Poisson Wavelets on \(n\)-Dimensional Spheres

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Abstract In this paper, Poisson wavelets on \(n\)-dimensional spheres, derived from Poisson kernel, are introduced and characterized. We compute their Gegenbauer expansion with respect to the origin of the sphere, as well as with respect to the field source. Further, we give recursive formulae for their explicit representations and we show how the wavelets are localized in space. Also their Euclidean limit is calculated explicitly and its space localization is described. We show that Poisson wavelets can be treated as wavelets derived from approximate identities and we give two inversion formulae.

Keywords Poisson wavelets · \(n\)-Spheres

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1 Introduction

Investigation of data on higher dimensional spheres has become more and more important in the last years. Statistical problems, computer vision, medical imaging, quantum chemistry, crystallography are some of the application areas. The most important problem by defining of continuous spherical wavelets is the lack of natural dilation operator. Several constructions have been proposed, each of them based on another idea how to overcome this difficulty.

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The probably most popular is the one based on group-theoretical approach, introduced in [4] (compare also [5] for the two-dimensional case). Dilation is performed in the tangent space to the sphere, onto which the wavelets are mapped via stereographic projection from the south pole. The construction is quite technical, but the wavelets have many nice properties, most of them investigated in the two-dimensional case, e.g., existence of fast algorithms based on FFT and of a directional wavelet transform [2,39,40,47], discrete wavelet frames [1,10,41] (however, the idea can be hardly generalized to $n$-dimensions since the discretization is performed on an equiangular—with respect to spherical variables—grid what causes a concentration of sampling points around the poles). On the other hand, the same wavelets are reintroduced and further developed in a more straightforward way in [46], it is also shown in that paper that the inverse stereographic projection of Euclidean wavelets leads to spherical ones.

Another idea to introduce spherical wavelet transforms is based on the theory of singular integrals and approximate identities [6]. Developed for the two-dimensional sphere in the 1990s [16–20], it was several years later generalized to the three-dimensional [7] and $n$-dimensional case [8,13,14,33]. Approximate identities yield zonal wavelets (the most important examples are Gauss–Weierstrass wavelet and Abel–Poisson wavelet), but in the recent years also the non-zonal case was considered [9,14,33]. The wavelets (under some additional conditions) have Euclidean limit property [33], i.e., for small scales they behave like wavelets over the tangent space, moreover, discrete frames exist [16,19,35], another advantage is their straightforward definition that does not require an extensive theoretical background. On the other hand, unlike in the case of wavelets defined in [4], a whole wavelet family must be given in order to perform dilation.

A very similar concept stays behind the definition utilized by Holschneider and his coworkers [12,30,31]—wavelet transform and wavelet synthesis are given by the same formulae as in the works of Freeden et al. and Bernstein et al., also dilation relies on the parameter choice in a wavelet family. The main difference lies in the background: whereas in the previous approach one derives wavelets from approximate identities and shows that in this case wavelet transform and inverse transform converge for any $L^2$-signal, here a family of functions is called admissible if it proves that the wavelet analysis and synthesis converge. An example of a wavelet family satisfying this definition is Poisson wavelet family introduced in [31], compare also [32], and to the author’s knowledge it is the only one which was investigated or applied. In this paper, we define Poisson wavelets on $n$-dimensional spheres, motivated by the facts that they proved to be useful in computations in the two-dimensional case [12,31] and that discrete frames [36] and directional counterpart exist [29]. However, in order to have a solid theoretical background, we prove that they satisfy the more rigorous definition from [14]. In [35] we show that a wavelet frame exists, i.e., the wavelet transform can be discretized.

Recently, a new class of wavelets was presented. Needlets, introduced by Narcowich et al. in [43] (compare also [42]), have excellent point-wise localization and approximation properties and they yield a tight frame on the sphere. An important difference to the above described constructions is that needlets have a compact spectrum.
Localization results from [43] can be easily adapted to the case of Poisson wavelets, as will be discussed in Sect. 6.

The construction proposed by Geller and Mayeli in [24,25] somehow resembles needlets (differences are discussed in [24, Sect. 1.1]), the wavelets are kernels of the convolution operator \( f(t \Delta^a) \), where \( 0 \neq f \in \mathcal{S}(\mathbb{R}_+) \), \( f(0) \neq 0 \), and \( \Delta^a \) denotes the Laplace–Beltrami operator on a manifold. In the case of the sphere this leads to zonal wavelets of the form

\[
K_{\rho}(\hat{e}, y) = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} f(\rho^2 l(2 \lambda)) \frac{\lambda + l}{\lambda} C^l_\lambda(y) \tag{1}
\]

(compare Sect. 2 for notation). As a particular example the authors investigate the wavelet given by \( f(s) = se^{-s} \). It is worth noting that this wavelet is exactly the linear Gauss–Weierstrass wavelet (indexed by \( \rho^2 \)) from the Freeden–Windheuser theory. A wavelet transform is called linear if the wavelet itself is not needed for reconstruction, otherwise, one has to do with the bilinear wavelet theory (this is the case in [24], compare Proposition 5.4). Thus, one cannot identify these constructions. In [38] the name Mexican needlets is introduced for kernels of the form (1) with \( f(s) = sr e^{-s} \), \( r \in \mathbb{N} \). Bilinear Gauss–Weierstrass could be treated as a Mexican needlet of order \( \frac{1}{2} \) if the theory was extended to rational exponents. Due to excellent localization properties, kernels (1) yield a nearly tight frame [25], however, not a tight one because of uncompactness of their spectra. Statistical properties and applications of Mexican needlets are discussed in [37,38], and their usefulness for characterization of Besov spaces in [23,27].

A generalization of this idea are needlet-type spin wavelets for investigation of sections of line bundles instead of scalar-valued functions [22]. Also in this case, nearly tight frames exist [26], their statistical properties are investigated e.g. in [21].

Unlike Gauss–Weierstrass wavelet, Poisson wavelets are not examples of needlets or wavelets defined in [24,25]. Thus, we cannot utilize results obtained by other authors, especially those concerning localization and frame building. We want to propose these functions as an alternative to other wavelets. A feature that distinguishes Poisson wavelets from other ones is their explicit representation, whereas other properties are comparable to e.g. Mexican needlets.

The paper is organized as follows. Section 2 contains basic information about analysis of functions on spheres. In Sect. 3 we introduce Poisson wavelets as derivatives of Poisson kernel located inside the unit ball, and we state their basic properties, such as Gegenbauer expansion with respect to the origin of the coordinate system. The Gegenbauer expansion of harmonic continuation of Poisson wavelets with respect to the location of the kernel is derived in Sect. 4. In Sect. 5 we give explicit expressions for Poisson wavelets as irrational functions of the first spherical coordinate \( \theta_1 \) (since they are zonal functions) and the distance \( r \) of the source location from the origin of coordinate system. Using them, in Sect. 6 we compute the space localization of wavelets, and using the representation from Sect. 4 we find explicit expressions for the Euclidean limit in Sect. 7. In the end, we show that Poisson wavelets are bilinear (Sect. 8) and linear (Sect. 9) wavelets according to definitions given in [33], and
consequently that they possess all the advantageous properties of these constructions. In particular, we give two inversion formulae for the wavelet transform with respect to Poisson wavelets.

2 Preliminaries

2.1 Functions on the Sphere

By $S^n$ we denote the $n$-dimensional unit sphere in $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma$ normalized so that

$$\Sigma_n = \int_{S^n} d\sigma = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda + 1)},$$

where $\lambda$ and $n$ are related by

$$\lambda = \frac{n - 1}{2}.$$

The surface element $d\sigma$ is explicitly given by

$$d\sigma = \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1} d\theta_1 d\theta_2 \cdots d\theta_{n-1} d\varphi,$$

where $(\theta_1, \theta_2, \cdots, \theta_{n-1}, \varphi) \in [0, \pi]^{n-1} \times [0, 2\pi)$ are spherical coordinates satisfying

$$x_1 = \cos \theta_1,$$
$$x_2 = \sin \theta_1 \cos \theta_2,$$
$$x_3 = \sin \theta_1 \sin \theta_2 \cos \theta_3,$$
$$\cdots$$
$$x_{n-1} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1},$$
$$x_n = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1} \cos \varphi,$$
$$x_{n+1} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1} \sin \varphi.$$

$(x, y)$ or $x \cdot y$ stand for the scalar product of vectors with origin in $O$ and endpoints on the sphere. As long as it does not lead to misunderstandings, we identify these vectors with points on the sphere.

Scalar product of $f, g \in L^2(S^n)$ is defined by

$$\langle f, g \rangle_{L^2(S^n)} = \frac{1}{\Sigma_n} \int_{S^n} f(x) g(x) d\sigma(x),$$

and by $\| \circ \|$ we denote the induced $L^2$-norm.
Gegenbauer polynomials $C^\lambda_l$ of order $\lambda, \lambda \in \mathbb{R}$, and degree $l \in \mathbb{N}_0$, are defined in terms of their generating function

$$
\sum_{l=0}^{\infty} C^\lambda_l(t) r^l = \frac{1}{(1 - 2tr + r^2)^{\lambda}}, \quad t \in [-1, 1], (2)
$$

and they are explicitly given by

$$
C^\lambda_l(t) = \sum_{k=0}^{[l/2]} (-1)^k \frac{\Gamma(l-k+\lambda)}{\Gamma(\lambda) k! (l-2k)!} (2t)^{l-2k}, (3)
$$

cf. [45, Sect. IX.3.1, formula (3)]. They are real-valued and for some fixed $\lambda \neq 0$ orthogonal to each other with respect to the weight function $(1 - r^2)^{\lambda - \frac{1}{2}}$, compare [28, formula 8.939.8].

Let $Q_l$ denote a polynomial on $\mathbb{R}^{n+1}$ homogeneous of degree $l$, i.e., such that $Q_l(az) = a^l Q_l(z)$ for all $a \in \mathbb{R}$ and $z \in \mathbb{R}^{n+1}$, and harmonic in $\mathbb{R}^{n+1}$, i.e., satisfying $\Delta Q_l(z) = 0$, then $Y_l(x) = Q_l(x), x \in S^n$, is called a hyperspherical harmonic of degree $l$. The set of hyperspherical harmonics of degree $l$ restricted to $S^n$ is denoted by $\mathcal{H}_l = \mathcal{H}_l(S^n)$. $\mathcal{H}_l$-functions are eigenfunctions of Laplace–Beltrami operator $\Delta^* := \Delta|_{S^n}$ with eigenvalue $-l(l+2\lambda)$, further, hyperspherical harmonics of distinct degrees are orthogonal to each other. The number of linearly independent hyperspherical harmonics of degree $l$ is equal to

$$
N = N(n, l) = \frac{(n + 2l - 1)(n + l - 2)!}{(n-1)!l!}.
$$

Every $\mathcal{L}^1(S^n)$-function $f$ can be expanded into Laplace series of hyperspherical harmonics by

$$
S(f; x) \sim \sum_{l=0}^{\infty} Y_l(f; x),
$$

where $Y_l(f; x)$ is given by

$$
Y_l(f; x) = \frac{\Gamma(\lambda)(\lambda + l)}{2\pi^{\lambda+1}} \int_{S^n} C^\lambda_l(x \cdot y) f(y) d\sigma(y) = \frac{\lambda + l}{\lambda} \langle C^\lambda_l(x \cdot \cdot), f \rangle.
$$

For zonal functions (i.e., those depending only on $\theta_1 = \langle \hat{e}, x \rangle$, where $\hat{e}$ is the north pole of the sphere $\hat{e} = (1, 0, \ldots, 0)$) we obtain the Gegenbauer expansion

$$
f(\cos \theta_1) = \sum_{l=0}^{\infty} \hat{f}(l) C^\lambda_l(\cos \theta_1)
$$

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with Gegenbauer coefficients
\[
\hat{f}(l) = c(l, \lambda) \int_{-1}^{1} f(t) C_\lambda^l(t) \left(1 - t^2\right)^{-1/2} dt,
\]
where \(c\) is a constant that depends on \(l\) and \(\lambda\). Such functions are identified with functions over the interval \([-1, 1]\), i.e., whenever it does not lead to misunderstandings, we write

\[
f(x) = f(\cos \theta).
\]

For \(f, g \in L^1(S^n)\), \(g\) zonal, their convolution \(f \ast g\) is defined by

\[
(f \ast g)(x) = \frac{1}{\Sigma_n} \int_{S^n} f(y) g(x \cdot y) d\sigma(y).
\]

With this notation we have

\[
Y_l(f; x) = \frac{\lambda + l}{\lambda} \left( f \ast C_\lambda^l \right)(x),
\]
i.e., the function

\[
K_\lambda^l = \frac{\lambda + l}{\lambda} C_\lambda^l
\]
is the reproducing kernel for \(\mathcal{H}_l\).

3 Definition and Basic Properties

In the next three sections, we introduce Poisson wavelets on \(n\)-spheres and prove their basic properties. It is a generalization of ideas from [31,32,34] to the case of \(n\)-dimensional spheres. Our motivation is the fact that zonal Poisson wavelets on \(S^2\) and also their directional counterpart have proven to be of great practical importance and very handful for implementation [12,29,31]. Since in the recent years the investigation of objects in higher dimensions has become popular, we want to propose an analytic tool for their analysis.

Let the Poisson kernel for the unit sphere be given,

\[
p_\zeta(y) = \frac{1}{\Sigma_n} \frac{1 - |\zeta|^2}{|\zeta - y|^{n+1}} = \frac{1}{\Sigma_n} \frac{1 - r^2}{(1 - 2r \cos \theta + r^2)^{(n+1)/2}},
\]
where \(\zeta, y \in \mathbb{R}^{n+1}\),

\[
r = |\zeta| < |y| = 1
\]
and $\theta$ is the angle between the vectors $\zeta$ and $y$, i.e.,

$$r \cos \theta = \zeta \cdot y,$$

compare [44]. Without loss of generality, suppose $\zeta$ lies on the positive $x_1$-axis, i.e.,

$$\zeta = r \hat{e}.$$

In this case, $\theta$ is the $\theta_1$-coordinate of $y$. Since

$$\frac{1 - r^2}{(1 - 2 r \cos \theta + r^2)^{\lambda + 1}} = \left(1 + \frac{r \partial_r}{\lambda}\right) \frac{1}{(1 - 2 r \cos \theta + r^2)^{\lambda}},$$

we obtain by the generating function equation (Eq. (2))

$$p_{\zeta}(y) = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} r^l K^\lambda_l(\cos \theta),$$

(5)

compare also Theorem IV.2.10 and Theorem IV.2.14 in [44]. Note that

$$\Psi^\lambda_x(x) := \frac{1}{\Sigma_n} \frac{1}{(|x|^2 - 2 x \cdot \zeta + |\zeta|^2)^{\lambda}}, \quad x \in \mathbb{R}^{n+1} \setminus \{\zeta\},$$

is the field caused by a monopole inside the unit ball,

$$\Delta \Psi_x = \delta_{\zeta},$$

where $\delta_{\zeta}$ is the Dirac measure located at $\zeta$, consequently,

$$p_{\zeta} = \Psi_x + \frac{1}{\lambda} \Psi^1_x,$$

(6)

where $\Psi^m_x$ denotes the field caused by a multipole,

$$\Psi^m_x = (r \partial_r)^m \Psi_x = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} l^m r^l C^\lambda_l, \quad \Delta \Psi^m_x = (r \partial_r)^m \delta_{\zeta}, \quad \zeta = r \hat{e}.$$

**Definition 3.1** The Poisson wavelet of order $m$, $m \in \mathbb{N}$, at a scale $\rho$, $\rho \in \mathbb{R}_+$, is given recursively by

$$g^1_{\rho} = r \rho^{\rho} \rho_{\rho} \hat{e}, \quad r = e^{-\rho},$$

$$g^{m+1}_{\rho} = r \rho^{\rho} g^m_{\rho},$$

(7)
Lemma 3.2  Gegenbauer expansion of Poisson wavelets is given by

\[ g^m_\rho(y) = \frac{1}{\sum_n} \sum_{l=0}^\infty \frac{\lambda + l}{\lambda} (\rho l)^m e^{-\rho l} C^\lambda_l (\cos \theta). \]

Proof  Use formula (5). \qed

Lemma 3.3  The Poisson wavelet of order m is a sum of fields caused by multipoles,

\[ g^m_\rho = \rho^m \left( \frac{\Psi^m_{r\hat{e}}}{\lambda} + \frac{1}{\lambda} \Psi^{m+1}_{r\hat{e}} \right) \quad (8) \]

Proof  Apply m times the operator \((r \partial_r)\) to the Eq. (6). \qed

4 Harmonic Continuation

A remarkable feature of Poisson wavelets is that they possess a representation as a finite sum of hyperspherical harmonics, however, such centered in the point where the field source (multipole) is located. Moreover, a harmonic continuation exists to functions over the space with the source point excluded.

Proposition 4.1  Poisson wavelets \(g^m_\rho, m \in \mathbb{N},\) can be uniquely harmonically continued to functions over \(\mathbb{R}^{n+1} \setminus \{r\hat{e}\}.\) They are given by

\[ g^m_\rho(x) = \frac{\rho^m}{\sum_n} \sum_{l=0}^{m+1} l! \left( \alpha^m_l + \frac{\alpha^{m+1}_l}{\lambda} \right) e^{-\rho l} \frac{C^\lambda_l (\cos \chi)}{|x - r\hat{e}|^{l+2\lambda}}, \quad (9) \]

where \(r = e^{-\rho},\)

\[ \cos \chi = \frac{x - r\hat{e}}{|x - r\hat{e}|} \cdot \hat{e} \]

and the coefficients \(\alpha^m_l\) are recursively given by

\[ \alpha^0_0 = 1, \]

\[ \alpha^m_0 = 0 \quad \text{for } m \geq 1, \]

\[ \alpha^m_l = 0 \quad \text{for } l > m, \]

\[ \alpha^{m+1}_l = l \alpha^m_l + \alpha^m_{l-1}. \]

The proof is analogous to the proof of [32, Proposition 1].
Proof  Let \( x \in \mathbb{R}^{n+1} \) be different from \( r \hat{e} \). For its distance from \( r \hat{e} \) we obtain the expression

\[
|x - r \hat{e}|^2 = |x|^2 \left| \frac{x - r \hat{e}}{|x|} \right|^2 = |x|^2 \left( \sin^2 \theta + \left( \cos \theta - \frac{r}{|x|} \right)^2 \right)
\]

for \( \hat{x} = \frac{x}{|x|} \) and \( \cos \theta = \hat{x} \cdot \hat{e} \). According to (2) we have

\[
\Sigma_n \Psi_{r \hat{e}}(x) = \frac{1}{|x|^{2\lambda}} \sum_{l=0}^{\infty} \left( \frac{r}{|x|} \right)^l C_l^\lambda \cos \theta,
\]

and therefore

\[
\Sigma_n \partial_r^m \Psi_{r \hat{e}}(x) \bigg|_{r=0} = m! \frac{C_m^\lambda \cos \chi}{|x|^{m+2\lambda}}.
\]

Consequently, for a field caused by a multipole located at \( \zeta = r \hat{e} \) we may write

\[
\Sigma_n \partial_r^m \Psi_{r \hat{e}}(x) = m! \frac{C_m^\lambda \cos \chi}{|x - r \hat{e}|^{m+2\lambda}}.
\]

We put this expression into (8) and obtain the representation (9) for the wavelet \( g^m_\rho \), where the coefficients \( \alpha^m_l \) are defined through

\[
(r \partial_r)^m = \sum_{l=1}^{m+1} \alpha^m_l r^l \partial_r \quad \text{for} \ l \leq m
\]

and

\[
\alpha^m_l = 0 \text{ for } l > m.
\]

The recursive formula follows from (11). \( \square \)

**Proposition 4.2**  The harmonically extended Poisson wavelet has the following expansion around 0:

\[
g^m_\rho(y) = \frac{\rho^m}{\Sigma_n |x|^{2\lambda}} \sum_{l=0}^{\infty} l^m \left( \frac{r}{|x|} \right)^l K_l^\lambda \cos \theta,
\]

\( y \in S^n, \ x \in \mathbb{R}^{n+1}, \ r = e^{-\rho}, \ \cos \theta = \langle \hat{e}, y \rangle. \)
Proof. Apply the operator
\[(r \partial_r)^m + \frac{(r \partial_r)^{m+1}}{\lambda}\]
to the Eq. (10).
\[\Box\]

5 Explicit Expressions

In this section we derive explicit formulae for Poisson wavelets as irrational functions of the first spherical variable. This is one of the features that makes them suited for applications. Note that Gauss–Weierstrass wavelets [16, Sect. 10], respectively Mexican needlets [38], as well as all the discrete wavelets investigated in [16, Sect. 11] are given only as Laplace series.

Proposition 5.1 Poisson wavelets of order \(m \in \mathbb{N}\) are represented by

\[g^m_\rho (y) = \frac{\rho^m}{\sum_n} D_{\lambda+m+1} \sum_{k=0}^m R^m_k (r) \cos k \theta, \quad (12)\]

where

\[D_j = D_j(r, \theta) = \frac{r}{(1 - 2r \cos \theta + r^2)^j}\]

and \(R^m_k\) are polynomials of degree \(2m - k + 1\), explicitly given by

\[R^m_k (r) = \sum_{j=0}^{[(2m-k+1)/2]} a^m_j r^{2j+(k-1)\text{mod}2}, \]

where the coefficients \(a^m_{j,k}\) satisfy the recursion

\[a^m_{j+1,0} = b^m_{j+1,0}, \quad j = 0, \ldots, m + 1, \quad k = 1, \ldots, m,\]

\[a^m_{j+1,k} = b^m_{j+1,k} + c^m_{j+1,k}, \quad j = 0, \ldots, m + 1 - \left\lfloor \frac{k - 1}{2} \right\rfloor,\]

\[a^m_{j+1,m+1} = c^m_{j+1,m+1}, \quad j = 0, \ldots, \left\lfloor \frac{m + 1}{2} \right\rfloor,\]

with

\[a^1_{0,0} = -(n + 3), \quad a^1_{1,0} = n - 1,\]

\[a^1_{0,1} = n + 1, \quad a^1_{1,1} = -(n - 3)\]
\[ b_{m+1,k}^0 = 2 \alpha_0^m, \]
\[ b_{j+1}^m = 2 (j+1) \alpha_j^m + 2 (j-\lambda - m - 1) \alpha_{j-1}^m, \quad j = 1, \ldots, m - k/2, \]
\[ b_{m+k/2}^m = -(2\lambda + k) \alpha_{m-k/2}^m, \]
\[ c_{m+1,k}^0 = 2 (\lambda + m) \alpha_0^m, \]
\[ c_{j+1}^m = 2 (\lambda + m) \alpha_j^m - 2 \cdot 2j \alpha_{j-1}^m \]
\[ = 2 (\lambda + m - 2j) \alpha_{j-1}^m, \quad j = 1, \ldots, m + 1 - k/2, \]

for an even \( k \) and

\[ b_{m+1,k}^0 = 2 \alpha_0^m, \]
\[ b_{j+1}^m = 2 (j+1) \alpha_j^m + 2 (j-\lambda - m - 3) \alpha_{j-1}^m, \quad j = 1, \ldots, m - [k/2], \]
\[ b_{m+[k/2]}^m = -(2\lambda + 2m + 1) \alpha_{m-[k/2]}^m + (2m - k + 1) \alpha_{m-[k/2]}^m \]
\[ = -(2\lambda + k) \alpha_{m-[k/2]}^m, \]
\[ c_{m+1,k}^0 = 0, \]
\[ c_{j+1}^m = 2 (\lambda + m - 2j + 1) \alpha_{j-1}^m, \quad j = 1, \ldots, m + 1 - [k/2], \]

for an odd \( k \).

The proof for the two-dimensional case is given in [34].

Proof For \( m = 1 \) we compute the wavelet as \( \rho r \partial_r \rho_r \hat{e} \) with \( \rho_r \hat{e} \) given by (4) and obtain

\[ g^1_\rho(y) = \frac{\rho r}{\sum_n (1 - 2r \cos \theta + r^2)^{(n+3)/2}} \cdot \left[ \left( - (n + 3)r + (n - 1)r^3 \right) + \left( (n + 1) - (n - 3)r^2 \right) \cos \theta \right]. \]

Suppose, \( g^m_\rho \) has the representation (12). Since

\[ \partial_r D_j = \frac{[1 - (2j - 1)r^2 + 2(j-1)r \cos \theta]}{r} \cdot D_{j+1}, \]

for \( m + 1 \) we obtain by (7)

\[ \frac{\sum_n}{\rho^{m+1}} g_{m+1}^m \cdot D_{\lambda + m + 2} \left[ 1 - (2\lambda + 2m + 1)r^2 + 2(\lambda + m)r \cos \theta \right] \sum_{k=0}^{m} R_k^m(r) \cos^k \theta \]

\[ + \quad D_{\lambda + m + 2} \left[ 1 + r^2 - 2r \cos \theta \right] r \sum_{k=0}^{m} R_k^m(r) \cos^k \theta. \]

\[ \text{Birkhäuser} \]
Collecting the powers of $\cos \theta$ leads to

$$\sum_{n} \rho^{m+1} \delta_{n}^{m+1} = D_{\lambda+m+2} \left[ B_{0}^{m+1} + \sum_{k=1}^{m} (B_{k}^{m+1} + C_{k}^{m+1}) \cos^{k} \theta + C_{m+1}^{m+1} \cos^{m+1} \theta \right],$$

where the coefficients

$$B_{k}^{m+1} = B_{k}^{m+1}(r) = \sum_{j=0}^{m+1-\left\lfloor \frac{k-1}{2} \right\rfloor} b_{j}^{m+1,k} r^{2j+(k-1) \mod 2}, \quad k = 0, \ldots, m,$$

and

$$C_{k}^{m+1} = C_{k}^{m+1}(r) = \sum_{j=0}^{m+1-\left\lfloor \frac{k-1}{2} \right\rfloor} c_{j}^{m+1,k} r^{2j+(k-1) \mod 2}, \quad k = 1, \ldots, m+1,$$

are given by

$$B_{k}^{m+1} = \left(1 - (2\lambda + 2m + 1)r^{2}\right) R_{k}^{m}(r) + (1 + r^{2}) r R_{k}^{m,r}(r),$$

$$C_{k}^{m+1} = 2 (\lambda + m) r R_{k}^{m}(r) - 2r^{2} R_{k}^{m,r}(r).$$

The derivative of $R_{k}^{m}$ is equal to

$$R_{k}^{m,r}(r) = \sum_{j=0}^{m-k/2} (2j + 1) a_{j}^{m,k} r^{2j}$$

for an even $k$ and

$$R_{k}^{m,r}(r) = \sum_{j=1}^{m-k/2} 2j a_{j}^{m,k} r^{2j-1},$$

for an odd $k$. Therefore, the for the coefficients of the polynomials $B_{k}^{m+1}$ and $C_{k}^{m+1}$ we obtain the formulae:

$$b_{0}^{m+1,k} = a_{0}^{m,k} + a_{0}^{m,k} = 2 a_{0}^{m,k},$$

$$b_{j}^{m+1,k} = a_{j}^{m,k} - (2\lambda + 2m + 1) a_{j-1}^{m,k} + (2j + 1) a_{j}^{m,k} + (2j - 1) a_{j-1}^{m,k} = 2 (j + 1) a_{j}^{m,k} + 2 (j - \lambda - m - 1) a_{j-1}^{m,k}, \quad j = 1, \ldots, m - k/2,$$

$$b_{m+1-k/2}^{m+1,k} = -(2\lambda + 2m + 1) a_{m-k/2}^{m,k} + (2m - k + 1) a_{m-k/2}^{m,k} = -(2\lambda + k) a_{m-k/2}^{m,k}. $$

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\begin{align*}
    c_{0}^{m+1,k} &= 2(\lambda + m) a_{0}^{m,k-1}, \\
    c_{j}^{m+1,k} &= 2(\lambda + m) a_{j}^{m,k-1} - 2 \cdot 2 j a_{j}^{m,k-1} \\
        &= 2(\lambda + m - 2j) a_{j}^{m,k-1}, \quad j = 1, \ldots, m + 1 - k/2,
\end{align*}

for an even \(k\) and

\begin{align*}
    b_{0}^{m+1,k} &= a_{0}^{m,k}, \\
    b_{j}^{m+1,k} &= a_{j}^{m,k} - (2\lambda + 2m + 1) a_{j-1}^{m,k} + 2 j a_{j}^{m,k} + 2 (j - 1) a_{j-1}^{m,k} \\
        &= (2j + 1) a_{j}^{m,k} - 2j a_{j-1}^{m,k}, \quad j = 1, \ldots, m - [k/2], \\
    b_{m+1-[k/2]} &= -(2\lambda + 2m + 1) a_{m-[k/2]}^{m,k} + (2m - k + 1) a_{m-[k/2]}^{m,k} \\
        &= -(2\lambda + k) a_{m-[k/2]}^{m,k}, \\
    c_{0}^{m+1,k} &= 0, \\
    c_{j}^{m+1,k} &= 2(\lambda + m) a_{j-1}^{m,k-1} - 2 \cdot (2j - 1) a_{j-1}^{m,k-1} \\
        &= 2(\lambda + m - 2j + 1) a_{j-1}^{m,k-1}, \quad j = 1, \ldots, m + 1 - [k/2],
\end{align*}

for an odd \(k\).

\hfill \Box

\section{Localization of the Wavelets}

Although quite useful in applications, Poisson wavelets seem not to have a solid theoretical base. The definition of spherical wavelets, introduced by Holschneider in [30] has two weaknesses that we want to point out. The first one is the \textit{ad hoc} choice of the scale parameter, an objection that has been made by many authors, see e.g. [3–5,11,15,39]. The other one is the fact that wavelets are not defined intrinsically. In the case of zonal wavelets, it is explicitly stated that the wavelet reconstruction formula is valid \textit{whenever the integral makes sense}, cf. [31, Sect. 2.2.2].

However, we are able to show (cf. Sects. 8 and 9) that the family of Poisson wavelets satisfies stronger conditions of definition from [14] or [33], being a generalization of definitions based on spherical singular integrals and approximate identities and used by many authors over the last two decades. In order to do it, we need to show that the wavelets are localized in space. More exactly, the following inequality

\[
    \rho^n g_{\rho}^{m} (\cos(\rho \theta)) \leq \frac{c \cdot e^{-\rho}}{\theta^{m+n}}, \quad \theta \in \left[0, \frac{\pi}{\rho}\right],
\]

holds uniformly in \(\rho\) for a constant \(c\). The lemmas, the theorem, and the corollary in this section are analogous and they can be proven along the same lines as those in [36, Sect. 5].
Lemma 6.1 Let $Q_m$, $m \in \mathbb{N}$, be a sequence of polynomials in two variables satisfying the recursion

$$
Q_{m+1}(r, t) = A_m(r, t) \cdot Q_m(r, t) + B(r, t) \cdot \frac{\partial}{\partial r} Q_m(r, t)
$$

with

$$
A_m(r, t) = 1 - (\alpha + 1)r^2 + \alpha rt \quad \text{for some positive } \alpha
$$

and

$$
B(r, t) = (1 + r^2 - 2rt) r,
$$

and such that

$$
Q_1(1, 1) = 0, \quad \text{and} \quad \frac{\partial}{\partial r} Q_1(r, 1) \bigg|_{r=1} \neq 0.
$$

Then the polynomial $Q_m(1, \cdot)$, $m \geq 2$, has an $\lceil (m + 1)/2 \rceil$-fold root in 1.

Lemma 6.2 Let $\{f_d\}$ be a family of functions over $(0, 1) \times [0, \pi]$ given by

$$
f_1(r, \theta) = \frac{r Q_1(r, \cos \theta)}{(1 + r^2 - 2r \cos \theta)^{1+\lambda}},
$$

$$
f_m+1(r, \theta) = r \frac{\partial}{\partial r} f_m(r, \theta),
$$

where $Q_1$ is a polynomial satisfying (15) and $\lambda$ is a positive integer or half-integer. Then, for any $k \geq 2[m/2] + 2\lambda$ there exists a constant $c$ such that

$$
|f_m(r, \theta)| \leq c \cdot \frac{r}{\theta^k}, \quad \theta \in (0, \pi],
$$

uniformly in $r$. For $m \geq 2$, the number $2[m/2] + 2\lambda$ is the smallest possible exponent $k$. If $Q_1(1, y)$ has a simple root in 1, then $2\lambda$ is the smallest possible exponent $k$ on the right-hand-side of (16) for $m = 1$.

Remark 1 For $\lambda = \frac{1}{2}$ the recursion (13) holds for $A_m$ given by (14) with $\alpha = 2m - 1$, i.e., the first statement in the proof of [36, Lemma 3] is not correct. However, it does not affect the multiplicity of the root of $E_d(1, \cdot)$ and, consequently, the forthcoming considerations.

Remark 2 In the proof of [36, Lemma 5], the function $F$ for $\lambda = 0$ should be defined by $F(0, \theta) = \theta^k \sin \theta E_d(0, \cos \theta)$ in order to be continuous on $[0, 1] \times [0, \pi]$.

Theorem 6.3 Let

$$
\Psi_{re}^m = \frac{1}{\Sigma_{l=0}^{\infty}} l^m r^l C_l^\lambda, \quad m \in \mathbb{N}_0,
$$
be the field on the sphere caused by the multipole (monopole for \( m = 0 \)) \( \mu = (r \partial_r)^m \delta_{r \hat{e}} \). For any \( k \geq 2 \left[ \frac{m}{2} \right] + 2 \lambda \) there exists a constant \( c \) such that

\[
|\Psi_r^m(r \hat{e}, \cos \theta)| \leq c \cdot \frac{r}{\theta^k}, \quad \theta \in (0, \pi],
\]

uniformly in \( r \). \( 2 \left[ \frac{m}{2} \right] + 2 \lambda \) is the smallest possible exponent on the right-hand-side of this inequality.

**Corollary 6.4** For any \( k \geq 2 \left[ \frac{m+1}{2} \right] + 2 \lambda \) there exists a constant \( c \) such that

\[
|g_\rho^m(\cos \theta)| \leq c \cdot \frac{\rho^m e^{-\rho}}{\theta^k}, \quad \theta \in (0, \pi],
\]

uniformly in \( r \). \( 2 \left[ \frac{m+1}{2} \right] + 2 \lambda \) is the smallest possible exponent on the right-hand-side of this inequality.

**Theorem 6.5** Let \( g_\rho^m \) be the Poisson wavelet of order \( m \). Then there exists a constant \( c \) such that

\[
|\rho^n g_\rho^m(\cos(\rho \theta))| \leq c \cdot \frac{e^{-\rho}}{\theta^{m+n}}, \quad \theta \in \left(0, \frac{\pi}{\rho}\right],
\]

uniformly in \( \rho \). \( m + n \) is the largest possible exponent in this inequality.

**Corollary 6.6** The functions \( (\rho, \theta) \mapsto \rho^n g_\rho^m(\cos \theta) \) are bounded by \( c \cdot e^{-\rho} \) uniformly in \( \theta \), and \( n \) is the smallest possible exponent.

**Remark** Similar results were obtained in [43] for kernels of the form

\[
K_{\rho} = \sum_{l=0}^{\infty} \kappa(\rho(l + \lambda)) \mathcal{K}_l, \quad \rho \in (0, 1],
\]

where \( \kappa \in C^k(\mathbb{R}) \), \( k \geq \max\{2, n - 1\} \), is an even function with

\[
|\kappa^{(r)}(t)| \leq C_{\kappa}(1 + |t|)^{-\alpha} \quad \text{for all } t \in \mathbb{R}, \ r = 0, \ldots, k,
\]

where \( \alpha > n + k \) and \( C_{\kappa} > 0 \) are fixed constants. Estimations on \( \kappa(\rho(l + \lambda)) \) derived in [43, Sect. 3.1] are valid also for \( \kappa(\rho l) \), however, for low values of indices of Poisson wavelets, i.e. \( m < n \), it is impossible to extend the generating function of the wavelet \( g_\rho^m \) to an even \( C^k(\mathbb{R}) \)-function with \( k = n - 1 \), and the evenness of \( \kappa \) plays an important role in the proof of estimations on (17). Therefore, an adaptation of the results from [43] to Poisson wavelets would require some further investigations.

Analogous estimations for kernels of \( f(\rho^2 \Delta^*) \), \( f(t) = t f_0(t), \ f_0 \in \mathcal{S}(\mathbb{R}_+) \), can be found in [24]. An application of the methods used there to Poisson wavelets would require a replacement of the Laplace–Beltrami operator by a particular first-order
pseudodifferential operator which is similar to \( \Delta^s \) and multiplies spherical harmonics of degree \( l \) by \( l^2 \) instead of \( l(l + 2\lambda) \), compare discussion in [24, Sect. 1.1]. In the proof estimations on the kernel of a pseudodifferential operator derived from \( \Delta^s \) are used, thus, the results cannot be applied to Poisson wavelets without detailed calculations.

Advantages of the approach presented here are big-scale-decay and sharper estimations for small \( \theta \) for even \( m \) (in this case, the minimal difference between the exponents in the denominator and in the numerator in Corollary 6.4 is equal to \( n - 1 \), whereas in [43] and in [24] it is equal to \( n \)). Moreover, the results are obtained with elementary analytical tools. On the other hand, estimations from the papers of Narcowich et al. admit a wider range of exponents (in the numerator), and the constants are given explicitly. In [24] also bounds for derivatives of the kernel are given.

7 Euclidean Limit

It is easy to verify that normalized Poisson wavelets satisfy conditions of [33, Theorem 3.4]. Consequently, they have the Euclidean limit property, i.e., there exists a function \( f : [0, \infty) \rightarrow \mathbb{C} \), square integrable with respect to \( x^{n-1} \, dx \), and such that

\[
\lim_{\rho \to 0} \rho^n g^m(\rho, \xi) = g^m(|\xi|),
\]

holds point-wise for every \( \xi \in \mathbb{R}^n \), where \( S^{-1} \) denotes the inverse stereographic projection. This means that the wavelets for small scales behave like wavelets over the Euclidean space, a feature that shows that the sphere is locally almost flat.

**Theorem 7.1** The Euclidean limits of Poisson wavelets are given by

\[
g^m(|\xi|) = \frac{1}{\Sigma_n \lambda} (m + 1)! C^\lambda_{m+1} \left(1/\sqrt{1+|\xi|^2}\right) \left(1+|\xi|^2\right)^{(m+n)/2}. \tag{18}
\]

**Proof** Analogous to the proof of [32, Theorem 1]. \( \Box \)

**Proposition 7.2** The functions \( g^n \) decay at infinity polynomially with degree

\[
m + n + (m + 1)_{\text{mod} \, 2}.
\]

**Proof** For an even \( m \) the lowest power of the argument in the polynomial \( C^\lambda_{m+1} \) is equal to 1, compare (3), consequently the behavior of the numerator in (18) for \( |\xi| \to \infty \) is governed by \( |\xi|^{-1} \). In the case of an even \( m \), the function \( C^\lambda_{m+1} \left(1/\sqrt{1+|\xi|^2}\right) \) behaves in infinity like a constant. For an alternative proof compare [32, Corollary 1, Sect. 4]. \( \Box \)

**Proposition 7.3** The functions \( g^m \) are of zero mean with respect to the measure

\[
d\nu(|\xi|) = \frac{4 (4|\xi|^2)^{2\lambda}}{(4+|\xi|^2)^{2\lambda+1}} d|\xi|.
\]
Proof Analogous to the proof of [34, Proposition 4].

8 The Bilinear Wavelet Transform

The most comprehensive work about wavelets derived from approximate identities, however only for the two-dimensional sphere, is [16]. Once Theorem 8.2 is proven, we know that Poisson wavelets have all the nice properties of bilinear wavelets discussed in [16, Sect. 10], such as least squares property or scale discretization of different types (their generalization to higher dimensions is straightforward). The same holds for the fully discretized wavelet transform, note, however, that for Poisson wavelets also another discretization is possible [35], where the main difference between the two approaches lies in the way how scale discretization is performed.

For our $n$-dimensional case we shall use the definition from [14, Sect. 5.2] (with weight $\alpha(\rho) = \frac{1}{\rho}$). Since Gegenbauer coefficients of Poisson wavelets satisfy

$$
\hat{g}^m_\rho(l) = \frac{\lambda + l}{\lambda} \psi(\rho l)
$$

for function $\psi(u) = u^m e^{-u}$, we need to verify whether conditions of the following definition are satisfied.

Definition 8.1 A family

$$\left\{ \Psi_\rho = \sum_{l=0}^{\infty} \psi(\rho l) K^\lambda_l \right\}_{\rho \in \mathbb{R}_+^+}
$$

of $L^1(S^n)$-functions is called spherical wavelet if it satisfies the following admissibility conditions:

(1) for $l \in \mathbb{N}_0$

$$\int_0^\infty |\psi(u)|^2 \frac{du}{u} = 1,
$$

(2) for $R \in \mathbb{R}_+$

$$\int_{-1}^1 \left| \int_R^\infty (\overline{\psi}_\rho \ast \psi_\rho)(t) \frac{d\rho}{\rho} \right| \left(1 - t^2\right)^{-1/2} dt \leq c \tag{20}
$$

with $c$ independent of $R$.

Theorem 8.2 Normalized Poisson wavelets

$$G^m_\rho = \sum_{l=0}^{\infty} \psi_m(\rho l) K^\lambda_l
$$
with
\[ \psi_m(u) = \frac{2^m}{\sqrt{\Gamma(2m)}} u^m e^{-u} \]
are wavelets in the sense of Definition 8.1.

In the proof a property of Poisson wavelets is imposed, which we would like to state on its own.

**Lemma 8.3** Poisson wavelets of order \( m \in \mathbb{N}_0 \) satisfy
\[ \int_{-1}^{1} \left| g^m_{\rho}(t) \right| \left( 1 - t^2 \right)^{\lambda - 1/2} dt \leq c e^{-\rho} \]
with a constant \( c \) independent of \( \rho \).

**Proof** We change the variables \( t = \cos \theta \) and divide the integration interval into two parts: \( 0 \leq \theta \leq a \) and \( a \leq \theta \leq 1 \). For small values of \( \theta \) we use the estimation from Corollary 6.6, and for big values of \( \theta \) the estimation from Theorem 6.5. Since \( \sin \theta \leq \theta \), we have
\[
\int_{0}^{\pi} |g(\cos \theta)| \sin^{2\lambda} \theta d\theta \leq \int_{0}^{a} \frac{\xi e^{-\rho}}{\rho^{2\lambda+1}} \theta^{2\lambda} d\theta + \int_{a}^{1} \frac{\xi e^{-\rho}}{\rho^{2\lambda+1}} \theta^{m+2\lambda+1} d\theta \leq c e^{-\rho}.
\]
\[ \square \]

**Proof of Theorem 8.2** \( \psi_m \) is normalized such that (19) holds. In order to prove that (20) is satisfied, note that by the reproducing property of \( K_\lambda^l \)
\[
G^m_{\rho} \ast G^m_{\rho} = \sum_{l=0}^{\infty} |\psi_m(\rho l)|^2 K_\lambda^l,
\]
and further
\[
\int_{R} \left( G^m_{\rho} \ast G^m_{\rho} \right) \frac{d\rho}{\rho} = \sum_{l=0}^{\infty} \varphi_m(Rl) K_\lambda^l \tag{21}
\]
with
\[
\varphi_m(Rl) = \int_{R} |\psi_m(\rho l)|^2 \frac{d\rho}{\rho} = W_m(Rl) e^{-2Rl},
\]
where \( W_m \) is a polynomial (of degree \( 2m - 1 \)). Consequently, (21) is a weighted sum of Poisson wavelets at the scale \( 2R \), and according to the Lemma 8.3, it is uniformly \( L^1 \)-bounded with respect to the weight \( (1 - t^2)^{\lambda - 1/2} \), as required. \[ \square \]
Remark

\[ G^m_\rho = \frac{2^m \Sigma_n}{\sqrt{\Gamma(2m)}} g^m_\rho. \]

Consequently, all the definitions and theorems from [14] and [33] concerning bilinear wavelets apply to Poisson wavelets. In particular, the wavelet transform is given by

\[ W^m f(\rho, x) = \langle G^m_\rho, x, f \rangle = f * G^m_\rho(x), \]

where \( g^m_{a,x} \) denotes the wavelet rotated to \( x \) and \( f \in \mathcal{X}(\mathbb{S}^n) \), and it can be inverted in \( \mathcal{X} \)-sense by

\[ f(x) = \frac{1}{\Sigma_n} \int_0^\infty \int_{\mathbb{S}^n} W^m f(\rho, y) G^m_{\rho,y}(x) \frac{d\sigma(y) d\rho}{\rho}, \tag{22} \]

where \( \mathcal{X} \) denotes the space of \( \mathcal{C} \) or \( \mathcal{L}^p, 1 \leq p < \infty \). Further, the image of the bilinear wavelet transform with respect to Poisson wavelets is a reproducing kernel Hilbert space with scalar product

\[ \langle F, G \rangle_{\mathcal{L}^2(\mathbb{S}^n \times \mathbb{R}_+)} = \frac{1}{\Sigma_n} \int_0^\infty \int_{\mathbb{S}^n} F(\rho, x) G(\rho, x) \frac{d\sigma(x) d\rho}{\rho} \]

and with reproducing kernel given by

\[ \Pi^m(\rho, x; \tau, y) = \frac{\sqrt{\Gamma(4m)}}{\Gamma(2m)} \frac{(\rho \tau)^m}{(\rho + \tau)^{2m}} G^{2m}_{\rho^2+\tau}(x \cdot y). \]

Note that Abel–Poisson wavelet introduced in [20] can be treated as the Poisson wavelet of order 1/2.

9 Linear Wavelet Transform

In a similar way we prove that Poisson wavelets yield linear wavelet analysis. Let us recall Definition 4.1 from [33].

**Definition 9.1** Let \( \{\Psi^L_\rho\}_{a \in \mathbb{R}_+} \) be a family of zonal \( \mathcal{L}^2 \)-functions such that the following admissibility conditions are satisfied:

1. for \( l \in \mathbb{N}_0 \)

\[ \int_0^\infty \frac{\Psi^L_\rho(t)}{\rho} \frac{d\rho}{\rho} = \frac{\lambda + l}{\lambda}, \]

2. for \( R \in \mathbb{R}_+ \)

\[ \int_{-1}^1 \left| \int_R^\infty \frac{\Psi^L_\rho(t)}{\rho} \frac{d\rho}{\rho} \right| \left( 1 - t^2 \right)^{\lambda-1/2} dt \leq c \]

with \( c \) independent of \( R \).
Then \( \{ \Psi^L_{\rho} \}_{\rho \in \mathbb{R}^+} \) is called a spherical linear wavelet. The associated wavelet transform is defined by
\[
W^L_{\Psi} f(\rho, x) = \left( \Psi^L_{\rho, x}, f \right)_{L^2(S^n)} = (f \ast \overline{\Psi^L_{\rho}})(x).
\]
for all \( f \in L^2(S^n) \).

**Theorem 9.2** Normalized Poisson wavelets

\[
\tilde{G}^m_{\rho} = \sum_{l=0}^{\infty} \gamma_m(\rho l) K_l^2
\]

with
\[
\gamma_m(t) = \frac{1}{\Gamma(m)} t^m e^{-t}
\]
are linear wavelets in the sense of Definition 9.1.

**Proof** Analogous to the proof of Theorem 8.2. \(\square\)

**Remark**

\[
\tilde{G}^m_{\rho} = \frac{\Sigma_n}{\sqrt{\Gamma(m)}} g^m_{\rho},
\]

Note that the wavelet transform is the same as in the bilinear case. The inversion formula, however, is given by
\[
f(x) = \frac{1}{\Sigma_n} \int_0^{\infty} W^L_{\lambda} f(\rho, x) \frac{d\rho}{\rho}
\]
in \( C \) and \( L^p \), \( 1 \leq p < \infty \), sense. This means that the wavelet transform with respect to Poisson wavelets defined in Sect. 3 can be inverted with any of the formulae (22) respectively (23) with constant \( \frac{1}{\Sigma_n} \) replaced by
\[
\frac{4^m \Sigma_n}{\Gamma(2m)} \text{ respectively } \frac{1}{\Gamma(m)}.
\]

Further, note that Abel–Poisson L-wavelet introduced in [20] is the Poisson wavelet of order 1.

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