Designing a Competitive Monotone Signaling Equilibrium∗

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Abstract

In this paper we consider a generalized competitive signaling model with two-sided matching. A decision maker (DM) sets the support of reactions that receivers can choose before senders and receivers sequentially choose their actions and reactions. Adopting the proposed methodology, the DM can build the optimal design of a unique stronger monotone signaling equilibrium, which maximizes the aggregate net surplus. Our analysis sheds light on how the trade-off between matching efficiency and signaling costs affects optimal equilibrium designing. We further clarify how the trade-off depends on the relative heterogeneity between receiver and sender types and on the (direct) productivity effect of the sender’s action. Specifically, the DM’s equilibrium design is most effective (i) when the receiver type distribution has the smallest mean and variance; and (ii) when the sender’s action has no productivity effect. The DM’s equilibrium design quickly loses its effectiveness as the mean/variance of the receiver type distribution increases or the productivity effect of the sender’s action increases.

Keywords: optimal signaling equilibrium design, matching efficiency, stronger monotone beliefs, firm size

JEL classification codes: D82, D86

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1 Introduction

We study an equilibrium design problem faced by the decision maker (DM) (e.g., a government or a policy maker) who can choose the set of feasible reactions before senders and receivers move in a generalized competitive signaling model with two-sided matching. We use the term, “equilibrium design” because the DM chooses the set of feasible reactions and it affects the endogenous formation of the belief on the sender’s type. In our model, there is a continuum of heterogeneous senders and receivers (e.g., sellers and buyers, workers and firms, and entrepreneurs and investors) in terms of their types. The DM is interested in maximizing the aggregate net surplus. She moves first by publicly announcing the set of feasible reactions that receivers can take. After that, senders take actions, followed by receivers’ reaction choices as they are matched with senders. For example, the policy maker may announce the set of feasible transfers that firms can make to their employees. After that but prior to entering the job matching market, workers (senders) make investments in observable characteristics such as education. Once firms and workers enter the market, they form one-to-one matches as a firm offers its employee a wage (reaction). In a competitive signaling equilibrium, the market wage function that specifies a worker’s wage conditional on her observable characteristics clears the matching market.

Signaling creates a trade-off in matching markets. It increases matching efficiency because separating induces assortative matching. On the other hand, it is costly in that senders need to choose inefficiently high levels of equilibrium actions in order to separate themselves. Because of the trade-off, there may be efficiency gains if the DM restricts the set of feasible reactions to prevent a separating equilibrium from happening in the first place. How can the DM find the optimal set of feasible reactions and what the optimal signaling equilibrium would look like as a result? We first provide a general methodology that the DM can use in designing the optimal signaling equilibrium.

Given the multiplicity of signaling equilibrium, the DM focuses on a stronger monotone signaling equilibrium where equilibrium actions, reactions, beliefs, and matching are monotone in the stronger set order (Shannon (1995)). We show that when utility functions satisfy monotonicity and single crossing properties, any competitive signaling equilibrium is stronger monotone if and only if it passes Criterion D1 (Cho and Kreps (1987), Cho and Sobel (1990), Banks and Sobel (1987)). The stronger monotonicity of beliefs makes it easy to derive any type of a stronger monotone equilibrium given any set of feasible
reactions chosen by the DM, even when a separating equilibrium does not exist.\footnote{The stronger monotonicity of beliefs is the full implication of Criterion D1 on beliefs. When no restrictions are imposed on feasible reactions, only a partial implication (e.g., Cho and Sobel monotonicity (1990)) is needed to show that a separating equilibrium is a unique D1 equilibrium. See Section 4 for more discussion.}

When utility functions are quasilinear, the DM can only focus on intervals as the set of feasible reactions that she chooses without loss of generality.\footnote{We allow the lower and upper bounds of the interval the DM chooses to be the same. In this case, the interval shrinks to a singleton.} We show that given any interval of feasible reactions that the DM may choose, a stronger monotone equilibrium is \textit{unique} and \textit{well-behaved}. A “well-behaved” equilibrium is characterized by the two threshold sender types. The lower threshold sender type specifies the lowest sender type who enters the market, whereas any sender above the upper threshold sender type pools their actions. Any sender between the two threshold types separates themselves. If the two threshold types are the same, it becomes a pooling equilibrium. If the upper threshold type is the supremum of the sender types and greater than the lower threshold type, it becomes a separating equilibrium. If the upper threshold type is less than the supremum of the sender types but greater than the lower threshold type, separating and pooling coexist in the well-behaved equilibrium. In the separating part of the equilibrium, matching is \textit{ assortative} in terms of sender action and receiver type (and hence in terms of sender type and receiver type), whereas in the pooling part, it is \textit{random}.

The aggregate net surplus is a function of the two threshold sender types in a unique stronger monotone equilibrium. We further show that choosing the two threshold sender types is equivalent to choosing the lower and upper bounds of the corresponding interval of feasible reactions in the sense that we can uniquely retrieve the two bounds of the reaction interval from any given two threshold sender types. Therefore, DM’s design problem for an optimal stronger monotone equilibrium comes down to the choice of the two threshold sender types.

For the optimal equilibrium design, we propose an approach that approximates the distribution of receiver types by two parameters, which are called “shift” ($k$) and “relative spacing” ($q$), given an arbitrary distribution of sender types. The relative heterogeneity of receiver types to sender types is parametrized by $q$. We first prove that regardless of the sender’s type distribution, there exists a range of feasible reactions which induces a non-separating stronger monotone equilibrium that is more efficient than the stronger monotone separating equilibrium when the relative heterogeneity of receiver types to sender types and the productivity of sender action are small. It implies that the unique (separating) equilibrium without any restrictions on feasible reactions is not optimal in
the classical Spencian model of pure signaling with no heterogeneity of receivers (Spence 1973).

For numerical analysis, we choose various beta distributions for sender type such that, given any sender type distribution, the mean and variance of the receiver type distribution increases as $q$ increases. This is particularly relevant to the recent empirical findings in Poschke (2018) who documents that the mean and variance of the firm size distribution are larger in rich countries and increased over time for US firms. For the concreteness, one may think of senders as workers and receivers as firms. A firm’s type is then its size, which can be measured by the amount of labour employed by it or its market value if it is publicly traded. A sender’s type can be her unobservable skill as a worker,\(^3\) whereas her action is her observable skill. We parametrize the (direct) productivity effect of sender action on generating the gross match surplus. We allow this productivity parameter to vary.

In a wide range of parameter values, the optimal stronger monotone equilibrium is strictly well-behaved in that it has a separating part below the upper threshold type and a pooling part above it. This implies that in the pooling part of the equilibrium, the cost savings associated with the pooled action choice by senders above the upper threshold type outweighs the decrease in matching efficiency due to random matching. The lower threshold type in the optimal stronger monotone equilibrium is always equal to the minimum of the support of the sender type distribution. As the mean/variance of the receiver type distribution (e.g., firm size distribution) or the productivity effect of sender action (e.g., observable skill) increases, the pooling part on the top in the optimal stronger monotone equilibrium is getting smaller (i.e., the upper threshold type increases).

Furthermore, as the mean and variance of the receiver type distribution increases (i.e., $q$ increases), the efficiency of the baseline separating equilibrium increases\(^4\) and the relative net gain in the optimal stronger monotone equilibrium decreases, regardless of the productivity effect of sender action and the sender type distribution (e.g., unobservable skill distribution). In addition, as the productivity effect of sender action increases, the rate of the increase in the efficiency of the baseline separating equilibrium with respect to an increase in $q$ is getting smaller and the relative surplus gain in the optimal stronger monotone equilibrium is smaller at every value of $q$.

\(^3\)In an entry-level job market, a worker’s unobservable skill could be her ability to understand a task given to her and to figure out how to complete it. In a managerial job market, a worker’s unobservable skill could be her ability to come up with new business idea or innovation.

\(^4\)This may suggest that the efficiency in rich countries is higher than in poor countries and that it also increases over time in the U.S., given the empirical findings in Poschke (2018).
Our result suggests that DM’s equilibrium design is most effective (i) when the receiver type distribution has the smallest mean and variance; and (ii) when sender action has no productivity effect. As the mean and variance of the receiver type distribution increase or the productivity parameter of sender action increases, the DM’s equilibrium design quickly loses its effectiveness. This highlights (i) how a trade-off between matching efficiency and the economic cost of signaling changes as the firm size distribution becomes more spread and the direct productivity of observable skill increases; and (ii) how does it impact on optimal equilibrium design.

Related literature Our paper opens a new research direction to the stronger monotone equilibrium design and contributes to the literature on several fronts. While the literature has studied monotone equilibrium, exploring complementarities between actions and types, they mostly focus on games with simultaneous moves and no signaling (Athey (2001), McAdams (2003), Reny and Zamir (2004), Reny (2011), Van Zandt and Vives (2007)). Our Stronger Monotone Signaling Equilibrium Theorem is the first fully-fledged monotone equilibrium theorem in a model with sequential moves and signaling.

Recently, Liu and Pei (2020) derive the monotonicity of a sender’s equilibrium strategy in a two-period signaling game between one sender and one receiver with an assumption similar to our assumption on the sender’s utility function. However, our paper differs from theirs because ours shows (i) the equivalence between Criterion D1 and stronger monotone beliefs and its implication and (ii) the monotonicity of equilibrium matching given a monotonicity and a single crossing assumption on the receiver’s utility. Mensch (2020) shows the existence of an equilibrium where players’ strategies and beliefs are both monotone in a multi-period signaling game with multiple players and totally ordered signal spaces. However, he does not show the relation between monotone beliefs and equilibrium refinement and its implication on deriving all stronger monotone equilibria. Not only do we establish the existence of a stronger monotone signaling equilibrium but we let the DM choose a set of feasible reactions before senders and receivers move, while Liu and Pei (2020) and Mensch (2020) do not. We propose a general methodology for the DM’s optimal design of a unique stronger monotone signaling equilibrium with his choice of a set of reactions.

Pre-match investment competition studies whether pre-match competition to match with a better partner can solve the hold-up problem of non-contractible pre-match investment that prevails when a match is considered in isolation (e.g. Grossman and Hart (1986) and Williamson (1986)). Cole, Mailath, and Postlewaite (1995), Rege (2008), and Hoppe,
Moldovanu, and Sela (2009) consider pre-match investment with incomplete information and non-transferable utility without monetary transfers (i.e., no reaction choice by a receiver). Therefore, the sender-receiver framework does not apply. Pre-match investment with incomplete information in Hopkins (2012) includes the transferable-utility case but with no restrictions on transfers. A separating equilibrium is their focus.

2 Preliminaries

There is a continuum of senders and receivers. They can be interpreted as sellers and buyers, workers and firms, or entrepreneurs and investors. Receivers and senders are all heterogeneous in terms of types. The sender’s type set is $Z$ and the receiver’s type set is $X$. The unrestricted set of feasible reaction is $R$. The DM can choose any subset of $R$ as the set of feasible reactions, denoted by $T$, that a receiver can choose. Thus, $P(R)$, the power set of $R$, is his choice set. In the example of workers and firms, $T$ is the set of feasible transfers that a firm (receiver) can make to his worker (sender).

The DM moves first by choosing $T \in P(R)$. Given $T$, each sender can take an (observable) action $s$ from a set $S$ prior to matching. As a sender and a receiver form a match, the receiver takes reaction $t$ from a set $T$ given his partner’s action. For example, workers (senders) choose education $s$ before entering market. A worker and a firm are matched as the worker accepts the firm’s wage offer $t$. When a sender of type $z$ chooses action $s$ and matches with a receiver of type $x$ who takes reaction $t$, the sender’s utility is $u(t, s, z)$ and the receiver’s utility is $g(t, s, z, x)$.

In the example with workers and firms, the utilities for a sender (worker) of type $z$ and a receiver (firm) of type $x$ are $u(t, s, z) = t - c(s, z)$ and $g(t, s, z, x) = v(x, s, z) - t$, respectively. Note that $t$ is the monetary transfer from a firm to his worker, $c(s, z)$ is the cost of choosing education $s \in S$ for a worker of type $z$, and $v(x, s, z)$ is the monetary value of the output produced by the worker in a match.

Assume that the measures of senders and receivers are one respectively. Let $G(z)$ and $H(x)$ denote cumulative distribution functions (CDFs) for sender types and receiver types respectively. The reservation utility for every agent corresponds to staying out of the market and it is equal to zero. We assume that a sender takes the null action $\eta \in S$ to stay out of the market such that $\eta < s$ for all $s \neq \eta$ (e.g., $\eta = 0$ if $S = \mathbb{R}_+$). Each of $S, T, X$, and $Z$ is a chain.

The rest of the paper is organized as follows. We define the notion of a competitive signaling equilibrium given $T$ chosen by the DM and analyze a monotone signaling equi-
librium in Sections 3 and 4. In Section 5, we characterize the unique stronger monotone equilibrium given each $T$ that the DM may choose. In Section 6, we conduct numerical analyses for the optimal design of the unique stronger monotone equilibrium. Section 7 concludes. Omitted proofs can be found in Appendix.

3 Competitive signaling equilibrium given $T$

While senders and receiver may randomize their actions and reactions given the set of feasible reactions $T$, we are interested in a competitive equilibrium where they make deterministic choices. However, when it comes to the DM’s choice considered in Section 5, allowing receivers to randomize reactions makes it possible for the DM to focus on only intervals as the set of feasible reactions $T$ that she chooses without loss of generality given the quasilinearity of utility functions.

After the DM publicly announces the set of feasible reactions $T$, senders and receivers make decisions over two periods. In the first period, senders choose actions given a reaction function $\tau$. In the application of workers and firms, the reaction function $\tau$ is a market wage function that specifies a wage for a worker (sender) conditional on her choice of an action (e.g., education) that is observed in equilibrium.

Let $\sigma(z)$ be the optimal action chosen by a sender of type $z$. Given $\sigma : Z \rightarrow S$, let $S^*$ denote the set of actions chosen by senders who enter the market for matching. The solution concept of a competitive signaling equilibrium given $T$ is based on the reaction function $\tau : S^* \rightarrow T$, which specifies a receiver’s reaction conditional on a sender’s equilibrium action $s$. In the second period, senders and receivers who enter the market form one-to-one matches given the senders’ action choices, and receivers take reactions upon forming a match with a sender.

We assume that all senders and receivers share a common belief, denoted by $\mu(s) \in \Delta(Z)$, on a sender’s type conditional on her action $s \in S$. Fix a reaction function $\tau : S^* \rightarrow T$. If $G(\{z|\sigma(z) = s\}) > 0$ for some $s \neq \eta$, then there must be the same measure of receivers who are matched with senders with $s$ in equilibrium. Since senders with $s$ is observationally identical, matching between them randomly occurs in equilibrium and the receiver’s (expected) utility is $E_{\mu(s)} [g(\tau(s), s, z, x)]$, where $E_{\mu(s)} [\cdot]$ is the expectation operator over $Z$ given the probability distribution $\mu(s)$. If $\{z|\sigma(z) = s\}$ is a singleton, then $\mu(s)$ becomes a degenerate probability distribution. In a separating equilibrium, $\mu(s)$ is a degenerate probability distribution for all $s \in S^*$.

Consider a sender’s action choice problem. Let $\sigma(z) \in S^*$ be the optimal action for a
sender of type z if (i) it solves the following problem,

\[ \max_{s \in S^*} u(\tau(s), s, z) \text{ s.t. } u(\tau(s), s, z) \geq 0, \]  

(1)

and (ii) there is no profitable sender deviation to an off-path action \( s' \notin \text{range } \sigma \). We define the notion of a profitable sender deviation in Definition 1 below. Note that \( \sigma(z) = \eta \) becomes the optimal action for a sender of type z if there is no solution for (1) and there is no profitable sender deviation to an off-path action \( s' \notin \text{range } \sigma \).

We now formulate the profitable sender deviation. Let \( X^* \subset X \) be the set of receivers who enter the market. For all \( s \in S^* \), let \( m(s) \in P(X^*) \) be the set of receiver types who are matched with a sender with \( s \), where \( P(X^*) \) is the power set of \( X^* \). Therefore, \( m : S^* \rightarrow P(X^*) \) is a set-valued matching function. For all \( x \in X^* \), \( m^{-1}(x) \in S^* \) denotes the action chosen by a sender with whom a receiver of type \( x \) is matched, i.e., \( x \in m(m^{-1}(x)) \).

**Definition 1** Given \( \{\sigma, \mu, \tau, m\} \), there is a profitable sender deviation to an off-path action if there exists \( z \) for which there are an action \( s' \notin \text{range } \sigma \) and a reaction \( t' \in T \) such that, for some \( x' \),

\[ (a) \ E_{\mu(s')} [g(t', s', z', x')] > E_{\mu(m^{-1}(x'))} [g(\tau(m^{-1}(x')), m^{-1}(x'), z', x')] \quad \text{and} \]

(2)

\[ (b) \ u(t', s', z) > u(\tau(\sigma(z)), \sigma(z), z) \quad \text{if } \sigma(z) \in S^*, \]

(3)

\[ u(t', s', z) > 0, \quad \text{otherwise.} \]

Note that \( z' \) on each side of (2) is the random variable governed by \( \mu(s') \) and \( \mu(m^{-1}(x')) \) respectively.

A receiver’s matching problem can be formulated as follows:

\[ \max_{s \in S^*} \mathbb{E}_{\mu(s)} [g(\tau(s), s, z, x)] \text{ s.t. } \mathbb{E}_{\mu(s)} [g(\tau(s), s, z, x)] \geq 0. \]  

(4)

We use the notation \( \xi(x) \) as the action of the sender whom the receiver of type \( x \) optimally chooses as his match partner. If (4) has a solution for \( x \in X \), \( \xi(x) \) is the solution. Otherwise, \( \xi(x) = \eta \). Note that \( X^* \) be the set of receiver types such that \( \xi(x) \) is a solution for (4).

**Definition 2** Given \( \{\sigma, \mu\} \), \( \{\tau, m\} \) is a stable matching outcome if (i) \( \tau \) clears the markets, i.e., for all \( A \in P(S^*) \) such that \( H(\{x | x \in m(\xi(x)) \}, \xi(x) \in A) = G(\{z | \sigma(z) \in A\}) \), (ii) \( m \) is stable, i.e., there is no pair of a sender with action \( s \) and a receiver of type
for some \( t' \in T \), some \( z \) with \( \sigma(z) = s \in S^* \),

\[
(a) \ E_{\mu(s)} [g(t', s, z', x)] > E_{\mu(m^{-1}(x))} [g(\tau(m^{-1}(x)), m^{-1}(x), z', x)], \quad (5)
\]

\[
(b) \ u(t', s, z) > u(\tau(s), s, z). \quad (6)
\]

Note that \( z' \) on each side of (5) is the random variable governed by \( \mu(s) \) and \( \mu(m^{-1}(x)) \) respectively. Condition (i) implies that the market-clearing reaction function \( \tau \) induces a measure preserving matching function \( m \). Condition (ii) implies that the induced \( m \) is stable where no two agents would like to block the outcome after every sender has chosen her action.

**Definition 3** \( \{\sigma, \mu, \tau, m\} \) constitutes a competitive signaling equilibrium (henceforth simply an equilibrium) with incomplete information if

1. for all \( z \in Z \), \( \sigma(z) \) is optimal
2. \( \mu \) is consistent:
   
   (a) if \( s \in \text{range} \sigma \) satisfies \( G(\{z|\sigma(z) = s\}) > 0 \), then \( \mu(s) \) is determined from \( G \) and \( \sigma \), using Bayes’ rule.

   (b) if \( s \in \text{range} \sigma \) but \( G(\{z|\sigma(z) = s\}) = 0 \), then \( \mu(s) \) is any probability distribution with \( \text{supp} \mu(s) = \text{cl} \{z|\sigma(z) = s\} \)

   (c) if \( s \notin \text{range} \sigma \), then \( \mu(s) \) is unrestricted.
3. given \( (\sigma, \mu) \), \( \{\tau, m\} \) is a stable matching outcome.

## 4 Criterion D1 and stronger monotone equilibrium

Given the indeterminacy of the off-equilibrium-path beliefs, an equilibrium refinement called Criterion D1 was developed by Cho and Kreps (1987) and Banks and Sobel (1987). It restricts the off-equilibrium-path beliefs. Following Ramey (1996), we define Criterion D1 as follows. Given an equilibrium \( \{\sigma, \mu, \tau, m\} \), we define type \( z \)'s equilibrium utility \( U(z) \). If \( \sigma(z) \in S^* \), then \( U(z) := u(\tau(\sigma(z)), \sigma(z), z) \); otherwise, \( U(z) = 0 \).

**Definition 4** (Criterion D1) Fix any \( s \notin \text{range} \sigma \) and any \( t \in T \). Suppose that there is a non-empty set \( Z' \subset Z \) such that the following is true: for each \( z \notin Z' \), there exists \( z' \) such that

\[
u(t, s, z) \geq U(z) \implies u(t, s, z') > U(z'). \quad (7)
\]
Then, the equilibrium is said to violate Criterion D1 unless it is the case that $\text{supp } \mu(s) \subset Z'$.

Intuitively, following the observation for an off-equilibrium-path action $s$, zero posterior weight is placed on a type $z$ whenever there is another type $z'$ that has a stronger incentive to deviate from the equilibrium in the sense that type $z'$ would strictly prefer to deviate for any given $t$ that would give type $z$ a weak incentive to deviate.

We can equivalently define Criterion D1 by the contrapositive of (7), that is

$$u(t, s, z') \leq U(z') \implies u(t, s, z) < U(z).$$

Upon observing an off-equilibrium action $s$, zero posterior weight is placed on a type $z$ whenever a type $z$ is strictly worse off by deviating for any $t$ that would make type $z'$ weakly worse with the same deviation.

For the monotonicity of the belief, we employ the notion of the stronger set order (Shannon (1995)), which is stronger than the strong set order (Veinnott (1989)).

**Definition 5 (Stronger set order)** Consider two sets $A$ and $B$ in the power set $P(Y)$ for $Y$ a lattice with the given relation $\geq$. We say that $A \leq_c B$, read “$A$ is completely lower than $B$” if for every $a \in A$ and $b \in B$, $a \leq b$. Given a partially ordered set $K$ with the given relation $\geq$, a set-valued function $M : K \to P(Y)$ is monotone non-decreasing in the stronger set order if $k' \leq k$ implies that $M(k') \leq_c M(k)$.

If a set-valued function is non-decreasing with respect to the stronger set order, then it is also non-decreasing with respect to the strong set order. For a single-valued function, the two set orders are identical. Importantly, if $M$ is monotone non-decreasing in the stronger set order, $M(k)$ and $M(k')$ have at most one element in common: for any ordered pair of $(k, k')$, $M(k) \cap M(k')$ is $\emptyset$ or a singleton.

Consider a belief function $\mu : S \to \Delta(Z)$. The monotonicity of a belief function is defined by the stronger set order on the supports of the probability distributions. A belief function is non-decreasing in the stronger set order if $s' \leq s$ implies $\text{supp } \mu(s') \leq_c \text{supp } \mu(s)$. We also use the stronger set order for the monotonicity of a matching function $m : S^* \to P(X^*)$. A matching function is non-decreasing in the stronger set order if $s' \leq s$ implies $m(s') \leq_c m(s)$. Now we define the stronger monotone equilibrium as follows.

**Definition 6 (Stronger Monotone Equilibrium)** An equilibrium $\{\sigma, \mu, \tau, m\}$ is stronger monotone if $\sigma$, $\mu$, $\tau$, and $m$ are non-decreasing in the stronger set order.
We impose the following assumptions for $u$.

**Assumption A** $u(t, s, z)$ is (i) decreasing in $s$, increasing in $t$ and $z$, and satisfies (ii) the strict single crossing property in $((t, s); z)$.\(^5\)

Given Assumption A, the stronger monotonicity of $\sigma$ and $\tau$ comes from Lemma 10 in Appendix. The stronger monotonicity of $\mu$ is equivalent to Criterion D1.

**Corollary 1** Let $\sigma$ and $\mu$ be a sender action function and a belief function in equilibrium, respectively. If Assumption A is satisfied, $\mu$ passes Criterion D1 if and only if it is non-decreasing in the stronger set order.

The stronger monotonicity of $\mu$ implies that for any $s$ in the interval of off-path sender actions induced by the discontinuity of $\sigma$ at an interior sender type $z$, the support of $\mu(s)$ is a singleton and it is $\{z\}$. This implication is satisfied if and only if $\mu$ satisfies Criterion D1 given Assumption A. Cho and Sobel monotonicity of $\mu$ does not lead to this implication although any belief function $\mu$ that passes Criterion D1 satisfies Cho and Sobel monotonicity.\(^6\)

**Corollary 2** According to Lemma 11 in Appendix, the support of the belief $\mu(s)$ conditional on $s \notin \text{range } \sigma$ is a singleton if it passes Criterion D1. This implies that if the unique type in the support of the belief $\mu(s)$ is weakly worse off by deviating to $s \notin \text{range } \sigma$, any other type is strictly worse off with the same deviation.

The proof of Corollary 2 is straightforward, so it is omitted. Corollaries 1 and 2 play a crucial role in deriving a non-separating stronger monotone equilibrium in the DM’s optimal stronger monotone equilibrium design.

To establish the stronger monotonicity of an equilibrium, $\{\sigma, \mu, \tau, m\}$, we still need to identify sufficient conditions under which $m$ is non-decreasing in the stronger set order. We impose Assumption B for $g$ and apply the Milgrom-Shannon Monotone Selection Theorem (Milgrom and Shannon (1994)).

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\(^5\)Let $A$ be a lattice, $\Theta$ be a partially ordered set and $f : A \times \Theta \rightarrow \mathbb{R}$. Then, $f$ satisfies the single crossing property in $(a; \theta)$ if for $a' > a''$ and $\theta' > \theta''$, (i) $f(a', \theta'') \geq f(a'', \theta'')$ implies $f(a', \theta') \geq f(a'', \theta')$ and (ii) $f(a', \theta') > f(a'', \theta'')$ implies $f(a', \theta') > f(a'', \theta')$. If $f(a', \theta'') \geq f(a'', \theta'')$ implies $f(a', \theta') > f(a'', \theta')$ for every $\theta' > \theta''$, then $f$ satisfies the strict single crossing property in $(a; \theta)$.

\(^6\)Suppose that an action $s$ is chosen by some sender type $z$ on the equilibrium path. Cho and Sobel monotonicity means that a receiver should believe that $s' > s$ is not chosen by a lower sender type than $z$. 
Assumption B (i) $g(t, s, z, x)$ is supermodular\(^7\) in $(t, s, z)$ and satisfies the single crossing property in $((t, s, z); x)$ and the strict single crossing property in $(z; x)$ at each $(s, t)$. (ii) $g(t, s, z, x)$ is increasing in $x$.

Theorem 1 (Milgrom-Shannon Monotone Selection Theorem) Let $f: A \times \Theta \rightarrow \mathbb{R}$, where $A$ is a lattice and $\Theta$ is a partially ordered set. If $f$ is quasisupermodular\(^8\) in $a$ and satisfies the strict single crossing property in $(a; \theta)$, then every selection $a^*(\theta)$ from arg max\(a \in A\) $f(a, \theta)$ is non-decreasing.

Theorem 2 (Stronger Monotone Signaling Equilibrium Theorem) Suppose that Assumptions A and B are satisfied. Then, an equilibrium $\{\sigma, \mu, \tau, m\}$ is stronger monotone if and only if it passes Criterion D1.

Proof. Given Assumption A, the stronger monotonicity of $\sigma, \mu$, and $\tau$ comes from Lemma 10 and Corollary 1, in the Online Appendix. Given the stronger monotonicity of $\sigma, \mu$, and $\tau$, consider a receiver’s matching problem that is max\(s \in S^*\) $V(s, x)$, where $V(s, x) := \mathbb{E}_{\mu(s)}[g(\tau(s), s, z, x)]$.

For any $s, s' \in S^*$ such that $s > s'$, we have that $\tau(s) > \tau(s')$ and $z \geq z'$ for any $z \in \text{supp} \mu(s)$ and $z' \in \mu(s')$. Therefore, the first three arguments in $g$ are linearly ordered with respect to $s$. Given Assumption B(ii), this implies that $V(s, x)$ satisfies the strict single crossing property.

Choose an arbitrary selection $\xi_o(x) \in \text{arg max}_{s \in S^*} V(s, x)$. Then, by Milgrom and Shannon’s Monotone Selection Theorem, $\xi_o(x)$ is non-decreasing in $x$. Note that max\(s \in S^*\) $V(s, x)$ is a maximization problem with no individual rationality.

For all $x \in X$, let

$$
\xi(x) = \begin{cases} 
\xi_o(x) & \text{if } V(\xi_o(x), x) \geq 0, \\
\eta & \text{otherwise.}
\end{cases}
$$

$V(s, x)$ is increasing in $x$ because of Assumption B(ii) and hence we have that $x < x'$ for any $x$ with $\xi(x) = \eta$ and any $x'$ with $\xi(x') \neq \eta$. This property and the non-decreasing property of $\xi_o(x)$ make $\xi(x)$ non-decreasing in $x$.

For any $s \in S^*$, the set of receiver types who are matched with senders with $s$ can be expressed as $m(s) = \xi^{-1}(s) := \{x | \xi(x) = s\}$. Because $\xi(x)$ is non-decreasing in $x$, $m$ is non-decreasing with respect to the stronger set order. \(\blacksquare\)

Without loss of generality, we can focus on stronger monotone equilibria to derive all D1 equilibria given Assumptions A and B.

---

\(^7\)Given a lattice $A$, $f: A \rightarrow \mathbb{R}$ is supermodular if $f(a \land b) + f(a \lor b) \geq f(a) + f(b)$ for all $a$ and $b$ in $A$. $f: A \rightarrow \mathbb{R}$ is strictly supermodular if $f(a \land b) + f(a \lor b) > f(a) + f(b)$ for all unordered $a$ and $b$ in $A$.

\(^8\)Given a lattice $A$, a function $f: A \rightarrow \mathbb{R}$ is quasisupermodular if (i) $f(a) \geq f(a \land b)$ implies $f(a \lor b) \geq f(b)$ and (ii) $f(a) > f(a \land b)$ implies $f(a \lor b) > f(b)$. If $f$ is supermodular, then it is quasisupermodular.
A couple of remarks are in order. First of all, Cho and Sobel monotonicity of beliefs\(^9\) (Cho and Sobel (1990)), a partial implication of Criterion D1, is instrumental for the selection of a separating equilibrium as a unique D1 equilibrium: Among those who chose the same action, the highest sender type always has a profitable upward deviation given Cho and Sobel monotonicity, so a pooled action cannot be sustained in a D1 equilibrium. However, this argument does not apply if a receiver cannot reward such an upward deviation with a higher reaction when the upper bound of feasible reactions is too low. Given any set of feasible reactions that the DM may choose, we can derive a unique D1 equilibrium using the stronger monotonicity of beliefs, which is the full implication of Criterion D1.

Secondly, one could find statements in the literature for games with totally ordered signal spaces of something like “single crossing implies monotonicity on the path if one imposes monotonicity off path beliefs.” Corollary 1 provides a stronger result in that it shows the equivalence between Criterion D1 and the stronger monotonicity of beliefs in a general model.

5 Unique Stronger Monotone Equilibrium given \(T\)

The sender’s type set is \(Z = [\underline{z}, \overline{z}] \subset \mathbb{R}\) and the receiver’s type set is \(X = [\underline{x}, \overline{x}] \subset \mathbb{R}\). \(X\) and \(Z\) do not have to be bounded. The equilibrium analysis with non-negative unbounded type sets can be analogously done. We assume that \(S = \mathbb{R}_+\). Let \(0 \in S\) be the null action. Utilities are transferable through a receiver’s reaction \(t\). A receiver’s utility is \(g(t, s, z, x) = v(x, s, z) - t\) and a sender’s utility is \(u(t, s, z) = t - c(s, z)\). \(v\) can be interpreted as gross match surplus and \(c\) is the cost of taking an action for senders.

Given that receivers can randomize their reactions and utility functions of both receivers and senders are quasilinear in reactions, choosing \(T\) is equivalent to choosing its convex hull for the DM’s perspective. This reduces the DM’s choice set without loss of generality. For simplicity of notation, we expand the whole set of feasible reactions to \(\mathbb{R}_+ \cup \{\infty\}\). The DM only needs to consider an interval, \([t_\ell, t_h]\) as the set of feasible reactions \(T\), where \(0 \leq t_\ell \leq t_h \leq \infty\) given that receivers can randomize their reactions. If \(t_\ell = 0\) and \(t_h = \infty\), then there is no restrictions on feasible reactions. If \(t_\ell = t_h\), then \([t_\ell, t_h]\) is a singleton that allows only one feasible reaction. Let us start with assumptions. Focusing on stronger monotone equilibria, the DM maximizes the aggregate net surplus.

---

\(^9\)Suppose that an action \(s\) is chosen by some sender type \(t\) on the equilibrium path. Cho and Sobel monotonicity means that a receiver should believe that \(s' > s\) is not chosen by a lower sender type than \(t\).
**Assumption 1.** (i) \(c(s, z)\) is increasing in \(s\) but decreasing in \(z\) and (ii) \(-c(s, z)\) is strictly supermodular in \((s, z)\).

It is easy to see that Assumption 1 implies Assumption A given the form of the utility function, \(u(t, s, z) = t - c(s, z)\).\(^{10}\)

**Assumption 2.** (i) \(v(x, s, z)\) is supermodular in \((x, s, z)\) and strictly supermodular in \((z, x)\), (ii) \(v\) is increasing in \(x\).

**Lemma 1** If Assumption 2 holds, then \(g\) satisfies Assumptions B.

Because Assumptions A and B are implied by Assumptions 1 and 2, Theorem 2 goes through in this section. Focusing on stronger monotone equilibria, the DM maximizes the aggregate net surplus.

We impose Assumptions 3, 4, 5, and 6 below for the differentiability of the separating part of a stronger monotone equilibrium and the existence of a stronger monotone equilibrium.

**Assumption 3** (i) \(v\) is non-negative, increasing in \(z\), and non-decreasing in action \(s\). (ii) \(v\) is differentiable and \(v_s\) and \(v_z\) are continuous.

**Assumption 4** \(c\) is differentiable with \(c(0, z) = 0\), \(\lim_{s \to \infty} c(s, z) = \infty\) for all \(z \in [\underline{z}, \overline{z}]\), and \(c_s\) is continuous.

**Assumption 5** If \(v(x, s, z)\) is increasing in \(s\), it is concave in \(s\) with \(\lim_{s \to 0} v_s(x, s, z) = \infty\) and \(\lim_{s \to \infty} v_s(x, s, z) = 0\) and \(c(s, z)\) is strictly convex in \(s\) with \(\lim_{s \to 0} c_s(s, z) = 0\) and \(\lim_{s \to \infty} c_s(s, z) = \infty\).

**Assumption 6** \(0 < G'(z) < \infty\) for all \(z \in [\underline{z}, \overline{z}]\) and \(0 < H'(x) < \infty\) for all \(x \in [\underline{x}, \overline{x}]\).

We define the function \(n\) as \(n \equiv H^{-1} \circ G\) so that \(H(n(z)) = G(z)\) for all \(z \in [\underline{z}, \overline{z}]\). A bilaterally efficient action \(\zeta(x, z)\) for type \(z\) given \(x\) maximizes \(v(x, s, z) - c(s, z)\) and

\[
v(x, \zeta(x, z), z) - c(\zeta(x, z), z) \geq 0. \tag{9}\]

We normalize \(\zeta(x, z)\) to 0. The reservation utility for each agent is zero. We assume that every agent enters the market if she can get at least her reservation utility by entering the market in equilibrium.

---

\(^{10}\)If the domain \(A\) of a real-valued function \(f\) is a subset of \(\mathbb{R}^N\), then the (strict) supermodularity of \(f\) is equivalent to non-decreasing (increasing) differences (Theorem 2.6.1 and Corollary 2.6.1 in Topkis (1998)), which in turn guarantees the (strict) single crossing property.
We introduce a well-behaved stronger monotone equilibrium, a type of stronger monotone equilibrium, that encompasses a separating equilibrium and a pooling equilibrium as well. A stronger monotone equilibrium is called well-behaved if it is characterized by two threshold sender types, \( z^\ell \) and \( z^h \) such that every sender of type below \( z^\ell \) stays out of the market, every sender in \([z^\ell, z^h]\) differentiates themselves with their unique action choice, and every sender in \([z^h, \bar{z}]\) pools themselves with the same action. If \( z^\ell < z^h = \bar{z} \), then a well-behaved equilibrium is separating. If \( z^\ell = z^h \), then a well-behaved equilibrium is pooling. If \( z^\ell < z^h < \bar{z} \), then it is (strictly) well-behaved with both separating and pooling parts in the equilibrium. We shall show that any stronger monotone equilibrium (i.e., any D1 equilibrium) is unique as well as well-behaved.

We first start with a stronger monotone separating equilibrium. Once we characterize it, the characterization of any stronger monotone well-behaved equilibrium comes naturally. For now, let us assume that the lower bound \( t^\ell \) of the interval \( T \) that the DM chooses for feasible reactions is less than the maximal value of \( v - c \) that can be created by the highest types \( \bar{x} \) and \( \bar{z} \). Otherwise all agents would stay out of the market. Let \( z^\ell \) be the lowest sender type who is matched in equilibrium and \( s^\ell \) her action. The following two inequalities must be satisfied at \((s^\ell, z^\ell)\):

\[
\begin{align*}
v(n(z), s, z) - t^\ell &\geq 0, \\
t^\ell - c(s, z) &\geq 0.
\end{align*}
\]

The two cases must be distinguished. If \( z^\ell = \bar{z} \), then all types are matched in equilibrium. This is the first case. In this case, if we have a separating part in equilibrium, there is no information rent in the lowest match between type \( \bar{z} \) and type \( \bar{x} \). Therefore, the equilibrium action \( s^\ell \) in the lowest match is bilaterally efficient, i.e., \( s^\ell = \zeta(\bar{x}, \bar{z}) = 0 \). In this case, we assume that the DM sets \( t^\ell \) to \( c(\zeta(\bar{x}, \bar{z}), \bar{z}) = 0 \).

If \( t^\ell \) is so high that type \( \bar{x} \) cannot achieve a non-negative value of \( v - t^\ell \) in a match with type \( \bar{z} \) who takes an action that costs her \( t^\ell \), then the lowest match must be between types \( z^\ell \) and \( x^\ell := n(z^\ell) \) in the interior of both type distributions. \((10)\) and \((11)\) must be also satisfied with equality at \((s^\ell, z^\ell)\). This is the second case. If either one of them, e.g., \((10)\), is positive, then a receiver whose type is below but arbitrarily close to \( x^\ell \) finds it profitable to be matched with type \( z^\ell \) instead of staying out of the market.

**Lemma 2** If there is a solution \((s^\ell, z^\ell)\) that solves \((10)\) and \((11)\), it is unique.

Now consider the upper bound \( t^h \) of the interval \( T \) that the DM chooses for feasible reactions. What happens if it is equal to \( \infty \)? In any stronger monotone equilibrium with
The first-order necessary condition for the sender’s equilibrium action choice that solves her problem in (1) would satisfy that for all \( z \in (z_\ell, \bar{z}) \)

\[
\tau'(\sigma(z)) - c_s(\sigma(z), z) = 0.
\]

(12)

On the other hand, the equilibrium reaction choice \( \tau(s) \) by the receiver who is matched with a sender with action \( s \) solves his problem in (4) and its first-order necessary condition must satisfy that for all \( s \in \text{Int}S^* \)

\[
\tau'(s) = v_s(m(s), s, \mu(s)) + v_z(m(s), s, \mu(s))\mu'(s),
\]

(13)

where \( m(s) = n(\mu(s)) \) is the type of the receiver who is matched with a sender with action \( s \). Note that the equilibrium matching function \( m \) is stronger monotone due to Theorem 2. Because all senders on the market differentiate themselves with unique action choices in a stronger monotone separating equilibrium, \( m \) is strictly increasing over \( S^* \) and the matching is assortative in terms of the receiver’s type and the sender’s action (and the receiver’s type and the sender’s type as well).

In our two-sided matching model, the market clearing conditions must be embedded into senders’ action choices and the belief on sender types. In Theorem 3, the differentiability of \( \tau \) comes from senders’ optimal action choices and the differentiability of \( \mu \) comes from receivers’ optimal choice of a sender, given the market-clearing condition and both the continuity of \( \sigma \) and the differentiability of \( \tau \) that we can derive from sender’s optimal action choices. Theorem 3 is the consequence of Assumptions 1 - 5. The proof of Theorem 3 is rather long but its differentiability results contribute to the literature.\(^\text{11}\)

**Theorem 3** In any well-behaved stronger monotone equilibrium with \( t_h = \infty \), (i) \( S^* \) is a compact real interval, \( [\sigma(\bar{z}), \sigma(\bar{z})] \), (ii) \( \tau : S^* \to T \) is increasing and continuous on \( S^* \) and has continuous derivative \( \tilde{\tau}' \) on \( \text{Int}S^* \), and (iii) \( \mu : S \to \Delta(Z) \) is increasing and continuous on \( S^* \) and has continuous derivative \( \mu' \) on \( \text{Int}S^* \).

Because \( \mu \) is the inverse of \( \sigma \), the differentiability of \( \sigma \) is immediate from Theorem

\(^{11}\)To apply the differentiability results in a model with one sender and one receiver (Mailath (1987)) to a two-sided matching model, Hopkins (2012) imposes the restriction that there is no complementary between receiver type \( x \) and sender action \( s \). This restriction gets rid of a matching effect on the marginal productivity of a sender’s action. This restriction is not needed for establishing our differentiability result in Theorem 3 above.
Combining (12) and (13) yields a function \( \phi(s, z) \) defined below:

\[
\phi(s, z) := -\left[ v_s(n(z), s, z) - c_s(s, z) \right] / v_z(n(z), s, z).
\]

This is the first-order ordinary differential equation, \( \mu' = \phi(s, \mu(s)) \) with the initial condition \((s_\ell, z_\ell)\).

**Lemma 3** If \( v \) and \( c \) are such that \( \phi \) defined in (14) is uniformly Lipshitz continuous, then the solution exists and it is unique.

Lemma 3 is the simple application of the Picard-Lindelof Theorem (See Teschl (2012)). Let \( \tilde{\mu} \) be the unique solution for the differential equation.

Given \( z_\ell \) induced by \( t_\ell \) and \( t_\ell < t_h = \infty \), \( \{\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{m}\} \) denotes a stronger monotone separating equilibrium.\(^{12}\)

Once we drive \( \tilde{\mu} \), we can construct the functions \( \tilde{\sigma}, \tilde{\tau}, \tilde{m} \) implied by \( \tilde{\mu} \). \( \tilde{\sigma}(z) \) for all \( z \in [z_\ell, \bar{z}] \) is determined by \( \tilde{\sigma}(z) = \tilde{\mu}^{-1}(z) \) for all \( z \in [z_\ell, \bar{z}] \), where \( \tilde{\mu}^{-1}(z) \) is the type of a sender that satisfies \( z = \tilde{\mu}(\tilde{\mu}^{-1}(z)) \) for all \( z \in [z_\ell, \bar{z}] \). For \( s \in [s_\ell, \tilde{\sigma}(\bar{z})] \), we can derive the matching function \( \tilde{m} \) according to \( \tilde{m}(s) = n(\tilde{\mu}(s)) \). Because \( \tilde{\mu} \) is continuous everywhere and differentiable at all \( s \in \text{Int } S^* \), integrating the right-hand-side of (13) with the initial condition with \( \tilde{\tau}(s_\ell) = t_\ell \) induces

\[
\tilde{\tau}(s) = \int_{s_\ell}^{s} \left[ v_s(\tilde{m}(y), y, \tilde{\mu}(y)) + v_z(\tilde{m}(y), y, \tilde{\mu}(y))\tilde{\mu}'(y) \right] dy + t_\ell.
\]

However, if \( t_h < \tilde{\tau}(\tilde{\sigma}(\bar{z})) \), then we have no separating equilibrium. In this case, let \( Z(s) \) denote the set of the types of senders who choose the same action \( s \).

**Lemma 4** If \( Z(s) \) has a positive measure in a stronger monotone equilibrium, then it is an interval with \( \max Z(s) = \bar{z} \).

Lemma 5 below shows that if there is pooling on the top of the sender side, the reaction to those senders pooled at the top must be the upper bound of feasible reaction \( t_h \).

**Lemma 5** If \( Z(s) \) has a positive measure in a stronger monotone equilibrium, then \( t_h \) is the reaction to the senders of types in \( Z(s) \).

\(^{12}\)The characterization of the stronger monotone separating equilibrium can be found in Theorem 11 in the Online Appendix.
We can establish Lemmas 4 and 5 using only Cho and Sobel monotonicity of \( \mu \) without relying on the stronger monotonicity of \( \mu \). However, we cannot derive a D1 equilibrium with Cho and Sobel monotonicity as explained after Theorem 6.

Using Lemmas 4 and 5, we can establish that the stronger monotone separating equilibrium \( \{\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{m}\} \) is a unique stronger monotone equilibrium if \( \tilde{\tau}(\tilde{\sigma}(\bar{z})) \leq t_h \).

**Theorem 4** Suppose that the DM chooses \( T = [t_\ell, t_h] \) such that \( 0 \leq t_\ell < \tilde{\tau}(\tilde{\sigma}(\bar{z})) \leq t_h \), a unique stronger monotone equilibrium is the well-behaved stronger monotone equilibrium and it is separating.

**Proof.** Lemma 3 leads to the existence of the unique stronger monotone separating equilibrium. The remaining question is whether there are other stronger monotone equilibria. Lemma 4 is still valid. Therefore, if there is bunching in sender action \( s \), it must be among senders in a type interval \( Z(s) \) with \( \bar{z} \) as its maximum. However, such bunching is not sustained because if we follow the logic in the proof of Lemma 5, we can show that there is a profitable small upward deviation from \( s \) for the sender of type-\( \bar{z} \). Therefore, there is no additional stronger monotone equilibrium. ■

Theorem 4 extends the uniqueness result in Cho and Sobel (1990) and Ramey (1996) to a two-sided matching model. If \( t_h < \tilde{\tau}(\tilde{\sigma}(\bar{z})) \), there are only two types of non-separating stronger monotone equilibria as shown in Lemma 6. The reason is that pooling can happen only among senders in an interval with \( \bar{z} \) being its maximum due to Lemma 4,

**Lemma 6** If there is an upper bound of reactions \( t_h < \tilde{\tau}(\tilde{\sigma}(\bar{z})) \), then, there are two possible stronger monotone equilibria: (i) a strictly well-behaved stronger monotone equilibrium and (ii) a stronger monotone pooling equilibrium.

Lemma 6 follows Theorem 2 (Stronger Monotone Equilibrium Theorem) and Lemmas 4 and 5. For the uniqueness of a stronger monotone equilibrium when \( t_h < \tilde{\tau}(\tilde{\sigma}(\bar{z})) \), we impose an additional assumption as follows.

**Assumption 7** \( \lim_{z \to \bar{z}} c(s, z) = \infty \) for all \( s > 0 \) and either (i) or (ii) is satisfied:

(i) \( v(x, s, z) = v(x, s', z) \) for all \( s, s' \in \mathbb{R}_+ \), \( v(x, s, z) = 0 \) for all \( s, z \), and \( v(x, s, z) > 0 \) for all \( x > \underline{x}, \) all \( z > \bar{z}, \) and all \( s \in \mathbb{R}_+ \).

(ii) \( v(x, 0, z) = 0 \) and \( v(x, s, z) \) is increasing in \( s \) for all \( x \) and \( z \), and \( v(x, 0, z) - c(0, z) \geq 0 \) for all \( x > \underline{x}, \) all \( z > \bar{z} \).
We first consider a stronger monotone pooling equilibrium. This is a type of stronger monotone equilibrium when \( t_{\ell} = t_{h} = t^* \). Every seller of type above \( z_{\ell} = z_{h} = z^* \) enters the market with the pooled action \( s^* \).

\[
t^* - c(s^*, z^*) \geq 0, \tag{16}
\]
\[
\mathbb{E} [v(n(z^*), s^*, z') | z' \geq z^*] - t^* \geq 0, \tag{17}
\]

where each condition holds with equality if \( z^* > z \).

**Theorem 5** For only a single feasible reaction \( t^* > 0 \), the only possible stronger monotone pooling equilibrium is a stronger monotone pooling equilibrium with \( z^* > z \) and \( s^* > 0 \) that satisfy (16) and (17), each with equality. For only a single feasible reaction, \( t^* = 0 \), the only possible stronger monotone pooling equilibrium is a stronger monotone pooling equilibrium with \( z^* = z \) and \( s^* = 0 \).

**Proof.** Fix \( t^* > 0 \). We first show that \( z^* > z \). On the contrary, suppose that \( z^* = z \). Then, \( s^* = 0 \). Otherwise (i.e., \( s^* > 0 \)), (16) is not satisfied because \( \lim_{z \to z^*} c(s, z) = \infty \) for all \( s > 0 \) in Assumption 7. If Assumption 7.(i) is satisfied, \( \mathbb{E} [v(n(z^*), s^*, z') | z' \geq z^*] = 0 \) because \( n(z^*) = x \). Therefore, (17) is not satisfied. If Assumption 7.(ii) is satisfied, \( s^* = 0 \) implies that \( \mathbb{E} [v(n(z^*), s^*, z') | z' \geq z^*] = 0 \). Therefore, (17) is not satisfied. It means that if \( t^* > 0 \), then \( z^* > z \).

Given \( t^* > 0 \), let \( z^* > z \) be the threshold sender type in a pooling equilibrium and hence (16) and (17), each with equality. It means that \( s^* > 0 \). If a sender reduces her action below \( s^* \), the stronger monotone belief implies that her type is believed to be \( z^* \) and no receiver is willing to match with her at \( t^* \). Furthermore, no sender wants to choose her action above \( s^* \) because the reaction is fixed to \( t^* \). Given binding (16) and (17), no agent who stays out of the market wants to enter it and vice versa no agent who enters the market stays out of the market.

If \( t^* = 0 \), then we must have \( s^* = 0 \). Otherwise the sender with \( s^* \) will have utility less than her reservation utility. Suppose that Assumption 7.(i) is satisfied. Then, every receiver of type above \( z \) gets positive (expected) utility by matching with a sender. Therefore, every receiver wants to enter the market. Then, every sender must enter the market as well. Therefore, \( z^* = z \). Suppose that Assumption 7.(ii) is satisfied. Because \( s^* = 0 \), any receiver who is matched with a sender with \( s^* = 0 \) gets the same utility as his reservation utility. All receivers and sender receive zero utility regardless of their decisions on market entry (So, the aggregate net surplus is always zero). Because agents
enter the market whenever they are indifferent between entering the market and staying out, agents enter the market, $z^* = \tilde{z}$. It is clear to see that no agent want to leave the market.

Now consider a (strictly) well-behaved equilibrium with both separating and pooling parts when $t_\ell < t_h < \tilde{\tau} (\tilde{\sigma}(\tilde{z}))$. The system of equations represented in (18) and (19) is the key to understand jumping and pooling in the upper tail of the match distribution with $t_h < \tilde{\tau}(\tilde{\sigma}(\tilde{z}))$:

$$t_h - c(s, z) = \tilde{\tau}(\tilde{\sigma}(z)) - c(\tilde{\sigma}(z), z),$$  \hspace{1cm} (18)

$$\mathbb{E}[v(n(z), s, z')|z' \geq z] - t_h = v(n(z), \tilde{\sigma}(z), z) - \tilde{\tau}(\tilde{\sigma}(z)).$$  \hspace{1cm} (19)

Let $(s_h, z_h)$ denote a solution of (18) and (19). Note that (18) makes the type $z_h$ sender indifferent between choosing $s_h$ for $t_h$ and $\tilde{\sigma}(z_h)$ for $\tilde{\tau}(\tilde{\sigma}(z_h))$. The expression on the left hand side of (18) is the equilibrium utility for the type $z_h$ receiver. The expression on the left hand side of (19) is the utility for the type $n(z_h)$ receiver who chooses a sender with action $s_h$ as his partner by choosing $t_h$ for her. This is the equilibrium utility for type $n(z_h)$. The expression on the right-hand side is his utility if he chooses a sender of type $z_h$ with action $\tilde{\sigma}(z_h)$ as his partner by choosing the reaction $\tilde{\tau}(\tilde{\sigma}(z_h))$.

**Lemma 7** If there is a solution $(s_h, z_h)$ of (18) and (19), it is unique.

Lemma 8 below shows that there is jumping in reactions and actions at the threshold sender type.

**Lemma 8** If there exists a solution $(s_h, z_h)$ of (18) and (19) with $z_\ell < z_h < \tilde{z}$, then $\tilde{\tau}(\tilde{\sigma}(z_h)) < t_h < \tilde{\tau}(\tilde{\sigma}(\tilde{z}))$ and $\tilde{\sigma}(z_h) < s_h < \tilde{\sigma}(\tilde{z})$.

We exploit Assumptions 1 and 2(i) for the proof of Lemma 8. Theorem 6 establishes the existence of a unique well-behaved stronger monotone equilibrium given $T = [t_\ell, t_h]$ with $0 \leq t_\ell < \tilde{\tau}(\tilde{\sigma}(\tilde{z})) < t_h$. Let $x_h := n(z_h)$. Note that Theorem 6 allows for the possibility of separating or pooling as well as strictly well-behaved.

**Theorem 6** Suppose that the DM chooses a set of feasible reactions $T = [t_\ell, t_h]$ with $0 \leq t_\ell < \tilde{\tau}(\tilde{\sigma}(\tilde{z})) < t_h$ under which $(z_\ell, s_\ell)$ is the lower threshold sender type and her equilibrium action and $(z_h, s_h)$ is the upper threshold sender type and her equilibrium action. Then, there exists a unique well-behaved stronger monotone equilibrium $\{\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{m}\}$. It is characterized as follows.
1. \( \hat{\sigma} \) follows (i) \( \hat{\sigma}(z) = 0 \) if \( z \in [z, z_\ell) \); (ii) \( \hat{\sigma}(z) = s_\ell \) if \( z = z_\ell \); (iii) \( \hat{\sigma}(z) \) satisfies that \( \hat{\tau}'(\hat{\sigma}(z)) - c_s(\hat{\sigma}(z), z) = 0 \) if \( z \in (z_\ell, z_h) \); (iv) \( \hat{\sigma}(z) = s_h \) if \( z \in [z_h, z] \). Further, \( \hat{\sigma}(z_h) < s_h \).

2. \( \hat{\mu} \) follows (i) \( \hat{\mu}(s) = G(z|z \leq z < z_\ell) \) if \( s = 0 \); (ii) \( \hat{\mu}(s) = z_\ell \) if \( s \in (0, \hat{\sigma}(z_\ell)) \); (iii) \( \hat{\mu}(s) = \hat{\sigma}^{-1}(s) \) if \( s \in [\hat{\sigma}(z_\ell), \hat{\sigma}(z_h)) \); (iv) \( \hat{\mu}(s) = z_h \) if \( s \in \lim_{z \to z_h} \hat{\sigma}(z), s_h \); (v) \( \hat{\mu}(s) = G(z|z_h \leq z \leq z) \) if \( s = s_h \); (vi) \( \hat{\mu}(s) = z \) if \( s > s_h \).

3. \( \hat{\tau}(s) \) with \( \hat{\tau}(s_\ell) = t_\ell \) satisfies (i) \( v_s(x, s, \hat{\mu}(s)) + v_z(x, s, \hat{\mu}(s)) \hat{\mu}'(s) - \hat{\tau}'(s) = 0 \) at \( s = \xi(x) \) for all \( x \in (x_\ell, x_h) \) and (ii) \( \hat{\tau}(s) = t_h \) if \( s \geq s_h \). Further, \( \hat{\tau}(\hat{\sigma}(z_h)) < t_h \).

4. \( \hat{m} \) follows that (i) \( \hat{m}(s) = n(\hat{\mu}(s)) \) if \( s \in [\hat{\sigma}(z_\ell), \hat{\sigma}(z_h)) \), (ii) \( \hat{m}(s) = [x_h, z] \) if \( s = s_h \).

If a well-behaved equilibrium has both separating and pooling, it follows the separating equilibrium with the same \( z_\ell \) before \( z \) hits \( z_h \) according to Conditions 1(i)–(iii), 2(i)–(iii), 3(i), and 4(i) in Theorem 6 above. As Condition 1 in Theorem 6 and Lemma 8 show, in a (strictly) well-behaved stronger monotone equilibrium, we have jumping in equilibrium sender actions at the threshold sender type \( z_h \), followed by pooling. In Figure 1, the equilibrium sender actions consist of the three different blue parts.\(^{13}\) Note that equilibrium matching is assortative in terms of sender action and receiver type (and therefore in terms of sender type and receiver type) in the separating part of the equilibrium but it is random in the pooling part of the equilibrium. Therefore, there is matching inefficiency in the pooling part but there may be potential savings in the signaling cost associated with the pooled action choice by senders above \( z_h \).

\(^{13}\)Note that \( \lim_{z \to z_h} \hat{\sigma}(z) = \hat{\sigma}(z_h) \) in Figure 1
Because $z_h$ is in the interior of the sender’s type interval in a strictly well-behaved stronger monotone equilibrium, Cho and Sobel monotonicity does not pin down the belief $\hat{\mu}(s)$ conditional on an off-path action $s \in [\lim_{z \rightarrow z_h} \hat{\sigma}(z), s_h)$, whereas the stronger monotonicity of the belief (see Corollary 1 in the Online Appendix) uniquely pins it down as one that puts all the probability weights on $z_h$ as specified in Condition 2(iv). Further, because the stronger monotonicity of the belief is equivalent to Criterion D1, we only need to show that the sender type $z_h$ has no incentive to deviate to an off-path action in $[\lim_{z \rightarrow z_h} \hat{\sigma}(z), s_h)$ in order to show that no sender has an incentive to deviate to such an off-path action.

Theorem 7 below shows that $\{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\}$ is a unique stronger monotone equilibrium if $0 \leq t_\ell < t_h < \tilde{\tau}(\tilde{\sigma}(z))$.

**Theorem 7** Suppose that the DM fixes a set of feasible reactions to $T = [t_\ell, t_h]$ with $0 \leq t_\ell < t_h < \tilde{\tau}(\tilde{\sigma}(z))$. A unique stronger monotone equilibrium is $\{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\}$.

**Proof.** Since there is no separating equilibrium with $T = [t_\ell, t_h]$ with $t_\ell < t_h < \tilde{\tau}(\tilde{\sigma}(z))$. Because of Lemma 6, it is sufficient to show that there is no pooling equilibrium. On contrary, suppose that there exists a pooling equilibrium. Because of Lemma 5, $t_h$ is the equilibrium reaction for senders with pooled action $s^*$. Therefore, $x^* = n(z^*)$ and the following system of equations is satisfied in a pooling equilibrium:

\begin{align}
t_h - c(s^*, z^*) &\geq 0, \\
\mathbb{E}[v(n(z^*), s^*, z') | z' \geq z^*] - t_h &\geq 0,
\end{align}

where both inequalities hold with equality if $z^* > z$.

Suppose that $z^* > z$. Then, (20) and (21) hold with equality. Further, because both $t_h$ and $z^*$ are positive, $s^*$ must be positive from (20) with equality. On the other hand, there should be no profitable downward deviation for senders. Therefore,

\[ v(n(z^*), s, z^*) - c(s, z^*) \leq \mathbb{E}[v(n(z^*), s^*, z') | z' \geq z^*] - c(s^*, z^*) \text{ for all } s < s^* \]

Because (20) and (21) hold with equality, this becomes

\[ v(n(z^*), s, z^*) - c(s, z^*) \leq 0 \text{ for all } s < s^*. \]

If $v(x, s, z)$ satisfies Assumption 7(i), then, $v(n(z^*), 0, z^*) - c(0, z^*) = v(n(z^*), 0, z^*) > 0$. Therefore, (22) is violated. If $v$ and $c$ are satisfies Assumption 7(ii), then there exists
\(s < s^*\) such that \(v(n(z^*), s, z^*) - c(s, z^*) > 0\). Therefore, (22) is violated.

Therefore, if there is a stronger monotone pooling equilibrium, it must be the case where \(z^* = \bar{z}\). In this case, \(s^* = 0\). Otherwise, the sender type \(z\) arbitrarily close to 0 will get negative utility because \(t_h < \infty\) and \(\lim_{z \to 0} c(s, z) = \infty\) for all \(s > 0\) (Assumption 7).

Given \(z^* = \bar{z}\) and \(s^* = 0\), every sender will get positive utility upon being matched. We distinguish the two cases. If Assumption 7(ii) is satisfied, then \(s^* = 0\) makes the surplus equal to zero upon being matched, so no receiver is willing to pay \(t_h > 0\). Therefore there is no pooling equilibrium. If Assumption 7(i) is satisfied, then we must have \(n(z^*) > \bar{x}\) in order to make (21) hold, because \(v(\bar{x}, 0, z) = 0\) for all \(z\). This implies that there are more senders than receivers, and hence the market clearing condition is not satisfied. Therefore, there is no pooling equilibrium. ■

Theorems 4, 5, and 7 establish the unique stronger monotone equilibrium given each type of the feasible reaction sets.

6 Optimal equilibrium design

Lemma 9 shows that the DM only needs to consider a well-behaved stronger monotone equilibrium because it also covers a stronger monotone separating equilibrium and a stronger monotone pooling equilibrium. Combining with Lemma 9, the next two propositions reduce the DM’s the design problem of an optimal stronger monotone equilibrium to the choice of \(z_\ell\) and \(z_h\) subject to \(z_\ell \in Z\) and \(z_h \geq z_\ell\) given that a unique well-behaved stronger monotone equilibrium is the only stronger monotone equilibrium. In other words, for the DM’s optimal equilibrium design, choosing the lower and upper bounds of the feasible reaction interval is equivalent to choosing the two threshold sender types \(z_\ell\) and \(z_h\), one for market entry and the other for pooling on the top.

Proposition 1 (i) For any given \(z_\ell \in [\underline{z}, \bar{z}]\), there exists a unique solution \((s_\ell, t_\ell)\) of (10) and (11). (ii) Suppose that the DM chooses \(t_\ell\) in (i) above Then, \((z_\ell, s_\ell)\) solves (10) and (11) given \(t_\ell\) and it is unique.

Note that \(s_\ell = \zeta(\underline{x}, \underline{z}) = 0\) given \(z_\ell = \underline{z}\). \(s_\ell\) is also determined uniquely by \(z_\ell\) when \(z_\ell > \underline{z}\) because it solves

\[v(n(z_\ell), s, z_\ell) - c(s, z_\ell) = 0,\] (23)

which is the sum of (10) and (11) with equality. Therefore, Proposition 1 implies that we can retrieve \(t_\ell\) that induces \((s_\ell, z_\ell)\) from (10) with equality when \(z_\ell = \underline{z}\) or either (10) or
(11), each with equality when \( z_\ell > \bar{z} \). Therefore, the DM’s point of view, choosing \( z_\ell \) is equivalent to choosing \( t_\ell \).

**Proposition 2** (i) For any given \( z_h \in (z_\ell, \bar{z}) \), there exists a unique \((s_h, t_h)\) of (18) and (19). (ii) Suppose that the DM chooses \( t_h \) in (i) above. Then, \((z_h, s_h)\) solves (18) and (19) given \( t_h \) and it is unique. 

Note that \( s_h \) is determined solely by \( z_h \) because it solves 

\[
\mathbb{E}[v(n(z_h), s, z')|z' \geq z_h] - c(s, z_h) = v(n(z_h), \hat{\sigma}(z_h), z_h) - c(\hat{\sigma}(z_h), \bar{z}_h),
\]

which is the sum of (18) and (19) at \( z_h \). Therefore, Proposition 2 implies that we can retrieve \( t_h \) that induces \((s_h, z_h)\) from either (18) or (19). The DM can first choose the threshold sender type \( z_h \) and retrieve the upper bound of feasible reactions \( t_h \) that induces \( z_h \) in a well-behaved equilibrium.

**Lemma 9** As \( z_h \to \bar{z} \), \( \{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} \) converges to the stronger monotone separating equilibrium with the same lower threshold sender type \( z_\ell \). As \( z_h \to z_\ell \), \( \{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} \) converges to the stronger monotone pooling equilibrium in which \( t_\ell \) is a single feasible reaction, \( z_\ell \) is the threshold sender type for market entry, and \( s_\ell \) is the pooled action for senders in the market.\(^{14}\)

Given Propositions 1–2 and Lemma 9, we can say that for the DM’s point of view, choosing an interval of feasible reactions \( T = [t_\ell, t_h] \) is equivalent to choosing the corresponding \( z_\ell \) and \( z_h \).

Because a unique stronger monotone equilibrium is always well-behaved, this implies that the solution for the DM’s unconstrained design problem of the optimal stronger monotone equilibrium is the same as the solution for the DM’s design problem of the optimal stronger monotone equilibrium where the DM chooses the lower and upper threshold sender types, \( z_\ell \) and \( z_h \), for a well-behaved stronger monotone equilibrium.

Given a well-behaved stronger monotone equilibrium \( \{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} \) with the lower and upper threshold sender types, \( z_\ell \) and \( z_h \), the aggregate net surplus is

\[
\Pi(z_\ell, z_h) := \int_{z_\ell}^{z_h} v(n(z), \hat{\sigma}(z), z)dG(z) - \int_{z_\ell}^{z_h} c(\hat{\sigma}(z), z)dG(z)
+ \int_{\bar{z}}^{z_\ell} \mathbb{E}[v(n(z), s_h(z_h), z')|z' \geq z_h]dG(z) - \int_{z_h}^{\bar{z}} c(s_h(z_h), z)dG(z),
\]

\(^{14}\)As \( z_h \to z_\ell \), (18) and (19) become (16) and (17), each with equality if \( z^* \geq \bar{z} \). From (16) and (17), we can also directly derive \((t^*, s^*)\) given each \( z^* \) or \((z^*, s^*)\) given each \( t^* \) for a pooling equilibrium.
where \( s_h(z_h) \) is the pooled action chosen by all sender types above \( z_h \) and it is unique given any \( z_h \in [z_\ell, \bar{z}] \) because of Proposition 2(i). Note that the first line in \( \Pi(z_\ell, z_h) \) is the aggregate net surplus in the separating part of the equilibrium where matching is \textit{assortative} in terms of types. The second line is the aggregate net surplus in the pooling part of the equilibrium with \textit{random} matching and hence matching efficiency is lower in this pooling part but there is potential savings in the cost due to the pooled action chosen by all senders above \( z_h \).

\textbf{Theorem 8} The solution for the DM’s unconstrained design problem of the optimal stronger monotone equilibrium is the same as the solution for the DM’s design following problem of the optimal stronger monotone equilibrium:

\[
\max_{\tau > z_\ell \geq z, \zeta \geq z_\ell} \Pi(z_\ell, z_h)
\]

If \( z_\ell < z_h < \bar{z} \), then the stronger monotone equilibrium is strictly well-behaved. If \( z_\ell = z_h < \bar{z} \), it is separating. If \( z_\ell = z_h < \bar{z} \), it is pooling.

Generally, the aggregate equilibrium surplus depends on \( v, c, G \), and \( H \). For the optimal equilibrium design, we propose an approach that approximates the distribution of receiver types with the “shift” and “relative spacing” parameters given an arbitrary distribution of sender types. Consider a gross match surplus function that follows the form of \( v(x, s, z) = As^a x z \) with \( 0 \leq a < 1 \). The cost of choosing an action \( s \) is \( c(s, z) = \beta s^2/2 \) for the sender of type \( z \), where \( \beta > 0 \). The lowest sender type is \( z_\ell = 0 \). Note that \( v, c, \) and \( z_\ell = 0 \) satisfy Assumption 7.

A sender’s type follows a probability distribution \( G \), whereas a receiver’s type follows \( H \). Recall that \( n \) is defined as \( H^{-1} \circ G \) so that \( H(n(z)) = G(z) \) for all \( z \). We assume that \( n \) takes the following form:

\[
n(z) = kz^q,
\]

where \( k > 0 \) and \( q \geq 0 \). Note that \( k \) is the “shift” parameter and \( q \) is the “relative spacing” parameter. The relative spacing parameter \( q \) shows the relative heterogeneity of receiver types to sender types. Recall that \( n(z) \) denotes the type of a receiver who is matched with the sender of type \( z \) in the stronger monotone separating equilibrium. This approach is general in the sense that it approximates the distribution of receiver types with the “shift” and “relative spacing” parameters for any arbitrary distribution of sender types.

To derive a well-behaved stronger monotone equilibrium, we first need to solve the first-order differential equilibrium \( \tilde{\mu}'(s) = \phi(s, \tilde{\mu}(s)) \) in (14) with the initial condition \((z_\ell, s_\ell)\). The value of \( s_\ell \) only depends on \( z_\ell \). If \( z_\ell = 0 \), then \( s_\ell(z_\ell) = \zeta(0, 0) = 0 \). If \( z_\ell > 0 \),
then \( s_\ell(z_\ell) \) is determined by (23) and it is \( s_\ell(z_\ell) = \left( \frac{A_k z_\ell q + 2}{b} \right)^{\frac{1}{2-a}} \). Note that \( s_\ell(z_\ell) \) is continuous everywhere including at \( z_\ell = 0 \).

**Proposition 3** Given any initial condition \((z_\ell, s_\ell(z_\ell))\), the solution for first-order differential equation \( \mu'(s) = \phi(s, \mu(s)) \) is

\[
\tilde{\mu}(s) = \begin{pmatrix}
\left( \frac{2\beta (2 + q)}{A_k} \right) \frac{s^{2-a}}{2 + a + aq} \\
+ \left( \frac{s_\ell(z_\ell)}{s} \right)^{a(2+q)} & \left[ \frac{z_\ell^q a}{A_k (2 + a + aq)} - \frac{\beta z_\ell^{q+2}}{A_k (2 + a + aq)} \right]
\end{pmatrix} \frac{1}{2 + q}.
\]

Note that \( \tilde{\sigma}(z) \) is the inverse of \( \tilde{\mu}(s) \), which is derived numerically as \( \tilde{\mu}(s) \) does not allow for a closed-form solution for its inverse. Given \( z_h, s_h(z_h) \) is unique and it solves (24), which is

\[
A k s_h^{q} E[z' | z' \geq z_h] - \beta s_h^2 z_h = A k \tilde{\sigma} (z_h)^a z_h^{1+q} - \beta \frac{\tilde{\sigma} (z_h)^2}{z_h}.
\]

We need to numerically derive \( s_h(z_h) \) as it does not allow for a closed form solution. Given a choice of \( z_\ell \) and \( z_h \), the aggregate net surplus is

\[
\Pi_w(z_\ell, z_h, q, a, G) = \int_{z_\ell}^{z_h} \left[ v(n(z), \tilde{\sigma}(z), z) - c(\tilde{\sigma}(z), z) \right] g(z) dz \\
+ \int_{z_h}^{\bar{z}} \left[ E[v(n(z), s_h(z_h), z') | z' \geq z_h] - c(s_h(z_h), z) \right] g(z) dz \\
= \int_{z_\ell}^{z_h} \left( A k z^{q+1} \tilde{\sigma}(z)^a - \beta \frac{\tilde{\sigma}(z)^2}{z} \right) g(z) dz \\
+ A k s_h(z_h)^a E[z' | z' \geq z_h] \int_{z_h}^{\bar{z}} z^q g(z) dz - \beta s_h(z_h)^2 \int_{z_h}^{\bar{z}} \frac{1}{z} g(z) dz,
\]

where \( \tilde{\sigma}(z) = \tilde{\sigma}(z) \) for \( z \in [z_\ell, z_h] \).

Given \((a, q, G)\), we can find the best well-behaved equilibrium through the following maximization problem:

\[
\max_{(z_\ell, z_h)} \Pi_w(z_\ell, z_h, q, a, G) \\
\text{subject to } 0 \leq z_\ell \leq z_h \leq \bar{z}.
\]
6.1 More efficient non-separating equilibria

Before turning to numerical analysis, we show that the DM can always increases the efficiency of the stronger monotone equilibrium by restricting the set of feasible reactions when \( q \) and \( a \) are small.

Suppose that the DM fixes the lower bound of feasible reactions such that \( z_\ell = 0 \) and hence \( s_\ell(z_\ell) = 0 \) (no sender stays out of the market). In this case, the belief function \( \tilde{\mu}(s) \) allows for the closed-form expression of its inverse, which is the sender’s equilibrium action function in the separating equilibrium:

\[
\tilde{\sigma}(z) = \left( \frac{A_kaq + a + 2}{2\beta \left( q + 2 \right)} \right)^{\frac{1}{2-a}} z^\frac{q+2}{2-a} \text{ for } z < z_h.
\]

The aggregate net surplus in the well-behaved stronger monotone separating equilibrium is

\[
\Pi_w(0, z_h, q, a, G) = \left( \left( A_k \right)^{\frac{2}{2-a}} \left( \frac{aq + a + 2}{2\beta (q + 2)} \right) \left( \frac{2}{2-a} \right) \right) \int_0^{z_h} z^{2q+2+a} dG(z)
\]

\[
+ Aks(z_h)^q \mathbb{E}[z | z \geq z_h] \int_{z_h}^z z^q g(z) dz - \beta s_h(z_h)^2 \int_{z_h}^\bar{z} \frac{1}{z} g(z) dz.
\]

As \( z_h \) approaches \( \bar{z} \), the maximum of the support of \( G \), \( \Pi_w(0, z_h, q, a, G) \) becomes the aggregate net surplus \( \Pi_s(a, q, G) = \Pi_w(0, \bar{z}, q, a, G) \) without any restrictions on feasible reactions (i.e., \( \Pi_s(a, q, G) \) is the aggregate net surplus in the baseline separating equilibrium). We show that, when the relative heterogeneity of receiver types \( (q) \) and the productivity of the sender action \( (a) \) are not too large, there is a strictly well-behaved equilibrium only with the binding upper bound of feasible reactions that is more efficient than the separating equilibrium without any restrictions on the feasible reactions regardless of \( G \).

**Theorem 9** There are \( \hat{q}, \hat{a} > 0 \) such that for any given \( (q, a) \in [0, \hat{q}] \times [0, \hat{a}] \), the DM can set up the interval of feasible reactions \([0, \hat{t}]\) that induces a unique strictly well-behaved stronger monotone equilibrium, which is more efficient than the stronger monotone separating equilibrium with no restrictions on feasible reactions. Given \([0, \hat{t}]\), it is a unique stronger monotone equilibrium.

**Proof.** First, we construct a (unique) strictly well-behaved stronger monotone equilib-
rium with $0 = z_ℓ < z_h < \text{supremum of the support of } G$. Let $s_h(z_h, a, q)$ be the value of $s_h$ that solves (26) at every $a$ and $q$. Because functions in (26) are continuous in $a$ and $q$, $s_h(z_h, a, q)$ is continuous in $a$ and $q$. Given (26), we have

$$
\lim_{q,a \to 0} \left( A \frac{s_h(z_h, q, a)^2}{z_h} k z_h q \mathbb{E} [z | z \geq z_h] - \beta \frac{s_h(z_h, q, a)^2}{z_h} \right)
= \lim_{q,a \to 0} \left( A k \tilde{\sigma} (z_h)^a z_h^{q+1} - \beta \frac{\tilde{\sigma} (z_h)^2}{z_h} \right).
$$

Therefore, we have that $\lim_{q,a \to 0} s_h(z_h, q, a) = \sqrt{z_h^2 A k (\mathbb{E} [z | z \geq z_h] - 1) / \beta}$. This implies that

$$
\lim_{q,a \to 0} \Pi_w(0, z_h, q, a, G) = \int_{z_h}^{\bar{z}} A k \frac{zdG(z)}{2} + \int_{z_h}^{\mathbb{E} [z | z \geq z_h]} A k \frac{zdG(z)}{2} - z_h^2 A k (\mathbb{E} [z | z \geq z_h] - 1) \frac{1}{z} dG(z).
$$

Taking the limit of $\lim_{q,a \to 0} \Pi_w(0, z_h, q, a, G)$ with respect to $z_h$ yields

$$
\lim_{z_h \to 0} \left[ \lim_{q,a \to 0} \Pi_w(0, z_h, q, a, G) \right] = \int_{0}^{\mathbb{E} [z | z \geq z_h]} A k zdG(z) = A k \mu_z,
$$

where $\mu_z$ is the unconditional mean of the sender type $z$.

On the other hand, the limit of the aggregate net surplus in the stronger monotone separating equilibrium is

$$
\lim_{q,a \to 0} \Pi_s(q, a, G) = \int_{0}^{\bar{z}} A k zdG(z) - \int_{0}^{\mathbb{E} [z | z \geq z_h]} A k zdG(z) = \frac{A k \mu_z}{2}.
$$

Therefore, we have that

$$
\lim_{z_h \to 0} \left[ \lim_{q,a \to 0} \Pi_w(0, z_h, q, a, G) \right] - \lim_{q,a \to 0} \Pi_s(q, a, G) = \frac{A k \mu_z}{2} > 0. \quad (27)
$$

Because $\Pi_w(0, z_h, q, a, G)$ and $\Pi_s(q, a, G)$ are continuous, there exists $\hat{q} > 0$, and $\hat{a} > 0$ and $\hat{z}_h(\hat{q}, \hat{a}) \in \text{Int } Z$ such that for every $(q, a) \in [0, \hat{q}] \times [0, \hat{a}]$ and every $z_h \in (0, \hat{z}_h(\hat{q}, \hat{a}))$, $\Pi_w(0, z_h, q, a, G) > \Pi_s(q, a, G)$. We can retrieve $t_h$ given $z_h \in (0, \hat{z}_h(\hat{q}, \hat{a})].$ ■

In the well-behaved stronger monotone equilibrium constructed above, a small fraction of senders and receivers on the low end of the type distribution follow their equilibrium sender actions, reactions, and assortative matching that would have occurred in the
stronger monotone separating equilibrium. The rest of senders and receivers are matched randomly because the rest of senders all choose the same action. We can also establish the same result for a stronger monotone pooling equilibrium as represented in Theorem 10 below.

**Theorem 10** There are $\tilde{q}, \tilde{a} > 0$ such that, for any given $(q, a) \in [0, \tilde{q}] \times [0, \tilde{a}]$, the DM can induce a unique stronger monotone pooling equilibrium that is more efficient than the stronger monotone separating equilibrium without restrictions on feasible reactions.

The intuition behind Theorem 10 is the same as that behind Theorem 9. A major difference is that a pooling equilibrium forces a small fraction of senders and receivers on the low end of type distribution to stay out of the market even though they can produce positive net surplus, whereas everyone is matched in the strictly well-behaved equilibrium identified in 9. Theorems 9 and 10 in fact show that a separating equilibrium is not optimal in the classical Spencian model (Spence 1973) of pure signaling with no heterogeneity of receivers (i.e., $a = q = 0$).

### 6.2 Numerical analysis

We turn our attention to numerical analysis. For the concreteness of our numerical analysis, one can think of senders as workers and receivers as firms. A sender’s type can be then viewed as unobservable skill and her action as observable skill. A firm’s type can be viewed as its size. In macro/development, firm size is measured by the amount of labour it employs; In finance, it is measured by the firm’s market value if it is publicly traded. In an entry-level job market, a worker’s unobservable skill could be her ability to understand a task given to her and to figure out how to complete it. In a managerial job market, a worker’s unobservable skill could be her ability to come up with new business idea or innovation.

Poschke (2018) documented that the mean and variance of the firm size distribution are larger in rich countries and increased over time for US firms when firm size is measured by the number of workers employed in a firm. While Poschke (2018) showed that a frictionless general equilibrium model of occupational choice with skill-biased change accounts for key aspects of the US experience, his model is mute to (a) the implication of such changes in the firm size distribution on efficiency and (b) the effectiveness of the DM’s policy choice over the feasible wages on improving efficiency. We are interested in addressing these questions, examining the impact of the distributional changes in firm size on workers’ investment in their observable skill, $s$.  

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For numerical illustrations, we consider a specific distribution of $G$ with various combinations of $q$ and $a$. The support of the worker’s unobservable skill $z$ is set to be $[0, 3]$ and is generated from the beta distribution multiplied by 3 with the following shape parameters:

$$\{(1, 1), (5, 5), (3, 5), (5, 3)\}.$$

Figure 2 shows the probability density functions with different shape parameter values. Note that Beta(1,1) corresponds to the uniform distribution and Beta(5,5) to a symmetric bell-shaped distribution. Beta(3,5) and Beta(5,3) correspond to right-skewed and left skewed unobservable skill distributions, respectively. The model parameters $q$ and $a$ vary over $\{0, 0.1, \ldots, 2\}$ and $\{0, 0.1, \ldots, 0.9\}$, respectively. Therefore, we compute the optimal well-behaved equilibrium (i.e., optimal stronger monotone equilibrium) for 840 ($= 4 \times 21 \times 10$) different specifications in total. For the remaining parameters, we set $A = 1$, $k = 1$, and $\beta = 0.5$. Note that both the mean and the variance of firm size $x = z^q$ increase in $q$ across all these settings, which reflects the empirical findings in Poschke (2018). Finally, we set the effective zero as $10^{-6}$.

**Figure 2: Probability Density Functions of the Beta Distribution**

| Beta(1,1) | Beta(5,5) | Beta(3,5) | Beta(5,3) |
|-----------|-----------|-----------|-----------|
| ![Graph](image1.png) | ![Graph](image2.png) | ![Graph](image3.png) | ![Graph](image4.png) |

Notes. The send type variable $z$ is generated by $3 \cdot \text{Beta}(\cdot, \cdot)$.

### 6.2.1 Optimal stronger monotone equilibrium

Figures 3–4 show the optimal solution paths and the relative surplus gains of the well-behaved equilibrium, respectively. It turns out that $z_\ell = 0$ in all specifications. Thus, in Figure 3, we report only $z_h$ that solves the optimization problem in each design. In the graph, the horizontal axis denotes the parameter value of $q$. To make the graph readable, we report the solution paths of $z_h$ for four different values of $a = 0, 0.3, 0.6, \text{ and } 0.9$.

We illustrate the well-behaved equilibrium with two examples. When $G$ follows Beta(1,1) and $(q, a) = (1.5, 0.9)$, the optimal well-behaved equilibrium is achieved at $(z_\ell, z_h) = (0, 2.5)$ as denoted by a circle point on the vertical line in the top-left graph.
Figure 3: Solutions $z_h$ for Different Parameter Specifications

Notes. We show only solutions $z_h$ since $z_l = 0$ for all designs. Each line represents solutions over $q \in [0, 2]$ for each $a$ value. For example, the circle point at $(1.5, 2.5)$ in Beta(1,1) denotes the best well-behaved equilibrium of $(z_l, z_h) = (0, 2.5)$ when $z$ follows $3 \times Beta(1, 1)$ and $(q, a) = (1.5, 0.9)$. 
of Figure 3. This implies that every worker enters the market and that the workers with $0 \leq z < 2.5$ differentiate themselves with unique observable skill choices. Furthermore, we can compute from (26) that those workers with $z \geq 2.5$ choose pooled observable skill $s_h = 27.1$. The upper threshold unobservable skill $z_h = 2.5$ is induced when the DM sets the upper bound of feasible reactions $t_h = 894.6$ (We can derive the value of $t_h$ from (18) or (19) given $z_h$ and $s_h$). From Theorem 8, we can conclude that this is a unique optimal stronger monotone equilibrium that maximizes the aggregate net surplus and that it is reached when the DM sets the set of feasible wages as $[t_\ell, t_h] = [0, 894.6]$.

The second example is in case that $G$ follows Beta(1,1) and $(q, a) = (0.2, 0.2)$. Then, the optimal well-behaved equilibrium is achieved at $(z_\ell, z_h) = (0, 0.3)$. Those workers with $z \geq 0.3$ choose pooled observable skill $s_h = 0.8$. This equilibrium is reached when the DM sets the set of feasible wages as $[t_\ell, t_h] = [0, 1.3]$.

We have some remarks on these numerical results. First, as we have discussed above, $z_\ell$ is equal to zero for all specifications. In the case of the separating equilibrium, it is clear that $z_\ell = 0$ is optimal since any positive $z_\ell$ does not improve efficiency. With $z_\ell > 0$, we lose some positive surplus that could have been created by lower matches. At the same time, it further increases the inefficiently high equilibrium action of every worker with unobservable skill $z > z_\ell$. However, this is not certain in the well-behaved equilibrium with $z_h < 3$. In this case, raising $z_\ell$ leads to an increase in the pooled observable skill $s_h$ chosen by worker with unobservable skill above $z_h$. Because $s_h$ is lower than the equilibrium observable skill chosen by for the worker with the highest unobservable skill level in a separating equilibrium, it is possible that $s_h$ may be even lower than the efficient level of observable skill for some workers on the top end of the unobservable skill distribution. In this case, raising $s_h$ through raising $z_\ell$ increases efficiency for those workers while decreasing efficiency for the other workers. Our numerical analysis shows that, when $z_\ell > 0$, the efficiency loss by lower types dominates any possible efficiency gains by higher types across all designs considered.

Second, $z_h$ is strictly increasing in $a$ for any given $q$. Also, $z_h$ is strictly increasing in $q$ for any given $a$ besides $a = 0$. When $a = 0$, (i.e., the worker’s observable skill is

\[ z_\ell = 3 \text{ is } \tilde{\sigma}(3) = 39.3 \text{ and the receiver’s feasible reaction for her is } \tilde{\tau}(\tilde{\sigma}(1)) = 3355.5 > t_h = 894.6. \]

The aggregate net surplus in the separating equilibrium is 26.2, whereas the aggregate net surplus in the optimal well-behaved equilibrium is 26.5. Therefore, the optimal well-behaved equilibrium increases the aggregate net surplus by 1.1%.

The aggregate net surplus in the optimal well-behaved equilibrium is 1.3, whereas the aggregate net surplus in the separating equilibrium is 1. Therefore, the optimal well-behaved equilibrium increases the aggregate net surplus by 30.5%.
not productive at all), we observe $z_h = 0$ in a range of $q$ values. This result implies that the optimal equilibrium becomes the pooling equilibrium. To make the well-behaved equilibrium deviate from the pooling equilibrium, the relative spacing parameter $q$ has to be larger than a threshold that depends on the distribution of $z$. Otherwise, the inefficiency associated with high costs of separating themselves among senders with lower unobservable skill dominates matching efficiency created by separating themselves and it is the optimal equilibrium design to force every worker not to choose any observable skill by setting $z_h = 0$.

Third, the optimal well-behaved equilibrium converges to the separating equilibrium as $a$ converges to 1 and $q$ increases. For example, when we conduct an additional analysis with Beta(1,1), $a = 0.99$ and $q = 25$, the optimal well-behaved equilibrium is $(z_{\ell}, z_h) = (0, 2.99)$, which is very close to the separating equilibrium with $z_{\ell} = 0$ and $z_h = 3$. However, this occurs only with extreme parameter values. The strictly well-behaved equilibrium is still the optimal equilibrium in a wide range of $(q, a)$, that is, the cost savings associated with the pooled observable skill choice by workers above $z_h$ outweighs the decrease in matching efficiency due to random matching in the pooling part of the equilibrium.
Notes. Each cell in the graph shows the difference of efficiency gains in the corresponding cell of beta distributions. Specifically, let $R(\beta_1) := 100 \times (\Pi_w(\beta_1) - \Pi_s(\beta_1))/\Pi_s(\beta_1)$ be the ratio of the net surplus gains given a beta distribution denoted by $\beta_1$. Then, the difference of ratios (ratio_diff) is defined as $R(\beta_1) - R(\beta_2)$ for two beta distributions $\beta_1$ and $\beta_2$, where $\beta_1$ first-order stochastically dominates $\beta_2$.

Fourth, the different shapes of the probability density function affect the curvature of the $z_h$ paths for all $a$ and the threshold of $q$ that makes the well-behaved equilibrium the optimal equilibrium for $a = 0$.

Finally, we show the relative surplus gains from the optimal well-behaved equilibrium in Figure 4. For each design, we compute the aggregate net surpluses for the optimal well-behaved and separating equilibria. Then, we compute the relative gain of the well-behaved equilibrium by $100 \times (\Pi_w - \Pi_s)/\Pi_s$. For example, under Beta(1,1), the relative gains are 52.8% and 0.7% when $(a,q)$ are $(0.1,0.1)$ and $(0.9,2.0)$, respectively. We also find that the surplus gains become larger as there are relatively more workers with high unobservable skill levels, i.e., more density weights on higher $z$ values (see, for example, Beta(3,5) and Beta(5,3) in Figure 4).

To elaborate the final point in detail, we ordered the Beta distributions according to the first-order stochastic dominance and check whether the efficiency gains are larger in the case of a stochastically dominating distribution. Note that Beta(5,3) dominates Beta(5,5), which again dominates Beta(3,5) as denoted in Figure 7 in the appendix. In Figure 5, we show the differences of relative surplus gains between the paired beta distributions. For example, in the figure of “Beta(3,5) vs. Beta(5,5)”, we subtract the heat map of Beta(3,5) from Beta(5,5). More specifically, let $R(\beta_i) := 100 \times (\Pi_w(\beta_i) - \Pi_s(\beta_i))/\Pi_s(\beta_i)$ be the ratio of the net surplus gains given a beta distribution denoted by $\beta_i$. Then, $R(\beta_1) - R(\beta_2)$ is the difference of ratios (ratio_diff) for two beta distributions $\beta_1$ and $\beta_2$, where $\beta_1$ first-order stochastically dominates $\beta_2$. The results show that the differences are positive across all $(a,q)$. This implies that the efficiency gains by the stochastically dominating distribution are uniformly larger than that by the dominated one.
6.2.2 Underlying efficiency and relative surplus gains

We now examine the change in the efficiency of the baseline separating equilibrium and the change in the relative net surplus gain in the optimal well-behaved equilibrium as $q$ increases (i.e., as the mean and variance of the firm size distribution increase). To define the efficiency measure, we first derive the maximum aggregate net surplus under complete information. It is based on the efficient choice of observable skill $s^*(z)$ by a worker with unobservable skill $z$ with a firm with size $n(z) = k z^q$ as her employee:

$$s^*(z) \in \arg \max \left[ A k s^q z^q + 1 - \beta \frac{s^2}{z} \right].$$

We have a unique $s^*(z) = \left[ a k z^q + 2 \right] \frac{1}{2 - \alpha}$. Then the maximum aggregate net surplus is

$$\Pi^*(a, q, G) = \int_{z} \Pi^*(a, q, G).$$

On the other hand, the separating equilibrium without the DM’s intervention is $\Pi_s(a, q, G) = \Pi_s(z, z, a, q, G)$. The efficiency of the separating equilibrium is defined as

$$E(a, q, G) = \frac{\Pi_s(a, q, G)}{\Pi^*(a, q, G)}.$$

Figure 6 shows the efficiency of the separating equilibrium on the first column and the relative net surplus gain of the optimal well-behaved equilibrium on the second as we change the value of $a \in \{0, 0.3, 0.6, 0.9\}$. Given any value of $a$ and any unobservable skill distribution, the efficiency of the separating equilibrium increases as $q$ increases. This may suggest that the efficiency in rich countries is higher than poor countries and that it also increases over time in the U.S., given the empirical findings in Poschke (2018). As the efficiency of the separating equilibrium increases in response to an increase in $q$, the relative net surplus gain in the optimal well-behaved equilibrium decreases. In addition, as the value of $a$ is increases, the rate of the increase in the efficiency with respect to an increase in $q$ is getting smaller and the relative gain is smaller at every value of $q$. For example, when $a = 0$, the relative net surplus gain is above 20% at $q = 1$ regardless of the beta distribution, whereas, with $a = 0.9$, it is less than 10% at $q = 1$. As $a$ and $q$ both increase, the relative net surplus gain quickly converges to zero.

Our result suggests that in terms of efficiency improvement, the DM’s equilibrium design is most effective (i) when the firm size distribution has the smallest mean and variance; and (ii) when the productivity parameter of observable skill ($a$ in our notation)

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Figures 8 - 10 in Appendix provide the full graphs based on all values of $a$.  

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Figure 6: Efficiency Measures and Relative Surplus Gains

| Eff. Measures | Rel. Sur. Gains |
|---------------|----------------|
| $a = 0.0$     |                |
| $a = 0.3$     |                |
| $a = 0.6$     |                |
| $a = 0.9$     |                |

Notes. The efficient measures are computed by $\Pi_s/\Pi^*$, where $\Pi^*$ is the aggregate net surplus without any restriction on feasible reactions. The relative surplus gains are computed by $100 \times (\Pi_w - \Pi_s)/\Pi_s$, where $\Pi_w$ is aggregate net surplus of the well-behaved equilibrium.
has the lowest value. As the mean and variance of the firm size distribution increase or
the productivity parameter of observable skill increases, our result shows that the DM’s
equilibrium design quickly loses its effectiveness. This highlights how a trade-off between
matching efficiency and (net) signaling costs changes as the firm size distribution becomes
more spread in terms of its mean and variance and the direct productivity of observable
skill increases and its impact on optimal equilibrium design.

7 Concluding remarks

In this paper we generalize Spencian competitive signaling (Spence (1973)) with two-
sided matching. A decision maker can choose a set of feasible reactions before senders
and receivers move. We characterize a unique stronger monotone equilibrium (unique D1
equilibrium) given each set of feasible reactions. We propose a general method that the
DM can use for the design of an optimal unique stronger monotone equilibrium and study
the optimal equilibrium design in various settings. Our analysis sheds light on the impact
of a trade-off between matching efficiency and signaling costs on optimal equilibrium
designing and how the trade-off depends on the relative heterogeneity of receiver types to
sender types, the distribution of sender types, and the productivity of the sender’s action.
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Appendix

A Lemma 10

Lemma 10 Consider an equilibrium \( \{\sigma, \mu, \tau, m\} \). If Assumptions A is satisfied, the equilibrium satisfies the following properties: (i) \( \sigma \) is non-decreasing in \( z \), (ii) \( \mu \) is non-decreasing in the subset of domain, range \( \sigma \), with respect to the stronger set order: for \( s, s' \in \text{range} \ \sigma \), \( s \geq s' \) implies \( \text{supp} \ \mu(s') \leq c \text{supp} \ \mu(s) \), (iii) \( \tau \) is increasing.

Proof. For any \( s \) and \( s' \) in \( S^* \) such that \( s > s' \), consider a sender who chooses \( s \) in equilibrium. Then, her utility must satisfy

\[
  u(\tau(s), s, z) \geq u(\tau(s'), s', z) \tag{A1}
\]

Because \( s > s' \) and \( u \) is decreasing in \( s \) and increasing in \( t \), (A1) implies \( \tau(s) > \tau(s') \) and hence \( \tau \) is monotone increasing.

Now we prove the monotonicity of \( \sigma \) by contradiction. Suppose that for \( z > z' \), type \( z \) chooses \( s' \in S^* \) and type \( z' \) chooses \( s \in S^* \) such that \( s > s' \) in equilibrium. This implies \( u(\tau(s), s, z') \geq u(\tau(s'), s', z') \). Because \( (\tau(s), s) > (\tau(s'), s') \), the strict single crossing property of \( u \) in \( ((t, s); z) \) implies that \( u(\tau(s), s, z) > u(\tau(s'), s', z) \) for any \( z > z' \). This contradicts that \( z \) chooses \( s' \). Therefore, \( \sigma \) is non-decreasing over types that choose actions in \( S^* \).

For the non-decreasing property of \( \sigma \), we only need to show that any \( z' \) with \( \eta \) is no higher than \( z \) with \( s \in S^* \) in equilibrium. By contradiction, suppose that there exist \( z' \) with \( \eta \) and \( z \) with \( s \in S^* \) such that \( z' > z \). Then, we have that \( 0 \leq u(\tau(s), s, z) < u(\tau(s), s, z') \), where the weak inequality holds because \( s \) is the optimal choice for type \( z \) and the strict inequality holds due to the monotonicity of \( u \) in type. This contradicts that taking no action (i.e., null action \( \eta \)) is optimal for type \( z' \). This completes the proof of the non-decreasing property of \( \sigma \).

We prove the monotonicity of \( \text{supp} \ \mu(s) \) in the subset of domain, range \( \sigma \), in the stronger set order by contradiction. Suppose that for \( s > s' \), there exist \( z \in \text{supp} \ \mu(s) \) and \( z' \in \text{supp} \ \mu(s') \) such that \( z' > z \). Because \( z \in \text{cl} \ \{\tilde{z}|\sigma(\tilde{z}) = s\} \) and \( z' \in \text{cl} \ \{\tilde{z}|\sigma(\tilde{z}) = s'\} \), it contradicts that \( \sigma \) is non-decreasing. Therefore, \( \mu \) is non-decreasing in the subset of domain, range \( \sigma \), with respect to the stronger set order. \( \blacksquare \)
Lemma 11

Let $\sigma$ and $\mu$ be a sender action function and a belief function in equilibrium respectively. If Assumption A is satisfied, the belief $\mu(s)$ conditional on $s \notin$ range $\sigma$ that passes Criterion D1 is unique and it is characterized as follows:

1. If $s$ belongs to the interval of off-path sender actions induced by the discontinuity of $\sigma$ at $z$, then $\supp \mu(s) = \{z\}$.

2. Let $\overline{z}$ be the least upper bound of $Z$ if it exists. If $s > \sigma(\overline{z})$, then $\supp \mu(s) = \{\overline{z}\}$.

3. Let $\underline{z}$ be the greatest lower bound of $Z$ if it exists. If $s < \sigma(\underline{z})$, then $\supp \mu(s) = \{\underline{z}\}$.

**Proof.** According to the proof of Lemma 10, $\sigma$ is non-decreasing if Assumption A is satisfied. If there are types who choose $\eta$ and stay out of the market, those types are lower than the types who choose actions in $S$ because of the non-decreasing property of $\sigma$. If this happens, let $z_\eta := \min \{z \in Z| \sigma(z) \in S^*\}$ (or $z_\eta := \inf \{z \in Z| \sigma(z) \in S^*\}$ if $\min \{z \in Z| \sigma(z) \in S^*\}$ does not exist).

At any discontinuity point $z$, let $\sigma(z_+) := \lim_{k \uparrow z} \sigma(k)$ and $\sigma(z_-) := \lim_{k \downarrow z} \sigma(k)$. For the proof of item 1, we first consider the case where a discontinuity occurs at $z > z_\eta$ and $\sigma$ is only right continuous at $z$, then, $\sigma(z_+) = \sigma(z)$. In this case, $[\sigma(z_-), \sigma(z)]$ is the interval of off-path sender actions due to the discontinuity at $z$. We show that Criterion D1 places zero posterior weight on $z' \neq z$.

**Case 1:** We show that $z'$ cannot be in the support of $\mu(s)$ for any $s \in [\sigma(z_-), \sigma(z)]$ if $z' > z$. On the contrary, suppose that $z' \in \supp \mu(s)$ for some $s \in [\sigma(z_-), \sigma(z)]$ when $z < z'$. If $z < z'$, then we have $z''$ such that $z < z'' < z'$. For the proof, it is sufficient that if type $z''$ is weakly worse off by deviating to $s \in [\sigma(z_-), \sigma(z)]$, then type $z'$ is strictly worse off with the same deviation. For a reaction $t$ chosen by the receiver after observing such $s$, let

$$u(t, s, z'') \leq u(\tau(\sigma(z'')), \sigma(z''), z''). \quad (A2)$$

Because $s \in [\sigma(z_-), \sigma(z)]$, we have that $s < \sigma(z)$. Because $\sigma$ is non-decreasing, we have that $\sigma(z) \leq \sigma(z'')$. These two inequality relations yield $s < \sigma(z'')$. Because the first part of Assumption A says that $u$ is decreasing in $s$ and increasing in $t$, we must have that $t < \tau(\sigma(z''))$ in order to satisfy $(A2)$. Because $s < \sigma(z'')$ and $t < \tau(\sigma(z''))$, we can use the strict single crossing property of $u$ in Assumption A to show that $(A2)$ implies that for $z' > z''$

$$u(t, s, z') < u(\tau(\sigma(z'')), \sigma(z''), z'). \quad (A3)$$
On the other hand, we have that

\[ u(\tau(\sigma(z'')), \sigma(z''), z') \leq u(\tau(\sigma(z')), \sigma(z'), z') \tag{A4} \]

in equilibrium. Combining (A3) and (A4) yields that for \( z' > z'' \),

\[ u(t, s, z') < u(\tau(\sigma(z')), \sigma(z'), z') , \]

which shows that type \( z' \) is strictly worse off with the same deviation. (8), the contra-
positive of (7) in the definition of Criterion D1 implies that any \( z' > z \) cannot be in the
support of \( \mu(s) \) for any \( s \in [\sigma(z_-, \sigma(z)) \).

Case 2: We now show that \( z' \) cannot be in the support of \( \mu(s) \) for any \( s \in [\sigma(z_-, \sigma(z)) \) if \( z' < z \). On the contrary, suppose that \( z' \in \text{supp} \mu(s) \) for some \( s \in [\sigma(z_-, \sigma(z)) \) when \( z' < z \). We work with the original condition (7). If \( z' < z \), then we have \( z'' \) such that \( z' < z'' < z \). For a reaction \( t \) chosen by the receiver after observing such \( s \), let

\[ u(t, s, z') \geq u(\tau(\sigma(z')), \sigma(z'), z') \tag{A5} \]

that is, type \( z' \) is weakly better off by deviating to some \( s \in [\sigma(z_-, \sigma(z)) \). On the other
hand, we have that

\[ u(\tau(\sigma(z')), \sigma(z'), z') \geq u(\tau(\sigma(z'')), \sigma(z''), z') \tag{A6} \]

in equilibrium. Combining (A5) and (A6) yields

\[ u(t, s, z') \geq u(\tau(\sigma(z'')), \sigma(z''), z') \tag{A7} \]

Because \( s > \sigma(z'') \), the monotonicity of \( u \) in Assumption A implies that \( t > \tau(\sigma(z'')) \) in
order to satisfy (A7). Then, applying the strict single crossing property of \( u \) in Assumption
A to (A7), we have that for \( z'' > z' \),

\[ u(t, s, z'') > u(\tau(\sigma(z'')), \sigma(z''), z'') , \]

which shows that the sender of type \( z'' \) is strictly better off with the same deviation. Crite-
rion D1 implies that any \( z' < z \) cannot be in the support of \( \mu(s) \) for any \( s \in [\sigma(z_-, \sigma(z)) \). Therefore, the only \( \mu(s) \) conditional on \( s \in [\sigma(z_-, \sigma(z)) \) that passes Criterion D1 puts
all the posterior weights on \( z \) and hence \( \text{supp} \mu(s) = \{ z \} \).
Item 1 can be proved similarly in the cases where \( \sigma \) is only left continuous at \( z \) or \( \sigma(z_-) < \sigma(z) < \sigma(z_+) \), or in the case where \( \sigma \) is discontinuous at \( z_\eta \). The only thing we need to be careful about is the case where \( \sigma \) is discontinuous at \( z_\eta \). If \( \sigma \) is only right continuous at \( z_h \), the discontinuity at \( z_h \) creates off-path actions \( s \in S \) such that \( s < \min S^* \) (or \( s \leq \inf S^* \) if \( \min S^* \) does not exist). We can show that any \( z' > z_h \) cannot be in the support of \( \mu(s) \) for any \( s < \min S^* \) following the logic of Case 1 above.

Consider \( z' < z_\eta \). If \( z' < z \), then we have \( z'' \) such that \( z' < z'' < z_h \). Note that \( \sigma(z') = \sigma(z'') = \eta \) and hence, equilibrium utilities for both types are zero. For a reaction \( t \) chosen by the receiver after observing \( s \in S \setminus S^* \), let \( u(t, s, z') \geq 0 \). By the increasing property of \( u \) in type in Assumption A.(i), \( u(t, s, z') \geq 0 \) implies that \( u(t, s, z'') > 0 \). This shows that any \( z' < z_\eta \) cannot be in the support of \( \mu(s) \) given Criterion D1. Therefore, the only \( \mu(s) \) conditional on \( s \in S \setminus S^* \) that passes Criterion D1 puts all the posterior weights on \( z \) and hence \( \text{supp} \mu(s) = \{z_h\} \). Item 1 can be proved similarly if only left continuous at \( z_h \) or \( \sigma_-(z_h) < \sigma(z_h) < \sigma_+(z_h) \).

For the proof of item 2, we can following the proof of Case 2 to show that the only \( \mu(s) \) conditional on \( s > \sigma(\bar{z}) \) that passes Criterion D1 puts all the posterior weights on \( \bar{z} \) and hence \( \text{supp} \mu(s) = \{\bar{z}\} \) for \( s > \sigma(\bar{z}) \). Similarly, for the proof of item 3, we can follow the proof of Case 1 above to show that only \( \mu(s) \) conditional on \( s < \sigma(\bar{z'}) \) that passes Criterion D1 puts all the posterior weights on \( \bar{z'} \) and hence \( \text{supp} \mu(s) = \{\bar{z}'\} \) for \( s < \sigma(\bar{z'}) \).

\[ \blacksquare \]

C Corollary 1

According to the proof of Lemma 10, \mu is non-decreasing in the subset of domain, range \sigma, with respect to the stronger set order if Assumption A is satisfied.

We first show that a non-decreasing \mu in the stronger set order passes Criterion D1. Consider \( s \notin \text{range} \sigma \). If \( s > \sigma(\bar{z}) \), a non-decreasing \mu in the stronger set order must have \( \{\bar{z}\} \) as \text{supp} \mu(s). On the contrary, suppose that \( z \in \text{supp} \mu(s) \) for \( s > \sigma(\bar{z}) \) and \( z < \bar{z} \). This implies that \( z < \bar{z} \) for \( z \in \text{supp} \mu(s) \) and \( \bar{z} \in \text{supp} \mu(\sigma(\bar{z})) \) but \( s > \sigma(\bar{z}) \): \text{supp} \mu(s) \not\subset \text{supp} \mu(\sigma(\bar{z})). \) This contradicts the monotonicity of \mu in the stronger set order and hence \text{supp} \mu(s) = \{\bar{z}\} for \( s > \sigma(\bar{z}) \). This passes Criterion D1, which requires it as in item 2 in Lemma 11. We can analogously show that if \( s < \sigma(\bar{z'}) \), a monotone non-decreasing \mu in the stronger set order must have \( \{\bar{z}'\} \) as \text{supp} \mu(s) and that it passes Criterion D1, which requires it as in item 3 in Lemma 11.

Consider the case where \( s \) belongs to the interval of off-path sender actions induced
by the discontinuity of \( \sigma \) at some \( z \). Consider the case where \( \sigma \) is only right-continuous at \( z \). Given the monotonicity of \( \sigma \), we have that

\[
\lim \sup_{k \uparrow z} \sup \mu(\sigma(k)) = \min \sup \mu(\sigma(z)) = z.
\]  

(A8)

If \( \mu \) is monotone non-decreasing in the stronger set order, \( \sup \mu(s') \) and \( \sup \mu(s'') \) for two different \( s' \) and \( s'' \) have at most one element in common. Therefore, (A8) implies that a non-decreasing \( \mu \) in the stronger set order must have \( \{z\} \) as \( \sup \mu(s) \) for any \( s \in [\sigma(z_-), \sigma(z)] \). This passes Criterion D1, which requires it as in item 1 in Lemma 11. We can analogously prove that a non-decreasing \( \mu \) in the stronger set order satisfies item 1 in Lemma 11 in the cases where \( \sigma \) is only left continuous at \( z \) or \( \sigma(z_-) < \sigma(z) < \sigma(z_+) \). It is straightforward to show that an equilibrium \( \mu \) that passes Criterion D1 is non-decreasing in the stronger set order.

D  Proof of Lemma 1

If Assumption 2 is satisfied, then \( g(t, s, z, x) = v(x, s, z) - t \) is supermodular in all arguments \((t, s, z, x)\). Because each of \( T, S, Z, \) and \( X \) is a lattice in \( \mathbb{R} \), we can invoke Theorem 2.6.1 in Topkis (1998) to show that the supermodularity of \( g(t, s, z, x) \) in all arguments implies non-decreasing differences in \((t, s, z, x)\), which in turn implies the single crossing property in \((t, s, z); x\). Because Assumptions 2.(i) implies that \( g(t, s, z, x) \) has increasing differences in \((z, x)\). Therefore, Assumption B.(i) is satisfied. Assumption B.(ii) is satisfied by Assumption 2.(ii).

E  Proof of Lemma 2

If \( t_\ell \leq v(x, \zeta(x, z), z) \), then \( z_\ell = z \) and \( s_\ell = \zeta(x, z) \). For the case with \( t_\ell > v(x, \zeta(x, z), z) \). Let \( \Lambda(s, z) := v(n(z), s, z) \), so that (10) becomes \( \Lambda(s, z) - t_\ell = 0 \). Consider the receiver’s indifference curve \( \{(s, z) \in \mathbb{R}_+ \times [\overline{z}, \overline{z}] : \Lambda(s, z) - t_\ell = 0 \} \). The slope of this indifference curve is

\[
\frac{dz}{ds} = -\frac{\Lambda_s}{\Lambda_z} = -\frac{v_s}{v_x n' + v_z} \leq 0.
\]  

(A9)

Given \( v_s \geq 0, v_x > 0, v_z > 0 \) (Assumptions 2.(ii), 3.(i), and 5), the sign above holds because of \( n' > 0 \). Note that \( n' > 0 \) holds because of Assumption 6 \( G'(z) > 0 \) for all \( z \) and \( H'(x) > 0 \) for all \( x \). On the other hand, \( \{(s, z) \in \mathbb{R}_+ \times [\overline{z}, \overline{z}] : t_\ell - c(s, z) = 0 \} \) is the
sender’s indifference curve based on (11) and its slope is

\[
\frac{dz}{ds} = -\frac{c_z}{c_s} > 0
\]

(A10)

where the sign holds due to Assumption 1.(i). (A9) and (A10) imply that the two indifference curves intersect at most once and the intersection becomes a unique solution for the system of equations, (10) and (11).

F Proof of Theorem 3

We first show that \(\sigma\) is continuous on \([z_\ell, \bar{z}]\). To prove that, we start by showing that \(\sigma(z) \geq \zeta(x, z)\), the bilaterally efficient level of type \(z\)’s action, where \(x\) is the type of the receiver who is matched with the sender of type \(z\) in equilibrium.

Lemma 12 For all \(z \in [z_\ell, \bar{z}]\), \(\sigma(z) \geq \zeta(x, z)\) in any stronger monotone separating equilibrium, where \(x\) is the type of the receiver who matches with type \(z\).

Proof. We prove by contradiction. Suppose that there exists \(z \in [z_\ell, \bar{z}]\) such that \(\sigma(z) < \zeta(x, z)\). There are two possible cases. The first case is when \(\sigma(z) < \zeta(x, z)\) and \(\zeta(x, z) \notin S^*\). Then, it is a profitable sender deviation by type \(z\) to an off-path action \(\zeta(x, z)\) if

\[
\mathbb{E}_{\mu(\zeta(x,z))} [v(x, \zeta(x, z), z')] - c(\zeta(x, z), z) > v(x, \sigma(z), z) - c(\sigma(z), z).
\]

(A11)

Because of the constrained efficiency of \(\zeta(x, z)\) given the strict concavity of \(v - c\) in \(s\) (Assumption 3), we have that

\[
v(x, \zeta(x, z), z) - c(\zeta(x, z), z) > v(x, \sigma(z), z) - c(\sigma(z), z).
\]

(A12)

Further, because \(\sigma(z) < \zeta(x, z)\), we have \(z' \geq z\) for all \(z' \in \text{supp} \mu(\zeta(x, z))\) due to the stronger monotonicity of \(\mu\). Therefore, we have

\[
\mathbb{E}_{\mu(\zeta(x,z))} [v(x, \zeta(x, z), z')] \geq v(x, \zeta(x, z), z).
\]

(A13)

Because of (A12) and (A13), (A11) holds.

Therefore, if there exists \(z \in [z_\ell, \bar{z}]\) such that \(\sigma(z) < \zeta(x, z)\), then it must be the second case where \(\zeta(x, z) \in S^*\). This implies that there exists \(z' > z\) such that \(\sigma(z') =

\( \zeta(x, z) < \zeta(x', z') \), where \( x' \) is the type of the receiver who matches with type \( z' \), given the increasing property of \( \sigma \) in a stronger monotone separating equilibrium (implication of Lemma 10.(i)). Because we have \( \sigma(z) < \zeta(x, z) \) and \( \sigma(z') < \zeta(x', z') \), there exists an interval \((z_1, z_2) \subset (z, z')\) such that for all \( z'' \in (z_1, z_2) \), \( \sigma(z'') < \zeta(x'', z'') \), where \( x'' \) is the type of the receiver who matches with type \( z'' \) in equilibrium.

\( \sigma \) is increasing on \((z_1, z_2)\) in any stronger monotone separating equilibrium. Further, \( \sigma \) is finite on \((z_1, z_2)\) because \( \sigma(z'') \leq \zeta(x_2, z_2) \), where \( x_2 \) is the type of the receiver who matches with type \( z_2 \) in equilibrium. One can invoke Theorem 7.21 in Wheeden and Zygmund (1977) to show that \( \sigma \) is differentiable with non-negative derivative \( \sigma' \) almost everywhere on \((z_1, z_2)\). Because \( \sigma \) is strictly increasing in \( s \in S^* \) in a stronger monotone separating equilibrium, \( \sigma' \) must be in fact positive almost everywhere on \((z_1, z_2)\).

Because \( \sigma \) is differentiable with positive \( \sigma' \) almost everywhere on \((z_1, z_2)\), we can find an interval \((z_1', z_2') \subset (z_1, z_2)\) such that \( \sigma \) is differentiable with positive \( \sigma' \) everywhere on \((z_1', z_2')\). Because \( \sigma \) is differentiable with positive \( \sigma' \) everywhere on \((z_1', z_2')\), \( \{\sigma(z'') : z'' \in (z_1', z_2')\} \) is an interval \((\sigma(z_1'), \sigma(z_2'))\) and \( \mu \) is differentiable with positive \( \mu' = 1/\sigma' \) everywhere on \((\sigma(z_1'), \sigma(z_2'))\). On the other hand, \( \tau : S^* \rightarrow [t_\ell, t_\bar{h}] \) is increasing according to Lemma 10.(iii). Invoking Theorem 7.21 in Wheeden and Zygmund (1977), we can show that \( \tau \) is differentiable with non-negative \( \tau' \) almost everywhere on \((\sigma(z_1'), \sigma(z_2'))\). Finally, we can pick an interval \((z_1^0, z_2^0) \subset (z_1', z_2')\) such that (i) \( \sigma \) is differentiable with positive \( \sigma' \) everywhere on \((z_1^0, z_2^0)\), (ii) \( \mu \) is differentiable with positive \( \mu' = 1/\sigma' \) everywhere on \((\sigma(z_1^0), \sigma(z_2^0))\) and (iii) \( \tau \) is differentiable with non-negative derivative everywhere on \((\sigma(z_1^0), \sigma(z_2^0))\). It implies that for all \( z'' \in (z_1^0, z_2^0) \), the following first-order condition must be satisfied:

\[
\tau'(\sigma(z'')) - c_s(\sigma(z''), z'') = 0. \tag{A14}
\]

Because (1) \( \mu(s) \) is differentiable everywhere in \((\sigma(z_1^0), \sigma(z_2^0))\) and (2) \( \tau \) is differentiable everywhere on \((\sigma(z_1^0), \sigma(z_2^0))\), and (3) \( \pi \) is differentiable everywhere on \((\sigma(z_1^0), \sigma(z_2^0))\). If type \( x'' \) chooses a sender with \( s \in (\sigma(z_1^0), \sigma(z_2^0)) \), the following first-order condition must be satisfied:

\[
\pi_s(x, x'') = v_s(x'', s, \mu(s)) + v_z(x, s, \mu(s)) \mu'(s'') - \tau'(s'') = 0 \tag{A15}
\]

Let type \( z'' \) choose \( s = \sigma(z'') \in (\sigma(z_1^0), \sigma(z_2^0)) \), which type \( x'' \) chooses. Combining (A14) and (A15) yields that \( v_s(x, s, \mu(s)) - c_s(\sigma(z''), z'') + v_z(x'', s, \mu(s)) \mu'(s) = 0 \), which cannot hold. The reason is that (a) \( v_s - c_s > 0 \) because \( v - c \) is strictly concave (Assumption 3) and \( \sigma(z'') < \zeta(x'', z'') \), (b) \( \mu' \geq 0 \) and (c) \( v_z > 0 \) (Assumption 5). Therefore, we cannot
have the case where \( \sigma(z) < \zeta(x, z) \), and \( \zeta(x, z) \in S^* \). This concludes the proof. ■

**Lemma 13** \( \sigma \) is continuous on \([z_\\ell, \bar{z}]\) and hence \( S^* = [s_\\ell, \sigma(\bar{z})] \) in any stronger monotone separating equilibrium.

**Proof.** We prove by contradiction. Suppose that \( \sigma \) is discontinuous at some \( z \in [z_\\ell, \bar{z}] \). We consider the case where \( \sigma \) is only right continuous at \( z \), so that \( \sigma(z_+) = \sigma(z) > \sigma(z_-) \).

Because \( \sigma(z) \geq \zeta(x, z) \) for all \( z \in [z_\\ell, \bar{z}] \) by Lemma 12, it implies that \( \sigma(z) > \zeta(x, z) \), where \( x \) is the type of a receiver who matches with type \( z \). This discontinuity creates an off-path action interval \([\sigma(z_-), \sigma(z)]\). The stronger monotone belief \( \mu \) implies that \( \mu(s) \) puts all the weights on \( z \) conditional on \( s \in [\sigma(z_-), \sigma(z)] \) because of Lemma 11.1.

Because \( \sigma(z) \) is inefficiently high (i.e., \( \sigma(z) > \zeta(x, z) \)), there exists \( s \in [\sigma(z_-), \sigma(z)] \) such that

\[
v(x, s, z) - c(s, z) > v(x, \sigma(z), z) - c(\sigma(z), z), \tag{A16}
\]
due to the strict concavity of \( v - c \) in \( s \) (Assumption 3). (A16) shows the existence of a profitable sender deviation by type \( z \) to an off path action \( s \in [\sigma(z_-), \sigma(z)] \). One can analogously show the existence of a profitable sender deviation by type \( z \) in the case where \( \sigma \) is only left continuous at \( z \) or \( \sigma(z_-) < \sigma(z) < \sigma(z_+) \). Therefore, \( \sigma \) is continuous at all \( z \in [z_\\ell, \bar{z}] \).

Because \( \sigma \) is increasing over \([z_\\ell, \bar{z}]\) in any stronger monotone separating equilibrium, the continuity of \( \sigma \) at all \( z \in [z_\\ell, \bar{z}] \) implies a compact real interval \( S^* = [s_\\ell, \sigma(\bar{z})] \) ■

**Lemma 14** \( \tau : S^* \rightarrow T \) is increasing and continuous on \( S^* \) and has continuous derivative \( \tau' \) on \( \text{Int } S^* \) in any stronger monotone separating equilibrium.

**Proof.** The increasing property of \( \tau \) is from Lemma 10.(iii). We prove the continuity of \( \tau \) by contradiction. Suppose that \( \tau \) is discontinuous at \( s \in S^* \). Consider the case where \( \tau \) is only right continuous at \( s \). Let \( z \) be the type of a sender who chooses \( s \) in equilibrium, i.e., \( \sigma(z) = s \). Because \( \tau(\sigma(z_-)) < \tau(\sigma(z)) \) and \( c \) and \( \sigma \) are continuous (Assumption 5.(i) and Lemma 13), there exists \( z' < z \) such that \( \tau(\sigma(z')) - c(\sigma(z'), z') < \tau(\sigma(z)) - c(\sigma(z), z') \), which contradicts the optimality of \( \sigma(z') \) for type \( z' \). We can analogously prove that the discontinuity of \( \tau \) contradicts the optimality of the sender’s action choice in the case where \( \tau \) is left right continuous at \( s \) or \( \tau(s_-) < \tau(s) < \tau(s_+) \). Therefore, \( \tau : S^* \rightarrow T \) is continuous everywhere on \( S^* \).

We prove the differentiability by contradiction as well. Suppose that \( \tau \) is not differentiable at some \( \hat{s} \in \text{Int } S^* = (s_\\ell, \sigma(\bar{z})) \).
Lemma 15 implies that if $S$ is in $\text{Int} \ S^*$ due to Theorem 7.21 in Wheeden and Zygmund (1977). This implies that if $\tau$ is not differentiable at $\hat{s}$, there exists two intervals $(s_1, \hat{s})$, $(\hat{s}, s_2) \subset \text{Int} \ S^*$ where $\tau$ is differentiable. Because $\tau$ is differentiable at any point in $(s_1, \hat{s}) \cup (\hat{s}, s_2)$ and $c$ is differentiable everywhere (Assumption 4), the optimality of $s = \sigma(z)$ implies that the first-order condition $\tau'(s) = c_\sigma(s, \mu(s))$ for all $s = \sigma(z) \in (s_1, \hat{s}) \cup (\hat{s}, s_2)$. Because $c_\sigma$ is continuous (Assumption 4) and $\mu = \sigma^{-1}$ is continuous on $S^*$, this implies that

$$\tau'(\hat{s}_-) = c_\sigma(\hat{s}, \mu(\hat{s})) = \tau'(\hat{s}_+).$$

Because $\tau$ is continuous, (A17) implies that $\tau$ is differentiable at $\hat{s}$, which contradicts the non-differentiability of $\tau$ at $\hat{s}$. Therefore, $\tau$ must be differentiable everywhere on $\text{Int} \ S^*$.

Because $\tau$ is differentiable everywhere on $\text{Int} \ S^*$, the first-order condition $\tau'(s) = c_\sigma(s, \mu(s))$ must be satisfied for all $s \in \text{Int} \ S^*$ in equilibrium. $\mu$ is continuous on $S^*$ because it is the inverse of $\sigma$ over $S^*$ and $\sigma$ is continuous (Lemma 13). Further $c_\sigma$ is continuous (Assumption 5.(i)). Therefore, $\tau'(s) = c_\sigma(s, \mu(s))$ is continuous on $\text{Int} \ S^*$.

Lemma 15 $\mu : S \to \Delta(Z)$ is increasing and continuous on $S^*$ and has continuous derivative $\mu'$ on $\text{Int} \ S^*$.

Proof. $\sigma$ is continuous on $Z$ and $S^*$ is a compact real interval $[\sigma(z_1), \sigma(z_2)]$ (Lemma 13 in Online Appendix). Given Lemma 10.(i), $\sigma$ is increasing over $Z$ in a stronger monotone separating equilibrium. Therefore, Lemma 13 implies that $\mu(s)$ (the support of $\mu(s)$ to be precise) for all $s \in S^*$ is the inverse of $\sigma(z)$ so that $\mu$ is increasing and continuous on $S^*$.

Because $\mu$ is increasing on $S^*$ and $\mu(s) \in Z$ for $s \in S^*$, we can apply Theorem 7.21 in Wheeden and Zygmund (1977) to show that $\mu$ is differentiable almost everywhere on $\text{Int} \ S^*$. Let us prove that $\mu$ is differentiable everywhere on $\text{Int} \ S^*$. Suppose that $\mu$ is not differentiable at $\hat{s} \in \text{Int} \ S^*$. Because $\mu$ is not differentiable at only finitely many points, there exists $s_1, s_2 \in \text{Int} \ S^*$ such that $\mu$ is differentiable everywhere on $(s_1, \hat{s})$ and $(\hat{s}, s_2)$.

Because (1) $v$ is differentiable with respect to $s$ and $z$ (Assumptions 3.(ii)), (2) $\mu$ is differentiable everywhere on $(s_1, \hat{s})$ and $(\hat{s}, s_2)$, and (3) $\tau$ is differentiable everywhere on $\text{Int} \ S^*$ (Lemma 14 in Online Appendix), $\pi(s, x) := v(x, s, \mu(s)) - \tau(s)$, $x$ is differentiable everywhere on $(s_1, \hat{s}) \cup (\hat{s}, s_2)$.

Let $x$ be the type of a receiver who matches with a sender with $s = \xi(x)$. For the receiver’s matching problem, the following first-order condition is satisfied: for all $s =
\[ \xi(x) \in (s_1, \bar{s}) \cup (\bar{s}, s_2): \]

\[ \pi_s(s, x) = v_s(x, s, \mu(s)) + v_z(x, s, \mu(s)) \mu'(s) - \tau'(s) = 0 \quad (A18) \]

Suppose that \( \xi(x) = \xi(x') = s \) for some \( s \in S^* \) with \( x > x' \). It implies that \( \xi(x'') = s \) for all \( x'' \in [x, x'] \) because \( \xi \) is non-decreasing in a stronger monotone equilibrium (Theorem 2) given Assumptions 1 and 2. Then, the market clearing condition is not satisfied because \( H([x, x']) > G(\{s\}) = 0 \). Therefore, \( \xi \) is increasing on \( X \).

Then, the market-clearing condition implies that \( \xi = \sigma \circ n^{-1} \). Because \( \sigma \) is continuous on \( Z \) (Lemma 13) and \( n^{-1} \) is continuous on \( X \) (implication of Assumption 6), \( \xi \) is continuous on \( [z_\ell, \bar{z}] \) because \( \xi \) is increasing and continuous on \( Z \), \( \xi^{-1} = n \circ \mu \) is increasing and continuous on \( S^* \).

Given \( v_z > 0 \) (Assumption 3.(i)), replacing \( x \) with \( \xi^{-1}(s) \) in (A18) yields that

\[ \mu'(s) = -\frac{v_s(\xi^{-1}(s), s, \mu(s)) - \tau'(s)}{v_z(\xi^{-1}(s), s, \mu(s))}, \forall s \in (s_1, \bar{s}) \cup (\bar{s}, s_2). \quad (A19) \]

In addition to the continuity of \( \xi^{-1} \) on \( S^* \), \( v_s, v_z, \tau' \), and \( \mu \) are continuous (Assumption 3.(ii) and Lemmas 13 and 14 in Online Appendix). Therefore, from (A19), we have that

\[ \mu'(\bar{s}_-) = -\frac{v_s(\xi^{-1}(\bar{s}), \bar{s}, \mu(\bar{s})) - \tau'(\bar{s})}{v_z(\xi^{-1}(\bar{s}), \bar{s}, \mu(\bar{s}))} = \mu'(\bar{s}_+) \quad (A20) \]

Because \( \mu \) is continuous on \( S^* \), (A20) implies that \( \mu \) is differentiable at \( \bar{s} \), which contradicts the non-differentiability of \( \mu \) at \( \bar{s} \). Therefore, \( \mu \) must be differentiable everywhere on \( \text{Int} \ S^* \) in any stronger monotone separating equilibrium. Further, the continuity of \( v_s, v_z, \tau', \xi^{-1}, \) and \( \mu \) implies that \( \mu' \) is continuous on \( \text{Int} \ S^* \). ■

**G Stronger monotone separating equilibrium**

Here we present the stronger monotone separating equilibrium \( \{\bar{\sigma}, \bar{\mu}, \bar{\tau}, \bar{m}\} \) given \( z_\ell \) induced by \( t_\ell \) and \( t_\ell < t_h = \infty \). Once we establish the stronger monotone equilibrium, it is convenient to establish the (strictly) well-behaved equilibrium with the same \( z_\ell \) but \( t_h < \bar{\tau}(\bar{\sigma}(\bar{z})) \).

**Theorem 11** The necessary and sufficient conditions for a stronger monotone separating equilibrium \( \{\bar{\sigma}, \bar{\mu}, \bar{\tau}, \bar{m}\} \) are

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1. $\tilde{\sigma}(z) = \zeta(x, \bar{z})$, and $\tilde{\sigma}(z)$ satisfies that $\tilde{\tau}'(\tilde{\sigma}(z)) - c_s(\tilde{\sigma}(z), z) = 0$ for all $z \in \text{Int} Z$.

2. For $s \in [0, \tilde{\sigma}(\bar{z})]$, $\tilde{\mu}(s) = \bar{z}$; for all $s \in S^*$, $\tilde{\mu}(s) = \tilde{\sigma}^{-1}(s)$, where $\tilde{\sigma}^{-1}(s)$ satisfies $\tilde{\sigma}(\tilde{\sigma}^{-1}(s)) = s$ for all $s \in S^*$; for all $s > \tilde{\sigma}(\bar{z})$, $\tilde{\mu}(s) = \bar{z}$.

3. $\xi(x)$ satisfies
   \[ v_s(x, s, \tilde{\mu}(s)) + v_z(x, s, \tilde{\mu}(s)) \tilde{\mu}'(s) - \tilde{\tau}'(s) = 0 \] (A21)
   at $s = \xi(x)$ for all $x \in \text{Int} X$.

4. $\tilde{\tau}$ with $\tilde{\tau}(\tilde{\sigma}(\bar{z})) = t_\ell$ clears the market given $\tilde{m}$ such that $\tilde{m}(s) = \xi^{-1}(s) = n(\tilde{\mu}(s))$ for all $s \in S^*$.

The proof is below.

Proof of Condition 1 If there are types who stay out of the market, they must be below $z_\ell$ given that $c$ is decreasing in $z$ (Assumption 1.(i)). Note that type $z_\ell$ is indifferent between staying out of the market and taking action $s_\ell$ because they satisfy 11. Since $c$ is decreasing in $z$ (Assumption 1.(i)), it means that any type in $[\bar{z}, z_\ell]$ is strictly better off by staying out of the market instead of taking action $s_\ell$.

Given that $\tau$ is continuous on $S^*$ and differentiable on $\text{Int} S^*$ (Theorem 3.(ii)) and $c$ is differentiable (Assumption 5.(i)), it is clear that $\tilde{\tau}'(\tilde{\sigma}(z)) - c_s(\tilde{\sigma}(z), z) = 0$ for all $z \in (z_\ell, \bar{z})$ is a necessary condition for $\tilde{\sigma}(z)$ to be an optimal action for type $z \in [z_\ell, \bar{z}]$ among all actions in $S^* = [\tilde{\sigma}(z_\ell), \tilde{\sigma}(\bar{z})]$. We show that it is also a sufficient condition for $\tilde{\sigma}(z)$ to be an optimal action for type $z \in [z_\ell, \bar{z}]$ among all actions in $S^* = [\tilde{\sigma}(z_\ell), \tilde{\sigma}(\bar{z})]$. We need to be careful about the boundary condition. A part of Condition 1 in Theorem 1 is that $\tilde{\sigma}(z)$ satisfies that

\[ \tilde{\tau}'(\tilde{\sigma}(z)) - c_s(\tilde{\sigma}(z), z) = 0 \text{ for all } z \in (z_\ell, \bar{z}). \] (A22)

Applying the strict supermodularity of $-c$ (Assumption 1.(ii)) to (A22) yields that

\[ \tilde{\tau}'(\tilde{\sigma}(z')) - c_s(\tilde{\sigma}(z'), z) \geq 0 \text{ if } z' \leq z, \forall z, z' \in (z_\ell, \bar{z}) \] (A23)

\[ \tilde{\tau}(\tilde{\sigma}(z)) - c(\tilde{\sigma}(z), z) > \tilde{\tau}(\tilde{\sigma}(z')) - c(\tilde{\sigma}(z'), z) \forall z, z' \in (z_\ell, \bar{z}) \text{ s.t. } z \neq z'. \] (A24)

(A23) implies that $\tilde{\sigma}$ is increasing in $z \in (z_\ell, \bar{z})$. Because $\tilde{\sigma}$ is continuous on $S^*$ (Lemmas 13), it implies that $\tilde{\sigma}$ is increasing over $[z_\ell, \bar{z}]$. Because $\tilde{\sigma}$ is increasing over $[z_\ell, \bar{z}]$ and $\tilde{\tau}$ and $c$ are continuous (Theorem 3.(ii) and Assumption 5.(i)), (A24) implies
that
\[
\bar{\tau}(\bar{\sigma}(z)) - c(\bar{\sigma}(z), z) > \bar{\tau}(\bar{\sigma}(z')) - c(\bar{\sigma}(z'), z) \quad \forall z, z' \in [z_\ell, \bar{z}] \text{ s.t. } z \neq z'. 
\] (A25)

(A25) shows that \( \bar{\sigma}(z) \) is be an optimal action for type \( z \in [z_\ell, \bar{z}] \) among all actions in \( S^* = [\bar{\sigma}(z_\ell), \bar{\sigma}(\bar{z})] \), ignoring the individual rationality.

To show the individual rationality, let \( \tilde{U}(z) := \bar{\tau}(\bar{\sigma}(z)) - c(\bar{\sigma}(z), z) \). Because of 11, we have that \( \tilde{U}(z_\ell) = \bar{\tau}(\bar{\sigma}(z_\ell)) - c(\bar{\sigma}(z_\ell), z_\ell) = 0 \). It is clear that \( \tilde{U}(z) > \bar{\tau}(\bar{\sigma}(z_\ell)) - c(\bar{\sigma}(z_\ell), z) > \tilde{U}(z_\ell) = 0 \) for all \( z \in (z_\ell, \bar{z}] \), where the first inequality holds because of (A25) and the second inequality holds because \( c \) is decreasing in \( z \) (Assumption 1.(i)).

We need to show that type \( z \) has no incentive to deviate to \( s \notin \) range \( \sigma \) to complete the proof that \( \bar{\sigma}(z) \) is an optimal action for type \( z \) among all actions in \( S \). We defer it to the end. First we start with the stronger monotone belief \( \bar{\mu} \)

Proof of Condition 2  Note that range \( \sigma = \{0\} \cup [s_\ell, \bar{\sigma}(\bar{z})] \). Therefore, we can derive the belief on the equilibrium path as follows: (i) for \( s = 0 \), \( \bar{\mu}(s) = G(z|z < z_\ell) \) and (ii) for all \( s \in [s_\ell, \bar{\sigma}(\bar{z})] \), \( \bar{\mu}(s) = \bar{\sigma}^{-1}(s) \), where \( \bar{\sigma}^{-1}(s) \) satisfies \( \bar{\sigma}(\bar{\sigma}^{-1}(s)) = s \). This is part of Condition 2 in Theorem 11 so that consistency is satisfied. There are two intervals of off path sender actions: \((0, s_\ell)\) and \((\bar{\sigma}(\bar{z}), \infty)\). The monotonicity of the belief in the stronger set order uniquely determines the belief conditional on \( s \notin \) range \( \sigma \): (iii) for \( s \in (0, s_\ell) \), \( \bar{\mu}(s) = z_\ell \) and (iv) for \( s \in (\bar{\sigma}(\bar{z}), \infty) \), \( \bar{\mu}(s) = \bar{z} \).

The belief function \( \bar{\mu} \) satisfying (i) - (iv), which is Condition 2 in Theorem 11, is the unique equilibrium monotone in the stronger set order given \( \sigma \) in Condition 1

Proof of Conditions 3  If there are types who stay out of the market, they must be below \( x_\ell \) given that \( v \) is increasing in \( x \) (Assumption 2.(ii)). Note that type \( x_\ell \) is indifferent between staying out of the market and matching with the sender of type \( z_\ell \) because they satisfy (10). Since \( v \) is increasing in \( x \) (Assumption 2.(ii)), it means that any type in \([x_\ell, x_\ell]\) is strictly better off by staying out of the market.

Consider the matching problem for type \( x \in [x_\ell, \bar{x}] \). Note that \( v, c, \gamma, \bar{\tau} \) and \( \bar{\mu} \) are continuous and differentiable (Assumptions 4.(ii) and 5.(i) and Theorem 3). Therefore, it is clear that for all \( x \in (x_\ell, \bar{x}) \),
\[
\pi_s(\xi(x), x) = v_s(x, \xi(x), \bar{\mu}(\xi(x))) + v_z(x, \xi(x), \bar{\mu}(\xi(x))) \bar{\mu}'(\xi(x)) - \bar{\tau}'(\xi(x)) = 0 \quad \text{(A26)}
\]
is a necessary condition for \( \xi(x) \) to be an optimal choice of a matching partner (in terms
of her action) for type \( x \in [x, \bar{x}] \) among all actions in \( S^* \).

We show that (A26) is also a sufficient condition for \( \xi(x) \) to be an optimal action of a matching partner for type \( x \in [x, \bar{x}] \) among all actions in \( S^* \). Applying the supermodularity of \( v \) (Assumptions 2.(i)) to (A26) yields that

\[
\pi_s(\xi(x'), x) \geq 0 \text{ if } x' \leq x \quad \forall x, x' \in (x, \bar{x}) ,
\]

(A27)

\[
\pi(\xi(x), x) > \pi(\xi(x'), x) \quad \forall x, x' \in (x, \bar{x}) \text{ s.t. } x \neq x'
\]

(A28)

(A27) implies that \( \xi(x) \) is increasing on \((x, \bar{x})\). Given the increasing property of \( \xi \) on \((x, \bar{x})\), \( \xi \) must be continuous on \([x, \bar{x}]\). Otherwise, senders in the interval created by a discontinuity of \( \xi \) are not matched in equilibrium and it violates the market clearing condition.

The continuity of \( \xi \) on \([x, \bar{x}]\) makes it increasing on \([x, \bar{x}]\) because \( \xi \) is increasing on \( x \in (x, \bar{x}) \). Together with the continuity of \( \xi \) over \([x, \bar{x}]\), the continuity of \( v \) and \( \tilde{\tau} \) (Assumptions 3.(ii) and 14) makes \( \pi(\xi(x'), x) \) continuous in \( x' \in [x, \bar{x}] \) and \( x \in [x, \bar{x}] \). Therefore, (A28) implies that

\[
\pi(\xi(x), x) > \pi(\xi(x'), x) \quad \forall x, x' \in [x, \bar{x}] \text{ s.t. } x \neq x'.
\]

(A29)

(A29) shows that \( \xi(x) \) is be an optimal choice of a matching partner for type \( x \in [z, \bar{z}] \) among all actions in \( S^* = [\tilde{\sigma}(z), \tilde{\sigma}(\bar{z})] \), ignoring the individual rationality.

To show the individual rationality, let \( \tilde{\mu}(x) := \pi(\xi(x), x) \). Because of (10), we have that \( \pi(\xi(x), x) = 0 \). It is clear that \( \pi(\xi(x), x) > \pi(\xi(x), x) > \pi(\xi(x), x) = 0 \) for all \( x \in (x, \bar{x}) \), where the first inequality holds because of (A29) and the second inequality holds because \( v \) is increasing in \( x \) (Assumption 2.(ii)).

**Proof of Condition 4**  It is straightforward that \( \tilde{m} \) in in Condition 4 is a unique measure-preserving matching function given that \( \xi \) and \( \tilde{\sigma} \) are both increasing.

**Proof of no profitable sender deviation to an off-path action**  Applying Lemma 11 and Corollary 1, the monotone belief in the stronger set order is the unique belief that pass Criterion D1. Because \( \tilde{\mu}(s) \) for \( s \notin \text{range} \sigma \) is a degenerate probability distribution with a singleton as its support, as suggested in Corollary 2, we only need to check if the type of the sender in that support has an incentive to deviate in order to check if any sender has an incentive to deviate to such \( s \).

Now let us prove no profitable sender deviation to an off-path action. Conditional on
s ∈ (0, s_ℓ), it is believed that the sender who chose s is \( \tilde{\mu}(s) = z_\ell \). According to Corollary 1, if the sender of type \( z_\ell \) has no profitable deviation to \( s \in (0, s_\ell) \), then no one else does. Therefore, we only need to check if the sender of type \( z_\ell \) has an profitable deviation to \( s \in (0, s_\ell) \). If she reduces her action down to \( s \in (0, s_\ell) \), no receiver wants her because he has to transfer at least \( t_\ell \) but he can be matched with a sender with \( s_\ell \) at \( t_\ell \). Therefore, there is no sender profitable deviation to s.

Now, let us examine if there is a profitable sender deviation to \( s \in (\bar{\sigma}(\mathcal{Z}), \infty) \). First, agents at the action choice stage expect \( \tilde{m} \) is increasing and the equilibrium transfer \( \tilde{\tau} : S^* \to T \) satisfies

\[
\tilde{\tau}'(s) = v_x(\tilde{m}(s), s, \tilde{\mu}(s)) + v_z(\tilde{m}(s), s, \tilde{\mu}(s))\tilde{\mu}'(s). \tag{A30}
\]

Because the support of \( \tilde{\mu}(s) \) for \( s > \bar{\sigma}(\mathcal{Z}) \) is a singleton, \( \mathcal{Z} \), we only need to check if the sender of type \( \mathcal{Z} \) has an incentive to choose \( s > \bar{\sigma}(\mathcal{Z}) \). Because there is a continuum of receivers with different actions and types, we need to check which receiver is willing to transfer the largest amount to the sender with \( s > \bar{\sigma}(\mathcal{Z}) \). The type x receiver’s maximum willingness to transfer is

\[
t(s, x) = v(x, s, \mathcal{Z}) - [v(x, \bar{\sigma}(n^{-1}(x))), n^{-1}(x)] - \tilde{\tau}(\bar{\sigma}(n^{-1}(x))) \tag{A31}.
\]

Because \( s > \bar{\sigma}(\mathcal{Z}) \) and \( \mathcal{Z} \geq n^{-1}(x) \), we have that \( t(s, x) > \tilde{\tau}(\bar{\sigma}(n^{-1}(x))) \).

Given \( \tilde{\tau}' \) in (A30), taking the derivative of \( t(s, x) \) with respect to \( x \) yields

\[
t_x(s, x) = v_x(x, s, \mathcal{Z}) - v_x(x, \bar{\sigma}(n^{-1}(x))), n^{-1}(x)) > 0. \tag{A31}
\]

Note that \( v_x(x, s, z) \) is non-decreasing in \( s \) and increasing in z, given Assumption 2.(i) - \( v(b, x, s, z) \) is supermodular in \((b, x, s, z)\) and strictly supermodular in \((z, x)\). Because \( s > \bar{\sigma}(n^{-1}(x)) \) and \( \mathcal{Z} \geq n^{-1}(x) \) for all \( x \geq x_\ell \), this implies that \( t_x(s, x) \) is positive for any \( s > \bar{\sigma}(\mathcal{Z}) \), as in (A31). It in turn implies that the maximum amount of transfer that the the receiver of type \( \mathcal{Z} \) is willing to make is the largest. Then, given Criterion D1, we only need to check if the sender of type \( \mathcal{Z} \) has a profitable deviation to \( s > \bar{\sigma}(\mathcal{Z}) \) while keeping her current match partner, the receiver of type \( \mathcal{Z} \).

There is a profitable sender deviation to \( s > \bar{\sigma}(\mathcal{Z}) \) for \( \mathcal{Z} \) if and only if for some \( s > \bar{\sigma}(\mathcal{Z}) \)

\[
v(\mathcal{Z}, s, \mathcal{Z}) - c(s, \mathcal{Z}) > v(\mathcal{Z}, \bar{\sigma}(\mathcal{Z})) - c(\bar{\sigma}(\mathcal{Z}), \mathcal{Z}) \tag{A32}.
\]

However, the inequality above is not satisfied for any \( s > \bar{\sigma}(\mathcal{Z}) \). The reason is that the
information rent, the last term in (A30), makes \( \tilde{\sigma}(z) \) larger than the constrained efficient action level for the sender of type \( z \). Therefore, if \( s > \tilde{\sigma}(z) \), then \( v - c \) is even smaller given the strict concavity of \( v - c \) in \( s \) (Assumption 3).

**H Proof of Lemma 4**

Let \( Z(s) \) be the set of the types of senders who choose the same action \( s \) and it has a positive measure. We start with the case where there exists \( \max Z(s) \). Let \( z^\circ := \max Z(s) \). We first show that \( z^\circ = z \). Let \( x^\circ := \max X(s) \), where \( X(s) \) be the set of types of receivers who are matched with a sender with \( s \) in equilibrium. We prove by contradiction. Suppose that bunching does not happen on the top, i.e., \( z^\circ < z \). Then we have that

\[
s = \sigma(z^\circ) \leq \lim_{z \searrow z^\circ} \sigma(z) \tag{A33}
\]

This is due to the monotonicity of \( \sigma \) in Lemma 10.(i). We like to show that (A33) holds with strict inequality, i.e., \( s < \lim_{z \searrow z^\circ} \sigma(z) \). In equilibrium, we have that for any \( z > z^\circ \),

\[
\tau(s) - c(s, z^\circ) \geq \lim_{z \searrow z^\circ} [\tau(\sigma(z)) - c(\sigma(z), z^\circ)] \tag{A34}
\]

\[
\mathbb{E}[v(x^\circ, s, z'| z' \in Z(s)] - \tau(s) \geq \lim_{z \searrow z^\circ} (\mathbb{E}[v(x^\circ, \sigma(z), z''| z'' \in Z(\sigma(z))] - \tau(\sigma(z))] \tag{A35}
\]

For any \( \sigma(z) \geq s \), we have that \( z'' \geq z^\circ = \max Z(s) \) for any \( z'' \in Z(\sigma(z)) \) because of the monotonicity of \( \sigma \) (Lemma 10.(i)). Further \( Z(s) \) has a positive measure. Therefore, the monotonicity of \( v \) in Assumption 3.(i) implies that, for any \( z > z^\circ \)

\[
\mathbb{E}[v(x^\circ, s, z'| z' \in Z(s)] < \mathbb{E}[v(x^\circ, \sigma(z), z''| z'' \in Z(\sigma(z))] \tag{A36}
\]

(A35) and (A36) imply that

\[
\tau(s) < \lim_{z \searrow z^\circ} \tau(\sigma(z)) \tag{A37}
\]

Because \( c \) is decreasing in \( s \) (Assumption 1.(i)), (A34) and (A37) induces that

\[
s = \sigma(z^\circ) < \lim_{z \searrow z^\circ} \sigma(z) \tag{A38}
\]

Therefore, any \( s' \in (s, \lim_{z \searrow z^\circ} \sigma(z)) \) is not chosen in equilibrium given that the monotonicity of \( \sigma \).

The support of \( \mu(s) \) is \( Z(s) \). On the other hand, we have that \( \lim_{z \searrow z^\circ} \inf \text{supp}(\mu(\sigma(z))) = 54 \)
This implies that there is the unique stronger monotone belief on the sender’s type conditional on any $s' \in (s, \lim_{\zeta \downarrow z} \sigma(z))$ and it is equal to $\mu(s') = z^\circ$.

Suppose that the sender of type $z^\circ$ deviates to action $s + \epsilon \in (s, \lim_{\zeta \downarrow z} \sigma(z))$. A receiver of type $x$ who is currently matched with a sender with $s$ receives the matching utility of $E[v(x, s, z|z \in Z(s)] - \tau(s)$. Note that $\tau(s) < t_h$ given (A37). Therefore, there is a profitable deviation for the sender of type $z^\circ$ if

$$v(x, s + \epsilon, z^\circ) - c(s + \epsilon, z^\circ) > E[v(x, s, z|z \in Z(s)] - c(s, z^\circ).$$

(A39)

Because $v$ and $c$ are continuous in the sender’s action, $v$ is increasing in $z$, and $Z(s)$ has a positive measure, we have that

$$\lim_{\epsilon \downarrow 0} (v(x, s + \epsilon, z^\circ) - c(s + \epsilon, z^\circ)) > E[v(x, s, z|z \in Z(s)] - c(s, z^\circ)$$

(A40)

Because $v$ and $c$ are continuous in the sender’s action, (A40) implies that there exists $\epsilon$ such that (A39) is satisfied. This contradicts that $s$ is an equilibrium chosen by all senders whose types are in $Z(s)$.

We can analogously prove that there exists a profitable sender deviation if $z^\circ < \bar{z}$ when $z^\circ$ is defined as sup $Z(s)$ rather than max $Z(s)$.

Assumption 6 implies that there is no atom in the sender type distribution. Therefore, $Z(s)$ is an interval with max $Z(s) = \bar{z}$ due to the monotonicity of $\sigma$ (Lemma 10.(i)).

## I Proof of Lemma 5

Let $z^*$ be the minimum of $Z(s)$ (We can analogously prove the lemma for the case where $z^*$ is infimum of $Z(s)$). If $Z(s)$ has a positive measure, we have that $z^* < \bar{z}$ given Assumption 6 on $G$. Let $t^*$ be the reaction to action $s$ chosen by the positive measure of senders. We prove by contradiction.

On the contrary, suppose that $t^* < t_h$ in a stronger monotone equilibrium. Because type $\bar{z}$ is one of senders who choose $s$ and $\bar{z}$ is the maximum of sender types, the stronger monotonicity of $\mu$ implies that $\mu(s') = \bar{z}$ for any $s' > s$. Suppose that the sender of type $\bar{z}$ deviates to $s + \epsilon$ for small $\epsilon > 0$. The type of this sender is believed to be $\bar{z}$. Suppose that the receiver of type $\bar{z}$ is matched with the sender with $s + \epsilon$. A profitable upward
deviation for a sender is equivalent to the existence of \( t \in [t_\ell, t_h] \) and \( \epsilon > 0 \) such that

\[
v(\overline{x}, s + \epsilon, \overline{z}) - t > \mathbb{E}[v(\overline{x}, s, z^*) | z^* \leq z' < \overline{z}] - t^*,
\]

(A41)

\[
t - c(s + \epsilon, \overline{z}) > t^* - c(s, \overline{z}),
\]

(A42)

which yield

\[
v(\overline{x}, s + \epsilon, \overline{z}) - c(s + \epsilon, \overline{z}) > \mathbb{E}[v(\overline{x}, s, z^*) | z^* \leq z' < \overline{z}] - c(s, \overline{z}).
\]

(A43)

Given \( G'(z) > 0 \) for all \( z \in Z \) (Assumption 6), the monotonicity of \( v \) in \( z \) (Assumption 3.(i)) implies that

\[
v(\overline{x}, s, \overline{z}) - c(s, \overline{z}) > \mathbb{E}[v(\overline{x}, s, z^*) | z^* \leq z' < \overline{z}] - c(s, \overline{z}).
\]

(A44)

Because \( v \) and \( c \) are continuous in the sender action, (A44) ensures the existence of \( \epsilon > 0 \) that satisfies (A43). Because \( t^* < t_h \), (A43) implies that there exists \( t \) such that \( t^* < t < t_h \) and it satisfies (A41) and (A42). Therefore, the only way to prevent such an upward deviation by the sender is \( t^* = t_h \).

### J Proof of Lemma 8

When all senders of type \( z_h \) or above choose the same action \( s_h \) in equilibrium, we have that for all \( z > z_h \)

\[
t_h - c(s_h, z) \geq \tilde{\tau}(\tilde{\sigma}(z_h)) - c(\tilde{\sigma}(z_h), z).
\]

(A45)

Since (18) holds at \((s_h, z_h)\), (18) and (A45) imply that for all \( z > z_h \),

\[-c(s_h, z) + c(\tilde{\sigma}(z_h), z) \geq -c(s_h, z_h) + c(\tilde{\sigma}(z_h), z_h),
\]

which implies that \( s_h \geq \tilde{\sigma}(z_h) \) by the strict supermodularity of \(-c\) (Assumption 1.(ii)). Because \( s_h \geq \tilde{\sigma}(z_h) \), both (18) and (A45) imply that \( t_h \geq \tilde{\tau}(\tilde{\sigma}(z_h)) \).

Because \( s_h \geq \tilde{\sigma}(z_h) \), we have that \( \mathbb{E}[v(n(z_h), s_h, z') | z' \geq z_h] > v(n(z_h), \tilde{\sigma}(z_h), z_h) \).

(19) at \((s, z) = (s_h, z_h)\) is written as \( \mathbb{E}[v(n(z_h), s_h, z') | z' \geq z_h] - t_h = v(n(z_h), \tilde{\sigma}(z_h), z_h) - \tilde{\tau}(\tilde{\sigma}(z_h)) \), which implies \( t_h > \tilde{\tau}(\tilde{\sigma}(z_h)) \) given \( \mathbb{E}[v(n(z_h), s_h, z') | z' \geq z_h] > v(n(z_h), \tilde{\sigma}(z_h), z_h) \).

If \( t_h > \tilde{\tau}(\tilde{\sigma}(z_h)) \), (18) at \((s_h, z_h)\) implies that \( s_h > \tilde{\sigma}(z_h) \).

For all \( z \), let \( s_h^s(z) \) be the value of \( s \) that satisfies \( t_h - c(s_h^s(z), z) = \tilde{\tau}(\tilde{\sigma}(z)) - c(\tilde{\sigma}(z), z) \).

Because \( t_h < \tilde{\tau}(\tilde{\sigma}(z)) \), \( s_h^s(z) < \tilde{\sigma}(z) \). On the other hand, \( s_h = s_h^s(z_h) < s_h^s(\tilde{z}) \).
because $z_h < \bar{z}$ and $s_h^s$ is increasing in $z$. Therefore, we have that $s_h < \bar{\sigma}(\bar{z})$.

**K Proof of Lemma 7**

Consider the following set for senders:

$$\{(s, z) \in \mathbb{R}^+ \times Z : t_h - c(s, z) = \tilde{\tau}(\bar{\sigma}(z)) - c(\bar{\sigma}(z), z), s > \bar{\sigma}(z)\} \quad (A46)$$

Because $(s_h, z_h)$ must satisfy (18) and $s_h > \bar{\sigma}(z_h)$ (Lemma 8), $(s_h, z_h)$ must belong to the set in (A46). Applying the envelope theorem for $\bar{\sigma}$, to the total differential of $t_h - c(s, z) = \tilde{\tau}(\bar{\sigma}(z)) - c(\bar{\sigma}(z), z)$ yields the slope of the equation as

$$\frac{dz}{ds} = -\frac{c_s(s, z)}{c_z(s, z) - c_z(\bar{\sigma}(z), z)} > 0 \text{ for all } s > \bar{\sigma}(z), \quad (A47)$$

where the sign holds because $c_s > 0$ and $c_z(s, z) - c_z(\bar{\sigma}(z), z) < 0$ due to Assumption 1.(ii).

Consider the following set for receivers:

$$\left\{(s, z) \in \mathbb{R}^+ \times Z : \mathbb{E}[v(n(z), s, z')|z' \geq z] - t_h = v(n(z), \bar{\sigma}(z), z) - \tilde{\tau}(\bar{\sigma}(z)), \quad s > \bar{\sigma}(z)\right\}. \quad (A48)$$

Because $(s_h, z_h)$ must satisfy (19) and $s_h > \bar{\sigma}(z_h)$ (Lemma 8), $(s_h, z_h)$ must belong to the set in (A48).

Applying the envelope theorem for $\bar{\sigma}$, and $\tilde{\tau}$ to the total differential of $\mathbb{E}[v(n(z), s, z')|z' \geq z] - t_h = v(n(z), \bar{\sigma}(z), z) - \tilde{\tau}(\bar{\sigma}(z))$, we can express the slope of the equation as

$$\frac{dz}{ds} = -\frac{\mathbb{E}v_s}{\mathbb{E}v_sn' + \frac{\partial \mathbb{E}[v|z' > z]}{\partial z} - (v_sn' + v_z)} \leq 0 \text{ for all } s > \bar{\sigma}(z), \quad (A49)$$

where the sign holds because $-\mathbb{E}v_s \leq 0 \ (v_s \geq 0 \text{ according to Assumption 3.(i)})$ and the denominator is positive due to Assumptions 2, 3.(i), and 3.(iv) given $n' > 0$.

(A47) and (A49) imply that the two sets in (A46) and (A48) have at most one element in common. This implies that if there is a solution $(s_h, z_h)$ that solves (A47) and (A49), it must be unique.
L Proof of Theorem 6

Note that when \( t_h < \tilde{\tau}(\tilde{\sigma}(z)) \), \( \{\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{m}\} \) follows the separating equilibrium \( \{\tilde{\sigma}, \tilde{\mu}, \tilde{\tau}, \tilde{m}\} \) with the same \( z_\ell \) before \( z \) hits \( z_h \). Therefore, we will use some of the proof of Theorem 11. Because there is a jump to \( s_h \) and every sender of type \( z_h \) or higher chooses the same action \( s_h \), \( \lim_{z \uparrow z_h} \tilde{\sigma}(z) = \tilde{\sigma}(z_h) \). Therefore, we will use \( \tilde{\sigma}(z_h) \) instead of \( \lim_{z \uparrow z_h} \tilde{\sigma}(z) \) for simplicity of notation.

It is straightforward to show that the beliefs in Condition 2 of Theorem 6 satisfies the consistency and the stronger monotonicity. There are three off-path sender action intervals, \((0, s_\ell), [\tilde{\sigma}(z_h), s_h), (s_h, \infty)\). The stronger monotonicity of beliefs (Lemma 11 and Corollary 1) uniquely pins down the singleton support of a belief \( \tilde{\mu}(s) \) conditional on \( s \) in each off-path sender action interval. Further, we only need to check the type-\( \tilde{\mu}(s) \) sender’s incentive to deviate to \( s \) in any off-path action interval, thanks to Corollary 2.

L.1 Sender’s optimal action choice

In subsection (a) below, we first show that there is no profitable deviation to an off-path action for every sender if they choose actions according to \( \tilde{\sigma} \).

In the remaining subsections, we show that \( \tilde{\sigma}(z) \) solves Problem 1 if Problem 1 admits a solution; \( \tilde{\sigma}(z) = 0 \) otherwise. Note that \( \tilde{\sigma}(z) \) solves Problem 1 for \( z \in S^*[s_\ell, \tilde{\sigma}(z_h)) \cup s_h \). When it does, the sender’s equilibrium utility \( \tilde{U}(z) \) is increasing and positive for \( z > z_\ell \) starting from \( \tilde{U}(z_\ell) = 0 \) due to the envelope theorem. Therefore, the constraint in Problem 1 is satisfied.

(a) No profitable sender deviation to an off-path action  There are three intervals of actions that are not observed in equilibrium: \((0, s_\ell), [\tilde{\sigma}(z_h), s_h), (s_h, \infty)\). First, consider a deviation to \( s > s_h \). Since the belief \( \tilde{\mu}(s) = \tilde{\tau}(z) \) for \( s > s_h \) passes Criterion D1, we only need to check if the sender of type \( \tilde{\tau} \) has an incentive to deviate to such \( s \) in order to establish that there is no profitable sender deviation to such \( s \). Suppose that the sender of type \( \tilde{\tau} \) increases her action above \( s_h \). The maximum transfer she can receive is \( t = t_h \). Because \( \tilde{\sigma}(\tilde{\tau}) < s \) and \( \hat{\tau}(\tilde{\sigma}(\tilde{\tau})) = t_h \), we have that \( \hat{\tau}(\tilde{\sigma}(\tilde{\tau})) - c(\tilde{\sigma}(\tilde{\tau}), \tilde{\tau}) > t_h - c(s, \tilde{\tau}) \). Therefore, the sender of type \( \tilde{\tau} \) cannot gain by changing her action to \( s > s_h \).

Second, consider a deviation to \( s \in (0, s_\ell) \). In this case, the belief is \( \tilde{\mu}(s) = z_\ell \). Note that \( \tilde{\mu}(s_\ell) = z_\ell \). Since a receiver can be matched with a sender with \( z_\ell \) whose type is believed to be \( z_\ell \), transferring the lower bound of transfers, \( t_\ell \) to her, no receiver wants a sender with \( s < s_\ell \) whose type is believed to be \( z_\ell \), transferring \( t_\ell \) to her. Therefore, there
is no profitable deviation to \( s \in (0, s_\ell) \).

Third, consider a deviation to \( s \in [\tilde{\sigma}(z_h), s_h) \). In this case, the belief is \( \tilde{\mu}(s) = z_h \). Suppose that the sender of type \( z_h \) decreases her action from \( s_h \) to \( s \in (\tilde{\sigma}(z_h), s_h) \). She can be matched with a receiver of type \( x \in [x_\ell, x_h) \) or a receiver of type \( x \in [x_h, \overline{x}] \).

We first show that there is no profitable sender deviation to \( s \in (\tilde{\sigma}(z_h), s_h) \), being matched with any receiver of type \( x \in [x_\ell, x_h) \). Let \( t(s, x) \) be the maximum amount of transfer that a receiver of type \( x \in [x_\ell, x_h) \) is willing to make to a sender with \( s \in (\tilde{\sigma}(z_h), s_h) \). Following the proof of no profitable sender deviation in the stronger monotone separating equilibrium, we can show that the receiver’s maximum willingness increases as \( x \) approaches \( x_h \) from the left. Therefore, the supremum of the amount of transfers to the sender with \( s \) is

\[
t(s, x_h) = v(x_h, s, z_h) - (v(x_h, \tilde{\sigma}(z_h), z_h) - \tilde{\tau}(\tilde{\sigma}(z_h))). \tag{A50}
\]

Because \( s > \tilde{\sigma}(z_h) \), we have that \( t(s, x_h) > \tilde{\tau}(\tilde{\sigma}(z_h)) \). Given the upper bound of transfers \( t_h \), the receiver of type \( x_h \) cannot transfer \( t(s, x_h) \) if \( t(s, x_h) > t_h \). Suppose that the receiver can always transfer \( t(s, x_h) \) as if there is no upper bound of transfers. If a sender of type \( z_h \) has no incentive to deviate to \( s \) when there is no upper bound of transfers, then she also has no incentive to deviate to \( s \) when there is the upper bound of transfers.

Therefore, the sender of type \( z_h \) has an incentive to deviate to \( s \in (\tilde{\sigma}(z_h), s_h) \) if and only if

\[
t(s, x_h) - c(s, z_h) > t_h - c(s_h, z_h) = \tilde{\tau}(\tilde{\sigma}(z_h)) - c(\tilde{\sigma}(z_h), z_h), \tag{A51}
\]

where the equality comes from (18). (A50) and (A51) together implies that the sender of type \( z_h \) has an incentive to deviate to \( s \in (\tilde{\sigma}(z_h), s_h) \) if and only if

\[
v(x_h, s, z_h) - c(s, z_h) > v(x_h, \tilde{\sigma}(z_h), z_h) - c(\tilde{\sigma}(z_h), z_h). \tag{A52}
\]

\( \tilde{\sigma}(z_h) \) is the action chosen by type \( z_h \) in the stronger monotone separating equilibrium. The first-order conditions for action choices by type \( z_h \) and type \( x_h \) in Conditions 1 and 4 in Theorem 11 imply that

\[
v_s(x_h, \tilde{\sigma}(z_h), z_h) + v_z(x_h, \tilde{\sigma}(z_h), z_h) \tilde{\mu}'(\tilde{\sigma}(z_h)) - c_s(\tilde{\sigma}(z_h), z_h) = 0.
\]

Because \( \tilde{\mu}' > 0 \) for all \( s \in \text{Int } S^* \) (implication of Theorem 3.(iii)), the equality above means that

\[
v_s(x_h, \tilde{\sigma}(z_h), z_h) - c_s(\tilde{\sigma}(z_h), z_h) < 0 \tag{A53}
\]
Because \( s > \bar{\sigma}(z_h) \) and \( v - c \) is strictly concave in \( s \) (Assumption 3), (A53) implies that (A52) is not satisfied. Therefore, there is no profitable sender deviation to \( s \in [\bar{\sigma}(z_h)), s_h) \), being matched with any receiver of type \( x \in [x_h, x_{\ell}) \).

Finally, we show that there is no profitable sender deviation to any off-path action \( s \) in \([\bar{\sigma}(z_h), s_h)\), followed by matching with a receiver in \([x_h, x])\). \([x_h, x])\) is the interval of receiver types who are matched with senders on the top. The maximum amount of the reaction that the receiver of type \( x \in [x_h, x])\) is willing to choose is

\[
T(s, x) = v(x, s, z_h) - (\mathbb{E}[v(x, s_h, z') | z' \geq z_h] - t_h)
\]  

(A54)

Because \( z_h \leq z' \) and \( s < s_h \), we have that \( T(s, x) < t_h \) and we can also apply Assumption 2.(i) to show that \( T(s, x) \) decreases in \( x \). Therefore, if and only if

\[
v(x_h, s, z_h) - c(s, z_h) \leq \mathbb{E}[v(x_h, s_h, z') | z' \geq z_h] - c(s_h, z_h) \text{ for all } s \in [\bar{\sigma}(z_h), s_h)
\]

is satisfied, the sender of type \( z_h \) has no profitable deviation to any \( s \in [\bar{\sigma}(z_h)), s_h) \), followed by matching with a receiver in \([x_h, x])\). Consequently, Corollary 2 implies that no sender has an incentive to deviate to any off-path action \( s \) in \([\bar{\sigma}(z_h), s_h)\) if and only if (A55) is satisfied for all \( s \in [\bar{\sigma}(z_h), s_h)\). Because (18) and (19) are satisfied in equilibrium, we have that

\[
\mathbb{E}[v(x_h, s_h, z') | z' \geq z_h] - c(s_h, z_h) = v(x_h, \bar{\sigma}(z_h), z_h) - c(\bar{\sigma}(z_h), z_h).
\]  

(A56)

Applying (A56) to (A55) yields that for \( s \in [\bar{\sigma}(z_h), s_h)\),

\[
v(x_h, s, z_h) - c(s, z_h) \leq v(x_h, \bar{\sigma}(z_h), z_h) - c(\bar{\sigma}(z_h), z_h),
\]  

(A57)

which is always satisfied given Assumption 5 (strict concavity of \( v - c \) in \( s \)) because \( \bar{\sigma}(z_h) \) is greater than the bilaterally efficient action \( \zeta(x_h, z_h) \). Therefore, there is no profitable sender deviation to any off-path action \( s \) in \([\bar{\sigma}(z_h), s_h)\), followed by matching with a receiver in \([x_h, x])\).

(a) **Action choice in \( S^* \) by the sender of type \( z_h \)** The sender’s equilibrium action is \( s_h \). Note that \( S^* = [s_{\ell}, \bar{\sigma}(z_h)) \cup s_h \). Because of (18), her utility is the same as \( \bar{\tau}(\bar{\sigma}(z_h)) - c(\bar{\sigma}(z_h), z_h) \), which is her utility in the stronger monotone separating equilibrium. Suppose that she chooses \( s \in [s_{\ell}, \bar{\sigma}(z_h)) \). Because the transfer schedule \( \bar{\tau} \) is the same as \( \bar{\tau} \) in the stronger monotone separating equilibrium, we can apply Theorem 11 to
show that the sender of type $z_h$ has no incentive to decrease her action to $s \in [s_\ell, \tilde{\sigma}(z_h))$. Therefore, $s_h$ solves Problem 1.

(b) **Action choice in $S^*$ by the sender of type** $z \in (z_h, z]$ The sender’s equilibrium action is $s_h$. From (a) above, we know that for all $s \in [s_\ell, \tilde{\sigma}(z_h))$,

$$t_h - c(s_h, z_h) \geq \hat{\tau}(s) - c(s, z_h).$$

(A58)

Applying Assumption 1.(ii) to (A58) yields that $t_h - c(s_h, z_h) > \hat{\tau}(s) - c(s, z_h)$ for all $z > z_h$ and all $s \in [s_\ell, \tilde{\sigma}(z_h))$, which shows that the sender of type $z > z_h$ has no incentive to change her action to any other action in $S^*$. Therefore, $s_h$ solves Problem 1.

(c) **Action choice in $S^*$ by the sender of type** $z \in [z_\ell, z_h)$ Because $z < z_h$, the sender’s action is lower than $\tilde{\sigma}(z_h)$. Because (i) the transfer schedule $\hat{\tau}$ for the action in $[s_\ell, \tilde{\sigma}(z_h))$ is the same as the one in the equilibrium with only the lower bound of transfers and (ii) the sender’s action $\tilde{\sigma}(z)$ is the same $\tilde{\sigma}(z)$ that she would have chosen in the stronger monotone separating equilibrium, we apply the proof of Condition 1 in Theorem 11 to show that for any $s \in [s_\ell, \tilde{\sigma}(z_h))$ and $s \neq \tilde{\sigma}(z)$

$$\hat{\tau}(z) - c(\tilde{\sigma}(z), z) > \hat{\tau}(s) - c(s, z).$$

(A59)

Therefore, the sender has no incentive to change her action to another action in $[s_\ell, \tilde{\sigma}(z_h))$.

Now suppose that the sender changes her action to $s_h$. We know that for the sender of type $z_h$,

$$t_h - c(s_h, z_h) = \hat{\tau}(\tilde{\sigma}(z_h)) - c(\tilde{\sigma}(z_h), z_h).$$

Because $s_h > \tilde{\sigma}(z_h)$ by Lemma 8 and $z_h > z$, applying Assumption 1.(ii) to the equation above yields that

$$t_h - c(s_h, z) < \hat{\tau}(\tilde{\sigma}(z_h)) - c(\tilde{\sigma}(z_h), z).$$

(A60)

Combining (A59) at $s = \tilde{\sigma}(z_h)$ and (A60) yields that $\hat{\tau}(s) - c(s, z) > t_h - c(s_h, z)$, which shows that the sender of type $z \in [z_\ell, z_h)$ has no incentive to increase her action to $s_h$. Therefore, $s_h$ solves Problem 1.

(d) **No action choice by the sender of type** $z \in [z, z_\ell)$ The sender chooses no action in the equilibrium: $\tilde{\sigma}(z) = 0$. Consider a change to $s_\ell$. We know that for the sender of
type $z_\ell$, $t_\ell - c(s_\ell, z_\ell) = 0$. Because $z < z_\ell$, applying Assumption 1.(ii) yields that
\[ t_\ell - c(s_\ell, z) < 0, \tag{A61} \]
which implies that the sender’s utility is lower than zero, so the sender cannot gain by increasing her action to $s_\ell$.

Consider a change to $s \in S^*$ with $s > s_\ell$. From the previous section, we know that for all $s \in S^*$ with $s > s_\ell$
\[ t_\ell - c(s_\ell, z) > \hat{\tau}(s) - c(s, z_\ell). \]
Because $z < z_\ell$, applying Assumption 1.(ii) to the inequality relation above yields that for all $s \in S^*$ with $s > s_\ell$
\[ t_\ell - c(s_\ell, z) > \hat{\tau}(s) - c(s, z). \tag{A62} \]
Combining (A61) and (A62) yields that $0 > \hat{\tau}(s) - c(s, z)$ for $s \in S^*$ with $s > s_\ell$, which shows that a change to any $s \in S^*$ with $s > s_\ell$ lowers the sender’s utility. Therefore, for the sender of type $z \in [z; z_\ell)$, there is no solution for Problem 1.

L.2 Receiver’s optimal matching choice

Applying the envelope theorem to the receiver’s equilibrium utility $\hat{\Pi}(x)$, we can show that $\hat{\Pi}(x)$ is increasing and positive for $x > x_\ell$ starting from $\hat{\Pi}(x_\ell) = 0$. The receiver’s matching problem can be seen as: which sender with an action in $S^*$ does he want to match with as formulated in (4)?

(a) Optimal matching choice by the receiver of type $x_h$ The equilibrium partner is a sender with $s_h$ as his partner. Suppose that the receiver wants to choose a sender with $s \in [s_\ell, \tilde{\sigma}(z_h))$ as his partner. According to (19), the receiver’s equilibrium utility with a sender with $s_h$ satisfies
\[ \mathbb{E}[v(x_h, s_h, z)| z \geq z_h] - t_h = v(x_h, \tilde{\sigma}(z_h), z_h) - \hat{\tau}(\tilde{\sigma}(z_h)). \tag{A63} \]

The proof of Condition 3 of Theorem 11 shows that in the stronger monotone separating equilibrium, we have that for any $s \in [s_\ell, \tilde{\sigma}(z_h))$,
\[ v(x_h, \tilde{\sigma}(z_h), z_h) - \hat{\tau}(\tilde{\sigma}(z_h)) > v(x_h, s, \tilde{\mu}(s)) - \hat{\tau}(s). \tag{A64} \]
Because $\hat{\mu}(s) = \check{\mu}(s)$ and $\check{\tau}(s) = \tilde{\tau}(s)$ for any $s \in [s_{\ell}, \tilde{\sigma}(z_h))$, (A63) and (A64) together show that for any $s \in [s_{\ell}, \tilde{\sigma}(z_h))$

$$E[v(x_h, s, z) | z \geq z_h] - t_h > v(x_h, s, \hat{\mu}(s)) - \hat{\tau}(s),$$

(A65)

which shows that the receiver has no incentive to change his action to be matched with a sender with $s \in [s_{\ell}, \tilde{\sigma}(z_h))$.

(b) Optimal matching choice by the receiver of type $x > x_h$  The equilibrium utility for the receiver of type $x$ with a sender with $s_h$ as his partner is $E[v(x, s_h, z) | z \geq z_h] - t_h$. Suppose that the receiver changes his partner to a sender with $s \in [s_{\ell}, \tilde{\sigma}(z_h))$.

Given $s \in [s_{\ell}, \tilde{\sigma}(z_h))$, the belief on the sender’s type is $\hat{\mu}(s) < z_h$ and his utility is $v(x, s, \hat{\mu}(s)) - \hat{\tau}(s)$. Therefore, we need to examine the sign of the utility difference:

$$E[v(x, s_h, z) | z \geq z_h] - t_h - [v(x, s, \hat{\mu}(s)) - \hat{\tau}(s)]$$

(A66)

Applying the envelope theorem for $b_e(x, s_h)$ and $\gamma(x, s, \hat{\mu}(s))$, we can express the partial derivative of (A66) with respect to $x$

$$E[v_x(x, s_h, z) | z \geq z_h] - v_x(x, s, \hat{\mu}(s)) > 0.$$  

(A67)

To show the positive sign in (A67), note that $s_h > s$. Therefore, Assumption 2.(i) implies that $v_x(x, s_h, z) > v_x(x, s, \hat{\mu}(s))$ for any $z \geq z_h > \hat{\mu}(s)$, which leads to $E[v_x(x, s_h, z) | z \geq z_h] > v_x(x, s, \hat{\mu}(s))$.

Because the utility difference in (A66) is zero at $x = x_h$ and $s = \tilde{\sigma}(z_h)$, (A67) implies that (A66) is positive for $x > x_h$, which means that the receiver of type $x > x_h$ has no incentive to change his partner to a sender with $s \in [s_{\ell}, \tilde{\sigma}(z_h))$.

(c) Optimal matching choice by the receiver of type $x \in [x_{\ell}, x_h)$  The equilibrium outcomes for receivers of type $x \in [x_{\ell}, x_h)$ and senders of types in $[z_{\ell}, z_h)$ are the same as the outcomes in the stronger monotone separating equilibrium, including actions, transfers and matching. Therefore, from the proof of Conditions 3 in Theorem 11, we know that the utility for the receiver will be lower by changing his partner to any sender with $s \in [s_{\ell}, \tilde{\sigma}(z_h))$.

Suppose that the receiver changes his partner to a sender with $s_h$. To see if the receiver prefers such a change, first note that
\[ \mathbb{E} [v(x_h, s_h, z) | z \geq z_h] - t_h = v(x_h, \tilde{\sigma} (z_h), z_h) - \tilde{\tau} (\tilde{\sigma} (z_h)) \]  

(A68)

Given \( s_h > \tilde{\sigma} (z_h) \) and \( z \geq z_h \), we can apply Assumption 2.(i) to show that for \( x < x_h \)

\[ \mathbb{E} [v(x, s_h, z) | z \geq z_h] - t_h < v(x, \tilde{\sigma} (z_h), z_h) - \tilde{\tau} (\tilde{\sigma} (z_h)). \]  

(A69)

From the proof of Conditions 3 in Theorem 11, we also know that for \( x < x_h \)

\[ v(x, \tilde{\sigma} (z_h), z_h) - \tilde{\tau} (\tilde{\sigma} (z_h)) < v(x, \tilde{\sigma} (n^{-1}(x)), n^{-1}(x)) - \tilde{\tau} (\tilde{\sigma} (n^{-1}(x))) \]  

(A70)

Combining (A69) and (A70) yields

\[ \mathbb{E} [v(x, s_h, z) | z \geq z_h] - t_h < v(n^{-1}(x), x, \tilde{\sigma} (n^{-1}(x)), n^{-1}(x)) - \tilde{\tau} (\tilde{\sigma} (n^{-1}(x))) \]  

(A71)

The expression on the right-hand side of (A71) is indeed the same as the equilibrium utility for the receiver of type \( x < x_h \) in the well-behaved equilibrium. Therefore, a receiver of type \( x \in [x_\ell, x_h) \) strictly prefers a sender with \( \tilde{\sigma} (n^{-1}(x)) = \tilde{\sigma} (n^{-1}(x)) \) as his partner.

(d) Optimal action choice by the receiver of type \( x \in [x, x_\ell) \) The receiver of type \( x \in [x, x_\ell) \) is unmatched in equilibrium. Suppose that the receiver decides to choose a sender with \( s_\ell \) as his partner. We know that \( v(x_\ell, s_\ell, \hat{\mu} (s_\ell)) - t_\ell = 0 \). This implies that for \( x \in [x, x_\ell), \)

\[ v(x, s_\ell, \hat{\mu} (s_\ell)) - t_\ell < 0. \]  

(A72)

Therefore, the receiver of type \( x \in [x, x_\ell) \) has no incentive to choose a sender with \( s_\ell \) as his partner.

Suppose that the receiver of type \( x \in [x, x_\ell) \) chooses a sender with \( s \in (s_\ell, \tilde{\sigma} (z_h)) \) as his partner. According to Subsection (c) above, we know that for any \( s \in (s_\ell, \tilde{\sigma} (z_h)), \)

\[ v(x_\ell, s, \hat{\mu} (s)) - \tau(s) < v(x_\ell, s_\ell, \hat{\mu} (s_\ell)) - t_\ell \]  

(A73)

Applying Assumption 2.(i) to (S52) yields that for any \( s \in (s_\ell, \tilde{\sigma} (z_h)) \) and any \( x \in [x, x_\ell) \)

\[ v(x, s, \hat{\mu} (s)) - \tau(s) < v(x, s_\ell, \hat{\mu} (s_\ell)) - t_\ell. \]  

(A74)

Because the expression on the right hand side of (S53) is the same as the expression on
the left hand side of (S51), we can conclude that \( v(x, s, \hat{\mu}(s)) - \tau(s) < 0 \) for \( x < x_\ell \) and \( s \in (s_\ell, \hat{\sigma}(z_h)) \), which show that the receiver’s utility becomes negative to choose a sender with \( s \in (s_\ell, \hat{\sigma}(z_h)) \) as his partner.

Finally, suppose that the receiver of type \( x \in [x, x_\ell) \) chooses a sender with \( s_h \) as his partner. According to Subsection (c) above, we know that

\[
\mathbb{E}[v(x, s_h, z) | z \geq z_h] - t_h < v(x_\ell, s_\ell, \hat{\mu}(s_\ell)) - t_\ell
\]

(A75)

Given \( s_h > s_\ell, \ z_h > \hat{\mu}(s_\ell) \), applying Assumption 2.(i) implies that for \( x < x_\ell \)

\[
\mathbb{E}[v(x, s_h, z) | z \geq z_h] - t_h < v(x, s_\ell, \hat{\mu}(s_\ell)) - t_\ell
\]

(A76)

Because the expression on the right hand side of (S55) is the same as the expression on the left hand side of (S51), we can conclude that \( \mathbb{E}[v(x, s_h, z) | z \geq z_h] - t_h < 0 \) for \( x < x_\ell \). This shows that the receiver’s utility is negative with a sender a sender with \( s_h \) as his partner. This concludes that no receiver of type \( x \in [x, x_\ell) \) wants to choose any sender in the market as his partner.

M Proof of Proposition 1

When \( z_\ell = z, \ s_\ell = \zeta(x, z) = 0 \) and \( t_\ell = c(\zeta(x, z), z) = 0 \). When \( t_\ell = 0, \ z_\ell = z \) and \( s_\ell = \zeta(x, z) = 0 \). Now consider the case with \( z_\ell \in (\underline{z}, \overline{z}) \). First consider case (i) in Assumption 7. For any given \( z_\ell \in (\underline{z}, \overline{z}) \), (10) and (11) induce the equation:

\[
v(n(z_\ell), s, z_\ell) - c(s, z_\ell) = 0.
\]

(A77)

If \( s = 0 \), then the left-hand side of (A77) is positive. As \( s \to \infty \), the left-hand side approaches \( -\infty \) because of Assumption 4. Given Assumption 7.(i), \( v(n(z_\ell), s, z_\ell) \) is positive and it is independent of \( s \). Because \( c \) is continuous in \( s \) (Assumption 4), it means that there exists a unique \( s_\ell \) satisfying ((A77)). Then, a unique \( t_\ell \) is determined by either (10) or (11) given \( s_\ell \) and \( z_\ell \).

Now consider case (ii) in Assumption 7. For any given \( z_\ell \in (\underline{z}, \overline{z}) \), the left hand side of (A77) is zero at \( s = 0 \). However, we cannot have \( s_\ell = 0 \). If \( s_\ell = 0 \), then \( t_\ell \) must be zero. Then every seller’s utility is zero by entering the market. This implies that every sender will enter the market so \( z_\ell \) cannot be greater than \( \overline{z} \) given our assumption that everyone enters the market if she is indifferent between entering the market and staying out of it.
Because of Assumptions 4 and 5, \( v - c \) is strictly concave and the left hand side of (A77) approaches \(-\infty\) as \( s \to \infty \). Since the left hand side of (A77) is zero at \( s = 0 \), this implies that there exists a unique positive \( s_\ell \) satisfying (A77). Then, a unique \( t_\ell \) is determined by either (10) or (11) given \( s_\ell \) and \( z_\ell \).

Suppose that the DM chooses \( t_\ell \), a part of the unique solution \((t_\ell, s_\ell)\) that solves (10) and (11) given \( z_\ell \). Then, \((z_\ell, s_\ell)\) is a unique solution that solves (10) and (11) because of Lemma 2.

### Proof of Proposition 2

First consider case (i) in Assumption 7. For any given \( z_h \in (z_\ell, \overline{z}) \), (18) and (19) induces the equation:

\[
\begin{align*}
  c(s, z_h) - c(\tilde{\sigma}(z_h), z_h) &= \mathbb{E}[v(n(z_h), s, z')|z' \geq z_h] - v(n(z_h), \tilde{\sigma}(z_h), z_h) - c(0, z_h) < v(n(z_h), \tilde{\sigma}(z_h), z_h). \quad \text{(A78)}
\end{align*}
\]

The right hand side of (A78) is positive because it is independent of \( s \) and \( \tilde{\sigma}(z_h) \) and \( z' \geq z_h \) for \( z_h < \overline{z} \). The left-hand side is continuous and increasing in \( s \). Because of Assumptions 1.(i) and 4, the left hand side is increasing in \( s \) with \( \lim_{s \to 0} [c(s, z_h) - c(\tilde{\sigma}(z_h), z_h)] < 0 \) and \( \lim_{s \to \infty} [c(s, z_h) - c(\tilde{\sigma}(z_h), z_h)] = \infty \). Therefore, we have a unique solution for \( s_h \) that solves (A78). Then, \( t_h \) can be uniquely derived from either (18) and (19). Therefore, for any given \( z_h \in (z_\ell, \overline{z}) \), there exists a unique \((t_h, s_h)\) that satisfies (18) and (19).

Now consider case (ii) in Assumption 7. For any given \( z_h \in (z_\ell, \overline{z}) \), (18) and (19) induce the equation:

\[
\begin{align*}
  \mathbb{E}[v(n(z_h), s, z')|z' \geq z_h] - c(s, z_h) &= v(n(z_h), \tilde{\sigma}(z_h), z_h) - c(\tilde{\sigma}(z_h), z_h) - c(0, z_h) < v(n(z_h), \tilde{\sigma}(z_h), z_h). \quad \text{(A79)}
\end{align*}
\]

The right hand side of (A79) is positive because it is the sum of the equilibrium utilities for the sender type \( z_h \) and the receiver type \( n(z_h) \) for \( z_h \in (z_\ell, \overline{z}) \) in the stronger monotone separating equilibrium and both equilibrium utilities for senders and receivers are increasing in types in the separating equilibrium. Because of Assumption 4 (\( c(0, z) = 0 \) for all \( z \)) and case (ii) in Assumption 7 (\( v(x, 0, z) = 0 \) for all \( x \) and \( z \)), we have that

\[
0 = \mathbb{E}[v(n(z_h), 0, z')|z' \geq z_h] - c(0, z_h) < v(n(z_h), \tilde{\sigma}(z_h), z_h) - c(\tilde{\sigma}(z_h), z_h) \quad \text{(A80)}
\]

Because of Assumption 5 (\( \lim_{s \to \infty} v_a(x, s, z) = 0 \) given any \( x \) and \( z \) and \( \lim_{s \to \infty} c_a(s, z) = \)).
\[ -\infty = \lim_{s \to \infty} \mathbb{E}[v(n(z_h), s, z')|z' \geq z_h] - c(s, z_h) < v(n(z_h), \sigma(z_h), z_h) - c(\tilde{\sigma}(z_h), z_h) \quad (A81) \]

When \( s = \tilde{\sigma}(z_h) \), we have that

\[ \mathbb{E}[v(n(z_h), \tilde{\sigma}(z_h), z')|z' \geq z_h] - c(\tilde{\sigma}(z_h), z_h) > v(n(z_h), \tilde{\sigma}(z_h), z_h) - c(\tilde{\sigma}(z_h), z_h). \quad (A82) \]

Because \( \mathbb{E}[v(n(z_h), \sigma(z_h), z')|z' \geq z_h] - c(s, z_h) \) is strictly concave in \( s \) due to Assumption 5, (A80), (A81), and (A82) imply that there are two values, \( s^0 \) and \( s^1 \) that satisfy (A79) with \( s^0 < \tilde{\sigma}(z_h) < s^1 \). Because \( \tilde{\sigma}(z_h) < s_h \) from Lemma 8, \( s_h \) is equal to \( s^1 \). Because \( t_h \) can be uniquely derived from either (18) and (19), there exists a unique \((t_h, s_h)\) that satisfies (18) and (19) for any given \( z_h \in (z_\ell, \bar{z}) \). Therefore, in both cases in Assumption 7, there exists a unique \( t_h \) and a unique \( s_h \) that satisfies (18) and (19) for any given \( z_h \in (z_\ell, \bar{z}) \). Suppose that the DM chooses \( t_h \), a part of the unique solution \((t_h, s_h)\) that solves (18) and (19) given \( z_h \in (z_\ell, \bar{z}) \). Then, \((z_h, s_h)\) is a unique solution that solves (18) and (19) because of Lemma 7.

Because all functions in (18) and (19) are continuous (Assumptions 3.(ii), 4.(i), 6.(ii) and Theorem 3), it is clear the solution \((t_h, s_h)\) is continuous in \( z_h \) and that \( z_h \), the part of the solution \((z_h, s_h)\) is also continuous in \( t_h \). Finally, \( \lim_{z_h \to z_\ell} t_h = t_\ell \) and \( \lim_{z_h \to z_\ell} s_h = \tilde{\sigma}(z_\ell) \) because (18) and (19) are satisfied only when \((t_h, s_h) = (t_\ell, \tilde{\sigma}(z_\ell))\) as \( z_h \to \bar{z} \).

### O  Proof of Lemma 9

It is clear that if \( z_h \to \bar{z} \), \( \{\tilde{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} \) converges to the stronger monotone separating equilibrium with the same lower threshold sender type \( z_\ell \) as the pooling part vanishes. Note that as \( z_h \to z_\ell \), \( \lim_{z_h \to z_\ell} t_h = t_\ell \) and \( \lim_{z_h \to z_\ell} s_h = \tilde{\sigma}(z_\ell) = s_\ell \) because (18) and (19) are satisfied only when \((t_h, s_h) = (t_\ell, \tilde{\sigma}(z_\ell))\). Therefore, combining (10) and (11) with (18) and (19) yields that

\[
\lim_{z_h \to z_\ell} [t_h - c(s_h, z_h)] = t_\ell - c(s_\ell, z_\ell) = 0,
\]

\[
\lim_{z_h \to z_\ell} \left[ \mathbb{E}[v(n(z_h), s_h, z')|z' \geq z_h] - t_h \right] = \mathbb{E}[v(n(z_\ell), s_\ell, z')|z' \geq z_\ell] - t_\ell \geq 0,
\]

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where the inequality holds with equality if $z_\ell > \bar{z}$. The second equality of the first line and the inequality of the second line are a consequence of Theorem 5. These imply that \{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} converges the stronger monotone pooling equilibrium where $t_\ell$ is the single feasible reaction, $z_\ell$ is the threshold sender type for market entry and $s_\ell$ is the pooled action for senders in the market. Note that when $z_\ell = \bar{z}$, we have $s_\ell = 0$ and $t_\ell = 0$ in \{\hat{\sigma}, \hat{\mu}, \hat{\tau}, \hat{m}\} due to Theorem 5.

\section{Proof of Proposition 3}

Consider the initial value problem:

\[
\begin{cases}
    v_s(n(\mu(s)), s, \mu(s)) + v_z(n(\mu(s)), s, \mu(s))\mu'(s) - c_s(s, \mu(s)) = 0 \\
    \mu(s_\ell) = z_\ell
\end{cases}
\]

Given $v(x, s, z) = As^a x z$, $c(s, z) = \beta s^2 z$, and $n(z) = kz^q$, we have that $v_z(x, s, z) = As^a x$, $v_s(x, s, z) = aAs^a x z$, and $c_s(s, z) = 2\beta s$. Therefore, the above IVP becomes

\[
aAs^{a-1}k\mu(s)^q\mu(s) + As^a k\mu(s)^q \mu'(s) - \frac{2\beta s}{\mu(s)} = 0.
\]  

(A83)

Rewriting (A83) gives

\[
\frac{Ak}{2\beta} \mu(s)^{1+q}\mu'(s) + \frac{aAk}{2\beta s} \mu(s)^{2+q} = s^{1-a}.
\]  

(A84)

Let $D =: \frac{Ak}{2\beta}$. Thus, (A84) becomes

\[
D\mu^{1+q}\mu' + \frac{aD}{s} \mu^{2+q} = s^{1-a}
\]  

(A85)

where we denote $\mu =: \mu(s)$ for simplicity. Let $v = \mu^{2+q}$. Then $v' = (2 + q)\mu^{1+q}\mu'$ and (A85) becomes

\[
v' + \frac{a(2 + q)}{s} v = \left(\frac{2 + q}{D}\right)s^{1-a}
\]  

(A86)

which is a first order linear differential equation with integrating factor $I(s) = s^{a(2+q)}$.  

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Therefore, (A86) is equivalent to
\[
\frac{d}{ds} \left\{ v s^{a(2+q)} \right\} = \left( \frac{2+q}{D} \right) s^{1+a+aq},
\]
which implies
\[
vs^{a(2+q)} = \left( \frac{2+q}{D} \right) \int s^{1+a+aq} ds + \kappa, \tag{A87}
\]
where \(\kappa\) is some integration constant. Equation (A87) implies
\[
\mu(s)^{2+q} = \left( \frac{2+q}{D} \right) \left( \frac{s^{2-a}}{2+a+aq} \right) + \frac{\kappa}{s^{a(2+q)}}, \tag{A88}
\]
given that \(v = \mu^{2+q}\). By using the initial condition \(\mu(s_\ell) = z_\ell\), we compute \(\kappa\) as follows:
\[
z_\ell^{2+q} = \left( \frac{2+q}{D} \right) \left( \frac{s_\ell^{2-a}}{2+a+aq} \right) + \frac{\kappa}{s_\ell^{a(2+q)}}, \tag{A89}
\]
which gives
\[
\kappa = \frac{s_\ell^{a(2+q)} \left[ D(2+a+aq)z_\ell^{2+q} - (2+q)s_\ell^{2-a} \right]}{D(2+a+aq)}. \tag{A90}
\]
Plugging (A90) into (A88) gives
\[
\tilde{\mu}(s) = \left[ \left( \frac{2\beta(2+q)}{Ak} \right) \frac{s^{2-a}}{2+a+aq} + \left( \frac{s_\ell}{s} \right)^{a(2+q)} \left[ Ak(2+a+aq)z_\ell^{2+q} - 2\beta(2+q)s_\ell^{2-a} \right] \right] \frac{1}{2+q}, \tag{A91}
\]
where \(s_\ell\) is determined by (23) and it is \(s_\ell(z_\ell) = \left( \frac{Ak}{\beta} \frac{q^{2+2}}{z_\ell^{2-a}} \right)^{\frac{1}{2-a}}\).

Q Proof of Proposition 10

First, we construct a (unique) stronger monotone pooling equilibrium with a positive single feasible reaction \(t^* > 0\). Because (16) and (17) must hold with equality by Theorem 5 with \(z^* > z\), the pooled action \(s^*(z^*, q, a)\) solves \(As^*sz^*qE[z|z \geq z^*] - \beta s^2 = 0\) and hence we have that \(s^*(z^*, q, a) = \left( \frac{z^*q + AkE[z|z \geq z^*]}{\beta} \right)^{\frac{1}{2-a}}\). Therefore, we have that
\[
\lim_{q,a \to 0} s^*(t, q, a) = \sqrt{z^* AkE[z|z \geq z^*]}/\beta.
\]

The aggregate net surplus in the stronger monotone pooling equilibrium is
\[
\Pi_p(z^*, q, a, G) := \int_{z^*}^{\bar{z}} As^*(z^*, q, a)^a k z^* q E[z|z \geq z^*] dG(z) - \beta s^*(z^*, q, a)^2 \int_{z^*}^{\bar{z}} \frac{1}{z} dG(z).
\]
This implies that
\[
\lim_{q,a \to 0} \Pi_p(z^*, q, a, G) = \int_{z^*}^{\bar{z}} Ak E[z|z \geq z^*] dG(z) - z^* Ak E[z|z \geq z^*] \int_{z^*}^{\bar{z}} \frac{1}{z} dG(z),
\]
\[
\lim_{z^* \to 0} \left[ \lim_{q,a \to 0} \Pi_p(z^*, q, a, G) \right] = \int_{0}^{z^*} Ak \mu_z dG(z) = Ak \mu_z
\]

where \(\mu_z\) is the unconditional mean of the sender type \(z\). Because \(\lim_{q,a \to 0} \Pi^*(q, a, G) = \frac{Ak \mu_z}{2}\), we have that
\[
\lim_{z^* \to 0} \left[ \lim_{q,a \to 0} \Pi_p(z^*, q, a, G) \right] - \lim_{q,a \to 0} \Pi^*(q, a, G) = \frac{Ak \mu_z}{2} > 0. \quad (A92)
\]

Because \(\Pi_p(z^*, q, a, G)\) and \(\Pi^*(q, a, G)\) are continuous, there exists \(\tilde{q} > 0\), and \(\tilde{a} > 0\) and \(z^*(\tilde{q}, \tilde{a}) \in \text{Int } Z\) such that for every \((q, a) \in [0, \tilde{q}] \times [0, \tilde{a}]\) and every \(z^* \in (0, z^*(\tilde{q}, \tilde{a})]\), \(\Pi_p(z^*, q, a, G) > \Pi^*(q, a, G)\). We can retrieve \(t^*\) given \(z^* \in (0, z^*(\tilde{q}, \tilde{a})\].

**R Additional Figures**

Figure 7: Cumulative Distribution Functions

![Cumulative Distribution Functions](image-url)
Figure 8: Efficiency Measures and Relative Surplus Gains

Eff. Measures

| $a$ = 0.0 |
|------------|

Rel. Sur. Gains

| $a$ = 0.1 |
|------------|

| $a$ = 0.2 |
|------------|

Notes. The efficient measures are computed by $\Pi_s/\Pi^*$, where $\Pi^*$ is the aggregate net surplus without any restriction on feasible reactions. The relative surplus gains are computed by $100 \times (\Pi_w - \Pi_s)/\Pi_s$, where $\Pi_w$ is aggregate net surplus of the well-behaved equilibrium.
Figure 9: Efficiency Measures and Relative Surplus Gains (Cont.)

Eff. Measures

\[ a = 0.3 \]

\[ a = 0.4 \]

\[ a = 0.5 \]

Rel. Sur. Gains
Figure 10: Efficiency Measures and Relative Surplus Gains (Cont.)

Eff. Measures

\[ a = 0.6 \]

Rel. Sur. Gains

\[ a = 0.7 \]

\[ a = 0.8 \]

\[ a = 0.9 \]