Rogers functions and fluctuation theory

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Lévy Processes 7
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...it would be worth studying Lévy processes whose jump measure has a completely monotone density, and in particular, the Wiener–Hopf factorization of such.
Outline

- L.C.G. Rogers’s result
- Extension and further results
- ‘Rogers functions’ and their properties
- Wiener–Hopf factorisation
- Further research

Rogers functions and fluctuation theory
In preparation
Credits

In connection with this presentation, I thank:

- **L.C.G. Rogers**  
  — for an inspiring article

- **A. Kuznetsov**  
  — for letting me know about it

- **K. Kaleta, T. Kulczycki, J. Małecki, M. Ryznar**  
  — for joint research in the symmetric case

- **P. Kim, Z. Vondraček**  
  — for a discussion of the non-symmetric case
**CM jumps**

\( X_t \) is a 1-D Lévy process with Lévy measure \( \nu(x)dx \)

Notation: **CM** = completely monotone

**Definition**

\( X_t \) has **CM jumps** \( \iff \) \( \nu(x) \) and \( \nu(-x) \) are **CM** on \((0, \infty)\):

\[
\nu(x) = \mathcal{L} \mu_+(x) = \int_{(0,\infty)} e^{-sx} \mu_+(ds) \quad (x > 0)
\]

\[
\nu(x) = \mathcal{L} \mu_-(x) = \int_{(0,\infty)} e^{sx} \mu_-(ds) \quad (x < 0)
\]

(see Bernstein’s theorem)

**Examples:**

- Stable processes: \( \nu(\pm x) = c_\pm x^{-1-\alpha} \)
- Tempered stable processes: \( \nu(\pm x) = c_\pm x^{-1-\alpha} e^{-m\pm x} \)
- Meromorphic processes
Plot of $\nu(x)$ for a sample process with CM jumps.
Rogers’s theorem

Notation: **CBF** = complete Bernstein function

**Definition**

\[ f \text{ is a CBF} \iff \frac{1}{f} = \mathcal{L} g \text{ for a CM } g \iff \frac{1}{f} = \mathcal{L} \mathcal{L} \mu \]

(there are many equivalent definitions)

**Theorem [Rogers, 1983]**

\[ X_t \text{ has CM jumps} \iff \kappa(\tau; \xi) \text{ and } \hat{\kappa}(\tau; \xi) \text{ are CBFs of } \xi \text{ for some/all } \tau \]

- \( \kappa(\tau; \xi) \text{ and } \hat{\kappa}(\tau; \xi) \) are Laplace exponents of the ladder processes for \( X_t \) (describe extrema of \( X_t \))

(\textit{more on this below})
Extension of Rogers’s theorem

Theorem [K., 2013]

\( X_t \) has **CM jumps** and is **balanced**

\[ \kappa(\tau; \xi) \text{ and } \hat{\kappa}(\tau; \xi) \text{ are CBFs of both } \tau \text{ and } \xi \]

Furthermore, the following are **CBF**s of \( \tau \) and \( \xi \):

\[
\begin{align*}
\kappa(\tau_1; \xi) & \quad \hat{\kappa}(\tau_1; \xi) \\
\kappa(\tau_2; \xi) & \quad \hat{\kappa}(\tau_2; \xi) \\
\kappa(\tau; \xi_1) & \quad \hat{\kappa}(\tau; \xi_1) \\
\kappa(\tau; \xi_2) & \quad \hat{\kappa}(\tau; \xi_2)
\end{align*}
\]

\( (0 \leq \tau_1 \leq \tau_2) \quad (0 \leq \xi_1 \leq \xi_2) \)

- The meaning of ‘**balanced**’ is explained later

  *(stable are balanced; tempered stable can be made balanced)*
Supremum functional

**Definition**

Supremum of $X_t$:

$$M_t = \sup_{s \in [0,t]} X_s$$

Time of supremum:

$$T_t \in [0, t] : M_t = X_{T_t}$$

**Theorem**

$$\int_0^\infty \left( E e^{-\sigma T_t - \xi M_t} \right) e^{-\tau t} dt dx = \frac{1}{\tau} \frac{\kappa(\tau; 0)}{\kappa(\sigma + \tau; \xi)}$$

That is:

$$\mathcal{L}_{t \mapsto \tau} P(T_t \in ds, M_t \in dx) = \frac{1}{\tau} \frac{\kappa(\tau; 0)}{\kappa(\sigma + \tau; \xi)}$$
Properties of the supremum

Corollary [Rogers, 1983]
If $X_t$ has **CM jumps**:
\[
\frac{d}{dx} \int_0^\infty e^{-\tau t} P(M_t < x) \, dt
\]
is **CM** in $x$

Corollary [K.]
If $X_t$ has **CM jumps** and is **balanced**:
\[
\frac{d}{ds} \int_0^\infty e^{-\tau t} P(T_t < s) \, dt
\]
is **CM** in $s$

Corollary [K.]
If $X_t$ has **CM jumps** and is **balanced**:
\[
\mathbb{E}e^{-\xi M_t}
\]
is **CM** in $t$
Space-only Laplace transform

**Theorem [K.]**

If $X_t$ has **CM jumps** and is **balanced**:

$$E e^{-\xi M_t} = \int_0^\infty e^{-tr} \frac{\xi \Re \psi^{-1}(r)}{|i\xi - \psi^{-1}(r)|^2} \frac{\psi^*_r(\xi)}{r} dr$$

where

$$\psi^*_r(\xi) = \exp \left( \frac{1}{\pi} \int_{\psi_r(0)}^\infty \arg \left( 1 - \frac{i\xi}{\psi^{-1}(s)} \right) \frac{ds}{s} \right)$$

and

$$\psi_r(\xi) = \frac{(\xi - \psi^{-1}(r))(\xi + \overline{\psi^{-1}(r)})}{\psi(\xi) - r}$$

($\psi$ is the Lévy–Khintchine exponent; more on this later)
Semi-explicit formula?

If one can justify the use of Fubini:

If $X_t$ has **CM jumps** and is **balanced**: 

$$P(M_t < x) = \int_0^\infty e^{-tr}F_r(x)dr$$

where 

$$F_r(x) = c_r e^{arx} \sin(\beta_rx + \vartheta_r) - \{\text{CM correction}\}$$

$$\alpha_r = \text{Im}(\psi^{-1}(r))$$

$$\beta_r = \text{Re}(\psi^{-1}(r))$$

$c_r, \vartheta_r$ and the **CM** correction are given semi-explicitly
Potential applications

- Semi-explicit expression for the distribution of $M_t$
- Asymptotic expansions and estimates of the above
- Eigenfunction expansion for $X_t$ in half-line

For the symmetric case, see:

1. K.
   *Spectral analysis of subordinate Brownian motions...*
   Studia Math. 206(3) (2011)

2. K., J. Małecki, M. Ryznar
   *Suprema of Lévy processes*
   Ann. Probab. 41(3B) (2013)

3. K., J. Małecki, M. Ryznar
   *First passage times for subordinate Brownian...*
   Stoch. Proc. Appl 123 (2013)
Lévy–Khintchine exponent

Definition

\[ \mathbb{E} e^{-i\xi X_t} = e^{-t\psi(\xi)} \]

Lévy–Khintchine formula

\[ \psi(\xi) = a\xi^2 - ib\xi + \int_{\mathbb{R}} (1 - e^{i\xi x} + i\xi x \mathbf{1}_{|x|<1}) \nu(x) \, dx \]

- \( \text{Re} \, \psi(\xi) \geq 0 \)
CM jumps revisited

Observation: If $X_t$ has **CM jumps**: $\nu(x) = \mathcal{L} \mu_+(x)$ (for $x > 0$)

$\nu(x) = \mathcal{L} \mu_-(x)$ (for $x < 0$)

then

$$\psi(\xi) = a\xi^2 - ib\xi + \int_{\mathbb{R}\setminus\{0\}} \left( \frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

with

$$\mu(E) = \mu_+(E \cap (0, \infty)) + \mu_-((-E) \cap (-\infty, 0))$$
Rogers functions

**Definition**

$f$ is a **Rogers function** if

$$f(\xi) = a\xi^2 - ib\xi + c + \int_{\mathbb{R}\setminus\{0\}} \left( \frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

for $a \geq 0$, $b \in \mathbb{R}$, $c \geq 0$, $\mu \geq 0$

- $f$ extends to $\mathbb{C} \setminus i\mathbb{R}$
- $f(-\bar{\xi}) = \overline{f(\xi)}$
- It suffices to consider $f$ on $\{\xi : \text{Re} \xi > 0\}$
Equivalent definitions

Proposition

The following are equivalent:

(a) for $a \geq 0$, $b \in \mathbb{R}$, $c \geq 0$, $\mu \geq 0$:

$$f(\xi) = a\xi^2 - ib\xi + c + \int_{\mathbb{R}\setminus\{0\}} \left( \frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

(b) for $k \geq 0$, $\varphi \in [0, \pi]$:

$$f(\xi) = k \exp \left( \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\xi}{\xi + is} - \frac{1}{1 + |s|} \right) \frac{\varphi(s)ds}{|s|} \right)$$

(c) $f$ is holomorphic in $\{\xi : \text{Re } \xi > 0\}$ and:

$$\text{Re} \frac{f(\xi)}{\xi} \geq 0 \quad \text{if} \ \text{Re } \xi > 0$$

*(that is, $f(\xi)/\xi$ is a Nevanlinna–Pick function)*
Real values

Theorem [K.]
If $f$ is a Rogers function, then:
(a) For $r > 0$ there is at most one solution of
$$f(\xi) = r \quad \text{ (Re} \, \xi > 0)$$
Write $\xi = f^{-1}(r)$
(b) $|f^{-1}(r)|$ is increasing

Definition
A Rogers function $f$ is **balanced** if
$$-\frac{\pi}{2} + \varepsilon \leq \arg(f^{-1}(r)) \leq \frac{\pi}{2} - \varepsilon$$
$X_t$ is **balanced** if $\psi$ is **balanced**
BM with drift stable tempered stable

Real lines $\{\xi : f(\xi) \in (0, \infty)\}$ for some Rogers functions
Extension

Definition

A Rogers function \( f \) is **nearly balanced** if

\[
    f \circ \Phi
\]

is **balanced** for some Möbius transformation \( \Phi \) which preserves \( \{\xi : \Re \xi > 0\} \) (e.g. vertical translation).

Theorem

Main results extend to **nearly balanced** processes.

Examples of **nearly balanced** processes:

- Non-monotone strictly stable and their mixtures
- Tempered strictly stable:
  \[
  \nu(\pm x) = c_{\pm} x^{-1-\alpha} e^{-m_{\pm} x}
  \]
- (Completely) subordinate to above

\*(that is, with a subordinator corresponding to a CBF)*
Real lines $\{\xi : f(\xi) \in (0, \infty)\}$ for some Rogers functions
### Analytical approach

**Wiener–Hopf method**

For $A \in \mathcal{S}'(\mathbb{R})$ write

\[
A = A^+ \ast A^- \quad \text{(or } \mathcal{F}A = \mathcal{F}A^+ \cdot \mathcal{F}A^-)\]

where \(\text{supp} A^+ \subseteq [0, \infty)\), \(\text{supp} A^- \subseteq (-\infty, 0]\)

- Fourier transform of $A^+$ extends to \(\{\xi : \text{Im} \xi > 0\}\):
  \[
  \log \mathcal{F}A^+(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \log \mathcal{F}A(z)dz
  \]

- Fourier transform of $A^-$ extends to \(\{\xi : \text{Im} \xi < 0\}\):
  \[
  \log \mathcal{F}A^-(\xi) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \log \mathcal{F}A(z)dz
  \]

  (these are principal value integrals at \(\pm\infty\))

- Developed to solve integral equations and PDEs with mixed boundary conditions on \((-\infty, 0)\) and \((0, \infty)\)
Wiener–Hopf in fluctuation theory

Wiener–Hopf factorization

\[
\frac{1}{\psi(\xi) + \tau} = \frac{1}{\kappa(\tau; -i\xi)} \cdot \frac{1}{\hat{\kappa}(\tau; i\xi)}
\]

- \( \frac{1}{\psi(\xi) + \tau} = \mathcal{F} U^\tau(\xi) \) with \( U^\tau(E) = \int_0^\infty e^{-\tau t} P(X_t \in E) \, dt \)
  (or \( \frac{\tau}{\psi(\xi) + \tau} \) is the Fourier transform of \( X_{e^t} \))

- \( \frac{1}{\kappa(\tau; -i\xi)} = \mathcal{F} V^\tau(\xi) \) and \( \frac{1}{\kappa(\tau; i\xi)} = \mathcal{L} V^\tau(\xi) \)
  (\( V^\tau(\, dx \) is the renewal measure of the ascending ladder height process for \( X_t \) killed at rate \( \tau \))

- \( U^\tau(E) = \int_\mathbb{R} V^\tau(x - E) V^\tau(\, dx) \)
Wiener–Hopf in fluctuation theory

Wiener–Hopf factorization

\[
\frac{1}{\Psi(\xi) + \tau} = \frac{1}{\kappa(\tau; -i\xi)} \cdot \hat{\kappa}(\tau; i\xi)
\]

- Baxter–Donsker-type formula:

\[
\log \frac{\kappa(\tau; \xi)}{\kappa(\tau; 1)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{1}{i\xi - z} - \frac{1}{i - z} \right) \log(\psi(z) + \tau) \, dz
\]

- Deform the contour of integration from \( \mathbb{R} \) to:

- Exponential **CBF** representation of \( \kappa(\tau; \xi) \) in \( \xi \) follows

(proving Rogers’s result)
Wiener–Hopf in fluctuation theory

Wiener–Hopf factorization

\[
\frac{1}{\Psi(\xi) + \tau} = \frac{1}{\kappa(\tau; -i\xi)} \cdot \frac{1}{\hat{\kappa}(\tau; i\xi)}
\]

- Baxter–Donsker-type formula:

\[
\log \frac{\kappa(\tau; \xi_1)}{\kappa(\tau; \xi_2)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{1}{i\xi_1 - z} - \frac{1}{i\xi_2 - z} \right) \log(\Psi(z) + \tau) \, dz
\]

- Deform the contour of integration from \( \mathbb{R} \) to

\[
\{ \xi \in \mathbb{C} : \Psi(\xi) \in (0, \infty) \}
\]

- Then \( \log(\Psi(z) + \tau) \) is holomorphic in \( \tau \in \mathbb{C} \setminus (-\infty, 0] \)
  (a major step towards the extension)
Non-balanced processes

Problem

Show that if $X_t$ has **CM jumps**, then:

\[
\begin{align*}
\kappa(\tau; \xi) &= \hat{\kappa}(\tau; \xi) \\
\kappa(\tau_1; \xi) &= \hat{\kappa}(\tau_1; \xi) \\
\kappa(\tau_2; \xi) &= \hat{\kappa}(\tau_2; \xi) \\
\kappa(\tau; \xi_1) &= \hat{\kappa}(\tau; \xi_1) \\
\kappa(\tau; \xi_2) &= \hat{\kappa}(\tau; \xi_2)
\end{align*}
\]

\[
(0 \leq \tau_1 \leq \tau_2) \quad (0 \leq \xi_1 \leq \xi_2)
\]

are **CBF**s of $\tau$ and $\xi$

Problem

When the above are **CBF**s of $\tau$ only?
Bivariate CBFs

Problem

Describe functions \( f(\xi, \eta) \) such that

\[
\begin{align*}
  f(\xi, \eta), \quad \frac{f(\xi, \eta_1)}{f(\xi, \eta_2)} \quad \text{and} \quad \frac{f(\xi_1, \eta)}{f(\xi_2, \eta)} \\
  (0 \leq \xi_1 \leq \xi_2) \quad (0 \leq \eta_1 \leq \eta_2)
\end{align*}
\]

are CBFs of \( \xi, \eta \)
Distribution of the supremum functional

**Problem**

Justify the use of Fubini for the formula for $P(M_t < x)$

**Problem**

Prove generalised eigenfunction expansion for $X_t$ killed upon leaving half-line

- Work in progress
Hitting time of a point

\[ P(\tau_X > t) \quad \text{with} \quad \tau_X = \inf\{t : X_t \geq x\} \]

Problem

Find a formula, estimates and asymptotic expansion of \( P(\sigma_X > t) \) for

\[ \sigma_X = \inf\{t : X_t = x\} \]

For the symmetric case, see:

1. K. Spectral theory for one-dimensional symmetric... Electron. J. Probab. 17 (2012)
2. T. Juszczyszyn, K. Hitting times of points for symmetric Lévy... In preparation