Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit

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Modelling traffic flow.

Practical goals.
- Rational planning and management of vehicle fluxes.
- Reduce environmental pollution and cities congestion.

Mathematical modelling.
- **Microscopic approach.** Each car is a ‘moving particle’ satisfying an ODE.
- **Macroscopic approach.** Averaged quantities satisfying PDEs.

Advantages of the macroscopic approach.
- Very powerful description of queues tails in terms of shocks.
- Suitable with very large number of vehicles.
- Easy to validate and implement (low number of parameters).
- Suitable to *real time* prediction, estimation, optimization and control.
Modelling traffic flow.

Practical goals.
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Mathematical modelling.
- **Microscopic approach.** Each car is a ‘moving particle’ satisfying an ODE.
- **Macroscopic approach.** Averaged quantities satisfying PDEs.

Continuum hypothesis is not satisfied!
The number of cars is far lower than that of molecules, for example, in gas dynamics (1 mole of gas contains $6 \times 10^{23}$ molecules), the continuum assumption is not justified and the macroscopic formulation is not a priori justified!

Link between macroscopic and microscopic approach.
It provides a validation of the macroscopic approach and of the use of data collection from GPS devises when the number of detected ‘reference cars’ is very large.
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The macroscopic variables.

The macroscopic variables are:

- $\rho$: density, \#cars per unit length of the road
- $v$: velocity, space covered per unit time by the cars
- $f$: flow, \#cars per unit time

The macroscopic variables satisfy:

- by definition: \( f = \rho v \)
- by conservation of \#cars: \( \rho_t + f_x = 0 \)

We have 3 variables and 2 equations!
Two macroscopic approaches.

- **First order** models close the system by giving an explicit expression of 1 of the 3 unknowns in terms of the remaining 2 (equation of state). Example: Lighthill-Whitham-Richards (1955, 1956).

\[ \rho_t + [\rho v(\rho)]_x = 0. \]

- Equilibrium models.
- Velocity as function of the density.

- **Second order** models close the system by adding a further PDE. Example: Aw-Rascle-Zhang (1999, 2002).

\[ \rho_t + (\rho v)_x = 0, \quad [v + p(\rho)]_t + v[v + p(\rho)]_x = 0. \]

- Continuum analogue of Newton’s law.
- Velocity evolves via a separate PDE.

**Justification of the macroscopic models.**

- **A posteriori.** Descriptive power (no physical laws).
- **A priori.** Validation via microscopic models (no continuum hypothesis).
The LWR model.

\[ \rho_t + [\rho v(\rho)]_x = 0, \quad \rho(t = 0) = \bar{\rho} \]

Relevant parameters.

- Maximum density of vehicles \( \rho_{\text{max}} > 0 \). We normalize \( \rho_{\text{max}} = 1 \).
- Maximum possible speed \( v_{\text{max}} > 0 \).
- Total length of the vehicles (constant in time)
  \[ L = \int_{\mathbb{R}} \rho(t, x) dx > 0. \]

Main assumptions on the velocity map \( v \).

- \( v \in C^1([0, 1]; [0, v_{\text{max}}]) \).
- \( v \) strictly decreasing on \([0, 1]\).
- \( v(0) = v_{\text{max}}, v(1) = 0 \).

Initial condition.

- \( \rho(t = 0) = \bar{\rho} \in L^\infty(\mathbb{R}), \bar{\rho} \geq 0, \bar{\rho} \) with compact support.
Examples of velocities (from empirical observations).

Greenshields 1935:

\[ v(\rho) = \nu_{\text{max}}(1 - \rho) \]

\[ f \]

\[ 1 \quad \rho \]

\[ v \]

\[ 1 \quad \rho \]
Examples of velocities (from empirical observations).

Greenberg 1959, ‘renormalized’ version

\[ v(\rho) = v_{\text{max}} \left[ \log \left( \frac{1 + \alpha}{\alpha} \right) \right]^{-1} \log \left( \frac{1 + \alpha}{\rho + \alpha} \right) \]

\[ \alpha > 0 \]
Examples of velocities (from empirical observations).

Pipes-Munjal 1967

\[ v(\rho) = v_{\text{max}} \left( 1 - \rho^\alpha \right) \quad \alpha > 0 \]

Example: \( \alpha = 0.2 \)
Examples of velocities (from empirical observations).

Pipes-Munjal 1967

\[ v(\rho) = v_{\text{max}} \left( 1 - \rho^\alpha \right) \quad \alpha > 0 \]

Example: \( \alpha = 2 \)
A quick review on the mathematical theory.

- Discontinuous solutions, weak solutions: for all $\phi \in C^1_c([0, +\infty) \times \mathbb{R})$,
  \[
  \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left[ \rho(t, x) \phi_t(t, x) + f(\rho(t, x)) \phi_x(t, x) \right] \, dt \, dx + \int_{\mathbb{R}} \bar{\rho}(x) \phi(0, x) \, dx = 0
  \]

- Non uniqueness of weak solutions. Uniqueness of entropy solutions (Oleinik 1963, Kružkov 1970): for all test functions $\phi \geq 0$ and for all $k \in \mathbb{R}$,
  \[
  \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left[ |\rho(t, x) - k| \phi_t(t, x) + \text{sgn}(\rho(t, x) - k) \left[ f(\rho(t, x)) - f(k) \right] \phi_x(t, x) \right] \, dt \, dx + \int_{\mathbb{R}} \phi(0, x)|\bar{\rho}(x) - k| \, dx \geq 0 \tag{1}
  \]

- Initial trace. Uniqueness if (1) is satisfied with $\phi(t = 0) = 0$ and initial trace is reached in the weak-$\ast$ $L^\infty$-topology, provided $f$ is not affine a.e. (Chen-Rascle 2000).

- Oleinik condition. Entropy solutions are characterised by
  \[ f'(\rho)x \leq \frac{1}{t}, \quad \text{in } D'((0, +\infty) \times \mathbb{R}). \]
Constructing entropy solutions.

Regularization strategy.
- Vanishing viscosity, see Dafermos 2000 and the references therein.

Numerical methods.
- Finite differences. Glimm 1965.
- Wave front tracking method. Dafermos 1972.

Mesoscopic approximation.
- Kinetic approximation. Lions-Perthame-Tadmor 1994.

Microscopic probabilistic approach.
- Exclusion processes (list incomplete!). Rost 1982, Ferrari-Fouque 1987.

Microscopic deterministic approach.
- Sticky particles. Brenier-Grenier 1998.
- Lagrangian, follow-the-leader type systems.
  - Formal derivation: Whitham 1974, Colombo-Rossi 2013.
  - Rigorous derivation: Di Francesco-Rosini 2015.
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The follow-the-leader particle approximation.

Fix the initial condition $\bar{\rho}$ and let $L$ be its total mass.

- Fix $n \in \mathbb{N}$ and let $\ell = \ell_n = L 2^{-n}$ be the length of each platoon of cars.
- Consider $N + 1 = N_n + 1 = 2^n + 1$ (ordered) reference cars $x_0, x_1, \ldots, x_N$ corresponding to the end points of the $N$ platoons.

The cars $x_0, \ldots, x_N$ evolve according to

$$
\dot{x}_N(t) = v_{\text{max}},
$$

$$
\dot{x}_i(t) = v \left( \frac{\ell}{x_{i+1}(t) - x_i(t)} \right), \quad i = N - 1, \ldots, 0.
$$

The initial conditions of the cars are taken by atomization of $\bar{\rho}$, i.e.

$$
x_0(t = 0) = \bar{x}_0 = \min(\text{spt}(\bar{\rho})),
$$

$$
x_i(t = 0) = \bar{x}_i = \sup \left\{ x \in \mathbb{R} : \int_{\bar{x}_{i-1}}^{x} \bar{\rho}(y) \, dy < \ell \right\}, \quad i = 1, \ldots, N.
$$
Atomization of the initial condition.

The initial condition $\bar{\rho}$ is split into $N$ parts with equal integral $\ell$. 
Large particle limit.

Empirical measure

\[ \tilde{\rho}^n(t) = \sum_{i=1}^{N} \ell_n \delta_{x_i(t)} \]

Discrete density

\[ \hat{\rho}^n(t, x) = \sum_{i=1}^{N} y_i^n(t) \chi_{[x_i(t), x_{i+1}(t)]}(x), \quad y_i^n(t) = \frac{\ell_n}{x_{i+1}(t) - x_i(t)} \]

Goal:
Prove that \( \tilde{\rho}^n(t) \) and \( \hat{\rho}^n(t, \cdot) \) converge to the unique entropy solution \( \rho \) of the LWR equation with \( \bar{\rho} \) as initial condition.
Empirical measure and discrete density.

\[ \tilde{\rho}(t) = \sum_{i=1}^{N} \ell \delta_{x_i(t)} \]

\[ \hat{\rho}(t, x) = \sum_{i=1}^{N} y_i(t) \chi_{[x_i(t), x_{i+1}(t))}(x) \]
Heuristic derivation.

- Let $\rho$ be the entropy solution of

$$
\rho_t + [\rho v(\rho)]_x = 0. \tag{2}
$$

- Let $F$ be the cumulative distribution of $\rho$:

$$
F(x, t) = \int_{-\infty}^{x} \rho(t, y) dy.
$$

- Let $X$ be the pseudo inverse of $F$:

$$
X(t, z) = \inf \{ x \in \mathbb{R} : F(x) > z \}, \quad z \in [0, L].
$$

- Formally, $X(t, z)$ satisfies $F(t, X(t, z)) = z$ and

$$
\begin{cases}
F_x = \rho \\
F_t = -\rho v(\rho)
\end{cases}
\Rightarrow \begin{cases}
1 = F(t, X(t, z))_z = F_x X_z = \rho X_z \\
0 = F(t, X(t, z))_t = F_t + F_x X_t = \rho (X_t - v(\rho))
\end{cases} \Rightarrow X_t = v \left( \frac{1}{X_z} \right). \tag{3}
$$

- Forward $z$-finite difference of (3) with step $\ell$ gives

$$
X_t(t, z) = v \left( \frac{\ell}{X(t, z + \ell) - X(t, z)} \right), \quad z = 0, \ldots, (N - 1)\ell. \tag{4}
$$

The follow-the-leader system (4) is the discrete Lagrangian version of (2).
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Convergence Theorem.

Let $\rho$ be the unique entropy solution of

$$
\rho_t + [\rho v(\rho)]_x = 0, \quad \rho(t = 0) = \bar{\rho},
$$

where $\bar{\rho}$ is in $\mathcal{M}_L \cap L^\infty(\mathbb{R})$, $v$ is in $C^1(\mathbb{R}_+)$, strictly decreasing with $v(0) = v_{\text{max}} > 0$.

Theorem (MDF and MDR, ARMA 2015.)

If

- $\bar{\rho} \in BV(\mathbb{R})$,

or

- $\mathbb{R}_+ \ni \rho \mapsto \rho v'(\rho) \in \mathbb{R}_+$ is non-increasing.

Then,

- the sequence $\hat{\rho}^n \to \rho$ almost everywhere and in $L^1_{\text{loc}}([0, +\infty) \times \mathbb{R})$.
- the sequence $\tilde{\rho}^n \to \rho$ in the topology $L^1_{\text{loc}}([0, +\infty); d_{L,1})$, where $d_{L,1}$ is the scaled 1-Wasserstein distance.

$$
\mathcal{M}_L = \{ \rho \text{ Radon measure on } \mathbb{R} \text{ with compact support}: \rho \geq 0, \rho(\mathbb{R}) = L \}
$$

$$
d_{L,1}(\rho_1, \rho_2) = L d_1(\rho_1/L, \rho_2/L) = \| F_{\rho_1} - F_{\rho_2} \|_{L^1(\mathbb{R})}
$$
Ingredients: cumulative distributions.

\[ x \mapsto \hat{F}^n(t, x) = \int_{-\infty}^x \hat{\rho}^n(t, y) dy \text{ is PLC} \]

\[ x \mapsto \tilde{F}^n(t, x) = \tilde{\rho}^n(t)((-\infty, x]) \text{ is PC} \]
Ingredients: pseudo inverses.

\( \hat{X}^n \) pseudo-inverse of \( \hat{F}^n \) is \textbf{PLC}

\( \tilde{X}^n \) pseudo-inverse of \( \tilde{F}^n \) is \textbf{PC}
Ingredients: discrete Lagrangian density.

\[ \rho^n = \hat{\rho}^n \circ \hat{X}^n \text{ is PC} \]
Strategy of the proof.

(i) Prove that \((\tilde{X}^n)_n\) has a strong limit \(X\) in \(L^1_{\text{loc}}([0, +\infty[ \times [0, L])\), equivalent to \((\tilde{\rho}^n)_n\) converging to a measure \(\rho\) in \(L^1_{\text{loc}}([0, +\infty[; d_{L,1})\).

Prove that \((\hat{X}^n)_n\) converges in \(L^1_{\text{loc}}([0, +\infty[ \times [0, L])\) to the same limit \(X\), i.e. \((\hat{\rho}^n)_n\) converges to \(\rho\) in \(L^1_{\text{loc}}([0, +\infty[; d_{L,1})\).

(ii) Prove that \(X\) has difference quotients bounded below by 1, i.e. \(\rho\) is actually in \(L^\infty\) and is a.e. bounded by 1.

This easily implies weak-* convergence of \((\tilde{\rho}^n)_n\) to a limit \(\tilde{\rho}\) in \(L^\infty\).

It remains to prove that \(\tilde{\rho} \circ F = \rho\), and that such limit is the unique entropy solution to LWR. This requires stronger estimates on \(\hat{\rho}^n\).

(iii) Case \(\bar{\rho} \in BV\): direct uniform \(BV\) estimate of \(\hat{\rho}^n\).

Case \(\bar{\rho} \in L^\infty\) + additional assumption on \(\nu\): uniform discrete Oleinik condition for \(\tilde{\rho}^n\), which implies uniform \(BV\) estimate for \(\tilde{\rho}^n\), i.e. for \(\hat{\rho}^n\).

(iv) Prove that \(\rho\) is a weak solution: it follows from letting \(n \to +\infty\) in the formulation of the FTL system

\[
\tilde{X}^n_t = \nu(\tilde{\rho}^n).
\]

(v) Prove that \(\rho\) is an entropy solution: in the discrete setting, use strong \(L^1\) compactness to pass it to the limit.
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Discrete maximum principle.

The global existence for the FTL system is guaranteed by the following Lemma (Discrete maximum principle):

For all \( i = 0, \ldots, N - 1 \), we have \( \ell \leq x_{i+1}(t) - x_i(t) \) for all times \( t \geq 0 \).

Proof.

- Replace \( \nu \) with its extension \( V = \nu \chi_{[0,1]} \), \( V \) is still Lipschitz.
- Assume by contradiction \( x_{i+1}(t_1) - x_i(t_1) = \ell \), and \( x_{i+1}(t) - x_i(t) < \ell \) on \( t \in (t_1, t_2) \).
- Integrate FTL on \([t_1, t]\):

\[
x_i(t) = x_i(t_1) + \int_{t_1}^{t} V \left( \frac{\ell}{x_{i+1}(\tau) - x_i(\tau)} \right) d\tau = x_i(t_1).
\]

- \( \ell > x_{i+1}(t) - x_i(t) \geq x_{i+1}(t_1) - x_i(t_1) = \ell \), contradiction!
- By uniqueness, the same holds for the system with \( \nu \).
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Strong compactness of $\hat{X}^n$ and $\tilde{X}^n$.

1-dimensional Wasserstein distance

$F_i$ cumulative distribution of $\rho_i$

$X_i$ pseudo-inverse of $F_i$

$\Rightarrow d_{L,1}(\rho_1, \rho_2) = \left\| X_1 - X_2 \right\|_{L^1([0, L])}$

Proposition

There exists a unique $X \in L_\infty(\mathbb{R}_+ \times [0, L])$, monotone non-decreasing and right continuous with respect to $z$, such that

$(\hat{X}^n)_n$ and $(\tilde{X}^n)_n$ converge to $X$ in $L_{loc}^1(\mathbb{R}_+ \times [0, L])$,

and for any $t, s > 0$

$$TV\left[ X(t) \right] \leq |\bar{x}_N - \bar{x}_0 + v_{\text{max}} t|,$$

$$\left\| X(t) \right\|_{L_\infty([0, L]; \mathbb{R})} \leq \max \left\{ |\bar{x}_0|, |\bar{x}_N + v_{\text{max}} t| \right\},$$

$$\int_{0}^{L} |X(t, z) - X(s, z)| \, dz \leq v_{\text{max}} L |t - s|.$$ (5c)

Moreover, $(\tilde{X}^n)_n$ converges to $X$ a.e. on $\mathbb{R}_+ \times [0, L]$. 
Strong compactness of $\hat{X}^n$ and $\tilde{X}^n$.

**Proof.**

Fix $T > t > s \geq 0$. Estimates (5a) and (5b) are immediately proven for $\tilde{X}^n$.

$$\int_0^L \left| \tilde{X}^n(t, z) - \tilde{X}^n(s, z) \right| \, dz = \sum_{i=0}^{N_n-1} \ell_n \left[ x_i^n(t) - x_i^n(s) \right]$$

$$= \sum_{i=0}^{N_n-1} \ell_n \left[ \int_s^t v (y_i^n(\tau)) \, d\tau \right] \leq \nu_{\text{max}} L (t - s).$$

By Helly's theorem, $\tilde{X}^n$ converges strongly as in the statement up to a subsequence. As $\tilde{X}^n$ is monotone in $n$, the whole sequence converges to a unique limit $X$.

$$\int_0^L \left| \hat{X}^n(t, z) - \tilde{X}^n(t, z) \right| \, dz = \sum_{i=0}^{N_n-1} y_i^n(t)^{-1} \int_{i \ell_n}^{(i+1) \ell_n} [z - i \ell_n] \, dz$$

$$= \frac{\ell_n}{2} \sum_{i=0}^{N_n-1} \left[ x_{i+1}^n(t) - x_i^n(t) \right] = \frac{\ell_n}{2} \left[ x_{N_n}^n(t) - x_0^n(t) \right] \leq \frac{\ell_n}{2} \left[ \bar{x}_{\text{max}} - \bar{x}_{\text{min}} + \nu_{\text{max}} T \right],$$

Hence, $\hat{X}^n$ and $\tilde{X}^n$ have the same limit.
$L^\infty$ bound for the limit measure.

- Let $0 \leq z_1 < z_2 \leq L$.
- For $n$ sufficiently large, let $i \ell_n \leq z_1 < (i + 1) \ell_n$ and $\ell_n j \leq z_2 < \ell_n (j + 1)$.
- The discrete maximum principle implies
  \[
  \frac{\tilde{X}^n(t, z_2) - \tilde{X}^n(t, z_1)}{z_2 - z_1} \geq \frac{x^n_j(t) - x^n_i(t)}{(j + 1) \ell_n - i \ell_n} \geq \frac{(j - i) \ell_n}{(j + 1) \ell_n - i \ell_n} = 1 - \frac{\ell_n}{z_2 - z_1}.
  \]
- By sending $n \to +\infty$, we get
  \[
  \partial_z X(t, \cdot) \geq 1 \quad \text{in } D'.
  \]
- Let $F(t, \cdot)$ be the pseudo inverse of $X(t, \cdot)$. Then $\rho = F_x$ satisfies
  \[
  \rho(t, x) \leq 1.
  \]
- $\tilde{\rho}^n$ and $\hat{\rho}^n$ converge to $\rho$ in $L^1_{\text{loc}}([0, +\infty); d_{L,1})$.
- $\tilde{\rho}^n = \hat{\rho}^n \circ \hat{X}^n$ converges to some limit $\tilde{\rho}$ up to a subsequence in the weak-$*$ $L^\infty$ topology.
In the sequel, we shall make an extensive use of the discrete equations

\[
\dot{y}_i^n(t) = -\frac{y_i^n(t)^2}{\ell_n} \left[ v(y_{i+1}^n(t)) - v(y_i^n(t)) \right], \quad i = 0, \ldots, N - 2,
\]

\[
\dot{y}_{N-1}^n(t) = -\frac{y_{N-1}^n(t)^2}{\ell_n} \left[ v_{\text{max}} - v(y_{N-1}^n(t)) \right].
\]
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**BV contraction for BV initial data.**

**Proposition (BV contractivity for BV initial data)**

Assume \( \bar{\rho} \in \text{BV} \). Then for any \( n \in \mathbb{N} \)

\[
\text{TV} \left[ \hat{\rho}^n(t) \right] = \text{TV} \left[ \tilde{\rho}^n(t) \right] \leq \text{TV} \left[ \bar{\rho} \right] \quad \text{for all } t \geq 0.
\]

**Proof.** (We omit the index \( n \) for simplicity)

\[
\begin{align*}
\frac{d}{dt} \text{TV} \left[ \hat{\rho}(t) \right] &= \frac{d}{dt} \left[ y_0 + y_{N-1} + \sum_{i=0}^{N-2} |y_i - y_{i+1}| \right] \\
&= \dot{y}_0 + \dot{y}_{N-1} + \sum_{i=0}^{N-2} \text{sgn} \left[ y_i - y_{i+1} \right] \left[ \dot{y}_i - \dot{y}_{i+1} \right] \\
&= \left[ 1 + \text{sgn} \left[ y_0 - y_1 \right] \right] \dot{y}_0 + \left[ 1 - \text{sgn} \left[ y_{N-2} - y_{N-1} \right] \right] \dot{y}_{N-1} \\
&\quad + \sum_{i=1}^{N-2} \left[ \text{sgn} \left[ y_i - y_{i+1} \right] - \text{sgn} \left[ y_{i-1} - y_i \right] \right] \dot{y}_i.
\end{align*}
\]
**BV contraction for BV initial data.**

**Proof (continued):** We analyse all the terms above, and we get

\[
\begin{align*}
1 + \text{sgn} [y_0 - y_1] \dot{y}_0 &= - \left[ 1 + \text{sgn} [y_0 - y_1] \right] \frac{y_0^2}{\ell} [v(y_1) - v(y_0)] \leq 0, \\
1 - \text{sgn} [y_{N-2} - y_{N-1}] \dot{y}_{N-1} &= - \left[ 1 - \text{sgn} [y_{N-2} - y_{N-1}] \right] \frac{y_{N-1}^2}{\ell} [v_{\text{max}} - v(y_{N-1})] \leq 0, \\
\text{sgn} [y_i - y_{i+1}] - \text{sgn} [y_{i-1} - y_i] \dot{y}_i &= \\
= - \left[ \text{sgn} [y_i - y_{i+1}] - \text{sgn} [y_{i-1} - y_i] \right] \frac{y_i^2}{\ell} [v(y_{i+1}) - v(y_i)] \leq 0.
\end{align*}
\]

Therefore, \( TV [\hat{\rho}(t)] \leq TV [\bar{\rho}] \) for all \( t \geq 0 \).
Discrete Oleinik condition.

**Lemma (Discrete Oleinik-type condition)**

Assume \( v \) satisfies the additional assumption \( \rho v' (\rho) \) non-increasing. Then, for any \( i = 0, \ldots, N_n - 2 \) we have

\[
 t y^n_i (t) \left[ v (y^n_{i+1} (t)) - v (y^n_i (t)) \right] \leq \ell_n \quad \text{for all } t \geq 0. \tag{6}
\]

**Remark**

Condition (6) in terms of \( x_i (t) \)

\[
\frac{v (y^n_{i+1} (t)) - v (y^n_i (t))}{x_{i+1} (t) - x_i (t)} \leq \frac{1}{t} \quad \text{for all } t \geq 0. \tag{7}
\]

(7) is a discrete analogous of

\[
v(\rho) x \leq \frac{1}{t}.
\]

However, the sharp form of the Oleinik condition for the scalar conservation law is (cf. Hoff 1983)

\[
f' (\rho) x = (v (\rho) + \rho v' (\rho)) x \leq \frac{1}{t}.
\]
Proof of the discrete Oleinik condition.

**Notation:** (we omit the dependence on \( n \) and \( t \))

\[
\begin{align*}
z_i & \doteq t y_i \left[ v(y_{i+1}) - v(y_i) \right], & i = 0, \ldots, N - 2, \\
z_{N-1} & \doteq t y_{N-1} \left[ v_{\text{max}} - v(y_{N-1}) \right].
\end{align*}
\]

The Lemma is proven once we provide the estimate \( z_i \leq \ell \) for \( i = 1, \ldots, N - 1 \).

**Step 0:** \( z_{N-1} \leq \ell \).

\[
\begin{align*}
\dot{z}_{N-1} &= y_{N-1} \left[ v_{\text{max}} - v(y_{N-1}) \right] + t \dot{y}_{N-1} \left[ v_{\text{max}} - v(y_{N-1}) \right] - t y_{N-1} v'(y_{N-1}) \dot{y}_{N-1} \\
&= y_{N-1} \left[ v_{\text{max}} - v(y_{N-1}) \right] - \frac{t y_{N-1}^2}{\ell} \left[ v_{\text{max}} - v(y_{N-1}) \right]^2 \\
&\quad + \frac{t v'(y_{N-1}) y_{N-1}^3}{\ell} \left[ v_{\text{max}} - v(y_{N-1}) \right] \\
&\leq y_{N-1} \left[ v_{\text{max}} - v(y_{N-1}) \right] \left[ 1 - \frac{z_{N-1}}{\ell} \right].
\end{align*}
\]

Since \( z_{N-1}(0) = 0 \), from the above estimate we get \( z_{N-1}(t) \leq \ell \) for all \( t \geq 0 \).
Proof of the discrete Oleinik condition (continued).

**Step 1:** \( z_{i+1} \leq \ell \Rightarrow z_i \leq \ell \).

\[
\dot{z}_i = y_i \left[ v(y_{i+1}) - v(y_i) \right] + t \dot{y}_i \left[ v(y_{i+1}) - v(y_i) \right] + t y_i \left[ v'(y_{i+1}) \dot{y}_{i+1} - v'(y_i) \dot{y}_i \right] \\
= y_i \left[ v(y_{i+1}) - v(y_i) \right] - \frac{ty_i^2}{\ell} \left[ v(y_{i+1}) - v(y_i) \right]^2 \\
+ t \frac{y_i}{\ell} \left[ -\frac{v'(y_{i+1}) y_{i+1}^2}{\ell} \left[ v(y_{i+2}) - v(y_{i+1}) \right] + \frac{v'(y_i) y_i^2}{\ell} \left[ v(y_{i+1}) - v(y_i) \right] \right] \\
= y_i \left[ v(y_{i+1}) - v(y_i) \right] - \frac{y_i}{\ell} \left[ v(y_{i+1}) - v(y_i) \right] z_i - \frac{v'(y_{i+1}) y_i y_{i+1}}{\ell} z_{i+1} + \frac{v'(y_i) y_i^2}{\ell} z_i.
\]

Since \( \text{sgn}_+ [z_i] = \text{sgn}_+ [v(y_{i+1}) - v(y_i)] = \text{sgn}_+ [y_i - y_{i+1}] \) for all \( t > 0 \), from the assumption on \( z_{i+1} \) we easily obtain

\[
\frac{d}{dt} [z_i]_+ = y_i \left[ v(y_{i+1}) - v(y_i) \right]_+ - \frac{y_i}{\ell} \left[ v(y_{i+1}) - v(y_i) \right]_+ [z_i]_+ \\
- \frac{v'(y_{i+1}) y_i y_{i+1}}{\ell} \text{sgn}_+ [z_i] z_{i+1} + \frac{v'(y_i) y_i^2}{\ell} [z_i]_+ \\
\leq y_i \left[ v(y_{i+1}) - v(y_i) \right]_+ \left[ 1 - \frac{[z_i]_+}{\ell} \right] - v'(y_{i+1}) y_i y_{i+1} \text{sgn}_+ [z_i] + \frac{v'(y_i) y_i^2}{\ell} [z_i]_+.
\]
Proof of the discrete Oleinik condition (continued).

The additional condition on \( v \) gives \(-v'(y_{i+1})y_{i+1} \leq -v'(y_i)y_i\) for \( y_i \geq y_{i+1} \), and then

\[
\frac{d}{dt} [z_i]_+ \leq y_i \left( v(y_{i+1}) - v(y_i) \right)_+ \left[ 1 - \frac{[z_i]_+}{\ell} \right] - v'(y_i)y_i^2 \operatorname{sgn}_+[z_i] + \frac{v'(y_i)y_i^2}{\ell} [z_i]_+
\]

\[
= y_i \left[ [v(y_{i+1}) - v(y_i)]_+ - v'(y_i)y_i \right] \left[ 1 - \frac{[z_i]_+}{\ell} \right].
\]

Now, as \( v' \leq 0 \), and since \( z_i(0) = 0 \), we get that \( z_i(t)_+ \leq \ell \) for all \( t \geq 0 \).

**Step 2:** \( z_{N-2} \leq \ell \). From analogous computations as in previous step, by using the monotonicity of \( y \mapsto y v'(y) \) and Step 0, we get

\[
\frac{d}{dt} [z_{N-2}]_+ \leq y_{N-2} \left[ [v(y_{N-1}) - v(y_{N-2})]_+ - v'(y_{N-2})y_{N-2} \right] \left[ 1 - \frac{[z_{N-2}]_+}{\ell} \right].
\]

Again, \( v' \leq 0 \) and \( z_{N-2}(0) = 0 \) imply that \( z_{N-2}(t)_+ \leq \ell \) for all \( t \geq 0 \).

**Conclusion.** The estimate (6) is proven recursively: Step 2 provides the first step with \( i = N-2 \), whereas Step 1 proves that the estimate holds for all \( i \in \{0, \ldots, N-3\} \).

**Remark:** The discrete Oleinik condition provides a uniform \( BV \) estimate away from \( t = 0 \). Here we use that solutions have compact support.
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A technical problem.

Typically (e.g. the wave-front-tracking algorithm for conservation laws) $L^1$-continuity of the approximating sequence gives the desired compactness via Helly’s Theorem. Here we are able to prove such an estimate for $\bar{\rho}^n$, but not for $\hat{\rho}^n$.

**Proposition (Uniform $L^1$-continuity in time of $\bar{\rho}^n$)**

For any $\delta > 0$ we have

$$\int_0^L |\bar{\rho}^n(t, z) - \bar{\rho}^n(s, z)| \; dz \leq C|t - s|$$

for all $t, s \geq \delta$,

with some $C$ depending on $\delta$.

**Proof.**

(Sketched) A direct computation of the l.h.s. and the discrete maximum principle give

$$\int_0^L |\bar{\rho}^n(t, z) - \bar{\rho}^n(s, z)| \; dz = \sum_{i=0}^{N_n-1} \ell_n |y_i^n(t) - y_i^n(s)| \leq \int_s^t \left[ TV \left( v \left( \bar{\rho}^n(\tau) \right) \right) + v_{\max} \right] \; d\tau.$$
Proposition (Uniform Wasserstein time continuity of $\hat{\rho}^n$)

For any $n \in \mathbb{N}$ we have

$$d_{L,1} \left( \hat{\rho}^n(t), \hat{\rho}^n(s) \right) \leq 2 L \, v_{\max} \, |t - s| \quad \text{for all } s, t \geq 0.$$

Proof.

$$d_{L,1} \left( \hat{\rho}^n(t), \hat{\rho}^n(s) \right) = \left\| \hat{X}^n(t) - \hat{X}^n(s) \right\|_{L^1([0,L];\mathbb{R})} = \sum_{i=0}^{N_n-1} \int_{i \ell_n}^{(i+1) \ell_n} \left[ \hat{X}^n(t, z) - \hat{X}^n(s, z) \right] \, dz$$

$$= \sum_{i=0}^{N_n-1} \ell_n \left[ x^n_i(t) - x^n_i(s) \right] + \sum_{i=0}^{N_n-1} \left[ y^n_i(t)^{-1} - y^n_i(s)^{-1} \right] \int_{i \ell_n}^{(i+1) \ell_n} (z - i \ell_n) \, dz$$

$$= \sum_{i=0}^{N_n-1} \ell_n \int_s^t v \left( y^n_i(\tau) \right) \, d\tau + \sum_{i=0}^{N_n-1} \frac{\ell_n^2}{2} \int_s^t \frac{d}{d\tau} \left[ y^n_i(\tau)^{-1} \right] \, d\tau$$

$$\leq L \, v_{\max} \, (t - s) + \frac{\ell_n}{2} \int_s^t \left[ v_{\max} - v \left( y^n_0(\tau) \right) \right] \, d\tau \leq 2 L \, v_{\max} \, (t - s).$$
A generalisation of Aubin-Lions Lemma.

The desired compactness then follows from the following

**Theorem (Generalized Aubin-Lions lemma, Rossi-Savaré 2003)**

Let $T, L > 0$ and $I \subset \mathbb{R}$ be a bounded open convex interval. Assume $w : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous and strictly monotone function. Let $(\rho^n)_{n \in \mathbb{N}}$ be a nonnegative sequence in $L^\infty((0, T) \times \mathbb{R}; \mathbb{R})$, with compact support and fixed mass $L > 0$, such that:

- $\rho^n : (0, T) \to L^1(\mathbb{R}; \mathbb{R})$ is measurable for all $n \in \mathbb{N}$;
- $\text{spt}(\rho^n(t)) \subseteq I$ for all $t \in ]0, T[$ and $n \in \mathbb{N}$;
- $\sup_{n \in \mathbb{N}} \int_0^T \left[ \| w(\rho^n(t)) \|_{L^1(I; \mathbb{R})} + \text{TV}[w(\rho^n(t))] \right] dt < +\infty$;
- There exists a constant $C$ depending only on $T$ such that $d_{L,1}(\rho^n(s), \rho^n(t)) \leq C |t - s|$ for all $s, t \in ]0, T[$ and $n \in \mathbb{N}$.

Then, $(\rho^n)_{n \in \mathbb{N}}$ is strongly relatively compact in $L^1((0, T) \times \mathbb{R}; \mathbb{R})$. 
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Another important property that we need to check is

$$\bar{\rho}(t, F(t, x)) = \rho(t, x), \quad \text{on spt}(\rho),$$

where

- $\bar{\rho}$ is the strong limit of $\bar{\rho}^n$,
- $\rho$ is the strong limit of $\hat{\rho}^n$,
- $F$ is the cumulative distribution of $\rho$.

This is ensured by the strong compactness of both $\bar{\rho}^n$ and $\hat{\rho}^n$. 
Weak solutions in the limit.

Proposition

The limit function \( \rho \) of \( \hat{\rho}^n \) is a weak solution of the LWR equation with i.c. \( \bar{\rho} \).

Proof.

Let \( \phi \in C_c^\infty ([0, +\infty[ \times \mathbb{R}; \mathbb{R}) \). We have

\[
\int_{\mathbb{R}^+} \int_0^L \left[ v(\hat{\rho}^n(t, z)) \phi_x(t, \tilde{X}^n(t, z)) \right] dz dt = \sum_{i=0}^{N_n-1} \int_{\mathbb{R}^+} \int_i^{(i+1)\ell_n} \left[ v(y^n_i(t)) \phi_x(t, x^n_i(t)) \right] dz dt
\]

\[
= \sum_{i=0}^{N_n-1} \int_{\mathbb{R}^+} \int_i^{(i+1)\ell_n} \left[ \frac{d}{dt} \phi(t, x^n_i(t)) - \phi_t(t, x^n_i(t)) \right] dz dt
\]

\[
= - \int_0^L \phi(0, \tilde{X}^n(0, z)) \, dz - \int_{\mathbb{R}^+} \int_0^L \phi_t(t, \tilde{X}^n(t, z)) \, dz dt.
\]

By the strong convergence of \( \tilde{X}^n \) and \( \hat{\rho}^n \), by chancing variable \( x = X(t, z) \), and by using \( \hat{\rho}(t, \hat{F}(t, x)) = \rho(t, x) \) a.e. on \( \text{sp}(\rho) \), we get the definition of weak solution for LWR.
Entropy solutions in the limit.

Conclusion of the proof.

- We need to prove that the limit $\rho$ is an entropy solution in the Kružkov sense.
- This is trivial in the uniformly concave case $f'' \leq -\epsilon < 0$, as in that case $f'(\rho) \approx v(\rho)$, and one can obtain the sharp Oleinik condition $f'(\rho)_x \leq 1/t$ in the limit.
- In the general case, we need to use the definition of entropy solution by Kružkov. This follows from the inequality

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[ \left| \hat{\rho}^n(t, x) - k \right| \phi_t(t, x) + \text{sgn}(\hat{\rho}^n(t, x) - k) \left[ f(\hat{\rho}^n(t, x)) - f(k) \right] \phi_x(t, x) \right] dx \, dt \geq o(1),$$

as $n \to +\infty$, which is very technical and is omitted here.
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Concluding remarks.

- We remark that our set of assumptions on \( v \) allows for degenerate concave fluxes at zero.

- In the case of linear velocity \( v \), e.g. \( v(\rho) = v_{\text{max}}(1 - \rho) \), the convergence to a weak solution can be obtained without the need of the BV estimates, as the velocity term in the pseudo-inverse PDE is linear. This is somehow intrinsic in using a Lagrangian description.

- In order to get continuity in time for the sequence \( \hat{\rho}^n \), the most natural try would be getting \( L^1 \)-continuity. Encouraged by the \( L^1 \) time equi-continuity of \( \hat{\rho}^n \), we have attempted at proving such a property in many ways without success. This is the reason why use the generalized Aubin-Lions lemma, which allows to take advantage of the Wasserstein equi-continuity of \( \hat{\rho}^n \), and still get the same \( L^1 \)-compactness in the end. The only drawback of this strategy is that we can’t get any \( L^1 \) time continuity for the limit.

- Our approach has the advantage of providing a piecewise constant approximation with a non increasing number of jumps. The price to pay for such a simplification is that we lose the classical shock structure at a microscopic level. Indeed, the explicit solution to the FTL system even for simple Riemann-type initial conditions is not immediate.
Future projects.

- Extending the results to Dirichlet boundary condition (phantom moving particles at the boundary).

- Use this strategy to attack the existence theory of similar models, e.g. with discontinuous flux.
Numerical simulation.
THANK YOU FOR YOUR ATTENTION