Diamond Subgraphs in the Reduction Graph of a One-Rule String Rewriting System

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Abstract

In this paper, we study a certain case of a subgraph isomorphism problem. We consider the Hasse diagram of the lattice $M_k$ (the unique lattice with $k+2$ elements and one anti-chain of length $k$) and want to find the maximal $k$ for which it is isomorphic to a subgraph of the reduction graph of a given one-rule string rewriting system. We obtain a complete characterization for this problem and show that there is a dichotomy. There are one-rule string rewriting systems for which the maximal such $k$ is 2 and there are cases where there is no maximum. No other intermediate option is possible.

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1 Introduction

The (directed) reduction graph of a string rewriting system (SRS) $S$ is the graph whose vertices are words, and whose edges are the one-step reductions. This graph plays a central role in the study of properties of $S$. In this paper we want to study the reduction graph of one-rule SRSs. Despite being very simple objects,
there are many open problems regarding one-rule string rewriting systems (see [6] and [4, Problems 95 and 21b]). Therefore, any progress in understanding this type of reductions is of value. One way to have a better understanding of a graph G is by finding basic graphs that are or aren’t isomorphic to a subgraph of G. The reduction graph of a one-rule SRS \( \langle A \mid u \rightarrow v \rangle \) is always a graded graph (in the sense that for any two vertices \( x, y \) any two paths from \( x \) to \( y \) has the same length), so clearly it has only graded subgraphs. One of the most basic graded graphs is the Hasse diagram of the lattice \( M_k \), where \( M_k \) is the lattice with \( k + 2 \) elements \( \{x, y, z_1, \ldots, z_n\} \) such that \( x \leq z_i \leq y \) for \( 1 \leq i \leq n \) and \( \{z_1, \ldots, z_n\} \) are pairwise incomparable. For the sake of simplicity, we use \( M_k \) also for the Hasse diagram of this lattice. Given a one-rule SRS \( S = \langle A \mid u \rightarrow v \rangle \) we denote its reduction graph by \( G_S \). We consider the question of whether \( M_k \) is isomorphic to a subgraph of \( G_S \). In other words, we want to know what is the maximal \( k \) for which \( M_k \) is embeddable in \( G_S \). It turns out that the answer is closely related to some other well-known notions and properties of one-rule systems. Neglecting few trivial cases (\( u = v \) or \( |A| = 1 \)) and assuming without loss of generality that \( |u| \leq |v| \), we divide the problem into several cases. In Section 3.1 we prove that if \( S \) is left (right) cancellative (i.e., \( u \) and \( v \) has different first (respectively, last) letters) then \( M_3 \) is not embeddable in \( G_S \), hence \( k = 2 \) is maximal. In Section 3.2 we generalize this to any system where \( v \) is not bordered with \( u \), i.e., \( u \) is not a prefix or not a suffix of \( v \). In Section 3.3 we discuss systems where \( u = 1 \) and prove that if \( v \neq b^n \) for every \( b \in A \) then \( M_k \) is embeddable in \( G_S \) for any natural \( k \). On the other hand, if \( v = b^n \) for some \( b \in A \) then \( k = 2 \) is again the maximum. In Section 3.4 we deal with the remaining case where \( u \neq 1 \) and \( v \) is bordered with \( u \). We use Adyan reduction [2] to reduce this case to a system of the form \( \langle \tilde{A} \mid 1 \rightarrow \tilde{v} \rangle \) which is the case solved in Section 3.3. In conclusion, we have obtained a dichotomy between cases where \( M_k \) is embeddable in \( G_S \) for every natural \( k \) and cases where \( k = 2 \) is the maximal value for which \( M_k \) is embeddable in \( G_S \).

2 Preliminaries

A directed graph is a tuple \((V, E, d, r)\) consists of a set (of vertices) \( V \), a set (of edges) \( E \) and two functions \( d, r : E \rightarrow V \) associating each edge \( e \in E \) with a domain vertex \( d(e) \) and a range vertex \( r(e) \). A subgraph \( G' = (V', E', d', r') \) of \( G \) is a graph such that \( V' \subseteq V, E' \subseteq E \) and \( d', r' : E' \rightarrow V' \) are the
corresponding restrictions of \( \mathbf{d} \) and \( \mathbf{r} \) (in particular, this requires that \( \mathbf{d}(E') \subseteq V' \) and \( \mathbf{r}(E') \subseteq V' \)). Let \( G_1 = (V_1, E_1, \mathbf{d}_1, \mathbf{r}_1) \) and \( G_2 = (V_2, E_2, \mathbf{d}_2, \mathbf{r}_2) \) be two graphs. A graph homomorphism \( f : G_1 \to G_2 \) consists of two functions \( f_V : V_1 \to V_2 \) and \( f_E : E_1 \to E_2 \) such that

\[
\mathbf{d}_2(f_E(e)) = f_V(\mathbf{d}_1(e)), \quad \mathbf{r}_2(f_E(e)) = f_V(\mathbf{r}_1(e))
\]

for every \( e \in E_1 \). We say that \( f \) is an embedding (so \( G_1 \) is embedded in \( G_2 \)) if \( f_E \) and \( f_V \) are injective functions.

The set of all words over an alphabet \( A \) is denoted by \( A^* \). We denote the empty word by \( 1 \) and the set of all non-empty words by \( A^+ \). Let \( u, v \in A^+ \) be some words. We say that \( u \) is a prefix (suffix) of \( v \) if there exists \( x \in A^* \) such that \( v = ux \) (respectively, \( v = xu \)). Also, \( u \) is called a factor of \( v \) if there exist \( x, y \in A^* \) such that \( v = xuy \). We say that \( v \) is bordered with \( u \) if \( u \) is both a prefix and a suffix of \( v \). Recall that the length of a word \( u \in A^* \) is the number of letters in \( u \) and it is denoted \( |u| \). Assume that \( u = xay \) where \( a \in A \) is a letter. We say that the letter \( a \) is at position \( i \) of \( u \) if \( |x| = i \).

Let \( A \) be some set and let \( R \) be a relation on \( A^* \). A tuple \( S = \langle A \mid R \rangle \) is called a string rewriting system (SRS). Elements of \( R \) are usually written in the form \( u_i \to v_i \) instead of \( (u_i, v_i) \). Let \( S = \langle A \mid R \rangle \) be an SRS. The single-step reduction relation induced by \( R \) is a relation on \( A^* \) denoted \( \to_R \) which is defined by \( w \to_R w' \) if \( w = xuy \) and \( w' = xvy \) for some \( x, y \in A^* \) and \( u \to v \in R \). If \( |x| = i \) we say that the rule \( u \to v \) is being used at position \( i \) in the reduction \( w \to_R w' \). We denote by \( G_S \) the reduction graph of \( S \). It is the (directed) graph defined as follows. The set of vertices of \( G_S \) is the set \( A^* \) of all words over \( A \). Given \( w, w' \in A^* \), edges \( w \to w' \) correspond to tuples \( (i, u \to v) \) where \( u \to v \) is a rule in \( R \) and \( w \to_R w' \) is a one-step reduction where \( u \to v \) is being used at position \( i \). If \( S \) has only one rule, we can identify an edge only with the position \( i \) where the unique rewrite rule is being used. A path in the reduction graph is called a reduction of \( S \).

3 The embeddability of \( M_k \) in the reduction graph of a one-rule SRS

**Definition 3.1.** Denote by \( M_k \) the directed graph whose set of vertices is \( \{x, y, z_1, \ldots, z_k\} \) and for every \( 1 \leq i \leq k \) there are two edges \( x \to z_i \) and \( z_i \to y \).
Note that $M_k$ is “diamond shaped”, for instance, $M_3$ is the Hasse diagram of the diamond lattice:

![Hasse diagram of the diamond lattice](image)

We want to consider the following question. Given a one-rule SRS $S = \langle A \mid u \rightarrow v \rangle$, what is the maximal $k$ for which $M_k$ is isomorphic to a subgraph of the reduction graph $G_S$?

We start with some simple observations. If $u = v$ then the reduction graph contains only loops and even $M_1$ is not embeddable in $G_S$ so from now on we assume $u \neq v$. If $|A| = 1$ then every connected component of $G_S$ with more than one vertex is just an (infinite) path graph. Therefore only $M_1$ is embeddable in $G_S$ and we can assume from now on that $|A| > 1$. Another simple observation is that $M_k$ is embeddable in $G_S$ for $S = \langle A \mid u \rightarrow v \rangle$ if and only if it is embeddable in $G_{SC}$ where $S^C$ is the converse system $S^C = \langle A \mid v \rightarrow u \rangle$. Therefore, without loss of generality we can assume that $|u| \leq |v|$.

If an SRS $S = \langle A \mid u \rightarrow v \rangle$ satisfy both $|A| > 1$ and $u \neq v$, it is easy to see that $M_2$ is embeddable in $G_S$. Indeed, choose a word $w \in A^*$ such that $uvw \neq vwv$ (for instance, if $\max\{|u|, |v|\} < l$ we can choose $w = a^l b^l$). The reduction graph of $S$ contains the subgraph

![Subgraph isomorphic to $M_2$](image)

which is isomorphic to $M_2$. The question left is whether there are other values of $k$ for which $M_k$ is embeddable in $G_S$? We split this question into several cases.

### 3.1 Left (right) cancellative SRSs

Let $S = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS such that $u, v \neq 1$. We say that $S$ is left cancellative if the first letter of $u$ and $v$ are different.
Remark 3.2. The term “left cancellative” comes from the well-known fact that the first letter of $u$ and $v$ are different if and only if the semigroup presented by $S$ is left cancellative, i.e., $ax = ay$ implies $x = y$ (see [1], Chapter II Theorem 2, also stated clearly in [3] Theorem 16).

In this section we will prove that $M_3$ is not embeddable in $G_S$ if $S$ is a left cancellative SRS.

Given a reduction of some SRS

$$x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n$$

we want a way to mark letters that are involved in the rewriting. For this we introduce a technical tool. Given a set of letters $A = \{a_1, \ldots, a_n\}$ we define a set of “decorated” copies $A^* = \{a_1^*, \ldots, a_n^*\}$. Let $u \in A^*$ and assume $u = u_1 \ldots u_k$ where every $u_i$ is a letter of $A$. We denote by $u^* = u_1^* \ldots u_k^*$ a decorated copy of the word $u$. Denote by $\pi : A \cup A^* \rightarrow A$ a function defined $\pi(a_i) = \pi(a_i^*) = a_i$ which clearly extends to a projection $\pi : (A \cup A^*)^* \rightarrow A^*$. Now we can define:

**Definition 3.3.** Let $S = \langle A \mid R \rangle$ be an SRS. Define a new SRS, denoted $\overline{S} = \langle \overline{A}, \overline{R} \rangle$, in the following way. The set of letters of $\overline{S}$ is $\overline{A} = A \cup A^*$. For every rule $u \rightarrow v$ in $R$ and for every word $\overline{u} \in (\overline{A} \cup \overline{A^*})^*$ such that $\pi(\overline{u}) = u$ the relation $\overline{R}$ will have the rule $\overline{u} \rightarrow \overline{v}^*$.

**Example 3.4.** If $S = \langle a, b \mid ab \rightarrow bba \rangle$ then the SRS $\overline{S}$ is

$$\overline{S} = \langle a, a^*, b, b^* \mid ab \rightarrow b^*b^*u^*, \quad a^*b \rightarrow b^*b^*a^*, \quad ab^* \rightarrow b^*b^*a^*, \quad a^*b^* \rightarrow b^*b^*a^* \rangle$$

It is obvious that every reduction

$$\overline{x}_1 \rightarrow \ldots \rightarrow \overline{x}_n$$

of $\overline{S}$ can be projected into a reduction of $S$

$$\pi(\overline{x}_1) \rightarrow \ldots \rightarrow \pi(\overline{x}_n)$$

by deleting all the “decorations”. Moreover, it is easy to see that every reduction of $S$

$$x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n$$
can be “lifted” into a reduction of \( \overline{S} \)

\[
\overline{x}_1 \rightarrow \overline{x}_2 \ldots \rightarrow \overline{x}_n
\]
such that \( \pi(\overline{x}_i) = x_i \) and \( \overline{x}_1 = x_1 \). The decorated letters in this reduction will be the letters that are “involved” in the reduction or “affected” by it.

**Example 3.5.** Consider the SRS \( S \) in example 3.4 and the reduction

\[
abaabb \rightarrow^{(3)} ababbab \rightarrow^{(2)} abbbbab \rightarrow^{(4)} abbbbbaab
\]

where the numbers over the arrows are the positions in which the rewrite is being done. This reduction can be lifted to the reduction

\[
abaabb \rightarrow^{(3,ab \rightarrow b \cdot b \cdot a)} abab\cdot a \cdot b \rightarrow^{(2,ab \rightarrow b \cdot b \cdot a)} ab\cdot b \cdot a \cdot b \rightarrow^{(4,ab \rightarrow b \cdot b \cdot a)} ab\cdot b \cdot b \cdot a \cdot b
\]

of the SRS \( \overline{S} \).

The following observation about reductions in \( \overline{S} \) will be useful.

**Lemma 3.6.** Let \( S = \langle A | u \rightarrow v \rangle \) be a one-rule SRS and consider a reduction

\[
x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n
\]
of \( S \) and its lifting

\[
\overline{x}_1 \rightarrow \overline{x}_2 \rightarrow \ldots \rightarrow \overline{x}_n
\]
to a reduction of \( \overline{S} \). Assume that the first decorated letter of \( \overline{x}_n \) is at position \( i \) then

1. No step in the reduction is carried out at position \( j \) for \( j < i \).
2. There is a step in the reduction carried out at position \( i \).
3. If \( S \) is left cancellative then the letter at position \( i \) of \( x_1 \) is the first letter of \( u \) and the letter at position \( i \) of \( x_n \) is the first letter of \( v \).

**Proof.** Statements (1) and (2) are clear so we will prove (3). Denote by \( a \) the first letter of \( u \) and by \( b \) the first letter of \( v \). Assume that in the step \( x_k \rightarrow x_{k+1} \) the rewrite rule is carried out at position \( i \) (such step exists by (2)). Therefore, the letter at position \( i \) of \( x_k \) is \( a \) and the letter at position \( i \) of \( x_{k+1} \) is \( b \). Since no step is carried out at position \( j \) for \( j < i \), the first letter of \( x_1 \) is also \( a \). In
addition, the first letter of $u$ and $v$ are different so we can not carry out any step at position $i$ in the reduction $x_{k+1} \rightarrow \ldots \rightarrow x_n$. Therefore, the letter at position $i$ of $x_n$ is $b$ as required.

Lemma 3.7. Let $S = \langle A \mid u \rightarrow v \rangle$ be a left cancellative SRS and let $x \rightarrow z_1 \rightarrow y$ and $x \rightarrow z_2 \rightarrow y$ be two reductions in $S$. Denote the corresponding “lifted” reductions in $\bar{S}$ by

$$\bar{x} \rightarrow \bar{z}_1 \rightarrow \bar{y}_1, \quad \bar{x} \rightarrow \bar{z}_2 \rightarrow \bar{y}_2.$$

(a priory, $\bar{y}_1 \neq \bar{y}_2$ because they might have different decorations). Then, the first decorated positions of $\bar{y}_1$ and $\bar{y}_2$ are equal.

Proof. Denote by $i_1$ ($i_2$) the first decorated position of $\bar{y}_1$ (respectively, $\bar{y}_2$). We continue to use $a$ for the first letter of $u$ and $b$ for the first letter of $v$. Assume without loss of generality that $i_1 < i_2$. Applying part (3) of Lemma 3.6 on the reduction $x \rightarrow z_1 \rightarrow y$, we obtain that $b$ is the letter at position $i_1$ of $y$ and $a$ is the letter at position $i_1$ of $x$. Applying part (1) of Lemma 3.6 on $x \rightarrow z_2 \rightarrow y$, we obtain that $a$ is the letter at position $i_1$ of $y$ (since there are no steps carried out in this reduction at position $j$ for $j < i_2$). This is a contradiction so $i_1 = i_2$ as required.

Proposition 3.8. Let $S = \langle A \mid u \rightarrow v \rangle$ be a left cancellative SRS. Then $M_3$ is not isomorphic to a subgraph of $G_S$.

Proof. Consider three reductions

$$x \rightarrow z_1 \rightarrow y, \quad x \rightarrow z_2 \rightarrow y, \quad x \rightarrow z_3 \rightarrow y$$

such that $z_1, z_2, z_3$ are all distinct and lift them into three reductions in $\bar{S}$

$$\bar{x} \rightarrow \bar{z}_1 \rightarrow \bar{y}_1, \quad \bar{x} \rightarrow \bar{z}_2 \rightarrow \bar{y}_2, \quad \bar{x} \rightarrow \bar{z}_3 \rightarrow \bar{y}_3.$$  

According to Lemma 3.7 the first decorated positions of $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$ are identical. Denote this position by $i$. Part (2) of Lemma 3.6 implies that in each one of the three reduction there is a rewrite step carried out at position $i$. Without loss of generality we assume that in the first reduction this is the first step

$$x \stackrel{(i)}{\rightarrow} z_1.$$
In the second reduction this cannot be the first step
\[ x^{(i)} \rightarrow z_2 \]
because this will imply \( z_1 = z_2 \) in contradiction to our assumption. Therefore, this must be the second step \( z_2^{(i)} \rightarrow y \).

For the third reduction we cannot have
\[ x^{(i)} \rightarrow z_3 \]
as this implies \( z_1 = z_3 \) and we cannot have
\[ z_3^{(i)} \rightarrow y \]
as this implies \( z_2 = z_3 \). This is a contradiction which finishes the proof. \( \square \)

**Remark 3.9.** Clearly, a dual result holds for right cancellative SRSs.

### 3.2 SRSs where \( v \) is not bordered with \( u \)

In this section we generalize the results of Section 3.1 to a wider class of SRSs.

**Proposition 3.10.** Let \( S = \langle A \mid u \rightarrow v \rangle \) be an SRS such that \( u \) is not a prefix of \( v \), then \( M_3 \) is not embeddable in \( G_S \).

**Proof.** Denote by \( p \) the maximal prefix of \( u \) which is also a prefix of \( v \). Therefore, we can write \( u = pu' \) and \( v = pv' \) for some words \( u', v' \). It might be the case that \( p = 1 \) (if \( S \) is left cancellative) but note that \( u' \neq 1 \) since \( u \) is not a prefix of \( v \) and \( v' \neq 1 \) since we are assuming \( |u| \leq |v| \). The maximality of \( p \) implies that the SRS defined by \( S' = \langle A \mid u' \rightarrow v' \rangle \) is left cancellative. Now, note that any reduction \( x \rightarrow y \) which is carried out using the rule \( pu' \rightarrow pv' \) can be carried out using the rule \( u' \rightarrow v' \). Therefore, \( G_S \) is a subgraph of \( G_{S'} \). Since \( M_3 \) is not embeddable in \( G_{S'} \) by Proposition 3.8, it is not embeddable in \( G_S \) as well. \( \square \)

Clearly, a dual result holds for SRSs where \( u \) is not a suffix of \( v \) so we can conclude:
Proposition 3.11. Let $S = \langle A \mid u \rightarrow v \rangle$ be an SRS. If $v$ is not bordered with $u$ (i.e., $u$ is not a prefix of $v$ or not a suffix of $v$) then $M_3$ is not embeddable in $G_S$.

3.3 Special one-rule SRSs

In this section we deal with SRSs of the form $S = \langle A \mid 1 \rightarrow \rangle$. We remark that SRSs of the form $\langle A \mid v_i \rightarrow 1 \rangle$ are called special (see [3, Definition 3.4.1]).

We have already mentioned that $M_k$ is embeddable in $G_S$ if and only if it is embeddable in $G_{SC}$ where $S^C$ is the converse system. So we can say that in this section we consider special one-rule SRSs. There are few subcases.

Lemma 3.12. If $v = b^n$ for some letter $b \in A$ then $M_3$ is not embeddable in $G_S$.

Proof. Any word $x \in A^*$ can be uniquely decomposed into

$$x = b^{m_0}a_{i_1}b^{m_1}a_{i_2}b^{m_2} \ldots b^{m_{l-1}}a_{i_l}b^{m_l}$$

where $a_{i_1}, \ldots, a_{i_l} \in A$ are letters distinct from $b$ and $m_0, \ldots, m_l$ are non-negative integers. If $x \rightarrow z$ is a one-step reduction then

$$z = b^{m'_0}a_{i_1}b^{m'_1}a_{i_2}b^{m'_2} \ldots b^{m'_{l-1}}a_{i_l}b^{m'_l}$$

such that $m'_i = m_i + n$ for some $i \in \{0, \ldots, l\}$ and $m'_j = m_j$ if $j \neq i$. It is clear that we can identify $x$ with the tuple $(m_0, \ldots, m_l)$ and a one-step reduction is equivalent to adding $n$ to one of the entries. Therefore, a two step reduction $x \rightarrow z_1 \rightarrow y$ is equivalent to adding $n$ to two of the entries (or twice to the same one). Now, it is clear that there could be at most one additional reduction $x \rightarrow z_2 \rightarrow y$ from $x$ to $y$ with $z_1 \neq z_2$. This finishes the proof.

Lemma 3.13. For any $k \in \mathbb{N}$, the graph $M_k$ is embeddable in $G_S$ for $S = \langle A \mid 1 \rightarrow ab \rangle$.

Proof. Choose $k \in \mathbb{N}$ and take $x = (aabb)^{k-1}$. For $0 \leq i \leq k - 1$ define $z_i = (aabb)^i ab (aabb)^{k-i-1}$. It is clear that $z_i$ is obtained from $x$ by applying the rewrite rule at position $4i$. Moreover, it is clear that $z_i \neq z_j$ for $i \neq j$. Now, applying the rewrite rule at position $4i + 1$ we obtain a reduction $z_i \rightarrow y$ where $y = (aabb)^k$. This yields a subgraph isomorphic to $M_k$ as required.
Lemma 3.14. Let $S = \langle A \mid 1 \to v \rangle$ be an SRS such that $v \neq b^n$ for every $b \in A$. Then, $M_k$ is embeddable in $G_S$ for every $k$.

Proof. Assume that the first letter of $v$ is $a$ so $v = av'$ where $v'$ contains at least one letter distinct from $a$. Define a monoid homomorphism $f : \{a, b\}^* \to A^*$ which is the extension of

$$f(a) = a, \quad f(b) = v'.$$

It is easy to see that $f$ is injective and that $f(ab) = av' = v$. Therefore, it induces a graph embedding

$$\hat{f} : G_T \to G_S$$

where $T = \langle a, b \mid 1 \to ab \rangle$. In particular, it embeds the subgraph of $G_T$ isomorphic to $M_k$ (which exists by Lemma 3.13) onto an isomorphic subgraph of $G_S$.

Combining Lemma 3.12 and Lemma 3.14 we conclude this section.

Proposition 3.15. Let $S = \langle A \mid 1 \to v \rangle$ be an SRS. If $v = b^n$ for some $b \in A$ then $k = 2$ is the maximal value such that $M_k$ is embeddable in $G_S$. Otherwise, $M_k$ is embeddable in $G_S$ for every natural $k$.

3.4 SRSs where $v$ is bordered with $u$

In this section we will show that any system $S = \langle A \mid u \to v \rangle$ where $v$ is bordered with $u$ can be reduced using Adyan reduction [2] into an SRS of the form $\tilde{S} = \langle \tilde{A} \mid 1 \to \tilde{v} \rangle$ such that $M_k$ is embeddable in $G_S$ if and only if it is a embeddable in $G_S$. Therefore, we can use Proposition 3.13 in order to determine whether $M_k$ is a subgraph of $S$. We remark that a similar approach of using Adyan reductions for other one-rule problems was used in [7] and [8, Section 6].

We start with some basic definitions required for the reduction.

Definition 3.16. Let $u \in A^*$ be some word. Its set of self-overlaps is defined by

$$\text{OVL}(u) = \{w \in A^+ \mid \exists x, y \in A^+ \quad u = xw = wy\}.$$ 

The word $u$ is called self-overlap-free if $\text{OVL}(u) = \varnothing$. 

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Let $T$ be a self-overlap-free word over some alphabet $A$. Enumerate all words in $A^*$ without $T$ as a factor by

$$R_1, R_2, \ldots$$

and let $B$ be an infinite set of new letters

$$B = \{b_1, b_2, \ldots\} \quad (B \cap A = \emptyset).$$

Denote the set of words bordered with $T$ by $\text{Bord}_T$ and note that every word $x \in \text{Bord}_T$ can be decomposed uniquely into

$$x = TR_{i_1}TR_{i_2} \cdots TR_{i_k}T.$$

Adyan and Oganesyan define a bijection $\varphi_T : \text{Bord}_T \to B^*$ inductively by

$$\varphi_T(x) = \begin{cases} 
1 & x = T \\
\varphi_T(x_1)b_i & x = x_1R_iT, \quad x_1 \in \text{Bord}_T.
\end{cases}$$

It is important to observe some properties of $\varphi_T$.

**Lemma 3.17.** For every $x \in \text{Bord}_T$ we have that $|\varphi_T(x)| < |x|$.

*Proof.* This can easily be proved by induction since $0 = |\varphi_T(T)| < |T|$ and $1 = |b_i| \leq |R_iT|$ even if $R_i$ is the empty word. \qed

**Lemma 3.18.** Let $u, v \in \text{Bord}_T$ such that $u$ is a prefix of $v$, then $\varphi_T(u)$ is a prefix of $\varphi_T(v)$.

*Proof.* It is clear from the definition of $\varphi_T$ that

$$\varphi_T(Tx_1Tx_2T) = \varphi_T(Tx_1T)\varphi_T(Tx_2T).$$

Therefore, if $u = T\overline{\pi}T$ and $v = T\overline{\pi}TwT$ then

$$\varphi_T(v) = \varphi_T(T\overline{\pi}TwT) = \varphi_T(T\overline{\pi}T)\varphi_T(TwT) = \varphi_T(u)\varphi_T(TwT)$$

so $\varphi_T(u)$ is indeed a prefix of $\varphi_T(v)$. \qed
A dual argument shows that if \( u \) is a suffix of \( v \) then \( \varphi_T(u) \) is a suffix of \( \varphi_T(v) \). Therefore, we obtain:

**Lemma 3.19.** Let \( u, v \in \text{Bord}_T \) be distinct words such that \( v \) is bordered with \( u \) then \( \varphi_T(v) \) is bordered with \( \varphi_T(u) \).

From now on we consider an SRS \( S = \langle A \mid u \to v \rangle \) such that \( u \neq v \) and \( v \) is bordered with \( u \). This implies that \( u \in \text{OVL}(v) \). Denote by \( T \) the shortest element of \( \text{OVL}(u) \) or \( T = u \) if \( \text{OVL}(u) = \emptyset \). Clearly, \( T \) is self-overlap-free and \( T \in \text{OVL}(v) \) so both \( u \) and \( v \) are bordered with \( T \). (A system \( S = \langle A \mid u \to v \rangle \) with this property is called *reducible* in [2].) We make some observations on the existence of a subgraph of \( G_S \) isomorphic to \( M_k \).

**Lemma 3.20.** If \( M_k \) is embeddable in \( G_S \) then it is also isomorphic to a subgraph of \( G_S \) whose vertices are in \( \text{Bord}_T \).

**Proof.** Assume

\[
x \to z \to y
\]

is a reduction in \( G_S \). Note that any word \( x \in A^* \) which contains \( T \) as a factor can be written uniquely as \( x = x'x'' \) where \( x' \in \text{Bord}_T \) and \( x', x'' \) do not contain \( T \) as a factor. Therefore, we can write the above reduction as

\[
x'x'' \to z'z'' \to y'y''.
\]

Since \( u \) and \( v \) are bordered with \( T \), it is clear that

\[
x' = z' = y', \quad x'' = z'' = y''
\]

and

\[
x' \to z' \to y'
\]

is also a reduction. Therefore, if we have \( k \) different reductions

\[
x \to z_1 \to y, \ldots, x \to z_k \to y
\]

there are \( k \) corresponding reductions

\[
x' \to z'_1 \to y'_1, \ldots, x' \to z'_k \to y'_k
\]

such that \( x'_i, y'_i, z'_1, \ldots, z'_k \in \text{Bord}_T \). Since the steps \( x \to z_i \) and \( x \to z_j \) are
carried out at different positions for \( i \neq j \) we know that \( x_i \rightarrow z_i \) and \( x_j \rightarrow z_j \) are carried out in different positions and hence \( z_i \neq z_j \). Therefore, we have a subgraph isomorphic to \( M_k \) such that all the vertices are bordered with \( T \) as required.

\[ \square \]

**Lemma 3.21.** Let \( S = \langle A \mid u \rightarrow v \rangle \) be an SRS such that \( v \) is bordered with \( u \) and let \( T \) be defined as above. Then \( M_k \) is embeddable in \( G_S \) if and only if it is embeddable in \( G_{\hat{S}} \) for \( \hat{S} = \langle B \mid \varphi_T(u) \rightarrow \varphi_T(v) \rangle \).

**Proof.** Recall that \( \varphi_T \) is a bijection \( \varphi_T : \text{Bord}_T \rightarrow B^* \). It is clear that \( \varphi_T^{-1} \) maps any subgraph of \( G_{\hat{S}} \) onto an isomorphic subgraph of \( G_S \). On the other direction, if \( G_S \) has a subgraph isomorphic to \( M_k \), then by Lemma 3.20 it has such subgraph whose vertices are elements of \( \text{Bord}_T \). Therefore, \( \varphi_T \) maps it onto a subgraph of \( G_{\hat{S}} \) isomorphic to \( M_k \) as required.

\[ \square \]

**Lemma 3.22.** Let \( B \) be an alphabet (perhaps infinite) and let \( S = \langle B \mid u \rightarrow v \rangle \) be an SRS. Let \( B' \subseteq B \) be the (finite) set of letters from \( B \) that occur in \( u \) and \( v \) and define \( S' = \langle B' \mid u \rightarrow v \rangle \). Then, \( M_k \) is embeddable in \( G_S \) if and only if it is embeddable in \( G_{S'} \).

**Proof.** It is clear that \( G_{S'} \) is a subgraph of \( G_S \) by inclusion so any subgraph of \( G_{S'} \) is a subgraph of \( G_S \). On the other direction denote by \( \pi \) the standard projection \( \pi : B^* \rightarrow (B')^* \) defined by

\[
\pi(b) = \begin{cases} 
  b & b \in B' \\
  1 & b \notin B'.
\end{cases}
\]

It is clear that if

\[ x \rightarrow y \]

is a reduction of \( G_S \) carried out at position \( i \) then

\[ \pi(x) \rightarrow \pi(y) \]

is also a reduction of \( G_{S'} \). Moreover, the letter at position \( i \) of \( x \) is a letter of \( B' \) (it is the first letter of \( u \)). Therefore, if

\[ x \rightarrow z_1 \rightarrow y, \ldots, x \rightarrow z_k \rightarrow y \]
are $k$ reductions in $G_S$ such that $z_i \neq z_j$ for $i \neq j$ then

$$\pi(x) \to \pi(z_1) \to \pi(y), \ldots, \pi(x) \to \pi(z_k) \to \pi(y)$$

are $k$ reductions in $G_{S'}$ such that $\pi(z_i) \neq \pi(z_j)$ for $i \neq j$. This finishes the proof. \hfill $\Box$

We can now state the main result of this section.

**Proposition 3.23.** Let $S = \langle A \mid u \rightarrow v \rangle$ be an SRS such that $v$ is bordered with $u$ then we can effectively construct another SRS $\tilde{S} = \langle \tilde{A} \mid 1 \rightarrow \tilde{v} \rangle$ such that $M_k$ is embeddable in $G_S$ if and only if it is embeddable in $G_{\tilde{S}}$.

**Proof.** Choose $T$ to be the shortest element of $OVL(u)$ (or $T = u$ if $OVL(u) = \emptyset$). Take $B'$ to be the set of letters from $B$ that occur in $\varphi_T(u)$ and $\varphi_T(v)$. Denote $A_1 = B'$, $u_1 = \varphi_T(u)$, $v_1 = \varphi_T(v)$ and $S_1 = \langle A_1 \mid u_1 \rightarrow v_1 \rangle$. By Lemma 3.21 and Lemma 3.22 $M_k$ is embeddable in $G_S$ if and only if it is embeddable in $G_{S_1}$. There is no reason to expect that $u_1 = 1$. However, by Lemma 3.19 $v_1$ is still bordered with $u_1$ so we choose $T_1$ to be the shortest element of $OVL(u_1)$ or $T_1 = u_1$ if $OVL(u_1) = \emptyset$. Now we can continue this process and construct $S_2 = \langle A_2 \mid u_2 \rightarrow v_2 \rangle$ with $u_2 = \varphi_{T_1}(u_1)$, $v_2 = \varphi_{T_1}(v_1)$ and so on. Since $|\varphi_T(x)| < |x|$ this process must terminate. It will terminate when $u_k = \varphi_{T_{k-1}}(u_{k-1}) = 1$. Then we can define $\tilde{A} = A_k$ and $\tilde{v} = v_k$ and obtain a system $\tilde{S} = \langle \tilde{A} \mid 1 \rightarrow \tilde{v} \rangle$ which satisfy the desired result. \hfill $\Box$

Proposition 3.23 is enough in order to solve the case of this section. Given an SRS $S = \langle A \mid u \rightarrow v \rangle$ such that $v$ is bordered with $u$ we can carry on the procedure described in Proposition 3.23 and obtain an SRS $\tilde{S} = \langle \tilde{A} \mid 1 \rightarrow \tilde{v} \rangle$ which is the case dealt with in Proposition 3.15.

**4 Conclusion**

In conclusion we obtain the following theorem which gives a complete answer to the question of whether $M_k$ is embeddable in the reduction graph of a one-rule SRS.
Theorem 4.1. Let $S = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS such that $u \neq v$, $|u| \leq |v|$ and $|A| > 1$. Then:

1. If $v$ is not bordered with $u$ then $k = 2$ is the maximal value such that $M_k$ is embeddable in $G_S$.

2. If $v$ is bordered with $u$ then we can use Adyan reductions as described in Proposition 3.23 and obtain an SRS $\tilde{S} = \langle \tilde{A} \mid 1 \rightarrow \tilde{v} \rangle$. In this case:
   
   (a) If $\tilde{v} = b^n$ for some $b \in \tilde{A}$ then $k = 2$ is the maximal value such that $M_k$ is embeddable in $G_S$.

   (b) If $\tilde{v} \neq b^n$ for every $b \in \tilde{A}$ then $M_k$ is embeddable in $G_S$ for every $k$.

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