Numerical Methods for Fractional-Order Fornberg-Whitham Equations in the Sense of Atangana-Baleanu Derivative

Naveed Iqbal, Humaira Yasmin, Akbar Ali, Abdul Bariq, M. Mossa Al-Sawalha, and Wael W. Mohammed

1Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2Department of Basic Sciences, Preparatory Year Deanship, King Faisal University, Al-Ahsa 31982, Saudi Arabia
3Department of Mathematics, Laghman University, Mehterlam, 2701 Laghman, Afghanistan
4Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Correspondence should be addressed to Abdul Bariq; abdulbariq.maths@lu.edu.af

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In this paper, we investigate the numerical solution of the Fornberg-Whitham equations with the help of two powerful techniques: the modified decomposition technique and the modified variational iteration technique involving fractional-order derivatives with Mittag-Leffler kernel. To confirm and illustrate the accuracy of the proposed approach, we evaluated in terms of fractional order the projected models. Furthermore, the physical attitude of the results obtained has been acquired for the fractional-order different value graphs. The results demonstrated that the future method is easy to implement, highly methodical, and very effective in analyzing the behavior of complicated fractional-order linear and nonlinear differential equations existing in the related areas of applied science.

1. Introduction

The analysis of the Fornberg-Whitham equation (FWE) is a significant mathematical equation of mathematical physics. The Fornberg-Whitham equation is defined as [1, 2]

\[ D_{\alpha}^{\mu} - D_{\psi}^{\mu} + D_{\chi}^{\mu} = \mu D_{\xi}^{\mu} - \mu D_{\zeta}^{\mu} + 3D_{\zeta}^{\mu}D_{\xi}^{\mu}. \]  

This model was invented to evaluate the nonlinear breaking dispersive ocean waves. The Fornberg-Whitham equation is shown to yield peakon solutions as a physical equation for waves of restricting height and the occurrences of wave breaking. Fractional calculus is now widely used and accepted, owing to its well-known uses in a variety of fields of seemingly disparate sectors of science and engineering [3, 4]. Many scholars, including Gupta and Singh [5] and Alderremy et al. [6], have examined the fractional of the Fornberg-Whitham equation relevant to the fractional Caputo derivative, Sunthrayuth et al. [7], Singh et al. [8], etc. Because of the singular kernel of the fractional Caputo derivative, its implementations are limited. Caputo and Fabrizio [9] created derivatives of any (real or complex) order with a nonsingular kernel. Caputo and Fabrizio's derivative has been used to a variety of real-world situations, including fractional nonhomogeneous heat models [10], El Nino-Southern fractional oscillations models [11], and arbitrary-order system of smoking models [12]. Atangana and Baleanu [13] devised a novel fractional-order derivative called the Atangana-Baleanu (AB) fractional derivative, which has the kernel of a Mittag-Leffler-type function. Kumar et al. [14] investigated the regularised long-wave equation with a Mittag-Leffler-type kernel incorporating the fractional operator. A Mittag-Leffler-type kernel is used in the chemical kinetics system connected with a fractional derivative which was investigated by Singh et al. [15]. Baleanu et al. [16] recently proposed optimal fractional models with nonsingular Mittag-Leffler kernels. As we all know, the Mittag-Leffler function is more beneficial in expressing physical difficulties than the power function or the exponential function; as a result, the AB fractional derivative is well suited to
unravelling material heterogeneities and structures or media with different scales.

George Adomian was introduced to and established a technique for “solving integro-differential, differential equations, delay differential, and partial differential equations” [17, 18]. The result is discovered as an infinite sequence that quickly converges to precise solution. This method has been shown to be effective in solving both linear and nonlinear models. The method for solving a nonlinear operator problem is to use decomposition equations in a series of functions. Each expression’s sequence is derived from a polynomial derived from the expansion of an approximate solution into power series. The Adomian decomposition method technique is really simple in theory, but the difficulty arises when it comes to determining polynomials and illustrating the convergence of a series of functions [19]. Lesnic [20] analyzed the convergent of the Adomian decomposition method when using heat and wave models for both backward and forward time evolution. Gaber and El-Sayed used the Adomian method of solving fractal-order partial differential equations on a finite domain in [21]. Ghoreishi et al. [22] investigated the Adomian decomposition method’s ability to investigate nonlinear wave problems with changing coefficients, demonstrating that the Adomian decomposition method can solve these equations without the need for dissertation, linearization, transformation, or perturbation.

The variational iteration approach [23, 24] was published in the late 1990s to solve a seepage flow with fractional derivatives and a nonlinear oscillator, and it has since been widely utilized as a primary analytical tool for solving a variety of nonlinear problems. It has fully grown into a fully fledged mathematical approach as a result of considerable research by a number of authors, including He [25, 26], Ganji and Sadighi [27], Ozis and Yildirim [28], and Noor and Mohyud-Din [29]. On November 24, 2018, we searched Clarivate’s Web of Science for “variational iteration approach” and got 3761 hits. The technique’s identification of the Lagrange multiplier necessitates the understanding of variational theory [30], and the technique’s sophisticated identification process may limit its implementation to real-world issues.

2. Preliminary Concepts

Definition 1. The Caputo fractional derivative is given as [31]

\[
C_0^\gamma D_\alpha^\gamma f(\zeta, \theta) = \frac{1}{(n - \gamma)} \int_0^\gamma (\zeta - \theta)^{n-\gamma-1} f^{(n)}(\zeta, \theta) d\theta, \quad n - 1 < \gamma \leq n.
\]  (2)

Definition 2. The Laplace transformation connected with fractional Caputo derivative \(LC D_\alpha^\gamma f(\zeta, \theta)\) is expressed by [31]

\[
\mathcal{L}[LC D_\alpha^\gamma f(\zeta, \theta)](s) = s^n \mathcal{L}[f(x, \zeta)](s) - s^{n-1} f(x, 0).
\]  (3)

Definition 3. In Caputo sense, the Atangana-Baleanu derivative is defined as [31]

\[
ABC D_\alpha^\gamma f(\zeta, \theta) = \frac{A(y)}{1 - y} \int_0^\gamma f^{(k)}(k)E_{\gamma}(\gamma(1-k)^y) dk,
\]  (4)

where \(A(y)\) is a normalization function such that \(A(0) = A(1) = 1, f \in H^1(a, b), b > a, \gamma \in [0, 1]\), and \(E_{\gamma}\) represent the Mittag-Leffler function.

Definition 4. The Atangana-Baleanu derivative in the Riemann-Liouville sense is defined as [31]

\[
AB D_\alpha^\gamma f(\zeta, \theta) = \frac{A(y)}{1 - y} \int_0^\gamma f^{(k)}(k)E_{\gamma}(\gamma(1-k)^y) dk.
\]  (5)

Definition 5. The Laplace transform connected with the Atangana-Baleanu operator is defined as [31]

\[
AB D_\alpha^\gamma f(\zeta, \theta) = \frac{A(y)\mathcal{L}[f^{(k)}(\zeta, \theta)](s) - s^{k-1} f(0)}{(1 - y)(\gamma(y(1 - y)))}.
\]  (6)

Definition 6. Consider \(0 < \gamma < 1\), and \(f\) is a function of \(y\); then, the fractional-order integral operator of \(y\) is given as [31]

\[
ABC I_\alpha^\gamma f(\zeta, \theta) = \frac{1 - y}{A(y)} f(\zeta, \theta) + \frac{\gamma}{A(y)\Gamma(\gamma)} \int_0^\gamma f(k)(\zeta - k)^{-1} dk.
\]  (7)

3. The Methodology of Variational Iteration Method

This section introduces the solution of fractional partial differential equations with the help of the variational iteration method.

\[
ABC D_\alpha^\gamma \nu(\zeta, \theta) + \mathcal{B}(\zeta, \theta) + \mathcal{N}(\zeta, \theta) - \mathcal{P}(\zeta, \theta) = 0, \quad \phi - 1 < \gamma \leq \phi.
\]  (8)

The initial condition is

\[
\nu(\zeta, 0) = g(\zeta),
\]  (9)

where \(ABC D_\alpha^\gamma = \partial^\gamma / \partial \zeta^\gamma\) is the fractional derivative Caputo order \(\gamma\). \(\mathcal{B}\) and \(\mathcal{N}\) are linear and nonlinear terms, respectively, and \(\mathcal{P}\) is the source function.

The Laplace transformation is applied to equation (8); we get

\[
L[ABC D_\alpha^\gamma \nu(\zeta, \theta)] + L[\mathcal{B}(\zeta, \theta) + \mathcal{N}(\zeta, \theta) - \mathcal{P}(\zeta, \theta)] = 0.
\]  (10)
The Lagrange multiplier iterative method is
\[ L\left[ABC D^\alpha_0 v(\zeta, \mathfrak{F})\right] + L\left[\mathcal{G}(\zeta, \mathfrak{F}) + \mathcal{N}(\zeta, \mathfrak{F}) - \mathcal{P}(\zeta, \mathfrak{F})\right] = 0. \] (11)

A Lagrange multiplier is as
\[ \lambda(s) = -\frac{(s^\gamma(1 - y) + y)}{s^\gamma}. \] (12)

Applying inverse Laplace transform \( L^{-1} \), equation (11) can be written as
\[ v_{\phi+1}(\zeta, \mathfrak{F}) = v_\phi(\zeta, \mathfrak{F}) - L^{-1}\left[\frac{(s^\gamma(1 - y) + y)}{s^\gamma} \left[-L\left\{\mathcal{G}(\zeta, \mathfrak{F}) + \mathcal{N}(\zeta, \mathfrak{F})\right\} - L\left[\mathcal{P}(\zeta, \mathfrak{F})\right]\right]\right]. \] (13)

### 4. The Conceptualization of MDM

In this section, we discuss the solution of fractional partial differential equations with the help of the modified decomposition method.

\[ ^{ABC}D^\alpha_0 v(\zeta, \mathfrak{F}) + \mathcal{G}(\zeta, \mathfrak{F}) + \mathcal{N}(\zeta, \mathfrak{F}) - \mathcal{P}(\zeta, \mathfrak{F}) = 0, \quad m - 1 < y \leq m. \] (14)

The initial condition is
\[ v(\zeta, 0) = g(\zeta), \] (15)

where \(^{ABC}D^\alpha_0\) is the fractional derivative of Caputo order \(y\), \(\mathcal{G}\) and \(\mathcal{N}\) are linear and nonlinear terms, respectively, and \(\mathcal{P}\) is the source term.

Using Laplace transformation to equation (14), we get
\[ L\left[^{ABC}D^\alpha_0 v(\zeta, \mathfrak{F})\right] + L[\mathcal{G}(\zeta, \mathfrak{F}) + \mathcal{N}(\zeta, \mathfrak{F}) - \mathcal{P}(\zeta, \mathfrak{F})] = 0. \] (16)

Taking the Laplace transform of differentiation property, we have
\[ L[v(\zeta, \mathfrak{F})] = \frac{1}{s} v(\zeta, 0) + \frac{(s^\gamma(1 - y) + y)}{s^\gamma} L[\mathcal{P}(\zeta, \mathfrak{F})] - \frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\left[\mathcal{G}(\zeta, \mathfrak{F}) + \mathcal{N}(\zeta, \mathfrak{F})\right]. \] (17)

MDM result of infinite series \(v(\zeta, \mathfrak{F})\),
\[ v(\zeta, \mathfrak{F}) = \sum_{\phi=0}^{\infty} v_\phi(\zeta, \mathfrak{F}). \] (18)

\(\mathcal{N}\) nonlinear function is defined as
\[ \mathcal{N}(\zeta, \mathfrak{F}) = \sum_{\phi=0}^{\infty} \mathcal{A}_\phi. \] (19)

The nonlinear terms can be analyzed with the aid of Adomian polynomials. So the Adomian polynomial formula is expressed as
\[ \mathcal{A}_\phi = \frac{1}{\phi!} \frac{\partial^\phi}{\partial \lambda^\phi} \left(\mathcal{N}\left(\sum_{\phi=0}^{\infty} \lambda^\phi v_\phi\right)\right) \bigg|_{\lambda=0}. \] (20)

Then, put equations (18) and (19) into (17), which gives
\[ L\left[\sum_{\phi=0}^{\infty} v_\phi(\zeta, \mathfrak{F})\right] = \frac{1}{s} v(\zeta, 0) + \frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\left\{\mathcal{P}(\zeta, \mathfrak{F})\right\} - \frac{(s^\gamma(1 - y) + y)}{s^\gamma} \left\{L\left[\sum_{\phi=0}^{\infty} v_\phi\right] + \sum_{\phi=0}^{\infty} \mathcal{A}_\phi\right\}\bigg]\right]. \] (21)

Applying the inverse Laplace transformation to equation (21), we get
\[ \sum_{\phi=0}^{\infty} v_\phi(\zeta, \mathfrak{F}) = L^{-1}\left[\frac{1}{s} v(\zeta, 0) + \frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\{\mathcal{P}(\zeta, \mathfrak{F})\} - \frac{(s^\gamma(1 - y) + y)}{s^\gamma} \left\{L\left[\sum_{\phi=0}^{\infty} v_\phi\right] + \sum_{\phi=0}^{\infty} \mathcal{A}_\phi\right\}\right]. \] (22)

Define the terms as follows:
\[ v_0(\zeta, \mathfrak{F}) = L^{-1}\left[\frac{1}{s} v(\zeta, 0) + \frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\{\mathcal{P}(\zeta, \mathfrak{F})\}\right], \]
\[ v_1(\zeta, \mathfrak{F}) = -L^{-1}\left[\frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\left\{\mathcal{G}(v_0) + \mathcal{A}_0\right\}\right]. \] (23)

In general, \(\phi \geq 1\) is defined as
\[ v_{\phi+1}(\zeta, \mathfrak{F}) = -L^{-1}\left[\frac{(s^\gamma(1 - y) + y)}{s^\gamma} L\left\{\mathcal{G}(v_{\phi}) + \mathcal{A}_\phi\right\}\right]. \] (24)

### 5. Application of Techniques

**Example 7.** Consider the time-fractional nonlinear Fornberg-Whitham equation
\[ D^\alpha_0 v - D_{\zeta\zeta}^\alpha v + D_\zeta v = vD_{\zeta\zeta} v - vD_\zeta v + 3D_\zeta vD_{\zeta\zeta} v, \quad 0 < y \leq 1, \] (25)

with the initial condition
Taking Laplace transformation of (25),
\[
\mathcal{L}^\phi \left( \frac{v(\zeta, 0)}{s^\phi(1-\gamma) + y} \right) \left\{ \mathcal{L} \left[ v(\zeta, 0) \right] - \frac{1}{s} v(\zeta, 0) \right\} = L \left[ D_{\zeta_{\zeta_{\zeta}}} v - D_\zeta v + v D_{\zeta_{\zeta}} v - \nu D_\zeta v + 3 D_\zeta v D_{\zeta_{\zeta}} v \right].
\]
(27)

Using inverse Laplace transformation
\[
v(\zeta, 3) = L^{-1} \left[ \frac{v(\zeta, 0)}{s} \right] = L^{-1} \left[ \frac{e^{(2)}(\gamma)}{s} \right],
\]
\[
v(\zeta, 3) = e^{(2)}(\gamma),
\]
\[
\sum_{n=0}^{\infty} v \phi_{n+1}(\zeta, 3) = L^{-1} \left[ \frac{(\phi^\gamma(1-\gamma) + y) L \left[ \sum_{n=0}^{\infty} (D_{\zeta_{\zeta}} v)_n - \sum_{n=0}^{\infty} (D_\zeta v)_n \right] + \sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + 3 \sum_{n=0}^{\infty} C_n \right], \quad \phi = 0, 1, 2, \ldots,
\]
\[
A_0 (v D_{\zeta_{\zeta}} v) = v_0 D_{\zeta_{\zeta}} v_0,
\]
\[
B_0 (v D_\zeta v) = v_0 D_\zeta v_0,
\]
\[
A_1 (v D_{\zeta_{\zeta_{\zeta}}} v) = v_0 D_{\zeta_{\zeta}} v_1 + v_1 D_{\zeta_{\zeta}} v_0,
\]
\[
B_1 (v D_\zeta v) = v_0 D_\zeta v_1 + v_1 D_\zeta v_0,
\]
\[
A_2 (v D_{\zeta_{\zeta_{\zeta}}} v) = v_1 D_{\zeta_{\zeta}} v_2 + v_2 D_{\zeta_{\zeta}} v_1 + v_1 D_{\zeta_{\zeta}} v_0,
\]
\[
B_2 (v D_\zeta v) = v_1 D_\zeta v_2 + v_2 D_\zeta v_1 + v_2 D_\zeta v_0,
\]
\[
C_0 (D_\zeta v D_{\zeta_{\zeta}} v) = D_\zeta v_0 D_{\zeta_{\zeta}} v_0,
\]
\[
C_1 (D_\zeta v D_{\zeta_{\zeta}} v) = D_\zeta v_0 D_{\zeta_{\zeta}} v_1 + D_\zeta v_1 D_{\zeta_{\zeta}} v_0,
\]
\[
C_2 (D_\zeta v D_{\zeta_{\zeta}} v) = D_\zeta v_1 D_{\zeta_{\zeta}} v_2 + D_\zeta v_2 D_{\zeta_{\zeta}} v_1 + D_\zeta v_1 D_{\zeta_{\zeta}} v_0,
\]
(29)

for \( \phi = 1, \)
\[
v(\zeta, 3) = L^{-1} \left[ \frac{(\phi^\gamma(1-\gamma) + y) L \left[ D_{\zeta_{\zeta_{\zeta}}} v_0 - D_\zeta v_0 + A_0 - B_0 + 3 C_0 \right] \right] = \frac{1}{2} e^{(2)}(\gamma) \left( 1 - \gamma \right) + \frac{\gamma^3 \Gamma(\gamma + 1)}{\Gamma(\gamma + 1)},
\]
(30)
Apply the variational method to obtain series form solution. The iteration formulas for equation (25), we get

\[ v_{\phi+1}(\zeta, \mathfrak{F}) = v_{\phi}(\zeta, \mathfrak{F}) - L^{-1} \left[ \frac{\left( \frac{\partial}{\partial \zeta} (1-y) + y \right)}{s^r} L \left\{ D_{\zeta}^\nu v_0 + D_\zeta \nu_0 ight\} ight], \]

where

\[ v_0(\zeta, \mathfrak{F}) = e^{(\zeta/2)}. \]  

(35)

For \( \phi = 0, 1, 2, \ldots \),

\[ v_1(\zeta, \mathfrak{F}) = v_0(\zeta, \mathfrak{F}) - L^{-1} \left[ \frac{\left( \frac{\partial}{\partial \zeta} (1-y) + y \right)}{s^r} L \left\{ D_{\zeta}^\nu v_0 + D_\zeta \nu_0 ight\} \right], \]

\[ v_1(\zeta, \mathfrak{F}) = e^{(\zeta/2)} - \frac{1}{2} e^{(\zeta/2)} \left( 1 - y + \frac{\zeta y}{2^{(y+y+1)}} \right), \]

\[ v_2(\zeta, \mathfrak{F}) = v_1(\zeta, \mathfrak{F}) - L^{-1} \left[ \frac{\left( \frac{\partial}{\partial \zeta} (1-y) + y \right)}{s^r} L \left\{ D_{\zeta}^\nu v_1 + D_\zeta \nu_1 ight\} \right], \]

\[ v_2(\zeta, \mathfrak{F}) = e^{(\zeta/2)} - \frac{1}{2} e^{(\zeta/2)} \left( 1 - y + \frac{\zeta y}{2^{(y+y+1)}} \right) - \frac{1}{8} e^{(\zeta/2)} \frac{3^{y-1}}{2^{(y+y+1)}}, \]

\[ v_3(\zeta, \mathfrak{F}) = v_2(\zeta, \mathfrak{F}) - L^{-1} \left[ \frac{\left( \frac{\partial}{\partial \zeta} (1-y) + y \right)}{s^r} L \left\{ D_{\zeta}^\nu v_2 + D_\zeta \nu_2 ight\} \right], \]

\[ v_3(\zeta, \mathfrak{F}) = e^{(\zeta/2)} - \frac{1}{2} e^{(\zeta/2)} \left( 1 - y + \frac{\zeta y}{2^{(y+y+1)}} \right) - \frac{1}{8} e^{(\zeta/2)} \frac{3^{y-1}}{2^{(y+y+1)}}, \]

\[ \cdots. \]  

(37)

The exact solution of equation (25) at \( y = 1 \),

\[ v(\zeta, \mathfrak{F}) = e^{(\zeta/2)} - \frac{1}{2} e^{(\zeta/2)} \left( 1 - y + \frac{\zeta y}{2^{(y+y+1)}} \right) - \frac{1}{8} e^{(\zeta/2)} \frac{3^{y-1}}{2^{(y+y+1)}}, \]

\[ + \frac{1}{4} e^{(\zeta/2)} \left( 1 - y + \frac{\zeta y}{2^{(y+y+1)}} \right) - \frac{1}{8} e^{(\zeta/2)} \frac{3^{y-1}}{2^{(y+y+1)}}, \]

\[ \frac{1}{32} e^{(\zeta/2)} \frac{3^{y-1}}{2^{(y+y+1)}}, \]

\[ \left\{ \left( 1 - y \right)^3 + \left( 1 - y \right) \left( 1 + y + 2y^2 \right) \frac{\zeta y}{2^{(y+y+1)}} \right\}, \]

\[ + \frac{3y^2(1-y)3^{y-1}}{2^{(y+y+1)}} \frac{\zeta y}{2^{(y+y+1)}} \right\}, \]

\[ \cdots. \]  

(38)

In Figure 1, the analytical results of MDM/MVITM example 1 graphs show close contact with each other at \( y = 1 \) and 0.8. It is investigated that analytical results are in close relation with the actual results of example 1. In Figure 2, the results of example 1 at different fractional-order of the derivative are plotted at \( y = 0.6 \) and 0.4. Figure 3 shows the different fractional of two and three dimensional. The graphical representation has shown the convergence phenomena of fractional-order results towards the result at integer-order of example 1.

**Example 8.** Consider the time-fractional nonlinear Fornberg-Whitham equation is given as

\[ D^\nu_v - D_{\zeta}^\nu v + D_\zeta v = vd_{\zeta}^\nu v + vd_\zeta v + 3D_\zeta vD_{\zeta} v, \quad \mathfrak{F} > 0, \ 0 < y \leq 1. \]

(39)

The initial condition is

\[ v(\zeta, 0) = \cosh^2 \left( \frac{\zeta}{4} \right). \]

(40)

Applying Laplace transformation of (39), we get

\[ \left( \frac{\partial}{\partial \zeta} (1-y) + y \right) \left\{ L[v(\zeta, \mathfrak{F})] - \frac{1}{s} v(\zeta, 0) \right\} = L \left[ D^\nu_v - D_\zeta v + vd_{\zeta}^\nu v + vd_\zeta v + 3D_\zeta vD_{\zeta} v \right]. \]

(41)

Using inverse Laplace transformation,

\[ v(\zeta, \mathfrak{F}) = L^{-1} \left[ \frac{v(\zeta, 0)}{s} - \frac{\left( \frac{\partial}{\partial \zeta} (1-y) + y \right)}{s^r} L \left\{ D_{\zeta}^\nu v - D_\zeta v ight\} + vd_{\zeta}^\nu v + vd_\zeta v + 3D_\zeta vD_{\zeta} v \right]. \]

(42)
Figure 1: The solution graph of MDM/MVITM at $\gamma = 1$ and 0.8 of example 1.

Figure 2: The solutions graph of MDM/MVITM at $\gamma = 0.6$ and 0.4 of example 1.

Figure 3: The different fractional-order graph of MDM/MVITM of example 1.
Using ADM procedure, we get

\[ v_0(\xi, \eta) = L^{-1} \left[ \frac{v(\xi, 0)}{s} \right] = L^{-1} \left[ \exp \left( \frac{\cosh^2(\xi/4)}{s} \right) \right], \]

\[ v_0(\xi, \eta) = \cosh^2 \left( \frac{\xi}{4} \right). \]

\[ \sum_{\phi=0}^{\infty} v_{\phi+1}(\xi, \eta) = L^{-1} \left[ \left( \frac{\sigma(1-y)}{s^2} + y \right) L \left[ \sum_{\phi=0}^{\infty} \left( D_{\zeta \zeta} v \right)_\phi - \sum_{\phi=0}^{\infty} \left( D_{\xi} v \right)_\phi \right] \right. \]

\[ \left. + \sum_{\phi=0}^{\infty} A_{\phi} - \sum_{\phi=0}^{\infty} B_{\phi} + 3 \sum_{\phi=0}^{\infty} C_{\phi} \right], \quad \phi = 0, 1, 2, \ldots, \]

(43)

for \( \phi = 0, \)

\[ v_1(\xi, \eta) = L^{-1} \left[ \frac{\sigma(1-y) + y}{s^2} L \left[ D_{\zeta \zeta} v_0 - D_{\xi} v_0 + A_0 - B_0 + 3C_0 \right] \right] \]

\[ = - \frac{11}{32} \sinh \left( \frac{\xi}{4} \right) \left( 1 - y \right) + \frac{y^3 y}{I(y+1)}, \]

(44)

for \( \phi = 1, \)

\[ v_1(\xi, \eta) = L^{-1} \left[ \frac{\sigma(1-y) + y}{s^2} L \left[ D_{\zeta \zeta} v_1 - D_{\xi} v_1 + A_1 - B_1 + 3C_1 \right] \right], \]

(45)

for \( \phi = 2, \)

\[ v_1(\xi, \eta) = L^{-1} \left[ \frac{\sigma(1-y) + y}{s^2} L \left[ D_{\zeta \zeta} v_1 - D_{\xi} v_1 + A_1 - B_1 + 3C_2 \right] \right], \]

(46)

The MDM solution of example (8) is

\[ v(\xi, \eta) = v_0(\xi, \eta) + v_1(\xi, \eta) + v_2(\xi, \eta) + v_3(\xi, \eta) + v_4(\xi, \eta) + \ldots. \]
\[ D_\gamma v_2 - 3 D_\gamma v_2 D_\gamma v_2, (\zeta, \Theta) = \cosh^2 \left( \frac{\zeta}{2} \right) - \frac{11}{32} \sinh \left( \frac{\zeta}{2} \right) \left( 1 - \gamma \right) + \frac{\gamma^3}{1(y + 1)} \left( 1 - y \right) + \frac{11}{32} \sinh \left( \frac{\zeta}{2} \right) \left( 1 - \gamma \right) + \frac{121}{1024} \cosh \left( \frac{\zeta}{2} \right) \left( 1 - \gamma \right) + \frac{y^3 \Gamma(2)(y + 1)}{\Gamma(y + 1)} \left( y + 1 \right) + \frac{\gamma^3}{1(y + 1)} \left( 1 - y \right)^2 + \frac{\gamma^3 \Gamma(2)(y + 1)}{\Gamma(y + 1)} \left( y + 1 \right) + \frac{2(1 - y)\gamma\Theta^2}{\Gamma(y + 1)} - \frac{1331}{1024} \sinh \left( \frac{\zeta}{2} \right) \left( 1 - y \right)^2 + \frac{y^3 \Gamma(2)(y + 1)}{\Gamma(y + 1)} \left( y + 1 \right) + \frac{\gamma^3}{1(y + 1)} \left( 1 - y \right)^3 + \frac{y^3 \Gamma(2)(y + 1)}{\Gamma(y + 1)} \left( y + 1 \right) + \frac{3y^2(1 - y)\gamma\Theta^3}{\Gamma(y + 1)} + \frac{y^3 \Gamma(2)(y + 1)\Theta^3}{\Gamma(y + 1)} + \cdots \]

The exact solution of equation (39) at \( \gamma = 1 \),

\[ v(\zeta, \Theta) = \cosh^2 \left( \frac{\zeta}{4} - \frac{11\Theta}{24} \right). \quad (52) \]

In Figure 4, the analytical results of MDM/MVITM example 2 graphs show close contact with each other at \( \gamma = 1 \) and 0.8. It is investigated that analytical results are in close relation with the actual results of example 2. In Figure 5, the results of example 2 at different fractional-order of the derivative are plotted at \( \gamma = 0.6 \) and 0.4. Figure 6 shows the different fractional of two and three dimensional. The graphical representation has shown the convergence phenomena of fractional-order results towards the result at integer-order of example 2.
6. Conclusion

In this paper, we have been successfully applied two modified methods to investigate the approximate solutions of fractional Fornberg-Whitham equations. Agreement between numerical results obtained by the modified decomposition method and modified variational iteration method involving fractional-order derivatives with Mittag-Leffler kernel with exact result appears very appreciable by means of illustrative results in figures. The proposed techniques are easy to implement, effective, and suitable for achieving the results of nonlinear fractional Fornberg-Whitham equations. Moreover, both the modified decomposition method and variational iteration method provide the convergent series results with easily calculated components without applying any linearization, perturbation, or limiting assumptions. Finally, we can conclude the suggested methods are more accurate and highly methodical and which can be applied to investigate nonlinear models that arise in applied sciences.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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