RESEARCH ARTICLE

Modified Green–Lindsay thermoelasticity wave propagation in elastic materials under thermal shocks

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Abstract

In this study, a nonlinear numerical method is presented to solve the governing equations of generalized thermoelasticity in a large deformation domain of an elastic medium subjected to thermal shock. The main focus of the study is on the modified Green–Lindsay thermoelasticity theory, solving strain and temperature rate-dependent model using finite strain theory. To warrant the continuity of the finding responses at the boundary after the applied shock, higher order elements are adopted. An analytical solution is provided to validate the numerical findings and an acceptable agreement between the two presented solutions is obtained. The findings revealed that stress and thermal waves have distinct interactions and a harmonic temperature variation may lead to a systematic uniform stress distribution. Besides, a notable difference in the results predicted by the modified Green–Lindsay model and classic theory is observed. It is also found that the modified Green–Lindsay theory is more efficient in determining the wave propagation phenomenon. Furthermore, the findings established that thermal shock induces tensile stresses in the structure immediately after the shock, and the perceived phenomenon mainly depends on the defined boundary conditions. The results show that the strain rate can have a significant influence on the displacement and stress wave propagation in a structure subjected to thermal shock and these impacts may be more considerable with mechanical loading.

Keywords: modified Green–Lindsay theory; strain and temperature rate-dependent model; wave propagation; thermal shock

1. Introduction

The first research on thermoelasticity can be traced back to the 1830s when Duhamel introduced the first coupled thermoelasticity equations (Todhunter, 2007). Neumann, Almansi, and Tedone also conducted much of the research in this area (Neumann, 1885; Almansi, 1897; Allgemeine, 1906). Their studies were dedicated to static problems. Manoach and Ribeiro (Ribeiro & Manoach, 2005; Ribeiro, 2007) analyzed the thermal reaction of isotropic, straight, and curved beams by using Newmark’s method to solve the nonlinear equations. Their results showed that straight beams are affected by temperature changes in a different way compared with the curved beams. Thermal stress in disks based on coupled thermomechanics equations was studied by Belhocine and Bouchetara (2013). They showed that the von Mises stress and the final deformations in a disk grow strongly when thermal and mechanical deformations are coupled. The influence of temperature-dependent material property of Functionally Graded material (FGM) cylinders

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under thermal load was studied by Abbas (2014a). The author also used the Laplace transform to study the thermoelastic response of a structure with spherical cavities (Abbas, 2014a). Shariyat and Niknami (2016) investigated thermal responses of plates and explored thermal and mechanical wave propagation and the influences of the strain-rate dependency of materials. Functionally Graded (FG) beam exposed to step heating was investigated by Malik and Kadoli (2018). Alshorbagy (2013) also proposed a numerical solution to study the effects of temperature on FG beam’s behavior. He found that the final mechanical response of the beam strongly depends on input temperature. The numerical response of nonlinear analysis of generalized thermoelasticity for an elastic half-space was presented by Abbas and Youssef (2013). For most engineering applications, using the classic model of thermoelasticity is responsive to study the effect of temperature on FG beam’s behavior. Hefound that the final mechanical response of the beam strongly depends on the constitutive parameters. Subsequently, other non-classical theories were proposed to improve the leakage related to this problem of the thermoelasticity model (Wankhade, Kundu, and Dasc 2018). Green and Lindsay model used by Zarnehr et al. (2018) presented the second generalization model that involves temperature rise as a dependent factor concerning two variables of relation times. Green and Naghdi model used by El-Attar (2019), suggested the new model without energy dissipation that the heat flux-temperature-displacement rate was applied in Fourier’s law. El-Attar, Hendy, and Ezzat (2019) and Kiani (2017a, b) presented new investigations on the thermoelasticity model without energy dissipation and found that nonlinearity effects appear strongly in situations with higher thermal shock and relaxation. Lofy and Gabr (2017) studied a 2D deformation model of a semi-infinite sheet subjected to heat flux laser. They analyzed the influences of coupling terms and thermal relaxation times on the system’s response. Shariyat, Jahanshahi, and Rahimi (2019) and, Shariyat and Ghafoorianam (2019) investigated the hydrothermoelastic responses of spheres subjected to thermomechanics shocks using the generalized theory of thermoelasticity and discussed the impacts of moisture absorption in the equations. Some research works presented an analytical solution for thermoelasticity problems. Esfahani, Kiani, and Esfami (2013) employed the generalized differential quadrature method to solve thermal equations of FG materials. Based on their results, the temperature dependency of the constituents has considerable impacts on final deflections. Swelam (2007) in 2006 used Adomian Decomposition Method (ADM) and Variation Iteration Method (VIM) and a numerical solution for analyzing the harmonic thermal response of an elastic half-space model. Gupta and Sharma (2013) discussed thermal gradient effects on the trapezoidal plate and found that the frequency reduces by increasing the temperature changes. Jalili, Ganji, and Nourazar (2018) and Hosseinzadeh, Jafari, and Ardaheia, and Ganji (2017) presented an analytical solution to the thermal analysis of different materials. Jalili employed the homotopy perturbation method (HPM) to solve the governing partial differential equations. Ghofoori et al. (2011) provided an analytical study using differential transformation method and the HPM. A generalized differential quadrature approach was applied by Kiani and Esfami (2016) to analyze an annular disk and thick spheres under thermal loads. Based on their results in a high range of temperatures, the nonlinear thermoelasticity equations should be applied. Zhang and Starzewsk (2019) used Maxwell–Cattaneo heat conduction to study thermoelastic waves in helical stands and found that the thermal conductivity equation has a significant impact on the thermal wave propagation. Kundu and Dewanjee (2015) presented a new method for the thermal response of layer tissues. Dynamic response of the shells subjected to thermal loads was studied by Hamzah, Jobair, Oday, and Hashim (2018). Rogovoi and Stolbova (2008) presented the coupled thermoelasticity equations using a variational perspective, to obtain the constitutive equations for a complex medium using the presumption of the intermediate configurations. Safari, Tahani, and Hosseini (2011) used a homotopy analysis method to investigate thermal waves in a cylinder subjected to thermal shock. They used the Laplace inverse transform method for transferring the Laplace domain resulting in time domain results. Kirane and Tatar (2001) and Jafari, Ghaderi, Golmankhaneh, and Baleanu (2013) compared HPM and VIM methods to solve thermoelasticity problems. They asserted that these methods are effective to several groups of equations; however, they considered boundary conditions as a function of displacement to find an analytical solution.

The presented literature survey indicates that despite many studies performed in the domain of analytical and numerical solutions for thermoelastic problems, the basic understanding of nonlinear thermoelasticity requires further investigation, particularly with respect to wave propagation. Previous works provide invaluable numerical data into the impacts of nonlinear terms on thermal responses. However, they did not consider all nonlinear terms simultaneously.

This work aims to fill the gap by providing quantitative and systematic analysis into solving the nonlinear thermoelasticity equations by adding strain rate and temperature rate to the governing equations of the Green–Lindsay model. In the present research, a numerical approach is proposed to investigate thermal wave propagation on a finite-length medium under a uniform heat flux. The nonlinear coupled formulations are derived based on the finite strain theory (FST) in the reference coordinate. The multiple-scale method is also employed to present an analytical solution to validate the accuracy of finite element results. In the driving process of the momentum and energy equations, the temperature variations compared with the reference temperature are considered, while these variations were ignored in the previous studies. Moreover, in contrast to the other analytical works that proposed unrealistic boundary conditions only to solve the relations, in this study a set of real boundary conditions is defined in the process of the analytical and numerical solution. The significance of this research is hence in its following core contributions, namely: (i) modified Green–Lindsay model is considered and its characteristics are compared with numerical results of the Green–Lindsay and classic theory, (ii) temperature variation compared with the reference temperature is considered in this research, while it has ignored in previous studies for simplification, (iii) the nonlinear form of the strain tensor is employed in terms of the displacement component using FST, and (iv) a numerical solution algorithm is developed based on the weak formulation including nonlinear terms and using high-order cubic elements.

2. Governing Formulation

The thermoelastic wave propagation phenomenon in a finite-length medium is investigated in this research. The nonlinear system of equations is described using the modified Green–Lindsay model of the generalized theory. The nonlinear governing equations of thermoelasticity are derived using the FSD theory for isotropic media.
The general form of Piola stress for material points \( \lambda \) in modified Green–Lindsay model is considered as follows (Yu, Xue, & Tian, 2018; Shivay & Mukhopadhyay, 2019):

\[
s_{ij} = \delta_{ij}\lambda \left( \epsilon_{pp}^E + \eta \beta \left( \frac{\partial \epsilon_{pp}^E}{\partial t} \right) \right) + 2\mu \left( \epsilon_{ij} + \eta \beta \left( \frac{\partial \epsilon_{ij}}{\partial t} \right) \right) - \beta \delta_{ij} \left( \theta + \bar{m} \frac{\partial \theta}{\partial t} \right),
\]

and in a 1D model can be presented as

\[
s_{11} = (\lambda + 2\mu) \left( \Dot{\epsilon}^2 + \frac{1}{2} \left( \frac{\partial \epsilon}{\partial x} \right)^2 + \eta \beta \frac{\partial^2 \epsilon}{\partial t \partial x} \right) + \eta \beta \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial^2 \epsilon}{\partial t \partial x} \right) - \beta \left( \theta + \bar{m} \frac{\partial \theta}{\partial t} \right).
\]

The governing equations of motion and energy in reference coordinate system are presented Equations (3) and (4) (Abbas & Dahab, 2014; Miranville, 2019; Mirparizi, Fotuhi, & Shariyat, 2020):

\[
\frac{\partial}{\partial X} [s(X, t)(1 + u(X, t))] = \rho \frac{\partial^2 u}{\partial t^2} (X, t)
\]

\[
\rho C \left( \frac{\partial \theta}{\partial t} + \bar{n} \frac{\partial^2 \theta}{\partial t^2} \right) + \beta (\theta + T_0) \left( \frac{\partial \tilde{E}}{\partial t} \right) + \eta \beta \left( \frac{\partial \tilde{E}}{\partial \tau} \right) \left( \frac{\partial^2 \tilde{E}}{\partial \tau^2} \right) - \rho \left( \frac{\partial^2 u}{\partial \tau^2} \right) = 0.
\]

In these equations, \( s(X, t) \) and \( \tilde{E} \) are second Piola–Kirchhoff stress and Green’s strain tensor, respectively. \( \bar{m} \) and \( \bar{n} \) are parameters related to strain and temperature rate and \( \eta \) is considered to express the modified Green–Lindsay model. Given that all equations are expressed in the original coordinate based on FST, Green’s strain tensor is employed as the following in a 1D thermoelasticity (Gu, Qu, Chen, Song, & Zhang, 2019):

\[
\tilde{E}_{11} = \left( 1 + \frac{1}{2} \frac{u_x}{u} \right) u_x.
\]

By substituting the stress and Green’s strain equations, the nonlinear 1D governing equations of thermoelasticity are described as Shaw (2019):

\[
(\lambda + 2\mu) \left[ \frac{\partial^2 \epsilon}{\partial x^2} + 3 \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial^2 \epsilon}{\partial x^2} \right) + \frac{3}{2} \left( \frac{\partial^2 \epsilon}{\partial x^2} \right)^2 \right]
\]

\[
+ \eta \beta (\lambda + 2\mu) \left[ \frac{\partial^3 \epsilon}{\partial x^2 \partial \tau} + 2 \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial^2 \epsilon}{\partial x^2} \right) \left( \frac{\partial^2 \epsilon}{\partial x \partial \tau} \right) + 2 \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial^2 \epsilon}{\partial x \partial \tau} \right) + \eta \beta \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial^2 \epsilon}{\partial x \partial \tau} \right) \right] - \beta \eta \beta \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial \epsilon}{\partial x} \right) \left( \frac{\partial \epsilon}{\partial \tau} \right) - \rho \left( \frac{\partial^2 u}{\partial \tau^2} \right) = 0.
\]

\[
\rho C \left( \frac{\partial \epsilon}{\partial t} + \bar{n} \frac{\partial^2 \epsilon}{\partial t^2} \right) + \beta (\theta + T_0) \left( \frac{\partial \tilde{E}}{\partial t} \right) + \eta \beta \left( \frac{\partial \tilde{E}}{\partial \tau} \right) \left( \frac{\partial^2 \tilde{E}}{\partial \tau^2} \right) - \rho \left( \frac{\partial^2 u}{\partial \tau^2} \right) = 0.
\]

Equations (6) and (7) represent nonlinear thermoelasticity equations in which no nonlinear terms are omitted. In comparison with previous works done by Yu et al. (2018) and Shivay and Mukhopadhyay (2019), where small temperature change was assumed (\( T_0 - T \approx 1 \)), in this study all range of temperature changes are taken into account (in all equations: \( \theta = (T - T_0) \), where \( T_0 \) is the ambient temperature). Besides, in this study, the nonlinear FST theory is employed to derive the nonlinear coupled formulations, while previous research works (Yu et al., 2018; Shariyat et al., 2019) used linear form of strain equation. Moreover, both strain and temperature-rate are adopted to derive these equations by taking \( \tilde{E}, \tilde{E}, \theta, \dot{\theta} \). Equations (1), (6), and (7) display the modified Green–Lindsay theory of generalized thermoelasticity. In these equations, the Green–Lindsay and classic theory can be achieved having \( \eta = 0 \) and \( \bar{m} = \bar{n} = 0 \), respectively.

### 3. Statement of Geometry and Boundary Conditions

In this study, the nonlinear mechanical and thermal wave propagation in the stainless steel medium subjected to the thermal shock shown in Fig. 1 is discussed. The end face of the solid is fixed and exposed to heat convection with the ambient. The 1D governing equations are functions of \( x \) and time and all variations are discarded in other coordinates.

As shown in Fig. 1, the medium is subjected to thermal shock in the stress-free surface at \( x = 0 \), which is expressed as follows:

\[
\begin{align*}
T(x, t) &= T(t) \\
\theta(0, t) &= 0 \\
U(0, 0) &= 0 \\
\frac{\partial T}{\partial x} \Big|_{x=0} &= -h(T - T_0)
\end{align*}
\]

And the initial conditions are selected as follows:

\[
\begin{align*}
u(X, t = 0) &= \frac{\partial u}{\partial t} (X, t = 0) = 0, \\
T(X, t = 0) &= T_0.
\end{align*}
\]
4. Proposed Solutions

4.1. Finite element procedure

Based on FST, strain, and temperature rate, as well as thermal nonlinear analysis, a set of nonlinear equations is obtained (considering the temperature variation compared with the reference temperature). Galerkin finite element process and Runge–Kutta time integration process are applied to present a numerical solution to the governing equations. Due to the existence of several nonlinear terms, an iterative algorithm is proposed that is explained in detail in the following. In the numerical solution process, it is demonstrated that the best precision can be achieved when the order of shape functions is chosen larger than the weak form of nonlinear equations (Eslami, 2014). For this purpose, the high-order Hermitian elements are selected. By extending a weighted-integral statement of differential Equations (6) and (7) and using dependent variables $u, \theta$ and weight function $W$, independent systems of equations will be obtained (Abbas & Yousef, 2012; Reddy, Raju, & Rao, 2018; Guo, Su, & Xiana, 2019).

\[
\begin{align*}
 u(X, t) &= \sum_{j=1}^{n} u_j^e(t) N_j(X), \quad \theta(X, t) = \sum_{j=1}^{n} \theta_j^e(t) N_j(X) \quad W(X) = \sum_{i=1}^{n} \phi_i
\end{align*}
\]

The weak form of the weighted residual equation is represented by

\[
 [M\ddot{\phi} + [C(\phi)]\dot{\phi} + [K(\phi)]\phi] = f + \ddot{f}
\]

where $\theta_j^e, u_j^e$ are the values of temperature rise and displacement at the defined nodes and $N_j^e(\alpha)$ are the Hermite shape functions expressed in terms of the local coordinate (Demir & Civalek, 2017):

\[
\begin{align*}
 N_1^e(\alpha) &= (1 - 3X/l_x)(1 - 3X/2l_x)(1 - X/l_x), \\
 N_2^e(\alpha) &= (9X/l_x)(1 - 3X/2l_x)(1 - X/l_x), \\
 N_3^e(\alpha) &= (-\frac{9}{2}X/l_x)(1 - (3X/l_x))(1 - (X/l_x)), \\
 N_4^e(\alpha) &= (X/l_x)(1 - (3X/l_x))(1 - (3X/2l_x)).
\end{align*}
\]

where $X$ is the local coordinate. The weak form of equations can be expanded from the coupled governing differential equations and can form Equation (11) (Abbas & Othman, 2011).

\[
\begin{align*}
 K_{11} &= -(\lambda + 2\mu) \int_{x^e} x^e \frac{\partial^2 \phi}{\partial X} \frac{\partial^2 \phi}{\partial X} \, dx + \int_{x^e} x^e \left( \frac{\partial \phi}{\partial X} \right)^2 + 3 \left( \frac{\partial \phi}{\partial X} \right) \frac{\partial^2 \phi}{\partial X} \, dx + (\lambda + 2\mu) \int_{x^e} x^e \frac{\partial \phi}{\partial X} \, dx + (\lambda + 2\mu) \int_{x^e} x^e \frac{\partial \phi}{\partial X} \, dx, \\
 K_{12} &= -\beta \int_{x^e} x^e \left( \frac{\partial \phi}{\partial X} \right) \frac{\partial \phi}{\partial X} \, dx + \int_{x^e} x^e \left( \frac{\partial^2 \phi}{\partial X^2} \right) + \int_{x^e} x^e \frac{\partial \phi}{\partial X} \, dx, \\
 K_{21} &= 0, \quad K_{22} = k \int_{x^e} x^e \frac{\partial \phi}{\partial X} \, dx
\end{align*}
\]
\[
C(\phi) = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}
\]

\[
C_{11} = \eta m(\lambda + 2\mu) \int_\Omega \left(1 + 2 \left(\frac{a u}{a x}\right)^2 + \left(\frac{a u}{a x}\right)^4\right) \phi_1 - \tau \phi_0 dX + \int_\Omega \left(2 \left(\frac{a u}{a x}\right)^2\right) \phi_1 - \tau \phi_0 dX
\]

\[
C_{12} = -\beta \tilde{m} \int_\Omega \left(1 + \left(\frac{a u}{a x}\right)^2\right) \phi_1 - \tau \phi_0 dX
\]

\[
C_{21} = \beta \int_\Omega (\theta + T_0) \left(1 + \frac{a u}{a x}\right) \phi_1 - \tau \phi_0 dX + \beta \int_\Omega (\theta + T_0) \left(\frac{a u}{a x^2}\right) \phi_1 - \tau \phi_0 dX
\]

\[
C_{22} = \rho c \int_\Omega \phi_1 - \tau \phi_0 dX
\]

(14)

4.2. Finite difference formulation

In this section, the finite difference technique is employed to convert the conventional differential equations into a group of algebraic equations. By this, \(\phi = \left(\frac{u}{h}\right)\) can be determined at \(t^{+1} = (t + \Delta t)\) using the obtained values of \(\phi\) at \(t\) (\(\Delta t\) is the time increment and in this research \(\Delta t = 0.01\,\text{ms}\)).

Utilizing Newmark’s numerical time integration process (Matle, 2017; Mirparizi and Fotuhi, 2020) for the \(t\) time step, it can be written as

\[
\phi_{i+1} = a_1 \phi_i - a_2 \phi_i - a_3 \phi_i.
\]

(17)

where

\[
a_1 = \hat{\alpha} \Delta t, \quad a_2 = (1 - \hat{\alpha}) \Delta t, \quad a_3 = \frac{2}{\dot{\chi}(\Delta t)^2}, \quad a_4 = \alpha_3 \Delta t, \quad a_5 = \frac{1}{\dot{\chi}} - 1.
\]

(19)

Finally, the following form of algebraic equation will be obtained using Equations (17) and (18).

\[
(K + a_3\chi^{-1} - a_1 a_3 C^{-1})\phi = \phi^0 + (a_2 - a_1 a_4 - a_3)\phi_i + \phi_i - a_1 a_3 \phi_i.
\]

(20)

The accuracy and stability of the procedure are dependent on the parameters of \(\hat{\alpha}\) and \(\dot{\chi}\). In this study, these parameters are chosen based on the second-order Runge–Kutta method.

\[
\hat{\alpha} = \dot{\chi} = 0.5
\]

(21)

Notice that the initial conditions are the prerequisite of finding the discussed matrices. In this regard, \(\phi^0, \phi^0_i\) are considered to be zero and \(\phi^0_i = \gamma^{-1}(\phi^0 + \phi^0_i)\).

In each iteration, Picard’s iterative procedure is applied to assess the convergence of the solution (Yang & Liu, 2014; Angaroni, Benenti, & Strini, 2018). It is clear that the achieved matrices are the combination of known shape functions and unknown terms including \((\frac{a u}{a x}), (\frac{a u}{a x}), (\frac{a u}{a x}), \theta\). To convert these matrices to a known one, an iterative algorithm is utilized, which employed the initial values to define unknown parameters at the first step, and subsequently it used the obtained results for the next step as initial values. This iterative process continues until obtaining an allowable value satisfying defined Picard’s iterative criteria \(((\lambda_{m+1} - \lambda_{m+1})/\lambda_{m+1}) < 0.0001)\).
5. Results and Discussion

5.1. The analytical solution to verification

In this section, the multiple-scale technique is employed to provide the analytical results and to validate the obtained results. The multiple-scale technique which is the development of the perturbation method uses an expansion that represents the response as a multiple-scale function. In this study, independent variables such as \( T_i = \epsilon^i t \) are considered, and the main functions are expanded similarly, which will be described in more detail.

At first, the applied boundary conditions are considered in a general form of Equations (8) and (9) for an isotropic medium exposed to surface temperature changes, and both sides of the solid are assumed to be fixed. As mentioned earlier, defining boundary conditions as a function of displacement is one of the weaknesses of the previous analytical studies that avoided in this research. The temperature rise and displacement are expressed using time-dependent functions and shape functions that satisfy the boundary conditions as follows:

\[
\begin{align*}
\omega(X, t) &= f(t) \sin(\omega X), \quad X > 0 \\
w &= \frac{\hat{\omega}}{T_i}, \\
\theta(X, t) &= g(t) \cos(\lambda X), \quad X > 0 \\
\lambda &= \frac{\hat{\lambda}}{T_i}.
\end{align*}
\]

Also, the initial conditions are considered as

\[
\begin{align*}
f(0) &= f(0) = g(0) = 0.
\end{align*}
\]

By substituting Equations (22)–(25) into Equations (6) and (7), considering \( \tilde{m} = \tilde{n} = 0 \), and employing the Galerkin integration, the governing ordinary differential equations can be obtained as:

\[
\begin{align*}
a_1 \frac{d^2 f(t)}{dt^2} + \tilde{a}_1 \hat{f}^2(t) + \tilde{a}_2 \hat{f}^2(t) + a_1 f(t) + \tilde{a}_3 f(t) g(t) + \tilde{a}_4 f(t) g(t) + a_0 g(t) &= 0, \\
a_0 g(t) + a_10 \frac{dg(t)}{dt} + \tilde{a}_11 \frac{df(t)}{dt} + \tilde{a}_12 \frac{df(t)}{dt} f(t) + a_0 \frac{df(t)}{dt} g(t) + \tilde{a}_2 f(t) g(t) &= 0.
\end{align*}
\]

All coefficients in Equations (27) and (28) are presented in Appendix A. Given that the simplifying assumption of \( \epsilon^3 \ll 1 \) is not employed, \( m \frac{d^2 f(t)}{dt^2} g(t) \) and \( r \frac{d^2 f(t)}{dt^2} g(t) \) appeared in the obtained equations. Using the multiple-scale method, \( f(t) \) and \( g(t) \) may be expressed in a power series (Dreyfus, Lastra, & Malek, 2019) as in Equation (29):

\[
\begin{align*}
f(t) &= \epsilon^0 f_0(t) + \epsilon^1 f_1(t) + \ldots \\
g(t) &= \epsilon^0 g_0(t) + \epsilon^1 g_1(t) + \ldots
\end{align*}
\]

where \( \epsilon \) is a small parameter representing the time scale, and \( T_i = \epsilon^i t \). Besides, \( f(t) \), \( g(t) \) are determined as a function of \( T_0, T_1, T_2, \ldots \) and using the chain rule, we will have (Nayfeh, 1993)

\[
\begin{align*}
\frac{d}{dt} &= \frac{\tilde{a}}{T_0} + \frac{\tilde{a}}{T_1} + \epsilon \frac{\tilde{a}}{T_2} + \ldots \\
\frac{d^2}{dt^2} &= \frac{\tilde{a}^2}{T_0^2} + 2 \epsilon \frac{\tilde{a}^2}{T_0 T_1} + \epsilon^2 \left( 2 \frac{\tilde{a}^2}{T_0^2 T_1} + \frac{\tilde{a}^2}{T_1^2} \right) + \ldots
\end{align*}
\]

Letting \( \epsilon \hat{a}_0 = a_0 \), \( \epsilon \hat{a}_1 = a_1 \), \( \epsilon \hat{a}_2 = a_2 \), \( \epsilon \hat{a}_3 = a_3 \), \( \epsilon \hat{a}_4 = a_4 \) and substituting Equations (27) and (28) into Equations (29) and (30) and equating each of \( \epsilon^i \) coefficients to zero:

Order \( \epsilon^0 \):

\[
\begin{align*}
a_1 \frac{d^2 f_0(T_0, T_1, \ldots)}{dT_0^2} f_0^2(T_0, T_1, \ldots) + a_0 f_0(T_0, T_1, \ldots) + a_0 g_0(T_0, T_1, \ldots) &= 0, \\
a_9 g_0^2(T_0, T_1, \ldots) + a_10 \frac{d}{dT_0} g_0^2(T_0, T_1, \ldots) &= 0
\end{align*}
\]

Order \( \epsilon^1 \):

\[
\begin{align*}
a_1 \frac{d^2 f_1(T_0, T_1, \ldots)}{dT_0^2} f_1^2(T_0, T_1, \ldots) + a_0 f_1(T_0, T_1, \ldots) + 2a_1 \frac{d^2 f_0(T_0, T_1, \ldots)}{dT_0^2} f_0^2(T_0, T_1, \ldots) + a_2 (f_0(T_0, T_1, \ldots))^3 + a_3 (f_0(T_0, T_1, \ldots))^2 \\
+ (a_5 + a_6) f_0(T_0, T_1, \ldots) g_0^2(T_0, T_1, \ldots) + a_9 g_0^2(T_0, T_1, \ldots) &= 0
\end{align*}
\]
The general solution of Equation (34) is
\[ \gamma = \frac{a_9}{a_{10}} g^9(T_0, T_1, ...) = A(T_1^1) e^{(-\gamma T)} e^{(-\gamma T)}. \] 
(35)

By substituting Equation (35) into Equation (31) and finding the ODE solution, Equation (36) is obtained as
\[ f^0(T_0, T_1, ...) = \tilde{\beta}(T_1) e^{(-\gamma T)} + \tilde{\beta}(T_1) e^{(-\gamma T)} = \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2}. \] 
(36)

where \( \tilde{\beta}(T_1) \) is the complex conjugate of \( \beta(T_1) \). Substituting \( f^0(T_0, T_1, ...) \), \( g^9(T_0, T_1, ...) \) into Equation (33), the following correlations are obtained:
\[
\begin{align*}
a_9 g^9(T_0, T_1, ...) + a_{10} \frac{d}{dT_0} g^9(T_0, T_1, ...) &= -a_{10} \frac{dA(T_1)}{dT_1} e^{(-\gamma T)} - a_{11} \left[ (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \\
&- a_{12} \left[ (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \\
&\times (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \\
&- m(A(T_1) e^{(-\gamma T)}). \left[ (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \\
&- r \left[ (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \right]. \right)
\end{align*}
\] 
(37)

To find a valid expansion, the secular terms \( \ldots e^{(-\gamma T)} \) in Equation (37) must cease to exist. Hence
\[ -a_{10} \frac{dA(T_1)}{dT_1} + \frac{a_{12} a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} - a_{11} a_9 A(T_1^1) e^{(-\gamma T)} = 0, \]
and
\[ A(T_1) = M e^{\left( \frac{b_1 T_0}{a_4 + a_1 \gamma^2} \right)}. \] 
(38)

Using Equations (35), (36), and (38), the solution of Equation (37) will be obtained (the final result is presented in Appendix B). By substitute Equations (33), (36), and (37) into Equation (33), we have
\[ a_1 \frac{d^2 f^1(T_0, T_1, ...)}{dT_0^2} + a_4 f^1(T_0, T_1, ...) = -2a_1 \left[ \frac{d\tilde{\beta}(T_1) e^{(-\gamma T)} + \tilde{\beta}(T_1) e^{(-\gamma T)}}{dT_1^1} \left( \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \right] \\
- a_{12} \left[ (i\tilde{\beta}(T_1) e^{(-\gamma T)} + (-i\tilde{\beta}(T_1) e^{(-\gamma T)} + \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \right) \\
- a_{11} a_9 A(T_1^1) e^{(-\gamma T)} - a_{12} a_9 A(T_1^1) e^{(-\gamma T)} = 0. \]
(39)

Since Equation (39) is valid for all \( T_0 \), the coefficients of \( e^{(-\gamma T)} \) and \( e^{(-\gamma T)} \) must be wiped out.
\[ -2a_1 \frac{d\tilde{\beta}(T_1) e^{(-\gamma T)}}{dT_1} + a_4 \tilde{\beta}(T_1) e^{(-\gamma T)} = \frac{a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2} \left( a_{12} a_9 A(T_1^1) e^{(-\gamma T)} \right) = 0. \] 
(40)

By letting \( \tilde{\beta}(T_1) = \zeta(T_1, e^{(i\phi(T_1))}) \) and separating real and imaginary parts, we have
\[ \zeta(T_1) = \Re, e^{(\frac{i a_{12} a_9 A(T_1^1) e^{(-\gamma T)}}{a_4 + a_1 \gamma^2})}, \] 
(41)
\[ \phi(T_1) = \frac{1}{2(\tilde{\alpha}^2 a_{10} + a_9 a_9)} \left( \frac{3\tilde{\alpha} a_2 a_2 a_2 a_2 + a_9 a_9}{a_{10} a_{10} a_{10} a_{10} a_{10}} \right) + \frac{3\tilde{\alpha} a_2 a_2 a_2 a_2 + a_9 a_9}{a_{10} a_{10} a_{10} a_{10} a_{10}} \right) \] 
(42)

Substituting \( \zeta(T_1), \phi(T_1) \) into Equation (39), we obtain \( f^1(T_0, T_1, ...) \) that is expressed in Appendix B. All parameters are determined in the previous steps except \( A, \chi_1 \) and \( \chi_2 \) that will be calculated using the initial conditions. By expanding \( f^1(T_0, T_1, ...) \), it is noticed...
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that the unreal parts of the complex number will be eliminated (Zafari, Jalili, & Mazidi, 2018). To present analytical and numerical results, \( T(t) \) in boundary condition (Equation (8)) is taken as

\[
T(t) = \vartheta \sin(\omega t). \tag{43}
\]

As mentioned, the stainless steel in Fig. 1 with the following material properties is chosen in this study (Euro Inox, 2007):

\[
E = 189.5 \times 10^9 \text{ (N/m\(^2\)}); \quad \rho = 7700 \text{ (kg/m\(^3\)}); \quad k = 16.5 \text{ (w/m\(^\circ\)K)}; \quad h = 100 \text{ (w/m\(^2\)K)}; \quad \varpi_T = 12 \times 10^{-6} \text{ (K}^{-1}); \quad c_v = 500 \text{ (J/kgK)}. \tag{44}
\]

The numerical and analytical results are obtained for \( l = 2, \varpi = 100, \vartheta = 100(\circ\text{C}). \) Variations of the displacement and temperature with time in the finite-length medium under the surface temperature changes are depicted in Figs 2 and 3.

As can be seen from the results shown in Figs 2 and 3, the displacement wave is affected by temperature change releases over time.

A comparison between the time history of displacement and its distribution along the length obtained from the numerical and analytical analysis is shown in Figs 5 and 6. The results show an acceptable agreement between these two presented techniques (finite element and multiple-scale methods) in this research. It is found that the output of the displacement increases with growing temperature changes, and because of the fixed boundaries, the maximum values of elastic wave amplitude can be seen in the middle of the domain.

The displacement profile is shown to be a harmonic function of time since the boundary temperature was considered a harmonic function. It is demonstrated that, due to the defined boundary conditions, the displacement is zero at boundaries, and follows the trend of the imposed temperature change at \( x = 0. \)
Figure 4: Spatial and time history of the: (a) temperature, (b) stress, (c) displacement in a medium subjected to harmonic temperature changes.

Figure 5: Comparison between the displacement distribution along the length obtained by the analytical and numerical analysis presented in this study.

5.2. Numerical results of modified Green–Lindsay model

In this section, the thermal shock \( T(t) \) in the defined boundary condition (Equations 7 and 8; Fig. 1) is considered in the form of an exponential function, so that higher temperature change in a short time is applied to the medium.

\[
T(t) = (\vartheta \times t) \exp(-0.2 \times t)
\]  

(45)
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In the applied numerical solution, the length is discretized into 50 nonlinear Hermitian elements. At first, another comparison is done between the present results and the results of Mirparizi et al. (2020), where thermoelastic waves under stress shock were discussed. The volume fraction exponent is considered to be zero in this research. Based on other defined boundary conditions in the reference, at \( x = 0 \) the boundary is exposed to step traction \( (s) \), \( s = 10^6 \text{ (pa)} \) \( t \leq 0.00006 \text{(s)} \). \( s = 0 \) \( t \geq 0.00006 \text{(s)} \) and both sides are subjected to the convection with the ambient. Figures 7 and 8 illustrate the time history of the stress and temperature for different points shown in results. As can be seen, the waves follow the same trend in two results. As the results show, after the elimination of imposed stress, the compressive wave moves through the length.

The wave propagation phenomenon in the medium under the heat flux is discussed using the modified Green–Lindsay model and the results of the numerical analysis are described accordingly. To show, the wave propagation phenomenon, the time histories of the Piola stress at different points will be presented. To study the present generalized thermoelasticity model, a 3D distribution of the stress for different relaxation parameters is presented, where the results of the modified Green–Lindsay model and classic theory are compared. In the following, the temperature and displacement distribution are also given. Finally, the modified Green–Lindsay model is compared with Green–Lindsay theory to determine the effects of strain and temperature rate on the thermal and mechanical wave behaviors. The stress wave propagation in the medium exposed to thermal shock is shown in Fig. 9. The results show that the external thermal shock leads to a tensile stress wave propagation in the body. It is evident that the beginning of the medium experiences the maximum stress, which is close to the external heat source \( (x = 0.1 \text{(m)}) \). Over time, this wave travels throughout the length of the body, and then by losing the energy, the wave’s amplitude (the maximum amount of the stress at any time) decreases. Furthermore,
the end surface is subjected to heat convection with the ambient temperature and there is no other external source to offset this energy shortage. The results of the stress waves are obtained for different modified Green–Lindsay parameters ($\tilde{m}, \tilde{n}$) and reveal that in this research, the most appropriate value considering time relaxation parameters is $\tilde{m}, \tilde{n} \approx 10^{-6}$. Based on the results, the wave propagation phenomenon is visible in the best way by choosing the mentioned value for $\tilde{m}, \tilde{n}$. For different values of $\tilde{m}$ and $\tilde{n}$, time histories for all points in the medium are depicted in Fig. 10 in three dimensions. It can be seen that the wave propagation in Fig. 10a and b is more visible than in Fig. 10c. Moreover, Fig. 10d describes the wave propagation in the case of $\tilde{m} = \tilde{n} = 0$ that implies classical thermoelasticity theory. As the results of Fig. 8 show, although the wave motion is well visible, the amplitude of the wave has remained constant over time. So, it can be concluded that considering strain and temperature rate in the modified Green–Lindsay theory leads to reasonable outcomes in the wave propagation analysis.

The results for displacement distribution against the time are shown in Fig. 11. These results are presented to compare the classic thermoelasticity model and modified Green–Lindsay theory from different aspects. The comparison reveals that the strain rate has significant effects on the system's responses and the difference between displacement distributions in both models increases over time. To have a complete perception of the behavior of displacement wave, a 3D plot of displacement variations is shown in Fig. 12a and b. Results demonstrated in this figure also confirm that the affected domain increases with growing time.
Figure 10: 3D plots showing the simultaneous propagation of the stress wave: modified Green–Lindsay: (a) $\tilde{m}, \tilde{n} \approx 10^{-6}$, (b) $\tilde{m}, \tilde{n} \approx 10^{-8}$, (c) $\tilde{m}, \tilde{n} \approx 10^{-4}$, and classic model: (d) $\vartheta = 100, \eta = 1$.

Figure 11: Comparison between the displacement results of the present modified Green–Lindsay model and classic theory.
The nonlinear equations of thermoelasticity (Equations 6 and 7) represent the model of Green–Lindsay by setting the values of $\eta = 0$. In the following, considerable differences in the stress and displacement wave propagation predicted by modified Green–Lindsay, Green–Lindsay, and classic models are discussed in detail. Responses based on the three mentioned models are compared in Figs 13 and 14. As can be seen, the stress and displacement obtained by modified Green–Lindsay theory and Green–Lindsay theory are smaller than the classical model, in that considering relaxation time in modified Green–Lindsay and Green–Lindsay increases the stiffness matrix (Equation 13) of the finite element process. Although, as the imposed load is a thermal shock rather than a mechanical one, the effects of strain rate are not well visible and its influences may be much remarkable in structures under mechanical loads.

The 3D plots of the stress wave propagation using Green–Lindsay theory for two values of $\tilde{m}$, $\tilde{n}$ and $\eta = 0$ are shown in Fig. 15. The results reveal that the applied thermal shock leads to tensile stress that travels along the length over time. As can be seen from the results presented in this figure, choosing $\tilde{m}$, $\tilde{n} \approx 10^{-6}$ leads to more reasonable results in the wave propagation phenomenon.

The findings of temperature wave propagation are depicted in Fig. 16. It is clear that the points in contact with the heat flux experience more temperature variations and the imposed flux has its maximum value at the initial times and reaches a minimum over time. Also, it can be seen that the maximum temperature wave moves during the time. These results show the differences in the predictions by the modified Green–Lindsay model as compared with the predictions by Green–Lindsay and classic model. It reveals that both Green–Lindsay and modified Green–Lindsay models have a similar outcome in thermally nonlinear analysis. Additionally, the results of the classic model determine that considering temperature and strain rate in governing equations may lead to significant effects on thermal wave propagation. The analysis of wave propagation throughout the length can be followed in Fig. 17. Figure 17a
Figure 14: Evaluation of the influences of the strain and temperature rate on displacement wave propagation (comparison of three theories: modified Green–Lindsay, Green–Lindsay, and classic), $\dot{\vartheta} = 100$.

Figure 15: 3D plots showing the simultaneous propagation of the stress wave in Green–Lindsay model for (a) $\tilde{m}, \tilde{n} \approx 10^{-6}$, (b) $\tilde{m}, \tilde{n} \approx 10^{-8}$, $\vartheta = 100$.

Figure 16: Evaluation of the influences of the strain and temperature rate on thermal wave propagation (comparison of three theories: modified Green–Lindsay, Green–Lindsay, and classic), $\vartheta = 100$. 


Figure 17: The thermoelastic wave propagation versus displacement arising from imposed thermal shock.
shows the stress wave distribution before removing the imposed pressure, and Fig. 17b describes the continuous moving of waves in the medium. Also, the displacement and thermal wave for considered theory are illustrated in Fig. 17c–e.

6. Conclusion

A modified Green–Lindsay theory integrating strain and temperature rate-dependent model was employed to examine the stress, displacement, and temperature wave propagation in the medium subjected to thermal shock. The employed equations have met the requirements for the strain and temperature-rate (modified Green–Lindsay) theories and can be simplified for the classic model. Hence, all nonlinear governing equations were rewritten based on strain and temperature rate of both the stress relation and energy law. To take into account the effects of large deformations, a nonlinear form of the strain tensor (Green’s tensor) was used in terms of the displacement component using FST. Based on the findings, even though the strain rate has insignificant influences on the temperature, it can have a notable effect on the stress and displacement wave propagation in a structure exposed to thermal shock and these impacts may be more significant in situations with mechanical loading.

The wave propagation phenomenon is evident in the best way by employing a modified Green–Lindsay theory compared with the classic model. The stress and thermal waves have noticeable interaction and a harmonic temperature change may lead to a harmonic stress distribution. Thermal shock induces tensile stresses in the medium immediately after the shock, and the perceived phenomenon depends on the defined boundary conditions. The outcome also revealed that the amplitude of wave propagation in the medium subjected to the heat flux decreases over time, which is mostly due to the backplane affected by convection heat transfer, and the trend is more visible in modified Green–Lindsay model results compared with classic theories.

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Appendix A.

\[ a_1 = \rho_0 \int_0^1 \sin(\omega x) dx \]

\[ a_2 = \frac{3}{2} (\lambda + 2\mu \int_0^1 \cos^2(\omega x) \sin(\omega x) dx \]

\[ a_3 = 3(\lambda + 2\mu) \int_0^1 \cos^2(\omega x) \sin(\omega x) dx \]

\[ a_4 = (\lambda + 2\mu) \int_0^1 \omega^2 \sin(\omega x) dx \]

\[ a_5 = -\omega (3\lambda + 2\mu) \int_0^1 \omega^2 \sin(\omega x) dx \]

\[ \omega = \frac{1}{\sqrt{A_1}} \]

\[ a_7 = -\omega (3\lambda + 2\mu) \int_0^1 (\omega \cos(\omega x)) (\lambda \sin(\omega x)) dx \]

\[ a_8 = -\omega (3\lambda + 2\mu) \int_0^1 \omega \sin(\omega x) dx \]

\[ a_9 = k_1 \int_0^1 \lambda (\sin(\omega x)) dx \]

\[ a_{10} = C_\omega \int_0^1 (\cos(\lambda x)) dx \]

\[ m = \beta \int_0^1 \omega (\cos(\lambda x)) (\cos(\omega x)) dx \]

\[ r = \beta \int_0^1 \omega^2 (\cos(\omega x))^2 (\cos(\lambda x)) dx \]

\[ a_{11} = \beta \int_0^1 \omega (\sin(\omega x)) dx \]

\[ a_{12} = \beta \int_0^1 \omega \omega (\sin(\omega x))^2 dx \]

Appendix B.

\[ g^I(T_0, T_1, ...) = -\frac{i A(T_1)^2 \beta(T_1)^2 r a e^{i(\omega x - \gamma y)}}{2 i \omega a + y a_10 - a_9} + \frac{i A(T_1)^2 \beta(T_1)^2 \gamma a (i \omega + \gamma) e^{i(\omega x - \gamma y)}}{(i \omega a + y a_10 - a_9)(\gamma^2 a_1 + a_4)} \]

\[ + \frac{A(T_1)^2 \beta(T_1)^2 \gamma a (i \omega - \gamma) e^{i(\omega x - \gamma y)}}{(i \omega a + y a_10 - a_9)(\gamma^2 a_1 + a_4)} \]

\[ - \frac{A(T_1)^2 \beta(T_1)^2 i a_2 \gamma^2 m a_1 + i a_2 \gamma^2 m a_2 - i a_2 \gamma^2 m a_12 - a_2 \gamma^2 m a_4) e^{i(\omega x - \gamma y)}}{(i \omega a + y a_10 - a_9)(\gamma^2 a_1 + a_4)a_4} \]

\[ - \frac{i A(T_1)^2 \beta(T_1)^2 a_1 \gamma e^{i(\omega x - \gamma y)}}{(i a_2 - a_9)a_4} - \frac{A(T_1)^3 a_1 \gamma e^{i(\omega x - \gamma y)}}{(2 i \omega a + a_9)(\gamma^2 a_1 + a_4)} \]

\[ \times \frac{A(T_1)^3 \gamma a (a_2 \gamma^2 m a_1 + a_2 \gamma^2 m - a_2 \gamma^2 m a_2) e^{i(\omega x - \gamma y)}}{a_2 (2 i \omega a + a_9)(\gamma^2 a_1 + a_4)} \]

(B.1)
\[ f_2(T_0, T_1) = -\frac{3A(T_1)2\alpha_0 b(T_2)^2 e^{\frac{i\pi}{2}}}{(4\alpha_1 y_1 + y^2 a_1 + 4\alpha_2 y_2 a_2)\alpha_1(y^2 a_1 + a_i)} + \frac{3A(T_1)2\alpha_0 b(T_2)^2 e^{\frac{i\pi}{2}}}{(4\alpha_1 y_1 + y^2 a_1 + 4\alpha_2 y_2 a_2)\alpha_1(y^2 a_1 + a_i)} \\
+ 12\alpha_2 a_1 A(T_1)^2 \left( \frac{4\alpha^2 + \alpha a_4}{(4\alpha^2 + \alpha^2)\alpha_1 + 2(\alpha + \gamma)(\alpha_1 a_1 + (a_i)^2)(y^2 a_1 + a_i)^2} \right) \\
- 12\alpha_2 a_1 A(T_1)^2 \left( \frac{4\alpha^2 + \alpha a_4}{(4\alpha^2 + \alpha^2)\alpha_1 + 2(\alpha + \gamma)(\alpha_1 a_1 + (a_i)^2)(y^2 a_1 + a_i)^2} \right) \\
- \frac{4(-\frac{1}{2} y^2 a_1 a_1 - \frac{1}{2} (\alpha + \beta) a_1^2 + \alpha a_1 a_4)A(T_1)\left((\alpha + \beta) y a + \frac{1}{2} \alpha_1^2 - 2(\alpha + \gamma)(\alpha_1 a_1 + (a_i)^2)(y^2 a_1 + a_i) \right)}{a_1(y^2 + \alpha a_4)^2 a_1^2 - 2(\alpha + \gamma)(\alpha_1 a_1 + (a_i)^2)(y^2 a_1 + a_i)} \\
+ \frac{a_1^2 a_1 a_4 (9\alpha^2 a_1 + a_i)(9\alpha^2 a_1 + a_i)}{a_1 (-4\alpha^2 a_1 + a_i)(9\alpha^2 a_1 + a_i)} + \frac{\alpha_2 B(T_1) \alpha_2 a_1 a_4 e^{\frac{i\pi}{2}} A(T_1)}{(y^2 a_1 + a_i)^2} \\
+ \frac{\alpha_2 B(T_1) \alpha_2 a_1 a_4 e^{\frac{i\pi}{2}} A(T_1)}{(y^2 a_1 + a_i)^2} (9\alpha^2 a_1 + a_i)(4\alpha^2 a_1 + a_i) \]