ABSTRACT. We prove that one can obtain natural bundles of Lie algebras on rank two $s$-Kähler manifolds, whose fibres are isomorphic to $\mathfrak{so}(s+1, s+1)$, $\mathfrak{su}(s+1, s+1)$ and $\mathfrak{sl}(2s+2, \mathbb{R})$. In the most rigid case (which includes complex tori and abelian varieties) these bundles have natural flat connections, whose flat global sections act naturally on cohomology. We also present several natural examples of manifolds which can be equipped with an $s$-Kähler structure with various levels of rigidity: complex tori and abelian varieties, cotangent bundles of smooth manifolds and moduli of pointed elliptic curves.

1. INTRODUCTION

In this paper we prove that one can obtain natural bundles of Lie algebras on rank two $s$-Kähler manifolds, with fibres isomorphic to $\mathfrak{so}(s+1, s+1)$ and to $\mathfrak{su}(s+1, s+1)$, $\mathfrak{sl}(2s+2, \mathbb{R})$. In the most rigid case (which includes complex tori and abelian varieties) these bundles have natural flat connections, whose flat global sections act naturally on cohomology.

An $s$-Kähler structure is a direct generalization (with $s$ distinct "Kähler forms") of the notion of Kähler structure, to which it reduces when $s = 1$. The original motivation for the introduction in [G1] of $s$-Kähler manifolds was the geometric study of the analytical theory of maps from (open subsets of) $\mathbb{R}^s$ to a given manifold. Then it was realized that this theory in the case $s = 2$ is well suited for the study of Mirror Symmetry (see [G2],[G3]), as it should be, given that for $s = 2$ we are considering maps from Riemann surfaces into general manifolds. In a further specialization of the general features of the theory, already in [G1] and [G2], and more in detail in [GG1], it was argued that, independently of the geometric motivations, one can observe a rich algebraic structure "living" on natural bundles on $s$-Kähler manifolds. We decided therefore to embark in a systematic study of these algebraic structures, because of their intrinsic interest, and with the strong belief that if we could master them well enough, we would then be able to apply this theory to interesting problems in algebra and in geometry.

The first result of this line of work is the paper [GG2], where we found a natural Lie superalgebra bundle on rank three 2-Kähler manifolds. We believe that these computations will be very relevant to the geometric study of Mirror Symmetry, and to the search of a geometrical interpolation between the various string theories (see [G2] for a more detailed introduction on this aspect). More specifically, in [G2] it was conjectured that the natural bundles of Lie (super) algebras and of their representations on certain 2-Kähler manifolds could provide the natural background on which to build Field Theories; these are rich from the representation theoretic point of view and, once quantized using the language of [G1], were conjectured to be the right playing field for the search of an M-theory (see for example [DOPW] for a similar approach to the Standard Model in particle physics). A direction more in line with this algebraic study (but actually strongly related to the previous one)
is the relationship with Higgs bundles and Hitchin systems (see for example [HT]). In the present paper we obtain results which are directly applicable to questions in Complex geometry and in Algebraic geometry. The basic reason for this is that a rank two s-Kähler manifold (or a naturally defined double cover of it, in some cases) has a canonical complex structure, with which it becomes Kähler of complex dimension $s + 1$. We prove that on these Kähler manifolds originating from s-Kähler geometry there are natural bundles of unitary Lie algebras of signature $(s + 1, s + 1)$, which act in various ways on differential forms. It is this natural way of representing "large" and well known unitary Lie algebras on differential forms which opens a wide range of geometric applications. For comparison, one should recall that the corresponding constructions for plain Kähler manifolds produce the "Lefschetz" action of $\mathfrak{sl}(2, \mathbb{C})$ on forms and on cohomology, which has a lot of geometric applications and consequences.

Let us now introduce more in detail the geometric and algebraic characters which will play a role. The initial object comes from a generalization and an unconventional point of view on the notion of jet space (see [G1] for details):

**Definition 1.1** ([G1], Definition 2.1 and Corollary 2.6). A polysymplectic manifold of rank $r$ is a smooth manifold of dimension $(s + 1)r + c$, together with $s$ smooth closed two-forms $\omega_1, \ldots, \omega_s$ such that for any $p \in X$ there is a (Darboux) coordinate system around $p$ of the form

$$x_1, \ldots, x_r, y_1^1, \ldots, y_r^1, \ldots, y_1^s, \ldots, y_s^s, z_1, \ldots, z_c$$

for which the forms have the canonical (local) expression

$$\omega_j = \sum_{i=1}^{r} dx_i \wedge dy_i^j$$

Of course, when $s = 1$ we recover the usual notion of (possibly degenerate) symplectic manifold. When one adds a Riemannian metric, and asks for the natural compatibility conditions with the polysymplectic data, one comes to our main object of study:

**Definition 1.2** ([G1], Definition 7.2). A smooth manifold $M$ of dimension $r(s + 1)$ together with a Riemannian metric $g$ and 2-forms $\omega_1, \ldots, \omega_s$ is s-Kähler (of rank $r$) if for each point of $M$ there exist an open neighborhood $U$ of $p$ and a system of coordinates $x_i, y_i^j, i = 1, \ldots, r$, $j = 1, \ldots, s$ on $U$ such that:

1) $\forall j \omega_j = \sum_{i=1}^{r} dx_i \wedge dy_i^j$,

2) $g(x, y) = \sum_{i=1}^{r} dx_i \otimes dx_i + \sum_{i,j} dy_i^j \otimes dy_i^j + O(2)$.

Any such system of coordinates is called standard (s-Kähler).

For $s = 1$ one recovers the usual notion of Kähler manifold. As in the case of Kähler manifolds, one can use the differential forms associated to the structure to build "wedge" operators on forms, and, using their adjoints, one gets natural Lie algebras. Again as in the case of Kähler manifolds, to build correctly a theory involving also the adjoints of such wedge operators, it is necessary to consider their pointwise action, and to recover the global operators on forms as global sections of corresponding bundles of Lie algebras. When $s = 1$ one obtains the classical $\mathfrak{sl}(2, \mathbb{C})$ action on the forms of a Kähler manifold (and on its cohomology using the Hodge identities). In the case $s > 1$ there is a qualitatively different situation, in that there are more natural differential two-forms than one could initially guess. Indeed, in addition to the structural forms $\omega_1, \ldots, \omega_s$ which generalize directly the Kähler form, there are also "mixed" forms $\omega_{jk}$ for any pair of indices $j, k \in \{0, \ldots, s\}$,
including the structural ones via the identifications
\[ \omega_j = \omega_{0j} \quad \text{for} \ j \in \{1, \ldots, s\} \]
The precise description of these derived natural forms will be given in the next section. Here however we can already use them to build corresponding "wedge" operators, Lefschetz style:

**Definition 1.3.** For \( \phi \in \Omega^i_X \) and \( j, k \in \{0, \ldots, s\} \) with \( j \neq k \),
\[ L_{jk}(\phi) = \omega_{jk} \wedge \phi = -L_{kj}(\phi) \]

Some canonical mutually orthogonal distributions \( W_i \ (i = 0, 1, 2, \ldots, s) \) are induced on \( T_p^* X \) by the forms \( \omega_{jk} \) (see Section 2). Therefore other natural operators, called \( V_i \ (i = 0, 1, 2, \ldots, s) \) arise from wedging with the local volume forms of these distributions.

One then uses all these operators, and their (pointwise) adjoints, to build a natural bundle of real Lie algebras \( \mathcal{L}_S \) (and its complexified bundle \( \mathcal{L}_C \)) acting on forms. To be precise, one can fix a point \( p \in X \), and on this point one can restrict the actions above, to obtain bundles of Lie algebras on the \( s \)-Kähler manifold. This approach has many advantages, among which the fact that these bundles will exist also in situations in which the single operators used to define them do not have global sections on all of \( X \).

Coming to a more detailed description of the contents of the present paper, in Section 2 we give the definition of almost \( s \)-Kähler structure, which is a weaker version of the definition of \( s \)-Kähler structure. Then we provide a first geometric description of an (almost) \( s \)-Kähler manifold \( X \): we discuss the existence of an (almost) complex structure, the natural distributions on the cotangent space, the group of local structure-preserving transformations and the orientability properties.

Section 3 is devoted to the definition of the bundles of Lie algebras \( \mathcal{L}_S \) and \( \mathcal{L}_C \) on \( X \) which constitute the main object to be studied in this paper. We also point out two other real forms \((sL^*, uL^*)\), defined in terms of geometric generators, of the bundle \( \mathcal{L}_C \). We then define \( \text{Lef}^* \) as the real sub-bundle of \( \mathcal{L}_C^* \) which is the direct generalization of the classical \( \text{sl}(2, \mathbb{C}) \) Lefschetz bundle of Kähler geometry.

The sections from 4 to 7 are a detailed study of the fibres of above mentioned Lie algebra bundles: in Section 4 the Lefschetz bundle \( \text{Lef}^* \) is studied in detail, by showing some fundamental relations among its generators; the fibres of the bundle turn out to be isomorphic to the orthogonal algebra \( \text{so}(s + 1, s + 1, \mathbb{R}) \) and Serre generators are presented in terms of simple brackets of geometric generators (Theorem 4.5).

Sections 5 and 6 are devoted to the complete description of the main complex bundle \( \mathcal{L}_C^* \): we use the Hodge decomposition on \( \bigwedge^* C{T}^* p X \) with respect to the (almost) complex structure and Clifford algebra techniques to show that the fibres of \( \mathcal{L}_C^* \) are isomorphic to \( \text{sl}(2s + 2, \mathbb{C}) \); furthermore, we characterize \( \mathcal{L}_S^* \) as the bundle of all quadratic elements of trace zero (compatible with the almost complex structure) of a Clifford algebra bundle (Theorem 6.2).

Section 7 focusses on the real forms of \( \mathcal{L}_S^* \) which turn out to be interesting both from the algebraic and the geometric point of view. In fact \( sL^* \) is proven to be the bundle of the real split form of \( \mathcal{L}_C^* \), while \( \mathcal{L}_S^* \) and \( uL^* \) are shown to be the real bundles of operators which preserve two natural non degenerate hermitian inner products on \( \bigwedge^* C{T}^* p X \) (Theorems 7.1, 7.5, 7.7). A superHermitean variant of one of these inner products was introduced in [GG2] to study rank three WSD structures. A computation of the signature shows that the fibres of \( \mathcal{L}_S^* \) and \( uL^* \) are unitary Lie algebras isomorphic to \( \text{su}(s + 1, s + 1) \). We observe that the complete description
of \( \mathcal{L}_R^s \) fully answers to the question (first raised in [G1] and then more precisely in the rank two case in [GG1]) on the nature of the algebraic bundles generated by the real canonical operators associated to an \( s \)-Kähler structure. Furthermore, at the end of the section, \( \text{Lef}^s \) is shown to coincide with \( \mathcal{L}_R^s \cap s \mathcal{L}_s^s \).

After this presentation of the natural Lie bundles of an \( s \)-Kähler structure, we devote Sections 8 and 9 to the construction of some examples of (full, almost, or pointwise) \( s \)-Kähler structures. In Section 8 we first recall the standard examples of [G1] built using iterated cotangent bundles of smooth Riemannian manifolds. We then observe how any real torus of dimension \( r(s+1) \) can be given many non-equivalent translation invariant \( s \)-Kähler structures. In Corollary 8.3 we prove that when \( X \) is compact orientable \( s \)-Kähler there is a natural action of the flat global sections of the bundles of Lie algebras \( \mathcal{L}_R^s, \mathcal{L}_C^s, s \mathcal{L}_s^s, u \mathcal{L}_s^s, \text{Lef}^s \) on cohomology.

Then, in Theorem 8.4, we characterize the rank two \( s \)-Kähler manifolds which are quotients of tori in terms of the Calabi-Yau condition. In Section 9 we put a pointwise rank two \( s \)-Kähler structure on the moduli space of elliptic curves with \( s + 2 \) punctures, which depends naturally on any chosen Kähler metric.

2. Introduction to the geometric setting

This section is again introductory in nature, but with a stronger emphasis on the geometric aspects of the theory. It would be too long to describe all the general facts on \( s \)-Kähler geometry here, so we will list here only the most relevant ones for our purposes, while referring to [G1] for a more thorough analysis. First, as will have been clear already to the reader, one can isolate the pointwise aspects of the definition of an \( s \)-Kähler manifold. The notion of almost \( s \)-Kähler manifold given below is actually a hybrid between pointwise and local properties, which was introduced in [G1] (in the nondegenerate case) in the belief that this mix could be best suited to the purposes of that paper:

**Definition 2.1.** A almost \( s \)-Kähler structure of rank \( r \) on a smooth manifold \( X \) of dimension \((s+1)r+c \) is given by a Riemannian metric \( g \) and a smooth differential two-form \( \omega_{jk} \) for any pair of indices \( j,k \in \{0 \ldots s\} \) such that \( \omega_{01}, \ldots, \omega_{0s} \) give a polysymplectic structure, and for any point \( p \in X \) there is an orthonormal basis of \( T^*_pX \) made of covectors \( v_{ij} \) for \( i \in \{1, \ldots, r\} \) and \( j \in \{0, \ldots, s\} \) and \( u_1, \ldots, u_c \) such that,

\[
\omega_{jk} = \sum_{i=1}^r v_{ij} \wedge v_{ik}
\]

The forms \( \omega_{jk} \) with \( j,k \in \{1, \ldots, s\} \) are called dualizing forms.

**Remark 2.2.** One can directly check that giving an almost \( s \)-Kähler structure is equivalent to giving the forms \( \omega_{01}, \ldots, \omega_{0s} \) and a compatible metric, which is exactly what is needed for an \( s \)-Kähler structure, except for the local condition 2), which is equivalent to the invariance of the forms with respect to the Levi-Civita connection. This is the way in which \( s \)-Kähler manifolds were introduced in [G1].

Clearly there are some redundancies in the definition given above: for example, one has always \( \omega_{jk} = -\omega_{kj} \). Observe also that an almost 1-Kähler manifold is simply an almost Kähler manifold, and for this reason in this paper we consider only the case \( s \geq 2 \) which is moreover the range where our constructions do exist. Recall also that an almost 2-Kähler manifold in which the structure forms are closed is a Weakly Self Dual manifold (see [G2], Definition 2.6), or WSD manifold for short. For the algebraic constructions to be discussed in this paper, all that is needed is a rank 2 almost \( s \)-Kähler structure (actually for the main construction we will need only the pointwise part of the definition).
The almost s-Kähler structure on a manifold $X$ splits its cotangent space as $T^*_pX = W_0 \oplus W_1 \oplus \cdots \oplus W_s$ where the $W_j$ are $s+1$ mutually orthogonal canonical distributions defined as:

$$W_j = \{ \phi \in T^*_pX \mid \phi \wedge \omega_{jk} = 0 \text{ for } k \in 0, \ldots, j, \ldots, s \}$$

The almost s-Kähler structure also determines canonical pairwise linear identifications among the spaces $W_j$, so that one can also write $T^*_pX \cong W_0 \otimes_{\mathbb{R}} \mathbb{R}^{s+1}$ or more simply

$$T^*_pX \cong W \otimes_{\mathbb{R}} \mathbb{R}^{s+1}$$

where $W = W_0 \cong W_1 \cong \cdots \cong W_s$.

Let us now come back to the canonical operators $L_{jk}$ mentioned in the Introduction. We now choose an orientation of $W_0$ at a fixed point $p \in X$, and a (non-canonical) orthonormal basis $\gamma_1, \gamma_2$ compatible with this orientation; this together with the standard identifications of the $W_j$ determines an orientation and an orthonormal basis for $T^*_pX$, which we write as $\{ \nu_{ij} = \gamma_i \otimes e_j \mid i = 1, 2, j = 0, \ldots, s \}$. We remark that the $\nu_{ij}$ are an adapted coframe for the almost s-Kähler structure, and therefore we have the explicit expressions:

$$\omega_{jk} = \nu_{1j} \wedge \nu_{1k} + \nu_{2j} \wedge \nu_{2k}$$

A different choice of the $\gamma_1, \gamma_2$ would be related to the previous one by an element in $O(2, \mathbb{R})$. The Lie algebra of the group $O(2, \mathbb{R})$ expressing the change from one adapted basis to another is generated point by point by the operator $J$, which is determined and determines a (pointwise, local or global if possible) orientation of the distribution $W_0$.

**Definition 2.3.** The operator $J \in End_{\mathbb{R}}(\wedge^* T^*_pX)$ associated to the standard basis $\nu_{ij}$ is defined as

$$J(\nu_{ij}) = \nu_{2j}, \quad J(\nu_{2j}) = -\nu_{1j} \quad \text{for } j \in \{0, 1, \ldots, s\}$$

and $J(\nu \wedge \nu) = J(\nu) \wedge \nu + \nu \wedge J(\nu)$ for $\nu, \nu \in \Lambda^* T^*_pX$.

**Remark 2.4.** As $J$ commutes with itself, and it is determined at every point $p \in X$ by an orientation of $(W_0)_p \subset T_pX$, it is always well defined locally. Of course, $J$ admits a global determination if and only if $W_0$ admits a global orientation. This happens for example if $X$ is orientable and $s$ is even.

Whenever we will need a local volume form on $X$, we will use the one induced by a local choice of $J$ which we will call $\Omega_p$ over the point $p \in X$. From the above considerations it follows the following fundamental remark:

**Remark 2.5.** An (almost) s-Kähler manifold of rank 2 is in particular an (almost) complex manifold of complex dimension $s+1$, when there is a global determination of $J$. This happens in particular when $X$ is orientable and $s$ is even.

For this reason, rank two (almost) 2-Kähler manifolds can be seen as a chapter in (almost) complex geometry. This allows on one hand to “import” the techniques of complex geometry into the realm of almost 2-Kähler geometry, and on the other hand allows one to apply the results of almost 2-Kähler geometry to the complex world. Summing up, we have that

$s$-Kähler $\implies$ almost $s$-Kähler $\implies$ Polysymplectic

and in the rank two case we have moreover that locally

(almost) $s$-Kähler $\implies$ (almost) Kähler

When the structure is $s$-Kähler, one has that all the structure forms are covariant constant with respect to the Levi-Civita connection associated to the metric. This
allows one to perform many of the same constructions that one usually performs in
the Kähler case. In particular, one recovers (the analog of) the Hodge identities, and
the adjoints of the canonical operators $L_{jk}$ operate on cohomology (see Theorem
8.2 and Corollary 8.3). This is the context in the case of Abelian varieties, which
in our opinion will provide many interesting applications of the constructions to be
detailed in the present paper.

For a general rank of the structure $r \geq 1$, many of the above considerations
generalize; for example the group of pointwise transformations which preserve the
structure is $O(r)$. As we have seen above, in the $r = 2$ case we obtain $O(2)$
whose algebra is generated by $J$, while the $r = 3$ case (in which comes into play
$O(3)$) was discussed in detail in [GG2]. Clearly however, not everything generalizes
to arbitrary rank: for example, a rank three s-Kähler manifold may be of (real)
dimension 9, which is odd and therefore it is impossible to have an almost complex
structure on such a manifold. Still in case $r = 3$, one has natural operators also in
odd degree, and therefore the natural algebras which come out of the geometry are
Lie superalgebras, instead of Lie algebras (see [GG2]).

3. Construction of the natural algebras

In this section we fix a point $p$ in an almost s-Kähler manifold $X$ and we mostly
work on tensor powers of $T_p X$.

As was mentioned in the previous sections, using the forms $\omega_{jk}$ of the almost
s-Kähler structure, we can build corresponding operators on forms, much in the
way as the $L$ operator is built on Kähler manifolds:

**Definition 1.3** For $\phi \in \Omega^*_p X$ and $j, k \in \{0, \ldots, s\}$ with $j \neq k$,

$$L_{jk}(\phi) = \omega_{jk} \wedge \phi = -L_{kj}(\phi)$$

The above operators restrict also to $\bigwedge \bigwedge^p X$ for any $p \in X$ where, using the chosen
(orthonormal) basis, one can define also corresponding (non canonical) wedge and
contraction operators:

**Definition 3.1.** Let $i \in \{1, 2\}$, $j \in \{0, 1, \ldots, s\}$ and $p \in X$. The operators $E_{ij}$
and $I_{ij}$ are respectively the wedge and the contraction operator with the form $v_{ij}$
on $\bigwedge \bigwedge^p X$ (defined using the given basis); we use the notation $\frac{\partial}{\partial v_{ij}}$ to indicate the
element of $T_p X$ dual to $v_{ij} \in T^*_p X$:

$$E_{ij}(\phi) = v_{ij} \wedge \phi, \quad I_{ij}(\phi) = \frac{\partial}{\partial v_{ij}} \rightarrow \phi$$

**Proposition 3.2.** The operators $E_{ij}, I_{ij}$ satisfy the following relations:

$$\forall i, j, k, l \quad E_{ij}E_{kl} = -E_{kl}E_{ij}, \quad I_{ij}I_{kl} = -I_{kl}I_{ij}$$

$$\forall i, j \quad E_{ij}I_{ij} + I_{ij}E_{ij} = \text{Id}$$

$$\forall (i, j) \neq (k, l) \quad E_{ij}I_{kl} = -I_{kl}E_{ij}$$

$$\forall i, j \quad E_{ij}^* = I_{ij}, \quad I_{ij}^* = E_{ij}$$

where $*$ is adjunction with respect to the metric.

**Proof** The proof is a simple direct verification, which we omit. \qed

It is then immediate to check that:

**Proposition 3.3.** $J$ can be expressed on the whole $\bigwedge \bigwedge^p X$ as

$$J = \sum_{j=0} (E_{2j}I_{1j} - E_{1j}I_{2j})$$
Definition 3.5. For forms of the distributions $W^V$ the pointwise action of the canonical wedge operators $J$

Remark 3.4. From this expression and the previous proposition one obtains that we indicate with the same symbols: for restrictions of corresponding global operators on smooth differential forms, which

Definition 3.6.

$J$ operator $J$ these spaces, which is implied for example by the choice of a determination for the volume forms of the spaces $W^o$

The smooth bundle of Lie algebras Lef$^\ast$ is the real sub-bundle of Lie algebras of $End_\mathbb{R}(\Omega^\ast(X))$ generated locally by the operators

Definition 3.7. The smooth bundle of Lie algebras $Lef^\ast$ is the real sub-bundle of Lie algebras of $End_\mathbb{R}(\Omega^\ast(X))$ generated locally by the operators

In the study of Kähler geometry, a central role is played by the Lie algebra generated by Lefschetz operator and its adjoint. The direct generalization of that algebra to the setting of (almost) s-Kähler manifolds is the following:

Definition 3.8. The smooth bundle of Lie algebras $\mathcal{L}_\mathbb{R}^\ast$ is the real sub-bundle of Lie algebras of $End_\mathbb{R}(\Omega^\ast(X))$ generated locally by the operators

for any fixed determination of $J$. The $\ast$-Lie algebra $\mathcal{L}_\mathbb{C}^\ast$ is $\mathcal{L}_\mathbb{R}^\ast \otimes_\mathbb{R} \mathbb{C}$. The $\ast$ operator on $\mathcal{L}_\mathbb{C}^\ast$ is induced by the adjoint with respect to the Hermitean metric induced by the Riemannian one via complexification.

As mentioned in the Introduction, in the present paper we will describe completely the structure of the fibers of the bundles $Lef^\ast$, $\mathcal{L}_\mathbb{R}^\ast$, $\mathcal{L}_\mathbb{C}^\ast$, and we will further describe two other real forms of $\mathcal{L}_\mathbb{C}^\ast$, which are especially significant from a geometric point of view. Here are their definitions:

Definition 3.9. The real form $s\mathcal{L}^\ast$ of the complex bundle of $\ast$-Lie algebras $\mathcal{L}_\mathbb{C}^\ast$ is generated (as a bundle of real Lie algebras) by the local operators:

$L_{jk}$, $iV_j$, $\Lambda_{jk}$, $iA_j$
Definition 3.10. The real form \( uL^s \) of the complex bundle of \( * \)-Lie algebra \( L^s_C \) is generated (as a bundle of real Lie algebras) by the local operators:

\[
iL_{jk}, \ iV_j, \ i\Lambda_{jk}, \ iA_j
\]

4. Clifford algebras and a natural presentation of \( Lef^s \) as a \( \text{so}(s + 1, s + 1, \mathbb{R}) \) bundle

In this section we will show that \( L^s_R \) lies inside a (real) Clifford algebra bundle over the \((4s + 4)\)-dimensional real bundle \( TX \oplus T^*X \); we will also point out that the natural bundle of Lie subalgebras \( Lef^s \subset L^s_R \) is isomorphic to the constant bundle having as fibre the orthogonal algebras \( \text{so}(s + 1, s + 1, \mathbb{R}) \). Notice that the above considerations do not apply to the \( s = 1 \) (Kähler) situation; \( Lef^s \) in that case is simply a constant \( \text{sl}(2, \mathbb{R}) \) bundle, as it is well known classically. Notice also that this global trivialization of \( Lef^s \) does not depend on a determination of the (almost) complex structure \( J \).

In the following we define some new operators, and in the meantime we introduce a unifying notation which concerns the \( L_{jk}, \Lambda_{jk} \). These operators will be shown in Corollary 4.3 to be (global) sections of \( L^s_R \).

Definition 4.1. For \( j, k \in \{0, \ldots, s\} \)

\[
L_{jk} = \sum_{i=1}^{2} E_{ij} E_{ik} \quad L_{jk} = \sum_{i=1}^{2} E_{ij} I_{ik}
\]

\[
L_{\bar{k}j} = \Lambda_{jk} = \sum_{i=1}^{2} I_{ik} I_{ij} \quad L_{\bar{j}k} = \sum_{i=1}^{2} I_{ij} E_{ik}
\]

In accordance with the notation introduced in [G2] Section 7, we will use the shortcuts \( L_{\alpha\beta} \) with \( \alpha, \beta \in \{0, \ldots, s, \bar{0}, \ldots, \bar{s}\} \), with the convention that \( \bar{s} = \alpha \).

Notice that with the above notation \( L_{\alpha\alpha} = 0 \) for any \( \alpha \in \{0, \ldots, s, \bar{0}, \ldots, \bar{s}\} \).

Lemma 4.2. Given \( \alpha, \beta, \gamma \in \{0, \ldots, s, \bar{0}, \ldots, \bar{s}\} \) with \( \alpha \neq \beta, \alpha \neq \overline{\gamma}, \gamma \neq \overline{\beta} \):

\[
[L_{\alpha\beta}, L_{\overline{\gamma}\overline{\beta}}] = L_{\alpha\gamma}
\]

Given \( \alpha \neq \beta \in \{0, \ldots, s, \bar{0}, \ldots, \bar{s}\} \):

\[
[L_{\alpha\beta}, L_{\overline{\beta}\overline{\pi}}] = L_{\alpha\pi} + L_{\beta\overline{\pi}}
\]

Given \( \alpha, \beta, \gamma, \delta \in \{0, \ldots, s, \bar{0}, \ldots, \bar{s}\} \) with \( \{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset \):

\[
[L_{\alpha\beta}, L_{\gamma\delta}] = 0
\]

Proof. We prove the first relations with \( \alpha = i, \beta = j, \gamma = \bar{k} \) and the second ones with \( \alpha = i, \beta = j \). The other cases of the first and second are proved exactly with the same passages. The third set of relations is straightforward due to the anticommutativity of the degree one operators which appear in the expressions of \( L_{\alpha\beta}, L_{\gamma\delta} \).

For the first set of relations, a direct computation which is based on the fundamental relations 3.2 among the operators \( E_{ij} \) and \( I_{rs} \) proves:

\[
[L_{ij}, L_{\overline{k}\bar{k}}] = \sum_r E_{rj} \sum_s I_{sj} I_{sk} - \sum_s I_{sj} I_{sk} \sum_r E_{rj} E_{ij} =
\]

\[
= \sum_r E_{rj} I_{ij} \sum_s I_{sj} I_{sk} - \sum_{s \neq r} E_{rj} E_{rj} I_{sj} I_{sk} - \sum_{s = r} I_{sj} I_{sk} E_{sj} E_{s} =
\]
For the last assertion, it follows from the first one and the fact that (according to Lemma 4.4.1) among the generators of the fibre of $L_{ij} - j$ belongs to $\Gamma(X, Lef^*) \subset \Gamma(X, Lef^s \cap sL^*)$. Furthermore, for every $j = 0, 1, 2, \ldots, s$, the elements $L_{ij}$ belong to $\Gamma(X, Lef^*) \subset \Gamma(X, Lef^s \cap sL^*)$.

Proof. For any fixed $p \in X$, the values of the elements $L_{jk}$ and $L_{ik}$ at $p$ are (maybe up to a sign) among the generators of the fibre of $L_{ik} \cap sL^*$ at $p$. To show that $L_{ik}$ is a section of $L_{ik} \cap sL^*$ we notice that, since $s \geq 2$, we can find an index $i \in \{0, 1, 2, \ldots, s\}$ which is different from both $j$ and $k$. Then we can use the lemma above and construct $L_{ij}$ as:

$$[L_{ij}, L_{ik}] = L_{ik}$$

The element $L_{ik}$ is equal to $-L_{ik}$ and therefore also is a section of $L_{ik} \cap sL^*$. As for the last assertion, it follows from the first one and the fact that (according to the above lemma) $[L_{ij}, L_{ik}] = L_{ij - k}$ and $[L_{ik}, L_{ij}] = L_{ij} - L_{ij}$.

The operators defined below give rise to a set of Serre generators for $\Gamma(X, Lef^*)$, as shown in the following Theorem.

**Definition 4.4.** Let $\{e_1, e_2, \ldots, e_{s+1}\}$ be the adjoint of $e_i$. Moreover, for every $i = 1, 2, \ldots, s+1$, let $f_i$ be the adjoint of $e_i$.

**Theorem 4.5.** The global operators $e_i$, $f_j$ and $h_i = [e_i, f_j]$ restrict to a set of Serre generators of $\text{Lef}_p^*$ for any $p \in X$, and $\text{Lef}_p^*$ is (canonically) a trivial Lie algebra bundle with fibre isomorphic to $\text{so}(s + 1, s + 1, \mathbb{R})$. 
From the previous corollary, the global operators $e_i$, $f_j$ and $h_i = [e_i, f_i]$ are sections of $\text{Lef}^*$. It is immediate, using Lemma 4.2, to check that these elements are also enough to produce a set of linear generators of $\text{Lef}^*_p$. We are left with the verification of the Serre relations for a root system of type $\text{D}_{s+1}$. We consider a basis of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{s-2}, \alpha_{s-1}, \alpha_s, \alpha_{s+1}$ indexed according to the labeled Dynkin diagram in Figure 1, and think of the operator $e_i$ (resp. $f_i$) as a generator of the root space associated to $\alpha_i$ (resp. $-\alpha_i$).

$$
\begin{array}{cccccc}
& 1 & 2 & \ldots & \ldots & s \\
1 & & & & & \\
& & & & & \\
s & & & & & \\
& & & & & \\
& & & & & s+1 \\
\end{array}
$$

Figure 1. The Dynkin diagram of type $\text{D}_{s+1}$ with labels.

Then we have to verify that the following relations hold in $\text{L}^*_p$: 

1. $[h_1, h_i] = 0$
2. $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$
3. $(ad e_i)^{1-\alpha_i(h_i)} e_j = 0$ and $(ad f_i)^{1-\alpha_i(h_i)} f_j = 0$ for $i \neq j$.
4. $[e_i, f_j] = 0$ for $i \neq j$.
5. $[h_i, e_j] = \alpha_i(h_j) e_j$, $[h_i, f_j] = -\alpha_i(h_j) f_j$

From the relations above it follows that the $h_1, \ldots, h_{s+1}$ span a Cartan subalgebra of the real Lie algebra generated by the $e_i, f_j$, with real eigenvalues. This proves that the algebra is the split real form $\text{so}(s+1, s+1, \mathbb{R})$ of $\text{so}(2s+2, \mathbb{C})$.

Concerning the proof of these relations, they are actually all consequence of Lemma 4.2. Relations of type (2), for instance, are all verified using the same computation, which we show in the example of $[h_1, e_1] = 2e_1$

This follows from the observation that $h_1 = [L_{1,\overline{1}}, L_{0,\overline{1}}] = L_{1,\overline{1}} - L_{0,\overline{1}}$ and then

$$
[h_1, e_1] = [L_{1,\overline{1}} - L_{0,\overline{1}}, L_{1,\overline{1}}] = L_{1,\overline{1}} - L_{0,\overline{1}} = 2e_1
$$

Among the last relations to be verified we show as final examples:

$$
[h_{s+1}, e_s] = [L_{s-1, s-1}, L_{s-1, s-1}] =
$$

\[-L_{s-1, s-1} + L_{s-1, s-1}] = L_{s-1} - L_{s-1} = 0

and, again by Lemma 4.2,

$$
[h_{s+1}, e_{s-1}] = -[L_{s-1, s-1}, L_{s-1, s-1}] = -[L_{s-1, s-1}, L_{s-1, s-1}] = 0 =
$$

\[-L_{s-1, s-1} = -e_{s-1}\]

Remark 4.6. We notice that Theorem 4.5 is in accordance with [GG1] and [GG2], where the specialization of these computations to the case of WSD manifolds of rank two and three led us to the description of a natural subalgebra isomorphic to $\text{sl}(4, \mathbb{R}) \cong \text{so}(2, 2, \mathbb{R})$. 

□
An alternative interpretation of the relations in Lemma 4.2 and of the appearance of $D_{s+1}$ is through the use of two different Clifford Algebras, which will play a prominent role in the rest of this paper. For the first one, generalizing to arbitrary $s$ the $s = 2$ case considered in \[GG1\], we define:

**Definition 4.7.** For $p \in X$, the Clifford algebra $\mathcal{C}_p$ is

$$\mathcal{C}_p = Cl(T_pX \oplus T_p^*X, q)$$

with the quadratic form $q$ induced by the metric

$$\forall i, j, h, k <v_{ij}, v_{hk}> = 0$$
$$\forall i, j, h, k <\frac{\partial}{\partial v_{ij}}, \frac{\partial}{\partial v_{hk}}> = 0$$
$$\forall (i, j) \neq (h, k) <v_{ij}, \frac{\partial}{\partial v_{ij}}> = 0$$
$$\forall i, j <v_{ij}, \frac{\partial}{\partial v_{ij}}> = -\frac{1}{2}$$

**Remark 4.8.** The Clifford algebras $\mathcal{C}_p$ for varying $p$ define a Clifford bundle $\mathcal{C}$ on $X$, as the definition of $\mathcal{C}_p$ is independent on the choice of a basis. Indeed, the quadratic form used to define it is simply induced by $-\frac{1}{2}$ times the natural bilinear pairing $T_pX \otimes T_p^*X \rightarrow \mathbb{R}$.

**Proposition 4.9.** The Clifford algebra $\mathcal{C}_p$ has a canonical representation $\rho_p$ on $\Lambda^2 T_p^*X$, induced by the wedge and contraction operators $E_{ij}$ and $I_{ij}$ via the map

$$\rho_p(v_{ij}) = E_{ij}, \quad \rho_p \left( \frac{\partial}{\partial v_{ij}} \right) = I_{ij}$$

**Proof** The Clifford relations

$$\phi \psi + \psi \phi = -2 < \phi, \psi >$$

are precisely the content of Proposition 3.2. \hfill $\square$

Abusing slightly the notation, we will identify $\mathcal{C}_p$ with its (faithful) image inside $End_{\mathbb{R}}(\Lambda^2 T_p^*X)$, and we will omit any reference to the map $\rho_p$ when it will not be necessary. Actually, as the representation above is a real analogue of the Spinor representation, it is easy to check that the map $\rho_p$ is an isomorphism of associative algebras. One then has:

**Definition 4.10.** The linear subspace $\mathcal{C}_2^2$ of $\mathcal{C}_p$ is the image of the natural map $\Lambda^2(T_pX \oplus T_p^*X) \rightarrow \mathcal{C}_p$. The linear subspace $\mathcal{C}_0^0$ of $\mathcal{C}_p$ is the subspace generated by $1$.

Recall (see for instance \[LM\]) that $\mathcal{C}_2^2$ is a Lie subalgebra of $\mathcal{C}_p$ (with the commutator bracket).

**Proposition 4.11.** The bundle of Lie algebras $\mathcal{L}_S$ is a sub-bundle of $\mathcal{C}^2$. Any local determination of the operator $J$ is a (local) section of $\mathcal{C}_p$.

**Proof** Let us fix $p \in X$. We consider the pointwise values of the operators $L_{\omega_2}$, the $V_j$ and the $A_j$; they all lie inside $\mathcal{C}_2^2 \oplus \mathcal{C}_0^0$ by Proposition 3.2 and by the fact that the forms $\omega_{ij}$ restrict to elements of $\Lambda^2 T_p^*X$. The space $<J>$ lies inside $\mathcal{C}_2^2 \oplus \mathcal{C}_0^0$ by Proposition 3.3. By definition the elements $\mathcal{C}_2^2$ are commutators, and therefore have trace zero in any representation, and hence also in the $\rho_p$. Moreover, again by inspection all the generators of the fibre in $p$ of $\mathcal{L}_S$ have trace zero once represented via $\rho_p$ (they are nilpotent), and therefore they must lie inside $\mathcal{C}_2^2$. Both pointwise determinations of operator $J$ are in the Lie algebra of the isometry group, and therefore they too have trace zero and hence sit inside $\mathcal{C}_2^2$. As $\mathcal{C}_2^2$ is closed under the commutator bracket of $\mathcal{C}_p$, and this commutator coincides with the composition bracket of operators, we have the conclusion. \hfill $\square$
Remark 4.12. For any \( p \in X \), the Clifford algebra \( \mathcal{C}_p \) is isomorphic to the standard Clifford Algebra \( \text{Cl}_{2s+2,2s+2} \), as the metric used to define it has signature \((2s + 2, 2s + 2)\). The previous proposition therefore shows that all the fibres of \( E^s \) are Lie subalgebras of \( \text{Cl}^2_{2s+2,2s+2} \cong \text{spin}^{2s+2,2s+2} \).

Remark 4.13. For any fixed \( p \in X \), giving degree \( 1 \) to the operators \( E_{ij} \) and degree \(-1\) to the operators \( I_{ij} \), we induce a \( \mathbb{Z}\)-degree on \( \mathcal{C}_p \). This degree coincides with the degree of the operators induced from the grading on the forms from \( \bigwedge^* T^*_p X \).

Similarly to Definition 4.7, for any \( p \in X \) one could define a Clifford Algebra

\[
\text{Cl}((\mathbb{R}^{s+1} \oplus (\mathbb{R}^{s+1})^*), q_{\text{nat}})
\]

where \( q_{\text{nat}} \) is the quadratic form induced by \((-\frac{1}{2}\) times) the natural paring and

\[
\mathbb{R}^{s+1} = <\bar{E}_0, ..., \bar{E}_s>, \quad (\mathbb{R}^{s+1})^* = <\bar{I}_0, ..., \bar{I}_s>
\]

One has also a natural representation on \( \bigwedge^* T^*_p X \) of the operators \([\bar{E}_j, \bar{E}_k], [\bar{E}_j, \bar{I}_k], [\bar{I}_j, \bar{I}_k]\) generating the degree two part of this Clifford Algebra, induced by the map which acts as follows:

\[
[\bar{E}_j, \bar{E}_k] \rightarrow 2L_{jk}\]
\[
[\bar{E}_j, \bar{I}_k] \rightarrow 2L_{j\bar{k}}\]
\[
[\bar{I}_j, \bar{I}_k] \rightarrow 2L_{\bar{j}\bar{k}}
\]

This gives directly the bundle \( Lef^* \) as a quotient of the \( \text{spin}^{s+1, s+1} \) Lie Algebra bundle of this Clifford bundle, proving again that its fibre is indeed \( \text{so}(s+1, s+1, \mathbb{R}) \).

5. Quadratic invariants and Hodge decomposition

Fixing \( p \in X \) and a determination \( J \) at \( p \), the complex structure \( J \) acts on all the Clifford algebra \( \mathcal{C}_p \) by adjunction with respect to the commutator bracket, and sends its quadratic part \( \mathcal{C}^2_p \) to itself from Proposition 4.11.

Definition 5.1. We call quadratic invariants the elements in \( \mathcal{C}^2_p \) which commute with \( J \). For varying \( p \), we obtain a bundle of quadratic invariants.

As usual, to decompose the representation \( \bigwedge^* T^* X \) with respect to the weight induced by \( J \), it is necessary to consider complexified forms (and algebras). The weight decomposition of the space \( T^*_p X \otimes \mathbb{C} \) is obtained introducing a new basis for each \( W_j \otimes \mathbb{C} = <v_{1j}, v_{2j}> \subset \mathbb{C}^2 \):

\[
w_j = \frac{1}{\sqrt{2}}(v_{1j} + tv_{2j}), \quad \bar{w}_j = \frac{1}{\sqrt{2}}(v_{1j} - tv_{2j})
\]

To describe explicitly the space of (complex) quadratic invariants in the Clifford algebra \( \mathcal{C}_p \), let us introduce the following notation, which gives a basis of eigenvectors for the (adjoint) action of \( J \):

Definition 5.2.

\[
E_{w_j} = \frac{1}{\sqrt{2}}(E_{1j} + tE_{2j}), \quad E_{\bar{w}_j} = \frac{1}{\sqrt{2}}(E_{1j} - tE_{2j})
\]
\[
I_{w_j} = \frac{1}{\sqrt{2}}(I_{1j} - tI_{2j}), \quad I_{\bar{w}_j} = \frac{1}{\sqrt{2}}(I_{1j} + tI_{2j})
\]

Lemma 5.3. The adjoint action of the complex structure operator \( J \) on \( E_{w_j}, I_{w_j}, E_{\bar{w}_j}, I_{\bar{w}_j} \) is:

\[
[J, E_{w_j}] = -iE_{w_j}, \quad [J, I_{w_j}] = iI_{w_j}, \quad [J, E_{\bar{w}_j}] = iE_{\bar{w}_j}, \quad [J, I_{\bar{w}_j}] = -iI_{\bar{w}_j}
\]
Proof It is enough to consider the corresponding \( J \)-weights of the \( w_i, \overline{w}_j \).

As \( \mathcal{L}^* \subset \mathbb{C}^2 \otimes \mathbb{C} \) from Proposition 4.11, in the following we show that \( \mathcal{L}^* \) lies inside the bundle of quadratic invariants. Immediately after we will give a basis for the space of quadratic invariants, thus providing a first upper bound for \( \mathcal{L}^* \) (which will be later shown to be off by only 1).

**Proposition 5.4.** The operator \( J \) commutes with all the elements in the fiber at \( p \) of \( \mathcal{L}^* \).

Proof We prove the statements by a direct computation. It is useful to rewrite \( \omega_{jk} \) (and hence \( L_{jk} \) which is wedge with \( \omega_{jk} \)) in terms of the basis generated by the \( w_j, \overline{w}_k \):

\[
\omega_{jk} = v_{1j} \wedge v_{1k} + v_{2j} \wedge v_{2k} = \frac{1}{2} (w_j \wedge \overline{w}_k + \overline{w}_j \wedge w_k)
\]

and therefore

\[
L_{jk} = \frac{1}{2} (\{E_{w_j}, E_{\overline{w}_k}\} - \{E_{\overline{w}_k}, E_{w_j}\})
\]

The vanishing \( [J, L_{jk}] = 0 \) then follows immediately from Lemma 5.3. Similarly, to show that \( [J, V_k] = 0 \) it is enough to observe that

\[
V_k = \frac{1}{2} \{E_{w_k}, E_{\overline{w}_k}\}
\]

The following proposition will show that, except for a toral part which will be discussed later, all the quadratic invariants of the Clifford bundle \( \mathcal{C} \) lie inside \( s\mathcal{L}^* \subset \mathcal{L}^* \). It will follow therefore that

\[
4(s+1)^2 - 2(s+1) \leq \dim_{\mathbb{R}} s\mathcal{L}^* \leq \dim_{\mathbb{C}} \mathcal{L}^*_s \leq 4(s+1)^2
\]

**Proposition 5.5.** The following \( 4(s+1)^2 \) operators are a linear basis for the quadratic \( J \)-invariants:

1. \( \{E_{w_i}, E_{\overline{w}_j}\} \) with \( i \neq j \).
2. \( \{I_{w_i}, I_{\overline{w}_j}\} \) with \( i \neq j \).
3. \( \{E_{w_i}, E_{\overline{w}_j}\} \) where \( i = 0, 1, \ldots, s \).
4. \( \{I_{w_i}, I_{\overline{w}_j}\} \) where \( i = 0, 1, \ldots, s \).
5. \( \{E_{w_i}, I_{w_j}\} \) with \( i \neq j \).
6. \( \{I_{w_i}, I_{\overline{w}_j}\} \) with \( i \neq j \).
7. \( \{E_{w_i}, I_{\overline{w}_j}\} \) where \( i = 0, 1, \ldots, s \).
8. \( \{I_{w_i}, I_{\overline{w}_j}\} \) where \( i = 0, 1, \ldots, s \).

The \( 4(s+1)^2 - 2(s+1) \) operators of type \( (1), (2), (3), (4), (5), (6) \) belong to the bundle of real algebras \( s\mathcal{L}^* \subset \mathcal{L}^* \).

Proof In this proof, we fix \( p \in \mathbb{X} \) and all the bundles and operators will be considered at this point. The \( J \)-weight of a bracket of \( J \)-homogeneous operators is the sum of the respective weights. The quadratic "monomials" (with respect to the bracket) in the \( E_{w_i}, I_{w_i}, E_{\overline{w}_j}, I_{\overline{w}_j} \) are all \( J \)-homogeneous, and therefore to find a basis of \( J \)-invariant quadratic operators it is enough to identify the \( J \)-invariant quadratic monomials. To be \( J \)-invariant means simply to have weight zero, and the computation of the \( J \)-weight of the quadratic monomials follows immediately from those of \( E_{w_i}, I_{w_i}, E_{\overline{w}_j}, I_{\overline{w}_j} \), which are respectively \( -i, i, i, -i \).

It remains to be shown that the monomials of type \( (1), (2), (3), (4), (5), (6) \) belong to \( s\mathcal{L}^* \). Let us consider \( \{E_{w_i}, E_{\overline{w}_j}\} \) with \( i \neq j \). Since \( E_{w_i} \) and \( E_{\overline{w}_j} \) anticommute, this is equal to \( 2E_{w_i}E_{\overline{w}_j} \). Then

\[
2E_{w_i}E_{\overline{w}_j} = (E_{1i} + \imath E_{2i})(E_{1j} - \imath E_{2j}) = E_{1i}E_{1j} + E_{2i}E_{2j} + \imath(E_{2i}E_{1j} - E_{1i}E_{2j}) =
\]
We have therefore to show that \(\eta(E_{21}E_{ij} - E_{1i}E_{2j})\) belongs to \(sL^n\).

We recall that, by Corollary 4.3, the elements \(L_{ij}\) belong to \(sL^n\) and notice that

\[
[L_{ij}, \eta V_j] = \eta(E_{1i}E_{1j} + E_{2i}E_{2j})E_{ij}E_{2j} - \eta E_{ij}E_{2j}(E_{1i}E_{1j} + E_{2i}E_{2j}) = \\
= \eta(E_{1i}E_{2j} - E_{2i}E_{2j})E_{ij}E_{2j} - E_{1i}E_{1j}E_{2j}E_{2j} - E_{1i}E_{1j}E_{1j}E_{2j}E_{2j}E_{2j}E_{2j}E_{2j}) = \\
= \eta(E_{1i}E_{2j} - E_{2i}E_{ij})
\]

which concludes the proof that the monomial \([E_{w_i}, E_{w_j}]\) lies in \(sL^n\). By adjunction, we immediately have that also \([I_{w_i}, I_{w_j}]\) lies in \(sL^n\).

Also the monomials \([E_{w_i}, E_{w_j}]\) belong to \(sL^n\); in fact they are imaginary multiples of the volume forms:

\[
2[E_{w_i}, E_{w_j}] = (E_{1i} + \eta E_{2i})(E_{1j} - \eta E_{2j}) - (E_{1i} - \eta E_{2i})(E_{1j} + \eta E_{2j}) = \\
= 2(E_{2i}E_{1j} - E_{1i}E_{2j}) = -4\alpha_i.
\]

As a consequence, we have, by adjunction, \(2[I_{w_i}, I_{w_j}] = -4\alpha_i\).

Let us consider the monomials of type (5) \([E_{w_i}, I_{w_j}]\) with \(i \neq j\). Since \(E_{w_i}\) and \(I_{w_j}\) anticommute, this is equal to \(2E_{w_i}I_{w_j}\). Then

\[
2E_{w_i}I_{w_j} = (E_{1i} + \eta E_{2i})(E_{1j} - I_{2j}) = E_{1i}E_{1j} + E_{2i}E_{2j} + \eta(E_{2i}E_{1j} - E_{1i}E_{2j}) = \\
= L_{ij} + \eta(E_{2i}E_{1j} - E_{1i}E_{2j})
\]

We have therefore to show that \(\eta(E_{2i}E_{1j} - E_{1i}E_{2j})\) belongs to \(sL^n\).

We notice that

\[
[L_{ij}, \eta V_j] = \eta(I_{1i}I_{1j} + I_{2i}I_{2j})E_{ij}E_{2j} - \eta E_{ij}E_{2j}(I_{1i}I_{1j} + I_{2i}I_{2j}) = \\
= \eta(I_{1i}E_{2j} - I_{1i}E_{2i}E_{2j} - I_{2i}E_{1j}E_{2j}I_{2j} - E_{1i}E_{2j}E_{1j}I_{2j} - E_{1i}E_{2j}E_{2j}E_{2j}E_{2j}E_{2j}E_{2j}E_{2j}E_{2j}) = \\
= \eta(I_{1i}E_{2j} - I_{2i}E_{1j})
\]

which by adjunction gives that also

\[
[L_{ij}, \eta A_j] = \eta(I_{1i}E_{2j} - E_{2i}I_{1j})
\]

This allows us to conclude that the monomial \([E_{w_i}, I_{w_j}]\) (as of course its conjugate \([E_{w_i}, I_{w_j}]\)) lies in \(sL^n\).

\[\Box\]

6. All the fibres of the bundle \(L^n\) are isomorphic to \(sl(2s+2, \mathbb{C})\).

In this section, we fix once and for all a determination of \(J\) at the point \(p\) and consider the Hodge decomposition of \(\Lambda^*_CT^*_pX\) with respect to the (almost) complex structure \(J\). We will use this information to first study the complex algebra \(L^n\), while in the next sections we will concentrate on its reals forms. In the rank 3 case this corresponds to performing the pluythysm with respect to the action of \(SO(3, \mathbb{R})\) (see [GG2], where we analyze the case \(s = 2\) in the context of WSD manifolds). The Hodge (type) decomposition of forms on \(X\) with respect to the complex structure \(J\)

\[
\Lambda^k CT^*_pX = \bigoplus_{r+t=k} \Lambda^r CT^*_pX
\]

can be described as usual explicitly as follows, using the \(J\)-homogeneous basis \(w_{ij}, \overline{w}_{ij}\):

\[
r \in \Lambda^r CT^*_pX = \langle w_{i_1} \wedge \cdots \wedge w_{i_r} \wedge \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_t} | i_1, \ldots, j_t \in \{0, 1, 2, \ldots, s\} \rangle_{\mathbb{C}}
\]

**Definition 6.1.** At a given point \(p \in X\), and with the chosen a determination of \(J\) at \(p\), we indicate with \(I_{\alpha}\) the subspace (isotypical component) of forms of \(J\)-weight \(\alpha (-s-1 \leq \alpha \leq s+1)\).
Here is for instance the Hodge "diamond" in the case $s = 4$ (we used the following notation: the symbol $\Omega^r_\tau$ indicates the space $\Lambda^{\tau} T^*_p X_p = \Omega^{\tau}_r X_p$ and specifies its dimension $m$). The Lie algebra bundle $L^r_\tau$ acts preserving the weight of forms, and therefore the spaces $I \alpha$ which are the columns in the Hodge diamond.

|       | $I_{-5}$ | $I_{-4}$ | $I_{-3}$ | $I_{-2}$ | $I_{-1}$ | $I_0$ | $I_1$ | $I_2$ | $I_3$ | $I_4$ | $I_5$ |
|-------|----------|----------|----------|----------|----------|-------|-------|-------|-------|-------|-------|
| $\Omega^r_{2,p}$ |          |          |          |          |          | $\Omega^0_1$ | $\Omega^0_1$ |          |          |          |          |
| $\Omega^r_{3,p}$ |          |          |          |          |          |          | $\Omega^1_2$ | $\Omega^0_1$ |          |          |          |
| $\Omega^r_{4,p}$ |          |          |          |          |          |          |          | $\Omega^2_3$ | $\Omega^1_2$ |          |          |
| $\Omega^r_{5,p}$ |          |          |          |          |          |          |          |          | $\Omega^3_4$ | $\Omega^2_3$ |          |
| $\Omega^r_{6,p}$ |          |          |          |          |          |          |          |          |          | $\Omega^4_4$ | $\Omega^3_4$ |
| $\Omega^{10}_p$ |          |          |          |          |          |          |          |          |          |          | $\Omega^5_5$ |

| Dim. | 1   | 10  | $\left(\frac{10}{2}\right)$ | $\left(\frac{10}{4}\right)$ | $\left(\frac{10}{5}\right)$ | $\left(\frac{10}{5}\right)$ | 10  | 1   |

Table 1. Hodge diamond (case $s = 4$)

**Theorem 6.2.** Let $X$ be a (almost, pointwise) $s$-Kähler manifold or rank two.

a) The Lie algebra bundle $L^r_\tau$ has fibre isomorphic to $\text{sl}(2s + 2, \mathbb{C})$.

b) At a given point $p \in X$, the direct sum of $L^r_{\tau,p}$ with the space spanned by the operator $J_p$ is the set of all quadratic invariants of $\mathcal{L}_p$.

c) At a given point $p \in X$, the restriction of $L^r_\tau$ to the $2s + 2$ dimensional space $I_{-s}$ of forms of $J$-weight $-s$ is faithful.

**Proof** We work at a fixed point $p$. The isotypical component $I_{-s}$ has dimension $2s + 2$ and has a basis $\{b_i\}$ ($0 \leq i \leq 2s + 2$) given by the following monomials:

- $b_i = w_0 \wedge \ldots \wedge \hat{w}_i \wedge \ldots \wedge w_s$, for $i \in \{0, 1, 2, \ldots, s\}$, where $\hat{w}_i$ means that $w_i$ is omitted and therefore the monomial has degree $s$.
- $b_{s+1+i} = w_0 \wedge \ldots \wedge w_s \wedge \overline{w}_i$, where $i \in \{0, 1, 2, \ldots, s\}$ and the monomial has degree $s + 2$. 
It is then immediate to check that, for instance,
\[
[E_{w_0}, I_{w_1}](b_0) = [E_{w_0}, I_{w_1}](w_1 \wedge \cdots \wedge w_s) = 2w_0 \wedge \widetilde{w_1} \wedge \cdots \wedge w_s = 2b_1
\]
\[
[E_{w_0}, E_{w_1}](b_s) = [E_{w_0}, E_{w_1}](w_0 \wedge \cdots \wedge w_{s-1}) = 2w_0 \cdots \wedge w_s \wedge \widetilde{w_0} = 2b_{s+1}
\]
\[
[E_{\overline{w}_s}, I_{w_1}](b_1) = [E_{\overline{w}_s}, I_{w_1}](w_0 \wedge \cdots \wedge w_s \wedge \overline{w_0}) = 2w_0 \cdots \wedge w_s \wedge \overline{w_1} = 2b_{s+2}
\]

Completely analogous computations show that, when we represent the action of \( L_{\mathbb{C},p} \) on the isotypical component \( I_{-s} \) using the above mentioned basis, all the elementary matrices \( e_{ij} \) (where \( i \neq j \) and \( e_{ij} \) is the matrix with all the entries equal to 0 except for the entry \((i,j)\) which is 1) are obtained using the quadratic invariants of type (1), (2), (3), (4), (5), (6) which in Proposition 5.5 were shown to lie in \( L_{\mathbb{C},p} \).

More precisely, we have the following identifications for the “positive” set of Serre generators \( e_{j+1,j} \):
\[
\begin{align*}
& e_{j+2,j+1} = \frac{1}{2}[E_{w_j}, I_{w_{j+1}}] \text{ for } 0 \leq j \leq s - 1; \\
& e_{s+2,s+1} = \frac{1}{2}[E_{w_s}, E_{w_0}]; \\
& e_{s+4+j,s+2+j} = \frac{1}{2}[E_{\overline{w}_{j+1}}, I_{w_j}] \text{ for } 0 \leq j \leq s - 1.
\end{align*}
\]

Therefore \( L_{\mathbb{C},p} \) acts as \( \mathfrak{sl}(2s + 2, \mathbb{C}) \) on \( I_{-s} \) (notice that, as the generators \( L_{ij}, \Lambda_{ij} = L_{j,i}, V_i, A_i \) of \( L_{\mathbb{C},p} \) are nilpotent, they still have trace zero when restricted to \( I_{-s} \)).

Summing up, the algebra \( L_{\mathbb{C},p} \) has a quotient isomorphic to the simple algebra \( \mathfrak{sl}(2s + 2, \mathbb{C}) \) and is embedded in the \( 4(s + 1)^2 \)-dimensional space of the quadratic invariants; now, since the quadratic invariant \( J_p \) doesn’t belong to \( L_{\mathbb{C},p} \) (in fact the trace of its restriction to \( I_{-s} \) is different from 0, since \( J_p \) acts on \( I_{-s} \) as multiplication by \(-i\)), we conclude that \( L_{\mathbb{C}} \) has dimension \( 4(s + 1)^2 - 1 \). Therefore the restriction to \( I_{-s} \) provides us with an isomorphism of \( L_{\mathbb{C},p} \) with \( \mathfrak{sl}(2s + 2, \mathbb{C}) \).

The decomposition \( T^s X = W_0 \oplus W_1 \oplus \cdots \oplus W_s \) induces naturally a multi-degree on \( \bigwedge T^s \) with values in \( \mathbb{Z}^{s+1} \), which we indicate with \( mdeg \). This follows from the equation
\[
\bigwedge T^s X \cong \bigoplus_{p_0 + p_1 + \cdots + p_s = n} \bigwedge (W_0 \otimes \mathbb{C}) \oplus \bigwedge (W_1 \otimes \mathbb{C}) \oplus \cdots \oplus \bigwedge (W_s \otimes \mathbb{C})
\]

We notice furthermore that the (complexified) decomposition is preserved by the operator \( J \), and therefore \( mdeg \) commutes with the action of \( \mathfrak{so}(2, \mathbb{R}) \). We will still call by \( mdeg \) the multidegree induced on the bundle \( L_{\mathbb{C}} \) by the previous one.

**Corollary 6.3.** For any fixed point \( p \in X \), the space of quadratic invariants with \( mdeg \) equal to \((0,0,0,\ldots,0)\) coincides with the \( 2s + 2 \)-dimensional space spanned by the quadratic monomials \([E_{w_i}, I_{w_i}]\) and by their complex conjugates \([E_{\overline{w}_i}, I_{\overline{w}_i}]\). This space can be expressed as the direct sum of a maximal toral subalgebra of \( L_{\mathbb{C},p} \cong \mathfrak{sl}(2s + 2, \mathbb{C}) \) plus the one dimensional subspace \(< J_p >\).

**Proof** The first assertion is trivial (by inspection of the basis of the quadratic invariants described in Proposition 5.5). The restriction of \( L_{\mathbb{C},p} \) to \( I_{-s} \) is an isomorphism by the previous theorem. Therefore, the diagonal matrices with trace zero (in the same basis \( \{b_i\} \) used in the proof of the theorem) provide a toral subalgebra of \( L_{\mathbb{C},p} \) formed by operators with \( mdeg \) equal to zero as they can be obtained as brackets of operators with opposite \( mdeg \). As the above mentioned basis is made up of \( mdeg \)-homogeneous elements, all the quadratic invariants with vanishing \( mdeg \) must be associated to diagonal matrices. Summing up, the quadratic invariants with vanishing \( mdeg \) are the toral elements of \( L_{\mathbb{C},p} \) plus \( J_p \) (which
has certainly vanishing \(m\text{deg}\), since it admits a global basis of \(m\text{deg}\)-homogeneous eigenvectors).

\[\square\]

7. The real forms of \(\mathcal{L}_C^s\)

In the previous section we described completely the complex bundle of Lie algebras \(\mathcal{L}_C^s\) (see Theorem 6.2). Here, we will give a complete description or the geometrically natural bundles of real Lie algebras \(s\mathcal{L}^s\) \(u\mathcal{L}^s\) and \(\mathcal{L}_R^s\).

Recall that the bundle of Lie algebras \(\mathcal{L}_C^s\) is generated by the Lefschetz operators \(L_{ij}\) and by the \(\mathcal{I}V_j\), \(\mathcal{A}_k\).

**Theorem 7.1.** The bundle of real Lie algebras \(\mathcal{L}_C^s\) has fibre isomorphic to the Lie algebra \(\text{sl}(2s+2,\mathbb{R})\).

**Proof** We identify the fibre of \(\mathcal{L}_C^s\) at a point \(p\) with \(\text{sl}(2s+2,\mathbb{C})\) using the faithful representation \(\mathcal{I}^{-s}\) and the basis \(\{b_i\}\) of \(\mathcal{I}^{-s}\) provided in the proof of Theorem 6.2.

As we already noticed in that proof, in this basis all the invariant monomials of type (1), (2), (3), (4), (5), (6) act via real matrices, and provide (up to a scalar) all the elementary matrices \(e_{ij}\) \((i \neq j)\). Therefore they generate over the real numbers the subalgebra \(\text{sl}(2s+2,\mathbb{R})\) of \(\text{sl}(2s+2,\mathbb{C})\). If we prove that the whole fibre at \(p\) of \(\mathcal{L}_C^s\) lies in \(\text{sl}(2s+2,\mathbb{R})\), then, since \(\mathcal{L}_C^s\) is a real form of \(\mathcal{L}_C^s\), we must have \(\mathcal{L}_C^s = \text{sl}(2s+2,\mathbb{R})\).

It suffices to notice that the generators \(L_{ij}\) and \(\mathcal{I}V_j\) (and therefore their adjoints) are in the real algebra generated by the above mentioned invariant monomials. Now, in the proof of Theorem 6.2, \(\mathcal{I}V_j\) was obtained as \(-\frac{1}{2}[E_{w_i},E_{\bar{w}_j}]\) and in Proposition 5.4 we showed that

\[\mathcal{I}V_j = 2L_{ij}\]

\[\square\]

On the complex bundle of vector spaces \(\bigwedge^* T^*X\) there is a natural hermitean inner product \(\langle\ ,\ \rangle\), obtained from the wedge operation on forms (cf. [GG2] where we used a superHermitean variant of this product for the rank 3 case), and defined below. Associated to this pairing, there is a natural notion of antihermitean operator. We will prove that the set of antihermitean operators inside \(\mathcal{L}_C^s\) is a real form for \(\mathcal{L}_C^s\), generated by operators naturally derived from the geometry and coinciding with \(u\mathcal{L}^s\).

**Definition 7.2.** For every \(p \in X\) there is a natural non degenerate Hermitean inner product \(\langle\ ,\ \rangle_p\) on \(\bigwedge^* T^*_p X\), defined using the natural (standard) Hermitean inner product \(\langle\ ,\ \rangle\) associated to the metric \(g\) and the (pointwise) volume form \(\Omega\) associated to the metric \(g\) and to the chosen determination of \(J\) at \(p\):

\[\langle\alpha,\beta\rangle_p = i^{\text{deg}(\alpha)\text{deg}(\beta)} (\alpha \wedge \bar{\beta}, \Omega)_p\]

We indicate with \(\langle\ ,\ \rangle\) the corresponding form with values in smooth functions.

Let us denote with \(\tau\) the (complex linear) operator obtained composing conjugation with the Hodge star associated to the metric.

**Proposition 7.3.** For every \(p \in X\), the pairing \(\langle\ ,\ \rangle_p\) satisfies the following properties:

\(a)\) \(\langle\alpha,\beta\rangle_p = i^{\text{deg}(\alpha)+\text{deg}(\beta)} (\alpha, \tau\beta)_p\)
b) \(<\alpha,\beta>_p\) is preserved by the operator \(J\) in derived sense, namely
\[
\forall \alpha, \beta \in \mathfrak{X}_p \quad <J\alpha, \beta>_p + <\alpha, J\beta>_p = 0
\]

c) \(<\alpha, \beta>_p\) is preserved by the operator \(\varpi\), namely
\[
\forall \alpha, \beta \in \mathfrak{X}_p \quad <\varpi\alpha, \varpi\beta>_p = <\alpha, \beta>_p
\]

d) The pure weight components \(\mathfrak{I}_k\) are mutually \(<\alpha, \beta>_p\)-orthogonal and \(<\alpha, \beta>_p\) is nondegenerate when restricted to any one of them.

Proof. The first three facts are standard. For the orthogonality statement in part d), we observe that, if \(\alpha \in \mathfrak{I}_k\) and \(\beta \in \mathfrak{I}_k\) with \(\text{deg} \alpha + \text{deg} \beta = \dim(X)\) then
\[
<\alpha, \beta>_p \Omega = (<\alpha \wedge \beta>_p)_{\mathfrak{I}_k}
\]
is a complex number times a form of \(J\)-weight zero, but from the right hand side it also has to have \(J\)-weight equal to \((h - k)\). Therefore if \(h \neq k\), it must be zero. Restricting to a single \(\mathfrak{I}_k\), notice that \(\varpi\) sends this component to itself (as it commutes with \(J\)), and then if \(\alpha \neq 0 \in \mathfrak{I}_k\), \(<\alpha, \varpi\alpha>_p\) is a power of \(i\) times \((\alpha, \alpha)_p\) by point \(a\), and is therefore nonzero. \(\square\)

We want now to characterize the operators inside \(\mathfrak{L}_p^*\), which preserve the form \(<\alpha, \beta>_p\). First we observe that, since the dimension of \(T^*_pX\) is even, \(\varpi\) is equal to the identity on the forms of even degree while \(\varpi = -J\) when restricted to the odd forms. Then, for fixed \(p \in X\), using the expression \(<\alpha, \beta>_p = (<\alpha, \beta>_p)_p\), we see that the “differential” condition for preservation of the form by the operator \(\phi\)
\[
\forall \alpha, \beta \in \mathfrak{X}_p \quad <\phi(\alpha), \beta>_p + <\alpha, \phi(\beta)>_p = 0
\]
is equivalent to \(\phi^* = -\varpi \phi \varpi\) on the even forms and to \(\phi^* = \varpi \phi \varpi\) on the odd forms.

The next two theorems show that the bundle \(\mathfrak{uL}_p\) (generated at any point by the value of the operators \(iL_{j,k}\) \((j \neq k)\), \(iV_i\) and their adjoints, see Definition 3.10) is precisely the bundle of Lie subalgebras given point by point by the operators which preserve the form \(<\alpha, \beta>_>\):

Theorem 7.4. The Lie algebra bundle \(\mathfrak{uL}_p\) preserves the form \(<\alpha, \beta>_>\).

Proof. As we observed before, the statement is equivalent to the fact that the condition \(\phi^*(\alpha) = (-1)^{\text{deg} \alpha + 1} \varpi \phi(\alpha)\) holds for all the sections \(\phi\) of \(\mathfrak{uL}_p^*\) and all the homogeneous elements \(\alpha \in \bigwedge^*_c T^*_pX\).

It is enough to check these equations for the generators of \(\mathfrak{uL}_p^*\), at a fixed point \(p \in X\), which as usual we omit from the notation for the operators when not strictly necessary.

Let \(\Psi\) be one of the generators \(iL_{j,k}\) or \(iV_i\); this means that \(\Psi\) is the operator given by the wedge with an even form \(\psi\), where \(\psi\) is real. One has, given \(\psi\) homogeneous of degree \(h\) and \(w\) in \(\bigwedge^*_c T^*_pX\) with degree of the same parity (which is the only possibly non-vanishing case):
\[
(\Psi(v), w)_p = (\psi \wedge v \wedge \varpi w, \Omega)_p = -(v \wedge \varpi \psi \wedge \varpi w, \Omega)_p = -(-1)^{h}(v \wedge \varpi (\varpi \psi \varpi))(w, \Omega)_p
\]

On the other hand,
\[
(\Psi(v), w)_p = (v, \Psi^*(w))_p = (v \wedge \varpi \Psi^*(w), \Omega)_p
\]

This implies
\[
\varpi \Psi^* = -(-1)^h \varpi (\varpi \psi \varpi)
\]

which is equivalent to
\[
\Psi^* = -(-1)^h \varpi \psi \varpi
\]
that is the relation we wanted to check. The adjoint of this equation immediately
proves the relation also for the generators \( iL_i, j \neq k \) and \( iA_i \). \( \square \)

**Theorem 7.5.** The Lie algebra bundle \( \mathfrak{uL}^s \) is the full real Lie subalgebra bundle
of \( \mathcal{L}^s_C \) of operators which preserve the form \( < , > \), and its fibre is isomorphic to
\( \mathfrak{su}(s+1, s+1) \).

**Proof** As usual, let us fix once and for all a point \( p \in X \), which will be omitted
from the notation when not strictly necessary.

In view of part c) of Theorem 6.2 we have to compute the signature of the form \( < , > \),
when restricted to \( \mathcal{I} \). It is convenient to use a basis \( \{ c_r \} \) of \( \mathcal{I} \),
which differs from the basis \( \{ b_r \} \) provided in the proof of Theorem 6.2 only for some signs.
Namely, \( \{ c_r \} \) \((0 \leq r \leq 2s+2)\) is given by the following monomials:

- \( c_r = b_r = w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \) for \( r \in \{ 0, 1, 2, \ldots, s \} \);
- \( c_{s+1+r} = w_0 \wedge \ldots \wedge w_r \wedge \hat{w}_r \wedge \ldots \wedge w_s, \) for \( r \in \{ 0, 1, 2, \ldots, s \} \).

By construction, for every \( r < j, c_r, c_j >_p = 0 \) unless \( j = s + 1 + r \) and in this case we have that

\[
< c_r, c_{s+1+r} >_p = \epsilon^{s+2}(w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \Omega)_p = \\
\epsilon^s w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \wedge \Omega)_p = \\
\epsilon^s w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \wedge \Omega)_p = \\
\epsilon^s \cdot (1)^{s+1}(w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \wedge \Omega)_p = \\
\epsilon^s \cdot (1)^{s+1}(1)^{s+1}(w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \wedge \Omega)_p = \\
\epsilon^s \cdot (1)^{s+1}(1)^{s+1}(1)^{s+1}(w_0 \wedge \ldots \wedge \hat{w}_r \wedge \ldots \wedge w_s, \wedge \Omega)_p = \\
\epsilon^s \cdot (1)^{s+1}(1)^{s+1}(1)^{s+1}(1)^{s+1}= \\
\epsilon^s \cdot (1)^{s+1}(1)^{s+1}(1)^{s+1}(1)^{s+1}
\]

Thus we notice that \( c_r, c_{s+1+r} >_p \) does not depend on the index \( r \), being equal
to \( \epsilon \) when \( s \) is even and to \( -\epsilon \) when \( s \) is odd. It follows that for every \( s \) the signature of \( < , >_p \) is \((s+1, s+1)\).

From Theorem 7.4 and the above remark on the signature one deduces that the fibre of \( \mathfrak{uL}^s \) at \( p \) can be identified with a subalgebra
of \( \mathfrak{su}(s+1, s+1) \subseteq \mathfrak{sl}(2s+2, \mathbb{C}) \cong \mathcal{L}^s_{C,p} \). Since, by construction, \( \mathfrak{uL}^s \) is a real form
of \( \mathcal{L}^s_C \), we can replace \( \subseteq \) with \( = \) in the inclusion above and the claim follows. \( \square \)

This theorem also provides us the key ingredient to understand the structure of
the bundle \( \mathcal{L}^s_C \), which is the most natural real Lie algebra bundle associated to the
(almost, pointwise) \( s \)-Kähler structure. Indeed, it is generated at every point
by the values of the operators \( L_{ij} \), \( V_k \) and their pointwise adjoints. The main tool will be a new hermitean inner product \( < , > \) defined starting from \( < , > \) on
the complex bundle of vector spaces \( \Lambda^C \wedge T^*X \) which we now introduce:

**Definition 7.6.** For every \( p \in X \) there is a natural non degenerate Hermitian
inner product \( < , >_p \) on \( \Lambda^C_t \wedge T^*X \), defined, on homogeneous elements \( \alpha, \beta \), as:

\[
< \alpha, \beta >_p = \epsilon^{s+1+\deg \beta} \alpha, \beta >_p
\]

We indicate with \( < , > \) the corresponding form with values in smooth functions.

**Theorem 7.7.** The Lie algebra bundle \( \mathcal{L}^s_C \) is the full real Lie subalgebra bundle
of \( \mathcal{L}^s_C \) of operators which preserve the form \( < , > \), and its fibre is isomorphic to
\( \mathfrak{su}(s+1, s+1) \).

**Proof** As usual, let us fix once and for all a point \( p \in X \). From Proposition 7.3 we
immediately deduce that the weight components \( \mathcal{I}_k \) are mutually \( < , >_p \)-orthogonal and \( < , >_p \) is nondegenerate when restricted to any one of them.
Therefore, for the first part of the claim it suffices to show that the generators of $\mathcal{L}^s_R$ preserve $\langle\langle \cdot, \cdot \rangle\rangle$, then a dimensional argument can be applied (since $\mathcal{L}^s_R$ is a proper real form of $\mathcal{L}^s_C$).

Let $\Gamma$ be any one of the generators $L_{jk}, V_j, A_{jk}, A_j$ of $\mathcal{L}^s_R$. Then $\alpha \Gamma$ is a generator of $\mathfrak{u} \mathfrak{c} \mathfrak{c}$ and, by Theorem 7.5, given two homogeneous elements $\alpha, \beta \in \Lambda^*_C T^*_p X$, it satisfies:

$$\langle\langle \alpha \Gamma(\alpha), \beta \rangle\rangle_p + \langle\langle \alpha, \alpha \Gamma(\beta) \rangle\rangle_p = 0$$

Therefore,

$$\langle\langle \alpha \Gamma(\alpha), \beta \rangle\rangle_p + \langle\langle \alpha, \alpha \Gamma(\beta) \rangle\rangle_p = i^{s+1+\deg \beta} \langle\langle \alpha \Gamma(\alpha), \beta \rangle\rangle_p + i^{s+1+\deg \beta+\deg \Gamma} \langle\langle \alpha, \alpha \Gamma(\beta) \rangle\rangle_p = i^{s+\deg \beta} \langle\langle \alpha \Gamma(\alpha), \beta \rangle\rangle_p + i^{s+\deg \beta} \langle\langle \alpha, \alpha \Gamma(\beta) \rangle\rangle_p = 0$$

since $\deg \Gamma$ is equal to 2 or $-2$.

For the second part of the claim, it suffices to compute the signature of the form $\langle\langle \cdot, \cdot \rangle\rangle$ when restricted to $\mathcal{I}_{-s}$. Using the basis $\{c_i\}$ of $\mathcal{I}_{-s}$ introduced in the proof of Theorem 7.5 we have:

$$\langle\langle c_r, c_{s+1+r} \rangle\rangle_p = i^{2s+3} \langle\langle c_r, c_{s+1+r} \rangle\rangle_p = i^{2s^2+2s+4} = i^{2(s+1)} = 1$$

which shows that the total signature is $(s+1, s+1)$. □

We want now to characterize explicitly the matrices of $\mathcal{L}^s_{\mathbb{R}, p}$ with respect to the natural basis $\{c_i\}$ of $\mathcal{I}_{-s}$ defined in the proof of Theorem 7.5. Notice that the basis is not real, but we will show that the matrices are nevertheless in the standard form for $\mathfrak{su}(s+1, s+1)$.

**Proposition 7.8.** With respect to the basis $\{c_i\}$ of $\mathcal{I}_{-s}$, the algebra $\mathcal{L}^s_{\mathbb{R}, p}$ is faithfully presented as the algebra of matrices

$$\begin{pmatrix} D & H_2 \\ H_1 & -D \end{pmatrix}$$

with $D$ an $(s+1) \times (s+1)$ complex matrix and $H_1, H_2$ two $(s+1) \times (s+1)$ complex antihermitean matrices.

**Proof.** We start by noticing that the operators of degree zero $L_{jk}^\pi$ ($j \neq k$) (which lie in $\text{Lef}^s_{\mathbb{C}, p} \subset \mathcal{L}^s_{\mathbb{R}, p}$) can be expressed in terms of the basis of quadratic invariant monomials as:

$$2L_{jk}^\pi = [E_{w_j}, I_{w_k}] + [E_{w_k}, I_{w_j}]$$

This allows us to compute, for $i = 0, 1, 2, \ldots, s$:

$$L_{jk}^\pi(c_i) = 0 \quad \text{if } i \neq j$$

$$L_{jk}^\pi(c_j) = \frac{1}{2} [E_{w_j}, I_{w_k}(c_j)] = -I_{w_k} E_{w_j}(c_j) = -(1)^{j+k} c_k$$

and

$$L_{jk}^\pi(c_{s+1+i}) = 0 \quad \text{if } i \neq k$$

$$L_{jk}^\pi(c_{s+1+k}) = \frac{1}{2} [E_{w_j}, I_{w_k}(c_{s+1+k})] = E_{w_j} I_{w_k}(c_{s+1+k}) = -(1)^{j+k} c_{s+1+j}$$

This means that the matrices of the degree 0 subalgebra generated by the operators $L_{jk}^\pi$ have real coefficients and, more precisely they are all the matrices with the following block-form:

$$\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$$
where $A$ is a real $(s+1) \times (s+1)$ matrix with trace zero. This explicitly establishes an isomorphism between $<L_j \mathcal{F}>_\mathbb{R}$ and $\mathfrak{sl}(s+1, \mathbb{R})$.

The computation of the matrices of the operators $V_j$ is made easier by the use of the relation contained in the proof of Proposition 5.4:

$$V_j = \frac{1}{2}[E_{w_j}, E_{\omega_j}]$$

We can now observe that, for $i = 0, 1, 2, \ldots, s$:

$$V_j(c_i) = 0 \quad \text{if} \quad i \neq j$$

$$V_j(c_j) = \frac{1}{2}2E_{w_j}E_{\omega_j}(c_j) = iC_{s+1+j}$$

This, together with the observation that

$$V_j(c_{s+1+i}) = 0 \quad \forall i = 0, \ldots, s$$

implies that the matrix of $V_j$ has the following block-form:

$$\begin{pmatrix}
0 & 0 \\
0 & iB
\end{pmatrix}$$

where $B$ is a real and symmetric (actually diagonal) $(s+1) \times (s+1)$ matrix.

It follows that all the matrices of the above form are in $\mathcal{L}_{s,R}^p$ since they provide an irreducible representation for the adjoint action of $<L_j \mathcal{F}>_\mathbb{R} \cong \mathfrak{sl}(s+1, \mathbb{R})$; notice that the action of a matrix with upper diagonal $A$ over one with lower left block $iB$ is as follows:

$$iB \rightarrow -i(BA + tAB)$$

As for the operators $L_{jk}$ ($j \neq k$) of degree two, as it has been shown in Proposition 5.4:

$$2L_{jk} = [E_{w_j}, E_{\omega_k}] - [E_{w_k}, E_{\omega_j}]$$

Therefore,

$$L_{jk}(c_i) = 0 \quad \text{if} \quad i \neq j, k$$

and

$$L_{jk}(c_j) = -E_{\omega_k}E_{w_j}(c_j) = (-1)^{j+k}C_{s+1+k}$$

$$L_{jk}(c_k) = E_{\omega_k}E_{w_j}(c_k) = (-1)^{j+k}C_{s+1+j}$$

This, together with the observation that

$$L_{jk}(c_{s+1+i}) = 0 \quad \forall i = 0, \ldots, s$$

implies that the matrix of $L_{jk}$ has the block-form:

$$\begin{pmatrix}
0 & 0 \\
0 & C
\end{pmatrix}$$

where $C$ is a real and antisymmetric $(s+1) \times (s+1)$ matrix. Then all the matrices of the above form are in $\mathcal{L}_{s,R}^p$ since they provide an irreducible representation for the action of $<L_j \mathcal{F}>_\mathbb{R} \cong \mathfrak{sl}(s+1, \mathbb{R})$ similarly as before.

In the same way, acting with $<L_j \mathcal{F}>_\mathbb{R} \cong \mathfrak{sl}(s+1, \mathbb{R})$ on the adjoint operators $A_{jk}$ and $A_{j}$, we can show that $\mathcal{L}_{s,R}^p$ contains all the matrices of the form

$$\begin{pmatrix}
0 & H \\
0 & 0
\end{pmatrix}$$

where $H$ is a complex antihermitean $(s+1) \times (s+1)$ matrix.

It is now immediate to check that the matrices constructed above generate the matrix algebra as in the claim. To conclude it is enough to use point c) of Theorem 6.2, which states that the restriction of $\mathcal{L}_{s,R}^p$ to $\mathcal{I}_{-s}$ is faithful. $\square$

The following corollary refines the result of Theorem 4.5:
Corollary 7.9. For a fixed \( p \in X \), the orthogonal algebra \( \text{Lef}_p^s \cong \text{so}(s + 1, s + 1) \) coincides with the intersection \( \mathcal{L}_{R.p}^s \cap s \mathcal{L}_p^s \).

Proof. On one side, by Theorem 4.5, we know that \( \text{Lef}_p^s \) lies in \( \mathcal{L}_{R.p}^s \cap s \mathcal{L}_p^s \) and is isomorphic to \( \text{so}(s + 1, s + 1) \).

On the other side, let us consider the associated matrix algebras with respect to the basis \( \{ e_i \} \) of \( \mathcal{L} \); see (the proof of Theorem 7.1), and the matrices of \( \mathcal{L}_{R.p}^s \) are as in Proposition 7.8 above.

\[
dim_{\mathbb{R}} \left( \mathcal{L}_{R,p}^s \cap s \mathcal{L}_p^s \right) \leq \dim_{\mathbb{R}} \{ \text{subspace of } \mathcal{L}_{R,p}^s \} \text{ of matrices with real coefficients} \]

\[
= (s + 1)^2 + s(s + 1) = \dim_{\mathbb{R}} \text{so}(s + 1, s + 1)
\]

8. Cotangent bundles, Tori and abelian varieties

Let \( M \) be a smooth Riemannian manifold with metric \( h \), and let

\[
X = \bigwedge^s T^\ast M \oplus_{\otimes_{\mathbb{R}}} \cdots \oplus_{\otimes_{\mathbb{R}}} \bigwedge^s T^\ast M.
\]

We will show that \( X \) is naturally almost \( s \)-Kähler. First of all, clearly \( h \) induces naturally a Riemannian metric \( g \) on \( X \). We then have to define the differential forms \( \omega_{jk} \). These will come in two sets, with different constructions: the ones in which \( j \) or \( k \) is equal to zero and the other ones. The first ones are the simplest to define: is \( \pi_j \) is the natural projection from \( X \) to the \( j \)-th copy of \( T^\ast M \), then we define

\[
\omega_{0j} = \pi_j^\ast \omega_{0t}
\]

where \( \omega_{0t} \) is the standard symplectic form on the cotangent bundle \( T^\ast M \). A proof that with these forms \( X \) becomes polysymplectic can be found in [G1], Example 2.3. The forms \( \omega_{jk} \) when \( j, k \neq 0 \) will be defined by a different method. First, observe that using the Levi-Civita connection associated to the metric (induced by \( g \) on the cotangent bundle of \( M \)) we have a natural identification at any point \( Q = (p, \phi_1, ..., \phi_s) \in X \)

\[
T_Q X \cong \bigwedge^s T_p M \oplus_{\otimes_{\mathbb{R}}} \bigwedge^s T_p M \oplus_{\otimes_{\mathbb{R}}} \cdots \oplus_{\otimes_{\mathbb{R}}} \bigwedge^s T_p M.
\]

Let us call \( W_{jk} \subset T_Q X \) the direct sum of the \( j \)-th and of the \( k \)-th summands among the copies of \( T_p^s M \) in the identification above:

\[
T_p^s M \oplus_{\otimes_{\mathbb{R}}} T_p^s M \cong W_{jk} \subset T_Q X
\]

Using the metric, we can define \( \omega_{ij} \in \bigwedge^2 T_Q X \) simply by defining a natural element in \( \bigwedge^2 W_{jk} \). To do so, it is enough to observe that the identity (bundle) map from \( T_p M \) to itself is an element \( 1d \in T_p^s M \otimes T_p M \). This space is naturally isomorphic (using the metric \( h \)) to \( T_p^s M \otimes_{\otimes_{\mathbb{R}}} T_p M \) and this last space maps naturally to

\[
\bigwedge^2 (T_p^s M \oplus_{\otimes_{\mathbb{R}}} T_p^s M) \cong \bigwedge^2 W_{jk} \subset \bigwedge^2 T_Q X
\]

where the last inclusion is again induced by the use of the metric. The proof that these forms satisfy the almost \( s \)-Kähler condition is a simple direct computation.

The following example is a direct generalization of Example 2.7 of [G2]. Let \( \Gamma_0, ..., \Gamma_s \subset \mathbb{R}^r \) be \( s + 1 \) lattices, and let

\[
X = \mathbb{R}^r / \Gamma_0 \times \cdots \times \mathbb{R}^r / \Gamma_s.
\]
Then \( X \) has a natural structure of almost \( s \)-Kähler manifold of rank \( r \). Indeed, the (flat) metric is clear from the definition. We have also a natural choice of global coordinates 
\[
\{ y_i^j \mid i \in \{1, \ldots, r\}, j \in \{0, \ldots, s\} \}
\]
using which one can give directly the expressions for the forms:
\[
\omega_{jk} = \sum_{i=1}^{r} dy_i^j \wedge dy_i^k
\]
With the above definitions it is immediate to check that we have a almost \( s \)-Kähler structure. More generally, one could take the above definitions of the \( \omega_{jk} \) as forms on \((\mathbb{R}^r)^{s+1}\) and define \( X \) as the quotient of this almost \( s \)-Kähler manifold by any (not necessarily maximal rank) lattice \( \Gamma \subset (\mathbb{R}^r)^{s+1} \):
\[
X = (\mathbb{R}^r)^{s+1}/\Gamma
\]
It is again immediate to check that in this way we obtain a almost \( s \)-Kähler manifold of rank \( r \), which is compact when \( \Gamma \) is of maximal rank. As the lattice \( \Gamma \) varies, we obtain different (and in general not isomorphic) almost \( s \)-Kähler structures. As mentioned above, the metric being flat, these manifolds are actually a step higher in the rigidity ladder: they are \( s \)-Kähler (see Example 8.3 of [G1]).

The argument of Theorem 3.2 in [G2] can be used to produce many more examples of almost \( s \)-Kähler manifolds, compact or otherwise, by performing fibred products of special lagrangian fibrations. If the structure is \( s \)-Kähler (see [G1], Definition 7.2) then it is automatically almost \( s \)-Kähler; \( s \)-Kähler manifolds are however extremely rigid and difficult to construct, even more than Calabi-Yau ones, and therefore although geometrically interesting they are certainly not the correct way to try and build almost \( s \)-Kähler ones.

**Proposition 8.1.** On all the bundles of Lie algebras \( \mathcal{L}^s_C, \mathcal{L}^s_R, \mathcal{S}^s, \mathcal{U} \mathcal{L}^s, \mathcal{Lef}^s \) there are natural flat connections induced by the almost \( s \)-Kähler structure and compatible with inclusions. In the special case of \( s \)-Kähler manifolds this follows from the observation that all the natural generators of these bundles of algebras are parallel tensors with respect to the Levi-Civita connection.

**Proof.** Once fixed on an open set \( U \) an orientation of the bundle \( W_0 \), one can choose over \( U \) determinations of all the operators \( V_j, A_k \). These, together with the restrictions to \( U \) of the global sections \( L_{jk}, \Lambda_{jk} \), generate over \( \mathbb{R} \) a (finite dimensional) Lie algebra which we define to be the set of flat sections over \( U \) for our connection on the bundle \( \mathcal{L}^s_{R} \). Of course, the set of restriction of these sections to a given point \( p \in X \) is exactly \( \mathcal{L}^s_{R,p} \). A different choice for the orientation of \( W_0 \) would determine a choice for the \( V_j \) (and hence \( A_j \)) which differs at most by a sign, thus the Lie algebra generated will not change, and we have a well defined set of sections over \( U \). It is immediate to check that in this way one obtains a locally constant sheaf of sections for \( \mathcal{L}^s_{R} \), and this determines a flat connection. The argument for the other natural bundles is the same. \( \Box \)

We now show that we have a representation of the flat sections of the bundles of \( \mathcal{L}^s_C, \mathcal{L}^s_R, \mathcal{S}^s, \mathcal{U} \mathcal{L}^s, \mathcal{Lef}^s \) on the cohomology of an \( s \)-Kähler manifold, induced by the representation on the space of forms. This will be done showing that the Laplacian \( \Delta_d \) commutes with the action of generators of these spaces of sections, as in Theorem 10.1 on page 46 of [G1].

**Theorem 8.2.** Let \((X, \omega_1, \ldots, \omega_s, g)\) be a compact orientable \( s \)-Kähler manifold. Then we have that if \( \phi \in \{L_{jk}\} \cup \{V_j\} \), and \( d \) is the de Rham differential:
1) \([\phi, d] = 0\)
2) If we define \( d^c := [\phi, d^*] \), we have that \( dd^c + d^c d = 0 \);
3) \( [\phi, \Delta_d] = [\phi^*, \Delta_d] = 0 \), where \( \Delta_d \) is the \( d \)-Laplacian relative to the metric \( g \) and to the orientation.

**Proof** We adapt the proof of Theorem 10.1 of [G1].

1) This equation follows immediately from the fact that the forms \( \omega_{jk} \) and the volume forms \( \text{Vol}(W_j) \) of the distributions \( W_j \) are covariant constant with respect to the Levi-Civita connection, and therefore closed.

2) If we write down the expression for \( d^c \) in standard \( s \)-Kähler coordinates centered at a point \( p \in X \), we see that no derivative of the metric appears. Therefore, when we write down the expression for \( dd^c + d^c d \), only the first derivatives of the metric are involved. We skip the details, as they are completely analogous to those of, for example, [GH, Pages 111-115].

It follows, as in the classical case of Kähler manifolds, that to prove the equation it is enough to reduce to the case of a constant metric. When the metric is flat, however, the equation is easily seen to be equivalent (using 1)) to \( [\phi, \Delta_d] = 0 \), which with a flat metric follows immediately from the fact that the two-form corresponding to \( \phi \) is constant in flat (orthonormal) coordinates.

3) The second equation is the adjoint of the first. The first one, once written down explicitly in terms of \( d \) and \( d^* \), follows immediately from points 1) − 2).

\[\Box\]

**Corollary 8.3.** Let \((X, \omega_1, ..., \omega_s, g)\) be a compact orientable \( s \)-Kähler manifold. Then there is a canonical representation of the Lie algebras of flat global sections of the bundles \( \mathcal{L}_C^*, \mathcal{L}_R^*, s\mathcal{L}, u\mathcal{L}, \text{Lef}^* \) on \( H^*(X, \mathbb{C}) \).

**Theorem 8.4.** For a compact rank two \( s \)-Kähler manifold \( X \) with a global determination of \( J \) the following are equivalent:

1) \( X \) is Calabi-Yau.

2) There are a complex torus \( T \) with a translation invariant \( s \)-Kähler structure and a holomorphic covering map \( f : T \to X \) compatible with the \( s \)-Kähler structures of \( T \) and \( X \).

**Proof** In the direction from 1) to 2), the main point in the proof is the observation that with a choice of \( J \) at \( p \in X \), the assignment of any one-form \( v_{10} \in W_0 \subset T^*_pX \) is enough to determine a complete adapted coframe. Indeed, let us call such a form \( v_{10} \), and let us call \( v_{20} \) its image under \( J \):

\[ v_{20} = Jv_{10} \]

Using the natural identifications between the various \( W_j \) induced by the structure forms and the metric, one obtains then corresponding forms \( v_{11}, v_{12}, ..., v_{1s}, v_{2s} \) and it is immediate to check that they form an adapted coframe at the point \( p \). Now, if we use these forms to build the corresponding \( w_0, ..., w_s \), whose wedge product is a nonzero form of type \((s + 1, 0)\). A different choice of the initial \( v_{10} \) could differ from the first one by a rotation of angle \( \theta \):

\[ v'_{10} = e^{2\pi i \theta J} v_{10} \]

This then is reflected in a modification as follows in the holomorphic volume form:

\[ w'_0 \wedge \cdots \wedge w'_s = e^{2\pi i \theta} w_0 \wedge \cdots \wedge w_s \]

Assuming now to have determined a global holomorphic volume form \( \Omega \), we see that for any point there are \( s + 1 \) choices of \( v_{10} \) which produce the equation

\[ w_0 \wedge \cdots \wedge w_s = \Omega \]

at the point \( p \). In other words, the set of all possible choices inside \( T^*X \) forms a \((s + 1)\)-sheeted covering \( \tilde{X} \) of \( X \), over which there are \( s + 1 \) global sections of the
pull back of the covering itself. Such a covering space moreover inherits all the local geometric properties of $X$: it is a compact $s$-Kähler manifold of rank two, with a global determination of $J$ and of a holomorphic volume form, and with a global smooth form $v_{10}$ determining the holomorphic volume form following the procedure described above. Correspondingly, there is a global determination of smooth forms $v_{10}, \ldots, v_2$, determining an adapted coframe at all points $p \in X$. It is clear that from the covariance of the holomorphic volume form under parallel transport one obtains that any corresponding smooth determination of this adapted coframe will be also covariant constant with respect to Levi-Civita, and therefore a global determination will determine $s(s + 1)$ covariant constant and orthonormal differential forms (and, dualizing, vector fields). This immediately shows that $X$ must be metrically flat, and being compact we have that it must me a torus. The global covariant orthonormal vector fields have therefore associated to them local coordinates which become global ones on the $\mathbb{R}^{2(s+1)}$ which is the universal covering space of $\tilde{X}$. This (and the fact that the forms associated to these coordinates determine an adapted coframe) guarantee that the $s$-Kähler structure is translation invariant with respect to the natural translation operation on $\tilde{X}$.

In the direction from 2) to 1), notice that you can always write $T$ as $\mathbb{C}^{s+1}/\Gamma$ (as a complex manifold), with $\Gamma$ a lattice. Then observe that $\mathbb{C}^{s+1}$ admits a translation invariant $s$-Kähler structure with the complex structure inducing the $J$ operator; this $s$-Kähler structure is therefore induced on $A$. This argument works equally well when one has a covering of an abelian variety.

\begin{remark}

The proof of the theorem shows in particular the following: if $A$ is a complex abelian variety of (complex) dimension $s + 1$, then it is possible to put on it an $s$-Kähler structure, with a translation invariant Kähler metric, and with the complex structure of $A$ giving a determination of the operator $J$.

The preceding remark shows that one can think of these $s$-Kähler structures as "decorations" or "enrichments" of the underlying Kähler structure. As such, one can use them to study the moduli problems for Abelian varieties by first studying the moduli problems of related $s$-Kähler manifolds.

\end{remark}

9. Moduli of pointed elliptic curves

In this section we make a first attempt to put in contact the theory of $s$-Kähler manifolds with that of moduli of pointed elliptic curves. We feel that there is (still partly hidden) much bigger interaction, which we intend to study in the future. Here, for $s \geq 2$, we construct a pointwise $(s-1)$-Kähler structure (of rank 2) on the (open) moduli space $M_{1,s+1}$ with a fixed Kähler structure. Let $E$ be an elliptic curve, with punctures $p_1, \ldots, p_{s+1}$. The tangent space at the point $[E, p_1, \ldots, p_{s+1}]$ of $M_{1,s+1}$ is $H^1(E, T_E(-\sum_i p_i))$, and there is a short exact sequence of coherent sheaves

$$0 \to T_E(-\sum_i p_i) \to T_E \to T_E/T_E(-\sum_i p_i) \to 0$$

from which one obtains the following exact sequence of complex vector spaces

$$0 \to H^0(E, T_E) \to \bigoplus_i T_{p_i}E \to T[E, p_1, \ldots, p_{s+1}]M_{1,s+1} \to H^1(E, T_E) \to 0$$

The first and the last one of these vector spaces have dimension one, while the two intermediate ones have both dimension $s + 1$. In particular, there are canonical
inclusions $T_{p_i}E \to T_{[E,p_1,\ldots,p_{s+1}]}M_{1,s+1}$, and moreover one has that (with respect to these inclusions)

$$V_{E,p_1,\ldots,p_{s+1}} := \text{Im} \left( \bigoplus_i T_{p_i}E \right) \subset T_{[E,p_1,\ldots,p_{s+1}]}M_{1,s+1}$$

has codimension one, with a permutation invariant syzygy among the images of a set of generators induced by a single translation invariant global holomorphic vector field on $E$. We indicate with $V$ the associated sub-bundle of $TM_{1,s+1}$. Using the metric on both sides, and taking the exterior power, we obtain a natural map

$$\bigwedge^2 \mathbb{R} \left( \bigoplus_i T_{p_i}^*E \right) \to \bigwedge^2 \mathbb{R} \left( T_{[E,p_1,\ldots,p_{s+1}]}^*M_{1,s+1} \right)$$

As $E$ is an elliptic curve, for any pair $p_j, p_k$ of its points one has a natural complex linear and isometric identification (given by translation by $p_j - p_k$) from $T_{p_j}E \to T_{p_k}E$, which can be equivalently seen as an element of $T_{p_j}^*E \otimes \mathbb{R} T_{p_k}E$, and also (using the metric) as an element of $T_{p_j}^*E \otimes \mathbb{R} T_{p_k}^*E$. Using the natural map

$$T_{p_j}^*E \otimes \mathbb{R} T_{p_k}^*E \to \bigwedge^2 \mathbb{R} \left( T_{p_j}^*E \otimes \mathbb{R} T_{p_k}^*E \right)$$

we have obtained a natural element

$$\omega_{E,p_j,p_k} \in \bigwedge^2 \mathbb{R} T_{p_i}E$$

Varying the curve and the points, and using the map between exterior powers described before, we obtain a natural two-form

$$\omega_{jk} \in \Omega^2_{\mathbb{R}}(M_{1,s+1})$$

These two-forms are all well defined, and we want to restrict the set of forms

$$\omega_{jk} : j,k \in \{1,\ldots,s\}$$

(excluding the index $s+1$) to the bundle $V \subset TM_{1,s+1}$. Using the metric of $M_{1,s+1}$ one can define a complementary (complex dimension one) subbundle $C \subset TM_{1,s+1}$, and induce on it a metric. On $V$ instead one induces a metric using the forms $\omega_{jk}$, and therefore in the end one obtains a pointwise $(s-1)$-Kähler structure (this is a pointwise verification) on $M_{1,s+1}$. As mentioned at the beginning of the section, this should be only part of the story: for instance, as a first step, it should be possible to go up to a full nondegenerate almost $s$-Kähler structure with a little more effort.

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