Diffusion as a damping mechanism for neutron-star oscillations

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Abstract. We study the effects of diffusion on damping of oscillations in the neutron star cores. This dissipation mechanism is usually ignored in the literature. As we show, the effect of diffusion is always smaller than viscous dissipation if the normal (nonsuperfluid and nonsuperconducting) matter of neutron stars is considered. However, we argue that for superconducting stars the role of diffusion may increase.

1. Introduction

The most frequently-addressed and well-investigated dissipative mechanisms operating in neutron stars are thermal conductivity, bulk and shear viscosities [1, 2]. Calculations of damping times for different oscillation modes [1, 2, 3] indicate that among these three mechanisms the shear viscosity is the leading one at not too high stellar temperatures.

Here in this paper we shall study the fourth possible dissipation mechanism related to diffusion of particles in dense stellar cores. To the best of our knowledge, this mechanism has not been studied in the context of neutron star physics yet. We shall compare the effect of diffusion and shear viscosity on damping of oscillations by analyzing sound waves in neutron-star matter. For the sake of simplicity, we restrict ourselves to the case of nonsuperfluid and nonsuperconducting matter.

In section 2 we derive a general formula for the damping time of sound waves due to diffusion. In section 3 we apply this formula to neutron-proton-electron matter of neutron-star cores. Finally, section 4 contains a brief summary of our results.

2. Diffusion damping time for sound waves

Let us consider a small periodic perturbation over a static background solution of hydrodynamic equations. This means that the velocity $\vec{v}$ of the perturbation is small as well as the departures of thermodynamic quantities from their equilibrium values, and thus we can write expansions in the form $f = f_0 + f'$ with $f_0$ being the equilibrium value of a quantity $f$ and $f'$ being the small deviation from equilibrium. When dissipation in the system is weak, the damping time $\tau$ is given by the formula [4]

$$\frac{1}{\tau} = -\frac{1}{2E_{\text{mech}}} \left\langle \frac{dE_{\text{mech}}}{dt} \right\rangle_P,$$

(1)
where

\[ E_{\text{mech}} = \left\langle \int w_0 v^2 c^2 \, dV \right\rangle_P, \]

\[ \left\langle \frac{dE_{\text{mech}}}{dt} \right\rangle_P = -c \left\langle \int T \partial_\mu s^\mu \, dV \right\rangle_P. \] (3)

In equations (1)–(3) \( E_{\text{mech}} \) is the total mechanical energy, stored in the perturbation; \( c \) is the speed of light; \( w_0 \) is the equilibrium enthalpy density; \( \langle \ldots \rangle_P \) denotes averaging over the period of the perturbation; spatial integration is performed over the whole volume of the system. Further, \( \langle dE_{\text{mech}}/dt \rangle_P \) is the average rate of energy loss; \( s^\mu \) is the entropy density four-current; \( \partial_\mu = (\partial/\partial(ct), \nabla) \) and \( T \) is the temperature.

A convenient formalism of relativistic dissipative hydrodynamics accounting for the effects of viscosity and diffusion has recently been developed in Dommes et. al. [5]. In particular, these authors derived the entropy generation equation, from which it follows, that in the case of vanishing magnetic field and non-relativistic velocities, the entropy generation due to diffusion and due to shear viscosity can be written as

\[ \langle \partial_\mu s^\mu \rangle_{\text{diff}} = D_{km} \vec{d}_k \cdot \vec{d}_m, \]

\[ \vec{d}_k = \nabla \left( \frac{\mu_k}{T} \right) - \frac{e_k E}{T}. \] (4)

\[ \langle \partial_\mu s^\mu \rangle_\eta = \frac{\eta}{2cT} \left( \partial_a v_b + \partial_b v_a - \frac{2}{3} \delta_{ab} \text{div} \vec{v} \right)^2. \] (5)

Here and hereafter Latin indices \( \{a, b, \ldots\} \) refer to tensor components, Latin indices \( \{i, j, \ldots\} \) refer to particle species present in the fluid and the summation over repeated indices is implied. Diffusion coefficients \( D_{km} \) constitute a positive-definite symmetric matrix, \( \eta \) is the non-negative coefficient of shear viscosity, \( e_k \) is the charge of particle species \( k \), \( \mu_k \) and \( E \) are the chemical potential of particle species \( k \) and the electric field, respectively.

Now we are in position to apply the procedure described above to the case of non-relativistic sound waves in quasineutral degenerate matter. Quasineutrality means that the electric current in the perturbed fluid should vanish to a very high precision, \( e_k \vec{j}_k = 0 \) [8], where

\[ \vec{j}_k = n_k \vec{v} - c D_{km} \vec{d}_m \] (6)

is the number density current for particle species \( k \). The vector \( \vec{d}_m \) depends on the self-consistent electric field (see Eq. 4). From the quasineutrality condition it follows that this field can be expressed as

\[ e_k \vec{j}_k = 0 \quad \Rightarrow \quad \frac{\vec{E}}{T} = \frac{e_k D_{km}}{e_r D_{rs} e_s} \nabla \left( \frac{\mu_m}{T} \right). \] (7)

Since hydrodynamic non-dissipative equations do not contain linear terms in the electric field, perturbations of the latter do not appear in linearised equations in homogeneous matter. In the linear approximation the magnetic field, generated in the course of oscillations, can also be ignored. Nevertheless, one should account for the perturbed electric field when calculating damping time due to diffusion.

The degeneracy of the matter means that the pressure \( p = p(n_k) \) and chemical potentials \( \mu_k = \mu_k(n_m) \) can be considered as functions of number densities only. Also for degenerate matter coefficients \( D_{km} \) can be expressed in terms of momentum transfer rates \( J_{ik} \) [5], introduced in [6]–[8] and, therefore, are known from the microscopic theory. It can be shown [5], that in this case \( D_{kn} \) satisfy an additional constraint, \( \mu_k \vec{D}_{kn} = 0 \).
Accounting for the degeneracy and quasineutrality of the matter it is easy to derive $\tau$ for sound waves due to diffusion and shear viscosity. Indeed, linearised hydrodynamic equations in homogeneous quasineutral matter,

$$\frac{w_0}{c^2} \frac{\partial \bar{v}}{\partial t} = -\bar{\nabla} p', \quad \frac{\partial n'_k}{\partial t} + n_{k0} \text{div} \bar{v} = 0, \quad p' = \left( \frac{\partial p}{\partial n_k} \right)_0 n'_k, \quad (8)$$

admit a solution of the form $\{\vec{v}, p', n'_k, \vec{E}'\} \sim e^{i(kx - \omega t)}$ with frequency $\omega$ and wave number $k$ related through the speed of sound $c_s$ as

$$\omega = c_s k, \quad c_s = \sqrt{\frac{1}{w_0} \left( \frac{\partial p}{\partial n_k} \right)_0} n_{k0}. \quad (9)$$

Now using (2)–(5) one can easily obtain

$$\frac{1}{\tau_{\text{diff}}} = \frac{c k^2}{2w_0 T} \left( \frac{c}{c_s} \right)^2 \cdot \tilde{D}_{km} \cdot \left( \frac{\partial p}{\partial n_k} \right)_0 \cdot \left( \frac{\partial p}{\partial n_m} \right)_0, \quad \frac{1}{\tau_{\eta}} = \frac{c^2 k^2 4\eta}{2w_0} 3, \quad (10)$$

where the modified diffusion coefficients $\tilde{D}_{km}$ are given by the formula

$$\tilde{D}_{km} = D_{kn} \left( \delta_{nm} - \frac{e_n e_g D_{qm}}{e_r D_{rs} e_s} \right). \quad (11)$$

3. Application: $npe$-matter

Let us illustrate these results by considering quasineutral degenerate matter composed of neutrons ($n$), protons ($p$) and electrons ($e$). In what follows, we employ BSk24 equation of state [9]; shear viscosity and momentum transfer rates are adopted from [10].

Figure 1. Damping times for sound waves due to diffusion (left panel) and shear viscosity (right panel) as functions of baryon number density, $n_b$, for a set of temperatures, $T$. 

![Figure 1](image-url)
Damping times due to diffusion and shear viscosity in npe-matter for different temperatures are plotted in figure 1. We see that diffusion leads to the damping on a typical timescale \( \sim (10^2 \div 10^3) \) years (for \( k = 10^{-6} \text{ cm}^{-1} \)) which is much larger than \( \tau \) for the shear viscosity. Therefore, in the normal npe-matter particle diffusion can be safely ignored.

However, under certain circumstances the situation may change. The point is we did not take into account the effects due to proton superconductivity which is very likely to take place in the neutron-star interiors [11]. In this case in formulas for \( D_{km} \), instead of proton number density, one should introduce the number density of Bogoliubov excitations, corresponding to the so-called ‘normal’ component of superconducting proton fluid. If all protons are superconducting, this number density vanishes. In addition, scattering on protons is also suppressed, so that \( J_{ep} \) and \( J_{np} \) become very small compared to \( J_{en} \). From the formulas, relating \( D_{km} \) to \( J_{ik} \) one can see that \( D_{km} \sim n_k n_m \) and therefore, if all protons are superconducting, we have only three nonzero coefficients: \( D_{ee}, D_{en} \) and \( D_{nn} \). Let us take a look, for example, at the coefficient \( D_{nn} \), which, according to [5], is given by the formula

\[
D_{nn} = \frac{n_e^2 T (J_{ep}(\mu_e n_e + \mu_p n_p)^2 + J_{en} \mu_p^2 n_e^2 + J_{np} \mu_e^2 n_p^2)}{c(J_{ep}(J_{en} + J_{np}) + J_{en} J_{np})(\mu_e n_e + \mu_p n_p + \mu_n n_n)^2}. \tag{12}
\]

Considering the case of completely superconducting protons and using \( \mu_k D_{kn} = 0 \) it is easy to show that non-zero diffusion coefficients reduce to

\[
D_{nn}^{(sf)} = \frac{n_e^2 T \mu_e^2 n_e^2}{c(\mu_e n_e + \mu_n n_n)^2 J_{en}}, \quad D_{en}^{(sf)} = -\frac{\mu_n}{\mu_e} D_{nn}^{(sf)}, \quad D_{ee}^{(sf)} = \frac{\mu_n^2}{\mu_e^2} D_{nn}^{(sf)}. \tag{13}
\]

In superconducting matter electrons are coupled to protons and move together as ‘neutral particles’, which means that, effectively, we have a two-component neutral fluid. In this case it is easy to show, that in (10) one should replace \( D_{km} \rightarrow D_{km}' \). Thus, we expect that the ratio of damping times due to diffusion in normal matter and in matter with superconducting protons is of the order of

\[
\frac{\tau_{\text{diff}}^{(n)}}{\tau_{\text{diff}}^{(sf)}} \sim \frac{(\partial p/\partial n_k)_0 D_{km}^{(sf)}(\partial p/\partial n_m)_0}{(\partial p/\partial n_k)_0 D_{km}(\partial p/\partial n_m)_0}. \tag{14}
\]

For \( n_b = 0.25 \text{ fm}^{-3} \) we find \( \frac{\tau_{\text{diff}}^{(n)}}{\tau_{\text{diff}}^{(sf)}} \sim 7.2 \times 10^4 \), i.e., in the case of superconducting protons diffusive damping can become much stronger and even more efficient than the viscous damping. This estimate is, of course, very rough since, strictly speaking, the formula (10) was derived for normal matter and will definitely be modified by the effects of proton superconductivity. However we tend to believe that accurate calculation within the framework of superfluid hydrodynamics will confirm this estimate.

4. Conclusion
We derived a formula for the damping time of sound waves, propagating in the degenerate nonsuperfluid and nonsuperconducting matter, and applied it to study sound waves in the neutron-star cores. We demonstrate that the effect of diffusion on damping is negligible in comparison to viscous damping, if the normal matter of neutron-star cores is considered. This result indicates that diffusion can likely be neglected in the studies of large-scale oscillation modes (such as \( f^- \), \( p^- \), \( g^- \), and \( r^- \)modes) in nonsuperfluid and nonsuperconducting neutron stars. However, we argue that it may become important if the protons in the system are superconducting.
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