Couplings of the $\eta$ and $\eta'$ Mesons to the Nucleon

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Abstract

The couplings of the $\eta$ and $\eta'$ mesons to the nucleon are obtained from the $U_A(1)$ Goldberger-Treiman relation. The chiral symmetry breaking corrections are very large and bring the calculated values of the coupling constants $G_{\eta NN}$ and $G_{\eta' NN}$ close to values obtained from potential models.

The coupling constants of the nucleon to the $\pi$ and to the $\eta$ and $\eta'$ mesons enter in the study of low energy hadronic physics especially in the description of nucleon-nucleon scattering data [1], $\eta$ photoproduction [2], in the estimates of the electric dipole moment of the neutron [3] or of the proton-neutron mass difference [4]. While the value of $G_{\pi NN}$ is known with reasonable precision, the values of $G_{\eta NN}$ and $G_{\eta' NN}$ are not [5]. For the pion, one has the well known Goldberger-Treiman relation [6].

\[ f_\pi G_{\pi NN} = \sqrt{2} m_N G^{(3)} \] (1)

Which involves the pion decay constant $f_\pi = 0.924$ GeV and the renormalized axial-vector coupling constant $G^{(3)} = (1.267 \pm 0.004)/\sqrt{2}$ defined through

\[ \langle N(p, s) | A^{(3)}_\mu | N(p, s) \rangle = G^{(3)} s_\mu/\sqrt{2} \] (2)

at vanishing momentum transfer.

The Goldberger-Treiman relation is satisfied to a very good accuracy [7]. This is not surprising because the approximations used to obtain eq. (1) involve an extrapolation in momentum transfer squared from 0 to $m_\pi^2$ which is a very small quantity on the hadronic scale.

The straightforward generalisation of eq. (1) to the $\eta$ and $\eta'$ channels was considered in [8] who obtained

\[ G_{\eta NN} = 3.4 \pm 0.5 \quad \text{,} \quad G_{\eta' NN} = 1.4 \pm 1.1 \] (3)
values which differ considerably from ones obtained in potential models [5]:

\[ G_{\eta NN} = 6.8, \quad G_{\eta' NN} = 7.3 \]  \tag{4}

The \( \eta \) and \( \eta' \) mesons are however very heavy and the extrapolation from the chiral limit to the values of the physical masses potentially introduces large corrections.

It is the purpose of this work to undertake such an extrapolation. This will be done under the sole assumption that corrections to chiral symmetry breaking arise mainly from the contributions of intermediate states in the isoscalar \( 0^- \) continuum with invariant mass squared in the interval extending from 1.5\( \text{GeV}^2 \) to 3.5\( \text{GeV}^2 \) which includes the resonances \( \eta(1295) \), \( \eta(1405) \), \( \eta(1475) \) as well as the newly discovered \( X(1835) \) which couples strongly to the nucleon [9].

It will be seen that, when chiral symmetry breaking is taken into account, the \( U_A(1) \) Goldberger-Treiman relation yields for \( G_{\eta NN} \) and \( G_{\eta' NN} \) values close to the ones appearing in eq. (4).

The isoscalar components of the octet of axial-vector currents couple to the \( \eta \) and \( \eta' \) mesons:

\[
\langle 0 | A^{(8)}_{\mu} | \eta(p) \rangle = i f_8 \cos \theta \ p_{\mu}, \quad \langle 0 | A^{(0)}_{\mu} | \eta(p) \rangle = i f_0 \sin \theta \ p_{\mu}\]

(5)

\( \theta \) is the singlet-octet mixing angle. In the \( SU(3) \) limit \( f_8 = f_\pi \). The axial-vector currents are expressed in terms of quark fields

\[
A^{(8)}_{\mu} = \frac{1}{2\sqrt{3}}(\bar{u}\gamma_\mu \gamma_5 u + \bar{d}\gamma_\mu \gamma_5 d - 2\bar{s}\gamma_\mu \gamma_5 s)
\]

\[
A^{(0)}_{\mu} = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu \gamma_5 u + \bar{d}\gamma_\mu \gamma_5 d + \bar{s}\gamma_\mu \gamma_5 s) \tag{6}
\]

A two angle description for the octet and singlet components, \( \theta_8 \) and \( \theta_0 \), has more recently been advocated [10]. We shall see that this description does not alter our numerical analysis.

When the divergence of the currents is taken, the singlet component picks up a gluon anomaly term

\[
D^{(8)} = \partial_\mu A^{(8)}_\mu = \frac{i}{\sqrt{3}}(m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d - 2m_s \bar{s}\gamma_5 s)
\]

\[
D^{(0)} = \partial_\mu A^{(0)}_\mu = i \sqrt{\frac{2}{3}}(m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d + m_s \bar{s}\gamma_5 s) + Q \tag{7}
\]

Where

\[
Q = \frac{1}{\sqrt{6}} \frac{3\alpha_s}{4\pi} G_{\mu\nu} \tilde{G}^{\mu\nu}
\]

(8)

\( G_{\mu\nu} \) being the gluonic field strength tensor and \( \tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma} \) it’s dual.
Consider the matrix element

\[
\langle N(p') \left| D^{(8)} / f_8 \right| N(p) \rangle = \Pi_8(t) \, \overline{\psi} \gamma_5 \psi
\]

with \( t = (p - p')^2 \). We have

\[
\Pi_8(t = 0) = \sqrt{2m_N G^{(8)}/f_8}
\]

where \( G^{(8)} \) is the isoscalar octet axial-vector coupling constant related to \( G^{(3)} \) through the \( F/D \) ratio obtained from Hyperon decay, \( G^{(8)} = .24 \).

In the low energy domain, \( \Pi_8(t) \) is dominated by the \( \eta \) and \( \eta' \) poles, i.e.

\[
\Pi_8(t) = \frac{m_\eta^2 \cos \theta}{m_\eta^2 - t} G_{\eta NN} + \frac{m_{\eta'}^2 \sin \theta}{m_{\eta'}^2 - t} G_{\eta' NN} + \text{other terms}
\]

\( \Pi_8(t) \) is an analytic function in the complex \( t \)-plane with poles at \( t = m_\eta^2 \) and \( t = m_{\eta'}^2 \) and a cut along the positive \( t \)-axis which starts at the \( (\eta + 2\pi) \) threshold.

An immediate consequence of Cauchy’s theorem is

\[
\frac{1}{2\pi i} \int_{C} \frac{dt}{t} \Pi_8(t) = \frac{1}{2\pi i} \int_{th} \frac{dt}{t} Disc \Pi_8(t) + \frac{1}{2\pi i} \oint \frac{dt}{t} \Pi_8(t) = \sqrt{2m_N G^{(8)/f_8}} - G_{\eta NN} \cos \theta - G_{\eta' NN} \sin \theta
\]

where \( C \) is a closed contour consisting of two straight lines immediately above and below the cut which starts effectively at \( t_1 = (m_{\eta'} + 2m_\pi)^2 \) and a circle of large radius in the complex \( t \)-plane (\( R \simeq 4 - 5 \text{GeV}^2 \)). The contribution of the non-resonant threshold part of the continuum extending from \( t_0 = (m_\eta + 2m_\pi)^2 \) to \( t_1 \) is small compared to the one of the \( \eta \) and \( \eta' \) mesons and is neglected.

Treating the \( SU(3) \) singlet amplitude \( \Pi_0(t) \) in a similar fashion gives

\[
\frac{1}{2\pi i} \int_{C} \frac{dt}{t} \Pi_0(t) = \sqrt{2m_N G^{(0)/f_0}} + G_{\eta NN} \sin \theta - G_{\eta' NN} \cos \theta
\]

If the left hand sides of eqs. (12) and (13), which represent the corrections due to the contributions of the \( 0^+ \) continuum were negligible, these equations would yield for the coupling constants \( G_{\eta NN} \) and \( G_{\eta' NN} \) the values obtained in ref. [8]. The neglect of the contributions of the continuum is however not justified, they are of order \( m_\eta^2 \) and \( m_{\eta'}^2 \) which are not small quantities on the hadronic scale.

In order to estimate these contributions, use will be made of the modified integral \( \frac{1}{2\pi i} \int_{C} dt \frac{1}{t - a_0 - a_1} \Pi_8(t) \) where \( a_0 \) and \( a_1 \) are so far arbitrary constants. The residue theorem yields

\[
G_{\eta NN} \cos \theta (1 - a_0 m_\eta^2 - a_1 m_\eta^4) + G_{\eta' NN} \sin \theta (1 - a_0 m_{\eta'}^2 - a_1 m_{\eta'}^4) - \sqrt{2m_N G^{(8)/f_8}}
\]

\[
= \frac{1}{2\pi i} \int_{th} dt \frac{1}{t - a_0 - a_1} Disc \Pi_8(t) + \frac{1}{2\pi i} \oint dt \frac{1}{t - a_0 - a_1} \Pi_8^{QCD}(t)
\]

(14)
In the first of the integrals above, over the cut, the main contribution is expected to arise from the interval $I : 1.5\text{GeV}^2 \leq t \leq 3.5\text{GeV}^2$ where the PDG [11] lists three $0^-$ resonances $\eta(1295)$, $\eta(1405)$ and $\eta(1475)$ in addition to the newly discovered $X(1835)$ [9]. The constants $a_0$ and $a_1$ are now chosen so as to practically annihilate the kernel $(\frac{1}{t} - a_0 - a_1 t)$ on the interval $I$. The choice:

$$a_0 = .879\text{GeV}^{-2}, \quad a_1 = -.177\text{GeV}^{-4} \quad (15)$$

obtained by a least square fit, reduces the integrand to a few percent of it’s initial value on $I$. This allows the neglect of the integral over the cut.

In the second integral, over the circle, $\Pi_8(t)$ has been replaced by $\Pi_{8}^{\text{QCD}}(t)$ an approximation which is good except possibly for a small region in the vicinity of the positive real $t$ axis. This integral is likewise negligible even though the kernel $(\frac{1}{t} - a_0 - a_1 t)$ is no longer small because $\Pi_{8}^{\text{QCD}}(t)$ itself is small, $\Pi_{8}^{\text{QCD}}(t) \sim m_{u,d}^2(\vec{q}q)$.

This can be seen by using the method of QCD sum-rules [12]. Because the nucleon currents do not involve the strange quark $s$, $\Pi_{8}^{\text{QCD}}(t)$ can be obtained from a study of the three point function

$$A(p,p',q) = i \frac{\sqrt{3}f_8}{3} \int \int \text{d}x \text{d}y e^{ipx} e^{ip'y} \langle 0 | T \eta(x) J_8(0) \bar{\eta}(y) | 0 \rangle$$

where $\eta$ is the nucleon current with coupling $\lambda_N$ to the nucleon, $J_8 = m_u \bar{u} \gamma_5 u + m_d \bar{d} \gamma_5 d$.

$A(p,p',q)$ has a double nucleon pole ($p = p'$)

$$A = i\gamma. q \gamma_5 \left[ \frac{\lambda_N^2 m_N^2}{(p^2 - m_N^2)^2} \Pi_6(q^2) + \ldots \right] + \text{other tensor structures} \quad (16)$$

In the deep euclidean region, $A_{\text{QCD}}$ is given by the operator product expansion.

$$A_{\text{QCD}} = c_u(p,p',q) \langle 0 | m_u \bar{u} | 0 \rangle + c_d(p,p',q) \langle 0 | m_d \bar{d} | 0 \rangle \quad (17)$$

$\Pi_{8}^{\text{QCD}}(t)$ is obtained by extrapolating to the nucleon mass shell, i.e. by taking the Borel (Laplace) transform of eqs. (10) and (17) with respect to the variable $p^2$ while keeping $t = q^2$ large and negative. Identifying terms shows that $\Pi_{8}^{\text{QCD}}(t)$ is proportional to $\langle 0 | m_q \bar{q} | 0 \rangle$.

Note that the contribution of the gluonic term $Q$ (which contributes only to $\Pi_0(t)$) on the circle is not necessarily small. For this reason $\Pi_0(t)$ will not be used in this calculation.

With our choice of the coefficients $a_0$ and $a_1$ the integrals on the r.h.s. of eq. (14) thus becomes negligible and we have

$$G_{\eta NN} \cos \theta \left( 1 - a_0 m_\eta^2 - a_1 m_\eta^4 \right) + G_{\eta' NN} \sin \theta \left( 1 - a_0 m_{\eta'}^2 - a_1 m_{\eta'}^4 \right) = \sqrt{2} m_N G^{(s)}/f_8 \quad (18)$$
The chiral symmetry breaking corrections now show up in the coefficients of the coupling constants.

More information is obtained by reiterating the same procedure with a quadratic fit to $\frac{1}{t}$ i.e. use the integral $\frac{1}{2\pi i} \int_c dt (\frac{1}{t} - b_0 - b_1 t - b_2 t^2) \Phi(t)$ with fit coefficients

$$b_0 = 1.380 \text{Gev}^{-2}, \quad b_1 = -0.607 \text{Gev}^{-4}, \quad b_2 = 0.085 \text{Gev}^{-6} \quad (19)$$

This yields

$$G_{\eta NN} \cos \theta \left( 1 - b_0 m_\eta^2 - b_1 m_\eta^4 - b_2 m_\eta^6 \right) + G_{\eta' NN} \sin \theta \left( 1 - b_0 m_{\eta'}^2 - b_1 m_{\eta'}^4 - b_2 m_{\eta'}^6 \right) = \sqrt{2} m_N G(8) / f_8 \quad (20)$$

Numerically eqs. (18) and (20) boil down to

$$0.752 G_{\eta NN} \cos \theta + 0.341 G_{\eta' NN} \sin \theta = \sqrt{2} m_N G(8) / f_8 \quad (21)$$

The coupling constant $f_8$ is not measured experimentally. It can be obtained from the two photon decay rate of the $\eta$ and $\eta'$ mesons

$$\Gamma(\eta \to 2\gamma) = \frac{\alpha^2 m_\eta^3}{192\pi^3} \left( \frac{\cos \theta}{f_8} - \frac{2\sqrt{2} \sin \theta}{f_0} \right)^2 (1 + \Delta_\eta)^2$$

$$\Gamma(\eta' \to 2\gamma) = \frac{\alpha^2 m_{\eta'}^3}{192\pi^3} \left( \frac{\sin \theta}{f_8} + \frac{2\sqrt{2} \cos \theta}{f_0} \right)^2 (1 + \Delta_{\eta'})^2 \quad (22)$$

$\Delta_\eta$ and $\Delta_{\eta'}$ represent chiral symmetry breaking correction factors which are often neglected in the literature but which have been evaluated in ref. [13] and found to be large:

$$\Delta_\eta = 0.77 \quad \text{and} \quad \Delta_{\eta'} = 6.0 \quad (23)$$

The corresponding values of $f_8, f_8 = 1.27 f_\pi, 1.55 f_\pi$ for $\theta = -18.5^0, -30.5^0$, respectively, are obtained from the measured values of the decay rates.

The coupling constants $G_{\eta NN}$ and $G_{\eta' NN}$ thus come out as a function of the mixing angle $\theta$.

Traditionally, analyses based on the Gell-Mann-Okubo mass formula led to the adoption of a small value for the mixing angle, $\theta \simeq -10^0$. Subsequent study of the axial-anomaly generated decays $\eta, \eta' \to 2\gamma$ and measurement of the decay rates led to the adoption of larger values for $\theta, -25^0 \lesssim \theta \lesssim 20^0$ [14]. Recently [15] the corrections due to chiral symmetry breaking to the Gell-Mann-oakes-Renner relation, to the calculation of the decay rate $\Gamma(\eta \to 3\pi)$ and to radiative decays involving $\eta$ and $\eta'$ were evaluated. It was found that good agreement with experiment and numerical stability is obtained for values of $\theta$ in the range $-30.5^0 \leq \theta \leq -18.5^0$.

Varying $\theta$ within these limits yields

$$G_{\eta NN} = 5.45 \quad , \quad G_{\eta' NN} = 10.90 \quad \text{for} \ \theta = -18.5^0$$

$$G_{\eta NN} = 4.95 \quad , \quad G_{\eta' NN} = 5.60 \quad \text{for} \ \theta = -30.5^0 \quad (24)$$
In order to get an estimate of the error and to check the stability of the calculation a cubic fit to \( \frac{1}{t} \) can be used, i.e. take for kernel in the dispersion integral \( (1/t - c_0 - c_1 t - c_2 t^2 - c_3 t^3) \) with

\[
\begin{align*}
  c_0 &= 1.85 \text{GeV}^{-2} , \\
  c_1 &= -1.263 \text{GeV}^{-4} , \\
  c_2 &= 0.375 \text{GeV}^{-6} , \\
  c_3 &= -0.041 \text{GeV}^{-8}
\end{align*}
\]  

resulting in

\[
.548 \, G_{\eta NN} \cos \theta + .103 \, G_{\eta' NN} \sin \theta = \sqrt{2} m_N G^{(8)} / f_8
\]  

For the values of the coupling constants \( G_{\eta NN} \) and \( G_{\eta' NN} \) we have obtained and for the range of variation of \( \theta \) considered, the discrepancy between the two sides of the equation above amounts to \( 2\% - 8\% \). This shows that the calculation is stable and gives an idea of the magnitude of the errors involved.

In the two angle description \([10]\) the mixing angles \( \theta_8 \) and \( \theta_0 \) accompany the octet and singlet components. Because only the octet component has been used in this calculation the corresponding results hold with \( \theta \) replaced by \( \theta_8 \). The values \( \theta_8 = -21.2^0 \), \( f_8 = 1.26 f_\pi \) resulting from the analysis of ref. \([8]\) which correspond to \( \theta = -21.2^0 \), \( f_8 = 1.31 f_\pi \) in my case yield practically identical results for the coupling constants.

It is seen that the values obtained for the coupling constants are close to those appearing in eq. \([4]\) which are obtained in potential models.

Chiral symmetry breaking thus bridges the gap between values of the coupling constants obtained from the Goldberger-Treiman relation and those obtained from potential models.

It is finally to be noted that while \( G_{\eta NN} \), as well as all physical quantities studied in \([15]\) are quite insensitive to the exact value of the mixing angle, this is not the case for \( G_{\eta' NN} \).
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