A WEAK TYPE INEQUALITY FOR NON-COMMUTATIVE MARTINGALES AND APPLICATIONS

NARCISSE RANDRIANANTOANINA

Abstract. We prove a weak-type (1,1) inequality for square functions of non-commutative martingales that are simultaneously bounded in $L^2$ and $L^1$. More precisely, the following non-commutative analogue of a classical result of Burkholder holds: there exists an absolute constant $K > 0$ such that if $\mathcal{M}$ is a semi-finite von Neumann algebra and $(\mathcal{M}_n)_{n=1}^{\infty}$ is an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ then for any given martingale $x = (x_n)_{n=1}^{\infty}$ that is bounded in $L^2(\mathcal{M}) \cap L^1(\mathcal{M})$, adapted to $(\mathcal{M}_n)_{n=1}^{\infty}$, there exist two martingale difference sequences, $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$, with $dx_n = a_n + b_n$ for every $n \geq 1$, $\left\| \left( \sum_{n=1}^{\infty} a_n^* a_n \right)^{1/2} \right\|_2 + \left\| \left( \sum_{n=1}^{\infty} b_n b_n^* \right)^{1/2} \right\|_2 \leq 2 \|x\|_2$, and $\left\| \left( \sum_{n=1}^{\infty} a_n^* a_n \right)^{1/2} \right\|_1 + \left\| \left( \sum_{n=1}^{\infty} b_n b_n^* \right)^{1/2} \right\|_1 \leq K \|x\|_1$.

As an application, we obtain the optimal orders of growth for the constants involved in the Pisier-Xu non-commutative analogue of the classical Burkholder-Gundy inequalities.

0. Introduction

Non-commutative (or quantum) probability has developed considerably in recent years. It provides many connections between several fields of mathematics such as mathematical physics, operator algebras, and classical probability theory. We refer to the book by Meyer [31] for general quantum probability, the book by Parthasarathy [35] for quantum stochastic calculus, and the book by Voiculescu et al. [44] for free probability.

In classical theory, martingale theory has played a significant role in the developments of various fields of analysis (see for instance [5, 17, 28]). In this paper, our main interest is on non-commutative martingales. As in the classical case, non-commutative martingales have connections with other area such as operator

2000 Mathematics Subject Classification. Primary: 46L53, 46L52. Secondary: 46L51, 60G42.

Key words and phrases. von Neumann algebras, non-commutative $L^p$-spaces, martingale inequalities, square functions.

Supported in part by NSF grant DMS-0096696.
algebra theory, operator space theory, and matrix valued harmonic analysis which includes among other things, operator valued Carleson measures, operator valued Hardy spaces, and operator valued Hankel operators (see for instance \[20, 33\]).

Alongside the general development of quantum probability theory, the subfield of non-commutative martingales has received considerable progress in recent years. Indeed, many of classical inequalities from the usual (commutative) martingale theory have been generalized to the non-commutative settings. Let us recall some sample contributions by several authors. For instance, pointwise convergence of non-commutative martingales was already considered by Dang-Ngok \[9\], Cuculescu \[8\], and Barnett \[1\] in the 70's and 80's. Pisier and Xu \[37\] proved the non-commutative analogue of the Burkholder-Gundy inequalities on square functions and non-commutative analogue of Stein’s inequality. It is their general functional analytic approach that led to the consideration of non-commutative analogue of several classical martingale inequalities. A non-commutative analogue of Doob’s maximal inequality was successfully formulated and proved by Junge in \[23\] and non-commutative analogues of Burkholder/Rosenthal inequalities on conditioned square functions were studied by Junge and Xu in \[25\] among many other related topics. These different results pave the way to the consideration of non-commutative martingale Hardy spaces and non-commutative martingale \textit{BMO} which are non-commutative generalizations of spaces that are central to the developments of classical harmonic analysis and interpolation theory. We note also a very recent result of Musat \[32\] on interpolation involving non-commutative \textit{BMO} and non-commutative \textit{L}^p\text{-spaces as endpoints.}

In most of the papers listed above, square functions played a very crucial role. Note however that in strong contrast with the classical case, square functions in the non-commutative case can take many different forms so it is very important to formulate the “right” square functions. Recall that if \(1 < p < 2\), and \(x = (x_n)_{n=1}^\infty\) is a non-commutative martingale (see the formal definition below), the \(\mathcal{H}^p\)-norm (\(\mathcal{H}^p\) being the Hardy space of non-commutative martingales) is given by:

\[
\|x\|_{\mathcal{H}^p} = \inf \left\{ \left\| \left( \sum_{n \geq 1} |d y_{n}|^2 \right)^{1/2} \right\|_{p} + \left\| \left( \sum_{n \geq 1} |d z_n^*|^2 \right)^{1/2} \right\|_{p} \right\} \tag{0.1}
\]

where the infimum runs over all decompositions \(x = y + z\), with \(y\) and \(z\) being martingales. That is, it depends on two different types of square functions (given by right and left moduli). The fact that one has to decompose the martingale \(x\) into two martingales was first discovered for non-commutative Khintchine inequalities (see \[29, 30\]) and this type of decomposition is often the source of the difficulties in extending classical results to non-commutative settings.

The main purpose of this paper is to study the square functions of non-commutative martingales for the case \(p = 1\) which is primarily motivated by the following classical result of Burkholder.
Theorem 0.1 ([4]). Let \((f_n)_{n=1}^{\infty}\) be a martingale on a probability space \((\Omega, \Sigma, P)\) and 
\[ S(f) = (\sum_{n=1}^{\infty} |f_n - f_{n-1}|^2)^{1/2}. \]
Then there exists an absolute constant \(M > 0\) such that for every \(\lambda > 0\),
\[ \lambda P(S(f) > \lambda) \leq M \sup_n E(|f_n|). \]

It is a natural question to consider whether Theorem 0.1 has non-commutative counterparts. We remark that Burkholder deduced the above result from the weak-type (1,1) boundedness of martingale transforms (also proved in [4]) via the classical Khintchine inequality. One can also use the classical Doob’s identity (see for instance [19, Chap. II]) to deduce Theorem 0.1 from the weak-type (1,1) boundedness of martingale transforms. We note that non-commutative martingale transforms are of weak-type (1,1) ([40]). However, unlike the classical case, a non-commutative analogue of Theorem 0.1 can not be deduced directly from the weak type (1,1) boundedness of martingale transforms via the classical techniques, as (at least at the time of this writing) there is no adequate Khintchine inequality for non-commutative weak-\(L^1\)-spaces.

In [42], a first attempt was made to generalize Theorem 0.1 to non-commutative settings. The \(p\)-norm of the square functions for the case \(1 < p < 2\) stated in (0.1) suggests that the formulation of the weak-\(L^1\) norm of square functions should require decompositions of the martingales involved. We obtained in [42, Theorem 2.1] a decomposition of any given martingale into two sequences in weak \(L^1\)-space where the corresponding weak \(L^1\)-norm of the square functions similar to the one stated in (0.1) is bounded by the \(L^1\)-norm of the corresponding martingale. The result from [42] prompted the question of whether or not such decomposition can be chosen to be martingales. For the finite case, our main result answers this positively for the case of \(L^2\)-bounded martingales (see Theorem 3.1 below). More precisely, there exists a constant \(K > 0\) such that if \(x = (x_n)_{n=1}^{\infty}\) is a non-commutative martingale, there exists two martingales \(y = (y_n)_{n=1}^{\infty}\) and \(z = (z_n)_{n=1}^{\infty}\) such that \(x = y + z\) and with the property:
\[ \left\| \left( \sum_{n \geq 1} |dy_n|^2 \right)^{1/2} \right\|_{1,\infty} + \left\| \left( \sum_{n \geq 1} |dz_n|^2 \right)^{1/2} \right\|_{1,\infty} \leq \|x\|_1. \quad (0.2)\]
Coupled with general interpolation techniques, our main result provides a solution to a problem left open in [25] (see also [45, Problem 8.2]) on optimal order of growth of the constants involved in the non-commutative Burkholder-Gundy inequalities when \(p \to 1\) (see Theorem 5.2 below).

In order to achieve the decomposition into two martingales, our method of proof (although it follows closely those taken in [42] and [40]) requires substantial adjustments. It depends heavily on a non-commutative version of the classical Doob’s maximal inequality obtained by Cuculescu [8], and weak-type (1,1) boundedness of triangular truncations relative to disjoint projections.
The paper is organized as follows: in Section 1 below, we set some basic preliminary background concerning non-commutative spaces and collect some results on triangular truncations. In Section 2, we recall the general setup of non-commutative martingale theory. Section 3 is devoted mainly to the statement of the main decomposition, the construction of the decomposition and a detailed proof of the weak-type inequality for the finite case. In Section 4, we will point out the adjustment needed to extend our main result from Section 3 to the semi-finite case. In the last section, we provide a new proof of one of the inequalities involved in the non-commutative Burkholder-Gundy inequality and deduce the optimal order of the constants involved.

1. NON-COMMUTATIVE SPACES AND PRELIMINARY RESULTS

We use standard notation in operator algebras. We refer to [26] and [43] for background on von Neumann algebra theory. In this section, we will recall some basic definitions that we will use throughout this paper. In particular, we will outline the general construction of non-commutative spaces and discuss triangular truncations with respect to sequence of disjoint projections.

Throughout, $\mathcal{M}$ is a semi-finite von Neumann algebra with a normal faithful semifinite trace $\tau$. The identity element of $\mathcal{M}$ is denoted by $1$. For $0 < p \leq \infty$, let $L^p(\mathcal{M}, \tau)$ be the associated non-commutative $L^p$-space (see for instance [10] and [34]). Note that if $p = \infty$, $L^\infty(\mathcal{M}, \tau)$ is just $\mathcal{M}$ with the usual operator norm; also recall that for $0 < p < \infty$, the (quasi)-norm on $L^p(\mathcal{M}, \tau)$ is defined by

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(\mathcal{M}, \tau),$$

where $|x| = (x^*x)^{1/2}$ is the usual modulus of $x$.

In order to ease the introduction of some of the spaces used in the sequel, we need the general scheme of symmetric spaces of measurable operators developed in [7, 11, 13, 46].

Let $H$ be a complex Hilbert space and $\mathcal{M} \subseteq B(H)$. A closed densely defined operator $a$ on $H$ is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary $u$ in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $a$ is a densely defined self-adjoint operator on $H$, and if $a = \int_{-\infty}^{\infty} sde_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{-\infty}^{\infty} \chi_B(s)de_s^a$. A closed densely defined operator $a$ on $H$ affiliated with $\mathcal{M}$ is said to be $\tau$-measurable if there exists a number $s \geq 0$ such that $\tau(\chi_{(s,\infty)}(|a|)) < \infty$.

The set of all $\tau$-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [34]. For $\varepsilon, \delta > 0$, let

$$N(\varepsilon, \delta) = \{x \in \overline{\mathcal{M}}: \text{for some projection } p \in \mathcal{M}, \|xp\| < \varepsilon \text{ and } \tau(1-p) \leq \delta\}.$$
The system \((N, \varepsilon, \delta))_{\varepsilon, \delta}\) forms a fundamental system of neighborhoods of the origin of the vector space \(\mathcal{M}\) and the translation-invariant topology induced by this system is called the \textit{measure topology}. Convergence in measure will be used in the sequel.

For \(x \in \mathcal{M}\), the generalized singular value function \(\mu(x)\) of \(x\) is defined by
\[
\mu_t(x) = \inf\{s \geq 0 : \tau(\chi_{(s,\infty)}(\|x\|)) \leq t\}, \quad \text{for } t \geq 0.
\]
The function \(t \to \mu_t(x)\) from the interval \([0, \tau(1)]\) to \([0, \infty)\) is right continuous, non-increasing and is the inverse of the distribution function \(\lambda(x)\), where \(\lambda_s(x) = \tau(\chi_{(s,\infty)}(\|x\|))\), for \(s \geq 0\). For an in depth study of \(\mu(\cdot)\) and \(\lambda(\cdot)\), we refer the reader to [18].

For the definition below, we refer the reader to [2, 28] for the theory of rearrangement invariant function spaces.

**Definition 1.1.** Let \(E\) be a rearrangement invariant (quasi-) Banach function space on the interval \([0, \tau(1)]\). We define the symmetric space \(E(\mathcal{M}, \tau)\) of measurable operators by setting:
\[
E(\mathcal{M}, \tau) = \{ x \in \mathcal{M} : \mu(x) \in E \} \quad \text{and} \quad \|x\|_{E(\mathcal{M}, \tau)} = \|\mu(x)\|_E, \text{ for } x \in E(\mathcal{M}, \tau).
\]

It is well known that \(E(\mathcal{M}, \tau)\) is a Banach space (respectively, quasi-Banach space) if \(E\) is a Banach space (respectively, quasi-Banach space). The space \(E(\mathcal{M}, \tau)\) is often referred to as the non-commutative analogue of the function space \(E\) and if \(E = L^p[0, \tau(1)]\), for \(0 < p \leq \infty\), then \(E(\mathcal{M}, \tau)\) coincides with the usual non-commutative \(L^p\)-space associated with \((\mathcal{M}, \tau)\). We refer to [7, 11, 13, 46] for more detailed discussions about these spaces. Of special interest in this paper are non-commutative weak \(L^1\)-spaces. The non-commutative weak \(L^1\)-space, denoted by \(L^{1,\infty}(\mathcal{M}, \tau)\), is defined as the linear subspace of all \(x \in \mathcal{M}\) for which the quasi-norm
\[
\|x\|_{1,\infty} := \sup_{t \geq 0} t \mu_t(x) = \sup_{\lambda > 0} \lambda \tau(\chi_{(\lambda,\infty)}(\|x\|)) \tag{1.1}
\]
is finite. Equipped with the quasi-norm \(\| \cdot \|_{1,\infty}\), \(L^{1,\infty}(\mathcal{M}, \tau)\) is a quasi-Banach space. It is easy to verify that as in the commutative space, \(\|x\|_{1,\infty} \leq \|x\|_1\) for all \(x \in L^1(\mathcal{M}, \tau)\).

For a complete, detailed, and up to date presentation of non-commutative integration and non-commutative spaces, we refer to the recent survey article [38].

The next lemma is probably well known. It will be used repeatedly in the sequel.

**Lemma 1.2.** Let \(a \) and \(b \) be operators in \(L^{1,\infty}(\mathcal{M}, \tau)\). For every \(\lambda > 0\), \(\alpha \in (0,1)\), and \(\beta \in (0,1)\),
\[
\tau(\chi_{(\lambda,\infty)}(|a + b|)) \leq \alpha^{-1} \tau(\chi_{(\beta\lambda,\infty)}(|a|)) + (1 - \alpha)^{-1} \tau(\chi_{((1-\beta)\lambda,\infty)}(|b|)).
\]

**Proof.** Using properties of generalized singular value functions \(\mu(\cdot)\) from [18], we have,
\[
\tau(\chi_{(\lambda,\infty)}(|a + b|)) = \int_0^1 \chi_{(\lambda,\infty)}(\{\mu_t(a + b)\}) \, dt.
\]
This follows from [18, Corollary 2.8] by approximating the characteristic function \( \chi_{(\lambda, \infty)}(\cdot) \) from below by sequences of continuous functions \( f \) on \([0, \infty)\) satisfying \( f(0) = 0 \). We can deduce the following estimate:

\[
\tau \left( \chi_{(\lambda, \infty)}([a + b]) \right) \leq \int_0^1 \chi_{(\lambda, \infty)}\{\mu_{at}(a) + \mu_{(1-a)t}(b)\} \, dt \\
\leq \int_0^1 \chi_{(\beta, \lambda, \infty)}\{\mu_{t}(a)\} \, dt + \int_0^1 \chi_{(1-\beta, \lambda, \infty)}\{\mu_{(1-a)t}(b)\} \, dt
\]

and by simple change of variables,

\[
\tau \left( \chi_{(\lambda, \infty)}([a + b]) \right) \leq \alpha^{-1} \int_0^1 \chi_{(\beta, \lambda, \infty)}\{\mu_{t}(a)\} \, dt + (1 - \alpha)^{-1} \int_0^1 \chi_{((1-\beta), \lambda, \infty)}\{\mu_{(1-a)t}(b)\} \, dt \\
= \alpha^{-1} \tau(\chi_{(\beta, \lambda, \infty)}([a])) + (1 - \alpha)^{-1} \tau(\chi_{((1-\beta), \lambda, \infty)}([b]))
\]

as stated in the lemma. \( \square \)

We end this section with a brief discussion on triangular truncations. This will be very crucial throughout the paper. Let \( \mathcal{P} = \{p_i\}_{i=1}^M \) be an arbitrary finite or infinite sequence of mutually orthogonal projections from \( \mathcal{M} \). We recall the triangular truncation on \( \overline{\mathcal{M}} \) (with respect to \( \mathcal{P} \)) by

\[
\mathcal{T}(\mathcal{P}) x := \sum_{j=1}^M \sum_{i \leq j} p_i x p_j, \quad x \in \overline{\mathcal{M}}.
\]

The diagonal projection \( D(\mathcal{P}) \) is defined on \( \overline{\mathcal{M}} \) by setting

\[
D(\mathcal{P}) x := \sum_{i=1}^M p_i x p_i, \quad x \in \overline{\mathcal{M}}.
\]

We also use the following operator on \( \overline{\mathcal{M}} \),

\[
\mathcal{H}(\mathcal{P}) x := -i(\mathcal{T}(\mathcal{P}) x - \mathcal{T}(\mathcal{P}) x^*), \quad x \in \overline{\mathcal{M}}.
\]

For convenience, we collect some properties of the operators introduced above that are useful for our presentation. In the following lemma, \( \overline{\mathcal{M}}_\mathcal{P} \) denotes the range of \( \mathcal{T}(\mathcal{P}) \).

**Lemma 1.3** ([15]). The operators defined above satisfy the following properties:

- (i) If \( 0 \leq x \in \overline{\mathcal{M}} \) and \( \lambda > 0 \), then \( \lambda 1 + x + i\mathcal{H}(\mathcal{P})(x) \) is invertible, with \( (\lambda 1 + x + i\mathcal{H}(\mathcal{P})(x))^{-1} \in \mathcal{M} \) and \( \| (\lambda 1 + x + i\mathcal{H}(\mathcal{P})(x))^{-1} \|_\infty \leq 1/\lambda \).
- (ii) \( D(\mathcal{P}) \mathcal{T}(\mathcal{P}) = \mathcal{T}(\mathcal{P}) D(\mathcal{P}) = D(\mathcal{P}) \).
- (iii) If \( x, y \in \overline{\mathcal{M}}_\mathcal{P} \), then \( D(\mathcal{P})(xy) = D(\mathcal{P})(x)D(\mathcal{P})(y) \).

If we assume that \( \sum_{i=1}^M p_i = 1 \), then:

- (iv) If \( x \in \overline{\mathcal{M}}_\mathcal{P} \) is invertible and \( D(\mathcal{P})(x) \) is self-adjoint, then \( x^{-1} \in \overline{\mathcal{M}}_\mathcal{P} \) and \( D(\mathcal{P})(x)^{-1} = D(\mathcal{P})(x^{-1}) \).
(v) If \( x \in \mathcal{M} \) is self-adjoint, then \( x + i\mathcal{H}(\mathcal{P})(x) = 2\mathcal{T}(\mathcal{P})(x) \).

(vi) If \( x \in L^1(\mathcal{M}, \tau) \), then \( \tau(D(\mathcal{P})(x)) = \tau(x) \).

The next result is a weak-type boundedness of “\( l^2 \)-sum” of finite family of triangular truncations.

**Proposition 1.4.** Let \( \{\mathcal{P}^{(n)}\}_{n=1}^N \) be a family of finite sequence of mutually disjoint projections with \( \mathcal{P}^{(n)} = \{p_{i,n}\}_{i=1}^M \) for each \( 1 \leq n \leq N \). If \( (x_n)_{n=1}^N \) is a finite sequence of positive operators in \( L^1(\mathcal{M}, \tau) \) then

\[
\left\| \left( \sum_{n=1}^N |\mathcal{T}(\mathcal{P}^{(n)})(x_n)|^2 \right)^{1/2} \right\|_{1,\infty} \leq 5\sqrt{2} \sum_{n=1}^N \|x_n\|_1.
\]

We remark that if \( N = 1 \), then the above proposition (at least for the finite case) is a particular case of the weak type \((1,1)\) boundedness of the Hilbert transform associated with finite subdiagonal subalgebra obtained in [39]. A more concise proof for the case \( N = 1 \) also appeared in [15, Theorem 1.4]. It is the presentation in [15] that we will adopt below to prove Proposition 1.4.

**Proof of Proposition 1.4.** For each \( 1 \leq n \leq N \), we will simply write \( \mathcal{T}_n, D_n, \) and \( \mathcal{H}_n \) for \( \mathcal{T}(\mathcal{P}^{(n)}), D(\mathcal{P}^{(n)}), \) and \( \mathcal{H}(\mathcal{P}^{(n)}) \), respectively.

Since \( \|(\sum_{i=1}^M p_{i,n})x(\sum_{i=1}^M p_{i,n})\|_1 \leq \|x\|_1 \) and \( \mathcal{T}_n((\sum_{i=1}^M p_{i,n})x(\sum_{i=1}^M p_{i,n})) = \mathcal{T}_n x \) for every \( x \in L^1(\mathcal{M}, \tau) \), it is clear that it is enough to consider the case where for each \( 1 \leq n \leq N \), \( x_n \) belongs to the space \( L^1((\sum_{i=1}^M p_{i,n})\mathcal{M}(\sum_{i=1}^M p_{i,n})) \). Therefore we may assume without loss of generality that for each \( 1 \leq n \leq N \), \( \sum_{i=1}^M p_{i,n} = 1 \). We will assume first that for each \( 1 \leq n \leq N \), \( x_n \in \mathcal{M} \cap L^1(\mathcal{M}, \tau) \).

For \( 1 \leq n \leq N \), set

\[
A_n := x_n + i\mathcal{H}_n(x_n).
\]

We will show first that for \( \lambda > 0 \),

\[
\tau \left( \chi_{(\lambda, \infty)} \left( \sum_{n=1}^N |A_n|^{2/2} \right) \right) \leq 4 \sum_{n=1}^N \|x_n\|_1 / \lambda. \tag{1.2}
\]

The main argument is to estimate the trace of the operator \( \sum_{n=1}^N |D_n(A_n(\lambda 1 + A_n)^{-1})| \) for \( \lambda > 0 \) from above and below.

Note that for \( 1 \leq n \leq N \), \( A_n \in \mathcal{M}_{\mathcal{P}(x)} \) and \( D_n(A_n) = D_n(x_n) \). We can deduce from Lemma 1.3(iii) that

\[
D_n(A_n(\lambda 1 + A_n)^{-1}) = D_n(A_n)D_n((\lambda 1 + A_n)^{-1}).
\]
The following estimate from above follows directly from Lemma 1.3(i):

\[
\sum_{n=1}^{N} \tau(|D_n(A_n(\lambda I + A_n)^{-1})|) \leq \sum_{n=1}^{N} \tau(D_n(x_n)) \| (\lambda I + A_n)^{-1} \|_{\infty}
\]

\[
= \sum_{n=1}^{N} \| x_n \|_1 \| (\lambda I + A_n)^{-1} \|_{\infty}
\]

\[
\leq \left( \sum_{n=1}^{N} \| x_n \|_1 \right) / \lambda.
\]

(1.3)

For the estimate from below, we first note from Lemma 1.3(iii), (iv), and (v) that for \( 1 \leq n \leq N \), the operator

\[
D_n(A_n(\lambda I + A_n)^{-1}) = D_n(A_n)D_n((\lambda I + A_n)^{-1}) = D_n(x_n)(\lambda I + D_n(x_n))^{-1}
\]

is self-adjoint and we clearly have,

\[
\sum_{n=1}^{N} \tau(|D_n(A_n(\lambda I + A_n)^{-1})|) \geq \sum_{n=1}^{N} \tau(D_n(A_n(\lambda I + A_n)^{-1})).
\]

For each \( 1 \leq n \leq N \), we will estimate \( \tau(D_n(A_n(\lambda I + A_n)^{-1})) \) exactly as in [15]. We include the argument for completeness.

\[
\tau(D_n(A_n(\lambda I + A_n)^{-1})) = \tau(\text{Re}D_n(A_n(\lambda I + A_n)^{-1}))
\]

\[
= \tau(D_n(\text{Re}A_n(\lambda I + A_n)^{-1}))
\]

\[
= \tau(\text{Re}A_n(\lambda I + A_n)^{-1}).
\]

Note that \( \text{Re}A_n(\lambda I + A_n)^{-1} = (\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1} \geq 2^{-1}((\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1}) \geq 2^{-1}((\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1}) \geq 2^{-1}[(\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1}] \). We have,

\[
\tau(D_n(A_n(\lambda I + A_n)^{-1})) \geq 2^{-1} \tau((\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1})
\]

\[
= 2^{-1} \tau((\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1})
\]

\[
= 2^{-1} \tau((\lambda I + A_n^*)^{-1}(\lambda I + A_n)^{-1}).
\]

Set \( y_n := |A_n|^2 + 2\lambda \text{Re}A_n \) and \( y := \sum_{n=1}^{N} y_n \). As \( \text{Re}A_n \geq 0 \), we have \( y_n \geq 0 \) and therefore for each \( n \geq 1, y_n \leq y \) and \( (y_n + \lambda^2 \mathbf{1})^{-1} \geq (y + \lambda^2 \mathbf{1})^{-1} \). Hence, we can deduce,

\[
\sum_{n=1}^{N} \tau(D_n(A_n(\lambda I + A_n)^{-1})) \geq 2^{-1} \sum_{n=1}^{N} \tau(y_n(\lambda + \mathbf{1})^{-1})
\]

\[
\geq 2^{-1} \sum_{n=1}^{N} \tau(y_n(\lambda + \mathbf{1})^{-1})
\]

\[
= 2^{-1} \tau(y(\lambda + \mathbf{1})^{-1}).
\]
Observe that $g(y + \lambda^2 1)^{-1} \geq \chi_{(\lambda^2, \infty)}(y). y(y + \lambda^2 1)^{-1} \geq 2^{-1} \chi_{(\lambda^2, \infty)}(y)$. We obtain,

$$\sum_{n=1}^{N} \tau(D_n(A_n(\lambda 1 + A_n)^{-1})) \geq 4^{-1} \tau(\chi_{(\lambda^2, \infty)}(y)).$$

Again as $\text{Re}A_n \geq 0$, $y \geq \sum_{n=1}^{N} |A_n|^2$, we have

$$\sum_{n=1}^{N} \tau(D_n(A_n(\lambda 1 + A_n)^{-1})) \geq 4^{-1} \tau \left( \chi_{(\lambda^2, \infty)}(\sum_{n=1}^{N} |A_n|^2) \right) = 4^{-1} \tau \left( \chi_{(\lambda, \infty)} \left( \sum_{n=1}^{N} |A_n|^2 \right)^{1/2} \right). \tag{1.4}$$

Combining (1.3) and (1.4), inequality (1.2) follows and hence

$$\left\| \left( \sum_{n=1}^{N} |A_n|^2 \right)^{1/2} \right\|_{1, \infty} \leq 4 \sum_{n=1}^{N} \|x_n\|_1.$$

Now, from Lemma 1.3(vi) and the elementary fact that for any operators $a$ and $b$, $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, we have:

$$\sum_{n=1}^{N} |T_n(x_n)|^2 \leq 2^{-1} \left( \sum_{n=1}^{N} |A_n|^2 + \sum_{n=1}^{N} |D_n(x_n)|^2 \right).$$

Using properties of singular values $\mu(\cdot)$ from [18], we have for $t > 0$,

$$t \mu_t \left\{ \left( \sum_{n=1}^{N} |T_n(x_n)|^2 \right)^{1/2} \right\} \leq \sqrt{2} \frac{t}{2} \mu_{t/2} \left\{ \sum_{n=1}^{N} |A_n|^2 \right\}^{1/2}$$

$$+ \sqrt{2} \frac{t}{2} \mu_{t/2} \left\{ \sum_{n=1}^{N} |D_n(x_n)|^2 \right\}^{1/2} \leq \sqrt{2} \left( \sum_{n=1}^{N} |A_n|^2 \right)^{1/2} + \sqrt{2} \left( \sum_{n=1}^{N} |D_n(x_n)|^2 \right)^{1/2} \leq 4 \sum_{n=1}^{N} \|x_n\|_1 + \sqrt{2} \left( \sum_{n=1}^{N} |D_n(x_n)|^2 \right)^{1/2} \|1, \infty.$$
We complete the proof of the proposition by noting that if \((x_n)_{n=1}^{N}\) is a finite sequence of positive operators in \(L^1(\mathcal{M}, \tau)\) then for each \(1 \leq n \leq N\), we can choose a sequence \((x_n^{(k)})_{k=1}^{\infty}\) in \(\mathcal{M} \cap L^1(\mathcal{M}, \tau)\) with \(0 \leq x_n^{(k)} \uparrow x_n\). Observe that for every \(1 \leq n \leq N\), \(T_n(x_n^{(k)}) \rightarrow k T_n(x_n)\) in \(L^{1,\infty}(\mathcal{M}, \tau)\). A fortiori, the sequence \(\{(\sum_{n=1}^{N} |T_n(x_n^{(k)})|^2)^{1/2}\}_{k=1}^{\infty}\) converges to \((\sum_{n=1}^{N} |T_n(x_n)|^2)^{1/2}\) for the measure topology (when \(k \rightarrow \infty\)). From [18, Lemma 3.1], we can conclude that for every \(t > 0\),

\[
\lim_{k \rightarrow \infty} \mu_t\left\{\left(\sum_{n=1}^{N} |T_n(x_n^{(k)})|^2\right)^{1/2}\right\} = \mu_t\left\{\left(\sum_{n=1}^{N} |T_n(x_n)|^2\right)^{1/2}\right\}.
\]

Hence, for \(t > 0\),

\[
t\mu_t \left\{\left(\sum_{n=1}^{N} |T_n(x_n)|^2\right)^{1/2}\right\} = \lim_{k \rightarrow \infty} t\mu_t \left\{\left(\sum_{n=1}^{N} |T_n(x_n^{(k)})|^2\right)^{1/2}\right\} \leq 5\sqrt{2} \lim_{k \rightarrow \infty} \sum_{n=1}^{N} \|x_n^{(k)}\|_1
\]

\[
= 5\sqrt{2} \sum_{n=1}^{N} \|x_n\|_1.
\]

Taking the supremum over \(t > 0\), the definition of \(\| \cdot \|_{1,\infty}\) provides the desired inequality and thus the proof of the proposition is complete. \(\square\)

2. Conditional expectations and non-commutative martingales

Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra and \(\mathcal{N}\) be a von Neumann subalgebra of \(\mathcal{M}\). A linear map \(E : \mathcal{M} \rightarrow \mathcal{N}\) is called a normal conditional expectation if it satisfies the following:

(i) \(E\) is a weak\(^*\)-continuous projection;
(ii) \(E\) is positive;
(iii) \(E(ab) = aE(b)\) for all \(a, b \in \mathcal{N}\) and \(x \in \mathcal{M}\);
(iv) \(\tau \circ E = \tau\).

Recall that such normal conditional expectation from \(\mathcal{M}\) onto \(\mathcal{N}\) exists if and only if the restriction of the trace of \(\mathcal{M}\) to \(\mathcal{N}\) remains semi-finite. For the case where \(\mathcal{M}\) is finite, such conditional expectations always exist. Indeed, if \(\mathcal{N}\) is a von Neumann subalgebra of a finite von Neumann algebra \(\mathcal{M}\), then the embedding \(\iota : L^1(\mathcal{N}, \tau|_{\mathcal{N}}) \rightarrow L^1(\mathcal{M}, \tau)\) is an isometry and the dual map \(E = \iota^* : \mathcal{M} \rightarrow \mathcal{N}\) yields a normal conditional expectation (see for instance, [43, Theorem 3.4]).

Since \(E\) is trace preserving, it extends as a contractive projection \(E : L^p(\mathcal{M}, \tau) \rightarrow L^p(\mathcal{N}, \tau|_{\mathcal{N}})\) for all \(1 \leq p \leq \infty\) satisfying the property:

\[
E(ab) = aE(b)
\]
when \( 1 \leq p, q, r \leq \infty \), \( 1/p + 1/q + 1/r \leq 1 \), \( a \in L^p(\mathcal{N}, \tau_{|\mathcal{N}}) \), \( b \in L^q(\mathcal{N}, \tau_{|\mathcal{N}}) \) and \( x \in L^r(\mathcal{M}, \tau) \). More generally, a simple interpolation argument would prove that if \( E \) is a rearrangement invariant Banach function space on \([0, \tau(1))\), then \( E \) is a contraction from \( E(\mathcal{M}, \tau) \) onto \( E(\mathcal{N}, \tau_{|\mathcal{N}}) \).

Let us recall the general setup for martingales. The reader is referred to \([16]\) and \([19]\) for the classical (commutative) martingale theory. Let \((\mathcal{M}_n)_{n=1}^{\infty}\) be an increasing sequence of von Neumann subalgebras of \(\mathcal{M}\) such that the union of \(\mathcal{M}_n\)'s is weak*-
dense in \(\mathcal{M}\). For each \(n \geq 1\), assume that there is a normal conditional expectation \(\mathcal{E}_n\) from \(\mathcal{M}\) onto \(\mathcal{M}_n\). It is clear that for every \(m\) and \(n\) in \(\mathbb{N}\), \(\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(m,n)}\).

The following definition isolates the main topic of this paper.

**Definition 2.1.** A non-commutative martingale with respect to the filtration \((\mathcal{M}_n)_{n=1}^{\infty}\) is a sequence \(x = (x_n)_{n=1}^{\infty}\) in \(L^1(\mathcal{M}, \tau)\) such that:

\[
\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.
\]

Similarly, if for all \(n \geq 1\), \(x_n\) is self-adjoint and \(\mathcal{E}_n(x_{n+1}) \leq x_n\) (respectively, \(\mathcal{E}_n(x_{n+1}) \geq x_n\)), then the sequence \((x_n)_{n=1}^{\infty}\) is called a supermartingale (respectively, submartingale).

If additionally, \(x \in L^p(\mathcal{M}, \tau)\) for some \(1 < p < \infty\), then \(x\) is called a \(L^p\)-martingale. In this case, we set

\[
\|x\|_p := \sup_{n \geq 1} \|x_n\|_p.
\]

If \(\|x\|_p < \infty\), then \(x\) is called a bounded \(L^p\)-martingale. The difference sequence \(dx = (dx_n)_{n=1}^{\infty}\) of a martingale \(x = (x_n)_{n=1}^{\infty}\) is defined by

\[
dx_n = x_n - x_{n-1}
\]

with the usual convention that \(x_0 = 0\).

Recall that a subset \(S\) of \(L^1(\mathcal{M}, \tau)\) is said to be uniformly integrable if it is bounded and for every sequence of projections \((p_n)_{n=1}^{\infty}\) with \(p_n \downarrow 0\) (for the strong operator topology), we have \(\lim_{n \to \infty} \sup \{\|h\|_1; h \in S\} = 0\) \([41]\). It can be easily verified that a martingale \(x = (x_n)_{n=1}^{\infty}\) in \(L^1(\mathcal{M}, \tau)\) is uniformly integrable if and only if there exists \(x_\infty \in L^1(\mathcal{M}, \tau)\) such that \(x_n = \mathcal{E}_n(x_\infty)\) for all \(n \geq 1\). In this case, the sequence \((x_n)_{n=1}^{\infty}\) converges to \(x_\infty\) in \(L^1(\mathcal{M}, \tau)\). In particular, if \(1 < p < \infty\), then every bounded \(L^p\)-martingale is of the form \((\mathcal{E}_n(x_\infty))_{n=1}^{\infty}\) for some \(x_\infty \in L^p(\mathcal{M}, \tau)\).

For some concrete natural examples of non-commutative martingales, we refer to \([37]\) and the recent survey in this topic \([45]\).

We will now describe square functions of non-commutative martingales. Following Pisier and Xu \([37]\), we will consider the following row and column versions of square functions: for a martingale \(x = (x_n)_{n=1}^{\infty}\), we denote by \(dx\) the difference sequence as defined above. For \(N \geq 1\), set

\[
S_{C,N}(x) = \left( \frac{1}{N} \sum_{k=1}^{N} |dx_k|^2 \right)^{1/2} \text{ and } S_{R,N}(x) = \left( \frac{1}{N} \sum_{k=1}^{N} |dx_k|^2 \right)^{1/2}.
\]
Let $E[0, \tau(1)]$ be a rearrangement invariant (quasi-) Banach function space on the interval $[0, \tau(1))$. For any finite sequence $a = (a_n)_{n \geq 1}$ in $E(\mathcal{M}, \tau)$, set

$$\|a\|_{E(\mathcal{M}; l^2_R)} = \left\| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \right\|_{E(\mathcal{M}, \tau)}, \quad \|a\|_{E(\mathcal{M}; l^2_C)} = \left\| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \right\|_{E(\mathcal{M}, \tau)}.$$

The difference sequence $dx$ belongs to $E(\mathcal{M}; l^2_C)$ (respectively, $E(\mathcal{M}; l^2_R)$) if and only if the sequence $(S_{C,n}(x))_{n=1}^{\infty}$ (respectively, $(S_{R,n}(x))_{n=1}^{\infty}$) is a bounded sequence in $E(\mathcal{M}, \tau)$. In this case, the limit $S_C(x) = (\sum_{k=1}^{\infty} |dx_k|^2)^{1/2}$ (respectively, $S_R(x) = (\sum_{k=1}^{\infty} |dx_k|^2)^{1/2}$) is an element of $E(\mathcal{M}, \tau)$. These two versions of square functions are very crucial in the subsequent sections.

3. **Main Results: The Finite Case**

In this section, we assume that $\mathcal{M}$ is a finite von Neumann algebra and $\tau$ is normalized normal faithful trace on $\mathcal{M}$.

We will retain all notations introduced in the previous two sections. In particular, all adapted sequences are understood to be with respect to a fixed filtration of von Neumann subalgebras of $\mathcal{M}$. The principal result of this paper is Theorem 3.1 below. It answers the problem raised in [42].

**Theorem 3.1.** There is an absolute constant $K > 0$ such that if $x = (x_n)_{n=1}^{\infty}$ is a $L^2$-bounded martingale, then there exist two sequences $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ such that:

1. $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are $L^2$-bounded martingales;
2. for every $n \geq 1$, $x_n = y_n + z_n$;
3. $\|dy\|_{L^2(\mathcal{M}; l^2_C)} + \|dz\|_{L^2(\mathcal{M}; l^2_R)} \leq 2\|x\|_2$;
4. $\|dy\|_{L^1(\mathcal{M}; l^2_C)} + \|dz\|_{L^1(\mathcal{M}; l^2_R)} \leq K\|x\|_1$.

As in the martingale transforms, our approach depends very heavily on a non-commutative version of the classical Doob weak type $(1,1)$ maximal inequality, due to Cuculescu [8] (which we will recall below). As noted in [42], the general case can be deduced easily from the special case of positive martingale. Hence, without loss of generality, we can and do assume that the martingale $x = (x_n)_{n=1}^{\infty}$ is a positive martingale and $\|x\|_1 = 1$.

We will divide the proof into two parts. In the first part, we will provide a detailed description of the concrete decomposition of the martingale $(x_n)_{n=1}^{\infty}$ and point out that (i), (ii), and (iii) are easily verified from the construction. In the second part, we will show that the decomposition satisfies the conclusion (iv) of the theorem.

- **Construction of the Martingales** $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$.

We start with the proposition (due to Cuculescu [8]) below which can be viewed as a substitute for the classical weak type $(1,1)$ boundedness of maximal functions.
We will state a version that incorporates the different properties that we need in the sequel. A short proof of the form stated below can be found in [40].

**Proposition 3.2 ([8]).** For every $\lambda > 0$, there exists a sequence of decreasing projections $(q_n^{(\lambda)})_{n=1}^{\infty}$ in $\mathcal{M}$ with:

(a) for every $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$;
(b) $q_n^{(\lambda)} = \chi_{[0,\lambda]}(q_{n-1}^{(\lambda)}x_nq_{n-1}^{(\lambda)})$. In particular, $q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)}x_nq_{n-1}^{(\lambda)}$;
(c) $q_n^{(\lambda)}x_nq_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$;
(d) if we set $q^{(\lambda)} = \bigwedge_{n=1}^{\infty} q_n^{(\lambda)}$ then $(1 - q^{(\lambda)}) \leq \lambda^{-1}$.

We consider collections of sequences of pairwise disjoint projections as follows: for $n \geq 1$, set

\[
\begin{align*}
    p_{0,n} &:= \bigwedge_{k=0}^{\infty} q_n^{(2^k)}, \quad \text{and} \\
    p_{i,n} &:= \bigwedge_{k=i}^{\infty} q_n^{(2^k)} - \bigwedge_{k=i-1}^{\infty} q_n^{(2^k)} \quad \text{for } i \geq 1.
\end{align*}
\]  

(3.1)

Similarly,

\[
\begin{align*}
    p_0 &:= \bigwedge_{k=0}^{\infty} q^{(2^k)}, \quad \text{and} \\
    p_i &:= \bigwedge_{k=i}^{\infty} q^{(2^k)} - \bigwedge_{k=i-1}^{\infty} q^{(2^k)} \quad \text{for } i \geq 1.
\end{align*}
\]  

(3.2)

Useful properties of the sequences $(p_{i,n})_{i=0}^{\infty}$ and $(p_i)_{i=0}^{\infty}$, that are relevant for our proof, are collected in the following proposition whose verification is straightforward and therefore is left to the reader.

**Proposition 3.3.** For $n \geq 1$, the sequence of projections $(p_{i,n})_{i=0}^{\infty}$ (respectively, $(p_i)_{i=0}^{\infty}$) are pairwise disjoint with the following properties:

(a) For every $n \geq 1$ and $i \geq 0$, $p_{i,n} \in \mathcal{M}_n$;
(b) $\sum_{i=0}^{\infty} p_{i,n} = 1$ and $\sum_{i=0}^{\infty} p_i = 1$ (for the strong operator topology);
(c) for every $n_0 \geq 1$, $\sum_{i=0}^{n_0} p_{i,n} \leq q_n^{(2^n_0)}$ and $\sum_{i=0}^{n_0} p_i \leq q^{(2^n_0)}$.

Since $\sum_{i=0}^{\infty} p_{i,n} = 1$, we have that $a = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} p_{i,n}a p_{j,n}$ for all $a \in L^2(\mathcal{M}, \tau)$ so clearly, $a = \sum_{j=0}^{\infty} \sum_{i\leq j} p_{i,n}a p_{j,n} + \sum_{j=0}^{\infty} \sum_{i>j} p_{i,n}a p_{j,n}$. Our construction is based in this simple fact.
Define the sequences \( y = (y_n)_{n=1}^{\infty} \) and \( z = (z_n)_{n=1}^{\infty} \) as follows:

\[
\begin{align*}
\{dy_1\} & := \sum_{j=0}^{\infty} \sum_{i \leq j} p_{i,1} dx_1 p_{j,1}; \\
\{dy_n\} & := \sum_{j=0}^{\infty} \sum_{i \leq j} p_{i,n-1} dx_n p_{j,n-1} \quad \text{for } n \geq 2; \\
\{dz_1\} & := \sum_{j=0}^{\infty} \sum_{i > j} p_{i,1} dx_1 p_{j,1}; \\
\{dz_n\} & := \sum_{j=0}^{\infty} \sum_{i > j} p_{i,n-1} dx_n p_{j,n-1} \quad \text{for } n \geq 2.
\end{align*}
\tag{3.3}
\]

Triangular truncations were also the main tools for the construction of the decomposition used in [42]. The new adjustment we need, in order to achieve the decomposition into two martingales, is the use of sequence of mutually disjoint projections \( (p_{i,n-1})_{i=0}^{\infty} \) from \( \mathcal{M}_{n-1} \) (when \( n \geq 2 \)) instead of \( (p_i)_{i=0}^{\infty} \) used in [42]. This was already clear since [42] but we were unable to verify Proposition B below at that time.

Clearly, \( dx_n = dy_n + dz_n \) for every \( n \geq 1 \), therefore \( x_n = y_n + z_n \) for every \( n \geq 1 \), hence (ii) is verified.

Let \( n \geq 2 \). Since \( (p_{i,n-1})_{i=0}^{\infty} \) are mutually disjoint projections in \( \mathcal{M}_{n-1} \), and triangular truncations are orthogonal projections in \( L^2(\mathcal{M}, \tau) \), for every \( a \in L^2(\mathcal{M}, \tau) \),

\[
\sum_{j=0}^{\infty} \sum_{i \leq j} p_{i,n-1} a p_{j,n-1} = \lim_{k \to \infty} \sum_{j=0}^{k} \sum_{i \leq j} p_{i,n-1} a p_{j,n-1}.
\]

We deduce that for every \( a \in L^2(\mathcal{M}, \tau) \),

\[
\mathcal{E}_{n-1} \left( \sum_{j=0}^{\infty} \sum_{i \leq j} p_{i,n-1} a p_{j,n-1} \right) = \sum_{j=0}^{\infty} \sum_{i \leq j} p_{i,n-1} \mathcal{E}_{n-1}(a) p_{j,n-1}.
\]

In particular, as \( dx_n \in L^2(\mathcal{M}_n, \tau_n) \) and \( \mathcal{E}_{n-1}(dx_n) = 0 \), it follows that \( dy_n \in L^2(\mathcal{M}_n, \tau_n) \) and \( \mathcal{E}_{n-1}(dy_n) = 0 \). A similar remark can be made for \( \mathcal{E}_{n-1}(dz_n) \). Hence, \( (dy_n)_{n=1}^{\infty} \) and \( (dz_n)_{n=1}^{\infty} \) are martingale difference sequences, which verifies (i).

To verify (iii), it is enough to note from the boundedness of the triangular truncations in \( L^2(\mathcal{M}, \tau) \) that

\[
\sum_{n=1}^{\infty} \|dy_n\|_2^2 \leq \sum_{n=1}^{\infty} \|dx_n\|_2^2 = \|x\|_2^2.
\]

Noting that a similar inequality is also valid for \( \sum_{n=1}^{\infty} \|y\|_2^2 \), the inequality (iii) follows. Thus the items (i), (ii), and (iii) of Theorem 3.1 are verified.

- **Proof of the Weak-Type Inequality** (iv).
In order to prove Theorem 3.1(iv), we will make several reductions. First, we remark that from the construction of the martingales \((y_n)_{n=1}^{\infty}\) and \((z_n)_{n=1}^{\infty}\), the square functions \((\sum_{n=1}^{\infty} |dz_n|^2)^{1/2}\) and \((\sum_{n=1}^{\infty} |dy_n|^2)^{1/2}\) have the same form. This can easily be seen from the following lemma whose verification is just a notational adjustment of [42, Lemma 2.2] and is left to the interested reader.

**Lemma 3.4.** For the sequences defined above, we have:

(a) \(|dy_1|^2 = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i \leq \min(l,j)} p_{i,n} dx_1p_{i,n} dx_{1,1}; \)
(b) \(|dy_n|^2 = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i \leq \min(l,j)} p_{i,n-1} dx_n p_{i,n-1} dx_{n,j,n-1}\) for \(n \geq 2; \)
(c) \(|dz_{n,1}^*|^2 = \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i \leq \min(l,j)} p_{i,n} dx_1p_{i,n} dx_{1,1}; \)
(d) \(|dz_n^*|^2 = \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \sum_{i \leq \min(l,j)} p_{i,n-1} dx_n p_{i,n-1} dx_{n,j,n-1}\) for \(n \geq 2;\)

where the sums are taken in the measure topology.

From the preceding lemma, we only have to show that there is an absolute constant \(C_1 > 0\) such that:

\[\|dy\|_{L^{1,\infty}(M;\ell_1^2)} \leq C_1.\]  

(3.4)

According to the definition of the quasi-norm \(\| \cdot \|_{1,\infty}\), this is equivalent to show the existence of a numerical constant \(C_1 > 0\) such that for every \(\lambda > 0\),

\[\tau(\chi_{(\lambda,\infty)}(S_C(y))) \leq C_1 \lambda^{-1}.\]

(3.5)

The proof basically follows the steps used in [42, 40] but some non-trivial adjustments had to be made.

\(\diamond\) First, we consider the particular case: \(\lambda = 2^{n_0}\) for some \(n_0 \geq 0\).

The proof of this case consists of three fundamental steps that will be highlighted in three separate propositions.

To avoid dealing with convergence, we will show that there is an absolute constant \(C_0 > 0\) such that for every \(N \geq 1\),

\[\tau(\chi_{(\lambda,\infty)}(S_{C,N}(y))) \leq C_0 2^{-n_0}.\]

(3.6)

Throughout the proof, \(N \geq 1\) is fixed. We will reduce first to the case of difference sequence of a bounded sequence in \(L^{\infty}(M, \tau)\). For notational purpose, we will simply write, throughout the proof, \((q_n)_{n=1}^{\infty}\) (respectively, \(q\)) for the projections \((q_n^{(2^{n_0})})_{n=1}^{\infty}\) (respectively, \(q^{(2^{n_0})}\)). Consider the projection

\[w_{n_0} = \sum_{i=0}^{n_0} p_i = \bigwedge_{k=n_0}^{\infty} q^{(2^k)},\]

(3.7)

and the operator

\[\gamma = \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,j} dx_1p_{i,j} dx_{1,1}^2 + \sum_{n=2}^{N} \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_n p_{i,n-1} dx_{n,j,n-1}^2,\]

(3.8)
The first step is to reduce the inequality from $S_{C,N}(y)$ to $\gamma$. The significance of such reduction is the fact that the triangular truncations used in $\gamma$ are formed from collections of finitely many projections.

**Proposition A.** For every $\alpha \in (0,1)$ and every $\beta \in (0,1)$,

$$
\tau \left( \chi_{(2^{n_0},\infty)}(S_{C,N}(y)) \right) \leq \alpha^{-1}\tau \left( \chi_{(3^{n_0},\infty)}(\gamma) \right) + 2(1-\alpha)^{-1}2^{-n_0}.
$$

*Proof.* Let $S = S_{C,N}(y)^2 = \sum_{n=1}^{N} |dy_n|^2$. Write $S^{1/2} = S^{1/2}w_{n_0} + S^{1/2}(1 - w_{n_0})$ and apply Lemma 1.2 to get

$$
\tau \left( \chi_{(2^{n_0},\infty)}(S_{C,N}(y)) \right) \leq \alpha^{-1}\tau \left( \chi_{(\sqrt{2^{n_0}},\infty)}(|S^{1/2}w_{n_0}|) \right) + (1-\alpha)^{-1}\tau \left( \chi_{((1-\sqrt{2})^{n_0},\infty)}(|S^{1/2}(1 - w_{n_0})|) \right).
$$

Since $\chi_{((1-\sqrt{2})^{n_0},\infty)}(|S^{1/2}(1 - w_{n_0})|)$ is a subprojection of $1 - w_{n_0}$, it follows that

$$
\tau \left( \chi_{(2^{n_0},\infty)}(S_{C,N}(y)) \right) \leq \alpha^{-1}\tau \left( \chi_{(\sqrt{2^{n_0}},\infty)}(|S^{1/2}w_{n_0}|) \right) + (1-\alpha)^{-1}\tau(1 - w_{n_0}).
$$

Note that $w_{n_0} = \sum_{i=0}^{n_0} p_i = \bigvee_{k=n_0}^{\infty} q^{(2^k)}$ so $1 - w_{n_0} = \bigvee_{k=n_0}^{\infty} (1 - q^{(2^k)})$. By Proposition 3.2(d), $\tau(1 - w_{n_0}) \leq \sum_{k=n_0}^{\infty} 2^{-k} = 2.2^{-n_0}$. Combining with the previous estimate, we conclude

$$
\tau \left( \chi_{(2^{n_0},\infty)}(S_{C,N}(y)) \right) \leq \alpha^{-1}\tau \left( \chi_{(\sqrt{2^{n_0}},\infty)}(|S^{1/2}w_{n_0}|) \right) + 2(1-\alpha)^{-1}2^{-n_0} = \alpha^{-1}\tau \left( \chi_{(3^{n_0},\infty)}(w_{n_0}S w_{n_0}) \right) + 2(1-\alpha)^{-1}2^{-n_0}.
$$

To complete the proof, we will show that $w_{n_0}S w_{n_0} = w_{n_0}\gamma w_{n_0}$.

In fact, from the form of $|dy_n|^2$ stated in Lemma 3.4, we can write:

$$
w_{n_0}^2S_{C,N}(y)w_{n_0} = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} w_{n_0} p_{l,1} dx_1 p_{l,1} dx_1 p_{j,1} w_{n_0} + \sum_{n=2}^{N} \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} w_{n_0} p_{l,n-1} dx_n p_{i,n-1} dx_n p_{j,n-1} w_{n_0}.
$$

We claim that all the sums taken in the expression of $w_{n_0}^2S_{C,N}(y)w_{n_0}$ above are finite sums. For this, we remark that if $l > n_0$ and $s \geq 1$, then $w_{n_0} p_{l,s} = p_{l,s} w_{n_0} = 0$.

In fact, as $p_{l,s} = \bigwedge_{k=l}^{\infty} q^{(2^k)} - \bigwedge_{k=l-1}^{\infty} q^{(2^k)}$ and $q^{(2^k)} \leq q^{(2^k)}$ for all $k \geq 1$, it is clear that $w_{n_0} = \bigwedge_{k=n_0}^{\infty} q^{(2^k)}$ is a subprojection of $\bigwedge_{k=l-1}^{\infty} q^{(2^k)}$ when $l > n_0$ and therefore
$w_{n_0} \perp p_{l,s}$. With this observation, we can write:

$$
\begin{align*}
{w_{n_0}}^2 S_{C,N}(y)w_{n_0} &= \sum_{l=0}^{n_0} \sum_{j=0}^{n_0} \sum_{i \leq \min(l,j)} w_{n_0} p_{l,1} dx_1 p_{i,1} dx_1 p_{j,1} w_{n_0} \\
+ \sum_{n=2}^{N} \sum_{l=0}^{n_0} \sum_{j=0}^{n_0} \sum_{i \leq \min(l,j)} w_{n_0} p_{l,n-1} dx_n p_{i,n-1} dx_n p_{j,n-1} w_{n_0} \\
&= w_{n_0} \gamma w_{n_0}.
\end{align*}
$$

We conclude the proof by noting that

$$
\begin{align*}
\tau \left( \chi_{(\beta 4n_0, \infty)}(w_{n_0} \gamma w_{n_0}) \right) &= \int_0^1 \chi_{(\beta 4n_0, \infty)}(\mu_t(w_{n_0} \gamma w_{n_0})) \ dt \\
&\leq \int_0^1 \chi_{(\beta 4n_0, \infty)}(\mu_t(\gamma)) \ dt \\
&= \tau \left( \chi_{(\beta 4n_0, \infty)}(\gamma) \right). 
\end{align*}
$$

The proof is complete. □

The next step is to estimate $\tau(\chi_{(\beta 4n_0, \infty)}(\gamma))$ using the $L^2$-norm of square function of a supermartingale. This is the most significant adjustment of the proof.

**Proposition B.** The sequence $(q_n x_n q_n)_{n=1}^\infty$ is a supermartingale in $L^2(\mathcal{M}, \tau)$ and if we set $K_1 = 4(\alpha^{-1} - 1)(1-\alpha)^{-1}(1-\beta)^{-1} + 10\sqrt{2}(1-\alpha)^{-2}(1-\beta)^{-1}(\sqrt{\beta})^{-1}$ for $\alpha \in (0,1)$ and $\beta \in (0,1)$, then the following inequality holds:

$$
\tau \left( \chi_{(\beta 4n_0, \infty)}(\gamma) \right) \leq 2\alpha^{-1} \beta^{-2} 4^{-n_0} \left( \|q_1 x_1 q_1\|^2 + \sum_{n=2}^{N} \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|^2 \right) + K_1 2^{-n_0}.
$$

**Proof.** The fact that the sequence $(q_n x_n q_n)_{n=1}^\infty$ is a supermartingale was already noted and proved in [40, Lemma 3.3] (see also, [42, Lemma 2.4]) so there is no need to repeat it here. To prove the estimate on $\tau(\chi_{(\beta 4n_0, \infty)}(\gamma))$, we note that since for $0 \leq j \leq n_0$ and $2 \leq n \leq N$, $p_{j,n-1} \leq q_{n-1}$ and $q_n \leq q_{n-1}$, we have $p_{j,n-1} = q_n p_{j,n-1} + (q_{n-1} - q_n)p_{j,n-1} = p_{j,n-1} q_n + p_{j,n-1} (q_{n-1} - q_n)$. We can decompose $\gamma$ as
follows:
\[
\gamma = \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_{1} p_{j,1} \right|^2 + \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} p_{j,n-1} \right|^2
\]
\[
= \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_{1} p_{j,1} \right|^2 + \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_n) p_{j,n-1} \right|^2
\]
\[
+ \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_{n-1} - q_n) p_{j,n-1} \right|^2.
\]

From the elementary inequality \(|a + b|^2 \leq 2|a|^2 + 2|b|^2\) for any operators \(a\) and \(b\), we have,
\[
\gamma \leq \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_{1} p_{j,1} \right|^2 + 2 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_n) p_{j,n-1} \right|^2
\]
\[
+ 2 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_{n-1} - q_n) p_{j,n-1} \right|^2.
\]

The last of the three terms above can be further decomposed to get,
\[
\gamma \leq \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_{1} p_{j,1} \right|^2 + 2 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_n) p_{j,n-1} \right|^2
\]
\[
+ 4 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} q_n dx_{n} (q_{n-1} - q_n) p_{j,n-1} \right|^2
\]
\[
+ 4 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} (q_{n-1} - q_n) dx_n (q_{n-1} - q_n) p_{j,n-1} \right|^2.
\]

Consider the following operators:
\[
\gamma_1 := \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_{1} p_{j,1} \right|^2 + 2 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_{n} (q_n) p_{j,n-1} \right|^2
\]
\[
\gamma_2 := 4 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} q_n dx_{n} (q_{n-1} - q_n) p_{j,n-1} \right|^2
\]
\[
\gamma_3 := 4 \sum_{n=2}^{N} \left| \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} (q_{n-1} - q_n) dx_n (q_{n-1} - q_n) p_{j,n-1} \right|^2. \tag{3.9}
\]
Clearly, $\gamma \leq \gamma_1 + \gamma_2 + \gamma_3$. The splitting technique from Lemma 1.2 can be applied to deduce that:

$$
\tau \left( \chi_{(\beta^2 4^{n_0} \alpha, \infty)}(\gamma) \right) \leq \alpha^{-1} \tau \left( \chi_{(\beta^2 4^{n_0} \alpha, \alpha)}(\gamma_1) \right) + (1 - \alpha)^{-1} \tau \left( \chi_{(1 - \beta^2 4^{n_0} \alpha, \alpha)}(\gamma_2 + \gamma_3) \right)
$$

$$
\leq \alpha^{-1} \tau \left( \chi_{(\beta^2 4^{n_0} \alpha, \alpha)}(\gamma_1) \right) + \alpha^{-1}(1 - \alpha)^{-1} \tau \left( \chi_{(1 - \beta^2 4^{n_0} \alpha, \alpha)}(\gamma_2) \right) + (1 - \alpha)^{-2} \tau \left( \chi_{(1 - \beta^2 4^{n_0} \alpha, \alpha)}(\gamma_3) \right)
$$

$$
= I + II + III.
$$

We will estimate the quantities $I$, $II$, and $III$ in separate three lemmas.

**Lemma 3.5.** $I \leq 2\alpha^{-1} \beta^{-2} 4^{-n_0} (\|q_1 x_1 q_1\|_2^2 + \sum_{n=2}^{N} \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2)$.

To prove this lemma, we remark first that

$$
I = \alpha^{-1} \tau \left( \chi_{(\beta^2 4^{n_0} \alpha, \alpha)}(\gamma_1) \right)
$$

$$
\leq \alpha^{-1} \beta^{-2} 4^{-n_0} (\sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_1 p_{j,1} \|_2^2 + 2 \sum_{n=2}^{N} \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_n q_n p_{j,n-1} \|_2^2).
$$

Note that since triangular truncations are contractive in $L^2(\mathcal{M}, \tau)$, the preceding inequality yields:

$$
I \leq \alpha^{-1} \beta^{-2} 4^{-n_0} (\sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_1 p_{j,1} \|_2^2 + 2 \sum_{n=2}^{N} \sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,n-1} dx_n q_n p_{j,n-1} \|_2^2)
$$

$$
\leq \alpha^{-1} \beta^{-2} 4^{-n_0} (\sum_{j=0}^{n_0} \sum_{i \leq j} p_{i,1} dx_1 p_{j,1} \|_2^2 + 2 \sum_{n=2}^{N} \sum_{i=0}^{n_0} p_{i,n-1} dx_n q_n \|_2^2).
$$

Since for $j \geq 1$, $\sum_{i=0}^{n_0} p_{i,j} = \bigwedge_{k=n_0}^{\infty} q_j^{(2k)} \leq q_j$ (Proposition 3.3(c)), we have

$$
I \leq \alpha^{-1} \beta^{-2} 4^{-n_0} (\|q_1 x_1 q_1\|_2^2 + 2 \sum_{n=2}^{N} \tau(q_n dx_n q_{n-1} dx_n q_n))
$$

$$
\leq 2\alpha^{-1} \beta^{-2} 4^{-n_0} (\|q_1 x_1 q_1\|_2^2 + \sum_{n=2}^{N} \tau(q_n dx_n q_{n-1} dx_n q_n)).
$$

To conclude the estimate on $I$, we will verify that for every $n \geq 2$,

$$
\tau(q_n dx_n q_{n-1} dx_n q_n) \leq \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2.
$$
This follows directly from the facts (Proposition 3.2) that \( q_n \leq q_{n-1} \) and \( q_n \) commutes with \( q_{n-1} x_n q_{n-1} \). In fact,

\[
\tau (q_n d x_n q_{n-1} d x_n q_n) = \tau (q_n (x_n - x_{n-1}) q_{n-1} (x_n - x_{n-1}) q_n) \\
= \tau (q_n [q_{n-1} x_n q_{n-1} - q_{n-1} x_{n-1} q_{n-1}]) [q_{n-1} x_n q_{n-1} - q_{n-1} x_{n-1} q_{n-1}] q_n \\
\leq \tau (q_n [q_{n-1} x_n q_n - q_{n-1} x_{n-1} q_{n-1}]) [q_{n-1} x_n q_{n-1} - q_{n-1} x_{n-1} q_{n-1}] q_n \\
\leq \| q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1} \|^2.
\]

This shows that \( I \leq 2 \alpha^{-1} \beta^{-2} 4^{-n_0} (\| q_1 x_1 q_1 \|^2_2 + \sum_{n=2}^{N} \| q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1} \|^2_2) \) as stated in the lemma.

**Lemma 3.6.** \( II \leq 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 2^{-n_0} \).

To prove this estimate, recall that \( \gamma_2 = 4 \sum_{n=2}^{N} \| \sum_{i=0}^{n} \sum_{j=0}^{n} p_{i,j,n-1} q_n d x_n (q_{n-1} - q_n) p_{j,n-1} \|^2 \). The proof rests upon the following elementary but crucial observation: for \( 2 \leq n \leq N \),

\[
q_n d x_n (q_{n-1} - q_n) = -q_n x_n (q_{n-1} - q_n).
\]

Indeed, from the fact that \( q_n \) commutes with \( q_{n-1} x_n q_{n-1} \), we have

\[
q_n x_n (q_{n-1} - q_n) = q_n (q_{n-1} x_n q_{n-1}) (q_{n-1} - q_n) = (q_{n-1} x_n q_{n-1}) q_n (q_{n-1} - q_n) = 0.
\]

We can now estimate \( II \) as follows:

\[
II = \alpha^{-1} (1 - \alpha)^{-1} \tau (\chi_{(1 - \beta)^{-2} 2^{-n_0}, \infty}) (\gamma_2)) \\
\leq 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 4^{-n_0} \sum_{n=2}^{N} \| \sum_{j=0}^{n} \sum_{i=0}^{j} p_{i,j,n-1} q_n d x_n (q_{n-1} - q_n) p_{j,n-1} \|^2_2 \\
\leq 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 4^{-n_0} \sum_{n=2}^{N} \| q_n d x_n (q_{n-1} - q_n) \|^2_2 \\
= 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 4^{-n_0} \sum_{n=2}^{N} \| q_n x_n (q_{n-1} - q_n) \|^2_2 \\
= 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 4^{-n_0} \sum_{n=2}^{N} \| q_n (q_{n-1} x_n q_{n-1}) (q_{n-1} - q_n) \|^2_2 \\
\leq 4 \alpha^{-1} (1 - \alpha)^{-1} (1 - \beta)^{-1} \beta^{-2} 4^{-n_0} \sum_{n=2}^{N} \| q_n (q_{n-1} x_n q_{n-1}) \|^2_\infty \| q_{n-1} - q_n \|^2_2.
\]
By Proposition 3.2(c), \( \|q_{n-1}x_{n-1}q_{n-1}\|_\infty \leq 2^{n_0} \) and therefore we have
\[
II \leq 4\alpha^{-1}(1-\alpha)^{-1}(1-\beta)^{-1}2^{-2}\sum_{n=2}^{N}\tau(q_{n-1}-q_n)
= 4\alpha^{-1}(1-\alpha)^{-1}(1-\beta)^{-2}\tau(q_1-q_N)
\leq 4\alpha^{-1}(1-\alpha)^{-1}(1-\beta)^{-2}2^{-n_0}.
\]
This completes the proof of the lemma.

**Lemma 3.7.** \( III \leq 10\sqrt{2}(1-\alpha)^{-2}(1-\beta)^{-1}2^{-n_0}. \)

The main tool for the proof is Proposition 1.4. For \( 2 \leq n \leq N \), let \( \mathcal{P}^{(n)} = \{p_{i,n-1}\}_{i=0}^{n_0} \). Using the notation of Proposition 1.4, \( \gamma_3 \) as defined in (3.9) can be expressed as:
\[
\gamma_3 = 4\sum_{n=2}^{N}\left|\sum_{j=0}^{n_0}\sum_{i\leq j}p_{i,n-1}(q_{n-1}-q_n)dx_n(q_{n-1}-q_n)p_{j,n-1}\right|^2
= 4\sum_{n=2}^{N}\left|\mathcal{T}\mathcal{P}^{(n)}[(q_{n-1}-q_n)dx_n(q_{n-1}-q_n)]\right|^2
= \sum_{n=2}^{N}\left|\mathcal{T}\mathcal{P}^{(n)}[2(q_{n-1}-q_n)dx_n(q_{n-1}-q_n)]\right|^2.
\]
The crucial fact here is that for \( 2 \leq n \leq N \),
\[
(q_{n-1}-q_n)dx_n(q_{n-1}-q_n) \geq 0.
\]
In fact, from the construction of the sequence of projection \( (q_n)_{n=1}^\infty \) from Proposition 3.2(b), we have \( (q_{n-1}-q_n)x_n(q_{n-1}-q_n) \geq 2^{n_0}(q_{n-1}-q_n) \). On the other hand, \( q_{n-1}x_{n-1}q_{n-1} \leq 2^{n_0}q_{n-1} \) and therefore, \( (q_{n-1}-q_n)x_{n-1}(q_{n-1}-q_n) \leq 2^{n_0}(q_{n-1}-q_n) \). Hence,
\[
(q_{n-1}-q_n)x_n(q_{n-1}-q_n) \geq 2^{n_0}(q_{n-1}-q_n) \geq (q_{n-1}-q_n)x_{n-1}(q_{n-1}-q_n)
\]
which shows that \( (q_{n-1}-q_n)dx_n(q_{n-1}-q_n) \geq 0 \). Therefore Proposition 1.4 applies to \( \gamma_3^{1/2} \). Hence, we have the following estimates:
\[
III = (1-\alpha)^{-2}\tau\left(\chi_{((1-\beta)^{-1}2^{n_0},\infty)}(\gamma_3)\right)
= (1-\alpha)^{-2}\tau\left(\chi_{((1-\beta)^{-1}2^{n_0},\infty)}(\gamma_3^{1/2})\right)
\leq 5\sqrt{2}(1-\alpha)^{-2}(1-\beta)^{-1}(\sqrt{\beta})^{-2}2^{-n_0}\sum_{n=2}^{N}\|2(q_{n-1}-q_n)dx_n(q_{n-1}-q_n)\|_1
\leq 10\sqrt{2}(1-\alpha)^{-2}(1-\beta)^{-1}(\sqrt{\beta})^{-2}2^{-n_0}\sum_{n=2}^{N}\tau((q_{n-1}-q_n)dx_n(q_{n-1}-q_n)).
\]
Note that for every \(2 \leq n \leq N\) (using the fact that \(\mathcal{E}_{n-1}\) is \(\tau\)-invariant),

\[
\tau \left( (q_{n-1} - q_n) dx_n (q_{n-1} - q_n) \right) = \tau \left( (q_{n-1} - q_n) (x_n - x_{n-1}) \right) \\
= \tau \left( q_{n-1} x_n - q_n x_n - q_{n-1} x_{n-1} + q_n x_{n-1} \right) \\
= \tau \left( q_{n-1} \mathcal{E}_{n-1}(x_n) - q_n x_n - q_{n-1} x_{n-1} + q_n x_{n-1} \right) \\
= \tau \left( -q_n x_n q_n + q_n x_{n-1} q_n \right) \\
\leq \tau \left( -q_n x_n q_n + q_{n-1} x_{n-1} q_{n-1} \right).
\]

Taking the sum, we conclude that

\[
III \leq 10 \sqrt{2} (1 - \alpha)^{-2} (1 - \beta)^{-1} (\sqrt{\beta})^{-1} 2^{-n_0} \tau (q_1 x_1 q_1 - q_N x_N q_N) \\
= 10 \sqrt{2} (1 - \alpha)^{-2} (1 - \beta)^{-1} (\sqrt{\beta})^{-1} 2^{-n_0} \tau ((q_1 - q_N) x_N) \\
\leq 10 \sqrt{2} (1 - \alpha)^{-2} (1 - \beta)^{-1} (\sqrt{\beta})^{-1} 2^{-n_0}.
\]

This completes the proof of the lemma.

The inequality in Proposition B follows by combining the estimates on \(I, II,\) and \(III\). The proof of the proposition is complete. \(\square\)

The last step is to estimate the \(L^2\)-norm of the square function of the supermartingale \((q_n x_n q_n)_{n=1}^\infty\). This was already achieved in [42] but we include below a much shorter simplification that produces a better bound.

**Proposition C.** The square function of the supermartingale from Proposition B is \(L^2\)-bounded with:

\[
\|q_1 x_1 q_1\|_2^2 + \sum_{n=2}^N \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2 \leq 2^{n_0+1}.
\]

**Proof.** We will use the elementary identity \((a - b)^2 = a^2 - b^2 + b(b - a) + (b - a)b\) for self-adjoint operators. With \(a = q_n x_n q_n\) and \(b = q_{n-1} x_{n-1} q_{n-1}\), we have for every \(n \geq 2\),

\[
\left\| q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1} \right\|^2 = \tau \left( (q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2 \right) \\
+ 2 \tau \left( q_{n-1} x_{n-1} q_{n-1} [q_{n-1} x_{n-1} q_{n-1} - q_n x_n q_n] \right) \\
= \tau \left( (q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2 \right) \\
+ 2 \tau \left( q_{n-1} x_{n-1} q_{n-1} [q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1} (q_n x_n q_n)] \right).
\]

By Proposition 3.2(c), \(\|q_{n-1} x_{n-1} q_{n-1}\|_\infty \leq 2^{n_0}\). Moreover, since the sequence \((q_n x_n q_n)_{n=1}^\infty\) is a supermartingale, \(q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1} (q_n x_n q_n) \geq 0\). Therefore, we get for every
\[ n \geq 2, \]
\[
\|q_n x_n q_n - x_{n-1} x_{n-1} q_{n-1}\|_2^2 \leq \tau \left( (q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2 \right) + 2^{n_0 + 1} \tau \left( q_{n-1} x_{n-1} q_{n-1} - E_{n-1}(q_n x_n q_n) \right)
\]

Thus the proof is complete.

Now, we take the summation over \( 1 \leq n \leq N \), we can conclude that
\[
\|q_1 x_1 q_1\|_2^2 + \sum_{n=2}^{N} \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2
\]
\[
\leq \|q_1 x_1 q_1\|_2^2 + \sum_{n=2}^{N} (\|q_n x_n q_n\|_2^2 - \|q_{n-1} x_{n-1} q_{n-1}\|_2^2) + 2^{n_0 + 1} \sum_{n=2}^{N} \tau \left( q_{n-1} x_{n-1} q_{n-1} - q_n x_n q_n \right)
\]
\[
= \|q_1 x_1 q_1\|_2^2 + \|q_N x_N q_N\|_2^2 - \|q_1 x_1 q_1\|_2^2 + 2^{n_0 + 1} \tau \left( (q_1 - q_N) x_N \right)
\]
\[
\leq 2^{n_0} \tau \left( q_N x_N \right) + 2^{n_0 + 1} \tau \left( (q_1 - q_N) x_N \right)
\]
\[
\leq 2^{n_0 + 1}.
\]

Thus the proof is complete. \( \square \)

We are now in a position to conclude the proof of the weak-type inequality (3.5) for the case \( \lambda = 2^{n_0} \). This is accomplished by applying successively Proposition A, Proposition B, and Proposition C above. Indeed,
\[
\tau \left( \chi_{(2^{n_0}, \infty)}(S_{C,N}(y)) \right) \leq \alpha^{-1} \tau \left( \chi_{(\beta^{4^{n_0}}, \infty)}(\gamma) \right) + 2(1 - \alpha)^{-1} 2^{-n_0}
\]
\[
\leq \alpha^{-1} \left[ 2\alpha^{-1} \beta^{-2} 4^{-n_0} \|q_1 x_1 q_1\|_2^2 + \sum_{n=1}^{N} \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2 \right] + K_1 2^{-n_0}
\]
\[
+ 2(1 - \alpha)^{-1} 2^{-n_0}
\]
\[
\leq \alpha^{-1} \left[ 2\alpha^{-1} \beta^{-2} 4^{-n_0} 2^{n_0 + 1} + K_1 2^{-n_0} \right] + 2(1 - \alpha)^{-1} 2^{-n_0}
\]
\[
= \left[ 4\alpha^{-2} \beta^{-2} + \alpha^{-1} K_1 + 2(1 - \alpha)^{-1} \right] 2^{-n_0}.
\]

If we set \( C_0 := \inf \{ 4\alpha^{-2} \beta^{-2} + \alpha^{-1} K_1 + 2(1 - \alpha)^{-1}; \alpha \in (0, 1), \beta \in (0, 1) \} \) then
\[
\tau \left( \chi_{(2^{n_0}, \infty)}(S_{C,N}(y)) \right) \leq C_0 2^{-n_0}.
\]

Hence taking the limit as \( N \) tends to \( \infty \), inequality (3.5) is verified for \( \lambda = 2^{n_0} \).

\( \diamond \) Assume now the more general case that \( 1 < \lambda < \infty \).
Fix \( n_0 \geq 0 \) such that \( 2^{n_0} < \lambda \leq 2^{n_0+1} \). We clearly have,
\[
\chi_{(\lambda,\infty)}(S_C(y)) \leq \chi_{(2^{n_0},\infty)}(S_C(y)).
\]
From the previous case, we can deduce,
\[
\tau \left( \chi_{(\lambda,\infty)}(S_C(y)) \right) \leq C_0 2^{-n_0} = 2C_0 2^{-(n_0+1)} \leq 2C_0 \lambda^{-1}.
\]
Hence inequality (3.5) is verified for \( \lambda \geq 1 \) with \( C_1 = 2C_0 \).

\[\diamond\] For the case \( 0 < \lambda \leq 1 \), we note that since \( \tau \) is normalized, \( \tau \left( \chi_{(\lambda,\infty)}(S_C(y)) \right) \leq 1 \). In particular, \( \tau \left( \chi_{(\lambda,\infty)}(S_C(y)) \right) \leq \lambda^{-1} \). Hence inequality (3.5) is satisfied with a constant equals to 1.

Combining the two cases \( \lambda \geq 1 \) and \( \lambda < 1 \), we can now conclude that
\[
\|dy\|_{L^1,\infty(M;\ell_2^R)} \leq C_1.
\]
From the similarity of \(|dy_n|^2\) and \(|dz_n|^2\) demonstrated in Lemma 3.4, we have
\[
\|dy\|_{L^1,\infty(M;\ell_2^R)} + \|dz\|_{L^1,\infty(M;\ell_2^R)} \leq 2C_1 = K_0.
\]
This completes the proof of Theorem 3.1(iv) for the case of normalized positive martingales.

The full generality as stated in the theorem is obtained with \( K = 8K_0 \) by writing the martingale as linear combinations of four positive martingales and normalization. Details are left to the interested reader. \( \Box \)

Recall that in general, triangular truncations are only of weak-type \((1,1)\) (see for instance, [15, Theorem 1.4]). The restriction to \( L^2 \)-bounded martingales is needed in order to verify that the sequences \((dy_n)_{n=1}^\infty\) and \((dz_n)_{n=1}^\infty\) are martingale difference sequences. This was possible from the boundedness of the triangular truncations. If we replace \( L^2(M,\tau) \) by any symmetric space of measurable operators on which triangular truncations are bounded, then \((dy_n)_{n=1}^\infty\) and \((dz_n)_{n=1}^\infty\), as constructed in equation (3.3), are still martingale difference sequences. However, the corresponding martingales may not be bounded. Before proceeding, we need to recall the notion of Boyd indices [28, p. 130]. Let \( E \) be a rearrangement invariant Banach function space on \([0,1)\). For \( s > 0 \), the dilation operator \( D_s : E \to E \) is defined by setting for any \( f \in E \),
\[
D_s f(t) = \begin{cases} 
  f(t/s), & t \leq \min(1,s) \\
  0, & s < t < 1 \ (s < 1). 
\end{cases}
\]
The lower and upper Boyd indices of \( E \) are defined by
\[
\underline{\alpha}_E := \lim_{s \to 0^+} \frac{\log \|D_s\|}{\log s}, \quad \overline{\alpha}_E := \lim_{s \to \infty} \frac{\log \|D_s\|}{\log s}.
\]
It is well known that \( 0 \leq \underline{\alpha}_E \leq \overline{\alpha}_E \leq 1 \) and if \( E = L^p[0,1] \) for \( 1 \leq p \leq \infty \) then \( \underline{\alpha}_E = \overline{\alpha}_E = 1/p \). If \( 0 < \underline{\alpha}_E \leq \overline{\alpha}_E < 1 \), we shall say that \( E \) has non-trivial Boyd indices. From [14, Theorem 3.3], we can state:
Theorem 3.8. There is an absolute positive constant $K > 0$ such that if $E$ be a rearrangement invariant Banach function space on $[0, 1]$ with the Fatou property and has non-trivial Boyd indices, and $x = (x_n)_{n=1}^{\infty}$ is a $L^1$-bounded martingale with $x_n \in E(\mathcal{M}, \tau)$ for all $n \geq 1$, then there exist two sequences $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ such that:

(α) for every $n \geq 1$, $x_n = y_n + z_n$;
(β) $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are martingales (not necessarily bounded);
(γ) $\|dy\|_{L^1(\mathcal{M}, F^2)} + \|dz\|_{L^1(\mathcal{M}, F^2)} \leq K \|x\|_1$.

Assume now that $\mathcal{M}$ is hyperfinite and $(\mathcal{M}_n)_{n=1}^{\infty}$ is a filtration consisting of finite dimensional von Neumann subalgebras of $\mathcal{M}$, then the above restriction is no longer needed. In fact, in the case where the $\mathcal{M}_n$’s are finite dimensional then for every $n \geq 2$, the mutually disjoint sequence $(p_{i,n-1})_{i \geq 0} \subset \mathcal{M}_{n-1}$ (used in the proof of Theorem 3.1) is a finite sequence. Therefore, the truncations used in the construction of the sequences $(dy_n)_{n=1}^{\infty}$ and $(dz_n)_{n=1}^{\infty}$ are done with finite sets of mutually disjoint projections and consequently, is bounded in $L^1(\mathcal{M}, \tau)$ (but not necessarily with uniform bound). In this particular case, we can state the following result as a complete non-commutative analogue of Theorem 0.1:

Theorem 3.9. There is an absolute constant $K$ such that if $\mathcal{M}$ is a finite hyperfinite von Neumann algebra and $(\mathcal{M}_n)_{n=1}^{\infty}$ is a filtration in $\mathcal{M}$ consisting of finite dimensional von Neumann subalgebras, then for every $L^1$-bounded martingale $x = (x_n)_{n=1}^{\infty}$, there exist two sequences $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ such that:

(α) for every $n \geq 1$, $x_n = y_n + z_n$;
(β) $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are $L^1$-martingales (not necessarily $L^1$-bounded);
(γ) $\|dy\|_{L^1(\mathcal{M}, F^2)} + \|dz\|_{L^1(\mathcal{M}, F^2)} \leq K \|x\|_1$.

Remark 3.10. Theorem 3.1 and Theorem 3.9 can be extended to square functions of non-commutative submartingales and non-commutative supermartingales. In this case, the decompositions are with submartingales (respectively, supermartingales). Details of such extension are done with just notational adjustments of the proof of [42, Corollary 2.11] and are left to the interested reader.

4. Generalization to the semi-finite case

In this section, we will consider the case where $\mathcal{M}$ is no longer assumed to be finite. We can extend Theorem 3.1 to the more general semi-finite case as follows:

Theorem 4.1. There is an absolute constant $M > 0$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale that is bounded in $L^2(\mathcal{M}, \tau) \cap L^1(\mathcal{M}, \tau)$, then there exist two sequences $v = (v_n)_{n=1}^{\infty}$ and $w = (w_n)_{n=1}^{\infty}$ such that:

(i) $(v_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are $L^2$-bounded martingales;
(ii) for every $n \geq 1$, $x_n = v_n + w_n$;
(iii) $\|dv\|_{L^2(\mathcal{M}, F^2)} + \|dw\|_{L^2(\mathcal{M}, F^2)} \leq 2 \|x\|_2$;
\[
(iv) \|dv\|_{L^{1,\infty}(\mathcal{M};\mathcal{F})} + \|dw\|_{L^{1,\infty}(\mathcal{M};\mathcal{F})} \leq M\|x\|_1.
\]

We will only outline the adjustments needed for the proof of Theorem 3.1 to cover the semi-finite case. For this, we consider the case where \( x = (x_n)_{n=1}^\infty \) is a positive martingale and \( \|x\|_1 = 1 \) (the general case follows from this case as noted at the end of the proof of Theorem 3.1). We will use the same notation as in the construction in Section 2 and Section 3. In particular, \((dy_n)_{n=1}^\infty\) and \((dz_n)_{n=1}^\infty\) are martingale difference sequences defined as in (3.3).

We remark first that the fact that the trace \( \tau \) being normalized, was used only to verify inequality (3.5) when \( 0 < \lambda < 1 \). That is, the proof of inequality (3.5) when \( \lambda \geq 1 \) still applies for the semi-finite case.

As already noted in [42], the only obstruction for proving inequality (3.5) without using the trace being normalized is the index \( j = 0 \) in the definition of \((dy_n)_{n=1}^\infty\). Indeed, if we set

\[
\begin{cases}
    ds_1 := \sum_{j=1}^\infty \sum_{i \leq j} p_{i,1} dx_1 p_{j,1}; \\
    ds_n := \sum_{j=1}^\infty \sum_{i \leq j} p_{i,n-1} dx_n p_{j,n-1} \quad \text{for } n \geq 2,
\end{cases}
\]

then \(dy_1 = p_{0,1} dx_1 p_{0,1} + ds_1\) and for \( n \geq 2\), \(dy_n = p_{0,n-1} dx_n p_{0,n-1} + ds_n\). Moreover, the sequence \((s_n)_{n=1}^\infty\) is a \(L^2\)-bounded martingale and satisfies the following weak-type inequality:

\textbf{Proposition 4.2.} If \( K \) is the positive constant from Theorem 3.1, then

\[
\|ds\|_{L^{1,\infty}(\mathcal{M};\mathcal{F})} + \|dz\|_{L^{1,\infty}(\mathcal{M};\mathcal{F})} \leq K.
\]

\textit{Proof.} As noted above, the fact that \( \lambda \tau (\lambda(\lambda,\infty)(S_C(s))) \leq K \) when \( \lambda \geq 1 \) is done exactly as in the finite case.

For \( 0 < \lambda < 1 \), we note that \(|ds_1|\) is supported by the projection \( 1 - p_{0,1} \) and for \( n \geq 2 \), \(|ds_n|\) is supported by the projection \((1 - p_{0,n-1})\). As \( p_0 \leq p_{0,l} \) for every \( l \geq 1 \), it is clear that \( S_C(s) \) is supported by \((1 - p_0)\) and we claim that \( \tau(1 - p_0) \leq 2 \). This can be seen directly from the definition of \( p_0 \). Indeed, \( \tau(1 - p_0) = \tau(1 - \bigwedge_{k=0}^\infty q^{(2^k)}) \leq \sum_{k=0}^\infty \tau(1 - q^{(2^k)}) \leq \sum_{k=0}^\infty 2^{-k} = 2 \). It follows that, for \( 0 < \lambda < 1 \), \( \lambda \tau (\lambda(\lambda,\infty)(S_C(\gamma))) \leq 2 \). The same observation on support applies to \( dz \) as well and thus the proof of the proposition is complete. \( \square \)

From Proposition 4.2, it is clear that we only need to provide the “right” decomposition of the martingale difference sequence \((dy_n - ds_n)_{n=1}^\infty\). As in the construction of \((dy_n)_{n=1}^\infty\) and \((dz_n)_{n=1}^\infty\) in (3.3), we will decompose the projections \( p_0 \) and \( p_{0,n} \)’s into
pairwise disjoint sequence of projections. For \( n \geq 1 \) and \( i \geq 0 \), we set

\[
\begin{align*}
\{ e_{i,n} \} &= i \bigg\{ \bigg( q_n^{(2-k)} \wedge p_{0,n} \bigg) - \bigg( q_n^{(2-k)} \wedge p_{0,n} \bigg); \\
\{ e_{i-1,n} \} &= i \bigg\{ \bigg( q_n^{(2-k)} \wedge p_{0,n} \bigg); \\
\{ e_{i-1} \} &= i \bigg\{ \bigg( q_n^{(2-k)} \wedge p_{0} \bigg).
\end{align*}
\]

Similarly,

\[
\begin{align*}
\{ e_{-1,n} \} &= i \bigg\{ \bigg( q_n^{(2-k)} \wedge p_{0,n} \bigg); \\
\{ e_{-1} \} &= i \bigg\{ \bigg( q_n^{(2-k)} \wedge p_{0} \bigg).
\end{align*}
\]

**Remarks 4.3.** We have the following immediate properties:

(a) For each \( n \geq 1 \), \( (e_{i,n})_{i=1}^{\infty} \) is a sequence of disjoint projections and for \( m \geq 1 \),

\[
\sum_{i=1}^{m} e_{i,n} = p_{0,n} - \bigwedge_{k=0}^{m+1} (q_n^{(2-k)} \wedge p_{0,n}) + \bigwedge_{k=0}^{\infty} (q_n^{(2-k)} \wedge p_{0,n}).
\]

In particular,

\[
\sum_{i=1}^{\infty} e_{i,n} = p_{0,n};
\]

(b) For every \( m \geq 1 \), \( \sum_{i=m}^{\infty} e_{i,n} = \bigwedge_{k=0}^{m+1} (q_n^{(2-k)} \wedge p_{0,n}) - \bigwedge_{k=0}^{\infty} (q_n^{(2-k)} \wedge p_{0,n}). \)

In particular, \( \sum_{k=m}^{\infty} e_{i,n} \leq q_n^{(2-m)}. \)

It is clear from Remark 4.3 that for every \( n \geq 2 \),

\[
\sum_{j=-1}^{\infty} \sum_{i=1}^{\infty} e_{i,n-1} d x_{n} e_{j,n-1} = p_{0,n-1} d x_{n} p_{0,n-1}.
\]

The decomposition of \( (d y_{n} - d s_{n})_{n=1}^{\infty} \) is done as in (3.3), using the triangular truncation

\[
\begin{align*}
\{ d \Xi_{1} \} &= \sum_{j=-1}^{\infty} \sum_{i=j}^{\infty} e_{i,1} d x_{1} e_{j,1}; \\
\{ d \Xi_{n} \} &= \sum_{j=-1}^{\infty} \sum_{i=j}^{\infty} e_{i,n-1} d x_{n} e_{j,n-1} \quad \text{for } n \geq 2; \\
\{ d \Psi_{1} \} &= \sum_{j=-1}^{\infty} \sum_{i=j}^{\infty} e_{i,1} d x_{1} e_{j,1}; \\
\{ d \Psi_{n} \} &= \sum_{j=-1}^{\infty} \sum_{i=j}^{\infty} e_{i,n-1} d x_{n} e_{j,n-1} \quad \text{for } n \geq 2.
\end{align*}
\]

Following the same line of argument used for the martingales \( (y_{n})_{n=1}^{\infty} \) and \( (z_{n})_{n=1}^{\infty} \), one can easily verify that \( (d \Xi_{n})_{n=1}^{\infty} \) and \( (d \Psi_{n})_{n=1}^{\infty} \) are martingale difference sequences.
Moreover, \( p_{0,1} dx_1 p_{0,1} = d\Xi_1 + d\Psi_1 \) and for every \( n \geq 2 \),
\[
p_{0,n-1} dx_n p_{0,n-1} = d\Xi_n + d\Psi_n.
\]

Theorem 4.1 can be deduced from the following property of the two martingales \( \Psi \) and \( \Xi \).

**Proposition 4.4.** There is a numerical constant \( C > 0 \) with:
\[
\|d\Psi\|_{L^1,\infty(\mathcal{M};l_2)} + \|d\Xi\|_{L^1,\infty(\mathcal{M};l_2)} \leq C.
\]

Indeed, if Proposition 4.4 is verified, then it is enough to set for \( n \geq 1 \),
\[
v_n = \Psi_n + s_n
\]
and \( w_n = \Xi_n + z_n \) and Theorem 4.1(iv) would follow immediately from Proposition 4.2 and Proposition 4.4.

**Sketch of the proof of Proposition 4.4.** First, we remark that as in Lemma 3.4,
\[
|d\Psi_n|^2 \quad \text{and} \quad |d\Xi_n|^2
\]
are of the same form. Therefore, as in the finite case, it is enough to verify that for every \( 0 < \lambda < \infty \),
\[
\lambda \tau \left( (\chi_{(\lambda,\infty)}(S_C(\Psi))) \right) \leq C. \tag{4.5}
\]

We will divide the proof into two cases.

\( \Diamond \) **Case 1:** \( \lambda \geq 1 \). For \( N \geq 1 \), one can verify as in Lemma 3.4 that
\[
S_{C,N}^2(\Psi) = \sum_{l=-1}^{\infty} \sum_{j=-1}^{\infty} \sum_{i \geq \max(l,j)} e_{i,j} d x_1 e_{i,j} d x_1 e_{i,j} + \sum_{n=2}^{N} \sum_{l=-1}^{\infty} \sum_{j=-1}^{\infty} \sum_{i \geq \max(l,j)} e_{l,n-1} d x_n e_{i,n-1} d x_n e_{i,n-1}.
\]

As \( \sum_{j=-1}^{\infty} e_{j,n-1} = p_{0,n-1} \leq q_{n-1}^{(1)} \), we can estimate \( S_{C,N}^2(\Psi) \leq \tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\gamma}_3 \) where the operators \( \tilde{\gamma}_1, \tilde{\gamma}_2, \) and \( \tilde{\gamma}_3 \) are defined as follows:
\[
\tilde{\gamma}_1 := \left| \sum_{j=-1}^{\infty} \sum_{i > j} e_{i,j} d x_1 e_{i,j} \right|^2 + 2 \sum_{n=2}^{N} \sum_{j=-1}^{\infty} \sum_{i > j} e_{i,n-1} d x_n (q_{n}^{(1)} - q_{n-1}^{(1)}) e_{j,n-1} \right|^2
\]
\[
\tilde{\gamma}_2 := 4 \sum_{n=2}^{N} \sum_{j=-1}^{\infty} \sum_{i > j} e_{i,n-1} d x_n (q_{n}^{(1)} - q_{n-1}^{(1)}) e_{j,n-1} \right|^2
\]
\[
\tilde{\gamma}_3 := 4 \sum_{n=2}^{N} \sum_{j=-1}^{\infty} \sum_{i > j} e_{i,n-1} (q_{n-1}^{(1)} - q_{n}^{(1)}) d x_n (q_{n}^{(1)} - q_{n-1}^{(1)}) e_{j,n-1} \right|^2.
\]
Using the splitting technique from Lemma 1.2, we can deduce that for every \( \alpha \in (0, 1) \) and \( \beta \in (0, 1) \):
\[
\tau \left( \chi_{(\lambda, \infty)}(S_{C,N}(\Psi)) \right) = \tau \left( \chi_{(\lambda^2, \infty)}(S_{C,N}^2(\Psi)) \right)
\leq \alpha^{-1} \tau \left( \chi_{(\beta \lambda^2, \infty)}(\gamma_1) \right) + \alpha^{-1} (1 - \alpha)^{-1} \tau \left( \chi_{(1 - \beta)\lambda^2, \infty)}(\gamma_2) \right)
+ (1 - \alpha)^{-2} \tau \left( \chi_{(\beta^2 \lambda^2, \infty)}(\gamma_3) \right)
\]
\[
= IV + V + VI.
\]
We can estimate \( IV, V, \) and \( VI \) separately following the proofs of Lemma 3.5, Lemma 3.6, and Lemma 3.7 to deduce that there are constants \( A_1 \) and \( A_2 \) (depending only on \( \alpha \) and \( \beta \)) such that
\[
\tau \left( \chi_{(\lambda, \infty)}(S_{C,N}(\Psi)) \right) \leq A_1 \lambda^{-2} (\|q_1^{(1)} x_1 q_1^{(1)}\|_2^2 + \sum_{n=2}^{\infty} \|q_n^{(1)} x_n q_n^{(1)} - q_{n-1}^{(1)} x_{n-1} q_{n-1}^{(1)}\|_2^2)
+ A_2 \lambda^{-1}.
\]
Now, we can apply Proposition C with \( n_0 = 0 \) to get
\[
\|q_1^{(1)} x_1 q_1^{(1)}\|_2^2 + \sum_{n=2}^{\infty} \|q_n^{(1)} x_n q_n^{(1)} - q_{n-1}^{(1)} x_{n-1} q_{n-1}^{(1)}\|_2^2 \leq 2.
\]
Combining the last two inequalities, we conclude that
\[
\tau \left( \chi_{(\lambda, \infty)}(S_{C}(\Psi)) \right) \leq 2 A_1 \lambda^{-2} + A_2 \lambda^{-1}
\]
and since \( \lambda \geq 1 \), (4.5) follows.

\( \diamond \) Case 2: For \( \lambda < 1 \), we will consider the special case \( \lambda = 2^{-n_0} \) for some \( n_0 \geq 1 \). Consider the following projection
\[
f_{n_0} = \sum_{i=n_0}^{\infty} e_i,
\]
and the operator
\[
\varphi_0 = \left| \sum_{j=0}^{\infty} \sum_{i \geq j} e_{i,1} dx_1 e_{j,1} \right|^2 + \sum_{n=2}^{N} \left| \sum_{i=n_0}^{\infty} \sum_{i \geq j} e_{i,n-1} dx_n e_{j,n-1} \right|^2.
\]
It is easy to verify that \( \tau(1 - f_{n_0}) \leq 2^{n_0+1} \). Moreover, \( f_{n_0} S_{C,N}^2(\Psi) f_{n_0} \leq f_{n_0} \varphi_0 f_{n_0} \)
This can be seen as follows: first, note that
\[
f_{n_0} S_{C,N}^2(\Psi) f_{n_0} = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i \geq \max(l,j)} f_{n_0} e_{i,1} dx_1 e_{i,1} dx_1 e_{j,1} f_{n_0}
+ \sum_{n=2}^{N} \sum_{l=0}^{\infty} \sum_{j \geq \max(l,j)} f_{n_0} e_{i,n-1} dx_n e_{i,n-1} dx_n e_{j,n-1} f_{n_0}.
\]
Remark that if \( s \geq 1 \) and \( l < n_0 \), then \( f_{l_0}e_{l,s} = e_{l,s}f_{n_0} = 0 \). In fact, we note that as \( e_{l,s} = \bigwedge_{k=0}^{l} (q_s^{(2^{-k})} \land p_{0,s}) - \bigwedge_{k=0}^{l+1} (q_s^{(2^{-k})} \land p_{0,s}) \), \( q_s^{(2^{-k})} \leq q_s^{(2^{-k})} \) for all \( k \geq 1 \) and \( p_0 \leq p_{0,n} \), it is clear that \( f_{n_0} = \bigwedge_{k=0}^{n_0+1} (q_s^{(2^{-k})} \land p_0) - \bigwedge_{k=0}^{\infty} (q_s^{(2^{-k})} \land p_0) \) is a subprojection of \( \bigwedge_{k=0}^{l+1} (q_s^{(2^{-k})} \land p_{0,s}) \) when \( l < n_0 \) and by the definition of \( e_{l,s} \), it follows that \( f_{n_0} \perp e_{l,s} \). Therefore,

\[
f_{n_0}S_{C,N}^2(\Psi)f_{n_0} = \sum_{l=n_0}^{\infty} \sum_{j=n_0}^{\infty} \sum_{i \geq \max(l,j)} f_{n_0}e_{l,1}dx_1 e_{i,1}dx_1 e_{j,1}f_{n_0}
\]

\[
+ \sum_{n=2}^{N} \sum_{l=n_0}^{\infty} \sum_{j=n_0}^{\infty} \sum_{i \geq \max(l,j)} f_{n_0}e_{l,n-1}dx_n e_{i,n-1}dx_n e_{j,n-1}f_{n_0}
\]

\[
= f_{n_0} \varphi_0 f_{n_0}.
\]

Write \( S_{C,N}(\Psi)^2 = f_{n_0}S_{C,N}^2(\Psi)f_{n_0} + f_{n_0}S_{C,N}^2(\Psi)(1 - f_{n_0}) + (1 - f_{n_0})S_{C,N}^2(\Psi) \). Using the splitting techniques as in Proposition A, we can make the following reduction:

**Lemma 4.5.** For \( \alpha \in (0, 1) \) and \( \beta \in (0, 1) \),

\[
\tau(\chi_{(2^{-n_0}, \infty)}(S_C(\Psi))) \leq \alpha^{-1}\tau(\chi_{(\beta 2^{-n_0}, \infty)}(\varphi_0)) + (1 - \alpha)^{-1}2^{n_0}.
\]

From Remark 4.3(iii), \( \sum_{i=n_0}^{\infty} e_{i,n-1} \leq q_n^{(2^{-n_0})} \), and therefore as above, we have \( \varphi_0 \leq \varphi_1 + \varphi_2 + \varphi_3 \) where

\[
\varphi_1 := \left| \sum_{j=n_0}^{\infty} \sum_{i \geq j} e_{i,1}dx_1 e_{j,1} \right|^2 + 2 \sum_{n=2}^{N} \left| \sum_{j=n_0}^{\infty} \sum_{i \geq j} e_{i,n-1}dx_n (q_n^{(2^{-n_0})}) e_{j,n-1} \right|^2
\]

\[
\varphi_2 := 4 \sum_{n=2}^{N} \left| \sum_{j=n_0}^{\infty} \sum_{i \geq j} e_{i,n-1}q_n^{(2^{-n_0})}dx_n (q_n^{(2^{-n_0})} - q_n^{(2^{-n_0})}) e_{j,n-1} \right|^2
\]

\[
\varphi_3 := 4 \sum_{n=2}^{N} \left| \sum_{j=n_0}^{\infty} \sum_{i \geq j} e_{i,n-1}(q_n^{(2^{-n_0})} - q_n^{(2^{-n_0})})dx_n (q_n^{(2^{-n_0})} - q_n^{(2^{-n_0})}) e_{j,n-1} \right|^2.
\]

Using the splitting technique from Lemma 1.2 a second time, we can deduce that for \( \alpha \in (0, 1) \) and \( \beta \in (0, 1) \):

\[
\tau(\chi_{(\beta 2^{-n_0}, \infty)}(\varphi_0)) \leq \alpha^{-1}\tau(\chi_{(\beta 2^{2^{-n_0}}, \infty)}(\varphi_1)) + \alpha^{-1}(1 - \alpha)^{-1}\tau(\chi_{(\beta 2^{(1-\beta)2^{-n_0}}, \infty)}(\varphi_2)) + (1 - \alpha)^{-2}\tau(\chi_{(\beta (1-\beta)2^{2^{-n_0}}, \infty)}(\varphi_3)) = A + B + C.
\]
We recall the definitions of martingale Hardy spaces. For $1 \leq p < \infty$, $\mathcal{H}^p_C(\mathcal{M})$ (respectively, $\mathcal{H}^p_R(\mathcal{M})$) is defined as the set of all $L^p$-martingales $x$ with respect to a filtration $(\mathcal{M}_n)_{n\geq 1}$ such that $dx \in L^p(\mathcal{M}; l^2_C)$ (respectively, $L^p(\mathcal{M}; l^2_R)$), and set

$$\|x\|_{\mathcal{H}^p_C(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l^2_C)} \quad \text{and} \quad \|x\|_{\mathcal{H}^p_R(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l^2_R)}.$$ 

Equipped with the previous norms, $\mathcal{H}^p_C(\mathcal{M})$ and $\mathcal{H}^p_R(\mathcal{M})$ are Banach spaces. The Hardy space of non-commutative martingale is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}^p_C(\mathcal{M}) + \mathcal{H}^p_R(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \inf \left\{ \|y\|_{\mathcal{H}^p_C(\mathcal{M})} + \|z\|_{\mathcal{H}^p_R(\mathcal{M})} \right\}$$

where the infimum is taken over all $y$ and $z$ with $x = y + z$, $y \in \mathcal{H}^p_C(\mathcal{M})$, and $z \in \mathcal{H}^p_R(\mathcal{M})$; and if $2 \leq p < \infty$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}^p_C(\mathcal{M}) \cap \mathcal{H}^p_R(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \max \left\{ \|x\|_{\mathcal{H}^p_C(\mathcal{M})}, \|x\|_{\mathcal{H}^p_R(\mathcal{M})} \right\}.$$ 

The aim of this section is to point out that using interpolation techniques, Theorem 3.1 provides a new proof of the non-commutative Burkholder-Gundy inequalities.

**Theorem 5.1** ([37]). Let $1 < p < \infty$. Let $x = (x_n)_{n=1}^\infty$ be an $L^p$-martingale. Then $x$ is bounded in $L^p(\mathcal{M}, \tau)$ if and only if $x$ belongs to $\mathcal{H}^p(\mathcal{M})$. If this is the case then,

$$\alpha_p^{-1}\|x\|_{\mathcal{H}^p(\mathcal{M})} \leq \|x\|_p \leq \beta_p\|x\|_{\mathcal{H}^p(\mathcal{M})}. \quad (BG_p)$$
For Clifford martingales, some particular cases of Theorem 5.1 was also obtained in [6]. Note that up until now, both the original proof in [37] and the alternative proof from [40] made use of the non-commutative Stein inequality (also proved in [37]) in order to achieve the decomposition into two martingales, as described in the definition of $\| \cdot \|_{H^p(M)}$ for $1 < p < 2$. Theorem 3.1 allows us to avoid the use of the non-commutative Stein inequality. This approach, which is probably more complex than the existing proofs, produces better estimates of the constants involved. Indeed, it allows us to deduce the optimal order of growth for the constant $\alpha_p$ (which is the same as in the case of commutative case) when $p \to 1$. This solves a question left open in [24] (see the remark after the main theorem of [24]).

We will write $a_p \approx b_p$ as $p \to p_0$ to abbreviate the statement that there are two absolute positive constants $K_1$ and $K_2$ such that

$$K_1 \leq \frac{a_p}{b_p} \leq K_2$$

for $p$ close to $p_0$.

The following theorem is the principal result of this section.

**Theorem 5.2.** We have the following estimates for the best constants in $(BG_p)$:

(i) $\alpha_p \approx (p - 1)^{-1}$ as $p \to 1$;

(ii) $\alpha_p \approx p$ as $p \to \infty$;

(iii) $\beta_p \approx 1$ as $p \to 1$;

(iv) $\beta_p \approx p$ as $p \to \infty$.

These are the optimal orders of growth of the constants $\alpha_p$ and $\beta_p$.

**Remarks 5.3.** (a) Compared with the commutative setting, the optimal orders of $\beta_p$ are the same as its commutative counterpart. However, $\alpha_p$ behaves differently. In the commutative case, $\alpha_p \approx \sqrt{p}$ when $p \to \infty$, and $\alpha_p \approx (p - 1)^{-1}$ when $p \to 1$.

(b) The only new result here is (i). The optimal orders as stated in (ii), (iii), and (iv) were obtained by combining results from [25], [24], and [40]. We also note that for the special case of even integers, (ii) was established in [36] for more general sequences called $p$-orthogonal sums.

We will use the real interpolation, namely the $J$-method. We will review the general theory of real interpolation. Our main reference for interpolation is the book of Bergh and Löfström [3] and the recent survey [27].

A pair of (quasi)-Banach spaces $(E_0, E_1)$ is called a compatible couple if they embed continuously into some topological vector space $X$. This allows us to consider the spaces $E_0 \cap E_1$ and $E_0 + E_1$ equipped with $\|x\|_{E_0 \cap E_1} = \max\{\|x\|_{E_0}, \|x\|_{E_1}\},$ $\|x\|_{E_0 + E_1} = \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x = x_0 + x_1, \ x_0 \in E_0, \ x_1 \in E_1\}$ respectively.

For a compatible couple $(E_0, E_1)$, we define for any $x \in E_0 \cap E_1$, and $t > 0$,

$$J(x, t; E_0, E_1) = \max\{\|x\|_{E_0}, t\|x\|_{E_1}\}.$$ 

If the compatible couple is clear from the context, we will simply write $J(x, t)$. 

To avoid dealing with measurability, we will be working with the discrete version of the J-method which we will now describe: for $0 < \theta < 1$ and $1 \leq p < \infty$, we denote by $\lambda^{\theta,p}$ the space of all sequences $(\alpha_\nu)_{\nu=-\infty}^{\infty}$ for which,

$$\|\alpha_\nu\|_{\lambda^{\theta,p}} = \left\{ \sum_{\nu \in \mathbb{Z}} (2^{-\nu \theta} |\alpha_\nu|)^p \right\}^{1/p} < \infty.$$ 

**Definition 5.4.** Let $(E_0, E_1)$ be a compatible couple and suppose that $0 < \theta < 1$, and $1 \leq p < \infty$. The interpolation space $(E_0, E_1)_{\theta,p,J}$ consists of elements $x \in E_0 + E_1$ which admits a representation:

$$x = \sum_{\nu \in \mathbb{Z}} u_\nu, \quad (\text{convergence in } E_0 + E_1), \quad (5.1)$$

with $u_\nu \in E_0 \cap E_1$ and such that

$$\|x\|_{\theta,p,J} = \inf \{ \|\{J(u_\nu, 2^\nu)\}\|_{\lambda^{\theta,p}} \} < \infty,$$

where the infimum is taken over all representations of $x$ as in (5.1).

For general information on interpolations of non-commutative spaces, we refer to [12] and [38, p. 1466].

**Proof of Theorem 5.2(i).** It is enough to consider positive $L^2$-bounded martingale $x = (x_n)_{n=1}^{\infty}$. Let $x_\infty \in L^2(\mathcal{M}, \tau)$ such that $x_n = \mathcal{E}_n(x_\infty)$ for every $n \geq 1$. Let $1 < p < 2$ and $0 < \theta < 1$ with $1/p = (1 - \theta) + \theta/2$. For $\varepsilon > 0$, fix $(u_\nu)_{\nu=-\infty}^{\infty}$ in $L^2(\mathcal{M}, \tau)$ such that

$$x_\infty = \sum_{\nu \in \mathbb{Z}} u_\nu \quad \text{and} \quad \|x_\infty\|_{\theta,p,J} + \varepsilon \geq \|\{J(u_\nu, 2^\nu)\}\|_{\lambda^{\theta,p}},$$

where the $J$-functional is relative to the interpolation couple $(L^1(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau))$.

For each $\nu \in \mathbb{Z}$, Theorem 3.1 guaranties the existence of an absolute constant $K > 0$, and two $L^2$-bounded martingales $y^{(\nu)}$ and $z^{(\nu)}$ such that:

(a) $\mathcal{E}_n(u_\nu) = y_n^{(\nu)} + z_n^{(\nu)}$ for all $n \geq 1$;

(b) $J(S_C(dy^{(\nu)}), t) \leq KJ(u_\nu, t)$ for every $t > 0$;

(c) $J(S_R(dz^{(\nu)}), t) \leq KJ(u_\nu, t)$ for every $t > 0$,

where the $J$-functionals in the left hand side of the inequalities in (b) and (c) above are taken relative to the interpolation couple $(L^{1,\infty}(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau))$. From this, we can deduce that,

$$\|\{J(S_C(dy^{(\nu)}), 2^\nu)\}\|_{\lambda^{\theta,p}} \leq K(\|x_\infty\|_{\theta,p,J} + \varepsilon).$$
Note from the definition of the \( J \)-functionals that,
\[
J(S_C(dy^{(\nu)}), 2^\nu) = \max \left\{ \|S_C(dy^{(\nu)})\|_{1,\infty}, 2^\nu \|S_C(dy^{(\nu)})\|_2 \right\}
\]
\[
= \max \left\{ \| \sum_n dy^{(\nu)}_n \otimes e_n, 1 \|_{L^1(\mathcal{M}\otimes B(l^2))}, 2^\nu \| \sum_n dy^{(\nu)}_n \otimes e_n, 1 \|_{L^2(\mathcal{M}\otimes B(l^2))} \right\}
\]
\[
= J \left( \sum_n dy^{(\nu)}_n \otimes e_n, 1, 2^\nu; L^1(\mathcal{M}\otimes B(l^2)), L^2(\mathcal{M}\otimes B(l^2)) \right).
\]

where \((e_{i,j})_{i,j}\) denotes the usual base of \( B(l^2) \), that is, \((dy^{(\nu)}_n)_n\) is viewed as a column vector with entries from \( L^2(\mathcal{M}, \tau) \). This implies that
\[
\left\| \left\{ J(\sum_n dy^{(\nu)}_n \otimes e_n, 1, 2^\nu) \right\} \right\|_{L^\theta,p} \leq K(\| x_\infty \|_{\theta,p;\downarrow} + \varepsilon). \tag{5.2}
\]

Let \( S \) be a finite subset of \( \mathbb{Z} \). By the definition of \( \| \cdot \|_{\theta,p;\downarrow} \),
\[
\left\| \sum_{\nu \in S} \sum_n dy^{(\nu)}_n \otimes e_n, 1 \right\|_{L^1(\mathcal{M}\otimes B(l^2)), L^2(\mathcal{M}\otimes B(l^2))}_{\theta,p;\downarrow} \leq K(\| x_\infty \|_{\theta,p;\downarrow} + \varepsilon). \tag{5.3}
\]

Similar argument on \((z^{(\nu)})_\nu\) also gives,
\[
\left\| \sum_{\nu \in S} \sum_n dz^{(\nu)}_n \otimes e_n, 1 \right\|_{L^1(\mathcal{M}\otimes B(l^2)), L^2(\mathcal{M}\otimes B(l^2))}_{\theta,p;\downarrow} \leq K(\| x_\infty \|_{\theta,p;\downarrow} + \varepsilon). \tag{5.4}
\]

Set
\[
y := \sum_{\nu \in \mathbb{Z}} y^{(\nu)} \text{ and } z := \sum_{\nu \in \mathbb{Z}} z^{(\nu)}.
\]

Note that since the inequalities (5.2) and (5.3) are valid for arbitrary finite subset \( S \) of \( \mathbb{Z} \), \( y \) and \( z \) are well-defined (that is, the series are unconditionally convergent in the Banach space \([L^1(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau)]_{\theta,p;\downarrow}\)). Clearly, \( y \) and \( z \) are martingales with \( x = y + z \) and from (5.2) and (5.3), we have:
\[
\| S_C(dy) \|_{\theta,p;\downarrow} + \| S_R(dz) \|_{\theta,p;\downarrow} \leq 2K(\| x_\infty \|_{\theta,p;\downarrow} + \varepsilon). \tag{5.4}
\]

To conclude the proof, we note from the general equivalence theorem on real interpolations that the same statement as in (5.4) can be made with any real interpolation method (with possible change on the absolute constant). It is well known that
\[
[L^1(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau)]_{\theta,p} = L^p(\mathcal{M}, \tau) \quad \text{(with equivalent norms)}
\]
and
\[
[L^1(\mathcal{M}, \tau), L^2(\mathcal{M}, \tau)]_{\theta,p} = L^p(\mathcal{M}, \tau) \quad \text{(with equivalent norms)}.
\]
From [38, Corollary 2.2, P. 1467], it is enough to track the order of the constants for the commutative case. Estimates on the order of growth of constants involved on the equivalent norms can be deduced from [22, Theorem 4.3] as follows: for \( f \in L^2 \),

\[
C(1 - \theta)^{-1/2} \|f\|_{L^p} \leq \|f\|_{\|L^1,\infty;L^2\|_{\theta,p}} \leq C^{-1}\theta^{-1/p}(1 - \theta)^{-1/p} \|f\|_{L^p}
\]  

(5.5)

and

\[
C\theta^{-1/p}(1 - \theta)^{-1/2} \|f\|_{L^p} \leq \|f\|_{\|L^1,\infty;L^2\|_{\theta,p}} \leq C^{-1}(1 - \theta)^{-1/p} \|f\|_{L^p}.
\]  

(5.6)

Combining (5.4), (5.5), and (5.6), we can conclude the existence of an absolute constant \( M > 0 \) such that:

\[
\|S_C(dy)\|_p + \|S_R(dz)\|_p \leq M(p - 1)^{-1}(|x|_p + \varepsilon).
\]

Taking the infimum over \( \varepsilon > 0 \), we get

\[
\|x\|_{\mathcal{H}^p(M)} \leq M(p - 1)^{-1} \|x\|_p,
\]

which shows that \( \alpha_p \leq M(p - 1)^{-1} \) for \( 1 < p < 2 \). The proof is complete. \( \square \)

**Remark 5.5.** The optimal orders as stated in Theorem 5.2 remain valid in the more general situation of Haagerup’s \( L^p \)-spaces using Haagerup’s approximation ([21]). This follows from a general deduction of the non-commutative Burkholder-Gundy inequalities from the finite case to the type III-case (with same constants) achieved by Junge and Xu (still unpublished notes).

For the next application, we recall the dual space of \( \mathcal{H}^p (1 < p < 2) \) studied in [25]. For \( 2 < q \leq \infty \), \( L^q_{C,MO}(\mathcal{M}) \) (\( MO \) stands for mean oscillation) is the space of all martingales \( x = (x_n)_n \) in \( L^2(\mathcal{M}, \tau) \) for which

\[
\|x\|_{L^q_{C,MO}(\mathcal{M})} = \sup_m \left\{ \sup_{n \leq m} \mathcal{E}_n \left( \sum_{k=n}^m |dx_k|^2 \right) \right\}^{1/2} < \infty.
\]

This was introduced as a non-commutative analogue of the \( q \)-norm of the classical sharp function. In the above, the suggestive notation introduced in [23] for the supremum is understood in the following sense: if \( 1 \leq r, r' \leq \infty \) and \( 1/r + 1/r' = 1 \) then for any sequence \( (a_n)_{n \geq 1} \) of positive operators in \( \mathcal{M} \),

\[
\left\{ \sup_n a_n \right\}_r = \sup \left\{ \sum_{n \geq 1} \tau(a_n b_n) : b_n \geq 0, \sum_{n \geq 1} b_n \right\}_r^r \leq 1.
\]

Similarly, \( L^q_{R,MO}(\mathcal{M}) \) is defined as the space of all martingales \( x \) such that \( x^* \in L^q_{C,MO}(\mathcal{M}) \), with norm given by \( \|x\|_{L^q_{R,MO}(\mathcal{M})} = \|x^*\|_{L^q_{C,MO}(\mathcal{M})} \) and as above,

\[
L^q_{MO}(\mathcal{M}) = L^q_{C,MO}(\mathcal{M}) \cap L^q_{R,MO}(\mathcal{M}),
\]

equipped with the usual intersection norm

\[
\|x\|_{L^q_{MO}(\mathcal{M})} = \max \left\{ \|x\|_{L^q_{C,MO}(\mathcal{M})}, \|x\|_{L^q_{R,MO}(\mathcal{M})} \right\}.
\]
Let $1 < p < 2$ and $p'$ the index conjugate to $p$. It was shown in [25, Theorem 4.1] that $(H^p(M))^* = L^{p'}(MO(M))$. As part of this characterization, they noted the inequality that for $2 < q < \infty$, there is a constant $\lambda_q > 0$ such that

$$\|a\|_q \leq \lambda_q' \|a\|_{L^{q}(MO(M))}, \quad a \in L^q(MO(M)).$$

By duality, we can answer a problem raised in [25, p. 972] using Theorem 5.2(i):

**Corollary 5.6.** $\lambda_q' \approx q$ as $q \to \infty$.

We end the paper with a short note on the class $L\log L$. Recall the Zygmund space $L\log L$. If $L^0(\Omega, \mathcal{F}, P)$ is the space of all (classes) of measurable functions on a given probability space $(\Omega, \mathcal{F}, P)$, the class $L\log L$ is defined by setting

$$L\log L = \left\{ f \in L^0(\Omega, \mathcal{F}, P); \int |f| \log^+ |f| \ dP < \infty \right\}.$$

Set $\|f\|_{L\log L} = \int |f| \log^+ |f| \ dP$. Note that $\| \cdot \|_{L\log L}$ is not a norm but is equivalent to a rearrangement invariant norm $\|f\|_{L\log L} = \int_0^1 f^*(t) \log(1/t) \ dt$.

Equipped with $\| \cdot \|_{L\log L}$, the spaces $L\log L$ is a rearrangement invariant Banach function space (see for instance [2, Theorem 6.4, pp. 246-247]) so its non-commutative analogue $L\log L(M, \tau)$ is well defined as described in Section 1. We note as in [40] that if a martingale $x$ is bounded in $L\log L(M, \tau)$ then it is uniformly integrable in $L^1(M, \tau)$ and therefore is of the form $x = (E_n(x_\infty))_{n=1}^\infty$ for some $x_\infty \in L\log L(M, \tau)$.

With $\alpha_p \approx (p - 1)^{-1}$ (when $p \to 1$), the elementary argument used in the proof of [40, Proposition 6.5] can be adjusted to deduce the following strengthening of [40, Proposition 6.5](which answers positively [40, Problem 6.4]).

**Theorem 5.7.** There is a constant $K > 0$ such that if $x = (x_n)_{n=1}^\infty$ is a martingale that is bounded in $L\log L(M, \tau)$, then

$$\|x\|_{L\log L(M, \tau)} \leq K + K \|x_\infty\|_{L\log L(M, \tau)}.$$

**Acknowledgements.** This project started during my visit to the Department of Mathematics at the Université de Franche-Comté in Spring 2003. I would like to express my gratitude to the department for its warm hospitality.

**References**

[1] C. Barnett, *Supermartingales on semifinite von Neumann algebras*, J. London Math. Soc. (2) 24 (1981), 175–181. MR 82h:46085
[2] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press Inc., Boston, MA, 1988. MR 89e:46001
[3] J. Bergh and J. Lofström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223. MR 58 #2349
[4] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. 37 (1966), 1494–1504. MR 34 #8456
[5] ______, Martingales and singular integrals in Banach spaces, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 233–269. MR 2003b:46009

[6] E. Carlen and P. Krée, On martingale inequalities in non-commutative stochastic analysis, J. Funct. Anal. 158 (1998), 475–508. MR 99g:81111

[7] V. I. Chilin and F. A. Sukochev, Symmetric spaces over semifinite von Neumann algebras, Dokl. Akad. Nauk SSSR 313 (1990), 811–815. MR 92a:46075

[8] I. Cuculescu, Martingales on von Neumann algebras, J. Multivariate Anal. 1 (1971), 17–27. MR 45 #4464

[9] N. Dang-Ngok, Pointwise convergence of martingales in von Neumann algebras, Israel J. Math. 34 (1979), 273–280 (1980). MR 81g:46091

[10] J. Dixmier, Formes linéaires sur un anneau d’opérateurs, Bull. Soc. Math. France 81 (1953), 9–39. MR 15,539a

[11] P. G. Dodds, T. K. Dodds, and B. de Pagter, Noncommutative Banach function spaces, Math. Z. 201 (1989), 583–597. MR 90j:46054

[12] ______, Remarks on noncommutative interpolation, Miniconference on Operators in Analysis (Sydney, 1989), Austral. Nat. Univ., Canberra, 1990, pp. 58–78. MR 91i:46073

[13] ______, Noncommutative Köthe duality, Trans. Amer. Math. Soc. 339 (1993), 717–750. MR 94a:46093

[14] P. G. Dodds, T. K. Dodds, B. de Pagter, and F. A. Sukochev, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, J. Funct. Anal. 148 (1997), 28–69. MR 98g:46098

[15] ______, Lipschitz continuity of the absolute value in preduals of semifinite factors, Integral Equations Operator Theory 34 (1999), 28–44. MR 2000d:46079

[16] J. L. Doob, Stochastic processes, John Wiley & Sons Inc., New York, 1953. MR 15,445b

[17] R. E. Edwards and G. I. Gaudry, Littlewood-Paley and multiplier theory, Springer-Verlag, Berlin, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90. MR 58 #29760

[18] T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, Pacific J. Math. 123 (1986), 269–300. MR 87h:46122

[19] A. M. Garsia, Martingale inequalities: Seminar notes on recent progress, W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam, 1973. Mathematics Lecture Notes Series. MR 56 #6844

[20] T. A. Gillespie, S. Pott, S. Treil, and A. Volberg, Logarithmic growth for matrix martingale transforms, J. London Math. Soc. (2) 64 (2001), 624–636. MR 2002k:47066

[21] U. Haagerup, Non-commutative integration theory, Lecture given at the Symposium in Pure Mathematics of the Amer. Math. Soc., Queens University, Kingston, Ontario, 1980.

[22] T. Holmstedt, Interpolation of quasi-normed spaces, Math. Scand. 26 (1970), 177–199. MR 54 #3440

[23] M. Junge, Doob’s inequality for non-commutative martingales, J. Reine Angew. Math. 549 (2002), 149–190. MR 2003k:46097

[24] M. Junge and Q. Xu, The optimal orders of growth of the best constants in some non-commutative martingale inequalities, preprint 2001.

[25] ______, Noncommutative Burkholder/Rosenthal inequalities, Ann. Probab. 31 (2003), no. 2, 948–995. MR 1 964 955

[26] K. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. I, Academic Press, New York, 1983, Elementary theory. MR 85j:46099

[27] N. Kalton and S. Montgomery-Smith, Interpolation of Banach spaces, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1131–1175. MR 1 999 193
[28] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. II, Springer-Verlag, Berlin, 1979, Function spaces. MR 81c:46001
[29] F. Lust-Piquard, Inégalités de Khintchine dans $C_p$ ($1 < p < \infty$), C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), 289–292. MR 87j:47032
[30] F. Lust-Piquard and G. Pisier, Noncommutative Khintchine and Paley inequalities, Ark. Mat. 29 (1991), 241–260. MR 94b:46011
[31] P. A. Meyer, Quantum probability for probabilists, Lecture Notes in Math., 1538, Springer-Verlag, Berlin, 1993. MR 94b:46011
[32] M. Musat, Interpolation between non-commutative $BMO$ and non-commutative $L^p$-spaces, J. Funct. Anal. 202 (2003), 195–225.
[33] F. Nazarov, G. Pisier, S. Treil, and A. Volberg, Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts, J. Reine Angew. Math. 542 (2002), 147–171. MR 2002m:47038
[34] E. Nelson, Notes on non-commutative integration, J. Funct. Anal. 15 (1974), 103–116. MR 50 #8102
[35] K. R. Parthasarathy, An introduction to quantum stochastic calculus, Monographs in Mathematics, vol. 85, Birkhäuser Verlag, Basel, 1992. MR 93g:81062
[36] G. Pisier, An inequality for $p$-orthogonal sums in non-commutative $L_p$, Illinois J. Math. 44 (2000), 901–923. MR 2001k:46101
[37] G. Pisier and Q. Xu, Non-commutative martingale inequalities, Comm. Math. Phys. 189 (1997), 667–698. MR 98m:46079
[38] ———, Non-commutative $L^p$-spaces, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1459–1517. MR 1 999 201
[39] N. Randrianantoanina, Hilbert transform associated with finite maximal subdiagon al algebras, J. Austral. Math. Soc. Ser. A 65 (1998), 388–404. MR 2000a:46109
[40] ———, Non-commutative martingale transforms, J. Funct. Anal. 194 (2002), 181–212. MR 2003m:46098
[41] ———, Sequences in non-commutative $L^p$-spaces, J. Operator Theory 48 (2002), 255–272. MR 2003b:46093
[42] ———, Square function inequalities for non-commutative martingales, Israel J. Math. 140 (2004), 333–365.
[43] M. Takesaki, Theory of operator algebras. I, Springer-Verlag, New York, 1979. MR 81e:46038
[44] D. V. Voiculescu, K. J. Dykema, and A. Nica, Free random variables, American Mathematical Society, Providence, RI, 1992. MR 94c:46133
[45] Q. Xu, Recent development on non-commutative martingale inequalities, preprint 2003.
[46] ———, Analytic functions with values in lattices and symmetric spaces of measurable operators, Math. Proc. Cambridge Philos. Soc. 109 (1991), 541–563. MR 92g:46036

Department of Mathematics, Miami University, Oxford, Ohio 45056 (USA)
E-mail address: randrin@muohio.edu