Turbulence and Multiscaling in the Randomly Forced Navier Stokes Equation

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We present an extensive pseudospectral study of the randomly forced Navier-Stokes equation (RFNSE) stirred by a stochastic force with zero mean and a variance $\sim k^{4-d-y}$, where $k$ is the wavevector and the dimension $d = 3$. We present the first evidence for multiscaling of velocity structure functions in this model for $y \geq 4$. We extract the multiscaling exponent ratios $\zeta_p/\zeta_2$ by using extended self similarity (ESS), examine their dependence on $y$, and show that, if $y = 4$, they are in agreement with those obtained for the deterministically forced Navier-Stokes equation (3dNSE). We also show that well-defined vortex filaments, which appear clearly in studies of the 3dNSE, are absent in the RFNSE.

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Kolmogorov's classic work (K41) on homogeneous, isotropic fluid turbulence focussed on the scaling behavior of velocity $\mathbf{v}$ structure functions $S_p(r) = \langle |\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})|^p \rangle$, where the angular brackets denote an average over the statistical steady state ($\mathbf{0}$). He suggested that, for separations $r = |\mathbf{r}|$ in the inertial range, which is substantial at large Reynolds numbers $Re$ and lies between the forcing scale $L$ and the dissipation scale $\eta_d$, these structure functions scale as $S_p \sim r^{\zeta_p}$, with $\zeta_p = p/3$. Subsequent experiments \cite{2} have suggested instead that multiscaling obtains with $p/3 > \zeta_p$, which turns out to be a nonlinear, monotonically increasing function of $p$; this has also been borne out by numerical studies of the three-dimensional Navier-Stokes equation forced deterministically (3dNSE) at large spatial scales \cite{3}. The determination of the exponents $\zeta_p$ has been one of the central, but elusive, goals of the theory of turbulence. One of the promising starting points for such a theory is the randomly forced Navier-Stokes equation (RFNSE) \cite{3,14,18}, driven by a Gaussian random force whose spatial Fourier transform $\hat{f}(k,t)$ has zero mean and a covariance $\langle \hat{f}_i(k,t)\hat{f}_j(k',t') \rangle = Ak^{4-d-y}P_{ij}(k)\delta(k + k')\delta(t - t')$; here $k,k'$ are wave numbers, $t,t'$ times, $i,j$ Cartesian components in $d$ dimensions, and $P_{ij}(k)$ the transpose projector which enforces the incompressibility condition. One-loop renormalization-group (RG) studies of this RFNSE yield \cite{14} a K41 energy spectrum, namely, $E(k) \sim k^2 S_2(k) = k^2 \langle |\mathbf{v}(k)|^2 \rangle \sim k^{-5/3}$, if we set $d = 3$ and $y = 4$; this has also been verified numerically \cite{14}. Nevertheless, these RG studies have been criticised for a variety of reasons \cite{2,14} such as using a large value for $y$ in a small-$y$ expansion and neglecting an infinity of marginal operators (if $y = 4$). These criticisms of the approximations used in these studies might well be justified; but they clearly cannot be used to argue that the RFNSE is in itself inappropriate for a theory of turbulence. It is our purpose here to check if, indeed, the RFNSE is a good starting point for such a theory. Specifically we want to test whether structure functions in the RFNSE display the same multiscaling as in the 3dNSE for some value of $y$; if they do, then we can argue that both equations are in the same universality class and the RFNSE can, defensibly, be used to develop a statistical theory of inertial-range multiscaling in homogeneous, isotropic fluid turbulence.

To achieve this end we have carried out an extensive pseudospectral study of the RFNSE and compared our results with earlier numerical studies \cite{11,14} of the 3dNSE and experiments \cite{2}. We find several interesting and new results: We show that structure functions in the RFNSE display multiscaling for $y \geq 4$. As in the 3dNSE, we obtain good estimates for ratios of the multiscaling exponents, such as $\zeta_p/\zeta_2$, by using the extended-self-similarity (ESS) procedure (Fig. 1) \cite{10}; we obtain $\zeta_2$ from $S_2(k)$ (Fig. 2). Next we investigate the $y$-dependence of $\zeta_p$ and find that it is close to the 3dNSE result (Fig. 3) for $y = 4$ at least for $p \leq 7$. Thus the RFNSE should be a good starting point for a theory of inertial-range multiscaling in the 3dNSE barring weak correction which do not affect ratios like $\zeta_p/\zeta_2$ (see below). Furthermore we show that the qualitative behaviors of the probability distributions $P(\delta v_n(r))$, where $\delta v_n(r) \equiv v_n(x) - v_n(x + r)$, are similar in the two models (Fig. 4). However, the shapes of constant-$|\omega|$ surfaces, where $\omega$ is the vorticity, are markedly different (Fig. 5); the stochastic force destroys well-defined filamentary structures that obtain in 3dNSE studies. This has implications for the She-Leveque \cite{1} formula for $\zeta_p$ as we discus below.

We use a pseudospectral method to solve the RFNSE numerically on a $64^3$ grid with a cubic box of linear size $L = 2\pi$ and periodic boundary conditions; we have checked in representative cases that our results are unchanged if we use an $80^3$ grid or aliasing. Aside from the stochastic forcing, our numerical scheme is the same as that used in Ref. \cite{14}. Our dissipation term, which is $\nu(H k^2)k^2 v(k)$ in wave-vector $(k)$ space includes both the viscosity $\nu$ and the hyperviscosity $\nu_H$; the exponents $\zeta_p$ are unaffected by $\nu_H$ if $\nu > 0$ \cite{14}. Our numerical study of the RFNSE differs from conventional studies of the 3dNSE in two important ways: (1) For a fixed grid size we can attain higher Taylor-microscale
Reynolds numbers $R_{\chi}$ in the RFNSE, and hence a larger inertial range, than in the 3dNSE ($R_{\chi} \approx 120$ compared to $R_{\chi} \approx 22$ in our study), as noted earlier \cite{5} for $y = 4$. (2) This advantage is reduced somewhat by the need to average statistical observables longer in the RFNSE than in the 3dNSE. In the latter case it normally suffices to average over a few box-size eddy turnover times $\tau_L$; this is not enough for the RFNSE since (a) $R_{\chi}$ fluctuates strongly over time scales considerably larger than $\tau_L$ (inset of Fig. 2) and (b) the length of the $\mathbf{f}(\mathbf{k}, t)$ time series required to obtain a specified variance for the stochastic force turns out to be quite large (the time series must be of length $\approx 6\tau_L$ to achieve the given variance within $1-2\%$). Thus, in our studies, we have collected data for averages over $25 - 33\tau_L$ (for different values of $y$), after initial transients have been allowed to decay (over times $\approx 10 - 20\tau_L$). Our $\tau_L \approx 10\tau_1$, the integral-scale time used in some studies \cite{12}; $\tau_1 \equiv L_1/v_{rms}$, where the integral scale $L_1 \equiv \left[\int dk k^2 E(k)/\int dk E(k)\right]^{-1}$ and $v_{rms}$ is the root-mean-square velocity.

We begin by investigating the inertial-range scaling of the $k$-space structure function $S_2(k) \sim k^{-\zeta_2}$. Given this power-law form, the exponent $\zeta_2$ is easily seen to be related to the $r$-space inertial-range exponent $\zeta_2$ by the formula $\zeta_2 = c'_2 - 3$. Our data in Fig. 1 for $4 \leq y$ are consistent with $c'_2 = 11/3$ (i.e., the K41 value since $E(k) \sim k^2 S_2(k) \sim k^{-5/3}$). For $y = 4$ this result has been reported earlier \cite{5}. The $y$-independence of $\zeta_2$ above some critical $y_c$ (our data suggest $y_c \approx 4$) is theoretically satisfying since the variance of the stochastic force in the RFNSE rises rapidly at small $k$, so we might expect that, for sufficiently large $y$, it approximates the conventional deterministic forcing of the 3dNSE at large spatial scales. This point of view has been explored in the $N \to \infty$ limit of an $N$-component generalization of the RFNSE \cite{8}; however, this study suggests $\zeta_2 = 7/2$ for $y_c = 4 \leq y$; given our error bars (Table 1) it is difficult to distinguish this from the $O(y)$ RG prediction $\zeta_2 = 11/3$ though our data are closer to the latter. For $0 < y < 3$ both the one-loop RG \cite{5} and the $N \to \infty$ theory \cite{15} predict $\zeta_2 \sim 1 + 2y/3 + O(y^2)$, which is in fair agreement with our numerical results, especially for small $y$ (Table 1). We note in passing that, for $0 < y < 4$, there is no invariant energy cascade as in conventional K41: The dominance of dissipation at large $k$ does lead to an energy cascade, but the energy flux depends on the length scale $r$; specifically $\Pi(r) \sim A r^{y-4}$, with $A$ the scale-independent part of the variance of the stochastic force in the RFNSE. A K41-type argument \cite{14} now yields an energy-transfer rate $\sim <\delta v^2> / r \sim r^{(y-4)}$, whence $S_3(r) \sim r^{(y-3)}$ and, if we assume simple scaling as in K41, $S_2(r) \sim r^{(y-3)/2}$, i.e., $\zeta_2 = 1 + 2y/3$, which is same as the $O(y)$ RG prediction mentioned above. This formula breaks down for $y < 0$; however, the RG predicts correctly that the linear-hydrodynamics result obtains in this regime.

Several precautions must be taken to ensure that systematic errors do not affect the determination of $\zeta_2$. If $k_{max}$ is the largest wave-vector magnitude in our numerical scheme, we find that $L_{max} = A y_{max}$ decreases with decreasing $y$; this shortens the inertial range of the energy spectrum which can be used to obtain $\zeta_2$. The lower the value of $y$ the more difficult it is to obtain a dissipation range free of systematic, finite-resolution errors. For $y < 4$, we define $k_d \equiv \eta_d^{-1}$ to be the inverse length scale at which the energy-transfer time $t_r \sim (\eta_d / v_r) \sim [A r^{(y-6)}]^{1/3}$ equals the diffusion time $t_D \sim [\nu k^2 + \nu_H k^4]^{-1/2}$; this yields

\begin{equation}
\nu_0 k^{2}_{d} + \nu_h k^{4}_{d} = [A k^{6-y}]^{1/3},
\end{equation}

which when solved numerically shows that, for fixed $A$, $k_d$ increases as $y$ decreases (Table 1). It is important to recognize that statistical steady states, with ill-resolved dissipation ranges that do not have a decaying tail \cite{9,8}, can be obtained by adjusting the amplitude $A$. In such cases $k_d \gg k_{max}$ and we get spurious results for $\zeta_2$. We find that, if we increase the hyperviscosity $\nu_H$, $k_d$ is sufficiently close to $k_{max}$ that we can resolve both inertial and dissipation ranges and obtain reliable values for $\zeta_2$. Table 1 shows the range over which we fit our data for $S_2(k)$. Since our data for $\zeta_2$ indicate that $y_c \approx 4$, we investigate multiscaling only for $y \geq 4$.

Our data for $\zeta_2$ in Table 1 suggest that naive estimates for the multiscaling exponents $\zeta_2$ require longer inertial ranges than are available in our studies. However, we find that, as in the 3dNSE, the extended-self-similarity (ESS) procedure \cite{8,12,13,14} can be used fruitfully here to extract the exponent ratios $\zeta_p / \zeta_2$ from the slopes of log-log plots of $S_p(r)$ versus $S_q(r)$ (see Fig. 2 for $p = 5$ and $q = 2$) since this extends the apparent inertial range. The ratios $\zeta_p / \zeta_2$ that we obtain from such ESS plots are compared
in Fig. 3 with the She-Leveque (SL) formula \[11\], which provides a convenient parametrization for the experimental values for \(\zeta_p\). (If we assume the power-law form for \(S_2(k)\) in the inertial range (Fig. 1), we get \(\zeta \simeq \zeta_2 - 3\) and thence all the exponents \(\zeta_p\).) Figure 3 shows clearly that, with \(y = 4\), our RFNSE exponent ratios lie very close to those for the 3dNSE and, to this extent, these two models are in the same universality class. The exponent ratios for \(y < 4\) lie away from the 3dNSE values as we might have anticipated from our results for \(S_2(k)\) (Fig. 1). These results also lead to the expectation that, if \(y\) is sufficiently large, the exponent ratio \(\zeta_p/\zeta_2\) should become independent of \(y\), for we find that \(\zeta_2(y = 4) \simeq \zeta_2(y = 6)\). However, for \(p > 3\), our data for \(\zeta_p/\zeta_2(y = 6)\) fall systematically below those for \(\zeta_p/\zeta_2(y = 4)\) or the SL line. We also find that that the probability distributions of \(P(\delta v_r)\) (Fig. 4) have non-Gaussian tails for \(r\) in the dissipation range; and for \(y > 4\) the deviations from a Gaussian distribution increase systematically with \(y\). Thus, at least at the resolution of our calculation, it seems that the RFNSES with \(y = 4\) and \(y = 6\) are in different universality classes. However, we wish to point out that our data for \(y = 6\) are more noisy than those for \(y = 4\), so longer runs with finer grids might well be required to settle this issue conclusively.

Strictly speaking the RFNSE with \(y = 4\) falls in the same universality class as the 3dNSE only in the ESS sense. For arbitrary \(y\) the energy flux through the \(k^{th}\) shell is \(\Pi_k \equiv \Pi(r = k^{-1}) \sim \int_{kL}^{\infty} |f(k)|^2 d^3k\), where \(r\) is in the inertial-range and we have used Novikov's theorem \[15\], i.e., \(\langle f(k) \cdot \nu(-k) \rangle \sim \langle |f(k)|^2 \rangle\). For \(y > 4\), \(\Pi_k\) saturates to a constant for \(kL \gg 1\); but for \(y = 4\), \(\Pi_k \sim \log(kL)\) in the RFNSE. This is to be contrasted, with the 3dNSE where \(\Pi_k = \text{constant}\). Thus the inertial-range behaviors of all correlation functions in the two models are not the same. A K41-type dimensional analysis suggests that for \(y = 4\) the energy flux \(\Pi_k \sim \delta v_r^3 / r \sim \log(r/L)\); if we further assume that there is no multiscaling, then \(S_p(r) \sim [r \log(r/L)]^{p/3}\). Multiscaling will clearly modify this simple prediction; but some weak deviation from the Von-Karman-Howarth form \(S_3(r) \sim r^{-1}\) must remain, since the standard derivation of the Von-Karman-Howarth relation \[15\] does not go through \[14\] with the RFNSE result for \(\Pi_k\). To the extent that our data show that the ESS procedure works for the RFNSE, it seems that these weak deviations cancel when we consider the ratios of structure functions; and, as noted above, for \(y = 4\) the exponent ratios \(\zeta_p/\zeta_2\) agree with the SL result for the 3dNSE.

Filamentary structures (Fig. 5) \[17\] in iso-\(|\omega|\) plots have been used as important ingredients in phenomenological models for multiscaling in fluid turbulence. For example, the SL formula \[11\] is obtained by postulating a hierarchical relation among the moments of the scale-dependent energy dissipation; this yields a difference equation for the exponents \(\tau_p\), which are simply related to the exponents \(\zeta_p\); one of the crucial boundary conditions used to solve this equation requires the codimension of the most intense structures. If these are taken to be vorticity filaments, their codimension is 2 and one gets the SL formula. Filaments have been ob-
TABLE I. The dissipation-scale wavenumber $k_d$ (determined from Eq. 1), the integral-scale wavenumber $k_I \equiv L_I^{-1}$, the apparent inertial range over which we fit our data for $S_2(k)$, the hyperviscosities $\nu_H$, the exponent $\zeta_2$ that we compute, and its $O(y)$ RG value, for $1 \leq y \leq 4$. The viscosity $\nu$ is $5 \times 10^{-6}$ in all these runs which use a $64^3$ grid.

| $y$ | $k_d$ | $k_I$ | Fitting Range | $\nu_H$ | $\zeta_2$ this study | $\zeta_2$ from $O(y)$ RG |
|-----|-------|-------|---------------|--------|---------------------|--------------------------|
| 4   | 49    | 1.16  | $(0.1 - 0.5)k_d$ | $10^{-6}$ | $3.6 \pm 0.1$ | $\geq 3.67$ |
| 3   | 38.7  | 1.90  | $(0.16 - 0.52)k_d$ | $3 \times 10^{-6}$ | $3.0 \pm 0.1$ | $\geq 3$ |
| 2   | 35.0  | 5.90  | $(0.17 - 0.63)k_d$ | $8 \times 10^{-6}$ | $2.3 \pm 0.1$ | $\geq 2.33$ |
| 1   | 35.4  | 10.3  | $(0.2 - 0.7)k_d$ | $8 \times 10^{-6}$ | $1.6 \pm 0.15$ | $\geq 1.67$ |

![FIG. 5. Iso-$|\omega|$ surfaces obtained from instantaneous snapshots of the vorticity fields showing filaments for the 3dNSE (left) and no filaments for the RFNSE with $y = 4$ (right).](image-url)

We have shown above that the exponent ratios $\zeta_p/\zeta_2$ that we obtain from the RFNSE with $y = 4$ agree with the SL formula. One might expect, therefore, that filamentary structures should appear in iso-$|\omega|$ plots for the RFNSE. However, this is not the case as can be seen from the representative plot shown in Fig. 5. The stochastic forcing seems to destroy the well-defined filaments observed in the 3dNSE without changing the multiscaling exponent ratios. Therefore, the existence of vorticity filaments is not crucial for obtaining these exponents, which is perhaps why simple shell models also yield good estimates for $\zeta_p$.

In summary, then, we have shown that the RFNSE with $y = 4$ exhibits the same multiscaling behavior as the 3dNSE, at least in the ESS sense. Probability distributions like $P(\delta \nu_v)$ (Fig. 4) are also qualitatively similar in the two models, in so far as they show deviations from Gaussian distributions for $r$ in the dissipation range. It would be interesting to see if the RFNSE model can be obtained as an effective, inertial-range equation for fluid turbulence. We have tried to do this by a coarse-graining procedure that has been used to map the Kuramoto-Sivashinsky(KS) equation onto the Kardar-Parisi-Zhang (KPZ) equation; however, it turns out that the 3dNSE $\rightarrow$ RFNSE mapping, if it exists, is far more subtle than the KS $\rightarrow$ KPZ mapping as we discuss elsewhere.

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