THE PROPERTY OF UNIQUE CONTINUATION FOR SECOND ORDER EVOLUTION PDE'S

MOURAD CHOULLI

ABSTRACT. We present a simple and self-contained approach to establish the property of unique continuation for some classical evolution equations of second order in a cylindrical domain. We namely discuss this property for wave, parabolic and Schrödinger operators with time-independent principal part. Our method is build on two-parameter Carleman inequalities combined with unique continuation argument across pseudo-convex hypersurfaces with respect to space variable uniformly in the time variable.

CONTENTS

1. Introduction 2
2. Preliminaries 3
  2.1. Main notations and assumptions 3
  2.2. Pseudo-convexity condition 5
  2.3. Carleman weights 5
  2.4. Pseudo-convex hypersurface 7
3. The wave equation 9
  3.1. Carleman inequality 9
  3.2. Geometric form of the Carleman inequality 20
  3.3. Unique continuation 21
  3.4. Observability inequality 24
4. Elliptic equations 26
  4.1. Carleman inequality 26
  4.2. Unique continuation 28
5. Parabolic equations 30
  5.1. Carleman inequality 30
  5.2. Unique continuation 33
  5.3. Final time observability inequality 34
6. Schrödinger equations 35
  6.1. Carleman inequality 35
  6.2. Unique continuation 39
  6.3. Observability inequality 40
References 41
1. Introduction

Let $D$ be a domain of $\mathbb{R}^d$, $d \geq 1$. We recall that a function $f \in C^\infty(D)$ is said real-analytic if its Taylor series around any arbitrary point of $D$ converges in a ball centered at this point. It is known that a real analytic function enjoy the property of unique continuation which means that if $f$ vanishes in a nonempty open subset $D_0$ of $D$ then $f$ must vanishes identically (see for instance [20, Theorem, page 65]). Functions satisfying this property of unique continuation are also called quasi-analytic.

A classical result shows that a solution, with some minimal smoothness, of an elliptic operator with Lipschitz principal coefficients is quasi-analytic. In the present work we consider the analogue of this property for second order evolution PDE’s. The right property in this context should be the following: if a solution of an evolution equation of second order in the cylindrical domain $D \times (t_1, t_2)$ vanishes in $D_0 \times (t_1, t_2)$, for some nonempty open subset $D_0$ of $D$, then this solution must vanishes identically.

Our aim is to provide a simple and self contained approach to show this property of unique continuation for wave, parabolic and Schrödinger equations. The approach we carry out is quite classical and it is based on two-parameter Carleman inequalities. The property of unique continuation as its is defined above is obtained as a consequence of the property of unique continuation across a non characteristic hypersurface satisfying in addition a pseudo-convexity condition in the case of wave or Schrödinger equations.

The core of our analysis consists in establishing two-parameter Carleman inequalities. We follow a classical scheme for obtaining this $L^2$-weighted energy estimates, essentially based on conjugating the original operator with a well chosen exponential function, splitting the resulting operator into its self-adjoint part and skew-adjoint part and finally making integrations by parts. The main assumption on the weight function is a pseudo-convexity condition with respect to the operator under consideration. A systematic approach was considered by Hörmander [17, Section 28.2, page 234] for a general operator $P$ of an arbitrary order $m$ where the pseudo-convexity condition is expressed in term of the principal symbol of $P$. The method we develop in this work is more simple and does not appeal to fine analysis of PDE’s and our results have no pretension nor for generality neither for optimality.

It is worth remarking that splitting the conjugated operator into self-adjoint and skew-adjoint parts is not the best possible way to get two-parameter Carleman inequalities for elliptic and parabolic operators. There is a particular way to split the conjugated operator into two parts which, in addition, requires only $C^2$-weight function instead of $C^4$-weight function which is necessary when the conjugated operator is decomposed into its self-adjoint and skew-adjoint parts. Unfortunately this special decomposition is not applicable for wave and Schrödinger equations. But this is not really surprising since solutions of wave and Schrödinger equations do not enjoy the same regularity properties of solutions of elliptic and parabolic equations.

We choose to start with the more subtle case corresponding to the wave equation. Since most calculations for getting Carleman inequalities are common for different type of equations. Carleman inequalities for parabolic and Schrödinger equations are obtained by making some modifications in the proof of the Carleman inequality.
for the wave equation. For sake of completeness we also added a short section for the elliptic case whose analysis is almost similar to that in the parabolic case.

One can find in the literature two-parameter Carleman inequalities with degenerate weight function. We refer for instance to [7, 11, 12, 13, 16, 22] for parabolic operators and [2, 3, 4, 23, 24] for Schrödinger operators.

A Carleman inequality for wave equations on compact Riemannian manifold can be found in [6] and quite recently Huang [18] proved a Carleman inequality for a general wave operators with time-dependent principal part. The interested reader is referred to [14] for a unified approach for establishing Carleman inequalities for second order PDE’s and their applications to control theory and inverse problems.

The property of unique continuation for elliptic and parabolic operators with unbounded lower order coefficients was obtained in [27, 28, 29, 30]. Their results combine both classical tools used for establishing the property of unique continuation together with interpolation inequalities. Uniqueness and non-uniqueness for general operators were discussed in [1, 32] (see also the references therein). We also mention [19, 25] as additional references on uniqueness of Cauchy problems.

We also discuss briefly observability inequalities which can be seen as quantification of unique continuation for the Cauchy problem associated to IBVP’s. We refer to [21] for general observability inequalities for wave and Schrödinger equations with arbitrary interior or boundary observation region. The reader can find in this work a detailed introduction to explain the main steps to get the property of unique continuation for an intermediate case between Holmgren (analytic case) and Hörmander (general case) for operators with partially analytic coefficients.

A more difficult problem consists in quantifying the property of unique continuation from an interior subdomain or the Cauchy data on a sub-boundary. The elliptic case is now almost completely solved with optimal results for \(C^{1,\alpha}\)-solutions and \(C^{0,1}\)-domains [10] or \(H^2\)-solutions and \(C^{1,1}\)-domains [8]. A non optimal result for \(H^2\)-solutions and \(C^{0,1}\)-domains was obtained in [9]. These kind of results can be obtained by a method based on three-sphere inequality which is deduced itself from two-parameter Carleman inequality. The case of parabolic and wave equations is extremely more difficult than in the elliptic case. Concerning parabolic equations, a first result was obtained in [7] with Cauchy data in a particular sub-boundary. This result is based on a global Carleman inequality. The general case was tackled in [12] where a non optimal result was established using a three-cylinder inequality. A partial result for the wave equation was recently proved in [5]. This result was obtained via Fourier-Bros-Iagolnitzer transform allowing to transfer the quantification of unique continuation of an elliptic equation to that of the wave equation.

2. Preliminaries

2.1. Main notations and assumptions. Throughout \(\Omega\) is bounded Lipschitz domain of \(\mathbb{R}^n\), \(n \geq 2\), with boundary \(\Gamma\), \(Q = \Omega \times (t_1, t_2)\) and \(\Sigma = \Gamma \times (t_1, t_2)\), where \(t_1, t_2 \in \mathbb{R}\) are fixed so that \(t_1 < t_2\).

\(A = (a_{k\ell})\) will denote a symmetric matrix with coefficients \(a_{k\ell} \in C^{0,1}(\Omega)\), \(1 \leq k, \ell \leq n\), and there exist two constants \(m > 0\) and \(\varkappa \geq 1\) so that

\[\varkappa^{-1}|\xi|^2 \leq \sum_{k,\ell=1}^{n} a_{k\ell}(x)\xi_k \xi_\ell \leq \varkappa|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,\]
and
\[
\|A\|_{C^{0.1}([\Omega, \mathbb{R}^n \times n])} \leq m.
\]

The set of such matrices will be denoted by \(\mathcal{M}(\Omega, \alpha, m)\).

It is worth mentioning that according to Rademacher's theorem \(C^{0,1}(\overline{\Omega})\) is continuously embedded in \(W^{1,\infty}(\Omega)\).

For \(\xi, \eta \in \mathbb{C}^n\), \((\xi|\eta)\) and \(\xi \otimes \eta\) are defined as usual respectively by
\[
(\xi|\eta) = \sum_{k=1}^{n} \xi_k \eta_k,
\]
\[
\xi \otimes \eta = (\xi_k \eta_{k})_{1 \leq k, \ell \leq n}.
\]

We use the following notations respectively for the Jacobian matrix and Hessian matrix
\[
U' = (\partial_{\ell} U_k), \quad U \in H^1(\Omega; \mathbb{C}^n),
\]
\[
\nabla^2 u = (\partial^2_{\ell \mu} u), \quad u \in H^2(\Omega; \mathbb{C}^n).
\]

If \(E\) is a Banach (resp. Hilbert) space then its natural norm (resp. scalar product) is always denoted by \(\| \cdot \|_E\) (resp. \((\cdot|\cdot)_E\)).

We will use in this text the anisotropic Sobolev space \(H^{2,1}(Q)\) which is defined as follows
\[
H^{2,1}(Q) = L^2((t_1, t_2); H^2(\Omega)) \cap H^1((t_1, t_2); L^2(\Omega)).
\]

The following notations will be useful in the sequel, where \(S = \Omega\) or \(S = \Gamma\),
\[
\nabla_A u(x) = A(x)^{1/2} \nabla u(x), \quad u \in H^1(\Omega; \mathbb{C}),
\]
\[
\text{div}_A U(x) = \text{div}(A(x)^{1/2} U(x)), \quad U \in H^1(S; \mathbb{C}^n),
\]
\[
\Delta_A u(x) = \Delta_A \nabla_A u(x) = \text{div}(A(x) \nabla u(x)), \quad u \in H^2(\Omega; \mathbb{C}),
\]
\[
(U|V)_A(x) = (A(x) U(x)|V(x)) = (U(x)|A(x)V(x)), \quad U, V \in L^2(S; \mathbb{C}^n),
\]
\[
|U|_{A}(x) = [(U|U)_A(x)]^{1/2}, \quad U \in L^2(S; \mathbb{C}^n).
\]

It is not hard to check that
\[
(\nabla_A u | \nabla_A v) = (\nabla u | \nabla v)_A, \quad u, v \in H^1(\Omega; \mathbb{C}),
\]

and Green's formula
\[
\int_{\Omega} \Delta_A u v dx = -\int_{\Omega} (\nabla_A u | \nabla_A v) dx + \int_{\Gamma} (\nabla u | \nu)_A v d\sigma
\]
\[
= -\int_{\Omega} (\nabla u | \nabla v)_A dx + \int_{\Gamma} (\nabla u | \nu)_A v d\sigma
\]
holds for any \(u \in H^2(\Omega; \mathbb{C})\) and \(v \in H^1(\Omega; \mathbb{C})\).

The following notations will be used in the sequel
\[
L^c_{A,0} = \Delta_A \quad \text{(elliptic operator)},
\]
\[
L^p_{A,0} = \Delta_A - \partial_t \quad \text{(parabolic operator)},
\]
\[
L^w_{A,0} = \Delta_A - \partial^2_t \quad \text{(wave operator)},
\]
\[
L^s_{A,0} = \Delta_A + i\partial_t \quad \text{(Schrödinger operator)}.
\]
The \(n \times n\) identity matrix will be denoted by \(I\).

We shall use for convenience the following notation
\[
[h]_{t=t_1}^{t_2} = h(\cdot, t_2) - h(\cdot, t_1), \quad h \in H^1((t_1, t_2); L^2(\Omega)).
\]
Finally, we equip $\partial Q$ with the following measure:
\[ d\mu(x,t) = 1_{\Gamma \times (t_1, t_2)}(x, t) d\sigma(x) dt + 1_{\Omega \times (t_1, t_2)}(x) dx dt, \]
where $d\sigma(x)$ is the Lebesgue measure on $\Gamma$ and $\delta_t$ is the Dirac measure at $t$.

2.2. Pseudo-convexity condition. Define for $A \in \mathcal{M}(\Omega, \kappa, m)$
\[ \Lambda^{m}_{k,\ell}(A)(x) = -\sum_{p=1}^{n} \partial_p a_{k\ell}(x)a_{pm}(x) + 2 \sum_{p=1}^{n} a_{kp}(x)\partial_p a_{m}(x) \]
where $x \in \overline{\Omega}$ and $1 \leq k, \ell, m \leq n$.

To $h \in C^1(\overline{\Omega})$ we associate the matrix $\Upsilon_A(h)$ given by
\[ (\Upsilon_A(h))_{k\ell}(x) = \sum_{m=1}^{n} \Lambda^{m}_{k,\ell}(A)(x) \partial_m h(x), \quad x \in \Omega. \]

Note that $\Upsilon_A(h)$ is not necessarily symmetric.

Inspired by the definition introduced in [17, Section 28.2, page 234] we consider the following one:

**Definition 2.1.** We say that $h \in C^2(\overline{\Omega})$ is $A$-pseudo-convex with constant $\kappa > 0$ in $\Omega$ if $\nabla h(x) \neq 0$ for any $x \in \overline{\Omega}$ and if
\[ (\Theta_A(h)(x))\xi(\xi) \geq \kappa|\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \]
where
\[ \Theta_A(h) = 2A\nabla^2 h A + \Upsilon_A(h). \]

It worth noticing that $A \mapsto \Theta_A$ is positively homogenous of degree two:
\[ \Theta_{\lambda A} = \lambda^2 \Theta_A, \quad \lambda > 0. \]

Since $\Theta_I(h) = 2\nabla^2 h$, $h$ is $I$-pseudo-convex in $\Omega$ if $\nabla h(x) \neq 0$ and $\nabla^2 h(x)$ is positive definite for any $x \in \overline{\Omega}$. In other words when $A = I$ the pseudo-convexity is reduced to local strict convexity.

2.3. Carleman weights. It will be convenient to define the notion of Carleman weight for different kind of operators we are interested in.

In the rest of this paper $\psi = \psi(x, t)$ is a function of the form
\[ \psi(x, t) = \psi_0(x) + \psi_1(t) \]
and $\phi = e^{\lambda \phi}$, $\lambda > 0$.

**Definition 2.2.** (a) Let $0 \leq \psi_0 \in C^2(\overline{\Omega})$. We say that $\psi_0 = e^{\lambda \phi_0}$, $\lambda > 0$, is a weight function for the elliptic operator $\mathcal{L}_{A,0}^p$ if $\nabla \psi_0(x) \neq 0$ for any $x \in \overline{\Omega}$.
(b) If $0 \leq \psi \in C^2(\overline{Q})$ and $\nabla \psi_0(x) \neq 0$ for any $x \in \overline{\Omega}$ we say that $\phi$ is a weight function for the parabolic $\mathcal{L}_{A,0}^p$.
(c) Assume that $0 \leq \psi \in C^1(\overline{Q})$. Then $\phi$ is said a weight function for the Schrödinger operator $\mathcal{L}_{A,0}^p$ if $\psi_0$ is $A$-pseudo-convex in $\Omega$.
(d) We say that $\phi$, with $0 \leq \psi \in C^4(\overline{Q})$, is a weight function for the wave operator $\mathcal{L}_{A,0}^w$ if $0 \leq \psi_0$ is $A$-pseudo-convex with constant $\kappa > 0$ in $\Omega$ and if in addition the following two conditions hold:
\[ \min_{\overline{Q}} \left[ |\nabla \psi_0|^2 - (\partial_t \psi_0)^2 \right] > 0, \]
\[ |\partial_t^2 \psi_1| \leq \kappa^{-1} \kappa / 4. \]
Example 2.1. Fix $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and set for $\gamma \in \mathbb{R}$

$$
\psi(x, t) = \left( |x - x_0|^2 + \gamma(t + t_0)^2 \right) / 2.
$$

In that case

$$(\Upsilon_A(\psi_0))_{\alpha\beta} = -\sum_{k, \ell=1}^n \partial_k a_{\alpha\beta} a_{k\ell}(x_{\ell} - x_{0,\ell}) + 2 \sum_{k, \ell=1}^n a_{\alpha k} \partial_k a_{\beta\ell}(x_{\ell} - x_{0,\ell}).$$

Let us first discuss $A$-pseudo-convexity condition of $\psi_0$ in different cases.

(i) Assume that $\Omega = B(0, r)$ and $x_0 \in B(0, 2r) \setminus \overline{B}(0, r)$. We can then choose $r$ sufficiently small in such a way that

$$(\Upsilon_A(\psi_0)\xi|\xi) \geq -\kappa^2 |\xi|^2,$$

from which we deduce that

$$(\Theta_A(\psi_0)\xi|\xi) \geq \kappa^2 |\xi|^2.$$

(ii) As the mapping

$$A \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n}) \mapsto \Upsilon_A(\psi_0) \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n})$$

is continuous in a neighborhood of $I$ and $\Upsilon_I(\psi_0) = 0$, we conclude that there exists $\mathcal{N}$, a neighborhood of $I$ in $C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n})$, so that for any $A \in \mathcal{N}$ we have

$$(A(x)\xi|\xi) \geq |\xi|^2 / 2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n,$n

$$(\Upsilon_A(\psi_0)\xi|\xi) \geq -|\xi|^2 / 4, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n.$$n

Whence

$$(\Theta_A(\psi_0)\xi|\xi) \geq |\xi|^2 / 4, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n,$n

provided that $A \in \mathcal{N}$.

(iii) Consider the particular case in which $A = aI$ with $a \in C^{0,1}(\overline{\Omega})$ satisfying $a \geq \kappa$. Simple computations then yield

$$\Upsilon_A(\psi_0) = -a(\nabla a|x - x_0|)I + 2a\nabla a \otimes (x - x_0).$$

In consequence

$$(\Theta_A(\psi_0)\xi|\xi) \geq \kappa(2\kappa - 3 |\nabla a||x - x_0|) |\xi|^2.$$n

Hence a condition guaranteeing that $\psi_0$ is $aI$-pseudo-convex is

$$|\nabla a||x - x_0| < 2\kappa / 3.$$n

This condition is achieved for instance if $\Omega$ has sufficiently small diameter and $x_0$ is close to $\Omega$ or $\nabla a$ is small enough.

Next, we discuss a bound on $\gamma$ for which (2.2) and (2.3) hold. If $d = \text{dist}(x_0, \overline{\Omega})$ and $\sigma = || t + t_0 ||_{L^\infty((t_1, t_2))}$, then (2.2) is satisfied whenever

$$|\nabla \psi|^2 - |\partial_t \psi|^2 \geq d^2 - \sigma^2 \gamma^2 > 0.$$n

As $\partial_t^2 \psi = \gamma$ both (2.2) and (2.3) are satisfied when

$$0 < |\gamma| < \min\left[d / \sigma, \kappa / (4\kappa)\right].$$
2.4. Pseudo-convex hypersurface. We begin by a lemma concerning the action of an orthogonal transformations on $A$. For an orthogonal transformation $O$ and $A \in \mathcal{M}(\Omega, \kappa, m)$ we set $A_O(y) = OA(O^t y)O^t$. Here $O^t$ denotes the transposed matrix of $O$.

**Lemma 2.1.** Let $A \in \mathcal{M}(\Omega, \kappa, m)$ and $O$ is an orthogonal transformation. Then $A_O \in \mathcal{M}(\Omega, \kappa, n^2m)$.

**Proof.** Clearly $\kappa$ is invariant under an orthogonal transform. On the other hand, we see that if $L_{k\ell}$ is the Lipschitz constant of $a_{k\ell}$ and if $A_O = (a_{k\ell}^O)$ then $L_{k\ell}^O$, the Lipschitz constant of $a_{k\ell}^O$, satisfies

$$L_{k\ell}^O \leq n^2 L_{k\ell}, \quad 1 \leq k, \ell \leq n,$$

from which we complete the proof in a straightforward manner. \hfill \Box

The gradient with respect to the variable $x' \in \mathbb{R}^{n-1}$ or $y' \in \mathbb{R}^{n-1}$ is denoted henceforward by $\nabla'$. Let $\theta$ be a $C^2$-function defined in a neighborhood $U$ of $\tilde{x}$ in $\Omega$ with $\nabla\theta(\tilde{x}) \neq 0$. Consider then the hypersurface

$$H = \{x \in U; \theta(x) = \theta(\tilde{x})\}.$$

Making a translation and change of coordinates we may assume that $\tilde{x} = 0$, $\theta(\tilde{x}) = 0$, $\nabla'\theta(0) = 0$ and $\partial_\alpha\theta(0) \neq 0$. With help of implicit function theorem $\theta(x) = 0$ near 0 may rewritten as $x_n = \vartheta(x')$ with $\vartheta(0) = 0$ and $\nabla'\vartheta(0) = 0$.

Let $\hat{A}$ be the matrix obtained after these transformations. According to Lemma 2.1, $\hat{A} \in \mathcal{M}(O(U + \tilde{x}), \kappa, n^2m)$ where $O$ is the orthogonal transformation corresponding to the above change of coordinates. Also, note that $\hat{I} = I$.

Consider in some neighborhood of 0

$$\varphi = \varphi_H : (x', x_n) \in \omega \mapsto (y', y_n) = (x', x_n - \vartheta(x') + |x'|^2).$$

Elementary calculations yield

$$\varphi'(x', x_n) = \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \ldots & 1 & 0 \\ g_1(x') & \ldots & g_{n-1}(x') & 1 \end{pmatrix}$$

with $g_k(x') = -\partial_k \vartheta(x') + 2x_k'$, $0 \leq k \leq n - 1$. Whence

$$(\varphi'(x', x_n) \xi|\xi) = |\xi'|^2 + (-\nabla' \vartheta(x')) + 2x' |\xi'| \xi_n + \xi_n^2, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n.$$

Since

$$|(-\nabla' \vartheta(x') + 2x'|\xi'|) \xi_n| \leq |(-\nabla' \vartheta(x') + 2x'|\xi'|)^2/2 + \xi_n^2/2$$
$$\leq | - \nabla' \vartheta(x') + 2x'|^2 |\xi'|^2/2 + 2\xi_n^2/2$$
$$\leq (|\nabla' \vartheta(x')|^2 + 4|x'|^2) |\xi'|^2 + \xi_n^2/2$$

there exists a neighborhood $\omega$ of 0, only depending of $\vartheta$, so that

$$|(-\nabla' \vartheta(x') + 2x'|\xi'|) \xi_n| \leq |\xi'|^2/2 + \xi_n^2/2 = |\xi|^2/2.$$

In consequence

$$\varphi'(x', x_n) \xi|\xi) \geq |\xi'|^2/2, \quad x = (x', x_n) \in \omega, \xi \in \mathbb{R}^n.$$
Whence (2.5) together with Cauchy-Schwarz inequality yields
\[ |(\varphi')^t(x', x_n)\xi|^2 \geq |\xi|^2/2, \quad x = (x', x_n) \in \omega, \quad \xi \in \mathbb{R}^n. \] (2.6)

Let \( \tilde{\varphi} = \varphi(\omega) \) (hence \( \varphi \) is a diffeomorphism from \( \omega \) onto \( \tilde{\omega} \)) and define
\[ (\varphi')^t(\varphi^{-1}(y)) \tilde{\varphi} (\varphi^{-1}(y)) (\varphi')^t(\varphi^{-1}(y)), \quad y \in \tilde{\omega}. \] (2.7)

In light of (2.6) we obtain
\[ (\tilde{\varphi}(y)\xi)(\xi) \geq \kappa |(\varphi')^t(\varphi^{-1}(y))\xi|^2 \geq \kappa/4, \quad y \in \tilde{\omega}. \]

Also, by straightforward computations we get
\[ \|\tilde{\varphi}\|_{W^{1,\infty}(\tilde{\omega}, \mathbb{R}^{n \times n})} \leq \tilde{m}, \]
with \( \tilde{m} \) only depending of \( n, m \) and \( \vartheta \).

We observe that the role of \( \varphi \) is to transform the hypersurface \( \{x_n = \vartheta(x')\} \) in a neighborhood of the origin into the convex hypersurface \( \{y_n = |y'|^2\} \) in another neighborhood of the origin.

Define \( \tilde{\psi}_0 \) as follows
\[ \tilde{\psi}_0(y) = (y_n - 1)^2 + |y'|^2. \]

The matrix \( \tilde{\varphi} \) appearing in (2.7) is denoted henceforward by \( A_H \).

The following definition is motivated by the classical procedure used to establish the property of unique continuation of an elliptic operator across the convex hypersurface \( \{y_n = |y'|^2\} \).

**Definition 2.3.** We say that the hypersurface \( H \) is pseudo-convex if \( \tilde{\psi}_0 \) is \( A_H \)-pseudo-convex in \( \tilde{\varphi} \).

**Lemma 2.2.** (a) There exists \( N \) a neighborhood of \( I \) in \( C^{0,1} \overline{(\Omega; \mathbb{R}^{n \times n})} \) so that for any \( A \in N \), \( H \) is \( A \)-pseudo-convex.
(b) There exists \( \tilde{N}_0 \) a neighborhood of \( I \) in \( C^{0,1} \overline{(\Omega; \mathbb{R}^{n \times n})} \) so that for any \( A \in \tilde{N}_0 \) and any orthogonal transformation \( O \) and \( A_O(y) = OA(O^ty)O^t \), we have \( A_O \in \tilde{N} \).

**Proof.** (a) Let us first discuss the case where \( A = I \). Note that it is not hard to check that
\[ \varphi^{-1}(y', y_n) = (y', y_n + \vartheta(y') - |y'|^2) \]
and
\[ I_H(y) = \tilde{I}(y) = (\tilde{a}_{ij}(y')) = \begin{pmatrix} 1 & \ldots & 0 & \tilde{g}_1(y') \\ \vdots & \ddots & 0 & \vdots \\ 0 & \ldots & 1 & \tilde{g}_{n-1}(y') \\ \tilde{g}_1(y') & \ldots & \tilde{g}_{n-1}(y') & \tilde{g}_n(y') \end{pmatrix} \]
with \( \tilde{g}_k(y') = \partial_k \vartheta(y') - 2y'_k, \quad 0 \leq k \leq n - 1 \) and \( \tilde{g}_n = |\nabla \vartheta(y') - 2y'|^2 + 1 \).

We have clearly \( A(0) = I \) and for \( 1 \leq p \leq n - 1 \)
\[ \partial_p \tilde{a}_{k\ell}(y') = \begin{cases} 0, & 1 \leq k, \ell \leq n - 1, \\ \partial^2_{p\ell} \vartheta(y') - 2 \delta_{p\ell}, & 1 \leq k \leq n - 1, \ell = n, \\ 2 \sum_{\alpha=1}^{n-1} (\partial_{\alpha \ell} \vartheta(y') - 2y'_\alpha) (\partial^2_{p\alpha} \vartheta(y') - 2 \delta_{p\alpha}), & k = n, \ell = n. \end{cases} \]
Therefore
\[
\partial_p \tilde{a}_{kl}(0) = \begin{cases} 
0, & 1 \leq k, \ell \leq n - 1, \\
\partial^2 p_k \vartheta(0) - 2 \delta_p k, & 1 \leq k \leq n - 1, \ell = n, \\
0 & k = n, \ell = n.
\end{cases}
\]

Let \( \tilde{\Lambda}^m_{k,\ell} \) given by
\[
\tilde{\Lambda}^m_{k,\ell}(y) = -\sum_{p=1}^n \partial_p \tilde{a}_{kl}(y) \tilde{a}_{pm}(y) + 2 \sum_{p=1}^n \tilde{a}_{kp}(y) \partial_p \tilde{a}_{\ell m}(y)
\]
and define \( \tilde{\Upsilon}(y) = (\tilde{\Upsilon}_{k\ell}(y)) \) as follows
\[
\tilde{\Upsilon}_{k\ell}(y) = \sum_{m=1}^n \tilde{\Lambda}^m_{k,\ell}(y) \partial_m \tilde{\psi}_0(y).
\]

It is then straightforward to check that
\[
\tilde{\Upsilon}_{k\ell}(0) = \sum_{m=1}^n \Lambda^m_{k,\ell}(0) \partial_m \tilde{\psi}_0(0) = -2 \Lambda^0_{k,\ell}(0) = 0, \quad 0 \leq k, \ell \leq n.
\]

Since
\[
\tilde{\Theta}(y) = \Theta(y) = 2 \nabla^2 \tilde{\psi}_0(y) + \tilde{\Upsilon}(y)
\]
we get
\[
\tilde{\Theta}(0) = 4I,
\]
and hence
\[
(\tilde{\Theta}(0) \xi | \xi) \geq 4 |\xi|^2, \quad \xi \in \mathbb{R}^n.
\]
Continuity argument first with respect to \( y \) and then with respect to \( A \) shows that, by reducing \( \tilde{\omega} \) if necessary,
\[
(\Theta_{,\tilde{A}}(\psi_0)(y) \xi | \xi) \geq 2 |\xi|^2, \quad y \in \tilde{\omega}, \xi \in \mathbb{R}^n.
\]
(b) Immediate from Lemma 2.1.

\[\square\]

### 3. The wave equation

#### 3.1. Carleman inequality

In this subsection \( \psi(x, t) = \psi_0(x) + \psi_1(t) \) is a weight function for the wave operator \( L^\infty_{A,0} \) with \( A \)-pseudo-convexity constant \( \kappa > 0 \). We set
\[
\delta = \min_\Omega [|| \nabla \psi ||^2_A + (\partial_x \psi)^2]^2 > 0
\]
and \( \phi = e^{\lambda \psi} \).

For notational convenience we use in the sequel \( \mathfrak{d} = (\Omega, t_1, t_1, \kappa, m, \kappa, \delta, b) \) with \( b \geq ||\psi||_{C^4(\overline{\Omega})} \), and \( D_A = (\nabla_A, \partial_t) \).

**Theorem 3.1.** There exist three constants \( \mathfrak{N} = \mathfrak{N}(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that
\[
(3.1) \quad \mathfrak{N} \int_Q e^{2\tau \phi} \left[ \tau^3 \lambda^4 \phi^3 u^2 + \tau \lambda \phi |D_A u|^2 \right] dxdt \\
\quad \leq \int_Q e^{2\tau \phi} (L^\infty_{A,0} u)^2 dxdt + \int_{\partial Q} e^{2\tau \phi} \left[ \tau^3 \lambda^3 \phi^3 u^2 + \tau \lambda \phi |D_A u|^2 \right] d\mu
\]
for any $\lambda \geq \lambda^*, \tau \geq \tau^*$ and $u \in H^2(Q, \mathbb{R})$.

**Proof.** In this proof, $\mathbb{R}_j$, $\lambda_j$ and $\tau_j$, $j = 0, 1, \ldots$, are generic constants only depending on $\varrho$.

Set $\Phi = e^{-\tau\phi}$ with $\tau > 0$. Elementary computations then give

$$
\partial_i \Phi = -\tau \partial_i \phi \Phi,
\partial_{ij} \Phi = (-\tau \partial_{ij}^2 \phi + \tau^2 \partial_i \phi \partial_j \phi) \Phi,
\partial_t \Phi = -\tau \partial_t \phi \Phi = -\tau \partial_t \phi \Phi,
\partial^2_t \Phi = (-\tau \partial^2_t \phi + \tau^2 (\partial_t \phi)^2) \Phi.
$$

Remark that the preceding two first formulas can be rewritten as

$$
\nabla \Phi = -\tau \nabla \phi, \\
\nabla^2 \Phi = \Phi \left(-\tau \nabla^2 \phi + \tau^2 \nabla \phi \otimes \nabla \phi \right).
$$

For $w \in H^2(Q; \mathbb{R})$, we obtain

$$
\Phi^{-1} \Delta_A (\Phi w) = \Delta_A w - 2\tau (\nabla w | \nabla \phi)_A + \left[\tau^2 |\nabla \phi|_A^2 - \tau \Delta_A \phi \right] w.
$$

Also

$$
\Phi^{-1} \partial^2_t (\Phi w) = \partial^2_t w - 2\tau \partial_t \phi \partial_t w \left(-\tau \partial^2_t \phi + \tau^2 (\partial_t \phi)^2 \right) w.
$$

We decompose $L = \Phi^{-1} L^w_{A,0} \Phi$ into its self-adjoint part and skew-adjoint part:

$$
L = L_+ + L_-.
$$

Simple calculations show that

$$
L_+ w = \Delta_A w - \partial^2_t w + aw, \\
L_- w = (B | \nabla w) + d \partial_t w + bw
$$

with

$$
a(x, t) = \tau^2 \left(|\nabla \phi|_A^2 - (\partial_t \phi)^2 \right), \\
b(x, t) = -\tau \left(\Delta_A \phi - \partial^2_t \phi \right), \\
B = -2\tau A \nabla \phi, \\
d = 2\tau \partial_t \phi.
$$

We write

$$
(L_+ w | L_- w)_{L^2(Q)} = \sum_{i=1}^{9} I_i,
$$

(3.2)
where

\[ I_1 = \int_Q \Delta_A w(\nabla w|B)dxdt, \]
\[ I_2 = \int_Q \Delta_A w\partial_twdxdt, \]
\[ I_3 = \int_Q \Delta_A bwwdxdt, \]
\[ I_4 = -\int_Q \partial^2_t w(\nabla w|B)dxdt, \]
\[ I_5 = -\int_Q \partial^2_t w\partial_twdxdt, \]
\[ I_6 = -\int_Q \partial^2_t bwwdxdt, \]
\[ I_7 = \int_Q aw(\nabla w|B)dxdt, \]
\[ I_8 = \int_Q adw\partial_twdxdt, \]
\[ I_9 = \int_Q abw^2dxdt. \]

Integration by parts with respect to the space variable we use in the rest of this proof are often based on Green’s formula (2.1).

A first integration by parts then yields

\[ I_1 = \int_Q \Delta_A w(\nabla w|B)dxdt = -\int_Q (\nabla w|\nabla(\nabla w|B))_A dxdt \]
\[ + \int_\Sigma (\nabla w|\nu)_A(\nabla w|B)dsdt. \]

Whence

\[ (3.3) \quad I_1 = -\int_Q (\nabla^2 wB + (B')^t\nabla w|\nabla w) A dxdt + \int_\Sigma (\nabla w|\nu)_A(\nabla w|B)dsdt. \]

Now as

\[ \int_Q \partial^2_{ij} wB_ja_{ik}\partial_kwdxdt = \int_Q \partial_iw\partial_j(B_ja_{ik})\partial_kw - \int_Q \partial_iwB_ja_{ik}\partial^2_{kj}w \]
\[ + \int_\Sigma \partial_iwB_j\nu_ja_{ik}\partial_kwsdsdt \]

we find

\[ (3.4) \quad 2\int_Q (\nabla^2 wB|\nabla w)_A dxdt = -\int_Q (C\nabla w|\nabla w) + \int_\Sigma (B|\nu)|\nabla w|^2A dsdt \]

where \( C = (\text{div}(a_{ij}B)). \)

Let

\[ D = C/2 - A(B')^t. \]
We get by putting (3.4) into (3.3)

\[ I_1 = \int_Q (D \nabla w | \nabla w) dx dt + \int_\Sigma \left[ (\nabla w | \nu)_A (\nabla w \cdot B) - 2^{-1} (B | \nu) | \nabla w |^2_A \right] d\sigma dt. \]

For \( I_2 \) we obtain by making integrations by parts

\[
I_2 = - \int_Q (\nabla w | \nabla (d \partial_t w))_A dx dt + \int_\Sigma (\nabla w | \nu)_A d\partial_t w d\sigma dt
\]

\[ = - \int_Q (\nabla w | \nabla d)_A \partial_t w dx dt - \int_Q d(\nabla w | \nabla \partial_t w) dx dt + \int_\Sigma (\nabla w | \nu)_A d\partial_t w d\sigma dt.
\]

As \( A \) is symmetric we have

\[ (\nabla w | \nabla \partial_t w)_A = \partial_t | \nabla w |^2_A / 2. \]

Hence

\[ I_2 = - \int_Q (\nabla w | \nabla d)_A \partial_t w dx dt - \int_\Sigma (\nabla w | \nu)_A d\partial_t w d\sigma dt - \int_\Omega [(d/2) | \nabla w |^2_A]_{t=t_1}^t dx. \]

We have also

\[ I_3 = \int_Q \Delta_A w b w = - \int_Q b | \nabla w |^2_A dx dt - \int_Q w (\nabla b | \nabla w)_A dx dt + \int_\Sigma (\nabla w | \nu)_A b w d\sigma dt
\]

\[ = - \int_Q b | \nabla w |^2_A dx dt + \int_Q \Delta_A (b/2) w^2 dx dt - \int_\Sigma (\nabla (b/2) | \nu)_A w^2 dx dt + \int_\Sigma (\nabla w | \nu)_A b w d\sigma dt.
\]

Let \( J_1 = I_1 + I_2 + I_3 \) and

\[ A_1 = D + [\partial_t (d/2) - b] A, \]

\[ a_1 = \Delta_A (b/2), \]

\[ B_1 (w) = - (\nabla w | \nabla d)_A \partial_t w, \]

\[ g_1 (w) = (\nabla w | \nu)_A (\nabla w \cdot B) - (B/2 | \nu) | \nabla w |^2_A + (\nabla w \cdot \nu)_A d\partial_t w \]

\[ - (\nabla (b/2) | \nu)_A w^2 dx + (\nabla w | \nu)_A b w, \]

\[ h_1 (w) = - [(d/2) | \nabla w |^2_A]_{t=t_1}^t. \]

Putting together (3.5) to (3.7) we find

\[ J_1 = \int_Q (A_1 \nabla w | \nabla w) dx dt + \int_Q B_1 (w) dx dt + \int_Q a_1 w^2 dx dt.
\]

\[ + \int_\Sigma g_1 (w) d\sigma dt + \int_\Omega h_1 (w) dx.
\]

Straightforward computations show that

\[ A_1 = 2 \tau A \nabla^2 \phi A + \tau Y_A (\phi). \]
and hence
\begin{equation}
(3.11) \quad \hat{\mathcal{Q}} = \frac{1}{\tau} \int_0^\tau \left( [2A \nabla^2 \phi A + Y_A(\phi)] \nabla w \nabla w \right) dx dt + \int_Q B_1(w) dx dt \\
+ \int_Q a_1 w^2 dx dt + \int_{\Sigma} g_1(w) d\sigma dt + \int_\Omega b_1(w) dx.
\end{equation}

In consequence
\begin{equation}
(3.9) \quad \hat{\mathcal{Q}} = \frac{1}{\tau} \int_0^\tau \partial_t w \partial_t (\nabla w | \mathcal{B}) dx dt - \int_\Omega \left[ \partial_t w (\nabla w | \mathcal{B}) \right]_{t=t_1}^{t_2} dx.
\end{equation}

We obtain by using again an integration by parts
\[ I_4 = \int_Q \partial_t w \partial_t (\nabla w | \mathcal{B}) dx dt - \int_\Omega \left[ \partial_t w (\nabla w | \mathcal{B}) \right]_{t=t_1}^{t_2} dx. \]

Hence
\[ I_4 = \int_Q \partial_t w (\nabla w | \partial_t \mathcal{B}) dx dt + \int_Q \partial_t w (\nabla \partial_t w | \mathcal{B}) dx dt - \int_\Omega \left[ \partial_t w (\nabla w | \mathcal{B}) \right]_{t=t_1}^{t_2} dx. \]

But
\begin{align*}
\int_Q \partial_t w (\nabla \partial_t w | \mathcal{B}) dx dt &= - \int_Q \text{div}(\partial_t w \mathcal{B}) \partial_t w dx dt + \int_\Sigma (\partial_t w)^2 (B | \nu) d\sigma dt \\
&= - \int_Q \partial_t w (\nabla \partial_t w | \mathcal{B}) dx dt - \int_Q (\partial_t w)^2 \text{div}(\mathcal{B}) dx dt \\
&\quad + \int_\Sigma (\partial_t w)^2 (B | \nu) d\sigma dt.
\end{align*}

Therefore
\begin{align*}
2 \int_Q \partial_t w (\nabla \partial_t w | \mathcal{B}) dx dt &= - \int_Q (\partial_t w)^2 \text{div}(\mathcal{B}) dx dt + \int_\Sigma (\partial_t w)^2 (B | \nu) d\sigma dt.
\end{align*}

In consequence
\begin{equation}
(3.9) \quad I_4 = \int_Q \partial_t w (\nabla w | \partial_t \mathcal{B}) dx dt - \int_Q (\partial_t w)^2 \text{div}(\mathcal{B}/2) dx dt \\
+ \int_\Sigma (\partial_t w)^2 (B/2 | \nu) d\sigma dt - \int_\Omega \left[ \partial_t w (\nabla w | \mathcal{B}) \right]_{t=t_1}^{t_2} dx.
\end{equation}

For \( I_5 \) we have
\begin{equation}
(3.10) \quad I_5 = - \int_Q (d/2) \partial_t (\partial_t w)^2 dx dt = \int_Q \partial_t (d/2) (\partial_t w)^2 dx dt \\
- \int_\Omega \left[ (d/2) (\partial_t w)^2 \right]_{t=t_1}^{t_2} dx.
\end{equation}

Also
\begin{align*}
I_6 &= - \int_Q \partial_t^2 w bw = \int_Q \partial_t bw \partial_t w dx dt + \int_Q b(\partial_t w)^2 dx dt - \int_\Omega [bw \partial_t w]_{t=t_1}^{t_2} dx \\
&= \int_Q \partial_t (b/2) \partial_t w^2 dx dt + \int_Q b(\partial_t w)^2 dx dt - \int_\Omega [bw \partial_t w]_{t=t_1}^{t_2} dx \\
\text{and hence}
\end{align*}

\begin{equation}
(3.11) \quad I_6 = - \int_Q \partial_t^2 (b/2) w^2 dx dt + \int_Q b(\partial_t w)^2 dx dt \\
- \int_\Omega [bw \partial_t w]_{t=t_1}^{t_2} dx + \int_\Omega [\partial_t bw]^2_{t=t_1} dx.
\end{equation}
Let $J_2 = I_4 + I_5 + I_6$ and define
\[ a_2 = -\div(B/2) + \partial_t(d/2) + b, \]
\[ a_2 = -\partial_t^2(b/2), \]
\[ B_2(w) = \partial_t w(\nabla w|\partial_t B), \]
\[ g_2(w) = (\partial_t w)^2(B/2|\nu), \]
\[ h_2(w) = -[\partial_t w(\nabla w|B)]_{i=1}^{t_2} - [d(\partial_t w)^2]_{i=1}^{t_2} - [bw\partial_t w]_{i=1}^{t_2} + [bw^2]_{i=1}^{t_2}. \]

A combination of (3.9) to (3.11) gives
\[ J_2 = \int_Q a_2(\partial_t w)^2 dxdt + \int_Q B_2(w)dxdt + \int_Q a_2w^2 dxdt \]
\[ + \int \sum g_2(w)dxdt + \int_\Omega h_2(w)dx. \]

Let us observe that we have by straightforward computations
\[ a_2 = 2\tau\partial_t^2 \phi, \quad B_2 = B_1. \]
Hence
\[ (3.12) \quad J_2 = 2\tau\int_Q \partial_t^2 \phi(\partial_t w)^2 dxdt + \int_Q B_1(w)dxdt + \int_Q a_2w^2 dxdt \]
\[ + \int \sum g_2(w)dxdt + \int_\Omega h_2(w)dx. \]

Let $\tilde{J} = J_1 + J_2$. In light of (3.5) and (3.12) we deduce that
\[ \tilde{J} = \tau\int_Q (2A\nabla^2 \phi A + \Upsilon_A(\phi)) \nabla w|\nabla w) dxdt + 2\tau\int_Q \partial_t^2 \phi(\partial_t w)^2 dxdt \]
\[ - 4\tau\int_Q \partial_t w(\nabla w|\nabla \partial_t \phi)A dxdt + \int_Q \tilde{a}w^2 dxdt \]
\[ + \int \sum \tilde{g}(w)dxdt + \int_\Omega \tilde{h}(w)dx \]

where
\[ \tilde{a} = a_1 + a_2, \quad \tilde{g} = g_1 + g_2, \quad \tilde{h} = h_1 + h_2. \]

As $\phi = e^{\lambda \psi}$ we have
\[ \nabla^2 \phi = \lambda^2 \phi(\nabla^2 \phi), \quad \nabla \partial_t \phi = \lambda \phi \nabla \psi, \]
\[ \partial_t^2 \phi = \lambda^2 \phi(\partial_t \phi)^2 + \lambda \phi \partial_t^2 \psi. \]

This and the fact that $\nabla \partial_t \psi = 0$ imply
\[ (\nabla^2 \phi A \nabla w|\nabla w)_A + \partial_t^2 \phi(\partial_t w)^2 - 2\partial_t w(\nabla w|\nabla \partial_t \phi)_A = \]
\[ \lambda \phi \left[ (\nabla^2 \psi A \nabla w|\nabla w)_A + \partial_t^2 \psi(\partial_t w)^2 \right] \]
\[ + \lambda^2 \phi \left[ (\nabla \psi|\nabla w)_A^2 + (\partial_t \psi)^2(\partial_t w)^2 - 2\partial_t \psi \partial_t w(\nabla \psi|\nabla w)_A \right]. \]

That is
\[ (\nabla^2 \phi A \nabla w|\nabla w)_A + \partial_t^2 \phi(\partial_t w)^2 - 2\partial_t w(\nabla w|\nabla \partial_t \phi)_A = \]
\[ \lambda \phi \left[ (\nabla^2 \psi A \nabla w|\nabla w)_A + \partial_t^2 \psi(\partial_t w)^2 \right] + \lambda^2 \phi \left[ (\nabla \psi|\nabla w)_A - \partial_t \psi \partial_t w \right]^2 \]
from which we deduce, by noting that \( \Upsilon_A(\phi) = \lambda \phi \Upsilon_A(\psi) \).

\[
\tilde{J} \geq \tau \lambda \int_Q \phi \left[ (A A^{1/2} \nabla w | A^{1/2} \nabla w) + \partial_t^2 \psi (\partial_t w)^2 \right] dx dt + \int_Q \tilde{a} w^2 dx dt \\
+ \int_{\Sigma} \tilde{g}(w) d\sigma dt + \int_{\Omega} \tilde{h}(w) dx.
\]

Here

\[
A = 2 A^{1/2} \nabla^2 \psi A^{1/2} + \Theta_A(\psi) A^{-1/2} = \Theta_A(\psi) A^{-1/2}.
\]

For \( I_7 \) we have

\[
I_7 = \int_Q a w (B | \nabla w) = \int_Q a (B/2 | \nabla w^2) = - \int_Q \text{div}(a B/2) w^2 dx dt \\
+ \int_{\Sigma} a (B/2 | \nu) w^2 d\sigma dt.
\]

Finally

\[
I_8 = \int_Q (a d/2) \partial_t w^2 dx dt = - \int_Q \partial_t (a d/2) w^2 dx dt + \int_{\Omega} [(a d/2) w^2]_{t=t_1}^t dx.
\]

Define \( \hat{J} = I_7 + I_8 + I_9 \) and

\[
\begin{align*}
\hat{a} &= -\text{div}(a B/2) - \partial_t (a d/2) + a b + \Delta A (b/2) - \partial^2_t (b/2), \\
\hat{g}(w) &= \tilde{g}(w) + a (B/2 | \nu) w^2, \\
\hat{h}(w) &= \tilde{h}(w) + [(a d/2) w^2]_{t=t_1}^t.
\end{align*}
\]

Then clearly we have from (3.13) to (3.15)

\[
(L+w | L-w)_{L^2(Q)} = \hat{J} + \tilde{J} \\
\geq \tau \lambda \int_Q \phi \left[ (A A^{1/2} \nabla w | A^{1/2} \nabla w) + \partial_t^2 \psi (\partial_t w)^2 \right] dx dt + \int_Q \tilde{a} w^2 dx dt \\
+ \int_{\Sigma} \tilde{g}(w) d\sigma dt + \int_{\Omega} \tilde{h}(w) dx.
\]

We prove (see details in the end of this proof) that

\[
\hat{a} \geq \tau^3 \lambda^4 \phi^3 \delta, \quad \lambda \geq \lambda_1, \quad \tau \geq \tau_1.
\]

Whence

\[
2(L+w | L-w)_{L^2(Q)} \geq 2 \kappa^{-1} \kappa \lambda \int_Q \phi | \nabla w |^2 dx dt + 2 \tau \lambda \int_Q \phi \partial_t^2 \psi (\partial_t w)^2 dx dt \\
+ 2 \tau^3 \lambda^4 \delta \int_Q \phi^3 w^2 dx dt + \mathcal{R}_0(w)
\]

for \( \lambda \geq \lambda_1 \) and \( \tau \geq \tau_1 \) with

\[
\mathcal{R}_0(w) = \int_{\Sigma} \tilde{g}(w) d\sigma dt + \int_{\Omega} \tilde{h}(w) dx.
\]
On the other hand we find by making twice integration by parts

\[
\int_Q (L + w)\phi w dx dt = -\int_Q \phi |\nabla w|_A^2 + \int_Q \phi (\partial_t w)^2 dx dt + \int_Q (a - b/2)w^2 dx dt + B_1(w)
\]

with

\[
B_1(w) = \int_\Sigma \phi (\nabla w|_A w d\sigma dt - \frac{1}{2} \int_\Sigma (\nabla \phi|_A w^2 dx dt + \int_\Omega -[\phi w \partial_t w]_{\Omega = t} dx + \frac{1}{2} \int_\Omega [w^2 \partial_t \phi]_{t_{-t}}^{t_t} dx.
\]

Let \(\epsilon > 0\) to be determined later. Then Cauchy-Schwarz’s inequality yields

\[
\int_Q (L + w)^2 dx dt + \epsilon^2 \tau^2 \lambda^2 \int_Q \phi^2 w^2 dx dt \geq -\frac{\epsilon \tau \lambda}{2} \int_Q \phi |\nabla w|_A^2 dx dt + \frac{\epsilon \tau \lambda}{2} \int_Q \phi (\partial_t w)^2 dx dt + \frac{\epsilon \tau \lambda}{2} \int_Q (a - b/2)w^2 dx dt + \frac{\epsilon \tau \lambda}{2} B_1(w).
\]

Using that \(a = \tau^2 \lambda^2 \phi^2 (|\nabla \psi|_A^2 - (\partial_t \psi)^2)\) we get

\[
\int_Q (L + w)^2 dx dt \geq -\frac{\epsilon \tau \lambda}{2} \int_Q \phi |\nabla w|_A^2 dx dt + \frac{\epsilon \tau \lambda}{2} \int_Q \phi (\partial_t w)^2 dx dt + \frac{\epsilon \tau \lambda}{2} \int_Q (a - b/2)w^2 dx dt + \frac{\epsilon \tau \lambda}{2} B_1(w).
\]

Hence

\[
2(L + w|L - w)_{L^2(Q)} + \int_Q (L + w)^2 dx dt \geq (2\epsilon^{-1} \kappa - \epsilon/2) \tau \lambda \int_Q \phi |\nabla w|_A^2 dx dt
\]

\[
+ \tau \lambda \int_Q \phi (\epsilon/2 + 2\partial_t^2 \psi)(\partial_t w)^2 dx dt + 2\tau^3 \lambda^4 \delta \int_Q \phi^3 w^2 dx dt + \frac{\epsilon \tau \lambda^3}{2} \int_Q \phi^3 (|\nabla \psi|_A^2 - (\partial_t \psi)^2) w^2 dx dt
\]

\[
- \epsilon^2 \tau^2 \lambda^2 \int_Q \phi^2 w^2 dx dt - \frac{\epsilon \tau \lambda^2}{2} \int_Q (b/2)w^2 dx dt + B(w), \quad \lambda \geq \lambda_1, \quad \tau \geq \tau_1.
\]

Here \(B(w) = B_0(w) + \frac{\epsilon \tau \lambda}{2} B_1(w)\).

We take \(\epsilon = 2\epsilon^{-1} \kappa\) in the preceding inequality and we use inequality (2.3).
We then obtain by noting that in the right hand side the fourth, fifth and sixth terms can be absorbed by the third term
\[
2(L_+, L_-) + \int_Q (L+w)^2 dx dt \geq \kappa_2 \kappa_3 \int Q \phi |\nabla w|^2_A + (\partial_t w)^2 dx dt
\]
\[+ \tau^3 \lambda^4 \delta \int_Q \phi^3 w^2 dx dt + B(w), \quad \lambda \geq \lambda_2, \tau \geq \tau_2.
\]
We find by making elementary calculations
\[
\mathcal{R}_3 |\mathcal{B}(w)| \leq \int_{\partial Q} e^{2\tau \phi} \left[ \tau^3 \lambda^3 \phi^3 u^2 + \tau \lambda \phi (|\nabla_A u|^2 + (\partial_t u)^2) \right] d\mu
\]
for \(\lambda \geq \lambda_3\) and \(\mu \geq \mu_3\).

This and
\[
\|Lw\|_{L^2(Q)}^2 \geq 2\langle L_+ w, L_- w \rangle_{L^2(Q)} + \|L_+ w\|_{L^2(Q)}^2
\]
imply
\[
(3.18) \quad \mathcal{R} \int_Q \left[ \tau^3 \lambda^3 \phi^3 w^2 + \tau \lambda \phi |D_A w|^2 \right] dx dt
\]
\[\leq \int_Q (L^w_{A,0} w)^2 dx dt + \int_{\partial Q} \left[ \tau^3 \lambda^3 \phi^3 w^2 + \tau \lambda \phi |D_A w|^2 \right] d\mu.
\]

Take in this inequality \(w = \Phi^{-1} u\) with \(u \in H^2(Q; \mathbb{R})\). In light of the identities
\[
\Phi^{-1} \nabla u = -\tau \lambda w \nabla \psi + \nabla w,
\]
\[
\Phi^{-1} \partial_t u = -\tau \lambda w \partial_t \psi + \partial_t w
\]
we obtain an inequality similar to (3.31) which leads to (3.31) by observing that the additional terms in the right hand side appearing in this intermediate inequality can be absorbed by the terms in left hand side.

Proof of (3.17). Set
\[
\chi = |\nabla \psi|^2_A - (\partial_t \psi)^2.
\]

We have
\[
\Delta_A \phi = \text{div}(A \nabla e^{\lambda \psi}) = \text{div}(\lambda \phi A \nabla \psi) = \lambda^2 \phi |\nabla \psi|^2_A + \lambda \phi \Delta_A \psi
\]
\[\partial_t^2 \phi = \partial_t (\lambda \phi \partial_t \psi) = \lambda^2 \phi (\partial_t \psi)^2 + \lambda \phi \partial_t^2 \psi.
\]

That is
\[
(3.19) \quad L^w_{A,0} \phi = \lambda^2 \phi \chi + \lambda \phi L^w_{A,0} \psi.
\]

In light of (3.19) we get
\[
a(x, t) = \tau^2 \left(|\nabla \phi|^2_A - (\partial_t \phi)^2\right) = \tau^2 \lambda^2 \phi^2 \chi,
\]
\[
b(x, t) = -\tau (\Delta_A \phi - \partial_t^2 \phi) = -\tau \lambda^2 \phi \chi - \tau \lambda \phi L^w_{A,0} \psi,
\]
\[
B = -2\tau A \nabla \phi = -2\tau \lambda \phi A \nabla \psi,
\]
\[
d = 2\tau \partial_t \phi = 2\tau \lambda \phi \partial_t \psi.
\]

Since
\[
-aB/2 = \tau^3 \lambda^3 \phi^3 A \nabla \psi
\]
we find
\[
(3.20) \quad -\text{div}(aB/2) = 3\tau^3 \lambda^4 \phi^3 \chi |\nabla \psi|^2 + \tau^3 \lambda^3 \phi^3 \text{div}(\chi A \nabla \psi).
\]
Also as

\[-ad/2 = -\tau^3 \lambda^3 \phi^3 \chi \partial_t \psi\]

we obtain

\(\partial_t (ad/2) = -3\tau^3 \lambda^4 \phi^3 \chi (\partial_t \psi)^2 - \tau^3 \lambda^3 \phi^3 \partial_t (\chi \partial_t \psi).\)

We get by putting together \((3.20)\) and \((3.21)\)

\(- \text{div}(aB/2) - \partial_t (ad/2) = 3\tau^3 \lambda^4 \phi^3 \chi^2 + 3\tau^3 \lambda^3 \phi^3 [\text{div}(\chi A \nabla \psi) - \partial_t (\chi \partial_t \psi)].\)

On the other hand

\(ab = -\tau^3 \lambda^4 \phi^3 \chi^2 - \tau^3 \lambda^3 \phi^3 \chi L_{A,0}^w \psi.\)

Set

\(\chi_1 = \text{div}(\chi A \nabla \psi) - \partial_t (\chi \partial_t \psi) - \chi L_{A,0}^w \psi = (\nabla \chi | \nabla \psi)_A - \partial_t \chi \partial_t \psi.\)

A combination of \((3.22)\) and \((3.23)\) yields

\(- \text{div}(aB/2) - \partial_t (ad/2) + ab = 2\tau^3 \lambda^4 \phi^3 \chi^2 + \tau^3 \lambda^3 \phi^3 \chi_1.\)

We have again from \((3.19)\)

\(L_{A,0}^w (b/2) = \left(-\tau \lambda^2 \phi \chi - \tau \lambda \phi L_{A,0}^w \psi \right)\)

\(= - \left(\lambda^2 \phi \chi + \lambda \phi L_{A,0}^w \psi \right) \left(\tau \lambda^2 \chi + \tau \lambda \left( L_{A,0}^w \right)^2 \psi \right)\)

Therefore

\(L_{A,0}^w (b/2) \geq -\tau \lambda^4 \phi \delta, \quad \lambda \geq \lambda_4, \quad \tau > 0,\)

where we used that \(\chi^2 \geq \delta.\)

Inequality \((3.17)\) then follows by combining \((3.24)\) and \((3.25)\) and using that

\(\hat{a} = -\text{div}(aB/2) - \partial_t (ad/2) + ab + L_{A,0}^w (b/2).\)

Let \(\nu\) be the unit normal vector field on \(\Gamma\) pointing outward \(\Omega\). Define \(\partial_{\nu A} \psi_0\) by

\(\partial_{\nu A} \psi_0 = (\nabla \psi_0 | \nu)_A\)

and set

\(\Gamma_+ = \Gamma_+^{\psi_0} = \{x \in \Gamma; \partial_{\nu A} \psi_0(x) > 0\}, \quad \Sigma_+ = \Sigma_+^{\psi_0} = \Gamma_+ \times (t_1, t_2).\)

Let \(w \in H^2(Q, \mathbb{R})\) satisfying \(w = 0\) on \(\Sigma\). In that case it is straightforward to check that \(\hat{g}(w)\) defined in the preceding proof takes the form

\(\hat{g}(w) = -\tau \lambda \phi (\partial_{\nu A} w)^2 |\nu|^2_{\nu A} \partial_{\nu A} \psi_0.\)

If in addition \(u = \Phi w\) then \(\partial_{\nu} u = \Phi \partial_{\nu} w.\) In light of these two identities slight modifications of the last part of the preceding theorem enable us to prove the following result.
Theorem 3.2. There exist three constants \( N = N(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
\begin{aligned}
N \int_Q e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] dx dt \\
\leq \int_Q e^{2r\phi} \left[ L_{A,0}^w u \right]^2 dx dt + \tau \lambda \int_{\Sigma^+} e^{2r\phi} \phi(\partial_\nu u)^2 d\sigma dt
\end{aligned}
\]  

for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^2(Q, \mathbb{R}) \) satisfying \( u = 0 \) on \( \Sigma \) and \( u = \partial_t u = 0 \) in \( \Omega \times \{ t_1, t_2 \} \).

Let us see that Theorem 3.1 remains valid whenever we add to \( L_{A,0}^w \) a first order operator. Consider then the operator

\[
L_A^w = L_{A,0}^w + q_0 \partial_t + \sum_{i=1}^n q_i \partial_i + p
\]

where \( q_0, \ldots, q_n \) and \( p \) belong to \( L^\infty(Q, \mathbb{C}) \) and satisfy

\[
\max_{0 \leq i \leq n} \| q_i \|_{L^\infty(Q)} \leq m, \quad \| p \|_{L^\infty(Q)} \leq m.
\]

Let \( u = v + iw \in H^2(Q, \mathbb{C}) \) and apply Theorem 3.1 to both \( v \) and \( w \). We obtain by adding side by side inequalities (3.31) for \( v \) and for \( w \)

\[
N \int_Q e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] dx dt \\
\leq \int_Q e^{2r\phi} \left| L_{A,0}^w u \right|^2 dx dt + \int_{\partial Q} e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] d\mu.
\]

Since

\[
|L_{A,0}^w u|^2 \leq 2|L_A^w u|^2 + 2(n + 2)m^2 (\tau |\nabla u|_{A}^2 + |\partial_t u|^2 + |u|^2)
\]

and the term

\[
2(n + 2)m^2 \int_Q e^{2r\phi} (\tau |\nabla u|_{A}^2 + |\partial_t u|^2 + |u|^2) dx dt
\]

can be absorbed by the left hand side of (3.27) we have the following result:

Corollary 3.1. We find three constants \( N = N(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
N \int_Q e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] dx dt \\
\leq \int_Q e^{2r\phi} \left| L_A^w u \right|^2 dx dt + \int_{\partial Q} e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] d\mu
\]

for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^2(Q, \mathbb{C}) \).

Finally, we note that the preceding arguments allow us to prove the following corollary.

Corollary 3.2. There exist three constants \( N = N(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
N \int_Q e^{2r\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |D_A u|^2 \right] dx dt \\
\leq \int_Q e^{2r\phi} \left| L_A^w u \right|^2 dx dt + \tau \lambda \int_{\Sigma^+} e^{2r\phi} |\partial_\nu u|^2 d\sigma dt
\]
for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^2(Q, \mathbb{C}) \) satisfying \( u = 0 \) on \( \Sigma \) and \( u = \partial_t u = 0 \) in \( \Omega \times \{t_1, t_2\} \).

We close this subsection by a local one-parameter Carleman inequality that we obtain as a special case of Corollary 3.1 in which we fixed \( \lambda \geq \lambda^* \).

**Theorem 3.3.** There exist two constants \( \mathcal{N} = \mathcal{N}(\mathcal{D}) > 0 \) and \( \tau^* = \tau^*(\mathcal{D}) > 0 \) so that

\[
\sum_{|\alpha| \leq 1} \tau^{2(2-|\alpha|)} \int_Q e^{2\tau|\partial^n u|^2} \, dx \, dt \leq \mathcal{N} \int_Q e^{2\tau|\mathcal{L}^*_A u|^2} \, dx \, dt
\]

for any \( \tau \geq \tau^* \) and \( u \in C^\infty_0(Q, \mathbb{C}) \).

### 3.2. Geometric form of the Carleman inequality.

Let \( (\alpha_{kl}(x)) = (a_{kl}(x))^{-1}, \quad x \in \overline{\Omega} \). Consider on \( \overline{\Omega} \) the Riemannian metric \( g \) defined as follows

\[
g_{kl}(x) = |\text{det}(A)|^{1/(n-2)} a_{kl}(x), \quad x \in \overline{\Omega}.
\]

Set then

\[
(g^{kl}(x)) = (g_{kl}(x))^{-1}, \quad |g(x)| = |\text{det}(g_{kl}(x))|, \quad x \in \overline{\Omega}.
\]

As usual define on \( T_x \overline{\Omega} = \mathbb{R}^n \) the inner product

\[
(X|Y)_g(x) = \sum_{k,\ell=1}^n g_{k\ell}(x) X_k Y_\ell, \quad X = \sum_{k=1}^n X_k \partial_k, \quad Y = \sum_{k=1}^n Y_\ell \partial_\ell \in \mathbb{R}^n,
\]

where \( (\partial_1, \ldots, \partial_n) \) is the dual basis of the Euclidean basis of \( \mathbb{R}^n \). Set

\[
|X|_g(x) = (X|X)^{1/2}_g(x), \quad X = \sum_{k=1}^n X_k \partial_k \in \mathbb{R}^n.
\]

For notational convenience we use \( (X|Y)_g \) and \(|X|_g\) instead of \((X|Y)_g(x)\) and \(|X|_g(x)\).

Recall that the gradient of \( u \in H^1(\Omega) \) is the vector field given by

\[
\nabla_g u(x) = \sum_{k,\ell=1}^n g_{k\ell}(x) \partial_k u(x) \partial_\ell, \quad x \in \Omega,
\]

and the divergence of a vector field \( X = \sum_{\ell=1}^n X_\ell \partial_\ell \) with \( X_\ell \in H^1(\Omega), 1 \leq \ell \leq n \), is defined as follows

\[
\text{div}_g(x) = \frac{1}{|g(x)|} \sum_{\ell=1}^n \partial_k (|g(x)| X_\ell(x)), \quad x \in \Omega.
\]

The usual Laplace-Betrami operator associated to the metric \( g \) is given for \( u \in H^2(\Omega) \) by

\[
\Delta_g u(x) = \text{div}_g \nabla_g u(x) = \frac{1}{|g(x)|} \sum_{k,\ell=1}^n \partial_k \left( \sqrt{|g(x)|} g^{k\ell}(x) \partial_\ell u(x) \right), \quad x \in \Omega.
\]

Straightforward computations show that \( \Delta_A u = \sqrt{|g|} \Delta_g u \) for which we deduce the following identity

\[
\Delta_g u(x) = \Delta_{\sqrt{|g|}^{-1} A} u - 2 \left( \nabla_{\sqrt{|g|}^{-1}} |\nabla u| \right)_A - u \Delta_A \sqrt{|g|}^{-1}
\]

whenever \( A \in \mathcal{M}(\Omega, \mathbb{R}, m) \cap W^{2,\infty}(\Omega) \) and \( |A|_{W^{2,\infty}(\Omega, \mathbb{R}^{n \times n})} \leq m \).

We assume in this subsection that \( \phi \) is a weight function for the wave operator \( \Delta_{\sqrt{|g|}^{-1} A} - \partial_t^2 \) with \( \sqrt{|g|}^{-1} A \)-pseudo convexity constant \( \kappa > 0 \).
There exist three constants \( \mathbb{N} = \mathbb{N}(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
\mathbb{N} \int_Q e^{2\tau^*} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |\mathbf{D}_g u|^2 \right] dx \leq \int_Q e^{2\tau^*} |\mathcal{L}_g u|^2 \, dx dt + \int_{\partial Q} e^{2\tau^*} \left[ \tau^3 \lambda^3 \phi^3 |u|^2 + \tau \lambda \phi |\mathbf{D}_g u|^2 \right] d\mu
\]

for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^2(Q, \mathbb{C}) \).

### 3.3. Unique continuation

We start with a unique continuation across a particular convex hypersurface. To this end we set

\[
E_+ (\bar{x}, c, r) = \{ x = (x', x_n) \in B(\bar{x}, r) : 0 \leq x_n - \bar{x}_n < c \text{ and } x_n - \bar{x}_n \geq |x' - \bar{x}'|^2/c \}
\]

with \( \bar{x} = (\bar{x}', \bar{x}_n) \in \Omega, c > 0 \) and \( r > 0 \).

**Theorem 3.5.** There exists \( c^* = c^*(\mathfrak{d}, m) > 0 \) with the property that for any \( 0 < c < c^* \) and \( \bar{x} \in \Omega \) we find \( 0 < r = r(c, \bar{d}) < \bar{d} \) and \( 0 < \rho = \rho(c, \bar{d}) < r \), where \( \bar{d} = \text{dist}(\bar{x}, \Gamma) \), so that if \( u \in H^2(Q; \mathbb{C}) \) satisfies \( \mathcal{L}_g u = 0 \) in \( Q \) and \( \text{supp}(u)(\cdot, t) \cap B(\bar{x}, r) \subset E_+ (\bar{x}, c, r) \) for each \( t \in (t_1, t_2) \) then \( u = 0 \) in \( B(\bar{x}, \rho) \times (t_1, t_2) \).

**Proof.** Let \( \bar{x} \in \Omega \) and \( \epsilon_n = (0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R} \). Set \( \bar{d} = \text{dist}(\bar{x}, \Gamma) \), \( x_0 = \bar{x} - \epsilon \), \( c > 0 \) and define

\[
\psi_0(x) = \psi_0(x', x_n) = |x - x_0|^2/2.
\]

As
\[
|x - x_0| \leq |x - \bar{x}| + |\bar{x} - x_0| \leq r + c,
\]
we find in a straightforward manner that
\[
(\Theta(\psi_0)(x) \xi |\xi| \geq 2x^2 - \mathfrak{n}(r + c), \quad x \in B(\bar{x}, r), \quad 0 < r < \bar{d},
\]
for some constant \( \mathfrak{n} = \mathfrak{n}(m) \).

We fix \( 0 < c < c^* = x^2/(28) \) and assume that \( r < \min(\bar{d}, c) \). With this choice of \( c \) and \( r \), we have
\[
(\Theta(\psi_0)(x) \xi |\xi| \geq x^2 |\xi|^2, \quad x \in B(\bar{x}, r), \quad \xi \in \mathbb{R}^n.
\]

It is straightforward to see that, where \( E_+ = E_+ (\bar{x}, c, r) \),
\[
E_+ \setminus \{ \bar{x} \} \subset \{ x \in B(\bar{x}, r) \setminus \{ \bar{x} \}; \psi_0(x) < \psi_0(\bar{x}) = c^2/2 \}.
\]
Pick $\chi \in C_0^\infty(B(\bar{x}, r))$ satisfying $\chi = 1$ in $B(\bar{x}, \rho_1)$ for some fixed $0 < \rho_1 < r$. Let then $\epsilon > 0$ so that
\[
E_+ \cap [B(\bar{x}, r) \setminus \overline{B}(\bar{x}, \rho_1)] \subset \{x \in B(\bar{x}, r); \psi_0(x) < \psi_0(\bar{x}) - \epsilon\}.
\]
Also, choose $0 < \rho_0 < \rho_1$ in such a way that
\[
E_+ \cap B(\bar{x}, \rho_0) \subset \{x \in B(\bar{x}, r); \psi_0(x) > \psi_0(\bar{x}) - \epsilon/2\}.
\]
Let $u \in H^2(Q; \mathbb{C})$ satisfying $L^w_A u = 0$ in $Q$ and $\text{supp}(\cdot, t) \cap B(\bar{x}, r) \subset E_+$, $t \in (t_1, t_2)$. We are going to apply Corollary 3.1 with $Q$ substituted by $Q = B(\bar{x}, r) \times (t_1, t_2)$. Let
\[
\psi(x, t) = \psi_0(x) + \gamma t^2 / 2,
\]
From Example 2.1 we can easily see that $\phi = e^{\lambda \psi}$ is a weight function for $L^w_A$ in $Q$ whenever
\[
0 < \gamma < \gamma_0 = \min(e/t, \kappa/4), \quad \text{with}, \quad t = \max(|t_1|, |t_2|).
\]
Fix $t_1 < t'_1 < t'_2 < t_2$ arbitrary and set $v = \chi(x)\vartheta(t) u$ with $\vartheta \in C_0^\infty((t_1, t_2))$ satisfying $\vartheta = 1$ in $[t'_1, t'_2]$. Let $\lambda^*$ and $\tau^*$ be as in Corollary 3.1. Fix then $\lambda \geq \lambda^*$ and set
\[
c_0 = e^{\lambda(c^2/2 - \epsilon/2)}, \quad c_1 = e^{\lambda(c^2/2 - \epsilon + \gamma t^2 / 2)}.
\]
Corollary 3.1 yields
\[
\int_{B(\bar{x}, \rho_0) \times (t'_1, t'_2)} |u|^2 \, dx \, dt = \int_{[B(\bar{x}, \rho_0) \cap E_+] \times (t'_1, t'_2)} |v|^2 \, dx \, dt \leq 8\tau^{-3} e^{(c_0 - c_1) t} \int_{E_+ \cap [(B(\bar{x}, r) \setminus \overline{B}(\bar{x}, \rho_1))] \times (t_1, t_2)} |L^w_A v|^2 \, dx \, dt
\]
for $\tau \geq \tau^*$.

Suppose in addition that $\gamma < \min(\gamma_0, \gamma_1)$ with $\gamma_1 = \epsilon/t^2$. This condition then implies $c_0 > c_1$. Passing then to the limit, as $\tau$ tends to $\infty$, in (3.32) in order to obtain that $u = 0$ in $B(\bar{x}, \rho_0) \times (t'_1, t'_2)$. As $t'_1$ and $t'_2$ are arbitrary chosen so that $t_1 < t'_1 < t'_2 < t_2$, we conclude that $u = 0$ in $B(\bar{x}, \rho_0) \times (t_1, t_2)$. $\square$

We now prove a global property of unique continuation. To $u \in H^2(Q; \mathbb{C})$ we associate
\[
\Omega_u = \bigcap_{t \in [t_1, t_2]} \left\{ \Omega \setminus \text{supp}(u(\cdot, t)) \right\}.
\]
Observe that if $\text{Int}(\Omega_u)$, the interior of $\Omega_u$, is nonempty then
\[
u = 0 \quad \text{in} \quad \text{Int}(\Omega_u) \times [t_1, t_2].
\]
Also, if $\Omega'$ is an open subset of $\Omega$ is so that
\[
u = 0 \quad \text{in} \quad \Omega' \times [t_1, t_2]
\]
then $\Omega' \subset \Omega \setminus \text{supp}(u(\cdot, t))$, $t_1 \leq t \leq t_2$, and hence $\Omega' \subset \Omega_u$. In consequence $\text{Int}(\Omega_u)$ is the maximal open subset $\Omega'$ satisfying (3.33).

The following definition is equivalent to the one we give in the introduction.

**Definition 3.1.** We will say that $L^w_A$ has the property of unique continuation if for any $u \in H^2(Q; \mathbb{C})$ satisfying $L^w_A u = 0$ in $Q$ and $\text{Int}(\Omega_u) \neq \emptyset$ then $u = 0$. 
**Theorem 3.6.** There exists a neighborhood $N$ of $I$ in $C^{0,1}(\overline{\Omega}; \mathbb{R}^n \times \mathbb{R}^n)$ so that $L_A^u$ has the property of unique continuation for any $A \in N$.

**Proof.** Set

$$H = \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; \ x_n = 0 \right\}.$$ 

Let $N_0$ be the neighborhood of $I$ in $C^{0,1}(\overline{\Omega}; \mathbb{R}^n \times \mathbb{R}^n)$ given by Lemma 2.2.

Pick $A \in N_0$ and let $u \in H^2(Q; \mathbb{C})$ satisfying $L_A^u = 0$ in $Q$ and $\Omega_0 = \text{Int}(\Omega_u) \neq 0$. Clearly to prove that $u = 0$ it is sufficient to prove that $\Omega \setminus \Omega_0 = \emptyset$. We proceed by contradiction. So we suppose that $\Omega \setminus \Omega_0 \neq \emptyset$. Fix $y \in \partial \Omega_0 \cap \Omega$ and $r > 0$ so that $B(y, r) \subset \Omega$. Pick then $y_0 \in \Omega_0 \cap B(y, r)$ sufficiently close to $y$ in such a way that $\partial B(y_0, d) \cap \partial \Omega_0 \neq \emptyset$, with $d = \text{dist}(y_0, \partial \Omega_0)$. Pick then $z \in \partial B(y_0, d) \cap \partial \Omega_0$.

Making a translation and change of coordinates we may assume that $z = 0$ and $B(0, d) \subset \{(y', y_n) \in \mathbb{R}^n; \ x_n < 0\}$. For convenience we still denote the new matrix obtained after this translation and change of coordinates by $A$. In that case, according to Lemma 2.2, $A$ belongs to the neighborhood $N$ appearing in this lemma. Whence, supp$(u(\cdot, t)) \cap B(z, \rho) \subset H_+$, for some $\rho > 0$, with

$$H_+ = \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; \ x_n \geq 0 \right\}.$$ 

Let $\varphi$ given by (2.4) with $\vartheta = 0$ and $\hat{A} = A_H$. If $v$ is defined in a neighborhood $\hat{\omega}$ by $v(y, \cdot) = u(\varphi^{-1}(y), \cdot)$ then straightforward computations give $L_A^u v = 0$ in $\hat{\omega}$ and supp$(v(\cdot, t)) \subset E_+$ with

$$E_+ = \{(y', y_n) \in \hat{\omega}; \ 0 < y_n < 1, y_n \geq |y'|^2\}.$$ 

As $\phi = e^{\lambda \psi}$ with

$$\psi(y, t) = (y_n - 1)^2 + |y'|^2 + \gamma t^2,$$ 

is a weight function for $L_A^u$ in $\hat{\omega}$ provided that $\gamma$ is sufficiently small, we can mimic the preceding proof in order to get $v = 0$ in $\hat{U} \times (t_1, t_2)$ is some neighborhood $\hat{U}$ of $0$. In consequence $u = 0$ in $\mathcal{U} \times (t_1, t_2)$, $\mathcal{U}$ a neighborhood of $z$ in $\Omega$. By maximality of $\Omega_0$ we deduce $\mathcal{U} \subset \Omega_0$ which leads to the expected contradiction. \qed

The property of unique continuation can serve to establish uniqueness of the Cauchy problem associated to $L_A^u$ as shows the following result.

**Corollary 3.3.** Let $N$ be as in Theorem 3.6 with $\Omega$ substituted by larger domain $\hat{\Omega} \equiv \Omega$. Let $\Gamma_0$ a nonempty open subset of $\Gamma$ and $\Sigma_0 = \Gamma_0 \times (t_1, t_2)$. Let $A \in N$, $u \in H^2(Q; \mathbb{C})$ satisfying $L_A^u = 0$ in $Q$ and u = $\partial_{\nu} u = 0$ on $\Sigma_0$. Then $u = 0$.

**Proof.** Pick $A \in N$, $u \in H^2(Q; \mathbb{C})$ satisfying $L_A^u = 0$ in $Q$ and $u = \partial_{\nu} u = 0$ on $\Sigma_0$. Then there exists $\mathcal{V} \subset \hat{\Omega}$, a neighborhood of a point in $\Gamma_0$, so that $\hat{u}$ the extension by of $u$ by zero $\mathbb{R}^n \setminus \mathcal{V}$ belongs to $H^2(\Omega' \times (t_1, t_2); \mathbb{C})$, with $\Omega' = \Omega \setminus \mathcal{V}$, satisfies $L_A^{\hat{u}} = 0$ in $\Omega' \times (t_1, t_2)$ and Int$(\Omega_0') \neq \emptyset$. Theorem 3.6 allows us to conclude that $\hat{u} = 0$ and hence $u = 0$. \qed

We end this subsection by remarking that we can proceed similarly to the preceding theorem to prove the property of unique continuation for $A$-pseudo-convex hypersurface.

**Theorem 3.7.** Let $H = \{x \in \omega; \ \theta(x) = \theta(\hat{x})\}$ be a $A$-pseudo-convex hypersurface defined in a neighborhood $\omega$ of $\hat{x} \in \Omega$ with $\theta \in C^2(\mathbb{R})$. Then there exists $\mathcal{B}$ a neighborhood of $\hat{x}$ so that if $u \in H^2(\omega \times (t_1, t_2))$ satisfies $L_A^{\hat{u}} = 0$ in $\omega$ and supp$(u(\cdot, t)) \subset H_+ = \{x \in \omega; \ \theta(x) \geq \theta(\hat{x})\}$ for any $t \in (t_1, t_2)$ then $u = 0$ in $\mathcal{B} \times (t_1, t_2)$. 
3.4. **Observability inequality.** We shall need in sequel the following technical lemma.

**Lemma 3.1.** Fix $0 < \alpha < 1$ and let $0 \leq \psi_0 \in C^1(\Omega)$ satisfying
\[
\min_{x \in \Omega} |\nabla \psi_0|^2 := \delta_0 > 0.
\]

Let $m = \|\psi_0\|_{L^\infty(\Omega)}$ and define for an arbitrary constant $C > 0$
\[
\psi(x, t) = \psi_0(x) - T^{-2+\alpha}(t - T/2)^2 + C, \quad x \in \Omega, \ t \in [0, T].
\]

If $T > T_0 = \max \left(\delta_0^{-1/2(1-\alpha)}, (64m/2)^{1/\alpha}\right)$ then
\[
\min_{\Omega} \left(\left|\nabla \psi\right|^2 - (\partial_t \psi)^2\right) := \delta > 0,
\]
\[
\psi(x, t) \geq -T^\alpha/64 + C, \quad (x, t) \in \Omega \times [3T/8, 5T/8],
\]
\[
\psi(x, t) \leq -2T^\alpha/64 + C, \quad (x, t) \in \Omega \times ([0, T/4] \cup [3T/4, T]).
\]

**Proof.** If $T \geq T_0$ then
\[
|\nabla \psi|^2 - (\partial_t \psi)^2 \geq \delta_0 - T^{-4+2\alpha}T^2 = \delta_0 - T^{-2(1-\alpha)} := \delta > 0.
\]
That is we proved (3.35).

Inequality (3.36) is straightforward. On the other hand we have
\[
\psi(x, t) \leq m - T^\alpha/16 + C < 2T^\alpha/64 - T^\alpha/16 + C = -2T^\alpha/64 + C
\]
if $(x, t) \in \Omega \times ([0, T/4] \cup [3T/4, T])$. That is we proved (3.37). \qed

In this subsection
\[
\mathcal{L}_A^w = \Delta_A - \partial_t^2 + p\partial_t + q,
\]
with $p, q \in L^\infty(\Omega; C)$, and consider the IBVP associated to $\mathcal{L}_A^w$:
\[
\left\{ \begin{array}{l}
\mathcal{L}_A^w = 0 \quad \text{in } Q = \Omega \times (0, T), \\
(u(\cdot, 0), \partial_t u(\cdot, 0)) = (u_0, u_1), \\
|u| = 0.
\end{array} \right.
\]

Here $\Sigma = \Gamma \times (0, T)$.

We recall that according the semigroup theory for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ the IBVP (3.38) admits unique solution
\[
u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)).
\]

We also know that $\partial_t \nu \in L^2(\Sigma)$ (hidden regularity).

From usual energy estimates for wave equations if
\[
\mathcal{E}_u(t) = \|D_A u\|_{L^2(\Omega; \mathcal{E}^{n+1})}
\]
then
\[
\mathcal{E}_u(t) \leq \mathcal{E}_u(0), \quad 0 \leq t \leq T.
\]

Here $\mathcal{N}_0 > 0$ is a constant only depending on $\Omega, A, T, p$ and $q$.

We apply (3.40) to $v(\cdot, t) = u(\cdot, s - t)$, with fixed $0 < t \leq s$. We find
\[
\mathcal{E}_u(s - t) = \mathcal{E}_u(t) \leq \mathcal{N}_1 \mathcal{E}_u(0) = \mathcal{N}_1 \mathcal{E}_u(s), \quad 0 \leq t \leq s,
\]
with $\mathcal{N}_1 > 0$ is a constant only depending on $\Omega, A, T, p$ and $q$. In particular
\[
\mathcal{E}_u(0) \leq \mathcal{N}_1 \mathcal{E}_u(s), \quad 0 \leq t \leq s.
\]
In light of (3.40) and (3.41) we have
\[(3.42) \quad \Re^{-1} \mathcal{E}(0) \leq \mathcal{E}(t) \leq \Re \mathcal{E}(0), \quad 0 \leq t \leq T,\]
for some \( \Re > 1 \) only depending on \( \Omega, A, T, p \) and \( q \).

**Theorem 3.8.** Fix \( 0 < \alpha < 1 \) and assume that \( 0 \leq \psi_0 \in C^4(\overline{\Omega}) \) is \( A \)-pseudo-convex with constant \( \kappa > 0 \) and let \( \Gamma_+ = \{ x \in \Gamma; \partial_{\nu_A} \psi_0(x) > 0 \} \). Then there exist two constants \( \Re \) and \( T_\alpha \), only depending \( \Omega, T, \kappa, \Gamma_+ \) and \( \alpha \), so that for any \( T \geq T_\alpha \) and \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega) \) we have
\[
\| (u_0, u_1) \|_{H^1_0(\Omega) \times L^2(\Omega)} \leq \Re \| \partial_v u \|_{L^2(\Sigma_+)}
\]
where \( \Sigma_+ = \Gamma_+ \times (0, T) \) and \( u \) is the solution of the IBVP (3.38) corresponding to \((u_0, u_1)\).

**Proof.** Let \( T_\alpha \) be as in Lemma 3.1 and set \( \tilde{T}_\alpha = \min \{ T_\alpha, (8\kappa/\kappa)^{1/(2-\alpha)} \} \). Fix \( T > \tilde{T}_\alpha \) and let \( \psi \) defined as in (3.34) in which the constant \( C > 0 \) is chosen sufficiently large to guarantee that \( \psi \geq 0 \). In that case we easily check that \( \phi = e^{\lambda \psi} \) is a weight function for the operator \( \mathcal{L}_A^w \) in \( Q \).

Pick \( \rho \in C^\infty_0(T/8, 7T/8) \) so that \( \rho = 1 \) in \([T/4, 3T/4]\).

Clearly a density argument shows that Corollary 3.2 remains valid for \( q \rho \) for any solution \( u \) of (3.38) with \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega) \). According to this Corollary we have for fixed \( \lambda \geq \lambda^* \)
\[(3.43) \quad \Re \int_Q e^{2\tau \phi} |D_A(qu)|^2 dxdt \leq \int_Q e^{2\tau \phi} |\mathcal{L}_A^w(qu)|^2 dxdt + \int_{\Sigma_+} e^{2\tau \phi} |\partial_v (q u)|^2 d\sigma dt \]
for any \( \tau \geq \tau^* \).

But
\[
\mathcal{L}_A^w(qu) = 2q'u + q''u.
\]
Hence (3.43) together with Poincaré’s inequality \((u(\cdot, t) \in H^1_0(\Omega))\) give
\[(3.44) \quad \Re \int_Q e^{2\tau \phi} |D_A(qu)|^2 dxdt \leq \int_{Q \cap \text{supp}(\rho)} e^{2\tau \phi} |D_A u|^2 dxdt + \int_{\Sigma_+} e^{2\tau \phi} |\partial_v u|^2 d\sigma dt.
\]

Define
\[
c_0 = e^{\lambda(-\gamma T^2/64+C)} \quad \text{and} \quad c_1 = e^{\lambda(-2\gamma T^2/64+C)}.
\]
If \( \mathcal{E}_u \) is given by (3.39) then we get from (3.36), (3.37) and (3.44)
\[
\Re e^{\tau c_0} \int_{3T/8}^{5T/8} \mathcal{E}_u(t) dt \leq e^{\tau c_1} \int_0^T \mathcal{E}_u(t) dt + \int_{\Sigma_+} e^{2\tau \phi} |\partial_v u|^2 d\sigma dt
\]
for any \( \tau \geq \tau^* \).

This inequality together with (3.42) imply
\[(3.45) \quad (\Re e^{\tau c_0} - e^{\tau c_1}) \mathcal{E}_u(0) \leq \int_{\Sigma_+} e^{2\tau \phi} |\partial_v u|^2 d\sigma dt, \quad \tau \geq \tau^*.
\]
As \( c_0 > c_1 \), we can choose \( \tau \) sufficiently large in such a way that \( \Re e^{\tau c_0} - e^{\tau c_1} > 0 \). The expected inequality follows then from (3.45). \( \Box \)
From the calculations in Example 2.1 when \( \psi_0(x) = |x - x_0|^2/2 \), with \( x_0 \in \mathbb{R}^n \setminus \overline{\Omega} \), there exists a neighborhood \( \mathcal{N} \) of \( I \) in \( C^{0,1}(\overline{\Omega}; \mathbb{R}^{n \times n}) \) so that that for any \( A \in \mathcal{N} \), \( \kappa = 1/2 \) and \( \psi_0 \) is \( A \)-pseudo-convex with constant \( \kappa = 1/4 \). In this case

\[
\tilde{T}_\alpha = \max \left( d_0^{-1/(1-\alpha)}, [16(d + d_0)]^{1/\alpha}, 16^{1/(2-\alpha)} \right)
\]

with \( d_0 = \text{dist}(x_0, \Omega) \) and \( d = \text{diam}(\Omega) \).

A result in the variable coefficients case was already established in [31, Theorem 1.1]. This result is based on a generalization of the multiplier method in which a vector field is used as an alternative to the multiplier. This vector field satisfies a certain convexity condition. Note however that the lower bound in \( \tilde{T} \) appearing [31, Theorem 1.1] is not easily comparable to that we used in Theorem 3.8.

4. Elliptic equations

We show briefly how we can modify the calculations we carried out for wave equations in order to retrieve Carleman inequalities for elliptic equations and the corresponding property of unique continuation.

In this section \( \mathcal{L}_A = \Delta_A + \sum_{\ell=1}^n p_\ell \partial_\ell + q \) where \( p_1, \ldots, p_n \) and \( q \) belong to \( L^\infty(\Omega; \mathbb{C}) \) and satisfy

\[
\|p_\ell\|_{L^\infty(\Omega)} \leq m, 1 \leq \ell \leq n \quad \text{and} \quad \|q\|_{L^\infty(\Omega)} \leq m.
\]

Also \( 0 \leq \psi \in C^2(\overline{\Omega}) \) is fixed so that

\[
|\nabla \psi| \geq \delta \quad \text{in} \quad \overline{\Omega}
\]

for some constant \( \delta > 0 \).

4.1. Carleman inequality. Let \( \phi = e^{\lambda \psi} \) and set \( \mathfrak{d} = (\Omega, \kappa, \delta, m) \).

**Theorem 4.1.** We find three constants \( \mathfrak{h} = \mathfrak{h}(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
\mathfrak{h} \int_{\Omega} e^{2\tau \phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda^2 \phi |\nabla u|^2 \right] \, dx
\]

\[
\leq \int_{\Omega} e^{\tau \phi} |\mathcal{L}_A u|^2 \, dx + \int_{\Gamma} e^{\tau \phi} \left[ \tau^3 \lambda^3 \phi^3 |u|^2 + \tau \lambda \phi |\nabla u|^2 \right] \, d\sigma
\]

for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^2(\Omega, \mathbb{C}) \).

**Proof.** In this proof \( \lambda_k \) and \( \mu_k \), \( k = 1, 2, \ldots \), are generic constants only depending on \( \mathfrak{d} \).

If \( \Phi = e^{-\tau \phi} \), \( \tau > 0 \), we have from the calculations of the preceding section

\[
L = \Phi^{-1} \Delta_A (\Phi w) = \Delta_A w - 2\tau (\nabla w |\nabla \phi)_A + \left[ \tau \phi^3 |\nabla \phi|^2 - \tau \Delta_A w \right] w.
\]

We decompose \( L \) in the following special form

\[
L = L_0 + L_1 + c
\]

with for \( w \in H^2(\Omega, \mathbb{R}) \)

\[
L_0 w = \Delta_A w + aw,
\]

\[
L_1 w = (B |\nabla w) + bw.
\]
The coefficients of \( L_0 \) and \( L_1 \) and \( c \) are given as follows

\[
\begin{align*}
a &= \tau^2 |\nabla \phi|^2_A, \\
b &= -2\tau \Delta A \phi, \\
c &= \tau \Delta A \phi, \\
B &= -2\tau A \nabla \phi.
\end{align*}
\]

We write then

\[
\langle L_0|L_1 \rangle_{L^2(\Omega)} = \sum_{k=1}^{4} I_k.
\]

Here

\[
\begin{align*}
I_1 &= \int_{\Omega} \Delta_A w(\nabla w|B)dx, \\
I_2 &= \int_{\Omega} \Delta_A bw^2dx, \\
I_3 &= \int_{\Omega} aw(\nabla w|B)dx, \\
I_4 &= \int_{\Omega} abw^2 dx.
\end{align*}
\]

Let

\[
D = C/2 - A(B')^t.
\]

Straightforward modifications of the computations of the preceding section yield

\[
\begin{align*}
I_1 &= \int_{\Omega} (D\nabla w|\nabla w)dxdt + \int_{\Gamma} \left[(\nabla w|\nu)_A(\nabla w \cdot B) - (B/2|\nu)|\nabla w|^2_A\right]d\sigma, \\
I_2 &= -\int_{\Omega} b|\nabla w|^2_A dxdt + \int_{\Omega} \Delta_A (b/2)w^2dx \\
&\quad - \int_{\Gamma} (\nabla (b/2)|\nu)_A w^2 d\sigma + \int_{\Gamma} (\nabla w|\nu)_A bw d\sigma, \\
I_3 &= -\int_{\Omega} \text{div}(aB/2)w^2dx + \int_{\Gamma} a(B/2|\nu)w^2 d\sigma.
\end{align*}
\]

Identities (4.3) to (4.5) in (4.2) give

\[
\langle L_0|L_1 \rangle_{L^2(\Omega)} = \int_{\Omega} (\mathfrak{A}\nabla w|\nabla w)dx + \int_{\Omega} aw^2 dx + \int_{\Gamma} g(w) d\sigma
\]

with

\[
\mathfrak{A} = D - bA, \\
a = ab + \Delta_A (b/2) - \text{div}(aB/2), \\
g(w) = (\nabla w|\nu)_A(\nabla w \cdot B) - (B/2|\nu)|\nabla w|^2_A \\
&\quad - (\nabla (b/2)|\nu)_A w^2 + (\nabla w|\nu)_A bw + a(B/2|\nu)w^2.
\]

We have

\[
(\mathfrak{A}\xi|\xi) = \tau \lambda^2 \left[|\nabla \psi|^2(A\xi|\xi) + (\nabla w|A\xi)^2\right] + \tau \lambda (\mathfrak{A}\xi|\xi)
\]
where \( \hat{A} \) is a matrix depending only on \( A \) and \( \psi \). Therefore
\[
(\mathcal{A} \xi | \xi) \geq \tau \lambda^2 \delta^2 \xi | \xi|^2 / 2, \quad \lambda \geq \lambda_1.
\]
(4.7)

We have also
\[
a = \tau^3 \lambda^4 \phi^3 |\nabla \psi|^4_A + \hat{a}
\]
where the reminder term \( \hat{a} \) contains as for the wave equation only terms with factors \( \tau^k \lambda^\ell \phi^m \), \( 1 \leq k, \ell, m \leq 3 \) and terms with factor \( \tau \lambda^4 \phi \).

Hence
\[
a \geq \tau^3 \lambda^4 \delta^4 \phi^3 / 2, \quad \lambda \geq \lambda_2, \quad \tau \geq \tau_2.
\]
(4.8)

The rest of the proof is almost similar to that of the wave equation. \( \square \)

**Remark 4.1.** The symbol of the principal part of the operator \( \mathcal{L}_A^c \) is given by
\[
p(x, \xi) = |\xi|^2 (x) = (A(x) |\xi|), \quad x \in \overline{\Omega}, \ \xi \in \mathbb{R}^n.
\]
Therefore if \( \phi \in C^2(\overline{\Omega}) \) then we have \( p(x, \xi + i \tau \nabla \phi) = p_0 + ip_1 \) with
\[
p_0 = |\xi|^2 - \tau^2 |\nabla \phi|_A, \quad p_1 = 2 \tau (\xi |\nabla \phi|_A).
\]

When \( \phi = e^{\lambda \psi} \) we find for \( \tau \geq 1 \)
\[
\{p_0, p_1\} := \sum_{j=1}^n \left[ \partial_{x_j} p_0 \partial_{x_j} p_1 - \partial_{x_j} p_0 \partial_{x_j} p_1 \right] = \tau^2 \left[ 2 \lambda^3 \phi^2 |\nabla \psi|^4_A + O(\lambda^2) \right]
\]

In consequence \( \phi \) satisfies the sub-ellipticity condition in [Theorem 8.3.1, page 190] if \( \lambda \) is sufficiently large and hence the following Carleman inequality holds: there exist \( N > 0 \) and \( \tau^* > 0 \) only depending on \( \Omega \) and bounds on coefficients of \( \mathcal{L}_A^c \) so that
\[
\sum_{|\alpha| \leq 1} \tau^{2(2-|\alpha|)} \int_{\Omega} e^{2 \tau \phi} |\partial^\alpha u|^2 dx \leq N \tau \int_{\Omega} e^{2 \tau \phi} |\mathcal{L}_A^c u|^2 dx, \quad u \in C^\infty_0(\Omega), \ \tau \geq \tau^*.
\]

In other words if \( \phi = e^{\lambda \psi} \) is a weight function for the elliptic operator \( \mathcal{L}_{A,0}^c \) then \( \phi \) possesses the sub-ellipticity condition for large \( \lambda \).

**4.2. Unique continuation.** We use similar method as for the wave equation. For sake of completeness we provide some details.

We start with a unique continuation result across a convex hypersurface. To this end set
\[
\psi(x', x_n) = (x_n - 1)^2 + |x'|^2.
\]
As \( |\nabla \psi(0, 0)| = 2 \), there exists \( r > 0 \) so that \( |\nabla \psi| \geq 1 \) in \( B(0, r) \). Consider then the set
\[
E_+ = \{(x', x_n) \in B(0, r); \ 0 \leq x_n < 1 \text{ and } x_n \geq |x'|^2 \}.
\]

We have clearly
\[
E_+ \setminus \{(0, 0)\} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; \ \psi(x', x_n) < \psi(0, 0) = 1\}.
\]

Pick \( \chi \in C^\infty_0(B(0, r)) \) satisfying \( \chi = 1 \) in \( B(0, \rho_1) \), where \( 0 < \rho_1 < r \) is fixed arbitrary. Let then \( \epsilon > 0 \) so that
\[
E_+ \cap [B(0, r) \setminus \overline{B}(0, \rho_1)] \subset \{(x', x_n) \in B(0, r); \ \psi(x', x_n) < \psi(0, 0) - \epsilon\}.
\]

**Lemma 4.1.** There exists \( 0 < \rho_0 < \rho_1 \) so that if \( u \in H^2(B(0, r); \mathbb{C}) \) satisfies \( \mathcal{L}_A^c u = 0 \) in \( B(0, r) \) and \( \text{supp}(u) \subset E_+ \) then \( u = 0 \) in \( B(0, \rho_0) \).
Proof. Let us choose $0 < \rho_0 < \rho_1$ in such a way that

$$E_+ \cap B(0, \rho_0) \subset \{ (x',x_n) \in B(0,r); \ \psi(x',x_n) > \psi(0,0) - \epsilon/2 \}. $$

Pick $u \in H^2(B(0,r);\mathbb{C})$ satisfying $\mathcal{L}_A^* u = 0$ in $B(0,r)$ and supp$(u) \subset E_+$. Let $v = \chi u$.

Let $\lambda^*$ and $\tau^*$ be as in Theorem 4.1. Fix then $\lambda \geq \lambda^*$ and set

$$c_0 = e^{\lambda(1-\epsilon/2)}, \quad c_1 = e^{\lambda(1-\epsilon)}. $$

Theorem 4.1 yields

$$\int_{B(0,\rho_0)} u^2 dxdt = \int_{B(0,\rho_0) \cap E_+} v^2 dx \
\leq 8\tau^{-3}e^{-\tau(c_0-c_1)} \int_{E_+ \cap (B(0,r) \setminus B(0,\rho_1))} (\mathcal{L}_A^* v)^2 dx, \quad \tau \geq \tau^*.$$ 

Noting that $c_0 > c_1$, we obtain that $u = 0$ in $B(0,\rho_0)$ by taking the limit as $\tau$ tends to $\infty$. \hfill \Box

Let $\vartheta = \vartheta(x')$ be in $C^2(\overline{B}(0,r))$ satisfying $\vartheta(0) = 0$. Let $\omega \subset B(0,r) \times \mathbb{R}$ be a neighborhood of $0$ in $\mathbb{R}^n$.

Consider

$$\varphi: (x',x_n) \in \omega \mapsto (y',y_n) = (x',x_n - \vartheta(x') + |x'|^2).$$

As det$(\varphi'(x,x_n)) = 1$ for any $(x',x_n) \in \omega$ we deduce that $\varphi$ is a diffeomorphism from $\omega$ onto $\tilde{\omega} = \varphi(\omega)$.

Pick $u \in H^2(\omega,\mathbb{C})$ satisfying $\mathcal{L}_A^* u = 0$ in $\omega$ and supp$(u) \subset \omega_+ = \{ (x',x_n) \in \omega; \ x_n \geq \vartheta(x') \}$.

Define $v$ by $v(y',y_n) = u(\varphi^{-1}(y',y_n)), \ (y',y_n) \in \tilde{\omega}$. Then it is straightforward to check that $\mathcal{L}_A^* v = 0$ in $\tilde{\omega}$. Here $\mathcal{L}_A^*$ is of the same form as $\mathcal{L}_A^*$ of the origin in $\omega$.

Given by

$$\mathcal{L}_A^* \varphi' \Delta \tilde{\omega}$$

with

$$\tilde{\Delta}(y) = \varphi' (\varphi^{-1}(y)) A (\varphi^{-1}(y)) (\varphi^{-1})^t (\varphi^{-1}(y)).$$

Moreover supp$(v) \subset \tilde{\omega}_+ = \{ (y',y_n) \in \tilde{\omega}; \ y_n \geq |y'|^2 \}$.

Similar calculations as in Subsection 2.4 show, by reducing $\omega$ if necessary, that

$$\tilde{\Delta}(y) \xi = \varphi' (\varphi^{-1}(y)) A (\varphi^{-1}(y)) (\varphi^{-1})^t (\varphi^{-1}(y)) \xi \geq \epsilon \xi^2/4, \quad \xi \in \mathbb{R}^n.$$ 

We apply Lemma 4.1 in order to get $u = 0$ in $\tilde{\omega}$, $\tilde{\omega}$ is a neighborhood of the origin, and hence $u = 0$ in $\mathcal{V}$ with $\mathcal{V} = \varphi^{-1}(\tilde{\omega})$.

In other words we proved the following result.

Lemma 4.2. There exists a neighborhood $\mathcal{V}$ of the origin in $\omega$ so that if $u \in H^2(\omega;\mathbb{C})$ satisfies $\mathcal{L}_A^* u = 0$ in $\omega$ and supp$(u) \subset \omega_+$ then $u = 0$ in $\mathcal{V}$.

The global uniqueness of continuation result is based on the following lemma.

Lemma 4.3. Let $\Omega_0 \subset \Omega$ so that $\partial \Omega_0 \cap \Omega \neq \emptyset$. There exists $z \in \partial\Omega_0 \cap \Omega$ and $\mathcal{V}$ a neighborhood of $z$ in $\Omega$ so that if $u \in H^2(\Omega;\mathbb{C})$ satisfies $\mathcal{L}_A^* u = 0$ in $\Omega$ together with $u = 0$ in $\Omega_0$ then $u = 0$ in $\mathcal{W}$. 


Proof. Fix $y \in \partial\Omega_0 \cap \Omega$ and $r > 0$ so that $B(y, r) \subset \Omega$. Pick then $y_0 \in \Omega_0 \cap B(y, r)$ sufficiently close to $y$ in such a way that $\partial B(y_0, d) \cap \partial\Omega_0 \neq \emptyset$, with $d = \text{dist}(y_0, \partial\Omega_0)$. Pick then $z \in \partial B(y_0, d) \cap \partial\Omega_0$. Making a translation we may assume that $z = 0$. As the $\partial B(y_0, d)$ can be represented locally by a graph $x_n = \vartheta(x')$. Making a change of coordinates we may assume that $\text{supp}(u) \subset \{(x', x_n); x_n \geq \vartheta(x')\}$. This orthogonal transformation modify $A$ but the new matrix has the same properties as $A$. We then complete the proof by using Lemma 4.2 with $A$ substituted by this new matrix. \hfill\Box

**Theorem 4.2.** Let $u \in H^2(\Omega; \mathbb{C})$ satisfying $\mathcal{L}_A^\nu u = 0$ in $\Omega$ and $u = 0$ in $\omega$ for some nonempty open subset $\omega$ of $\Omega$. Then $u = 0$ in $\Omega$.

**Proof.** Let $\Omega_0$ be the maximal domain in which $u = 0$. If $\Omega \setminus \overline{\Omega}_0 \neq \emptyset$ then we would have $\partial\Omega_0 \cap \Omega \neq \emptyset$. Therefore we would find by Lemma 4.3 $z \in \partial\Omega_0 \cap \partial\Omega$ and $\mathcal{W}$ a neighborhood of $z$ in $\Omega$ so $u = 0$ in $\mathcal{W}$. That is $u = 0$ in $\Omega_0 \cup \mathcal{W}$ which contains strictly $\Omega_0$ and hence contradicts the maximality of $\Omega_0$. \hfill\Box

As an immediate consequence of Theorem 4.2 we get uniqueness result of elliptic Cauchy problems.

**Corollary 4.1.** Assume that $\mathcal{L}_A^\nu$ is defined in $\hat{\Omega} \ni \Omega$. Let $\Gamma_0$ be an arbitrary nonempty open subset of $\Gamma$. If $u \in H^2(\Omega; \mathbb{C})$ satisfies $\mathcal{L}_A^\nu u = 0$ in $\Omega$ together with $u = \partial_\nu u = 0$ in $\Gamma_0$ then $u = 0$ in $\Omega$.

**Proof.** If $u \in H^2(\Omega; \mathbb{C})$ satisfies $u = \partial_\nu u = 0$ in $\Gamma_0$ then we find $\mathcal{V} \subset \hat{\Omega}$ a neighborhood of a point in $\Gamma_0$ so that $\tilde{u}$ the extension by of $u$ by zero in $\mathbb{R}^n \setminus \overline{\mathcal{V}}$ belongs to $H^2(\Omega \cap \mathcal{V})$, satisfies $\mathcal{L}_A^\nu \tilde{u}$ in $\Omega \cap \mathcal{V}$ and $\tilde{u} = 0$ in $\mathcal{V}$. Theorem 4.2 allows us to conclude that $\tilde{u} = 0$ and hence $u = 0$. \hfill\Box

### 5. Parabolic Equations

In this section $0 \leq \psi \in C^2(\overline{\Omega})$ is fixed so that $\psi(x, t) = \psi_0(x) + \psi_1(t)$ with

$$|\nabla \psi_0| \geq \delta \quad \text{in} \quad \overline{\Omega}$$

for some constant $\delta > 0$.

Let $\phi = e^{\lambda \psi}$, $\lambda > 0$, and consider the parabolic operator

$$\mathcal{L}_A^p = \Delta_A - \partial_t + \sum_{\ell=1}^n p_\ell \partial_\ell + q$$

where $A \in \mathcal{M}(\Omega, \mathbb{R}, m)$, $p_1, \ldots, p_n$ and $q$ belongs to $L^\infty(\overline{\Omega}; \mathbb{C})$ and satisfy

$$\|p_\ell\|_{L^\infty(\overline{\Omega})} \leq m, \quad 1 \leq \ell \leq n \quad \text{and} \quad \|q\|_{L^\infty(\overline{\Omega})} \leq m.$$

#### 5.1. Carleman Inequality

Recall that $\mathcal{L}_A^{p,0}$ represents the principal part of $\mathcal{L}_A^p$:

$$\mathcal{L}_A^{p,0} = \Delta_A - \partial_t.$$

Henceforward $\mathfrak{d} = (\Omega, t_1, t_2, \mathbb{R}, \delta, m)$. 
Theorem 5.1. There exist three constants $\aleph = \aleph^*(\mathfrak{d})$, $\lambda^* = \lambda^*(\mathfrak{d})$ and $\tau^* = \tau^*(\mathfrak{d})$ so that

$$
\aleph \int_Q e^{2\tau \phi} \left[ \tau^3 \lambda^3 \phi^3 |u|^2 + \tau \lambda^2 \phi |\nabla u|^2 \right] dxdt
\leq \int_Q e^{2\tau \phi} |L^0_A u|^2 dxdt + \int_{\partial Q} e^{2\tau \phi} \left[ \tau^3 \lambda^3 \phi^3 |u|^2 + \tau \lambda \phi |\nabla u|^2 \right] d\mu
+ \int_{\Sigma} e^{2\tau \phi} (\tau \lambda \phi)^{-1} |\partial_t u|^2 d\sigma dt
$$

(5.1)

for any $\lambda \geq \lambda^*$, $\tau \geq \tau^*$ and $u \in H^{2,1}(Q, \mathbb{C})$.

Proof. As for the wave equation for $\tau > 0$ we set $\Phi = e^{-\tau \phi}$ and we recall that

$$
\partial_i \Phi = -\tau \partial_i \phi \Phi = -\tau \partial_i \phi \Phi,
\partial_{ij} \Phi = \left( -\tau \partial_{ij}^2 \phi + \tau^2 \partial_i \phi \partial_j \phi \right) \Phi,
\partial_t \Phi = -\tau \partial_t \phi \Phi = -\tau \partial_t \phi \Phi.
$$

We have for $w \in H^{2,1}(Q, \mathbb{R})$

$$
\Phi^{-1} \Delta_A (\Phi w) = \Delta_A w - 2\tau (\nabla w | \nabla \phi)_A + \left[ \tau^2 |\nabla \phi|^2_A - \tau \Delta_A w \right] w.
$$

Also

$$
\Phi^{-1} \partial_t (\Phi w) = \partial_t w - \tau \partial_t \phi w.
$$

We decompose $L = \Phi^{-1} L^p_{A,0} \Phi$ as in the elliptic case. That is we write

$$
L = L_0 + L_1 + c
$$

with

$$
L_0 w = \Delta_A w + aw,
L_1 w = (B|\nabla w) - \partial_t w + bw
$$

where

$$
a(x, t) = \tau^2 |\nabla \phi|^2_A,
b(x, t) = -2\tau \Delta_A \phi,
c(x, t) = \tau \Delta_A \phi + \tau \partial_t \phi,
B = -2\tau A|\nabla \phi|.
$$

We have

$$
(L_0 w|L_1 w)_{L^2(Q)} = \sum_{i=1}^6 I_i.
$$
Here $I_k$, $1 \leq k \leq 6$, are given as follows

\[
I_1 = \int_Q \Delta_A w(\nabla w|B)dxdt,
\]
\[
I_2 = -\int_Q \Delta_A w\partial_twdxdt,
\]
\[
I_3 = \int_Q \Delta_A bwdxdt,
\]
\[
I_4 = \int_Q aw(\nabla w|B)dxdt,
\]
\[
I_5 = -\int_Q aw\partial_twdxdt,
\]
\[
I_6 = \int_Q abw^2dxdt.
\]

Let $C = (\text{div}(a_{ij}B))$ and $D = C/2 - A(B')^t$.

We already proved that

\begin{equation}
(5.3) \quad I_1 = \int_Q (D\nabla w|\nabla w)dxdt + \int_\Sigma [(\nabla w|\nu)_A(\nabla w \cdot B) - (B/2|\nu)|\nabla w|_A^2] \, d\sigma dt.
\end{equation}

On the other hand inequality (3.6) with $d = -1$ gives

\begin{equation}
(5.4) \quad I_2 = \int_\Sigma (\nabla w|\nu)_A \partial_twdxdt + \int_\Omega \left[\frac{|\nabla w|_A^2}{2}\right]_{t=t_1}^{t_2} \, dx.
\end{equation}

$I_3$ is the same as in (3.7):

\begin{equation}
(5.5) \quad I_3 = -\int_Q b|\nabla w|^2dxdt + \int_Q \Delta_A (b/2)w^2dxdt
\end{equation}

\[
- \int_\Sigma (\nabla (b/2)|\nu)_A w^2d\sigma dt + \int_\Sigma (\nabla w|\nu)_A bwd\sigma dt.
\]

Let $J_1 = I_1 + I_2 + I_3$ and

\[
\mathfrak{A} = D - bA,
\]
\[
a_1 = \Delta_A (b/2),
\]
\[
g_1(w) = (\nabla w|\nu)_A(\nabla w \cdot B) - (B/2|\nu)|\nabla w|_A^2 - (\nabla w|\nu)_A \partial_t w
\]
\[
- (\nabla (b/2)|\nu)_A w^2d\sigma + (\nabla w|\nu)_A bw,
\]
\[
h_1(w) = \left[\frac{|\nabla w|_A^2}{2}\right]_{t=t_1}^{t_2}.
\]

Putting together (5.3) to (5.5) we find

\begin{equation}
(5.6) \quad J_1 = \int_Q (\mathfrak{A}\nabla w|\nabla w)dxdt + \int_\Omega a_1w^2dxdt.
\end{equation}

\[
+ \int_\Sigma g_1(w)d\sigma dt + \int_\Omega h_1(w)dx.
\]
We next recall that \( I_4 \) was calculated in (3.14). Precisely we have
\[
I_4 = - \int_Q \text{div}(aB/2)w^2 dx dt + \int_{\Sigma} a(B/2|\nu|)w^2 d\sigma dt.
\]

On the hand an integration by parts in \( t \) gives
\[
I_5 = - \int_Q (a/2)\partial_t(w^2) dx dt = \int_Q \partial_t(a/2)w^2 dx dt + \int_{\Omega} [(a/2)w^2]_{t=t_1}^{t_2} dx.
\]

Set
\[
a_2 = -\text{div}(aB/2) + \partial_t(a/2) + ab,
\]
\[
g_2(w) = a(B/2|\nu|)w^2,
\]
\[
h_2(w) = [(a/2)w^2]_{t=t_1}^{t_2}
\]
and let \( J_2 = I_4 + I_5 + I_6 \). In light of (5.7) and (5.8) we get
\[
J_2 = \int_Q a_2 w^2 dx dt + \int_{\Omega} g_2(w) d\sigma dt + \int_{\Omega} h_2(w) dx.
\]

Putting together (5.2), (5.6) and (5.9) in order to obtain
\[
\langle L_0 w|L_1 w \rangle_{L^2(Q)} = 2\tau \int_Q (A \nabla w|\nabla w) dx dt + \int_{\Omega} a w^2 dx dt + \int_{\Omega} g(w) d\sigma dt + \int_{\Omega} h(w) dx.
\]

where \( a = a_1 + a_2 \), \( g = g_1 + g_2 \) and \( h = h_1 + h_2 \).

As \( \partial_t a = 0 \) (which is a consequence of \( \partial_t \nabla \psi = 0 \)), we see that \( A \) and \( a \) has exactly the same form as in the elliptic case. Therefore we can mimic the proof of the elliptic case to complete the proof.

We already defined \( \Gamma_+ = \Gamma^{\psi_0}_+ = \{ x \in \Gamma; \partial_{\nu_0} \psi_0 > 0 \} \) and
\[
\Sigma_+ = \Sigma^{\psi_0}_+ = \Gamma_+ \times (t_1, t_2).
\]

Similarly to the wave equation we have the following result.

**Theorem 5.2.** There exist three constants \( \mathcal{N} = \mathcal{N}(\mathfrak{A}) \), \( \lambda^* = \lambda^*(\mathfrak{A}) \) and \( \tau^* = \tau^*(\mathfrak{A}) \) so that
\[
\mathcal{N} \int Q e^{2\tau^*} \left( \tau^* \lambda^* \phi^3 |u|^2 + \tau \lambda^* \phi |\nabla u|^2 \right) dx dt
\leq \int_Q e^{2\tau^*} |\mathcal{L}_A^\phi u|^2 dx dt + \tau \lambda \int_{\Sigma_+} e^{2\tau^*} \phi |\partial_t u|^2 d\sigma dt
\]
for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^{2,1}(Q, \mathbb{C}) \) satisfying \( u = 0 \) on \( \Sigma \) and \( u(\cdot, t) = 0 \), \( t \in \{ t_1, t_2 \} \).

**5.2. Unique continuation.** An adaptation of the proof in the case of wave equations enables us to establish the following result.

**Theorem 5.3.** Let \( u \in H^{2,1}(Q) \) satisfying \( \mathcal{L}_A^\phi u = 0 \) in \( Q \) and \( u = 0 \) in \( Q_0 = \omega \times (t_1, t_2) \) for some nonempty open subset \( \omega \) of \( \Omega \). Then \( u = 0 \) in \( Q \).

Similarly to the case of wave equations Theorem 5.3 can serve to prove uniqueness of the parabolic Cauchy problem.
Corollary 5.1. Assume that \( L^p_\Omega \) is defined in \( \hat{\Omega} \supset \Omega \). Let \( \Gamma_0 \) be an arbitrary nonempty open subset of \( \Gamma \). If \( u \in H^1((t_1, t_2), H^2(\Omega)) \) satisfies \( L^p_A u = 0 \) in \( Q \) and \( u = \partial_t u = 0 \) in \( \Sigma_0 = \Gamma_0 \times (t_1, t_2) \) then \( u = 0 \) in \( Q \).

5.3. Final time observability inequality. Consider the IBVP

\[
\begin{cases}
\Delta A u - \partial_t u = f & \text{in } Q = \Omega \times (0, T), \\
u(\cdot, 0) = u_0, \\
u|_\Sigma = 0.
\end{cases}
\]  

(5.12)

Define the unbounded operator \( \mathcal{A} : L^2(\Omega) \to L^2(\Omega) \) by

\[ \mathcal{A} u = -\Delta_A u, \quad D(\mathcal{A}) = H^1_0(\Omega) \cap H^2(\Omega). \]

It is known that \( -\mathcal{A} \) generates an analytic semigroup \( e^{-t\mathcal{A}} \). In particular for any \( (u_0, f) \in L^2(\Omega) \times L^1((0, T); L^2(\Omega)) \) the IBVP (6.10) has a unique (mild) solution \( u = \mathcal{S}(u_0, f) \in C([0, T]; L^2(\Omega)) \) so that

\[
\|u(\cdot, t)|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^1((0, T); L^2(\Omega))}, \quad 0 \leq t \leq T.
\]

(5.13)

This solution is given by Duhamel’s formula

\[ u(t) = e^{-t\mathcal{A}} u_0 + \int_0^t e^{-(t-s)\mathcal{A}} f(s)ds, \quad 0 \leq t \leq T. \]

(5.14)

Clearly if \( u_0 \in D(\mathcal{A}) \), then \( u = \mathcal{S}(u_0, 0) \) satisfies

\[ u \in C([0, T]; D(\mathcal{A})) \cap C^1([0, T]; L^2(\Omega)). \]

We refer to [26, Chapter 11] for a concise introduction to semigroup theory.

Lemma 5.1. Let \( f \in L^2(Q) \) and \( \zeta \in C_0^\infty([0, T)) \). Then \( u(T) = \mathcal{S}(0, \zeta f)(T) \in H^1_0(\Omega) \) and

\[
\|u(T)\|_{H^1_0(\Omega)} \leq \|\zeta f\|_{L^2(Q)}
\]

(5.15)

where \( \mathcal{N} \) is a constant only depending on \( \Omega, A, T \) and \( \zeta \).

Proof. From Fujiiwara [15] we know that \( D(\mathcal{A}^{1/2}) = H^1_0(\Omega) \). Therefore according to (5.14) we have

\[
\mathcal{A}^{1/2} u(T) = \int_0^T \mathcal{A}^{1/2} e^{-(T-s)\mathcal{A}} (\zeta(s) f(s))ds.
\]

But

\[
\left\| \mathcal{A}^{1/2} e^{-t\mathcal{A}} \right\|_{L^2(\Omega)} \leq \mathcal{N}_0 t^{-1/2}, \quad t > 0,
\]

where the constant \( \mathcal{N}_0 \) only depends on \( \Omega, A \). If \( \text{supp}(\zeta) \subset [0, T - \epsilon] \), \( \epsilon > 0 \), we find

\[
\left\| \mathcal{A}^{1/2} u(T) \right\|_{L^2(\Omega)} \leq \mathcal{N}_0 \int_0^{T-\epsilon} (T - s)^{-1/2}\|\zeta(s) f(s)\|_{L^2(\Omega)}ds.
\]

Whence Cauchy-Schwarz’s inequality yields

\[
\left\| \mathcal{A}^{1/2} u(T) \right\|_{L^2(\Omega)} \leq \mathcal{N}_0 \left( \int_0^{T-\epsilon} (T - s)^{-1}ds \right)^{1/2}\|\zeta f\|_{L^2(Q)}
\]

\[
\leq \mathcal{N}_0 \ln \left( \frac{T}{\epsilon} \right)^{1/2}\|\zeta f\|_{L^2(Q)}.
\]

The expected inequality then follows. \( \square \)
Theorem 5.4. Let \( 0 \leq \psi_0 \in C^2(\Omega) \) with no critical point in \( \overline{\Omega} \). Set \( \Gamma_+ = \{ x \in \Gamma : \partial_x \psi_0(x) > 0 \} \) and \( \Sigma_+ = \Gamma_+ \times (0, T) \). For any \( u = e^{-t\partial_\tau} u_0 \) with \( u_0 \in D(\mathcal{A}) \) we have

\[
\|u(T)\|_{H^2_0(\Omega)} \leq \mathcal{N}\|\partial_\tau u\|_{L^2(\Sigma_+)}. 
\]

Here \( \mathcal{N} \) is a constant only depending on \( \Omega, A \) and \( T \).

Proof. Let \( \chi \in C^\infty_0((0, T)) \) satisfying \( \chi = 1 \) in \([T/4, 3T/4]\). We apply Theorem 5.2 to \( \chi u \) in order to obtain

\[
\|u(T)\|_{L^2(\Omega \times (T/4, 3T/4))} \leq \mathcal{N}\|\partial_\tau u\|_{L^2(\Sigma_+)}. 
\]

Pick \( \varphi \in C^\infty((0, T]) \) so that \( \varphi = 0 \) in \([0, T/4]\) and \( \varphi = 1 \) in \([3T/4, T]\). We easily check that \( \varphi u = \mathcal{A}(0, \varphi' u) \). In light of Lemma 5.1 we then conclude that

\[
\|u(T)\|_{H^2_0(\Omega)} \leq \mathcal{N}\|\varphi' u\|_{L^2(Q)} = \mathcal{N}\|\varphi' u\|_{L^2(\Omega \times (T/4, 3T/4))}. 
\]

This and (5.16) imply the expected inequality. □

6. Schrödinger equations

Let \( \phi = e^{\lambda \psi} \) be a weight function for the Schrödinger operator

\[
\mathcal{L}^*_{A,0} = \Delta_A + i\partial_t 
\]

and set

\[
\mathcal{L}_A = \Delta_A + i\partial_t + \sum_{\ell=1}^n p_\ell \partial_\ell + q
\]

where \( p_1, \ldots p_n \) and \( q \) belongs to \( L^\infty(Q; \mathbb{C}) \) and satisfy

\[
\|p_\ell\|_{L^\infty(Q)} \leq m, \quad 1 \leq \ell \leq n \quad \text{and} \quad \|q\|_{L^\infty(Q)} \leq m.
\]

6.1. Carleman inequality. In the sequel

\[
\delta = \min_Q |\nabla \psi|_A
\]

and \( \mathfrak{d} = (\Omega, t_1, t_2, \kappa, \delta, m) \).

Theorem 6.1. There exist three constants \( \mathcal{N} = \mathcal{N}(\mathfrak{d}) \), \( \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that

\[
\mathcal{N} \int_Q e^{2\tau\phi} \left[ \tau^3 \lambda^4 \phi^3 |u|^2 + \tau \lambda \phi |\nabla u|^2 \right] \, dxdt 
\]

\[
\leq \int_Q e^{2\tau\phi} |\mathcal{L}^*_A u|^2 \, dxdt + \int_{\partial Q} e^{2\tau\phi} \left[ \tau^3 \lambda^3 \phi^3 |u|^2 + \tau \lambda \phi |\nabla u|^2 \right] \, d\mu 
\]

\[
+ \int_\Sigma e^{2\tau\phi} (\tau \lambda \phi)^{-1} |\partial_\nu u|^2 d\sigma dt dx.
\]

for any \( \lambda \geq \lambda^* \), \( \tau \geq \tau^* \) and \( u \in H^{2,1}(Q, \mathbb{C}) \).

Proof. In this proof \( \mathcal{N}, \lambda_j, \tau_j, j = 1, 2, \ldots \), are positive generic constants only depending on \( \mathfrak{d} \).

As in the preceding section if \( \Phi = e^{-\tau \phi}, \tau > 0 \), then

\[
\partial_\nu \Phi = -\tau \partial_\nu \phi \Phi = -\tau \partial_\nu \phi \Phi 
\]

\[
\partial_j \Phi = (\tau \partial_j^2 \phi + \tau^2 \partial_\nu \phi \partial_j \phi) \Phi
\]

\[
\partial_\nu \Phi = -\tau \partial_\nu \phi \Phi = -\tau \partial_\nu \phi \Phi
\]
We have for $w \in H^2(Q, \mathbb{C})$
\[ \Phi^{-1} \Delta_A(\Phi w) = \Delta_A w - 2\tau(\nabla w|\nabla \phi)_A + \left[\tau^2|\nabla \phi|_A^2 - \tau \Delta_A \phi\right] w \]
and
\[ i\Phi^{-1} \partial_t(\Phi w) = i\partial_t w - i\tau \partial_t \phi w. \]

We decompose $L = \Phi^{-1} L_{A,0}^* \Phi$ as follows
\[ L = L_0 + L_1 + c \]
with
\[
L_0 w = \Delta_A w + i\partial_t w + aw, \\
L_1 w = (B|\nabla w) + bw
\]
where we set
\[
a = \tau^2|\nabla \phi|_A^2, \\
b = -\tau \Delta_A \phi, \\
c = i\partial_t \phi, \\
B = -2\tau A \nabla \phi.
\]

We write
\[
\langle L_0 w|L_1 w \rangle_{L^2(Q)} = \int_Q L_0 w \overline{L_1 w} \ dx \ dt = \sum_{i=1}^6 I_i
\]
with
\[
I_1 = \int_Q \Delta_A w(\nabla w|B) dx \ dt, \\
I_2 = \int_Q \Delta_A w b \ dx \ dt, \\
I_3 = i \int_Q \partial_t w(\nabla w|B) dx \ dt, \\
I_4 = i \int_Q \partial_t w b \ dx \ dt, \\
I_5 = \int_Q aw(\nabla w|B) dx \ dt, \\
I_6 = \int_Q ab|w|^2 dx \ dt.
\]

Most parts of the proof are quite similar to that in the case of the wave equation and therefore we omit their details.

We have
\[
I_1 = \int_Q (D\nabla w|\nabla w) dx \ dt + \int_\Sigma \left[ (\nabla w|\nu)_A(\nabla w|B) - (B/2|\nu)|\nabla w|_A^2 \right] d\sigma \ dt
\]
where
\[ D = C/2 - (B')^t \]
with $C = (\text{div}(a_{ij}B))$. 
Also
\[ \Re I_2 = \int_Q \nabla_A w b w = - \int_Q b|\nabla w|^2_A dxdt - \Re \int_Q \nabla b|\nabla w|_A dxdt \]
\[ + \Re \int_\Sigma (\nabla w|\nu)_A b \nu d\sigma dt. \]

But \( \Re(\nabla w) = \nabla |w|^2/2 \). Therefore
\[ \Re I_2 = - \int_Q b|\nabla w|^2_A dxdt + \int_Q \Delta_A (b/2)|w|^2 dxdt \]
\[ - \int_\Sigma (\nabla (b/2)|\nu)_A |w|^2 d\sigma dt + \Re \int_\Sigma (\nabla w|\nu)_A b \nu d\sigma dt. \]

Let \( J = \Re(I_1 + I_2) \) and define
\[ A = D - bA, \]
\[ a_1 = \Delta_A (b/2), \]
\[ g_1(w) = \Re[(\nabla w|\nu)_A (\nabla w|B) + (\nabla w|\nu)_A b \nu] \]
\[ - (\nabla (b/2)|\nu)_A |w|^2 - (B/2)|\nu| \nabla w|^2. \]

We combine (6.2) and (6.3) in order to obtain
\[ J = \int_Q \Re(A \nabla w|\nabla w) dxdt + \int_Q a_1 |w|^2 dxdt + \int_\Sigma g_1(w) d\sigma dt. \]

We have again from the calculations we done for the wave equation
\[ A = 2\tau A \nabla^2 \phi^A + \tau \Upsilon_A(\phi(\cdot, t)). \]

This identity together with the following ones
\[ \nabla^2 \phi = \lambda^2 \phi(\nabla \psi_0 \otimes \nabla \psi_0) + \lambda \phi \nabla^2 \psi_0, \]
\[ \Upsilon_A(\phi(\cdot, t)) = \lambda \phi \Upsilon_A(\psi_0) \]

imply
\[ A = \Theta_A(\psi_0) + 2\tau \lambda^2 \phi A(\nabla \psi_0 \otimes \nabla \psi_0) A. \]

As \( A(\nabla \psi_0 \otimes \nabla \psi_0)A \) is non negative and \( \psi_0 \) is \( A \)-pseudo-convex with constant \( \kappa > 0 \) we get
\[ \Re(A \nabla w|\nabla w) \geq \kappa \kappa^2 |\nabla w|^2. \]

This inequality in (6.4) yields
\[ J \geq \kappa \kappa^2 \int_Q |\nabla w|^2 dxdt + \int_Q a_1 |w|^2 dxdt + \int_\Sigma g_1(w) d\sigma dt. \]

We find by making an integration by parts with respect to \( t \) and then with respect to \( x \)
\[ \int_Q \partial_t w(\nabla w|B) dxdt = \int_Q \partial_t (\nabla w|B) dxdt - \int_Q \partial_t w \nabla \text{div}(B) dxdt \]
\[ + \int_Q (\nabla w|\partial_t B) \nu d\sigma dt \]
\[ - \int_\Sigma \partial_t w \nabla w|\nu| d\sigma dt + \int_{t_1}^{t_2} [w(\nabla w|B)]_{t=t_1}^t dx. \]
We then obtain by noting that $\text{div}(B) = 2b$

$$
\int_Q \partial_t w(\nabla w|B) dx dt = \int_Q \partial_t \nabla w(\nabla w|B) dx dt - 2 \int_Q \partial_t w \nabla w dx dt
+ \int_Q (\nabla w|\partial_t B) \nabla w dx dt
- \int_\Sigma w \partial_t \nabla(B|\nu) d\sigma dt + \int_\Omega [w(\nabla w|B)]|_{t=t_1}^t dx.
$$

From the identity

$$\partial_t w(\nabla w|B) - \partial_t \nabla w(\nabla w|B) = 2i \Im[\partial_t \nabla w(\nabla w|B)]$$

we deduce that

$$2i \Im \int_Q \partial_t w(\nabla w|B) dx dt = -2 \int_Q \partial_t w \nabla w dx dt + \int_Q (\nabla w|\partial_t B) \nabla w dx dt
- \int_\Sigma w \partial_t \nabla(B|\nu) d\sigma dt + \int_\Omega [w(\nabla w|B)]|_{t=t_1}^t dx.$$

Or equivalently

$$-\Im \int_Q \partial_t w(\nabla w|B) dx dt = -\Re \left( i \int_Q \partial_t w \nabla w dx dt \right) - \Im \int_Q (\nabla w|\partial_t B/2) \nabla w dx dt
+ \Im \int_\Sigma w \partial_t \nabla(B/2|\nu) d\sigma dt - \Im \int_\Omega [w(\nabla w|B/2)]|_{t=t_1}^t dx.$$

Observing that

$$\Re I_3 = -\Im \int_Q \partial_t w(\nabla w|B) dx dt \quad \text{and} \quad I_4 = i \int_Q \partial_t w \nabla w dx dt$$

we conclude that

$$\Re (I_3 + I_4) = -\Im \int_Q (\nabla w|\partial_t B/2) \nabla w dx dt + \int_\Sigma g_2(w) d\sigma dt + \int_\Omega h(w) dx.$$

with

$$g_2(w) = \Im (w \partial_t \nabla(B/2|\nu)), \quad h(w) = -\Im \left( [w(\nabla w|B/2)]|_{t=t_1}^t \right).$$

We find by using again the identity $\Re w \nabla w = \nabla |w|^2/2$

$$\Re I_5 = - \int_Q \text{div}(aB/2)|w|^2 + \int_\Sigma a(B/2|\nu)|w|^2 d\sigma dt$$

and hence

$$\Re (I_5 + I_6) = \int_Q a_2 |w|^2 dx dt + \int_\Sigma g_3(w) d\sigma dt$$

with

$$a_2 = -\text{div}(aB/2) + ab, \quad g_3(w) = a(B/2|\nu)|w|^2.$$

Let

$$a = a_1 + a_2 = -\text{div}(aB/2) + ab + \Delta_A(b/2).$$
We can carry out the same calculations as for the wave equation in order to obtain
\[ a \geq \tau^3 \lambda^4 \phi^3 \delta^4, \quad \lambda \geq \lambda_1, \quad \tau \geq \tau_1. \]
We end up getting by combining (6.1), (6.5), (6.6) and (6.7)
\begin{align*}
(6.8) \quad \Re \langle L_0 w | L_1 w \rangle_{L^2(Q)} & \geq \tau \lambda \kappa \int_Q |\nabla w|^2 dx \, dt + \tau^3 \lambda^4 \phi^3 \int_Q |\phi|^2 |w|^2 dx \, dt \\
& \quad - 3 \int_Q (\nabla w) \partial_t B/2 \sigma dx \, dt + \int_{\Sigma} g(w) d\sigma dt + \int_{\Omega} h(w) dx
\end{align*}
where we set \( g = g_1 + g_2. \)

Let \( \epsilon > 0. \) Then an elementary convexity inequality yields
\[ |(\nabla w) \partial_t B(w)| \leq \Re \phi (\epsilon \tau \lambda |\nabla w|^2 + \epsilon^{-1} \tau \lambda^2 |w|^2). \]

In consequence the third term in (6.8) can be absorbed by the first two ones, provided \( \lambda \geq \lambda_2 \) and \( \tau \geq \tau_2. \) That is we have
\begin{align*}
\Re \langle L_0 w | L_1 w \rangle_{L^2(Q)} & \geq \tau \lambda \kappa \int_Q |\nabla w|^2 dx \, dt + \tau^3 \lambda^4 \phi^3 \int_Q |\phi|^2 |w|^2 dx \, dt \\
& \quad + \int_{\Sigma} g(w) d\sigma dt + \int_{\Omega} h(w) dx.
\end{align*}
The rest of the proof is almost similar to that of the wave equation. \( \square \)

Recall that \( \Gamma_+ = \Gamma^{\psi}_+ = \{ x \in \Gamma; \partial_{\nu A} \psi_0 > 0 \} \) and \( \Sigma_+ = \Sigma^{\psi}_+ = \Gamma_+ \times (t_1, t_2). \)

As for the wave equation we have

**Theorem 6.2.** There exist three constants \( \mathcal{N} = \mathcal{N}(\mathfrak{d}), \lambda^* = \lambda^*(\mathfrak{d}) \) and \( \tau^* = \tau^*(\mathfrak{d}) \) so that
\begin{align*}
(6.9) \quad \mathcal{N} \int_Q e^{2\tau \phi} \left[ \tau^3 \lambda^4 \phi^3 |w|^2 + \tau \lambda \phi |\nabla u|^2 \right] dx \, dt \\
& \quad \leq \int_Q e^{2\tau \phi} |L_A u|^2 dx \, dt + \tau \lambda \int_{\Sigma_+} e^{2\tau \phi} |\partial_{\nu} u|^2 d\sigma dt
\end{align*}
for any \( \lambda \geq \lambda^*, \tau \geq \tau^* \) and \( u \in H^{2,1}(Q, \mathbb{C}) \) satisfying \( u = 0 \) on \( \Sigma \) and \( u(\cdot, t) = 0, \) \( t \in \{ t_1, t_2 \}. \)

**6.2. Unique continuation.** We can proceed similarly as for the wave equation. We limit ourself to give the statement of the results. Recall first that
\[ E_+(\tilde{x}, c, r) = \{ x = (x', x_n) \in B(\tilde{x}, r); 0 \leq x_n - x_n < c \}
\]
and
\[ x_n - \tilde{x}_n \geq |x' - \tilde{x}'|^2 / c \]
with \( \tilde{x} \in \Omega, \) \( c > 0 \) and \( r > 0. \)

**Theorem 6.3.** There exists \( c^* = c^*(\mathfrak{m}) \) with the property that for any \( 0 < c < c^* \) and \( \tilde{x} \in \Omega \) we find \( 0 < r = r(c, \tilde{d}) < \tilde{d} \) and \( 0 < \rho = \rho(c, \tilde{d}) < r, \) where \( \tilde{d} = \text{dist}(\tilde{x}, \Gamma), \) so that if \( u \in H^{2,1}(Q) \) satisfies \( L_A u = \text{in } Q \) and \( \text{supp}(u(\cdot, t)) \cap B(\tilde{x}, r) \subset E_+(\tilde{x}, c, r), \) \( t \in \{ t_1, t_2 \}, \) then \( u = 0 \) in \( B(\tilde{x}, \rho) \times (t_1, t_2). \)
We say that $\mathcal{L}_A^s$ has the property of unique continuation if for any $u \in H^2(Q)$ satisfying $\mathcal{L}_A^s u = 0$ in $Q$ and $u = 0$ in $\Omega_0 \times (t_1, t_2)$ for some non-empty open subset $\Omega_0 \subset \Omega$ then $u$ must be identically equal to zero in $Q$.

**Theorem 6.4.** There exists a neighborhood $\mathcal{N}$ of $I$ in $C^{0,1}(\overline{\Omega}; \mathbb{R}^n \times \mathbb{R}^n)$ so that $\mathcal{L}_A^s$ has the property of unique continuation for any $A \in \mathcal{N}$.

Concerning the uniqueness for the Cauchy problem associated to $\mathcal{L}_A^s$, we have the following corollary.

**Corollary 6.1.** Let $\mathcal{N}$ be as in Theorem 6.4 with $\Omega$ substituted by larger domain $\hat{\Omega} \supseteq \Omega$. Let $\Gamma_0$ a non-empty open subset of $\Gamma$ and $\Sigma_0 = \Gamma_0 \times (t_1, t_2)$. For $A \in \mathcal{N}$ let $u \in H^{2,1}(Q)$ satisfying $\mathcal{L}_A^s u = 0$ in $Q$ and $u = \partial_\nu u = 0$ on $\Sigma_0$. Then $u = 0$.

The property of unique continuation across a $A$-pseudo-convex hypersurface is given in the following theorem.

**Theorem 6.5.** Let $H = \{ x \in \omega; \theta(x) = \theta(\hat{x}) \}$ be a $A$-pseudo-convex hypersurface defined in a neighborhood of $\hat{x} \in \Omega$ with $\theta \in C^2(\omega)$. Then there exists $B$ a neighborhood of $\hat{x}$ so that if $u \in H^{2,1}(\omega \times (t_1, t_2))$ satisfies $\mathcal{L}_A^s u = 0$ in $\omega \times (t_1, t_2)$ and $\text{supp}(u(\cdot, t)) \subset H_+ = \{ x \in \omega; \theta(x) \geq \theta(\hat{x}) \}$, $t \in (t_1, t_2)$, then $u = 0$ in $B \times (t_1, t_2)$.

### 6.3 Observability Inequality

Let $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ be the unbounded operator introduced in the preceding section. That is

$$\mathcal{A} u = -\Delta_A u, \quad D(\mathcal{A}) = H^1_0(\Omega) \cap H^2(\Omega).$$

It is known that $u(t) = e^{it\mathcal{A}} u_0, u_0 \in L^2(\Omega)$ is the solution of the following IBVP

$$\begin{align*}
\Delta_A u + i\partial_t u &= 0 \quad \text{in } Q, \\
u u &= 0 \quad \text{on } \Sigma, \\
\mathcal{A} u &= 0.
\end{align*}$$

(6.10)

Moreover $u$ belongs to $C([0, T]; D(\mathcal{A})) \cap C^1([0, T]; L^2(\Omega))$ whenever $u_0 \in D(\mathcal{A})$ and for $0 \leq t \leq T$ we have

$$\|u(\cdot, t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}, \quad \|\nabla_A u(\cdot, t)\|_{L^2(\Omega)} = \|\nabla_A u_0\|_{L^2(\Omega)}.$$ 

(6.11)

**Theorem 6.6.** Suppose that $0 \leq \psi_0 \in C^4(\overline{\Omega})$ is $A$-pseudo-convex with constant $\kappa > 0$ and let $\Gamma_+ = \{ x \in \Gamma; \partial_\nu \psi_0(x) > 0 \}$. Then there exists a constant $K$ only depending $\Omega$, $T$, $\kappa$, $\kappa$ and $\Gamma_+$, so that for any $u_0 \in D(\mathcal{A})$ we have

$$\|u_0\|_{H^1_0(\Gamma_+)} \leq K\|\partial_\nu u\|_{L^2(\Gamma_+)},$$

where $\Gamma_+ = \Gamma_+ \times (0, T)$ and $u = e^{it\mathcal{A}} u_0$.

**Proof.** In light of Theorem 6.1 and identities (6.11) the expected inequality can be proved by modifying slightly that of the wave equation. □

**Remark 6.1.** It worth mentioning that the results for the elliptic, wave and Schrödinger equations can be extended to the case where $\Delta_A$ is substituted by the associated magnetic operator defined by

$$\Delta_{A, b} u = \sum_{k, \ell} (\partial_k + ib_k) a_{k\ell} (\partial_\ell + ib_\ell) u$$

with $b = (b_1, \ldots, b_n) \in W^{1, \infty}(\Omega; \mathbb{R}^n)$.

Note that $\Delta_{A, b} u$ can rewritten in the following form

$$\Delta_{A, b} u = \Delta_A u + 2i(\nabla u|b)_A + (-|b|^2_A + \text{div}(Ab)) u.$$
References

[1] S. Alinhac, Non-unicité du problème de Cauchy, Ann. Math. (2) 117 (1983), 77-108.

[2] L. Baudouin and J.-P. Puel, Détermination du potentiel dans l'équation de Schrödinger à partir de mesures sur une partie du bord, C. R. Math. Acad. Sci. Paris 334 (11) (2002), 967-972.

[3] L. Baudouin and J.-P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, Inverse Problems 18 (6) (2002), 1537-1554.

[4] L. Baudouin and J.-P. Puel, Corrigendum: ‘Uniqueness and stability in an inverse problem for the Schrödinger equation’ [Inverse Problems 18 (6) (2002), 1537-1554], Inverse Problems 23 (3) (2007), 1327-1328.

[5] M. Bellassoued and M. Choulli, Global logarithmic stability of the Cauchy problem for anisotropic wave equations, arXiv:1902.05878.

[6] M. Bellassoued, and M. Yamamoto, Carleman estimates and applications to inverse problems for hyperbolic systems, Springer Monographs in Mathematics. Springer, Tokyo, 2017, xii+260 pp.

[7] L. Bourgeois, Quantification of the unique continuation property for the heat equation, Math. Control and Related fields, 7 (3) (2017), 347-367.

[8] L. Bourgeois, About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$-domains, M2AN Math. Model. Numer. Anal. 44 (4) (2010), 715-735.

[9] M. Choulli, New global logarithmic stability result for the Cauchy problem for elliptic equations, Bull. Aust. Math. Soc. 101 (1) (2020) 141-145.

[10] M. Choulli, Applications of elliptic Carleman inequalities to Cauchy and inverse problems, SpringerBriefs in Mathematics, BCAM SpringerBriefs. Springer, Bilbao, 2016. ix+81 pp.

[11] M. Choulli, Une introduction aux problèmes inverses elliptiques et paraboliques, Mathématiques & Applications 65, Springer-Verlag, Berlin, 2009, xxii+249 pp.

[12] M. Choulli and M. Yamamoto, Logarithmic global stability of parabolic Cauchy problems, arXiv:1702.06299.

[13] E. Fernández-Cara and S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability, SIAM J. Control Optim. 45 (4) (2006), 1399-1446.

[14] X. Fu, Q. Lü and X. Zhang, Carleman estimates for second order partial differential operators and applications. A unified approach, SpringerBriefs in Mathematics, BCAM SpringerBriefs. Springer, 2019. xi+127 pp.

[15] D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad. 43 (1967), 82-86.

[16] A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, Lecture Notes Series, Seoul National Univ., 1996.

[17] L. Hörmander, The analysis of linear partial differential operators IV. Fourier integral operators, reprinted from the 1994 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2009, viii+352 pp.

[18] X. Huang, Carleman Estimate for a general second-order hyperbolic equation, Inverse Problems and related Topics, Springer Proceedings in Mathematics and Statistics, Springer, Singapore, 2020, 149-165.

[19] V. Isakov, Inverse problems for partial differential equations, third edition, Applied Mathematical Sciences 127, Springer, Cham, 2017, xv+406 pp.

[20] F. John Partial differential equations, fourth edition, Applied Mathematical Sciences 1, Springer, New York, 1986, x+429 pp.

[21] C. Laurent and M. Léautaud, Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves, J. European Math. Soc. 21(4) (2019), 957-1069.

[22] J. Le Rousseau and G. Lebeau, On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations, ESAIM Control Optim. Calc. Var. 18 (3) (2012), 712-747.

[23] A. Mercado, Alberto, A. Osses and L. Rosier, Carleman inequalities and inverse problems for the Schrödinger equation, C. R. Math. Acad. Sci. Paris 346 (1-2) (2008), 53-58.

[24] A. Mercado, Alberto, A. Osses and L. Rosier, Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights, Inverse Problems 24 (1) (2008), 015017, 18 pp.
[25] L. Nirenberg, Uniqueness in Cauchy problems for differential equations with constant leading coefficients, Comm. Pure Appl. Math. 10 (1957), 89-105.

[26] M. Renardy and R. C. Rogers, An introduction to partial differential equations, Texts in Applied Mathematics, 13 Springer-Verlag, New York, 1993. xiv+428 pp.

[27] J.-C. Saut et B. Scheurer, Un théorème de prolongement unique pour des opérateurs elliptiques dont les coefficients ne sont pas localement bornés, C. R. Acad. Sci. Paris Sér. A-B 290 (13) (1980), A595-A598.

[28] J.-C. Saut et B. Scheurer, Sur l’unicité du problème de Cauchy et le prolongement unique pour des équations elliptiques à coefficients non localement bornés, J. Differential Equations 43 (1) (1982), 28-43.

[29] J.-C. Saut et B. Scheurer, Remarques sur un théorème de prolongement unique de Mizohata, C. R. Acad. Sci. Paris Sér. I Math. 296 (6) (1983), 307-310.

[30] J.-C. Saut et B. Scheurer, Unique continuation for some evolution equations, J. Differential Equations 66 (1) (1987), 118-139.

[31] P.-F. Yao, On the observability inequalities for exact controllability of wave equations with variable coefficients, SIAM J. Control Optim. 37 (5) (1999), 1568-1599.

[32] C. Zuily, Uniqueness and nonuniqueness in the Cauchy problem, Progress in Mathematics, 33, Birkhäuser Boston, Inc., Boston, MA, 1983. xi+168 pp.

Université de Lorraine, 34 cours Léopold, 54052 Nancy cedex, France
E-mail address: mourad.choulli@univ-lorraine.fr