THE MODULAR ISOMORPHISM PROBLEM AND ABELIAN DIRECT FACTORS

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Abstract. Let \( p \) be a prime and let \( G \) be a finite \( p \)-group. We show that the isomorphism type of the maximal abelian direct factor of \( G \), as well as the isomorphism type of the group algebra over \( \mathbb{F}_p \) of the non-abelian remaining direct factor, are determined by \( \mathbb{F}_p G \), generalizing the main result in [MSS21] over the prime field. In order to do this, we address the problem of finding characteristic subgroups of \( G \) such that their relative augmentation ideals depend only on the \( k \)-algebra structure of \( kG \), where \( k \) is any field of characteristic \( p \), and relate it to the modular isomorphism problem, extending and reproving some known results.

1. Introduction

Let \( k \) be a field of characteristic \( p \), and \( G \) and \( H \) finite \( p \)-groups. The modular isomorphism problem (MIP) asks whether the existence of an isomorphism of \( k \)-algebras \( kG \cong kH \) implies the existence of an isomorphism of groups \( G \cong H \). The most classical version of this question also assumes that \( k = \mathbb{F}_p \), the field with \( p \) elements. Indeed, it is already mentioned by Brauer in [Bra63] as a possibly much easier particular case of the general isomorphism problem for group rings ([Bra63, Problem 2]). Despite Brauer’s optimistic observation, only partial positive results for MIP have been obtained under quite severe restrictions on the structure of \( G \) and \( H \). For example, MIP is known to have positive answer if the groups are abelian (Deskins [Des56]), metacyclic (Bagiński [Bag88], and Sandling [San96]), or have class 2 and elementary abelian derived subgroup (Sandling [San89]). Some more recent positive results and approaches can be found in [Sak20, MM22, BdR21, MS22], and an up to date state of the art, in [Mar22]. The modular isomorphism problem is now known to have negative answer in general, as it is shown in [GLMdR22] that there exist non-isomorphic finite 2-groups with isomorphic group algebras over every field of characteristic 2. This counterexample makes it even more interesting to investigate which properties (weaker than the isomorphism type) of \( G \) and \( H \) must necessarily coincide when \( kG \cong kH \). In any case, the original problem remains open for \( p \) an odd prime.

On the other hand, and despite all the attention MIP received, an approach that surprisingly seems to not have been exploited until very recently is to reduce the problem from the vast class of all finite \( p \)-groups to some smaller (but maybe as vast and complicated) subclass of groups. In [MSS21] it is shown that the modular isomorphism problem can be reduced to the same problem over groups without elementary abelian direct factor. We generalize this result by dropping the ‘elementary’ hypothesis, i.e., showing that MIP can be reduced to the problem over groups without abelian direct factors, with no restrictions on the exponent. We formalize this as follows: For a finite \( p \)-group \( G \), consider a decomposition \( G = \text{Ab}(G) \oplus \text{NAb}(G) \), where \( \text{Ab}(G) \) and \( \text{NAb}(G) \) are subgroups of \( G \) such that \( \text{Ab}(G) \) is abelian, and maximal such that a decomposition like that exists. From the Krull-Remak-Schmidt theorem it follows that the isomorphism types of \( \text{Ab}(G) \) and \( \text{NAb}(G) \) do not depend on the chosen decomposition, so they are group-theoretical invariants of \( G \). Our main theorem states that we can disregard the direct factor \( \text{Ab}(G) \) in the study of MIP. Formally:

**Theorem A.** Let \( k = \mathbb{F}_p \) and \( G \) and \( H \) be finite \( p \)-groups. Then

\[
kG \cong kH \quad \text{if and only if} \quad k(\text{NAb}(G)) \cong k(\text{NAb}(H)) \quad \text{and} \quad \text{Ab}(G) \cong \text{Ab}(H).
\]

As an immediate corollary, we can extend non-trivially some of the classes of groups for which MIP is known to have a positive answer. We summarize some of them in the following corollary.

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Corollary B. Let $k = \mathbb{F}_p$ and $A$ and $G$ be finite $p$-groups such that $A$ is abelian and at least one of the following holds:

1. $G$ is metacyclic.
2. $G$ is 2-generated of nilpotency class 2.
3. $G$ is 2-generated with nilpotency class 3 with elementary abelian derived subgroup.
4. $G$ is elementary abelian-by-cyclic.
5. $G$ has a cyclic subgroup of index $p^2$.
6. The third term of the Jennings series of $G$ is trivial.
7. The fourth term of the Jennings series of $G$ is trivial and $p > 2$.
8. $G$ has order at most $p^3$.

If $H$ is another group such that $kH \cong k(G \oplus A)$, then $H \cong G \oplus A$.

Observe that none of these properties is closed under taking direct products with abelian groups, so our result is indeed non-trivial. The primality of the field is important in the proof of Theorem A; however, for an arbitrary field of characteristic $p$ we can still recover the isomorphism type of the maximal abelian direct factor.

Proposition C. Let $G$ and $H$ be finite $p$-groups and $k$ be a field of characteristic $p$. Then $kG \cong kH$ implies that $\text{Ab}(G) \cong \text{Ab}(H)$.

Hence naturally the following question arises

Question D. Let $k$ be a field of characteristic $p$, and $G$ and $H$ be finite $p$-groups. Does $kG \cong kH$ imply that $k(\text{NAb}(G)) \cong k(\text{NAb}(H))$?

We recall that the indecomposable decomposition of a finite group $G$, i.e., the indecomposable direct factors of $G$ arising from the Krull-Remak-Schmidt theorem, is unique up to isomorphism and reordering. Since the groups in [GLMdR22] are indecomposable, it makes also sense to ask if Theorem A can be extended so that the modular isomorphism problem is reduced to the same problem over indecomposable finite $p$-groups, carrying the reduction approach (in the sense of direct factor decompositions) to its ideal conclusion. In other words:

Question E. Let $k = \mathbb{F}_p$ and $G$ and $H$ be finite $p$-groups. Does $kG \cong kH$ imply that the terms of the indecomposable decompositions of $G$ and $H$ have pairwise isomorphic group algebra over $k$?

Lemma 4.9 seems to suggest that this problem is not completely hopeless; however the remainder of techniques used in Section 4 rely heavily on the fact that the direct factors we disregard are abelian (and hence contained in the center of the group), so new ideas and techniques would be needed to follow this path.

The paper is organized as follows. In Section 2 we set notation and present some well-known results related to the modular isomorphism problem. In Section 3 we address the problem of finding normal subgroups $N$ of $G$ such that their relative augmentation ideals depend only on the algebra structure of $kG$, and relate this problem to the MIP. Finally, in Section 4 we prove, with the help of the results in the previous section, both Proposition C and Theorem A.

2. Notation and preliminaries

Throughout the paper, $p$ will denote a prime number, $k$ a field of characteristic $p$, $\mathbb{F}_p$ the field with $p$ elements and $G$ and $H$ finite $p$-groups. The group algebra of $G$ over $k$ is denoted by $kG$ and its augmentation ideal is denoted by $I(G)$. It is a classical result that $I(G)$ is also the Jacobson ideal of $kG$. For every normal subgroup $N$ of $G$, we write $I(N;G)$ for the relative augmentation ideal $I(N)kG$, i.e., the two-sided ideal generated by the elements of the form $n-1$ with $n \in N$. It is well-known that this ideal is just the kernel of the natural projection $kG \to k(G/N)$, i.e., the homomorphism of algebras extending the natural projection $G \to G/N$. Moreover, $I(N;G) \cap (G-1) = N-1$. For these basic facts about the augmentation ideals we refer to [Pas77, Section 1.1]. We denote by $[kG,kG]$ the vector subspace of $kG$ generated by the elements of the form $xy - yx$, with $x, y \in kG$. Given a subspace $U$ of a vector space $V$ over $k$, we write $\text{codim}_V(U) = \dim(V) - \dim(U)$ to denote the codimension of $U$ in $V$. 

Our group theoretic notation is mostly standard. We write $\oplus$ both for internal and external direct products of groups, and also for the direct sum of vector spaces. For $n \geq 1$, we denote by $C_n$ the cyclic group of order $n$. Given $n, r \geq 1$, a homocyclic group of exponent $n$ and rank $r$ is a group isomorphic to $(C_n)^r = C_n \oplus \cdots \oplus C_n$ ($r$ times). We let $|G|$ denote the order of the finite $p$-group $G$, $Z(G)$ its center, $\{g_i(G)\}_{i \geq 1}$ its lower central series, $G' = g_2(G)$ its commutator subgroup, $\Phi(G)$ its Frattini subgroup and $d(G) = \min\{|X| : X \subseteq G$ and $G = \langle X \rangle\}$ its minimum number of generators. It is well known that $G/\Phi(G)$ is an elementary abelian $p$-group, and $d(G) = d(G/\Phi(G))$. If $A$ is a homocyclic $p$-group, then $d(A)$ equals the rank of $A$. If moreover $A$ is elementary abelian, it can be seen as a vector space over $\mathbb{F}_p$ with dimension $d(A)$. A Burnside basis of $G$ is a minimal set of generators of $G$, i.e., a subset of $G$ such that its image in $G/\Phi(G)$ is a basis as vector space. We define the omega series $(\Omega_n(G))_{n \geq 0}$ and the agemo series $(\Omega_n(G))_{n \geq 0}$ of $G$ by:

$$\Omega_n(G) = \left\{ g \in G : g^{p^n} = 1 \right\} \quad \text{and} \quad \Omega_n(G) = \left\{ g^{p^n} : g \in G \right\}.$$ 

If $N$ is a normal subgroup of $G$ and $n \geq 1$, we also write

$$\Omega_n(G : N) = \left\{ g \in G : g^{p^n} \in N \right\},$$

that is, $\Omega_n(G : N)$ is the unique subgroup of $G$ containing $N$ such that $\Omega_n(G : N)/N = \Omega_n(G)/N$. The Jennings series $(D_n(G))_{n \geq 1}$ of $G$ is defined by

$$D_n(G) = \left\{ g \in G : g - 1 \in I(G)^n \right\} = \prod_{i \geq n} \Omega_i(g(G))$$

In particular $D_1(G) = G$ and $D_2(G) = \Omega_1(G)G' = \Phi(G)$. A property of these series that we will use is that if $G$ is abelian, then the orders of the terms completely determine the isomorphism type of $G$. For more on the Jennings series, see for instance [Pas77, Section 11.1] and [Seh78, Section III.1].

The next proposition collects some well-known results about MIP that will have some relevance for our results; a complete list can be consulted in [Mar22].

**Proposition 2.1.** Let $k = \mathbb{F}_p$ be the field with $p$ elements, and let $G$ and $H$ be finite $p$-groups. Suppose that $G$ satisfies at least one of the following conditions:

1. $G$ is abelian [Des56].
2. $G$ is metacyclic [Bag88, San96].
3. $G$ is 2-generated with nilpotency class 2 [BdR21].
4. $G$ is 2-generated with elementary abelian derived subgroup and nilpotency class 3 [MM22, Bag99].
5. $G$ has a cyclic subgroup of index $p^2$ [BK07].
6. $G$ is elementary abelian-by-cyclic [Bag99].
7. $D_4(G) = 1$ [PS72].
8. $D_4(G) = 1$ and $p > 2$ [Her07].
9. $|G| \leq p^5$ [Pas65, SS95].

If $kG \cong kH$, then $G \cong H$.

Let $\phi : kG \to kH$ be an isomorphism preserving the augmentation. Then the ideal of $kG$ generated by $[kG, kG]$ is exactly $I(G'; G)$. As $\phi([kG, kG]) = [kH, kH]$, it follows that

$$\phi(I(G'; G)) = I(H'; H).$$

This is a classical result that already appears in [Col62], and plays a role in the proofs of a great part of the known results about MIP (e.g., the ones involving the small group algebra such as [San89]). In Section 3 we will be interested in subgroups satisfying the property in (2.1). More concretely, given a family $F$ of groups, we say that a map $N_* : F \to C$, where $C$ is the class of all groups, is a subgroup assignment if given a group $G$ in $F$ it returns a normal subgroup $N_G$ of $G$. Partially following [Sal93, Section 3.1.2], we say that the subgroup assignment $N_*$ is $k$-canonical over $F$, if

$$\phi(I(N_G; G)) = I(N_H; H)$$

for $G, H \in F$ and every isomorphism $\phi : kG \to kH$ preserving the augmentation. Clearly if a subgroup assignment is $k$-canonical, then the image subgroup is a characteristic subgroup of the original group. Unless stated otherwise, in the rest of this paper all the $k$-canonical assignations will be over the family $F$ of all the finite $p$-groups. In the next section we provide specific ways to obtain group-theoretical invariants of $G$.
determined by \( kG \) from \( k \)-canonical assignations, and also to obtain new \( k \)-canonical assignations from the known ones.

3. The \( k \)-canonical subgroup problem and a transfer lemma

The following general fact is widely known, as it is mentioned in the proof of [San85, Lemma 6.10]. A particular case of it is stated and reproved in [MSS21, Lemma 2.7] restricted to \( L = \mathbb{Z}(G) \) and \( N = \Omega_t(G)G' \) for any positive integer \( t \), though their proof works in general.

**Lemma 3.1.** Let \( k \) be a field of characteristic \( p \) and \( G \) a \( p \)-group. If \( N \) and \( L \) are normal subgroups of \( G \), then \( I(L;G) \cap I(N;G) = I(L \cap N;G) \).

All the items in the following lemma seem to be known, at least for some specific \( k \)-canonical subgroup assignation (mostly \( G \mapsto G' \)), but to the best of our knowledge it had never been stated in general.

**Lemma 3.2.** Let \( k \) be a field of characteristic \( p \), let \( G \) and \( H \) be finite \( p \)-groups and let \( L_G \) and \( L_H \) be normal subgroups of \( G \) and \( H \), respectively. Assume that there is an isomorphism \( \phi : kG \rightarrow kH \) such that \( \phi (I(L_G;G)) = I(L_H;H) \). Then

1. For each \( i \geq 1 \), one has that \( D_i(L_G)/D_{i+1}(L_G) \cong D_i(L_H)/D_{i+1}(L_H) \).
2. If \( L_G \) and \( L_H \) are both abelian, then \( L_G \cong L_H \).
3. If \( N_\Gamma \) is a subgroup of \( \Gamma \) containing \( \Gamma' \) for \( \Gamma \in \{ G, H \} \) such that \( \phi (I(N_G;G)) = I(N_H;H) \), then \( G/L_G N_G \cong G/L_H N_H \) and \( L_G N_G/L_G \cong L_H N_H/L_H \).
4. \( G/L_G' \cong H/L_H' \) and \( G/L_G''/G' \cong H/L_H''/H'' \).
5. \( \mathbb{Z}(G) \cap L_G \cong \mathbb{Z}(H) \cap L_H \) and \( \mathbb{Z}(G)L_G/L_G \cong \mathbb{Z}(H)L_H/L_H \).

**Proof.** (1) is proven for \( k = \mathbb{F}_p \) in [San85, Lemma 6.26], although the proof also works for an arbitrary field \( k \) of characteristic \( p \). See also the proof of [Bag88, Lemma 2] for \( N = G' \). (2) is an immediate consequence of (1), since the isomorphism type of an abelian group is determined by the orders of the terms of its Jennings series. As for (3), let \( \phi : kG \rightarrow kH \) an isomorphism of \( k \)-algebras. Then

\[
\phi (I(L_G N_G;G)) = \phi (I(L_G;G) + I(N_G;G)) = I(L_H;H) + I(N_H;H) = I(L_H N_H;H).
\]

This implies that \( \phi \) induces an isomorphism \( k(G/L_G N_G) \cong k(H/L_H N_H) \), thus the isomorphism \( G/L_G N_G \cong H/L_H N_H \) follows from (2) because both groups are abelian, as \( \Gamma' \subseteq N_\Gamma \). For the last isomorphism in (3), observe that \( \phi \) also induces an isomorphism \( \hat{\phi} : k(G/N_G) \rightarrow k(H/N_H) \). Moreover, if \( \pi_N : k\Gamma \rightarrow k(\Gamma/N_\Gamma) \) is the natural projection, then \( \pi_{N_G} (I(L_G N_G;G)) = I(L_G N_G;G/N_\Gamma) \). Thus \( \phi (I(L_G N_G;G/N_G)) = I(L_H N_H;H/N_H) \). Then applying item (1) it follows that

\[
D_i(L_G N_G;G)/D_{i+1}(L_G N_G;G) \cong D_i(L_H N_H;H)/D_{i+1}(L_H N_H;H)
\]

for each \( i \geq 1 \). Hence, as the groups \( L_G N_G;G \) and \( L_H N_H;H \) are abelian, they must be isomorphic. Now (4) follows immediately using (2.1).

The underlying idea behind (5) is once again the proof of [San85, Lemma 6.10], which states the same result for \( L_\Gamma = \Gamma' \); similar ideas are used in [MSS21] to prove the result specialized to \( L_G = \Omega_t(G)G' \), for any positive integer \( t \). It is well-known (see [San85, Lemma 6.10]) that

\[
Z(kG) = I(Z(G)) \oplus [kG, kG] \cap Z(kG),
\]

where \([kG, kG] \cap Z(kG)\) is an ideal of \( Z(kG) \) and \( I(Z(G)) \) is a \( k \)-algebra. Moreover Lemma 3.1 for \( N = Z(G) \) and \( L = L_G \) yields that \( I(Z(G)) \cap I(L_G;G) = I(Z(G) \cap L_G) \). Therefore

\[
Z(kG) \cap I(L_G;G) + [kG, kG] \cap Z(kG) = I(Z(G) \cap L_G;Z(G)) \oplus [kG, kG] \cap Z(kG).
\]

Since \( \phi : kG \rightarrow kH \) maps \([kG, kG] \cap Z(kG)\) to \([kH, kH] \cap Z(kH)\), we deduce that the restriction of \( \phi \) to \( Z(kG) \cap I(G) \) induces an isomorphism \( \hat{\phi} : I(Z(G)) \rightarrow I(Z(H)) \) such that \( \hat{\phi} (I(Z(G) \cap L_G;Z(G))) = I(Z(H) \cap L_H;Z(H)) \); hence the isomorphism \( Z(G) \cap L_G \cong Z(H) \cap L_H \) follows from (2). Moreover \( \hat{\phi} \) induces an isomorphism \( k (Z(G)/Z(G) \cap L_G) \cong k (Z(H)/Z(H) \cap L_H) \), so the last isomorphism also follows from (2). \( \Box \)

Lemma 3.2 gives an idea about how useful to find new group-theoretical invariants of the group algebra \( kG \) is to find \( k \)-canonical subgroup assignations over the family of finite \( p \)-groups. This would be also an interesting problem by itself, as it seems to be the most natural way to study how the normal subgroup structure of the group is reflected inside the group algebra.
Let us compare the situation with the case of group algebra with integral coefficients: Let $\Gamma_1$ and $\Gamma_2$ be two finite groups such that there is an isomorphism $\phi : \mathbb{Z}\Gamma_1 \to \mathbb{Z}\Gamma_2$ preserving the augmentation. We adopt temporarily (only for this paragraph) the notations $I(\Gamma_i)$ and $I(N; \Gamma_i)$ for the augmentation ideal of $\mathbb{Z}\Gamma_i$ and the augmentation ideal of $\mathbb{Z}\Gamma_i$ relative to a normal subgroup $N$ of $\Gamma_i$. By the so called Normal Subgroup Correspondence in the integral case (see [Seh78, Theorem III.4.17]), $\phi$ induces an isomorphism $\phi^*$ between the lattices of normal subgroups of $\Gamma_1$ and $\Gamma_2$. Furthermore, by [Seh78, Theorem III.4.26] this isomorphism satisfies $\phi(I(N; \Gamma_1)) = I(\phi^*(N); \Gamma_2)$.

Back to the modular case, no such Normal Subgroup Correspondence exists in general. However, there exists a limited version of this correspondence, restricted to the sublattice of the lattice of normal subgroups of $G$ formed by the $k$-canonical subgroups of $G$ (more properly, evaluations in $G$ of $k$-canonical assignations). This obvious correspondence is given by $\phi^*(N_G) = N_H$, for each $k$-canonical assignation $N$. Thus the real problem is to identify the $k$-canonical subgroups and determine how large this sublattice can be. In general, one might ask the following

**Question 3.3.** Given a field $k$ and a family $\mathcal{F}$ of groups, which are the $k$-canonical subgroup assignations $N_\mathcal{F} : \mathcal{F} \to \text{Grp}$?

Unfortunately, if $\mathcal{F}$ is the family of finite $p$-groups, the list of subgroups assignations which are known to be $k$-canonical is limited to $G'$ and to some examples appearing in [Sal93] and [BK07], which we will mention later. Even more unfortunately, a choice of subgroup assignation as natural as the center of the group is known to fail to satisfy this property in general, as shown by Baginski and Kurdics [BK19, Example 2.1], by virtue of a group of the group is known to fail to satisfy this property in general, as shown by Baginski and Kurdics [BK19, Example 2.1], by virtue of a group $G$ of order 81 and maximal class, for which there exists an automorphism $\phi : F_3^2G \to F_3^2G$ such that $\phi(I(Z(G); G)) \neq I(Z(G); G)$. Since $G$ is of maximal class, also $Z(G) = \gamma_3(G)$, so the terms of the lower central series in general (aside from $G' = \gamma_2(G)$) are neither candidates to be $k$-canonical subgroups assignations.

However, though limited by the existence of this counterexample, it is still possible that the search of this kind of subgroups could lead to new invariants and MIP-related results, or at least to shed some new light on the existing ones. The remainder of the section is devoted to extend this list, and to that end the following pair of easy general facts will be useful. The first one corresponds to [Seh78, Proposition III.6.1].

**Lemma 3.4.** Let $A$ be an abelian finite $p$-group, $k$ be a field, $p$ be a positive integer, and $\lambda : kA \to k\hat{U}_1(A)$ be the homomorphism given by $x \mapsto x^p$. Then

$$\ker \lambda = I(\Omega(A); A).$$

Moreover, the $k$-linear hull of the image of $\lambda$ equals $k\hat{U}_1(A)$.

**Proof.** Observe that the $p^i$-power map $A \to \check{U}_i(A)$, $x \mapsto x^{p^i}$ is a surjective homomorphism of groups whose kernel is $\Omega(A)$, so that there is an isomorphism of groups $\kappa : A/\Omega(A) \to \check{U}_1(A)$ which extends to an algebra isomorphism $\kappa : k(A/\Omega(A)) \to k\check{U}_1(A)$. Denote $\sigma : kA \to k\check{U}_1(A)$ the $k$-linear extension of the $p^i$-power map $A \to \check{U}_i(A)$, and $\pi : kA \to k(A/\Omega(A))$ the canonical projection. Then $\sigma$ factors as $\sigma = \kappa \circ \pi$. Now let $\tau : kA \to kA$ be the ring homomorphism extending the identity on $A$ and the map $x \mapsto x^{p^i}$ on $k$. Then $\lambda$ factors as $\lambda = \tau \circ \sigma = \tau \circ \kappa \circ \pi$. Since both $\tau$ and $\kappa$ are injective, $\ker \lambda = \ker \pi = I(\Omega(A); A)$.

For the last statement it suffices to observe that the image of $\lambda$ contains $\check{U}_1(A)$.

The following observation is trivial.

**Lemma 3.5.** Let $k$ be a field of characteristic $p$ and $G$ a finite $p$-group. Let also $N$ and $L$ be normal subgroups of $G$ with $N \subseteq L$, and let $\pi_N : kG \to k(G/N)$ be the canonical projection. Then

$$I(L; G) = \pi_N^{-1}(I(L/N; G/N)).$$

We close this section with a lemma that allows us to obtain new $k$-canonical subgroup assignations from known ones, and a series of examples connecting this with other results. A version of this lemma (comprehending only the first item), appears in [GLdRS22] with the name of Transfer Lemma. We choose to reprove it here, as the method of proof is strongly related to the one of the other items.

**Lemma 3.6 (Transfer Lemma).** Let $p$ be a prime number and $G$ and $H$ be finite $p$-groups. Let $t$ be a positive integer and for $\Gamma \in \{G, H\}$ let $N_\Gamma$ and $L_\Gamma$ be normal subgroups of $\Gamma$ such that $\Gamma^t \subseteq N_\Gamma$. Let $k$ be a field of characteristic $p$ and $\phi : kG \to kH$ be a $k$-algebra isomorphism preserving the augmentation such that $\phi(I(N_G; G)) = I(N_H; H)$ and $\phi(I(L_G; G)) = I(L_H; H)$. Then
(1) \( \phi(I(\Omega_t(G : N_G); G)) = I(\Omega_t(H : N_H); H) \).

(2) \( \phi(I(\Omega_t(L_G)N_G; G)) = I(\Omega_t(L_H)N_H; H) \).

(3) \( \phi(I(\Omega_t(Z(G))N_G; G)) = I(\Omega_t(Z(H))N_H; H) \).

Proof. Throughout the proof, for a normal subgroup \( K \) of \( \Gamma \), we denote by \( \pi_K : k\Gamma \to k(\Gamma/K) \) the natural projection, with kernel \( I(K; \Gamma) \). The hypothesis \( \phi(I(N_G; G)) = I(N_H; H) \) yields that the isomorphism \( \phi : k(G/N_G) \to k(H/N_H) \) induced by \( \phi \) makes the following square commutative

\[
\begin{array}{ccc}
kG & \xrightarrow{\pi_{NG}} & k(G/N_G) \\
\downarrow{\phi} & & \downarrow{\tilde{\phi}} \\
kH & \xrightarrow{\pi_{NH}} & k(H/N_H)
\end{array}
\]  

(3.2)

Let also \( \lambda_T : k(\Gamma/N_T) \to k\tilde{\Omega}_t(\Gamma/N_T) \subseteq k(\Gamma/N_T) \) be the \( p' \)-power map, which is a ring homomorphism because of the commutativity of \( \Gamma/N_T \). There is a commutative square

\[
\begin{array}{ccc}
k(G/N_G) & \xrightarrow{\lambda_T} & k\tilde{\Omega}_t(G/N_G) \\
\downarrow{\tilde{\phi}} & & \downarrow{\tilde{\phi}'} \\
k(H/N_H) & \xrightarrow{\lambda_H} & k\tilde{\Omega}_t(H/N_H)
\end{array}
\]  

(3.3)

where the vertical arrow on the right is just the restriction of \( \tilde{\phi} \).

(1) Taking the kernels of the maps \( \lambda_T \), the commutativity of (3.3) and Lemma 3.4 yield that

\[
\tilde{\phi}(I(\Omega_t(G/N_G); G/N_G)) = I(\Omega_t(H/N_H); H/N_H),
\]

which can be rewritten as

\[
\tilde{\phi}(I(\Omega_t(G : N_G); G/N_G)) = I(\Omega_t(H : N_H); H/N_H).
\]

Now, since by Lemma 3.5 we have that \( \pi_{NH}^{-1}(I(\Omega_t(\Gamma : N_T)/N_T)) = I(\Omega_t(\Gamma : N_T); \Gamma) \), the commutativity of (3.2) yields that

\[
\phi(I(\Omega_t(G : N_G); G)) = I(\Omega_t(H : N_H); H).
\]

(2) Observe that the hypotheses imply that \( \phi(I(L_G N_G; G)) = I(L_H N_H; H) \); moreover in general \( \tilde{\Omega}_t(L_T)N_T = \tilde{\Omega}_t(L_T N_T)N_T \). Thus we can assume without loss of generality that \( N_T \subseteq L_T \). Now observe that \( \pi_{NH}^{-1}(I(L_T; \Gamma)) = I(L_T/N_T; \Gamma/N_T) \). Hence, by the commutativity of (3.2),

\[
\tilde{\phi}(I(L_G N_G; G/N_G)) = I(L_H N_H; H/N_H).
\]

Observe also that \( \lambda_T(I(L_T/N_T; \Gamma/N_T)) \) generates \( I(\tilde{\Omega}_t(L_T/N_T; \Gamma/N_T) = I(\tilde{\Omega}_t(L_T)N_T/N_T; \Gamma/N_T) \) as an ideal of \( k(\Gamma/N_T) \). Hence by the commutativity of (3.3),

\[
\tilde{\phi}(I(\tilde{\Omega}_t(G)N_G/N_G; G/N_G)) = I(\tilde{\Omega}_t(H)N_H/N_H; H/N_H).
\]

Now the result follows from Lemma 3.5 and the commutativity of (3.2) as in the previous item.

(3) Using the decomposition of the center (3.1), we get that

\[
Z(k\Gamma) \cap \ker \lambda_T = kZ(\Gamma) \cap \ker \lambda_T + [k\Gamma, k\Gamma] \cap Z(k\Gamma) \cap \ker \lambda_T
\]

\[
= I(\Omega_t(Z(\Gamma)); Z(\Gamma)) + [k\Gamma, k\Gamma] \cap Z(k\Gamma) \cap \ker \lambda_T,
\]

where the second equality is due to Lemma 3.4. As \( [k\Gamma, k\Gamma] \subseteq I(\Gamma^G; \Gamma) \subseteq I(N_\Gamma; \Gamma) \), it follows that the ideal of \( k\Gamma \) generated by \( Z(k\Gamma) \cap \ker \lambda_T \) and \( I(N_\Gamma; \Gamma) \) is exactly \( I(\Omega_t(Z(\Gamma))N_\Gamma; \Gamma) \). Thus \( \phi(Z(kG) \cap \ker \lambda_G) = Z(kH) \cap \ker \lambda_H \) and \( \phi(I(N_G; G)) = I(N_H; H) \) imply the result.

□

Examples 3.7. We illustrate some uses of the Transfer Lemma and relate it to some (recent and classical) known results, as a unified way to approach them. Let \( k \) be a field of characteristic \( p \), \( G \) and \( H \) finite \( p \)-groups, and \( \phi : kG \to kH \) be an isomorphism preserving the augmentation, and fix \( t \geq 1 \).
(1) Taking $t = 1$ and $N_G = G'$, Lemma 3.6(2) yields that
\[ \phi(I(Z(G))G'; G) = I(Z(H)H'; H) \] (3.4)
This was already proven in [BK07] in order to show that if $G$ is of nilpotency class 2 and $KG \cong kH$, then $H$ also has nilpotency class 2.

(2) Taking $N_T = \Gamma'$ and $L_T = \Gamma$, Lemma 3.6(2) yields that
\[ \phi(I(\mathcal{O}_t(G); G)) = I(\mathcal{O}_t(H)H'; H), \] (3.5)
which is [Sal93, Theorem 3.2]. In particular for $t = 1$ this yields $\phi(I(\Phi(G); G)) = I(\Phi(H); H)$. For $N_T = G'$, Lemma 3.6(1) becomes $\phi(I(\Omega_t(G : G'); G)) = I(\Omega_t(H : H'); H)$, which is [Sal93, Lemma 3.5].

(3) We also observe that [MSS21, Theorem B] follow as special cases of the previous lemmas. Indeed, applying Lemma 3.2(5) to (3.5) we obtain that $Z(G) \cap \mathcal{O}_t(G)G' \cong Z(H) \cap \mathcal{O}_t(H)H'$ and $Z(G)\mathcal{O}_t(G)G'/\mathcal{O}_t(G)G' \cong Z(H)\mathcal{O}_t(H)H'/\mathcal{O}_t(H)H'$. Furthermore, for $N_T = \Gamma'$, Lemma 3.6(3) yields that
\[ \phi(I(\Omega_t(Z(G))G'; G)) = I(\Omega_t(Z(H))H'; H), \] (3.6)
and hence applying Lemma 3.2(4) we get that $G/\Omega_t(Z(G))G' \cong H/\Omega_t(Z(H))H'$ and $\Omega_t(Z(G))G'/G' \cong \Omega_t(Z(H))/H'/H'$. The two remaining invariants. Some other invariants resembling the previous ones but not appearing in the mentioned theorem also follow readily. For example, for $L_T = Z(\Gamma)\Gamma'$ and $N_T = \Gamma'$, Lemma 3.6(2) yields $\phi(I(\mathcal{O}_t(Z(G)); G)) = I(\mathcal{O}_t(Z(H)); H')$. So that applying Lemma 3.2(4) we derive that
\[ G/\mathcal{O}_t(Z(G))G' \cong H/\mathcal{O}_t(Z(H))H' \quad \text{and} \quad \mathcal{O}_t(Z(G))G'/G' \cong \mathcal{O}_t(Z(H))H'/H'. \]
These two last invariants, as far as we know, were never considered before.

(4) For $N_T = Z(G)G'$, Lemma 3.6(1) and (3.4) yield that $\phi(I(\Omega_t(G : Z(G)G'); G)) = I(\Omega_t(H : Z(H)H'); H)$. This leads, with the help of Lemma 3.2, to a number of invariants of the group algebra, which, to the best of our knowledge, were used for the first time in [GLdRS22]. There it is shown (see [GLdRS22, Lemma 4.1]) that for $p$-groups with cyclic derived subgroup and $p > 2$ there exists an integer $t$ depending only on $kG$ such that $C_G(G') = \Omega_t(G : Z(G)G')$, so over this family of groups the subgroup assignation $G \mapsto C_G(G')$ is $k$-canonical.

4. Abelian direct factors

This section is devoted to prove our main results, and, after a short introduction, it is divided in two parts: the first one is focused on Proposition C, while the second one contains the proof of Theorem A.

Let $G$ be a finite $p$-group. A homocyclic decomposition $\mathfrak{d}$ of $G$ is an internal direct product decomposition
\[ \mathfrak{d} : \quad G = U_1 \oplus \cdots \oplus U_l \oplus A_1 \oplus \cdots \oplus A_k \] (4.1)
where $U_i, A_i$ are subgroups of $G$, $U_i$ is non-abelian and indecomposable, and $A_i$ is homocyclic of exponent $p^i$ and rank $r_i$, i.e., $A_i \cong (C_{p^i})^{r_i}$. Here we allow $r_i = 0$, in which case $A_i = 1$. Such a decomposition always exists by the Krull-Remak-Schmidt theorem. Given a decomposition (4.1), denote $H^\circ(G) = A_i$. Observe that the isomorphism type of $H^\circ(G)$ does not depend on $\mathfrak{d}$, also by the Krull-Remak-Schmidt theorem. Sometimes we are only interested in the isomorphism type of $H^\circ(G)$, and in such case we drop $\mathfrak{d}$ from the notation. We say that $H(G)$ is the homocyclic component of $G$ of exponent $p^i$. Moreover, if we can express $G$ as an internal direct product $G = S \oplus T$, where $T$ is homocyclic of exponent $p^i$, then $G$ has a homocyclic decomposition $\mathfrak{d}$ satisfying $T \subseteq H^\circ(G)$. In this case we say that $\mathfrak{d}$ extends the decomposition $G = S \oplus T$.

With the notation above, we also denote $Ab^\circ(G) = A_1 \oplus \cdots \oplus A_k$ and $NAb^\circ(G) = U_1 \oplus \cdots \oplus U_l$. The same considerations using the Krull-Remak-Schmidt theorem yield that the isomorphism types of these subgroups do not depend on $\mathfrak{d}$. Hence we can drop it from the notation and write
\[ G \cong Ab(G) \oplus NAb(G), \]
so this notation agrees with the one used in the introduction.

For the rest of the section, let $t$ be a positive integer. The map
\[ \lambda_{t-1} : \frac{\Omega_t(Z(G))\Phi(G)}{\Phi(G)} \rightarrow \frac{\mathcal{O}_{t-1}(G)G'}{\mathcal{O}_t(G)G'}, \quad x\Phi(G) \mapsto x^{p^t-1}\mathcal{O}_t(G)G', \]
which is a homomorphism of elementary abelian $p$-groups, will play an important role in the proof of Theorem A as a tool to detect homocyclic components. We continue by listing some elementary properties of the homocyclic components.

**Lemma 4.1.** Let $\mathfrak{d}$ be a homocyclic decomposition of $G$. Then

1. $H^p_d(G) \subseteq \Omega_d(Z(G))$.
2. $d(H^p_d(G)) = d(H^p_d(G)\Phi(G)/\Phi(G))$.
3. If $U$ is a subgroup of $G$ such that $G = U \oplus H^p_d(G)$, then $\mathcal{U}_d(G)G' = \mathcal{U}_d(U)U'$.
4. $H^p_d(G) \cap \mathcal{U}_d(G)G' = 1$.
5. The map $\lambda^p_G$ restricted to $H^p_d(G)\Phi(G)/\Phi(G)$ is injective.

**Proof.** (1) and (2) are obvious. Let $U$ be a subgroup of $G$ such that $G = U \oplus H^p_d(G)$. As $H^p_d(G)$ is abelian, it follows that $G' = U'$; similarly $\mathcal{U}_d(G) = \mathcal{U}_d(U)$ because $H^p_d(G)$ has exponent $p'$, and (3) follows. In particular $\mathcal{U}_d(G)G' \cap H^p_d(G) \subseteq U \cap H^p_d(G) = 1$. This proves (4). Finally, observe that, in general, for a homocyclic group $H$ of exponent $p'$, every element in $H \setminus \Phi(H)$ has order $p'$. Therefore if $x \in H^p_d(G)$ is such that $xp^{e-1} \in \mathcal{U}_d(G)G'$, then by (2) we have that $xp^{e-1} = 1$, and hence $x \in \Phi(H^p_d(G)) \subseteq \Phi(G)$.

4.1. **Proof of Proposition C.**

**Lemma 4.2.** Let $T$ be a homocyclic subgroup of $\Omega_d(Z(G))$ with exponent $p'$ such that $d(T) = d(T\Phi(G)/\Phi(G))$ and $T \cap \mathcal{U}_d(G)G' = 1$. Then $G = S \oplus T$, for some subgroup $S$ of $G$.

**Proof.** To simplify the notation, we set $N = \mathcal{U}_d(G)G'$. The proof will rely on the following

**Claim.** Let $\{x_1, \ldots, x_n, y\}$ be a Burnside basis of $G$ and assume that $y \in \Omega_d(Z(G))$ has order $p'$ and verifies $\langle y \rangle \cap N = 1$. Then

$$G = \langle x_1', \ldots, x_n', y \rangle,$$

where $x_i' = x_i^{w_i}y^{e_i}$, for some integers $w_i$ and $e_i$ with $p \nmid w_i$, for each $i$.

**Proof of the claim:** We set $S_0 = 1$ and construct $x_j' = x_j^{w_j}y^{e_j}$ recursively such that $S_jN \cap \langle y \rangle = 1$, where $S_j = \langle x_1', \ldots, x_j' \rangle$. Suppose that $j \geq 0$ and $x_1', \ldots, x_j'$ have been already constructed with $S_jN \cap \langle y \rangle = 1$. If $\langle S_j, x_{j+1} \rangle N \cap \langle y \rangle = 1$, then set $x_{j+1}' = x_{j+1}' (i.e., e_{j+1} = 0$ and $w_{j+1} = 1)$. Thus $S_{j+1} = \langle S_j, x_{j+1}' \rangle$, and continue.

Assume on the contrary that $\langle S_j, x_{j+1} \rangle N \cap \langle y \rangle = \langle y^{p'} \rangle \neq 1$. In particular $e < t$. Then taking quotients modulo $S_jN$ one has that $\langle x_{j+1}S_jN \rangle \cap \langle yS_jN \rangle = \langle y^{p'}S_jN \rangle \neq 1$, since if $y^{p'} \in S_jN$ then $y^{p'} = 1$ by the hypothesis $S_jN \cap \langle y \rangle = 1$. Thus there is an integer $wp^{e'}$ with $p \nmid w$ such that $xp^{e'}y^{-p'^{-e}+w} \in S_jN$. If $s \geq t$, then $y^{p'} \in S_jN$, a contradiction. Thus $s < t$. If $s < e$ then $xp^{e'-e}y^{-p'^{-e}+w} \in S_jN$, a contradiction. Thus $s \leq e$. Then set $x_{j+1}' = x_{j+1}^{p'}$ and $S_{j+1} = \langle S_j, x_{j+1}' \rangle$. The previous argument shows that $S_{j+1}N \cap \langle y \rangle = 1$. This finishes the recursive construction of the $x_i'$s. Then $S_n \cap \langle y \rangle = 1$ and hence $G = \langle x_1, \ldots, x_n, y \rangle = \langle x_1', \ldots, x_n', y \rangle = S_n \oplus \langle y \rangle$. This finishes the proof of the claim.

Now we are ready to prove the lemma. Let $\{y_1, \ldots, y_r\}$ be a Burnside basis of $T$. Clearly each $y_i$ has order $p'$, as $T$ is homocyclic with exponent $p'$. Since $d(T\Phi(G)/\Phi(G)) = d(T) = r$, we can extend it to a Burnside basis $\{x_1, \ldots, x_n, y_1, \ldots, y_r\}$ of $G$. Then we can apply the claim $r$ times to obtain a decomposition

$$G = ((\langle x_1', \ldots, x_n' \rangle \oplus \langle y_1' \rangle) \oplus \langle y_2' \rangle) \cdots \oplus \langle y_r' \rangle = \langle x_1', \ldots, x_n', y_1, \ldots, y_r \rangle,$$

where, for $1 \leq i < r$, $y_i' = y_i^{w_{i,j}}x_{i,j}^{-e_i^{-1+1}} \cdots y_r^{w_{i,r}}$, for suitable integers $e_{i,j}$ and $w_{i,j}$, with $p \nmid w_{i,j}$. This implies that $\langle y_1', \ldots, y_{r-1}', y_r \rangle = \langle y_1, \ldots, y_r \rangle$, so the lemma follows taking $S = \langle x_1', \ldots, x_n' \rangle$.

**Lemma 4.3.** One has

$$H_t(G) \cong H_t\left( \frac{\Omega_d(Z(G))\mathcal{U}_d(G)G'}{\mathcal{U}_d(G)G'} \right).$$

**Proof.** Fix a homocyclic decomposition $\mathfrak{d}$ of $G$. Let $\pi : G \to G/\mathcal{U}_d(G)G'$ be the natural projection. By Lemma 4.1(4) the map $\pi$ restricted to $H^p_d(G)$ is injective. Let $V$ be a subgroup of $G$ such that $G = V \oplus H^p_d(G)$, and write $U = V \cap \Omega_d(Z(G))$. Then $\Omega_d(Z(G)) = U \oplus H^p_d(G)$. Moreover, $\pi(\Omega_d(Z(G))) = \pi(U) \oplus \pi(H^p_d(G))$. Indeed, it suffices to prove that $\pi(U) \cap \pi(H^p_d(G)) = 1$, and to show this we observe that $\mathcal{U}_d(G)G' =
Observe that

\[ \text{Proof.} \]

Proof of Theorem A.

Hence we can apply Lemma 4.2 to derive the result.

\[ \text{Lemma 4.4. One has} \]

\[ Ab(G) \cong \bigoplus_{i \geq 1} H_t \left( \frac{\Omega_t(Z(G))\bar{\Omega}_t(G)G'}{\bar{\Omega}_t(G)G'} \right). \]

**Proof.** Since \( Ab(G) \cong \bigoplus_{i \geq 1} H_t(G) \), Lemma 4.3 yields the result.

The following is the same as Proposition C.

**Proposition 4.5.** Let \( G \) and \( H \) finite p-groups such that \( kG \cong kH \). Then \( Ab(G) \cong Ab(H) \).

**Proof.** Let \( \phi : kG \to kH \) be an isomorphism preserving the augmentation. By (3.5) we can apply Lemma 3.6(3) for \( N_\Gamma = \bar{\Omega}_t(G)^\Gamma' \) and derive that \( \phi \left( \langle \Omega_t(Z(G))\bar{\Omega}_t(G)G'(G) \rangle \right) = \langle \Omega_t(Z(H))\bar{\Omega}_t(H)H'(G) \rangle \). By the same reason, we can apply Lemma 3.2(3) with \( N_\Gamma = \bar{\Omega}_t(G)^\Gamma' \) and \( L_\Gamma = \Omega_t(Z(G))\bar{\Omega}_t(G)^\Gamma' \) to deduce that

\[ \frac{\Omega_t(Z(G))\bar{\Omega}_t(G)G'}{\bar{\Omega}_t(G)G'} \cong \frac{\Omega_t(Z(H))\bar{\Omega}_t(H)H'}{\bar{\Omega}_t(H)H'} \]

for each \( t \geq 1 \). In particular

\[ H_t \left( \frac{\Omega_t(Z(G))\bar{\Omega}_t(G)G'}{\bar{\Omega}_t(G)G'} \right) \cong H_t \left( \frac{\Omega_t(Z(H))\bar{\Omega}_t(H)H'}{\bar{\Omega}_t(H)H'} \right). \]

Thus the result follows from Lemma 4.4.

In particular the previous lemma is equivalent to \( H_t(G) \cong H_t(H) \) for each \( t \), provided that \( kG \cong kH \). Hence \( |H_t(G)| = |H_t(H)| \).

### 4.2. Proof of Theorem A.

We will need another pair of lemmas about homocyclic components, as well as to recover some ideas from [MSS21].

**Lemma 4.6.** Let \( T \) be a subgroup of \( \Omega_t(Z(G)) \) such that \( d(T) = d(T\Phi(G)/\Phi(G)) \) and such that the restriction of \( \lambda_{G}^{-1} \) to \( T\Phi(G)/\Phi(G) \) is injective. Then \( T \) is homocyclic of exponent \( p' \) and \( G = S \oplus T \) for some subgroup \( S \) of \( G \).

**Proof.** Observe that \( d(T) = d(T\Phi(G)/\Phi(G)) \) implies that \( \Phi(G) \cap T = \Phi(T) \). As \( T \subseteq \Omega_t(Z(G)) \), to show that it is homocyclic of exponent \( p' \) it suffices to prove that every element in \( T \setminus \Phi(T) = T \setminus \Phi(G) \) has order greater than \( p'^{-1} \), and that is a direct consequence of the injectivity of \( \lambda_{G}^{-1} \).

We claim that \( T \cap \bar{\Omega}_t(G)^G = 1 \). Indeed, if \( 1 \neq x \in T \cap \bar{\Omega}_t(G)^G \) then there exist some \( y \in G \) and \( z \in G' \) such that \( x = y^p \cdot z \). As \( x \in T \subseteq \Omega_t(Z(G)) \), there is some integer \( e \) and some element \( g \in T \setminus \Phi(T) = T \setminus \Phi(G) \) such that \( g^p = x \), with \( e < t \). Therefore \( g^{p^{t-1}} = x^{p^{t-1}-e} = (y^p)^{e^{t'-1}} = y^{p^{t'-1}}z' \) for some \( z' \in G' \), that is, \( g^{p^{t-1}} \in \bar{\Omega}_t(G)^G \), thus \( 1 \neq \Phi(G) \) belongs to the kernel of \( \lambda_{G}^{-1} \), a contradiction. So the claim follows, and we can apply Lemma 4.2 to derive the result.

**Lemma 4.7.** Let \( T \) be a subgroup of \( \Omega_t(Z(G)) \) such that \( d(T) = d(T\Phi(G)/\Phi(G)) \). Then the following conditions are equivalent:
\(1\) \(\Omega_t(Z(G))\Phi(G)/\Phi(G)\) admits the following direct product decomposition
\[
\frac{\Omega_t(Z(G))\Phi(G)}{\Phi(G)} = \frac{T\Phi(G)}{\Phi(G)} \oplus \ker \lambda^{t-1}_G.
\]

\(2\) There exists a homocyclic decomposition \(\mathfrak{d}\) of \(G\) such that \(T = H^p_t(G)\).

**Proof.** Assume \((1)\). This implies that the restriction of \(\lambda^{t-1}_G\) to \(T\Phi(G)/\Phi(G)\) is injective. Thus by Lemma 4.6 we have that \(G = S \oplus T\) for some subgroup \(S\) of \(G\), and \(T\) is homocyclic of exponent \(p^f\). Hence there is a homocyclic decomposition \(\mathfrak{d}\) of \(G\) such that \(T \subseteq H^p_t(G)\). Suppose by contradiction that the inclusion is strict. Then, observing that the elementary abelian group \(\Omega_t(Z(G))\Phi(G)/\Phi(G)\) can be seen as a vector space over \(\mathbb{F}_p\), by counting dimensions it is clear that \((H^p_t(G))\Phi(G)/\Phi(G)) \cap \ker \lambda^{t-1}_G \neq 1\). Hence \(\lambda^{t-1}_G\) is not injective over \(H^p_t(G)\), contradicting Lemma 4.1(5).

Conversely, assume that \(T = H^p_t(G)\) for some homocyclic decomposition \(\mathfrak{d}\) of \(G\). Since \(\lambda^{t-1}_G\) restricted to \(T\Phi(G)/\Phi(G)\) is injective by Lemma 4.1(5), we have that \((T\Phi(G)/\Phi(G))\cap \ker \lambda^{t-1}_G = 1\). Hence \((T\Phi(G)/\Phi(G))\oplus \ker \lambda^{t-1}_G \subseteq \Omega_t(Z(G))\Phi(G)/\Phi(G)\). It suffices to show that the inclusion is in fact an equality, and to see this we count dimension as \(\mathbb{F}_p\)-spaces. Take a direct complement \(U\) of \(\ker \lambda^{t-1}_G\) in \(\Omega_t(Z(G))\Phi(G)/\Phi(G)\). Let \(x_1, \ldots, x_r\) be a basis of \(U\). Then we can take elements \(y_1, \ldots, y_r \in \Omega_t(Z(G))\) such that \(y_i\Phi(G) = x_i\). Set \(S = (y_1, \ldots, y_r)\). Then \(S \subseteq \Omega_t(Z(G))\), \(d(S) = r = d(S\Phi(G)/\Phi(G))\) and \(S\Phi(G)/\Phi(G) \oplus \ker \lambda^{t-1}_G = \Omega_t(Z(G))\Phi(G)/\Phi(G)\). Thus by the previous paragraph there is a homocyclic decomposition \(\mathfrak{d}\) of \(G\) such that \(S = H^p_t(G)\). Thus
\[
\dim(T\Phi(G)/\Phi(G)) \leq \dim(\Omega_t(Z(G))\Phi(G)/\Phi(G)) - \dim(\ker \lambda^{t-1}_G) = \dim(H^p_t(G)\Phi(G)/\Phi(G)).
\]

By Lemma 4.1(2) the first and the last terms in the previous expression equal \(d(H^p_t(G)) = d(H_t(G))\), so the inequality is in fact an equality and the result follows.

\[\square\]

**Lemma 4.8.** [MSS21, Lemma 4.2] Let \(N\) and \(L\) be subgroups of \(G\) such that \(G\) is the direct product \(G = N \times L\). Then for each positive integer \(n\) the following equality holds:
\[
I(G)^n = (N)I(G)^{n-1} \oplus I(L)^n.
\]

The following lemma extracts the idea, with essentially the same proof, of [MSS21, Lemma 4.5].

**Lemma 4.9.** Let \(N\) and \(L\) be subgroups of \(G\) such that \(G\) is the direct product \(G = N \times L\). Let \(J\) be a proper ideal of \(kG\) such that:
\((1)\) \(\text{codim}_{kG}(J) = |L|\).
\((2)\) \(J + I(G)^2 = I(N)kG + I(G)^2\).

Then \(kG = J \oplus kL\).

**Proof.** Observe that \((2)\) easily implies that for each \(n \geq 1\)
\[
J(I(G))^{n-1} + I(G)^{n+1} = I(N)I(G)^{n-1} + I(G)^{n+1}.
\]

We claim that
\[
I(G)^n = J(I(G))^{n-1} + I(L)^n
\]
for each \(n \geq 1\). Let \(c\) the smallest integer such that \(I(G)^c = 0\). If \(n \geq c\) then both terms in the equality are zero since \(J \subseteq I(G)\), thus it holds trivially. Assume by reverse induction that \(I(G)^{n+1} = J(I(G))^n + I(L)^{n+1}\). Then
\[
I(G)^n = I(G)^n + I(G)^{n+1} = I(N)I(G)^{n-1} + I(L)^n + I(G)^{n+1} = J(I(G))^{n-1} + I(L)^n + I(G)^{n+1} = J(I(G))^{n-1} + I(L)^n + J(I(G))^n + I(L)^{n+1} = J(I(G))^{n-1} + I(L)^n + J(I(G))^n + I(L)^{n+1}.
\]

Thus the claim follows. In particular, for \(n = 1\) we have that \(I(G) = J + I(L)\). So that \(kG = J + kL\), and \((1)\) guarantees that the sum is direct.

\[\square\]

**Lemma 4.10.** If \(N\) is a normal subgroup of \(G\) then \((1 + I(N; G) + I(G)^n) \cap G = D_n(G)N\).
Proof. The right-to-left inclusion is trivial, so we prove the converse one. Let \( \pi_N : kG \to k(G/N) \) be the natural projection and assume that \( g \in (1 + I(N;G) + I(G)^n) \cap G \). Then \( \pi_N(G) \in (1 + I(G/N)^n) \cap (G/N) = D_n(G/N) \subseteq D_n(G)N/N \), so that \( g \in D_n(G)N \).

We will use that the following map
\[
\Lambda_G^{t-1} : \frac{I(G)}{I(G)^2} \to \frac{I(G)^{p^{t-1}}}{I(G)^{p^{t-1}+1}}, \quad x + I(G)^2 \mapsto x^{p^{t-1}} + I(G)^{p^{t-1}+1},
\]
already introduced in \([\text{Pas65}]\), is well defined. Moreover there are injective maps
\[
\psi_n : \frac{D_n(G)}{D_{n+1}(G)} \to \frac{I(G)^n}{I(G)^{n+1}}, \quad xD_{n+1}(G) \mapsto x - 1 + I(G)^{n+1}.
\]

From now on we assume that \( k = \mathbb{F}_p \). Then \( \psi_1 \) is an isomorphism (see, for example, \([\text{Seh78, Proposition III.1.15}]\)). Let \( \tilde{N} \) be a normal subgroup of \( G \). Therefore \( \psi_1(N\Phi(G)/\Phi(G)) = (I(N;G) + I(G)^2)/I(G)^2 \). Furthermore, from Lemma 4.10 it follows easily that
\[
\psi_N : \frac{D_n(G)N}{D_{n+1}(G)N} \to \frac{I(G)^n + I(N;G)}{I(G)^{n+1} + I(N;G)}, \quad xD_{n+1}(G)N \mapsto x - 1 + I(G)^{n+1} + I(N;G)
\]
is also injective. Therefore we have a commutative diagram
\[
\begin{array}{ccc}
D_{p^{t-1}+1}(G)N & \xrightarrow{\Lambda_G^{t-1}} & \frac{D_{p^{t-1}+1}(G)N}{D_{p^{t-1}+1}(G)N} \\
\downarrow{\psi} & & \downarrow{\psi_N} \\
\frac{I(G)}{I(G)^2} & \xrightarrow{\lambda_G^{t-1}} & \frac{I(G)^{p^{t-1}+1} + I(N;G)}{I(G)^{p^{t-1}+1} + I(N;G)}
\end{array}
\]
where \( \lambda_G^{t-1} \) is given by the \( p^{t-1} \)-power map in the group, \( x\Phi(G) \mapsto x^{p^{t-1}}D_{p^{t-1}+1}(G)N \). Observe that the image of \( \lambda_G^{t-1} \) is contained, by definition, in \( \Omega_{t-1}(G) \). Moreover, restricting this diagram to \( \Omega_{t}(Z(G))\Phi(G)/\Phi(G) \) we have that
\[
\begin{array}{ccc}
\Omega_{t}(Z(G))\Phi(G) & \xrightarrow{\lambda_G^{t-1}} & \Omega_{t-1}(G)N \\
\downarrow{\psi} & & \downarrow{\psi_N} \\
\frac{I_G^2}{I_G^2} & \xrightarrow{\lambda_G^{t-1}} & \frac{I(G)^{p^{t-1}+1} + I(N;G)}{I(G)^{p^{t-1}+1} + I(N;G)}
\end{array}
\]
commutes, where \( \psi \) is still an isomorphism. Now set \( N = \Omega_{t-1}(G)G' \). As \( \Omega_{t-1}(G) \subseteq \Omega_{t-1}(G') \) and \( D_{p^{t-1}+1}(G) \subseteq \Omega_{t-1}(G)G' \), we have that \( \lambda_G^{t-1} = \lambda_G^{t-1} \) and the previous diagram becomes:
\[
\begin{array}{ccc}
\Omega_{t}(Z(G))\Phi(G) & \xrightarrow{\lambda_G^{t-1}} & \Omega_{t-1}(G)G' \\
\downarrow{\psi} & & \downarrow{\psi_N} \\
\frac{I_G^2}{I_G^2} & \xrightarrow{\lambda_G^{t-1}} & \frac{I(G)^{p^{t-1}+1} + I(\Omega_{t-1}(G)G'G')}{I(G)^{p^{t-1}+1} + I(\Omega_{t-1}(G)G'G')}.
\end{array}
\]

Observe that by the election of \( N \) the diagram can be seen as constructed from the commutative group algebra \( k(G/G') \), so that both \( \lambda_G^{t-1} \) and \( \Lambda_G^{t-1} \) are homomorphisms of elementary abelian groups/vector spaces over \( k \).

Lemma 4.11. Let \( V_G \) be a subspace of \( I(G) \) containing \( I(G)^2 \). With the notation above, the following conditions are equivalent:

(1) There is a direct sum decomposition
\[
\frac{V_G}{I(G)^2} \oplus \ker \Lambda_G^{t-1} = \frac{I(\Omega_{t}(Z(G))G';G) + I(G)^2}{I(G)^2}.
\]
Lemma 4.12. Let \( \phi : kG \to kH \) be an isomorphism preserving the augmentation, and \( \mathfrak{d} \) be a homocyclic decomposition of \( G \). Then there exists a homocyclic decomposition \( \mathfrak{d} \) of \( H \) such that if \( H = S \oplus H^p(H) \), then \( kH = \phi(I(H^p(G);G)) \oplus kS \).

Proof. Set \( V_G = I(H^p(G);G) + I(G)^2 \), so that by Lemma 4.11 we have that
\[
\frac{V_G}{I(G)^2} \oplus \ker \Lambda_G^t = \frac{I(\Omega_t(Z(G)))G^t;G + I(G)^2}{I(G)^2}.
\]

It follows from (3.5) and (3.6) that \( \phi \) induces isomorphisms \( \hat{\phi} \) and \( \tilde{\phi} \) such that the following diagram commutes:
\[
\begin{array}{ccc}
I(\Omega_t(Z(G)))G^t;G + I(G)^2 & \xrightarrow{\Lambda_G^t} & I(G)^p^{t-1} + I(\Omega_t(G)^t)G^t;G \\
\frac{V_G}{I(G)^2} & \xrightarrow{\hat{\phi}} & \frac{I(\Omega_t(Z(H))H^t;H) + I(H)^2}{I(H)^2} \\
\frac{I(\Omega_t(Z(H))H^t;H) + I(H)^2}{I(H)^2} & \xrightarrow{\tilde{\phi}} & \frac{I(\Omega_t(Z(H))H^t;H) + I(H)^2}{I(H)^2}
\end{array}
\]

In particular this shows that \( \hat{\phi}(\ker \Lambda_G^t) = \ker \Lambda_H^t \), and therefore that
\[
\hat{\phi}(V_G) \oplus \ker \Lambda_H^t = \frac{I(\Omega_t(Z(H))H^t;H) + I(H)^2}{I(H)^2}.
\]

Applying once more Lemma 4.11, we derive the existence of a homocyclic decomposition \( \mathfrak{d} \) of \( H \) verifying
\[
\hat{\phi}(V_G) = I(H^p(H);H) + I(H)^2.
\]

Now \( \phi(I(H^p(G);G)) \) is an ideal of \( kH \) verifying the following conditions:

1. \( \text{codim}_{kH}(\phi(I(H^p(G);G))) = \text{codim}_{kG}(I(H^p(G);G)) = |G|/|H^p(G)| = |H|/|H^p(H)| \), where the last equality is due to Proposition 4.5;
2. \( \phi(I(H^p(G);G)) + I(H)^2 = \phi(I(H^p(G);G) + I(G)^2) = \phi(V_G) = I(H^p(H);H) + I(H)^2 \).

Thus Lemma 4.9 yields that for any subgroup \( S \) of \( H \) such that \( H = S \oplus H^p(H) \) one has that \( kH = \phi(I(H^p(G);G)) \oplus kS \), as desired.

Now Theorem A follows easily:

**Theorem 4.13.** Let \( G \) and \( H \) be finite p-groups and \( k = \mathbb{F}_p \). Then
\[
kG \cong kH \quad \text{if and only if} \quad k(\text{NAb}(G)) \cong k(\text{NAb}(H)) \quad \text{and} \quad \text{Ab}(G) \cong \text{Ab}(H).
\]

**Proof.** If \( k(\text{NAb}(G)) \cong k(\text{NAb}(H)) \) and \( \text{Ab}(G) \cong \text{Ab}(H) \), then also \( k(\text{Ab}(G)) \cong k(\text{Ab}(H)) \), so that
\[
kG \cong k(\text{NAb}(G)) \otimes_k k(\text{Ab}(G)) \cong k(\text{NAb}(H)) \otimes_k k(\text{Ab}(H)) \cong kH.
\]

Conversely, assume that \( \phi : kG \to kH \) is an isomorphism preserving the augmentation. Then by Proposition 4.5 we have that \( \text{Ab}(G) \cong \text{Ab}(H) \). Consider a sequence \( \langle G_i, H_i, \phi_i \rangle \), defined recursively as follows. Set
\(G_0 = G, \mathcal{H}_0 = H\) and \(\phi_0 = \phi\). Assume that for \(i \geq 0\) the tuple \((\mathcal{G}_i, \mathcal{H}_i, \phi_i)\) is defined such that \(\phi_i : k\mathcal{G}_i \to k\mathcal{H}_i\) is an isomorphism preserving the augmentation. Choose a homocyclic decomposition \(\bar{H}_i\) of \(\mathcal{G}_i\) and take a subgroup \(\mathcal{G}_{i+1}\) of \(\mathcal{G}_i\) such that \(\mathcal{G}_i = \mathcal{G}_{i+1} \oplus H^\delta_i(\mathcal{G}_i)\); then by Lemma 4.12 there is a homocyclic decomposition \(\bar{H}_i\) of \(\mathcal{H}_i\) such that, given a subgroup \(\mathcal{H}_{i+1}\) of \(\mathcal{H}_i\) satisfying \(\mathcal{H}_i = \mathcal{H}_{i+1} \oplus H^\delta_i(\mathcal{H}_i)\), one has that

\[ k\mathcal{H}_i = \phi(I(H^\delta_i(\mathcal{G}_i); \mathcal{G}_i)) \oplus k\mathcal{H}_{i+1}. \]

Since \(k\mathcal{G}_i = I(H^\delta_i(\mathcal{G}_i); \mathcal{G}_i) \oplus k\mathcal{G}_{i+1}\) by Lemma 4.8 for \(n = 1\), it follows that \(\phi_i\) induces an isomorphism \(\phi_{i+1} : k\mathcal{G}_{i+1} \to k\mathcal{H}_{i+1}\), and we can advance to the next step of the sequence. If \(i\) is large enough (for example if \(p^i\) exceeds the exponent of \(G\)), by the Krull-Remak-Schmidt theorem \(\mathcal{G}_i \cong \text{NAb}(G)\) and \(\mathcal{H}_i \cong \text{NAb}(H)\). Thus we have an isomorphism \(\phi_i : k(\text{NAb}(G)) \to k(\text{NAb}(H))\) as desired. \(\square\)

**Remarks 4.14.** Observe that both Proposition C and Theorem A rely on the results in Section 3: indeed, the former depends on the fact that the subgroup assignations \(G \mapsto \Omega_k(Z(G))Z_k(G)G'\) and \(G \mapsto Z_k(G)G'\) are \(k\)-canonical to show that the isomorphism type of the quotient \(\Omega_k(Z(G))Z_k(G)G'/Z_k(G)G'\) is determined by \(kG\) in Proposition 4.5, while the latter, in addition, uses that \(G \mapsto \Omega_k(Z(G))G'\) is \(k\)-canonical to guarantee that the arguments involving the commutative diagram (4.6) make sense.

In the light of Theorem A, Corollary B follows easily. Indeed, it suffices to observe that if a non-abelian finite \(p\)-group \(G\) satisfies one of the properties (1)-(8) in Corollary B, so does \(\text{NAb}(G)\), and then apply Proposition 2.1.

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