GRADED SELF-INJECTIVE ALGEBRAS “ARE” TRIVIAL EXTENSIONS

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Abstract. For a positively graded artin algebra \( A = \oplus_{n \geq 0} A_n \) we introduce its Beilinson algebra \( b(A) \). We prove that if \( A \) is well-graded self-injective, then the category of graded \( A \)-modules is equivalent to the category of graded modules over the trivial extension algebra \( T(b(A)) \). Consequently, there is a full exact embedding from the bounded derived category of \( b(A) \) into the stable category of graded modules over \( A \); it is an equivalence if and only if the 0-th component algebra \( A_0 \) has finite global dimension.

1. Introduction

Let \( R \) be a commutative artinian ring and let \( A = \oplus_{n \geq 0} A_n \) be a positively graded artin \( R \)-algebra. Set \( c = \max \{ n \geq 0 \mid A_n \neq 0 \} \). Throughout we will assume that \( A \) is nontrivially graded, that is, \( c \geq 1 \). We define the Beilinson algebra \( b(A) \) of the graded algebra \( A \) to be the following upper triangular matrix algebra

\[
b(A) = \begin{pmatrix}
A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\
0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_0 & A_1 \\
0 & 0 & \cdots & 0 & A_0
\end{pmatrix}.
\]

Here the multiplication of \( b(A) \) is induced from the one of \( A \). This concept originates from the following example: let \( A \) be the exterior algebra over a field with the usual grading, then its Beilinson algebra \( b(A) \) appeared in Beilinson’s study on the bounded derived category of projective spaces (see [3], also see the algebras in [2] Example 4.1.2, [9] p.90 and [13] Corollary 2.8); note that the algebra \( b(A) \) is different from the Beilinson algebra in [11] (also see [4] and [2] p.332, Remark), while they are derived equivalent.

Denote by \( A\text{-gr} \) the category of finitely generated graded left \( A \)-modules with morphisms preserving degrees. It is well known that the algebra \( A \) is self-injective (as an ungraded algebra) if and only if every projective object in \( A\text{-gr} \) is injective (by [12] Theorem 2.8.7 or [8]). In this case, we say that the graded artin algebra \( A \) is graded self-injective (compare [15]).

Let \( T(b(A)) = b(A) \oplus D(b(A)) \) be the trivial extension algebra of \( b(A) \), where \( D \) is the Matlis duality on finitely generated \( R \)-modules ([11] Chapter II §3]). Note that \( T(b(A)) \) is a graded algebra such that \( \deg b(A) = 0 \) and \( \deg b(A) = 1 \), and that

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We say that the graded artin algebra $A$ is left well-graded if for each nonzero idempotent $e \in A_0$, $eA_0 \neq 0$. Dually one has the notion of right well-graded algebras. We say that the graded algebra $A$ is well-graded provided that it is both left and right well-graded. For example if the 0-th component algebra $A_0$ is local, then $A$ is well-graded. Note that for a graded self-injective algebra $A$ it is left well-graded if and only if it is right well-graded, thus well-graded, see Lemma 2.2. Clearly the trivial extension algebra $T(b(A))$ is well-graded, since $D(b(A))$ is a faithful (left and right) $b(A)$-module.

The following result is inspired by [9] Chapter II, Example 5.1, and it somehow justifies the title.

**Theorem 1.1.** Let $A = \oplus_{n \geq 0} A_n$ be a well-graded self-injective algebra. Then we have an equivalence of categories $A\text{-gr} \simeq T(b(A))\text{-gr}$.

Note that the equivalence above may not be the graded equivalence of algebras in the sense of [8] (compare [6, 16]), that is, in general it does not commute with the degree-shift automorphisms.

Denote by $A\text{-gr}$ the stable category with respect to projective modules. It has a natural triangulated structure ([9] Chapter I, section 2]). Denote by $b(A)\text{-mod}$ the category of finitely generated left $b(A)$-modules, $D(b(A))\text{-mod}$ its bounded derived category. The following generalizes a result by Orlov [13 Corollary 2.8], which might be traced back to [3] [5] [4] (consult [2]).

**Corollary 1.2.** Let $A = \oplus_{n \geq 0} A_n$ be a well-graded self-injective algebra. Then we have a full exact embedding of triangulated categories $A\text{-gr} \hookrightarrow D(b(A))\text{-mod}$. Moreover, it is an equivalence if and only if the 0-th component algebra $A_0$ has finite global dimension.

**Proof.** Note that by [11] p.78, Proposition 2.7, the algebra $A_0$ has finite global dimension if and only if so does the Beilinson algebra $b(A)$. Theorem 1.1 implies the natural equivalence $A\text{-gr} \simeq T(b(A))\text{-gr}$ of triangulated categories (by [9] Chapter I, 2.8]). Thus the corollary follows immediately from a result by Happel ([10] Theorem 2.5)).\qed

2. The Proof of Theorem 1.1

Let $R$ be a commutative artinian ring, and let $D = \text{Hom}_R(\cdot, E)$ be the Matlis duality with $E$ the minimal injective $R$-cogenerator ([11] p.37-39]). Let $B$ be an artin $R$-algebra and let $B \times X_B$ be a $B$-bimodule such that $R$ acts on $X$ centrally and $X$ is finitely generated both as a left and right $B$-module. The trivial extension $B \times X$ of $B$ by the bimodule $X$ is defined as follows: as an $R$-module $B \times X = B \oplus X$, and the multiplication is given by $(b, m)(b', m') = (bb', bm' + mb')$ ([11] p.78]). Then $B \times X$ is a positively graded $R$-algebra such that $\deg B = 0$ and $\deg X = 1$. We will denote by $B \times X\text{-gr}$ the category of finitely generated graded left $B \times X$-modules.

Consider the regular $B$-bimodule $B B_B$ and its dual $B$-bimodule $D(B) = D(B B_B)$, and thus the $B$-bimodule structure on $D(B)$ is given such that for each $b \in B$ and $f \in D(B) = \text{Hom}_R(B, E)$, $(bf)(x) = f(xb)$ and $(fb)(x) = f(bx)$ for all $x \in B$. The trivial extension $T(B) = B \times D(B)$ is simply referred as the trivial extension algebra of $B$. It is a symmetric algebra, thus self-injective ([11] p.128, Proposition 3.9]). More generally, given an automorphism $\sigma : B \longrightarrow B$ of $R$-algebras, consider the twisted $B$-bimodule $\sigma B_B$ such that the left $B$-module structure is given by the multiplication as usual and the right $B$-module structure is given by $xb := x\sigma(b)$,
for all \( b \in B \) and \( x \in \mathcal{B} \). Note that since \( \sigma \) is an \( R \)-algebra automorphism, \( R \) acts on the \( B \)-bimodule \( \mathcal{B} \) centrally. Denote by \( D(B^*) = D(B, \mathcal{B}) \) the dual \( B \)-bimodule and the corresponding trivial extension \( T(B^*) = B \rtimes D(B^*) \) is called the twisted trivial extension algebra of \( B \) with respect to \( \sigma \). Note that \( T(B^*) \) is self-injective, in general not symmetric (see Example (4) in [7]).

We observe the following result.

**Lemma 2.1.** Use the notation above. We have an isomorphism of categories \( T(B) \text{-gr} \simeq T(B^*) \text{-gr} \).

**Proof.** Note that as \( R \)-modules \( T(B^*) = B \oplus D(B) \), and its multiplication is given by \((b, f) \star (b', f') = (bb', \sigma(b)f' + fb')\). Given a graded \( T(B) \)-module \( M = \oplus_{n \in \mathbb{Z}} M_n \), we endow a \( T(B^*) \)-action on it as follows: given a homogeneous element \( m \in M \), define

\[(b, f) \star m = \sigma^{|m|}(b)m + (f \circ \sigma^{-|m|})m,\]

where \(|m|\) denotes the degree of \( m \), and \( f \circ \sigma^{-|m|} : B^* \rightrightarrows B \xrightarrow{f} E \in D(B) \) means the composite. It is direct to check that \( \star \) gives \( M \) a graded \( T(B^*) \)-module structure. Furthermore this gives an isomorphism (more than an equivalence) of categories \( T(B) \text{-gr} \simeq T(B^*) \text{-gr} \). \( \square \)

Let \( A = \oplus_{n \geq 0} A_n \) be a positively graded artin algebra and let \( c = \max\{n \geq 0 \mid A_n \neq 0\} \). As in the introduction we always assume that \( c \geq 1 \). Consider the category \( A \text{-gr} \) of finitely generated graded left \( A \)-modules. For a graded \( A \)-module \( M = \oplus_{n \in \mathbb{Z}} M_n \), its width \( w(M) \) is defined to be \( \max\{n \mid M_n \neq 0\} - \min\{n \mid M_n \neq 0\} + 1 \) (for \( M = 0 \), set \( w(M) = 0 \)). For example \( w(A) = c + 1 \), here we regard \( A \) as a graded \( A \)-module via the multiplication such that the identity \( 1_A \) is at the 0-th component. For a graded \( A \)-module \( M = \oplus_{n \in \mathbb{Z}} M_n \), denote by \( M(1) \) its shifted module which is the same as \( M \) as ungraded modules, and which is graded such that \( M(1)_n = M_{n+1} \). This gives rise to the degree-shift automorphism \( (1) : A \text{-gr} \longrightarrow A \text{-gr} \). Denote by \( (d) \) the \( d \)-th power of \( (1) \) for each \( d \in \mathbb{Z} \) ([12]). Recall that each indecomposable projective object in \( A \text{-gr} \) is of the form \( Ae(d) \), where \( e \in A_0 \) is a primitive idempotent and \( d \in \mathbb{Z} \); dually each indecomposable injective object is of the form \( D(eA)(d) \), where \( D(eA) \) is graded such that \( D(eA)_n = D(eA_{n-d}) \). For details, see [8] section 5.

**Lemma 2.2.** Let \( A = \oplus_{n \geq 0} A_n \) be a graded self-injective algebra. Assume that it is left well-graded. Then it is right well-graded.

**Proof.** Note that \( A \) is left well-graded if and only if \( w(Ae) = c+1 \) for each primitive idempotent \( e \in A_0 \), thus if and only if \( w(P) = c+1 \) for each indecomposable projective object \( P \) in \( A \text{-gr} \). Since \( A \) is graded self-injective, the indecomposable injective graded module \( D(eA) \) is projective, and thus by above \( w(D(eA)) = c+1 \). Note that \( w(D(eA)) = w(eA) \), where \( eA \) is considered as a graded right \( A \)-module. Hence for each primitive idempotent \( e \in A_0 \) we have \( w(eA) = c+1 \), and this shows that \( A \) is right well-graded. \( \square \)

We will divide the proof of Theorem 1.1 into several easy results. Let \( A = \oplus_{n \geq 0} A_n \) be a graded artin algebra and let \( b(A) \) be its Beilinson algebra. Consider the following \( R \)-module

\[
\mathbf{x}(A) = \begin{pmatrix}
A_c & 0 & \cdots & 0 & 0 \\
A_{c-1} & A_c & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_2 & A_3 & \cdots & A_c & 0 \\
A_1 & A_2 & \cdots & A_{c-1} & A_c
\end{pmatrix}.
\]
Note that there is a natural $b(A)$-bimodule structure on $x(A)$, induced from matrix multiplication and the multiplication of $A$; moreover, $\hat{R}$ acts on $x(A)$ centrally. Consider the trivial extension $t(A) = b(A) \oplus x(A)$, which is a graded algebra as above.

**Lemma 2.3.** There is an equivalence of categories $A\text{-gr} \simeq t(A)\text{-gr}$. Moreover, $A$ is left well-graded if and only if $t(A)$ is.

**Proof.** Define a functor $\Phi : A\text{-gr} \to t(A)\text{-gr}$ as follows: for $M = \oplus_{n \in \mathbb{Z}} M_n \in A\text{-gr}$, set $\Phi(M) = \oplus_{n \in \mathbb{Z}} \Phi(M)_n$ with $\Phi(M)_n = \bigoplus_{i = n}^{(n+1)c-1} M_i$, and there is a natural graded $t(A)$-module structure on $\Phi(M)$ (using the multiplication rule of matrices on column vectors; here the elements in $\Phi(M)_n$ are viewed as column vectors of size $c$, and note that $c \geq 1$); the action of $\Phi$ on morphisms is the identity. To construct the inverse, for each $0 \leq r \leq c-1$, set $e_{rr} \in b(A)$ to be the elementary matrix having the $(r+1, r+1)$ entry 1 and elsewhere 0. Define a functor $\Psi : t(A)\text{-gr} \to A\text{-gr}$ sending a graded $t(A)$-module $N = \oplus_{n \in \mathbb{Z}} N_n$ to $\Psi(N) = \oplus_{n \in \mathbb{Z}} \Psi(N)_n$ such that $\Psi(N)_{ic+rr} = e_{rr} N_i$ for $i \in \mathbb{Z}$ and $0 \leq r \leq c-1$. Then $\Psi(N)$ is a natural graded $A$-module structure. Then it is direct to check that $\Phi$ and $\Psi$ are mutually inverse to each other. Note that one may have a more conceptual proof of the equivalence above by [14 Theorem 2.12] (compare [16 Example 3.10] and [6]).

For the second statement, take $1_{A_0} = \sum_{i=1}^l e_i$ to be a decomposition of unity into primitive idempotents, and thus every primitive idempotent of $A_0$ is conjugate to one of $e_i$’s. Hence $A$ is left well-graded if and only if $e_i A_\sigma \neq 0$ for each $1 \leq i \leq l$. However, $1_{b(A)} = \sum_{r=0}^{c-1} \sum_{i=1}^l e_{rr} e_i$ is a decomposition of unity in $b(A)$ into primitive idempotents, and hence $t(A)$ is left well-graded if and only if $e_{rr} e_i x(A) \neq 0$ for each $0 \leq r \leq c-1$ and $1 \leq i \leq l$. Note that $e_{rr} e_i x(A) = e_i A_\sigma$ and then we are done. \qed

Consider the trivial extension $T = B \ltimes X$ of an artin $R$-algebra $B$ by a (nonzero) $B$-bimodule $X$ as above. Take $e \in B$ to be an idempotent such that $eBe$ is the basic algebra associated to $B$ ([1, p.35]). Thus $eXe$ has the induced $eBe$-bimodule structure and we have an identification of (graded) algebras $eTe = eBe \ltimes eXe$. The following result is immediate from the Morita equivalence between the algebras $B$ and $eBe$.

**Lemma 2.4.** Use the notation above. We have an equivalence of categories $T\text{-gr} \simeq eTe\text{-gr}$. Moreover $T$ is left well-graded if and only if so is $eTe$.

The key observation is as follows.

**Lemma 2.5.** Let $T = B \ltimes X$ be a trivial extension as above. Assume that $B$ is a basic algebra and $T$ is well-graded self-injective. Then there is an isomorphism of $B$-bimodules $X \simeq D(B^\sigma)$ for some $R$-automorphism $\sigma$ on $B$. In particular, there is an isomorphism $T \simeq T(B^\sigma)$ of graded algebras.

**Proof.** Take $1_B = \sum_{i=1}^l e_i$ to be a decomposition of unity into primitive idempotents. Since $B$ is basic, the set $\{Te_i(d) \mid 1 \leq i \leq l, d \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic projective objects in $T$-gr. Dually, $\{D(e_i T)(d) \mid 1 \leq i \leq l, d \in \mathbb{Z}\}$ forms a complete set of pairwise non-isomorphic injective objects in $T$-gr. Since $T$ is well-graded, all these modules have width 2. Since $T$ is graded self-injective, we have an isomorphism of graded $T$-modules $Te_i \simeq D(e_{s(i)} T)(-1)$, where $s : \{1, \cdots, l\} \to \{1, \cdots, l\}$ forms a permutation. In particular, we have isomorphisms $Xe_i \simeq D(e_{s(i)} B)$ of left $B$-modules for each $1 \leq i \leq l$. Since $B$ is basic, we deduce an isomorphism of left $B$-modules $B X \simeq D(B_B)$. Similarly we have an isomorphism $X_B \simeq D(B_B)$ of right $B$-modules.

Consider the dual $B$-bimodule $M = D(B X_B)$. We have isomorphisms $BM \simeq B B$ and $M_B \simeq B_B$. It is a good exercise to deduce from these isomorphisms that there
2.1 We deduce that \( T \) isomorphism of graded algebras \( T \) is well-graded self-injective. By Lemma 2.5, we have an isomorphism of \( (b(A))e \) and we have a natural \( B \)-bimodule isomorphism \( D(B) \simeq eD(b(A))e \), and thus we get the desired equivalence \( A \)-gr \( \cong T(b(A)) \)-gr.

\[ \square \]

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