FIBRED COARSE EMBEDDINGS, A-T-MENABILITY AND THE COARSE ANALOGUE OF THE NOVIKOV CONJECTURE

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Abstract. The connection between the coarse geometry of metric spaces and analytic properties of topological groupoids is well known. One of the main results of Skandalis, Tu and Yu is that a space admits a coarse embedding into Hilbert space if and only if a certain associated topological groupoid is a-T-menable. This groupoid characterisation then reduces the proof that the coarse Baum-Connes conjecture holds for a coarsely embeddable space to known results for a-T-menable groupoids. The property of admitting a fibred coarse embedding into Hilbert space was introduced by Chen, Wang and Yu to provide a property that is sufficient for the maximal analogue to the coarse Baum-Connes conjecture and in this paper we connect this property to the traditional coarse Baum-Connes conjecture using a restriction of the coarse groupoid and homological algebra. Additionally we use this results to give a characterisation of the a-T-menability for residually finite discrete groups.

Keywords: Fibred coarse embeddings, topological groupoids, negative type functions, coarse Baum-Connes conjecture.

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1. Introduction

The application of coarse methods to algebraic topological problems is well known [23, 32]. One possible method utilised for such problems is a higher index theorem that allows us to calculate refined large scale information from small scale topological or analytic data. This process can be succinctly phrased using the language of K-theory and K-homology and can be encoded into the coarse geometric version of the well-known Baum-Connes conjecture, which asks whether or not a certain assembly map:

$$\mu_* : KX_* (X) = \lim_{R \to 0} K_*(P_R(X)) \to K_* (C^*(X))$$

is an isomorphism for all uniformly discrete spaces X with bounded geometry. One approach to this conjecture for suitable metric spaces X is via the concept of a coarse embedding into Hilbert space. A seminal paper by Yu [32] first showed the importance of coarse embeddings into Hilbert space by proving that this is a sufficient condition for the coarse Baum-Connes assembly map $\mu_*$ to be an isomorphism.

In this paper we study the relationship between fibred coarse embedding into Hilbert space, first introduced by Chen, Wang and Yu [6], and the coarse analogue of the strong Novikov conjecture. Intuitively, a space admits a fibred coarse embedding into Hilbert space if for each scale it is acceptable to forget bounded portions of the space and embed what remains locally and compatibly into Hilbert space. This is made precise in Definition 6. This property was used in [6] to prove a maximal analogue of the work of [32], that is the existence of a fibred...
coarse embedding into Hilbert space implies that the maximal coarse Baum-Connes assembly map is an isomorphism for any uniformly discrete metric space with bounded geometry.

Another approach to these questions was considered in [9], in which a conjecture known as the boundary coarse Baum-Connes conjecture was defined for uniformly discrete bounded geometry spaces. This conjecture, defined via groupoids, was designed to capture the structure of a space at infinity. We explain the relationship between this conjecture and the work of [6] and give a new method to prove the maximal coarse Baum-Connes conjecture in this instance that has additional consequences. We explore these consequences in Section 5.

The strategy of this paper is to convert coarse assembly problems into groupoid assembly problems was pioneered in [25]:

**Theorem 1.** Let $X$ be a uniformly discrete space with bounded geometry that admits a fibred coarse embedding into Hilbert space. Then the associated boundary groupoid $G(X)_{|∂βX}$ is a-T-menable.

We define the boundary groupoid $G(X)_{|∂βX}$ in Definition 13.

This result gives us access to the tools developed in [9] concerning the boundary coarse Baum-Connes conjecture and in in Section 5 we provide applications of Theorem 1 to the coarse analogue of the strong Novikov conjecture for uniformly discrete spaces with bounded geometry (Theorem 33), as well as considering the situation concerning ghost operators within Roe algebras associated to coarsely embeddable spaces (Corollary 36). Lastly we give a characterisation of a-T-menability for residually finite groups via box spaces (Theorem 40).

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2. Fibred coarse embeddings and groupoids associated to coarse spaces

We recap now the important details surrounding Theorem 5.4 from [25] and the notion of a fibred coarse embedding [6].

2.1. Coarse and fibred coarse embeddings. We first make precise the metric spaces that we will study in this paper.

**Definition 2.** Let $X$ be a metric space. Then $X$ is said to be uniformly discrete if there exists $c > 0$ such that for every pair of distinct points $x \neq y \in X$ the distance $d(x, y) > c$. Additionally, $X$ is said have bounded geometry if for every $R > 0$ there exists $N_R > 0$ such that for every $x \in X$ the cardinality of the ball of radius $R$ about $x$ is smaller than $N_R$.

And now recall the concept of a coarse embedding, to motivate the definition of a fibred coarse embedding.

**Definition 3.** A metric space $X$ is said to admit a coarse embedding into Hilbert space $\mathcal{H}$ if there exist maps $f : X \to \mathcal{H}$ and non-decreasing $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}$ such that:

1. for every $x, y \in X$, $\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y))$;
2. for each $i$, we have $\lim_{r \to \infty} \rho_i(r) = +\infty$.
One application of admitting a coarse embedding into Hilbert space is the main result of 
[32, 25].

**Theorem 4.** Let $X$ be a uniformly discrete metric space with bounded geometry that admits a coarse embedding into Hilbert space. Then the coarse Baum-Connes conjecture holds for $X$, that is the assembly map $\mu_*$ is an isomorphism.

Many metric spaces admit coarse embeddings; the property is a suitably flexible one as it is implied by many other coarse properties such as finite asymptotic dimension [21, Example 11.5], or the weaker properties of finite decomposition complexity [10] and property A [32]. The primary application is to finitely generated discrete groups via the descent principle that is originally due to Higson [11], then adapted by Skandalis, Tu and Yu in [25]:

**Corollary 5.** Let $\Gamma$ be a finitely generated discrete group admitting a coarse embedding into Hilbert space. Then the (strong) Novikov conjecture holds for $\Gamma$. □

For more information on the strong Novikov conjecture and its connections to coarse geometry see [8, 25].

There are spaces that do not admit a coarse embedding into Hilbert space, such as expander graphs [18]. Certain types of expander graph are known to be counterexamples to the Baum-Connes conjecture [13, 12, 30, 31, 19] and within this class, it is possible to prove some positive results under hypotheses on the structure of the expander; it is known that certain families do satisfy the coarse analogue of the Novikov conjecture and the maximal coarse Baum-Connes conjecture [19, 30]. The notion of a fibred coarse embedding was introduced in [6] where the authors gave results that partially explained these phenomena. The following is Definition 2.1 from [6].

**Definition 6.** A metric space $X$ is said to admit a **fibred coarse embedding** into Hilbert space if there exists

- a field of Hilbert spaces $\{H_x\}_{x \in X}$ over $X$;
- a section $s : X \to \sqcup_{x \in X} H_x$ (i.e $s(x) \in H_x$);
- two non-decreasing functions $\rho_1, \rho_2$ from $[0, \infty)$ to $(-\infty, +\infty)$ such that $\lim_{r \to \infty} \rho_i(r) = \infty$ for $i = 1, 2$;
- a reference Hilbert space $H$

such that for any $r > 0$ there exists a bounded subset $K_r \subset X$ and a trivialisation:

$$t_C : \sqcup_{x \in C} H_x \to C \times H$$

for each $C \subset X \setminus K_r$ of diameter less than $r$. We ask that this map $t_C|_{H_x} = t_C(x)$ is an affine isometry from $H_x$ to $H$, satisfying:

1. for any $x, y \in C$, $\rho_1(d(x, y)) \leq \|t_C(x)(s(x)) - t_C(y)(s(y))\| \leq \rho_2(d(x, y))$;
2. for any two subsets $C_1, C_2 \subset X \setminus K_r$ of diameter less than $r$ and nonempty intersection $C_1 \cap C_2$ there exists an affine isometry $t_{C_1, C_2} : H \to H$ such that $t_{C_1}(x)t_{C_2}(x)^{-1} = t_{C_1, C_2}$ for all $x \in C_1 \cap C_2$.

Let $F = \{K_r\}_r$. In this instance we say $X$ fibred coarsely embeds into Hilbert space with respect to $F$.

It is possible to improve this definition in the context of **sequences of finite metric spaces** and the infinite metric spaces defined from them using the following construction:
Definition 7. Let \( \{X_i\}_{i \in \mathbb{N}} \) be a sequence of finite metric spaces that are uniformly discrete with bounded geometry uniformly in the index \( i \), such that \( |X_i| \to \infty \) in \( i \). Then we can form the coarse disjoint union \( X \) with underlying set \( \sqcup X_i \), metric \( d \) given by the metric on each component and \( d(X_i, X_j) \to \infty \) as \( i + j \to \infty \). Any such metric is proper and unique up to coarse equivalence.

In this case, we can refine Definition 6 to the following:

Definition 8. The coarse disjoint union \( X \) of a sequence of finite metric spaces \( \{X_i\} \) is said to admit a fibred coarse embedding into Hilbert space if there exists

- a field of Hilbert spaces \( \{H_x\}_{x \in X} \) over \( X \);
- a section \( s: X \to \sqcup_{x \in X} H_x \) (i.e. \( s(x) \in H_x \));
- two non-decreasing functions \( \rho_1, \rho_2 \) from \([0, \infty)\) to \((\infty, +\infty)\) such that \( \lim_{r \to \infty} \rho_i(r) = \infty \) for \( i = 1, 2 \);
- a non-decreasing sequence of numbers \( 0 \leq l_0 \leq l_1 \leq ... \leq l_i \leq ... \) with \( \lim_{i \to \infty} l_i = \infty \);
- a reference Hilbert space \( H \).

such that for each \( x \in X_i \) there exists a trivialisation:

\[
t_x : \sqcup_{y \in B_{l_i}(x)} H_y \to B_{l_i}(x) \times H
\]

such that the map \( t_x(z) \) is an affine isometry from \( H_z \to H \), satisfying:

1. \( \rho_1(d(z, z')) \leq \|t_x(z)(s(z)) - t_x(z')(s(z'))\| \leq \rho_2(d(z, z')) \) for any \( z, z' \in B_{l_i}(x), x \in X_i, i \in \mathbb{N} \);
2. for any \( x, y \in X_i \) such that \( B_{l_i}(x) \cap B_{l_i}(y) \neq \emptyset \) there exists an affine isometry \( t_{xy} : H \to H \) such that \( t_x(z) = t_{xy} t_y(z) \) for all \( z \in B_{l_i}(x) \cap B_{l_i}(y) \).

In particular many expanders are known to satisfy this property, such as those coming from \(\alpha\)-T-menable discrete groups or those of large girth [6, 19].

The main application of this property is the main result of [6]:

Theorem 9. Let \( X \) be a bounded geometry metric space admitting a fibred coarse embedding into Hilbert space. Then the maximal coarse Baum-Connes conjecture holds for \( X \).

We give a different proof of this in Section 5.1.1

2.2. Groupoids and coarse properties. Groupoids play an integral role in the constructions we adapt from [24]. Below we recap some basic properties of groupoids before giving an outline of the construction of the coarse groupoid associated to a coarse space \( X \).

A groupoid \( \mathcal{G} \) is a topological groupoid if both \( \mathcal{G} \) and \( \mathcal{G}^{(0)} \) are topological spaces, and the maps \( r, s, s^{-1} \) and the composition are all continuous. A Hausdorff, locally compact topological groupoid \( \mathcal{G} \) is proper if \( (r, s) \) is a proper map and étale or \( r \)-discrete if the map \( r \) is a local homeomorphism. When \( \mathcal{G} \) is étale, \( s \) and the product are also local homeomorphisms, and \( \mathcal{G}^{(0)} \) is an open subset of \( \mathcal{G} \).

Let \( X \) be a uniformly discrete bounded geometry metric space. We want to define a groupoid with the property that it captures the coarse information associated to \( X \). To do this effectively we need to define what we mean by a coarse structure that is associated to a metric. The details of this can be found in [21].

Definition 10. Let \( X \) be a set and let \( \mathcal{E} \) be a collection of subsets of \( X \times X \). If \( \mathcal{E} \) has the following properties:

1. \( \mathcal{E} \) is closed under finite unions;
Then we say $E$ is a coarse structure on $X$ and we call the elements of $E$ entourages. If in addition $E$ contains all finite subsets then we say that $E$ is weakly connected.

Example 11. Let $X$ be a metric space. Then consider the collection $S$ of the $R$-neighbourhoods of the diagonal in $X \times X$; that is, for every $R > 0$ the set:

$$\Delta_R = \{(x, y) \in X \times X | d(x, y) \leq R\}$$

Let $E$ be the coarse structure generated by $S$. This is called the metric coarse structure on $X$.

If $X$ is a uniformly discrete bounded geometry metric space this coarse structure is uniformly locally finite, proper and weakly connected [25].

Let $X$ be a uniformly discrete metric space with bounded geometry. We denote by $\beta X$ the Stone-\v{C}ech compactification of $X$ (similarly with $\beta (X \times X)$).

Define $G(X) := \bigcup_{R > 0} \Delta_R \subseteq \beta(X \times X)$. Then $G(X)$ is a locally compact, Hausdorff topological space. To equip it with a product and inverse we would ideally consider the natural extension of the pair groupoid product on $\beta X \times \beta X$. We remark that the map $(r, s)$ from $X \times X$ extends first to an inclusion into $\beta X \times \beta X$ and universally to $\beta (X \times X)$, giving a map $(r, s) : \beta(X \times X) \to \beta X \times \beta X$. We can restrict this map to each entourage allowing us to map the set $G(X)$ to $\beta X \times \beta X$. The following is Corollary 10.18 [21]

Lemma 12. Let $X$ be a uniformly discrete bounded geometry metric space, let $E$ be any entourage and let $\overline{E}$ its closure in $\beta(X \times X)$. Then the inclusion $E \to X \times X$ extends to a topological embedding $\overline{E} \to \beta X \times \beta X$ via $(\overline{r}, \overline{s})$.

Using this Lemma, we can conclude that the pair groupoid operations on $\beta X \times \beta X$ restrict to give continuous operations on $G(X)$; we equip $G(X)$ with this induced product and inverse.

Given this construction of $G(X)$ as an extension of the pair groupoid, it is easy to see that the set $X$ is an open saturated subset of $\beta X$. In particular, this also means that the Stone-Čech boundary $\partial \beta X$ is saturated.

Definition 13. The boundary groupoid associated to $X$ is the restriction groupoid $G(X)|_{\partial \beta X}$.

3. Negative type functions on groupoids

The role of positive and conditionally negative type kernels within group theory is well known and plays an important role in studying both analytic and representation theoretic properties of groups [2] [14]. These notions were considered for groupoids by Tu in [26] and we use his conventions here. Let $\mathcal{G}$ be a locally compact, Hausdorff groupoid.

Definition 14. A continuous function $F : \mathcal{G} \to \mathbb{R}$ is said to be of negative type if

1. $F|_{\mathcal{G}(0)} = 0$;
2. $\forall x \in \mathcal{G}, F(x) = F(x^{-1})$;
3. Given $x_1, \ldots, x_n \in \mathcal{G}$ all having the same range and $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ such that $\sum \sigma_i = 0$ we have $\sum_{j,k} \sigma_j \sigma_k F(x_j^{-1} x_k) \leq 0$.

The important feature of functions of this type is their connection to a-T-menability for locally compact, $\sigma$-compact groupoids, in fact the following are equivalent by work of Tu [26]:
(1) There exists a locally proper negative type function on $G$.
(2) There exists a continuous field of Hilbert spaces over $G^{(0)}$ with a proper affine action of $G$.

We remark that in the situation we are considering, étale groupoids with compact Hausdorff base space, local properness is equivalent to properness. We also remark that an alternative proof of this is given by Renault in Section 2.4 of [20].

We can now make precise what we mean by “$G(X)$ captures the coarse geometry of $X$” in our particular case:

**Theorem 15.** Let $X$ be a uniformly discrete space with bounded geometry and let $G(X)$ be its coarse groupoid. Then $X$ admits a coarse embedding into Hilbert space if and only if $G(X)$ is a-T-menable [25, Theorem 5.4].

4. Main Theorem

We dedicate this section to proving Theorem 1. More precisely we construct, given a fibred coarse embedding into Hilbert space, a proper conditionally negative type function on the boundary groupoid $G(X)|_{\partial \beta X}$.

**Definition 16.** Let $\mathcal{F} := \{K_R\}$ be a family of bounded subsets of $X$. Let $A_R$ be the restricted entourage $\Delta_R \cap ((X \setminus K_R) \times (X \setminus K_R))$.

**Remark 17.** Let $X$ admit a fibred coarse embedding into Hilbert space.

1. $A_R$ is an entourage with the same corona as $\Delta_R$, that is $\overline{A_R} \setminus A_R = \overline{\Delta_R} \setminus \Delta_R$. Assume that $X$ admits a fibred coarse embedding into Hilbert space with respect to $\mathcal{F}$. Then for each $R > 0$ the following function is defined on $A_R$:

   \[ k_R(x, y) = \|t_x(x)(s(x)) - t_x(y)(s(y))\| \]

   where $t_x := t_{B_R(x)} : H_x \to H$ is the affine isometry provided by the fibred coarse embedding.

2. In the special case that $X$ is a coarse disjoint union constructed from a sequence of finite graphs $\{X_i\}$, the $A_R$’s defined above can be taken to have the form: $A_R = \sqcup_{i \geq i_R} \Delta^i_R$ for some $i \geq i_R$ and where $\Delta^i_R$ denotes the $R$–neighbourhood of the diagonal in $X_i$.

3. For $G(X)|_{\partial \beta X} = \bigcup_{R > 0} \overline{\Delta_R} \setminus \Delta_R$ there is a smallest $R > 0$ such that $\gamma \in \overline{\Delta_R} \setminus \Delta_R$. We denote this by $R_\gamma$.

Using this data we wish to construct a proper, negative type function $f$ on $G(X)|_{\partial \beta X}$. The main idea is to extend each $k_R$ to a function $\tilde{k}_R$ using the universal property of $\beta A_R \cong A_R$. We then show that these extensions $\tilde{k}_R$ piece together to form a single function $f$ on $G(X)|_{\partial \beta X}$. The next section gives the technical details of how to do this supposing that the following additional criterion is satisfied.

**Definition 18.** A family of kernels $\{k_R : A_R \to \mathbb{R}_+\}$ is scale independent if for every $S > R > 0$, we have that $k_R|_{A_R \cap A_S} = k_S|_{A_R \cap A_S}$.

This notion will allow us to piece the kernels on every scale together as prescribed above.
4.1. The technical steps. In this section, we give the details of the construction outlined above.

Proposition 19. Let \( X \) be a uniformly discrete bounded geometry metric space that fibred coarsely embeds into Hilbert space, and suppose additionally that the kernels \( k_R \) coming from this embedding are scale independent. Then the function \( f \) obtained by patching together the \( k_R \) is well defined and continuous on \( G(X)|_{\partial \beta X} \).

Proof. We have two considerations:

1. Extending \( k_R \) to \( \tilde{k}_R \) on the closure of \( A_R \) for each \( R > 0 \).
2. Each \( \tilde{k}_R \) pieces together; for any \( S > R \) we have that \( \tilde{k}_S(\gamma) = \tilde{k}_R(\gamma) \).

We first prove (1). Under the assumption that \( X \) fibred coarsely embeds into Hilbert space, we know that each \( k_R \) is a bounded function on \( A_R \), hence extends to a continuous function \( \tilde{k}_R \) on the Stone-Čech compactification of \( A_R \), which in this context is homeomorphic to its closure in \( \beta(X \times X) \).

For the proof of (2) consider the sets \( A_R \) and \( A_R \cap A_S \) for \( S > R \). Using the compatibility properties of a fibred coarse embedding and the scale independence of Definition 18 we can see that the function \( k_R \) and \( k_S \) restricted to \( A_R \cap A_S \) agree. Using the observation that for all \( R > 0 \) and \( S > R \) the set \( A_R \cap A_S \) has the same Stone-Čech boundary as \( A_R \) we can deduce that \( \tilde{k}_R \) and \( \tilde{k}_S \) restricted to \( \overline{A_R \setminus A_R} \) agree. Hence we can define, for any \( \gamma \in G(X)|_{\partial \beta X} \), \( f(\gamma) = \tilde{k}_R(\gamma) \), which is the natural continuous function defined on the union \( \cup_{R>0} \overline{A_R} \).

Lemma 20. The function \( f \) is proper.

Proof. To see \( f \) is proper it is enough to prove that the preimage of an interval \( [0, r] \) is contained in \( \overline{A_R} \) for some \( R > 0 \). This is as each interval \( [0, r] \) is a closed subset of \( \mathbb{R} \), the map \( f \) is continuous and hence \( f^{-1}([0, r]) \) would be a closed subset of a compact set, hence would itself be compact.

We now assume for a contradiction that the preimage of \( [0, r] \) contains elements \( \gamma \) with the \( R_\gamma \) defined in Remark 17(4) being arbitrarily large. Then from the definition of \( f \) and the fact that \( X \) admits a fibred coarse embedding, we can see:

\[
\rho_-(\gamma^{i}_R)^2 \leq f(\gamma) \leq \rho_+(\gamma^{i}_R)^2.
\]

As \( \rho_-(S) \) tends to infinity as \( S \) does, we can find an \( S > 0 \) such that \( \rho_-(S) > r \). By assumption there exists \( \gamma \in f^{-1}([0, r]) \) with \( R_\gamma \) as large as we like: in particular \( \rho_-(\gamma^{i}_R) > r \), which is impossible. Whence there exists an \( R > 0 \) such that \( f^{-1}([0, r]) \subset \overline{A_R} \).

Lemma 21. The function \( f \) is of negative type.

Proof. This relies on the ideas of [25, Theorem 5.4]. Let \( \gamma_1, ..., \gamma_n \in G(X)|_{\partial \beta X} \) such that \( r(\gamma_1) = ... = r(\gamma_n) := \omega \) and let \( \sigma_1, ..., \sigma_n \in \mathbb{R} \) with sum 0. We need to prove that:

\[
\sum_{i,j} \sigma_i \sigma_j f(\gamma_i^{-1} \gamma_j) \leq 0.
\]

As there are only finitely many \( \gamma_i \), there exists a smallest \( R > 0 \) such that each \( \gamma_i \) and each product \( \gamma_j^{-1} \gamma_i \) are elements of \( \overline{A_R} \). Let \( (x_{i,\lambda}, y_{i,\lambda}) \) be nets within \( \overline{A_R} \) that converge to \( \gamma_i \) respectively. As the ranges of the \( \gamma_i \) are all equal without loss of generality we can assume that \( y_{\lambda, i} = y_{\lambda} \) is equal in each net. To see this is possible take an approximation of \( \omega \) by a net \( y_{\lambda} \) and then use the fact that each \( \gamma_i \) belongs to the closure of the graph of a partial translation \( t_i \) [21]. Now use \( x_{\lambda, i} = t_i^{-1}(y_{\lambda}) \): \( \gamma_i = \lim_{\lambda}(x_{\lambda, i}, y_{\lambda}) \).
Hence, for each \( \lambda \), we know that \( (x_{\lambda,i}, y_{\lambda}) \) and \( (x_{\lambda,j}, x_{\lambda,i}) \in \Delta_R \), and that \( x_{\lambda,i}, x_{\lambda,j} \in B_R(y_{\lambda}) \).

We wish to compute \( \sum_{i,j} \sigma_i \sigma_j k_R(x_{\lambda,j}, x_{\lambda,i}) \) by using an affine isometry to work relative to \( y_{\lambda} \): denote by \( t_{y_{\lambda}} \) the local trivialisation \( t_{BR(y_{\lambda})} \) and by \( k_R^{y_{\lambda}} \) the kernel defined using the trivialisation \( t_{y_{\lambda}} \) as in Remark 17.1. Let \( t_{x_{\lambda,i},y_{\lambda}} \) be the unique affine isometry such that \( t_{x_{\lambda,i},y_{\lambda}} t_{y_{\lambda}} = t_{x_{\lambda,j},y_{\lambda}} \) on the intersections of the corresponding balls, which we know exist from the definition of a fibred coarse embedding.

Hence:

\[
\sum_{i,j} \sigma_i \sigma_j k_R(x_{\lambda,j}, x_{\lambda,i}) = \sum_{i,j} \sigma_i \sigma_j \| t_{x_{\lambda,j},y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})) - t_{x_{\lambda,j},y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \|^2 \\
= \sum_{i,j} \sigma_i \sigma_j \| t_{x_{\lambda,j},y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})) - t_{x_{\lambda,j},y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \|^2 \\
= \sum_{i,j} \sigma_i \sigma_j k_R^{y_{\lambda}}(x_{\lambda,j}, x_{\lambda,i}) \text{ as } t_{x_{\lambda,i},y_{\lambda}} \text{ is an isometry.}
\]

This reformulation allows us to directly compute using a standard argument (although with worse notation):

\[
\sum_{i,j} \sigma_i \sigma_j k_R^{y_{\lambda}}(x_{\lambda,j}, x_{\lambda,i}) = \sum_{i,j} \sigma_i \sigma_j \| t_{y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})) - t_{y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \|^2 \\
= \sum_{i,j} \sigma_i \sigma_j \| t_{y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})) \|^2 + \| t_{y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \|^2 - 2 \langle t_{y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})), t_{y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \rangle \\
= (\sum_{i} \sigma_i \| t_{y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})) \|^2) \sum_{i} \sigma_i + (\sum_{i} \sigma_i \| t_{y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \|^2) \sum_{j} \sigma_j \\
- 2 \sum_{j} \sigma_j t_{y_{\lambda}} (x_{\lambda,j}) (s(x_{\lambda,j})), \sum_{i} \sigma_i t_{y_{\lambda}} (x_{\lambda,i}) (s(x_{\lambda,i})) \rangle \leq 0.
\]

This holds for each \( \lambda \) in the net. Taking a limit in \( \lambda \):

\[
\sum_{i,j} \sigma_i \sigma_j f(\gamma^{-1}_{i,j}) = \sum_{i,j} \sigma_j \sigma_i \lim_{\lambda} k_R^{y_{\lambda}}(x_{\lambda,j}, x_{\lambda,i}) \leq 0
\]

\( \square \)

The preceding Lemmas will now be used to prove the following Theorem.

**Theorem 22.** Let \( X = \bigsqcup X_i \) be a coarse disjoint union of finite metric spaces such that the space \( X \) is uniformly discrete and has bounded geometry. If \( X \) admits a fibred coarse embedding into Hilbert space then the boundary groupoid \( G(X)|_{0\partial X} \) is a-T-menable.

**Proof.** By assumption the fibred coarse embedding can be chosen such that the family of kernels satisfies the restriction property of Definition 13 (this follows from the fact that such spaces admit the nicer Definition 8 which is Definition 5.1 of [6]). The Lemmas above now combine to prove the result. \( \square \)

The remainder of this section is to show how Theorem 22 can be adapted to the situation that \( X \) is not a coarse disjoint union but is fibred coarsely embeddable into Hilbert space.
4.2. **What happens if the space is not a disjoint union of finite metric spaces?**

In general, the property of scale independence of Definition 18 will not hold for an arbitrary fibred coarse embedding (at least not obviously). However, in this section we outline a method of “coarse” decomposition that will still allow us to prove the analogue of Theorem 22 for general uniformly discrete spaces with bounded geometry. We will use [29, Chapter 3] as a common reference for results concerning the relationship between coarse embeddings and kernels in this section.

**Definition 23.** Let \( Z \) be a uniformly discrete bounded geometry metric space and let \( z_0 \in Z \). Then an **annular decomposition** of \( Z \) is a covering \( Y_0 \cup Y_1 = Z \), where \( Y_0, Y_1 \) and \( Y_0 \cap Y_1 \) are unions of annular regions around \( z_0 \).

The next Lemma, the details of which appear in [6], illustrates it is always possible to find a decomposition that is very well spaced, i.e a decomposition in which each of \( Y_0, Y_1 \) and \( Y_0 \cap Y_1 \) is a coarse disjoint union of finite metric spaces.

**Lemma 24.** Let \( Z \) be a uniformly discrete bounded geometry metric space. Then \( Z \) admits an annular decomposition that is coarsely excisive.

**Proof.** We give the construction here and the remainder is implicit from the proof of Theorem 1.1, from page 19 of [6]. Let \( z_0 \in Z \), then for \( n \in \mathbb{N} \) let

\[
Z_n = \{ z \in X | n^3 - n \leq d(z, z_0) \leq (n + 1)^3 + (n + 1) \}.
\]

Now, set \( Y_0 \) to be the union over even \( n \) and \( Y_1 \) to be the union over odd \( n \). \( \square \)

So, for any uniformly discrete bounded geometry metric space \( Z \) we have two decompositions:

1. \( Z = Y_0 \cup Y_1 \), where the two pieces are themselves unions of finite (annular) metric spaces;
2. \( Z = \bigcup_n Z_n \), where each \( Z_n \) are finite but not of uniformly bounded cardinality in \( n \).

We can still use the \( Z_n \) as a covering of \( X \) to define partition of unity and because they form the sets \( Y_0 \) and \( Y_1 \), we can use the favourable notion of fibred coarse embedding from Definition 8 when we work with them. To get a partition of unity we use Proposition 4.1 of [7]:

**Proposition 25.** Let \( U \) be a cover of a metric space \( X \) with multiplicity at most \( k + 1 \) and Lebesgue number \( L > 0 \). For \( U \in U \) define:

\[
\phi_U(x) = \frac{d(x, X \setminus U)}{\sum_{V \in U} d(x, X \setminus V)}
\]

then \( \{ \phi_U \}_{U \in \mathcal{U}} \) is a partition of unity on \( X \) subordinated to the cover \( \mathcal{U} \). Moreover, each \( \phi_U \) satisfies:

\[
|\phi_U(x) - \phi_U(y)| \leq \frac{2k + 3}{L} d(x, y) \text{ for every } x, y \in X.
\]

\( \square \)

Let \( X \) be a uniformly discrete space, which we will later take to be fibred coarsely embeddable. We will now attempt to apply Proposition 25 to the covering of \( X \) obtained by applying Lemma 24 that is \( X = \bigcup_n Z_n \). Define, \( X_{>n} \) to be the union \( \bigcup_{m>n} Z_m \subset X \).
We observe that by the construction in Lemma 24 the multiplicity of the covering by the $Z_n$ is 2. Furthermore, we can control the Lebesgue number of this cover away from a bounded subset using the following observation:

**Lemma 26.** Let $x, y$ belong to the symmetric difference of $Z_n$ and $Z_{n+1}$. If $x \in Z_n$ and $y \in Z_{n+1}$ then $d(x, y) \geq 2(n+1)$.

**Proof.** If $x \in Z_n$ and $y \in Z_{n+1}$ then by the definition of $Z_n$ and the fact that $x \not\in Z_{n+1}$ tells us that $d(z_0, x) \leq (n+1)^3 - (n + 1)$. Similarly, as $y \in Z_{n+1}$ but $y \in Z_n$, we know that $d(z_0, y) \geq (n + 1)^3 + (n + 1)$. Combining these inequalities with the triangle inequality gives the desired estimate. \hfill \Box

An immediate consequence of this is that for $L > 0$ there is an $n_L$ such that the covering restricted to $X > n_L$ has Lebesgue number $\leq L$. To see this, consider the situation that there is a point $x \in Z_n \subset Z$ such that the ball $B_L(x)$ is not contained in some element $Z_m$ of the covering. As the multiplicity of the cover is 2, it is immediate that the ball cannot be contained in any sets other than $Z_{n-1}$, $Z_n$ or $Z_{n+1}$. We can conclude that, without loss of generality, there is a $y \in Z_{n+1}$ such that there is a point $y \in Z_{n+1}$ that does not belong to $Z_n$ but does belong to the ball $B_L(x)$. If $x \not\in Z_{n+1}$, then Lemma 26 implies that $L \geq 2(n + 1)$. So by choosing $n_L$ to be the least integer above $\frac{L}{2} - 1$ we see that this situation cannot occur. What remains is the case $x \in Z_{n+1}$, but a mirror of this argument reduces this to the previous case. Hence on the space $X > n_L$, the covering has Lebesgue number at least $L$.

Suppose now that $X$ fibred coarsely embeds into Hilbert space with respect to a family $\{K_R\}$. Then for every $R > 0$ there is a smallest natural number such that the bounded set $K_R$ is contained within some finite union of annuli. Denote this by $m_R$. Observe also that the above statement can be modified to take into account the two bounded sets that will occur after we break $X$ into $Y_0$ and $Y_1$. In this case, we define $m_R$ to be the maximum of the natural numbers required on each piece.

This allows us to control, up to scale, the asymptotic behaviour of kernel functions on the spaces $Y_0$ and $Y_1$ when gluing them back together. To make this formal, we need first to recall the result of Schoenberg [24], where the statement here is taken from [29]:

**Theorem 27.** Let $X$ be a set and let $k$ be a negative type kernel on $X$. Then for every $t > 0$, the kernel $F_t(x, y) := e^{-tk(x, y)}$ is of positive type. \hfill \Box

On each of the spaces $Y_i$ we have a family of scale independent negative type kernels $\{k_{i, R}\}_R$ defined on $A_R$ that is well controlled by a common pair of functions $\rho_{\pm}$. By applying Theorem 27 we now have a family, controlled by $R$ and $t$, of scale independent kernels of positive type that are also well controlled (c.f [29] Chapter 3 for a full treatment). Recall that a kernel $k$ is said to have $(R, \epsilon)$-variation if whenever $(x, y) \in \Delta_R$, we have that $|1 - k(x, y)| \leq \epsilon$.

The following result collects these ideas and is a modification of an argument shared with the author by Rufus Willett.

**Lemma 28.** Suppose $X$ is fibred coarsely embeddable into Hilbert space. Then there is a family of normalised positive type kernels $\{F_{R, t} : A_R \to \mathbb{R}\}_{R, t}$ such that the following are satisfied:

- $\{F_R\}_R$ is a scale independent family for every $t$;
- for every $R, \epsilon > 0$ there is a bounded subset $D$ that contains $K_R$ and a real number $t$ such that there is a positive type kernel $F_{R, t}$ with $(R, \epsilon)$-variation when restricted to $A_R \cap (X \setminus D)^2$;
for all \( t, \epsilon > 0 \) there exists \( S > 0 \) such that for all \( R > S \) and all \((x, y) \in A_R \) that do not belong to \( A_S \) we have \( |F_{R,t}(x,y)| < \epsilon \).

Proof. The proof proceeds in a few steps:

1. We define, from Proposition 25, an \( \ell^2 \)-partition of unity.
2. Using this partition of unity for the cover \( U = \{Z_n\} \) and the families of kernels on \( Y_i \) provided by Theorem 27 and the assumption of fibred coarse embeddability we define a family of new kernels for \( X \). This family will be scale independent.
3. We show that this kernel can be constructed such that for any \( R \) and \( \epsilon \), there is a natural number \( o_{R,\epsilon} \) and a \( t \) such that the kernel \( F_{R,t} \) restricted to the set \( X > o_{R,\epsilon} \) has \((R, \epsilon)\)-variation.
4. Finally, we prove the asymptotic control estimates using observations concerning the control functions that come from the original fibred coarse embedding.

Let \( n_R \) and \( m_R \) be defined as during the discussion prior to the statement of the Lemma. We now work through the steps outlined above:

1. Consider the new partition of unity defined using \( \Phi_n = \sqrt{\phi_{X_n}} \). This will have \( \ell^2 \)-sum equal to 1 for each \( x \), and satisfies, for any pair \( x, y \in X \):

\[
|\Phi_n(x) - \Phi_n(y)| \leq \sqrt{\frac{5}{L}d(x, y)}.
\]

2. Now, on each scale we restrict this partition of unity to the subspace \( X \setminus D_R \) and observe that for all but finitely many \( x \in X \), the sum: \( \sum_{n > \max(n_R, m_R)} \Phi_n(x) = \sum_n \Phi_n(x) \). Let \( F_{R,t}^{[n]} \) be the kernel output by Schoenbergs theorem that arises from the kernel \( k_{[n], R} \), where \( [n] \) represents the class of \( n \) modulo 2, and consider for each \( R > 0 \) the kernel \( F_{R,t} \) defined pointwise by:

\[
F_{R,t}(x,y) = \sum_n \Phi_n(x) F_{R,t}^{[n]}(x,y) \Phi_n(y)
\]

By construction this family of kernels is scale independent for each \( t \) as each \( F_{R,t}^{[n]} \) are, and the same partition of unity is used and it was constructed globally. This completes Step 2.

3. By combining observations from above and estimates from Theorem 3.2.8(2) in [29] we can see that for every \( R, \epsilon > 0 \) there exists an \( o_{R,\epsilon} \) such that the covering restricted to \( X > o_{R,\epsilon} \) has Lebesgue number \( S := \frac{180R}{\epsilon^2} \) and \( K_R \subseteq \cup_{n \leq o_{R,\epsilon}} Z_n := D \).

Let \( t = \frac{4}{3(1 + \rho_+(R)^2)} \). Then each of the kernels \( F_{R,t}^{[n]} \) satisfies

\[
1 - F_{R,t}^{[n]}(x,y) \leq 1 - e^{-t\rho_+(d(x,y))} \leq 1 - e^{-\frac{4}{3}} \leq \frac{\epsilon}{3}
\]
for every \( x, y \in A_R \). Now, consider the following for every \( x, y \in A_R \cap (X \setminus D)^2 \):

\[
1 - F_{R,t}(x,y) = \sum_n \Phi_n(x) - \Phi_n(x) F^{[n]}_{R,t}(x,y) \Phi_n(y)
\]

\[
= \Phi_n(x)^2 + \Phi_{n+1}(x)^2 - \Phi_n(x) F^{[n]}_{R,t}(x,y) \Phi_n(y) - \Phi_{n+1}(x) F^{[n+1]}_{R,t}(x,y) \Phi_{n+1}(y)
\]

\[
\leq \Phi_n(x) \Phi_n(y) + \Phi_n(x)(\Phi_n(x) - \Phi_n(y)) + \Phi_{n+1}(x)(\Phi_{n+1}(x) - \Phi_{n+1}(y)) - \Phi_n(x) F^{[n]}_{R,t}(x,y) \Phi_n(y) - \Phi_{n+1}(x) F^{[n+1]}_{R,t}(x,y) \Phi_{n+1}(y)
\]

\[
< \Phi_n(x) \Phi_n(y)(1 - F^{[n]}_{R,t}(x,y)) + \Phi_{n+1}(x) \Phi_{n+1}(y)(1 - F^{[n+1]}_{R,t}(x,y)) + \frac{\epsilon}{3}.
\]

Where \([n]\) denotes \( n \) modulo 2, \( n \) depends only on \((x,y)\) and the \( \frac{\epsilon}{3} \) term comes from the choice of \( o_{R,t} \) such that each \(|\Phi_n(x) - \Phi_n(y)| < \sqrt{\frac{2R}{S}} = \frac{\epsilon}{6}\). To complete the proof, observe also that \(|\Phi_n(x) \Phi_n(y)(1 - F^{[n]}_{R,t}(x,y))| \leq \frac{\epsilon}{3}\) by the choice of \( S \) and \( t \).

(4) We now use the fact that these kernels are scale independent; the \((R,\epsilon)\)-variation property above holds only on a large piece of \( A_R \) and outside this piece the kernels \( F^{[n]}_{R,t} \) satisfy \( F^{[n]}_{R,t}(x,y) = e^{-tk(x,y)} \leq e^{-t\rho_-(d(x,y))} \), which clearly tends to 0 as \( d(x,y) \to \infty \). Patching this together, we see that \( F_{R,t} \) will also tend to 0 as \( d(x,y) \to \infty \), and so for fixed \( t \) and \( \epsilon \) we pick \( S \) large enough such that \( e^{-t\rho_-(S)} \) is less than \( \epsilon \), which proves point 3 in the claim.

\[ \square \]

The important point of Lemma 28 is that the family of kernels \( \{F_{R,t}\}_{R} \) is extendable to the boundary groupoid of \( X \) for every \( t \), so we can now prove an analogue of Theorem 3.2.8(4) of [29] at infinity using these families.

**Theorem 29.** Let \( X \) be a uniformly discrete metric space that admits a fibred coarse embedding into Hilbert space. Then the boundary groupoid \( G(X)|_{\partial \beta X} \) is \( a\)-\( T \)-menable.

**Proof.** We proceed as in Theorem 3.2.8 of [29].

(1) For every \( n \in \mathbb{N} \) define, using Lemma 28, a family of kernels \( F_n \) that satisfy \( 1 - F_n(x,y) < 2^{-n} \) whenever \( d(x,y) \leq n \).

(2) We now consider the extensions of \( F_n \) to the boundary groupoid \( G(X)|_{\partial \beta X} \), which is possible as they are each scale independent.

(3) Define \( k(\gamma) = \sum_{n=1}^{\infty} (1 - F_n(\gamma)) \). This is the limit of the partial sums defined on the boundary of the closure of \((X \setminus D_{n,2^{-n}})^2\). This converges as for any \( \gamma \in G(X)|_{\partial \beta X} \) there are only finitely many \( n \) for which \( \gamma \not\in \Delta_n \setminus \Delta_n \) and let \( n_\gamma \) denote the largest such \( n \). Now the tail of this sum will be bounded above by \( \sum_{n>n_\gamma} 2^{-n} \). The function \( k \) has negative type, as each term in the sum does and this property is closed under pointwise limits and positive sums (The extension point follows from nothing other than the argument for Lemma 21).

(4) It is now enough to show that this is proper (and we do this by showing it is controlled by two functions that tend to infinity \( \tau_\pm \)). Observe that for each \( \gamma \in \Delta_R \setminus \Delta_\emptyset \) we have:

\[
k(\gamma) \leq \sum_{n=1}^{\lfloor R \rfloor} 1 + \sum_{n=\lfloor R+1 \rfloor}^{\infty} 2^{-n} \leq R + 1
\]
so we can take \( \tau_+(R) = R + 1 \). For the lower bound, let \( h_n(R) := e^{-t_n \rho - (R)^2} \), where \( t_n \) is the parameter that realises the kernel \( F_n \). Then for each \( N \in \mathbb{N} \), there exists an \( S_N \) such that for all \( n \leq N \) and \( S > S_N \), we have \( h_n(S) < \frac{1}{2} \). We can find such an \( S_N \) as each \( h_n \) tends to 0 at infinity. Now, for every \( \gamma \in G(X)_{|\partial B} \) that is a limit of pairs in \( A_{S_K}^\infty \), we have:

\[
k(\gamma) \geq \sum_{n=1}^{N} 1 - F_n(\gamma) \geq \sum_{n=1}^{N} 1 - h_n(S_N) \geq \frac{N}{2}.
\]

Finally, choose:

\[
\tau_-(s) = \frac{1}{2} \max\{N|s \geq s_N\}.
\]

With this choice we have \( \tau_-(R) \leq k(\Delta_R \setminus \Delta_R) \leq \tau_+(R) \). Hence, \( k \) is a proper, negative type function on \( G(X)_{|\partial B} \).

\[
\square
\]

5. Applications

5.1. The boundary coarse Baum-Connes conjecture and the coarse Novikov conjecture. Throughout this section let \( A_\partial \) denote the quotient \( C^* \)-algebra \( l^\infty(X,K)/C_0(X,K) \). Recall from [9] the boundary coarse Baum-Connes conjecture.

**Conjecture** (Boundary Coarse Baum-Connes Conjecture). Let \( X \) be a uniformly discrete bounded geometry metric space. Then the assembly map:

\[
\mu_{\text{bdry}} : K_*^{\text{top}}(G(X)_{|\partial B}, A_\partial) \rightarrow K_*(A_\partial \rtimes_r G(X)_{|\partial B})
\]

is an isomorphism.

This conjecture also has a maximal form [9, Section 4] that is equivalent to the maximal coarse Baum-Connes conjecture at infinity from [6]. If \( X \) is a uniformly discrete bounded geometry metric space we can see that the algebra at infinity defined in [6] and the groupoid crossed product algebra \( A_\partial \rtimes_m G(X)_{|\partial B} \) are isomorphic. Proceeding via this conjecture we can appeal to the machinery of Tu [26] concerning \( \sigma \)-compact, locally compact a-T-menable groupoids to conclude results about the (maximal) coarse Baum-Connes conjecture.

In particular, we can use this conjecture and homological algebra to conclude Theorem [9]

**Theorem 30.** [9, Theorem 1.1] Let \( X \) be a uniformly discrete space with bounded geometry that fibred coarse embeds into Hilbert space. Then the maximal coarse Baum-Connes assembly map is an isomorphism for \( X \).

**Proof.** We have a short exact sequence of maximal groupoid \( C^* \)-algebras:

\[
0 \rightarrow K \rightarrow C_m^*(G(X)) \rightarrow C_m^*(G(X)_{|\partial B}) \rightarrow 0.
\]

This gives us the following diagram, arising from the long exact sequence in K-theory and suitable Baum-Connes conjectures (omitting the coefficients):

\[
\begin{array}{cccccccc}
K_1(C^*(G(X)_{|\partial B})) & \longrightarrow & K_0(K) & \longrightarrow & K_0(C_m^*(G(X))) & \longrightarrow & K_0(C_m^*(G(X)_{|\partial B})) & \longrightarrow & K_1(K) \\
\downarrow \quad \beta \downarrow & & \downarrow \quad \beta \downarrow & & \downarrow \mu_{\text{bdry}} \downarrow & & \downarrow \quad \beta \downarrow \\
K_1^{\text{top}}(G(X)_{|\partial B}) & \longrightarrow & K_0^{\text{top}}(X \times X) & \longrightarrow & K_0^{\text{top}}(G(X)) & \longrightarrow & K_0^{\text{top}}(G(X)_{|\partial B}) & \longrightarrow & K_1^{\text{top}}(X \times X)
\end{array}
\]

By Corollary [9] the maximal boundary assembly map is an isomorphism. The result now follows from the Five lemma. 

\[
\square
\]
To understand the reduced assembly map requires a more delicate approach.

**Definition 31.** Let $X$ be as above. We say that $X$ has an *infinite coarse component* if there exists $E \in \mathcal{E}$ such that $P_E(X)$, the Rips complex over $E$, has an unbounded connected component. Otherwise we say that $X$ *only has finite coarse components*. In the metric coarse structure, this condition becomes: if there exists an $R > 0$ such that $P_R(X)$ has an unbounded connected component.

**Example 32.** (Space of Graphs) Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of finite graphs such that $|X_i| \to \infty$ in $i$. Then we can form the coarse disjoint union (or *space of graphs*) $X$ with underlying set $\sqcup X_i$, as in Definition 7. The space $X$ is then a prototypical example of a space with only finite coarse components.

**Lemma 33.** Let $X$ be a uniformly discrete bounded geometry metric space, $i : \mathcal{K} \hookrightarrow C^*X$ be the inclusion of the compact operators into the Roe algebra of $X$ and $j : X \times X \to G(X)$ be the canonical inclusion. Then:

1. If $X$ has an infinite coarse component then the map from $K^{\text{top}}(X \times X) \to K^{\text{top}}(G(X))$ is the 0 map.
2. If $X$ is a space with only finite coarse components then $i_* : K_*(\mathcal{K}) \to K_*(C^*X)$ is injective.

**Proof.** Claim (1) relies on an argument involving groupoid equivariant KK-theory [3]. We remark that it is useful in this instance to decompose $G(X) = \beta X \rtimes \mathcal{G}_A$, where $\mathcal{G}_A$ is some second countable, Hausdorff étale groupoid that contains the pair groupoid $X \times X$ as an open subgroupoid. This is possible by Lemma 3.3b) of [25]. We then identify $KK_{\mathcal{G}_A}$ with $KK_{G(X)}$ where appropriate using Theorem 3.8 from [28].

Let $Z = P_E(\mathcal{G}_A)$, a proper $\mathcal{G}_A$-space, and $U$ be the preimage of $X$ under the anchoring map and $Z_x$ be the preimage of the singleton $x \in X$. We consider the following diagram, with notation following [25] [3]:

$$
\begin{array}{ccc}
KK_{\mathcal{G}_A}(C_0(U), C_0(X, \mathcal{K})) & \xrightarrow{1} & KK_{\mathcal{G}_A}(C_0(Z), \ell^\infty(X, \mathcal{K})) \\
& \xrightarrow{2} & \xrightarrow{3} & \xrightarrow{4} & \xrightarrow{5} & KK(\mathbb{C}, \mathbb{C})
\end{array}
$$

with maps at the level of cycles given by:

1. $(E, \phi, F) \mapsto (E \otimes_{\alpha} \ell^\infty(X, \mathcal{K}), \phi \otimes 1, F \otimes 1)$
2. $(\mathbb{C}, 1, 0) \mapsto (C_0(X, \mathcal{K}), \tilde{\psi}, 0)$
3. $(\mathbb{C}, 1, 0) \mapsto (\ell^\infty(X, \mathcal{K}), \tilde{\psi}, 0)$
4. $(E, \phi, F) \mapsto (E|_x, \phi|_x, F|_x)$
5. $(\mathbb{C}, 1, 0) \mapsto (\mathcal{K}, \psi, 0)$

Some remarks now about these maps:

- Map 1 is the map induced by extending a cycle from $A \triangleleft B$ to $B$ using the interior tensor product construction given on page 38 of [16]. We apply this construction to the natural inclusion map, denoted $\alpha : C_0(X, \mathcal{K}) \subseteq \ell^\infty(X, \mathcal{K})$; where $\ell^\infty(X, \mathcal{K})$ is considered as compact adjointable operators over itself considered as a Hilbert $C^*$-module. To construct the left action of $C_0(Z)$ on $E \otimes_{\alpha} \ell^\infty(X, \mathcal{K})$ we take the
composition of $\psi : C_0(U) \to \mathcal{B}(E)$ and $\alpha_* : \mathcal{B}(E) \to \mathcal{B}(E \otimes_\alpha \ell^\infty(X,K))$, then extend them using Proposition 2.1 from [16]. This has the desired properties as $\alpha_*$ maps the compacts on $E$ into the compacts on the interior tensor product over $\alpha$ in this instance (by Lemma 4.6 in [16]).

- 4 is the restriction map induced in equivariant KK-theory by the inclusion of a point $\{x\}$ as an open subgroupoid of $G(X)$. This is an isomorphism by Lemma 4.7 from [25].

- We recall the construction of the natural inverse to 4 defined in Lemma 4.7 of [25] denoted here by $\Xi$. Given a cycle $(E, \phi, F)$, we can construct a cycle for $KK_{\phi A}^\top(C_0(Z), \ell^\infty(X,K))$ by taking the module $\ell^\infty(X,K)$, and the representation and operator, $\tilde{\phi}$ and $\tilde{F}$, as follows:

$$ (\tilde{\phi}(a)\xi)(x) := \phi(a(x))\xi(x), \quad (\tilde{F}\xi)(x) := F\xi(x). $$

We now define $\Xi$ to be the map that sends the cycle $(E, \phi, F)$ to $(\ell^\infty(X,K), \tilde{\phi}, \tilde{F})$. This map is the same as the map defined in the proof of Lemma 4.7 of [25] after stabilising the cycles by the degenerate standard cycle $(\tilde{H},0,0)$ (where $\tilde{H}$ is the standard Hilbert module over $\ell^\infty(X,K)$).

- We remark that 5 is the map induced on K-homology by including a point into $X$. In particular it maps the generating cycle $(\mathbb{C},1,0)$ to $(K,\psi,0)$, where $\psi : C_0(Z) \to K$ constructed as follows: let $q$ be a rank one projection in $K$ and fix a point $x \in X$, then define $\psi(f) = f(x)q$. This turns the triple $(K,\psi,0)$ into a Kasparov cycle.

- Combining these last two points it is possible to describe the maps 2,3 and their compatibility with 5: We observe that 3 maps $(\mathbb{C},1,0)$ to $\Xi \circ 5$, which is $(\ell^\infty(X,K),\psi,0)$. We can now take 2 similarly as $\psi$ restricts to $C_0(X,K)$ and so we can define 2 by mapping $(\mathbb{C},1,0)$ to $(C_0(X,K),\psi,0)$. This cycle maps using 1 to $(\ell^\infty(X,K),\psi,0)$ (this is seen by considering the interior tensor product defined using an inclusion).

- We remark also that by the points above it is clear that this diagram commutes.

We now take the limit through the directed set of entourages $E \in \mathcal{E}$:

$$ K^\top(X \times X,C_0(X,K)) \xrightarrow{\cong} K^\top(\mathcal{G}_A, \ell^\infty(X,K)) \cong KK(\mathbb{C},\mathbb{C}) $$

where Theorem 3.8 [25] allows us to identify $K^\top(X \times X,C_0(X,K))$ with $K^\top(\mathcal{G}_A,C_0(X,K))$ and $K^\top(\mathcal{G}(X), \ell^\infty(X,K))$ with $K^\top(\mathcal{G}_A, \ell^\infty(X,K))$.

After this identification, we can observe that map 2 will induce an isomorphism in the limit at the level of KK-groups (as identified in the diagram). To verify this we first remark that $X$, as an $(X \times X)$-space, is an example of a cocompact classifying space for proper actions and so it is enough to compute the group $KK_{X \times X}(C_0(X),C_0(X,K))$.

We do this using the restriction map induced by including $\{x\}$ into $X \times X$, which splits the map 2 at the level of cycles. Then a straight line homotopy of operators shows that every cycle for $KK_{X \times X}(C_0(X),C_0(X,K))$ is homotopic to a cycle that is constant in each fiber. This shows 2 is an isomorphism in the limit.

Finally, armed with this diagram we can then conclude the result by observing that having an infinite coarse component in $X$ implies that including a point via $\Xi$ factors through the coarse K-homology of a ray and hence is the zero map.
We now prove Claim (2). Let \( X \) have only finite coarse components and let \( \mathcal{E} \) denote the metric coarse structure on \( X \). Now for every entourage \( E \in \mathcal{E} \), we can decompose \( E \) as \( \sqcup_{i \in I} E_i \), where each \( E_i \) is finite, in the following way. Consider the Rips complex \( P_{E,X} \). We know this decomposes into countably many finite, disjoint complexes \( P_{E,i,X} \) as we have assumed that we have no ray in any such Rips complex \( P_{E,X} \). By considering the 0-skeleton of this complex, which we denote by \( X_1 \), we obtain a disjoint decomposition \( X = \sqcup_{i \in I} X_i \), and a further decomposition of \( E = \sqcup E_i \), where \( E_i := E \cap (X_i \times X_i) \).

We now consider the coarse structure generated by \( E \), which is a substructure of \( \mathcal{E} \). Denote the algebraic Roe algebra \( C_E[X] \) of finite propagation operators in the coarse structure generated by \( E \) and by \( C_{E,x}^{\ast} \) its closure as bounded operators on \( \ell^2(X) \). Now we observe \( C_{E,x}^{\ast} \) is the closure of elements supported precisely in each finite \( E_i \), that is the uniform part of the product \( \prod_i C_{E,i}^{\ast} \). Hence it sits naturally in the following diagram:

\[
0 \to \bigoplus_i M_{n_i} \to C_{E,x}^{\ast} \to \prod_i M_{n_i}
\]

where \( n_i \) is the size of the \( E_i \) component of the zero skeleton of the Rips complex \( P_{E,X} \). We remark also that the sum \( \bigoplus_i M_{n_i} \) is the intersection of the compact operators \( K(\ell^2(X)) \) with \( C_{E,x}^{\ast} \). Denote this ideal, to keep track of objects clearly in a limiting argument, by \( K_E \).

The inclusion of the sum into the product induces an injection at the level of K-theory; so, for each \( E \in \mathcal{E} \), we now have an injection on K-theory induced by the map \( K_E \) into \( C_{E,x}^{\ast} \). To complete the proof consider the directed system of diagrams \( \{ K_E \to C_{E,x}^{\ast} \}_{E \in \mathcal{E}} \), which converge to the diagram \( K \to C^{\ast}X \). These induce natural K-theory maps, which are injections for every \( E \in \mathcal{E} \). Whence the map induced by the inclusion \( K \to C^{\ast}X \) is injective at the level of K-theory.

We now have the tools in Lemma 33 to prove the following Theorem:

**Theorem 34.** Let \( X \) be a uniformly discrete bounded geometry metric space that admits a fibred coarse embedding into Hilbert space. Then the coarse Baum-Connes assembly map for \( X \) is injective.

**Proof.** For a uniformly discrete space \( X \) with bounded geometry we know that the ghost ideal \( I_G \) fits into the sequence:

\[
0 \to I_G \to C^{\ast}X \to A_\emptyset \rtimes G(X)|_{\partial_B X} \to 0.
\]

This addition makes the sequence exact and thus it gives rise to the ladder in K-theory and K-homology where the rungs are the assembly maps defined in [21] (omitting coefficients):

\[
\begin{array}{cccccc}
& K_1(C^{\ast}_r(G(X)|_{\partial_B X})) & \longrightarrow & K_0(I_G) & \longrightarrow & K_0(C^{\ast}_r(G(X))) \\
\mu_{\text{bdry}} & \downarrow{2} & & & \downarrow{\mu} & \\
& K_1^{\text{top}}(G(X)|_{\partial_B X}) & \longrightarrow & K^{\text{top}}(X \times X) & \longrightarrow & K_0^{\text{top}}(G(X)) \\
\mu_{\text{bdry}} & \downarrow{0} & & & \downarrow{0} & \\
& 0 & & & 0 &
\end{array}
\]

Corollary 1 allows us to conclude that \( \mu_{\text{bdry}} \) is an isomorphism. Now we treat cases. If \( X \) has an infinite coarse component then Lemma 33 (1) implies the map \( K^{\text{top}}(X \times X) \to K^{\text{top}}(G(X)) \) is the zero map. Now assume that \( x \in K^{\text{top}}(G(X)) \) maps to 0 in \( K_0(C^{\ast}_r(G(X)) \). Then it maps to 0 in \( K^{\text{top}}(G(X)|_{\partial_B X}) \) as \( \mu_{\text{bdry}} \) is an isomorphism. As the second line is exact, this implies it comes an element in \( K^{\text{top}}(X \times X) \). As the map labelled 1 is the zero map, \( x \) must be 0.
If $X$ has only finite coarse components, then by Lemma 33 the map $2$ is injective and so we can conclude injectivity of $\mu$ by the Five Lemma.

We can also describe the obstructions to $\mu_*$ being an isomorphism when $X$ admits a fibred coarse embedding into Hilbert space.

**Proposition 35.** Let $X$ be a uniformly discrete metric space with bounded geometry such that $X$ admits a fibred coarse embedding into Hilbert space. Then the inclusion of $K$ into $I_G$ induces an isomorphism on $K$-theory if and only if $\mu_*$ is an isomorphism. In addition, if $X$ has only finite coarse components then every ghost projection in $C^*X$ is compact if and only if $\mu_0$ is an isomorphism.

**Proof.** We consider the diagram from the proof of Theorem 34.

\[
\begin{align*}
    &\quad K_1(C^*_r(G(X)|\partial^i X)) \xrightarrow{\mu^{bdr}} K_0(I_G) \quad K_0(C^*_r(G(X))) \xrightarrow{\mu} K_0(C^*_r(G(X)|\partial^i X)) \quad K_1(I_G) \\
    &\quad \xrightarrow{2} K_1^{top}(G(X)|\partial^i X) \quad \xrightarrow{\mu^{bdr}} K^{top}(X \times X) \quad K_0^{top}(G(X)) \quad \xrightarrow{\mu} K_0^{top}(G(X)|\partial^i X) \quad 0
\end{align*}
\]

When $X$ fibred coarsely embeds into Hilbert space, we know that $\mu^{bdr}$ is an isomorphism. The result then follows from the Five Lemma.

Now assume $X$ has only finite coarse components: As in Lemma 33 we will work using the generators of the metric coarse structure by considering the algebra $C^*_R X := C^*_R(\Lambda R)$ and its intersection, $K_R$, with the ideal of compact operators on $\ell^2 X$. We now want to construct a tracelike map, similar to the maps defined in [1, 3, 30], on each scale $R > 0$.

For any $T \in C[X]$ of propagation $R$, as in the proof of Lemma 33(2), we have a natural decomposition $T = \prod_{i \in \Lambda_R} T_i$, where each $T_i \in C^*_R X_i$ is the component of $T$ restricted to each of $X_i$ (Recall $X_i$ denotes the $i$-th component of the $0$-skeleton of the Rips complex $P_E X$).

We define a map from $C^*_R X$ to $\prod_{i \in \Lambda_R} C^*_R X_i$ that sends each $T$ to $\prod T_i$ and we denote the map it induces on $K$-theory by $\overline{\text{Tr}}_R$.

\[
\text{Tr}_R : K_0(C^*_R X) \rightarrow \prod_{i \in \Lambda_R} K_0(C^*_R X_i).
\]

To complete the proof, let $p$ be a noncompact ghost projection in $C^*_R X$, let $\{T_n\}_n$ be operators of propagation $R_n$ that approximate $p$ and let $\epsilon < \frac{1}{2}$. Observe that for some large $n$, there are quasiprojections $T_n$ such that $\|T_n - p\| \leq \epsilon$ that determine a $K$-theory class for $C^*_R(X)$, which maps to a non-zero class under $\text{Tr}_{R_n}$. Suppose that $p$ was equivalent to some compact operator $q$ then we take representatives $q_n \in K_{R_n}$ with $\|q_n - q\| \leq \epsilon$. Now, $\overline{\text{Tr}}_{R_n}([q]) = 0$ but $\text{Tr}_{R_n}([T_n]) \neq 0$: this gives the desired contradiction.

If $\mu$ is an isomorphism, it follows that every ghost projection is equivalent to a compact operator on $K$-theory, that is $K_*(\Lambda) \cong K_*(I_G)$. Hence, any ghost projection in $C^*(X)$ vanishes under $\overline{\text{Tr}}_R$ for a cofinal subset of $R > 0$. This happens if and only if the ghost projection is compact [30].

A natural corollary of this concerns coarsely embeddable spaces, where we can now show:

**Corollary 36.** If $X$ coarsely embeds into Hilbert space then $K_*(I_G) \cong K_*(\Lambda)$. □

**Remark 37.** This is motivated by some remarks of Roe [21, Chapter 11], where he asks in what way coarse embeddability interacts with the existence of ghost operators. This was settled by Roe and Willett in [22], where it was shown that the existence of non-compact...
ghost operators is equivalent to not having property A. Corollary 36 shows however that for spaces that do not have property A, but are coarsely embeddable into Hilbert space (such as spaces constructed in [1, 15]) that the ideals $\mathcal{K}$ and $I_G$ are not equal, but do have the same K-theory. For the spaces constructed in [1, 15] we can go further and say that whilst $I_G \neq \mathcal{K}$, there are no non-compact ghost projections in $I_G$ (This follows from Proposition 35, as the spaces referenced above have only finite coarse components).

5.2. **Box spaces of residually finite discrete groups.** Let $\Gamma$ be a finitely generated residually finite discrete group, and let $\mathcal{N} := \{N_i\}_{i \in \mathbb{N}}$ be a family of nested finite index normal subgroups with trivial intersection. Fix a generating set $S$ for $\Gamma$. Then we can construct a metric space $\Box \Gamma$, called the *box space* of $\Gamma$ with respect to the family $\mathcal{N}$ by considering the coarse disjoint union of the sequence: $\{\text{Cay}(\frac{\Gamma}{N_i}, S)\}_i$.

It is well known [21, Proposition 11.26] that a coarse embedding of $\Box \Gamma$ into Hilbert space implies that $\Gamma$ is a-T-menable. Using Theorem 2.2 from [6], it is possible to show that if $\Gamma$ is a-T-menable, then any box space of $\Gamma$ admits a fibred coarse embedding. The following Lemma will allow us to prove the converse of Theorem 2.2 from [6] using Corollary 1.

**Lemma 38.** Let $\Box \Gamma$ be a box space of a finitely generated residually finite group $\Gamma$. Then the boundary algebra $C(\partial \beta X)$ admits a $\Gamma$-invariant state.

**Proof.** Let $\omega \in \partial \beta \mathcal{N}$ and consider the function:

$$\mu(f) = \lim_{\omega} \frac{1}{|X_i|} \sum_{x \in X_i} f(x_i).$$

Where the limit above is the ultralimit in $\omega$. Clearly, $\mu(1_{\beta X}) = 1$, $\mu$ is linear and positive, whence $\mu$ is a state on $C(\partial \beta X)$. Let $g \in \Gamma$, now we check invariance:

$$\mu(g \circ f) = \lim_{\omega} \frac{1}{|X_i|} \sum_{x \in X_i} f(gx_i).$$

After relabelling the elements $x \in X_i$ by $g^{-1}x'$, we now see that $\mu(f) = \mu(g \circ f)$. The result now follows. \qed

It is well known that the boundary groupoid associated to a box space $\Box \Gamma$ decomposes as $\partial \beta \Box \Gamma \rtimes \Gamma$, for a proof see [9]. We now recall Corollary 5.12 of [4]:

**Proposition 39.** Let $\Gamma$ be a discrete group acting on a space $X$ with an invariant probability measure $\mu$. Then the action is a-T-menable if and only if $\Gamma$ is a-T-menable. \qed

Now Lemma 38 plus the above Proposition prove:

**Theorem 40.** Let $\Gamma$ be a residually finite discrete group. If the box space $\Box \Gamma$ admits a fibred coarse embedding into Hilbert space then the group $\Gamma$ is a-T-menable. \qed

The author would like to remark that that Theorem 40 was proved independently using different methods in [5].

**References**

[1] Goulnara Arzhantseva, Erik Guentner, and Ján Špakula. Coarse non-amenability and coarse embeddings. *Geom. Funct. Anal.*, 22(1):22–36, 2012.

[2] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan’s property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
[3] Bruce Blackadar. *K*-theory for operator algebras, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[4] Nathanial P. Brown and Erik P. Guentner. New C*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.*, 45(6):1181–1193, 2013.

[5] Xiaoman Chen, Qin Wang, and Xianjin Wang. Characterization of the Haagerup property by fibred coarse embedding into Hilbert space. *Bull. Lond. Math. Soc.*, 45(5):1091–1099, 2013.

[6] Xiaoman Chen, Qin Wang, and Guoliang Yu. The maximal coarse Baum-Connes conjecture for spaces which admit a fibred coarse embedding into Hilbert space. *Adv. Math.*, 249:88–130, 2013.

[7] Marius Dadarlat and Erik Guentner. Uniform embeddability of relatively hyperbolic groups. *J. Reine Angew. Math.*, 612:1–15, 2007.

[8] Steven C. Ferry and Shmuel Weinberger. A coarse approach to the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 147–163. Cambridge Univ. Press, Cambridge, 1995.

[9] Martin Finn-Sell and Nick Wright. The coarse Baum-Connes conjecture, boundary groupoids and expander graphs. *Adv. Math.*, 259C:306–338, 2014.

[10] Erik Guentner, Romain Tessera, and Guoliang Yu. A notion of geometric complexity and its application to topological rigidity. *Invent. Math.*, 189(2):315–357, 2012.

[11] N. Higson. Bivariant *K*-theory and the Novikov conjecture. *Geom. Funct. Anal.*, 10(3):563–581, 2000.

[12] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[13] Nigel Higson. Counterexamples to the coarse Baum-Connes conjecture. *Preprint*, 1999.

[14] Nigel Higson and Gennadi Kasparov. Operator *K*-theory for groups which act properly and isometrically on Hilbert space. *Electron. Res. Announc. Amer. Math. Soc.*, 3:131–142 (electronic), 1997.

[15] A. Khukhro. Box spaces, group extensions and coarse embeddings into Hilbert space. *J. Funct. Anal.*, 263(1):115–128, 2012.

[16] E. C. Lance. *Hilbert *C*-modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[17] Pierre-Yves Le Gall. Théorie de Kasparov équivariante et groupoides. I. *K*-Theory, 16(4):361–390, 1999.

[18] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2010. With an appendix by Jonathan D. Rogawski, Reprint of the 1994 edition.

[19] Hervé Oyono-Oyono and Guoliang Yu. *K*-theory for the maximal Roe algebra of certain expanders. *J. Funct. Anal.*, 257(10):3239–3292, 2009.

[20] Jean Renault. Groupoid cocycles and derivations. http://arxiv.org/abs/1201.4599, 2012.

[21] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[22] John Roe and Rufus Willett. Ghostbusting and property A. *J. Funct. Anal.*, 266(3):1674–1684, 2014.

[23] J. Rosenberg. *C*-algebras, positive scalar curvature and the Novikov conjecture. II. 123:341–374, 1986.

[24] I. J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, 44(3):522–536, 1938.

[25] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topoogy*, 41(4):807–834, 2002.

[26] Jean-Louis Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.

[27] Jean-Louis Tu. The Baum-Connes conjecture for groupoids. In *C*-algebras (Münster, 1999), pages 227–242. Springer, Berlin, 2000.

[28] Jean-Louis Tu. The coarse Baum-Connes conjecture and Groupoids. II. *New York Journal of Mathematics*, 18, 2012.

[29] Rufus Willett. Some notes on property A. In *Limits of graphs in group theory and computer science*, pages 191–281. EPFL Press, Lausanne, 2009.

[30] Rufus Willett and Guoliang Yu. Higher index theory for certain expanders and Gromov monster groups, I. *Adv. Math.*, 229(3):1380–1416, 2012.

[31] Rufus Willett and Guoliang Yu. Higher index theory for certain expanders and Gromov monster groups, II. *Adv. Math.*, 229(3):1762–1803, 2012.
[32] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

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