(Dual) Hoops Have Unique Halving

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Abstract. Continuous logic extends the multi-valued \L ukasiewicz logic by adding a halving operator on propositions. This extension is designed to give a more satisfactory model theory for continuous structures. The semantics of these logics can be given using specialisations of algebraic structures known as hoops and coops. As part of an investigation into the metatheory of propositional continuous logic, we were indebted to Prover9 for finding proofs of important algebraic laws.

1 Introduction

(Like its title, this chapter begins with a parenthesis concerning notation. It is common practice to order truth-values by decreasing logical strength, but the opposite, or dual, convention is used in the literature that motivates the present work. So in this chapter $A \geq B$ means that $A$ is logically stronger than $B$. Accordingly, in the algebraic structures we will study, 0 models truth rather than falsehood and conjunction corresponds to an operation written as addition rather than multiplication. The halves alluded to in the title would otherwise be square roots.)

Around 1930, \L ukasiewicz and Tarski [16] instigated the study of logics admitting models in which the truth values are real numbers drawn from some subset $T$ of the interval $[0, 1]$. In these models, with the notational conventions discussed above, conjunction is capped addition: $x + y = \inf\{x + y, 1\}$. Boolean logic is the special case when $T = \{0, 1\}$. These \L ukasiewicz logics have been widely studied, e.g., as instances of fuzzy logics [11].

In recent years, Ben Yaacov has used a Lukasiewicz logic with an infinite number of truth values as a building block in what is called continuous logic [3]. Continuous logic unifies work of Henson and others [14] that aims to overcome shortfalls of classical first-order model theory when applied to continuous structures such as metric spaces and Banach spaces. A detailed discussion of these shortfalls would be out of place here, but a few remarks are in order. In functional analysis there is a well-accepted notion of ultraproduct that takes into account metric structure and is an important tool for constructing Banach spaces. By contrast, the class of Banach spaces is not closed under the standard model-theoretic notion of ultraproduct. Continuous logic aims to capture properties that are preserved under the good notion of ultraproduct for continuous
structures [14]. From another point of view, continuous logic mitigates the fact that ordinary first-order logic for continuous structures tends to be unexpectedly strong, the first-order theory of Banach spaces being strictly stronger than second-order arithmetic [20].

The motivation for ordering truth values by increasing logical strength in continuous logic stems from the fact that in a metric space with metric \( d \), \( x = y \) iff \( d(x, y) = 0 \). In first-order continuous logic, one wishes to treat \( d \) as a two-place predicate symbol analogous to equality in classical first-order logic. Representing truth by 0 is then the natural choice.

A difficulty with both the Lukasiewicz logics and continuous logic is that it requires considerable ingenuity to work with the known axiomatisations of their propositional fragments. Work on algebraic semantics for Lukasiewicz logic begun by Chang [8,9] has helped greatly with this, but basic algebraic laws in the algebras involved are often quite difficult to prove. This chapter reports on ongoing work to gain a better understanding of both the proof theory and the semantics of continuous logic that is benefitting from the use of automated theorem proving to find counterexamples and to derive algebraic properties.

![Fig. 1: Eight Logics and the Relationships between Them](image)

Our work began with the observation that both Lukasiewicz logic, \( \text{LL}_c \), and Ben Yaacov’s continuous logic, \( \text{CL}_c \), are extensions of a very simple intuitionistic substructural logic \( \text{AL}_i \). In Section 2 of this chapter we show how \( \text{CL}_c \) may be built up via a system of extensions of \( \text{AL}_i \). We also show how the Brouwer-Heyting intuitionistic propositional logic, \( \text{IL} \), and Boolean logic, \( \text{BL} \), fit into this picture. The relationships between the eight logics in this system of extensions are depicted in Figure 1. In Section 3 we describe a class of monoids called pocrimbs that have been quite widely studied in connection with \( \text{AL}_i \) and sketch a proof of a theorem asserting that each of the eight logics is sound and complete with respect to an appropriate class of pocrimbs. The sketch is easy to complete apart from one tricky lemma concerning the continuous logics.

In Section 4 we discuss our use of Bill McCune’s Mace4 and Prover9 to assist in these investigations, in particular to prove the lemma needed for the theorem of Section 2. Our application seems to be a “sweet spot” for this kind
of technology: the automatic theorem prover found a proof of a difficult problem that can readily be translated into a human readable form.

In Section 5 we discuss some other results that Prover9 has proved for us. Section 6 gives some concluding remarks.

2 The Logics

We work in a language \( \mathcal{L}_1 \) whose atomic formulas are the propositional constants 0 (truth) and 1 (falseness) and propositional variables drawn from the set \( \text{Var} = \{ P, Q, \ldots \} \). If \( A \) and \( B \) are formulas of \( \mathcal{L}_1 \) then so are \( A \rightarrow B \) (implication), \( A \& B \) (conjunction) and \( A/2 \) (halving). We adopt the convention that implication associates to the right and has lower precedence than conjunction, which in turn has lower precedence than halving. So, for example, the brackets in \((A \& (B/2)) \rightarrow (C \rightarrow (D \& F))\) are all redundant, while those in \(((A \rightarrow B) \rightarrow C) + D)/2\) are all required. We denote by \( \mathcal{L}_1 \) the language without halving. We write \( A\perp \) as an abbreviation for \( A \rightarrow 1 \), a form of negation.

The judgements of the eight logics that we will consider are sequents \( \Gamma \vdash A \), where \( A \) is an \( \mathcal{L}_1 \)-formula and \( \Gamma \) is a multiset of \( \mathcal{L}_1 \)-formulas. The inference rules are the introduction and elimination rules for the two binary connectives shown in Figure 2.

The axiom schemata for the logics are selected from those shown in Figure 3. These are the axiom of assumption \([\text{ASM}]\), ex-falso-quodlibet \([\text{EFQ}]\), double negation elimination \([\text{DNE}]\), commutative weak conjunction \([\text{CWC}]\), commutative strong disjunction \([\text{CSD}]\), the axiom of contraction \([\text{CON}]\), and two axioms giving lower and upper bounds for the halving operator: \([\text{HLB}]\) and \([\text{HUB}]\).

\([\text{ASM}], [\text{EFQ}], [\text{DNE}]\) and \([\text{CON}]\) are standard axioms of classical logic. \([\text{CON}]\) asserts that \( A \) is a strong as \( A \& A \) and is equivalent to the rule of contraction allowing us to infer \( \Gamma, A \vdash B \), from \( \Gamma, A, A \vdash B \). \([\text{CON}]\) allows one to think of the contexts \( \Gamma \) as sets rather than multisets. The significance of \([\text{CWC}], [\text{CSD}], [\text{HLB}]\) and \([\text{HUB}]\) will be explained below as we introduce the logics that include them and as we give the semantics for those logics.

The definitions of the eight logics are discussed in the next few paragraphs and are summarised in Table 1. In all but \( \text{CL}_1 \) and \( \text{CL}_c \), halving plays no rôle

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\(1\) Omitting disjunction from the logic greatly simplifies the algebraic semantics. While it may be unsatisfactory from the point of view of intuitionistic philosophy, disjunction defined using de Morgan’s law is adequate for our purposes.
and the logical language may be taken to be the sublanguage $\mathcal{L}_1$ in which halving does not feature.

Intuitionistic affine logic \[4\], $\mathbf{AL}_1$, has for its axiom schemata $[\text{ASM}]$ and $[\text{EFQ}]$. All our other logics include $\mathbf{AL}_1$. The contexts $\Gamma$, $\Delta$ are multisets because we wish to keep track of how many times each of the assumptions in $\Gamma$ is used in order to derive the conclusion $A$ in $\Gamma \vdash A$. This is not relevant if formulas can be duplicated or contracted (i.e. if $A$ is equivalent to $A \otimes A$). We will, however, mainly work with so-called substructural logics where such equivalences are not valid in general. $\mathbf{AL}_1$ serves as a prototype for such substructural logics.

Under the Curry-Howard correspondence between proofs and $\lambda$-terms, the proof system $\mathbf{AL}_1$ corresponds to a $\lambda$-calculus with pairing and paired abstractions, so in this calculus, if $t$, $u$ and $v$ are terms, then so are $(t, u)$, $(t, (u, v))$, $\lambda(x, y) \cdot t$, $\lambda((x, y), z) \cdot u$, $\lambda(x, (y, z)) \cdot v$ etc. where $x$, $y$ and $z$ are variables. Proofs in $\mathbf{AL}_1$ then correspond to affine $\lambda$-terms: terms in which each variable is used at most once. So for example $\lambda f \cdot \lambda x \cdot \lambda y \cdot f(x, y)$ is an affine $\lambda$-term corresponding to a proof of the sequent $\vdash (A \otimes B \rightarrow C) \rightarrow A \rightarrow B \rightarrow C$.

| Logic | Axioms | Models |
|-------|--------|--------|
| $\mathbf{AL}_1$ | $[\text{ASM}] + [\text{EFQ}]$ | bounded pocrims |
| $\mathbf{AL}_c$ | $\mathbf{AL}_1 + [\text{DNE}]$ | bounded involutive pocrims |
| $\mathbf{IL}_1$ | $\mathbf{AL}_1 + [\text{CWC}]$ | bounded hoops |
| $\mathbf{IL}_c$ | $\mathbf{AL}_1 + [\text{CSD}]$ | bounded Wajsberg hoops |
| $\mathbf{IL}$ | $\mathbf{AL}_1 + [\text{CON}]$ | bounded idempotent pocrims |
| $\mathbf{BL}$ | $\mathbf{IL} + [\text{DNE}]$ | bounded involutive idempotent pocrims |
| $\mathbf{CL}_1$ | $\mathbf{IL}_1 + [\text{HLB}] + [\text{HUB}]$ | bounded coops |
| $\mathbf{CL}_c$ | $\mathbf{IL}_c + [\text{HLB}] + [\text{HUB}]$ | bounded involutive coops |

Table 1: The Logics and their Models
Classical affine logic \([10]\), \(\mathbf{AL}_C\), extends \(\mathbf{AL}_i\) with the axiom schema \([\text{DNE}]\). It can also be viewed as the extension of the so-called multiplicative fragment of Girard’s linear logic by allowing weakening and the axiom schema \([\text{EFQ}]\).

What we will call intuitionistic Lukasiewicz logic, \(\mathbf{LL}_i\), extends \(\mathbf{AL}_i\) with the axiom schema \([\text{CWC}]\). \(\mathbf{LL}_i\) is known by a variety of names in the literature. The name we use reflects its position in Figure 1. For any formulas \(A\) and \(B\), \(A \otimes (A \to B)\) implies both \(A\) and \(B\) and so can be thought of as a weak form of conjunction. In \(\mathbf{LL}_i\) we have commutativity of this weak conjunction. \([\text{CWC}]\) turns out to be a surprisingly powerful axiom. However, it often requires considerable ingenuity to use it.

Classical Lukasiewicz logic \([13]\), \(\mathbf{LL}_C\), extends \(\mathbf{AL}_i\) with the axiom schema \([\text{CSD}]\). Just as \(A \otimes (A \to B)\) can be viewed as a form of conjunction, \((A \to B) \to B\) can be viewed as a form of disjunction that may be stronger than the one defined by the usual intuitionistic rules for disjunction. In \(\mathbf{LL}_C\) we have commutativity of this strong disjunction. This gives the widely-studied multi-valued logic of Lukasiewicz. Like \([\text{CWC}]\), \([\text{CSD}]\) is powerful but not always easy to use.

Intuitionistic propositional logic, \(\mathbf{IL}\), extends \(\mathbf{AL}_i\) with the axiom schema of contraction \([\text{CON}]\). This gives us the conjunction-implication fragment of the well-known Brouwer-Heyting intuitionistic propositional logic.

Classical propositional logic (or boolean logic), \(\mathbf{BL}\), extends \(\mathbf{IL}\) with the axiom schema \([\text{DNE}]\). This is the familiar two-valued logic of truth tables.

What we have termed intuitionistic continuous logic, \(\mathbf{CL}_i\), allows the halving operator and extends \(\mathbf{LL}_i\) with the axiom schemas \([\text{HLB}]\) and \([\text{HUB}]\), which effectively give lower and upper bounds on the logical strength of \(A/2\). They imply the surprisingly strong condition that \(A/2\) is equivalent to \(A/2 \to A\). This is an intuitionistic version of the continuous logic of Ben Yaacov \([3]\).

Classical continuous logic, \(\mathbf{CL}_C\) extends \(\mathbf{CL}_i\) with the axiom schema \([\text{DNE}]\). This gives Ben Yaacov’s continuous logic. The motivating model takes truth values to be real numbers between 0 and 1 with conjunction defined as capped addition.

Our initial goal was to gain insight into \(\mathbf{CL}_C\) by investigating the relations amongst \(\mathbf{AL}_i, \mathbf{LL}_C\) and \(\mathbf{CL}_C\). The other logics came into focus when we tried to decompose the somewhat intractable axiom \([\text{CSD}]\) into a combination of \([\text{DNE}]\) and an intuitionistic component. It can be shown that the eight logics are related as shown in Figure 1. In the figure, an arrow from \(T_1\) to \(T_2\) means that \(T_2\) extends \(T_1\), i.e., the set of provable sequents of \(T_2\) contains that of \(T_1\). In each square, the north-east logic is the least extension of the south-west logic that contains the other two. For human beings, at least, the proof of this fact is quite tricky for the \(\mathbf{AL}_i-\mathbf{LL}_C\) square, see \([11, \text{chapters 2 and 3}]\).

The routes in Figure 1 from \(\mathbf{AL}_i\) to \(\mathbf{IL}\) and \(\mathbf{BL}\) have been quite extensively studied, as may be seen from \([5, 18, 15]\) and the works cited therein. We are not aware of any work on \(\mathbf{CL}_i\), but it is clearly a natural object of study in connection with Ben Yaacov’s continuous logic. It should be noted that \(\mathbf{IL}\) and \(\mathbf{CL}_i\) are
incompatible: as we will see at the end of this section, any formula is provable given the axioms [CON], [HLB] and [HUB].

3 Algebraic Semantics

We give algebraic semantics for the logics of Section 2 using pocrim: partially ordered, commutative, residuated, integral monoids.

Definition 1 A pocrim is a structure for the signature (0, +, →; ≥) of type (0, 2, 2; 2) satisfying the following laws:

\[(x + y) + z = x + (y + z)\]  \[m_1\]
\[x + y = y + x\]  \[m_2\]
\[x + 0 = x\]  \[m_3\]
\[x ≥ x\]  \[o_1\]
\[\text{if } x ≥ y \text{ and } y ≥ z, \text{ then } x ≥ z\]  \[o_2\]
\[\text{if } x ≥ y \text{ and } y ≥ x, \text{ then } x = y\]  \[o_3\]
\[\text{if } x ≥ y, \text{ then } x + z ≥ y + z\]  \[o_4\]
\[x ≥ 0\]  \[b\]
\[x + y ≥ z \iff x ≥ y → z\]  \[r\]

Intuitively, → is the semantic counterpart of the syntactic implication →, whereas + corresponds to the syntactic conjunction ⊗. As with the syntactic connectives, we adopt the convention that → associates to the right and has lower precedence than +. Note that = and ≥ are predicate symbols and so necessarily have lower precedence than the function symbols → and +: the only valid reading of \(a → b ≥ c + d\) is \((a → b) ≥ (c + d)\).

Let \(M = (M, 0, +; →; ≥)\) be a pocrim. The laws \([m_1], [o_2]\) and \([b]\) say that \((M, 0, +; ≥)\) is a partially ordered commutative monoid with the identity 0 as least element. Law \([r]\), the residuation property, says that for any \(x\) and \(z\) the set \(\{y \mid x + y ≥ z\}\) is non-empty and has \(x → z\) as least element. \(M\) is said to be bounded if it has a (necessarily unique) annihilator, i.e., an element \(1\) such that for every \(x\) we have:

\[1 = x + 1\]  \[\text{ann}\]

Let us assume \(M\) is bounded. Then \(1 = x + 1 ≥ x ≥ 0\) for any \(x\) and \((M; ≥)\) is indeed a bounded ordered set. Let \(α : \text{Var} → M\) be an interpretation of logical variables as elements of \(M\) and extend \(α\) to a function \(ν_α : L → M\) by interpreting 0, 1, ⊗ and → as 0, 1, + and → respectively. If \(Γ = C_1, \ldots, C_n\), we say that \(α\) satisfies the sequent \(Γ ⊢ A\), iff \(ν_α(C_1) + \ldots + ν_α(C_n) ≥ ν_α(A)\). We say that \(Γ ⊢ A\) is valid in \(M\) if it is satisfied by every assignment \(α : \text{Var} → M\). We say \(M\) is a model for a logic \(L\) if every sequent provable in \(L\) is valid in \(M\).

If \(C\) is a class of pocrim, we say \(Γ ⊢ A\) is valid if it is valid in every \(M ∈ C\).

\[2\] Strictly speaking, this is a dual pocrim, since we order it by increasing logical strength and write it additively.
We will need some special classes of pocrim. We write $\neg x$ as an abbreviation for $x \rightarrow 1$, a semantic analogue of the derived syntactic operator $\perp$. We say a bounded pocrim is involutive if it satisfies $\neg \neg x = x$. We say a pocrim is idempotent if it is idempotent as a monoid, i.e., it satisfies $x + x = x$.

**Definition 2 (Büchi & Owens)** A hoop is a pocrim that is naturally ordered, i.e., whenever $x \geq y$, there is $z$ such that $x = y + z$.

It is a nice exercise in the use of the residuation property to show that a pocrim is a hoop iff it satisfies the identity

$$x + (x \rightarrow y) = y + (y \rightarrow x) \quad [\text{cwc}]$$

In any pocrim, $x \leq x + (x \rightarrow y) \geq y$, so we can view $x + (x \rightarrow y)$ as a weak form of conjunction, but in general this conjunction is not commutative and there need be no least $z$ such that $x \leq z \geq y$. In a hoop, the weak conjunction is commutative and $x + (x \rightarrow y)$ can be shown to be the least upper bound of $x$ and $y$.

**Definition 3 (Blok & Ferreirim)** A Wajsberg hoop is a hoop satisfying the identity

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \quad [\text{csd}]$$

We may view $(x \rightarrow y) \rightarrow y$ as a form of disjunction. In a Wajsberg hoop this disjunction is commutative and can be shown to give a greatest lower bound of $x$ and $y$. See [5] for more information on hoops and Wajsberg hoops.

**Definition 4** A continuous hoop, or coop, is a hoop where for every $x$ there is a unique $y$ such that $y = y \rightarrow x$. In this case we write $y = x/2$.

In a coop, for any $x$, we have $x \geq x/2 \rightarrow x = x/2$, whence, by [cwc], $x = x + 0 = x + (x \rightarrow x/2) = x/2 + (x/2 \rightarrow x) = x/2 + x/2$, justifying our choice of notation. Here, as with the syntactic connectives, we take halving to have higher precedence than conjunction.

If $M$ is a coop, we extend the function $v_\alpha : \mathcal{L}_1 \rightarrow M$ induced by an interpretation $\alpha : \text{Var} \rightarrow M$ to a function $v_\alpha : \mathcal{L}_2 \rightarrow M$ by interpreting $A/2$ as $v_\alpha(A)/2$. The notions of validity and satisfaction extend to interpretations of $\mathcal{L}_2$ in a coop in the evident way.

We say that a logic $L$ is sound for a class of pocrims $C$ if every sequent that is provable in $L$ is valid in $C$. We say that $L$ is complete for $C$ if the converse holds. We then have:

**Theorem 1** Each of the logics $\text{AL}_l$, $\text{AL}_c$, $\text{LL}_l$, $\text{LL}_c$, $\text{IL}$, $\text{BL}$, $\text{CL}_l$, and $\text{CL}_c$ is sound and complete for the class of pocrims listed for it in the column headed “Models” in Table 1.

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[3] Büchi and Owens write of hoops that “their importance ... merits recognition with a more euphonious name than the merely descriptive “commutative complemented monoid””. Presumably they chose “hoop” as an euphonious companion to “group” and “loop”.

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**Proof:** The proof follows a standard pattern and, with one exception, filling in the details is straightforward. Soundness is a routine exercise. For the completeness, one defines an equivalence relation \( \equiv \) on formulas such that \( A \equiv B \) holds iff both \( A \vdash B \) and \( B \vdash A \) are provable in the logic. One then shows that the set of equivalence classes becomes a pocrim in the indicated class, the *term model*, under operators \(+\) and \(\to\) induced on the equivalence classes by \(\otimes\) and \(\triangleright\). As the only sentences valid in the term model are those provable in the logic, completeness follows. The difficult detail is showing that the term models for the continuous logics satisfy our definition of a coop: it is easy to see that for any \( x = [A] \), one has that \( y = [A/2] \) satisfies \( y = y \to x \), but is this \( y \) unique? We shall answer this question in the affirmative in the next section. If the equation \( y = y \to x \) did not uniquely determine \( y \), halving would not be well-defined on the term model and the completeness proof would fail.

Using Theorem \[\square\] we can give an algebraic proof of the claim made earlier that \( IL \) and \( CI_4 \) are incompatible. By dint of the theorem, this is equivalent to the claim that a bounded idempotent coop is the trivial coop \( \{0\} \). We may prove this as follows: if \( a \) is an element of a coop and \( a/2 \) is idempotent, so that \( a/2 = a/2 + a/2 \), then \( a/2 \geq a/2 + a/2 = a \), so by the residuation property, \( a/2 \rightarrow a = 0 \). Now \( a/2 = a/2 \rightarrow a \) by the definition of a coop, so we have \( a = a/2 + a/2 = (a/2 \rightarrow a) + (a/2 \rightarrow a) = 0 + 0 = 0 \).

### 4 Automated Proofs and Counterexamples

In our early attempts to understand the relationships represented in Figure \[\square\] we spent some time devising finite pocrims with interesting properties. This can be a surprisingly difficult and error-prone task. Verifying associativity, in particular, is irksome. Having painstakingly accumulated a small stock of examples, a conversation with Alison Pease reminded us of the existence of Bill McCune’s Mace4 tool [17] that automatically searches for finite counter-examples to conjectures in a finitely axiomatised first-order theory.

It was fascinating to see Mace4 recreate examples similar to those we had already constructed. The following input asks Mace4 to produce a counterexample to the conjecture that all bounded pocrims are hoops:

```plaintext
op(500, infix, "==>").
formulas(assumptions).
  (x + y) + z = x + (y + z). \quad \% monoid law 1
  x + y = y + x. \quad \% monoid law 2
  x + 0 = x. \quad \% monoid law 3
  x >= x. \quad \% ordering law 1
  x >= y & y >= z -> x >= z. \quad \% ordering law 2
  x >= y & y >= x -> x = y. \quad \% ordering law 3
  x >= y -> x + z >= y + z. \quad \% ordering law 4
  x >= 0. \quad \% boundedness law
  x + 1 = 1. \quad \% annihilator law
```

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8
Here we use ‘==>' and ‘=>' to represent ‘→’ and ‘≥’ in the pocrim and ‘&’, ‘->’ and ‘<->’ are Mace4 syntax for logical conjunction, implication and bi-implication. Given the above, Mace4 quickly prints out the diagram of a pocrim on the ordered set 0 < p < q < 1 with x + y = 1 whenever \( \{x, y\} \subseteq \{p, q, 1\} \), a counter-example which we had already come up with over the course of an afternoon. That led us to test Mace4 on yet other conjectures which we had already refuted with some small counter-examples. Mace4, again and again, came up with similar counter-models to the ones we had contrived.

Some weeks later we wanted to show that the two axiom schemata \( [HLB] \) and \( [HUB] \) uniquely determine the halving operator over the logic \( \mathbf{LL}_1 \), which would conclude the proof of Theorem 1. That would give us an intuitionistic counterpart \( \mathbf{CL}_1 \) to continuous logic \( \mathbf{CL}_C \). In logical terms, we wanted to show that the rule shown in Figure 4 is derivable in \( \mathbf{LL}_1 \):

\[
\begin{array}{c}
A \rightarrow B \vdash A \\
A \vdash A \rightarrow B \\
C \vdash C \rightarrow B \\
C \rightarrow B \vdash C
\end{array} \Rightarrow A \vdash C
\]

Fig. 4: A Conjectured Inference Rule

After several failed attempts to find a proof, we had started to wonder whether the rule was not derivable. That is when we thought of using Prover9 to look for a proof. We gave Prover9 the input shown below comprising the laws for a hoop, the assumptions \( a \rightarrow b = a \) (corresponding to \( A \rightarrow B \vdash A \) and \( A \vdash A \rightarrow B \)) and \( c \rightarrow b = c \) (corresponding to \( C \vdash C \rightarrow B \) and \( C \rightarrow B \vdash C \)) and the goal \( a = c \). (Because the conjectured inference rule is symmetric in \( A \) and \( C \), if the rule is valid, then the antecedents imply that \( A \) and \( C \) are equivalent).
\begin{align*}
\text{a} &\implies \text{b} = \text{a}. \quad \% \text{assumption 1} \\
\text{c} &\implies \text{b} = \text{c}. \quad \% \text{assumption 2} \\
\end{align*}
end_of_list.
formulas(goals).
\begin{align*}
\text{a} &\equiv \text{c}.
\end{align*}
end_of_list.

To our surprise Prover9 took just a few seconds to produce the proof shown in the appendix. The proof that Prover9 found seems perplexingly intricate at first glance, but after studying it for a little while, we found we could edit it into a form fit for human consumption. From a human perspective, the proof involves the 9 intermediate claims given in the following lemma. Once these are proved, we will see that the desired result is an easy consequence of claim (9).

**Lemma 2** Let $\textbf{M} = (M, 0, +, \rightarrow; \geq)$ be a hoop and let $a, b, c, x, y \in M$. Assume that, (i), $a \rightarrow b = a$ and, (ii), $c \rightarrow b = c$. Then the following hold:

1. $b \geq a \text{ and } b \geq c$,
2. $a + a = b$,
3. $a \rightarrow (a \rightarrow c) = 0$,
4. $(x \rightarrow y) + z \geq x \rightarrow (y + (y \rightarrow x) + z)$,
5. $c \rightarrow (a + a + x) \geq c$,
6. $c \rightarrow a \geq a \rightarrow c$,
7. $c \rightarrow a = a \rightarrow c$,
8. $c + (c \rightarrow a) + ((a \rightarrow c) \rightarrow a) = b$,
9. $a + c = b$.

**Proof:** In the proof below (in)equalities which are not labelled as following from one of the assumptions (i) and (ii) or an earlier part of the lemma follow immediately from the axioms of a pocrim.

1. We have $b \geq a \rightarrow b$ and, by (i), $a \rightarrow b = a$. So $b \geq a$ and similarly $b \geq c$ using (ii).
2. By (1) we have $b \rightarrow a = 0$. Therefore
   \begin{align*}
   a + a &= a + (a \rightarrow b) \\
   &= b + (b \rightarrow a) \\
   &= b.
   \end{align*}
   \hfill (i)

3. By (i) and (1) we have $a = a \rightarrow b \geq a \rightarrow c$ and hence $0 \geq a \rightarrow (a \rightarrow c)$, which implies (3).
4. By \textbf{cwc} $x + (x \rightarrow y) + z = y + (y \rightarrow x) + z$, whence (4) follows.
5. We have
   \begin{align*}
   c \rightarrow (b + x) \geq c \rightarrow b \\
   = c
   \end{align*}
   \hfill (ii)
and then using (2) we obtain (5).

(6) By (5), as \((c \rightarrow a) + a \geq c \rightarrow (a + a)\), we have \((c \rightarrow a) + a \geq c\) and hence (6).

(7) Our assumptions are symmetric in \(a\) and \(c\). Hence, (6) holds with \(a\) and \(c\) interchanged, i.e., \(a \rightarrow c \geq c \rightarrow a\), which taken with (6) gives (7).

(8) We have
\[
\begin{align*}
c + (c \rightarrow a) + ((a \rightarrow c) \rightarrow a) &= a + (a \rightarrow c) + ((a \rightarrow c) \rightarrow a) \quad \text{[cwc]} \\
&= a + a + (a \rightarrow (a \rightarrow c)) \quad \text{[cwc]} \\
&= b + (a \rightarrow (a \rightarrow c)) \quad (2) \\
&= b.
\end{align*}
\]

(9) We have
\[
\begin{align*}
b &= c + (c \rightarrow a) + ((a \rightarrow c) \rightarrow a) \quad (8) \\
&= c + (a \rightarrow c) + ((a \rightarrow c) \rightarrow a) \quad (7) \\
&= c + a + (a \rightarrow (a \rightarrow c)) \quad \text{[cwc]} \\
&= c + a. \quad (3)
\end{align*}
\]

This completes the proof of the lemma.

It is interesting to note the complexity of the proof in terms of uses of \([\text{cwc}]\) (used 6 times!) and the important sub-lemma (2) (used twice) as depicted in the outline proof tree shown in Figure 5.

Finally, from part (9) of Lemma 2 we have the theorem that the equation \(a \rightarrow b = a\) uniquely determines \(a\) in terms of \(b\):

**Theorem 3** In any hoop, if \(a \rightarrow b = a\) and \(c \rightarrow b = c\) then \(a = c\).

**Proof:** Since the assumptions are symmetric in \(a\) and \(c\) it is enough to show \(c \geq a\), from which we can immediately conclude \(a \geq c\) and hence \(a = c\). By Lemma 2 (9) we have \(c \geq a \rightarrow b\) and hence \(c \geq a\).

We already have the part of Theorem 1 that gives soundness and completeness of \(\mathbb{L}\mathbb{L}_4\) for bounded hoops. Theorem 3 now gives us that the continuous logic
axioms $|\text{HLB}|$ and $|\text{HUB}|$ uniquely determine halving given the other axioms of $\text{LL}_1$ and that is exactly what we need to complete the proof of Theorem 1.

5 Subsequent Work

The importance of Theorem 3 is that it provides a powerful method for proving statements of the form $a = b/2$ in a coop: to prove $a = b/2$, one proves that $a = a \rightarrow b$. Very frequently one has to prove statements of the forms $a \geq b/2$ and $a \leq b/2$. The result on equality suggests that sufficient conditions for these should be $a \geq a \rightarrow b$ and $a \leq a \rightarrow b$ respectively. In logical terms, this means that it is valid to omit either the first or the last of the antecedents in the inference rule of Figure 4. Encouraged by our success with Theorem 3, we presented these two problems to Prover9, which, in just under 4 minutes and just over 20 minutes respectively, found proofs, that turned out to be even simpler than that of Theorem 3. Once one has these basic tools for reasoning about the halving operator, a deeper investigation of the algebra of coops becomes possible. One finds for example, that a coop is simple (in the sense of universal algebra) iff it is isomorphic to a coop of real numbers under capped addition. See [2] for more information and for the lovely proofs found by Prover9 of the rules for $a \geq b/2$ and $a \leq b/2$.

Prover9 has also found some other intricate proofs in this area. For example, it can prove a lemma on pocrims implying that the axiom schemata $[\text{CWC}] + [\text{DNE}]$ is equivalent to $[\text{CSD}]$ over intuitionistic affine logic $\text{AL}_1$. This implies the aforementioned result that in the $\text{AL}_1-\text{LL}_c$ square of Figure 4 the north-east logic $\text{LL}_c$ is the least extension of the south-west logic $\text{AL}_1$ that contains the other two logics $\text{AL}_c$ and $\text{LL}_1$. Prover9 is able to prove analogous results for each square in Figure 4. To complement this, Mace4 can also produce the examples needed to show that the various logics are distinct, with the exception of the logics in the right-hand column: a non-trivial model of continuous logic is necessarily infinite and hence not within the scope of Mace4.

A selection of the problems that Prover9 has solved for us will be included in a forthcoming release of the TPTP Problem Library [21]. As can be seen from the CPU times in Table 2, some of the proof problems are quite challenging. The timings were taken on an Apple iMac with a 3.06 GHz Intel Core 2 Duo processor using Prover9’s “auto” settings. The only tuning we have done is with the choice of axiomatization. Most of the problems use a straightforward translation into first-order logic of the various equations and Horn clauses given above as the axioms for pocrims, hoops etc. For hoops, a purely equational axiomatization is known and, in one case (LCL897+1.p), we were unable to obtain a proof using the Horn axiomatization but obtained a proof very rapidly with the equational axioms. In other cases (LCL894+1.p, LCL895+1.p), the Horn axiomatization gives quicker results.

The three axiomatizations we tried for the rule for proving $a \leq b/2$ displayed an interesting phenomenon: in the first axiomatization we tried (LCL890+1.p), we included the annihilator axiom $1 + x = x$, but the proof, which has 53
steps and was found in about 20 minutes, makes no use of this. When we tried again without the unnecessary axiom (LCL891+1.p), the search took an order of magnitude longer and found a proof with 154 steps. When we put the axiom back in, but this time at the end of the list of axioms (LCL892+1.p), the search took over 14 hours and gave a proof with 283 steps. Presumably, in our fortunate first attempt the annihilator axiom had a beneficial influence on the subsumption process and eliminated a lot of blind alleys.

When the TPTP formulation of the problems were tried on a selection of automated theorem provers, only Prover9 was able to find a proof for the first two problems in less than 300 seconds. Each problem has been proved by at least one other prover given enough time. From our perspective as users of this technology, this is very remarkable: Prover9 delivered a proof of a key lemma (LCL888+1.p) in just over 3 seconds. Encouraged by that, we were prepared to be patient when we tried the two important refinements of that lemma (LCL889+1.p and LCL890+1.p). These three lemmas have been invaluable in our subsequent theoretical work on the algebra of coops. We suspect our progress would have been very different if the first lemma had severely tested our patience.

## 6 Final Remarks

We are by no means the first to apply automated theorem proving technology in the area of Łukasiewicz logics. In 1990, a conjecture of Łukasiewicz was proposed by Wos as a challenge problem in automated theorem proving [23] that was successfully attacked by Anantharaman and Bonacina [16]. Others to apply
automated theorem proving to Lukasiewicz logics include Harris and Fitelson [12] and Slaney [19]. Veroff and Spinks [22] used Otter to find a remarkable direct algebraic proof of a property of idempotent elements in hoops that had previously only been proved by indirect model-theoretic methods.

Clearly our application is one to which technology such as Mace4 and Prover9 is well suited. It is nonetheless a ringing tribute to the late Bill McCune that the accessibility and ease of use of these tools have enabled two naive users to get valuable results with very little effort.

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Appendix

Formal proof of Theorem 3 as output by Prover9:

1. \( x \geq y \land y \geq z \rightarrow x \geq z \) # label(non_clause). [assumption].
2. \( x \geq y \land y \geq x \rightarrow x = y \) # label(non_clause). [assumption].
3. \( x + z \geq y \leftrightarrow z \geq x \rightarrow y \) # label(non_clause). [assumption].
4. \( x \geq y \rightarrow x + z \geq y + z \) # label(non_clause). [assumption].
5. \( x \geq y \rightarrow z \geq x \rightarrow z \geq y \) # label(non_clause). [assumption].
6. \( y = y \rightarrow x + z = z \rightarrow x \rightarrow y = z \) # label(non_clause) # label(goal). [goal].
7. \((x + y) + z = x + (y + z)\). [assumption].
8. \(x + y = y + x\). [assumption].
9. \(x + 0 = x\). [assumption].
10. \(0 + x = x\). [assumption].
11. \(1 \geq x\). [assumption].
12. \(-x \geq y \lor -(y \geq z) \land x \geq z\). [clausify(1)].
13. \(-x \geq y \lor -(y \geq x) \land y = x\). [clausify(2)].
14. \(-x + y \geq z \lor y \geq x \rightarrow z\). [clausify(3)].
15. \(x + y \geq z \rightarrow -(y \geq x \rightarrow z)\). [clausify(3)].
16. \(x \geq 0\). [assumption].
17. \(-x \geq y \lor x + z \geq y + z\). [clausify(4)].
18. \(-x \geq y \lor y \geq z \rightarrow x \geq z\). [clausify(5)].
19. \(-x \geq y \lor z \geq x \rightarrow y\). [clausify(6)].
20. \(x + (x \geq y) = y + (y \geq x)\). [assumption].
21. \(c_1 \leftrightarrow c_2 = c_1\). [deny(7)].
22 \( c_3 \implies c_2 = c_3 \). \[\text{[deny(7)]}.\]
23 \( c_3 \notimplies c_1 \). \[\text{[deny(7)]}.\]
24 \( x + (y + z) = y + (x + z) \). \[\text{[para(9(a,1),8(a,1,1)), rewrite([8(2)])]}\]
27 \( 0 + x = x \). \[\text{[para(10(a,1),9(a,1))}, \text{flip(a)}\]
28 \( x \implies y \implies (y + x) \). \[\text{[hyper(14,a,11,a)]}\]
30 \( -(x + y) := z \) \( x \implies y \implies z \). \[\text{[para(9(a,1),14(a,1))]}\]
31 \( -(x \implies y) \implies 0 \implies x \implies y \). \[\text{[para(10(a,1),14(a,1))]}\]
32 \( x + (x \implies y) \implies y \). \[\text{[hyper(15,b,11,a)]}\]
33 \( x \implies y \implies 0 \). \[\text{[hyper(14,a,16,a)]}\]
34 \( x + y \implies y \). \[\text{[hyper(17,a,16,a), rewrite([27(3)])]}\]
35 \( 0 \implies x \implies y \implies x \). \[\text{[hyper(18,a,16,a)]}\]
36 \( x + ((x \implies y) + z) = y + ((y \implies x) + z) \). \[\text{[para(20(a,1),8(a,1,1)), rewrite([8(3)])]}\]
41 \( c_3 + x := c_2 | -(x := c_3) \). \[\text{[para(22(a,1),15(b,2))]}\]
43 \( -(x + y) := u | x + z := y \implies u \). \[\text{[para(24(a,1),14(a,1))]}\]
46 \( x := 0 = 0 \). \[\text{[para(27(a,1),20(a,1))]}\]
52 \( x := 0 = 0 \). \[\text{[hyper(13,a,16,a,b,33,a)}, \text{flip(a)}\]
53 \( x := x \). \[\text{[back_rewrite(46), rewrite([52(4),10(4)])]}\]
54 \( x \implies y \implies x \). \[\text{[back_rewrite(35), rewrite([53(2)])]}\]
55 \( x \implies (y + z) := x \implies z \). \[\text{[hyper(19,a,34,a)]}\]
70 \( x := y \implies (x + y) \). \[\text{[para(9(a,1),28(a,2,2))]}\]
81 \( c_2 \implies c_1 \). \[\text{[para(21(a,1),54(a,2))]}\]
82 \( c_2 \implies c_3 \). \[\text{[para(22(a,1),54(a,2))]}\]
86 \( x + c_2 := c_1 \). \[\text{[hyper(12,a,34,a,b,81,a)]}\]
89 \( x := c_2 \implies x := c_3 \). \[\text{[hyper(19,a,82,a)]}\]
127 \( x := c_2 \implies c_1 \). \[\text{[hyper(30,a,86,a)]}\]
171 \( c_2 := c_1 = 0 \). \[\text{[hyper(13,a,16,a,b,127,a)}, \text{flip(a)}\]
180 \( c_1 + c_1 = c_2 \). \[\text{[para(171(a,1),20(a,1,2)), rewrite([9(3),27(3),21(5)])}, \text{flip(a)}\]
205 \( c_1 + (x + c_1) = x + c_2 \). \[\text{[para(180(a,1),8(a,2,2)), rewrite([9(4)])]}\]
271 \( x + ((x \implies y) + ((y \implies x) \implies z)) = y + ((z + (z \implies (y \implies x))) \implies x) \). \[\text{[para(20(a,1),36(a,1,2)), flip(a)]}\]
275 \( x \implies y \implies z := x \implies (y + ((y \implies x) + z)) \). \[\text{[para(36(a,1),28(a,2,2))]}\]
418 \( c_1 \implies c_1 \implies c_3 \). \[\text{[para(21(a,1),89(a,1))]}\]
419 \( 0 \implies c_1 \implies c_3 \). \[\text{[hyper(31,a,418,a)]}\]
609 \( c_3 + (x + (x \implies c_3)) := c_2 \). \[\text{[hyper(41,b,32,a)]}\]
895 \( c_3 \implies (x + c_2) := c_3 \). \[\text{[para(22(a,1),55(a,2))]}\]
996 \( c_1 \implies (c_1 \implies c_3) = 0 \). \[\text{[hyper(13,a,16,a,b,419,a)}, \text{flip(a)}\]
5220 \( c_3 \implies (c_1 + (x + c_1)) := c_3 \). \[\text{[para(205(a,2),895(a,2))]}\]
10398 \( c_3 + (x \implies c_3) := x \implies c_2 \). \[\text{[hyper(43,a,609,a)]}\]
16713 \( c_3 + ((c_3 \implies c_1) + ((c_1 \implies c_3) \implies c_1)) = c_2 \). \[\text{[para(996(a,1),271(a,2,2,2)), rewrite([9(15),27(15),180(14)])]}\]
20059 \( c_1 + (c_3 \implies c_1) \implies c_3 \). \[\text{[hyper(12,a,275,a,b,5220,a)}, \text{rewrite([9(5)])]}\]
20066 \( c_3 \implies c_1 \implies c_3 \). \[\text{[hyper(14,a,20059,a)]}\]
20564 \( c_3 + (c_1 \implies c_3) \implies c_1 \). \[\text{[para(21(a,1),10398(a,2))]}\]
20570 \( c_1 \implies c_3 \implies c_3 \implies c_3 \). \[\text{[hyper(14,a,20564,a)]}\]
20614 \( c_3 \implies c_1 = c_1 \implies c_3 \). \[\text{[hyper(13,a,20066,a,b,20570,a)}, \text{flip(a)}\]
20625 \( c_1 + c_3 = c_2 \). \[\text{[back_rewrite(16713), rewrite([20614(4),20(10),996(7),9(4),27(4),9(3)])]}\]
20634 \( c_3 \implies c_1 \). \[\text{[para(20625(a,1),28(a,2,2)), rewrite([21(4)])]}\]

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c1 >= c3. [para(20625(a,1),70(a,2,2)), rewrite([22(4)])].
-(c1 >= c3). [ur(13,b,20634,a,c,23,a)].
$F$. [resolve(20793,a,20637,a)].