ON OPTIMAL DECAY ESTIMATES FOR ODES AND PDES WITH MODAL DECOMPOSITION

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Abstract. We consider the Goldstein-Taylor model, which is a 2-velocity BGK model, and construct the “optimal” Lyapunov functional to quantify the convergence to the unique normalized steady state. The Lyapunov functional is optimal in the sense that it yields decay estimates in $L^2$-norm with the sharp exponential decay rate and minimal multiplicative constant. The modal decomposition of the Goldstein-Taylor model leads to the study of a family of 2-dimensional ODE systems. Therefore we discuss the characterization of “optimal” Lyapunov functionals for linear ODE systems with positive stable diagonalizable matrices. We give a complete answer for optimal decay rates of 2-dimensional ODE systems, and a partial answer for higher dimensional ODE systems.

1. Introduction

This note is concerned with optimal decay estimates of hypocoercive evolution equations that allow for a modal decomposition. The notion hypocoercivity was introduced by Villani in [15] for equations of the form $\frac{d}{dt}f = -Lf$ on some Hilbert space $H$, where the generator $L$ is not coercive, but where solutions still exhibit exponential decay in time. More precisely, there should exist constants $\lambda > 0$ and $c \geq 1$, such that

\begin{equation}
\|e^{-Lt}f\|_{\tilde{H}} \leq c e^{-\lambda t}\|f\|_{\tilde{H}} \quad \forall f \in \tilde{H},
\end{equation}

where $\tilde{H}$ is a second Hilbert space, densely embedded in $(\ker L)^\perp \subset H$.

The large-time behavior of many hypocoercive equations have been studied in recent years, including Fokker-Planck equations [3,4,15], kinetic equations [11] and BGK equations [1,2]. Determining the sharp (i.e. maximal) exponential decay rate $\lambda$ was an issue in some of these works, in particular [1,24]. But finding at the same time the smallest multiplicative constant $c \geq 1$, is so far an open problem. And this is the topic of this note. For simple cases we shall describe a procedure to construct the “optimal” Lyapunov functional that will imply (1.1) with the sharp constants $\lambda$ and $c$.

For illustration purposes we shall focus here only on the following 2-velocity BGK-model (referring to the physicists Bhatnagar, Gross and Krook [7]) for the two functions $f_{\pm}(x,t) \geq 0$ on the one-dimensional torus $x \in \mathbb{T}$

\begin{equation}
$f_{\pm}(x,t) = \frac{1}{2}(f_1 + \alpha f_2) \pm \frac{1}{2}(f_1 - \alpha f_2) e^{-\lambda t}$
\end{equation}

\begin{equation}f_{\pm}(x,t) \geq 0 \quad \forall x \in \mathbb{T}, \forall t \geq 0
\end{equation}

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and for \( t \geq 0 \). It reads

\[
\begin{align*}
\partial_t f_+ &= -\partial_x f_+ + \frac{1}{2} (f_- - f_+) , \\
\partial_t f_- &= \partial_x f_- - \frac{1}{2} (f_+ - f_-) .
\end{align*}
\]

This system of two transport-reaction equations is also called Goldstein-Taylor model.

For initial conditions normalized as

\[
\int_0^{2\pi} \left[ f_I^+ + (x) + f_I^- (x) \right] \, dx = 2\pi ,
\]

the solution

\[
f(t) = (f_+ (t), f_- (t))^\top \]

converges to its unique (normalized) steady state with

\[
f^\infty_+ = f^\infty_- = \frac{1}{2} .
\]

The operator norm of the propagator for (1.2) can be computed explicitly from the Fourier modes, see [13]. By contrast, the goal of this paper and of [1,11] is to refrain from explicit computations of the solution and to use Lyapunov functionals instead. Following this strategy, an explicit exponential decay rate of this two velocity model was shown in [11, §1.4]. The sharp exponential decay estimate was found in [1, §4.1] via a refined functional, yielding the following result:

**Theorem 1.1** ([1, Th. 6]). Let \( f^I \in L^2(0, 2\pi; \mathbb{R}^2) \). Then the solution to (1.2) satisfies

\[
\| f(t) - f^\infty \|_{L^2(0, 2\pi; \mathbb{R}^2)} \leq c e^{-\lambda t} \| f^I - f^\infty \|_{L^2(0, 2\pi; \mathbb{R}^2)} , \quad t \geq 0 ,
\]

with the optimal constants \( \lambda = \frac{1}{2} \) and \( c = \sqrt{3} \).

**Remark 1.2.**

a) Actually, the optimal \( c \) was not specified in [1], but will be the result of Theorem 3.7 below.

b) As we shall illustrate in §5 it does not make sense to optimize these two constants at the same time. The optimality in Theorem 1.1 refers to first maximizing the exponential rate \( \lambda \), and then to minimize the multiplicative constant \( c \).

The proof of Theorem 1.1 is based on the spatial Fourier transform of (1.2), cf. [1,11]. We denote the Fourier modes in the discrete velocity basis \( \{ (1) , (1) \} \) by \( u_k (t) \in \mathbb{C}^2 , k \in \mathbb{Z} \). They evolve according to the ODE systems

\[
\frac{d}{dt} u_k = -C_k u_k , \quad C_k = \begin{pmatrix} 0 & ik \\ ik & 1 \end{pmatrix} , \quad k \in \mathbb{Z} ,
\]

and their (normalized) steady states are

\[
u_0^\infty = \frac{1}{2} ; \quad u_k^\infty = \frac{1}{2} , \quad k \neq 0 .
\]

In the main body of this note we shall construct appropriate Lyapunov functionals for such ODEs, in order to obtain sharp decay rates of the form (1.1). In the context of the BGK-model (1.2), combining such decay estimates for all modes \( u_k \) then yields Theorem 1.1 as they are uniform in \( k \). We remark that the construction of Lyapunov functionals to reveal optimal decay rates in ODEs was already included in the classical textbook [6, §22.4], but optimality of the multiplicative constant \( c \) was not an issue there.

In this article we shall first review, from [1,2], the construction of Lyapunov functionals for linear first order ODE systems that reveal the sharp decay rate. They are quadratic functionals represented by some Hermitian matrix \( P \). As these functionals are not uniquely determined, we shall
then discuss a strategy to find the “best Lyapunov” functional in §3—by minimizing the condition number $\kappa(P)$. The method of §3 always yields an upper bound for the minimal multiplicative constant $c$ and the sharp constant in certain subcases (see Theorem 3.7). The refined method of §4 covers another subclass (see Theorem 4.1). Overall we shall determine the optimal constant $c$ for 2-dimensional ODE systems, and give estimates for it in higher dimensions. In the final section §5 we shall illustrate how to obtain a whole family of decay estimates—with suboptimal decay rates, but improved constant $c$. For small time this improves the estimate obtained in §3.

2. Lyapunov Functionals for Hypocoercive ODEs

In this section we review decay estimates for linear ODEs with constant coefficients of the form

$$\frac{df}{dt} = -Cf, \quad t \geq 0,$$

$$f(0) = f^I \in \mathbb{C}^n,$$

for some (typically non-Hermitian) matrix $C \in \mathbb{C}^{n \times n}$. To ensure that the origin is the unique asymptotically stable steady state, we assume that the matrix $C$ is hypocoercive (i.e. positive stable, meaning that all eigenvalues have positive real part). Since we shall not require that $C$ is coercive (meaning that its Hermitian part would be positive definite), we cannot expect that all solutions to (2.1) satisfy for the Euclidean norm: $\|f(t)\|_2 \leq e^{-\tilde{\lambda}t} \|f^I\|_2$ for some $\tilde{\lambda} > 0$. However, such an exponential decay estimate does hold in an adapted norm that can be used as a Lyapunov functional.

The construction of this Lyapunov functional is based on the following lemma:

**Lemma 2.1** ( [1, Lemma 2], [4, Lemma 4.3]). For any fixed matrix $C \in \mathbb{C}^{n \times n}$, let $\mu := \min\{\Re(\lambda)\mid \lambda$ is an eigenvalue of $C\}$. Let $\{\lambda_j \mid 1 \leq j \leq j_0\}$ be all the eigenvalues of $C$ with $\Re(\lambda_j) = \mu$. If all $\lambda_j$ ($j = 1, \ldots, j_0$) are non-defective, then there exists a positive definite Hermitian matrix $P \in \mathbb{C}^{n \times n}$ with

$$C^*P + PC \geq 2\mu P,$$

but $P$ is not uniquely determined.

Moreover, if all eigenvalues of $C$ are non-defective, examples of such matrices $P$ satisfying (2.2) are given by

$$P := \sum_{j=1}^n b_j w_j^* \otimes w_j^T,$$

where $w_j \in \mathbb{C}^n$ ($j = 1, \ldots, n$) denote the (right) normalized eigenvectors of $C^*$ (i.e. $C^*w_j = \lambda_j w_j$), and $b_j \in \mathbb{R}^+$ ($j = 1, \ldots, n$) are arbitrary weights.

For $n = 2$ all positive definite Hermitian matrices $P$ satisfying (2.2) have the form (2.3), but for $n \geq 3$ this is not true (see Lemma 3.1 and Example 4.1 respectively).

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1 An eigenvalue is defective if its geometric multiplicity is strictly less than its algebraic multiplicity.
In this article, for simplicity, we shall only consider the case when all eigenvalues of $C$ are non-defective. For the extension of Lemma 2.1 and of the corresponding decay estimates to the defective case we refer to [3, Prop. 2.2] and [5].

Due to the positive stability of $C$, the origin is the unique and asymptotically stable steady state $f^\infty = 0$ of (2.1). Due to Lemma 2.1, there exists a positive definite Hermitian matrix $P \in \mathbb{C}^{n \times n}$ such that $C^*P + PC \geq 2\mu P$ where $\mu = \min \Re(\lambda_j) > 0$. Thus, the time derivative of the adapted norm $\|f\|_P := \langle f, Pf \rangle$ along solutions of (2.1) satisfies
\[
\frac{d}{dt}\|f(t)\|_P^2 \leq -2\mu\|f(t)\|_P^2.
\]
Hence the evolution becomes a contraction in the adapted norm:
\[
\|f(t)\|_P^2 \leq e^{-2\mu t}\|f(0)\|_P^2, \quad t \geq 0.
\]
Clearly, this procedure can yield the sharp decay rate $\mu$, only if $P$ satisfies (2.2).

Next we translate this decay in $P$-norm into a decay in the Euclidean norm:
\[
\|f(t)\|_E^2 = \left(\lambda_{\min}^P\right)^{-1}\|f(t)\|_P^2 \leq \left(\lambda_{\min}^P\right)^{-1}e^{-2\mu t}\|f(t)\|_P^2 \leq \kappa(P)e^{-2\mu t}\|f(t)\|_P^2, \quad t \geq 0,
\]
where $0 < \lambda_{\min}^P \leq \lambda_{\max}^P$ are, respectively, the smallest and largest eigenvalues of $P$, and $\kappa(P) = \lambda_{\max}^P/\lambda_{\min}^P$ is the (numerical) condition number of $P$ with respect to the Euclidean norm. While (2.4) is sharp, (2.5) is not necessarily sharp: Given the spectrum of $C$, the exponential decay rate in (2.5) is optimal, but the multiplicative constant not necessarily. For the optimality of the chain of inequalities in (2.5) we have to distinguish two scenarios: Does there exist an initial datum $f^I$ such that each inequality will be (simultaneously) an equality for some finite $t_0 \geq 0$? Or is this only possible asymptotically as $t \to \infty$? We shall start the discussion with the former case, which is simpler, and defer the latter case to §4. The first scenario allows to find the optimal multiplicative constant for $C \in \mathbb{R}^{2 \times 2}$, based on (2.5). But in other cases it may only yield an explicit upper bound for it, as we shall discuss in §4.

Concerning the first inequality of (2.5), a solution $f(t_0)$ will satisfy $\|f(t_0)\|_P^2 = (\lambda_{\min}^P)^{-1}\|f(t_0)\|_P^2$ for some $t_0 \geq 0$ only if $f(t_0)$ is in the eigenspace associated to the eigenvalue $\lambda_{\min}^P$ of $P$. Moreover, the initial datum $f^I$ satisfies $\|f^I\|_P^2 = \lambda_{\max}^P\|f^I\|_E^2$ if $f^I$ is in the eigenspace associated to the eigenvalue $\lambda_{\max}^P$ of $P$. Finally we consider the second inequality of (2.5): If the matrix $C$ satisfies, e.g., $\Re \lambda_j = \mu > 0; j = 1, ..., n$, with all eigenvalues non-defective, then we always have
\[
\|f(t)\|_P^2 = \lambda_{\min}^P\|f(t)\|_E^2 \quad \forall t \geq 0,
\]
since (2.2) is an equality then. This is the case for our main example (1.3) with $k \neq 0$.

Since the matrix $P$ is not unique, we shall now discuss the choice of $P$ as to minimize the multiplicative constant in (2.5). To this end we need to find the matrix $P$ with minimal condition number that satisfies (2.2). Clearly,
the answer can only be unique up to a positive multiplicative constant, since 
\( P := \tau P \) with \( \tau > 0 \) would reproduce the estimate (2.3).

As we shall prove in §3, the answer to this minimization problem is very easy in 2 dimensions: The best \( P \) corresponds to equal weights in (2.3), e.g.
choosing \( b_1 = b_2 = 1 \).

3. Optimal Constant via Minimization of the Condition Number

In this section, we describe a procedure towards constructing “optimal” Lyapunov functionals: For solutions \( f(t) \) of ODE (2.1) they will imply
\[
\| f(t) \|_2 \leq c e^{-\mu t} \| f \|_2
\]
with the sharp constant \( \mu \) and partly also the sharp constant \( c \).

We shall describe the procedure for ODEs (2.1) with positive stable matrices \( C \). For simplicity we confine ourselves to diagonalizable matrices \( C \) (i.e. all eigenvalues are non-defective). In this case, Lemma 2.1 states that there exist positive definite Hermitian matrices \( P \) satisfying the matrix inequality (2.2). Following (2.5), \( \sqrt{\kappa(P)} \) is always an upper bound for the constant \( c \) in (3.1). Our strategy is now to minimize \( \kappa(P) \) on the set of all admissible matrices \( P \). We shall prove that this actually yields the minimal constant \( c \) in certain cases (see Theorem 3.7). In 2 dimensions this minimization problem can be solved very easily thanks to Lemma 3.1 and Lemma 5.3:

**Lemma 3.1.** Let \( C \in \mathbb{C}^{2 \times 2} \) be a diagonalizable, positive stable matrix. Then all matrices \( P \) satisfying (2.2) are of the form (2.3).

**Proof.** We use again the matrix \( W \) whose columns are the normalized (right) eigenvectors of \( C^* \) such that
\[
C^* W = WD^*,
\]
with \( D = \text{diag}(\lambda_1 C, \lambda_2 C) \) where \( \lambda_j C \) \( (j \in \{1, 2\}) \) are the eigenvalues of \( C \). Since \( W \) is regular, \( P \) can be written as
\[
P = WBW^*,
\]
with some positive definite Hermitian matrix \( B \). Then the matrix inequality (2.2) can be written as
\[
2\mu WBW^* \leq C^* WBW^* + WBW^* C = W (D^* B + BD) W^*.
\]
This matrix inequality is equivalent to
\[
0 \leq (D^* - \mu I) B + B (D - \mu I).
\]
Next we order the eigenvalues \( \lambda_j C \) \( (j \in \{1, 2\}) \) of \( C \) increasingly with respect to their real parts, such that \( \Re(\lambda_1 C) = \mu \). Moreover, we consider
\[
B = \begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
\]
where \( b_1, b_2 > 0 \) and \( \beta \in \mathbb{C} \) with \( |\beta|^2 < b_1 b_2 \). Then the right hand side of (3.3) is
\[
(D^* - \mu I) B + B (D - \mu I) = \begin{pmatrix} 0 & (\lambda_2 C - \lambda_1 C) \beta \\ (\lambda_2 C - \lambda_1 C) \beta & 2 b_2 \Re(\lambda_2 C - \lambda_1 C) \beta \end{pmatrix}
\]
with
\[
\text{Tr}[(\mathbf{D}^* - \mu \mathbf{I})\mathbf{B} + \mathbf{B}(\mathbf{D} - \mu \mathbf{I})] = 2b_2 \Re(\lambda_2^C - \lambda_1^C)
\]
and
\[
\det[(\mathbf{D}^* - \mu \mathbf{I})\mathbf{B} + \mathbf{B}(\mathbf{D} - \mu \mathbf{I})] = -|\lambda_2^C - \lambda_1^C|^2 |\beta|^2.
\]
Condition (3.3) is satisfied if and only if
\[
\text{Tr}[(\mathbf{D}^* - \mu \mathbf{I})\mathbf{B} + \mathbf{B}(\mathbf{D} - \mu \mathbf{I})] \geq 0
\]
which holds due to our assumptions on \(\lambda_2^C\) and \(b_2\), and
\[
\det[(\mathbf{D}^* - \mu \mathbf{I})\mathbf{B} + \mathbf{B}(\mathbf{D} - \mu \mathbf{I})] \geq 0.
\]
The last condition holds if and only if
\[
\lambda_2^C = \lambda_1^C \quad \text{or} \quad \beta = 0.
\]
In the latter case \(\mathbf{B}\) is diagonal and hence \(\mathbf{P}\) is of the form (2.3). In the former case, (3.2) shows that \(\mathbf{C} = \lambda_1^C \mathbf{I}\), and the inequality (2.2) is trivial. Now any positive definite Hermitian matrix \(\mathbf{P}\) has a diagonalization \(\mathbf{P} = \mathbf{VEV}^*\), with a diagonal real matrix \(\mathbf{E}\) and an orthogonal matrix \(\mathbf{V}\), whose columns are—of course—eigenvectors of \(\mathbf{C}\). Thus, \(\mathbf{P}\) is again of the form (2.3).

Example 3.2. Consider the matrix \(\mathbf{C} = \text{diag}(1, 2, 3)\). Then, all matrices

\[
(3.5) \quad \mathbf{P}(b_1, b_2, b_3, \beta) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & \beta \\ 0 & \beta & b_3 \end{pmatrix}
\]

with positive \(b_j\) \((j \in \{1, 2, 3\})\) and \(\beta \in \mathbb{R}\) such that \(8b_2b_3 - 9\beta^2 \geq 0\), are positive definite Hermitian matrices and satisfy (2.2) for \(\mathbf{C} = \text{diag}(1, 2, 3)\) and \(\mu = 1\). But the eigenvectors of \(\mathbf{C}^*\) are the canonical unit vectors. Hence, matrices of form (2.3) would all be diagonal.

Restricting the minimization problem to admissible matrices \(\mathbf{P}\) of form (2.3) we find: Defining a matrix \(\mathbf{W} := (w_1 | \ldots | w_n)\) whose columns are the (right) normalized eigenvectors of \(\mathbf{C}^*\) allows to rewrite formula (2.3) as

\[
(3.6) \quad \mathbf{P} = \sum_{j=1}^{n} b_j w_j \otimes w_j^* = \mathbf{W} \text{diag}(b_1, b_2, \ldots, b_n) \mathbf{W}^*
\]

\[
= (\mathbf{W} \text{diag}(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n})) (\mathbf{W} \text{diag}(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}))^*
\]

with positive constants \(b_j\) \((j = 1, \ldots, n)\). The identity

\[
\mathbf{W} \text{diag}(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}) = (\sqrt{b_1} w_1 | \ldots | \sqrt{b_n} w_n)
\]

shows that the weights are just rescalings of the eigenvectors. Finally, the condition number of \(\mathbf{P}\) is the squared condition number of \((\mathbf{W} \text{diag}(\sqrt{b_1}, \sqrt{b_2}, \ldots, \sqrt{b_n}))\).

Hence, to find matrices \(\mathbf{P}\) of form (3.6) with minimal condition number, is equivalent to identifying (right) precondition matrices among the positive definite diagonal matrices which minimize the condition number of \(\mathbf{W}\). This minimization problem can be formulated as a convex optimization problem [9] based on the result [14]. Due to [10, Theorem 1], the minimum is attained (i.e., an optimal scaling matrix exists) since our matrix \(\mathbf{W}\) is nonsingular. (Note that its column vectors form a basis of \(\mathbb{C}^n\).) The convex optimization problem can be solved by standard software providing also the exact scaling matrix which minimizes the condition number of \(\mathbf{P}\), see the discussion and references in [9]. For more information on convex optimization and numerical solvers, see e.g., [8].
We return to the minimization of $\kappa(P)$ in 2 dimensions:

**Lemma 3.3.** Let $C \in \mathbb{C}^{2 \times 2}$ be a diagonalizable, positive stable matrix. Then the condition number of the associated matrix $P$ in (2.3) is minimal by choosing equal weights, e.g. $b_1 = b_2 = 1$.

**Proof.** A diagonalizable matrix $C$ has only non-defective eigenvalues. Up to a unitary transformation, we can assume w.l.o.g. that the eigenvectors of $C^*$ are

$$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} \alpha \\ \sqrt{1 - \alpha^2} \end{pmatrix}$$

for some $\alpha \in [0, 1)$.

This unitary transformation describes the change of the coordinate system. To construct the new basis, we choose one of the normalized eigenvectors $w_1$ as first basis vector, and recall that the second normalized eigenvector $w_2$ is only determined up to a scalar factor $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. The right choice for the scalar factor $\gamma$ allows to fulfill the above restriction on $\alpha$.

We use the representation of the positive definite matrix $P$ in (3.6):

$$P = W \text{diag}(b_1, b_2)W^*$$

with $W = \begin{pmatrix} \alpha \\ \sqrt{1 - \alpha^2} \end{pmatrix}$.

Since $P$ and $\tau P$ have the same condition number, we consider w.l.o.g. $b_1 = 1/b$ and $b_2 = b$. Thus, we have to determine the positive parameter $b > 0$ which minimizes the condition number of

$$P(b) = W \text{diag}(1/b, b)W^* = \begin{pmatrix} \frac{1}{b} + b\alpha^2 & b\alpha \sqrt{1 - \alpha^2} \\ b\alpha \sqrt{1 - \alpha^2} & b(1 - \alpha^2) \end{pmatrix}.$$

The condition number of matrix $P(b)$ is given by

$$\kappa(P(b)) = \lambda_+^P(b)/\lambda_-^P(b) \geq 1,$$

where

$$\lambda_{\pm}^P(b) = \frac{\text{Tr} P(b) \pm \sqrt{\left(\text{Tr} P(b)\right)^2 - 4 \text{det} P(b)}}{2}$$

are the (positive) eigenvalues of $P(b)$. We notice that $\text{Tr} P(b) = b + 1/b$ is independent of $\alpha$ and is a convex function of $b \in (0, \infty)$ which attains its minimum for $b = 1$. Moreover, $\text{det} P(b) = 1 - \alpha^2$ is independent of $b$. This implies that the condition number

$$\kappa(P(b)) = \frac{\lambda_+^P(b)}{\lambda_-^P(b)} = \frac{1 + \sqrt{1 - \frac{4 \text{det} P(b)}{(\text{Tr} P(b))^2}}}{1 - \sqrt{1 - \frac{4 \text{det} P(b)}{(\text{Tr} P(b))^2}}}$$

attains its unique minimum at $b = 1$, taking the value

$$\kappa_{\text{min}} = \frac{1 + \alpha}{1 - \alpha}.$$

This 2D-result does not generalize to higher dimensions. In dimensions $n \geq 3$ there exist diagonalizable positive stable matrices $C$, such that the matrix $P$ with equal weights $b_i$ does not yield the lowest condition number among all matrices of form (2.3). We give a counterexample in 3 dimensions:
Example 3.4. For some $C^*$, consider its eigenvector matrix

$$W := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{diag} \left( 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right),$$

which has normalized column vectors. We define the matrices $P(b_1, b_2, b_3) := W \text{diag}(b_1, b_2, b_3) W^*$ for positive parameters $b_1$, $b_2$ and $b_3$, which are of form $\text{(2.3)}$ and hence satisfy the inequality $\text{(2.2)}. In case of equal weights $b_1 = b_2 = b_3$ the condition number is $\kappa(P(b_1, b_1, b_1)) \approx 15.1285876$. But using $\text{[12, Theorem 3.3]}$, the minimal condition number $\min \kappa(P(b_1, b_2, b_3)) \approx 13.92820324$ is attained for the weights $b_1 = 2$, $b_2 = 4$ and $b_3 = 3$. □

Combining Lemma $\text{[3.1]}$ and Lemma $\text{[3.3]}$, we have

Corollary 3.5. Let $C \in \mathbb{C}^{2 \times 2}$ be a diagonalizable, positive stable matrix. Then the condition number is minimal among all matrices $P$ satisfying $\text{(2.2)}$, if $P$ is of form $\text{(2.3)}$ with equal weights, e.g. $b_1 = b_2 = 1$.

This 2D-result does not generalize to higher dimensions. Extending the conclusion of Example $\text{[3.4]}$, we shall now show that $P$ does not necessarily have to be of form $\text{(2.3)}$, if its condition number should be minimal:

Example 3.6. We consider a special case of Example $\text{[3.4]}$, with

$$\tilde{C} = (W^*)^{-1} \text{diag}(1, 2, 3) W^*$$

with $W$, the eigenvector matrix of $\tilde{C}^*$, given by $\text{(3.11)}$. Then the matrices $\tilde{C}$ and

$$\tilde{P}(b_1, b_2, b_3, \beta) := WP(b_1, b_2, b_3, \beta) W^*$$

with matrix $P(b_1, b_2, b_3, \beta)$ in $\text{(3.3)}$ satisfy the matrix inequality $\text{(2.2)}$ with $\mu = 1$. But $\tilde{P}$ is not of form $\text{(2.3)}$ if $\beta \neq 0$. Nevertheless, the condition number $\kappa(\tilde{P}(b_1, b_2, b_3, \beta)) \approx 5.82842780720132$ for the weights $b_1 = 2$, $b_2 = 4$, $b_3 = 3$, and $\beta = -2.45$, is much lower than with $\beta = 0$ (i.e. $\kappa(\tilde{P}(2, 4, 3, 0)) \approx 13.92820324$, cf. Example $\text{[3.4]}$). □

Lemma $\text{[3.3]}$ and inequality $\text{(2.5)}$ show that $\sqrt{\kappa_{\min}}$ from $\text{(3.10)}$ is an upper bound for the best constant in $\text{(3.1)}$ for the 2D case. For matrices with eigenvalues that have the same real part, it actually yields the minimal multiplicative constant $c$, as we shall show now. Other cases will be discussed in $\text{[4]}

For a diagonalizable matrix $C \in \mathbb{C}^{2 \times 2}$ with $\lambda_1^C = \lambda_2^C$, it holds that $\|f(t)\|_2 = e^{-\Re \lambda_2^C t} \|f(t)\|_2$. And for the general case we have:

Theorem 3.7. Let $C \in \mathbb{C}^{2 \times 2}$ be a diagonalizable, positive stable matrix with eigenvalues $\lambda_1^C \neq \lambda_2^C$, and associated eigenvectors $v_1$ and $v_2$, resp. If the eigenvalues have identical real parts, i.e. $\Re \lambda_1^C = \Re \lambda_2^C$, then the condition number of the associated matrix $P$ in $\text{(2.3)}$ with equal weights, e.g. $b_1 = b_2 = 1$, yields the minimal constant in the decay estimate $\text{(3.1)}$ for the ODE $\text{(2.1)}$:

$$c = \sqrt{\kappa(P)} = \sqrt{\frac{1 + \alpha}{1 - \alpha}} \text{ where } \alpha := \left\langle \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\rangle.$$
Proof. With the notation from the proof of Lemma 3.3 we have
\[
P(1) = \begin{pmatrix} 1 + \alpha^2 & \alpha \sqrt{1 - \alpha^2} \\ \alpha \sqrt{1 - \alpha^2} & 1 - \alpha^2 \end{pmatrix},
\]
with the eigenvectors \( y_+^P = (\sqrt{1 - \alpha^2}, 1 - \alpha)^\top, \) \( y_-^P = (\sqrt{1 - \alpha^2}, -1 - \alpha)^\top. \)

According to the discussion after (4.2) we choose the initial condition \( f^I = y_+^P. \) From the diagonalization (4.2) of \( C \) we get
\[
f(t) = (W^*)^{-1} e^{-D^t} W^* f^I.
\]
Using (3.8) and \( W^* y_\pm^P = \sqrt{1 - \alpha^2} (1 \pm i) \) we obtain directly that
\[
f(t_0) = e^{-\lambda^C t_0} y_\pm^P \quad \text{with} \quad t_0 = \frac{\pi}{|3(\lambda^C_2 - \lambda^C_1)|}.
\]
Hence, also the first inequality in (2.5) is sharp at \( t_0. \) Sharpness of the whole chain of inequalities then follows from (2.6), and this finishes the proof. \( \square \)

This theorem now allows us to identify the minimal constant \( c \) in Theorem 4.1 on the Goldstein-Taylor model: The eigenvalues of the matrices \( C_k, k \neq 0 \) from (1.13) are \( \lambda = \frac{1}{2} \pm i \sqrt{k^2 - \frac{1}{4}}. \) The corresponding transformation matrices \( P_k \) with \( b_1 = b_2 = 1 \) are given by \( P_0 = I \) and
\[
P_k = \begin{pmatrix} 1 & -\frac{\pi}{2k} \\ \frac{\pi}{2k} & 1 \end{pmatrix}, \quad \text{with} \quad \kappa(P_k) = \frac{|k| + 1}{|k| - 1}, \quad k \neq 0.
\]
Combining the decay estimates for all Fourier modes \( u_k(t) \) shows that the minimal multiplicative constant in Theorem 4.1 is given by \( c = \sqrt{\kappa(P_{\pm 1})} = \sqrt{3}. \) For a more detailed presentation how to recombine the modal estimates we refer to \( \S 4.1 \) in [1].

4. Optimal Constant for 2D Systems

The optimal constant \( c \) in (3.1) for \( C \in \mathbb{C}^{2 \times 2} \) with \( \Re \lambda^C_1 = \Re \lambda^C_2 \) was determined in Theorem 3.11. In this section we shall discuss the remaining 2D cases. We start to derive the minimal multiplicative constant \( c \) for matrices \( C \) with eigenvalues that have distinct real parts but identical imaginary parts.

**Theorem 4.1.** Let \( C \in \mathbb{C}^{2 \times 2} \) be a diagonalizable, positive stable matrix with eigenvalues \( \lambda^C_1 \) and \( \lambda^C_2, \) and associated eigenvectors \( v_1 \) and \( v_2, \) resp. If the eigenvalues have distinct real parts \( \Re \lambda^C_1 < \Re \lambda^C_2 \) and identical imaginary parts \( \Im \lambda^C_1 = \Im \lambda^C_2, \) then the minimal multiplicative constant \( c \) in (3.1) for the ODE (2.1) is given by
\[
c = \frac{1}{\sqrt{1 - \alpha^2}} \quad \text{where} \quad \alpha := \left| \left\langle \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \right\rangle \right|.
\]

**Proof.** We use again the unitary transformation as in the proof of Lemma 3.3 such that the eigenvectors \( w_1 \) and \( w_2 \) of \( C \) are given in (3.7). If \( f(t) \) is a solution of (2.1), then \( \tilde{f}(t) := e^{\Im \lambda^C t} f(t) \) satisfies
\[
\frac{d}{dt} \tilde{f}(t) = -\tilde{C} \tilde{f}(t), \quad \tilde{f}(0) = f^I,
\]
with
\[ \tilde{C} := (C - i\Re\lambda_1^C I) = (W^*)^{-1} \begin{pmatrix} \Re\lambda_1^C & 0 \\ 0 & \Re\lambda_2^C \end{pmatrix} W^*. \]

The multiplication with \( e^{i\Re\lambda_1^C t} \) is another unitary transformation and does not change the norm, i.e. \( \|f(t)\|_2 = \|\tilde{f}(t)\|_2 \). Therefore, we can assume w.l.o.g. that matrix \( C \) has real coefficients and distinct real eigenvalues. Then, the solution \( f(t) \) of the ODE (2.1) satisfies \( Re f(t) = \tilde{f}_{re}(t) \) and \( Im f(t) = \tilde{f}_{im}(t) \) where \( \tilde{f}_{re}(t) \) and \( \tilde{f}_{im}(t) \) are the solutions of the ODE (2.1) with initial data \( \Re f^I \) and \( \Im f^I \), resp. Altogether, we can assume w.l.o.g. that all quantities are real valued:

Considering a matrix \( C \in \mathbb{R}^{2\times2} \) with two distinct real eigenvalues \( \lambda_1 < \lambda_2 \) and real eigenvectors \( v_1 \) and \( v_2 \), then the associated eigenspaces span \( \{v_1\} \) and span \( \{v_2\} \) dissect the plane into four sectors

\[ S^{\pm} := \{ z_1 v_1 + z_2 v_2 \mid z_1 \in \mathbb{R}^\pm, \ z_2 \in \mathbb{R}^\mp \}, \]

see Fig.1. A solution \( f(t) \) of ODE (2.1) starting in an eigenspace will approach the origin in a straight line, such that

\[ \|f(t)\|^2 = e^{-2\Re\lambda_1^C t} \|f^I\|^2 \quad \forall t \geq 0. \]

If a solution starts instead in one of the four (open) sectors \( S^{\pm} \), it will remain in that sector while approaching the origin. In fact, since \( \lambda_1^C < \lambda_2^C \), if \( f^I = z_1(v_1 + \gamma v_2) \) for some \( z_1 \in \mathbb{R} \setminus \{0\} \) and \( \gamma \in \mathbb{R} \), then the solution

\[ f(t) = z_1(e^{-\lambda_1^C t} v_1 + \gamma e^{-\lambda_2^C t} v_2) = z_1 e^{-\lambda_1^C t} (v_1 + \gamma e^{-(\lambda_2^C - \lambda_1^C) t} v_2) \]

of the ODE (2.1) will remain in the sector

\[ S^{\pm} := \{ z_1(v_1 + \gamma v_2) \mid z_1 \in \mathbb{R}^\pm, \ \gamma \in [\min(0, \gamma), \max(0, \gamma)] \}, \]

see Fig.1. For a fixed \( f^I = z_1(v_1 + \gamma v_2) \), let \( S \) be the corresponding sector \( S^{\pm} \). Then estimate (2.3) can be improved as follows

\[ \|f(t)\|^2 \leq \frac{1}{\lambda_{\text{min},S}^P} \|f(t)\|^2 \leq e^{-2\mu t} \|f^I\|^2 \leq c_S(P) e^{-2\mu t} \|f^I\|^2, \quad t \geq 0, \]

where

\[ \lambda_{\text{min},S}^P := \inf_{x \in S} \langle x, P x \rangle, \quad \lambda_{\text{init},S}^P := \frac{\langle f^I, P f^I \rangle}{\langle f^I, f^I \rangle}, \quad c_S(P) := \frac{\lambda_{\text{init},S}^P}{\lambda_{\text{min},S}^P}. \]

Note that, in the definition of \( \lambda_{\text{init},S}^P \) the sector \( S \in \{ S^\pm \mid \gamma \in \mathbb{R} \} \) also determines corresponding initial conditions \( f^I \in \partial S \) via \( f^I = z_1(v_1 + \gamma v_2) \) (up to the constant \( z_1 \neq 0 \) which drops out in \( \lambda_{\text{init},S}^P \)).

For (4.6) to hold for all trajectories and one fixed constant on the right hand side, we have to take the supremum over all initial conditions or, equivalently, over all sectors \( S \in \{ S^\pm \mid \gamma \in \mathbb{R} \} \). Although \( f^I = z_2 v_2 \) is not included in any sector \( S^\pm \), its corresponding multiplicative constant 1 (see (4.1)) is still covered. Then, the minimal multiplicative constant in (3.1) using (4.3) is

\[ \tilde{c} = \sqrt{\inf_S \sup_P c_S(P)}, \]
Figure 1. The blue (black) lines are the eigenspaces span\{v_1\} and span\{v_2\} of matrix C. The red (grey) curve is a solution f(t) of the ODE (2.1) with initial datum f^I. The shaded regions are the sectors S^+_\gamma, S^-_\gamma with the choice \gamma = 1/2. Note: The curves are colored only in the electronic version of this article.

where P ranges over all matrices of the form (2.3).

Step 1 (computation of \lambda_{min, S^+_\gamma}^P for \gamma fixed): To find an explicit expression for this minimal constant c, we first determine c_S(P) for a given admissible matrix P. As an example of sectors, we consider only S^+_\gamma for fixed \gamma \leq 0 and compute

\[
\lambda_{min, S^+_\gamma}^P = \inf_{x \in S^+_\gamma} \frac{\langle x, Px \rangle}{\|x\|^2} = \inf_{z_1 \in \mathbb{R}^+, z_2 \in [\gamma, 0]} \frac{\langle z_1(v_1 + z_2v_2), P(z_1(v_1 + z_2v_2)) \rangle}{\|z_1(v_1 + z_2v_2)\|^2} = \inf_{z_2 \in [\gamma, 0]} \frac{\langle v_1 + z_2v_2, P(v_1 + z_2v_2) \rangle}{\|v_1 + z_2v_2\|^2}.
\]

This also shows that \lambda_{min, S^-_\gamma}^P = \lambda_{min, S^+_\gamma}^P for any fixed \gamma \in \mathbb{R}. Next, we use the result of Lemma 3.1 and (3.6), stating that the only admissible matrices are P = W \text{diag}(b_1, b_2) W^* for b_1, b_2 > 0. Since c_S(bP) = c_S(P) for all
b > 0, we consider w.l.o.g. $b_1 = 1/b$ and $b_2 = b$ for $b > 0$. Then, we deduce
\[
\lambda_{\text{min}, S^+_\gamma} = \inf_{z \in [\gamma, 0]} \frac{\langle v_1 + zv_2, P(v_1 + zv_2) \rangle}{\|v_1 + zv_2\|^2} = \inf_{z \in [\gamma, 0]} \frac{\langle W^*(v_1 + zv_2), \text{diag}(1/b, b)W^*(v_1 + zv_2) \rangle}{\|v_1 + zv_2\|^2}.
\]

In our case of a real matrix $C$ with distinct real eigenvalues, the left and right eigenvectors are related as follows: Up to a change of orientation, $\langle w_j, v_k \rangle = \delta_{jk}$ $(j, k \in \{1, 2\})$. Considering $\langle w_j, v_j \rangle = 1$ for $j = 1, 2$, implies that the vectors $w_j$ and $v_j$ can be normalized simultaneously only if matrix $C$ is symmetric. Therefore, using a coordinate system such that the normalized eigenvectors of $C^*$ are given as (3.7) and $V := (v_1|v_2) = (W^*)^{-1}$ yields
\[
v_1 = \frac{1}{\sqrt{1 - \alpha^2}} \begin{pmatrix} \sqrt{1 - \alpha^2} \\ -\alpha \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{1 - \alpha^2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\text{ for } \alpha \in (\mathbb{R}).
\]

Finally, we obtain
\[
\lambda_{\text{min}, S^+_\gamma} = \inf_{z \in [\gamma, 0]} \frac{\langle W^*(v_1 + zv_2), \text{diag}(1/b, b)W^*(v_1 + zv_2) \rangle}{\|v_1 + zv_2\|^2} = \inf_{z \in [\gamma, 0]} g(z)
\]
and $\lambda_{\text{init}, S^+_\gamma} = g(\gamma)$ with
\[
g(z) := \frac{(1 - \alpha^2)(\frac{1}{b} + bz^2)}{1 - 2\alpha z + z^2}.
\]

**Step 2 (extrema of the function $g$):** The function $g$ has local extrema at
\[
z_{\pm} = \frac{1}{2\alpha b} \left( b - \frac{1}{b} \pm \sqrt{(b - \frac{1}{b})^2 + 4\alpha^2} \right)
\]
which satisfy $z_- < 0 < z_+$. Writing $g'(z) = h_1(z)/h_2(z)$ with $h_1(z) := (-2\alpha bz^2 + 2(b - \frac{1}{b})z + \frac{1}{b}\alpha)$ and $h_2(z) := (1 - 2\alpha z + z^2)^2/(1 - \alpha^2) > 0$, we derive
\[
g''(z_{\pm}) = \frac{h_1'(z_{\pm})}{h_2(z_{\pm})} = \mp 2 - \frac{1}{h_2(z_{\pm})} \sqrt{(b - \frac{1}{b})^2 + 4\alpha^2}.
\]
In fact, the function $g$ attains its global minimum on $\mathbb{R}$ (and on $\mathbb{R}_-^+$) at $z_-$, and its global maximum on $\mathbb{R}$ at $z_+$. The global supremum of $g(z)$ on $\mathbb{R}^-$ exists and satisfies
\[
\sup_{z \in \mathbb{R}^-} g(z) = \begin{cases} 
  g(0) = (1 - \alpha^2)/b & \text{if } b \in (0, 1), \\
  g(0) = \lim_{z \to -\infty} g(z) = 1 - \alpha^2 & \text{if } b = 1, \\
  \lim_{z \to -\infty} g(z) = (1 - \alpha^2)b & \text{if } b \in (1, \infty).
\end{cases}
\]

**Step 3 (optimization of $c_{S^{z^\pm}}(P)$ w.r.t. $\gamma$):** We obtain
\[
c_{S^{\pm}}(P(b)) = \frac{g(\gamma)}{\lambda_{\text{min}, S^+_\gamma}^{P(b)}} = \begin{cases} 
  1 & \text{if } z_- \leq \gamma < 0, \\
  \frac{1}{g(\gamma)/g(z_-)} & \text{if } \gamma \leq z_+.
\end{cases}
\]
Finally, we derive
\[
\sup_{\gamma \in \mathbb{R}^-} c_{S^+_\gamma}(P(b)) = \lim_{\gamma \to -\infty} \frac{g(\gamma)}{g(z_-)} = \frac{(1 - \alpha^2)b}{g(z_-)}.
\]
and in a similar way,

\begin{equation}
\sup_{\gamma \in \mathbb{R}^+} c_{S_{\gamma}^+}(P(b)) = \frac{g(z_+)}{g(0)} = \frac{bg(z_+)}{1 - \alpha^2}.
\end{equation}

To finish this analysis we note that \( c_{S_\gamma^+}(P(b)) = 1 \), due to (4.11) and \( f^I = z_1v_1 \).

Step 4 (minimization of \( \sup_S c_S(P) \) w.r.t. \( P \)): We obtain

\[
\inf_P \sup_{S \in \mathbb{R}} c_S(P) = \inf_{b \in (0, \infty)} \sup_{\gamma \in \mathbb{R}} c_{S_{\gamma}^+}(P(b)) = \inf_{b \in (0, \infty)} \sup_{\gamma \in \mathbb{R}} \max \left\{ \frac{(1 - \alpha^2)b}{g(z_-)}, 1, \frac{bg(z_+)}{1 - \alpha^2} \right\}.
\]

Taking into account the \( b \)-dependence of \( z_\pm \), the functions \( \frac{(1 - \alpha^2)b}{g(z_-)} \) and \( \frac{bg(z_+)}{1 - \alpha^2} \) are monotone increasing in \( b \), since

\[
\frac{\partial}{\partial b} \left( 1 - \alpha^2 \right)b = 0, \quad \frac{\partial}{\partial b} \frac{bg(z_+)}{1 - \alpha^2} > 0.
\]

Therefore we have to study their limits as \( b \to 0 \): We derive

\begin{equation}
\lim_{b \to 0} \frac{(1 - \alpha^2)b}{g(z_-)} = 1 \quad \text{using} \quad \lim_{b \to 0} z_-(b) = -\infty, \quad \lim_{b \to 0} \frac{bg(z_+)}{1 - \alpha^2} = 1 \quad \text{using} \quad \lim_{b \to 0} z_+(b) = \alpha.
\end{equation}

Hence, \( \inf_{b \in (0, \infty)} \sup_{\gamma \in \mathbb{R}} c_{S_{\gamma}^+}(P(b)) \) is realized by the sector \( S_{\gamma}^+ \) with \( \gamma = z_+(b) > 0 \) and in the limit \( b \to 0 \). Altogether we obtain

\[
\tilde{c} = \sqrt{\inf_P \sup_S c_S(P)} = \frac{1}{\sqrt{1 - \alpha^2}},
\]

where the first equality holds since we discussed all solutions. This finishes the proof.

Step 5: Finally we have to verify that \( \tilde{c} \) is minimal in (3.11). We shall show that it is attained asymptotically (as \( t \to \infty \)) for a concrete trajectory: For fixed \( b \in (0, \infty) \), the minimal multiplicative constant in (4.11) is attained for the solution with initial datum \( f^I = v_1 + z_+(b)v_2 = y_+^{P(b)} \), which is the eigenvector pertaining to the largest eigenvalue of \( P(b) \) (cp. to the proof of Theorem 3.7). The formula for \( f^I \) holds since \( \sup_S c_S(P(b)) = bg(z_+(b))/(1 - \alpha^2) \). This can be verified by a direct comparison of (4.10) and (4.11). For \( b \) small it also follows from (4.12). In the limit \( b \to 0 \), \( P(b) \) in (3.9) approaches a multiple of \( w_1 \otimes w_1^* \) and

\[
f^I = v_1 + z_+(b)v_2 \to v_1 + \alpha v_2 = w_1.
\]

The solution \( f(t) \) of the ODE (2.11) with \( f^I = w_1 \) satisfies

\begin{equation}
f(t) = e^{-Ct}w_1 = V \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} W^*w_1 = e^{-\lambda_1 t}v_1 + \alpha e^{-\lambda_2 t}v_2.
\end{equation}

This implies

\[
e^{R\lambda_1 t}\|f(t)\|_2 \leq \|v_1 + \alpha e^{-R(\lambda_2 - \lambda_1)t}v_2\|_2 \xrightarrow{t \to \infty} \|v_1\|_2 = \frac{1}{\sqrt{1 - \alpha^2}}.
\]

and it finishes the proof. \( \square \) \( \square \)
After the analysis in Theorems 3.7 and 4.1 we are left with the case of a matrix $C \in \mathbb{C}^{2 \times 2}$ with eigenvalues $\lambda_1$ and $\lambda_2$ such that the real and imaginary parts are distinct. This case can not occur for real matrices $C$.

The proof of Lemma 3.3 gives an upper bound $\sqrt{\frac{1+i}{1-i}}$ for the multiplicative constant in (3.1). On the other hand, the solution of (2.1) with $f^I = w_1$ satisfies (4.13), hence,

$$\|f(t)\|_2 = e^{-2\Re \lambda_1 t} \|v_1 + \alpha e^{-(\lambda_2-\lambda_1)t} v_2\|_2^2$$

$$= \frac{1}{1-\alpha^2} e^{-2\Re \lambda_1 t} \left( 1 - 2\alpha^2 e^{-2\Re (\lambda_2-\lambda_1)t} \cos (\Im (\lambda_2 - \lambda_1) t) + \alpha^2 e^{-2\Re (\lambda_2-\lambda_1)t} \right).$$

The expression in the bracket is bigger than 1, e.g. at time $t = \pi/\Im (\lambda_2 - \lambda_1)$. Thus the minimal multiplicative constant $c$ is definitely bigger than $\frac{1}{\sqrt{1-\alpha^2}}$, which is the best constant for $3\lambda_1 = 3\lambda_2$ (see Theorem 4.1).

Next, we derive the upper and lower envelopes for the norm of solutions $f(t)$ of ODE (2.1) in order to determine the sharp constant $c$. For a diagonalizable matrix $C \in \mathbb{C}^{2 \times 2}$ with $\lambda_1^C = \lambda_2^C$ it holds that $\|f(t)\|_2 = e^{-\Re \lambda_1^C t} \|f\|_2$. And for the general case we have:

**Proposition 4.2.** Let $C \in \mathbb{C}^{2 \times 2}$ be a diagonalizable, positive stable matrix with eigenvalues $\lambda_1^C \neq \lambda_2^C$, and associated eigenvectors $v_1$ and $v_2$, resp. Then the norm of solutions $f(t)$ of ODE (2.1) satisfies

$$h_-(t) \|f\|_2^2 \leq \|f(t)\|_2^2 \leq h_+(t) \|f\|_2^2, \quad \forall t \geq 0,$$

where the envelopes $h_\pm(t)$ are given by

$$h_\pm(t) := e^{-2\Re \lambda_{\pm}^C t} m_\pm(t)$$

with

$$m_\pm(t) := \pm e^{-\gamma t} \left( \sqrt{\frac{\cosh(\gamma t) - \alpha^2 \cos(\delta t)}{(1-\alpha^2)^2}} - 1 \pm \frac{\cosh(\gamma t) - \alpha^2 \cos(\delta t)}{1-\alpha^2} \right),$$

where $\gamma := \Re (\lambda_2^C - \lambda_1^C)$, $\delta := \Im (\lambda_2^C - \lambda_1^C)$, $\alpha := \left| \langle v_1, v_2 \rangle / \|v_1\| \right|$ and $\alpha \in [0, 1]$.

While the rest of the article is based on estimating Lyapunov functionals, the following proof will use the explicit solution formula of the ODE.

**Proof.** We use again the unitary transformation as in the proof of Lemma 3.3 such that the eigenvectors $w_1$ and $w_2$ of $C^*$ are given in (3.7). If $f(t)$ is a solution of (2.1), then $\tilde{f}(t) = e^{\overline{C} t} f(t)$ satisfies

\begin{equation}
\frac{d}{dt} \tilde{f}(t) = -\overline{C} \tilde{f}(t), \quad \tilde{f}(0) = f^I,
\end{equation}

with $\overline{C} = (C - \lambda_1^C I) = (W^*)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2^C - \lambda_1^C \end{pmatrix} W^*$.

The explicit solution $\tilde{f}(t)$ of (4.14) is

$$\tilde{f}(t) = (W^*)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{-(\gamma + i\delta)t} \end{pmatrix} W^* f^I = \frac{e^{-(\gamma + i\delta)t}}{\sqrt{1-\alpha^2}} (e^{-(\gamma + i\delta)t} - 1) f_1^I + e^{-(\gamma + i\delta)t} f_2^I,$$
where $\gamma = \Re(\lambda_2^C - \lambda_1^C)$ and $\delta = \Im(\lambda_2^C - \lambda_1^C)$. If the initial data $f^I$ lies in $\mathbb{R} \times \mathbb{C}$ then the solution will satisfy $\tilde{f}(t) \in \mathbb{R} \times \mathbb{C}$ for all $t \geq 0$. The multiplication with $\tilde{f}_2^I/|\tilde{f}_2^I|$ is another unitary transformation and does not change the norm. Therefore, to compute the envelope for the norm of solutions $\tilde{f}(t)$ of ODE (4.14) we assume w.l.o.g. that

$$f^I_{\phi, \theta} = \left( \frac{\cos(\phi)}{\sin(\phi)} e^{i\theta} \right) \in \mathbb{R} \times \mathbb{C}, \quad \text{where } \phi, \theta \in [0, 2\pi),$$

such that $\|f^I_{\phi, \theta}\| = 1$. We consider the solution $\tilde{f}_{\phi, \theta}(t)$ for (4.14) with $f^I = f^I_{\phi, \theta}$. To compute the envelopes (for fixed $t$), we solve $\partial_t \|\tilde{f}_{\phi, \theta}\|^2 = 0$ and $\partial_\theta \|\tilde{f}_{\phi, \theta}\|^2 = 0$ in terms of $\phi$ and $\theta$. Evaluating $\|\tilde{f}_{\phi, \theta}(t)\|^2$ at $\phi = \phi(t)$ and $\theta = \theta(t)$ yields the envelopes for the norm of solutions $\tilde{f}(t)$ of ODE (4.14). Consequently, we derive the envelopes $h_{\pm}(t)\|f^I\|^2$ for the original problem, since $\|f(t)\|_2 = e^{-\Re(\lambda^C)|t|}\|\tilde{f}(t)\|_2$. □ □

**Corollary 4.3.** Let $C \in \mathbb{C}^{2 \times 2}$ be a diagonalizable, positive stable matrix. Then the minimal multiplicative constant $c$ in (3.1) for the ODE (2.1) is given by

$$c = \sqrt{\sup_{t \geq 0} m_+(t)},$$

where $m_+(t)$ is the function given in Proposition 4.2.

In general we could not find an explicit formula for $\sup_{t \geq 0} m_+(t)$.

5. A Family of Decay Estimates for Hypocoercive ODEs

In this section we shall illustrate the interdependence of maximizing the decay rate $\lambda$ and minimizing the multiplicative constant $c$ in estimates like (3.1). For the ODE-system (2.1), the procedure described in Remark 1.2(b) yields the optimal bound for large time, with the sharp decay rate $\mu := \min\{\Re(\lambda)|\lambda\text{ is an eigenvalue of } C\}$. But for non-coercive $C$ we must have $c > 1$. Hence, such a bound cannot be sharp for short time. As a counterexample we consider the simple energy estimate (obtained by premultiplying (2.1) with $f^*$)

$$\|f(t)\|_2 \leq e^{-\mu_s t}\|f^I\|_2, \quad t \geq 0,$$

with $C_s := \frac{1}{2}(C + C^*)$ and $\mu_s := \min\{\lambda|\lambda\text{ is an eigenvalue of } C_s\}$.

The goal of this section is to derive decay estimates for (2.1) with rates in between this weakest rate $\mu_s$ and the optimal rate $\mu$ from (2.5). It holds that $\mu_s \leq \mu$. At the same time we shall also present lower bounds on $\|f(t)\|_2$. The energy method again provides the simplest example of it, in the form

$$\|f(t)\|_2 \geq e^{-\nu_s t}\|f^I\|_2, \quad t \geq 0,$$

with $\nu_s := \max\{\lambda|\lambda\text{ is an eigenvalue of } C_s\}$. Clearly, estimates with decay rates outside of $[\mu_s, \nu_s]$ are irrelevant.

We present our main result only for the two-dimensional case, as the best multiplicative constant is not yet known explicitly in higher dimensions (cf. [3]):
Proposition 5.1. Let \( \mathbf{C} \in \mathbb{C}^{2 \times 2} \) be a diagonalizable positive stable matrix with spectral gap \( \mu := \min\{\Re(\lambda_j^C)\mid j = 1, 2\} \). Then, all solutions to (2.1) satisfy the following upper and lower bounds:

\[
\|f(t)\|_2 \leq c_1(\tilde{\mu}) e^{-\tilde{\mu}t} \|f\|_2, \quad t \geq 0, \quad \mu_s \leq \tilde{\mu} \leq \mu,
\]

with

\[
c_1^2(\tilde{\mu}) = \kappa_{\text{min}}(\beta(\tilde{\mu}))
\]

given explicitly in (5.8) below. There, \( \alpha \in [0, 1) \) is the cos of the (minimal) angle of the eigenvectors of \( \mathbf{C}^* \) (cf. the proof of Lemma 3.1), and \( \beta(\tilde{\mu}) = \max(-\alpha, -\beta_0) \), with \( \beta_0 \) defined in (5.6), (5.7) below.

\[
\|f(t)\|_2 \geq c_2(\tilde{\mu}) e^{-\tilde{\mu}t} \|f\|_2, \quad t \geq 0, \quad \nu \leq \tilde{\mu} \leq \nu_s,
\]

with \( \nu := \max\{\Re(\lambda_j^C)\mid j = 1, 2\} \). The maximal constant

\[
c_2^2(\tilde{\mu}) = \kappa_{\text{min}}(\beta(\tilde{\mu}))^{-1}
\]

is given again by (5.8), with \( \alpha, \beta(\tilde{\mu}) \) defined as in Part (a).

Proof. Part (a): For a fixed \( \tilde{\mu} \in [\mu_s, \mu] \) we have to determine the smallest constant \( c_1 \) for the estimate (5.1), following the strategy of proof from [3]. To this end, we use a unitary transformation of the coordinate system and write \( \mathbf{P}(\tilde{\mu}) = \mathbf{W} \mathbf{B}_u \mathbf{W}^* \) with

\[
\mathbf{W} = \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\sqrt{1 - \alpha^2}} \end{pmatrix}, \quad \mathbf{B}_u = \begin{pmatrix} 1/b & \beta(\tilde{\mu}) \\ \beta(\tilde{\mu}) & b \end{pmatrix},
\]

where we set w.l.o.g. \( b_1 = 1/b, b_2 = b \) with \( b > 0 \). Moreover, \( |\beta|^2 < 1 \) has to hold. Now, we have to find the positive definite Hermitian matrix \( \mathbf{B}_u \), such that the analog of (4.3), (5.3) holds, i.e.:

\[
\mathbf{A} := \begin{pmatrix} 2(\Re(\lambda_1^C) - \tilde{\mu})/b & (\lambda_1^C + \lambda_2^C - 2\tilde{\mu})\beta \\ (\lambda_1^C + \lambda_2^C - 2\tilde{\mu})\beta & 2(\Re(\lambda_2^C) - \tilde{\mu})b \end{pmatrix} \geq 0,
\]

As in the proof of Lemma 3.1 we assume that the eigenvalues of \( \mathbf{C} \) are ordered as \( \Re(\lambda_2^C) \geq \Re(\lambda_1^C) = \mu \geq \tilde{\mu} \). Hence, \( \text{Tr} \mathbf{A} \geq 0 \). For the non-negativity of the determinant to hold, i.e.

\[
\det \mathbf{A} = 4(\Re(\lambda_2^C) - \tilde{\mu})(\Re(\lambda_2^C) - \tilde{\mu}) - |\lambda_1^C + \lambda_2^C - 2\tilde{\mu}|^2|\beta|^2 \geq 0,
\]

we have the following restriction on \( \beta \):

\[
|\beta|^2 \leq \beta_0^2 := \frac{4(\Re(\lambda_2^C) - \tilde{\mu})(\Re(\lambda_2^C) - \tilde{\mu})}{|\lambda_1^C + \lambda_2^C - 2\tilde{\mu}|^2}.
\]

If \( \lambda_1^C + \lambda_2^C - 2\tilde{\mu} = 0 \), we conclude \( \lambda_1^C = \lambda_2^C \) and that we have chosen the sharp decay rate \( \tilde{\mu} = \mu \). As the associated, minimal condition number \( \kappa(\mathbf{P}) \) was already determined in Lemma 3.1 we shall not re-discuss this case here. But to include this case into the statement of the theorem, we set

\[
\beta_0 := 1, \quad \text{if } \lambda_1^C = \lambda_2^C \text{ and } \tilde{\mu} = \mu.
\]
From (5.6) we conclude that $\beta_0 \in [0, 1]$. Note that $\beta_0 = 1$ is only possible for $\bar{\mu} = \mu$ and $\lambda^C_1 = \lambda^C_2$, i.e. the case that we just sorted out. For the rest of the proof we hence assume that condition (5.6) holds with $\beta_0 \in [0, 1)$.

For admissible matrices $B_u$ (i.e. with $b > 0$ and $|\beta| \leq \beta_0$) it remains to determine the matrix

$$P(b, \beta) = WB_uW^* = \begin{pmatrix} \frac{1}{2} + 2\alpha\Re \beta + ba^2 & (\beta + ba)\sqrt{1 - \alpha^2} \\ (\beta + ba)\sqrt{1 - \alpha^2} & b(1 - \alpha^2) \end{pmatrix},$$

(with $W$ and $B_u$ given in (5.3)), having the minimal condition number $\kappa(P(b, \beta)) = \lambda^P_+(b, \beta)/\lambda^P_-(b, \beta)$. Here

$$\lambda^P_\pm(b, \beta) = \frac{\Tr P(b, \beta) \pm \sqrt{(\Tr P(b, \beta))^2 - 4 \det P(b, \beta)}}{2}$$

are the (positive) eigenvalues of $P(b, \beta)$.

As a first step we shall minimize $\kappa(P(b, \beta))$ w.r.t. $b$ (and for $\beta$ fixed), since $\arg\min_{b>0} \kappa(P(b, \beta))$ will turn out to be independent of $\beta$. We notice that $\Tr P(b, \beta) = b + 2\alpha\Re \beta + 1/b$ is a convex function of $b \in (0, \infty)$ which attains its minimum for $b = 1$. Moreover, $\det P(b, \beta) = (1 - \alpha^2)(1 - |\beta|^2) > 0$ is independent of $b$. This yields the condition number

$$\kappa_{\text{min}}(\beta) = \frac{\lambda^P_+(1, \beta)}{\lambda^P_-(1, \beta)} = \frac{1 + \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)} - 1 - \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)}}{1 - \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)}}.$$

As a second step we minimize $\kappa_{\text{min}}(\beta)$ on the disk $|\beta| \leq \beta_0$. To this end, the quotient $\frac{(1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)}$ should be as large as possible. For any fixed $|\beta| \leq \beta_0$, this happens by choosing $\beta = -|\beta|$, since $\alpha \in [0, 1]$. Hence it remains to maximize the function $g(\beta) := \frac{1 - \beta^2}{(1 + \alpha|\beta|^2)}$ on the interval $[-\beta_0, 0]$. It is elementary to verify that $g$ is maximal at $\beta := \max(-\alpha, -\beta_0)$. Then, the minimal condition number is

$$\kappa_{\text{min}}(\beta) = \kappa(P(1, \beta)) = \frac{1 + \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)} - 1 - \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)}}{1 - \frac{1 - (1 - \alpha^2)(1 - |\beta|^2)}{(1 + \alpha|\beta|^2)}}.$$

Part (b): Since the proof of the lower bound is very similar to Part (a), we shall just sketch it. For a fixed $\bar{\mu} \in [\nu, \nu_s]$ we have to determine the largest constant $c_2$ for the estimate (5.2). To this end we need to satisfy the inequality

$$C^*P + PC \leq 2\bar{\mu}P$$

with a positive definite Hermitian matrix $P$ with minimal condition number $\kappa(P)$. In analogy to (2) this would imply

$$\frac{d}{dt}\|f(t)\|^2_P \geq -2\bar{\mu}\|f(t)\|^2_P,$$

and hence the desired lower bound

$$\|f(t)\|^2_2 \geq (\lambda^{\text{max}}_P)^{-1}\|f(t)\|^2_P \geq (\lambda^{\text{max}}_P)^{-1}e^{-2\bar{\mu}t}\|f(t)\|^2_P \geq (\kappa(P))^{-1}e^{-2\bar{\mu}t}\|f(t)\|^2_2.$$
For minimizing $\kappa(P)$, we again use a unitary transformation of the coordinate system and write $P$ as $P(\tilde{\mu}) = WB_l W^*$, with $W$ from (5.3) and the positive definite Hermitian matrix

$$B_l = \begin{pmatrix} 1/b & \beta(\tilde{\mu}) \\ \beta(\tilde{\mu}) & b \end{pmatrix},$$

with $b > 0$ and $|\beta|^2 < 1$. Then, the matrix $A$ from (5.4) has to satisfy $A \leq 0$. Since we chose the eigenvalues of $C$ to be ordered as $\Re(\lambda_C^1) \leq \Re(\lambda_C^2) = \nu \leq \tilde{\mu}$, we have $\text{Tr} A \leq 0$. The necessary non-negativity of its determinant again reads as (5.5).

In the special case $\lambda_C^1 + \lambda_C^2 - 2\tilde{\mu} = 0$, we conclude again $\lambda_C^1 = \lambda_C^2$ and $\tilde{\mu} = \nu$. Hence $A = 0$. Since $\beta$ is then only restricted by $|\beta| < 1$, we can again set $\beta_0 = 1$ and obtain the minimal $\kappa(P)$ for $\beta(\nu) = -\alpha$, as in Part (a).

In the generic case, the minimal $\kappa(P)$ is obtained for $\tilde{\beta} = \max(-\alpha, -\beta_0)$ with $\beta_0$ given in (5.6). Hence, the maximal constant in the lower bound (5.2) is $c_2(\tilde{\mu}) = \kappa_{\text{min}}(\tilde{\beta})^{-1}$ where $\kappa_{\text{min}}$ is given by (5.8). This finishes the proof. \hfill \Box

We illustrate the results of Proposition 5.1 with two examples.

**Example 5.2.** We consider ODE (2.1) with the matrix

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues $\lambda_{\pm} = (1 \pm i\sqrt{3})/2$, and some normalized eigenvectors of $C^*$ are, e.g.

$$w_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \lambda_- \end{pmatrix}, \quad w_- = \frac{1}{\sqrt{2}} \begin{pmatrix} -\lambda_- \\ 1 \end{pmatrix}.$$  

(5.9)

The optimal decay rate is $\mu = 1/2$, whereas the minimal and maximal eigenvalues of $C_s$ are $\mu_s = 0$ and $\nu_s = 1$, respectively. To bring the eigenvectors of $C^*$ in the canonical form used in the proof of Proposition 5.1, we fix the eigenvector $w_+$, and choose the unitary multiplicative factor for the second eigenvector $w_-$ as in (5.9) such that $\langle w_+, w_- \rangle$ is a real number. Finally, we use the Gram-Schmidt process to obtain a new orthonormal basis such that the eigenvectors of $C^*$ in the new orthonormal basis are of the form (3.7) with $\alpha = 1/2$. Then, the upper and lower bounds for the Euclidean norm of a solution of (2.1) are plotted in Fig. 2 and Fig. 3. For both the upper and lower bounds, the respective family of decay curves does *not* intersect in a single point (see Fig. 3). Hence, the whole family of estimates provides a (slightly) better estimate on $\|f(t)\|_2$ than if just considering the two extremal decay rates. For the upper bound this means

$$\|f(t)\|_2 \leq \min_{\tilde{\mu} \in [\mu_s, \mu]} c_1(\tilde{\mu}) e^{-\tilde{\mu} t} \|f^I\|_2 \leq \min\{1, c_1(\mu) e^{-\mu t}\} \|f^I\|_2, \quad t \geq 0,$$

and for the lower bound

$$\|f(t)\|_2 \geq \max_{\tilde{\nu} \in [\nu, \nu_s]} c_2(\tilde{\nu}) e^{-\tilde{\nu} t} \|f^I\|_2 \geq \max\{c_2(\nu) e^{-\nu t}, c_2(\nu_s) e^{-\nu_s t}\} \|f^I\|_2, \quad t \geq 0.$$
Figure 2. The red (grey) curves are the squared norm of solutions $f(t)$ for ODE (2.1) with matrix $C = [1, -1; 1, 0]$ and various initial data $f^I$ with norm 1. The blue (black) curves are the lower and upper bounds for the squared norm of solutions. Note: The curves are colored only in the electronic version of this article.

Figure 3. Zoom of Fig. 2. The curves are the lower bounds for the squared norm of solutions for ODE (2.1) with matrix $C = [1, -1; 1, 0]$ and various initial data $f^I$ with norm 1. This plot shows that these lower bounds do not intersect in a single point.

Note that the upper bound $\sqrt{3}e^{-t/2}$ with the sharp decay rate $\mu = \frac{1}{2}$ carries the optimal multiplicative constant $c = \sqrt{3}$, as it touches the set of solutions (see Fig. 2). But this is not true for the estimates with smaller...
decay rates (except of \( \bar{\mu} = 0 \)). The reason for this lack of sharpness is the fact that the inequality \( \| f(t) \|_P^2 \leq e^{-2\bar{\mu}t} \| f \|_P^2 \) used in the proof of Proposition 5.1 is, in general, not an equality (in contrast to (2.3)). \( \square \)

In the next example we consider a matrix \( C \in \mathbb{R}^{2\times2} \) with \( \Re \lambda_1 \neq \Re \lambda_2 \), which corresponds to the case analyzed in Theorem 4.1. For such cases the strategy of Proposition 5.1 (based on minimizing \( \kappa(P) \)) could be improved in the spirit of Theorem 4.1 but we shall not carry this out here. Hence, the estimates of the following example will not be sharp, see Fig. 4.

**Example 5.3.** We consider ODE (2.1) with the matrix

\[
C = \begin{pmatrix} 19/20 & -3/10 \\ 3/10 & -1/20 \end{pmatrix}
\]

which has the eigenvalues \( \lambda_1 = 1/20 \) and \( \lambda_2 = 17/20 \), and some normalized eigenvectors of \( C^* \) are, e.g.,

\[
w_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad w_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.
\]

The optimal decay rate is \( \mu = 1/20 \), whereas the minimal and maximal eigenvalues of \( C_s \) are \( \mu_s = -1/20 \) and \( \nu_s = 19/20 \), respectively. Since the matrix \( C \) and its eigenvalues are real valued, the eigenvectors of \( C^* \) are already in the canonical form used in the Gram-Schmidt process to obtain a new orthogonal basis such that the eigenvectors of \( C^* \) in the new basis are of the form (3.7) with \( \alpha = 3/5 \). Then, the upper and lower bounds for the Euclidean norm of a solution of (2.1) are plotted in Fig. 4. Since \( \mu_s < 0 \), solutions \( f(t) \) to this example may initially increase in norm. \( \square \)

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Figure 4. The red (grey) curves are the squared norm of solutions $f(t)$ for ODE (2.1) with matrix $C = [19/20, -3/10; 3/10, -1/20]$ and various initial data $f^I$ with norm 1. The blue (black) curves are the lower and upper bounds for the squared norm of solutions derived from Proposition 5.1. The green (black) dash-dotted curve is the upper bound for the squared norm of solutions derived from Theorem 4.1. Note: The curves are colored only in the electronic version of this article.

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