UNIPOTENT GENERATORS FOR ARITHMETIC GROUPS

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Abstract. We sketch a simplification of proofs of old results on the arithmeticity of the group generated by opposing integral unipotent radicals contained in higher rank arithmetic groups.

1. Introduction

A well known theorem of Jacques Tits [Tits1] says that if \( n \geq 3, k \geq 1 \) are integers, then the group generated by upper and lower triangular unipotent matrices in the principal congruence subgroup \( SL_n(k\mathbb{Z}) \) of level \( k \), has finite index in \( SL_n(\mathbb{Z}) \). This theorem admits a generalisation which will be described below (for definitions of the terms involved, see section 2).

Let \( G \subset SL_n \) be a semi-simple \( \mathbb{Q} \)-simple algebraic group defined over \( \mathbb{Q} \) and let \( Q \subset G \) be a proper parabolic \( \mathbb{Q} \)-subgroup with unipotent radical \( U^+ \). Let \( U^- \) be the opposite unipotent radical and for an integer \( k \geq 1 \), denote by \( E_Q(k) \) the subgroup generated by \( U^+ \cap SL_n(k\mathbb{Z}) \) and \( U^- \cap SL_n(k\mathbb{Z}) \). The aim of this note is to provide a proof (which is perhaps simpler and more uniform than the existing ones in the literature) of the following result.

**Theorem 1.** If \( \mathbb{R} - \text{rank}(G) \geq 2 \), then the group \( E_Q(k) \) is an arithmetic subgroup of \( G(\mathbb{Q}) \), i.e. has finite index in \( G(\mathbb{Z}) = G \cap SL_n(\mathbb{Z}) \).

**Remark.** Every semi-simple \( \mathbb{Q} \)-simple algebraic group \( G \) is the group obtained by (Weil) restriction of scalars, of an absolutely simple algebraic group \( \mathcal{G} \) defined over a number field \( K : G = R_{K/\mathbb{Q}}(\mathcal{G}) \). Theorem 1 is due to [Tits1] If \( \mathcal{G} \) is a Chevalley group with \( K - \text{rank}(\mathcal{G}) \geq 2 \). For most of the classical groups, Theorem 1 was proved by Vaserstein [Vaserstein1], and [Raghunathan1] proved it for general groups of \( \mathbb{Q} - \text{rank} \) at least two. The remaining cases were proved in [V1].
These references prove Theorem 1 when \( Q \) is a minimal parabolic \( Q \)-subgroup; however, as observed in \([V3]\) and \([Oh2]\), the general case follows easily from this case.

Remark. A generalisation of Theorem 1 to the case when the arithmetic group \( G(\mathbb{Z}) \) is replaced by any Zariski dense discrete subgroup of \( G(\mathbb{R}) \) is proved in \([Oh1]\), \([Benoist-Oh]\), \([Benoist-Oh2]\) and \([Benoist-Miquel]\); the proofs of these results are of a very different nature and we do not consider this situation. In fact, the proofs in these references make use of Theorem 1.

Remark. The proof of Theorem 1 given here is uniform; however, this is based on Theorem 4 whose proof is not quite uniform but works especially well (see Section 3) when the group \( G(\mathbb{R}) \) is not a product of simple Lie groups of real rank one. In case \( G(\mathbb{R}) \) is a product of rank one groups, a more complicated argument is needed and we give the proof in Section 4.

We now describe another result from which Theorem 1 will be derived. Let \( G \) be a \( Q \)-simple group with \( Q - \text{rank}(G) \geq 1 \). Let \( P \subset G \) be a proper maximal parabolic \( Q \)-subgroup. Denote by \( U^+ \) (resp. \( U^- \)) the unipotent radical (the “opposite unipotent radical”) of \( P \) and let \( P = LU^+ \) be a Levi-decomposition of \( P \); set \( P^- = LU^- \), the parabolic subgroup of \( G \) “opposite” to \( P \). Clearly \( L \) normalises \( U^\pm \).

Denote by \( M \) the connected component of the Zariski closure of the group \( L(\mathbb{Z}) \) of integer points of \( L \). The group \( M \) is non-trivial if and only if \( \mathbb{R} - \text{rank}(G) \geq 2 \) (Lemma 8). We denote by \( V^\pm \) the commutator group \([M, U^\pm]\).

For each \( k \geq 1 \), denote by \( F(k) \) the group generated by \( V^\pm(k\mathbb{Z}) \) and \( M(k\mathbb{Z}) \). Denote by \( Cl(F(k)) \) the closure of \( F(k) \) in the group \( G(k_f) \) of finite adeles \( A_f \) over \( \mathbb{Q} \). Let \( \Gamma_k = G(\mathbb{Q}) \cap Cl(F(k)) \). The group \( \Gamma_k \) has finite index in \( G(\mathbb{Z}) \) (Lemma 11); it is the smallest congruence subgroup of \( G(\mathbb{Z}) \) containing \( F(k) \). We prove:

**Theorem 2.** If \( \mathbb{R} - \text{rank}(G) \geq 2 \), then \( F(k) \) contains the commutator subgroup \([\Gamma_k, \Gamma_k]\):

\[
[\Gamma_k, \Gamma_k] \subset F(k).
\]

We now show that Theorem 2 implies Theorem 1. By the Margulis normal subgroup theorem, the commutator \([\Gamma_k, \Gamma_k]\) has finite index in the higher rank arithmetic group \( \Gamma_k \) and is therefore an arithmetic group; therefore, by Theorem 2, the group

\[
F(k) = \langle V^+(k), V^-(k), M(k) \rangle
\]
is an arithmetic group. Since $M$ normalises $V^+$ and $V^-$ it follows that $F(k)$ normalises the group $E'_P(k) = \langle V^+(k), V^-(k) \rangle$ generated by $V^\pm(k)$. Therefore, again by the normal subgroup theorem, $E'_P(k)$ is an arithmetic group. Since $E'_P(k)$ is contained in the group $E_P(k) = \langle U^+(k), U^-(k) \rangle$ it follows that the group $E_P(k)$ is an arithmetic group for every maximal parabolic $\mathbb{Q}$-subgroup $P$ of $G$.

Let $Q \subset G$ be a parabolic $\mathbb{Q}$-subgroup as in Theorem 1. Fix a maximal parabolic $\mathbb{Q}$-subgroup $P$ of $G$ containing $Q$. Then $U^\pm_Q \supset U^\pm_P$. Hence $E_Q(k) = \langle U^+(k), U^-(k) \rangle$ contains the group $E_P(k) = \langle U^+_P(k), U^-_P(k) \rangle$. By the preceding paragraph, $E_P(k)$ is arithmetic and hence so is $E_Q(k)$; this proves Theorem 1.

Theorem 2 is deduced from a result on the centrality of the kernel for a map between two completions of the group $G(\mathbb{Q})$. This centrality is somewhat analogous to that of the centrality of the congruence subgroup kernel (in the case $R - \text{rank}(G) \geq 2$), except that the congruence subgroup kernel is a compact (profinite) group. In our case, it is not clear, a priori, that $C$ is even locally compact (it will follow after the fact that $C$ is in fact finite). The details of the construction of the relevant completion will be given in Section 2. We briefly describe the construction here.

 Equip the group $G(\mathbb{Q})$ with the topology $\mathcal{T}$ generated by the various cosets $\{gF(k) : g \in G(\mathbb{Q}), k \geq 1\}$ where $F(k)$ is as in Theorem 2. Then we prove in Section 2 the following proposition (note that even in the proposition the assumption of higher real rank is necessary).

**Proposition 3.** If $R - \text{rank}(G) \geq 2$, then the group $G$, equipped with the topology $\mathcal{T}$ is a topological group.

The topological group $(G, \mathcal{T})$ then admits a (two-sided) completion $\hat{G}$ which can be shown to map onto $\overline{G} \subset G(\mathbb{A}_f)$ where $\overline{G}$ is the closure of $G(\mathbb{Q})$ in the finite adelic group $G(\mathbb{A}_f)$ (also referred to as the congruence completion of $G(\mathbb{Q})$); if $G$ is simply connected, then by strong approximation, $\overline{G} = G(\mathbb{A}_f)$. Then we get an exact sequence

$$1 \to C \to \hat{G} \to \overline{G} \to 1$$

of topological groups; the map $\hat{G} \to \overline{G}$ can be shown to be an open map. The kernel $C$ is closed in $\hat{G}$; however, it is not clear a-priori that $C$ is even compact). The main result of the paper is

**Theorem 4.** If $R - \text{rank}(G) \geq 2$, then the kernel $C$ is central in $\hat{G}$. 

It can now be seen why this centrality implies Theorem 2. By the definition of the group $\Gamma_k$ (as the smallest congruence subgroup containing $F(k)$), the groups $F(k)$ and $\Gamma_k$ have the same closure in $G(A_f)$. The openness of the map $\hat{G} \to \hat{G}$ then implies that the closures $\hat{\Gamma}_k, \hat{F}(k)$ in $\hat{G}$ of the groups $\Gamma_k, F(k)$ have the property that $\hat{\Gamma}_k \subset C\hat{F}(k)$. Therefore, by the centrality of $C$ (Theorem 4) we see that
\[
[\Gamma_k, \Gamma_k] \subset [\hat{\Gamma}_k, \hat{\Gamma}_k] \subset [\hat{F}(k), \hat{F}(k)] \subset \hat{F}(k).
\]
Therefore, we get
\[
[\Gamma_k, \Gamma_k] \subset G(\mathbb{Q}) \cap \hat{F}(k).
\]
From Lemma 6 we have that $G(\mathbb{Q}) \cap \hat{F}(k) = F(k)$; therefore we get $[\Gamma_k, \Gamma_k] \subset F(k)$, proving Theorem 2.

The centrality of $C$ is deduced in Section 3 when the group $M$ is not abelian; this is shown to be a simple consequence of strong approximation. When the group $M$ is not abelian, the proof is more complicated and is dealt with in Section 4; this involves the analogues of some results which are essentially proved (but stated only for the congruence subgroup kernel, instead of our group $C$) in [V2].

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2. Preliminaries

2.1. **Topological Groups.** Let $G$ be a group and $\mathcal{C} = \{W\}$ a (countable) collection of subgroups $W$ with $\cap_{W \in \mathcal{C}} W = \{1\}$, and such that for any finite set $F \subset \mathcal{C}$ there exists a subgroup $V \in \mathcal{C}$ such that $V \subset \cap_{W \in F} W$.

Let $\mathcal{T}$ be the topology on $G$ generated by the cosets $xW$ with $x \in G$ and $W \in \mathcal{C}$.

**Lemma 5.** The pair $(G, \mathcal{T})$ is a topological group if and only if for any $x \in G$ and $W \in \mathcal{C}$ there exists a subgroup $V \in \mathcal{C}$ such that $xWx^{-1} \supset V$.

**Proof.** Suppose for $x \in G$ and $W \in \mathcal{C}$, there exists $V \in \mathcal{C}$ such that $xWx^{-1} \supset V$. Let $x, y \in G$ and put $z = xy$; let $U$ be a neighbourhood of $z$. There exists $W \in \mathcal{C}$ such that $zW$ is a neighbourhood of $z$ and $zW \subset U$. By our assumptions on $\mathcal{C}$, there exists a $V \in \mathcal{C}$ such that $V \subset yWy^{-1} \cap W$. We then get

$$xVy = xyVy \subset xyWW = xyW = zW,$$

proving that the multiplication map $(x, y) \mapsto xy = z$ is continuous at $(x, y)$. Moreover, $x^{-1}W \supset (Wx)^{-1} = (xx^{-1}Wx)^{-1} \supset (xV)^{-1}$ for some $V \in \mathcal{C}$, proving the continuity of $x \mapsto x^{-1}$. Therefore, $(G, \mathcal{T})$ is a topological group.

If the pair $(G, \mathcal{T})$ is a topological group, then the map $x \mapsto x^{-1}$ is continuous, and hence given $W \in \mathcal{C}$ and $x \in G$, the group $xWx^{-1}$ is an open subgroup and hence contains some $V \in \mathcal{C}$. \[\Box\]

**Example.** Take $G \subset SL_n(\mathbb{Q})$ to be a $\mathbb{Q}$-algebraic subgroup and $\mathcal{C}$ to be the collection of principal congruence subgroups $G(k\mathbb{Z}) := G \cap SL_n(k\mathbb{Z})$ of integral matrices in $G \cap SL_n(\mathbb{Z})$ congruent to the identity matrix modulo $k$ with $k \geq 1$. Then $(G, \mathcal{T})$ becomes a topological group and the topology on $G(\mathbb{Q})$ is the **congruence topology**.

**Remark.** If the condition of Lemma 5 is satisfied, we say that a sequence $x_p$ of elements of $G$ converges to an element $y$, if given a subgroup $W \in \mathcal{C}$, there exists an integer $p(W)$ such that for all $p \geq p(W)$, we have $x_p^{-1}y \in W$.

2.2. **Completions of Topological Groups.** Let $G$ be a topological group and $\mathcal{C} = \{W\}$ a collection of subgroups as in 2.1. We will say that a sequence $\{x_n\}_{n \geq 1}$ in $G$ is a (two sided) Cauchy sequence if given a subgroup $W \in \mathcal{C}$, there exists an integer $n(W)$ such that

$$x_n^{-1}x_{n+m} \in W, \quad x_{n+m}x_n^{-1} \in W \quad (\forall \ n \geq n(W), \ \forall \ m \geq 1).$$
We will say that two Cauchy sequences \( \{x_n\}, \{y_n\} \) are equivalent if given \( W \in \mathcal{C} \), there exists an integer \( n(W) \) such that
\[
x_n^{-1}y_n \in W, \quad y_nx^{-1}_n \in W \quad \forall \quad n \geq n(W).
\]
Denote by \( \hat{G} \) the set of equivalence classes of Cauchy sequences; elements of the original group \( G \) may be thought of as the set of constant Cauchy sequences. The resulting map \( G \to \hat{G} \) is an embedding (this follows from the assumption that the intersection \( \cap W \in \mathcal{C} = \{1\} \)). If \( x = \{x_n\}, y = \{y_n\} \in \hat{G} \) denote by \( xy \) and \( x^{-1} \) respectively the sequences \( \{x_ny_n\} \) and \( \{x_n^{-1}\} \); it is routine to see that these sequences are Cauchy and we then get the structure of a group on \( \hat{G} \).

Given \( W \in \mathcal{C} \), denote by \( \hat{W} \) the set of Cauchy sequences \( x = (\{x_n\}) \) such that for some integer \( n(W) \) we have \( x_n \in W \quad \forall \quad n \geq n(W) \). Then \( \hat{W} \) is a subgroup of \( \hat{G} \). Write \( \hat{\mathcal{C}} \) for the collection of sets \( \{\hat{W} : W \in \mathcal{C}\} \), and by \( \hat{T} \) the topology on \( \hat{G} \) generated by the the collection of cosets \( \{x\hat{W} : x \in \hat{G}, \ W \in \mathcal{C}\} \). Then \( \hat{G} \) and \( \hat{\mathcal{C}} \) satisfy the conditions of 2.1 and hence \( (\hat{G}, \hat{T}) \) is a topological group, referred to as the (two sided) completion of \( (G, T) \) (see [Bour], Chapter III, Section 3, Exercise 6). It follows from the definitions that \( \hat{G} \) is complete in the sense that Cauchy sequences in \( (\hat{G}, \hat{T}) \) converge to an element of \( \hat{G} \).

We first note an easy consequence of the definitions.

**Lemma 6.** If \( W, G \) is as before then the intersection \( \hat{W} \cap G = W \).

**Proof.** Let \( g = (g, g, g, \cdots) \in G \) and suppose it lies in \( \hat{W} \). Therefore, there exists a Cauchy sequence \( \{x_n\}_{n \geq 1} \) such that \( x_n \in W \) for large enough \( n \) which is equivalent to the constant sequence \( g \). Therefore, for \( m \) large enough, the elements \( gx_m^{-1}, x_m^{-1}g \) lie in \( W' \) for any given \( W' \in \mathcal{C} \); in particular, if we take \( W' = W \), we then see that for \( m \) large,
\[
g = (gx_m^{-1})x_m \in W.
\]
\[ \square \]

**Notation.** Given elements \( x, y \) of a group \( \Gamma \), we write
\[
x(y) = xyx^{-1}, \quad [x, y] = xyx^{-1}y^{-1}.
\]
If \( \Delta \subset \Gamma \) is a subgroup, we write \( x(\Delta) = x\Delta x^{-1} \). If \( A, B \subset \Gamma \) are subgroups, then \([A, B]\) denotes the subgroup generated by the commutators \([a, b]\) with \( a \in A \quad b \in B \).

**Example.** Let \( G \subset SL_n \) be a linear algebraic group defined over \( \mathbb{Q} \). Consider the collection \( G(k\mathbb{Z}) = G \cap SL_n(k\mathbb{Z}) \) of congruence subgroups
in $G(\mathbb{Q})$. This collection satisfies the hypotheses of 2.1 and hence we get a completion denoted $\overline{G}$ of $G(\mathbb{Q})$, referred to as the congruence completion of $G(\mathbb{Q})$. This is also the closure of $G(\mathbb{Q})$ embedded as a subgroup of $G(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adeles.

We may also consider the collection of subgroups of $G(\mathbb{Q})$ commensurable to $G(\mathbb{Z})$ (these are referred to as arithmetic subgroups of $G(\mathbb{Q})$); the collection of arithmetic subgroups also satisfy the hypotheses of 2.1. We therefore get a completion $\hat{G}$ of $G(\mathbb{Q})$, referred to as the arithmetic completion of $G(\mathbb{Q})$. We then have a surjective open map $\hat{G} \to G$ of topological groups split over $G(\mathbb{Q})$; the kernel $\mathcal{C}_G$ is seen to be a profinite (compact) group, called the congruence subgroup kernel.

The foregoing facts are well known ([BMS]).

2.3. Isotropic Algebraic Groups over $\mathbb{Q}$. In what follows, $G \subset SL_n$ is a $\mathbb{Q}$-simple linear algebraic group defined over $\mathbb{Q}$. It is said to be $\mathbb{Q}$-isotropic if there exists a torus $\mathbb{Q}$-isomorphic to the multiplicative group $\mathbb{G}_m$ embedded in $G$; let $S$ in $G$ be a maximal $\mathbb{Q}$-split torus and $\Phi = \Phi(g, S)$ be the roots (characters of $S$ written additively) of $S$ occurring in the Lie algebra $\mathfrak{g}$ of $G$ under the adjoint action of $S$. Denote the root space of $\alpha \in \Phi$ by $\mathfrak{g}_\alpha$, the subspace of $\mathfrak{g}$ on which the torus $S$ acts by the character $\alpha \in \Phi$. Fix a positive system of roots $\Phi^+ \subset \Phi$; then $\Phi = \Phi^+ \cup (-\Phi^+)$. We write $\alpha > 0$ if $\alpha \in \Phi^+$.

Denote by $P_0$ the connected subgroup of $G$ whose Lie algebra is the direct sum $\mathfrak{g}^S \oplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}^S$ denotes the subspace of vectors in $\mathfrak{g}$ fixed by the split torus $S$ (the Lie algebra of the centraliser of $S$ in $G$). Then $P_0$ is a minimal parabolic $\mathbb{Q}$-subgroup of $G$. Let $P \subset P_0$ be a maximal parabolic $\mathbb{Q}$-subgroup of $G$ and $U^+$ its unipotent radical with Lie algebra $\mathfrak{u}^+$. Then $\mathfrak{u}^+ \subset \oplus_{\alpha>0} \mathfrak{g}_\alpha$ and is a sum $\mathfrak{u}^+ = \oplus_{\alpha \in X} \mathfrak{g}_\alpha$ of root spaces for some subset $X \subset \Phi^+$ of positive roots. There is a decomposition (the Levi decomposition) of $P$ as a product $P = LU^+$, where $L$ is a connected subgroup of $G$ containing $S$, whose Lie algebra is the direct sum of the root spaces $\mathfrak{g}_{\pm \alpha}$ with $\alpha \notin X$ and $\mathfrak{g}^S$.

Let $\mathfrak{u}^- = \oplus_{\alpha \in X} \mathfrak{g}_{-\alpha}$ and $U^-$ the connected (in fact unipotent) subgroup of $G$ with Lie algebra $U^-$. This is called the opposite of $U$; the group $P^- = U^- L$ is a maximal parabolic $\mathbb{Q}$-subgroup called the opposite of $P$. The multiplication map $U^- \times P \to G$ given by $(v, p) \mapsto vp$ identifies the product space $U^- \times P$ as a Zariski dense open set $U = U^- P$ in $G$ defined over $\mathbb{Q}$. If $H \subset SL_n$ is a $\mathbb{Q}$-subgroup,
and \( \mathbb{Z}_p \) is the ring of \( p \)-adic integers, we write \( H(k\mathbb{Z}_p) \) for the subgroup of elements of \( H \cap SL_n(\mathbb{Z}_p) \) viewed as \( n \times n \)-matrices which are congruent to the identity matrix modulo \( k \).

**Lemma 7.** Let \( k \geq 1 \) be an integer and for a prime \( p \), consider the set \( U(k\mathbb{Z}_p) = U^-(k\mathbb{Z}_p)P(k\mathbb{Z}_p) \) where \( \mathbb{Z}_p \) is the ring of \( p \)-adic integers. There exists a compact open subgroup \( K_p(k) \) of \( G(\mathbb{Z}_p) \) contained in \( U(k\mathbb{Z}_p) \).

**Proof.** The set \( U(k\mathbb{Z}_p) \) is an open subset of \( G(\mathbb{Z}_p) \) containing 1. A fundamental system of neighbourhoods of identity in \( G(\mathbb{Z}_p) \) is given by open subgroups and hence the lemma follows. \( \Box \)

2.4. **The Groups** \( M, V^+ \) and \( V^- \). The group \( L(\mathbb{Z}) \) of integer points of the Levi subgroup \( L \) of subsection 2.3 is not Zariski dense in \( L \) (since the \( \mathbb{Q} \)-split central torus of \( L \) has only a finite number of integer points). Denote by \( M \) the connected component of the Zariski closure of \( L(\mathbb{Z}) \). The group \( M \) does not change if we replace \( L(\mathbb{Z}) \) by a finite index subgroup. Since \( L(\mathbb{Q}) \) commensurates \( M(\mathbb{Z}) \), it follows that its Zariski closure \( L \) normalises \( M \).

**Lemma 8.** The dimension of \( M \) is positive if and only if \( \mathbb{R} - \text{rank}(G) \geq 2 \).

**Proof.** By definition, \( M \) is the connected component of identity of the Zariski closure of \( L(\mathbb{Z}) \); hence its dimension is zero if and only if \( L(\mathbb{Z}) \) is finite. Since \( L \) is the Levi subgroup of a maximal parabolic \( \mathbb{Q} \)-subgroup \( P \) of \( G \), we may write \( L = S_1L_1L_2 \) as a product where \( S_1 \) is a one dimensional \( \mathbb{Q} \)-split torus, \( L_1 \) is the product of \( \mathbb{Q} \)-isotropic simple factors of \( L \), and \( L_2 \) is a \( \mathbb{Q} \)-anisotropic group. (To see this, we write the semisimple part \( L^{ss} \) of \( L \) as a product \( L_1L_3 \) where \( L_1 \) is a product of \( \mathbb{Q} \)-isotropic simple groups \( L_1 \) and \( L_3 \) is a product of \( \mathbb{Q} \)-anisotropic simple groups; then \( L = S_1T_1L^{ss} \) where \( S_1 \) is \( \mathbb{Q} \) split torus in the centre of \( L \), and \( T_1 \) is a \( \mathbb{Q} \)-anisotropic part of the centre of \( L \). We may take \( L_2 = L_3T_1 \).

If \( L(\mathbb{Z}) \) is finite, then \( L_2(\mathbb{Z}) \) is finite and since \( L_2 \) is \( \mathbb{Q} \)-anisotropic, by the Godement criterion, \( L_2(\mathbb{R})/L_2(\mathbb{Z}) \) is compact and hence \( L_2(\mathbb{R}) \) is compact and has real rank zero. Since \( L_1(\mathbb{Z}) \) is also finite but \( L_1 \) is a product of \( \mathbb{Q} \) simple groups, it follows that \( L_1 \) is trivial and hence \( L = S_1L_2 \) where \( L_2 \) has real rank 0. Consequently the real rank of \( L \) is the dimension of \( S_1 \) which is one and hence \( \mathbb{R} - \text{rank}(G) = \mathbb{R} - \text{rank}(L) = 1 \).

Conversely, if the real rank of \( G \) is one, then the group \( L(\mathbb{R}) = S_1(\mathbb{R})L_1(\mathbb{R})L_2(\mathbb{R}) \) has real rank one and hence \( L_1 \) is trivial and \( L_2 \)
is anisotropic over $\mathbb{R}$; hence $L_2(\mathbb{R})$ is compact. Therefore, $L(\mathbb{Z}) \simeq S_1(\mathbb{Z})L_1(\mathbb{Z}) = \{\pm 1\}L_2(\mathbb{Z})$ is finite and hence $M$ has dimension zero. □

Denote by $V^\pm$ the group $[M, U^\pm]$ generated by the commutators $mum^{-1}u^{-1}$ with $m \in M$ and $u \in U^\pm$. Since $M$ is normal in $L$ (and $U^\pm$ are normalised by $L$, it follows that $V^\pm$ is normalised by $L$. It is clear that $V^\pm$ are unipotent subgroups of $U^\pm$.

Lemma 9. Suppose $G$ is $\mathbb{Q}$-simple and $\mathbb{Q}$-isotropic.

1. The group $U^\pm$ normalises $V^\pm$.

2. If $\mathbb{R} - \text{rank}(G) \geq 2$ then $G$ is generated by the groups $V^+, V^-$ and $M$.

Proof. The action of the reductive group $M$ on the Lie algebra $u^\pm$ is completely reducible. Consequently, the lie algebra $u$ splits into the space $(u^\pm)^M$ of $M$ invariants and the space of non-invariants i.e. the span of $mXm^{-1} - X$ with $X \in u^\pm$ and $m \in M$. Since the non-invariants all lie in the Lie algebra $\text{Lie}(V^\pm)$, it follows that $u^\pm = (u^\pm)^M + (\text{Lie} V)^\pm$ (it is possible that the Lie algebra generated by the non-invariants picks up invariant vectors; hence the sum may not be direct). If $X \in (u^\pm)^M$, $m \in M$ and $Y \in u^\pm$, then we have

$$[X, m(Y) - Y] = [m(X), m(Y)] - [X, Y] = m([X, Y]) - [X, Y] \in (\text{Lie} V)^\pm.$$

Therefore, $U^\pm$ normalises $(\text{Lie} V)^\pm$ proving the first part.

Let $G'$ be the group generated by $V^\pm$ and $M$; since all these groups are connected, so is $G'$; let $g'$ be its Lie algebra. We will show that $g$ normalises $g'$; the $\mathbb{Q}$-simplicity of $G$ then implies that $g' = g$ and hence that $G' = G$. Since the group $L$ normalises $U^\pm$ and also $M$, clearly $L$ normalises $g'$ and hence its Lie algebra $l$ normalises $g'$. We therefore need to check that $u^\pm$ normalises $g'$. We first note that since $M$ is (connected and) normal in $M$, we have $m(Z) - Z \in \text{Lie}(M) \subset g'$ if $Z \in l$. Since $[M, U^\pm] = V^\pm$, it follows that if $Z \in u^\pm$ then $m(Z) - Z \in \text{Lie} V^\pm \subset g'$. Since the whole Lie algebra $g$ is spanned by $u^\pm$ and $l$, we have

$$m(Z) - Z \in g' \quad \forall \quad Z \in g.$$

We have proved in the proof of the first part of the lemma, that $u^\pm = (u^\pm)^M + \text{Lie}(V^\pm)$ The latter spaces $\text{Lie}(V^\pm)$ are already contained in $g'$. Therefore, in order to verify that the sub-algebras $u^\pm$
normalise $\mathfrak{g}'$, it is enough to check that the $M$-invariants in $u^\pm$ normalise $\mathfrak{g}'$.

Suppose $X \in (u^+)^M$ and $Y \in u^\pm$. Fix $m \in M$. We compute the bracket

$$[X, m(Y) - Y] = [X, m(Y)] - [X, Y] = [m(X), m(Y)] - [X, Y],$$

where the last equality follows because $X$ is invariant under $m \in M$. Hence $[X, m(Y) - Y] = m(Z) - Z$ with $Z = [X, Y]$. By equation (11), the bracket $[X, m(Y) - Y] = m(Z) - Z \in \mathfrak{g}'$. We have thus proved that $\langle [u^+, \text{Lie}(V^\pm)] \rangle \subseteq \mathfrak{g}'$. If $Z \in \text{Lie}(M)$, then $\langle [u^+, Z] \rangle = 0 \subseteq \mathfrak{g}'$. Since $\mathfrak{g}'$ is generated by $\text{Lie}(V^\pm)$ and $\text{Lie}(M)$ and each of these spaces, upon taking brackets with elements of $(u^+)^M$ lie in $\mathfrak{g}'$, it follows that $\langle [u^+, \mathfrak{g}'] \rangle \subseteq \mathfrak{g}'$: the $M$-invariants in $u^+$ normalise $\mathfrak{g}'$. Similarly the $M$-invariants in $u^-$ also normalise $\mathfrak{g}'$. By the last remark of the preceding paragraph, $u^\pm$ normalises $\mathfrak{g}'$.

On the other hand I is contained in the normaliser of $\mathfrak{g}'$. Therefore all of $\mathfrak{g}$ normalises $\mathfrak{g}'$ and the lemma follows. \hfill $\Box$

If $H \subset SL_n$ is an algebraic $\mathbb{Q}$-subgroup, we write $H(k\mathbb{Z})$ for the intersection $H \cap SL_n(k\mathbb{Z})$, where $SL_n(k\mathbb{Z})$ is the group of $n \times n$ matrices in $SL_n(\mathbb{Z})$ congruent to the identity matrix modulo $k$. Denote by $F(k)$ the group generated by $V^\pm(k\mathbb{Z})$ and $M(k\mathbb{Z})$. We note a corollary of lemma 9.

**Corollary 1.** If $\mathbb{R} - \text{rank}(G) \geq 2$, then $F(k)$ is Zariski dense in $G$.

Proof. Since $V^\pm$ are unipotent $\mathbb{Q}$-groups, it is clear that $V^\pm(k\mathbb{Z})$ are Zariski dense in $V^\pm$. Moreover, $M(k\mathbb{Z}) \simeq L(k\mathbb{Z})$ is Zariski dense in $M$. Therefore, the Zariski closure of $F(k)$ contains $V^\pm$ and $M$. By Lemma 9, the Zariski closure of $F(k)$ is equal to $G$. \hfill $\Box$

2.5. **Strong Approximation.** We recall some well known results on strong approximation. Suppose $H \subset SL_n$ be a simply connected semi-simple $\mathbb{Q}$-simple algebraic group with $\mathbb{R} - \text{rank}(G) \geq 1$. We work with the fixed embedding $H \subset SL_n$. Set $H(\mathbb{Z}) = H \cap SL_n(\mathbb{Z})$. Then strong approximation says that $H(\mathbb{Z})$ is dense in the group $H(\hat{\mathbb{Z}})$ where $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$ where $p$ runs through all primes. Let $a, b$ be coprime integers and $H(a\mathbb{Z}) = H(\mathbb{Z}) \cap SL_n(a\mathbb{Z})$ where $SL_n(a\mathbb{Z})$ are integral matrices in $SL_n(\mathbb{Z})$ congruent to the identity matrix modulo the integer $a$. Then, as before, $H(a\mathbb{Z})$ is called the principal congruence subgroup of level $a$. 
Lemma 10. If $a, b$ are co-prime integers, and $H \subset SL_n$ is a simply connected $\mathbb{Q}$-simple with $\mathbb{R} - \text{rank}(H) \geq 1$, then $H(a\mathbb{Z})$ and $H(b\mathbb{Z})$ generate $H(\mathbb{Z})$.

The same conclusion holds if $H$ is semisimple, simply connected $\mathbb{Q}$ group which is product of $\mathbb{Q}$-simple (simply connected) groups $H_i$ with $\mathbb{R} - \text{rank}(H_i) \geq 1$ for each $i$.

Proof. This lemma and the proof are well known consequences of strong approximation. For the sake of completeness of the exposition, we recall the proof.

The definitions imply that $H(a\mathbb{Z}) = H(\mathbb{Z}) \cap H(\hat{a}\mathbb{Z})$ is dense in $H(\hat{a}\mathbb{Z})$. Thus the group $\Gamma^*$ generated by the two principal congruence groups $H(a\mathbb{Z}), H(b\mathbb{Z})$ is a also congruence group dense in the group $H^*$ generated by $H(\hat{a}\mathbb{Z}) = \prod_p H(a\mathbb{Z}_p)$ and $H(\hat{b}\mathbb{Z}) = \prod_p H(b\mathbb{Z}_p)$; since $a, b$ are coprime, at each prime $p$, the group generated by $H(a\mathbb{Z}_p)$ and $H(b\mathbb{Z}_p)$ is $H(\mathbb{Z}_p)$ and hence $H^*$ is all of $H(\hat{\mathbb{Z}})$. If two congruence subgroups of $H(\mathbb{Z})$ have the same closure in the congruence completion $H(\hat{\mathbb{Z}})$, then they are the same; hence $\Gamma^* = H(\mathbb{Z})$ and first part of the lemma follows.

The second part readily follows from the first part applied to each $H_i$. □

Lemma 11. The intersection $\Gamma_k = G(\mathbb{Q}) \cap \overline{F(k)}$ where $\overline{F(k)}$ is the closure of the group $F(k)$ in the congruence completion $\overline{G}$ of $G(\mathbb{Q})$ is an arithmetic group (called the congruence closure of $F(k)$).

Proof. A theorem of Nori and Weisfeiler ([Nori], [W]) says, in particular, that if $\Gamma \subset G(\mathbb{Z})$ is a Zariski dense subgroup (and $G$ is $\mathbb{Q}$-simple, $\mathbb{Q}$-isotropic), then the closure of $\Gamma$ in the congruence completion (in this case $G(\mathbb{A}_f)$ is open). It is not difficult to extend this to the case when $G$ is not necessarily simply connected (but the congruence completion $\overline{G}$ of $G$ may not be all of $G(\mathbb{A}_f)$). Thus the intersection of $G(\mathbb{Q})$ with the closure of $\Gamma$ is a congruence (arithmetic) subgroup of $G(\mathbb{Q})$; it is the smallest congruence subgroup of $G(\mathbb{Z})$ containing $\Gamma$ and is called the congruence closure of $\Gamma$.

Applying this to the Zariski dense subgroup $F(k)$ (Corollary [I]), we see that the congruence closure $\Gamma_k$ of $F(k)$ has finite index in $G(\mathbb{Z})$ (in fact, in this case, one can prove directly by a somewhat lengthy argument that the closure of $F(k)$ in $\overline{G}$ is open, without using Nori-Weisfeiler).
Remark. Consider the group $P^\pm_F = MV^\pm$ where $V^\pm = [M, U^\pm]$. We have seen (Lemma 9) that $U^\pm$ normalises $V^\pm$; it then follows that $U^\pm$ normalises $MV^\pm$ as well, since

$$u(mv) = [u, m]u(v) = m^{[m^{-1}(u), m]}u(v) \in MV.$$  

The groups $M$ and $V^\pm$ are normalised also by $L$; hence $P^\pm = LU^\pm$ normalises $P^\pm_F$. Since $P^\pm_F$ is the semi-direct product of the $\mathbb{Q}$ groups $M$ and $V^\pm$, it follows from definitions that for varying integers $k$, $M(k)V^\pm(k)$ is a fundamental system of congruence subgroups of $P^\pm_F(\mathbb{Q})$; in particular given $p \in P(\mathbb{Q})$ and an integer $k \geq 1$, there exists an integer $l$ such that $p(M(k)V^\pm(k))p^{-1} \supset M(l)V^\pm(l)$.

**Lemma 12.** Given a Zariski dense subset $D \subset U^-(\mathbb{Q})$ and an integer $k \geq 1$, there exists a finite set $F$ in $D$ and an integer $l \geq 1$ such that the group $M(l)V^-(l)$ is contained in $B$ where $B$ is the group generated by the conjugates $v(M(k)) := v(M(k))v^{-1}$ as $v$ runs through elements of the finite set $F$.

**Proof.** Fix an element $v^* \in D$. Consider the algebraic group $V'$ generated by the elements of the form

$$\phi(u) = (v^*)^{-1}v \in [m^{-1}, u]) = [v^*m^{-1}(v^*)^{-1}]v^*umu^{-1}(v^*)^{-1} = (m^{-1})^u(m)$$

with $u = (v^*)^{-1}v$ varying through the dense set $D' = (v^*)^{-1}D$ as $v$ varies in $D$ and $m$ varies in $M$. Since $D'$ is Zariski dense in $U^-$, it follows that this group $V'$ is the group $v^*([M, U^-] = v^* (V^-) = V^-$ since $U^-$ normalises $V^-$.

For reasons of dimension, there exists a finite set of these elements $v$ such that the elements $\phi(u)$ generate a Zariski dense subgroup of the unipotent group $U^-$ as $m$ varies in $M(l)$ and $v$ varies in $F$. Since these finite set of elements $u$ are all rational, by choosing the congruence level $l'$ suitably, we may assume that $\phi(u)$ are all elements in $U^-(k)$ for all $m \in M(l')$ and all $v \in F$. But a Zariski dense subgroup of integral elements in a unipotent group (namely $V^-$) contains $V^-(l'')$ for some congruence level $l''$. Moreover, since $\phi(u) = v^*(m)^u(m^{-1})$, the group $v^* (M(k))$ together with $V^-(l'')$ generates a congruence subgroup containing $M(l)V^-(l)$ for some $l$. □

**Proposition 13.** Assume $G$ is a $\mathbb{Q}$-simple $\mathbb{Q}$ isotropic algebraic group with $\mathbb{R} - \text{rank}(G) \geq 2$. Given $x \in G(\mathbb{Q})$ and $k \geq 1$, there exists an integer $l = l(k, x)$ such that

$$x(F(k)) \supset F(l).$$
Proof. For every \( \theta \in F(k) \) we have \( x(F(k)) = x^\theta(F(k)) \). Since \( F(k) \) is Zariski dense in \( G \), we may assume, by replacing \( x \) by \( x^\theta \) if necessary, that \( x \in U^P = U \). Write \( x = v_p \) accordingly with \( v \in U^-(Q) \) and \( p \in P(Q) \) with \( v = v(x) \) depending algebraically on \( x \in U \). Then,

\[
x(F(k)) \supset x(M(k)V^+(k)) \supset x(M(k)V^+(k)) \cap M(k)V^-(k) = \supset^v (M(l_1)V^+(l_1)) \cap M(k)V^-(k) \supset^v (M(l_1)V^+(l_1) \cap M(l_1)V^-(l_1)),
\]

for some integer \( l_1 \) (since \( v \in P^-(Q) \) normalises \( MV^- \)). Since \( MV \cap MV^- = M \) we get:

\[
x(F(k)) \supset^v (M(l_2))
\]

for some integer \( l_2 \) with \( v = v(x) \). Replacing \( x \) by any \( x\gamma \) with \( \gamma \in F(k) \), we see that \( x(F(k)) \supset^v(x\gamma) (M(l_1)) \) for some integer \( l_1 \). By Lemma 12 for some finite set \( F \) of these \( \gamma \)'s, the group generated by the conjugates

\[
v(M(l_2)), v(x\gamma) (M(l_1)) \quad (\gamma \in F),
\]

contains a congruence subgroup of the form \( M(l) \cap V^-(-l) \) for some integer \( l \) and therefore, \( x(F(k)) \supset M(l) \cap V^-(-l) \) for some \( l \); similarly, \( x(F(k)) \supset M(l) \cap V^+(l) \) for some \( l \) and hence \( x(F(k)) \) contains the group \( F(l) \) generated by \( V^\pm(l) \) and \( M(l) \).

\[\square\]

2.6. The Group \( C \). By (2.1) and by Proposition 13 we get the following. If \( G \) is a \( \mathbb{Q} \)-isotropic \( \mathbb{Q} \)-simple algebraic group with \( \mathbb{R} - \text{rank}(G) \geq 2 \), denote by \( \mathcal{T} \) the topology on \( G(Q) \) generated by the cosets \( xF(k) : x \in G(Q), k \geq 1 \). Then \( (G(Q), \mathcal{T}) \) gets the structure of a topological group. By (2.2) the topological group \( (G, \mathcal{T}) \) admits a two sided completion \( (\hat{G}, \hat{\mathcal{T}}) \). If, as before, \( \overline{G} \subset G(A_f) \) denotes the congruence completion of \( G(Q) \), we get a surjective homomorphism \( \hat{G} \to \overline{\mathcal{T}} \). This proves Proposition 5.

Since the group \( F(k) \) lies in \( G(kZ) \) the principal congruence subgroup of \( G(Z) = G \cap SL_n(Z) \) of level \( k \), and since the \( G(kZ) : k \geq 1 \) form a fundamental system of neighbourhoods of identity, it follows that any congruence subgroup of \( G(Q) \) contains \( G(kZ) \) for some \( k \) and hence contains \( F(k) \). Since the group \( \Gamma_k \) is the smallest congruence subgroup of \( G(Q) \) containing \( F(k) \), it follows that the \( \Gamma_k \) form a fundamental system of neighbourhoods of identity in \( G(Q) \) for the congruence topology. Since \( F(k) \) is dense in \( \Gamma_k \), it follows that if \( l \) is a multiple of \( k \), then the quotient set \( F(k)/F(l) \) maps onto the finite congruence quotient set \( \Gamma_k/\Gamma_l \). Taking inverse limits, it follows that \( \hat{F}(k) \) maps onto \( Cl(\Gamma_k) = Cl(F(k)) \) and the latter is an open subgroup
of $\overline{G}$. Thus the map $\widehat{G} \to \overline{G}$ is an *open map*, with kernel $C$, say.

The kernel $C$ is the inverse image of the completions $\widehat{\Gamma}_k$ as $k$ varies. Moreover, the inverse limit of $\widehat{F}(k)$ is trivial. Hence we get

$$(2) \quad C = \varprojlim \widehat{F}(k) \backslash \widehat{\Gamma}_k / \widehat{F}(k) = \varprojlim F(k) \backslash \Gamma_k / F(k).$$

Note that the group $M(\mathbb{Z})$ normalises $V^±(k\mathbb{Z})$ and $M(k\mathbb{Z})$ and hence normalises $F(k), \Gamma_k$. Since $C$ is normal in $\widehat{G}$, it is normalised by $G(\mathbb{Q}) \supset M(\mathbb{Z})$; the above expression \textsuperscript{2} of $C$ as the inverse limit of the double cosets $F(k) \backslash \Gamma_k / F(k)$ respects this $M(\mathbb{Z})$ action.
3. When $M$ is not abelian

We now prove the centrality of the kernel $C$ (Theorem 4 in the case when $M$ is not abelian. Since $M$ is connected reductive and is (the connected component of identity of) the Zariski closure of $L(\mathbb{Z})$, it follows that $M(\mathbb{Z})$ is Zariski dense in $M$; hence the commutator subgroup $S = [M, M]$ is a (non-trivial) semi-simple $\mathbb{Q}$-group with $S(\mathbb{Z})$ being Zariski dense in $S$. Let $S^*$ denote the simply connected cover of $S$. Let $S^* = \prod S_i$ be a product of $\mathbb{Q}$-simple groups $S_i$. Since $S^*(\mathbb{Z})$ is Zariski dense in $S^*$, we have that $S_i(\mathbb{Z})$ is Zariski dense in each $S_i$, and hence each $S_i(\mathbb{R})$ is non-compact; i.e. $\mathbb{R} - \text{rank} (S_i) \geq 1$ for each $i$.

Consider an element $x$ in the double coset $C_k = F(k)\backslash \Gamma_k/F(k)$. Since $F(k)$ is Zariski dense in $G$, we may choose a representative $x \in \Gamma_k$ with $x = vp$ with $v \in U^-(\mathbb{Q})$ and $p \in P(\mathbb{Q})$. Moreover, since the closure of $F(k)$ in the congruence topology on $G(\mathbb{Q})$ is open, we may choose $x$ so that for all primes $p$ dividing the level $k$, $x$ lies in the open neighbourhood $U^-(k\mathbb{Z}_p)P(k\mathbb{Z}_p)$ of identity in $G(\mathbb{Z}_p)$. Thus the rational matrices $v, p$ have a common denominator, say $a$; but the elements $v, p$ are integral at all primes $p$ dividing $k$; in other words, $a$ is coprime to $k$.

Fix an element $m \in M(a^N\mathbb{Z})$ for some large $N$. Since the group $U^-$ is normalised by $M$ and $v \in U^-(\mathbb{Q})$ has denominator dividing $a$, the commutator $[m, v] = mvm^{-1}v^{-1}$ is integral and is divisible by $k$ at all primes $p$ dividing $k$. Moreover, since $(m, v) \mapsto mvm^{-1}v^{-1}$ is a polynomial in the entries of $v$ and $m$ with integer coefficients, for $N$ large enough, $mvm^{-1}v^{-1}$ is integral at all primes dividing $a$; in other words, $[m, v] \in V^-(k\mathbb{Z})$, where, we recall, $V^\pm = [M, U^\pm]$. Similarly, the commutator $[p^{-1}, m] \in (MV)(k\mathbb{Z})$.

We now consider the conjugate $mxm^{-1}$. We have written $x = vp$ with $v \in U^-(\mathbb{Q})$ and $p \in P(\mathbb{Q})$. Hence

$$mxm^{-1} = mvm^{-1}mpm^{-1} = [m, v]vp[p^{-1}, m] = [m, v]x[p^{-1}, m].$$

Thus, if $m \in M(a^N\mathbb{Z})$, then from the discussion in the preceding paragraph, as an element of the double coset $C_k = F(k)\backslash \Gamma_k/F(k)$, $mxm^{-1} \in F(k)xF(k)$, i.e. $mxm^{-1} = x$ as double cosets. In other words, the group $M(a^N\mathbb{Z})$ acts trivially, under conjugation, on the element $x \in C_k$.

The double coset $x \in C_k$ may be replaced (since $F(k)$ has open closure in the congruence completion of $G(\mathbb{Q})$), by an element $y \in \Gamma_k$
such that for each prime $p$ dividing $a$, the element $y \in U^-(\mathbb{Z}_p)P(\mathbb{Z}_p)$. In other words, if $y = v'p'$ is written as a product of $v' \in U^-(\mathbb{Q}), p' \in P(\mathbb{Q})$, then $v' \in U^-(\mathbb{Z}_p), p' \in P(\mathbb{Z}_p)$. In other words, the elements $v', p'$ are integral at all primes dividing $a$. Therefore, the common denominator (say $b$) of the rational matrices $v', p'$ is co-prime to $a$. From the conclusion of the preceding paragraph, the group $M(b^N\mathbb{Z})$ acts trivially on the coset representative $y$ of $x$.

Since $S^*(\mathbb{Q})$ acts on $C_k$ via its image $S(\mathbb{Q})$ in $M(\mathbb{Q})$, we see from the last two paragraphs that, both $S^*(a^N\mathbb{Z})$ and $S^*(b^N\mathbb{Z})$ act trivially on the double coset $x \in C_k$, hence so does the group generated by these. By Lemma 10, the group generated by these subgroups is $S^*(\mathbb{Z})$, and hence $S^*(\mathbb{Z})$ acts trivially on each coset $x$ in $C_k$. Hence $S^*(\mathbb{Z})$ acts trivially on $C_K$ and by taking inverse limits, it follows that $S^*(\mathbb{Z})$ acts trivially on the kernel $C$. But all of $G(\mathbb{Q})$ acts on the kernel $C$, and the infinite group $S^*(\mathbb{Z})$ acts trivially. In view of the simplicity of $G(\mathbb{Q})$ modulo its centre, it follows that $G(\mathbb{Q})$, and hence $\hat{G}$, act trivially on $C$: $C$ is central in $\hat{G}$ and Theorem 4 is proved.
4. When $M$ is abelian

When $M$ is abelian, the proof of Theorem 4 is more involved. We will use some results from [V2]. Since $M$ is abelian and $P$ is a maximal parabolic subgroup, this means that the semisimple part of $L$ has $\mathbb{Q}$-rank zero, and hence $\mathbb{Q} - \text{rank}(G) = 1$. We state them now.

[1] For each prime $p$, denote by $G_p \subset \hat{G}$ the subgroup generated by $U^\pm(\mathbb{Q}_p)$. The group $C$ is central if and only if, for every pair $p, q$ of distinct primes, the groups $G_p, G_q$ commute. In [V2] this is proved only in the case $C$ is a (compact) profinite group (since the main application was to the centrality of the congruence subgroup kernel), but the proof works in general and does not use the compactness of $C$.

[2] There exists a morphism $\phi : H = SL_2 \to G$ of $\mathbb{Q}$-algebraic groups such that $\phi(U^\pm_H) \subset U^\pm$. Here $U^\pm_H$ is the group of upper (resp lower) triangular unipotent matrices in $SL_2$. Further, the conjugates $\{s \phi(U^+_H), s \in L(\mathbb{Q})\}$ generate the group $U^+$.

[3] There exists an infinite subgroup $\Delta \subset M(\mathbb{Z})$ such that for every triple $a, b, k$ of mutually coprime integers, the group generated by the collection $\{M(azk + b) : z \in \mathbb{Z}\}$ of subgroups contains this fixed group $\Delta$, and such that the commutator $[\Delta, \phi(SL_2)]$ contains $\phi(SL_2)$.

Assume the above facts, and write $H = SL_2$. For each integer $k$, write $F_H(k), \Gamma_{H,k}$ for the intersections $F(k) \cap H$, $\Gamma_k \cap H$. Denote by $C_H(k)$ the double coset $F_H(k) \backslash \Gamma_{H,k}/F_H(k)$ and fix an element $x \in C_H(k)$. Then $F_H(k)$ has the same closure in the congruence completion of $H(\mathbb{Q})$ as $\Gamma_{H,k}$. If $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $c \neq 0$ (which we may assume after replacing $x$ by a left translation by a suitable element of $F(k)$) we may write $x = vp$ with $v = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}$ and hence the common denominator of $v, p$ is the integer $a$. For a suitable power $N$ which depends only on the embedding $\phi : H \to G$, we have that for $m \in M(a^N)$, the commutator $[m, \phi(v)] = m\phi(v)m^{-1}\phi(v)^{-1} \in V^-(k\mathbb{Z}) \subset F(k)$, and the commutator $[(\phi(p))^{-1}, m] \in (MV)(k\mathbb{Z}) \subset F(k)$. Consider the conjugate $m\phi(x)m^{-1}$. Writing $x = vp$ we get

$$m\phi(x)m^{-1} = m\phi(v)m^{-1}m\phi(p)m^{-1} =$$

$$= [m, \phi(v)]x[\phi(p)^{-1}, m] \in F(k)\phi(x)F(k).$$
That is, \(m\phi(x)m^{-1} = \phi(x)\) as double cosets. Thus the group \(M(a)\) fixes the image of the element \(x \in C_H(k)\) under \(\phi\) in the double coset \(C_k = F(k) \backslash \Gamma_k/F(k)\) where \(a\) is the top left entry of the matrix \(x\).

We may replace \(x\) by an element \(y = x\gamma\) with \(\gamma \in F(k)\). We choose \(\gamma = \begin{pmatrix} 1 & 0 \\ kz & 1 \end{pmatrix}\) for some integer \(z\). Then \(y = \begin{pmatrix} a + bkz & b \\ c + dkz & d \end{pmatrix}\) has top left entry \(a + bkz\). By the conclusion of the preceding paragraph, the group \(M(a + bkz)\) also fixes the element \(y\) viewed as a double coset in \(C_k\). But by construction \(x = y\) as double cosets, and hence both the groups \(M(a)\) and \(M(a + bkz)\) fix \(x\) for all \(z \in \mathbb{Z}\). Thus the group \(M_{a,b,k}\) generated by the collection \(\{M(a + bkz)\mathbb{Z}: z \in \mathbb{Z}\}\) fixes the double coset \(x\). By [3] of the listed facts, there is a fixed infinite subgroup \(\Delta \subset M_{a,b,k}\) for every \(a, b, k\). Hence \(\Delta\) fixes \(x\) for every \(x \in C_H(k)\) and hence \(C_H(k)\) is fixed by \(\Delta\) for every \(k\). By taking inverse limits, we see that the image of \(C_H\) under the map \(\phi\) is fixed by all of \(\Delta\).

However, the image \(\phi(C_H)\) of \(C_H\) is invariant under the action of \(SL_2 = H\). Again by the second part of [3], \(\phi H \subset [\Delta, \phi(H)]\) acts trivially on \(\phi(C_H)\) and hence \(\phi(C_H)\) is a central extension of \(\phi(SL_2(\mathbb{A}_f))\). Therefore, by fact [1], for each pair of distinct primes \(p, q\) the groups \(\phi(U^+_H(\mathbb{Q}_p))\) and \(\phi(U^-(\mathbb{Q}_q))\) commute.

Let \(s \in L(\mathbb{Q})\) be arbitrary, and write \(s = (s_p) \in L(\mathbb{A}_f)\). Being the linear action, the adjoint action of \(L(\mathbb{Q})\) on \(U^\pm(\mathbb{A}_f)\) in the topological group \(\widehat{G}\) factors through the finite adelic group \(L(\mathbb{A}_f)\). Hence, for each \(p, q, u \in U^+_H(\mathbb{Q}_p)\) and \(v \in U^-_H(\mathbb{Q}_q)\), we have
\[
s\phi(u)s^{-1} = s_p\phi(u)s^{-1}_p, \quad s\phi(v)s^{-1} = s_q\phi(v)s^{-1}_q.
\]

Furthermore, by weak approximation ([Gille]), \(L(\mathbb{Q})\) is dense in \(L(\mathbb{Q}_p)\times L(\mathbb{Q}_q)\). Since \(\phi(u)\) and \(\phi(v)\) commute by the conclusion of the preceding paragraph, we see (by taking limits of elements in \(L(\mathbb{Q})\subset L(\mathbb{Q}_p)\times L(\mathbb{Q}_q)\)) that for every \(s_p \in L(\mathbb{Q}_p)\) and every \(s_q \in L(\mathbb{Q}_q)\), the elements \(s_p\phi(u)s^{-1}_p\) and \(s_q\phi(v)s^{-1}_q\) commute for all \(u, v\). But by fact [2], \(\phi\) may be so chosen that the collection \(s_p\phi(u)s^{-1}_p\) with \(s_p \in L(\mathbb{Q}_p), u \in U^+_H(\mathbb{Q}_p)\) generates all of \(U^+(\mathbb{Q}_p)\); similarly for \(U^-(\mathbb{Q}_q)\). Hence \(U^+(\mathbb{Q}_p)\) commutes with \(U^-(\mathbb{Q}_q)\) for each pair \(p, q\) of distinct primes. By fact [1], this means that \(C\) is central. Thus we have proved Theorem [4] in all cases.
Since we have already shown in the introduction that Theorem 4 implies Theorems 2 and 3, we have also proved Theorem 1 in all cases.
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