Abstract

An affine Lie algebra acts on cohomology groups of quiver varieties of affine type. A Heisenberg algebra acts on cohomology groups of Hilbert schemes of points on a minimal resolution of a Kleinian singularity. We show that in the case of type $A$ the former is obtained by Frenkel-Kac construction from the latter.

1 Introduction

2 Preliminaries

2.1 Notations for indices

2.2 Notations for Young diagrams

2.3 Notations for Maya diagrams

2.4 Young diagrams and Maya diagrams

2.5 Cores and quotients

3 Frenkel-Kac construction

3.1 Boson-fermion correspondence

3.2 Frenkel-Kac construction

3.3 Explicit formula for Frenkel-Kac construction

4 Quiver varieties

4.1 Quiver varieties

4.2 Minimal resolution

4.3 $\mathbb{Z}/l\mathbb{Z}$-equivariant Hilbert scheme ($\zeta_0$-case)

4.4 Hilbert scheme of the minimal resolution ($\zeta_\infty$-case)

4.5 Correspondence of fixed points

5 Representations on equivariant cohomologies

5.1 Equivariant cohomology groups

5.2 Equivariant cohomology groups of quiver varieties

5.3 Representation of the affine Lie algebra

5.4 Representation of the Heisenberg algebra

5.5 Main theorem for equivariant cohomologies

6 Representations on ordinary cohomologies

6.1 Representations on ordinary cohomologies

6.2 Main theorem for ordinary cohomologies
1 Introduction

In this paper we study representation theory associated with the quiver varieties of type $\hat{A}$.

A quiver variety is a moduli space of representations of a quiver ([Nak94]). Since it is defined as a geometric invariant theory quotient ([MFK94]), we need to fix a parameter $\zeta$ of the stability condition in order to define a quiver variety $M_{\zeta}$. The space of parameters of stability conditions has a chamber structure. A wall consists of parameters where the stability condition and the semistability condition are not equivalent. When the parameter cross a wall, the variety is changed by a flop, as is typical in the geometric invariant theory ([Tha96]). But in our situation it is also known that the underlying $C^\infty$-manifold is not changed ([Nak94]). In particular, there exists a canonical isomorphism between cohomology groups of quiver varieties associated with different chambers.

One can construct a level-1 integrable highest weight representation of a Kac-Moody Lie algebra on a direct sum of cohomology groups of quiver varieties associated with a specific parameter $\zeta_0$ ([Nak98]). On the other hand, one can also construct the Fock space representation of a Heisenberg algebra on a direct sum of cohomology groups of Hilbert schemes of points on a surface ([Nak97]). When the surface is a minimal resolution of a Kleinian singularity, the Hilbert schemes of points on it can be described as quiver varieties of affine type, which are associated with another specific parameter $\zeta_\infty$.

An affine Lie algebra contains the Heisenberg algebra as its subalgebra. One can reconstruct a level-1 integrable representation of an affine Lie algebra from a level-1 representation of the Heisenberg algebra ([Frenkel-Kac construction FK81]).

So both cohomology groups of $M_{\zeta_0}$ and $M_{\zeta_\infty}$ are endowed with level-1 integrable highest weight actions of an affine Lie algebra. Our purpose is to show that the canonical isomorphism of cohomology groups induced by the diffeomorphism intertwines those actions.

In this paper we will deal with the case of type $A$ only. Let us explain our method.

The quiver varieties are endowed with specific $S^1$-actions. The quiver varieties associated with different chambers are $S^1$-equivariantly diffeomorphic. The actions of the affine Lie algebra and the Heisenberg algebra on the cohomology groups can be lifted to the $S^1$-equivariant cohomology groups. We will show the isomorphism induced by the $S^1$-equivariant diffeomorphism intertwine the two actions on the $S^1$-equivariant cohomology groups, which will be followed by the result for the ordinary cohomology groups.

It is an advantage that the $S^1$-equivariant cohomology groups have specific bases indexed by the $S^1$-fixed points. Moreover the $S^1$-fixed points of $M_{\zeta_0}$ and $M_{\zeta_\infty}$ can be parametrized in term of Young diagrams. We will exhibit the two actions of the affine Lie algebra with respect to these bases and show that the operation called ”taking cores and quotients of Young diagrams” induces an isomorphism as representations.

It seems to have been known or believed that the fixed points of the quiver varieties associated with different stability conditions are related by the operation taking cores and quotients. For example, in §7.2.4 of [Hai03] it is tried
to characterize the wreath Macdonald polynomials in term of cores and quotients, where the wreath Macdonald polynomials are conjectured to exist, to correspond to fixed points of quiver varieties, and so, to depend on the chamber of the stability conditions. We will prove that this relation can be realized by the diffeomorphism. We should mention that this result is already used in §6.3 of [Lic], but neither proof nor reference is given there.

The paper is organised as follows. In §2 we are devoted to combinatorial preliminaries. In §3 we review Frenkel-Kac construction and give an explicit formula for the resulting representation. In §4 we study the geometry of the quiver varieties and show the correspondence of the fixed points is given by the operation taking cores and quotients. In §5 we review Nakajima’s geometrical construction of the representations of the affine Lie algebra and the Heisenberg algebra, and describe the representations on the $S^1$-equivariant cohomology groups explicitly using the bases derived from the fixed points. The results of §3–§5.4 are combined to give the theorem for the equivariant cohomology groups in §5.5. In §6 we deduce the result for the ordinary cohomology groups.

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2 Preliminaries

2.1 Notations for indices

Fix an integer $l$ larger than 2. In this paper we sometimes use half integers in order to describe everything symmetrically. Symbols $k$ and $h$ are used for half integers.

We set $I = \{0, \ldots, l - 1\}$. We sometimes identify $I$ with the set of modulo $l$ equivalent classes of integers. We also identify $I$ with the set of vertices of the cyclic quiver.

We set $\tilde{I} = \{1, 2, \ldots, l - 1\}$. We sometimes identify $\tilde{I}$ with the set of modulo $l$ equivalent classes of half integers. We also identify $\tilde{I}$ with the set of edges of the cyclic quiver.

2.2 Notations for Young diagrams

Let $\Pi$ denote the set of all Young diagrams. Identify a Young diagram with a subset of $(\mathbb{Z}_{\geq 0})^2$. A node is an element of $(\mathbb{Z}_{\geq 0})^2$. The content of a node $(a, b)$ is the number $a - b$. A node is called $i$-node if its content equals to $i$ modulo $l$. For $\lambda \in \Pi$ we define

$$n_j(\lambda) = \sharp\{(a, b) \in \lambda \mid a - b = j\}, \quad f_\lambda(z) = \sum_{j \in \mathbb{Z}} n_j(\lambda)z^j \in \mathbb{Z}[z^\pm]$$
and

$$v_i(\lambda) = \sharp\{(a, b) \in \lambda \mid a - b \equiv i \pmod{l}\}, \quad v(\lambda) = (v_i(\lambda)) \in \mathbb{Z}^l.$$
For $\lambda \in \Pi$, a node $(a, b)$ is called \textit{addable} if $(a, b) \notin \lambda$ and $(a-1, b), (a, b-1) \in \lambda$. A node $(a, b)$ is called \textit{removable} if $(a, b) \in \lambda$ and $(a+1, b), (a, b+1) \notin \lambda$.

Let $A_{\lambda, i}$ (resp. $R_{\lambda, i}$) denote the set of all addable (removable) $i$-nodes for $\lambda$.

Let $X$ be a removable or an addable node of $\lambda$. We set
$$\eta^+(\lambda, i, X) = \sharp\{\text{addable} \ i\text{-node to the right of} \ X\} - \sharp\{\text{removable} \ i\text{-node to the right of} \ X\},$$
$$\eta^-(\lambda, i, X) = \sharp\{\text{addable} \ i\text{-node to the left of} \ X\} - \sharp\{\text{removable} \ i\text{-node to the left of} \ X\}.$$ 

\subsection*{2.3 Notations for Maya diagrams}

A \textit{Maya diagram} is an increasing sequence of half integers $m = (k_j)_{j \geq 1}$ such that $k_{j+1} = k_j + 1$ for sufficiently large $j$. Let $\mathcal{M}$ denote the set of all Maya diagrams.

Note that a Maya diagram can be identified with a map $\mathbb{Z} + \frac{1}{2} \to \{\pm 1\}$ such that
$$m(h) = \begin{cases} 1 & \text{for } h \gg 0, \\ -1 & \text{for } h \ll 0. \end{cases}$$

We define the \textit{charge} of $m$ by
$$c(m) = \sharp\{h < 0 \mid m(h) = 1\} - \sharp\{h > 0 \mid m(h) = -1\}.$$ 

For $c \in \mathbb{Z}$, we define $m^{(c)}$ by $m^{(c)}(h) = m(h+c)$. Note that $c(m^{(c)}) = c(m) - c$.

\subsection*{2.4 Young diagrams and Maya diagrams}

\subsubsection*{2.4.1}

For $\lambda \in \Pi$, let us define $m_\lambda \in \mathcal{M}$. First, note that
$$n_{k-\frac{1}{2}}(\lambda) - n_{k+\frac{1}{2}}(\lambda) = \begin{cases} -1 \text{ or } 0 & (k < 0), \\ 0 \text{ or } 1 & (k > 0). \end{cases}$$

Then we define
$$m_\lambda(k) = \begin{cases} 1 & (k < 0, \ n_{k-\frac{1}{2}}(\lambda) - n_{k+\frac{1}{2}}(\lambda) = -1), \\ -1 & (k < 0, \ n_{k-\frac{1}{2}}(\lambda) - n_{k+\frac{1}{2}}(\lambda) = 0), \\ 1 & (k > 0, \ n_{k-\frac{1}{2}}(\lambda) - n_{k+\frac{1}{2}}(\lambda) = 0), \\ -1 & (k > 0, \ n_{k-\frac{1}{2}}(\lambda) - n_{k+\frac{1}{2}}(\lambda) = 1). \end{cases}$$
2.4.2
An integer $j$ such that $(m_\lambda(j - \frac{1}{2}), m_\lambda(j + \frac{1}{2})) = (1, -1)$ (resp. $(-1, 1)$) corresponds to an addable (resp. a removable) node of $\lambda$. Its content equals to $j$.

For an addable or a removable node with content $j$ we have

$$η^-(X, j, λ) = \sharp \left\{ j' \equiv j \mid j' < j, \left( m_\lambda \left( j - \frac{1}{2} \right), m_\lambda \left( j' + \frac{1}{2} \right) \right) = (1, -1) \right\}$$

$$- \sharp \left\{ j' \equiv j \mid j' < j, \left( m_\lambda \left( j - \frac{1}{2} \right), m_\lambda \left( j' + \frac{1}{2} \right) \right) = (-1, 1) \right\}$$

$$= \sharp \left\{ h \equiv j - \frac{1}{2} \mid h < j, m_\lambda(h) = 1 \right\} - \sharp \left\{ h \equiv j + \frac{1}{2} \mid h < j, m_\lambda(h) = 1 \right\}.$$

2.4.3
For $λ ∈ Π$ we have $c(m_\lambda) = 0$. Conversely given $m ∈ M$ with charge 0, there exists a unique Young diagram $λ$ such that $m_λ = m$. Consequently, we have the bijection

$$F: \mathbb{Z} × Π \rightarrow M, (c, λ) \mapsto m^{(-c)}.$$

Let us define $q(m) ∈ Π$ by $F^{-1}(m) = (c(m), q(m))$.

2.5 Cores and quotients

2.5.1
For $k ∈ \tilde{I}$ and $m ∈ M$, we define $m_k ∈ M$ by

$$m_k(h) = m \left( l \left( h - \frac{1}{2} \right) + k \right).$$

Note that $m$ can be recovered from $\{m_k\}_{k ∈ I}$. We have $c(m) = \sum c(m_k)$.

For $λ ∈ Π$, we set $c_k(λ) = c(m_λ,k)$ and $q_k(λ) = q(m_λ,k)$. We define the $l$-core of $λ$ by $c(λ) = (c_k(λ)) ∈ (\mathbb{Z})^0 = \{(c_1, \ldots, c_l) ∈ \mathbb{Z}^l \mid \sum c_k = 0\}$. We also define the $l$-quotient of $λ$ by $q(λ) = q_k(λ) ∈ \Pi^l$. We have the bijection

$$CQ = c × q: Π \rightarrow (\mathbb{Z}^l)^0 × Π^l.$$

2.5.2
Lemma.

$$c_k(λ) = v_{k-\frac{1}{2}}(λ) - v_{k+\frac{1}{2}}(λ).$$
Proof. Note that for $k \in \overline{I}$ and $h \in \mathbb{Z} + \frac{1}{2}$ the sign of $h$ and of $l \left( h - \frac{1}{2} \right) + k$ coincide.

\[
c_k(\lambda) = c(m_{\lambda,k}) = \begin{cases} 
- \sum_{j} & \text{if } k(h_j + 1) + \frac{1}{2} > 0 \\
+ \sum_{j} & \text{if } k(h_j + 1) + \frac{1}{2} < 0 \\
0 & \text{otherwise}
\end{cases}
\]

Thus
\[
\sum_{k \in \overline{I}} c_k(\lambda) = \frac{1}{2} \sum_{j} (l(c_k(\lambda) + j - 1) + k)
\]

and $k(h) = k_+^\circ (h)$ if $h \neq l(c_k(\lambda) + j - 1) + k, l(c_k(\lambda) + j) + k$. This means

\[
n_j^-(\lambda) - n_j^+(\lambda) = \begin{cases} 
1 & \text{if } l(c_k(\lambda) + j - 1) + k + \frac{1}{2} \leq j' \leq l(c_k(\lambda) + j) + k + \frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then the statement follows by induction on $\sum |q_k(\lambda)|$.

\[
\square
\]

2.5.3 Proposition. For $\lambda \in \Pi$ we set $\lambda = CQ^{-1}(c(\lambda), (0, \ldots, 0))$. Then we have

\[
f_\lambda(z) = f_\lambda(z) + \sum_k z^{\ell_\lambda(k)} f_{q_k(\lambda)}(z^{\lambda}) \left( z^{k-l+\frac{1}{2}} + \cdots + z^{k-\frac{1}{2}} \right).
\]

Proof. Let $\lambda^+$ be a Young diagram obtained from $\lambda$ by adding a node with content $j \in \mathbb{Z}$ to $q_k(\lambda)$. Since $k_{q_k(\lambda)}(h) = k_\lambda \left( l(c_k(\lambda) + h - \frac{1}{2}) + k \right)$ we have

\[
k_\lambda (l(c_k(\lambda) + j - 1) + k) = 1,
k_\lambda (l(c_k(\lambda) + j) + k) = -1,
k_\lambda^+ (l(c_k(\lambda) + j - 1) + k) = -1,
k_\lambda^+ (l(c_k(\lambda) + j) + k) = 1,
\]

and $k_\lambda(h) = k_\lambda^+(h)$ if $h \neq l(c_k(\lambda) + j - 1) + k, l(c_k(\lambda) + j) + k$. This means

\[
n_j^-(\lambda^+) - n_j^+(\lambda) = \begin{cases} 
1 & \text{if } l(c_k(\lambda) + j - 1) + k + \frac{1}{2} \leq j' \leq l(c_k(\lambda) + j) + k + \frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then the statement follows by induction on $\sum |q_k(\lambda)|$.

\[
\square
\]

3 Frenkel-Kac construction

In this section we review Frenkel-Kac construction.

Let $Q$ denote the finite root lattice of type $A_{l-1}$ and $H_{l-1} = H_{\tilde{\lambda}_l}$ denote the Heisenberg algebra of type $A_{l-1}$. Let $B^{\otimes l}$ be a vector space with a basis indexed by $\Pi^l$. This is endowed with an action of $H_{l-1}$. By Frenkel-Kac construction, we get an action $\tilde{\lambda}_l$ on $CQ \otimes B^{\otimes l}$. By the operation taking $l$-cores and $l$-quotients, $Q \times \Pi^l$ is bijective to $\Pi$. Thus $CQ \otimes B^{\otimes l}$ has a basis indexed by $\Pi$. We will
give an explicit formula of the action of \( \hat{a}_0 \) with respect to this basis (Theorem 3.3.2). This formula seems to be well-known to many people, but I cannot find any references. We will prove the formula by reducing to the boson-fermion correspondence.

3.1 Boson-fermion correspondence

3.1.1 The Heisenberg algebra \( \mathcal{H} \) is the \( \mathbb{C} \)-algebra generated by \( p(m), p(-m) (m \in \mathbb{Z}_{>0}) \) and the central element \( c \) with relation

\[
[p(m), p(n)] = \delta_{0,n+m} c.
\]

The Heisenberg algebra \( \mathcal{H} \) acts on the polynomial ring \( \mathbb{C}[p_1, p_2, \ldots] \) with infinitely many variables by

\[
p(-m) \cdot f = mp_m f, \quad p(m) \cdot f = \frac{\partial}{\partial p_m} f, \quad c \cdot f = f.
\]

Identify \( p_m \) with the power sum symmetric function, then \( \mathbb{C}[p_1, p_2, \ldots] \) is isomorphic to the ring of symmetric functions with infinitely many variables. Identify the Schur function \( s_\lambda \) with a formal element \( b_\lambda \), then the ring of symmetric functions with infinitely many variables is isomorphic to \( \mathcal{B} = \bigoplus_{\lambda \in \Pi} \mathbb{C} b_\lambda \) as vector space.

Hence we have the action of \( \mathcal{H} \) on \( \mathcal{B} \). This representation is called the bosonic Fock space.

3.1.2 The Clifford algebra \( \mathcal{C} \) is the \( \mathbb{C} \)-algebra generated by \( \psi(k), \psi^*(k) (k \in \mathbb{Z} + \frac{1}{2}) \) and the central element \( c \) with relation

\[
\{ \psi(k), \psi(h) \} = \{ \psi^*(k), \psi^*(h) \} = 0, \quad \{ \psi(k), \psi^*(h) \} = \delta_{k,h} c.
\]

The Clifford algebra \( \mathcal{C} \) acts on \( \mathcal{F} = \bigoplus_{m \in \mathcal{M}} \mathbb{C} m \) by

\[
\psi_k(m) = \begin{cases} (-1)^i (-1, \ldots, k_{i-1}, k_{i+1}, \ldots) & \text{if } k_i = k \text{ for some } i, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\psi^*_k(m) = \begin{cases} (-1)^i (-1, \ldots, k_i, k_{i+1}, \ldots) & \text{if } k_i < k < k_{i+1} \text{ for some } i, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( c = 1 \). This representation is called the fermionic Fock space.

3.1.3 Let us identify \( \mathbb{C} \mathcal{Z} \otimes \mathcal{B} \) and \( \mathcal{F} \) by the bijection \( F \) given in 2.4.3. The action of \( \mathcal{H} \) on \( \mathcal{B} \) is naturally extended to the action on \( \mathbb{C} \mathcal{Z} \otimes \mathcal{B} \). Consider generating functions

\[
\Gamma^+(z) = \exp \left( \sum_{m=1}^{\infty} \frac{z^m}{m} p(-m) \right), \quad \Gamma^-(z) = \exp \left( \sum_{m=1}^{\infty} \frac{-z^m}{m} p(m) \right),
\]
\[ \Psi(z) = \sum_{j=-\infty}^{\infty} z^{-j} \tilde{\psi}_j, \quad \Psi^*(z) = \sum_{j=-\infty}^{\infty} z^{-j} \tilde{\psi}_j^* . \]

Let \( e^K, D \) denote the shift and degree operators on \( \mathbb{C}Z \otimes \mathcal{B} \) defined as follows:

\[ e^K([c] \otimes b) = [c+1] \otimes b, \quad D([c] \otimes b) = c([c] \otimes b) \quad ([c] \otimes b \in \mathbb{C}Z \otimes \mathcal{B}) . \]

**Theorem.** (See [Kac90] Theorem 14.10)

\[ \Psi(z) = \Gamma^+(z) \Gamma^-(z)^{-1} e^K z^D, \]
\[ \Psi^*(z) = \Gamma^+(z)^{-1} \Gamma^-(z) e^{-K} z^{-D} . \]

**Remark.** Our notation is different from [Kac90]'s. They do not use half integers.

### 3.2 Frenkel-Kac construction

#### 3.2.1

Let \( \mathcal{H}_{l-1} \) denote the \( \mathbb{C} \)-algebra generated by \( p_i(m), p_i(-m) \) \( (i \in \{1, \ldots, l-1\}, m > 0) \) and a central element \( c \) with relations

\[ [p_i(m), p_j(n)] = \delta_{i+n+m} a_{ij} c \]

where \( a_{ij} \) is the Cartan matrix of type \( A_{l-1} \). We say a representation of \( \mathcal{H}_{l-1} \) is level-1 if \( c \) acts by the identity map.

The affine Lie algebra of type \( A_{l-1} \) is \( \tilde{\mathfrak{h}}_l = \mathfrak{h}_l \otimes \mathbb{C}[x^{\pm 1}] \oplus \mathbb{C} \). The subalgebra generated by \( h_i \otimes x^m \) \( (i \in \{1, \ldots, l-1\}, m \neq 0) \) and \( c \) is isomorphic to \( \mathcal{H}_{l-1} \).

Let \( Q \) be the root lattice of type \( A_{l-1} \). We can regard \( Q \) as the subset of \( \mathfrak{h} \) by the isomorphism \( \mathfrak{h} \cong \mathfrak{h}^* \). We can also identify \( Q \) with \( (\mathbb{Z}^l)_0 = \{ (c_1, \ldots, c_{l-1}) \in \mathbb{Z}^l \mid \sum c_k = 0 \} \) so that simple roots \( \alpha_i \) \( (i \in \{1, \ldots, l-1\}) \) correspond to \( e_{i-\frac{1}{2}} - e_{i+\frac{1}{2}} \), where \( e_k \)'s are the coordinate vector in \( \mathbb{Z}^l \). Then the set of positive roots is described as \( P_+ = \{ e_k - e_{k'} \mid \frac{1}{2} \leq k < k' \leq l - \frac{1}{2} \} \). For a positive root \( \alpha = e_{k} - e_{k'} \) let \( e_{\alpha} \) (resp. \( f_{\alpha} \)) denote the matrix unit \( E_{k+\frac{1}{2}, k'+\frac{1}{2}} \) (resp. \( E_{k'+\frac{1}{2}, k+\frac{1}{2}} \)) in \( \mathfrak{sl}_l \).

#### 3.2.2

Let \( B \) be a level-1 \( \mathcal{H}_{l-1} \)-module. Assume that for any \( b \in B \) there exists an integer \( m(b) \) such that

\[ p_{k_1}(m_1) \cdots p_{k_a}(m_a) b = 0 \quad \text{if} \quad m_i > 0, \quad \sum m_i > m(b) . \]

Set \( F = \mathbb{C}Q \otimes B \). For \( \alpha \in Q \), we define a generating function \( X(\alpha, z) \) of operators on \( F \) by

\[ X(\alpha, z) = \exp \left( \sum_{m=1}^{\infty} \frac{z^m}{m} \alpha(-n) \right) \exp \left( - \sum_{m=1}^{\infty} \frac{z^{-m}}{m} \alpha(n) \right) \exp (z \cdot \alpha(0) + \alpha) \]
where $\exp(\log z \cdot \alpha(0) + \alpha)$ is the operator defined by

$$
\exp(\log z \cdot \alpha(0) + \alpha)([\beta] \otimes b) = z^{\frac{1}{2}(\alpha, \alpha) + (\alpha, \beta)}([\alpha + \beta] \otimes b) \quad ([\beta] \otimes b \in \mathbb{C}Q \otimes B).
$$

Let $X_m(\alpha)$ denote the operator given by

$$
X(\alpha, z) = \sum_{m \in \mathbb{Z}} X_m(\alpha) z^m.
$$

We define a map $\varepsilon : \mathbb{Q} \times \mathbb{Q} \to \{\pm 1\}$ by

$$
\varepsilon(\alpha_i, \alpha_j) = \begin{cases} 
-1 & (j = i, i + 1) \\
1 & \text{otherwise}
\end{cases}
$$

and $\varepsilon(\alpha + \alpha', \beta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha', \beta)$, $\varepsilon(\alpha, \beta + \beta') = \varepsilon(\alpha, \beta)\varepsilon(\beta, \beta')$.

**Theorem.** (FK81) The vector space $F = \mathbb{C}Q \otimes B$ has an $\hat{sl}_l$-module structure such that

$$(h_i \otimes 1)([\beta] \otimes b) = \langle \alpha, \beta \rangle ([\beta] \otimes b), \quad (h_i \otimes t^m)([\beta] \otimes b) = [\beta] \otimes p_i(m)b,$

and for a positive root $\alpha$

$$(e_{\alpha} \otimes t^m)([\beta] \otimes b) = \varepsilon(\alpha, \beta)X_m(\alpha)([\beta] \otimes b),
\quad (f_{\alpha} \otimes t^m)([\beta] \otimes b) = \varepsilon(\beta, \alpha)X_{-m}(-\alpha)([\beta] \otimes b)$$

and $c = 1$, $d = 0$.

**Remark.** Identify $Q$ with $(\mathbb{Z}^l)_o$ then we have

$$
\varepsilon \left( \alpha_i, (c_{\frac{1}{2}}, \ldots, c_{l-\frac{1}{2}}) \right) = (-1)^{c_{i+\frac{1}{2}}} \quad \varepsilon \left( (c_{\frac{1}{2}}, \ldots, c_{l-\frac{1}{2}}), \alpha_i \right) = (-1)^{c_{i+\frac{1}{2}}}. 
$$

### 3.3 Explicit formula for Frenkel-Kac construction

#### 3.3.1

We define a level-1 $\mathcal{H}_{l-1}$-action on $B^\otimes \tilde{f}$ by

$$
p_i(m) = p(m)_{i-\frac{1}{2}} - p(m)_{i+\frac{1}{2}}
$$

where $p(m)_k$ means the action of $p(m)$ on the $k$-th factor of $B^\otimes \tilde{f}$.

We will give an explicit formula for the representation of $\hat{sl}_l$ obtained from this $\mathcal{H}_{l-1}$-action on $B^\otimes \tilde{f}$ by Frenkel-Kac construction.

#### 3.3.2

By the map $CQ$ defined in 2.5.1, we can identify $\mathbb{C}Q \otimes B^\otimes \tilde{f}$ and $B$.

**Theorem.** The action of $\hat{sl}_l$ on $B$ obtained by Frenkel-Kac construction from the $\mathcal{H}_{l-1}$-module $B^\otimes \tilde{f}$ is given as follows:

- $\hat{e}_i b_\lambda = (-1)^{v_+(\lambda) + v_+}(\lambda) \sum (-1)^{\gamma-(\lambda, \mu \cdot i, \lambda)} b_{\mu}$, where the summation runs over all $\mu$ obtained from $\lambda$ by removing a removable $i$-node,
\[ \hat{f}_h b_\lambda = (-1)^{u_{-1}(\lambda) + u_0(\lambda)} \sum (-1)^{v(\mu \setminus \lambda, i, \lambda)} b_\mu, \] where the summation runs over all \( \mu \) obtained from \( \lambda \) by adding an addable \( i \)-node, and

\[ \hat{h}_i b_\lambda = (|A_{\lambda, i}| - |R_{\lambda, i}|) b_\lambda. \]

**Proof.** Here we only check the two actions of \( \hat{e}_i \) coincide. Note that \( \hat{e}_i = e_i \otimes 1 (i = 1, \ldots, l - 1), \hat{e}_0 = (-\sum e_i) \otimes t \). Since \( p(m)_k \) and \( p(m')_k \) commute each other for any \( m \) and \( m' \) if \( k \neq k' \), we have

\[
\exp \left( \sum_{m=1}^{\infty} \frac{z^{m}}{m} (p(-m)_{i} - p(-m)_{i+\frac{1}{2}}) \right) \exp \left( -\sum_{m=1}^{\infty} \frac{z^{-m}}{m} (p(m)_{i} - p(m)_{i+\frac{1}{2}}) \right)
\]

\[
= \Gamma^-_{i+\frac{1}{2}}(z) \circ \Gamma^+_i(z) \circ \Gamma^-_{i+\frac{1}{2}}(z) \circ \Gamma^-_{i+\frac{1}{2}}(z)
\]

\[
= \Gamma^-_{i+\frac{1}{2}}(z) \circ \Gamma^-_{i+\frac{1}{2}}(z) \circ \Gamma^+_i(z) \circ \Gamma^-_{i+\frac{1}{2}}(z)
\]

\[
= (\Psi_{i+\frac{1}{2}}(z)z^{-D_i - K_i} e^{-K_i \frac{1}{2}}) \circ (\Psi_{i+\frac{1}{2}}(z)z^{D_i + K_i \frac{1}{2}})
\]

\[
= \Psi_{i+\frac{1}{2}}(z) \circ \Gamma^-_{i+\frac{1}{2}}(z) z^{-D_i - K_i - D_i - K_i \frac{1}{2}} e^{-K_i \frac{1}{2}}
\]

By definition we have

\[
\exp (\log z \cdot \alpha_i(0) + \alpha_i) = e^{K_i} e^{-K_i - D_i} e^{1 + D_i - D_i}.
\]

Thus we get

\[
X(\alpha_i, z) = \Psi_{i+\frac{1}{2}}(z) \Psi_{i+\frac{1}{2}}^+(z) z
\]

\[= \sum_{m \in \mathbb{Z}} \left( \sum_{h \in \mathbb{Z} + \frac{1}{2}} \psi_{m+h, i+\frac{1}{2}} \psi_{m+h, i+\frac{1}{2}}^* \right) z^m,
\]
in particular,

\[
X_0(e_i) = \sum_{h \in \mathbb{Z} + \frac{1}{2}} \psi_{h, i+\frac{1}{2}} \psi_{h, i+\frac{1}{2}}^*,
\]

\[
X_1 \left( -\sum e_i \right) = \sum_{h \in \mathbb{Z} + \frac{1}{2}} \psi_{1+h, i+\frac{1}{2}} \psi_{h, i+\frac{1}{2}}^*.
\]

By 2.4.2 and 3.1.2 we have

\[
\psi_{h, i+\frac{1}{2}} \psi_{h, i+\frac{1}{2}}^* (b_\lambda)
\]

\[= \begin{cases} 
\eta^- (X, i, \lambda) \cdot b_{\lambda \setminus X} & (\lambda \text{ has a removable node } X \text{ with content } l(h - \frac{1}{2}) + i) \\
0 & \text{(otherwise)}.
\end{cases}
\]

and similar equations hold for \( \psi_{1+h, i+\frac{1}{2}} \psi_{h, i+\frac{1}{2}}^* \). Combine with Lemma 2.5.2 then the claim follows. \( \square \)
4 Quiver varieties

In this section we study some properties of quiver varieties of type $\hat{A}$.

We have an $S^1$-action on the quiver varieties, so that the fixed points are isolated. The quiver varieties associated with different generic parameters are $S^1$-equivariantly diffeomorphic to each other, and so there exists a natural bijection of fixed points.

For the construction of the representation of the affine Lie algebra, we use an "ordinary" parameter $\zeta_0$. On the other hand, the Hilbert scheme of points on the minimal resolution corresponds to a parameter $\zeta_\infty$ in a chamber adjacent to the level-0 hyperplane.

The fixed points of the quiver varieties of type $\hat{A}$ associated with an "ordinary" parameter are parametrized by $\Pi$. The fixed points of the Hilbert schemes of points on the minimal resolution of type $A$ are parametrized by $\Pi^l$. The main results of this section is that the bijection described above is nothing but the operation $l$-core and $l$-quotient (Theorem 4.5).

4.1 Quiver varieties

4.1.1 Let $(I,H)$ be a quiver, namely $I$ is a set of vertices and $H$ is a set of oriented edges. Assume we have a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$, $\Omega \cap \bar{\Omega} = \emptyset$ where $\bar{\Omega}$ means reversing orientations of edges. For $v, w \in \mathbb{Z}^I_{\geq 0}$ and $\zeta = (\zeta_C, \zeta_R) \in \mathbb{C}^I \oplus \mathbb{R}^I$, we define the quiver variety $M_{\zeta}(v,w)$ as follows ([Nak94]). Here we assume $v \in \mathbb{Z}^I_>$, since otherwise we need a slight and non-essential modification.

Let $V, W$ be $I$-graded vector spaces such that $\dim V_i = v_i$, $\dim W_i = w_i$. Then we set

$$M(v,w) = \left( \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \right) \oplus \left( \bigoplus_{i \in I} \text{Hom}(W_i, V_i) \oplus \text{Hom}(V_i, W_i) \right),$$

where $h \in H$ is drawn from $\text{out}(h)$ to $\text{in}(h)$. Note that $\text{GL}_V = \prod \text{GL}(V_i)$ acts on $M(v,w)$ by

$$(g_i) \cdot (B_h, a_i, b_i) = \left( g_{\text{in}(h)} B_h g_{\text{out}(h)}^{-1}, g_i a_i, b_i g_i^{-1} \right)$$

for $g_i \in \text{GL}(V_i)$, $B_h \in \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)})$, $a_i \in \text{Hom}(W_i, V_i)$ and $b_i \in \text{Hom}(V_i, W_i)$.

The moment map $\mu_{\zeta}$ is given by

$$\mu_{\zeta}(B, a, b) = \left( \sum_{\text{in}(h)=i} \varepsilon(h) B_h B_h + a_i b_i \right) \in \bigoplus_{i \in I} \text{gl}(V_i) = \text{gl}_V,$$

where $\varepsilon(h) = 1$ and $\varepsilon(h) = -1$ for $h \in \Omega$. Note that the center of $\text{gl}_V$ is canonically identified with $\mathbb{C}^I$.

The quiver variety associated to $v, w$ and $\zeta$ is

$$\mathcal{M}_{\zeta}(v,w) = (\mu_{\zeta}^{-1}(\zeta_C))^{ss} \parallel \text{GL}_V.$$

By the index "ss", we mean the set of $\zeta$-semistable objects in $\mu_{\zeta}^{-1}(\zeta_C)$. We omit the definition of $\zeta$-semistability which is not required in the rest of this paper.
4.1.2

Let $A = (a_{ij})_{i,j \in I}$ be the adjacency matrix of the quiver $(I, H)$, that is to say $a_{ij}$ is the number of elements of $H$ which is drawn from $i$ to $j$. Let $C = 2 \cdot \text{id} - A$ be the Cartan matrix. Then we consider the set of positive roots

$$R_+ = \{ \theta = (\theta_i) \in (\mathbb{Z}_{\geq 0})^I \mid \theta C \theta \leq 2 \} \setminus \{0\},$$

and for $v \in \mathbb{Z}^I$ we set

$$R_+(v) = \{ \theta \in R_+ \mid \theta_i \leq v_i, \forall i \in I \}.$$ 

The element $\zeta \in \mathbb{C}^I \oplus \mathbb{R}^I$ is called generic with respect to $v$ if for any $\theta \in R_+(v)$

$$\zeta \notin \theta C \theta \otimes \mathbb{R}^3 \subset \mathbb{R}^I \otimes \mathbb{R}^3 \simeq \mathbb{C}^I \oplus \mathbb{R}^I.$$

where

$$D_\theta = \left\{ \zeta = (\zeta') \in \mathbb{R}^I \mid \sum \zeta' \theta_i = 0 \right\} \subset \mathbb{R}^I.$$

It is known that

- $\mathcal{M}_\zeta(v, w)$ is smooth if $\zeta$ is generic with respect to $v$. Since $\mathbb{R}^I \otimes \mathbb{R}^3 \setminus \cup_\theta D_\theta \otimes \mathbb{R}^3$ is connected, $\mathcal{M}_\zeta(v, w)$ and $\mathcal{M}_\zeta'(v, w)$ are diffeomorphic for generic $\zeta$, $\zeta'$.

- Let us fix $\zeta' \in \mathbb{C}^I$. Then we have a chamber structure on $\pi^{-1}(\zeta')$ defined by $D_\theta$'s, where $\pi$ is the projection $\mathbb{C}^I \oplus \mathbb{R}^I \to \mathbb{C}^I$. If $\zeta$ and $\zeta'$ are in a same chamber, $\mathcal{M}_\zeta(v, w)$ and $\mathcal{M}_\zeta'(v, w)$ are isomorphic.

- For generic $\zeta$, $\mathcal{M}_\zeta(v, w)$ is a fine moduli space of representations of the quiver. Namely, on $\mathcal{M}_\zeta(v, w)$ there exists a universal family of representations of the quiver $(\mathfrak{B}, \mathfrak{M}, \mathfrak{B}, a, b)$.

Hereafter we sometimes identify a point of a quiver variety with the corresponding representation of the quiver.

4.1.3

We define an $S^1$-action on $\mathcal{M}_\zeta(v, w)$ by

$$t * [(B_{\Omega}, B_{\Omega}, a, b)] = [(tB_{\Omega}, t^{-1}B_{\Omega}, a, b)].$$

Note that this action may or may not be trivial. For generic $\zeta$ and $\zeta'$, $\mathcal{M}_\zeta(v, w)$ and $\mathcal{M}_\zeta'(v, w)$ are $S^1$-equivariantly diffeomorphic.

Assume the $S^1$-fixed points are isolated for generic parameters. Then for generic $\zeta$ and $\zeta'$ we have a canonical bijection between the $S^1$-fixed points of $\mathcal{M}_\zeta(v, w)$ and of $\mathcal{M}_\zeta'(v, w)$ induced by the $S^1$-equivariant diffeomorphism.

4.1.4

For $[(B_{\Omega}, B_{\Omega}, a, b)] \in \mathcal{M}_\zeta(v, w)^{S^1}$, there exists a group homomorphism $\rho: S^1 \to \text{GL}_v$ such that

$$t * (B_{\Omega}, B_{\Omega}, a, b) = \rho(t)(B_{\Omega}, B_{\Omega}, a, b)$$

($\rho$ is uniquely determined because of the stability condition). This induces an $S^1$-action on $V$. The bijection described above preserve $V \in R(S^1)$ where $R(S^1) \simeq \mathbb{Z}[z^\pm]$ is the representation ring of $S^1$. 

12
4.2 Minimal resolution

Let \((I, \Omega)\) be the quiver of type \(\hat{\mathbb{A}}_{l-1}\) with cyclic orientation. We set \(w_0 = (1, 0, 0, \ldots, 0)\), \(\delta = (1, \ldots, 1) \in \mathbb{Z}^I\).

From now on we will consider the quiver varieties \(\mathfrak{M}_{(0, \zeta R)}(v, w_0)\) of type \(\hat{\mathbb{A}}_{l-1}\) only. We write simply \(\zeta\) and \(\mathfrak{M}_{\zeta}(v)\) for \((0, \zeta R)\) and \(\mathfrak{M}_{(0, \zeta R)}(v, w_0)\).

4.2.1 Theorem. ([Kro89]) Assume \(\zeta\) is generic with respect to \(\delta\), then \(\mathfrak{M}_{\zeta}(\delta)\) is isomorphic to the minimal resolution of \(\mathbb{C}^2/\langle \mathbb{Z}/l\mathbb{Z} \rangle\).

Remark. For any generic \(\zeta\)'s, \(\mathfrak{M}_{\zeta}(\delta)\)'s are isomorphic to each other, but the universal family depends on the chamber.

4.2.2 We set \(C_0 = \{\zeta \in \mathbb{R}^I \mid \zeta^i < 0 \text{ for all } i \in I\}\).

For \(\zeta_0 \in C_0\), \(\zeta_0\)-semistability condition is equivalent to that there exists no proper subspace of \(V\) which is \(B_i\)-invariant and contains \(\text{Im} a\).

Proposition. For \(\zeta_0 \in C_0\), \(\mathfrak{M}_{\zeta_0}(\delta)\) has \(l S^1\)-fixed points \(P_1, P_2, \ldots, P_{l-1}\) and \(P_k\) corresponds to the following representation:

\[
B_{i,i+1} = \begin{cases} 
1 & (i = 0, \ldots, k - \frac{3}{2}) \\
0 & (i = k - \frac{3}{2}, \ldots, l - 1), 
\end{cases}
\]

\[
B_{i,i-1} = \begin{cases} 
0 & (i = 1, \ldots, k + \frac{1}{2}) \\
1 & (i = k + \frac{1}{2}, \ldots, l), 
\end{cases}
\]

\[a = 1, \ b = 0.\]

Here we identify the Hom spaces with \(\mathbb{C}\) by certain bases of 1-dimensional spaces \(V_i\) and \(W_i\).s.

Proof. For \(i, j \in \mathbb{Z}\) we define \(B_{i,j} \in \text{Hom}(V_i, V_j)\)

\[
B_{i,j} = \begin{cases} 
B_{j-1,j} \cdots B_{i,i+1} & (i < j), \\
1 & (i = j), \\
B_{j+1,j} \cdots B_{i,i-1} & (i > j).
\end{cases}
\]

Assume that \(B_{0,k-\frac{3}{2}} \neq 0\) for \(k \in \hat{I}\). We can consider the ratio \(B_{l,k-\frac{3}{2}}/B_{0,k-\frac{3}{2}} \in \mathbb{C}\) of the two elements in \(\text{Hom}(V_0, V_{k-\frac{3}{2}}) = \text{Hom}(V_l, V_{k-\frac{3}{2}}) \simeq \mathbb{C}\). For a fixed point we have

\[
B_{l,k-\frac{3}{2}}/B_{0,k-\frac{3}{2}} = t^* \left( B_{l,k-\frac{3}{2}}/B_{0,k-\frac{3}{2}} \right) \\
= t^l \cdot B_{l,k-\frac{3}{2}}/B_{0,k-\frac{3}{2}} \quad (t \in S^1),
\]

and so \(B_{l,k-\frac{3}{2}}\) should be 0.

By the parallel argument we also have \(B_{0,l} = 0\). So without loss of generality we may assume further \(B_{0,k+\frac{1}{2}} = 0\). Since \(B_{0,k-\frac{3}{2}} \neq 0\) we have \(B_{k-\frac{3}{2},k+\frac{1}{2}} = 0\).
The stability condition assures $B_{l,k+\frac{1}{2}} \neq 0$. Since $B_{l,k-\frac{1}{2}} = 0$ we have $B_{k+\frac{1}{2},k-\frac{1}{2}} = 0$.

Vanishing of the value of the moment map assures $B_{i,i+1} = 0$ ($i = k + \frac{1}{2}, \ldots, l - 1$) and $B_{i,i-1} = 0$ ($i = 1, \ldots, k - \frac{1}{2}$).

By the $GL_v$-action we get the normalized forms as in the statement. □

4.2.3

Proposition. We have a coordinate $(x,y)$ of $\mathbb{M}_\delta(\delta)$ on a neighborhood of $P_k$ such that the representation corresponding to a point $(x,y)$ is given by

$$B_{i,i+1} = \begin{cases} 1 & (i = 0, \ldots, k - \frac{3}{2}) \\ x & (i = k - \frac{1}{2}) \\ xy & (i = k + \frac{1}{2}, \ldots, n - 1), \end{cases}$$

$$B_{i,i-1} = \begin{cases} xy & (i = 1, \ldots, k - \frac{1}{2}) \\ y & (i = k + \frac{1}{2}) \\ 1 & (i = k + \frac{3}{2}, \ldots, n), \end{cases}$$

$a = 1, b = 0$.

In this coordinate, the $S^1$-action is described as $t^a(x, y) = (t^a x, t^{-a} y)$.

Proof. Note that $B_{i,i+1}$ ($i = 0, \ldots, k - \frac{3}{2}$) and $B_{i,i-1}$ ($i = k + \frac{3}{2}, \ldots, l$) do not vanish around $P_k$ and we can normalize them to 1.

Vanishing of the value of the moment map assures that the values of $B_{k+\frac{1}{2},k-\frac{1}{2}}$ determine the representation. We can see different values of $B_{k+\frac{1}{2},k-\frac{1}{2}}$ induce non-isomorphic representations. So the statement follows. □

4.2.4

Example. (the case $l = 4$)

The following figure exhibits the representations corresponding to the fixed point $P_3$ and a point $(x,y)$ in the local coordinate around $P_3$.

The three curves exhibit the exceptional curves in the minimal resolution, and the four dots exhibit the fixed points.

We have five dots in each box. The top dot represents the basis of $W$ and the others represent the basis of $V$. The numbering of the vertices of the quiver is given counterclockwise.
4.3 \( \mathbb{Z}/l\mathbb{Z} \)-equivariant Hilbert scheme (\( \zeta_0 \)-case)

4.3.1

Let \((C^2)^{[n]} \) denote the Hilbert scheme of \( n \) points on \( C^2 \):

\[
(C^2)^{[n]} = \{ J \subset \text{ideal } C[z_1, z_2] \mid \dim C[z_1, z_2]/J = n \}.
\]

The cyclic group \( \mathbb{Z}/l\mathbb{Z} \) acts on \( C^2 \) by \( r \cdot (z_1, z_2) = (rz_1, z_2) \) (\( r \in \mathbb{Z}/l\mathbb{Z} \)). This action induces an action of \( \mathbb{Z}/l\mathbb{Z} \) on \((C^2)^{[n]} \). Let \(( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}} \) be the set of fixed points. For \( J \in ( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}} , C[z_1, z_2]/J \) has a canonical \( \mathbb{Z}/l\mathbb{Z} \)-module structure. Let \(( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}, v} \subset ( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}} \) denote the set of points such that the corresponding \( \mathbb{Z}/l\mathbb{Z} \)-module is isomorphic to \( \bigoplus_i (C(z(i))^{\mathbb{Z}/v_i}) \), where \( C(z(i)) \) is the 1-dimensional representation of \( \mathbb{Z}/l\mathbb{Z} \) with weight \( i \).

**Theorem.** ([Nak99] Theorem 4.4)

(i) For \( \zeta_0 \in C_0 \), \( \mathfrak{M}_{\zeta_0}(v) \) is isomorphic to \( ( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}, v} \).

(ii) The representation \((V(J), W(J), B(J), a(J), b(J))\) of the quiver corresponding to \( J \in ( (C^2)^{[n]} )^{\mathbb{Z}/l\mathbb{Z}} \) is given as follows:

\[
V(J)_i = \text{Hom}_{\mathbb{Z}/l\mathbb{Z}}(C(z(i)), C[z_1, z_2]/J), \quad W(J)_0 = C
\]

and

\[
B(J)_{i,i+1} = z_1|_{V_i}: V_i \to V_{i+1}, \quad B(J)_{i,i-1} = z_2|_{V_i}: V_i \to V_{i-1},
\]

\[
a(J)(1) = [1] \in C[z_1, z_2]/J, \quad b(J) = 0,
\]

where \( z_1 \) and \( z_2 \) represents the multiple operators on \( C[z_1, z_2]/J \).
4.3.2
Let us consider the $S^1$-action on $C^2$ given by $t \cdot (z_1, z_2) = (tz_1, t^{-1}z_2)$. This induces a $S^1$-action on $(C^2)^m$. The embedding $\mathfrak{m}_{\mathfrak{g}_0}(v) \cong \left((C^2)^m\right)^{\mathbb{Z}/l\mathbb{Z}} \subset (C^2)^m$ is $S^1$-equivariant. So we get $\mathfrak{m}_{\mathfrak{g}_0}(v)^{S^1} = \mathfrak{m}_{\mathfrak{g}_0}(v) \cup \left((C^2)^m\right)^{S^1}$.

The $S^1$-fixed points of $(C^2)^m$ are parametrized by Young diagrams (Nak99). For $\lambda \in \Pi$ the corresponding ideal $J_\lambda \in \left(C^2\right)^{[\lambda]}$ is the ideal generated by $\{z_1^a z_2^b \mid (a, b) \not\in \lambda\}$. Then $\{z_1^a z_2^b \in \mathbb{C}[z_1, z_2]/J_\lambda \mid (a, b) \in \lambda\}$ forms a basis of $\mathbb{C}[z_1, z_2]/J_\lambda$. Since $r \cdot [z_1^a z_2^b] = r^a b [z_1^a z_2^b]$ for $r \in \mathbb{Z}/l\mathbb{Z}$ we have $\mathfrak{m}_{\mathfrak{g}_0}(v)^{S^1} = \{J_\lambda \mid v(\lambda) = v\}.$

4.3.3
Proposition. The representation $(V(J_\lambda), W(J_\lambda), B(J_\lambda), a(J_\lambda), b(J_\lambda))$ of the quiver corresponding to a fixed point $J_\lambda$ is described as follows:

$$V(J_\lambda)_i = \bigoplus_{a-b=1, (a,b) \in \lambda} \mathbb{C} v(a,b), \quad W(J_\lambda)_0 = \mathbb{C} w,$$

and

$$B(J_\lambda)_{i,i+1}(v_{a,b}) = \begin{cases} v_{a+1,b} & \text{if } (a+1, b) \in \lambda, \\ 0 & \text{if } (a+1, b) \not\in \lambda, \end{cases}$$

$$B(J_\lambda)_{i,i-1}(v_{a,b}) = \begin{cases} v_{a,b+1} & \text{if } (a, b+1) \in \lambda, \\ 0 & \text{if } (a, b+1) \not\in \lambda, \end{cases}$$

$$a(J_\lambda)(w) = v_{(0,0)}, \quad b(J_\lambda) = 0.$$

Proof. Apply Theorem 4.3.1(ii) for $\mathbb{C}[x,y]/J_\lambda.$

Corollary.

$$V(J_\lambda) = f_\lambda(z) \in R(S^1).$$

Proof. We can see $\rho(t)(v_{a,b}) = t^a b v_{a,b}$. So the claim follows.

4.3.4
The character of the tangent space of $(C^2)^m$ at fixed point $J_\lambda$ is

$$T_{J_\lambda}(C^2)^m = \sum_{(a,b) \in \lambda} t^h(a,b) + t^{-h(a,b)}$$

where $h(a,b)$ is the hook length of the hook associated with a node $(a,b)$ (Nak99). Further we can see

$$T_{J_\lambda} \mathfrak{m}_{\mathfrak{g}_0}(v) = \sum_{(a,b)} t^h(a,b) + t^{-h(a,b)}$$

where the summation runs over all $(a,b) \in \lambda$ such that $h(a,b) \equiv 0 \pmod{l}$.

Remark. In term of Maya diagrams, such a hook corresponds to

$$(k, k+nl) \in (\mathbb{Z} + 1/2)^2$$

such that $n > 0$, $m(k) = 1$, $m(k+nl) = -1$ and its hook length is $nl$. 

16
4.4 Hilbert scheme of the minimal resolution ($\zeta_\infty$-case)

4.4.1

The hyperplane $D_\delta = \{ \zeta \in \mathbb{R}^l \mid \sum \zeta_i = 0 \}$ of $\mathbb{R}^l$ is called the level 0 hyperplane. We have the chamber structure on this hyperplane defined by $D_\alpha$’s, where $\alpha$ is a root of finite root system, and

$$C = \{ \zeta \in D_\delta \mid \zeta_i > 0 \ (i = 1, \ldots, l-1) \} \subset D_\delta$$

is one of the chambers.

For $\mathbb{C}$ torsion free sheaf $\alpha$ where $\tilde{\mathcal{E}}$ means the natural morphism $E$ free sheaf $\tilde{\mathcal{E}}$ action compatible with the one on $M$.

Remark. (i) For a representation $\chi(\delta)$ of $\zeta_\infty$ $\in C_\infty(\chi)$ the quiver variety $\mathcal{M}_{\zeta_\infty}(\chi)$ is a certain moduli space of torsion free sheaves on $M$. In next subsection we will review this result very briefly. The reader can refer [Nak] for detail.

4.4.2

Let us consider the action of $\mathbb{Z}/l\mathbb{Z}$ on $\mathbb{P}^2$ given by $r[x : y : z] = [rx : r^{-1}y : z]$. Let $\bar{M}$ denote the orbifold which is a compactification of $M$ given by resolving the singular point $[0 : 0 : 1]$ of $\mathbb{P}^2/(\mathbb{Z}/l\mathbb{Z})$. Note that $\bar{M}$ has the natural $S^1$-action compatible with the one on $M$. We set $l_\infty = \bar{M} \setminus M$.

Let $\bar{V} = \oplus \bar{V}_i$ be the universal bundle on $\bar{M}$ and $\bar{V} = \oplus \bar{V}_i$ be its extension to $\bar{M}$. Note that $\bar{V}$ has the unique $S^1$-equivariant structure such that the restriction of the action to $l_\infty$ is trivial.

Theorem. For $\zeta_\infty \in C_\infty(\chi)$ the quiver variety $\mathcal{M}_{\zeta_\infty}(\chi)$ is the fine moduli space of rank 1 torsion free sheaves $E$ on $\bar{M}$ such that $E|_{l_\infty}$ is trivial and $c_1(E)$ and $c_2(E)$ take values given in (1.8) of [Nak].

Remark. (i) For a representation $(V, W, B_\Omega, B_\Theta, a, b)$, corresponding torsion free sheaf $E(V, W, B_\Omega, B_\Theta, a, b)$ is given as the cohomology of the following complex of sheaves:

$$V \otimes \bar{V} \oplus$$

$$C^\bullet(V, W, B_\Omega, B_\Theta, a, b) : V \otimes \bar{V}(l_\infty) \overset{\alpha}{\rightarrow} V \otimes \bar{V} \overset{\beta}{\rightarrow} V \otimes \bar{V}(l_\infty) \oplus W \otimes \bar{V}$$

where

$$\alpha = \begin{pmatrix} B_\Omega \otimes \bar{1} - 1 \otimes \bar{B}_\Omega \\ B_\Omega \otimes \bar{1} - 1 \otimes \bar{B}_\Omega \\ a \otimes \bar{1} \end{pmatrix}, \quad \beta = \begin{pmatrix} -B_\Theta \otimes \bar{1} + 1 \otimes \bar{B}_\Theta, B_\Theta \otimes \bar{1} - 1 \otimes \bar{B}_\Theta, b \otimes \bar{1} \end{pmatrix}.$$

Here $\bar{B}$ is the extension of universal family $B : \mathcal{V} \rightarrow \mathcal{V}$ of maps to $M$ and $\bar{1}$ means the natural morphism $\bar{V}(l_\infty) \rightarrow \bar{V}$ and $\bar{V} \rightarrow \bar{V}(l_\infty)$.

(ii) For a torsion free sheaf $E$, the representation spaces $V(E)$ and $W(E)$ are $H^1(E \otimes \bar{V})$ and the fiber of $E$ on a point in $l_\infty$ respectively. Although we omit the constructions of the maps, we mention that they are functorial.
4.4.3
For $E \in \mathcal{M}_\infty(v)$ its double dual $E^{\vee\vee}$ is a line bundle on $\tilde{M}$ such that its restriction to $l_\infty$ is trivial and $c_1(E^{\vee\vee}) = c_1(E)$. Note that $E^{\vee\vee}$ is determined uniquely by these conditions. In fact, since $c_1(E) = \sum_{i \neq 0} u_i c_1(\tilde{V}_i)$ where $u = -C v$ ((1.8) of [Nak]) and $\{c_1(\tilde{V}_i)\}$ is the dual basis of $\{[C_i]\}$ ([KN90] Proposition 2.2), we can check $E^{\vee\vee} = \mathcal{O}(\sum_{i=1}^{l-1} (v_i - v_0) C_i)$.

The quotient sheaf $E^{\vee\vee}/E$ is supported at finitely many points on $M$ and its length equals to $n = \text{ch}_2(E^{\vee\vee}) - \text{ch}_2(E)$, which we can calculate from (1.8) of [Nak]. The map $E \mapsto E^{\vee\vee}/E$ induces the isomorphism between $\mathcal{M}_\infty(v)$ and the Hilbert scheme $M^{[n]}$ of $n$ points on $M$. Since dim $\mathcal{M}_\infty(v) = v C^1 v + 2v_0$ we have $n = vC^1 v/2 + v_0$.

For $c = (c_0, \ldots, c_l) \in Q$ and $n \in \mathbb{Z}_{\geq 0}$ let $v \in \mathbb{Z}^l$ be the unique elements such that $c_k = v_{k-1} - v_{k+1}$ and $n = vC^1 v/2 + v_0$. Let $\phi_{c,n} = \phi_v$ denote the isomorphism between $M^{[n]}$ and $\mathcal{M}_\infty(v)$.

4.4.4
Let $\text{Sym}^n(M)$ be the $n$-th symmetric product of $M$ and $\pi: M^{[n]} \to \text{Sym}^n(M)$ be the Hilbert-Chow morphism.

The $S^1$-action on $M$ induces $S^1$-actions on $M^{[n]}$ and $\text{Sym}^n(M)$ so that $\pi$ is $S^1$-equivariant. We can see

$$\text{Sym}^n(M)^{S^1} = \left\{ \sum n_k [P_k] \mid \sum n_k = n \right\}.$$ 

There exists a neighborhood of $\sum n_k [P_k]$ which is isomorphic to some open set in $\prod \text{Sym}^n(C^2)$. The inverse image for $\pi$ of this open set is isomorphic to some open set in $\prod (C^2)^{[n]}$. So $(M^{[n]})^{S^1}$ is parametrized by $l$-tuple of Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_l) \in \Pi^l$ such that $\sum |\lambda_k| = n$. Let $J_\lambda$ denote the corresponding fixed point.

4.4.5
Lemma. The $S^1$-action on the moduli space $\mathcal{M}_\infty(v)$ induced by the $S^1$-action on $M$ coincides with the one given in 4.1.3

Proof. The $S^1$-equivariant structure of $\tilde{V}$ induces the isomorphism $\text{Hom}(\tilde{V}, \tilde{V}) = \text{Hom}(t^* \tilde{V}, t^* \tilde{V})$ for $t \in S^1$. Under this isomorphism we have $t^* \mathcal{O}_\tilde{M} = t^{-1} \mathcal{O}_\tilde{M}$ and $t^* \mathcal{O}_O = t \mathcal{O}_O$. Thus the complex $C^\bullet(V, W, t^* \mathcal{O}_O, t^{-1} \mathcal{O}_O, a, b)$ is isomorphic to the complex $t^* C^\bullet(V, W, \mathcal{O}_O, a, b)$. Take the cohomology of the complex, then the claim follows.

The isomorphism $\phi_{c,n}$ given in 4.4.3 is $S^1$-equivariant. For $c \in Q$ and $\lambda \in \Pi^l$, we set $E_{c,\lambda} = \phi_{c,n}(J_\lambda)$. This gives a bijection between $Q \times \Pi^l$ and $\mathcal{M}_\infty(v)^{S^1}$.

4.4.6
For $E \in \mathcal{M}_\infty(v)^{S^1}$, $E^{\vee\vee} = \mathcal{O}(\sum_{i=1}^{l-1} (v_i - v_0) C_i)$ has the unique $S^1$-equivariant structure such that the restriction of the action on $l_\infty$ is trivial, and so is $E$. 

18
Lemma. The $S^1$-action on $V(E) = H^1(E \otimes \tilde{V})$ induced by the $S^1$-equivariant structures on $\tilde{V}$ and $E$ coincides with the one given in 4.1.4.

Proof. The $S^1$-action on $W(E)$ induced by the $S^1$-equivariant structures on $\tilde{V}$ and $E$ is trivial. Since every construction is functorial with respect to $E$, the homomorphism $S^1 \to GL(V(E))$ satisfies the condition of $\rho$ described in 4.1.4. □

4.4.7

Proposition. For $c \in Q$ we set $\lambda(c) = CQ^{-1}(c, (0, \ldots, 0))$. Then we have

$$H^1(E_c, \tilde{V}) = f_{M(c)}(z) + \sum_k z^{lc_k} f_{\lambda_k}(z) \left( z^{k-t+c} + \ldots + z^{k-\frac{c}{2}} \right) \in R(S^1).$$

Proof. The quiver variety $\mathfrak{M}_\infty(\nu(\lambda(c)))$ is one point and the point corresponds to $E^{\\nu\nu}$. So we have $H^1(E^{\\nu\nu} \otimes \tilde{V}) = f_{M(c)}(z) \in R(S^1)$.

Since $H^0(\tilde{M}, E^{\\nu\nu} \otimes \tilde{V}) = 0$ ([Nak], 5(ii)) we have the following exact sequence of $S^1$-module:

$$0 \to H^0(\tilde{M}, E^{\\nu\nu} \otimes \tilde{V}) \to H^1(E \otimes \tilde{V}) \to H^1(E^{\\nu\nu} \otimes \tilde{V}) \to 0.$$

Let $(E^{\\nu\nu}/E)_k$ denote the direct summand of $E^{\\nu\nu}/E$ supported on $P_k$. On the neighborhood of $P_k$ given in 4.2.3 we have

$$(E^{\\nu\nu}/E)_k \otimes \tilde{V} = \mathcal{O}/J_{\lambda_k} \otimes E^{\\nu\nu} \otimes \tilde{V} \cong \mathbb{C}[x,y]/J_{\lambda_k} \otimes E^{\\nu\nu}_{P_k} \otimes \tilde{V}_{P_k}.$$

Recall that the $S^1$-action on this neighborhood is given by $t \ast (x, y) = (t^l x, t^{-l} y)$. We have $\mathcal{O}/J_{\lambda_k} = f_{\lambda_k}(z)$ as in 4.4.4. Since $E^{\\nu\nu} = \mathcal{O}/(v_k - v_0)(x\text{-axis}) + (v_k + \frac{c}{2} - v_0)(y\text{-axis})$ on the neighborhood of $P_k$, the weight of $E^{\\nu\nu}_{P_k}$ is $l(v_k - \frac{1}{2} - v_k + \frac{1}{2}) = lc_k$. Using the description in 4.2.2 we can see $\tilde{V}_{P_k} = z^{k-t+c} + \ldots + z^{k-\frac{c}{2}}$. Then the claim follows. □

4.5 Correspondence of fixed points

Theorem. The following diagram is commutative:

\[
\begin{array}{ccc}
\Pi \mathfrak{M}_\infty(\nu)^{S^1} & \longrightarrow & \Pi \\
\big\downarrow^{4.1.3} & & \big\downarrow^{4.4.3} \\
\Pi \mathfrak{M}_\infty(\nu)^{S^1} & \longrightarrow & Q \times \Pi \tilde{f}.
\end{array}
\]

Proof. Note that the map $\Pi \to \mathbb{Z}[z^\pm]$ given by $\lambda \mapsto f_\lambda(z)$ is injective. By 4.1.3 it is enough to check $V(J_{\lambda}) = V(E_{c(\lambda)} q(\lambda)) \in R(S^1)$. This follows from Proposition 2.5.3 and Proposition 4.4.7. □
5 Representations on equivariant cohomologies

In this section we study the representations of the affine Lie algebra and the Heisenberg algebra on the middle degree $S^1$-equivariant cohomology groups of the quiver varieties.

First we see that the middle degree $S^1$-equivariant cohomology groups of quiver varieties has bases indexed by the $S^1$-fixed points (Proposition 5.2.5).

The affine Lie algebra $\hat{sl}_n$ acts on $\oplus_v H_{S^1}^{\text{mid}}(M_{\zeta_0}(v))$ and the Heisenberg algebra $H_{l-1}$ acts on $\oplus_n H_{S^1}^{\text{mid}}(M_{\zeta_\infty}(n\delta))$. The main purpose of this section is to describe these actions with respect to above bases.

For the affine Lie algebra the argument works parallel with the one in [VV99], dealing with the equivariant K-groups (Proposition 5.3.3). For the Heisenberg algebra we can find a formula in [QW] (Proposition 5.4.2).

5.1 Equivariant cohomology groups

We review some general results about $S^1$-equivariant cohomology groups. For more details, the reader can refer to [Aud04] for example.

5.1.1

We take $\mathbb{C}$ as the coefficient ring of cohomology groups. Let $ES^1 \to BS^1$ be the universal $S^1$-bundle which is given as the inductive limit of the Hopf fibration $S^{2n+1} \to \mathbb{CP}^n$. Note that the cohomology ring of $BS^1 \cong \mathbb{CP}^\infty$ is the polynomial ring $\mathbb{C}[t]$ with a generator $t \in H^2(BS^1)$.

For a topological space $X$ with an $S^1$-action, we define the $S^1$-equivariant cohomology group of $X$ by

$$H^*_{S^1}(X) = H^*(ES^1 \times_{S^1} X).$$

5.1.2

Let $X, Y$ be $S^1$-equivariant topological spaces and $f: X \to Y$ be an $S^1$-equivariant map. Then we can define the following operators

$$U: H^*_{S^1}(X) \otimes_{\mathbb{C}} H^*_{S^1}(X) \to H^*_{S^1}(X) \quad \text{(cup product)},$$

$$f^*: H^*_{S^1}(Y) \to H^*_{S^1}(X) \quad \text{(pullback)}.$$  

Note that pullbacks preserve cup products.

Let $p: X \to \{pt\}$. We have the action of $\mathbb{C}[t] = H^*_{S^1}(\{pt\})$ on $H^*_{S^1}(X)$ induced by $p_*$ and $\cup$. We can see cup products and pullbacks commute with the $\mathbb{C}[t]$-actions.

5.1.3

Let $Y$ a smooth $S^1$-manifold and $X$ be an $S^1$-invariant codimension $d$ smooth submanifold of $Y$. Let $i: X \to Y$ denote the embedding. Then we can define a map

$$i_*: H^*_{S^1}(X) \to H^{*+d}_{S^1}(Y).$$

(see [Aud04] VI.4.c.). We set

$$[X] = i_*(1_X) \in H^d_{S^1}(Y).$$
where \(1_X \in H^0_{S^1}(X)\).

Let \(W\) be a smooth \(S^1\)-manifold and \(\pi: Z \to W\) be an \(S^1\)-equivariant fibre bundle whose fibre is a \(d\)-dimensional compact smooth manifold. Then we can define a map

\[
\pi_*: H^*_{S^1}(Z) \to H^{* - d}_{S^1}(W).
\]

(see [Aud04] VI.4.c.). Even in the case the fibre is not compact, if an element \(\alpha \in H^*_{S^1}(Z)\) has the compact support on each fibre then we can define the element \(\pi_*(\alpha) \in H^{* - d}_{S^1}(W)\).

**Remark.** Roughly speaking, \(i_*\) is induced by the Thom isomorphism with respect to the normal bundle on \(X\) to \(Y\) and \(\pi_*\) is induced by the integration along fibres.

### 5.1.4

For an \(S^1\)-equivariant vector bundle \(Z\) on an \(S^1\)-manifold \(X\), we define the \(S^1\)-equivariant Euler class by

\[
e_{S^1}(Z) = e(ES_1 \times_{S^1} Z) \in H^*(ES_1 \times_{S^1} X) = H^*_{S^1}(X)
\]

where \(e(\cdot)\) represents the Euler class of a vector bundle. To be precise, we define as the limit of \(e(S^{2n+1} \times_{S^1} Z)\).

**Lemma.** Let the all manifolds and morphisms below be smooth.

(i) Let \(i: X \to Y\) be an \(S^1\)-equivariant embedding and \(\nu\) denote the normal bundle on \(X\) to \(Y\). Then we have

\[
i^* i_* \alpha = \alpha \cup e_{S^1}(\nu) \quad (\alpha \in H^*_{S^1}(X)).
\]

(ii) (projection formula)

Let \(i: X \to Y\) be an \(S^1\)-equivariant embedding. Then we have

\[
i_* (\alpha \cup i^* \beta) = i_* \alpha \cup \beta \quad (\alpha \in H^*_{S^1}(X), \beta \in H^*_{S^1}(Y)).
\]

(iii) Let \(i: X \to Y\) be an \(S^1\)-equivariant embedding and \(Z\) be an \(S^1\)-manifold. If \(\alpha \in H^*_{S^1}(X \times Z)\) has the compact support on each fibre of \(p_X: X \times Z \to X\), then \((i \times \text{id})_* \alpha \in H^*_{S^1}(Y \times Z)\) also has the compact support on each fibre of \(p_Y: Y \times Z \to Y\) and

\[
i_*(p_X^*(\alpha)) = p_Y^*((i \times \text{id})_* (\alpha)).
\]

(iv) (Thom class is Poincare dual of the zero section)

Let \(\pi: Z \to W\) be an \(S^1\)-equivariant vector bundle and \(s: W \to Z\) be the zero section. Let \(p: W \to \{pt\}\). If \(w \in H^*_{S^1}(Z)\) has the compact support, then we have

\[
p_* \circ \pi_*(w \cup s_*(1_Z)) = p_* \circ s^*(w).
\]

**Proof.** We can prove those claims by calculations of differential forms on the finite dimensional approximation \(S^{2n+1} \times_{S^1} X\). For (i), see Proposition VI.4.6 in [Aud04]. The reader can refer to Proposition 6.15 and Proposition 6.24 in [BT82] for non-equivariant version of (ii) and (iv).

\[\square\]
5.1.5

We set \( R = \mathbb{C}(t) \) and \( H^*_S(X) = H^*_S(X) \otimes_{\mathbb{C}[t]} R \).

Let \( X \) be an \( S^1 \)-manifold and \( i \) denote the inclusion \( X^{S^1} \hookrightarrow X \). Let \( \{Z_\lambda\}_{\lambda \in \Lambda} \) denote the set of connected components of the fixed point set, \( i_\lambda \) denote the inclusion \( Z_\lambda \hookrightarrow X \) and \( \nu_\lambda \) denote the normal bundle on \( Z_\lambda \) to \( X \). We can check that \( e_{S^1}(\nu_\lambda) \) is invertible in \( H^*_S(X) \).

**Theorem.** (AB87) The map \( i_*: H^*_S(X^{S^1}) \to H^*_S(X) \) is isomorphism. In particular, for \( x \in H^*_S(X) \) we have

\[
x = \sum_\lambda i^*_{\lambda} \left( \frac{i^*_\lambda x}{\varepsilon_{S^1}(\nu_\lambda)} \right).
\]

5.2 Equivariant cohomology groups of quiver varieties

5.2.1

**Lemma.**

\[
H^m(M_\zeta(v)) = 0, \quad (m \text{ is odd, or } m > n = \frac{1}{2} \dim M_\zeta(v)).
\]

**Proof.** A quiver variety is endowed with a symplectic structure and an \( S^1 \)-action which preserve the symplectic form. The moment map is a perfect Morse function (Nak99 §5.1) and all the indices are even. So the odd degree cohomology groups vanish.

A quiver variety is homotopy equivalent to a certain Lagrangean sub variety (Nak94 Corollary 5.5). So the cohomology groups with degree larger than the half of the dimension vanish. \( \square \)

5.2.2

**Proposition.** There exists a (non-canonical) isomorphism as graded \( \mathbb{C}[t] = H^*_S(\{pt\}) \)-module

\[
H^*_S(M_\zeta(v)) \simeq H^*_S(\{pt\}) \otimes H^*(M_\zeta(v)).
\]

**Proof.** Since \( BS^1 \) is simply connected and the odd degree cohomologies of \( BS^1 \) and \( M_\zeta(v) \) vanish, the spectral sequence associated with the fibration \( ES^1 \times_{S^1} M_\zeta(v) \to BS^1 \) degenerates at \( E_2 \)-term. So the claim follows. \( \square \)

**Corollary.** The forgetful map \( H^*_S(M_\zeta(v)) \to H^*(M_\zeta(v)) \) is surjective.

**Proof.** Note that the forgetful map can be described as \( i^* \) where \( i: M_\zeta(v) \hookrightarrow ES^1 \times_{S^1} M_\zeta(v) \) is an inclusion of a fiber. Then the claim follows by Theorem 5.9 and Theorem 5.10 of McC01. \( \square \)
5.2.3

Proposition. The following map is isomorphism:

\[ H^n_{S_1}(\mathcal{M}_\zeta(v)) \xrightarrow{x} H^{2n}_{S_1}(\mathcal{M}_\zeta(v)) \]

where \( n = \frac{1}{2} \dim(\mathcal{M}_\zeta(v)) \).

Proof. This follows from Lemma 5.2.1 and Proposition 5.2.2. \qed

5.2.4

Let us write \( H^{\text{mid}}_{S_1}(\mathcal{M}_\zeta(v)) \) for \( H^n_{S_1}(\mathcal{M}_\zeta(v)) \), where \( n = \frac{1}{2} \dim(\mathcal{M}_\zeta(v)) \).

Definition. For \( P \in (\mathcal{M}_\zeta(v))^{S_1} \) we define

\[ \xi_P = t^{-n}.\{P\} \in H^{\text{mid}}_{S_1}(\mathcal{M}_\zeta(v)). \]

5.2.5

For a finite dimensional \( C S^1 \)-module \( M \), we define a number \( g(M) \) by the product of all the weight of \( M \). Note that \( e_{S^1}(M) = g(M) \cdot t^{\dim M} \) when we regard \( M \) as a \( S^1 \)-equivariant vector bundle on a point.

For \( P \in \mathcal{M}_\zeta(v)^{S_1} \) let \( T_P \) denote the tangent space of \( \mathcal{M}_\zeta(v) \) at \( P \).

Proposition. The set \( \{\xi_P\} \) forms a basis of \( H^{\text{mid}}_{S_1}(\mathcal{M}_\zeta(v)) \).

Proof. Linear independency follows directly from Theorem 5.1.5. For \( \alpha \in H^{\text{mid}}_{S_1}(\mathcal{M}_\zeta(v)) \) we have \( i^*_P(\alpha) = c_P \cdot t^n \) for some \( c_P \in \mathbb{C} \). So we have

\[ \alpha = \sum_P \frac{c_P \cdot t^n} {g(T_P) \cdot t^{2n}} = \sum_P \frac{c_P} {g(T_P)} \xi_P. \]

Thus the claim follows. \qed

5.3 \ Representation of the affine Lie algebra

5.3.1

Let \( e_i \) denote the \( i \)-th coordinate vector of \( Z^I \). For \( v \in Z^I \) we define the subvariety

\[ B_i(v) = \{(J_1, J_2) \in \mathcal{M}_{\zeta^0}(v) \times \mathcal{M}_{\zeta^0}(v + e_i) \mid J_1 \text{ is a subrepresentation of } J_2\}. \]

of \( \mathcal{M}_{\zeta^0}(v) \times \mathcal{M}_{\zeta^0}(v + e_i) \). This is called Hecke correspondence. This is smooth and \( 2|v| + 1 \)-dimensional (\cite{Nak98}).

Let \( p_\varepsilon \) be a projection from \( \mathcal{M}_{\zeta^0}(v) \times \mathcal{M}_{\zeta^0}(v + e_i) \) to the \( \varepsilon \)-th factor. We define operators \( e_i \) and \( f_i \) on \( \mathcal{H}^{\text{mid}}_{S_1}(\mathcal{M}_{\zeta^0}(v)) \) by

\[ e_i(\alpha) = (-1)^v p_{1*} (p_1^*(\alpha) \cup B_i(v)) \quad (\alpha \in H^{\text{mid}}_{S_1}(\mathcal{M}_{\zeta^0}(v + e_i))), \]
\[ f_i(\alpha) = (-1)^v p_{2*} (p_1^*(\alpha) \cup B_i(v)) \quad (\alpha \in H^{\text{mid}}_{S_1}(\mathcal{M}_{\zeta^0}(v))). \]

These operators give an representation of \( \hat{\mathfrak{g}} \).
5.3.2
We define a bilinear form $\langle \cdot , \cdot \rangle$ on $H^\text{mid}_{S, R}(\mathcal{M}_G(v))$ by

$$
\langle \alpha, \beta \rangle = p_*(i_*)^{-1}(\alpha \cup \beta) \in H^*_S, R(\{pt\})
$$

where $i: \mathcal{M}_G(v)^S \hookrightarrow \mathcal{M}_G(v)$ and $p: \mathcal{M}_G(v)^S \to \{pt\}$.

For $\lambda \in \Pi$ we write simply $\xi_\lambda$ and $T_\lambda$ for $\xi_{\lambda, \lambda}$ and $T_{\lambda, \lambda}$. For $\lambda$ and $\mu$ such that $(J_\lambda, J_\mu) \in B_\lambda(v)$, let $N_{\lambda, \mu}$ denote the fiber of the normal bundle on $B_\lambda(v)$ to $\mathcal{M}(v) \times \mathcal{M}(v + e_1)$ at $(J_\lambda, J_\mu)$.

Proposition.

$$
\langle \xi_\lambda, \xi_\mu \rangle = \delta_{\lambda, \mu} \cdot g(T_\lambda), \quad \langle e_i \xi_\lambda, \xi_\mu \rangle = (-1)^{v_i - 1 + v} \delta((J_\mu, J_\lambda) \in B_i(v)) : g(N_{\mu, \lambda}).
$$

Proof.

$$
\langle \xi_\lambda, \xi_\mu \rangle = t^{-\mid\lambda \mid - \mid\mu\mid} \langle i_{\lambda, \lambda}(1), i_{\mu, \mu}(1) \rangle
$$

(Lemma 5.1.4 (ii))

$$
= t^{-\mid\lambda \mid - \mid\mu\mid} \langle 1, i_{\lambda, \lambda} \circ i_{\mu, \mu}(1) \rangle
$$

(Lemma 5.1.4 (i))

$$
= \xi(T_\lambda)
$$

$$
(-1)^{v_i - 1 + v_i} \langle e_i \xi_\lambda, \xi_\mu \rangle
$$

$$
= t^{-\mid\lambda \mid - \mid\mu\mid} \langle p_1 \circ i_{\lambda, \lambda}(1) \cup [B_\lambda(v)], i_{\mu, \mu}(1) \rangle
$$

(Lemma 5.1.4 (ii))

$$
= t^{-\mid\lambda \mid - \mid\mu\mid} \langle i_{\mu} \circ p_1, (p_2 \circ i_{\lambda, \lambda}(1) \cup [B_\lambda(v)], 1) \rangle
$$

(Lemma 5.1.4 (iii))

$$
= t^{-\mid\lambda \mid - \mid\mu\mid} \langle p_\ast(i_{\lambda, \lambda}(1) \cup i_{\mu} \ast([B_\lambda(v)]) \cup 1) \rangle
$$

(Lemma 5.1.4 (iv))

$$
= t^{-\mid\lambda \mid - \mid\mu\mid} \langle i_{\lambda, \lambda}(1) \cup i_{\mu} \ast([B_\lambda(v)]) \cup 1 \rangle
$$

(Lemma 5.1.4 (i))

$$
= \delta((J_\lambda, J_\mu) \in B_\lambda(v)) \cdot g(N_{\mu, \lambda}),
$$

where $p: X_2 \to \{J_\mu\}$ and $i_{\lambda}: X_2 \simeq \{J_\mu\} \times X_2 \to X_1 \times X_2$. \hfill \Box

Corollary.

$$
e_i \xi_\lambda = (-1)^{v_i - 1 + v_i} \sum_{\mu} \xi(N_{\mu, \lambda} - T_\lambda) \xi_\mu.
$$

where The summation runs over all $\mu$ such that $(J_\mu, J_\lambda) \in B_i(v)$, which is equivalent to that $\mu$ is obtained by removing a removable $i$-node from $\lambda$.

5.3.3
For a $\mathbb{Z}/2\mathbb{Z}$-module $M$, we set $M_i = \text{Hom}_{\mathbb{Z}/2\mathbb{Z}}(C_{(i)}, M)$ where $C_{(i)}$ is the 1-dimensional representation of $\mathbb{Z}/2\mathbb{Z}$ with weight $i$.

For two nodes $X = (a, b)$ and $X' = (a', b')$, we set $l(X, X') = a - b - a' + b'$. 

24
Thus the claim follows.

Note that $V_0$ such that $n > 0$

Proposition.

We write simply $V_\lambda$ from $\lambda - \mu$ where the summation runs over all $\mu$ obtained by removing a removable $i$-node from $\lambda$.

Proof. We write simply $V_\lambda$ for $V(J_\lambda)$. For a node $X = (a, b)$ we set $V_X = t^{a-b}$.

Using the description in [Nak98] (Corollary 3.12. and §5), we have

$$T_\mu = \left((t + t^{-1} - 2) V_\mu V_\lambda + V_\mu + V_\lambda^*\right)_0$$

$$N_{\mu, \lambda} = \left((t + t^{-1} - 2) V_\mu V_\lambda + V_\lambda + V_\mu^* - 1\right)_0.$$

Thus we have

$$N_{\mu, \lambda} - T_\lambda = \left((t + t^{-1} - 2) V_\mu + 1\right)_i \cdot V_\lambda_{i\mu} - 1.$$

On the other hand, we can verify

$$\left((t + t^{-1} - 2) V_\mu + 1\right)_i = \sum_{A \in A_{\mu, i}} V_A - \sum_{R \in R_{\mu, i}} V_R.$$

Substituting this we get

$$N_{\mu, \lambda} - T_\lambda = \sum_{A \in A_{\mu, i}} V_{\lambda\mu} V_A^* - \sum_{R \in R_{\mu, i}} V_{\lambda\mu} V_R^* - 1$$

$$= \sum_{A \in A_{\mu, i}} V_{\lambda\mu} V_A^* - \sum_{R \in R_{\mu, i}} V_{\lambda\mu} V_R^*.$$}

Note that the weight of $V_\lambda V_\mu^*$ equals to $l(X_i' X)$, so the claim follows. \qed

5.3.4

Let $L_\lambda$ be the product of all the negative weights of $T_\lambda$. Normalize the basis by $b_\lambda = L_\lambda^{-1} \xi_\lambda$. We identify $\oplus_{\nu} H^{\text{mid}}_S(\mathfrak{M}_G(\nu))$ and $B$ as vector spaces.

Theorem. The action on $\oplus_{\nu} H^{\text{mid}}_S(\mathfrak{M}_G(\nu))$ coincides with representation defined in 

Proof. We will check only for $c_i$’s.

By [13.3] we have $L_\mu = \prod_{k,a} (-n)$ where the product runs through all $k$ and $n > 0$ such that $m_k(k) = -1, m_k(k + n) = 1$. So we have

$$L_\lambda / L_\mu = \prod_{A \in A_{\lambda, i}} -|l(\lambda, \mu, A)| \prod_{R \in R_{\mu, i}} -|l(\lambda, \mu, A)|^{-1}$$

$$= (-1)^{\eta^+(\lambda, i, \lambda)} \prod_{A \in A_{\lambda, i}} -l(\lambda, \mu, A) \prod_{R \in R_{\mu, i}} -l(\lambda, \mu, A).$$

Note that

$$\eta^+(\lambda, i, \lambda) + \eta^-(\lambda, i, \lambda) \equiv c_{i-1}(\lambda) + c_{i+1}(\lambda) + 1$$

$$= v_{i-1} + v_i + v_{i+1}(\lambda).$$

Thus the claim follows. \qed
5.4 Representation of the Heisenberg algebra

5.4.1
Recall that \( \mathcal{M} = \mathcal{M} \zeta_0(\delta) \) is isomorphic to the minimal resolution of \( \mathbb{C}^2/(\mathbb{Z}/l \mathbb{Z}) \).

The exceptional fiber has \( l - 1 \) connected component \( \mathcal{L}_i \) (\( i = 1, \ldots, l - 1 \)) and \( \mathcal{L}_i \) contains \( P_i - 1 \) and \( P_i + 1 \).

For \( m \in \mathbb{Z}_{>0} \) we consider subvarieties \( \mathcal{L}_i(m) = \{ (I, J) \in \mathcal{M}[n] \times \mathcal{M}[n+m] | I \supset J, \text{ supp}(I/J) = \{ x \} \text{ for some } x \in \mathcal{L}_i \} \) and define \( p_i(m) \) by

\[
p_i(m)(\alpha) = (-1)^mp_2^*(p_1^*(\alpha) \cup [\mathcal{L}_i(m)]) \quad (\alpha \in H_{S_1}^{\text{mid}}(\mathcal{M}[n])) .
\]

These operators satisfy the relations of \( \mathcal{H}_{l-1} \) (Nak97).

5.4.2
For \( \vec{\lambda} \in \Pi_{\widetilde{I}} \) let \( \mathcal{L}_{\vec{\lambda}} \) denote the product of all the negative weights of \( T_{\vec{\lambda}} \). We set

\[
b_{\vec{\lambda}} = \prod_{\vec{\lambda}} \mathcal{L}_{\vec{\lambda}} .
\]

Let us identify \( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \) and \( B^\otimes \) as vector spaces.

Proposition. ([QW] Lemma 3.3) The action of \( \mathcal{H}_{l-1} \) on \( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \) coincides with the action defined in 3.2.1.

5.5 Main theorem for equivariant cohomologies
The \( S^1 \)-equivariant diffeomorphism induces the isomorphism

\[
\oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_0}(\vec{\nu})) \simeq \oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_\infty}(\vec{\nu})).
\]

The quiver variety \( \mathcal{M}_{\zeta_\infty}(\vec{\nu}) \) is isomorphic to the Hilbert scheme and we have the isomorphism

\[
\oplus_{\vec{\nu}} \phi_{\vec{\nu}}^* : \oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_\infty}(\vec{\nu})) \simeq CQ \otimes \left( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \right).
\]

The \( \mathfrak{sl}_l \)-action on \( \oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_0}(\vec{\nu})) \) is given in 5.3.1. Apply the Frenkel-Kac construction for the \( \mathcal{H}_{l-1} \)-action on \( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \) given in 5.4.1, then we have the \( \mathfrak{sl}_l \)-action on \( CQ \otimes \left( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \right) \).

Theorem. The composition of the two isomorphisms

\[
\oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_0}(\vec{\nu})) \simeq \oplus_{\vec{\nu}} H_{S_1}^{\text{mid}}(\mathcal{M}_{\zeta_\infty}(\vec{\nu})) \simeq CQ \otimes \left( \oplus_n H_{S_1}^{\text{mid}}(\mathcal{M}[n]) \right).
\]

intertwines the \( \mathfrak{sl}_l \)-actions.

Proof. This follows from Theorem 3.3.2, Theorem 4.5, Theorem 5.3.4 and Proposition 5.4.2. \( \square \)
6 Representations on ordinary cohomologies

6.1 Representations on ordinary cohomologies

The constructions of representations of the affine Lie algebra and the Heisenberg algebra on the equivariant cohomology groups in 5.3.1 and 5.4.1 can be applied to the ordinary cohomology groups too.

Theorem. (Nak98) The affine Lie algebra \( \hat{\mathfrak{sl}} \) acts on \( \oplus_v H^\text{mid}(\mathcal{M}_{\zeta_0}(v)) \), and this is a level-1 integrable highest weight representation.

Theorem. (Nak97) The Heisenberg algebra \( \mathcal{H}_{l-1} \) acts on \( \oplus_n H^\text{mid}(\mathcal{M}_{\zeta_0}(n\delta)) \), and this is the Fock space representation.

6.2 Main theorem for ordinary cohomologies

As the case of equivariant cohomology groups in §5.5, we have the canonical isomorphisms of ordinary cohomology groups and the \( \hat{\mathfrak{sl}} \)-actions on \( \oplus_v H^\text{mid}(\mathcal{M}_{\zeta_0}(v)) \) and \( C\mathbb{Q} \otimes (\oplus_n H^\text{mid}(M^[n])) \).

Theorem. The composition of the two isomorphisms

\[
\oplus_v H^\text{mid}(\mathcal{M}_{\zeta_0}(v)) \cong \oplus_v H^\text{mid}(\mathcal{M}_{\zeta_\infty}(v)) \cong C\mathbb{Q} \otimes \left( \oplus_n H^\text{mid}(M^[n]) \right).
\]

intertwines the \( \hat{\mathfrak{sl}} \)-actions.

Proof. Note that the forgetful map is compatible with the actions and the isomorphisms. Then the claim follows from Corollary 5.2.2 and Theorem 5.5.

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28