Frozen deconfined quantum criticality

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There is a number of contradictory findings with regard to whether the theory describing easy-plane quantum antiferromagnets undergoes a second-order phase transition. The traditional Landau-Ginzburg-Wilson approach suggests a first-order phase transition, as there are two different competing order parameters. On the other hand, it is known that the theory has the property of self-duality which has been connected to the existence of a deconfined quantum critical point (DQCP). The latter regime suggests that order parameters are not the elementary building blocks of the theory, but rather consist of fractionalized particles that are confined in both phases of the transition and only appear — deconfine — at the critical point. Nevertheless, many numerical Monte Carlo simulations disagree with the claim of a DQCP in the system, indicating instead a first-order phase transition. Here we establish from exact lattice duality transformations and renormalization group analysis that the easy-plane CP1 antiferromagnet does feature a DQCP. We uncover the criticality starting from a regime analogous to the zero temperature limit of a certain classical statistical mechanics system which we therefore dub “frozen”. At criticality our bosonic theory is dual to a fermionic one with two massless Dirac fermions, which thus undergoes a second-order phase transition as well.

Quantum antiferromagnets that possess a global SU(2) symmetry and an emergent U(1) gauge symmetry can give rise to exotic phases of matter, like spin liquids and valence-bond solid states [1, 2]. An interesting scenario occurs when at low temperatures the system features a quantum critical point at a value gc of some effective coupling constant. For instance, such a quantum critical point can separate the magnetically ordered (Néel) state, which breaks the SU(2) symmetry, from a dimerized paramagnetic state breaking the lattice symmetries. The latter finds a paradigmatic realization in the valence-bond solid (VBS) phase [2, 3]. An effective field theory formulation with an emergent U(1) gauge symmetry is achieved in this context by rewriting the unit vector field n representing the direction of the magnetization in terms of a doublet of complex fields, za (a = 1, 2), n = za σ ab za, where |z1|^2 + |z2|^2 = 1 and σ is a vector of Pauli matrices and summation over repeated indices is implied. A global O(3) symmetry becomes henceforth a global O(4) one under this mapping. It also makes the U(1) symmetry manifest, since n is invariant under the local gauge transformation za(x) → e^iθ za(x). This map, which is also referred to as a CP1 representation, leads to a lattice gauge theory of quantum antiferromagnets. Under some very precise circumstances the magnetization falls apart in such a theory, liberating more elementary modes — spinon fields za. This regime leads to a special type of universal behavior, governed by the so called deconfined quantum criticality (DQC) [4, 5].

A salient property of the DQC paradigm is that it describes a universality class that cannot be derived from a Landau-Ginzburg-Wilson type of approach. In the latter a transition between phases breaking different symmetries (competing orders), be it at zero or finite temperature, is a first-order one, which implies that the theory should not feature a critical point. Within the DQC theory, on the other hand, a second-order phase transition is predicted to occur. Furthermore, DQC predicts a large anomalous dimension ηA for the correlator G(x) = ⟨n(x) · n(0)⟩ at the critical point [4, 5], this result being a consequence of the composite field character of n underlying the CP1 representation. This prediction has been confirmed by multiple computer simulations on a number of specific lattice models proposed to describe DQC [6–9].

A paradigmatic model argued to exhibit DQC features an easy-plane anisotropy that reduces the global O(4) symmetry to an Abelian one, namely, U(1) × U(1) [4, 5]. The question of whether the anisotropic quantum antiferromagnets feature a deconfined critical point has been open since the creation of the field [9–21]. This model has the advantage of being analytically tractable to a certain extent and has been demonstrated to exhibit a self-dual regime [22]. Interestingly, from the symmetry point of view, this model can also describe the phase transition in two-component superconductors [16]. Early computer simulations [13, 14] have failed to find evidence of a second-order phase transition in this case. Nevertheless, it has been recently suggested in the context of bosonization dualities that DQC can be achieved in this self-dual model [18]. Also, more recently, other numerical works [9, 17] on easy-plane systems concluded that a second-order phase transition occurs. However, the controversy persists, as a recent numerical work [19] for the easy-plane J − Q model favors a first-order phase transition.

Here we demonstrate by purely analytical means that the easy-plane model in CP1 representation features...
DQC in quite a specific regime. The key observation is that by considering the lattice theory as a classical statistical mechanics model, we identify the coupling constant $g$ as playing the role of temperature in the action. From considering the “frozen” $g \to 0$ regime, where we show that a quantum critical point exists, we construct a duality that allows us to derive DQC in the more general case ($g \neq 0$). The standard duality transformation in the spirit of Refs. [23–25] is performed in the frozen regime leads to a $U(1) \times U(1)$-symmetric Higgs theory featuring two gauge fields and no Maxwell terms. To access the $g \neq 0$ case, however, we dualize only a single $U(1)$ sector, obtaining the same model but with Maxwell terms and different gauge couplings. A subsequent renormalization group (RG) analysis then establishes the existence of a quantum critical point. Importantly, the obtained critical regime is the same as in the frozen limit. We demonstrate that at criticality the theory is topologically ordered and is dual to a theory of two massless Dirac fermions in the infrared (IR) limit. From the derived bosonization duality we conclude that also the fermionic theory possesses a quantum critical regime.

The natural starting point to investigate DQC lies in quantum antiferromagnetic systems, whose behavior can be described in terms of two spinon fields $z_I, I = 1, 2$, via the CP$^1$ representation of the nonlinear $\sigma$ model [26],

$$S = -\frac{1}{g} \sum_{(ij),i} (z^*_I z_J e^{i\theta_{ij}} + c.c.) \sum_j (\epsilon_{\mu\nu\lambda} \Delta_\mu a_{\nu j} a_{\lambda j})^2,$$

with a local constraint $|z_1|^2 + |z_2|^2 = 1$. The first term in $S$ describes nearest-neighbor hopping of the CP$^1$ fields on a square lattice and $a_{\nu j}$ is an emergent gauge field. The second term represents a Maxwell Lagrangian in the lattice. The so-called deep easy-plane limit fixes the spinon amplitudes to be equal, which leads to $|z_1|^2 = |z_2|^2 = 1/2$ due to the initial constraint. In this way the action takes the form,

$$S = -\frac{1}{2g} \sum_{(ij),i} \cos (\theta_{ij} - \theta_I - a_{ij}) + \frac{1}{2e^2} \sum_j (\epsilon_{\mu\nu\lambda} \Delta_\mu a_{\nu j} a_{\lambda j})^2,$$

where $\theta_{ij}$ arises from the polar representation of the spinons $z_{ij} = \rho_I e^{i\theta_{ij}}$ with $\rho_I = 1/\sqrt{2}$. We recognize the theory as a gauged version of a two-component \textit{XY} model. When interpreting this Euclidean action as a classical statistical physics Hamiltonian, the coupling constant $g$ plays a role analogous to the temperature. The action (2) can be well approximated in the form of a $U(1) \times U(1)$ lattice Villain system [22],

$$S = \frac{1}{2} \sum_j \left[ \frac{1}{g} \sum_{I=1,2} (\Delta_\mu \theta_{Ij} - 2\pi n_{1j\mu} - a_{j\mu})^2 + \frac{1}{e^2} (\epsilon_{\mu\nu\lambda} \Delta_\nu a_{\lambda j})^2 \right],$$

where $\Delta_\mu$ is the lattice derivative, $n_{1j\mu}$ are integer-valued lattice fields and $\theta_{Ij} \in [-\pi, \pi]$. The action (3) has besides the usual gauge invariance $\theta_{Ij} \to \theta_{Ij} + \alpha_j$, $a_{j\mu} \to a_{j\mu} + \Delta_\mu \alpha_j$, two $\mathbb{Z}$ gauge symmetries, $n_{1j\mu} \to n_{1j\mu} + \Delta_\mu K_{Ij}$, $\theta_{Ij} \to \theta_{Ij} + 2\pi K_{Ij}$, for integers $K_{Ij}$.

In order to obtain the dual theory, we use the Poisson summation formula [23, 27] to introduce a new integer-valued field,

$$\sum_{\{n_{1j\mu}\}} e^{-\frac{1}{2g} (\Delta_\mu \theta_{Ij} - 2\pi n_{1j\mu} - a_{j\mu})^2} \sim \sum_{\{N_{1j\mu}\}} e^{\frac{1}{2} N_{1j\mu}^2 + i N_{1j\mu} (\Delta_\mu \theta_{Ij} - a_{j\mu})},$$

which allows us to integrate out $\theta_{Ij}$ to obtain the constraints $\Delta_\mu N_{1j\mu} = 0$. Thus, after the constraints are solved by $N_{1j\mu} = \epsilon_{\mu\nu\lambda} \Delta_\nu M_{Ij\lambda}$ and $a_{j\mu}$ is integrated out, we obtain the dual action in the form [5, 28],

$$S = \sum_j \left\{ \sum_{I=1,2} \left[ \frac{g}{2} (\epsilon_{\mu\nu\lambda} \Delta_\mu b_{Ij\lambda})^2 - 2\pi i m_{1j\mu} b_{Ij\mu} \right] + \frac{e^2}{2} (b_{j\mu} + b_{j\nu})^2 \right\},$$

where we have used the Poisson formula once more to promote the integer-valued fields $M_{Ij\lambda}$ to real-valued fields $b_{j\mu}$ and the constraints $\Delta_\mu m_{1j\mu} = 0$ hold. Physically the fields $m_{1j\mu}$ represent vortices and the zero divergence constraints imply that all vortex lines form loops [29].

We will consider now the “zero temperature” limit ($g \to 0$) of the obtained dual model. This causes the Maxwell terms to vanish. After integrating out the gauge field $b_{j\mu}$ and solving the constraint $\Delta_\mu m_{2j\mu} = 0$ via the integral representation of the Kronecker $\delta$, the following action is obtained through the Poisson summation,

$$\tilde{S} = \sum_j \left\{ \frac{e^2}{8\pi^2} (\Delta_\mu \tilde{\theta}_j - 2\pi \tilde{n}_{j\mu} - 2\pi b_{j\mu})^2 - 2\pi i m_{1j\mu} b_{j\mu} \right\},$$

where the Poisson summation formula was applied to promote the integer-valued field to be real-valued.

By performing a shift (“Higgsing”) $b_{j\mu} \to b_{j\mu} + (\Delta_\mu \tilde{\theta}_j - 2\pi \tilde{n}_{j\mu})/(2\pi)$, the action of Eq. (6) becomes simply

$$\tilde{S} = \sum_j \left( e^2 (\tilde{b}_{j\mu})^2 - 2\pi i m_{1j\mu} \tilde{n}_{j\mu} \right),$$

since the zero divergence constraint on $m_{1j\mu}$ makes $\tilde{\theta}_j$ disappear and the term $2\pi i m_{1j\mu} \tilde{n}_{j\mu}$ does not contribute as its exponential yields the unity. After integrating $b_{j\mu}$ out a vortex loop gas representation of the \textit{XY} model is obtained in a way akin to the one considered in Refs. [23, 24]. In this case $e^2/(2\pi)^2$ plays the role of the inverse temperature. Therefore, the frozen regime $g \to 0$ has a quantum critical point in the inverted \textit{XY} universality class [24].

Let us now show that the quantum critical regime associated to the frozen limit exhibits topological order. This
is in contrast to the standard inverted XY universality class, where no such an order arises. To demonstrate the topological order underlying the action of Eq. (6), we first solve the zero divergence constraint on $m_{1j\mu}$ via $m_{1j\mu} = \epsilon_{\mu\nu\lambda}\Delta_\nu \tilde{N}_{j\lambda}$, and promote the field $\tilde{N}_{j\mu}$ to be real-valued via the Poisson formula, to obtain,
\[
\tilde{S}' = \sum_j \left\{ \frac{e^2}{8\pi^2} \left( \Delta_\mu \tilde{\theta}_j - 2\pi \tilde{n}_{j\mu} - 2\pi b_{1j\mu} \right)^2 
\right. \\
\left. - 2\pi ib_{1j\mu} \epsilon_{\mu\nu\lambda} \Delta_\nu b_{j\lambda} \right\}.
\]
(7)
The field equation for $b_{1j\mu}$ expresses the fact that the Noether current $J_{j\mu} = (e^2/4\pi^2)(\Delta_\mu \tilde{\theta}_j - 2\pi \tilde{n}_{j\mu} - 2\pi b_{1j\mu})$ is topological, since $J_{j\mu} = 2\pi i \epsilon_{\mu\nu\lambda} \Delta_\nu b_{j\lambda}$.

We now add a term $\varepsilon \tilde{m}_{j\mu}^2 / 2$, where $\varepsilon$ represents the vortex core energy [29], which can also be viewed as a chemical potential for the vortex loops [23–25, 30]. Introducing a phase field $\tilde{\varphi}_j$ from the integral representation of the Kronecker delta constraint on $\tilde{m}_{j\mu}$, we arrive at the following action,
\[
\tilde{S}' = \sum_j \left\{ \frac{e^2}{8\pi^2} \left( \Delta_\mu \tilde{\theta}_j - 2\pi \tilde{n}_{j\mu} - b_{1j\mu} \right)^2 
\right. \\
\left. + \frac{1}{2\varepsilon} (\Delta_\mu \tilde{\varphi}_j - 2\pi \tilde{n}_{j\mu} - b_{2j\mu})^2 
\right. \\
\left. - \frac{i}{2\pi} b_{1j\mu} \epsilon_{\mu\nu\lambda} \Delta_\nu b_{2j\lambda} \right\},
\]
(8)
where a rescaling, $b_{1j\mu} \rightarrow b_{1j\mu}/(2\pi)$ has been made. This formulation of the dual action in the frozen regime has a number of interesting features. First, we note that thanks to the so called BF term [the last term in Eq. (8)], both Noether currents are topological in view of the field equations for both $b_{1j\mu}$ and $b_{2j\mu}$. Second and also in view of the property just mentioned, the dual action (8) can be regarded as a theory for topologically ordered superconductors in 2+1 dimensions [31]. In this interpretation, one of the Noether currents is associated to the quasi-particle currents while the other one describes the vortex current. We therefore conclude that such a topologically ordered system undergoes a second-order phase transition governed by the inverted XY universality class.

So far, using the exact duality transformations in the frozen limit, we showed the existence of an XY critical point and demonstrated its topological nature. Grounded in these findings, we will now expand the criticality claim to the case where the coupling $g$ is finite. In order to do so, we develop a new strategy where only one $U(1)$ sector of the easy-plane CP$^1$ model is dualized. This approach is motivated by the intuition we developed considering the frozen dual model. Indeed, we note that the frozen limit causes the dual model to have one vortex loop field suppressed, as is seen from Eq. (6). The procedure will allow us to demonstrate the existence of a quantum critical point starting from a finite $g$.

Returning to the easy-plane CP$^1$ model of Eq. (3), we repeat the step discussed in Eq. (4) but only for one phase variable (we choose $\theta_{2j}$). This leads to
\[
\tilde{S}'' = \sum_j \left[ \frac{1}{2g} (\Delta_\mu \theta_{1j} - 2\pi n_{1j\mu} - a_{j\mu})^2 
\right. \\
\left. + \frac{1}{2\varepsilon^2} (\epsilon_{\mu\nu\lambda} \Delta_\nu a_{j\lambda})^2 + ia_{j\mu} \epsilon_{\mu\nu\lambda} \Delta_\nu b_{j\lambda} 
\right. \\
\left. + \frac{g}{2} (\epsilon_{\mu\nu\lambda} \Delta_\nu b_{j\lambda})^2 - 2\pi ib_{j\mu} m_{j\mu} \right],
\]
(9)
where $b_{j\mu}$ is a new gauge field and $m_{j\mu}$ is a lattice vortex loop field.

Similarly to our previous calculations, the constraint $\Delta_\mu m_{j\mu} = 0$ allows us to introduce a new phase field $\varphi_{j}$. Adding the vortex core energy and using the Poisson summation formula, we arrive at the following action,
\[
\tilde{S}'' = \sum_j \left[ \frac{1}{2g} (\Delta_\mu \theta_{1j} - 2\pi n_{j\mu} - a_{j\mu})^2 
\right. \\
\left. + \frac{1}{2\varepsilon} (\Delta_\mu \varphi_{j} - 2\pi n_{j\mu} - b_{j\mu})^2 
\right. \\
\left. + \frac{g}{8\pi^2} (\epsilon_{\mu\nu\lambda} \Delta_\nu b_{j\lambda})^2 + \frac{1}{2\varepsilon^2} (\epsilon_{\mu\nu\lambda} \Delta_\nu a_{j\lambda})^2 
\right. \\
\left. + \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} a_{j\mu} \Delta_\nu b_{j\lambda} \right],
\]
(10)
where the new gauge field was rescaled, $b_{j\mu} \rightarrow b_{j\mu}/(2\pi)$. The model above is reminiscent of the one obtained in the frozen regime in Eq. (8) as both actions contain two Higgs terms and two gauge fields coupled via a topological BF term. The crucial difference, however, lies in the fact that in Eq. (10) the Maxwell terms for both $a_{j\mu}$ and $b_{j\mu}$ are present. Hence, these models actually represent different physical pictures, as we will discuss in more detail below.

Let us first put into perspective the physical significance of the action (10) and recapitulate what we have achieved so far. We started with the $U(1) \times U(1)$ gauge theory of Eq. (3) (or any of its equivalent forms), and derive the exact dual action seen in Eq. (5) which features two (dual) gauge fields. Then we show that in the frozen limit the dual action can be cast in the form (8) with the gauge fields coupled via a BF term. This theory describes an ensemble of two types of vortex loops having the same gauge charge, as the particle-vortex duality has been performed in both $U(1)$ sectors. By contrast, Eq. (10) results from performing the particle-vortex duality in only one $U(1)$ sector. This naturally implies that the gauge charge of the particles in one $U(1)$ sector is attached to the flux resulting from particle-vortex duality in the other $U(1)$ sector. Indeed, now the gauge coupling of the dualized $U(1)$ sector corresponds to the phase stiffness of the original particles, while the $U(1)$ sector that has not been dualized still retains its original "electric" charge $e$. In this sense, the action of Eq. (10) represents rather an electric-magnetic duality in 2+1 dimen-
sions. Note that due to the presence of Maxwell terms, Noether currents are no longer topological, in contrast to the frozen regime. This causes the gauge potentials to be gapped, similarly to the situation of (2+1)-dimensional superconductors where the topological (BF) action has to be supplemented with Maxwell terms in order to account for the plasmon modes [31]. Here the mutually dual Maxwell terms appear quite naturally as a consequence of the duality transformation.

From Eq. (10) we infer the continuum field theory Lagrangian in imaginary time,

$$\tilde{L}'' = [(\partial_{\mu} - i e \sigma_{\mu}) \phi_1]^2 + m^2 |\phi_1|^2 + \frac{u_1}{2} |\phi_1|^4$$

$$+ [(\partial_{\mu} - i \frac{2\pi}{\sqrt{g}} b_\mu) \phi_2]^2 + m^2 |\phi_2|^2 + \frac{u_2}{2} |\phi_2|^4$$

$$+ \frac{1}{2} (\epsilon_{\mu\nu\lambda} \partial_{\mu} a_\lambda)^2 + \frac{1}{2} (\epsilon_{\mu\nu\lambda} \partial_{\nu} b_\lambda)^2$$

$$+ \frac{ie}{\sqrt{g}} a_\mu \epsilon_{\mu\nu\lambda} \partial_{\nu} b_\lambda,$$

where the gauge fields were rescaled as $a_\mu \rightarrow ea_\mu$ and $b_\mu \rightarrow \frac{2\pi}{\sqrt{g}} b_\mu$. This two-component Lagrangian features two different charges: $e$ from the original model and $\tilde{e} = \frac{2\pi}{\sqrt{g}}$ obtained for the dual $U(1)$ sector.

The RG analysis performed on the Lagrangian (21) yields the interesting result that the theory features a critical regime which belongs to the same universality class as the frozen model with two dual $U(1)$ sectors (full details of calculations can be found in the Supplemental Material [32]). We define the renormalized dimensionless couplings of the $|\phi|^4$-interactions as $
abla I = u_{I1} m^{-\epsilon} r$ for $I = 1, 2$, where $m_R$ is the renormalized mass. Here we have also introduced $\epsilon = 4 - d$ for a spacetime dimension $2 < d < 4$ in order to obtain a perturbative fixed point of $O(\epsilon)$. The one-loop $\beta$ functions for both $\nabla I$ have the same form and are given by $\beta_{\nabla I} = -\epsilon \nabla I + 5 \nabla I^2 / (8\pi)$. The IR stable fixed points, $\nabla I^* = (8\pi) e / 5$, obtained from the RG equations are consistent with the XY universality class. Let us mention that an RG analysis of the Dasgupta-Halperin dual model also implies an XY fixed point [33]. In both cases, this occurs due to a gapped gauge field. However, the mechanism by which the gauge fields of Eq. (10) become massive is quite different from the one described in Ref. [33]. In fact, the theory above is gauge invariant and the gap follows from the presence of a topological BF term.

The critical point is reached when both dimensionless counterparts of the renormalized couplings $\tilde{e}^2_R$ and $\tilde{e}^2_R$ flow to their fixed points as well as $\nabla I \rightarrow (8\pi) e / 5$. The $\beta$ functions for the gauge couplings calculated from the one-loop vacuum polarization have the same general form, $\beta_f = -\epsilon f + f^2 / (24\pi)$, where $f^2$ is a dimensionless renormalized coupling corresponding to either $f = e^2 R m^{-\epsilon} R$ or $f = \tilde{e}^2 R m^{-\epsilon} R$. From the $\beta$ functions it is straightforward to find the IR stable fixed points, $\tilde{e}^2 = 24\pi e$ and $1/\tilde{g} = \pi e / 6$, where $\tilde{e}^2$ and $\tilde{g}$ are dimensionless couplings. When the $\beta$ functions vanish, the RG flows of $\tilde{e}^2$ and $\tilde{g}$ are dual with respect to each other, $\beta_{\tilde{g}} / \tilde{g} = -\beta_{\tilde{e}} / \tilde{e}$. This leads to a Dirac-like relation, $\tilde{e}^2 \tilde{g} = 4\pi^2$, which is satisfied at the fixed point. Importantly, from this analysis it follows that at criticality the Maxwell terms in the Lagrangian (21) become RG irrelevant. Consequently, we conclude that this critical theory belongs to the same universality class as a continuous version of the frozen dual model in Eq. (8) where both $U(1)$ sectors are dualized. Hence, a continuum field theory implied by Eq. (8) can be readily identified to Eq. (21) with the Maxwell terms absent and $\tilde{e}^2 \tilde{g} = 4\pi^2$. Incidentally, since an RG analysis in terms of bare rather than renormalized parameters leads to the same critical behavior [34], we obtain that a dimensionless bare coupling defined by $g_0 = g \Lambda$ causes $g$ to flow to zero as the fixed point $g_0 \neq 0$ is approached when the ultraviolet cutoff $\Lambda \rightarrow \infty$. Although this is the same fixed point we have obtained for the dimensionless renormalized coupling (as implied by scale invariance), the result that $g \rightarrow 0$ as $\Lambda \rightarrow \infty$ highlights the role played by the frozen regime.

From the one-loop RG analysis it follows that the correlation length critical exponent $\nu \approx 0.625$ (after setting $\epsilon = 1$), which is precisely the one-loop value for the XY universality class. Furthermore, the salient critical property of DQC is the large anomalous scaling dimension of order parameters. From the irrelevance of the Maxwell terms near the IR fixed points, we see that the correlation function of the VBS order parameter at the quantum critical point can be represented as a bound state between a vortex and a particle (here a spinon) operator. The correlation function associated to the VBS order parameter is the gauge invariant correlation function $C_{\text{VBS}}(x) = \langle \phi_I^*(x) \phi(x) \phi_0(0) \phi_0^*(0) \rangle$. The anomalous dimension $\tilde{\eta}$ is defined via the large distance behavior $C(x) \sim 1 / |x|^{1 + \tilde{\eta}}$ at the critical point. We obtain that $\tilde{\eta} = 1 - 2\eta_1$, where $\eta_1$ is the anomalous dimension of the gauge-invariant operator $\phi_I^*(x) \phi(x)$ [34]. At one-loop order we obtain $\eta_1 = -\eta_4 / (8\pi)$ and, therefore, $\tilde{\eta}^{(1)} = 1.4$. The result shows that we are dealing with a modified XY’ universality class, akin to the so called XY, discussed in Ref. [35], where an anomalous dimension $\eta_4 = 1.493$ is numerically obtained for a lattice boson model exhibiting fractionalized excitations.

So far we have demonstrated that actions describing the frozen DQC naturally contain a topological BF term linked to a topological order arising at the critical point. We will now explore the interesting fact that such a BF term flux attachment allows one to derive a duality within a bosonization framework [8, 18, 36]. Using this technique, we show in the SM [32] that the bosonic two-component model with dynamical gauge fields coupled via a BF term is dual to the theory of two massless
Dirac fermions coupled via a shared gauge field,
\[
\mathcal{L}_b = \sum_{I=1,2} \left[ \left( \partial_\mu - i b_{1\mu} \right) \bar{\psi}_I \right]^2 + m^2 |\bar{\psi}_I|^2 + \frac{g}{2} |\bar{\psi}_I|^4
\]
+ \frac{i}{2\pi} \epsilon_{\mu\nu\lambda} b_{1\mu} \partial_\nu b_{2\lambda}
\]
\[
\mathcal{L}_f = \sum_{I=1,2} \bar{\psi}_I (\bar{\psi} - i \lambda) \psi_I.
\]
Thus, the duality integrates the topologically ordered $U(1) \times U(1)$ Abelian Higgs model into a wider duality web. As the bosonization duality leads to the expectation that critical behavior on both sides is the same, we conclude that the fermionic side of the duality also undergoes a second-order phase transition. If we now are to consider the fermionic theory as an intermediate step, we obtain a boson-boson duality between the easy-plane antiferromagnet by exploring the interplay between duality transformations and the RG scaling.

In summary we have analyzed the DQC paradigm for the easy-plane antiferromagnet by exploring the interplay between duality transformations and the RG scaling behavior. We have identified a quantum critical regime given by a modified XY universality class, where at the fixed point $\tilde{\epsilon}^2 \tilde{g} = (2\pi)^2$. Furthermore, at the critical point the topological order which arises in the frozen regime is recovered.

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SUPPLEMENTAL MATERIAL

Halperin-Lubensky-Ma for the dual easy-plane $\text{CP}^1$ model

The Halperin-Lubensky-Ma (HLM) mean-field theory [10] is actually a calculation where mean-field theory is applied to an effective Higgs theory action where the gauge fields were integrated out exactly, something that it is only possible in the case of an Abelian Higgs model. An instance of it already existed in 3+1 dimension [37], corresponding to a mechanism of inducing spontaneous symmetry breaking by quantum fluctuations. This symmetry breaking mechanism typically implies a first-order phase transition. In 2+1 dimensions it generates a non-analytical term in the effective potential, since assuming that the scalar field $\phi$ is uniform and integrating out the gauge field yields [10],

$$
\text{Tr} \ln(-\partial^2 + 2e^2|\phi|^2) = \frac{2\Lambda e^2}{\pi^2}|\phi|^2 - \frac{\sqrt{2}e^3}{\pi}|\phi|^3, \tag{13}
$$

where $\Lambda$ is the UV cutoff, here assumed to be such that $\Lambda^2 \gg 2e^2|\phi|^2$. The term $\sim |\phi|^3$ is the mentioned non-analytic term that causes the second-order phase transition from the Landau theory to turn into a first-order one. However, the presence of such a non-analytic term reveals that the essence of this problem is non-perturbative. The Dasgupta-Halperin duality [24] posits that in the strong coupling regime one actually finds a second-order phase transition. In order to see this, let us recall the continuum version of the dual model [38],

$$
\mathcal{L}_{\text{dual}} = \frac{1}{2}((\epsilon_{\mu\nu\lambda}\partial_\mu b_\lambda)^2 + \frac{M^2}{2}b_\mu^2 + |(\partial_\mu - iM\bar{e}b_\mu)\bar{\phi}|^2 + m^2|\bar{\phi}|^2 + \frac{u}{2}|\bar{\phi}|^4, \tag{14}
$$

where the scalar field $\bar{\phi}$ is dual to the original Higgs field $\phi$ and $\bar{e} = 2\pi/e$ is the dual gauge coupling. This dual Lagrangian features a massive vector field $b_\mu$. Upon integrating out $b_\mu$, a term $\sim -(M^2 + 2e^2|\phi|^2)^{3/2}$ is generated. The latter leads to an analytic Landau expansion in $|\bar{\phi}|^2$. Due to the mass $M$, the interaction between vortex loops is screened and circumvents the first-order transition scenario from the HLM mean-field theory.

The situation described above changes considerably for the $U(1) \times U(1)$ Abelian Higgs model. From the dual lattice action in Eq. (5) of the main text, one can infer a continuous field theory with the following Lagrangian,

$$
\tilde{\mathcal{L}} = \sum_{I=1,2} \left[ \frac{1}{2}(\epsilon_{\mu\nu\lambda}\partial_\mu b_{I\lambda})^2 + |(\partial_\mu - i\bar{e}b_{I\mu})\bar{\phi}|^2 \right]
+ \frac{M^2}{2}(b_{1\mu} + b_{2\mu})^2 + m^2(|\phi_1|^2 + |\phi_2|^2)
+ \frac{u}{2}(|\phi_1|^4 + |\phi_2|^4) + v|\phi_1|^2|\phi_2|^2, \tag{15}
$$
where for the two-component case we have introduced a new dual bare gauge coupling \( \bar{e} = \sqrt{2\pi/g} \) and \( M^2 = e^2/g \). The theory features two complex scalar fields \( \phi_1 \) and \( \phi_2 \) and two gauge fields, \( b_{1\mu} \) and \( b_{2\mu} \), along with a term \( M^2(b_{1\mu} + b_{2\mu})^2/2 \). In fact, integrating out both \( b_{1\mu} \) and \( b_{2\mu} \) yields an effective potential,

\[
U_{\text{eff}}(\phi_1, \phi_2) = \frac{\Delta \bar{e}^2 M^2}{3\pi^2} (|\phi_1|^2 + |\phi_2|^2) - \frac{\bar{M}^4}{2\pi} \sum_{s=\pm} [M_s^2(\phi_1, \phi_2)]^{3/2} + \ldots, \tag{16}
\]

where,

\[
M_s^2(\phi_1, \phi_2) = 1 + \bar{e}^2(|\phi_1|^2 + |\phi_2|^2) \pm \sqrt{1 + \bar{e}^2(|\phi_1|^2 - |\phi_2|^2)^2}. \tag{17}
\]

Hence, up to a constant term, attempting to perform a Landau expansion gives us,

\[
U_{\text{eff}}(\phi_1, \phi_2) = \frac{M^2 \bar{e}^2}{\pi} \left( \frac{\Lambda}{3\pi} - 3M^2 \sqrt{2\pi} \right) (|\phi_1|^2 + |\phi_2|^2) - \frac{3M^4 \bar{e}^4}{16\pi^2} \left[ 5(|\phi_1|^4 + |\phi_2|^4) - 6|\phi_1|^2|\phi_2|^2 \right] - \frac{M^6 \bar{e}^6}{2\pi} (|\phi_1|^2 + |\phi_2|^2)^{3/2} + \ldots, \tag{18}
\]

which also yields the non-analytic term characteristic of the first-order phase transition in the HLM mean-field theory. Thus, the easy-plane model features a non-analytic term both in the original and in the dual models. This result reflects the self-duality of the model. The reason why this happens can be easily understood by diagonalizing the gauge field matrix via the fields \( b_{\pm\mu} = b_{1\mu} \pm b_{2\mu} \). Only \( b_{-\mu} \) is gapped and contributes to screening of vortex loops, while \( b_{+\mu} \) is gapless, leading to a HLM mean-field behavior like the one obtained from the original model by integrating out \( a_{\mu} \). There is henceforth a self-duality of the weak first-order transition described by the Halperin-Lubensky-Ma mechanism [10].

Going one step further and accounting for the scalar field fluctuations, at one-loop order the RG equations for dimensionless couplings \( \hat{u} \) and \( \hat{v} \) yield

\[
\frac{\mu}{d\mu} \frac{d\hat{u}}{d\mu} = -(4 - d) \hat{u} + 2 \left[ (N + 4) \hat{u}^2 + N \hat{v}^2 + 2(d - 1) f^2 \right] \tag{19}
\]

\[
\frac{\mu}{d\mu} \frac{d\hat{v}}{d\mu} = -(4 - d) \hat{v} + 4 \hat{v}^2 + 4(N + 1) \hat{u},
\]

where we used a notation \( f = \bar{e}^2 \mu^{d-4} \) and \( \mu \) is a renormalization scale. The dimensionless gauge coupling \( f \) has the following \( \beta \) function,

\[
\frac{\mu}{d\mu} \frac{df}{d\mu} = -(4 - d) f + \frac{f^2}{24\pi}. \tag{20}
\]

In our case of \( N = 1 \) and \( d = 3 \), there are no real solutions for this system of equations if \( f \) is nonzero. A runaway flow is obtained and no second-order phase transition occurs, similarly to Ref. [10].

### Renormalization group analysis of the dual model

Here we will perform an RG analysis of the continuous Lagrangian of the dual model presented in Eq. (11) of the main text,

\[
\tilde{L}'' = \left[ (\partial_\mu - ie a_\mu) \phi_1 \right]^2 + m^2 |\phi_1|^2 + \frac{u_1}{2} |\phi_1|^4
\]

\[
+ \left[ (\partial_\mu - \bar{e} \bar{b}_\mu) \phi_2 \right]^2 + m^2 |\phi_2|^2 + \frac{u_2}{2} |\phi_2|^4
\]

\[
+ \frac{1}{2} (\bar{e} \bar{\epsilon}_{\mu\nu}\partial_\mu a_\nu) + \frac{1}{2} (\bar{e} \bar{\epsilon}_{\mu\nu}\partial_\mu b_\nu)^2
\]

\[
+ \frac{ie}{\sqrt{g}} a_\mu \bar{e} \bar{\epsilon}_{\mu\nu} \partial_\nu b_\lambda,
\]

where the charge \( \bar{e} \) is defined in terms of the original coupling as \( \bar{e} = 2\pi/\sqrt{g} \).

Integrating out the gauge fields in the Lagrangian (21), we calculate a matrix gauge field propagator,

\[
D_{\mu\nu}(p) = \left[ \frac{1}{p^2 + M^2} \left( \delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) \right] M \left( \frac{\bar{e} \ddot{e} \epsilon_{\mu\nu\lambda}}{p^2 + M^2} \right), \tag{22}
\]

where \( M^2 = e^2/g \) and we used the Landau gauge. The diagonal element of the matrix propagator allows us to calculate the contribution from the bubble diagram (Fig. 1) and a self-energy (Fig. 2) that enters the wave function renormalization.

As the gauge fields have different charges, the gauge field bubble diagrams provide contributions evaluated by the integral of the following form,

\[
2s_1 h^4 \int \frac{1}{(p^2 + M^2)^2} = \frac{s_1 h^4}{4\pi M}, \tag{23}
\]

where \( h^2 \) plays the role of \( e^2 \) or \( \bar{e}^2 \) and \( s_1 \) is a symmetry factor of the diagram that is found to be equal to 2. Therefore, the gauge field \( a_\mu \) bubble diagram results in \( e^4\sqrt{3}/(2\pi) \), while for \( b_\mu \) the contribution is equal to \( 8\pi^3/(eg^{3/2}) \).

---

**FIG. 1:** Gauge field bubble diagram contributing to the coupling \( u_\nu \). External lines represent either \( \phi_1 \) or \( \phi_2 \).

The wiggle represents either \( b_{1\mu} \) or \( b_{2\mu} \).

The self-energy diagram (Fig. 2) contributes to the wave function renormalization through the expansion up to a \( p^2 \) term. The diagram corresponds to the integral

\[
-4h^2 \int_k \frac{p_{\mu} p_{\nu}}{[(p - k)^2 + m^2] (k^2 + M^2)} \left( \delta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right), \tag{24}
\]
there is no fixed point of $O$.

Sis employed the fixed dimension approach pioneered by the same result would have followed. In principle consider the $\epsilon$ be resummed anyway at higher orders. Furthermore, we actual concern, since the the perturbation series has to be found in the Appendix A.2 of [39]. The wave function renormalizations corresponding to the two gauge fields have the same form,

$$Z = 1 + \frac{2h^2}{3\pi \frac{1}{m_R + M}},$$

where to evaluate the contributions for $a_\mu$ or $b_\mu$, one has to substitute the coupling $h$ with $e$ or $\tilde{e}$, respectively.

To express the renormalized couplings $u_{1R}$ and $u_{2R}$, one needs to calculate the so called fish diagram. In the case of both couplings the contribution is equal to $(s_a u_1^2)/(8\pi m_R)$, where $I = 1, 2$ and $s_2 = 5$ is a symmetry factor of the diagram.

Eventually, one obtains a renormalized coupling $u_{1R}$ and can define the dimensionless coupling $u_1$,

$$\hat{u}_1 = \frac{u_{1R}}{m_R} = Z_1^2 \left( \frac{u_1}{m_R} - \frac{5u_1^2}{8\pi m_R} - \frac{e^3 \sqrt{g}}{2\pi m_R} \right) \approx \frac{u_1}{m_R} \left( 1 + \frac{4e^2}{3\pi} \frac{1}{m_R + \frac{1}{\sqrt{g}}} \right) - \frac{5u_1^2}{8\pi m_R^2} - \frac{e^3 \sqrt{g}}{2\pi m_R}.$$

Calculating the $\beta$ function for $\hat{u}_1$, one obtains

$$m_R \frac{d\hat{u}_1}{dm_R} = -\frac{u_1}{m_R} \left( 1 + \frac{4e^2}{3\pi} \frac{1}{m_R + \frac{1}{\sqrt{g}}} \right) + \frac{5u_1^2}{4\pi m_R^2} + \frac{e^3 \sqrt{g}}{2\pi m_R} \approx \frac{u_1}{m_R} \left( 1 + \frac{4e^2}{3\pi} \frac{1}{m_R + \frac{1}{\sqrt{g}}} \right) - \frac{5u_1^2}{8\pi m_R^2} - \frac{e^3 \sqrt{g}}{2\pi m_R} + \frac{5u_1^2}{8\pi m_R^2}$$

$$= -\hat{u}_1 + \frac{5u_1^2}{8\pi m_R^2} \approx -\hat{u}_1 + \frac{5\hat{u}_1^2}{8\pi}. \tag{27}$$

In a similar fashion, we evaluate the $\beta$ function for $\hat{u}_2$,

$$m_R \frac{d\hat{u}_2}{dm_R} = -\frac{u_2}{m_R} \left( 1 + \frac{16\pi}{3g} \frac{1}{m_R + \frac{1}{\sqrt{g}}} \right) + \frac{10u_2^2}{8\pi m_R^2} + \frac{8\pi^3}{eg^{3/2} m_R} \approx \frac{u_2}{m_R} \left( 1 + \frac{16\pi}{3g} \frac{1}{m_R + \frac{1}{\sqrt{g}}} \right) - \frac{5u_2^2}{8\pi m_R^2} - \frac{8\pi^3}{eg^{3/2} m_R} + \frac{5u_2^2}{8\pi m_R^2}$$

$$= -\hat{u}_2 + \frac{5u_2^2}{8\pi m_R^2} \approx -\hat{u}_2 + \frac{5\hat{u}_2^2}{8\pi}. \tag{28}$$

This way we obtain $\beta_{\hat{u}_I}$ which vanish at $\hat{u}_I = 0$ and $\hat{u}_I = (8\pi)/5$, where $I = 1, 2$. Note that this RG analysis employed the fixed dimension approach pioneered by Parisi [40], which is at first sight less controlled, since there is no fixed point of $O(\epsilon)$. However, this is not an actual concern, since the the perturbation series has to be resummed anyway at higher orders. Furthermore, we could in principle consider the $\epsilon$ expansion as well, and the same result would have followed.

Bosonization through flux attachment

To see how the existence of the critical point in the easy-plane CP\(^1\) model has consequences for fermionic systems, we turn to the flux attachment technique to derive a fermionized version of the bosonic model considered in the main text. It was demonstrated that the frozen limit ($g \to 0$) of the model with both $U(1)$ sectors dualized almost completely coincides with the partially dual model where $g$ is kept finite. At the critical point the Maxwell terms arising in the partially dual model become irrelevant and so both field theories have the same

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**FIG. 2:** Scalar field self-energy
To make the fermionic side of duality more symmetrical, we perform a shift in the fermionic side of the duality in Eq. (34). We arrive at the expression,

\[ Z_{\text{IQED+flux}}[A] = Z_{\text{bQED}}[A] e^{S_{\text{CS}}[A]} \]

written in the imaginary time formalism. The flux attachment used in the conjecture has the form of,

\[ Z_{\text{IQED+flux}}[A] = \int D\alpha \sum_{\mu} Z_{\text{IQED}}[\alpha] e^{-\frac{i}{2} S_{\text{CS}}[\alpha] - S_{\text{BF}}[\alpha; A]} \]

We start with the following well known conjecture \[18, 36, 41\],

\[ Z_{\text{IQED+flux}}[A_1] Z_{\text{IQED+flux}}[A_2] e^{-S_{\text{CS}}[A_1] - S_{\text{CS}}[A_2]} = Z_{\text{bQED}}[A_1] Z_{\text{bQED}}[A_2]. \]  \hfill (33)

The technique we use here has recently been applied to find a fermionic dual of the topological version of the easy-plane CP\(^1\) model \[21, 42\]. The results from the flux attachment were shown to agree with the exact duality transformations in the spirit of Refs. \[23, 24\].

Since the bosonic theory is not supposed to contain any CS terms, the latter now appear on the fermionic side of the conjecture.

We multiply both sides of the expression above by \( \exp (-S_{\text{BF}}[A_1; A_2]) \) and promote the background fields \( A_1 \) and \( A_2 \) to be dynamical \( b_1 \) and \( b_2 \). This promotion requires introducing two new background fields, which we will denote as \( C_1 \) and \( C_2 \). Then, the expression (33) takes the form,

\[ Z_{\text{IQED+flux}}[A] Z_{\text{IQED+flux}}[A] e^{-S_{\text{CS}}[A_1] - S_{\text{CS}}[A_2]} = Z_{\text{bQED}}[A_1] Z_{\text{bQED}}[A_2]. \]  \hfill (33)

Using the definition of the fermionic flux attachment, we integrate out the dynamic gauge fields \( b_1 \) and \( b_2 \) on the fermionic side of the duality in Eq. (34). We arrive at the expression,

\[ Z_{\text{IQED+flux}}[A_1] Z_{\text{IQED+flux}}[A_2] e^{-S_{\text{CS}}[b_1+b_2]+S_{\text{BF}}[b_1;C_1]+S_{\text{BF}}[b_2;C_2]} = \int D\alpha D\beta Z_{\text{IQED}}[\alpha] Z_{\text{IQED}}[\beta] e^{-S_{\text{BF}}[\alpha;\beta] + S_{\text{BF}}[\alpha;C_1] + S_{\text{BF}}[\beta;C_2]}. \]  \hfill (34)

To make the fermionic side of duality more symmetrical, we perform a shift \( a \rightarrow a + (C_1 - C_2)/2 \),

\[ \int D\alpha Z_{\text{IQED}}[\alpha] Z_{\text{IQED}}[\alpha + (C_1 - C_2)/2] e^{-\frac{i}{2} S_{\text{BF}}[\alpha;C_1] - S_{\text{CS}}[C_1] + S_{\text{CS}}[C_1]} \]

\[ = \int D\alpha Z_{\text{IQED}}[\alpha] Z_{\text{IQED}}[\alpha + (C_1 - C_2)/2] e^{-\frac{i}{2} S_{\text{BF}}[\alpha;C_1] - S_{\text{CS}}[C_1] + S_{\text{CS}}[C_1] + S_{\text{BF}}[C_1;C_2]}. \]  \hfill (35)

And so, the bosonization duality relates a bosonic theory with two interacting dynamical gauge fields to the theory.
of two massless Dirac fermions coupled via the same gauge field,

\[ \mathcal{L}_b = \sum_{I=1,2} \left[ \left( \partial_\mu - ib_{1\mu} \right) \phi_I \right]^2 + m^2 |\phi_I|^2 + \frac{u}{2} |\phi_I|^4 \] 

\[ \mathcal{L}_f = \sum_{I=1,2} \bar{\psi}_I (\partial - i\gamma) \psi_I, \]  

(37)

where we put the background fields to zero.