Integrable Systems and Geometry of Riemann Surfaces

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Abstract

We give a self-contained introduction to the relations between Integrable Systems and the Geometry of Riemann Surfaces. We start from a historical introduction to the topic of integrable systems. Afterwards, we study the polynomial solutions to the stationary KdV equation, giving concrete examples of the computations of the solutions on small degree ($N = 0, 1, 2$). We discussed the geometry of the so-called square eigen-functions for those cases. Later, we present the solutions in the general case, presenting a formula for the hyper-elliptic curve of genus $N$ that parametrizes the solutions, the $N$-solitons. We relate also our equations to the Lax hierarchy, and analyze the time evolution for the KdV. Using the scalar operator for the NLS equation, computed by Kamchatnov, Krankel and Umarov [17, 18, 19], we analyze such an equation following a similar approach as the one we used for the KdV equation in the first part of the manuscript. We present some original results obtained in that case.

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Introduction

Many equations of the mathematical-physics are obiquous and appear in many applications and in the physics of many nature phenomena. To understand nature, more sophisticated models are needed. However, *integrable systems* are good minimal models to describe and understand more complicated phenomena in nature, such as wave propagation. This is why we are interested in Integrable Systems, which can be solved by means of the inverse scattering transform (IST).

Among the most notable completely integrable partial differential equations we find the Korteweg-de Vries (KdV), the Non-linear Schrödinger (NLS), the Derivative Non-linear Schrödinger (dNLS) and the sine-Gordon (sG) equations.

In the early 1830s, the scottish engineer Scott Russell observed a *large solitary elevation* in a narrow Edinburgh channel, now called the *Scott Russell’s Aqueduct*. Russell observed the phenomenon years before the equation that describes this wave motion were discovered, and certainly he did not have any idea of the impact that his observation and experiments would have in physics, mathematics and modern technology [1].

Namely, the Korteweg-deVries equation is given, in its standard form, by

$$\frac{\partial q}{\partial t} + 6q \frac{\partial q}{\partial x} + \frac{\partial^3 q}{\partial x^3} = 0,$$

which describes the profile of a wave propagating in shallow water. The *one-soliton* solution derived by D.J. Korteweg and G. de Vries (1895) is given by:

$$q(x, t) = \frac{c}{2} \text{sech}^2 \left[ \frac{\sqrt{c}}{2} (x - ct - x_0) \right].$$

This is a solution in the infinite line, with infinite period. Korteweg and de Vries also found the traveling wave solution for the periodic case, and they found the solution expressed as an elliptic function,

$$q(x, t) = f(x - ct)$$

$$= -f_2 + (f_2 - f_3) \text{cn}^2 \left[ (x - ct - x_0) \sqrt{\left\{ \frac{f_1 - f_3}{2} \right\}} \right] m \quad (1)$$

where \( c \) is the propagation velocity, and \( f_1, f_2 \) and \( f_3 \) are roots of the polynomial \( F(f) := -f^3 + \frac{3}{2}f^2 + Af + B \), where \( A, B \) are constants of integration.
In part I of the manuscript, we consider the solutions of the periodic (in $x$) KdV equation as worked by Dubrovin, Novikov, Its, Krichever and Matveev [3, 4, 5, 6], but following Flaschka and Newell [14, 15]. The analysis is performed through the scalar eigenvalue problem associated to the KdV equation, namely, the stationary Schrödinger equation with potential $q = q(x)$, where $q(x)$ is a periodic solution of the stationary KdV equation [14, 15]. The case for the non-stationary solution is briefly discussed for the KdV equation in section 9.

In part II we develop the theory studied in part I but applied to the NLS equation. As opposed to the standard matrix approach of the Zakharov-Shabat (ZS) operator, we write this operator in scalar form as Kamchatnov, Kraenkel and Umarov in [17, 18, 19], where the authors express some matrix eigenvalue problems (associated to Lax’s operators) as a scalar eigenvalue problem, and proceed as in part I. We recover the infinite set of NLS fluxes and constants of motion as a recursion formula as it was done in the literature [16] and, at each step of recursion, we find two fluxes. One of them corresponds to the constants of motion, and the other one to the Lax hierarchy [22]. We also write the Riemann hyperelliptic curve associated to this equation.

In part III of this work, we include three appendices which are needed for the results in the parts I and II. The appendix A is the proof of one of the main results Theorem 7.6. The appendix B is the proof of a well known identity of symmetric functions, which is usually used in the literature, but here you can find a novel and elementary proof of that identity. Appendix C contains the computations of the derivatives of $E$, which are used to simplify expressions in the NLS computations in part II.

Part I

The KdV equation

1 KdV equation and the Complex Torus

Consider the KdV equation $q_t + 6qq_x + q_{xxx} = 0$ with the extra assumption that the solution is a fixed wave that maintains its shape $q(x, t) = f(x - ct)$.
(i.e. it is a travelling wave). Set \( s = x - ct \). Thus, the KdV equation can be expressed just in terms of \( f \) and its derivatives and it becomes \(-cf' + 6ff' + f''' = 0\). Integrating, we obtain \(-cf + 3f^2 + f'''' = A\), where \( A \) is a constant. Now, multiplying by \( f' \) and integrating again, it follows the equation \(-\frac{c}{2}f^2 + f^3 + \frac{1}{2}(f')^2 = Af + B\). The last equation corresponds to a Complex Torus or an Elliptic Curve \( E \) with coordinates \((f, f') \in \mathbb{C}^2\).

Using that \( f' = \frac{df}{ds} = \sqrt{2(-f^3 + \frac{c}{2}f^2 + Af + B)} \) and integrating we have that

\[
x - ct = s = \int_{P_0}^{P} \frac{df}{\sqrt{2(-f^3 + \frac{c}{2}f^2 + Af + B)}},
\]

with \( P \) a moving point and \( P_0 \) a chosen initial point in \( E \), and the inverse of this function is given as in equation (1).

## 2 The Lax pair for KdV equation

As it is well known from the literature, or the introductory texts [15, 26, 25], the Korteweg-deVries equation

\[
\frac{\partial q}{\partial t} + 6q\frac{\partial q}{\partial x} + \frac{\partial^3 q}{\partial x^3} = 0,
\]

(2)

is a completely integrable equation [27], and it is given by the compatibility condition of the two scalar differential equations [24]:

\[
\widehat{L}y = \lambda y,
\]

(3)

\[
\frac{\partial y}{\partial t} = \widehat{P}y,
\]

(4)

where \( \widehat{L} \) and \( \widehat{P} \) is a pair of differential operators,

\[
\widehat{L} = \frac{\partial^2}{\partial x^2} + q(x, t),
\]

(5)

\[
\widehat{P} = 4\frac{\partial^3}{\partial x^3} + 6q(x, t)\frac{\partial}{\partial x} + 3q_x(x, t),
\]

(6)

and \( \lambda \) is the spectral parameter. This pair of operators is known as the "Lax’s pair" for the KdV equation. They were called this way in honor to P. Lax after his discovering in 1968 [28].
Equation (3) is an eigenvalue problem for the function $y(x,t)$, which turns to be (Miraculously!) the stationary Schrödinger equation of Quantum Mechanics. Equation (4) is the evolution in time equation, or the $t$-flow equation.

Now, $q(x,t)$ in equation (5) is the solution to the KdV equation (2) if and only if

$$\frac{d\lambda}{dt} = 0,$$

We then say that the KdV equation is an *isospectral flow*.

## 3 Traveling wave solutions of the KdV equation

In this section, we consider stationary solutions of the KdV equation. We will study first the infinite and finite periodic solutions in traveling wave form. Then, we will consider the "0-soliton", "1-soliton", and the "2-soliton" solutions of the periodic KdV. Afterwards, we will consider time-dependent solutions. All the computations here are following [14, 15], from the original work of [3, 4, 5].

In the original paper by Korteweg and de Vries [2] the authors construct the equation for waves in shallow water, and also found the periodic and infinite-periodic solutions we are going to show here. They proceeded as follows, which is also a standard computation [25]. Consider the KdV equation (2), and consider the change of variable $u(x,t) = -q(x,t)$ to obtain the equation:

$$u_t - 6uu_x + u_{xxx} = 0$$

Consider thus, the solutions of the form

$$u(x,t) = f(\xi),$$

donde $\xi = x - ct$, with $c$ is a constant and it is the velocity of propagation. We get,

$$-cf' - 6ff' + f''' = 0.$$ 

Integrating twice,

$$\frac{1}{2}(f')^2 = \frac{c}{2} f^2 + f^3 + Af + B,$$

where $A$ and $B$ are constants of integration. Assuming $f, f', f'' \to 0$ as $|\xi| \to \infty$, we get $A = B = 0$ and

$$\frac{1}{2}(f')^2 = \frac{c}{2} f^2 + f^3.$$
which is a separable equation
\[ \frac{d\xi}{df} = \pm \frac{1}{f\sqrt{2f + c}}, \]
assuming that \(2f + c > 0\) to get real solutions. Integrating, we obtain the "1-soliton" solution
\[ q(x, t) = -u(x, t) = \frac{1}{2}c \text{ sech}^2 \left[ \frac{1}{2}\sqrt{c}(x - ct - x_0) \right], \]
which is a "bell-shaped" profile, as observed by Scott Russell in 1835.

If \(A \neq 0, B \neq 0\), we have equation (8), which is also separable, to get
\[ \frac{d\xi}{df} = \pm \frac{1}{\sqrt{2F(f)}}, \]
with \(F(f) = f^3 + \frac{2}{3}f^2 + Af + B\) and \(F(f) \neq 0\), with solution
\[ \xi = \xi_0 \pm \int_{f_0}^f \frac{1}{\sqrt{2F(f)}} d\hat{f}, \]
with \(f_0 = f(\xi_0)\), where \(\xi_0\) is an initial position for the limit of integration. This is an elliptic integral.

Now, if we consider that \(F(f) = 0\) has three distinct real roots, order as \(f_3 < f_2 < f_1\). And, if we take as our initial limit of integration \(f = f_3(\xi_3) = f_3\), we obtain:
\[ \xi = \xi_3 \pm \int_{f_3}^f \frac{1}{\sqrt{2(\hat{f} - f_1)(\hat{f} - f_2)(\hat{f} - f_3)}} d\hat{f}, \]
Under the change of variables,
\[ \hat{f} = f_3 + (f_2 - f_3) \sin^2 \theta, \quad (f_2 - f_3 \neq 0), \]
we obtain
\[ \xi(\phi) = \xi_3 \pm \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta, \]
with \(m = (f_2 - f_3)/(f_1 - f_3)\) and
\[ f = f_2 - (f_2 - f_3) \cos^2 \phi \]
relates the upper limits of the integrals (10) and (11).

Now, if in the integral
\[ \int_{0}^{\phi} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta, \]
we invert \( \xi = \xi(\phi) \) to \( \phi = \phi(\xi) \), and we define the "cnoidal-cosine",
\[ \text{cn}(\xi|m) = \cos(\phi(\xi)). \]

Then, substituting into equation (12) we obtain the periodic solution found by Korteweg and de Vries:
\[ u(x, t) = f(x - ct) = f_2 - (f_2 - f_3) \text{cn}^2 \left( \frac{\sqrt{f_1 - f_3}}{f} (x - ct - x_0) \right) m \] (13)

The period of the solution is
\[ P = 2 \int_{f_3}^{f_2} \frac{1}{\sqrt{2F(\hat{f})}} d\hat{f} = 2K(m)\sqrt{\frac{2}{f_1 - f_3}}, \]
where
\[ K(m) := 2 \int_{0}^{\pi} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta. \]

This is the only solution that can be computed by means of the traveling wave solution assumption. This is actually a single-phase solution of the KdV equation, with phase \( \theta = x - ct - x_0 \). With the work developed in [17, 18, 19] by Dubrovin, Novikov, Its, Faddeev, Matveev, more solutions can be developed, in particular, multi-phase solutions. We are going to follow a general scheme starting from the spectral problem (3) for the KdV equation and we develop multiphase solutions. In particular, we will recover equation (9), see section 6.1.

4 Stationary solutions of the KdV equation

Consider again the Korteweg-de Vries equation (2),
\[ q_t + 6qq_x + q_{xxx} = 0. \] (14)
We know that, to the KdV equation, we have an associated linear eigenvalue problem, equations (3) and (5), which imply the Schrödinger equation

\[ y'' + qy = \lambda y. \tag{15} \]

Here, we consider stationary solutions of the KdV equations, \( q = q(x, t_0) \) for a fixed time \( t_0 \), and we will recover solutions to equation (14) from the associated Riemann surface. Next, we will consider how this Riemann surface evolves in time. We start following the ideas of Flaschka [14].

Look for solutions of (15) of the form

\[ \phi = y^2. \tag{16} \]

These functions are called "squared eigenfunctions". In section 7, we obtain the following differential equation for \( \phi \)

\[ \phi''' + 4q\phi + 2q'\phi = 4\lambda\phi'. \tag{17} \]

We assume that the function \( \phi \) is also a polynomial function of \( \lambda \), i.e., \( \lambda \) just appears as a parameter. To set ideas, set \( \phi_N = \phi_N(x; \lambda) \) a polynomial of degree \( N \),

\[ \phi_N(x; \lambda) = (4\lambda)^N F_{-1} + (4\lambda)^{N-1} F_0 + (4\lambda)^{N-2} F_1 + \cdots + (4\lambda)^F_{N-2} + F_{N-1}, \]

thus, we prove the following lemma.

**Lemma 4.1.** The function \( \phi_N(x; \lambda) \) solves (17) if, and only if, \( F_N \) is constant (in \( x \)), i.e.,

\[ F_N = F_N(q(x), q'(x), q''(x), q'^3(x), \ldots, q^{(2N)}) = K \]

is independent of \( x \),

For a detailed proof, see Corollary 7.7.

In the meanwhile, we will show how the result works for \( N = 0, 1, 2 \).

## 5 The 0-, 1- and 2-soliton solutions for the KdV equation

### 5.1 The 0–soliton solution: \( N = 0 \)

In this instance,

\[ \phi_0(x; \lambda) = A_0(x), \]
is a zero-degree polynomial in \( \lambda \) (here, \( F_{-1} = A_0(x) \)). Substitute into (17), to get

\[
A''_0 + 4qA'_0 + 2q'A_0 = 4\lambda A'_0.
\]

On the right-hand-side of this equation, we have a first degree polynomial in \( \lambda \). Then:

\[
A'_0 = 0,
\]

\[
A'''_0 + 4qA'_0 + 2q'A_0 = 0.
\]

Hence, \( A_0(x) = A_0 \) is a constant, and \( 2q'A_0 = 0 \). Assuming \( A_0 \neq 0 \), then we must have

\[
q(x) = q_0
\]

is also a constant, which is the trivial solution to the KdV equation. This is to say, this is the 0—soliton solution.

### 5.2 The 1-soliton solution: \( N = 1 \)

Consider

\[
\phi_1(x; \lambda) = \lambda A_0 + A_1(x),
\]

where \( A_1(x) \) has to be found. Substitute \( \phi_1 \) into equation (17), to get

\[
P[\phi_1] = 4\lambda \phi'_1,
\]

where

\[
P = \frac{d^3}{dx^3} + 4q \frac{d}{dx} + 2q_x.
\]

We then have,

\[
A'''_1 + 4qA'_1 + 2q'(\lambda A_0 + A_1(x)) = 4\lambda A'_1.
\]

At \( O(\lambda) \), \( 2A'_1 = q'A_0 \), setting \( A_0 = 2 \), we have

\[
A'_1 = q'
\]

\[
A_1(x) = q(x) + q_1,
\]

with \( q_1 \) is a constant of integration. Hence:

\[
\phi_1(x; \lambda) = 2\lambda + (q(x) + q_1),
\]
At $O(\lambda^0) = O(1)$,
\[ A_1'' + 4qA_1' + 2q'A_1 = 0, \]
\[ q'' + 4qq' + 2q'(q + q_1) = 0, \]
i.e.,
\[ q'' + 6qq' + 2q_1q' = 0. \]
which is the stationary KdV equation! Notice this equation can be written as:
\[ q'' + 3q^2 + 2q_1q = K = \text{constant}, \tag{23} \]
which is a constant flux for the stationary KdV equation.

### 5.3 The 2-soliton solution: $N = 2$

We define
\[ \phi_2(x; \lambda) = \alpha_2\lambda \phi_1 + A_2(x), \tag{24} \]
and this formula will be used as a recursion formula. Here, we require to find $A_1(x)$ and the appropriate constant $\alpha_2$. To set ideas, consider $q_1 = 0$ in (22).

Use (18) in (24) to get:
\[ \phi_2(x; \lambda) = \alpha_2 A_0\lambda^2 + \alpha_2\lambda A_1(x) + A_2(x). \]

Substitute this $\phi_2$ into (17), to get
\[
P[\phi_2] = 4\lambda\phi_2'
\]
\[
P[\alpha_2 A_0\lambda^2 + \alpha_2\lambda A_1 + A_2] = 4\lambda(\alpha_2 A_0 + \alpha_2\lambda A_1 + A_2)'.
\]

Since $A_0$ = constant, and by linearity of $P$,
\[
\alpha_2\lambda^2 P[A_0] + \alpha_2\lambda P[A_1] + P[A_2] = 4\alpha_2\lambda^2 A_1' + 4\lambda A_2'.
\]

Now, we have to determine $A_2(x)$ and $\alpha_2$. Consider terms of the order $O(\lambda^2)$ in the previous equation. We have $\alpha_2 P[A_0] = 4\alpha_2 A_1'$. Since $A_0 = 2$ is a constant, and assuming $\alpha_2 \neq 0$, we have $q_x = A_1'$, which holds identically since $A_1(x) = q(x) + q_1$.

At order $O(\lambda)$, we have $\alpha_2 P[A_1] = 4A_1'$. Since $A_1(x) = q(x) + q_1$, and using the definition of $P$, equation (19), we have
\[
\alpha_2(q'' + 4qq' + 2q'[q + q_1]) = 4A_2'.
\]
Choose $\alpha_2 = 4$. Hence $q'' + 6qq' + 2q_1q' = A'_2$ i.e., $A'_2 = (q'' + 3q^2 + 2q_1q)'$. Hence:

$$A_2 = q'' + 3q^2 + 2q_1q + q_2,$$

where $q_2$ is another constant. This is again a constant flux for the KdV equation, as found in equation (23).

At order $O(\lambda^0) = O(1)$, this is to say, $P[A_2] = 0$, i.e.,

$$A'''_2 + 4qA'_2 + 2q_2A_2 = 0.$$  \hspace{1cm} (26)

Substitute (25) into the last equation (26), to get

$$q^{(5)} + 10qq''' + 20q'q'' + 30q^2q' + 2q_1(q''' + 6qq') + 2q_2q' = 0.$$  \hspace{1cm} (27)

The last two terms correspond to the KdV flow and the traveling wave flow. Then, setting $q_1 = q_2 = 0$, we don not lose any information. Hence,

$$q^{(5)} + 10qq''' + 20q'q'' + 30q^2q' = 0,$$

which is the fifth order KdV equation! Or the third order KdV in the Lax’s hierarchy

$$q^{(4)} + 5(q')^2 + 10qq'' + 10q^3 = \text{constant},$$

which is also a KdV constant flux. We then have.

$$\phi_2(x, \lambda) = 8\lambda^2 + 4\lambda(q(x) + q_1) + (q'' + 3q^2 + 2q_1q + q_2)$$  \hspace{1cm} (28)

If $q_1 = q_2 = 0$, this function corresponds to Flaschka’s solution [14].

### 6 Recovering the solutions of the KdV

In this section we will study how to recover the solution to the (stationary) KdV equation out of its corresponding Riemann Surface. We will do it for the cases $N = 0, 1$ and 2. For the $N = 0$ case, there is no much to do, $q(x) = q_0 = \text{constant}$, and this is the trivial solution to the KdV equation.
6.1 The 1–soliton solution: $N = 1$

For the $N = 1$ case, the recovering is so straightforward that the general procedure cannot be really appreciated. Setting $q_1 = 0$, we have,

$$\phi_1(x; \lambda) = 2 \left( \lambda + \frac{q(x)}{2} \right). \quad (29)$$

Then, the roots, in $\lambda$, for $\phi_1(x; \lambda) = 0$ are $\lambda = \lambda_1(x)$ (here, we just have one), and they are functions of $x$, namely,

$$\phi_1(x; \lambda) = 2(\lambda - \lambda_1(x)). \quad (30)$$

So, if we can compute the zeros of $\phi_1(x; \lambda) = 0$, we can compute the solution to the KdV equation:

$$q(x) = -2\lambda_1(x). \quad (31)$$

This may look quite trivial. But we will see how to find $\lambda_1(x)$ so that we can use equation (31) to solve the KdV equation.

Consider the equation for $\phi$, (17), and multiply it by $\phi$ itself and, after an integration by parts, we obtain

$$\phi \phi'' - \frac{1}{2} (\phi')^2 + 2(q - \lambda)\phi^2 =: R(\lambda), \quad (32)$$

where the right-hand-side is a polynomial in $\lambda$. Substitute (29) into (32) above, to get

$$R_3(\lambda) := -\frac{1}{8}R(\lambda) = \lambda^3 - \frac{1}{4}\lambda(q'' + 3q^2) + \frac{1}{8} \left( (q^3 + \frac{1}{2}(q')^2) - q(q'' + 3q^2) \right). \quad (33)$$

(Here, $R_3(\lambda)$ represents a polynomial of degree 3).

Assume we substitute the initial conditions for the KdV equation, $q = q(x, 0)$, into (33), to have functions of $x$ only in the coefficients of the polynomial $R_3(\lambda)$. How can we recover $q(x)$ from these functions of $x$? We proceed as follows.

From (29), it follows that $\phi|_{\lambda=\lambda_1} = 0$. Hence, evaluate (32) at $\lambda = \lambda_1$ to get

$$-\frac{1}{2} (\phi')^2 \Bigg|_{\lambda=\lambda_1(x)} = -8R_3(\lambda_1).$$
which represents a Riemann surface (an elliptic curve) of genus $g = 1$ in $(\lambda, \phi') \in \mathbb{C}^2$. On the other hand, from equation (30),

$$\frac{d\phi}{dx} = -2 \frac{d\lambda_1}{dx},$$

so that from (34) we obtain,

$$-2 \frac{d\lambda_1}{dx} \bigg|_{\lambda=\lambda_1(x)} = 2^2 \sqrt{R_3(\lambda_1)}.$$

i.e.,

$$\frac{\lambda'_1}{\sqrt{R_3(\lambda_1)}} \bigg|_{\lambda=\lambda_1(x)} = -2,$$

where we have considered the positive root. This is a separable equation for $\lambda_1(x)$. Since in this instance $R_3(\lambda)$ is a cubic equation in $\lambda$, it turns out that equation (35) is precisely equation (9). Solving for $\lambda_1(x)$, we then are able to recover $q(x)$,

$$q(x) = -2\lambda_1(x),$$

which is a stationary solution of the KdV equation. How to incorporate the time-dependence, will be studied later.

### 6.2 The 2–soliton solution: $N = 2$

In this case, we proceed as in the previous case. The idea to follow is exactly the same, although the computations are a little bit more involved.

We have equation (28), with $q_1 = q_2 = 0$ (just to set ideas):

$$\phi_2(x, \lambda) = 8\lambda^2 + 4\lambda(q(x)) + (q'' + 3q^2).$$

(36)

Substitute into (32) to get

$$\phi_2\phi_2'' - \frac{1}{2}(\phi_2')^2 + 2(q - \lambda)\phi_2^2 = -128R_5(\lambda),$$

(37)

where,

$$R_5(\lambda) = \lambda^5 - \frac{1}{16}F_2\lambda^3 - \frac{1}{32}K_2\lambda - \frac{1}{128}L_2.$$
is a fifth order polynomial in \( \lambda \); and

\[
F_2 = q^{(4)} + 10qq'' + 5(q')^2 + 10q^3 ,
\]

\[
K_2 = qq^{(4)} - q'q^{(3)} + 10q^2q^{(2)} + \frac{1}{2}(q^{(2)})^2 + 10q(q')^2 + \frac{25}{2}q^4,
\]

\[
L_2 = q^{(4)}(q^{(2)} + 3q^2) - q^{(3)}\left(\frac{1}{2}q^{(3)} + 6qq'\right) + q^{(2)}(8q^{(2)}q + 6(q')^2 + 30q^3) + 18q^5.
\]

are constants along trajectories (i.e., solutions) of the 5th order KdV flow (27). Now, we can write equation (36) as:

\[
\phi_2(x, \lambda) = 8(\lambda - \lambda_1(x))(\lambda - \lambda_2(x)),
\]

so that

\[
\phi_2|_{\lambda=\lambda_1(x)} = \phi_2|_{\lambda=\lambda_2(x)} = 0.
\]

Evaluating at \( \lambda = \lambda_j(x) \) \((j = 1, 2)\) equation (37),

\[
\phi'_2|_{\lambda=\lambda_j(x)} = 16\sqrt{R_5(\lambda_j)}, \quad j = 1, 2.
\]

From (38), it follows that

\[
\phi'_2 = 8\left( - \lambda'_1(x)(\lambda - \lambda_2(x)) - \lambda'_2(x)(\lambda - \lambda_1(x)) \right),
\]

so that

\[
\phi'_2|_{\lambda=\lambda_1(x)} = -8\lambda'_1(x)(\lambda_1(x) - \lambda_2(x)),
\]

\[
\phi'_2|_{\lambda=\lambda_2(x)} = -8\lambda'_2(x)(\lambda_2(x) - \lambda_1(x)),
\]

and using (39), we obtain

\[
\frac{\lambda'_1(x)}{\sqrt{R_5(\lambda_1)}}(\lambda_1(x) - \lambda_2(x)) = -2, \quad (40)
\]

\[
\frac{\lambda'_2(x)}{\sqrt{R_5(\lambda_1)}}(\lambda_1(x) - \lambda_2(x)) = 2.
\]

Adding both equations, and dividing by \( \lambda_1 - \lambda_2 \neq 0 \), we get

\[
\frac{\lambda'_1}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda'_2}{\sqrt{R_5(\lambda_2)}} = 0. \quad (41)
\]
Hence,
\[
\frac{\lambda_1'}{\sqrt{R_5(\lambda_1)}} = -\frac{\lambda_2'}{\sqrt{R_5(\lambda_2)}}
\]
and
\[
\frac{\lambda_1\lambda_1'}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2\lambda_2'}{\sqrt{R_5(\lambda_2)}} = \frac{\lambda_1\lambda_1'}{\sqrt{R_5(\lambda_1)}} - \frac{\lambda_2\lambda_1'}{\sqrt{R_5(\lambda_1)}} = \frac{\lambda_1'}{\sqrt{R_5(\lambda_1)}}(\lambda_1 - \lambda_2) = -2
\]  
(42)

Putting together equations (41-42), we obtain a system of differential equations for \( \lambda_1 \) and \( \lambda_2 \), namely,
\[
\frac{\lambda_1'}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2'}{\sqrt{R_5(\lambda_2)}} = 0,
\]  
(43)
\[
\frac{\lambda_1\lambda_1'}{\sqrt{R_5(\lambda_1)}} + \frac{\lambda_2\lambda_2'}{\sqrt{R_5(\lambda_2)}} = -2.
\]  
(44)

If we integrate,
\[
\int_{\lambda_1}^{\lambda_1} \frac{d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \int_{\lambda_2}^{\lambda_2} \frac{d\lambda_2}{\sqrt{R_5(\lambda_2)}} = C_1,
\]  
(45)
\[
\int_{\lambda_1}^{\lambda_1} \frac{\lambda_1 d\lambda_1}{\sqrt{R_5(\lambda_1)}} + \int_{\lambda_2}^{\lambda_2} \frac{\lambda_2 d\lambda_2}{\sqrt{R_5(\lambda_2)}} = -2x + C_2,
\]  
(46)

where \( C_1, C_2 \) are constants of integration. If we are able to integrate and invert these equations, we should be able to get
\[
\lambda_1 = \lambda_1(x),
\]
\[
\lambda_2 = \lambda_2(x).
\]

How can we recover the solution \( q(x) \) of the KdV equation out of these functions? From equation (38)
\[
\phi_2(x, \lambda) = 8 \left( \lambda^2 - (\lambda_1(x) + \lambda_2(x))\lambda + \lambda_1(x)\lambda_2(x) \right),
\]
and comparing to equation (36), we find
\[
q(x) = -\frac{1}{2}(\lambda_1(x) + \lambda_2(x)),
\]
which is the solution of the KdV equation. This is the way we solve the periodic KdV equation.
7 Generalizations in Schrödinger equation

Consider a solution $y(x, t)$ of the Schrödinger equation $y'' + qy = \lambda y$ with $q(x)$ and $\lambda(t)$, functions only of $x$ and $t$, respectively.

Setting $\phi = y^2$, and taking the derivative with respect to $x$, we obtain

$$\phi' = 2yy'$$

and therefore

$$(y')^2 = \frac{(\phi')^2}{4\phi}.$$  

Similarly, taking a second derivative of $\phi$, we have

$$\phi'' = 2(y')^2 + 2yy''$$

$$= \frac{(\phi')^2}{2\phi} + 2yy''.$$  

Hence,

$$yy'' = \frac{1}{2}\phi'' - \frac{(\phi')^2}{4\phi}.$$  

Now, multiplying the Schrödinger equation by $y$ and using (48), we get the following differential equation in terms of $\phi$:

$$\frac{1}{2}\phi'' - \frac{(\phi')^2}{4\phi} = (\lambda - q)\phi.$$  

A straight-forward computation shows that

$$\left(\frac{(\phi')^2}{4\phi}\right)_x = \frac{\phi'}{\phi} \left(\frac{1}{2}\phi'' - \frac{(\phi')^2}{4\phi}\right)$$

$$= \frac{\phi'}{\phi}(\lambda - q)\phi = (\lambda - q)\phi'.$$

Hence, differentiating equation (49), we obtain

$$\frac{1}{2}\phi''' - (\lambda - q)\phi' = [(\lambda - q)\phi]_x.$$  

Multiplying by 2 and re-arranging terms, we finally get the differential equation:

$$\phi''' + 4q\phi' + 2q'\phi = 4\lambda\phi'.$$  

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Define the linear operator

\[ B = \frac{d^3}{dx} + 4q \frac{d}{dx} + 2q'; \]

thus, equation (52) can be re-written as

\[ B(\phi) = 4\lambda \phi'. \] (53)

We are interested in solutions to (53) of polynomial form in \( \lambda \):

\[ \phi_n(x, t) = A_n \lambda^n(t) + A_{n-1} \lambda^{n-1}(t) + \cdots + A_0 \] (54)

where \( A_n \) is constant, and \( A_i(x) \) for \( 0 \leq i < n \), functions of \( x \).

Define recursively the following conservation laws \( F_n(x) \):

\[ F_{-1} = \frac{1}{2} \]

\[ F_{n+1} = \int B(F_n) dx. \]

The first three terms are \( F_{-1} = \frac{1}{2}, F_0 = q, F_1 = q'' + 3q^2 \). Also, sometimes for convinience, we write the recursion among conservations laws in its differential form \( F_{n+1}' = B(F_n) \).

**Theorem 7.1.**

Let \( \phi_n = (4\lambda)^n F_{-1} + (4\lambda)^{n-1} F_0 + \cdots + F_{n-1} \). Thus \( B(\phi_n) - 4\lambda \phi_n' = B(F_{n-1}) \).

**Proof.** First, observe

\[ \phi_n := (4\lambda) \left[(4\lambda)^{n-1} F_{-1} + (4\lambda)^{n-2} F_0 + \cdots + F_{n-2}\right] + F_{n-1} = 4\lambda \phi_{n-1} + F_{n-1}. \] (55)

The result is clear for \( n = 0 \), because \( \phi_0 = F_{-1} = \frac{1}{2} \) and thus \( \phi_0' = 0 \) and \( B(\phi_0) = B(F_{-1}) \).

Now, let’s assume the result is valid for \( n \), hence the calculation

\[ B(\phi_{n+1}) - 4\lambda \phi_{n+1}' = B(4\lambda \phi_n + F_n) - 4\lambda [4\lambda \phi_n + F_n]' \]

\[ = 4\lambda B(\phi_n) + B(F_n) - (4\lambda)^2 \phi_n' - 4\lambda F_n' \]

\[ = (4\lambda) [B(\phi_n) - 4\lambda \phi_n'] + B(F_n) - 4\lambda B(F_{n-1}) \]

\[ = 4\lambda B(F_{n-1}) + B(F_n) - 4\lambda B(F_{n-1}) = B(F_n). \] (56)

shows it for \( n + 1 \). \( \square \)
Corollary 7.2. \( B(\phi_n) = 4\lambda\phi'_n \) if and only if \( F_n \) is constant.

Proof. Since \( F'_n = B(F_{n-1}) \), it is an immediate consequence from Theorem 7.1. \( \square \)

The definition of the polynomials \( \phi_n \) in Theorem 7.1 may seem arbitrary, but they are enough to understand the polynomial solutions in \( \lambda \) to (52). The following lemma is useful to understand this claim.

Lemma 7.3. Let \( \psi_n = A_{-1}(4\lambda)^n + A_0(4\lambda)^{n-1} + \cdots + A_{n-2}(4\lambda) + A_{n-1} \) with \( A_{-1} \) constant and \( A_i \) functions of \( x \) for \( 0 \leq i \leq n-1 \).

If \( B(\psi_n) = 4\lambda\psi'_n \), thus \( B(A_{i-1}) = A'_i \) for \( 0 \leq i \leq n-1 \) and \( B(A_{n-1}) = 0 \).

Proof. On the one hand, using linearity of the operator \( B \), we have

\[
B(\psi_n) = B(A_{-1})(4\lambda)^n + B(A_0)(4\lambda)^{n-1} + \cdots + B(A_{n-2})(4\lambda) + B(A_{n-1}), \quad (57)
\]

on the other hand, taking derivative of \( \phi_n \) and multiplying by \( 4\lambda \), we get

\[
(4\lambda)\psi'_n = A'_0(4\lambda)^n + A'_1(4\lambda)^{n-1} + \cdots + A'_{n-2}(4\lambda)^2 + A'_{n-1}(4\lambda). \quad (58)
\]

Hence, the condition \( B(\psi_n) = 4\lambda\psi'_n \) gives the equality of two polynomials of degree \( n \) in \( 4\lambda \); since terms with the same power should coincide, we obtain that \( B(A_{i-1}) = A'_i \) and \( B(A_{n-1}) = 0 \). \( \square \)

Definition 7.1. A function of the form

\[
\psi_n = A_{-1}(4\lambda)^n + A_0(4\lambda)^{n-1} + \cdots + A_{n-2}(4\lambda) + A_{n-1}
\]

with \( A_{-1} \) constant and \( A_i(x) \) functions of \( x \) for \( 0 \leq i \leq n-1 \) which is solution of \( B(\psi_n) = 4\lambda\psi'_n \) is called an \( n \)-soliton.

Theorem 7.4. 1. Each \( n \)-soliton \( \psi_n \) can be written as a linear combination of the basic solitons: \( \phi_n, \phi_{n-1}, \ldots, \phi_0 \).

2. The linear combination

\[
\psi_n = \alpha_n\phi_n + \alpha_{n-1}\phi_{n-1} + \cdots + \alpha_0\phi_0
\]

(with \( \alpha_i \)'s constant and \( \alpha_n \neq 0 \)) is a \( n \)-soliton if and only if

\[
\alpha_nF_n + \alpha_{n-1}F_{n-1} + \cdots + \alpha_0F_0
\]

is constant.
Proof. 1. By lemma (7.3), assuming that \( \psi_n \) is written as in (??), then

\[
B(A_{i-1}) = A'_i
\]

for \( 0 \leq i \leq n - 1 \).

Now, since \( A_{-1} \) and \( F_{-1} \) are constants not zero, set \( K_{-1} \) such that

\[
A_{-1} = K_{-1}F_{-1}.
\]

Also, integrating

\[
A_0 = \int B(A_{-1})dx = \int B(K_{-1}F_{-1})dx = K_{-1} \int B(F_{-1})dx = K_{-1}F_{0} + C_0 = K_{-1}F_{0} + K_0F_{-1},
\]

where \( C_0 \) is a constant and \( K_0 \) chosen such that \( C_0 = K_0F_{-1} \).

Proceeding by induction, we find constants \( K_j \), such that

\[
A_i = K_{-1}F_i + K_0F_{i-1} + \cdots + K_{i-1}F_0 + K_iF_{-1} = \sum_{j=0}^{i+1} K_{j-1}F_{i-j}
\]

for \( -1 \leq i \leq n - 1 \).

Hence,

\[
\psi_n = \sum_{i=0}^{n} (4\lambda)^i A_{n-1-i} = \sum_{i=0}^{n} (4\lambda)^i \sum_{j=0}^{n-i} K_{j-1}F_{n-1-i-j} = \sum_{j=0}^{n} K_{j-1} \sum_{i=0}^{n-j} (4\lambda)^i F_{n-1-i-j} = \sum_{j=0}^{n} K_{j-1} \phi_{n-j}
\]

(60)
2. Now, if

\[ \psi_n = \sum_{j=0}^{n} \alpha_j \phi_j, \]

satisfies \( B(\psi_n) = 4\lambda \psi_n' \), using linearity

\[
B(\psi_n) - 4\lambda \psi_n' = \sum_{j=0}^{n} \alpha_j B(\phi_j) - 4\lambda \left[ \sum_{j=0}^{n} \alpha_j \phi_j' \right]
= \sum_{j=0}^{n} \alpha_j \left[ B(\phi_j) - 4\lambda \phi_j' \right]
= \sum_{j=0}^{n} \alpha_j B(F_j) = 0
\]

(61)

Integrating, we prove that \( \sum_{j=0}^{n} \alpha_j F_j \) is constant. 

\[ \square \]

**Definition 7.2.** Define

\[ \mathcal{H}(\phi) := \int \phi B(\phi) \, dx - 2\lambda \phi^2 = \phi \phi'' - \frac{1}{2} (\phi')^2 + 2(q-\lambda)\phi^2. \]

**Proposition 7.5.** If \( \phi \neq 0 \), then \( \mathcal{H}(\phi) \) is a constant if and only if \( B(\phi) = 4\lambda \phi' \).

**Proof.** If \( B(\phi) = 4\lambda \phi' \), then clearly \( \phi B(\phi) - 4\lambda \phi' \phi = 0 \) and integrating \( \mathcal{H}(\phi) = \int \phi B(\phi) \, dx - 2\lambda \phi^2 \) is a constant. If \( \phi \neq 0 \), the “only if” part is also true. 

\[ \square \]

Set \( \mathcal{H}_n := \mathcal{H}(\phi_n) \). Observe that since \( \phi_n \) is a polynomial in \( \lambda \), \( \mathcal{H}_n \) is also a polynomial in \( \lambda \). The following result gives an explicit description of \( \mathcal{H}_n(\lambda) \) in terms of the conservation laws \( F_{-1}, F_0, \ldots, F_n \).

**Theorem 7.6.**

\[
\mathcal{H}_n(\lambda) = -\left( \frac{(4\lambda)^{2n+1} F^2}{2} \right) + (4\lambda)^n F_{-1} F_n + (4\lambda)^{n-1} \left[ F_0 F_n - \int F'_0 F_n \, dx \right] + (4\lambda)^{n-2} \left[ F_1 F_n - \int F'_1 F_n \, dx \right] + \cdots + (4\lambda)^0 \left[ F_{n-1} F_n - \int F'_{n-1} F_n \, dx \right]
\]

(62)
The proof of Theorem 7.6 is on Appendix ??.

**Corollary 7.7.** If \( \phi_n \) is a solution of \( B(\phi_n) = 4\phi_n' \), then the polynomial \( \mathcal{H}_n = H(\phi_n) \) has constant coefficients with respect to \( x \).

**Proof.** The conservation law \( F_n \) is constant by Corollary 7.2.

Since \( F_{-1} = \frac{1}{2} \), the coefficients of \( \lambda^{2n+1} \) and \( \lambda^n \) are clearly constant.

For the other coefficients with \( 0 \leq j \leq n - 1 \), since \( F_n \) is constant, we have that

\[
F_j F_n - \int F'_j F_n \, dx = F_n \left[ F_j - \int F'_j \, dx \right]
\]

is also constant, by the Fundamental Theorem of Calculus. 

\( \square \)

### 8 Geometry of Solutions

In this section, the explicit expressions of \( \phi_n \) and \( \mathcal{H}_n \) as polynomials of \( \lambda \) obtained in section 7 will be used to analyze the geometry in the solutions of equation (53).

The solutions of equation (53) by Theorem 7.1 have the explicit form

\[
\phi_n = \frac{4^n}{2} \lambda^n + 4^{n-1} F_0 \lambda^{n-1} + \cdots + 4 F_{n-1} \lambda + F_{n-1}
\]

provided that \( F_n \) is constant.

But, the conservation laws \( F_0, F_1, \ldots, F_{n-1} \) are not constant, they depend on the position \( x \) and the time \( t \). Therefore, \( \phi_n(x, t) \) is a polynomial which their coefficients vary depending on \( (x, t) \). Nevertheless, we can factorize \( \phi_n \) over \( \mathbb{C} \) for fixed values of \( x \) and \( t \).

\[
\phi_n = \frac{4^n}{2} \prod_{i=1}^{n} \left[ \lambda - \lambda_i(x, t) \right].
\]

Taking the derivative of the above expression with respect to \( x \), we obtain

\[
\phi'_n = -\frac{4^n}{2} \sum_{j=1}^{n} \lambda'_j \prod_{i \neq j} \left[ \lambda - \lambda_i(x, t) \right],
\]

\( 63 \)

\( 64 \)
evaluating for $\lambda = \lambda_k(x, t)$, we finally get

$$\phi'_n|_{\lambda=\lambda_k} = -\frac{4n}{2} \lambda'_k \prod_{i \neq k} [\lambda_k(x, t) - \lambda_i(x, t)].$$  \hspace{1cm} (65)$$

Now, using that

$$\mathcal{H}_n(\lambda) = \mathcal{H}(\phi_n) = \phi_n \phi''_n - \frac{1}{2}(\phi'_n)^2 + 2(q - \lambda)\phi_n^2$$

and that $\phi_n|_{\lambda=\lambda_k(x,t)} = 0$, we obtain

$$\mathcal{H}_n(\lambda_k) = -\frac{1}{2}(\phi'_n|_{\lambda=\lambda_k})^2$$ \hspace{1cm} (66)$$

Now, since $\mathcal{H}_n$ is a polynomial of degree $2n + 1$ with constant coefficients with respect to the variable $x$, the equation

$$\mathcal{H}_n(X) = -\frac{1}{2}Y^2$$ \hspace{1cm} (67)$$

is of an hyperelliptic curve $\mathcal{H}_n$ of genus $n$ and each $P_k = (\lambda_k, \phi'_n|_{\lambda=\lambda_k})$ with $1 \leq k \leq n$ represents a point on it. Hence, if we vary $x$, but fixed $t$, a soliton solution defines $n$ real curves on $\mathcal{H}_n$. These curves on the tangent space define a system of linear differentials.

In fact, combining equations (65) and (66), we have that

$$\sqrt{-2\mathcal{H}_n(\lambda_k)} = \phi'_n|_{\lambda=\lambda_k}$$

$$= -\frac{4n}{2} \lambda'_k \prod_{i \neq k} [\lambda_k(x, t) - \lambda_i(x, t)].$$ \hspace{1cm} (68)$$

But, since $F_2^1 = \frac{1}{4}$, the leading coefficient of $\mathcal{H}_n$ is

$$-\frac{4^{2n}}{2}.$$

Hence, setting

$$P_n = \frac{-2\mathcal{H}_n}{4^{2n}},$$

we obtain

$$\frac{\lambda'_k}{\sqrt{P_n(\lambda_k)}} = \frac{-2}{\prod_{i \neq k} [\lambda_k(x, t) - \lambda_i(x, t)]}$$ \hspace{1cm} (69)$$
Now, the differentials
\[
\omega_\mu = \frac{X^{\mu-1}dX}{\sqrt{P_n(X)}}, \quad 1 \leq \mu \leq n
\] (70)
form a basis of the differential space \(\Omega^1(\mathcal{H}_n)\).

Evaluating those differentials in the points \(P_k\), we obtain
\[
\omega_\mu(P_k) = \omega_\mu(\lambda_k) = \frac{\lambda_k^{\mu-1}\lambda_k' dx}{\sqrt{P_n(X)}}
\] (71)

\[= \frac{-2\lambda_k^{\mu-1}}{\prod_{i \neq k} [\lambda_k(x,t) - \lambda_i(x,t)]}\]

Adding up over all points \(P_k\) and using the main proposition [3.1] in Appendix ??, we get
\[
\sum_{k=1}^{n} \omega_\mu(P_k) = \begin{cases} 0 & \text{if } 1 \leq \mu < n \\ -2 & \text{if } \mu = n \end{cases}
\] (72)

Now, the Abel map from a Riemann surface \(\mathcal{H}_n\) to its Jacobian \(J(\mathcal{H}_n)\) (choosing a fixed point \(P_0\)) is defined by
\[
A_{P_0}(Q) := (A_{P_0}^{(1)}(Q), A_{P_0}^{(2)}(Q), \ldots, A_{P_0}^{(n)}(Q))
\] (73)
where
\[A_{P_0}^{(k)}(Q) := \int_{P_0}^{Q} \omega_k.\]

If \(S_n = \bigwedge^n \mathcal{H}_n\) is the \(n\)-symetic power of \(\mathcal{H}_n\), the Abel is the extended map
\[\tilde{A}_{P_0} : S_n \rightarrow J(\mathcal{H}_n)\]
given by the addition formula
\[
\tilde{A}_{P_0}(Q_1, Q_2, \ldots, Q_n) = \sum_{k=1}^{n} A_{P_0}(Q_k)
\] (74)

Hence, for a soliton \(\phi_n(x,t)\), using equation (72), taking
\[P_k(x,t) = (\lambda_k(x,t), \phi'_n|_{\lambda=\lambda_k}(x,t)),\]
and considering \(P_0 = (\lambda_0, \mu_0)\) any other point in \(\mathcal{H}_n\), we obtain that
\[
\tilde{A}_{P_0}(P_1(x,t), P_2(x,t), \ldots, P_n(x,t)) = (C_1(t), C_2(t), \ldots, C_{n-1}(t), C_n(t) - 2x)
\] (75)
where the functions \(C_i(t)\) are constant with respect to \(x\).
9 Time Evolution

The Schrödinger differential operator

\[ \mathcal{L} = \frac{d^2}{dx} + q \]

has an associated operator

\[ P = -4 \frac{d^3}{dx} - 6q \frac{d}{dx} - 3q' , \]

so that they form the Lax pair of the Korteweg-deVries equation.

Define the commutator operator \([\mathcal{L}, P] := \mathcal{L}P - P\mathcal{L}.\) The following result describes the relationship among \(\mathcal{L}\) and \(P\).

**Lemma 9.1.** For \(\mathcal{L}\) and \(P\) as above, we have:

\[ [\mathcal{L}, P] = q_{xxx} + 6qq_x \] \hspace{1cm} (76)

**Proof.** To simplify our notation, set

\[ D := \frac{d}{dx}. \]

If \(f\) is a differentiable function over some domain \(\Omega\), \(D\) acts according to the formalism:

\[ Df = f_x + fD. \]

In particular, it can be proven by induction that:

\[ D^n f = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}D^i. \]

Thus, using this notation, we have that

\[ [\mathcal{L}, P] = [D^2 + q, -4D^3 - 6qD - 3q_x] \]

\[ = -\{6[D^2, qD] + 3[D^2, q_x] + 4[q, D^3] + 6[q, qD]\}, \]

since \([D^2, D^3] = 0\) and \([q, q_x] = 0\). Now,

\[ 6[D^2, qD] = 6\{(D^2q)D - qD^2\} = 6\{(D^2q)D - qD^3\} \]

\[ = 6\{(q_{xx} + 2q_xD + qD^2)D - qD^3\} = 6q_{xx}D + 12q_xD^2 \] \hspace{1cm} (78)
\[3[D^2, q_x] = 3\{D^2(q_x) - (q_x)D^2\}
= 3\{(q_{xxx} + 2q_{xx}D + q_xD^2) - q_xD^2\} = 3q_{xxx} + 6q_{xx}D \tag{79}\]

\[4[q, D^3] = 4\{qD^3 - D^3 q\} = 4\{qD^3 - (q_{xxx} + 3q_{xx}D + 3q_xD^2 + qD^3)\}
= -4q_{xxx} - 12q_{xx}D - 12q_xD^2 \tag{80}\]

\[6[q, qD] = 6\{q^2D - qDq\} = 6\{q^2D - q(q_x + qD)\}
= -6q_{xx} \tag{81}\]

Adding up the equations (78–81), we obtain the result:

\[[\mathcal{L}, P] = q_{xxx} + 6qq_x\]

\[\square\]

The Schrödinger equation, at the beginning of section 7, is the spectral or eigenfunction problem

\[\mathcal{L}y = \lambda y, \tag{82}\]

while the \(P\) operators gives the time evolution condition

\[Py = y_t = \frac{dy}{dt}. \tag{83}\]

As in section 7 if we consider \(\phi = y^2\), we can obtain a differential equation for \(\phi_t\). In fact,

\[\phi_t = 2yy_t = 2yPy
= -8yy''' - 12qyy' - 6q'y^2
= -8yy''' - 6q\phi' - 6q\phi
= -8yy''' - 6(q\phi)' \tag{84}\]

But, using that \(yy''' = (yy'')' - y'y''\) and that \(y'' = (\lambda - q)y\) (Schrödinger equation), we obtain that

\[yy''' = ((\lambda - q)y^2)' - (\lambda - q)y'y
= ((\lambda - q)\phi)' - (\lambda - q)\phi' \tag{85}\]

\[= -q\phi + \frac{(\lambda - q)\phi'}{2}\]

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Hence,
\[
\phi_t = -8\left[-q'\phi + \frac{(\lambda - q)\phi'}{2}\right] - 6(q\phi')^2
\]
\[
= 2q'\phi - 2q\phi' - 4\lambda\phi'
\]
But, using that \(4\lambda\phi' = B(\phi) = \phi'' + 4q\phi' + 2q'\phi\), we obtain the linearized Kdv equation
\[
\phi_t = -\phi''' - 6q\phi'
\]  
(87)

Now, let’s assume that \(\phi\) is the \(n\)-soliton \(\phi_n\). Hence, evaluating \(\phi_t\) in \(\lambda = \lambda_k\), similarly as in section 8 with \('\) and using (86), we obtain that
\[
\phi_t |_{\lambda = \lambda_k} = -2(q + 2\lambda)\phi' |_{\lambda = \lambda_k}
\]
(88)

A similar computation as the one in equation (65) gives
\[
\phi_t |_{\lambda = \lambda_k} = -\frac{4^n}{2} \frac{\partial \lambda_k}{\partial t} \prod_{i \neq k} \left[\lambda_k(x, t) - \lambda_i(x, t)\right].
\]
(89)

Hence, equation (88) simplifies to the following relation of the partial derivatives of \(\lambda_k\)
\[
\frac{\partial \lambda_k}{\partial t} = -2(q + 2\lambda)\frac{\partial \lambda_k}{\partial x}
\]
(90)

Part II

The NLS equation

10 The Lax pair for the NLS equation. Analogies with the KdV equation.

We know that the Nonlinear Schrödinger (NLS) equation
\[
i\frac{\partial q}{\partial t} + \frac{1}{2} \frac{\partial^2 q}{\partial x^2} + \sigma|q|^2q = 0,
\]
(91)
(where \(\sigma = \pm 1\) is the focusing/defocusing parameter) is a completely integrable system by means of the inverse scattering transform [16, 15, 25]. In
this context, the NLS admits a Lax’s pair in matrix form

\[
\mathcal{L} = i\lambda D + N, \quad \text{(92)}
\]

\[
\mathcal{P} = i\lambda \mathcal{L} + \frac{1}{2} D(N_x - N^2) \quad \text{(93)}
\]

where \( D \) and \( N \) are two \( 2 \times 2 \) matrices over \( \mathbb{C} \), defined by

\[
D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & q \\ -\sigma \bar{q} & 1 \end{pmatrix},
\]

with \( q = q(x, t) \) is solution to the NLS equation and \( \bar{q} \) denotes the complex conjugate of \( q \).

The Lax’s pair, \( \mathcal{L}, \mathcal{P} \), defines an overdetermined system of ordinary differential equations,

\[
\frac{\partial \Psi}{\partial x} = \mathcal{L} \Psi, \quad \text{(spectral problem)} \quad \text{(94)}
\]

\[
i\frac{\partial \Psi}{\partial t} = \mathcal{P} \Psi, \quad \text{(t-flow)} \quad \text{(95)}
\]

whose compatibility condition, \((\Psi_x)_t = (\Psi_t)_x\) (cross-differentiation holds) becomes the NLS equation, equation (91) above.

\section{The scalar spectral problem for the squared eigenfunctions}

Computations made from Sections 4 to 7 can be generalized under the hypothesis of having a linear multiplicative operator \( \hat{\mathcal{L}} \) satisfying the differential equation:

\[
\frac{\partial^2 y}{\partial x^2} = \hat{\mathcal{L}} y \quad \text{(96)}
\]

Multiplying by \( y \) and using that \( \hat{\mathcal{L}} \) is a multiplicative operator, we get

\[
y \frac{\partial^2 y}{\partial x^2} = \hat{\mathcal{L}} y^2 \quad \text{(97)}
\]

Hence, setting \( \phi = y^2 \) and using equation (48), we obtain

\[
\frac{1}{2} \frac{\phi''}{\phi} - \frac{(\phi')^2}{4\phi} = \hat{\mathcal{L}} \phi \quad \text{(98)}
\]
Taking derivatives in both sides, we obtain:

\[
\frac{1}{2} \phi''' - \left( \frac{(\phi')^2}{4\phi} \right)_x = \left[ \hat{\mathcal{L}} \phi \right]_x
\]

(99)

But, now repeating computations made in (50) and (51), we obtain

\[
\left( \frac{(\phi')^2}{4\phi} \right)_x = \frac{\phi'}{\phi} \left( \frac{1}{2} \phi'' - \frac{(\phi')^2}{4\phi} \right) = \frac{\phi'}{\phi} \hat{\mathcal{L}} \phi = \hat{\mathcal{L}} \phi'
\]

(100)

and

\[
\frac{1}{2} \phi''' - \hat{\mathcal{L}} \phi' = \left[ \hat{\mathcal{L}} \phi \right]_x
\]

(101)

After multiplication by 2, we obtain the third order differential equation

\[
\phi''' - 2\hat{\mathcal{L}}_x \phi - 4\hat{\mathcal{L}} \phi' = 0.
\]

(102)

In the case of KdV, setting \( \hat{\mathcal{L}} = \lambda - q \), the equation (52) is obtained, which is the basis for the KdV’s recursion formulæ.

In what follows, we will use a similar approach as the one used in Sections 4-7, to obtain a recursion formulæ for the NLS equation:

\[
iq_t + \frac{1}{2} q_{xx} + \sigma \|q\|^2 q_x = 0.
\]

(103)

The basic assumption is, as already mentioned, the fact that we have a scalar multiplicative operator \( \hat{\mathcal{L}} \). We will make use of the work of Kamchatnov, Kraenkel and Umarov \[17, 18, 19\], where the authors find the scalar multiplicative operator \( \hat{\mathcal{L}} \) associated to the NLS equation.

The scalar multiplicative operator is:

\[
\hat{\mathcal{L}} = - \left( \lambda - \frac{iq_x}{2q} \right)^2 - \sigma \|q\|^2 - \left( \frac{q_x}{2q} \right)_x = -\lambda^2 + E\lambda + F
\]

(104)

with

\[
E = \frac{iq_x}{q}
\]

(105)

and

\[
F = -\frac{1}{4} E^2 - \sigma \|q\|^2 + \frac{i}{2} E'
\]

(106)
which is a polynomial expresion of degree two in $\lambda$.

To simplify our computations, define the following bilinear operator:

$$\langle \psi, \phi \rangle = (\psi \phi)_x + \psi \phi_x$$

$$= \psi_x \phi + 2\psi \phi_x.$$  \hspace{1cm} (107)

Hence, equation (102) can be written as:

$$\phi''' - 2\langle \hat{L}, \phi \rangle = 0.$$ \hspace{1cm} (108)

Using linearity, and that $\lambda$ is constant with respect to $x$, (102) becomes:

$$\phi''' - 2 \left[ -\lambda^2 \langle 1, \phi \rangle + \lambda \langle E, \phi \rangle + \langle F, \phi \rangle \right] = 0.$$ \hspace{1cm} (109)

But, using that $\langle 1, \phi \rangle = 2\phi'$, we obtain:

$$\phi''' + 4\lambda^2 \phi' - 2 \left[ \lambda \langle E, \phi \rangle + \langle F, \phi \rangle \right] = 0.$$ \hspace{1cm} (110)

This equation is a scalar spectral problem for the squared-eigenfunction $\phi$.

12 Recursion Formulae

Now, as we did in the case of the KdV equation, we will assume that $\phi$ is polynomial in $\lambda$, and that $\lambda$ is constant with respecto to $x$. Set

$$\phi = A_0 \lambda^n + A_1 \lambda^{n-1} + \cdots + A_{n-1} \lambda + A_n,$$

without lost of generality, we assume $A_0$ is constant. (In fact, it is an easy exercise, to verify that if $\phi$ satisfies the equation (110) then $A_0$ is constant.)

Hence, taking derivatives, we have

$$\lambda^2 \phi' = A'_1 \lambda^{n+1} + A'_2 \lambda^n + \cdots + A'_{n-1} \lambda^3 + A'_n \lambda^2$$

and

$$\phi''' = A'''_1 \lambda^{n-1} + A'''_2 \lambda^{n-2} + \cdots + A'''_{n-1} \lambda + A'''_n.$$  

Now, using that the operator $\langle \_, \_ \rangle$ is bilinear, we obtain

$$\lambda \langle E, \phi \rangle = \lambda^{n+1} \langle E, A_0 \rangle + \lambda^n \langle E, A_1 \rangle + \cdots + \lambda^2 \langle E, A_{n-1} \rangle + \lambda \langle E, A_n \rangle$$
\[ \langle F, \phi \rangle = \lambda^n \langle F, A_0 \rangle + \lambda^{n-1} \langle F, A_1 \rangle + \cdots + \lambda \langle F, A_{n-1} \rangle + \langle F, A_n \rangle \]

Next, we will compare orders in equation (110).

Comparing in order \( \lambda^{n+1} \):

\[ 2A_1' = \langle E, A_0 \rangle \] (111)

and thus,

\[ A_1 = \frac{\int \langle E, A_0 \rangle dx}{2} \] (112)

Comparing in order \( \lambda^n \):

\[ 2A_2' = \langle E, A_1 \rangle + \langle F, A_0 \rangle \] (113)

and hence,

\[ A_2 = \frac{1}{2} \int \left( \langle E, A_1 \rangle + \langle F, A_0 \rangle \right) dx \] (114)

Comparing in order \( \lambda^{n-1} \), we obtain:

\[ A_3 = \frac{1}{2} \int \left( \langle E, A_2 \rangle + \langle F, A_1 \rangle \right) dx - \frac{1}{4} A''_1 \] (115)

In general, comparing in order \( \lambda^{n-k} \) for \( 1 \leq k \leq n-2 \), we obtain

\[ A_{k+2} = \frac{1}{2} \int \left[ \langle E, A_{k+1} \rangle + \langle F, A_k \rangle \right] dx - \frac{1}{4} A''_k \] (116)

Notice that the last comparison for \( k = n - 2 \) occurs in order \( \lambda^2 \).

If we shift the indexes by \(-2\), we obtain the recursion formula:

\[ A_j = \frac{1}{2} \int \left[ \langle E, A_{j-1} \rangle + \langle F, A_{j-2} \rangle \right] dx - \frac{1}{4} A''_{j-2} \] (117)

If we define \( A_{-1} = 0 \), this recursion formula is valid for \( j = 1, \ldots, n \).

Finally, comparing terms in \( \lambda \) and 1, we obtain two conditions for \( \phi \) to be a solution of the NLS equation:

\[ A''_{n-1} = 2 \int \left[ \langle E, A_n \rangle + \langle F, A_{n-1} \rangle \right] dx \quad (\text{Condition A}) \] (118)

\[ A''_n = 2 \int \langle F, A_n \rangle dx \quad (\text{Condition B}) \] (119)
13 NLS N-solitons.

As in the KDV case, we define some basic polynomials using the recursion formula (117) to characterize all polynomial solutions to equation (110).

Set

\[ \phi_0 := A_0 = 2. \]

Thus,

\[
A_1 = \frac{\int \langle E, 2 \rangle dx}{2} = \frac{1}{2} \int 2E_x dx = E + C
\]

with \( C \) a constant. If we set \( C = 0 \), we obtain

\[ \phi_1 := 2\lambda + E. \]

Continuing with the recursion, we obtain

\[
A_2 = \frac{1}{2} \int [(E, E) + (F, 2)]
= \frac{1}{2} \int [3EE' + 2F']
= \frac{3}{4} E^2 + F + C
\]

Taking again \( C = 0 \), we get

\[ \phi_2 := 2\lambda^2 + E\lambda + \frac{3}{4} E^2 + F \]

In general, we define the basic NLS N-soliton \( \phi_n \) by using the recursion formula (117), setting \( A_0 = 2 \) and taking all constants of integration equal to zero.

Next terms are

\[
\phi_3 := 2\lambda^3 + E\lambda^2 + \left( \frac{3}{4} E^2 + F \right) \lambda + \frac{5}{8} E^3 + \frac{3}{2} FE - \frac{1}{4} E''
\]

\[
\phi_4 := 2\lambda^4 + E\lambda^3 + \left( \frac{3}{4} E^2 + F \right) \lambda^2 + \left( \frac{5}{8} E^3 + \frac{3}{2} FE - \frac{1}{4} E'' \right) \lambda
+ \left( \frac{35}{64} E^4 + \frac{15}{8} E^2 F + \frac{3}{4} F^2 - \frac{5}{16} (E')^2 - \frac{5}{8} EE'' - \frac{1}{4} F'' \right)
\]
Now, suppose that
\[ \psi_n(\lambda) = B_0 \lambda^n + B_1 \lambda^{n-1} + \cdots + B_{n-1} \lambda + B_n \]
(with \( B_0 \) a constant and \( B_i \) function of \( x \) and \( t \)) is a solution of equation \( \Box \). We will call such a function a NLS N-soliton. The following theorem characterize such functions. It is an analogue of theorem \( \Box \) for the NLS-equation.

**Theorem 13.1.** 1. Any \( \psi_N \) can be written as a linear combination of the basic NLS-N solitons: \( \phi_N, \phi_{N-1}, \ldots, \phi_0 \).

2. Moreover, if
\[ \psi_n = \alpha_0 \phi_n + \alpha_1 \phi_{n-1} + \cdots + \alpha_n \phi_0 \]
then
\[ B_i = \sum_{j=0}^{i} \alpha_j A_{i-j} \]
where the \( B_i 's \) are the coefficients of \( \psi_N \) and the \( A_i 's \) are the coefficients of the \( \phi_k 's \). Hence, the conditions (A) and (B) can be written as
\[ \sum_{k=0}^{n} \alpha_k A_{n-k} = 0 \] (Condition A)
and
\[ \sum_{k=0}^{n} \alpha_k B_{n-k} = 0 \] (Condition B)
where
\[ A_i = A''_i - 2 \int (\{E, A_i\} + \{F, A_{i-1}\}) \, dx \]
and
\[ B_i = A''_i - 2 \int \{F, A_i\} \, dx \]
for \( i = 0, 1, \ldots, n \). Assuming \( A_{-1} = 0 \).

14 The 0-, 1- and 2- soliton solutions for the NLS equation

In this section, we explain how the recursion formulas obtained in the previous section are used. We will develop the cases when \( n = 0, 1, 2 \).
14.1 0-soliton for NLS

We will assume $\phi = A$ is constant and $\hat{L} = -\lambda^2 + E\lambda + F$. Thus, equation \[111\] gives

$$\langle E, A_0 \rangle = A_0 E' = 0$$

Hence,

$$E = \frac{i q_x}{q} = i (\ln q)' = k$$

is constant. So, $q$ satisfies the linear equation

$$q_x = -ikq.$$ 

Thus, it follows that

$$q = Ce^{-ikx},$$

with $C$ and $k$ constants.

Now, equation \[113\] gives

$$\langle F, A_0 \rangle = A_0 F' = 0$$

Hence, $F$ is constant. But,

$$F = -\frac{1}{4} E^2 - \sigma \|q\|^2 + \frac{i}{2} E'$$

$$= -\frac{1}{4} k^2 - \sigma \|q\|^2. \quad (123)$$

Hence, $\|q\|^2$ is constant, since $F$, $E = k$ and $\sigma$ are constants.

Thus, we can conclude that $q = Ce^{-ikx}$ with $k \in \mathbb{R}$.

14.2 1-soliton for NLS

For $n = 1$, equation \[113\] gives Condition (A):

$$\langle E, A_1 \rangle + \langle F, A_0 \rangle = E_x A_1 + 2 E A'_1 + F_x A_0 = 0 \quad (124)$$

And, Condition (B) in its differential form is

$$A''_1 - 2\langle F, A_1 \rangle = A''_1 - 2F_x A_1 - 4FA'_1 = 0 \quad (125)$$
But, from the recursion formula (112)

\[ A_1 = \int \langle E, A_0 \rangle dx \]
\[ = \frac{1}{2} \int A_0 E_x dx \]  

(126)

Hence, taking \( A_0 = 2 \) and integrating, we obtain:

\[ A_1 = E \]

Hence, Condition (A) in (124) becomes

\[ 3EE_x + 2F_x = 0. \]  

(127)

Integrating with respect to \( x \), we obtain

\[ \frac{3}{2} E^2 + 2F = C, \]  

(128)

where \( C \) is a constant. Now, using (106, 176, 177), we can compute

\[ 2F = -\frac{1}{2} E^2 - 2\sigma \|q\|^2 + iE' \]
\[ = -\frac{1}{2} E^2 - 2\sigma \|q\|^2 + i [E_{(2)} + iE^2] \]  

(129)

\[ = -\frac{3}{2} E^2 - 2\sigma \|q\|^2 + iE_{(2)} \]

Hence, setting \(-2\omega = C\), we get

\[ \frac{3}{2} E^2 + 2F = -2\sigma \|q\|^2 - \frac{q_{xx}}{q} = -2\omega. \]  

(130)

Multiplying by \(-\frac{3}{2}\), we obtain the Stacionary Non-linear Shrodinger equation:

\[ \sigma \|q\|^2 q + \frac{1}{2} q_{xx} = \omega q. \]

Similarly, Condition B in (125) becomes

\[ E''' - 2F_x E - 4FE' = 0 \]  

(131)
using that $F' = -\frac{3}{2}EE'$ and $F = -\omega - \frac{3}{4}E^2$, we obtain

$$E''' + 6E^2E' + 4\omega E' = 0$$

(132)

Integrating:

$$E'' + 2E^3 + 4\omega E = C_1$$

(133)

Multiplying by $E'$ and integrating again:

$$\frac{1}{2}(E')^2 + \frac{1}{2}E^4 + 2\omega E^2 = C_1E + C_2$$

(134)

Hence, we can parametrize the solutions as pair of points $(E, E')$ in the curve of genus 1

$$y^2 = -x^4 + 4\omega x^2 + C_1x + C_2$$

(135)

In fact, the above equation defines a family of elliptic curves, since $C_1$ and $C_2$ are constants.

### 14.3 2-soliton for NLS

Using the recursion formulas, we can take $A_0 = 2$ and $A_1 = E$, thus, by (114)

$$A_2 = \frac{1}{2} \int [(E, E) + (F, 2)]$$

$$= \frac{1}{2} \int [3EE' + 2F']$$

$$= \frac{3}{4}E^2 + F = -\omega$$

(136)

But, this $\omega$ is not constant, it is variable with respect to $x$.

Now, Condition (A) is

$$E'' = 2 \int [(E, -\omega) + (F, E)]$$

$$= 2 \int \left[ \langle E, \frac{3}{4}E^2 + F \rangle + \langle F, E \rangle \right]$$

$$= 2 \int \left[ \frac{3}{4} \langle E, E^2 \rangle + \langle E, F \rangle + \langle F, E \rangle \right]$$

(137)

But,

$$\int [(E, F) + \langle F, E \rangle] = \int 3(Fe) = 3FE$$

(138)
\[
\int \langle E, E^2 \rangle = \int \left[ E_x E^2 + 2E(2E \cdot E_x) \right] = 5 \int (E^2 E_x) = \frac{5}{3} E^3
\] (139)

Hence, Condition (A) simplifies to

\[
E'' = \frac{5}{2} E^3 + 6FE
\] (140)

Now, using the explicit expression (106) for \( F \) in terms of \( E \), we obtain:

\[
E'' = E^3 - 6\sigma \|q\|^2 E + 3iE'E
\] (141)

But, using that \( E' = E_{(2)} + iE^2 \), equation (177), we obtain

\[
E'' = -6\sigma \|q\|^2 E + 3iE_{(2)}E - 2iE^3
\] (142)

Comparing with expression for \( E'' \) in (180), we finally get

\[
E_{(3)} = -6\sigma \|q\|^2 E,
\] (143)

which can be express in terms of \( q \) and its derivatives, after multiplication by \( q \) and considering a constant of integration \( C \) by the equation:

\[
iq_{xxx} + 6\sigma \|q\|^2 q_x = Cq.
\] (144)

15 Summary: NLS

We follow a similar approach as the one used in the KDV equation.

For the NLS equation \( iq_t + \frac{1}{2}q_{xx} + \sigma \|q\|^2 q_x = 0 \), the operator

\[
\hat{L} = -\left( \lambda - \frac{iqx}{2q} \right)^2 - \sigma \|q\|^2 - \left( \frac{q_x}{2q} \right)_x
\]

is the scalar multiplicative operator for the equation:

\[
y'' = \hat{L}y.
\]
If we write $\hat{L}$ as a polynomial in $\lambda$, we obtain

$$\hat{L} = -\lambda^2 + E\lambda + F,$$

where $E = \frac{iq}{q}$ and $F = -\frac{1}{4}E^2 - \sigma\|q\|^2 + \frac{1}{2}E_x$.

Using again the geometric condition $\phi = q^2$, assuming $\phi$ is of polynomial in $\lambda$, we solve the equation:

$$(G) \quad \phi''' - 4\hat{L}\phi' - 2\hat{L}'\phi = 0$$

depending on the parameters $E$ and $F$.

(Condition (G) follows from equation (A) and the geometric condition)

Solving as in the case of KDV for a polynomial function in $\lambda$ of degree $n$: 

$$\phi_n = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n,$$

we obtain for the solutions of (G):

1. A recursion formulae:

$$A_j = \frac{1}{2} [EA_{j-1} + FA_{j-2}] + \frac{1}{2} \int [EA_{j-1}' + FA_{j-2}'] \, dx$$

with the initial conditions $A_{-1} = 0$ and $A_0 = 2$. The choice of the constant $A_0 = 2$ makes computations easier, but one can consider any other constant $A_0$. Besides this recursion

2. Two Extra Conditions in the coefficients:

Defining the bilinear operator

$$\langle \psi, \phi \rangle = (\psi\phi)_x + \psi\phi_x,$$

the conditions are:

**Condition A**

$$A_{n-1}''' - 2\langle E, A_n \rangle - 2\langle F, A_{n-1} \rangle = 0$$

**Condition B**

$$A_n''' - 2\langle F, A_n \rangle = 0$$

Conditions (A) and (B) can also be expressed in an integral form, useful in some computations.
15.1 Computations.

We made some computations to illustrate how this formulas work.

1. Case $n = 0$
   Condition (A) implies that $q = e^{-iC_x}$.
   Condition (B) gives $\|q\|^2$ constant. Hence, $C \in \mathbb{R}$.

2. Case $n = 1$
   Condition (A) gives the Stationary Non-linear Shrodinger equation:
   $$\sigma \|q\|^2 q + \frac{1}{2} q_{xx} = \omega q.$$  
   Condition (B) gives an equation in terms of $E$ and $E'$, which may be solved by integration on an elliptic curve (Riemann Surface of genus 1):
   $$(E')^2 = -E^4 + 4\omega E^2 + C_1 E + C_2$$  
   where $C_1$ and $C_2$ are constants.

3. Case $n = 2$
   From equation A, we obtain the equation:
   $$iq_{xxx} + 6i\sigma \|q\|^2 q_x = Cq$$  
   where $C$ is a constant.
   Condition (B), in this case, is more complicated.
   The equation in terms of E and F is the following:
   $$\frac{9}{2} E'E'' + \frac{3}{2} EE''' + F''' - \frac{3}{2} E^2 F_x - 6F_x F - 6EFE_x = 0$$
Part III
Appendixes

A Proof of Theorem \[7.6\]

We need some definitions and lemmas.

**Definition A.1.** Define the following sums

\[ S_{l,k} = \sum_{s=k}^{l-k} F_s F'_{l-s} \]  \hspace{1cm} (145)

and

\[ T_{l,k} = \sum_{s=k}^{l-k} F_s F_{l-s} \]  \hspace{1cm} (146)

**Lemma A.1.**

\[ \int S_{l,k} = \frac{T_{l,k}}{2} \]  \hspace{1cm} (147)

**Proof.** Integrating by parts

\[ \int S_{l,k} = \sum_{s=k}^{l-k} \int F_s F'_{l-s} = \sum_{s=k}^{l-k} F_s F_{l-s} - \sum_{s=k}^{l-k} \int F'_s F_{l-s} \]

But,

\[ \sum_{s=k}^{l-k} F'_s F_{l-s} = \sum_{s=k}^{l-k} F_s F'_{l-s} = S_{l,k} \]

and, therefore

\[ 2 \int S_{l,k} = \sum_{s=k}^{l-k} F_s F_{l-s} = T_{l,k}. \]

**Lemma A.2.**

\[ T_{l,k} - T_{l,k+1} = 2F_k F_{l-k} \]  \hspace{1cm} (148)
Proof. Follows from the explicite computation

\[ \sum_{s=k}^{l-k} F_s F_{l-s} - \sum_{s=k+1}^{l-k-1} F_s F_{l-s} = F_k F_{l-k} + F_{l-k} F_k. \]

\[ \square \]

Lemma A.3.

\[ \phi_n^2 = \sum_{m=n}^{2n} (4\lambda)^m T_{2n-m-2,-1} + \sum_{m=0}^{n-1} (4\lambda)^m T_{2n-m-2,n-m-1} \quad (149) \]

Proof. Now, since

\[ \phi_n^2 = \left( \sum_{r=0}^{n} (4\lambda)^r F_{n-r-1} \right) \left( \sum_{s=0}^{n} (4\lambda)^s F_{n-s-1} \right), \]

the terms of degree \( m = r + s \) are of the form

\[ (4\lambda)^{r+s} F_{n-r-1} F_{n-s-1} \]

and therefore, the sum of the subindexes of \( F \) is \( 2n - m - 2 \). Now, since \( s = m - r \), we can write this terms only with the variables \( m \) and \( r \) as

\[ (4\lambda)^m F_{n-r-1} F_{n-m+r-1}, \]

where \( n \geq r \geq 0 \) or equivalently \( n - 1 \geq n - r - 1 \geq -1 \). Now, the restriction on the posible values of \( r \) given by the degrees in the polynomial \( \phi_n \), is also valid for \( s = m - r \). Hence, it must be satisfied also that \( n \geq m - r \geq 0 \) and equivalently \( n + 1 \geq n + m + r - 1 \geq -1 \).

The proof of the lemma is divided in two cases.

1. If \( m \geq n \), we list the terms in a descending way from \( r = n \) to \( r = m-n \).

   Notice that when \( r = m-n \), the number \( n - m + r - 1 \) = -1, its smallest value (or biggest for \( s, s = n \)). Hence, we have the sum

   \[ F_{-1} F_{2n-m-1} + F_0 F_{2n-m-1} + \cdots + F_{2n-m-1} F_{-1} = T_{2n-m-2,-1} \]

2. If \( n > m \), we list the terms in an ascending way from \( r = 0 \) to \( m \), to obtain the sum

   \[ F_{n-1} F_{n-m-1} + F_{n-2} F_{n-m} + \cdots + F_{n-m-1} F_{-1} = T_{2n-m-2,n-m-1} \]

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Both cases prove the formula.

As in definition (A.1), but now with the operator $B$, we define the sum

**Definition A.2.**

$$B_{l,k} = \sum_{s=k}^{l-k} F_s B (F_{l-s}) \quad (150)$$

**Lemma A.4.**

$$B_{l,k} = S_{l+1,k} - F_{l+1-k} F'_k \quad (151)$$

**Proof.**

$$B_{l,k} = \sum_{s=k}^{l-k} F_s B (F_{l-s}) = \sum_{s=k}^{l-k} F_s F'_{l-s+1}$$

$$= \left( \sum_{s=k}^{l+1-k} F_s F'_{l-s+1} \right) - F_{l+1-k} F'_k = S_{l+1,k} - F_{l+1-k} F'_k \quad (152)$$

Now, we use lemma (A.4) to prove a formula for $\phi_n B(\phi_n)$ similar to (149).

**Lemma A.5.**

$$\phi_n B(\phi_n) = \frac{2n}{m=n} (4\lambda)^m S_{2n-m-1, -1} + \sum_{m=0}^{n-1} (4\lambda)^m \left[ S_{2n-m-1, n-m-1} - F_n F'_{n-m-1} \right] \quad (153)$$

**Proof.** First, notice that the same arguments used in the proof of lemma (A.3) can be used to prove that

$$\phi_n B(\phi_n) = \sum_{m=n}^{2n} (4\lambda)^m B_{2n-m-2,-1} + \sum_{m=0}^{n-1} (4\lambda)^m B_{2n-m-2,n-m-1} \quad (154)$$

But, by lemma (A.4), we have

$$B_{2n-m-2,-1} = S_{2n-m-1,-1} - F_{2n-m} F'_{-1} = S_{2n-m-1,-1},$$

(since $F_{-1}$ is constant) and

$$B_{2n-m-2,n-m-1} = S_{2n-m-1,n-m-1} - F_n F'_{n-m-1}.$$
Now, we are in condition to giving the proof of Theorem (7.6)

**Proof.** Using lemmas (A.5) and (A.1), we have that

\[
\int \phi_n B(\phi_n) dx = \sum_{m=n}^{2n} (4\lambda)^m \frac{T_{2n-m-1,n-1}}{2} + \sum_{m=0}^{n-1} (4\lambda)^m \left[ \frac{T_{2n-m-1,n-1}}{2} - \int F_n F'_{n-m-1} dx \right],
\]

and multiplying lemma (A.3) by 2λ

\[
2\lambda \phi_n^2 = \sum_{m=n}^{2n} (4\lambda)^{m+1} \frac{T_{2n-m-2,n-1}}{2} + \sum_{m=0}^{n-1} (4\lambda)^{m+1} \frac{T_{2n-m-2,n-1}}{2}.
\]

Substracting

\[
\mathcal{H}_n = \int \phi_n B(\phi_n) dx - 2\lambda \phi_n^2
\]

\[
= -\frac{(4\lambda)^{2n+1}}{2} T_{-2,-1} + (4\lambda)^n \frac{T_{n-1,n-1} - T_{n-1,0}}{2} + \sum_{m=1}^{n-1} (4\lambda)^m \left[ \frac{T_{2n-m-1,n-1} - T_{2n-m-1,n-m}}{2} - \int F_n F'_{n-m-1} dx \right]
\]

\[
+ \left[ \frac{T_{2n-1,n-1}}{2} - \int F_n F'_{n-1} \right]
\]

(155)

Now, using lemma (A.2) to compute substractions of T’s, and that \(T_{-2,-1} = F_{-2}^2\) and \(T_{2n-1,n-1} = 2F_{n-1}F_n\), we finally conclude

\[
\mathcal{H}_n = -\frac{(4\lambda)^{2n+1}}{2} F_{-1}^2 + (4\lambda)^n F_{-1} F_n + \sum_{m=1}^{n-1} (4\lambda)^m \left[ F_{n-m-1} F_n - \int F_n F'_{n-m-1} dx \right]
\]

(156)

\[
+ F_{n-1} F_n - \int F_n F'_{n-1}
\]

\(\square\)
B One rational symmetric identity

The main goal of this appendix is to prove the following result. It is important to mention that this theorem appears in [15] without a proof. The authors here provided a rigorous proof of this fact.

**Theorem B.1.** Given a set of different values \( \lambda_1, \lambda_2, \ldots, \lambda_{n+1} \), the following identities are satisfied:

\[
\sum_{k=1}^{n+1} \frac{\lambda_\mu}{\prod_{j \neq k} (\lambda_k - \lambda_j)} = \begin{cases} 
0 & \text{if } 0 \leq \mu \leq n - 1 \\
1 & \text{if } \mu = n
\end{cases} \quad (157)
\]

Define the following polynomial of degree \( n \) on \( x \).

\[
K_n(x; \lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{i=1}^{n} (x - \lambda_i). \quad (158)
\]

Now, by induction define the following polynomial in \( n + 1 \) variables.

\[
P_{n+1}(x_1, x_2, \ldots, x_{n+1}) = K_n(x_1; x_2, x_3, \ldots, x_{n+1})P_n(x_2, x_3, \ldots, x_{n+1}), \quad (159)
\]

starting with \( P_1(x_1) = 1 \) and \( P_0 = 1 \).

These polynomials can also be expressed by the product formula:

\[
P_{n+1}(x_1, x_2, \ldots, x_{n+1}) = \prod_{1 \leq i < j \leq n+1} (x_i - x_j) \quad (160)
\]

**Example B.1.** Next of these polynomials are:

\[
P_2(x_1, x_2) = K_1(x_1; x_2) = x_1 - x_2
\]

\[
P_3(x_1, x_2, x_3) = K_2(x_1; x_2, x_3)P_2(x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)
\]

**Lemma B.2.** The polynomial \( P_{n+1} \) is anti-symmetric. (If we interchange any two variables, there is a change of sign on the polynomial.)

**Proof.** It is sufficient to prove the lemma for two adjacent variables. Now, given an index \( 1 \leq k \leq n \), we observe that when interchanging the variable \( x_k \) with \( x_{k+1} \)

1. the terms \( x_j - x_k \) and \( x_j - x_{k+1} \) are interchanged, for \( j < k \).

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2. the terms \( x_k - x_j \) and \( x_{k+1} - x_j \) are interchanged, for \( j > k + 1 \).

3. the term \( x_k - x_{k+1} \) becomes \( x_{k+1} - x_k = -(x_k - x_{k+1}) \).

Hence, \( P_{n+1} \) changes only one sign.

\[ \square \]

We need to introduce some notation. If \((x_1, x_2, \ldots, x_n)\) denotes a vector, we will denote the vector interchange of the variables \( x_j \) and \( x_k \), with \( j < k \) (leaving the other variables fixed) by \((x_1, \ldots, \widehat{x_j}, \ldots, \widehat{x_k}, \ldots, x_n)\). If the two variables are adjacent, we will just denote \((x_1, \ldots, \widehat{x_j}, x_{j+1}, \ldots, x_n)\).

**Corollary B.3.** If \( x_i = x_j \) for some pair \( i < j \), then \( P_n(x_1, x_2, \ldots, x_n) = 0 \).

**Proof.** Interchanging \( x_i \) with \( x_j \), does not change \( P_n(x_1, x_2, \ldots, x_n) \). Hence,

\[
P(x_1, \ldots, \widehat{x_j}, \ldots, x_k, \ldots, x_n) = P_n(x_1, x_2, \ldots, x_n)
\]

and also

\[
P(x_1, \ldots, \widehat{x_j}, x_j, \ldots, x_k, \ldots, x_n) = -P_n(x_1, x_2, \ldots, x_n)
\]

\[ \square \]

Also, denote \((x_1, \ldots, \widehat{x_i}, \ldots, x_n)\), the vector obtained from \((x_1, x_2, \ldots, x_n)\), after removing the \( x_i \) variable.

**Definition B.1.** Define the polynomials

\[
Q_{n+1,j}(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_j^i P_n(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}).
\] (161)

Observe that, since the polynomial \( P_n \) has degree \( n-1 \) in the first variable \( x_1 \), the polynomial \( Q_{n+1,j} \) has degree at most \( n-1 \), if \( j < n \), and has degree exactly \( j \), if \( j \geq n \) in the first variable \( x_1 \).

**Lemma B.4.** The distinct values \( \lambda_2, \lambda_3, \ldots, \lambda_{n+1} \) are roots of the polynomial \( Q_{n+1,j}(X, \lambda_2, \ldots, \lambda_{n+1}) \).

**Proof.** Evaluating in each \( \lambda_k \),

\[
Q_{n+1,j}(\lambda_k, \lambda_2, \ldots, \lambda_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \lambda_k^i P_n(\lambda_k, \ldots, \widehat{\lambda_i}, \ldots, \lambda_{n+1})
\]

\[= \lambda_k^i P_n(\lambda_2, \lambda_3, \ldots, \lambda_{n+1}) + (-1)^{i+1} \lambda_k^i P_n(\lambda_k, \lambda_2, \lambda_3, \ldots, \widehat{\lambda_k}, \ldots, \lambda_{n+1})
\] (162)
since all the terms with two $\lambda_k$ are equal to zero by B.3.

Now,

$$P_n(\lambda_k, \lambda_2, \lambda_3, \ldots, \hat{\lambda}_k, \ldots, \lambda_{n+1}) = (-1)^{k-2}P_n(\lambda_2, \lambda_3, \ldots, \lambda_{n+1}),$$

thus

$$Q_{n+1,j}(\lambda_k, \lambda_2, \ldots, \lambda_{n+1}) = P_n(\lambda_2, \lambda_3, \ldots, \lambda_{n+1}) + (-1)^{2k-1}P_n(\lambda_2, \lambda_3, \ldots, \lambda_{n+1}) = 0.$$  \hspace{1cm} (163)

**Corollary B.5.** If $j < n$, then

$$Q_{n+1,j}(x_1, \ldots, x_{n+1}) = 0.$$  \hspace{1cm} (164)

**Proof.** If $\lambda_2, \lambda_3, \ldots, \lambda_{n+1}$ are $n$ distinct values, $Q_{n+1,j}(x_1, \lambda_2, \lambda_2, \ldots, \lambda_{n+1})$ is a polynomial in $x_1$ of degree $n - 1$ with $n$ roots by B.4. Hence, $Q_{n+1,j}(x_1, \lambda_2, \lambda_2, \ldots, \lambda_{n+1}) = 0$. \hspace{1cm} \hfill \Box

A few inductive identities regarding the polynomial $P_n(x_1, \ldots, x_n)$ can be deduced using B.5.

**Lemma B.6.** The following identity is true

$$P_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i+n} x_1 \cdots \hat{x}_i \cdots x_n P_{n-1}(x_1, \ldots, \hat{x}_i, \ldots, x_n)$$  \hspace{1cm} (165)

**Proof.** First, notice that

$$P_n(0, x_2, \cdots, x_n) = (-1)^{n-1}x_2 \cdots x_n P_{n-1}(x_2, \cdots, x_n).$$

Hence,

$$Q_{n+1,0}(0, x_2, \ldots, x_{n+1}) = P_n(x_2, \ldots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^{i+1}P_n(0, x_2, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

$$= P_n(x_2, \ldots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^{i+n}x_2 \cdots \hat{x}_i \cdots x_n P_n(0, x_2 \ldots \hat{x}_i \ldots x_{n+1})$$

$$= 0.$$  \hspace{1cm} (166)

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Therefore,
\[ P_n(x_2, \ldots, x_{n+1}) = \sum_{i=2}^{n+1} (-1)^{i-1+n} x_2 \cdots \hat{x}_i \cdots x_n P_n(0, x_2, \ldots, \hat{x}_i, \ldots, x_{n+1}), \]
which gives the identity of the lemma, shifting by $-1$ the index on the variables.

\[\square\]

**Lemma B.7.** The following identity is true
\[ P_n(x_1, \ldots, x_n) = Q_{n,n-1}(x_1, \ldots, x_n) \]
\[ = \sum_{i=1}^{n} (-1)^{i+1} x_{i}^{n-1} P_{n-1}(x_1, \ldots, \hat{x}_i, \ldots, x_n) \] (167)

**Proof.** First, notice that
\[ Q_{1,0}(x) = (-1)^2 P_0 = 1 = P_1(x) \]
\[ Q_{2,1}(x_1, x_2) = (-1)^2 x_1 P_1(x_2) + (-1)^3 x_2 P_1(x_1) = x_1 - x_2 = P_2(x_1, x_2). \]

To simplify the writing of the proof, we will define:
\[ \sum_{j=a}^{A} c_j = 0 \text{ if } a > A \]

We will prove the identity for $n+1$, assuming it is valid for $n$.

Using lemma B.6 we have
\[ P_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_1 \cdots \hat{x}_i \cdots x_{n+1} P_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \] (168)

Now, by the induction hypothesis, each term $P_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$ is equal to
\[ \sum_{j=1}^{i-1} x_j^{n-1} P_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_n) + \sum_{j=i+1}^{n+1} x_j^{n-1} P_{n-1}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) \]
Combining with (168), we obtain

\[ P_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} \sum_{j=1}^{i-1} A_{i,j} + \sum_{i=1}^{n+1} \sum_{j=i+1}^{n+1} B_{i,j} \] (169)

where

\[ A_{i,j} = x_j^n (-1)^{n+i+j} x_1 \cdots \hat{x}_j \cdots \hat{x}_i \cdots x_n P_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_n) \]

and

\[ B_{i,j} = x_j^n (-1)^{n+i+j+1} x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n P_{n-1}(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) \]

Interchanging the order of summation,

\[ P_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{j=1}^{n+1} \sum_{i=j+1}^{n+1} A_{i,j} + \sum_{j=1}^{n+1} \sum_{i=1}^{j-1} B_{i,j} \]

(170)

Notice that in the first double summation of (170), when \( j = n + 1 \), the term

\[ \sum_{i=j+1}^{n+1} A_{i,j} = \sum_{i=n+2}^{n+1} A_{i,j} = 0 \]

by definition, but this agrees with the fact that in \( A_{i,j} \), \( n + 1 = j < i \), which gives no term \( i \), since there are only \( n + 1 \) variables. A similar situation occurs in the second summation when \( j = 1 \), then

\[ \sum_{i=1}^{j-1} B_{i,j} = \sum_{i=1}^{0} B_{i,j} = 0, \]

and now \( B_{i,j} \) satisfies that \( i < j = 1 \), but we start labeling variables at 1.

Now, factoring out \((-1)^{j+1}x_j^n\) from the terms \( A_{i,j} \) and \( B_{i,j} \), we obtain
\[ P_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j+1} x_j^n C_{i,j} \]  

(171)

with

\[ C_{i,j} = \sum_{i=j+1}^{n+1} (-1)^{n+i-1} x_1 \cdots \hat{x}_j \cdots \hat{x}_i \cdots x_n P_{n-1}(x_1, \ldots, \hat{x}_j \ldots \hat{x}_i \ldots, x_n) + \]
\[ + \sum_{i=1}^{j-1} (-1)^{n+i} x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n P_{n-1}(x_1, \ldots, \hat{x}_i \ldots \hat{x}_j \ldots, x_n). \]  

(172)

But, lemma B.6 gives

\[ C_{i,j} = P_n(x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}). \]

Hence,

\[ P_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j+1} x_j^n P_n(x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}) \]
\[ = Q_{n+1,n}(x_1, \ldots, x_{n+1}). \]  

(173)

Now, we are ready to proof Theorem B.1.

**Proof.** First, notice

\[ K_n(\overrightarrow{\lambda_1 \ldots \lambda_k \ldots, \lambda_{n+1}}) = \prod_{j \neq k} (\lambda_k - \lambda_j). \]

Hence,

\[ \sum_{k=1}^{n+1} \frac{\lambda_k^\mu}{\prod_{j \neq k} (\lambda_k - \lambda_j)} = \sum_{k=1}^{n+1} \frac{\lambda_k^\mu}{K_n(\overrightarrow{\lambda_1 \ldots \lambda_k \ldots, \lambda_{n+1}})}. \]  

(174)

Now, permuting \( x_1 \) with \( x_k \) in equation (159) we obtain
\[ P_{n+1}(\vec{x}_1 \ldots \vec{x}_k \ldots, x_{n+1}) = K_n(\vec{x}_1 \ldots \vec{x}_k \ldots, x_{n+1})P_n(x_2 \ldots, x_{k-1}, x_1, x_{k+1} \ldots, x_{n+1}) \]

But, now moving the variable \( x_1 \) at the beginning in the \( P_n \) term on the right of the above equation, we have

\[ P_n(x_2 \ldots, x_{k-1}, x_1, x_{k+1} \ldots, x_{n+1}) = (-1)^{k-2}P_n(x_1, \ldots, \hat{x}_k \ldots, x_n) \]

And, using that

\[ P_{n+1}(\vec{x}_1 \ldots \vec{x}_k \ldots, x_{n+1}) = -P_{n+1}(x_1, \ldots, x_{n+1}), \]

we obtain

\[ P_{n+1}(x_1, \ldots, x_{n+1}) = (-1)^{k-1}K_n(\vec{x}_1 \ldots \vec{x}_k \ldots, x_{n+1})P_n(x_1, \ldots, \hat{x}_k \ldots, x_n). \]

Hence, equation (174) becomes

\[ \sum_{k=1}^{n+1} \frac{\lambda_k^\mu}{\prod_{j \neq k} (\lambda_k - \lambda_j)} = \frac{\sum_{k=1}^{n+1}(-1)^{k-1}\lambda_k^\mu P_n(\lambda_1, \ldots, \hat{\lambda}_k \ldots, \lambda_n)}{P_{n+1}(\lambda_1, \ldots, \lambda_{n+1})} \]

(175)

where

\[ Q_{n+1, \mu}(\lambda_1, \ldots, \lambda_{n+1}) = \frac{P_{n+1}(\lambda_1, \ldots, \lambda_{n+1})}{\prod_{j \neq k} (\lambda_k - \lambda_j)} \]

Therefore, the above sum is 0 if \( 0 \leq \mu \leq n-1 \) (corollary B.5) and 1 if \( \mu = n \) (lemma B.7).

\[ \square \]

C  \textbf{Note on successive derivatives of } \( E = \frac{iq_x}{q} \)

The first derivative of \( E \) is

\[ E' = \frac{iq_{xx}}{q} - \frac{iq_x^2}{q^2} \]

Setting

\[ E_{(2)} = \frac{iq_{xx}}{q}, \]

(176)
we can write the first derivative as:

$$E' = E_{(2)} + iE^2$$  \hspace{1cm} (177)

Now, define:

$$E_{(n)} = \frac{iq^{(n)}}{q}.$$

Hence, we easily compute:

$$E'_{(n)} = E_{(n+1)} + iE_{(n)}E$$ \hspace{1cm} (179)

Using this notation, we can easily compute derivatives of $E$ of superior order. For example,

$$E'' = E'(2) + 2iE \cdot E'$$

$$= E_{(3)} + iE_{(2)}E + 2iE \cdot \left( E_{(2)} + iE^2 \right)$$ \hspace{1cm} (180)

and

$$E''' = E'(3) + 3i \left( E_{(2)}E \right)' - 6E^2E'$$

$$= E_{(4)} + iE_{(3)}E + 3i \left[ E'_{(2)}E + E_{(2)}E' \right] - 6E^2 \cdot \left[ E_{(2)} + iE^2 \right]$$

$$= E_{(4)} + iE_{(3)}E - 6E^2E_{(2)} - 6iE^4 + 3iE'_{(2)}E + 3iE_{(2)}E'$$

$$= E_{(4)} + iE_{(3)}E - 6E^2E_{(2)} - 6iE^4 + 3iE'_{(2)}E + 3iE_{(2)} \left( E_{(2)} + iE^2 \right)$$

$$= E_{(4)} + iE_{(3)}E - 9E^2E_{(2)} - 6iE^4 + 3iE'_{(2)}E + 3iE_{(2)}E$$

$$= E_{(4)} + iE_{(3)}E - 9E^2E_{(2)} - 6iE^4 + 3iE^2_{(2)} + 3iE \left( E_{(3)} + iE_{(2)}E \right)$$

$$= E_{(4)} + 4iE_{(3)}E - 12E^2E_{(2)} - 6iE^4 + 3iE^2_{(2)}$$ \hspace{1cm} (181)

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