THE GENERALISED NILRADICAL OF A LIE ALGEBRA

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Abstract

A solvable Lie algebra $L$ has the property that its nilradical $N$ contains its own centraliser. This is interesting because gives a representation of $L$ as a subalgebra of the derivation algebra of its nilradical with kernel equal to the centre of $N$. Here we consider several possible generalisations of the nilradical for which this property holds in any Lie algebra. Our main result states that for every Lie algebra $L$, $L/Z(N)$, where $Z(N)$ is the centre of the nilradical of $L$, is isomorphic to $\text{Der}(N^*)$ where $N^*$ is an ideal of $L$ such that $N^*/N$ is the socle of a semisimple Lie algebra.

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1 Introduction

Throughout, $L$ will be a finite-dimensional Lie algebra, over a field $F$, with nilradical $N$ and radical $R$. If $L$ is solvable, then $N$ has the property that $C_L(N) \subseteq N$. This property supplies a representation of $L$ as a subalgebra of $\text{Der}(N)$ with kernel $Z(N)$. The purpose of this paper is to seek a larger ideal for which this property holds in all Lie algebras. The corresponding problem has been considered for groups (see, for example, Aschbacher [1, Chapter 11]). In group theory, the quasi-nilpotent radical (also called by some the generalised Fitting subgroup), $F^*(G)$, of a group $G$ is defined to be $F(G) + E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the set of components of $G$: that is, the quasi-simple subnormal subgroups of the group. It is also equal to the socle of $C_G(F(G))/F(G)$. The generalised Fitting subgroup, $\tilde{F}(G)$, is defined to be the socle of $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of $G$ (see, for example, [8]). Here we consider various possible analogues for Lie algebras.

First we introduce some notation that will be used. The centre of $L$ is $Z(L) = \{x \in L : [x,y] = 0$ for all $y \in L\}$; if $S$ is a subalgebra of $L$, the centraliser of $S$ in $L$ is $C_L(S) = \{x \in L : [x,S] = 0\}$; the Frattini ideal, $\phi(L)$, of $L$ is the largest ideal contained in all of the maximal subalgebras of $L$; we say that $L$ is $\phi$-free if $\phi(L) = 0$; the socle of $S$, $\text{Soc} S$, is the sum of all of the minimal ideals of $S$; and the $L$-socle of $S$, $\text{Soc}_L S$, is the sum of all of the minimal ideals of $L$ contained in $S$. The symbol ‘⊕’ will be used to denote an algebra direct sum, whereas ‘\dot{+}’ will denote a direct sum of the vector space structure alone.

We call $L$ quasi-simple if $L^2 = L$ and $L/Z(L)$ is simple. Of course, over a field of characteristic zero a quasi-simple Lie algebra is simple, but that is not the case over fields of prime characteristic. For example, $A_n$ where $n \equiv -1(modp)$ is quasi-simple, but not simple. This suggests using the quasi-simple subideals of a Lie algebra $L$ to define a corresponding $E(L)$. However, first note that quasi-simple subideals of $L$ are ideals of $L$. This follows from the following easy lemma.

**Lemma 1.1** If $I$ is a perfect subideal (that is, $I^2 = I$) of $L$ then $I$ is a characteristic ideal of $L$.

**Proof.** If $I$ is perfect then $I = I^n$ for all $n \in \mathbb{N}$. It follows that $[L,I] = [L,I^n] \subseteq L$ (ad $I^n$) $\subseteq I$ for some $n \in \mathbb{N}$, and hence that $I$ is an ideal of $L$. But now, if $D \in \text{Der}(L)$, then $D([x_1,x_2]) = [x_1,D(x_2)] + [D(x_1),x_2] \in I$ for all $x_1,x_2 \in I$. Hence $D(I) = D(I^2) \subseteq I$. □
Combining this with the preceding remark we have the following.

**Lemma 1.2** Let $L$ be a Lie algebra over a field of characteristic zero. Then $I$ is a quasi-simple subideal of $L$ if and only if it is a simple ideal of $L$.

We say that an ideal $A$ of $L$ is *quasi-minimal* in $L$ if $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Clearly a quasi-simple ideal is quasi-minimal. Over a field of characteristic zero, an ideal $A$ of $L$ is quasi-minimal if and only if it is simple. So an alternative is to define $E(L)$ to consist of the quasi-minimal ideals of $L$. We investigate these two possibilities in sections 3 and 5.

In sections 4 and 6 our attention turns to two further candidates for a generalised nilradical: the $L$-socle of $(N + C_L(N))/N$ and the socle of $L/\phi(L)$. All of these possibilities turn out to be related, but not always equal.

## 2 Preliminary results

Let $L$ be a Lie algebra over a field $F$ and let $U$ be a subalgebra of $L$. If $F$ has characteristic $p > 0$ we call $U$ *nilregular* if the nilradical of $U$, $N(U)$, has nilpotency class less than $p - 1$. If $F$ has characteristic zero we regard every subalgebra of $L$ as being nilregular. We say that $U$ is *characteristic in* $L$ if it is invariant under all derivations of $L$. Then we have the following result.

**Theorem 2.1** (i) If $I$ is a nilregular ideal of $L$ then $N(I) \subseteq N(L)$.

(ii) If $I$ is a nilregular subideal of $L$ and every subideal of $L$ containing $I$ is nilregular, then $N(I) \subseteq N(L)$.

**Proof.**

(i) We have that $N(I)$ is characteristic in $I$. This is well-known in characteristic zero, and is given by [2 Corollary 1] in characteristic $p$. Hence it is a nilpotent ideal of $L$ and the result follows.

(ii) Let $I = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = L$ be a chain of subalgebras of $L$ with $I_j$ an ideal of $I_{j+1}$ for $j = 0, \ldots, n - 1$. Then $N(I) \subseteq N(I_1) \subseteq \ldots \subseteq N(I_n) = N(L)$, by (i).
Similarly, we will call the subalgebra $U$ \textit{solregular} if the underlying field $F$ has characteristic zero, or if it has characteristic $p$ and the (solvable) radical of $U$, $R(U)$, has derived length less than $\log_2 p$. Then we have the following corresponding result.

\textbf{Theorem 2.2} \quad (i) If $I$ is a solregular ideal of $L$ then $R(I) \subseteq R(L)$.

(ii) If $I$ is a solregular subideal of $L$ and every subideal of $L$ containing $I$ is solregular, then $R(I) \subseteq R(L)$.

\textbf{Proof.} This is similar to the proof of Theorem 2.1 using \cite[Theorem 2]{9}.

We also have the following result which we will improve upon below, but by using a deeper result than is required here.

\textbf{Theorem 2.3} Let $L$ be a Lie algebra over a field $F$, and let $I$ be a minimal non-abelian ideal of $L$. Then either

(i) $I$ is simple or

(ii) $F$ has characteristic $p$, $N(I)$ has nilpotency class greater than or equal to $p - 1$, and $R(I)$ has derived length greater than or equal to $\log_2 p$.

\textbf{Proof.} Let $I$ be a non-abelian minimal ideal of $L$ and let $J$ be a minimal ideal of $I$. Then $J^2 = J$ or $J^2 = 0$. The former implies that $J$ is an ideal of $L$ by Lemma 1.1 and hence that $I$ is simple. So suppose that $J^2 = 0$. Then $N(I) \neq 0$ and $R(I) \neq 0$. But if $I$ is nilregular we have that $N(I) \subseteq N(L) \cap I = 0$, since $I$ is non-abelian, a contradiction. Similarly, if $I$ is solvregular, then $R(I) \subseteq R(L) \cap I = 0$, a contradiction. The result follows.

As a result of the above we will call the subalgebra $U$ \textit{regular} if it is either nilregular or solregular; otherwise we say that it is \textit{irregular}. Then we have the following corollary.

\textbf{Corollary 2.4} Let $L$ be a Lie algebra over a field $F$. Then every minimal ideal of $L$ is abelian, simple or irregular.

Block’s Theorem on differentiably simple rings (see \cite{3}) describes the irregular minimal ideals as follows.
Theorem 2.5 Let $L$ be a Lie algebra over a field of characteristic $p > 0$ and let $I$ be an irregular minimal ideal of $L$. Then $I \cong S \otimes O_n$, where $S$ is simple and $O_n$ is the truncated polynomial algebra in $n$ indeterminates. Moreover, $N(I)$ has nilpotency class $p - 1$ and $R(I)$ has derived length $\lceil \log_2 p \rceil$.

Proof. Every non-abelian minimal ideal $I$ of $L$ is $\text{ad}|_I (L)$-simple, so the first assertion follows from [3, Theorem 1]. Now $N(I) = R(I) \cong S \otimes O_n^+$, where $O_n^+$ is the augmentation ideal of $O_n$. It is then straightforward to check that the final assertion holds. □

Note that if $N$ and $S$ are the classes of Lie algebras that are themselves nilregular and solregular respectively, then $N \not\subseteq S$ and $S \not\subseteq N$, as the following examples show.

Example 2.1 Let $L$ be a filiform nilpotent Lie algebra of dimension $n$ over a field $F$. Then $L$ has nilpotency class $n - 1$ and derived length $2$. Thus, if $F$ has characteristic $p > 3$, and $n \geq p$, then $L$ has nilpotency class greater than or equal to $p - 1$, and so is not nilregular. However, it is solregular, since $2 < \log_2 p$.

Example 2.2 Let $L = Fe_1 + Fe_2$ with product $[e_1, e_2] = e_2$ and let $F$ have characteristic 3. The $N(L) = Fe_2$ has nilpotency class $1 < p - 1$ and so $L$ is nilregular. But $R(L) = L$, so $L$ has derived length $2 > \log_2 p$ and is not solregular.

For every Lie algebra $L^{(n)} \subset L^{2^n}$, so any nilregular nilpotent Lie algebra of nilpotency class $2^n$ is solregular, since $2^n < p - 1 < p$ implies that $n < \log_2 p$. However, it is not true generally that a nilpotent nilregular Lie algebra is solregular, as the following example shows.

Example 2.3 Let $L$ be the seven-dimensional Lie algebra over a field $F$ of characteristic $p = 7$ with basis $e_1, \ldots, e_7$ and products $[e_2, e_1] = e_4, [e_3, e_1] = e_5, [e_3, e_2] = e_5, [e_4, e_3] = -e_6, [e_5, e_1] = e_7, [e_5, e_2] = 2e_6, [e_5, e_4] = e_7, [e_6, e_1] = e_7$ and $[e_6, e_2] = e_7$ (see [4, page 87]). Then $L$ has nilpotency class $5 < p - 1$ and so is nilregular, but its derived length is $3 > \log_2 p$, so it is not solregular.

We also have the following result.

Corollary 2.6 If $L$ is a Lie algebra and $A$ is a regular ideal of $L$, then $A$ is quasi-minimal in $L$ if and only if it is a quasi-simple ideal of $L$. 

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However, the above result is not true for all ideals, as the following example shows.

**Example 2.4** Let $L = \mathfrak{sl}(2) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$, where $\mathcal{O}_m$ is the truncated polynomial algebra in $m$ indeterminates, $\mathcal{D}$ is a non-zero solvable subalgebra of $\text{Der}(\mathcal{O}_m)$, $\mathcal{O}_m$ has no $\mathcal{D}$-invariant ideals, and the ground field is algebraically closed of characteristic $p > 5$. Then $L$ is semisimple and $A = \mathfrak{sl}(2) \otimes \mathcal{O}_m$ is the unique minimal ideal of $L$ (see [18, Theorem 6.4]). Since $Z(A) = 0$, $A$ is clearly quasi-minimal but not quasi-simple.

If $S$ is a subalgebra of $L$, we denote by $R_c(S)$ the (solvable) characteristic radical of $S$; that is, the sum of all of the solvable characteristic ideals of $L$ (see Seligman [11]).

**Theorem 2.7** Let $L$ be a Lie algebra over any field $F$. Then $R_c(C_L(N)) = Z(N)$. Moreover, if $C_L(N)$ is regular, then $R_c(C_L(N)) = R(C_L(N))$.

**Proof.** Let $Z = Z(N)$, $\mathcal{T} = L/Z$ and $H = R_c(C_L(N))$. Then $H$ is a characteristic ideal of $C_L(N)$, and hence an ideal of $L$. Assume that $\overline{H} \neq 0$. Then there exists $k \geq 1$ such that $H^{(k+1)} \subseteq Z$ but $X = H^{(k)} \not\subseteq Z$. Then $X^2 \subseteq Z$ and $X^3 \subseteq [N, C_L(N)] = 0$, since $X \subseteq C_L(N)$. It follows that $X$ is a nilpotent ideal of $L$, and hence that $X \subseteq N$. But $[X, N] = 0$, giving $X \subseteq Z$, a contradiction.

Now suppose that $C_L(N)$ is nilregular. Then, clearly, $R_c(C_L(N)) \subseteq R(C_L(N))$. Suppose that $R(C_L(N)) \neq Z$. Let $A/Z$ be a minimal ideal of $C_L(N)/Z$ with $A \subseteq R(C_L(N))$. Then $A^3 = 0$ and so $A \subseteq N(C_L(N)) \subseteq N(L)$, by Theorem 2.1 (i). Hence $A = Z$, a contradiction.

Finally, suppose that $C_L(N)$ is solregular. Then $R(C_L(N)) = R(L) \cap C_L(N)$ is an ideal of $L$, and arguing as in the first paragraph of this proof shows that $R(C_L(N)) = Z(N)$. $\square$

This has the following useful corollary.

**Corollary 2.8** Let $L$ be a Lie algebra over a field $F$, let $N$ be its nilradical and let $C = C_L(N)$ be regular. Then

(i) if $\phi(C) \cap Z(N) = 0$, $C = Z(N) + B$ where $B$ is a semisimple subalgebra of $L$ and $B^2$ is an ideal of $L$;

(ii) if $\phi(L) \cap Z(N) = 0$, $C = Z(N) \oplus B$ where $B$ is a maximal semisimple ideal of $L$; and
(iii) if \( F \) has characteristic zero, then \( C = Z(N) \oplus S \) where \( S \) is the maximal semisimple ideal of \( L \).

Proof.

(i) Suppose that \( \phi(C) \cap Z(N) = 0 \). Then \( C = Z(N) + B \) for some subalgebra \( B \) of \( C \), by [14, Lemma 7.2]. Moreover, \( B \cong C/Z(N) \) is semisimple, by Theorem 2.7, and \( B^2 = C^2 \) is an ideal of \( L \).

(ii) Suppose that \( \phi(L) \cap Z(N) = 0 \). The \( L = Z(N) + U \) for some subalgebra \( U \) of \( L \), by [14, Lemma 7.2] again. It follows that \( C = Z(N) \oplus B \) where \( B = C \cap U \), which is an ideal of \( L \), and \( B \) is semisimple. Moreover, if \( S \) is a semisimple ideal of \( L \) with \( B \subseteq S \), then \( [S, N] \subseteq S \cap N = 0 \), so \( S \subseteq C \). Hence \( S = B \).

(iii) So suppose now that \( F \) has characteristic zero. Then \( C = Z(N) + B \) where \( B \) is a Levi factor of \( C \). Also, \( B = B^2 = C^2 \) is an ideal of \( L \), so \( C = Z(N) \oplus B \). Moreover, if \( S \) is the maximal semisimple ideal of \( L \), then \( B \subseteq S \) and \( [S, N] \subseteq S \cap N = 0 \), so \( S \subseteq C \). It follows that \( S = B \).

\( \square \)

Finally, the following straightforward results will prove useful.

**Lemma 2.9** Let \( K \) be an ideal of \( L \) with \( K \subseteq C_L(N) \). Then \( Z(K) = Z(N) \cap K \).

**Proof.** Clearly \( Z(K) \) is an abelian ideal of \( L \), so \( Z(K) \subseteq N \). Moreover, \([Z(K), N] \subseteq [K, N] = 0 \), so \( Z(K) \subseteq Z(N) \cap K \). Also \([Z(N) \cap K, K] \subseteq [N, K] = 0 \), so \( Z(N) \cap K \subseteq Z(K) \). \( \square \)

**Lemma 2.10** Let \( L \) be any Lie algebra and suppose that \( A \) is an ideal of \( L \) with \( A^2 = A \). Then \( Z(A) \subseteq \phi(L) \). If \( A \) is a quasi-minimal ideal of \( L \), then \( Z(A) = A \cap \phi(L) \).

**Proof.** Suppose that \( Z(A) \not\subseteq \phi(L) \). Then there is a maximal subalgebra \( U \) of \( L \) such that \( L = Z(A) + U \). Thus \( A = Z(A) + U \cap A \) and \( U \cap A \) is an ideal of \( L \). It follows that \( A = A^2 = (U \cap A)^2 \subseteq U \cap A \subseteq A \), whence \( Z(A) \subseteq U \), a contradiction. Hence \( Z(A) \subseteq \phi(L) \).

Suppose now that \( A \) is a quasi-minimal ideal of \( L \). Then \( Z(A) \subseteq A \cap \phi(L) \subseteq A \), so \( A \cap \phi(L) = A \) or \( Z(A) \). The former implies that \( A \subseteq \phi(L) \), which is impossible since \( \phi(L) \) is nilpotent. Hence \( A \cap \phi(L) = Z(A) \). \( \square \)
3 The quasi-minimal radical

Here we construct a radical by adjoining the quasi-minimal ideals of $L$ to its nilradical $N$.

**Lemma 3.1** Quasi-minimal ideals of $L$ are characteristic in $L$.

**Proof.** This follows from Lemma 1.1. □

**Lemma 3.2** Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$. Then $A = A^2 + Z(A)$ and $A^2$ is quasi-minimal in $L$.

**Proof.** Let $P = A^2$ and $\mathfrak{N} = L/Z(A)$. Then $\mathfrak{N}$ is an ideal of $\mathfrak{N}$ and $\mathfrak{N}$ is minimal, so $\mathfrak{N} = 0$ or $\mathfrak{N}$. The former implies that $\mathfrak{N}$ is abelian, a contradiction. Hence $\mathfrak{N} = A$, so $A = P + Z(A) = A^2 + Z(A)$. Also, $P = A^2 = P^2$ and $[Z(P), A] = [Z(P), P] + [Z(P), Z(A)] = 0$, so $P \cap Z(A) = Z(P)$. Thus $P/Z(P) = P/P \cap Z(A) \cong P + Z(A)/Z(A) = A/Z(A)$ is a minimal ideal of $L/Z(P)$. □

**Proposition 3.3** Let $A$ be quasi-minimal in $L$ and $B$ be an ideal of $L$. Then either $A \subseteq B$ or $A \subseteq C_L(B)$.

**Proof.** Clearly $A \cap B + Z(A)/Z(A)$ is an ideal of $L/Z(A)$ contained in $A/Z(A)$, so $A \cap B + Z(A) = A$ or $A \cap B + Z(A) = Z(A)$. The former implies that $A = A^2 \subseteq A \cap B \subseteq A$, whence $A = A \cap B$ and $A \subseteq B$. The latter yields that $A \cap B \subseteq Z(A)$, giving $[A, B] = [A^2, B] \subseteq [A, [A, B]] \subseteq [A, A \cap B] \subseteq [A, Z(A)] = 0$ and so $A \subseteq C_L(B)$. □

The quasi-minimal components of $L$ are its quasi-minimal ideals. Write $\text{MComp}(L)$ for the set of quasi-minimal components of $L$, and let $E^\dagger(L)$ be the subalgebra generated by them. Then $E^\dagger(L)$ is a characteristic ideal of $L$, by Lemma 1.1.

**Corollary 3.4** $E^\dagger(L) \subseteq C_L(R)$.

**Proof.** Let $A \in \text{MComp}(L)$ and put $B = R$ in Proposition 3.3. Then either $A \subseteq R$ or $A \subseteq C_L(R)$. But the former is impossible, since $A^2 = A$, whence $A \subseteq C_L(R)$. □

**Corollary 3.5** Distinct quasi-minimal components of $L$ commute, so

$$E^\dagger(L) = \sum_{P \in \text{MComp}(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in \text{MComp}(L)$. 

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**Proof.** This first assertion follows directly from Proposition 3.3. But then \( P \cap Q \subseteq Z(P) \cap Z(Q) \subseteq N \) and \([P, R] = [Q, R] = 0\), using Corollary 3.4. Hence \( P \cap Q \subseteq Z(R) \). □

**Lemma 3.6** If \( B \) is an ideal of \( L \), then \( \text{MComp}(B) \subseteq \text{MComp}(L) \cap B \). Moreover, if \( B \) is regular, then this is an equality.

**Proof.** Let \( A \) be a quasi-minimal ideal of \( B \). Then \( A \) is a quasi-minimal ideal of \( L \), by Lemma 3.1. Thus \( \text{MComp}(B) \subseteq \text{MComp}(L) \cap B \).

Now suppose that \( B \) is regular, and let \( A \in \text{MComp}(L) \cap B \), so \( A \) is a quasi-minimal ideal of \( L \) and \( A \subseteq B \cap C(L(N)) \), by Corollary 3.4. Let \( C/Z(A) \) be a minimal ideal of \( B/Z(A) \) with \( C \subseteq A \). Then \( C^2 \subseteq Z(A) \) or \( C^2 + Z(A) = C \). The former implies that \( C^3 = 0 \), and hence that \( C \) is a nilpotent ideal of \( B \). If \( B \) is nilregular, it follows from Theorem 2.1 that \( C \subseteq N \), whence \([C, A] = 0 \) and \( C \subseteq Z(A) \), a contradiction. Similarly, if \( B \) is solregular, then \( C \subseteq R(B) \subseteq R(L) \), by Theorem 2.2. But then \([C, A] = 0 \), by Corollary 3.4 since \( A \in E^+ \), leading to the same contradiction. Hence \( C^2 + Z(A) = C \). But now

\[
[L, C] = [L, C^2 + Z(A)] \subseteq [[L, C], C] + Z(A) \subseteq [B, C] + Z(A) \subseteq C,
\]

so \( C \) is an ideal of \( L \). But \( A/Z(A) \) is a minimal ideal of \( L/Z(A) \), so \( C = Z(A) \) or \( C = A \). It follows that \( A/Z(A) \) is a minimal ideal of \( B/Z(A) \) and \( A^2 = A \). Thus \( A \in \text{MComp}(B) \). □

**Example 3.1** Note that if \( B \) is not regular then the inclusion in Lemma \( 3.6 \) can be strict. For, let \( L \) be as in Example \( 2.4 \). Then \( \mathcal{O}_m \) has a unique maximal ideal \( \mathcal{O}_m^+ \) and \( A^+ = \mathfrak{sl}(2) \otimes \mathcal{O}_m^+ \) is the unique maximal ideal of \( A \) (and is nilpotent). Hence \( \text{MComp}(A) \subseteq A^+ \neq A \), whereas \( \text{MComp}(L) = A \).

**Proposition 3.7** Let \( L \) be a Lie algebra in which \( C_L(N) \) is regular. Put \( Z = Z(N), \overline{L} = L/Z, \overline{S} = \text{Soc}(C_L(N)) \). Then \( E^\dagger(L) = S^2 \) and \( S = E^\dagger(L) + Z \).

**Proof.** Let \( H = C_L(N) \). Then \( R(\overline{H}) = 0 \), by Theorem 2.7. Hence each minimal ideal of \( \overline{H} \) is quasi-minimal in \( \overline{H} \), and so is a quasi-minimal component of \( \overline{H} \). Thus \( \overline{S} \subseteq E^\dagger(\overline{H}) \). Let \( \overline{F} \in \text{MComp}(\overline{H}) \subseteq \text{MComp}(\overline{L}) \), by Lemma 3.6. Hence \( K/Z(K) \) is a quasi-minimal ideal of \( L/Z(K) \), by Lemma 2.9. Then \( K = K^2 + Z \) with \( K^2 \) quasi-minimal in \( L \), since \( Z(K) = Z \) by Lemma 3.2. Hence \( K^2 \in \text{MComp}(L) \), so \( S \subseteq E^\dagger(L) + Z \).

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Let $P \in \text{MComp}(L)$. Then $P \subseteq H$ since $E^\dagger(L) \subseteq H$, by Corollary 3.4. Hence $P \in \text{MComp}(L) \cap H = \text{MComp}(H)$, by Lemma 3.6. Hence $P$ is a minimal ideal of $H$, so $P \subseteq S$. Thus $S = E^\dagger(L) + Z$ and $E^\dagger(L) = S^2$. □

We define the quasi-minimal radical of $L$ to be $N^\dagger(L) = N + E^\dagger(L)$. From now on we will denote $N^\dagger(L)$ simply by $N^\dagger$. Then this has the property we are seeking.

**Theorem 3.8** If $L$ is a Lie algebra, over any field $F$, with nilradical $N$, then $C_L(N^\dagger) = Z(N)$. In particular, $C_L(N^\dagger) \subseteq N^\dagger$.

**Proof.** Let $C = C_L(N^\dagger)$. Then $Z(N) \subseteq C$, by Corollary 3.4. Suppose that $Z(N) \neq C$ and let $A/Z(N)$ be a minimal ideal of $L/Z(N)$ with $A \subseteq C$. Then $[A,Z(N)] \subseteq [C,N^\dagger] = 0$, so $Z(N) \subseteq Z(A)$. Thus $A = Z(A)$ or $Z(A) = Z(N)$. The former implies that $A \subseteq N$. But $[A,N] \subseteq [C,N^\dagger] = 0$, so $A \subseteq Z(N)$, a contradiction. The latter implies that $A^2 \subseteq E^\dagger \subseteq N^\dagger$, by Lemma 3.2. Hence $A^3 \subseteq [C,N^\dagger] = 0$, so $A \subseteq N$, which leads to the same contradiction as before. The result follows. □

**Proposition 3.9** Let $L$ be a Lie algebra in which $N^\dagger$ is regular. Then $N^\dagger(N^\dagger) = N^\dagger$.

**Proof.** Clearly $N^\dagger(N^\dagger) \subseteq N^\dagger$. But $E^\dagger(L) \subseteq E^\dagger(N^\dagger)$, by putting $B = N^\dagger$ in Lemma 3.6 and, clearly, $N \subseteq N(N^\dagger)$, giving the reverse inclusion. □

**Example 3.2** Again, Proposition 3.9 does not hold if $N^\dagger$ is not regular. For, let $L$ be as in Example 2.4. Then $N^\dagger = A$, but $N^\dagger(N^\dagger) = A^\dagger$.

Next we investigate the behaviour of $N^\dagger$ with respect to factor algebras, direct sums and ideals.

**Proposition 3.10** Let $L$ be a Lie algebra over any field, and let $I$ be an ideal of $L$. Then

\[
\frac{N^\dagger(L) + I}{I} \subseteq N^\dagger\left(\frac{L}{I}\right).
\]

**Proof.** Clearly $N(L) + I/I \subseteq N(L/I)$. Let $A$ be a quasi-minimal ideal of $L$, so $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Put $C = C_L(A+I/I)$. Then $Z(A) \subseteq C \cap A \subseteq A$, so $C \cap A = A$ or $C \cap A = Z(A)$. The former implies that $A = A^2 \subseteq I$, whence $A + I/I \subseteq N(L/I)$. If the latter holds,
then \( C = C \cap (A + I) = C \cap A + I = Z(A) + I \) and \( A \cap I \subseteq A \cap C = Z(A) \), whence

\[
\frac{A + I}{Z(A + I)} \cong \frac{A + I}{C} = \frac{A + I}{Z(A) + I} \cong \frac{A}{Z(A) + A \cap I} = \frac{A}{Z(A)}
\]

and

\[
\left( \frac{A + I}{I} \right)^2 = \frac{A + I}{I}.
\]

Thus \( A + I/I \) is a quasi-minimal ideal of \( L/I \) and

\[
E\dagger(L) + I/I \subseteq E\dagger \left( \frac{L}{I} \right).
\]

The result follows. □

The above inclusion can be strict, as we shall see later.

**Proposition 3.11** Let \( L \) be a Lie algebra over any field, and suppose that \( L = I \oplus J \), where \( I, J \) are ideals of \( L \). Then \( N\dagger(L) = N\dagger(I) \oplus N\dagger(J) \).

**Proof.** It is easy to see that \( N\dagger(I) \oplus N\dagger(J) \subseteq N\dagger(L) \). Let \( \pi_I, \pi_J \) be the projection maps onto \( I, J \) respectively. Then \( N(L) = \pi_I(N(L)) \oplus \pi_J(N(L)) \).

Clearly \( \pi_I(N(L)) \subseteq N(I) \) and \( \pi_J(N(L)) \subseteq N(J) \), so \( N(L) \subseteq N(I) \oplus N(J) \).

Let \( A \) be a quasi-minimal ideal of \( L \), so \( A/Z(A) \) is a minimal ideal of \( L \) and \( A^2 = A \). Then

\[
A = A^2 \subseteq [A, I \oplus J] = [A, I] \oplus [A, J] \subseteq A,
\]

so \( A = [A, I] \oplus [A, J] \). Since \( A = A^2 = [A, I]^2 + [A, J]^2 \), we also have that \( [A, I]^2 = [A, I] \) and \( [A, J]^2 = [A, J] \). Now \( [A, I] + Z(A) = Z(A) \) or \( A \).

The former implies that \( [A, I] \subseteq Z(A) \), which gives that \( [A, I] = [A, I]^2 = 0 \). The latter yields that \( [A, I] \subseteq Z(A) \), which gives that \( [A, I] = [A, I]^2 = 0 \).

The former implies that \( [A, I] \subseteq Z([A, I]) \), so \( Z([A, I]) = [A, I] \) or \( Z(A) \cap [A, I] \). The former gives \( [A, I] = [A, I]^2 = 0 \) again, whereas the latter yields that \( [A, I]/Z([A, I]) \) is quasi-minimal and \( [A, I] \in E\dagger(I) \).

Similarly \( [A, J] = 0 \) or else \( [A, J] \in E\dagger(J) \). It follows that \( E\dagger(L) \subseteq E\dagger(I) \oplus E\dagger(J) \), whence the result. □

**Proposition 3.12** Let \( L \) be a Lie algebra over any field, and let \( I \) be a nilregular ideal of \( L \). Then \( N\dagger(I) \subseteq N\dagger(L) \).
**Proof.** Since $I$ is nilregular, we have that $N(I) \subseteq N(L)$, by Theorem 2.1 (i). Also, $E^\dagger(I) \subseteq E^\dagger(L)$, by Lemma 3.6 whence the result. □

The following result describes the ideals of $L$ contained in $E^\dagger$.

**Proposition 3.13** Let $A$ be an ideal of $L$ with $A \subseteq E^\dagger(L)$. Then $A = P_1 + \ldots + P_k + Z(A)$, where $P_i$ is a quasi-minimal component of $L$ for $1 \leq i \leq k$.

**Proof.** Let $E^\dagger(L) = P_1 + \ldots + P_n$, where $P_i$ is a quasi-minimal component of $L$ for each $1 \leq i \leq n$. Then $P_i \subseteq A$ or $P_i \subseteq C_L(A)$ for each $i = 1, \ldots, n$, by Proposition 3.3. Let $P_i \subseteq A$ for $1 \leq i \leq k$ and $P_i \not\subseteq A$ for $k + 1 \leq i \leq n$. Then $A \cap (P_{k+1} + \ldots + P_n) \subseteq Z(A)$, so $A = (P_1 + \ldots + P_k) + Z(A)$. □

Finally we give two further characterisations of $N^\dagger$, valid over any field. Recall that $A/B$ is a chief factor of $L$ if $B$ is an ideal of $L$ and $A/B$ is a minimal ideal of $L/B$.

**Theorem 3.14** Let $L$ be a Lie algebra, over any field $F$, with radical $R$. Then

$N^\dagger = \cap \{ A + C_L(A/B) \mid A/B \text{ is a chief factor of } L \}$.

**Proof.** Denote the given intersection by $I$, let $A/B$ be a chief factor of $L$ and let $P$ be a quasi-minimal component of $L$. Then $P \subseteq A$ or $P \subseteq C_L(A)$, by Proposition 3.3. Hence $E^\dagger \subseteq I$. Moreover, $N \subseteq I$, by [2, Lemma 4.3], so $N^\dagger \subseteq I$.

If $P$ is a quasi-minimal component of $L$ then $P/Z(P)$ is a chief factor of $L$. Also, if $C = C_L(P/Z(P))$ we have $[C, P] = [C, P^2] \subseteq \{ [C, P], P \} \subseteq [Z(P), P] = 0$, so $C = C_L(P)$ and $N \subseteq C$, by Corollary 3.4. Hence $I \subseteq P + C_L(P/Z(P)) = P + C_L(P)$. Now, if $P, Q$ are quasi-minimal components of $L$, then

$$(P + C_L(P)) \cap (Q + C_L(Q)) = P + Q + C_L(P) \cap C_L(Q),$$

since $P \subseteq C_L(Q)$ and $Q \subseteq C_L(P)$. It follows that $I \subseteq N^\dagger + C_L(E^\dagger)$ and $I = N^\dagger + I \cap C_L(E^\dagger)$.

If

$$0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$$

is part of a chief series for $L$ then $I \subseteq \cap_{i=1}^k C_L(N_i/N_{i-1})$, so $I$ acts nilpotently on $N$. Suppose that $N \subset I \cap C_L(E^\dagger)$. Let $A/N$ be a minimal ideal of $L/N$ with $A \subseteq I \cap C_L(E^\dagger)$. Then $A^2 \subseteq N$ or $A^2 + N = A$. The former
implies that $A \subseteq N$, since $A$ acts nilpotently on $N$, a contradiction. Hence $A = A^2 + N \subseteq A^r + N$ for all $r \geq 1$. But now

$$[A, N] \subseteq [A^r + N, N] \subseteq N(\text{ad}, A)^r + N^r,$$

so $[A, N] = 0$, whence $A \subseteq C_L(E^\dagger) \cap C_L(N) = C_L(N^\dagger) = Z(N)$, by Theorem 3.8, a contradiction again. Thus $I \cap C_L(E^\dagger) = N$ and $I = N^\dagger$. □

We put

$$I_L(A/B) = \{x \in L \mid \text{ad}(x + B)|_{A/B} = \text{ad}(a + B)|_{A/B} \text{ for some } a \in A\}.$$

The map $\text{ad}(x + B)|_{A/B}$ is called the inner derivation induced by $x$ on $A/B$. Then $I_L(A/B) = A + C_L(A/B)$, by [16, Lemma 1.4 (i)], so we have the following corollary.

**Corollary 3.15** Let $L$ be a Lie algebra over any field $F$. Then $N^\dagger$ is the set of all elements of $L$ which induce an inner derivation on every chief factor of $L$.

4 The generalised nilradical of $L$

We define the generalised nilradical of $L$, $N^*(L)$, by

$$\frac{N^*(L)}{N} = \text{Soc}_{L/N} \left( \frac{N + C_L(N)}{N} \right)$$

As usual we denote $N^*(L)$ simply by $N^*$. The following result shows that this is, in fact, the same as the quasi-nilpotent radical.

**Theorem 4.1** Let $L$ be a Lie algebra with nilradical $N$ over any field. Then $N^* = N^\dagger$.

**Proof.** Put $C = C_L(N)$. Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$ for which $A^2 = A$. Then $Z(A) \subseteq A \cap N$, so $A \cap N = A$ or $A \cap N = Z(A)$. the former implies that $A \subseteq N$, which is a contradiction, so the latter holds. It follows that $(A + N)/N \cong A \cap N = A/Z(A)$, so $(A + N)/N$ is a minimal ideal of $L/N$. Moreover, $[A, N] = [A^2, N] \subseteq [A, [A, N]] \subseteq [A, Z(A)] = 0$, so $A \subseteq C$ and $(A + N)/N \subseteq N^*/N$. Hence $N^\dagger \subseteq N^*$.

Now let $A/N$ be a minimal ideal of $L/N$ with $A \subseteq N + C$. Then $A = N + A \cap C$. Now $Z(A \cap C) = Z(N)$, by Lemma 2.7, so $A/N \cong A\cap C/N \cap C = A\cap N/Z(N) = A\cap C/Z(A\cap C)$. It follows that $A\cap N/Z(A\cap C)$
is a minimal ideal of $L/Z(A \cap C)$. Thus $(A \cap C)^2$ is a quasi-minimal ideal of $L$, by Lemma 3.2. Moreover, $(A \cap C)^2 + Z(N) = Z(N)$ or $A \cap C$. The former implies that $(A \cap C)^2 \subseteq Z(N)$, which yields that $(A \cap C)^3 = 0$ and $A \cap C \subseteq N$, a contradiction. Hence $A \cap C = (A \cap C)^2 + Z(N) \subseteq N^\dagger$, and so $A \subseteq N^\dagger$. This shows that $N^* \subseteq N^\dagger$. □

This last result together with Theorem 3.8 gives the following.

**Theorem 4.2** Let $L$ be a Lie algebra over any field $F$. Then $L/Z(N)$ is isomorphic to a subalgebra of $\text{Der}(N^*)$, and $N^*/N$ is a direct sum of minimal ideals of $L/N$ which are simple or irregular.

**Proof.** The isomorphism results from the map $\theta : L \to \text{Der}(N^*)$ given by $\theta(x) = \text{ad} x|_{N^*}$. Let $A/N$ be a minimal ideal of $L/N$ with $A \subseteq A + C$. The $A = N + A \cap C$ and, as in the second paragraph of the proof of Theorem 4.1, $(A \cap C)^2$ is quasi-minimal in $L$, which implies that $A/N$ cannot be abelian. It follows from Corollary 2.4 that $A/N$ is simple or irregular. □

**Proposition 4.3** Let $L$ be a Lie algebra with nilradical $N$ over a field $F$, and suppose that $C_L(N)$ is nilregular in $L$. Then

$$\frac{N^*}{N} = \text{Soc} \left( \frac{N + C_L(N)}{N} \right).$$

**Proof.** Put $C = C_L(N)$, $D = N + C$. Let $A/N$ be a minimal ideal of $D/N$. Then $A^2 + N = N$ or $A$. The former implies that $A^2 \subseteq N$, whence $A^3 \subseteq [N, N + C] \subseteq N^2$, and an easy induction shows that $A^{n+1} \subseteq N^n = 0$ for some $n \in \mathbb{N}$. It follows that $A$ is a nilpotent ideal of $D$, which is an ideal of $L$, and thus that $A \subseteq N(D) = N + N(C) \subseteq N$, by Theorem 2.1, a contradiction. Hence $A = A^2 + N$ and

$$[L, A] = [L, A^2 + N] \subseteq [[L, A], A] + [L, N] \subseteq [D, A] + N \subseteq A,$$

so $A/N$ is a minimal ideal of $L/N$ inside $D/N$.

Now suppose that $B/N$ is a minimal ideal of $L/N$ inside $D/N$, and let $A/N$ be a minimal ideal of $D/N$ inside $B/N$. Then, by the argument in the paragraph above, $A/N$ is an ideal of $L/N$, and so $A = B$. The result follows. □

**Proposition 4.4** (i) If $C_L(N)$ is regular and $\phi(L) \cap Z(N) = 0$ then $N^*(L) = N(L) \oplus S$, where $S$ is the socle of a maximal semisimple ideal of $L$.  

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(ii) Over a field of characteristic zero, $N^*(L) = N(L) \oplus S = N(L) + C_L(N)$, where $S$ is the biggest semisimple ideal of $L$.

**Proof.** This follows from Corollary 2.8 $\square$

**Proposition 4.5** Let $L$ be a Lie algebra over a field of characteristic zero and let $I \subseteq N^*(L)$ be an ideal of $L$. Then

$$\frac{N^*(L)}{I} \subseteq N^* \left( \frac{L}{T} \right).$$

**Proof.** This is a special case of Proposition 3.10 $\square$

As a result of Example 3.2 we define, for each non-negative integer $n$, $N^*_n$, inductively by

$$N^*_0(L) = L \text{ and } N^*_n = N^*(N^*_{n-1}(L)) \text{ for } n > 0.$$  

Clearly the series

$$L = N^*_0(L) \supseteq N^*_1(L) \supseteq \ldots$$

will terminate in an equality, so we put $N^*_\infty(L)$ equal to the minimal subalgebra in this series. It is easy to see that $N^*_\infty(N^*_\infty(L)) = N^*_\infty(L)$. Then we have

**Proposition 4.6** Let $n \in \mathbb{N} \cup \{0\}$, and let $I, J$ be ideals of the Lie algebra $L$ over the field $F$. Then

(i) if $N^*_k(I)$ is a nilregular ideal of $N^*_k(L)$ then $N^*_{k+1}(I)$ is a characteristic ideal of $N^*_k(L)$ for $k \geq 0$;

(ii) if $I \subseteq N^*_n(L))$ is an ideal of $L$ then $N^*_{n+1}(L)/I \subseteq N^*_n+1(L/I)$.

(iii) if $L = I \oplus J$, then $N^*_k(L) = N^*_k(I) \oplus N^*_k(J)$ for all $k \geq 0$.

**Proof.**

(i) This follows from Theorem 2.1 (i) and Lemma 3.1.

(ii) The case $n = 1$ is given by Proposition 4.5. So suppose that the case $n = k$ holds, where $k \geq 1$, and let $I \subseteq N^*_k(L)$. Then $I \subseteq N^*_{k-1}(L)$.

Hence

$$\frac{N^*_{k+1}(L)}{I} = \frac{N^*(N^*_k(L))}{I} \subseteq N^* \left( \frac{N^*_k(L)}{I} \right) \subseteq N^* \left( \frac{N_k^*(L)}{I} \right) = N^* \left( \frac{L}{I} \right) = N^*_k \left( \frac{L}{I} \right).$$

The result now follows by induction.
(iii) This is a straightforward induction proof: the case \( k = 1 \) is given by Proposition \ref{prop:inductionbasecase}.

\[ \square \]

**Corollary 4.7** Let \( n \in \mathbb{N} \), and let \( I, J \) be ideals of the Lie algebra \( L \) over the field \( F \). Then

(i) if \( N_\infty^*(I) \) is nilregular, it is a characteristic ideal of \( N_\infty^*(L) \);

(ii) if \( I \subseteq N_\infty^*(L) \) is an ideal of \( L \) then \( N_\infty^*(L)/I \subseteq N_\infty^*(L/I) \).

(iii) if \( L = I \oplus J \), then \( N_\infty^*(L) = N_\infty^*(I) \oplus N_\infty^*(J) \).

## 5 The quasi-nilpotent radical

Here we construct a radical by adjoining the quasi-simple ideals of \( L \) to the nilradical \( N \). Since quasi-simple ideals are quasi-minimal they are characteristic in \( L \).

**Lemma 5.1** Let \( L/Z(L) \) be simple. Then \( L = L^2 + Z(L) \) and \( L^2 \) is quasi-simple.

**Proof.** Let \( P = L^2 \) and \( \overline{L} = L/Z(L) \). Then \( \overline{P} \) is an ideal of \( \overline{L} \) and \( \overline{L} \) is simple, so \( \overline{P} = 0 \) or \( \overline{L} \). The former implies that \( \overline{L} \) is abelian, a contradiction. Hence \( \overline{P} = \overline{L} \), and so \( L = P + Z(L) = L^2 + Z(L) \). Also, \( P = L^2 = P^2 \) and \( P/Z(P) = P/P \cap Z(L) \cong (P + Z(L))/Z(L) = L/Z(L) \) is simple. \( \square \)

**Lemma 5.2** Let \( A \) be a quasi-simple ideal of \( L \) and \( B \) an ideal of \( L \). Then either \( A \subseteq B \) or \( A \subseteq C_L(B) \).

**Proof.** Since quasi-simple ideals are quasi-minimal the result follows from Proposition \ref{prop:quasisimplecharacteristic}. \( \square \)

The *quasi-simple components* of \( L \) are its quasi-simple ideals. We will write \( \text{SComp}(L) \) for the set of quasi-simple components of \( L \), and put \( \hat{E}(L) = \langle \text{SComp}(L) \rangle \), the subalgebra generated by the quasi-simple components of \( L \). Clearly \( \text{SComp}(L) \subseteq \text{MComp}(L) \), \( \hat{E}(L) \subseteq E^\dagger(L) \) and \( \hat{E}(L) \) is characteristic in \( L \).

**Lemma 5.3** If \( B \) is an ideal of \( L \), then \( \text{SComp}(B) = \text{SComp}(L) \cap B \).
Proof. If $A$ is a quasi-simple ideal of $B$, it is an ideal of $L$ since it is characteristic in $B$, and so $\text{SComp}(B) \subseteq \text{SComp}(L) \cap B$. The reverse inclusion is clear. □

**Proposition 5.4** Let $P \in \text{SComp}(L)$ and let $B$ be an ideal of $L$. Then $P \in \text{SComp}(B)$ or $[P, B] = 0$.

Proof. Suppose that $[P, B] \neq 0$. We have that $P$ is a quasi-simple ideal of $L$, so $P \subseteq B$, by Lemma 5.2. Hence $P \in \text{SComp}(B)$, by Lemma 5.3 □

**Corollary 5.5** Distinct quasi-simple components of $L$ commute, so

$$\hat{E}(L) = \sum_{P \in \text{SComp}(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in \text{SComp}(L)$.

Proof. This follows easily as in Corollary 3.5. □

**Theorem 5.6** Suppose that $L$ is a Lie algebra in which $E^\dagger(L)$ is regular, then $\hat{E}(L) = E^\dagger(L)$.

Proof. Let $P$ be a quasi-simple ideal of $L$. Then $N(P)$ and $R(P)$ are ideals of $E^\dagger(L)$, by Corollary 5.5. It follows that $P$ is a regular ideal of $L$ and the result follows from Corollary 2.6. □

Clearly, if $L$ is as in Example 2.4 we have $\hat{E}(L) = 0 \neq A = E^\dagger(L)$, so Theorem 5.6 does not hold for all Lie algebras.

**Corollary 5.7** Let $L$ be a Lie algebra in which $E^\dagger(L)$ and $C_L(N)$ are regular. Put $Z = Z(N)$, $\overline{L} = L/Z$, $\overline{S} = \text{Soc}(C_L(N))$. Then $\hat{E}(L) = S^2$ and $S = \hat{E}(L) + Z$.

Proof. This follows from Proposition 3.7 and Theorem 5.6. □

We define the quasi-nilpotent radical of $L$ to be $\hat{N}(L) = N + \hat{E}(L)$. From now on we will denote $\hat{N}(L)$ simply by $\hat{N}$. The following is an immediate consequence of Theorems 3.8 and 5.6.

**Corollary 5.8** Suppose that $L$ is a Lie algebra in which $N^\dagger(L)$ is regular. Then $C_L(\hat{N}) = Z(N)$. In particular $C_L(\hat{N}) \subseteq \hat{N}$. 
Once more, Example 2.4 shows that the above result does not hold without some restrictions. For, if \( L \) is as in that example, then \( \hat{N}(L) = 0 \) and \( C_L(\hat{N}(L)) = L \).

**Proposition 5.9** Let \( L \) be a Lie algebra a field \( F \), and let \( B \) be a nilregular ideal of \( L \). Then \( \hat{N}(B) \subseteq \hat{N} \).

**Proof.** Under the given hypotheses \( N(B) \) is a characteristic ideal of \( B \) (see [7]), so \( N(B) \subseteq N \). Moreover, \( \hat{E}(B) \subseteq \hat{E}(L) \) by Lemma 5.3. □

**Proposition 5.10** Let \( L \) be a Lie algebra over any field. Then \( \hat{N}(\hat{N}) = \hat{N} \).

**Proof.** Clearly \( \hat{N}(\hat{N}) \subseteq \hat{N} \). But \( \hat{E}(\hat{N}) = \hat{E}(L) \), by Lemma 5.3 and, clearly, \( N \subseteq N(\hat{N}) \), giving the reverse inclusion. □

**Proposition 5.11** Let \( L \) be a Lie algebra over any field, and let \( I \) be an ideal of \( L \). Then \( \hat{N}(L) + I \subseteq \hat{N}(L)\).

**Proof.** This follows exactly as in Proposition 3.10. □

### 6 Another generalisation of the nilradical

We put \( \tilde{N}(L)/\phi(L) = \text{Soc}(L/\phi(L)) \). We write \( \tilde{N}(L) \) simply as \( \tilde{N} \). Then we see that this radical also has our desired property.

**Theorem 6.1** Let \( L \) be a Lie algebra over any field, with nilradical \( N \). Then \( C_L(\tilde{N}) \subseteq Z(N) \subseteq \tilde{N} \).

**Proof.** Put \( C = C_L(\tilde{N}) \). Suppose first that \( \phi(L) = 0 \). Then \( L = N + U \) where \( N = \text{Asoc}L \) and \( U \) is a subalgebra of \( L \), by [14] Theorems 7.3 and 7.4. Then \( C = N + C \cap U \) and \( C \cap U \) is an ideal of \( L \). Suppose that \( C \cap U \neq 0 \) and let \( A \) be a minimal ideal of \( L \) with \( A \subseteq C \cap U \). Then \( A \subseteq \tilde{N} \), so \( A^2 \subseteq [\tilde{N}, C] = 0 \). Hence \( A \subseteq N \cap U = 0 \), a contradiction. It follows that \( C = N \).

If \( \phi(L) \neq 0 \) we have

\[
\frac{C + \phi(L)}{\phi(L)} \subseteq C_L/\phi(L) \left( \frac{\tilde{N}}{\phi(L)} \right) \subseteq \frac{N}{\phi(L)}.
\]

Hence \( C \subseteq N \), which yields \( C \subseteq Z(N) \). □
Theorem 6.2 Let $L$ be a $\phi$-free Lie algebra over any field $F$ and suppose that $\tilde{N}(L)$ is nilregular. Then $L/C_L(\tilde{N}(L))$ is isomorphic to a subalgebra of

$$\mathcal{M}_r^- \oplus \left( \bigoplus_{i=1}^s \text{Der}(A_i) \right)$$

where $\mathcal{M}_r$ is the set of $r \times r$ matrices over $F$, $r$ is the dimension of the nilradical, and $A_1, \ldots, A_s$ are the simple minimal ideals of $L$.

Proof. Since $L$ is $\phi$-free we have that $\tilde{N}(L) = N(L) \oplus (\bigoplus_{i=1}^s A_i)$ where $A_1, \ldots, A_s$ are the non-abelian minimal ideals of $L$. Also, each $A_i$ is nilregular and hence simple, by Corollary 2.4. The map $\theta : L \to \text{Der}(\tilde{N}(L))$ given by $\theta(x) = \text{ad} x \mid_{\tilde{N}(L)}$ is a homomorphism with kernel $C_L(\tilde{N}(L))$. But $N(L)$ is characteristic, since it is nilregular, and the $A_i$’s are characteristic, since they are perfect, so

$$\text{Der}(\tilde{N}(L)) = \text{Der}(N(L)) \oplus \left( \bigoplus_{i=1}^s \text{Der}(A_i) \right),$$

whence the result. □

Proposition 6.3 $N^* \subseteq \tilde{N}$.

Proof. There is a subalgebra $U/\phi(L)$ of $L/\phi(L)$ such that $L/\phi(L) = N(\phi(L)) \oplus (\bigoplus_{i=1}^s A_i)$, by [14] Theorems 7.3 and 7.4. Let $A/N$ be a minimal ideal of $L/N$ with $A \subseteq N + C_L(N)$. Then $A = N + A \cap U$, so $[N, A] = [N, N + A \cap U] \subseteq \phi(L)$ and $A \cap U/\phi(L)$ is a minimal ideal of $L/\phi(L)$. Moreover, $N/\phi(L) \subseteq \text{Soc}(L/\phi(L))$, by [14] Theorem 7.4. Hence $A/N \subseteq \text{Soc}(L/\phi(L))$, and so $N^* \subseteq \tilde{N}$. □

In general we can have $N^* \subset \tilde{N}$ and $\tilde{N}(\tilde{N}) \subset \tilde{N}$, as we will show below. Recall that the category $O$ is a mathematical object in the representation theory of semisimple Lie algebras. It is a category whose objects are certain representations of a semisimple Lie algebra and morphisms are homomorphisms of representations. The formal definition and its properties can be found in [5]. As in other artinian module categories, it follows from the existence of enough projectives that each $M \in O$ has a projective cover $\pi : P \to M$. Here $\pi$ is an epimorphism and is essential, meaning that no proper submodule of the projective module $P$ is mapped onto $M$. Up to isomorphism the module $P$ is the unique projective having this property (see [5] page 62]).
**Example 6.1** So let $S$ be a finite-dimensional simple Lie algebra over a field $F$ of prime characteristic, let $P$ be the projective cover for the trivial irreducible $S$-module and let $R$ be the radical of $P$. Then $R$ is a faithful irreducible $S$-module and $P/R$ is the trivial irreducible $S$-module. Let $T = P \rtimes S$ be the semidirect sum of $P$ and $S$. Then $T^2 = R \rtimes S$ is a primitive Lie algebra of type 1 and $\dim(T/T^2) = 1$, say $T = T^2 + Fx$. Put $L = T + Fy$ where $[x, y] = y$ and $[T^2, y] = 0$.

Then $\phi(T) \subseteq T^2$, so $\phi(T)$ is an ideal of $L$ and $\phi(T) \subseteq \phi(L)$, by $[14]$, Lemma 4.1. But $\phi(L) \subseteq T$ and, if $M$ is a maximal subalgebra of $T$ then $M + Fy$ is a maximal subalgebra of $L$, so $\phi(L) = \phi(T) = R$. Also $\text{Soc}(L/R) = (T^2 + Fy)/R$, so $\tilde{N}(L) = T^2 \oplus Fy$. However, $N(L) = R \oplus Fy$ and $C_L(N(L)) = N(L)$, so $N^*(L) = N(L) \neq \tilde{N}(L)$.

Moreover, $\phi(\tilde{N}(L)) = 0$, so $\tilde{N}(\tilde{N}(L)) = \text{Soc}(\tilde{N}(L)) = R \oplus Fy \neq \tilde{N}(L)$. Notice that we also have $N^*(L)/\phi(L) = N(L)/R \cong Fy$, whereas $N^*(L/\phi(L)) = T^2 + Fy/R$. Hence the inclusions in Propositions 3.10, 4.5, 4.6 and Corollary 4.7 can be strict.

Note that a similar example can be constructed in characteristic $p$. Let $L$ be a finite-dimensional restricted Lie algebra over a field $F$ of prime characteristic, and let $u(L)$ denote the restricted universal enveloping algebra of $L$. Then every restricted $L$-module is a $u(L)$-module and vice versa, and so there is a bijection between the irreducible restricted $L$-modules and the irreducible $u(L)$-modules. In particular, as $u(L)$ is finite-dimensional, every irreducible restricted $L$-module is finite-dimensional. So, in the above example we could take $S$ to be a restricted simple Lie algebra, as the projective cover of the trivial $S$-module again exists.

**Proposition 6.4** If $I$ is an ideal of $L$ then

$$\frac{\tilde{N} + I}{I} \subseteq \tilde{N} \left( \frac{L}{T} \right).$$

Moreover, if $I \subseteq \phi(L)$, then $\tilde{N}(L)/I = \tilde{N}(L/I)$.

**Proof.** Let $A/\phi(L)$ be a minimal ideal of $L/\phi(L)$. Then

$$\frac{A + I/I}{\phi(L) + I/I} \cong \frac{A + I}{\phi(L) + I} \cong \frac{A}{A \cap (\phi(L) + I)}.$$  

Now $\phi(L) \subseteq A \cap (\phi(L) + I)$, so $A \cap (\phi(L) + I) = A$ or $\phi(L)$. But $\phi(L) + I/I \subseteq \phi(L/I)$, so the former implies that $A + I/I = \phi(L/I)$ and $A + I/I \subseteq \tilde{N}(L/I)$.
If the latter holds then \( A \cap I \subseteq \phi(L) \). But now, \( \phi(L)/A \cap I = \phi(L/A \cap I) \), by \([14\text{, Proposition 4.3}]\), so

\[
\frac{A/A \cap I}{\phi(L)/A \cap I} = \frac{A/A \cap I}{\phi(L)/A \cap I} \cong \frac{A}{\phi(L)}.
\]

It follows that \( A/A \cap I \subseteq \tilde{N}(L/A \cap I) \), whence \( A + I/I \subseteq \tilde{N}(L/I) \).

The second assertion follows from the definition of \( \tilde{N} \) and the fact that \( \phi(L/I) = \phi(L)/I \). \( \square \)

**Proposition 6.5** \( \tilde{N}(L)/\phi(L) = N^*(L/\phi(L)) \).

**Proof.** Suppose first that \( \phi(L) = 0 \). Then \( \tilde{N}(L) \) is the socle of \( L \). Now \( N(L) = \text{Asoc}(L) \), by \([14\text{, Theorem 7.4}]\). Also, if \( A \) is a minimal ideal of \( L \) with \( A \not\subseteq N(L) = N \), then \([A,N] \subseteq A \cap N = 0 \), so \( A \subseteq C_L(N) \). Hence \( \tilde{N}(L) \subseteq N^*(L) \).

If \( \phi(L) \neq 0 \) the above shows that \( \tilde{N}(L/\phi(L)) \subseteq N^*(L/\phi(L)) \). The result now follows from Propositions \([6.3\text{ and }6.4]\). \( \square \)

**Proposition 6.6** If \( L = I \oplus J \), then \( \tilde{N}(L) = \tilde{N}(I) \oplus \tilde{N}(J) \).

**Proof.** We have that \( N(L) = N(I) \oplus N(J) \) and \( \phi(L) = \phi(I) \oplus \phi(J) \) by \([14\text{, Theorem 4.8}]\). Let \( A/\phi(L) \) be a minimal ideal of \( L/\phi(L) \) and suppose that \( A \not\subseteq N(L) \). Then \( A = A^2 + \phi(L) \). But \( \phi(L) = \phi(I) \oplus \phi(J) \), by \([14\text{, Theorem 4.8}]\), so

\[
A = A^2 + \phi(I) + \phi(J) = [A,I] + \phi(I) + [A,J] + \phi(J).
\]

Hence

\[
\frac{A}{\phi(L)} \cong \frac{[A,I] + \phi(I)}{\phi(I)} \oplus \frac{[A,J] + \phi(J)}{\phi(J)}.
\]

It is easy to see that the direct summands are minimal ideals of \( I/\phi(I) \) and \( J/\phi(J) \) respectively, so \( \tilde{N}(L) \subseteq \tilde{N}(I) \oplus \tilde{N}(J) \). Also, if \( A/\phi(I) \) is a minimal ideal of \( I/\phi(I) \), then \( A + \phi(J)/\phi(L) \) is a minimal ideal of \( L/\phi(L) \), so \( \tilde{N}(I) \subseteq \tilde{N}(L) \). Similarly \( \tilde{N}(J) \subseteq \tilde{N}(L) \), which gives the result. \( \square \)

As a result of Example \([6.1]\) we define, for each non-negative integer \( n \), \( \tilde{N}_n(L) \) inductively by

\[
\tilde{N}_0(L) = L \text{ and } \tilde{N}_n(L) = \tilde{N}(\tilde{N}_{n-1}(L)) \text{ for } n > 0.
\]

Clearly the series

\[
L = \tilde{N}_0(L) \supseteq \tilde{N}_1(L) \supseteq \ldots
\]

will terminate in an equality, so we put \( \tilde{N}_\infty(L) \) equal to the minimal subalgebra in this series. It is easy to see that \( \tilde{N}_\infty(\tilde{N}_\infty(L)) = \tilde{N}_\infty(L) \).

\[21\]
Proposition 6.7 Let \( n \in \mathbb{N} \cup \{0\} \), and let \( I, J \) be ideals of the Lie algebra \( L \) over a field \( F \).

(i) If \( I \subseteq \phi(\tilde{N}_{n-1}(L)) \) then \( \tilde{N}_n(L/I) = \tilde{N}_n(L)/I \).

(ii) \( N(\tilde{N}_n(L)) \subseteq N(\tilde{N}_{n+1}(L)) \) for each \( n \geq 0 \).

(iii) If \( \tilde{N}_\infty(L) \) is nilregular, then \( \phi(\tilde{N}_{n+1}(L)) \subseteq \phi(\tilde{N}_n(L)) \) for each \( n \geq 0 \).

(iv) If \( \tilde{N}_\infty(L) \) is nilregular then \( N(\tilde{N}_n(L)) = N(L) \) and \( \tilde{N}_n(L) \) is an ideal of \( L \) for all \( n \geq 0 \).

(v) If \( N^*(L) \) is nilregular then \( N^*(L) \subseteq \tilde{N}_n(L) \) for each \( n \geq 0 \).

(vi) If \( \tilde{N}_n(L) \) is nilregular and \( \phi(\tilde{N}_n(L)) = 0 \) then \( \tilde{N}_{n+1}(L) = N^*(L) \).

(vii) If \( N^*(L) \) is nilregular then \( C_L(\tilde{N}_n(L)) = Z(N(L)) \).

(viii) If \( F \) has characteristic zero, then \( \tilde{N}_n(I) \subseteq \tilde{N}_n(L) \).

(ix) If \( F \) has characteristic zero, then \( \tilde{N}_n(L) + I/I \subseteq \tilde{N}_n(L/I) \).

(x) If \( L = I \oplus J \) then \( \tilde{N}_n(L) = \tilde{N}_n(I) \oplus \tilde{N}_n(J) \).

Proof.

(i) The case \( n = 1 \) is given by Proposition 6.4 A straightforward induction argument then yields the general case.

(ii) We have that \( N(L) \subseteq \tilde{N}(L) \), by [14] Theorem 7.4, whence \( N(L) \subseteq N(\tilde{N}(L)) \). Thus \( N(N(L)) \subseteq N(\tilde{N}_2(L)) \), and a simple induction argument gives the general result.

(iii) Put \( \tilde{N}_i = \tilde{N}_i(L) \). Then

\[
\frac{\tilde{N}_{n+1}}{\phi(\tilde{N}_n)} = \bigoplus_{i=1}^r A_i \frac{A_i}{\phi(\tilde{N}_n)}.
\]

where each direct summand is a minimal ideal of \( \tilde{N}_n/\phi(\tilde{N}_n) \). Now

\[
N \left( \frac{A_i}{\phi(\tilde{N}_n)} \right) \subseteq N \left( \frac{\tilde{N}_n}{\phi(\tilde{N}_n)} \right) = \frac{N(\tilde{N}_n)}{\phi(\tilde{N}_n)}
\]

and \( N(\tilde{N}_n) \subseteq N(\tilde{N}_\infty) \) by (ii), so the direct summands are nilregular, and hence are abelian or simple, by Corollary 2.4. It follows that they are \( \phi \)-free, and thus, so is \( \tilde{N}_{n+1}/\phi(\tilde{N}_n) \). The result follows.

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(iv) Consider the first assertion: it clearly holds for \( n = 0 \). Suppose that \( \tilde{N}_\infty(L) \) is nilregular and that the result holds for \( k \leq n \) (\( n \geq 0 \)). Then \( \tilde{N}_k(L) \) is nilregular for all \( k \geq 0 \), by (iii). It follows from [9, Corollary 1] that \( N(\tilde{N}_n(L)) \) is a characteristic ideal of \( \tilde{N}_n(L) \), and hence an ideal of \( \tilde{N}_{n-1}(L) \). Thus \( N(\tilde{N}_n(L)) = N(\tilde{N}_{n-1}(L)) \), and so \( N(\tilde{N}_n(L)) = N(L) \) by the inductive hypothesis, which proves the first assertion.

Put \( \tilde{N}_n = \tilde{N}_n(L) \), \( \phi_n = \phi(\tilde{N}_n) \) and let \( A/\phi_n \) be a minimal ideal of \( \tilde{N}_n/\phi_n \). If \( A \not\subseteq N(\tilde{N}_n) \), then \( A/\phi_n \) is a perfect subideal of \( L/\phi_n \) and so an ideal of \( L/\phi_n \), by Lemma 1.1. The result follows.

(v) The case \( n = 1 \) is Proposition 6.3. So suppose that \( N^*(L) \subseteq \tilde{N}_k(L) \) for some \( k \geq 1 \). Then

\[
N^*(L) = N^*(N^*(L)) \subseteq N^*(\tilde{N}_k(L)) \subseteq \tilde{N}_{k+1}(L),
\]

by Propositions 3.9, 3.12 and 6.3.

(vi) If \( \phi(\tilde{N}_n(L)) = 0 \) then

\[
\tilde{N}_{n+1}(L) \subseteq N^*(\tilde{N}_n(L)) \subseteq N^*(L) \subseteq \tilde{N}_{n+1}(L),
\]

since \( \tilde{N}_n(L) \) is nilregular (and hence so is \( N^*(L) \)), by Propositions 6.5, 3.12 and (v) above.

(vii) Using (v) above we have that \( C_L(\tilde{N}_n(L)) \subseteq C_L(N^*(L)) = Z(N) \), by Theorem 3.8.

(viii) We have \( \phi(I) \subseteq \phi(L) \), by [14, Corollary 3.3], so \( \tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I) \). Now

\[
\tilde{N}(I)/\phi(I) = N^*(I)/\phi(I) \subseteq N^*(L/\phi(I)) \subseteq \tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I),
\]

by Propositions 6.3, 3.12 and 6.3. Hence \( \tilde{N}(I) \subseteq \tilde{N}(L) \). Then a simple induction proof shows that \( \tilde{N}_n(I) \subseteq \tilde{N}_n(L) \).

(ix) The case \( n = 1 \) is given by Proposition 6.4. Suppose it holds for some \( k \geq 1 \). Then

\[
\frac{\tilde{N}_{k+1}(L) + I}{I} = \frac{\tilde{N}(\tilde{N}_k(L)) + I}{I} \subseteq \frac{\tilde{N}(\tilde{N}_k(L) + I) + I}{I}
\]

\[
\subseteq \tilde{N} \left( \frac{\tilde{N}_k(L) + I}{I} \right) \subseteq \tilde{N} \left( \frac{\tilde{N}_k \left( \frac{L}{I} \right)}{I} \right) = \tilde{N}_{k+1} \left( \frac{L}{I} \right),
\]

by (viii) and Proposition 6.4.
(x) The case \( n = 1 \) is given by Proposition 6.6. A straightforward induction argument then gives the general result.

□

**Corollary 6.8** Let \( I, J \) be ideals of \( L \).

(i) If \( I \subseteq \phi(\tilde{N}_\infty(L)) \) then \( \tilde{N}_\infty(L/I) = \tilde{N}_\infty(L)/I \).

(ii) If \( \tilde{N}_\infty(L) \) is nilregular the \( N(\tilde{N}_\infty(L)) = N(L) \) and \( \tilde{N}_\infty(L) \) is an ideal of \( L \).

(iii) If \( N^*(L) \) is nilregular then \( N^*(L) \subseteq \tilde{N}_\infty(L) \).

(iv) If \( \tilde{N}_\infty(L) \) is nilregular and \( \phi(\tilde{N}_\infty(L)) = 0 \) then \( \tilde{N}_\infty(L) = N^*(L) \).

(v) If \( N^*(L) \) is nilregular then \( C_L(\tilde{N}_\infty(L)) = Z(N(L)) \).

(vi) If \( F \) has characteristic zero, then \( \tilde{N}_\infty(I) \subseteq \tilde{N}_\infty(L) \).

(vii) If \( F \) has characteristic zero, then \( \tilde{N}_\infty(L) + I/I \subseteq \tilde{N}_\infty(L/I) \);

(viii) If \( L = I \oplus J \) then \( \tilde{N}_\infty(L) = \tilde{N}_\infty(I) \oplus \tilde{N}_\infty(J) \).

If \( S \) is a subalgebra of \( L \) the core of \( S \), \( S_L \), is the biggest ideal of \( L \) contained in \( S \). The following is an analogue of a result for groups given by Vasil’ev et al. in [17].

**Theorem 6.9** Let \( L \) be a Lie algebra over any field. Then the core of the intersection of all maximal subalgebras such that \( L = M + \tilde{N}(L) \) is equal to \( \phi(L) \).

**Proof.** Put \( P \) equal to the intersection of all maximal subalgebras such that \( L = M + \tilde{N} \). Clearly \( \tilde{N} \nsubseteq \phi(L) \) and \( \phi(L) \subseteq P_L \). Factor out \( \phi(L) \) and suppose that \( P_L \neq 0 \). Let \( A \) be a minimal ideal of \( L \) contained in \( P_L \). Then \( A \subseteq \tilde{N}(L) \).

Since \( \phi(L) = 0 \) there is a maximal subalgebra of \( L \) such that \( A \nsubseteq M \). If \( L = \tilde{N}(L) + M \) we have \( A \subseteq P_L \subseteq M \), a contradiction. If not, then \( A \subseteq \tilde{N}(L) \subseteq M \), a contradiction again. Hence \( P_L = 0 \).

It follows that \( P_L \subseteq \phi(L) \), whence the result.

□
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