Modeling of normal contact of elastic bodies with surface relief taken into account

I G Goryacheva and I Yu Tsukanov∗
Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia
E-mail: ∗ivan.yu.tsukanov@gmail.com

Abstract. An approach to account the surface relief in normal contact problems for rough bodies on the basis of an additional displacement function for asperities is considered. The method and analytic expressions for calculating the additional displacement function for one-scale and two-scale wavy relief are presented. The influence of the micrelief geometric parameters, including the number of scales and asperities density, on additional displacements of the rough layer is analyzed.

1. Introduction
Contact problems accounting surface microrelief are used in many engineering applications. Since the contact stresses and displacements can reach significant values, they should be accounted in design of curvilinear bearings and liners in machines and instruments. This problem is also actual for thermal and electrical contacts between curvilinear surfaces, where the thermal and electric conductivities depend on the real contact area and the value of the gap between the surfaces. Another field of application deals with probe methods of materials testing, where it is necessary to indent unprepared surfaces.

Many experimental investigations of the contact stiffness of the machine and instrument parts showed that there is a certain difference between the experimental and computational results obtained for smooth elastic bodies. To remove this discrepancy, it is required to formulate and solve contact problems for rough elastic bodies. The first formulation of such problem was done in [1]. The further research was related to both experimental determination of the influence of the roughness parameters on the contact characteristics [2, 3] and formulation of new contact problems for bodies of various macro- and microgeometry interacting under specified boundary conditions [4–8].

The influence of rough layer on the contact interaction of elastic bodies can be quantitatively estimated on the basis of discrete contact mechanics. Here the two basic approaches – statistical and deterministic – can be applied. The statistical approach assumes interaction of two single asperities of regular shape with a number of contacting asperities depending on the probability of their heights. The displacements of asperities are summarized with the displacements of the whole curvilinear elastic body. Then, as a rule, the contact problem is solved neglecting the elastic interaction between asperities. One of the first study in this direction was published in [9]. A comparative review of statistical approach is given in [10]. In the present paper, we consider the deterministic approach for evaluation of additional displacement due to the surface
roughness in contact problems, which allows us to take the elastic interaction between asperities into account. On the basis of this approach, the additional displacement as a function of nominal pressure is analyzed for various types of the surface microgeometry.

2. Modeling of the surface roughness effect by an additional displacement function

The plane contact problem for rough elastic bodies was first posed by Schtaierman [1]. In his work, it was assumed that under the action of a rough body of specified macrogeometry \( f(x) \), the elastic displacements at each point of the nominal contact area are defined by a sum of basic displacements \( u_{z1} \) of the half-plane under the action of nominal pressure \( p(x) \) and additional displacements \( u_{z2} \) related to local deformations of roughness. In [1], it is also assumed that this additional displacement is proportional to the acting nominal pressure, i.e., \( u_{z2} = kp(x) \), where \( k \) is a coefficient depending on the surface microgeometry. The nominal contact pressure \( p(x) \) during the penetration of a body of a given macroshape \( f(x) \) into the elastic half-plane is determined by the integral equation [1]

\[
- \frac{\partial f(x)}{\partial x} = k \frac{\partial p(x)}{\partial x} + \frac{2(1 - \nu^2)}{\pi E} \int_{-a}^{a} \frac{p(\xi)}{x - \xi} d\xi.
\]

(1)

Schtaierman proposed to define the coefficient \( k \) in equation (1) by tests. Attempts to determine the additional displacement experimentally were made in [2]. The experiments were carried out on the developed test bench for determining the contact separation of circular specimens under the first and successive loadings. The bench design allows avoiding the influence of the specimen bulk deformation and determining only the displacements of the surface asperities. However, no general technique of experimental determination of the additional displacement function has yet been developed. The research results [2] show that, at small displacements, this dependence is better described by the power function \( u_{z2} = k(p(x, y))^\alpha \) than by a linear one.

Goryacheva [11–15] proposed derivation of an equation similar to (1) on the basis of modeling the discrete contact of bodies with specified macro- and microgeometry. The contact problem for a rigid punch, having a specified macrogeometry \( f(x, y) \) and a rough surface, on an elastic half-space was formulated and solved in the linear statement under the assumption that elastic deformations were infinitesimally small. This assumption is satisfied if the heights of asperities are much smaller than their dimensions in the longitudinal and transverse directions. The real contact area consists of a finite number \( n \) of discrete subareas \( \omega_i \ (i = 1, 2, \ldots, n) \). The displacements of the elastic half-space boundary are written as [11, 12]

\[
u_z(x, y) = \frac{1 - \nu^2}{\pi E} \sum_{i=1}^{n} \iint_{\omega_i} \frac{p_i(x, y) \, d\xi \, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}}.
\]

(2)

Here \( p_i(x, y) \) are real contact pressures which satisfy the equilibrium equation

\[
P = \sum_{i=1}^{n} \iint_{\omega_i} p_i(x, y) \, dx \, dy.
\]

(3)

Here \( P \) is the total load applied to the rigid body which penetrates into the elastic half-space.

The idea of this approach consists in the following. Let us assume that we have many contact spots and, in an arbitrarily small vicinity \( \Omega_0 \) of each point \((x, y)\) of the nominal contact area, the microgeometry of the surface can be described by the function \( F(x, y) \). There are two basic length scales in the problem under consideration (figure 1): the macrolevel which corresponds to the nominal contact area \( \Omega \) and to the nominal (average) pressure \( p(x, y) \) and the microlevel which corresponds to a subarea \( \Omega_0 \) containing \( N \) real contact spots \( \omega_i \) (\( N \ll n \)).
Figure 1. Scheme of contact of a body with a given macroshape \( f(x, y) \) and microgeometry \( F(x, y) \).

As \( \Omega_0 \ll \Omega \), the curvature of the surface macrogeometry \( f(x, y) \) can be neglected if the discrete contact problem is solved within \( \Omega_0 \). It can be assumed that the uniformly distributed pressure \( p(x, y) \) is applied in the subarea \( \Omega_0 \). Then, as was proved in [13] using the localization principle [11], the additional displacements due to the rough layer can be calculated by the formula

\[
C[p(x, y)] = -F(x_0, y_0) + \frac{1 - \nu^2}{\pi E} \left[ \sum_{i=1}^{N} \int_{\omega_i} \frac{p_i(\xi, \eta) \, d\xi \, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} - \int_{\Omega_0} \frac{p(x, y) \, d\xi \, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} \right]. \tag{4}
\]

Here \( F(x_0, y_0) \) is the height of the highest asperity in the subarea \( \Omega_0 \).

As follows from equation (4), the additional displacement function depends on the distribution of the contact pressure at real contact spots, which is calculated with regard to the shape and location of asperities in the subarea \( \Omega_0 \) and to the coupling of contact spots.

Then the equation for determining the nominal contact pressure distribution \( p(x, y) \) and the nominal area of contact \( \Omega \) with regard to the effect of microgeometry parameters has the form [13]

\[
C[p(x, y)] + \frac{1 - \nu^2}{\pi E} \int_{\Omega} \frac{p(x, y) \, d\xi \, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} = \delta - f(x, y). \tag{5}
\]

Here \( \delta \) is the punch penetration into the elastic half-space. To determine this value and the nominal contact area \( \Omega \), the equilibrium equation should be added to equation (5):

\[
\int_{\Omega} p(x, y) \, dx \, dy = P. \tag{6}
\]

In addition, for a smooth function \( f(x, y) \), the condition of a continuity of nominal pressures on the boundary \( \partial \Omega \) of the nominal contact area \( \Omega \) must be taken into account:

\[
p(x, y) \bigg|_{(x, y) \in \partial \Omega} = 0. \tag{7}
\]

It should be noted that equation (5) is a generalization of equation (1), proposed by Schtaierman, to the case of an arbitrary configuration of the body (three-dimensional problem) and to the case of a nonlinear additional displacement function.

Methods for solving integral equations like (1) and (5) are developed in [6–8]. Particular cases of linear and power function \( C[p(x, y)] \) are considered in these papers.

For certain types of surfaces with regular microrelief, the function of additional displacement can be determined analytically or by approximating the numerical results of solving the corresponding contact problems on microscale level.
Goryacheva [13] developed an approach for determining the additional displacement function by modeling the contact interaction of a rough nominally flat body, having a regular periodic surface relief, with an elastic half-space. The contact problems of interaction of a system of periodically located asperities, having regular shape (spherical, cylindrical, etc.) and various heights, with an elastic half-space were solved in [12, 14]. Such model well describes the microrelief if the longitudinal and transversal roughnesses are commensurable (e.g., at grinding, polishing).

3. Additional displacement functions for surfaces with regular waviness

In another limiting case where the surface has clearly defined direction and periodicity of roughness (e.g., at finishing milling or turning), the surface is modeled by regular waviness of various shape neglecting the transverse roughness. Meanwhile, in finishing processing, the period of waviness is much higher than its amplitude, and so the linear elasticity can be applied.

For determining the additional displacement function for this type of model roughness, the periodic contact problem of plane elasticity should be solved. In that problem, a rigid body with regular relief, described by smooth function $F(x)$, penetrates into an elastic half-plane. The basic integral equation of the considered problem for the normal contact without friction is [1, 16]:

$$-\frac{E}{2(1-\nu^2)} \frac{\partial F(x)}{\partial x} = \frac{1}{2\pi} \int_{-a}^a p(\xi) \cot \frac{x-\xi}{2} d\xi,$$

where $a$ is the contact zone half-length and $p(x)$ is the contact pressure.

The wavy surface penetrates into the elastic half-plane under the nominal pressure $p_\infty$. Its value is determined by the ratio of linear load $P$ on a single period to the value $\lambda$ of the period, $p_\infty = P/\lambda$. It follows from the equilibrium equation that

$$P = \int_{-a}^a p(x) dx.$$

According to the theory described in Section 1, the additional displacement function for the wavy surface is determined by the expression

$$C[p_\infty] = -F(x_0) + \frac{2(1-\nu^2)}{\pi E} \left[ \int_{-\lambda/2}^{\lambda/2} p(\xi) \ln |x-\xi| d\xi - p_\infty \int_{-\lambda/2}^{\lambda/2} \ln |x-\xi| d\xi \right] + C_1 - C_2. \quad (10)$$

The constants $C_1$ and $C_2$ depend on the selected displacement datum point. If the surface relief has a periodic distribution of asperities heights and the highest point is located at the origin of coordinates, then the additional displacements should be determined at the point $x_0 = 0$.

Thus, the additional displacement function can be written in the form

$$C[p_\infty] = -[\bar{u}_{z\lambda}(0) - \bar{u}_{z\lambda\infty}(0)], \quad (11)$$

where $\bar{u}_{z}(0)$ is the displacement of the asperities peak under the pressure distribution $p(x)$ acting on the period $\lambda$. The value of $\bar{u}_{z}$ for an arbitrary coordinate $x$ is defined by the expression

$$\bar{u}_{z\lambda}(x) = \frac{2(1-\nu^2)}{\pi E} \int_{-\lambda/2}^{\lambda/2} p(\xi) \ln |x-\xi| d\xi + C_1,$$

and $\bar{u}_{z\lambda\infty}(0)$ is the displacement of the asperities peak from the nominal pressure $p_\infty$. For an arbitrary coordinate $x$, it is given by the expression

$$\bar{u}_{z\lambda\infty}(x) = \frac{2(1-\nu^2)}{\pi E} p_\infty \int_{-\lambda/2}^{\lambda/2} \ln |x-\xi| d\xi + C_2. \quad (13)$$
Figure 2. Waviness profiles on a single period for different values of $m$: $m = 0$ (curve 1), $m = 1$ (curve 2), and $m = 4$ (curve 3).

To calculate equation (11) numerically, we use the following additional condition, which is implemented in numerical integration of equations (12) and (13): 

$$
\lim_{x \to \infty} [\bar{u}_z(x) - \bar{u}_z(\infty)] = 0.
$$

(14)

It follows from equation (10) that the additional displacement function is significantly influenced by the contact pressure distribution $p(x)$ on each period. The pressure distribution is, in turn, determined by the shape $F(x)$ of the wavy surface, as it follows from equation (8).

The surface waviness can generally be represented as a Fourier series. It can have different number of scales (harmonics of specified frequencies). In the simplest case, the waviness is defined by the sinusoid, i.e., $F(x) = \Delta [1 - \cos(2\pi x/\lambda)]$, where $\Delta$ is an amplitude and $\lambda$ is a period. The contact problem for such a surface penetrating into an elastic half-plane was first solved by Westergaard [17].

To describe the waviness, other than the sinusoidal one, the close-form solution of the plane contact problem was found in [18] for the waviness with arbitrary shape of waveform defined by the function

$$
f(x, m) = \Delta \left[ 1 - \frac{(m + 1) \cos(2\pi x/\lambda)}{m \cos(2\pi x/\lambda) + 1} \right],
$$

(15)

where $m$ is a shape parameter. At $m = 0$, equation (15) becomes a simple sinusoid $f(x) = \Delta [1 - \cos(2\pi x/\lambda)]$; the radii of curvature of the profile peak and valley increase with increasing $m$ (figure 2).

For determining the normal pressure in contact between the surface described by equation (15) and an elastic half-space, the following close-form solution was found [18]:

$$
p(x, m) = -J(x, m) \sqrt{\frac{2\pi \Delta (1 - \nu^2)}{\lambda E}} \cos \frac{\pi x}{\lambda} \sqrt{\cos \frac{2\pi x}{\lambda} - \cos \frac{2\pi a}{\lambda}},
$$

(16)

where $J(x, m)$ is defined by

$$
J(x, m) = (m + 1)^2 \left( m \cos \frac{2\pi x}{\lambda} + 1 \right)^{-2} \left( m \cos \frac{2\pi a}{\lambda} + 1 \right)^{-1}.
$$

(17)

Substituting (16) in (11)–(14) and calculating the integrals numerically, we determine the additional displacement function for the surface waviness described by the periodic function (15).

The graphs of the dimensionless additional displacement function for waviness of various shape $m$ at $\Delta = 1$ mm and $\lambda = 10$ mm as a function of the dimensionless nominal pressure $p_\infty/p^*$, where $p^* = -\pi \Delta_1 E/[(1 - \nu^2)\lambda_1]$, are presented in figure 3.
Figure 3. Additional displacement function, related to $\lambda$, for wavy profile (15) for $m = 0$ (curve 1), $m = 0.5$ (curve 2), and $m = 0.8$ (curve 3).

Figure 4. Example of profile with two-scale waviness.

The results illustrate that the additional displacement of asperities decreases as the parameter $m$ increases. It should also be noted that the additional displacement increases linearly with the amplitude $\Delta$ and decreases nonlinearly as the period $\lambda$ increases. The influence of the waviness period on the additional displacements is stronger than that of the amplitude. Meanwhile, the dependence $C[p_\infty]$ can be approximated by a power function for small values of $p_\infty$.

The considered model is a one-scale model, however the actual waviness (roughness) profile is multiscale. As a first approximation, it can be represented by a sum of sinusoids with various amplitudes and periods. In [19], a contact problem for a rigid body with two-scale surface wavy profile and an elastic half-plane is considered. The two-scale profile was modeled by the function

$$f(x) = \Delta_1 \left[ 1 - \cos \frac{2\pi x}{\lambda_1} + \frac{1}{k} \left( 1 - \cos \frac{2\pi n x}{\lambda_1} \right) \right],$$

Here $k = \Delta_1/\Delta_2$ is the ratio of the long-wave to short-wave harmonic amplitude of waviness; $n = \lambda_1/\lambda_2$ is the ratio of their periods. The graph of waviness profile at $k = 10$ and $n = 11$ is presented in figure 4.

For $n/k < 0.5$, for determining of normal pressure in periodic contact of the surface described by equation (18) with an elastic half-plane, the integral equation was reduced and solved in [19] using a variable transform. The analytical expression for the contact pressure obtained in [19]
Figure 5. Additional displacement function related to $\lambda_1$ for wavy profile (18): single-scale profile (curve 1), $n = 5$ (curve 2), $n = 7$ (curve 3), and $n = 11$ (curve 4).

has the form

$$p(x) = p_1(x) + p_2(x),$$  \hspace{1cm} (19)

$$p_1(x) = -\sqrt{2\pi E\Delta_1} \frac{\cos \frac{\pi x}{\lambda_1}}{\left(1 - \nu^2\right)\lambda_1} \left[\sqrt{\cos \frac{2\pi x}{\lambda_1} - \cos \frac{2\pi a}{\lambda_1}}\right],$$  \hspace{1cm} (20)

$$p_2(x) = -\frac{2E}{(1 - \nu^2)\lambda_1 k} \left[1 - \left\{\frac{\tan(\pi x/\lambda_1)}{\tan(\pi a/\lambda_1)}\right\}^2 \sum_{j=1}^{\infty} U_{j-1} \left\{\frac{\tan(\pi x/\lambda_1)}{\tan(\pi a/\lambda_1)}\right\} \right] \int_{-1}^{1} \frac{\varphi(s,a)T_j(s)}{\sqrt{1 - s^2}} ds, \hspace{1cm} (21)

$$\varphi(s,a) = \frac{\tan(\pi a/\lambda_1)s}{1 + [\tan(\pi a/\lambda_1)s]^2} U_{n-1} \left\{1 - \left[\frac{\tan(\pi a/\lambda_1)s}{1 + [\tan(\pi a/\lambda_1)s]^2}\right]^2\right\},$$  \hspace{1cm} (22)

where $U_n$ are Chebyshev polynomials of the second kind of degree $n$.

The additional displacement function in dimensionless form for the two-scale waviness at $\Delta_1 = 1$ mm, $\lambda_1 = 10$ mm, $k = 40$, calculated using equations (11)–(14) and (19–22) is shown in figure 5 for various values of the ratio $n$ of long to short harmonic period.

It follows from the results (figure 5), that the main features of the additional displacement function for two-scales and single scale profiles are similar. The influence of amplitude and period of long-wave asperities also has the same character, as for single-scale waviness. However, the existence of the short wave harmonics leads to discontinuity of the derivative of the additional displacement function $C[p_\infty]$. This effect increases with the increasing ratio $n$ of long to short harmonic period. The ratio of amplitudes $k$ linearly influences the additional displacements, as it follows from equations (20) and (21).

The additional displacement function calculated for various waviness parameters can be used in equations of the macroscale contact problem after substituting the function $C[p(x,y)] = C[p_\infty]$ in (5).

Conclusions

The approach for calculating the contact characteristics (pressure distribution, nominal contact area, and penetration) in normal contact of elastic bodies with regard to their surface roughness is discussed. The approach is based on introducing an additional displacement function which describes the additional compliance due to the surface roughness.
The method for calculating the additional displacement function is presented and used to analyze the influence of the surface waviness parameters on the dependence of the additional displacement due to the surface microgeometry on the nominal pressure in periodic contact problems for elastic half-planes. The analytical determination of the additional displacement function allows us to take into account the effect of the asperities of the contacting surfaces on the nominal contact characteristics, providing a unified approach to contact problems for rough bodies with microgeometry of various types and avoiding labor-consuming experiments. The form of the additional displacement function obtained in the present work for various types of the wavy surfaces, is comparable with the results for a three-dimensional system of spherical asperities [11, 12]. An analysis of the additional displacement functions for some particular types of surface microgeometry shows that as the nominal pressure $p_\infty$ and hence the real contact area increase, the rate of change of additional displacement function decreases. The growth of the density of contact spots and the number of scales of asperities leads to an increase in the additional displacements. It should also be noted that the power function approximation of the additional displacement function describes its behavior with an adequate accuracy only at small nominal pressures and does not reflect the transition from partial to full contact.

Acknowledgments
The research was supported by Russian Science Foundation (project No. 14-29-00198).

References
[1] Schtaierman I Y 1949 Contact Problem of Theory of Elasticity (Moscow: Gostekhizdat) [in Russian]
[2] Kragelsky I V 1965 Friction and Wear (London: Butterworths)
[3] Kragelsky I V, Dobychin M N, and Kombalov V S 1982 Friction and Wear: Calculation Methods (Pergamon)
[4] Rabinovich A S 1979 On solving the contact problems for rough bodies Izv. Akad. Nauk SSSR. Mekh. Tverd. Tela No. 1 52–7
[5] Teplyi M I 1981 The problem of internal compression of cylindrical bodies with a surface layer of increased pliability Sov. Mater. Sci. 17 (2) 185–8
[6] Galanov B A 1985 The method of boundary equations of the hammerstein-type for contact problems of the theory of elasticity when the regions of contact are not known J. Appl. Math. Mech. 49 (5) 634–40
[7] Galanov B A 1984 Spatial contact problems for rough elastic bodies under elastoplastic deformations of the unevenness J. Appl. Math. Mech. 48 (6) 750–7
[8] Goryacheva I G 1979 Plane and axisymmetric contact problems for rough elastic bodies J. Appl. Math. Mech. 43 (1) 99–105
[9] Greenwood J A and Tripp J H 1967 The elastic contact of rough spheres Trans. ASME. Ser. E. J. Appl. Mech. 34 153–9
[10] Sviridenok A I, Chizhik S A, and Petrokovets M I 1990 Mechanics of discrete frictional contact (Minsk: Science and Engineering) [in Russian]
[11] Goryacheva I 1997 Contact Mechanics in Tribology (Kluwer)
[12] Goryacheva I G 2001 Mechanics of frictional interaction (Moscow: Nauka) [in Russian]
[13] Goryacheva I G 2006 Mechanics of discrete contact Trib. Int. 39 (5) 381–6
[14] Goryacheva I G 1998 The periodic contact problem for an elastic half-space J. Appl. Math. Mech. 62 (6) 959–66
[15] Goryacheva I G 1999 Determination of contact characteristics with the account of the parameters of surface macro- and microgeometries Frict. Wear 20 (3) 239–48
[16] Block J M and Keer L M 2008 Periodic contact problems in plane elasticity J. Mech. Mater. Struct. 3 (7) 1207–37
[17] Westergaard H M 1939 Bearing pressures and cracks Trans. ASME. J. Appl. Mech 6 49–53
[18] Tsukanov I Yu 2017 Effects of shape and scale in mechanics of elastic interaction of regular wavy surfaces Proc. Inst. Mech. Engineers. Part J. J. Engng Trib. 231 (3) 332–40
[19] Tsukanov I Yu 2018 Periodic contact problem for surface with two-scale waviness Prikl. Mat. Mekh. (in print)