \( \varepsilon > 0 \), we also conclude that there exists \( K^0 \in \{ K \} \) such that 
\[
\hat{\nu}_K \in \mathcal{E}_f^{(1+\varepsilon)\sigma_K}(K, g) \text{ for all } K \in \{ K \} \text{ that follow } K^0. 
\]
This yields
\[
G_f(\hat{\nu}_K) \geq G_f^{(1+\varepsilon)\sigma_K}(K, g). 
\] (4.5)

In view of the arbitrary choice of \( \nu \), substituting (4.5) into (4.4) gives
\[
G_f^\sigma(\Sigma, g) \geq \lim_{K \uparrow \Sigma} G_f^{(1+\varepsilon)\sigma_K}(K, g) \geq G_f^{(1+\varepsilon)\sigma}(\Sigma, g),
\]
the latter inequality being a consequence of the monotonicity of \( G_f \).

Letting here \( \varepsilon \to 0 \) and applying (b), we obtain
\[
G_f^\sigma(\Sigma, g) = \lim_{\varepsilon \to 0} \left[ \lim_{K \uparrow \Sigma} G_f^{(1+\varepsilon)\sigma_K}(K, g) \right] = \lim_{K \uparrow \Sigma} G_f^\sigma(K, g),
\]
and (4.3) is thus proved. Since obviously \( \lambda^\sigma_K \in \mathcal{E}_f^\sigma(\Sigma, g) \), relation (4.3), in turn, implies that the net \( (\lambda^\sigma_K)_{K \in \{ K \}} \) is minimizing and, hence, strongly fundamental.

Further, according to (b), for every \( K \in \{ K \} \) one can choose \( \beta^*_K \in (1, \infty) \) so that \( \beta^*_K \downarrow 1 \) as \( K \uparrow \Sigma \) and, for all \( \beta_K \in [1, \beta^*_K] \),
\[
\lim_{K \in \{ K \}} \left\| \lambda^{\beta_K\sigma_K} - \lambda^\sigma_K \right\|^2 = 0, 
\] (4.6)
\[
\lim_{K \in \{ K \}} \left[ G_f(\lambda^{\beta_K\sigma_K}) - G_f(\lambda^\sigma_K) \right] = 0. 
\] (4.7)

Then combining (4.3) and (4.7) gives (2.3), while (4.6) together with the strong fundamentality of \( (\lambda^\sigma_K)_{K \in \{ K \}} \) shows that \( (\lambda^{\beta_K\sigma_K})_{K \in \{ K \}} \) is strongly fundamental as well. Hence, according to Theorem 5.1 there exists a unique \( \nu_0 \) which is the strong limit of \( (\lambda^{\beta_K\sigma_K})_{K \in \{ K \}} \) and belongs to \( \mathcal{E}^{(1+\delta)\sigma}(\Sigma, g) \) for every \( \delta > 0 \); therefore, \( \nu_0 \in \mathcal{E}^\sigma(\Sigma, g) \). On account of Lemma 3.2 and (2.3), this yields
\[
G_f^\sigma(\Sigma, g) \leq G_f(\nu_0) \leq \lim_{s \in \mathcal{S}} G_f(\lambda^{\beta_K\sigma_K}) = G_f^\sigma(\Sigma, g).
\]

Consequently, \( \nu_0 = \lambda^\sigma_K \), and the proof is complete. \( \square \)

5. Proof of Theorem 2.3

Assume (i) to hold. Since \( G_f(\lambda) \) is finite, so is \( \langle W^f_\lambda, \lambda \rangle \). We start by showing that
\[
W^f_\lambda(x) \geq \langle W^f_\lambda, \lambda \rangle g(x) \text{ n.e. in } S_{\sigma - \lambda}.
\] (5.1)

On the contrary, let \( C(N) > 0 \), where \( N := \{ x \in S_{\sigma - \lambda} : W^f_\lambda(x) < \langle W^f_\lambda, \lambda \rangle g(x) \} \).

It follows from [3. Th. 4.2] that then one can choose \( n \in \mathbb{N} \) and a compact set \( K \subset N \) with \( C(K) > 0 \) so that \( W^f_\lambda(x)/g(x) \leq \langle W^f_\lambda, \lambda \rangle - n^{-1} \) for all \( x \in K \). Write \( \tau := \beta(\sigma - \lambda)_K \), where \( \beta := 1/\langle g, (\sigma - \lambda)_K \rangle \). Then \( \tau \) belongs to \( \mathcal{E}^+ \), is \( \neq 0 \), and
\[
\langle W^f_\lambda, \tau \rangle < \langle W^f_\lambda, \lambda \rangle.
\] (5.2)

Since \( \kappa(\lambda, \tau) \) is finite, this yields \( \langle f, \tau \rangle < \infty \). A straightforward verification also shows that \( \tau_h := (1 - h)\lambda + h\tau \leq \sigma \) for any \( h \in (0, 1] \). Consequently, \( \tau_h \in \mathcal{E}^f_\lambda(\Sigma, g) \) and, by Lemma 3.3 \( \langle W^f_\lambda, \tau_h - \lambda \rangle = h\langle W^f_\lambda, \tau - \lambda \rangle \geq 0 \), which contradicts (5.2).
Thus, according to \((2.4)\), \(W^f_\lambda/g(x)\) is bounded from below n.e. in \(S_{\sigma-\lambda}\); this implies \((2.4)\) with \(w_\lambda = L\), where \(L\) is defined by \((2.0)\). In turn, \((2.4)\) yields \(L < \infty\), because \(C(S_{\sigma-\lambda} \cap \Sigma_0) > 0\). Hence, \(\infty > L \geq (W^f_\lambda, \lambda) > -\infty\).

We proceed by establishing \((2.5)\) with \(w_\lambda = L\). Having denoted (cf. \([7]\))

\[
E^+(w) := \{ x \in \Sigma : W^f_\lambda(x)/g(x) > w \}, \quad E^-(w) := \{ x \in \Sigma : W^f_\lambda(x)/g(x) < w \},
\]

where \(w \in \mathbb{R}\) is arbitrary, we assume on the contrary that \((2.5)\) for \(w_\lambda = L\) does not hold. Then \(\lambda(E^+(L)) > 0\); hence, \(\lambda(E^+(w_1)) > 0\) for some \(w_1 \in (L, \infty)\).

At the same time, as \(w_1 > L\), relation \((2.4)\) yields \((\sigma - \lambda)(E^-(w_1)) > 0\). Therefore, there is a compact set \(F \subset E^-(w_1)\) such that \(\xi := (\sigma - \lambda)_F\) is nonzero. Since \(\xi \in E^+\) and \(\langle W^f_\lambda, \xi \rangle \leq w_1(g, \xi) < \infty\), we get \(\langle f, \xi \rangle < \infty\). A direct verification also shows that

\[
\gamma := \lambda - \lambda_{E^+(w_1)} + \alpha \xi \leq \sigma, \quad \text{where} \quad \alpha := \langle g, \lambda_{E^+(w_1)} \rangle/\langle g, \xi \rangle.
\]

Consequently, \(\gamma \in \mathcal{E}_f^+(\Sigma, g)\). On the other hand, it also follows from the above that

\[
\langle W^f_\lambda, \gamma - \lambda \rangle = \langle W^f_\lambda - w_1 g, \gamma - \lambda \rangle = -\langle W^f_\lambda - w_1 g, \lambda_{E^+(w_1)} \rangle + \alpha \langle W^f_\lambda - w_1 g, \xi \rangle < 0,
\]

which is, however, impossible (see Lemma \([3.3]\)). Thus, \((i) \Rightarrow (ii)\).

Furthermore, since \(L\) is finite, \((2.5)\) with \(w_\lambda = L\) yields \(\ell \leq L\). To complete the proof of \((iii)\), it remains to observe that \(\ell > -\infty\), which is obtained from \((2.5)\) with \(w_\lambda = \ell\) due to the fact that \(W^f_\lambda \neq -\infty\) n.e. in \(\Sigma\). Hence, \((i) \Rightarrow (iii)\).

Next, assume \(\lambda \in \mathcal{E}_f^+(\Sigma, g)\) to satisfy \((2.4)\) and \((2.5)\) for some \(w_\lambda \in \mathbb{R}\). Then actually \(\lambda \in \mathcal{E}_f^+(\Sigma, g)\), which is seen from \((2.5)\) when integrated with respect to \(\lambda\). Given \(\nu \in \mathcal{E}_f^+(\Sigma, g)\), we also conclude from \((2.4)\) and \((2.5)\) that

\[
\langle W^f_\lambda, \nu - \lambda \rangle = \langle W^f_\lambda - w_\lambda g, \nu - \lambda \rangle
\]

\[
= \langle W^f_\lambda - w_\lambda g, \nu_{E^+(w_\lambda)} \rangle + \langle W^f_\lambda - w_\lambda g, (\nu - \sigma)_{E^-(w_\lambda)} \rangle \geq 0,
\]

which establishes \((i)\) according to Lemma \([3.3]\). Thus, \((ii) \Rightarrow (i)\).

Since \((iii)\) obviously yields \((ii)\) for any \(w_\lambda \in [\ell, L]\), the proof is complete. \(\Box\)

6. Examples

The following easily verified fact is used in this section: if \(\lambda_*\) gives a solution to the **unconstrained** \(f\)-weighted minimal energy problem over a closed set \(\Sigma_*\), i.e.,

\[
\lambda_* \in \mathcal{E}_f^+(\Sigma_*, g), \quad G_f(\lambda_*) = \min_{\nu \in \mathcal{E}_f^+(\Sigma_*, g)} G_f(\nu),
\]

then \(\lambda_*\) also serves as an equilibrium measure \(\lambda^{\ast}_{\Sigma_*}\), provided the closed set \(\Sigma\) and the constraint \(\sigma \in \mathcal{M}_f^+(\Sigma)\) satisfy the assumptions \(S_{\lambda_*} \subset \Sigma \subset \Sigma_*\) and \(\sigma \geq \lambda_*\).

In the examples below, the collection of all \(w_\lambda\) for whom both \((2.4)\) and \((2.5)\) hold forms the whole non-degenerated interval \([\ell, L]\). The kernels from the examples are perfect, so that every equilibrium measure is determined uniquely.
Example 1. Let \( X = \mathbb{R}^n, n \geq 3, g = 1, f = 0, \kappa(x, y) = |x - y|^\alpha - n \), where \( \alpha \in (2, n) \) is given, and let \( \Sigma := S(0, 1) \cup S(0, r) \), where \( S(0, R) := \{ x : |x| = R \} \) and \( r < 1 \). Consider \( \sigma \in \mathcal{E}^+(\Sigma) \) such that \( \sigma_{S(0, 1)} \) is the rotationally symmetric probability measure, while \( \sigma_{S(0, r)} \) is an arbitrary nonzero measure. Then \( \lambda^\sigma_{\Sigma} = \sigma_{S(0, 1)} \), because \( \sigma_{S(0, 1)} \) minimizes \( \|\nu\|^2 \) among all probability measures supported by the closed unit ball (see [6]). Since the potential of \( \sigma_{S(0, 1)} \) takes constant values \( c_1 \) and \( c_r \) on \( S(0, 1) \) and \( S(0, r) \), respectively, and \( c_r > c_1 \) (see [6]), we get \( L = c_r > c_1 = \ell \).

A crucial assumption in Example 1 is that \( \kappa \) does not satisfy the maximum principle. As is seen from Example 2, this restriction is not necessary in case \( f \neq 0 \).

Example 2. Let \( X = \mathbb{R}^n, n \geq 3, g = 1, \kappa(x, y) = |x - y|^\alpha - n \), where \( \alpha \in (0, 2] \), \( f(x) = |x - a|^\alpha - n \), where \( a \in S(0, 1) \) is fixed, and let \( \lambda_\bullet \) minimize \( G_f(\nu) \) among the probability measures supported by \( S(0, 1) \). Then there are a constant \( q \) and a closed neighborhood \( U \) of \( a \) on \( S(0, 1) \) such that \( W_{\lambda_\bullet} f(x) = q \) a.e. in \( S_\lambda \) and \( W_{\lambda_\bullet} |f| > 2q \) (see [3]). We define \( \sigma \) to be \( \lambda_\bullet \) on \( S_\lambda \), any nonzero \( \nu \in \mathcal{E}^+ \) on \( U \), and 0 elsewhere, and let \( \Sigma := S_\lambda \cup U \). Then \( \lambda^\sigma_{\Sigma} = \lambda_\bullet \) and, consequently, \( \ell = q < 2q \leq L \).

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