Quantum Recurrence of a Subspace and Operator-Valued Schur Functions

J. Bourgain
1, F. A. Grünbaum
2, L. Velázquez
3, J. Wilkening
2

1 School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA. E-mail: bourgain@math.ias.edu
2 Department of Mathematics, University of California, Berkeley, CA 94720, USA. E-mail: grunbaum@math.berkeley.edu; wilken@math.berkeley.edu
3 Departamento de Matemática Aplicada, Universidad de Zaragoza & IUMA, Zaragoza, Spain. E-mail: velazque@unizar.es

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Abstract: A notion of monitored recurrence for discrete-time quantum processes was recently introduced in Grünbaum et al. (Commun Math Phys (2), 320:543–569, 2013) taking the initial state as an absorbing one. We extend this notion of monitored recurrence to absorbing subspaces of arbitrary finite dimension.

The generating function approach leads to a connection with the well-known theory of operator-valued Schur functions. This is the cornerstone of a spectral characterization of subspace recurrence that generalizes some of the main results in Grünbaum et al. (Commun Math Phys (2), 320:543–569, 2013). The spectral decomposition of the unitary step operator driving the evolution yields a spectral measure, which we project onto the subspace to obtain a new spectral measure that is purely singular if the subspace is recurrent, and consists of a pure point spectrum with a finite number of masses precisely when all states in the subspace have a finite expected return time.

This notion of subspace recurrence also links the concept of expected return time to an Aharonov–Anandan phase that, in contrast to the case of state recurrence, can be non-integer. Even more surprising is the fact that averaging such geometrical phases over the absorbing subspace yields an integer with a topological meaning, so that the averaged expected return time is always a rational number. Moreover, state recurrence can occasionally give higher return probabilities than subspace recurrence, a fact that reveals once more the counter-intuitive behavior of quantum systems.

All these phenomena are illustrated with explicit examples, including as a natural application the analysis of site recurrence for coined walks.

1. Introduction

The study of return properties of discrete-time random processes goes back at least to Pólya [22], who proved that only in dimension one and two is the simplest unbiased random walk recurrent, returning to the starting point with probability one. Since then, the notion of recurrence has been key in the theory of classical random processes.
Concerning the analog of recurrence for discrete-time quantum systems, two different approaches have been recently proposed. In 2008, Štefaňák et al. [32–34] resort to an ensemble of identically prepared independent systems that are discarded after the measurement to avoid changing the dynamics due to the collapse. A more recent proposal, from A.H. Werner, R.F. Werner and two of the authors [15], treats the perturbation of the evolution due to the monitoring as an essential ingredient in the definition of quantum recurrence. This second definition of recurrence is based on performing a projective measurement after each step of the quantum evolution, an idea that has some precursors in the context of quantum walks with absorbing boundary conditions [2,3].

The ideas developed in [15] deal with the probability of returning to the initial state, which corresponds to having an absorbing one-dimensional subspace generated by this initial state. A natural generalization, which is the aim of this paper, is to consider an absorbing subspace of arbitrary finite dimension which includes the initial state. This is the case, for instance, when we are interested in the return of a coined walk to the initial site, regardless of the internal state. This generalization also allows us to ask about the probability of returning to a finite set of sites from which an initial state was chosen. We will use the terminology “state recurrence” to refer to the notion of recurrence defined in [15], and “subspace recurrence” for the extension of this concept developed in the present paper.

The spectral characterization of subspace recurrence is the principal objective of this paper, which emerges from the identification of a key object that best codifies subspace recurrence, namely the operator-valued Schur function of the spectral measure of the absorbing subspace. This connects the subject of quantum recurrence with well-known areas of complex analysis: the theory of matrix Schur functions [5,14] and the theory of orthogonal polynomials on the unit circle [27,28]. See also [13] for a more detailed discussion of the matrix version of these orthogonal polynomials and its relation with matrix Schur functions.

It turns out that the recurrent subspaces—those whose states all return with probability one to the subspace—are characterized by inner Schur functions. This identifies the recurrent subspaces as those contained in the singular subspace of the unitary step operator driving the evolution. Furthermore, the recurrent subspaces whose states all possess a finite expected return time are those whose Schur functions are rational inner. Such subspaces are those contained in finite sums of eigenspaces of the unitary step. These results are the natural extensions of Theorems 1 and 2 in [15].

We stress that the aim of the paper is not limited to the generalization to subspace recurrence of the main results already known for state recurrence. The paper also shows important differences and unexpected relations between subspace and state recurrence. For instance, in contrast to state recurrence, the existence of states that return with probability one to a subspace of finite dimension greater than one does not require the existence of a singular subspace for the unitary step. This is true even if the state returns to the subspace in a finite expected time.

Besides, in the case of state recurrence, the expected return time is always infinite or an integer. This is because the expected return time has a topological interpretation as the degree of the related inner Schur function [15]. These results are no longer true for subspace recurrence because the expected return time of a state is related to a geometrical (rather than a topological) invariant in this case, namely the Aharonov–Anandan geometrical phase of the curve obtained by applying the (operator) boundary values of the Schur function on the unit circle to the state. As a consequence, the expected return time to a subspace can be any real number not smaller than one.