AN INTRODUCTION TO CONSTRUCTIVE DESINGULARIZATION.

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1. INTRODUCTION

Resolution of singularities has been an area of intense research since the late eighties. Particularly in simplification of the theory, but also in the task of implementations.

In these notes, intended for non-specialist, we present this new approach to the subject. So here we prove two important Theorems of algebraic geometry over fields of characteristic zero:

1) Desingularization (or Resolution of singularities).
2) Embedded Principalization or Log-Resolution of ideals.

Both results, stated in Theorems 2.2 and 2.3, are due to Hironaka. We focus here on the proof in [15], which is more elementary than that of Hironaka. In fact, it avoids the use of Hilbert Samuel functions, and of normal flatness.

Theorem 2.3, of Embedded Principalization, plays a fundamental role in the study of morphisms, and particularly on the elimination of base points of linear systems.

Hironaka’s proof of both theorems is existential; he proves that every singular variety, over a field of characteristic zero, can be desingularized. Our proof of the theorems is constructive, in the sense that we provide an algorithm to achieve such desingularization. We refer to [5] and to [16] for two computer implementations. Bodnár-Schicho’s implementation available at http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup

There are several other proofs of these two theorems, which also provide an algorithm: [3], [10], [12], [25], and [28].

It is natural to ask why is it interesting to study algorithms of resolution of singularities. Usually we simply need to know the formulation of a theorem in order to apply it. But sometimes a proof of a theorem can be strong enough to be useful as a tool. This is the purpose of developing algorithms to achieve resolution of singularities; a theorem with many applications in algebraic geometry. A very natural application arises, for example, when we want to classify singularities by the way that they can be desingularized. To this end it is not enough to know that singularities can be resolved, it is necessary to have an explicit manner to resolve them. This is an advantage of a constructive (or algorithmic) proof over an existential proof.

These notes are written as an introduction to the subject, and includes the contents of various one weeks courses on the subject (see also [27]). Resolution of singularities is based on a peculiar form of induction. In the case of resolution of hypersurfaces this form of induction was stated clear and explicitly by Abhyankar, in what is called a Tschirnhausen transformation.

We will focus on this point in Part I, where we discuss examples of this form of induction, with some indication on how it provides inductive invariants. These invariants are gathered in our resolution functions, and we prove the two Main Theorems 2.2 and 2.3 by extracting natural properties

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from these functions. In Part II we prove results which were motivated through examples in the first Part. In Part III we introduce the resolution functions. A mild technical aspect appears in Part II, where the behavior of derivations and monoidal transformations are discussed. But essentially the first three parts are intended to provide a conceptual (non-technical) and self-contained introduction to desingularization.

Technical aspects appear in the last Part IV, where we present the algorithm in full detail. This will allow the reader to understand also other algorithms, and will hopefully encourage the search for new ones.

These notes follow the notation in [26] (basic objects, and general basic objects). In that paper we prove that the algorithm of desingularization in [25] is equivariant, and that it also desingularizes schemes in étale topology. But the algorithm in [25] (and in [26]) provide embedded desingularizations which makes use of Hironaka’s invariants (of Hilbert-Samuel functions); whereas in these notes we discuss an algorithm in which such invariants are avoided. Hence the outcome is, in general, a different embedded desingularization. It turns out, however, that both algorithms coincide when it comes to the case of embedded desingularization of hypersurfaces. For this reasons we refer to the examples in [26], such as the desingularization of the Whitney Umbrella, or for examples that illustrate equivariance of the desingularization of embedded hypersurfaces.

The algorithm in these notes is also equivariant, and also extends to étale topology. However we do not study these properties in these introductional notes, and we refer to [8] and [14] for the study of these and of further properties of this proof. Among these further properties discussed in those cited papers, there is a new and remarkable formulation of embedded desingularization, with a strong algebraic flavor, obtain in [10] (see 5.4 in these notes).

We finally refer to the notes of D. Cutkosky [11], H. Hauser [18], and K. Matsuki [23], for other introductions to desingularization theorems.

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2. First definitions and formulation of Main Theorem.

The set of regular points, of a reduced scheme of finite type over a field, is a dense open set.

**Definition 2.1.** We say that a birational morphism of reduced irreducible schemes

\[(2.1.1) \quad X \xrightarrow{\pi} X'\]

is a desingularization of \(X\) if:

i) \(\pi\) defines an isomorphism over the open set \(U = \text{Reg}(X)\) of regular points.

ii) \(\pi\) is proper, and \(X'\) is regular.

We will prove the existence of desingularizations, over fields of characteristic zero, by proving a theorem of embedded desingularization in Theorem [2.2]. There we view an irreducible scheme as a closed subscheme in a smooth scheme \(W\).

Let \(W_1 \xrightarrow{\pi} W_2\) be a proper birational morphism of smooth schemes of dimension \(n\). If a closed point \(x_2 \in W_2\) maps to \(x_1 \in W_1\), there is a linear transformation of \(n\)-dimensional tangent spaces, say \(T_{W_2,x_2} \rightarrow T_{W_1,x_1}\). The set of points \(x_2 \in W_2\) for which \(T_{W_2,x_2} \rightarrow T_{W_1,x_1}\) is not an isomorphism defines a hypersurface \(H\) in \(W_2\), called the jacobian or exceptional hypersurface. It turns out that there is an open set \(U \subset W_1\) such that \(U \xrightarrow{\pi^{-1}} \pi^{-1}(U)\) is an isomorphism, and \(\pi^{-1}(U) = W_2 - H\).

Examples of proper birational morphisms of this kind are the monoidal transformations, defined by
blowing up a closed and smooth subscheme $Y$ in a smooth scheme $W_1$. In such case $H = \pi^{-1}(Y)$ is a smooth hypersurface. Let

$$W_0 \leftarrow (W_1, E_1 = (H_1)) \leftarrow (W_2, E_2 = (H_1, H_2)) \cdots \leftarrow (W_r, E_r = (H_1, H_2, \ldots, H_r))$$

be a composition of monoidal transformations, where each $Y_j \subset W_j$ is closed and smooth, $H_j \subset W_j$ is the exceptional hypersurface of $W_{j-1} \leftarrow W_j$ (the blow up at $Y_{j-1}$), and where $\{H_1, H_2, \ldots, H_r\}$ denote the strict transforms of the $H'_r$ in $W_r$. The composite $W_0 \leftarrow W_r$ is a proper birational morphism of smooth schemes, and $H = \bigcup_{1 \leq i \leq r} H_i$ is the exceptional hypersurface.

**Theorem 2.2 (Embedded Resolution of Singularities).** Given $W_0$ smooth over a field $k$ of characteristic zero, and $X_0 \subset W_0$ closed and reduced, there is a sequence (2.1.2) such that

(i) $\bigcup_{i=1}^r H_i$ have normal crossings in $W_r$.

(ii) $W_0 = \text{Sing}(X_0) \cong W_r \setminus \bigcup_{i=1}^r H_i$, and hence it induces a square diagram

$$
\begin{array}{ccc}
W_0 & \overset{π_r}{\longrightarrow} & W_r \\
\cup & & \cup \\
X_0 & \overset{π_r}{\longrightarrow} & X_r
\end{array}
$$

of proper birational morphisms, where $X_r$ denotes the strict transform of $X_0$.

(iii) $X_r$ is regular and has normal crossings with $E_r = \bigcup_{i=1}^r H_i$.

In particular $\text{Reg}(X_0) \cong \prod_{r}^{-1}(\text{Reg}(X_0)) \subset X_r$ and $X_0 \overset{π_r}{\longleftarrow} X_r$ is a desingularization (2.1).

**Theorem 2.3 (Embedded Principalization of ideals).** Given $I \subset O_{W_0}$, a non-zero sheaf of ideals, there is a sequence (2.1.2) such that:

(i) The morphism $W_0 \leftarrow W_r$ defines an isomorphism over $W_0 \setminus V(I)$.

(ii) The sheaf $IO_{W_r}$ is invertible and supported on a divisor with normal crossings, i.e.,

$$L = IO_{W_r} = I(H_1)^{c_1} \cdots I(H_s)^{c_s},$$

where $E' = \{H_1, H_2, \ldots, H_s\}$ are regular hypersurfaces with normal crossings, $c_i \geq 1$ for $i = 1, \ldots, s$, and $E' = E_r$ if $V(I)$ has no components of codimension 1.

**Part I**

Throughout these notes $W$ will denote a smooth scheme of finite type over a field $k$ of characteristic zero. We first recall here some definitions used in the formulation of the previous theorems.

**Definition 2.4.** Fix $y \in W$, and let $\{x_1, \ldots, x_d\}$ be a regular system of parameters (r.s. of p.) in the local regular ring $O_{W,y}$.

1) $Y(\subset W)$, defined by $I(Y) \subset O_W$, is **regular at** $y \in Y$, if there is a r. s. of p. such that $I(Y)_y = \langle x_1, \ldots, x_s \rangle$ in $O_{W,y}$.

2) A set $\{H_1, \ldots, H_r\}$ of hypersurfaces in $W$ has normal crossings at $y$ if there is a r.s. of p. such that $\cup H_i = V(\langle x_{j_1}, x_{j_2}, \ldots, x_{j_s} \rangle)$ locally at $y$, for some $j_i \in \{1, \ldots r\}$.

3) A closed subscheme $Y$ has normal crossings with $E$ at $y$, if there is a r.s. of p. such that, locally at $y$:

$$I(Y)_y = \langle x_1, \ldots, x_s \rangle \quad \text{and} \quad \cup H_i = V(\langle x_{j_1}, x_{j_2}, \ldots, x_{j_s} \rangle).$$

$Y$ is said to be regular if it is regular at any point; and $E = \{H_1, \ldots, H_r\}$ is said to have normal crossings if the condition holds at any point.
Remark 2.5. If
\[ W_0 \xleftarrow{\pi} W_1 \supset H = \pi^{-1}(Y), \]
denotes a monoidal transformation with a closed and regular center \( Y \subset W_0 \), then:
1) \( \pi \) is proper and \( W_1 \) smooth.
2) \( H = \pi^{-1}(Y) \) is a smooth hypersurface in \( W_1 \).
3) \( W_0 - Y \cong W_1 - H \) (i.e. \( \pi \) is birational).

Definition 2.6. The order of an non-zero ideal \( J \) in a local regular ring \((R, M)\) is the biggest integer \( b \geq 0 \) such that \( J \subset M^b \).

Remark 2.7. Assume that \( Y \) in 2.5 is irreducible with generic point \( y \in W \), and let \( h \in W_1 \) be the generic point of \( H \). Note that \( O_{W, y} \) is a local regular ring, and that \( O_{W_1, h} \) is a discrete valuation ring. Let \( M_y \) denote the maximal ideal of \( O_{W, y} \).

Set \( W_0 \xleftarrow{\pi} W_1 \) and \( H \subset W_1 \) as above. Then, for an ideal \( J \subset O_W \), the following are equivalent:

a) \( J_y \subset M_y^b \) (i.e. the order of \( J \) at \( O_{W, y} \) is \( \geq b \))
b) \( J \odot_{W_1} = I(H)^b \cdot J_1 \) for some \( J_1 \) in \( O_{W_1} \).
c) \( J \odot_{W_1} \) has order \( \geq b \) at \( O_{W_1, h} \).

Definition 2.8. Given a sheaf of ideals \( J \subset O_X \) and a morphism of schemes, \( X \xleftarrow{\pi} Y \), the sheaf of ideals \( \pi_\ast J \) is called the total transform of \( J \) in \( Y \). In the previous remark we considered the total transform by a monoidal transformation, and we do not assume \( b \) to be the order of \( J \) at the generic point of \( Y \). When such condition holds, then \( b \) is the highest integer for which an expression \( \pi_\ast J \) can be defined; and in such case \( J_1 \) is called the proper transform of \( J \).

The following result will be used to ensure that \( E_r \) has normal crossings in a sequence of monoidal transformations (2.1.2).

Proposition 2.9. Let \( W \) be smooth over \( k \), and let \( E = \{H_1, \ldots, H_s\} \) be a set of smooth hypersurfaces with normal crossings. Assume that \( Y \subset W \) is closed, regular, and has normal crossings with \( E = \{H_1, \ldots, H_s\} \), and set the monoidal transformation
\[ (W, E = \{H_1, \ldots, H_s\}) \xleftarrow{\pi} (W_1, E_1 = \{H'_1, \ldots, H'_s, H_{s+1} = \pi^{-1}(Y)\}) \]
where \( H'_i \) denotes the strict transform of \( H_i \). Then \( E_1 \) has normal crossings in \( W_1 \).

3. Examples: Tschirnhausen and a form of induction on resolution problems.

A variety, or an ideal, is usually presented by equations in a certain number of variables. A key point in resolution problems is to argue by induction on the number of variables involved. In order to illustrate the precise meaning of this form of induction we first consider the polynomial \( f = Z^2 + 2 \cdot X \cdot Z + X^2 + X \cdot Y^2 \in k[Z, X, Y] \), defining a hypersurface \( X \subset \mathbb{A}_k^3 \), where \( k \) denotes here an algebraically closed field of characteristic zero. We will see that all points in this hypersurface are of multiplicity at most two.
**Question:** How to describe the closed set of points of multiplicity 2?, say $\mathcal{F}_2 \subset \mathbb{X}$.

Recall first two definitions:

**Definition 3.1.** Set $p \in \mathbb{X} = V((f)) \subset \text{Spec}(k[Z,X,Y])$. We say that the hypersurface $\mathbb{X}$ has multiplicity $b$ at $p$, or that $p$ is a $b$-fold point of the hypersurface, if $(f)$ has order $b$ at the local regular ring $k[Z,X,Y]_p$. We will denote by $\mathcal{F}_b$ the set of points in $\mathbb{X}$ with multiplicity $b$.

There are now two ways in which we can address our question.

**Approach 1):** Consider the extension of the ideal $J = (f)$, say:

$$J(1) = (f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}).$$

Clearly $V(J(1)) = \mathcal{F}_2$. In fact, by taking Taylor expansions at any closed point $q$ we conclude that $q \in V(J(1))$ if and only if the multiplicity of $\mathbb{X}$ at $q$ is at least 2. Note also that $\mathbb{X}$ has no closed point of multiplicity higher than 2 since $\frac{\partial f}{\partial Z}$ is a unit. So the hypersurface $\mathbb{X}$ has only closed points of multiplicity one and two.

As for the non-closed points of $\mathbb{X}$, recall first that in a polynomial ring any prime ideal is the intersection of all maximal ideals containing it. On the other hand the multiplicity defines an upper-semi-continuous function on the hypersurface. So the multiplicity at a non-closed point, say $y \in \mathbb{X}$, coincides with the multiplicity at closed points in an non-empty open set of the closure $\overline{y}$. This settles our question.

**3.2.** Approach 2) (linked to the previous): Set $Z_1 = Z + X$. At $k[Z_1,X,Y] = k[Z,X,Y]$:

$$f = Z_1^2 + X \cdot Y^2.$$ (3.2.1)

2i) Note first that $Z_1 \in J(1)$, and hence $\mathcal{F}_2 \subset \overline{W}$, where $\overline{W} = V(Z_1)$ is a smooth hypersurface.

2ii) Set $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}}$. We claim that $\mathcal{F}_2 \subset \overline{W}$ is also defined as the set of points $q \in \overline{W}$ where the order of $J^*$, at the local regular ring $\mathcal{O}_{\overline{W},q}$, is at least 2.

In fact, if $q \in \text{Spec}(k[Z,X,Y])$ is a point (a prime ideal) of order 2, then $J(1) \subset q$, so $Z_1 \in q \subset k[Z_1,X,Y]$.

It is clear that among the prime ideals containing $Z_1$, those where $Z_1^2 + X \cdot Y^2$ has order 2, are those where $X \cdot Y^2$ has order at least 2. So the claim follows by setting $\overline{W} = V(Z_1)$ and $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_{\overline{W}}$ as before.

**3.3.** We will see that the answer to our earlier Question, provided in Approach 2, is better adapted to resolution problems, at least over fields of characteristic zero.

We started by asking for those points where the ideal $(f) \subset k[Z,X,Y]$ has order at least 2. So we fixed an ideal $J$ ($J = (f)$ in this case), and a positive integer $b$ ($b = 2$ in this case), and we considered the closed set $\mathcal{F}_2$ of points where this ideal has order 2. We ended up with a new ideal, $J^* = \langle X \cdot Y^2 \rangle$ in the ring of functions in $\overline{W}$, where

$$\overline{W} = (\text{Spec}(k[X,Y])) = \text{Spec}(k[Z_1,X,Y]/(Z_1)) \subset \text{Spec}(k[X,Y,Z]),$$

together with an integer $b_1 = 2$, describing the same closed set $\mathcal{F}_2$, but involving one variable less.

**Definition 3.4.** Fix a scheme $\mathcal{W}$, smooth over a field of characteristic zero. A *couple* will be an ideal $J \subset \mathcal{O}_W$ and an integer $b$, and will be denoted by $(J,b)$. 
The set described by the couple will be the set of points \( \{ x \in W/\nu_x(J) \geq b \} \), where \( \nu_x(J) \) denotes the order of \( J \) at the local regular ring \( \mathcal{O}_{W,x} \).

**3.5.** The set described by the couple \( (J = (Z_1^2 + X \cdot Y^2), 2) \) in \( \mathbb{A}_k^3 \) is included in a smooth hypersurface \( \overline{W} = V(Z_1) \). The dimension of \( \overline{W} \) is of course one less than that of \( W \). This inclusion is called the local inductive principal. Note that this closed set is also defined by the couple \( (J^*, 2) \) \( (J^* = (X \cdot Y^2) \subset \mathcal{O}_{\overline{W}}) \).

**Example 3.6.** The fact that \( J^* \subset \mathcal{O}_{\overline{W}} \) is principal just a coincidence of the previous example. Let now \( \mathbb{Y} \subset \mathbb{A}_k^3 \) be the hypersurface defined by \( g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z, X, Y] \). Define

\[
J(2) = \langle g, \frac{\partial g}{\partial x_i}, \frac{\partial^2 g}{\partial x_i \partial x_j} \rangle \quad \text{where } x_1 = X, x_2 = Y, x_3 = Z
\]

so \( V(J(2)) = F_3 \) is the set of points of multiplicity at least 3. The pattern of this equation is

\[
Z^3 + a_2 \cdot Z + a_3 \quad \text{with } a_2, a_3 \text{ in } k[X, Y].
\]

One can check that \( Z \in J(2) \), and that \( \mathbb{Y} \) has at most points of multiplicity 3 since \( \frac{\partial^3 g}{\partial x^3} \) is a unit.

We can argue as in Approach 2 to show that if \( q \in \text{Spec}(k[Z, X, Y]) \) is a point (a prime ideal) of multiplicity 3, then \( J(2) \subset q \). So

\[
Z \in q \subset k[Z, X, Y],
\]

and among all prime ideals \( q \) containing \( Z \), the polynomial \( Z^3 + X \cdot Y^2 \cdot Z + X^5 \) has order 3 at \( k[Z, X, Y]_q \) if and only if \( X \cdot Y^2 \) has order at least 2, and \( X^5 \) has order at least 3. In fact \( Z \) has order one at \( k[Z, X, Y]_q \), and \( Z, X, Y \) are independent variables.

Set now \( \overline{W} = V(Z), \overline{a_2} = X \cdot Y^2, \overline{a_3} = X^5 \) (the class of \( a_2 \) and \( a_3 \) in \( \mathcal{O}_{\overline{W}} \)), and note that

\[
F_3 = \{ x \in \overline{W}/\nu_x(\overline{a_2}) \geq 2; \nu_x(\overline{a_3}) \geq 3 \};
\]

where \( \nu_x(\pi) \) denotes the order of \( \pi \) at the local regular ring \( \mathcal{O}_{\overline{W},x} \).

Set

\[
(3.6.1) \quad (J^*, 6), \quad \text{where } J^* = \langle (\overline{a_2})^3, (\overline{a_3})^2 \rangle \subset \mathcal{O}_{\overline{W}}.
\]

Finally check that \( F_3 \subset \overline{W} \) (local inductive principal \( 3.5 \)), and note that we use this fact to show that the closed set \( F_3 \) is also defined by the couple \( (J^*, 6) \).

**Remark 3.7.** Transformations of couples and stability of inductive principal.

Let \( \mathbb{Y} \subset \mathbb{A}_k^3 \) be the hypersurface defined by \( g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z, X, Y] \), as in Example 3.6. The origin \( \overline{0} \in \mathbb{A}_k^3 \) is clearly a point of the closed set defined by \( (J, 3) \). We now define:

\[
(3.7.1) \quad \mathbb{A}_k^3 \leftarrow W_1
\]

as the blowup at \( \overline{0} \). Let \( W_1 \) be the strict transform of \( W \), \( \mathbb{Y}_1 \) the strict transform of \( \mathbb{Y} \), and \( H \) the exceptional hypersurface. By restriction of the morphism to the subschemes we obtain

\[
(3.7.2) \quad \overline{W} \leftarrow \overline{W}_1,
\]

which is also the monoidal transformation at the point \( \overline{0} \in \overline{W} \), with exceptional hypersurface \( \overline{H} = H \cap \overline{W}_1 \).

Note that there is a well defined factorization of the form

\[
(3.7.3) \quad JO_{W_1} = I(H)^3 \cdot J_1
\]
for an ideal \( J_1 \subset O_{W_1} \), defined in terms of \((3.7.1)\); and a factorization
\[
(3.7.4) \quad J^*O_{\overline{W}_1} = I(\overline{H})^6 \cdot J_1^*
\]
for \( J_1^* \subset O_{\overline{W}_1} \), defined in terms of \((3.7.2)\). These factorizations hold because \( \overline{U} \) is a point of the closed set defined by \((J,3)\), thus of the closed set in \( \overline{W} \) defined by \((J^*,6)\).

Since \( \overline{U} \) is a point of order 3 of \( J \) (a point of multiplicity 3 of the hypersurface \( Y \)), \( J_1 \subset O_{W_1} \) is the ideal defining the strict transform \( Y_1 \).

**Claim:** The set of 3-fold points of the hypersurface \( Y_1 \), or say the closed set of points defined by \((J_1,3)\), is included in \( W_1 \) and coincides with the closed set defined by \((J^*_1,6)\).

In other words, we claim that the role played by \( W \) and \((J^*,6)\) for the hypersurface \( Y \) (the local inductive principal \((3.5)\)), is now played by \( W_1 \) and \((J^*_1,6)\) for the hypersurface \( Y_1 \). We call this the **stability** of the local inductive principal.

To check this claim note first that \( W \) can be covered by three charts:
\[
U_X = \text{Spec}(k[Z/X, X, Y/X]) = \mathbb{A}^3_k
\]
\[
U_Y = \text{Spec}(k[Z/Y, X/Y, Y]) = \mathbb{A}^3_k
\]
\[
U_Z = \text{Spec}(k[Z, X/Z, Y/Z]) = \mathbb{A}^3_k
\]

The morphism: \( \mathbb{A}^3_k \leftarrow U_Y = \text{Spec}(k[Z/Y, X/Y, Y]) = \mathbb{A}^3_k \), induced by \((3.7.1)\), is defined by the inclusion \( k[Z, X, Y] \to k[Z/Y, X/Y, Y] \).

At this chart \( I(H) = \langle Y \rangle \), the factorization in \((3.7.3)\) is
\[
g = Z^3 + X \cdot Y^2 \cdot Z + X^5 = Y^3 \cdot ((Z/Y)^3 + (X/Y) \cdot Y \cdot (Z/Y) + (X/Y)^5 \cdot Y^2),
\]
and \( I(\overline{W}_1 \cap U_Y) = \langle Z/Y \rangle \).

Note that \( g_1 = (Z/Y)^3 + (X/Y) \cdot Y \cdot (Z/Y) + (X/Y)^5 \cdot Y^2 \in k[Z/Y, X/Y, Y] \) has the same general pattern as \( g \), namely: \((Z/Y)^3 + b_2 \cdot (Z/Y) + b_3 \), with \( b_2, b_3 \) in \( k[X/Y, Y] \). So the same argument applied to \( g \) asserts that:

1) The set of 3-fold points of \( Y_1 \cap U_Y \) is included in \( V(\langle Z/Y \rangle) \), or say in
\[
\overline{W}_1 \cap U_Y = \text{Spec}(k[Z/Y, X/Y, Y]/\langle Z/Y \rangle) = \text{Spec}(k[X/Y, Y]).
\]

2) The set of 3-fold points \( Y_1 \) in \( U_Y \) is the closed set in \( \overline{W}_1 \cap U_Y \) defined by \((A,6)\), where
\[
A = \langle (b_2)^3, (b_3)^2 \rangle \subset k[X/Y, Y].
\]

We are finally ready to address the main property of our form of induction in the number of variables, namely the compatibility of induction with transformations. To this end note that
\[
\overline{W} \leftarrow \overline{W}_1 \cap U_Y
\]
is defined by \( k[X, Y] \to k[X/Y, Y] \), and the transform of the couple \((J^*,6)\) in \((3.6.1)\), defined in \((3.7.4)\), is such that
\[
J^*_1 O_{\overline{W}_1 \cap U_Y} = A.
\]
A similar argument applies for $\mathbb{A}^3 \leftarrow U_X$. To study our claim for $\mathbb{A}^3 \leftarrow W_1$ it suffices to check at the charts $U_X, U_Y$. In fact, $U_X \cup U_Y$ cover all of $W_1$ except for one point (the origin at $U_Z = \mathbb{A}^3$), which is not a point of $\mathbb{Y}_1$. So $U_Z$ can be ignored for our purpose.

### 3.8. Summarizing: Stability of inductive principal.

Our previous discussion showed that the set of 3-fold points of $\mathbb{Y} \subset \mathbb{A}^3$ (defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z,X,Y]$) is included in a smooth hypersurface $\mathcal{W}$ (defined by $Z \in k[Z,X,Y]$) \footnote{3.5}. From this fact we conclude that the set is also defined by $(J^*, 6)$, where $J^*$ is an ideal in the surface $\mathcal{W}$. The property that links $\mathcal{W}$ with 3-fold points of $\mathbb{Y}$ goes beyond this fact. A transformation at a 3-fold point of $\mathbb{Y}$ defines a strict transform $\mathbb{Y}_1$. It also induces a transformation $\mathcal{W} \leftarrow \mathcal{W}_1$, together with a transformation of $(J^*, 6)$, say $(J^*_1, 6)$. $\mathcal{W}_1$ is the strict transform of $\mathcal{W}$, and the property is that the set of three fold points of $\mathbb{Y}_1$ is included in $\mathcal{W}_1$. This is what we call the stability of the inductive principal. Furthermore, $(J^*_1, 6)$ defines the closed set of 3-fold points of $\mathbb{Y}_1$. In particular, if $J^*_1$ would not have points of order 6 (which is not the case in our example), then $\mathbb{Y}_1$ would not have 3-fold points. Here we have analyzed this stability for one quadratic transformation, but it turns out that the same argument holds for any sequence of monoidal transformations: Defining a sequence of transformations, say

\begin{equation}
\begin{array}{cccccc}
\mathbb{A}^3 & \leftarrow & W_1 & \leftarrow & \cdots & \leftarrow & W_k \\
\mathcal{W} & \leftarrow & \mathcal{W}_1 & \leftarrow & \cdots & \leftarrow & \mathcal{W}_k \\
\end{array}
\end{equation}

where each $\pi_{i+1}$ is a blow-up at a closed and smooth centers included in the 3-fold points of $\mathbb{Y}_i$, the strict transform of $\mathbb{Y}_{i-1}$, is equivalent to the definition of a sequence of transformations

\begin{equation}
\begin{array}{cccccc}
\mathbb{Y} & \leftarrow & \mathbb{Y}_1 & \leftarrow & \cdots & \leftarrow & \mathbb{Y}_k \\
(J^*, 6) & \leftarrow & (J^*_1, 6) & \leftarrow & \cdots & \leftarrow & (J^*_k, 6) \\
\end{array}
\end{equation}

where each $J^*_i \subset \mathcal{O}_{\mathcal{W}_i}$, and $(J^*_i, 6)$ is defined in terms of $(J^*_{i-1}, 6)$ as in \footnote{3.7.4}. Moreover, each $\mathcal{W}_i$ is a smooth hypersurface in $W_i$, and the closed set defined by $(J^*_i, 6)$ in the hypersurface $\mathcal{W}_i$ is the set of 3-fold points of $\mathbb{Y}_i$. In particular, if the second sequence is defined with the property that $J^*_k$ has no points of order 6 in $\mathcal{W}_k$, then the hypersurface $\mathbb{Y}_k$ has at most points of multiplicity 2.

This is induction on the dimension of the ambient space, where the lowering of the highest order of an ideal in a smooth scheme of dimension 3 is equivalent to a related problem in a smooth scheme of dimension 2. This property of the smooth hypersurface $\mathcal{W}$ will be discussed in Section 4.

### 3.9. Tschirnhausen.

Set $f = Z^b + a_1 Z^{b-1} + \cdots + a_b \in k[Z, X_1, \ldots, X_n]$, with $a_i \in k[X_1, \ldots, X_n]$ for $i = 1, \ldots, b$. If the characteristic of $k$ is zero set $Z_1 = Z + 1/a_1$. Check that $k[Z, X_1, \ldots, X_n] = k[Z_1, X_1, \ldots, X_n]$, and that $f = Z_1^b + c_1 Z_1^{b-2} + \cdots + c_b$, with $c_j \in k[X_1, \ldots, X_n]$ and $c_1 = 0$. One can argue as in Example 3.6 to show that the $b$-fold points of $\mathbb{Y}$ are included in the hypersurface $\mathcal{W} = V(Z_1)(\subset \mathbb{A}^{n+1})$(local inductive principle \footnote{3.5}). Furthermore, $\mathcal{W}$ will have the stability property discussed above, where the role of $(J^*, 6)$ in Remark 3.7 (in (3.8.2)) is now played by $(J^*, b!)$, where

\[ J^* = \langle c_i^b, i = 2, 3, \ldots, b \rangle \subset \mathcal{O}_{\mathcal{W}}. \]

### 4. Resolution functions and the main resolution theorems.

Our proofs of the two main theorems \footnote{2.2} and \footnote{2.3} will be constructive, as opposed to the original existential proofs of Hironaka. We introduce here the notion of resolution algorithm, or resolution functions. Constructive resolutions will be defined in terms of these functions, and the main purpose in this Section is to show how both proofs follow easily from natural properties of these functions.
4.1. In 3.6 we study the transform of a hypersurface in $\mathbb{A}^3$ by a monoidal transformation at a 3-fold point. Note that (3.7.3) is an example of a proper transform of an ideal, as defined in 2.8. However the ideal $J^*$ has order 9 at the center of the monoidal transformation, so $J^*_1$ in (3.7.4) is not a proper transform. This shows that our form of induction will lead us to transformations, defined by expressions of the form $J\mathcal{O}_{W_1} = I(H)^b \cdot J_1$, even when $b$ is not the highest possible integer in such expression.

We have defined couples as pairs $(J, b)$, where $J \subset \mathcal{O}_W$ is a non-zero sheaf of ideals, and $b \in \mathbb{N}$ is a positive integer. We introduce now two notions related to couples:

- The closed set attached to $(J, b)$:
  \[ \text{Sing}(J, b) = \{ x \in W \mid \nu_x(J) \geq b \}, \]
  namely the set of points in $W$ where $J$ has order at least $b$. This is closed in $W$ (see 6.4, ii)).

- Transformation of $(J, b)$:
  Let $Y \subset \text{Sing}(J, b)$ be a closed and smooth subscheme, and let
  \[ W \leftarrow^\pi Y \]
  be the monoidal transformation at $Y$. Since $Y \subset \text{Sing}(J, b)$, the total transform $J\mathcal{O}_{W_1}$ can be expressed as a product:
  \[ J\mathcal{O}_{W_1} = I(H)^b J_1 (\subset \mathcal{O}_{W_1}) \]
  for a uniquely defined $J_1$ in $\mathcal{O}_{W_1}$. The new couple $(J_1, b)$ is called the transform of $(J, b)$, and the transformation is denoted by
  \[ \begin{array}{c}
  W \\
  (J, b)
  \end{array} \leftarrow^\pi \begin{array}{c}
  W_1 \\
  (J_1, b)
  \end{array} \]

A sequence of transformations will be denoted as

\[ \begin{array}{c}
  W \\
  (J, b)
  \end{array} \leftarrow^\pi_1 \begin{array}{c}
  W_1 \\
  (J_1, b)
  \end{array} \leftarrow^\pi_2 \cdots \leftarrow^\pi_k \begin{array}{c}
  W_k, \\
  (J_k, b)
  \end{array} \]

Note that in such case

\[ J\mathcal{O}_{W_k} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_k)^{c_k} \cdot J_k \]

for suitable exponents $c_2, \ldots, c_k$, and $c_1 = b$. Furthermore, all $c_i = b$ if for any index $i < k$ the center $Y_i$ is not included in $\bigcup_{j \leq i} H_j \subset W_i$ (the exceptional locus of $W \leftarrow W_i$).

**Example 4.2.** The ideal $J = \langle x^2 - y^5 \rangle \subset k[x, y]$ has a unique 2-fold point at the origin $(0, 0) \in \mathbb{A}^2$. Let $W = \mathbb{A}^2 \leftarrow W_1$ be the blow up at the origin. The strict transform of the curve has a unique 2-fold point, say $q \in W_1$. Set $W_1 \leftarrow W_2$ by blowing-up $q$. This defines a sequence,

\[ \begin{array}{c}
  W \\
  (J, b)
  \end{array} \leftarrow \begin{array}{c}
  W_1 \\
  (J_1, b)
  \end{array} \leftarrow \begin{array}{c}
  W_2 \\
  (J_2, b)
  \end{array} \]

Here $J\mathcal{O}_{W_2} = I(H_1)^2 \cdot I(H_2)^4 \cdot J_2$ provides an expression of the total transform of $J$ involving $J_2$. 
Remark 4.3. The ideal $J_1$ in the previous example is the proper transform of $J$, and $J_2$ is the proper transform of $J_1$ (Def 2.8). In particular $J_2$ does not vanish along $H_1$ or $H_2$. Recall however that this is not a general fact as indicated in 4.1.1. Set now $K = J$, and note the same sequence as before defines $(K, 1)$; $(K_1, 1)$; $(K_2, 1)$ and $K\mathcal{O}_{W_2} = I(H_1)^1 \cdot I(H_2)^2 \cdot K_2$.

In this case the ideal $K_2$ does vanish along the exceptional hypersurface $H_i$, in fact there is a unique and well defined expression, say

$$K_2 = I(H_1)^a \cdot I(H_2)^b \cdot \overline{K_2}$$

in $\mathcal{O}_{W_2}$, so that $\overline{K_2}$ does not vanish along the exceptional hypersurfaces. It follows from 4.2 that $a = 1$, $b = 2$ and $\overline{K_2} = J_2$.

Definition 4.4. Fix $J \subset \mathcal{O}_W$, $W$ smooth over a field of characteristic zero, and a couple $(J, b)$. A sequence of transformations as in 4.1.2 is said to be a resolution of $(J, b)$ if:

i) $\text{Sing}(J_k, b) = \emptyset$.

ii) The exceptional locus of $W \leftarrow W_k$, namely $\cup_{1 \leq i \leq k} H_i$, is a union of hypersurfaces with normal crossings.

4.5. We define a pair, denoted by $(W, E = \{H_1, ..., H_r\})$, to be a set of smooth hypersurfaces $H_1, ..., H_r$ with normal crossings in a smooth scheme $W$.

Let $W \leftarrow W_1$ be a monoidal transformation at a closed and smooth center $Y$. If $Y$ has normal crossings with $\cup H_i$, we say that $Y$ is permissible for the pair $(W, E)$, and that

$$(W, E = \{H_1, ..., H_r\}) \leftarrow (W_1, E_1 = \{H_1, ..., H_r, H_{r+1}\})$$

is a transformation of pairs (see Prop 2.9).

We define a basic object to be a pair $(W, E = \{H_1, ..., H_r\})$ together with a couple $(J, b)$, with the condition that $J_x \neq 0(\subset \mathcal{O}_{W,x})$ at any point $x \in W$. We indicate this basic object by

$$(W, (J, b), E).$$

If a smooth center $Y$ defines a transformation of the pair $(W, E)$, and in addition $Y \subset \text{Sing}(J, b)$, then a transform of the couple $(J, b)$ is defined. In this case we say that

$$(W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1)$$

is a transformation of the basic object. A sequence of transformations

$$(W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (W_s, (J_s, b), E_s)$$

is a resolution of the basic object if $\text{Sing}(J_s, b) = \emptyset$.

In such case

$$J \cdot \mathcal{O}_{W_s} = I(H_{r+1})^{c_1} \cdot I(H_{r+2})^{c_2} \cdots I(H_{r+s})^{c_s} \cdot J_s$$

for some integer $c_i$, where $J_s$ is a sheaf of ideals of order at most $b - 1$, and the $H_j$ have normal crossings.

Definition 4.6. Let $X$ be a topological space, and $(T, \geq)$ a totally ordered set. A function $g : X \rightarrow T$ is said to be upper semi-continuous if: i) $g$ takes only finitely many values, and, ii) for any $\alpha \in T$ the set

$$\{x \in X \mid g(x) \geq \alpha\}$$

is closed in $X$. 

Then largest value achieved by $g$ will be denoted by $\max g$.

Clearly the set
\[
\text{Max } g = \{ x \in X : g(x) = \max g \}
\]
is a closed subset of $X$.

4.7. Resolution functions. We now show why constructive resolutions of basic objects will lead us to simple proofs of both Main Theorems 2.2 and 2.3.

In 3.2 we defined an upper semi-continuous function, say $h_3 : \text{Spec}(k[Z_1, X, Y]) \to \mathbb{Z}$, defined by taking order of the ideal $J = \langle Z_1^2 + X \cdot Y^2 \rangle$. It was shown that $\max h_3 = 2$, and that $\text{Max } h_3(= F_2) \subset W$, where $W = V(Z_1)$ is a smooth hypersurface isomorphic to $\text{Spec}(k[X, Y])$. Furthermore, an ideal $J^* = \langle X \cdot Y^2 \rangle \subset \mathcal{O}_W$ was attached to $\text{Max } h_3$. We may take now $h_2 : \text{Spec}(k[X, Y]) \to \mathbb{Z}$, defined by taking order of the ideal $J^*$, so that $\text{Max } h_2$ is included in a smooth hypersurface; and ultimately define a function $h_1$ with values at $Z$.

In this frame of mind it is conceivable to assign a copy of $Z$ for each dimension, namely $Z \times Z \times Z$, with lexicographic order, and a function, say $h = (h_3, h_2, h_1)$ with values at this ordered set, so that $h$ is upper semi-continuous. This is not exactly the way we will proceed, but we will define a totally ordered set for each dimension, and then take the product of copies of this set, one for each dimension.

We will fix an integer $d$, and define a totally ordered set $(I^d, \geq)$. Moreover, for any basic object $B : (W, (J, b), E)$, dimension of $W = d$, an upper semi-continuous function $f_B : \text{Sing}(J, b) \to I^d$ will be defined with the property that $\text{Max } f_B$ is a smooth subscheme of $\text{Sing}(J, b)$, and a permissible center for the pair $(W, E)$. Thus, a transformation of the basic object can be defined with center $\text{Max } f_B$.

In this way a unique sequence (4.5.1) is defined inductively, by setting centers $\text{Max } f_{B_i}$. In addition, this sequence defined by the functions will be a resolution of the basic object. In fact, for some index $s$ (depending on $B$) $\text{Sing}(J_s, b) = \emptyset$.

In other words, the set $(I^d, \geq)$ will be fixed, and the functions on this set defined so as to provide a resolution for any basic object of dimension $d$. We now state the properties that will hold for such sequence:

Properties:

P1) For each $l$, $\text{Max } f_l$ is closed regular and has normal crossings with $\bigcup_{H_i \in E_l} H_i$.

P2) For some index $k_0$, depending on the basic object $B$, $\text{Sing}(J_{k_0}, b) = \emptyset$.

If $p \in \text{Sing}(J_k, b)$, and $p \notin \text{Max } f_k$, then $p$ can be identified with a point in $W_{k+1}$. Furthermore, $p \in \text{Sing}(J_{k+1}, b)$, and:

P3) $f_k(p) = f_{k+1}(p)$.

Of particular interest will be the case of basic objects with $b = 1$. In such case $\text{Sing}(J_0, 1)$ is the underlying topological space of $V(J_0)$ (the subscheme defined by the sheaf of ideals $J_0$).

P4) There is fixed value $R \in I^d$, and whenever $p \in \text{Sing}(J_0, 1)$ is a point where the subscheme defined by $J_0$ is smooth, then $f_0(p) = R$ (where $f_0 : \text{Sing}(J_0, 1) \to I^d$).
The definition of \((I^d, \geq)\), and of the functions \(f\), will be discussed in Part III, and studied exhaustively in Part IV. We now prove our two Main Theorems \(2.2\) and \(2.3\) using the the properties of resolution functions.

4.8. Proof of Theorem \(2.3\). Fix \(I \subset \mathcal{O}_W\) as in Theorem \(2.3\) and consider the basic object
\[
(W, (J, 1), E = \emptyset),
\]
with \(J = I\), and the resolution defined by the resolution functions. Property \(P2\) asserts that \(\text{Sing}(J_{k_0}, 1) = \emptyset\) for some index \(k_0\). It follows that \(J_{k_0} = \mathcal{O}_{W_{k_0}}\), namely that
\[
J\mathcal{O}_{W_{k_0}} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_{k_0})^{c_{k_0}}.
\]
It is easy to check now that the conditions of the Theorem are fulfilled for \(W \leftarrow W_{k_0}\).

4.9. Proof of Theorem \(2.2\). Let \(J \subset \mathcal{O}_{W_0}\) be the sheaf of ideals defining \(X \subset W_0\) in Theorem \(2.2\) and consider, as above, the resolution of the basic object \(\text{(4.8.1)}\) defined by the resolution functions. So again \(J_{k_0} = \mathcal{O}_{W_{k_0}}\), and hence \(J\mathcal{O}_{W_{k_0}} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_{k_0})^{c_{k_0}}\).

Let \(V = W_0 - \text{Sing}(X)\) be the complement of the singular locus of \(X\). Note that \(V\) is an open set, dense in \(W_0\), and \(f_0(p) = R\) for any \(p \in V \cap \text{Sing}(J, 1)\). Here \(X = \text{Sing}(J, 1)\), and \(V \cap \text{Sing}(J, 1)\) is dense in \(\text{Sing}(J, 1)\) since \(X\) is reduced. Furthermore, \(f_0(p) = R\) for any \(p \in V \cap \text{Sing}(J, 1)\) \((P4)\). So \(\max f_0 \geq R\).

If \(\max f_0 = R\), then \(\text{Sing}(J, 1) = \max f_0\) and \(X\) is smooth in \(W_0\) \((P1)\).

If \(\max f_0 > R\), then \(V\) can be identified with an open set, say \(V_1\), in \(W_1\), and \(f_1(p) = R\) for any \(p \in V_1 \cap \text{Sing}(J_{1, 1})\) \((P3)\).

If \(\max f_1 = R\), then the strict transform of \(X\) is a union of components of \(\max f_1\), so the strict transform defines an embedded desingularization \((P1)\).

If \(\max f_1 > R\) then \(V\) can be identified with an open subset \(V_2\) in \(W_2\).

Note that that there must be an index \(k\), for some \(k < k_0\), so that \(\max f_k = R\). In fact this follows from \(P4\), \(P2\), and the fact that \(\text{Sing}(J_{k_0}, 1) = \emptyset\). Note that \(V\) can be identified with an open set, \(V_k \subset W_k\), and that the strict transform of \(X\) in \(W_k\) fulfills the conditions of the Theorem.

5. ON THE NOTION OF STRICT TRANSFORMS OF IDEALS.

5.1. The notion of strict transform of embedded schemes appears in the very formulation of our Main Theorem \(2.2\). A subscheme of a given schemes is defined by a sheaf of ideals. Given a blow-up at the scheme, there is a notion of strict transform of ideals, corresponding to the notion of strict transform of embedded schemes.

A novel aspect of the proof of Theorem \(2.2\) given in \(4.9\) as compared to the proof of Hironaka and from previous constructive proofs (\(3\), \(26\)), is that we do not consider, within this algorithmic procedure, the notion of strict transform of ideals. In fact, let \(J \subset \mathcal{O}_W\) be the sheaf of ideals defining \(X \subset W_0\), and let
\[
(W_0, (J, 1), E_0) \leftarrow (W_1, (J_1, 1), E_1)
\]
be a transformation with center \(Y \subset \text{Sing}(J, 1)\). We show here that, in general, \(J_1\) is not the sheaf of ideals defining the strict transform of \(X\) in \(W_1\) (i.e. is not the strict transform of \(J\)). Let \(H \subset W_1\) denote the exceptional locus of \(W \leftarrow W_1\).
so that \( W - Y = W_1 - H \). The strict transform of \( X \) is the smallest subscheme of \( W_1 \) containing \( X - Y \), via the identification \( W_1 - H = W - Y \). In other words, it is the closure of \( X - Y \) in \( W_1 \) by this identification.

Such smallest subscheme is defined by the biggest sheaf of ideals, say \( \tilde{J}_1 \subset \mathcal{O}_{W_1} \), which coincide with \( J \) when restricted to \( W_1 - H \). We claim that the biggest sheaf ideal that fulfills this condition is that defined by the increasing union of colon ideals:

\[
\tilde{J}_1 = \cup_k (J\mathcal{O}_{W_1} : I(H)^k).
\]

To check this, set \( U = \text{Spec}(A) \), an open affine set of \( W_1 \), so that the hypersurface \( H \cap U \) is defined by an element \( a \in A \). Let \( K \) denote the ideal defined by restriction of \( \tilde{J}_1 \) to \( U \). The localization \( K \cdot A_a \) is also a restriction of the sheaf of ideals \( J \) to \( U_a = \text{Spec}(A_a) \).

Note that \( K \cdot A_a \cap A \) is the biggest ideal in \( A \) defining \( K \cdot A_a \) at \( U_a = \text{Spec}(A_a) \). On the other hand \( K \cdot A_a \cap A = \cup_k (K : a^k) \). Since this holds for an affine covering of \( W_1 \), it turns out that \( \tilde{J}_1 \) is the biggest sheaf of ideals with the previous condition.

The ideal \( K \) (the restriction of \( \tilde{J}_1 \) to \( U \)), is a finite intersection of \( p \)-primary ideals, called the \( p \)-primary components. The ideal \( K \cdot A_a \cap A \) is obtained from \( K \) by neglecting, in the previous intersection, those \( p \)-primary components corresponding to prime ideals containing the element \( a \in A \) (i.e. with closure of \( p \) included in the exceptional hypersurface \( H \)).

It is not hard to check that \( J_1 \subset \tilde{J}_1 \), in fact \( J_1 = (J\mathcal{O}_{W_1} : I(H)^1) \) according to the definition of transformation of basic objects.

If \( W_1 \) arises from blowing up \( W = A^3_3 \) at the origin, and \( J = (X, Y, Z) \subset k[X, Y, Z] \), then \( V(J_1) \cap H \) is a line, whereas \( V(\tilde{J}_1) \) (the strict transform of the curve), intersects \( H \) at a unique point. So \( J_1 \neq \tilde{J}_1 \) in this case.

### 5.2. Resolution of singularities

Resolution of singularities is defined by a proper birational morphism, defined in a step by step procedure, each step consisting of a suitably defined monoidal transformation. So given equations defining the ideal \( J \), and a monoidal transformation as above, Hironaka provides equations defining the strict transform ideal \( \tilde{J}_1 \). This turns out being, in general, a very difficult task. In fact a major part of the proof of Hironaka is devoted to address this particular point; he introduces the notions of Hilbert-Samuel functions and of normal flatness with this purpose. An important conceptual simplification of constructive desingularization, presented in [3], relies on the fact that it provides a proof avoiding all these notions. In fact, we prove resolution by means of elementary transformations (defining \( J_1 \)), avoiding the use of the strict transform ideal \( \tilde{J}_1 \).

**Example 5.3.** The following example illustrates a situation in which both notions of transformations discussed in 5.1 coincide (i.e. where \( J_1 = \tilde{J}_1 \)).

Let \( X \subset W \) be a closed and smooth subscheme of \( W \). Set \( J = I(X) \), and note that \( \text{Sing}(J, 1) = X \), and that the order of \( J \) at \( \mathcal{O}_{W,x} \) is one at any \( x \in X \).

Let \( W \leftarrow W_1 \) be the monoidal transformation with center \( Y \) which defines a transformation, say: \( (J, 1) ; (J_1, 1) \). In other words, assume that \( Y \subset \text{Sing}(J, 1) \) (so that \( J\mathcal{O}_{W_1} = J_1 \cdot I(H) \), where \( H \subset W_1 \) denotes the exceptional locus). We claim now the following holds:

1. \( \text{Sing}(J_1, 1) = V(J_1) \) is the strict transform of \( X \).
2. The subscheme \( X_1 \subset W_1 \), defined by \( J_1 \), is smooth.
Note that (2) follows from (1). In fact the induced morphism $X \leftarrow X_1$ is the blowup of $X$ at $Y$, and the blowup of a smooth scheme in a smooth subscheme is smooth. To prove 1) note that at any point $x \in W$, there is a regular system of parameters $\{x_1, \ldots, x_n\}$ such that $J_x = \langle x_1, \ldots, x_r \rangle$ and $I(Y)_x = \langle x_1, \ldots, x_s \rangle$ for $r \leq s$. The fiber over $x \in W$ can be covered by $\text{Spec}(\mathcal{O}_W[x_1/x_1, x_2, \ldots, x_s/x_{i_1}, x_{i_1+1}, \ldots, x_n])$ for $i = 1, 2, \ldots, s$. Finally (1) can be checked directly at the charts corresponding to indices $r + 1 \leq i \leq s$.

5.4. There is a stronger formulation of embedded desingularization than that in [2,2] which was proved in [10]. That theorem proves that given $W_0$ smooth over a field $k$ of characteristic zero, and $X_0 \subset W_0$ closed and reduced, there is a sequence of monoidal transformations, say

$$W_0 \leftarrow (W_1, E_1 = \{H_1\}) \leftarrow (W_2, E_2 = \{H_1, H_2\}) \cdots \leftarrow (W_r, E_r = \{H_1, H_2, \ldots, H_r\}),$$

such that, in addition to the three conditions i), ii), and iii) in [2,2] it also holds that:

iv) $I(X_0)\mathcal{O}_{W_r} = I(H_1)^{c_1} \cdot I(H_2)^{c_2} \cdots I(H_r)^{c_r} \cdot I(X_r)$

where $X_r$ denotes the strict transform of $X$.

Consider the particular case in which $X$ is an irreducible subscheme in $W_0 = \text{Spec}(k[X_1, \ldots, X_n])$ defined by a prime ideal $P$ of height $h$. In this case the theorem says that at any point $x \in W_r$ there is a regular system of parameters $\{Z_1, \ldots, Z_n\}$ at $\mathcal{O}_{W_r,x}$, such that:

\begin{itemize}
  \item i) $P \cdot \mathcal{O}_{W_r,x} = \langle Z_{11}, \ldots, Z_{h} \rangle \cdot Z_{12}^{a_1} \cdot Z_{22}^{a_2} \cdots Z_{ij}^{a_i}$ if $x$ is a point of the strict transform $X_r$, and
  \item ii) $P \cdot \mathcal{O}_{W_r,x} = \langle Z_{j1}^{a_1} \cdot Z_{j2}^{a_2} \cdots Z_{js}^{a_s} \rangle$ (is an ideal spanned by a monomial in these coordinates) if $x$ is not in $X_r$.
\end{itemize}

This result does not hold, in general, for desingularizations which make use of invariants such as Hilbert Samuel functions (which we avoid in our proof). This algebraic formulation of embedded desingularization is not a consequence of the theorem of desingularization as proved by Hironaka.

Part II

In [3,3] we discussed a strong link between the set of 3-fold points of the hypersurface $Y \subset \mathbb{A}^3$, defined by $g = Z^3 + X \cdot Y^2 \cdot Z + X^5 \in k[Z, X, Y]$, and the smooth hypersurface $\overline{W}$ defined by $Z \in k[Z, X, Y]$. The link showed that the reduction of 3-fold points of $Y$, by means of monoidal transformations, was equivalent to a related problem for a suitable ideal in the smooth subscheme $\overline{W}$ (see also [3,4]).

This is the key for induction in resolution Theorems. In this second Part we justify the discussion in [3,3] (see Example [7,15]), and generalize this main property in Section 7. In section 6 we study an important preliminary: the behavior of derivations with monoidal transformations.

6. Derivations and monoidal transformations on smooth schemes.

In this Section we study behavior of derivations when applying monoidal transformations. This will be used in the next Section 7 where the inductive properties discussed in [3,3] will be clarified.

Fix $W$ smooth over a field $k$, and $y \in W$ a closed point. Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters at $\mathcal{O}_{W,y}$.

We define an operator $\Delta_y$ on ideals in $\mathcal{O}_{W,y}$ by setting, for $J_y = \langle f_1, f_2, \ldots, f_s \rangle$ in $\mathcal{O}_{W,y}$:

$$\Delta_y(J_y) = \langle f_1, f_2, \ldots, f_s, \frac{\partial f_j}{\partial x_i} / 1 \leq i \leq n; 1 \leq j \leq s \rangle.$$
Note that \( \Delta_y(\Delta_y(\mathcal{J}_y)) = \langle f_1, f_2, \ldots, f_s, \frac{\partial f_i}{\partial x_i}, \frac{\partial^2 f_i}{\partial x_i \partial x_j} / 1 \leq i \leq n; 1 \leq j \leq s \rangle. \) The whole point of restriction to fields of characteristic zero relies on the following property:

6.1. Characteristic zero. If \( k \) is a field of characteristic zero and \( (b \geq 1) \), then \( \mathcal{J}_y \) has order \( b \) at \( \mathcal{O}_{W,y} \) iff \( \Delta_y(\mathcal{J}_y) \) has order \( b - 1 \).

Example 6.2. Let \( \mathcal{O}_{W,y} = k[x_1, x_2, x_3]\langle x_1, x_2, x_3 \rangle. \)

\[ \mathcal{J}_y = \langle x_1^4 + x_2^4 + x_3^4 \rangle. \]

Note that, if \( k \) is of characteristic zero, the orders of these ideals drop by one: 3,2,1,0.

6.3. Further properties of the operator \( \Delta_y \) are:

i) \( \mathcal{J}_y \subseteq \Delta_y(\mathcal{J}_y) \subseteq \Delta_y(\Delta_y(\mathcal{J}_y)) = \Delta_y^2(\mathcal{J}_y) \subseteq \Delta_y^3(\mathcal{J}_y) \subseteq \ldots \)

ii) \( \mathcal{J}_y \subset \mathcal{O}_{W,y}, \) has order \( b \geq 1 \) iff \( \Delta_y^{b-1}(\mathcal{J}_y) \) has order 1.

iii) The order of \( \mathcal{J}_y \subset \mathcal{O}_{W,y}, \) is \( s \) iff \( \Delta_y^{s-1}(\mathcal{J}_y) \) is a proper ideal in \( \mathcal{O}_{W,y}. \)

6.4. On the \( \Delta \) operator. The locally defined operators \( \Delta_y \) can be globalized in the following sense. Fix \( W \) smooth over a field \( k \), there is an operator \( \Delta \) on the class of all \( \mathcal{O}_{W} \)-ideals, such that:

\[ \mathcal{J} \subseteq \Delta(\mathcal{J})(\subset \mathcal{O}_{W}), \]

and at any closed point \( y \in W \):

\[ \Delta(\mathcal{J})_y = \Delta_y(\mathcal{J}_y). \]

Furthermore, the following properties hold:

i) \( \mathcal{J} \subseteq \Delta(\mathcal{J}) \subseteq \Delta^2(\mathcal{J}) \subseteq \ldots \) (hence \( V(\Delta) \supseteq V(\Delta^2) \supseteq \ldots \))

ii) \( V(\Delta^{s-1}(\mathcal{J})) = \text{Sing}(J, s). \) In fact \( V(\Delta^{s-1}(\mathcal{J})) \) is the closed set of points in \( W \) where \( J \) has order \( \geq s \) (i.e. \( (\Delta^{s-1}(\mathcal{J}))_y = \Delta_y^{s-1}(\mathcal{J})_y \subseteq \mathcal{O}_{W,y} \)) if the order of \( \mathcal{J}_y \mathcal{O}_{W,y} \) is \( \geq s \).

iii) If \( b \) is the biggest order of \( J \), \( V(\Delta^b(J)) = \emptyset \) and \( V(\Delta^{b-1}(J)) \) is locally included in a smooth hypersurface.

Proof of iii) If \( b \) is the biggest order of \( J \), \( \Delta^b(J) = \mathcal{O}_{W} \) and \( \Delta^{b-1}(J) \) has order at most 1. So if \( y \in V(\Delta^{b-1}(J)) \), \( \Delta^{b-1}(J) \mathcal{O}_{W,y} \) has order 1 at \( \mathcal{O}_{W,y}. \) If \( y \in V(\Delta^{b-1}(J)) \) has order 1 at \( \mathcal{O}_{W,y}, \) then:

\[ \overline{W} = V(< g >) \supset V(\Delta^{b-1}(J)), \]

and \( \overline{W} \) is a smooth hypersurface in a neighborhood of \( y. \)

Example 6.5. Set \( W = \mathcal{A}^3 = \text{Spec}(k[X, Y, Z]), F = Z^3 + XY^2 Z + X^5, \) and \( J = < F >, \) as in 3.8

Then:

\[ \Delta(J) = < 3Z^2 + XY^2, Y^2 Z + 5X^4, 2XY Z, F > \subset \Delta^2(J) = < Z, XY, Y^2, X^3 > \subset \Delta^3(J) = k[X, Y, Z]. \]

So, as indicated in 3.8, the 3-fold points of the hypersurface \( Y \subset \mathcal{A}^3 \) defined by \( V(< J >) \) are included in smooth hypersurface \( \overline{W} = V(< Z >). \)

6.6. We now address the compatibility of \( \Delta \) operators with monoidal transformations. So fix a couple \((J, b), \) and consider a transformation

\[
\begin{align*}
W & \xleftarrow{\pi} W_1 \\
(J, b) & \mapsto (J_1, b).
\end{align*}
\]
Lemma 6.7. Given $(J,b)$ $(J \subset O_W)$ and a transformation $[6.6.1]$, then:

1) If $b \geq 2$, $[6.6.1]$ induces a transformation of $(\Delta(J),b-1)$:

$$
\begin{array}{c}
W \\ \leftarrow \pi \\
(\Delta(J), b-1) \\
W_1 \\
((\Delta(J))_1, b-1).
\end{array}
$$

2) $(\Delta(J))_1 \subset \Delta(J_1)$.

Proof: Let $Y \subset W$ be the center of the monoidal transformation, and let $H \subset W_1$ be the exceptional locus. By assumption $Y \subset \text{Sing}(J,b)$, so $J \cdot O_W = I(H)^b \cdot J_1$. It follows from [6.4 ii] that for general $b$, $\text{Sing}(J,b) \subset \text{Sing}(\Delta(J), b-1)$. In particular $Y \subset \text{Sing}(\Delta(J), b-1)$, which proves 1).

In order to prove 2) we first note that if $U \subset W$ is open, a sheaf of ideals in $W$ induces a sheaf of ideals in $U$, and the $\Delta$ operators (on $W$ and on $U$) are compatible with restrictions. On the other hand note that the pull-back of $\delta$ to $U$, say $U_1$, is an open set, and the induced morphism $U \leftarrow U_1$ fulfills the conditions in 1) for the restriction of $J$ to $U$.

If we can prove that 2) holds over $U$ (at $U \leftarrow U_1$), for all $U$ in an open covering of $W$, then it is clear that 2) holds. Therefore we may argue locally.

Let $\xi \in W$ be a closed point and choose a regular system of parameters $\{x_1, \ldots, x_n\} \subset O_W, \xi$ so that the center of the monoidal transformation is locally defined by $\langle x_1, \ldots, x_s \rangle$. Now consider an affine neighborhood $U$ of $\xi$ such that $x_1, \ldots, x_s$ are global sections of $O_U$, and such that $J$ is generated by global sections, say $f_1, \ldots, f_r$. We may also assume that $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ are global derivations, and that $\Delta(J)$ is generated by the global sections $\{f_k\}_{k=1}^r \cup \{\frac{\partial f_k}{\partial x_j}\}_{k=1,\ldots, r}$.

By the previous discussion we may assume that $U = W$. The scheme $W_1$ is defined by patching the affine rings

$$
A_i = O_W[x_1/x_i, \ldots, x_s/x_i], \quad i \in \{1, \ldots, s\},
$$

and $I(H) = \langle x_i \rangle$ at $A_i$. For each index $k \in \{1, \ldots, r\}$ there is a factorization $f_k = x_i^{b_k} g_i^{(k)}$, and $\{g_i^{(1)}, g_i^{(2)}, \ldots, g_i^{(r)}\}$ generate the restriction of $J_1$ to $\text{Spec}(A_i)$, say $J_1^{(i)}$. In order to prove 2) we must show that, for each index $k \in \{1, \ldots, r\}$:

a) $\frac{f_k}{x_i^{b_k-1}} \in \Delta(J_1^{(i)})$, and

b) $\frac{\partial f_k}{x_i^{b_k-1}} \in \Delta(J_1^{(i)})$.

The assertion in a) is clear since $\frac{f_k}{x_i^{b_k-1}} = x_i g_i^{(k)} \in J_1^{(i)} \subset \Delta(J_1^{(i)})$. We now address b). In what follows we fix an index $k \in \{1, \ldots, r\}$ and set $f = f_k$. We also fix an index $j \in \{1, \ldots, n\}$ and set $\delta = \frac{\partial}{\partial x_j}$ which is a global derivation on $U$.

Note that

$$
\delta \left( \frac{x_j}{x_i} \right) = \frac{\delta(x_j)}{x_i} - \frac{x_j \delta(x_i)}{x_i},
$$

and that

$$
I(H) \cdot \delta|_{\text{Spec}(A_i)} = x_i \cdot \delta : A_i \rightarrow A_i,
$$

and hence $\mathcal{I}(H) \cdot \delta$ is an invertible sheaf of derivations on $W_1$. 

Now in $A_i$ consider the factorization $f = x_i^b g_i$, so $g_i \in J_1(i) \subset A_i$, and $x_i \cdot \delta$ is a derivation on $A_i$. Finally check that $\frac{\delta(f)}{x_i^{b-1}} = \frac{x_i \delta(x_i^b \cdot g_i)}{x_i^b} = \frac{x_i \delta(x_i^b)}{x_i^b} g_i + \frac{x_i^b (x_i \delta)(g_i)}{x_i^b} = b \cdot \delta(x_i) \cdot g_i + (x_i \delta)(g_i)$.

This already proves b) since the right hand side is in $\Delta(J_1(i))$. Our argument also shows that this equality is stable by any sequence of transformations (see 6.9).

**Remark 6.8.** Fix $K \subset J$ two ideals in $\mathcal{O}_W$, and couples $(J, b)$ and $(K, b)$. Then clearly:

a) $\text{Sing}(J, b) \subset \text{Sing}(K, b)$.

b) Any transformation, as in (6.6.1), of $(J, b)$, induces the transformation

$$\begin{align*}
W & \xrightarrow{\pi^1} W_1 \\
(K, b) & \xrightarrow{\pi} (K_1, b)
\end{align*}$$

and $K_1 \subset J_1$.

**6.9.** We finally extend the previous result to study the behavior of $\Delta$ operators with an arbitrary sequence of transformations.

**Corollary 6.10.** Fix a couple $(J, b)$ $(J \subset \mathcal{O}_W)$ and a sequence of transformations

$$W \xrightarrow{\pi_1} W_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_r} W_r,$$

(6.10.1)

1) If $b \geq 2$, then (6.10.1) induces a sequence of transformations

$$W \xrightarrow{\pi_1} W_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_r} W_r,$$

((\Delta(J), b - 1) \subset ((\Delta(J))_1, b - 1) \subset ((\Delta(J))_r, b - 1),

and

2) $(\Delta(J))_r \subset \Delta(J_r)$.

**Proof.** The case when $r = 1$ is in 6.7. Consider now the case $r = 2$, namely

$$W \xrightarrow{\pi_1} W_1 \xrightarrow{\pi_2} W_3.$$

Then 6.7 asserts that $\pi_1$ defines a transform of $(\Delta(J), b - 1)$, $(\Delta(J))_1$, and that $(\Delta(J))_1 \subset \Delta(J_1)$. The same result says that $\pi_2$ defines a transform of $(\Delta(J_1), b - 1)$, $(\Delta(J))_1$, and that $(\Delta(J_1))_1 \subset \Delta(J_2)$. The statement follows in this case from 6.8.

The general case $r \geq 2$ follows similarly, by induction.

**Corollary 6.11.** Fix a couple $(J, b)$ $(J \subset \mathcal{O}_W)$ and, as before, a sequence of transformations (6.10.1). Assume that $b \geq 2$. Then, for each index $1 \leq j \leq b - 1$:

1) The sequence (6.10.1) induces a sequence of transformations $((\Delta^{(j)}(J)), b - 1 - (j - 1))$, say

$$W \xrightarrow{\pi_1} W_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_r} W_r,$$

and

$$((\Delta^{(j)}(J))_1, b - 1 - (j - 1)) \subset ((\Delta^{(j)}(J))_r, b - 1 - (j - 1))$$

and
2) \((\Delta^{(j)}(J))_r \subset \Delta^{(j)}(J_r)\).

Proof. Note that for \(j = 1\), \(\Delta^{(j)} = \Delta\) and we obtain the previous corollary. So we prove now the statement for \(j\) assuming that it holds \(j - 1\). Set \(J^* = \Delta^{(j-1)}(J)\) and \(b^* = b - 1 - (j - 2)\).

By induction:

i) The sequence of transformations (6.10.1) induces transformations of \((J^*, b^*)\), say:

\[
W \quad \leftarrow \quad W_1 \quad \leftarrow \quad \ldots \quad \leftarrow \quad W_r,
\]

and

\[
(J^*, b^*) \quad \leftarrow \quad (J^*_1, b^*) \quad \leftarrow \quad \ldots \quad \leftarrow \quad (J^*_r, b^*)
\]

and

ii) \(J^*_r \subset \Delta^{(j-1)}(J_r)\).

Applying our previous Corollary 6.10 to i), we get:

i') The sequence in i) induces transformations of \((\Delta(J^*), b^* - 1)\):

\[
W \quad \leftarrow \quad W_1 \quad \leftarrow \quad \ldots \quad \leftarrow \quad W_r,
\]

and

\[
((\Delta(J^*))_1, b^* - 1) \quad \leftarrow \quad \ldots \quad \leftarrow \quad ((\Delta(J^*))_r, b^* - 1)
\]

and

ii') \((\Delta(J^*))_r \subset \Delta(J^*_r)\).

Here \(\Delta(J^*) = \Delta^{(j)}(J)\) and i') is statement 1). On the other hand, applying \(\Delta\) to ii) we get

\[
\Delta(J^*_r) \subset \Delta^{(j)}(J_r),
\]

which together with ii') proves 2).

In the next Section we shall apply Corollary 6.11 basically in the case \(j = b - 1\). The reader might want to look into Example 7.15 to get have an overview of this application of the Corollary.

7. Simple couples and a form of induction on resolution problems.

7.1. The purpose of this Section is the study of simple couples \((J, b)\) \((J \subset \mathcal{O}_W)\). Examples of simple couples appear already in Section 3. They will play a central role in our inductive arguments (induction on the dimension of the ambient space). The main results of this Section are Theorem 7.6 and Proposition 7.13 where the notion of stability of induction discussed in 3.8 is formalized.

7.2. Fix \(J \subset \mathcal{O}_W\), assume that \(J_x \neq 0(\subset \mathcal{O}_{W,x})\) for any \(x \in W\), and define a function

\[(7.2.1) \quad \text{ord}_J : W \to \mathbb{N},\]

where \(\text{ord}_J(x)\) denotes the order of \(J_x\) in the local ring \(\mathcal{O}_{W,x}\).

Note that \(\text{ord}_J\) is upper-semi-continuous (4.6). In fact, for any positive integer \(s\):

\[
\{x \in W/\text{ord}_J(x) \geq s\} = V(\Delta^{s-1}(J)) \quad (\text{see 6.4}).
\]
Remark 7.3. The following conditions are equivalent:
1) \( \max - \text{ord}_J = b \) (where, as in 4.6, \( \max - \text{ord}_J \) denotes the maximum value achieved).
2) \( V(\Delta^{b-1}(J)) \neq \emptyset \) and \( V(\Delta^b(J)) = \emptyset \).
3) \( \max - \text{ord}_{\Delta^{b-1}(J)} = 1. \)

The equivalence follows from the properties of the \( \Delta \) operator discussed in 6.4.

Definition 7.4. We say that \((J, b)\) is a simple couple if the previous conditions hold for \( J \) and \( b \).

The following theorem is a central result in this section.

Theorem 7.5. If \((J, b) (J \subset O_W)\) is a simple couple, and \( W \xleftarrow{\pi} W_1 \) is a transformation, then either \( \text{Sing}(J_1, b) = \emptyset \) or \((J_1, b)\) is a simple couple.

The case \( b = 1 \) will be proved in Proposition 7.8, and the case \( b \geq 2 \) in Proposition 7.9.

We shall first draw attention to the case of simple couples of the form \((J, 1)\).

Remark 7.6. The following conditions are equivalent:
1) \( \max - \text{ord}_J = 1. \)
2) \( V(J) \neq \emptyset \) and \( V(\Delta(J)) = \emptyset \).
3) There is an open covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( W \), and for each \( \lambda \) a smooth hypersurface \( W_\lambda \) in \( U_\lambda \) such that \( I(W_\lambda) \subset J_\lambda \), where \( J_\lambda \) denotes the restriction of \( J \) to \( U_\lambda \).

For the proof of 3), note that an ideal of order one in a local regular ring \( O_{W,x} \) contains an element of order one; and that element defines a smooth hypersurface in some open neighborhood of \( x \in W \).

Remark 7.7. Fix, as before, an open covering of \( W \), say \( \{U_\lambda\}_{\lambda \in \Lambda} \), and a monoidal transformation with center \( Y \subset W \), say \( W \xleftarrow{\pi} W_1 \). For each index \( \lambda \) set \( U^{(1)}_\lambda \subset W_1 \) as the pull-back of \( U_\lambda \). In this way we get
\[
U_\lambda \xleftarrow{\pi_\lambda} U^{(1)}_\lambda
\]
which is either a monoidal transformation (in case \( Y \cap U_\lambda \neq \emptyset \)), or the identity map (if \( Y \cap U_\lambda = \emptyset \)). Note also that \( \{U^{(1)}_\lambda\}_{\lambda \in \Lambda} \) is an open cover of \( W_1 \).

Proposition 7.8. Fix \( J \subset O_W \) with maximum order 1, and a sequence of transformations
\[
(\text{7.8.1}) \quad W \xleftarrow{\pi_1} W_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_r} W_r
\]
then the maximum order of \( J_r \) is either 1 or 0 (i.e. \( J_r = O_{W_r} \) in the last case).

Proof. Define an open covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( W \), and inclusions
\[
(\text{7.8.2}) \quad I(W_\lambda) \subset J_\lambda
\]
where \( W_\lambda \) is a smooth hypersurface in \( U_\lambda \), as indicated in Remark 7.6.3).

The sequence (7.8.1) defines, for each index \( \lambda \), a sequence of transformations:
\[
U_\lambda \xleftarrow{\pi_\lambda} U^{(1)}_\lambda \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_r} U^{(r)}_\lambda
\]
and also

\[
\begin{align*}
(I(W_\lambda), 1) & \xleftarrow{\pi_1} U_\lambda^1 & \xleftarrow{\pi_2} \cdots & \xleftarrow{\pi_r} U_\lambda^r, \\
((I(W_\lambda)), 1) & \xleftarrow{\pi_1} U_\lambda^1 & \xleftarrow{\pi_2} \cdots & \xleftarrow{\pi_r} ((I(W_\lambda)), r, 1).
\end{align*}
\]

Furthermore

\[(I(W_\lambda)), r \subset (J_\lambda)_r\]

by Remark 6.8. Let \(W_\lambda^{(r)} \subset U_\lambda^r\) denote the strict transform of \(W_\lambda\). Since \(W_\lambda\) is smooth in \(U_\lambda\), Example 5.3 asserts that \(W_\lambda^{(r)}\) is smooth, and defined by the ideal \((I(W_\lambda)), r\). In particular \((I(W_\lambda)), r\) has maximum order at most one, and hence the same holds for \((J_\lambda)_r\). Since the open sets \((U_\lambda)^{(r)}\) cover \(W_\lambda\) it follows that \(J_\lambda\) has order at most 1.

**Proposition 7.9.** Fix \(J \subset O_W\) with maximum order \(b \geq 2\), and consider a sequence of transformations

\[
(7.9.1) \quad W \xleftarrow{\pi_1} W_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} W_r.
\]

Then then the maximum order of \(J_r(\subset O_{W_r})\) is at most \(b\).

**Proof.** From 6.4 we conclude that the maximum order of \(\Delta^{b-1}(J) (\subset O_W)\) is 1. Corollary 6.11 applied for \(j = b - 1\) says that (7.9.1) defines the sequence of transformations

\[
(7.9.2) \quad W \xleftarrow{\pi_1} W_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} W_r
\]

and that \((\Delta^{b-1}(J)), r \subset \Delta^{b-1}(J_r)\). On the other hand Proposition 7.8 asserts that \((\Delta(J)), r\) has order at most 1, and hence \(\Delta^{b-1}(J_r)\) has order at most one. From this and 6.4 we conclude that \(J_r\) has order at most \(b\).

**Remark 7.10.** There is a stronger outcome that follows from the proof of Proposition 7.9 that relates to induction in the dimension of the ambient space. Note that \(J\) has highest order \(b\), so \(\Delta^{b-1}(J)\) has highest order one. We can argue as in the proof of Proposition 7.8 and define an open cover \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(W\), and for each index \(\lambda\), a smooth hypersurface \(W_\lambda \subset U_\lambda\), defined by

\[
(7.10.1) \quad I(W_\lambda) \subset (\Delta^{b-1}(J))_{\lambda}.
\]

Now use the compatibility of the \(\Delta\) operator with restriction to open sets and check that \((\Delta^{b-1}(J)), \lambda = (\Delta^{b-1}(J_\lambda))\). Note also that \(\text{Sing}(J, b) \cap U_\lambda \subset W_\lambda\). Recall that 7.9.2 defines, for each index \(\lambda\), a sequence of transformations of \((\Delta^{b-1}(J)), 1\), say:

\[
U_\lambda \xleftarrow{\pi_1} U_\lambda^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} U_\lambda^r
\]

and also

\[
(I(W_\lambda), 1) \xleftarrow{\pi_1} U_\lambda^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} ((I(W_\lambda)), r, 1).
\]

Furthermore, \((I(W_\lambda)), r \subset (\Delta^{b-1}(J)), r\) and \((I(W_\lambda)), r\) defines a smooth hypersurface \(W_\lambda^{(r)} \subset U_\lambda^{(r)}\) which is the strict transform of \(W_\lambda\). We finally note that \(\{U_\lambda^{(r)}\}_{\lambda \in \Lambda}\) is a cover of \(W^{(r)}\), and
taking restriction of the inclusion \((\Delta^{b-1}(J))_r \subset \Delta^{b-1}(J_r)\), we get that:

\[((\Delta^{b-1}(J))_\lambda)_r = ((\Delta^{b-1}(J))_r)_\lambda \subset (\Delta^{b-1}(J_r))_\lambda,\]

and hence \((I(W_\lambda))_r \subset (\Delta^{b-1}(J_r))_\lambda\). In particular

\[(\text{Sing}((J_r)_b) \cap U^{(r)}_\lambda = \text{Sing}((J_\lambda)_b) \subset W^{(r)}_\lambda).\]

**Lemma 7.11.** Fix \(J \subset O_W\) with maximum order \(b\). There is an open covering, say \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(W\), and for each index \(\lambda\) a smooth hypersurface \(\overline{W}_\lambda \subset U_\lambda\), such that the following properties hold:

**P1** \(\text{Sing}(J_\lambda, b) \subset \overline{W}_\lambda\).

**P2** For any sequence

\[(7.11.1)\]

\[
\begin{array}{cccccccc}
W & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & W_r \\
(J, b) & & (J_1, b) & & & & (J_r, b)
\end{array}
\]

and setting by restriction, for each \(\lambda\), say:

\[(7.11.2)\]

\[
\begin{array}{cccccccc}
U_\lambda & \xleftarrow{\pi_1} & U^{(1)}_\lambda & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & U^{(r)}_\lambda \\
(J_\lambda, b) & & ((J_\lambda)_1, b) & & & & ((J_\lambda)_r, b)
\end{array}
\]

then \(\{U^{(r)}_\lambda\}_{\lambda \in \Lambda}\) is an open covering of \(W_r\), and

\[(7.11.3)\]

\[
\text{Sing}(J_r, b) \cap U^{(r)}_\lambda = \text{Sing}((J_\lambda)_r, b) \subset W^{(r)}_\lambda,
\]

where \(\overline{W}_\lambda^{(r)}\) is the smooth hypersurface defined by the strict transform of \(\overline{W}_\lambda\).

**Proof.** The case \(b = 1\) (in which \(\text{Sing}(J, 1) = V(J)\)) is in the proof of Proposition 7.8. The case \(b \geq 2\) is in Remark 7.10 and relies entirely on the inclusion (7.10.1).

7.12. Let \(\overline{W}_\lambda^{(i)}\) denote the strict transform of \(\overline{W}_\lambda^{(0)}\) in \(U_\lambda^{(i)}\) (see (7.11.2)). A consequence of (7.11.3) is that all the centers of monoidal transformations involved in (7.11.2) are included in \(\overline{W}_\lambda^{(i)}\); hence (7.11.2) defines a sequence of monoidal transformations

\[(7.12.1)\]

\[
\overline{W}_\lambda \leftarrow \overline{W}_\lambda^{(1)} \leftarrow \ldots \leftarrow \overline{W}_\lambda^{(r)}.
\]

**Proposition 7.13.** Fix \(J \subset O_W\) with maximum order \(b\). There is an open covering, say \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(W\), and for each index \(\lambda\) a closed and smooth hypersurface \(\overline{W}_\lambda \subset U_\lambda\), and a couple \((K_\lambda^{(0)}, b!\) with \(K_\lambda^{(0)} \subset O_{\overline{W}_\lambda}\), such that, in addition to **P1** and **P2** (7.11), the following property holds:

**P3** The sequence (7.12.1) defined by (7.11.1) as above, induces a sequence of transformations

\[(7.13.1)\]

\[
\begin{array}{cccccccc}
\overline{W}_\lambda & \xleftarrow{\pi_1} & \overline{W}_\lambda^{(1)} & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & \overline{W}_\lambda^{(r)} \\
(K_\lambda, b!) & & ((K_\lambda)_1, b!) & & & & ((K_\lambda)_r, b!)
\end{array}
\]

and

\[(7.13.2)\]

\[
\text{Sing}((J_\lambda)_r, b) = \text{Sing}((K_\lambda)_r, b!)(\subset \overline{W}_\lambda^{(r)}).
\]
Remark 7.14. On the converse. Set \( W = U_\lambda \) so that \((J, b) = (J_\lambda, b)\). The equality in (7.13.4) asserts, by induction on \( r \), that any sequence (7.13.1) induces a sequence (7.11.1). And furthermore, if (7.13.1) is a resolution, so is (7.11.1).

We are interested mainly in this converse, since we will argue by increasing induction on the dimension of the ambient space. If we accept, by induction, that there is a resolution (7.13.1) for each index \( \lambda \), then we will have defined a resolution (7.11.2) for each \( \lambda \). We will define these resolutions so that they patch to a resolution (7.11.1).

Full details of the proof of Proposition 7.13 will be given in Part IV, however the following example illustrates the basic idea of the proof.

Example 7.15. In Example 6.5 we considered the case \( W = \text{Spec}(k[X, Y, Z]) \), and
\[
J = \langle Z^3 + XY^2Z + X^5 \rangle,
\]
an ideal of maximum order \( b = 3 \). In such example we noted that \( Z \in \Delta^2(J) = \langle Z, XY, Y^2, X^3 \rangle \), and we considered the smooth hypersurface \( \overline{W} = V(\langle Z \rangle) \). This is a particular example of Lemma 7.11 where there is no need to consider the open covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( W \). In fact here the Lemma applies globally in \( W \). In this example \( b! = 6 \), and Proposition 7.13 applies by setting \( K = J^* \) as in (3.6.1).

A similar situation holds, more generally, in 3.9, for \( K = J^* = \langle c^*_i, i = 2, 3, \ldots, b \rangle \).

Remark 7.16. The compatibility of the \( \Delta \) operator with open restrictions has played an important role in the proofs in this section. If the transformation in Theorem 7.5 is defined with center \( Y \subset W \), and if \( H \subset W_1 \) denotes the exceptional locus, then \( J_1 \mathcal{O}_{W_1} = I(H)^b \cdot J_1 \), and \( J_1 \) has at most order \( b \). Suppose now that the highest order of \( J \) along points in \( W \) is bigger than \( b \), but that we simply know that the order of \( J \) is constant and equal to \( b \) along any point of the center \( Y \). Since the order of \( J \) along points in \( W \) defines an upper-semi-continuous function on \( W \), then there is an open neighborhood \( U \subset W \) of \( Y \), so that \( b \) is the highest order of the restriction \( J_U \). In particular there is an open neighborhood \( U_1 \) of \( H \) in \( W_1 \) so that the restriction \( (J_1)_{U_1} \) has highest order \( b \).

Remark 7.17. The compatibility of the \( \Delta \) operator with open restrictions will also play a role in our proof of Proposition 7.13 and this will allow us to present the ideals \( K_\lambda \) so that they are also compatible with a restriction of \( W \) to an open set \( U \), at least if the restricted ideal \( J_U \) is again of highest order \( b \).

There is yet another context in which there is a natural compatibility of the operator \( \Delta \), which are not open restrictions, but will also play a role in the proof of Proposition 7.13. In fact, set \( W \leftarrow W_1 = W \times A^1_k \) where \( A^1_k \) denotes the affine line and the map is the projection on the first coordinate. Note that if \( J \) is an ideal in \( \mathcal{O}_W \), and if \( \Delta_1 \) denotes the operator on the smooth scheme \( W_1 \), then
\[
\Delta_1(J\mathcal{O}_{W_1}) = \Delta(J)\mathcal{O}_{W_1}.
\]
Note that a covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( W \) induces by pull-back, a covering of \( W_1 \). The setting of Proposition 7.8 and the inclusions (7.8.2) are compatible with pull-backs; and so are the setting of Proposition 7.9 and the inclusions (7.10.1). This will guarantee the compatibility of all our development for this particular kind of projection.

Part III
8. On how the algorithm works. Examples.

We finally sketch the main ideas and invariants involved in our definition of Resolution Functions in \[4.7\] which lead us to the simple proofs of the Main Theorems in \[4.8\] and \[4.9\]. Recall the notion of permissible sequence of transformations of pairs, say

\[
W_{0} \leftarrow (W_{1}, E_{1} = \{H_{1}\}) \leftarrow \cdots \leftarrow (W_{k}, E_{k} = \{H_{1}, H_{2}, \ldots, H_{k}\}),
\]

in which we require that each monoidal transformation \(W_{i} \leftarrow W_{i+1}\) be defined so that all exceptional hypersurfaces introduced have normal crossings (Prop. \[2.9\]).

Given \(J \subset O_{W}\), there is an expression of the total transform (Def. \[2.8\]), say

\[
J O_{W_{k}} = I(H_{1})^{a_{1}} I(H_{2})^{a_{2}} \cdots I(H_{k})^{a_{k}} \cdot A_{k}.
\]

This factorization is unique if we require the \(a_{i}\) to be the highest possible exponents in any such expression. In \[4.8\] we want to achieve \(A_{k} = O_{W_{k}}\) with the conditions stated in Theorem \[2.3\]. We will argue in steps to achieve the proof of that theorem, each step will introduce an exceptional hypersurface, and this will lead us to consider a pair \((W, E) = \{H_{1}, \ldots, H_{r}\}\), rather than simply \(W\), and also permissible transformations of pairs

\[
(8.0.1) \quad (W, E) \leftarrow (W_{1}, E_{1}) \leftarrow \cdots \leftarrow (W_{k}, E_{k});
\]

always in the conditions of Prop. \[2.9\].

In \[4.3\] we have defined a basic object as a couple \((J, b), J \subset O_{W}\), together with a pair \((W, E)\). A sequence of transformations, say

\[
(8.0.2) \quad (W, (J, b), E) \leftarrow (W_{1}, (J_{1}, b), E_{1}) \leftarrow \cdots \leftarrow (W_{k}, (J_{k}, b), E_{k}),
\]

is a sequence of transformations of couples, say

\[
W \leftarrow W_{1} \leftarrow \cdots \leftarrow W_{k},
\]

\[
(J, b) \leftarrow (J_{1}, b) \leftarrow \cdots \leftarrow (J_{k}, b),
\]

(see \[1.1.2\]), which also defines a sequence of transformations of pairs, as in \[8.0.1\].

We shall say that \(8.0.2\) is a resolution of \((W, (J, b), E)\) if \(V(\Delta^{j-1}(J_{k})) = \emptyset\). Note that \(V(\Delta^{j-1}(J_{k})) = \emptyset\) is equivalent to \(Sing(J_{k}, b) = \emptyset\), and also to the condition \(\max -ord J_{k} < b\).

So the resolution would provide an expression of the form:

\[
J O_{W_{k}} = I(H_{1})^{a_{1}} I(H_{2})^{a_{2}} \cdots I(H_{k})^{a_{k}} \cdot J_{k}, \text{ and max } -ord J_{k} = b' < b.
\]

If \(b' = 0\) we have achieved what is stated in Theorem \[2.3\]. If not we repeat the argument, and try to produce a resolution of \((J_{k}, b')\) and \((W_{k}, E_{k})\). It is clear that ultimately we come to the case \(b' = 0\).

Our task is to produce a resolution of \((J, b)\) and \((W, E)\), in some explicit manner, in which centers of monoidal transformations are defined by an upper-semi-continuous function. In some particular cases this will be clear from the data involved (see \[8.3\]). But, in general, the strategy will be to reduce to the case in which \(b = \max -ord J\), namely to the case of simple couples \[7.4\].

In case \(b = \max -ord J\), Theorem \[7.13\] says that there is \(\mathbb{W} \subset W\), at least locally, and that \(8.0.2\) induces

\[
(8.0.3) \quad (\mathbb{W}, (K, d), E = \emptyset) \leftarrow (\mathbb{W}_{1}, (K_{1}, d), E_{1}) \leftarrow \cdots \leftarrow (\mathbb{W}_{k}, (K_{k}, d), E_{k})
\]
such that \(V(\Delta^{d-1}(K_k)) = \emptyset\). It is important to point out here that we will argue by induction, and hence we would like to reverse the argument; namely, to define (8.0.2) in terms of (8.0.3). We now indicate the difficulties to overcome.

**The three difficulties for an inductive argument:**

**D1** \((K, d)\) encodes information of \((J, b)\), but not of the set of hypersurfaces \(E\) in \(W\). Theorem [8.13] asserts that, after restriction to an open subset of \(W\), \([8.03]\) will define a sequence of transformations of couples, say

\[
\begin{align*}
W &\leftarrow W_1 &\leftarrow \cdots &\leftarrow W_k \\
(J, b) &\leftarrow (J_1, b) &\leftarrow \cdots &\leftarrow (J_k, b),
\end{align*}
\]

such that \(V(\Delta^{b-1}(J_k)) = \emptyset\). However this sequence might not define a sequence \([8.02]\). In fact, it might not be permissible over \((W, E)\) because of the presence of hypersurfaces of \(E\).

This is an important point to overcome. As indicated above, since we will argue in steps, we introduce hypersurfaces with normal crossings (those in \(E\)), and we want to preserve this condition of normal crossings in all exceptional hypersurfaces to be introduced in forthcoming steps.

**D2** The couple \((K, d)\) might not be a simple couple (might not be such that \(d = \max -ord_K\)). Take for example the case \(J = (z^3 - x^2 \cdot y^2)\) and the couple \((J, 3)\) in the affine 3-space. Clearly \(3 = \max -ord_J\) so the couple is simple. Since \(z \in \Delta^2(J)\), we may take \(\overline{W}\) as the affine plane, and \((K, d) = ((x^2 \cdot y^2), 3)\). Note that \(\max -ord_K = 4\), so that \((K, d)\) is not a simple couple [7.4].

**D3** If \((J, b)\) is a simple couple (i.e. if \(\max -ord_J = b\)), then \(\overline{W}\) is defined by choosing, locally at a point \(x \in V(\Delta^{b-1}(J))\), an element of order one in \(\Delta^{b-1}(J)_x\). In general this choice is not unique, and the definition of \((K, d)\) \((K \subset \mathcal{O}_{\overline{W}})\) is local at \(x\). Our form of induction should provide a resolution \([8.02]\) with independence of open restrictions and of choices of \(\overline{W}\).

8.1. Set \(J \subset \mathcal{O}_W\) and \((W, E)\) as before. Assume, in accordance with D2), that \(b \geq \max -ord_J\). So here \((J, b)\) might not be simple. Consider a sequence of transformations, say:

\[
(W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (W_s, (J_s, b), E_s).
\]

We claim that this provides a factorization of \(J_s\), say

\[
J_s = I(H_1)^{b_1} \cdot I(H_2)^{b_2} \cdots I(H_s)^{b_s} \cdot \overline{J}_s
\]

so that \(\overline{J}_s\) does not vanish along \(H_i\), \(1 \leq i \leq s\). In this manner we may consider \((J_s, b)\), together with this factorization of \(J_s\). This extra structure on \((J_s, b)\) will allow us to overcome D2), namely to reduce the general case to the case of simple couples.

**Example 8.2.** Set \(J = \langle x_1, x_2^2 >^4, W = \mathbb{A}_k^2\)

\[
(W, (J, 3), E = \emptyset) \leftarrow (W_1, (J_1, 3), E_1 = \{H_1\}) \leftarrow (W_2, (J_2, 3), E_2 = \{H_1, H_2\})
\]

\[
\begin{align*}
J_{\mathcal{O}_{W_1}} &= I(H_1)^4 \cdot \mathcal{M}_p^4 \\
J_1 &= I(H_1)^2 \cdot \mathcal{M}_p^4 \\
J_2 &= I(H_1)^2 \cdot I(H_2)^3 \\
J_3 &= I(H_1)^2 \cdot I(H_2)^4.
\end{align*}
\]

Here \(W \leftarrow W_1\) is the blow-up at \(0 \in \mathbb{A}_k^2, p \in W_1\) is a point in the exceptional line \(H_1, \mathcal{M}_p\) is the sheaf of functions that vanish at \(p\), and finally \(W_1 \leftarrow W_2\) is the blow-up at \(p\).
Remark 8.3. If $\mathcal{T}_s = \mathcal{O}_{W_s}$, we say that $(J_s, b)$ is within the monomial case. In this case it is easy to extend \[8.1.1\] to a resolution; namely, to define for some $k \geq s$:

\[(8.3.1) \quad (W, (J, b), E) \leftarrow \cdots \leftarrow (W_s, (J_s, b), E_s) \leftarrow \cdots \leftarrow (W_k, (J_k, b), E_k)\]

so that $V(\Delta^{b-1}(J_k)) = \emptyset$. The following example illustrates this fact. Note that in the previous example $\mathcal{T}_2 = \mathcal{O}_{W_2}$.

Example 8.4. Consider transformations with centers $Y_j$:

\[
(W_2, (J_2, 3), E_2 = \{H_1, H_2\}) \xleftarrow{id} (W_3, (J_3, 3), E_3 = \{H_1, H_2\}) \leftarrow (W_4, (J_4, 3), E_4 = \{H_1, H_2, H_4\})
\]

$J_2 = I(H_1)^2 \cdot I(H_2)^3$ \quad $J_3 = I(H_1)^2 \cdot I(H_2)$ \quad $J_3 = I(H_1)^2 \cdot I(H_4) \cdot I(H_2)$

$Y_2 = H_2$ \quad $Y_3 = H_1 \cap H_2$ \quad $V(\Delta^2(J_4)) = \emptyset$.

The first transformation is defined with center at the hypersurface $H_2$. So the morphism is the identity map, but we take here $H_2 \in E_2$ to be the exceptional locus. Note that $J_3$ is not $J_2$.

8.5. On the function $v$-ord.

Given a sequence of transformations of basic objects, say \[8.1.1\], we have defined an expression:

\[J_s = I(H_1)^{b_1} I(H_2)^{b_2} \cdots I(H_s)^{b_s} \cdot \mathcal{T}_s\]

so that $\mathcal{T}_s$ does not vanish along $H_i$, $1 \leq i \leq s$. Define now:

\[v$-ord$_s : V(\Delta^{b-1}(J_s)) \rightarrow \mathbb{N}\]

\[v$-ord$_s(x) = \nu_x(\mathcal{T}_s), \text{ (the order of } (\mathcal{T}_s)_x \text{ at } \mathcal{O}_{W_s,x})\]

Note that:

1) The function is upper-semi-continuous. In particular $\text{Max } v$-ord is closed.

2) For any index $i \leq s$, there is an expression

\[J_i = I(H_1)^{b_i} I(H_2)^{b_2} \cdots I(H_i)^{b_i} \cdot \mathcal{T}_i,\]

and hence a function $v$-ord$_i : V(\Delta^{b-1}(J_i)) \rightarrow \mathbb{N}$ can be defined.

Another property of these functions is:

3) If each step $(W_i, E_i) \leftarrow (W_{i+1}, E_{i+1})$ in \[8.1.1\] is defined with center $Y_i \in \text{Max } v$-ord$_i$, then

\[\text{max } v$-ord $\geq \text{max } v$-ord$_1 \geq \cdots \geq \text{max } v$-ord$_s.\]

Property 3) follows from the fact that, if $\text{max } v$-ord$_s = b'$, then $\text{Max } v$-ord$_s = V(\Delta^{b'-1}(\mathcal{T}_s))$ (the closed set of $\langle \mathcal{T}_s, b' \rangle$), where $\langle \mathcal{T}_s, b' \rangle$ is, by definition, a simple couple.

Example 8.6. Set $(J, 1)$; $J = x^2 - y^5$, and $W = \mathbb{A}^2_k$. Let $C$ denote the curve defined by $J$.

\[\begin{align*}
(W, (J, 1), E = \emptyset) & \leftarrow \cdots \leftarrow (W_1, (J_1), E_1 = \{H_1\}) & \leftarrow \cdots & \leftarrow (W_3, (J_3, 1), E_3) \\
J_1 = I(H_1) \cdot I(C') & \leftarrow J_2 = I(H_1)^1 \cdot I(H_2)^1 \cdot I(C''') & \leftarrow & \leftarrow J_3 = I(H_1)^1 \cdot I(H_2)^1 \cdot I(H_3)^2 \cdot I(C''') \\
Y_0 = 0 \in \mathbb{A}^2_k & \quad Y_1 = C' \cap H_1 & \quad Y_2 = H_1 \cap H_2.
\end{align*}\]
Here the $Y_i$ are the centers of the monoidal transformations, and $C', C''$, and $C'''$ are strict transforms of $C$. In this example
\[
\max v\text{-ord}_J = 2; \quad \max v\text{-ord}_{J_1} = 1; \quad \max v\text{-ord}_{J_2} = 1; \quad \max v\text{-ord}_{J_3} = 1;
\]
and the sequence is defined by setting
\[
Y = \max v\text{-ord}_J = Y; \quad Y_1 = \max v\text{-ord}_{J_1}, \quad \text{and} \quad Y_2 = \max v\text{-ord}_{J_2}.
\]

### 8.7. On the inductive function $t$

Consider, as before, a sequence
\[
(8.7.1) \quad (W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (W_s, (J_s, b), E_s),
\]
where each $W_i \leftarrow W_{i+1}$ is defined with center $Y_i \subset \max v\text{-ord}_i$, so that:
\[
\max v\text{-ord} \geq \max v\text{-ord}_1 \geq \cdots \geq \max v\text{-ord}_s.
\]
Set $s_0 \leq s$ such that
\[
\max v\text{-ord} \geq \cdots \geq \max v\text{-ord}_{s_0-1} > \max v\text{-ord}_{s_0} = \max v\text{-ord}_{s_0+1} = \cdots = \max v\text{-ord}_s,
\]
and
\[
E_s = E_s^+ \sqcup E_s^- \quad \text{(disjoint union)},
\]
where $E_s^-$ are the strict transform of hypersurfaces in $E_{s_0}$. Define
\[
t_s : V(\Delta^{b-1}(J_s)) \to \mathbb{N} \times \mathbb{N} \quad \text{(ordered lexicographically).}
\]
\[
t_s(x) = (v\text{-ord}_s(x), n_s(x))
\]
\[
n_s(x) = \sharp \{H_i \in E_s^-, x \in H_i\}
\]

One can check that:
1) the function is upper-semi-continuous. In particular $\max t_s$ is closed.
2) There is a function $t_i$ for any index $i \leq s$.

Example 8.8 illustrates the following properties which also hold for this function:

- If each $(W_i, E_i) \leftarrow (W_{i+1}, E_{i+1})$ in (8.7.1) is defined with center $Y_i \subset \max t_i$, then
\[
\max t \geq \max t_1 \geq \cdots \geq \max t_s.
\]
- If $\max t_s = (b', r)$ (here $\max v\text{-ord}_s = b'$) then $\max t_s \subset \max v\text{-ord}_s$.
- If $\max t_s$ has codimension 1 in $W_s$, then it is smooth. Moreover, in such case $Y_s = \max t_s$ is a permissible center, defining
\[
(W_s, (J_s, b), E_s) \leftarrow (W_{s+1}, (J_{s+1}, b), E_{s+1}),
\]
and $\max t_s > \max t_{s+1}$ (hence $\max v\text{-ord}_s \geq \max v\text{-ord}_{s+1}$).

#### Example 8.8.

0) Consider $(J, 1); \quad J = \langle x^2 - y^3 \rangle$ defining a curve $C \subset W = \mathbb{A}_k^2$.

Here $t(x) = (1, 0)$ at any $x \in C$ except at $\bar{0} \in \mathbb{A}_k^2, t(\bar{0}) = (2, 0)$. So
\[
\max t = (2, 0) \quad \text{and} \quad \max t = \bar{0} \in \mathbb{A}_k^2.
\]
Let now
\[
(W, (J, 1), E = \emptyset) \leftarrow (W_1, (J_1, 1), E_1 = \{H_1\})
\]
be the quadratic transformation at $\bar{0} \in \mathbb{A}_k^2$. 
1) Let \( C' \subset W_1 \) denote the strict transform of \( C \). Here
\[
J_1 = I(H_1) \cdot \overline{J}_1
\]
where \( \overline{J}_1 = I(C') \), and \( t_1(x) = (1, 0) \) at any \( x \in C' \) except for \( p = C' \cap H_1 \), where \( t_1(p) = (1, 1) \). So \( \max t_1 = (1, 1) \) and \( \text{Max} t_1 = p \).

Set
\[
(W_1, (J_1, 1), E_1) \leftarrow (W_2, (J_2, 1), E_2 = \{H_1, H_2\})
\]
with center at \( p \in W_1 \).

2) If \( C'' \subset W_2 \) denotes the strict transform of \( C \),
\[
J_2 = I(H_1) \cdot I(H_2) \cdot \overline{J}_2
\]
where \( \overline{J}_2 = I(C'') \).

Now \( t_2(x) = (1, 0) \) at any \( x \in C'' \) except for \( q = C'' \cap H_1 \cap H_2 \), where \( t_2(q) = (1, 1) \). So \( \max t_2 = (1, 1) \) and \( \text{Max} t_2 = q \).

Set
\[
(W_2, (J_2, 1), E_2 = \{H_1, H_2\}) \leftarrow (W_3, (J_3, 1), E_3 = \{H_1, H_2, H_3\})
\]
with center at \( q \in W_2 \).

3) Now
\[
J_3 = I(H_1) \cdot I(H_2) \cdot I(H_3)^2 \cdot \overline{J}_3
\]
where \( \overline{J}_3 = I(C''') \) (ideal of the strict transform). Finally check that \( t_3(x) = (1, 0) \) at any \( x \in C''' \). So \( \max t_3 = (1, 0) \) and \( \text{Max} t_3 = C''' \).

This is a case in which \( \text{Max} t \) has codimension 1. Note that \( \text{Max} t_3 \) is a smooth hypersurface, and the blow-up at \( \text{Max} t_3 \) defines a permissible transformation (the identity map):
\[
(W_3, (J_3, 1), E_3 = \{H_1, H_2, H_3\}) \leftarrow (W_3, (J_4, 1), E_3 = \{H_1, H_2, H_3\})
\]
with \( J_4 = I(H_1) \cdot I(H_2) \cdot I(H_3)^2 \).

8.9. Overcoming difficulties D1) and D2)

We finally indicate a further property of the function \( t_s \), which leads to constructive desingularization by induction. To this end set:

\[
(W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (W_s, (J_s, b), E_s)
\]
so that
\[
\max v\text{-ord} \geq \max v\text{-ord}_1 \geq \cdots \geq \max v\text{-ord}_s .
\]

And define, as before, the function
\[
t_s : V(\Delta^{b-1}(J_s)) \rightarrow \mathbb{N} \times \mathbb{N}.
\]

This last property can be stated as follows:

There is a couple \( (J''_s, b'') \) with the following properties:

- \( V(\Delta^{b''-1}(J'')) = \text{Max} t_s \), and \( \max ord_{J''} = b'' \) (i.e. the couple is a simple couple).
Let \( \overline{W}_s \) be a smooth hypersurface containing \( V(\Delta^{d-1}(J^{m})) \), and set \((K, d) (K \subset \mathcal{O}_{\overline{W}})\) as in Proposition 7.13. Then any resolution, say:

\[
(\text{8.9.2}) \quad (\overline{W}_s, (K, d), E_s = \emptyset) \leftarrow (\overline{W}_1, (K_1, d), E_{s+1}) \leftarrow \cdots \leftarrow (\overline{W}_k, (K_k, d), E_{s+k})
\]

\((V(\Delta^{d-1}(K_k)) = \emptyset)\), induces an extension of \((\text{8.9.1}), \) say:

\[
(\text{8.9.3}) \quad (W_s, (J_s, b), E_s) \leftarrow (W_{s+1}, (J_{s+1}, b), E_{s+1}) \leftarrow \cdots \leftarrow (W_{s+k}, (J_{s+k}, b), E_{s+k}),
\]

such that

\[
\max t_s = \max t_{s+1} = \cdots = \max t_{s+k-1} > \max t_{s+k}.
\]

Furthermore

\[
\text{Max } t_{s+i} = V(\Delta^{d-1}(K_i)) = \text{Sing}(K_i, d)
\]

for \( i = 0, \cdots, k - 1 \).

**8.10. Example of constructive resolution.**

**Example 8.11.** The curve \( C \) defined by \( J = \ll x^2 - y^5 > \) in \( W = \mathbb{A}^2_k \) is irreducible, in particular reduced. We attach to it the basic object

\[
(W, (J, 1), E = \emptyset),
\]

and the function

\[
t : V(\Delta^0(J)) = V(J) \rightarrow \mathbb{N} \times \mathbb{N}.
\]

Here \( t(x) = (1, 0) \) except at the origin \( \overline{0} \in \mathbb{A}^2_k, t(\overline{0}) = (2, 0) \).

Note that in Example 8.6

- \( \max t = (2, 0) \) and \( Y = \text{Max } t = \overline{0}; \)
- \( \max t_1 = (1, 1) \) and \( Y_1 = \text{Max } t_1; \)
- \( \max t_2 = (1, 1) \) and \( Y_2 = \text{Max } t_2; \)
- \( \max t_3 = (1, 0), \text{Max } t_3 = C^m, \) is a smooth hypersurface (see 8.9). Thus, this defines an embedded desingularization.

Compare with the proof of Theorem 2.2.

**Example 8.12.** The hypersurface \( Z^2 + X^2 + Y^3 = 0 \) is irreducible with an isolated singularity at \( \overline{0} \in \mathbb{A}^3_k \). Set \( W = \mathbb{A}^2_k, J = \ll Z^2 + X^2 + Y^3 > \). According to the proof of Theorem 2.2 in 4.9 desingularization is achieved at some intermediate step of the resolution of the basic object:

\[
(W, (J, 1), E = \emptyset).
\]

The function \( t : V(J) \rightarrow \mathbb{N} \times \mathbb{N} \) takes value \( t(x) = (1, 0) \) except at the singular point, \( t(\overline{0}) = (2, 0) \). In this case, and following the notation in 8.9

- \( \max t = (2, 0); \)
- \( (J^m, b^m) \) can be defined as \( (J, 2); \)
- \( \overline{W} = V(\ll Z \rr) \) (in fact \( Z \in \Delta^1(J) \).
- \( (K, d) \) can be defined by \( \ll X^2 + Y^3 >, 2). \)

Here \( \overline{W} = \mathbb{A}^2_k \), and the blow-up at \( \overline{0} \in \mathbb{A}^2_k \) defines a resolution, namely

\[
(W, (K, 2), E = \emptyset) \leftarrow (W_1, (K_1, 2), E_1 = \{H_1\})
\]

and \( V(\Delta(K_1)) = \emptyset \). According to 8.9 this defines

\[
(W, (J, 1), E = \emptyset) \leftarrow (W_1, (J_1, 1), E_1 = \{H_1\}) \quad \text{and} \quad \max t > \max t_1.
\]
In fact \( \max t_1 = (1, 1) \). So again, we argue as in \( \text{[8.1.3]} \) and attach a couple \((J''', b''')\) to the value \( \max t_1 = (1, 1) \). Moreover, a smooth hypersurface \( \overline{W} \) and a couple \((K, d)\) can be defined so that a resolution, say:

\[(8.12.1) \quad (\overline{W}, (K, d), \overline{E}_s = \emptyset) \leftarrow (\overline{W}_1, (K_1, d)\overline{E}_{s+1}) \leftarrow \cdots \leftarrow (\overline{W}_k, (K_k, d), \overline{E}_{s+k}) \]

(such that \( V(\Delta^{d-1}(K_k)) = \emptyset \)), induces:

\[(8.12.2) \quad (W_1, (J_1, 1), E_1) \leftarrow (W_2, (J_2, 1), E_2) \leftarrow \cdots \leftarrow (W_s, (J_s, 1), E_s) \]

such that

\[(1, 1) = \max t_1 = \max t_2 = \cdots = \max t_{s-1} > \max t_s = (1, 0).\]

Note that \( J_s \) is the sheaf of ideals of the strict transform of the hypersurface, that \( \text{Max} \ t_s = V(J_s) \). So \( \text{Max} \ t_s \) is a hypersurface, and the last property in \( \text{[8.7]} \) says that this is an embedded desingularization.

**Part IV**

In this Part we will address constructive resolution in detail. Part III was devoted to give an overview of the invariants involved, and examples of constructive resolution. This last Part IV can be read independently of the previous one, so we will introduce all invariants, and prove resolution theorems in full generality.

### 9. TCHIRNHAUSEN REVISITED.

The objective of this Section is to prove Proposition \( \text{[7.13]} \) (see also \( \text{[9.3]} \)), which is the form of induction that leads to resolution. This form of induction is that suggested by the examples in Section \( \text{[8]} \).

In Example \( \text{[3.6]} \) we treated a case of a simple basic object where \( W = A^3 \), and \( b = 3 \). There the covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) is trivial (i.e. \( U_\lambda = W \)), and \( Z \in \Delta^{(2)}(J) \) defines a smooth hypersurface \( \overline{W} = V(\{ < Z > \})(\subset W) \). Moreover, in that example the couple \((J^*, b)\) \(( J^* \subset \mathcal{O}_W \text{ in } \text{[3.6.1]} \) plays the role of \((K_\lambda, b!)\) with property P3) in Proposition \( \text{[7.13]} \) to be defined in \( \text{[9.3]} \).

**Remark 9.1.** We will assume here that the setting of Remark \( \text{[7.10]} \) holds for \( U_\lambda = W \), but in a more general form, where the role of the smooth hypersurface \( \overline{W} \) is played now by an arbitrary smooth subscheme, say \( Z \subset W \). In other words, assume that \( b \) is the highest order of \( J \subset \mathcal{O}_W \), and that for any sequence of transformations of couples, say

\[(9.1.1) \quad W \leftarrow W^{(1)} \leftarrow \cdots \leftarrow W^{(r)} \]

then

\[(9.1.2) \quad \text{Sing}((J)_r, b) \subset Z^{(r)},\]

where \( Z^{(r)} \) is the smooth subscheme in \( W^{(r)} \) defined by the strict transform of \( Z \).

Note, in particular, that \( \text{[9.1.1]} \) induces a sequence of monoidal transformations

\[(9.1.3) \quad Z \leftarrow Z^{(1)} \leftarrow \cdots \leftarrow Z^{(r)}.\]

Let \( \xi \in Z \) be a closed point, and let \( \{z_1, \ldots, z_r, x_1, \ldots, x_n\} \) be a regular system of parameters in \( \mathcal{O}_{W, \xi} \) such that \( \mathcal{I}(Z)_\xi = (z_1, \ldots, z_r) \). Consider the isomorphisms

\[\hat{\mathcal{O}}_{W, \xi} \cong k(\xi)[[z_1, \ldots, z_r, x_1, \ldots, x_n]], \quad \hat{\mathcal{O}}_{Z, \xi} \cong k(\xi)[[x_1, \ldots, x_n]],\]
where the right hand sides are rings of formal series. Given \( f \in \mathcal{O}_{W, \xi} \), let \( \hat{f} \) denote the image at \( \hat{O}_{W, \xi} \), say:

\[
\hat{f} = \sum_{i_1, \ldots, i_r = 0}^{\infty} a_{i_1, \ldots, i_r} x_1^{i_1} \cdots x_r^{i_r},
\]

where each \( a_{i_1, \ldots, i_r} \in k(\xi)[[x_1, \ldots, x_n]] \). Note that

\[
(i_1! \cdots i_r!) a_{i_1, \ldots, i_r} = \varphi \left( \frac{\partial^{i_1 + \cdots + i_r} f}{\partial z_1^{i_1} \cdots \partial z_r^{i_r}} \right),
\]

where \( \varphi : k(\xi)[[z_1, \ldots, z_r, x_1, \ldots, x_n]] \rightarrow k(\xi)[[x_1, \ldots, x_n]] \) is the quotient map induced by the inclusion \( Z \subset W \) at \( \xi \). Note also that, for a fixed integer \( b \),

\[
\nu_{\xi}(f) \geq b \iff \nu_{\xi}(a_{i_1, \ldots, i_r}) \geq b - (i_1 + \cdots + i_r),
\]

for all \( i_1, \ldots, i_r \) with \( 0 \leq i_1 + \cdots + i_r < b \) (here the left hand side is the order at \( \mathcal{O}_{W, \xi} \), and the right hand side is the order at \( \mathcal{O}_{Z, \xi} \)). Set now

\[
I(f, b) = \{(a_{i_1, \ldots, i_r}) \in \mathbb{Z}^{i_1 + \cdots + i_r} : 0 \leq i_1 + \cdots + i_r < b\}
\]

and reformulate (9.1.5) by means of the equivalence

\[
\nu_{\xi}(f) \geq b \iff \nu_{\xi}(I(f, b)) \geq b!.
\]

**Lemma 9.2.** Assume now that:

1) \( f \) and \( \{z_1, \ldots, z_r, x_1, \ldots, y_n\} \) are global sections of \( \mathcal{O}_W \) and \( J = \langle f \rangle \),

2) the sheaf of differentials \( \Omega_W \) is free with basis \( \{d(z_1), \ldots, d(z_r), d(y_1), \ldots, d(y_n)\} \),

3) \( I(Z) = \langle z_1, \ldots, z_r \rangle \) and \( Z \) fulfills the property expressed in (9.1.2) for \( \langle f, b \rangle \).

Then there is a couple \( (J^* , d) \), with \( J^* \subset \mathcal{O}_Z \), such that any sequence of transformations of \( (J, b) \), say (9.1.4), induces a sequence of transformations for the couple \( (J^*, d) \), say

\[
(J^*, d) \leftarrow (J_1^*, d) \leftarrow \cdots \leftarrow (J_r^*, d),
\]

and

\[
\text{Sing}((J)_r, b) = \text{Sing}((J^*)_r, d) \subset Z(r)
\]

Conversely, any sequence (9.2.7) induces a sequence (9.1.1).

**Proof.** For the converse stated in the last line see (9.1.4).

If \( g \) is a global section in \( \mathcal{O}_W \), let \( \overline{g} \) denote the class in \( \mathcal{O}_Z \). Set

\[
J^* = \left\{ \left( \frac{1}{i_1! \cdots i_r!} \left( \frac{\partial^{i_1 + \cdots + i_r} f}{\partial z_1^{i_1} \cdots \partial z_r^{i_r}} \right) \right)_{i_1 + \cdots + i_r < b} \right\}
\]

Fix a closed point \( \xi \in Z \) with residue field \( k(\xi) \), and let \( a_i \in k(\xi) \) denote the class of \( y_i \) at the point. Set \( \{z_1, \ldots, z_r, x_1, \ldots, x_n\} \subset \hat{\mathcal{O}}_{W, \xi} \), where \( x_i = y_i - a_i \). Note also that, despite the change of coordinates, the global derivations \( \frac{\partial}{\partial z_i} \) (defined in terms of \( \{z_1, \ldots, z_r, y_1, \ldots, y_n\} \)) induces the derivation \( \frac{\partial}{\partial z_i} \) on \( \hat{\mathcal{O}}_{W, \xi} \) defined in terms of \( \{z_1, \ldots, z_r, x_1, \ldots, x_n\} \).

It follows now from (9.1.4) and (9.1.7) that, setting \( d = b! \), \( \text{Sing}(J, b) = \text{Sing}(J^*, d) \subset Z \).
Fix a closed point $\xi_r \in Z^{(r)}$, and set $\xi_k$ as the image of $\xi_r$ in $Z^{(k)}$. In particular, $\xi_0 \in Z^{(0)} = Z$.

We may assume, by induction, that:

1) there is a regular system of parameters $\{z_{k-1,1}, \ldots, z_{k-1,r}, x_{k-1,1}, \ldots, x_{k-1,n}\}$ at $\hat{O}_{W_{k-1}, \xi_{k-1}} \cong R_{k-1} = k(\xi_{k-1})[[z_{k-1,1}, \ldots, z_{k-1,r}, x_{k-1,1}, \ldots, x_{k-1,n}]]$,

2) $I(Z^{(k-1)}) = \langle z_{k-1,1}, \ldots, z_{k-1,r} \rangle$, and

3) there is a generator $\hat{f}_{k-1}$ of $(J)_{k-1}$, together with an expression:

$$\hat{f}_{k-1} = \sum_{i_1, \ldots, i_r = 0}^{\infty} a_{k-1,i_1,\ldots,i_r} z_{k-1,1}^{i_1} \cdots z_{k-1,r}^{i_r},$$

with $a_{k-1,i_1,\ldots,i_r} \in k(\xi_{k-1})[[z_{k-1,1}, \ldots, z_{k-1,r}, x_{k-1,1}, \ldots, x_{k-1,n}]]$. This particular kind of change of coordinates in $\hat{O}_{W_{k-1}, \xi_{k-1}}$ fixes the ideal in 2), and modifies the expression in 3) by changing each coefficient $a_{k-1,i_1,\ldots,i_r} \in k(\xi_{k-1})[[x_{k-1,1}, \ldots, x_{k-1,n}]]$.

A change of coordinates in the subring $R_{k-1}$, extends to a change of coordinates at $\hat{O}_{W_{k-1}, \xi_{k-1}}$ by fixing $\{z_{k-1,1}, \ldots, z_{k-1,r}\}$.

This particular kind of change of coordinates in $\hat{O}_{W_{k-1}, \xi_{k-1}}$ fixes the ideal in 2), and modifies the expression in 3) by changing each coefficient $a_{k-1,i_1,\ldots,i_r} \in k(\xi_{k-1})[[x_{k-1,1}, \ldots, x_{k-1,n}]]$.

Note here that $\hat{O}_{Z_{k-1}, \xi_{k-1}}$ is compatible with our definition of the ideal $I(\hat{f}_{k-1}, b)$, defined in terms of expression 3). The point is that, after enlarging $k(\xi_{k-1})$ to $k(\xi_k)$, and taking a suitable change of coordinates as before, we may choose

1') coordinates $\{z_{k,1}, \ldots, z_{k,r}, x_{k,1}, \ldots, x_{k,n}\}$ in $\hat{O}_{W_{k}, \xi_{k}}$ with

$$z_{k,i} = \frac{z_{k-1,i}}{x_{k-1,1}}, \quad i = 1, \ldots, r, \quad x_{k,1} = x_{k-1,1}, \quad x_{k,i} = \frac{x_{k-1,i}}{x_{k-1,1}}, \quad i = 1, \ldots, n,$$

so that

2') $I(Z_{k})_{\xi_{k}} = \langle z_{k,1}, \ldots, z_{k,r} \rangle$. Set the expression

$$\hat{f}_{k} = \frac{f_{k-1}}{x_{k-1,1}} = \sum_{i_1, \ldots, i_r = 0}^{\infty} a_{k,i_1,\ldots,i_r} z_{k,1}^{i_1} \cdots z_{k,r}^{i_r}, \quad \text{where} \quad a_{k,i_1,\ldots,i_r} = \frac{a_{k-1,i_1,\ldots,i_r}}{x_{k-1,1}^{i_1 + \cdots + i_r}}.$$

Note here that $\hat{f}_{k}$ is a generator of $(J)_{k}$. Furthermore, since

$$\left(a_{k,i_1,\ldots,i_r}ight)^{\frac{b}{i_1 + \cdots + i_r}} = \left(a_{k-1,i_1,\ldots,i_r}ight)^{\frac{b}{i_1 + \cdots + i_r} \left(x_{k-1,1}^{i_1 + \cdots + i_r}\right)^{-1}}$$

it follows that the transform of the couple $(I(\hat{f}_{k-1}, b), b)$ $(I(\hat{f}_{k-1}, b) \subset \hat{O}_{Z_{k-1}, \xi_{k-1}})$ is $I(\hat{f}_{k}, b) \subset \hat{O}_{Z_{k}, \xi_{k}}$ and the Lemma follows now by (9.1.6).

9.3. **Proof of 7.13** We first consider a covering $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of $W$, so that there is a closed and smooth hypersurface $W_{\lambda} \subset U_{\lambda}$, and $I(W_{\lambda}) \subset (\Delta^{b-1}(J))_{\lambda}$ as in 7.10. After suitable refinement we may assume that, for each $\lambda$, the conditions of Lemma 9.2 hold for $Z = W_{\lambda}$, and $J = \langle f_j \rangle$ in 7.10 where $\{f_1, \ldots, f_j, \ldots, f_l\}$ are global sections in $O_{U_{\lambda}}$ that span $J_{U_{\lambda}} = \langle f_1, \ldots, f_l \rangle$. 
Finally, one can check that a couple \((K^{(0)}_\lambda, b)\), with property P3 in Proposition 7.13 is defined by setting:

\[
K^{(0)}_\lambda = \left\{ \left( \frac{1}{i_1! \cdots i_r!} \frac{\partial^{i_1 + \cdots + i_r} f_j}{\partial z_1^{i_1} \cdots \partial z_r^{i_r}} \right) \right\} / 0 \leq i_1 + \cdots + i_r < b ; \ j = 1, \ldots, l. \]

10. On resolution functions I.

10.1. In this, and in the next Section 11 we show that resolution of basic objects can be achieved once we know how to define resolution for a simple class of basic objects.

Definition 10.2. We will say that a basic object \((W, (J, b), E)\) is a simple basic object, if \((J, b)\) is a simple couple \((7.4)\), \(J \neq 0(\subseteq \mathcal{O}_W)\), and \(E = \emptyset\) (or, more generally, if \(H_i \cap \text{Sing}(J, b) = \emptyset\) for any \(H_i \subseteq E\)).

The following result indicates the relevance of simple basic objects for inductive arguments.

Proposition 10.3. Fix a simple basic object \((W, (J, b), E)\). Set \(\text{Sing}(J, b) = \bigcup_{1 \leq i \leq s} Z_i\), where each \(Z_i\) denotes an irreducible component of this closed set, and let \(R(1)(\text{Sing}(J, b))\) be the union of those \(Z_i\) of codimension one in \(W\). Then

a) \(R(1)(\text{Sing}(J, b))\) is open and closed in \(\text{Spec}(J, b)\) (i.e. a union of connected components), and it is a closed and smooth hypersurface in \(W\). Moreover, no other component of \(\text{Sing}(J, b)\) meets \(R(1)(\text{Sing}(J, b))\).

b) If \((W, (J, b), E) \rightarrow (W_1, (J_1, b), E_1)\) is defined with center \(R(1)(\text{Sing}(J, b))\), then \(W_1 = W\) and \(\text{Sing}(J_1, b) = \text{Sing}(J, b) - R(1)(\text{Sing}(J, b))\).

In particular \((W_1, (J_1, b), E_1)\) is a simple basic object and \(R(1)(\text{Sing}(J_1, b)) = \emptyset\).

Proof. a) If \(Z_1\) is of codimension one, and if \(x \in Z_1 \cap Z_i\) for some other component \(Z_i\), then \(\text{Spec}(J, b)\) cannot be included in a smooth hypersurface locally at \(x\), in contradiction with property P1) in 7.11.

The same property insures that \(R(1)(\text{Sing}(J, b))\) is regular.

b) The blow-up on a hypersurface is the identity map, so \(W_1 = W\). The second assertion follows from property P2) in 7.11. In fact, locally at a point \(x \in R(1)(\text{Sing}(J, b))\) there is a smooth hypersurface \(\overline{W}\), such that locally at \(x \in W\), \(\text{Sing}(J, b) = \overline{W}\). Moreover, locally at \(x \in W_1 = W\), \(\text{Sing}(J_1, b) \subset \overline{W}_1\), where \(\overline{W}_1\) is the strict transform of \(\overline{W}\) by blowing up at the center \(\overline{W}\). So \(\overline{W}_1\), and hence \(\text{Sing}(J_1, b)\), are empty locally at \(x\).

10.4. Resolution functions and the principle of Patching. If \((W, (J, b), E = \emptyset)\) is a simple basic object, Proposition 10.3 says that, after blowing-up at the center \(R(1)\text{Sing}(J, b)\), we may assume that \(R(1)\text{Sing}(J, b) = \emptyset\); and point is that in this setting we can profit of the form of induction on the dimension \(d\) in Proposition 7.13. In fact, if the simple basic object is such that \(R(1)\text{Sing}(J, b) = \emptyset\), then there is a covering \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(W\), and for each index \(\lambda\) a \(d - 1\) dimensional basic object \(E^{d-1}_\lambda = (\overline{W}_\lambda, (K^{(0)}_\lambda, b), \emptyset)\), such that \(\text{Sing}(J, b) \cap U_\lambda = \text{Sing}(K^{(0)}_\lambda)\). Note that \(R(1)\text{Sing}(J, b) = \emptyset\) asserts that the ideals \(K^{(0)}_\lambda\) are non-zero; a condition required in our definition of basic object.

Assume that \((I^{d-1}_\beta, \geq)\) has been defined, together with the functions defining, as in 7.11 resolutions of \(d - 1\) dimensional basic objects. We will require that

\[
I^{d-1}_\lambda = I^{d-1}_\beta
\]
along points in \( \text{Sing}(J, b) \cap U_\lambda \cap U_\beta \) (condition of patching). In such case we can define a function
\[
f_B^{d-1} : \text{Sing}(J, b) \rightarrow I^{d-1}
\]
simply by patching the functions \( f_{B_\delta}^{d-1} \). The function \( f_B^{d-1} \) is upper-semi-continuous, and \( U_\delta \cap \text{Max} f_B^{d-1} = \text{Max} f_{B_\delta}^{d-1} \) whenever \( U_\delta \cap \text{Max} f_B^{d-1} \neq \emptyset \). Therefore \( \text{Max} f_B^{d-1} \) is a center defining
\[
(W, (J, b), E = \emptyset) \leftarrow (W_1, (J_1, b), E_1).
\]

Assume, for simplicity, that \( U_\delta \cap \text{Max} f_B^{d-1} \neq \emptyset \) for each index \( \delta \), then \( W_1 \) can be covered by \( \{U^{(1)}_\lambda\}_{\lambda \in \Lambda} \) (notation as in \( \ref{33} \)), and for each \( \lambda \) we obtain
\[
\bar{B}_\lambda^{d-1} = (\overline{W}_\lambda, (K^{(0)}_\lambda, b!), \emptyset) \leftarrow (\bar{B}_\lambda^{(1)}, (K^{(1)}_\lambda, b!), \bar{E}_1)
\]
with \( \overline{W}_\Lambda^{(1)} \subset U_\Lambda^{(1)} \). Furthermore, the closed set \( \text{Sing}(J_1, b) \subset W_1 \) is such that \( \text{Sing}(J_1, b) \cap U_\lambda^{(1)} = \text{Sing}(K^{(1)}_\lambda, b!) \). We will require that
\[
f_B^{d-1}(J_1, b) = f_B^{d-1}(J_1, b),
\]
along points in \( \text{Sing}(J_1, b) \cap U_\lambda^{(1)} \cap U_\beta^{(1)} \) so as to define \( f_B^{d-1} : \text{Sing}(J_1, b) \rightarrow I^{d-1} \) (requirement of patching); and in such case \( \text{Max} f_B^{d-1} \) defines a transformation of \( (W_1, (J_1, b), E_1) \).

The point is that if all these requirements of patching hold again and again, the resolutions of the different basic objects \( \bar{B}_\lambda^{d-1} = (\overline{W}_\lambda, (K^{(0)}_\lambda, b!), \emptyset) \), defined in terms of the functions on \( (I^{d-1}, \geq) \), patch so as to define a resolution of \( (W, (J, b), \emptyset) \). This would provide resolution of simple basic objects of dimension \( d \).

**Conclusion:** Resolution of simple basic objects \( (W, (J, b), \emptyset) \) can be achieved by blowing up successively at \( \text{Max} f_B^{d-1} \), for \( f_B^{d-1} : \text{Sing}(J_1, b) \rightarrow I^{d-1} \) defined as above, if the condition of patching holds.

**General strategy for resolution of basic objects:**

1. Define the functions so that the patching principle holds.
2. Reduce the problem of resolution of a basic object to that of simple basic objects \( \ref{33} \).

**10.5.** Fix a basic object and a sequence of transformations

\[
(10.5.1) \quad B_0 = (W, (J, b), E) \leftarrow B_1 = (W_1, (J_1, b), E_1) \leftarrow B_k = (W_k, (J_k, b), E_k)
\]
where \( E = \{H_1, \ldots, H_r\} \) and \( E_k = \{H_1, \ldots, H_r, \ldots, H_{r+k}\} \). There is an expression relating \( J_k \) with the total transform, say:

\[
(10.5.2) \quad J_{O_{W_k}} = I(H_{r+1})^{e_1} I(H_{r+2})^{e_2} \cdots I(H_{r+k})^{e_k} \cdot J_k
\]
as in \( \ref{33} \). Note here that \( J_k \) might vanish along some of the exceptional hypersurfaces \( \{H_{r+1}, \ldots, H_{r+k}\} \); and, as indicated in the example in \( \ref{33} \), there is also a well defined expression:

\[
(10.5.3) \quad J_k = I(H_{r+1})^{a_{r+1}} I(H_{r+2})^{a_{r+2}} \cdots I(H_{r+k})^{a_{r+k}} \cdot \overline{J_k}
\]
at \( O_{W_k} \), so that \( \overline{J_k} \) does not vanish along any exceptional hypersurface. Set \( d = \dim W_k \), and

\[
\text{ord}^{d_k}_x : \text{Sing}(J_k, b) \rightarrow \mathbb{Q}
\]
\[
x \quad \rightarrow \nu_{J_k}(x)
\]
$w$-ord$_k^d : \text{Sing}(J_k, b) \to \mathbb{Q}$

$x \mapsto \frac{\nu_{J_k}(x)}{b}$,

where $\nu_{J_k}(x)$ denotes the order of the ideal $J \mathcal{O}_{W_k,x}$ at $J \mathcal{O}_{W_k,x}$. Both functions are upper semi-continuous and, since $J_0 = J_0 = J$, they coincide for $k = 0$.

We will also define an upper semi-continuous function by setting

$\overline{a}_{r+1} : \text{Sing}(J_k, b) \to \mathbb{Q}$

where $\overline{a}_{r+1}(x) = \frac{a_{r+1}(x)}{b}$ and $a_{r+1}(x) = a_{r+1}$ in (10.5.3) if $x \in H_{r+1}$, and $a_{r+1}(x) = 0$ otherwise.

The role of the denominator $b$ is of no use for the moment and the reader might want to ignore it. We will justify the presence of $b$ in (10.11).

**Remark 10.6.** Assume that $\max w$-ord$_k^d = \frac{b'}{b}$, and that $(W_k, (J_k, b), E_k) \leftarrow (W_{k+1}, (J_{k+1}, b), E_{k+1})$ is defined with center, say $Y_k \subset \text{Max } w$-ord$_k^d$. So the function $w$-ord$_k^d$ takes only the value $\frac{b'}{b}$ along points of $Y_k$. Then the expression (10.5.3), corresponding now to $J_{k+1}$, is:

\begin{equation}
 J_{k+1} = I(H_{r+1})^{a_1} I(H_{r+2})^{a_2} \cdots I(H_{r+k})^{a_k} \cdot I(H_{r+k+1})^{a_{k+1}} \cdot \mathcal{J}_{k+1}.
\end{equation}

Furthermore:

1) $(\mathcal{J}_k, b')$ is a simple couple (7.3), and $\text{Max } w$-ord$_k = \text{Sing}(\mathcal{J}_k, b') \cap \text{Sing}(J_k, b)$.
2) $(\mathcal{J}_{k+1}, b')$ is the transform of the simple couple $(\mathcal{J}_k, b')$, and $\max w$-ord$_{k+1} \leq \frac{b'}{b}$.

In fact, if $\mathcal{J}_{k+1}$ is not to vanish along $H_{r+k+1}$ it must be defined as the proper transform of $\mathcal{J}_k$ (2.8). The first assertion follows from our choice of center and the second from Theorem 7.5.

**10.7.** We will impose conditions on a sequence (10.5.1). Set $\frac{b'}{b} = \max w$-ord$_i$. Assume all centers

$Y_i \subset \text{Max } w$-ord$_i$, for $i = 0, \ldots, k$;

and hence, that

$\max w$-ord$_{k+1} \geq \max w$-ord$_{k+1} \geq \ldots \geq \max w$-ord$_{k+1}$

(namely $\frac{b_i}{b} \geq \frac{b_i}{b} \geq \ldots \geq \frac{b_{i+1}}{b} \geq \frac{b_i}{b}$) by the previous remark. Let $k_0$ be the smallest index such that

$\frac{b_i}{b} = \frac{b_i}{b} = \ldots = \frac{b_i}{b}$.

For each index $k_0 \leq j \leq k$ define a partition on the set of hypersurfaces in $E_j$, say $E_j = E^-_j \cup E^+_j$, where $E^-_j = \{H_1, \ldots, H_r, \ldots H_{r+k_0}\}$ and $E^+_j = \{H_{r+k_0+1}, \ldots H_{r+j}\}$. So for $j = k_0$ $E_{k_0} = E^-_{k_0}$; and for $j > k_0$, $E^-_j$ consists of the strict transforms of hypersurfaces in $E_{k_0}$. We finally order $\mathbb{Q} \times \mathbb{N}$ lexicographically, and set

$t^d_k : \text{Sing}(J_k, b) \to \mathbb{Q} \times \mathbb{N}$

$x \mapsto (w$-ord$_k^d(x), n_k^d(x))$

\[ n_k^d(\xi) = \begin{cases} 
\# \{H \in E_k \mid \xi \in H\} & \text{if } w$-ord$_k^d(\xi) < \frac{b_i}{b} \\
\# \{H \in E^-_k \mid \xi \in H\} & \text{if } w$-ord$_k^d(\xi) = \frac{b_i}{b}.
\end{cases} \]

To see that $t_k^d$ is upper-semi-continuous we argue coordinate-wise: fix integers $m$, $n$, and note that

$G_{(m, n)} = \{\xi \in \text{Sing}(J_k, b)/t_k^d(\xi) \geq \frac{m}{b}, n)\}$
is closed. The statement is clear if \( \frac{m}{b} = \frac{1}{b} \). If not, set \( G_{(m,n)} = F_1 \cup F_2 \), for

\[
F_1 = \{ \xi / \text{w-ord}_k(\xi) \geq \frac{m+1}{b} \}, \quad \text{and} \quad F_2 = \{ \xi / \text{w-ord}_k(\xi) \geq \frac{m}{b}; \# \{ H \in E_k^- \mid \xi \in H \} \geq n \}.
\]

**Remark 10.8.** Because of the lexicographic order on \( \mathbb{Q} \times \mathbb{N} \) we see that \( \text{max} t_k^d = (\text{max} \text{w-ord}_k^d, a) \) for some integer \( 0 \leq a \leq \dim W = d \); and hence that \( \text{Max} t_k^d \subset \text{Max} \text{w-ord}_k^d \). With notation as in [10.7] at most a hypersurface of \( E_k^- \) cut at a point of \( \text{Max} \text{w-ord}_k^d \), and \( \text{Max} t_k^d \) are the points with this condition.

If a transformation \((W_k, (J_k, b), E_k) \leftarrow (W_{k+1}, (J_{k+1}, b), E_{k+1})\), is defined with center \( Y_k \subset \text{Max} t_k^d \), then \( \text{w-ord}_k \geq \text{max} \text{w-ord}_{k+1} \) [10.3.2].

If \( \text{max} \text{w-ord}_k > \text{max} \text{w-ord}_{k+1} \), then \( \text{max} t_k^d > \text{max} t_{k+1}^d \). On the other hand, if \( \text{max} \text{w-ord}_k = \text{max} \text{w-ord}_{k+1} \), then \( E_{k+1}^- \) will consist on the strict transform of hypersurfaces in \( E_k^- \). It is clear that in such case \( \text{max} t_k^d \geq \text{max} t_{k+1}^d \).

**10.9. Projections of Basic Objects.** So far we have only considered transformations on basic objects defined by monoidal transformations. Set \( W_{k+1} = W_k \times A^1 \) (the affine line), \( W_k \leftarrow W_{k+1} \) the projection, and define

\[
(W_k, (J_k, b), E_k) \leftarrow (W_{k+1}, (J_{k+1}, b), E_{k+1})
\]

where \( E_{k+1} \) is the pull-back of hypersurfaces in \( E_k \), and \( J_{k+1} = J_k \cdot \mathcal{O}_{W_{k+1}} \). We call this a *projection* of basic objects. Projections will play a key role when proving the patching conditions discussed in [10.4].

Note that if a point \( x_{k+1} \in W_{k+1} \) maps to a point \( x_k \in W_k \), then the order of \( J \) at \( \mathcal{O}_{W_k, x_k} \) is the same as the order of the basic object \( J_{k+1} \) at \( \mathcal{O}_{W_{k+1}, x_{k+1}} \).

Here that dimension of \( W_{k+1} = \dim W_k + 1 \), but ignoring superscripts, the functions \( \text{w-ord}_k, n_k \) and \( t_k \) can also be extended to functions \( \text{w-ord}_{k+1}, n_{k+1} \) and \( t_{k+1} \) at \( \text{Sing}(J_{k+1}, b) \) (pull-back of \( \text{Sing}(J, b) \)). Furthermore, if \( x_{k+1} \in \text{Sing}(J_{k+1}, b) \) maps to \( x_k \in \text{Sing}(J, b) \), then \( \text{w-ord}_{k+1}(x_{k+1}) = \text{w-ord}_k(x_k) \), and \( t_{k+1}(x_{k+1}) = t_k(x_k) \).

In other words, given a sequence

\[
B_0 = (W, (J, b), E) \leftarrow B_1 = (W_1, (J_1, b), E_1) \leftarrow \ldots \leftarrow B_{k_0} = (W_{k_0}, (J_{k_0}, b), E_{k_0})
\]

where each index \( i < k \) \( (W_i, (J_i, b), E_i) \leftarrow (W_{i+1}, (J_{i+1}, b), E_{i+1}) \) is defined by

1. a monoidal transformation with center \( Y_i \subset \text{Max} \text{w-ord}_i \); or by
2. a projection of basic objects;

we have that

\[
\text{max} \text{w-ord}_0 \geq \text{max} \text{w-ord}_1 \geq \ldots \geq \text{max} \text{w-ord}_k.
\]

In particular, the partitions of \( E_i = E_i^+ \cup E_i^- \), and the functions \( t_i^d : \text{Sing}(J, b) \to \mathbb{Q} \times \mathbb{N} \) defined in [10.7] can also be defined in this setting. Furthermore, if all centers \( Y_i \subset \text{Max} t_i^d \subset \text{Max} \text{w-ord}_i \) then

\[
\text{max} t_0 \geq \text{max} t_1 \geq \ldots \geq \text{max} t_{k_0}.
\]

**Proposition 10.10.** Consider a sequence [10.9.2] (of transformations and projections), and assume that either \( k_0 = 0 \) or that \( \text{max} t_0^d \geq \text{max} t_1^d \ldots \geq \text{max} t_{k_0-1}^d > \text{max} t_{k_0}^d \). Then there is a simple basic object \((W_{k_0}, (K_{k_0}, c), \emptyset)\) (Def. [10.2]) such that any sequence

\[
(W_{k_0}, (K_{k_0}, c), \emptyset) \leftarrow (W_{k_0}, (K_{k_0+1}, c), E_1^d) \leftarrow \ldots \leftarrow (W_{k_0}, (K_{k_0+m}, c), E_m^d)
\]
(of transformations and projections) induces a sequence over \((W_k, (J_{k}, b), E_k)\), say:

\[
(10.10.2) \quad (W, (J, b), E) \ldots \leftarrow (W_k, (J_k, b), E_k) \leftarrow \ldots \leftarrow (W_{k+m}, (J_{k+m}, b), E_{k+m}).
\]

Furthermore, \((10.10.1)\) and \((10.10.2)\) are related by the following properties:

**P1** \(\max t_{k_0}^d = \ldots = \max t_{k_0+m-1}^d\) in \((10.10.2)\).

**P2** \(\max t_{k_0+j}^d = \text{Sing}(K_{k_0+j}, c)\) for \(0 \leq j \leq m - 1\).

**P3** If \(\text{Sing}(K_{k_0+m}, c) = \emptyset\), then \(\text{Sing}(J_{k_0+m}, b) = \emptyset\) or \(\max t_{k_0+m-1}^d > \max t_{k_0+m}^d\).

**P4** If \(\text{Sing}(K_{k_0+m}, c) \neq \emptyset\), then \(\max t_{k_0+m-1}^d = \max t_{k_0+m}^d\) and \(\text{Sing}(K_{k_0+m}, c) = \max t_{k_0+m}^d\).

We begin by the following two remarks, needed to sketch a proof of this Proposition.

**Remark 10.11.** Fix a basic object \(B = (W, (J, b), E)\) and a positive integer \(m \geq 1\). Set \(J^m \subset \mathcal{O}_W\) the \(m\)-th power of \(J\), and consider \(B_m = (W, (J^m, m \cdot b), E)\). Note that

\[
\text{Sing}(J, b) = \text{Sing}(J^m, m \cdot b).
\]

In particular, a smooth center \(Y\) defines a transformation of one basic object iff it defines a transformation of both, say:

\[
(W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1) \quad \text{and} \quad (W, (J^m, m \cdot b), E) \leftarrow (W_1, ((J^m)_1, m \cdot b), E_1).
\]

Since the total transform of \(J^m\), namely \(J^m \cdot \mathcal{O}_{W_1}\), is the \(m\)-th power of the total transform of \(J\), it follows that \((J^m)_1\) is the \(m\)-th power of \(J_1\) (i.e. \((J^m)_1 = J_1^m\)). The same holds after any sequence of transformations. Therefore a resolution of \(B\) induces a resolution of \(B_m\), and the other way around. It will turn out that the resolution of \(B\), defined by the resolution functions in \(1.7\) will coincide with the resolution of \(B_m\) defined by the resolution functions. For the time being note that at a point \(\xi \in \text{Sing}(J, b) = \text{Sing}(J^m, m \cdot b), \nu_{J}(\xi) = \nu_{J^m}(\xi)\), where \(\nu_{J}(x)\) denotes the order of \(J\) at \(\mathcal{O}_{W,x}\).

**Remark 10.12.** Given two basic objects \((W, (J, b), E)\) and \((W, (K, c), E)\) (same \((W,E))\), note that

\[
\text{Sing}(J, b) \cap \text{Sing}(K, c) = \text{Sing}(J^c, c \cdot b) \cap \text{Sing}(K^b, b \cdot c) = \text{Sing}(J^c + K^b, b \cdot c).
\]

If \(Y \subset \text{Sing}(J^c + K^b, b \cdot c)\) defines \((W, (J^c + K^b, b \cdot c), E) \leftarrow (W_1, ((J^c + K^b)_1, b \cdot c), E_1)\), then \(Y\) also defines \((W, (J, b), E) \leftarrow (W_1, (J_1, b), E_1)\) and \((W, (K, c), E) \leftarrow (W_1, (K_1, c), E_1)\). Check finally that \((J^c + K^b)_1 = J_1^c + K_1^b\), and hence that

\[
\text{Sing}(J_1, b) \cap \text{Sing}(K_1, c) = \text{Sing}((J^c + K^b)_1, c \cdot b).
\]

So, by induction, a sequence of transformations of \((W, (J^c + K^b, b \cdot c), E)\), induces sequences of transformations of \((W, (J, b), E)\) and of \((W, (K, c), E)\), and for each index \(i\)

\[
\text{Sing}(J_i, b) \cap \text{Sing}(K_i, c) = \text{Sing}((J^c + K^b)_i, c \cdot b).
\]

Because of this property we will set formally:

\[
(10.12.1) \quad (W, (J, b), E) \cap (W, (K, c), E) = (W, (J^c + K^b, b \cdot c), E).
\]

**Example 10.13.** Let \(X_0\) be an curve in a smooth surface \(W_0\), analytically irreducible at a closed point \(\xi_0 \in W_0\). These data allow us to define, for each integer \(k\), a sequence of \(k\) quadratic transformations over \(W_0\). In fact, if \(W_0 \leftarrow W_1\) is defined with center \(\xi_0\), the strict transform \(X_1\) intersects the exceptional locus \(H_1\) at a unique point, say \(\xi_1\). Set \(W_1 \leftarrow W_0\) with center \(\xi_1\). By iteration we get \(W_i \leftarrow W_{i+1}\), with exceptional hypersurface \(H_{i+1}\), and \(\xi_{i+1} = H_{i+1} \cap X_{i+1} \in W_{i+1}\).
Set \( X_0 = V(< x^4 - y^5 >) \subset W_0 = \text{Spec}(k[x, y]). \) For any \( k, \) the sequence of length \( k \) defined by this curve induces a sequence of transformations of \((W_0, J = < x^4 - y^5 >, 1, E_0 = \emptyset). \) For \( k = 1: \)

\[(W_0, J = < x^4 - y^5 >, 1, E_0 = \emptyset) \leftarrow (W_1, (J_1, 1, E_1)).\]

Check first that \( J_1 = I(H_1)^3J_1, \) and that \( t_0^2 = (4, 0) > \max t_0^2 = (1, 1). \) So for \( k = 1, \) \( k_0 = 1 \) in the setting of Proposition 10.10. Show that \((W_1, (J_1, 1, \emptyset) \cap (W_1, (I(H_1), 1, \emptyset)) \) (see (10.12.1)) plays the role of \((W_{k_0}, (K_{k_0}, c, \emptyset)) \) in the Proposition. Note finally that for the sequence of five quadratic transformations defined by the curve (i.e. for \( k = 5): \)

\[
\max t_1^3 = \max t_2^3 = \max t_3^2 > \max t_5^2.
\]

**Proof.** Of Prop 10.10. We define \((W_{k_0}, (K_{k_0}, c, \emptyset)) \) with those properties, and we do so by taking suitable intersections (10.12).

If \( t_{k_0}^d = (\frac{b_{k_0}}{d}, n_{k_0}), \) \( b_{k_0} \) is the highest order of \( J_{k_0} \) along points in \( \text{Sing}(J_{k_0}). \) Set

\[
(W_{k_0}, (A_{k_0}, c, \emptyset)) = (W_{k_0}, (J_{k_0}, b, \emptyset)) \cap (W_{k_0}, (J_{k_0}, b_{k_0}, \emptyset))
\]

so that \( \text{Sing}(A_{k_0}, c) \) is \( \text{Max} \) w-order \( k_0. \) By assumption \( E_{k_0} = E_{k_0}^{-}, \) and at most \( n_{k_0} \) hypersurfaces of \( E_{k_0}^{-}, \) can come together at a point of \( \text{Max} \) w-order \( k_0. \) For a subset \( S \subset E_{k_0}^{-}, \) with \( n_{k_0} \) hypersurfaces, set \( F_S = \text{Max} \) w-order \( \cap \{H_i \in S H_i). \) Given two such subsets \( S_1 \neq S_2 \) note that \( F_{S_1} \cap F_{S_2} = \emptyset \) since \( n_{k_0} \) is a maximum. Furthermore \( \max t_{k_0} = \cup F_S \) for all \( S \) as before. Recall that each \( H_i \in E_{k_0} \) is a smooth hypersurface, and set \((W_{k_0}, (I(H_i), 1, \emptyset) \cap (W_{k_0}, (I(H_j), 1, \emptyset)) = (W_{k_0}, (I(H_i) + I(H_j), 1, \emptyset)). \) Finally set \( B_{k_0} = \sum_{S} \sum_{H_i \in S} I(H_i), \) and

\[
(W_{k_0}, (K_{k_0}, c, \emptyset)) = (W_{k_0}, (A_{k_0}, c, \emptyset)) \cap (W_{k_0}, (B_{k_0}, 1, \emptyset))
\]

and check that \( \text{Sing}(K_{k_0}, c) = \text{Max} t_{k_0}^d. \)

If \( Y_{k_0} \subset \text{Sing}(K_{k_0}, c) \) is a center of transformation for this basic object, then, for any \( H_i \in E_{k_0}, \) either \( Y_{k_0} \subset H_i \) or \( Y_{k_0} \cap H_i = \emptyset. \) In particular \( Y_{k_0} \) has normal crossing with \( E_{k_0}^{-}, \) and defines a transformation of \((W_{k_0}, (J_{k_0}, b, E_{k_0})). \) Furthermore, using 5.3 and the previous Remarks, we conclude that either \( t_{k_0}^d > \max t_{k_0+1}^d, \) in which case \( \text{Sing}(K_{k_0+1}, c) = \emptyset, \) or \( t_{k_0}^d = \max t_{k_0+1}^d, \) in which case \( \text{Sing}(K_{k_0+1}, c) = \text{Max} t_{k_0+1}^d \) (notation as in (10.11.1) and (10.12.1)).

In the last case \( E_1' = (10.11.1) \) is \( E_{k_0+1}^+ \) in (10.12.2). If \( Y_{k_0+1} \) is a center that defines a transformation of \((W_{k_0+1}, (J_{k_0+1}, c), E_1', \) then \( Y_{k_0+1} \) must have normal crossing with \( E_{k_0+1}^+, \) and on the other hand, for any hypersurface \( H_i \in E_{k_0+1}^+, \) either \( Y_{k_0+1+1} \subset H_i \) or \( Y_{k_0+1} \cap H_i = \emptyset. \) This insures that \( Y_{k_0+1} \) has normal crossing with \( E_{k_0+1}, \) and defines a transformation of \((W_{k_0+1}, (J_{k_0+1}, b, E_{k_0+1}). \)

All properties in Proposition 10.10 follow by iteration of this argument.

We end this proof by showing that \((W_{k_0}, (K_{k_0}, c, \emptyset)) \) is a simple basic object. To check this note that if \( J \) has highest order \( b, \) then \( J^c + K^b \) has highest order \( b \cdot c \) in (10.12.1). So it suffices to check that \( B_{k_0} \) has highest order 1, which is clear.

\section*{11. On resolution functions II: the Monomial Case}

**11.1.** Proposition 10.10 asserts that if we knew how to define resolutions of simple basic objects, then we could define an extension 10.10.2, so that \( \max t_{k_0}^d = \ldots = \max t_{k_0+m-1}^d > \max t_{k_0+m}^d. \)

Since 10.10.2 is a sequence of transformations of \((W, (J, b), E), \) the first coordinate of max \( t^d \) is of the form \( \frac{k}{n} \) for a positive integer \( l; \) and the second coordinates is at most the dimension of \( W. \) So
by iteration of resolutions of the simple basic objects in Proposition 10.10, we could force the last value \( \max t^d \) to drop again and again. Ultimately, we come to the case in which either 10.10.2 is a resolution, or the first coordinate of \( \max t^d \) is zero. This last case is called the monomial case. In Proposition 11.5 we will provide resolution for this case. Note also that in the monomial case \( J_k = \mathcal{O}_{\mathbb{W}_k} \) in 10.5.3, locally at any point in \( \text{Sing}(J_k,b) \), so \( J_k \) is locally spanned by a monomial.

**Example 11.2.** Let \( W = \text{Spec}(k[X_1,X_2,X_3]) \) and \( (W,(J,5),\emptyset) \), \( J = \langle X_1^6 \cdot X_2^2 \cdot X_3^4 \rangle \). The singular locus is a union of two hypersurfaces \( \langle X_1 \rangle \cup \langle X_2 \rangle \). Blowing up \( \langle X_2 \rangle \) we get \( W_1 = W \) and \( (J_1,5) \), \( J_1 = \langle X_1^6 \cdot X_2^2 \cdot X_3^4 \rangle \). The singular locus is a union of a hypersurface with a line. Blowing up at the hypersurface we get \( W_2 = W \) and \( J_2 = \langle X_1^1 \cdot X_2^2 \cdot X_3^4 \rangle > \) where the singular locus is a line. A resolution is finally achieved by blowing up such line.

It is simple to establish a general strategy, in the monomial case, so that, as in this example, resolution is achieved by blowing up at maximal dimension components of the singular locus.

Note that, for a monomial basic object, the closed set \( \text{Sing}(J,b) \) is the union of some of the irreducible components of intersections of the hypersurfaces \( H_i \). In fact, consider the functions \( a_i_1, \ldots, a_i_p \) defined in 10.5.2 and an irreducible component \( C \) of the intersection \( H_i_1 \cap \cdots \cap H_i_p \); then the functions \( a_i_1, \ldots, a_i_p \) are constant on \( C \), and \( C \) is included in \( \text{Sing}(J,b) \) if and only \( a_i_1 + \cdots + a_i_p \geq b \) along \( C \).

**Definition 11.3.** Let \( (W,(J,b),E) \) be a monomial basic object. Define the function:

\[
\begin{align*}
\mathcal{h} : \text{Sing}(J,b) & \rightarrow \Gamma = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}^N \\
\mathcal{h}(\xi) &= (-p(\xi), \omega(\xi), \ell(\xi)).
\end{align*}
\]

where, if \( \xi \in \text{Sing}(J,b) \), the values \( p(\xi), \omega(\xi) \) and \( \ell(\xi) \) are defined as follows:

\[
\begin{align*}
(11.3.1) \quad p(\xi) &= \min \left\{ q \mid \exists i_1, \ldots, i_q, \quad a_{i_1}(\xi) + \cdots + a_{i_q}(\xi) \geq b, \quad \xi \in H_{i_1} \cap \cdots \cap H_{i_q} \right\}, \\
(11.3.2) \quad \omega(\xi) &= \max \left\{ \frac{a_{i_1}(\xi) + \cdots + a_{i_q}(\xi)}{b} \mid q = p(\xi), \ a_{i_1}(\xi) + \cdots + a_{i_q}(\xi) \geq b, \ \xi \in H_{i_1} \cap \cdots \cap H_{i_q} \right\}, \\
(11.3.3) \quad \ell(\xi) &= \max \left\{ (i_1, \ldots, i_q, 0, \ldots) \in \mathbb{Z}^N \mid q = p(\xi), \ \frac{a_{i_1}(\xi) + \cdots + a_{i_q}(\xi)}{b} = \omega(\xi), \ \xi \in H_{i_1} \cap \cdots \cap H_{i_q} \right\}.
\end{align*}
\]

In the last formula we consider the lexicographical order in \( \mathbb{Z}^N \).

Fix a point \( \xi \in \text{Sing}(J,b) \) and let \( C_1, \ldots, C_s \) be the irreducible components of \( \text{Sing}(J,b) \) at \( \xi \).

- The first coordinate of \( h(\xi) \) is \( -p(\xi) \), where \( p(\xi) \) is the minimal codimension of \( C_1, \ldots, C_s \).
- Denote by \( C'_1, \ldots, C'_{s'} \) the components with minimum codimension \( p(\xi) \) (i.e. of highest dimension at the point \( \xi \)).
The second coordinate of \( h(\xi) \) is \( \omega(\xi) = \frac{b'}{b} \), where \( b' \) is the maximum order of \( J \) along the components \( C_1', \ldots, C_m' \).
Denote by \( C_1'', \ldots, C_n'' \), the components with maximum order.

The last coordinate of \( h(\xi) \), \( \ell(\xi) \), corresponds to one \( C_j'' \), for some index \( j \).

So for a fixed point \( \xi \), with \( p(\xi) \) we have selected the irreducible components of \( \text{Sing}(J, b) \), at \( \xi \), of highest dimension. With \( \omega(\xi) \) we have select, among the previous components, those where the order of \( J \) is maximum. Finally with \( \ell(\xi) \) we select a unique component containing \( \xi \).

11.4. Now one can check that the function \( h \) is upper-semi-continuous, and that the closed set \( \text{Max} h \) is regular. In fact if \( \max h = (-p_0, w_0, (i_1, \ldots, i_{p_0}, 0, \ldots)) \), then \( \text{Max} h \) is a union of connected components of the regular scheme \( H_{i_1} \cap \cdots \cap H_{i_{p_0}} \).

It is clear that \( \text{Max} h \) is a permissible center for the basic object \((W, (J, b), E)\). Let

\[
(W, (J, b), E) \leftarrow \Pi (W_1, (J_1, b), E_1)
\]

be the transformation with center \( \text{Max} h \), and let \( E_1 = \{H_1, \ldots, H_r, H_r+1\} \), where, by abuse of notation, \( H_i \) denotes the strict transform of \( H_i \), for \( i = 1, \ldots, r \), and \( H_{r+1} \) is the exceptional divisor of \( \Pi \). The basic object \((W_1, (J_1, b), E_1)\) is also monomial, in fact for \( \xi \in \text{Sing}(J_1, b) \) we have

\[
J_{\xi} = I(H_1)_{\xi} a_1(\Pi(\xi)) \cdots I(H_r)_{\xi} a_r(\Pi(\xi)) I(H_{r+1})_{\xi} a_{r+1}(\Pi(\xi)),
\]

where the functions \( a_i \) are given by:

\[
a_i(\xi) = a_i(\Pi(\xi)) \quad \forall \xi \in H_i \quad \text{and} \quad i = 1, \ldots, r;
\]

\[
a_{r+1}(\xi) = a_{r+1}(\Pi(\xi)) + \cdots + a_{r+0}(\Pi(\xi)) - b \quad \forall \xi \in H_{r+1}.
\]

As in Definition 11.3, a function \( h_1 \) has been associated to the basic object \((W_1, (J_1, b), E_1)\), and one can check that the maximum value has dropped:

\[
\max h > \max h_1.
\]

In fact, for any point \( \xi \in \text{Sing}(J_1, b) \):

\[
h_1(\xi) = h(\Pi(\xi)) \quad \text{if} \quad \Pi(\xi) \in \text{Max} h
\]

\[
h_1(\xi) < h(\Pi(\xi)) \quad \text{if} \quad \Pi(\xi) \in \text{Max} h.
\]

It is not hard to check now that this function \( h \) defines a resolution in the monomial case:

**Proposition 11.5.** Consider a sequence \([10.9,2]\) (of transformations and projections), and assume that \( \max t_0^d \geq \max t_1^d \geq \cdots \geq \max t_{k_0-1}^d > \max t_{k_0}^d \), and that \( \max \text{w-ord}_{k_0} = 0 \). A resolution

\[
(W_{k_0}, (J_{k_0}, b), E_{k_0}) \leftarrow \cdots \leftarrow (W_{k_0+m}, (J_{k_0+m}, b), E_{k_0+m}).
\]

is defined by the functions \( h_i : \text{Sing}(J_{k_0+i}, b) \to \Gamma \), by blowing up successively at \( Y_{k_0+i} = \text{Max} h_i \).

12. General basic objects and resolution functions.

In [10.4] we already discussed the need to generalize the notion of basic object in order to profit from a form of induction on the dimension of basic objects, which would enable us to achieve resolutions of basic objects. This leads us to the notion of general basic objects which will be developed in this section. Recall that in the setting of [10.4], namely the case of a simple basic object \((W, (J, b), E = \emptyset)\) (in which \( \text{dim} W = d \), and where \( R(1)(\text{Sing}(J, b)) = \emptyset \)), there is a form of induction on the dimension \( d \). In fact, in such case there is a covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( W \), and for each index \( \lambda \) a
$d - 1$ dimensional basic object $\overline{B}_{\lambda}^{d-1} = (\overline{W}_{\lambda}, (K_{\lambda}^{(0)}, b!), \emptyset)$, such that $\text{Sing}(J, b) \cap U_{\lambda} = \text{Sing}(K_{\lambda}^{(0)}, b!)$. The outcome of the previous sections is to show that resolutions of simple basic objects implies resolutions of arbitrary basic objects. However in doing so, we expect to argue inductively by defining the functions $w$-ord, $n$ (see (4.7)), and $h$ (see (14.3)), for these locally defined basic objects $\overline{B}_{\lambda}^{d-1}$. In this Section we provide a precise formulation of these locally defined basic objects. The key point, that will ultimately allow us to define the functions $w$-ord, $n$, and $h$ in this more ample context, is the fact that the singular loci of these $d$-1 dimensional basic objects, namely the sets $\text{Sing}(K_{\lambda}^{(0)}, b!)$, patch and define the closed set $\text{Sing}(J, b)$. In fact $\text{Sing}(J, b) \cap U_{\lambda} = \text{Sing}(K_{\lambda}^{(0)}, b!)$. Furthermore, this form of patching will also hold after transformations; a concept that will be made precise in the following definition. The covering $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of $W$ and the $d-1$ dimensional basic object $\overline{B}_{\lambda}^{d-1} = (\overline{W}_{\lambda}, (K_{\lambda}^{(0)}, b!), \emptyset)$ will define, in the sense of the following definition, a $d-1$ dimensional general basic object.

**Definition 12.1.** A $d$-dimensional general basic object over a pair $(W, E)$ ($W$ smooth, $E = \{H_1, \ldots, H_s\}$ as in (13), consists of an open covering of $W$, say $\{U_{\alpha}\}_{\alpha \in \Lambda}$; and setting $(U_{\alpha}, E_{\alpha})$ as the restriction of $(W, E)$ to $U_{\alpha}$, there is:

(i) A collection of basic objects. For every $\alpha \in \Lambda$ there is a closed and smooth $d$-dimensional subscheme $\tilde{W}_{\alpha} \subset U_{\alpha}$, which intersects transversally all hypersurfaces $E_{\alpha}$, in the sense that $H_\alpha \cap \tilde{W}_{\alpha} = (H_\alpha)$, is either empty or a smooth hypersurface of $\tilde{W}_{\alpha}$, defining a pair $(\tilde{W}_{\alpha}, \tilde{E}_{\alpha}) = \{(H_\alpha)\}_1, \ldots, (H_\alpha)\}_s$. And, for each $\alpha$ there is a basic object

$$(\tilde{W}_{\alpha}, (B_{\alpha}, d_\alpha), \tilde{E}_{\alpha}).$$

Obviously, for each index $\alpha$ the closed set $\text{Sing}(B_{\alpha}, d_\alpha) \subset U_{\alpha}$ is locally closed in $W$.

(ii) A patching condition. There is a closed subset $F \subset W$ such that

$$F \cap U_{\alpha} = \text{Sing}(B_{\alpha}, d_\alpha)$$

for every $\alpha \in \Lambda$.

(iii) Stability of patching (I). Let

$$(W, E) \leftarrow (W_1, E_1)$$

be a permissible transformation with center $Y \subset F$ (13), let $\{U_{\alpha,1}\}$ be the pullback of $\{U_{\alpha}\}_{\alpha \in \Lambda}$ to $W_1$, and for each $\alpha \in \Lambda$ let

$$(\tilde{W}_{\alpha}, (B_{\alpha}, d_\alpha), \tilde{E}_{\alpha}) \leftarrow (\tilde{W}_{\alpha,1}, (B_{\alpha,1}, d_\alpha), \tilde{E}_{\alpha,1}).$$

be the corresponding transformation of basic objects. Then there is a closed set $F_1 \subset W_1$ so that

$$F_1 \cap U_{\alpha,1} = \text{Sing}(B_{\alpha,1}, d_\alpha)$$

for each index $\alpha \in \Lambda$.

(iv) Stability of patching (II). Let $W \leftarrow W_1 = W \times A^1$ be the projection and let

$$(W, E) \leftarrow (W_1, E_1)$$

where $E_1$ is defined as the set of pull-backs of hypersurfaces in $E$. Let $\{U_{\alpha,1}\}$ be the pullback of $\{U_{\alpha}\}_{\alpha \in I}$ to $W_1$, and for each $\alpha \in \Lambda$ set

$$(\tilde{W}_{\alpha}, (B_{\alpha}, d_\alpha), \tilde{E}_{\alpha}) \leftarrow (\tilde{W}_{\alpha,1}, (B_{\alpha,1}, d_\alpha), \tilde{E}_{\alpha,1}),$$
where \( \tilde{W}_{\alpha,1} = \tilde{W}_\alpha \times A^1, \tilde{E}_{\alpha,1} \) is the pull-back of hypersurfaces in \( \tilde{E}_\alpha \), and \( B_{\alpha,1} = B_\alpha \mathcal{O}_{\tilde{E}_{\alpha,1}} \).

Then there is a closed set \( F_1 \subset W_1 \) such that, for each index \( \alpha \in \Lambda \)

\[
F_1 \cap U_{\alpha,1} = \text{Sing}(B_{\alpha,1}, d_\alpha).
\]

(v) **Stability of patching (III).** The patching condition defined in (iii) and (iv) holds after any sequence of transformations: Given a sequence of transformations of pairs,

\[
(W_0, E_0) \leftarrow (W_1, E_1) \leftarrow \ldots \leftarrow (W_r, E_r) \leftarrow (W_{r+1}, E_{r+1})
\]

\[
F_0 \quad F_1 \quad \ldots \quad F_r
\]

where for \( i = 0, 1, \ldots, r \), \( W_{i+1} \to W_i \) is defined either by:

1. blowing up at centers \( Y_i \), permissible for the pair \((W_i, E_i)\), and \( Y_i \) included in the inductively defined closed sets \( F_i \subset W_i \), or
2. a projection \( p : W_{i+1} \to W_i \), there is an open covering \( \{U_{\alpha,r+1}\} \) of \( W_{r+1} \) (the pull back of \( \{U_\alpha\} \)), a sequence of transformations of basic objects,

\[
(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha) \leftarrow \tilde{W}_{\alpha,1}, (B_{\alpha,1}, d_\alpha), \tilde{E}_{\alpha,1}) \leftarrow \ldots \leftarrow \tilde{W}_{\alpha,r+1}, (B_{\alpha,r+1}, d_\alpha), \tilde{E}_{\alpha,r+1}),
\]

and a closed set \( F_{r+1} \subset W_{r+1} \), such that for each \( \alpha \in \Lambda \),

\[
F_{r+1} \cap U_{\alpha,r+1} = \text{Sing}(B_{\alpha,r+1}, d_\alpha).
\]

(vi) **Restriction to open sets.** If \( V \subset W \) is an open set, consider the restriction of all data to \( V \): the open covering \( \{U_\alpha \cap V\}_{\alpha \in \Lambda} \), the basic objects \( (\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)_V \), and the closed set \( F_V = F \cap V \). Then we require that all properties (i), (ii), (iii) (iv) and (v) hold for the restriction.

Last condition (vi) could be avoided if we assume desingularization. In fact, if \( Y \subset F \cap V \) is a smooth center, the closure of \( Y \) in \( W \) might be singular. If we assume desingularization we may assume that the closure is regular, and that the transformation over \( V \) is a restriction of a transformation over \( W \). However we want to prove desingularization, so we impose condition (vi).

A general basic object will be denoted by \((\mathcal{F}, (W, E))\), the restriction to an open set \( V \) will be denoted by \((\mathcal{F}_V, (V, E_V))\). Note that we have defined two notions of transformations of general basic objects: one as in (12.1.iii) (by a monoidal transformations), and another one as in (12.1.iv), by a projection. This last transformation increases the dimension by one.

We denote a sequence (of transformations and projections) as

\[
(\mathcal{F}_0, (W_0, E_0)) \leftarrow \ldots \leftarrow (\mathcal{F}_r, (W_r, E_r)) \leftarrow (\mathcal{F}_{r+1}, (W_{r+1}, E_{r+1}))
\]

\[
F_0 \quad F_1 \quad \ldots \quad F_r \quad F_{r+1}
\]

**Remark 12.2.** If \((\mathcal{F}, (W, E))\) is \( d \)-dimensional, then \( d \) can be different from \( \dim W \).

1. A basic object \((W, (J, b), E)\) defines a \( d \)-dimensional general basic object \((\mathcal{F}, (W, E))\), with the trivial open covering and \( d = \dim W \).
2. A simple basic object \((W, (J, b), E = \emptyset)\), with \( \dim W = d \) and \( R(1)(\text{Sing}(J, b)) = \emptyset \), also defines a general basic object \((\mathcal{F}, (W, E))\) of dimension \( d - 1 \).
This follows from Proposition 7.13 as was indicated in 10.4; see also 7.17 for the case of projections.

Remark 12.3. A general basic object can be described by giving two different open coverings. What is important here are the closed sets $F$ that it defines. That is why in the notation for general basic objects $(\mathcal{F}, (W, E))$, defined in terms of an open cover $\{U_\alpha\}$ of $W$ and basic objects $(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)$, is said to be a simple general basic object, when all the basic objects $(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)$ are simple (10.2).

We now extend the result in Proposition 10.3 to the case of general basic objects.

Let $R(1)(\mathcal{F})$ be the union of $d-1$ dimensional components of $\mathcal{F}$ (so that $R(1)(\mathcal{F}) \cap U_\alpha = R(1)(\text{Sing}(B_\alpha, d_\alpha)))$.

a) $R(1)(\mathcal{F})$ is open and closed in $\mathcal{F}$ (i.e. a union of connected components), and smooth in $W$.
b) Setting $(\mathcal{F}, (W, E)) \leftarrow (\mathcal{F}_1, (W_1, E_1))$ with center $R(1)(\mathcal{F})$, then $W_1 = W$ and $F_1 = F - R(1)(\mathcal{F})$; in particular:
c) $(\mathcal{F}_1, (W_1, E_1))$ is simple and $R(1)(\mathcal{F}_1) = \emptyset$.

Finally, one can generalize 12.2, 2), to show that if c) holds, then $(\mathcal{F}_1, (W_1, E_1))$ has a structure of $d-1$ dimensional general basic object (where $d =$ dimension of $(\mathcal{F}, (W, E)))$.

Definition 12.4. A resolution of a general basic object $(\mathcal{F}_0, (W_0, E_0))$ is a sequence of transformations as in (12.1.2) which fulfills the following two conditions:

(i) The sequence involves only monoidal transformations (12.1(iii)).
(ii) The closed set $F_{r+1}$ is empty.

Note that if $\{U_\alpha\}$ is an open covering of $W$ as in Definition 12.1, then for any $\alpha$ we obtain a resolution of the basic object $(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)$ as defined in 4.5.

12.5. We will assign, to each general basic object $(\mathcal{F}, (W, E))$, an upper semi-continuous function $f_\mathcal{F} : F \to (T, \geq)$ (on the closed set $F \subset W$ as in 12.1(ii)). Such functions will be defined so that they are compatible with open restrictions. In other words, if $V$ is an open subset of $W$, the closed set of the restriction $(\mathcal{F}_V, (V, E_V))$ is $F \cap V$, and we require that the restriction of $f_\mathcal{F}$ to $F \cap V$ be $f_{\mathcal{F}_V}$. The following is an example.

Lemma 12.6. Let $(\mathcal{F}, (W, E))$ be a general basic object, let $\{U_\alpha\}_{\alpha \in \Lambda}$ be the corresponding open covering of $W$, and let $(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)$ be the collection of $d$-dimensional basic objects associated to $(\mathcal{F}, (W, E))$. Then the functions $\text{ord}_\alpha^d : (F \cap U_\alpha =) \text{Sing}(B_\alpha, d_\alpha) \to \mathbb{Q}$ patch so as to define $\text{ord}_\mathcal{F}^d : F \to \mathbb{Q}$.

The proof of this Lemma will be developed in Section 13. It is an example of the principle of patching of functions (10.4). Indeed it is the main example, and the proof in Section 13 will clarify why projections were considered in 12.1(iv) (and in 10.9).
12.7. Define \((F_0, (W_0, E_0))\) as before, by the covering \(\{U_\alpha\}_{\alpha \in \Lambda}\) and basic objects \((\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha)\). Recall that a sequence of transformations

\[
(F_0, (W_0, E_0)) \leftarrow \ldots \leftarrow (F_r, (W_r, E_r)) \leftarrow (F_{r+1}, (W_{r+1}, E_{r+1}))
\]

(12.7.1)

induces, for each index \(\alpha\), a sequence of transformations of basic objects

\[
(\tilde{W}_\alpha, (B_\alpha, d_\alpha), \tilde{E}_\alpha) \leftarrow (\tilde{W}_\alpha)_1, ((B_\alpha)_1, d_\alpha), (\tilde{E}_\alpha)_1) \leftarrow \ldots
\]

\[
\ldots (\tilde{W}_\alpha)_r, ((B_\alpha)_r, d_\alpha), (\tilde{E}_\alpha)_r) \leftarrow ((\tilde{W}_\alpha)_{r+1}, ((B_\alpha)_{r+1}, d_\alpha), (\tilde{E}_\alpha)_{r+1})
\]

(12.7.2)

and for each index \(k\), set

\[
(B_\alpha)_k = I(H_{\alpha,1})^{a_{\alpha,1}} \cdot I(H_{\alpha,2})^{a_{\alpha,2}} \ldots I(H_{\alpha,k})^{a_{\alpha,k}} \cdot (B_\alpha)_k
\]

(12.7.3)

as in [10.5.3].

**Lemma 12.8.** Assume that sequence (12.7.1) is such that, for each index \(0 \leq k \leq r\):

1) The functions

\[\text{w-ord}_d^{\alpha,k} : (F_k \cap (U_\alpha)_k = \text{Sing})((B_\alpha)_k, d_\alpha) \to \mathbb{Q}\] and \(\overline{\alpha}_{r,i} : \text{Sing}(((B_\alpha)_k, d_\alpha)) \to \mathbb{Q}\)

patch to define functions

\[\text{w-ord}_k : F_k \to \mathbb{Q}\] and \(\overline{\alpha}_i : F_k \to \mathbb{Q}\).

2) If \((F_k, (W_k, E_k)) \leftarrow (F_{k+1}, (W_{k+1}, E_{k+1}))\) is defined by a monoidal transformation, assume that the center \(Y_k\) is such that:

\[Y_k \subset \max \text{w-ord}_k(\subset W_k)\]

Then, under assumptions 1) and 2), the functions defined in terms the expression (12.7.3) for the index \(r + 1\), namely the functions

\[\text{w-ord}_d^{\alpha,r+1} : \text{Sing}((B_\alpha)_{r+1}, d_\alpha) \to \mathbb{Q}\] and \(\overline{\alpha}_{r,i} : \text{Sing}(((B_\alpha)_{r+1}, d_\alpha)) \to \mathbb{Q}\)

( see [10.5], patch, and define functions

\[\text{w-ord}_{r+1} : F_{r+1} \to \mathbb{Q}\] and \(\overline{\alpha}_r : F_{r+1} \to \mathbb{Q}\).

**Proof.** A) Assume that \((F_r, (W_r, E_r)) \leftarrow (F_{r+1}, (W_{r+1}, E_{r+1}))\) is defined by a projection \(W_r \leftarrow W_{r+1} = W_r \times A^1\). Then for each \(\alpha\), the expression (12.7.3) for the index \(r + 1\) is the pull-back of the expression for index \(r\). In such case the patching of functions with index \(r + 1\) follows from the case of index \(r\).

B) Assume that \((F_r, (W_r, E_r)) \leftarrow (F_{r+1}, (W_{r+1}, E_{r+1}))\) is defined by a center \(Y_r \subset F_r\) and let \(H_{r+1} \subset W_{r+1}\) denote the exceptional locus. Choose a point \(x \in F_{r+1} \subset W_{r+1}\) and assume that \(x \in U_{\alpha_1,r+1} \cap U_{\alpha_2,r+1}\). Consider the two expressions:

\[
(B_{\alpha_j})_k = I(H_{\alpha_j,1})^{a_{\alpha_j,1}} \cdot I(H_{\alpha_j,2})^{a_{\alpha_j,2}} \ldots I(H_{\alpha_j,k})^{a_{\alpha_j,k}} \cdot (B_{\alpha_j})_k
\]

(12.7.4)

We want to prove that \(\text{w-ord}_{\alpha_j,r+1}(x) = \text{w-ord}_{\alpha_2,r+1}(x)\), and that \(\overline{\alpha}_{\alpha_1,i}(x) = \overline{\alpha}_{\alpha_2,i}(x)\) for each \(H_i \in E_{r+1}\). If \(x \in F_{r+1} \cap H_{r+1}\) then \(x\) can be identified with a point in \(F_r\), and the equalities follow by assumption. So assume that \(x \in F_{r+1} \cap H_{r+1}\). Note that

\[
\text{ord}_{\alpha_j}(x) = \sum \overline{\alpha}_{\alpha_j,i}(x) + \text{w-ord}_{\alpha_j,r+1}(x).
\]
Lemma 12.8 asserts that \(\operatorname{ord}_{a_1}(x) = \operatorname{ord}_{a_2}(x)\); and by assumption, we also know that \(\overline{a}_{\alpha_1, i}(x) = \overline{a}_{\alpha_2, i}(x)\) for each hypersurface \(H_i\) with index \(i < r + 1\). So it suffices to prove that

\[
(12.8.2) \quad \overline{a}_{\alpha_1, r+1}(x) = \overline{a}_{\alpha_2, r+1}(x).
\]

Note that \(x \in H_{r+1} \subset W_{r+1}\) maps to a point \(x' \in Y_r \subset W_r\). Let \(y\) be the generic point of the irreducible component of \(Y_r\) containing \(x'\). To settle (12.8.2), note that

\[
\overline{a}_{\alpha_j, r+1}(x) = \sum_{H_i \in \mathcal{E}_r} \overline{a}_{\alpha_j, r}(y) + \operatorname{w-ord}_{\alpha_j, r}(y),
\]

and, by assumption, all terms are independent of \(j\).

**Remark 12.9.** Lemma 12.8 was proved under some assumptions on (12.7.1) (for each index \(0 \leq k \leq r\)). Note that such assumptions hold for \(r = 0\). As in 10.7 we see that

\[
\max \operatorname{w-ord}_{k+1} \geq \max \operatorname{w-ord}_{k+1} \geq \ldots \geq \max \operatorname{w-ord}_{k+1}
\]

since \(Y_i \subset \operatorname{Max} \operatorname{w-ord}_i \subset F_i\). Set \(b' = \max \operatorname{w-ord}_i\), so \(b' \geq b' \geq \ldots \geq b' \geq b'\), and let \(k_0\) be the smallest index such that \(b' = b' = \ldots = b'\).

For each index \(k_0 \leq j \leq k\) define a partition on the set of hypersurfaces in \(E_j\), say \(E_j = E_j^- \cup E_j^+\), where \(E_j^- = \{H_1, \ldots, H_r, \ldots H_{r+k_0}\}\) and \(E_j^+ = \{H_{r+k_0+1}, \ldots H_{r+j}\}\).

For \(j = k_0\), \(E_{k_0} = E_{k_0}^\cdot\), and for \(j > k_0\), \(E_{j}^\cdot\) are the strict transforms of hypersurfaces in \(E_{k_0}\). Order \(\mathbb{Q} \times \mathbb{N}\) lexicographically, and set

\[
t^d_k : F_k \rightarrow \mathbb{Q} \times \mathbb{N} \quad \text{and} \quad n^d_k(\xi) = \left\{ \begin{array}{ll}
\# \{H \in E_k | \xi \in H\} & \text{if } \operatorname{w-ord}_k(\xi) < b' \\
\# \{H \in E_k^- | \xi \in H\} & \text{if } \operatorname{w-ord}_k(\xi) = b'.
\end{array} \right.
\]

One can check, as for the case of basic objects, that this function is upper-semi-continuous.

We now extend 10.10 to the setting of general basic objects.

**Proposition 12.10.** Consider a sequence of transformations

\[
(F_0, (W_0, E_0)) \leftarrow \ldots \leftarrow (F_{k_0-1}, (W_{k_0-1}, E_{k_0-1})) \leftarrow (F_{k_0}, (W_{k_0}, E_{k_0}))
\]

(12.10.1)

where each transformation is either a projection, or transformation with centers \(Y_i \subset \operatorname{Max} t_i\); and assume that \(\max t_{i-1} \geq \max t_i\). Assume, in addition, that \(\max t_{k_0-1} > \max t_{k_0}^\cdot\), or that \(k_0 = 0\).

Then, there is a simple general basic object \((G_{k_0}, (W_{k_0}, E_{k_0}^\cdot))\), such that any resolution

\[
(G_{k_0}, (W_{k_0}, E_{k_0}^\cdot)) \leftarrow (G_{k_0+1}, (W_{k_0+1}, E_{k_0+1}^\cdot)) \leftarrow \ldots \leftarrow (G_{k_0+m}, (W_{k_0+m}, E_{k_0+m}^\cdot))
\]

(12.10.2)

induces a sequence of transformations

\[
(F_0, (W_0, E_0)) \leftarrow \ldots \leftarrow (F_{k_0}, (W_{k_0}, E_{k_0})) \leftarrow \ldots \leftarrow (F_{k_0+m}, (W_{k_0+m}, E_{k_0+m})).
\]

(12.10.3)
Furthermore, this sequence has the following two properties:

**P1** max $t_{k_0}^d = \ldots = t_{k_0+m-1}^d$; and $F_{k_0+m} = \emptyset$ or $\max t_{k_0+m-1}^d > t_{k_0+m}^d$.

**P2** $\max t_{k_0+j}^d = G_{k_0+j}$ for $0 \leq j \leq m - 1$.

**Proof.** Note that the properties in Proposition 10.10 assert that $(G_{k_0}, (W_{k_0}, E_{k_0}'))$ is indeed a general basic object.

**Proposition 12.11.** Assume that (12.10) is such that $\max t_{k_0-1}^d > t_{k_0}^d$ and that $\max w$-ord$_{k_0} = 0$. Then, there are upper-semi-continuous functions $h_i: F_{k_0+i} \to \Gamma$, and a resolution

$$(F_{k_0}, (W_{k_0}, E_{k_0})) \leftarrow (F_{k_0+1}, (W_{k_0+1}, E_{k_0+1})) \leftarrow \cdots \leftarrow (F_{k_0+m}, (W_{k_0+m}, E_{k_0+m})).$$

The resolution defined by blowing up successively on $\max h_i$.

**Proof.** This is an extension of 11.6 to the case of general basic object. The fact that $h_i$ are well defined as functions on $F_{k_0+i}$ follows from Lemma 12.8.

**Theorem 12.12.** (Theorem (d)) Fix a positive integer $d$. There is a totally ordered set $I^d$, and for each $d$-dimensional general basic object $(F_0, (W_0, E_0))$, a function $f_{F_0}: F_0 \to I^d$. The functions defined so that:

i) $f_{F_0}$ is upper-semi-continuous, and $\max f_{F_0}$ is a smooth permissible center for $(F_0, (W_0, E_0))$.

ii) For each $(F_0, (W_0, E_0))$, there is a resolution $R_{F_0}$ (Def 12.7), say

$$\begin{align*}
(F_0, (W_0, E_0)) &\leftarrow \cdots \leftarrow (F_r, (W_r, E_r)) \leftarrow (F_{r+1}, (W_{r+1}, E_{r+1})) \leftarrow \cdots \leftarrow (F_{k_0+m}, (W_{k_0+m}, E_{k_0+m})),
\end{align*}$$

(12.12.1)

obtained by blowing up successively at $\max f_{F_r}$ ($f_{F_r}: F_r \to I^d$).

**Proof.** The proof is based on inductive argument, so we first show why Theorem 12.12 holds for $0$-dimensional general basic objects: Note that in such case, each $(W_0, (B_0, d_0), (E_0))$ is zero dimensional, so we can assume that each $\bar{W}_\alpha$ is a point, and hence, each $B_0$ is a non-zero ideal in a field. Therefore, Sing$(B_0, d_0) = \emptyset$, and hence, $F_0 = \emptyset$. Here we can take $I^0$ to be a point; it plays no role in any case.

Set $T^d = \{\infty\} \sqcup (Q \times Z) \sqcup \Gamma$ with $\Gamma$ as in 11.3. This disjoint union is totally ordered by setting $\infty$ as the biggest element, and $\alpha < \beta$ if $\beta \in (Q \times Z)$ and $\alpha \in \Gamma$. We now set $I^d = T^d \times I^{d-1}$ ordered lexicographically. In our proof, upper-semi-continuous functions $f_{i}^d: F_i \to I_d$ will be defined with the property stated in the theorem; namely, that a resolution (12.12.1) will be achieved by taking successive monoidal transformations with centers $Y_i = \max f_{i+1}^d(\subset F_i)$.

For the index $i = 0$ we know that the locally defined functions ord$_{i}^d: \text{Sing}(B_0, d_0) \to Q$ and $n_{i}^d: \text{Sing}(B_0, d_0) \to Z$ patch so as to define functions

$$\text{ord}_{F}^d: F \to Q \quad \text{and} \quad n_{F}^d: F \to Z,$$

and also an upper-semi-continuous function

$$t_0 = (\text{ord}_{F}^d, n_{F}^d): F_0 \to Q \times Z$$

as in 12.9 (recall that $w$-ord$_0 = \text{ord}_0$).
We attach to the value $\text{max } t^d_0$ the simple general basic object $(G_0, (W_0, E'_0))$ as in (12.10) so that $G_0 = \text{Max } t^d_0$. We now define a resolution of this simple general basic object:

$$(12.12.2) \quad (G_0, (W_0, E'_0)) \leftarrow (G_1, (W_1, E'_1)) \leftarrow \ldots (G_m, (W_m, E'_m))$$

and, in order to define this resolution, we first apply the transformation with center $R(1)(G_0)$, if not empty, so as to obtain a $d-1$ general basic object (see (12.3)). We then proceed to define the resolution (12.12.2) by induction (i.e. by blowing up successively at Max $W^{-\text{ord}}$).

We define a resolution of $(G_0, (W_0, E'_0))$ as in 12.11. Hence, a resolution ($12.12.2$) is obtained by blowing up at Max $W^{-\text{ord}}$. We now define a resolution of this simple general basic object:

$$(12.12.3) \quad (F_0, (W_0, E'_0)) \leftarrow (F_1, (W_1, E'_1)) \leftarrow \ldots (F_m, (W_m, E'_m))$$

and functions $t^d_i : F_i \rightarrow \mathbb{Q} \times \mathbb{Z}$ for $i = 0, \ldots, m$. Furthermore, $G_i = \text{Max } t^d_i$ for $i = 0, \ldots, m-1$, and

$$(12.13) \quad \text{max } t^d_0 = \text{max } t^d_1 = \ldots = \text{max } t^d_{m-1}; \text{ and either } F_m = \emptyset \text{ or } \text{max } t^d_{m-1} > \text{max } t^d_m.$$
Since \( \text{Max} f_i^d \subset G_i \subset F_i \), a point \( x \in F_i - G_i \) can be identified with a point, say \( x \in F_{i+1} \). Furthermore, since \( \text{(12.12)} \) is a resolution, there is smallest index \( i_0 > i \) such that \( x \) can be identified with a point, say again \( x \in G_{i_0} \subset F_{i_0} \). Define
\[
f_i(x) = f_{i_0}(x).
\]
Note that \( t_i^d(x) = t_{i_0}^d(x) \) (if \( \text{w-ord}_i(x) > 0 \)), that \( h_i(x) = h_{i_0}(x) \) (if \( \text{w-ord}_i(x) = 0 \)); and that an open neighborhood of \( x \) in \( F_i \) can be identified with a neighborhood of \( x \) in \( G_{i_0} \). Finally argue coordinate-wise (as in 10.7) to show that the extended functions \( f_i^d : F_i \to I^d = T^d \times I^{d-1} \) are in fact upper-semi-continuous.

The compatibility with open restrictions of the functions \( t_i^d \) and \( h_i \), and also that of \( f_i^{d-1} \) (by induction), insure that the same property holds for the functions \( f_i^d \) (see 12.5).

12.14. On Resolution functions and Proof of 4.7. Recall that basic objects are, in particular, general basic objects \( \text{(12.2)} \), so Theorem 12.12 provides, for each dimension \( d \), resolution functions as in 10.7. Here
\[
I^d = T^d \times I^{d-1} = T^d \times T^{d-1} \times \cdots \times T^0
\]
and \( f_i^d(x) \) can be expressed with \( d + 1 \) coordinates. For instance case i) in 12.13 is:

i) \( f_0^d(x) = (\max t^d, \infty, \infty, \ldots, \infty) \in I^d \) if \( x \in R(1)(G_0) \).

If \( W \) is smooth of dimension \( d \), and \( X \subset W \) is a smooth hypersurface, then the basic object \( (W, (J, 1), E = \emptyset) \) defines a \( d \)-dimensional general basic object. In this case \( t_0^d(x) = (1, 0) \), and \( f_0^d(x) = ((1, 0), \infty, \infty, \ldots, \infty) \) for any \( x \in \text{Sing}(J, 1) \).

If \( X \subset W \) is smooth of codimension two, then \( f_0^d(x) = ((1, 0), (1, 0), \infty, \ldots, \infty) \) for any \( x \in \text{Sing}(J, 1) \).

If \( X \subset W \) is smooth of codimension \( r \), then \( f_0^d(x) = R = ((1, 0), (1, 0), \ldots, (1, 0), \infty, \ldots, \infty) \) \( (r \text{ copies of } (1, 0)) \) for any \( x \in \text{Sing}(J, 1) \); and if \( X \subset W \) is reduced, pure dimensional and of codimension \( r \), then \( ((1, 0), (1, 0), \ldots, (1, 0), \infty, \ldots, \infty) \) \( (r \text{ copies of } (1, 0)) \) is the value \( R \) in property P4 of 12.7.

13. On Hironaka’s trick and proof of Lemma 12.76

The purpose of this Section is to prove Lemma 12.76 which states that the function \( \text{ord} \), introduced in 10.3 for basic objects, can be naturally defined in the setting of general basic objects.

Let \( (\mathcal{F}, (W, E)) \) be an \( d \)-dimensional general basic object, and set an open covering \( \{U_\alpha\}_{\alpha \in \Lambda} \) of \( W \) as in Definition 12.21.

Recall that \( (\mathcal{F}, (W, E)) \) defines a closed set \( F(\subset W) \), and that for each index \( \alpha \) there is a closed smooth \( d \)-dimensional subscheme \( \overline{W}_\alpha \subset U_\alpha \), and a basic object \( (\overline{W}_\alpha, (B_\alpha, d_\alpha), \overline{E}_\alpha) \) such that
\[
F \cap U_\alpha = \text{Sing}(B_\alpha, d_\alpha).
\]

Assume that a point \( x \in F \) appears in two such charts, namely \( x \in F \cap U_\alpha \cap U_\beta \). In order to simplify notation set
\[
(\overline{W}_\alpha, (B_\alpha, d_\alpha), \overline{E}_\alpha) = (W', (B', d'), E')
\]
and
\[
(\overline{W}_\beta, (B_\beta, d_\beta), \overline{E}_\beta) = (W'', (B'', d''), E'').
\]
So \( x \in \operatorname{Sing}(B',d') = \operatorname{Sing}(B'',d'') \) and the claim in Lemma 12.6 is that:

\[
\frac{\nu_{B'}(x)}{d'} = \frac{\nu_{B''}(x)}{d''},
\]

(notation as in 10.5).

**Proof.** Set \( \omega' = \nu_{B'}(x) \) and \( \omega'' = \nu_{B''}(x) \). We shall prove the Lemma by constructing infinitely many sequences of transformations of general basic objects. A sequence

\[(13.0.1) \quad (F, (W,E)) \overset{\Pi_0}{\leftarrow} (F_0, (W_0, E_0)) \overset{\Pi_1}{\leftarrow} (F_1, (W_1, E_1)) \overset{\Pi_2}{\leftarrow} \cdots \overset{\Pi_k}{\leftarrow} (F_k, (W_k, E_k))
\]

of transformations of general basic objects defines sequences of transformations of basic objects, say:

\[(13.0.2) \quad (W', (B', d'), E') \overset{\Pi_0'}{\leftarrow} (W'_0, (B'_0, d'), E'_0) \overset{\Pi_1'}{\leftarrow} (W'_1, (B'_1, d'), E'_1) \overset{\Pi_2'}{\leftarrow} \cdots \]

\[
\cdots \overset{\Pi_k'}{\leftarrow} (W'_k, (B'_k, d'), E'_k),
\]

and

\[(13.0.3) \quad (W'', (B'', d''), E'') \overset{\Pi_0''}{\leftarrow} (W''_0, (B''_0, d''), E''_0) \overset{\Pi_1''}{\leftarrow} (W''_1, (B''_1, d''), E''_1) \overset{\Pi_2''}{\leftarrow} \cdots \]

\[
\cdots \overset{\Pi_k''}{\leftarrow} (W''_k, (B''_k, d''), E''_k).
\]

We take the first transformation \( \Pi_0 \) of (13.0.1) to be a projection (as in 12.1 iv)), so the first transformations of (13.0.2) and (13.0.3) are projections too. All the other transformation will be permissible transformations (as in (as in 12.1 iii)). For each index \( k > 0 \), sequence (13.0.1) will be defined as follows:

1. Identify \( L_0 = \Pi_0^{-1}(x) \) with \( A^4_k \) and set \( x_0 = 0 \in L_0 \). Note that \( L_0 \subset F_0 \), the singular locus of \((F_0, (W_0, E_0))\).
2. Given an index \( s \geq 0 \), a line \( L_s \subset F_s \) and a point \( x_s \in L_s \), define the transformation \( \Pi_{s+1} \) with center \( x_s \). Now set:
   - i: \( L_{s+1} \subset F_{s+1} \) as the strict transform of \( L_s \);
   - ii: \( H_{s+1} \subset E_{s+1} \) as the exceptional locus of \( \Pi_{s+1} \);
   - iii: \( x_{s+1} = H_{s+1} \cap L_{s+1} \).

In this way (1) together with (2) provide a rule to construct a sequence (13.0.1) of length \( s \), for any \( s \geq 1 \). In this sequence \( L_s \subset F_s \) for any \( s \), so in particular \( x_s \in F_s \), and by assumption:

\[
x_s \in \operatorname{Sing}(B'_s, d') = \operatorname{Sing}(B''_s, d'') \quad \forall s \geq 0.
\]

Locally at \( x_s \) there are expressions, as in (10.5.3), say:

\[(13.0.4) \quad (B'_s)_{x_s} = I(H'_s)_{x_s}^{d'_s}, (B''_s)_{x_s} = I(H''_s)_{x_s}^{d''_s}.
\]

Note that here \( H'_s = H_s \cap W'_s \) and \( H''_s = H_s \cap W''_s \). On may check, by induction on \( s \), that

\[
\omega'_s = s(\omega' - d') \quad \omega''_s = s(\omega'' - d'').
\]

Since only the first term of this sequence is a projection, for \( s \geq 1 \), \( \dim(W'_s) = \dim(W''_s) = d + 1 \). It follows that

\[
\dim(F_s \cap H_s) = d \iff a'_s = s(\omega' - d') \geq d' \\
\quad \iff a''_s = s(\omega'' - d'') \geq d''.
\]
Note that \( \dim H'_s = \dim H''_s = d \), so if \( \dim(F_s \cap H_s) = d \) then \( F_s \cap H_s = H'_s = H''_s \).

Furthermore if \( \dim(F_s \cap H_s) = d \), then \( F_s \cap H_s \) is a permissible center for the general basic object. In such case, set \( F_s \cap H_s \) as a center of a transformation \( \Pi_{s+1} \). It turns out that in (13.0.4),

\[
a'_s = s(\omega' - d') - d', \quad a''_s = s(\omega'' - d'') - d''.
\]

Fix the index \( s \) and set, if possible, the center of transformations \( \Pi_{s+j} \) to be \( F_{s+j} \cap H_{s+j} \), for \( j \geq 0 \).

Note that \( \dim(F_{s+j} \cap H_{s+j}) = d \iff a'_s + j \geq d' \iff a''_s + j \geq d'' \).

And we conclude that

\[
\dim(F_{s+j} \cap H_{s+j}) = d \text{ (in which case is a permissible center)} \iff j \leq \ell'_s \quad \iff j \leq \ell''_s,
\]

where

\[
\ell'_s = \left\lfloor \frac{s(\omega' - d')}{d'} \right\rfloor, \quad \ell''_s = \left\lfloor \frac{s(\omega'' - d'')}{d''} \right\rfloor
\]

and \( \lfloor \cdot \rfloor \) denotes the integer part.

Finally note that

\[
\frac{\nu_{B'}(x)}{d'} = \frac{w'}{d'} = \lim_{s \to \infty} \frac{1}{s} \ell'_s + 1, \quad \frac{\nu_{B''}(x)}{d''} = \frac{w''}{d''} = \lim_{s \to \infty} \frac{1}{s} \ell''_s + 1.
\]

This expresses the rational numbers \( \frac{\nu_{B'}(x)}{d'} \) and \( \frac{\nu_{B''}(x)}{d''} \) in terms of permissible sequences of the general basic object \((F, (W, E))\). Hence \( \frac{\nu_{B'}(x)}{d'} = \frac{\nu_{B''}(x)}{d''} \). \( \square \)

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