Random-matrix theory of Majorana fermions and topological superconductors

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The theory of random matrices originated half a century ago as a universal description of the spectral statistics of atoms and nuclei, dependent only on the presence or absence of fundamental symmetries. Applications to quantum dots (artifical atoms) followed, stimulated by developments in the field of quantum chaos, as well as applications to Andreev billiards — quantum dots with induced superconductivity. Superconductors with topologically protected subgap states, Majorana zero-modes and Majorana edge modes, provide for a new arena of applications of random-matrix theory. We review these recent developments, with an emphasis on electrical and thermal transport properties that can probe the Majoranas.

Contents

I. Introduction 1

II. Superconducting quasiparticles 2
A. Particle-hole symmetry 2
B. Majorana fermions 2
C. Majorana zero-modes 2
D. Time-reversal and chiral symmetries 3

III. Hamiltonian ensembles 3
A. The ten-fold way 3
B. Midgap spectral peak 3
C. Energy level repulsion 5

IV. Scattering matrix ensembles 5
A. Fundamental symmetries 5
B. Chaotic scattering 6
C. Circular ensembles 7
D. Topological quantum numbers 7

V. Electrical conduction 8
A. Majorana nanowire 8
B. Counting Majorana zero-modes 8
C. Conductance distribution 9
D. Weak antilocalization 10
E. Andreev resonances 11

VI. Thermal conduction 12
A. Topological phase transitions 12
B. Heat transport by Majorana edge modes 12
C. Thermopower and time-delay matrix 14

VII. Josephson junctions 15
A. Fermion parity switches 15
B. \(4\pi\)-periodic Josephson effect 16

VIII. Discussion and outlook 16

Acknowledgments 16

References 17

I. INTRODUCTION

The theory of random matrices goes back to the 1960’s [1, 2] and has found applications in many branches of physics [3–5]. In condensed matter physics, random-matrix theory can describe the universal properties of disordered metals and superconductors [6, 7], dependent only on the presence or absence of fundamental symmetries in 10 symmetry classes (the so-called “ten-fold way”) [8]. It was recently discovered that 5 out of these 10 symmetry classes have a topological invariant [9, 10], which identifies distinct states of matter. Some of these topological superconductors and insulators have been realized in the laboratory [11, 12]. In this review we will discuss how random-matrix theory can be extended to account for topological properties.

Topological invariants count the number of protected subgap states, either bound to a defect or propagating along a boundary. In a superconductor these are Majorana fermions, described by a real rather than a complex wave function [13, 14]. The absence of complex phase factors fundamentally modifies the random-matrix description, notably, the scattering matrix at the Fermi level is real orthogonal rather than complex unitary. The circular ensemble of random orthogonal matrices, and the corresponding Gaussian ensemble of real antisymmetric matrices, was treated in text books [4, 5] for its mathematical elegance — without applications in quantum physics. Topological superconductors have changed that.

Experimentally, the search for Majorana zero-modes (bound to a vortex core or to the end of a superconducting wire) and Majorana edge modes (propating along the boundary of a two-dimensional superconductor) is still at an initial stage [15]. But much is understood from the theoretical point of view [16–19], so the time seems right for this review of the random-matrix theory of Majorana fermions.

II. SUPERCONDUCTING QUASIPARTICLES

The fermionic excitations \(\Psi\) of a superconductor are called Bogoliubov quasiparticles [20]. Unlike the electron and hole excitations \(\psi_e\) and \(\psi_h\) of a normal metal, the state \(\Psi\) has no definite charge — it is a coherent superposition of \(\psi_e\) (negatively charged, filled state at energy
The single-particle Hamiltonian \( H \) of the Bogoliubov-De Gennes (BdG) Hamiltonian [21],

\[
H = \begin{pmatrix} E_0 - E_F & -i \sigma_y \Delta \\ i \sigma_y \Delta & E_F - E_0 \end{pmatrix}.
\]

The single-particle Hamiltonian \( H_0 \) acts on \( \psi_e = (\psi_{e\uparrow}, \psi_{e\downarrow}) \) in the upper left block, while \( -H_0^* \) acts on \( \psi_h = (\psi_{h\uparrow}, \psi_{h\downarrow}) \) in the lower right block, with \( \uparrow \downarrow \) labeling the spin bands. The off-diagonal blocks couple electrons and holes in opposite spin bands (switched by the Pauli matrix \( \sigma_y \)), through the (complex) potential \( \Delta \) of an s-wave, spin-singlet superconductor.\(^1\) Each eigenfunction \( \Psi \) of \( H \) at energy \( E > 0 \) has a copy \( \tau_y \Psi \) at \( -E \). (The Pauli matrix \( \tau_y \) switches electrons and holes.) The corresponding symmetry of \( H \),

\[
H = -CHC^{-1} = -\tau_y H^* \tau_x,
\]

is called particle-hole symmetry. The charge conjugation operator \( C = \tau_y \mathcal{K} \), with \( \mathcal{K} \) the operator of complex conjugation, is anti-unitary and squares to \(+1\). If \( H_0 \) is spin independent, then the BdG Hamiltonian (1) decouples into the two blocks

\[
H_{\pm} = \begin{pmatrix} H_0 - E_F & \pm \Delta \\ \pm \Delta^* & E_F - H_0^* \end{pmatrix},
\]

acting separately on \( (\psi_{e\uparrow}, \psi_{h\uparrow}) \) and \( (\psi_{e\downarrow}, \psi_{h\downarrow}) \). The charge conjugation operator \( C = i\tau_y \mathcal{K} \) for each block now squares to \(-1\).

### A. Particle-hole symmetry

Particle-hole symmetry is a property of the Bogoliubov-De Gennes (BdG) Hamiltonian [21],

\[
H = \begin{pmatrix} E_0 - E_F & -i \sigma_y \Delta \\ i \sigma_y \Delta^* & E_F - E_0 \end{pmatrix}.
\]

In this so-called Majorana basis the particle-hole symmetry relation (2) reads simply

\[
C = K, \quad H = -H^*,
\]

so \( H \) is purely imaginary. The BdG equation

\[
H \Psi(r, t) = -i\hbar \frac{\partial}{\partial t} \Psi(r, t)
\]

becomes a real wave equation, because the \( i \) cancels. The resulting real wave function \( \Psi(r, t) \) produces a selfconjugate field operator

\[
\hat{\Psi}(r, t) = \hat{\Psi}^\dagger(r, t).
\]

This identity of the Bogoliubov creation and annihilation operators means that particle and antiparticle are one and the same.

All of this refers to the four-component BdG Hamiltonian (1), with \( C^2 = +1 \), so \( C \mapsto K \) can be achieved by a unitary transformation. The wave equation for the reduced two-component BdG Hamiltonian (3), with \( C^2 = -1 \), cannot be brought to a real form by any unitary transformation.

Superconductor quasiparticles are typically probed in the energy domain, rather than in the time domain. The Fourier transform

\[
\hat{\Psi}_E(r) = \int dt e^{iEt/\hbar} \hat{\Psi}(r, t) = \hat{\Psi}_{-E}(r)
\]

is selfconjugate at \( E = 0 \), so for quasiparticles at the Fermi level. Transport experiments at small voltage and low temperature can therefore probe the Majorana character of Bogoliubov quasiparticles, as we will explore in this review.

### C. Majorana zero-modes

Bogoliubov quasiparticles can be bound by a magnetic vortex or an electrostatic defect [26]. Particle-hole symmetry requires that the bound states come in pairs at \( \pm E \), with the possibility of an unpaired state at \( E = 0 \). The creation and annihilation operators are related by

\[
a_E = a_{-E}^\dagger,
\]

so they are identical at \( E = 0 \) (at the Fermi level). This self-conjugate bound state, \( a_0 = a_0^\dagger \), is called a Majorana

---

\(^1\) The BdG Hamiltonian (1) is sometimes given in the alternative basis \( (\psi_{e\uparrow}, \psi_{e\downarrow}, -\psi_{h\uparrow}, \psi_{h\downarrow}) \), when it has the form

\[
\hat{H} = \begin{pmatrix} H_0 - E_F & \Delta \\ \Delta^* & E_F - H_0 \end{pmatrix},
\]

with a scalar off-diagonal block. The charge conjugation operator then equals \( C = (\sigma_y \otimes \tau_y) \mathcal{K} \). We prefer the equivalent representation (1) because of the simpler \( C = \tau_y \mathcal{K} \). Note that both \( C \) and \( \hat{C} \) square to \(+1\).
zero-mode or Majorana bound state (or sometimes just “Majorana fermion”, when no confusion with unbound Bogoliubov quasiparticles can arise).

A Majorana zero-mode has a certain stability, it cannot be displaced away from the Fermi level without breaking the $\pm E$ symmetry of the spectrum [27]. If a vortex contains a nondegenerate state at $E = 0$, then it will remain pinned to the Fermi level if we perturb the system. This robustness is called “topological protection” and a superconductor that supports Majorana zero-modes is called a topological superconductor (or a “topologically nontrivial” superconductor). For an overview of the ongoing search for Majorana zero-modes in superconductors, see Refs. 16–19.

The ground state of $2n$ vortices containing Majorana zero-modes is $2^n$-fold degenerate and the exchange of pairs of vortices is a unitary operation on the ground-state manifold [28, 29]. Such non-commuting exchange operations (“non-Abelian statistics”) are at the basis of proposals to store and manipulate quantum information by means of a magnetic vortex [26]. If spin-rotation symmetry is broken by spin-orbit coupling, the system is systematic) labeling of the symmetry classes, shown in the top row of the table, originates from differential geometry [34]. Majorana zero-modes appear in the three symmetry classes $AIII$, $CII$, and $DIII$ in which the particle-hole operator $\mathcal{C}$ squares to +1.

For a random-matrix approach the Hamiltonian operator is represented by an $\mathcal{N} \times \mathcal{N}$ Hermitian matrix $H = H^\dagger$. In the Gaussian ensemble the Hamiltonian has the probability distribution

$$P(H) \propto \exp \left\{- \frac{c}{N} \text{Tr} H^2 \right\},$$

(12a)

$$\begin{cases} \frac{\pi^2 \beta_E}{8 \delta_0^2} \times \frac{2}{\mathcal{N}} & \text{in class A, AI, AII}, \\
1 & \text{in the other classes}, \end{cases}$$

(12b)

where $\delta_0$ is the mean level spacing of $H$ in the bulk of the spectrum and $\beta_E \in \{1, 2, 4\}$ describes the strength of the level repulsion (see Sec. III.C). The Gaussian form is chosen for mathematical convenience, in the large-$\mathcal{N}$ limit the spectral correlations depend only on the fundamental symmetries of $H$.

D. Time-reversal and chiral symmetries

Anti-unitary symmetries come in two types, the Hamiltonian $H$ may commute or anti-commute with an anti-unitary operator. The particle-hole symmetry discussed in Sec. II.A is the anti-commutation, $HC = -\mathcal{C}H$, while the commutation $HT = \mathcal{T}H$ is called time-reversal symmetry. The physical operation of time reversal should reverse the spin, $T \sigma_k T^{-1} = -\sigma_k$, as well as the momentum, $T p T^{-1} = -p$. The corresponding operator $T = i\sigma_z \mathcal{K}$ squares to $-1$. The Hamiltonian (1) commutes with $T$ if $\Delta$ is real and

$$H_0 = T H_0 T^{-1} = \sigma_y H_0^\dagger \sigma_y.$$ \hspace{1cm} (10)

For a real Hamiltonian we can take $T = \mathcal{K}$ squaring to $+1$. The combination of this fake time-reversal symmetry with the particle-hole symmetry (2) implies that

$$H \sigma_z = -\sigma_z H.$$ \hspace{1cm} (11)

Such anti-commutation of the Hamiltonian with a unitary operator, $HC T = -\mathcal{C}TH$, is called a chiral symmetry. (The name originates from particle physics [33].)

III. HAMILTONIAN ENSEMBLES

A. The ten-fold way

The classification of fermionic Hamiltonians on the basis of particle-hole and time-reversal symmetry was introduced by Altland and Zirnbauer in a seminal 1997 paper [8]. It is a useful classification in the random-matrix theory of disordered systems, because these anti-unitary symmetries tend to be more robust than unitary symmetries (which are typically broken by disorder). There is a total of 10 symmetry classes, the celebrated “ten-fold way” of random-matrix theory [3–5].

In Table I we summarize the six symmetry classes with particle-hole symmetry. For completeness, we also include in this table the four additional symmetry classes without particle-hole symmetry. The (seemingly unsystematic) labeling of the symmetry classes, shown in the top row of the table, originates from differential geometry [34]. Majorana zero-modes appear in the three symmetry classes $AIII$, $BDI$, and $DIII$ in which the particle-hole operator $\mathcal{C}$ squares to +1.

The $\pm E$ particle-hole symmetry modifies the spectral correlations near $E = 0$. This is the Fermi level, in the middle of the superconducting gap, so to allow for states near $E = 0$ one needs to locally close the gap, for example by means of a magnetic vortex [26]. If spin-rotation symmetry is broken by spin-orbit coupling, the system is in symmetry class $D$. The particle-hole symmetry relation (5) requires that $H = iA$ is purely imaginary in the Majorana basis, and since it is also Hermitian it must be an antisymmetric matrix: $A_{nm} = -A_{mn} = A^*_{nm}$. There are no other symmetry constraints in class $D$.

In the Gaussian ensemble the upper-diagonal matrix elements $A_{nm}$ $(n > m)$ of the real antisymmetric matrix $A$ all have identical and independent distributions

$$P(\{A_{nm}\}) \propto \prod_{n,m, n > m} \exp \left(- \frac{\pi^2 A_{nm}^2}{2N \delta_0^2} \right),$$

(13)

cf. Eq. (12) with $\beta_E = 2$. As for any antisymmetric matrix, the eigenvalues come in $\pm E$ pairs, so if $\mathcal{N}$ is odd

2 The symmetry classes $AIII$ and $CII$ also support zero-modes, but these do not correspond to a self-conjugate Majorana operator.

3 The factor-of-two difference in the coefficient $c$ is there on account of the $\pm E$ symmetry of the spectrum in the classes with particle-hole or chiral symmetry, see Ref. 35. The mean level spacing $\delta_0$ refers to distinct levels, not counting degeneracies.
The primed product $\nu$ (Uncoupled spin bands are not included in the degeneracy count.) The integer $\nu$ counts the number of $d_E$-fold degenerate, topologically protected zero-modes (Majorana in class D, BDI, and DIII).

![Diagram](image)

FIG. 1 Panels a) and b) show the spectrum of a vortex core in a class-D superconductor. The $\pm E$ symmetric spectrum may or may not have an unpaired Majorana zero-mode (red horizontal line). The ensemble-averaged density of states (16) is plotted in panel c). The delta-function contribution from the zero-mode is accompanied by a dip in the smooth part $\rho_-$ of the density of states. Without the zero-mode there is a midgap spectral peak $\rho_+$. The density of states in symmetry class C is given by $\rho_-$ without the zero-mode contribution.

then $H$ necessarily has one eigenvalue pinned at zero — a Majorana zero-mode, see Sec. II.C. For the nonzero eigenvalue pairs $\pm E_n$, the Gaussian ensemble gives the probability distribution [4]

$$P(E_n) \propto \prod_{i<j} (E^2 - E_j^2)^2 \prod_k E_k^{2\nu} \exp \left( -\frac{\pi^2 E_n^2}{2(N \delta_E)} \right).$$

The primed product $\prod'$ is a reminder that only positive energies are included. The number $\nu$ indicates the presence or absence of a Majorana zero-mode: $\nu = 1$ if $N$ is odd and $\nu = 0$ if $N$ is even.

In either case $\nu = 0, 1$ the ensemble-averaged density of states $\rho(E)$ has a peak at $E = 0$ [8, 36, 37],

$$\rho(E) = \begin{cases} \rho_+(E) & \text{if } \nu = 0, \\ \rho_-(E) + \delta(E) & \text{if } \nu = 1, \end{cases}$$

$$\rho_\pm(E) = \delta_0^{-1} \pm \frac{\sin(2\pi E/\delta_0)}{2\pi E},$$

see Fig. 1.

A physical realization of a class D, $\nu = 1$ vortex is offered by the surface of a three-dimensional topological insulator (such as Bi$_2$Te$_3$) covered by an s-wave superconductor [38]. The small level spacing $\delta_0 \simeq \Delta_0/E_F$ in the vortex core (with superconducting gap $\Delta_0$ much smaller than the Fermi energy $E_F$) complicates the detection of the midgap spectral peak at experimentally accessible temperatures [39]. Notice that $\rho_+$ and $\rho_- + \delta(E)$ have identical spectral weight of 1/2, so a thermally smeared density of states is not a prominent signature of a Majorana zero-mode. (The transport signatures discussed in Sec. V are more reliable for that purpose.)

The midgap spectral peak does serve as an unambiguous distinction between symmetry classes C and D, with and without spin-rotation symmetry. Particle-hole symmetry of the class-C Hamiltonian (3) can be expressed as

$$H_\pm = -\tau_y H^*_\pm \tau_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the subscript $\pm$ labels the spin degree of freedom and the subblocks refer to the electron-hole degree of freedom. Because of the spin degeneracy it is sufficient to consider $H_+ = iQ$.

Eq. (17) implies that the matrix elements of the $\mathcal{N} \times \mathcal{N}$ anti-Hermitian matrix $Q$ are quaternion numbers, of the
Level crossings are avoided away from the Fermi level, but at the Fermi level pairs of levels may cross. (The physical significance of the level crossings is discussed in Sec. VII.A.)

For reference we record the complete expressions [8, 37] for the probability distributions of energy levels in the ten symmetry classes from Table I:

\[
P(E_n) \propto \prod_{i<j}^N |E_i - E_j|^\beta_E \prod_{k=1}^{N/d_E} \exp \left( -\frac{\pi^2 \beta_E d_E E_k^2}{4N \delta_0^2} \right)
\]

in class A, AI, AII,

\[
P(E_n) \propto \prod_{k=1}^{(N-\nu d_E)/2d_E} |E_k|^{\alpha_E + \nu \beta_E} \exp \left( -\frac{\pi^2 \beta_E d_E E_k^2}{4N \delta_0^2} \right)
\]

\[
\times \prod_{1<i<j} |E_i^2 - E_j^2|^\beta_E,
\]

in the other classes. (21)

The \(d_E\)-fold degenerate levels are included only once in each product, and the primed product indicates that only positive energies are included (excluding also the \(\nu d_E\) zero-modes). The effect of the repulsion factor \(|E|^\alpha_E + \nu \beta_E\) on the density of states in the Altland-Zirnbauer ensembles is shown in Fig. 3.

IV. SCATTERING MATRIX ENSEMBLES

A. Fundamental symmetries

We seek to probe Majorana fermions by means of electrical or thermal transport at low voltages and temperatures, near the Fermi level where the Majorana operators (8) are selfconjugate. These transport properties are determined by a quantum mechanical scattering problem, in which a set of incident and outgoing wave amplitudes \(\psi^{\text{in}}_n, \psi^{\text{out}}_n\), \(n = 1, 2, \ldots, N\), is linearly related by

\[
\psi^{\text{out}}_n(E) = \sum_{m=1}^N S_{nm}(E)\psi^{\text{in}}_m(E).
\]

(22)

The scattering is elastic, so incident and outgoing states are at the same energy \(E\), and it is conservative, so \(\sum_n |\psi^{\text{out}}_n|^2 = \sum_n |\psi^{\text{in}}_n|^2\), and \(S \in \mathbb{U}(N)\) is a unitary \(N \times N\) matrix,

\[
S^{-1}(E) = S^\dagger(E) \Leftrightarrow \sum_k S_{nk}(E)S^*_{mk}(E) = \delta_{nm}.
\]

(23)

The Cayley transform

\[
S(E) = \frac{1 - i\pi \mathcal{G}(E)}{1 + i\pi \mathcal{G}(E)}
\]

relates the unitary scattering matrix \(S(E)\) to the Green function \(\mathcal{G}(E) = W(E - H)^{-1}W^\dagger\), projected onto the

FIG. 2 Model calculation of the excitation spectrum of a disordered InAs-GaSb quantum well with superconducting electrodes [40]. It is a quantum spin Hall (QSH) insulator, withconducting edges that connect the quantum dot (induced by a gate voltage) to superconducting electrodes. This class-D system has level crossings at the Fermi level \((E = 0)\), signaling a fermion-parity switch of the superconducting condensate (see Sec. VII.A).
scattering states by a coupling matrix $W$. We assume that $W$ commutes both with the charge-conjugation operator $C$ and with the time-reversal operator $T$. (See Ref. 41 for a more general treatment of the symmetry constraints on the scattering.) Particle-hole symmetry $H = -\mathcal{C}HC^{-1}$ of the Hamiltonian translates into the scattering-matrix symmetry

$$S(-E) = CS(E)C^{-1} = \begin{cases} S^*(E) & \text{if } C = \mathcal{K}, \\ \tau_yS^*(E)\tau_y & \text{if } C = i\tau_y\mathcal{K}. \end{cases}$$

(25)

At the Fermi level, $E = 0$, this implies that $S(0) \in O(N)$ is a real orthogonal matrix for $C^2 = +1$, while $S(0) \in \text{Sp}(N)$ is a unitary symplectic matrix\(^4\) for $C^2 = -1$.

Time-reversal symmetry $H = THT^{-1}$ translates into

$$S(E) = TS^\dagger(E)T^{-1} = \begin{cases} S^T(E) & \text{if } T = \mathcal{K}, \\ \sigma_yS^T(E)\sigma_y & \text{if } T = i\sigma_y\mathcal{K}. \end{cases}$$

(26)

where the superscript $T$ denotes the transpose. This may be written more succinctly, upon a change of basis $S \mapsto i\sigma_y S$ of the outgoing modes, as a condition of symmetry or antisymmetry,

$$S(E) = \pm S^T(E),$$

depending on whether the anti-unitary operator $T$ squares to $+1$ or $-1$.

The symmetry requirements on the scattering matrix are summarized in Table I, for each of the ten symmetry classes.

B. Chaotic scattering

The approach of random-matrix theory applies if the scattering is “chaotic”. Chaotic scattering is a concept that originates from classical mechanics, referring to the exponential sensitivity of a trajectory to a slight change in initial condition [42]. In the 1990’s Blümel and Smilansky transferred this concept to quantum mechanics [43], by considering — at a fixed energy — the ensemble of scattering matrices produced by slight deformations of the scattering potential. Chaotic scattering then refers to a uniform distribution\(^5\) of this ensemble in the unitary group,

$$P(S) = \text{constant}, \quad S \in U(N).$$

(28)

This so-called Circular Unitary Ensemble (CUE) was introduced by Dyson in the early days of random-matrix theory [46], long before the advent of quantum chaos. It has found many applications in the context of microwave cavities [47], and electronic quantum dots [6].

The constraint (27) on the scattering matrix imposed by time-reversal symmetry restricts $S$ to a subset of $U(N)$. A symmetric scattering matrix $S = UU^T = S^T$ applies to electrons when their spin is conserved by the scattering potential. The ensemble generated by the uniform distribution of $U \in U(N)$ then describes chaotic scattering. Dyson [46] called this ensemble the Circular Orthogonal Ensemble (COE), because unitary symmetrictic matrices form the coset $U(N)/O(N)$ of the orthogonal group $O(N)$.

In the presence of spin-orbit coupling the constraint of time-reversal symmetry reads $S = U\sigma_y U^T\sigma_y$. The uniform distribution of $U \in U(N)$, with $N$ even, then produces Dyson’s Circular Symplectic Ensemble (CSE), thus called because $S$ is in the coset $U(N)/\text{Sp}(N)$ of the unitary symplectic group $\text{Sp}(N)$. Equivalently, upon a change of basis $S \mapsto i\sigma_y S$, we may describe the CSE

\(^4\) Orthogonal and symplectic matrices are both unitary, but while the matrix elements of an orthogonal matrix are real numbers, the matrix elements of a symplectic matrix are quaternions, of the form (18).

\(^5\) Uniformity in the unitary group is defined in terms of the Haar measure $dU = d(UU_0)$ for any fixed $U_0 \in U(N)$. See Ref. 44 for how one can generate random matrices with this uniform distribution, and Ref. 45 for how one can perform integrals $\int dU$ of polynomials of $U$. 

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**FIG. 3** Average density of states in the four Altland-Zirnbauer ensembles, calculated numerically [35] for Hamiltonians of dimensionality $N = 60$ in class $C$ and $C_I$, $N = 60 + \nu$ in class $D$, and $N = 120 + 2\nu$ in class $DIII$. The delta-function singularity of the zero-mode for $\nu = 1$ is not plotted. Except for class $D$ with $\nu = 0$, the density of states vanishes at the Fermi level as $|\nu - \nu_0| = 0$, this implies that $\rho(E) = 0$, the density of states vanishes at the Fermi level. We assume that $\rho$ commutes both with the charge-conjugation operator $C$ and with the time-reversal operator $T$. (See Ref. 41 for a more general treatment of the symmetry constraints on the scattering.)
by the set of unitary antisymmetric matrices \([48]\), \(S = U\sigma_y U^T = -S^T\).

**C. Circular ensembles**

Superconductivity introduces a new type of scattering process, Andreev scattering \([49]\), which is the conversion of an electron-like excitation at \(E_F + E\) into a hole-like excitation at \(E_F - E\). At the Fermi level, for excitation energy \(E \to 0\), electrons and holes have complex conjugate wave functions. A linear superposition produces quasiparticles with a real wave function, the selfconjugate Majorana fermions discussed in Sec. II.B.

Andreev scattering does not conserve charge (the missing charge is accounted for by the superconducting condensate), but it does conserve particle number. The scattering matrix \(S(E)\) therefore remains a unitary matrix, of dimension \(2N \times 2N\) to accommodate the \(N\) electron and \(N\) hole degrees of freedom. The constraint of a real scattering amplitude of Majorana fermions restricts \(S(0)\) to the orthogonal subgroup \(O(2N)\). Chaotic scattering then implies a uniform distribution,

\[
P(S) = \text{constant}, \quad S \in O(2N). \tag{29}
\]

This extension of Dyson’s circular ensembles to include Andreev scattering was introduced by Altland and Zirnbauer \([8]\).

A few words about nomenclature. The name “circular orthogonal ensemble” for the distribution \((29)\) would be most logical, but this name is already taken for the ensemble of unitary symmetric matrices (COE). We have become used to calling it the Circular Real Ensemble (CRE) — another name found in the literature \([50]\) is “Haar orthogonal ensemble”. An alternative name could be “class D” ensemble, referring to the mathematical labeling of symmetric spaces \([34]\), but as Zirnbauer has argued one should distinguish the symmetry of the matrix space from the uniformity of the matrix ensemble \([51]\).

The restriction \(S = \pm S^T\) to symmetric or antisymmetric orthogonal matrices produces two further ensembles, which we will refer to as \(T_+\)CRE (symmetry class BDI) and \(T_-\)CRE (symmetry class DIII). This is analogous to how the CUE produces the COE and CSE, but the physics is different. A Majorana zero-mode is a coherent superposition of electrons and holes from the same spin band, while Andreev scattering couples opposite spin bands. Spin-orbit coupling is therefore needed to mix the spin bands and realize the CRE — while the CUE can exist with or without spin-orbit coupling. As a consequence, time-reversal symmetry can realize only the ensemble \(T_-\)CRE of antisymmetric orthogonal matrices. The symmetry that is responsible for the ensemble \(T_+\)CRE of symmetric orthogonal matrices is the “chiral” symmetry discussed in Sec. II.D.

In Table II we summarize the three scattering matrix ensembles that support Majorana zero-modes. The first row lists the name of the ensemble and the second row lists the name of the corresponding symmetric space. The last row lists the topological invariant, discussed next.

**D. Topological quantum numbers**

Typically, whenever the orthogonal group appears in a physics problem, it is sufficient to consider only matrices with determinant \(\text{Det} S = +1\) — the so-called special orthogonal group (denoted SO or \(O_+\)). The remaining orthogonal matrices in \(O_-\) have \(\text{Det} S = -1\), they are disconnected from the identity matrix and would seem unphysical.

The advent of topological superconductors \([9, 11, 12]\) has provided for a physical realization of orthogonal scattering matrices with \(\text{Det} S = -1\) \([36, 52]\). More generally, the three ensembles from Table II each decompose into disjoint sub-ensembles, distinguished by an integer \(Q\) called “topological quantum number” or “topological invariant”. In terms of the scattering matrix, this number is represented by \([53]\)

\[
Q = \text{Det} S = \pm 1 \text{ in the CRE}, \tag{30a}
\]

\[
Q = \frac{1}{2} \text{Tr} S \in \{0, \pm 1, \pm 2, \ldots \pm N\} \text{ in the } T_+\text{CRE}, \tag{30b}
\]

\[
Q = \text{Pf} S = \pm 1 \text{ in the } T_-\text{CRE}. \tag{30c}
\]

In the CRE the two sub-ensembles correspond to a uniform distribution of the orthogonal matrix \(S\) in \(O_+(2N)\). The orthogonal antisymmetric matrices in the \(T_-\)CRE can be decomposed as

\[
S = O\Sigma O^T, \quad O \in O_+(2N), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{31}
\]

where each block of \(J\) has dimension \(N \times N\). Again, a uniform distribution of \(O \in O_+(2N)\) produces two sub-ensembles, distinguished by the Pfaffian of the scattering matrix,\(^6\)

\[
\text{Pf} S = (\text{Pf} J)(\text{Det} O) = (-1)^{N(N-1)/2} \text{Det} O = \pm 1. \tag{32}
\]

Finally, in the \(T_+\)CRE the scattering matrix is both orthogonal and symmetric, so its eigenvalues are \(\pm 1\) and it has the decomposition

\[
S = O\Sigma O^T, \quad O \in O_+(2N), \quad \Sigma = \text{diag} (\pm 1, \pm 1, \ldots \pm 1). \tag{33}
\]

The matrix \(\Sigma\) is a so-called signature matrix and the number of \(-1\)’s on the diagonal represents the signature \(\nu(S)\) of \(S\). Now we may take \(O \in O_+(2N)\) without loss of generality, while the sub-ensembles are distinguished by the trace (or the signature) of the scattering matrix,

\[
\frac{1}{2} \text{Tr} S = \frac{1}{2} \text{Tr} \Sigma = N - \nu(S) = 0, \pm 1, \ldots \pm N. \tag{34}
\]

\[^6\] The Pfaffian entry in Wikipedia contains a useful collection of formulas. Computer algorithms for the evaluation of Pfaffians can be obtained from Ref. 54.
The invariant $Q$ that distinguishes the CRE and $T_-$ CRE sub-ensembles is called a $Z_2$ topological quantum number, because it can take on only two values, while the $T_+\ C R E$ has a $Z$ topological quantum number, because as $N$ is varied $Q$ ranges over all integer values. These invariants first appeared as winding numbers of Fermi surfaces, hence the adjective “topological” [9, 11, 12, 55]. In the present context of scattering matrices, where $Q$ results from an operation in linear algebra [53], the name “algebraic invariant” might be more natural.

\section*{V. ELECTRICAL CONDUCTION}

\subsection*{A. Majorana nanowire}

A semiconducting layer on a superconducting substrate is in symmetry class D when both time-reversal symmetry and spin-rotation symmetry are broken. The Bogoliubov-De Gennes Hamiltonian has the form of Eq. (1), with

$$H_0 = \frac{p^2}{2m_{\text{eff}}} + U(r) + \frac{\alpha_{so}}{\hbar} (\sigma_x p_y - \sigma_y p_x) + \frac{1}{2} g_{\text{eff}} \mu_B B \sigma_z. \tag{35}$$

The first two terms give the kinetic energy and electrostatic potential energy. In the third term the momentum $p = -i\hbar \partial / \partial r$ in the $x$-$y$ plane of the layer is coupled to spin by the Rashba effect, breaking spin-rotation symmetry with characteristic length $l_{so} = \hbar^2 (m_{\text{eff}} \alpha_{so})^{-1} \approx 100 \text{ nm}$ and energy $E_{so} = m_{\text{eff}} (\alpha_{so} / \hbar)^2 \simeq 0.1 \text{ meV}$. The last term describes the Zeeman effect of a magnetic field $B \hat{z}$, parallel to the layer, breaking time-reversal symmetry with characteristic energy $V_2 = g_{\text{eff}} \mu_B B \simeq 1 \text{ meV}$ at $B = 1 \text{T}$.

Without the term $\sigma_x p_y$ the Hamiltonian would be real and hence the chiral symmetry of Sec. II.D would promote the system from class D to class BDI [56]. Model calculations in a wire geometry (width $W$ in the $y$-direction), demonstrate that the chiral symmetry is effectively unbroken for $W \lesssim l_{so} / 2$ [57]. Experimentally realized InSb nanowires [58], of the type shown in Fig. 4, have $l_{so}$ in the range 100–200 nm, so the crossover from class D to class BDI happens when the wire becomes narrower than about 100 nm.

As discovered by Lutchyn, Sau & Das Sarma [59] and by Oreg, Refael & von Oppen [60], the nanowire enters into a topologically nontrivial phase, with Majorana zero-modes at the end points, once the Zeeman energy $V_Z$ exceeds the superconducting gap $\Delta$ (induced by the proximity effect). This theoretical prediction was a strong motivation for the experiments reviewed in Refs. 16–19, as well as for the development of the random-matrix theory reviewed here.

\subsection*{B. Counting Majorana zero-modes}

The topological quantum number $Q$ from Sec. IV.D counts the number $\nu$ of stable Majorana zero-modes at each end of the $N$-mode nanowire, at most one in class D and up to $N$ in class BDI. This number is fully determined by the $2N \times 2N$ matrix $r$ of Fermi-level reflection amplitudes from the end of the nanowire [53].

The reflection matrix $r$ is a unitary matrix if the wire is sufficiently long that transmission to the other end can be neglected. It has a block structure of $N \times N$ submatrices,

$$r = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}. \tag{36}$$

Andreev reflection (from electron to hole or from hole to electron) is described by the off-diagonal blocks, while
the diagonal blocks describe normal reflection (without change of charge).

At the Fermi level \((E = 0)\) the particle-hole symmetry, operative in both class D and BDI, is expressed by \(r = \tau_x r^* \tau_x\), while the chiral symmetry (or fake time-reversal symmetry) of class BDI is \(r = r^T\). The corresponding symmetry operators \(C = \tau_x \mathcal{K}\) and \(\mathcal{T} = \mathcal{K}\) both square to \(+1\), in accordance with Table I. In terms of the submatrices, this corresponds to

\[
\begin{align*}
    r_{ee} &= r_{hh}^* , \quad r_{he} = r_{eh}^* , \quad \text{in class D and BDI}, \\
    r_{ee} &= r_{he}^T , \quad r_{he} = r_{he}^1 , \quad \text{in class BDI only}.
\end{align*}
\]

The determinant of a unitary matrix lies on the unit circle in the complex plane, while \(r = \tau_x r^* \tau_x\) implies that the determinant is real, hence equal to \(\pm 1\). In class BDI the unitary matrix \(\tau_x r\) squares to the unit matrix, \((\tau_x r)^2 = r^* r = r^T r = I\), so its \(2N\) eigenvalues are \(\pm 1\). The corresponding topological quantum numbers are

\[
\begin{align*}
    Q &= \text{Det} \ r = \pm 1 , \quad \nu = \frac{1}{2} (1 - Q) , \quad \text{in class D}, \\
    Q &= \frac{1}{2} \text{Tr} (\tau_x r) = \text{Tr} r_{he} \in \{0, \pm 1, \pm \ldots N\}, \\
    \nu &= |Q| , \quad \text{in class BDI}.
\end{align*}
\]

### C. Conductance distribution

The electrical conductance \(G\) in the nanowire geometry of Fig. 4 is determined by the Andreev reflection eigenvalues \(A_n\) [61],

\[
G/G_0 = 2 \text{Tr} r_{he}^l r_{he}^\dagger = 2 \sum_{n=1}^N A_n ,
\]

where \(G_0 = e^2/h\) is the conductance quantum. The factor of two in front of the sum accounts for the fact that Andreev reflection of an electron doubles the current. The eigenvalues \(A_n\) of the Hermitian matrix product \(r_{he}^l r_{he}\) lie in the interval \([0,1]\). The \(A_n\)’s different from 0 and 1 are twofold degenerate (Béri degeneracy [62, 63]).\(^8\)

Both the conductance (39) and the number of Majorana zero-modes (38) are given by the same reflection matrix, so we can try to relate them. The Béri degeneracy enforces the upper and lower bounds [57]

\[
2\nu \leq G/G_0 \leq 2(N - \zeta),
\]

where \(\zeta = 0\) if \(N - \nu\) is even and \(\zeta = 1\) if \(N - \nu\) is odd.

For \(N = 1\) this immediately gives \(G/G_0 = 2\nu\), but for \(N > 1\) there is no one-to-one relation between the two quantities. If we assume that \(r\) is distributed according to the circular ensemble, a statistical dependence of \(G\) on \(\nu\) can be obtained.

For that purpose we need the probability distribution of the \(M = \frac{1}{2}(N - \nu - \zeta)\) twofold degenerate Andreev reflection eigenvalues in the class D or BDI circular ensemble. It is given by [57, 64]

\[
\begin{align*}
    P_D &\propto \prod_{1=i<j}^M (A_i - A_j)^4 \prod_{k=1}^M A_k^{2\zeta} (1 - A_k)^{2\nu}, \\
    P_{BDI} &\propto \prod_{1=i<j}^M (A_i - A_j)^2 \prod_{k=1}^M A_k^{\zeta - 1/2} (1 - A_k)^{\nu}.
\end{align*}
\]

These twofold degenerate \(A_n\)’s are free to vary in the interval \((0,1)\). In addition, there are \(\nu\) Andreev reflection eigenvalues pinned at 1 and \(\zeta\) pinned at 0. The resulting dependence of the conductance distribution \(P(G)\) on \(\nu\) is plotted in Fig. 5, for the case \(N = 3\).

The sensitivity of \(P(G)\) to Majorana zero-modes becomes weaker and weaker with increasing \(N\). This happens in a particularly striking (nonperturbative) way in the circular real ensemble of class D, where the \(p\)-th cumulant of the conductance becomes completely independent of the topological quantum number for \(N > p\) [64]. Indeed, the red and blue solid curves in Fig. 5 have different skewness but identical average and variance, as expected for \(N = 3\).

Fig. 5 also includes histograms of conductance from a microscopic calculation in the Majorana nanowire of

FIG. 5 Probability distribution of the electrical conductance for \(N = 3\) modes, in the circular ensemble of class D (solid curves) and class BDI (dashed curves), either without any Majorana zero-modes (\(\nu = 0\), red curves) or with one zero-mode (\(\nu = 1\), blue curves). The curves follow by integration of the probability distribution (41) of the Andreev reflection eigenvalues. The histograms are the results of a microscopic calculation [64] for the Rashba-Beeneman Hamiltonian (35), in a three-mode nanowire of width \(W = L_0 = 100\) nm (in class D, but close to the crossover into class BDI at \(W \lesssim L_0/2\)).

\(^7\) In Table II the class BDI topological quantum number is defined without the \(\tau_x\) matrix, because there the scattering matrix is taken in the Majorana basis, while here we use the electron-hole basis.

\(^8\) The Béri degeneracy of the Andreev reflection eigenvalues \(A_n \neq 0,1\) in class D and BDI is a consequence of particle-hole symmetry, which is an anti-unitary symmetry that squares to \(+1\). This distinguishes it from the more familiar Kramers degeneracy, resulting from an anti-unitary symmetry that squares to \(-1\).
Sec. V.A, where the ensemble is generated by varying the disorder potential. The agreement with the predictions from the circular ensemble is quite reasonable, with two reservations. The first is that the nanowire was near the class-D-to-BDI crossover, with a partially broken chiral symmetry. The second is that the diffusive scattering produced by the disorder potential is not the chaotic scattering of the circular ensembles — the scattering channels are not uniformly mixed by disorder.

D. Weak antilocalization

The presence or absence of a Majorana zero-mode is a topological property of the nanowire, irrespective of how the wire is terminated. In particular, the lower bound $G \geq 2me^2/h$ holds whether or not there is any tunnel barrier to confine the Majorana at the end of the wire. The barrier serves a purpose in providing a resonant peak in the differential conductance $G(V) = dI/dV$ around zero voltage [65–69]. This resonant peak, reported in several experiments [58, 70–73], is shown in the computer simulation of Fig. 6 [74].

One sees from that simulation (based on the Rashba-Zeeman Hamiltonian of Sec. V.A), that a broader and smaller zero-bias peak appears also in the disorder-averaged conductance of a topologically trivial nanowire — without any Majorana zero-modes (compare blue and red curves). In that case the origin of the peak is the weak-antilocalization effect: the destructive interference of phase-conjugate scattering sequences. In normal metals this interference effect requires time-reversal symmetry, but in the presence of a superconductor particle-hole symmetry suffices [75, 76].

The two distinct origins of a zero-bias conductance peak in the average conductance can be compared in a random-matrix model, by including the effect of a tunnel barrier on the circular ensemble. In the zero-voltage limit, so at the Fermi level, we take for the reflection matrix $r_0$ without the barrier a uniform distribution in $O_+(2N)$ for the topologically trivial system (no Majoranas, $\nu = 0$) and in $O_-(2N)$ for the nontrivial system (with a Majorana zero-mode, $\nu = 1$). This is the circular real ensemble (CRE) of symmetry class D, in the Majorana basis (see Table II). Away from the Fermi level, at voltages large compared to the Thouless energy, the constraint from particle-hole symmetry is ineffective and $r_0$ is distributed uniformly over the entire unitary group $U(2N)$. This is the circular unitary ensemble (CUE).

The tunnel barrier (transmission probability $T$ per mode) transforms $r_0$ into

$$r = \sqrt{1 - T} + Tr_0(1 + \sqrt{1 - T} r_0)^{-1}. \quad (42)$$

The resulting nonuniform distribution of $r$ is known as the Poisson kernel of the circular ensemble [77, 78],

$$P(r) \propto |\text{Det}(1 - \sqrt{1 - T} r)|^{-p}. \quad (43)$$

The exponent equals $p = 4N$ in the CUE and $p = 2N - 1$ in the CRE.

The formula (39) for the conductance in the electron-hole basis can be rewritten in the Majorana basis, by carrying out the unitary transformation

$$r \mapsto \Omega r \Omega^\dagger, \quad \Omega = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \quad (44)$$

The result is

$$G/G_0 = N - \frac{1}{2} \text{Tr} \tau_x \tau_y r \tau_y \tau_y, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (45)$$
ε are encoded by the poles and remain pinned to \( V = 0 \) but there is also a Y-shaped pattern of peaks that merge the Fermi level. Level crossings produce an X-shaped resonant Andreev reflection from quasibound states near nanowire geometry are shown in Fig. 8. These are due to ductance.

Sample-specific field values.

The conductance peak indicated by the dotted line is pinned to zero voltage over a range of magnetic field values because of the accumulation of reflection matrix poles on the imaginary axis in a class-D superconductor, see Fig. 9.

The zero-bias conductance peak is then given by the difference \( \delta G = \langle G \rangle_{\text{CRE}} - \langle G \rangle_{\text{CUE}} \) of the average of \( r_0 \) over \( O_{\Delta}(2N) \) (for the CRE) and over \( U(2N) \) (for the CUE). Results [74] are shown in Fig. 7. The large-\( N \) limit has the \( \nu \)-independent value [8]

\[
\frac{\delta G}{G_0} = 1 - T + \mathcal{O}(N^{-1}).
\]

E. Andreev resonances

The weak antilocalization effect explains the appearance of a zero-bias peak in the disorder-averaged conductance. Sample-specific zero-bias peaks in the same nanowire geometry are shown in Fig. 8. These are due to resonant Andreev reflection from quasibound states near the Fermi level. Level crossings produce an X-shaped pattern when two resonant peaks meet and split again, but there is also a Y-shaped pattern of peaks that merge and remain pinned to \( V = 0 \) over a range of magnetic field values.

The center \( E \) and width \( 2\gamma \) of the Andreev resonances are encoded by the poles \( \varepsilon = E - i\gamma \) of the reflection matrix in the complex energy plane. Referring to the Green’s function expression (24), which can also be written as

\[
r(E) = 1 - 2\pi i W(E - H + i\pi W^\dagger W)^{-1} W^\dagger,
\]

the reflection matrix poles are eigenvalues of the \( N \times N \) non-Hermitian matrix

\[
\mathcal{H} = H - i\pi W^\dagger W.
\]

Because the coupling matrix product \( W^\dagger W \) is positive definite, the poles all lie in the lower half of the complex plane (\( \gamma > 0 \)), as required by causality. Particle-hole symmetry requires that the poles are symmetrically arranged around the imaginary axis (\( \varepsilon \) and \( -\varepsilon^* \) are both poles).

Fig. 9 is a scatter plot of the eigenvalues of \( \mathcal{H} \) for the Gaussian distribution (12) of the Hamiltonian \( H \), with coupling matrix [7]

\[
W_{nm} = w_n \delta_{nm}, \quad 1 \leq n \leq 2N, \quad 1 \leq m \leq N,
\]

\[
|w_n|^2 = \frac{N\delta_0}{\pi^2 T_n} (2 - T_n - 2\sqrt{1 - T_n}),
\]

representing a tunnel barrier with transmission probability \( T_n \). (The plot is for a mode-independent \( T_n \approx 0.2 \).) The two Altland-Zirnbauer ensembles with broken time-reversal symmetry are contrasted, with and without spin-rotation symmetry (class C and class D). For \( |E| \gtrsim \delta_0 \) the poles have a uniform density in a strip parallel to the real axis, familiar from the Wigner-Dyson ensembles [79]. For smaller \( |E| \) the poles are repelled from the imaginary axis in class C, while in class D they accumulate on that axis [35].

As pointed out in Ref. 80, a nondegenerate pole \( \varepsilon = -i\gamma \) on the imaginary axis has a certain stability, it cannot acquire a nonzero real part \( E \) without breaking the \( \varepsilon \leftrightarrow -\varepsilon^* \) symmetry imposed by particle-hole conjugation. To see why this stability is not operative in class C, we note that on the imaginary axis \( \gamma \) is a real eigenvalue of a matrix \( i\mathcal{H} \) that commutes with the charge-conjugation operator: \( Ci\mathcal{H} = -i\mathcal{H} = i\mathcal{H}C \). In class C the anti-unitary operator \( C \) squares to \(-1\), see Table I, so Kramers theorem\(^9\) forbids nondegenerate poles on the

\(^9\) The usual Kramers degeneracy refers to the eigenvalues of a
imaginary axis. In class D, in contrast, the operator $C$ squares to $+1$, Kramers degeneracy is inoperative and a number $N_Y$ of nondegenerate poles is allowed on the imaginary axis.

Because

$$(-1)^{N_Y} = \lim_{E \to 0} \text{Det} r(E) \equiv Q \quad (51)$$

is the same class-D topological quantum number as in Eq. (38a), the nanowire is topologically trivial or non-trivial depending on whether $N_Y$ is even or odd. One can now distinguish two types of transitions [80, 81]: At a topological phase transition $N_Y$ changes by $\pm 1$, which requires closing of the excitation gap in the nanowire and breaking of the unitarity of the reflection matrix $r$. At a “pole transition” $N_Y$ changes by $\pm 2$, the excitation gap remains closed and $r$ remains unitary. Both types of transitions produce the same Y-shaped conductance profile of two peaks that merge and stick together for a range of parameter values — distinct from the X-shaped profile that happens without a change in $N_Y$.

One way to distinguish the pole transition from the topological phase transition is to measure the conductance at both ends of the nanowire and search for correlations: The $\pm 2$ changes in $N_Y$ at the two ends are uncorrelated, while $\pm 1$ changes should happen jointly at both ends. Alternatively, one might try to suppress the pole transitions by raising the tunnel barrier at the end of the nanowire.

VI. THERMAL CONDUCTION

A. Topological phase transitions

While the electrical conductance $G$ of a superconducting nanowire gives information on the topological quantum number $Q$, the thermal conductance $\kappa$ signals the topological phase transitions, where $Q$ changes from one value to another. This is illustrated in Fig. 10, for the class-D Majorana nanowire of Sec. V.A. The peak in $\kappa$ at the topological phase transition has the quantized value

$$\kappa_0 = \frac{\pi^2 k_B T_0}{6h}, \quad (52)$$

without any finite-size or disorder corrections [52].

The quantization follows directly from the relation

$$\kappa/\kappa_0 = 2N - \text{Tr} r r^\dagger = \sum_{n=1}^{2N} (1 - R_n) \quad (53)$$

between the thermal conductance and the eigenvalues $R_n$ of the reflection matrix product $r r^\dagger$. Far from the topological phase transition the reflection matrix $r$ is unitary, hence $R_n = 1$ for all $n$ and $\kappa$ vanishes. In a finite system the reflection matrix is a continuous function of external parameters, such as the Fermi energy $E_F$, so if the class-D topological quantum number $Q = \text{Det} r$ changes sign, it must go through zero: $\text{Det} r = 0 \Rightarrow \prod_n R_n = 0$ at the transition point. Generically, one single $R_n$ will vanish, producing a peak in $\kappa$ of amplitude $\kappa_0$.

In class BDI the argument is similar [53]: A change in the topological quantum number $Q = \frac{1}{2} \text{Tr} \tau_x r$ by one unit happens when one eigenvalue of $\tau_x r$ switches between $\pm 1$, so it must go through zero, hence $\text{Det} \tau_x r = 0 \Rightarrow \text{Det} r = 0 \Rightarrow \prod_n R_n = 0$ at the transition point, again resulting in a quantized thermal conductance peak.

B. Heat transport by Majorana edge modes

Two-dimensional topological superconductors have gapless edge modes that allow for thermal transport. In the absence of time-reversal symmetry these are chiral (unidirectional) edge modes of Majorana fermions, analogous to the chiral electronic edge modes in the quantum Hall effect. One speaks of the thermal quantum Hall effect [28, 82–84]. The quantization of the thermal conductance $\kappa$ in units of $\kappa_0 = \pi^2 k_B^2 T_0 / 6h$ is the superconducting analogue of the quantization of the electrical conductance $G$ in units of $G_0 = e^2 / h$.

Chiral Majorana edge modes require particle-hole symmetry without time-reversal symmetry, so they exist in symmetry classes D and C (see Table I). The difference
between these two symmetry classes is that spin-rotation symmetry is broken in class D and preserved in class C. The corresponding symmetry of the superconducting pair potential is spin-triplet $p_x \pm i p_y$-wave pairing in class D (possibly realized in strontium ruthenate [85, 86]) and spin-singlet $d_{x^2-y^2} \pm id_{xy}$ pairing in class C (possibly realized in heavily doped graphene [87, 88]).

Edge modes of opposite chirality can meet at a domain wall [89], as illustrated in Fig. 11. Disorder at the boundary will mix the modes and remove the conductance quantization. Under the assumption of uniform mode mixing the probability distribution of the thermal conductance can be obtained from a circular ensemble [91]: the Circular Real Ensemble (CRE) of random orthogonal matrices in class D and the Circular Quaternion Ensemble (CQE) of random symplectic matrices in class C (possibly realized in strontium ruthenate [85, 86]) and C (see Table III).

For $M$ chiral Majorana modes at each edge the scattering matrix $S$ has dimension $2M \times 2M$, with $M \times M$

![FIG. 11 Thermal and electrical transport by chiral edge modes. Panel a) (from Ref. 89) shows the chiral Majorana modes at the edge of a topological superconductor, with either spin-triplet $p$-wave or spin-singlet $d$-wave pairing. The shaded strip at the center represents a disordered boundary between two domains of opposite chirality of the order parameter. Panel b) shows the chiral edge modes of the quantum Hall effect in graphene, where modes of opposite chirality meet at a bipolar junction between electron-doped and hole-doped regions [90]. The probability distributions (57) of the conductance in the corresponding circular ensembles are plotted at the bottom [91].](image)

| Ensemble name | CRE | CQE | CUE |
|---------------|-----|-----|-----|
| Symmetry class | D   | C   | A   |
| $S$-matrix elements | real | quaternion | complex |
| $S$-matrix space | orthogonal | symplectic | unitary |
| $d_T$ | 1 | 2 | 1 |
| $\alpha_T$ | $-1$ | 2 | 0 |
| $\beta_T$ | 1 | 4 | 2 |

**TABLE III** The three ensembles that support chiral edge modes, Majorana modes in classes D and C, and electronic modes in class A.

reflection and transmission subblocks,

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}. \quad (54)$$

Because of unitarity the transmission matrix products $tt^\dagger$ and $t't'^\dagger$ have the same set of eigenvalues $T_1, T_2, \ldots T_M$. These determine the thermal conductance

$$\kappa_0 = \frac{\sum_{n=1}^{M} T_n}{\kappa_{\text{ref}}}. \quad (55)$$

We denote the degeneracy of the transmission eigenvalues by $d_T$, without counting spin degeneracy (since we may exclude that from the very beginning by considering only a single spin band for electron and hole).

Notice the distinction between the degeneracy factors $d_T$ for eigenvalues of $tt^\dagger$ and $d_E$ for eigenvalues of $H$, as listed in Table I. In class D these are equal, $d_E = d_T = 1$, but in class C we have $d_E = 1$ while $d_T = 2$ because of a twofold Kramers degeneracy of the electron-hole degree of freedom. Kramers degeneracy of the $T_n$’s appears when $tt^\dagger$ commutes with $C$ squaring to $-1$. Because the Hamiltonian $H$ does not commute with $C$ (it anticommutes), there is no Kramers degeneracy of the energy levels in class C.

The probability distribution of the $M/d_T$ independent transmission eigenvalues in the CRE and CQE is given by [91]

$$P(\{T_n\} \propto \prod_{1 \leq i < j}^M |T_i - T_j|^{\beta_T} \prod_{k=1}^M T_k^{\beta_T/2 - 1} (1 - T_k)^{\alpha_T/2},$$

with exponents $\alpha_T, \beta_T$ listed in Table III. As one can see from comparison with Table I, these exponents for transmission eigenvalue repulsion are different from the exponents $\alpha_E, \beta_E$ that govern the repulsion of energy eigenvalues in Eq. (21).\textsuperscript{10}

\[10\] In the context of differential geometry, the parameters $\alpha_E + 1 \equiv m_y$ and $\beta_E \equiv m_a$ are root multiplicities characterizing the symmetric space of Hamiltonians, while $\alpha_T + 1$ and $\beta_T$ do the same for transfer matrices [92].
A. Thermopower and time-delay matrix

The thermopower geometry is shown in Fig. 12. In a scattering formulation two matrices enter, the scattering matrix at the Fermi level $S_0 = S(E = 0)$ and the Wigner-Smith time-delay matrix [94–96]

$$D = -i\hbar \lim_{E \to 0} S^\dagger \frac{dS}{dE}. \quad (58)$$

We define transmission and reflection submatrices as in Eq. (54), where the transmission matrices $t, t'$ couple the $N'$ Majorana edge modes to the $N$ electron-hole modes in the point contact, mediated by quasibound states in the quantum dot.

In the circular ensembles the joint distribution of $S_0$ and $D$ follows from the invariance [97, 98]

$$P[S(E)] = P[U \cdot S(E) \cdot U'] \quad (59)$$

of the distribution functional $P[S(E)]$ upon multiplication of the scattering matrix by a pair of energy-independent matrices $U, U'$, restricted by symmetry to a subset of the full unitary group (see Table III).

The time-delay matrix $D$ is a positive-definite Hermitian matrix. Its eigenvalues $D_n > 0$ are the delay times, and $\gamma_n \equiv 1/D_n$ are the corresponding rates, each with the same degeneracy $d_T$ as the transmission eigenvalues. It follows from the invariance (59) that $D$ and $S_0$ are independent, so the distribution of the $D_n$'s can be considered separately from the distribution (56) of the $T_n$'s. The distribution of the $\mathcal{N} = (N + N')/d_T$ distinct delay times is given by [98]

$$P(\{\gamma_n\}) \propto \prod_{k=1}^{\mathcal{N}} \Theta(\gamma_k) \gamma_k^{\alpha_T + \mathcal{N} \beta_T/2} \exp\left(-\frac{1}{2} \frac{\delta_T}{t_0} \gamma_k\right) \times \prod_{1 \leq i < j}^{\mathcal{N}} |\gamma_i - \gamma_j|^{\beta_T}, \quad t_0 = \frac{d_E}{\delta_T} \frac{2\pi\hbar}{\delta_0}, \quad (60)$$

with coefficients from Table III. The unit step function $\Theta(\gamma)$ ensures that the probability vanishes if any $\gamma_n$ is negative.

The Cutler-Mott formula for the thermopower [99]

$$\frac{S}{S_0} = -\lim_{E \to 0} \frac{1}{G} \frac{dG}{dE}, \quad S_0 = \frac{\pi^2 k_B^2 T_0}{3e}, \quad (61)$$

can be written in terms of the matrices $S_0$ and $D$,

$$\frac{S}{S_0} = i\hbar^{-1} \frac{\text{Tr} \tau_2 S_0 (DP - PD) S_0^\dagger}{N - \text{Tr} \tau_2 S_0 P \tau_2 S_0^\dagger}. \quad (62)$$

The Pauli matrix $\tau_2$ acts on the electron-hole degree of freedom, while $P$ projects onto the $N$ modes at the point contact. The commutator of $D$ and $P$ in the numerator ensures a vanishing thermopower in the absence of gapless modes in the superconductor, because then the projector $P$ is just the identity.

The resulting thermopower distributions, shown in Fig. 13 for a single-channel point contact, are qualitatively different for Majorana edge modes in class C ($d$-wave pairing) or class D ($p$-wave pairing). Like the thermal conductance of Sec. VI.B, the thermopower does not feel the presence or absence of a Majorana zero-mode in the quantum dot — to probe that one needs the electrical conductance (Sec. V.C).
VII. JOSEPHSON JUNCTIONS

The electrical and thermal conductance discussed in the previous two sections probe the system out of equilibrium, in response to a voltage or temperature difference. A superconductor can also support a persistent electrical current $I(\phi)$ in equilibrium, in response to a phase difference $\phi = \Phi e/\hbar$ across the junction. The nanowire geometry of Fig. 4, with a single superconducting electrode, can be converted into a Josephson junction by adding a second superconductor, as shown in Fig. 14. The segment of the wire between the two superconductors forms a quantum dot, a confined region with quasiparticle excitation spectrum $0 < E_0 < E_1 < E_2 < \cdots$. The spectrum depends on the phase difference $\phi$ of the pair potential across the junction, which can be controlled by the magnetic flux $\Phi$. Because a $2\pi$ increment of $\phi$ corresponds to a variation of $\Phi$ by $h/2e$, the excitation spectrum has the flux periodicity $E_n(\Phi) = E_n(\Phi + h/2e)$.

The presence of Majorana zero-modes in the quantum dot induces a period-doubling of the flux dependence, with a free energy $F(\Phi)$ that has $h/e$ rather than $h/(2e)$ flux-periodicity, or equivalently, a $4\pi$ rather than $2\pi$ phase-periodicity \[30\]. The mechanism behind the period doubling is a switch in the fermion parity of the superconducting condensate, as we now discuss.

A. Fermion parity switches

We mentioned in Sec. III.C that a quantum dot in symmetry class D (no time-reversal or spin-rotation symmetry) supports level crossings at the Fermi energy. The level crossings may appear upon variation of the chemical potential, as in Fig. 2, or they may appear when we vary the phase difference $\phi$ of the superconducting electrodes, as in Fig. 15. We denote by $E_{\pm}(\phi)$ the smooth (adiabatic) $\phi$-dependence of the pairs of crossing levels, arranged such that $E_+ < E_- = E_0$ at $\phi = 0$.

Level crossings are a signal of a conserved quantity, preventing transitions between the levels that would convert the crossing into an avoided crossing. In this case the conserved quantity is the parity $P = \pm 1$ of the number of electrons in the quantum dot. At low temperatures and for small charging energy\[11\] ($k_B T$ and $e^2/C \ll \Delta_0$), the quantum dot can only exchange pairs of electrons with the superconducting electrodes, so $P$ is conserved. A transition between $E_+$ and $E_-$ would add or remove an unpaired quasiparticle, which is forbidden and hence the crossing is protected.

\[11\] Existing experiments \[100, 101\] have mainly explored the opposite regime $e^2/C > \Delta_0$ of large charging energy.
The free energy $F_P$ of the quantum dot depends on whether the number of electrons is even ($P = +1$) or odd ($P = -1$) [102, 103]. In the zero-temperature limit, this dependence can be written as [104]

$$F_P = \frac{1}{2}(E_P - E_1 - E_2 - \cdots).$$  \hspace{1cm} (63)

(The factor $1/2$ ensures that the switch from $E_0$ to $E_\pm$ properly introduces an excitation energy $E_0$ rather than $2E_0$.) At $\phi = 0$ the level $E_\pm$ lies below $E_0$, so the superconducting condensate favors fully paired electrons and is said to be of even fermion parity. At the first crossing $E_\pm$ drops below $E_0$, so now a single unpaired electron is favored (odd fermion parity). The fermion-parity switch signaled by a level crossing is a topological phase transition of the superconducting condensate [30].

Sequences of fermion-parity switches are not independent. As illustrated in Fig. 16 for an InSb Josephson junction, the level crossings show an anti-bunching effect, with a spacing distribution that vanishes linearly at small spacings [40]. So while in symmetry class D there is no energy level repulsion at $E = 0$, there is a “level crossing repulsion”.

This effect can be connected to a classic problem in non-Hermitian random-matrix theory [105, 106]: How many eigenvalues of a real random matrix are real? The connection is made by identifying the phase $\phi_n$ of a level crossing with the real eigenvalue $\varepsilon_n = \tan(\phi_n/2)$ of the matrix

$$\mathcal{M} = (1 - O)(1 + O)^{-1}J,$$  \hspace{1cm} (64)

with $J$ defined in Eq. (31) and $O \in \text{SO}(2N)$. The $2N \times 2N$ orthogonal matrix $O = r_L r_R$ is the product of the reflection matrices from the left and right ends of the Josephson junction, each end supporting $N$ electron modes and $N$ hole modes. The matrix $O$ is real because both $r_L$ and $r_R$ are taken in the Majorana basis (44), and $\text{Det} O = +1$ because $\text{Det} r_L = \text{Det} r_R = \pm 1$.

Fig. 16 shows good agreement between the spacing distribution from a random-matrix ensemble for $O$ and from a computer simulation of the InSb Josephson junction. Once the spacing is normalized by the average spacing, there are no adjustable parameters in this comparison. The linearly vanishing spacing distribution is reminiscent of the Wigner distribution in the Gaussian orthogonal ensemble [1], but for larger spacings the distribution has approximately the Poisson form of uncorrelated eigenvalues [40, 107]. This difference can be understood as a “screening” by intervening complex eigenvalues of $\mathcal{M}$ of the eigenvalue repulsion on the real axis.

B. 4$\pi$-periodic Josephson effect

One can distinguish topologically trivial from nontrivial Josephson junctions by counting the number $N_X$ of level crossings when $\phi$ is incremented by $2\pi$. For the spectrum shown in Fig. 15a the number $N_X = 4$ is even, this is the topologically trivial case. Alternatively, if $N_X$ odd the Josephson junction is topologically nontrivial, see Fig. 15b.

The free energy (63) is $2\pi$-periodic when $N_X$ is even and $4\pi$-periodic when $N_X$ is odd. This period doubling can be observed via the supercurrent flowing through the ring,

$$I_\pm(\phi) = \frac{2e}{\hbar} \frac{dF_\pm}{d\phi} = \frac{e}{\hbar} \frac{dE_\pm}{d\phi} + 2\pi$ \text{-periodic terms}. \hspace{1cm} (65)$$

The dependence on the enclosed flux $\Phi = \phi \hbar/e$ has $\hbar/2e$ periodicity when $N_X$ is even, but a doubled $\hbar/e$ periodicity when $N_X$ is odd. This is the 4$\pi$-periodic Josephson effect [30, 108].

Because $I_+(\phi) = I_-(\phi + 2\pi)$, the $2\pi$ periodicity is restored when a quasiparticle can enter or leave the junction during the measurement time, which severely complicates the observation of the effect [109].

VIII. DISCUSSION AND OUTLOOK

To be added later, when feedback on this first version has been incorporated.

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