COMMON INVARIANT SUBSPACE AND COMMUTING MATRICES

GERALD BOURGEOIS

Abstract. Let $K$ be a perfect field, $L$ be an extension field of $K$ and $A, B \in M_n(K)$. If $A$ has $n$ distinct eigenvalues in $L$ that are explicitly known, then we can check if $A, B$ are simultaneously triangularizable over $L$. Now we assume that $A, B$ have a common invariant proper vector subspace of dimension $k$ over an extension field of $K$ and that $\chi_A$, the characteristic polynomial of $A$, is irreducible over $K$. Let $G$ be the Galois group of $\chi_A$. We show the following results

(i) If $k \in \{1, n-1\}$, then $A, B$ commute.
(ii) If $1 \leq k \leq n-1$ and $G = S_n$ or $G = A_n$, then $AB = BA$.
(iii) If $1 \leq k \leq n-1$ and $n$ is a prime number, then $AB = BA$.

Yet, when $n = 4, k = 2$, we show that $A, B$ do not necessarily commute if $G$ is not $S_4$ or $A_4$. Finally we apply the previous results to solving a matrix equation.

1. Introduction

Throughout this paper, $K$ denotes a perfect field, and $\overline{K}$ an algebraic closure of $K$. Recall that a field $K$ is said to be perfect if every irreducible polynomial over $K$ has only simple roots in $K$.

For $M \in M_n(K)$, the set of $n \times n$ matrices with entries in $K$, $\sigma(M)$ denotes its spectrum, that is the set of its eigenvalues in $K$. Two matrices $A, B \in M_n(K)$ are said to be simultaneously triangularizable (denoted by ST) over $K$ if there exists a matrix $P \in GL_n(K)$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular. Thus such matrices have common invariant subspaces that form a complete flag over $K$.

In Section 2, we consider $A, B \in M_n(K)$ and we assume that $A$ has $n$ distinct eigenvalues in $L$, an extension field of $K$, and that $\sigma(A)$ is explicitly known. We give an algorithm which allows to check whether or not $A$ and $B$ are ST over $L$. Moreover, when $A$ and $B$ are ST we obtain a basis of $L$ that diagonalizes $A$ and triangularizes $B$.

In Section 3, we assume that $A, B \in M_n(K)$ have a common invariant proper vector subspace of dimension $k$ over $L$. We recall some criteria for the existence of common invariant proper subspaces of matrices. Shemesh gives this efficient criterion, when $k = 1$, in [7].

Theorem. Let $A, B \in M_n(\mathbb{C})$. Then $A$ and $B$ have a common eigenvector if and only if

$$\bigcap_{p, q=1}^{n-1} \ker([A^p, B^q]) \neq \{0\}.$$
Note that the complexity of this test is in \(O(n^5)\).
When \(k \geq 2\), a particular case, that is sufficient for our purpose, is treated in [4],[8] as follows. If \(U \in M_n(K)\), then \(U^{(k)}\) denotes its \(k\)-th compound.

**Theorem 1.** Let \(A, B \in M_n(\mathbb{C})\), \(k \in [2, n-1]\) be such that \(A\) has distinct eigenvalues. The following are equivalent

(i) \(A, B\) have a common invariant subspace \(W\) of dimension \(k\).

(ii) There exists \(s \in \mathbb{C}\) such that \((A + sI_n)^{(k)}\) has distinct eigenvalues and \((A + sI_n)^{(k)}, (B + sI_n)^{(k)}\) are invertible and have a common eigenvector in \(\mathbb{C}^k\). Moreover this eigenvector is decomposable in the exterior product of \(k\) vectors that constitute a basis of \(W\).

We can show that the complexity of this test is at most (when \(k = n/2\)) in \(O(2^{5n/n^2})\).

**Remark 1.** The previous two results are also valid over a field that is algebraically closed and has characteristic 0.

In the sequel, we work on a perfect field \(K\) and we will use the following notation

**Notation.** Let \(P \in K[x]\) be an irreducible polynomial of degree \(n\). The splitting field \(S_P\) of \(P\) is \(K(x_1, \ldots, x_n)\) where \(x_1, \ldots, x_n\) are the roots of \(P\) in \(\overline{K}\). The Galois group of \(P\) is the set of the \(K\)-automorphisms of \(S_P\), that is

\[\text{Gal}(S_P/K) = \{\tau \in \text{Aut}(S_P) \mid \forall t \in K, \tau(t) = t\}\]

it is isomorphic to a subgroup of \(S_n\), the group of all the permutations of \(\{1, \ldots, n\}\). If \(M \in M_n(K)\) and \(\chi_M\), the characteristic polynomial of \(M\), is irreducible, then \(G_M\) denotes the Galois group of \(\chi_M\).

Assume that \(A, B \in M_n(K)\) have a common invariant proper subspace of dimension \(k\) over an extension field \(L\) of \(K\) and that \(\chi_A\) is irreducible over \(K\). We consider conditions that imply that \(A, B\) commute. We show the following results.

(i) If \(k \in \{1, n-1\}\), then \(A, B\) commute.

(ii) If \(1 \leq k \leq n-1\) and \(G_A = S_A\) or \(G_A = A_n\), then \(AB = BA\).

(iii) If \(1 \leq k \leq n-1\) and \(n\) is a prime number, then \(AB = BA\).

The idea is as follows: let \(F = [u_1, \ldots, u_k]\) be a \(A\)-invariant vector space where the \((u_i)_{i \leq k}\) are eigenvectors of \(A\) associated to the eigenvalues \(E = (\alpha_i)_{i \leq k} \subset \sigma(A)\). We seek elements of \(G_A\) so that their orbits contain elements of \(E\) and elements of \(\sigma(A) \setminus E\). We consider \(Bu_1 \in F\) and we show that it is colinear to \(u_1\).

In Section 4, we consider the case when \(n = 4, k = 2\) and we show that the conclusion of (ii) may be false if we drop the hypothesis \(G_A = S_4\) or \(G_A = A_4\).

In Section 5, we use (i) and the simultaneous triangularization to solving the matrix equation \(AX = XA = X^a\) in a particular case.

2. **An algorithm checking ST property**

**Proposition 1.** Let \(A, B \in M_n(K)\) that are ST over \(L\), an extension field of \(K\). We assume that \(A\) has distinct eigenvalues over \(L\). Then there exists \(S \in GL_n(L)\) such that \(S^{-1}AS\) is diagonal and \(S^{-1}BS\) is upper triangular.

**Proof.** There exists \(P \in GL_n(L)\) such that \(P^{-1}AP = T, P^{-1}BP = U\) where \(T\) and \(U\) are upper triangular. Note that \(\sigma(A) \subset L\). The principal minors of \(T\) are diagonalizable over \(L\). By induction, we can construct a \(T\)-eigenvectors basis of \(L^n\) such that the associated change of basis matrix is an upper triangular matrix \(Q \in GL_n(L)\). Let \(S = PQ\). Then \(S^{-1}AS\) is diagonal and \(S^{-1}BS = Q^{-1}UQ\) is upper triangular. \(\square\)
Remark 2. We may replace each column of $S$ with a proportional column.

The previous result leads to an algorithm to check whether two such matrices are ST or not. Its complexity is in $O(n^4)$.

**Proposition 2.** Let $A, B \in \mathcal{M}_n(K)$. We assume that $A$ has $n$ distinct eigenvalues in $L$, an extension field of $K$, and that we know explicitly $\sigma(A)$. Then we can decide if whether or not $A$ and $B$ are ST over $L$. If $A$ and $B$ are ST over $L$, then we obtain explicitly a matrix $S \in GL_n(L)$ that diagonalizes $A$ and triangularizes $B$.

**Proof.** Since $A$ has distinct eigenvalues in $L$, we can calculate, from $\sigma(A)$, a $A$-eigenvectors basis of $L^n$. Let $R$ be the associated matrix and $Z = R^{-1}BR = [z_{i,j}]$.

Case 1. The matrices $A, B$ are ST. According to the proof of Proposition 1 and Remark 2 there exists a permutation matrix $D$ such that $S = RD$, then $S^{-1}BS = D^{-1}ZD$ is upper triangular.

We consider the following algorithm

$$ U := \{1, \cdots, n\}, $$

For every $i \leq n$,

- if the $i$th column of $Z$ is zero, then $\alpha_i := n$
- else $\alpha_i := n - \sup \{j \leq n \mid z_{i,j} \neq 0\}$.

For $r$ from 1 to $n$, do

- find $i_r$ such that $\alpha_{i_r} = \sup_{i \leq n} \alpha_i$.
- if $\alpha_{i_r} < n - r$ then BREAK
- $d_r := i_r$.
- $U := U \setminus \{i_r\}$.

If $r = n$ then OUTPUT $: = (d_1, \cdots, d_n)$
else OUTPUT $: = NULL$.

The output $(d_1, \cdots, d_n)$ constitutes a convenient permutation.

Case 2. The matrices $A, B$ are not ST. The previous algorithm gives the output NULL.

Remark 3. When $A$ has multiple eigenvalues or $\sigma(A)$ is unknown, to find an efficient algorithm is hard. A finite rational algorithm which allows to check whether two given $n \times n$ complex matrices are ST is exposed in [1] Theorem 6]. The study of complexity of the presented algorithm is omitted in [1] and, as the author shows in [2], this test is impractical for $n \geq 6$.

3. Common invariant subspace and commutativity

**Proposition 3.** Let $A \in \mathcal{M}_n(K)$ such that $A$ has $n$ distinct eigenvalues in an extension field $L$ of $K$ and let $Z = \{B \in \mathcal{M}_n(K) \mid A, B$ have a common eigenvector in $L^n\}$. Then $Z$ is the union of $n$ subspaces of $\mathcal{M}_n(K)$, each of them containing the commutant of $A$.

**Proof.** Let $\alpha \in \sigma(A)$, $L_\alpha = K[\alpha]$ and $[L_\alpha : K] = k_\alpha$. Let $u \in L_\alpha \setminus \{0\}$ be such that $Au = \alpha u$. If $B = [b_{i,j}] \in Z$, then the condition “$Bu$ and $u$ are linearly dependent” can be written in the form of $n - 1$ $L_\alpha$-linear conditions on the $(b_{i,j})_{i,j}$, that is $k_\alpha \times (n - 1)$ $K$-linear conditions on the $(b_{i,j})_{i,j}$. Thus $B$ is in a $K$-vector space of dimension at least $n^2 - k_\alpha(n - 1)$ that contains the commutant of $A$. Finally $B$ goes through the union of $n$ such subspaces.

Remark 4. One has several interesting properties when $\chi_A$ is irreducible over $K$

i) The endomorphism $A$ has no invariant proper subspaces over $K$.

ii) Since $K$ is a perfect field, $A$ has simple eigenvalues in $S_{\chi_A}$ and its commutant
is $K[A]$ and has dimension $n$.

(iii) According to [3, p. 51] and ii), any $A$-invariant subspace of dimension $k$ over $K$ is spanned by $k$ $A$-eigenvectors.

From now on, we suppose that $A$ and $B$ have a proper common invariant subspace of dimension $k$ over an extension field of $K$.

**Theorem 2.** Let $n \geq 2$. Let $A, B \in \mathcal{M}_n(K)$ be such that they have a common eigenvector over an extension field of $K$. We assume that the characteristic polynomial of $A$ is irreducible over $K$. Then $AB = BA$.

**Proof.** Let $u$ be a common eigenvector and put $Au = \alpha u$, $Bu = \beta u$. Recall that $G_A$ is a transitive group, that is, there exist $(\tau_i)_{i=1,\ldots,n-1} \in G_A$ such that
\[ \sigma(A) = \{ \alpha, \tau_1(\alpha), \ldots, \tau_{n-1}(\alpha) \}. \]
Moreover, $\tau_i(u)$ is defined componentwise and $A(\tau_i(u)) = \tau_i(Au) = \tau_i(\alpha)\tau_i(u)$. Finally $\{u, \tau_1(u), \ldots, \tau_{n-1}(u)\}$ is an associated basis of eigenvectors of $A$. Thus $B(\tau_i(u)) = \tau_i(Bu) = \tau_i(\beta)\tau_i(u)$ and $AB = BA$. \hfill $\Box$

We can slightly improve the previous result as follows.

**Lemma 1.** If $A, B \in \mathcal{M}_n(L)$ have a common invariant subspace of dimension $k$ over $L$, then $A^T$ and $B^T$ have a common invariant subspace of dimension $n-k$ over $L$.

**Proof.** The common invariant subspace of dimension $k$, can be written $V = \{ X \in L^n \mid \Lambda X = 0 \}$ where $\Lambda \in \mathcal{M}_{n-k,n}(L)$ has maximal rank $n-k$. Since $\ker(A) \subset \ker(\Lambda A)$, there exists $Z \in \mathcal{M}_{n-k}(L)$ such that $\Lambda A = ZA$, that is $A^T\Lambda^T = \Lambda^TZ^T$. The $n-k$ columns of $A^T$ span a vector space of dimension $n-k$ that is invariant for $A^T$. \hfill $\Box$

**Corollary 1.** Let $A, B \in \mathcal{M}_n(K)$ be such that they have a common invariant hyperplane over an extension field of $K$. We assume that the characteristic polynomial of $A$ is irreducible over $K$. Then $AB = BA$.

**Proof.** According to Lemma 1, $A^T$ and $B^T$ have a common eigenvector and by Theorem 2 $A^TB = B^TA^T$, that implies $AB = BA$. \hfill $\Box$

Now we consider the case where $A$ and $B$ have a common invariant proper subspace of dimension $\geq 2$. Recall that $A_n$, the group of even permutations of $\{1, \ldots, n\}$, contains the cycles of odd length.

**Theorem 3.** Let $n \geq 3$ and $A, B \in \mathcal{M}_n(K)$ be such that they have a common invariant proper vector subspace over an extension field of $K$. We assume that $\chi_A$ is irreducible over $K$ and $G_A = S_n$ or $G_A = A_n$. Then $AB = BA$.

**Proof.** Since $\chi_A = \chi_{AT}$ and according to Lemma 1, we may change $k$ with $n-k$ and assume that $k \leq \frac{n}{2}$, that implies $k+2 \leq n$. Let $F$ be a common invariant subspace of dimension $\bar{k} \geq 2$ for $A, B$. According to Remark 1 iii), the subspace $F$ is generated by certain eigenvectors $u_1, \ldots, u_k$ of $A$ respectively associated to the pairwise distinct eigenvalues of $A$: $\alpha_1, \ldots, \alpha_k$. Let $\sigma(A) = \{ \alpha_1, \ldots, \alpha_k, \ldots, \alpha_n \}$. There exists $\tau \in A_n \subset G_A$, a cycle of length $r = k+1$ if $k$ is even (resp. $r = k+2$ if $k$ is odd) such that, for every $1 \leq i \leq r-1$, $\alpha_{i+1} = \tau(\alpha_i)$. Note that $F = [u_1, \ldots, \tau^{k-1}(u_1)]$ and $\tau^k(u_1) \notin F$. Assume that $Bu_1 = \sum_{i=0}^{q-1} \lambda_i \tau^i(u_1)$ where $q \in [1, k-1]$, for every $i$, $\lambda_i \in \mathcal{S}_\chi$ and $\lambda_q \neq 0$. Therefore
\[ Bu_{k-q+1} = B(\tau^{k-q}(u_1)) = \sum_{i=0}^{q-1} \tau^{k-q}(\lambda_i)\tau^{k-q+i}(u_1) + \lambda_q \tau^k(u_1) \in F. \]
Then $\lambda = 0$, that is a contradiction. Finally $Bu_1 = \lambda_0 u_1$ and we conclude by Theorem 2.

\[\square\]

We can wonder if we still get the same conclusion of Theorem 2 when dropping the hypothesis $G_A = S_n$ or $G_A = A_n$. The answer is no in general but is yes if $n$ is a prime.

**Theorem 4.** Assume that $n \geq 3$ is a prime number and let $A, B \in M_n(K)$ be such that $\chi_A$ is irreducible over $K$. If $A$ and $B$ have a proper common invariant subspace, then $AB = BA$.

**Proof.** Let $F$ be a common invariant subspace of dimension $k \in [2, n - 1]$ for $A, B$. Let $u \in F$ be an eigenvector of $A$ associated to $\alpha \in \sigma(A)$. Note that $n$ divides the cardinality of $G_A$. Since $n$ is prime and according to Cauchy’s theorem, there exist $\tau \in G_A$ of order $n$. Necessarily the permutation $\tau$ is a cycle of length $n$ and $\sigma(A) = \{\alpha, \cdots, \tau^{n-1}(\alpha)\}$. Moreover, $\{u, \cdots, \tau^{n-1}(u)\}$ is a basis of eigenvectors of $A$ and some among these vectors constitute a basis of $F$. Put $Bu = \lambda_0 u + \sum_{0 \leq i < n} \lambda_i \tau^i(u)$ where the $(\lambda_i)$ are in $K$. Assume that there exists $p \in [1, n - 1]$ such that $\lambda_p \neq 0$. Since $n$ is prime and $k < n$, there exists an integer $q$ such that $\tau^q(u) \in F$ and $\tau^q p(u) \notin F$. Therefore

\[B(\tau^q(u)) = \tau^q(\lambda_0)\tau^q(u) + \sum_{0 < i < n, i \neq p} \tau^q(\lambda_i)\tau^{q+1}(u) + \tau^q(\lambda_p)\tau^{q+p}(u) \in F.\]

Thus $B(\tau^q(u))$ is written as a linear combination of the basis $\{u, \cdots, \tau^{n-1}(u)\}$ and the coefficients of the vectors that are not in $F$ are zero. Consequently $\lambda_p = 0$, that is a contradiction. Finally $Bu = \lambda_0 u$ and we conclude by Theorem 2. \[\square\]

**Remark 5.** Consider $A, B \in M_{35}(Q)$ such that $AB \neq BA$ (the verification is easy) and $G_A = S_{35}$ or $A_{35}$ (the verification is easy with the “Magma” software). Then, by Theorem 3, we deduce that $A, B$ admit no common invariant proper subspaces (the direct verification is impossible because the algorithm associated to Theorem 7 is impractical for $n > 12$).

4. The case $n = 4$

Assume that $A, B \in M_4(K)$ have a common invariant subspace of dimension $k \in \{1, 2, 3\}$ and that $\chi_A$ is irreducible over $K$. If $k = 1, 3$, then from Theorem 2 and Corollary 1 $AB = BA$. From Theorem 3 we obtain the same conclusion if $k = 2$ and $G_A = S_4$ or $A_4$. It remains to study the cases where $A$ admits an invariant plane $\Pi$ and $G_A = C_4$, the cyclic group with four elements, $C_4^2$ or $D_4$, the dihedral group with eight elements. Of course, if $K$ is a finite field, then necessarily $G_A = C_4$.

Let $A \in M_n(K)$ be such that $\chi_A$ is irreducible over $K$ and $\Pi$ be a $A$-invariant plane. We denote by $r_A(\Pi)$ the dimension of the $K$-vector space of the matrices $B \in M_n(K)$ such that $\Pi$ is a $B$-invariant plane. We will see that $r_A(\Pi)$ does not depend only on $k$ and $G_A$.

**Proposition 4.** Let $A \in M_4(K)$ such that $\chi_A$ is irreducible, $G_A = D_4$ and $\Pi$ is an $A$-invariant plane. Then $r_A(\Pi) = 4$ or $8$.

**Proof.** There exist $\alpha_1, \alpha_2 \in \sigma(A)$ such that $\Pi = \ker((A - \alpha_1 I_4)(A - \alpha_2 I_4))$. Let $L = K(\alpha_1, \alpha_2)$ and $H = \{\tau \in G_A \mid \tau(\alpha_1) = \alpha_2\}$ has two elements. Let $u$ be an eigenvector of $A$ associated to $\alpha_1$. If $\tau \in H$, then $\{u, \tau(u)\}$ is a basis of $\Pi$. Let $B \in M_4(K)$ such that $\Pi$ is $B$-invariant. Therefore $Bu = \lambda u + \mu \tau(u)$ where $\lambda, \mu \in L$. 

\[\square\]
Proof. We use the notations of Proposition 4. Here $A$ and $B$ are invariant planes. Then $B(r(u)) = r\lambda r(u)$ and $AB = BA$.

Case 2. The elements of $H$ have order 2. Then

$B(r(u)) = r\lambda r(u) + r\mu u \in \Pi$.

Let $(\tau_i)_{i=3,4} \in G_A$ such that $\tau_i(\alpha_1) = \alpha_i$. Clearly, $\{u, \tau(u), \tau_3(u), \tau_4(u)\}$ is a basis of eigenvectors of $A$ and, for $i = 3, 4$, $B(\tau_i(u)) = \tau_i(Bu)$ depends only on $Bu$. Finally $B$ depends only on $\lambda, \mu \in L$ and $r_A(\Pi) = 2[L : K]$. Necessarily $r_A(\Pi) < 16$ and $|L : K| = 4$ or 8. Therefore $r_A(\Pi) = 8$.

\[ \square \]

Proposition 5. Let $A \in M_4(K)$ such that $\chi_A$ is irreducible, $G_A = C_4$ and $\Pi$ is a $A$-invariant plane. Then $r_A(\Pi) = 4$ or 8.

Proof. We use the notations of Proposition 4. Here $[L : K] = 4$ and $H$ has a unique element $\tau$.

- Case 1. $\tau$ is a generator of $G_A$. As in the proof of Proposition 4, Case 1, we show that $AB = BA$.
- Case 2. $\tau$ has order 2. As in the proof of Proposition 4, Case 2, we show that $r_A(\Pi) = 2[L : K] = 8$.

\[ \square \]

Proposition 6. Let $A \in M_4(K)$ such that $\chi_A$ is irreducible, $G_A = C_2 \times C_2$ and $\Pi$ is a $A$-invariant plane. Then $r_A(\Pi) = 8$.

Proof. Again we use the notations of Proposition 4. Here $[L : K] = 4$, $H$ has a unique element $\tau$ and $\tau$ has order 2. As in the proof of Proposition 4, Case 2, we show that $r_A(\Pi) = 2[L : K] = 8$.

\[ \square \]

Example.

- We consider the following instance where $K = \mathbb{Q}$, $\chi_A(x) = x^4 + x^3 + x^2 + x + 1$ and $\Pi_\epsilon = \ker (A^2 + \frac{1 + \epsilon \sqrt{5}}{2} A + I_4)$ where $\epsilon = \pm 1$. Here $G_A = C_4$, the element of $H$ has order 2 and, according to Proposition 4, $r_A(\Pi_\epsilon) = 8$. In particular, the following pair $(A, B)$ is such that the planes $\Pi_\epsilon$ are invariant for $A, B$ and yet, $A$ and $B$ are not ST.

\[
A = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1
\end{pmatrix},
B = \begin{pmatrix}
0 & -1 & 0 & 2 \\
-1 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

where $G_B = D_4$.

With the help of Theorem 7, we show that $A, B$ admits only the planes $\Pi_\epsilon$ as proper common invariant subspaces over $\mathbb{C}$.

1) Applying the Shemesh’s criterion to the couples $(A, B)$ and $(A^T, B^T)$, we conclude that there are no solutions in dimensions 1 or 3.

2) We prove easily that $(A + I_4)^{(2)}$ and $(B + I_4)^{(2)}$ have two common eigenvectors

\[
u_\epsilon = [1, \frac{-\epsilon \sqrt{5} + 1}{2}, 1, \frac{-\epsilon \sqrt{5} + 1}{2}, \frac{-\epsilon \sqrt{5} + 1}{2}, 1]^T.
\]
An easy calculation shows that \( u_x \) is the exterior product of the vectors of a basis of \( \Pi_x \).

- Now we assume \( K = \mathbb{Z}/7\mathbb{Z} \) and \( \chi_A \) is as above. Then \( \chi_A \) is irreducible and since \( K \) is finite, \( G_A = C_4 \). Moreover 5 is not a square in \( K \) and we can define \( \sqrt{5} \) in an extension field of \( K \) of dimension 2. Then the planes \( \Pi_x \), as above, are \( A \)-invariant. By the previous reasoning, we obtain \( r_A(\Pi_x) = 8 \). The matrices \( A, B \), as above, admit the planes \( \Pi_x \), as common invariant subspaces. We can show that \( A, B \) have no other proper common invariant subspaces. Note that \( \chi_B(x) = (x^2 - x + 4)(x^2 + 2x + 2) \) and \( B \) admits two invariant planes over \( K \).

5. Solving a matrix equation

We give an application of Section 3 using the following known result

Theorem. (McCoy’s theorem) Let \( L \) be an algebraically closed field and \( A, B \in M_n(L) \). Then \( A, B \) are ST over \( L \) if and only if for any polynomial \( p(\lambda, \mu) \) in non-commuting indeterminates, \( p(A, B)(AB - BA) \) is nilpotent.

Proposition 7. Let \( A = \begin{pmatrix} U & 0_{q,p} \\ 0_{p,q} & V \end{pmatrix} \in M_{p+q}(K) \) be such that \((U, V) \in M_p(K) \times M_q(K), \chi_U \text{ and } \chi_V \) are distinct irreducible polynomials over \( K \). If \( B \in M_{p+q}(K) \), then \( A, B \) are ST over \( K \) if and only if \( B \) is in the form

\[
B = \begin{pmatrix} f(U) & Q \\ 0_{q,p} & g(V) \end{pmatrix}, \text{ respectively } B = \begin{pmatrix} f(U) & 0_{p,q} \\ 0_{p,q} & g(V) \end{pmatrix}
\]

where \( Q \in M_{p,q}(K) \), respectively \( Q \in M_{q,p}(K) \) and \( f, g \in K[x] \) are arbitrary.

Proof. \((\Rightarrow)\) Clearly \( \sigma(A) = \{\sigma(U), \sigma(V)\} \) and \( A \) has \( p + q \) distinct eigenvalues. The eigenvectors of \( A \) are in the form \([u, 0]^T\) where \( u^T \) is an eigenvector of \( U \) or \([0, v]^T\) where \( v^T \) is an eigenvector of \( V \). Note that \( A, B \) have a common eigenvector and assume, for instance, that it is in the form \([u, 0]^T\) with \( Uu^T = \alpha u^T \). We adapt the proof of Theorem 2 there exist \((\tau_i)_{i=1}^p \in G_U \) such that \( \sigma(U) = \{\alpha, \tau_1(\alpha), \cdots, \tau_{p-1}(\alpha)\} \). We deduce that the \( ([\tau_i(u), 0]^T)_{i \leq p} \) are eigenvectors of \( B \) and \( B \) is in the form \( B = \begin{pmatrix} P & Q \\ 0_{q,p} & R \end{pmatrix} \) where \( UP = PU \), that is \( P = f(U) \in K[U] \).

Then \( AB - BA = \begin{pmatrix} 0_p & UQ - QV \\ 0_{p,q} & VR - RV \end{pmatrix} \) and, more generally,

\[
p(A, B)(AB - BA) = \begin{pmatrix} 0_p & 0_{p,q} \\ 0_{p,q} & p(V, R)(VR - RV) \end{pmatrix}
\]

where \( p(\lambda, \mu) \) is any polynomial in non-commuting indeterminates \( \lambda, \mu \). According to the McCoy’s theorem, \( A, B \) are ST over \( K \) if and only for any polynomial \( p(\lambda, \mu) \) in non-commuting indeterminates, \( p(V, R)(VR - RV) \) is nilpotent, that is equivalent to: \( V, R \) are ST. Then \( V, R \) have a common eigenvector and, according to Theorem 2 \( VR = RV \), that is \( R \) is a polynomial in \( V \).

\((\Leftarrow)\) Again using the McCoy’s theorem, the converse is clear. \( \square \)

Finally we apply Proposition 7 to solving a matrix equation.

Proposition 8. Let \( p, q \) be distinct positive integers. Let \( A \in M_{p+q}(K) \) be such that \( \chi_A = \Phi \Psi \) where \( \Phi \) and \( \Psi \) are polynomials of degree \( p \) and \( q \), irreducible over \( K \). Let \( \alpha \) be a positive integer. Then the equation, in the unknown \( X \in M_{p+q}(K) \),

\[
(1) \quad AX - XA = X^\alpha
\]

admits the unique solution \( X = 0 \).
Proof. We may assume that \( A = \begin{pmatrix} U & 0_{p,q} \\ 0_{q,p} & V \end{pmatrix} \) where \( U, V \) are the companion matrices of \( \Phi, \Psi \). Since \( X \) satisfies Equation (1), \( A \) and \( X \) are ST over \( K \) (cf. [3]). According to Proposition 7, necessarily \( X \) has two possible forms, for instance this one
\[
X = \begin{pmatrix} f(U) & Q \\ 0_{p,q} & g(V) \end{pmatrix}
\]
and consequently \( AX -XA = \begin{pmatrix} 0_p & UQ - QV \\ 0_{q,p} & 0_q \end{pmatrix} \).

i) Assume \( \alpha = 1 \). Equation (1) reduces to
\[
f(U) = 0, \ g(V) = 0, \ UQ - QV = Q.
\]
The last equation can be rewritten \( \phi(Q) = Q \) where \( \phi = U \otimes I_q - I_p \otimes V^T \) is the sum of two linear functions that commute. Therefore
\[
\sigma(\phi) = \{ \lambda - \mu \mid \lambda \in \sigma(U), \mu \in \sigma(V) \}.
\]
If there are non-zero solutions, then there exist \( \lambda \in \sigma(U), \mu \in \sigma(V) \) such that \( \lambda - \mu = 1 \). Since \( \chi_U \) is the minimum polynomial of \( \lambda \) over \( K \), then \( \chi_U(x+1) \) is the minimum polynomial of \( \mu \) over \( K \) and \( \chi_U(x+1) = \chi_V(x) \). That implies \( p = q \), a contradiction.

ii) Assume \( \alpha > 1 \). Equation (1) reduces to
\[
f(U)^\alpha = 0, \ g(V)^\alpha = 0, \ UQ - QV = 0.
\]
Let \( \sigma(U) = (\lambda_i)_{i \leq p} \). Then \( (f(\lambda_i))_{i \leq p} = \sigma(f(U)) = \{0\} \). Since \( f \) is a unitary polynomial of degree \( p \), \( f(x) = (x-\lambda_1) \cdots (x-\lambda_p) \). By Cayley-Hamilton Theorem, \( f(U) = 0 \) and, in the same way, \( g(V) = 0 \). By the reasoning used in i), for every \( \lambda \in \sigma(A), \mu \in \sigma(B), \lambda - \mu \neq 0 \) and \( \phi \) is a linear bijection. We conclude that \( Q = 0 \).

Acknowledgements. The author thanks David Adam and Roger Oyono for many valuable discussions. The author thanks the referee for helpful comments.

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GÉRALD BOURgeois, GAATI, UNIVERSITÉ DE LA POLYNÉSIE FRANÇAISE, BP 6570, 98702 FAATI, TAHITI, POLYNÉSIE FRANÇAISE.

E-mail address: bourgeois.gerald@gmail.com