Measures on contour, polymer or animal models.

A probabilistic approach

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Abstract. We present a new approach to study measures on ensembles of contours, polymers or other objects interacting by some sort of exclusion condition. For concreteness we develop it here for the case of Peierls contours. Unlike existing methods, which are based on cluster-expansion formalisms and/or complex analysis, our method is strictly probabilistic and hence can be applied even in the absence of analyticity properties. It involves a Harris graphical construction of a loss network for which the measure of interest is invariant. The existence of the process and its mixing properties depend on the absence of infinite clusters for a dual oriented percolation process which we dominate by a multitype branching process. Within the region of subcriticality of this branching process the approach yields: (i) exponential convergence to the equilibrium measure, (ii) clustering and finite-effect properties of the contour measure, (iii) a particularly strong form of the central limit theorem, and (iv) a Poisson approximation for the distribution of contours at low temperature.

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1 Introduction

Contours were introduced by Peierls (1936) to prove the existence of a first-order phase transition for the Ising model in 2 or more dimensions. His argument, later put on a rigorous mathematical basis by Dobrushin (1965) and Griffiths (1964), used contours only as an auxiliary device to estimate spin correlations. Polymer models, in the sense of interest here, were introduced later by Gruber and Kunz (1971). These are abstract general models of possibly extended objects that interact only by volume exclusion. They include both contour ensembles and ensembles formed by the open walks (“polymers”) or surfaces obtained in high-temperature expansions. Gruber and Kunz (1971) were the first to treat these models as probability ensembles of their own, and to ask genuinely probabilistic questions such as existence and properties of the corresponding probability measure. The formalism of cluster expansions, whose use in mathematical physics started with a paper by Glimm, Jaffe and Spencer (1976), soon established itself as the technique of choice to study these type of systems and questions [Malyshev (1980), Seiler (1982), Brydges (1984)]. The formalism was extended by Kotecký and Preiss (1986) to objects obeying generalized exclusion laws defined by compatibility relations. This extension was taken up by Dobrushin (1996, 1996a) who proposed to call animal models to such general systems and introduced a new approach to the construction of the expansions.

The cluster-expansion technology, being designed to construct and study distributions of general systems with exclusions, seems potentially very useful for probabilists in general. Nevertheless, its use has so far remained confined to the mathematical physics community working in statistical mechanics and quantum field theory. This unfortunate situation has been pointed out by Dobrushin (1996a), who attributed it to two reasons: (1) “its analytical and combinatorial complexity”, and (2) “the absence . . . [of a] systematical exposition oriented to mathematicians”. He addressed both issues in his posthumous review, Dobrushin (1996a), where he presented an exposition geared towards “probabilistic interpretations and applica-
tions”, based on his new approach that avoids “tremendous combinatorial considerations”. In fact, his approach does not resort to cluster expansions at all.

In our opinion there is still an additional aspect that explains the lack of popularity, among probabilists, of this powerful technique: All the existing formulations transcend the probabilistic framework. First, the expansions used are a bit unnatural from the measure-theoretical point of view. Cluster expansions were, in fact, originally introduced to control the pressure of gases with exclusions. The rigorous proof that they converge and have nice mathematical properties required highly nontrivial combinatorial estimates which took a reasonable form only with the insight of Cammarotta (1982) [clearly described in Brydges (1984) and Pfister (1991)]. The existence of a measure is proven by using pressure-like expansions for numerator and denominator and cancelling out terms. It is, therefore, a rather indirect approach whose mathematical bottleneck refers to an object —the pressure— that from the probabilistic point of view is just auxiliary. Dobrushin’s approach, on the other hand, avoids the use of the pressure and the cluster expansion but its use of complex analysis reveals that the corresponding hypotheses and results go beyond probability. Furthermore, for actual computations one needs to go back to the traditional approach and its explicit expressions for the correlation functions (or the pressure).

A second manifestation of “probabilistic unnaturalness” is conveyed by the results themselves. Indeed, the existing formulations require the absolute convergence of the expansions involved. As a consequence, besides existence and mixing properties, they yield analyticity of the correlation functions with respect to different parameters, for instance with respect to the exponential of minus the inverse temperature. Though analyticity is a very nice property to have —in particular it allows Dobrushin to produce an amazingly simple proof of the central limit theorem— it is also a symptom that these approaches are too strong and not optimal from the probabilist point of view. This is not just an academic remark. The most interesting recent applications of cluster-expansion methods fall outside these formulations, as they involve measures that are known or suspected to have non-analytical behavior: Measures at intermediate temperatures [Olivieri (1988), Olivieri and Picco(1990)], measures for annealed disordered systems [von Dreifus, Klein and Perez (1995)], measures for long-range interactions [Bricmont
and Kupiainen (1996)] and infinite-dimensional Sinai-Ruelle-Bowen measures [Bricmont and Kupiainen (1997)].

In this paper we present a novel approach to the study of animal models which presents a number of advantages regarding these issues. For concreteness we discuss the case of usual Peierls contours; a more general treatment will be presented in Fernández, Ferrari and García (1998a). Here is an overview of the main features of our approach.

1. The approach is purely probabilistic, no cluster expansion or complex analysis is involved. The measure is obtained as the unique stationary measure of a Markov process. The condition of validity of our theory is stated in terms of a backwards oriented percolation process. The theory holds when percolation is absent.

2. The range of validity of the theory exceeds that of previous approaches [see comment after (2.10)]. Within this range we obtain all the properties yielded by the latter — existence, uniqueness, exponential mixing, central limit theorem— with one conspicuous, and expected, exception: analyticity.

3. We obtain a rather nice version of the central limit theorem [stronger than that in Dobrushin (1996a)].

4. The approach allows us to show that the rescaled distribution of contours of a fixed length convergence towards a Poisson process. We are not aware of similar results in the literature.

5. The construction constitutes, in fact, a simulation scheme that converges to equilibrium exponentially fast. Hence, it has the potential to become a very efficient computational tool.

6. The avoidance of series expansions for the pressure makes our approach more direct to compute general properties of the equilibrium measure, but limits its use for the estimation of “thermodynamic” quantities. For instance, the approach does not seem to be suitable for the study of “surface corrections” to the pressure. Bounds on these corrections...
are crucial for several applications of contour ensembles [see eg. Zahradníc (1984), Borgs and Imbrie (1989)].

In this paper we present a careful statement of these results and a sketch of their proofs. We aim at providing a streamlined exposition free of inessential technicalities that may obscure the natural form of the construction. Nevertheless, we present enough details for an educated probabilist to reconstruct most of the missing links. The full argument will be presented in Fernández, Ferrari and Garcia (1998), theretofore referred as FFG.

2 Contour distribution and loss networks. Results.

2.1 Contours

The contours for the ferromagnetic Ising model with “+1” boundary conditions, in dimensions \(d \geq 2\), are surfaces constructed with \((d - 1)\)-dimensional unit cubes —traditionally known as plaquettes— centered at points of \(\mathbb{Z}^d\) and perpendicular to the edges of the dual lattice \(\mathbb{Z}^d + (\frac{1}{2}, \ldots, \frac{1}{2})\). We shall identify a plaquette with its center and denote \(x \in \gamma\) if the plaquette centered at \(x\) is in \(\gamma\). Two plaquettes are adjacent if they have a common \((d - 2)\)-dimensional face. A collection of plaquettes forms a connected surface if for every two plaquettes \(x, y\) one can find a finite sequence of plaquettes, starting at \(x\) and ending at \(y\), such that two consecutive plaquettes of the sequence are adjacent. A closed surface has every \((d - 2)\)-dimensional face shared by 2 or 4 plaquettes. A contour, \(\gamma\), is a connected and closed family of plaquettes. We say that two contours \(\gamma\) and \(\theta\) are incompatible, and denote \(\gamma \cap \theta \neq \emptyset\), if they have adjacent plaquettes. We use the notation \(|x - y|\) for the minimal number of plaquettes needed to link, in a connected fashion, \(x\) with \(y\) (this is also known as “Manhattan distance”).

For \(\Lambda \subset \mathbb{Z}^d\), denote by \(G(\Lambda)\) the set of contours whose plaquettes have centers in \(\Lambda\). A configuration of contours \(\eta \in \mathbb{N}^{G(\Lambda)}\) is a function that at each contour \(\gamma\) assigns a natural number \(\eta(\gamma)\) indicating the number of contours \(\gamma\) present in \(\eta\). The subset \(\mathcal{X}(\Lambda) \subset \mathbb{N}^{G(\Lambda)}\) of compatible-contour configurations is defined as

\[
\mathcal{X}(\Lambda) = \{\eta \in \{0, 1\}^{G(\Lambda)}; \eta(\gamma) \eta(\theta) = 0 \text{ if } \gamma \cap \theta \neq \emptyset\}
\]
that is, a configuration of contours is compatible if it contains at most one copy of each contour and does not contain two intersecting contours.

For each fixed $\beta \in \mathbb{R}^+$, a parameter usually called the inverse temperature and for each finite $\Lambda \subset \mathbb{Z}^d$ define the measure $\mu^\Lambda$ on $\mathcal{X}(\Lambda)$ by

$$
\mu^\Lambda(\eta) = \frac{\exp\left(-\beta \sum_{\gamma; \eta(\gamma)=1} |\gamma|\right)}{Z^\Lambda}
$$

where $|\gamma|$ is the area (=number of plaquettes) of the contour $\gamma$ and $Z^\Lambda$ is a normalization constant making $\mu^\Lambda$ a probability.

### 2.2 Loss network of contours

We introduce a birth-and-death dynamics on the set of compatible contours. This process is known in the literature as loss network, see Kelly (1991) and references therein.

We define the process $\eta^\Lambda_t$ as a Markov process on $\mathcal{X}(\Lambda)$ with generator given by:

$$
A^\Lambda f(\eta) = \sum_{\gamma \in \mathcal{G}(\Lambda)} e^{-\beta|\gamma|} 1\{\eta^{+\gamma} \in \mathcal{X}(\Lambda)\} [f(\eta^{+\gamma}) - f(\eta)] + \sum_{\gamma \in \mathcal{G}(\Lambda)} \eta(\gamma) [f(\eta^{-\gamma}) - f(\eta)]
$$

for $f : \mathcal{X}(\Lambda) \to \mathbb{R}$, where $1\{ \cdot \}$ denotes the characteristic function of the set $\{ \cdot \}$ and for $\gamma \in \mathcal{G}(\Lambda)$,

$$
\eta^{\pm\gamma}(\theta) = \begin{cases} 
\eta(\theta) & \text{if } \theta \neq \gamma \\
\eta(\gamma) \pm 1 & \text{if } \theta = \gamma 
\end{cases}
$$

It is immediate to check that the measure $\mu^\Lambda$ is reversible for $\eta^\Lambda_t$.

In terms of loss network language, the above process can be described as follows. Consider a network consisting of a finite number of links represented by plaquettes with vertices in $\Lambda \subset \mathbb{Z}^d$, each link comprising one circuit. Calls are offered to this network along routes $\gamma \in \mathcal{G}(\Lambda)$ according to independent Poisson processes with rate $e^{-\beta|\gamma|}$. A call accepted on route $\gamma$ holds all links along this route for an exponential holding time with mean 1 and on completion of the service releases all these circuits simultaneously. All arrival streams and holding times are mutually independent. A call is accepted along route $\gamma \in \mathcal{G}(\Lambda)$ if $\gamma$ is not compatible with
other calls already in progress. Hence \( \eta_t^A = (\eta_t^A(\gamma))_{\gamma \in \mathcal{G}(\Lambda)} \) where \( \eta_t^A(\gamma) \) is the number of calls in progress on route \( \gamma \) at time \( t \), then a call is accepted along route \( \gamma \) at time \( t \) if

\[
\sum_{\gamma' : \gamma' \cap \gamma \neq \emptyset} \eta_t^A(\gamma') = 0.
\]

We can represent this model as a solution of the following system of equations:

\[
\eta_t^A(\gamma) = \eta_0^A(\gamma) + \int_0^t \mathbf{1} \{ \sum_{\gamma' : \gamma' \cap \gamma \neq \emptyset} \eta_t^A(\gamma') = 0 \} dN^+_{\gamma}(e^{-\beta|\gamma|}) - N^-_{\gamma} \left( \int_0^t \eta_t^A(\gamma) \right) ds \tag{2.5}
\]

where \( N^+_{\gamma} e N^-_{\gamma} \) are independent unit Poisson processes; \( N^+_{\gamma} \) creates new contours and \( N^-_{\gamma} \) destroys them.

### 2.3 Range of validity of the approach

Let \( \mathcal{X} = \{ \eta \in \{0,1\}^{\mathcal{G}(\mathbb{Z}^d)} : \eta(\gamma)\eta(\theta) = 0 \text{ if } \gamma \cap \theta \neq \emptyset \}. \) Since \( \mathcal{G}(\mathbb{Z}^d) \) is countable, \( \mathcal{X} \) is compact in the product topology. Let \( f \) be a continuous function on \( \mathcal{X} \). The infinite-volume loss network on \( \mathcal{X} \) has formal generator given by

\[
Af(\eta) = \sum_{\gamma \in \mathcal{G}} e^{-\beta|\gamma|} \mathbf{1}\{\eta^{+\gamma} \in \mathcal{X}\} [f(\eta^{+\gamma}) - f(\eta)] + \sum_{\gamma \in \mathcal{G}} \eta(\gamma) [f(\eta^{\gamma}) - f(\eta)] \tag{2.6}
\]

where \( \eta^{+\gamma} \) was defined in (2.4).

We use a graphical construction to show that a sufficient condition for the existence of a process \( \eta_t \) on \( \mathcal{X} \) with generator \( A \) is

\[
\lambda_\beta = \sum_{\gamma \in \mathcal{G}} |\gamma| e^{-\beta|\gamma|} < \infty. \tag{2.7}
\]

Using the fact that \( \mathcal{X} \) is compact, abstract nonsense imply that, under (2.7), there exists an invariant measure \( \mu \) for \( \eta_t \). However, the way of proving existence is so general that we are not able to show any further property of this measure. We remark that, as pointed out by Aizenman, Bricmont and Lebowitz (1987), (2.7) defines a “Peierls” inverse temperature,

\[
\beta_P = \inf \{ \beta : \lambda_\beta < \infty \}, \tag{2.8}
\]
above which, with probability one, only a finite number of contours surround any given site (a fact that, for the Ising model, implies existence of spontaneous magnetization). The results of this paper, however, apply to the more limited regime

$$\beta > \beta_M,$$  \hspace{1cm} (2.9)

where

$$\beta_M = \inf\{\beta : \lambda\beta < 1/(d-1)\}. \hspace{1cm} (2.10)$$

For the Peierls contours the best estimations of the range of validity of “traditional” cluster-expansion approaches follow from Proposition 5.6 in Dobrushin (1996a), which has been stated in its most precise form by Lebowitz and Mazel (1997). As a matter of fact, these authors present their estimations in a form slightly different to ours: They consider contours with a given site of the dual lattice in its interior, rather than contours containing a given plaquette as we do. The final expressions obtained in these two cases are not directly comparable because they involve differently-aimed upper bounds. For a meaningful comparison we have either to transcribe our approach in terms of interior sites, or to write theirs in terms of anchoring plaquettes. The latter policy leads to a bound

$$\sum_{\gamma \ni 0} e^{c|\gamma|} e^{-\beta|\gamma|} \leq \frac{c}{d-1}, \hspace{1cm} (2.12)$$

for some constant $c > 0$. This bound can be read off the work of Lebowitz and Mazel (1997) [who obtain $c = \beta e^{-d\beta/4}$], where in fact all the hard estimates [from their formula (2.7) till the end of their paper] refer to contours containing a fixed plaquette. On the other hand, our condition (2.9)–(2.10) implies

$$\sum_{\gamma \ni 0} |\gamma| e^{-\beta|\gamma|} \leq \frac{1}{d-1}, \hspace{1cm} (2.12)$$

which is strictly weaker than (2.11) because $e^x > x$ for $x \geq 1$.

Lebowitz and Mazel show that, defining $\beta_{LM}$ as the infimum of $\beta$ satisfying (2.11),

$$\beta_{LM} \geq 64 \frac{\log d}{d},$$  \hspace{1cm} (2.13)

where (2.12) plus their counting method, yields

$$\beta_M \geq 6 \frac{\log d}{d}. \hspace{1cm} (2.14)$$
On the other hand, Aizenmann, Bricmont and Lebowitz (1987) show that the Peierls temperature defined by (2.8) satisfies
\[ \beta_P \geq \log d \frac{\log d}{2d}. \tag{2.15} \]

These three temperatures mark, therefore, limits where different properties can be proven by perturbation arguments. For \( \beta \geq \beta_P \), each site of the dual lattice is surrounded by a finite number of contours. In spin language, this means lack of percolation of minority spins (which, in turn, implies symmetry breaking and, by FKG, non zero magnetization). For \( \beta \geq \beta_M \), in addition, properties R1—R5 listed below can be proven by cluster-expansion-like methods. Finally when \( \beta \geq \beta_{LM} \) methods of this type also yield analytic temperature dependence.

### 2.4 Results

We say that \( f \) has support in \( \Upsilon \subset \mathbb{Z}^d \) if \( f \) depends only on contours intersecting \( \Upsilon \) (not necessarily contained in \( \Upsilon \)). Let \( |\text{Supp}(f)| = \min\{|\Upsilon| : f \text{ has support in } \Upsilon\} \). When we write \( \text{Supp}(f) \) we mean any \( \Upsilon \) such that \( |\Upsilon| = |\text{Supp}(f)| \) and \( f \) has support in \( \Upsilon \). For instance, if \( f(\eta) = \eta(\gamma) \), \( \text{Supp}(f) \) may be set as \( \{x\} \) for any \( x \in \gamma \).

A closer analysis of the graphical construction allows us to show that for \( \beta > \beta_M \) the following results hold. These are our main results.

**R1.** Reversibility and uniqueness: there exists a unique invariant measure \( \mu \) for \( \eta_t \). Furthermore, \( \mu \) is reversible for the process \( \eta_t \).

**R2.** The rate of convergence to the invariant measure is exponential. Let \( \delta_\xi S(t) \) be the distribution of the process at time \( t \) when the initial configuration is \( \xi \). For measurable \( f \) we prove
\[ \sup_{\xi \in \mathcal{X}} |\mu f - \delta_\xi S(t)f| \leq \|f\|_\infty |\text{Supp}(f)| e^{-M_0 t} \tag{2.16} \]
for any \( M_0 < (1 - (d - 1)\lambda_\beta)/(2 - (d - 1)\lambda_\beta) \).

**R3.** Infinite-volume limit: Let \( \Lambda \) be a (finite or infinite) subset of \( \mathbb{Z}^d \) and \( f \) a measurable
function depending on contours contained in $\Lambda$. Then

$$\left| \mu f - \mu^\Lambda f \right| \leq \|f\|_{\infty} M_2 \sum_{x \in \text{Supp}(f)} e^{-M_3 d(x, \Lambda^c)}$$

(2.17)

where $M_2 = e^{(\beta - \beta_M)/(d-1)}$ and $M_3 \geq (\beta - \beta_M)/(d-1)$. We denoted $d(x, \Lambda^c) = \min\{|x-y| : y \in \Lambda^c\}$.

R4. Clustering. For measurable functions $f$ and $g$ depending on contours contained in an arbitrary set $\Lambda \subset \mathbb{Z}^d$:

$$\left| \mu^\Lambda (fg) - \mu^\Lambda f \mu^\Lambda g \right| \leq 2 \|f\|_{\infty} \|g\|_{\infty} (M_2)^2 \sum_{x \in \text{Supp}(f), \ y \in \text{Supp}(g)} |x - y| e^{-M_3 |x-y|}$$

(2.18)

where $M_2$ and $M_3$ are the same of (2.17). This includes the infinite-volume measure $\mu^\mathbb{Z}^d = \mu$.

R5. Central limit theorem. Let $f$ be a measurable function on $\mathcal{X}$ with finite support such that $\mu f = 0$ and $\mu(|f|^{2+\delta}) < \infty$ for some $\delta > 0$. Assume $D = \sum_x \mu(\tau_x f) > 0$. Then $D < \infty$ and

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{x \in \Lambda} \tau_x f \xrightarrow{\Lambda \to \mathbb{Z}^d} \text{Normal}(0, D)$$

(2.19)

where the double arrow means convergence in distribution. This result generalizes (the central limit) Theorem 7.4 of Dobrushin (1996a). In the latter, only functions depending on a finite number of contours are considered.

For the following result we write $\mu_\beta$ to stress the $\beta$ dependence of $\mu$.

R6. Poisson approximation. Let $\eta^\beta$ distributed with $\mu_\beta$. For each measurable $V \subset \mathbb{R}^d$ let

$$V(a) = \{x \in \mathbb{Z}^d : x/a \in V\}$$

(2.20)

For each $j$, the process $N_{j,\beta}$ defined by

$$N_{j,\beta}(V) = \sum_{\gamma \subset V(e^{j\beta}), |\gamma| = j} \eta^\beta(\gamma).$$

(2.21)

converges weakly to a unit Poisson process on $\mathbb{R}^d$ as $\beta \to \infty$. The rate of convergence is exponential in $\beta$. 


The key to the proof of the above results is a graphical construction of the process starting from a marked Poisson process in $\mathbb{Z}^d \times \mathbb{R}$. The marks determine random cylinders whose bases are the contours and the heights are exponentially distributed random times. The exclusion condition is imposed through the study of the “ancestors” of each cylinder (Section 3). These ancestors determine a (backwards) oriented percolation process, and our construction is feasible if there is no such percolation. This is the meaning of the condition $\beta > \beta_M$. All our results follow from the estimation of the spatial and temporal extension of the cluster of a (finite number of) cylinders(s) (Section 4 and 5). This estimation is done through a domination of the percolation process by a multiple branching process which, in the regime $\beta > \beta_M$ has exponential moments (Section 6). The proof of R6, also based in the above properties, is omitted here. A complete proof is presented in FFG.

Ferrari and Garcia (1998) used space-time percolation to show ergodicity of loss networks under low arrival-rate of calls.

3 Graphical construction. The BO-cluster

3.1 Finite volume

To each contour $\gamma \in \mathcal{G}$ we associate a Poisson process of rate $e^{-\beta|\gamma|}$, and to each time event $T_k(\gamma)$ of the Poisson process we associate an independent exponentially distributed time $S_k(\gamma)$ of mean one. The collection $\mathbf{C} = (T_k(\gamma), S_k(\gamma))_{\gamma \in \mathcal{G}, k \in \mathbb{Z}}$ is a family of double-sided independent marked Poisson processes, with the convention $T_{-1}(\gamma) < 0 < T_0(\gamma)$. The $k$th attempt of birth of a contour $\gamma$ occurs at time $T_k(\gamma)$; $S_k(\gamma)$ corresponds to the lifetime of the contour. Each triplet $(\gamma, T_k(\gamma), S_k(\gamma))$ is called a cylinder of basis $\gamma$ birth-time $T_k(\gamma)$ and lifetime $S_k(\gamma)$. To each contour $\theta$ present in the initial configuration $\eta_0 = \eta$ we independently associate an exponential time $S(\theta)$ and cylinder $(\theta, 0, S(\theta))$. The collection of initial cylinders is called $\mathbf{C}(0)$. We realize the dynamics $\eta_t$ as a (deterministic) function of $\mathbf{C}$ and $\mathbf{C}(0)$.

When the number of possible contours is finite, the construction for $t > 0$ is as follows. We construct inductively $\mathbf{K}_{[0,t]}$, the set of kept cylinders. The complementary set corresponds to
erased cylinders. First include all cylinders of \( C(0) \) in \( K_{[0,t]} \). Then, move forward in time and consider the first Poisson mark: The corresponding cylinder is erased if it intersects any of the cylinders already in \( K_{[0,t]} \), otherwise it is kept. This procedure is successively performed mark by mark until all cylinders born before \( t \) are considered. Define \( \eta_t \in \mathcal{X}(\Lambda) \) as

\[
\eta_t(\gamma) = \eta_0(\gamma) \mathbf{1}\{S(\gamma) > t\} + \mathbf{1}\{\exists k : (T_k(\gamma), T_k(\gamma) + S_k(\gamma)) \ni t \text{ and } (\gamma, T_k(\gamma), S_k(\gamma)) \text{ is kept}\},
\]

that is, \( \eta_t \) signals all contours which are basis of a kept cylinder that is alive at time \( t \).

It is tedious but easy to show that \( \eta_t \) has as generator an operator defined as \( A \), but with the sums restricted to the finite set of contours involved. In particular, when the contours are contained in a finite region \( \Lambda \), we obtain the process \( \eta_t^\Lambda \) with generator \( A^\Lambda \).

The above finite-volume construction can also be performed in \((-\infty, \infty)\). Indeed, \( \eta_t^\Lambda \) is an irreducible Markov process in a finite state space. Hence, with probability one there exists a sequence of ordered random times \( t_k(C) \) such that no cylinder in \( C \) is alive by time \( t_k(C) \). Furthermore \( \mathbb{E}(t_{k+1} - t_k) < \infty \). Therefore one can apply the above construction independently in each interval \([t_k(C), t_{k+1}(C))\). In this case the cylinders of \( C(0) \) play no role. This procedure is time-translation invariant and so is the distribution of \( \eta_t \). This distribution is precisely given by the measure \( \mu^\Lambda \).

### 3.2 Infinite volume

For infinite volume, the Poisson processes are indexed by an infinite set of contours. Hence, it is not possible to decide which is the first mark in time. The construction must be performed more carefully. There are two alternatives.

The first alternative is to divide the time interval \([0, t]\) in successive intervals of small length \( h \) and perform the construction in each one of those intervals. Under (2.7) and for small \( h \), it is possible to partition \( \mathbb{Z}^d \) in finite regions such that each contour born in \([0, h]\) is contained in exactly one of these regions. To show this, one considers the percolation of (projected) contours and dominates the area occupied by the contours by a branching process. Such a construction is at the heart of Harris (1972) original graphical construction of particle systems.
and it is reviewed by Durrett (1995). The mark-by-mark construction described above can be performed in each of these finite regions to construct the process in the time interval \([0, h]\). The same procedure can then be applied in the interval \([h, 2h]\), etc.

The second alternative is the one we really use. In order to know whether a cylinder \(C \in \mathbf{C}\) is kept, one has to look at the set of cylinders \(C'\) (born before \(C\) and) alive at the birth-time of \(C\) whose basis intersects the basis of \(C\). This set is called the first generation of ancestors of \(C\). The second generation of ancestors of \(C\) consists, previsibly, of the ancestors of the ancestors, that is those cylinders that are in the first generation of ancestors of some \(C'\) in the first generation of ancestors of \(C\). Recursively we construct in this way the \(n\)th generation of ancestors of \(C\). The set of ancestors (of any generation) of \(C\) is called the \(BO-\) (backwards oriented) cluster of \(C\) and it is denoted by \(A(C)\). This set may contain cylinders in \(\mathbf{C}(0)\). [We remark that this BO-cluster is a cluster of space-time cylinders, it is different from the usual cluster of contours considered in the classical works on cluster expansions.] If for some \(C\) the BO-cluster of \(C\) has a finite number of cylinders, then we can decide whether \(C\) is kept or not by looking at \(A_{[0,t]}(C) = \{C' \in A(C) : C' \text{ is born in } [0, t]\}\), the set cylinders in the BO-cluster of \(C\) born in \([0, t]\). This is done in the following way. First, those \(C' \in A_{[0,t]}(C)\) that have no ancestors are kept. Then we look to the remaining cylinders in \(A_{[0,t]}(C)\) and erase those that have a kept cylinder in its first generation. We repeat these two steps for the cylinders in \(A_{[0,t]}(C)\) that have not already declared to be kept or erased, and continue in this way until we reach \(C\). The end result is a partition of \(A_{[0,t]}(C)\) in two subsets formed, respectively, by kept and erased cylinders. In particular, the subset to which \(C\) belongs decides its status. In fact, under \((2.7)\) one can prove that all cylinders have a finite number of ancestors born in the interval \([0, t]\), and, thus, the process \(\eta_t\) can be constructed following, BO-cluster by BO-cluster, the steps of the finite case \((3.1)\). In the next section, we sketch the proof of the finiteness of the number of ancestors and give further details of the construction.

It is natural, and it turns out to be convenient, to extend the notion of ancestors of a cylinder to that of ancestors of a space-time point \((x, t)\), \(x \in \mathbb{Z}^d\) and \(t \in \mathbb{R}\): Let the first generation of ancestors of \((x, t)\) be the set of cylinders in \(\mathbf{C}\) whose basis contains \(x\) and are alive at time \(t\). The \(n\)th generation of ancestors of \((x, t)\) is then formed by the \((n - 1)\)-th generation of
ancestors of the cylinders in the first generation. The union of all the generations of ancestors is the BO-cluster \( A(x,t) \) of \((x,t)\). More generally, the set of ancestors of \( \Upsilon \subset \mathbb{Z}^d \) at time \( t \) is defined by
\[
A(\Upsilon, t) = \bigcup_{x \in \Upsilon} A(x, t).
\] (3.2)

4 Existence of \( \mu \) and exponential convergence

4.1 Backwards percolation

To perform the construction described in the previous section, every cylinder \( C \in \mathcal{C} \) must have a finite number of ancestors. If this is the case, we say that there is no (backwards oriented) percolation in \( C \). Hence, if with probability one there is no backwards percolation, the double infinite construction holds and we have a process \((\eta_t)_{t \in \mathbb{R}}\) that is time-translation invariant. The marginal distribution of \( \eta_t \) does not depend on \( t \) and it is called \( \mu \). By construction \( \mu \) is an invariant measure for \( \eta_t \). This shows the existence of \( \mu \) in a constructive way. In contrast, the existence of \( \mu \) under (2.7) uses a fixed point theorem.

The condition \( \beta > \beta_M \) implies that there is no percolation with probability one. This is shown by dominating the number of plaquettes in the bases of the cylinders in a BO-cluster by a branching process. The number
\[
(d - 1) \lambda \beta
\] (4.1)
is an upperbound on the mean number of branches of the process. That is, the mean number of plaquettes born from the branching (= incompatible contours) of each single plaquette. The process is subcritical if this number is less than one, thus the condition \( \beta > \beta_M \). This argument, sketched in Section 6 below is inspired by Hall (1985), who dominated a continuum percolation process by a branching process. We sketch this domination in Section 6 below. If there is no percolation, the number \(|A(x,t)|\) is finite for all \((x,t)\). As a consequence, there exists a function \( \Phi : (f, A(\text{Supp }f), t) \mapsto \Phi(f, A(\text{Supp }f), t) \) such that for any \( f \) with finite support
\[
f(\eta_t) = \Phi(f, A(\text{Supp }f), t).
\] (4.2)
For instance, to decide whether a contour $\gamma$ is present at time $t$ it suffices to look at the BO-cluster of $(x, t)$ for some $x \in \gamma$. The function $\Phi$ is the one that decides which cylinders are kept and indicates the presence/absence of $\gamma$ at time $t$.

4.2 Time length and space width of the BO-cluster

Most of the stated properties—uniqueness of $\mu$, exponential clustering and finite-volume effects, and exponential convergence to equilibrium of the loss network—follow from the observation that for $\beta > \beta_M$ both the time length and the space width of the BO-cluster of any given site decay exponentially. More precisely, let us introduce $\text{Proj}(\mathbf{A}(x, t)) \subset \mathbb{Z}^d$, the spatial projection of the BO-cluster, defined as the set of points in $\mathbb{Z}^d$ belonging to the basis of some cylinder in the BO-cluster:

$$\text{Proj}(\mathbf{A}(x, t)) = \bigcup_{\gamma \in \text{a}(x, t)} \{x \in \gamma\}$$

where $\text{a}(x, t) = \{\gamma : (\gamma, T_k(\gamma), S_k(\gamma)) \in \mathbf{A}(x, t) \text{ for some } k\}$, is the set of bases of the cylinders of the BO-cluster of $(x, t)$. The cardinality of this set will be bounded by the cumulative number of points:

$$\|\mathbf{A}(x, t)\| = \sum_{\gamma \in \text{a}(x, t)} |\gamma| \ .$$

Indeed, it is clear that

$$|\text{Proj}(\mathbf{A}(x, t))| \leq \|\mathbf{A}(x, t)\| \ .$$

We then have:

1. Let $E_2(t)$ be the set of $\mathbf{C}$ for which the BO-cluster of $(0, 0)$ has time-length larger than $t$:

$$E_2(t) = \{\mathbf{C} : \mathbf{C} \text{ is alive at time } -t \text{ for some } C \in \mathbf{A}(0, 0)\} .$$

Then, for $\beta > \beta_M$

$$\mathbb{P}(E_2(t)) \leq M_1 e^{-t(1-(d-1)\lambda_0)}$$

with $M_1 > 0$. 

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2. Let \( E_3(w) \) be the set of \( C \) for which the projection of the BO-cluster of \((0,0)\) is not contained in \([-w, w]^d\):

\[
E_3(w) = \left\{ C : \text{Proj} (A(0,0)) \not\subset [-w, w]^d \right\}.
\] (4.8)

Then, for \( \beta > \beta_M \)

\[
\mathbb{P}(E_3(w)) \leq M_2e^{-M_3w}
\] (4.9)

where \( M_2, M_3 \) are as in (2.17).

The proof of (4.7) and (4.9) are sketched in Section 6. To prove (4.7) we dominate \( A(0,0) \) by a continuous-time branching process. On the other hand, to prove (4.9) we dominate \( \|A(0,0)\| \) by the total population of a branching process.

### 4.3 Proof of R1 and R2

The exponentially fast time-convergence (2.16) is a consequence of (4.7) and (4.9). We use the same Poisson marks to construct simultaneously the stationary process \( \eta_t \) and a process starting at time zero with an arbitrary initial configuration \( \xi \). The second process is called \( \xi_t \), where \( \xi_0 = \xi \). The process \( \xi_t \) ignores the cylinders in \( C \) with birth-times less than 0 and considers cylinders in \( C(0) \) with basis in \( \xi \) and birth-time zero. The process \( \eta_t \) ignores the cylinders in \( C(0) \). Hence for any \( \gamma \ni 0 \),

\[
|\eta_t(\gamma) - \xi_t(\gamma)| \leq 1\{A(0,t) \neq \tilde{A}(0,t)\}
\] (4.10)

where \( \tilde{A}(0,t) \) is the cluster constructed in \( C_{[0,t]} \cup C(0) \). In FFG it is shown, via a coupling argument, how (4.8) and (4.9) imply that the expectation of the right hand side of (4.10) decays as \( \exp(-M_0 t) \). The exponential decay of length and width of the cluster implies exponential decay of the probability that it contains a cylinder of the initial configuration \( C(0) \).

The uniqueness of \( \mu \) follows immediately from (2.10).

Reversibility follows from the facts that \( (\eta_t^\Lambda) \) converges in distribution to \( (\eta_t) \), \( \mu^\Lambda \) converges to \( \mu \) and \( \mu^\Lambda \) is reversible for \( \eta_t^\Lambda \). From the construction, under \( \beta > \beta_M \), it is possible to show that \( (\eta_t^\Lambda) \) converges almost surely to \( (\eta_t) \). Some details are given in the next sections.
5 Space-time mixing and the central limit theorem

5.1 The key facts

The mixing properties of the measure $\mu$ are a consequence of the following space-time mixing properties of $C$.

- Let $f$ be a function depending on contours contained in a finite set $\Lambda$. Let $\eta^\Lambda_t$ be the loss network process constructed in $\Lambda$. Then
  \[ |\mathbb{E}(f(\eta_0)) - \mathbb{E}(f(\eta^\Lambda_0))| \leq 2 \|f\|_\infty \mathbb{P}(A(\text{Supp } (f), 0) \neq A^\Lambda(\text{Supp } (f), 0)) \tag{5.1} \]
  where $A^\Lambda(\text{Supp } (f), t)$ is the cluster of $(\text{Supp } (f), t)$ constructed with cylinders in
  \[ C^\Lambda = \{ (\gamma, T_k(\gamma), S_k(\gamma)) \in C : \gamma \subset \Lambda, k \in \mathbb{Z} \} \tag{5.2} \]
  the subset of cylinders whose basis is in $\Lambda$.

- For arbitrary measurable functions $f$ and $g$,
  \[
  |\mathbb{E}(f(\eta_0)g(\eta_0)) - \mathbb{E}(f(\eta_0)) \mathbb{E}(g(\eta_0))| 
  \leq 2 \|f\|_\infty \|g\|_\infty \mathbb{P}(C' \cap C'' \neq \emptyset \text{ for some } C' \in A(\text{Supp } (f), 0) \text{ and } C'' \in \hat{A}(\text{Supp } (g), 0)) \tag{5.3}
  \]
  where $\hat{A}(\text{Supp } (g), t)$ has the same distribution as $A(\text{Supp } (g), t)$ but is independent of $A(\text{Supp } (f), t)$.

The proof of (5.1) follows rather straightforwardly from the space-time construction. Using (4.2) we get
  \[ f(\eta_0) - f(\eta^\Lambda_0) = \left[ \Phi(f, A(\text{Supp } (f), 0)) - \Phi(f, A^\Lambda(\text{Supp } (f), 0)) \right] \times 1\{A(\text{Supp } (f), 0) \neq A^\Lambda(\text{Supp } (f), 0)\} \tag{5.4} \]
  As, by definition, $|\Phi(f, A(\text{Supp } (f), t))| \leq \|f\|_\infty$, taking expectations and absolute values in (5.4) we get (5.1).

The proof of (5.3) is similar in spirit but requires a somewhat more delicate argument based on the coupling of two continuous-time versions of the backwards percolation process. See details in FFG.
5.2 Proof of R3 and R4

To prove the finite-volume effects (2.17) we use the space-time representation (3.1) and get

\[ \mu f - \mu^\Lambda f = \mathbb{E} f(\eta_0) - \mathbb{E} f(\eta_0^\Lambda). \]  

(5.5)

By (5.1) it is enough to bound

\[ \mathbb{P}\left( A(\text{Supp}(f), 0) \neq A^\Lambda(\text{Supp}(f), 0) \right), \]  

(5.6)

which as in (4.9) is bounded by

\[ M_2 \sum_{x \in \text{Supp}(f)} e^{-M_3 d(x, \Lambda^c)}. \]  

(5.7)

This proves the decay stated in (2.17).

The proof of exponential mixing (2.18) is similar but using instead the bound (5.3).

While we have not yet done a careful study, we believe that (2.17) and (2.18) lead to sharper inequalities than those obtained via the use of “duplicated variables” [von Dreifus, Klein and Perez (1995), Bricmont and Kupiainen (1996)]. The reason is that clusters formed by superposition of two systems of contours have larger probabilities of intersection than our single-system clusters.

5.3 Proof of the central limit theorem

We use the results for stationary mixing random fields of Bolthausen (1982). Let \( X_x = \tau_x f \).

By hypothesis, \( \|X_x\|_{2+\delta} < \infty \). Under this conditions, Bolthausen (1982) shows that if

\[ \sum_{n=1}^{\infty} n^{d-1}(\alpha_{2,\infty}(n))^{\delta/(2+\delta)} < \infty \]  

(5.8)

then \( D < \infty \) and (2.19) holds. Here \( \alpha_{2,\infty}(n) \) measures the dependence between functions depending on the sigma algebra generated by \( X_0 \) and \( X_y \) and the sigma algebra generated by \( \{X_x : x \in \Lambda\} \) for \( |\Lambda| = \infty \) and \( \min\{|x|, |y - x| : x \in \Lambda\} > n \). In FFG we use (2.18) to show that

\[ \alpha_{2,\infty}(n) \leq (M_2)^2 |\text{Supp}(f)| \sum_{|y| \geq n-2|\text{Supp}(f)|} e^{-M_3 |y|} \]  

(5.9)
Hence, $\alpha_{2,\infty}(n)$ decreases exponentially fast with $n$. This shows the central limit theorem.

6 Length and width of the BO-cluster

To conclude, let us sketch the arguments behind the bounds (4.7) and (4.9). In both cases we rely on dominating branching processes.

6.1 Time length

To show (4.7) we consider a continuous time multitype Markov branching process $b_t$ on $\mathbb{N}^G$. In this process, each contour $\gamma$ lives an mean-one exponential time after which it dies and gives birth to $k_\theta$ contours $\theta$, $\theta \in \mathcal{G}$, with probability

$$\prod_{\theta} e^{\mu(\gamma, \theta)} (\mu(\gamma, \theta))^{k_\theta} k_\theta!$$

(6.1)

for $k_\theta \geq 0$. These are independent Poisson distributions of mean $\mu(\gamma, \theta) = 1\{\gamma \cap \theta \neq \emptyset\} e^{-\beta|\theta|}$.

Fix $b_0(\gamma) = |\{k : (\gamma, T_k(\gamma), S_k(\gamma)) \text{ is alive at time } 0\}| 1\{\gamma \ni 0\}$ and zero otherwise. Under this initial condition it is possible to couple $(b_t)_{t \geq 0}$ and $A(0,0)$ in such a way that

$$E_2(t) \subset \left\{ \sum_{\theta} b_t(\theta) = 0 \right\}.$$  

(6.2)

Using the backwards Kolmogorov equation for $R_t = \mathbb{E} \sum_{\theta} b_t(\theta)$, one can show that

$$\mathbb{P}\left( \sum_{\theta} b_t(\theta) > 0 \right) \leq e^{((d-1)\lambda_{\beta}-1)t}.$$  

(6.3)

6.2 Space width

Define a Galton-Watson branching process $Z_n \in \mathbb{N}$ as follows. Let $Y_i^n$ be i.i.d. non negative integer valued random variables with the same distribution as

$$Y := \sum_{\gamma \ni 0} (d-1)|\gamma|X_\gamma$$

(6.4)
where $X_\gamma$ are independent integer valued random variables with Poisson distribution of mean $e^{-\beta|\gamma|}$. Define $Z_0 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_i^n$$

(6.5)

(with the convention $\sum_{i=1}^{0} Y_i^n = 0$). It is possible to couple the BO-cluster $A(x,t)$ and $(Z_n)_{n \geq 0}$ in such a way that the number of plaquettes in the bases of the cylinders in the $n$th generation of ancestors of $(x,t)$ is less than or equal to $Z_n$:

$$\|A(x,t)\| \leq \sum_{n \geq 0} Z_n.$$  

(6.6)

Hence, to show (4.9) it suffices to prove

$$P(Z > k) \leq M_2 e^{-M_3 k}$$

(6.7)

where $Z = \sum_{n \geq 0} Z_n$. Call $F(b)$ the generating function of $Z$, we will prove that if $\beta > \beta_M$, $\bar{b} = \sup\{b : F(b) < \infty\} > 1$.

The generating function of $Y$ is given by

$$f(a) = E a^Y = \prod_{\gamma \geq 0} E a^{(d-1)|\gamma|X_\gamma} = \exp \left( \sum_{\gamma \geq 0} e^{-\beta|\gamma|}(a^{(d-1)|\gamma|} - 1) \right).$$

(6.8)

The radius of convergence of $f(a)$ is given by $\exp(\beta - \beta_P)$, where $\beta_P$ is defined in (2.8). For $\beta > \beta_M(> \beta_P)$, the radius of convergence is strictly larger than 1. The mean number of offsprings $E Z_1$ is given by

$$f'(a)|_{a=1} = E Y = (d - 1) \sum_{\gamma \geq 0} |\gamma|e^{-\beta|\gamma|} < 1$$

(6.9)

for $\beta > \beta_M$. Hence, our branching process is subcritical,

$$1 = f(1) \text{ and } x = f(x) \text{ implies } x \geq 1$$

(6.10)

i.e. the smallest solution of the equation $x = f(x)$ is 1.

By (13.3) of Harris (1963) $F(b)$, the generating function of $Z$, must satisfy the equation

$$F(b) = bf(F(b)).$$

(6.11)
The largest solution of this is

\[ \bar{b} = \frac{a}{f(a)} \]  

(6.12)

where \( \bar{a} \) is the solution of

\[ f'(a) = \frac{f(a)}{a} \]  

(6.13)

In this case, it is easy to see that

\[ f'(a) = \frac{f(a)}{a} (d - 1) \sum_{\gamma \geq 0} |\gamma| e^{-\beta |\gamma|} a^{(d - 1) |\gamma|} \]  

(6.14)

and \( \bar{a} \) is the solution of

\[ \sum_{\gamma \geq 0} |\gamma| e^{-\beta |\gamma|} a^{(d - 1) |\gamma|} = \frac{1}{d - 1} \]  

(6.15)

which gives us

\[ \bar{a} = e^{(\beta - \beta_M)/(d - 1)} \]  

(6.16)

Therefore,

\[ \bar{b} = \exp\left\{ \frac{\beta - \beta_M}{(d - 1)} + \sum_{\gamma \geq 0} e^{\beta_M |\gamma|/(d - 1)} (1 - e^{(\beta - \beta_M) |\gamma|/(d - 1)}) \right\}. \]  

(6.17)

By exponential Chebichev, fixing \( M_2 = \mathbb{E}\bar{b}^2 \) and \( M_3 = \log \bar{b} \), we get (6.7).

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References

[1] Aizenman, M.; Bricmont, J. and Lebowitz, J.L. (1987) Percolation of minority spins in high-dimensional Ising models. J. Statist. Phys. 49, 859–865.

[2] Bolthausen, E. (1982) On the central limit theorem for stationary mixing random fields. Ann. Probab. 10, 1047–1050.
[3] Borgs, C. and Imbrie, J. Z. (1989) A Unified Approach to Phase Diagrams in Field Theory and Statistical Mechanics. *Commun. in Math. Phys.*, **123**, 305–328.

[4] Bricmont, J. and Kupiainen, A. (1996) High temperature expansions and dynamical systems. *Commun. Math. Phys.*, **178**, 703–732.

[5] Bricmont, J. and Kupiainen, A. (1997) Infinite-dimensional SRB measures. Lattice dynamics. *Physica D*, **103**, 18–33.

[6] Brydges, D.C. (1984) A short cluster in cluster expansions. In *Critical Phenomena, Random Systems, Gauge Theories*, Osterwalder, K. and Stora, R. (eds.), Elsevier, pages 129–183.

[7] Cammarota, C. (1982) Decay of correlations for infinite range interactions in unbounded spin systems. *Commun. Math. Phys.*, **85**, 517–528.

[8] Dobrushin, R.L. (1965) Existence of a phase transition in the two-dimensional and three-dimensional Ising models, *Th. Prob. Appl.*, **10**, 193-213. (Russian original: *Soviet Phys. Doklady*, **10**, 111–113.)

[9] Dobrushin, R.L. (1996) Estimates of semiinvariants for the Ising model at low temperatures. *Topics in Statistics and Theoretical Physics*, Amer. Math. Soc. Transl. (2), **177**, 59–81.

[10] Dobrushin, R.L. (1996a) Perturbation methods of the theory of Gibbsian fields. In *Ecole d'Eté de Probabilités de Saint-Flour XXIV – 1994*, Springer-Verlag (Lecture Notes in Mathematics **1648**), Berlin–Heidelberg–New York, 1–66.

[11] von Dreifus, H., Klein, A., Perez, J. F. (1995) Taming Griffiths' singularities: infinite differentiability of quenched correlation functions. *Comm. Math. Phys.* **170**, 21–39.

[12] Durrett, R. (1995) Ten lectures on particle systems. *Lecture Notes in Mathematics* **1608**, 97–201, Springer-Verlag, Berlin–Heidelberg–New York.

[13] Fernández, R., Ferrari, P. A. and Garcia, N. (1998) Loss network representation of Peierls contours. In preparation.

[14] Fernández, R., Ferrari, P. A. and Garcia, N. (1998a) Space-time representation of Gibbs measures with soft-core interaction. In preparation.

[15] Ferrari, P. A. and Garcia, N. (1998) One-dimensional loss networks and conditioned $M/G/\infty$ queues. To appear in *J. Appl. Probab*.

[16] Glimm, J., Jaffe, A. and Spencer, T. (1976) A convergent expansion about mean field theory. I. *Ann. Phys. (NY)*, **101**, 610–630.

[17] Griffiths, R.B. (1964) Peierls' proof of spontaneous magnetization in a two dimensional Ising ferromagnet, *Phys. Rev.*, **136A**, 437-439.

[18] Gruber, C. and Kunz, H. (1971) General properties of polymer systems. *Commun. Math. Phys.*, **22**, 133-161.

[19] Hall, P. (1985) On continuum percolation *Ann. Probab.* **13** 4:1250–1266.

[20] Harris, T. E. (1963) *The theory of branching processes*, Die Grundlehren der Mathematischen Wissenschaften, Bd. 119 Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J.
[21] Harris, T. E. (1972) Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.* **9**, 66–89.

[22] Kelly, F.P. (1991) Loss Networks. *Ann. Appl. Probab.*, **1**, 319-378.

[23] Kotecký R. and Preiss, D. (1986) Cluster expansion for abstract polymer models. *Commun. Math. Phys.*, **103**, 491–498.

[24] Lebowitz, J.L. and Mazel, A.E. (1997) Improved Peierls argument for high dimensional Ising models. Preprint.

[25] Malyshev, V.A. (1980) Cluster expansions in lattice models of statistical physics and quantum theory of fields. *Russian Mathematical Surveys*, **35**, 1–62.

[26] Olivieri, E. (1988) On a cluster expansion for lattice spin systems: a finite-size condition for the convergence, *J. Statist. Phys.*, **50**, 1179–1200.

[27] Olivieri, E.; Picco, P. (1990) Cluster expansion for $d$-dimensional lattice systems and finite-volume factorization properties. *J. Statist. Phys.*, **59**, 221–256.

[28] Peierls, R. (1936) On Ising’s model of ferromagnetism, *Proc. Cambridge Phil. Soc.*, **32**, 477-481.

[29] Pfister, C.-E. (1991) Large deviations and phase separation in the two-dimensional Ising model. *Helvetia Physica Acta*, **64**, 953–1054.

[30] Seiler, E. (1982) *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics **159**, Springer-Verlag, Berlin–Heidelberg–New York.

[31] Zahradník, E. (1984) An alternate version of Pirogov-Sinai theory. *Commun. Math. Phys.*, **93**, 559–5581.