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Some results on pseudo-Q algebras

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Abstract. The notions of a dual pseudo-Q algebra and a dual pseudo-QC algebra are introduced. The properties and characterizations of them are investigated. Conditions for a dual pseudo-Q algebra to be a dual pseudo-QC algebra are given. Commutative dual pseudo-QC algebras are considered. The interrelationships between dual pseudo-Q/QC algebras and other pseudo algebras are visualized in a diagram.

1. Introduction

G. Georgescu and A. Iorgulescu [6] and independently J. Rachůnek [15], introduced pseudo-MV algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV algebras, pseudo-BL algebras [7] and pseudo-BCK algebras [8] were introduced and studied by G. Georgescu and A. Iorgulescu. A. Walendziak [18] gave a system of axioms defining pseudo-BCK algebras. W.A. Dudek and Y.B. Jun defined pseudo-BCI algebras as an extension of BCI-algebras [5]. Y.H. Kim and K.S. So [11] discussed on minimal elements in pseudo-BCI algebras. G. Dymek studied p-semisimple pseudo-BCI algebras and then defined and investigated periodic pseudo-BCI algebras [3].

A. Walendziak [19] introduced pseudo-BCH algebras as an extension of BCH-algebras and studied ideals in such algebras.

The notion of BE-algebras was introduced by H.S. Kim and Y.H. Kim [10].

B.L. Meng [13] introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCK/BCI/BCH-algebras. R.A. Borzooei et al. defined and studied pseudo-BE algebras which are a generalization of BE-algebras [1]. A. Rezaei et al. introduced the notion of pseudo-CI algebras as a generalization

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of pseudo-BE algebras and proved that the class of commutative pseudo-CI alge-
bras coincides with the class of commutative pseudo-BCK algebras [16]. Recently,
Y.B. Jun et al. defined and investigated pseudo-Q algebras [9] as a generalization
of Q-algebras [14].

In this paper, we define dual pseudo-Q and dual pseudo-QC algebras. We in-
vestigate the properties and characterizations of them. Moreover, we provide some
conditions for a dual pseudo-Q algebra to be a dual pseudo-QC algebra. We also
consider commutative dual pseudo-QC algebras and prove that the class of such
algebras coincides with the class of commutative pseudo-BCI algebras. Finally, the
interrelationships between dual pseudo-Q/QC algebras and other pseudo algebras
are visualized in a diagram.

2. Preliminaries

In this section, we review the basic definitions and some elementary aspects
that are necessary for this paper.

**Definition 2.1** ([5])
An algebra \( X = (X; \rightarrow, \leadsto, 1) \) of type \((2, 2, 0)\) is called a \textit{pseudo-BCI algebra} if it
satisfies the following axioms: for all \( x, y, z \in X \),

\[
(\text{psBCI}_1) \quad (x \rightarrow y) \leadsto ((y \rightarrow z) \leadsto (x \rightarrow z)) = 1,
\]

\[
(\text{psBCI}_2) \quad (x \leadsto y) \rightarrow ((y \leadsto z) \rightarrow (x \leadsto z)) = 1,
\]

\[
(\text{psBCI}_3) \quad x \rightarrow ((x \rightarrow y) \leadsto y) = 1 \quad \text{and} \quad x \leadsto ((x \leadsto y) \rightarrow y) = 1,
\]

\[
(\text{psBCI}_4) \quad x \rightarrow x = x \leadsto x = 1,
\]

\[
(\text{psBCI}_5) \quad x \rightarrow y = y \leadsto x = 1 \implies x = y,
\]

\[
(\text{psBCI}_6) \quad x \rightarrow y = 1 \iff x \leadsto y = 1.
\]

Every pseudo-BCI algebra \( X \) satisfying, for every \( x \in X \), condition

(\text{psBCK}) \quad x \rightarrow 1 = 1

is said to be a \textit{pseudo-BCK algebra} ([12]).

From [4] it follows that a pseudo-BCI-algebra \( X = (X; \rightarrow, \leadsto, 1) \) has the fol-
lowing property (for all \( x, y \in X \))

(\text{psEx}) \quad x \rightarrow (y \leadsto z) = y \leadsto (x \rightarrow z).

**Definition 2.2** ([17])
A (dual) pseudo-BCH algebra is an algebra \( X = (X; \rightarrow, \leadsto, 1) \) of type \((2, 2, 0)\) verifying
the axioms \( \{\text{psBCI}_1\}, \{\text{psBCI}_2\}, \{\text{psBCI}_3\}, \{\text{psEx}\} \).

**Remark 2.3**
Obviously, every pseudo-BCI algebra is a pseudo-BCH algebra.
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Definition 2.4 (16)
An algebra $X = (X; \rightarrow, \leadsto, 1)$ of type $(2, 2, 0)$ is called a pseudo-CI algebra if, for all $x, y, z \in X$, it satisfies the following axioms:

$(\text{psCl}_1)$ $x \rightarrow x = x \leadsto x = 1$,

$(\text{psCl}_2)$ $1 \rightarrow x = 1 \leadsto x = x$,

$(\text{psCl}_3)$ $x \rightarrow (y \leadsto z) = y \leadsto (x \rightarrow z)$,

$(\text{psCl}_4)$ $x \rightarrow y = 1 \iff x \leadsto y = 1$.

Remark 2.5
Since every pseudo-BCH algebra satisfies $(\text{psCl}_1)$–$(\text{psCl}_4)$, pseudo-BCH algebras are contained in the class of pseudo-CI algebras.

A pseudo-CI algebra $X = (X; \rightarrow, \leadsto, 1)$ verifying condition

$(\text{psBE})$ $x \rightarrow 1 = x \leadsto 1 = 1$,

for all $x \in X$, is said to be a pseudo-BE algebra (see [1]).

Proposition 2.6 (2)
Any pseudo-BCK algebra is a pseudo-BE algebra.

In a pseudo-CI algebra $X$ we can introduce a binary relation “$\leq$” by

$x \leq y \iff x \rightarrow y = 1 \iff x \leadsto y = 1$ for all $x, y \in X$.

An algebra $X = (X; \rightarrow, \leadsto, 1)$ of type $(2, 2, 0)$ is called commutative if for all $x, y \in X$, it satisfies the following identities:

(i) $(x \rightarrow y) \leadsto y = (y \rightarrow x) \leadsto x$,

(ii) $(x \leadsto y) \rightarrow y = (y \leadsto x) \rightarrow x$.

From [2] (see Theorem 3.4) it follows that any commutative pseudo-BE algebra is a pseudo-BCK algebra. By Theorem 3.9 of [16], any commutative pseudo-CI algebra is a pseudo-BE algebra. Therefore we obtain

Proposition 2.7
Commutative pseudo-CI algebras coincide with commutative pseudo-BE algebras and with commutative pseudo-BCK algebras (hence also coincide with commutative pseudo-BCI algebras and with commutative pseudo-BCH algebras).

Definition 2.8 (9)
An algebra $X = (X; \ast, \circ, 0)$ of type $(2, 2, 0)$ is called a pseudo-Q algebra if, for all $x, y, z \in X$, it satisfies the following axioms:

$(\text{psQ}_1)$ $x \ast x = x \circ x = 0$,

$(\text{psQ}_2)$ $x \ast 0 = x \circ 0 = x$,

$(\text{psQ}_3)$ $(x \ast y) \circ z = (x \circ z) \ast y$. 
3. Dual pseudo-Q algebras

**Definition 3.1**
An algebra $X = (X; \rightarrow, \leadsto, 1)$ of type $(2, 2, 0)$ is called a dual pseudo-Q algebra if, for all $x, y, z \in X$, it verifies the following axioms:

\[(dpsQ_1)\] $x \rightarrow x = x \leadsto x = 1,$

\[(dpsQ_2)\] $1 \rightarrow x = 1 \leadsto x = x,$

\[(dpsQ_3)\] $x \rightarrow (y \leadsto z) = y \leadsto (x \rightarrow z).$

In a dual pseudo-Q algebra, we can introduce two binary relations $\leq \rightarrow$ and $\leq \leadsto$ by

$x \leq \rightarrow y \iff x \rightarrow y = 1$ and $x \leq \leadsto y \iff x \leadsto y = 1.$

**Proposition 3.2**
Let $X = (X; \rightarrow, \leadsto, 1)$ be a dual pseudo-Q algebra. Then $X$ is a pseudo-CI algebra if and only if $\leq \rightarrow = \leq \leadsto$.

**Example 3.3**
(i) Let $X = \{1, a, b, c, d\}$. Define binary operations $\rightarrow$ and $\leadsto$ on $X$ by the following tables (16):

|   | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | c | c | 1 |
| b | d | d | 1 | 1 | d |
| c | 1 | 1 | d | 1 | 1 |
| d | 1 | 1 | c | 1 | 1 |

and

|   | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | b | c | 1 |
| b | d | d | 1 | 1 | d |
| c | d | d | d | 1 | 1 |
| d | d | b | c | 1 | 1 |

Then $X = (X; \rightarrow, \leadsto, 1)$ is a dual pseudo-Q algebra which is not a pseudo-BCI algebra, since $b \neq c$ and $b \rightarrow c = c \leadsto b = 1$ (that is, $[psBCI_5]$ does not hold in $X$).

(ii) Let $X = \{1, a, b, c\}$. Define binary operations $\rightarrow$ and $\leadsto$ on $X$ by the following tables:

|   | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | b | c |
| b | 1 | 1 | 1 | 1 |
| c | 1 | 1 | a | 1 |

and

|   | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | 1 | 1 | c | c |
| b | 1 | 1 | 1 | 1 |
| c | 1 | 1 | c | 1 |

Then $X = (X; \rightarrow, \leadsto, 1)$ is a dual pseudo-Q algebra which is not a pseudo-CI algebra, because $b \rightarrow c = 1$ but $b \leadsto c = c$.

By definition, we have

**Proposition 3.4**
Any pseudo-CI algebra is a dual pseudo-Q algebra.
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REMARK 3.5
The converse of Proposition 3.4 does not hold. See Example 3.3 (ii).

PROPOSITION 3.6
Let \( X \) be a dual pseudo-Q algebra. If one of the following identities:

1. \((y \rightarrow x) \rightarrow x = y \rightarrow x,\)
2. \((y \rightarrow x) \leftrightarrow x = y \leftrightarrow x,\)
3. \((y \leftrightarrow x) \rightarrow x = y \rightarrow x,\)
4. \((y \leftrightarrow x) \leftrightarrow x = y \leftrightarrow x,\)
5. \((y \leftrightarrow x) \leftrightarrow x = y \leftrightarrow x,\)
6. \((y \rightarrow x) \rightarrow x = y \rightarrow x,\)
7. \((y \rightarrow x) \rightarrow x = y \rightarrow x,\)
8. \((y \rightarrow x) \rightarrow x = y \rightarrow x,\)

holds in \( X \), then \( X \) is a trivial algebra.

Proof. Suppose, for example, that (1) is satisfied. Let \( x \in X \). Applying \((\text{dpsQ}_1)\) and \((\text{dpsQ}_2)\) we have

\[ 1 = x \rightarrow x = (x \rightarrow x) \rightarrow x = 1 \rightarrow x = x. \]

Thus \( X \) is a trivial algebra.

PROPOSITION 3.7
Let \( X \) be a dual pseudo-Q algebra. If one of the following identities:

1. \((y \rightarrow x) \rightarrow x = x \rightarrow y,\)
2. \((y \rightarrow x) \leftrightarrow x = x \leftrightarrow y,\)
3. \((y \leftrightarrow x) \rightarrow x = x \rightarrow y,\)
4. \((y \leftrightarrow x) \leftrightarrow x = x \leftrightarrow y,\)
5. \((y \leftrightarrow x) \leftrightarrow x = x \leftrightarrow y,\)
6. \((y \rightarrow x) \rightarrow x = x \rightarrow y,\)
7. \((y \rightarrow x) \rightarrow x = x \rightarrow y,\)
8. \((y \rightarrow x) \rightarrow x = x \rightarrow y,\)

holds in \( X \), then \( X \) is a trivial algebra.

Proof. The proof is similar to the proof of Proposition 3.6.

PROPOSITION 3.8
In a dual pseudo-Q algebra \( X \), for all \( x, y, z \in X \), we have:

1. if \( 1 \leq \rightarrow x \) or \( 1 \leq \rightarrow x \), then \( x = 1, \)
(2) \( x \leq \to y \to z \iff y \leq \to x \to z \),

(3) \( x \to 1 = x \to 1 \),

(4) \((x \to y) \to 1 = (x \to 1) \to (y \to 1)\) and \((x \to y) \to 1 = (x \to 1) \to (y \to 1)\),

(5) if \( x \leq \to y \), then \( x \to 1 = y \to 1 \),

(6) if \( x \leq \to y \), then \( x \to 1 = y \to 1 \),

(7) \( y \to ((y \to x) \to x) = 1 \) and \( y \to ((y \to x) \to x) = 1 \),

Proof. (1) Let \( 1 \leq \to x \). Then \( 1 \to x = 1 \). Now, by \((dpsQ)_{2}\) we obtain \( x = 1 \). Similarly, if \( 1 \leq \to x \), then \( x = 1 \).

(2) Let \( x, y, z \in X \). By \((dpsQ)_{3}\),

\[
x \to (y \to z) = 1 \iff y \to (x \to z) = 1.
\]

Consequently, \((2)\) holds.

(3) We have \( x \to 1 = x \to (x \to x) = x \to (x \to x) = x \to 1 \).

(4) Let \( x, y, z \in X \). Then

\[
(x \to y) \to 1 = (x \to y) \to [(x \to 1) \to (x \to 1)]
\]

\[
= (x \to 1) \to [(x \to y) \to (x \to 1)]
\]

\[
= (x \to 1) \to [(x \to y) \to (x \to (y \to y))] 
\]

\[
= (x \to 1) \to [(x \to y) \to (y \to (x \to y))] 
\]

\[
= (x \to 1) \to [(y \to ((x \to y) \to (x \to y))] 
\]

\[
= (x \to 1) \to (y \to 1).
\]

The proof of the second part is similar.

(5) Let \( x \leq \to y \). Then \( x \to y = 1 \) and so \( y \to 1 = y \to 1 = (x \to y) = x \to (y \to y) = x \to 1 \). Thus \( y \to 1 = x \to 1 \).

(6) The proof is similar to the proof of \((5)\).

(7) By \((dpsQ)_{3}\) and \((dpsQ)_{1}\) we get

\[
y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1
\]

and

\[
y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1.
\]

A dual pseudo-Q algebra \( X = (X; \to, \to, 1) \) satisfying the conditions \((psBCI)_{1}\) and \((psBCI)_{2}\) is said to be a dual pseudo-QC algebra. The following example shows that there exist pseudo-Q algebras which do not satisfy \((psBCI)_{1}\) or \((psBCI)_{2}\).
Example 3.9

(i) Dual pseudo-Q algebra from Example 3.3 (ii) satisfies $(psBCI_2)$ but it does not satisfy $(psBCI_1)$ since

\[(a \to b) \leadsto ((b \to c) \leadsto (a \to c)) = b \leadsto (1 \leadsto c) = c \neq 1.\]

(ii) Let $X = \{1, a, b, c, d, e, f, g, h\}$. We define the binary operations $\to$ and $\leadsto$ on $X$ as follows:

\[
\begin{array}{cccccccc}
\to & 1 & a & b & c & d & e & f & g & h \\
1 & 1 & a & b & c & d & e & f & g & h \\
a & 1 & 1 & 1 & 1 & d & e & f & g & h \\
b & 1 & c & 1 & 1 & d & e & f & g & h \\
c & 1 & c & b & 1 & d & e & f & g & h \\
d & d & d & d & 1 & g & h & e & f & g \\
e & e & e & e & h & 1 & g & f & d & e \\
f & f & f & f & g & h & 1 & d & e & f \\
g & h & h & h & e & f & d & 1 & g & h \\
h & g & g & g & g & f & d & e & h & 1 \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
\leadsto & 1 & a & b & c & d & e & f & g & h \\
1 & 1 & a & b & c & d & e & f & g & h \\
a & 1 & 1 & 1 & 1 & d & e & f & g & h \\
b & 1 & c & 1 & 1 & d & e & f & g & h \\
c & 1 & c & b & 1 & d & e & f & g & h \\
d & d & d & d & 1 & g & h & e & f & g \\
e & e & e & e & g & 1 & h & d & f & e \\
f & f & f & f & h & g & 1 & e & d & f \\
g & h & h & h & f & d & e & 1 & g & h \\
h & g & g & g & g & c & f & d & h & 1 \\
\end{array}
\]

Then $X = (X; \to, \leadsto, 1)$ is a dual pseudo-Q algebra which does not satisfy $(psBCI_1)$ and $(psBCI_2)$. Indeed,

\[(c \to a) \leadsto ((a \to b) \leadsto (c \to b)) = c \leadsto (1 \leadsto b) = c \leadsto b = b \neq 1\]

and

\[(c \leadsto a) \to ((a \leadsto b) \to (c \leadsto b)) = c \to (1 \to b) = c \to b = b \neq 1.\]

(iii) Let $X = \{1, a, b, c\}$. Define binary operations $\to$ and $\leadsto$ on $X$ by the following tables:

\[
\begin{array}{cccc}
\to & 1 & a & b & c \\
1 & 1 & a & b & c \\
a & 1 & 1 & b & c \\
b & 1 & a & 1 & c \\
c & 1 & 1 & 1 & 1 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
\leadsto & 1 & a & b & c \\
1 & 1 & a & b & c \\
a & 1 & 1 & b & c \\
b & 1 & a & 1 & a \\
c & 1 & 1 & 1 & 1 \\
\end{array}
\]

Then $X = (X; \to, \leadsto, 1)$ is a dual pseudo-QC algebra.

Lemma 3.10

Let $X = (X; \to, \leadsto, 1)$ be a dual pseudo-QC algebra and $x, y \in X$. Then $x \to y = 1$ if and only if $x \leadsto y = 1$.

Proof. Let $x \to y = 1$. Using $(dpsQ_2)$ and $(psBCI_1)$ we obtain

\[x \leadsto y = x \leadsto (1 \leadsto y) = (1 \to x) \leadsto ((x \to y) \leadsto (1 \to y)) = 1.\]

Similarly, if $x \leadsto y = 1$, then $x \to y = 1$.

From Lemma 3.10 we have
Proposition 3.11
Any dual pseudo-QC algebra is a pseudo-CI algebra.

Remark 3.12
The converse of Proposition 3.11 does not hold. See Example 3.9 (ii).

Proposition 3.13
Every pseudo-BCI algebra is a dual pseudo-QC algebra.

Proof. Let $X$ be a pseudo-BCI algebra. It is easy to see that $X$ satisfies \( (dpsQ_1) \)–\( (dpsQ_3) \), that is, it is a dual pseudo-Q algebra. Moreover, $X$ obviously satisfies \( (psBCI_1) \) and \( (psBCI_2) \). Consequently, $X$ is a dual pseudo-QC algebra.

Remark 3.14
In a dual pseudo-QC algebra, $\leq \rightarrow = \leq \Rightarrow$. Set $\leq = \rightarrow (= \Rightarrow)$.

Proposition 3.15
Let $X$ be a dual pseudo-QC algebra and $x, y, z \in X$. Then:

1. if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \Rightarrow z \leq x \Rightarrow z$,

2. if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \Rightarrow x \leq z \Rightarrow y$.

Proof. Let $x \leq y$. Then $x \rightarrow y = 1$. By \( (dpsQ_2) \) and \( (psBCI_1) \) we have

\[
(y \rightarrow z) \Rightarrow (x \rightarrow z) = 1 \Rightarrow ((y \rightarrow z) \Rightarrow (x \rightarrow z)) = (x \rightarrow y) \Rightarrow ((y \rightarrow z) \Rightarrow (x \rightarrow z)) = 1.
\]

Hence $y \rightarrow z \leq x \rightarrow z$. The proof of the second part is similar.

Let $x \leq y$. Hence $x \rightarrow y = 1$. Applying \( (dpsQ_2) \) and \( (psBCI_1) \) we obtain

\[
(z \rightarrow x) \rightarrow (z \rightarrow y) = 1 \Rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = (x \rightarrow y) \Rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = (z \rightarrow x) \Rightarrow ((x \rightarrow y) \Rightarrow (z \rightarrow y)) = 1.
\]

Hence $z \rightarrow x \leq z \rightarrow y$. Similarly, $z \Rightarrow x \leq z \Rightarrow y$.

Theorem 3.16
Let $X$ be a dual pseudo-Q algebra. Then $X$ is a pseudo-QC algebra if and only if it satisfies the following implications:

\( (*) \) $y \Rightarrow z \implies x \rightarrow y \leq x \rightarrow z$,

\( (**) \) $y \Rightarrow z \implies x \rightarrow y \leq x \rightarrow z$.

Proof. If $X$ is a pseudo-QC algebra, then it satisfies \( (*) \) and \( (**) \) by Proposition 3.15. Conversely, suppose that implications \( (*) \) and \( (**) \) hold for all $x, y, z \in X$. By Proposition 3.8 (7), $y \Rightarrow (y \rightarrow z) \Rightarrow z$. Hence $(x \rightarrow y) \Rightarrow ((x \rightarrow y) \Rightarrow (y \rightarrow z) \Rightarrow z) = 1$. Applying \( (psEx) \) we obtain $(x \rightarrow y) \Rightarrow ((y \rightarrow z) \Rightarrow (x \rightarrow z)) = 1$, that is, \( (psBCI_1) \) holds. Similarly, using \( (**) \) we have $\{psBCI_2\}$.
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Proposition 3.17
Let $\mathfrak{X}$ be a dual pseudo-QC algebra. Then $\mathfrak{X}$ is a pseudo-BCI algebra if and only if it verifies $\left[ \text{psBCI}_5 \right]$.  

Proof. Let $\mathfrak{X}$ be a dual pseudo-QC algebra satisfying $\left[ \text{psBCI}_5 \right]$. Clearly, $\mathfrak{X}$ verifies $\left[ \text{psBCI}_1 \right], \left[ \text{psBCI}_2 \right]$ and $\left[ \text{psBCI}_4 \right]$. The axiom $\left[ \text{psBCI}_3 \right]$ follows from Proposition 3.8 (7). By Lemma 3.10 $\left[ \text{psBCI}_6 \right]$ holds in $\mathfrak{X}$. Therefore, $\mathfrak{X}$ is a pseudo-BCI algebra.

The converse is obvious.

Proposition 3.18
Let $\mathfrak{X}$ be a dual pseudo-QC algebra and $x, y, z \in X$ such that $x \leq y$ and $y \leq z$. Then $x \leq z$.

Proof. Applying $\left[ \text{dpsQ}_2 \right]$ and $\left[ \text{psBCI}_1 \right]$ we get
\[
x \rightarrow z = 1 \leadsto (x \rightarrow z) \\
= 1 \leadsto ((x \rightarrow z)) \\
= (x \rightarrow y) \leadsto ((y \rightarrow z) \leadsto (x \rightarrow z)) \\
= 1,
\]
and therefore $x \leq z$.

Corollary 3.19
If a dual pseudo-QC algebra $\mathfrak{X}$ satisfies the condition $\left[ \text{psBCI}_5 \right]$ then $(X, \leq)$ is a poset.

Theorem 3.20
If $\mathfrak{X}$ is a commutative dual pseudo-QC algebra, then it is a pseudo-BCI algebra.

Proof. It is sufficient to prove that $\left[ \text{psBCI}_5 \right]$ holds in $\mathfrak{X}$. Let $x, y \in X$ and $x \rightarrow y = y \leadsto x = 1$. Then
\[
x = 1 \rightarrow x = (y \leadsto x) \rightarrow x = (x \rightarrow y) \rightarrow y = 1 \rightarrow y = y.
\]
Therefore, $\mathfrak{X}$ satisfies $\left[ \text{psBCI}_5 \right]$. Thus $\mathfrak{X}$ is a pseudo-BCI algebra.

From Theorem 3.20 it follows

Corollary 3.21
Commutative dual pseudo-QC algebras coincide with commutative pseudo-BCI algebras.

4. Conclusion

Denote by $\text{psBCK}$, $\text{psBCI}$, $\text{psBCH}$, $\text{psCI}$, $\text{psBE}$, $\text{dpsQ}$, and $\text{dpsQC}$ the classes of pseudo-BCK, pseudo-BCI, pseudo-BCH, pseudo-Cl, pseudo-BE, dual pseudo-Q, and dual pseudo-QC algebras respectively. By definition, $\text{psBCK} \subset \text{psBCI}$ and $\text{psBE} \subset \text{psCl} \subset \text{dpsQ}$. From Remarks 2.3 and 2.5 we obtain $\text{psBCI}$...
Moreover, that $\text{psBCI} \subset \text{dpsQC} \subset \text{psCI}$ follows from Propositions 3.13 and 3.11.

By Proposition 2.7 and Corollary 3.21, commutative pseudo-QC algebras coincide with commutative algebras pseudo-BCK, -BCI, -BCH, -CI, -BE.

Now, in the following diagram we summarize the results of this paper and the previous results in this field. An arrow indicates proper inclusion, that is, if $X$ and $Y$ are classes of algebras, then $X \rightarrow Y$ denotes $X \subset Y$. The mark $X \xrightarrow{C} Y$ means that every commutative algebra of $X$ belongs to $Y$.

Problem 4.1
Is it true that every commutative dual pseudo-Q algebra is a pseudo-BCK algebra?

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