COUNTING THE LATTICE RECTANGLES INSIDE AZTEC DIAMONDS AND SQUARE BISCUITS

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Abstract. We are counting the lattice rectangles that can be constructed inside several planar shapes and identify the corresponding sequences in the OEIS.

1. Introduction

A point \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) is called a lattice point. A set \([x, x'] \times [y, y'] \subseteq \mathbb{R}^2\), with \(x < x'\), \(y < y'\) and \(x, x', y, y' \in \mathbb{Z}\) is called a lattice rectangle. In particular, a lattice rectangle having all sides of length 1 is called a unit lattice square.

Let \(n\) be a positive integer. An Aztec diamond of order \(n\) \([1, \text{p. 277}]\) is obtained by stacking \(2n\) rows of consecutive unit lattice squares, with the centers of rows vertically aligned and consisting successively of \(2, 4, \ldots, 2n, 2n, \ldots, 4, 2\) squares. This shape is symmetric with respect to some lattice point \((p, q)\) (that will be called the center of the Aztec diamond) and can also be described as the union of those unit lattice squares that lie inside the tilted square \(\{(x, y) \in \mathbb{R}^2 : |x - p| + |y - q| \leq n + 1\}\).

Figure 1. Aztec diamonds of order up to 4 and their corresponding centers marked in orange.

The square biscuit of order \(n\) is defined in a similar fashion, by stacking \(2n - 1\) rows with their centers vertically aligned which consist successively of \(1, 3, \ldots, 2n - 3, 2n - 1, 2n - 3, \ldots, 2, 1\) consecutive unit lattice squares. The coordinates of its center are some half integers \(p + \frac{1}{2}\) and \(q + \frac{1}{2}\), respectively, so the square biscuit is the union of the unit lattice squares inside the tilted square

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\[
\{(x, y) \in \mathbb{R}^2 : |x - p - \frac{1}{2}| + |y - q - \frac{1}{2}| \leq n\}. \text{ The lattice point } (p, q) \text{ will be called the quasi-center of the square biscuit.}
\]

Figure 2. Biscuits of order up to 4 and their corresponding quasi-centers marked in orange.

A staircase of order \(n\) is obtained by stacking \(n\) rows of consecutive unit lattice squares, aligned either to the left or to the right, which consist of \(1, 2, 3, \ldots, n\) squares and which are stacked either in the increasing or in the decreasing order of their lengths. Although there are four\(^1\) types of staircases, depending on the alignment of the rows and the ordering in the stack, they are identical up to a rotation.

Splitting an Aztec diamond, either vertically or horizontally, through the center gives two halves that are identical up to a rotation. In the case of a square biscuit\(^2\), a vertical or a horizontal splitting through the quasi-center produces two different halves, one larger by one row than the other. Going further, both an Aztec diamond and a square biscuit\(^3\) of order \(n\) can be split into four staircases, one of each type. The staircases in the former case have all the same order \(n\), while for the latter there is one staircase of order \(n\), two of order \(n - 1\) and one of order \(n - 2\) (see Figure 3).

Figure 3. Aztec diamonds, biscuits and staircases.

\(^{1}\)for \(n = 1\), there is only one type
\(^{2}\)of order at least 2
\(^{3}\)not too small
2. Problems

Main problems: Find the number of lattice rectangles included in:

1. an Aztec diamond of order \( n \);
2. a square biscuit of order \( n \).

Intermediate problems: Find the number of lattice rectangles included in:

3. a staircase of order \( n \);
4. a half of an Aztec diamond of order \( n \);
5. the larger half of a square biscuit of order \( n \).

We will denote the answers to these five problems by \( a(n) \), \( b(n) \), \( s(n) \), \( a_{1/2}(n) \) and \( b_{1/2}(n) \), respectively.

Problems 2, 3 and 5 were studied by the authors in [2]. Here, we present alternative solutions to Problems 2 and 5 that make the connection with Problems 1 and 4.

3. Solutions to the intermediate problems

Solution to Problem 3. Consider a staircase of order \( n \), positioned in the plane as shown in Figure 4. A lattice rectangle \([a, b] \times [c, d]\) is included in the staircase if and only if \( 0 \leq a < b \leq n, 1 < c < d \leq n + 2 \) and \( b + 1 \leq c \), where the last condition means that the bottom right corner of the rectangle does not lie below the line \( y = x + 1 \). Concluding, any lattice rectangle that lies inside the staircase is uniquely determined by a quadruple \((a, b, c, d)\) of integers that satisfy \( 0 \leq a < b < c < d \leq n + 2 \). There are \(\binom{n+3}{4}\) such quadruples, hence \( s(n) = \binom{n+3}{4} \).

Figure 4. Finding the relations between the coordinates of a lattice rectangle inside a staircase of order \( n \).
Solution to Problem 4. Fix $n \geq 2$. Denote by $A_{1/2}$ the upper half of an Aztec diamond of order $n$ and let $\Delta$ be its vertical axis of symmetry that splits $A_{1/2}$ into two staircases of order $n$.

There are $2s(n)$ lattice rectangles inside the Aztec diamond that lie entirely either in the left staircase or in the right staircase.

For the remaining lattice rectangles (see Figure 5), $\Delta$ splits each rectangle into two smaller ones (that will be individually referred to as the left part and the right part of the rectangle). We say that a rectangle is of type $L$, $R$ or $C$ if its left part is larger, smaller or equal in size to its right part, respectively.

We count the type–$L$ rectangles by the following bijective argument. Each such rectangle can be uniquely identified with the difference between the reflection of its left part with respect to $\Delta$ and its right part (in Figure 6, the orange type–$L$ rectangle is transformed into the blue-filled rectangle). The result of this transformation is always a rectangle that is included in the staircase of order $n - 1$, obtained by the vertical split of $A_{1/2}$ one unit to the right of $\Delta$ and taking the right part. It is easy to check that the transformation is uniquely reversible, hence bijective, which gives the number of type–$L$ rectangles to be $s(n - 1)$.

By symmetry, the number of type–$R$ rectangles is equal to the number of type–$L$ ones.

\[\text{These notations are abbreviations of the position of the rectangle with respect to } \Delta: \text{ left, right or centered}\]
Also, every type–C rectangle can be uniquely described by its right part, which is a rectangle included in the staircase of order $n$ to the right of $\Delta$, having the left side on $\Delta$. The number of such rectangles is $s(n) - s(n-1)$, since there are $s(n-1)$ rectangles inside the right staircase of order $n$ that do not have the left side on $\Delta$.

Concluding, there are $2s(n-1) + s(n) - s(n-1) = s(n) + s(n-1)$ lattice rectangles included in $A_{1/2}$ whose interior is intersected by $\Delta$, so

$$a_{1/2}(n) = 2s(n) + s(n) + s(n-1) = 3s(n) + s(n-1) = 3\left(\frac{n+3}{4}\right) + \left(\frac{n+2}{4}\right) = \frac{n(n+1)(n+2)^2}{6}$$

for all $n \geq 2$ and with $a_{1/2}(1) = 3$ also satisfying the general formula (consider that $s(0) := 0$). □

Solution to Problem 5. Fix $n \geq 2$. Denote by $B_{1/2}$ the larger upper half of a square biscuit of order $n$ and by $\delta$ its vertical axis of symmetry. There are $2s(n-1)$ lattice rectangles included in $B_{1/2}$ that lie entirely either to the left or to the right of $\delta$.

When counting the remaining lattice rectangles whose interior is intersected by $\delta$ (see Figure 7), it is enough to expand $B_{1/2}$ to half of an Aztec diamond (denote it by $A_{1/2}$), by inserting a middle column of $n$ unit lattice squares to the left (see Figure 8). Naturally, the rectangles that intersect $\delta$ will also expand one column to the left of $\delta$.

![Figure 7. Half of a square biscuit and a rectangle intersected by $\delta$.](image1)

![Figure 8. Expanding half of a square biscuit to half of an Aztec diamond of the same order, by inserting a middle column of height $n$.](image2)
rectangles included in $A_{1/2}$ whose interior is intersected by $\Delta$; their number $s(n) + s(n - 1)$ was found in the solution of Problem 4.

Concluding,

$$b_{\frac{3}{2}}(n) = 2s(n - 1) + s(n) + s(n - 1) = s(n) + 3s(n - 1) = \binom{n + 3}{4} + 3\binom{n + 2}{4} = \frac{n^2(n + 1)(n + 2)}{6}$$

for all $n \geq 2$ and with $b_{\frac{3}{2}}(1) = 1$ also satisfying the general formula.

\[\square\]

4. Solutions to the main problems

Solution to Problem 1. Fix $n \geq 2$. Let $A$ be an Aztec diamond of order $n$ and $\Delta$ be its vertical axis of symmetry which splits $A$ into two equal halves. There are $2a_{\frac{3}{2}}(n)$ lattice rectangles inside the Aztec diamond that lie entirely either in the left half or in the right half.

The remaining lattice rectangles, split in two by $\Delta$, are counted using the same approach as in the solution of Problem 4, with the staircases replaced by the halves of the diamond that lie entirely either in the left half or in the right half.

We leave to the interested reader to check that the arguments presented in the solution of Problem 4.

Concluding,

$$a(n) = 2a_{\frac{3}{2}}(n) + a_{\frac{3}{2}}(n) + a_{\frac{3}{2}}(n - 1) = 3a_{\frac{3}{2}}(n) + a_{\frac{3}{2}}(n - 1)$$

$$= 9\binom{n + 3}{4} + 6\binom{n + 2}{4} + \binom{n + 1}{4} = \frac{n(n + 1)(4n^2 + 12n + 11)}{6}$$

for all $n \geq 2$. A direct count gives $a(1) = 9$, which also agrees with the general formula.

\[\square\]

Solution to Problem 2. Fix $n \geq 2$. Denote by $B$ a square biscuit of order $n$ and by $\delta$ its vertical axis of symmetry. There are $2b_{\frac{3}{2}}(n - 1)$ lattice rectangles included in $B$ that lie entirely either to the left or to the right of $\delta$. We leave to the interested reader to check that the arguments presented in the previous proofs can be easily adapted in counting the lattice rectangles included in $B$ and intersected by $\delta$, leading to $b_{\frac{3}{2}}(n) + b_{\frac{3}{2}}(n - 1)$ rectangles.

Concluding,

$$b(n) = 2b_{\frac{3}{2}}(n - 1) + b_{\frac{3}{2}}(n) + b_{\frac{3}{2}}(n - 1) = b_{\frac{3}{2}}(n) + 3b_{\frac{3}{2}}(n - 1)$$

$$= \binom{n + 3}{4} + 6\binom{n + 2}{4} + 9\binom{n + 1}{4} = \frac{n(n + 1)(4n^2 - 4n + 3)}{6}$$

for all $n \geq 2$. Also, $b(1) = 1$ which agrees with the general formula.

\[\square\]

5. Identifying the results in the On-Line Encyclopedia of Integer Sequences

The sequences $a_{\frac{3}{2}}, b_{\frac{3}{2}}, a$ and $b$ appear in the On-Line Encyclopedia of Integer Sequences (OEIS) [3] as A004320, A002417, A330805 and A213840, respectively. At the time this manuscript was typeset, there was no mention in the OEIS about the combinatorial problems studied in this paper in connection to the sequences A004320 and A213840. The OEIS connects the sequences A002417 and A330805 to the corresponding problems studied in this paper (though using different terminology), but provides no reference to a proof.

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\[9\] the same line as in the solution of Problem 5; it is not a lattice line