A finite dimensional proof of the Verlinde formula

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Abstract  We prove two recurrence relations among dimensions

$$D_g(r, d, \omega) := \dim H^0(\mathcal{U}_C, \omega, \Theta_{\mathcal{U}_C, \omega})$$

of spaces of generalized theta functions on the moduli spaces $\mathcal{U}_C, \omega$. By using these recurrence relations, an explicit formula (the Verlinde formula) of $D_g(r, d, \omega)$ is proved (see Theorem 4.3).

Keywords  moduli space, parabolic sheave, generalized theta function, Verlinde formula

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1 Introduction

Let $C$ be a smooth projective curve and $J^d_C$ be the moduli space of rank 1 vector bundles of degree $d$ on $C$ (i.e., the Jacobian of $C$). It is a classical theorem that the space $H^0(J^d_C, \Theta_{J^d_C})$ of theta functions of order $k$ on $J^d_C$ has dimension $k^3$. A natural question is to find a formula of dimension for the space $H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ of generalized theta functions of order $k$ on the moduli space $\mathcal{U}_C$ of semistable vector bundles on $C$ with rank $r$ and degree $d$. It seems impossible for mathematicians to guess such a formula without the help of rational conformal field theories (RCFT) (see [26]). RCFT is defined to be a functor which associates a finite dimensional vector space $V_C(I, \{\mathbf{a}(x)\}_{x \in I})$ to a marked projective curve $(C, I, \{\mathbf{a}(x)\}_{x \in I})$ satisfying certain axioms (A0–A4) (see [3] for the detail). The axioms, in particular the factorization rules (A2 and A4), can be encoded in a finite dimensional Z-algebra (the so-called fusion ring of the theory). An explicit formula (the so-called Verlinde formula) for the dimension of $V_C(I, \{\mathbf{a}(x)\}_{x \in I})$ can be obtained in terms of the characters of the fusion ring (see [3, Proposition 3.3]).

An important example of RCFT was constructed for a Lie algebra $\mathfrak{g}$ in [25] (Wess-Zumino-Witten (WZW)-models) by associating a space $V_C(\mathfrak{g}, I, \{\mathbf{a}(x)\}_{x \in I})$ of conformal blocks to a marked projective curve $(C, I, \{\mathbf{a}(x)\}_{x \in I})$. It is this example that relates RCFT to algebraic geometry when the space of conformal blocks was proved to be the spaces of generalized theta functions on moduli spaces of parabolic $G$-bundles ($\text{Lie}(G) = \mathfrak{g}$) (see [4, 8, 16]). Then the characters of its fusion ring are determined in terms of representations of $\mathfrak{g}$ (see [2] for $\mathfrak{g} = \mathfrak{sl}(r)$, $\mathfrak{sp}(r)$ and [8] for all classical algebras). Thus an explicit formula

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(the Verlinde formula) for the dimension of spaces of generalized theta functions is proved. This kind of proof was called \textit{infinite dimensional proof} in [1].

Let $\mathcal{U}_{C,\omega}$ be moduli spaces of semistable parabolic bundles of rank $r$ and degree $d$ on smooth curves $C$ of genus $g \geq 0$ with parabolic structures determined by $\omega = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in \mathcal{I}})$. It is natural to ask if one can prove an explicit formula of $D_g(r, d, \omega) := \dim H^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$ without using conformal blocks, which is called \textit{finite dimensional proof} in [1]. There exist proofs due to Bertram, Szemes, Thaddeus, Zagier, Donaldson, Witten, Narasimhan and Ramadas for $r = 2$ (see [1] for the survey and references). For $r > 2$, by intersection theory on $\mathcal{SU}(r, \mathcal{L})$ (the moduli space of stable bundles with fixed determinant $\mathcal{L}$), one can compute $\chi(\Theta_{\mathcal{SU}(r, \mathcal{L})})$ in the case when $(r, d) = 1$ and $I = \emptyset$ (see [13,14]), which gives a Verlinde formula since $H^1(\Theta_{\mathcal{SU}(r, \mathcal{L})}) = 0$ ($i > 0$) in this case. Jeffrey [12] generalized this approach to the case of parabolic bundles under the conditions of $(r, d) = 1$ and the weights $\{\bar{a}(x)\}_{x \in \mathcal{I}}$ are very small (both conditions are essentially needed since the paper needs that $\mathcal{U}_{C,\omega} \to \mathcal{U}_{C}(r, d)$ is a flag bundle). When $I \neq \emptyset$, Bismut and Labourie [5] proved that $\chi(\Theta_{\mathcal{U}_{C,\omega}})$ equals the index of a Dirac operator when $k > 0$ is sufficiently large (their result holds for any compact, connected, simple, and simply connected Lie group $G$). As it was pointed out by Meinrenken (see the review of [5]), this result combined with recent vanishing results of [23] (see also [24,27]) gives a new proof of the formula of $\dim H^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$ when $k$ is sufficiently large.

As far as we know, a \textit{finite dimensional proof} of the Verlinde formula remains open for $r > 2$ and $I \neq \emptyset$ (even if $I = \emptyset$, the proof of [14] and [13] do not cover the case of $(r, d) \neq 1$). In fact, even for $r = 2$, we know only one such proof due to Narasimhan and Ramadas [15], which covers the case of parabolic bundles (see also [6,7] for an analytic proof when $g \geq 2$). When $r = 2$, by the result of [15], Ramadas [17] proved a formula of $D_g(2, d, \omega)$ by reducing it to the case $g = 0$ and using a formula of $D_0(2, d, \omega)$. Unfortunately, the formula of $D_0(2, d, \omega)$ was taken from [10] where its proof is not algebraic. In fact, one of our main results is the computation of $D_0(r, d, \omega)$ (i.e., when $C = \mathbb{P}^1$), which is of course trivial when $I = \emptyset$. By using the following recurrence relations (1.1) and (1.2), we are reduced to compute $D_0(r, d, \omega)$ with 3 parabolic points (i.e., $|I| = 3$), which is almost trivial when $r = 2$. But it is nontrivial when $r > 2$ and we are not able to find any reference for such computation. In fact, the general case of $|I| = 3$ is further reduced to the computation of $D_0(r, 0, \{\omega_y, \lambda_y, \lambda_z\})$ by the recurrence relation (1.2) (see Proposition 4.8). Then we have to determine the moduli spaces $\mathcal{U}_{\mathbb{P}^1}(r, 0, \{\omega_y, \lambda_y, \lambda_z\})$ and $\mathcal{U}_{\mathbb{P}^1}(r, 0, \{\omega_y, \lambda_y, \lambda_z\})$ (see Lemma 4.7), which is combinatorially complicated when $r > 2$.

Another main result of this article is a proof of the following recurrence relations of $D_g(r, d, \omega)$.

\textbf{Theorem 1.1} (See Theorems 3.6 and 3.12). \textit{For the partitions $g = g_1 + g_2$ and $I = I_1 \cup I_2$, let $W_k = \{\lambda = (\lambda_1, \ldots, \lambda_r) \mid 0 = \lambda_r \leq \lambda_{r-1} \leq \cdots \leq \lambda_1 \leq k\}$ and}

$$W_k' = \left\{\lambda \in W_k \mid \left( \sum_{x \in I_1} \sum_{i=1}^{t_x} d_i(x) r_i(x) + \sum_{i=1}^{r} \lambda_i \right) \equiv 0 \pmod{r} \right\}.$$  

Then we have the following recurrence relation:

$$D_g(r, d, \omega) = \sum_{\mu} D_{g-1}(r, d, \omega^\mu), \quad (1.1)$$

$$D_g(r, d, \omega) = \sum_{\lambda \in W_k'} D_{g_1}(r, 0, \omega_1^\lambda) \cdot D_{g_2}(r, d, \omega_2^\lambda), \quad (1.2)$$

where $\mu = (\mu_1, \ldots, \mu_r)$ runs through $0 \leq \mu_r \leq \cdots \leq \mu_1 < k$ and $\omega^\mu$, $\omega_1^\lambda$ and $\omega_2^\lambda$ are explicitly determined by $\mu$ and $\lambda$.

The idea is to consider a family $\{C_t\}_{t \in \Delta}$ of curves degenerating to a curve $C_0$ with exactly one node. A factorization theorem

$$H^0(\mathcal{U}_{C_0,\omega_0}, \Theta_{\mathcal{U}_{C_0,\omega_0}}) = \bigoplus_{\mu} H^0(\mathcal{U}_{C_t,\omega_0}, \Theta_{\mathcal{U}_{C_t,\omega_0}})$$
for irreducible $C_0$ was proved in [19] (see [15] for $r = 2$) where $\tilde{C}_0$ is the normalization of $C_0$. Finally, one has to show that $\dim H^0(U_{C_t,\omega_t}, \Theta_{U_{C_t,\omega_t}})$ is independent of $t \in \Delta$, which follows from

$$H^1(U_{C_t,\omega_t}, \Theta_{U_{C_t,\omega_t}}) = 0.$$  

When $C_0$ is irreducible, the vanishing theorem was proved under the assumption that $g \geq 3$ (see [15,19]). Although we have shown in [21] that $\dim H^0(U_{C_t,\omega_t}, \Theta_{U_{C_t,\omega_t}})$ is constant for $t \neq 0$ without using vanishing theorem (see [21, Corollary 4.8]), the vanishing theorems for singular curves $C_0$ are needed in order to show

$$\dim H^0(U_{C_t,\omega_t}, \Theta_{U_{C_t,\omega_t}}) = \dim H^0(U_{C_0,\omega_0}, \Theta_{U_{C_0,\omega_0}}) \quad (\forall t \in \Delta).$$

When $r = 2$, Ramadas [17] proved $H^1(U_{C_t,\omega_t}, \Theta_{U_{C_t,\omega_t}}) = 0$ for $g \geq 0$ and irreducible $C_0$ (thus the recurrence relation (1.1) for $r = 2$). When $C_0 = C_1 \cup C_2$ is the union of two smooth curves, a factorization theorem

$$H^0(U_{C_0,\omega_0}, \Theta_{U_{C_0,\omega_0}}) = \bigoplus_{\mu} H^0(U_{C_1,\omega_{\mu}'}, \Theta_{U_{C_1,\omega_{\mu}'}}) \otimes H^0(U_{C_2,\omega_{\mu}'}, \Theta_{U_{C_2,\omega_{\mu}'}})$$

was proved in [20] for $r \geq 2$. Thus, to prove the recurrence relations (1.1) and (1.2), we need firstly to prove $H^1(U_{C_0,\omega_0}, \Theta_{U_{C_0,\omega_0}}) = 0$ for both cases that $C_0$ is irreducible or reducible. The argument of [17] seems not work for these general cases, and our proof of vanishing theorems uses the main results of [22] that the modulo $p$ reduction of moduli spaces are globally $F$-regular for almost $p$ (such varieties are called of globally $F$-regular type). Moreover, in order to obtain the recurrence relation (1.2), we have to study the behavior of $D_g(r, d, \omega)$ under Hecke transformations, which is one of the technical parts in Section 3.

We describe the content of our article briefly. In Section 2, we recall the notion of globally $F$-regular type varieties and the main results of [22]. In Section 3, we prove firstly the vanishing theorem as follows:

**Theorem 1.2** (See Theorems 3.1, 3.3 and 3.4). When $C$ is smooth, for any ample line bundle $L$ on $U_{C,\omega}$, we have

$$H^i(U_{C,\omega}, L) = 0, \quad \forall i > 0.$$

When $C$ is irreducible with at most one node, we have

$$H^1(U_{C,\omega}, \Theta_{U_{C,\omega}}) = 0,$$

where $\Theta_{U_{C,\omega}}$ is the theta line bundle. When $C$ is reducible with at most one node, for any ample line bundle $L$ on $U_{C,\omega}$, we have

$$H^1(U_{C,\omega}, L) = 0.$$

Then we prove Theorem 1.1, where the recurrence relation (1.1) follows from Theorem 1.2 and the factorization theorem in [19]. But the recurrence relation (1.2) is obtained by using the factorization theorem in [20] and the Hecke transformation. In Section 4, by the recurrence relations (1.1) and (1.2), we reduce the Verlinde formula to the study of the moduli spaces $U_{P^1}(r, 0, \{ \lambda_i \}_{i \in I})$ when $|I| \leq 3$. Then we determine these moduli spaces in Lemma 4.7 and Proposition 4.8, which together with the recurrence relations (1.1) and (1.2) give a self-contained exposition of computation of $D_g(r, d, \omega)$ (see Theorem 4.3 for the detail).

**Theorem 1.3.** For any given data $\omega = (k, \{ \tilde{a}(x), \tilde{a}(x) \}_{x \in I})$, we have

$$D_g(r, d, \omega) = (-1)^{d(r-1)} \left( \frac{k}{r} \right)^g (r(r + k)^{-1})^{g-1} \times \sum_{g} \exp(2\pi i (\frac{d}{r} - \frac{|\omega|}{r(r+k)}) \sum_{i=1}^r v_i) S_\omega(\exp 2\pi i \frac{\bar{v}}{r+k}) \prod_{i<j} (2 \sin \pi \frac{\bar{v}_i - \bar{v}_j}{r+k})^{2(g-1)},$$

where $\bar{v} = (v_1, v_2, \ldots, v_r)$ runs through the integers

$$0 = v_r < \cdots < v_2 < v_1 < r + k.$$
2 Globally $F$-regular type of moduli spaces

Let $X$ be a variety over a perfect field $k$ of $\text{char}(k) = p > 0$,

$$F : X \to X$$

be the Frobenius map and $F^e : X \to X$ be the $e$-th iterate of the Frobenius map. When $X$ is normal, for any (Weil) divisor $D \in \text{Div}(X)$,

$$\mathcal{O}_X(D)(V) = \{f \in K(X) | \text{div}_V(f) + D|_V \geq 0\}, \quad \forall V \subset X$$

is a reflexive sub-sheaf of the constant sheaf $K = K(X)$. In fact, we have

$$\mathcal{O}_X(D) = j_* \mathcal{O}_{X^{\text{sm}}}(D),$$

where $j : X^{\text{sm}} - \to X$ is the open set of smooth points, and $\mathcal{O}_X(D)$ is an invertible sheaf if and only if $D$ is a Cartier divisor.

**Definition 2.1.** A normal variety $X$ over a perfect field is called stably Frobenius $D$-split if $\mathcal{O}_X \to F^e_* \mathcal{O}_X(D)$ is split for some $e > 0$. $X$ is called globally $F$-regular if $X$ is stably Frobenius $D$-split for any effective divisor $D$. A variety $X$ is called Frobenius split if $\mathcal{O}_X \to F_* \mathcal{O}_X$ is split. In particular, any globally $F$-regular variety is Frobenius split.

For any scheme $X$ of finite type over a field $K$ of characteristic zero, there is a finitely generated $\mathbb{Z}$-algebra $A \subset K$ and an $A$-flat scheme

$$X_A \to S = \text{Spec}(A)$$

such that $X_K = X_A \times_S \text{Spec}(K) \cong X$. $X_A \to S = \text{Spec}(A)$ is called an integral model of $X/K$, and a closed fiber $X_s = X_A \times_S \text{Spec}(k(s))$ is called modulo $p$ reduction of $X$, where $p = \text{char}(k(s)) > 0$.

**Definition 2.2.** A variety $X$ over a field of characteristic zero is said of globally $F$-regular type (resp. $F$-split type) if its modulo $p$ reduction of $X$ are globally $F$-regular (resp. $F$-split) for a dense subset of closed points $s \in S$.

An equivalent definition of globally $F$-regular type for a projective variety $X$ is that its modulo $p$ reductions (for a dense set of $p$) are stably Frobenius $D$-split along any effective Cartier divisor $D$, which do not require normality of its modulo $p$ reductions prior to the definition. Projective varieties of globally $F$-regular type have many nice properties and a good vanishing theorem of cohomology.

**Theorem 2.3** (See [18, Corollaries 5.3 and 5.5]). Let $X$ be a projective variety over a field of characteristic zero. If $X$ is of globally $F$-regular type, then we have

1. $X$ is normal, Cohen-Macaulay with rational singularities. If $X$ is $\mathbb{Q}$-Gorenstein, then $X$ has log terminal singularities.
2. For any nef line bundle $\mathcal{L}$ on $X$, we have $H^i(X, \mathcal{L}) = 0$ when $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ whenever $i > 0$.

In [22], we have proved that moduli spaces of parabolic bundles and generalized parabolic sheaves with a fixed determinant on a smooth curve are of globally $F$-regular type. To state it, we recall firstly the notions of moduli spaces of parabolic bundles and generalized parabolic sheaves.

Let $C$ be an irreducible projective curve of genus $g \geq 0$ over an algebraically closed field $K$ of characteristic zero, which has at most one node $x_0 \in C$. Let $I$ be a finite set of smooth points of $C$, and $E$ be a coherent sheaf of rank $r$ and degree $d$ on $C$ (the rank $r(E)$ is defined to be the dimension of $E_\xi$ at the generic point $\xi \in C$, and $d = \chi(E) - r(1 - g)$).

**Definition 2.4.** By a quasi-parabolic structure of $E$ at a smooth point $x \in C$, we mean a choice of flag of the quotients

$$E_x \to Q_{t_0}(E)_x \to \cdots \to Q_1(E)_x \to Q_0(E)_x = 0$$
of the fibre $E_x$, and $n_i(x) = \dim(\ker\{Q_i(E)_x \to Q_{i-1}(E)_x\})$ $(1 \leq i \leq l_x)$ together with
\[ n_{l_x+1}(x) := r(E) - \dim Q_{l_x}(E)_x \]
are called the type of the flags. If, in addition, a sequence of the integers
\[ 0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k \]
are given, we call that $E$ has a parabolic structure of the type
\[ \vec{a}(x) = (n_1(x), n_2(x), \ldots, n_{l_x+1}(x)) \]
and the weight $\vec{a}(x) = (a_1(x), a_2(x), \ldots, a_{l_x+1}(x))$ at $x \in C$.

**Definition 2.5.** For any sub-sheaf $F \subset E$, let $Q_i(E)^F_x \subset Q_i(E)_x$ be the image of $F$ and
\[ n_i^F(x) = \dim(\ker\{Q_i(E)_x^F \to Q_{i-1}(E)_x^F\}) \]
\[ n_{l_x+1}^F(x) := r(F) - \dim Q_{l_x}(E)_x^F \]
are called semistable (resp. stable) for $\omega = (k, \{\vec{a}(x), \vec{a}(x)\}_{x \in I})$ if for any nontrivial $E' \subset E$ such that $E/E'$ is torsion free, one has
\[ \text{par}\chi(E') \leq \frac{\text{bar}\chi(E)}{r} \cdot r(E') \quad \text{(resp. <)}. \]

**Theorem 2.6** (See [21, Theorem 2.13]). There exists a seminormal projective variety
\[ U_{C, \omega} := U_C(r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}), \]
which is the coarse moduli space of $S$-equivalence classes of the semistable parabolic sheaves $E$ of rank $r$ and $\chi(E) = \chi = d + r(1-g)$ with parabolic structures of the type $\{\vec{n}(x)\}_{x \in I}$ and the weights $\{\vec{a}(x)\}_{x \in I}$ at the points $\{x\}_{x \in I}$. If $C$ is smooth, then it is normal, with only rational singularities.

Recall the construction of $U_{C, \omega} = U_C(r, d, \omega)$. Fix a line bundle $O(1) = O_C(c \cdot y)$ on $C$ of $\deg(O(1)) = c$, let $\chi = d + r(1-g)$, $P$ denote the polynomial $P(m) = cm + \chi$, $O_C(-N) = O(1)^{-N}$ and $V = C^P(N)$. Let $Q$ be the Quot scheme of the quotients $V \otimes O_C(-N) \to F \to 0$ (of rank $r$ and degree $d$) on $C$. Thus there is on $C \times Q$ a universal quotient
\[ V \otimes O_C(-N) \to F \to 0. \]
Let $F_x = F \mid_{\{x\} \times Q}$ and $\text{Flag}_{\vec{n}(x)}(F_x) \to Q$ be the relative flag scheme of the type $\vec{n}(x)$. Let
\[ \mathcal{R} = \times_{x \in I} \text{Flag}_{\vec{n}(x)}(F_x) \to Q, \]
on which reductive group $SL(V)$ acts. The data $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I})$, more precisely, the weight $(k, \{\vec{a}(x)\}_{x \in I})$ determines a polarisation
\[ \Theta_{\mathcal{R}, \omega} = (\det R\pi_{\mathcal{R}}^*E)^{-k} \otimes \bigotimes_{x \in I} \left( \bigotimes_{i=1}^{l_x} \det(Q_{\{x\} \times \mathcal{R}, i})^{d_i(x)} \right) \otimes \bigotimes_q \det(E_q)^{\ell} \]
on $\mathcal{R}$ such that the open set $\mathcal{R}_{\mathcal{R}, \omega}^*$ (resp. $\mathcal{R}_{\mathcal{R}, \omega}^*$) of GIT (where GIT stands for geometric invariant theory) semistable (resp. GIT stable) points are precisely the set of semistable (resp. stable) parabolic sheaves.
on $C$ (see [21]), where $E$ is the pullback of $\mathcal{F}$ (under $C \times \mathcal{R} \to C \times \mathcal{Q}$), $\det R\pi_\mathcal{R} E$ is determinant line bundle of cohomology,

$$\mathcal{E}_x \to \mathcal{Q}_{\{x\} \times \mathcal{R}, t_x} \to \mathcal{Q}_{\{x\} \times \mathcal{R}, t_x-1} \to \cdots \to \mathcal{Q}_{\{x\} \times \mathcal{R}, 1} \to 0$$

are universal quotients on $\mathcal{R}$ of the type $\bar{n}(x)$, $d_i(x) = a_{i+1}(x) - a_i(x)$ and

$$\ell := k\chi - \sum_{x \in I} \sum_{i=1}^{t_x} d_i(x)r_i(x), \quad r_i(x) = r(Q_{\{x\} \times \mathcal{R}, i}).$$

Then $\mathcal{U}_{C, \omega}$ is the GIT quotient $\mathcal{R}_{\omega}^ss \xrightarrow{\psi} \mathcal{U}_{C, \omega} := \mathcal{U}_C(r, d, \omega)$ and $\Theta_{\mathcal{R}^{ss}, \omega}$ descends to an ample line bundle $\Theta_{\mathcal{U}_{C, \omega}}$ on $\mathcal{U}_{C, \omega}$ when $\ell$ is an integer.

**Definition 2.7.** When $C$ is a smooth projective curve, let

$$\det : \mathcal{U}_{C, \omega} \to J_0^d, \quad E \mapsto \det(E) := \bigwedge^r E$$

be the determinant map. Then, for any $L \in J_0^d$, the fiber

$$\det^{-1}(L) := \mathcal{U}_{C, \omega}^L$$

is called moduli space of parabolic bundles with a fixed determinant.

**Theorem 2.8** (See [22, Theorem 3.7]). The moduli spaces $\mathcal{U}_{C, \omega}$ are of globally $F$-regular type. If $J_0^d$ of $C$ is of $F$-split type, so is $\mathcal{U}_{C, \omega}$.

When $C$ is irreducible with one node $x_0 \in C \setminus I$, let $\pi : \bar{C} \to C$ be the normalization and $\pi^{-1}(x_0) = \{x_1, x_2\} \subset \bar{C}$.

Then the normalization $\mathcal{P}_\omega$ of $\mathcal{U}_{C, \omega}$ is the moduli space of generalized parabolic sheaves on $\bar{C}$. A *generalized parabolic sheaf* (GPS) $(E, Q)$ of rank $r$ and degree $d$ on $\bar{C}$ consists of a sheaf $\bar{E}$ of degree $d$ on $\bar{C}$, torsion free of rank $r$ outside $\{x_1, x_2\}$ with parabolic structures at the points of $I$ and an $r$-dimensional quotient $E_{x_1} \oplus E_{x_2} \rightarrow Q \to 0$.

**Definition 2.9.** A GPS $(E, Q)$ on an irreducible smooth curve $\bar{C}$ is called *semistable* (resp. *stable*), if for every nontrivial sub-sheaf $E' \subset E$ such that $E/E'$ is torsion free outside $\{x_1, x_2\}$, we have

$$\chi(E') - \dim(Q^{E'}) \leq r(E') \cdot \frac{\chi(E) - \dim(Q)}{r(E)} \quad \text{(resp. <)},$$

where $Q^{E'} = q(E'_{x_1} \oplus E'_{x_2}) \subset Q$.

**Theorem 2.10** (See [21, Theorem 2.24]). For any $\omega = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in I})$, there exists a normal projective variety $\mathcal{P}_\omega$ with at most rational singularities, which is the coarse moduli space of $S$-equivalence classes of semi-stable GPS on $\bar{C}$ with parabolic structures at the points of $I$ given by the data $\omega$.

Recall the construction of $\mathcal{P}_\omega$. Let $\operatorname{Grass}_x(F_{x_1} \oplus F_{x_2}) \to Q$ and

$$\bar{\mathcal{R}} = \operatorname{Grass}_x(F_{x_1} \oplus F_{x_2}) \times_Q \mathcal{R} \xrightarrow{\ell} \mathcal{R},$$

$\omega = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in I})$ determines a polarization, which linearizes the $\operatorname{SL}(V)$-action on $\bar{\mathcal{R}}$, such that the open set $\bar{\mathcal{R}}_{ss}^\omega$ (resp. $\bar{\mathcal{R}}_{ss}^\omega$) of GIT semistable (resp. GIT stable) points are precisely the set of semistable (resp. stable) GPS on $C$ (see [21]). Then $\mathcal{P}_\omega$ is the GIT quotient

$$\bar{\mathcal{R}}_{ss}^\omega \cong \bar{\mathcal{R}}_{ss}^\omega / / \operatorname{SL}(V) := \mathcal{P}_\omega.$$  

**(2.1)**

**Notation 2.11.** Let $\mathcal{H} \subset \bar{\mathcal{R}}$ be the open subscheme parametrizing the generalized parabolic sheaves $E = (E, E_{x_1} \oplus E_{x_2} \rightarrow Q)$ satisfying

1. the torsion $\operatorname{Tor} E$ of $E$ is supported on $\{x_1, x_2\}$ and
   $$q : (\operatorname{Tor} E)_{x_1} \oplus (\operatorname{Tor} E)_{x_2} \rightarrow Q;$$

2. if $N$ is large enough, then $H^1(E(N)(-x - x_1 - x_2)) = 0$ for all $E$ and $x \in C$. 

Then $\mathcal{H}$ is reduced, normal, Gorenstein with at most rational singularities (see [19, Proposition 3.2 and Remark 3.1]). Moreover, for any data $\omega$, we have $\overline{R}_w^{ss} \subset \mathcal{H}$ and by [19, Lemma 5.7], there is a morphism $\text{Det}_H : \mathcal{H} \to J^d_{\overline{C}}$ which extends determinant morphism on the open set $\overline{R}_F \subset \mathcal{H}$ of locally free sheaves, and induces a flat morphism

$$\text{Det} : \mathcal{P}_\omega \to J^d_{\overline{C}}.$$ (2.2)

**Notation 2.12.** For $L \in J^d_{\overline{C}}$, let

$$\mathcal{H}^L = \text{Det}^{-1}(L) \subset \mathcal{H}, \quad \overline{R}_F^L = \text{Det}^{-1}(L) \subset \overline{R}_F, \quad (\overline{R}_w^{ss})^L = \text{Det}^{-1}(L) \subset \overline{R}_w^{ss}.$$ Then $\mathcal{P}_\omega^L = \text{Det}^{-1}(L) \subset \mathcal{P}_\omega$ is the GIT quotient

$$(\overline{R}_w^{ss})^L \xrightarrow{\psi} \mathcal{P}_\omega^L = (\overline{R}_w^{ss})^L / / \text{SL}(V).$$

**Theorem 2.13 (See [22, Theorem 4.7]).** For any $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I})$, the moduli space $\mathcal{P}_\omega^L$ is of globally $F$-regular type.

When $C = C_1 \cup C_2$ is reducible with two smooth irreducible components $C_1$ and $C_2$ of the genus $g_1$ and $g_2$ meeting at only one point $x_0$ (which is the only node of $C$), we fix an ample line bundle $\mathcal{O}(1)$ of degree $c$ on $C$ such that $\text{deg}(\mathcal{O}(1)|_{C_1}) = c_i > 0$ ($i = 1, 2$). For any coherent sheaf $E$, $P(E, n) := \chi(E(n))$ denotes its Hilbert polynomial, which has degree 1. We define the rank of $E$ to be

$$r(E) := \frac{1}{\text{deg}(\mathcal{O}(1))} \cdot \lim_{n \to \infty} \frac{P(E, n)}{n}.$$ Let $r_i$ denote the rank of the restriction of $E$ to $C_i$ ($i = 1, 2$). Then

$$P(E, n) = (c_1 r_1 + c_2 r_2)n + \chi(E), \quad r(E) = \frac{c_1}{c_1 + c_2} r_1 + \frac{c_2}{c_1 + c_2} r_2.$$ We say that $E$ is of rank $r$ on $C$ if $r_1 = r_2 = r$, otherwise it will be said of rank $(r_1, r_2)$. Fix a finite set $I = I_1 \cup I_2$ of smooth points on $C$, where $I_i = \{x \in I \mid x \in C_i\}$ ($i = 1, 2$), and the parabolic data $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x \in I}$ with

$$\ell := \frac{k\chi - \sum_{x \in I} \sum_{i=1}^I d_i(x) r_i(x)}{r}$$

(recall $d_i(x) = a_{i+1}(x) - a_i(x), r_i(x) = n_1(x) + \cdots + n_i(x)$). Let

$$n_j^{\omega} = \frac{1}{k} \left( r \frac{c_j}{c_1 + c_2} \ell + \sum_{x \in I_j} \sum_{i=1}^I d_i(x) r_i(x) \right) \quad (j = 1, 2).$$ (2.3)

**Definition 2.14.** For any coherent sheaf $F$ of rank $(r_1, r_2)$, let

$$m(F) := \frac{r(F) - r_1}{k} \sum_{x \in I_1} a_{i+1}(x) + \frac{r(F) - r_2}{k} \sum_{x \in I_2} a_{i+1}(x).$$

The modified parabolic Euler characteristic and slop of $F$ are

$$\text{par}(F) := \text{par}(F) + m(F), \quad \text{par}(F) := \frac{\text{par}(F)}{r(F)}.$$ A parabolic sheaf $E$ is called semistable (resp. stable) if, for any sub-sheaf $F \subset E$ such that $E/F$ is torsion free, one has, with the induced parabolic structure,

$$\text{par}(F) \leq \frac{\text{par}(E)}{r(F)} r(F) \quad \text{(resp. <)}.$$
Theorem 2.15 (See [20, Theorem 1.1] or [21, Theorem 2.14]). There exists a reduced, seminormal projective scheme

$$U_C := U_C(r, d, \mathcal{O}(1), \{k, \tilde{n}(x), \tilde{a}(x)\}_{x \in I_1 \cup I_2}),$$

which is the coarse moduli space of $S$-equivalence classes of the semistable parabolic sheaves $E$ of rank $r$
and $\chi(E) = \chi = d + r(1 - g)$ with parabolic structures of the type $\{\tilde{n}(x)\}_{x \in I}$ and the weights $\{\tilde{a}(x)\}_{x \in I}$
at points $\{x\}_{x \in I}$. The moduli space $U_C$ has at most $r + 1$ irreducible components.

The normalization of $U_C$ is a moduli space of semistable GPS on $\tilde{C} = C_1 \cup C_2$ with parabolic structures
at points $x \in I$. Recall, a GPS $(E, Q)$ of rank $r$ and degree $d$ on $\tilde{C} = C_1 \cup C_2$ consists of a sheaf $E$ of
degree $d$ on $\tilde{C}$, torsion free of rank $r$ outside $\{x_1, x_2\}$ with parabolic structures at the points of $I$ and an
$r$-dimensional quotient $E_{x_1} \oplus E_{x_2} \xrightarrow{\varphi} Q \to 0$.

Definition 2.16. A GPS $(E, E_{x_1} \oplus E_{x_2} \xrightarrow{\varphi} Q)$ is called semistable (resp. stable), if for every nontrivial
sub-sheaf $E' \subset E$ such that $E/E'$ is torsion free outside $\{x_1, x_2\}$, we have, with the induced parabolic
structures at points $\{x\}_{x \in I},$

$$\text{par} \chi_m(E') - \dim(Q^{E'}) \leq r(E') \cdot \frac{\text{par} \chi_m(E) - \dim(Q)}{r(E)} \quad \text{(resp. <),}$$

where $Q^{E'} = q(E'_{x_1} \oplus E'_{x_2}) \subset Q$.

Theorem 2.17 (See [20, Theorem 2.1] or [21, Theorem 2.26]). For any data

$$\omega = (\{k, \tilde{n}(x), \tilde{a}(x)\}_{x \in I_1 \cup I_2}, \mathcal{O}(1)),
$$
the coarse moduli space $\mathcal{P}_\omega$ of $S$-equivalence classes of semi-stable GPS on $\tilde{C}$ with parabolic structures
at the points of $I$ given by the data $\omega$ is a disjoint union of at most $r + 1$ irreducible, normal projective varieties
$\mathcal{P}_{\chi_1, \chi_2}(\{E_i \cup \chi_i\}, W_i = \mathcal{O}_{C_i}(-N), V_i = \mathcal{O}_{V_i}^{P_i(N)},
\text{where } \mathcal{O}_{C_i}(1) = \mathcal{O}(1)_{C_i}, \text{has degree } c_i. \text{ Consider the Quot schemes } Q_i = \text{Quot}(V_i \otimes W_i, P_i),
\text{the universal quotient } V_i \otimes W_i \to F^1 \to 0 \text{ on } C_i \times Q_i \text{ and the relative flag scheme}$$

$$R_i = \times_{x \in I_i} \text{Flag}_{\tilde{n}(x)}(F^k_{x}) \to Q_i.$$

Let $F = F^1 \oplus F^2$ denote the direct sum of pullbacks of $F^1$ and $F^2$ on

$$\tilde{C} \times (Q_1 \times Q_2) = (C_1 \times Q_1 \times Q_2) \sqcup (C_2 \times Q_1 \times Q_2).$$

Let $E$ be the pullback of $F$ to $\tilde{C} \times (R_1 \times R_2)$, and

$$\rho : \tilde{R} = \text{Grass}_{r}(E_{x_1} \oplus E_{x_2}) \to R = R_1 \times R_2 \to Q = Q_1 \times Q_2.$$

For the given $\omega = (\{k, \tilde{n}(x), \tilde{a}(x)\}_{x \in I_1 \cup I_2}, \mathcal{O}(1)), let \tilde{R}_{\omega}^{ss} \text{(resp. } \tilde{R}_{\omega}^{s} \text{ denote the open set of GIT semistable (resp. GIT stable) points under action of } G = (GL(V_1) \times GL(V_2)) \cap SL(V_1 \oplus V_2) \text{ on } \tilde{R} \text{ respect to the polarization determined by } \omega. \text{ Let } H \subset \tilde{R} \text{ be the open set defined in Notation 2.11. Then for any data } \omega \text{ we have } \tilde{R}_{\omega}^{s} \subset \tilde{R}_{\omega}^{ss} \subset H. \text{ The moduli space in Theorem 2.17 is nothing but the GIT quotient}$$

$$\psi : \tilde{R}_{\omega}^{ss} \to \mathcal{P}_\omega := \tilde{R}_{\omega}^{ss} \cap G.$$

There exists a morphism $\operatorname{Det}_H : H \to J_{C_1}^{d_1} \times J_{C_2}^{d_2}$, which extends

$$\operatorname{Det}_F : H_F \to J_{C_1}^{d_1} \times J_{C_2}^{d_2}, \quad (E, Q) \mapsto (\det(E|_{C_1}), \det(E|_{C_2})).$$
on the open set $\mathcal{H}_F \subset \mathcal{H}$ of GPB (i.e., GPS $(E, Q)$ with $E$ locally free) and induces a flat determinant morphism

$$\text{Det}_P : P_\omega \to J^{d_1}_{C_1} \times J^{d_2}_{C_2}$$

(see [21, p. 46] for detail). For any $L \in J^{d_1}_{C_1} \times J^{d_2}_{C_2}$, let

$$P^L_\omega := \text{Det} P^1_1(L) \subset P_\omega.$$  \hfill (3.2)

**Theorem 2.18** (See [22, Theorem 4.15]). For any data

$$\omega = \{ (k, \{ n(x), \bar{n}(x) \}_{x \in I}, \bar{a}(x) \}_{x \in I}, \mathcal{O}(1))$$

and the integers $\chi_1$ and $\chi_2$ satisfying $\chi_1 + \chi_2 = \chi + r$, $n_j^\omega \leq \chi_j \leq n_j^\omega + r (j = 1, 2)$, let $P^L_\omega$ be the coarse moduli space of $S$-equivalence classes of semi-stable GPS $E = (E_1, E_2)$ on $\tilde{C}$ with fixed determinant $L$, $\chi(E_j) = \chi_j$ and parabolic structures at the points of $I$ given by the data $\omega$. Then $P^L_\omega$ is of globally $F$-regular type.

### 3 Vanishing theorems on moduli spaces and recurrence relations

In this section, we prove vanishing theorems on moduli spaces of parabolic sheaves on curves with at most one node and establish the recurrence relations of dimension of generalized theta functions. As an immediate application of globally $F$-regular type of moduli spaces of parabolic sheaves on a smooth projective curve $C$, we have the following vanishing theorem

**Theorem 3.1.** Let $\mathcal{U}_{C, \omega}$ be the moduli space of semistable parabolic bundles of rank $r$ and degree $d$ on a smooth projective curve $C$ with parabolic structures determined by $\omega = (k, \{ n(x), \bar{n}(x) \}_{x \in I})$. Then

$$H^i(\mathcal{U}_{C, \omega}, \mathcal{L}) = 0, \quad \forall i > 0$$

for any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{C, \omega}$.

**Proof.** Let $\text{Det} : \mathcal{U}_{C, \omega} \to J^d_C$ and $\mathcal{U}^L_{C, \omega} = \text{Det}^{-1}(L)$. Then the morphism

$$J^d_C \times \mathcal{U}^L_{C, \omega} \to \mathcal{U}_{C, \omega}, \quad (L_0, E) \mapsto L_0 \otimes E$$

is an $r^{2g}$-fold cover. Then it is enough to show $H^i(J^d_C \times \mathcal{U}^L_{C, \omega}, \mathcal{L}) = 0$ for $i > 0$ and any ample line bundle $\mathcal{L}$. Since $\mathcal{U}^L_{C, \omega}$ is of globally $F$-regular type, we have $H^1(J^d_C, \mathcal{O}_{\mathcal{U}^L_{C, \omega}}) = 0 (\forall L \in J^d_C)$ by Theorem 2.3(2), which implies that $\mathcal{L}_y$ does not depend on $y \in J^d_C$. Then, by the see-saw lemma, we have $\mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2$, where $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is an ample line bundle on $J^d_C$ (resp. $\mathcal{U}^L_{C, \omega}$). Thus

$$H^i(J^d_C \times \mathcal{U}^L_{C, \omega}, \mathcal{L}) = H^i(J^d_C, \mathcal{L}_1) \otimes H^0(\mathcal{U}^L_{C, \omega}, \mathcal{L}_2) = 0.$$

This completes the proof. \hfill $\square$

For any irreducible curve $C$ with at most one node $x_0 \in C$, there is an algebraic family of ample line bundles $\Theta_{\mathcal{U}_{C, \omega}}$ on $\mathcal{U}_{C, \omega}$ when

$$\ell := k \chi - \sum_{x \in I} \sum_{i=1}^{d_j} d_i(x)r_i(x)$$

is an integer

(see [21, Theorem 3.1]). Then Theorem 3.1 implies that the number

$$D_y(r, d, \omega) = \dim H^0(\mathcal{U}_{C, \omega}, \Theta_{\mathcal{U}_{C, \omega}})$$

is independent of $C$, parabolic points $x \in I$ (of course, depending on the number $|I|$ of parabolic points) and the choice of $\Theta_{\mathcal{U}_{C, \omega}}$ in the algebraic family when $C$ is smooth.
When $C$ has one node $x_0 \in C$, the moduli spaces $\mathcal{U}_{C,\omega}$ are only seminormal (see [19, Theorem 4.2]) and its normalization

$$
\phi : \mathcal{P}_\omega \to \mathcal{U}_{C,\omega}
$$

is the coarse moduli space $\mathcal{P}_\omega$ of $S$-equivalence classes of semi-stable GPS on $\tilde{C} \to C$ with generalized parabolic structures on $\pi^{-1}(x_0) = x_1 + x_2$ and parabolic structures at the points of $\pi^{-1}(I)$ given by the data $\omega$ (see [19, Proposition 2.1] or [20, Proposition 3.1]).

**Theorem 3.3.** For any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{C,\omega}$,

$$
\phi^* : H^1(\mathcal{U}_{C,\omega}, \mathcal{L}) \to H^1(\mathcal{P}_\omega, \phi^* \mathcal{L})
$$

is injective.

**Proof.** Let $\text{Det} : \mathcal{P}_\omega \to J^d_{\omega}$ be the flat morphism defined in (2.2). Then $H^i(\mathcal{P}_\omega, \Theta_{\mathcal{P}_\omega}) = 0$ follows $R^i\text{Det}_* \Theta_{\mathcal{P}_\omega} = 0$ by Theorem 2.13 and $H^i(J^d_{\omega}, \text{Det}_* \Theta_{\mathcal{P}_\omega}) = 0$ by a decomposition of $\text{Det}_* \Theta_{\mathcal{P}_\omega}$ (see [19, Remark 4.2] or a more precise version in [21, Lemma 5.2]).

When $C = C_1 \cup C_2$ is a reducible one nodal curve, we have a stronger vanishing theorem on $\mathcal{U}_{C,\omega}$ and $\mathcal{P}_\omega$.

**Theorem 3.4.** When $C$ is a reducible one nodal curve with two smooth irreducible components, let $\mathcal{P}_\omega$ be the moduli spaces of semi-stable GPS on $\tilde{C}$ with parabolic structures determined by $\omega$. Then, for any ample line bundle $\tilde{\mathcal{L}}$ on $\mathcal{P}_\omega$ and $i > 0$, we have $H^i(\mathcal{P}_\omega, \tilde{\mathcal{L}}) = 0$. In particular,

$$
H^1(\mathcal{U}_{C,\omega}, \mathcal{L}) = 0
$$

holds for any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{C,\omega}$.

**Proof.** By Lemma 3.2, it is enough to show $H^i(\mathcal{P}_\omega, \tilde{\mathcal{L}}) = 0$ for any ample line bundle $\tilde{\mathcal{L}}$ and $i > 0$.

When $C = C_1 \cup C_2$, the moduli space $\mathcal{P}_\omega$ is a disjoint union of

$$
\{ \mathcal{P}_{d_1,d_2} \}_{d_1 + d_2 = d},
$$

where $\mathcal{P}_{d_1,d_2}$ consists of GPS $(E, Q)$ with $d_i = \text{deg}(E |_{C_i})$. It is enough to consider $\mathcal{P}_\omega = \mathcal{P}_{d_1,d_2}$, and thus we have the flat morphism

$$
\text{Det} : \mathcal{P}_\omega \to J^d_{C_1} \times J^d_{C_2}
$$

and the Jacobian $J^0_{\omega}$, which is isomorphic to $J^0_{C_1} \times J^0_{C_2}$ (see [11, p. 250]), acts on $\mathcal{P}_\omega$ by

$$
((E, Q), \mathcal{N}) \mapsto (E \otimes \pi^* \mathcal{N}, Q \otimes \mathcal{N}_x),
$$

where $\pi : \tilde{C} \to C$ is the normalization of $C$. Let $\mathcal{P}^L_{\omega} = \text{Det}^{-1}(L)$ and consider the morphism $f : \mathcal{P}^L_{\omega} \times J^0_{C} \to \mathcal{P}_\omega$, which is a finite morphism (see the proof of Lemma 6.6 in [21] where we figure out a line bundle $\Theta$ on $\mathcal{P}_\omega$ such that its pullback $f^*(\Theta)$ is ample). Thus it is enough to prove the vanishing theorem on $\mathcal{P}^L_{\omega} \times J^0_{C}$, which follows from the same arguments in the proof of Theorem 3.1 by using Theorem 2.18.

**Notation 3.5.** For $\mu = (\mu_1, \ldots, \mu_r)$ with $0 \leq \mu_r \leq \cdots \leq \mu_1 < k$, let

$$
\{ d_i = \mu_{r_i} - \mu_{r_{i+1}} \}_{1 \leq i \leq \ell}
$$

be the subset of nonzero integers in $\{ \mu_i - \mu_{i+1} \}_{i=1, \ldots, r-1}$. We define

$$
\begin{align*}
r_i(x_1) &= r_i, & d_i(x_1) &= d_i, & l_{x_1} &= l, \\
r_i(x_2) &= r - r_{i-1}, & d_i(x_2) &= d_{i-1}, & l_{x_2} &= l,
\end{align*}
$$

for $i = 1, \ldots, \ell$. Theorem 3.1 then holds for $\{ d_i(x_1), d_i(x_2) : 1 \leq i \leq \ell \}$.
and for \( j = 1, 2 \), we set
\[
\bar{a}(x_j) = \left( \mu_r, \mu_r + d_1(x_j), \ldots, \mu_r + \sum_{i=1}^{l_{x_j}-1} d_i(x_j), \mu_r + \sum_{i=1}^{l_{x_j}} d_i(x_j) \right),
\]
\[
\bar{a}(x_j) = (r_1(x_j), r_2(x_j) - r_1(x_j), \ldots, r_{l_{x_j}}(x_j) - r_{l_{x_j}-1}(x_j), r - r_{l_{x_j}}(x_j)).
\]

**Theorem 3.6.** For any \( \omega = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in I}) \) such that:
\[
\ell := \frac{k \chi - \sum_{x \in I} \sum_{i=1}^{l_{x_j}} d_i(x)r_i(x)}{r} \text{ is an integer},
\]
where \( \chi = d + r(1 - g) \), let \( D_g(r, d, \omega) = \dim H^0(U_c, \Theta^{d/g}_{U_c, \omega}) \). Then, for any positive integers \( c_1 \) and \( c_2 \) and the partitions \( I = I_1 \cup I_2 \), \( g = g_1 + g_2 \) such that \( \ell_j = \frac{c_1 \ell}{c_1 + c_2} \) \((j = 1, 2)\) are integers, we have
\[
D_g(r, d, \omega) = \sum_{\mu} D_{g_1}(r, d_1^\mu, \omega_1^\mu) : D_{g_2}(r, d_2^\mu, \omega_2^\mu),
\]
where \( \mu = (\mu_1, \ldots, \mu_r) \) runs through \( 0 \leq \mu_r \leq \cdots \leq \mu_1 < k \) and
\[
\omega_1^\mu = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in I_1 \cup \{x_j\}}), \quad \omega_2^\mu = (k, \{\bar{n}(x), \bar{a}(x)\}_{x \in I_2 \cup \{x_j\}})
\]
with \( \bar{n}(x_1), \bar{a}(x_1) \) \((j = 1, 2)\) determined by \( \mu \) (see Notation 3.5) and
\[
d_1^\mu = n_1^\mu + \frac{1}{k} \sum_{i=1}^{r} \mu_i + r(g_1 - 1), \quad d_2^\mu = n_2^\mu + r - \frac{1}{k} \sum_{i=1}^{r} \mu_i + r(g_2 - 1),
\]
\[
n_j^\mu = \frac{1}{k} \left( r \frac{c_1}{c_1 + c_2} \ell + \sum_{x \in I_j} \sum_{i=1}^{l_{x_j}} d_i(x)r_i(x) \right) \quad (j = 1, 2).
\]

**Proof.** Consider a flat family of projective \(|I|\)-pointed curves \( \mathcal{X} \rightarrow T \) and a relative ample line bundle \( \mathcal{O}_{\mathcal{X}}(1) \) of relative degree \( c \) such that a fiber \( \mathcal{X}_{t_0} := \mathcal{X} \times T \rightarrow \{t_0\} \) is a connected curve with only one node \( x_0 \in \mathcal{X} \) and \( \mathcal{X}_t = C \) \((t \neq t_0)\) are smooth curves with a fiber \( \mathcal{X}_{t_0} = C \). Then one can associate a family of moduli spaces \( \mathcal{M} \rightarrow T \) and a line bundle \( \Theta \) on \( \mathcal{M} \) such that each fiber \( \mathcal{M}_t = \mathcal{U}_{\mathcal{X}_t} \) is the moduli space of semi-stable parabolic sheaves on \( \mathcal{X}_t \) and \( \Theta |_{\mathcal{M}_t} = \Theta_{\mathcal{U}_{\mathcal{X}_t}} \). By degenerating \( C \) to a reducible curve \( X = C_1 \times C_2 \) with \( g(C_i) = g_i \) and choose the relative ample line bundle \( \mathcal{O}_{\mathcal{X}}(1) \) such that \( c_i = \deg(\mathcal{O}_{\mathcal{X}}(1)|_{C_i}) \), by using Theorems 3.1 and 3.4, the recurrence relation (3.4) is exactly the Factorization theorem of [19]. If we degenerate \( C \) to a reducible curve \( X = C_1 \times C_2 \) with \( g(C_i) = g_i \) and choose the relative ample line bundle \( \mathcal{O}_{\mathcal{X}}(1) \) such that \( c_i = \deg(\mathcal{O}_{\mathcal{X}}(1)|_{C_i}) \), by using Theorems 3.1 and 3.4, the recurrence relation (3.4) is exactly the Factorization theorem of [20].

In the recurrence relation (3.4), degree \( d_1^\mu \) varies with \( \mu \) and \( D_{g_1}(r, d_1^\mu, \omega_1^\mu) \) makes sense only when \( d_1^\mu \) is an integer, which are not convenient for applications. To remedy it, we are going to study the behavior of \( D_g(r, d, \omega) \) under the Hecke transformation.

Given a parabolic bundle \( E \) with the quasi-parabolic structure
\[
E_z \rightarrow Q_{t_1}(E)_z \rightarrow \cdots \rightarrow Q_1(E)_z \rightarrow Q_0(E)_z = 0
\]
of the type \( \bar{n}(z) = (n_1(z), \ldots, n_{l_z}(z)) \) at \( z \in I \) and the weights
\[
0 = a_1(z) < a_2(z) < \cdots < a_{l_z}(z) < k,
\]
let \( F_j(E)_z = \ker\{E_z \rightarrow Q_j(E)_z\} \) and \( E' = \ker\{E \rightarrow Q_1(E)_z\} \). Then, at \( z \in I \), \( E' \) has a natural quasi-parabolic structure
\[
E'_z \rightarrow F_1(E)_z \rightarrow Q_{l_z-1}(E')_z \rightarrow \cdots \rightarrow Q_1(E')_z \rightarrow 0
\]
(3.5)
of the type \( \vec{n}'(z) = (n'_1(z), \ldots, n'_{i+1}(z)) \) = \( (n_2(z), \ldots, n_{i+1}(z), n_1(z)) \), where

\[ Q_i(E'_z) \subset Q_{i+1}(E)_z \]

is the image of \( F_1(E)_z \) under \( E_z \to Q_{i+1}(E)_z \). It is easy to see

\[ Q_i(E'_z) \cong F_1(E)_z/F_{i+1}(E)_z. \]

**Definition 3.7.** The parabolic bundle \( E' \) with the given weight

\[ 0 = a'_1(z) < \cdots < a'_{i+1}(z) < k \]

is called the Hecke transformation of the parabolic bundle \( E \) at \( z \in I \), where \( a'_{i+1}(z) = k - a_2(z) \) and \( a'_i(z) = a_i(z) - a_2(z) \) for \( 2 \leq i \leq l_z \).

**Lemma 3.8.** The parabolic bundle \( E' \) is semistable (resp. stable) if and only if \( E \) is semistable (resp. stable).

**Proof.** \( E' \) is defined by the exact sequence of the sheaves

\[ 0 \to E' \xrightarrow{\iota} E \xrightarrow{\delta} Q_1(E)_z \to 0 \]

such that \( E_z \xrightarrow{\delta} Q_1(E)_z \) is the surjective homomorphism

\[ E_z \to Q_{l_z}(E)_z \to \cdots \to Q_1(E)_z. \]

For any sub-bundle \( F \subset E \), let \( Q_i(E)_z \subset Q_{l_z}(E)_z \) be the image of \( F_z \subset E_z \) under \( E_z \to Q_{l_z}(E)_z \to \cdots \to Q_1(E)_z \), and the sub-sheaf \( F' \subset E' \) is defined by the exact sequence

\[ 0 \to F' \xrightarrow{\iota} F \xrightarrow{\delta} Q_1(E)_z \to 0. \]

Let \( Q_i(E')_{z'} \subset F_1(E)_z \) and \( Q_i(E')_{z'} \subset Q_1(E')_z \) (\( 1 \leq i \leq l_z \)) be the image of \( F'_z \subset E'_z \) under the quotient maps \( E'_z \to F_1(E)_z \) and \( E'_z \to Q_1(E)_z \), in (3.5). Since \( Q_i(E')_{z'} = \ker \{ F_z \xrightarrow{\delta} Q_1(E)_z \} \),

\[ \ker \{ Q_i(E')_{z'} \to Q_{i-1}(E')_{z'} \} = \ker \{ Q_{i+1}(E)_{z'} \to Q_i(E)_{z'} \} \]

for \( 1 \leq i \leq l_z \). In particular, \( n_i^{F'}(z) = n_{i+1}(z) \), \( n_i^{F'}(z) = n_i^{F}(z) \),

\[ \param(\chi'(F')) = \param(\chi(F)) - \frac{r_1}{k} a_2(z), \quad \param(\chi'(E')) = \param(\chi(E)) - \frac{r}{k} a_2(z). \]

Thus \( \param(\mu(F')) - \param(\mu(E')) = \param(\mu(F)) - \param(\mu(E)) \), which proves the lemma. \( \square \)

**Lemma 3.9.** For the parabolic data \( \omega = (k, \{ \vec{n}(x), \vec{a}(x) \}_{x \in I}) \), let

\[ \omega' = (k, \{ \vec{n}(x), \vec{a}(x) \}_{x \neq z \in I} \cup \{ \vec{a}'(z), \vec{n}'(z) \}) \]

where \( \vec{n}'(z) = (n'_1(z), \ldots, n'_{i+1}(z)) = (n_2(z), \ldots, n_{i+1}(z), n_1(z)) \),

\[ \vec{a}'(z) = (0, a'_2(z), \ldots, a'_{i+1}(z)), \quad a'_{i+1}(z) = k - a_2(z) + a_1(z) \]

and \( a'_i(z) = a_i(z) - a_2(z) + a_1(z) \) for \( 2 \leq i \leq l_z \). Then

\[ D_g(r, d, \omega) = D_g(r, d - n_1(z), \omega'). \]

**Proof.** One can also define the Hecke transformation of a family of parabolic bundles (flat family yielding flat family, and preserve semistability). Thus, for \( z \in I \), we have a morphism

\[ H_z : U_{C, \omega} = U_C(r, d, \omega) \to U_C(r, d - n_1(z), \omega') = U_{C, \omega}. \]
such that $H^*_\mathcal{U}_{\mathcal{C}^\omega} = \Theta_{\mathcal{U}_{\mathcal{C}^\omega}}$. In fact, $H_z$ is an isomorphism. For any parabolic bundle $E'$ with the quasi-parabolic structure of the type $\vec{a}^*(z)$, let

$$F_i(E')_z = \ker\{E'_z \rightarrow Q_i(E')_z\} \quad (1 \leq i \leq l_z).$$

Then there exists a bundle $E$ and a homomorphism $E' \xrightarrow{\tau} E$ such that $F_{l_z}(E')_z = \ker\{E'_z \xrightarrow{\tau} E\}$. Let $F_1(E)_z = i_z(E'_z) \subset E_z$ and

$$F_{i+1}(E)_z = i_z(F_i(E')_z).$$

Then the quasi-parabolic structure of $E$ at $z \in I$ given by

$$0 = F_{i+1}(E)_z \subset F_i(E)_z \subset F_{i-1}(E)_z \subset \cdots \subset F_1(E)_z \subset E_z$$

is of the type $\vec{a}(z) = (n_1(z), \ldots, n_{l_z+1}(z))$ and the weights $\vec{a}(z)$ determined by $\vec{a}''(z)$ (let $a_1(z) = 0$, $a_2(z) = k - a'_{l_z+1}(z)$ and $a_{l_z+1}(z) = a'_z(z) + k - a'_{l_z+1}(z)$ for $2 \leq i \leq l_z$). The construction can be applied to a family of parabolic bundles, which induces $H_{l_z}^{-1}$.

In the following Lemma 3.10, the data $\omega''$ has a point $z \in I$ with $a_{l_z+1}(z) - a_1(z) = k$. The moduli spaces $\mathcal{U}_{\mathcal{C}^\omega}$ and theta line bundles $\Theta_{\mathcal{U}_{\mathcal{C}^\omega}}$ are constructed in [21] for such $\omega''$, and the vanishing theorems can be generalized to this case. We remind that the semistable parabolic sheaf $E$ may have torsion at $z \in C$ and stable parabolic bundles are not GIT stable in this case.

**Lemma 3.10.** For $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in \mathbb{I}})$, if $n_1(z) > 1$, let

$$\omega'' = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \notin \mathbb{I}} \cup \{\vec{a}''(z), \vec{n}''(z)\}),$$

where $\vec{a}''(z) = (0, a_2(z), \ldots, a_{l_z+1}(z), k)$ (we assume $a_1(z) = 0$) and $\vec{n}''(z) = (n_1(z) - m, n_2(z), \ldots, n_{l_z+1}(z), m), 1 < m < n_1(z)$.

Then $D_g\{r, d - n_1(z), \omega'\} = D_g\{r, d - m, \omega''\}$.

**Proof.** For an $\omega''$-semistable parabolic sheaf $E = (E, Q_{l_z+1}(E)_z)$, its quasi-parabolic structure at $z \in I$ is given by $Q_{l_z+1}(E)_z$:

$$E_z \rightarrow Q_{l_z+1}(E)_z \rightarrow Q_{l_z}(E)_z \rightarrow \cdots \rightarrow Q_1(E)_z \rightarrow Q_0(E)_z = 0,$$

where $n_i''(z) = \dim Q_i(E)_z = n_1(z) - m, r - \dim Q_{l_z+1}(E)_z = m,$ and $n_i''(z) = \dim \ker(Q_i(E)_z \rightarrow Q_{i-1}(E)_z) = n_i(z) \quad (2 \leq i \leq l_z + 1)$.

Let $E' = \ker\{E \rightarrow Q_1(E)_z\}$. Then $E'$ has the quasi-parabolic structure

$$Q_{l_z+1}(E')_z : E'_z \rightarrow Q_{l_z}(E')_z \rightarrow \cdots \rightarrow Q_1(E')_z \rightarrow Q_0(E')_z = 0,$$

of the type $\vec{n}''(z) = (n_2(z), \ldots, n_{l_z+1}(z), n_1(z))$ at $z \in I$, where

$$Q_i(E')_z = \ker\{Q_{i+1}(E)_z \rightarrow Q_i(E)_z\} \quad (1 \leq i \leq l_z).$$

Since $E$ is $\omega''$-semistable, by [21, Remark 2.4], $E$ is torsion free outside $z \in I$ and quotient homomorphism injects its torsion $E''_z$ to $Q_1(E)_z$. Thus $E'$ must be a vector bundle. Then we show that $E' = (E', Q_{l_z+1}(E')_z)$ is an $\omega'$-semistable parabolic bundle if and only if $E$ is an $\omega''$-semistable parabolic sheaf. In fact, we have

$$\chi(E) + \frac{1}{k} \sum_{i=1}^{l_z+2} a''_i(z)n''_i(z) = r \frac{a_2(z) + \chi(E') + \frac{1}{k} \sum_{i=1}^{l_z+1} a'_i(z)n'_i(z),$$

which implies that $\text{par}_{\omega''}(E) = a''_2(z) \chi(E')$. For any sub-sheaf $F' \subset E$ with $E/F$ torsion free, let $F' \subset E'$ be the sub-bundle such that

$$0 \rightarrow F' \rightarrow F \rightarrow Q_1(E)^F_z \rightarrow 0$$
is an exact sequence of sheaves (every sub-bundle of $E'$ can be obtained by this way). Then
\[
\text{par}_{\omega'} \chi(F) = \text{par}_{\omega'} \chi(F') + \frac{a_2(z)}{k} r(F')
\]
(if $F$ is the torsion sub-sheaf, we have $\text{par}_{\omega'} \chi(F) = 0$ since $a_1'(z) = 0$ and $n_{t+1}F(z)$ is defined to be $r(F) - \dim QT_z(E)\ell_z$). The construction can be applied to a family of parabolic sheaves, which induces
\[
\mathcal{U}_C, \omega' = \mathcal{U}_C(r, d - m, \omega') \xrightarrow{\varphi} \mathcal{U}_C, \omega = \mathcal{U}_C(r, d - n_1(z), \omega')
\]
such that $\varphi^* \Theta_{\mathcal{U}_C, \omega'} = \Theta_{\mathcal{U}_C, \omega}$. To show that $\varphi$ is a bijective morphism, which implies that $\varphi$ is an isomorphism since $\mathcal{U}_C, \omega'$ and $\mathcal{U}_C, \omega'$ are normal projective varieties, we note
\[
Q_1(E')_z = Q_{t+1}(E)E'_z \subset Q_{t+1}(E)_z
\]
is the image of $E' \to E \to Q_{t+1}(E)_z$. Then $(E', Q_\bullet(E')_z)$ can be considered as a parabolic sub-sheaf $(E', Q_{t+1}(E)E'_z)$:
\[
E' \to Q_{t+1}(E)E'_z \to Q_1(E)E'_z \to \cdots \to Q_1(E)_z \to Q_0(E)E'_z = 0
\]
of $(E, Q_{t+1}(E)_z)$, where $Q_1(E)E'_z = 0$. We have the exact sequence
\[
0 \to (E', Q_{t+1}(E)E'_z) \to (E, Q_{t+1}(E)_z) \to (zQ_1(E)_z, Q_1(E)z_{t+1}) \to 0
\]
of parabolic sheaves, where
\[
Q_1(E)z_{t+1} : Q_1(E)_z \to Q_1(E)_z \to \cdots \to Q_1(E)_z \to 0,
\]
and $\text{par}_{\omega'} \mu((E', Q_{t+1}(E)E'_z)) = \text{par}_{\omega'} \mu((E, Q_{t+1}(E)_z))$ (by direct computations). Thus $(E, Q_{t+1}(E)_z)$ is $S$-equivalent to
\[
(E', Q_{t+1}(E)E'_z) \oplus (zQ_1(E)_z, Q_1(E)z_{t+1}),
\]
which implies the injectivity of $\varphi$. To see that $\varphi$ is surjective, given an $\omega'$-semistable parabolic bundle $(E', Q_\bullet(E')_z)$, let $(E', Q_{t+1}(E')_z)$ be obtained from $(E', Q_\bullet(E')_z)$ by adding one term $\to 0$, and
\[
(zC^n_{z}(z) - m)_{t+1} : zC^n_{z}(z) - m \to zC^n_{z}(z) - m \to \cdots \to zC^n_{z}(z) - m \to 0.
\]
Then $(E', Q_{t+1}(E')_z) \oplus (zC^n_{z}(z) - m)_{t+1}$ is an $\omega''$-semistable parabolic sheaf whose image is $(E', Q_\bullet(E')_z)$. 

\textbf{Remark 3.11.} (1) The proof of Lemma 3.10 shows that every $\omega''$-semistable parabolic sheaf $E$ satisfies $0 \to E' \to E \to zC^n_{z}(z) - m \to 0$ with $\text{par}_{\omega'} \mu(E') = \text{par}_{\omega'} \mu(E)$. For the fixed $E'$, one can construct another extension $0 \to E' \to \tilde{E} \to zC^n_{z}(z) - m \to 0$ of parabolic sheaves such that $\tilde{E}$ is locally free. Thus every $\omega''$-semistable parabolic sheaf $E$ is $S$-equivalent to an $\omega''$-semistable parabolic bundle $\tilde{E}$. Moreover, if $E'$ is $\omega'$-stable, then $\tilde{E}$ is $\omega''$-stable, which is $S$-equivalent to an $\omega''$-semistable parabolic sheaf $E' \oplus zC^n_{z}(z) - m$. This shows that an $\omega''$-stable parabolic bundle $E$ does not correspond to a GIT-stable point. Indeed, if we take $F = E' \subset E$ in [21, Proposition 2.9], then $\epsilon(H) = s(F) = 0$ where $H \subset V$ is the inverse image of $H_0^{\bullet}(F(N))$ under the isomorphism $V \to H_0^{\bullet}(F(N))$ of vector spaces (see [21, Propositions 2.6 and 2.9]) for the definition of $s(F)$ and $\epsilon(H)$), which implies that the corresponding point $(V \otimes O(-N) \to E, Q_{t+1}(E)_z) \in R^n_{\omega''}$ is not GIT-stable by [21, Proposition 2.6]. In fact, for an $\omega''$-stable parabolic bundle $E$, we have shown in [21, Proposition 2.9] that the sub-sheaf $F \subset E$ with $s(F) = 0$ (which destroys the GIT-stability) must have $r(F) = r(E)$ and $E/F = 0$ outside of $z \in I$. Unfortunately, such sub-sheaf $F \subset E$ does not destroy $\omega''$-stability of $E$ according to its definition (since $E/F$ has torsion). When there exists $x \in I$ such that $a_{t+1}(x) - a_1(x) = k$, one may modify the definition of semistability (resp. stability) of parabolic sheaves: A parabolic sheaf $E$ is called semistable (resp. stable) if, for any sub-sheaf $F \subset E$ with $E/F$ torsion free outside of $x \in I$ where $a_{t+1}(x) - a_1(x) = k$, we have
\[
\text{par} \chi(F) \leq (\text{resp. } <) \frac{r(F)}{r} \text{par} \chi(E).
\]
Then there is no $\omega''$-stable parabolic sheaf in our case, which coincide with the fact that there is no GIT-stable point in our construction of moduli spaces $\mathcal{U}_{G',\omega''}$.  

(2) Let $\omega'' = H^m_z(\omega)$, and we will simply call $H^m_z(\omega)$ a Hecke transformation of $\omega$ at $z \in I$. Then

$$D_g(r, d, \omega) = D_g(r, d - m, H^m_z(\omega)).$$

(3.8)

Now we can prove another version of the recurrence relation (3.4), in which degree $d$ is kept unchanged.

**Theorem 3.12.** For any partitions $g = g_1 + g_2$ and $I = I_1 \cup I_2$, let

$$W_k = \{ \lambda = (\lambda_1, \ldots, \lambda_r) | 0 = \lambda_r \leq \lambda_{r-1} \leq \cdots \leq \lambda_1 \leq k \},$$

$$W'_k = \left\{ \lambda \in W_k \left| \left( \sum_{x \in I_1} \sum_{i=1}^{l_x} d_i(x)r_i(x) + \sum_{i=1}^{r} \lambda_i \right) \equiv 0 \pmod{r} \right. \right\}.$$

Then we have the following recurrence relation:

$$D_g(r, d, \omega) = \sum_{\mu \in W'_k} D_{g_1}(r, 0, \omega_1^\mu) \cdot D_{g_2}(r, d, \omega_2^\mu).$$

(3.9)

**Proof.** Let $P_k = \{ \mu = (\mu_1, \ldots, \mu_r) | 0 \leq \mu_r \leq \cdots \leq \mu_1 < k \}$. By the recurrence relation (3.4), we have

$$D_g(r, d, \omega) = \sum_{\mu \in Q_k} D_{g_1}(r, d_1^\mu, \omega_1^\mu) \cdot D_{g_2}(r, d_2^\mu, \omega_2^\mu),$$

where $Q_k = \{ \mu = (\mu_1, \ldots, \mu_r) \in P_k \mid d_1^\mu \in \mathbb{Z} \}$. Recall the definition of $d_j^\mu$ in Theorem 3.6, which are integers such that $d_1^\mu + d_2^\mu = d$ and

$$k(d_1^\mu + r) = k \cdot n_1^\alpha + |\mu|, \quad |\mu| = \sum_{i=1}^{r} \mu_i,$$

(3.10)

where $n_1^\alpha$ is the rational number defined in Theorem 3.6.

For $\mu = (\mu_1, \ldots, \mu_r)$, $0 \leq \mu_r \leq \cdots \leq \mu_1 < k$, let

$$H^1(\mu) = (k - \mu_{r-1} + \mu_r, \mu_1 - \mu_{r-1}, \mu_2 - \mu_{r-1}, \ldots, \mu_{r-2} - \mu_{r-1}, 0),$$

$$H^m(\mu) := H^1(H^{m-1}(\mu)) \text{ for } 2 \leq m \leq r. \text{ Then, when } 1 \leq m < r,$$

$$H^m(\mu)_j = \begin{cases} k - \mu_{r-m} + \mu_{r-m+j}, & \text{when } 1 \leq j \leq m, \\
\mu_{j-m} - \mu_{r-m}, & \text{when } j > m, \end{cases}$$

and $H^r(\mu) = (\mu_1 - \mu_r, \mu_2 - \mu_r, \ldots, \mu_{r-1} - \mu_r, 0)$. Moreover,

$$|H^m(\mu)| = \begin{cases} k \cdot m - r \cdot \mu_r + |\mu|, & \text{when } m < r, \\
r \cdot \mu_r + |\mu|, & \text{when } m = r. \end{cases}$$

(3.11)

Let $0 \leq i^\mu < r$ be the unique integer such that $d_1^\mu \equiv i^\mu \pmod{r}$, and let

$$\phi(\mu) := H^{r-i^\mu}(\mu).$$

Then, by (3.11), it is easy to see that we have a map

$$\phi : Q_k \rightarrow W'_k.$$

(3.12)

One can check that $\omega^{\phi(\mu)}_i$ is a Hecke transformation of $\omega^\mu_i$ ($i = 1, 2$), and thus

$$D_{g_1}(r, d_1^\mu, \omega_1^\mu) = D_{g_1}(r, 0, \omega_1^{\phi(\mu)}), \quad D_{g_2}(r, d_2^\mu, \omega_2^\mu) = D_{g_2}(r, d, \omega_2^{\phi(\mu)}).$$
by Lemmas 3.9 and 3.10. To prove the recurrence relation (3.9), it is enough to show that \( \phi \) is bijective.

To prove the injectivity of \( \phi \), let \( \phi(\mu) = \phi(\mu') \), and it is enough to show \( i^\mu = i^{\mu'} \). If both \( i^\mu \) and \( i^{\mu'} \) are nonzero, note \( |\phi(\mu)| = k(r - i^\mu) - r \mu_{i^\mu} + |\mu| \), by \( \phi(\mu) = \phi(\mu') \) and (3.10), there exists a \( q \in \mathbb{Z} \) such that

\[
r \cdot (\mu_{i^\mu} - \mu_{i^{\mu'}}) = k(d_1^{\mu'} - i^{\mu'}) - (d_1^\mu - i^\mu)) = k \cdot r \cdot q.
\]

Thus \( k > |\mu_{i^\mu} - \mu_{i^{\mu'}}| = k|q| \), which implies \( q = 0 \) and \( \mu_{i^\mu} = \mu_{i^{\mu'}} \). If \( i^\mu \neq i^{\mu'} \), let \( a = i^\mu - i^{\mu'} > 0 \), and then the formula

\[
\phi(\mu)_j = \begin{cases} 
- k - \mu_{i^\mu} + \mu_{j+\mu'}, & \text{when } 1 \leq j \leq r - i^\mu, \\
\mu_{j-r+i^\mu} - \mu_{i^\mu}, & \text{when } j > r - i^\mu
\end{cases}
\]

implies \( \mu_a = k + \mu_r' \geq k \), which leads to a contradiction since \( \mu \in Q_k \). If \( i^\mu = 0 \), \( i^{\mu'} \) must be zero. Otherwise, the same arguments imply \( \mu_{i^\mu'} = \mu_r \) and \( \mu_j = k + \mu_{i^{\mu'}+j} \) for all \( 1 \leq j \leq r - i^{\mu'} \).

To prove that \( \phi \) is surjective, by using (3.11), (3.10) becomes

\[
\frac{k \cdot n^\mu_r + |\phi(\mu)|}{r} = \begin{cases} 
k \cdot \frac{d_1^\mu + 2r - i^\mu}{r} - \mu_{i^\mu}, & \text{when } i^\mu > 0, \\
k \cdot \frac{d_1^\mu + r}{r} - \mu_r, & \text{when } i^\mu = 0.
\end{cases}
\]

For any \( \lambda = (\lambda_1, \ldots, \lambda_{r-1}, 0) \in W_k^r \), there are the unique integers \( q^\lambda \) and \( 0 \leq r^\lambda < k \) such that

\[
\frac{k \cdot n^\mu + |\lambda|}{r} = k \cdot q^\lambda - r^\lambda.
\]

If \( \lambda_1 + r^\lambda < k \), let \( \mu = (\lambda_1 + r^\lambda, \ldots, \lambda_{r-1} + r^\lambda, r^\lambda) \in P_k \), and then \( d_1^\mu = r(q^\lambda - 1) \) by (3.10). Thus \( i^\mu = 0 \) and \( \phi(\mu) = \lambda \). If \( \lambda_1 + r^\lambda \geq k \), since \( \lambda_r + r^\lambda < k \), there exists a unique \( 1 \leq i_0 \leq r - 1 \) such that

\[
\lambda_i + r^\lambda \geq k, \quad \lambda_{i+1} + r^\lambda < k.
\]

Let \( \mu_j = \lambda_{i_0+j} + r^\lambda (1 \leq j \leq r - i_0) \) and \( \mu_{r-i_0-j} = \lambda_j + r^\lambda - k (1 \leq j \leq i_0) \). Then \( \mu = (\mu_1, \ldots, \mu_r) \in Q_k \) with \( d_1^\mu = r(q^\lambda - 1) - i_0 \) and \( i^\mu = r - i_0 \). It is easy to see that \( \phi(\mu) = \lambda \).

\section{4 Proof of the Verlinde formula}

As an application of the recurrence relation (3.3) and (3.9), we prove a closed formula of \( D_\sigma(r, d, \omega) \) (the so-called Verlinde formula). Recall

\[
S_\lambda(z_1, \ldots, z_r) = \frac{\det(z_{i+j}^{\lambda_i+r-i})}{\det(z_{i+j}^{\lambda_i+r-i})} = \frac{\det(z_{i+j}^{\lambda_i+r-i})}{\Delta(z_1, \ldots, z_r)}
\]

is the so-called Schur polynomial of \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0) \),

\[\Delta(z_1, \ldots, z_r) = \prod_{i<j}(z_i - z_j).\]

We give here a detailed proof of some identities of Schur polynomials.

\textbf{Proposition 4.1.} \ For \( \vec{v} = (v_1, \ldots, v_r) \), \( 0 \leq v_r < \cdots < v_1 < r+k \), let

\[
S_\lambda\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right) = S_\lambda(e^{2\pi i \frac{v_1}{r+k}}, \ldots, e^{2\pi i \frac{v_r}{r+k}}),
\]

\[
P_k = \{ \mu = (\mu_1, \ldots, \mu_r) \mid 0 \leq \mu_r \leq \cdots \leq \mu_1 < k \} \text{ and } |\mu| := \sum \mu_i. \text{ Then }
\]

\[
\sum_{\mu \in P_k} S_\mu\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right) \cdot S_{\mu^*}\left(\exp 2\pi i \frac{-\vec{v}}{r+k}\right) = \exp\left(2\pi i \frac{k}{r+k} |\vec{v}| \right) \cdot \frac{k(r+k)^{r-1}}{\prod_{i<j}(2 \sin \pi \frac{v_i-v_j}{r+k})^2}.
\]
where $\mu^* := (k - \mu_r, \ldots, k - \mu_1)$. Let

$$W_k = \{ \mu = (\mu_1, \ldots, \mu_r) | 0 = \mu_r \leqslant \mu_{r-1} \leqslant \cdots \leqslant \mu_1 \leqslant k \}.$$

We have

$$\sum_{\mu \in W_k} S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) \cdot S_{\mu^*} \left( \exp 2\pi \frac{r}{r+k} \right) = \exp \left( 2\pi \frac{k}{r+k} |\vec{v}| \right) \cdot \frac{r(r+k)^{-1}}{\prod_{i<j}(2 \sin \frac{\nu_i - \nu_j}{r+k})^2} \tag{4.3}$$

and, if $\vec{v} \sim \vec{v}'$ (here, $\vec{v} \sim \vec{v}' \iff \vec{v} - \vec{v}' = (a, \ldots, a)$ for some $a \in \mathbb{Z}$),

$$\sum_{\mu \in W_k} \exp 2\pi \frac{-|\mu| \cdot |\vec{v}|}{r(r+k)} \cdot \exp 2\pi \frac{-|\mu^*| \cdot |\vec{v}'|}{r(r+k)} \cdot S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) \cdot S_{\mu^*} \left( \exp 2\pi \frac{r}{r+k} \right) = 0. \tag{4.4}$$

(4.2) and (4.3) are invariant under the equivalence relation $\vec{v} \sim \vec{v}'$.

Proof. To prove (4.2), since $S_{\mu^*}(V) = \det(V)^k \otimes S_{\mu}(V^*)$, we have

$$S_{\mu^*} \left( \exp 2\pi \frac{r}{r+k} \right) = S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) \exp \left( 2\pi \frac{k}{r+k} \sum_{i=1}^r v_i \right).$$

Thus it is enough to show that

$$\sum_{\mu \in P_k} S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) \cdot S_{\mu^*} \left( \exp 2\pi \frac{r}{r+k} \right) = \frac{k(r+k)^{-1}}{\prod_{i<j}(2 \sin \frac{\nu_i - \nu_j}{r+k})^2}. \tag{4.5}$$

For $\lambda = (\lambda_1, \ldots, \lambda_r)$, the functions $e^{\tau(\vec{v})}$ and $J(e^{\vec{v}})$ are defined by

$$e^{\tau(\vec{v})}(\text{diag}(t_1, \ldots, t_r)) := t_1^{\lambda_{r(1)}} \cdots t_r^{\lambda_{r(r)}},$$

$$J(e^{\vec{v}})(\text{diag}(t_1, \cdots, t_r)) := \sum_{\tau \in G_r} e(\tau) e^{\tau(\vec{v})}(\text{diag}(t_1, \ldots, t_r)),$$

where $\tau(\vec{v}) = (\lambda_{r(1)}, \ldots, \lambda_{r(r)})$, and $G_r$ is the symmetric group. Let

$$\Delta(\vec{v}) = \prod_{i<j} (e^{2\pi \frac{\nu_i - \nu_j}{r+k}} - e^{2\pi \frac{\nu_i}{r+k}})$$

and $\rho = (r-1, r-2, \ldots, 0)$. By expansion of determinant, we have

$$S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) = \frac{1}{\Delta(\vec{v})} \sum_{\tau \in G_r} e(\tau) e^{2\pi \frac{\nu_k - \nu_r - 1}{r+k} v_{r(1)}} \cdots e^{2\pi \frac{\nu_k - \nu_r}{r+k} v_{r(r)}} \cdot \frac{1}{\Delta(\vec{v})} J(e^{\vec{v}})(t_\mu),$$

where $t_\mu = \exp 2\pi \frac{\vec{v} + \vec{v}'}{r+k}$. By

$$\Delta(\vec{v}) \Delta(\vec{v}') = \prod_{i<j} \left( 2 \sin \frac{\nu_i - \nu_j}{r+k} \right)^2,$$

we have

$$S_\mu \left( \exp 2\pi \frac{r}{r+k} \right) \cdot S_{\mu^*} \left( \exp 2\pi \frac{r}{r+k} \right) = \frac{1}{\prod_{i<j}(2 \sin \frac{\nu_i - \nu_j}{r+k})^2} J(e^{\vec{v}})(t_\mu) \cdot J(e^{\vec{v}})(t_\mu). \tag{4.6}$$

Let

$$T_k = \{ t = \text{diag}(e^{\frac{2\pi i}{r+k} t_1}, \ldots, e^{\frac{2\pi i}{r+k} t_r}) | 0 \leqslant t_i < r+k \} \subset \text{GL}(r)$$
be a finite subgroup and $T_k^\text{reg} = \{ t \in T_k \mid t_i \neq t_j \text{ if } i \neq j \}$. The group $\mathfrak{S}_r$ acts on $T_k$ by
\[
\tau(t) = \text{diag}(e^{2\pi i t_{(1)}}, \ldots, e^{2\pi i t_{(r)}})
\]
and the functions
\[
J(e^\bar{\lambda})(t) = \sum_{\tau \in \mathfrak{S}_r} \epsilon(\tau)e^{2\pi i \frac{\bar{\lambda}(t_{(1)})t_1}{r+k} \ldots e^{2\pi i \frac{\bar{\lambda}(t_{(r)})t_r}{r+k}}
\]
for any $\lambda = (\lambda_1, \ldots, \lambda_r)$ are ant-symmetric functions, and thus $J(e^\bar{\lambda})(t) = 0$ if $t \notin T_k^\text{reg}$. It is clear that $\mathfrak{S}_r$ acts on $T_k^\text{reg}$ freely and
\[
T_k^\text{reg} = \bigcup_{\mu \in \bar{P}_k} \mathfrak{S}_r \cdot t_\mu, \quad t_\mu = \exp 2\pi i \frac{\mu + \rho}{r+k}.
\]
The right-hand side of (4.6) is a symmetric function, and we have
\[
\sum_{\mu \in \bar{P}_k} S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right) \cdot S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right)
\]
\[
= \frac{1}{|\mathfrak{S}_r|} \prod_{1 < j} \left( 2 \sin \pi \frac{v_i - v_j}{r+k} \right)^{-2} \sum_{t \in T_k} J(e^{\bar{v}})(t)J(e^{\bar{v}})(t),
\]
where $\bar{P}_k = \{ \mu = (\mu_1, \ldots, \mu_r) \mid 0 \leq \mu_r \leq \cdots \leq \mu_1 \leq k \}$. To compute
\[
\sum_{t \in T_k} J(e^{\bar{v}})(t)J(e^{\bar{v}})(t) = \sum_{\tau, \sigma \in \mathfrak{S}_r} \epsilon(\tau) \cdot \epsilon(\sigma) \sum_{t \in T_k} e^{\tau(\bar{v})}(t) \cdot e^{\sigma(\bar{v})}(t),
\]
note that $e^{\tau(\bar{v})}$ and $e^{\sigma(\bar{v})}$ are different character of $T_k$ when $\tau \neq \sigma$, and then
\[
\sum_{t \in T_k} J(e^{\bar{v}})(t)J(e^{\bar{v}})(t) = |\mathfrak{S}_r| \cdot |T_k|
\]
by the first orthogonality relation of characters. Thus
\[
\sum_{\mu \in \bar{P}_k} S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right) \cdot S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right) = \frac{(r+k)^r}{\prod_{1 < j} (2 \sin \pi \frac{v_i - v_j}{r+k})^2}.
\]
For $\mu \in P'_k := \bar{P}_k \setminus P_k$, let
\[
t'_\mu = \text{diag}(1, e^{2\pi i \frac{\mu_2 + \cdots + \mu_k - 1}{r+k}}, \ldots, e^{2\pi i \frac{\mu_1 + \mu_2 + \cdots + \mu_k - 1}{r+k}})
\]
and
\[
T'_k = \{ t = \text{diag}(1, e^{2\pi i t_2}, \ldots, e^{2\pi i t_r}) \mid 0 \leq t_i < r+k \} \subset T_k
\]
be the subgroup and $T_k^\text{reg} = T'_k \cap T_k^\text{reg}$. Then
\[
T_k^\text{reg} = \bigcup_{\mu \in P'_k} \mathfrak{S}_{r-1} \cdot t'_\mu.
\]
Note $J(e^{\bar{v}})(t_\mu) = e^{-2\pi i \frac{\bar{v}_\mu}{r+k}} J(e^{\bar{v}})(t'_\mu)$ and $J(e^{\bar{v}})(t) = 0$ if $t \notin T_k^\text{reg}$. We have
\[
\sum_{\mu \in P'_k} S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right) \cdot S_\mu \left( \exp 2\pi i \frac{\bar{v}}{r+k} \right)
\]
\[
= \frac{1}{|\mathfrak{S}_{r-1}|} \prod_{1 < j} \left( 2 \sin \pi \frac{v_i - v_j}{r+k} \right)^{-2} \sum_{t \in T_k} J(e^{\bar{v}})(t)J(e^{\bar{v}})(t),
\]
Theorem 4.3. For any given data $\omega = (k, \{\hat{n}(x), \vec{a}(x)\}_{x \in I})$, let

$$S_\omega(z_1, \ldots, z_r) = \prod_{x \in I} S_{\lambda_x}(z_1, \ldots, z_r), \quad |\omega| = \sum_{x \in I} |\lambda_x|,$$

Then it is equivalent to prove that, when $\vec{v} \sim \vec{v}'$, we have

$$\sum_{\mu \in W_k} G_\mu(\vec{v})G_\mu(\vec{v}') = 0. \quad (4.9)$$

Let $\lambda^\mu = (\lambda_1^\mu, \ldots, \lambda_r^\mu)$ with $\lambda_i^\mu = \mu_i + r - i - \frac{|\mu|+|\nu|}{r+k}$. Then

$$G_\mu(\vec{v}) = \frac{\exp 2\pi i \frac{|\mu|+|\nu|}{r+k}}{\Delta(\vec{v})} \sum_{\tau \in \mathfrak{S}_r} e(\tau)e^{2\pi i \frac{\lambda^\mu}{r+k} v_{\tau(1)} \cdot v_{\tau(r)}} = \frac{\exp 2\pi i \frac{|\mu|+|\nu|}{r+k}}{\Delta(\vec{v})} \sum_{\tau \in \mathfrak{S}_r} e(\tau)e^{2\pi i \frac{\lambda^\mu}{r+k} v_{\tau(1)} \cdot v_{\tau(r)}}.$$

Since $e^{\pi i(\nu)}$ and $e^{\pi i(\nu')}$ for all $\nu, \nu' \in \mathfrak{S}_r$ are different characters of a subgroup

$$T_k = \left\{ t = \text{diag}(e^{\frac{2\pi i}{r+k} t_1}, \ldots, e^{\frac{2\pi i}{r+k} t_r}) \left| \sum t_i = 0, t_i - t_j \in \mathbb{Z}\right. \right\} \subset \text{GL}(r)$$

whenever $\vec{v} \sim \vec{v}'$, we have

$$\sum_{\mu \in W_k} G_\mu(\vec{v})G_\mu(\vec{v}') = \frac{\exp 2\pi i \frac{|\mu|+|\nu|}{r+k}}{\Delta(\vec{v})\Delta(\vec{v}')} \sum_{\mu \in W_k} J(e^{\vec{v}})(t_{\lambda^\mu}) \cdot J(e^{\vec{v}'})(t_{\lambda^\nu}) = 0.$$
where \(S_{\lambda_x}(z_1, \ldots, z_r)\) are Schur polynomials and \(|\lambda_x|\) denotes the total number of boxes in a Young diagram associated with \(\lambda_x\). Then

\[
D_g(r, d, \omega) = (-1)^{d(r-1)} \left( \frac{k}{r} \right)^g \left( r(r + k)^{g-1} \right) \\
\times \sum_{\vec{v}} \exp(2\pi i \frac{d'}{r} - \frac{\omega}{r(r+k)}) \sum_{i=1}^r v_i S_{\omega^i}(\exp 2\pi i \frac{\vec{v}}{r+k}) \\
\prod_{i<j} (2 \sin \pi \frac{v_i - v_j}{r+k})^{2(g-1)},
\]

(4.11)

where \(\vec{v} = (v_1, v_2, \ldots, v_r)\) runs through the integers

\[0 = v_r < \cdots < v_2 < v_1 < r + k.\]

**Proof.** Let \(V_g(r, d, \omega)\) denote the right-hand side of (4.11) (the Verlinde number). When \(|I| = 0\), we define \(V_g(r, d, \omega)\) to be

\[
(-1)^{d(r-1)} \left( \frac{k}{r} \right)^g \left( r(r + k)^{g-1} \right) \sum_{\vec{v}} \exp(2\pi i \frac{d'}{r} \sum_{i=1}^r v_i) \\
\prod_{i<j} (2 \sin \pi \frac{v_i - v_j}{r+k})^{2(g-1)}.
\]

Note that both \(V_g(r, d, \omega)\) and \(D_g(r, d, \omega)\) (even the moduli space \(U_{C, \omega}\) and the theta line bundle \(\Theta_{U_{C, \omega}}\) on it) are invariant under the equivalence relation: \(\lambda_x \sim \lambda_x' \Leftrightarrow \lambda_x - \lambda_x' = (a_x, a_x, \ldots, a_x)\) for some integer \(a_x \in \mathbb{Z}\). Assume that \(D_g(r, d, \omega) = V_g(r, d, \omega)\) when \(|I| \leq 3\) (we will prove it later). Then the proof is completed by the following lemmas.

**Lemma 4.4.** If the formula (4.11) holds when \(g = 0\), then it holds for any \(g > 0\).

**Proof.** Recalling the recurrence relation (3.3), we have

\[
D_g(r, d, \omega) = \sum_{\mu} D_{g-1}(r, d, \omega^\mu),
\]

(4.12)

where \(\omega^\mu = (k, \{\bar{a}(x), \bar{a}(x)\}_{x \in \Omega(x_1, x_2)})\) was defined in Notation 3.5 and \(\mu = (\mu_1, \ldots, \mu_r)\) runs through the integers \(0 \leq \mu_r \leq \cdots \leq \mu_1 < k\).

It is easy to check that

\[
\lambda_{x_2} = (k - \mu_r, \ldots, k - \mu_1)
\]

and

\[
\lambda_{x_1} = (\mu_1, \ldots, \mu_r) + (\mu_1 + \mu_r - k, \mu_1 + \mu_r - k, \ldots, \mu_1 + \mu_r - k)
\]

(in Notation 4.2). Thus, without loss of generality, we can assume

\[
\lambda_{x_1} = \mu = (\mu_1, \ldots, \mu_r), \quad \lambda_{x_2} = \mu^* = (k - \mu_r, \ldots, k - \mu_1).
\]

Assume that formula (4.11) holds for \(g - 1\). Then

\[
D_{g-1}(r, d, \omega^\mu) = (-1)^{d(r-1)} \left( \frac{k}{r} \right)^{g-1} \left( r(r + k)^{g-1} \right)^{g-2} \\
\times \sum_{\vec{v}} \exp(2\pi i \frac{d'}{r} - \frac{\omega^\mu}{r(r+k)}) \sum_{i=1}^r v_i S_{\omega^i}(\exp 2\pi i \frac{\vec{v}}{r+k}) \\
\prod_{i<j} (2 \sin \pi \frac{v_i - v_j}{r+k})^{2(g-2)},
\]

(4.13)

where \(|\omega^\mu| = |\omega| + k \cdot r\) and \(S_{\omega^\mu} = S_{\omega} \cdot S_{\mu} \cdot S_{\mu^*}\). By (4.12) and (4.13),

\[
D_g(r, d, \omega) = (-1)^{d(r-1)} \left( \frac{k}{r} \right)^g \left( r(r + k)^{g-1} \right)^{g-1} \\
\times \sum_{\vec{v}} \exp(2\pi i \frac{d'}{r} - \frac{\omega}{r(r+k)}) \sum_{i=1}^r v_i S_\omega(\exp 2\pi i \frac{\vec{v}}{r+k}) \\
\prod_{i<j} (2 \sin \pi \frac{v_i - v_j}{r+k})^{2(g-1)} \\
\times \exp \left( -2\pi i \frac{k}{r+k} \sum_{i=1}^r v_i \right) \prod_{i<j} (2 \sin \pi \frac{v_i - v_j}{r+k})^2 \frac{1}{k(r+k)^{g-1}} \\
\times \sum_{\mu} S_\mu(\exp 2\pi i \frac{\vec{v}}{r+k}) \cdot S_{\mu^*}(\exp 2\pi i \frac{\vec{v}}{r+k}).
\]
Then the formula (4.11) holds by the identity (4.2) in Proposition 4.1.

**Lemma 4.5.** If the formula (4.11) for $D_0(r, d, \omega)$ holds when $|I| \leq 3$, then it holds for all $D_0(r, d, \omega)$.

**Proof.** The proof is by induction on the number of parabolic points. By Theorem 3.12, let $I = I_1 \cup I_2$ with $|I_1| = 2$. We have

$$D_0(r, d, \omega) = \sum_{\mu \in W_k} V_0(r, 0, \omega_1) \cdot V_0(r, d, \omega_2).$$

It is not difficult to check that $V_0(r, 0, \omega_1) = 0$ for $\mu \in W_k \setminus W_k'$. Thus

$$D_0(r, d, \omega) = \sum_{\mu \in W_k} V_0(r, 0, \omega_1') \cdot V_0(r, d, \omega_2')$$

and we are done by (4.3) and (4.4) of Proposition 4.1. Here, we remark that $\vec{v} \sim \vec{v}'$ if and only if $\vec{v} \neq \vec{v}'$ since our $\vec{v}$ and $\vec{v}'$ satisfy $v_r = v'_r = 0$.

Finally, we prove $D_0(r, d, \omega) = V_0(r, d, \omega)$ when $|I| \leq 3$. The data

$$\omega = (k, \{\tilde{n}(x), \tilde{a}(x)\}_{x \in I})$$

is encoded in the partitions $\omega = \{\lambda_x\}_{x \in I}$ (see Notation 4.2), where

$$\lambda_x = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)), \quad k \geq \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x) \geq 0.$$

Thus, for convenience of computations, we will use notations

$$D_g(r, d, \{\lambda_x\}_{x \in I}) := D_g(r, d, \omega), \quad V_g(r, d, \{\lambda_x\}_{x \in I}) := V_g(r, d, \omega),$$

$$\omega_s := (1, \ldots, 1, 0, \ldots, 0) \quad (1 \leq s \leq r - 1).$$

Let $V$ be the standard representation of $GL_r(\mathbb{C})$ and $1 \leq s \leq r - 1$. Then

$$S_{\lambda}(V) \otimes S_{\omega_s}(V) = \bigoplus_{\mu \in \mathcal{Y}(\lambda, \omega_s)} S_\mu(V), \quad (4.14)$$

where the Young diagrams of the partitions $\mu \in Y(\lambda, \omega_s)$ are obtained from $\lambda$ by adding $s$ boxes with no two in the same row (see [9, p. 79, (6.9)]). In the rest of the article, without loss of generality, we assume $\lambda_r(x) = 0$ for all partitions $\lambda_x$ ($x \in I$). We compute $V_0(r, 0, \{\lambda_x\}_{x \in I})$ firstly for special partitions.

**Lemma 4.6.**

1. When $|I| = 0$, $V_0(r, 0, \{\lambda_x\}_{x \in I}) = 1$
2. $V_0(r, 0, \lambda_x)$ equals 1 if $\lambda_x = 0$ and zero otherwise.
3. $V_0(r, 0, \lambda_x, \lambda_y)$ equals 1 if $\lambda_x \sim \lambda_y^*$ and zero otherwise.
4. Let $Y(\lambda_y, \omega_s)$ be the set defined in (4.14). Then

$$V_0(r, 0, \omega_s, \lambda_y) = \begin{cases} 1, & \text{when } \lambda_z^* \sim \mu \in Y(\lambda_y, \omega_s), \\ 0, & \text{when } \lambda_z^* \not\sim \mu \in Y(\lambda_y, \omega_s). \end{cases}$$

Note that for any $\mu, \mu' \in Y(\lambda_y, \omega_s)$, $\mu \sim \mu' \iff \mu = \mu'$. 


**Proof.** (1) When $|I| = 0$ and $r \mid d$, recalling from (4.11), we have
\[
V_0(r, d, \{\lambda_x\}_{x \in I}) = \frac{1}{r(r + k)^{r-1}} \sum_{\vec{v}} \Delta(\vec{v}) \Delta(\vec{v}),
\]
where $\vec{v} = (v_1, v_2, \ldots, v_r)$ runs through the integers
\[
0 = v_0 < v_1 < \cdots < v_2 < v_1 < r + k.
\]
Let $\rho = (\rho_1, \ldots, \rho_r) = (r - 1, \ldots, 1, 0)$. By expansion of determinant,
\[
\Delta(\vec{v}) = \sum_{\tau \in \mathcal{G}_r} e(\tau) e^{2\pi i \tau(r)} \cdots e^{2\pi i \tau(r)} = J(e^{\vec{v}}) \left( \exp 2\pi i \frac{\vec{v}}{r + k} \right).
\]
Then the same computations in the proof of Proposition 4.1 imply that
\[
\sum_{\vec{v}} \Delta(\vec{v}) \Delta(\vec{v}) = \frac{1}{|\mathcal{G}_r|} \sum_{t \in T'_r} J(e^{\vec{v}})(t) J(e^{\vec{v}})(t)
= \frac{1}{|\mathcal{G}_r|} \sum_{\tau, \sigma \in \mathcal{G}_r} e(\tau) e(\sigma) \sum_{t \in T'_r} e^{\tau(\vec{v})}(t) \cdot e^{\sigma(\vec{v})}(t)
= r(r + k)^{r-1},
\]
where $T'_r = \{ t = \text{diag}(e^{\frac{2\pi i}{r} t_1}, \ldots, e^{\frac{2\pi i}{r} t_r}, 1) \mid 0 \leq t_i < r + k \}$.
To prove (2) and (3), we note that (3) implies (2) since
\[
V_0(r, 0, \{\lambda_x\}) = V_0(r, 0, \{\lambda_x, \lambda_y\})
\]
when $\lambda_y = 0$. Thus it is enough to show that
\[
\sum_{\vec{v}} \frac{\exp(2\pi i(-\frac{\lambda_x + \lambda_y}{r + k}))(\sum_{i=1}^r v_i) S_{\lambda_x, \lambda_y}(\exp 2\pi i \frac{\vec{v}}{r + k})}{\prod_{i < j} (2 \sin \pi \frac{v_i - v_j}{r + k})^{-2}} = r(r + k)^{r-1}
\]
when $\lambda_x \sim \lambda^*_y$ and zero otherwise. Write $\vec{v} = \rho + \mu (\mu \in W_k)$. Then
\[
S_{\lambda_x}(\exp 2\pi i \frac{\vec{v}}{r + k}) = \Delta(\vec{v}) \frac{\Delta(\vec{v})}{\Delta(\vec{v})} S_{\mu}(\exp 2\pi i \frac{\vec{v}}{r + k}),
\]
\[
S_{\lambda_y}(\exp 2\pi i \frac{\vec{v}}{r + k}) = \Delta(\vec{v}) \frac{\Delta(\vec{v})}{\Delta(\vec{v})} S_{\mu}(\exp 2\pi i \frac{\vec{v}}{r + k}),
\]
\[
S_{\lambda_y}(\exp 2\pi i \frac{\vec{v}}{r + k}) = S_{\lambda^*_y}(\exp 2\pi i \frac{\vec{v}}{r + k}) \cdot \exp \left( 2\pi i \frac{k}{r + k} \sum_{i=1}^r v_i \right).
\]
Thus we have
\[
\sum_{\vec{v}} \frac{\exp(2\pi i(-\frac{\lambda_x + \lambda_y}{r + k}))(\sum_{i=1}^r v_i) S_{\lambda_x, \lambda_y}(\exp 2\pi i \frac{\vec{v}}{r + k})}{\prod_{i < j} (2 \sin \pi \frac{v_i - v_j}{r + k})^{-2}}
= \frac{\Delta(\vec{v})}{\exp(2\pi i \frac{k}{r + k} \lambda^*_y + \rho)} \sum_{\mu \in W_k} e^{2\pi i \left( \frac{|\lambda_x + \lambda_y|}{r + k} \right) |\mu + \rho|} S_{\mu}(\exp 2\pi i \frac{\vec{v}}{r + k}) \cdot S_{\mu^*}(\exp 2\pi i \frac{\vec{v}}{r + k}),
\]
which equals 1 when $\lambda_x \sim \lambda^*_y$ by (4.3) of Proposition 4.1 and otherwise zero if $\lambda_x \sim \lambda^*_y$ by (4.4) of Proposition 4.1.
In order to prove (4), by using (4.14), we have
\[
S_{\omega_x}(\exp 2\pi i \frac{\vec{v}}{r + k}) \cdot S_{\omega_y}(\exp 2\pi i \frac{\vec{v}}{r + k}) = \sum_{\mu \in Y(\lambda_y, \omega_x)} S_{\mu}(\exp 2\pi i \frac{\vec{v}}{r + k}).
\]
Since $|\mu| = |\lambda_y| + |\omega_3|$ for any $\mu \in Y(\lambda_y, \omega_3)$, we have

$$V_0(r, 0, \{\omega_3, \lambda_y, \lambda_z\}) = \frac{1}{r(r + k)} \sum_{i} \exp(-2\pi i \frac{\mu_i}{r + k}) S_{\lambda_i, \lambda_y, \lambda_z} \left( \exp(2\pi i \frac{r}{r + k}) \right) \prod_{i < j} (2\sin \frac{\pi v_i - v_j}{r + k})^{-2}$$

which together with (3) imply (4). \qed

Let $U_0(r, 0, \{\lambda_x\} \in I)$ be the moduli space of semi-stable parabolic bundles of rank $r$ and degree 0 on $\mathbb{P}^2$ with parabolic structures given by $\{\lambda_x\} \in I$. Recall the condition in (3.1) (the necessary condition to define theta line bundle on $U_0(r, 0, \{\lambda_x\} \in I)$: $\sum_{i \in I} |\lambda_x| \in \mathbb{Z}$). We will assume this condition (in the case it is needed), otherwise $V_0(r, 0, \{\lambda_x\} \in I) = 0$ and we can define $D_0(r, 0, \{\lambda_x\} \in I) = 0$.

Lemma 4.7. (1) When $|I| = 0$, $U_0(r, 0, \{\lambda_x\} \in I)$ consists of one point.

(2) When $|I| = 1$, $U_0(r, 0, \lambda_x)$ consists of one point if $\lambda_x = 0$ and is empty otherwise.

(3) When $|I| = 2$, $U_0(r, 0, \{\lambda_x, \lambda_y\})$ consists of one point if $\lambda_x \sim \lambda_y$ and is empty otherwise.

(4) When $|I| = 3$, $U_0(r, 0, \{\omega_3, \lambda_y, \lambda_z\})$ (1 ≤ $s \leq r - 1$) consists of one point if $\lambda_s \sim \mu \in Y(\lambda_y, \omega_3)$ and is empty otherwise.

Proof. (1) is clear. For other statements, recall (see Notation 4.2) that the parabolic structure of $E$ at $x \in I$ determined by $\lambda_x$ is given by a flag

$$E_x = Q_{l_{x+1}}(E)_x \rightarrow Q_{l_x}(E)_x \rightarrow \cdots \rightarrow Q_1(E)_x \rightarrow Q_0(E)_x = 0$$

and $0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \leq k$ (here, we have to include the case of $a_{l_x+1}(x) - a_1(x) = k$ because of the Hecke modifications) such that

$$\lambda_x = (k - a_1(x), \ldots, k - a_{l_x+1}(x), k - a_{l_x+1}(x)), \ldots, k - a_{l_x+1}(x), \ldots, k - a_{l_x+1}(x),$$

where $a_i(x) = r_i(x) - r_{i-1}(x)$ (1 ≤ $i \leq l_x + 1$) and $r_i(x) = \dim Q_i(E)_x$. For any sub-sheaf $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of $F_x$ and $n_i^F(x) = r_i^F(x) - r_{i-1}^F(x)$, $r_i^F(x) = \dim Q_i(E)_x^F$. Let

$$\text{pardeg}(E) := \deg(E) + \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} a_i(x) n_i(x),$$

$$\text{pardeg}(F) := \deg(F) + \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} a_i(x) n_i^F(x).$$

Then $E$ is called semistable (resp. stable) for $\omega = (k, \{\tilde{\alpha}(x), \tilde{\alpha}(x)\} \in I)$ if for any nontrivial sub-bundle $E' \subset E$, one has

$$\text{par}(\omega) := \frac{\text{pardeg}(E')}{{r(E')}^2} \leq \frac{\text{pardeg}(E)}{r(E)} =: \text{par}(\omega) \quad (\text{resp. } <).$$

To show (2), when $\lambda_x$ is nontrivial (i.e., $0 < a_1(x) < r$), we have

$$\text{par}(\omega) \leq \frac{1}{r} \sum_{i=1}^{l_x+1} a_{l_x+1}(x) - \frac{(l_x + 1 - i)}{k} n_i(x) \leq \frac{a_{l_x+1}(x)}{k}. $$
Thus any semistable parabolic bundle must have $E = \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ and the evaluation map $H^0(\mathbb{P}^1, E) \to E_x$ is an isomorphism. Then there is a line bundle $F \subset E$ of degree zero such that $n_{l_x+1}(x) = 1$, which implies

$$\operatorname{par} \mu(F) = \frac{a_{l_x+1}(x)}{k} > \operatorname{par} \mu(E)$$

and $\mathcal{U}_p(r, 0, \lambda_x)$ is empty.

To prove (3), we consider firstly the case $\lambda_x \sim \lambda_x^*$, and then

$$l_y = l_x, \quad n_i(y) = n_{t_x-i+2}(x), \quad a_i(y) + a_{t_x-i+2}(x) = a$$

and $\operatorname{par} \mu(E) = \frac{a}{k}$. To show that $\mathcal{U}_p(r, 0, \{\lambda_x, \lambda_y\})$ consists of one point, recall that it is a GIT quotient $\psi : R^{ss}_{\{\lambda_x, \lambda_y\}} \to \mathcal{U}_p(r, 0, \{\lambda_x, \lambda_y\})$ where $R^{ss}_{\{\lambda_x, \lambda_y\}} \subset R$ is a (maybe empty) open subset of an irreducible quasi-projective variety $R$. Let $R^0 \subset R$ be the (non-empty) open subset of the parabolic bundles $E$ with a trivial underlying bundle. We will show that $R^0 \cap R^{ss}_{\{\lambda_x, \lambda_y\}}$ is non-empty and $\psi(R^0 \cap R^{ss}_{\{\lambda_x, \lambda_y\}})$ is a point, which implies that $\mathcal{U}_p(r, 0, \{\lambda_x, \lambda_y\})$ consists of one point. Clearly, the following semistable parabolic bundle:

$$l_y+1 \bigoplus_{i=1}^{l_y+1} \left( \mathcal{O}_{\mathbb{P}^1}, \{a_{t_x-i+2}(x), a_i(y)\} \right)^{\oplus n_i(y)}$$

(4.15)

defines a point of $\mathcal{U}_p(r, 0, \{\lambda_x, \lambda_y\})$, where $(\mathcal{O}_{\mathbb{P}^1}, \{a_{t_x-i+2}(x), a_i(y)\})$ is a parabolic line bundle with the weight $\{a_{t_x-i+2}(x), a_i(y)\}$. On the other hand, any $E \in R^0 \cap R^{ss}_{\{\lambda_x, \lambda_y\}}$ is $S$-equivalent to the parabolic bundle defined in (4.15). In fact, since $H^0(\mathbb{P}^1, E) \to E_x$ is an isomorphism, there is an $L_1 = \mathcal{O}_{\mathbb{P}^1} \subset E$ such that $n_{l_x+1}(x) = 1$. Then the semistability of $E$ implies $n_{l_x+1}(y) = 1$ and $L_1 = \mathcal{O}_{\mathbb{P}^1}, \{a_{l_x+1}(x), a_1(y)\}$ has $\operatorname{par} \mu(L_1) = \operatorname{par} \mu(E)$. Let $E' = E/L_1$ be the quotient parabolic bundle. Then $E$ is $S$-equivalent to $L_1 \oplus E'$, where the parabolic structures of $E'$ are defined by the partitions

$$\lambda'_x = \left( \underbrace{k-a_1(x), \ldots, k-a_1(x)}_{n_1(x)-1}, \underbrace{k-a_{t_x+1}(x), \ldots, k-a_{t_x+1}(x)}_{n_{t_x+1}(x)-1} \right),$$

$$\lambda'_y = \left( \underbrace{k-a_1(y), \ldots, k-a_1(y)}_{n_1(y)-1}, \underbrace{k-a_{t_y+1}(y), \ldots, k-a_{t_y+1}(y)}_{n_{t_y+1}(y)} \right).$$

Clearly, $\lambda'_x \sim \lambda_y^*$ and, by induction of $r(E)$, $E'$ is $S$-equivalent to

$$(\mathcal{O}_{\mathbb{P}^1}, \{a_{l_x+1}(x), a_1(y)\})^{\oplus (n_1(y)-1)} \bigoplus_{i=2}^{l_y+1} \left( \mathcal{O}_{\mathbb{P}^1}, \{a_{t_x-i+2}(x), a_i(y)\} \right)^{\oplus n_i(y)}.$$

Thus $E$ is $S$-equivalent to the parabolic bundle defined in (4.15).

To prove $\mathcal{U}_p(r, 0, \{\lambda_x, \lambda_y\})$ is empty when $\lambda_x \sim \lambda_y^*$, it is enough to prove $R^0 \cap R^{ss}_{\{\lambda_x, \lambda_y\}}$ is empty. Let $E \in R^0$. Recall its flag at $x$ and $y$,

$$E_x = Q_{l_x+1}(E)_{x} \to Q_{l_x}(E)_{x} \to \cdots \to Q_1(E)_{x} \to Q_0(E)_{x} = 0,$$

$$E_y = Q_{l_y+1}(E)_{y} \to Q_{l_y}(E)_{y} \to \cdots \to Q_1(E)_{y} \to Q_0(E)_{y} = 0$$

and $r_i(x) := \dim(Q_i(E)_x)$, $r_i(y) := \dim(Q_i(E)_y)$. For $1 \leq m < r$, let $m_x = \max\{i \mid r_i(x) \leq m\}$, $m_y = \max\{i \mid r_i(y) \leq m\}$,

$$d_x(m) := \sum_{i=1}^{m_x} \frac{a_i(x)}{k} - n_i(x) + (m - r_{m_x}(x)) \frac{a_{m_x+1}(x)}{k},$$

$$d_y(m) := \sum_{i=1}^{l_y+1} \frac{a_i(y)}{k} - n_i(y) - d_x(m),$$
and let \( d_y(m) \) and \( \tilde{d}_y(m) \) be defined similarly. If \( \lambda_x \sim \lambda_y^* \), we claim that there is an integer \( 1 \leq m < r \) such that either
\[
\frac{d_x(m) + \tilde{d}_y(m')}{m} > \mu(E)
\]
or \( \frac{d_x(m) + d_y(m')}{m'} > \mu(E) \) where \( m' = r - m \). If the claim is true, without loss of generality, we assume
\[
\frac{d_x(m) + \tilde{d}_y(m')}{m} > \mu(E)
\]
holds for some \( 1 \leq m < r \). Note \( K_{m_y' + 1} := \ker(Q_{m_y' + 1}(E)_y \to Q_{m_y'}(E)_y) \) has the dimension
\[
r_{m_y' + 1}(y) - r_{m_y'}(y) \geq r_{m_y' + 1}(y) - m'
\]
and, for any subspace \( W \subset K_{m_y' + 1} \) of the dimension \( r_{m_y' + 1}(y) - m' \), there is a sub-bundle \( F = O_{P_i}^{\oplus m'} \subset E \) such that \( Q_{m_y' + 1}(E)_y = W \). Then
\[
n_{m_y' + 1}(y) = r_{m_y' + 1}(y) - m', \quad n_i^F(y) = n_i(y) \quad (m_y' + 2 \leq i \leq l_y + 1),
\]
\[
n_i^F(y) = 0 \quad (1 \leq i \leq m_y'), \quad \tilde{d}_y(m') = \sum_{i=1}^{l_y + 1} \frac{a_i(y)}{k} n_i^F(y)
\]
and
\[
\sum_{i=1}^{l_y + 1} \frac{a_i(x)}{k} n_i^F(x) - d_x(m) = \sum_{i=m_y' + 1}^{l_x + 1} \frac{a_i(x)}{k} n_i^F(x) - \sum_{i=1}^{m_y} \frac{a_i(x)}{k} (n_i(x) - n_i^F(x)) - (m - r_{m_y}(x)) \frac{a_{m_y} + 1(x)}{k} \geq \left( \sum_{i=1}^{l_y + 1} n_i^F(x) - m \right) \frac{a_{m_y} + 1(x)}{k} = 0,
\]
which imply \( \mu(F) \geq \frac{d_x(m) + \tilde{d}_y(m')}{m} > \mu(E) \). Thus \( E \) is not semistable.

To prove the claim, assume both \( \frac{d_x(m) + \tilde{d}_y(m')}{m} \leq \mu(E) \) and \( \frac{d_x(m) + d_y(m')}{m'} \leq \mu(E) \) hold for all \( 1 \leq m < r \). We will show
\[
\frac{a_i(x)}{k} + \frac{a_{i-y-1+2}(y)}{k} = \mu(E), \quad n_i(y) = n_{l_y - i + 2}(y) \quad (4.16)
\]
for \( 1 \leq i \leq \min\{l_x + 1, l_y + 1\} \), which implies \( \lambda_x \sim \lambda_y^* \). Indeed, if both \( \frac{d_x(m) + \tilde{d}_y(m')}{m} \leq \mu(E) \) and \( \frac{d_x(m) + d_y(m')}{m'} \leq \mu(E) \) hold for all \( 1 \leq m < r \), we must have the equalities (for all \( 1 \leq m < r \))
\[
\frac{d_x(m) + \tilde{d}_y(m')}{m} = \mu(E), \quad \frac{d_x(m) + d_y(m')}{m'} = \mu(E) \quad (4.17)
\]
since \( d_x(m) + \tilde{d}_y(m') + d_x(m) + d_y(m') = (m + m') \mu(E) \). We will prove (4.16) by taking different \( 1 \leq m < r \) in (4.17). Check (4.16) for \( i = 1 \) firstly. By taking \( m = 1 \), we have \( d_x(m) = \frac{a_1(x)}{k}, \tilde{d}_y(m') = \frac{a_{l_y}(y)}{k} \) and \( \frac{a_1(x)}{k} + \frac{a_{l_y}(y)}{k} = \mu(E) \). Take \( m = r_1(x) \) and \( m' = r - r_1(x) \). Then \( d_x(m) = \frac{a_1(x)}{k} r_1(x) \) and
\[
\tilde{d}_y(m') = \sum_{i=m_y'+2}^{l_y + 1} \frac{a_i(y)}{k} n_i(y) + (r_{m_y' + 1}(y) - m') \frac{a_{m_y' + 1}(y)}{k}
\]
\[
\leq \frac{a_{l_y}(y)}{k} n_{l_y + 1}(y) + (r_{m_y' + 1}(y) - m') \frac{a_{m_y' + 1}(y)}{k} \quad (\text{if } m' < l_y)
\]
\[
< \frac{a_{l_y}(y)}{k} n_{l_y + 1}(y) + (r_{l_y}(y) - m') \frac{a_{l_y + 1}(y)}{k} = \frac{a_{l_y + 1}(y)}{k} r_1(x),
\]
which imply \( \varpi_m(E) = \frac{a_1(x) + a_{i+1}(y)}{m} < \frac{a_1(x)}{k} + \frac{a_{i+1}(y)}{k} = \varpi_m(E) \). Thus we must have \( m_y = l_y \), i.e., \( r_{l_y} \leq m' = r - r_1(x) \), which means that \( n_{l_y+1}(y) = r - r_{l_y} \geq r_1(x) = n_1(x) \). In fact, \( n_1(x) = n_{l_y+1}(y) \). Otherwise, take \( m = n_1(x) + 1 \leq n_{l_y+1}(y) \) (which implies \( m' = l_y \)). Then
\[
d_x(m) = \frac{a_1(x)}{k} n_1(x) + \frac{a_2(x)}{k}, \quad \tilde{d}_y(m') = \frac{a_{l_y+1}(x)}{k} m
\]
and (by \( a_1(x) < a_2(x) \)) we get a contradiction: \( \varpi_m(E) = \frac{d_x(m) + \tilde{d}_y(m')}{m} < \frac{a_1(x)}{k} + \frac{a_{l_y+1}(y)}{k} = \varpi_m(E) \).

Assume \( l < i_0 \leq \min\{l_x + 1, l_y + 1\} \) such that (4.16) holds for all \( i < i_0 \). We show (4.16) holds for \( i = i_0 \). Take \( m = r_{i_0-1}(x) + 1 \). Then \( m' = r - r_{i_0-1}(x) - 1 = r_{l_y-i_0+2}(y) - 1, m' = l_y - i_0 + 1 \),
\[
d_x(m) = \sum_{i=1}^{i_0-1} \frac{a_i(x)}{k} t_i(x) + \frac{a_{i_0}(x)}{k} \quad \text{and} \quad \tilde{d}_y(m') = \sum_{i=i_0}^{l_y-i_0+3} \frac{a_i(y)}{k} n_{i}(y) + \frac{a_{l_y-i_0+2}(y)}{k}.
\]
By (4.17) and \( \sum_{i=1}^{i_0-1} \frac{a_i(x)}{k} n_{i}(x) + \sum_{i=i_0}^{l_y-i_0+3} \frac{a_i(y)}{k} n_{i}(y) = r_{i_0-1}(x) \varpi_m(E) \), we have \( \frac{a_{i_0}(x)}{k} + \frac{a_{l_y-i_0+2}(y)}{k} = \varpi_m(E) \).

If \( n_{i_0}(x) > n_{l_y-i_0+2}(y) \), take \( m = r_{i_0}(x) \), then \( m' = r_{l_y-i_0+2}(y) - n_{i_0}(x) \) and \( m' < l_y - i_0 \) (otherwise \( n_{i_0}(x) < n_{l_y-i_0+2}(y) \)), which leads to a contradiction
\[
d_x(m) + \tilde{d}_y(m') = \sum_{i=1}^{i_0-1} \frac{a_i(x) + a_{l_y-i_0+2}(y)}{k} n_{i}(x) + \frac{a_{i_0}(x)}{k} n_{i_0}(x)
\]
\[
+ \sum_{i=i_0}^{l_y-i_0+2} \frac{a_i(y)}{k} n_{i}(y) + \frac{a_{l_y-i_0+2}(y)}{k} n_{i_0}(x) - m' \frac{a_{i_0}(y)}{k} n_{i_0}(x)
\]
\[
< r_{i_0-1}(x) \varpi_m(E) + \frac{a_{i_0}(x)}{k} n_{i_0}(x) + \frac{a_{l_y-i_0+2}(y)}{k} n_{i_0}(x)
\]
\[
= r_{i_0}(x) \varpi_m(E).
\]

If \( n_{i_0}(x) < n_{l_y-i_0+2}(y) \), take \( m = r_{i_0}(x) + 1 \), then
\[
m' = r - r_{i_0}(x) - 1 = r_{l_y-i_0+2}(y) - (n_{i_0}(x) + 1) \geq r_{l_y-i_0+1}(y).
\]
Thus \( m'_y = l_y - i_0 + 1 \) and
\[
d_x(m) + \tilde{d}_y(m') = \sum_{i=1}^{i_0} \frac{a_i(x) + a_{l_y-i_0+2}(y)}{k} n_{i}(x) + \frac{a_{i_0}(x)}{k} + \frac{a_{l_y-i_0+2}(y)}{k}
\]
\[
= r_{i_0}(x) + 1) \varpi_m(E) + \frac{a_{i_0}(x)}{k} - \frac{a_{i_0}(x)}{k},
\]
which leads to a contradiction since \( a_{i_0+1}(x) > a_{i_0}(x) \). We must have \( n_{i_0}(x) = n_{l_y-i_0+2}(y) \).

To prove (4), let
\[
\mu = (\mu_1, \ldots, \mu_1, \ldots, \mu_1, \ldots, \mu_1) \in Y(\varpi, \omega_\psi)
\]
and recall that the parabolic structure at \( x \in \mathbb{P}^1 \) determined by \( \omega_\psi \) is
\[
E_x = Q_2(E_x) - Q_1(E_x) + Q_0(E_x)z = 0
\]
with \( n_1(x) = s, n_2(x) = r - s, a_1(x) = k - 1 \) and \( a_2(x) = k \). For any \( E \in \mathcal{R}_0 \), it is easy to compute that \( \varpi_m(E) = 2 - \frac{1}{2}(s + |\lambda_y| - |\lambda_x^s|) \).

When \( \lambda_x^s \sim \mu \), we have \( l = l_x, n_i = n_{l_x-i_2}(z) \) and there is a constant \( a \in \mathbb{Z} \) such that \( \mu_i - a_{l_x-i_2}(z) = a \ (1 \leq i \leq l_x + 1) \), which implies \( \varpi_m(E) = 2 - \frac{a}{k} \). We are going to prove that \( \mathcal{R}^s \cap \mathcal{R}^s_{(\omega_\psi, \lambda_y, \lambda_x)} \) is nonempty and any \( E \in \mathcal{R}^s \cap \mathcal{R}^s_{(\omega_\psi, \lambda_y, \lambda_x)} \) is \( S \)-equivalent to a direct sum of parabolic line bundles. If \( \mu_1 = k - a_1 + 1 \), let \( \lambda_i := k - a_i(y) - 1 \),
\[
\begin{align*}
\lambda_x^s &= (\lambda_0(y), \ldots, \lambda_0(y), \ldots, \lambda_0(y), \ldots, \lambda_{l_y+1}(y), \ldots, \lambda_{l_y+1}(y), \lambda_{l_y+1}(y), \ldots, \lambda_{l_y+1}(y),) \\
n_{l_x-i_2}(z) - 1 &= (\lambda_1(y), \lambda_1(y), \ldots, \lambda_1(y), \lambda_1(y), \lambda_1(y), \lambda_1(y), \lambda_1(y),) \\
n_{l_x-i_2}(z) &= (\lambda_1(y), \lambda_1(y), \lambda_1(y), \lambda_1(y), \lambda_1(y), \lambda_1(y),)
\end{align*}
\]
and recall that the parabolic structure at \( x \in \mathbb{P}^1 \) determined by \( \omega_\psi \) is
\[
E_x = Q_2(E_x) - Q_1(E_x) + Q_0(E_x)z = 0
\]
and $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, \{k - 1, a_1(y), a_{l+1}(z)\})$. Note that

$$
\lambda_z^* \sim \mu' = (\mu_1, \ldots, \mu_1, \ldots, \mu_{l+1}, \ldots, \mu_{l+1}) \in Y(\lambda'_y, \omega_{s-1}).
$$

We have $\mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}} \neq \emptyset$ by induction. Let

$$
E' = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1(r-1)}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\}) \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}
$$

and $E = \mathcal{L} \oplus E'$. Then $E \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$ since

$$
\par \mu(\mathcal{L}) = \frac{k - 1 + a_1(y) + a_{l+1}(z)}{k} = \par \mu(E') = 2 - \frac{a}{k}.
$$

Conversely, for any $E \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$, there is an $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, E \subset \mathcal{L})$ such that $n^c_{l+1}(z) = 1$. The semi-stability of $E$ implies $n^c_1(x) = 1$ and $n^c_2(y) = 1$. Then $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, \{k - 1, a_1(y), a_{l+1}(z)\})$ is a parabolic sub-bundle of $E = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\})$ with $\par \mu(\mathcal{L}) = \par \mu(E)$ and its quotient parabolic bundle $E' = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1(r-1)}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\})$ is semi-stable. By induction, $E$ is $S$-equivalent to a direct sum of parabolic line bundles.

If $\mu_1 = k - a_1(y)$, we have $a_1(y) + a_{l+1}(z) = k - a$,

$$
\lambda_z^* \sim \mu' = (\mu_1, \ldots, \mu_1, \ldots, \mu_{l+1}, \ldots, \mu_{l+1}) \in Y(\lambda'_y, \omega_{s-1})
$$

and there exists $E' = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1(r-1)}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\}) = \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$ by induction assumption on the rank. Then $E' \oplus \mathcal{L} \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$ where $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, \{k - 1, a_1(y), a_{l+1}(z)\})$ with

$$
\par \mu(\mathcal{L}) = \frac{k + a_1(y) + a_{l+1}(z)}{k} = 2 - \frac{a}{k} = \par \mu(E) = \par \mu(E').
$$

On the other hand, $\forall E \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$, let $E' = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1(r-1)}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\}) \subset E$ such that

$$
E'_y \supset K_y = \ker E_y = Q_{l+1}(E)_y \to Q_1(E)_y.
$$

Then the induced flag $E'_y \to Q_{l}(E)_y \to \cdots \to Q_1(E)_y$ must have $\dim(Q_i(E)_y) - 1 (1 \leq i \leq l + 1)$. Let $0 \to E' \to E \to \mathcal{L} \to 0$ and $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, \{a_i(x), a_1(y), a_{l+1}(z)\})$ be the induced parabolic quotient line bundle. Then $\par \mu(\mathcal{L}) = \par \mu(E) = \frac{k}{2} - \frac{a}{k}$ by semi-stability of $E$. But

$$
\par \mu(\mathcal{L}) = \frac{a_i(x) + a_1(y) + a_{l+1}(z)}{k} = 2 - \frac{a}{k} = \frac{a_i(x) - k + a_{l+1}(z) - a_{l+1}(z)}{k} \leq 2 - \frac{a}{k},
$$

the equality holds if and only if $a_i(x) = k$ and $a_{l+1}(z) = a_{l+1}(z)$. Thus $E$ is $S$-equivalent to $E' \oplus (\mathcal{O}_\mathfrak{p}_1, \{k, a_1(y), a_{l+1}(z)\})$, where

$$
E' = (\mathcal{O}_{\mathfrak{p}_1}^{\mathfrak{p}_1(r-1)}, \{\omega_{s-1}, \lambda'_y, \lambda'_z\}) \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}.
$$

When $\lambda_z^* \sim \mu$ for any $\mu \in Y(\lambda'_y, \omega_{s-1})$, we prove $\mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}} = \emptyset$. In fact, if there is an $E \in \mathcal{R}^0 \cap \mathcal{R}^s_{\{\omega_{s-1}, \lambda'_y, \lambda'_z\}}$, we will prove that there exists $\mu \in Y(\lambda'_y, \omega_{s-1})$ such that $\lambda_z^* \sim \mu$. Let $a = \frac{x + |\lambda_z^*| - |\lambda'_z|}{r} \in \mathbb{Z}$. Then $\par \mu(E) = \frac{k}{2} - \frac{a}{k}$ and $k - 1 + a_1(y) + a_{l+1}(z) \leq 2k - a$. Otherwise, let $\mathcal{L} = (\mathcal{O}_\mathfrak{p}_1, \{k, a_1(y), a_{l+1}(z)\})$ be a sub-bundle such that $n^c_{l+1}(z) = 1$, which implies

$$
\par \mu(\mathcal{L}) = \frac{a_i(x) + a_1(y) + a_{l+1}(z)}{k} \geq \frac{a_i(x) + a_1(y) + a_{l+1}(z)}{k} > 2 - \frac{a}{k}.
$$

Similarly, if $k + a_1(y) + a_{l+1}(z) < 2k - a$, we find a parabolic quotient line bundle $\mathcal{L}$ with

$$
\par \mu(\mathcal{L}) = \frac{a_i(x) + a_1(y) + a_{l+1}(z)}{k} \leq \frac{k + a_1(y) + a_{l+1}(z)}{k} < 2 - \frac{a}{k}.
$$
Thus $k-1+a_1(y)+a_{t+1}(z) \leq 2k-a \leq k+a_1(y)+a_{t+1}(z)$, which implies either $\frac{k-1+a_1(y)+a_{t+1}(z)}{k} = 2 - \frac{a}{k}$ or $\frac{k+a_1(y)+a_{t+1}(z)}{k} = 2 - \frac{a}{k}$. Then we have either $R^0 \cap R^3_{\{\omega_1, \lambda_1', \lambda_1\}} \neq \emptyset$ or $R^0 \cap R^3_{\{\omega_1, \lambda_1', \lambda_1\}} \neq \emptyset$ (see the proof of the case when $\lambda_2^* \sim \mu$). By induction, we have either $\lambda_2^* \sim \mu' \in Y(\lambda_2', \omega_1)$ or $\lambda_2^* \sim \mu \in Y(\lambda_2', \omega_1)$. \hfill $\square$

**Proposition 4.8.** $D_0(r, d, \{\lambda_x\}_{x \in I}) = V_0(r, d, \{\lambda_x\}_{x \in I})$ if $|I| < 3$.

**Proof.** We treat firstly the case $r | d$. By Lemmas 4.6 and 4.7, we can assume $d = 0$ and $|I| = 3$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, let

$$s(\lambda) = \max \{i \mid \lambda_i - \lambda_{i+1} > 0\}, \quad m(\lambda) = \sum_{i=1}^{s(\lambda)} \lambda_i$$

and $s(\lambda) = 0$ if $\lambda_1 = \lambda_2 = \cdots = \lambda_r$. When $s(\lambda) = 0$, $\lambda$ defines the trivial parabolic structure at $x \in I$ and the proof reduces to the case of $|I| = 2$. Thus, to prove

$$D_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\}) = V_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\}),$$

we can assume $s(\lambda_x) > 0$. We will prove it by induction on $m(\lambda_x) - s(\lambda_x)$.

When $m(\lambda_x) - s(\lambda_x) = 0$, $\lambda_x$ must be $\omega_{s(\lambda_x)}$ and the equality

$$D_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\}) = V_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\})$$

(4.18) holds by Lemmas 4.6(4) and 4.7. Assume the equality (4.18) holds for any $\lambda_y$ and $\lambda_z$ when $m(\lambda_x) - s(\lambda_x) < N$. For any $\lambda_x$ with $m(\lambda_x) - s(\lambda_x) = N$, let $\lambda_x' = \lambda_x - \omega_{s(\lambda_x)}$. Then $\mu \in Y(\lambda_x', \omega_{s(\lambda_x)}) - \{\lambda_x\}$ is obtained by adding $t < s(\lambda_x)$ boxes to the first $s(\lambda_x)$ rows and one box for each row running from the $(s(\lambda_x) + 1)$-th to the $(2s(\lambda_x) - t)$-th row. Thus

$$m(\mu) - s(\mu) = m(\lambda_x) - s(\lambda_x) - (s(\lambda_x) - t) < N.$$

Then, by the recurrence relation (3.9) and Lemma 4.7(4), we have

$$D_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_x', \lambda_y, \lambda_z\})$$

$$= \sum_{\mu^* \in W^r_0} D_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_x', \mu^*\}) \cdot D_0(r, 0, \{\mu, \lambda_y, \lambda_z\})$$

$$= \sum_{\mu \in Y(\lambda_x', \omega_{s(\lambda_x)})} D_0(r, 0, \{\mu, \lambda_y, \lambda_z\})$$

$$= D_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\}) + \sum_{\mu \in Y(\lambda_x', \omega_{s(\lambda_x)}) \setminus \{\lambda_x\}} V_0(r, 0, \{\mu, \lambda_y, \lambda_z\}).$$

By Lemmas 4.6 and 4.7 and the recurrence relation in (3.9) again,

$$D_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_y, \lambda_z\})$$

$$= \sum_{\mu \in W^r_0} D_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_y, \mu\}) \cdot D_0(r, 0, \{\lambda_x, \lambda_z, \mu^*\})$$

$$= \sum_{\mu \in W^r_0} V_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_y, \mu\}) \cdot V_0(r, 0, \{\lambda_x, \lambda_z, \mu^*\})$$

$$= V_0(r, 0, \{\omega_{s(\lambda_x)}, \lambda_x', \lambda_y, \lambda_z\}),$$

where $D_0(r, 0, \{\lambda_x', \lambda_y, \mu^*\}) = V_0(r, 0, \{\lambda_x', \lambda_y, \mu^*\})$ by induction assumption (since either $m(\lambda_x') - s(\lambda_x')$
Recall that for any $N < N$ or $s(\lambda') = 0$. Thus

$$D_0(r, 0, \{\lambda_x, \lambda_y, \lambda_z\}) + \sum_{\mu \in Y(\lambda'_0, \nu(\lambda'_0)) \setminus \{\lambda_x\}} V_0(r, 0, \{\mu, \lambda_y, \lambda_z\})$$

$$= V_0(r, 0, \{\nu(\lambda'_0), \lambda'_x, \lambda_y, \lambda_z\})$$

$$= \frac{1}{r(r + k)} \sum_{\vartheta} \exp(-2\pi i \frac{|\nu| + |\lambda_0| + |\lambda_z|}{r(r + k)}) S_{(\lambda'_0), \lambda'_x, \lambda_y, \lambda_z}(\exp 2\pi i \frac{\vartheta}{r + k})$$

where $H^{r-d}(\lambda_z)$ is the Hecke transformation of $\lambda_z$ (which is the inverse of $H_z$ considered in Remark 3.11).

Note that, even if $|I| = 0$, we can add a trivial parabolic structure $\lambda_z = \{k, k, \ldots, k\}$ which does not change the numbers $D_g(r, d, \{\lambda_x\}_{x \in I})$ and $V_g(r, d, \{\lambda_x\}_{x \in I})$. Thus, to finish our proof, we only need to show

$$V_g(r, 0, \{\lambda_x\}_{x \in I}) = V_0(r, 0, \{\lambda_x\}_{x \in I} \cup \{H^{r-d}(\lambda_z)\}).$$

(4.19)

Recall that for any $\mu = (\mu_1, \ldots, \mu_r), 0 \leq \mu_r \leq \cdots \leq \mu_1 \leq k$, we define

$$H^1(\mu) = (k - \mu_{r-1} + \mu_r, \mu_1 - \mu_{r-1}, \mu_2 - \mu_{r-1}, \ldots, \mu_{r-2} - \mu_{r-1}, 0)$$

and $H^m(\mu) := H^1(H^{m-1}(\mu))$ for $2 \leq m \leq r$. It is enough to show

$$V_g(r, d, \{\lambda_x\}_{x \in I}) = V_g(r, d + 1, \{\lambda_x\}_{x \in I} \cup \{H^1(\lambda_z)\}).$$

(4.20)

Note that $|H^1(\mu)| = k - r \mu_{r-1} + |\mu|$ and

$$S_{H^1(\mu)}(\exp 2\pi i \frac{\vartheta}{r + k}) = (-1)^{r-1} \exp 2\pi i \left( - \frac{\mu_{r-1} + 1}{r + k} \sum_{i=1}^r \nu_i \right) S_{\mu}(\exp 2\pi i \frac{\vartheta}{r + k}).$$

It is easy to check (4.20) and we are done.

\[\Box\]

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