Recovering Hölder smooth functions from noisy modulo samples

Michaël Fanuel  
Univ. Lille, CNRS, Centrale Lille, UMR 9189, CRISAM  
F-59000 Lille, France  
michael.fanuel@univ-lille.fr

Hemant Tyagi*  
Inria, Univ. Lille, CNRS, UMR 8524, Laboratoire Paul Painlevé  
F-59000 Lille, France  
hemant.tyagi@inria.fr

Abstract—In signal processing, several applications involve the recovery of a function given noisy modulo samples. The setting considered in this paper is that the samples corrupted by an additive Gaussian noise are wrapped due to the modulo operation. Typical examples of this problem arise in phase unwrapping problems or in the context of self-reset analog to digital converters. We consider a fixed design setting where the modulo samples are given on a regular grid. Then, a three-stage recovery strategy is proposed to recover the ground truth signal up to a global integer shift. The first stage denoises the modulo samples by using local polynomial estimators. In the second stage, an unwrapping algorithm is applied to the denoised modulo samples on the grid. Finally, a spline based quasi-interpolant operator is used to yield an estimate of the ground truth function up to a global integer shift. For a function in Hölder class, uniform error rates are given for recovery performance with high probability. This extends recent results obtained by Fanuel and Tyagi for Lipschitz smooth functions wherein $k$-NN regression was used in the denoising step.

Index Terms—modulo samples, non parametric regression, phase unwrapping

I. INTRODUCTION

Various signal processing applications deal with noisy modulo samples of a signal. A typical example is “phase unwrapping” where the data is often available in the form of modulo $2\pi$ samples and which arises during the estimation of the depth map of a terrain (e.g., [1], [2]), or in the context of biomedical applications [3], [4]. Another application involves self-reset analog to digital converters [5]–[7] in a context where the reset counts are not used but only modulo samples are available. This is closely related to the works [8], [9] where a folding of the signal is deliberately injected.

II. PROBLEM SETUP

Let $f : [0, 1]^d \rightarrow \mathbb{R}$ be an unknown function, we assume that we are given noisy modulo 1 samples of $f$ on a uniform grid, i.e.,

$$y_i = (f(i/n) + \eta_i) \mod 1; \quad i = 1, \ldots, n.$$  

(II.1)

Here, $\eta_i \sim \mathcal{N}(0, \sigma^2)$ are i.i.d Gaussian noise samples, with $\sigma$ denoting the noise level. It will be useful to denote $x_i = i/n$ from now on. We will assume that $f \in C^{1,\alpha}([0, 1], M)$ which denotes the Hölder class of functions, defined below.

* The authors are listed in alphabetical order.

Definition 1. For $l \in \mathbb{N}_0$, $\alpha \in (0, 1]$ and $M > 0$, the Hölder class $C^{l,\alpha}([0, 1], M)$ consists of $l$ times continuously differentiable functions $f : [0, 1] \rightarrow \mathbb{R}$ whose $l$th derivative $f^{(l)}$ satisfies $|f^{(l)}(x) - f^{(l)}(y)| \leq M |x - y|^\alpha \quad \forall x, y \in [0, 1]$. Moreover, $\beta := l + \alpha > 0$ denotes the smoothness of this class.

Given $(y_i)_{i \in [n]}$, our aim is to obtain an estimate $\hat{f} : [0, 1] \rightarrow \mathbb{R}$ of $f$. Clearly, we can only hope to recover $f$ up to a global integer shift. To this end, we will follow a three-stage strategy considered recently in [10] for estimating Lipschitz $f$. We recall the steps below.

1) (Stage 1: Denoise modulo samples) This involves mapping the noisy modulo samples onto the unit complex circle $T_1$ as

$$z_i = \exp(2\pi i y_i) = h_i \exp(2\pi \eta_i),$$  

(II.2)

with $h_i = h(x_i) = \exp(2\pi f(x_i))$ denoting the clean modulo samples with $i = 1, \ldots, n$, and where $h : [0, 1] \rightarrow T_1$ is defined as $h(x) = \exp(2\pi f(x))$.

The idea in [10] was to note that if $f$ is Lipschitz smooth, then it implies that $h$ is also Lipschitz. This motivated a nearest neighbor based denoising procedure for uniformly estimating $h_i$ via $h_i \in T_1$, for all $i$. That in turn lead to estimates $\hat{f}_i \mod 1 = \frac{1}{2\pi} \arg(h_i)$ of $f_i \mod 1 = \frac{1}{2\pi} \arg(h_i)$ with a uniform bound

$$d_u(\hat{f}_i \mod 1, f_i \mod 1) \lesssim \delta(n), \quad \forall i \in [n].$$

2) (Stage 2: Unwrap denoised modulo samples) Given the estimates $\hat{f}_i \mod 1$ from the first stage, we then perform an unwrapping procedure reminiscent of Itho’s method from phase unwrapping. Denoting $L$ to be the Lipschitz constant of $f$, if $\delta(n) + \frac{L}{\pi} \lesssim 1$, then this procedure returns estimates $\tilde{f}(x_i)$ such that (see [10] Lemma 5)

$$|\tilde{f}(x_i) + q^* - f(x_i)| \lesssim \delta(n), \quad \forall i \in [n].$$

3) (Stage 3: Obtain $\tilde{f}$ via quasi-interpolants) Given the estimates $(\tilde{f}(x_i))_{i \in [n]}$ from the second stage, the estimate $\tilde{f} : [0, 1] \rightarrow \mathbb{R}$ is finally obtained by applying a suitable quasi-interpolant operator on these estimates.
Additionally, one can readily show the error bound (see [10] Theorem 5)
\[
\left\| \hat{f} + q^* - f \right\|_\infty \leq C_1 n^{-1} + C_2 \delta(n) \tag{II.3}
\]
where \(C_1, C_2\) are absolute constants, and \(q^* \in \mathbb{Z}\) is some integer.

The setup in [10] assumes \(f\) to be Lipschitz continuous which results in \(\delta(n) = O((\log n/n^{1/3}))\) for univariate functions.\footnote{In [10] the rate \(O((\log n/n)^{(d+2)})\) was derived for \(d\)-variate \(f\).
}

For the estimation of \(f\), this leads to the \(L_\infty\) error rate \((\log n/n^{1/3})\) which matches the optimal \(L_\infty\) rate for estimating univariate Lipschitz functions for the model \((\ref{II.1})\) without the modulo operation (see \((\ref{II.1})\)).

Our goal: We aim to extend this result to the more general setting where \(f \in C^{l,\alpha}(\{0, 1\}, M)\). We will show this by modifying the denoising procedure in Stage 1 where we will instead consider a local polynomial estimator of order \(l\). Such estimators are classical in the nonparametric regression literature, we will adapt the analysis in the book of Tsybakov [12, Chapter 1] to our setting and show that \(\delta(n) = O((\log n/n^{1 - \beta}))\) where \(\beta := l + \alpha > 0\). Due to \((\ref{II.3})\), this will then imply the \(L_\infty\) error rate \((\log n/n^{2 - \beta/l})\) for estimating \(f\) which matches the optimal \(L_\infty\) rate for estimating functions lying in \(C^{l,\alpha}(\{0, 1\}, M)\) for the model \((\ref{II.1})\) without the modulo operation (see \((\ref{II.1})\)).

Notations: We denote \([n] = \{1, \ldots, n\}\). The imaginary unit is \(i\) such that \(i^2 = -1\). The product of unit circles is denoted by \(\mathbb{T}_n = \mathbb{T}_1 \times \cdots \times \mathbb{T}_1\) with \(\mathbb{T}_1 := \{u \in \mathbb{C} : |u| = 1\}\). We define the projection of \(u \in \mathbb{C}^n\) on \(\mathbb{T}_n\) as
\[
\left( \frac{u}{|u|} \right)_i = \begin{cases} \frac{u_i}{|u_i|} & \text{if } u_i \neq 0, \\ 1 & \text{otherwise,} \end{cases}
\]
for all \(i \in [n]\). The commonly used wrap around metric \(d_w : [0, 1] \to [0, 1/2] \) is defined as \(d_w(t, t') := \max \{|t - t'|, 1 - |t - t'|\}\). For \(1 \leq p \leq \infty\), \(\|x\|_p\) is the \(\ell_p\) norm of \(x \in \mathbb{C}^n\). For any \(a, b \geq 0\), we write \(a \lesssim b\) if there is \(C > 0\) such that \(a \leq Cb\). Moreover, we write \(a \asymp b\) if \(a \lesssim b\) and \(b \lesssim a\).

III. Denoising modulo samples via local polynomial estimators

Denoting \(h_R(x), h_I(x)\) to be the real and imaginary parts of \(h(x) = \exp(i2\pi f(x))\) for any \(x \in [0, 1]\), a crucial observation that we use is that if \(f \in C^{l,\alpha}([0, 1], M)\), and if \(f^{(l)}\) are uniformly bounded for all \(0 \leq l \leq l\), then it implies \(h_R, h_I \in C^{l,\alpha}([0, 1], M')\) for some constant \(M' > 0\).

**Proposition 1.** Suppose \(f \in C^{l,\alpha}([0, 1], M)\) for some \(l \in \mathbb{N}_0, \alpha \in (0, 1]\) and \(M > 0\). Further, assume that \(\|f^{(l)}\|_\infty \leq \kappa\) for some \(\kappa > 0\) and for all integers \(0 \leq l \leq l\). Then there exists \(M' > 0\) depending only on \(l, M, \kappa\) such that for the functions \(h_R(x) = \cos(2\pi f(x))\) and \(h_I(x) = \sin(2\pi f(x))\), we have that \(h_R, h_I \in C^{l,\alpha}([0, 1], M')\).

The proof of Proposition \(\text{II}\) and all other results in the paper are outlined in the appendix. Our estimator \(\hat{h}(x) = \exp(i2\pi f(x))\) will constructed via a local polynomial estimator of order \(l\), namely \(LP(l)\). Before introducing the estimator, let us first define some additional quantities following the notation in [12, Section 1.6].

- \(K : \mathbb{R} \to \mathbb{R}\) denotes a kernel, and \(b > 0\) its bandwidth.
- For any \(u \in \mathbb{R}\) and integer \(l \geq 0\), \(U(u) = (1, u, u^2/2!, \ldots, u^l/l!)^T \in \mathbb{R}^{l+1}\).
- For any \(x \in [0, 1]\), \(B_{nx} \in \mathbb{R}^{(l+1)\times (l+1)}\) denotes the matrix 
  \[
  B_{nx} = \frac{1}{nb} \sum_{i=1}^n U \left( \frac{x_i - x}{b} \right) U^\top \left( \frac{x_i - x}{b} \right) K \left( \frac{x_i - x}{b} \right).
  \]
  a) Local polynomial estimator: Denoting \(z = (z_1, \ldots, z_n)^T \in \mathbb{T}_n\) with \(z_i \in \mathbb{T}_1\) as in \((\ref{II.2})\), we first compute
  \[
  \hat{\theta}(x) = \arg \min_{\theta \in \mathbb{C}^{l+1}} \sum_{i=1}^n \left| z_i - \theta^T U \left( \frac{x_i - x}{b} \right) \right|^2 K \left( \frac{x_i - x}{b} \right).
  \]
  Denote \(\hat{\theta}_{n,R}, \hat{\theta}_{n,I} \in \mathbb{R}^n\) to be the real and imaginary parts of \(\hat{\theta}(x)\). Then, we obtain a preliminary estimate
  \[
  \tilde{h}(x) = U^\top(0)\hat{\theta}_{n,R}(x) + iU^\top(0)\hat{\theta}_{n,I}(x) =: \tilde{h}_R(x) + i\tilde{h}_I(x). \tag{III.1}
  \]
  Here, \(\tilde{h}_R(x), \tilde{h}_I(x)\) are \(LP(l)\) estimators of \(h_R(x)\) and \(h_I(x)\) respectively (see [12, Definition 1.8]).
  b) Projection step: Next, we project \(\tilde{h}(x)\) onto \(\mathbb{T}_1\) to obtain \(\hat{h}(x)\), and hence \(f(x)\) mod 1, as
  \[
  \hat{h}(x) := \frac{\tilde{h}(x)}{||\tilde{h}(x)||}, \quad f(x) \text{ mod } 1 = \frac{1}{2\pi} \arg(\hat{h}(x)). \tag{III.2}
  \]

**Remark 1.** If \(B_{nx} > 0\) then \(\hat{h}(x)\) can be uniquely written as 
\[
\hat{h}(x) = \tilde{h}_R(x) + i\tilde{h}_I(x) = \sum_{i=1}^n z_i W_{ni}^*(x), \tag{III.3}
\]
where \(W_{ni}^*(x) \in \mathbb{R}\) is given by [12]
\[
W_{ni}^*(x) = \frac{1}{nb} U^\top(0)B_{nx}^{-1}U \left( \frac{x_i - x}{b} \right) K \left( \frac{x_i - x}{b} \right).
\]

Our aim is to show that \(\left| \hat{h}(x_1) - \hat{h}(x_2) \right|\) is uniformly bounded for all \(i \in [n]\). To show this, we will need some preliminary tools from [12, Section 1.6].

A. Preliminaries

Firstly, we recall [12, Proposition 1.12] which states that under mild assumptions, the \(LP(l)\) estimator reproduces polynomials of degree less than or equal to \(l\).

**Proposition 2** ( [12, Proposition 1.12]). Let \(x\) be such that \(B_{nx} > 0\) and let \(Q\) be a polynomial of degree \(\leq l\). Then the
LP(1) weights $W_n^*_i$ are such that $\sum_{i=1}^{n} Q(x_i)W_n^*(x_i) = Q(x)$. In particular, 
\[ \sum_{i=1}^{n} W_n^*_i(x) = 1, \quad \sum_{i=1}^{n} (x_i - x)^k W_n^*_i(x) = 0 \text{ for } k = 1, \ldots, l. \]

Next, we need to establish conditions under which $B_{nx} > 0$ for all $x \in [0, 1]$.

**Lemma 1 (Lemma 1.5).** Suppose there exist $K_{\text{min}} > 0$ and $\Delta > 0$ such that $|K(u)| \leq K_{\text{max}} 1_{\{|u| \leq \Delta\}}$, $\forall u \in \mathbb{R}$. Let $b = b_n$ be a sequence satisfying $b_n \to 0$ and $nb_n \to \infty$ as $n \to \infty$. Then there exist $\lambda_0, n_0 > 0$ such that $\lambda_{\text{min}}(B_{nx}) \geq \lambda_0$ for all $n \geq n_0$ and any $x \in [0, 1]$.

Finally, we recall the following result from **[12] Lemma 1.3** which states useful properties for the weights $W_n^{*i}$.

**Lemma 2 (Lemma 1.3).** Suppose that the kernel $K$ has compact support in $[-1, 1]$ and there exists a number $K_{\text{max}} < \infty$ such that $|K(u)| \leq K_{\text{max}}, \forall u \in \mathbb{R}$. Then under the assumptions of Lemma 2 and with $b \geq 1/(2n)$, the following is true.

1. $\sup_{x,i} |W_n^*_i(x)| \leq C_n b_n$.
2. $\sum_{i=1}^{n} |W_n^*_i(x)| \leq C_n^* \beta q$.
3. $W_n^*_i(x) = 0$ if $|x_i - x| > b$.

where $C_n^* = \frac{8K_{\text{max}}}{\lambda_0}$.

**B. Analysis**

We now derive conditions so that with high probability, $|\hat{h}(x_i) - h(x_i)| = O(\left(\frac{\log n}{n}\right)^{1/2})$ holds for all $i \in [n]$. The main tool is the following lemma which details the bias-variance trade-off in the estimation error.

**Lemma 3.** Assume $h_R, h_I \in C^{l,\alpha}([0, 1], M^l)$ and denoting $\beta = l + \alpha$, let $\hat{h}_R, \hat{h}_I$ be their respective LP(1) estimators as in (III.2). Under the notation and conditions of Lemma 7 and 2 and assuming $nb \geq \log n$, it holds with probability at least $1 - 4n^{1-\epsilon}$ (for any $\epsilon \geq 2$) that the estimator $\hat{h}$ in (III.2) satisfies 
\[ |\hat{h}(x_i) - h(x_i)| \leq q_1 b \beta + q_2 \left(\frac{\log n}{nb}\right)^{1/2}, \quad \forall i \in [n], \]

with $q_1 = \frac{4M \epsilon_c}{\sqrt{\pi}}, q_2 = \sqrt{\pi} \alpha$, $\epsilon_c$ = $A(\alpha)$ where $A(\alpha) := \frac{1}{\sqrt{\pi} e^{\frac{\alpha}{2}}}$.

We now obtain the following result by instantiating Lemma 3 for the best choice of $b$. This is our main denoising result which we had set out to prove in this section.

**Theorem 1.** Let $f \in C^{l,\alpha}([0, 1], M)$ with $\|f\|_{l,\alpha} \leq K$ for some $\alpha > 0$ and for all integers $0 \leq l \leq l$. Consider the estimator $\hat{h}(x) = \text{exp}(2\pi f(x))$ as in (III.2), at any $x \in [0, 1]$. Denoting $\beta = l + \alpha$, for any $\epsilon \geq 2$, suppose that 

1. there exist constants $K_{\text{min}}, K_{\text{max}}, \Delta > 0$ so that $K_{\text{min}} 1_{\{|u| \leq \Delta\}} \leq K(u) \leq K_{\text{max}} 1_{\{|u| \leq 1\}}, \quad \forall u \in \mathbb{R};$
2. $b = b^* = \left(\frac{\sqrt{\epsilon_c}}{\sqrt{\pi} e^{\frac{\alpha}{2}}}, \frac{\alpha}{2} \pi \beta, \frac{\log n}{\sqrt{\pi} e^{\frac{\alpha}{2}}}\right)$ with $A(\alpha)$ as in Lemma 3 and $M^* > 0$ depending only on $M, l, \kappa$;
3. $n \geq n_0, \frac{\log n}{n} \geq \left(\frac{32M^*}{e^{\frac{\alpha}{2}}\lambda_0}\right)^{1/\beta}$ with $n_0$ as in Lemma 7.

Then with probability at least $1 - 4n^{1-\epsilon}$ we have for all $i \in [n], |
\begin{align*}
\hat{h}(x_i) - h(x_i) & \leq \left(\frac{32M^* K_{\text{max}}}{\sqrt{\pi} \lambda_0}\right)^{1/\beta} \left(\frac{64cK_{\text{max}}A(\alpha)}{\lambda_0}\right)^{1/\beta} \\
& \quad \times \left((2\beta)^{-\frac{1}{2\beta} + 1} + (2\beta)^{-\frac{1}{2\beta} + 1}\right) \left(\frac{\log n}{n}\right)^{1/\beta} \\
& =: \delta(n),
\end{align*}
\]

where $\lambda_0$ is as in Lemma 7. Furthermore, if $\delta(n)$ holds and if $\delta(n) \leq 2$, then 
\[ d_n(f(x_i) \mod 1, f(x) \mod 1) \leq \frac{\delta(n)}{4}, \quad \forall i \in [n]. \]

**Proof.** Denoting $F(b)$ to be the bound in (III.3), we easily verify that the global minimizer of $F$ is $b^* = \left(\frac{\sqrt{\epsilon_c}}{\sqrt{\pi} e^{\frac{\alpha}{2}}}, \frac{\alpha}{2} \pi \beta, \frac{\log n}{\sqrt{\pi} e^{\frac{\alpha}{2}}}\right)$ with $q_1, q_2$ as in Lemma 3. Clearly $b_n \to 0$ as $n \to \infty$, and $b_n^* \geq \log n$ for the stated condition on $n/\log n$. Plugging $b = b^*$ in (III.4) leads to the stated bound in (III.5) after some simplifications. The bound in (III.6) follows directly using [10] Fact 4.

IV. ERROR RATE FOR RECOVERING $f$

Given the denoised estimates of $f(x_i) \mod 1$ for each $i$, we can now recover $f$ following the steps described in [10]. Indeed, we first recover estimates $f(x_i)$ for the samples $f(x_i) \mod 1$ using the sequential unwrapping procedure outlined as Algorithm 2 in [10], for the univariate setting $d = 1$. Denoting $\hat{g}(x_i) = f(x_i) \mod 1$, we recall from [10] that $f(x_i) = \hat{g}(x_i)$ and 
\[ \hat{f}(x_i) = \hat{f}(x_i-1) + \begin{cases} d_i & \text{if } |d_i| < 1/2, \\ 1 + d_i & \text{if } d_i < -1/2, \\ -1 + d_i & \text{if } d_i > 1/2. \end{cases} \]

for $d_i = \hat{g}(x_i) - \hat{g}(x_i-1)$ with $i \geq 2$. We also know that there exists $L > 0$ such that 
\[ |f(x) - f(y)| \leq L |x - y|^{\min(\beta, 1)}, \]

where $L = M$ if $\beta \leq 1$, and $L = \kappa$ otherwise. Then, following the steps in [10] Lemma’s 2.2 2.3, it is easy to verify that if (III.6) holds along with the condition $\delta(n) + \frac{2L}{\min(\beta, 1)} < 1$ then for some integer $q^*$, 
\[ f(x_i) + q^* - f(x_i) \leq \frac{\delta(n)}{4}, \quad \forall i \in [n]. \]

Next, we use these estimates to obtain an estimate of $f$ via spline-based quasi-interpolators (QI). QI’s are linear operators $Q_n : C([0, 1]) \to C([0, 1])$, where $Q_n(g)$ depends only on the values $g(x_i)$ for $i = 1, \ldots, n$. These objects are classical in the literature, see for e.g. [13], [14] for a detailed overview including the construction of these operators. It is well known (see [10] Remark 2.6) that there exists a constant $C_{l,\alpha,M} > 0$ (depending only on $l, \alpha, M$) such that 
\[ \|Q_n(g) - g\|_{l,\alpha} \leq C_{l,\alpha} n^{-l(1+\alpha)} = C_{l,\alpha} n^{-\beta}, \]
for all \( g \in C^{l,\alpha}([0,1],M) \). Denoting \( \tilde{f} \in C([0,1]) \) to function which takes the values \( f(x_i) \) for each \( i \), the estimate \( \tilde{f} \) of \( f \) is obtained as \( f = Q_n(\tilde{f}) \). The complete procedure outlined as Algorithm 1 below.

Algorithm 1 Function recovery from noisy modulo samples

1. Input: noisy modulo samples \( y_i \in [0,1] \) for each \( x_i = i/n, i = 1, \ldots, n; l \in \mathbb{N}_0 \).
2. Output: Estimate \( \hat{f} : [0,1] \to \mathbb{R} \) of \( f \).
3. Denoising step: Form \( z_i = \exp(\sqrt{2}y_i) \) for \( i = 1, \ldots, n \).
4. for \( i \in [n] \) do
   5. Compute LP(l) estimators \( \hat{h}(x_i) \in \mathbb{C} \) as in (III.1).
   6. Obtain estimate \( \hat{g}(x_i) = f(x_i) \mod 1 \) as in (III.2).
7. end for
8. Unwrapping step: Use \( (\hat{g}(x_i))_{i \in [n]} \) to obtain \( (\hat{f}(x_i))_{i \in [n]} \) as in (IV.1).
9. Recovering \( f \): Output \( \hat{f} = Q_n(\hat{f}) \) where \( Q_n \) is a spline-based quasi interpolant operator.

Main result for recovering \( f \): The following theorem provides a \( L\infty \) error bound for recovering \( f \in C^{l,\alpha}([0,1],M) \) up to a global integer shift. The proof follows in the same manner as that of [10] Theorem 2.4 and is omitted.

**Theorem 2.** Under the notations in Theorem [7] for any \( f \in C^{l,\alpha}([0,1],M) \) with \( \beta = l + \alpha \), suppose that \( f, n, b \) and the kernel \( K : [-1,1] \to \mathbb{R} \) satisfy the conditions of Theorem [7]. Recalling \( \delta(n) \) from (III.5), which is defined for a given constant \( c \geq 2 \), suppose additionally that \( \delta(n) + \frac{2L}{n^{1/2}} < 1 \) with \( L \) as in (IV.2). If the IQ operator \( Q_n \) satisfies (IV.4), then with probability at least \( 1 - 4n^{1-c} \), it holds that the estimate \( \hat{f} = Q_n(\hat{f}) \) satisfies (for some integer \( q^* \)) the bound

\[
\| \hat{f} + q^* - f \|_{\infty} \leq C_{l,\alpha,M}n^{-\beta} + C\delta(n).
\]

Here \( C > 0 \) is an absolute constant while \( C_{l,\alpha,M} \) is the constant in (IV.4).

Theorem 2 suggests that \( \| \hat{f} + q^* - f \|_{\infty} = O\left( \frac{\log(n)/n}{\sqrt{n}} \right) \). As remarked earlier, this rate matches the optimal \( L\infty \) rate for estimating functions lying in \( C^{l,\alpha}([0,1],M) \), for the model (I.I) without the modulo operation (see (I.I)).

V. NUMERICAL SIMULATIONS

We consider the function

\[
f(x) = 4x \cos(2\pi x)^2 - 2\sin(2\pi x)^2 + 4.7 \quad (V.1)
\]

and the noise model (I.I) with \( \sigma = 0.12 \). The output of Algorithm 1 is illustrated in Figure 1. The following other methods are considered: kNN denoising [10] (kNN), an unconstrained quadratic program [13] (UCQP) and a trust region subproblem [16] (TRS). A comparison between these methods is given in Figure 2 where the Root Mean Square Error (RMSE) between the ground truth and the recovered samples is displayed. We choose the parameters in the following way. For the local polynomial estimator, we take \( l = 2, \beta = 2.4 \) and \( b = 0.1(\log(n)/n)^{2/3+1} \). The number of neighbours of kNN is \( k = \lceil k^* \rceil \) with \( k^* = 0.09n^{\frac{3}{4}}(\log n) \); see [10]. We follow the analysis of (UCQP) and (TRS) by Tyagi [15] Corollary 4 and Corollary B which gives \( \lambda \approx (n^{10/3})^{1/4} \) up to multiplicative constants. Hence, we choose \( \lambda = 0.04n^{10/12} \). The latter methods use a graph \( G = ([n], E) \) which is simply here a path graph \( E = \{\{i,i+1\}: i = 1, \ldots, n-1\} \). From the numerical results in Figure 2 we observe that all the methods have a similar behavior given the error bars, whereas we conjecture that a fine-tuning of the parameters could boost the performance for a small sample size \( n \).

ACKNOWLEDGMENT

M.F. acknowledges support from ERC grant BLACKJACK (ERC-2019-STG-851866, PI: R. Bardenet).

2The code is available at https://github.com/mrfanuel/denosing-modulo-samples-local-poly.
REFERENCES

[1] L. C. Graham, “Synthetic interferometer radar for topographic mapping,” 
Proceedings of the IEEE, vol. 62, no. 6, pp. 763–768, 1974.
[2] H. Zebker and R. Goldstein, “Topographic mapping from interferometric 
synthetic aperture radar observations,” Journal of Geophysical Research: 
Solid Earth, vol. 91, no. B5, pp. 4993–4999, 1986.
[3] M. Hedley and D. Rosenfeld, “A new two-dimensional phase unwrapping 
algorithm for mri images,” Magnetic Resonance in Medicine, 
vol. 24, no. 1, pp. 177–181, 1992.
[4] P. Lauterbur, “Image formation by induced local interactions: examples 
employing nuclear magnetic resonance,” Nature, vol. 242, pp. 190–191, 
1973.
[5] W. Kester, “Mt-025 tutorial adc architectures vi: Folding adcs,” 2009, 
analog Devices, Tech. report.
[6] J. Rhee and Y. Joo, “Wide dynamic range cmos image sensor with pixel 
level adc,” Electronics Letters, vol. 39, no. 4, pp. 360–361, 2003.
[7] T. Yamaguchi, H. Takehara, Y. Sunaga, M. Haruta, M. Motoyama, 
Y. Ohta, T. Noda, K. Sasagawa, T. Tokuda, and J. Ohta, “Implantable 
self-reset cmos image sensor and its application to hemodynamic re-
response detection in living mouse brain,” Japanese Journal of Applied 
Physics, vol. 55, no. 4S, p. 04EM02, 2016.
[8] A. Bhandari, F. Krahmer, and R. Raskar, “On unlimited sampling and 
reconstruction,” IEEE Transactions on Signal Processing, pp. 1–1, 2020.
[9] ——, “Unlimited sampling of sparse sinusoidal mixtures,” in 2018 IEEE 
International Symposium on Information Theory (ISIT), 2018, pp. 336– 
340.
[10] M. Fanuel and H. Tyagi, “Denoising modulo samples: k-NN 
regression and tightness of SDP relaxation,” Information and 
Inference: A Journal of the IMA, 10 2021. [Online]. Available: 
https://doi.org/10.1093/imaiai/iaab022.
[11] A. Nemirovski, “Topics in non-parametric statistics,” Ecole d’Eté de 
Probabilités de Saint-Flour, vol. 28, p. 85, 2000.
[12] A. B. Tsybakov, Introduction to Nonparametric Estimation, 1st ed. 
Springer Publishing Company, Incorporated, 2008.
[13] R. A. DeVore and G. G. Lorentz, Constructive Approximation, ser. 
Grundlehren der mathematischen Wissenschaften. Springer, 1993, vol. 
303.
[14] C. de Boor, “Quasiinterpolants and approximation power of multivariate 
splines,” in Computation of Curves and Surfaces, 1990, pp. 313–345.
[15] H. Tyagi, “Error analysis for denoising smooth modulo signals on a 
graph,” arXiv preprint arXiv:2009.04859, 2020.
[16] M. Cucuringu and H. Tyagi, “Provably robust estimation of modulo 1 
samples of a smooth function with applications to phase unwrapping,” 
Journal of Machine Learning Research, vol. 21, no. 32, pp. 1–77, 2020.
[17] H. Liu, M.-C. Yue, and A. Man-Chao So, “On the estimation performance 
and convergence rate of the generalized power method for phase 
synchronization,” SIAM Journal on Optimization, vol. 27, no. 4, pp. 
2426–2446, 2017.
Starting with the observation that for any \(x \in [0,1]\), \(\tilde{h}(x) = \overline{h}(x)/|\overline{h}(x)| = e^{2\pi^2\sigma^2} \tilde{h}(x)/|e^{2\pi^2\sigma^2} \tilde{h}(x)|\), we obtain using Proposition 3.3 the bound
\[
|\overline{h}(x) - h(x)| \leq 2 e^{2\pi^2\sigma^2} |\tilde{h}(x) - h(x)|. \tag{A.1}
\]

Henceforth, we will focus on bounding the quantity \(e^{2\pi^2\sigma^2} \tilde{h}(x) - h(x)\). Since \(B_{nx} > 0\) for all \(x \in [0,1]\), we know from Proposition 2 that \(\sum_{i=1}^n W^*_{ni}(x) = 1\), and so using (A.3), we can write
\[
e^{2\pi^2\sigma^2} \tilde{h}(x) - h(x) = \sum_i (e^{2\pi^2\sigma^2} z_i - h_i) W^*_{ni}(x) + \sum_i (h_i - h(x)) W^*_{ni}(x),
\]
where we used Lemma 2. Moreover, it is easy to verify the bound
\[
V(x) \leq 4|\log n|_{\tilde{h}} \left(\frac{\log n}{\tilde{h}}\right)^{1/2} < 4n^{-c}
\]
which together with (A.1) readily yields the statement of the lemma after a union bound (with \(x = x_i\)) over \(i \in [n]\).

Bounding \(|\text{Bias}(x)|\):

We start by writing
\[
\text{Bias}(x) = \sum_{i=1}^n (h_{R,i} - h_{R}(x)) + \iota(h_{I,i} - h_{I}(x)) W^*_{ni}(x).
\]
Since \(h_{R}, h_{I} \in C_{f,\alpha}([0,1], M')\), hence we have using Proposition 2 that
\[
|\text{Bias}(x)| \leq 2 M' \sum_{i} |x_i - \bar{x}_i|^{\beta} |W^*_{ni}(x)| = 2 M' \sum_{i} |x_i - \bar{x}_i|^{\beta} |W^*_{ni}(x)| 1_{|x_i - \bar{x}_i| \leq \beta} \leq 2 M' C_{b} \beta
\]
where we used Lemma 2.

Bounding \(|V(x)|\):

We start by writing
\[
V(x) = \sum_{i} (e^{2\pi^2\sigma^2} z_{R,i} - h_{R,i}) W^*_{ni}(x) + \iota \sum_{i} (e^{2\pi^2\sigma^2} z_{I,i} - h_{I,i}) W^*_{ni}(x) =: V_R(x) + \iota V_I(x).
\]
Since \(|V(x)| \leq |V_R(x)| + |V_I(x)|\), we will bound the two RHS terms using Bernstein’s inequality. Its enough to bound \(|V_R(x)|\) since the same bound will apply for \(|V_I(x)|\). To this end, first note that each term in the summation is uniformly bounded as
\[
e^{2\pi^2\sigma^2} z_{R,i} - h_{R,i}, \quad |W^*_{ni}(x)| \leq (1 + e^{2\pi^2\sigma^2}) C_{n \beta},
\]
where we used Lemma 2. Moreover, it is easy to verify the bound
\[
E \left[ |e^{2\pi^2\sigma^2} z_{R,i} - h_{R,i}|^2 \right] = e^{4\pi^2\sigma^2} E \left[ |z_i - e^{2\pi^2\sigma^2} h_i|^2 \right] = e^{4\pi^2\sigma^2} - 1.
\]
Hence, it follows using Lemma 2 that
\[
\sum_{i} E[(e^{2\pi^2\sigma^2} z_{R,i} - h_{R,i})^2](W^*_{ni}(x))^2 \leq (e^{4\pi^2\sigma^2} - 1) \sum_{i}(W^*_{ni}(x))^2
\]
\[
= (e^{4\pi^2\sigma^2} - 1) \sum_{i}(W^*_{ni}(x))^2 1_{|x_i - x| \leq \beta} \leq (e^{4\pi^2\sigma^2} - 1) \frac{C_{n \beta}}{n^2} \sum_{i} 1_{|x_i - x| \leq \beta} \leq (e^{4\pi^2\sigma^2} - 1) \frac{C_{n \beta}}{n^2} \text{max} \{2nb, 1\} = 2(e^{4\pi^2\sigma^2} - 1) \frac{C_{n \beta}}{nb} (\text{ if } 2nb \geq 1).
Then, by using Bernstein’s inequality, we obtain for any \( c \geq 2 \) that with probability at least \( 1 - 2n^{-c} \),
\[
|V_B(x)| \leq 4c \log(n) (e^{4\pi^2\sigma^2} - 1) \frac{C^2}{nb} + 2cC_\ast \left( \frac{\log n}{nb} \right) (1 + e^{2\pi^2\sigma^2}) \\
\leq 2cC_\ast A(\sigma) \left( \frac{\log n}{nb} \right)^{1/2} \quad \text{(if } nb \geq \log n). 
\]
This leads to the bound on \( |V(x)| \) in (A.2) after a union bound, which also concludes the proof.

**Appendix B**

**Proof of Proposition 1**

Define for convenience \( \phi_s(x) = \cos(x - s\pi/2) \) with \( s \in \{0, 1\} \) so that \( \phi_0(x) = \cos(x) \) and \( \phi_1(x) = \sin(x) \). We now consider functions of the form
\[
\phi_s(g(x)) \quad \text{where } g(x) = 2\pi f(x) \quad \text{with } f \in C^{\ell,\alpha}([0,1],M),
\]
so that \( g \in C^{\ell,\alpha}([0,1],2\pi M) \). The boundedness assumption on \( \|f^{(\ell)}\|_\infty \) implies \( \|g^{(\ell)}\|_\infty \leq 2\pi \kappa \) for all \( 0 \leq \ell \leq l \). In particular, we have \( \phi_0(g(x)) = h_R(x) \) and \( \phi_1(g(x)) = h_I(x) \). For simplicity, we omit the dependence on \( s \) and simply work with \( \phi(x) \).

Our objective is to show that there exists a constant \( M' > 0 \) such that
\[
|\phi^{(l)}(g(x)) - \phi^{(l)}(g(y))| \leq M'|x - y|^{\beta - l},
\]
for all \( x, y \in [0,1] \).

**Preliminary results**

We start by stating useful elementary results. First, we give a simple result based on Taylor’s theorem.

**Lemma 4.** Let \( g \in C^{\ell,\alpha}([0,1],2\pi M) \) and \( \beta = \ell + \alpha \). Assume \( l \geq 1 \). Then, for all \( 0 \leq p < l \), the following inequality
\[
|g^{(p)}(x) - g^{(p)}(y)| \leq \frac{2\pi M}{(l-p)!} |x - y|^{\beta - p} + \sum_{k=1}^{l-p} \frac{|g^{(k+p)}(y)|}{k!} |x - y|^k,
\]
holds for all \( x, y \in [0,1] \).

**Proof.** By using the order \( q - 1 \) Taylor expansion of a \( q \)-times differentiable function \( z(x) \) at \( y \) and the remainder theorem, we have
\[
z(x) = \sum_{k=0}^{q} \frac{z^{(k)}(y)}{k!} (x - y)^k + \frac{(x - y)^q}{q!} \left( z^{(q)}(\xi) - z^{(q)}(y) \right),
\]
for some \( \xi \) between \( x \) and \( y \), thanks to the remainder theorem. Now choose \( z(x) = g^{(p)}(x) \) and \( q = l - p \). This gives
\[
g^{(p)}(x) = \sum_{k=0}^{l-p} \frac{g^{(p+k)}(y)}{k!} (x - y)^k + \frac{(x - y)^{l-p}}{(l-p)!} \left( g^{(l)}(\xi) - g^{(l)}(y) \right).
\]
Hence, by using the definition of the Hölder class, we find
\[
\left| g^{(p)}(x) - g^{(p)}(y) - \sum_{k=1}^{l-p} \frac{g^{(p+k)}(y)}{k!} (x - y)^k \right| \leq \frac{2\pi M}{(l-p)!} |x - y|^{l-p} |\xi - y|^{\beta - l} \leq \frac{2\pi M}{(l-p)!} |x - y|^{\beta - p},
\]
where we used \( |\xi - y| \leq |x - y| \) since \( \xi \) is between \( x \) and \( y \). Next, we use the reverse triangle inequality to yield
\[
\left| g^{(p)}(x) - g^{(p)}(y) - \sum_{k=1}^{l-p} \frac{g^{(p+k)}(y)}{k!} (x - y)^k \right| \leq \frac{2\pi M}{(l-p)!} |x - y|^{\beta - p}.
\]
Hence, it holds
\[
\left| g^{(p)}(x) - g^{(p)}(y) - \sum_{k=1}^{l-p} \frac{g^{(p+k)}(y)}{k!} (x - y)^k \right| \leq \frac{2\pi M}{(l-p)!} |x - y|^{\beta - p},
\]
and the desired result follows by using once again the triangle inequality.
A simple consequence of Lemma 4 is given below.

**Lemma 5.** Let \( g \in C^{l,\alpha}([0,1],2\pi M) \) and denote \( \beta = l + \alpha \). Suppose \( l \geq 1 \) and let \( p \) be an integer such that \( 0 \leq p < l \). Let \( \kappa > 0 \) a constant such that \( |g^{(l)}(x)| \leq 2\pi \kappa \) for all \( x \in [0,1] \) and \( 0 \leq \ell \leq l \). Then, there exists a constant \( C_2(M,p,\kappa) > 0 \) such that

\[
|g^{(p)}(x) - g^{(p)}(y)| \leq C_2|x - y|^{\beta - l}.
\]

**Proof.** Recall the following fact: if \( x, y \in [0,1] \), then \( |x - y| \leq |x - y|^s \) for all \( s \in (0,1] \). Then, the result follows by using Lemma 4 and \( |g^{(l)}(y)| \leq 2\pi \kappa \) for all \( y \in [0,1] \) and for all \( p + 1 \leq \ell \leq l \).

The following fact is a property of the sine function.

**Fact 1.** Let \( p \in \mathbb{N}_0 \). Then, it holds \( |\phi^{(p)}(x)| \leq 1 \) for all \( x \in [0,1] \).

Another useful identity in given in Fact 2 which is a simple algebraic fact that is readily proved by adding and subtracting identical terms.

**Fact 2.** Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) be real numbers with \( n \geq 2 \). Then, we have

\[
\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = (a_n - b_n) \prod_{i=1}^{n-1} a_i + \sum_{\ell=1}^{n-2} \left( (a_n - \ell) - (b_n - \ell) \right) \prod_{i=n-\ell+1}^{n} b_i \prod_{j=1}^{n-\ell} a_j + (a_1 - b_1) \prod_{i=2}^{n} b_i.
\]

The next statement is standard about polynomial factorization.

**Fact 3.** Let \( a, b \in \mathbb{R} \). For all integer positive \( n \geq 1 \), there exists a polynomial \( p_{n-1}(a, b) \) of order \( n - 1 \) such that

\[
a^n - b^n = (a - b)p_{n-1}(a, b).
\]

Finally, we recall the well-known Faà di Bruno formula for derivatives of composed functions.

**Fact 4 (Faà di Bruno formula).** Let \( l \in \mathbb{N}_0 \) and let \( g, \bar{g} \) be real valued \( l \)-times differentiable functions. We have

\[
(\bar{g} \circ g)^{(l)}(x) = \sum_{k_1, \ldots, k_l} \frac{l!}{k_1! \cdots k_l!} \bar{g}^{(k_1)}(g(x)) \left( \frac{g^{(1)}(x)}{1!} \right)^{k_1} \cdots \left( \frac{g^{(l)}(x)}{l!} \right)^{k_l},
\]

with \( k = k_1 + \cdots + k_l \) and where the sum goes over \( k_1, \ldots, k_l \) such that \( k_1 + 2k_2 + \cdots + lk_l = l \).

**Main part of the proof of Proposition 7.**

The proof mainly relies on the triangle inequality applied to the Faà di Bruno formula (see, Fact 4 with \( \bar{g} = \phi \)) and Fact 2. Let us also recall that \( |g^{(l)}(x)| \leq 2\pi \kappa \) for all \( x \in [0,1] \) and \( 0 \leq \ell \leq l \). Using Faà di Bruno formula with the triangle inequality, we obtain

\[
\left| (\phi \circ g)^{(l)}(x) - (\phi \circ g)^{(l)}(y) \right| \leq \sum_{k_1, \ldots, k_l} \frac{l!}{k_1! \cdots k_l!} |\Delta_{k_1, \ldots, k_l}(x) - \Delta_{k_1, \ldots, k_l}(y)|,
\]

with \( k = k_1 + \cdots + k_l \) and where the sum goes over \( k_1, \ldots, k_l \) such that \( k_1 + 2k_2 + \cdots + lk_l = l \). Here, we defined

\[
\Delta_{k_1, \ldots, k_l}(x) = \phi^{(k_1)}(g(x)) \left( \frac{g^{(1)}(x)}{1!} \right)^{k_1} \cdots \left( \frac{g^{(l)}(x)}{l!} \right)^{k_l}.
\]

Notice that the above product includes at most \( l + 1 \) non trivial factors. Next, we use Fact 2 and obtain

\[
\Delta_{k_1, \ldots, k_l}(x) - \Delta_{k_1, \ldots, k_l}(y)
\]

\[
= \left( \frac{g^{(1)}(x)}{1!} \right)^{k_1} - \left( \frac{g^{(1)}(y)}{1!} \right)^{k_1} \phi^{(k_1)}(g(x)) \left( \frac{g^{(1)}(x)}{1!} \right)^{k_1} \cdots \left( \frac{g^{(l-1)}(x)}{(l-1)!} \right)^{k_{l-1}}
\]

\[
+ \sum_{\ell=1}^{l-1} \left( \frac{g^{(\ell)}(x)}{(\ell)!} \right)^{k_{\ell}} - \left( \frac{g^{(\ell)}(y)}{(\ell)!} \right)^{k_{\ell}} \phi^{(k)}(g(x)) \left( \frac{g^{(1)}(x)}{1!} \right)^{k_1} \cdots \left( \frac{g^{(\ell)}(x)}{(\ell)!} \right)^{k_{\ell}} i \prod_{i=1}^{l-\ell+1} \left( \frac{g^{(i)}(x)}{i!} \right)^{k_i} \prod_{j=1}^{l-\ell-1} \left( \frac{g^{(j)}(y)}{j!} \right)^{k_j}.
\]
A triangle inequality and the bound $|g^{(\ell)}(x)| \leq 2\pi\kappa$ for all $x \in [0,1]$ and $0 \leq \ell \leq l$ yields

$$|\Delta_{k_1,\ldots,k_l}(x) - \Delta_{k_1,\ldots,k_l}(y)| \leq \left| \left(\frac{g^{(l)}(x)}{l!}\right)^{k_l} - \left(\frac{g^{(l)}(y)}{(l-\ell)!}\right)^{k_l} \right| \leq \left| \left(\frac{2\pi\kappa}{l!}\right)^{k_l} \cdots \left(\frac{2\pi\kappa\cdot(l-1)!}{l-\ell)!}\right)^{k_l}\right|$$

$$+ \sum_{\ell=1}^{l-1} \left| \left(\frac{g^{(l-\ell)}(x)}{(l-\ell)!}\right)^{k_{l-\ell}} - \left(\frac{g^{(l-\ell)}(y)}{(l-\ell)!}\right)^{k_{l-\ell}} \right| \leq \left| \frac{2\pi\kappa}{l!}\right|^{k_l} \cdots \left| \frac{2\pi\kappa\cdot(l-1)!}{l-\ell)!}\right|^{k_l}$$

For the second term, in the same way, we can upper bound each of the terms in the finite sum, i.e., for a given $\ell$,

$$\left| \phi^{(k)}(g(x)) - \phi^{(k)}(g(y)) \right| \leq \left| \frac{2\pi\kappa}{l!}\right|^{k_l} \cdots \left| \frac{2\pi\kappa\cdot(l-1)!}{l-\ell)!}\right|^{k_l},$$

where we used Lemma 5 for the last inequality. Here, $\phi^{(k)}(g(x))$ is some positive polynomial of $\kappa, l$.

Consider each of the three terms on the RHS of the last inequality.

For the first term, we can assume that $k_l \geq 1$, otherwise it vanishes. By using the factorization result in Fact 3, we have

$$\left| \left(\frac{g^{(l)}(x)}{l!}\right)^{k_l} - \left(\frac{g^{(l)}(y)}{(l-\ell)!}\right)^{k_l} \right| \leq \left| g^{(l)}(x) - g^{(l)}(y) \right| \cdot \left| p_{k_l-1}(g^{(l)}(x), g^{(l)}(y)) \right|$$

$$\leq \left| g^{(l)}(x) - g^{(l)}(y) \right| \cdot \tilde{C}_0(\kappa, l)$$

$$\leq 2\pi M |x - y|^{\beta-1} \tilde{C}_0(\kappa, l),$$

where we used a triangle inequality to bound $|p_{k_l-1}(g^{(l)}(x), g^{(l)}(y))|$ and where $\tilde{C}_0(\kappa, l)$ is some positive polynomial of $\kappa, l$.

For the second term, in the same way, we can upper bound each of the terms in the finite sum, i.e., for a given $\ell$,

$$\left| \left(\frac{g^{(l-\ell)}(x)}{(l-\ell)!}\right)^{k_{l-\ell}} - \left(\frac{g^{(l-\ell)}(y)}{(l-\ell)!}\right)^{k_{l-\ell}} \right| \leq \tilde{C}_1(\kappa, l) \left| g^{(l-\ell)}(x) - g^{(l-\ell)}(y) \right| \leq \tilde{C}_1(\kappa, l) C_2(M, \kappa, l - \ell)|x - y|^{\beta-1},$$

where we used Lemma 5 for the last inequality. Here, $\tilde{C}_1(\kappa, l)$ is some positive polynomial of $\kappa, l$ and $C_2(M, \kappa, l - \ell) > 0$ depends only on the indicated terms.

For the third term, remark that $|\phi(x) - \phi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. By definition of $\phi(x)$, this implies that

$$\left| \phi^{(k)}(g(x)) - \phi^{(k)}(g(y)) \right| \leq |g(x) - g(y)|,$$

for any non-negative integer $k$. Also, by Lemma 5 $|g(x) - g(y)| \leq C_2(M, \kappa) |x - y|^{\beta-1}$, which yields

$$\left| \phi^{(k)}(g(x)) - \phi^{(k)}(g(y)) \right| \leq C_2(M, \kappa) |x - y|^{\beta-1},$$

for all $x, y \in [0,1]$ and all non-negative integer $k$.

Finally, putting everything together, we obtain

$$|\Delta_{k_1,\ldots,k_l}(x) - \Delta_{k_1,\ldots,k_l}(y)| \leq C(M, \kappa, l)|x - y|^{\beta-1},$$

where $C(M, \kappa, l) > 0$ depends only on $M, \kappa$ and $l$. Plugging this within (B.1), we obtain the result.