A Quadruple Integral Involving Chebyshev Polynomials $T_n(x)$: Derivation and Evaluation

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Abstract: The aim of the current document is to evaluate a quadruple integral involving the Chebyshev polynomial of the first kind $T_n(x)$ and derive in terms of the Hurwitz-Lerch zeta function. Special cases are evaluated in terms of fundamental constants. The zero distribution of almost all Hurwitz-Lerch zeta functions is asymmetrical. All the results in this work are new.

Keywords: Chebyshev polynomial; quadruple integral; Hurwitz-Lerch zeta function; Apéry’s constant

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1. Significance Statement

Named for the Russian mathematician Pafnuty Lvovich Chebyshev (1821–1894), this type of polynomial function is detailed in [1]. Chebyshev polynomials appear in almost every branch of numerical analysis, and they are especially important in recent breakthroughs in orthogonal polynomials, polynomial approximation, numerical integration, and spectral approaches [2]. Definite integrals involving Chebyshev polynomials are investigated in the works of Wang and Hu [3], where the authors derived an identity containing the integral Chebyshev Polynomials of the First-Kind. The Mellin transform of Chebyshev Polynomials is tabled in the book of Oberhettinger [4]. The definite integral of special functions is investigated in the work by McClure and Wong [5] where the authors derived an expansion for a multidimensional integral containing the Bessel function, which is used in the study of crystallography. Watson [6], investigated a quadruple integral where he expressed it in terms of the ratio of Gamma functions.

Based on current literature, there exists work conducted on definite integrals of the Chebyshev Polynomial, and also present in current literature is work conducted on quadruple integrals involving special functions though sparse. In this work we propose a systematic approach to deriving a quadruple integral involving a special function and derived it in terms of the Hurwitz-Lerch zeta function. This approach is based on the contour integral method in [7]. The goal is to expand upon work on multiple integrals involving special functions with the aim of assisting researchers where this work is useful. In this present work, we investigate the quadruple integral of the Chebyshev polynomial $T_n(x)$ and derive an integral transform which is invariant under the index $n$ with respect to the Hurwitz-Lerch zeta function.

2. Introduction

The Chebyshev polynomial $T_n(x)$ see Section (22:3) in [1] has an algebraic definition given by:

$$T_n(x) = \frac{1}{2} \left[ (x + i \sqrt{1 - x^2})^n + (x - i \sqrt{1 - x^2})^n \right],$$
the polynomial has a Gauss hypergeometric function representation given by
\[ T_n(1 - 2x) = \sum_{j=0}^{n} \frac{(-n)_j (n)_j}{(1/2)_j (1)_j} x^j, \]
the polynomial can be expressed by the Rodrigue’s formula
\[ T_n(1 - 2x) = (-1)^n \sqrt{1 - x^2} \frac{d^n}{dx^n} (1 - x^2)^{(2n-1)/2} \]
and has a contour integral representation given in [8] by
\[ T_n(1 - 2x) = \frac{1}{4\pi i} \oint \frac{(1 - t^2)^{n-1}}{(1 + 2tx + t^2)} dt \]
where the contour encircles the origin and no other singular points.

The quadruple definite integral derived in this work is
\[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 x^{m-1} T_n(x) \log^{-m-1} \left( \frac{1}{t} \right) \sqrt{1 - x^2} \log^{(m+n-1)} \left( \frac{1}{t} \right) \log^{(m+n+1)} \left( \frac{1}{t} \right) \log^{(k)} \left( \frac{ax}{\log(z)} \right) \sqrt{\log(\frac{1}{t})} \right) dx dy dz dt, \]
where the parameters \( k, a, n, m \) are general complex numbers and \( \Re(n) < \Re(m) \). This definite integral will be used to derive special cases in terms of special functions and fundamental constants.

3. Definite Integral of the Contour Integral
3.1. The Generalized Hankel Contour

An alternate derivation of Equation (2) in [9] when \( Re(m + w) \leq 0 \) can be achieved by recalling a variant of Hankel’s formula involving the Gamma function:
\[ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{t} \frac{dt}{t^2} \quad (|\arg t| < \pi, z \in \mathbb{C}), \]
where the contour starts from \( -\infty \) along the lower side of the negative real axis, encircles the origin once in the positive (counter-clockwise) direction, and then goes back to \( -\infty \) along the upper side of the negative real axis. Replacing \( t \) by \( tu \) in (2) gives
\[ \frac{u^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i} \int_{C_u} e^{tu} \frac{dt}{t^2} \quad (|\arg t| < \pi, z \in \mathbb{C}) \]
Here, \( u \in \mathbb{C} \) with suitable restriction on \( C_u \) is another contour of the Hankel type depending on the \( \arg u = \theta \) of the polar form \( u = |u| e^{i\theta} \). For instance,
(i) If \( \theta = 0 \), then
\[ C_u = \int_{-\infty}^{(0+)} ; \]
(ii) If \( \theta = \frac{\pi}{2} \), then
\[ C_u = \int_{-\infty}^{(0+)} ; \]
(iii) If \( \theta = \pi \), then
\[ C_u = \int_{+\infty}^{(0+)} ; \]
(iv) If \( \theta = \frac{3\pi}{2} \), then
\[
C_{\theta} = \int_{\rho_{\theta}}^{(0+)} \; ;
\]

3.2. The Present Case of the Contour Integral

The derivations follow the method used in [7,9]. In the present case the cut approaches the origin from the interior of the first quadrant and the cut lies on opposite sides of the cut going round the origin with zero radius. The variable of integration in the contour integral is \( \alpha = w + m \). Using a generalization of Cauchy’s integral formula we form the quadruple integral by replacing \( y \) by \( \log \left( \frac{\alpha \sqrt{\log(\frac{\alpha}{y})}}{\log(\frac{\alpha}{y})} \right) \) and multiplying by \( x^{m-1} T_{n}(x) \log^{-m} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m-n-1)} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m+n-1)} \left( \frac{1}{2} \right) \) then taking the definite integral with respect to \( x \in [0,1], y \in [0,1], z \in [0,1] \) and \( t \in [0,1] \) to obtain

\[
\frac{1}{12\pi i} \int_{C} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x^{m-1} T_{n}(x) \log^{-m} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m-n-1)} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m+n-1)} \left( \frac{1}{2} \right)}{\sqrt{1-x^2}} dxdydzdt
\]

\[
= \frac{1}{8m} \int_{C} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{a^{m+n-1} T_{n}(x) x^{m+n-1} \log^{-m-n-1} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m-n-1)} \left( \frac{1}{2} \right) \log^{\frac{1}{2}(m+n-1)} \left( \frac{1}{2} \right)}{\sqrt{1-x^2}} dwdxdydzdt
\]

\[
= \frac{1}{8m} \int_{C} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{\log^{\frac{1}{2}(m+n+1)} \left( \frac{1}{2} \right)}{\sqrt{1-x^2}} dwdxdydzdw = \frac{1}{8m} \int_{C} \frac{\pi^2 a^{m+n-1} 2^{-m-n} \csc(\pi (m+w))}{\sqrt{1-x^2}} dwdxdydzdw
\]

from Equation (1.9.95) in [4] and Equation (4.215.1) in [10] where \( \Re(m+w) > 0, \Re(n) < \Re(m) \) and using the reflection Formula (8.334.3) in [10] for the Gamma function. We are able to switch the order of integration over \( x, y, z \) and \( t \) using Fubini’s theorem for multiple integrals see (9.112) in [11] since the integrand is of bounded measure over the space \( \mathbb{C} \times [0,1] \times [0,1] \times [0,1] \times [0,1] \).

4. The Hurwitz-Lerch Zeta Function and Infinite Sum of the Contour Integral

In this section, we use Equation (2) in [9] to derive the contour integral representations for the Hurwitz-Lerch zeta function. The significance of this section is to derive a special function equivalent to the definite integral of the contour integral derived in Section 3 in terms of the same contour integral.

4.1. The Hurwitz-Lerch Zeta Function

The Hurwitz-Lerch zeta function in [12,13] has a series representation given by

\[
\Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^{n}}{(n+v)^{s}}
\]

where \( v \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; s \in \mathbb{C} \) when \( |z| < 1; \Re(s) > 1 \) when \( |z| = 1 \). Here \( \mathbb{Z}_{\leq 0} \) denotes the set of non-positive integers and is continued analytically by its integral representation given by

\[
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}e^{-(v-1)t}}{e^{t} - z} dt
\]
where \( \Re(v) > 0 \), and either \( |z| \leq 1, z \neq 1, \Re(s) > 0 \), or \( z = 1, \Re(s) > 1 \).

4.2. Infinite Sum of the Contour Integral

Using Equation (2) in [9] and replacing \( y \) by \( \log(a) + i\pi(2y + 1) - \log(2) \) then multiplying both sides by \(-i\pi^22^{1-m}\epsilon/m(2y+1)\) taking the infinite sum over \( y \in [0, \infty) \) and simplifying in terms of the Hurwitz-Lerch zeta function we obtain

\[
\frac{1}{\Gamma(k+1)}\int_{1}^{s} k^{p} + 2 - \sum_{n=0}^{\infty} \int_{C} i\pi^2a^w w^{-k} - 2 - m - w + 1 \epsilon/m(2y+1)(m+w) dw
\]

\[
= -i \frac{1}{\pi^2} \int_{C} \pi^2a^w w^{-k} - 2 - m - w + 1 \epsilon/m(2y+1)(m+w) dw
\]

\[
= \frac{1}{\pi^2} \int_{C} \pi^2a^w w^{-k} - 2 - m - w \csc(\pi(m+w)) \, dw
\]

from Equation (1.323.3) in [10] where \( \Im(\pi(m+w)) > 0 \) in order for the sum to converge.

5. Definite Integrals and Invariant Index Forms

In this section we evaluate Equation (8) such that the index of the Chebyshev polynomials \( T_n(x) \) is independent of the right-hand side. These types of integrals could involve orthogonal constants and other special functions. We also derive special cases of Equation (8) in terms of fundamental constants and other special functions.

**Theorem 1.** For all \( k, a, n, m \in \mathbb{C}, \Re(n) < \Re(m) < 1, \)

\[
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x^{m-1}T_n(x) \log^{-m}(\frac{1}{2}) \log^{\frac{1}{2}(m-n)}(\frac{1}{2}) \log^{\frac{1}{2}(m+n)}(\frac{1}{2})}{\sqrt{1-x^2}} \log^k \left( \frac{ax \sqrt{\log(\frac{1}{2}) \log(\frac{1}{2})}}{\log(\frac{1}{2})} \right) \, dx \, dy \, dz \, dt
\]

\[
= \frac{k^{-1}}{\pi^2} e^{\pi/m} 2^{k} \epsilon/m(2k-1) \Phi \left( e^{\pi/m} \left( -k, -i \log(a) + i \log(2) + \pi \right) \right)
\]

**Proof.** The right-hand sides of relations (4) and (7) are identical, hence their left-hand sides are also identical. The required result is obtained by simplifying with the Gamma function. Note the invariance of the index \( n \) of the Hurwitz-Lerch zeta function.

**Example 1.** The degenerate case.

\[
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x^{m-1}T_n(x) \log^{-m}(\frac{1}{2}) \log^{\frac{1}{2}(m-n)}(\frac{1}{2}) \log^{\frac{1}{2}(m+n)}(\frac{1}{2})}{\sqrt{1-x^2}} \log^k \left( \frac{ax \sqrt{\log(\frac{1}{2}) \log(\frac{1}{2})}}{\log(\frac{1}{2})} \right) \, dx \, dy \, dz \, dt
\]

\[
= \pi^2 2^{m-n} \csc(\pi/m)
\]

**Proof.** Use Equation (8) and set \( k = 0 \) and simplify using entry (2) in Table below (64:12.7) in [1].

**Example 2.** The zeta function of Riemann \( \zeta(s) \),

\[
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{T_n(x) \log^{-\frac{1}{2}+\frac{1}{2}}(\frac{1}{2}) \log^{\frac{1}{2}+\frac{1}{2}}(\frac{1}{2})}{\sqrt{1-x^2} \log(\frac{1}{2})} \log^k \left( \frac{-2x \sqrt{\log(\frac{1}{2}) \log(\frac{1}{2})}}{\log(\frac{1}{2})} \right) \, dx \, dy \, dz \, dt
\]

\[
= -\pi^2 2^{k+\frac{1}{2}} \left( 2^{k+1} - 1 \right) \pi^{k+2} \zeta(-k)
\]
**Proof.** Use Equation (8) and set \( m = 1/2, a = -2 \) and simplify using entry (4) in Table below (64:12:7) and entry (2) in Table below (64:7) in [1].

**Example 3.** The fundamental constant \( \log(2) \),

\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{T_n(x) \log^{2 - \frac{1}{2}} \left( \frac{1}{2} \right) \log^{2 - \frac{1}{4}} \left( \frac{1}{4} \right)}{\sqrt{1 - x^2} \log(\frac{1}{4})} \log \left( -\frac{2e^{\log(\frac{1}{4})/\log(\frac{1}{4})}}{\log(\frac{1}{4})} \right) dx \, dy \, dz \, dt = -\frac{i\pi \log(2)}{\sqrt{2}}
\]

**Example 4.** Apéry’s constant \( \zeta(3) \),

\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{T_n(x) \log^{2 - \frac{1}{2}} \left( \frac{1}{2} \right) \log^{2 - \frac{1}{4}} \left( \frac{1}{4} \right)}{\sqrt{1 - x^2} \log(\frac{1}{4})} \log \left( -\frac{2e^{\log(\frac{1}{4})/\log(\frac{1}{4})}}{\log(\frac{1}{4})} \right) dx \, dy \, dz \, dt = \frac{3\zeta(3)}{16\sqrt{2\pi}}
\]

**Example 5.**

\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{T_n(x) \log^{2 - \frac{1}{2}} \left( \frac{1}{2} \right) \log^{2 - \frac{1}{4}} \left( \frac{1}{4} \right)}{\sqrt{1 - x^2} \log(\frac{1}{4})} \log \left( -\frac{2e^{\log(\frac{1}{4})/\log(\frac{1}{4})}}{\log(\frac{1}{4})} \right) dx \, dy \, dz \, dt = \pi 2^{-m} e^{\pi i m} \Phi \left( e^{i2\pi}, 1, \frac{\pi + i\log(2)}{2\pi} \right) - \pi 2^{-m} e^{i\pi n} \Phi \left( e^{i2\pi}, 1, \frac{\pi + i\log(2)}{2\pi} \right)
\]

**Example 6.**

\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{T_n(x) \left( \frac{\sqrt{\log(\frac{1}{4})} - \sqrt{\pi} \sqrt{\log(\frac{1}{4})} \sqrt{\log(\frac{1}{4})} \sqrt{\log(\frac{1}{4})} \sqrt{\log(\frac{1}{4})} \right)}{\sqrt{1 - x^2} \log(\frac{1}{4}) \log(\frac{1}{4}) \log(\frac{1}{4}) \log(\frac{1}{4}) \log(\frac{1}{4})} dx \, dy \, dz \, dt = \pi \left( \Phi \left( -1, 1, \frac{\pi + i\log(2)}{2\pi} \right) - \sqrt{-\frac{2}{\pi}} \Phi \left( -1, 2/3, 1, \frac{\pi + i\log(2)}{2\pi} \right) \right)
\]

**6. Discussion**

We provide a new approach for deriving a new integral containing the Chebyshev polynomial \( T_n(x) \) in this study. Several intriguing definite integrals can be obtained using this contour integral method. For both real and imaginary and complex values of the parameters in the integrals, the findings were mathematically validated using Wolfram Mathematica.
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