Definability Equals Recognizability for $k$-Outerplanar Graphs

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Abstract

One of the most famous algorithmic meta-theorems states that every graph property that can be defined by a sentence in counting monadic second order logic (CMSOL) can be checked in linear time for graphs of bounded treewidth, which is known as Courcelle’s Theorem [7]. These algorithms are constructed as finite state tree automata, and hence every CMSOL-definable graph property is recognizable. Courcelle also conjectured that the converse holds, i.e. every recognizable graph property is definable in CMSOL for graphs of bounded treewidth. We prove this conjecture for $k$-outerplanar graphs, which are known to have treewidth at most $3k - 1$ [2].

1 Introduction

A seminal result from 1990 by Courcelle states that for every graph property $P$ that can be formulated in a language called counting monadic second order logic (CMSOL), and each fixed $k$, there is a linear time algorithm that decides $P$ for a graph given a tree decomposition of width at most $k$ [7] (while similar results were discovered by Arnborg et al. [1] and Borie et al. [5]). Counting monadic second order logic generalizes monadic second order logic (MSOL) with a collection of predicates testing the size of sets modulo constants. Courcelle showed that this makes the logic strictly more powerful [7]. The algorithms constructed in Courcelle’s proof have the shape of a finite state tree automaton and hence we can say that CMSOL-definable graph properties are recognizable (or, equivalently, regular or finite-state).

† The research was done while the first author was a student at Utrecht University.
‡ The research of the second author was partially funded by the Networks programme, funded by the Dutch Ministry of Education, Culture and Science through the Netherlands Organisation for Scientific Research.
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Courcelle’s Theorem generalizes one direction of a classic result in automata theory by Büchi, which states that a language is recognizable, if and only if it is MSOL-definable \[6\]. Courcelle conjectured in 1990 that the other direction of Büchi’s result can also be generalized for graphs of bounded treewidth in CMSOL, i.e. that each recognizable graph property is CMSOL-definable.

This conjecture is still regarded to be open. Its claimed resolution by Lapoire \[18\] is not considered to be valid by several experts. In the course of time proofs were given for the classes of trees and forests \[7\], partial 2-trees \[8\], partial 3-trees and \(k\)-connected partial \(k\)-trees \[16\]. A sketch of a proof for graphs of pathwidth at most \(k\) appeared at ICALP 1997 \[15\]. Very recently, one of the authors proved, in collaboration with Heggernes and Telle, that Courcelle’s Conjecture holds for partial \(k\)-trees without chordless cycles of length at least \(\ell\) \[3\].

By the results presented in this paper, we add the class of \(k\)-outerplanar graphs to this list. In particular, we first prove the conjecture for 3-connected \(k\)-outerplanar graphs and then generalize this result to all \(k\)-outerplanar graphs, based on the decomposition of a connected graph into its 3-connected components, discovered by Tutte \[20\] and shown to be definable in monadic second order logic by Courcelle \[11]\.

The rest of the paper is organized as follows. In Section 2 we give the basic definitions and review the concepts involved in our proofs. We present the main result in Section 3 and conclude in Section 4.

## 2 Preliminaries

### 2.1 Graphs and Tree Decompositions

Throughout the paper, a graph \(G = (V,E)\) with vertex set \(V\) and edge set \(E\) is undirected, connected and simple. We denote the subgraph relation by \(G \subseteq H\) and for a set \(W \subseteq V\), \(G[W]\) denotes the induced subgraph over \(W\) in \(G\), so \(G[W] = (W,E \cap (W \times W))\). We call a set \(C \subseteq V\) a cut of \(G\), if \(G[V \setminus C]\) is disconnected. An \(\ell\)-cut of \(G\) is a cut of size \(\ell\). A set \(S \subseteq V\) is said to be incident to an \(\ell\)-cut \(C\), if \(C \subseteq S\). We call a graph \(\ell\)-connected, if it does not contain a cut of size at most \(\ell - 1\).

We now define the class of \(k\)-outerplanar graphs and some central notions used extensively throughout the rest of the paper.

**Definition 2.1** ((Planar) Embedding). A drawing of a graph in the plane is called an embedding. If no pair of edges in this drawing crosses, then it is called planar.

**Definition 2.2** (\(k\)-Outerplanar Graph). Let \(G = (V,E)\) be a graph. \(G\) is called a planar graph, if there exists a planar embedding of \(G\). An embedding of a graph \(G\) is 1-outerplanar, if it is planar, and all vertices lie on the exterior face. For \(k \geq 2\), an embedding of a graph \(G\) is \(k\)-outerplanar, if
it is planar, and when all vertices on the outer face are deleted, then one obtains a \((k-1)\)-outerplanar embedding of the resulting graph. If \(G\) admits a \(k\)-outerplanar embedding, then it is called a \(k\)-outerplanar graph.

The following definition will play a central role in many of the proofs of Section 3.

**Definition 2.3** (Fundamental Cycle). Let \(G = (V,E)\) be a graph with maximal spanning forest \(T = (V,F)\). Given an edge \(e = \{v, w\}, e \in E \setminus F\), its fundamental cycle is a cycle that is formed by the unique path from \(v\) to \(w\) in \(F\) together with the edge \(e\).

**Definition 2.4** (Tree Decomposition, Treewidth). A tree decomposition of a graph \(G = (V,E)\) is a pair \((T,X)\) of a tree \(T = (N,F)\) and an indexed family of vertex sets \((X_t)_{t \in N}\) (called bags), such that the following properties hold.

(i) Each vertex \(v \in V\) is contained in at least one bag.

(ii) For each edge \(e \in E\) there exists a bag containing both endpoints.

(iii) For each vertex \(v \in V\), the bags in the tree decomposition that contain \(v\) form a subtree of \(T\).

The width of a tree decomposition is the size of the largest bag minus 1 and the treewidth of a graph is the minimum width of all its tree decompositions. We might sometimes refer to graphs of treewidth at most \(k\) as partial \(k\)-trees.

To avoid confusion, in the following we will refer to elements of \(N\) as nodes and elements of \(V\) as vertices. Sometimes, to shorten the notation, we might not differ between the terms node and bag in a tree decomposition.

We use the following notation. If \(P\) denotes a graph property (e.g. a graph has a Hamiltonian cycle), then by ‘\(P(G)\)’ we express that a graph \(G\) has property \(P\).

### 2.2 Monadic Second Order Logic of Graphs

We now define counting monadic second order logic of graphs \(G = (V,E)\), using terminology from \([5]\) and \([10]\). Variables in this predicate logic are either single vertices(edges) or vertex/edge sets. We form predicates by joining atomic predicates (vertex equality \(v = w\), vertex membership \(v \in V\), edge membership \(e \in E\) and vertex-edge incidence \(\text{Inc}(v,e)\)) via negation \(\neg\), conjunction \(\land\), disjunction \(\lor\), implication \(\rightarrow\) and equivalence \(\leftrightarrow\) together with existential quantification \(\exists\) and universal quantification \(\forall\) over variables.

\footnote{For several characterizations of graphs of treewidth at most \(k\), see e.g. \([2]\) Theorem 1}
in our domain $V \cup E$. To extend this monadic second order logic (MSOL) to counting monadic second order logic (CMSOL), one additionally allows the use of predicates $\text{mod}_{p,q}(S)$ for sets $S$, which are true, if and only if $|S| \mod q = p$, for constants $p$ and $q$ (with $p < q$).

Let $\phi$ denote a predicate without unquantified (so-called free) variables constructed as explained above and $G$ be a graph. We call $\phi$ a sentence and denote by $G \models \phi$ that $\phi$ yields a truth assignment when evaluated with the graph $G$.

**Definition 2.5.** Let $P$ denote a graph property. We say that $P$ is $(C)MSOL$-definable, if there exists a $(C)MSOL$-sentence $\phi_P$ such that $P(G)$ if and only if $G \models \phi$.

We distinguish between two types of free variables. Consider a predicate $\phi$ with free variables $x_1, \ldots, x_p$. A subset of $x_1, \ldots, x_p$, say $x_1, \ldots, x_a$ (where $a \leq p$), can be considered its arguments, and the variables $x_{a+1}, \ldots, x_p$ are its parameters. We denote this predicate as $\phi(x_1, \ldots, x_a)$, i.e. its parameters do not appear in the notation. We illustrate the difference between arguments and parameters in the following example.

**Example 2.6.** Let $P$ denote the property that a graph has a $k$-coloring and $\phi_{col}(v, w)$ a predicate, which is true, if and only if a vertex $v$ has a lower numbered color than $w$ in a given coloring. Then $\phi_{col}$ has two arguments, vertices $v$ and $w$, and $k$ parameters, the $k$ color classes. Clearly, the choice of the parameters influences the evaluation of $\phi_{col}$, but in most applications of parameters for predicates, it is sufficient to show that one can guess some variables of the evaluation graph to define a property.

Now, let $R(x_1, \ldots, x_r)$ denote a relation with arguments $x_1, \ldots, x_r$. We say that $R$ is $(C)MSOL$-definable, if there exists a parameter-free predicate $\phi_R(x_1, \ldots, x_r)$, encoding the relation $R$. Furthermore we call $R$ existentially $(CMSOL)$-definable, if there exists a predicate $\phi_R(x_1, \ldots, x_r)$ with parameters $x_1, \ldots, x_p$, which, after substituting the parameters by fixed values in the evaluation graph, encodes the relation $R$.

A central concept used in this paper is an implicit representation of tree decompositions in monadic second order logic, as we cannot refer to its bags and edges as variables in MSOL directly. We have to define predicates, which encode the construction of a tree decomposition of each member of a given graph class. We require two types of predicates. The Bag-predicates will allow us to verify whether a vertex is contained in some bag and whether any vertex set in the graph constitutes a bag in its tree decomposition. Each bag will be associated with either a vertex or an edge in the underlying graph (its witness) together with some type, whose definition depends on the graph class under consideration. The Parent-predicate allows for identifying edges in the tree decomposition, i.e. for any two vertex sets $S_p$ and $S_e$, this predicate will be true if and only if both $S_p$ and $S_e$ are bags in the tree decomposition and $S_p$ is the bag corresponding to the parent node of $S_e$. 
Definition 2.7 (MSOL-definable tree decomposition). A tree decomposition \((T = (N, F), X)\) of a graph \(G = (V, E)\) is called \textit{existentially MSOL-definable}, if the following are existentially MSOL-definable (with parameters \(x_1, \ldots, x_p\) for some constant \(p\)).

(i) Each bag \(X_p, p \in N\) in the tree decomposition is associated with either a vertex \(v \in V\) or an edge \(e \in E\) (called its \textit{witness}) and can be identified by one of the following predicates (where \(S \subseteq V\) and \(s\) and \(t\) are constants).

(a) \(\text{Bag}_{\tau_1}(v, S), \ldots, \text{Bag}_{\tau_t}(v, S)\): The vertex set \(S\) forms a bag in the tree decomposition of \(G\), i.e. \(S = X_p\) for some \(p \in N\), it is of type \(\tau_i\) \((1 \leq i \leq t)\) and its witness is \(v\).

(b) \(\text{Bag}_{\sigma_1}(e, S), \ldots, \text{Bag}_{\sigma_s}(e, S)\): The vertex set \(S\) forms a bag in the tree decomposition of \(G\), i.e. \(S = X_p\) for some \(p \in N\), it is of type \(\sigma_i\) \((1 \leq i \leq s)\) and its witness is \(e\).

(ii) Each edge in \(F\) can be identified with a predicate \(\text{Parent}(S_p, S_c)\), where \(S_p, S_c \subseteq V\): The vertex sets \(S_p\) and \(S_c\) form bags in \((T, X)\), i.e. \(S_p = X_p\) and \(S_c = X_c\) for some \(p, c \in N\), and \(p\) is the parent node of \(c\) in \(T\).

Lemma 2.8. Let \((T, X)\) be an existentially MSOL-definable tree decomposition with parameters \(x_1, \ldots, x_p\). There exists a predicate \(\phi\) with zero parameters and \(p\) arguments, which is true if and only if the predicates \(\text{Bag}_{\tau_1}, \ldots, \text{Bag}_{\tau_t}, \text{Bag}_{\sigma_1}, \ldots, \text{Bag}_{\sigma_s}\) and \(\text{Parent}\) describe a width-\(k\) rooted tree decomposition of an evaluation graph \(G\).

Proof. The proof can be done analogously to the proof of Lemma 4.7 in [16].

A fundamental result about definable graph properties, which we use extensively throughout our proofs, states that one can define any edge orientation of partial \(k\)-trees in MSOL. For an in-depth study of MSOL-definable edge orientations on graphs, see [10].

Lemma 2.9 (Lemma 4.8 in [16]). Any direction over a subset of the edges of an undirected graph of treewidth at most \(k\) is existentially MSOL-definable with \(k + 2\) parameters.

The idea of the proof of Lemma 2.9 is to find a \((k + 1)\)-coloring \(\gamma : V \rightarrow \{1, \ldots, k + 1\}\) (expressed in MSOL by \(k + 1\) vertex sets) of the graph and an edge set \(F\), such that an edge \(e = \{v, w\}\) is directed from \(v\) to \(w\), if and only if \(\gamma(v) < \gamma(w)\) and \(e \in F\) or \(\gamma(v) > \gamma(w)\) and \(e \notin F\). Hence, by the choice of the set \(F\) we can define any orientation on the edges of a graph in MSOL, if some \((k + 1)\)-vertex coloring of the graph can be fixed.
2.3 Tree Automata for Graphs of Bounded Treewidth

We briefly review the concept of tree automata and recognizability of graph properties for graphs of bounded treewidth. For an introduction to the topic we refer to [14, Chapter 12]. For the formal details of the following notions, the reader is referred to [16].

A tree automaton $A$ is a finite state machine accepting as an input a tree structure over an alphabet $\Sigma$ as opposed to words in classical word automata. Formally, $A$ is a triple $(Q, Q_{\text{Acc}}, f)$ of a set of states $Q$, a set of accepting states $Q_{\text{Acc}} \subseteq Q$ and a transition function $f$, deriving the state of a node in the input tree $T$ from the states of its children and its own symbol $s \in \Sigma$. $T$ is accepted by $A$, if the state of the root node of $T$ is an element of the accepting states $Q_{\text{Acc}}$ (after a run of $A$ with $T$ as an input).

To recognize a graph property on graphs of treewidth at most $k$, one encodes a rooted width-$k$ tree decompositions as a labeled tree over a special type of alphabet, in the following denoted by $\Sigma_k$ (see Definition 3.5, Proposition 3.6 in [16]). We say that a tree automaton over such an alphabet processes width-$k$ tree decompositions.

**Definition 2.10 (Recognizable Graph Properties).** Let $P$ denote a graph property. We call $P$ recognizable (for graphs of treewidth $k$), if there exists a tree automaton $A_P$ processing width-$k$ tree decompositions, such that the following are equivalent.

(i) $(T, X)$ is a width-$k$ tree decomposition of a graph $G$ with $P(G)$.

(ii) $A_P$ accepts (the labeled tree over $\Sigma_k$ corresponding to) $(T, X)$.

Kaller has shown that Courcelle’s Conjecture follows immediately from the construction of an MSOL-definable tree decomposition.

**Lemma 2.11 (Lemma 5.4 in [16]).** Let $P$ denote a graph property, which is recognizable for graphs of bounded treewidth. Suppose that there is an MSOL-definable tree decomposition of width at most $k$ for any partial $k$-tree $G$. Then, one can write a CMSOL-sentence $\Phi$, such that $G \models \Phi$ if and only if $P(G)$.

3 The Main Result

In this section we investigate Courcelle’s Conjecture in the context of $k$-outerplanar graphs (see Definition 2.2). Bodlaender has shown that every $k$-outerplanar graph has treewidth at most $3k - 1$ [2, Theorem 83], using the following properties of maximal spanning forests of a graph.

**Definition 3.1 (Vertex and Edge Remember Number).** Let $G = (V, E)$ be a graph with maximal spanning forest $T = (V, F)$. The vertex remember
number of $G$ (with respect to $T$), denoted by $vr(G,T)$, is the maximum number over all vertices $v \in V$ of fundamental cycles (in $G$ given $T$) that use $v$. Analogously, we define the edge remember number, denoted by $er(G,T)$.

In particular, Bodlaender gave a constructive proof that the treewidth of a graph is bounded by at most $\max\{vr(G,T), er(G,T) + 1\}$ \cite[Theorem 71]{bodlaender96}. The idea of the proof is to create a bag for each vertex and edge in the spanning tree, containing the vertex itself (or the two endpoints of the edge, respectively) and one endpoint of each edge, whose fundamental cycle uses the corresponding vertex/edge. The tree structure of the decomposition is inherited by the structure of the spanning tree. He then showed, that in a $k$-outerplanar graph $G$ one can split the vertices of degree $d > 3$ into a path of $d - 2$ vertices of degree three without increasing the outerplanarity index of $G$ (the so-called vertex expansion step, see Figure 1). In this expanded graph $G'$ one can find a spanning tree of vertex remember number at most $3k - 1$ and edge remember number at most $2k$ \cite[Lemmas 81 and 82]{bodlaender96}. Using \cite[Theorem 71]{bodlaender96}, this yields a tree decomposition of width at most $3k - 1$ for $G'$ and by simple replacements one finds a tree decomposition for $G$ of the same width. A constructive version of this proof was given by Katsikarelis \cite{katsikarelis08}. The expansion step is the major challenge in defining a tree decomposition of a $k$-outerplanar graph in monadic second order logic, since we cannot use these newly created vertices as variables. We find an implicit representation of this step in Section 3.1. We show how to construct an existentially MSOL-definable tree decomposition of a 3-connected $k$-outerplanar graph in Section 3.2 and for the general case of $k$-outerplanar graphs in Section 3.3.

### 3.1 An Implicit Representation of the Vertex Expansion Step

As outlined before, the central step in constructing a width-$(3k - 1)$ tree decomposition of a $k$-outerplanar graph $G$ is splitting the vertices of degree $d > 3$ into a path of $d - 2$ vertices of degree 3 without increasing the outerplanarity index of the graph $G$ (see above). Since we cannot mimic this expansion step in MSOL directly, we have to find another characterization of this method, the first step of which is to partition the vertices of a $k$-outerplanar graph into its stripping layers.

![Figure 1: Expanding a vertex $v$, where $f_1$ is a layer with lowest layer number.](image)
**Definition 3.2** (Stripping Layer of a $k$-Outerplanar Graph). Let $G$ be a $k$-outerplanar graph. Removing the vertices on the outer face of an embedding of $G$ is called a *stripping step*. When applied repeatedly, the set of vertices being removed in the $i$-th stripping step is called the $i$-th *stripping layer* of $G$, where $1 \leq i \leq k$.

**Lemma 3.3.** Let $G = (V, E)$ be a $k$-outerplanar graph. The partition of $V$ into the stripping layers of $G$ is existentially MSOL-definable with $k$ parameters.

*Proof.* We first introduce another characterization of stripping layers of $k$-outerplanar graphs, which we can use later to define our predicates.

**Proposition 3.4.** Let $G = (V, E)$ be a $k$-outerplanar graph. A partition $V_1, \ldots, V_k$ of $V$ represents its stripping layers, if and only if:

(i) $G[V_i]$ is an outerplanar graph for all $i = 1, \ldots, k$.

(ii) For each vertex $v \in V_i$, all its adjacent vertices are contained in either $V_{i-1}, V_i$ or $V_{i+1}$.

*Proof.** ($\Rightarrow$) Since in each step we remove the vertices on the outer face of the graph, it is easy to see that (i) holds. For (ii), suppose not. Wlog. assume that $v \in V_i$ has a neighbor $w$ in $V_{i+2}$. Before stripping step $i$, $v$ lies on the outer face. Now, for $w$ to not lie on the outer face after stripping step $i$, there needs to be a cycle crossing the edge $\{v, w\}$, hence the embedding of $G$ is not planar and we have a contradiction.

($\Leftarrow$) We use induction on $k$. The case $k = 1$ is trivial. Now assume that $G = (V, E)$ is an $\ell$-outerplanar graph with a partition of $V$ into $V_1, \ldots, V_\ell$ such that our claim holds. Let $V_{\ell+1}$ be a set of vertices with neighbors only in $V_{\ell+1}$ and $V_\ell$. We denote the corresponding edge set by $E_{\ell+1}$. Clearly, placing the vertices in $V_\ell$ on the outer face results in an $(\ell+1)$-outerplanar embedding of the graph $G' = (V \cup V_{\ell+1}, E \cup E_{\ell+1})$. However, some vertices in $V_\ell$ might still lie on the outer face. Denote this vertex set by $V_\ell^O$. We let $V_{\ell+1}' = V_{\ell+1} \cup V_\ell^O$ and $V_\ell' = V_\ell \setminus V_\ell^O$. Then, the partition $V_1, \ldots, V_{\ell-1}, V_\ell', V_{\ell+1}'$ satisfies our claim and the result follows (reversing the indices of the sets in the partition).

It is well known that a graph is outerplanar if it does not contain $K_4$, the clique of four vertices, and $K_{2,3}$, the complete bipartite graph on two and three vertices, as a minor (cf. [13, p. 112], [19]). Borie et al. showed that the fixed minor relation is MSOL-definable [5, Theorem 4], so in our definition we use the predicates Minor$_{K_4}$ and Minor$_{K_{2,3}}$ for stating the respective minor containment. The rest can be done in a straightforward way according to Proposition 3.4. The details of the predicates can be found in Appendix A.2 which conclude the proof of Lemma 3.3. \[\square\]
Figure 2: A spanning tree of a planar graph with some additional edges (dashed lines). The remember number of the face $f$, bounded by $bd(f) = \{v, w, x\}$, is 3 in this graph, since the fundamental cycles of the edges $e_1$, $e_2$ and $e_3$ intersect with $bd_E(f)$.

**Definition 3.5** (Layer Number). Let $G = (V, E)$ be a planar graph. The *layer number* of a face is defined in the following way. The outer face gets layer number 0. Then, for each other face, we let the layer number be one higher than the minimum layer number of all its adjacent faces.²

**Proposition 3.6.** Let $G = (V, E)$ be a $k$-outerplanar graph, $V_1, \ldots, V_k$ its stripping layers and $v \in V_i$. Each face $f$ incident to $v$ has either layer number $i$ or $i - 1$. Furthermore, $f$ has layer number $i - 1$, if the boundary of $f$ contains a vertex $w$ with $w \in V_{i-1}$.

**Proof.** We observe that removing all vertices on the outer face makes a face of layer number $i$ become a face of layer number $i - 1$ and our claim follows.

The expansion step does not preserve facial adjacency, so in order to not increase the outerplanarity index of the graph, one makes sure that all faces are adjacent to a face with lowest layer number. We illustrate the expansion step of a vertex in Figure 1. Following the ideas of the proofs given in [2, Section 13], we define another type of *remember number* to implicitly represent the expansion step for creating a tree decomposition of a $k$-outerplanar graph.

**Definition 3.7** (Face Remember Number). Let $G = (V, E)$ be a planar graph with a given embedding $E$ and $T = (V, F)$ a maximal spanning forest of $G$. The *face remember number* of $G$ w.r.t. $T$, denoted by $fr(G, T)$, is the maximum number of fundamental cycles $C$ of $G$ given $T$, such that $bd_E(f) \cap E(C) \neq \emptyset$, where $bd_E(f)$ denotes the boundary edges of a face $f$, over all faces $f$ in $E$, excluding the outer face.

For an illustration of face remember numbers, see Figure 2. Now, consider the vertex $v_1$ in Figure 1 and let $e$ be an edge whose fundamental

²Unless stated otherwise, we call to faces adjacent, if they share an incident vertex.
cycle $C_v$ uses $v_1$ in some spanning tree of $G'$. We observe that $C_v$ intersects with one of the face boundaries of $f_1$, $f_2$ or $f_3$. Since $v_1$ is a vertex in the expanded graph, we know that in each tree decomposition based on a spanning tree of $G'$ there will be a bag containing one endpoint of each edge, whose fundamental cycle intersects with the face boundary of $f_1$, $f_2$ or $f_3$. Using this observation, we can also show that one can find a tree decomposition of a planar graph, whose width is bounded by the face remember number of a maximal spanning forest, without explicitly expanding vertices.

**Lemma 3.8.** Let $G = (V, E)$ be a planar graph with maximal spanning forest $T = (V, F)$. The treewidth of $G$ is at most $\max\{\text{er}(G, T) + 1, 3 \cdot \text{fr}(G, T)\}$.

**Proof.** Recall the vertex expansion step and see Figure 1 for an illustration. In the following, we will construct a tree decomposition $(T, X)$ of the unexpanded graph $G$, imitating the ideas of the expansion step. That is, for each vertex $v \in V$ we create a path in $(T, X)$ in the following way. First, we add $v$ to each of these bags. Let $f_i$ denote a face with lowest layer number of all faces incident to $v$ and let all face indices be as depicted in Figure 1a. Let $C(f_i)$ denote the set, containing one endpoint of each edge $e \in E \setminus F$, whose fundamental cycle $C_e$ intersects with the edge set of the boundary of the face $f_i$, i.e. $bd_E(f_i) \cap E(C_e) \neq \emptyset$. Let $\deg(v) = d$. We create bags containing the vertices in $C(f_1) \cup C(f_2) \cup C(f_3)$, where $i = 2, \ldots, d - 1$. (For an edge $e_i$ incident to $v$, $f_i$ and $f_{i+1}$ are its incident faces.) We make two bags adjacent, if they share two sets $C(f_i)$ and $C(f_j)$ and belong to the same vertex. Note that this way we precisely imitate the construction of bags for the artificially created vertices during the expansion step.

Furthermore, for each edge $e_i \in F$, we create a bag containing both its endpoints and one endpoint of each edge $e_{fc} \in E \setminus F$, whose fundamental cycle uses $e$. We observe that the set $C(f_i) \cup C(f_j)$ contains precisely one vertex for each such edge $e_{fc}$, where $f_i$ and $f_j$ are the two faces incident to $e_i$. We then make this bag adjacent to each bag created in the step before, which corresponds to both $C(f_i)$ and $C(f_j)$ and one more set $C(f')$. For each incident vertex there will always be precisely one such bag and hence, each edge bag will have two neighbors in the tree decomposition (one for each endpoint). For an illustration of the constructed part of the tree decomposition, see Figure 3.

One can verify that this construction yields a tree decomposition of $G$, and since we know that by definition $|C(f)| \leq \text{fr}(G, T)$ for all faces $f$ (except the outer face) we know that its width is bounded by $\max\{\text{er}(G, T) + 1, 3 \cdot \text{fr}(G, T)\}$. 

To apply this result to a $k$-outerplanar graph $G$, we show that we can find a maximal spanning forest of $G$ of bounded edge and face remember number.

\[\text{Note that by by Proposition 3.6, this number will be either } i \text{ or } i - 1, \text{ if } v \in V_i.\]
Figure 3: A part of a tree decomposition corresponding to a vertex, as used in the proof of Lemma 3.8 (assuming, for explanatory purposes, that all incident edges of $v$ are contained in the maximal spanning forest of the graph).

Lemma 3.9. Let $G = (V, E)$ be a $k$-outerplanar graph. There exists a maximal spanning forest $T = (V, F)$ of $G$ with $er(G, T) \leq 2k$ and $fr(G, T) \leq k$.

Proof. The proof can be done analogously to the proof of Lemma 81 in [2].

3.2 3-Connected $k$-Outerplanar Graphs

We now show that the construction of the tree decomposition given in the proofs of Lemmas 3.8 and 3.9 is existentially MSOL-definable for 3-connected $k$-outerplanar graphs. In particular we will make use of the fact that the face boundaries of a 3-connected planar graph can be defined by a predicate in monadic second order logic. We will then define an ordering of all incident edges of a vertex to create a path in the tree decomposition as described in the proof of Lemma 3.8.

A classic result by Whitney states that every 3-connected planar graph has a unique embedding [23] (up to the choice of the outer face). Reconstructing this proof, Diestel has shown that the face boundaries of this embedding can be characterized in strictly combinatorial terms.

Proposition 3.10 (Proposition 4.2.7 in [13]). The face boundaries in a 3-connected planar graph are precisely its non-separating induced cycles.

We immediately have the following.

Proposition 3.11. The face boundaries of a 3-connected planar graph are MSOL-definable.

Proof. We use Proposition 3.10 and define a predicate, which is true if and only if a vertex set $V'$ is the face boundary of a 3-connected planar graph.
Figure 4: A vertex $v$ with two edges $e_i$ and $e_j$, such that $\text{nb}_< (e_i, e_j)$ as described in the proof of Lemma 3.13 defining a clockwise ordering on the incident edges of $v$. Note that paths in the other direction starting at $e_A$ do not exists, since $e'_A$ cannot be included in such a path.

in the following straightforward way.

\[
\text{FaceBd}_3(V') \iff \text{Cycle}(V', \text{IncE}(V')) \land \text{Conn}(V \setminus V', E \setminus \text{IncE}(V'))
\]

We can use this predicate to define this notion in terms of edge sets as well.

\[
\text{FaceBd}_3(E') \iff \text{FaceBd}_3(\text{IncV}(E'))
\]

Using these observations, we can define predicates encoding the above mentioned ordering on the incident edges of each vertex. We first need another definition.

**Definition 3.12** (Face-Adjacency of Edges). Let $G = (V, E)$ be a planar graph and $v \in V$. We call two incident edges $e, f \in E$ of $v$ face-adjacent, if there is a face-boundary containing both $e$ and $f$.

**Lemma 3.13.** Let $G = (V, E)$ be a 3-connected $k$-outerplanar graph, $v \in V$ with $\deg(v) > 3$ and $e_A$ an incident edge of $v$, called its anchor. There exists an ordering $\text{nb}_<(e, f)$, which mimics a clockwise (or counter-clockwise) traversal (in the unique embedding of $G$) on all incident edges of $v$, starting at $e_A$, which is existentially MSOL-definable with two parameters $e_A$ and $e'_A$.

**Proof.** We first observe an important property of 2-connected planar graphs, which we will use to define the ordering later in the proof.

**Proposition 3.14.** Let $G = (V, E)$ be a 2-connected planar graph and $v \in V$. Then, all faces incident to $v$ are pairwise different.
Proof. Suppose not. Then \( \{ v \} \) is a separator of \( G \).

Let \( e'_A \) be another incident edge of \( v \), which is also face-adjacent to \( e_A \). (Note that there are exactly two such edges in \( G \), the choice of which decides whether the ordering is clockwise or counter-clockwise.) For any pair of incident edges of \( v \), \( e_i \) and \( e_j \), we let \( \text{nb}_<(e_i, e_j) \), if and only if we can find sets of edges \( E_i \) and \( E_j \) with the following properties. Let \( \text{Inc}(v) \) denote the set of incident edges of \( v \).

(i) For \( \ell = i, j \), the set \( E_\ell \) consists of the edge \( e_\ell, e_A \) and a subset of \( \text{Inc}(v) \setminus \{ e'_A \} \) and contains precisely all pairs of face-adjacent edges that, according to face-adjacency, form a path from \( e_A \) to \( e_\ell \).

(ii) \( E_i \subset E_j \).

For an illustration of the meaning of these edge sets see Figure 4. We now turn to defining this ordering in MSOL. By Proposition 3.14, we know that all faces adjacent to \( v \) are pairwise different and hence, we can use Proposition 3.11 to define paths in terms of face-adjacency in the unique embedding of \( G \) between two incident edges of \( v \). The predicates given in Appendix A.2.1 complete the proof.

Note that one can lead an alternative proof of Lemma 3.13, using the notion of rotation systems, introduced in \cite{12}. Furthermore one can see that the relation \( \text{nb}_<(e, f) \) is existentially MSOL-definable for a graph \( G \) (as opposed to a single vertex, as stated in the Lemma) by replacing the parameters in the formulation of Lemma 3.13 with the corresponding edge set equivalents.

Defining the Tree Decomposition

Lemma 3.15. Let \( G = (V, E) \) be a 3-connected \( k \)-outerplanar graph. \( G \) admits an existentially MSOL-definable tree decomposition of width at most \( 3k \) and maximum degree 3 with \( 4k + 4 \) parameters.

Proof. We mimic the construction given in the proof of Lemma 3.8 and use the same notation. We first prove the definability of the spanning tree, upon which the construction of our tree decomposition is based.

Proposition 3.16. Let \( G = (V, E) \) be a 3-connected \( k \)-outerplanar graph. There exists a spanning tree \( T = (V, F) \) of \( G \) with \( er \leq 2k \) and \( fr(G, T) \leq k \), which is existentially MSOL-definable with one parameter, the edge set \( F \) of \( T \).

Proof. By Lemma 3.9 we know that such a spanning tree \( T \) exists. We can use Proposition 3.11 to define \( T \) in MSOL, see Appendix A.2.2.
We direct the spanning tree $T$ of Proposition 3.16 as shown in Lemma 2.9 to be a rooted tree, using a 3k-coloring $\Gamma_G$ of $G$. Note that two colors would already suffice, but we will later use these color sets to impose an (arbitrary) orientation on the edges in $E \setminus F$ as well.

We now choose the set of anchor and co-anchor edges $E_A$ and $E'_A$, respectively, to fix an ordering on the incident edges of a vertex as shown in Lemma 3.13. For a vertex $v$, let $e_{\ell_1}$ and $e_{\ell_2}$ denote the edges bounding a face $f_\ell$ with lowest layer number. (If there is more than one face with lowest layer number, we choose the one whose boundary has a shortest face-adjacency path from the unique incoming edge in the spanning tree $T$.) We then add $e_{\ell_1}$ to $E_A$ and $e_{\ell_2}$ to $E'_A$. Hence, we have that nb$_< (e_{\ell_1}, e)$, for all incident edges $e$ of $v$.

We define three types of bag predicates, all associated with edges. The first type, $\sigma$, contains the endpoints of an edge $e \in F$ in the spanning tree of $G$ and one endpoint of each edge, whose fundamental cycle uses $e$. Note for the following that we can identify an incident face of lowest layer number of each vertex by using Proposition 3.6 (for details see Appendix A.2.2). We fix an arbitrary orientation on all edges in $E \setminus F$ using the coloring $\Gamma_G$ together with the empty edge set (see Lemma 2.9). Then we define two more types of bags, $\sigma_H$ and $\sigma_T$ for each edge $e_i \in \text{Inc}(v) \setminus \{e_{\ell_1}, e_{\ell_2}\}$ for all $v \in V$. Let $e_i = \{v, w\}$ with orientation from $v$ to $w$, where $f_i$ and $f_{i-1}$ denote the incident faces of $e_i$. Then, we create a bag of type $\sigma_H$, containing $v$ and one endpoint of each edge in $C(v, f_i) \cup C(v, f_{i-1}) \cup C(v, f_i)$ meaning that $\sigma_H$ is a type associated with the head vertex of an edge. We similarly define a type associated with the tail vertex of an edge, $\sigma_T$, which is created in the same way as $\sigma_H$, except that it contains the tail vertex instead of the head vertex of $e_i$ (in this case: $w$).

We now turn to defining the Parent-predicate. For an illustration of any of the below mentioned cases, we refer the reader to Figure 5, which gives an example of a part of a tree decomposition constructed for a vertex.

First we consider bags of type $\sigma$. Let $e = \{v, w\} \in F$ such that $v$ is its tail vertex and denote the corresponding $\sigma$-bag by $X$. Then, we make $X$ the parent of the bag $Y$ of type $\sigma_T$ for the edge $e$. If $v$ is the head vertex of $e$, then we make the bag $Y$ of type $\sigma_H$ for the edge $e$ the parent of the bag $X$. As mentioned above, we do not create bags of type $\sigma_H$ and $\sigma_T$ for the two edges bounding the fixed face with lowest layer number $f_\ell$ (for details see the proof of Lemma 3.8). Let $e_\ell \in \{e_{\ell_1}, e_{\ell_2}\}$. Then, we make the bag $X$ of type $\sigma$ corresponding to $e_\ell$ the parent of a bag $Y$ of type $\sigma_T$ corresponding to an edge $e$, if $e$ and $e_\ell$ bound a face together, which is adjacent (in this case, sharing an edge) to the face $f_\ell$. Analogously, we make $Y$ the parent of

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4As opposed to the notation in the proof of Lemma 3.8, we use the vertex $v$ as an argument for sets $C$ as well to clarify that the faces we are considering in this step are incident faces of $v$. 

---
Figure 5: A component of a definable tree decomposition as described in the proof of Lemma [3.15], corresponding to a vertex $v$ with a clockwise ordering on its edges, anchored at $e_{\ell_1}$, where $f_{\ell}$ is a face with lowest layer number of all incident faces of $v$.

$x$, if $X$ is of type $\sigma_H$ for such an edge $e_{\ell}$.

Furthermore, we need to add edges between bags of types $\sigma_T$ and $\sigma_H$ as well. Note that by now, the only bag, which already has a parent is the bag of type $\sigma_T$ for the unique incoming edge $e^* \in F$ in the spanning tree of $G$. We use the ordering $\text{nb}_<(e,f)$ of the incident edges of a vertex $v$ to make sure that the resulting tree decomposition is rooted. Let $\text{nb}_<(e,f)$ express that two incident edges $e,f$ of $v$ are direct neighbors in the ordering $\text{nb}_<(e,f)$. Suppose that $X^*$ is the $\sigma_T$-bag for the edge $e^*$ and $Y$ is either a $\sigma_H$- or $\sigma_T$-bag for an edge $f$ with either $\text{nb}_<(e^*,f)$ or $\text{nb}_<(f,e^*)$. In all of these cases, we make $X^*$ the parent of $Y$, since $X^*$ already has a parent bag. We observe that we have to direct the remaining edges in such a way that they point away from the bag $X^*$. Let $e,f \in \text{Inc}(v)\setminus\{e^*, e_{\ell_1}, e_{\ell_2}\}$ with $\text{nb}_<(e,f)$, $X$ the $\sigma_H/\sigma_T$-bag of $e$ and $Y$ the $\sigma_H/\sigma_T$-bag of $f$. We have to analyze two cases. Note that always precisely one of the two holds.

(i) If $\text{nb}_<(e^*,e)$, then make $X$ the parent of $Y$.

(ii) If $\text{nb}_<(f,e^*)$, then make $Y$ the parent of $X$.

This completes existentially defining the tree decomposition as constructed in the proof of Lemma [3.8] in monadic second order logic for a 3-connected $k$-outerplanar graph.

We now count the parameters used in this proof. To find a face with lowest layer number for each vertex, we need the partition into its stripping layers as shown in Lemma [3.3]. For this step we need $k$ parameters. As explained above, for directing the edges of $G$ we use $3k$ color sets ($G$ has treewidth at most $3k - 1$ [2]) and one edge set (see Lemma [2.9]). We fix edge sets for the spanning tree and the anchors $E_A$ and co-anchors $E'_A$ of the edge ordering $\text{nb}_<(e,f)$. Hence, total number of parameters is $4k + 4$.

The predicates given in Appendix [A.2.2] complete the proof. □
3.3 Implications of Hierarchical Graph Decompositions to Courcelle’s Conjecture

A block decomposition of a connected graph $G$ is a tree decompositions, whose bags contain either the endpoints of a single edge or maximal 2-connected subgraphs $^{5}$ of $G$ (called the blocks of $G$) or a cut-vertex of $G$ (called the cuts) by making a block-bag adjacent to a cut-bag $\{v\}$ if the block bag contains $v$ (see e.g. Section 2.1 in [13]).

Analogously, Tutte showed that given a 2-connected graph (or a block of a connected graph) one can find a 3-block decomposition into its 2-cuts and 3-blocks, the latter of which are either 3-connected graphs or cycles (but not necessarily subgraphs of $G$, see below), which can be joined in a tree structure in the same way [20 Chapter 11] [21 Section IV.3]. Courcelle showed that both of these decompositions of a graph are MSOL-definable [11] and also proved that one can find an MSOL-definable tree decomposition of width 2, if all 3-blocks of a graph are cycles [11 Corollary 4.11]. In this section, we will use these methods to prove Courcelle’s Conjecture for k-outplanar graphs by showing that the results of the previous section can be applied to define tree decompositions of 3-connected 3-blocks of a k-outplanar graph.

As many of our proofs make explicit use of the structure of Tutte’s decomposition of a 2-connected graph into its 3-connected components, we will now review this concept more closely.

Definition 3.17 (3-Block). Let $G = (V, E)$ be a 2-connected graph, $S$ a set of 2-cuts of $G$ and $W \subseteq V$. A graph $H = (W, F)$ is called a 3-block, if it can be obtained by taking the induced subgraph of $W$ in $G$ and for each incident 2-cut $S = \{x, y\} \in S$, adding the edge $\{x, y\}$ to $F$ (if not already present), plus one of the following holds.

(i) $H$ is a cycle of at least three vertices (referred to as a cycle 3-block).

(ii) $H$ is a 3-connected graph (referred to as a 3-connected 3-block).

Definition 3.18 (Tutte Decomposition). Let $G = (V, E)$ be a 2-connected graph. A tree decomposition $(T = (N, F), X)$ is called a Tutte decomposition of $G$, if the following hold. Let $S$ denote a set of 2-cuts of $G$.

(i) For each $t \in N$, $X_t$ is either a 2-cut $S \in S$ (called the cut bags) or the vertex set of a 3-block (called the block bags).

(ii) Each edge $f \in F$ is incident to precisely one cut bag.

(iii) Each cut bag is adjacent to precisely two block bags.

---

$^{5}$Let $G = (V, E)$ be a graph and $W \subseteq V$. $H = G[W]$ is called a maximal 2-connected subgraph of $G$, if $G[W]$ is 2-connected and for all $W' \supset W$, $G[W']$ is not 2-connected.
(iv) Let \( t \in T \) denote a cut node with vertex set \( X_t \). Then, \( t \) is adjacent to each block node \( t' \) with \( X_t \subseteq X_{t'} \).

Tutte has shown that additional restrictions can be formulated on the choice of the set of 2-cuts, such that the resulting decomposition is unique for each graph (for details see the above mentioned literature). In the following, when we refer to the Tutte decomposition of a graph, we always mean the one that is unique in this sense, which is also the one that Courcelle defined in his work [11]. Similarly, by a 3-connected 3-block (cycle 3-block, 2-cut etc.) of a graph \( G \) we mean a 3-connected 3-block in the Tutte decomposition of a block of \( G \).

We will now state a property of Tutte decompositions, which will be useful in later proofs.

**Definition 3.19** (Adhesion). Let \((T = (N,F), X)\) be a tree decomposition. The **adhesion** of \((T, X)\) is the maximum over all pairs of adjacent nodes \( t, t' \in N \) of \(|X_t \cap X_{t'}|\).

**Proposition 3.20.** Each Tutte decomposition has adhesion 2.

**Proof.** The claim follows directly from Definition 3.18 (ii) and (iv). \( \square \)

For the proof of the next lemma, we need the notion of \( W \)-paths.

**Definition 3.21** (\( W \)-Path). Let \( G = (V,E) \) be a graph, \( W \subseteq V \) and \( x, y \in V \). Then, a path \( P_{xy} = (V_P, E_P) \) between \( x \) and \( y \) is called a \( W \)-path, if \( x, y \in W \) and \( V_P \cap W = \{x, y\} \), i.e. \( P_{xy} \) avoids all vertices in \( W \) except its endpoints.

**Lemma 3.22.** Let \( G = (V,E) \) be a 2-connected graph with Tutte decomposition \((T = (N,F), X)\). If \( G \) is \( k \)-outerplanar, then all 3-connected 3-blocks \( C = (W,F) \) of \((T, X)\) are at most \( k \)-outerplanar.

**Proof.** We know that \( W = X_t \) for some \( t \in N \). Let \( S = \{x, y\} \) denote a 2-cut of \( G \), which is incident to \( W \). If \( \{x, y\} \in E \), we do not have to consider \( S \) any further, so in the following, if we refer to a 2-cut \( S \), we always assume that \( \{x, y\} \notin E \). Since each such pair \( \{x, y\} \) appears in precisely two 3-blocks (Definition 3.18 (iii)), we know that there is always at least one \( W \)-path between \( x \) and \( y \) in \( G \).

**Proposition 3.23.** Let \((T = (N,F), X)\) be a tree decomposition of adhesion 2 and \( t \in T \). Let \( P_1 \) and \( P_2 \) denote two \( X_t \)-paths. If \( P_1 \) and \( P_2 \) share an internal vertex, then \( P_1 \) and \( P_2 \) have the same endpoints.

**Proof.** Let \( t \in N \). Then, all internal vertices of an \( X_t \)-path \( P \) are contained in a set of bags of a unique component \( T_t \) of \( T[N \setminus \{t\}] \). Let \( t' \in T_t \) be a neighbor of \( t \). Then, the endpoints of \( P_1 \) and \( P_2 \) are contained in \( X_t \cap X_{t'} \). Since \((T, X)\) has adhesion 2, both paths have to have the same endpoints. \( \square \)
Figure 6: A 2-connected graph $G$ with induced subgraph $G[W]$ over the vertex set of a 3-connected 3-block of $G$ with incident 2-cuts $\{a, b\}$ and $\{x, y\}$. The dashed lines indicate that there might be several edges between a vertex and the depicted set and dotted lines represent ($W$-)paths in $G$.

Let $G' = G[W]$ denote the induced subgraph of $G$ over the vertex set $W$. For each 2-cut $S$ incident to $W$ we add one $W$-path from $G$ to $G'$, connecting the two vertices in $S$. Since $G$ is planar and $G'$ is a subgraph of $G$, we know that $G'$ is planar. Since $(T, X)$ has adhesion 2 (Proposition 3.20), we know by Proposition 3.23 that there is no pair of $W$-paths corresponding to two different incident 2-cuts, sharing an internal vertex. Hence, we can contract each of these paths to a single edge such that the embedding of $G'$ stays planar. Clearly, $G'$ is isomorphic to $C$ after contraction and the outerplanarity index of $G'$ is less than or equal to $k$.

For an illustration of the proof of Lemma 3.22 see Figure 6. The ideas in this proof can be applied to more general graph classes as well and we have the following consequence. For the proof of statement (ii), we need the following definition.

**Definition 3.24 (Safe Separator [4])**. Let $G = (V,E)$ be a connected graph with separator $S \subseteq V$. $S$ is called a safe separator, if the treewidth of $G$ is at most the maximum of the treewidth of all connected components $W$ of $G[V \setminus S]$, by making $S$ a clique in $G[W]$.

**Corollary 3.25.** Let $G$ be a 2-connected graph with Tutte decomposition $(T, X)$.

(i) If $G$ is planar, then the 3-connected 3-blocks of $(T, X)$ are planar.

(ii) If $G$ is a partial $k$-tree, then the 3-connected 3-blocks of $(T, X)$ are partial $k$-trees (for $k \geq 2$).

(iii) If $G$ is $\mathcal{H}$-minor free, then the 3-connected 3-blocks of $(T, X)$ are $\mathcal{H}$-minor free, where $\mathcal{H}$ is a set of fixed graphs.
Proof. (i) and (iii) follow from the same argumentation (and, clearly, (i) is a consequence of (iii) by Wagner’s Theorem [22].) For (ii), we observe the following. By [11, Corollary 4.12] we know that each cut bag \(S = \{x, y\}\) is a safe separator of \(G\) and hence, there is a width-\(k\) tree decomposition of \(G\) which has a bag \(X_{xy}\) containing both \(x\) and \(y\). Subsequently, adding the edge between \(x\) and \(y\) does not increase the treewidth of a 3-connected 3-block \(B_3\). (One simply performs a short case analysis of whether \(X_{xy}\) is contained in the tree decomposition of \(B_3\) or not.)

Replacing Edge Quantification by Vertex Quantification

As discussed above, a 3-block is in general not a subgraph of a graph \(G\), as we add edges between the 2-cuts of the Tutte decomposition to turn the 3-blocks into cycles or 3-connected graphs. Since these absent edges cannot be used as variables in MSOL-predicates (which would make our logic non-monadic), we need to find another way to quantify over them.

In [9], Courcelle discusses several structures over which one can define monadic second order logic of graphs, which we will now review.

Definition 3.26 (cf. Definition 1.7 in [9]). Let \(G = (V, E)\) be a graph. We associate with \(G\) two relational structures, denoted by \(|G|_1 = \langle V, \text{edg} \rangle\) and \(|G|_2 = \langle V \cup E, \text{edg}' \rangle\).

(i) All MSOL-sentences and -predicates over \(|G|_1\) only use vertices or vertex sets as variables and we have that \(\text{edg}(x, y)\) is true for \(x, y \in V\), if and only if there is some edge \(\{x, y\} \in E\). MSOL-sentences and -predicates over \(|G|_2\) use both vertices and edges and vertex and edge sets as variables. Furthermore, \(\text{edg}'(e, x, y)\) is true if and only if \(e = \{x, y\}\) and \(e \in E\).

(ii) If we can express a graph property in the structure \(|G|_1\), we call it 1-definable and if we can express a graph property in the structure \(|G|_2\), we call it 2-definable.

Clearly, the monadic second order logic we are using throughout this paper is the one represented by the structure \(|G|_2\). We use both vertex and edge quantification and one simply rewrites \(\text{Inc}(v, e)\) to \(\exists w \ \text{edg}'(e, v, w)\). Since every 1-definable property is trivially also 2-definable, we can conclude that both 1-definability and 2-definability imply MSOL-definability in our sense. Some of the main results of [9] can be summarized as follows.

Theorem 3.27 ([9]). 1-Definability equals 2-definability for

(i) planar graphs.

(ii) partial \(k\)-trees.
(iii) $\mathcal{H}$-minor free graphs, where $\mathcal{H}$ is a set of fixed graphs.

Hence, by Theorem 3.27 we know that we can rewrite each formula using vertex and edge quantification to one only using vertex quantification, if a graph is a member of one of these classes. We will now show that this result can be used to implicitly quantify over virtual edges of a graph, if these virtual edges can be expressed by an (existentially) MSOL-definable relation. (For a similar application of this result, see [11, Problem 4.10].)

**Lemma 3.28.** Let $G = (V, E)$ be a graph which is a member of a graph class $C$ as stated in Theorem 3.27 and let $P$ denote a graph property, which is 2-definable by a predicate $\phi_P$. Let $E' \subseteq V \times V$ denote a set of virtual edges, such that there is a predicate $\text{edg}_\text{Virt}(v, w)$, which is true if and only if $\{v, w\} \in E'$. Then, $P$ is 1-definable for the graph $G' = (V, E \cup E')$, if $G'$ is a member of $C$.

**Proof.** By Theorem 3.27 $P$ is 1-definable for the graph $G$. Let $\phi_{P1}$ denote the predicate expressing $P$ in $|G|_1$. We replace each occurrence of $\text{edg}(x, y)$' in $\phi_{P1}$ by $\text{edg}(x, y) \lor \text{edg}_\text{Virt}(x, y)$' and denote the resulting predicate by $\phi'_{P1}$, which expresses the property $P$ for the graph $G'$ in $|G'|_1$. Since $G' \in C$, one can replace quantification over sets of virtual edges (or mixed sets of edges and virtual edges) by vertex set quantification in the same way as for $G$. \(\square\)

For the specific case of $k$-outerplanar graphs, we can now derive the following.

**Corollary 3.29.** Let $G = (V, E)$ be a $k$-outerplanar graph and $P$ a graph property, which is (C)MSOL-definable for 3-connected $k$-outerplanar graphs. Let $B_3$ denote a 3-block of $G$, including the virtual edges between all incident 2-cuts of $B_3$. Then, $P$ is (C)MSOL-definable for $B_3$.

**Proof.** By [11, Section 3] we know that there is a predicate $\phi_{c_2}(x, y)$, which is true, if and only if $\{x, y\}$ is a 2-cut in the Tutte decomposition of (a block of) $G$. We know that $B_3$ (including the virtual edges) is still $k$-outerplanar (Lemma 3.22). Hence let $\text{edg}_\text{Virt}(x, y) = \phi_{c_2}(x, y)$ and apply Lemma 3.28. \(\square\)

Note that the statements of Lemma 3.28 and Corollary 3.29 also hold for existential definability.

**Defining the Tree Decomposition of a $k$-Outerplanar Graph**

By Corollary 3.29 we now know that every graph property, which can be defined for a 3-connected $k$-outerplanar graph, can also be defined for a 3-block of any $k$-outerplanar graph $G$ (including its virtual edges).
Figure 7: An example hierarchical decomposition of a graph $G$. A bag labeled $C_1$ contains a cut-vertex of $G$, $C_2$ a 2-cut of $G$. Bags labeled $B_2$ contain a 2-block (a single edge or a maximal 2-connected component). If a 2-block contains a maximal 2-connected component of $G$, it is decomposed further into its 2-cuts and 3-blocks, labeled by $B_3$, which contain either a cycle or a 3-connected 3-block.

To apply these results to any $k$-outerplanar graph $G$, we first show how to construct an existentially definable tree decomposition of $G$, assuming that there exist predicates existentially defining bounded width tree decompositions for the 3-connected 3-blocks of (the Tutte decomposition of the 2-blocks of) $G$. For an illustration of the proof idea of the following Lemma, see Figure 7, which shows that we can fix a parent-child ordering of the hierarchical graph decomposition of $G$. After replacing the 3-blocks of $G$ by their corresponding tree decompositions (taking into account the direction of the edges in the hierarchical decomposition), one can see that we have a bounded width tree decomposition of the entire graph $G$.

Remark 3.30. Note that in the proofs of the following results, one fixes a root vertex $r \in V$ of a $k$-outerplanar graph $G = (V, E)$, which will be used to induce a parent-relation on the bags of the hierarchical decomposition of $G$ (see Figure 7). In a later proof, one guesses a rooted spanning tree of $G$, from which one derives a set of edges that contains a spanning tree of each 3-connected 3-block of $G$ (see Lemma 3.36). The root of this spanning tree will be precisely this vertex $r$, hence ensuring that we have a conflict-free parent-child relation in the resulting tree decomposition of $G$.

Lemma 3.31. Let $G = (V, E)$ be a $k$-outerplanar graph with Tutte decompositions $(T, X)$ of its 2-connected blocks. Then, $G$ admits an existentially MSOL-definable tree decomposition of width at most $3k + 3$ with a constant number of parameters, if there exist predicates existentially defining width-3k tree decompositions for the 3-connected 3-blocks of $G$ with a constant number of parameters.

Proof. Recall the decomposition of a graph into its 3-connected components described in the beginning of Section 3.3 and see Figure 7 for an illustration. We will first show how to construct a rooted tree decomposition

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(\mathcal{T} = (N, \mathcal{F}), X) of G width at most 3k + 3 and then prove that (\mathcal{T}, X) is indeed MSOL-definable. Naturally, the description of the tree decomposition is already aimed at providing straightforward methods to define its predicates in MSOL.

**I. Constructing the tree decomposition.** We use the following notation. \( C_1 \) denotes the set of singletons containing a cut-vertex of G and \( C_2 \) denotes the set of 2-cuts in all Tutte decompositions of the 2-connected blocks of G. Furthermore, \( B_2 \) denotes the set of blocks of G, \( B^E_2 \) the set of blocks that are single edges and \( B_3 \) denotes the set of 3-blocks of (\mathcal{T}, X). Let \( \Theta_{B_3} = \{\Theta_1, \ldots, \Theta_r\} \) denote the set of tree decompositions of all elements in \( B_3 \). Then, we create a bag in (\mathcal{T}, X) for all elements in \( C_1 \), \( C_2 \), \( B^E_2 \) and all bags of each \( \Theta_i \) in \( \Theta_{B_3} \), where \( 1 \leq i \leq r \). Note that if a 3-block \( B_3 \in B_3 \) is a cycle, one can find a tree decomposition of \( B_3 \) of width 2 directly. We will later study how to find an MSOL-definable tree decomposition of such a cycle in a more detailed way.

(In the following, keep Remark 3.30 in mind.) We add an edge to \( \mathcal{F} \) between all pairs of adjacent bags originating from a tree decomposition \( \Theta_i \) with the same orientation. To make \( \mathcal{T} \) a directed tree, we add edges to \( \mathcal{F} \) between the above mentioned components in the following way. First, we fix an arbitrary root \( r \in V \) of the graph, which is not a member of a cut or a 2-cut of G. For each vertex \( x \in V \), we let \( P_x \) denote the paths from \( r \) to \( x \) in \( \mathcal{S} \) (and sometimes, slightly abusing notation, we might denote it as if it was one path, if the meaning of the corresponding statement is clear from the context).

Let \( B_2 \in B_2 \setminus B^E_2 \) with Tutte decomposition \((T = (N, F), X)\). We know that the bags of \((T, X)\) either contain a 2-cut \( C_2 \in C_2 \) or a 3-block \( B_3 \in B_3 \) with tree decomposition \( \Theta_i \in \Theta_{B_3} \) for some \( i \) with \( 1 \leq i \leq r \). We now show which edges we need to add to \( \mathcal{F} \) and how to direct them to obtain a rooted tree decomposition of \( B_2 \) of width at most \( 3k + 2 \). We know that each edge in \( \mathcal{F} \) is incident to one cut bag and one block bag (Definition 3.18(ii), cf. Figure 7). Let \( C_2, B_3 \) and \( \Theta_i \) be as above and additionally \( C_2 \subset B_3 \). By Definition 3.18(iv) we know that there has to be an edge in \( \mathcal{F} \) between \( C_2 \) and one bag in \( \Theta_i \), as there is an edge in \( \mathcal{F} \) between \( C_2 \) and \( B_3 \) in the Tutte decomposition \((T, X)\). We use the following (MSOL-definable) properties to create a rooted tree decomposition of a 2-block of G.

**Proposition 3.32.** Let \( C_2 = \{x, y\} \subset C_2 \) and denote by \( B_3(C_2) \) its (two) neighbors in the corresponding Tutte decomposition and \( r \in V \) an arbitrarily chosen but fixed root vertex, which is not a member of a 2-cut. Then, for each of the following two statements, there is precisely one 3-block \( B_3 \) which satisfies it.

(i) For all \( v \in B_3 \): \( P_x \sqsubset P_v \) or \( P_y \sqsubset P_v \).

(ii) There exists at least one \( v \in B_3 \), such that \( P_v \sqsubset P_x \) or \( P_v \sqsubset P_y \).
Proof. Observe that $C_2$ separates $G$ into two components, say $G^+$ and $G^-$, where $r \in V(G^+)$. Then it immediately follows that $[i]$ holds for the component $B_3^+ \in B_3(C_2)$ with $B_3^+ \subseteq V(G^-)$. Now, let $B_3^- \in B_3(C_2) \setminus \{B_3^+\}$. Clearly, $B_3^- \subseteq V(G^+)$. Denote by $C_2(B_3^+)$ the neighbors of $B_3^+$ in $C_2$. Then, there is a 2-cut $C'_2 \subseteq C_2(B_3^+)$, such that $[i]$ holds for $C'_2$ w.r.t. $B_3^+$. By definition, we know that there is a vertex $z \in C_2 \setminus C'_2$ (where ‘\’ denotes the symmetric difference) and $z$ is also contained in $B_3^+$ (again, by definition). Hence, $B_3^+$ satisfies $[ii]$ (with $v = z$).

In case $[i]$ we let $C_2 = \{x, y\}$ be the parent bag of $B_3$. Recall that $\Theta_i$ denotes a tree decomposition of $B_3$. We add both $x$ and $y$ to all bags in $\Theta_i$ and make $C_2$ the parent bag of the root of $\Theta_i$.

In case $[ii]$ we let $B_3$ be the parent of $C_2$. Note that while a cut bag is always the parent of precisely one block bag, a block bag can be the parent of any number of cut bags (cf. Figure 7). Hence, adding all vertices of these 2-cuts to the tree decomposition $\Theta_i$ could increase the width of $\Theta_i$ to a non-constant number. Instead, we observe the following. Since there is a (virtual or non-virtual) edge between $x$ and $y$ in $B_3$, we know that there is at least one bag containing both $x$ and $y$. Denote the set of such bags by $X_{xy}$. Since we have to choose precisely one bag in this set to make it a parent of $C_2$, we observe the following. Either, there is a bag $X^* \in X_{xy}$, whose parent does not contain both $x$ and $y$ or both $x$ and $y$ are contained in the root bag of $\Theta_i$. In the latter case, we let $X^*$ be the root of $\Theta_i$. We then make $X^*$ the parent of $C_2$.

One can verify that this yields a rooted tree decomposition of width at most $3k + 2$ for any $B_2 \in B_2 \setminus B_2^F$.

To finish the construction of the rooted tree decomposition $(T, \mathcal{X})$, we need to show, which edges to add to $\mathcal{F}$ between bags in $C_1$ and (tree decompositions of elements in) $B_2$. We use the same idea as before, based on a fixed root vertex $r$ in $G$. In the following let $C_1 = \{x\} \in C_1$ and $B_2 \in B_2$ with $C_1 \subseteq B_2$. Since $C_1$ is a separator of $G$, one of the following holds for all $v \in B_2$, $v \neq x$.

(i) $P_x \subseteq P_v$.

(ii) $P_v \subseteq P_x$.

Again, in case $[i]$ we make $C_1$ the parent bag of $B_2$. We add $x$ to all bags in the tree decomposition of $B_2$ and make $X'_i$ the parent of a bag $X'_r$, where $X'_r$ is a bag with $X'_r = B_2$ in case $B_2 \in B_2^F$ and if $B_2 \in B_2 \setminus B_2^F, X'_r$ is the root bag of the tree decomposition of $B_2$, constructed as described above.

In case $[ii]$ we make $B_2$ the parent bag of $C_1$. If $B_2 \in B_2^F$, we simply let the bag $X'_i$ with $X'_i = B_2$ be the parent of the bag $X'_r$ with $X'_r = C_1$. If $B_2 \in B_2 \setminus B_2^F$, we observe the following. Since $x$ is a cut vertex of $G$, no 2-cut of a block of $G$ can contain $x$. Hence we know that there exists one
unique 3-block $B_3^* \in B_3$ with $x \in B_3^*$. We denote its tree decomposition by $\Theta_i^*$. Again, we find a bag $X_i$ in $\Theta_i^*$, such that its parent does not contain $x$. If no such bag exists, we let $X_i$ be the root of $\Theta_i^*$. We again let $X_i$ be the bag with $X_i = C_1$ and make $X_i$ the parent of $X_i$.

One can verify that now $(T, X)$ is a rooted tree decomposition and since in the last stage we introduced at most one vertex to each bag of a tree decomposition of an element in $B_2$, its width is at most $3k + 3$.

II. Definability. For defining all necessary predicates for the tree decomposition $(T, X)$, we will refer to $G$ as the graph after adding all virtual edges of its Tutte decomposition. We might write down predicates quantifying over virtual edges or having virtual edges as free variables, and by Corollary 3.29 we know that all these predicates can be defined only using vertex quantification as well.

By some trivial definitions, the statement of the lemma, and the results of [II] we know that the predicates listed below exist.

Proposition 3.33 (cf. [II]). Let $G = (V, E)$ be a $k$-outerplanar graph, for whose 2-blocks all Tutte decompositions are known. Let $G' = (V, E \cup E')$ denote the graph obtained by adding all corresponding virtual edges $E'$ to $G$ and $\gamma : V \to \{1, 2, \ldots, 3k + 1\}$ a coloring of $V$ in $G'$. The following predicates are MSOL-definable.

(I) $\text{Bag}_{C_1}(v, X)$: $X \in C_1$ and $X = \{v\}$.

(II) $\text{Bag}_{B_2^E}(e, X)$: $X \in B_2^E$ and $X = \{v, w\}$, where $e = \{v, w\}$.

(III) $\text{2-Conn}_{B_2 \setminus B_2^E}(X)$: $X$ is the vertex set of a 2-connected 2-block of $G$.

(IV) $\text{Bag}_{C_2}(v, X)$: $X \in C_2$, $v \in X$ and for $w \in X$, $v \neq w$, we have $\gamma(v) < \gamma(w)$.

(V) $\text{3-Conn}_{B_3}(X)$: $X$ is the vertex set of a 3-connected 3-block of $G$.

(VI) $\text{Cycle}_{B_3}(X)$: $X$ is a set of vertices forming a cycle block in a 2-block of $G$.

(VII) $\text{Bag}_{R_1}(v, X), \ldots, \text{Bag}_{R_2}(v, X)$, $\text{Bag}_{R_3}(e, X), \ldots, \text{Bag}_{R_4}(e, X)$: The Bag-predicates of the tree decompositions of the 3-connected 3-blocks of $G$.

(VIII) $\text{Parent}_{B_3}(X, Y)$: The Parent-predicate of the tree decompositions of the 3-connected 3-blocks of $G$.

Proof. (I) and (II) follow from [II] Lemma 2.1, (III) from [II] Section 2 and (IV) from [II] Section 3 and Corollary 3.29. (V) is shown in [II] Corollary 4.8 and a proof of (VI) can done with the same argument. Finally, (VII) and (VIII) are part of the statement of the lemma. 

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We now turn to defining tree decompositions for the cycle 3-blocks of a graph, after which we only need to show that gluing together all components of our construction explained above is MSOL-definable.

**Proposition 3.34.** Let \( G = (V, E) \) be a graph and \( C = (W, F) \) a cycle 3-block of \( G \) (including virtual edges). There is an existentially definable predicate \( \text{Bag}_{Cyc}(e, X) \), which is true if and only if \( X \) is a bag of a tree decomposition of \( C \) associated with a (possibly virtual) edge \( e \) and an existentially definable predicate \( \text{Parent}_{Cyc}(X, Y) \) encoding a parent-relation of a tree decomposition of \( C \).

**Proof.** Recall that for orienting the edges of our tree decomposition, we first fix a root vertex \( r \) in the graph \( G \) and note that by Proposition 3.33(V), \( W \) is MSOL-definable. To create a definable tree decomposition of \( C \), we now find a root \( r_C \in W \) of \( C \). If \( r \in W \), we let \( r_C = r \), otherwise we know that there is one incident parent cut \( C_P \in C_1 \cup C_2 \) of \( C \) in \( G \). \( C_P \) can be identified by checking for all 1- and 2-cuts \( C_C \), which are incident to \( W \), if all paths in \( S \) from \( r \) to the vertices \( w \in W \) pass through (at least one of the vertices in) \( C_C \). This can be defined in a straightforward way and one can see that there is always precisely one such cut. If \( C_P = \{x\} \in C_1 \), then we let \( r_C = x \) and if \( C_P = \{x, y\} \in C_2 \), then we let \( r_C = x \), if \( \gamma(x) < \gamma(y) \) in a fixed coloring \( \gamma \) of \( C \). We create a bag \( X \) for each edge \( f = \{v, w\} \in F \), which is not incident to \( r_C \) and let \( X = \{r_C, v, w\} \). Hence, the predicate \( \text{Bag}_{Cyc}(e, X) \) is also definable in a straightforward way.

We then orient the edges in \( F \) in such a way that \( C \) is a directed cycle. Note that one can find a conflict-free ordering for all cycle blocks in the graph \( G \). (Otherwise, we might violate the cardinality constraint of MSOL.) The predicate \( \text{Parent}_{Cyc}(X, Y) \) is true, if and only if the following hold.

(i) There are two edges \( e, f \in F \), such that \( \text{Bag}_{Cyc}(e, X) \) and \( \text{Bag}_{Cyc}(f, Y) \) (and \( e \) and \( f \) are contained in the same cycle).

(ii) The directed path from \( r_C \) to \( \text{tail}(e) \) in \( C \) is a strict subpath of the path from \( r_C \) to \( \text{tail}(f) \).

(iii) \(|X \cap Y| = 2|\).

Note that we only need one additional parameter, the edge set defining the edge orientation of \( F \), since we already have a coloring for the entire graph \( G \) (see Proposition 3.33). The details of the predicates in Appendix A.3.1 complete the proof.

To unify the parent-relations for all tree decompositions of 3-blocks, we can write

\[
\text{Parent}'_{B_3}(X, Y) \iff \text{Parent}_{B_3}(X, Y) \lor \text{Parent}_{Cyc}(X, Y).
\]
As described above, to create the according parent-relation between blocks of the hierarchical decomposition of $G$, we need to add a number of vertices to some of the bags of the final tree decomposition $(T, \mathcal{X})$. The details for the changes in those definitions are presented in Appendix A.3.2. We can define a $\text{Parent}$-predicate for $(T, \mathcal{X})$ by using the ideas explained above to add edges between blocks and cut-bags. Let $\text{Parent}_{BC}(X, Y)$ denote such a predicate. Then, we have that

$$\text{Parent}(X, Y) \iff \text{Parent}_{B3}^l(X, Y) \lor \text{Parent}_{BC}(X, Y).$$

To show that the number of parameters that we need to define the above mentioned predicates is constant, we note that we only use constructions of previous results with constant numbers of parameters. (For the exact number see the corresponding result.) Note that for the cycle components one additional parameter is as well enough (see the proof of Proposition 3.34) to turn all cycles into directed cycles, since they are connected in a tree structure in the Tutte decomposition of $G$. Hence, fixing the direction of one cycle will always yield the possibility to direct adjacent (i.e. sharing a 2-cut) cycles in a conflict-free manner.

The details for the predicate $\text{Parent}_{BC}(X, Y)$ are given in Appendix A.3.2 and complete the proof of Lemma 3.31.

As mentioned in the previous proof, another obstacle in applying Lemma 3.15 to define a tree decomposition for $G$ using its (definable) hierarchical graph decomposition is the cardinality constraint of MSOL. We illustrate this problem with an example.

Example 3.35. Let $G = (V, E)$ be a $k$-outerplanar graph with $O(n/\log n)$ 3-connected 3-blocks of size $O(\log n)$. Let $P$ denote a graph property, which is definable for 3-connected $k$-outerplanar graphs by a predicate $\phi_P$. Suppose that $\phi_P$ uses a constant number of parameters. When applying $\phi_P$ to all 3-connected 3-blocks of $G$, this might result in a predicate using $O(n/\log n)$ parameters and hence, $P$ not definable in this straightforward way for $G$.

However, for the case of defining a tree decomposition of a $k$-outerplanar graph, we can avoid this problem. When defining a tree decomposition for a 3-connected $k$-outerplanar graph in MSOL, one first guesses a rooted spanning tree of $G$. To avoid guessing a non-constant number of spanning trees, we will find a set of edges $S_E$, which contains a spanning tree with bounded edge and face remember number for each 3-connected 3-block of $G$. Furthermore we guess one set $R_V$, containing one unique vertex for each 3-connected 3-block of $G$, which we will use as the root of its spanning tree. We need to make some observations about such candidate sets $S_E$ and $R_V$. We first prove the existence of these sets and then their MSOL-definability.

Lemma 3.36. Let $G = (V, E)$ be a planar graph and $G = (V, E \cup E')$ the graph obtained by adding the virtual edges $E'$ of the Tutte decompositions
Figure 8: A forest $T_{B_3}$ of a 3-connected 3-block of an example graph. The dashed lines indicate the paths in $T$ between two endpoints of a incident cut of $B_3$. Here, $\{v, w\}$ is the root cut of $B_3$ and $\{x, y\}$ a child cut. Note that by Propositions 3.39 and 3.40 this small example is already somewhat general.

of the 2-connected blocks of $G$ to $G$. Let $T = (V, F)$ be a spanning tree of $G$ with $er(G, T) \leq \lambda$ and $fr(G, T) \leq \mu$. Let $B_3 = (V_{B_3}, E_{B_3}) \in B_3$ be a 3-connected 3-block of $G'$ (including virtual edges) and $T_{B_3} = T[V_{B_3}]$. One can construct from $T_{B_3}$ a spanning tree $T'_{B_3}$ of $B_3$ with $er(B_3, T'_{B_3}) \leq \lambda$ and $fr(B_3, T'_{B_3}) \leq \mu$ by adding edges from $E \cup E'$ to $T_{B_3}$.

Proof. Clearly, $T_{B_3} = (V_{B_3}, E_{B_3})$ is a forest in $B_3$ and in the following we denote its tree components by $F_1 = (V_{F_1}, E_{F_1}), \ldots, F_c = (V_{F_c}, E_{F_c})$. We will now show how to connect these components to a tree. Let $v, w \in V_{B_3}$ and consider the unique path $P_{vw}$ between $v$ and $w$ in $T$. There are two cases: (I) The path $P_{vw}$ is completely contained in $B_3$ and $v$ and $w$ belong to the same connected component. (II) Suppose that they do not and let $F_j$ denote the component with $v \in V_{F_j}$ and $F_j$ the component with $w \in V_{F_j}$. Let $x$ and $y$ be the vertices on the path $P_{vw}$ with $x, y \in V_{B_3}$ (and $x \neq y$), such that $x$ has a neighbor $x' \notin V_{B_3}$ and $y$ has a neighbor $y' \notin V_{B_3}$ (both in $P_{vw}$). Denote this subpath by $P_{xy}$. Then, $P_{xy}$ is a $B_3$-path in $G$. Hence, there is a unique component in $T'_{T} = T_T[N_T \setminus \{t\}]$ containing all internal vertices of $P_{xy}$. Since the neighbor of $t$ in $T'_{T}$ is a cut-bag, we know that it has to contain both $x$ and $y$ and hence $\{x, y\} \in C_2'$.

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By Proposition 3.37 we know that we can find a subset of incident 2-cuts of each 3-connected 3-block to turn $T_{B_3}$ into a tree. We now prove that adding these edges does not increase the edge and face remember number. Consider a 2-cut $C_2 = \{x, y\} \in C_2'$, such that $\{x, y\} \notin F$. Since $T$ is a spanning tree of $G$, we know that there is one unique path $P_{xy}$ between $x$ and $y$ in $T$. Let $T'_{B_3} = (V'_{B_3}, F'_{B_3})$ denote the tree obtained by adding the above described paths between the components of $T_{B_3}$. Then, $T'_{B_3}$ is a spanning tree of the graph $G'_{B_3} = (V'_{B_3}, E_{B_3} \cup F'_{B_3})$ with $er(G'_{B_3}, T'_{B_3}) \leq \lambda$ and $fr(G'_{B_3}, T'_{B_3}) \leq \mu$, since $G'_{B_3} \subseteq G$ and no edges, which are not members of $T'_{B_3}$, are introduced in $G'_{B_3}$. Subsequently, replacing each path $P_{xy}$ by a single edge in $T'_{B_3}$ does not increase the edge and face remember number as well and after these replacements, we have that $T'_{B_3} = T^*_B$ and our claim follows. For an illustration of this proof see Figure 8a.

**Lemma 3.38.** The statement of Lemma 3.36 also holds, if one replaces the term spanning tree by rooted spanning tree. Furthermore there is a set $R_V \subseteq V$, which contains precisely one vertex acting as a root for a spanning tree for each 3-connected 3-block of $G$.

**Proof.** We use the same notation as in the proof of Lemma 3.36. Since $T = (V, F)$ is a rooted spanning tree, we know that its components $F_1, \ldots, F_c$ in $B_3$ are rooted trees as well, see Figure 8b for an illustration. Since the direction between block and cut bags of a Tutte decomposition of a block of $G$ are based on the root of the spanning tree $T$ (see Remark 3.30 and the proof of Lemma 3.31), we observe the following. Let $C_2 = \{x, y\} \in C_2$ denote an incident 2-cut of $B_3$ with $\{x, y\} \notin F$. There are two cases we have to consider. Either, $C_2$ is the parent cut of $B_3$ or it is a child cut.

**Proposition 3.39.** Let $C_2$ be a child cut of $B_3$. Wlog. $x$ is a vertex in a tree $F_i$ and $y$ is the root of a tree $F_j$.

**Proof.** Suppose not. We know that there is a path $P_{xy}$ between $x$ and $y$ in $T$. If $y$ is a non-root vertex in $F_j$, then we cannot direct the edges of $P_{xy}$ in $T$ such that every vertex has precisely one parent. Hence, $T$ is not a directed tree and we have a contradiction.

**Proposition 3.40.** Let $C_2$ be the parent cut of $B_3$. Then, $x$ and $y$ are roots of two trees $F_i$ and $F_j$.

**Proof.** For any vertex $v \in V_{B_3}$, we know by definition (see the proof of Lemma 3.31) that for every vertex $v \in V_{B_3}$, the directed path from the root $r$ of $T$ to $v$ in $T$ is either a subpath of the directed path from $r$ to $x$ or from $r$ to $y$. Hence, neither $x$ nor $y$ can have a parent in $T_{B_3}$.

We can direct the additional edges using Propositions 3.39 and 3.40. In the case that $C_2$ is a child cut, we can always direct the edge $\{x, y\}$ from
x to y (using the notation of Proposition 3.39). If \( C_2 \) is the parent cut, we know by Proposition 3.40 that we can orient \( \{x, y\} \) arbitrarily. There are two cases we need to analyze to make sure we do not create a conflicting orientation of \( S_E \). In the first case, the edge \( \{x, y\} \) has been added to \( S_E \) by the parent block of \( C_2 \). We then use the same orientation. In the second case, if \( \{x, y\} \notin S_E \), we can choose the direction arbitrarily.

We now turn to finding the set of roots \( R_V \). If \( B_3 \) is the root block according to the spanning tree of \( G \) with root \( r_G \), then we add \( r_G \) to \( R_V \) as the root of \( B_3 \). Otherwise, we find its parent cut \( C_2 = \{x, y\} \). Assume wlog. that the edge \( \{x, y\} \) is directed from \( x \) to \( y \) according to the construction explained above. Then we add \( x \) to \( R_V \). Since each cut-bag has precisely one child block bag (Definition 3.18(ii)), we know that this vertex is unique for each 3-block \( B_3 \).

**Lemma 3.41.** The sets \( S_E \) and \( R_V \) of Lemmas 3.36 and 3.38 are existentially MSOL-definable with \( 3k + 2 \) parameters.

**Proof.** Let \( G = (V, E) \) denote a \( k \)-outerplanar graph, such that the virtual edges introduced by the Tutte decompositions of its 2-connected blocks are already included in \( E \). On a high level, for defining \( R_V \) and \( S_E \), we need to encode is the following:

(i) There are sets \( R_V \subseteq V \), \( F \subseteq E \) and \( F' \subseteq E \) with \( S_E = F \cup F' \).

(ii) Guess a root \( r_T \in V \), such that \( F \) is the edge set of a rooted spanning tree in \( G \).

(iii) An edge \( e = \{x, y\} \) is possibly (but not necessarily) a member of \( F' \), if \( \{x, y\} \in C_2 \) and \( e \notin F \).

(iv) For all \( B_3 \in B_3 \), the graph \( T_{B_3}^* = (B_3, S_E \cap (B_3 \times B_3)) \) is a spanning tree of the graph \( G_{B_3} = G[B_3] \) with \( er(G_{B_3}, T_{B_3}^*) \leq 2k \) and \( fr(G_{B_3}, T_{B_3}^*) \leq k \).

(v) A vertex \( v \in V \) is possibly (but not necessarily) a member of \( R_V \), if it is a member of a 2-cut \( \{v, w\} \in C_2 \).

(vi) For each 3-connected 3-block \( B_3 \in B_3 \), there is a vertex \( r_{B_3} \in R_V \), such that \( T_{B_3}^* \) can be rooted at \( r_{B_3} \) (without altering the edge direction of any other edge in \( S_E \)).

The existence of such sets \( R_V \) and \( S_E \) is shown in Lemmas 3.36 and 3.38, so we do not need to encode all details mentioned in the corresponding proofs explicitly. Property \([iv]\) is MSOL-definable by Proposition 3.16, since \( G_{B_3} \) is 3-connected.

As parameters we have the edge set of the spanning tree and again a 3k-coloring and one edge set to fix the orientation of the edges in \( S_E \).
The details of the predicates encoding the rest of the properties are given in Appendix A.3.3 and complete the proof.

We can now use the above results to conclude that we can find predicates defining tree decompositions of 3-connected 3-blocks of $k$-outerplanar graphs.

**Corollary 3.42.** Let $G = (V, E)$ be a $k$-outerplanar graph. Then, there exist predicates existentially defining tree decompositions of width at most $3k$ for each 3-connected 3-block of $G$ with a constant number of parameters.

**Proof.** By Lemma 3.15 we know that a 3-connected $k$-outerplanar graph admits an MSOL-definable tree decomposition of width $3k$, based on a rooted spanning tree of the graph. By Corollary 3.29 we can define such a tree decomposition in a structure, which also includes the virtual edges of a 3-block in $G$ (and by Lemma 3.22 we know that this graph is still $k$-outerplanar). Finally, by Lemmas 3.36, 3.38 and 3.41 we know that we can find definable edge and vertex sets which contain the edges of spanning trees for each 3-connected 3-block with the required bound on their vertex and edge remember numbers without violating the cardinality constraint of monadic second order logic. Similarly, we can find sets containing anchor and co-anchor edges for all 3-connected 3-blocks in a straightforward way. Hence, also for defining the ordering of all incident edges of all vertices in a 3-connected 3-block, two sets are sufficient. Subsequently, the number of parameters involved is bounded by a constant. For the exact bounds see the corresponding result.

Combining Lemma 3.31 and Corollary 3.42 yields that $k$-outerplanar graphs admit existentially MSOL-definable tree decompositions of width at most $3k + 3$. It then follows from Lemma 2.11 that recognizability implies CMSOL-definability for $k$-outerplanar graphs. In the light of Courcelle’s Theorem [7], we have the main result of this paper.

**Theorem 3.43.** CMSOL-definability equals recognizability for $k$-outerplanar graphs.

### 4 Conclusion

In this paper we have shown that recognizability implies definability in counting monadic second order logic for $k$-outerplanar graphs, resolving a special case of a conjecture by Courcelle [7]. Starting at the more restrictive case of 3-connected $k$-outerplanar graphs, we proved that one can use hierarchical graph decompositions to define tree decompositions for general $k$-outerplanar graphs in monadic second order logic. We have also given indications that this technique might be applicable for other graph classes.
as well (see Corollary 3.25), depending on how their tree decompositions are defined in MSOL. 3-Connected graphs often have favorable properties when it comes to defining graph properties in MSOL. For example, in our proof we used the fact that the face boundaries of a 3-connected can be expressed in strictly combinatorial terms and are definable in a straightforward way (see Propositions 3.10 and 3.11). Hence, we believe that the techniques presented in this paper can be helpful in resolving the conjecture in its general statement.

References

[1] Stefan Arnborg, Jens Lagergren, and Detlef Seese. Easy problems for tree-decomposable graphs. Journal of Algorithms, 12(2):308–340, 1991.

[2] Hans L. Bodlaender. A partial $k$-arboretum of graphs with bounded treewidth. Theoretical Computer Science, 209(1-2):1–45, 1998.

[3] Hans L. Bodlaender, Pinar Heggernes, and Jan Arne Telle. Recognizability equals definability for graphs of bounded treewidth and bounded chordality. In Proceedings EUROCOMB 2015, Electronic Notes in Discrete Mathematics. Elsevier, 2015.

[4] Hans L. Bodlaender and Arie M.C.A. Koster. Safe separators for treewidth. Discrete Mathematics, 306(3):337 – 350, 2006.

[5] Richard B. Borie, R. Gary Parker, and Craig A. Tovey. Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families. Algorithmica, 7(1-6):555–581, 1992.

[6] J. Richard Büchi. Weak second-order arithmetic and finite automata. Mathematical Logic Quarterly, 6(1-6):66–92, 1960.

[7] Bruno Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. Information and Computation, 85(1):12–75, 1990.

[8] Bruno Courcelle. The monadic second-order logic of graphs V: On closing the gap between definability and recognizability. Theoretical Computer Science, 80(2):153–202, 1991.

[9] Bruno Courcelle. The monadic second order logic of graphs VI: On several representations of graphs by relational structures. Discrete Applied Mathematics, 54(23):117 – 149, 1994.

[10] Bruno Courcelle. The monadic second-order logic of graphs VIII: Orientations. Annals of Pure and Applied Logic, 72(2):103–143, 1995.
[11] Bruno Courcelle. The monadic second-order logic of graphs XI: Hierarchical decompositions of connected graphs. *Theoretical Computer Science*, 224(12):35 – 58, 1999.

[12] Bruno Courcelle. The monadic second-order logic of graphs XII: Planar graphs and planar maps. *Theoretical Computer Science*, 237(12):1 – 32, 2000.

[13] Reinhard Diestel. *Graph Theory*. Number 173 in Graduate Texts in Mathematics. Springer, 4 edition, 2012. Corrected reprint.

[14] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[15] Valentine Kabanets. Recognizability equals definability for partial $k$-paths. In *Proceedings ICALP 1997*, volume 1256 of *LNCS*, pages 805–815. Springer, 1997.

[16] Damon Kaller. Definability equals recognizability of partial 3-trees and $k$-connected partial $k$-trees. *Algorithmica*, 27(3-4):348–381, 2000.

[17] Ioannis Katsikarelis. Computing bounded-width tree and branch decompositions of k-outerplanar graphs, 2013.

[18] Denis Lapoire. Recognizability equals monadic second-order definability for sets of graphs of bounded tree-width. In *Proceedings STACS 1998*, volume 1373 of *LNCS*, pages 618–628. Springer, 1998.

[19] Maciej M. Sysło. Characterizations of outerplanar graphs. *Discrete Mathematics*, 26(1):47 – 53, 1979.

[20] William T. Tutte. *Connectivity in Graphs*. University of Toronto Press, 1966.

[21] William T. Tutte. *Graph Theory*, volume 21 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, 1984.

[22] Klaus Wagner. Über eine Eigenschaft der ebenen Komplexe. *Mathematische Annalen*, 114(1):570–590, 1937.

[23] Hassler Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54:150–168, 1932.
A Monadic Second Order Predicates and Sentences

We build sentences in monadic second order logic from a collection of predicates. Once we defined these predicates they will be the building blocks of more complex expressions, joined by MSOL-connectives and/or quantification of its declared variables. Hence, we follow the ideas of the work of Borie et al. \[5\], who also give a large list of predicates and their definitions. Note that the length of our sentences and formulas always has to be bounded by some constant, independent of the size of the input graph.

We will denote single element variables by small letters, where \(v, w, v', w'\), \(e, f, e', f'\), \(\ldots\) typically represent vertices and \(e, f, e', f'\), \(\ldots\) edges. Set variables will be denoted by capital letters. Unless stated otherwise explicitly, \(V\) always denotes the vertex set of some input graph \(G\) and \(E\) its edge set. Since we always assume our predicates to appear in the context of such a graph we might drop these two variables as an argument of a predicate.

By some trivial definition, the following predicates are MSOL-definable (see also Theorem 1 in [5]). In our text we might refer to them as the atomic predicates of monadic second order logic over graphs.

(I) \(v = w\) (Vertex equality)

(II) \(Inc(e, v)\) (Vertex-edge incidence)

(III) \(v \in V\) (Vertex membership)

(IV) \(e \in E\) (Edge membership)

Note that to shorten our notation we might omit statements such as \(v \in V\) or \(e \in E\) when quantifying over a variable. In this case we are referring to some vertex/edge in the whole graph and the interpretation of the variables will always be obvious from the context or the notational conventions explained above.

From the atomic predicates, one can directly derive the following:

- \(Adj(v, w, E)\) (Adjacency of \(v\) and \(w\) in \(E\))
- \(Edge(e, v, w)\) (\(e = \{v, w\}\))

In a straightforward way (and by Theorem 4 in [5]), one can see that the following are MSOL-definable:

- \(V = V' \cup V''\), \(V = V' \setminus V''\), \(V = V' \cap V''\) (plus the edge set equivalents)
- \(V' = IncV(E') [E' = IncE(V')]\) (\(V' [E']\) is the set of incident vertices [edges] of \(E' [V']\))
- \(\deg(v, E) = k\) (\(v\) has degree \(k\) in \(E\), where \(k\) is a constant)
- \(Conn(V, E), Conn_k(V, E), Cycle(V, E), Tree(V, E), Path(V, E)\)
- \(Minor_H\) (A graph contains a minor \(H\) of fixed size)
A.1 Bounded Vertex and Edge Remember Number

In this section we show how to define tree decompositions of graphs for which we can find a spanning tree with bounded vertex and edge remember number. Note that this immediately implies a bounded-width tree decomposition for bounded degree $k$-outerplanar graphs. First, we are going to show how to identify an edge set as a spanning tree with vertex remember number less than or equal to $\kappa$ and edge remember number less than or equal to $\lambda$, both constant.

$$\exists E_T (\text{Tree}(V,E_T) \land vr(E_T) \leq \kappa \land er(E_T) \leq \lambda)$$

$$vr(E_T) \leq \kappa \Leftrightarrow (\forall v \in V)(\forall e_1 \in E \setminus E_T) \ldots (\forall e_{\kappa+1} \in E \setminus E_T)
\left( \left( \bigwedge_{i=1}^{\kappa+1} \text{FundCyc}(v,e_i) \right) \rightarrow \bigvee_{1 \leq i < j \leq \kappa+1} e_i = e_j \right)$$

$$er(E_T) \leq \lambda \Leftrightarrow (\forall e \in E)(\forall e_1 \in E \setminus E_T) \ldots (\forall e_{\lambda+1} \in E \setminus E_T)
\left( \left( \bigwedge_{i=1}^{\lambda+1} \text{FundCyc}(e,e_i) \right) \rightarrow \bigvee_{1 \leq i < j \leq \lambda+1} e_i = e_j \right)$$

In the following, assume that $E_T$ is the edge set of the spanning tree of $G$ (as shown above), which additionally has edge orientations, defined in MSOL by predicates head and tail.

$$\text{Bag}_V (v, X) \Leftrightarrow \forall v' \in X \rightarrow (v' = v \lor (\exists e \in E \setminus E_T) (\text{Inc}(v', e) \land \text{FundCyc}(v, e)))$$

$$\text{Bag}_E (e, X) \Leftrightarrow \forall v' \in X \rightarrow (\exists e \in E \setminus E_T) (\text{Inc}(v', e') \lor (\exists e' \in E \setminus E_T) (\text{Inc}(v', e') \land \text{FundCyc}(e, e'))$$

$$\text{Parent}(X_p, X_c) \Leftrightarrow \exists v (\exists e \in E_T) ((\text{Bag}_V (v, X_p) \land \text{Bag}_E (e, X_c) \land \text{head}(v, e)) \lor (\text{Bag}_V (v, X_c) \land \text{Bag}_E (e, X_p) \land \text{tail}(v, e)))$$

A.2 $k$-Outerplanar Graphs

Using the forbidden minors ($K_4$ and $K_{2,3}$), we can define a predicate for verifying whether a graph is outerplanar in a straightforward way.

$$\text{Outerpl}(V', E') \Leftrightarrow \neg (\text{Minor}_{K_4}(V', E') \lor \text{Minor}_{K_{2,3}}(V', E'))$$
Following the argumentation in the proof of Lemma 3.3, we can define our predicate as follows.

\[
\exists V_1 \cdots \exists V_k \left( \text{Part}_V(V, V_1, \ldots, V_k) \land \text{Outerpl}(V_1, \text{IncE}(V_1)) \right.
\]
\[
\land \cdots \land \text{Outerpl}(V_k, \text{IncE}(V_k)) \land \forall v \left( \bigwedge_{i=1, \ldots, k} v \in V_i \rightarrow \forall w \forall e (\text{Edge}(e, v, w) \rightarrow (w \in V_{i-1} \lor w \in V_i \lor w \in V_{i+1})) \right) \)
\]

A.2.1 3-Connected \(k\)-Outerplanar Graphs

We first give the necessary definition of defining the ordering \(\text{nb}<\) as described in Lemma 3.13. The first step is to define face-adjacency of two edges.

\[
\text{Adj}_F(e, f) \iff \exists v (\text{Inc}(v, e) \land \text{Inc}(v, f)) 
\]
\[
\land (\exists E' \subseteq E)(\text{FaceBd}_3(E') \land e \in E' \land f \in E')
\]

Next, we define a set to check whether a set of edges is a face-adjacency path from the one to the other, if they both share a vertex \(v\). Intuitively speaking, this predicate states that each edge in the candidate set \(E'\) has precisely one neighbor in it, if the edge is either \(e\) or \(f\) and precisely two otherwise. Furthermore, \(E'\) has to consist of a subset of the incident edges of \(v\), without \(e'_A\) (see the proof of Lemma 3.13) and it has to contain both \(e\) and \(f\).

\[
\text{Path}_F(E', e, f) \iff (\exists E'' \subseteq (\text{IncE}(v) \setminus e'_A))(E' = E'' \cup \{e, f\}) 
\]
\[
\land e_1 \in E' \leftrightarrow \left( (e_1 = e \lor e_1 = f) \land (\exists e_2 \in E')(\text{Adj}_F(e_1, e_2) 
\]
\[
\land (\forall e_3 \in E')(\neg e_2 = e_3 \rightarrow \neg \text{Adj}_F(e_1, e_3)) \right) 
\]
\[
\lor \left( (e_1 = e \lor e_1 = f) \land (\exists e_2 \in E') \land (\exists e_3 \in E') \left( \text{Adj}_F(e_1, e_2) 
\right. 
\]
\[
\land \text{Adj}_F(e_1, e_3) \land (\forall e_4 \in E')(\neg (e_4 = e_2 \lor e_4 = e_3)) 
\]
\[
\rightarrow \neg \text{Adj}_F(e_1, e_4) \right) \right)
\]

We are now ready to define the predicate for the ordering \(\text{nb}<\).

\[
\text{nb}< (e, f) \iff \exists E_e \exists E_f (\text{Path}_F(E_e, e_A, e) \land \text{Path}_F(E_f, e_A, f) \land E_e \subset E_f)
\]

A.2.2 Tree Decompositions for 3-Connected \(k\)-Outerplanar Graphs

We first show how to define that a spanning tree with edge set \(F\) has bounded face remember number \(\nu\) in a 3-connected planar graph \(G = (V, E)\), which
completes the proof of Proposition 3.16. Intuitively speaking, this predicate checks that for each combination of a vertex and a face boundary $FB$, the number of edges, whose fundamental cycle uses both $v$ and some edge in $FB$, is bounded by $\nu$.

$$fr(V, E, F) \leq \nu \Leftrightarrow \forall v(\forall E_{FB} \subseteq E)(\forall e_1 \in E \setminus F) \cdots (\forall e_{\nu+1} \in E \setminus F)
$$

$$\bigg(\big(\bigwedge_{1 \leq i \leq \nu+1} (\exists E_C \subseteq E)(\text{FundCyc}(e_i, C_e) \wedge \neg (C_E \cap E_{FB} = \emptyset) \wedge \text{Inc}(v, E_C))\big) \rightarrow \bigg(\bigvee_{1 \leq i < j \leq \nu+1} e_i = e_j\bigg)\bigg)$$

Next, we will define the edge sets $C(v, f_i)$, as used in the proof of Lemma 3.15.

$$E' = C(v, E_{FB}, F) \iff e \in E' \iff (\exists E_C \subseteq E)(\text{FundCyc}(e, E_C)$$

$$\wedge \neg (E_C \cap E_{FB} = \emptyset) \wedge \text{Inc}(v, E_C))$$

We furthermore denote by $C(v, e, F)$ the union of the sets $C(v, f_i)$ and $C(v, f_j)$ of the two faces $f_i$ and $f_j$, whose face boundaries contain $e$ (such that $e$ is incident to $v$).

We now define a predicate identifying a unique face boundary with lowest layer number for each vertex.

$$\text{Layer}_i(E_{FB}) \Leftrightarrow \text{FaceBd}_3(E_{FB}) \wedge \exists v(\text{Inc}(v, E_{FB}) \wedge v \in V_i)$$

$$E' = E_{f_i}(v) \Leftrightarrow (\exists e \in E')(\text{Inc}(v, e) \wedge \bigwedge_{i=1, \ldots, k} \left( v \in V_i \rightarrow \left( (\text{Layer}_{i-1}(E'))
$$

$$\wedge \neg ((\exists f \exists E_f)(\text{Layer}_{i-1}(E_f) \wedge f \in E_f \wedge \text{Inc}(v, f) \wedge \text{nb}_< (f, e)))\right) \wedge (\text{Layer}_i(E') \wedge (\exists E_f(\text{Layer}_{i-1}(E_f) \wedge \text{Inc}(v, E_f)))\right)$$

$$\wedge (\neg ((\exists f \exists E_f)(\text{Layer}_i(E_f) \wedge f \in E_f \wedge \text{Inc}(v, f) \wedge \text{nb}_< (f, e))))\right)\bigg)$$

We are now ready to define the Bag-predicates of our tree decomposition. Note that the bag type $\sigma$ can be defined in the same way as for bounded degree $k$-outerplanar graphs, hence we refer to Appendix A.1 for the details. The types $\sigma_H$ can be defined using the predicates given above. We assume that we are given an arbitrary but fixed orientation on the edges as described in the proof of Lemma 3.15.

$$\text{Bag}_{\sigma_H}(e, X) \Leftrightarrow v \in X \leftrightarrow \text{head}(v, e) \lor (\exists e' \in (C(v, e, F) \cup C(v, E_{f_i}(\text{head}(e)), F))$$

$$(\text{Inc}(v,e') \wedge \forall w(\neg (v = w) \wedge \text{Inc}(w, e') \rightarrow \text{col}(v) < \text{col}(w))$$

We can define the bag type $\sigma_T$ by replacing 'head' by 'tail' in the above predicate.
We now define the set of anchor edges $E_A$ and co-anchor edges $E'_A$. For each vertex $v$ we need to find a face with lowest layer number $f_\ell$. Let $e_{\ell_1}$ and $e_{\ell_2}$ denote the incident edges of $v$ bounding $f_\ell$. Then, $e_{\ell_1}$ has to be contained in $E_A$ and $e_{\ell_2}$ in $E'_A$. Note that this choice is arbitrary and that we have to choose precisely one such face for each vertex in the graph.

$$E' = E_A \iff \forall v \exists e (e \in E' \land \text{Inc}(v, e) \land e \in E_{f_\ell}(v)$$

$$\land \forall e'((\text{Inc}(v, e') \land \neg e = e') \rightarrow \neg(e' \in E'))$$

$$E' = E'_A \iff (\forall e \in E_A) \forall v \exists e' (e' \in E' \land \text{Inc}(v, e) \land \text{Inc}(v, e') \land e \in E_{f_\ell}(v) \land e' \in E_{f_\ell}(v)$$

$$\land \forall e''((\text{Inc}(v, e'') \land \neg e'' = e') \rightarrow \neg(e'' \in E'))$$

We now turn to defining the Parent-predicate and begin by defining the case when a bag of type $\sigma$ is a bag of type $\sigma_T$.

$$\text{Parent}_{\sigma \sigma_T}(X, Y) \iff (\exists e \in F) (\text{Bag}_\sigma(e, X) \land \text{Bag}_{\sigma_T}(e, Y))$$

$$\lor (\exists e \in F) ((\exists e_{\ell} \in E_{f_\ell}(\text{tail}(e)) \cap \text{Inc}(\text{tail}(e)))(\text{Adj}_F(e, e_{\ell})$$

$$\land \text{Bag}_\sigma(e_{\ell}, X) \land \text{Bag}_{\sigma_T}(e, Y))$$

Similarly, we can define the case when a bag of type $\sigma_H$ is the parent of a bag of type $\sigma$.

$$\text{Parent}_{\sigma_H \sigma}(X, Y) \iff (\exists e \in F) (\text{Bag}_{\sigma_H}(e, X) \land \text{Bag}_\sigma(e, Y))$$

$$\lor (\exists e \in F) ((\exists e_{\ell} \in E_{f_\ell}(\text{head}(e)) \cap \text{Inc}(\text{head}(e)))(\text{Adj}_F(e, e_{\ell})$$

$$\land \text{Bag}_{\sigma_H}(e, X) \land \text{Bag}_{\sigma}(e_{\ell}, Y))$$

We now consider edges between bags of type $\sigma_H/\sigma_T$. In the following, we define the case when all bags involved are $\sigma_T$-bags and note that the other cases can be defined by the obvious replacements. We first define the outgoing edges of the $\sigma_T$-bag corresponding to the unique incoming edge in the directed spanning tree $T = (V, F)$.

$$\text{Parent}^f_{\sigma_T \sigma_T}(X, Y) \iff (\exists e^* \in F) (\exists e \in E) (\text{Bag}_{\sigma_T}(e^*, X) \land \text{Bag}_{\sigma_T}(e, Y)$$

$$\land \text{tail}(e^*) = \text{tail}(e) \land (\text{nb}_<(e, e^*) \lor \text{nb}_<(e^*, e)))$$

We now define the rest of the edges. We denote by $e^*(v)$ the edge which satisfies $e^* \in F \land \text{tail}(e^*) = v$.

$$\text{Parent}^R_{\sigma_T \sigma_T}(X, Y) \iff \exists e \exists f (\text{tail}(e) = \text{tail}(f) \land \text{nb}_<(e, f)$$

$$\land ((\text{Bag}_{\sigma_T}(e, X) \land \text{Bag}_{\sigma_T}(f, Y) \land \text{nb}_<(e^*(\text{tail}(e)), e))$$

$$\lor (\text{Bag}_{\sigma_T}(f, X) \land \text{Bag}_{\sigma_T}(e, Y) \land \text{nb}_<(f, e^*(\text{tail}(f))))))$$

Unifying all above defined predicates (plus the omitted similar cases) yields the Parent$(X, Y)$-predicate for our tree decomposition.
A.3 Hierarchical Graph Decompositions for $k$-Outerplanar Graphs

In this section we provide details for the predicates used in proofs of Section 3.3. First we show how to define the parent-relation between blocks in our hierarchical decomposition as explained in the proof of Lemma 3.31. We assume that we are given a graph $G = (V,E)$ with a spanning tree $S = (V,F)$, which is rooted at an (arbitrary) vertex $r \in V$.

Let $\text{Block}(X)$ denote a predicate which is true if and only if a set $X \subseteq V$ is a block in the hierarchical decomposition of $G$. $\text{Block}(X)$ is definable by (cf. also Proposition 3.33). This predicate both encodes the cases of the edges between 2-cuts and 3-blocks (see Proposition 3.32) and of edges between 1-cuts and 2-blocks.

$$\text{Parent}_{\text{Block}}(X,Y) \Leftrightarrow (\text{Block}(X) \land \text{Block}(Y) \land (X \cap Y = X \lor X \cap Y = Y))$$

$$\land \left( (X \subset Y) \rightarrow (\forall v \in Y)(\exists x \in X) \land E_{P_x} \subset E_{P_v} \right)$$

$$\land \left( (Y \subset X) \rightarrow (\exists v \in Y)(\exists x \in X) \land E_{P_x} \subset E_{P_v} \right)$$

A.3.1 Defining a Cycle Block

We now show how to define the predicates for tree decompositions of a cycle block $C = (W,E_C)$ as used in the proof of Proposition 3.34. First, we find the root $r_C \in W$ of the cycle.

$$v = r_C \Leftrightarrow (r \in W \land v = r) \lor \left( (\exists C \subset V)(\text{Parent}_{\text{Block}}(C_P, C) \land \exists v ((\text{Bag}_{\text{Cyc}}(v,C_P) \lor \text{Bag}_{\text{Cyc}}(v,C_P)) \land v = r_C)) \right)$$

Now we can define the predicate $\text{Bag}_{\text{Cyc}}$ straightforwardly.

$$\text{Bag}_{\text{Cyc}}(e,X) \Leftrightarrow \neg \text{Inc}(e,r_C) \land (v \in X \leftrightarrow (\text{Inc}(v,e) \lor v = r_C))$$

Furthermore we can define the predicate $\text{Parent}_{\text{Cyc}}(X,Y)$ as described in the proof of Proposition 3.34.

$$\text{Parent}_{\text{Cyc}}(X,Y) \Leftrightarrow \exists e \exists f \left( \text{Bag}_{\text{Cyc}}(e,X) \land \text{Bag}_{\text{Cyc}}(f,Y) \land |X \cap Y| = 2 \right.$$

$$\land (\exists Z \subseteq V) \left( \text{Clock}_{\text{Cyc}}(Z) \land \text{Inc}(e,Z) \land \text{Inc}(f,Z) \right.$$

$$\land (\exists E \subseteq \text{Inc}(Z)) (\exists P_e \subseteq \text{Inc}(Z)) (\exists P_f \subseteq \text{Inc}(Z))$$

$$\land (\text{Path}_\rightarrow (r_C, \text{tail}(e), P_e) \land \text{Path}_\rightarrow (r_C, f, P_f) \land P_e \subset P_f)) \right)$$

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A.3.2 Defining the Parent-predicate for \((T, X)\)

We now complete the proof of Lemma 3.31 by defining the parent-relation in all bags of the resulting tree decomposition \((T, X)\) of the graph \(G\). During this step we also modify some of the \textit{Bag}-predicates, since, as explained in the proof, a number of vertices might be added to each bag in the tree decomposition. A vertex \(v\) is added to a bag \(X\), when it is a member of a tree decomposition of a 2-connected 2-block or a 3-block and \(v\) is contained in the parent cut bag of \(X\) in the hierarchical decomposition of \(G\). We show how to define such a predicate for an arbitrary case.

\[
\text{Bag}'_c(X) \iff (\exists X' \subseteq X) \left( \text{Bag}_c(X') \land v \in X \setminus X' \iff \exists Y \exists Z \left( X' \subseteq Z \land (2\text{-Conn}_B(Z) \lor 3\text{-Conn}_B(Z) \lor \text{Cycle}_B(Z)) \land \text{Parent}_B(Y, Z) \land v \in Y \right) \right)
\]

In the following, we indicate that we refer to these modified bags by using the notation ‘\textit{Bag}' ...’ instead of ‘\textit{Bag} ...’. We define two cases: One, in which a \(C_1\)- or \(C_2\)-block is a parent of a \(B_3\)-block and vice versa. The cases for \(C_1\)- and \(B_2\)-blocks can be defined by the obvious replacements. Note that the predicate \textit{Root}_B can be defined straightforwardly using the \textit{Bag}_B(X)- and \textit{Parent}_B(X,Y)-predicates.

\[
\text{Parent}_{CB_3}(X, Y) \iff (\text{Bag}'_{C_1}(X) \lor \text{Bag}'_{C_2}(X)) \land \text{Bag}_B(Y) \land X \subseteq Y \land \text{Root}_B(Y)
\]

\[
\text{Parent}_{BC_3}(X, Y) \iff \text{Bag}_B(X) \land (\text{Bag}'_{C_2}(Y) \lor \text{Bag}'_{C_1}(Y)) \land X \subseteq Y
\]

\[
\land \exists Z(X \subseteq Z \land \text{Parent}_B(X \setminus Z, Z))
\]

\[
\land \neg (\exists X'(\text{Parent}_B(X', X) \land X' \subseteq Y))
\]

The \textit{Parent}_{SC}(X,Y)-predicate can now be defined as a unification of all these cases.

A.3.3 Defining Tree Decompositions for 3-Connected 3-Blocks

We now show how to define the predicates for defining the sets \(S_E\) and \(R_V\) as outlined in the proof of Lemma 3.41. To shorten our notation, we will
use the symbol $E[B_3, S_E]$ instead of the term \( \text{IncE}(B_3) \cap S_E \).

\([i]\) \((\exists \mathcal{R}_V \subseteq V)(\exists F \subseteq E)(\exists F' \subseteq E)(\exists S_E \subseteq E)(S_E = F \cup F') \ldots \)

\([ii]\) \((\exists r_T \in V)(\text{Tree}_\rightarrow (r_T, F)) \ldots \)

\([iii]\) \(e \in F' \rightarrow \exists x \exists y (\neg x = y \land \text{Inc}(x, e) \land \text{Inc}(y, e) \land \neg e \in F \)
\(\land \exists X (\text{Bag}_{C_2}(X) \land x \in X \land y \in X)) \ldots \)

\([iv]\) \((\forall B_3 \subseteq V) \left(3:\text{Conn}_{B_3}(B_3) \rightarrow \left(\text{er}(B_3, \text{IncE}(B_3), E[B_3, S_E]) \leq 2k \right. \right. \land \left. \left. fr(B_3, \text{IncE}(B_3), E[B_3, S_E]) \leq k \right) \right) \)

\([v]\) \(v \in \mathcal{R}_V \rightarrow \exists X (\text{Bag}_{C_2}(X) \land v \in X) \)

\([vi]\) \((\forall B_3 \subseteq V) \left(3:\text{Conn}_{B_3}(B_3) \rightarrow (\exists r_{B_3} \in \mathcal{R}_V) \left(\text{Tree}_\rightarrow (r_{B_3}, B_3, E[B_3, S_E]) \right) \right) \)