ON THE PROPAGATION OF REGULARITY FOR SOLUTIONS OF THE ZAKHAROV-KUZNETSOV EQUATION

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ABSTRACT. In this work, we study some special properties of smoothness concerning to the initial value problem associated with the Zakharov-Kuznetsov (ZK) equation in the $n-$ dimensional setting, $n \geq 2$.

It is known that the solutions of the ZK equation in the $2d$ and $3d$ cases verify special regularity properties. More precisely, the regularity of the initial data on a family of half-spaces propagates with infinite speed. Our objective in this work is to extend this analysis to the case in that the regularity of the initial data is measured on a fractional scale. To describe this phenomenon we present new localization formulas that allow us to portray the regularity of the solution on a certain class of subsets of the euclidean space.

1. Introduction

In this work we are interested in to describe some regularity properties of solutions to the initial value problem (IVP) associated to the Zakharov-Kuznetsov (ZK) equation

\begin{equation}
\begin{cases}
\partial_t u + \partial_{x_1} \Delta u + u \partial_{x_1} u = 0, & t \in \mathbb{R} \\
u(x,0) = u_0(x), & x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, n \geq 2,
\end{cases}
\end{equation}

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$ is the $n$-dimensional Laplacian.

This equation was deduced by Zakharov and Kuznetsov [26] to describe the ionic-acoustic waves uniformly magnetized plasma in the two dimensional and three dimensional cases. More precisely, this equation was derived as a long wave small amplitude limit of the Euler-Poisson system in the "cold-plasma" approximation. Later on, this long wave limit was rigorously described by Lannes, Linares and Saut [14]. Also the ZK equation has been derived from the Vlasov-Poisson system in a combined cold ions and long wave limit by Kwan [4].

The ZK equation has a Hamiltonian structure and it has at least formally three conserved quantities, namely

\begin{align*}
\mathcal{I}_1[u](t) &= \int_{\mathbb{R}^n} u(x,t) \, dx = \mathcal{I}_1[u](0), \\
\mathcal{I}_2[u](t) &= \int_{\mathbb{R}^n} (u(x,t))^2 \, dx = \mathcal{I}_2[u](0) \\
\mathcal{I}_3[u](t) &= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 \, dx - \frac{1}{3} \int_{\mathbb{R}^n} (u(x,t))^3 \, dx = \mathcal{I}_3[u](0).
\end{align*}

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Due to its physical relevance the ZK equation have called the attention in the recent years. Nevertheless, before describe the main goal in this work we require to describe the space solution where the property to be described has sense from the mathematical point of view. In this direction, we give a brief description of the Initial Value Problem (IVP) issues associated to (1.1).

Since the IVP for the ZK equation have been broadly studied in the recent years, the ZK literature have been increasing more and more. So that, trying to describe the major part of the results associated to (1.1) is a difficult task. Thus, we present a short review that summarize the IVP issues according to the physical dimension. In the particular case that the physical dimension is \( n = 2 \), Faminskii [2] proved global well-posedness in \( H^j(\mathbb{R}^2), j \in \mathbb{Z}^+, j \geq 1 \), later Linares and Pastor [15] proved global well-posedness in \( H^s(\mathbb{R}^2), s > 3/4 \), additionally and independently and simultaneously, Grünrock and Herr [3] and Molinet and Pilod [20] proved local well posedness in \( H^{3/4} + (\mathbb{R}^2) \). Additionally, it was proved recently Kinoshita [13] local well posedness in \( H^{-\frac{1}{4} +}(\mathbb{R}^2) \) that according to the scaling argument it is optimal up to the end-point.

Concerning to the case in which the dimension is \( n = 3 \), Linares and Saut [17] proved local well-posedness in \( H^s(\mathbb{R}^3), s > 9/8 \). Later, [17] Ribaud and Vento [24] proved local well-posedness in \( H^s(\mathbb{R}^2), s > 1 \), Molinet and Pilod [20] prove global that the results in [24] can be extended globally in time. More recently, Herr and Kinoshita [5] have proved that for dimension \( n \geq 3 \), the IVP associated to (1.1) is locally well-posed on \( H^n(\mathbb{R}^n), s > \frac{n}{2} - 2 \). Additionally, Herr and Kinoshita [5] also establish that in dimension \( n = 3 \) and for real solutions, the IVP associated to (1.1) is globally well posed in \( L^2(\mathbb{R}^3) \) and in dimension \( n = 4 \) it is globally well-posed for real-valued initial data in \( H^1(\mathbb{R}^4) \) with sufficiently small \( L^2(\mathbb{R}^4) \)-norm.

Although of the improvements concerning to the local and global theory, the regularity properties associated to the solutions of (1.1) we intend to describe in this work depends strongly on having a priori- estimates for \( \|\nabla u\|_{L^4} < \infty \). So that, before we firstly describe the space where the properties to be described have sense. We recall a result that becomes a direct consequence of combining energy estimates, the commutator estimates in [11], the Sobolev embedding and the arguments in [1].

\textbf{Theorem 1.2.} Given \( u_0 \in H^s(\mathbb{R}^n) \) with \( s > \frac{n}{2} + 1 \), there exist \( T = T (\|u_0\|_{H^s}) > 0 \), and a unique solution \( u = u(x,t) \) of the IVP 1.1 such that

\begin{equation}
(1.3) \quad u \in C \left( [0,T] : H^s(\mathbb{R}^n) \right).
\end{equation}

Moreover, the map data-solution \( u_0 \mapsto u(x,t) \) from \( H^s(\mathbb{R}^n) \) into \( C \left( [0,T] : H^s(\mathbb{R}^n) \right) \) is locally continuous.

The Theorem 1.2 is the basis space to describe the properties we intend to describe in this work since it allow us to guarantee that \( \nabla u \in C \left( [0,T] : H^{s-1}(\mathbb{R}^n) \right) \subset L^1 \left( [0,T] : L^\infty (\mathbb{R}^n) \right) \).

Since the space solution has been set we proceed to establish the main goal of this work, that is mainly based in to extend the study of propagation of regularity found by Linares and Ponce [16] in solutions of the ZK equation in the 2–dimensional case as well as in the 3d–case to a more general context where the regularity to be considered be fractional. Roughly speaking, the propagation of regularity phenomena describe the behavior of the regularity of the solution when the initial data
enjoy of some extra smoothness on a particular class of subsets of the physical space. Specifically, this class of sets are strips and half-spaces, that in our work will be indicated according to the following notation: For \( \sigma \) a non-null vector in \( \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \) the half-space \( \mathcal{H}(\sigma, \alpha) \) will be indicated by

\[
\mathcal{H}(\sigma, \alpha) := \{ x \in \mathbb{R}^n \mid \sigma \cdot x > \alpha \},
\]

where \( \cdot \) denotes the canonical inner product in \( \mathbb{R}^n \). Additionally, for \( \gamma, \beta \in \mathbb{R} \) with \( \gamma < \beta \) we define the strip

\[
\mathcal{Q}(\gamma, \alpha, \beta) := \{ x \in \mathbb{R}^n \mid \gamma < \sigma \cdot x < \beta \}.
\]

Formally, the description of the propagation of regularity phenomena in solutions of the ZK equation is summarized in the following theorem.

**Theorem 1.4** ([16]). Let \( u_0 \in H^{\frac{5}{2}}(\mathbb{R}^3) \). If for some \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \) with

\[
\sigma_1 > 0, \quad \sigma_2, \sigma_3 \geq 0 \quad \text{and} \quad 3\sqrt{\sigma_1} > \sqrt{\sigma_2^2 + \sigma_3^2},
\]

and for some \( j \in \mathbb{Z}^+ \), \( j \geq 3 \)

\[
\mathcal{N}_j := \sum_{|\alpha| = j} \int_{\mathcal{H}(\sigma, \alpha)} (\partial_x^{\alpha}u_0(x))^2 \, dx < \infty,
\]

then the corresponding solution of the IVP for the ZK equation (1.1) satisfies that for any \( \nu \geq 0 \), \( \varepsilon > 0 \) and \( \tau > 4\varepsilon \)

\[
\sup_{0 \leq t \leq T} \sum_{|\alpha| \leq j} \int_{\mathcal{H}(\sigma, \alpha)} (\partial_x^{\alpha}u(x, t))^2 \, dx
\]

\[
+ \sum_{|\alpha| = j+1} \int_0^T \int_{\mathcal{Q}(\sigma, \alpha, \nu, \tau)} (\partial_x^{\alpha}u(x, t))^2 \, dx \, dt
\]

\[
\leq c = c \left( \| u_0 \|_{H^\nu}; \{ \mathcal{N}_l : 1 \leq l \leq j \}; j; \sigma; \nu; T; \varepsilon; \tau \right).
\]

**Remark 1.1.** Similar results holds in the 2--dimensional case, for a more detailed description see [16].

The property described in Theorem 1.4 is inherent to some nonlinear dispersive models, e.g. in the one dimensional case this issue have been verified in solutions of the KdV and the Benjamin-Ono equation by Isaza, Linares, Ponce see [7] and [8] resp. Also, Kenig, Linares, Ponce and Vega [12] studied this subject in solutions of the KdV by considering initial data with fractional regularity. Additionally, combining the approach in [8] and [12] it was verified that the dispersive generalized Benjamin-Ono [18] as well as the fractional KdV equation [19] also satisfy this property. Recently, Muñoz, Ponce and Saut [21] proved that the solutions of the Intermediate long-wave equation satisfy this property.

As shows Theorem 1.4, this property is not only inherent to one-dimensional nonlinear dispersive models, but on the contrary, its validity has been established in multidimensional nonlinear dispersive models such as: Kadomtsev-Petviashvili II (KP-II) (see Isaza, Linares, Ponce [9]) as well as in the fifth order Kadomtsev-Petviashvili II (KP5-II) and the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equations by Nascimento [22], [23] resp.
Note that the case in which the additional regularity of the initial data on the half-space $\mathcal{H}_{(\sigma,\beta)}$ (see (1.5)) is given on a fractional scale, it does not fall under the scope of Theorem 1.4. The study of this case constitutes the main objective in this work. More precisely, we managed to show that even in the case in which the additional regularity of the data is measured on a fractional scale this is propagated with infinite speed.

In summary, our main result reads as follows:

**Theorem A.** Let $u_0 \in H^s(\mathbb{R}^n)$ with $s > s_n := \frac{n+2}{2}$. If for some $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n$, $n \geq 2$ with

\begin{equation}
\sigma_1 > 0, \quad \sigma_2, \ldots, \sigma_n \geq 0 \quad \text{and} \quad \sqrt{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2} < \sqrt{3} \sigma_1,
\end{equation}

and for some $s \in \mathbb{R}, s > s_n$

\begin{equation}
\|J^s u_0\|_{L^2(\mathcal{H}_{(\sigma,\beta)})} < \infty \tag{1.8}
\end{equation}

then the corresponding solution $u = u(x,t)$ of the IVP (1.1) satisfies: For any $\nu \geq 0, \epsilon > 0$ and $\tau \geq 5\epsilon$

\begin{equation}
\sup_{0 \leq t \leq T} \int_{\mathcal{H}_{(\sigma,\beta)}} (J^\nu u(x,t))^2 \, dx \leq c^* \tag{1.9}
\end{equation}

for any $r \in (0,s]$ with $c^* = c^* \left(\epsilon; \sigma; T; \nu; \|u_0\|_{H^\nu}; \|J^\nu u_0\|_{L^2(\mathcal{H}_{(\sigma,\beta)})}\right) > 0$.

In addition, for any $\nu \geq 0, \epsilon > 0$ and $\tau \geq 5\epsilon$

\begin{equation}
\int_0^T \int_{Q_{(\sigma,\beta)}} (J^{\nu+1} u(x,t))^2 \, dx \, dt \leq c^*, \tag{1.10}
\end{equation}

with $c^* = c^* \left(\epsilon; \tau; \sigma; T; \nu; \|u_0\|_{H^\nu}; \|J^\nu u_0\|_{L^2(\mathcal{H}_{(\sigma,\beta)})}\right) > 0$.

The proof of the Theorem A is mainly based in combining the ideas used in the proof of Theorem 1.4, as well as the ideas used in the study of propagation of regularity for solutions of the KdV equation [12]. More precisely, it combines an inductive argument together with weighted energy estimates, where the class of weights used enjoy of some particular properties that allow to capture the information related to the regularity in certain subsets of $\mathbb{R}^n$, $n = 2, 3$. Also, the method of proof uses strongly the *Kato’s smoothing effect*, this is a property found originally by Kato in the KdV context (see Kato [10]).

In this sense, our contribution is mainly based on establishing certain localization formulas on half-spaces and strips for the operator $J^s, s > 0$. Similarly, we show that under certain conditions it is possible to establish relationships between $J^s, s \geq 1$ and $\partial^\alpha_x, \alpha \in (\mathbb{Z}^+)^n$, when we are restricted to certain class of subsets of the Euclidean space, for a more detailed description see Lemma 4.4.

Although the proof of the Theorem A is somewhat technical, the properties it describes are quite intuitive and have its particular flavor. In this sense, we will present a geometric description of these when we restrict ourselves to dimension $n = 2$ and $n = 3$ due to its physical relevance.

In dimension $n = 2$ two situations arise according to Theorem A. The first one, namely $\sigma = (\sigma_1, 0)$ with $\sigma_1 > 0$. Under this condition, the extra regularity of the initial data $u_0$ in the half space $\mathcal{H}_{(\sigma,\beta)}$ see (1.8)), that in this case is fractional (cf.(1.5)). This dynamics is exemplified in figure 1, where the arrows indicate the
sense of propagation (to the left), as well as two gray zones denoting the regions $Q_{\sigma, \epsilon - \nu t, \tau + \beta - \nu t}$ and $H_{\sigma, \epsilon - \nu t}$, denoting the set of propagation and the strip that carry out the smoothing effect resp. The set $H_{\sigma, \epsilon + \beta - \nu t}$ denotes the moving region of the plane that carries out the information corresponding to the propagation of regularity, that despite being fractional it is propagated to the left with infinite speed. Instead, the set $Q_{\sigma, \epsilon + \beta - \nu t, \tau + \beta - \nu t}$ corresponds to the region where the solution is smoother by one local-derivative. More precisely, in this region is present the Kato’s smoothing effect. Unlike the previous situation, the case $\sigma = (\sigma_1, \sigma_2)$

![Figure 1. Sense of propagation of regularity in the case $\sigma_1 > 0, \sigma_2 = 0$.](image1)

with $\sigma_1, \sigma_2 > 0$ is more involved, since the effects on the $y$-variable yield a change in the geometry of the propagation. In this case, the dynamics is carried out in a diagonal sense as is indicated by the arrows in the figure 2. To give a geometrical

![Figure 2. Sense of propagation of regularity in the 2-dimensional case with $\sigma_1, \sigma_2 > 0$, whence $\eta_1(t) := \epsilon + \beta - \nu t$ and $\eta_2(t) := \tau + \beta - \nu t$.](image2)

idea about the propagation of regularity phenomena in the three-dimensional case it is even more involved than in the 2d case since it entails to describe many more sub-cases that we do not intend to describe them by entirely, but instead to fix the ideas we only focus on describing the case $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1, \sigma_2, \sigma_3 > 0$. 

The consideration of a third variable yield to a more complex dynamics. In this
tsituation, the sense of propagation occurs in a diagonal sense as indicates the dashed
arrow in the figure 3. As in the previous situations, the sets \( Q_{(v, \epsilon + \beta - vt, \tau + \beta - vt)} \) and
\( H_{(v, \epsilon + \beta - vt)} \) denotes moving regions with infinite speed of propagation indicating
the regularity propagated as well as the smoothing effect. Note that the dashed
triangles enclose the channel \( Q_{(v, \epsilon + \beta - vt, \tau + \beta - vt)} \) and the upper triangle denotes
the border of the region \( H_{(v, \epsilon + \beta - vt)} \) when we restrict to the first octant.

Additionally to the properties described in Theorem A we also provide informa-
tion in the case when we give privilege to a specific direction, this constitute our
second main result and it is summarized in the following theo rem.

**Theorem B.** Let \( u_0 \in H^{s \frac{n}{2}} (\mathbb{R}^n) \). If for some \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n \) with \( n \geq 2 \);
\[ \sigma_1 > 0, \sigma_2, \ldots, \sigma_n \geq 0 \quad \text{and} \quad \sqrt{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2} < \sqrt{3} \sigma_1, \]
and for some \( s \in \mathbb{R}, s > s_n \)
\[ \left\| J_{x_j}^s u_0 \right\|_{L^2(H_{(v, \epsilon, \beta)})} < \infty \]
for some \( j \in \{1, 2, \ldots, n\} \), then the corresponding solution \( u = u(x, t) \) of the IVP
(1.1) satisfies: For any \( v \geq 0, \epsilon > 0 \) and \( \tau \geq 5 \epsilon \)
\[ \sup_{0 \leq t \leq T} \int_{H_{(v, \epsilon + \epsilon - vt)}} \left( J_{x_j}^{r} u(x, t) \right)^2 \, dx \leq c^*, \]
for any \( r \in (0, s) \) with \( c^* = c^* \left( \epsilon; \sigma; T; \nu; \| u_0 \|_{H^{s \frac{n}{2}}(\mathbb{R}^n)}; \| J_{x_j}^r u_0 \|_{L^2(H_{(v, \epsilon, \beta)})} \right) > 0. \)
In addition, for any $\nu \geq 0$, $\epsilon > 0$ and $\tau \geq 5\epsilon$,

$$
\sum_{1 \leq m \leq n, m \neq j} \int_0^T \int_{Q_{(\beta-\nu t+\epsilon, -\nu t-\epsilon)}} \left( \partial_{x_m} f_{x_j}^{\xi} u \right)^2 \, dx \, dt \\
+ \int_0^T \int_{Q_{(\beta-\nu t+\epsilon, -\nu t-\epsilon)}} \left( f_{x_j}^{\xi+1} u(x, t) \right)^2 \, dx \, dt \leq c
$$

with $c = c \left( \epsilon; \sigma; \tau; T; \nu; \| u_0 \|_{H^{n+1}}; \| f_{x_j}^{\xi} u_0 \|_{L^2} \left( \mathcal{H}_{(\epsilon, \beta)} \right) \right) > 0$.

An argument quite similar to the one given for the proof of Theorem B also applies for a proof of Theorem A.

A quite similar description of the phenomena presented in the Theorem B can be given in geometrical terms as we did for Theorem A when we restrict ourselves to dimension $n = 2$ and $n = 3$ see figure 1 and figure 3 resp.

Finally, as a by product we present a consequence of Theorem A that describes the behavior of the solution and its derivatives in the remainder part of the half-space described in Theorem A.

**Corollary 1.1.** Let $u \in C \left( [-T, T] : H^{n+1}(\mathbb{R}^n) \right)$ be a solution of the equation in (1.1) described by theorem A. Then, for any $t \in (0, T)$ and $\delta > 0$, the following inequality holds:

$$
\int_{\mathbb{R}^n} \frac{1}{\langle (\sigma \cdot x - \beta) \rangle + \delta} \left( J_{x_j}^{s} u(x, t) \right)^2 \, dx \lesssim_{\delta, \nu} \frac{1}{T},
$$

where $x_- = \max\{0, -x\}$.

1.0.1. *Organization of the paper.* In the section 2 we present a full description of the notation to be used throughout the document. In what concerns to section 3 we present a short review of several well known results about pseudo-differential operators. Additionally, in section 3 we deduce new localization formulas for the operator $J_{x_j}^{s}, s > 0$, that will be used extensively through the proof of Theorem A. Finally, in the section 4 we present the proof of theorem A, and some of its consequences.

2. Notation

In this section we introduce the notation to be used throughout this document. We adopt the following convention for the *Fourier transform*

$$
\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) \, dx.
$$

For $x \in \mathbb{R}^n$, we denote $\langle x \rangle := \left( 1 + |x|^2 \right)^{1/2}$. Additionally, for any $s \in \mathbb{R}$ we define the operator $J^s$ via its Fourier transform as $\hat{J^s f}(\xi) = (\xi, \xi^{\nu} f(\xi))$. In the particular case that be required to emphasize the action over a specific variable we write

$$
\hat{J_{x_j}^{s} f}(\xi) := (\xi_j)^{s} \hat{f}(\xi), \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n,
$$

where $j \in \mathbb{Z}^+$ with $1 \leq j \leq n$. 
The set of the *Schwarz functions* will be denoted by \( \mathcal{S}(\mathbb{R}^n) \), and its dual, the set of the *Ååtempered distributions* will be denoted by \( \mathcal{S}'(\mathbb{R}^n) \). The *dual pair* between \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{S}(\mathbb{R}^n) \) will be indicated as usual by \( \langle \cdot, \cdot \rangle_{\mathcal{S}', \mathcal{S}} \).

For \( 1 \leq p \leq \infty \), \( L^p(\mathbb{R}^n) \) Ååis the usual ÅåLebesgue space Ååwith the norm Åå\( \| f \|_{L^p} \) Åå. Additionally, for \( s \in \mathbb{R} \), Ååwe consider the *Sobolev space* \( H^s(\mathbb{R}^n) \) Ååthat is defined Ååas Åå
\[
H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{L^2} < \infty \right\}.
\]

Let \( f = f(x,t) \) Ååbe a function Åådefined for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( t \) in the time interval \([0, T]\), with \( T > 0 \) Ååor in the hole line \( \mathbb{R} \), then, if \( A \) denotes any of the spaces defined above, we define Ååthe spaces Åå\( \widehat{\mathcal{A}} L^p_T A \) Ååand \( L^p_T A \) by the norms
\[
\| f \|_{L^p_T A} := \left( \int_0^T \| f(\cdot, t) \|_A^p \, dt \right)^{1/p} \quad \text{and} \quad \| f \|_{L^p_T A} := \left( \int_{\mathbb{R}} \| f(\cdot, t) \|_A^p \, dt \right)^{1/p},
\]
for \( 1 \leq p \leq \infty \), with the Åånatural modification in the case \( p = \infty \).

For \( A, B \) operators Ååwe will denote Ååthe *commutator* between \( A \) and \( B \) by
\[
[A; B] := AB - BA.
\]

For two quantities \( \widehat{\mathcal{A}} A \) and \( B \), we denote \( \widehat{\mathcal{A}} A \lesssim B \widehat{\mathcal{A}} A \) if \( A \) for some constant \( c > 0 \). Similarly, \( A \gtrsim B \widehat{\mathcal{A}} A \gtrsim cB \) for some \( c > 0 \). Åå\( \mathcal{A} \mathcal{A} \) Ååalso for two positive quantities, \( \mathcal{A} \mathcal{A} \mathcal{A} \) Ååwe say that are *comparable* if \( \mathcal{A} \mathcal{A} \mathcal{A} \leq B \) and \( B \leq \mathcal{A} \mathcal{A} \mathcal{A} \), when such condition be satisfied we will indicate it by \( A = B \). ÅåThe dependence of the constant \( c \) Ååon other parameters or constants are usually clear from the context and we will often suppress this dependence whenever it be possible.

For any real number \( a \), we denote Åå\( \mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A} a \) Ååthe quantity \( a + c \) for Ååany \( c > 0 \).

3. Pseudo-differential Operators

In the following section it is our intention to provide a brief summary about some well known facts about pseudo-differential operators as well as several properties in Sobolev spaces, that will be relevant in our analysis.

**Definition 3.1.** Let \( m \in \mathbb{R} \). Let \( \mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n) \) denote the set of functions \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) such that for all \( \alpha \) and all \( \beta \) multi-index
\[
\left| \partial_\xi^\alpha \partial_\eta^\beta \hat{a}(x, \xi) \right| \lesssim_{\alpha, \beta} (1 + |\xi|)^{m-|\beta|}, \quad \text{for all} \quad x \in \mathbb{R}^n.
\]

An element \( a \in \mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n) \) is called a *symbol of order* \( m \).

**Remark 3.1.** For the sake of simplicity in the notation from here on we will suppress the dependence of the space \( \mathbb{R}^n \) when we make reference to a symbol in a particular class.

**Definition 3.2.** A *pseudo-differential operator* is a mapping \( f \mapsto \Psi f \) given by
\[
(\Psi f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi,
\]
where \( a(x, \xi) \) is the symbol of \( \Psi \).

**Remark 3.2.** In order to emphasize the role of the symbol \( a \) we will often write \( \Psi_a \).
Definition 3.3. If $a(x, \xi) \in \mathcal{S}^m$, the operator $\Psi_a$ is said to belong to $\text{OPS}^m$. More precisely, if $\Sigma$ is any symbol class and $a(x, \xi) \in \Sigma$, we say that $\Psi_a \in \text{OP}\Sigma$.

A quite remarkable property that pseudo-differential operators enjoy is the existence of the adjoint operator, that is described below in terms of its asymptotic decomposition.

Theorem 3.3. Let $a \in \mathcal{S}^m$. Then, there exist $a^* \in \mathcal{S}^m$ such that $\Psi_a^* = \Psi_{a^*}$, and for all $N \geq 0$,

$$a^*(x, \xi) = \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \xi_\alpha \xi_\alpha \hat{a}(x, \xi) \in \mathcal{S}^{m-N}.$$ 

Proof. See Stein [25] chapter VI.

Remark 3.3. In fact from the formula above it follows that $\Psi_a^* = \Psi_{\overline{a}} \text{ mod } \text{OPS}^{m-1}$.

Also, the existence of the adjoint allow to extend the action of $\Psi_a$ over a wider class of objects, such as tempered distributions.

Theorem 3.4. If $a \in \mathcal{S}^m$, then $\Psi_a$ defined in (3.2) is a continuous operator $\Psi_a : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$. Additionally, the map can be extended to a continuous map

$$\Psi_a : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n).$$

More precisely, for $u \in \mathcal{S}'$ and $v \in \mathcal{S}$

$$\langle \Psi_a u, v \rangle_{\mathcal{S}', \mathcal{S}} := \langle u, \overline{\Psi_a v} \rangle_{\mathcal{S}', \mathcal{S}}.$$

In fact the proof of this theorem is based in the following lemma that condensates several properties to be used later.

Lemma 3.1. Let $a \in \mathcal{S}^m$, $v \in \mathcal{S}(\mathbb{R}^n)$. Then for all $\xi, \eta \in \mathbb{R}^n$,

$$\left| \int_{\mathbb{R}^n} v(x) a(x, \xi) e^{2\pi i x \cdot \eta} \, dx \right| \leq C_N (1 + |\xi|)^m (1 + |\eta|)^{-N},$$

for all $N \geq 0$.

Proof. The proof follows combining Leibniz rule and definition 3.1. □

A quite interesting and useful remark about this extension is related with the representation of pseudo-differential operators. More precisely, for $u, v \in \mathcal{S}(\mathbb{R}^n)$, real valued functions

$$\langle \Psi_a u, v \rangle_{\mathcal{S}', \mathcal{S}} = \int_{\mathbb{R}^n} v(x) \Psi_a u(x) \, dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i x \cdot \xi} v(x) \hat{u}(\xi) \, d\xi \, dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i (x-y) \cdot \xi} v(x) u(y) \, dy \, d\xi \, dx$$

$$= \langle K, uv \rangle_{\mathcal{S}', \mathcal{S}}$$

whence we get after interpreting properly as a distributional integral that

$$K(x, x-y) := \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi.$$
This brief description of pseudo-differential operators in terms of kernels is summarized in the following theorem.

**Theorem 3.6** (Realization of Pseudo-differential operators as singular integrals). Let \( a \in S^m \), \( m \in \mathbb{R} \) and \( \Psi_a \) its corresponding pseudo-differential operator associated. Then, there exist a kernel \( k_a \in C^\infty (\mathbb{R}^n \times \mathbb{R}^n - \{0\}) \) satisfying the following properties:

(i) The operator \( \Psi_a \) admits the following representation

\[
(\Psi_a f)(x) = \int_{\mathbb{R}^n} k_a(x, x - y) f(y) \, dy, \quad \text{if} \quad x \notin \text{supp}(f);
\]

(ii) for all \( \alpha, \beta \) multi-index and all \( N \geq 0 \),

\[
\left| \tilde{\partial}_x^\alpha \tilde{\partial}_\xi^\beta k_a(x, z) \right| \lesssim_{\alpha, \beta, N, \delta} |z|^{-n-m-|\beta|-N}, \quad |z| \geq \delta,
\]

if \( n + N + m + |\beta| > 0 \).

**Proof.** See Stein [25], chapter VI. \( \square \)

Additionally the product \( \Psi_a \Psi_b \) of two operators with symbols \( a(x, \xi) \) and \( b(x, \xi) \) respectively is a pseudo-differential operator \( \Psi_c \) with symbol \( c(x, \xi) \). More precisely, the description of the symbol \( c \) is summarized in the following theorem:

**Theorem 3.9.** Suppose \( a \) and \( b \) symbols belonging to \( S^m \) and \( S^r \) respectively. Then, there is a symbol \( c \) in \( S^{m+r} \) so that

\[
\Psi_c = \Psi_a \circ \Psi_b.
\]

Moreover,

\[
c \sim \sum_{|\alpha|} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \tilde{\partial}_x^\alpha \tilde{\partial}_\xi^\alpha a \tilde{\partial}_x^\alpha b,
\]

in the sense that

\[
c = \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \tilde{\partial}_x^\alpha a \tilde{\partial}_\xi^\alpha b \in S^{m+r-N}, \quad \text{for all integer} \quad N, N \geq 0.
\]

**Proof.** For the proof see Stein [25] chapter VI. \( \square \)

**Remark 3.4.** Note that \( c - ab \in S^{m+r-1} \). Moreover, each symbol of the form \( \tilde{\partial}_x^\alpha a \tilde{\partial}_\xi^\alpha b \) lies in the class \( S^{m+r-|\alpha|} \).

A direct consequence of the decomposition above is that it allows to describe explicitly up to an error term, operators such as commutators between pseudo-differential operators as is described below:

**Proposition 1.** For \( a \in S^m \) and \( b \in S^r \) we define the commutator \([\Psi_a; \Psi_b]\) by

\[
[\Psi_a; \Psi_b] = \Psi_a \circ \Psi_b - \Psi_b \circ \Psi_a.
\]

Then, the operator \([\Psi_a; \Psi_b] \in OP S^{m+r-1}\), has by principal symbol the Poisson bracket, i.e.,

\[
\sum_{|\alpha|=1} \frac{1}{2\pi i} \left( \tilde{\partial}_x^\alpha a \tilde{\partial}_\xi^\alpha b - \tilde{\partial}_x^\alpha a \tilde{\partial}_\xi^\alpha b \right) \mod S^{m+r-2}.
\]
Also, certain class the pseudo-differential operators enjoy of some continuity properties as the described below.

**Theorem 3.10.** Suppose that \( a \) is symbol with \( a \in S^0 \). Then, the operator \( \Psi_a \) given by
\[
(\Psi_a f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi,
\]
initially defined on \( S(\mathbb{R}^n) \), extends to a bounded operator from \( L^2(\mathbb{R}^n) \) to itself.

**Proof.** The proof can be consulted in Stein [25] Chapter VI, Theorem 1. \( \square \)

Also, thorough our analysis it will be necessary to provide upper bound for a certain commutator expressions. More precisely, we will require the use of the following result.

**Theorem 3.11** (Kato & Ponce [11]). Let \( s > 0 \), and \( p \in (1, \infty) \). Then, for \( f, g \in S(\mathbb{R}^n) \) the following inequalities hold:
\[
\| [J^s; f] g \|_{L^2} \lesssim \| \nabla f \|_{L^p} \| f^{-1} g \|_{L^2} + \| J^s f \|_{L^2} \| g \|_{L^\infty}, \tag{3.12}
\]
and
\[
\| J^s f g \|_{L^2} \lesssim \| f \|_{L^p} \| J^s g \|_{L^2} + \| J^s f \|_{L^2} \| g \|_{L^\infty}. \tag{3.13}
\]

**Proof.** See [11]. \( \square \)

4. Localization Tools

In this section we intend to provide the necessary tools to describe our results. The representation of pseudo-differential operators as the displayed in (3.7) together with the decay property (3.8) will find a wide range of applicability in our analysis.

**Lemma 4.1.** Let \( \Psi_a \in \text{OPS}' \). Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a multi-index with \( |\alpha| \geq 0 \). If \( f \in L^2(\mathbb{R}^n) \) and \( g \in L^p(\mathbb{R}^n) \), \( p \in [2, \infty] \) with
\[
dist(\text{supp}(f), \text{supp}(g)) \geq \delta > 0,
\]
then,
\[
\| g \partial^\alpha_x \Psi_a f \|_{L^2} \lesssim \| g \|_{L^p} \| f \|_{L^2},
\]
where \( \partial^\alpha_x := \partial^\alpha_{x_1} \ldots \partial^\alpha_{x_n} \).

**Proof.** The proof follows as a direct application of the representation theorem 3.6. In the following we will consider \( f \in S(\mathbb{R}^n) \), the general case can be obtained by mollifying \( f \).

So that, in virtue of (4.1) and Theorem 3.6, the following representation holds
\[
g(x) \Psi_a \partial^\alpha_x f(x) = \int_{\mathbb{R}^n} g(x) k_a(x, x - z) \partial^\alpha_x f(z) \, dz,
\]
where \( k_a \) is the distributional kernel associated to \( \Psi_a \).

Next, integration by parts produce
\[
g(x) \partial^\alpha_x \Psi_a f(x) = (-1)^{|\alpha|} \int_{\{|x-z| \geq \delta\}} g(x) \partial^\alpha_x k_a(x, x - z) f(z) \, dz.
\]
Finally, we combine Young’s inequality to obtain
\[
\|g \hat{\mathcal{A}}_{\alpha} \Psi f\|_{L^2} \lesssim_{\mathbf{n}, \mathbf{r}, N} \|g \left( \frac{\mathbf{1}_{\{1 \geq \delta\}}}{|n + r + |a| + N|} \right) * f \|_{L^2}
\]
\[
\lesssim \beta_{\mathbf{n}, \mathbf{r}, N} \|g\|_{L^p_1} \left\| \frac{\mathbf{1}_{\{1 \geq \delta\}}}{|n + r + |a| + N|} \right\|_{L^p_2}
\]
\[
\lesssim_{\mathbf{n}, \mathbf{r}, N} \|g\|_{L^p_1} \|f\|_{L^2}
\]
where the indexes \( p_1, p_2, \) and \( p_3 \) satisfy:

\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2} + \frac{1}{p_2} = \frac{1}{p_3},
\]

with \( 2 \leq p_1, p_2 \leq \infty \) and \( 1 \leq p_3 \leq 2 \).

Hence, after choosing \( N \) properly we get finally
\[
\|g \hat{\mathcal{A}}_{\alpha} \Psi f\|_{L^2} \lesssim_{\mathbf{n}, \mathbf{r}, \alpha} \|g\|_{L^p_1} \|f\|_{L^2}.
\]

The first result that incorporates the use of the previous considerations becomes summarized in the following lemma.

**Lemma 4.2.** Let \( \Psi \in \mathcal{C}_0^\infty \). Assume that \( f \in H^s(\mathbb{R}^n), s < 0 \). If \( \theta_1 f \in L^2(\mathbb{R}^n) \), then
\[
\theta_2 \Psi f \in L^2(\mathbb{R}^n).
\]

**Proof.** Let \( \theta_1, \theta_2 \in \mathcal{C}_c(\mathbb{R}^n) \) with bounded derivatives of all orders, such that: \( 0 \leq \theta_1, \theta_2 \leq 1 \), and their respective supports satisfy:
\[
\text{dist} (\text{supp } (1 - \theta_1), \text{supp } (\theta_2)) \geq \delta,
\]
for some \( \delta > 0 \).

First, we decompose \( f \) by incorporating \( \theta_1 \) and \( \theta_2 \) as follows:
\[
\langle \theta_2 \Psi f, g \rangle_{S', S} = \langle \theta_2 \Psi ((1 - \theta_1) f), g \rangle_{S', S} + \langle \theta_2 \Psi (\theta_1 f), g \rangle_{S', S} + \langle \theta_2 \Psi (1 - \theta_1) f, g \rangle_{S', S}
\]
for all \( g \in \mathcal{S}(\mathbb{R}^n) \).

In view that \( \Psi \in \mathcal{C}_0^\infty \) and \( \theta_1 f \in L^2(\mathbb{R}^n) \), it is clear that combining Theorem 3.10 and hypothesis, it follows that
\[
\|\theta_2 \Psi (\theta_1 f)\|_{L^2} \lesssim \|\theta_1 f\|_{L^2} < \infty.
\]

Notice that the second term in (4.3) condenses all the information of \( f \) that does not behave as a function. In this sense, we proceed as we described previously in (3.5), that is, for \( g \in \mathcal{S}(\mathbb{R}^n) \)

\[
\langle \theta_2 \Psi ((1 - \theta_1) f), g \rangle_{S', S} = \langle \Psi ((1 - \theta_1) f), \theta_2 g \rangle_{S', S}
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i(x - y) \cdot \xi} \theta_2(x) \hat{g}(x)(1 - \theta_1)(y) f(y) \, dy \, d\xi \, dx
\]
so that, after interpreting properly in the distributional sense together with Theorem 3.6 yield
\[
\theta_2(x)\Psi_a((1 - \theta_1)f)(x) = \int_{\mathbb{R}^n} \theta_2(x)k_a(x, x - z)(1 - \theta_1)(z)f(z)\,dz
\]
being the last equality above a consequence of (4.2).

Next, for any \(m \in \mathbb{Z}\) such that \(s > 2m\), then it is clear that \(f \in H^{-2m}(\mathbb{R}^n)\). So that, after applying integration by parts we get
\[
\int_{B_r(\delta)}^{} \theta_2(x)k_a(x, x - z)(1 - \theta_1)(z)f(z)\,dz
\]
\[
= \int_{B_r(\delta)} J^{2m} (\theta_2(x)k_a(x, x - z)(1 - \theta_1)(z)) J^{-2m}f(z)\,dz
\]
\[
= \sum_{j, \beta, \gamma} c_{\beta, m, j, \beta, \gamma} \int_{B_r(\delta)}^{} \theta_2(x)\xi_\gamma k_a(x, x - z)\xi_{\delta - \gamma} \frac{1}{1 - \theta_1}(z)J^{-2m}f(z)\,dz,
\]
whence \(B_r(\delta)\) denotes the open ball with center at \(x\) and radius \(\delta > 0\).

The proof finish after combining Young’s convolution inequality and the representation theorem 3.6, whence we get finally that \(\theta_2 \Psi_a((1 - \theta_1)f) \in L^2(\mathbb{R}^n)\). \(\square\)

In the next part we will describe the operator \(J^s, s > 0\) when we restrict on a certain class of half-spaces.

**Lemma 4.3.** Let \(f \in L^2(\mathbb{R}^n)\) and \(\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n\) such that \(\sigma_1 > 0\), for \(j = 1, 2, \ldots, n\). Also assume that
\[
J^s f \in L^2(\mathcal{H}_{\sigma, \alpha}), \quad s > 0.
\]
Then, for any \(\epsilon > 0\) and any \(r \in (0, s]\)
\[
J^r f \in L^2(\mathcal{H}_{\sigma, \alpha + \epsilon}).
\]

**Proof.** Let \(\epsilon > 0\). In the following \(\theta_1, \theta_2\) are smooth functions with bounded derivatives satisfying: \(0 \leq \theta_1, \theta_2 \leq 1\),
\[
\theta_1(x) = \begin{cases} 1 & x \in \mathcal{H}_{\sigma, \beta + \epsilon} \\ 0 & x \in \mathcal{H}_{\sigma, \beta} \end{cases} \quad \text{and} \quad \theta_2(x) = \begin{cases} 1 & x \in \mathcal{H}_{\sigma, \beta + \epsilon} \\ 0 & x \in \mathcal{H}_{\sigma, \beta} \end{cases},
\]
so that, by construction it follows that
\[
\text{dist} (\text{supp} (1 - \theta_1), \text{supp} (\theta_2)) = \frac{\epsilon}{4|\sigma|} > 0.
\]

Next, for \(z \in \mathbb{C}\), we define the function
\[
F(z) := \theta_2 z^s f, \quad z = \alpha + i\tau, \quad \alpha, \tau \in \mathbb{R}.
\]
The function \(F\) defines a continuous function at \(\Omega := \{z \in \mathbb{C} : 0 < \text{Re}(z) < 1\}\), as well as analytic in its interior.

First, for \(z = i\tau, \tau \in \mathbb{R}\), \(F(i\tau) = \theta_2 z^{i\tau} f\), that combined with theorem 3.10 implies that \(F(i\tau) \in L^2\).
Instead in the case \( z = 1 + i\tau, \tau \in \mathbb{R} \) we have that \( F(1 + i\tau) = \theta_2 f^{i\tau} f \), then, by lemma 4.2 we get that \( F(1 + i\tau) \in L^2 \).

Finally, by the three lines lemma we obtain
\[
\theta_2 f^{i\tau} f \in L^2(\mathbb{R}^n), \quad \text{for any} \quad \tau \in (0, 1).
\]

\( \square \)

**Remark 4.1.** The Lemma 4.3 represents a extension to the \( n \)– dimensional case, \( n \geq 2 \) of a one-dimensional version proved in [12].

In our analysis we will encounter repeatedly the operators \( J^s, s > 0 \) and \( \partial_x^\alpha \), \( \alpha \in (\mathbb{Z}^+) \) and it will be essential for us to establish a relationship between them when we restrict ourselves to an specific class of subsets of \( \mathbb{R}^n \).

**Lemma 4.4 (Localization formulas).** Let \( f \in L^2(\mathbb{R}^n) \). Let \( \sigma = (\sigma_1, \sigma_2, \ldots , \sigma_n) \in \mathbb{R}^n \) a non-null vector such that \( \sigma_j \geq 0, j = 1, 2, \ldots , n \). Let \( \varepsilon > 0 \), we consider the function \( \varphi_{\sigma, \varepsilon} \in C^{\infty}_0(\mathbb{R}^n) \) to satisfy: \( 0 \leq \varphi_{\sigma, \varepsilon} \leq 1 \),
\[
\varphi_{\sigma, \varepsilon}(x) = \begin{cases} 
0 & \text{if} \quad x \in \mathcal{H}_{\sigma, \varepsilon}^c \setminus \mathcal{H}_{\sigma, \varepsilon} \\
1 & \text{if} \quad x \in \mathcal{H}_{\sigma, \varepsilon}
\end{cases}
\]
and the following increasing property: for every multi-index \( \alpha \) with \( |\alpha| = 1 \)
\[
\partial_x^\alpha \varphi_{\sigma, \varepsilon}(x) \geq 0, \quad x \in \mathbb{R}^n.
\]

(I) If \( m \in \mathbb{Z}^+ \) and \( \varphi_{\sigma, \varepsilon} J^m f \in L^2(\mathbb{R}^n) \), then for all \( \varepsilon' > 2\varepsilon \) and all multi-index \( \alpha \) with \( 0 \leq |\alpha| \leq m \), the derivatives of \( f \) satisfy
\[
\varphi_{\sigma, \varepsilon} J^m f \in L^2(\mathbb{R}^n).
\]

(II) If \( m \in \mathbb{Z}^+ \) and \( \varphi_{\sigma, \varepsilon} \partial_x^\alpha f \in L^2(\mathbb{R}^n) \) for all multi-index \( \alpha \) with \( 0 \leq |\alpha| \leq m \), then for all \( \varepsilon' > 2\varepsilon \)
\[
\varphi_{\sigma, \varepsilon} J^m f \in L^2(\mathbb{R}^n).
\]

(III) If \( s > 0 \), and \( J^s(\varphi_{\sigma, \varepsilon} f) \in L^2(\mathbb{R}^n) \), then for any \( \varepsilon' > 2\varepsilon \)
\[
\varphi_{\sigma, \varepsilon} J^s f \in L^2(\mathbb{R}^n).
\]

(IV) If \( s > 0 \), and \( \varphi_{\sigma, \varepsilon} J^s f \in L^2(\mathbb{R}^n) \), then for any \( \varepsilon' > 2\varepsilon \)
\[
J^s(\varphi_{\sigma, \varepsilon} f) \in L^2(\mathbb{R}^n).
\]

**Proof of Lemma 4.4.** In the following we will assume that \( f \in \mathcal{S}(\mathbb{R}^n) \), and the general case follows by using a regularization device.

**Proof of (I):**

Notice that for \( \alpha \)– multi-index with \( |\alpha| \leq m \), the partial derivatives satisfy
\[
\hat{\partial_x^\alpha f}(x) = \left( \frac{2\pi i \xi^\alpha}{\left( 1 + |\xi|^2 \right)^{\frac{s}{2}}} \right) \hat{f}(\xi) \, d\xi;
\]
where
\[
\Psi_{m, \alpha} f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \left( \frac{(2\pi i)^{\alpha}}{\left( 1 + |\xi|^2 \right)^{\frac{s}{2}}} \right) \hat{f}(\xi) \, d\xi.
\]
whence \( \Psi_{m,\alpha} \in \mathcal{O}^{1/|\alpha| - m} \subset \mathcal{O} \).

Since,
\[
\text{dist} \left( \text{supp} \left( q_{\sigma,\epsilon'} \right), \text{supp} \left( 1 - q_{\sigma,\epsilon'} \right) \right) \geq \frac{\epsilon' - 2\epsilon}{|\sigma|} > 0,
\]
we rewrite \( q_{\sigma,\epsilon'} \bar{c}_x^\alpha f \) as follows:
\[
q_{\sigma,\epsilon'} \bar{c}_x^\alpha f = \Psi_{m,\alpha} \left( q_{\sigma,\epsilon'} I^m f \right) - \left[ \Psi_{m,\alpha}, q_{\sigma,\epsilon'} \right] q_{\sigma,\epsilon'} I^m f + q_{\sigma,\epsilon'} \Psi_{m,\alpha} \left( (1 - q_{\sigma,\epsilon'}) I^m f \right)
= I + II + III.
\]

It is straightforward to obtain from the hypothesis that
\[
q_{\sigma,\epsilon'} I^m f \in L^2(\mathbb{R}^n), \quad \text{for all } \epsilon' > 2\epsilon.
\]

Hence, in virtue of Theorem 3.10
\[
\|I\|_{L^2} = \|\Psi_{m,\alpha} \left( q_{\sigma,\epsilon'} I^m f \right)\|_{L^2} \lesssim \|q_{\sigma,\epsilon'} I^m f\|_{L^2} < \infty.
\]

With respect to \( II \) we shall remark that \( \left[ \Psi_{m,\alpha}, q_{\sigma,\epsilon'} \right] \in \mathcal{O}^{1/|\alpha| - m - 1} \subset \mathcal{O} \), thus by Theorem 3.10 it is clear that
\[
\|II\|_{L^2} \lesssim \|q_{\sigma,\epsilon'} I^m f\|_{L^2},
\]
being the term in the r.h.s bounded by hypothesis.

To estimate \( III \) we notice that after applying an argument similar to the one used in the proof of lemma 4.2 we get \( \|III\|_{L^2} < \infty \), so that, for the sake of brevity we will omit the details.

Finally, we gather the estimates above whence we obtain
\[
q_{\sigma,\epsilon'} \bar{c}_x^\alpha f \in L^2(\mathbb{R}^n), \quad \text{for all } \epsilon' > 2\epsilon,
\]
for all multi-index \( \alpha \) satisfying \( |\alpha| \leq m \).

**Proof of (ii):**

First, notice that for \( f \in \mathcal{S}(\mathbb{R}^n) \) the operator \( I^m \) is defined as
\[
I^m f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \left( 1 + |\xi|^2 \right)^m \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n.
\]

Since the symbol associated to the operator \( I^m \) in the Fourier space corresponds to \( \langle \xi \rangle^m \), it is clear that
\[
I^m f(x) = \sum_{|\alpha| \leq m} c_{m,\alpha} \Psi_{m,\alpha} \hat{g}_\alpha f(x), \quad (4.4)
\]
where
\[
\Psi_{m,\alpha} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \left( \frac{(2\pi i \xi)^\alpha}{(1 + |\xi|^2)^{\frac{m}{2}}} \right) \hat{g}(\xi) \, d\xi, \quad g \in \mathcal{S}(\mathbb{R}^n).
\]

Notice that for any multi-index \( \alpha \in (\mathbb{Z}^+)^n \), the condition \( |\alpha| - m \leq 0 \) implies that \( \Psi_{m,\alpha} \in \mathcal{O}^{1/|\alpha| - m} \subset \mathcal{O} \).

So that, in virtue of Theorem 3.10 it is clear that \( \Psi_{m,\alpha} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \).
Now, we combine (4.4) with the hypothesis whence we obtain for \( \epsilon' > 2\epsilon \),
\[
q_{\sigma, \epsilon'} J^m f = \sum_{|\alpha| \leq m} c_{\alpha, m} \Psi_{m, \alpha} (\partial_x^\alpha q_{\sigma, \epsilon'} f) - \sum_{|\alpha| \leq m} c_{\alpha, m} \Psi_{m, \alpha} (\partial_x^\alpha q_{\sigma, \epsilon'} f)
\]
\[
+ \sum_{|\alpha| \leq m} c_{\alpha, m} q_{\sigma, \epsilon'} \Psi_{m, \alpha} ((1 - q_{\sigma, \epsilon'}) \partial_x^\alpha f).
\]
By hypothesis \( q_{\sigma, \epsilon'} \partial_x^\alpha f \in L^2(\mathbb{R}^n) \), then by \( L^2 \)-continuity of \( \Psi_{m, \alpha} \) we get
\[
\| \Psi_{m, \alpha} (\partial_x^\alpha q_{\sigma, \epsilon'} f) \|_{L^2} \lesssim \| q_{\sigma, \epsilon'} \partial_x^\alpha f \|_{L^2} < \infty.
\]
Since \( \Psi_{m, \alpha} q_{\sigma, \epsilon'} \in \text{OPS}[|\alpha|^{-m-1} \subset \text{OPS}^0 \), then by theorem 3.10 it follows that
\[
\| \Psi_{m, \alpha} q_{\sigma, \epsilon'} \partial_x^\alpha f \|_{L^2} \lesssim \| q_{\sigma, \epsilon'} \partial_x^\alpha f \|_{L^2} < \infty.
\]
being this last inequality a consequence of our hypothesis.

Finally, we focus our attention in to bound the third term in the r.h.s of (4.5), for that we will take hand of the following fact
\[
\text{dist} (\operatorname{supp} (q_{\sigma, \epsilon'}), \operatorname{supp} (1 - q_{\sigma, \epsilon'})) \geq \frac{\epsilon' - 2\epsilon}{|\sigma|} > 0.
\]
Hence, an argument similar to the one used in the proof of lemma 4.2 allow us to obtain that \( q_{\sigma, \epsilon'} \Psi_{m, \alpha} ((1 - q_{\sigma, \epsilon'}) \partial_x^\alpha f) \in L^2(\mathbb{R}^n) \).

In summary we have proved that for all \( \epsilon' > 2\epsilon \),
\[
q_{\sigma, \epsilon'} J^m f \in L^2(\mathbb{R}^n).
\]

**Proof of (ii):**

Since \( J^\alpha (q_{\sigma, \epsilon} f) \in L^2(\mathbb{R}^n) \), it is clear that
\[
q_{\sigma, \epsilon'} J^\alpha f = q_{\sigma, \epsilon'} J^\alpha (q_{\sigma, \epsilon} f + (1 - q_{\sigma, \epsilon}) f)
\]
\[
= q_{\sigma, \epsilon'} J^\alpha (q_{\sigma, \epsilon} f) + q_{\sigma, \epsilon'} J^\alpha ((1 - q_{\sigma, \epsilon}) f).
\]
So that, to handle the remainder term in (4.7) it is sufficiently to consider the relation between the supports of the function involved. More precisely,
\[
\operatorname{supp}(q_{\sigma, \epsilon'}) \subset \mathcal{H}_{\{\sigma, \epsilon'\}} \quad \text{and} \quad \operatorname{supp}(f(1 - q_{\sigma, \epsilon})) \subset \mathcal{H}_{\{\sigma, \epsilon\}},
\]
which implies
\[
\text{dist} (\operatorname{supp} (q_{\sigma, \epsilon'}), \operatorname{supp} (1 - q_{\sigma, \epsilon})) \geq \frac{\epsilon' - 2\epsilon}{2|\sigma|} > 0.
\]
Thus, by means of Lemma 4.1 we obtain that
\[
\| q_{\sigma, \epsilon'} J^\alpha ((1 - q_{\sigma, \epsilon}) f) \|_{L^2} \lesssim \| q_{\sigma, \epsilon'} \|_{L^\infty} \| (1 - q_{\sigma, \epsilon}) f \|_{L^2} \lesssim \| f \|_{L^2}.
\]
Finally, gathering the bounds above we get
\[
q_{\sigma, \epsilon'} J^\alpha f, \quad \text{for all} \quad \epsilon' > 2\epsilon,
\]
which finish the proof of (iii).

**Proof of (iv):**
Without loss of generality we can assume that \( s \in [m, m + 1) \) where \( m \in \mathbb{N}_0 \).

First, we rewrite \( J^s(\varphi_{e,e'}) \) as

\[
J^s(\varphi_{e,e'}) = \varphi_{e,e'} f + \Psi\overline{\varphi_{e,e'}} f
\]

where \( \Psi\overline{\varphi_{e,e'}} := [f; \varphi_{e,e'}] \).

The decomposition above shows that in order to obtain a bound for \( J^s(\varphi_{e,e'}) \) we require to estimate the commutator term.

In the following we will decompose the operator \( \Psi\overline{\varphi_{e,e'}} \), whence

\[
\Psi\overline{\varphi_{e,e'}} g(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \overline{\varphi_{e,e'}}(x, \xi) \widehat{g}(\xi) \, d\xi,
\]

where \( g \in \mathcal{S}(\mathbb{R}^n) \).

Next, by proposition 1 we get

\[
\overline{\varphi_{e,e'}}(x, \xi) = \sum_{1 \leq |a| \leq m} \frac{(2\pi i)^{|a|}}{a!} \left\{ \partial_x^a (\langle \xi \rangle^s) \partial_{\xi}^a \varphi_{e,e'} \right\} + \kappa_{s-m-1}(x, \xi)
\]

where \( \kappa_{s-m-1} \in \mathcal{S}^{m+1-s} \subset \mathcal{S}^0 \), and for this symbol we consider the operator

\[
\Theta_{\kappa_{s-m-1}} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \kappa_{s-m-1}(x, \xi) \widehat{g}(\xi) \, d\xi,
\]

where \( g \in \mathcal{S}(\mathbb{R}^n) \).

In view that \( \kappa_{s-m-1} \in \mathcal{S}^0 \), then by Theorem 3.10 is clear that \( \| \Theta_{\kappa_{s-m-1}} f \|_{L^2} \lesssim \| f \|_{L^2} \).

Also, for multi-index \( \alpha, \beta \) with \( \beta \leq \alpha \) we define the symbol

\[
\eta_{\alpha, \beta}(x, \xi) = \frac{(2\pi i \xi)^{\beta}}{(1 + |\xi|^2)^{\frac{\alpha - \beta}{2}}},
\]

\( x, \xi \in \mathbb{R}^n \).

Then, it is clear that \( \Psi_{\eta_{\alpha, \beta}} \in \text{OPS}^0 \), whence

\[
\Psi_{\eta_{\alpha, \beta}} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \eta_{\alpha, \beta}(x, \xi) \widehat{g}(\xi) \, d\xi,
\]

where \( g \in \mathcal{S}(\mathbb{R}^n) \).
with this notation at hand, we have

\[
\Psi_{\tilde{\eta},\tilde{\epsilon},\tilde{\epsilon}'}(x) = \sum_{j=1}^{m} \sum_{|\alpha|=j} c_{\alpha} \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \tilde{f}_{\epsilon,\epsilon'} \tilde{f}(x) + \Theta_{\eta_{\epsilon,-1}} \tilde{f}(x).
\]

Notice that from (4.10) the problem becomes reduced to localize \(f^{s-|\alpha|}f\), for every \(\alpha \in (\mathbb{N}_0)^n\). In this sense, we claim that there exist a smooth function \(\theta_{\epsilon,\epsilon'}\) such that:

\[
supp(\theta_{\epsilon,\epsilon'}) \subset \mathcal{H}_{(\epsilon', \epsilon' + 2\epsilon)}.
\]

as well as

\[
0 \leq \theta_{\epsilon,\epsilon'} \leq 1 \quad \text{and} \quad \theta_{\epsilon,\epsilon'} \equiv 1 \quad \text{on} \quad \mathcal{H}_{(\epsilon', \epsilon' + 2\epsilon)}.
\]

The interested reader can verify that it is enough to consider \(\rho \in C_0^\infty(\mathbb{R}), \rho \geq 0\), even, with \(supp(\rho) \subseteq (-1,1)\) and \(\|\rho\|_{L^1} = 1\). Then, by defining

\[
v_{\epsilon,\epsilon'}(y) = \begin{cases} 
0 & \text{if} \quad y < \frac{\epsilon'}{8}, \\
4 \left( \frac{8y - \epsilon' - 6\epsilon}{\epsilon' - 2\epsilon} \right) & \text{if} \quad \frac{\epsilon'}{8} \leq y < \frac{3\epsilon' + 2\epsilon}{8}, \\
1 & \text{if} \quad y \geq \frac{3\epsilon' + 2\epsilon}{8},
\end{cases}
\]

and \(\rho_{\epsilon,\epsilon'}(y) := \frac{16}{\epsilon' - 2\epsilon} \rho \left( \frac{16y}{\epsilon' - 2\epsilon} \right)\) for all \(y \in \mathbb{R}\) and \(\epsilon' > 2\epsilon\), we see that

\[
\theta_{\epsilon,\epsilon'}(x) = (\rho_{\epsilon,\epsilon'} \ast v_{\epsilon,\epsilon'})(\sigma \cdot x), \quad x \in \mathbb{R}^n
\]

satisfy the claimed properties.

Hence,

\[
\begin{align*}
\tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \tilde{f}_{\epsilon,\epsilon'} \tilde{f}^{s-|\alpha|}f &= - \left[ \tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \theta_{\epsilon,\epsilon'} f^{s-|\alpha|}f + \tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \left( 1 - \theta_{\epsilon,\epsilon'} \right) f^{s-|\alpha|}f \right] \\
&= I + II + III.
\end{align*}
\]

The first and the third term in the r.h.s above are bounded in the \(L^2\)–norm. More precisely,

\[
\|I\|_{L^2} = \left\| \tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \theta_{\epsilon,\epsilon'} f^{s-|\alpha|}f \right\|_{L^2} \lesssim \|\theta_{\epsilon,\epsilon'} f^{s-|\alpha|}f\|_{L^2} < \infty,
\]

being the last inequality a consequence of combining lemma 4.3 and Theorem 3.10.

The third term is handled by using that \(\Psi_{\tilde{\eta},\tilde{\epsilon},\tilde{\epsilon}'} \in \text{OPS}^0\), so that, combining lemma 4.3, (4.11)-(4.12) and Theorem 3.10 we obtain

\[
\|III\|_{L^2} = \left\| \tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} \left( 1 - \theta_{\epsilon,\epsilon'} \right) f^{s-|\alpha|}f \right\|_{L^2} \lesssim \|\tilde{c}_{\alpha}' \tilde{\zeta}_{\alpha} \tilde{\eta}_{\alpha,\beta} f^{s-|\alpha|}f\|_{L^2} \lesssim \|\tilde{f}^{s-|\alpha|}f\|_{L^2} < \infty.
\]
Notice that by construction, for any multi-index $\alpha$ with $|\alpha| \geq 1$ the following relationship holds

$$\text{dist} \left( \text{supp} \left( (1 - \theta_{e,e'}) \right), \text{supp} (\hat{c}_x^\alpha \varphi_{e,e'}) \right) \geq \frac{\left( e' - 2e \right)}{16 |\sigma|} > 0;$$

this property will so that by using a similar argument to the used in the proof of lemma 4.2 it is possible to prove

For any multi-index $\alpha$ with $|\alpha| \leq s$, we set $g_{s,\alpha} := \int^{s-|\alpha|} f$, then, it is clear that $g_{s,\alpha} \in H^{-s-|\alpha|}(\mathbb{R}^n)$.

Thus, combining (4.13) and the Fourier transform properties we obtain

$$\hat{c}_x^\alpha \varphi_{e,e'}(x) \Psi_{\eta_{a,b}} ((1 - \theta_{e,e'}) g_{s,\alpha}) (x)$$

$$= \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{c}_x^\alpha \varphi_{e,e'}(x) \eta_{a,b}(x, \xi) \left( (1 - \theta_{e,e'}) g_{s,\alpha} \right) (\xi) \, d\xi$$

$$= \int_{B_{\mu}(\mu(\varepsilon))} \hat{c}_x^\alpha \varphi_{e,e'}(x) k_{a,b}(x, x - y) \left( (1 - \theta_{e,e'}(y)) f^{2q} f^{-2q} g_{s,\alpha}(y) \right) \, dy,$$

where $q$ is chosen in such a way that $q \in \mathbb{Z}^+$ and

$$q \geq \max \left\{ 1, \left[ \frac{s - |\alpha|}{2} \right] \right\},$$

thus, integration by parts yield

$$\int_{B_{\mu}(\mu(\varepsilon))} \hat{c}_x^\alpha \varphi_{e,e'}(x) k_{a,b}(x, x - y) \left( (1 - \theta_{e,e'}(y)) f^{2q} f^{-2q} g_{s,\alpha} \right) (x) \\ = \sum_{j,k,\gamma} c_{\beta,\gamma,j} \hat{c}_x^\alpha \varphi_{e,e'}(x) \left( k_{a,b,\gamma,\beta}(\mu(\varepsilon), x, \cdot) * \left( \left( \hat{c}_y^{\beta - \gamma} (1 - \theta_{e,e'}) \right) \int f^{-2q} g_{s,\alpha} \right) \right) (x),$$

where $k_{a,b,\gamma,\beta}(\mu(\varepsilon), x, \cdot) := \left( \mathbb{1}_{|\cdot| > \mu(\varepsilon)} \right) \hat{c}_y^{\beta} k_{a,b}(x, \cdot)$.

Nevertheless, at this point two cases have to be distinguished.

**Case : $\gamma = \beta$**

For this case we combine Young’s inequality and (4.14) to obtain

$$\left\| \hat{c}_x^\alpha \varphi_{e,e'} \right\|_{L^2} \lesssim_{\mu(\varepsilon)} \left( \sum_{j=1}^{2q} \sum_{|\beta| \leq 2} c_{\beta,\eta,j} \gamma \right) \left\| \hat{c}_x^\alpha \varphi_{e,e'} \right\|_{L^\infty} \| f \|_{L^2} < \infty.$$
Combining (4.14), Young’s inequality and theorem 3.10 imply that
\[
\left\| \partial_x^\alpha \varphi_{s,\sigma,\epsilon'} \Psi_{\eta_{a,\beta}} \left( (1 - \theta_{\epsilon,\epsilon'}) \mathcal{G}_{s,a} \right) \right\|_{L^2} 
\]
\[
\lesssim \mu(\epsilon) \sum \sum \sum c_{\beta,\gamma} \| \partial_x^\alpha \varphi_{s,\epsilon'} \|_{L^\infty} \| \partial_y^{\alpha-\gamma} (1 - \theta_{\epsilon,\epsilon'}) \|_{L^\infty} \| f \|_{L^2}. \tag{4.16}
\]

Gathering the estimates in (4.15) and (4.16) we get
\[
\| II \|_{L^2} \lesssim_{\mu(\epsilon), n, \sigma} \| \partial_x^\alpha \varphi_{s,\sigma,\epsilon'} \|_{L^\infty} \| f \|_{L^2}.
\]

We shall remark that we are looking for an upper bound in the $L^2$–norm of $\widehat{\Psi_{s_{\sigma,\epsilon'}} f}$.

In this sense, we get after going back into (4.10) the following:
\[
\left\| \Psi_{s_{\sigma,\epsilon'}} f \right\|_{L^2} 
\]
\[
\lesssim \sum \sum \sum \sum \left| c_{\alpha,\beta} \right| \| \partial_x^\alpha \varphi_{s,\epsilon'} \|_{L^\infty} \| \theta_{\epsilon,\epsilon'} f \|_{L^2} \left( \| \partial_x^\alpha \varphi_{s,\sigma} \|_{L^\infty} + 1 \right) 
\]
\[
+ \| f \|_{L^2} < \infty,
\]

then,
\[
\left\| f^s(\varphi_{s,\epsilon'}, f) \right\|_{L^2} 
\]
\[
\lesssim_{\mu(\epsilon), n, \sigma} \| \varphi_{s,\epsilon'} f \|_{L^2} + \| f \|_{L^2} 
\]
\[
+ \sum \sum \sum \left| c_{\alpha,\beta} \right| \left( \| \theta_{\epsilon,\epsilon'} f \|_{L^2} + \| f \|_{L^2} \right) \left( \| \partial_x^\alpha \varphi_{s,\sigma} \|_{L^\infty} + 1 \right) < \infty.
\]

So that, we have proved that $\varphi_{s,\epsilon'} f \in L^2$ implies that
\[
f^s(\varphi_{s,\epsilon'}, f) \in L^2(\mathbb{R}^n), \quad \text{for all } \epsilon' > 2 \epsilon. \quad \square
\]

**Remark 4.2.** At this point several comments about this lemma should be presented due to its relevance.

(i) To our knowledge the first version of this "localization formulas" were presented in [12] in the study of propagation of regularity in solutions of the $k$–generalized KdV equation.

(ii) Also we shall emphasize that the lemma above also holds when we consider a “direction” $\sigma$ with different conditions to the emphasized above. As we shall see later a more general version of this lemma also holds when we consider different regions of the space instead of just merely half-spaces.
Remark 4.3. A more general version of this lemma also hold when we consider a wider class of domains where is required to localize the regularity. More precisely, the following localization result is also true.

**Lemma 4.5.** Let \( f \in L^2(\mathbb{R}^n) \). If \( \theta_1, \theta_2 \in C^\infty(\mathbb{R}^n) \) are functions such that: \( 0 \leq \theta_1, \theta_2 \leq 1 \), their respective supports satisfy
\[
\text{dist}(\text{supp}(1 - \theta_1), \text{supp}(\theta_2)) \geq \delta,
\]
for some positive number \( \delta \), and for all multi-index \( \beta \), the functions \( \partial_x^\beta \theta_1, \partial_x^\beta \theta_2 \in L^\infty(\mathbb{R}^n) \).

Then, the following identity holds:

1. If \( m \in \mathbb{Z}^+ \) and \( \theta_1 \partial_1^m f \in L^2(\mathbb{R}^n) \), then for all multi-index \( \alpha \) with \( 0 \leq |\alpha| \leq m \), the derivatives of \( f \) satisfy
\[
\theta_2 \partial_1^\alpha f \in L^2(\mathbb{R}^n).
\]

2. If \( m \in \mathbb{Z}^+ \) and \( \theta_1 \partial_x^\alpha f \in L^2(\mathbb{R}^n) \) for all multi-index \( \alpha \) with \( 0 \leq |\alpha| \leq m \), then
\[
\theta_2 \partial_1^m f \in L^2(\mathbb{R}^n).
\]

3. If \( s > 0 \), and \( J^s(\theta_1 f) \in L^2(\mathbb{R}^n) \), then
\[
\theta_2 J^s f \in L^2(\mathbb{R}^n).
\]

4. If \( s > 0 \), and \( \theta_1 J^s f \in L^2(\mathbb{R}^n) \), then
\[
J^s(\theta_2 f) \in L^2(\mathbb{R}^n).
\]

**Proof.** The proof follows by using an argument similar to the one used in the proof of lemma 4.4, so that for the sake of brevity we omit the details. \( \square \)

5. **Kato’s smoothing effect**

In the following section we present he Kato’s smoothing effect version satisfied by the solutions of the ZK equation in the \( n \)-dimensional setting, \( n \geq 2 \). By using the Kato’s approach for the KdV model [10], we are able to prove that the solutions associated to the ZK equation gain one local spatial derivative for each direction.

**Lemma 5.1.** Let \( \psi \) be a smooth function such that \( \psi, \psi' \geq 0 \). Let \( u \) be a function such that \( u \in C([0, T] : H^\infty(\mathbb{R}^n)) \) is a solution of the IVP (1.1). Assume also that \( s \geq 0 \), \( r > n/2 \) and \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n \), with
\[
\sigma_1 > 0 \quad \text{and} \quad \sqrt{3} \sigma_1 > \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.
\]

Then,
\[
\sum_{m=1}^n \int_0^T \int_{\mathbb{R}^n} (\partial_x^m J^s u(x,t))^2 \psi'(\sigma \cdot x) \, dx \, dt 
\leq_{u, \sigma} (1 + T + \| \nabla u \|_{L^1 L^\infty} + T \| u \|_{L^\infty H^r}^2) \| u \|_{L^\infty H^r}^2.
\]
Proof. For $\sigma$ as in (5.1) we define the function

$$\psi_{\sigma}(x) := \psi(\sigma \cdot x), \quad x \in \mathbb{R}^n.$$  

To obtain (5.2) we apply $f^s$ to the equation in (1.1) followed by a multiplication by $f^s \psi_{\sigma}$, so that after integrating in the spatial variable yield

$$\frac{d}{dt} \int_{\mathbb{R}^n} \left( f^s u(x, t) \right)^2 \psi_{\sigma}(x) \, dx + \sigma_1 \sum_{m=1}^n \int_{\mathbb{R}^n} \left( \partial_{x_m} f^s u(x, t) \right)^2 \psi'(\sigma \cdot x) \, dx$$

$$+ 2 \sum_{m=1}^n \sigma_m \int_{\mathbb{R}^n} \left( (\partial_{x_m} f^s u)(\partial_{x_1} f^s u) \right) (x, t) \psi'(\sigma \cdot x) \, dx$$

$$- \sigma_1 \sum_{m=1}^n \sigma_m^2 \int_{\mathbb{R}^n} \left( f^s u(x, t) \right)^2 \psi'''(\sigma \cdot x) \, dx$$

$$+ \int_{\mathbb{R}^n} (f^s u \partial_{x_1} u) f^s u \right) (x, t) \psi_{\sigma}(x) \, dx = 0.$$  

(5.3)

At this point we use the approach used in [16], to obtain the smoothing effect. More precisely, for $m$ a positive integer we define

$$\mu_m^2 := \int_{\mathbb{R}^n} \left( \partial_{x_m} f^s u \right)^2 \psi'(\sigma \cdot x) \, dx.$$  

We claim that

$$\sigma_1 \sum_{m=1}^n \mu_m^2 + 2 \sum_{m=1}^n \sigma_m \int_{\mathbb{R}^n} \left( (\partial_{x_m} f^s u)(\partial_{x_1} f^s u) \right) (x, t) \psi'(\sigma \cdot x) \, dx$$

$$= \sigma \sum_{m=1}^n \mu_m^2.$$  

(5.4)

For our proposes it is necessary to decouple the term involving the interactions of the derivatives in (5.4), this can be achieved by using Cauchy-Schwarz inequality. More precisely, due to $\psi' \geq 0$, it is clear that

$$\left| \int_{\mathbb{R}^n} \left( (\partial_{x_m} f^s u)(\partial_{x_1} f^s u) \right) (x, t) \psi'(\sigma \cdot x) \, dx \right| \leq \mu_m \mu_1, \quad \text{for} \quad m \in \{1, 2, \ldots, n\}.$$  

(5.5)

So that, for $\sigma \in \mathbb{R}^n$ with

$$\sigma_1 > 0 \quad \text{and} \quad \sqrt{3} \sigma_1 > \sqrt{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2},$$

there exist $\lambda = \lambda(\sigma) > 0$, such that

$$\sigma_1 \sum_{m=1}^n \mu_m^2 + 2 \sum_{m=1}^n \sigma_m \int_{\mathbb{R}^n} \left( \partial_{x_m} f^s u \right) \left( \partial_{x_1} f^s u \right) \psi'(\sigma \cdot x) \, dx \geq \lambda(\sigma) \sum_{m=1}^n \mu_m^2.$$  

(5.6)

The opposite inequality follows as an straightforward application of Young’s inequality in (5.5), that is

$$\sigma_1 \sum_{m=1}^n \mu_m^2 + 2 \sum_{m=1}^n \sigma_m \int_{\mathbb{R}^n} \left( \partial_{x_m} f^s u \right) \left( \partial_{x_1} f^s u \right) \psi'(\sigma \cdot x) \, dx \leq \lambda(\sigma) \sum_{m=1}^n \mu_m^2.$$  

(5.7)
Finally, we gather the inequalities (5.5) and (5.7) to obtain
\[
\sigma_1 \sum_{m=1}^{n} \mu_m^2 + 2 \sum_{m=1}^{n} \sigma_m \int_{\mathbb{R}^n} (\partial_{x_m} J^s u) (\partial_{x_1} J^s u) \psi'(\sigma \cdot x) \, dx = \sigma \sum_{m=1}^{n} \mu_m^2.
\]
Instead, the nonlinear part is handled by means of integration by parts, Hölder’s inequality and (3.12), whence we get
\[
(5.8) \int_{\mathbb{R}^n} J^s (u \partial_{x_1} u) f^s u \psi \psi' \, dx = \int_{\mathbb{R}^n} \psi \psi' f^s u \partial_{x_1} u \, dx - \frac{\sigma_1}{2} \int_{\mathbb{R}^n} \partial_{x_1} u (J^s u)^2 \psi \psi' \, dx
\]
\[
\lesssim \sigma \left( \| \nabla u \|_{L^\infty} + \| u \|_{L^\infty} \right) \| u \|_{H^s}.
\]
Therefore, we deduce after gathering (5.3), (5.6), (5.8) and integrating in time that
\[
\sum_{m=1}^{n} \int_0^T \int_{\mathbb{R}^n} |\partial_{x_m} J^s u(x,t)|^2 \psi' \psi \, dx \, dt \lesssim \sigma \left( 1 + T + \| \nabla u \|_{L^\infty} + \| u \|_{L^\infty} \right) \| u \|_{H^s}.
\]
From the last inequality above it is straightforward to obtain (5.3) by means of Sobolev embedding. \qed

6. Proof of Theorem A

In this section we provide a proof of theorem A, nevertheless before provide the proof scheme we will describe the weighted functions to be used in our analysis.

More precisely, for \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \) we define the following families of functions
\[
\chi_{\epsilon, \tau}, \phi_{\epsilon, \tau}, \psi_{\epsilon} \in C^\infty(\mathbb{R}),
\]
satisfying the conditions indicated below:

(i) \( \chi_{\epsilon, \tau}(x) = \begin{cases} 1 & x \geq \tau \\ 0 & x \leq \epsilon \end{cases} \),

(ii) \( \text{supp}(\chi_{\epsilon, \tau}) \subset [\epsilon, \tau] \),

(iii) \( \chi_{\epsilon, \tau}'(x) \geq 0 \),

(iv) \( \chi_{\epsilon, \tau}'(x) \geq \frac{1}{10(\tau - \epsilon)} \mathbb{1}_{[2\epsilon, \tau - 2\epsilon]}(x) \),

(v) \( \text{supp}(\phi_{\epsilon, \tau}) \subset [\frac{\epsilon}{4}, \tau] \),

(vi) \( \phi_{\epsilon, \tau}(x) = \phi_{\epsilon, \tau}(x) = 1, \text{ if } x \in [\frac{\epsilon}{2}, \epsilon] \),

(vii) \( \text{supp}(\phi_{\epsilon}) \subset (-\infty, \frac{\epsilon}{4}] \).

(viii) For all \( x \in \mathbb{R} \) the following quadratic partition of the unity holds
\[
\chi_{\epsilon, \tau}^2(x) + \phi_{\epsilon, \tau}^2(x) + \psi_{\epsilon}(x) = 1,
\]

(ix) also for all \( x \in \mathbb{R} \)
\[
\chi_{\epsilon, \tau}(x) + \phi_{\epsilon, \tau}(x) + \psi_{\epsilon}(x) = 1, \quad x \in \mathbb{R}.
\]

For a more detailed construction of these families of weighted functions see [7].
Proof of theorem A. Formally we apply the operator $J^s$ to the equation in (1.1), followed by a multiplication by $J^s u \chi_{\epsilon, \tau, \nu}^2(x, t)$ to obtain

$$(J^s \partial_t u + J^s \partial_{x_1} \Delta u + J^s (u \partial_{x_1} u) J^s u) \chi_{\epsilon, \tau, \nu}^2(x, t) = 0,$$

where

$$\chi_{\epsilon, \tau, \nu}(x, t) := \chi_{\epsilon, \tau}(\sigma \cdot x + vt - \beta), \quad x \in \mathbb{R}^n, t \geq 0,$$

similarly are defined the functions $\phi_{\epsilon, \tau, \nu}(x, t), \phi_{\epsilon, \tau, \nu}'(x, t)$ and $\chi_{\epsilon, \tau, \nu}(x, t)$.

Without loss of generality we will assume that $\beta = 0$.

So that, after integrating by parts we obtain the following identity:

$$(6.1) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (J^s u(x, t))^2 \chi_{\epsilon, \tau, \nu}^2(x, t) \, dx - \nu \int_{\mathbb{R}^n} (J^s u(x, t))^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx$$

$$+ \sigma_1 \sum_{m=1}^n \int_{\mathbb{R}^n} (\partial_{x_m} J^s u(x, t))^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx$$

$$+ 2 \sum_{m=1}^n \sigma_m \int_{\mathbb{R}^n} (\partial_{x_m} J^s u(x, t)) (\partial_{x_1} J^s u(x, t)) (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx$$

$$- 3\sigma_1 |\sigma|^2 \int_{\mathbb{R}^n} (J^s u(x, t))^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx$$

$$- \sigma_1 |\sigma|^2 \int_{\mathbb{R}^n} (J^s u(x, t))^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx$$

$$+ \int_{\mathbb{R}^n} J^s (u(x, t) \partial_{x_1} u(x, t)) J^s u(x, t) \chi_{\epsilon, \tau, \nu}^2(x, t) \, dx = 0.$$

Case: $s \in (s_n, s_n + 1)$.

For $\epsilon > 0$ and $\tau \geq 5\epsilon$, there exist $\delta > 0$ and $\delta > 0$ such that

$$\chi_{\epsilon, \tau, \nu}(x, t) \chi_{\epsilon, \tau}'(\sigma \cdot x + vt) \leq c \mathbf{1}_{Q_{(r, -\delta, \delta)}}(x) \quad x \in \mathbb{R}^n, t \in [0, T].$$

Hence,

$$\int_0^T |A_1(t)| \, dt \leq |\nu| \int_0^T \int_{\mathbb{R}^n} (J^s u(x, t))^2 (\chi_{\epsilon, \tau} \chi_{\epsilon, \tau}') (\sigma \cdot x + vt) \, dx \, dt$$

$$\leq c |\nu| \int_0^T \int_{Q_{(r, -\delta, \delta)}} (J^s u(x, t))^2 \, dx \, dt.$$
Nevertheless, by constructing $\psi$ properly in lemma 5.1 it can be deduced that there exist $\delta_1 > 0$, such that

$$\int_0^T \int_{Q_{(r,-\delta_1,\delta_1)}} \left( J^r u(x,t) \right)^2 \, dx \, dt \leq c, \quad \text{for any} \quad r \in [0, \beta],$$

where $\beta = s_n + 1$.

Thus, as a particular case we get that $A_1 \in L^1_T$.

By using a similar argument we deduce that

$$\int_0^T |A_4(t)| \, dt \leq 3\sigma_1 |\sigma|^2 \int_0^T \int_{\mathbb{R}^n} \left( f^r u(x,t) \right)^2 \left( \chi_{e,\tau} \chi_{e,\tau}'' \right) (\sigma \cdot x + vt) \, dx \, dt \leq c_{\sigma,\epsilon},$$

and

$$\int_0^T |A_5(t)| \, dt \leq 3\sigma_1 |\sigma|^2 \int_0^T \int_{\mathbb{R}^n} \left( f^r u(x,t) \right)^2 \left| \chi_{e,\tau}'' \right| (\sigma \cdot x + vt) \, dx \, dt \leq c_{\sigma,\epsilon}.$$

Next, for $m$ a positive integer with $1 \leq m \leq n$, we set

$$\mu_m^2(t) := \int_{\mathbb{R}^n} (\partial_{x_m} f^r u(x,t))^2 \left( \chi_{e,\tau} \chi_{e,\tau}' \right) (\sigma \cdot x + vt) \, dx, \quad t \in (0, T).$$

By using an argument similar to that one used in the proof of lemma 5.1, we obtain that there exist a positive number $\lambda = \lambda(\sigma)$, such that

$$\sum_{m=1}^n \lambda \mu_m^2 \leq 2 \sum_{m=1}^n \sigma_1 \int_{\mathbb{R}^n} (\partial_{x_m} f^r u(x,t)) (\partial_{x_1} f^r u(x,t)) \left( \chi_{e,\tau} \chi_{e,\tau}' \right) (\sigma \cdot x + vt) \, dx$$

$$+ \sigma_1 \sum_{m=1}^n \mu_m^2.$$  

Finally, we show how to control the nonlinear part. First, notice that by Cauchy-Schwarz inequality it is enough to estimate $\chi_{e,\tau,\sigma} f^r (u \partial_{x_1} u)$ in the $L^2-$norm. In this sense, the following relationships are very useful

$$\chi_{e,\tau,\sigma}(x,t) + \phi_{e,\tau,\sigma}(x,t) + \psi_{e,\sigma}(x,t) = 1, \quad x \in \mathbb{R}^n, t \in \mathbb{R}$$

and

$$\chi_{e,\tau,\sigma}^2(x,t) + \phi_{e,\tau,\sigma}^2(x,t) + \psi_{e,\sigma}(x,t) = 1, \quad x \in \mathbb{R}^n, t \in \mathbb{R}.$$  

So that, in order to integrate these decomposition into the non-linear part we write

$$\chi_{e,\tau,\sigma} f^r (u \partial_{x_1} u)$$

$$= -\frac{1}{2} \left[ f^r; \chi_{e,\tau,\sigma} \right] \partial_{x_1} \left( u \chi_{e,\tau,\sigma} \right)^2 + \left( u \phi_{e,\tau,\sigma} \right)^2 + u^2 \psi_{e,\sigma}$$

$$+ \left[ f^r; u \chi_{e,\tau,\sigma} \right] \partial_{x_1} \left( u \chi_{e,\tau,\sigma} \right) + \left[ f^r; u \chi_{e,\tau,\sigma} \right] \partial_{x_1} \left( u \phi_{e,\tau,\sigma} \right)$$

$$+ \left[ f^r; u \chi_{e,\tau,\sigma} \right] \partial_{x_1} \left( u \psi_{e,\sigma} \right) + \chi_{e,\tau,\sigma} u f^r \partial_{x_1} u$$

$$= B_{6,1} + B_{6,2} + B_{6,3} + B_{6,4} + B_{6,5} + B_{6,6} + B_{6,7}.$$

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First, we denote by $\Psi_\xi$ the operator $\Psi_\xi := [F^s; \chi_{e,\tau,\sigma}]$, that according to proposition 1

\[
\zeta(x, \xi) \sim \sum_{\alpha} \left( \frac{2\pi i}{\alpha!} \right)^{-|\alpha|} \left( \partial^{\alpha \xi} \chi_{e,\tau,\sigma}(x, t) \right)
\]

for any $t \geq 0$.

Nevertheless, we have previously obtained similar decomposition for this operator in the proof of lemma 4.4, so that, for the sake of brevity we will omit the details.

Therefore, we claim that there exist a pseudo-differential operator $\Psi_{\kappa_{s-m-1}} \in \text{OPS}^{s-m-1}$, such that for any $f \in \mathcal{S}$ the following identity holds

\[
[F^s; \chi_{e,\tau,\sigma}] f(x) = \sum_{j=1}^m \sum_{|\alpha|=j} \sum_{\beta} c_{\alpha,\beta} \partial_{x,\tau,\sigma} \left( \partial_{x,\tau,\sigma}^{\alpha \beta} \chi_{e,\tau,\sigma}(x, t) \Psi_{\eta_{\beta,\delta}} f^{s-|\alpha|} f(x) \right)
\]

where $m = \left[ \frac{n}{2} \right] + 1$, and $\Psi_{\eta_{\beta,\delta}} \in \text{OPS}^{\beta-|\alpha|} \subset \text{OPS}^0$ is defined as in (4.9).

Hence, combining theorem 3.10, interpolation and inequality (3.13), we obtain

\[
\|B_{6.1}\|_2 \leq \left\| [F^s; \chi_{e,\tau,\sigma}] \partial_{x,\tau,\sigma} \left( \left( u\chi_{e,\tau,\sigma} \right)^2 \right) \right\|_2
\]

\[
\leq \sum_{j=1}^m \sum_{|\alpha|=j} \sum_{\beta} c_{\alpha,\beta} \partial_{x,\tau,\sigma} \left( \partial_{x,\tau,\sigma}^{\alpha \beta} \chi_{e,\tau,\sigma}(x, t) \right) \left\| \Psi_{\eta_{\beta,\delta}} f^{s-|\alpha|} \left( \partial_{x,\tau,\sigma} \left( \left( u\chi_{e,\tau,\sigma} \right)^2 \right) \right) \right\|_2
\]

\[
+ \left\| \Psi_{\kappa_{s-m-1}} \left( \partial_{x,\tau,\sigma} \left( \left( u\chi_{e,\tau,\sigma} \right)^2 \right) \right) \right\|_2
\]

\[
\lesssim e, \tau, \sigma, m \| u \|_{L^\infty} \left\| F^s \left( u\chi_{e,\tau,\sigma} \right) \right\|_2.
\]

Similarly,

\[
\|B_{6.2}\|_2 = \left\| [F^s; \chi_{e,\tau,\sigma}] \partial_{x,\tau,\sigma} \left( \left( u\psi_{e,\tau,\sigma} \right)^2 \right) \right\|_2
\]

\[
\lesssim e, \tau, \sigma, m \| u \|_{L^\infty} \left\| F^s \left( u\psi_{e,\tau,\sigma} \right) \right\|_2.
\]

For the term $B_{6.3}$ it is not required the use of the previous machinery to obtain some upper bounds. Instead, we notice that for any $t \geq 0$,

\[
\text{dist} \left( \text{supp} \left( \chi_{e,\tau,\sigma}(\cdot, t) \right), \text{supp} \left( \psi_{e,\sigma}(\cdot, t) \right) \right) \geq \frac{\epsilon}{2|\sigma|}.
\]

So, we fall in the hypothesis of lemma 4.1, therefore it is clear that

\[
\|B_{6.3}\|_2 = \left\| \chi_{e,\tau,\sigma} F^s \partial_{x,\tau,\sigma} \left( u^2 \psi_{e,\sigma} \right) \right\|_2 \lesssim e, \tau \| u \|_{L^2} \| u \|_{L^\infty}.
\]

Also, from the remark indicated in (6.11) and lemma 4.1 it also follows that

\[
\|B_{6.6}\|_2 = \left\| u \chi_{e,\tau,\sigma} F^s \left( u\psi_{e,\sigma} \right) \right\|_2 \lesssim e, \sigma \| u \|_{L^\infty} \| u \|_{L^2}.
\]

Concerning to the terms $B_{6.4}$ and $B_{6.5}$, the scenario is quite different, and the argument used above can be avoided by using the Kato-Ponce commutator estimate (3.12). More precisely,

\[
\left\| [F^s; u\chi_{e,\tau,\sigma}] \partial_{x,\tau,\sigma} \left( u\chi_{e,\tau,\sigma} \right) \right\|_2 \lesssim \left\| F^s \left( u\chi_{e,\tau,\sigma} \right) \right\|_2 \| \partial \left( u\chi_{e,\tau,\sigma} \right) \|_{L^\infty}.
\]
and
\[ \| [J^s; u \chi_{e, \tau, s}] \partial_x u \phi_{e, \tau, s} \|_{L^2} \lesssim \| J^s (u \phi_{e, \tau, s}) \|_{L^2} \| \nabla (u \chi_{e, \tau, s}) \|_{L^\infty} + \| J^s (u \chi_{e, \tau, s}) \|_{L^2} \| \partial_x (u \phi_{e, \tau, s}) \|_{L^\infty}. \]

Next, we bound the term \( B_{6,7} \), for this we notice that up to constants, this term is in essence upper bounded by \( \| u(t) \|_{L^\infty} \) times \( A_1(t) \), more precisely,
\[ |B_{6,7}(t)| \lesssim \| u(t) \|_{L^\infty} \| A_1(t) \|. \]

Since it was proved previously that \( A_1 \in L^1 \) is bounded, for the sake of brevity we will omit the proof, so that, we shall only guarantee that \( \| u(t) \|_{L^\infty} < \infty \) for almost all \( t \in [0, T] \). Nevertheless, a combination of the local theory and the Sobolev embedding clearly imply that \( \| u(t) \|_{L^\infty} < \infty \).

Instead, to handle the term \( B_{6,8} \), we only require to use Sobolev’s embedding, this in order to guarantee that \( \nabla u \in L^1 ([0, T] : L^\infty (\mathbb{R}^n)) \). More precisely,
\[ |B_{6,8}(t)| \lesssim \| \nabla u(t) \|_{L^\infty} \int_{\mathbb{R}^n} (J^s u(x, t))^2 \chi_{e, \tau, s}^2(x, t) \, dx, \]
notice that the quantity to be estimated by Gronwall’s inequality corresponds to the integral expression in the r.h.s above.

Finally, we turn our attention to several terms that remains to estimate in several inequalities above e.g. the term \( \| J^s (u \chi_{e, \tau, s}) \|_{L^2} \) present in (6.9), (6.14), (6.15). Nevertheless, after taking into consideration the following decomposition
\[ J^s (u \chi_{e, \tau, s}) = \chi_{e, \tau, s} J^s u + [J^s; \chi_{e, \tau, s}] (u \chi_{e, \tau, s} + u \phi_{e, \tau, s} + u \psi_{e, \tau, s}) = I + II + III + IV. \]

The term \( I \) in the r.h.s above is the quantity to be estimated by Gronwall’s inequality; instead the term \( IV \) can be easily handled by using lemma 4.1. More precisely, we get
\[ IV \|_{L^2} = \| \chi_{e, \tau, s} J^s (\phi_{e, \tau} u) \|_{L^2} \lesssim \| \phi_{e, \tau} u \|_{L^2}. \]

Instead the terms, \( \| J^s (u \phi_{e, \tau, s}) \|_{L^2} \), \( \| J^s (u \psi_{e, \tau, s}) \|_{L^2} \) can be estimated by using lemma 4.4.

However, to bound the terms \( II \) and \( III \) we require to decompose the commutator in a similar manner to that one in (6.7)-(6.8). So that, we get
\[ II \|_{L^2} \lesssim \| \chi_{e, \tau, s} J^{s-1} (u \chi_{e, \tau, s}) \|_{L^2} \]
and
\[ III \|_{L^2} \lesssim \| \chi_{e, \tau, s} J^{s-1} (u \phi_{e, \tau, s}) \|_{L^2}. \]

Later, after integrating in time, we get that
\[ \| II \|_{L^2 L^2}^2 \lesssim \| \chi_{e, \tau, s} T \| J^{s-1} (u \chi_{e, \tau, s}) \|_{L^2 L^2}^2 < \infty, \]
the last inequality above follows as a direct application of theorem ??.

Similarly,
\[ \| III \|_{L^2 L^2}^2 \lesssim \| \chi_{e, \tau, s} T \| J^{s-1} (u \phi_{e, \tau, s}) \|_{L^2 L^2}^2 < \infty. \]
Finally, this step conclude by gathering the estimates above combined with Gronwall’s inequality and integration in time whence we obtain that:

for any $\nu \geq 0$, $\epsilon > 0$, $\tau \geq 5\epsilon$,

\[
(6.17) \quad \sup_{0 \leq t \leq T} \left\| f^s u \right\|_{L^2(H_{(\nu,\epsilon,-\nu)})}^2 + \sum_{m=1}^{n} \lambda(\sigma) \left\| \partial_{x_m} f^s u \right\|_{L^2_{\nu}L^2(Q_{(\nu,\epsilon,-\nu,\tau,-\nu)})}^2 \leq c^*_{(s)},
\]

where $c^*_{(s)}$ is a positive constant depending on the following quantities

\[
c^*_{(s)} = c^*_{(s)} \left( \epsilon; \tau; \lambda; n; \nu; s; T; \left\| f^s u_0 \right\|_{L^2(H_{(\nu,\epsilon)})}; \left\| u_0 \right\|_{L^2_{\nu}H} \right) > 0.
\]

Notice that the second term in the left hand side above provides a gain of one local derivative in all directions, being this indicated by the presence of the operator $\nabla f^s$. However, our method of proof requires a modified version of the smoothing effect obtained in (6.17). Roughly speaking, we shall indicate the smoothing effect in terms of the operator $f^{s+1}$ instead of $\nabla f^s$.

So that, we will undertake this modification by means of a chains of claims that will imply the smoothing effect required.

**Claim 1:**

If for any $\nu \geq 0$, $\epsilon_1 > 0$ and $\tau_1 \geq 5\epsilon_1$

\[
(6.18) \quad \sum_{m=1}^{n} \lambda(\sigma) \left\| \partial_{x_m} f^s u \right\|_{L^2_{\nu}L^2(Q_{(\nu,\epsilon_1,-\nu,\tau_1,-\nu)})}^2 \leq c^*_{(s)},
\]

for some positive constant $c^*_{(s)}$, then there exist a constant $c^*_{(s)} > 0$, such that for any $\nu \geq 0$, $\epsilon > 0$, and $\tau \geq \epsilon$,

\[
(6.19) \quad \left\| f^{s+1} u \right\|_{L^2_{\nu}L^2(Q_{(\nu,\epsilon,-\nu,\tau,-\nu)})} \leq c^*_{(s)},
\]

where $c^*_{(s)} = c^*_{(s)} \left( \epsilon; \tau; n; c^*_{(s)}; \lambda(\sigma); \left\| u_0 \right\|_{L^2} \right)$.

**Proof of Claim 1.** We claim that there exist functions $\theta_1, \theta_2 \in C^\infty(\mathbb{R}^n)$, with bounded derivatives such that:

\[
\text{supp} (\theta_1) \subseteq Q_{\{\nu, \tau, \epsilon\}} \quad \text{with} \quad \theta_1 \equiv 1 \quad \text{on} \quad \overline{Q_{\{\nu, \tau, \epsilon\}}},
\]

and

\[
\text{supp} (\theta_2) \subseteq Q_{\{\nu, \tau, \epsilon\}} \quad \text{with} \quad \theta_2 \equiv 1 \quad \text{on} \quad \overline{Q_{\{\nu, 2\epsilon, \tau - 2\epsilon\}}}.
\]

Next, notice that for any $s > 0$, the following representation holds

\[
f^{s+1} f = f^{s-1} f + \sum_{m=1}^{n} c_m \Psi_m \partial_{x_m} f^{s} f.
\]

Roughly speaking, up to constant the operator $\Psi_m$ is nothing more than $\partial_{x_m} f^{-1}$, for $m = 1, 2, \ldots, n$. Besides, $\Psi_m \in \text{OPS}^0$, so that, in virtue of theorem 3.10 it maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

Notice that notwithstanding we recover in theory all the derivatives in the channel of propagation, it is not so useful for our proposes, due to the perturbation by the non-local pseudo-differential operator $\Psi_m$. So that, in order to reconcile all these
issues, we introduce into our analysis the functions \( \theta_1, \theta_2 \), by means of the following decomposition:

\[
\theta_2 \Psi_m \partial_{x_n} f^s u = [\Psi_m; \theta_2] \theta_1 \partial_{x_n} f^s u + \theta_2 \Psi_m \left( (1 - \theta_1) \partial_{x_n} f^s f \right) + \Psi_m (\theta_2 \partial_{x_n} f^s u).
\]

Nevertheless, in order to incorporate this decomposition properly into the argument, it is necessary to define the following functions

\[
\theta_1(x, t) := \theta_1 \left( x + \frac{vt}{|\sigma|^2} \right) \quad \text{and} \quad \theta_2(x, t) := \theta_2 \left( x + \frac{vt}{|\sigma|^2} \right), \quad x \in \mathbb{R}^n, t \in \mathbb{R}.
\]

Therefore, combining lemma 4.2, the continuity of \( \Psi_m \) and the properties of the weighted functions \( \theta_1, \theta_2 \), we get

\[
\left\| f^{s+1} u \right\|_{L^2(Q_{(r,\varepsilon-v,t,v-t)}} \leq \left\| \theta_2 f^{s+1} u \right\|_{L^2} \\
\leq \sum_{m=1}^n \left\| \theta_2 \Psi_m \partial_{x_n} f^s u \right\|_{L^2} + \left\| \theta_2 f^{s-1} u \right\|_{L^2} \\
= \sum_{m=1}^n \gamma_m \left\{ \left\| \Psi_m; \partial_1 \right\| \partial_2 \partial_{x_n} f^s u \right\|_{L^2} + \left\| \partial_2 \Psi_m \left( (1 - \theta_1) \partial_{x_n} f^s f \right) \right\|_{L^2} \\
+ \left\| \Psi_m (\theta_2 \partial_{x_n} f^s u) \right\|_{L^2} \right\} + \left\| \partial_2 f^{s-1} u \right\|_{L^2} \\
\lesssim_{\varepsilon, \tau, n} \sum_{m=1}^n \gamma_m \left\{ \left\| \partial_2 \partial_{x_n} f^s u \right\|_{L^2} \left( Q_{(r,\varepsilon-v,t-v,\varepsilon-v)} \right) + \left\| u_0 \right\|_{L^2} \\
+ \left\| \partial_{x_n} f^s u \right\|_{L^2} \left( Q_{(r,\varepsilon-v,t-v,\varepsilon-v)} \right) \right\} + \left\| \partial_2 f^{s-1} u \right\|_{L^2}.
\]

Hence, in virtue of (6.17) and (6.18), it is clear that

\[
(6.20) \quad \left\| f^{s+1} u \right\|_{L^2(Q_{(r,\varepsilon-v,t,v-t)}} \lesssim_{\varepsilon, \tau, n} T \left( \left\| u_0 \right\|_{L^2}^2 + c^s_{(s-1)} \right) + \left( \frac{c^s_{(s)}}{\lambda(s)} \right).
\]

Gathering the estimates above we get that: for all \( \nu \geq 0, \varepsilon > 0 \) and \( \tau \geq 5\varepsilon \), the following inequality holds

\[
\left\| f^{s+1} u \right\|_{L^2(Q_{(r,\varepsilon-v,t,v-t)}} \lesssim_{\varepsilon, \tau, n} c^{**},
\]

whence \( c^{(s)}_{(s)} := c \left( T \left( \left\| u_0 \right\|_{L^2}^2 + c^s_{(s-1)} \right) + \left( \frac{c^s_{(s)}}{\lambda(s)} \right) \right)^{1/2} \) and \( c = c(\varepsilon, \tau, n) \) is a positive constant depending on the parameters indicated.

\[\square\]

Claim 2:
If for any $\nu \geq 0$, $\epsilon > 0$ and $\tau \geq 5\epsilon$

\begin{equation}
(6.21) \quad \left\| J^{s+1}u \right\|_{L^2_t L^2_x(Q_{(\sigma,\tau+k\tau-vt)})} \leq c^{**}(\sigma),
\end{equation}

then for any $\epsilon > 0$, $\nu > 0$ and $\tau \geq 5\epsilon$, there exist a positive constant such that

\begin{equation}
(6.22) \quad \left\| J^r u \right\|_{L^2_t L^2_x(Q_{(\sigma,\tau-k\tau-vt)})} \leq c^{***}(r), \quad \text{for any } r \in (0, s + 1].
\end{equation}

**Proof of Claim 2:** The proof follows by using an argument quite similar to the one used in the proof of lemma 4.3. Nevertheless, we will indicate some details about the proof.

First that all we shall give us enough room to handle the operators involved in the smoothing effect (6.21). For that, notice that the inequality (6.21) holds for any $\epsilon > 0$ and $\tau \geq 5\epsilon$ in particular if we choose $(\epsilon, \tau) = \left(\frac{\epsilon}{2}, \frac{5\epsilon}{2}\right)$.

Next, we consider $\theta_1, \theta_2$ smooth functions satisfying: $0 \leq \theta_1, \theta_2 \leq 1$, with bounded derivatives, verifying the conditions indicated below

\[ \text{supp} \left( \theta_1 \right) \subseteq Q_{\left\{ \sigma, \frac{\tau}{2}, \tau + \frac{\epsilon}{2} \right\}} \text{ with } \theta_1 \equiv 1 \text{ on } Q_{\left\{ \sigma, \frac{\tau}{2}, \tau + \frac{\epsilon}{2} \right\}}, \]

and

\[ \text{supp} \left( \theta_2 \right) \subseteq Q_{\left\{ \sigma, \frac{\tau}{2}, \tau + \frac{\epsilon}{2} \right\}} \text{ with } \theta_2 \equiv 1 \text{ on } Q_{\left\{ \sigma, \tau, \tau + \frac{\epsilon}{2} \right\}}. \]

From these conditions it is clear that the following chain of inequalities holds

\begin{equation}
(6.23) \quad \int_0^T \left\| J^{s+1}u(\cdot,t)\theta_2(\sigma \cdot + vt) \right\|^2_{L^2_x} \ dt \leq \int_0^T \left\| J^{s+1}u(\cdot,t)\theta_1(\sigma \cdot + vt) \right\|^2_{L^2_x} \ dt
\end{equation}

\[ \leq \int_0^T \int_{Q_{\left\{ \sigma, \frac{\tau}{2}, \tau + \frac{\epsilon}{2} \right\}}} \left| J^{s+1}u(x,t) \right|^2 \ dx \ dt \leq c^{**}(\sigma). \]

Next, for $t \in (0, T)$ (fixed) we define the function

\[ F_t(z) = \theta_2 \left( x + \frac{vt \sigma}{|\sigma|^2} \right) J^s u(x,t), \quad z = \alpha + i\beta \in \mathbb{C}, \quad \alpha \in [0, s + 1] \quad \text{and} \quad \beta \in \mathbb{R}. \]

Notice that from theorem 3.10 it is clear that

\[ F_t(i\beta) = \theta_2 \left( \cdot + \frac{vt \sigma}{|\sigma|^2} \right) J^s u(\cdot,t) \in L^2(\mathbb{R}^n). \]

Instead, in the case we evaluate $F_t$ at $z = \alpha + i\beta$, we notice that it falls on the scope of lemma 4.2, so that,

\[ F_t(\alpha + i\beta) = \theta_2 \left( \cdot + \frac{vt \sigma}{|\sigma|^2} \right) J^{s+1}u(\cdot,t) = \theta_2 \left( \cdot + \frac{vt \sigma}{|\sigma|^2} \right) J^{s+1}u(\cdot,t) \in L^2(\mathbb{R}^n). \]

Hence, by the three lines lemma and Young’s inequality, we get for fixed $t$, the following:

\begin{equation}
(6.24) \quad \left\| \theta_2 \left( \cdot + \frac{vt \sigma}{|\sigma|^2} \right) J^r u(\cdot,t) \right\|_{L^2_x} \lesssim \left\| u_0 \right\|_{L^2_x} + \left\| \theta_2 \left( \cdot + \frac{vt \sigma}{|\sigma|^2} \right) J^s u(\cdot,t) \right\|_{L^2_x}, \quad \text{for } r \in (0, s].
\end{equation}
A careful inspection of the constant involved in the inequalities (6.24) show that these one’s do not depend on the temporal variable. So that, after integrating both sides in (6.24) we finally obtain: For any \( \epsilon > 0 \), \( \tau \geq 5\epsilon \) and \( \nu > 0 \),

\[
\int_0^T \int_{Q_{(x,\epsilon,\epsilon,\nu,\tau,\nu)}} |f' u(x,t)|^2 \, dx \, dt \lesssim T^2 \|u_0\|_{L^2}^2 + c_{(s)}^{\epsilon \nu} \quad r \in (0, s+1].
\]

Since the technical details necessary has been clarified, we turn back our attention to the inductive process.

**Case**: \( s \in (l, l+1) \), \( l \in \mathbb{N}, l > s_R + 1 \).

As usual, our starting point is the identity (6.1), so that, to follow with the inductive argument we estimate the corresponding terms coming from such identity.

Firstly, we handle \( A_1 \). So, the main idea is to use the smoothing effect obtained from the former case i.e., \( s \in (l-1, l] \), that as we have seen it provides one extra local derivative.

Nevertheless, combining the inductive hypothesis, claim 1 and claim 2 we obtain that: For \( \nu > 0 \), \( \epsilon > 0 \), and \( \tau \geq 5\epsilon \)

(6.25) \[
\int_0^T \int_{Q_{(x,\epsilon,\epsilon,\nu,\tau,\nu)}} (f' u(x,t))^2 \, dx \, dt \lesssim c_{(r)}^{\epsilon \nu}, \quad \text{for any} \quad r \in (0, l+1].
\]

So that, with this remark at hand it is enough to combine the properties of the weighted function and (6.25) to finally obtain

\[
\int_0^T \int_{Q_{(x,\epsilon,\epsilon,\nu,\tau,\nu)}} (f' u(x,t))^2 \, dx \, dt
\lesssim_{\epsilon, \tau} \int_0^T \int_{\mathbb{R}^n} (\chi e/3 + \epsilon \chi e/3, \tau + \epsilon) (\sigma \cdot x + \nu t) (f' u(x,t))^2 \, dx \, dt.
\]

Therefore, for \( \epsilon > 0 \) and \( \tau \geq 5\epsilon \) we obtain that

\[
\int_0^T |A_1(t)| \, dt = c \int_0^T \int_{Q_{(x,\epsilon,\epsilon,\nu,\tau,\nu)}} (f' u(x,t))^2 \, dx \, dt
\lesssim_{\epsilon, \tau, \nu} \left( \frac{c_{(s-1)}^{\epsilon \nu}}{\lambda(\sigma)} \right).
\]

Next, we have that

\[
\int_0^T |A_4(t)| \, dt = 3c_1 c_1 |\sigma|^2 \int_0^T \int_{Q_{(x,\epsilon,\epsilon,\nu,\tau,\nu)}} (f' u(x,t))^2 \, dx \, dt
\lesssim_{\epsilon, \tau, \sigma} \left( \frac{c_{(s-1)}^{\epsilon \nu}}{\lambda(\sigma)} \right),
\]

being the last inequality a consequence of (6.25) with \( r = s - 1 < l \).
Next, by combining the properties of the weighted functions we obtain

\[
\int_0^T |A_\xi(t)| \, dt \lesssim_{\epsilon, \tau, \sigma} \int_0^T \int_{Q_{1,1}} (f^\xi u(x, t))^2 \, dx \, dt \\
\lesssim_{\epsilon, \tau, \sigma} \left( \epsilon^{**} \overline{\lambda}(\sigma) \right),
\]

being the last inequality a consequence of (6.25) with \( r = s - 1 < l \).

In what concerns to the terms \( A_{2,m} \) and \( A_{3,m} \), an analysis similar to the used in (6.4)-(6.5) imply the existence of a positive constant \( \lambda = \lambda(\sigma) \), such that

\[
(6.26) \quad \lambda(\sigma) \sum_{m=1}^{n} \int_{\mathbb{R}^n} (\partial_{x_m} f^\xi u(x, t))^2 \left( \chi_{\epsilon, \tau} \chi'_{\epsilon, \tau} \right) (\sigma \cdot x + vt) \, dx
\lesssim \sigma_1 \sum_{m=1}^{n} \int_{\mathbb{R}^n} (\partial_{x_m} f^\xi u(x, t))^2 \left( \chi_{\epsilon, \tau} \chi'_{\epsilon, \tau} \right) (\sigma \cdot x + vt) \, dx
\lesssim \sigma_1 \sum_{m=1}^{n} \int_{\mathbb{R}^n} (\partial_{x_m} f^\xi u(x, t)) (\partial_{x_m} f^\xi u(x, t)) \left( \chi_{\epsilon, \tau} \chi'_{\epsilon, \tau} \right) (\sigma \cdot x + vt) \, dx.
\]

Notice that the terms in the l.h.s above are positive, and after integrating in time these will provide the smoothing effect.

Next, we handle the non-linear part. In this sense, we will denote the commutator term \( \Psi_{\xi,\tau,\sigma} = [f^\xi; \chi_{\epsilon, \tau, \sigma}] \in O\mathcal{P}S^{s-1} \).

According to Proposition 1, the symbol \( \xi_{\epsilon, \tau, \sigma} \) admits the following decomposition:

\[
\xi_{\epsilon, \tau, \sigma}(x, \xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \left( \frac{\partial^\alpha}{\partial^\alpha} \left( \chi_{\epsilon, \tau, \sigma}(x, t) \right) \right) \quad \text{for any} \quad t \in \mathbb{R}.
\]
More precisely,\n\[
\tilde{\zeta}_{e,\tau,s,\sigma}(x, \xi) = \sum_{1 \leq |\alpha| \leq t} (2\pi i)^{-|\alpha|} a! \left\{ \partial^{|\alpha|}_{\xi} \left( \langle \xi \rangle^{|\alpha|} \right) \partial^{|\alpha|}_{\epsilon} \left( \chi_{e,\tau,s,\sigma}(x, t) \right) \right\} + \kappa_{s-1-1}(x, \xi)
\]
\[
= \sum_{j=1}^{l} \sum_{|\alpha| = j} (2\pi i)^{-|\alpha|} \frac{a!}{a!} \left\{ \partial^{|\alpha|}_{\xi} \left( \langle \xi \rangle^{|\alpha|} \right) \partial^{|\alpha|}_{\epsilon} \left( \chi_{e,\tau,s,\sigma}(x, t) \right) \right\} + \kappa_{s-1-1}(x, \xi)
\]
\[
= c_1(s) \sum_{|\alpha| = 1} (2\pi i)^s a! \langle \xi \rangle^{s-2} \chi_{e,\tau}^{(1)}(\sigma \cdot x + vt) + \sum_{|\alpha| = 2} \delta_{a_1, a_2} \frac{a!}{a!} \chi_{e,\tau}^{(2)}(\sigma \cdot x + vt) \langle \xi \rangle^{s-2} \]
\[
+ c_2(s) \sum_{|\alpha| = 3} \frac{a!}{a!} \left( \delta_{a_1, a_2} (2\pi i)^{a_3} + \delta_{a_3, a_1} (2\pi i)^{a_2} + \delta_{a_3, a_2} (2\pi i)^{a_1} \right) \langle \xi \rangle^{s-4} \chi_{e,\tau}^{(3)}(\sigma \cdot x + vt)
\]
\[
+ \cdots + \sum_{|\alpha| = l} (2\pi i)^{-|\alpha|} \frac{a!}{a!} \left\{ \partial^{|\alpha|}_{\xi} \left( \langle \xi \rangle^{|\alpha|} \right) \chi_{e,\tau}^{(|\alpha|)}(\sigma \cdot x + vt) \right\} + \kappa_{s-1-1}(x, \xi),
\]

where \( \kappa_{s-1-1} \in S^{l+1-s} \subset S^0 \). Precisely, we associate to the symbol \( \kappa_{s-1-1} \) the operator
\[
\Psi_{\kappa_{s-1-1}} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \kappa_{s-1-1}(x, \xi) \hat{g}(\xi) \, d\xi, \quad g \in \mathcal{S}(\mathbb{R}^n).
\]

In view that \( \kappa_{s-1-1} \in S^0 \) and Theorem 3.10 it is clear that \( \Psi_{\kappa_{s-1-1}} \) maps \( L^2(\mathbb{R}^n) \) into itself i.e.,
\[
\| \Psi_{\kappa_{s-1-1}} f \|_{L^2} \lesssim \| f \|_{L^2}
\]

for \( f \) in an appropriated class of functions.

Also, for multi-index \( \alpha, \beta \) with \( \beta \leq \alpha \) we define
\[
\eta_{\alpha, \beta}(x, \xi) := \frac{(2\pi i)^\beta}{(1 + |\xi|^2)^{|\alpha|}}, \quad x, \xi \in \mathbb{R}^n,
\]
whence \( \eta_{\alpha, \beta} \in S^{\beta-|\alpha|} \subset S^0 \), and we associated to it the operator
\[
\Psi_{\eta_{\alpha, \beta}} g(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \eta_{\alpha, \beta}(x, \xi) \hat{g}(\xi) \, d\xi, \quad g \in \mathcal{S}(\mathbb{R}^n),
\]
that according to Theorem 3.10 it satisfies
\[
\| \Psi_{\eta_{\alpha, \beta}} g \|_{L^2} \lesssim \| g \|_{L^2}.
\]
Now, by rearranging the terms above in the decomposition of the symbol $\zeta_{\varepsilon, \tau, s, \sigma}$, we obtain

$$\Psi_{\varepsilon, \tau, s, \sigma} f(x) = \sum_{j=1}^{l} \sum_{|\alpha|=j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, s} \partial_x^\alpha \chi_{\varepsilon, \tau, s}(x, t) \Psi_{\eta, \beta} f^{\varepsilon, \eta} + \Psi_{\kappa_{s-1}} f(x),$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $\omega_{\alpha, \beta, s, \sigma}$ denotes a constant depending on the parameters indicated.

Now, we turn back our attention to the terms involving this commutator term. So that, combining interpolation and inequality (3.13) we get

(6.28)

$$\|B_{6,1}\|_{L^2} \lesssim \sum_{j=1}^{l} \sum_{|\alpha|=j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, s} \left\| \partial_x^\alpha \chi_{\varepsilon, \tau, s}(\cdot, t) + v t \right\|_{L^p} \left\| \Psi_{\eta, \beta} f^{\varepsilon, \eta} \right\|_{L^2}$$

$$+ \left\| \Psi_{\kappa_{s-1}} \right\|_{L^2} \lesssim_{e, \tau, s, \sigma, \eta} \|u\|_{L^\infty} \left\| f^s (u\chi_{\varepsilon, \tau, s}) \right\|_{L^2}.$$  

Analogously,

(6.29)

$$\|B_{6,2}\|_{L^2} \lesssim \sum_{j=1}^{l} \sum_{|\alpha|=j} \sum_{\beta \leq \alpha} \omega_{\alpha, \beta, s} \left\| \partial_x^\alpha \chi_{\varepsilon, \tau, s}(\cdot, t) \right\|_{L^p} \left\| \Psi_{\eta, \beta} f^{\varepsilon, \eta} \right\|_{L^2}$$

$$+ \left\| \Psi_{\kappa_{s-1}} \right\|_{L^2} \lesssim_{e, \tau, s, \sigma, \eta} \|u\|_{L^\infty} \left\| f^s (u\phi_{\varepsilon, \tau}) \right\|_{L^2}.$$  

Since we have finished estimating the terms that require more effort, we focus our attention on the remaining terms.

In the first place, we note that the same argument used in the previous case for the terms $B_{6,3}$ and $B_{6,6}$ produce

$$\|B_{6,3}\|_{L^2} \lesssim_{e, \sigma} \varepsilon_0 \|u\|_{L^2} \|u\|_{L^\infty} \quad \text{and} \quad \|B_{6,6}\|_{L^2} \lesssim \|u\|_{L^\infty} \|u\|_{L^2}.$$  

In the second place, the inequality (3.12) imply that

(6.30)  

$$\left\| [f^s; u\chi_{\varepsilon, \tau, s}] \partial_x \chi_{\varepsilon, \tau, s} (u\chi_{\varepsilon, \tau, s}) \right\|_{L^2} \lesssim \left\| f^s (u\chi_{\varepsilon, \tau, s}) \right\|_{L^2} \left\| \partial_x (u\chi_{\varepsilon, \tau, s}) \right\|_{L^\infty},$$

and

(6.31)  

$$\left\| [f^s; u\chi_{\varepsilon, \tau, s}] \partial_x (u\phi_{\varepsilon, \tau, s}) \right\|_{L^2} \lesssim \left\| f^s (u\phi_{\varepsilon, \tau, s}) \right\|_{L^2} \left\| \nabla (u\chi_{\varepsilon, \tau, s}) \right\|_{L^\infty}$$

$$+ \left\| f^s (u\chi_{\varepsilon, \tau, s}) \right\|_{L^2} \left\| \partial_x (u\phi_{\varepsilon, \tau, s}) \right\|_{L^\infty}.$$  

As in the previous case, our analysis requires to estimate several terms in (6.28)-(6.31) for which we have not provided with upper bounds. To finish our argument we estimate

$$f^s (u\chi_{\varepsilon, \tau, s}) = \chi_{\varepsilon, \tau, s} f^s u + [f^s; \chi_{\varepsilon, \tau, s}] (u\chi_{\varepsilon, \tau, s} + u\phi_{\varepsilon, \tau, s} + u\psi_{\varepsilon, \tau, s})$$

$$= I + II + III + IV.$$
Notice that $I$ is the quantity to be estimated. Instead, the remainder terms are of lower order and after integrating in the time variable these can be bounded by combining (6.25) and lemma 4.5.

Similarly, the terms $\left\| J^s (u \varphi_{e, \tau, \sigma}) \right\|_{L^2}^2$, $\left\| J^s (u \varphi_{e, \tau, \sigma}) \right\|_{L^2}$ can be bounded by combining lemma 4.5 and (6.25).

Finally, we gather the information relative to this step, followed by an application of Gronwall’s inequality and integration in time whence we get that for any $\nu \geq 0, \epsilon > 0, \tau \geq 5\epsilon$,

$$\sup_{0 \leq t \leq T} \left\| J^s u \right\|_{L^2(H(\sigma, t-v))}^2 + \sum_{m=1}^{n} \lambda(\sigma) \left\| \partial_x^m J^s u \right\|_{L^2}^2 \leq c^{****},$$

where $c^{****}$ is a positive constant depending on the following quantities

$$c^{****} = c^{****} \left( \epsilon; \tau; \sigma; \lambda; \nu; n; s; T; \left\| J^s u_0 \right\|_{L^2(H(\sigma, e))}; \left\| u \right\|_{L^2(H_n^{m+1})} \right) > 0.$$

This last inequality finish the inductive argument.

Notice that from (6.32) it can be deduced (1.9) after combining lemma 4.3 and the properties of the weighted functions. Instead, to obtain (1.10) it can be used an argument quite similar to the one used in the proof of claim 1.

Finally we gather the estimates obtained to conclude the proof of (1.9) and (1.10).

Next, we focus our attention in to understand the behavior of the $J^s u$ when we restrict to the remainder part of the space, instead of the half-space where the propagation occurs. For that, we first present previous results that will help us to provide the bounds required in the proof of Corollary 1.1.

**Corollary 6.1.** Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n$ with $\sigma_1 > 0$, $\sigma_n, \ldots, \sigma_n \geq 0$. Let $f : H(\sigma, 0) \rightarrow [0, \infty)$ be a continuous function, such that for every $\alpha > 0$,

$$\int_{Q(\sigma, 0, \alpha)} f(x) \, dx \leq c \alpha^q,$$

for some $q > 0$ and some positive constant $c$.

Then, for every $\delta > 0$,

$$\int_{H(\sigma, 0)} \frac{f(x)}{\langle \sigma \cdot x \rangle^{q+\delta}} \, dx \leq c(\delta, \sigma, q).$$

**Proof.** Firstly, we make the following change of variable

$$\int_{H(\sigma, 0)} \frac{f(x)}{\langle \sigma \cdot x \rangle^{q+\delta}} \, dx = \frac{1}{\sigma_1} \int_{H(\epsilon_1, 0)} \frac{f(Ay)}{\langle y_1 \rangle^{q+\delta}} \, dy,$$

where $A \in M^{n \times n}(\mathbb{R})$. More precisely,

$$A = \begin{pmatrix}
\frac{1}{\epsilon_1} & -\frac{\epsilon_2}{\epsilon_1} & \ldots & -\frac{\epsilon_n}{\epsilon_1} \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}.$$
Next, notice that

(6.35) \[ \mathcal{H}_{\{e_1,0\}} \subset \mathcal{Q}_{\{e_1,-1,1\}} \cup \bigcup_{k \in \mathbb{N}} \mathcal{Q}_{\{e_1,2^{k-1},2^k+1\}}. \]

So that, in virtue of the decomposition indicated above, we consider \( \{\psi_k\}_{k \geq 0} \) be a smooth partition of unity of \( \mathbb{R}^+ \) such that

\[ \text{supp}(\psi_k) \subseteq [2^{k-1}, 2^{k+1}] \quad \text{for} \quad k = 1, 2, \ldots, \]

and \( \text{supp}(\psi_0) \subseteq [-1, 1] \). Under the conditions specified above we have that

(6.36) \[ 1 = \psi_0(x) + \sum_{k \in \mathbb{N}} \psi\left(2^{-k}x\right) \quad \text{for all} \quad x \in \mathbb{R}, x \geq 0. \]

Then, combining the hypothesis (6.33) and (6.36)-(6.35) we obtain

\[
\int_{\mathcal{H}_{\{e,0\}}} \frac{f(x)}{\langle \sigma \cdot x \rangle^{q+\delta}} \, dx = \frac{1}{\sigma_1} \int_{\mathcal{H}_{\{e_1,0\}}} \frac{f(Ax)}{\langle x_1 \rangle^{q+\delta}} \, dx \\
\leq \frac{1}{\sigma_1} \int_{\mathcal{Q}_{\{e_1,0\}}} \psi_0(x_1) \frac{\psi_0(x_1)}{\langle x_1 \rangle^{q+\delta}} f(Ax) \, dx \\
+ \frac{1}{\sigma_1} \sum_{k=1}^{\infty} \int_{\mathcal{Q}_{\{e_1,2^{k-1},2^k+1\}}} \psi_0(x_1) \frac{\psi_0(x_1)}{\langle x_1 \rangle^{q+\delta}} f(Ax) \, dx \\
\leq c \frac{1}{\sigma_1} \left(1 + \left(\frac{2^{2q+\delta}}{2^q - 1}\right)\right),
\]

for all \( \delta > 0 \).

Therefore,

\[
\int_{\mathcal{H}_{\{e,0\}}} \frac{f(x)}{\langle \sigma \cdot x \rangle^{q+\delta}} \, dx \lesssim c(\delta, \sigma, q), \quad \text{for all} \quad \delta > 0. 
\]

\[ \square \]

Remark 6.1. At this point several issues have to be emphasized and clarified.

(i) We shall remark that the corollary also applies when integrating a non-negative function on the set \( \mathcal{Q}_{\{e,-(\sigma+\epsilon),-\epsilon\}} \) which implies decay on the complement of the half-space \( \mathcal{H}_{\{e,0\}} \).

(ii) It is also important to emphasize that the constant \( c \) appearing in (6.33) also appears implicitly in (6.34), as it was evidenced in the proof of corollary 6.1.

Proof of Corollary 1.1. Without loss of generality we will assume that \( \beta = 0 \) in Theorem A. So that, it is clear that for \( t \in (0, T) \) (fixed), and for all \( \epsilon > 0 \),

\[
\int_{\mathcal{H}_{\{e,\epsilon-v\}}} (f^e u(x,t))^2 \, dx = \int_{\mathcal{Q}_{\{e,\epsilon-v\}}} (f^e u(x,t))^2 \, dx + \int_{\mathcal{H}_{\{e,\epsilon\}}} (f^e u(x,t))^2 \, dx \\
\lesssim c^\epsilon.
\]
Notice that the second term in the r.h.s above is bounded, to see this it is enough to take $\nu = 0$ in Theorem A. So that, it only remains to estimate the missing term above.

Since,

$$\int_{Q_{(\epsilon, \epsilon - \epsilon)}(x, t)} (J^s u(x, t))^2 \, dx = \int_{Q_{(\epsilon, \epsilon - \epsilon)}(x, t)} \left( J^s u \left( x + \frac{2\epsilon}{|\sigma|^2} \sigma, t \right) \right)^2 \, dx \lesssim c^s t^{-1} \text{ for } t \in (0, T)$$

and $\nu > 0$.

So that, combining corollary 6.1 and the remark 6.1 with $\alpha = t$ and $q = s$, we obtain

$$\int_{\mathcal{H}_{(\epsilon, \epsilon - \epsilon)}^c} \frac{1}{\langle \sigma \cdot x \rangle^{\delta + \epsilon}} \left( J^s u \left( x + \frac{2\epsilon}{|\sigma|^2} \sigma, t \right) \right)^2 \, dx \lesssim_{\delta, s, \epsilon} \frac{1}{t} \text{ for } \delta > 0.$$ 

Finally, we gather the estimates above to obtain

$$\int_{\mathbb{R}^n} \frac{1}{\langle (\sigma \cdot x) \rangle^{\delta + \epsilon}} (J^s u(x, t))^2 \, dx \lesssim_{\delta, s, \epsilon} \frac{1}{t} \text{ for all } t \in (0, T).$$

\[ \square \]

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