Resumming perturbative series in the presence of monopole bubbling effects

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(Dated: November 30, 2017)

Monopole bubbling effect is screening of magnetic charges of singular Dirac monopoles by regular ’t Hooft-Polyakov monopoles. We study properties of weak coupling perturbative series in the presence of monopole bubbling effects as well as instantons. For this purpose, we analyze supersymmetric ’t Hooft loop in four dimensional \(\mathcal{N} = 2\) supersymmetric gauge theories with Lagrangians and non-positive beta functions. We show that the perturbative series of the ’t Hooft loop is Borel summable along positive real axis for fixed instanton numbers and screened magnetic charges. It turns out that the exact result of the ’t Hooft loop is the same as the sum of the Borel resummations over instanton numbers and effective magnetic charges. We also obtain the same result for supersymmetric dyonic loops.

I. INTRODUCTION

Progress on quantum field theory (QFT) hinges on understanding non-perturbative effects such as instantons, monopoles, vortices and so on. One of much less understood non-perturbative effects is monopole bubbling effect, which is screening of magnetic charges of singular Dirac monopoles \(^1\) by regular ’t Hooft-Polyakov monopoles \(^2\) \(^3\) \(^4\). Aim of this paper is to understand properties of weak coupling perturbative series in the presence of monopole bubbling effects as well as instantons.

The monopole bubbling effects typically appear in ’t Hooft loop \(^4\), which is magnetic version of Wilson loop and detects Higgs phase by area law, while area law of Wilson loop implies confinement \(^5\). The ’t Hooft loop can be defined as a partition function with singular boundary conditions for gauge fields given by Dirac monopoles \(^6\). For example, a straight ’t Hooft line along the Euclidean time at spatial origin in \(\mathbb{R}^4\) is described by

\[
F = \frac{B}{4} \epsilon_{\mu
u}\frac{x^\mu}{|x|^2} dx^\nu \wedge dx^\sigma, \tag{1}
\]

where \(B\) is a flux on \(S^2\) surrounding the loop \(B = \frac{1}{2\pi} \oint_{S^2} F\). It is known that ’t Hooft loop receives not only instanton corrections but also monopole bubbling effects. In this paper we study perturbative series of half-BPS ’t Hooft loops in 4d \(\mathcal{N} = 2\) supersymmetric (SUSY) gauge theories. To preserve SUSY, we need an additional boundary condition for the adjoint scalar \(\Phi\) in the \(\mathcal{N} = 2\) vector multiplet: \(\Phi = \frac{B}{2|x|^2}\). Here we analyze the SUSY ’t Hooft loops put on a curved space.

Before going to our setup in detail, let us recall general expectations on perturbative series in QFT. Perturbative series in QFT is typically not convergent \(^6\). In mathematics, there is a standard way to resum non-convergent series called Borel resummation. Given a perturbative series \(\sum_{\ell=0}^\infty \epsilon^\ell g^{a+\ell}\) of a quantity \(I(g)\), its Borel resummation along \(\mathbb{R}_+\) is defined by

\[
\mathcal{S}_0 I(g) = \int_0^\infty dt \, e^{-\frac{2}{\epsilon} t} I(t), \tag{2}
\]

where \(I(t)\) is an analytic continuation of the formal Borel transformation \(\sum_{\ell=0}^\infty \frac{\epsilon^\ell}{(a+\ell)!} I^{a+\ell}\) after the summation. Perturbative series in typical QFT is expected to be non-Borel summable due to singularities along \(\mathbb{R}_+\) of \(I(t)\) and Borel resummation formula has ambiguities since the integral depends on how to avoid the singularities. However, it is generally unclear when perturbative series in QFT are Borel summable. Another important question is in the case that we do not have Borel ambiguities, how the resummation is related to exact results.

In \(^6\) one of the authors initiated to address these questions. It has been proven that perturbative series in 4d \(\mathcal{N} = 2\) and 5d \(\mathcal{N} = 1\) SUSY gauge theories with Lagrangians are Borel summable along \(\mathbb{R}_+\) for various observables in sectors with fixed instanton numbers \(^9\). This result for the 4d \(\mathcal{N} = 2\) case is expected from a recent proposal \(^11\) on a semi-classical realization of IR renormalons \(^12\), where the semi-classical solution does not exist in the 4d \(\mathcal{N} = 2\) theories \(^13\) (see also \(^14\)). In this paper we study the ’t Hooft loop in the 4d \(\mathcal{N} = 2\) theories, which receives monopole bubbling effects in addition to instanton effects.

Now we explain details of our setup. Let us consider 4d \(\mathcal{N} = 2\) theories with Lagrangians and non-positive beta functions. In this class of theories we study the supersymmetric ’t Hooft loop placed at a circle in a squashed sphere \(S^4_0\), which is defined as the hypersurface in \(\mathbb{R}^5\)

\[
X_0^2 + b^{-2}(X_2^2 + X_3^2) + b^2(X_4^2 + X_5^2) = r^2, \tag{3}
\]

In this geometry, we can place half-BPS line operators on the circles \(S^2_0\) and \(S^2_{b-1}\), which are given by \(X_3 = X_4 = 0\).
and $X_1 = X_2 = 0$ at fixed $X_0$, respectively. Though we focus on $S^4$ at $X_0 = 0$, the result for $S^4_{b^{-1}}$ is simply obtained by the replacement $b \to b^{-1}$, which does not change our main result.

The 't Hooft loop receives both instantons and monopole bubbling effects [13, 23]:

$$T(B) = \sum_{k, \bar{k}, v} e^{-kS_{\text{inst}} - \bar{k}S_{\text{inst}} - \text{tr}(v^2)S_{\text{mono}}} T(k, \bar{k}, v)(g, \theta),$$

where $v$ denotes a screened monopole charge and

$$S_{\text{inst}} = -2\pi i \tau, \quad S_{\text{inst}} = 2\pi i \bar{\tau}, \quad S_{\text{mono}} = -\frac{\pi b^2}{2g},$$

with a complex gauge coupling $\tau = \theta/2\pi + i/g$. Eq. (4) is schematic in the sense that theories with product gauge group have multiple groupings and multiple $(k, \bar{k}, v)$. Here we are interested in weak expansion by (square) Yang-Mills coupling $g$:

$$T(k, \bar{k}, v)(g, \theta) \simeq \sum_{\ell} c^{\ell}_{k, \bar{k}, v}(\theta) g^{\sharp+\ell},$$

where $\sharp$ is a leading order exponent. In this paper we show that the perturbative series in the sector with fixed $(k, \bar{k}, v)$ is Borel summable around $\Re_e \bar{g}$ in the case that explicit expressions for instantons and monopole bubbling effects in SUSY localization formula [16] are available. Our main result is

$$T(k, \bar{k}, v)(g, \theta) = S_0 T(k, \bar{k}, v)(g, \theta).$$

Thus we can rewrite the whole exact result in terms of the Borel resummation:

$$T(B) = \sum_{k, \bar{k}, v} e^{-kS_{\text{inst}} - \bar{k}S_{\text{inst}} - \text{tr}(v^2)S_{\text{mono}}} T(k, \bar{k}, v)(g, \theta).$$

This equation can be understood as “semi-classical decoding” [13] of the 't Hooft loop in 4d $\mathcal{N} = 2$ theories though the resurgence structure itself is trivial for this case.

II. 'T HOOFT LOOP AND BOREL RESUMMATION

**Exact result**

Instead of a path integral expression, we use a conjectural finite dimensional integral representation for the 't Hooft loop [18, 20] which has passed highly nontrivial tests. There are exact results for the 't Hooft loop in the 4d $\mathcal{N} = 2$ theories on round $S^4$ [19], $S^1 \times \mathbb{R}^3$ [18] and $S^1 \times S^3$ [21] by SUSY localization [16]. Although there is no explicit computation for our $S^4_b$ case, one can find the reasonable expression for the exact result by combining the results for round $S^4$ [19], $S^1 \times \mathbb{R}^3$ [18] and the partition function on $S^4_b$ [22] (see also [23]).

consistent with the exact results on round $S^4$ for $b = 1$, the partition function on $S^4_b$ for $B = 0$, and AGT relation [24] for some theories.

Let us consider the 4d $\mathcal{N} = 2$ SUSY gauge theories with a semi-simple gauge group $G = G_1 \times \cdots \times G_n$, which are coupled to hyper multiplets of representations $(\mathbf{R}_1, \cdots, \mathbf{R}_n)$. According to [15, 20], the 't Hooft loop is given by [22]

$$T(B) = \sum_v \int dG a Z_{\text{cl}}^v Z_{\text{inst}}^v Z_{\text{NP}}^v,$$

where $v$ describes the screened magnetic charges. The classical contribution $Z_{\text{cl}}^v(a)$ is

$$Z_{\text{cl}}^v(a) = \frac{n}{p=1} \exp \left[ -\frac{\text{tr} p a^2}{g_p} - b \theta_p \text{tr}(p \cdot v) \right],$$

where $g$ is proportional to square of one-loop effective Yang-Mills coupling at scale $1/r$ and tr$_p$ denotes trace in the gauge group $G_p$. The one-loop determinant $Z_{\text{inst}}^v(a)$ has contributions both from poles and equator of $S^4_b$ [40]:

$$Z_{\text{inst}}^v(a) = |Z_N(a_N)|^2 Z_{\text{eq}}(a),$$

$$Z_N(a) = \left[ \prod_{i=1}^{N_f} \prod_{\rho_i \in \mathcal{R}_m} \Upsilon(i\rho \cdot a + Q) \right]^{1/2},$$

$$Z_{\text{eq}}(a) = \frac{\prod_{\rho_i \in \mathcal{R}_m} \prod_{k=0}^{\rho_i \cdot v - 1} \cosh^2 \frac{\pi b}{2} \left[ \pi b \left( \rho_i \cdot a + i b \left( k - \frac{|\rho_i \cdot v| - 1}{2} \right) \right) \right]}{\prod_{\alpha \in \Delta} \prod_{k=0}^{\alpha \cdot v - 1} \sinh^2 \left[ \pi b \left( \alpha \cdot a + i b \left( k - \frac{\alpha \cdot v}{2} \right) \right) \right]}.$$
of

\[ T^{(k,\bar{k},v)}(g,\theta) = \int_{-\infty}^{\infty} d^G a \, Z(\nu) \, Z_{1\text{loop}}(k,v), \]

where \( Z_{1\text{loop}}(k,v) = Z_{\text{inst}}(\nu) Z_{\text{mono}}(\nu), \)

\[ Z_{\text{inst}}(\nu) = Z_{\text{inst}}(\nu). \]

SU(\( N \)) superconformal QCD

We first discuss 4d \( N = 2 \) SU(\( N \)) superconformal QCD for simplicity of explanations which is SUSY QCD (SQCD) with 2\( N \) fundamental hyper multiplets. We will consider more general theories later. The \( \theta \)-Hooft loop of our SQCD in the sector \( (k,\bar{k},v) \) is

\[ T_{\text{SQCD}}^{(k,\bar{k},v)}(g,\theta) = \int d^N a \, d^N\nu \, \prod_{1 \leq j < \leq N} \left| Y(ia_{ij},\nu_{ij}) \right|^2 \sinh \left( \pi b \left( a_{ij} + i(b - |\nu_{ij}|/2) \right) \right), \]

\[ \prod_{j=1}^N \frac{1 - |\nu_j|}{\cosh N} \left| Y(ia_{ij},\nu_{ij}) \right|^2 \sinh \left( \pi b \left( a_{ij} + i(b - |\nu_{ij}|/2) \right) \right), \]

where \( a_{ij} = a_i - a_j \) and \( \nu_{ij} = \nu_i - \nu_j \). Let us study small-g expansion of this object. Instead of explicitly computing perturbative coefficients, we explicitly find Borel transformation, somehow already hidden in eq. (17) as in [8, 27]. To see this, first we make a change of the variable \( x = (\hat{x}_1, \ldots, \hat{x}_N) \) is the unit vector in \( \mathbb{R}^N \). Then, we rewrite the \( \theta \)-Hooft loop as

\[ T_{\text{SQCD}}^{(k,\bar{k},v)}(g,\theta) = \int_{-\infty}^{\infty} dt e^{-\frac{t}{4} f^{(k,\bar{k},v)}(t,\theta)}, \]

where

\[ f^{(k,\bar{k},v)}(t) = t^2 \int_{S^{N-1}} d^{N-1} \hat{x} \left( \sum_j \hat{x}_j \right) \left( h^{(k,\bar{k},v)}(t,\hat{x}), \right), \]

\[ h^{(k,\bar{k},v)}(t,\hat{x}) = e^{-b \theta} \sum_j \nu_j a_j Z_{\text{loop}}(k,v) Z_{\text{mono}}(k,v) \]

The exponent \( t \) is a constant depending on \( N \) and \( \nu_{ij} \), whose detail is not important for our purpose [11]. The functions \( f^{(k,\bar{k},v)} \) and \( h^{(k,\bar{k},v)} \) depend on \( \theta \) but we do not often write \( \theta \) explicitly in the arguments for simplicity. Since the expression [18] is the form of the Laplace transformation and similar to the Borel resummation, one may wonder whether the function \( f^{(k,\bar{k},v)}(t) \) of the perturbative series is the Borel transformation. Indeed, we can prove that this is the case as in [8, 27]:

\[ f^{(k,\bar{k},v)}(t,\theta) = B^{(k,\bar{k},v)}_{\text{SQCD}}(t,\theta). \]

The proof takes the following three steps. (1) We show that the integrand \( h^{(k,\bar{k},v)}(t,\hat{x}) \) is identical to analytic continuation of a convergent power series of \( t \). (2) We exchange the order of the power series expansion of \( h^{(k,\bar{k},v)}(t,\hat{x}) \) and integration over \( \hat{x} \). This is rigorously justified by proving uniform convergence of the small-\( t \) expansion. (3) It is easily seen that the coefficient of the perturbative series of \( f^{(k,\bar{k},v)}(t) \) is given by \( c_{\theta/(4\pi^2)}^{(k,\bar{k},v)} \) as guaranteed by the Laplace transformation [18].

First, let us focus on perturbative sector \( (k,\bar{k},v) = (0,0,B) \). We will consider non-perturbative effects later. To prove the uniform convergence of the small-\( t \) expansion of \( h^{(k,\bar{k},v)}(t,\hat{x}) \), it is convenient to use Weierstrass M-test, i.e. we find a sequence \( \{ M_t \} \) such that \( |h^{(0)}(\hat{x})| < M_t \) and \( \sum_{\ell=0}^{\infty} M_t \epsilon^{\ell} < \infty \) for fixed \( t \). We can easily find such a series as in [8, 27] by replacing all the sources of negative contributions to the small-\( t \) expansion by positive definite larger values [42]. As an example, the following function generates \( M_t \):

\[ P(t) e^{\theta t} \sum_{j=1}^{N} \sum_{\ell=0}^{\infty} \left( 5N^2 - 4N \right) \epsilon^t \]

\[ (1 - b^2) t^{N^2} \sum_{\ell=0}^{\infty} \left( 2t \pm \sqrt{7Q} \right)^{2(N^2 - N)}, \]

where \( P(t) \) is an appropriate finite order polynomial. Thus, \( f^{(0,0,B)}(t) \) is actually the Borel transformation.

Now we can explicitly study analytic properties of the Borel transformation. Since we have expressed the Borel transformation in terms of the one-loop determinant \( Z_{\text{loop}}^{(v)} \), this problem boils down to analytic properties of the one-loop determinant, whose details are studied in app. A for general case. As a result, the one-loop determinant in our SQCD has degree-\( 2N \) poles at

\[ a_j = \pm \left( m_1 + \frac{v_j + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1}, \]

where \( m_1, m_2 \in \mathbb{Z}_{\geq 0} \). We emphasize that all the apparent branch cuts are canceled in the whole expression. Most important point here is that the one-loop determinant does not have singularities for \( a \in \mathbb{R} \). Thus, the Borel transformation has singularities (poles) only along \( \mathbb{R}^- \) for \( b \in \mathbb{R} \) and the perturbative series is Borel summable along \( \mathbb{R}^+ \).

General \( \mathcal{N} = 2 \) theories with Lagrangian

We can easily generalize the above analysis to general \( \mathcal{N} = 2 \) theories with Lagrangian. Taking the polar coordinate \( a_i = \sqrt{t} x_i \) with \( x_i \in S_{G/\rho}^{N-1} \), we find:

\[ T^{(k,\bar{k},v)}(g,\theta) = \int_{0}^{\infty} d^m t \, e^{-\frac{t}{4} f^{(k,\bar{k},v)}(t)}, \]
where \( f(k,k,v)(t) = t^k \int_{\text{sphere}} d\hat{x} \ h^{(k,k,v)}(t, \hat{x}), \)
\[
h^{(k,k,v)}(t, \hat{x}) = e^{-b \sum_p \theta_{k,v}^{(p)-1} Z_{1\text{loop}}^{(p)}(\hat{x})} |a_p^{(p)} = \sqrt{t \rho^{(p)}}|^2.
\]
(24)

where \( t^k = \prod_{p} t_p^k \) with a constant \( t_p \) unimportant for our purpose \( [43] \). Again, eq. (23) is similar to the form of Borel resummation generalized to multiple couplings. Indeed, one can show that the function \( f(k,k,v)(t) \) is nothing but the Borel transformation of the perturbative series. Let us focus on the sector with \( (k,k,v) = (0,0,0) \) again. As in the previous case, we can always show that \( f(k,k,v)(t) \) is the Borel transformation:
\[
BT^{(k,k,v)}(t, \theta) = f^{(k,k,v)}(t, \theta).
\]
(25)

Let us look at analytic properties of the Borel transformation. According to app. [A] the one-loop determinant has singularities at
\[
\rho \cdot a = \pm i \left[ (m_1 + \frac{|a| + 1}{2}) b + (m_2 + \frac{1}{2}) b^{-1} \right].
\]
(26)

This shows that the one-loop determinant does not have singularities for real \( a \) and therefore the perturbative series is Borel summable along positive real axis.

**Instanton corrections**

Generalization to non-zero instanton sector is also straightforward because \( Z_{\text{Nek}}(a) \) with fixed instanton number and omega background parameters \((\epsilon_1, \epsilon_2) = (b, b^{-1})\) is a rational function, which does not have poles for real \( a \) unless \( m_1 b + m_2 b^{-1} \) is purely imaginary. Therefore, eq. (23) still holds in nonzero instanton sector.

For example, Nekrasov partition function for \( U(N) \) SQCD with \( N_f \) fundamental hyper is given by
\[
Z_{\text{Nek}}(a) = \sum_{Y} e^{-|Y| S_{\text{inst}}} \left( \prod_{j=1}^{N} n_{ij}^{V}(a, Y) \right)^{N_f},
\]
(27)

where \( Y = (Y_1, \cdots, Y_N) \) denotes a set of Young tableau in \( U(N) \), \( |Y| \) is the total number of boxes of \( Y \), and
\[
n_{ij}^{V}(a, Y) = \prod_{s \in Y_j} E_{ij}(a, s) (Q - E_{ij}(a, s)),
\]
\[
E_{ij}(a, s) = -b A_{ij}(s) + b^{-1}(L_{Y_j}(s) + 1) - ia_{ij},
\]
\[
n_{ij}^{V}(a, Y) = \prod_{s \in Y_j} \phi_{ij}(a, s) (Q - E_{ij}(a, s)),
\]
\[
\phi_{ij}(a, s) = a_{ij} + b(s_h - 1) + b^{-1}(s_h - 1). \]
(28)

The indices \( s = (s_h, s_e) \) label a box in Young diagram at \( s_h \)-th column and \( s_e \)-th row, and \( L_{Y_j}(s) \) is leg (arm) length of Young tableau \( Y \) at \( s \). The vector contribution gives poles at
\[
a_{ij} = -ib \left\{ \frac{m}{2} - A_{ij}(s) \right\} - ib^{-1} \left\{ L_{Y_j}(s) + 1 \right\}, \]
(29)

and
\[
a_{ij} = ib \left\{ 1 - \frac{m}{2} + A_{ij}(s) \right\} - ib^{-1} L_{Y_j}(s). \]
(30)

Although the contribution from each Young diagram may have poles at \( a_{ij} = 0 \), this type of poles are canceled after summing over Young diagrams with the same instanton number \([14]\). Thus the Borel transformation has singularities only along \( \mathbb{R}_{-} \) and especially perturbative series is Borel summable along \( \mathbb{R}_{+} \) even if we include the instanton corrections.

**Monopole bubbling effects**

Now let us add monopole bubbling effects. As in the instanton corrections, generalization is straightforward as long as we know explicit expressions of \( Z_{\text{mono}}^{(v)}(a) \) because the monopole bubbling effect is described by a ratio of finite products of hyperbolic functions, whose poles are not on real axis. Hence, (23) still holds in the presence of the monopole bubbling effects.

For example, for \( SU(N) \) or \( U(N) \) gauge theory with fundamentals and adjoints, the contribution from the monopole bubbling effects is given by
\[
Z_{\text{mono}}^{(v)}(a) = \sum_{Y} Z_{Y}^{\text{rec}}(a) \prod_{I} Z_{I}^{\text{tri}}(a),
\]
(31)

where
\[
Z_{Y}^{\text{rec}}(a) = \frac{1}{\prod_{i,j,s \in Y_i} \sinh^{2} \frac{2\pi ba_{ij} + ib^{2}(A_{ij}(s) - L_{Y_j}(s) \pm 1)}{2}},
\]
\[
Z_{Y}^{\text{adj}}(a) = \prod_{i,j,s \in Y_i} \cosh^{2} \frac{2\pi ba_{ij} + ib^{2}(A_{ij}(s) - L_{Y_j}(s))}{2},
\]
\[
Z_{Y}^{\text{fund}}(a) = \prod_{j,s \in Y_j} \cosh^{2} \frac{2\pi ba_{j} + i\pi b^{2}(s_{e} + s_{h}) - 1}{2}. \]
(32)

The sum in (31) is over a set of Young tableau with the total number of boxes \( \frac{1}{2} \text{tr}(B^{2} - v^{2}) \). Note also that we have put the symbol \( \cdot \) for the sum and product which means the sum over \( Y \) and product over \( s \) are constrained. Namely, we include only \( Y \) and \( s \) satisfying the constraints in contrast to the instanton corrections. We explain details on this in app. [B] because the constraints are quite complicated and nevertheless their details are not so important for our purpose.

The most important thing for us is singularity structure of \( Z_{\text{mono}}^{(v)}(a) \). For the contribution from single set of Young diagram \( Y \), the vector contribution gives poles at
\[
\text{pol}\ = \frac{im}{2} - \frac{ib}{2} A_{ij}(s) + L_{Y_j}(s) \pm 1. \]
(33)
When $A_Y(s) - L_Y(s) = \pm 1 \neq 0$ or $m \neq 0$, the poles are not located along real axis unless $b^2$ is purely imaginary. When $A_Y(s) - L_Y(s) = \pm 1 = 0$, we have the poles at $\alpha_{ij} = 0$ but this type of poles are canceled from other Young diagrams as long as we consider well-defined ‘t Hooft loops. Thus, $Z_{\text{mono}}^{(v)}(a)$ does not have singularities along real axis and the perturbative series is Borel summable along $\mathbb{R}_+$. Furthermore, the Borel transformation has singularities only along $\mathbb{R}_-$ for $b \in \mathbb{R}$ including all the non-perturbative corrections.

Dyonic loop

Supersymmetric dyonic loop can be computed by putting SUSY Wilson loop on $S^4$ in the setup of the ‘t Hooft loop. If we assume the conjectural expression $\mathcal{D}$ for the ‘t Hooft loop, then the exact result for the dyonic loop should be given by

$$D = (\text{tr}_{\mathbb{R}^b} e^{ib \rho})_{\text{M.M.}},$$

where $\langle \cdots \rangle_{\text{M.M.}}$ denotes unnormalized expectation value in the matrix model $\mathcal{D}$. Since this is just insertion of the function of $a$ without singularities, this does not prevent us from application of the above technique to this case. Then the Borel transformation is simply given by $f(k, k, v)$ in [24] with the extra insertion of $\text{tr}_{\mathbb{R}^b} e^{ib \rho}$ to the integrand. Thus, the analytic properties of $h(k, k, v)$ do not change and the perturbative series is still Borel summable along $\mathbb{R}_+$ including the instantons and the monopole bubbling effects.

III. CONCLUSION AND DISCUSSIONS

In this paper we have studied weak coupling perturbative series in the presence of monopole bubbling effects as well as instanton effects. We have shown that the perturbative series of the ‘t Hooft loop in the 4d $\mathcal{N} = 2$ supersymmetric gauge theories are Borel summable along $\mathbb{R}_+$. It has turned out that the exact result is the same as the sum of the Borel resummations over instanton-anti-instanton numbers $(k, \bar{k})$ and screened magnetic charges $v$. Our result is also non-trivially consistent with the conjecture that 4d $\mathcal{N} = 2$ theories do not have IR renormalon type singularities [11, 13].

There is a confusing point on our results. According to Lipatov’s argument [28] and topological selection rule [11, 27] (see also [30]), it is expected that Borel singularities come from saddle points with the same topological number. Hence, one may expect that there are Borel singularities coming from instantons-anti-instantons with the same $k - \bar{k}$ in our setup. However, we have shown that we do not have such singularities. Absence of this type of singularities as well as IR renormalons lead the Borel summability along $\mathbb{R}_+$ and makes the perturbative series in every sector isolated in contrast to usual resurgence scenario in quantum mechanics [31] and QFT [11, 27, 24, 22]. It is interesting to find physical interpretations for this point. One of possible scenarios would be “Chesire cat resurgence”, which has recently appeared in some SUSY theories [33]. Another possibility is that there is something like a generalization of topological selection rule particular for our setup.

We have found that the Borel transformation has infinitely many singularities in Borel plane. It is interesting to find their physical interpretations. Technically, these singularities come from singularities of the one-loop determinant, Nekrasov partition functions, and monopole bubbling effects. At least for those from the one-loop determinant, we expect that they can be explained by complexified SUSY solutions as in 3d $\mathcal{N} = 2$ case [27, 34], which formally satisfy SUSY conditions but may not be on the original path integral contour.

The formula for the ‘t Hooft loop, which we have used, has not been explicitly computed by the localization despite it has the passed nontrivial checks. It is nice to derive the localization formula directly.

It is known that the ‘t Hooft loop operator is S-dual to SUSY Wilson loop, whose perturbative expansion is also Borel summable along $\mathbb{R}_+$ [8]. It would be illuminating to study whether there are implications of the S-duality for structures of Borel singularities.

Acknowledgments

We thank Takuya Okuda for his correspondence to our questions on [19]. M. H. would like to thank Centro de Ciencias de Benasque, CERN, Fudan University, KITP, RIKEN, and YITP for hospitality. D. Y. would like to thank Tokyo Institute of Technology for hospitality. This research was supported in part by the National Science Foundation under Grant No. NSF PHY-1125915. D. Y. was supported by the ERC Starting Grant N. 304806, “The Gauge/Gravity Duality and Geometry in String Theory”.

Appendix A: Analytic property of one-loop determinant

We discuss details on analytic properties of the one-loop determinant $Z_{\text{1loop}}^{(v)}(a)$ as a function of $a$.

Hyper multiplet contribution

We study an analytic property of a contribution from a weight vector $\rho$:

$$\prod_{k=0}^{|\rho|, v-1} \cosh \frac{\pi b (\rho \cdot a + ib (k - |\rho|, v-1))}{2} \frac{\mathcal{Y} (i \rho \cdot a_N + Q \rho)}{2}.$$ (A1)
Using the infinite product representation \([12]\) of \(\Upsilon(x)\), we have the following convenient identity:

\[
\left| \Upsilon \left( i \left( x + \frac{ibv}{2} \right) + \frac{Q}{2} \right) \right|^2 = 
\prod_{k=0}^{\lfloor |v|/2 \rfloor} \prod_{m_2 \geq 0} \left( \left( m_2 + \frac{1}{2} \right)^2 - b^2 - \left( x + ib \left( k - \frac{|v| - 1}{2} \right) \right)^2 \right)
\prod_{m_1, m_2 \geq 0} \left[ \left( \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1} \right)^2 + x^2 \right]^2.
\]

(A2)

Then, recalling \(\cosh (\pi x) = \prod_{n=1}^{\infty} \left( 1 + \frac{4x^2}{(2n-1)^2} \right)\), the first term cancels the equator contribution and we find that \([A1]\) is proportional to

\[
\prod_{m_1, m_2 \geq 0} \frac{1}{\left( \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1} \right)^2 + (\rho \cdot a)^2}.
\]

(A3)

Thus, we do not have branch cuts but have simple poles at

\[
\rho \cdot a = \pm i \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1}.
\]

(A4)

**Vector multiplet contribution**

Next we study an analytic property of each positive root contribution:

\[
\prod_{m_1, m_2 \geq 0} \frac{1}{\left( \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1} \right)^2 + (\rho \cdot a)^2}.
\]

(A3)

For this case, it is convenient to use the identity

\[
\left| \Upsilon \left( i \left( x + \frac{ibv}{2} \right) \right) \Upsilon \left( -i \left( x + \frac{ibv}{2} \right) \right) \right|^2
\]

\[
\propto \prod_{k=0}^{\lfloor |v|/2 \rfloor} \prod_{m_2 \geq 0} \sinh \left[ \pi b \left( \pm \alpha \cdot a + ib \left( k - \frac{|v| - 1}{2} \right) \right) \right]
\prod_{m_1, m_2 \geq 0} \left[ \left( \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1} \right)^2 + \left( \pm x + \frac{ibQ}{2} \right)^2 \right]^2.
\]

(A6)

Then the first term cancels the equator contribution and \([A5]\) becomes proportional to

\[
\prod_{m_1, m_2 \geq 0} \left[ \left( \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1} \right)^2 + \left( \pm x + \frac{ibQ}{2} \right)^2 \right]^2,
\]

with \(\alpha \in \Delta\). (A7)

Thus, the vector one-loop determinant does not have any singularities. Note also that this has simple zeroes at

\[
\alpha \cdot a = \frac{ibQ}{2} \pm i \left( m_1 + \frac{|v| + 1}{2} \right) b + \left( m_2 + \frac{1}{2} \right) b^{-1}.
\]

(A8)

Thus the whole one-loop determinant does not have branch cuts and is meromorphic function of \(a\) whose poles are not located on real axis. This point is directly connected to Borel summability along \(\mathbb{R}_+\).

**Appendix B: Details on monopole bubbling effects**

We explain details of the monopole bubbling effect \([11]\). As we mentioned, the sum over \(Y\) in \([31]\) and product over \(s\) in \([32]\) are constrained. \(Y\) in \([31]\) is a set of Young diagrams \((Y = \{Y_1, \cdots, Y_N\})\), and the sum is over all possible configurations of \(p\) boxes distributed to \(Y\). The number of the boxes \(p\) is determined by the dimension of a vector \(K = (K_1, \cdots, K_p)\), which is specified by a following equation \([21]\)

\[
\text{tr} (\alpha^B) = \text{tr} (\alpha^v) + \alpha^{-1} (\alpha - 1)^2 \text{tr} (\alpha^K),
\]

(B1)

where \(\alpha\) is arbitrary element of \(U(1)\) and and the trace is the sum over all components of the vector. Taking \(\alpha = e^{i\epsilon}\) and comparing \(O(\epsilon^2)\), this condition uniquely determines \(p\) as \(p = \frac{1}{2} \text{tr} (B^2 - v^2)\). Especially when \(v = B\), we do not have solutions and therefore the monopole bubbling effect is trivial as mentioned in \([11]\). The products in terms of \(i, j, s\) in \([32]\) have two constraints. One is

\[
v_{i(s)} + s_h - s_v \in \{K_i\}_{i=1}^p,
\]

(B2)

where \(i(s)\) is a gauge group index of Young tableau that \(s\) belongs to. The other constraint depends on the representation. The constraint for vector and adjoint representations is

\[
v_{ij} + L_{Y_j}(s) + A_{Y_i}(s) + 1 = 0,
\]

(B3)

while the one for fundamental representation is

\[
v_{i(s)} - s_v + s_h = 0.
\]

(B4)

In order to find \((B, v)\) for \(SU(N)\), it is convenient to start with the vectors \((\vec{B}, \vec{v})\) satisfying the above constraints for \(U(N)\) and impose the traceless condition:

\[
B = \vec{B} - \frac{1}{N} \sum_{i=1}^N \vec{B}_i, \quad v = \vec{v} - \frac{1}{N} \sum_{i=1}^N \vec{v}_i.
\]

(B5)
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\[
\begin{align*}
\alpha & \rightarrow \Delta \\
\alpha \rightarrow \alpha + \sum_{i<j} \delta_{ij}
\end{align*}
\]

Explicitly, \( \alpha \rightarrow r \alpha \). 

[43] Note that because of \(|v| \leq |B|\), the contributions from the sector \(|v| \neq |B|\) are relatively exponentially suppressed corrections.

[44] See [4, 11] for earlier checks in some examples.

[45] Explicitly, \( \sum_{\alpha} \delta_{\alpha,0} \). 

[46] For example, it is convenient to make the replacements 

\[
\begin{align*}
\pm \hat{x}_1 & \rightarrow 1, \pm \hat{x}_2 \rightarrow 2, \zeta(\ell > 1) \rightarrow 2 \\
1/|m_1 b + m_2 b^{-1}|^2 + x^2 & \rightarrow 1/[|m_1 b + m_2 b^{-1}|^2 + x^2].
\end{align*}
\]

[47] If \( G_p \) is SU, then we insert the traceless constraint by the delta function into \( h_{\ell v} \).

[48] For example, \( k, \ell, v \) stands for \( k = \{k_1, \ldots, k_p\}, \ell = \{\ell_1, \ldots, \ell_p\} \), and \( v = \{v^{(1)}, \ldots, v^{(p)}\} \).

[49] Explicitly, \( \bar{G} \rightarrow G \cdot \alpha \rightarrow \sum_{i<j} \delta_{ij} \delta_{\alpha,0} \).

[50] Note that \( \alpha \cdot \delta \cdot \rho \cdot \gamma \) are integers by Dirac quantization conditions. 

[51] Explicitly, \( \gamma \rightarrow \Delta + \sum_{\alpha} \delta_{\alpha,0} \). 

[52] For example, we can recover the \( \Delta \)-dependence by taking \( r \rightarrow 0 \).

[53] We take \( r \rightarrow 0 \). We can recover \( r \)-dependence by taking \( \Delta \rightarrow \alpha \cdot \delta \cdot \rho \cdot \gamma \).

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