Conformal Triangles and Zig-Zag Diagrams

S.Derkachov\textsuperscript{a}, A.P.Isaev\textsuperscript{a,b}, L.Shumilov\textsuperscript{a}

\textsuperscript{a}St.Petersburg Department of the Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023, St.Petersburg, Russia.

\textsuperscript{b}Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia.

Abstract

A convenient operator representation for zig-zag four-point and two-point planar Feynman diagrams relevant to the bi-scalar $D$-dimensional "fishnet" field theory is obtained. This representation gives a possibility to evaluate exactly diagrams of the zig-zag series in special cases. In particular, we give a fairly simple proof of the Broadhurst-Kreimer conjecture about the values of zig-zag multi-loop two-point diagrams which make a significant contribution to the renormalization group $\beta$-function in the 4-dimensional $\phi^4$ theory.

Keywords: Feynman diagrams, Conformal symmetry, Quantum field theory, Mathematical physics methods

1. Introduction

The 4-dimensional $\phi^4$ field theory (and its multicomponent generalizations) serves the Brout-Englert-Higgs mechanism and thus is an essential part of the Standard Model of particle physics. It was shown by explicit evaluation (in MS scheme) of the Gell-Mann-Low $\beta$-function in 4-dimensional $\phi^4$ theory that special Feynman diagrams (so-called zig-zag diagrams depicted in eqs. (12) and (13) below) give 44%, 46% and 47% of numerical contributions, respectively, to the 3, 4 and 5 loop orders of $\beta$ (see also [2, 3, 4] and references therein for the explicit expression of the seven loop $\beta$-function in the $\phi^4_{D=4}$ theory). The $M$-loop zig-zag diagram gives the $(M+1)$-loop contribution to the $\beta$-function. The two-point function, which is represented by the $M$-loop zig-zag diagram (12), (13), has the general form

$$G_2(x, y) = \frac{\pi^{2M}}{(x - y)^2} Z(M + 1),$$

where $\pi^{2M}$ is the normalization factor (we discuss this factor at the end of the paper), $x, y \in \mathbb{R}^4$ and $Z(M + 1)$ is a constant that contributes to the $\beta$-function in the $\phi^4_{D=4}$ theory in the $(M + 1)$-loop order. The first nontrivial terms $Z(3) = 6\zeta_3$ and $Z(4) = 20\zeta_5$ were analytically evaluated in [5] and [6], respectively. The constant $Z(5) = \frac{441}{32} \zeta_7$ of the zig-zag graph (12) with 4 loops was calculated by D.Kazakov [7] in 1983 (see also [8]). The 5 loop zig-zag diagram contribution $Z(6)$ was found by D.Broadhurst [9] in 1985 and confirmed by N.Ussyukina [10] in 1991. Then D.Broadhurst and D.Kreimer in 1995 [1] (see also [11]) evaluated $Z(n)$ numerically up to $n = (M + 1) = 10$ loops, and based on these data they formulated a remarkable conjecture that the constant $Z(M + 1)$ is given by the following expression

$$Z(M + 1) = 4C_M \sum_{p=1}^{\infty} \frac{(-1)^{(p-1)(M+1)}}{p^{2(M+1)-3}} = \begin{cases} 4C_M \zeta_{2M-1} & \text{for } M = 2N + 1, \\ 4C_M (1 - 2^{2(1-M)}) \zeta_{2M-1} & \text{for } M = 2N, \end{cases}$$

where $M$ is the number of loops in the zig-zag diagrams (12), (13) and $C_M = \frac{(M-1)!}{(2M)!}$ is the Catalan number. The proof of the Broadhurst-Kreimer conjecture was found in [12, 13]. The proof of (12, 13) is based on the results of works [14, 15].

We used a rather general approach to analytical evaluation of the 2-point and 4-point zig-zag diagrams. This approach leads to a fairly simple proof...
of the Broadhurst-Kreimer conjecture that is different from the proof of [13]. Here we make use of the operator formalism [16, 17] and methods of [18] based on the Euclidean multi-dimensional conformal quantum field theories (see [19, 20, 21, 22, 23], and references therein).

Our approach is partially inspired by the papers [24, 25, 26, 27] devoted to the representation of separable processes and the corresponding graph-building operators, such that the whole problem is essentially reduced to the problem of diagonalization of this operator. A complete basis of the corresponding eigenfunctions in the two-dimensional D-dimensional Heisenberg algebras $\mathcal{H} \subset \mathcal{H}^{(n)}$}

$$\hat{q}_a^\mu |x_a\rangle = x_a^\mu |x_a\rangle, \quad \hat{p}_a^\mu |k_a\rangle = k_a^\mu |k_a\rangle,$$

and form the basis in the space $V_a$, where the sub-algebra $\mathcal{H}_a$ acts. The whole algebra $\mathcal{H}^{(n)}$ acts in the space $V_1 \otimes \cdots \otimes V_n$ with the basis elements $|x_1, \ldots, x_n\rangle := |x_1\rangle \otimes \cdots \otimes |x_n\rangle$. We also introduce the dual states $|x_a\rangle$ and $|k_a\rangle$ such that the orthogonality and completeness conditions are valid

$$\langle x_a|x'_a\rangle = \delta^D(x_a - x'_a), \quad \langle k_a|k'_a\rangle = \delta^D(k_a - k'_a),$$

$$\int d^Dx_a |x_a\rangle \langle x_a| = I_a = \int d^Dk_a |k_a\rangle \langle k_a|,$$

and $I_a$ is the unit operator in $V_a$. Relations (3), (4) are consistent if we have

$$k_a^\mu |x_a\rangle = (x_a^\mu \hat{p}_a^\mu |k_a\rangle = -i \frac{\partial}{\partial x_a^\mu} |x_a\rangle \Rightarrow \langle x_a|k_a\rangle = \frac{1}{(2\pi)^D/2} e^{-ik^\mu x_a^\mu},$$

where there are no summations over the repeated index $a$, and the normalization constant $(2\pi)^{-D/2}$ is fixed by (4). Below we use the operators $(\hat{q}_a)_{-2\alpha} = (\sum_\mu \hat{q}_a^\mu \hat{q}_a^\mu)^{-\alpha}$ and $(\hat{p}_a)_{-2\beta} = (\sum_\mu \hat{p}_a^\mu \hat{p}_a^\mu)^{-\beta}$ with non-integer powers $\alpha$ and $\beta$. These operators are understood as integral operators defined via their integral kernels $\langle x|\hat{q}^{-2\alpha} \hat{p}^{-2\beta} |y\rangle = (x-y)^{-2\alpha} \delta^D(x-y)$ and

$$\langle x|\frac{1}{(\hat{p})^{2\beta}} |y\rangle = \int d^Dk (\hat{p})^{2\beta} |k\rangle \langle k|y\rangle = \int d^Dk e^{ik(x-y)} \frac{1}{(2\pi)^D/2} \frac{\Gamma(\beta')}{\Gamma(\beta)} , \quad \beta' := D/2 - \beta.$$

Now we consider the case of the algebra $\mathcal{H}^{(2)} = 1 + \mathcal{H}_2$ and introduce

$$\hat{Q}^{(2)}_{12} := \frac{1}{a(\beta)} \mathcal{P}_{12} (\hat{q}_{12})^{-2\beta} (\hat{q}_{12})^{-2\beta} ,$$

where $\hat{q}_{12} = \hat{q}_1^a - \hat{q}_2^a$ and $\mathcal{P}_{12}$ is the permutation

$$\mathcal{P}_{12} \hat{q}_1 = \hat{q}_2 \mathcal{P}_{12} , \quad \mathcal{P}_{12} \hat{p}_1 = \hat{p}_2 \mathcal{P}_{12} ,$$

$$\mathcal{P}_{12} |x_1, x_2\rangle = |x_2, x_1\rangle , \quad (\mathcal{P}_{12})^2 = I.$$
The 4-dimensional analog of the kernel \((9)\) (for even loops)
for odd loops

\[ Q_{12}^{(\alpha)} = \frac{1}{\mathcal{P}_{12}} (\hat{p}_1)^{-2\beta} (\hat{q}_{12})^{-2\beta} |y_1, y_2 \rangle \]

where

\[ x_1 \cdots \frac{1}{(x_1 - x_2)^{2\beta}} \delta_D (\nu \cdot y) \cdot y_2 \]
\[ x_1 \frac{\beta}{x_2} (x_1 - x_2)^{-2\beta} \]

The 4-dimensional analog of the kernel \((9)\) (for \(\beta = 1\)) was considered in \([18]\) and denoted there as \(H_3\). Note that \(Q_{12}^{(\beta)}\) is the graph building operator for the planar zig-zag Feynman graphs:

\text{for even loops}

\[ x_1 \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} y_1 \]
\[ x_2 \frac{s^1}{y_2} \frac{s^2}{y_2} \cdots \frac{s^1}{y_2} \frac{s^2}{y_2} \]

\[ = \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = y_1 \]
\[ = \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = y_2 \]

\text{for odd loops}

\[ x_1 \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} y_1 \]
\[ x_2 \frac{s^1}{y_2} \frac{s^2}{y_2} \cdots \frac{s^1}{y_2} \frac{s^2}{y_2} \]

\[ = \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = y_1 \]
\[ = \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = y_2 \]

Here the bold face vertices denote the integration over \(\mathbb{R}^D\). We stress that the Feynman integrals, which correspond to the matrix elements \([10], [11]\), represent the contribution to the 4-point correlation functions in the bi-scalar \(D\)-dimensional "fishnet" theory (see \([18]\) and references therein). For clarity, in the right hand sides of \([10]\) and \([11]\), we present the zig-zag diagrams in the form of spiral graphs having the cylindrical topology \([18]\). We also stress that integral kernels \([10]\) and \([11]\), in the case of \(D = 4\) and \(\beta = 1\), contribute to Green’s functions of the standard \(\phi^4\) field theory.

The next important statement is that \(Q_{12}^{(\beta)}\), given in \([7]\), is the graph building operator for the integrals of the planar zig-zag two-point Feynman graphs:

\text{for even loops}

\[ x_1 \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} y_1 \]
\[ x_2 \frac{s^1}{y_2} \frac{s^2}{y_2} \cdots \frac{s^1}{y_2} \frac{s^2}{y_2} \]

\[ = \int d^D x_1 d^D y_2 \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (x_1 - x_2)^{2\beta} \]

\text{for odd loops}

\[ x_1 \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} \cdots \frac{s^1}{x_2} \frac{s^2}{x_2} y_1 \]
\[ x_2 \frac{s^1}{y_2} \frac{s^2}{y_2} \cdots \frac{s^1}{y_2} \frac{s^2}{y_2} \]

\[ = \int d^D x_1 d^D y_2 \langle x_1, x_2 | (Q_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (x_1 - x_2)^{2\beta} \]

In the next sections, we use the operator representations \([10] - [13]\) to evaluate exactly the corresponding class of 2-point and 4-point Feynman diagrams.

Finally, we note that the elements \(H_\beta := \mathcal{P}_{12} Q_{12}^{(\beta)} \equiv (\hat{p}_1)^{-2\beta} (\hat{q}_{12})^{-2\beta}\) form a commutative set of operators \([H_\alpha, H_\beta] = 0 \ (\forall \alpha, \beta)\). This fact can be easily demonstrated by means of the operator version \([10]\) of the star-triangle relation. Special forms of the elements \(H_3\) were used in \([16]\) as graph-building operators for ladder diagrams.

3. Eigenfunctions for the graph building operator \(\hat{Q}_{12}\). Scalar product and completeness

To find eigenvectors for the graph building operator \([7]\) we consider the standard 3-point correlation function of three fields \(O_{\Delta_1}, O_{\Delta_2}\) and \(O_{\Delta_{\beta}}^{\mu_1 \cdots \mu_n}\) in a conformal field theory. Here \(O_{\Delta_1}\) and \(O_{\Delta_2}\) are two scalar fields with conformal dimensions \(\Delta_1\) and \(\Delta_2\), while \(O_{\Delta_{\beta}}^{\mu_1 \cdots \mu_n}\) is a tensor field with conformal dimension \(\Delta\). The single conformally-invariant tensor structure of this correlation function (up to a normalization) is well known

\[ u^{\mu_1} \cdots u^{\mu_n} \langle O_{\Delta_1} (y_1) O_{\Delta_2} (y_2) O_{\Delta_{\beta}}^{\mu_1 \cdots \mu_n} (y) \rangle = \]
\[ = \frac{(u_{\mu_1} - y_{\mu_1}) (u_{\mu_n} - y_{\mu_n})}{(y_1 - y_2)^2 (y - y_1)^{2\Delta_1} (y - y_2)^{2\Delta_2} \cdots (y - y_2)^{2\Delta_2}} \]

(14)
where
\[ A = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta + n), \quad A_1 = \frac{1}{2}(\Delta_1 + \Delta - \Delta_2 - n), \]
\[ A_2 = \frac{1}{2}(\Delta_2 + \Delta - \Delta_1 - n). \]

Here and below we make formulas concise by using an auxiliary complex vector \( u \in \mathbb{C}^D \) such that \( (u, u) = u^* u = 0 \). We need the special form of the 3-point function \([13]\) when parameters \( A, A_1, A_2 \) are related to two numbers \( \alpha \in \mathbb{C}, \beta \in \mathbb{R} \):
\[
A = \alpha, \quad A_1 = \alpha':= \frac{D}{2} - \alpha, \quad A_2 = (\alpha + \beta)' = \frac{D}{2} - (\alpha + \beta) \Rightarrow \Delta_1 = \frac{D}{2}, \quad \Delta_2 = \frac{D}{2} - \beta, \quad \Delta = D - 2\alpha - \beta + n, \tag{15}\]
so we have
\[
\langle y_1, y_2 | \Psi^{(n, u)}_{\alpha, \beta}(y) \rangle := \frac{(u, y-y_1)(y, y-y_2)}{(y_1 - y_2)^{2n}(y - y_1)^{2\alpha}(y - y_2)^{2(\alpha + \beta)}}, \tag{16}\]
and, following the paper \([33]\), we call this function as a conformal triangle. It is a remarkable fact that \( |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle = u^\dagger \cdots u^\dagger \Phi^{\mu_1 \cdots \mu_n}(y)\rangle \), defined in \([16]\), is the eigenvector for the graph building operator \([7]\)
\[
Q_{12}^{(\beta)} |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle = \tau(\alpha, \beta, n) |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle, \tag{17}\]
with the eigenvalue
\[
\tau(\alpha, \beta, n) = (-1)^n \frac{\pi^{D/2} \Gamma(\beta) \Gamma(\alpha + (\alpha + \beta)' + n)}{\Gamma'(\beta) \Gamma(\alpha' + n) \Gamma(\alpha + \beta)} \tag{18}\]
An analogous statement, for \( D = 4 \) and \( \beta = 1 \), was announced in \([18]\).

Note that with respect to the standard Hermitian scalar product
\[
\langle \Psi | \Phi \rangle = \int d^D x_1 d^D x_2 \langle x_1, x_2 | \Psi \rangle \langle x_1, x_2 | \Phi \rangle = \int d^4 x_1 d^4 x_2 \Psi^*(x_1, x_2) \Phi(x_1, x_2), \tag{19}\]
the operator \([7]\) for \( \beta \in \mathbb{R} \) is Hermitian up to the equivalence transformation:
\[
(Q_{12}^{(\beta)})^\dagger = \frac{1}{a(\beta)} (\hat{q}_{12})^{-2\beta} (\hat{p}_1)^{-2\beta} \mathcal{P}_{12} = U \mathcal{Q}_{12}^{(\beta)} U^{-1}, \tag{20}\]
where
\[
U := \mathcal{P}_{12} (\hat{q}_{12})^{-2\beta} = (\hat{q}_{12})^{-2\beta} \mathcal{P}_{12}. \tag{21}\]
Therefore, we modify the scalar product \([19]\)
\[
\langle \Psi | U | \Phi \rangle = \int d^4 x_1 d^4 x_2 \frac{\Psi^*(x_2, x_1) \Phi(x_1, x_2)}{(x_1 - x_2)^{2\beta}}, \tag{22}\]
and with respect to this scalar product the operator \([7]\) is Hermitian. In \([21]\), a special conjugation for the vectors was introduced
\[
\langle \Psi | := \langle \Psi | U = \langle \Psi | (\hat{q}_{12})^{-2\beta} \mathcal{P}_{12}, \tag{23}\]
and the operator \( U \) in \([21]\) plays the role of the metric in the space \( V_1 \otimes V_2 \). It is evident that the graph building operator \([7]\) commutes with
\[
\hat{D} = \frac{i}{2} \sum_{\alpha=1}^2 (\hat{q}_\alpha \hat{p}_\alpha + \hat{p}_\alpha \hat{q}_\alpha) + \frac{1}{2} (\hat{y}_\alpha \partial_{y_\alpha} + \partial_{y_\alpha} \hat{y}_\alpha) - \beta, \tag{24}\]
which acts on the eigenvector \( |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle \) as follows:
\[
\hat{D} |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle = (2\alpha + \beta - 2D - n) |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle, \tag{25}\]
and, for \( \beta \in \mathbb{R} \), satisfies \( \hat{D}^\dagger = -U \hat{D} U^{-1} \). Thus, the operator \( \hat{D} \) is anti-Hermitian with respect to the scalar product \([21]\), and the corresponding condition on its eigenvalue gives
\[
2(\alpha + \alpha) = 2n + D - 2\beta \Rightarrow \alpha = \frac{1}{2} (n + D/2 - \beta) - i\nu, \quad \nu \in \mathbb{R}. \tag{26}\]
It is a remarkable fact that under this condition the eigenvalue \([18]\) is real
\[
\tau(\alpha, \beta, n) = (-1)^n \frac{\pi^{D/2} \Gamma(\beta) \Gamma(\alpha + (\alpha + \beta)' + n)}{\Gamma'(\beta) \Gamma(\alpha' + n) \Gamma(\alpha + \beta)} \times \frac{\Gamma(D/2 + \frac{n}{2} - \frac{\beta}{2} - i\nu)}{\Gamma(D/2 + \frac{n}{2} + \frac{\beta}{2} + i\nu)}, \tag{27}\]
and the parameter \( \Delta \) in \([15]\) acquires the form \( \Delta = \frac{n}{2} + 2i\nu \). In view of this, we denote the eigenvector \([16]\) as
\[
|\Psi^{(n, u)}_{\alpha, \beta, \nu}(y)\rangle := |\Psi^{(n, u)}_{\alpha, \beta}(y)\rangle = u^{\mu_1} \cdots u^{\mu_n} |\Phi^{\mu_1 \cdots \mu_n}(y)\rangle, \tag{28}\]
\[
|\Psi^{(n, u)}_{\alpha, \beta, \nu}(x_1, x_2)\rangle := \langle x_1, x_2 | \Psi^{(n, u)}_{\alpha, \beta, \nu}(y)\rangle. \tag{29}\]
Since the eigenvalue \([20]\) is real (it is invariant under the transformation \( \nu \rightarrow -\nu \)), two eigenvectors
\[ |\psi^{(n,u)}_{\lambda,\beta,x}\rangle \text{ and } |\psi^{(n,v)}_{\lambda,\beta,x}\rangle, \text{ having different eigenvalues} \]

(e.g. \(n \neq m\) and \(\lambda \neq \pm \nu\), should be orthogonal to each other with respect to the scalar product).

Indeed, we have the following orthogonality condition for two conformal triangles (see, e.g., [19, 20. 34, 18]):

\[ \langle \Psi^{(m,v)}_{\lambda,\beta,y} | \Psi^{(n,u)}_{\lambda,\beta,x} \rangle = \int d^D x \int d^D y \langle \Psi^{(m,v)}_{\lambda,\beta,y} | U \langle x_1, x_2 \rangle | \Psi^{(n,u)}_{\lambda,\beta,x} \rangle = \]

\[ = \int d^D x \int d^D y \frac{\langle \Psi^{(m,v)}_{\lambda,\beta,y} | (x_2, x_1) \rangle * \langle \Psi^{(n,u)}_{\lambda,\beta,x} | (x_1, x_2) \rangle}{(x_1 - x_2)^{2D-2\Delta_1 - 2\Delta_2}} = \]

\[ = C_1(n, \nu) \delta_{nm} \delta(\nu - \lambda) \delta^D(x - y) \langle u, v \rangle \]

+ \(C_2(n, \nu) \delta_{nm} \delta(\nu + \lambda) \frac{(u, v) - 2(\nu - \lambda) \delta^D(x - y) (x - y)^2(2D + 2\nu)}{(x - y)^{2D + 2\nu}} \)

\[ \tag{28} \]

where \((u, v) = u^\mu v^\nu\), \(\beta = D - \Delta_1 - \Delta_2 = \Delta_1 + \Delta_2\)

and

\[ C_1(n, \nu) = \frac{(-1)^n n! \pi^{D/2 + 1} n!}{\Gamma \left( \frac{D}{2} + n \right) \Gamma \left( \frac{D}{2} + n - 1 \right) + 4\nu^2} \]

\[ \times \frac{\Gamma \left( 2\nu \right) \Gamma \left( -2\nu \right)}{\Gamma \left( \frac{D}{2} + 2\nu - 1 \right) \Gamma \left( \frac{D}{2} - 2\nu - 1 \right)} \]

\[ \tag{29} \]

We note that the coefficient \(C_1\) is independent of \(\beta\) and plays the important role as the inverse of the Plancherel measure used in the completeness condition (see below). In contrast to this, the coefficient \(C_2\) in \(28\) depends on \(\beta\), but the explicit form for \(C_2\) will not be important for us. Respectively, completeness (or resolution of unity \(I\) for the basis of eigenfunctions \(27\)) is written as (see, e.g., [10, 20, 34, 18])

\[ I = \sum_{n=0}^{\infty} \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ = \sum_{n=0}^{\infty} \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \tag{30} \]

Applying to this relation the vector \(|y_1, y_2\rangle\) from the right and the vector \(|x_1, x_2\rangle\) from the left and using the formulas \(\langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | (x_1, x_2) \rangle = (\Psi^{(n,\alpha)}_{\nu,\beta,x} | (x_1, x_2) \rangle)^* = \Psi^{(n,\alpha)}_{\nu,\beta,x} | (x_1, x_2) \rangle\), we write the resolution of unity \(30\) in terms of the integral kernels (see e.g. eq. (A.7) in [18]).

4. Four-point and two-point correlation functions for zig-zag diagrams

Substitution of the resolution of unity \(30\) into expressions \(10, 11\) for zig-zag 4-point Feynman graphs gives

\[ G_4^{(M)}(x_1, x_2; y_1, y_2) = \]

\[ = \langle x_1, x_2 \rangle \langle Q^{(\beta)}_{12} \rangle^M \langle y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = \]

\[ = \sum_{n=0}^{\infty} \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle x_1, x_2 \rangle \langle Q^{(\beta)}_{12} \rangle^M \]

\[ \times \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \langle y_1, y_2 \rangle (y_1 - y_2)^{2\beta} = \]

\[ = \sum_{n=0}^{\infty} \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \times \int d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \quad \tag{31} \]

where the integral over \(x\) in the right-hand side of \(31\) is evaluated in terms of conformal blocks \([21, 22, 34]\) (in the four-dimensional case, this integral was considered in detail in [18]).

Making use of the standard relations between the 4-point zig-zag functions \(G_4^{(M)}(x_1, x_2; y_1, y_2)\) constructed in \(31\) and 2-point zig-zag functions \(G_2^{(M)}(x_2, y_1)\) (the graphs for these functions are presented in \([10 - 13]\)), we write explicit expressions for the 2-point \(M\)-loop zig-zag diagrams as follows:

\[ G_2^{(M)}(x_2, y_1) = \]

\[ \sum_{n=0}^{\infty} \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \times \int d^D x_1 d^D y_2 d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \]

\[ = \]

\[ = \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \times \int d^D x_1 d^D y_2 d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \]

\[ \times \int d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \quad \tag{32} \]

where we apply the two-point master integral

\[ \int d^D x_1 d^D y_2 d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \]

\[ = \]

\[ = \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \times \int d^D x_1 d^D y_2 d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \]

\[ = \]

\[ = \int_0^\infty \frac{d\nu}{C_1(n, \nu)} \int d^D x \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | U \rangle \]

\[ \times \int d^D x_1 d^D y_2 d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \]

\[ \times \int d^D x \langle x_1, x_2 \rangle \langle \Psi^{(n,\alpha)}_{\nu,\beta,x} | \Psi^{(n,\alpha)}_{\nu,\beta,x} | y_2, y_1 \rangle \quad \tag{33} \]

where the integral over \(x\) in the right-hand side of \(33\) is evaluated in terms of conformal blocks \([21, 22, 34]\) (in the four-dimensional case, this integral was considered in detail in [18]).

5
In the last equality in (37) we used the integral
\[ \int d^4x_1 d^4y_2 d^4x \frac{(x_1, x_2) \psi_{\nu, x}}{(x_1 - x_2)^2 (y_1 - y_2)^2} = \]
\[ = (-1)^n \frac{n+1}{2^n} \tau_3(\nu, n) \frac{1}{(x_2 - y_1)^2}, \]
where \( (x_1, x_2) \psi_{\nu, x} := \psi_{\nu, \beta, n}(x_1, x_2) \) and (see (26))
\[ \zeta \]
\[ \text{Finally, we substitute (34) – (36) into (32) and obtain} \]
\[ C_2^{(M)}(x_2, y_1)_{D=4, \beta=1} = \]
\[ = (\frac{1}{M+1}) \frac{1}{M} \tau_3(\nu, n) \int_0^\infty \frac{\nu^2}{(1 + n)^2 + 4\nu^2} \] \[ = 4\pi^2 M \frac{C_M}{(x_2 - y_1)^2} \]
\[ \frac{(-1)^{n(M+1)} (n+1)^2}{(n+1)^2 M^{-1}}. \]
Finally, we substitute (34) – (36) into (32) and obtain
\[ C_2^{(M)}(x_2, y_1)_{D=4, \beta=1} = \]
\[ = \frac{1}{M+1} \frac{1}{M} \tau_3(\nu, n) \int_0^\infty \frac{\nu^2}{(1 + n)^2 + 4\nu^2} \] \[ = 4\pi^2 M \frac{C_M}{(x_2 - y_1)^2} \]
\[ \frac{(-1)^{n(M+1)} (n+1)^2}{(n+1)^2 M^{-1}}. \]
where \( C_M = \frac{1}{M+1} \frac{1}{M} \) is the Catalan number. In the last equality in (37) we used the integral
\[ \int_0^\infty \frac{\nu^2}{(4\nu^2 + (1 + n)^2)^{M+2}} = \]
\[ = \frac{1}{2^{n(M+1) M+1}} \Gamma \left( \frac{M+1}{2} \right) \Gamma \left( \frac{M+2}{2} \right) \] \[ \Gamma \left( \frac{M+1}{2} \right) \Gamma \left( \frac{M+2}{2} \right) \]
\[ \Gamma \left( \frac{M+1}{2} \right) \Gamma \left( \frac{M+2}{2} \right) \]
and applied the identity \[ \frac{\Gamma \left( \frac{M+1}{2} \right) \Gamma \left( \frac{M+2}{2} \right)}{\Gamma(M+2)} = \]
\[ \frac{(2M)! \pi}{2^{2M+2} (M+1)!}. \]
Relation (37) is equivalent to (1), (2). Thus, we have derived the Broadhurst and Kreimer formula [1].

Finally, we note that D.Broadhurst and D.Kreimer fixed in their paper [1] the loop measure for each integration over loop momenta \( k \) as \( \frac{d^d k}{k^2} \).
Expression (37) is related to the \( M \) loop zig-zag diagram (it corresponds to the \( (M+1) \) loop contribution to the \( \beta \)-function of the \( \phi^4_{\beta=1} \) theory). Therefore, we have to divide our answer in (37) by \( (\pi^2)^M \). In this case, our result (37) justifies the normalization factor \( (\pi^2)^M \) in relation (1), which together with (2) states the Broadhurst and Kreimer conjecture [1].

5. Conclusions

In this letter, we demonstrated how the recent progress in the investigations of the multidimensional conformal field theories (CFT) can be applied, e.g., in the analytical evaluations of massless Feynman diagrams. We believe that the approach described here gives the universal method of the evaluation of contributions into correlation functions and critical exponents in various CFT. We also wonder if it is possible to apply our \( D \)-dimensional generalizations to evaluation similar 4-points functions (with fermions) that arise in the generalized "fishnet" model, in double scaling limit of \( \gamma \)-deformed \( N = 4 \) SYM theory.

Acknowledgements

We thank G.Arutyunov, M.Kompaniets and V.Kazakov for valuable discussions. This work is supported by the Russian Science Foundation project No 19-11-00131. API is grateful to L.Euler International Mathematical Institute in Saint Petersburg for kind hospitality.

References

[1] D. J. Broadhurst, D. Kreimer, Knots and numbers in \( \varphi^4 \) theory to 7 loops and beyond, Int. J. Mod. Phys. C 6 (1995) 519-524. \url{arXiv:hep-ph/9504352}
[2] O. Schnetz, Numbers and Functions in Quantum Field Theory, Phys. Rev. D 97 (8) (2018) 085018. \url{arXiv:1605.08598}
[3] M. V. Kompaniets, E. Panzer, Minimally subtracted six loop renormalization of \( O(n) \)-symmetric \( \varphi^4 \) theory and critical exponents, Phys. Rev. D 96 (3) (2017) 036016. \url{arXiv:1705.08483}
