EVERY THREE SPHERE OF POSITIVE RICCI CURVATURE CONTAINS A MINIMAL EMBEDDED TORUS

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One of the most celebrated theorems of differential geometry is the 1929 theorem of Lusternik and Schnirelmann, which states that for every riemannian metric on the 2-sphere there exist at least three simple closed geodesics. Jurgen Jost [J] (following important work of Pitts [P] and Simon and Smith [SS]) has recently generalized this result by showing that for every riemannian metric on \( S^3 \), there exist at least 4 minimal embedded 2-spheres. This is optimal in that there are metrics for which the number of embedded minimal 2-spheres is exactly four [W2,4.5]. However, one can also ask about surfaces of higher genus, and here our knowledge is very incomplete. On the one hand, Lawson [L] showed that \( S^3 \) with its standard metric contains embedded minimal surfaces of every orientable topological type, and recently Pitts and Rubinstein [PR] have discovered many new infinite families of examples. But for general metrics on \( S^3 \), no known theorem asserts the existence of any minimal surface other than a sphere. The present paper takes a first step in this direction by proving:

**Theorem 3.** For every \( C^4 \) riemannian metric \( \gamma \) of positive ricci curvature on \( S^3 \), there exists at least one minimal embedded torus.

I conjecture that every metric on \( S^3 \) admits at least 5 minimal embedded tori, but I can prove it (by a perturbation argument [W2,4.4]) only for metrics that are close to the standard metric.

**Preliminaries.** The proof uses the following facts about the space of all minimal surfaces for varying riemannian metrics. The facts are proved in [W2] using the implicit function theorem. Let \( N \) be a compact 3-manifold, \( \Gamma \) be an open set of \( C^4 \) metrics on \( N \), and \( M \) be the set of pairs \( (\gamma, S) \) where \( \gamma \in \Gamma \) and \( S \subset N \) is a smooth embedded \( \gamma \)-minimal surface.

**Theorem 1** [W2, 2.1,2.2,5.1]. The set \( M \) is a smooth Banach manifold, and the map

\[
\Pi : M \to \Gamma \\
\Pi : (\gamma, S) \mapsto \gamma
\]

is a smooth map. Almost every (in the sense of Baire category) \( \gamma \) is a regular value of \( \Pi \), i.e., each element of \( \Pi^{-1}(\gamma) \) is a nondegenerate critical point of the area functional.
Let \( M_0 \) be the union of one or more connected components of \( M \). If \( \Gamma \) is connected and if \( \Pi : M_0 \to \Gamma \) is a proper map (inverse images of compact sets are compact), then \( \Pi|_{M_0} \) has a mapping degree \( d \) such that for each regular value \( \gamma \),

\[
d = \sum_{(\gamma,S) \in M_0} (-1)^{\text{index}(S)}.
\]

**Remark 1.** If for some \( \gamma \), \( \Pi^{-1}(\gamma) = \emptyset \), then \( \gamma \) is a regular value of \( \Pi \) and so \( d \) would have to be 0 by (1).

**Remark 2.** This theorem remains true for simple immersed minimal surfaces. An immersion is simple if there is an \( x \in N \) that is covered exactly once.

If \( (\gamma,S) \in M_0 \) and \( S \) has nullity 0, then \( S \) is a nondegenerate critical point of the area functional, and \( (\gamma,S) \) is isolated in \( \Pi^{-1}(\gamma) \) and contributes \( (-1)^{\text{index}(S)} \) to the degree. It sometimes happens that \( \Pi^{-1}(\gamma) \) contains a compact \( k \)-dimensional manifold \( \Sigma \) of surfaces. Of course the surfaces in \( \Sigma \) are degenerate critical points, but if each of them has nullity equal to \( k \), then \( \Sigma \) is said to be a nondegenerate critical manifold and has nice properties.

**Theorem 2 [W2,5.1].** Suppose \( \Pi^{-1}(\gamma_0) \) contains a nondegenerate critical manifold \( \Sigma \). Then there is a neighborhood \( \Gamma_0 \subset \Gamma \) of \( \gamma_0 \) and a connected component \( M_0 \) of \( \Pi^{-1}(\Gamma_0) \) such that

\[
\Pi^{-1}(\gamma_0) \cap M_0 = \Sigma
\]

and

\[
\deg(\Pi|_{M_0}) = \chi(\Sigma)(-1)^{\text{index}(\Sigma)}
\]

where \( \chi(\Sigma) \) is the euler characteristic of \( \Sigma \) and \( \text{index}(\Sigma) \) is equal to the index of \( S \) for each \( (\gamma,S) \in \Sigma \).

For example consider the clifford tori. (A clifford torus is the set of points in \( S^3 \subset R^4 \) equidistant from a pair of orthogonal planes through the origin.) The set of all such tori (or, equivalently, the set of pairs of orthogonal planes in \( R^4 \)) is topologically the 4-manifold \( RP^2 \times RP^2 \). Straightforward calculations show that each clifford torus is a minimal surface with nullity 4 and index 1. Thus the hypotheses of Theorem 2 are satisfied.

**The Theorem.**

**Theorem 3.** For every \( C^4 \) metric \( \gamma \) of positive ricci curvature on \( S^3 \), there exists at least one embedded torus that is minimal with respect to \( \gamma \).

**Proof.** Let \( \gamma_0 \) be the standard metric on \( S^3 \) and let \( S^1 = \{ z \in C : |z| = 1 \} \) act on \( S^3 \subset R^4 = C^2 \) by complex multiplication. Let \( R_n : S^3 \to S^3 \) be multiplication by \( e^{2\pi i/n} \). We first prove the following:

**Lemma.** There is an \( n_0 \) such that if \( n \geq n_0 \) and if \( M \subset S^3 \) is an embedded \( \gamma_0 \)-minimal torus with \( R_n(M) = M \), then \( M \) is a clifford torus.
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PROOF. Suppose not. Then there is a sequence \( n(i) \to \infty \) of integers and a sequence \( M_i \) of embedded minimal tori, none of which is a Clifford torus, such that \( R_{n(i)}(M_i) = M_i \).

By the compactness theorem of Choi and Schoen [CS], there is a convergent subsequence, which we may assume to be the original sequence: \( M_i \to M \). Fix a real number \( t > 0 \) and let \( k(i) \) be the greatest integer less than or equal to \( tn(i) \), so that

\[
\frac{k(i)}{n(i)} \to t.
\]

By hypothesis

\[
e^{\frac{2\pi i}{n(i)}}(M_i) = M_i.
\]

Thus

\[
e^{2\pi i t}(M_i) = \left( e^{\frac{2\pi i}{n(i)}} \right)^{k(i)}(M_i) = M_i
\]

so, letting \( i \to \infty \),

\[
e^{2\pi i t}(M) = M.
\]

Since this holds for each \( t \), \( M \) is invariant under the \( S^1 \) action. Let

\[
h : S^3 \subset C^2 \to S^2 \cong C \cup \{\infty\}
\]

be the Hopf map

\[
h(z, w) = z/w.
\]

In more geometrical terms, \( h : S^3 \subset C^2 \to CP^1 \cong S^2 \) maps each point to the complex line that contains it. The \( S^1 \) invariance of \( M \) means that there is a curve \( T \subset S^2 \) such that

\[
M = h^{-1}(T).
\]

The area of the inverse image (under \( h \)) of a curve is a constant times the length of the curve. Thus the minimality of \( M \) implies that \( T \) is a union of geodesics. Since \( M \) is embedded, \( T \) is a single great circle and so \( M \) is a Clifford torus. It follows from Theorem 2 (and the example given after it) that for sufficiently large \( i \), \( M_i \) is a Clifford torus. \( \Box \)

Now to prove the theorem, let \( \Gamma \) be the set of \( C^4 \) metrics of positive Ricci curvature on \( S^3 \), \( \mathcal{M} \) be the Banach manifold of Theorem 1, and \( \mathcal{M}_0 \) be the set of \( (\gamma, S) \in \mathcal{M} \) such that \( S \) is a torus. The space \( \Gamma \) is connected by a theorem of Hamilton [H]. By the compactness theorem of Choi and Schoen [CS], \( \Pi|_{\mathcal{M}_0} \) is proper and therefore (by Theorem 1) has a mapping degree; we will show that the degree is not zero. Let \( p \) be a prime number greater than the \( n_0 \) of the lemma. Let \( \gamma_0 \) be the standard metric on \( S^3 \) and let \( \Gamma_0 \subset \Gamma \) be a neighborhood of \( \gamma_0 \) such that \( \Pi^{-1}(\Gamma_0) \) contains a connected component \( \mathcal{M}_{\text{cliff}} \) such that

\[
\Pi^{-1}(\gamma_0) \cap \mathcal{M}_{\text{cliff}}
\]

is precisely the set of Clifford tori. (This is possible by Theorem 2.) Let

\[
\mathcal{M}_{\text{other}} = \Pi^{-1}(\Gamma_0) \cap \mathcal{M}_0 \setminus \mathcal{M}_{\text{cliff}}.
\]
By the lemma we may choose $\Gamma_0$ small enough that if $(\gamma, S) \in M_{\text{other}}$ then $R_p(S) \neq S$. Note that since $p$ is prime, if $T \subset S^3$ is an embedded submanifold that is not $R_p$ invariant, then there is an $x \in T$ such that $x \notin (R_p)^k(T)$ for $1 \leq k < p$. That is, $T$ becomes a simply immersed submanifold in the quotient space $S^3/R_p$. According to Theorem 1, for almost every metric $\gamma$ on $S^3/R_p$, each simple $\gamma$-minimal surface has nullity 0. Equivalently, for almost every $R_p$ invariant metric $\gamma$ on $S^3$, each $\gamma$-minimal surface that is not $R_p$ invariant has nullity 0. Fix any such $R_p$ invariant metric $\gamma \in \Gamma_0$.

Now if $(\gamma, S) \in M_{\text{other}}$, then so is $(\gamma, (R_p)^k(S))$ for $1 \leq k < p$. Furthermore, these $p$ surfaces are distinct and all have the same index (and nullity 0). Thus

$$ \deg(\Pi|_{M_{\text{other}}}) = \sum_{(\gamma, S) \in M_{\text{other}}} (-1)^{\text{index}(S)} \equiv 0 \mod p. $$

On the other hand, by Theorem 2 and the example following it:

$$ \deg(\Pi|_{M_{\text{cliff}}}) = \chi(RP^2 \times RP^2) (-1)^1 = -1. $$

Thus

$$ \deg(\Pi|_{M_0}) = \deg(\Pi|_{M_{\text{cliff}}}) + \deg(\Pi|_{M_{\text{other}}}) \equiv -1 \mod p. $$

Since this holds for arbitrarily large $p$, in fact $\deg(\Pi|_{M_0})$ must be $-1$. The theorem follows immediately (see Remark 1 after Theorem 1). \(\square\)

**Remark 1.** The argument above for $M_{\text{other}}$ also shows

**Theorem 4.** Let $M_g$ be the Banach manifold of pairs $(\gamma, S)$ where $\gamma$ is a $C^4$ metric of positive ricci curvature on $S^3$ and $S \subset S^3$ is an embedded surface of genus $g + 1$ that is minimal with respect to $\gamma$. Then

$$ \deg(\Pi|_{M_g}) = 0 $$

unless $g = 0$.

**Remark 2.** Mapping degrees were first applied to minimal surface theory by Tomi and Tromba [TT]. Additional applications were given in [W1]. Theorem 3 is the first result for which it is necessary to use the integral mapping degree rather than the simpler mod 2 degree.

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