Gravity of superheavy higher-dimensional global defects

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Abstract

Numerical solutions of Einstein’s and scalar-field equations are found for a global defect in a higher-dimensional spacetime. The defect has a $(3 + 1)$-dimensional core and a “hedgehog” scalar-field configuration in $n = 3$ extra dimensions. For sufficiently low symmetry-breaking scales $\eta$, the solutions are characterized by a flat worldsheet geometry and a constant solid deficit angle in the extra dimensions, in agreement with previous work. For $\eta$ above the higher-dimensional Planck scale, we find that static-defect solutions are singular. The singularity can be removed if the requirement of staticity is relaxed and defect cores are allowed to inflate. We obtain an analytic solution for the metric of such inflating defects at large distances from the core. The

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three extra dimensions of the nonsingular solutions have a “cigar” geometry. Although our numerical solutions were obtained for defects of codimension $n = 3$, we argue that the conclusions are likely to apply to all $n \geq 3$.

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I. INTRODUCTION

According to the “brane-world” picture, our universe is a (3 + 1)-dimensional brane floating in a higher-dimensional bulk spacetime [1–3]. The branes can be thought of as fundamental $D$-branes, or they can arise as topological defects in a higher-dimensional field theory. It may well be that $D$-branes could also eventually find some dual field-theory description. In the simplest, codimension-one models, the brane can be pictured as a domain wall propagating in a 5D spacetime [1,4]. Higher codimensions with both gauge and global defects have also been considered. In particular, models have been discussed where the field configuration in the directions orthogonal to the brane is that of a cosmic string [5], a monopole [6,7], or a texture [8]. In these models, the bulk curvature produced by the defects plays a major role in localizing gravity on the brane and in solving the mass-hierarchy problem.

The solutions of Einstein’s equation describing higher dimensional defects with a global charge found in Ref. [6] are very simple. In the absence of a bulk cosmological constant, and for codimension $n > 2$, the spacetime is flat along the brane worldsheet and is characterized by a constant solid angle deficit in the transverse directions. The angle deficit grows as the symmetry breaking scale $\eta$ of the defect is increased, and eventually consumes the entire solid angle at some critical value $\eta_c$, which is comparable to the higher-dimensional Planck scale. For $\eta = \eta_c$, the transverse metric becomes that of a cylinder.

The solution of Ref. [6] were obtained in the asymptotic region outside the defect core. In the core region, analytic solutions cannot be found, and one of the purposes of the present paper is to check numerically that the asymptotic solutions of Ref. [6] can be matched to appropriate core solutions.

Another important question is what happens if $\eta$ is greater than $\eta_c$. One of the motivations for looking into this regime is the interesting work by Dvali, Gabadadze, and Shifman [9], suggesting that brane-world models of codimension $n > 2$, with a very low value of the bulk Planck scale, can provide a solution to the cosmological constant problem.
It was conjectured in Ref. [9] that defect solutions in such models may exhibit de Sitter inflation of the worldsheet, with the expansion rate inversely proportional to the brane tension. The very low-accelerated expansion rate observed today could then be obtained with a very large-brane tension.

The main purpose of the present paper is to solve Einstein’s and scalar-field equations for super-critical global defects. Specifically, we consider a codimension-3 global monopole. In the following Section, we introduce the model and review the asymptotic solutions of Ref. [6]. The numerical solutions assuming a static (noninflating) worldsheet are presented in Sec. III, for both sub- and super-critical regimes. We find that super-critical solutions have a curvature singularity at a finite distance from the monopole core. Inflating defect solutions are discussed in Sec. IV, where we show that the expansion rate $H$ can be adjusted so that the singularity is removed. We interpret the resulting nonsingular geometries as the physical defect solutions. Our conclusions are summarized in Sec. V.

II. FIELD EQUATIONS AND ASYMPTOTIC SOLUTIONS

The action for our model is

$$S = \int d^7x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_A \phi^a \partial^A \phi^a - \frac{\lambda}{4} (\phi^a \phi^a - \eta^2)^2 \right],$$

where $\kappa^2 = 1/M^{2+n}$, $M$ is the 7D Planck mass, and $\eta$ is the symmetry-breaking scale. The scalar-triplet field $\phi^a$ is assumed to be in a global monopole “hedgehog” configuration in $n = 3$ extra transverse dimensions, $\phi^a = \phi(r)\hat{x}^a$.

The general static form of the seven dimensional metric with spherical symmetry in the extra three dimensions is

$$ds^2 = B^2(r)ds_4^2 + dr^2 + C^2(r)r^2d\Omega_2^2,$$

where $ds_4^2$ represents the 4D world-volume metric. The corresponding Einstein equations are
\[-G_\mu^\nu = -\frac{3}{B} \left( \frac{B'}{B} \right)^2 - \frac{6}{B} \frac{B'}{Br} - \frac{6}{C} \frac{C'}{BC} - \frac{2}{C} \frac{C''}{Cr} - \frac{1}{C^2 r^2} - \frac{1}{r^2} + \frac{1}{4} \frac{\bar{R}}{B^2} \]

where \( \bar{R} \) is the Ricci scalar of the 4D world volume.

The field equation for the scalar field is

\[
\phi'' + 2 \left( \frac{2}{B} \frac{B'}{B} + \frac{C'}{C} + \frac{1}{r} \right) \phi' - \frac{2}{C^2 r^2} \phi - \frac{\lambda}{\kappa^2} (\phi^2 - \eta^2) = 0. 
\]

The solution obtained in Ref. [6], assuming a flat 4D world-volume, \( \bar{R} = 0 \), and \( \phi = \eta \) in the asymptotic region, is of the form

\[
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + (1 - \kappa^2 \eta^2) r^2 d\Omega_2^2. 
\]

The solid angle deficit is \( \Delta \Omega = 4\pi \kappa^2 \eta^2 \), and the critical symmetry-breaking scale is

\[
\eta_c = 1/\kappa. 
\]

It was shown in Ref. [6] that for this value of \( \eta \), the Einstein equations have a cylindrical solution

\[
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + C_0^2 d\Omega_2^2, 
\]

where \( C_0 \) is an arbitrary constant. The expectation was that this constant radius of the cylinder can be determined by matching to an appropriate-core solution, so that the full 3D geometry will be that of a “cigar”. In the next section, we shall see that (i) this expectation for the critical solution is not supported by the numerical analysis, but (ii) cigar-type geometries do nonetheless make their appearance in super-critical monopole solutions.
III. STATIC SOLUTIONS

We numerically solved the Einstein and scalar-field equations (3)-(6) with boundary conditions $B(0) = C(0) = 1$, $B'(0) = C'(0) = 0$, $\phi(0) = 0$, and $\phi = \eta$ at large $r$. For static solutions, we assumed a flat worldsheet, $ds_4^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, and $\bar{R}^{(4)} = 0$.

The solutions for the scalar field $\phi$ and the metric of a sub-critical monopole ($\kappa \eta = 0.2$) are shown in Fig. 1. As expected, $\phi$ rapidly approaches $\eta$, and the metric coefficients $B$ and $C$ approach constants at large $r$. The asymptotic value of the angle-deficit coefficient $C$ is in agreement with Eq. (7). As $\eta$ gets closer to $\kappa^{-1}$, the approach to this asymptotic value becomes increasingly slow.

In the critical case, $\kappa \eta = 1$, $C \to 0$ as $r \to \infty$. However, the radius of the extra dimensions $Cr$ is a growing function of $r$ (see Fig. 2). Thus, the asymptotic behavior of the critical-monopole solution is not described by the cylindrical metric (9). With $\phi \approx \eta$, we were able to find a power-law analytic solution of Einstein equations,

\begin{align*}
B &\propto r^{(2-\sqrt{10})/12} \approx r^{-0.097}, \\
C &\propto r^{(-5+\sqrt{10})/6} \approx r^{-0.306},
\end{align*}

which gives a good fit to our numerical solution at large $r$.

In the super-critical case, $\kappa \eta > 1$, we find that static solutions develop a curvature singularity at a finite distance from the defect core. When $\eta$ is only slightly above the critical value, this distance can be very large. The solution initially behaves as the “cigar” solution (9), but eventually the radius of extra dimensions blows up, $C \to \infty$, and the metric coefficient $B$ plummets to zero at the same finite value of $r = r_s$. This behavior is illustrated in Fig. 3 for $\kappa \eta = 1.06723$. The nature of this singularity is similar to that of a static 4D global string [10].

As the symmetry-breaking scale $\eta$ is increased, the singularity gets closer to the core of the monopole. We performed numerical calculations for a number of values of $\eta$; the resulting dependence $r_s(\eta)$ is plotted in Fig. 4. For $\eta \gtrsim 1.2$, the singularity gets too close to
the center, so that the scalar field does not fully relax to its asymptotic value, \( \phi = \eta \), before the singularity forms.

**IV. INFLATING SOLUTIONS**

It is well known that the singularity of some static-defect solutions can be removed by relaxing the requirement of staticity and allowing the defect worldsheet to inflate. Examples are domain walls [11,12] and global strings [13,14] in (3 + 1) dimensions. It is, therefore, reasonable to check if the same strategy works in higher dimensions.

The 4D metric of an inflating defect worldsheet is the de Sitter metric,

\[
d_{\bar{s}}^2 = -dt^2 + e^{2HT}d\mathbf{x}^2,
\]

and the 4D curvature scalar is \( \bar{R}^{(4)} = 12H^2 \). We solved the equations numerically for different values of \( H \), and indeed, for values of \( \eta \) above, but not too far above the critical, we were able to adjust \( H \) so that the singularity is removed. Two super-critical, nonsingular solutions are shown in Fig. 5.

We see that the extra dimensions in these solutions have a “cigar” geometry, with \( Cr \rightarrow \text{const} \) at large \( r \), and that the scalar field quickly approaches a constant value, which, however, is somewhat below \( \eta \). The latter fact can be easily understood from the field equation (6) for \( \phi \). With \( \phi' = 0 \) and \( \sqrt{\lambda \eta}Cr \equiv C_0 = \text{const} \), it gives

\[
\frac{\phi^2}{\eta^2} = 1 - \frac{2}{C_0^2}.
\]

We were able to find an exact analytic solution of our system of equations, which describes the asymptotic behavior of these nonsingular solutions:

\[
\phi = \phi_0 = \frac{\sqrt{5 - \kappa^2\eta^2}}{2\kappa},
\]

\[
\sqrt{\lambda \eta}B = \frac{H}{k} \sin(\sqrt{\lambda \eta}kr),
\]

\[
\sqrt{\lambda \eta}Cr = C_0 = \sqrt{\frac{8}{5\kappa^2 \eta^2 - 1}},
\]
where \( k = \sqrt{5/128(\kappa^2\eta^2 - 1)}/\kappa\eta \).

In order to solve the equations numerically, we had to impose the boundary condition \( \phi = \phi_0 \) at large \( r \). The actual sequence of steps that led to the solution was that, after experimenting with numerical solutions for different values of \( H \) with our old boundary condition, \( \phi(r \to \infty) = \eta \), we guessed that the nonsingular solution should have a cigar geometry. We then found the analytic solutions (14)-(16) and used (14) to set the boundary condition.

In the asymptotic solution (15), the expansion rate \( H \) can be absorbed by rescaling the time coordinate \( t \) in Eq. (12); then the solution takes the form

\[
(\lambda\eta^2)ds^2 = k^{-2}(\sin^2 \chi d\bar{s}_+^2 + d\chi^2) + C_0^2 d\Omega_2^2,
\]

where \( \chi = \sqrt{\lambda}\eta kr \) and \( d\bar{s}_+^2 \) is the 4D de Sitter metric with \( H = 1 \). The above metric is essentially the same as the one found in Ref. [6] as a solution for \( \phi = \eta \) and a positive bulk cosmological constant. In our model, the role of the cosmological constant is played by the scalar-field potential, \( V(\phi_0) > 0 \).

The solution (17) has an apparent singularity at \( \chi = \pi \), but as it was noted in Ref. [6], the first two terms in Eq. (17) describe a 5D de Sitter space. The 5D inflation rate is \( H_5 = \sqrt{\lambda}\eta k \), and the surface \( \chi = \pi \) corresponds to the de Sitter horizon. This shows that the space outside the monopole core inflates not only along the 4D worldsheet, but also in one of the transverse directions, while two of the extra dimensions remain compactified in a sphere. This behavior is reminiscent of the topological-inflation scenario [15,16], where inflation occurs in the cores of topological defects in 4D. There are, however, important differences. First, the defect core has a fixed radius and is not expanding in the radial direction, and second, two of the extra dimensions remain compactified, also at a fixed radius.

Although the 4D expansion rate \( H \) drops out of the asymptotic solution, it is a meaningful parameter for the full spacetime. With our boundary conditions at \( r = 0 \), it gives the inflation rate in the core of the monopole. As we saw in the preceding section, the static
$H = 0$ solutions with $\kappa \eta > 1$ are singular. As $H$ is increased, the singularity gets milder, and at a certain value of $H$ it disappears completely, resulting in a cigar geometry. We interpret the corresponding solution as the physical solution describing the spacetime of a superheavy defect.

Our conjecture is that for each $\eta$, only one particular value of $H$ gives a nonsingular solution. Of course, it is impossible to find this precise value numerically, and therefore, all super-critical numerical solutions that we considered eventually developed a singularity. The nature of the singularity is similar to that of a static solution in Fig. 3, with $B \to 0$ and $C \to \infty$. By adjusting the value of $H$, we were able to shift the onset of the singularity to larger and larger radii, so that the solution got closer and closer to the asymptotic solutions (14)-(16). In the end, however, $C(r)$ invariably started to diverge, at some point very close to the would-be de Sitter horizon, $r \approx \pi/\sqrt{\lambda \eta k}$. We believe that our results are very suggestive, but they do not, of course, constitute a proof that all values of $H$ except one yield singular solutions. It may be possible to provide a proof using the dynamical-systems method employed in Ref. [13].

Our numerical results indicate that, for the nonsingular solution, $H$ grows as we increase $\eta$, and the growth is nearly linear, at least, for $\eta$ sufficiently close to $\eta_c$ (see Fig. 6). Above some value, $\kappa \eta > 1.1$, it is difficult to get a cigar solution numerically. As $\eta$ ($H$) increases, $Cr$ becomes more wiggly in the asymptotic region unless $\eta$ and $H$ are tuned very delicately to get a cigar solution. This causes numerical difficulties in getting a cigar solution for large $\eta$.

From Eq. (14) we see that the asymptotic solution exists only for $1 < \kappa \eta < \sqrt{5}$. At the end of this interval, $\kappa \eta = \sqrt{5}$, the scalar field is in the unbroken-symmetry state, $\phi = 0$. Then the source term in the Einstein’s equation is pure vacuum, $T^{\nu}_{\mu} = V(0)\delta^{\nu}_{\mu}$, and the solution (17) takes the form

$$((\lambda \eta^2)ds^2 = 8(\sin^2 \chi d\bar{s}_+^2 + d\chi^2) + 2d\Omega_2^2).$$

(18)

The de Sitter horizon radius of the inflating five dimensions in this solution is twice the...
radius of the remaining compactified two dimensions. Since the field $\phi$ in this solution has the same value, $\phi = 0$, as in the monopole core, the metric (18) describes the entire spacetime for $\kappa \eta = \sqrt{5}$. For higher values of $\eta$, we expect that $\phi$ remains equal to zero, and the metric is still given by (18).

V. CONCLUSIONS

In this paper, we continued the investigation of the gravitational field of higher-dimensional global defects that was started in Ref. [6]. We have obtained numerical solutions of Einstein’s and scalar-field equations for a defect with a (3+1)-dimensional core, which has a global-monopole-like field configuration in the extra three dimensions. We have verified that, for symmetry-breaking scales below the higher-dimensional Planck scale, $\eta < 1/\kappa$, the spacetime of the defect worldsheet is flat, and the geometry of extra dimensions is that of a generalized cone, in agreement with the asymptotic solution obtained in Ref. [6]. The solid angle deficit grows with $\eta$ and consumes the entire solid angle at the critical value $\eta_c = 1/\kappa$.

Our main goal in this paper has been to investigate supercritical defects with $\eta > \eta_c$. First, we have found that static supercritical solutions have a curvature singularity. It has been known for some time that singularities of static-defect solutions in 4D can be removed by relaxing the staticity requirement and allowing the defect cores to inflate. We have shown that the same is true for our 7D global monopoles. We found numerical solutions in which the defect core has the geometry of de Sitter space inflating at some rate $H$. As $H$ is increased, for each $\eta > \eta_c$, there is a value of $H$ for which the spacetime becomes nonsingular. The extra dimensions of these nonsingular solutions have a “cigar” geometry.

At large distances from the core, the nonsingular solutions have a very simple form, and we have found an exact analytic solution of the field equations describing this asymptotic behavior. In this solution, the five dimensional subspace, formed by the four dimensions tangent to the defect worldsheet and the radial dimension in the extra space, has the geometry of a 5D de Sitter space. Thus, inflation occurs not only along the defect, but also in
one extra dimension. (Note, however, that the radius of the defect core is smaller than the
horizon and remains fixed during the course of inflation.) The remaining two extra dimen-
sions are compactified as a sphere of a fixed radius. Far away from the core, the scalar field
approaches a constant value $\phi_0$, which, however, is somewhat smaller than the flat-space
expectation value $\eta$.

As the symmetry-breaking scale $\eta$ is increased, the inflation rate $H$ grows and the asymp-
totic scalar-field value $\phi_0$ decreases. For $\eta \geq \sqrt{5}\kappa^{-1}$, the scalar field vanishes in the entire
spacetime, and the defect solution turns into a vacuum solution of Einstein equations with
a cosmological term. We interpret this as indicating that, for higher values of $\eta$, defect
solutions with a core of a fixed transverse size do not exist.

We focused in this paper on a global-monopole-type defect in a 7D spacetime, but since
the gravitational properties of all global defects of codimension $n > 2$ are rather universal,
we expect our conclusions to apply to all such defects, with only minor modifications. In
particular, the analogues of the $n = 3$ asymptotic solutions (14)-(16) can be easily found for
$n > 3$.

As already noted, the inflation rate $H$ grows as $\eta$ is increased. This behavior is opposite
to what is needed for solving the cosmological constant problem. This indicates that global
defects cannot be used to implement the Dvali-Gabadadze-Shifman idea [9]. We shall discuss
the gravity of higher-dimensional gauge defects in a forthcoming paper.

When this work was in progress, we learned that Ruth Gregory has independently studied
inflating black-brane solutions.

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FIG. 1. Scalar and gravitational fields of a sub-critical monopole, $\kappa \eta = 0.2$. The scalar field rapidly approaches $\eta$. (The lower part of the graph is not shown.) The metric coefficients $B^2$ and $C^2$ approach constants. $C^2$ exhibits the expected solid angle deficit in the asymptotic region.
FIG. 2. Metric coefficients $B$ and $C$ for the critical monopole, $\kappa \eta = 1.0$. The asymptotic power-law solutions are shown by dashed lines.
FIG. 3. Metric coefficients $B$ and $C$ for a super-critical monopole, $\kappa \eta = 1.06723$. The metric becomes singular at a finite distance from the center.
FIG. 4. The location of the singularity $r_s$ as a function of $\eta$. 
FIG. 5. Scalar and gravitational fields of the “cigar” solutions with $\kappa \eta = 1.02361$ ($H = 0.10543\sqrt{\lambda \eta}$; solid line) and $\kappa \eta = 1.03766$ ($H = 0.15565\sqrt{\lambda \eta}$; dashed line). In the asymptotic region, the scalar field and $C_r$ are nearly constant, and $B$ behaves as a sine function. The scalar field is scaled to have a better discrimination.
FIG. 6. The worldsheet expansion rate $H$ as a function of $\eta$. A linear fit is also shown.