The effect of quadrature rules on finite element solutions of Maxwell variational problems

Consistency estimates on meshes with straight and curved elements

Rubén Aylwin · Carlos Jerez-Hanckes

Received: 14 July 2019 / Revised: 28 September 2020 / Accepted: 30 January 2021 / Published online: 27 February 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH, DE part of Springer Nature 2021

Abstract
We study the effects of numerical quadrature rules on error convergence rates when solving Maxwell-type variational problems via the curl-conforming or edge finite element method. A complete a priori error analysis for the case of bounded polygonal and curved domains with non-homogeneous coefficients is provided. We detail sufficient conditions with respect to mesh refinement and precision for the quadrature rules so as to guarantee convergence rates following that of exact numerical integration. On curved domains, we isolate the error contribution of numerical quadrature rules.

Mathematics Subject Classification 35Q61 · 65N30 · 65N12

1 Introduction
We provide a complete error analysis on the effects of numerical integration when approximating Maxwell solutions via finite elements (FEs). Specifically, we consider a range of problems set in bounded domains with perfectly conducting boundary conditions (PEC). Through our analysis, we find conditions for quadrature rules to guarantee orders of convergence, with respect to the mesh-size, of the error associated with numerical approximation of exact integration.

This work was supported in part by Fondecyt Regular 1171491 and doctoral grant Conicyt-PFCHA 2017-21171791.

Carlos Jerez-Hanckes
Carlos.jerez@uai.cl
Rubén Aylwin
rdaylwin@uc.cl

1 School of Engineering, Pontificia Universidad Católica de Chile, Santiago, Chile
2 Faculty of Engineering and Sciences, Universidad Adolfo Ibáñez, Santiago, Chile
Strang-type lemmas have long been available for different types of problems: elliptic \[5,10–12\]; non-linear elliptic \[1\]; fourth-order elliptic problems \[6\]; and, eigenvalue problems \[3,4,27\]. However, and to our knowledge, similar results for Maxwell-type variational formulations are unavailable, thus motivating the present work.

Our main results are Theorems 1 and 4, in Sects. 3 and 4, respectively. The latter presents an estimate for the error convergence rate between fully discrete and continuous solutions on polygonal domains, specifying sufficient the conditions on the quadrature rules to ensure the same convergence rate one would obtain with exact integration. The former drops the assumption that the domain be polygonal and gives conditions on quadrature rules to ensure a desired convergence rate of the error terms that spawn from considering numerical integration. As is to be expected, the conditions on the quadrature rules will depend on the polynomial degree of FE approximation spaces and the degree of precision used to mesh the domain when not polygonal. Smoothness of parameters and of the continuous solution will only limit the maximum possible convergence rate.

In \[2\], we showed error estimates for fully discrete solutions of a Maxwell-type problem with inhomogeneous coefficients on a tetrahedral and quasi-uniform sequence of affine meshes, which was of importance in the uncertainty quantification (UQ) setting there considered. Our present results can be regarded as a generalization of Theorem 3.17 in \[2\] to account for inhomogeneous and/or anisotropic materials as well as for the implementation of meshes with curved elements (cf. \[23, Sec. 8.3\] and references therein). As in \[2\], a key tool throughout our analysis on affine meshes—with straight tetrahedrons as elements—will be the quasi-interpolation operators developed in \[16\]. These operators require very low smoothness: no greater than \(L^1\) from the interpolated function, whereas the canonical interpolation operator requires a minimum smoothness (cf. \[15,16,23\]). Coupling these results with standard estimates for the error convergence of FE solutions allow us to present a complete analysis of the convergence of fully discrete solutions of Maxwell equations on polyhedral domains.

We shall not consider an analogous result on curved meshes, as their use proves advantageous only when the solution has some minimum smoothness. As interpolation on curved elements lies beyond the scope of this work, we refer to \[6,11,20,22\] and references therein as examples of strategies when dealing with curved boundaries. Incidentally, we will mainly use the strategy presented in \[11\] to estimate the perturbations generated by the introduction of quadrature rules. Hence, we shall isolate the impact of numerical integration on the error convergence rate of fully discrete solutions and seek to find convergence rates for those specific contributing terms.

The structure of the article is as follows. In Sect. 2 we set notation to be used throughout, introduce Maxwell equations and fix the general structure of the variational problems considered. Sections 3 and 4 concern themselves with the analysis of the convergence rates of fully discrete solutions, i.e., when considering numerical integration of the previously introduced problems on polygonal domains and on domains with curved boundaries, respectively. Numerical examples are presented in Sect. 5 and are followed by concluding remarks in Sect. 6.
2 General definitions and Maxwell variational problems

We start by setting the notation used in the following sections, and continue by introducing the general form of the variational problems analysed.

2.1 Notation

For \( d = 1, 2, 3 \) we consider \( \Omega \subset \mathbb{R}^d \) an open and bounded Lipschitz domain. For \( m \in \mathbb{N}, C^m(\Omega) \) denotes the set of real valued functions with \( m \)-continuous derivatives on \( \Omega \). For \( k \in \mathbb{N}, \mathbb{P}_k(\Omega; \mathbb{C}^q) \) denotes the space of functions from \( \Omega \) to \( \mathbb{C}^q \) with polynomials of degree less than or equal to \( k \) in their \( q \) components and \( \mathbb{P}_k(\Omega; \mathbb{C}^q) \) denotes the space of elements of \( \mathbb{P}_k(\Omega; \mathbb{C}^q) \) of degree exactly \( k \) in their \( q \) components.

Let \( p \geq 1 \) and \( s \in \mathbb{R} \), then \( L^p(\Omega) \) and \( W^{p,s}(\Omega) \) denote the class of \( p \)-integrable functions on \( \Omega \) with values in \( \mathbb{C} \) and the standard Sobolev spaces, respectively. If \( p = 2 \), we use the standard notation \( H^s(\Omega) := W^{2,s}(\Omega) \). Boldface symbols will be used to differentiate general scalar valued function spaces from their vector valued counterparts.

Norms and seminorms over a general Banach space \( Y \) are indicated by subscript (\( \| \cdot \|_Y \) and \( | \cdot |_Y \)). We make an exception for \( H^s(\Omega) \), whose norm and seminorm will be written as \( \| \cdot \|_{s,\Omega} \) and \( | \cdot |_{s,\Omega} \). The dual of the Banach space \( Y \) is denoted \( Y' \). For a Hilbert space \( X \) (real or complex) its inner product is denoted as \( (\cdot, \cdot)_X \), while duality products are denoted by \( \langle \cdot, \cdot \rangle_{X' \times X} \). The same exception is made for Sobolev spaces as in the case of norms and seminorms. Both duality and inner products are understood in the sesquilinear sense.

2.2 Functional spaces

We shall require the following functional space of vector valued functions with integrable curl and divergence:

\[
H(\text{curl}; \Omega) := \left\{ U \in L^2(\Omega) : \text{curl} \, U \in L^2(\Omega) \right\},
\]

\[
H(\text{div}; \Omega) := \left\{ U \in L^2(\Omega) : \text{div} \, U \in L^2(\Omega) \right\},
\]

which are Hilbert spaces when paired with the following inner products

\[
(U, V)_{H(\text{curl}; \Omega)} := (U, V)_{0,\Omega} + (\text{curl} \, U, \text{curl} \, V)_{0,\Omega},
\]

\[
(U, V)_{H(\text{div}; \Omega)} := (U, V)_{0,\Omega} + (\text{div} \, U, \text{div} \, V)_{0,\Omega}.
\]

For \( s > 0 \), we introduce the following extension to \( H(\text{curl}; \Omega) \) of functions in \( H^s(\Omega) \) with curl in \( H^s(\Omega) \) [23, Sec. 3.5.3]:

\[
H^s(\text{curl}; \Omega) := \left\{ U \in H^s(\Omega) : \text{curl} \, U \in H^s(\Omega) \right\}
\]
with norm
\[ \| U \|_{H^s(\text{curl}; \Omega)} := \left( \| \text{curl} \ U \|_{s, \Omega}^2 + \| U \|_{s, \Omega}^2 \right)^{\frac{1}{2}}. \]

We also introduce the following subspace of \( H(\text{curl}; \Omega) \),

\[ H(\text{curl curl}; \Omega) := \left\{ U \in H(\text{curl}; \Omega) : \text{curl curl} \ U \in L^2(\Omega) \right\} \]

and following trace spaces [7,9,23]:

\[ H^{-\frac{1}{2}}(\text{div}; \partial \Omega) := \left\{ U \in H^{-\frac{1}{2}}(\partial \Omega) : U \cdot n = 0, \ \text{div}_{\partial \Omega} U \in H^{-\frac{1}{2}}(\partial \Omega) \right\}, \]

\[ H^{-\frac{1}{2}}(\text{curl}; \partial \Omega) := \left\{ U \in H^{-\frac{1}{2}}(\partial \Omega) : U \cdot n = 0, \ \text{curl}_{\partial \Omega} U \in H^{-\frac{1}{2}}(\partial \Omega) \right\} \]

where \( n \) is the outward normal vector from \( \Omega \), \( \text{div}_{\partial \Omega} \) is the surface divergence operator and \( \text{curl}_{\partial \Omega} \) is the surface scalar curl operator, respectively (cf. [7,9]). Also, by [8, Thm. 2] note that

\[ H^{-\frac{1}{2}}(\text{curl}; \partial \Omega) = \left( H^{-\frac{1}{2}}(\text{div}; \partial \Omega) \right)' \]

**Definition 1** Let \( U \in C^\infty(\overline{\Omega}) \), then

\[ \gamma_D U := n \times (U \times n)|_{\partial \Omega}, \quad \gamma_D^\times U := (n \times U)|_{\partial \Omega}, \]

\[ \gamma_N U := (n \cdot U)|_{\partial \Omega}, \quad \text{and} \quad \gamma_N U := (n \times \text{curl} U)|_{\partial \Omega} \]

are the Dirichlet trace, flipped Dirichlet trace, normal trace and Neumann trace, respectively.

The Dirichlet trace operators in Definition 1 can be extended to linear and continuous operators from \( H(\text{curl}; \Omega) \) to \( H^{-\frac{1}{2}}(\partial \Omega) \) [23, Thms. 3.29 and 3.31]. Specifically, one sees that

\[ \text{Im}(\gamma_D) = H^{-\frac{1}{2}}(\text{curl}; \partial \Omega), \quad \text{Im}(\gamma_D^\times) = H^{-\frac{1}{2}}(\text{div}; \partial \Omega) \]

allowing us to endow this spaces with the corresponding graph norms given by the trace operators. Similarly, the normal trace operator can be extended to a linear and continuous operator [23, Thm. 3.24]:

\[ \gamma_N : H(\text{div}; \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \]

while the Neumann trace may be extended as [7, Thm. 3.2]

\[ \gamma_N : H(\text{curl curl}; \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega). \]

With the trace operators \( \gamma_D^\times \) and \( \gamma_N \), we define

\[ \odot \text{ Springer} \]
\[ H_0(\text{curl}; \Omega) := \{ U \in H(\text{curl}; \Omega) : \gamma_D^X U = 0 \text{ on } \partial \Omega \} , \tag{2.1} \]
\[ H_0(\text{div}; \Omega) := \{ U \in H(\text{div}; \Omega) : \gamma_D U = 0 \text{ on } \partial \Omega \} . \tag{2.2} \]

By continuity of \( \gamma_D^X \), \( H_0(\text{curl}; \Omega) \) is a closed subspace of \( H(\text{curl}; \Omega) \) (analogously, \( H_0(\text{div}; \Omega) \) is a closed subspace of \( H(\text{div}; \Omega) \)). Finally, for \( U \) and \( V \in H(\text{curl}, \Omega) \) there holds [7, Eq. (27)]:

\[ (U, \text{curl} V)_\Omega - (\text{curl} U, V)_\Omega = -\langle \gamma_D^X U, \gamma_D V \rangle_{\partial \Omega} \tag{2.3} \]

where \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) denotes the duality between \( H^{-\frac{1}{2}}_{\text{div}}(\partial \Omega) \) and \( H^{-\frac{1}{2}}_{\text{curl}}(\partial \Omega) \).

### 2.3 Maxwell equations

We consider an open bounded Lipschitz domain \( D \subset \mathbb{R}^3 \) with boundary \( \Gamma = \partial D \) as well as a time-harmonic dependence \( e^{i \omega t} \) with circular frequency \( \omega > 0 \). We write \( E \) and \( H \) for the complex-valued electric and magnetic fields, respectively. Harmonic Maxwell equations on \( D \) read

\[ \text{curl } E + i \omega \mu H = 0, \tag{2.4} \]
\[ i \omega \varepsilon E - \text{curl } H = -J \]

where \( \mu \) and \( \varepsilon \) are assumed to be symmetric matrix-valued functions with coefficients in \( L^\infty(D) \), and \( J \) is an imposed current, usually—but not necessarily—compactly supported in \( D \).

**Assumption 1** *(Basic assumptions on the parameters)* Both \( \mu \) and \( \varepsilon \) are complex symmetric matrix-valued functions with coefficients in \( L^\infty(D) \). Furthermore, \( \mu \) has a pointwise inverse, denoted \( \mu^{-1} \), almost everywhere on \( D \).

The Maxwell system (2.4) is commonly reduced to a second order partial differential equation by removing either \( E \) or \( H \). We consider the following reduction:

\[ \text{curl } \mu^{-1} \text{curl } E - \omega^2 \varepsilon E = -i \omega J. \tag{2.5} \]

The system is completed by imposing PEC boundary conditions on the surface \( \Gamma \)

\[ \gamma_D^X E = 0. \tag{2.6} \]

### 2.4 Variational formulation

We proceed as in [2] and introduce the sesquilinear and antilinear forms associated to equations (2.5) and (2.6), defined for \( U \) and \( V \in H_0(\text{curl}; D) \)

\[ \Phi(U, V) := \int_D \mu^{-1} \text{curl } U \cdot \text{curl } V - \omega^2 \varepsilon U \cdot V \, dx, \tag{2.7} \]
\[ F(V) := -i \omega \int_D \mathbf{J} \cdot \nabla \mathbf{V} \, dx, \quad (2.8) \]

both continuous on \( H_0(\text{curl}; D) \).

**Problem 1 (Continuous variational problem)** Find \( E \in H_0(\text{curl}; D) \) such that

\[ \Phi(E, V) = F(V), \]

for all \( V \in H_0(\text{curl}; D) \).

Since we are only interested in the effect of numerical integration when discretizing Problem 1, we assume the sesquilinear form in (2.7) satisfies all necessary conditions for there to be a unique solution of Problem 1 that depends continually on the data.

**Assumption 2 (Wellposedness)** We assume the sesquilinear form \( \Phi \) in (2.7) to satisfy the following conditions:

\[
\begin{align*}
\sup_{U \in H_0(\text{curl}; D) \setminus \{0\}} |\Phi(U, V)| > 0 & \quad \forall V \in H_0(\text{curl}; D) \setminus \{0\}, \\
\inf_{U \in H_0(\text{curl}; D) \setminus \{0\}} \left( \sup_{V \in H_0(\text{curl}; D) \setminus \{0\}} \frac{|\Phi(U, V)|}{\|U\|_{H(\text{curl}; D)} \|V\|_{H(\text{curl}; D)}} \right) & \geq C > 0.
\end{align*}
\]

By Assumption 2 and the continuity of \( \Phi \) and \( F \) in (2.7) and (2.8), there exists a unique solution \( E \in H_0(\text{curl}; D) \) for Problem 1. For examples of variational problems with a structure analogous to that of \( \Phi \) in (2.7) we refer to [17], where two different problems concerning Maxwell equations are found to be coercive—i.e. \( |\Phi(U, U)| / \|U\|^2_{H(\text{curl}; D)} \geq \alpha > 0 \) for all \( U \in H_0(\text{curl}; D) \)—[23, Chap. 4.7] and references therein (incidentally, the problem analysed in [2] in the context of UQ is one of the problems in [17]).

### 3 Finite Elements and Consistency Error Estimates for Polyhedral Domains

In what follows, we concern ourselves with discretizations of Problem 1. We shall construct a sequence of meshes \( \{\tau_h\}_{h>0} \), from which we construct discrete subspaces of \( H_0(\text{curl}; D) \) in order to approximate the solution of Problem 1. We begin our analysis by assuming \( D \) to be polyhedral, so that meshes \( \tau_h \) constructed from tetrahedrons cover \( D \) exactly. We shall extend our analysis to curved domains and consider non-affine meshes on the following section.

**Assumption 3 (Polyhedral domain)** The open domain \( D \) is polyhedral.

#### 3.1 Finite elements

Let \( \{\tau_h\}_{h \in \mathbb{N}} \) be a sequence of quasi-uniform meshes constructed from disjoint, matching tetrahedrons—\( K \in \tau_h \) for each mesh \( \tau_h \) in the sequence—that cover \( D \) exactly,
where the subindex \( h > 0 \) refers to the mesh-size of each mesh in the sequence and where \( h_i \to 0 \) as \( i \in \mathbb{N} \) grows to infinity.

**Assumption 4 (Assumptions on the meshes)** The meshes in \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \) are affine, quasi-uniform and cover \( D \) exactly.

**Definition 2 (Reference element)** We define \( \tilde{K} \) as the tetrahedron with vertices \( 0, e_1, e_2 \) and \( e_3 \); and refer to it as the reference element.

**Definition 3** For any \( i \in \mathbb{N} \) and each \( K \in \tau_{h_i} \) we define \( T_K : \tilde{K} \mapsto K \) as affine, bijective mappings from the reference tetrahedron to arbitrary \( K \in \tau_{h_i} \). We denote the Jacobians of these mappings as \( J_K \).

The elements from the mesh, i.e. \( K \in \tau_h \), may be considered as constructed from the reference tetrahedron \( \tilde{K} \) through the mappings \( T_K \) introduced in Definition 3.

**Definition 4 (Finite elements)** We will consider finite elements as triples \( (K, P_K, \Sigma_K) \), with \( K \in \tau_h \), \( P_K \) a space of functions over \( K \) (usually polynomials) and \( \Sigma_K := \{ \sigma_i \}_{i=1}^n \}, n \in \mathbb{N} \) a set of linear functionals acting on \( P_K \) (cf. [23]).

Let \( k \in \mathbb{N} \). Since we are considering only Maxwell equations, we will only work with the finite element \( (K, P^c_K, \Sigma^c_K) \) as defined in [23, Chap. 5] and corresponding to curl-conforming elements,

\[
P^c_K := \mathbb{P}_{k-1}(K; \mathbb{R}^3) \oplus \{ p \in \mathbb{P}_k(K, \mathbb{R}^3) : x \cdot p = 0 \}.
\]

For completeness, we also introduce the function spaces for grad-, div and \( L^2 \)-conforming finite elements:

\[
\begin{align*}
P^g_K & := \mathbb{P}_k(K; \mathbb{C}), \\
P^d_K & := \mathbb{P}_{k-1}(K; \mathbb{C}^3) \oplus \{ xp : p \in \mathbb{P}_{k-1}(K, \mathbb{C}) \}, \\
P^b_K & := \mathbb{P}_{k-1}(K; \mathbb{C}).
\end{align*}
\]

We refer to [23, Chap. 5] for definitions of the degrees of freedom \( \Sigma^g_K, \Sigma^c_K \) and \( \Sigma^d_K \), corresponding to the spaces \( P^g_K, P^c_K \) and \( P^d_K \) respectively.

From here onwards, let \( k \in \mathbb{N} \) be fixed as the polynomial degree of our approximation spaces and let \( \tau_h \) be an arbitrary mesh in the sequence \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \), where the subindex \( h \) represents the size of the mesh, as before. Discrete spaces on \( \tau_h \) are constructed as follows

\[
\begin{align*}
P^g(\tau_h) & := \left\{ v_h \in H^1(\mathbb{D}) : v_h|_K \in P^g_K \right\}, \\
P^c(\tau_h) & := \left\{ V_h \in H(\text{curl}; \mathbb{D}) : V_h|_K \in P^c_K \quad \forall K \in \tau_h \right\}, \\
P^d(\tau_h) & := \left\{ V_h \in H(\text{div}; \mathbb{D}) : V_h|_K \in P^d_K \quad \forall K \in \tau_h \right\}, \\
P^b(\tau_h) & := \left\{ v_h \in L^2(\mathbb{D}) : v_h|_K \in P^b_K \quad \forall K \in \tau_h \right\}.
\end{align*}
\]
Homogeneous essential boundary conditions are accounted for by imposing the conditions at the boundary:

\[ P_0^g(\tau_h) := P^g(\tau_h) \cap H^1_0(D), \]
\[ P_0^c(\tau_h) := P^c(\tau_h) \cap H_0^\text{curl}(\Omega), \]
\[ P_0^d(\tau_h) := P^d(\tau_h) \cap H_0^\text{div}(\Omega). \]

We introduce, for every \( K \in \tau_h \) the following pullbacks to the reference element \( \tilde{K} \):

\[ \psi^g_K(v) := v \circ T_K, \quad \psi^c_K(V) := \int_{\tilde{K}} (V \circ T_K), \]
\[ \psi^d_K(V) := \det(\int_{\tilde{K}}) \int_{\tilde{K}}^{-1} (V \circ T_K), \quad \psi^b_K(v) := \det(\int_{\tilde{K}}) (v \circ T_K), \tag{3.1} \]

where \( v \in L^p(K) \) and \( V \in L^p(K) \), \( p \geq 1 \). We continue by stating some useful properties of the pullbacks in (3.1) and refer to [15] for their proofs. First, the mappings in (3.1) commute with the differential operators, i.e.

\[ \nabla \psi^g_K(v) = \psi^g_K(\nabla v), \quad \text{curl } \psi^c_K(V) = \psi^d_K(\text{curl } V), \quad \text{div } \psi^d_K(V) = \psi^b_K(\text{div } V), \]

for all \( K \in \tau_h \), for all functions \( v \) with well defined gradient, and \( V \) with well defined curl or divergence, respectively. Furthermore, the finite element spaces for functions with well defined gradient, curl and divergence are invariant with respect to their respective pullback to \( \tilde{K} \) so that, for all \( K \in \tau_h \) and \( j \in \{g, c, d, b\} \), it holds that

\[ \psi^j_K : P^j_K \to P^j_{\tilde{K}}, \quad (\psi^j_K)^{-1} : P^j_{\tilde{K}} \to P^j_K. \]

Under Assumption 4, there exist uniform positive constants \( c^g \) and \( c^b \) such that:

\[ |\det(\int_{\tilde{K}})| = |K| \left| \int_{\tilde{K}} \right|^{-1}, \quad \|\int_{\tilde{K}}\|_{\mathbb{R}^{3x3}} \leq c^g h, \quad \left\| \int_{\tilde{K}}^{-1} \right\|_{\mathbb{R}^{3x3}} \leq c^b h^{-1}. \tag{3.2} \]

As in [16,17], we continue by summarizing the linear mappings defined in (3.1) as

\[ \psi_K(v) = \mathbb{A}_K(v \circ T_K), \quad \text{or } \psi_K(V) = \mathbb{A}_K(V \circ T_K), \tag{3.3} \]

to avoid repeating the properties (3.4) and (3.5), where \( \mathbb{A}_K = 1, \int_{\tilde{K}}, \det(\int_{\tilde{K}}) \int_{\tilde{K}}^{-1} \) or \( \det(\int_{\tilde{K}}) \) and \( q = 1 \) or \( 3 \) depending on the choice of \( \mathbb{A}_K \). Then, for all \( K \in \tau_h \), \( p \geq 1 \) and \( l \in \mathbb{N}_0 \) the mappings in (3.1) satisfy

\[ |\psi_K|_{L^1(W^{l,p}(K);W^{l,p}(\tilde{K}))} \leq c \left\| \mathbb{A}_K \right\|_{\mathbb{R}^{q \times q}} \left\| \int_{\tilde{K}} \right\|_{\mathbb{R}^{3 \times 3}} |\det(\int_{\tilde{K}})|^{-\frac{1}{p}}, \tag{3.4} \]
\[ |\psi^{-1}_K|_{L^1(W^{l,p}(K);W^{l,p}(\tilde{K}))} \leq c \left\| \mathbb{A}_K^{-1} \right\|_{\mathbb{R}^{q \times q}} \left\| \int_{\tilde{K}}^{-1} \right\|_{\mathbb{R}^{3 \times 3}} |\det(\int_{\tilde{K}})|^{\frac{1}{p}}. \tag{3.5} \]

We continue by stating some Lemmas that will be required during the proof of the main result of this section.
Lemma 1 (Lemma 1.138 in [15]) Let $K \in \tau_h$ and $m, l \in \mathbb{N}_0$ such that $m \leq l$. Then, under Assumption 4, for $\phi$ in either $P^c_K$, $P^d_K$ or $P^b_K$,

$$\| \phi \|_{W^{l-1,p}(K;\mathbb{R}^q)} \leq ch^{m-l} \| \phi \|_{W^{m,p}(K;\mathbb{R}^q)},$$

for a positive constant $c$ independent of $K$ and the mesh $\tau_h$ and $q = 1$ or $3$ depending on the finite element.

Lemma 2 (Quasi-interpolation operator [16]) Let $K \in \tau_h$. For each one of the function spaces $P^c(K)$ and $P^d(K)$ there exists a quasi-interpolation operator, denoted $I_{K}^{\#,c}$ and $I_{K}^{\#,d}$ respectively, and a positive constant $c > 0$ independent of $K$ and $h$ such that for all $V \in W^{m,p}(K)$, $I_{K}^{\#,c}(V) \in P^c(K)$, $I_{K}^{\#,d}(V) \in P^d(K)$,

$$\| V - I_{K}^{\#,c}(V) \|_{W^{m,p}(K)} \leq c h^{r-m} \| V \|_{W^{r,p}(K)}$$

and

$$\| V - I_{K}^{\#,d}(V) \|_{W^{m,p}(K)} \leq c h^{r-m} \| V \|_{W^{r,p}(K)},$$

for all $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $p \in \mathbb{R}$ with $p \geq 1$ and $m \leq r \leq k$, where $h_K$ is the diameter of $K$. Furthermore, $P^c(K)$ and $P^d(K)$ are invariant with respect to their corresponding quasi-interpolation operators.

Lemma 3 (Lemma 3.5 in [2]) Let $K \in \tau_h$ and $I_{K}^{\#}$ denote either $I_{K}^{\#,c}$ or $I_{K}^{\#,d}$. Then there exists a constant $c > 0$ independent of $K$ and $h$ such that

$$\| I_{K}^{\#}(V) \|_{W^{m,p}(K)} \leq c \| V \|_{W^{m,p}(K)},$$

(3.6)

$$\| I_{K}^{\#}(V) \|_{W^{m,p}(K)} \leq c h^{-1} \| V \|_{W^{m-1,p}(K)},$$

(3.7)

for all $m \in \mathbb{N}$ with $m \leq k$ and $p \in [1, \infty]$. Furthermore, for $r \in \mathbb{R}$ such that $r \leq k$ it holds

$$\| I_{K}^{\#}(V) \|_{H^r(K)} \leq ch^{r-[r]} \| V \|_{H^r(K)}.$$  

(3.8)

Proof The estimates in (3.6) and (3.7) are consequence of the error estimate in Lemma 2, while (3.8) follows from both (3.6) and (3.7) by performing real interpolation between Sobolev spaces (cf. [26, Lem. 22.3]).
3.2 Discrete variational problem

With the previous definitions of curl-conforming discrete spaces at hand, we can now state the discrete version of Problem 1.

**Problem 2** *(Discrete variational problem on affine meshes)* Find \( E_h \in P_0^c(\tau_h) \) such that

\[
\Phi(E_h, V_h) = F(V_h),
\]

for all \( V_h \in P_0^c(\tau_h) \).

As with Assumption 2, we assume our framework to be such that a unique discrete solution \( E_h \in P_0^c(\tau_h) \) exists for all meshes \( \tau_h \in \{\tau_{hi}\}_{i \in \mathbb{N}} \).

**Assumption 5** *(Wellposedness on \( P_0^c(\tau_h) \)) We assume the sesquilinear form \( \Phi \) in (2.7) to satisfy the following:

\[
\sup_{U_h \in P_0^c(\tau_h) \backslash \{0\}} |\Phi(U_h, V_h)| > \alpha > 0 \quad \forall V_h \in P_0^c(\tau_h) \backslash \{0\},
\]

\[
\inf_{U_h \in P_0^c(\tau_h) \backslash \{0\}} \left( \sup_{V_h \in P_0^c(\tau_h) \backslash \{0\}} \frac{|\Phi(U_h, V_h)|}{\|U_h\|_{H(\text{curl};D)} \|V_h\|_{H(\text{curl};D)}} \right) \geq C > 0,
\]

on all meshes \( \tau_h \in \{\tau_{hi}\}_{i \in \mathbb{N}} \).

From Assumptions 2 and 5, the continuity of the sesquilinear and antilinear forms in (2.7) and (2.8) and the fact that \( P_0^c(\tau_h) \subset H_0(\text{curl};D) \) we see that both Problem 1 and 2 have unique solutions \( E \in H_0(\text{curl};D) \) and \( E_h \in P_0^c(\tau_h) \), respectively. Furthermore, if \( E_{hi} \in P_0^c(\tau_{hi}) \) solves Problem 2 on \( P_0^c(\tau_{hi}) \) then

\[
\|E - E_{hi}\|_{H(\text{curl};D)} \leq C \inf_{U_{hi} \in P_0^c(\tau_{hi})} \|E - U_{hi}\|_{H(\text{curl};D)},
\]

for some \( C > 0 \) independent of the mesh-size.

3.3 Numerical integration and main results

We now introduce quadrature rules for the numerical computation of the linear system terms associated with Problem 2.

**Definition 5** For \( L \in \mathbb{N} \), we define \( Q_{\tilde{K}} \), a quadrature rule over \( \tilde{K} \), as a linear functional acting on \( \phi \in C(\tilde{K}) \) in the following way:

\[
Q_{\tilde{K}}(\phi) := \sum_{l=1}^{L} \tilde{w}_l \phi(\tilde{b}_l),
\]
where \( \{ \bar{w}_l \}_{l=1}^L \subset \mathbb{R} \) is a set of quadrature weights and \( \{ \bar{b}_l \}_{l=1}^L \subset \mathcal{K} \) denotes a set of quadrature points.

Quadratures over arbitrary elements \( K \in \tau_h \) are built from those in Definition 5, for \( \phi \in \mathcal{C}(K) \), as follows

\[
Q_K(\phi) := \sum_{l=1}^L w_{l,K} \phi(\bar{b}_{l,K}) \quad \text{with} \quad w_{l,K} := |\det(\mathbb{J}_K)| \bar{w}_l \quad \text{and} \quad b_{l,K} := \mathbf{T}_K(\bar{b}_l),
\]

for \( T_K \) and \( \mathbb{J}_K \) as in Definition 3.

**Definition 6 (Numerical sesquilinear and antilinear forms)** Let \( Q^1_K, Q^2_K \) and \( Q^3_K \) be three distinct quadrature rules as in Definition 5. We denote by \( \tilde{\Phi}_h(\cdot, \cdot) \) and \( \tilde{\mathbf{F}}_h(\cdot) \) the perturbed, discrete, sesquilinear and antilinear forms over fields in \( P^c(\tau_h) \), where exact integration is replaced by numerical integration:

\[
\tilde{\Phi}_h(U_h, V_h) := Q^1_K(\mu^{-1} \text{curl} \ U_h \cdot \text{curl} \ \nabla h) + Q^2_K(-\omega^2 \epsilon U_h \cdot V_h),
\]

\[
\tilde{\mathbf{F}}_h(V_h) := Q^3_K(-i \omega \mathbf{J} \cdot \nabla h),
\]

where, for \( i \in \{1, 2, 3\}, Q^i_K \) is built from \( Q^i_K \) as in (3.9).

**Problem 3 (Discrete numerical problem)** Find \( \tilde{E}_h \in P^c_0(\tau_h) \) such that,

\[
\tilde{\Phi}(\tilde{E}_h, V_h) = \tilde{\mathbf{F}}(V_h),
\]

for all \( V_h \in P^c_0(\tau_h) \).

Our objective is to obtain error convergence rates estimates for the solution of Problem 3 with respect to the solution of Problem 1. As such, Strang’s Lemma (cf. [24, Sect. 4.2.4]) will be key throughout our analysis on this and the following section.

**Lemma 4** (Strang’s Lemma. Theorem 4.2.11 in [24]) Let \( \Phi \) in (2.7) satisfy Assumptions 2 and 5 and let \( \mathbf{E} \) and \( \mathbf{E}_{h_i} \) be the solutions of Problems 1 and 2, respectively. If the sequence of sesquilinear forms \( \{ \tilde{\Phi}_{h_i} \}_{i \in \mathbb{N}} \) given by Definition 6 satisfies:

\[
|\Phi(U_{h_i}, V_{h_i}) - \tilde{\Phi}_{h_i}(U_{h_i}, V_{h_i})| \leq c h_i^{\ell} \| U_{h_i} \|_{H^r(\text{curl}; D)} \| V_{h_i} \|_{H(\text{curl}; D)},
\]

\[
\forall \ U_{h_i}, V_{h_i} \in P^c_0(\tau_{h_i}),
\]

for a fixed and positive constant \( c \) independent of the mesh-size, then there is some \( \ell \in \mathbb{N} \) such that for all the meshes in the sequence \( \{ \tau_{h_i} \}_{i \in \mathbb{N}}, i > \ell \) there exists a unique solution to Problem 3, \( \mathbf{E}_{h_i} \in P^c_0(\tau_{h_i}) \) satisfying
\[ \| E - \tilde{E}_{hi} \|_{H(\text{curl}; D)} \leq C_S \left( \| E - E_{hi} \|_{H(\text{curl}; D)} + \sup_{\mathbf{v}_{hi} \in P_0^c(\tau_{hi}) \setminus \{0\}} \frac{\Phi(E_{hi}, \mathbf{v}_{hi}) - \tilde{\Phi}_{hi}(E_{hi}, \mathbf{v}_{hi})}{\| \mathbf{v}_{hi} \|_{H(\text{curl}; D)}} \right) + \sup_{\mathbf{v}_{hi} \in P_0^c(\tau_{hi}) \setminus \{0\}} \frac{| \Phi(E_{hi}, \mathbf{v}_{hi}) - \tilde{\Phi}_{hi}(E_{hi}, \mathbf{v}_{hi}) |}{\| \mathbf{v}_{hi} \|_{H(\text{curl}; D)}} \]

for a fixed positive constant \( C_S \), independent of the mesh-size.

We are now ready to state the main result of this section whose proof is provided later on.

**Theorem 1** (Error estimate in affine meshes. Main result of Sect. 3) Let \( E \) be the unique solution to Problem 1 and assume that \( E \) and problem data satisfy

\[ E \in H^r(\text{curl}; D), \quad J \in W[^{\lceil r \rceil}]^q(D), \quad \text{and} \quad \epsilon_{i,j}, (\mu^{-1})_{i,j} \in W[^{\infty}]^q(D), \quad \forall \ i, j \in \{1, 2, 3\}, \]

for some positive \( r \in \mathbb{R} \) and \( q \in \mathbb{R} \) such that

\[ r \leq k, \quad q > 2 \text{ and } q > \frac{\lceil r \rceil}{3}. \]

Then, if quadrature rules used to build \( \tilde{\Phi} \) and \( \tilde{F} \) are such that:

- \( Q^1_K \) is exact for polynomials of degree \( k + \lceil r \rceil - 2 \),
- \( Q^2_K \) is exact for polynomials of degree \( k + \lceil r \rceil - 1 \) and
- \( Q^3_K \) is exact for polynomials of degree \( k + \lceil r \rceil - 1 \),

there exists some \( \ell \in \mathbb{N} \) such that, for all \( i > \ell \), a unique solution \( \tilde{E}_{hi} \in P_0^c(\tau_{hi}) \) exists to Problem 3 with

\[ \| E - \tilde{E}_{hi} \|_{H(\text{curl}; D)} \leq C_1 h_i^{r'} \| E \|_{H^{r'}(\text{curl}; D)} + C_2 h_i^{\lceil r \rceil}, \]

where the positive constants \( C_1 \) and \( C_2 \) are independent of the mesh-size, but depend on the parameters of Problem 1 (\( \mu, \epsilon, \omega, J \) and \( D \)).

### 3.4 Consistency error estimates and proof of Theorem 1

We now find error estimates for the quadrature approximation given in Definition 5 of the integrals defining the sesquilinear and antilinear forms in (2.7) and (2.8), respectively.

We begin by stating the Bramble-Hilbert lemma (cf. [12, Thm. 4.1.3]), which shall be required to give error estimates to the approximation of exact integration by numerical quadrature.
Lemma 5 (Bramble-Hilbert) Let \( q \in \mathbb{N} \) and \( \mathcal{O} \) be an open subset of \( \mathbb{R}^q \) with a Lipschitz-continuous boundary. For some integer \( k \geq 0 \) and \( p \in [1, \infty] \), let \( f \) be a continuous linear form on \( W^{k+1,p}(\mathcal{O}) \) with the property that

\[
    f(q) = 0 \quad \forall \quad q \in \mathbb{P}_k(\mathcal{O}; \mathbb{C}^3).
\]

Then, there exists a constant \( C_{\mathcal{O}} \), depending on the domain such that for all \( V \in W^{k+1,p}(\mathcal{O}) \),

\[
    |f(V)| \leq C_{\mathcal{O}} \|f\|_{(W^{k+1,p}(\mathcal{O}))'} \|V\|_{W^{k+1,p}(\mathcal{O})}.
\]

The following Lemmas provide local error estimates for the quadrature rules over arbitrary tetrahedrons \( K \in \tau_h \). Their proofs are analogous to those in [12, Chap. 4] for grad-conforming finite elements.

Lemma 6 Let \( K \in \tau_h \), \( m \in \mathbb{N} \) and \( M = (M_{i,j})_{i,j=1}^3 \), with \( M(x) \in \mathbb{C}^{3 \times 3} \), be such that \( M_{i,j} \in W^{m,\infty}(K) \) for all \( i, j \in \{1, 2, 3\} \). If \( Q_K \) is a quadrature rule as in Definition 5 such that it is exact for polynomials of degree \( k + m - 1 \), then the local quadrature error (for \( Q_K \) as in (3.9))

\[
    E_K(MU_h \cdot V_h) := \int_K MU_h \cdot V_h \, dx - Q_K(MU_h \cdot V_h),
\]

is such that for all \( U_h, V_h \in \mathbb{P}_k(K; \mathbb{C}^3) \)

\[
    |E_K(MU_h \cdot V_h)| \leq C_{M} h^m \|U_h\|_{m,K} \|V_h\|_{0,K}, \tag{3.10}
\]

for a positive constant \( C \) independent of \( h, K \) and \( M \) and

\[
    C_M := \sum_{i,j=1}^{3} \|M_{i,j}\|_{W^{m,\infty}(K)}.
\]

Proof Let \( \phi \in W^{m,\infty}(\bar{K}) \) and \( V_h \in \mathbb{P}_k(\bar{K}; \mathbb{C}^3) \), then,

\[
    |E_{\bar{K}}(\phi \cdot V)| \leq C_{E} \|\phi \cdot V_h\|_{L^{\infty}(\bar{K})} \leq C_{E} \|\phi\|_{L^{\infty}(\bar{K})} \|V_h\|_{L^{\infty}(\bar{K})} \leq C_{E} \|\phi\|_{W^{m,\infty}(\bar{K})} \|V_h\|_{0,\bar{K}},
\]

for some positive \( C_E \)—depending only on \( \bar{K} \)—where the last inequality follows from the norm equivalence over \( \mathbb{P}_k(K; \mathbb{C}^3) \). For a fixed \( V_h \in \mathbb{P}_k(K; \mathbb{C}^3) \), the form \( E_{\bar{K}}(\phi \cdot V_h) \) is linear and bounded on \( \phi \in W^{m,\infty}(\bar{K}) \) and satisfies \( E_{\bar{K}}(\phi \cdot V_h) = 0 \) for all \( \phi \in \mathbb{P}_{m-1}(\bar{K}; \mathbb{C}^3) \). By the Bramble-Hilbert Lemma (Lemma 5) there exists a positive constant \( C_{\bar{K}} \) such that

\[
    |E_{\bar{K}}(\phi \cdot V_h)| \leq C_{\bar{K}} \|\phi\|_{W^{m,\infty}(\bar{K})} \|V_h\|_{0,\bar{K}}.
\]
Then, for any $K \in \tau_h$ and $U_h, V_h \in \mathbb{P}_k(K; \mathbb{C}^3)$, one has

$$|E_K(U_h \cdot V_h)| \leq C\bar{K} |\det(\bar{J}_K)| |(U_h) \circ T_K|_{W^{m, \infty}(\bar{K})} \|V_h \circ T_K\|_{0, \bar{K}}.$$  

We begin by bounding $|(U_h) \circ T_K|_{W^{m, \infty}(\bar{K})}$,

$$|(U_h) \circ T_K|_{W^{m, \infty}(\bar{K})} \leq c \sum_{i=1}^{3} \sum_{j=1}^{3} \left| (M_{i,j} \circ T_K)(U_{h,j} \circ T_K) \right|_{W^{m, \infty}(\bar{K})},$$

$$\leq c \sum_{i,j=1}^{3} \left| (M_{i,j} \circ T_K)(U_{h,j} \circ T_K) \right|_{W^{m, \infty}(\bar{K})},$$

$$\leq c \sum_{i,j=1}^{3} \sum_{n=0}^{m} \left| \psi_{K}^i(M_{i,j}) \right|_{W^{m-n, \infty}(\bar{K})} \left| \psi_{K}^j(U_{h,j}) \right|_{W^{n, \infty}(\bar{K})},$$

$$\leq c \sum_{i,j=1}^{3} \sum_{n=0}^{m} \left| \psi_{K}^i(M_{i,j}) \right|_{W^{m-n, \infty}(\bar{K})} \left| \psi_{K}^j(U_{h,j}) \right|_{n, \bar{K}},$$

(3.11)

$$\leq c \|\bar{J}_K\|_{3 \times 3}^{m} \left| \det(\bar{J}_K) \right|^{-\frac{1}{2}} \sum_{i,j=1}^{3} \sum_{n=0}^{m} \left| M_{i,j} \right|_{W^{m-n, \infty}(\bar{K})} \left| U_{h,j} \right|_{n, \bar{K}},$$

(3.12)

$$\leq c \|\bar{J}_K\|_{3 \times 3}^{m} \left| \det(\bar{J}_K) \right|^{-\frac{1}{2}} \left| U_{h} \right|_{m, \bar{K}} \sum_{i,j=1}^{3} \| M_{i,j} \|_{W^{m, \infty}(\bar{K})},$$

(3.13)

where the positive constant $c$ is independent of $K$ and may change at each step, (3.11) employs the equivalence of norms in spaces of finite dimension and (3.12) is a consequence of (3.4). A similar bound for $\left| V_h \circ T_K \right|_{0, \bar{K}}$ may be obtained analogously:

$$\|V_h \circ T_K\|_{0, \bar{K}} \leq c \left| \det(\bar{J}_K) \right|^{-\frac{1}{2}} \|V_h\|_{0, \bar{K}},$$

(3.14)

for a positive constant $c$ as before. Then,

$$|E_K(U_h \cdot V_h)| \leq C\bar{K} |\det(\bar{J}_K)| |(U_h) \circ T_K|_{W^{m, \infty}(\bar{K})} \|V_h \circ T_K\|_{0, \bar{K}}$$

$$\leq C\bar{K} C_M c \|\bar{J}_K\|_{3 \times 3}^{m} \left| U_{h} \right|_{m, \bar{K}} \|V_h\|_{0, \bar{K}}$$

$$\leq C C_M h^m \|U_{h} \|_{m, \bar{K}} \|V_h\|_{0, \bar{K}},$$

where $c$ is a positive constant independent of $K$ and $M$ and $C$ follows from combining $c$ and $C_{\bar{K}}$.  

\[\square\]

**Lemma 7** Let $K \in \tau_h$, $m \in \mathbb{N}$ and $q \in \mathbb{R}$ such that

$$q \geq 2 \quad \text{and} \quad q > \frac{3}{m},$$

(3.15)
and $Q^1_K$ be a quadrature rule as in Definition 5 such that it is exact on polynomials of degree $k + m - 1$. Then, if $J \in W^{m,q}(K)$, the local quadrature error $E_K (J \cdot V_h)$ (as defined in Lemma 6) is such that for all $V_h \in P_k(K; \mathbb{C}^3)$

$$|E_K (J \cdot V_h)| \leq Ch^m |K|^\frac{1}{2} \|J\|_{W^{m,q}(K)} \|V_h\|_{0,K},$$

for a positive constant $C$ independent of $h$ and $K$.

**Proof.** The proof is exactly as that of Lemma 6 upon realizing that, thanks to (3.15), it holds that

$$\|J \circ T_K\|_{L^\infty(\tilde{K})} \leq c \|J \circ T_K\|_{W^{m,q}(K)}$$

for some positive $c$ independent of $K$ and the meshsize (cf. [25, Thm. 2.5] or [23, Thm. 3.6]).

With the previous estimates at hand, we can now prove the following Theorems providing the necessary conditions for us to employ Strang’s Lemma (Lemma 4) in the proof of Theorem 1.

**Theorem 2** (Consistency error for the sesquilinear form) Recall $k \in \mathbb{N}$ as the polynomial degree of our approximation spaces. Let $m \in \mathbb{N}$ and assume the following of the quadrature rules defining $\tilde{\Phi}_{h_i}$ in Definition 6:

- The quadrature rule $Q^1_K$ is exact for polynomials of degree $k + m - 2$.
- The quadrature rule $Q^2_K$ is exact for polynomials of degree $k + m - 1$.

Then, under Assumptions 3 and 4 and if the coefficients of $\mu^{-1}$ and $\epsilon$ belong to $W^{m,\infty}(D)$, it holds that

$$|\Phi(U_{h_i}, V_{h_i}) - \tilde{\Phi}_{h_i}(U_{h_i}, V_{h_i})|$$

$$\leq C_{\Phi} h_i^m \sum_{K \in \tau_{h_i}} \left(C_{\mu^{-1}} \|\text{curl } U_{h_i}\|_{m,K} \|\text{curl } V_{h_i}\|_{0,K} + \omega^2 C_{\epsilon} \|U_{h_i}\|_{m,K} \|V_{h_i}\|_{0,K}\right)$$

for all $U_{h_i}, V_{h_i} \in P_0^c(\tau_{h_i})$, where $C_{\mu^{-1}}$ and $C_{\epsilon}$ are positive constants depending on $\mu^{-1}$ and $\epsilon$, and $C_{\Phi}$ is a positive constant independent of the mesh sizes $\{h_i\}_{i \in \mathbb{N}}$.

**Proof.** The result comes from noticing that, for all $U_h$ and $V_h \in P_0^c(\tau_h)$ and all $K \in \tau_h$,

$$V_h|_K \in P_k(K; \mathbb{C}^3), \quad \text{curl } V_h|_K \in P_{k-1}(K; \mathbb{C}^3),$$

and

$$|\Phi(U_h, V_h) - \tilde{\Phi}(U_h, V_h)| \leq \sum_{K \in \tau_h} |E_K (\mu^{-1} \text{curl } U_h \cdot \text{curl } V_h)| + \omega^2 |E_K (\epsilon U_h \cdot V_h)|.$$
where $E_K(\cdot, \cdot)$ is as in Lemma 6 and

$$C_{\mu^{-1}} := \sum_{i,j=1}^{3} \| (\mu^{-1})_{i,j} \|_{W^{m,\infty}(D)} \quad \text{and} \quad C_\epsilon := \sum_{i,j=1}^{3} \| \epsilon_{i,j} \|_{W^{m,\infty}(D)} . \quad \Box$$

**Theorem 3** (Consistency error for the antilinear form) Recall $k \in \mathbb{N}$ as the polynomial degree of our approximation spaces. Let $m \in \mathbb{N}$ and assume the quadrature rule $Q^3_K$ from Definition 6 to be exact for polynomials of degree $k + m - 1$. Then, under Assumptions 3 and 4 and if $J$ is such that $J \in W^{m,q}(D)$ for some $q \in \mathbb{R}$ such that $q > \frac{3}{m}$ and $q \geq 2$, one has

$$\| F(V_{hi}) - \tilde{F}(V_{hi}) \| \leq C_F \omega h_i \| D \|^{1 - \frac{1}{q}} \| J \|_{W^{m,q}(D)} \| V_{hi} \|_{0,D}$$

for all $V_{hi} \in P^c_0(\tau_{hi})$, for a positive constant $C_F$ independent of the mesh sizes $\{h_i\}_{i \in \mathbb{N}}$.

**Proof** The result follows analogously from that of Theorem 2 and Hölder’s inequality. □

We are now ready to present a proof for the main result of this section.

**Proof of Theorem 1** From our assumptions and Theorem 2 we can employ Strang’s Lemma (Lemma 4), since

$$\| \Phi(U_{hi}, V_{hi}) - \tilde{\Phi}(U_{hi}, V_{hi}) \| \leq C_{\phi} h_i \| D \|^{\frac{1}{2} - \frac{1}{q}} \| J \|_{W^{m,q}(D)} \| V_{hi} \|_{0,D}$$

$$+ C_\epsilon \| U_{hi} \|_{[r],K} \| V_{hi} \|_{0,K}$$

$$\leq C_{\phi} (C_{\mu^{-1}} + C_\epsilon) h_i \| \Phi(U_{hi}, V_{hi}) - \tilde{\Phi}(U_{hi}, V_{hi}) \| \leq C_{\phi} (C_{\mu^{-1}} + C_\epsilon) h_i \| \Phi(U_{hi}, V_{hi}) - \tilde{\Phi}(U_{hi}, V_{hi}) \|_{H^{[r]},D} \| V_{hi} \|_{H^{[r]},D} .$$

Hence, there is some $\ell \in \mathbb{N}$ so that, for all $i \in \mathbb{N}$ with $i \geq \ell$, there exists a unique solution to Problem 3 $\tilde{E}_{hi} \in P^c_0(\tau_{hi})$ that satisfies

$$\| E - \tilde{E}_{hi} \|_{H^{[r]},D} \leq C_S \left( \| E - E_{hi} \|_{H^{[r]},D} \right)$$

$$+ \sup_{V_{hi} \in P^c_0(\tau_{hi})} \| \Phi(E_{hi}, V_{hi}) - \tilde{\Phi}(E_{hi}, V_{hi}) \| \leq C_{\phi} (C_{\mu^{-1}} + C_\epsilon) h_i \| \Phi(U_{hi}, V_{hi}) - \tilde{\Phi}(U_{hi}, V_{hi}) \|_{H^{[r]},D} \| V_{hi} \|_{H^{[r]},D} .$$
where $C_S$ follows from Strang’s Lemma, $C_F$ follows from Theorem 3 and $E_{h_i} \in P_0(\tau_{h_i})$ is the unique discrete solution to Problem 2. By [17, Thm. 3.3], we see that
\[
\|E - E_{h_i}\|_{H(\text{curl}; D)} \leq c h_i^r \|E\|_{H^r(\text{curl}; D)},
\]
for some positive constant $c > 0$ independent of the mesh-size. We continue by bounding the error between $\Phi$ and $\tilde{\Phi}_{h_i}$, first noticing that, for any $V_{h_i} \in P_0(\tau_{h_i})$, there holds
\[
|\Phi(E_{h_i}, V_{h_i}) - \tilde{\Phi}_{h_i}(E_{h_i}, V_{h_i})| \\
\leq C_\Phi h_i^{[r]} \sum_{K \in \tau_{h_i}} \left( C \mu^{-1} \|\text{curl } E_{h_i}\|_{[r], K} \|\text{curl } V_{h_i}\|_{0, K} \\
+ \omega^2 C_e \|E_{h_i}\|_{[r], K} \|V_{h_i}\|_{0, K} \right).
\]
For arbitrary $K \in \tau_{h_i}$ and sequentially employing Lemmas 2, 3 and 1, it holds that
\[
\|E_{h_i}\|_{[r], K} \leq \|E_{h_i} - I_{h_i}^{\#c}(E)\|_{[r], K} + \|I_{h_i}^{\#c}(E)\|_{[r], K} \\
\leq \|I_{h_i}^{\#c}(E_{h_i} - E)\|_{[r], K} + c_1 h_i^{-[r]} \|E\|_{r, K} \\
\leq c_2 h_i^{-[r]} \|I_{h_i}^{\#c}(E_{h_i} - E)\|_{0, K} + c_1 h_i^{-[r]} \|E\|_{r, K} \\
\leq c_2 h_i^{-[r]} \|(E_{h_i} - E)\|_{0, K} + c_1 h_i^{-[r]} \|E\|_{r, K},
\]
where $c = c_1 + c_2$ and the positive constants $c_1$ and $c_2$ come from the previously referenced Lemmas and are independent of both the mesh-size and $r \leq k$. Analogously and since $\text{curl } E_{h_i} \in P_d(\tau_{h_i})$ [23, Lem. 5.40], we have
\[
\|\text{curl } E_{h_i}\|_{[r], K} \leq c \left( h_i^{-[r]} \|\text{curl}(E_{h_i} - E)\|_{0, K} + h_i^{-[r]} \|\text{curl } E\|_{r, K} \right),
\]
for some positive $c$ independent of $h$ and $r$. Therefore,
\[
|\Phi(E_{h_i}, V_{h_i}) - \tilde{\Phi}_{h_i}(E_{h_i}, V_{h_i})| \\
\leq C_\Phi (C \mu^{-1} + \omega^2 C_e) h_i^{[r]} \sum_{K \in \tau_{h_i}} \left( \|\text{curl } E_{h_i}\|_{[r], K} \|\text{curl } V_{h_i}\|_{0, K} \\
+ \|E_{h_i}\|_{[r], K} \|V_{h_i}\|_{0, K} \right) \\
\leq C_\Phi (C \mu^{-1} + \omega^2 C_e) c h_i^r \sum_{K \in \tau_{h_i}} \left( \|\text{curl } E\|_{[r], K} \|\text{curl } V_{h_i}\|_{0, K} \\
+ \|E\|_{[r], K} \|V_{h_i}\|_{0, K} \right).
\]
\[ + C_\Phi (C_\mu^{-1} + \omega^2 C_\epsilon) \sum_{K \in \tau_{h_i}} \left( \| \text{curl}(E - E_{h_i}) \|_{0,K} \right. \| \text{curl} V_{h_i} \|_{0,K} \\
+ \| E - E_{h_i} \|_{0,K} \| V_{h_i} \|_{0,K} \right) \]
\[ \leq C_\Phi (C_\mu^{-1} + \omega^2 C_\epsilon) \left( h_i \| E \|_{H^r(\text{curl}; D)} \right. \| V_{h_i} \|_{H(\text{curl}; D)} \\
+ \| E - E_{h_i} \|_{H(\text{curl}; D)} \| V_{h_i} \|_{H(\text{curl}; D)} \right) \]
\[ \leq C_\Phi (C_\mu^{-1} + \omega^2 C_\epsilon) c h_i \| E \|_{H^r(\text{curl}; D)} \| V_{h_i} \|_{H(\text{curl}; D)}, \]

where the last inequality follows from [17, Thm. 3.3] and the positive constant \( c \), independent of \( h \) and \( r \), may vary at each step. Finally,

\[ \| E - \tilde{E}_{h_i} \|_{H(\text{curl}; D)} \]
\[ \leq C_S \left( c h_i \| E \|_{H^r(\text{curl}; D)} + C_\Phi (C_\mu^{-1} + \omega^2 C_\epsilon) c h_i \| E \|_{H^r(\text{curl}; D)} \right) \\
+ C_F \omega h_i \| J \|_{W^{(r),q}(D)} \]

as stated. \( \square \)

### 4 Finite elements and consistency error estimates for smooth curved domains

We now drop the requirement that \( D \) be polyhedral (Assumption 3). As a direct consequence of this, it will prove impractical to generate meshes that cover \( D \) exactly and we shall instead consider a sequence of meshes that approximate \( D \) as the mesh-size \( h \) decreases.

**Assumption 6** The bounded domain \( D \) is of class \( C^M \) for some \( M \in \mathbb{N} \).

#### 4.1 Finite elements

We begin by introducing a sequence of polyhedral meshes constructed from disjoint, matching tetrahedrons that approximate \( D \), \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \). As in [20,22] and [15, Sect. 1.3.2], we will require some assumptions from \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \).

**Assumption 7** (Assumptions on the polyhedral meshes.) The meshes in \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \) are affine and quasi-uniform. Boundary nodes are located on \( \Gamma \) and the polyhedral domain generated by each mesh, denoted \( D_{h_i}^{\text{poly}} \) (with boundary \( \Gamma_{h_i}^{\text{poly}} \)), approximates \( D \) so that

\[ \lim_{i \to \infty} \text{dist}(D, D_{h_i}^{\text{poly}}) = 0. \]

As in the previous Section, the elements of the meshes \( \{ \tau_{h_i} \}_{i \in \mathbb{N}} \) may be constructed through affine transformations as in Definition 3. We continue by introducing curved
meshes that approximate D, which shall be constructed from the polyhedral meshes \( \{\tau_i\}_{i \in \mathbb{N}} \).

**Definition 7** *(Approximated meshes)* For each polyhedral mesh \( \tau_h \in \{\tau_i\}_{i \in \mathbb{N}} \), we consider \( \tilde{\tau}_h \) to be the *approximated mesh*, which shares its nodes with \( \tau_h \), but is composed of curved tetrahedrons that cover a domain \( D_h \) (that approximates D) exactly. For a given \( K \in \tau_h \) we refer to the element of \( \tilde{\tau}_h \) that shares its nodes with \( K \) as \( \tilde{K} \) and consider the bijective mappings \( T_{\tilde{K}} : \tilde{\mathcal{K}} \mapsto \tilde{\mathcal{K}} \) to be polynomial.

Henceforth, let \( \tau_h \) and \( \tilde{\tau}_h \) be an arbitrary meshes in \( \{\tau_i\}_{i \in \mathbb{N}} \) and \( \{\tilde{\tau}_i\}_{i \in \mathbb{N}} \), respectively. All numerical computations are to be done on the approximated meshes \( \{\tilde{\tau}_i\}_{i \in \mathbb{N}} \). As such, we shall require numerical approximations of \( \Phi \) and \( F \) on the approximated meshes. Also notice that \( D_h \setminus D \) need not be empty, so we will also require to assume \( \mu^{-1} \), \( \epsilon \) and \( J \) to be well defined outside of \( D \).

**Assumption 8** There exists a bounded domain \( D^H \subset \mathbb{R}^3 \) of class \( C^\infty \), referred to as *hold-all domain*, such that

\[
D \subset D^H \quad \text{and} \quad D_{hi} \subset D^H \quad \forall \ i \in \mathbb{N}.
\]

Moreover, \( \mu^{-1} \), \( \epsilon \) and \( J \) belong to \( C^0(D^H) \).

**Assumption 9** *(Assumptions on the approximated and exact elements)* Let \( \bar{\mathfrak{R}} \in \mathbb{N} \) with \( \bar{\mathfrak{R}} < \mathfrak{M} \) from Assumption 6. The family of approximate meshes \( \{\tilde{\tau}_h\}_{i \in \mathbb{N}} \) is assumed to be \( \bar{\mathfrak{R}} \)-regular, i.e. the mappings \( T_{\tilde{K}}, \tilde{K} \in \tilde{\tau}_h \), are \( C^{\bar{\mathfrak{R}}+1} \)-diffeomorphisms that belong to \( P_{\tilde{\mathfrak{R}}}(\tilde{K}; \tilde{K}) \). Also, the following bounds for derivatives of these transformations hold for all \( \tilde{K} \in \tilde{\tau}_h \)

\[
\sup_{x \in \tilde{\mathcal{K}}} \| D^n T_{\tilde{K}}(x) \| \leq C_n h^n \quad \text{and} \quad \sup_{x \in \tilde{\mathcal{K}}} \| D^n \left( T_{\tilde{K}}^{-1}\right)(x) \| \leq C_{-n} h^{-n}
\]

\[\forall \ n \in \{1, \ldots, \bar{\mathfrak{R}} + 1\},\]

where \( C_n \) and \( C_{-n} \) are positive constants independent from the mesh-size for all integers \( n \leq \bar{\mathfrak{R}} + 1 \), \( D^n T_{\tilde{K}} \) is the Fréchet derivative of order \( n \) of \( T_{\tilde{K}} \) and \( \| D^n T_{\tilde{K}}(x) \| \) is the appropriate induced norm, with functional spaces omitted for the sake of brevity. Furthermore, we assume that

\[
\det \mathbb{J}_{\tilde{K}}(x) > 0,
\]

for all \( x \in \tilde{K} \) and that there exists some positive \( \theta \in \mathbb{R} \), independent of the mesh-size, such that, for all \( \tilde{K} \in \tilde{\tau}_h \), the following condition holds

\[
\frac{1}{\theta} \leq \frac{\det \mathbb{J}_{\tilde{K}}(x)}{\det \mathbb{J}_{\tilde{K}}(y)} \leq \theta \quad \forall \ x, y \in \tilde{K}.
\]
In Assumption 9, \( K \) represents the degree of the polynomial approximation of \( D \). We expect the rate with which the sequence of domains \( \{D_{h_i}\}_{i \in \mathbb{N}} \) converges to \( D \) to be dependent on \( K \), so that larger \( K \) will imply faster convergence, e.g.

\[
\text{dist}(D, D_{h_i}) \leq C_{\mathcal{R}} h_i^{f(K)},
\]

for some strictly increasing positive function \( f : \mathbb{N} \rightarrow \mathbb{R} \) and a positive constant \( C_{\mathcal{R}} \) that may depend on \( K \). Furthermore, notice that for any curved tetrahedron \( \tilde{K} \) and any \( y \in \tilde{K} \)

\[
|\tilde{K}| = \int_{\tilde{K}} 1 \, dx = \int_{\tilde{K}} \det \mathbb{J}_{\tilde{K}}(x) \, dx.
\]  

(4.1)

**Lemma 8** Under Assumption 9 and for any \( y \in \tilde{K} \), it holds that

\[
\frac{1}{\theta} \leq \frac{|\tilde{K}|}{\det \mathbb{J}_{\tilde{K}}(y)} \leq \theta.
\]

**Proof** Notice that for any \( y \in \tilde{K} \), one has

\[
\int_{\tilde{K}} \det \mathbb{J}_{\tilde{K}}(x) \, dx = \det \mathbb{J}_{\tilde{K}}(y) \int_{\tilde{K}} \frac{\det \mathbb{J}_{\tilde{K}}(x)}{\det \mathbb{J}_{\tilde{K}}(y)} \, dx.
\]

The proof then follows from our assumed bounds for \( \det \mathbb{J}_{\tilde{K}}(x) \) and (4.1). \( \square \)

As before, we consider finite elements on curved tetrahedrons \( \tilde{K} \in \mathring{\tau}_h \) as triples \((\tilde{K}, P_{\tilde{K}}, \Sigma_{\tilde{K}})\), so that Definition 4 remains valid on curved tetrahedrons. We define the curl-conforming element on a curved tetrahedron \( \tilde{K} \) as

\[
P_{\mathcal{C}, \tilde{K}} := \{ p : \mathbb{J}_{\tilde{K}}^\top (p \circ T_{\tilde{K}}) \in P_{\mathcal{C}, \tilde{K}} \}.
\]

The function spaces for grad- and div-conforming finite elements are defined in a similar manner

\[
P_{\mathcal{G}, \tilde{K}} := \{ p : p \circ T_{\tilde{K}} \in P_{\mathcal{G}, \tilde{K}} \}, \quad P_{\mathcal{D}, \tilde{K}} := \{ p : \text{det}(\mathbb{J}_{\tilde{K}})\mathbb{J}_{\tilde{K}}^{-1}(p \circ T_{\tilde{K}}) \in P_{\mathcal{D}, \tilde{K}} \}.
\]

Discrete spaces on curved meshes are then defined as in the previous section:

\[
P_{\mathcal{G}, \tilde{h}} := \left\{ v_h \in H^1(D) : v_h|_{\tilde{K}} \in P_{\mathcal{G}, \tilde{K}} \ \forall \ \tilde{K} \in \mathring{\tau}_h \right\},
\]

\[
P_{\mathcal{G}, 0, \tilde{h}} := P_{\mathcal{G}, \tilde{h}} \cap H^1_0(D),
\]

\[
P_{\mathcal{C}, \tilde{h}} := \left\{ V_h \in H(\text{curl}; D) : V_h|_{\tilde{K}} \in P_{\mathcal{C}, \tilde{K}} \ \forall \ \tilde{K} \in \mathring{\tau}_h \right\},
\]

\[
P_{\mathcal{C}, 0, \tilde{h}} := P_{\mathcal{C}, \tilde{h}} \cap H_0(\text{curl}; D),
\]

\( \oplus \) Springer
\[ P^d(\tau_h) := \left\{ \mathbf{v}_h \in H(\text{div}; D) : \mathbf{v}_h|_{\tilde{K}} \in P^d_{\tilde{K}} \quad \forall \tilde{K} \in \tau_h \right\}, \]
\[ P^d_0(\tau_h) := P^d(\tau_h) \cap H_0(\text{div}; D), \]
\[ P^b(\tau_h) := \left\{ v_h \in L^2(D) : v_h|_{\tilde{K}} \in P^b_{\tilde{K}} \quad \forall \tilde{K} \in \tau_h \right\}. \]

Pullbacks from functions defined on curved tetrahedrons and triangles are defined analogously as those in (3.1). We continue by stating a property analogous to that in (3.5) in the context of curved meshes.

**Lemma 9** (Lemma 1 in [11]) Let \( p \in \mathbb{R} \) and \( l \in \mathbb{N}_0 \) be such that \( p \geq 1 \) and \( l \leq K + 1 \). Then, for a given \( \tilde{u} \in W^{s,q}(\tilde{K}) \) and a curved tetrahedron \( \tilde{K} \in \tau_h \), the function \( u \) defined as

\[ u := \psi^{-1}_{\tilde{K}}(\tilde{u}) = \tilde{u} \circ T^{-1}_{\tilde{K}} \]

belongs to \( W^{l,p}(\tilde{K}) \) and

\[ |\tilde{u}|_{W^{l,p}(\tilde{K})} \leq C \inf_{x \in \tilde{K}} |\det J_{\tilde{K}}(x)|^{-\frac{1}{p}} h^l \|u\|_{W^{l,p}(\tilde{K})}, \]

for a positive constant \( C \) independent of \( \tilde{K} \) and the mesh-size.

### 4.2 Discrete variational problem

We continue by introducing appropriate modifications of the sesquilinear and antilinear forms considered in the previous sections. In particular, for each \( i \in \mathbb{N} \) and all \( \mathbf{U}, \mathbf{V} \in H(\text{curl}; D_{h_i}) \), we shall use

\[ \Phi_{h_i}(\mathbf{U}, \mathbf{V}) := \int_{D_{h_i}} \mu^{-1} \text{curl} \mathbf{U} \cdot \text{curl} \mathbf{V} - \omega^2 \epsilon \mathbf{U} \cdot \mathbf{V} \, dx, \quad (4.2) \]
\[ \mathbf{F}_{h_i}(\mathbf{V}) := -i\omega \int_{D_{h_i}} \mathbf{J} \cdot \mathbf{V} \, dx. \quad (4.3) \]

**Problem 4** (Discrete variational problem on curved meshes) Find \( \mathbf{E}_h \in P^c_0(\tau_h) \) such that

\[ \Phi_h(\mathbf{E}_h, \mathbf{V}_h) = \mathbf{F}_h(\mathbf{V}_h), \]

for all \( \mathbf{V}_h \in P^c_0(\tau_h) \).

As with Assumptions 2 and 5, we assume our framework to be such that a unique discrete solution \( \mathbf{E}_h \in P^c_0(\tau_h) \) exists for all meshes in \( \{\tau_{h_i}\}_{i \in \mathbb{N}} \).
Assumption 10 (Wellposedness on $P^c_0(\tilde{\tau}_h)$) We assume the sesquilinear forms $\left\{ \Phi_{h_i} \right\}_{i \in \mathbb{N}}$ to satisfy the following:

$$\sup_{U_{h_i} \in P^c_0(\tilde{\tau}_h) \setminus \{0\}} |\Phi_{h_i}(U_{h_i}, V_{h_i})| > \alpha > 0 \quad \forall \; V_{h_i} \in P^c_0(\tilde{\tau}_h) \setminus \{0\},$$

$$\inf_{U_{h_i} \in P^c_0(\tilde{\tau}_h) \setminus \{0\}} \left( \sup_{V_{h_i} \in P^c_0(\tilde{\tau}_h) \setminus \{0\}} \frac{|\Phi_{h_i}(U_{h_i}, V_{h_i})|}{\|U_{h_i}\|_{H(\text{curl}; D_{h_i})} \|V_{h_i}\|_{H(\text{curl}; D_{h_i})}} \right) \geq C > 0,$$

for all $i \in \mathbb{N}$.

4.3 Numerical integration

We follow closely the steps taken in the previous section and consider the numerical computation of the integrals in Problem 2. We recall the definition of our quadrature rule over $\tilde{\tau}_h$ in Definition 5 and introduce numerical integration on curved elements $\tilde{\tau}_h$ as

$$Q_{\tilde{\tau}_h}(\phi) := \sum_{l=1}^L w_{l,\tilde{\tau}_h} \phi(b_{l,\tilde{\tau}_h}) \quad \text{where} \quad w_{l,\tilde{\tau}_h} = |\det \tilde{J}_{\tilde{\tau}_h}(\tilde{b}_l)| \tilde{w}_l \quad \text{and} \quad b_{l,\tilde{\tau}_h} := T_{\tilde{\tau}_h}^{\tilde{\tau}_h} \tilde{b}_l.$$

(4.4)

Definition 8 (Numeric sesquilinear and antilinear form) Let $Q^1_{\tilde{\tau}_h}, Q^2_{\tilde{\tau}_h}$ and $Q^3_{\tilde{\tau}_h}$ be three distinct quadrature rules as in Definition 5. For each $\tilde{\tau}_h \in \{\tilde{\tau}_h\}_{i \in \mathbb{N}}$, we denote by $\tilde{\Phi}_h(\cdot, \cdot)$ and $\tilde{\Phi}_h(\cdot)$ the perturbed, discrete, sesquilinear and antilinear forms over fields in $P^c(\tilde{\tau}_h)$, where exact integration is replaced by numerical integration

$$\tilde{\Phi}_h(U_h, V_h) := \sum_{\tilde{\tau}_h \in \tilde{\tau}_h} Q^1_{\tilde{\tau}_h}(\mu^{-1}\text{curl } U_h \cdot \text{curl } V_h) + Q^2_{\tilde{\tau}_h}(-\omega^2 \epsilon U_h \cdot V_h),$$

$$\tilde{\Phi}_h(V_h) := \sum_{\tilde{\tau}_h \in \tilde{\tau}_h} Q^3_{\tilde{\tau}_h}(-i \omega \mathbf{J} \cdot V_h),$$

where, for $i \in \{1, 2, 3\}, Q^i_{\tilde{\tau}_h}$ is built from $Q^i_{\tilde{\tau}_h}$ as in (4.4).

Problem 5 (Discrete numerical problem on curved meshes) Find $\tilde{E}_h \in P^c_0(\tilde{\tau}_h)$ such that,

$$\tilde{\Phi}_h(\tilde{E}_h, V_h) = \tilde{\Phi}_h(V_h),$$

for all $V_h \in P^c_0(\tilde{\tau}_h)$.

Let $E, E_h$ and $\tilde{E}_h$ be the respective solutions of Problems 1, 4 and 5. Our current objective is to study the differences between the convergence rates of $E_h$ and $\tilde{E}_h$ to $E$ for an appropriate extension onto the hold-all domain $\tilde{D}$. The following modification of Strang’s Lemma (Lemma 4) will prove useful. Its proof comes from slight modifications to that of [24, Thm. 4.2.11], and so we omit it for brevity. We shall emulate
and consider a sequence of mappings \( \{ \mathcal{J}_{h_i} \}_{i \in \mathbb{N}} \) such that, for all \( i \in \mathbb{N} \), it holds that
\[
\mathcal{J}_{h_i} : H(\text{curl}; D) \mapsto H(\text{curl}; D_{h_i}).
\] (4.5)

These mappings need not be linear, but it is assumed \( \mathcal{J}_{h_i} \) possesses some information on \( \mathcal{E} \), so that the computation of
\[
\| \mathcal{J}_{h_i}(\mathcal{E}) - \tilde{\mathcal{E}}_{h_i} \|_{H(\text{curl}; D_{h_i})}
\] (4.6)
is in some way meaningful [13, Rmk. 9]. Indeed, the estimates in [11] may be interpreted for the specific choice of \( \mathcal{J}_{h_i} \) as an extension to \( \mathcal{D}^{H} \) of \( \mathcal{E} \) for all \( i \in \mathbb{N} \). Notice that in [13], the authors assume
\[
\mathcal{J}_{h_i} : H(\text{curl}; D_{h_i}) \mapsto P_{c_0}(\tilde{\tau}_{h_i}),
\]
which they require to estimate (4.6).

**Lemma 10** (Modified Strang’s Lemma) Let \( \Phi \) in (2.7) satisfy Assumption 2 and 5 and let \( \mathcal{E} \) and \( \mathcal{E}_{h_i} \) be the solutions to Problems 1 and 4. If the sequence of sesquilinear forms \( \{ \bar{\Phi}_{h_i} \}_{i \in \mathbb{N}} \) given by Definition 8 satisfies, for all \( U_{h_i}, V_{h_i} \in P_{0}(\tilde{\tau}_{h_i}) \), the bound
\[
| \Phi_{h_i}(U_{h_i}, V_{h_i}) - \bar{\Phi}_{h_i}(U_{h_i}, V_{h_i}) | \leq c h_{i}^{m} \| U_{h_i} \|_{H^{m}(\text{curl}; D_{h_i})} \| V_{h_i} \|_{H(\text{curl}; D_{h_i})},
\] for \( m \in \mathbb{N} \) and a fixed and positive constant \( c \) independent of the mesh-size, then there is some \( \ell \in \mathbb{N} \) such that, for all the meshes in the sequence \( \{ \tilde{\tau}_{h_i} \}_{i \geq \ell} \), there exists a unique solution to Problem 5, \( \tilde{\mathcal{E}}_{h_i} \in P_{0}(\tilde{\tau}_{h_i}) \) with
\[
\| \mathcal{J}_{h_i}(\mathcal{E}) - \tilde{\mathcal{E}}_{h_i} \|_{H(\text{curl}; D_{h_i})} \leq C_S \left( \| \mathcal{J}_{h_i}(\mathcal{E}) - \mathcal{E}_{h_i} \|_{H(\text{curl}; D_{h_i})} + \sup_{V_{h_i} \in P_{0}(\tilde{\tau}_{h_i}) \setminus \{0\}} \left| \frac{| \Phi_{h_i}(\mathcal{E}_{h_i}, V_{h_i}) - \bar{\Phi}_{h_i}(\mathcal{E}_{h_i}, V_{h_i}) |}{\| V_{h_i} \|_{H(\text{curl}; D_{h_i})}} \right| \right)
\] (4.7)
for a fixed positive constant \( C_S \), independent of the mesh-size, and where the sequence of mappings \( \{ \mathcal{J}_{h_i} \}_{i \in \mathbb{N}} \) is such that (4.5) holds.

We will focus on providing estimates for the last two terms in the right-hand side of (4.7). The error induced by the approximation of \( D \) by \( D_{h_i} \) (i.e. the first term in (4.7)) lies beyond the scope of this article. We continue by stating the main result of this section.

**Theorem 4** (Error estimate in affine meshes. Main result of Sect. 4) Let \( \mathcal{E} \) be the unique solution to Problem 1 and suppose the following of the data of Problem 1:
\[
\mathcal{J} \in W^{m.d}(D^{H}), \quad \epsilon_{i,j}, (\mu^{-1})_{i,j} \in W^{m,\infty}(D^{H}) \quad \forall i, j \in \{1, 2, 3\},
\]
for some positive \( m \in \mathbb{N} \) and \( q \in \mathbb{R} \) such that

\[
m > 1, \quad q > 2 \quad \text{and} \quad q > \frac{m}{3}.
\]

Then, if the quadrature rules used to build \( \{ \tilde{\Phi}_h \}_i \in \mathbb{N} \) and \( \{ \tilde{F}_h \}_i \in \mathbb{N} \) in Definition 8 are such that:

- \( Q^1_K \) is exact for polynomials of degree \( k + s + m - 3 \),
- \( Q^2_K \) is exact for polynomials of degree \( k + 2s + m - 3 \) and
- \( Q^3_K \) is exact for polynomials of degree \( k + 2s + m - 3 \),

there exists some \( \ell \in \mathbb{N} \) such that for all \( i > \ell \) there exists a unique solution \( \tilde{E}_h \in P_0^c(\tau_h) \) to Problem 3 and the solutions satisfy

\[
\| J_{h_i}(E) - \tilde{E}_h_i \|_{H(curl;D_{h_i})} \leq C_S \left( \| J_{h_i}(E) - E_{h_i} \|_{H(curl;D_{h_i})} + C_1 h^m \| E_{h_i} \|_{H^m(curl;D_{h_i})} + C_2 h^m \right),
\]

where the positive constants \( C_1 \) and \( C_2 \) are independent of the mesh-size, but depend on the parameters of Problem 1 (\( \mu, \epsilon, \omega, J \) and \( D \)).

### 4.4 Consistency error estimates and proof of Theorem 4

As in Sect. 3, we seek error estimates for the integrals over curved tetrahedrons \( \tilde{K} \in \tilde{\tau}_h \). The most notorious difference in the proofs for the following estimates and those presented in the previous section are due to the fact that, if \( U_h \in P^c_{\tilde{K}} \) for some curved tetrahedron \( \tilde{K} \), then \( U_h \circ T_{\tilde{K}} \) is, in general, not a polynomial, so we can not apply the Bramble-Hilbert Lemma straightforwardly. We will see, however, that in our case \( U_h \circ T_{\tilde{K}} \det J_{\tilde{K}} \) will be a polynomial of a certain degree (higher than \( k \)) which will allow us to proceed as before.

**Lemma 11** Let \( \tilde{K} \in \tilde{\tau}_h \), \( q \geq 1 \), \( q' = \frac{q}{q-1} \), \( m \in \mathbb{N} \) with \( m > \frac{3}{q} \) and \( M(x) \in C^{3 \times 3} \), with \( M = (M_{i,j})_{i,j=1}^3 \), such that \( M_{i,j} \in W^{m,\infty}(\tilde{K}) \) for all \( i, j \in \{1, 2, 3\} \). If \( Q_{\tilde{K}} \) is a quadrature rule as in Definition 5, exact for polynomials of degree \( k + 2s + m - 3 \), then, for all \( U_h, V_h \in P^c_{\tilde{K}} \), the quadrature error

\[
\mathcal{E}_{\tilde{K}}(MU_h \cdot V_h) := \int_{\tilde{K}} MU_h \cdot V_h \, dx - Q_{\tilde{K}}(MU_h \cdot V_h)
\]

is such that

\[
|\mathcal{E}_{\tilde{K}}(MU_h \cdot V_h)| \leq C_C M h^m \| U_h \|_{W^{m,q}(\tilde{K})} \| V_h \|_{L^{q'}(\tilde{K})},
\]

\( \square \) Springer
where
\[ C_M := \sum_{i,j=1}^{3} \| M_{i,j} \|_{W^{m,\infty}((\mathcal{K}))}, \]

and \( C \) is a positive constant independent of \( h, K \) and \( M \).

**Proof** Let \( \phi \in W^{m,\infty}(\mathcal{K}) \) and \( V_h \in P_{k+2}(\mathcal{K}; \mathbb{C}^3) \). Then,
\[
|E_K(\phi \cdot V_h)| \leq C_E \| \phi \cdot V_h \|_{L^{\infty}(\mathcal{K})} \leq C_E \| \phi \|_{L^\infty(\mathcal{K})} \| V_h \|_{L^\infty(\mathcal{K})}
\]
\[
\leq C_E \| \phi \|_{W^{m,q}(\mathcal{K})} \| V_h \|_{L^{q'}(\mathcal{K})},
\]
for some positive \( C_E \)—depending only on \( \mathcal{K} \)—and \( \frac{1}{q} + \frac{1}{q'} = 1 \). Also, since the error is zero for all \( \phi \in P_{m-1}(\mathcal{K}; \mathbb{C}^3) \), by the Bramble-Hilbert Lemma (Lemma 5) we have that
\[
|E_K(\phi \cdot V_h)| \leq C_{\mathcal{K}} |\phi|_{W^{m,q}(\mathcal{K})} \| V_h \|_{L^{q'}(\mathcal{K})} \quad \forall \, \phi \in W^{m,q}(\mathcal{K}),
\] (4.8)

for some positive constant \( C_{\mathcal{K}} \) depending only on \( \mathcal{K} \) and \( C_E \). Notice that
\[
\int_{\mathcal{K}} M U_h \cdot V_h \, dx = \int_{\mathcal{K}} (M \circ T_{\mathcal{K}})(U_h \circ T_{\mathcal{K}}) \cdot (V_h \circ T_{\mathcal{K}}) \det \tilde{J} \, dx,
\]
\[
Q_{\mathcal{K}} (M U_h \cdot V_h) = Q_{\mathcal{K}} ((M \circ T_{\mathcal{K}})(U_h \circ T_{\mathcal{K}}) \cdot (V_h \circ T_{\mathcal{K}}) \det \tilde{J}).
\]

Hence, we need only find a bound for the term:
\[
E_K ((M \circ T_{\mathcal{K}})(U_h \circ T_{\mathcal{K}}) \cdot (V_h \circ T_{\mathcal{K}}) \det \tilde{J}).
\]

From the last equation, we require \( (V_h \circ T_{\mathcal{K}}) \det \tilde{J} \) to be a polynomial. Indeed, one has
\[
V_h \circ T_{\mathcal{K}} \det \tilde{J} = \det \tilde{J}^T \tilde{J}^{-1} \left( \tilde{J}^T (V_h \circ T_{\mathcal{K}}) \right)
\]
\[
= \text{Co}(\tilde{J}) \left( \tilde{J}^T (V_h \circ T_{\mathcal{K}}) \right) \in P_{k+2}(\mathcal{K}; \mathbb{C}^3),
\] (4.9)

where \( \text{Co}(\tilde{J}) \) is the pointwise cofactor matrix of \( \tilde{J} \). Then, from (4.8) and (4.9) we get
\[
E_K ((M \circ T_{\mathcal{K}})(U_h \circ T_{\mathcal{K}}) \cdot (V_h \circ T_{\mathcal{K}}) \det \tilde{J}) \leq C_{\mathcal{K}} \| \det \tilde{J} \|_{L^{\infty}(\mathcal{K})} \| (M U_h) \circ T_{\mathcal{K}} \|_{W} \| V_h \circ T_{\mathcal{K}} \|_{L^{q'}(\mathcal{K})}.
\] (4.10)
We continue by bounding each term in (4.10), beginning with the $L^{q'}(\tilde{K})$-norm of $V_h \circ T_{\tilde{K}}$, which is easily done through a change of variables:

$$\left\| V_h \circ T_{\tilde{K}} \right\|_{L^{q'}(\tilde{K})} \leq \left\| V_h \right\|_{L^{q'}(\tilde{K})} \frac{1}{\inf_{x \in \tilde{K}} \left| \det \nabla \right|_{\tilde{K}}^{\frac{1}{q'}}}.$$ 

To bound $\left\| (M U_h) \circ T_{\tilde{K}} \right\|_{W}$, we proceed as in Lemma 6 and employ Lemma 9 as follows

$$\left\| (M U_h) \circ T_{\tilde{K}} \right\|_{W} \leq c \sum_{i,j=1}^{3} \sum_{n=0}^{m} \left| M_{i,j} \right|_{W^{m-n,\infty}(\tilde{K})} \left\| U_{h,j} \circ T_{\tilde{K}} \right\|_{W^{n,q}(\tilde{K})}$$

$$\leq c \sum_{i,j=1}^{3} \sum_{n=0}^{m} h^{m-n} \left| M_{i,j} \right|_{W^{m-n,\infty}(\tilde{K})} \frac{1}{\inf_{x \in \tilde{K}} \left| \det \nabla \right|_{\tilde{K}}^{\frac{1}{q'}}} \left\| U_{h,j} \right\|_{W^{n,q}(\tilde{K})},$$

where the positive constant $c$ is independent of $\tilde{K}$ and the mesh-size and may change from line to line. Then, a combination of our computed bounds, (4.10) and Assumption 9 leads to

$$\left| E_{\tilde{K}}(M U_h \cdot V_h) \right| \leq C \theta C_{M} h^{m} \left\| U_{h} \right\|_{W^{m,q}(\tilde{K})} \left\| V_h \right\|_{L^{q'}(\tilde{K})},$$

for some positive $C$, independent of $\tilde{K}$ and the mesh-size.

Notice that if $m = 1$ in Lemma 11, we are unable to extract $L^{2}$-norms of $U_{h}$ and $V_{h}$, hence our requirement that $m > 1$ in Theorem 4. Also, observe the differences between the previous proof and that of Lemma 6. Not only do we require stronger conditions from our quadrature rules, we are also unable to bound $\left\| U_{h} \circ T_{\tilde{K}} \right\|_{W^{n,q}(\tilde{K})}$ as before, owing to the fact that $U_{h} \circ T_{\tilde{K}}$ fails to be a polynomial (this is also the reason why we require to introduce $q$ and $q'$).

Lemma 12 Let $\tilde{K} \in \tilde{T}_{h}$, $q \geq 1$, $q' = \frac{q}{q-1}$, $m \in \mathbb{N}$ with $m > \frac{3}{q}$ and $M(x) \in C^{3 \times 3}$, with $M = (M_{i,j})^{3}_{i,j=1}$, such that $M_{i,j} \in W^{m,\infty}(\tilde{K})$ for all $i, j \in \{1, 2, 3\}$. If $Q_{\tilde{K}}$ is a quadrature rule as in Definition 5 such that it is exact for polynomials of degree $k + \tilde{K} + m - 3$, then, for all $U_{h}, V_{h} \in P_{k \tilde{K}}$, the quadrature error

$$E_{\tilde{K}}(\text{Mcurl } U_{h} \cdot \text{curl } V_{h}) := \int_{\tilde{K}} \text{Mcurl } U_{h} \cdot \text{curl } V_{h} \, dx - Q_{\tilde{K}}(\text{Mcurl } U_{h} \cdot \text{curl } V_{h}),$$

with $Q_{\tilde{K}}$ as in (4.4), is such that

$$\left| E_{\tilde{K}}(\text{Mcurl } U_{h} \cdot \text{curl } V_{h}) \right| \leq C \theta C_{M} h^{m} \left\| \text{curl } U_{h} \right\|_{W^{m,q}(\tilde{K})} \left\| \text{curl } V_{h} \right\|_{L^{q'}(\tilde{K})},$$

where

$$C_{M} := \sum_{i,j=1}^{3} \left\| M_{i,j} \right\|_{W^{m,\infty}(\tilde{K})},$$

\(\square\) Springer
and $C$ is a positive constant independent of $h$, $K$ and $M$.

**Proof** We begin by noticing that for $V \in H(\text{curl}; \tilde{K})$ [21, Lem. 2.2]

$$\text{curl} \left( J^T_K(V \circ T_{\tilde{K}}) \right) = \text{det} J^T_K \text{curl} V \circ T_{\tilde{K}}.$$

Thus, one obtains

$$\text{det} J^T_K \text{curl} V \circ T_{\tilde{K}} = J^T_K \text{curl} \left( J^T_K(V \circ T_{\tilde{K}}) \right) \in P_{k+\rho-2}(\tilde{K} \mathbb{C}^3).$$

The proof proceeds as that for Lemma 11. □

**Lemma 13** Let $\tilde{K} \in \mathcal{T}_h$, $m \in \mathbb{N}$ and $q \in \mathbb{R}$ such that

$$q \geq 2 \quad \text{and} \quad q > \frac{3}{m}, \quad (4.11)$$

and $Q_{\tilde{K}}$ be a quadrature rule as in Definition 5 such that it is exact on polynomials of degree $k + 2\rho + m - 3$. Then, if $J \in W^{m,q}(\tilde{K})$, the local quadrature error $E_{\tilde{K}}(J \cdot V_h)$ (defined in Lemma 11) is such that, for all $V_h \in P_{c,\tilde{K}}$, it holds that

$$|E_{\tilde{K}}(J \cdot V_h)| \leq Ch^m \left| \text{det} J^T_{\tilde{K}} \right|^{\frac{1}{\theta} - \frac{1}{q}} \left| J \right|_{W^{m,q}(\tilde{K})} \left| V_h \right|_{0,\tilde{K}},$$

for a positive constant $C$ independent of $\tilde{K}$, $J$ and the mesh-size.

**Proof** We take cue from an analogous expression to that in (4.8), and notice that by similar arguments we get

$$|E_{\tilde{K}}(\phi \cdot V_h)| \leq C_{\tilde{K}} |\phi|_{W^{m,q}(\tilde{K})} \left| V_h \right|_{0,\tilde{K}} \quad \forall \phi \in W^{m,q}(\tilde{K}), \ V_h \in P_{c,\tilde{K}}.$$

Then, from our assumptions and Lemmas 8 and 9

$$E_{\tilde{K}}(J \cdot V_h) = E_{\tilde{K}} \left( J \circ T_{\tilde{K}} \cdot V_h \circ T_{\tilde{K}} \text{det} \tilde{J}_{\tilde{K}} \right)$$

$$\leq C_{\tilde{K}} \left\| \text{det} \tilde{J}_{\tilde{K}} \right\|_{L^\infty(\tilde{K})} \left\| J \circ T_{\tilde{K}} \right\|_{W} \left\| V_h \circ T_{\tilde{K}} \right\|_{0,\tilde{K}}$$

$$\leq c \left\| \text{det} \tilde{J}_{\tilde{K}} \right\|_{L^\infty(\tilde{K})} h^m \inf_{x \in \tilde{K}} \left\| J \right\|_{W} \inf_{x \in \tilde{K}} \left\| V_h \right\|_{0,\tilde{K}}$$

$$\leq c \theta \left| \text{det} J^T_{\tilde{K}} \right|^{\frac{1}{\theta} - \frac{1}{q}} \left| J^T_{\tilde{K}} \right|^{\frac{1}{q} - \frac{1}{2}} h^m \left| J \right|_{W} \left| V_h \right|_{0,\tilde{K}},$$

for a positive constant $c$ independent of the mesh-size and $K$, which may change from line to line. □

As before, the computed estimates will enable us to prove, based on our assumptions, consistency error estimates for the respective sesquilinear and antilinear forms.
considered in this section. The proofs of the following two Theorems (yielding said consistency estimates) are analogous to the proofs of Theorems 2 and 3 and are thus omitted.

**Theorem 5** (Consistency error for the sesquilinear forms \(\{\tilde{\Phi}^{hi}\}_{i \in \mathbb{N}}\)) Let \(k \in \mathbb{N}\) denote the polynomial degree of our approximation spaces. Let \(m \in \mathbb{N}\) and assume the following for the quadrature rules defining the sesquilinear forms in \(\{\tilde{\Phi}^{hi}\}_{i \in \mathbb{N}}\) in Definition 8:

- The quadrature rule \(Q^{1}_{\tilde{K}}\) is exact for polynomials of degree \(k + \bar{K} + m - 3\).
- The quadrature rule \(Q^{2}_{\tilde{K}}\) is exact for polynomials of degree \(k + 2\bar{K} + m - 3\).

Then, under Assumptions 3 and 4 and if the coefficients of \(\mu^{-1}\) and \(\epsilon\) belong to \(W^{m,\infty}(D^H)\) and for \(\Phi^{hi}\) as in (4.2),

\[
|\Phi^{hi}(U^{hi}, V^{hi}) - \tilde{\Phi}^{hi}(U^{hi}, V^{hi})| \leq C_{\Phi} h^{m \bar{K}} \sum_{\tilde{K} \in \tau^{hi}} (C\mu^{-1} \|\text{curl} U^{hi}\|_{W^{m,q}(\tilde{K})} \|\text{curl} V^{hi}\|_{L^{q'}(\tilde{K})} + \omega^2 C\epsilon \|U^{hi}\|_{W^{m,q}(\tilde{K})} \|V^{hi}\|_{L^{q'}(\tilde{K})})
\]

for all \(U^{hi}, V^{hi} \in P^{c}_{0}(\tau^{hi})\), where \(q\) and \(q'\) are such that \(q > 1, q' = \frac{q}{q-1}\) and \(q > \frac{3}{m}\), \(C\mu^{-1}\) and \(C\epsilon\) are positive constants depending on \(\mu^{-1}\) and \(\epsilon\), and \(C_{\Phi}\) is a positive constant independent of the mesh sizes \(\{h_{i}\}_{i \in \mathbb{N}}\).

**Theorem 6** (Consistency error for the antilinear forms \(\{\tilde{F}^{hi}\}_{i \in \mathbb{N}}\)) Let \(k \in \mathbb{N}\) be the polynomial degree of the approximation spaces. Let \(m \in \mathbb{N}\) and assume the quadrature rule \(Q^{3}_{\tilde{K}}\) from Definition 6 to be exact for polynomials of degree \(k + 2\bar{K} + m - 3\). Then, under Assumptions 3 and 4 and if \(J\) is such that \(J \in W^{m,q}(D)\) for \(q > \frac{3}{m}\) and \(q \geq 2\), it holds that

\[
|F^{hi}(V^{hi}) - \tilde{F}^{hi}(V^{hi})| \leq C_{F} \omega h^{m \bar{K}} |D^{hi}|^{\frac{1}{2}-\frac{1}{q}} \|J\|_{W^{m,q}(D^{hi})} \|V^{hi}\|_{\mathcal{L}^{0,D^{hi}}}
\]

for all \(V^{hi} \in P^{c}_{0}(\tau^{hi})\), for a positive constant \(C_{F}\) independent of the mesh-size.

**Proof of Theorem 4** Our main result follows by combining Theorems 5 and 6 with Lemma 10. \(\square\)

### 5 Numerical examples

We test our main results on a number of simple numerical examples. In order to isolate the effects that quadrature rules have on the observed rates of convergence in Theorems 1 and 4, we focus only on the case of a polygonal domain, namely the cube \(D := [-1, 1]^3 \subset \mathbb{R}^3\), and consider a coercive problem with the following sesquilinear
and antilinear forms:

\[
\Phi(U, V) := \int_D (\mu_0)^{-1} \text{curl } U \cdot \text{curl } V - \omega^2 (\epsilon_0) U \cdot V \, dx,
\]

(5.1)

\[
F(V) := -i \omega \int_D J \cdot V \, dx,
\]

(5.2)

where \( I \in \mathbb{R}^{3 \times 3} \) is the identity matrix, \( \omega = 1, \mu_0 \) and \( \epsilon_0 \) are strictly positive and negative scalars, respectively, \( J \) is given by

\[
J(x) = \frac{i}{\omega} \mu_0^{-1} (4 - 2x_2^2 - x_3^2) [1, 0, 0]^T
\]

\[
- \omega \epsilon_0 (x_2^2 - 1)(x_3^2 - 1) [1, 0, 0]^T \in W^{2,q}(D) \quad \forall q > 2,
\]

(5.3)

and the solution \( E \) to Problem 1 is given by

\[
E(x) = (x_2^2 - 1)(x_3^2 - 1) [1, 0, 0]^T \in H_0(\text{curl}; D) \cap H^2(\text{curl}; D).
\]

(5.4)

Experiments were carried out using GETDP [14] (version 3.3.0) and modifications to its source code corresponding to unimplemented quadrature rules. The points and weights of the quadrature rules available in GETDP may be examined in the file Kernel/Gauss_Tetrahedron.h of the source code. We also used GMSH [19] to generate the required meshes of the domain \( D \).

### 5.1 Convergence estimate for first order finite elements (\( k = 1 \))

We consider first order curl-conforming finite elements and set the parameters in (5.1), (5.2) and (5.3) to \( \mu_0 = 10 \) and \( \epsilon_0 = -10 \). We construct two numerical variations for both the sesquilinear and antilinear forms, given by

\[
\tilde{\Phi}_{h,0}(U_h, V_h) := \sum_{K \in \tau_h} Q^1_K ((\mu_0)^{-1} \text{curl } U_h \cdot \text{curl } V_h) + Q^1_K (\omega^2 (\epsilon_0) U_h \cdot V_h),
\]

\[
\tilde{\Phi}_{h,1}(U_h, V_h) := \sum_{K \in \tau_h} Q^1_K ((\mu_0)^{-1} \text{curl } U_h \cdot \text{curl } V_h) + Q^2_K (\omega^2 (\epsilon_0) U_h \cdot V_h),
\]

\[
\tilde{F}_{h,0}(V_h) := \sum_{K \in \tau_h} Q^3_K (-i \omega J \cdot V_h),
\]

\[
\tilde{F}_{h,1}(V_h) := \sum_{K \in \tau_h} Q^3_K (-i \omega J \cdot V_h),
\]

where \( Q^1_K \) and \( Q^3_K \) are one-point quadrature rules with arbitarily different chosen quadrature points in \( K \)—exact on polynomials of degree zero—and \( Q^2_K \) is a one-point Gaussian quadrature rule—exact on polynomials of degree one. Quadratures \( Q^1_K, Q^2_K \) and \( Q^3_K \), for \( K \in \tau_h \), are then built from \( Q^1_{\tilde{K}}, Q^2_{\tilde{K}} \) and \( Q^3_{\tilde{K}} \) as indicated in (3.9). Hence, \( \tilde{\Phi}_{h,1} \) and \( \tilde{F}_{h,1} \) satisfy the requirements of Theorem 1 (with \( r = 1 \)) while
Fig. 1 Error convergence in the $H(\text{curl}; D)$-norm for the solutions of Problem 3 to $E$ (given in (5.4)) depending on the implemented numerical variations of the sesquilinear and antilinear forms, indicated by the legend. The implementation considering $\tilde{\Phi}_{h,e}$ and $\tilde{F}_{h,e}$ attains a rate of convergence of $-\frac{1}{3}$ with respect to the number of degrees of freedom, as predicted by Theorem 1. On the other hand, implementations with at least one term not satisfying the conditions of Theorem 1 suffer from a degenerated rate between $-0.7 \times \frac{1}{3}$ and $-0.8 \times \frac{1}{3}$, but convergence is still observed.

$\tilde{\Phi}_{h,0}$ and $\tilde{F}_{h,0}$ do not. Figure 1 displays the convergence in the $H(\text{curl}; D)$-norm of the solution to Problem 3 corresponding to the different numerical implementations of the sesquilinear and antilinear forms. Nine meshes with 168, 228, 1,242, 2,810, 9,188, 38,782, 119,134, 500,300 and 1,265,246 degrees of freedom were employed.

5.2 Convergence estimate for second order finite elements ($k = 2$)

We extend our previous experiment to second order curl-conforming finite elements. Again, we consider $\mu_0 = 10$ and $\epsilon_0 = -10$, and construct three numerical variations for the sesquilinear form:

$$\tilde{\Phi}_{h,0}(U_h, V_h) := \sum_{K \in \mathcal{T}_h} Q^1_K ((\mu_0) I - \frac{1}{\omega^2} \text{curl} U_h \cdot \text{curl} V_h) + Q^1_K (\omega^2 (\epsilon_0) U_h \cdot V_h),$$

$$\tilde{\Phi}_{h,1}(U_h, V_h) := \sum_{K \in \mathcal{T}_h} Q^1_K ((\mu_0) I - \frac{1}{\omega^2} \text{curl} U_h \cdot \text{curl} V_h) + Q^2_K (\omega^2 (\epsilon_0) U_h \cdot V_h),$$

$$\tilde{\Phi}_{h,2}(U_h, V_h) := \sum_{K \in \mathcal{T}_h} Q^2_K ((\mu_0) I - \frac{1}{\omega^2} \text{curl} U_h \cdot \text{curl} V_h) + Q^2_K (\omega^2 (\epsilon_0) U_h \cdot V_h),$$

where $Q^1_K$ is a $2 \times 2 \times 2$ tensorized Gauss-Legendre quadrature rule—exact on polynomials of degree one on $K$—and $Q^2_K$ is a five-point Gaussian quadrature rule—exact on polynomials of degree three. Hence, $\tilde{\Phi}_{h,1}$ and $\tilde{\Phi}_{h,2}$ satisfy the requirements of...
The effect of quadrature rules on finite element… 933

Fig. 2 Error convergence in the $H(\text{curl}; D)$-norm for the solutions of Problem 3 to $E$, given in (5.4), depending on the implemented numerical variations of the sesquilinear and antilinear forms, indicated by the legend. The implementation considering $\tilde{\Phi}_{h,2}$ and $\tilde{\Phi}_{h,1}$ attain their predicted rates of convergence of $-\frac{2}{3}$ and $-\frac{1}{3}$, respectively. The implementations considering $\tilde{\Phi}_{h,0}$ suffers from a degenerated rate close to $-0.7 \times \frac{1}{3}$, though convergence is still observed.

Theorem 1 with $r = 1$ and 2, respectively. The right-hand side is implemented with a 15-point Gaussian quadrature and is left undisturbed throughout the experiments in this section. Figure 2 displays the convergence of the solution to Problem 3 corresponding to the different numerical implementations of the sesquilinear form. Eight meshes with 1,184, 1,688, 7,936, 1,7492, 54,480, 223,652, 674,676 and 2,454,312 degrees of freedom were employed.

Remark 1 GETDP does not include an implementation of the second order curl-conforming finite elements defined in Sect. 3.1. However, an implementation of the second order Webb basis functions [18,28,29] is available. Since they are contained in $P_2(\tilde{K})$, the consistency estimates in Theorems 2 and 3 remain valid, so our numerical examples are still meaningful when using this alternative basis.

5.3 Effect of quadrature precision on preasymptotic convergence

We investigate the effect that quadrature precision has on $\ell \in \mathbb{N}$ in Theorem 1, i.e., the duration of the preasymptotic regime before convergence is observed at the predicted rate. We consider $\mu_0 = 10$ (as before) and $\epsilon_0 = -10 - 9 \sin(m \pi x_3)$ for two cases $m = 10$ and 20. The solution to Problem 1 is still given by (5.4) for a modified right-hand side. We construct our numerical variations for the sesquilinear form as follows.
Fig. 3 Error convergence in the $H(\text{curl}; D)$-norm for solutions of Problem 3 to $E$ (given in (5.4)) for the two cases of $\epsilon_0$ and depending on the implemented numerical variations of the sesquilinear form. In both cases, we observe a marked preasymptotic regime when only one quadrature point is implemented, where no convergence is observed before reaching $\approx 10^4$ degrees of freedom and the mesh is able to resolve the oscillatory term in $\epsilon_0$. Improving the quadrature rule to 4 points quickly corrects this issue on the case in (a) though a preasymptotic regime is still observed on the case displayed in (b) which is improved on by a further increase in precision to 15 quadrature points.

\[
\tilde{\Phi}_{h,n}(U_h, V_h) := \sum_{K \in \tau_h} Q^1_K((\mu_0)^{-1} \text{curl } U_h \cdot \text{curl } V_h) + Q^2_{K,n}(-\omega^2(\epsilon_0 I)U_h \cdot V_h),
\]

where $Q^1_K$ is as before and $Q^2_{K,n}$ is a Gaussian quadrature over $\tilde{K}$ with $n = 1, 4$ and 15 points. The right-hand side is implemented with a 29 point Gaussian quadrature. Figure 3 displays the convergence of the solution of Problem 3 depending on the number of quadrature points used in the implementation of the sesquilinear form. The employed meshes were as in Sect. 5.1.

6 Concluding Remarks

Our two main results (Theorems 1 and 4) yield sufficient conditions to ensure convergence rates for the errors induced by quadrature rules used when solving Maxwell Equations via the FE method with inhomogeneous coefficients and on meshes with curved elements (tetrahedrons). Interestingly, Theorem 1 confirms the presumptions of P. Monk in the penultimate paragraph of Sect. 8.3 in [23], where it is stated that quadrature rules exact on polynomials of degree $2k - 1$ are expected to yield convergence rates of order $h^k$.

Unlike our result in Sect. 3, Theorem 4 analyses only the quadrature effect in implementation and does not present convergence estimates for the fully discrete solution to the real solution. The result does, however, set aside the issue of numerical integration, so that only the variational crime of the approximation of the real domain is left to be analysed. Notice as well, that choosing $\mathcal{R} = 1$ in Sect. 4 yields the same conditions for the quadrature rules in our two main results, which is of course to be expected.
Numerical examples in Sect. 5 not only confirm our results, but also display the necessity of the conditions of Theorems 1 and 4, since implementations that do not satisfy said conditions attain lower convergence rates than implementations that do.

Lastly, though we consider the smoothness of our parameters $\epsilon$, $\mu$ and $J$ to be global—belonging to some Sobolev space on the whole domain—one could quite easily accommodate our results to consider parameters with piecewise smoothness on a finite set of sub-domains of $D$, by requesting that the two dimensional surfaces across which they fail to possess the required degree of smoothness do not cross any element of the mesh. In other words, that each of the sub-domains is meshed so that the parameters are smooth on all elements of the mesh. An analogous consideration holds if the domain $D$ fails to possess a certain degree of smoothness at a finite number of points on its boundary.

References

1. Abdulle, A., Vilmart, G.: A priori error estimates for finite element methods with numerical quadrature for nonmonotone nonlinear elliptic problems. Numerische Mathematik 121(3), 397–431 (2012)
2. Aylwin, R., Jerez-Hanckes, C., Schwab, C., Zech, J.: Domain uncertainty quantification in computational electromagnetics. SIAM/ASA J. Uncertainty Quantification 8(1), 301–341 (2020)
3. Banerjee, U.: A note on the effect of numerical quadrature in finite element eigenvalue approximation. Numerische Mathematik 61(1), 145–152 (1992)
4. Banerjee, U., Osborn, J.E.: Estimation of the effect of numerical integration in finite element eigenvalue approximation. Numerische Mathematik 56(8), 735–762 (1989)
5. Banerjee, U., Suri, M.: The effect of numerical quadrature in the p-version of the finite element method. Math. Comput. 59(199), 1–20 (1992)
6. Bhattacharyya, P.K., Nataraj, N.: On the combined effect of boundary approximation and numerical integration on mixed finite element solution of 4th order elliptic problems with variable coefficients. ESAIM: Math. Model. Numer. Anal. 33(4), 807–836 (1999)
7. Buffa, A., Costabel, M., Sheen, D.: On traces for $H(\text{curl},\Omega)$ in Lipschitz domains. J. Math. Anal. Appl. 276(2), 845–867 (2002)
8. Buffa, A., Hiptmair, R.: Galerkin boundary element methods for electromagnetic scattering. In: Topics in computational wave propagation, volume 31 of Lecture Notes in Computational Science and Engineering, pp. 83–124. Springer, Berlin (2003)
9. Buffa, A., Hiptmair, R., von Petersdorff, T., Schwab, C.: Boundary element methods for Maxwell transmission problems in Lipschitz domains. Numerische Mathematik 95(3), 459–485 (2003)
10. Ciarlet, P.G.: Basic error estimates for elliptic problems. In: Finite Element Methods (Part 1), volume 2 of Handbook of Numerical Analysis, pp. 17–351. Elsevier (1991)
11. Ciarlet, P.G., Raviart, P.-A.: The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, pp. 409–474. Elsevier (1972)
12. Ciarlet, P.G.: The finite element method for elliptic problems. Society for Industrial and Applied Mathematics (2002)
13. Di Pietro, D.A., Droniou, Jérôme: A third strang lemma and an aubin-nitsche trick for schemes in fully discrete formulation. Calcolo 55(3), 40 (2018)
14. Dulac, P., Geuzaine, C.: GetDP reference manual: the documentation for GetDP, a general environment for the treatment of discrete problems. http://getdp.info
15. Ern, A., Guermond, J.L.: Theory and practice of finite elements, vol. 159. Springer Science & Business Media (2004)
16. Ern, A., Guermond, J.L.: Finite element quasi-interpolation and best approximation. Mathematical Modelling and Numerical Analysis 51(4), 1367–1385 (2017)
17. Ern, A., Guermond, J.L.: Analysis of the edge finite element approximation of the maxwell equations with low regularity solutions. Computers & Mathematics with Applications 75(3), 918–932 (2018)
18. Geuzaine, C., Meys, B., Dular, P., Legros, W.: Convergence of high order curl-conforming finite elements [for em field calculations]. IEEE Trans. Magn. 35(3), 1442–1445 (1999)
19. Geuzaine, C., Remacle, J.F.: Gmsh: A 3-d finite element mesh generator with built-in pre-and post-processing facilities. Int. J. Numer. Meth. Eng. 79(11), 1309–1331 (2009)
20. Hernández, E., Rodríguez, R.: Finite element approximation of spectral problems with neumann boundary conditions on curved domains. Math. Comput. 72(243), 1099–1115 (2003)
21. Jerez-Hanckes, C., Schwab, C., Zech, J.: Electromagnetic wave scattering by random surfaces: Shape holomorphy. Mathematical Models and Methods in Applied Sciences 27(12), 2229–2259 (2017)
22. Lenoir, M.: Optimal isoparametric finite elements and error estimates for domains involving curved boundaries. SIAM J. Numer. Anal. 23(3), 562–580 (1986)
23. Monk, P.: Finite element methods for Maxwell’s equations. Oxford University Press (2003)
24. Sauter, S. A., Schwab, C.: Boundary element methods, volume 39 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original
25. Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Springer Science & Business Media (2007)
26. Tartar, L.: An introduction to Sobolev spaces and interpolation spaces, vol. 3. Springer Science & Business Media (2007)
27. Vanmaele, M., Ženíšek, A.: The combined effect of numerical integration and approximation of the boundary in the finite element method for eigenvalue problems. Numerische Mathematik 71(2), 253–273 (1995)
28. Webb, J.P.: Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements. IEEE Trans. Antennas Propag. 47(8), 1244–1253 (1999)
29. Webb, J.P., Forgahani, B.: Hierarchal scalar and vector tetrahedra. IEEE Trans. Magn. 29(2), 1495–1498 (1993)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.