Breakdown of correspondence in chaotic systems: Ehrenfest versus localization times

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Breakdown of quantum-classical correspondence is studied on an experimentally realizable example of one-dimensional periodically driven system. Two relevant time scales are identified in this system: the short Ehrenfest time \( t_h \sim \ln(\hbar^{-1}) \) and the typically much longer localization time scale \( T_L \sim h^{-2} \). It is shown that surprisingly weak modification of the Hamiltonian may eliminate the more dramatic symptoms of localization without effecting the more subtle but ubiquitous and rapid loss of correspondence at \( t_h \).

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Quantum realizations of classically chaotic systems behave differently than their fully classical counterparts. The breakdown of quantum-classical correspondence occurs on two distinct time scales: the short Ehrenfest time

\[
t_h \cong \frac{1}{\chi} \ln \left( \frac{\Delta p}{\hbar} \right) \cong \frac{1}{\chi} \ln \left( \frac{A}{\hbar} \right) \tag{1}
\]

gives the time scale at which the quantum minimal wavepacket spreads sufficiently over a macroscopic part of the phase-space to feel nonlinearities in the potential \( A \). In (1) \( \Delta p \) denotes the initial uncertainty in momentum, \( \chi \cong \sqrt{\partial^2 V/\partial x^2} \) is a typical scale on which nonlinearities in the potential \( V \) are significant. Ehrenfest time \( t_h \) marks the moment when quantum corrections become important in the time evolution of the Wigner function \( W(x,p) \), governed by the Wigner transform of the von Neumann bracket, known as the Moyal bracket \( \{ \} \):

\[
\frac{d}{dt} W = \{ H, W \}_{MB} \tag{2}
\]

\[
= \{ H, W \} + \sum_{n=1}^\infty \frac{(-1)^n \hbar^{2n}}{2^n (2n + 1)!} \frac{\partial^{2n+1} V}{\partial x^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}.
\]

The Moyal bracket appearing above is defined as

\[
\{ H, W \}_{MB} = -\frac{2}{\hbar} H \sin \left( \frac{\hbar}{2} \left( \frac{\partial}{\partial p} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial p} \right) \right) W.
\]

The arrows indicate the direction of differentiation.

In a chaotic system initially localized distribution will be stretched in the unstable directions. In a Hamiltonian system, this stretching must be matched by shrinking in the stable directions as mandated by the Liouville theorem. Consequently, the typical width of the effective support of classical probability distribution shrinks as \( \exp(-\gamma t) \). That leads to an exponential increase of higher order derivatives of \( W \) in (3) so after \( t_h \) quantum corrections become comparable to terms given by the Poisson bracket \( \{ \} \).

There is evidence that at \( t_h \) also the averages over the phase space show symptoms of correspondence loss \( \{ \} \) although not everyone agrees \( \{ \} \). The first goal of our paper is then to investigate the dependence of the onset of the loss of correspondence in the averages on \( \hbar \), and to show that it is consistent with (1).

In addition to the rapid correspondence loss on the logarithmic time scale \( t_h \) there is plentiful evidence for the dynamical localization in a class of quantum chaotic systems \( \{ \} \). It sets in on a much longer localization time scale:

\[
T_L \cong \frac{d}{\hbar^2} \tag{3}
\]

where \( d \) is diffusion constant characterizing initial growth of variance in momentum of a quantum system. Tutorial example of localization is described in \( \{ \} \). It is of obvious interest to consider systems in which both mechanisms of correspondence loss are present, and to investigate their behavior with varying parameters. The system we investigate is a straightforward modification of experimentally studied system based on an optical lattice. Therefore, it should be simple to verify our predictions in the laboratory.

We study quantum-classical correspondence in a system described by the Hamiltonian

\[
H(x, p) = \frac{p^2}{2m} - \kappa \cos(x - l \sin t) + ax^2. \tag{4}
\]

For \( a = 0 \) this Hamiltonian reduces to a model recently investigated both theoretically \( \{ \} \) and experimentally \( \{ \} \) corresponding to motion of cold atoms in a quasi-resonant standing wave with periodically moving nodal pattern. Classical behavior of the system is of the mixed type with large regions of phase space dominated by a chaotic motion (e.g. Lyapunov exponent of
the chaotic component is $\lambda = 0.2$ for chosen parameter values $m = 1$, $\kappa = 0.36$, $i = 3$, $a = 0$, [3].

To investigate quantum-classical correspondence one can start by looking at the Wigner function. Structures which appear in $W$ are known to be different from (but not unrelated to) the classical probability distribution in the phase space. One can especially note existence of the interference fringes that saturate on small scales associated with action $s \sim (2\pi)^2/S$, where $S$ is the action of the system – e.g., the phase space volume over which $W$ can spread [10].

One might nevertheless suppose that the quantum structures appearing on such small in the Wigner function will have little effect on the averages. In particular, averages typically concern large scales, and, therefore, should not be directly affected by small-scale interference. This is, however, only partially true.

![Figure 1](image1.png)

**FIG. 1.** Difference between classical, $\Delta^2_{cl}$, and quantum, $\Delta^2_{qm}$, variances for three values of $\hbar$ and for the same initial condition i.e. coherent state centered at (0,0) in the phase space. In the insert, $t_h$ dependence on log($1/\hbar$) averaged over few initial conditions is shown. Error bars indicate standard deviation of the results for given value of $\hbar$. Criterion used to estimate Ehrenfest time scale compares magnitude of Poisson bracket and a quantum correction (the rest of the Moyal bracket in [4]).

Figure 1 shows the difference between classical, $\Delta^2_{cl}$, and quantum, $\Delta^2_{qm}$, averages of a typical quantity ($\Delta^2 = \langle p^2 \rangle - \langle p \rangle^2$) for $\hbar = 0.16$, 0.064 and 0.0256 obtained in the time evolution resulting from Hamiltonian (4) with $a = 0$. Initial classical distribution is a two-dimensional Gaussian in phase space and corresponds to a Gaussian wavepacket used for quantum evolution. The loss of correspondence is apparent: early on, a noticeable difference between the two averages develops. The smaller the value of $\hbar$, the later it appears. The insert shows that the discrepancies occur after a time which seems indeed logarithmic in $\hbar$. They persist without a significant change in their magnitude until localization (which occurs at a $T_L \sim \hbar^{-2}$, a time scale much more sensitive to changes of $\hbar$ than the logarithmic $t_h$) begins to dominate. Vertical error bars in the insert arise from the fact that the local Lyapunov exponents (i.e. evaluated over a time scale of order $t_h$, (3),) differ for various initial conditions at given $\hbar$. This in turn affects value of $t_h$ (see Fig. 2).

![Figure 2](image2.png)

**FIG. 2.** Difference between classical and quantum variances for $\hbar = 0.0256$ and four initial conditions [coherent states centered at points (0,0), $(\pi/2, 0)$, $(-\pi/2, 0)$ and $(\pi, 0)$ in the phase space] is plotted using black, red, green and blue lines respectively. The insert shows the discrepancy between classical and quantum averages as defined by (5) as a function of $\hbar$. The dependence seems to be logarithmic.

It is intriguing to enquire about the magnitude of the discrepancy magnitude of the typical discrepancy between the quantum and classical averages in the time interval between $t_h$ and $T_L$. To address this question we have evaluated

$$\delta(\hbar) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |\Delta^2_{cl} - \Delta^2_{qm}| dt,$$

for $t_h < t_1 < t_2 < T_L$ as a function of $\hbar$. The results (averaged over several initial conditions) show that $\delta(\hbar)$ decreases with decreasing $\hbar$ significantly slower than linearly (compare the insert in Fig 3). While our data do not allow for a definite conclusion, they point out towards a logarithmic $\hbar$ dependence of $\delta(\hbar)$. The results obtained for $\delta(\hbar)$ depend slightly on the choice of $t_1$ and $t_2$. That, together with the size of the error bars, does not allow us to be more definitive. The logarithmic dependence is not in disagreement with results of [17] (who claim that largest deviations are $O(\hbar)$) obtained for a model of two interacting spins. Our data indicate a similar behavior for a realistic, experimentally accessible system. Still, the latest result for the interacting spin system [18] gives a square root rather than the logarithmic behavior.

To the best of our knowledge, $t_h$ has not been yet detected experimentally. On the other hand, it is known
that random external noise \([19,20]\) as well as decoherence \([3,6,21,22]\) suppress evidence of the breakdown at \(t_h\) or at \(T_L\), restoring quantum-classical correspondence.

This happens when the coherence length \([22,23]\)

\[
l_c = \hbar \sqrt{\lambda/2D}
\]

maintained by the chaotic system in the presence of decoherence (which can be parametrized under certain conditions \([22,23]\) by the addition of the diffusive term \(\sim D\partial^2_W\) to the evolution equation, \((6)\)) becomes small compared to the scale of nonlinearities \(\chi\)

\[
l_c \ll \chi
\]

The price for the restoration of quantum-classical correspondence by decoherence is irreversibility \([3,22]\).

Let us now consider the localization. At the onset of localization (clearly visible in Fig. 3 for \(a = 0\)) the difference between the classical and quantum variances begins to rapidly increase, in a manner approximately consistent with \(\Delta^2_{cl} - \Delta^2_{qm} \approx d(t - T_L)\). This is to be expected. In systems effected by localization wave functions exhibit characteristic exponential form, as exemplified in Fig. 4(a).

Localization can be suppressed by noise and decoherence, as was recently shown by two groups using similar experimental setups \([23–25]\). Figure 3 shows that a much simpler mechanism for suppression of localization exists: When the driven periodic potential is supplemented by even a weak harmonic potential, \(\text{controlled by the parameter } a\) in \((4)\) localization disappears while chaotic character of motion persists, as illustrated by Poincaré surfaces of section in Fig. 5.

Our explanation of this effect is simple and general. Numerical studies show that the suppression of localization occurs when quarter of the period of the harmonic oscillator is approximately the localization time scale \(T_L\). Localization can be maintained only when the "swapping" between \(x\) and \(p\) generated by the harmonic evolution happens on a time scale longer than \(T_L\). (After all, localization in \(p\) is incompatible with localization in \(x\), since the wave function cannot have the exponential form in both complementary observables.) In fact, we...

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Difference between classical and quantum variance for four values of parameter \(a\), as indicated in the figure, and for \(\hbar = 0.16\). Addition of harmonic potential destroys dynamical localization present for \(a = 0\). All initial conditions are the same.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Time average probability density in momentum representation: (a) exponentially localized for \(a = 0\), (b) Gaussian for \(a = 0.01\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{Stroboscopic Poincaré surface of sections for the system with parameter \(a = 0\) (left), and for \(a = 0.01\) (right).}
\end{figure}
have seen disappearance of localization when the period of the harmonic oscillator is about $16T_L$. So, the "swapping" does not need to be very frequent. This simple explanation of suppression of localization is model independent and may be the key result of our paper.

Experimental verification of our two key results concerning (i) $t_h$ and (ii) $T_L$ should be possible. It will be more difficult to check for the intriguing prediction of the correspondence breakdown on short time scale $t_h$. One would ideally want to have otherwise identical two systems, one of them quantum, the other classical. In our quantum Universe this is unfortunately impossible. One remedy would be then to compare real experimental results with the classical computer experiment for the same values of all relevant parameters.

There is, however, something disingenuous about comparing a classical computer simulation with a quantum experiment in an attempt to find a breakdown of quantum classical correspondence. We shall therefore suggest an alternative strategy: Effective value of $\hbar$ can be adjusted in the experiments \textsuperscript{[12]} \textsuperscript{[23]}. Therefore, it may be possible to look for differences between some average quantities, say $\Delta_2$, for two otherwise identical systems which are quantum to a different extent that is adjusted by selecting different values of $\hbar$. It is obvious from Fig.\textsuperscript{[1]} that significant differences between two systems endowed with different effective value $\hbar$ would appear, and it should be possible to check for the validity of \textsuperscript{[1]}.

Destruction of localization by introduction of the harmonic trap into experiments which test quantum chaos in optical lattices should be much easier to verify. It may be desirable to superpose optical lattice of the sort used to investigate quantum chaos \textsuperscript{[11],23} in a magneto-optical trap anyway (e.g. to confine the system such as BEC, where investigation of chaos may be of interest in its own right \textsuperscript{[26]}). It should be then relatively easy to see if our prediction of the loss of localization is indeed borne out by the experiments.

Both of our key conclusions, while investigated numerically in a specific system selected for its relevance to experiment should have a broad validity. In particular, destruction of localization due to rotation of phase space as a whole by the background (e.g. harmonic) potential on a time scale comparable to the localization time $T_L$ is justified by a simple physical argument. This effect is dramatic and should be easily accessible to experiments. The earlier and more subtle phenomena that occur at $t_h$ are robust to the addition of an overall potential, providing that the chaotic character of the system is preserved. Our suggestion for investigation of the phenomena relies on the ability to adjust the relevant effective $\hbar$ in experiments. Moreover, we have found indication that such quantum-classical discrepancies in the averages diminish more slowly than $O(\hbar)$, which -- if confirmed in experiments we suggest -- would be of major importance to our understanding of the classical limit of quantum theory.

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