EXTENDED DOUBLE COVERS OF NON-SYMMETRIC ASSOCIATION SCHEMES OF CLASS 2

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Dedicated to the memory of Igor Faradžev

Abstract. In this paper, we give a method to construct non-symmetric association schemes of class 3 from non-symmetric association schemes of class 2. This construction is a non-symmetric analogue of the construction of Taylor graphs as an antipodal double cover of a complete graph. We also mention how our construction interact with doubling introduced by Pasechnik.

1. Introduction

The first eigenmatrix of a non-symmetric association scheme of class 2 on \( m \) points is given by

\[
\begin{pmatrix}
1 & \frac{m-1}{2} & \frac{m-1}{2} \\
1 & \frac{-1+\sqrt{-m}}{2} & \frac{-1-\sqrt{-m}}{2} \\
1 & \frac{-1-\sqrt{-m}}{2} & \frac{-1+\sqrt{-m}}{2}
\end{pmatrix},
\]

where \( m \equiv 3 \) (mod 4) (see, for example [8]). The digraphs defined by a nontrivial relation of such an association scheme are known as doubly regular tournaments, and the existence of such a digraph is equivalent to that of a skew Hadamard matrix. See [7] for details.

In this paper, it is shown that a non-symmetric association scheme with the first eigenmatrix (1) gives rise to an association scheme of class 3 with the first eigenmatrix

\[
\begin{pmatrix}
1 & m & m \\
1 & -\sqrt{-m} & -\sqrt{-m} & 1 \\
1 & -\sqrt{-m} & \sqrt{-m} & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

We call the resulting association scheme with the first eigenmatrix (2), the extended double cover of the original association scheme of class 2. The construction is analogous to that of Taylor graphs (see [2] Sect. 1.5)). In fact, the association scheme

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defined by a Taylor graph has the first eigenmatrix

\[
\begin{pmatrix}
1 & m & m & 1 \\
1 & \sqrt{m} & m & -1 \\
1 & -\sqrt{m} & -m & 1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

The extended double cover is a non-symmetric fission of a cocktail party graph, and self-dual. According to S. Y. Song [9, (5.3) Lemma], a self-dual non-symmetric fission of a complete multipartite graph has the first eigenmatrix

\[
\begin{pmatrix}
1 & k_1 & k_1 & k_2 \\
1 & -k_1 & -k_1 & -1 \\
1 & -\sqrt{2}k_1 & \sqrt{2}k_1 & -1 \\
1 & -\frac{k_2+1}{2} & \frac{k_2+1}{2} & k_2
\end{pmatrix},
\]

and our association scheme with the first eigenmatrix (2) is a special case where \(k_2 = 1\).

According to I. A. Faradžev, M. H. Klin, and M. E. Muzichuk [4, Theorem 2.6.6], D. Pasechnik invented a construction of a non-symmetric association scheme of class 2 on \(2m + 1\) points with the first eigenmatrix

\[
\begin{pmatrix}
1 & m & m & \frac{1}{2}m & \frac{1}{2}m \\
1 & -1+\sqrt{-(2m+1)} & \frac{1}{2}m & \frac{1}{2}m & -1-\sqrt{-(2m+1)} \\
1 & -1-\sqrt{-(2m+1)} & \frac{1}{2}m & \frac{1}{2}m & -1+\sqrt{-(2m+1)} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}m & \frac{1}{2}m
\end{pmatrix},
\]

provided that there exists an association scheme with first eigenmatrix (1). This construction leads to the doubling of skew-Hadamard matrices (see [7, Theorem 14]). We will give a direct description of the extended double cover of the doubling in Theorem 8.

One may wonder if an association scheme with first eigenmatrix (2) is related to a skew-Hadamard matrix. However, by [6], such an association scheme does not contain a (complex) Hadamard matrix in its Bose–Mesner algebra.

The organization of this paper is as follows. In Section 2 we introduce necessary notation and give useful properties of non-symmetric association schemes of class 2. In Section 3 we construct the adjacency matrices needed in our main theorem, and prove their multiplication formulas. In Section 4 we present our main results.

2. Preliminaries

For fundamentals of the theory of association schemes, we refer the reader to [1]. Let \(\mathfrak{X} = (X, \{R_i\}_i^{d=0})\) be a commutative association scheme of class \(d\) on \(n\) points. Let \(A_0, A_1, \ldots, A_d\) be the adjacency matrices of \(\mathfrak{X}\). The intersection numbers \(p_{i,j}^\ell\) are defined by

\[A_iA_j = \sum_{\ell=0}^dp_{i,j}^\ell A_\ell,\]

and the intersection matrices \(\{B_i\}_i^{d=0}\) are defined by \((B_i)_j^\ell = p_{i,j}^\ell\). The linear span \(\mathcal{A} = \langle A_0, A_1, \ldots, A_d\rangle\) is called the Bose–Mesner algebra of \(\mathfrak{X}\), and it has primitive
idempotents \( E_0 = \frac{1}{|X|} J, E_1, \ldots, E_d \). The first eigenmatrix \( P = (P_{i,j})_{0 \leq i, j \leq d} \) is defined by

\[
(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d) P,
\]

and \( Q = |X| P^{-1} \) is called the second eigenmatrix of \( \mathfrak{X} \). Then we have

\[
|X|(E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d) Q.
\]

Let \( k_i \ (i = 0, 1, \ldots, d) \) and \( m_i \ (i = 0, 1, \ldots, d) \) be the valencies and the multiplicities of \( \mathfrak{X} \), respectively. Then the intersection numbers \( p^\ell_{i,j} \) are given by

\[
(5) \quad p^\ell_{i,j} = \frac{1}{nk^\ell} \sum_{\nu=0}^{d} m_\nu P_{\nu,i} P_{\nu,j} P_{\nu,\ell}
\]

Now, assume that \( \mathfrak{X} \) is a non-symmetric association scheme of class 2 on \( m \) points. Then \( k_1 = k_2 = m_1 = m_2 = (m-1)/2 \). By (11) and (5) we have

\[
(6) \quad A_1^2 = \frac{m-3}{4} A_1 + \frac{m+1}{4} A_2,
\]

\[
(7) \quad A_2^2 = \frac{m+1}{4} A_1 + \frac{m-3}{4} A_2,
\]

\[
(8) \quad A_1 A_2 = \frac{m-1}{2} A_0 + \frac{m-3}{4} (A_1 + A_2).
\]

Then by (6), (7), and (8) we have

\[
(9) \quad A_1^2 + A_2^2 = \frac{m-1}{2} (A_1 + A_2),
\]

\[
(10) \quad J + 2A_1 A_2 = mA_0 + \frac{m-1}{2} (A_1 + A_2).
\]

3. Construction of adjacency matrices

**Definition 1.** Let \( \mathfrak{X} \) be a non-symmetric association scheme of class 2 on \( m \) points, and denote its adjacency matrices by \( A_0 = I_m, A_1, A_2 \). Define

\[
(11) \quad C_0 = I_{2(m+1)},
\]

\[
(12) \quad C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_1 & A_2 & 1^T \\ 1^T & A_2 & A_1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
(13) \quad C_2 = C_1^T,
\]

\[
(14) \quad C_3 = J - C_0 - C_1 - C_2,
\]

where \( 1 \) is the all-one row vector of length \( m \). The association scheme defined by the set of adjacency matrices \( \{C_0, C_1, C_2, C_3\} \) is called the extended double cover of \( \mathfrak{X} \).

We will show in the next section that \( \{C_0, C_1, C_2, C_3\} \) is indeed the set of adjacency matrices of an association scheme of class 3.
By (11)–(14), we have

\[
C_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1^\top & A_2 & A_1 & 0 \\
0 & A_1 & A_2 & 1^\top \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

(15)

\[
C_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & I_m & 0 \\
0 & I_m & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

(16)

Remark 2. In Definition 1, if \(A_1\) and \(A_2\) define isomorphic digraphs, then it is easy to see that \(C_1\) and \(C_2\) define isomorphic digraphs.

The next lemma is necessary in the next section in order to establish our main result.

Lemma 3. We have the following:

\[
C_1^2 = C_2^2 = \frac{m-1}{2}(C_1 + C_2) + mC_3,
\]

(17)

\[
C_1C_2 = C_2C_1 = mC_0 + \frac{m-1}{2}(C_1 + C_2),
\]

(18)

\[
C_1C_3 = C_3C_1 = C_2,
\]

(19)

\[
C_2C_3 = C_3C_2 = C_1,
\]

(20)

\[
C_3^2 = C_0.
\]

(21)

Proof. We can easily see that (19) and (21) hold. Then (20) follows immediately from (19) and (21).

Since

\[
(C_1^2)_{1,1} = (C_2^2)_{4,4} = 0,
\]

\[
(C_1^2)_{1,4} = (C_2^2)_{4,1} = m,
\]

\[
(C_1^2)_{1,2} = (C_2^2)_{1,3} = (C_2^2)_{4,2} = (C_2^2)_{4,3} = \frac{m-1}{2}1,
\]

\[
(C_1^2)_{2,1} = (C_2^2)_{2,4} = (C_2^2)_{3,1} = (C_2^2)_{3,4} = \frac{m-1}{2}1^\top,
\]

\[
(C_1^2)_{2,2} = (C_2^2)_{3,3} = A_1^2 + A_2^2 = \frac{m-1}{2}(A_1 + A_2) \quad \text{(by (12))},
\]

\[
(C_1^2)_{3,2} = (C_2^2)_{2,3} = J + 2A_1A_2 = mI_m + \frac{m-1}{2}(A_1 + A_2) \quad \text{(by (10))},
\]

we have \(C_1^2 = \frac{m-1}{2}(C_1 + C_2) + mC_3\). Similarly, we have \(C_2^2 = \frac{m-1}{2}(C_1 + C_2) + mC_3\).

Finally, (18) follows from (17) and (19)–(21).

\[\square\]
4. Main results

**Theorem 4.** The extended double cover of a non-symmetric association scheme of class 2 on \( m \) points is an association scheme with the first eigenmatrix \((2)\).

**Proof.** Suppose \( \tilde{X} \) is a non-symmetric association scheme of class 2 on \( m \) points. Let \( C_0, C_1, C_2, C_3 \) be the matrices given in Definition 1, and let \( \mathcal{A} = (C_0, C_1, C_2, C_3) \) be their linear span over the field of complex numbers. First we observe that \( \mathcal{A} \) is closed under multiplication by Lemma 3. Thus \( \mathcal{A} \) is the Bose–Mesner algebra of an association scheme \( \tilde{X} \) of class 3.

Secondly we compute the first eigenmatrix of \( \tilde{X} \). By Lemma 3 the intersection matrices \( B_1, B_2, B_3 \) of \( \tilde{X} \) are given by

\[
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & \frac{m-1}{2} & \frac{m-1}{2} & m \\
\frac{m-1}{2} & m & \frac{m-1}{2} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & \frac{m-1}{2} & \frac{m-1}{2} & 0 \\
\frac{m-1}{2} & m & \frac{m-1}{2} & m \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

Let

\[
E_0 = \frac{1}{2(m+1)}(C_0 + C_1 + C_2 + C_3),
\]

\[
E_1 = \frac{1}{4}(C_0 - \frac{1}{\sqrt{m}}(C_1 - C_2) - C_3),
\]

\[
E_2 = \frac{1}{4}(C_0 + \frac{1}{\sqrt{m}}(C_1 - C_2) - C_3),
\]

\[
E_3 = \frac{1}{2(m+1)}(mC_0 - (C_1 + C_2) + mC_3).
\]

Then by \((22)-(24)\) we have \( E_i E_j = \delta_{i,j} E_i \). By \((25)-(28)\) the second eigenmatrix of \( \tilde{X} \) is given by

\[
\begin{bmatrix}
1 & m+1 & m+1 & m \\
1 & \frac{2}{\sqrt{m}} & \frac{2}{\sqrt{m}} & -1 \\
1 & \frac{m+1}{2} & \frac{m+1}{2} & -1 \\
1 & \frac{m+1}{2} & \frac{m+1}{2} & -1
\end{bmatrix}.
\]

Then the first eigenmatrix of \( \tilde{X} \) is given by \((2)\).  \qed

**Remark 5.** In the database of \([5]\), as08[6], as16[11], and as24[14] can be constructed by Theorem 4.
Remark 6. Let \( \{A_0, A_1, A_2\} \) be the set of the adjacency matrices of a symmetric association scheme of class 2 with \( k = 2\mu \) on \( m \) points, and

\[
D_0 = I_{2(m+1)},
\]

\[
D_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1^\top & A_1 & A_2 & 0 \\
0^\top & A_2 & A_1 & 1^\top \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & A_2 & A_1 & 1^\top \\
1^\top & A_1 & A_2 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
D_3 = J - D_0 - D_1 - D_2.
\]

Then, \( D_1 \) is the adjacency matrix of a Taylor graph (see [2, Sect. 1.5]), and its first eigenmatrix is given by (3). In this sense, the extended double cover can be regarded as a non-symmetric analogue of Taylor graphs.

The following construction is due to D. Pasechnik (announced in [3]; see [4, Theorem 2.6.6]).

Definition 7. Let \( \mathfrak{X} \) be a non-symmetric association scheme of class 2 on \( m \) points, and denote its adjacency matrices by \( A_0 = I_m, A_1, A_2 \). Define

\[
\tilde{A}_0 = I_{2m+1}, \quad \tilde{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & A_1 & A_2 + I_m \\ 1^\top & A_2 & A_2 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1^\top & A_2 & A_1 \\ 0 & A_1 + I_m & A_1 \end{bmatrix},
\]

where \( A_2^\top = A_1 \). The association scheme with adjacency matrices \( \{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2\} \) is called the doubling of \( \mathfrak{X} \).

In Definition 7, even if \( A_1 \) and \( A_2 \) define isomorphic digraphs, the matrices \( \tilde{A}_1 \) and \( \tilde{A}_2 \) may define non-isomorphic digraphs.

Since the doubling of a non-symmetric association scheme of class 2 on \( m \) points is a non-symmetric association scheme of class 2 on \( 2m + 1 \) points, we can construct the extended double cover on \( 4(m+1) \) points by Theorem 4. Then we have the following.

Theorem 8. Let \( \{C_0, C_1, C_2, C_3\} \) be set of the adjacency matrices of the extended double cover of a non-symmetric association scheme \( \mathfrak{X} \) of class 2 on \( m \) points. Then the set of adjacency matrices of the extended double cover of the doubling of \( \mathfrak{X} \) is given
by \( \{C'_0, C'_1, C'_2, C'_3\} \), where

\[ C'_0 = I_{4(m+1)}, \]
\[ C'_1 = \begin{bmatrix} C_1 & C_2 + I_{2(m+1)} \\ C_2 + C_3 & C_2 \end{bmatrix}, \]
\[ C'_2 = \begin{bmatrix} C_2 & C_1 + C_3 \\ C_1 + I_{2(m+1)} & C_1 \end{bmatrix}, \]
\[ C'_3 = \begin{bmatrix} C_3 & 0 \\ 0 & C_3 \end{bmatrix}. \]

Proof. Substituting the adjacency matrices \( \tilde{A}_0, \tilde{A}_1, \tilde{A}_2 \) in Definition 7 into (12) directly, we have

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & A_1 & A_2 + I_m & 1^\top & A_2 & A_1 & 1^\top \\
0 & 1^\top & A_2 & A_2 & 0 & A_1 + I_m & A_1 & 1^\top \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1^\top & 1^\top & A_2 & A_1 & 0 & A_1 & A_2 + I_m & 0 \\
1^\top & 0 & A_1 + I_m & A_1 & 1^\top & A_2 & A_2 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

Rearranging the rows and columns in the order \([5, 6, 3, 2, 1, 7, 4, 8]\), we obtain

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & A_1 & A_2 & 1^\top & 1^\top & A_2 + I_m & A_1 & 0 \\
1^\top & A_2 & A_1 & 0 & 0 & A_1 & A_2 + I_m & 1^\top \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1^\top & A_2 & A_1 + I_m & 0 & 1^\top & A_2 & A_1 & 0 \\
0 & A_1 + I_m & A_2 & 1^\top & 0 & A_1 & A_2 & 1^\top \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix} C_1 & C_2 + I_{2(m+1)} \\ C_2 + C_3 & C_2 \end{bmatrix}
= C'_1.
\]

Hence we have (30). By (15) and (30) we have (31). By (14), (29)–(31) we have (32). \( \square \)

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