Hamiltonian perturbation theory in $f(R)$ gravity

Tuomas Multamäki, Jaakko Vainio, and Iiro Vilja

Department of Physics and Astronomy, University of Turku, FIN-20014, Finland

Abstract

Hamiltonian perturbation theory is used to analyse the stability of $f(R)$ models. The Hamiltonian equations for the metric and its momentum conjugate are written for $f(R)$ Lagrangian in the presence of perfect fluid matter. The perturbations examined are perpendicular to $R$. As perturbations are added to the metric and momentum conjugate to the induced metric instabilities are found, depending on the form of $f(R)$. Thus the examination of these instabilities is a way to rule out certain $f(R)$ models.
I. INTRODUCTION

The question of dark energy has been at the heart of cosmology since the discovery of accelerating expansion of the universe [1]. The traditional picture of general relativity with ordinary relativistic or non-relativistic matter in homogeneous and isotropic universe meets severe problems when accommodating it to current cosmological observations. The conflicting observational evidence comes mainly from supernova light curves [1, 2], CMB anisotropies [3, 4] and large scale structures [5, 6]. This has lead to several suggested remedies. Perhaps the most popular way is to add some non-conventional matter to the universe. Among these the simplest possibility is no doubt to use the cosmological constant. A review of the subject can be found in [7]. In any case, the key aspect is the negative pressure of the new matter which boosts the expansion of the universe. Other considerations include more general distribution of matter, i.e. non-homogeneous or non-isotropic universe (see e.g. [8]).

Besides these two, a lot of effort has been put into studies on generalizations and modifications of General Relativity. For example metric-affine theories (see e.g. [9]), scalar-tensor theory (see e.g. [10, 11]), brane-world gravity (see e.g. [12]) and more general Lagrangians have been considered. In the present paper we are especially interested in \( f(R) \) gravity models in which the Einstein-Hilbert action is replaced by a function of the curvature scalar \( R \) [13, 14, 15, 16, 17]. None of these modifications is free of problems and this is indeed the case of \( f(R) \) gravity as well. As for any model, the cosmological observations issue some constraints (see e.g. [18, 19]) as do the observations in the solar system (see e.g. [20, 21, 22, 23, 24]). The opinions are still divided on the viability of \( f(R) \) theories of gravity. There are numerous approving studies (see e.g. [25, 26]) as well as sceptical ones (see e.g. [27, 28]).

As the actual universe is not homogeneous and isotropic but contains local perturbations, additional challenges for \( f(R) \) theories emerge from stability analysis [29, 30, 31]. An acceptable cosmological model has to be stable against perturbations in the metric and the mass distribution. However, stability analysis is customarily done only in the direction of \( R \), i.e. only curvature perturbations are considered. This is motivated in particular in the case of General Relativity, where the relation between space-time curvature and the matter density is a simple one: the trace of Einstein equations imply \( R \propto \rho \). This in turn implicates a simple and direct relation between the perturbations in matter and curvature. This is not the only possibility. In a \( f(R) \) model the model the relation is more complicated due to appearance of function \( f(R) \) and higher derivative terms in the field equations. The phase space is considerably larger and metrics corresponding a given matter distribution ambiguous. The physical acceptability, however, of a model requires general stability; also stability
against perturbations which keep curvature constant, perpendicular to \( R \).

The Hamiltonian formulation of general relativity has been around since the work of Arnowitt, Deser and Misner \[32\]. Hamiltonian formulation has also surfaced in the works of Ashtekar \[33\]. The first papers on the subject often neglected the boundary terms, however, later works have clarified these details (e.g. \[34, 35\]). Hamiltonian formulation has not received too much interest in contemporary papers. In particular and to our knowledge the use of Hamiltonian formulation on perturbations of \( f(R) \) theories has not been studied so far. The main interest has been in specific choices for the function \( f(R) \).

In the present paper we look into perturbations using Hamiltonian formalism of \( f(R) \) theories. While the technique has not yet been applied to general \( f(R) \) theories with perturbations it is a useful tool in studying the stability of \( f(R) \) models: with it is simple to study perturbations perpendicular to \( R \). As in classical mechanics the Hamiltonian is written as a functional of the fields and their canonical momenta. However, in a geometric theory like General Relativity and \( f(R) \) theories, some complications appear due to constraints between field components. The two main aspects of the canonical Hamiltonian formalism are that the field equations are of the first order in the time derivatives and that time is distinguished from other coordinates. For writing the Hamiltonian equations we must thus foliate the region of space-time with space-like hypersurfaces. Finally the resulting field equations for the perturbations are then analyzed for instabilities. The conventions and details of the formalism can be found in \[36\].

The paper is organized as follows. In section II we write the Hamiltonian field equations. The 3+1 decomposition and foliation of the space-time are also presented. The first order perturbations are added to the system in section III. We also take a look at second order perturbations in section IV. In section V we summarize our results.

II. HAMILTONIAN FORMULATION

In this section we mainly follow the treatment of \[36\]. Another overview of Hamiltonian (and Lagrangian) formulation of general relativity can be found in \[37\]. As \( f(R) \) theories of gravity can be written as scalar tensor theories \[20\], we start by writing the \( f(R) \) action as

\[
S = S_G + S_M = \frac{1}{16\pi} \int_V f(R) \sqrt{-g} \, d^4x + S_s + S_M
\]

\[
= \frac{1}{16\pi} \int_V \sqrt{-g} (f(\varphi) + f'(\varphi)R - f'(\varphi)\varphi))d^4x + S_s + S_M,
\]

\[1\] The ideas were first seen in the long out of print Gravitation: an introduction to current research. The authors have later on released the article on arXiv as cited.
where $V$ is a volume of space-time, $S_s$ is a surface term which we will cover later on, $S_M$ is the matter term and $\varphi$ is the auxiliary scalar field. Variation with respect to $\varphi$ would lead to equation $f''(\varphi)(R - \varphi) = 0$. Therefore $R = \varphi$ unless $f''(\varphi) = 0$. If $f''(\varphi) = 0$ the two forms of action are trivially equal. From here on we make the assumption $f''(\varphi) = 0$ unless stated otherwise.

For the purposes of writing the action in terms of the Hamiltonian a 3+1 decomposition is needed. We foliate the space-time volume $V$ with space-like hypersurfaces $\Sigma_t$ of constant time. We shall use the Greek alphabet for space-time and the Latin alphabet for space. By decomposing the line element to (three) scalar, (three) vector and (three) tensor parts as

$$ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt),$$

one defines the lapse $N$, the shift $N_a$, and the induced metric $h_{ab} = g_{\mu\nu}e^a_\mu e^b_\nu$ on the hypersurface $\Sigma_t$. By using these introduced quantities the invariant volume element reads

$$\sqrt{-g} d^4x = N \sqrt{h} dt d^3x.$$ The Ricci scalar can be written in terms of the extrinsic curvature $K_{ab}$, $n_a$ the unit normal to the boundary $\partial V$ and $\tilde{R}$, the induced Ricci scalar of the (three dimensional) metric $h_{ab}$ as

$$R = \tilde{R} + K_{ab}K^{ab} - K^2 - 2\nabla(\nabla n^a n^b - n^a \nabla n^b).$$

Extrinsic curvature is the measure of shrinkage and deformation of an object upon being moved a unit interval of proper time into the enveloping space-time. It can be written as a function of the induced metric, the lapse and the shift, which appear to be the fields we are finally interested in,

$$K_{ab} = \frac{1}{2N} (\dot{h}_{ab} - N_a b - N_b a).$$

The surface terms of the action are not of special interest in this paper. However, it is not trivial that these parts do not affect the results. Generally, the surface term must be added to the action in order to avoid the need for further boundary conditions (e.g. [38]).

By choosing the space-time volume $V$ so that, its boundary can be written as a union of two space-like hypersurfaces $\Sigma_{t_2}, -\Sigma_{t_1}$ with normals pointing outwards and a time-like hypersurface $\mathcal{B}$, i.e. $\partial V = \Sigma_{t_2} \cup (-\Sigma_{t_1}) \cup \mathcal{B}$. The surface term reads

$$S_s = \frac{1}{8\pi} \int_{\partial V} \epsilon f'(\varphi)K|\sqrt{h}| d^3y - S_0 = \frac{1}{8\pi} \left( \int_{\Sigma_{t_1}} f'(\varphi)K \sqrt{h} d^3y - \int_{\Sigma_{t_2}} f'(\varphi)K \sqrt{h} d^3y + \int_{\mathcal{B}} f'(\varphi)K \sqrt{-\gamma} d^3y \right) - S_0,$$

where $\epsilon = n^a n_a$. Here $S_0 = \frac{1}{8\pi} \int_{\partial V} \epsilon K_0 \sqrt{|h|} d^3y$ is a non-dynamical subtraction term the purpose of which is to prevent the integral from diverging in the limit when the spatial boundary $S_t$ is pushed to the infinity. The constant $K_0$ is the extrinsic curvature of the
boundary $\partial V$ embedded in flat space-time. In the last term $\gamma$ is the induced metric on $B$ and $K$ is the extrinsic curvature scalar of $B$. However, this is not the only term contributing to the surface part. The term $f'(\varphi)R$ from (1) produces surface and volume terms, namely

$$
\int_V f'(\varphi)R\sqrt{-g} \, d^4x = \int_{t_1}^{t_2} dt \int_{\Sigma_t} f'(\varphi)(\tilde{R} + K_{ab}K_{ab} - K^2)N\sqrt{h} \, d^3y
$$

$$
- 2 \oint_{\partial V} f'(\varphi)(\nabla n^\alpha n^\beta - n^\alpha \nabla n^\beta) \, d\Sigma_\alpha.
$$

When combining these surface contributions the first two terms in (5) are eliminated. The only surface term left from (6) is

$$
-2 \int_B f'(\varphi)(\nabla n^\alpha n^\beta - n^\alpha \nabla n^\beta) d\Sigma_\alpha = -2 \int_B f'(\varphi)(\nabla n^\alpha) n^3 r_\alpha \sqrt{-\gamma} \, d^3y
$$

$$
= 2 \int_B f'(\varphi)(\nabla r_\alpha) n^3 n^\alpha \sqrt{-\gamma} \, d^3y,
$$

where $r_\alpha$ is the perpendicular unit vector of the boundary $S_t$ of $\Sigma_t$, i.e. $r^\alpha r_\alpha = 1$ and $r^\alpha n_\alpha = 0$. Summing the remaining surface terms we obtain

$$
S_S = 2 \oint_{S_t} (k - k_0)f'(\varphi)N\sqrt{\sigma} \, d^2\theta.
$$

We have also introduced the induced metric on the boundary $S_t$, $\sigma_{AB} = h_{ab}e^A_A e^B_B$ and $\sigma$ is its trace. The extrinsic curvature of $S_t$ embedded in $\Sigma_t$ is $k_{AB}e^A_A e^B_B \nabla b r_\alpha$, $k$ is its trace and similarly $k_0$ with the embedding in flat space. The constant $k_0$ comes from the subtraction term.

We now have the surface part of the action ready for construction of the Hamiltonian. We shall see later on that the surface term (8) is indeed cancelled in the process of calculating the field equations. Many of the technical details were omitted and we refer the reader to [36]. The generalization to $f(R)$ is easy.

In the Hamiltonian formulation field equations are found for fields and their momentum conjugates. Here the fields are $h_{ab}$, $N$, $N_a$ and $\varphi$. It turns out that in the case of $f'(R)$ gravity we need only the momentum conjugate to the induced metric $h_{ab}$. This can be written using the extrinsic curvature

$$
p^{ab} = \frac{\partial K_{cd}}{\partial h_{ab}} \frac{\partial}{\partial h_{ab}} (\sqrt{g}L_G) = \frac{\sqrt{h}f'(\varphi)(K^{ab} - Kh^{ab})}{16\pi}.
$$

For evaluating $\partial K_{cd}/\partial h_{ab}$ the extrinsic curvature was written as a function of the induced metric given in the formula (4).

For writing the Hamiltonian density $H = p^{ab}h_{ab} - \sqrt{-g}L$ we still need the volume part of the action. We write the gravitation part of the action without the surface part (which
we include later on) as
\[
S_{GV} = \frac{1}{16\pi} \int_T dt \left\{ \int_{\Sigma_t} [f(\varphi) + f'(\varphi)(K^{ab}K_{ab} + \tilde{R} - K^2) - f'(\varphi)\varphi]N\sqrt{h}\ d^3x \right\}. \tag{10}
\]

After some manipulations the volume part of the Hamiltonian density can be cast to the form
\[
\mathcal{H}_G = p^{ab}\dot{h}_{ab} - \sqrt{-g}\mathcal{L}_G
\]
\[
= \frac{N\sqrt{h}}{16\pi} \left\{ f'(\varphi)\left[K^{ab}K_{ab} - K^2 - \tilde{R} + \varphi\right] - f(\varphi) \right\}
+ \frac{\sqrt{h}f'(\varphi)}{8\pi} \left[(K^{ab} - Kh_{ab})\mathcal{N}_a\right]_b - \frac{\sqrt{h}f'(\varphi)}{8\pi} (K^{ab} - Kh_{ab})_b\mathcal{N}_a. \tag{11}
\]

To express the Hamiltonian density in terms of adequate variables, i.e. induced metric \(h_{ab}\) and its conjugate momentum \(p_{ab}\), we need to rewrite the extrinsic curvature. By inverting \(\ref{9}\) we get
\[
\sqrt{h}K^{ab}f'(\varphi) = 16\pi (p^{ab} - \frac{1}{2}ph^{ab}) \equiv \hat{p}^{ab} - \frac{1}{2}\hat{p}^2 h^{ab}. \tag{12}
\]

Using this equation the Hamiltonian can be written as a function of the momentum conjugate. Now the volume part of the gravitational Hamiltonian is obtained by integrating \(\mathcal{H}_G\) over the hypersurface \(\Sigma_t\)
\[
H_G = \frac{1}{16\pi} \int_{\Sigma_t} \left\{ N\sqrt{h}\left[f'(\varphi) - f(\varphi) - \tilde{R}f'(\varphi)\right] + \frac{N}{\sqrt{h}f'(\varphi)} \left(\hat{p}_{ab}\hat{p}^{ab} - \frac{\hat{p}^2}{2}\right) \right\}
\]
\[
- 2\sqrt{h}\mathcal{N}_a \left(\frac{\hat{p}^{ab}}{\sqrt{h}}\right)_b\right\}d^3x. \tag{13}
\]

Similarly we get the surface part of the gravitational Hamiltonian by taking the appropriate terms and integrating over the hypersurface \(\Sigma_t\):
\[
H_S = \frac{1}{8\pi} \int_{S_t} \left[ N(k - k_0) - \frac{N_a\hat{p}^{ab}r_b}{\sqrt{h}} \right] f'(\varphi)\sqrt{\sigma}d^2\theta. \tag{15}
\]

The latter term is produced by applying Gauss theorem to the middle term of \(\ref{11}\) when integrating over the density. We obtain the field equations by varying the action with respect to \(N, N^a, h_{ab}, p_{ab}\) and \(\varphi\). As we can see in the action the only time derivatives are those of the induced metric. Thus the only dynamic field is \(h_{ab}\) and the only momentum conjugate needed is \(p_{ab}\). We have the normal boundary conditions for the variations vanishing on the boundary
\[
\delta N = \delta N^a = \delta h_{ab} = \delta \varphi = 0. \tag{16}
\]

The full Hamiltonian \(H\) includes both surface and volume parts as well as a matter part \(S_M\). Since we can write variation of the action as
\[
\delta S = \int_{t_1}^{t_2} dt \left[ \int_{\Sigma_t} \left( p^{ab}\delta h_{ab} + \dot{h}_{ab}\delta p^{ab}\right) d^3y - \delta H \right] \tag{17}
\]
the Hamiltonian equations are of the form
\[
\dot{h}_{ab} = \frac{\partial H_G}{\partial p}, \quad \dot{\rho}_{ab} = -\frac{\partial H_G}{\partial h} + \delta S_M, \quad \frac{\partial H_G}{\partial N_a} = 0, \quad \frac{\partial H_G}{\partial N} = \delta S_M, \quad \frac{\partial H_G}{\partial \varphi} = 0.
\] (18)

To simplify the field equations, we can choose the foliation to be such that \( N_a = 0 \) and hence \( h_{ab} = g_{ab} \), when the effects of the surface terms vanish. This choice removes one field equation, that of \( N_a \), and the other ones are much simplified. After tedious calculations we end up with equations
\[
16\pi \sqrt{h} \rho = \left( \tilde{R} + K^2 - K_{ab}K^{ab} - \varphi \right) f'(\varphi) \sqrt{h} + f(\varphi) \sqrt{h},
\] (19b)
\[
\dot{h}_{ab} = \frac{2N}{\sqrt{h} f'(\varphi)} \left( p_{ab} - \frac{1}{2} p h_{ab} \right),
\] (19c)
\[
\varphi - \tilde{R} = \frac{\dot{\rho}^2}{\sqrt{h} \ f'(\varphi)^2},
\] (19d)

where \( \tilde{G}_{ab} = \tilde{R} - \frac{1}{2} \tilde{R} h_{ab} \). For technical details we refer the reader to [36] which can straightforwardly be generalized to the \( f(R) \) case. Note, however, that in the derivation of the equation (19d) further use is made of the assumption \( f''(R) \neq 0 \). Otherwise, we would get a trivial equality. As can be seen in equations (19a) and (19b) we have also added matter
\[
\delta S_m \delta h_{ab} = -\sqrt{g} T_{ab} \delta g^{00} \delta h = -\sqrt{h} \rho,
\] (20)
\[
\delta S_m \delta h_{ab} = -\sqrt{g} T_{ab} \delta g_{ab} \delta h = -\frac{N \sqrt{h}}{2} P h_{ab},
\] (21)

which is of the perfect fluid form.

Even though we assumed from the start that \( f''(R) \neq 0 \) it is worthwhile to take a look at the case of Einstein-Hilbert Lagrangian. If in Eq. (1) we choose \( f(R) = R \), the equality is trivial, and only a variation of a constant resulting in a trivial field equation. As the assumption of \( f''(R) \neq 0 \) is needed in the field equations only in (19d) the equations would stay the same except for this one equation which is irrelevant. From equation (19b) we get the familiar Friedmann equation for the background
\[
\mathcal{H}^2 = \frac{8\pi \rho_0}{3a},
\] (22)
in a matter dominated universe (\( \rho = a^{-3} \rho_0 \)). Here \( \mathcal{H} = a'(\eta)/a(\eta) \) is the conformal Hubble parameter. We will need this background result later on when we insert the asymptotic background solution into the equations. Namely, we can solve \( a'(\eta) \) from this equation.
In this section we add first order perturbations to the metric and the momentum conjugate. In general relativity the trace equation connects curvature and matter density (for fixed equation of state $p = p(\rho)$) by a simple relation $R = \kappa(\rho - 3P)$, where $\kappa = 8\pi G$. The perturbations would be connected correspondingly: $\delta R = \kappa(\delta \rho - 3\delta P)$. As the trace equation in $f(R)$ gravity is $f'(R)R - 2f(R) + 3f'(R) = \kappa(\rho - 3P)$ there are more freedom in metrics that produce a given mass configuration. Indeed, the relation between curvature and matter distribution is no more an algebraic one, but defined by a differential equation. Thus the phase space of metrics is larger and there are perturbations keeping $R$ and thus $\rho$ fixed. This is manifested by the statement that Birkhoff’s theorem is no more valid in the traditional form in $f(R)$ theories. Since there are number of studies of the perturbations along $R$ (e.g. [29]) we are now interested in the opposite and do not introduce perturbations to matter but perturbations perpendicular to $R$ only, i.e. $\delta R = \delta \rho = 0$.

We may add the most general first order perturbations to the metric. These include scalar, vector and tensor perturbations. In light of the recent observations and for simplicity we examine the case of spatially flat FRW metric. The perturbations in first order can now be written as

\begin{align*}
    g_{00} &= \tilde{g}_{00} - 2a^2 \Phi, \\
    g_{0a} &= \tilde{g}_{0a} + a^2(\partial_a \omega + \omega_a), \\
    g_{ab} &= \tilde{g}_{ab} + a^2 \left( -2\Psi \delta_{ab} + \nabla_a \nabla_b \chi + \partial_a \chi_b + \partial_b \chi_a + \chi_{ab} \right),
\end{align*}

where tilde denotes the background part and the vectors $\omega^a$ and $\chi^a$ are transverse (i.e. $\partial^a \omega_a = 0$, $\partial^a \chi_a = 0$) and $\chi_{ab}$ is trace free and symmetric tensor (i.e. $\partial^a \chi_{ab} = 0$, $\chi^a_a = 0$).

Comparing the elements in (23) and the line element (2) to find the perturbations in the first order for lapse, shift and the induced metric we obtain

\begin{align*}
    N &= \tilde{N} + a \Phi, \\
    N_a &= \tilde{N}_a + a^2(\partial_a \omega + \omega_a) \equiv \tilde{N}_a + a^2 \dot{\omega}_a, \\
    h_{ab} &= \tilde{h}_{ab} + a^2 \left( -2\Psi \delta_{ab} + \nabla_a \nabla_b \chi + \partial_a \chi_b + \partial_b \chi_a + \chi_{ab} \right) \\
    &\equiv \tilde{h}_{ab} + a^2 \left( -2\Psi \delta_{ab} + \dot{\chi}_{ab} \right).
\end{align*}

The standard practice of splitting the perturbations into scalar, vector and tensor parts is motivated by the reason that in a linear theory these modes decouple. Moreover

\footnote{Birkhoff proved the so called Birkhoff theorem in 1923 [39]. However, two years earlier a less known Norwegian physicist Jebsen presented the idea in [40]. The history of the theorem is examined in [41].}
each of them has a clear physical interpretation. The first order vector perturbations are not generated in the presence of scalar perturbations and dissipate over time. Tensor perturbations cause gravitational waves which do not couple to first order scalar perturbations. Therefore, we may omit vector and tensor perturbations in the first order case and assume $\omega_a = 0$, $\chi_{ab} = 0$. We can further simplify the metric for our purposes by choosing an appropriate gauge. We choose to use the Poisson gauge which is a generalization of the much used longitudinal gauge. The gauge conditions are

$$\nabla \cdot \hat{\chi} = 0, \quad (25a)$$
$$\nabla \cdot \hat{\omega} = 0. \quad (25b)$$

Since $\omega^a$ and $\chi_a$ are transverse vectors and $\chi_{ab}$ is a symmetric, transverse and trace-free tensor we have $\omega = \chi = \chi_a = 0$. Along with the physical meaning of the perturbations discussed above the perturbed metric simplifies to

$$N = \tilde{N} + a\Phi, \quad (26a)$$
$$N_a = \tilde{N}_a, \quad (26b)$$
$$h_{ab} = \tilde{h}_{ab} - 2a^2\Psi\delta_{ab}. \quad (26c)$$

As the dynamical components of the metric are coupled to their momentum conjugates, we are to add perturbations also to the conjugates. Only the induced metric $h_{ab}$ has a conjugate $p_{ab}$, and hence for perturbed one we write

$$p_{ab} = \tilde{p}_{ab} + \Theta\delta_{ab} \quad (27)$$

having same structure as (26c).

In the following we work mostly, unless otherwise stated, in conformal time instead of standard coordinate time. So we have $ds^2 = -a(\eta)^2d\eta^2 + a(\eta)^2\delta_{ab}dx^adx^b$, where the conformal time $\eta$ is related to standard coordinate time by $d\eta = a^{-1}dt$. Prime denotes derivatives with respect to the conformal time and dot denotes derivatives with respect to the coordinate time. This choice of background metric corresponds to $\tilde{R} = \tilde{G}_{ab} = 0$ and $\sqrt{\tilde{h}} = a^3$. Also, we now have $\tilde{p}_{ab} = -2f'(\varphi)a^3\varphi'$. Since we wrote the $f(R)$ theory using a scalar in (11) we have $\varphi \sim R$. Perturbing $\varphi$ would produce perturbations parallel to $R$ which we are not interested in.

The Eqs. (19) for the chosen background metric and scalar field are now given in a fairly
simple form. This reads

\[
16\pi P a^4 = 2(a')^2 f'(\varphi) - a^4 \left(f(\varphi) - \varphi f'(\varphi)\right) - 4a \left(a'' f'(\varphi) + a' \varphi' f''(\varphi)\right),
\]

(28a)

\[
16\pi \rho a^3 = \frac{f'(\varphi) \left(6(a')^2 - a^4 \varphi\right)}{a} + a^3 f(\varphi),
\]

(28b)

\[
\varphi = \frac{6(a')^2}{a^4}.
\]

(28c)

We get only three non-trivial equations as (19c) produces only a trivial identity. These equations, satisfied for any acceptable matter are used to simplify the perturbation equations derived later. In the following we assume a matter filled universe with \(P = 0\) and \(\rho = \rho_0/a^3\).

By adding the perturbations introduced in (24) and (27) to the equations of motion (19) we get three equations for the large scale perturbations (i.e. space independent perturbations)

\[
\Psi = \Theta \frac{\Theta}{10a^5 a' f'(\varphi)},
\]

(29a)

\[
\Theta' = \left(\frac{3a' a''}{a} + \frac{12a' f''(\varphi)(aa'' - 2(a')^2)}{a^5 f'(\varphi)}\right)\Theta,
\]

(29b)

\[
\Phi = 0,
\]

(29c)

where we in (29b) the background equations (28a) and (28c) were applied to simplify the equation. We immediately notice, that there remains only one dynamic equation while the other two are algebraic. The background equation for the induced metric can be used to eliminate the second time derivative of the scale parameter. Eq. (29b) is thus written as

\[
\Theta' = \left(\frac{5a'}{a} - \frac{4\pi \rho_0}{a' f'(\varphi)}\right)\Theta.
\]

(30)

The behaviour of perturbation is clearly dependent on form of the function \(f(\varphi)\) explicitly via its derivatives. Moreover, it is found that the time derivative of \(\Psi\) is zero and therefore by equation (29a) we can write

\[
\Theta = Ca^3 a' f'(\varphi)
\]

(31)

where \(C\) is a constant. So, in a universe with growing \(a(\eta)\) the perturbations in momentum conjugate increase. The perturbations of metric tensor, however, behave differently: the temporal part vanishes and the spatial perturbations are constant. So, the system leaves the linear perturbative regime and ultimately suffers linear instability.

Although asymptotic analysis is ultimately irrelevant for a linearized unstable system, we take a look to some examples to get a better feeling of the evolution. As known, the simple function \(f(R) = R - \mu^4/R\) results asymptotic Einstein-de-Sitter behaviour. Now \(a(t) = e^{\Lambda t},\)
and coordinate and conformal times are related by $\eta = -e^{-\Lambda t}/\Lambda + c$ so that $a(\eta) = \Lambda^{-1}(c-\eta)^{-1}$. In the high curvature limit we get

$$\Theta(\eta) = \hat{C} \frac{36\Lambda^4 + \mu^4}{36(c-\eta)^5\Lambda^8}$$

where $\hat{C}$ is a constant. The result can also be written more intuitively in coordinate time as

$$\Theta(t) = C e^{4\Lambda t} \left(1 + \frac{e^{4\Lambda t}\mu^4}{36\Lambda^4}\right),$$

and thus the perturbations increase as time goes to infinity. Here $C$ is another constant. Ultimately the first order perturbation theory breaks down; it is not applicable in this case. Similar behaviour can seen explicitly for another often used $f(R) = R - \mu^2 R^2$.

Even though we have not included perturbations in matter it is worthwhile to check what would happen if we did include these perturbations. For a moment we consider $\rho = \tilde{\rho} + \sigma$, where $\sigma$ is a perturbation. It turns out that no density perturbations are present, i.e. perturbation equation is $\sigma = 0$. This is not surprising as the matter perturbations are coupled to the temporal perturbation of the metric which is also zero. These vanish unless $\varphi$ (which is essentially $R$) is perturbed, too.

As we have found, the only dynamical equation is (29b) for the momentum conjugate, while the two other equations determine, how metric perturbations follow it; they are constraint equations. If these constraints were to be discarded, we would end up with non-diagonal perturbations in the metric. Moreover non-existence of temporal perturbations is connected with the orthogonality of perturbations to curvature. As it appears that the spatial perturbations in the metric do not grow or vanish in time, there is a flat direction of phase space, where any spatial first order perturbation is possible and stable.

**IV. SECOND ORDER PERTURBATIONS**

We have now seen that the first order perturbation predicts that $f(R)$ theories suffer instability which invalidates first order expansion; equation (30) reveals that we cannot use first order perturbation theory. The next check would be to consider second order perturbations, which might give us further understanding of the perturbations involved. We
first write the most general form of the metric and the conjugate momentum as

\[ N = \tilde{N} + a(\Phi^{(1)} + \Phi^{(2)}), \]  
\[ N^a = \tilde{N}^a + a \sum_{r=1}^{2} (\partial_a \omega^{(r)} + \omega_a^{(r)}), \]  
\[ h_{ab} = \tilde{h}_{ab} + a^2 \left[ - 2\Psi^{(1)} + \Psi^{(2)} \right] \delta_{ab} + \sum_{r=1}^{2} \left( \nabla_a \nabla_b \chi^{(r)} + \partial_a \chi_b^{(r)} + \partial_b \chi_a^{(r)} + \chi^{(r)}_a \right) \],  
\[ p_{ab} = \tilde{p}_{ab} + (\Theta^{(1)} + \Theta^{(2)}) \delta_{ab}, \]

where the upper index \( i \) denotes the order of the perturbation. As we have chosen to work in the Poisson gauge \[45\] we have \( \omega^{(r)} = \chi^{(r)} = \chi_a^{(r)} = 0 \). The vector perturbations \( \omega_a^{(r)} \) and \( \chi_a^{(r)} \) still remain, however, and some extra attention has to be paid to them. In general the scalar, vector and tensor perturbations do not decouple any more in the second order perturbation theory. First order vector perturbations contribute to the second order scalar perturbations by terms like \( \omega_a \omega_a \) and vice versa. However, first order perturbations do not manifest themselves if not present initially. Since we are now interested in to show the instability of the system, it is sufficient that some initial condition reveals unstable behaviour. In particular we are free to choose initial condition \( \omega_a(0) = 0 \) for the vector perturbations. First order tensor perturbations can omit them as well. Note, that if we were trying to show the stability of the system, the burden of proof would be much heavier: we should show, that any choice of initial conditions leads to stable system.

As mentioned, vector and tensor perturbations in second order cannot be discarded by similar arguments. They are strongly affected by first order scalar perturbations. However, the second order scalar perturbations are again independent of the tensor and scalar perturbations of the second order. Therefore, for our purposes, it is sufficient to study only second order scalar perturbations, which can be performed rather simply. We write the relevant perturbation equations for second order in the same manner as in the previous section. We obtain

\[ \Psi^{(2)} = 0, \]  
\[ \Theta^{(2)} = -\frac{3}{20a^3 a' f'(\varphi)} (\Theta^{(1)})^2 + 5a^3 a' f'(\varphi) \Psi^{(2)}, \]  
\[ \Phi^{(2)} = 0. \]

Thus metric perturbation \( \Phi^{(2)} \) still vanishes and \( \Psi^{(2)} \) is again constant related to the perturbation of the momentum conjugate \( \Theta^{(2)} \) by \(33b\). The perturbation in the momentum conjugate is still depending on the form of the \( f(R) \). For \( f(R) = R - \mu^4/R \) the result is the same as in the first order; the perturbation of the temporal part disappear, the spatial
part remains constant and the momentum conjugate is has the only dynamical equation. It is clear that the instabilities in the first order propagates to the second order as the metric perturbations behave in exactly the same way in both first and second order. Thus $f(R)$ models may be inherently unstable up to second order when examining perturbations perpendicular to $R$. Because of the similar form of the first and the second order scalar perturbations one might conjecture that it is a more general feature of the theory.

V. DISCUSSION

Traditionally the stability analysis is performed in the Lagrangian formalism and the analysis parallel to $R$ has been carried out before in several papers (e.g. [29]). Many of the interesting $f(R)$ models have been found to be inherently unstable in the past [31, 47]. However, stability analysis has not yet been used to the full extent as long as the studies concentrate on curvature perturbations only. By using the Hamiltonian instead of Lagrangian formulation we examined the large scale cosmological perturbations perpendicular to $R$ with non-relativistic matter. These perturbations are fairly easy to examine with the Hamiltonian formulation. The found instabilities are noticeably different to those of previous works (e.g. [31]). Because of the constraint $\dot{R} = 0$ diagonal perturbations of metric and conjugate momentum are related to each other. The temporal part of the metric showed up to be constrained by the conjugate momentum one. Moreover, the spatial part of the metric is forced to vanish. If these constraints were not satisfied we would have non-trivial perturbations of non-diagonal elements of the metrics like $g_{0i}$.

The perturbations of the momentum conjugate turn out to be the most interesting ones. The equation depends explicitly on the form of the function $f(R)$. Some choices of $f(R)$ lead clearly unstable cosmological model, but seemingly not all. We have studied some well-known $f(R)$ functions and find them unstable. Albeit the physical interpretation of the perturbation momentum conjugate is unfortunately not as clear as that of the metric perturbations, equation (9) demonstrates the relation between the momentum conjugate and the extrinsic curvature. In the 3+1 decomposition the intrinsic curvature $\tilde{R}$ defines how the hypersurface is curved whereas the extrinsic curvature defines how each slice is curved relative to the enveloping space-time.

As the perturbations were not well-behaved in this context further studies would be relevant in order to find the limits of these constraints. Fruitful directions would likely to be investigating the effects of more other types of matter. Also, it would be prudent to examine the case where the metric can include shift (i.e. $N_a \neq 0$). It is clear from the form of (11) that such a generalization would affect the following equations of motion deeply as the last term would be non-zero. This is understandable as the metric would now include spatio-
temporal elements. It is also possible to study more general theories with the Lagrangian depending also on for example $R_{\mu\nu}R^{\mu\nu}$ or Gauss-Bonnet term.

It appears that with Hamiltonian formulation of perturbations can be used to constrain the spectrum of cosmologically acceptable $f(R)$ theories. While there are several physical arguments to judge the $f(R)$ theories like cosmological observations and solar system behaviour, stability analysis is one important tool to rule out ill-behaved models out of numerous possible modified theories of gravity. With continued studies it is possible to find the ones best describing the observed behaviour of the universe.

Acknowledgments

This project has been partly funded by the Academy of Finland project no. 8111953. JV is also supported by the Magnus Ehrnroothin säätiö foundation.

[1] A. G. Riess et al. (Supernova Search Team), Astron. J. 116, 1009 (1998), arXiv:astro-ph/9805201
[2] S. Perlmutter et al. (Supernova Cosmology Project), Astrophys. J. 517, 565 (1999), arXiv:astro-ph/9812133
[3] C. L. Bennett et al. (WMAP), Astrophys. J. Suppl. 148, 1 (2003), arXiv:astro-ph/0302207
[4] C. B. Netterfield et al. (Boomerang), Astrophys. J. 571, 604 (2002), arXiv:astro-ph/0104460
[5] S. Perlmutter, M. S. Turner, and M. J. White, Phys. Rev. Lett. 83, 670 (1999), arXiv:astro-ph/9901052
[6] E. F. Bunn and M. J. White, Astrophys. J. 480, 6 (1997), arXiv:astro-ph/9607060
[7] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003), arXiv:astro-ph/0207347
[8] H. Alnes, M. Amarzguioui, and O. Gron, Phys. Rev. D73, 083519 (2006), arXiv:astro-ph/0512006
[9] T. P. Sotiriou and S. Liberati, Annals Phys. 322, 935 (2007), arXiv:gr-qc/0604006
[10] V. Faraoni, Cosmology in Scalar-Tensor Gravity (Kluwer Academic Publishers Group, Norwell, MA, USA, and Dordrecht, The Netherlands, 2004)
[11] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961)
[12] R. Maartens, Living Rev. Rel. 7, 7 (2004), arXiv:gr-qc/0312059
[13] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D70, 043528 (2004), arXiv:astro-ph/0306438
[14] S. M. Carroll et al., Phys. Rev. D71, 063513 (2005), arXiv:astro-ph/0410031
[15] X. Meng and P. Wang, Class. Quant. Grav. 21, 951 (2004), arXiv:astro-ph/0308031
[16] G. Allemandi, A. Borowiec, and M. Francaviglia, Phys. Rev. D70, 103503 (2004), arXiv:hep-th/0407090
[17] T. P. Sotiriou and V. Faraoni (2008), arXiv:0805.1726 [gr-qc]
[18] S. Tsujikawa, Phys. Rev. D77, 023507 (2008), arXiv:0709.1391 [astro-ph]
[19] A. A. Starobinsky, JETP Lett. 86, 157 (2007), arXiv:0706.2041 [astro-ph]
[20] T. Chiba, Phys. Lett. B575, 1 (2003), astro-ph/0307338
[21] W. Hu and I. Sawicki, Phys. Rev. D76, 064004 (2007), arXiv:0705.1158 [astro-ph]
[22] G. Magnano and L. M. Sokolowski, Phys. Rev. D50, 5039 (1994), arXiv:gr-qc/9312008
[23] T. Multamäki and I. Vilja, Physics Letters B 659, 843 (2008), arXiv:0709.3422
[24] K. Henttunen, T. Multamäki, and I. Vilja, Phys. Rev. D 77, 024040 (2008), arXiv:0705.2683 [gr-qc]
[25] V. Faraoni, Phys. Rev. D74, 023529 (2006), arXiv:gr-qc/0607016
[26] G. J. Olmo, Phys. Rev. D75, 023511 (2007), arXiv:gr-qc/0612047
[27] T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, Phys. Rev. D76, 063505 (2007), arXiv:astro-ph/0612569
[28] A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D74, 121501 (2006), arXiv:astro-ph/0610483
[29] A. D. Dolgov and M. Kawasaki, Phys. Lett. B573, 1 (2003), arXiv:astro-ph/0307285
[30] M. E. Soussa and R. P. Woodard, Gen. Rel. Grav. 36, 855 (2004), arXiv:astro-ph/0308114
[31] V. Faraoni, Phys. Rev. D72, 124005 (2005), arXiv:gr-qc/0511094
[32] R. L. Arnowitt, S. Deser, and C. W. Misner (1962), arXiv:gr-qc/0405109
[33] A. Ashtekar, Phys. Rev. D36, 1587 (1987)
[34] J. D. Brown and J. York, James W., Phys. Rev. D47, 1407 (1993), arXiv:gr-qc/9209012
[35] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. 13, 1487 (1996), arXiv:gr-qc/9501014
[36] E. Poisson, A Relativist’s Toolkit (Cambridge University Press, Cambridge, UK, 2004)
[37] R. Wald, General Relativity (Chicago University Press, Chicago, 1984)
[38] L. Querella, Variational principles and cosmological models in higher-order gravity, Ph.D. thesis, Université de Liège (1998), arXiv:gr-qc/9902044
[39] G. D. Birkhoff, Relativity and Modern Physics (Harvard University Press, Cambridge, MA, USA, 1923)
[40] J. Jebsen, Ark. Mat. Ast. Fys. 15 (1921)
[41] N. Voje Johansen and F. Ravndal, Gen. Rel. Grav. 38, 537 (2006), arXiv:physics/0508163
[42] K. A. Bronnikov and V. N. Mehlkov, Gen. Rel. Grav. 27, 465 (1995), arXiv:gr-qc/9403063
[43] H.-J. Schmidt, Grav. Cosmol. 3, 185 (1997), arXiv:gr-qc/9709071
[44] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, *Phys. Rept.* **402**, 103 (2004), arXiv:astro-ph/0406398

[45] E. Bertschinger (1993), arXiv:astro-ph/9503125

[46] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Phys. Rept.* **215**, 203 (1992)

[47] K. Kainulainen, J. Piilonen, V. Reijonen, and D. Sunhede, *Phys. Rev.* **D76**, 024020 (2007), arXiv:0704.2729 [gr-qc]