AN INITIAL-BOUNDARY VALUE PROBLEM IN A STRIP FOR
TWO-DIMENSIONAL ZAKHAROV–KUZNETSOV–BURGERS
EQUATION

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Abstract. An initial-boundary value problem in a strip with homogeneous
Dirichlet boundary conditions for two-dimensional Zakharov–Kuznetsov–
Burgers equation is considered. Results on global well-posedness and long-
time decay of solutions in $H^s$ for $s \in [0, 2]$ are established.

1. Introduction. Description of main results

The goal of this paper is to study global well-posedness and large-time decay of
solutions for an initial-boundary value problem on a strip $\Sigma = \mathbb{R} \times (0, L) = \{(x, y) : x \in \mathbb{R}, 0 < y < L\} \times \mathbb{R}$ of a given width $L$ for an equation
\[
 u_t + u_{xxx} + u_{yy} + uu_x - \delta(u_{xx} + u_{yy}) = 0, \quad \delta = \text{const} > 0, \tag{1.1}
\]
with an initial condition
\[
 u(0, x, y) = u_0(x, y), \quad (x, y) \in \Sigma, \tag{1.2}
\]
and homogeneous Dirichlet boundary conditions
\[
 u(t, x, 0) = u(t, x, L) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.3}
\]

This equation is referred as Zakharov–Kuznetsov–Burgers equation because it originates from Zakharov–Kuznetsov equation
\[
 u_t + u_{xxx} + uu_x = 0,
\]
supplemented with parabolic terms as in Burgers equation. Zakharov–Kuznetsov equation is a multi-dimensional generalization of Korteweg–de Vries equation
\[
 u_t + uu_x = 0,
\]
and is considered as a model equation for non-linear waves propagating in dispersive media in the preassigned direction $x$ with deformations in the transverse direction $y$. For the first time it was derived in [16] for ion-acoustic waves in magnetized plasma. Equation (1.1) can be treated as as a model equation for non-linear wave processes including both dispersion and dissipation.

The theory of well-posedness for Zakharov–Kuznetsov equation is most developed for the initial value problem and for initial-boundary value problems on domains of a type $I \times \mathbb{R}$, where $I$ is an interval (bounded or unbounded) on the

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variable \( x \), that is the variable \( y \) varies in the whole line (see, for example, references in \([7, 14]\)).

However, it seems more natural from the physical point of view to consider domains, where the variable \( y \) varies in a bounded interval. In fact, it turned out that it is more difficult to study such problems than the aforementioned ones and there are only a few results on the matter.

In \([12]\) an initial-boundary value problem for Zakharov–Kuznetov equation in the strip \( \Sigma \) with periodic boundary conditions is considered and local well-posedness result is established in the spaces \( H^s \) for \( s > 3/2 \). Initial-boundary value problems in the strip \( \Sigma \) with homogeneous boundary conditions of different types: Dirichlet, Neumann or periodic are studied in \([2]\) and results on global existence and uniqueness in classes of weak solutions with power weights at \( +\infty \) are established. For Dirichlet boundary conditions these results are supplemented in \([7]\) with results of exponential long-time decay of small solutions in \( L^2 \) spaces with exponential weights at \( +\infty \). In \([9, 10]\) an initial-boundary value problem in a half-strip \( \mathbb{R}_+ \times (0, L) \) with homogeneous Dirichlet boundary conditions is studied and global well-posedness in spaces \( L^2 \) and \( H^1 \) with exponential weights when \( x \to +\infty \) as well as exponential decay as \( t \to +\infty \) of small solutions are proved. For a bounded rectangle global well-posedness results can be found in \([6, 14]\) and exponential long-time decay of small solutions in \( L^2 \) — in \([6]\).

Of course, the presence of any regularization can improve results on well-posedness and long-time decay of solutions.

In \([7]\) the problem in the strip \( \Sigma \) with initial and boundary conditions \((1.2), (1.3)\) is studied for an equation

\[
\frac{\partial u}{\partial t} + u_{xxx} + u_{xyy} + uu_x - (a_1(x,y)u)_x - (a_2(x,y)u)_y + a_0(x,y)u = 0.
\]

The functions \( a_1, a_2 \) are assumed to be non-negative, that is the parabolic damping can degenerate. Certain results on global existence and uniqueness of weak solutions in \( L^2 \) and \( H^1 \) spaces (possibly weighted at \( +\infty \)) as well as their long-time decay in \( L^2 \) (not only for small solutions) are established when the parabolic damping is effective either at both infinities or only at \( +\infty \) or only at \( -\infty \) or even be absent. For example, for \( u_0 \in L^2(\Sigma) \) and \( a_j \in L^\infty(\Sigma) \) satisfying inequalities

\[
a_2(x,y) \geq \beta_2(x) \geq 0, \quad a_0(x,y) \geq \beta_1(x) \quad \forall (x,y) \in \Sigma,
\]

\[
\pi^2 \beta_2(x) \frac{L^2}{2} + \beta_0(x) \geq \beta = \text{const} > 0 \quad \forall x \in \mathbb{R}
\]

(that is either dissipation or absorption must be effective at every point) there exists a global solution such that

\[
\|u(t, \cdot, \cdot)\|_{L^2(\Sigma)} \leq e^{-\beta t} \|u_0\|_{L^2(\Sigma)} \quad \forall t \geq 0.
\]

If dissipation is effective at both infinities, that is \( a_1, a_2 \geq a = \text{const} > 0 \) for \( |x| \geq R \), the problem is globally well-posed and similar exponential decal in \( L^2(\Sigma) \) is valid without any absorption (\( a_0 \geq 0 \)).

In \([11]\) the same initial-boundary value problem is studied for an equation

\[
\frac{\partial u}{\partial t} + u_{xxx} + u_{xyy} + uu_x - u_{xx} = 0
\]

and results on global well-posedness in certain class of weak and regular solutions (decaying exponentially as \( x \to +\infty \)) as well as their exponential long-time decay in \( L^2 \) (for weak solutions) and \( H^1 \) (for regular ones) norms are obtained.
Note that without any additional damping of Zakharov–Kuznetsov equation the long-time decay of solutions to the considered problem is impossible even in $L_2$ because of the conservation law

$$\int\int_{\Sigma} u^2(t, x, y) \, dxdy = \text{const.}$$

In the present paper results on global well-posedness and long-time decay of solutions to problem (1.1)–(1.3) are established in the spaces $H^s(\Sigma)$ for $s \in [0, 2]$.

Introduce the following notation. For an integer $k \geq 0$ let

$$|D^k \varphi| = \left( \sum_{k_1+k_2=k} (\partial_x^{k_1} \partial_y^{k_2} \varphi)^2 \right)^{1/2}, \quad |D \varphi| = |D^1 \varphi|.$$ 

Let $L_p = L_p(\Sigma)$ for $p \in [1, +\infty]$ , $H^s = H^s(\Sigma)$ , $H_0^s = H_0^s(\Sigma)$ for $s \in \mathbb{R}$.

For any $T_1 < T_2$ let $\Pi_{T_1, T_2} = (T_1, T_2) \times \Sigma$ , let $\Pi_T = \Pi_{0, T}$ , $\Pi = \Pi_{+\infty} = \mathbb{R} \times \Sigma$.

For $s \geq 0$ define a functional space

$$X^s(\Pi_{T_1, T_2}) = C(\Pi_{T_1, T_2}; H^s) \cap L_2(\Pi_{T_1, T_2}; H^{s+1} \cap H_0^1).$$

We construct solutions to the considered problem lying in spaces $X^s(\Pi_T)$ for any $T > 0$ if $s \in [0, 2]$.

The main result of the paper is the following theorem.

**Theorem 1.1.** Let $u_0 \in H^s$ for a certain $s \in [0, 2]$ and, in addition, $u_0(x, 0) = u_0(x, L) \equiv 0$ if $s > 1/2$ and $(y^{-1/2} + (L-y)^{-1/2})u_0 \in L_2$ if $s = 1/2$. Then there exists a unique solution to problem (1.1)–(1.3) $u \in X^s(\Pi_T)$ for any $T > 0$.

Moreover, there exist a constant $\beta(s) > 0$ and a function $\sigma_s(\theta)$ nondecreasing with respect to $\theta \geq 0$ such that

$$\|u(t, \cdot, \cdot)\|_{H^s} \leq \sigma_s(\|u_0\|_{H^s}) e^{-\beta(s)t} \|u_0\|_{H^s} \quad \forall \ t \geq 0,$$

where $s_0 = 0$ if $s \in [0, 1]$ , $s_0 = 1$ if $s \in (1, 2]$.

Compare this result with the one-dimensional case from [4] (in fact, the present paper is inspired by that one), where the initial value problem is considered for damped Korteweg–de Vries–Burgers equation

$$u_t + u_{xxx} + uu_x - u_{xx} + a(x)u = 0.$$ 

The function $a \equiv a_0 + a_1$, where $a_0 = \text{const} > 0$ , $a_1 \in H^1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ for $p \in [1, +\infty)$ and $\|a_1\|_{L_\infty(\mathbb{R})}$ is small in some sense (depending on $p$ and $a_0$). Then for the initial data from $H^s(\mathbb{R})$ , $s \in [0, 3]$ , the considered problem is globally well-posed with exponential decay as $t \to +\infty$ of solutions also in the space $H^s(\mathbb{R})$. Note that the function $a$ is allowed to change sign but, of course, the presence of certain absorption is provided by the constant $a_0$.

Moreover, for pure Korteweg–de Vries–Burgers equation ($a = 0$) exponential decay of solutions to the initial value problem is in general case impossible even in $L_2(\mathbb{R})$ , because it is proved in [1] that for $u_0 \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ a corresponding solution to the initial value problem satisfies an inequality

$$\|u(t, \cdot)\|_{L_2(\mathbb{R})} \leq c(1+t)^{-1/4} \quad \forall t \geq 0$$

and this result is sharp, so here dissipation without absorption ensures only power decay.

The idea that homogeneous Dirichlet boundary conditions in the horizontal strip of a finite width provide internal dissipation for Zakharov–Kuznetsov equation
which, in particular, yields exponential long-time decay of solutions was for the first time used in [9].

Note that it is shown in [3] that the exponential long-time decay of solutions in $L_2(\mathbb{R})$ holds for the initial value problem for damped Korteweg–de Vries equation

$$u_t + uu_{xx} + uu_x + a(x) = 0$$

even in the case of a localized absorption, that is if $a(x) \geq 0 \ \forall x \in \mathbb{R}, \ a(x) \geq a_0 = \text{const} > 0$ for $|x| \geq R$.

Long-time behavior of solutions at $H^2(\mathbb{R})$ level for the initial value problem for generalized Korteweg–de Vries–Burgers equation with constant absorption with the use of the global attractors theory is studied in [5].

Further let $\eta(x)$ denotes a cut-off function, namely, $\eta$ is an infinitely smooth non-decreasing on $\mathbb{R}$ function such that $\eta(x) = 0$ when $x \leq 0$, $\eta(x) = 1$ when $x \geq 1$, $\eta(x) + \eta(1 - x) \equiv 1$.

We omit limits of integration in integrals over the whole strip $\Sigma$.

The following interpolating inequality from [8] is crucial for the study.

**Lemma 1.2.** Let $k$ be natural, $m \in [0, k] - \text{integer}$, $q \in [2, +\infty]$ if $k - m \geq 2$ and $q \in [2, +\infty)$ in other cases. Then there exists a constant $c > 0$ such that for every function $\varphi(x, y) \in H^k$ the following inequality holds

$$\|D^m \varphi\|_{L_q} \leq c \|D^k \varphi\|_{L_q}^{2s} \|\varphi\|_{L_q}^{1-2s} + c \|\varphi\|_{L_q},$$

where $s = \frac{m + 1}{2k} - \frac{1}{kq}$.

The use of nonlinear interpolation in this paper is based on the following result from [15].

**Lemma 1.3** (Tartar). Let $B_j^1 \text{ and } B_j^2$ be Banach spaces such, that $B_j^1 \subset B_j^2$ with continuous inclusion mappings, $j = 1, 2$. Let $B_\theta^3 = (B_\theta^0, B_\theta^1)_{\theta,2}$, $\theta \in (0, 1)$, be a space, constructed by the method of real interpolation. Assume, that an operator $A$ maps $B_1^0$ into $B_1^2$, $B_1^1$ into $B_1^2$, where for any $f, g \in B_1^0$

$$\|Af - A\theta g\|_{B_2^3} \leq c_1(\|f\|_{B_1^0}, \|g\|_{B_1^0})\|f - g\|_{B_1^0},$$

and for any $h \in B_1^1$

$$\|Ah\|_{B_2^2} \leq c_2(\|h\|_{B_1^0})\|h\|_{B_1^1}.$$

Then for any $\theta \in (0, 1)$ the operator $A$ maps $B_\theta^3$ into $B_\theta^2$ and for any $f \in B_\theta^3$

$$\|Af\|_{B_2^2} \leq c(\|f\|_{B_1^0})\|f\|_{B_1^1},$$

where all the functions $c_1, c_2, c$ are nondecreasing with respect to their arguments.

For the decay results we need Steklov inequality in such a form: for $\psi \in H_0^1(0, L)$

$$\int_0^L \psi^2(y) \, dy \leq \frac{L^2}{\pi^2} \int_0^L (\psi'(y))^2 \, dy.$$  \hspace{1cm} (1.6)

The paper is organized as follows. Auxiliary linear problems are considered in Section 2. Section 3 is devoted to global well-posedness of the original problem. Decay of solutions is studied in Section 4.
2. An auxiliary linear problem

In an arbitrary layer $\Pi_T$ consider a linear equation

$$u_t + u_{xxx} + u_{xyy} - \delta(u_{xx} + u_{yy}) = f(t, x, y)$$  \hspace{1cm} (2.1)

and set initial and boundary conditions (1.2), (1.3).

Introduce certain additional function spaces. Let $S(\Sigma)$ be a space of infinitely smooth in $\Sigma = \mathbb{R} \times [0, L]$ functions $\varphi(x, y)$ such that $(1 + |x|)^n |\partial_x^k \partial_y^l \varphi(x, y)| \leq c(n, k, l)$ for any integer non-negative $n, k, l$ and all $(x, y) \in \Sigma$.

**Lemma 2.1.** Let $u_0 \in S(\Sigma)$, $f \in C^\infty([0, T]; S(\Sigma))$ and for any integer $j \geq 0$

$$\partial_y^{2j} u_0|_{y=0} = \partial_y^{2j} f|_{y=L} = 0, \quad \partial_y^{2j} u_0|_{y=0} = \partial_y^{2j} f|_{y=L} = 0.$$

Then there exists a unique solution to problem (2.1), (1.2), (1.3) $u \in C^\infty([0, T]; S(\Sigma))$.

**Proof.** For any natural $l$ let $\psi_l(y) \equiv \sqrt{\frac{2}{L}} \sin \frac{\pi y}{L}$, $\lambda_l = \left(\frac{\pi}{L}\right)^2$. Then a solution to the considered problem can be written as follows:

$$u(t, x, y) = \frac{1}{2\pi} \int_\mathbb{R} \sum_{i=1}^{+\infty} e^{i\xi x} \psi_l(y) \tilde{u}(t, \xi, l) \, d\xi,$$

where

$$\tilde{u}(t, \xi, l) \equiv \tilde{u}_0(\xi, l) e^{(i(\xi^2 + \xi \lambda_l) - \delta(\xi^2 + \lambda_l))t} + \int_0^t \tilde{f}(\tau, \xi, l) e^{(i(\xi^2 + \xi \lambda_l) - \delta(\xi^2 + \lambda_l))\tau} \, d\tau,$$

$$\tilde{u}_0(\xi, l) = \iint e^{-i\xi x} \psi_l(y) u_0(x, y) \, dx dy,$$

$$\tilde{f}(t, \xi, l) = \iint e^{-i\xi x} \psi_l(y) f(t, x, y) \, dx dy,$$

and, obviously, $u \in C^\infty([0, T], S(\Sigma))$. \hfill \square

Next, consider generalized solutions. Let $u_0 \in S'(\Sigma)$, $f \in \left(C^\infty\left([0, T]; S(\Sigma)\right)\right)'$.

**Definition 2.2.** A function $u \in \left(C^\infty\left([0, T]; S(\Sigma)\right)\right)'$ is called a generalized solution to problem (2.1), (1.2), (1.3), if for any function $\phi \in C^\infty\left([0, T]; S(\Sigma)\right)$, such that $\phi|_{t=T} = 0$ and $\phi|_{y=0} = \phi|_{y=L} = 0$, the following equality holds:

$$\langle u, \phi_t + \phi_{xxx} + \phi_{xyy} + \delta \phi_{xx} + \delta \phi_{yy} \rangle + \langle f, \phi \rangle + \langle u_0, \phi|_{t=0} \rangle = 0. \hspace{1cm} (2.2)$$

**Lemma 2.3.** A generalized solution to problem (2.1), (1.2), (1.3) is unique.

**Proof.** The proof is implemented by standard Hölmgren's argument on the basis of Lemma 2.1. \hfill \square

**Lemma 2.4.** Let $u_0 \in L_2$, $f \equiv f_0 + f_1 + f_2$, where $f_0 \in L_1(0, T; L_2)$, $f_1, f_2 \in L_2(\Pi_T)$. Then there exists a (unique) generalized solution to problem (2.1), (1.2), (1.3) $u \in X^0(\Pi_T)$. Moreover, for any $t \in (0, T]$

$$\|u\|_{X^0(\Pi_t)} \leq c(T, \delta) \left[\|u_0\|_{L_2} + \|f_0\|_{L_1(0, t; L_2)} + \|f_1\|_{L_2(\Pi_t)} + \|f_2\|_{L_2(\Pi_t)}\right]. \hspace{1cm} (2.3)$$

and
\[
\iint u^2(t, x, y) \, dx \, dy + 2\delta \int_0^t \iint (u_x^2 + u_y^2) \, dx \, dy \, d\tau = \iint u_0^2 \, dx \, dy \\
+ 2 \int_0^t \iint (f_0 u - f_1 u_x - f_2 u_y) \, dx \, dy \, d\tau. \tag{2.4}
\]

**Proof.** It is sufficient to consider smooth solutions from Lemma 2.1 because of linearity of the problem.

Multiplying (2.1) by \(2u(t, x, y)\) and integrating over \(\Sigma\) we obtain an equality
\[
\frac{d}{dt} \iint u^2 \, dx \, dy + 2\delta \int_0^t \iint (u_x^2 + u_y^2) \, dx \, dy \, d\tau = 2 \int_0^t \iint (f_0 u - f_1 u_x - f_2 u_y) \, dx \, dy, \tag{2.5}
\]
whence (2.3) and (2.4) are immediate. \(\square\)

**Lemma 2.5.** Let \(u_0 \in H_0^1\), \(f \equiv f_0 + f_1\), where \(f_0 \in L_1(0, T; H_0^1)\), \(f_1 \in L_2(\Pi_T)\). Then there exists a (unique) generalized solution to problem (2.1), (1.2), (1.3) \(u \in X^1(\Pi_T)\). Moreover, for any \(t \in (0, T]\)
\[
\|u\|_{X^1(\Pi_t)} \leq c(T, \delta) \left[\|u_0\|_{H^1} + \|f_0\|_{L_1(0, T; H^1)} + \|f_1\|_{L_2(\Pi_t)}\right] \tag{2.6}
\]
and
\[
\iint (u_x^2 + u_y^2) \, dx \, dy + 2\delta \int_0^t \iint (u_x^2 + 2u_x u_y + u_y^2) \, dx \, dy \, d\tau = \iint (u_0^2 + u_{0y}^2) \, dx \, dy \\
+ 2 \int_0^t \iint (f_{0x} u_x + f_{0y} u_y - f_1 u_{xx} - f_1 u_{yy}) \, dx \, dy \, d\tau. \tag{2.7}
\]

**Proof.** In the smooth case multiplying (2.1) by \(-2(u_{xx}(t, x, y) + u_{yy}(t, x, y))\) and integrating over \(\Sigma\) one obtains an equality
\[
\frac{d}{dt} \iint (u_x^2 + u_y^2) \, dx \, dy + 2\delta \int_0^t \iint (u_x^2 + 2u_x u_y + u_y^2) \, dx \, dy \\
= 2 \int_0^t \iint (f_{0x} u_x + f_{0y} u_y) \, dx \, dy - 2 \int_0^t \iint f_1 (u_{xx} + u_{yy}) \, dx \, dy, \tag{2.8}
\]
whence (2.6) and (2.7) follows. \(\square\)

**Lemma 2.6.** Let the hypothesis of Lemma 2.5 be satisfied. Then for the solution to problem (2.1), (1.2), (1.3) \(u \in X^1(\Pi_T)\) for any \(t \in (0, T]\)
\[
-\frac{1}{3} \iint u^3(t, x, y) \, dx \, dy + 2 \int_0^t \iint u u_x(u_{xx} + u_{yy}) \, dx \, dy \, d\tau \\
+ \delta \int_0^t \iint u^2(u_{xx} + u_{yy}) \, dx \, dy \, d\tau = -\frac{1}{3} \iint u_0^3 \, dx \, dy - \int_0^t \iint f u^2 \, dx \, dy \, d\tau. \tag{2.9}
\]

**Proof.** In the smooth case multiplying (2.1) by \(-u^2(t, x, y)\) and integrating one instantly obtains equality (2.9).

In the general case we obtain this equality via closure. Note that by virtue of (1.5) if \(u \in C([0, T]; L_p)\), \(u_x, u_y \in L_2(0, T; L_p)\) for any \(p \in [2, +\infty)\) \tag{2.10}
and this passage to the limit is easily justified. \(\square\)
Lemma 2.7. Let $u_0 \in H^2 \cap H^1_0$, $f \in L_1(0,T;H^2 \cap H^1_0)$. Then there exists a (unique) generalized solution to problem (2.1), (1.2), (1.3) $u \in X^2(\Pi_T)$. Moreover, for any $t \in (0,T]$

$$\|u\|_{X^2(\Pi_T)} \leq c(T,\delta) \left[\|u_0\|_{H^2} + \|f\|_{L_1(0,t;H^2)}\right] \quad (2.11)$$

and

$$\iint (u_{xx}^2 + u_{xy}^2 + u_{yy}^2) \, dx \, dy + 2\delta \int_0^t \iint (u_{xxx}^2 + 2u_{xxy}^2 + 2u_{xyy}^2 + u_{yyy}^2) \, dx \, dy \, d\tau = \iint (u_{0xx}^2 + u_{0xy}^2 + u_{0yy}^2) \, dx \, dy + 2\int_0^t \iint (f_{xx}u_{xx} + f_{xy}u_{xy} + f_{yy}u_{yy}) \, dx \, dy \, d\tau. \quad (2.12)$$

Proof. In the smooth case multiplying (2.1) by $2(u_{xxx}(t,x,y) + u_{xyy}(t,x,y) + u_{yyy}(t,x,y))$ and integrating over $\Sigma$ one obtains an equality

$$\frac{d}{dt} \iint (u_{xx}^2 + u_{xy}^2 + u_{yy}^2) \, dx \, dy + 2\delta \iint (u_{xxx}^2 + 2u_{xxy}^2 + 2u_{xyy}^2 + u_{yyy}^2) \, dx \, dy = 2\iint (f_{xx}u_{xx} + f_{xy}u_{xy} + f_{yy}u_{yy}) \, dx \, dy, \quad (2.13)$$

whence (2.11) and (2.12) follows. Note also that the functions $u_0$ and $f$ can be approximated by corresponding functions satisfying the hypothesis of Lemma 2.1.

\[\square\]

3. Global well-posedness

Definition 3.1. Let $u_0 \in L_2$. A function $u \in L_\infty(0,T;L_2) \cap L_2(0,T;H^1)$ is called a weak solution to problem (1.1)–(1.3) in a layer $\Pi_T$ for some $T > 0$ if for any function $\phi \in L_2(0,T;H^3 \cap H^1_0)$, such that $\phi_t \in L_2(\Pi_T)$ and $\phi|_{t=T} = 0$, the following equality holds:

$$\iint_{\Pi_T} \left[u(\phi_t + \phi_{xxx} + \phi_{xyy}) + \frac{1}{2} u^2 \phi_x - \delta u_x \phi_x - \delta u_y \phi_y\right] \, dx \, dy \, d\tau + \int_0^T u_0 \phi|_{t=0} \, dx \, dy = 0. \quad (3.1)$$

If $u$ is a weak solution to this problem in $\Pi_T$ for any $T > 0$ it is called a weak solution to problem (1.1)–(1.3) in the layer $\Pi$.

Remark 3.2. By virtue of (1.5) for any function $u \in L_\infty(0,T;L_2) \cap L_2(0,T;H^1)$

$$\|u^2\|_{L_2(\Pi_T)} \leq c \left[\int_0^T \left(\iint (|Du|^2 + u^2) \, dx \, dy \int u^2 \, dx \, dy \right) \, dt\right]^{1/2} \leq \|u\|_{L_2(0,T;H^1)} \|u\|_{L_\infty(0,T;L_2)} < \infty, \quad (3.2)$$

therefore $u^2 \phi_x \in L_1(\Pi_T)$.

Theorem 3.3. Let $u_0 \in L_2$. Then problem (1.1)–(1.3) has a unique weak solution $u$ in $\Pi$ such that $u \in X^0(\Pi_T)$ for any $T > 0$. The mapping $u_0 \mapsto u$ is
Lipschitz continuous on any ball in the norm of the mapping from $L^2$ into $X^0(\Pi_T)$. Moreover, the function $\|u(t, \cdot, \cdot)\|_{L^2}^2$ is absolutely continuous for $t \geq 0$ and

$$\frac{d}{dt} \int \int u^2(t, x, y) \, dx \, dy + 2 \delta \int \int \left( u_x^2(t, x, y) + u_y^2(t, x, y) \right) \, dx \, dy = 0 \quad \text{for a.e. } t > 0.$$  

(3.3)

Proof. Consider first an auxiliary initial-boundary value problem in $\Pi$ with initial and boundary conditions (1.2), (1.3) for an equation

$$u_t + u_{xxx} + u_{xxy} - \delta(u_{xx} + u_{yy}) + (g_h(u))_x = 0,$$  

(3.4)

where for $h \in (0,1]$

$$g_h(u) = \int_0^u \left[ \theta \eta(2 - h|\theta|) + \frac{2 \text{sign} \theta}{h} \eta(h|\theta| - 1) \right] \, d\theta.$$  

(3.5)

Note that $g_h(u) = u^2/2$ if $|u| \leq 1/h$, $|g'_h(u)| \leq 2/h \, \forall u \in \mathbb{R}$ and $|g_h(u)| \leq 2|u|$ uniformly with respect to $h$.

We use the contraction principle to prove well-posedness of this problem in the space $X^0(\Pi_T)$ for any $T > 0$.

Fix $T > 0$. For $t_0 = (0,T]$ define a mapping $\Lambda$ on a set $X^0(\Pi_{t_0})$ as follows: $u = \Lambda v, v \in X^0(\Pi_{t_0})$ is a generalized solution to a linear problem

$$u_t + u_{xxx} + u_{xxy} - \delta(u_{xx} + u_{yy}) = -(g_h(v))_x$$  

(3.6)

in $\Pi_{t_0}$ with initial and boundary conditions (1.2), (1.3).

Note that $|g_h(v)| \leq 2|v|/h$ and, therefore, $g_h(u) \in L^2(\Pi_{t_0})$. According to Lemma 2.4 the mapping $\Lambda$ exists. Moreover, for functions $v, \tilde{v} \in X^0(\Pi_{t_0})$

$$\|g_h(v) - g_h(\tilde{v})\|_{L^2(\Pi_{t_0})} \leq \frac{2}{h} \|v - \tilde{v}\|_{L^2(\Pi_{t_0})} \leq \frac{2T^{1/2}}{h} \|v - \tilde{v}\|_{C([0,t_0];L^2)}.$$  

Inequality (2.3) yields that

$$\|\Lambda v - \Lambda \tilde{v}\|_{X^0(\Pi_{t_0})} \leq \frac{c(T, \delta)}{h} T^{1/2} \|v - \tilde{v}\|_{X^0(\Pi_{t_0})},$$

that is for small $t_0$, depending only on $T$, $\delta$ and $h$, the mapping $\Lambda$ is the contraction in $X^0(\Pi_{t_0})$. Since $t_0$ is uniform with respect to $\|u_0\|_{L^2}$ by the standard argument we construct a solution to problem (3.4), (1.2), (1.3) $u_h \in X^0(\Pi_T)$.

Now establish appropriate estimates for functions $u_h$ uniform with respect to $h$. Equality (2.4) (where $f_0 = f_2 \equiv 0, f_1 \equiv -g_h(u)$) provides that

$$\int \int u_h^2(t, x, y) \, dx \, dy + 2 \delta \int_0^T \int \int (u_{hx}^2 + u_{hy}^2) \, dx \, dy \, d\tau = \int \int u_0^2 \, dx \, dy + 2 \int_0^T \int \int g_h(u_h) u_{hx} \, dx \, dy \, d\tau.$$  

(3.7)

Since the last integral is obviously equal to zero it follows from (3.7) that uniformly with respect to $h$

$$\|u_h\|_{X^0(\Pi_T)} \leq c.$$  

(3.8)

Therefore, uniformly with respect to $h$

$$\|g_h(u_h)\|_{L^2(\Pi_T)} \leq \|u_h^2\|_{L^2(\Pi_T)} \leq c.$$  

(3.9)
From estimates (3.8), (3.9) and equation (3.4) itself follows that uniformly with respect to $h$

$$
\|u_{ht}\|_{L^2(0,T;H^{-2})} \leq c.
\tag{3.10}
$$

Inequalities (3.2), (3.8)–(3.10) by the standard argument provide existence of a weak solution $u$ to problem (1.1)–(1.3) in $L^\infty(0,T;L^2) \cap L^2(0,T;H^{-2})$.

Next, Lemma 2.3 (where $f_0 = f_2 \equiv 0$, $f_1 \equiv -u^2/2 \in L_2(\Pi_T)$) provides that (after possible change on a set of the zero measure) $u \in C([0,T];L^2)$ and similarly to (3.7)

$$
\iint u^2(t,x,y) \, dx \, dy + 2\delta \int_0^t \iint |Du|^2 \, dx \, dy \, d\tau = \iint u_0^2 \, dx \, dy.
\tag{3.11}
$$

In particular, equality (3.11) yields that the function $\|u(t,\cdot,\cdot)\|_{L^2}^2$ is absolutely continuous and equality (3.3) is satisfied.

Finally, establish properties of uniqueness and continuous dependence. Let $u$ and $\tilde{u}$ be two solutions in the considered space corresponding to initial data $u_0$ and $\tilde{u}_0$, $v \equiv u - \tilde{u}$, $v_0 \equiv u_0 - \tilde{u}_0$. Then the function $v$ is a weak solution to a linear problem

$$
v_t + v_{xxx} + v_{xxy} - \delta(v_{xx} + v_{yy}) = \frac{1}{2}(\tilde{u}^2 - u^2)_x, \quad v|_{t=0} = v_0, \quad v|_{y=0} = v|_{y=L} = 0.
\tag{3.12}
$$

Obviously the hypothesis of Lemma 2.3 are satisfied for this problem and equality (2.4) provides that

$$
\iint v^2(t,x,y) \, dx \, dy + 2\delta \int_0^t \iint |Dv|^2 \, dx \, dy \, d\tau = \iint v_0^2 \, dx \, dy
+ \int_0^t \iint (u + \tilde{u})vv_x \, dx \, dy \, d\tau.
$$

Here

$$
\iint |uvv_x| \, dx \, dy \leq \left(\iint u^4 \, dx \, dy\right)^{1/4} \left(\iint v^4 \, dx \, dy\right)^{1/4} \left(\iint v_x^2 \, dx \, dy\right)^{1/2}
\leq c \left(\iint (|Du|^2 + u^2) \, dx \, dy\right)^{1/4} \left(\iint u^2 \, dx \, dy\right)^{1/4}
\times \left(\iint (|Dv|^2 + v^2) \, dx \, dy\right)^{3/4} \left(\iint v^2 \, dx \, dy\right)^{1/4}
\leq \varepsilon \iint |Dv|^2 \, dx \, dy + c(\varepsilon) \iint (|Du|^2 + u^2) \, dx \, dy \iint v^2 \, dx \, dy,
$$

where $\varepsilon > 0$ can be chosen arbitrarily small. With use of (3.11) we finish the proof of the theorem.

\[\square\]

**Theorem 3.4.** Let $u_0 \in H^1_0$. Then problem (1.1)–(1.3) has a unique weak solution $u$ in $\Pi$ such that $u \in X^1(\Pi_T)$ for any $T > 0$. The mapping $u_0 \mapsto u$ in Lipschitz continuous on any ball in the norm of the mapping from $H^1$ into $X^1(\Pi_T)$ and

$$
\|u\|_{X^1(\Pi_T)} \leq \kappa_1(T,\|u_0\|_{L^2})\|u_0\|_{H^1},
\tag{3.14}
$$
where the positive function \( \kappa_1 \) is nondecreasing with respect to its arguments. Moreover, the function \( \| Du(\cdot, t, \cdot) \|_{L^2}^2 \) is absolutely continuous for \( t \geq 0 \) and

\[
\frac{d}{dt} \int (u_x^2 + u_y^2) \, dx \, dy + 2 \delta \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) \, dx \, dy = 2 \int uu_x(u_{xx} + u_{yy}) \, dx \, dy \quad \text{for a.e. } t > 0. \tag{3.15}
\]

**Proof.** As in the proof of Theorem 3.3 we first apply the contraction principle but for the original problem. To this end consider an initial-boundary value problem for a linear equation

\[
u_t + u_{xxx} + u_{xxy} - \delta(u_{xx} + u_{yy}) = -vv_x \tag{3.16}
\]

with initial and boundary conditions (1.2), (1.3). Again fix \( T > 0 \). Let \( v \in X^1(\Pi_t) \) and \( u = \Lambda t \) be a solution to this problem from the space \( X^1(\Pi_t) \) also. Note that by virtue of (1.5)

\[
\| vv_x \|_{L^2(\Pi_t)} \leq \left[ \int_0^T \left( \sup_{(x,y) \in \Sigma} v^2 \int v_x^2 \, dx \, dy \right) \, dt \right]^{1/2} \leq c \left[ \int_0^T \left( \int \left( |D^2 v|^2 + v^2 \right) \, dx \, dy \right) \int v^2 \, dx \, dy \right]^{1/2} \sup_{t \in (0,T)} \left( \int v_x^2 \, dx \, dy \right)^{1/2} \leq c t_0^{1/4} \| v \|_{L^2(0,t_0;H^2)}^{1/2} \| v \|_{C(0,t_0;H^1)}^{3/2} \leq c t_0^{1/4} \| v \|_{X^1(\Pi_t)} \tag{3.17}
\]

and similarly

\[
\| vv_x - \bar{v}v_x \|_{L^2(\Pi_t)} \leq c t_0^{1/4} \left( \| v \|_{X^1(\Pi_t)} + \| \bar{v} \|_{X^1(\Pi_t)} \right) \| v - \bar{v} \|_{X^1(\Pi_t)}. \tag{3.18}
\]

In particular, the hypothesis of Lemma 2.5 is satisfied (for \( f_0 \equiv 0, f_1 \equiv -vv_x \)) and, therefore, the mapping \( \Lambda \) exists. Moreover, inequalities (2.6), (3.17), (3.18) provide that

\[
\| \Lambda v \|_{X^1(\Pi_t)} \leq c(T, \delta) \left( \| u_0 \|_{H^1} + t_0^{1/4} \| v \|_{X^1(\Pi_t)} \right), \tag{3.19}
\]

\[
\| \Lambda v - \lambda \bar{v} \|_{X^1(\Pi_t)} \leq c(T, \delta) \left( \| u_0 - \bar{u}_0 \|_{H^1} + t_0^{1/4} \| v \|_{X^1(\Pi_t)} + \| \bar{v} \|_{X^1(\Pi_t)} \right) \| v - \bar{v} \|_{X^1(\Pi_t)}. \tag{3.20}
\]

Local well-posedness of problem (1.1)–(1.3) on the time interval \((0, t_0)\) depending on \( \| u_0 \|_{H^1} \) follows from (3.19), (3.20) by the standard argument.

In order to extend this local solution to an arbitrary time interval establish the corresponding a priori estimate. Let \( u \in X^1(\Pi_T) \) be a solution to problem (1.1)–(1.3). Again apply Lemma 2.5, where \( f_0 \equiv 0, f_1 \equiv -uu_x \). It follows from equality (2.7) that

\[
\int \int \left( u_x^2 + u_y^2 \right) \, dx \, dy + 2 \delta \int_0^T \int \left( u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right) \, dx \, dy \, dt = \int \int \left( u_{x}^2 + u_{y}^2 \right) \, dx \, dy \, dt + 2 \int_0^T \int uu_x(u_{xx} + u_{yy}) \, dx \, dy \, dt. \tag{3.21}
\]
Next, apply Lemma 2.6, then equality (2.9) yields that
\[
-\frac{1}{3} \int \int u^3(t, x, y) \, dxdy + 2 \int_0^t \int \int \partial_x(u_{xx} + u_{yy}) \, dxdydt \\
+ \delta \int_0^t \int \int u^2(u_{xx} + u_{yy}) \, dxdydt = -\frac{1}{3} \int \int u_0^3 \, dxdy + \int_0^t \int \int u_3 \, dxdydt. \tag{3.22}
\]
Summing (3.21) and (3.22) provides an equality
\[
\int \int (u_x^2 + u_y^2 - \frac{1}{3} u^3) \, dxdy + 2\delta \int_0^t \int \int (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) \, dxdydt \\
+ \delta \int_0^t \int \int u^2(u_{xx} + u_{yy}) \, dxdydt = \int \int \left( u_{0,x}^2 + u_{0,y}^2 - \frac{1}{3} u_0^3 \right) \, dxdy. \tag{3.23}
\]
By virtue of (1.5) and (3.11)
\[
\int \int |u|^3 \, dxdy \leq c \left( \int \int (|Du|^2 + u^2) \, dxdy \right)^{1/2} \int \int u^2 \, dxdy \\
\leq \varepsilon \int \int |Du|^2 \, dxdy + c(\varepsilon) \left( \|u_0\|_{L_2}^2 + \|u_0\|_{L_2}^4 \right), \tag{3.24}
\]
\[
\left| \int \int u^2(u_{xx} + u_{yy}) \, dxdy \right| \leq c \left( \int \int |D^2u|^2 \, dxdx \right)^{1/2} \left( \int \int u^4 \, dxdy \right)^{1/2} \\
\leq c_1 \left( \int \int (|D^2u|^2 + u^2) \, dxdx \right)^{3/4} \left( \int \int u^2 \, dxdx \right)^{3/4} \\
\leq \varepsilon \int \int |D^2u|^2 \, dxdy + c(\varepsilon) \left( \|u_0\|_{L_2}^4 + \|u_0\|_{L_2}^6 \right), \tag{3.25}
\]
where $\varepsilon > 0$ can be chosen arbitrarily small. Combining (3.23)–(3.25) yields an inequality
\[
\sup_{t \in (0, T')} \int \int |Du|^2 \, dxdy + \int_0^{T'} \int \int |D^2u|^2 \, dxdydt \leq c(T') \left( 1 + \|u_0\|_{L_2}^2 \right) \|u_0\|_{H^1}^2. \tag{3.26}
\]
This estimate provides the desired global well-posedness and, moreover, estimate (3.14).

Finally, note that for the solution $u \in X^1(\Pi_T)$ similarly to (2.10) $uu_x \in L_2(\Pi_T)$ and, therefore, $\int \int uu_x(u_{xx} + u_{yy}) \, dxdy \in L_1(0, T)$. As a result, it follows from (3.21) that $\|Du|(t, \cdot, \cdot)\|_{L_2^2}$ is absolutely continuous and equality (3.15) holds. \hfill \Box

**Corollary 3.5.** Let $u_0 \in H^s$ for certain $s \in (0, 1)$ and, in addition, $u_0|_{y=0} = u_0|_{y=L} = 0$ if $s > 1/2$ and $(y^{-1/2} + (L - y)^{-1/2})u_0 \in L_2$ if $s = 1/2$. Then problem (1.1)–(1.3) has a unique weak solution $u$ in $\Pi$ such that $u \in X^s(\Pi_T)$ for any $T > 0$. Moreover,
\[
\|u\|_{X^s(\Pi_T)} \leq \kappa_s(T, \|u_0\|_{L_2})\|u_0\|_{H^s}, \tag{3.27}
\]
where the positive function $\kappa_s$ is nondecreasing with respect to its arguments.

**Proof.** Results from [13] ensure that under the hypothesis of the corollary the function $u_0$ belong to spaces which form the real interpolation scale $(\cdot , \cdot )_{0, 2}$. The spaces
Lemma 1.3 and Theorems 3.3 and 3.4.

**Theorem 3.6.** Let $u_0 \in H^2 \cap H^1_0$. Then problem (1.1)–(1.3) has a unique weak solution $u$ in $\Pi$ such that $u \in X^2(\Pi_T)$ for any $T > 0$. The mapping $u_0 \mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping from $H^2$ into $X^2(\Pi_T)$ and

$$\|u\|_{X^2(\Pi_T)} \leq \kappa_\Sigma(T, \|u_0\|_{H^2})\|u_0\|_{H^2},$$

(3.28)

where the positive function $\kappa_\Sigma$ is nondecreasing with respect to its arguments. Moreover, the function $\|D^2u(t, \cdot, \cdot)\|^2_{L_2}$ is absolutely continuous for $t \geq 0$ and

$$\frac{d}{dt} \int \int (u^2_x + u^2_y + u^2_y) \, dx \, dy + 2\delta \int \int (u^2_x + 2u^2_{xy} + 2u^2_{xy} + u^2_{yy}) \, dx \, dy$$

$$= -2 \int \int ((uu_x)_{xx}u_{xx} + (uu_x)_{xy}u_{xy} + (uu_x)_{yy}u_{yy}) \, dx \, dy \quad \text{for a.e. } t > 0.$$  

(3.29)

**Proof.** As in the proof of Theorem 3.4 consider linear initial-boundary value problem (3.16), (1.2), (1.3). For $T > 0$ and $t_0 \in (0, T]$ let $v \in X^2(\Pi_{t_0})$ and $u = \Lambda v$ be a solution to this problem from the space $X^2(\Pi_{t_0})$. In order to apply Lemma 2.7 we have to estimate $f = -vu_x$ in $L_1(0, t_0; H^2)$. For example, by virtue of (1.5)

$$\|uv_{xxx}\|_{L_1(0, t_0; L_2)} \leq \int_0^{t_0} \sup_{(x, y) \in \Sigma} |v| \left( \int \int v^2_{xxx} \, dx \, dy \right)^{1/2} \, dt$$

$$\leq c \int_0^{t_0} \left( \int \int (|D^2v|^2 + v^2) \, dx \, dy \right)^{1/2} \left( \int \int v^2_{xxx} \, dx \, dy \right)^{1/2} \, dt$$

$$\leq c t_0^{1/2} \|v\|_{L_2(0, t_0; H^2)} \|v\|_{L_2(0, t_0; H^2)} \leq c t_0^{1/2} \|v\|_{X^2(\Pi_{t_0})}^2,$$

(3.30)

$$\|v_xv_{xx}\|_{L_1(0, t_0; L_2)} \leq \int_0^{t_0} \sup_{(x, y) \in \Sigma} |v_x| \left( \int \int v^2_{xx} \, dx \, dy \right)^{1/2} \, dt$$

$$\leq c \int_0^{t_0} \left( \int \int (|D^3v|^2 + v^2) \, dx \, dy \right)^{1/2} \left( \int \int v^2_{xx} \, dx \, dy \right)^{1/2} \, dt$$

$$\leq c t_0^{1/2} \|v\|_{L_2(0, t_0; H^2)} \|v\|_{L_2(0, t_0; H^2)} \leq c t_0^{1/2} \|v\|_{X^2(\Pi_{t_0})}^2.$$  

(3.31)

Other terms can be estimated in a similar way and, therefore, the hypothesis of Lemma 2.7 is satisfied and the mapping $\Lambda$ exists. Moreover, inequalities (2.11), (3.30), (3.31) provide that

$$\|\Lambda v\|_{X^2(\Pi_{t_0})} \leq c(T, \delta) \left[ \|u_0\|_{H^2} + t_0^{1/2} \|v\|_{X^2(\Pi_{t_0})}^2 \right].$$

(3.32)

Moreover, one can similarly show that

$$\|\Lambda v - \lambda \Lambda v\|_{X^2(\Pi_{t_0})} \leq c(T, \delta) \left[ \|u_0 - \bar{u}_0\|_{H^2} + t_0^{1/2} (\|v\|_{X^2(\Pi_{t_0})} + \|\bar{v}\|_{X^2(\Pi_{t_0})}) \right].$$

(3.33)

Local well-posedness of problem (1.1)–(1.3) on the time interval $(0, t_0)$ depending on $\|u_0\|_{H^2}$ follows from (3.32), (3.33) by the standard argument.

In order to extend this local solution to an arbitrary time interval establish the corresponding a priori estimate. Let $u \in X^2(\Pi_{t_0})$ be a solution to problem (1.1)–(1.3). Again apply Lemma 2.7, where $f = -uu_x$. It follows from equality (2.12)
that
\[
\iint (u_{xx}^2 + u_{xy}^2 + u_y^2) \, dx \, dy + 2\delta \int_0^{t_0} \iint (u_{xxx}^2 + 2u_{xxy}^2 + 2u_{xxy}^2 + u_{yyy}^2) \, dx \, dy \, dt
\]
\[
= \iint (u_0_{xx}^2 + u_0_{xy}^2 + u_0_{yy}^2) \, dx \, dy
\]
\[
- 2 \int_0^{t_0} \iint ((uu_x)_{xx} u_{xx} + (uu_x)_{xy} u_{xy} + (uu_x)_{yy} u_{yy}) \, dx \, dy \, dt. \tag{3.34}
\]

Here by virtue of (1.5) and (3.14)
\[
\iint |u_{ux} u_{xxx}| \, dx \, dy \leq \left( \iint u^4 \, dx \, dy \right)^{1/4} \left( \iint u_{xx}^2 \, dx \, dy \right)^{1/2}
\]
\[
\leq c_1 \left( \iint (|Du|^2 + |u|^2) \, dx \, dy \right)^{1/2} \left( \iint u_{xx}^2 \, dx \, dy \right)^{1/4} \left( \iint (|D^3 u|^2 + |D^2 u|^2) \, dx \, dy \right)^{3/4}
\]
\[
\leq \varepsilon \iint |D^3 u|^2 \, dx \, dy + c(\varepsilon) \left( \iint (|Du|^2 + |u|^2) \, dx \, dy \right)^{1/2} \iint |D^2 u|^2 \, dx \, dy
\]
\[
\leq \varepsilon \iint |D^3 u|^2 \, dx \, dy + c(\varepsilon, \|u_0\|_{H^1}) \iint |D^2 u|^2 \, dx \, dy, \tag{3.35}
\]
\[
\iint |u_x| u_{xx}^2 \, dx \, dy \leq \left( \iint u_x^2 \, dx \, dy \right)^{1/2} \left( \iint u_{xx}^2 \, dx \, dy \right)^{1/2}
\]
\[
\leq c \left( \iint u_x^2 \, dx \, dy \right)^{1/2} \left( \iint u_{xx}^2 \, dx \, dy \right)^{1/2} \left( \iint (|D^3 u|^2 + |D^2 u|^2) \, dx \, dy \right)^{1/2}
\]
\[
\leq \varepsilon \iint |D^3 u|^2 \, dx \, dy + c(\varepsilon) \iint u_x^2 \, dx \, dy \iint |D^2 u|^2 \, dx \, dy
\]
\[
\leq \varepsilon \iint |D^3 u|^2 \, dx \, dy + c(\varepsilon, \|u_0\|_{H^1}) \iint |D^2 u|^2 \, dx \, dy, \tag{3.36}
\]

where \( \varepsilon > 0 \) can be chosen arbitrarily small. Other terms in the right side of (3.34) are estimated in a similar way. Combining (3.34)–(3.36) yields an inequality
\[
\sup_{t \in (0, T')} \iint |D^2 u|^2 \, dx \, dy + \int_0^{T'} \iint |D^3 u|^2 \, dx \, dy \, dt \leq c(T', \|u_0\|_{H^1}) \|u_0\|_{H^2}^2. \tag{3.37}
\]

This estimate provides the desired global well-posedness and, moreover, estimate (3.28).

Finally, note that for the solution \( u \in X^2(\Pi_T) \) estimates (3.35), (3.36) ensure that \( \iint ((uu_x)_{xx} u_{xx} + (uu_x)_{xy} u_{xy} + (uu_x)_{yy} u_{yy}) \, dx \, dy \in L_1(0, T) \). Therefore, it follows from (3.34) that \( \|D^2 u(t, \cdot, \cdot)\|_{L_2}^2 \) is absolutely continuous and equality (3.29) holds.

\[\Box\]

**Corollary 3.7.** Let \( u_0 \in H^s \cap H_0^1 \) for a certain \( s \in (1, 2) \). Then problem (1.1)–(1.3) has a unique weak solution \( u \) in \( \Pi \) such that \( u \in X^s(\Pi_T) \) for any \( T > 0 \). Moreover,
\[
\|u\|_{X^s(\Pi_T)} \leq \kappa_s(T, \|u_0\|_{H^1}) \|u_0\|_{H^s}, \tag{3.38}
\]

where the positive function \( \kappa_s \) is nondecreasing with respect to its arguments.
Proof. The hypothesis of the corollary provides that the considered spaces for \( u \) form the real interpolation scale \( (\cdot, \cdot)_{p_2} \). The spaces \( X^s(\Pi_T) \) also form the same interpolation scale. Then the corollary succeeds from Lemma 1.3 and Theorems 3.4 and 3.6.

**Corollary 3.8.** Let the hypothesis of Theorem 3.3 be satisfied. Consider the unique weak solution to problem (1.1)–(1.3) \( u \) in \( \Pi \) such that \( u \in X^0(\Pi_T) \) for any \( T > 0 \). Then for any \( T > 0 \) and \( t_0 \in (0, T) \) the function \( u \in X^2(\Pi_{t_0, T}) \).

**Proof.** Since \( u \in L_2(0, T; H^1) \) for any \( t_0 \in (0, T) \) there exists \( t_1 \in (0, t_0) \) such that \( u(t_1, \cdot, \cdot) \in H^1_0 \). Consider the function \( u \) as a weak solution to an initial-boundary value problem in \( \Pi_{t_1, T} \) for equation (1.1) with initial data \( u_0 = u(t_1, \cdot, \cdot) \) and boundary condition (1.3). The hypothesis of Theorem 3.4 are satisfied for this problem, therefore, \( u \in X^1(\Pi_{t_1, T}) \).

Similarly there exists \( t_2 \in (t_1, t_0) \) such that \( u(t_2, \cdot, \cdot) \in H^2 \cap H^1_0 \). Now consider \( u \) as a weak solution to a similar initial-boundary value problem but in \( \Pi_{t_2, T} \), then according to Theorem 3.6 \( u \in X^2(\Pi_{t_2, T}) \).

4. Long-time decay

**Lemma 4.1.** Let \( u_0 \in L_2 \). Then a weak solution to problem (1.3)–(1.3) from the space \( X^0(\Pi_T) \) for any \( T > 0 \) satisfies inequality

\[
\|u(t, \cdot, \cdot)\|_{L_2} \leq e^{-\delta t^2} \|u_0\|_{L_2} \quad \forall t \geq 0. \tag{4.1}
\]

**Proof.** Consider equality (3.3). With the use of inequality (1.6) we derive that

\[
\iint u_y^2 \, dx \, dy \geq \frac{\pi^2}{L^2} \iint u^2 \, dx \, dy \tag{4.2}
\]

and it follows (3.3) that

\[
\frac{d}{dt} \iint u^2 \, dx \, dy + \frac{2\delta^2}{L^2} \iint u^2 \, dx \, dy \leq 0, \tag{4.3}
\]

which yields (4.1).

**Lemma 4.2.** Let \( u_0 \in H^1_0 \). Then a weak solution to problem (1.3)–(1.3) from the space \( X^1(\Pi_T) \) for any \( T > 0 \) satisfies inequality (1.4) for \( s = 1 \).

**Proof.** Consider equality (3.15). By virtue of (1.5)

\[
\left| \iint u_{xx}(u_{xx} + u_{yy}) \, dx \, dy \right| \leq c \sup_{(x, y) \in \Sigma} |u| \left( \iint u_x^2 \, dx \, dy \right)^{1/2} \left( \iint |D^2u|^2 \, dx \, dy \right)^{1/2}
\]

\[
\leq c_1 \iint (|D^2u|^2 + u^2) \, dx \, dy \iint u^2 \, dx \, dy. \tag{4.4}
\]

Choose \( T_1 = T_1(\|u_0\|_{L_2}) > 0 \) such that according to (4.1)

\[
\iint u^2(t, x, y) \, dx \, dy \leq \min \left( \frac{\delta}{2c_1}, \frac{\delta^2}{2c_2L^2} \right) \quad \forall t \geq T_1, \tag{4.5}
\]

where \( c_1 \) is the constant from (4.4). Then summing (3.15), (3.3) and applying (1.6) yields

\[
\frac{d}{dt} \iint (|Du|^2 + u^2) \, dx \, dy + \delta \iint |Du|^2 \, dx \, dy + \frac{\delta^2}{2L^2} \iint u^2 \, dx \, dy \leq 0 \quad \forall t \geq T_1. \tag{4.6}
\]
Since according to (3.14)
\[ \|u(T_1, \cdot, \cdot)\|_{H^1} \leq c(T_1, \|u_0\|_{L^2})\|u_0\|_{H^1} \]  
(4.7)
inequality (4.6) provides the desired result. □

**Lemma 4.3.** Let \( u_0 \in H^2 \cap H^1_0 \). Then a weak solution to problem (1.3)–(1.3) from the space \( X^2(\Pi_T) \) for any \( T > 0 \) satisfies inequality (1.4) for \( s = 2 \).

**Proof.** Consider equality (3.29). Similarly to (3.35), (3.36)
\[
\int \int |u u_{xxx}| \, dx \, dy + \int \int |u_x|^2 \, dx \, dy \\
\leq \varepsilon \int \int |D^3 u|^2 \, dx \, dy + c(\varepsilon) \int \int (|D u|^2 + u^2) \, dx \, dy \int \int |D^2 u|^2 \, dx \, dy,
\]  
where \( \varepsilon > 0 \) can be chosen arbitrarily small. Other terms in the right side of (3.29) are estimated in the same way. Therefore, it follows from this equality that
\[
\frac{d}{dt} \int \int |D^2 u|^2 \, dx \, dy \leq c_2 \int \int (|D u|^2 + u^2) \, dx \, dy \int \int |D^2 u|^2 \, dx \, dy.
\]  
(4.8)
Summing this inequality with (3.3), (3.15) and applying (1.6), (4.4) yields that
\[
\frac{d}{dt} \int \int (|D^2 u|^2 + |D^2 u|^2 + u^2) \, dx \, dy + \delta \int \int (2|D^2 u|^2 + |D u|^2) \, dx \, dy + \frac{\delta \pi^2}{L^2} \int \int u^2 \, dx \, dy \\
\leq c_1 \int \int (|D^2 u|^2 + u^2) \, dx \, dy \int \int u^2 \, dx \, dy \\
+ c_2 \int \int (|D u|^2 + u^2) \, dx \, dy \int \int |D^2 u|^2 \, dx \, dy.
\]  
(4.9)
Choose \( T_2 = T_2(\|u_0\|_{H^1}) \) such that according to (1.4) for \( s = 1 \)
\[
\int \int (|D u|^2 + u^2) \, dx \, dy \leq \min \left( \frac{\delta}{2c_1}, \frac{\delta \pi^2}{2c_1L^2}, \frac{\delta}{2c_2} \right) \quad \forall \ t \geq T_2,
\]
then it follows from (4.9) that
\[
\frac{d}{dt} \int \int (|D^2 u|^2 + |D u|^2 + u^2) \, dx \, dy + \delta \int \int (|D^2 u|^2 + |D u|^2) \, dx \, dy + \frac{\delta \pi^2}{2L^2} \int \int u^2 \, dx \, dy \leq 0 \quad \forall t \geq T_2.
\]  
(4.10)
Since according to (3.28)
\[
\|u(T_2, \cdot, \cdot)\|_{H^2} \leq c(T_2, \|u_0\|_{H^1})\|u_0\|_{H^2}
\]  
(4.11)
inequality (4.10) provides the desired result. □

Now we can finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** It still remains to prove exponential decay of solutions for \( s \in (0, 1) \) and \( s \in (1, 2) \).

Let \( s \in (0, 1) \). By virtue of Corollary 3.5 it is sufficient to prove (1.4) for \( t \geq 1 \). Then Corollary 3.8 yields that \( u \in X^1(\Pi_{1,T}) \) for any \( T > 1 \). We have with use of
whence (1.4) follows. The case \( s \in (1, 2) \) is considered in a similar way with use of (1.4) for \( s = 1 \), \( s = 2 \) and (3.28).

\[ \square \]

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