A MAXIMAL FUNCTION CHARACTERISATION
OF THE HARDY SPACE
FOR THE GAUSS MEASURE.

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Abstract. An atomic Hardy space $H^1(\gamma)$ associated to the Gauss measure $\gamma$ in $\mathbb{R}^n$ has been introduced by the first two authors. We first prove that it is equivalent to use $(1,r)$- or $(1,\infty)$-atoms to define this $H^1(\gamma)$. For $n = 1$, a maximal function characterisation of $H^1(\gamma)$ is found. In arbitrary dimension, we give a description of the nonnegative functions in $H^1(\gamma)$ and use it to prove that $L^p(\gamma) \subset H^1(\gamma)$ for $1 < p \leq \infty$.

1. Introduction

Denote by $\gamma$ the Gauss measure on $\mathbb{R}^n$, i.e., the probability measure with density $\gamma_0(x) = \pi^{-n/2} e^{-|x|^2}$ with respect to the Lebesgue measure $\lambda$. Harmonic analysis on the measured metric space $(\mathbb{R}^n, d, \gamma)$, where $d$ denotes the Euclidean distance on $\mathbb{R}^n$, has been the object of many investigations. In particular, efforts have been made to study operators related to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators [30, 16, 24, 13, 19], Riesz transforms [26, 14, 25, 29, 27, 17, 15, 8, 9, 10, 23, 32, 7, 21], functional calculus [11, 12, 18, 22] and, recently, tent spaces [20].

In [21] the first two authors defined an atomic Hardy-type space $H^1(\gamma)$ and a space $\text{BMO}(\gamma)$ of functions of bounded mean oscillation, associated to $\gamma$. We briefly recall their definitions. A closed Euclidean ball $B$ is called admissible at scale $s > 0$ if

$$r_B \leq s \min\{1, 1/|c_B|\};$$

here and in the sequel $r_B$ and $c_B$ denote the radius and the centre of $B$, respectively. We denote by $B_s$ the family of all balls admissible at scale $s$. For the sake of brevity, we shall refer to balls in $B_1$ simply as admissible balls. Further, $B$ will be called maximal admissible if $r_B = \min(1, 1/|c_B|)$.

Now let $r \in (1, \infty]$. A Gaussian $(1, r)$-atom is either the constant function 1 or a function $a$ in $L^r(\gamma)$ supported in an admissible ball $B$ and such that

$$\int a \, d\gamma = 0 \quad \text{and} \quad \|a\|_r \leq \gamma(B)^{1/r - 1};$$

here and in the whole paper, $\|\cdot\|_r$ denotes the norm in $L^r(\gamma)$. In the latter case, we say that the atom $a$ is associated to the ball $B$. The space $H^{1,r}(\gamma)$ is then the vector space of all functions $f$ in $L^1(\gamma)$ that admit a decomposition of the form $\sum_j \lambda_j a_j$.
where the \( a_j \) are Gaussian \((1, r)\)-atoms and the sequence of complex numbers \( \{\lambda_j\} \) is summable. The norm of \( f \) in \( H^{1,r}(\gamma) \) is defined as the infimum of \( \sum_j |\lambda_j| \) over all representations of \( f \) as above.

In [21] the spaces \( H^{1,r}(\gamma) \) were defined and proved to coincide for all \( 1 < r < \infty \), with equivalent norms. In Section 2 we complement this by proving that they coincide also with the space \( H^{1,\infty}(\gamma) \). Once this is established, we shall denote the space by \( H^1(\gamma) \) and use the \( H^{1,\infty}(\gamma) \) norm. Further, we shall frequently write \( \lambda \)-atom for \((1, \infty)\)-atom.

The space \( BMO(\gamma) \) consists of all functions \( f \) in \( L^1(\gamma) \) such that

\[
\sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \, d\gamma < \infty,
\]

where \( f_B \) denotes the mean value of \( f \) on \( B \), taken with respect to the Gauss measure. The norm of a function in \( BMO(\gamma) \) is

\[
\|f\|_{BMO(\gamma)} = \|f\|_1 + \sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \, d\gamma.
\]

If, in the definitions of \( H^1(\gamma) \) and \( BMO(\gamma) \), we replace the family \( \mathcal{B}_1 \) of admissible balls at scale 1 by \( \mathcal{B}_s \) for any fixed \( s > 0 \), we obtain the same spaces with equivalent norms, see [21]. We remark that a similar \( H^1 - BMO \) theory for more general measured metric spaces has been developed by A. Carbonaro and the first two authors in [11, 2, 20].

The main motivation for introducing these two spaces was to provide endpoint estimates for singular integrals associated to the Ornstein-Uhlenbeck operator \( \mathcal{L} = -(1/2)\Delta + x \cdot \nabla \), a natural self-adjoint Laplacian on \( L^2(\gamma) \). Indeed, in [21] the first two authors proved that the imaginary powers of \( \mathcal{L} \) are bounded from \( H^1(\gamma) \) to \( L^1(\gamma) \) and from \( L^\infty(\gamma) \) to \( BMO(\gamma) \) and that Riesz transforms of the form \( \nabla^\alpha \mathcal{L}^{-[\alpha]} \) and of any order are bounded from \( L^\infty(\gamma) \) to \( BMO(\gamma) \). In a recent paper [23], the authors proved that boundedness from \( H^1(\gamma) \) to \( L^1(\gamma) \) and from \( L^\infty(\gamma) \) to \( BMO(\gamma) \) holds for any first-order Riesz transform in dimension one, but not always in higher dimension.

The definition of the space \( H^1(\gamma) \) closely resembles the atomic definition of the classical Hardy space \( H^1(\lambda) \) on \( \mathbb{R}^n \) endowed with the Lebesgue measure \( \lambda \), but there are two basic differences. First, the measured metric space \((\mathbb{R}^n, d, \gamma)\) is non-doubling. Further, except for the constant atom, a Gaussian atoms must have “small support”, i.e., support contained in an admissible ball. Despite these differences, \( H^1(\gamma) \) shares many of the properties of \( H^1(\lambda) \). In particular, the topological dual of \( H^1(\gamma) \) is isomorphic to \( BMO(\gamma) \), an inequality of John-Nirenberg type holds for functions in \( BMO(\gamma) \) and the spaces \( L^p(\gamma) \) are intermediate spaces between \( H^1(\gamma) \) and \( BMO(\gamma) \) for the real and the complex interpolation methods.

It is well known that the classical Hardy space \( H^1(\lambda) \) can be defined in at least three different ways: the atomic definition, the maximal definition and the definition based on Riesz transforms [4, 51].

As shown in [24], in higher dimension the first-order Ornstein-Uhlenbeck Riesz transforms \( \partial_j \mathcal{L}^{-1/2} \) are unbounded from \( H^1(\gamma) \) to \( L^1(\gamma) \); here \( \partial_j = \partial/\partial x_j \). Thus \( H^1(\gamma) \) does not coincide with the space of all functions in \( L^1(\gamma) \) such that \( \partial_j \mathcal{L}^{-1/2} f \in L^1(\gamma) \) for \( j = 1, \ldots, n \).
This paper arose from the desire to find a maximal characterisation of the space \( H^1(\gamma) \). We recall that the classical space \( H^1(\lambda) \) can be characterised as the space of all functions \( f \) in \( L^1(\lambda) \) whose grand maximal function
\[
\mathcal{M}f(x) = \sup \{ |\phi_t \ast f(x)| : \phi \in \Phi, \, t > 0 \}
\]
is also in \( L^1(\lambda) \). Here \( \Phi = \{ \phi \in C_c^1(\Omega(0, 1)) : |D^n\phi| \leq 1 \text{ for } |\alpha| = 0, 1 \} \) and \( \phi_t(x) = t^{-n}\phi(x/t) \).

To characterise \( H^1(\gamma) \), we introduce the local grand maximal function defined on \( L^1_{loc}(\mathbb{R}^n, \gamma) \) by
\[
\mathcal{M}_{loc}f(x) = \sup \{ |\phi_t \ast f(x)| : \phi \in \Phi, \, 0 < t < \min(1, 1/|x|) \}.
\]
In Section 3 we shall prove that, in arbitrary dimension, \( f \in H^1(\gamma) \) implies \( \mathcal{M}_{loc}f \in L^1(\gamma) \). Moreover, in dimension one \( H^1(\gamma) \) can be characterised as the space of all functions \( f \) in \( L^1(\gamma) \) satisfying \( \mathcal{M}_{loc}f \in L^1(\gamma) \) and the following additional global condition
\[
E(f) = \int_0^\infty x \left( \int_{|x|}^\infty f \, d\gamma \right) \, d\lambda(x) < \infty.
\]
This is Theorem 3.3 below.

Roughly speaking, if we interpret a function \( f \) as a density of electrical charge on the real line, this global condition says that the positive and negative charges nearly balance out, so that the net charges inside the intervals \(( -\infty, -x ) \) and \(( x, \infty ) \) decay sufficiently fast as \( x \) approaches \(+\infty\). The condition is violated when the distance between the positive and the negative charges increases too much or the charges do not decay sufficiently fast at infinity. For instance, let \( (a_n)^\infty_1 \) and \( (a'_n)^\infty_1 \) be increasing sequences in \( (2, \infty) \) such that
\[
a_n + 2/a_n < a'_n \quad \text{and} \quad a'_n + 2/a'_n < a_{n+1} < 2a_n
\]
for all \( n \). Then set
\[
f = \sum_{1}^{\infty} c_n \left( \frac{\chi(a_n, a_n + 1/a_n)}{\gamma(a_n, a_n + 1/a_n)} - \frac{\chi(a'_n, a'_n + 1/a'_n)}{\gamma(a'_n, a'_n + 1/a'_n)} \right)
\]
for some \( c_n > 0 \). One easily verifies that \( \mathcal{M}_{loc}f \in L^1(\gamma) \) if and only if \( \sum c_n < \infty \). But the global condition \( E(f) < \infty \) is equivalent to \( \sum c_n(a'_n - a_n) < \infty \), which is here a stronger condition.

We have not been able to find a similar characterisation of \( H^1(\gamma) \) in higher dimension. However, in Section 4 we prove in all dimensions that if \( \mathcal{M}_{loc}f \in L^1(\gamma) \) and the function \( f \) satisfies the stronger global condition
\[
E_+(f) = \int |x|^2 |f(x)| \, d\gamma(x) < \infty,
\]
then \( f \in H^1(\gamma) \). Observe that for \( n = 1 \) and \( f \geq 0 \), Fubini’s theorem implies that the conditions \( E(f) < \infty \) and \( E_+(f) < \infty \) are equivalent. In arbitrary dimension, \( E_+(f) \) can be used to characterise the nonnegative functions in \( H^1(\gamma) \); see Theorem 1.2. This also leads to a simple proof of the inclusions \( L^p(\gamma) \subset H^1(\gamma) \) and \( BMO(\gamma) \subset L^p(\gamma) \) for \( 1 < p \leq \infty \).

We end the introduction with some technical observations and notation. In the following we use repeatedly the fact that on admissible balls at a fixed scale \( s \), the
Gauss and the Lebesgue measures are equivalent, i.e., there exists a constant $C(s)$ such that for every measurable subset $E$ of $B \in B_s$

\begin{equation}
C(s)^{-1} \gamma(E) \leq \gamma_0(c_B) \lambda(E) \leq C(s) \gamma(E).
\end{equation}

In particular this implies that the Gauss measure is doubling on balls in $B_s$, with a constant that depends on $s$ (see [21, Prop. 2.1]). Further, it is straightforward to see that if $B' \subset B$ are two balls and $B \in B_s$ then $B'$ is also in $B_s$.

Given a ball $B$ in $\mathbb{R}^n$ and a positive number $\rho$, we shall denote by $\rho B$ the ball with the same centre and with radius $\rho r_B$.

In the following $C$ denotes a constant whose value may change from occurrence to occurrence and which depends only on the dimension $n$, except when otherwise explicitly stated.

2. Coincidence of $H^{1,\infty}(\gamma)$ and $H^{1,r}(\gamma)$

First we need a lemma which will play a role also in the maximal characterisation. It deals with the classical Hardy space $H^1(\lambda)$ with respect to the Lebesgue measure and the associated standard $(1, \infty)$-atoms, called Lebesgue atoms below. The result is probably well known, but we include a proof because we have not been able to find a reference in the literature. Some related results can be found in [4] and [5].

**Lemma 2.1.** If $g \in H^1(\lambda)$ and the support of $g$ is contained in a ball $B$, then $g$ has an atomic decomposition $g = \sum \lambda_k a_k$ where the $a_k$ are Lebesgue $(1, \infty)$-atoms associated to balls contained in $2B$ and

\begin{equation}
\sum_k |\lambda_k| \leq C \|g\|_{H^1(\lambda)}.
\end{equation}

**Proof.** In this proof, all atoms are Lebesgue $(1, \infty)$-atoms. We claim that the grand maximal function of $g$ satisfies

\begin{equation}
\mathcal{M}g(x) \leq \frac{C \|g\|_1}{|B|}, \quad \forall x \notin 2B.
\end{equation}

To prove this, take $x \notin 2B$ and observe that

$$
\mathcal{M}g(x) = \sup_{\phi \in \Phi} \sup_{t \geq d(x,B)} |\phi_t * g(x)|,
$$

since $\phi_t * g(x) = 0$ for $0 < t < d(x, B)$. Because of the vanishing integral of $g$, one has for $\phi \in \Phi$ and $t \geq d(x, B)$

\begin{align*}
|\phi_t * g(x)| &\leq \int_B |\phi_t(x-y) - \phi_t(x-c_B)| |g(y)| \lambda(y) \\
&\leq Ct^{-n-1} \int_B |y-c_B| |g(y)| \lambda(y) \\
&\leq \frac{C \tau_{r_B}}{d(x,B)^{n+1}} \|g\|_1 \\
&\leq \frac{C}{|B|} \|g\|_1.
\end{align*}
For each integer \( k \), denote by \( \Omega_k \) the level set \( \{ x : M g(x) > 2^k \} \). Then (2.2) implies that

\[
\Omega_k \subset 2B \quad \text{when} \quad 2^k \geq C \frac{\| g \|_1}{|B|}.
\]

Let \( \Omega_k = \bigcup_i Q_i^k \) be a Whitney decomposition of \( \Omega_k \) into closed cubes \( Q_i^k \), \( i \in \mathbb{N} \), whose interiors are disjoint, and whose diameters are comparable to \( \delta \) times their distances from \( \Omega_k^c \), where \( \delta \) is a (small) positive constant to be chosen later. Define \( \tilde{Q}_i^k \) as the cube with the same centre as \( Q_i^k \) and side length expanded by a factor 2. Then \( \bigcup_i \tilde{Q}_i^k = \Omega_k \) and the family \( \{ \tilde{Q}_i^k : i \in \mathbb{N} \} \) will have the bounded overlap property, uniformly in \( k \), provided that \( \delta \) is small enough. Proceeding as in the proof of the atomic decomposition for \( H^1(\lambda) \) in [31], p. 107–109], one shows that there exists a decomposition

\[
g = \sum_{k,i} A_{k,i}^k
\]

with the following properties.

(i) Each function \( A_{k,i}^k \) is supported in a ball \( B_i^k \) that contains the cube \( \tilde{Q}_i^k \) as well as those \( \tilde{Q}_m^k \) that intersect \( \tilde{Q}_i^k \). Moreover, if \( \delta \) is sufficiently small, the ball \( B_i^k \) is contained in \( \Omega_k \) and for each \( k \) the family \( \{ B_i^k \} \) has the bounded overlap property.

(ii) \( A_{k,i}^k = \lambda_{k,i} a_{i}^k \), where \( a_{i}^k \) is a Lebesgue atom associated to the ball \( B_i^k \) and

\[
\sum_{k,i} |\lambda_{k,i}| \leq C \| g \|_{H^1(\lambda)}.
\]

(iii) \( |A_{k,i}^k| \leq C 2^k \) for each \( k \) and \( i \).

We split the sum in (2.4) in two parts

\[
g = \sum_{B_i^k \not\subset 2B} A_{k,i}^k + \sum_{B_i^k \subset 2B} A_{k,i}^k = \Sigma_1 + \Sigma_2.
\]

Clearly, \( \Sigma_2 \) is an atomic decomposition with atoms associated to balls contained in \( 2B \). Thus, it suffices to show that \( \Sigma_1 \) is a multiple of an atom associated to \( 2B \). And indeed, \( \Sigma_1 \) is supported in \( 2B \) and has integral zero, because it is the difference of \( g \) and \( \Sigma_2 \), both of which have these two properties. Moreover, by (i) and (iii)

\[
\left| \sum_i A_{i}^k \right| \leq C 2^k.
\]

Hence by (2.3)

\[
|\Sigma_1| \leq \sum_{2^k < C \| g \|_1 / |B|} \left| \sum_i A_{i}^k \right| \leq C \frac{\| g \|_1}{|B|} \leq C \frac{g \|_{H^1(\lambda)}}{|2B|}.
\]

Thus \( \Sigma_1 \) is a multiple of an atom associated to \( 2B \). We thus have the desired atomic decomposition of \( g \), and the norm estimate (2.1) also follows.

\[\square\]

**Theorem 2.2.** For every \( r \) in \((1, \infty)\), the spaces \( H^{1,r}(\gamma) \) and \( H^{1,\infty}(\gamma) \) coincide, with equivalent norms.
Hence, $a_{\gamma_0} = \sum_j \lambda_j a_j$, where each $\alpha_j$ is a Lebesgue $(1, \infty)$-atom associated to a ball $B_j$ contained in $2B$. Moreover
\[
\sum_j |\lambda_j| \leq C,
\]
and each $B_j$ is admissible at scale 2. Define $a_j = a_{\gamma_0}^{-1}$. Then $\int a_j \, d\gamma = 0$, and by the equivalence of the Gauss and Lebesgue measures on $B_j$
\[
\|a_j\|_\infty \leq C\gamma(B_j)^{-1}.
\]
Thus the $a_j$ are multiples of Gaussian $(1, \infty)$-atoms. Since $a = \sum_j \lambda_j a_j$, we conclude that $a \in H^{1,\infty}(\gamma)$ and
\[
(2.5) \quad \|a\|_{H^{1,\infty}(\gamma)} \leq C \sum_j |\lambda_j| \leq C.
\]

3. THE CHARACTERISATION OF $H^{1}(\gamma)$ IN $\mathbb{R}$

In this section, we shall prove that $f \in H^{1}(\gamma)$ implies $\mathcal{M}_{\text{loc}} f \in L^{1}(\gamma)$ and that, in dimension one, functions in $H^{1}(\gamma)$ can be characterised by the two conditions $\mathcal{M}_{\text{loc}} f \in L^{1}(\gamma)$ and $E(f) < \infty$. We start with a simple but useful lemma dealing with the support of the local grand maximal function.

**Lemma 3.1.** If $f \in L^{1}(\gamma)$ is supported in the admissible ball $B$, then $\text{supp} \mathcal{M}_{\text{loc}} f$ is contained in the ball $B' = B(c_B, R)$, where $R = 4 \min(1, 1/|c_B|)$.

**Proof.** Let $x \in \text{supp} \mathcal{M}_{\text{loc}} f$. We write $\rho = |x|$ and $c = |c_B|$, so that $B \subset B(c_B, \min(1, 1/c))$. The balls $B$ and $B(x, \min(1, 1/\rho))$ must intersect, and so
\[
(3.1) \quad |x - c_B| \leq \min(1, 1/c) + \min(1, 1/\rho).
\]
To prove the lemma, it is enough to show that
\[
(3.2) \quad \min(1, 1/\rho) \leq 3 \min(1, 1/c),
\]
since it then follows that $x \in B'$. Now $c - \rho \leq |x - c_B|$, so that $\min(1, 1/c) \leq 3 \min(1, 1/\rho)$ implies
\[
(3.3) \quad \max(1, c) - \min(1, 1/c) \leq \max(1, \rho) + \min(1, 1/\rho).
\]
Considering the cases $c \leq 1$ and $c > 1$, we conclude from this that
\[
(3.4) \quad \max(1, c) - \min(1, 1/c) \leq \max(1, \rho) + \min(1, 1/\rho).
\]
The function \( t \mapsto t^{-1} - t, \ t > 0 \), and its inverse are clearly decreasing. Considering the values of this function at \( t = \min(1, 1/c) \) and \( \min(1, 1/\rho)/3 \), we see that (3.2) is equivalent to

\[
\max(1, c) - \min(1, 1/c) \leq 3 \max(1, \rho) - \frac{1}{3} \min(1, 1/\rho).
\]

Because of (3.3), this inequality follows if

\[
\max(1, \rho) + \min(1, 1/\rho) \leq 3 \max(1, \rho) - \frac{1}{3} \min(1, 1/\rho)
\]
or equivalently \( \frac{2}{3} \min(1, 1/\rho) \leq 2 \max(1, \rho) \), which is trivially true. We have proved (3.2) and the lemma.

**Lemma 3.2.** If \( f \) is in \( H^1(\gamma) \), then \( M_{\text{loc}} f \in L^1(\gamma) \) and

\[
\| M_{\text{loc}} f \|_1 \leq C \| f \|_{H^1(\gamma)}.
\]

**Proof.** We shall prove that for any Gaussian atom \( a \)

\[
(3.4) \quad \| M_{\text{loc}} a \|_1 \leq C,
\]

from which the lemma follows.

Since (3.4) is obvious if \( a \) is the constant function 1, we assume that \( a \) is associated to an admissible ball \( B \). By the preceding lemma, \( \text{supp} M_{\text{loc}} f \) is contained in the ball denoted \( B' \).

The integral of \( M_{\text{loc}} a \) over \( 2B \) with respect to \( \gamma \) is no larger than \( C \), since \( M_{\text{loc}} a \leq C \sup|a| \leq C/\gamma(B) \). To estimate \( M_{\text{loc}} a \) at a point \( x \) in the remaining set \( B' \setminus 2B \), we take \( \phi \in \Phi \) and \( 0 < t < \min(1, 1/|x|) \) and estimate \( \phi_t \ast a(x) \). We can assume that \( t > d(x, B) \) so that \( t > |x - c_B|/2 \), since otherwise \( \phi_t \ast a(x) \) will vanish. Write

\[
(3.5) \quad \phi_t \ast a(x) = t^{-n} \int \phi(x / t, y) - \phi(x / t, c_B) \ a(y) \ dy + t^{-n} \phi(x / t, c_B) \int a(y) \ dy.
\]

Here the first term to the right can be estimated in a standard way by

\[
C t^{-n-1} \int_B |y - c_B| \ |a(y)| \ dy \leq C \ |x - c_B|^{-n-1} r_B \gamma_0(c_B)^{-1}.
\]

To deal with the second term, we estimate \( \int a(y) \ dy \), knowing that the integral of \( a \) against \( \gamma \) vanishes. Thus

\[
\int a(y) \ dy = \int a(y) \frac{\gamma_0(c_B) - \gamma_0(y)}{\gamma_0(c_B)} \ dy.
\]

The fraction appearing here is

\[
(3.6) \quad e^{c_B^2} \left( e^{-|c_B|^2} - e^{-|y|^2} \right) = 1 - e^{(c_B - y) \cdot (c_B + y)},
\]

and the last exponent stays bounded for \( y \in B \). Thus the modulus of the right-hand side of (3.6) is at most \( C |c_B - y| |c_B + y| \leq C r_B (1 + |c_B|) \). Since \( \int |a| \ d\gamma \leq 1 \), this implies

\[
| \int a(y) \ dy | \leq C r_B (1 + |c_B|) \gamma_0(c_B)^{-1}.
\]

For the last term in (3.6), we thus get the bound \( C |x - c_B|^{-n} r_B (1 + |c_B|) \gamma_0(c_B)^{-1} \).
Putting things together, we conclude that for $x \in B' \setminus 2B$
\[ M_{\text{loc}}(x) \leq C|x - c_B|^{-n-1}r_B\gamma_0(c_B)^{-1} + C|x - c_B|^{-n}r_B(1 + |c_B|)\gamma_0(c_B)^{-1}. \]
An integration with respect to $d\gamma$, or equivalently $\gamma_0 \, d\lambda$, then leads to
\[ \int_{B' \setminus 2B} M_{\text{loc}}(x) \, d\gamma(x) \leq C + Cr_B(1 + |c_B|)\log \frac{\min(1, 1/|c_B|)}{r_B} \leq C, \]
and (3.4) is proved. \hfill \Box

**Theorem 3.3.** Let $n = 1$, and suppose that $f$ is a function in $L^1(\gamma)$. Then $f$ is in $H^1(\gamma)$ if and only if $M_{\text{loc}}f \in L^1(\gamma)$ and $E(f) < \infty$. The norms $\|f\|_{H^1(\gamma)}$ and $\|M_{\text{loc}}f\|_{L^1(\gamma)} + E(f)$ are equivalent.

**Proof.** Suppose that $f \in H^1(\gamma)$. Then $M_{\text{loc}}f \in L^1(\gamma)$ by Lemma 3.2. To prove the necessity of the condition $E(f) < \infty$, it suffices to show that $E(a) < C$ for all Gaussian atoms $a$. This is obvious for the exceptional atom 1. If $a$ is associated to a ball $B \in B_1$, it follows from the inequality
\[ \int_{-\infty}^{-x} a \, d\gamma + \int_{x}^{\infty} a \, d\gamma \leq 1_B(x). \]
Conversely, assume that $f$ is a function in $L^1(\gamma)$ such that $M_{\text{loc}}f \in L^1(\gamma)$ and $E(f) < \infty$. We shall prove that $f \in H^1(\gamma)$, by constructing a Gaussian atomic decomposition $f = \sum j \lambda_j a_j$ such that $\sum j |\lambda_j| \leq C(\|M_{\text{loc}}f\|_{1} + E(f))$.

Most of the following argument, up to the decomposition (3.15), works also in the $n$-dimensional setting. Since we shall need it in the next section, we carry out that part in $\mathbb{R}^n$.

By subtracting a multiple of the exceptional atom 1, we may without loss of generality assume that
\[ (3.7) \quad \int f \, d\gamma = 0. \]
Let $\{B_j\}$ be a covering of $\mathbb{R}^n$ by maximal admissible balls. We can choose this covering in such way that the family $\{1/4B_j\}$ is disjoint and $\{4B_j\}$ has bounded overlap \[10, \急速 \text{Lemma 2.4}. \] Fix a smooth nonnegative partition of unity $\{\eta_j\}$ in $\mathbb{R}^n$ such that $\text{supp} \eta_j \subset B_j$ and $\eta_j = 1$ on $1/4B_j$ and verifying $|\nabla \eta_j| \leq C/r_{B_j}$. Thus $f = \sum j f\eta_j$. We now need the following lemma.

**Lemma 3.4.** For $g$ in $L^1_{\text{loc}}(\gamma)$ and $x \in \mathbb{R}^n$ one has
\[ (3.8) \quad M_{\text{loc}}(g\eta_j \gamma_0)(x) \leq C \gamma_0(c_{B_j}) M_{\text{loc}}g(x) 1_{4B_j}(x) \quad \forall j. \]

**Proof.** Since the support of $\eta_j$ is contained in $B_j$, the support of $M_{\text{loc}}(g\eta_j \gamma_0)$ is contained in the ball $4B_j$, because of Lemma 3.1. Moreover, for $\tilde{\phi} \in \Phi$ and $x \in 4B_j$
\[ \phi_t * (g\eta_j \gamma_0)(x) = \gamma_0(c_{B_j}) \tilde{\phi}_t * g(x), \]
where $\tilde{\phi}(z) = \phi(z)\eta_j(x - tz)\gamma_0(x - tz)/\gamma_0(c_{B_j})$. Thus, to prove (3.8) it suffices to show that there exists a constant $C$ such that $\tilde{\phi} \in C\Phi$ for $x \in 4B_j$ and $0 < t < \min(1, 1/|x|)$. The support of $\tilde{\phi}$ is contained in $B(0, 1)$ and
\[ \left| \tilde{\phi}(z) \right| \leq \frac{\gamma_0(x - tz)}{\gamma_0(c_{B_j})} \leq C, \]
because for $|z| \leq 1$
\[ |x - tz - c_{B_j}| \leq |x - c_{B_j}| + |tz| \leq C \min(1, 1/|c_{B_j}|). \]

Similarly $|\nabla \phi(z)| \leq C$, because the gradients $\nabla_z \eta_j(x-tz)$ and $\nabla_z \gamma_0(x-tz)/\gamma_0(c_{B_j})$ give the factors $t(1 + |c_{B_j}|)$ and $t|x - tz| \gamma_0(x - tz)/\gamma_0(c_{B_j})$, respectively, both of which are bounded. This concludes the proof of Lemma 3.4.

Continuing the proof of Theorem 3.3, we define $b_j \in C$ for each $j \in \mathbb{N}$ by
\[(3.9) \quad \int_{-\infty}^{\infty} (f - b_j) \eta_j \, d\gamma = 0.\]

Note that since $\eta_j = 1$ on $\frac{1}{2} B_j$,
\[(3.10) \quad |b_j| = \left| \frac{\int f \eta_j \, d\gamma}{\int \eta_j \, d\gamma} \right| \leq C \frac{1}{\gamma(B_j)} \int_{B_j} |f| \, d\gamma.\]

We now apply Lemma 3.4 with $g = f - b_j$ and use the subadditivity of $\mathcal{M}_{\text{loc}}$ combined with (3.10), to get
\[(3.11) \quad \int \mathcal{M}_{\text{loc}}((f - b_j)\eta_j \gamma_0) \, d\lambda \leq C \int_{4B_j} \mathcal{M}_{\text{loc}} f \gamma_0(c_{B_j}) \, d\lambda + C \frac{\gamma(4B_j)}{\gamma(B_j)} \int_{B_j} |f| \, d\gamma \quad \leq C \int_{4B_j} \mathcal{M}_{\text{loc}} f \, d\gamma.\]

**Lemma 3.5.** The function $(f - b_j)\eta_j \gamma_0$ is in $H^1(\lambda)$ and
\[(3.12) \quad \|(f - b_j)\eta_j \gamma_0\|_{H^1(\lambda)} \leq \int_{4B_j} \mathcal{M}_{\text{loc}} f \, d\gamma.\]

**Proof.** By the maximal characterisation of the classical space $H^1(\lambda)$, it suffices to show that
\[(3.13) \quad \int \mathcal{M}((f - b_j)\eta_j \gamma_0) \, d\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\text{loc}} f \, d\gamma.\]

Because of (3.11), all that needs to be verified is that
\[(3.14) \quad \int \sup_{\phi \in \Phi} \sup_{t \geq \min(1, 1/|x|)} \left| ((f - b_j)\eta_j \gamma_0) * \phi_t(x) \right| \, d\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\text{loc}} f \, d\gamma.\]

To prove (3.14), we split the integral in the left-hand side into the sum
\[
\int_{4B_j} \cdots \, d\lambda(x) + \int_{(4B_j)^c} \cdots \, d\lambda(x).
\]

If $x \in 4B_j$, then for $\phi \in \Phi$ and $t \geq \min(1, 1/|x|)$
\[
|\phi_t * ((f - b_j)\eta_j \gamma_0)(x)| \leq t^{-n} \int_{B_j} |f(y) - b_j| \, d\gamma(y) \leq (1 + |x|)^n \int_{B_j} (|f(y)| + |b_j|) \, d\gamma(y) \leq C (1 + |c_{B_j}|)^n \int_{B_j} |f(y)| \, d\gamma(y),
\]

where $C$ is a constant independent of $j$.
the last inequality because of (3.10). Hence
\[ \int_{4B_j} \cdots d\lambda(x) \leq C |4B_j| (1 + |c_{B_j}|)^n \int_{B_j} |f| \, d\gamma \leq C \int_{B_j} {\mathcal M}_{loc} f \, d\gamma. \]

If \( x \in (4B_j)^c \), we take \( \phi \) and \( t \) as before and observe that we can assume that \( t > d(x, B_j) \), since otherwise the convolution in (3.14) will vanish. In view of (3.9) and (3.10), we then get
\[
|\phi_t * ((f - b_j)\eta_j\gamma_0)(x)| \leq \int_{B_j} |\phi_t(x - y) - \phi_t(x - c_{B_j})| |f(y) - b_j| \eta_j(y) \, d\gamma(y)
\leq C t^{-n-1} \int_{B_j} |y - c_{B_j}| |f(y) - b_j| \, d\gamma(y)
\leq C \frac{1}{d(x, B_j)^{n+1} r_{B_j}} \int_{B_j} |f(y)| \, d\gamma(y).
\]
This implies that
\[
\int_{(4B_j)^c} \cdots d\lambda(x) \leq C \int_{B_j} |f| \, d\gamma \leq C \int_{B_j} {\mathcal M}_{loc} f \, d\gamma.
\]
We have proved (3.14) and the lemma. \( \square \)

We can now finish the proof of Theorem 3.3. By Lemmata 3.5 and 2.1 each function \( (f - b_j)\eta_j\gamma_0 \) has an atomic decomposition \( \sum_k \lambda_{jk} a_{jk} \) where the \( a_{jk} \) are Lebesgue atoms with supports in \( 2B_j \) and
\[
\sum_k |\lambda_{jk}| \leq C \int_{4B_j} {\mathcal M}_{loc} f \, d\gamma.
\]
As we saw in the proof of Theorem 2.2 each \( a_{jk} = \gamma_0^{-1} a_{jk} \) is a multiple of a Gaussian atom, with a factor which is independent of \( j \) and \( k \). Thus
\[
(3.15) \quad f = \sum_j (f - b_j)\eta_j + \sum_j b_j\eta_j = \sum_j \sum_k \lambda_{jk} a_{jk} + \sum_j b_j\eta_j.
\]
and
\[
\sum_{j, k} |\lambda_{jk}| \leq \sum_j \int_{4B_j} {\mathcal M}_{loc} f \, d\gamma \leq C \| {\mathcal M}_{loc} f \|_{L^1(\gamma)}.
\]
To complete the proof of Theorem 3.3 we need to find an atomic decomposition of \( \sum_j b_j\eta_j \). It is here that we must restrict ourselves to the one-dimensional case and that the global condition \( E(f) < \infty \) plays a role.

Choose the intervals \( I_0 = (-1, 1) \), \( I_j = (\sqrt{J - 1}, \sqrt{J + 1}) \) for \( j \geq 1 \) and \( I_j = -I_{|j|} \) for \( j \leq -1 \). The intervals \( I_j \) have essentially the same properties as the balls \( B_j \) introduced above, and we can use them instead of the \( B_j \) to construct \( \eta_j \) and \( b_j \) as before. To decompose now \( \sum_j b_j\eta_j \), we first normalise the functions \( \eta_j \), letting
\[
\hat{\eta}_j = \frac{\eta_j}{\int \eta_j \, d\gamma}.
\]
Then \( b_j\eta_j = \int f\eta_j \, d\gamma \, \hat{\eta}_j \), and clearly
\[
\sum_{j \geq k} \int f\eta_j \, d\gamma = \int f \mu_k \, d\gamma, \quad k \in \mathbb{Z},
\]
where $\mu_k(x) = \sum_{j \geq k} \eta_j(x)$. Notice that $\int f \mu_k \, d\gamma \to 0$ as $k \to \pm \infty$, in view of (3.7).

A summation by parts now yields

\begin{equation}
\sum_{j \in \mathbb{Z}} \int f \eta_j \, d\gamma \eta_j = \sum_{k \in \mathbb{Z}} \int f \mu_k \, d\gamma (\eta_k - \eta_{k-1}).
\end{equation}

But $\eta_k - \eta_{k-1} = C$ times a Gaussian atom, if we use admissible balls at some scale $s > 1$ in the definition of atoms. Thus (3.16) is our desired atomic decomposition of $\sum_j b_j \eta_j$, provided we can estimate the coefficients by showing that

\begin{equation}
\sum_{k \in \mathbb{Z}} \left| \int f \mu_k \, d\gamma \right| \leq C \left( \|f\|_1 + E(f) \right).
\end{equation}

To this end, observe that

$$\int f \mu_k \, d\gamma = \int f(x) \int_{-\infty}^\infty \mu_k'(y) \, d\lambda(y) \, d\gamma(x) = \int \mu_k'(y) \int_y^\infty f(x) \, d\gamma(x) \, d\lambda(y).$$

Since the support of $\mu_k'$ is contained in $I_k$ and

$$\|\mu_k'\| \leq \frac{C}{|I_k|} \leq C \left( 1 + |c_{I_k}| \right),$$

we obtain, using also the bounded overlap of the $I_j$,

$$\sum_{k \in \mathbb{Z}} \left| \int f \mu_k \, d\gamma \right| \leq C \sum_{k} \int_{I_k} \left( 1 + |c_{I_k}| \right) \left| \int_y^\infty f(x) \, d\gamma(x) \right| \, d\lambda(y)$$

\begin{align*}
&\leq C \int_{-\infty}^\infty \left( 1 + |y| \right) \left| \int_y^\infty f \, d\gamma \right| \, d\lambda(y) \\
&= C \int_0^\infty \left( 1 + y \right) \left( \left| \int_y^\infty f \, d\gamma \right| + \left| \int_{-y}^0 f \, d\gamma \right| \right) \, d\lambda(y) \\
&\leq C \left( \|f\|_1 + E(f) \right);
\end{align*}

here we used (3.7). This concludes the proof of Theorem 3.8

\section{A Characterisation of Nonnegative Functions in $H^1(\gamma)$}

The dimension $n$ is now arbitrary. The following lemma will be needed.

\begin{lemma}
Let $\phi_0 = \gamma(B(0,1))^{-1} 1_{B(0,1)}$. If $g \in L^\infty$ is supported in a maximal admissible ball $B$, then

$$\|g - \int B \phi_0 \|_{H^1(\gamma)} \leq C \left( 1 + \|B\|^2 \right) \gamma(B) \|g\|_{L^\infty}.$$

\end{lemma}

\begin{proof}
We shall construct atoms whose supports form a chain connecting $B(0,1)$ to $B$. First we define a finite sequence of maximal admissible balls

$$\tilde{B}_0 = B(0,1), \tilde{B}_1, \ldots, \tilde{B}_N,$$

all with centres $c_{\tilde{B}_j}$ on the segment $[0, c_B]$. The absolute values $\rho_j = |c_{\tilde{B}_j}|$ shall be increasing in $j$, and the boundary $\partial \tilde{B}_j$ shall contain $c_{\tilde{B}_{j-1}}$ for $j = 1, \ldots, N - 1$, which means that

\begin{equation}
\rho_j - \frac{1}{\rho_j} = \rho_{j-1}, \quad j = 1, \ldots, N - 1,
\end{equation}

\end{proof}
and \( \rho_0 = 0, \ \rho_1 = 1. \) Finally, \( N \) is defined so that \( \tilde{B}_{N-1} \) is the first ball of the sequence that contains \( c_B, \) and \( \tilde{B}_N = B. \) Squaring (4.1), we get

\[
\rho_j^2 - \rho_{j-1}^2 = 2 - \frac{1}{\rho_j^2} \geq 1,
\]

so that \( \rho_{N-1}^2 \geq N - 1. \) It follows that

(4.2)

\[
N \leq |c_B|^2 + 1.
\]

Next, we denote by \( B_j, \ j = 1, \ldots, N, \) the largest ball contained in \( \tilde{B}_j \cap \tilde{B}_{j-1}. \) Notice that the three balls \( \tilde{B}_j, \tilde{B}_{j-1} \) and \( B_j \) have comparable radii and comparable Gaussian measures. Define now functions \( \phi_j \) and \( g_j \) by setting

\[
\phi_j = \gamma(B_j)^{-1/2}1_{B_j}, \quad j = 1, \ldots, N,
\]

\[
g_j = \int g \, d\gamma (\phi_j - \phi_{j-1}), \quad j = 1, \ldots, N,
\]

\[
g_{N+1} = g - \int g \, d\gamma \phi_N.
\]

Clearly,

(4.3)

\[
g - \int g \, d\gamma \phi_0 = \sum_{j=1}^{N+1} g_j.
\]

Each function \( g_j \) is a multiple of an atom. Indeed, its integral against \( \gamma \) vanishes. Moreover, if \( 1 \leq j \leq N, \) the support of \( g_j \) is contained in \( B_{j-1} \) and

\[
\|g_j\|_\infty \leq (\gamma(B_j)^{-1} + \gamma(B_{j-1})^{-1}) \int |g| \, d\gamma \leq C \gamma(B_{j-1})^{-1} \gamma(B) \|g\|_{L^\infty}.
\]

The support of \( \phi_{N+1} \) is contained in \( B \) and

\[
\|g_{N+1}\|_\infty \leq \|g\|_\infty + \gamma(B)^{-1} \int |g| \, d\gamma \leq C \|g\|_{L^\infty}.
\]

Thus

\[
\|g_j\|_{H^1(\gamma)} \leq C \gamma(B) \|g\|_{L^\infty}, \quad j = 1, \ldots, N+1.
\]

Summing the coefficients in the atomic decomposition (4.3), we then obtain via (4.2)

\[
\|g - \int g \, d\gamma \phi_0\|_{H^1(\gamma)} \leq C (N+1) \gamma(B) \|g\|_{L^\infty} \leq C(1 + |c_B|^2) \gamma(B) \|g\|_{L^\infty}.
\]

The proof of the lemma is complete. \( \square \)

**Theorem 4.2.** Suppose that \( f \) is a function in \( L^1(\gamma). \) If \( \mathcal{M}_{\text{loc}} f \) is in \( L^1(\gamma) \) and

(4.4)

\[
E_+(f) = \int |x|^2 |f(x)| \, d\gamma(x) < \infty,
\]

then \( f \) is in \( H^1(\gamma) \) and

\[
\|f\|_{H^1(\gamma)} \leq C \|\mathcal{M}_{\text{loc}} f\|_1 + C E_+(f).
\]

If \( f \) is nonnegative, the conditions \( \mathcal{M}_{\text{loc}} f \in L^1(\gamma) \) and \( E_+(f) < \infty \) are also necessary for \( f \) to be in \( H^1(\gamma). \)
Proof. Let $f$ be a function in $L^1(\gamma)$ such that $M_{\text{loc}} f \in L^1(\gamma)$ and $E_+(f) < \infty$. Write $f = c(f) + f_0$, where $c(f) = \int f \, d\gamma$. Since $c(f)$ is a multiple of the exceptional atom, it suffices to find an atomic decomposition of $f_0$. Note that $f_0$ satisfies

$$
M_{\text{loc}} f_0 \in L^1(\gamma) \quad \text{and} \quad \int |x|^2 |f_0(x)| \, d\gamma(x) < \infty.
$$

Let $\{B_j\}$ be the covering of $\mathbb{R}^n$ by maximal admissible balls and $\{\eta_j\}$ the corresponding partition of unity introduced in the proof of Theorem \ref{thm:atom-decomposition}. As there, we choose numbers $b_j \in \mathbb{C}$ such that

$$
\int_{-\infty}^{\infty} (f_0 - b_j)\eta_j \, d\gamma = 0 \quad \forall j.
$$

Then the argument leading to \ref{eq:atomic-decomposition} shows that

$$
f_0 = \sum_j \sum_k \lambda_{jk} a_{jk} + \sum_j b_j\eta_j,
$$

where the $a_{jk}$ are Gaussian atoms supported in $4B_j$ and

$$
\sum_{j,k} |\lambda_{jk}| \leq C \|M_{\text{loc}} f_0\|_{L^1(\gamma)}.
$$

It remains only to prove that $\sum_j b_j\eta_j$ is in $H^1(\gamma)$. We write $g_j = b_j\eta_j$ and observe that

$$\int \sum_j g_j \, d\gamma = 0
$$

because $f_0$ and the $a_{ij}$ have integrals zero. Thus

$$
\sum_j g_j = \sum_j \left( g_j - \int g_j \, d\gamma \phi_0 \right),
$$

where $\phi_0$ is as in Lemma \ref{lem:atom-approximation}. Since \ref{eq:hardy-littlewood} remains valid for $f_0$, we have

$$
\|g_j\|_\infty \leq C \frac{1}{\gamma(B_j)} \int_{B_j} |f_0| \, d\gamma.
$$

Lemma \ref{lem:atomic-approximation} thus applies to each $g_j$, and using also the bounded overlap of the $B_j$ we conclude

$$
\|\sum_j g_j\|_{H^1(\gamma)} \leq C \sum_j (1 + |c_{B_j}|^2) \int_{B_j} |f_0| \, d\gamma \leq C \int (1 + |x|^2) |f_0| \, d\gamma.
$$

This concludes the proof of the sufficiency and the norm estimate.

The necessity of the condition $M_{\text{loc}} f \in L^1(\gamma)$ was obtained in Lemma \ref{lem:hardy-littlewood}.

To prove the necessity of \ref{eq:bmo-condition}, let $0 \leq f \in H^1(\gamma)$. We first observe that the function $x \mapsto |x|^2$ is in $BMO(\gamma)$. Indeed, its oscillation on any admissible ball is bounded. Since $BMO(\gamma)$ is a lattice, the functions $g_k(x) = \min \{ |x|^2, k \}$ are in $BMO(\gamma)$, uniformly for $k \geq 1$. By the monotone convergence theorem and the duality between $H^1(\gamma)$ and $BMO(\gamma)$,

$$
\int |x|^2 f(x) \, d\gamma(x) = \lim_k \int g_k(x)f(x) \, d\gamma(x) \leq C \|f\|_{H^1(\gamma)}.
$$

The theorem is proved. \hfill \Box
The following result is a noteworthy consequence of Theorem 4.2.

**Corollary 4.3.** For $1 < p \leq \infty$, one has continuous inclusions $L^p(\gamma) \subset H^1(\gamma)$ and $\text{BMO}(\gamma) \subset L^{p'}(\gamma)$, where $p' = p/(p-1)$.

**Proof.** We claim that the operator $M_{\text{loc}}$ is bounded on $L^p(\gamma)$ for $1 < p \leq \infty$. Deferring momentarily the proof of this claim, we complete the proof of the corollary. Suppose that $f$ is in $L^p(\gamma)$. Then $M_{\text{loc}}f$ is in $L^1(\gamma)$, because
\[
\|M_{\text{loc}}f\|_1 \leq \|M_{\text{loc}}f\|_p \leq C\|f\|_p < \infty,
\]
since $\gamma(\mathbb{R}^n) = 1$. Moreover, $E_+(f) \leq \|\|x\|^p\|_p \|f\|_p < \infty$, by Hölder’s inequality. Thus $f \in H^1(\gamma)$ by Theorem 4.2. It also follows that the inclusion $L^p(\gamma) \subset H^1(\gamma)$ is continuous, and by duality we get the continuous inclusion $\text{BMO}(\gamma) \subset L^{p'}(\gamma)$.

It remains to prove the claim. We shall use again the covering $\{B_j\}$ from the proof of Theorem 3.3. First we observe that the inequality
\[
(4.7) \quad \|M_{\text{loc}}g\|_p \leq C\|g\|_p
\]
holds when $\text{supp} \, g \subset B_j$, with a constant $C$ independent of $j$. Indeed, $M_{\text{loc}}$ is bounded on $L^p(\lambda)$, and $M_{\text{loc}}g$ is supported in the ball $4B_j$, where the Gaussian measure is essentially proportional to $d\lambda$.

Given a function $f \in L^p(\gamma)$, we write it as a sum $f = \sum f_j$ with $\text{supp} \, f_j \subset B_j$ and with the sets $\{f_j \neq 0\}$ pairwise disjoint. We can then apply (4.7) to each $f_j$ and sum.

\[
\square
\]

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