Homogeneous and Inhomogeneous Integral Formulations of Nonrelativistic Potential Scattering

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Abstract

Advantage is taken of the arbitrariness in energy reference to consider anew integral transcriptions of Schrödinger’s equation in the presence of potentials which at infinity acquire constant, non-vanishing values. It is found possible to present for the probability amplitude $\psi$ a linear integral equation which is entirely devoid of explicit reference to the wave function incident from infinity, and thus differs markedly from the prevailing inhomogeneous formulation. Identity of the homogeneous equation with an inhomogeneous statement which is at the same time available is affirmed in general terms with the aid of the Fourier transformation, and is then still further reinforced by application of both formalisms to the particular example of a spherical potential barrier/well. Identical, closed-form outcomes are gotten in each case for wave function eigenmode expansion coefficients on both scatterer interior and exterior. Admittedly, the solution procedure is far simpler in the inhomogeneous setting, wherein it exhibits the aspect of a direct, leapfrog advance, unburdened by any implicit algebraic entanglement. By contrast, the homogeneous path, of considerably greater length, insists, at each mode index, upon an exterior/interior coefficient entanglement, an entanglement which, happily, is no more severe than that of a non-singular two-by-two linear system. Each such two-by-two linear system reproduces of course the output already gotten under the inhomogeneous route, and is indeed identical to the two-by-two system encountered during the routine procedure wherein continuity is demanded at the barrier/well interface of both $\psi$ and its radial derivative.

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1 Introduction

A transcription of the time-independent Schrödinger differential equation in the form of a linear inhomogeneous integral equation represents the cornerstone of most discussions of quantum mechanical potential scattering. In these discussions the reference of energy is implicitly understood to vanish at infinity, \( V_\infty = 0 \). The potentials \( V(r) \) are assumed to be of bounded support or else to approach their asymptotic null value with sufficient rapidity to assure convergence of all integrals involved. Suppose however that the reference of energies is displaced so that \( V_\infty \neq 0 \). We wish to indicate that under these circumstances one has available a choice of integral transcriptions, one homogeneous while the other inhomogeneous, which are entirely equivalent to each other. The inhomogeneous equation follows from a comparison of all energies with \( V_\infty \). It requires that the incident particle wave be explicitly displayed and permits an easy transition to the limit in which \( V_\infty \to 0 \). Construction of the homogeneous equation, on the other hand, relies critically upon the fact that \( V_\infty \neq 0 \), but otherwise demands no explicit provision for the incident field. Equivalence of these formulations is difficult to establish directly in configuration space, but it does yield to an easy demonstration when the integral equations are first subjected to a Fourier transformation. That equivalence is then further illustrated by an application of the integral equations to the simple concrete example involving scattering by a spherically symmetric potential barrier/well.

2 A homogeneous/inhomogeneous (h/i) integral equation duality

We represent the potential \( V(r) \) in the form

\[
V(r) = v(r) + V_\infty ,
\]

with \( V_\infty \neq 0 \) and \( v(r) \to 0 \) as \( r \) grows without bound in all directions. A plane wave energy-momentum eigenstate

\[
\psi_{\text{inc}}(r) = \exp(ik \cdot r),
\]

with

\[
k^2 = \frac{2m}{\hbar^2} \left( E - V_\infty \right) > 0,
\]

is incident from infinity upon the scattering potential \( v(r) \). Symbols \( m, E, \) and \( k \) have their usual meanings as the particle mass, total energy, and vector momentum, the latter being measured in units of Planck’s constant \( \hbar \).

In order to place the ensuing homogeneous formulation in a proper perspective, we recapitulate briefly a statement of the inhomogeneous integral equation. This latter depends upon explicit decomposition of the wave function \( \psi \) into incident and scattered contributions

\[
\psi = \psi_{\text{inc}} + \psi_{\text{scatt}}.
\]

One obtains for \( \psi_{\text{scatt}} \) the equation
\[
\left( \nabla^2 + k^2 \right) \psi_{\text{scatt}} = \frac{2m}{\hbar^2} v \psi, \tag{5}
\]

with the transcription
\[
\psi_{\text{scatt}}(r) = \frac{2m}{\hbar^2} \int G(|r - r'|) v(r) \psi(r') \, dr', \tag{6}
\]

wherein
\[
G(r) = -\frac{e^{i kr}}{4\pi r} \tag{7}
\]
is the Green’s function for the Helmholtz equation which involves expanding waves alone. The composite wave function \( \psi \) from (4) therefore satisfies
\[
\psi(r) = \psi_{\text{inc}}(r) + \frac{2m}{\hbar^2} \int G(|r - r'|) v(r') \psi(r') \, dr', \tag{8}
\]

the integrals in (8) and elsewhere being taken over all of space and differential \( dr \) being a standard shorthand for the product \( dxdydz \). To this point the sole modification of the usual integral formulation reposes in the representation (3) which follows from the displacement \( V_\infty \) in energy reference. Transition to the limit \( V_\infty \to 0 \) in (8) is automatic, provided of course that energy eigenvalue \( E \) shifts accordingly so as to keep the difference on the right in (3) fixed, an invariance which respects the fact that the operational datum here consists entirely of the incoming kinetic energy and direction of particle flight.

Erection of a homogeneous integral equation for \( \psi \) becomes possible if at the outset one forgoes the decomposition (4). The Schrödinger equation in the form
\[
\left( \nabla^2 + k_0^2 \right) \psi = \frac{2m}{\hbar^2} V \psi, \tag{9}
\]

with
\[
k_0^2 = \frac{2m}{\hbar^2} E, \tag{10}
\]
entails a source term which, unlike that in (5), persists to infinity. From (9) one therefore obtains
\[
\psi(r) = \frac{2m}{\hbar^2} \int G_0(|r - r'|) V(r') \psi(r') \, dr', \tag{11}
\]
as the homogeneous counterpart of (8). In (11) the suffix zero serves to indicate that \( G_0 \) is obtained from (7) by the replacement \( k \to k_0 \).

### 3 Demonstration of equivalence

All reasonable attempts to supply a direct, configuration space demonstration of identity for (8) and (11) encounter what seem to be insuperable difficulties. The required identity, however, emerges quite simply

\[\textit{We hasten to remark that, on physical grounds, a positivity requirement constrains only (3) on its right, and that for a sufficiently negative choice of } V_\infty \text{ there is nothing to prevent having total energy } E \text{ itself negative. Should that occur, we become required of course to take } k_0 \text{ as positive imaginary, viz., } k_0 = i \sqrt{2m|E|/\hbar^2}, \text{ so as to retain integral convergence at infinity.}\]
under Fourier transformation, which for definiteness is taken here in the asymmetric form

\[ F(\cdots) = \frac{1}{(2\pi)^3} \int \exp(-iK \cdot r) \cdots dr, \quad (12) \]

its outcome being designated by a tilde placed above and transform argument \( K \) replacing its heritage antecedent \( r \). The underlying unknown then becomes the scattered wave function transform \( \tilde{\psi}(K) \). Identity of the transformed integral equations for \( \tilde{\psi}(K) \) is understood to constitute a demonstration of equivalence for (8) and (11).

Fourier transformation is immediate if one exploits the transformation law for spatial convolution products and takes into account the subordinate results

\[ F\left((r \exp(-ikr))^{-1}\right) = \left\{2\pi^2(K^2 - k^2)\right\}^{-1}, \]

\[ F(\psi_{\text{inc}}(r)) = \delta(K - k) \quad (13) \]

with \( \delta(K - k) \) being the Dirac delta function in three dimensions. In the neighborhood of the shell \( K = k \) the first of the structures in (13) assumes the usual form

\[ \frac{1}{2\pi^2(K^2 - k^2)} = \frac{1}{2\pi^2} \left(\frac{P}{K^2 - k^2} + \frac{i\pi}{2k} \delta(K - k)\right) \quad (14) \]

wherein \( P \) represents Cauchy’s principal value whereas \( \delta(K - k) \) is the one-dimensional analogue of \( \delta(K - k) \). Abbreviate also by writing

\[ S(K) = F(v\psi_{\text{scatt}}) = \int \tilde{v}(K - K')\tilde{\psi}_{\text{scatt}}(K')dK'. \quad (15) \]

Then on the basis of (8) one obtains

\[ \tilde{\psi}_{\text{scatt}}(K) = -\frac{2m}{\hbar^2(K^2 - k^2)} \left(\tilde{v}(K - k) + S(K)\right), \quad (16) \]

an inhomogeneous integral equation in its own right, but now in reciprocal, \( K \) space.

Analogous steps undertaken in the context of (11) then yield

\[ \delta(K - k) + \tilde{\psi}_{\text{scatt}}(K) = -\frac{2m}{\hbar^2(K^2 - k_0^2)} \left(\tilde{v}(K - k) + S(K) + V_\infty \left\{\delta(K - k) + \tilde{\psi}_{\text{scatt}}(K)\right\}\right). \quad (17) \]

\[ ^2 \text{The plus sign prefacing Dirac’s delta is associated with our implicit understanding that wave number } k \text{ is to be regarded as the limit of } k + i\epsilon \text{ when } \epsilon \downarrow 0^+, \text{a standard, formal device contrived to assure integral convergence at infinity. One then finds that } (k + i\epsilon)^2 = k^2 - \epsilon^2 + 2i\epsilon k \text{ likewise falls above the real axis, circumstance which ultimately leads to a plus sign being attached to Dirac’s delta in (14).} \]

\[ ^3 \text{The divisions by } K^2 - k_0^2 \text{ appearing on the right in (17) and on the left in (18) are sufficiently tame to cause no real trouble: if } k_0^2 < 0, \text{there is clearly no problem, whereas if } k_0^2 > 0, \text{then a vanishing, positively signed imaginary accompaniment to } k_0 \text{ is understood as before with } k, \text{leading to an obvious analogue of (14).} \]
Here
\[
\frac{2mV_\infty}{\hbar^2(K^2 - k_0^2)} \delta(K - k) = \frac{2mV_\infty}{\hbar^2(k^2 - k_0^2)} \delta(K - k)
= -\delta(K - k) \tag{18}
\]
when note is taken of (3) and (10), so that the delta functions in (17) cancel identically. A simple transposition of terms suffices next to display (17) in the form of (16). This concludes the formal demonstration of equivalence.

One small trace of this equivalence is already afforded by noting that \(\psi_{inc}\) as given in (2) itself satisfies (11) when that latter is specialized by setting \(v = 0\). The first of the Fourier transforms (13) is an essential ingredient in this subsidiary aspect of equivalence.

4 The limit \(V_\infty \to 0, k_0 \to k\)

The equivalence of (8) and (11) just now established provides an \textit{a priori} assurance that the limit of (11) as \(V_\infty \to 0, k_0 \to k\) must coalesce with that of (8). Inasmuch as in this limit there necessarily occurs a transition from homogeneous to inhomogeneous integral equations, a more direct discussion is of some independent interest. It bears repeating perhaps that energy eigenvalue \(E \) slides up or down in unison with \(V_\infty\), subject only to the overriding demand that the right hand side of (3) remain invariant.

Consider (11) written out in full as
\[
\psi(r) = \frac{2mV_\infty}{\hbar^2} \int G_0(|r - r'|)\psi_{inc}(r')dr'
+ \frac{2mV_\infty}{\hbar^2} \int G_0(|r - r'|)\psi_{scatt}(r')dr'
+ \frac{2m}{\hbar^2} \int G_0(|r - r'|)v(r')\psi(r')dr'. \tag{19}
\]

From the concluding remark of the previous section the first term on the right is simply \(\psi_{inc}(r)\), independently of \(V_\infty\). All that remains to display identity of limits for (8) and (11) therefore is to show that the second line vanishes together with \(V_\infty\). This behavior in turn is guaranteed as soon as freedom from divergence is exhibited for the integral
\[
\int G_0(|r - r'|)\psi_{scatt}(r')dr'. \tag{20}
\]

Now the only possible source of divergence for (20) is integration in the asymptotic region \(r' \to \infty\). Over this domain one writes as always, with \(\hat{r}' = r'/r\),
\[
G_0(|r - r'|) \approx -\frac{e^{ik_0r'}}{4\pi r'} \exp\left(-i k_0 \hat{r}' \cdot r\right), \tag{21}
\]
while for $\psi_{\text{scatt}}$ the corresponding representation

$$
\psi_{\text{scatt}}(\mathbf{r}') \approx -\left( \frac{m}{2\pi \hbar^2} \right) \left( \frac{e^{ikr'}}{r'} \right) \int \exp \left( -ik\mathbf{r}' \cdot \mathbf{r}'' \right) v(\mathbf{r}'') \psi(\mathbf{r}'') d\mathbf{r}''
$$

provided by (8) is employed. In view of the established equivalence of (8) and (11) this appeal to (8) is entirely legitimate. The convergence of (20) is thus seen to depend upon that of

$$
\int_0^\infty e^{i(k_0+k)r'} dr' = \frac{i}{k_0+k}
$$

and hence is amply assured by virtue of all previous remarks concerning value ranges admissible to $k_0$ and $k$. Accordingly, (20) is indeed a convergent integral and the limits of (8) and (11) are indeed identical.

5 A simple example: spherical potential barrier/well

While the preëminent utility of integral scattering formulations rests upon the easy access which they provide to iterative approximation schemes, it is possible also to base upon them exact solutions of simple scattering problems. In such approaches one avoids all questions of continuity at geometric boundaries separating regions of differing potential, and requires merely that the integral equations be satisfied both interior and exterior to such boundaries. This requirement is completely adequate to determine all expansion coefficients when mode decompositions are employed.

Consider for example a spherical region $0 \leq r \leq a$ throughout which $V$ equals a constant $V_1 \neq V_\infty$. For $r > a$ the potential assumes the uniform value $V_\infty \neq 0$. Otherwise put, $v(\mathbf{r}) = V_1 - V_\infty$ whenever $0 \leq r \leq a$ and is zero otherwise. Positive $v$ betokens a repulsive barrier, negative $v$ an attractive well. Then for $r > a$ one writes in terms of Legendre polynomials $P_l$ and spherical Hankel functions $h_l^{(1)}$ in standard notation

$$
\psi_{\text{scatt}}(\mathbf{r}) = \sum_{l=0}^\infty A_l^> h_l^{(1)}(k r) P_l(\hat{k} \cdot \hat{r}) ,
$$

while for $0 \leq r \leq a$ there holds for the total wave field a similar decomposition

$$
\psi(\mathbf{r}) = \sum_{l=0}^\infty A_l^< j_l(k_1 r) P_l(\hat{k} \cdot \hat{r})
$$

in terms of spherical Bessel functions $j_l$ governed by an interior propagation constant

$$
k_1 = \sqrt{\frac{2m}{\hbar^2} (E - V_1)} .
$$

Classical inaccessibility of the region $0 \leq r \leq a$ corresponds as always to a purely imaginary value for $k_1$. Our task is to determine the interior/exterior expansion coefficients $\{A_l^>\}_{l=0}^\infty$ and $\{A_l^<\}_{l=0}^\infty$ on the basis of Eqs. (8) and (11).
One employs also the standard representations
\[ \psi_{\text{inc}}(r) = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\hat{k} \cdot \hat{r}) , \]  
(27)
\[ G(|r - r'|) = \frac{k}{4\pi i} \sum_{l=0}^{\infty} (2l+1) j_l(kr_<) h_l^{(1)}(kr_>) P_l(\hat{r} \cdot \hat{r}') , \]  
(28)
and similarly for \( G_0 \). We adhere to common practice by writing \( r_> \) and \( r_< \) respectively for the greater and lesser of \( r \) and \( r' \), which is to say \( r_> = \max(r, r') = (r + r' - |r - r'|)/2 \), \( r_< = \min(r, r') = (r + r' - |r - r'|)/2 \). In (27), as opposed to what is both physically plausible and mathematically desirable in (24) and (28), wave number \( k \) must be taken strictly real so as to assure an incoming wave field that remains bounded at infinity, both fore and aft.

5.1 Solution based on the inhomogeneous form (8)

We examine first an application of the inhomogeneous form (8) to a determination of the expansion coefficients \( \{ A_l^\geq \}_{l=0}^\infty \) and \( \{ A_l^\leq \}_{l=0}^\infty \). The requirement that the integral equation hold true for \( r > a \) yields
\[ \sum_{l=0}^{\infty} A_l^\geq h_l^{(1)}(kr) P_l(\hat{k} \cdot \hat{r}) = \frac{mk}{2\hbar^2 \pi i} (V_1 - V_\infty) \sum_{l,n=0}^{\infty} A_l^\leq (2n+1) h_n^{(1)}(kr) \int_{r' \leq a} P_l(\hat{k} \cdot \hat{r}') P_n(\hat{r} \cdot \hat{r}') j_l(k_1 r') j_n(kr') dr' . \]  
(29)
Integrations over the angular coordinates \( (\theta', \varphi') \) of \( r' \) can be effected by appealing to the addition theorem for Legendre polynomials. It is then easily found that the matrix of integrals in (29) is diagonal in its indices \( l \) and \( n \), with the element at spot \((l, l)\) proportional to \( P_l(\hat{k} \cdot \hat{r}) \). Both members of (29), left and right, assume the form of single series in the functions \( h_l^{(1)}(kr) P_l(\hat{k} \cdot \hat{r}) \) and so force equality of corresponding coefficients. Taking note also of a standard (Lommel) quadrature formula for Bessel function pairs having a common index, and of the relation
\[ V_1 - V_\infty = \frac{\hbar^2}{2m} \left( k^2 - k_1^2 \right) , \]  
(30)
\[ A_l^\geq = \frac{\pi a}{2i} \sqrt{\frac{k}{k_1}} \left( k_1 J_{l+\frac{1}{2}}(k_1 a) J_{l+\frac{1}{2}}(ka) - k J_{l+\frac{1}{2}}(k_1 a) J_{l+\frac{1}{2}}(ka) \right) A_l^\leq , \]  
(31)
with spherical Bessel functions restored to their half-index antecedents. The spherical weight factor \( r'^2 \) in the quadratures on the right in (29) has of course been taken into account.

To enforce the validity of (8) for \( 0 \leq r \leq a \) one writes
\[ \sum_{l=0}^{\infty} \left( A_l^\leq j_l(k_1 r) - i^l (2l+1) j_l(kr) \right) P_l(\hat{k} \cdot \hat{r}) = k \left( \frac{k^2 - k_1^2}{4\pi i} \right) \sum_{l,n=0}^{\infty} A_l^\leq (2n+1) \int_{r' \leq a} P_l(\hat{k} \cdot \hat{r}') P_n(\hat{r} \cdot \hat{r}') j_l(k_1 r') j_n(kr_<) h_n^{(1)}(kr_>) dr' . \]  
(32)
It is necessary here to readjust the radial integration at \( r' = r < a \) in conformity with the fact that the decomposition (28) for the Green's function is sensitive to the relative order among magnitudes \( r, r' \). After some rearrangement the right member of (32) becomes

\[
\frac{\pi}{2i} \sqrt{\frac{k}{k_1}} \sum_{l=0}^{\infty} A_l^\leq P_l(\hat{k} \cdot \hat{r}) \left[ r \sqrt{kk_1} \ j_l(k_1 r) \left\{ J_{l+\frac{1}{2}}(kr) H_{l+\frac{1}{2}}^{(1)'}(kr) - J_{l+\frac{1}{2}}'(kr) H_{l+\frac{1}{2}}^{(1)}(kr) \right\} 
+ a j_l(kr) \left\{ k_1 J_{l+\frac{1}{2}}(k_1 a) H_{l+\frac{1}{2}}^{(1)}(ka) - k J_{l+\frac{1}{2}}(k_1 a) H_{l+\frac{1}{2}}^{(1)'}(ka) \right\} \right].
\] (33)

The coefficients of the various functions \( j_l(k_1 r) \) in expression (33) are seen to contain a well known Wronskian with value \( 2i/\pi kr \), which is independent of index \( l \). This value assures exact balance between terms involving the functions \( j_l(k_1 r) \) in (32). A requirement that the remaining series in \( P_l(\hat{k} \cdot \hat{r}) j_l(kr) \) agree term-by-term supplies at length the fully explicit evaluations

\[
A_l^\leq = \frac{2l+1}{\pi a} \sqrt{\frac{k_1}{k}} (2l+1) \left( k_1 J_{l+\frac{1}{2}}'(k_1 a) H_{l+\frac{1}{2}}^{(1)}(ka) - k J_{l+\frac{1}{2}}'(k_1 a) H_{l+\frac{1}{2}}^{(1)'}(ka) \right)^{-1}.
\] (34)

It is then easily verified that (31) and (34) together point toward results in complete accord with those which follow from the usual demand that both \( \psi \) and its radial derivative be continuous at the potential barrier/well boundary.

We encounter here a recurring pattern of great power, a decisive leitmotif in the inner/outer solution to inhomogeneous equations such as (8). As already embodied in expression (33) and Eq. (34), this pattern asserts itself within the scattering interior, wherein it provides an exact cancellation, left and right, of the total field (radial behavior guided by functions \( j_l(k_1 r) \)). Interior expansion coefficients \( A_l^\leq \) are gotten then explicitly and without further ado from a balance along exterior eigenfunctions (radial behavior guided by the \( j_l(kr) \)). The exterior coefficients \( A_l^\geq \) follow then directly from the \( A_l^\leq \) as a sort of afterthought, as in (31), the link having been forged beforehand by insisting that the integral equation be satisfied on the scattering exterior.

This pattern of interior field cancellation recurs over and over again in applications considerably more robust than the present quantum mechanical, strictly scalar setting. Such applications are listed in a concluding section. We pause here only to stress the analytic economy of this two-tier, completely explicit solution process.

### 5.2 Solution based on the homogeneous form (11)

Scattering solution via the homogeneous form (11) follows similar lines, although the computations are considerably more laborious since now the necessity to segment radial integrations into distinct patterns above and below observation radius \( r \) intrudes not only for \( r < a \) but also on scatterer exterior \( r > a \). Isolation of coefficients \( A_l^\geq \) and \( A_l^\leq \) proceeds as before by way of a term-by-term series identification, as follows.
One begins by setting down as master relations corresponding respectively to (29) for \( r > a \)
\[
\sum_{l=0}^{\infty} \left( A^>_l h_l^{(1)}(kr) + i^l(2l + 1)j_l(kr) \right) P_l(\hat{k} \cdot \hat{r}) =
\]
\[
\frac{mk_0 V_1}{2\hbar^2 \pi i} \sum_{l,n=0}^{\infty} A^<_l (2n + 1) h_n^{(1)}(k_0 r) \int_{r' \leq a} P_l(\hat{k} \cdot \hat{r'} ) P_n(\hat{r} \cdot \hat{r'}) j_l(k_1 r') j_n(k_0 r') \, dr' +
\]
\[
\frac{mk_0 V_{\infty}}{2\hbar^2 \pi i} \sum_{l,n=0}^{\infty} (2n + 1) \int_{r' > a} P_l(\hat{k} \cdot \hat{r'} ) P_n(\hat{r} \cdot \hat{r'}) \times
\]
\[
\times \left( A^>_l h_l^{(1)}(kr') + i^l(2l + 1)j_l(kr') \right) j_n(k_0 \rho_<) h_n^{(1)}(k_0 \rho_>) \, dr',
\]
and to (32) for \( 0 \leq r \leq a \)
\[
\sum_{l=0}^{\infty} A^<_l j_l(k_1 r) P_l(\hat{k} \cdot \hat{r}) =
\]
\[
\frac{mk_0 V_1}{2\hbar^2 \pi i} \sum_{l,n=0}^{\infty} A^<_l (2n + 1) \int_{r' \leq a} P_l(\hat{k} \cdot \hat{r'} ) P_n(\hat{r} \cdot \hat{r'}) j_l(k_1 r') j_n(k_0 \rho_-) h_n^{(1)}(k_0 \rho_>) \, dr' +
\]
\[
\frac{mk_0 V_{\infty}}{2\hbar^2 \pi i} \sum_{l,n=0}^{\infty} (2n + 1) j_n(k_0 \rho_-) \int_{r' > a} P_l(\hat{k} \cdot \hat{r'} ) P_n(\hat{r} \cdot \hat{r'}) \times
\]
\[
\times \left( A^>_l h_l^{(1)}(kr') + i^l(2l + 1)j_l(kr') \right) h_n^{(1)}(k_0 \rho_-) \, dr'.
\]

Here we utilize of course the Green’s function analog of (28) gotten under the replacement of wave number \( k \) by \( k_0 \). Since Eqs. (35)-(36) share a comparable level of complexity, it suffices perhaps to follow the reduction of just one of them, say for definiteness (35).

5.2.1 Analytic reductions

And so as before in connection with (29) an integration over the solid angle of \( r' \) in (35) diagonalizes indices \( l \) and \( n \) with
\[
\sum_{l=0}^{\infty} \left( A^>_l h_l^{(1)}(kr) + i^l(2l + 1)j_l(kr) \right) P_l(\hat{k} \cdot \hat{r}) =
\]
\[
\frac{2mk_0 V_1}{\hbar^2 i} \sum_{l=0}^{\infty} A^<_l P_l(\hat{k} \cdot \hat{r}) h_l^{(1)}(k_0 r) \int_{r' \leq a} j_l(k_1 r') j_l(k_0 r') r'^2 \, dr' +
\]
\[
\frac{2mk_0 V_{\infty}}{\hbar^2 i} \sum_{l=0}^{\infty} P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} \left( A^>_l h_l^{(1)}(kr') + i^l(2l + 1)j_l(kr') \right) j_l(k_0 \rho_-) h_l^{(1)}(k_0 \rho_>) r'^2 \, dr'
\]
as an intermediate result. Here the self-field is conveyed by coefficients \( A^>_l \), and one can initiate its ultimate left/right cancellation, akin to that revealed in (33), by writing, for the relevant portion of the right-hand side,
\[
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} h_l^{(1)}(kr') j_l(k_0 r_<) h_l^{(1)}(k_0 r_>) r'^2 dr' = \\
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) h_l^{(1)}(k_0 r) \int_{a}^{r'} h_l^{(1)}(kr') j_l(k_0 r') r'^2 dr' + \\
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) j_l(k_0 r) \int_{r}^{\infty} h_l^{(1)}(kr) h_l^{(1)}(k_0 r') r'^2 dr'.
\]

Application of the Bessel/Lommel quadrature, previously cited, converts this into

\[
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} h_l^{(1)}(kr') j_l(k_0 r_<) h_l^{(1)}(k_0 r_>) r'^2 dr' = \\
\frac{\pi mk_0 V_\infty}{ih^2 (k_0^2 - k^2) \sqrt{k k_0}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) h_l^{(1)}(k_0 r) \left[ r' \left( k J_{l+\frac{1}{2}}(k_0 r') H_{l+\frac{1}{2}}^{(1)'}(kr') - k_0 J'_{l+\frac{1}{2}}(k_0 r') H_{l+\frac{1}{2}}^{(1)}(kr') \right) \right]_{r' = a}^{r' = \infty} + \\
\frac{\pi mk_0 V_\infty}{ih^2 (k_0^2 - k^2) \sqrt{k k_0}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) j_l(k_0 r) \left[ r' \left( k H_{l+\frac{1}{2}}^{(1)'}(k_0 r) H_{l+\frac{1}{2}}^{(1)'}(kr') - k_0 H_{l+\frac{1}{2}}^{(1)'}(k_0 r') H_{l+\frac{1}{2}}^{(1)'}(kr') \right) \right]_{r' = a}^{r' = \infty}.
\]

On taking note from (3) and (10) of the fact, akin to (30), that \(k_0^2 - k^2 = 2mV_\infty / \hbar^2\) we arrive at the next intermediate stage

\[
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} h_l^{(1)}(kr') j_l(k_0 r_<) h_l^{(1)}(k_0 r_>) r'^2 dr' = \\
\frac{\pi^{3/2}}{2i \sqrt{2kr}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) H_{l+\frac{1}{2}}^{(1)'}(k_0 r) \left[ r' \left( k J_{l+\frac{1}{2}}(k_0 r') H_{l+\frac{1}{2}}^{(1)'}(kr') - k_0 J'_{l+\frac{1}{2}}(k_0 r') H_{l+\frac{1}{2}}^{(1)}(kr') \right) \right]_{r' = a}^{r' = \infty} + \\
\frac{\pi^{3/2}}{2i \sqrt{2kr}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) J_{l+\frac{1}{2}}(k_0 r) \left[ r' \left( k H_{l+\frac{1}{2}}^{(1)'}(k_0 r) H_{l+\frac{1}{2}}^{(1)'}(kr') - k_0 H_{l+\frac{1}{2}}^{(1)'}(k_0 r') H_{l+\frac{1}{2}}^{(1)'}(kr') \right) \right]_{r' = a}^{r' = \infty},
\]

a stage which admits a more perspicuous rearrangement of terms so as to bring into focus once again a recurring appearance by our most useful Bessel/Hankel Wronskian. Thus

\[
\frac{2mk_0 V_\infty}{\hbar^2} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} h_l^{(1)}(kr') j_l(k_0 r_<) h_l^{(1)}(k_0 r_>) r'^2 dr' = \\
\frac{\pi^{3/2}}{2i \sqrt{2kr}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) H_{l+\frac{1}{2}}^{(1)}(kr) k_0 r \left( J_{l+\frac{1}{2}}(k_0 r) H_{l+\frac{1}{2}}^{(1)'}(k_0 r) - J'_{l+\frac{1}{2}}(k_0 r) H_{l+\frac{1}{2}}^{(1)}(k_0 r) \right) - \\
\frac{\pi^{3/2}}{2i \sqrt{2kr}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) H_{l+\frac{1}{2}}^{(1)'}(k_0 r') a \left( k J_{l+\frac{1}{2}}(k_0 a) H_{l+\frac{1}{2}}^{(1)'}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0 a) H_{l+\frac{1}{2}}^{(1)}(ka) \right) - \\
\frac{i \pi a}{2} \sqrt{\frac{k_0}{k}} \sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) h_l^{(1)}(k_0 r) \left( k J_{l+\frac{1}{2}}(k_0 a) H_{l+\frac{1}{2}}^{(1)'}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0 a) H_{l+\frac{1}{2}}^{(1)}(ka) \right) + \\
\sum_{l=0}^{\infty} A_l^r P_l(\hat{k} \cdot \hat{r}) h_l^{(1)}(kr)
\]
which leads of course to an exact cancellation, \textit{vis-à-vis} Eq. (37), of the self-field $\sum_{l=0}^{\infty} A_l^\infty P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr)$.

The remaining ingredients of Eq. (37) are processed in similar fashion. The first term on the right is the easiest. It gives

\[
\begin{align*}
2mk_0 V_1 \sum_{l=0}^{\infty} A_l^\infty P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr) & \int_{r' \geq a} j_l(kr') j_l(k0r') r'^2 dr' = \\
\frac{\pi ma V_1}{\hbar^2 i(k_0^2 - k_f^2)^{1/2}} & \sum_{l=0}^{\infty} A_l^\infty P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr) \left\{ k_1 J_{l+\frac{1}{2}}(k0a)(l+\frac{1}{2})(k0a) - k_0 J_{l+\frac{1}{2}}(k0a)(l+\frac{1}{2})(k1a) \right\} \\
& = -\frac{i\pi a}{2} \sum_{l=0}^{\infty} A_l^\infty P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr) \left\{ k_1 J_{l+\frac{1}{2}}(k0a)(l+\frac{1}{2})(k1a) - k_0 J_{l+\frac{1}{2}}(k0a)(l+\frac{1}{2})(k1a) \right\}, \tag{42}
\end{align*}
\]

wherein $V_1$ as an overt factor vanishes by virtue of Eqs. (10) and (26).

For the final term on the right in (37), involving the incoming excitation, we are forced to emulate steps (38)-(41). Thus

\[
\begin{align*}
2mk_0 V_\infty \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) & \int_{r' \geq a} j_l(kr') j_l(k0r') h_l^{(1)}(k0r') r'^2 dr' = \\
2mk_0 V_\infty & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr) \int_a^{r'} j_l(kr') j_l(k0r') r'^2 dr' + \\
2mk_0 V_\infty & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) j_l(k0r) \int_r^{\infty} j_l(kr') h_l^{(1)}(k0r') r'^2 dr'. \tag{43}
\end{align*}
\]

Without further ado there follows next a cascade of three steps. First

\[
\begin{align*}
2mk_0 V_\infty & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) \int_{r' \geq a} j_l(kr') j_l(k0r') h_l^{(1)}(k0r') r'^2 dr' = \\
\frac{\pi mk_0 V_\infty}{\hbar^2 (k_0^2 - k^2)^{1/2}} & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) h_l^{(1)}(kr) \left[ \frac{r'}{k J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k0r') - k_0 J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r')} \right]_{r'=a}^{r'=r} + \\
\frac{\pi mk_0 V_\infty}{\hbar^2 (k_0^2 - k^2)^{1/2}} & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) j_l(k0r) \left[ \frac{r'}{k J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r') - k_0 J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r')} \right]_{r'=a}^{r'=\infty},
\end{align*}
\]

then

\[
\begin{align*}
2mk_0 V_\infty & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) \int_{r' \geq a} j_l(kr') j_l(k0r') h_l^{(1)}(k0r') r'^2 dr' = \\
\frac{\pi^{3/2}}{2i/k r} & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) J_{l+\frac{1}{2}}^{(1)}(k0r) \left[ \frac{r'}{k J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r') - k_0 J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r')} \right]_{r'=a}^{r'=r} + \\
\frac{\pi^{3/2}}{2i/k r} & \sum_{l=0}^{\infty} j_l^*(2l+1) P_l(\mathbf{k} \cdot \mathbf{r}) J_{l+\frac{1}{2}}^{(1)}(k0r) \left[ \frac{r'}{k J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r') - k_0 J_{l+\frac{1}{2}}(k0r')(l+\frac{1}{2})(k1r')} \right]_{r'=a}^{r'=\infty}, \tag{45}
\end{align*}
\]
and finally

\[
\frac{2mk_0V_{\infty}}{\hbar^2 i} \sum_{l=0}^{\infty} l^2 (2l+1) P_l(\hat{k} \cdot \hat{r}) \int_{r' > a} j_l(kr') j_l(k_0r) h^{(1)}_l(k_0r) r' dr' =
\]

\[
\frac{\pi^{3/2}}{2i\sqrt{2kr}} \sum_{l=0}^{\infty} l^2 (2l+1) P_l(\hat{k} \cdot \hat{r}) J_{l+\frac{1}{2}}(kr) k_0r \left\{ J_{l+\frac{1}{2}}(k_0r) H^{(1)'}_{l+\frac{1}{2}}(k_0r) - J'_{l+\frac{1}{2}}(k_0r) H^{(1)}_{l+\frac{1}{2}}(k_0r) \right\}
\]

\[
- \frac{\pi^{3/2}}{2i\sqrt{2kr}} \sum_{l=0}^{\infty} l^2 (2l+1) P_l(\hat{k} \cdot \hat{r}) H^{(1)}_{l+\frac{1}{2}}(k_0r) a \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) J'_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) J_{l+\frac{1}{2}}(ka) \right\}
\]

(46)

\[
= \frac{i\pi a}{2} \sqrt{\frac{k_0}{k}} \sum_{l=0}^{\infty} l^2 (2l+1) P_l(\hat{k} \cdot \hat{r}) j_l(kr),
\]

which assures yet another self-cancellation against the left side in (37), but accompanied now by a drive contribution itself proportional to \( h^{(1)}_l(k_0r) \), just as in (41) preceding.\[4\]

Lastly, on putting together the remaining pieces from (41), (42), and (46) we arrive at

\[
\sum_{l=0}^{\infty} P_l(\hat{k} \cdot \hat{r}) h^{(1)}_l(k_0r) \left[ A_l^> \sqrt{\frac{\pi}{k}} \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) H^{(1)'}_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) H^{(1)}_{l+\frac{1}{2}}(ka) \right\} -
\right.

\left. A_l^< \sqrt{\frac{1}{k_1}} \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) J'_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) J_{l+\frac{1}{2}}(ka) \right\} \right] = 0.
\]

a global statement which splinters of course into an infinity of subsidiary demands

\[
A_l^> \sqrt{\frac{\pi}{k}} \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) H^{(1)'}_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) H^{(1)}_{l+\frac{1}{2}}(ka) \right\} -
\]

\[
A_l^< \sqrt{\frac{1}{k_1}} \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) J'_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) J_{l+\frac{1}{2}}(ka) \right\} +
\]

(47)

\[
i^l(2l+1) \sqrt{\frac{\pi}{k}} \left\{ k_1 J_{l+\frac{1}{2}}(k_0a) J'_{l+\frac{1}{2}}(ka) - k_0 J'_{l+\frac{1}{2}}(k_0a) J_{l+\frac{1}{2}}(ka) \right\} = 0.
\]

holding good \( \forall \ l \geq 0 \).

\[4\]In arriving at (46) we have glibly discarded the Lommel quadrature contribution at \( r' = \infty \) in (45). Such omission is predicated on having wave number \( k_0 \) in Green’s function (28) endowed with a small positive imaginary component, already announced beneath Footnote 3, admittedly a \textit{deus ex machina} artifice which, in physical terms, implies some sort of particle disappearance, evaporation, if you would, on its outward flight to infinity, a concept altogether foreign to Schrödinger’s equation as normally received. In the absence of such an imaginary component the Lommel contribution at infinity is “\textit{infinite oscillatory},” and thus mathematically indigestible. It is perhaps better to simply accept, and learn to live with the outbound “\textit{particle probability dissipation}.” Of course, in view of the still earlier Footnote 1, there does hover the redemptive possibility of \( k_0 \) actually being, in its own right, positive imaginary at any magnitude. Similar anxieties regarding an undeclared quadrature omission at infinity when passing from (40) to (41) are softened even more decisively by the fact that one confronts there a pair of outgoing Hankel functions, each one of which is protected by a dissipative mechanism.
A reduction of comparable complexity is likewise available for Eq. (36), its quadrature disjunction at \( r' = r < a \) affecting only terms from the first summation on its right. It permits us to adjoin to (48) the independent statements

\[
A_l^> \sqrt{\frac{k}{k_1}} \left\{ k H_{l+\frac{1}{2}}^{(1)}(k_0a) H_{l+\frac{1}{2}}^{(1)\prime}(ka) - k_0 H_{l+\frac{1}{2}}^{(1)}(k_0a) H_{l+\frac{1}{2}}^{(1)}(ka) \right\} - \\
A_l^< \sqrt{\frac{1}{k_1}} \left\{ k_0 H_{l+\frac{1}{2}}^{(1)}(k_0a) J_{l+\frac{1}{2}}^{(1)}(ka) - k_0 H_{l+\frac{1}{2}}^{(1)}(k_0a) J_{l+\frac{1}{2}}^{(1)}(ka) \right\} + \\
v_l^i (2l + 1) \sqrt{\frac{k}{k_1}} \left\{ k H_{l+\frac{1}{2}}^{(1)}(k_0a) J_{l+\frac{1}{2}}^{(1)}(ka) - k_0 H_{l+\frac{1}{2}}^{(1)}(k_0a) J_{l+\frac{1}{2}}^{(1)}(ka) \right\} = 0
\]

holding good as before \( \forall \ l \geq 0 \).

5.2.2 Final analytic synopsis

Equations (48)-(49) are more concisely rendered in a matricial format governing coefficient vectors \( A_l \) defined by

\[
A_l = \begin{bmatrix} A_l^> \\ A_l^< \end{bmatrix}.
\]

One then finds for all indices \( l \geq 0 \) that

\[
M_l \left( N_l A_l - B_l \right) = 0,
\]

with matrices

\[
M_l = \begin{bmatrix} J_{l+\frac{1}{2}}(k_0a) & -k_0 J_{l+\frac{1}{2}}^{(1)}(k_0a) \\ H_{l+\frac{1}{2}}^{(1)}(k_0a) & -k_0 H_{l+\frac{1}{2}}^{(1)\prime}(k_0a) \end{bmatrix},
\]

\[
N_l = \begin{bmatrix} -\sqrt{k} H_{l+\frac{1}{2}}^{(1)\prime}(ka) & \sqrt{k} J_{l+\frac{1}{2}}^{(1)}(k_1a) \\ -H_{l+\frac{1}{2}}^{(1)}(ka)/\sqrt{k} & J_{l+\frac{1}{2}}^{(1)}(k_1a)/\sqrt{k_1} \end{bmatrix},
\]

and

\[
B_l = \frac{v_l^i (2l + 1)}{\sqrt{k}} \begin{bmatrix} k J_{l+\frac{1}{2}}^{(1)}(ka) \\ J_{l+\frac{1}{2}}^{(1)}(ka) \end{bmatrix}.
\]

Our ubiquitous Wronskian provides the nonvanishing evaluation \( \det M_l = 2/\pi i a \neq 0 \), and hence assures that the matrix factor \( M_l \) in (51) is in fact irrelevant and may be freely detached. The ensuing requirement that the remaining structure \( N_l A_l - B_l \) on the left in (51) vanish then easily leads to results consonant in every detail with those contained within (31) and (34). The stated Wronskian participates yet again when agreement is first sought on behalf of \( A_l^< \).
Indeed, the two-by-two linear system $N_1 A_l - B_l = 0$ is found to be identical as the revealed harmony of $A_l$ outcomes now clearly requires it to be, with that which expresses continuity at barrier/well interface $r = a$ for both $\psi$ and its radial derivative $\partial \psi / \partial r$.

One draws much reassurance from this agreement, since it provides clear, specific evidence of an otherwise globally proclaimed duality. On the other hand, since so much turgid algebra is stirred up along this path, one would hesitate to advocate on behalf of the homogeneous version as an efficient computational tool.

6 An aperçu on h/i integral equations in scattering theory

Sporadic sightings of integral equations, both homogeneous and inhomogeneous, have been encountered in the scattering literature, both electromagnetic and quantum mechanical. We can leave to the side the entire subgenre of Born and allied, Neumann-like iterative approximation schemes which figure so prominently in quantum field theory and are abundantly documented.

An early example of a homogeneous integral equation in the service of electromagnetics can be found in [1], offered therein as a vehicle for field/eigenfrequency determination in a cavity which encloses a dielectric object. Such offer, alas, is bereft of any further examples as to its use.

On an entirely different tack, now in solid state physics, there appeared the exquisitely beautiful paper of Saxon and Hutner [2], which extended and superseded the foundational Kronig-Penney treatment of one-dimensional quantum mechanical electron movement in a periodic crystal lattice. The Schrödinger equation was recast there under a homogeneous integral equation form, the lattice obstacles being modeled via Dirac deltas and with the Bloch ansatz firmly in mind.

The Saxon-Hutner one-dimensional achievement was soon to be eclipsed by the full, three-dimensional method attributed to Kohn, Korringa, and Rostoker, the so-called KKR method [3], so named after its originators. Their primary contributions are found in [4-5].

The homogeneous/inhomogeneous duality now on view seems first to have surfaced quite some time ago in a short note [6] devoted to electromagnetic scattering by dielectric obstacles. It turned out then that the vector attribute of the electromagnetic field was an inessential complication and could easily be accommodated on the road to duality demonstration. On the other hand, the brevity of that note, and the paucity of detailed development at that time, precluded the setting out of any specific application, such as that of the present spherical scatterer.

Over the ensuing decades occasional use was made to great effect of the self-field cancellation phenomenon as a bridge to concrete solutions. A first indication thereof appeared in [7] and dealt with electromagnetic reflection from and transmission into a lossy dielectric half space.

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5Save for an inessential row interchange and an overall minus sign.
6It should perhaps be mentioned in passing that the self-field cancellation phenomenon is reminiscent of the so called Ewald-Oseen extinction viewpoint encountered in optics. We, however, refrain from belaboring its physically obvious association with the concept of an incoming versus an internal field exchange, and view it simply as a useful, systematic route to problem solution.
Self-field cancellation showed its great power even in a magnetized plasma environment wherein dielectric properties had risen to a tensorial level. A pattern of such applications, originating in work by the undersigned, can be found in [8-9].

Several decades later, aided now by Laplace transformation, a still broader arena of time-dependent scattering was found to fit easily beneath a self-field cancellation purview [10]. And lastly, more recent self-field cancellation efforts have yielded a fully vectorial solution for reflection from and penetration across a dielectric slab [11], and an inroad at least into the heretofore intractable problem of electromagnetic diffraction by a dielectric wedge [12].

7 References

[1] Julian Schwinger (1943). The Theory of Obstacles in Resonant Cavities and Waveguides, MIT Radiation Laboratory Report 43-34, p. 16, eqs. (58)-(59).

[2] D. S. Saxon, R. A. Hutner (1949). Some Electronic Properties of a One-dimensional Crystal Model, Philips Research Reports, No. 4, pp. 82-122.

[3] J. M. Ziman (1972). Principles of the Theory of Solids, Second Edition, Cambridge University Press, pp. 106-108.

[4] J. Korringa (1947). On the calculation of the energy of a Bloch wave in a metal, Physica, Vol. XIII, Nos. 67, pp. 392-400.

[5] W. Kohn, N. Rostoker (1954). Solution of the Schrödinger Equation in Periodic Lattices with an Application to Metallic Lithium, Phys. Rev., Vol. 94, No. 5, pp. 1111-1120.

[6] J. Grzesik (1966). Note on Homogeneous and Inhomogeneous Integral Equations in the Theory of Electromagnetic Scattering by Dielectric Obstacles, Proc. IEEE, Vol. 54, No. 12, pp. 2028-2029.

[7] J. Grzesik (1980). Field Matching through Volume Suppression, IEE Proc., Vol. 127, Pt. H, No. 1, pp. 20-26.

[8] Leif Ulstrup and Jan Grzesik (1984). Vector Green’s Function Coil Code: Numerical Results for a Half-Turn Loop in a Circular Waveguide, Twenty-sixth Annual Meeting, APS Division of Plasma Physics, Boston.

[9] Hiroshi Agravante (1987). Plasma Separation Process: Vector Green’s Function Coil Code User’s Manual, TRW Space and Technology Group, One Space Park, Redondo Beach, CA 90278, Report PSP-R1-1307.
[10] J. A. Grzesik (2007). EM Pulse Transit across a Uniform Dielectric Slab, 7th Workshop on Computational Electromagnetics in Time-Domain, Perugia, Italy.

[11] J. A. Grzesik (2018). Dielectric Slab Reflection/Transmission as a Self-Consistent Radiation Phenomenon, Progress in Electromagnetics Research B, Vol. 82, pp. 31-48.

[12] J. A. Grzesik (2019). Dielectric Wedge Scattering: An Analytic Inroad, Progress in Electromagnetics Research B, Vol. 84, pp. 43-60.