ON THE QUESTION OF GENERICITY OF HYPERBOLIC KNOTS

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ABSTRACT. A well-known conjecture in knot theory says that the percentage of hyperbolic knots amongst all of the prime knots of \( n \) or fewer crossings approaches 100 as \( n \) approaches infinity. In this paper, it is proved that this conjecture contradicts several other plausible conjectures, including the 120-year-old conjecture on additivity of the crossing number of knots under connected sum and the conjecture that the crossing number of a satellite knot is not less than that of its companion.

1. INTRODUCTION

William Thurston proved in 1978 that every non-torus non-satellite knot is a hyperbolic knot. Computations show that the overwhelming majority of prime knots with small crossing number are hyperbolic knots. The following table gives the number of hyperbolic, prime satellite, and torus knots of \( n \) crossings for \( n = 3, \ldots, 16 \) (see [HTW98] or the sequences A002863, A052408, A051765, and A051764 in the Sloane’s encyclopedia of integer sequences).

| type       | \( n = 3 \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|------------|-----------|---|---|---|---|---|---|----|----|----|----|----|----|----|
| all prime  | 1         | 1 | 1 | 2 | 3 | 7 | 21| 49 | 165| 552| 2 176| 9 988| 46 972| 253 293| 1 388 705|
| hyperbolic | 0         | 0 | 0 | 0 | 0 | 0 | 0 | 2  | 2  | 6  | 10 | 2  | 1  | 2  |
| prime satellite | 0   | 0 | 0 | 0 | 0 | 0 | 0 | 2  | 2  | 6  | 10 | 2  | 1  | 2  |
| torus      | 1         | 0 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 2  | 2  | 1  | 1  | 2  |

(A part of) these data gave rise to the following conjecture (see [Ad94b] p. 119).

Conjecture 1. The percentage of hyperbolic knots amongst all of the prime knots of \( n \) or fewer crossings approaches 100 as \( n \) approaches infinity.

In the present paper, we show that Conjecture 1 contradicts several other long standing conjectures, including the following one.

Conjecture 2. The crossing number of knots is additive with respect to connected sum.

See, e.g., [Ad94b] p. 69, [Kir97] Problem 1.65, and [La09] for comments and related results. Another related conjecture is as follows.

Conjecture 3. The crossing number of a satellite knot is bigger (a weaker variant: not less) than that of its companion.

See [Ad94b] p. 118, [Kir97] Problem 1.67 (attributed to de Souza), and [La14]. It is remarked in [Kir97] Problem 1.67 concerning Conjecture 3 that ‘Surely the answer is yes, so the problem indicates the difficulties of proving statements about
the crossing number’. Since a composite knot is a connected sum of its factors and, at the same time, is a satellite of each of its factors, the ‘intersection’ of Conjectures 2 and 3 yields the following.

**Conjecture 4.** The crossing number of a composite knot is bigger (a weaker variant: not less) than that of each of its factors.

Let us denote by $\text{cr}(X)$ the crossing number of a knot $X$. If $P$ is a prime knot and $\lambda$ is a real number, we say that $P$ is $\lambda$-regular if we have $\text{cr}(K) \geq \lambda \cdot \text{cr}(P)$ whenever $P$ is a factor of a knot $K$. In this terminology, Conjecture 4 says that each prime knot is $1$-regular. Lackenby [La09] proved that each knot is $\frac{1}{152}$-regular. Our considerations involve the following conjecture.

**Conjecture 5.** Each prime knot is $\frac{2}{3}$-regular.

We also consider the following weakening of Conjecture 5.

**Conjecture 6.** There exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of $\frac{2}{3}$-regular knots amongst all of the hyperbolic knots of $n$ or fewer crossings is at least $\varepsilon$.

We have the following obvious implications.

\[
\text{Conj. 2} \iff \text{Conj. 4} \implies \text{Conj. 5} \implies \text{Conj. 6} \\
\text{Conj. 3}
\]

The main result of this paper is the following theorem.

**Theorem 1.** Conjecture 1 contradicts (each of) Conjectures 2, 3, 4, 5, and 6.

The paper is organised as follows. Section 2 contains remarks concerning Conjectures 1–6. In Section 3, we present the key idea of the proof of Theorem 1 and reduce Theorem 1 to Proposition 1 consisting of three assertions. Sections 4–8 contain the proof of Proposition 1. In Sections 4 and 5, we prove the first two assertions of Proposition 1. Section 6 contains a combinatorial lemma used in the proof of the last assertion of Proposition 1. Section 7 contains preliminaries on tangles. In Section 8, we prove the last assertion of Proposition 1. In Section 9, we introduce a new property of knots (strong property PT) and prove Theorem 8 strengthening Theorem 1. In Section 10, we show that an assumption that Conjectures 2, 5 has many strong counterexamples contradicts Conjecture 1 as well. In Section 11, we show that certain assumptions concerning unknotting numbers of knots contradict Conjecture 1.

The paper should be interpreted as being in either the PL or smooth category. For standard definitions we mostly use the conventions of [BZ06] and [BZH14]. There will be a certain abuse of language in order to avoid complicating the notation. In particular, a knot $K$ will be a circle embedded in a 3-sphere $S^3$, a pair $(S^3, K)$, or a class of homeomorphic pairs (cf. [BZ06, p. 1]). No orientations on knots and spaces are placed if not otherwise stated.

The author is grateful to Ivan Dynnikov, Evgeny Fominykh, Aleksandr Gaifullin, Vadim Kaimanovich, Maksim Karev, Paul Kirk, Vladimir Nezhinskij, Semën Podkorytov, Józef Przytycki, Alexey Sleptsov, and Andrei Vesnin for helpful comments and suggestions.
2. Remarks

We list certain results related to Conjectures 1-6.

Predominance of hyperbolic objects. In recent years, a number of results have been obtained showing the predominance of hyperbolic objects in various cases. We refer to the works of Ma [Ma14] and Ito [Ito15, Theorem 2] for results concerning genericity of hyperbolic knots and links. See also [Mah10a, LMW14, Theorem 2], [LusMo12], [Riv14], and [Ito15, Theorem 1] for results on genericity of hyperbolic 3-manifolds. Related results show that pseudo-Anosovs prevail (in various senses) in mapping class groups of surfaces. We refer to [Riv08, Riv09, Riv10, Riv12, Riv14], [Kow08], [Mah10b, Mah11, Mah12], [AK10, Sis11, Mal12, LubMe12, MS13], and also [Car13, CW13, Wi14] (non-random approach) for precise statements and detailed discussions. See also [GTT16] for the genericity of loxodromic isometries for actions of hyperbolic groups on hyperbolic spaces. Other examples of hyperbolicity predominance can be found in extensive literature on exceptional Dehn fillings (see [Thu79] etc.) and in [Gro87, 0.2.A], [GhH90, p. 20], [Ch91] and [Ch95, Ols92], [Gro93, Zuk03, Oll04, Oll05], where viewpoints are given from which it appears that a generic finitely presented group is word hyperbolic. Apparently, combining approaches developed by [Ito15] and [Ma14] with results of [Car13, CW13, Wi14] (the same for [LusMo12]) one can obtain more viewpoints where generic knot will be hyperbolic.

Predominance of non-hyperbolic objects. As for natural models where it is proved that hyperbolic objects are rare, we have the standard methods of generating knots as polygons in $\mathbb{R}^3$. Under this approach, composite knots prevail and prime knots (including hyperbolic ones) are asymptotically scarce. See Sumners and Whittington [SW88], Pippenger [Pip89], and also Soteros, Sumners, and Whittington [SSW92] for the case of self-avoiding random polygons on the simple cubic lattice; see [Oetal94] and [Sot98] for such polygons in specific subsets of the lattice; see [DPS94] and [Jun94] for local and global knotting in Gaussian random polygons; see [Di95] and [DNS01] for local and global knotting in equilateral random polygons; see also [Ken79] for knotting of Brownian motion and [Sum99, MMO11] for more references. An interesting idea has appeared in [Ad05, p. 4] and [Cr04, p. 95] that prime satellite knots should prevail over hyperbolic ones when we consider Gaussian random polygons. In both [Ad05] and [Cr04], however, the idea was apparently inspired by a misinterpretation of results in [Jun94].

Crossing number additivity. Murasugi [Mur87, Corollary 6] proved that Conjecture 2 is valid for alternating knots. (This follows from the proof of Tait conjecture that reduced alternating projections are minimal; this Tait conjecture was also proved, independently, by Kauffman [Kau87] and Thistlethwaite [Thi87].) Conjecture 2 is valid for adequate knots (see [LT88]). Diao [Diol04] and Gruber [Gru03] independently proved that Conjecture 2 is valid for torus knots and certain other particular classes of knots. Results of [Mur87], [Kau87], and [Thi87] imply that alternating knots are 1-regular.\footnote{It is shown in [Mou87], [Kau87], [Thi87] that (i) for each knot $K$ we have $\text{span} V_K(t) \leq cr(K)$, and (ii) for alternating $K$ we have $\text{span} V_K(t) = cr(K)$, where $V_K(t)$ is the Jones polynomial of $K$ and $\text{span} V_K(t)$ denotes the difference between the maximal and minimal degrees of $V_K(t)$. (It is known that $V_K(t) \neq 0$ so that span $V_K(t)$ is well-defined. See [Jon85, Theorem 15].) Then 1-regularity of alternating knots follows because, for any knots $K_1$ and $K_2$, we have $\text{Jon85}$.} Diao [Diol04, Theorem 3.8] showed that...
torus knots are 1-regular. In [PZ15], the authors introduce a telescopic family of conjectures concerning monotonic simplification of link diagrams and provide supporting evidence for (the strongest of) these conjectures. Each of Petronio–Zanellati conjectures implies Conjecture 2.

Torus knots. Murasugi [Mur91, Proposition 7.5] proved that the torus link of type \((p, q)\), where \(2 \leq p \leq q\), has crossing number \((p - 1)q\). Taking into account that the number of all prime knots of \(n\) crossings grows exponentially in \(n\) (see [ES87, Wel92]), this implies that the percentage of torus knots amongst all of the prime knots of \(n\) or fewer crossings approaches 0 as \(n\) approaches infinity. Thus, only satellite knots pose a danger to Conjecture 1.

Hyperbolic knots. Several interesting classes of knots are known to consist of hyperbolic and torus knots only. In particular, amongst these classes are:– prime alternating knots, including 2-bridge knots (see [Men84]),– prime almost alternating knots (see [Aetal92]),– prime toroidally alternating knots (see [Ad94a]),– arborescent knots, including 2-bridge knots, pretzel knots, and Montesinos knots (see [BS10, Theorem 1.5 and subsequent discussion in [FG09]], etc.

More families of hyperbolic knots, links, and tangles are listed in [Ad05]. See [Ito11, IK12, Theorems 8.3, 8.4] for new examples of huge classes of hyperbolic knots, links, and 3-manifolds.

3. The Idea of the Proof of Theorem 1

Our proof of Theorem 1 uses a specific way of constructing satellite knots. For brevity, we use the term \(\gamma\)-knots for the satellite knots constructed in this way.

Definition. \(\gamma\)-Knots. Let \(K\) be a knot in a 3-sphere \(S^3\), and let \(V\) be an unknotted solid torus in \(S^3\) such that \(K\) is contained in the interior of \(V\). Let \(\psi : V \to W \subset S^3\) be a homeomorphism onto a tubular neighbourhood \(W\) of a hyperbolic knot. Recall that the winding number of \(K\) in \(V\) is the absolute value of the algebraic intersection number of \(K\) with a meridional disk in \(V\). Assume that the winding number of \(K\) in \(V\) is at least 2 and that \(\psi\) maps a longitude of \(V\) to a longitude of \(W\). Then we say that the knot \(\psi(K) \subset S^3\) is a \(\gamma\)-knot over \(K\).

A method of constructing a \(\gamma\)-knot is given in Fig. 1. Assume that a diagram \(D'\) of a knot \(K'\) is obtained from a diagram \(D\) of a knot \(K\) by local move as in Fig. 1 (See Fig. 2 for an example.) Our definitions imply that if two arrows on arcs in Fig. 1(a) indicate the same orientation on \(K\), then \(K'\) is a \(\gamma\)-knot over \(K\). Here, the winding number is 2 while the companion hyperbolic knot is the figure-eight knot. (In order to check that the condition on longitudes is also fulfilled, we observe that each arc in Fig. 1 has zero total curvature.)

Theorem 6

\[\text{span } V_{K_1,tK_2}(t) = \text{span } V_{K_1}(t) + \text{span } V_{K_2}(t).\]

Ivan Dynnikov (private communication) found a counterexample to Petronio–Zanellati conjectures.

If a solid torus \(U\) is embedded in a 3-sphere \(S^3\), then there exists an essential curve in \(\partial U\) that bounds a 2-sided surface in \(S^3 \setminus \text{int}(U)\) (a Seifert surface). This curve is unique up to isotopy on \(\partial U\) and is called a longitude of \(U\) in \(S^3\) (see, e.g., [BZ00, Theorem 3.1]).
We deduce Theorem 1 from the following proposition on $\gamma$-knots.

**Proposition 1.**

(i) Each $\gamma$-knot is a satellite knot.

(ii) The sets of $\gamma$-knots over distinct non-satellite knots are disjoint.

(iii) If $P$ is a $\frac{2}{3}$-regular prime knot, then there exists a prime $\gamma$-knot $P'$ over $P$ with $cr(P') \leq cr(P) + 17$.

**Remark.** Assertion (iii) of Proposition 1 is not obvious because a $\gamma$-knot over a prime knot is not necessarily prime (see Fig. 2).

**Proposition 1 implies Theorem 1.** We introduce the following notation. Let $p_n$ (resp., $h_n$, $s_n$) denote the number of prime (resp., hyperbolic, prime satellite) knots with crossing number $n$. We set $P_n = \sum_{k=1}^{n} p_n$, $H_n = \sum_{k=1}^{n} h_n$, and $S_n = \sum_{k=1}^{n} s_n$.

Since each of Conjectures 2, 3, 4, and 5 implies Conjecture 6 (see the diagram before Theorem 1), it suffices to prove only that Conjectures 6 and 1 are incompatible. If Conjecture 6 is true, then there exist $\varepsilon_0 > 0$ and $N_0 > 0$ such that, for all $n > N_0$, the number of $\frac{2}{3}$-regular hyperbolic knots of $n$ or fewer crossings is at least $\varepsilon_0 H_n$. Obviously, in this case assertions (i), (ii), and (iii) of Proposition 1 imply that (for all $n > N_0$) we have

$$S_{n+17} \geq \varepsilon_0 H_n.$$
Therefore, we have
\[ P_{n+17} \geq H_{n+17} + \epsilon_0 H_n. \]
This is equivalent to the following inequality
\[ 1 \geq \frac{H_{n+17}}{P_{n+17}} + \epsilon_0 \frac{H_n}{P_n}. \]
If Conjecture (1) is true, then both sequences \( \frac{H_{n+17}}{P_{n+17}} \) and \( \frac{H_n}{P_n} \) tend to 1. In this case, Eq. (1) implies that
\[ \lim_{n \to +\infty} P_{n+17} = +\infty. \]

Consequently, for each \( B > 0 \) we have \( P_n > B^n \) for all sufficiently large \( n \). (We consider subsequences of the form \( P_{n_0+17i}, i \in \mathbb{N} \).) In other words, we have
\[ \limsup_{n \to +\infty} \frac{P_{n+17}}{P_n} < +\infty. \]
However, it is shown in [Wel92] that
\[ \limsup_{k \to \infty} \frac{P_{n+17}}{P_n} < +\infty, \]
which implies that there exists \( B > 0 \) such that \( p_n < B^n \) for all \( n \in \mathbb{N} \). Then, for each \( n \in \mathbb{N} \) we have \( P_n < (B + 1)^n \) whence it follows that
\[ \limsup_{k \to \infty} \frac{P_{n+17}}{P_n} \leq B + 1 < +\infty. \]
This contradicts (2). The obtained contradiction completes the proof. \( \square \)

4. Proof of assertion (i) of Proposition 1

We recall definitions of satellite knots. A knot \( K \) in \( S^3 \) is a satellite knot if \( S^3 \) contains a non-trivial knot \( C \) such that \( K \) lies in the interior of a regular neighbourhood \( V \) of \( C \), \( V \) does not contain a 3-ball containing \( K \), and \( K \) is not a core curve of the solid torus \( V \). The knot \( K \) is a satellite knot if and only if \( K \) contains an incompressible, non-boundary parallel torus in its complement. (For a proof, see [BZH14, Remark 16.1, p. 335].)

Let \( K \) be a \( \gamma \)-knot in \( S^3 \). Then the definition of \( \gamma \)-knots implies that \( K \) lies in a knotted solid torus \( W \subset S^3 \) such that the winding number of \( K \) in \( W \) is at least 2. Since the winding number of \( K \) in \( W \) is at least 2, it follows that \( W \) does not contain a 3-ball containing \( K \), and \( K \) is not a core curve of \( V \). This means by the above definition that \( K \) is a satellite knot.

5. Proof of assertion (ii) of Proposition 1

We show that the sets of \( \gamma \)-knots over distinct non-satellite knots are disjoint. Suppose to the contrary that there exist a knot \( K \) and two distinct non-satellite knots \( H_1 \) and \( H_2 \) such that \( K \) is a \( \gamma \)-knot both over \( H_1 \) and over \( H_2 \). By the definition of \( \gamma \)-knots, this means that there exist embedded solid tori \( V_1 \) and \( V_2 \) in \( S^3 \) and re-embeddings \( \phi_1 : V_1 \to S^3 \) and \( \phi_2 : V_2 \to S^3 \) such that, for each \( i \in \{1, 2\} \), the following conditions hold:
- \( V_i \) is a tubular neighbourhood of a hyperbolic knot,
- \( K \) lies in the interior of \( V_i \) and the winding number of \( K \) in \( V_i \) is at least 2,
- the solid torus \( \phi_i(V_i) \) is unknotted,
- \( \phi_i \) maps a longitude of \( V_i \) to a longitude of \( \phi_i(V_i) \),
we have $\phi_i(K) = H_i$.

**Claim 1.** The tori $\partial V_1$ and $\partial V_2$ are both incompressible in $S^3 \setminus K$.

Since the winding number of $K$ in $V_i$ is non-zero, it follows that no 3-ball in $V_i$ contains $K$. If a knotted solid torus $U$ in a 3-sphere $S^3$ contains a knot $L$ in its interior while no 3-ball in $U$ contains $L$, then $\partial U$ is sometimes called a companion torus of $L$. It is well known that, in this case, $\partial U$ is incompressible in $S^3 \setminus L$. (See, e.g., [BZH14] Propositions 3.10 and 3.12, and E 2.9.) This implies Claim 1.

**Claim 2.** There exists an isotopy of $\partial V_1$ in $S^3 \setminus K$ that moves $\partial V_1$ to a position where $\partial V_1 \cap \partial V_2 = \emptyset$.

It may be assumed that $\partial V_1$ intersects $\partial V_2$ transversely in simple closed curves. If the intersection $\partial V_1 \cap \partial V_2$ contains a curve that is inessential in $\partial V_2$, let $C$ be an innermost of such curves and let $d$ be the open disk in $\partial V_2 \setminus \partial V_1$ bounded by $C$. Then $C$ is inessential in $\partial V_1$ because $\partial V_1$ is incompressible in $S^3 \setminus K$ (Claim 1). Let $\delta$ be the open disk in $\partial V_1$ bounded by $C$ ($\delta$ may intersect $\partial V_2$). Then the sphere $d \cup \delta \cup C$ bounds a ball (say, $B$) in $S^3 \setminus K$. We have $B \cap \partial V_1 = \delta \cup C$. It follows that we can eliminate $C$ (together with $\delta \cap \partial V_2$, if nonempty) by an isotopy of $\partial V_1$ in a neighborhood of $B$. Therefore, we can eliminate all components of $\partial V_1 \cap \partial V_2$ that are inessential in $\partial V_2$. The remaining curves of $\partial V_1 \cap \partial V_2$ are essential in $\partial V_1$ as well. (For if $C$ is an innermost of inessential curves from $\partial V_1 \cap \partial V_2$ on $\partial V_1$, then $C$ is inessential in $\partial V_2$ because $\partial V_2$ is incompressible in $S^3 \setminus K$ by Claim 1.) Now, if $\partial V_1 \cap \partial V_2$ is still nonempty, the space $\partial V_1 \setminus \partial V_2$ is a collection of annuli. It is known that every incompressible properly embedded annulus in the closure of the complement of a hyperbolic knot is boundary parallel (see, e.g., [BZ06], Lemma 15.26). Applying this to the space $S^3 \setminus \text{int}(V_2)$, we see that there exists an isotopy of $\partial V_1$ in $S^3 \setminus K$ moving $\partial V_1$ in $S^3 \setminus \partial V_2$. Claim 2 is proved.

The classical Isotopy Extension Theorem (for smooth manifolds) says that if $A$ is a compact submanifold of a manifold $M$ and $F: A \times I \to M$ is an isotopy of $A$ with $F(A \times I) \subset \text{int}(M)$, then $F$ extends to an ambient isotopy (i.e., a diffeotopy of $M$) having compact support (see, e.g., [Hit76] p. 179). Applying this theorem to the isotopy of $\partial V_1$ in $S^3 \setminus K$ from Claim 2 yields the following.

**Claim 3.** There exists an ambient isotopy of $S^3$, fixing $K$ pointwise, that moves $V_1$ to a position in which $\partial V_1 \cap \partial V_2 = \emptyset$.

Thus, we can assume without loss of generality that $\partial V_1 \cap \partial V_2 = \emptyset$ (while $V_1$ and $V_2$ satisfy all properties listed at the beginning of the proof). Now, let $M_1$ and $M_2$ denote the closures of the complements $S^3 \setminus V_1$ and $S^3 \setminus V_2$ respectively.

**Claim 4.** $M_1$ and $M_2$ are disjoint.

In order to prove Claim 4 we need the following assertion.

**Claim 5.** There is no isotopy between $\partial V_1$ and $\partial V_2$ in $S^3 \setminus K$.

Suppose to the contrary that such an isotopy exists. Then the Isotopy Extension Theorem (see above) implies that there exists an ambient isotopy of $S^3$, fixing $K$ pointwise, that moves $\partial V_1$ to $\partial V_2$. This yields an isotopy between $V_1$ and $V_2$ that fixes $K$ pointwise. Then the triples $(V_1, K, \ell_1)$ and $(V_2, K, \ell_2)$, where $\ell_i$ is a longitude of $V_i$, $i = 1, 2$, are homeomorphic, i.e., there exists a homeomorphism $\tau: V_1 \to V_2$ such that $\tau(K) = K$ and $\tau(\ell_1) = \ell_2$. This implies that
the pairs \((S^3, \phi_1(K))\) and \((S^3, \phi_2(K))\) are homeomorphic. Indeed, we observe that \((S^3, \phi_1(K))\) is obtained from \((V_i, K)\) by a Dehn filling along \(\ell_i\), that is, \((S^3, \phi_1(K))\) is obtained by attaching a solid torus \(V\) to \(V_i\) by a gluing homeomorphism \(\sigma_i: V \to \partial V_i\) such that \(\sigma_i^{-1}(\ell_i)\) bounds a meridional disk of \(V\). Thus, the homeomorphism \(\tau: V_i \to V_2\) extends to a homeomorphism \(S^3 \to S^3\) that maps \(\phi_1(K)\) to \(\phi_2(K)\). This means that the knots \(H_1\) and \(H_2\) are equivalent because we have \(\phi_i(K) = H_i\) by construction. This contradicts the assumption that \(H_1\) and \(H_2\) are distinct. Claim 5 is proved.

Now, we pass to the proof of Claim 4. Observe that neither \(M_1\) contains \(\partial M_2 = \partial V_2\) nor \(M_2\) contains \(\partial M_1 = \partial V_1\) because an incompressible torus in a hyperbolic knot complement is boundary parallel by Thurston’s hyperbolization theorem, while \(\partial M_1 = \partial V_1\) and \(\partial M_2 = \partial V_2\) are not parallel by Claim 5. Obviously, this implies that \(M_1\) and \(M_2\) are disjoint.

Another fact that we need is implied by the following proposition.

**Proposition 2.** Let \(C_1, C_2, \ldots, C_n\) be \(n\) disjoint submanifolds of \(S^3\) such that for all \(i \in \{1, 2, \ldots, n\}\), \(K_i = \text{clo}(S^3 \setminus C_i)\) is a non-trivially embedded solid-torus in \(S^3\). Then there exists \(n\) disjointly embedded 3-balls \(B_1, B_2, \ldots, B_n \subset S^3\) such that \(C_i \subset B_i\) for all \(i \in \{1, 2, \ldots, n\}\). Moreover, each \(B_i\) can be chosen to be \(C_i\) union a 2-handle which is a tubular neighbourhood of a meridional disk for \(K_i\).

**Proof.** See [38][2] Proposition 2.1] and references therein for earlier proofs.\(\square\)

Applying Proposition 2 to \(M_1\) and \(M_2\), we obtain the following claim.

**Claim 6.** There exists a meridional disk \(D_2\) for \(V_2\) such that \(D_2 \subset V_1\).

Now, since we have \(M_2 \subset V_1\) (Claim 4), the image \(\phi_1(M_2)\) is well defined. We consider the complement \(W := S^3 \setminus \phi_1(\text{int}(M_2))\). Due to Alexander’s theorem on embedded torus in \(S^3\), we observe that \(W\) is a knotted solid torus because we know that the boundary \(\partial W = \partial \phi_1(M_2) = \phi_1(\partial M_2) = \phi_1(\partial V_2)\) is a torus, while the complement \(S^3 \setminus W = \phi_1(\text{int}(M_2))\) is homeomorphic to \(\text{int}(M_2)\), which is the complement of the knotted solid torus \(V_2\). (Of course, by the Gordon–Luecke theorem we know, moreover, that \(W\) is a tubular neighbourhood of a hyperbolic knot.) We see that \(W\) contains \(\phi_1(K)\) by construction. Finally, we see that the winding number of \(\phi_1(K)\) in \(W\) is equal to the winding number of \(K\) in \(V_2\) because there exists a meridional disk \(D_2\) for \(V_2\) such that \(D_2 \subset V_1\) so that \(\phi_1\) maps \(D_2\) to a meridional disk of \(W\). Therefore, \(\phi_1(K)\) is contained in a knotted solid torus \(W\) and the winding number of \(\phi_1(K)\) in \(W\) is at least 2. This means that \(\phi_1(K)\) is a satellite knot. Since we have \(H_1 = \phi_1(K)\), this contradicts the assumption that \(H_1\) is not a satellite knot. This contradiction completes the proof of assertion (ii) of Proposition 2.

6. A COMBINATORIAL LEMMA

The present section contains a lemma which is used in the proof of assertion (iii) of Proposition 2.

**Definitions.** Let \(K\) be a knot in the 3-sphere \(S^3 = \mathbb{R}^3 \cup \{\infty\}\), and let \(D \subset S^2\) be a projection of \(K\) on a 2-sphere \(S^2 = \mathbb{R}^2 \cup \{\infty\}\) in \(S^3\). A knot projection is said to be regular if its only singularities are transversal double points. If \(D\) is a regular knot projection, an edge in \(D\) is the closure of a component of the set.
$D \setminus V$, where $V$ is the set of double points of $D$. We say that two edges $I$ and $J$ of $D$ are neighboring edges or neighbors if there exists a component $Q$ of $S^2 \setminus D$ such that the boundary $\partial Q$ contains both $I$ and $J$. We say that two edges $I$ and $J$ of $D$ are consecutive if the union $I \cup J$ is the image of a (connected) arc of the knot. We will denote by $\rho$ the maximal metric on the set $E(D)$ of edges of $D$ in the class of metrics satisfying the condition

$$\rho(I, J) = 1 \text{ if } I \text{ and } J \text{ are consecutive edges of } D.$$  

**Lemma 1.** Any regular knot projection with $n > 0$ double points has a pair of neighboring edges $I$ and $J$ with $\rho(I, J) \geq 2n/3$.

**Proof.** Let $D \subset S^2$ be a regular knot projection with $n$ double points. We consider the case with $n \geq 2$ (the case $n = 1$ is obvious). Observe that $D$ has $2n$ edges. Put $k := \lfloor 2n/3 \rfloor$, the largest integer not greater than $2n/3$, and split the set $E(D)$ of edges of $D$ in three parts, $E_1$, $E_2$, and $E_3$, such that each part is a chain of consecutive edges, two parts consist of $k$ edges each, and the third part consists of $2n - 2k$ edges. (Note that $2n - 2k \in \{k, k + 1, k + 2\}$; in particular, we have $2n - 2k \geq k$, that is, each part consists of at least $k$ edges. No part is empty since we assume $n \geq 2$.) Let $D_i \subset D, i = 1, 2, 3$, be the union of edges from $E_i$. Observe that each $D_i$ is compact and connected and $D = D_1 \cup D_2 \cup D_3$. Let us smoothly embed $S^2$ in $\mathbb{R}^3$ as a sphere of radius 1 and let $\text{dist}$ denote the metric on $S^2$ induced by the Euclidean metric in $\mathbb{R}^3$. For each $i \in \{1, 2, 3\}$ we set

$$R_i := \{x \in S^2 : \text{dist}(x, D_i) = \text{dist}(x, D)\}.$$  

Observe that $R_1 \cup R_2 \cup R_3 = S^2$ because $D = D_1 \cup D_2 \cup D_3$. We see that for each $i \in \{1, 2, 3\} \setminus R_i$ the set $R_i$ is closed because $D_i$ is compact (consider a convergent sequence of points in $R_i$). Also, we see that for each $i \in \{1, 2, 3\}$ the set $R_i$ is connected. Indeed, if $p \in R_i$, then due to compactness of $D_i$ there exists a point $q \in D_i$ such that $\text{dist}(p, q) = \text{dist}(p, D)$. Then the geodesic segment between $p$ and $q$ is in $R_i$ by the triangle inequality. Therefore, $R_i$ is connected because $D_i$ is connected. Finally, we see that for any $\{i, j\} \subset \{1, 2, 3\}$ the intersection $R_i \cap R_j$ is not empty because $D_i \subset R_i$ and $D_j \subset R_j$, while $D_i \cap D_j$ is not empty.

Thus, the sets $R_1$, $R_2$, and $R_3$ satisfy assumptions of Lemma 2 below. Lemma 2 implies that $R_1$, $R_2$, and $R_3$ have a common point $x$. Clearly, $x$ is not an inner point of an edge of $D$, so we have two possible cases:

1) $x$ is a double point of $D$,

2) $x \notin D$.

Suppose $x$ is a double point of $D$. Then there exists a triple $\{J_1, J_2, J_3\}$ of edges of $D$ incident to $x$ such that $J_i \in E_i$ for all $i \in \{1, 2, 3\}$. Without loss of generality we can and will assume that $J_1$ and $J_3$ are consecutive. Then $J_1$ and $J_2$ are neighbors, and $J_2$ and $J_3$ are neighbors. It is easily seen that we have $\rho(J_1, J_2) \geq k$ and if $\rho(J_1, J_2) = k$ then $\rho(J_2, J_3) = k + 1$, and the theorem follows.

Suppose $x \in S^2 \setminus D$. Let $Q$ be the component of $S^2 \setminus D$ containing $x$. Observe that the set

$$\{y \in D : \text{dist}(x, y) = \text{dist}(x, D)\}$$  

is contained in $\partial Q \subset D$ and contains no double points of $D$ (due to smoothness of embedding $S^2 \to \mathbb{R}^3$). Therefore, since $x \in R_1 \cap R_2 \cap R_3$, for each $i \in \{1, 2, 3\}$ the set $\partial Q \cap D_i$ contains at least one edge of $D$. This means that there exists a triple $\{J_1, J_2, J_3\}$ of pairwise neighboring edges of $D$ such that we have $J_i \in E_i$.
for all $i \in \{1, 2, 3\}$. It is an easy exercise to check that this triple contains a pair
$\{I, J\}$ with
$$\rho(I, J) \geq \lceil 2n/3 \rceil \geq 2n/3. \quad \Box$$

**Lemma 2.** If a triple of pairwise intersecting closed connected sets cover a simply
connected space, then these three sets have a common point.

**Proof.** This follows, e. g., from Theorem 5 of [Bog02] in the case $m = 1$. \quad \Box

7. **Tangles**

Our proof of assertion (iii) of Proposition \ref{prop:2-string-tangle} uses tangles. The present section contains some preliminaries on tangles.

**Definitions.** A $k$-string **tangle**, where $k \in \mathbb{N}$, is a pair $(B, t)$ where $B$ is a 3-ball and
$t$ is the union of $k$ disjoint arcs in $B$ with $t \cap \partial B = \partial t$. We mostly interested
in the cases where $k \in \{1, 2\}$. Two tangles, $(B, t)$ and $(A, s)$, are **equivalent** if there is
a homeomorphism of pairs from $(B, t)$ to $(A, s)$. A tangle $(B, t)$ is **trivial** if $B$ contains a properly embedded disk containing $t$. A tangle $(B, t)$ is **locally knotted** if $B$ contains a ball $B'$ such that $(B', B' \cap t)$ is a nontrivial 1-string tangle. A 2-string tangle $(B, t)$ is **prime** if it is neither locally knotted nor trivial. If $(B, t)$ and $(A, s)$ are $k$-string tangles and $f: (\partial B, \partial t) \to (\partial A, \partial s)$ is a homeomorphism,
then $f$ is referred to as a **sum** of the two tangles. If $(B, t)$ is a 1-string tangle and $(A, s)$ is the trivial 1-string tangle, then there is a unique (up to a homeomorphism of pairs) knot which is a sum of $(B, t)$ and $(A, s)$. This knot is called the **closure** of $(B, t)$. We say that a 2-string tangle $(B, t)$ is a **cable**
tangle if there exists an embedding $f: I \times I \to B$ such that $f(I \times I) \cap \partial B = I \times \partial I$ and
$t = f(\partial I \times I)$, where $I := [0, 1]$. (We treat the trivial 2-string tangle as a cable tangle.) Clearly, each tangle $(B, t)$ can be embedded in $\mathbb{R}^3$ in such a way that $B$ becomes a Euclidean ball while the endpoints $\partial t$ lie on a great circle of this ball and $t$ is in general position with respect to the projection onto the flat disc bounded by the great circle. The projection, with additional information of over- and undercrossings, then gives us a **tangle diagram**. Examples of tangle diagrams are given in Figs. \ref{fig:1} \& \ref{fig:2} and \ref{fig:3}.

**Theorem 2** ([Lick81, Theorem 1]). A sum of two 2-string prime tangles is a prime link.

**Lemma 3.** Each nontrivial cable 2-string tangle is prime.

**Proof.** (See [Lick81, Examples (a) and (b)].) It is enough to observe that we can, in an obvious manner, add the trivial 2-string tangle to any cable 2-string tangle so as to create the trivial knot, which proves that the initial tangle has no local knots (this follows by the Unique Factorization Theorem by Schubert [Schu49]). \quad \Box

**Lemma 4.** No composite knot is a sum of a nontrivial cable 2-string tangle with the trivial 2-string tangle.

**Proof.** Suppose that a knot $K$ in $S^3$ is presented as a sum
$$(S^3, K) = (B, t) \cup_f (A, s), \quad f: (\partial A, \partial s) \to (\partial B, \partial t),$$

\footnote{We use notation $\lceil 2n/3 \rceil$ for the smallest integer not less than $2n/3$.}
of a nontrivial cable 2-string tangle \((B, t)\) with a trivial 2-string tangle \((A, s)\). Let
\[f_0: (\partial A, \partial s) \to (\partial B, \partial t)\]
yields an obvious ‘trivializing’ sum for \((B, t)\), that is, the sum \((B, t) \cup f_0 (A, s)\) is the trivial knot.\(^5\) (See left side of Fig. 3) Let \(M_0\) denote the double cover of the 3-sphere \(B \cup f_0 A\) branched over the trivial knot \(t \cup f_0 s\), and let \(M_1\) be the double cover of the 3-sphere \(B \cup f A\) branched over the knot \(t \cup f s = K\). Then \(M_0\) is homeomorphic to the 3-sphere, while \(M_0\) and \(M_1\) are related by a Dehn surgery along the solid torus covering \((A, s)\). We observe that the solid torus \(V_A \subset M_0 = S^3\) that covers \((A, s)\) is knotted as a composite knot. Indeed, the definition of cable tangles imply that there is an obvious ambient isotopy of \(B \cup f_0 A\) that moves \(t \cup f_0 s\) and \(A\) to a position in which \(t \cup f s\) is a geometric circle and \(A\) is a closed regular neighborhood of a ‘knotted diameter’ of this circle. See Fig. 3.

![Figure 3](image.png)

**Figure 3.** For the proof of Lemma 4

This clearly implies that \(V_A\) is a regular neighborhood of a composite knot. (This composite knot is a sum of two copies of the 1-string tangle \((B, t_1)\), where \(t_1\) is a component of \(t\).) It is known that a nontrivial Dehn surgery on a composite knot in \(S^3\) yields an irreducible (hence prime) manifold (see \[Gor83, Theorem 7.1\]). It is known that if the double cover of \(S^3\) branched over a knot \(R\) is prime then \(R\) is prime (see \[Wal69\]; see also \[KT80, Corollary 4\] for the inverse implication). Consequently, \(K\) is a prime knot if nontrivial. \(\square\)

**Remarks.**
1. Lemma 4 also follows from results of \[E-M86\] (see also \[E-M88, Theorem 6\]) or equivalently from the fact that only integral Dehn surgeries can yield reducible manifolds \[GL87\]. This way of proof uses the fact that cable knots are prime (see \[Schu53, p. 250, Satz 4\], \[Gra91\] Cor. 2).

2. Lemma 4 is used in the proof of Proposition 4 (which in its turn is used in the proof of assertion (iii) of Proposition 1), where it covers the case of 2-bridge knots. It is known (see \[Web92\]) that the percentage of 2-bridge knots amongst all of the prime knots of \(n\) or fewer crossings approaches 0 as \(n\) approaches infinity. Thus, in the proof of Theorem 1 we can discard 2-bridge knots together with Lemma 4.\(^5\)

5In fact, the results of \[BSS86, BSS88\] imply that there is essentially unique way to create the trivial knot as a sum of a given prime 2-string tangle and a trivial 2-string tangle. In particular, if \(\phi: (\partial A, \partial s) \to (\partial B, \partial t)\) is a homeomorphism such that \((B, t) \cup \phi (A, s)\) is the trivial knot then the map \(f_0^{-1} \circ \phi: (\partial A, \partial s) \to (\partial A, \partial s)\) extends to a map \(F: (A, s) \to (A, s)\) such that \(F(s) = s\).
Claim 7. We have $\text{cr}(K_1) \leq \frac{2}{3} \text{cr}(P) - 1$ for $i \in \{1, 2\}$.

Proof. The diagram $\delta \cap D_P$ of the tangle $(B,t)$ is formed by two curves, $c_1$ and $c_2$ say, corresponding to the components $t_1$ and $t_2$, respectively, of $t$. We denote by $\text{cr}(c_i)$ the number of double points of $c_i$. Since a diagram of $K_1$ can be obtained from $c_1$ by adding a simple arc in $d$, it follows that we have

$$\text{cr}(K_1) \leq \text{cr}(c_1). \tag{3}$$

Observe that by construction we have

$$\text{cr}(P) = \text{cr}(D_P) = \text{cr}(c_1) + \text{cr}(c_2) + \text{card}(c_1 \cap c_2). \tag{4}$$

By the definition of $\rho$ (this definition is given at the beginning of Sec. 3) we have

$$\rho(I,J) = \min\{2 \text{cr}(c_1) + \text{card}(c_1 \cap c_2), 2 \text{cr}(c_2) + \text{card}(c_1 \cap c_2)\}. \tag{5}$$
Since $\rho(I, J) \geq \frac{2\text{cr}(P)}{3}$, it follows from [3], [11], and [5] that
\[ \text{cr}(K_1) \leq \text{cr}(c_1) \leq \frac{2}{3} \text{cr}(P) - \frac{\text{card}(c_1 \cap c_2)}{2}. \]

Since $D_P$ is a minimal diagram of a prime knot and $I \neq J$, it follows that $c_1 \cap c_2 \neq \emptyset$. Assuming that $c_1$ intersects $c_2$ in a unique point $(q, \text{say})$ implies that $q$ is a cutpoint of $D_P$. However, no minimal diagram of a knot has a cutpoint. This implies that $\text{card}(c_1 \cap c_2) \geq 2$ and $\text{cr}(K_1) \leq \frac{2}{3} \text{cr}(P) - 1$, as required. The case of $K_2$ is analogous.

**Claim 8.** The 2-string tangle $(B, t)$ represented by the diagram $\delta \cap D_P$ is either prime or trivial.

**Proof.** Suppose on the contrary that $(B, t)$ is neither prime nor trivial. Then $(B, t)$ is locally knotted, that is, $B$ contains a ball $A$ such that the pair $(A, A \cap t)$ is a nontrivial 1-string tangle. Let $t_i$, where $i \in \{1, 2\}$, be the component of $t$ that meets $A$. We denote by $L$ the knot that is the closure of the 1-string tangle $(A, A \cap t)$. Then $L$ is a factor of $P$. Since $P$ is prime and $L$ is nontrivial, it follows that $L$ and $P$ are equivalent. At the same time, $L$ is a factor of $K_i$ (as defined above, $K_i$ is the closure of the 1-string tangle $(B, t_i)$). Since $L$ and $P$ are equivalent, while $P$ is assumed to be $\frac{4}{3}$-regular, we have $\text{cr}(K_i) \geq \frac{4}{3} \text{cr}(P)$, which contradicts Claim 7. The obtained contradiction proves that $(B, t)$ is either prime or trivial. □

Thus, all requirements from the definition of weak property PT are fulfilled. Consequently, $D_P$ has weak property PT. Proposition 3 is proved. □

**Proposition 4.** If $P$ is a knot with weak property PT, then there exists a prime $\gamma$-knot $P'$ over $P$ with $\text{cr}(P') \leq \text{cr}(P) + 17$.

**Proof.** By definition, $P$ has a minimal crossing diagram $D_P$ with weak property PT. This means that there exists a disk $d \subset S^2$ such that
- the boundary $\partial d$ intersects $D_P$ transversely in four points;
- the intersection $d \cap D_P$ consists of two simple non-intersecting arcs;
- the tangle diagram $\delta \cap D_P$, where $\delta := S^2 \setminus \text{int}(d)$, represents either prime or trivial 2-string tangle $(B, t)$.

Without loss of generality we can identify the pair $(d, d \cap D_P)$ with the tangle diagram in Fig. 1(a). We have the following two cases:

(a) two arrows on the arcs in Fig. 1(a) indicate the same orientation on $P$,
(b) two arrows on the arcs in Fig. 1(a) induce opposite orientations on $P$.

In case (a), let $D_\alpha$ be the diagram obtained from $D_P$ by local move as in Fig. 1 and let $P_\alpha$ be the knot represented by $D_\alpha$. Since the figure-eight knot is hyperbolic, an easy argument shows that $P_\alpha$ is a $\gamma$-knot over $P$. We check that $P_\alpha$ has all of the desired properties. First, the obtained diagram $D_\alpha$ of $P_\alpha$ has $\text{cr}(P_\alpha) \leq \text{cr}(P) + 16$ crossings. This means that $\text{cr}(P_\alpha) \leq \text{cr}(P) + 16$. Next, we prove that $P_\alpha$ is prime. We observe that, by construction, $P_\alpha$ is a sum of the cable tangle of Fig. 1(b) and the tangle $(B, t)$, which is prime or trivial. Each nontrivial cable tangle is prime (see Lemma 3). If $(B, t)$ is prime then $P_\alpha$ is prime by Theorem 2.

If $(B, t)$ is trivial then $P_\alpha$ is prime by Lemma 4 and assertion (i) of Proposition 1. (Lemma 3) implies that $P_\alpha$ is either prime or trivial if $(B, t)$ is trivial; assertion (i)
implies that $P_\alpha$ is a satellite knot and hence nontrivial). Thus, $P_\alpha$ is a prime $\gamma$-knot over $P$ with $\text{cr}(P_\alpha) \leq \text{cr}(P) + 16$, as required.

In case ($\beta$), let $D_\beta$ be the diagram obtained from $D_P$ by local move as in Fig. 4 and let $P_\beta$ be the knot represented by $D_\beta$.

![Figure 4. Type I Reidemeister move plus double figure-eight move](image)

The local move in Fig. 4 is the composition of a type I Reidemeister move and the move shown in Fig. 1. This implies that $P_\beta$ is a $\gamma$-knot over $P$. Obviously, $D_\beta$ has $\text{cr}(P) + 1 + 16$ crossings. This means that $\text{cr}(P_\beta) \leq \text{cr}(P) + 17$. The primeness of $P_\beta$ follows by the same argument as in case ($\alpha$) because $P_\beta$ is a sum of a nontrivial cable tangle and the tangle $(B,t)$. Thus, $P_\beta$ is a prime $\gamma$-knot over $P$ with $\text{cr}(P_\beta) \leq \text{cr}(P) + 17$, as required. $\Box$

Assertion (iii) of Proposition 1 readily follows from Proposition 3.

9. Addendum I: Strong property PT

In addition to weak property PT defined in Sec. 8 we introduce strong property PT.

**Definition. Strong property PT.** Let $D$ be a knot diagram on the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. We say that a tangle $(B,t)$ is **represented by a connected subdiagram of $D$** if there exists a 2-disk $\delta \subset S^2$ such that the intersection $\delta \cap D$ is connected and the pair $(\delta, \delta \cap D)$, with information of under- and overcrossings inherited from $D$, is a diagram of $(B,t)$. We say that $D$ has **strong property PT** if every 2-string tangle represented by a connected subdiagram of $D$ is either prime or trivial. We say that a knot has **strong property PT** if all of its minimal diagrams have strong property PT.

**Proposition 5.**

1. Each minimal diagram of each 1-regular prime knot has strong property PT. In particular, each 1-regular prime knot has strong property PT.

2. Each minimal diagram with strong property PT has weak property PT. In particular, each knot with strong property PT has weak property PT.

**Proof.** 1. Assume to the contrary that a non-prime non-trivial 2-string tangle $(B,t)$ is represented by a connected subdiagram $\delta \cap D_P$ in a minimal diagram $D_P$ of a 1-regular prime knot $P$. This implies in particular that $(B,t)$ is locally knotted, that is, $B$ contains a ball $B'$ such that $(B',B' \cap t)$ is a nontrivial 1-string tangle. Let $K_1$ denote the knot obtained by the closure of $(B',B' \cap t)$. Then $K_1$ is a factor of $P$, which is a prime knot, so that we have $K_1 = P$. (This follows by
the Unique Factorization Theorem by Schubert \cite{Schu49}. On the other hand, the knot $K_1 = P$ is a factor of the knot $K_2$ obtained as the closure of $(B, t_1)$, where $t_1$ is the component of $t$ that meets $B'$. Observe that we have $cr(K_2) \leq cr(P) - 1$ because, since the diagram $\delta \cap D_P$ representing $(B, t)$ is connected, the projection of $t_1$ has at least one crossing with the projection of the second component of $t$.

The inequality $cr(K_2) \leq cr(P) - 1$ implies that $K_2 \neq P$. Therefore, $K_2$ is a composite knot, $P$ is a factor of $K_2$, and $cr(K_2) \leq cr(P) - 1$. This contradicts the assumption that $P$ is a 1-regular knot.

2. Let $D$ be a minimal diagram with strong property PT. If $D$ is a circle with no double points then $D$ has weak property PT (obvious). Assume that $D$ has double points. We take a double point $x$ of $D$ and consider a disk $d \subset S^2$ in a small neighborhood of $x$ such that the intersection $d \cap D$ consists of two non-intersecting arcs (as on the left side of Fig. 1). Since $D$ is a minimal diagram, $x$ is not a cutpoint of $D$. This easily implies that the intersection $\delta \cap D$, where $\delta := S^2 \setminus \text{int}(d)$, is connected. Since $D$ has strong property PT, it follows that the 2-string tangle represented by the connected subdiagram $\delta \cap D$ is either prime or trivial. This means that $D$ has weak property PT. □

Propositions 5 and 8 give the following dependence for properties of prime knots.

\[
\begin{align*}
1\text{-regularity} & \implies \quad \frac{2}{3}\text{-regularity} \\
\updownarrow & \\
\text{strong property PT} & \implies \quad \text{weak property PT}
\end{align*}
\]

This implications can be treated in terms of conjectures. We consider the following conjectures.

**Conjecture 7.** Each prime knot has strong property PT.

**Conjecture 8.** Each prime knot has weak property PT.

**Conjecture 9.** There exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, the percentage of knots with weak property PT amongst all of the hyperbolic knots of $n$ or fewer crossings is at least $\varepsilon$.

We have the following implications.

Conj. 2

\[\implies\]

Conj. 4

\[\implies\]

Conj. 5

\[\implies\]

Conj. 6

\[\implies\]

Conj. 3

\[\implies\]

Conj. 7

\[\implies\]

Conj. 8

\[\implies\]

Conj. 9

The implication Conj. 4 \implies Conj. 7 follows from assertion 1 of Proposition 5. The implication Conj. 7 \implies Conj. 8 follows from assertion 2 of Proposition 5. The implications Conj. 5 \implies Conj. 8 and Conj. 6 \implies Conj. 9 follow from Proposition 3. The implication Conj. 8 \implies Conj. 9 is obvious.

Theorem 1 can be strengthened in the following way.

**Theorem 3.** Conjecture 7 contradicts (each of) Conjectures 8, 9.
Proof. Since each of Conjectures 2–8 implies Conjecture 9 (see the system of implications before Theorem 3), it suffices to show that Conjecture 9 contradicts Conjecture 1. In order to prove this, we repeat verbatim the reduction of Theorem 1 to Proposition 1 up to replacing $\frac{2}{3}$-regularity with weak property PT and assertion (iii) of Proposition 1 with Proposition 4.

10. Addendum II: Non-$\frac{1}{4}$-regular knots

The main theorem of the present paper states that Conjecture 1 concerning predominance of hyperbolic knots contradicts the conjecture on additivity of the crossing number (of knots under connected sum) as well as several weaker conjectures. In this section, we show that Conjecture 1 also contradicts an assumption that the conjecture on additivity has many strong counterexamples.

We say that a knot $P$ is non-$\lambda$-regular, $\lambda \in \mathbb{R}$, if there exists a knot $K$ such that $P$ is a factor of $K$ while $\text{cr}(K) < \lambda \cdot \text{cr}(P)$. In this section, we prove the following theorem.

**Theorem 4.** If there exist $\varepsilon_0 > 0$ and $N_0 > 0$ such that, for all $n > N_0$, the number of non-$\frac{1}{4}$-regular knots of $n$ or fewer crossings is at least $\varepsilon_0 H_n$, where $H_n$ is the number of hyperbolic knots of $n$ or fewer crossings, then Conjecture 1 does not hold.

Proof. Suppose that the assumption of the theorem holds true, denote by $\mathcal{M}_{\frac{1}{4}}$ the set of all non-$\frac{1}{4}$-regular knots, and let $f$ be a map with domain $\mathcal{M}_{\frac{1}{4}}$ sending $K \in \mathcal{M}_{\frac{1}{4}}$ to a composite knot $f(K)$ with factor $K$ such that

$$\text{cr}(f(K)) < \frac{1}{4} \text{cr}(K).$$

Then the result of Lackenby [La09] stating that for any knots $K_1, \ldots, K_n$ in the 3-sphere we have

$$\frac{\text{cr}(K_1) + \cdots + \text{cr}(K_n)}{152} \leq \text{cr}(K_1 \# \cdots \# K_n)$$

implies that for each knot $L$ in the codomain $f(\mathcal{M}_{\frac{1}{4}})$ we have

$$\text{card}(f^{-1}(L)) < 152/4 = 38.$$

Indeed, let $K$ be a knot with $f(K) = L$ having the smallest crossing number among the elements of $f^{-1}(L)$. Then (7) implies that

$$\frac{\text{card}(f^{-1}(L)) \cdot \text{cr}(K)}{152} \leq \text{cr}(L).$$

Obviously, (6) and (9) imply (8).

Since all of the knots in $f(\mathcal{M}_{\frac{1}{4}})$ are composite, it follows by (9) and (8) that for all $n \in \mathbb{N}$ we have

$$\text{card}\{K \in \mathcal{M}_{\frac{1}{4}} : \text{cr}(K) \leq 4n\} \leq 38 \cdot n \leq C_n,$$

where $C_n$ is the number of composite knots of $n$ or fewer crossings. At the other hand, by the assumption of the theorem, for all $m > N_0$ we have

$$H_m \varepsilon_0 \leq \text{card}\{K \in \mathcal{M}_{\frac{1}{4}} : \text{cr}(K) \leq m\}.$$
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Then (10) and (11) imply that for all $n > N_0/4$ we have

$\varepsilon_0 \frac{38}{4} H_{4n} < C_n.\quad (12)$

Now, we observe that each knot $K$ in the 3-sphere obviously has a two-strand cable knot $J_K$ with $cr(J_K) \leq 4\, cr(K) + 1$. Since a cable knot over a nontrivial knot is a prime satellite knot (see [Schu53, p. 250, Satz 4], [Gra91, Cor. 2]), while cable knots over distinct knots are distinct (Lemma 5 below), it follows by (12) that for all $n > N_0/4$ we have

$\varepsilon_0 \frac{38}{4} H_{4n} < C_n < S_{4n+1}$,

where $S_m$ denotes the number of all prime satellite knots of $m$ or fewer crossings. Consequently, since the sequences $(H_i)_{i \in \mathbb{N}}$ and $(S_i)_{i \in \mathbb{N}}$ are monotonically increasing, for all $m > N_0$ we have

$\varepsilon_0 \frac{38}{4} H_m < S_{m+4}$.

As is shown in Section 3 (see deduction of Theorem 1 from Proposition 1), conditions of this kind contradict Conjecture 1. □

**Lemma 5.** Cable knots over distinct knots are distinct.

**Proof.** By Corollary 2 of [FW78], the group of a cable knot $J(p, q; K)$ determines the numbers $|p|$ and $|q|$ and the topological type of $K$’s complement. By the Gordon–Luecke theorem [GL89], the knot complement determines the knot. □

11. **Addendum III: Weak property PT and unknotting numbers**

This section deals with a relation between weak property PT and the unknotting number of knots. The unknotting number of a knot $K$ is denoted by $u(K)$.

**Definitions.** Let us say that a knot $P$ is weakly U-regular if we have $u(P) \leq u(K)$ whenever $P$ is a factor of a knot $K$. We say that a knot $P$ is strictly U-regular if we have $u(P) < u(K)$ whenever $P$ is a factor of a knot $K \neq P$. We say that a knot $P$ has weak BJ-property if by altering one of the crossings in a minimal diagram of $P$ we obtain a knot $J \neq P$ with $u(J) \leq u(P)$. We say that a knot $P$ has strict BJ-property if by altering one of the crossings in a minimal diagram of $P$ we obtain a knot $J$ with $u(J) < u(P)$.

**Remarks.** 1. The conjecture that all knots are strictly U-regular is weaker than the old conjecture on additivity of the unknotting number of knots under connected sum (see, e.g., [Ad94b, p. 61], [Kir97, Problem 1.69]). At the moment, no counterexample seems to be known to the latter conjecture. Thus, no examples of non-U-regular knots are known up to now. The theorem of Scharlemann [Sch85] saying that unknotting number one knots are prime (together with the Unique Factorization Theorem by Schubert [Sch97]) implies that all knots with unknotting number one are strictly U-regular, while all knots with unknotting number two are weakly U-regular.

2. The so-called Bernhard–Jablan conjecture (see [Be94], [Ja98], and [JS07]) is equivalent to the conjecture that all knots have strict BJ-property. Kohm’s conjecture [Koh91, Conjecture 12] (which can be viewed as a particular case of the Bernhard–Jablan conjecture) is equivalent to the conjecture that all knots with
unknotting number one have strict BJ-property. The set of knots with strict BJ-property contains the set of knots satisfying the Bernhard–Jablan conjecture. At the moment, no counterexample seems to be known to the Bernhard–Jablan conjecture. Available results concerning unknotting numbers show that many small knots and some specific classes of knots satisfy the Bernhard–Jablan conjecture, hence have strict BJ-property. For example, results of [KrM93] and [Mur91] imply that all torus knots have strict BJ-property. Results of McCoy [McC13] imply that alternating knots with unknotting number one have strict BJ-property.

Proposition 6. 1. Each weakly U-regular prime knot with strict BJ-property has weak property PT.

2. Each strictly U-regular prime knot with weak BJ-property has weak property PT.

Proof. If \( P \) is a weakly [resp., strictly] U-regular prime knot with strict [resp., weak] BJ-property, then there exists a minimal diagram \( D_P \) of \( P \) (on the 2-sphere \( S^2 = \mathbb{R}^2 \cup \{\infty\} \)) with a crossing \( X_1 \) such that the change of the crossing yields a diagram of a knot \( J \) with \( u(J) = u(P) - 1 \) [resp., a knot \( J \neq P \) with \( u(J) \leq u(P) \)]. Let \( d \) be a disk in \( S^2 \) containing \( x_1 \) such that the intersection \( d \cap D_P \) is homeomorphic to \( \times \) while \( \partial d \) intersects \( D_P \) transversally in four points. Let \( \delta \) denote the disk \( S^2 \setminus \text{int}(d) \), and let \( (B, t) \) be the 2-string tangle represented by the diagram \( \delta \cap D_P \).

We show that \( (B, t) \) has no local knots. Suppose on the contrary that \( (B, t) \) is locally knotted, that is, \( B \) contains a ball \( A \) such that the pair \( (A, A \cap t) \) is a nontrivial 1-string tangle. We denote by \( L \) the knot that is the closure of the 1-string tangle \( (A, A \cap t) \). Then \( L \) is a factor of \( P \). Since \( P \) is prime and \( L \) is nontrivial, it follows that \( L \) and \( P \) are equivalent. At the same time, \( L(= P) \) is a factor of \( J \). Then we have \( u(P) \leq u(J) \) because \( P \) is weakly U-regular [resp., \( u(P) < u(J) \) because \( P \) is strictly U-regular while \( J \neq P \)]. However, \( u(J) = u(P) - 1 \) [resp., \( u(J) \leq u(P) \)]. The obtained contradiction proves that \( (B, t) \) has no local knots.

Now, we take a subdisk \( d' \) in \( d \) such that the intersection \( d' \cap D_P \) consists of two subarcs on two distinct legs of \( \times = d \cap D_P \) (while \( \partial d' \) intersects \( D_P \) transversely in four points):

\[ \includegraphics[width=0.2\textwidth]{diagram} \]

Let \( \delta' \) denote the disk \( S^2 \setminus \text{int}(d') \). Obviously, the diagram \( \delta' \cap D_P \) represents the same 2-string tangle \( (B, t) \), which has no local knots. Thus, the requirements from the definition of weak property PT are fulfilled. Consequently, \( P \) has weak property PT.

Corollary 2. If there exist \( \varepsilon > 0 \) and \( N > 0 \) such that, for all \( n > N \), the percentage of weakly U-regular knots with strict BJ-property amongst all of the hyperbolic knots of \( n \) or fewer crossings is at least \( \varepsilon \), then Conjecture [1] does not hold.
Proof. By Proposition \ref{prop:1}, the assumption of the corollary implies Conjecture \ref{conj:2} (which concerns the set of knots having weak property PT). By Theorem \ref{thm:3} Conjecture \ref{conj:2} contradicts Conjecture \ref{conj:1}.

\begin{corollary}
If there exist \(\varepsilon > 0\) and \(N > 0\) such that, for all \(n > N\), the percentage of strictly \(U\)-regular knots with \(\text{weak BJ}\)-property amongst all of the hyperbolic knots of \(n\) or fewer crossings is at least \(\varepsilon\), then Conjecture \ref{conj:1} does not hold.
\end{corollary}

\textbf{Proof.} See the proof of Corollary \ref{cor:2}.

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