Carleman estimate and application to an inverse source problem for a viscoelasticity model in anisotropic case

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Abstract
We consider an anisotropic hyperbolic equation with memory term:

\[
\partial_t^2 u(x,t) = \sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j u) + \int_0^t \sum_{|\alpha| \leq 2} b_\alpha(x,t,\eta) \partial_\alpha^\eta u(x,\eta)d\eta + R(x,t)f(x)
\]

for \(x \in \Omega\) and \(t \in (0,T)\), which is a simplified model equation for viscoelasticity. The main result is a both-sided Lipschitz stability estimate for an inverse source problem of determining a spatial varying factor \(f(x)\) of the force term \(R(x,t)f(x)\). The proof is based on a Carleman estimate and due to the anisotropy, the existing transformation technique does not work and we introduce a new transformation of \(u\) in order to treat the integral terms.

Keywords: inverse source problem, viscoelasticity, anisotropic media, Carleman estimate

1. Introduction and main results

Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary \(\partial\Omega\) and the unit outward normal vector \(\nu = (\nu_1, \ldots, \nu_n)\) to \(\partial\Omega\). For \(T > 0\), we consider the following class of integro-differential hyperbolic equations,
\[
\partial_t^2 u(x,t) = \sum_{i,j=1}^n \partial_i a_{ij}(x) \partial_j u + \int_0^T \sum_{|\alpha| \leq 2} b_{\alpha}(x,t,n) \partial_n^{|\alpha|} u(x,n) \, dn + R(x,t) f(x),
\]
\[
x \in \Omega, \ 0 < t < T,
\]
\[
u \big|_{\partial \Omega} = 0, \quad 0 < t < T,
\]
\[
u(x,0) = \partial_n u(x,0) = 0, \quad x \in \Omega.
\]
Here and henceforth let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \) be a multi-index and we set \(|\alpha| = \alpha_1 + \cdots + \alpha_n\). \( \partial_i = \frac{\partial}{\partial x_i}, 1 \leq i \leq n \), \( \partial_\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \).

Regarding the coefficients \( a_{ij}(x) \) and \( b_{\alpha}(x,t,\eta) \), throughout this paper, we assume
\[
\left\{ \begin{aligned}
a_{ij} & \in C^2(\overline{\Omega}), \quad 1 \leq i,j \leq n, \\
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j & \geq \mu_0 \sum_{i=1}^n \xi_i^2, \quad x \in \overline{\Omega}, \quad \xi_1, \ldots, \xi_n \in \mathbb{R}
\end{aligned} \right. 
\]
and
\[
\left\{ \begin{aligned}
b_{\alpha} & \in C^2([0,T]^2; C(\overline{\Omega})), \\
\partial_t^{|\alpha|} b_{\alpha} & \in C(\overline{\Omega} \times [0,T]^2), \quad |\alpha| \leq 2.
\end{aligned} \right. 
\]
There is a possibility of reducing the regularity assumption on \( b_{\alpha} \), but we do not argue details.

First of all let us explain the motivation to study the above partial differential equation with memory. As well-known for some viscoelastic materials, the effects of memory cannot be neglected without failing the analysis, as observed by Volterra [38]. Indeed, he embraced the Boltzmann model, according to which the stress has to depend linearly on strain history. Those argumentations lead to the so-called integro-differential equations, that have been largely studied in the framework of the viscoelasticity. For example, we cite the book by Renardy et al [35] and the references therein.

In particular, in his pioneering work [12], Dafermos studied an abstract Volterra equation in Hilbert spaces, giving as an application the case of anisotropic viscoelastic equations, where the solution \( u \) is a vector and the matrix of the coefficients becomes a fourth-rank tensor. Also in [1] the authors considered an abstract version of the equation of motion for non-stationary linear viscoelastic solids with the same operator \( \Lambda \) in the leading part and under integral. Our integro-differential equation (1.1) can be regarded as an example of the abstract Volterra equation studied by Dafermos, and moreover a generalisation of the case contemplated in [1].

It is noteworthy to mention that several papers treated the isotropic \( n \)-dimensional viscoelastic equation, where the operator is the Laplacian and the solutions are scalar functions, see e.g. [8, 10, 34]. In this framework our equation can represent a generalisation to anisotropic materials, that is, (1.1) can serve as a model for describing the viscoelastic properties of those materials whose properties are different along different directions.

In this paper, we consider

**Inverse source problem.**

Let \( T > 0, \ R = R(x,t) \) and a subboundary \( \Gamma \subset \partial \Omega \) be suitably given. Then determine \( f(x), \ x \in \Omega \) by \( \partial_n u \) on \( \Gamma \times (0,T) \), where
\[
\partial_n u := \sum_{i=1}^n a_{ij}(\partial_j u) \nu_i \quad \text{on} \ \partial \Omega.
\]
Inverse problems of determining coefficients $a_{ij}$ and/or $b_{\alpha}$ are not only theoretically challenging, but also important from the practical point of view. This inverse source problem is the essential first step towards the coefficient inverse problems because the inverse source problem can be regarded as a linearization of the coefficient inverse problem. Moreover we emphasize that the inverse source problem is practically significant because it is concerned with the identification of spatial components of the external source causing the current action. The form $R(x,t)f(x)$ seems special but in applications we often model the external force in a more special form $F(x,t) = \lambda(t)f(x)$ where $\lambda(t)$ is the time changing ratio and $f$ is the spatial distribution of the external force.

For example, in the case of impulsive forces, we can take $f(x) = \sum_{j=1}^{N} \delta(x - a_j)$, where $a_j \in \Omega$ is the center of the impulsive force and $\delta$ is Dirac’s delta function. In this paper, we assume that $f$ has more regularity.

For the statement of the Carleman estimate, we need to introduce notations. Henceforth $(x \cdot x')$ denotes the scalar product of $x, x' \in \mathbb{R}^n$. We set

$$a(x, \zeta) = \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j, \quad x \in \Omega, \quad \zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n.$$  

Given functions $p(x, \zeta)$ and $q(x, \zeta)$, we define the Poisson bracket by

$$\{ p, q \}(x, \zeta) = \sum_{j=1}^{n} \left( \frac{\partial p}{\partial \zeta_j} \frac{\partial q}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial q}{\partial \zeta_j} \right)(x, \zeta).$$  

We set

$$d(x) = |x - x_0|^2, \quad x \in \mathbb{R}^n$$  

with fixed $x_0 \in \mathbb{R}^n \setminus \Omega$. In addition to (1.4), throughout this paper, we assume that there exists a constant $\mu_1 > 0$ such that

$$\{ a, \{ a, d \} \}(x, \zeta) \geq \mu_1 |\zeta|^2, \quad x \in \Omega, \quad \zeta \in \mathbb{R}^n \quad (1.6)$$

(e.g. Bellassoued and Yamamoto [6, 7]). For proving a Carleman estimate, it is known that we need some condition like (1.6), which is called the pseudo-convexity (e.g. Hörmander [14]).

We assume that the subboundary $\Gamma \subset \partial \Omega$ satisfies

$$\Gamma \supset \{ x \in \partial \Omega; (x - x_0) \cdot \nu(x) \geq 0 \}. \quad (1.7)$$

We assume that

$$R \in H^3(0, T; W^{2, \infty}(\Omega)), \quad |R(x, 0)| \neq 0, \quad x \in \Omega. \quad (1.8)$$

Our main result is the stability for the inverse source problem.

**Theorem.** We assume (1.6)–(1.8). Then we can choose a constant $\beta > 0$ small for the constant $\mu_1 > 0$ given in (1.6) and have the following estimate: If

$$T > \frac{\max_{x \in \partial \Omega} |x - x_0|}{\sqrt{\beta}}, \quad (1.9)$$

then there exists a constant $C > 0$ such that

$$C^{-1}||\partial_\nu u||_{H^{0}(0,T;L^2(\partial\Omega))} \leq ||f||_{H^{0}(\Omega)} \leq C||\partial_\nu u||_{H^{0}(0,T;L^2(\Gamma))} \quad (1.10)$$
for each \( f \in H^2_0(\Omega) \) and each solution \( u \) satisfying (1.1)–(1.3) and
\[
F(x, t) := R(x, t) f(x).
\]

The following lemma is proved and the proof is a modification of Imanuvilov and Yamamoto [24].

**Lemma 1.** There exists a constant \( C > 0 \) such that
\[
\sum_{k=0}^{2} \| \partial_t^k u(\cdot, t) \|_{H^5(\Omega)} \leq C \| f \|_{H^2(\Omega)}, \quad 0 \leq t \leq T
\]
for \( u \) satisfying (1.1)–(1.3) and (1.11) with \( f \in H^2_0(\Omega) \).
Proof of lemma 1.

First Step.

We set

\[ E(t) = \int_\Omega \left( |\partial_t u(x,t)|^2 + \sum_{i,j=1}^n a_{ij}(x) \partial_i u(x,t) \partial_j u(x,t) \right) \, dx, \quad 0 \leq t \leq T. \] (2.1)

Then, using \( a_{ij} = a_{ji} \) and integrating by parts, by (1.1) we have

\[ \frac{dE}{dt}(t) = 2 \int_\Omega \sum_{i,j=1}^n a_{ij}(\partial_i \partial_j u) \partial_i u \, dx + 2 \int_\Omega (\partial_t u) \partial_t^2 u \, dx \]

\[ = -2 \int_\Omega \sum_{i,j=1}^n (\partial_t u) \partial_i (a_{ij} \partial_j u) \, dx + 2 \int_\Omega (\partial_t u) \partial_t^2 u \, dx = \int_\Omega \partial_t u(x,t) \partial_t^2 u + Au(x,t) \, dx. \]

Therefore (1.1) yields

\[ \frac{dE}{dt}(t) = 2 \int_\Omega (\partial_t u)(x,t) F(x,t) \, dx \]

\[ + 2 \int_\Omega (\partial_t u)(x,t) \left( \sum_{|\alpha| \leq 2} \int_0^t b_{\alpha}(x,t,\eta) \partial_\eta^\alpha u(x,\eta) \, d\eta \right) \, dx. \]

Writing \( \sum_{|\alpha| \leq 2} b_{\alpha} \partial_\eta^\alpha u = \sum_{j=1}^n \partial_t (\tilde{b}_j \partial_j u) + \sum_{j=1}^n \tilde{b}_j \partial_j u + \tilde{b}_0 u \) and integrating over \((0,t)\), we have

\[ E(t) = E(0) + 2 \int_0^t \int_\Omega \partial_\xi u(x,\xi) F(x,\xi) \, d\xi \]

\[ + 2 \int_0^t \int_\Omega \partial_\xi u(x,\xi) \left( \sum_{j=1}^n \int_0^\xi \partial_t (\tilde{b}_j(x,\xi,\eta) \partial_j u(x,\eta) \, d\eta \right) \]

\[ + \sum_{j=1}^n \int_0^\xi \left( \tilde{b}_j(x,\xi,\eta) \partial_j u(x,\eta) + \tilde{b}_0(x,\xi,\eta) u(x,\eta) \right) \, d\xi d\eta. \] (2.2)

By the Cauchy–Schwarz inequality and the Poincaré inequality by \( u|_{\partial\Omega} = 0 \), we obtain

\[ \left| 2 \int_0^t \int_\Omega \partial_\xi u(x,\xi) F(x,\xi) \, d\xi d\eta \right| \]

\[ + 2 \int_0^t \int_\Omega \left| \partial_\xi u(x,\xi) \sum_{j=1}^n \int_0^\xi (\tilde{b}_j(x,\xi,\eta) \partial_j u(x,\eta) + \tilde{b}_0(x,\xi,\eta) u(x,\eta)) \, d\eta \right| \, d\xi d\eta \]

\[ \leq C ||F||_{L^2(0,T;L^2(\Omega))}^2 + C \int_0^t \int_\Omega |\nabla_x u(x,\xi)|^2 \, d\xi d\eta. \] (2.3)

Next, integrating by parts in \( x \) and \( t \) and using \( u|_{\partial\Omega} = 0 \), we have
2 \int_0^t \left[ \frac{\partial \xi u(x, \xi)}{\partial \xi} \left( \sum_{j=1}^n \int_0^\xi \frac{\partial_j (\tilde{b}_j (x, \xi, \eta) \partial_j u(x, \eta))}{\partial \xi} \, d\eta \right) \right] \, dx \, d\xi \\
= -2 \sum_{i,j=1}^n \int_0^t \left[ \int_0^\xi \frac{\partial_i \partial_j u(x, \xi)}{\partial \xi} \left( \int_0^\xi \tilde{b}_j (x, \xi, \eta) \partial_j u(x, \eta) \, d\eta \right) \, d\xi \right] \, dx \\
= -2 \sum_{i,j=1}^n \int_0^\xi \left[ \frac{\partial_i u(x, \xi)}{\partial \xi} \int_0^\xi \tilde{b}_j (x, \xi, \eta) \partial_j u(x, \eta) \, d\eta \right] \, dx \\
+ 2 \sum_{i,j=1}^n \int_0^\xi \left[ \int_0^\xi \partial_i u(x, \xi) \frac{\partial_j (\tilde{b}_j (x, \xi, \eta) \partial_j u(x, \eta))}{\partial \xi} \, d\xi \right] \, dx.

Given \( \varepsilon > 0 \) arbitrarily, we see that

\[
\left| \sum_{i,j=1}^n \int_0^\xi \frac{\partial_i u(x, t)}{\partial \xi} \left( \int_0^t \tilde{b}_j (x, t, \eta) \partial_j u(x, \eta) \, d\eta \right) \, dx \right| \\
\leq \varepsilon \int_\Omega |\partial_i \partial_j u(x, t)|^2 \, dx + \frac{1}{\varepsilon} \sum_{i,j=1}^n \int_\Omega \left| \int_0^t \tilde{b}_j (x, t, \eta) \partial_j u(x, \eta) \, d\eta \right|^2 \, dx,
\]

so that we can easily obtain

\[
\left| \int_0^t \int_\Omega \left[ \sum_{i,j=1}^n \int_0^\xi \frac{\partial_i \partial_j (\tilde{b}_j (x, \xi, \eta) \partial_j u(x, \eta))}{\partial \xi} \, d\xi \right] \, dx \, d\xi \right| \\
\leq \varepsilon \int_\Omega |\nabla u(x, t)|^2 \, dx + \frac{C}{\varepsilon} \int_0^T \int_\Omega |\nabla u(x, \eta)|^2 \, dx \, d\eta.
\]

(2.4)

We note \( C^{-1} E(t) \leq \int_\Omega |\nabla_i \partial_j u(x, t)|^2 \, dx \leq CE(t) \) for \( 0 \leq t \leq T \), where \( C > 0 \) is independent of \( t \) and choices of \( u \). Hence from (2.2)–(2.4), we derive

\[
E(t) \leq CE(0) + \|F\|_{L^2(0,T;L^2(\Omega))}^2 + CE(t) + \frac{C}{\varepsilon} \int_0^T E(\eta) \, d\eta
\]

for \( 0 \leq t \leq T \). Choosing \( \varepsilon > 0 \) sufficiently small, we apply the Gronwall inequality to reach

\[
E(t) \leq C(CE(0) + \|F\|_{L^2(0,T;L^2(\Omega))}^2), \quad 0 \leq t \leq T.
\]

(2.5)

**Second Step.**

We set

\[
u_1 = \partial_t u, \quad u_2 = \partial_t^2 u, \quad u_3 = \partial_t^3 u
\]

and

\[
\begin{cases}
b^{(0)}_t = b, \\
b^{(k+1)}_t (x, t, \eta) = b^{(k)}_t (x, t, \eta) + \int_0^\xi \partial \partial_j b^{(k)}_t (x, t, \xi) \, d\xi, \quad k = 0, 1, 2.
\end{cases}
\]
Noting that \( u(x, t) = \int_0^t u_1(x, \xi) d\xi \), etc, by (1.3), we can directly verify
\[
\begin{align*}
\begin{cases}
\partial_t^2 u_1(x, t) = Au_1 + \int_0^t \sum_{|\alpha| \leq 2} b_0^{(1)}(x, t, \eta) \partial_\alpha^x u_1(x, \eta) d\eta + (\partial_t R) f, \\
u_1(0, x) = 0, & \partial_t u_1(0, x) = R(x, 0) f(x), & x \in \Omega, \\
u_{1|\partial\Omega} = 0,
\end{cases}
\end{align*}
\tag{2.6}
\]
\[
\begin{align*}
\begin{cases}
\partial_t^2 u_2(x, t) = Au_2 + \int_0^t \sum_{|\alpha| \leq 2} b_0^{(2)}(x, t, \eta) \partial_\alpha^x u_2(x, \eta) d\eta + (\partial_t^2 R) f, \\
u_2(0, x) = R(x, 0) f(x), & \partial_t u_2(0, x) = \partial_t R(x, 0) f(x), & x \in \Omega, \\
u_{2|\partial\Omega} = 0
\end{cases}
\end{align*}
\tag{2.7}
\]
and
\[
\begin{align*}
\begin{cases}
\partial_t^2 u_3(x, t) = Au_3 + \int_0^t \sum_{|\alpha| \leq 2} b_0^{(3)}(x, t, \eta) \partial_\alpha^x u_3(x, \eta) d\eta + (\partial_t^3 R) f \\
+ \sum_{|\alpha| \leq 2} b_0^{(4)}(x, t, \eta) \partial_\alpha^x (R(x, 0) f), \\
u_3(0, x) = (\partial_t R(x, 0)) f, & \partial_t u_3(0, x) = A(R(x, 0) f) + (\partial_t^3 R)(x, 0) f, & x \in \Omega, \\
u_{3|\partial\Omega} = 0
\end{cases}
\end{align*}
\tag{2.8}
\]

By (1.4), we note that \( b_0^{(k)} \in L^\infty(\Omega \times (0, T)^2) \) for \( k = 0, 1, 2, 3 \).

Applying (2.5) to (2.6), we have
\[
\sum_{k=0}^1 \| \partial_t^{k+1} u(\cdot, t) \|_{L^2(\Omega)} + \| \partial_t^k u(\cdot, t) \|_{H^1(\Omega)} \leq C \| f \|_{H^1(\Omega)}, \quad 0 \leq t \leq T.
\]

For \( u_2 \) and \( u_3 \), we argue in the same way to obtain
\[
\sum_{k=0}^3 \| \partial_t^{k+1} u(\cdot, t) \|_{L^2(\Omega)} + \| \partial_t^k u(\cdot, t) \|_{H^1(\Omega)} \leq C \| f \|_{H^1(\Omega)}, \quad 0 \leq t \leq T. \tag{2.9}
\]

Next we have to estimate \( \| \partial_t^k u(\cdot, t) \|_{H^1(\Omega)}, \quad k = 0, 1, 2 \). Since \( \partial_t u(\cdot, t) = \int_0^t \partial_t^2 u(\cdot, \eta) d\eta \) and \( u(\cdot, t) = \int_0^t (t-\eta) \partial_t^2 u(\cdot, \eta) d\eta \) by (1.3), it suffices to estimate \( \| \partial_t^2 u(\cdot, t) \|_{H^1(\Omega)} \). By (2.9) we have
\[
\| \partial_t^2 u(\cdot, t) \|_{L^2(\Omega)} \leq C \| f \|_{H^1(\Omega)}, \quad 0 \leq t \leq T. \tag{2.10}
\]

Therefore (2.7) implies
\[
A(\partial_t^2 u) = \partial_t^3 u(x, t) - \int_0^t \sum_{|\alpha| \leq 2} b_0^{(2)}(x, t, \eta) \partial_\alpha^x \partial_t u(x, \eta) d\eta + (\partial_t^3 R) f.
\]

Since \( \partial_t^2 u(\cdot, t)|_{\partial\Omega} = 0 \), we apply the \textit{a priori} estimate (e.g. Gilbarg and Trudinger [15]) for the elliptic boundary value problem, by (2.10) we obtain
\[
\| \partial_t^2 u(\cdot, t) \|_{H^1(\Omega)} \leq C \| f \|_{H^1(\Omega)} + C \int_0^t \| \partial_t^2 u(\cdot, \eta) \|_{H^1(\Omega)} d\eta, \quad 0 \leq t \leq T.
\]
The Gronwall inequality yields
\[ \| \partial^2_t u(\cdot, t) \|_{H^2(\Omega)} \leq C \| f \|_{H^2(\Omega)}, \quad 0 \leq t \leq T. \]

Thus the proof of lemma 1 is complete.

\[ \square \]

3. Proof of theorem

For concise description, we set
\[ Au := \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j u), \quad B(t, \eta)u := \sum_{|\alpha| \leq 2} b_\alpha(x, t, \eta) \partial_\alpha u(x, \eta) \]
and
\[ F(x, t) := R(x, t) f(x), \quad Q := \Omega \times (-T, T). \]

The proof is composed of the two main steps:

(i) a Carleman estimate

(ii) a combination of Carleman estimate with energy estimates

The argument in Step (ii) was created by Bukhgeim and Klibanov [9], Klibanov [27]. Here we apply a modified argument by Imanuvilov and Yamamoto [19].

Now we explain Step (i). For the diagonal case
\[ \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j u) = a\Delta \]
and
\[ \sum_{|\alpha| \leq 2} b_\alpha(x, t, \eta) \partial_\alpha u = b\Delta, \]
the simple transformation
\[ v(x, t) = a(x)u(x, t) + \int_0^t b(x, t, \eta)u(x, \eta)d\eta \quad (3.1) \]
reduces the hyperbolic equation in the form with the integral terms to a simple hyperbolic equation, so that we can easily obtain a Carleman estimate (e.g. Cavaterra et al [11]). However, in our case (1.1), in general, due to the non-commutativity of the terms
\[ \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j) \text{ and } \sum_{|\alpha| \leq 2} b_\alpha(x, t, \eta) \partial_\alpha \]
we cannot have a similar transformation to (3.1).

Thus in our case, the derivation of a Carleman estimate for (1.1) requires more works and we have to introduce other transformation involving the second-order operators \( A \) and \( B(t, \eta) \).

This is done at the first step. The total proof is divided into five steps.

First Step. Transformation of the equation.

Usually for Carleman estimate, the function \( u \) under consideration has to vanish at \( t = T \). Therefore we first need to define a cut-off function as follows. Let \( \chi \in C^\infty(\mathbb{R}) \) satisfy
\[ 0 \leq \chi \leq 1 \]
and
\[ \chi(t) = \begin{cases} 1, & |t| \leq T - 2\varepsilon, \\ 0, & |t| \geq T - \varepsilon, \end{cases} \quad (3.2) \]
where \( \varepsilon > 0 \) is a sufficiently small constant and is chosen later. We note that \( \chi u \) and its derivatives vanish near \( t = T \).

For overcoming the difficulty caused by the anisotropy, we define the key transformation
\[ v(x, t) = \chi(t)Au(x, t) + \chi(t) \int_0^t B(t, \eta)u(x, \eta)d\eta, \quad (x, t) \in \Omega \times (0, T). \quad (3.3) \]
Then from (1.1) we readily see that
\[ v(x, t) = \chi(t)\partial_t^2 u(x, t) - \chi(t)F(x, t), \quad (x, t) \in \Omega \times (0, T). \tag{3.4} \]

Next we derive a partial differential equation with respect to \( v \). We set
\[ \tilde{B}(t) := B(t, t) \quad \text{in} \ \Omega \times (0, T). \]

Then from (3.3) we have
\[ \partial_t v(x, t) = \chi A \partial_t u + \chi \tilde{B}(t) u \]
\[ + \chi \int_0^t \partial_t B(t, \eta) u d\eta + \chi' A u + \chi'' \int_0^t B(t, \eta) u d\eta, \tag{3.5} \]
and so
\[ \partial_t^2 v(x, t) = \chi \left( A \partial_t^2 u + \tilde{B} \partial_t u + (\partial_t \tilde{B}) u + (\partial^2 B)(t, t) u + \int_0^t \partial^2_t B(t, \eta) u(x, \eta) d\eta \right) \]
\[ + 2\chi'(t) \left( A \partial_t u + \tilde{B} u + \int_0^t (\partial_t B(t, \eta)) u d\eta \right) + \chi''(t) \left( A u + \int_0^t B(t, \eta) u d\eta \right). \tag{3.6} \]

Here we set
\[ S_1(x, t) = 2\chi'(t) \left( A \partial_t u + \tilde{B} u + \int_0^t (\partial_t B(t, \eta)) u d\eta \right) + \chi''(t) \left( A u + \int_0^t B(t, \eta) u d\eta \right). \tag{3.7} \]

Next, since
\[ A v(x, t) = \chi(t)A \left( Au(x, t) + \int_0^t B(x, t, \eta) u(x, \eta) d\eta \right), \]
we obtain
\[ \partial_t^2 v - A v \]
\[ = \chi A \left( \partial_t^2 u - A u(x, t) - \int_0^t B(x, t, \eta) u(x, \eta) d\eta \right) \]
\[ + \chi \left( \tilde{B} \partial_t u + (\partial_t \tilde{B}) u + (\partial_t B)(t, t) u + \int_0^t (\partial_t^2 B(t, \eta)) u d\eta \right) \]
\[ + S_1(x, t) \]
\[ = \chi A F + S_1 + S_2 \quad \text{in} \ \Omega \times (0, T). \]

Here we set
\[ S_2(x, t) = \chi \left( \tilde{B} \partial_t u + (\partial_t \tilde{B}) u + (\partial_t B)(t, t) u + \int_0^t (\partial_t^2 B(t, \eta)) u d\eta \right). \tag{3.8} \]

Moreover, since \( u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \) in \( \Omega \), it follows from (3.3) and (3.5) that \( v(\cdot, 0) = \partial_t v(\cdot, 0) = 0 \) in \( \Omega \).
By $F|_{\partial \Omega} = 0$ and (3.4), we see that $v|_{\partial \Omega} = 0$. Thus
\[
\begin{cases}
\partial^2_t v - Av = \chi AF + S_1 + S_2, & \text{in } \Omega \times (0, T), \\
v(v, 0) = \partial_t v(v, 0) = 0, & \text{in } \Omega, \\
\partial_t^j v(., T) = 0 & \text{in } \Omega, \quad j = 0, 1, \\
v|_{\partial \Omega} = 0.
\end{cases}
\]

For a Carleman estimate, it is more convenient to discuss over the time interval $(-T, T)$, not $(0, T)$. Thus we make the even extension of $v$ to $(-T, 0)$:
\[
v(x, t) = \begin{cases}
v(x, t), & 0 < t < T, \\
v(x, -t), & -T < t < 0.
\end{cases}
\]

Accordingly we make the even extensions of $\chi AF + S_1 + S_2$. Then, since $v(x, 0) = \partial_t v(x, 0) = 0$ for $x \in \Omega$, by (3.4) and (1.11), we can prove that
\[
v \in C^\ell([-T, T]; L^2(\Omega)) \cap C^\ell([-T, T]; H^1_0(\Omega)) \cap C([-T, T]; H^2(\Omega)) \cap H^\ell(-T, T; L^2(\Omega)) \cap H^\ell(-T, T; L^2(\Omega)).
\]

Moreover, since we can see by (1.11) that $S_1 + S_2 \in H^1(0, T; L^2(\Omega))$, we have
\[
\chi AF + S_1 + S_2 \in H^1(-T, T; L^2(\Omega)).
\]

Here and henceforth we denote the extended $v$, $\chi AF$, $S_1$, $S_2$, etc by the same notations.

Then by (1.4) and (1.5), taking into consideration of the even extensions to $(-T, 0)$, we can directly estimate $\partial^2_{tt} S_0$ with $k = 0, 1$ and $\ell = 1, 2$:
\[
\begin{cases}
(S_1 + S_2)(x, t) \leq C(|a'(t)| + |a''(t)|)U_1(x, |t|) + C|\chi(t)|U_1^1 U_1(x, |t|), \\
|\partial_t (S_1 + S_2)(x, t)| \leq C(|a'(t)| + |a''(t)| + |a'''(t)|)U_2^1 (x, |t|) + C|\chi(t)|U_2^1 (x, |t|).
\end{cases}
\]

where we set
\[
U_\ell(x, t) = \sum_{|\alpha| = 2} \sum_{k = 0}^{\ell} \left| \frac{\partial^{\alpha}_t \partial^k u(x, t)}{\partial t^{|\alpha|}} \right|^2 + \left( \sum_{|\alpha| = 2} \int_0^T \left| \frac{\partial^{|\alpha|}_t u(x, \eta)}{\partial t^{|\alpha|}} \right| d\eta \right)^2, \quad (x, t) \in \Omega \times (0, T)
\]

for $\ell = 0, 1, 2$.

Therefore we obtain
\[
\partial^2_{tt} v - Av = \chi AF + S_1 + S_2 \quad \text{in } Q,
\]

and
\[
\begin{cases}
\partial_t^j v(., \pm T) = 0 & \text{in } \Omega, \quad j = 0, 1, \\
v(., 0) = \partial_t v(., 0) = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega \times (-T, T).
\end{cases}
\]

**Second Step. Carleman estimates for $v$ and $\partial_t v$.**

Before the argument for obtaining Carleman estimates for (1.1), we describe a key idea. The equation (3.12) still contains integral terms $S_1$ and $S_2$, but in terms of $v$, we can regard $S_1, S_2$ as lower-order terms. Indeed (3.3) means
\( v = \chi Au + \text{integral terms of } \partial_x^\alpha u \text{ with } |\alpha| \leq 2 \) \hspace{1cm} (3.14)

and (3.7) and (3.8) yield

\[
(S_1 + S_2)(x,t) = \sum_{|\alpha|\leq 2} \sum_{k=0}^1 c_{\alpha k} \partial_x^\alpha \partial_t^k u(x,t) + \text{integral terms of } \partial_x^\alpha u \text{ with } |\alpha| \leq 2, \quad (3.15)
\]

where \( c_{\alpha k}, \alpha k \in L^\infty(Q) \) with \(|\alpha| \leq 2\). Neglecting the integral terms in (3.14) and (3.15), we have \( v \sim \chi Au \) and \( S_1 + S_2 \sim \sum_{|\alpha|\leq 2} \sum_{k=0}^1 c_{\alpha k} \partial_x^\alpha \partial_t^k u \). Since \( u|_{\partial \Omega} = 0 \) and \( \chi A \) is an elliptic operator, the elliptic regularity may yield the equivalence of the norms:

\[
\|\partial_x^k u(\cdot, t)\|_{H^2(\Omega)} \sim \|\partial_x^k v(\cdot, t)\|_{L^2(\Omega)}, \quad k = 0, 1.
\]

Therefore we can expect

\[
\| (S_1 + S_2)(\cdot, t) \|_{L^2(\Omega)} \sim \| \partial_x^k v(\cdot, t) \|_{L^2(\Omega)} + \| v(\cdot, t) \|_{L^2(\Omega)},
\]

that is, the term \( S_1 + S_2 \) in (3.12) corresponds to first-order terms of \( v \), and we can regard that the principal part of (3.12) is a hyperbolic operator \( \partial_t^2 - A \), and \( \chi AF \) is a non-homogeneous term.

For fixed \( x_0 \in \mathbb{R}^n \setminus \overline{\Omega} \), we set

\[

\psi(x, t) = |x - x_0|^2 - \beta t^2, \quad \varphi(x, t) = e^{\gamma \psi(x, t)}, \quad (x, t) \in Q := \Omega \times (-T, T),
\]

where \( \beta > 0 \) and \( \gamma > 0 \) are specified later.

First we show a Carleman estimate for a hyperbolic equation (e.g. Bellassoued and Yamamoto [6, 7]).

**Lemma 2.** We assume (1.4) and (1.6). Then we can choose a constant \( \beta > 0 \) for the constant \( \mu_1 \geq 0 \) in (1.6), so that we have the following Carleman estimate with the weight \( \varphi(x, t) = \exp(\gamma (|x - x_0|^2 - \beta t^2)) \): there exists a constant \( \gamma_0 > 0 \) such that for \( \gamma > \gamma_0 \), we can choose constants \( s_0 = s_0(\gamma) > 0 \) and \( C = C(\gamma) > 0 \) such that

\[
\int_Q \left( s \varphi \gamma |\nabla \varphi| w^2 + s^3 \varphi \gamma^3 |w|^2 \right) e^{2\varphi \gamma} \, dx dt \\
\leq C \int_Q \left( |(\partial_t^2 - A)w|^2 e^{2\varphi \gamma} + Ce^C \|\partial_x w\|_H^2 \right) \quad (x, t) \in \Gamma \times (-T, T)
\]

for all \( s > s_0 \) and \( w \in C([-T, T]; H^1_H(\Omega)) \cap C^1([-T, T]; L^2(\Omega)) \) satisfying \( \partial_t^2 w - Aw \in L^2(Q) \), \( w|_{\partial \Omega} = 0 \) and \( \partial_t^j w(\cdot, \pm T) = 0 \) in \( \Omega \), \( j = 0, 1 \).

As for hyperbolic Carleman estimates, we can refer also to Imanuvilov [16], Romanov [36]. Henceforth we assume that \( s > 0 \) and \( \gamma > 0 \) are sufficiently large.

Next we proceed to the formulation of the Carleman estimate for (1.1). The assumption (1.9) implies

\[
\psi(x, \pm T) = |x - x_0|^2 - \beta T^2 < 0, \quad x \in \overline{\Omega},
\]

and

\[
\psi(x, 0) = |x - x_0| > 0, \quad x \in \overline{\Omega}
\]

by \( x_0 \notin \overline{\Omega} \). Therefore there exist \( \varepsilon_0 > 0 \) and \( \varepsilon \in \left( 0, \frac{T}{4} \right) \) such that

\[
\begin{cases}
\varphi(x, t) \leq 1 - \varepsilon_0, & x \in \overline{\Omega}, T - 2\varepsilon \leq |t| \\
\varphi(x, t) \geq 1 + \varepsilon_0, & x \in \overline{\Omega}, |t| \leq 2\varepsilon.
\end{cases}
\]

(3.16)
Further we put
\[ \Phi_0 = \max_{x \in \Omega, 0 \leq t \leq T} \varphi(x, t). \]

We recall that \( u \) satisfies (1.1)–(1.3) in \( \Omega \times (0, T) \), while \( v \) is defined by (3.3) and extended to \((-T, 0)\) as an even function in \( t \).

The purpose of this step is to prove the following Carleman estimates for (1.1).

**Proposition.**

(i) There exists a constant \( \gamma_0 > 0 \) such that for \( \gamma > \gamma_0 \), we can choose constants \( s_0 = s_0(\gamma) > 0 \) and \( C > 0 \) such that

\[
\int_Q (s \gamma \varphi |\nabla_x \varphi|^2 + s^3 \gamma^3 |\varphi|^2) e^{2s\varphi} \, dxdt \leq C \int_Q |A|e^{2s\varphi} \, dxdt + C\|u\|_{H^2(0,T;H^2(\Omega))}^2 \Phi_0 e^{2(1-\epsilon_0)} + C\|\partial_v u\|_{H^2(0,T;L^2(\Gamma))}^2
\]

for all \( s > s_0 \) and \( u \in L^2(\Omega \times (0, T)) \) satisfying \( \partial_v^\alpha u \in H^2(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) for all \( |\alpha| \leq 2 \) and \( u|_{\partial\Omega} = 0 \).

(ii) Let \( F = 0 \). Then there exists a constant \( \gamma_0 > 0 \) such that for \( \gamma > \gamma_0 \), we can choose constants \( s_0 = s_0(\gamma) > 0 \) and \( C > 0 \) such that

\[
\int_Q (s \gamma \varphi |\nabla_x \varphi|^2 + s^3 \gamma^3 |\varphi|^2) e^{2s\varphi} \, dxdt \leq C \int_Q (|A| + |A(\partial_v F)|^2) e^{2s\varphi} \, dxdt + C\|u\|_{H^2(0,T;H^2(\Omega))}^2 \Phi_0 e^{2(1-\epsilon_0)} + C\|\partial_v u\|_{H^2(0,T;L^2(\Gamma))}^2
\]

and

\[
\int_0^T \int_\Omega \left( \sum_{|\alpha| \leq 2} (s \varphi |\partial_v^\alpha u|^2 + |\partial_v^\alpha \partial_v u|^2 + |\partial_v^\alpha \partial_v^2 u|^2) \right) e^{2s\varphi} \, dxdt \leq C \int_Q (|A| + |A(\partial_v F)|^2) e^{2s\varphi} \, dxdt + C\|u\|_{H^2(0,T;H^2(\Omega))}^2 \Phi_0 e^{2(1-\epsilon_0)} + C\|\partial_v u\|_{H^2(0,T;L^2(\Gamma))}^2
\]

for all \( s > s_0 \) and \( u \in L^2(\Omega \times (0, T)) \) satisfying \( \partial_v^\alpha u, \partial_v u \in H^2(0, T; L^2(\Omega)) \) for all \( |\alpha| \leq 2 \) and \( u|_{\partial\Omega} = 0 \).

Inequalities (3.17)–(3.19) hold uniformly for sufficiently large \( s > 0 \) in the sense that the constant \( C > 0 \) is independent of all large \( s > 0 \). Such inequalities are called Carleman estimates. The Carleman estimate is an essential tool for our inverse problem. The right-hand sides of (3.17)–(3.19) contain the non-local terms \( \|u\|_{H^2(0,T;H^2(\Omega))}^2 \) with \( k = 1 \) or \( = 2 \), and such non-local terms appear by the integral term in (1.1) and the cut-off function \( \chi \). Our Carleman estimates look different from usual Carleman estimates. However in applying to the inverse problem, \( e^{2(1-\epsilon_0)} \) is a minor factor, so that the term does not matter (e.g. the second term on the right-hand side of (3.41) later).

**Proof of proposition.** The proof of the proposition is a combination of hyperbolic and elliptic Carleman estimates (lemmata 3 and 3) with a weighted estimate of an integral (lemma 4).
First we have a Carleman estimate for the elliptic operator $A$ without the extra conditions on $a_y$.

**Lemma 3.** Let $p \in \mathbb{R}$ be given. There exists a constant $\gamma_0 > 0$ such that for $\gamma > \gamma_0$, we can choose constants $s_0 > 0$ and $C > 0$ such that

$$
\int_0^T \int_{\Gamma} \left( s^p \varphi^p \sum_{|\alpha|=2} |\partial^\alpha y|^2 + s^{p+2} \gamma^2 \varphi^{p+2} |\nabla y|^2 + s^{p+4} \gamma^4 \varphi^{p+4} |y|^2 \right) e^{2s\varphi} \mathrm{d}x \mathrm{d}t \\
\leq C \int_0^T \int_{\Gamma} s^{p+1} \varphi^{p+1} A |\nabla y|^2 + Ce^{Cy} \|\partial_y y\|^2_{L^2(\Gamma \times (0,T))}
$$

for all $s > s_0$ and $y \in L^2(0, T; H^2(\Omega)) \cap H^1_0(\Omega))$.

In the case of $p = -1$, for arbitrarily fixed $t \in [0, T]$ the Carleman estimate is classical (e.g. [14]). The integration over $0 < t < T$ yields the conclusion for $p = -1$. For general $p \in \mathbb{R}$, the result follows by applying the result in the case of $p = -1$ to $y \varphi^{p+1}$ (e.g. lemma 7.2 in [7]). We omit the details.

Applying lemma 2 to (3.12) with (3.13) and using (3.11), we have

$$
\int_Q (\gamma \varphi |\nabla x|^2 + s^3 \varphi^3 |x|^2) e^{2s\varphi} \mathrm{d}x \mathrm{d}t \leq C \int_Q \chi^2 |AF|^2 e^{2s\varphi} \mathrm{d}x \mathrm{d}t
$$

(3.20)

$$
+ C \int_Q |S_1| + |S_2|^2 e^{2s\varphi} \mathrm{d}x \mathrm{d}t + Ce^{Cy} \|\partial_y y\|^2_{L^2(\Gamma \times (-T,T))}
$$

$$
\leq C \int_Q |AF|^2 e^{2s\varphi} \mathrm{d}x \mathrm{d}t
$$

$$
+ C \int_Q (|\chi'(t)|^2 + |\chi''(t)|^2) U_1(x, |t|) e^{2s\varphi} \mathrm{d}x \mathrm{d}t + C \int_Q |\chi(t)|^2 U_1(x, |t|) e^{2s\varphi} \mathrm{d}x \mathrm{d}t
$$

for $s > s_0$.

For estimating the integral terms with the weight $e^{2s\varphi}$ which are created by $U_1$ in (3.20), we need to prove

**Lemma 4.** Let $q \geq 0$. Then

$$
\int_0^T \int_{\Gamma} \chi^2(t) (s\varphi)^q \left( \int_0^t |w(x, \eta)| \mathrm{d}\eta \right)^2 e^{2s\varphi} \mathrm{d}x \mathrm{d}t \leq C \int_0^T \int_{\Omega} s^{q-1} \gamma^{q-1} |\varphi^{q-1} \chi^2| w(x, t)^2 e^{2s\varphi} \mathrm{d}x \mathrm{d}t
$$

$$
+ Cs^{q-1} \Phi_0^{q-1} e^{2s(1-\alpha)} \|w\|^2_{L^2(\Omega \times (0,T))}
$$

This type of inequality is essential for applications of Carleman estimates to inverse problems (Bukhgeim and Klibanov [9], Klibanov [27]) and the inequality not involving the cut-off function $\chi$, is proved in [27, 28]. Our version of the inequality includes the cut-off function $\chi(t)$ and the proof of the lemma is provided in Appendix.

From lemma 4, we directly derive the following lemma which will be used repeatedly.
Lemma 5.  

(i)

\[
\int_0^T \int_\Omega \left( |\chi'(t)|^2 + |\chi''(t)|^2 + |\chi'''(t)|^2 \right) \left( |w_1(x, t)|^2 + \left( \int_0^t |w_2(x, \eta)| \text{d} \eta \right)^2 \right) e^{2q\varphi} \text{d}x \text{d}t \\
\leq C e^{2(1-\epsilon_0)}(|w_1|_{L^2(\Omega \times (0, T))}^2 + |w_2|^2_{L^2(\Omega \times (0, T))}).
\]

(ii) Let \( q \in \mathbb{N} \). Then

\[
\int_0^T \int_\Omega s^p q^q |\gamma(\eta)|^2 \left( |w_1(x, t)|^2 + \left( \int_0^t |w_2(x, \eta)| \text{d} \eta \right)^2 \right) e^{2q\varphi} \text{d}x \text{d}t \\
\leq C \int_0^T \int_\Omega s^p q^q |\gamma(\eta)|^2 \left( |w_1(x, t)|^2 + |w_2(x, t)|^2 \right) e^{2q\varphi} \text{d}x \text{d}t \\
+ C e^{2(1-\epsilon_0)-1} |w_2|^2_{L^2(\Omega \times (0, T))}.
\]

The proof is direct and is given in Appendix for completeness.

Applying lemma 5 to the second and the third terms on the right-hand side of (3.20) to obtain

\[
\int_0^T \int_\Omega s^p q^q |\gamma(\eta)|^2 \left( |w_1(x, t)|^2 + \left( \int_0^t |w_2(x, \eta)| \text{d} \eta \right)^2 \right) e^{2q\varphi} \text{d}x \text{d}t \\
\leq C \int_0^T \int_\Omega s^p q^q |\gamma(\eta)|^2 \left( |w_1(x, t)|^2 + |w_2(x, t)|^2 \right) e^{2q\varphi} \text{d}x \text{d}t \\
+ C e^{2(1-\epsilon_0)-1} |w_2|^2_{L^2(\Omega \times (0, T))}.
\]

Proof of (3.17). We have to estimate the second term on the right-hand side of (3.21) by lemma 3. First, by (3.3), for each \( t \in [0, T] \), we have

\[
\begin{align*}
A(\chi(t) u(\cdot, t)) &= \mathcal{B}(\cdot, t) + \mathcal{B}(t, \eta) u(x, \eta) \text{d} \eta \quad \text{in } \Omega, \\
\chi(t) u(\cdot, t) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Now we apply lemma 3 with \( p = 1 \), we obtain

\[
\int_0^T \int_\Omega s^p q^q \sum_{|\alpha| \leq 2} |\partial_x^\alpha (\gamma u)|^2 e^{2q\varphi} \text{d}x \text{d}t \\
\leq C \int_0^T \int_\Omega s^p q^q |\gamma|^2 e^{2q\varphi} \text{d}x \text{d}t \\
+ C e^{2q(1-\epsilon_0)} |u|^2_{H^1_0(\Omega \times (0, T))}.
\]

Next we apply lemma 4 with \( q = 2 \) to the second term on the right-hand side of (3.22) and, by (3.10) we obtain
\[
\begin{align*}
&\int_0^T \int_\Omega s^2 \varphi^2 \chi^2(t) \left( \int_0^t |B(t, \eta)| d\eta \right)^2 e^{2 \varphi} dt dr \\
&\leq C \int_0^T \int_\Omega s^2 \varphi^2 \chi^2(t) \left( \int_0^t |\partial_t \varphi| d\eta \right)^2 e^{2 \varphi} dt dr \\
&\leq C \sum_{|a| \leq 2} \int_0^T \int_\Omega s^2 \varphi^2 \chi^2(t) \left( \int_0^t |\partial_t \varphi| d\eta \right)^2 e^{2 \varphi} dt dr \\
&\leq C \sum_{|a| \leq 2} \int_0^T \int_\Omega s^2 \varphi^2 \chi^2(t) \left( \int_0^t |\partial_t \varphi| d\eta \right)^2 e^{2 \varphi} dt dr
\end{align*}
\]

Therefore (3.22) yields
\[
\int_0^T \int_\Omega s \varphi \sum_{|a| \leq 2} |\chi \partial_t^a \varphi| e^{2 \varphi} dt dr \\
\leq C \int_0^T \int_\Omega s^2 \varphi^2 |\varphi| e^{2 \varphi} dt dr + C \sum_{|a| \leq 2} \int_0^T \int_\Omega s^2 \varphi \chi^2 |\partial_t^a \varphi| e^{2 \varphi} dt dr
\]
\[
+C \varphi e^{2(1-\varepsilon_0)} \left( \|u\|_{L^2(0, T; H^1(\Omega))}^2 + C \sum_{|a| \leq 2} \|\partial_t^a \varphi\|_{L^2(\Gamma \times (0, T))}^2 \right)
\]

Therefore, choosing \(\gamma > 0\) large, we can absorb the second term on the right-hand side of (3.23) into the left-hand side, so that we obtain
\[
\int_0^T \int_\Omega s \varphi \sum_{|a| \leq 2} |\chi \partial_t^a \varphi| e^{2 \varphi} dt dr \leq C \int_0^T \int_\Omega s^2 \varphi^2 |\varphi| e^{2 \varphi} dt dr
\]
\[
+C \varphi e^{2(1-\varepsilon_0)} \left( \|u\|_{L^2(0, T; H^1(\Omega))}^2 + C \sum_{|a| \leq 2} \|\partial_t^a \varphi\|_{L^2(\Gamma \times (0, T))}^2 \right)
\]

Now we estimate \(\int_0^T \int_\Omega \sum_{|a| \leq 2} |\chi \partial_t^a \varphi| e^{2 \varphi} dt dr\). By (3.5) we have
\[
A(\partial_t u) = \partial_t \bar{\varphi} - \chi \bar{\varphi} u - \chi \int_0^t (\partial_t \bar{\varphi})(t, \eta) d\eta - \chi \bar{\varphi} u - \chi \int_0^t B(t, \eta) d\eta.
\]

Apply lemma 3 with \(p = 0\), and we obtain
\[
\int_0^T \int_\Omega \sum_{|a| \leq 2} |\partial_t^a (\chi \partial_t u)|^2 e^{2 \varphi} dt dr
\]
\[
\leq C \int_0^T \int_\Omega s \varphi |\partial_t^a \varphi|^2 e^{2 \varphi} dt dr + C \int_0^T \int_\Omega s \varphi \chi^2 |\bar{\varphi} u|^2 e^{2 \varphi} dt dr
\]
\[
+ C \int_0^T \int_\Omega s \varphi \chi^2 \left( \int_0^t (\partial_t \bar{\varphi})(t, \eta) d\eta \right)^2 e^{2 \varphi} dt dr + C \int_0^T \int_\Omega \chi^2 |\varphi u|^2 e^{2 \varphi} dt dr
\]
\[
+ C \int_0^T \int_\Omega \chi^2 |s \varphi|^2 \left( \int_0^t B(\eta) d\eta \right)^2 e^{2 \varphi} dt dr + C \varphi e^{2(1-\varepsilon_0)} \left( \|u\|_{L^2(0, T; H^1(\Omega))}^2 + C \sum_{|a| \leq 2} \|\partial_t^a \varphi\|_{L^2(\Gamma \times (0, T))}^2 \right)
\]
Applying lemma 4 with \( q = 1 \) to the third term and lemma 5 to the fourth and the fifth terms on the right-hand side, we have
\[
\int_0^T \int_\Omega \sum_{|\alpha| \leq 2} |\partial^\alpha_x (\chi \partial_t u)|^2 e^{2t} v^2 dx dt \\
\leq C \int_0^T s \varphi |\partial_t v|^2 e^{2t} v^2 dx dt + C \sum_{|\alpha| \leq 2} \int_0^T \int_\Omega \gamma^{-1} \chi^2 (t) |\partial^\alpha_x u|^2 e^{2t} v^2 dx dt \\
+ C \sum_{|\alpha| \leq 2} \int_0^T \int_\Omega s \varphi^2 \gamma^{-1} \chi^2 |\partial^\alpha_x u|^2 e^{2t} v^2 dx dt \\
+ C \varepsilon \delta^2 \|\partial_t \partial_t u\|^2_{L^2(\Gamma \times (-T,T))}.
\]
that is,
\[
\int_0^T \int_\Omega \sum_{|\alpha| \leq 2} |\chi \partial^\alpha_x \partial_t u|^2 e^{2t} v^2 dx dt \\
\leq C \int_0^T s \varphi |\partial_t v|^2 e^{2t} v^2 dx dt + C \int_0^T \int_\Omega s \varphi \gamma^{-1} \chi^2 |\partial^\alpha_x u|^2 e^{2t} v^2 dx dt \\
+ C \varepsilon \delta^2 \|\partial_t \partial_t u\|^2_{L^2(\Gamma \times (-T,T))}.
\]
Here we absorbed \( \int_0^T \int_\Omega \sum_{|\alpha| \leq 2} \gamma^{-1} |\chi \partial^\alpha_x u|^2 e^{2t} v^2 dx dt \) into the second term on the right-hand side of (3.26) by choosing \( s, \gamma > 0 \) large.

Applying (3.24) and (3.26) in (3.21), we obtain
\[
\int_0^T \left( s \varphi |\nabla \varphi|^2 + s^3 \varphi^3 |v|^2 \right) e^{2t} v^2 dx dt \\
\leq C \int_0^T |AF|^2 e^{2t} v^2 dx dt + \int_0^T \left( s \varphi |\partial_t v|^2 + s^2 \varphi^2 |v|^2 \right) e^{2t} v^2 dx dt \\
+ C \varepsilon \delta^2 \|\partial_t \partial_t u\|^2_{L^2(\Gamma \times (-T,T))}.
\]
(3.27)
By (3.4) and \( \partial_n F = 0 \) on \( \partial \Omega \), we see that \( \partial_n v = \chi \partial_n \partial_t u \) on \( \partial \Omega \), and so \( \|\partial_n \partial_t u\|^2_{L^2(\Gamma \times (-T,T))} = \|\partial_n \partial_t u\|^2_{H^1(\Gamma \times (0,T))} \leq C \|\partial_t \partial_t u\|^2_{L^2(\Gamma \times (0,T))} \). Choosing \( s, \gamma > 0 \) large, we can absorb the second term on the right-hand side of (3.27) into the left-hand side, we complete the proof of (3.17).

**Proof of (3.18).** Setting \( v_1 = \partial_t v \) and differentiate (3.12), we have
\[
\begin{cases}
\partial^j_t v_1 = A v_1 + \chi A (\partial_t F) + \chi' A F + \partial_j (S_1 + S_2) & \text{in } Q, \\
\partial^j_t v_1 (\cdot, \pm T) = 0 & \text{in } \Omega, \ j = 0, 1, \\
v_1 = 0 & \text{on } \partial \Omega \times (-T,T).
\end{cases}
\]
Repeating the arguments in the proof of (3.17), for the norms of the right-hand side of (3.17) we take one more \( t \)-derivatives to complete the proof of (3.18). □

**Proof of (3.19).** By (3.25), we have

\[
\begin{aligned}
&\left\{ A(\chi \partial_t^2 u) = \partial_t^2 v - \chi \left( \bar{B} \partial_t u + (\partial_t \bar{B})(t)u + (\partial_t B)(t,u) + \int_0^t (\partial_t^2 B)u \, dt \right) \right.
\end{aligned}
\]

\[
-2\chi'(t) \left( A\partial_t u + \bar{B}u + \int_0^t (\partial_t B)(t,\eta) \, d\eta \right) - \chi''(t) \left( Au + \int_0^t B(t,\eta)u(x,\eta) \, d\eta \right) \quad \text{in } Q,
\]

\[
\chi \partial_t^2 u \mid_{\partial\Omega} = 0.
\]

(3.29)

Applying lemma 3 with \( p = 0 \) to (3.29) and recalling (3.11), we obtain

\[
\begin{aligned}
&\gamma \int_0^T \int_\Omega |\chi \partial_t^r \partial_t^s u|^2 \, e^{2\gamma p} \, dx \, dt \\
&\quad + C \int_0^T \int_\Omega s \varphi |\chi \bar{B} \partial_t u|^2 \, e^{2\gamma p} \, dx \, dt + C \int_0^T \int_\Omega s \gamma |\chi \partial_t u|^2 (x,|t|) \, e^{2\gamma p} \, dx \, dt \\
&\quad + C \int_0^T \int_\Omega s \gamma (|\chi'|^2 + |\chi''(t)|^2) U_1(x,|t|) \, e^{2\gamma p} \, dx \, dt + Ce^{C \gamma} \|\partial_t \chi \partial_t u\|_{L^2(T \times (0,T))}^2
\end{aligned}
\]

(3.30)

We estimate the second term on the right-hand side as follows. Since \( \bar{B} = B(t, t) \) is independent of the variable \( \eta \), by \( \partial_t u(\cdot, 0) = 0 \) in \( \Omega \), we have

\[
\chi \bar{B} \partial_t u(x, t) = \chi \bar{B} \int_0^t \partial_t^2 u(x, \eta) \, d\eta = \chi \int_0^t \partial_t^2 u(x, \eta) \, d\eta.
\]

Lemma 4 with \( q = 1 \) yields

\[
\begin{aligned}
&\int_0^T \int_\Omega s \gamma \varphi |\chi \bar{B} \partial_t u|^2 \, e^{2\gamma p} \, dx \, dt = \int_0^T \int_\Omega s \gamma \varphi |\chi \bar{B} \partial_t u|^2 \, dx \, dt \\
&\quad \leq C \int_0^T \int_\Omega \gamma^2 |\chi \bar{B} \partial_t u|^2 \, dx \, dt + Ce^{2(1-\varepsilon_0)} \|\bar{B} \partial_t u\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\quad \leq C \int_0^T \sum_{|\alpha| \leq 2} |\chi \partial_t^\alpha \partial_t^s u|^2 \, dx \, dt + Ce^{2(1-\varepsilon_0)} \|u\|_{L^2(0,T;H^2(\Omega))}^2
\end{aligned}
\]

(3.31)

Applying lemma 5 to the third and fourth terms on the right-hand side of (3.30), by (3.24) we have

\[
\begin{aligned}
&\text{[the third and the fourth terms]} \\
&\quad \leq C \int_0^T \sum_{|\alpha| \leq 2} |\chi \partial_t^\alpha u|^2 \, dx \, dt + C\gamma s \Phi_0 e^{2s(1-\varepsilon_0)} \|u\|_{L^2(0,T;H^2(\Omega))}^2 \\
&\quad \leq C \int_0^T \int_\Omega s \gamma \sum_{|\alpha| \leq 2} |\chi \partial_t^\alpha u|^2 \, dx \, dt + Ce^{\gamma \Phi_0} \gamma e^{2s(1-\varepsilon_0)} \|u\|_{L^2(0,T;H^2(\Omega))}^2 + Ce^{C \gamma} \|\partial_t u\|_{L^2(\Gamma \times (0,T))}^2
\end{aligned}
\]

With (3.30) and (3.31), we obtain
\[
\int_0^T \int_\Omega \sum_{|\alpha| \leq 2} |\chi \partial_\alpha \partial_\tau^2 u|^2 e^{2\nu \tau} \mathrm{d}x \mathrm{d}t
\leq C \int_0^T \int_\Omega (s\gamma \partial_\alpha^0 \partial_\tau^2 u)^2 e^{2\nu \tau} \mathrm{d}x \mathrm{d}t + C \int_0^T \int_\Omega \sum_{|\alpha| \leq 2} |\chi \partial_\alpha \partial_\tau^2 u|^2 e^{2\nu \tau} \mathrm{d}x \mathrm{d}t
+ Cs\Phi_0 \gamma e^{2(1-\nu \tau)} ||u||_{H^2(0,T;H^2(\Omega))}^2 + C e^{C\gamma} ||\partial_\nu u||_{H^2(0,T;L^2(\Gamma))}^2.
\]

Choosing \(\gamma > 0\) large, we can absorb the second term on the right-hand side into the left-hand side and applying (3.18) to the first term on the right-hand side, we see
\[
\sum_{|\alpha| \leq 2} \gamma \int_0^T \int_\Omega |\chi \partial_\alpha \partial_\tau^2 u|^2 e^{2\nu \tau} \mathrm{d}x \mathrm{d}t \leq C \int_0^T (|AF|^2 + |A(\partial_\tau F)|^2) e^{2\nu \tau} \mathrm{d}t
\]
\[
+ Cs\Phi_0 \gamma e^{2(1-\nu \tau)} ||u||_{H^2(0,T;H^2(\Omega))}^2 + C e^{C\gamma} ||\partial_\nu u||_{H^2(0,T;L^2(\Gamma))}^2.
\]

Dividing by \(\gamma\) and again choosing \(\gamma\) large, we can obtain the desired estimate of
\[
\sum_{|\alpha| \leq 2} \int_0^T \int_\Omega |\chi \partial_\alpha \partial_\tau^2 u|^2 e^{2\nu \tau} \mathrm{d}x \mathrm{d}t \text{ in (3.19).}
\]
Combining (3.24) and (3.26) with (3.17), we complete the proof of the proposition.

**Third Step. Application of the Carleman estimate to \(\partial_\nu\).**
When proving the stability for the inverse problem, according to the classical method (e.g. [9, 19, 27]), it is a common strategy to keep unknown \(f\) in initial data. For it, we differentiate (3.12) with respect to \(t\).

We see that \(v_1 := \partial_\nu\) satisfies (3.28) and
\[
v_1(x, 0) = 0, \quad \partial_\tau v_1(x, 0) = AF(x, 0), \quad x \in \Omega.
\]
(3.32)

Here we used that \(\partial_\tau v_1(x, 0) = \chi AF(x, 0)\) by \(\partial_\nu v_1 = \partial_\tau^2 v\) and (3.12).

We set
\[
z = (\partial_\nu v)e^{\nu \tau} = v_1 e^{\nu \tau} \quad \text{in } Q.
\]
We write (3.28) in terms of \(z\). Henceforth we fix the parameter \(\gamma > 0\) large, so that in the proposition, we can omit the dependency on \(\gamma\) and \(\varphi\), and only the dependency on \(s\) is essential. Also we include \(\Phi_0^2\) into \(C > 0\).

First we have
\[
\partial_\tau z = (\partial_\nu v_1)e^{\nu \tau} + s(\partial_\tau \varphi)v_1 e^{\nu \tau}
\]
and
\[
\partial_\tau^2 z = (\partial_\nu^2 v_1)e^{\nu \tau} + 2s(\partial_\nu \varphi)(\partial_\tau v_1)e^{\nu \tau} + s(\partial_\tau^2 \varphi)v_1 e^{\nu \tau} + s^2(\partial_\nu \varphi)^2 v_1 e^{\nu \tau}.
\]
Moreover
\[
\partial_\tau z = (\partial_\nu v_1)e^{\nu \tau} + s(\partial_\nu \varphi)v_1 e^{\nu \tau},
\]
and so
\[
\partial_\tau \partial_\nu z = (\partial_\nu \partial_\nu v_1)e^{\nu \tau} + s[(\partial_\nu \varphi)(\partial_\nu v_1) + (\partial_\nu \varphi)(\partial_\tau v_1)]e^{\nu \tau}
+ \{s(\partial_\nu \varphi) + s^2(\partial_\nu \varphi) \partial_\nu \varphi\} v_1 e^{\nu \tau}, \quad 1 \leq i, j \leq n.
\]
Hence
\[ Az = \sum_{ij=1}^n a_{ij} \partial_i \partial_j z + \sum_{ij=1}^n (\partial_i a_{ij}) \partial_j z \]
\[ = \sum_{ij=1}^n a_{ij} (\partial_i \partial_j v_1) e^{\nu \varphi} + \sum_{ij=1}^n s a_{ij} (\partial_i v_1) \partial_j \varphi + (\partial_i \varphi) \partial_j e^{\nu \varphi} \]
\[ + \sum_{ij=1}^n s a_{ij} (\partial_i \varphi) \partial_j v_1 e^{\nu \varphi} + s^2 \sum_{ij=1}^n a_{ij} (\partial_i \varphi)^2 v_1 e^{\nu \varphi} \]
\[ + \sum_{ij=1}^n \partial_i a_{ij} (\partial_j v_1) e^{\nu \varphi} + \sum_{ij=1}^n (\partial_i a_{ij}) (\partial_j \varphi) v_1 e^{\nu \varphi}. \]

Using \( a_{ij} = a_{ji} \) in the second term on the right-hand side, we obtain
\[ Az = e^{\nu \varphi} A v_1 + 2s \sum_{ij=1}^n a_{ij} (\partial_i \varphi) (\partial_j v_1) e^{\nu \varphi} \]
\[ + s v_1 e^{\nu \varphi} A \varphi + s^2 v_1 e^{\nu \varphi} \sum_{ij=1}^n a_{ij} (\partial_i \varphi) \partial_j \varphi. \]

Thus, by noting \( \chi(0) = 1 \), equations (3.28) and (3.32) yield
\[ \partial_t^2 z - Az = \chi e^{\nu \varphi} A \partial_t F + \chi e^{\nu \varphi} AF + \partial_t (S_1 + S_2) e^{\nu \varphi} \] (3.33)
\[ + 2s \left( (\partial_i \varphi) \partial_j v_1 - \sum_{ij=1}^n a_{ij} (\partial_j v_1) \partial_j \varphi \right) e^{\nu \varphi} + s (\partial_i^2 \varphi - A \varphi) v_1 e^{\nu \varphi} \]
\[ + s^2 \left( (\partial_i \varphi)^2 - \sum_{ij=1}^n a_{ij} (\partial_i \varphi) \partial_j \varphi \right) v_1 e^{\nu \varphi} \quad \text{in } Q. \]

\[ \begin{aligned}
\partial t z(x, 0) &= A(R(x, 0) f(x)) e^{\nu \varphi(x, 0)}, \\
z(x, 0) &= 0, \quad x \in \Omega,
\end{aligned} \] (3.34)

and
\[ z|_{\partial \Omega} = 0. \] (3.35)

We apply the proposition to obtain (3.18). Now we rewrite (3.18) in terms of \( z = v_1 e^{\nu \varphi} \).

First \( s^3 |z|^2 = s^3 |v_1|^2 e^{2\nu \varphi} \) and then \( \partial_t z = (\partial_t v_1) e^{\nu \varphi} + s (\partial \varphi) z \), so that
\[ s |\partial_t z|^2 \leq C s^3 |z|^2 + C s |\partial_t v_1|^2 e^{2\nu \varphi}, \]
and we have similar estimates for \( s |\nabla z|^2 \). Hence
\[ \int_Q (s |\nabla_x z|^2 + s^3 |z|^2) dx dt \leq C \int_Q (s |\nabla_x v_1|^2 + s^3 |v_1|^2) e^{2\nu \varphi} dx dt, \]
and so the proposition yields
\[ \int_Q (s |\nabla_x z|^2 + s^3 |z|^2) dx dt \leq C \int_Q (|A|^2 + |A \partial_t F|^2) e^{2\nu \varphi} dx dt. \]
for \( s > s_0 \). Henceforth we set
\[
D^2 = \|\partial_t u\|_{H^2(0,T;L^2(\Gamma))}^2.
\]

We apply lemma 1 to the second term on the right-hand side of (3.36), and obtain
\[
\int_Q \left( |\nabla x|^2 + s^3 |z|^2 \right) dx dt \leq C \int_Q \left( |AF|^2 + |A\partial_t F|^2 \right) e^{2v_\epsilon} dx dt
\]
(3.37)

for \( s > s_0 \).

**Fourth Step. Energy estimate.**

We will establish a weighted energy estimate. We multiply (3.33) with \( \partial_t z \) and integrate by parts over \( \Omega \times (-T,0) \). Then
\[
\int_{-T}^0 \int_\Omega (\partial_t^2 z - Az) \partial_t z dx dt = \frac{1}{2} \int_{-T}^0 \partial_t \left( \int_\Omega |\partial_t z|^2 dx \right) dt - \int_{-T}^0 \int_\Omega (Az) \partial_t z dx dt.
\]

Here by \( z|_{\partial\Omega} = 0 \) and \( a_{ij} = a_{ji} \), integrating by parts, we see
\[
- \int_\Omega (Az) \partial_t z dx = \sum_{i,j=1}^n \int_\Omega a_{ij}(\partial_t \partial_j z) \partial_i z dx
\]
\[
= \int_\Omega \left\{ \sum_{i<j} a_{ij}((\partial_i z)(\partial_j z) + (\partial_j z)(\partial_i z)) + \sum_{i=1}^n a_{ii}(\partial_i z)(\partial_i z) \right\} dx
\]
\[
= \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(\partial_i z)(\partial_j z) dx = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(\partial_i z)(\partial_j z) dx.
\]

Using \( z(\cdot,0) = z(\cdot,-T) = 0 \) in \( \Omega \) and noting that \( a_{ij} \) is independent of \( t \), we obtain
\[
\int_{-T}^0 \int_\Omega (\partial_t^2 z - Az) \partial_t z dx dt = \frac{1}{2} \int_\Omega |\partial_t z(x,0)|^2 dx
\]
\[
= \frac{1}{2} \int_\Omega |A(R(x,0)f(x))|^2 e^{2v_\epsilon(x,0)} dx.
\]

(3.38)

Next, by the Cauchy–Schwarz inequality, we have
\[
\left| \int_{-T}^0 \int_\Omega [\text{the right-hand side of (3.33)}] \times \partial_t z dx dt \right|
\]
(3.39)
\[
\left[ \int_{-T}^{0} \int_{\Omega} (\chi A \partial_1 F + \chi' AF) e^{\xi_2} \partial_1 z dx \, dt + \int_{T}^{0} \int_{\Omega} \partial_1 (S_1 + S_2) e^{\xi_2} \partial_1 z dx \, dt \right] + 2s \int_{-T}^{0} \int_{\Omega} (\partial_2 \varphi) \partial_1 v_1 - \sum_{j=1}^{n} a_j (\partial_j v_1) \partial_2 \varphi \right) e^{\xi_2} \partial_1 z dx \, dt \\
+ s \int_{-T}^{0} \int_{\Omega} (\partial^2 \varphi - A \varphi) v_1 e^{\xi_2} \partial_1 z dx \, dt \\
+ s^2 \int_{-T}^{0} \int_{\Omega} \left( (\partial_2 \varphi)^2 - \sum_{j=1}^{n} a_j (\partial_j \varphi) (\partial_2 \varphi) \right) v_1 e^{\xi_2} \partial_1 z dx \, dt \\
\leq C \int_{Q} (|A \partial_1 F|^2 + |AF|^2) e^{2\xi_2} dx \, dt + C \int_{Q} |\partial_2 z|^2 dx \, dt \\
+ C \int_{Q} |\partial_1 (S_1 + S_2)|^2 e^{2\xi_2} dx \, dt + C \int_{Q} \left( s |\nabla_x \partial_1 v_1|^2 + s^3 |\partial_2 v|^2 \right) e^{2\xi_2} dx \, dt \\
+ C \int_{Q} s |\partial_2 z|^2 dx \, dt.
\]

Here we extended the integral domain \( \Omega \times (-T, 0) \) to \( \Omega \times (-T, T) \) := \( Q \) and we used

\[
\int_{Q} s |\nabla_x \partial_1 v_1 e^{\xi_2} | \partial_2 z | dx \, dt \leq 2 \int_{Q} s (|\nabla_x \partial_1 v_1|^2 e^{2\xi_2} + |\partial_2 z|^2) | dx \, dt
\]

and

\[
\int_{Q} s^2 |v_1 e^{\xi_2} | \partial_2 z | dx \, dt = \int_{Q} s^2 |v_1 e^{\xi_2} s^2 | \partial_2 z | dx \, dt \leq 2 \int_{Q} \left( s^3 |v_1|^2 e^{2\xi_2} + s |\partial_2 z|^2 \right) | dx \, dt.
\]

Moreover, by (3.10), (3.11), (3.19) and lemmata 1 and 5, we have

\[
\int_{Q} |\partial_1 (S_1 + S_2)|^2 e^{2\xi_2} dx \, dt
\]

\[
\leq C \int_{0}^{T} \int_{\Omega} \sum_{k=0}^{2} \sum_{|\alpha| \leq 2} \left| \chi \partial_1^2 \partial_2 \partial_1 u \right|^2 e^{2\xi_2} dx \, dt + Ce^{2(1-t_0)} \|u\|_{H^2(0,T;H^1(\Omega))}^2 \]

\[
\leq C \int_{Q} (|AF|^2 + |A(\partial_1 R)|^2) e^{2\xi_2} dx \, dt + Ce^{2(1-t_0)} \|f\|_{H^1(\Omega)}^2 + Ce^{2t_0} \|f\|_{H^1(\Gamma)}^2.
\]

Applying them to the third term and (3.18) and (3.37) to the second, the fourth and the fifth terms on the right-hand side of (3.39) and using lemma 1 for estimating \( \|u\|_{H^2(0,T;H^1(\Omega))}^2 \) by \( \|f\|_{H^1(\Omega)}^2 \), we obtain

\[
\int_{-T}^{0} \int_{\Omega} \left[ \text{the right-hand side of (3.33)} \right] \times \partial_2 z dx \, dt
\]

\[
\leq C \int_{Q} (|AF|^2 + |A(\partial_1 R)|^2) e^{2\xi_2} dx \, dt \\
+ Ce^{2(1-t_0)} \|f\|_{H^1(\Gamma)}^2 + Ce^{2t_0} \|\partial_2 u\|_{H^1(0,T;L^2(\Gamma))}^2.
\]
Therefore, combining (3.40) with (3.38) and noting $F(x,t) = R(x,t)f(x)$, we have
\[
\int_{\Omega} |A(R(x,0)f(x))|^2 e^{2t\varphi} \, dx \, dt
\leq C \int_0^T \left( |A((\partial_t R)f)|^2 + |A(Rf)|^2 \right) e^{2t\varphi} \, dx \, dt + C\|f\|_{H^2(\Omega)}^2 e^{2(1-\varepsilon_0)} + Ce^{Ct}D^2
\]
(3.41)
for $s > s_0$.

**Fifth Step. Completion of the proof.**

We complete the proof by an elliptic Carleman estimate. Since
\[
A((\partial_t^k R)(x,0)f(x)) = (\partial_t^k R)(Af) + \sum_{i=1}^n (\partial_i \partial_t^k R)a_i(\partial_t^k f)
\]
\[
+ \sum_{i=1}^n \partial_i(a_i(\partial_t^k R)f), \quad k = 0, 1,
\]
by (1.8) we estimate
\[
|A((\partial_t^k R)f)|^2 \leq C(|Af|^2 + |\nabla f|^2 + |f|^2),
\]
and
\[
|A(R(x,0)f)| \geq |R(x,0)f| - C(|\nabla f| + |f|)
\]
in $\Omega$, and (3.41) implies
\[
\int_{\Omega} |R(x,0)f|^2 e^{2t\varphi(x,0)} \, dx = C \int_0^T (|\nabla f|^2 + |f|^2) e^{2t\varphi(x,0)} \, dx
\leq C \int_{\Omega} \left( \int_{-T}^T |Af|^2 e^{2t\varphi} \, dt \right) \, dx + C\|f\|_{H^2(\Omega)}^2 e^{2(1-\varepsilon_0)} + Ce^{Ct}D^2.
\]
By $\varphi(x,t) \leq \varphi(x,0)$, we absorb the second term into the right-hand side by the second term on the left-hand side, and we apply $|R(\cdot,0)| > 0$ on $\overline{\Omega}$ by (1.8). Therefore
\[
\int_{\Omega} |Af|^2 e^{2t\varphi(x,0)} \, dx \leq C \int_{\Omega} |Af|^2 e^{2t\varphi(x,0)} \left( \int_{-T}^T e^{2t(\varphi(x,t) - \varphi(x,0))} \, dt \right) \, dx
\]
(3.42)
\[
+ C \int_{\Omega} (|\nabla f|^2 + |f|^2) e^{2t\varphi(x,0)} \, dx + C\|f\|_{H^2(\Omega)}^2 e^{2(1-\varepsilon_0)} + Ce^{Ct}D^2.
\]
Since
\[
2s(\varphi(x,t) - \varphi(x,0)) = 2s(e^{\gamma|x-s_0|^2} - e^{\gamma|x-s_0|^2})
\]
\[
= 2s(e^{\gamma|x-s_0|^2} - 1) \leq 2s(e^{-\gamma|\beta|^2} - 1)
\]
and $e^{-\gamma|\beta|^2} - 1 < 0$ for $t \neq 0$, the Lebesgue theorem yields
\[
\int_{-T}^T e^{2t(\varphi(x,t) - \varphi(x,0))} \, dt \leq \int_{-T}^T e^{2t(e^{-\gamma|\beta|^2} - 1)} \, dt = o(1)
\]
as $s \to \infty$. 


Therefore by choosing $s > 0$ sufficiently large, we can absorb the first term on the right-hand side of (3.42) into the left-hand side, we obtain
\[
\int_{\Omega} |Af|^2 e^{2s\varphi(x,0)} \, dx \leq C \int_{\Omega} ((|\nabla f|^2 + |f|^2)) e^{2s\varphi(x,0)} \, dx + C\|f\|_{H^1(\Omega)}^2 e^{2s(1-\varepsilon_0)} + Ce^{C_2D^2}
\]
for $s > s_0$. By $\nabla \varphi(x,0) = 2\gamma(x-x_0)/\varphi \neq 0$ for $x \in \Omega$, we apply the Carleman estimate for the elliptic operator $A$ of the second order which is similar to lemma 3 with $p = -1$ (here we fix $\gamma$), we have
\[
\int_{\Omega} \left( \frac{1}{s} \sum_{|\alpha| = 2} |\partial_\alpha f|^2 + s|\nabla f|^2 + s^2|f|^2 \right) e^{2s\varphi(x,0)} \, dx \\
\leq C \int_{\Omega} ((|\nabla f|^2 + |f|^2)) e^{2s\varphi(x,0)} \, dx + C\|f\|_{H^1(\Omega)}^2 e^{2s(1-\varepsilon_0)} + Ce^{C_2D^2}
\]
for $s > s_0$. Again choosing $s > 0$ sufficiently large, we can absorb the first term on the right-hand side into the left-hand side, multiplying with $s$ and replacing $se^{C_2t}$ by $e^{C_2t}$, we have
\[
\int_{\Omega} \sum_{|\alpha| = 2} |\partial_\alpha^t f|^2 e^{2s\varphi(x,0)} \, dx \leq C\|f\|_{H^1(\Omega)}^2 e^{2s(1-\varepsilon_0)} + Ce^{C_2D^2}
\]  \hspace{1cm} (3.43)
for $s > s_1$.

Hence, by (3.16) we obtain
\[
e^{2(1+\varepsilon_0)} \int_{\Omega} \sum_{|\alpha| = 2} |\partial_\alpha^t f|^2 \, dx \leq C\|f\|_{H^1(\Omega)}^2 e^{2(1-\varepsilon_0)} + Ce^{C_2D^2},
\]
that is,
\[
e^{2(1+\varepsilon_0)} (1 - Cs^2 e^{-4C_2t})\|f\|_{H^1(\Omega)}^2 \leq Ce^{C_2D^2}
\]
for all $s > s_0$. We choose $s > 0$ sufficiently large so that $1 - Cs^2 e^{-4C_2t} > 0$. Then $\|f\|_{H^1(\Omega)}^2 \leq Ce^{C_2D^2}$, and we complete the proof of the second inequality of (1.10).

Finally we have to prove the first inequality in (1.10). We set $w = \partial_3^t u$. Similarly to (3.12), we can obtain
\[
\begin{align*}
\partial_3^t w(x,t) &= A w + S_3(x,t) + (\partial_3^t R) f, \quad x \in \Omega, \ 0 < t < T, \\
w(x,0) &= (\partial_3^t R)(x,0) f(x), \\
\partial_3 w(x,0) &= A(R(x,0)f(x)) + (\partial_3^t R)(x,0) f(x), \quad x \in \Omega, \\
w|_{\partial_3^1 \times (0,T)} &= 0,
\end{align*}
\]
where $\tilde{B}(t) = B(t)$ and
\[
S_3(x,t) = \{ (\partial_3^t \tilde{B})(t) + \partial_3((\partial_3 B)(t,t)) + (\partial_3^2 B)(t,t) \} u(x,t) \\
+ \{ 2(\partial_3 \tilde{B})(t) + (\partial_3 B)(t,t) \} \partial_3^t u(x,t) \\
+ \tilde{B}(t) \partial_3^2 u(x,t) + \int_0^t \partial_3^t B(t,\eta) u(x,\eta) d\eta.
\]
Therefore by (1.5) we see
for \( x \in \Omega \) and \( 0 < t < T \). We apply the hidden regularity of \( \partial_t w \) (e.g. Komornik [30]) to obtain

\[
\| \partial_t w \|_{L^2(0,T;L^2(\partial\Omega))} \leq C \left( \| w(\cdot,0) \|_{H^2(\Omega)} + \| \partial_t w(\cdot,0) \|_{L^2(\Omega)} + \| S_3 + (\partial^3_t R) f \|_{L^2(0,T;L^2(\Omega))} \right).
\]

Hence

\[
\| \partial_t w \|_{L^2(0,T;L^2(\partial\Omega))} \leq C \left( \| f \|_{H^6(\Omega)} + \| S_3 \|_{L^2(0,T;L^2(\partial\Omega))} \right) + C \left( \| f \|_{H^6(\Omega)} + \| u \|_{H^2(0,T;H^2(\Omega))} + \int_0^T \left( \int_0^t \| u(\cdot,\eta) \|_{H^2(\Omega)}^2 d\eta \right) dt \right)
\]

\[
\leq C \left( \| f \|_{H^6(\Omega)} + \| u \|_{H^2(0,T;H^2(\Omega))} \right).
\]

Lemma 1 estimates the second term on the right-hand side, so that

\[
\| \partial_t^2 u \|_{L^2(0,T;L^2(\partial\Omega))} \leq C \| f \|_{H^6(\Omega)}.
\]

Finally, since

\[
\partial_t^2 u(x,0) = \partial_t (R(x,0) f) = 0, \quad \partial_t \partial_t u(x,0) = \partial_t u(x,0) = 0, \quad x \in \partial \Omega
\]

by (3.4) and \( f \in H^6_0(\Omega) \) and \( u(x,0) = \partial_t u(x,0) = 0 \) for \( x \in \Omega \), we have

\[
\partial_t^k \partial_t u(x,t) = \int_0^t \partial_t^{k-1} \partial_t u(x,\xi) d\xi, \quad k = 0, 1, 2.
\]

Consequently \( \| \partial_t^2 u \|_{H^0(0,T;L^2(\partial\Omega))} \leq C \| f \|_{H^6(\Omega)} \). Thus the proof of the first inequality, and so the theorem is completed.

4. Concluding remarks

1. Main result.

In this paper, we establish the Lipschitz stability in determining a spatially varying component of the external force term for hyperbolic integro-differential equation (1.1) where the principal part \( \sum_{i=1}^n a_{ij}(x) \partial_j \partial_i u \) is a general elliptic operator and general second-order derivatives \( \sum_{i=1}^n b_{ij} \partial_i^2 u \) are involved in the integral term. Moreover our estimate is both-sided, which implies that the estimate is the best possible. For our stability, we need some extra regularity (1.11) for the solution \( u \).

2. Technical novelty.

Due to the anisotropy caused by \( a_{ij} \) and \( b_{ij} \), the existing techniques for the inverse problem for integro-differential equations cannot work. Hence we introduce a new transformation (3.3) by increasing the orders of the Sobolev spaces which we use for the estimation for the inverse problem.

3. Future problems.

The current article has established the key Carleman estimate, and as natural applications, we will consider
• observability inequality and then the exact controllability.
• coefficient inverse problem of determining all and/or some of $a_{ij}(x)$ and/or spatial varying factors of $b_{\alpha}$.

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Appendix

Proof of lemma 4. By the Cauchy–Schwarz inequality, we have

$$\int_{\Omega} \int_{0}^{T} \chi^2(t) (s\varphi)^q \left( \int_{0}^{t} |w(x, \eta)| d\eta \right)^2 e^{2s\varphi} dx dt \leq \int_{\Omega} \int_{0}^{T} \chi^2(t) \left( \int_{0}^{t} |w(x, \eta)|^2 d\eta \right) s^q \varphi^q t e^{2s\varphi} dx dt.$$  \hspace{1cm} (A.1)

Noting that

$$s^q \varphi^q t e^{2s\varphi(x,t)} = -\frac{\partial_t (e^{2s\varphi(x,t)})}{4\gamma \beta} (s\varphi)^{q-1},$$

by integration by parts, we obtain

$$\int_{\Omega} \int_{0}^{T} \chi^2(t) \left( \int_{0}^{t} |w(x, \eta)|^2 d\eta \right) s^q \varphi^q t e^{2s\varphi} dx dt = \int_{\Omega} \int_{0}^{T} \chi^2(t) \left( \int_{0}^{t} w^2 d\eta \right) \left( \frac{e^{2s\varphi(x,t)}}{4\gamma \beta} (s\varphi)^{q-1} \chi^2(t) \right) \left( \int_{0}^{t} w^2 d\eta \right) d\eta dt$$

which is concluded.
Here we used

\[ \int_\Omega \left[ \frac{e^{2s\varphi(x,t)}}{4\gamma\beta} (s\varphi)^{q-1} \chi^2(t) \right]^{t=T}_{t=0} \, dx = 0 \]

by (3.2).

Therefore we can shift the first term on the right-hand side of (A.2) into the left-hand side, we have

\[
\int_\Omega \int_0^T ts^q \varphi^q \left( 1 - \frac{|q - 1|}{2} \right) \chi^2(t) \left( \int_0^t |w(x, \eta)|^2 \, d\eta \right) e^{2s\varphi} \, dx \, dt
\leq \int_\Omega \int_0^T ts^q \varphi^q \left( 1 + \frac{q - 1}{2} \right) \chi^2(t) \left( \int_0^t |w(x, \eta)|^2 \, d\eta \right) e^{2s\varphi} \, dx \, dt
= \int_\Omega \int_0^T \frac{(s\varphi)^{q-1}}{4\gamma\beta} \chi^2(t) \left( \int_0^t \left| w(x, \eta) \right|^2 \, d\eta \right) e^{2s\varphi} \, dx \, dt
\]

Choosing \( \gamma > 0 \) and \( s > 0 \) sufficiently large and noting that \( \varphi = e^{\gamma \psi} \) and \( \psi \geq 0 \) in \( Q \), we can obtain \( 1 - \frac{|q - 1|}{2} \chi^2(t) \geq \frac{1}{2} \). Therefore (A.1) yields

\[
\int_\Omega \int_0^T \chi^2(t) (s\varphi)^q \, \left| \int_0^t |w(x, \eta)|^2 \, d\eta \right|^2 e^{2s\varphi} \, dx \, dt
\leq C \int_\Omega \int_0^T \left( \frac{(s\varphi)^{q-1}}{\gamma} \right) \left| \partial_t \chi^2 \right| e^{2s\varphi} \left( \int_0^t \left| w^2 \right| \, d\eta \right) \, dx \, dt
+ C \int_\Omega \int_0^T \frac{e^{2s\varphi}}{\gamma} (s\varphi)^{q-1} \chi^2(t) \, w^2 \, dx \, dt.
\]

By (3.2), we see that the first term on the right-hand side is bounded from the above by

\[
C \int_\Omega \int_0^T (s\varphi)^{q-1} \chi^2 \, e^{2s\varphi} \left| w(x, \cdot) \right|^2_{\mathcal{L}^2(0, T)} \, dx \, dt. \]

Therefore by (3.16), it is bounded from the above by

\[
C \theta^{q-1} \varphi^{q-1} e^{2(1-\epsilon_0)} \| w \|^2_{\mathcal{L}^2(\Omega \times 0, T)}. \tag{A.3}
\]

Thus the proof of lemma 4 is complete.

\[ \square \]

**Proof of lemma 5.** The proof of lemma 5 (i) is the same as the argument in deriving the bound (A.3). The estimate of \( w_1 \) in lemma 5 (ii) is straightforward. As for \( w_2 \), lemma 4 yields the desired inequality.

\[ \square \]
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