Abstract

The mixing between the $f_2(1270)$, the $f_2(1525)$, and the $2^{++}$ glueball is determined and tested. The mass and the hadronic decay widths of the $G_2$ and the branching ratio $B(J/\psi \to \gamma G_2)$ are predicted.
1 Introduction

The glueball states are solutions of nonperturbative QCD. The study of $2^{++}$ glueball has a long history. The MIT bag model predicts $m(2^{++}) = 1.29\text{GeV}$ [1]. In Ref. [2] the mass of the $2^{++}$ glueball state is predicted in the range of $1.45 - 1.87\text{GeV}$. In Refs. [2,3] it is argued that there are glueball components in the $f_2(1270)$ and the $f_2(1525)$ mesons. In Ref. [4] the smallness of the ratio of the helicity amplitude $y = \frac{I_2}{I_0}$ of the decay $J/\psi \to \gamma f_2(1270)$ is explained by that the meson $f_2(1270)$ contains substantial component of $2^{++}$ glueball. There are many studies on $2^{++}$ glueball [5]. The mass of the $2^{++}$ glueball has been calculated by quenched lattice QCD to be about $2.39\text{GeV}$ [6]. On the other hand, many $2^{++}$ isoscalar states have been discovered [7]: $f_2(1430), f_2(1565), f_2(1640), f_2(1810), f_2(1910), f_2(1950), f_2(2010), f_2(2150),...$. Some of them are radial excitations of the $f_2(1270), f'(1525)$. It is very possible that a $2^{++}$ glueball is among them. Of course, a physical glueball state contains both $|q\bar{q}>$ and $|gg>$ states.

In this paper the mixing of the $f_2(1270), f'(1525)$ and the $2^{++}$ glueball is studied and tested. The mass and the hadronic decay width of the $2^{++}$ glueball $G_2$ and the branching ratio of $J/\psi \to \gamma G_2$ are predicted. These results can be used to identify the $2^{++}$ glueball.
2 Mixing of the $f_2(1270)$, $f'(1525)$ and the $G_2$ glueball
and the mass of the $G_2$

According to QCD, two expansions are applied in this study. One is the chiral expansion, the expansion of the current quark mass, $m_q$, and the second is the $N_C$ expansion. In QCD it is known $g^2 \sim \frac{1}{N_C}$, where $g$ is the coupling constant of gluons and quarks. The $f_2(1270)$, $f'(1525)$ and the $G_2$ glueball are the eigen states of the mass matrix of the $f_8$, $f_0$ and the pure $2^{++}$ glueball $g_2$, where $f_8$, $f_0$ are the $2^{++}$ octet and singlet states. The mass matrix is expressed as

$$
\begin{pmatrix}
    m_1 & \Delta_2 \\
    \Delta_1 & m_2 & \Delta_3 \\
    \Delta_2 & \Delta_3 & m_3
\end{pmatrix},
$$

where $m_1 = m_{f_8}^2$, $m_2 = m_{f_0}^2$, $m_3 = m_{g_2}^2$, and $\Delta_1 = \langle f_8|m^2|f_0 \rangle$, $\Delta_2 = \langle f_8|m^2|g_2 \rangle$, $\Delta_3 = \langle f_0|m^2|g_2 \rangle$. From the quark model, we obtain

$$
m_1 = \frac{1}{3}(4m_{K^*}^2 - m_{a_2}^2),
$$
$$
m_2 = \frac{1}{3}(2m_{K^*}^2 + m_{a_2}^2).
$$

Using the two expansions, we have

$$
\Delta_1 \sim O(m_q), \quad \Delta_2 \sim O(m_q \frac{1}{N_C}), \quad \Delta_3 \sim O(\frac{1}{N_C})
$$
The masses of the $f_8$ and $f_0$ (2) are up to $O(m_\pi)$. The study presented in this paper is up to either $O(m_\pi)$ or $O(1/N_C)$. Obviously, the $\Delta_2$ is at higher order in the two expansions and it can be ignored

$$\Delta_2 = 0.$$  \hspace{1cm} (4)

$m_3, \Delta_1, \Delta_3$ are the three undetermined parameters. The $m^2_{f_2}$ and $m^2_{f'_2}$ are taken as inputs. Therefore, one more input is required.

The branching ratios of $f'_2(1525) \to K\bar{K}, \pi\pi$ are listed [9] as

$$B(f'_2(1525) \to K\bar{K}) = (88.7 \pm 2.2)\%,$$

$$B(f'_2(1525) \to \pi\pi) = (8.2 \pm 1.5) \times 10^{-3}.$$  \hspace{1cm} (5)

The $B(f'_2(1525) \to K\bar{K})$ is larger than the $B(f'_2(1525) \to \pi\pi)$ by two order of magnitudes. On the other hand, both are d-wave decays. The phase space of the $\pi\pi$ channel is much larger than the one of the $K\bar{K}$ channel

$$\frac{(1 - \frac{4m^2}{m^2_{f'_2}})^\frac{5}{2}}{(1 - \frac{4m^2}{m^2_{f'_2}})^\frac{5}{2}} = 3.61.$$  

Therefore, the magnitude of the amplitude of the $f'_2(1525) \to K\bar{K}$ is about 20 times of the one of the $f'_2(1525) \to \pi\pi$. The physical state of the $f'_2(1525)$ contains both the $q\bar{q}$ and the gluon-gluon components. It is reasonable to assume that the $\pi\pi$ is from the gluon-gluon component of the $f'_2(1525)$. Therefore, the $q\bar{q}$ component is dominated by the $s\bar{s}$. The
physical state of the $f_2'(1525)$ is expressed as

$$f_2'(1525) = a_2 f_8 + b_2 f_0 + c_2 g,$$  
(6)

$$f_8 = \frac{1}{\sqrt{3}} (u \bar{u} + d \bar{d} - 2 s \bar{s}),$$  
(7)

$$f_0 = \frac{1}{\sqrt{3}} (u \bar{u} + d \bar{d} + s \bar{s}).$$  
(8)

The $s \bar{s}$ dominance in the $f_2'(1525)$ leads to

$$a_2 = -b_2.$$  
(9)

Eq. (9) is another input for determining the mixing.

Now the mass of the physical glueball state $G_2$ and the mixing of the $f_2, f_2', G_2$ can be determined. After input the values of the $m_{f_8}^2, m_{f_0}^2, m_{f_2}^2, m_{f_2'}^2$ the three eigen equations are obtained

$$m_{G_2}^2 = m_3 + 0.1326,$$  
(10)

$$m_{G_2}^2 = 1.2345 + 7.5643(\Delta_1^2 + \Delta_3^2),$$  
(11)

$$7.5643(\Delta_1^2 + \Delta_3^2)\Delta_1^2 - 1.7604\Delta_1^2 - 0.7186\Delta_3^2 + 0.08447 = 0.$$  
(12)

Using the Eqs. (9,10-12), it is determined

$$\Delta_1 = m_1 - m_{f_2'}^2 = -0.1819 \text{ GeV}^2.$$  
(13)
Substituting Eq. (13) into Eqs. (11-12), we obtain

\[ \Delta_3 = 0.07368 \text{ GeV}^2, m_{G_2} = 1.429 \text{ GeV}, m_{g_2} = 1.382 \text{ GeV}. \]  

(14)

The mass of the physical $2^{++}$ glueball state is predicted. In Ref. [7] a state $I^G(J^{PC}) = 0^+(2^{++}) f_2(1430)$ which is listed. The value of $m_{G_2}$ predicted in this study is close to the results presented in Refs. [1,2], but lower than the value obtained by the quenched Lattice QCD [6].

The expressions of the three physical $2^{++}$ states are determined from the three eigen equations of the mass matrix (1)

\[ f_2(1270) = 0.246 f_8 + 0.7002 f_0 - 0.6702 g_2, \]  

(15)

\[ f_2'(1525) = -0.6421 f_8 + 0.6421 f_0 + 0.4189 g_2, \]  

(16)

\[ G_2 = 0.618 f_8 + 0.3451 f_0 + 0.7064 g_2. \]  

(17)

The $G_2$ state contains substantial $q\bar{q}$ components, the glueball component in the $f_2$ is large and in $f_2'$ is not negligible. In this study the $g_2$ is a pure glueball state and the $G_2$ is a new $2^{++}$ state. Without the pure glueball state $g_2$ the $G_2$ doesn’t exist.
3 The decays $f_2 \to \pi\pi$, $K\bar{K}$, $\eta\eta$ and $f_2' \to K\bar{K}$, $\eta\eta$

As a test of these results (15-17), the decays of $f_2(1270) \to \pi\pi$, $K\bar{K}$ are studied in the chiral limit. They are d wave decays and the decay widths are expressed as

$$\Gamma(f_2 \to \pi\pi) = |T|^2 m_{f_2} (a_1 + b_1)^2 (1 - \frac{4m^2_\pi}{m^2_{f_2}})^{\frac{3}{2}},$$  \hspace{1cm} (18)$$

$$\Gamma(f_2 \to K\bar{K}) = |T|^2 m_{f_2} \frac{1}{3} (2b_1 - a_1)^2 (1 - \frac{4m^2_K}{m^2_{f_2}})^{\frac{3}{2}},$$  \hspace{1cm} (19)$$

where $a_1$ and $b_1$ are the coefficients of Eq. (15) and $a_1 = 0.246$, $b_1 = 0.7002$. It is known that the pion and the kaon are Goldstone bosons and $m^2_\pi \propto m_u + m_d$ and $m^2_K \propto m_s + \frac{1}{2}(m_u + m_d)$. In the chiral limit the $|T|^2$ is independent of the current quark masses. This study predicts

$$\frac{\Gamma(f_2 \to K\bar{K})}{\Gamma(f_2 \to \pi\pi)} = 0.0548.$$  \hspace{1cm} (20)$$

The experimental value of this ratio is [7] $0.054(1 \pm 0.12)$. Theory agrees with the data very well.

Because of the mixing between $\eta$, $\eta'$ and the $0^{-+}$ glueball [8] the decay mode $\eta\eta$ is more complicated. However, it is known that the octet component dominates the $\eta$ state. In this study the $\eta$ is taken as an octet to calculate $\Gamma(f_2 \to \eta\eta)$. The decay width of the $f_2 \to \eta\eta$ is derived as

$$\Gamma(f_2 \to \eta\eta) = |T|^2 m_{f_2} \frac{1}{3} (b_1 - a_1)^2 (1 - \frac{4m^2_\eta}{m^2_{f_2}})^{\frac{3}{2}},$$  \hspace{1cm} (21)$$

$$\frac{\Gamma(f_2 \to \eta\eta)}{\Gamma(f_2 \to K\bar{K})} = 0.055.$$  \hspace{1cm} (22)$$
The experimental data \([7]\) is \(0.087(1\pm0.29)\). As mentioned above in Eq. (22) there are mixing between \(\eta, \eta'\) and the \(0^{-+}\) glueball. The theoretical result agrees with the experimental data within the experimental error reasonably well. Similarly, the ratio

\[
\frac{\Gamma(f_2' \to \eta\eta)}{\Gamma(f_2' \to \bar{K}K)} = 0.29
\]

(23)

is obtained. The data of this ratio are following

\(0.069(1 \pm 0.17)[9]\) \(0.33(1 \pm 0.10)[10]\), \(0.12(1 \pm 0.23)[7]\).

4 The decays \(G_2 \to \pi\pi, K\bar{K}, \eta\eta\)

Now we need to study the decays of \(G_2(1429) \to \pi\pi, K\bar{K}, \eta\eta\). Because of \(g^2 \sim \frac{1}{N_C}\) the hadronic decays of the glueball components \(|gg\rangle\) of these states are in higher order in the \(N_C\) expansion and suppressed. Therefore, the hadronic decays are the decays of their \(q\bar{q}\) components only. Eq. (17) shows that the \(G_2(1429)\) state contains large \(|q\bar{q}\rangle\) components. The hadronic decay width of the \(G_2(1429)\) state won’t be small. It is reasonable to assume that in the chiral limit the \(|T|^2\) in Eqs. (18,19,21) are about the same. This assumption can be tested by calculating the \(\Gamma(f_2' \to K\bar{K})\) by inputting the \(\Gamma(f_2 \to K\bar{K})\). Replacing the quantities of the \(f_2\) in Eqs. (18,19) by the ones of the \(f_2'\), the corresponding decay widths
for the $f'_2$ are determined.

$$\frac{\Gamma(f'_2 \to K\bar{K})}{\Gamma(f_2 \to K\bar{K})} = 8.59$$

(24)

is obtained. The data[7] is 7.63(1 ± 0.23). Theory agrees with data within the experimental errors. Now the same $|T|^2$ is used to calculate the decay widths of $\Gamma(G_2 \to \pi\pi, K\bar{K}, \eta\eta)$. The formulas of the decay widths are obtained

$$\Gamma(G_2 \to \pi\pi) = |T|^2m_{G_2}(a_3 + b_3)^2(1 - \frac{4m^2_{\pi}}{m^2_{G_2}})^\frac{5}{2},$$

(25)

$$\Gamma(G_2 \to K\bar{K}) = |T|^2m_{G_2}\frac{1}{3}(2b_3 - a_3)^2(1 - \frac{4m^2_K}{m^2_{G_2}})^\frac{5}{2},$$

(26)

$$\Gamma(G_2 \to \eta\eta) = |T|^2m_{G_2}\frac{1}{3}(b_3 - a_3)^2(1 - \frac{4m^2_{\eta}}{m^2_{G_2}})^\frac{5}{2}.$$  

(27)

The $|T|^2$ is determined by $\Gamma(f_2 \to \pi\pi) = 158.1(1 \pm 0.04)\text{MeV}$ [7]. The numerical results are

$$\Gamma(G_2 \to \pi\pi) = 189(1 \pm 0.04)\text{MeV}, \quad \Gamma(G_2 \to K\bar{K}) = 0.23(1 \pm 0.04)\text{MeV},$$

$$\Gamma(G_2 \to \eta\eta) = 1.82(1 \pm 0.04)\text{MeV}.$$  

(28)

The total decay width of the $G_2(1429)$ is 191 MeV. Because the coefficients, $2b_3 - a_3, b_3 - a_3,$ and the phase space of the $K\bar{K}$ and $\eta\eta$ channels are smaller the decay modes of $K\bar{K}$ and $\eta\eta$ are strongly suppressed. The $\pi\pi$ decay mode dominates the hadronic decays of the $G_2$ state. In Ref. [11] a $f_2(1430)$ state with the width of 150 ± 50MeV has been reported.
5 The decay $J/\psi \to \gamma G_2$

In QCD the radiative decay of the $J/\psi$ is described as $J/\psi \to \gamma gg$. Therefore, a state with larger glueball component should have larger production rate in $J/\psi$ radiative decay. Using the mixing (15-17), we can study the branching ratios of $J/\psi \to \gamma f_2, \gamma f'_2, \gamma G_2$. In Ref. [4] the expression of the decay width of $J/\psi \to \gamma g_2$ is presented

$$\Gamma(J/\psi \to \gamma g_2) = \frac{128\pi\alpha}{81}\alpha_s^2(m_c)G^2(0)\psi^2_f(0)c^2\frac{1}{m_c^4}(1 - \frac{m^2}{m_J^2})\{T_0^2 + T_1^2 + T_2^2\},$$

where $\psi_J(0)$ is the wave functions of $J/\psi$ at origin, $G(0)$ is a parameter related to the 2++ glueball $g_2$ [4], $m$ is the mass of the physical state whose branching ratio is going to calculate, and $c$ is the coefficient of the glueball component of the states of $f_2, f'_2$ and $g_2$ respectively.

$$T_0 = -\frac{2}{\sqrt{6}}(A_2 + p^2A_1), \quad T_1 = -\frac{\sqrt{3}}{m_J}(EA_2 + mp^2A_3), \quad T_2 = -2A_2,$$

$$E = \frac{1}{2m}(m_J^2 + m^2), \quad p = \frac{1}{2m}(m_J^2 - m^2),$$

$$A_1 = -a\frac{2m^2 - m_J(m_J - 2m_c)}{m_c m_J[m_c^2 + \frac{1}{4}(m_J^2 - 2m^2)]}, \quad A_2 = -a\frac{1}{m_c}\{\frac{m^2}{m_J} - m_J + 2m_c\},$$

$$A_3 = -a\frac{m^2 - \frac{1}{2}(m_J - 2m_c)^2}{m_c m_J[m_c^2 + \frac{1}{4}(m_J^2 - 2m^2)]}, \quad a = \frac{16\pi}{3\sqrt{3}}\frac{\sqrt{m_J}}{m_c^2}. \quad (30)$$

Replacing $m_c, m_J, m_Q$ by $m_b, m_\Upsilon$, and $Q_b$ respectively in Eqs. (29-30), the decay $\Upsilon(1S) \to \gamma f_2$ is studied[12]. Theory agrees with data very well.

The ratios of the helicity amplitudes of the $J/\psi \to \gamma f_2$

$$x = \frac{T_1}{T_0}, \quad y = \frac{T_2}{T_0},$$
have been measured. The early measurements show

\[ x = 0.88 \pm 0.11, \ y = 0.04 \pm 0.14 \ \text{CrystalBall}[13], \]

\[ x = 0.81 \pm 0.16, \ y = 0.02 \pm 0.15 \ \text{MarKII}[13], \]

\[ x = 0.6 \pm 0.3, \ y = 0.3 \pm 0.6 \ \text{Pluto}[13]. \]

The BES Collaboration has reported following results [14]

\[ x = 0.89 \pm 0.02 \pm 0.10, \ y = 0.46 \pm 0.02 \pm 0.19. \]

The values of x obtained by BES [14] is consistent with other measurements [13]. However, the value of y obtained by BES Collaboration is much larger. As pointed in Ref. [4], the value of y and the decay rate (29,30) are very sensitive to the value of \( m_c \). In Ref. [4] small \( y = 0.04 \) and \( x = 0.66 \) are obtained by taking \( m_c = 1.3 \text{GeV} \).

The branching ratios of \( J/\psi \rightarrow \gamma f_2, \ \gamma f'_2 \) are measured [7]

\[ B(\ J/\psi \rightarrow \gamma f_2) = (1.43 \pm 0.11) \times 10^{-3}, \ B(\ J/\psi \rightarrow \gamma f'_2) = (4.5^{+0.7}_{-0.4}) \times 10^{-4}. \]

Using the Eqs. (29,30) and choosing the value of the \( m_c \) we obtain

1. Taking \( m_c = 1.3 \text{GeV} \) and inputting \( B(\ J/\psi \rightarrow \gamma f_2) \), we obtain

for the decay \( J/\psi \rightarrow \gamma f_2 \)

\[ x = 0.66, \ y = 0.04; \]
for the decay $J/\psi \rightarrow \gamma f'_2$

$$x = 0.79, \ y = 0.28;$$

for the decay $J/\psi \rightarrow \gamma G_2$

$$x = 0.74, \ y = 0.20;$$

$$B(J/\psi \rightarrow \gamma f'_2) = 1.04(1 \pm 0.08) \times 10^{-3},$$

$$B(J/\psi \rightarrow \gamma G_2) = 2.45(1 \pm 0.08) \times 10^{-3}. \ \ \ (31)$$

The branching ratio of $J/\psi \rightarrow \gamma f'_2$ is greater than the data $(4.5^{+0.7}_{-0.4}) \times 10^{-4}$ [6].

2. Taking $m_c = 1.5\text{GeV}$ and inputting $B(J/\psi \rightarrow \gamma f_2)$, we obtain

for the decay $J/\psi \rightarrow \gamma f_2$

$$x = 0.7, \ y = 0.37,$$

they are consistent with the data of Ref. [14];

for the decay $J/\psi \rightarrow \gamma f'_2$

$$x = 0.84, \ y = 0.55;$$

for the decay $J/\psi \rightarrow \gamma G_2$

$$x = 0.79, \ y = 0.47;$$
\[
B(J/\psi \to \gamma f'_2) = 0.67(1 \pm 0.08) \times 10^{-3},
\]
\[
B(J/\psi \to \gamma G_2) = 1.77(1 \pm 0.08) \times 10^{-3}.
\]

(32)

The branching ratio of \(J/\psi \to \gamma f'_2\) is closer to the data [6]. There are other two measurements

\[
B(J/\psi \to \gamma f'_2) = (5.6 \pm 1.4 \pm 0.9) \times 10^{-3}[15], \quad (6.8 \pm 1.6 \pm 1.4) \times 10^{-3}[16].
\]

The values of \(x\) and \(y\) of \(J/\psi \to \gamma f_2\) and \(B(J/\psi \to \gamma f'_2)\) obtained favors \(m_c = 1.5\)GeV.

In both cases the \(B(J/\psi \to \gamma G_2)\) predicted is greater than \(B(J/\psi \to \gamma f_2)\).

6 Summary

Using the argument of the \(N_C\) and chiral expansions, the mixing between \(f_8, f_0,\) and a \(2^{++}\) glueball state is determined. The mass of a new \(2^{++}\) state \(G_2\) is predicted. The predicted branching ratios of \(f_2 \to K \bar{K}, \eta \eta\) and \(f'_2 \to \eta \eta\) agree with data reasonably well. The hadronic decay widths of the new state \(G_2\) are predicted. The \(\pi \pi\) decay mode is dominant and the \(K \bar{K}\) and the \(\eta \eta\) modes are strongly suppressed. The predicted branching ratio of \(J/\psi \to \gamma G_2\) is larger than the branching ratio of \(J/\psi \to \gamma f_2\).
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