We study the pattern formation in a lattice of coupled phase oscillators with quenched disorder. In the synchronized regime concentric waves can arise, which are induced and increase in regularity by the disorder of the system. Maximal regularity is found at the edge of the synchronization regime. The emergence of the concentric waves is related to the symmetry breaking of the interaction function. An explanation of the numerically observed phenomena is given in a one-dimensional chain of coupled phase oscillators. Scaling properties, describing the target patterns are obtained.

The study of coupled oscillators is one of the fundamental problems in theoretical physics and has led to many insights into the mechanisms of spatio-temporal pattern formation in oscillatory media \([1,2,3]\), with applications in a variety of systems such as arrays of Josephson junctions \([4]\) or oscillating chemical reactions \([5]\). Renewed interest stems from its possible role in many biological systems like cardiac tissue \([6]\), neural systems \([7]\) and ecological systems \([8]\). However, nearly all theoretical studies have been carried out with idealized systems of identical oscillators, whereas not much is known about the dynamics and pattern formation in heterogeneous oscillatory media \([4]\). Here we show that spatial disorder leads to the emergence of quasi regular concentric waves.

Target waves are one of the most prominent patterns in oscillatory media and are usually associated with the presence of local impurities or defects in the system \([1,10,11]\). These pacemakers change the local oscillation frequency and are able to enslav all other oscillators in the medium, which finally results in regular ring waves \([1,12]\). However, the assumption of a discrete set of localized pacemaker regions in an otherwise homogeneous medium is somewhat artificial. Especially biological systems are often under the constraint of large heterogeneity. In such a disordered system no point can be distinguished as a pace-maker and it is not clear whether such a system can sustain highly regular target patterns and where they should originate.

In this paper we investigate the influence of quenched disorder on the pattern formation in a lattice of coupled phase oscillators. As we show the random nature of the medium itself plays a key role in the formation of the patterns. As the disorder in a rather homogeneous synchronized medium is increased we observe the formation of quasi regular target waves, which result from an intricate interplay between the heterogeneity and a symmetry breaking of the coupling function.

We study a system of \(N\) coupled phase oscillators \([1]\)

\[
\dot{\theta}_i = \omega_i + \epsilon \sum_{j \in N_i} \Gamma(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{1}
\]

Here, \(\theta_i\) represents the phase of oscillator \(i\), which is coupled with strength \(\epsilon\) to a set of nearest neighbors \(N_i\) in a one- or two-dimensional lattice. The natural frequencies \(\omega_i\) are fixed in time, uncorrelated and taken from a distribution \(\rho(\omega)\). A scaling of time and a transformation into a rotating reference frame can always be applied so that \(\epsilon = 1\) and the ensemble mean frequency \(\overline{\omega}\) is equal to zero. We refer to the variance \(\sigma^2 = \text{var}(\omega_i)\) of the random frequencies as the disorder of the medium.

The effects of coupling are represented by an interaction function \(\Gamma\) which, in general, is a 2\(\pi\)-periodic function of the phase difference. For weakly coupled, weakly nonlinear oscillators \(\Gamma\) has the universal form \([1,17]\)

\[
\Gamma(\phi) = (\sin(\phi) + \gamma(1 - \cos(\phi))) \tag{2}
\]

This function may be regarded as the first terms in a Fourier-expansion of \(\Gamma(\phi)\) with the constraint that \(\Gamma(0) = 0\). The symmetry breaking parameter \(\gamma\) describes the nonisochronicity of the oscillations \([1]\).

It is well known that for sufficiently small disorder the oscillators eventually become entrained to a common locking frequency \(\Omega\) \([1,2,3]\). Since the oscillators are nonidentical, even in this synchronized state they are usually separated by fixed phase differences. These can sum up over the whole lattice to produce spatio-temporal patterns, which are characterized by a stationary phase profile, \(\theta_1 - \theta_1\). This is demonstrated in Fig. 1 where we have simulated system \([1]\) with interaction \([2]\) in a two-dimensional lattice. If the heterogeneity is small the oscillations across the lattice synchronize to a homogeneous phase profile (Fig. 1d). However, by increasing the disorder we observe the formation of target waves with decreasing wave length (Fig. 1d-c). The emergence of the concentric waves is due to the symmetry breaking in the interaction function. For example, by reducing \(\gamma\) in Eq. (2) the pattern becomes more irregular (Fig. 1d).

It is the counterintuitive observation that the isotropic medium with random frequency distributions of no spatial correlation (see Fig. 1 insets) can generate and sustain very regular wave patterns. Since the equations \([1]\) generally approximate the phase dynamics for coupled limit cycle oscillators this effect is not restricted to phase equations \([1]\). We have observed similar disorder induced target patterns in lattices of a variety of oscillator types, including predator-prey and neural systems, chemical reactions and even chaotic oscillators \([5]\).
In the synchronized state all oscillators rotate with the constant locking frequency \( \dot{\theta}_i = \Omega \), so that system (1) becomes a set of \( N \) equations, which have to be solved self consistently for the phases \( \theta_i \) and \( \Omega \) under some imposed boundary conditions. To determine \( \Omega \) suppose first that the coupling function \( \Gamma \) is fully antisymmetric \( \Gamma(-\phi) = -\Gamma(\phi) \), e.g. \( \gamma = 0 \) in Eq. 2. In this case, by summing up all equations in (1) we obtain \( \Omega = \gamma = 0 \) in the rotating frame. Thus, nontrivial locking frequencies \( \Omega \neq 0 \) only arise if \( \Gamma(\phi) \) has a symmetric part \( \Gamma_S(\phi) = \frac{1}{2} (\Gamma(\phi) + \Gamma(-\phi)) \),

\[
\Omega = \frac{1}{N} \sum_{i,j \in N_i} \Gamma_S(\theta_j - \theta_i). \tag{3}
\]

For any coupling function \( \Gamma \) given a realization of the natural frequencies \( \omega_i \) we ask for the resulting phase profile \( \theta_i \). Note, that the inverse problem is easy to solve: for any regular phase profile \( \theta_i \) we can calculate \( \Omega \) from Eq. 3, which after inserting into Eq. 1 yields the frequencies \( \omega_i \).

Insights into the pattern formation can be gained from a one-dimensional chain of phase oscillators (16, 18). Here we use \( \phi_i = \theta_{i+1} - \theta_i \) for the phase differences between neighboring oscillators. We assume open boundary conditions \( \phi_0 = \phi_N = 0 \). The self consistency problem is trivial for an antisymmetric \( \Gamma \) where \( \Omega = 0 \).

In this case the \( \Gamma(\phi_i) \) simply describe a random walk \( \Gamma(\phi_i) = -\sum_{j=1}^{i} \omega_j \). Thus, for small \( |\phi| \) the phase profile \( \theta_i \) essentially is given by a double summation, i.e. a smoothening, over the disorder \( \omega_i \) (see Fig.2a,b). Note, that synchronization can only be achieved as long as the random walk stays within the range of \( \Gamma \). Thus, with increasing system size \( N \) synchronization becomes more and more unlikely.

The emergence of target waves is connected to a breaking of the coupling symmetry. To explore this we study a unidirectional coupling with respect to \( \phi \)

\[
\Gamma(\phi) = f(\phi) \Theta(\phi), \quad \text{for } |\phi| \ll 1, \tag{5}
\]

with the Heaviside function \( \Theta(\phi) \) and \( f(\phi) > 0 \). Here, the phase of oscillator \( i \) is only influenced from neighboring oscillators which are ahead of \( i \). If the solution are small phase differences we are not concerned about the periodicity of \( \Gamma(\phi) \) as the coupling is only required to be unidirectional close to zero. For open boundaries \( \phi_0 = \phi_N = 0 \) the solution to (1) is given by

\[
\phi_i = \begin{cases} 
    f^{-1}(\Omega - \omega_i), & \text{if } i < m \\
    f^{-1}(\Omega - \omega_{i+1}), & \text{if } i \geq m.
\end{cases} \tag{6}
\]

Here, the index \( m \) is the location of the oscillator with the largest natural frequency, which also sets the synchronization frequency

\[
\Omega = \omega_m = \max_i(\omega_i). \tag{7}
\]

The phase differences (6) are positive to the left of the fastest oscillator, \( i < m \), and negative to the right \( i \geq m \). As a consequence, the phase profile has a tent shape with a mean slope that is given by averaging (6) with respect to the frequency distribution (Fig.2c,d). We call this solution type a quasi-regular concentric wave. This example illustrates, that the asymmetry of the coupling function increases the influence of faster oscillators and effectively creates pacemakers with the potential to entrain the whole system. Note, that the solution (7) is not possible without disorder, i.e. for \( \sigma = 0 \). Further, in contrast to the antisymmetric coupling, here synchronization can be achieved for chains of arbitrary length.

In general, the coupling function \( \Gamma \) will interpolate between the two extremes of fully antisymmetry and unidirectional coupling in the vicinity of zero, e.g. Eq. 2 with \( \gamma \neq 0 \). As shown in Fig.2e this also gives rise to quasi regular concentric waves, very similar to the exactly solvable system Fig.2c. To further investigate the origin of these patterns note that for any given \( \Omega \) system (1) implicitly defines two transfer maps, \( T_{\Omega_1} : \{\phi_{i-1}, \omega_i\} \mapsto \phi_i \) and \( T_{\Omega_1}^{-1} : \{\phi_i, \omega_i\} \mapsto \phi_{i-1}, \) which describe the evolution of the phase differences into the right or the left direction of the chain, respectively. The random frequencies \( \omega_i \) can be seen as noise acting on the map (see Fig.2f)

\[
\phi_i = T_{\Omega_1}(\phi_{i-1}, \omega_i) = \Gamma^{-1}[\Omega - \omega_i - \Gamma(-\phi_{i-1})]. \tag{8}
\]
γ > n solution of the selfconsistency problem (4) is build up when iterating to the left. As a consequence, the general verted for φ is necessarily unstable. These stability properties are in-

trust that the fluctuations ζ are concentrated around φ* (filled circles).

It is easy to see that the breaking of symmetry leads to a pair of fixed points, φ* and −φ*, in the noisy maps

\[ \phi = T_0(\phi, \omega) = \Gamma^{-1} \left( \frac{\Omega}{2} \right) \, . \]  

The transfer map can be linearized at the fixed points so that

\[ T_0(\phi^* + \zeta, \omega) \approx \phi^* + a\zeta - b\omega \]  

with \( a = \frac{\varphi(r(\phi^*))}{\varphi(r(\phi^*)))} \) and \( b = \frac{\varphi(r(\phi^*))}{\varphi(r(\phi^*)))} \). While one fixed point, φ* in the case (2) with γ > 0, is linearly stable (a ≤ 1) the other fixed point is necessarily unstable. These stability properties are in-

verted for \( T^{-1}_0 \). Thus, when iterating to the right of the chair the φi are concentrated around φ* and around −φ* when iterating to the left. As a consequence, the general solution of the selfconsistency problem (1) is build up from two branches around the two fixed points ±φ*, superimposed by autocorrelated fluctuations ζi (see Fig.2)

\[ \phi_i = \pm \phi^* + \zeta_i \, . \]  

After summation this leads to the quasi regular tent shape of the phase profile \( \theta \). In linear approximation the \( \zeta \) describe an AR(1) process

\[ \zeta_i = a\zeta_{i-1} - b\omega_i \, . \]  

We want to stress that the fluctuations ζi are an essential ingredient of the solution. Although the emerging concentric waves seem to be regular the underlying hetero-

genecity of the system does not permit analytical traveling wave solutions \( \theta(t) = \Omega t - k|i - m| \).

The general solution (10) allows for very different phase profiles (see Fig.2). The regularity of the wave pattern depends on the relative influence of the mean slope φ* compared to the fluctuations ζ and can be measured by the quality factor \( Q_\phi = \phi^2/\varphi(\phi_i) \) and the auto-
correlation r of the φi. As demonstrated in Fig.3 both \( Q_\phi \) and r only depend on the product γσ (see below). For γσ → 0 we find \( Q_\phi → 0 \) and r → 1, and the solution is essentially a random walk (see Fig.2a,b). With increasing values of γσ the correlations r are reduced and eventually become negative. Furthermore \( Q_\phi \) increases with the product γσ, and for γ > 1 can rise drastically (see 6). Thus, with increasing disorder of the system we obtain more regular patterns until synchronization is lost.

A straightforward integration of system (11) can be problematic due to the long transients. During the formation of the tent-profile initially several locally synchronized clusters appear, with frequencies \( \Omega_i \) that are deter-

mined by the \( \omega_i \) in the vicinity to the cluster centers. Upon collision these clusters compete, where for γ > 1 a higher-frequency cluster will suppress a slower one (12). The transient time for extinction goes with \( T \sim 1/\Delta \Omega \).

In the disordered system with uniform \( p(\omega) \) many local maxima with nearly identical frequencies exist, so that the frequency differences \( \Delta \Omega \) can become arbitrary small with increasing system size.

Another approach, which also applies for two dimensional lattices, relies on the Cole-Hopf transformation of system (11). Assume that the \( \phi_i \) are small so that it is possible to approximate the coupling function \( \varphi \) around zero by \( \varphi(\phi) = \frac{1}{\gamma} (\gamma \phi - 1) + O(\phi^3) \). After the Cole-Hopf transformation \( \theta_i = \frac{1}{\gamma} \ln q_i \) the synchronized lattice (11) is reduced to a linear system (11) (12).

\[ q_i = \gamma \sigma \eta_i \cdot q_i + \sum_{j \in N_i} (q_j - q_i) \]  

where the random frequencies \( \eta_i = \omega / \sigma \) are of zero mean and variance one and \( E = \gamma \Omega \) is some eigenvalue. System (12) is known as the tight binding model for a particle in a random potential on a lattice (20). The eigenvector \( q_{max} \) corresponding to the largest eigenvalue \( E_{max} \) will, in the re-transformed system of angles, outgrow the con-

tribution of all other eigenvectors to the time dependent solution linearly in time. If the largest eigenvalue is non degenerate, the unique synchronized solution is

\[ \theta_i(t) - \theta_i(t) = \frac{1}{\gamma} \log \left( \frac{q_{max}}{q_i} \right) \, . \]  

Eq. (13) is well defined since the components of \( q_{max} \) do not change sign. Anderson localization theory (20) predicts exponentially decaying localized states with some localization length \( l \), which after applying the reverse...
The approximation breaks down when the correlation Eq. (12) can approximate the transfer map reasonably. The quality factor of the phase differences becomes negative. In this regime an increasing wavelength the effect in two dimensions is strongly depends on both $\gamma \sigma$ and $\gamma$. The noise term $\omega_i$ in (11) can become very small with increasing $\gamma$. This regime, which is not described by the Anderson approximation, can produce very regular concentric waves near the border of desynchronization.

The constructive role of noise has often been studied. It has been shown that in spatially extended excitable systems noise can enhance the pattern formation and for example is able to promote traveling waves. Further, it is well known that disordered systems can synchronize faster. Here, we have investigated the influence of quenched noise on the pattern formation in the oscillatory regime. Whereas local coupling tends to synchronize the oscillators, the imposed disorder tends to desynchronize the array. We have shown that the tension between these two opposing forces can lead to concentric target patterns.

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