COADJOINT ORBITS IN REPRESENTATION THEORY
OF PRO-LIE GROUPS

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ABSTRACT. We present a one-to-one correspondence between equivalence classes
of unitary irreducible representations and coadjoint orbits for a class of pro-Lie
groups including all connected locally compact nilpotent groups and arbitrary
infinite direct products of nilpotent Lie groups.

1. INTRODUCTION

In this paper we sketch an approach to unitary representation theory for a class
of projective limits of Lie groups, in the spirit of the method of coadjoint orbits from
representation theory of Lie groups. (See [BZ17] for more details.) The importance
of this method stems from the fact that the groups under consideration here are
not locally compact in general, hence they may not have a Haar measure, and
therefore it is not possible to model their representation theory in the usual way,
using Banach algebras or $C^*$-algebras.

By way of motivation, we discuss a simple example (cf. [BZ17, Ex. 4.10]), which
shows that the usual $C^*$-algebraic approach to group representation theory does
not work for topological groups which are not locally compact. Let $G = (\mathbb{R}^\mathbb{N}, +)$
be the abelian group which is the underlying additive group of the vector space of all
sequences of real numbers. Since the linear dual space $(\mathbb{R}^\mathbb{N})^* = \mathbb{R}^{(\mathbb{N})}$ is the vector
space of all finitely supported sequences of real numbers, it easily follows that there
exists a bijection $\Psi_G: \hat{G} \to \mathbb{R}^{(\mathbb{N})}$ (compare also Corollary 4.6). Specifically, for
every $\lambda = (\lambda_j)_{j \in \mathbb{N}} \in \mathbb{R}^{(\mathbb{N})}$, $\Psi^{-1}_G(\lambda) \in \hat{G}$ is the equivalence class of the 1-dimensional
representation

$$\chi_\lambda: G \to U(1), \quad \chi_\lambda((x_j)_{j \in \mathbb{N}}) := \exp(i \sum_{j \in \mathbb{N}} \lambda_j x_j)$$

where $U(1) := \{ z \in \mathbb{C} \mid |z| = 1 \}$. However, as the vector space $\mathbb{R}^{(\mathbb{N})}$ is infinite
dimensional, it is not locally compact, hence it is not homeomorphic to the spectrum
of any $C^*$-algebra. Consequently, the irreducible representation theory of $G$ cannot
be exhaustively described via any $C^*$-algebra.

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2. Preliminaries

**Lie theory.** We use upper case Roman letters to denote Lie groups, and their corresponding lower case Gothic letters to denote the Lie algebras. We will also use the notation $L$ for the Lie functor which associates to each Lie group its Lie algebra, hence for any Lie group $G$ one has $L(G) = \mathfrak{g}$. We denote the exponential map of a Lie group $G$ by $\exp_G : \mathfrak{g} \to G$, and if this map is bijective, then we denote its inverse by $\log_G : G \to \mathfrak{g}$. For any morphism of Lie groups $q : G \to H$, its corresponding morphism of Lie algebras is denoted by $L(q) : \mathfrak{g} \to \mathfrak{h}$, hence one has the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{L(q)} & \mathfrak{h} \\
\downarrow{\exp_G} & & \downarrow{\exp_H} \\
G & \xrightarrow{q} & H
\end{array}
$$

The coadjoint action of a Lie group is denoted by $\text{Ad}^*_G : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, and its corresponding set of coadjoint orbits is denoted by $\mathfrak{g}^*/G$ or $L(G)^*/G$. If $q : G \to H$ is a surjective morphism of Lie groups, then one has a map $L(q)^* : \mathfrak{h}^* \to \mathfrak{g}^*$ such that for every coadjoint $H$-orbit $O \in \mathfrak{h}^*/H$ its image $L(q)^*(O)$ is a coadjoint $G$-orbit, and one thus obtains a map

$$
L(q)^*_\text{Ad} : \mathfrak{h}^*/H \to \mathfrak{g}^*/G, \quad O \mapsto L(q)^*(O).
$$

**Representation theory.** For any topological group $G$ we denote by $\hat{G}$ its unitary dual, that is, its set of unitary equivalence classes $[\pi]$ of unitary irreducible representations $\pi : G \to \mathbb{B}(\mathcal{H})$. If $q : G \to H$ is a continuous surjective morphism of topological groups, then we define

$$
\hat{q} : \hat{H} \to \hat{G}, \quad [\pi] \mapsto [\pi \circ q].
$$

**Proposition 2.1.** Let $G$ be any connected nilpotent Lie group with its universal covering $p : \tilde{G} \to G$, and denote $\Gamma := \text{Ker} p \subseteq \tilde{G}$. We define

$$
\mathfrak{g}^*_\mathbb{Z} := \{ \xi \in \mathfrak{g}^* \mid (\xi \circ L(p) \circ \log_{\tilde{G}})({\Gamma}) \subseteq \mathbb{Z} \}.
$$

Then the following assertions hold:

1. The set $\Gamma$ is a discrete subgroup of the center of $\tilde{G}$.
2. The set $\mathfrak{g}^*_\mathbb{Z}$ is invariant to the coadjoint action of $G$.
3. There exists an injective correspondence $\Psi_G : \tilde{G} \to \mathfrak{g}^*/G$, whose image is exactly the set of all coadjoint $G$-orbits contained in $\mathfrak{g}^*_\mathbb{Z}$, such that if $H$ is any other connected nilpotent Lie group with a surjective morphism of Lie groups $q : G \to H$, then one has the commutative diagram

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\Psi_G} & \mathfrak{g}^*/G \\
\uparrow{\tilde{q}} & & \\
\hat{H} & \xrightarrow{\Psi_H} & \mathfrak{h}^*/H
\end{array}
$$

**Proof.** See [BZ17, Prop. A.3].
3. Pro-Lie groups and their Lie algebras

The main results that we will give below (see Theorem 4.3 and its corollaries) are applicable to pro-Lie groups and are stated in terms of Lie algebras and coadjoint orbits of these groups. Therefore we discuss these notions in this section. Our general reference for pro-Lie groups is the monograph [HM07], and we also refer to the paper [HN09] for the relation between pro-Lie groups and infinite-dimensional Lie groups.

Any topological group in this paper is assumed to be Hausdorff by definition. A Cauchy net in a topological group \( G \) is a net \( \{g_j\}_{j \in J} \) in \( G \) with the property that for every neighborhood \( V \) of \( 1 \in G \) there exists \( J_0 \in J \) such that for all \( i, k \in J \) with \( j \geq j_0 \) and \( k \geq j_0 \) one has \( g_j g_k^{-1} \in V \). A topological group \( G \) is called complete if every Cauchy net in \( G \) is convergent. Every locally compact group is complete by [HM07, Rem. 1.31].

For any topological group \( G \) we denote by \( \mathcal{N}(G) \) the set of its co-Lie subgroups, that is, the closed normal subgroups \( N \subseteq G \) for which \( G/N \) is a finite-dimensional Lie group. We say that \( G \) is a pro-Lie group if it is complete and for every neighborhood \( V \) of \( 1 \in G \) there exists \( N \in \mathcal{N}(G) \) with \( N \subseteq V \) (cf. [HM07, Def. 3.25]). If this is the case, then \( \mathcal{N}(G) \) is closed under finite intersections, hence it is a filter basis (cf. [HM07, page 148]).

Pro-Lie groups can be equivalently defined as the limits of projective systems of Lie groups, by [HM07, Th. 3.39].

**Definition 3.1.** For any pro-Lie group \( G \), its set of continuous 1-parameter subgroups

\[
\mathbf{L}(G) := \{ X \in C(\mathbb{R}, G) \mid (\forall t, s \in \mathbb{R}) \quad X(t + s) = X(t)X(s) \}
\]

is endowed with its topology of uniform convergence on the compact subsets of \( \mathbb{R} \). Then the topological space \( \mathbf{L}(G) \) has the structure of a locally convex Lie algebra over \( \mathbb{R} \), whose scalar multiplication, vector addition and bracket satisfy the following conditions for all \( t, s \in \mathbb{R} \) and \( X_1, X_2 \in \mathbf{L}(G) \):

\[
\begin{align*}
(t \cdot X_1)(s) &= X_1(ts); \\
(X_1 + X_2)(t) &= \lim_{n \to \infty} (X_1(t/n)X_2(t/n))^n; \\
[X_1, X_2](t^2) &= \lim_{n \to \infty} (X_1(t/n)X_2(t/n)X_1(-t/n)X_2(-t/n))^n,
\end{align*}
\]

where the convergence is uniform on the compact subsets of \( \mathbb{R} \). (See for instance [BB11, Ex. 2.7(4)].) One also has the dual vector space

\[
\mathbf{L}(G)^* := \{ \xi : \mathbf{L}(G) \to \mathbb{R} \mid \xi \text{ is linear and continuous} \}
\]

endowed with its locally convex topology of pointwise convergence on \( \mathbf{L}(G) \). The adjoint action is \( \text{Ad}_G : G \times \mathbf{L}(G) \to \mathbf{L}(G), \ (g, X) \mapsto \text{Ad}_G(g)X := gX(\cdot)g^{-1} \), and this defines by duality the coadjoint action

\[
\text{Ad}_{G^*} : G \times \mathbf{L}(G)^* \to \mathbf{L}(G)^*, \quad (g, \xi) \mapsto \text{Ad}_{G^*}(g)\xi := \xi \circ \text{Ad}_G(g^{-1}).
\]

We denote by \( \mathbf{L}(G)^*/G \) the set of all coadjoint orbits, that is, the orbits of the above coadjoint action.

In the following proposition we summarize a few basic properties of Lie algebras of connected locally compact groups. A pro-Lie group \( G \) is called pronilpotent if for every \( N \in \mathcal{N}(G) \) the finite-dimensional Lie group \( G/N \) is nilpotent. (See [HM07, Def. 10.12].)
Proposition 3.2. If $G$ is a connected locally compact group, then the following assertions hold:

1. $G$ is a pro-Lie group and its Lie algebra $L(G)$ is the direct product of a finite-dimensional Lie algebra, an abelian (possibly infinite-dimensional) Lie algebra, and a (possibly infinite) product of simple compact Lie algebras.

2. The following conditions are equivalent:
   (a) The group $G$ is pronilpotent.
   (b) The Lie algebra $L(G)$ is the product of a finite-dimensional nilpotent Lie algebra and an abelian (possibly infinite-dimensional) Lie algebra.
   (c) The Lie algebra $L(G)$ is nilpotent (possibly infinite-dimensional).
   (d) The group $G$ is nilpotent.

Proof. The first assertion follows by [Gl57, Th. 4] or [La57, Cor. 4.24]. See also [BoCzRu81, Th. 2.1.2.2].

For the second assertion, we first recall from [HM07, Th. 10.36 and Def. 7.42] that the group $G$ is pronilpotent if and only if its Lie algebra $L(G)$ is pronilpotent, that is, every finite-dimensional quotient algebra of $L(G)$ is nilpotent. Therefore, in view of Assertion 1, one has

$$ (2a) \iff (2b) \iff (2c). $$

Moreover, one clearly has $(2d) \implies (2b)$.

We now prove $(2b) \implies (2d)$. To this end let $\pi_G: \tilde{G} \to G$ be the universal morphism defined in [La57, Def. 4.20] and [HM07, page 259]. Then $L(\pi_G): L(\tilde{G}) \to L(G)$ is an isomorphism of Lie algebras and the image of $\pi_G$ is dense in $G$ by [HM07, Th. 6.6 (i) and (iv)]. It follows at once by condition $(2b)$ and [La57, Th. 4.23] that the group $G$ is nilpotent. Then, as the image of $\pi_G$ is dense in $G$, we obtain $(2d)$, and this completes the proof. □

4. Main results

Theorem 4.4 below provides an exhaustive description of the unitary dual of a class of topological groups that are not locally compact. As we discussed in the introduction, unitary dual spaces of non-locally-compact groups in general cannot be described in terms of representation theory of $C^*$-algebras.

For the following definition we recall that if $X$ is an arbitrary nonempty set, then a filter basis on $X$ is a nonempty set $B$ whose elements are nonempty subsets of $X$ having the property that for any $X_1, X_2 \in B$ there exists $X_0 \in B$ with $X_0 \subseteq X_1 \cap X_2$. If $X$ is moreover endowed with a topology, then one says that the filter basis $B$ converges to a point $x_0 \in X$ if for every neighborhood $V$ of $x_0$ one has $X_0 \subseteq V$.

Example 4.1. Here are some basic examples of filter bases.

1. Every neighborhood basis at any point of a topological space is a filter basis converging to that point.

2. If $G$ is a group endowed with the discrete topology and $B$ is a set of subgroups of $G$ such that the trivial subgroup $G_0 := \{1\}$ is an element of $B$, then $B$ is a filter basis on $G$ converging to $1 \in G$ since for any $G_1, G_2 \in B$ one has $G_0 \subseteq G_1 \cap G_2$ and on the other hand $G_0$ is contained in any neighborhood of $1 \in G$. 
(3) If $G$ is a topological group with the property that for every neighborhood $V$ of $1 \in G$ there exists a co-Lie subgroup $N \in \mathcal{N}(G)$ with $N \subseteq V$, then $\mathcal{N}(G)$ is a filter basis on $G$ converging to $1 \in G$ since in fact for every $N_1, N_2 \in \mathcal{N}(G)$ one has $N_1 \cap N_2 \in \mathcal{N}(G)$. (See [HM07, page 148].) In particular, this holds true for pro-Lie groups.

**Definition 4.2.** An amenable filter basis on a topological group $G$ is a filter basis $\mathcal{N} \subseteq \mathcal{N}(G)$ converging to $1 \in G$ such that every topological group $N \in \mathcal{N}$ is amenable.

**Example 4.3.** Here are two examples of amenable filter basis that are needed in Corollaries 4.5–4.6:

1. If $G$ is a connected locally compact group, then $\mathcal{N}(G)$ is an amenable filter basis. In fact, every $N \in \mathcal{N}(G)$ is compact hence amenable, and on the other hand $\mathcal{N}(G)$ converges to $1 \in G$ by the theorem of Yamabe. (See for instance [BoCzRu81, Th. 0.1.5].)

2. Let $\{G_j\}_{j \in J}$ be an infinite family of nilpotent Lie groups with their direct product topological group $G := \prod_{j \in J} G_j$. Denote by $\mathcal{N}$ the set of all subgroups of $G$ of the form $N_F := \prod_{j \in F} N_j$ associated to any finite subset $F \subseteq J$, with $N_j = \{1\} \subseteq G_j$ if $j \in F$ and $N_j = G_j$ if $j \in J \setminus F$. It is clear that every $N_F$ of this form has the following properties: $N_F$ is a closed normal subgroup of $G$ that is isomorphic to $\prod_{j \in J \setminus F} G_j$ hence $N_F$ is amenable by [BZ17, Prop. 3.8], and moreover $G/N_F$ is isomorphic to $\prod_{j \in F} G_j$, which is a Lie group since $F$ is a finite set, hence $N_F \in \mathcal{N}(G)$. For any finite subsets $F_1, F_2 \subseteq J$ one clearly has $N_{F_1} \cap N_{F_2} = N_{F_1 \cup F_2}$, where $F_1 \cup F_2$ is again a finite subset of $J$, hence $\mathcal{N}$ is a filter basis on $G$. Moreover, by the definition of an infinite direct product of topologies, it follows that the filter basis $\mathcal{N}$ converges to $1 \in G$. Consequently, $\mathcal{N}$ is an amenable filter basis on $G$.

**Theorem 4.4.** Let $G$ be a complete topological group with an amenable filter basis $\mathcal{N}$ for which $G/N$ is a connected nilpotent Lie group for every $N \in \mathcal{N}$. Then there exists a well-defined bijective correspondence

$$\Psi_G : \hat{G} \to L(G)^*/G, \quad [\pi] \mapsto O^\pi$$

between the equivalence classes of unitary irreducible representations of $G$ and the set of all coadjoint $G$-orbits contained in the $G$-invariant set

$$L(G)_\mathbb{Z} := \{ \xi \in L(G)^* \mid (\exists N \in \mathcal{N})(\exists \eta \in L(G/N)^*_\mathbb{Z}) \quad \xi = \eta \circ L(p_N) \}.$$

Every unitary irreducible representation $\pi : G \to \mathcal{B}(\mathcal{H})$ is thus associated to the coadjoint $G$-orbit $O^\pi := L(p_N)^*(\mathcal{O}_0) \subseteq L(G)^*_\mathbb{Z}$, where $N \in \mathcal{N}$ and $\mathcal{O}_0 \subseteq L(G/N)^*_\mathbb{Z}$ is the coadjoint $(G/N)$-orbit associated with a unitary irreducible representation $\pi_0 : G/N \to \mathcal{B}(\mathcal{H})$ satisfying $\pi_0 \circ p_N = \pi$.

**Proof.** See [BZ17, Th. 4.6].

In connection with the following corollary we note that the Lie algebras of connected locally compact nilpotent groups can be described as in Proposition 3.2.
Corollary 4.5. If \( G \) is a connected locally compact nilpotent group, then there is a bijective correspondence \( \Psi_G: \hat{G} \to L(G)^*/G \) onto the set of all coadjoint \( G \)-orbits contained in a certain \( G \)-invariant subset \( L(G)^*_G \subseteq L(G) \). For any filter basis \( \mathcal{N} \subseteq N(G) \) converging to the identity one has
\[
L(G)^*_G := \{ \xi \in L(G)^* \mid (\exists \eta \in L(G/N)^*_G) (\exists N \in \mathcal{N}) (\exists \eta \in L(G/N)^*_G) \xi = \eta \circ L(p_N) \}.
\]

Proof. See \( \text{[BZ17, Cor. 4.7]} \). \( \square \)

We now draw a corollary of Theorem 4.4 that applies to pro-Lie groups which are not locally compact.

Corollary 4.6. If \( \{G_j\}_{j \in J} \) is a family of connected nilpotent Lie groups, with their direct product topological group \( G := \prod_{j \in J} G_j \), then there is a bijective correspondence \( \Psi_G: \hat{G} \to L(G)^*/G \) onto the set of all coadjoint \( G \)-orbits contained in the \( G \)-invariant subset \( L(G)^*_G \subseteq L(G) \). Here we define
\[
L(G)^*_G := \{ \xi \in L(G)^* \mid (\exists F \in \mathcal{F}) (\exists \eta \in L(G/F)^*_G) \xi = \eta \circ L(p_F) \}
\]
where \( \mathcal{F} \) is the set of all finite subsets \( F \subseteq J \), and for every \( F \in \mathcal{F} \) we define \( G_F := \prod_{j \in F} G_j \) and \( p_F: G \to G_F \) is the natural projection.

Proof. See \( \text{[BZ17, Cor. 4.9]} \). \( \square \)

Remark 4.7. The amenability hypotheses of Theorem 4.4 may actually be removed, using some results of \( \text{[Ne10]} \).

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