A Generalized Newton Method for Subgradient Systems

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Abstract. This paper proposes and develops a new Newton-type algorithm to solve subdifferential inclusions defined by subgradients of extended-real-valued prox-regular functions. The proposed algorithm is formulated in terms of the second-order subdifferential of such functions that enjoys extensive calculus rules and can be efficiently computed for broad classes of extended-real-valued functions. Based on this and on metric regularity and subregularity properties of subgradient mappings, we establish verifiable conditions ensuring well-posedness of the proposed algorithm and its local superlinear convergence. The obtained results are also new for the class of equations defined by continuously differentiable functions with Lipschitzian gradients (C^{1,1} functions), which is the underlying case of our consideration. The developed algorithms for prox-regular functions and its extension to a structured class of composite functions are formulated in terms of proximal mappings and forward-backward envelopes. Besides numerous illustrative examples and comparison with known algorithms for C^{1,1} functions and generalized equations, the paper presents applications of the proposed algorithms to regularized least square problems arising in statistics, machine learning, and related disciplines.

Keywords. gradient and subgradient systems; Newton methods; variational analysis; second-order generalized differentiation; metric regularity and subregularity; tilt stability in optimization; prox-regular functions; superlinear convergence; regularized least square problems

1 Introduction and Overview

Recall that, given a function \( \varphi: \mathbb{R}^n \rightarrow \mathbb{R} \), which is twice continuously differentiable (C^2-smooth) around some point \( \bar{x} \in \mathbb{R}^n \), the classical Newton method to solve the nonlinear gradient system

\[
\nabla \varphi(x) = 0
\]

constructs the iterative procedure

\[
x^{k+1} := x^k + d^k \quad \text{for all} \quad k \in \mathbb{N} := \{1, 2, \ldots \},
\]

where \( x^0 \in \mathbb{R}^n \) is a given starting point, and where \( d^k \) is a solution to the linear system

\[
- \nabla \varphi(x^k) = \nabla^2 \varphi(x^k) d^k, \quad k = 0, 1, \ldots,
\]

written in terms of the Hessian matrix \( \nabla^2 \varphi(x^k) \) of \( \varphi \) at \( x^k \). As known in classical optimization, Newton’s algorithm in (1.2) and (1.3) is well-defined (i.e., the equations in (1.3) are solvable for \( d^k \)), and the sequence of its iterations \( \{x^k\} \) superlinearly (even quadratically) converges to a solution \( \bar{x} \) of (1.1) if \( x^0 \) is chosen sufficiently close to \( \bar{x} \) and if the Hessian matrix \( \nabla^2 \varphi(\bar{x}) \) is positive-definite. Note also that, besides being a necessary condition for local minimizers of \( \varphi \), the gradient system (1.1) is important for its own sake and holds not only for local minimizers and local maximizers of \( \varphi \), but in essentially larger settings. Furthermore, a counterpart of the classical Newton method has been developed for solving more general nonlinear equations of the type \( f(x) = 0 \), where \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuously differentiable (C^1-smooth) mapping, and where \( \nabla^2 \varphi(x^k) \) in (1.3) is replaced by the Jacobian matrix of \( f \) at the points in question. We are not going to deal with the latter method and its extensions in this paper while being fully concentrated on the gradient systems (1.1) and their appropriate subgradient counterparts.

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Concerning the gradient systems of type (1.1) where $\varphi$ may not be $C^2$-smooth around $\bar{x}$, we mention that the enormous literature has been devoted to developing various versions of the (generalized) Newton method; see, e.g., the books by Dontchev and Rockafellar [13], Facchinei and Pang [16], Izmailov and Solodov [27], Klatte and Kummer [32], Ulbrich [62], and the references therein. The vast majority of such extensions deals with functions $\varphi$ in (1.1) of class $C^{1,1}$ (or $C^{1,1}$ in the notation of Rockafellar and Wets [57]) around $\bar{x}$, which consists of continuously differentiable functions with locally Lipschitzian derivatives. The most popular generalized Newton method to solve (1.1) for functions of this type is known as the semismooth Newton method initiated independently by Kummer [33] and by Qi and Sun [55]. In the semismooth Newton method, the Hessian matrix of $\varphi$ in (1.3) is replaced by the (Clarke) generalized Jacobian (collection of matrices) of the gradient mapping $\nabla \varphi$. Then the corresponding Newton iterations are well-defined around $\bar{x}$ and exhibit a local superlinear convergence to the solution $\bar{x}$ of (1.1) provided that each matrix from the generalized Jacobian is nonsingular and the gradient mapping $\nabla \varphi$ is semismooth around $\bar{x}$ in the sense of Mifflin [36]. The latter property has been well investigated and applied in variational analysis and optimization, not only in connection with the semismooth Newton method. Besides the aforementioned books and papers, we refer the reader to Burke and Qi [5], Henrion and Outrata [23], and Meng et al. [35] among many other publications on the theory and applications of such functions.

In the case of $C^{1,1}$ functions $\varphi$, our Newton-type algorithm proposes replacing (1.3) by

$$-\nabla \varphi(x^k) \in \partial^2 \varphi(x^k)(d^k), \quad k = 0, 1, \ldots,$$

(1.4)

where $\partial^2 \varphi$ stands for second-order subdifferential/generalized Hessian of $\varphi$ introduced by Mordukhovich [41] for arbitrary extended-real-valued functions. This construction reduces to the classical Hessian operator for $C^2$-smooth functions while maintaining key properties of the latter for important classes of functions in broad generality; see below. In what follows we obtain efficient conditions ensuring the solvability of the inclusions in (1.4) and superlinear convergence of iterates $\{x^k\}$ if the starting point $x^0$ is sufficiently close to $\bar{x}$. As shown in this paper, the obtained conditions allow us to use the proposed algorithm (1.4) to solve systems (1.1) with $C^{1,1}$ functions $\varphi$ in the situations where the semismooth Newton method cannot be applied.

Observe that algorithm (1.4) has been recently introduced and developed, in an equivalent form, in the paper by Mordukhovich and Sarabi [48] to find tilt-stable local minimizers for functions $\varphi$ of class $C^{1,1}$. We’ll discuss tilt stability of local minimizers, the notion introduced by Poliquin and Rockafellar [54], in the corresponding place below with a detailed comparison of the results obtained in this paper and in the paper by Mordukhovich and Sarabi. Note that here we do not assume that $\bar{x}$ is a local minimizer of $\varphi$, not even talking about its tilt stability. Observe also that both the latter paper and the current one employ the semismooth* property of $\nabla \varphi$ that has been recently introduced and developed by Gfrerer and Outrata [19] for set-valued mappings as an improvement of the standard semismoothness for locally Lipschitzian functions used before.

The main thrust of this paper is on developing a generalized Newton method to solve subgradient inclusions of the following type:

$$0 \in \partial \varphi(x),$$

(1.5)

where $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ is an extended-real-valued function belonging to a broad class of prox-regular and subdifferentially continuous, which overwhelmingly appear in variational analysis and optimization. The subdifferential operator used in (1.5) is understood as the (Mordukhovich) limiting subdifferential of extended-real-valued functions that agrees with the classical gradient for $C^1$-smooth functions and the subdifferential of convex analysis when $\varphi$ is
convex. In very general settings, the limiting subdifferential enjoys comprehensive calculus rules that can be found in the books by Mordukhovich \cite{Mordukhovich, Mordukhovich2} and by Rockafellar and Wets \cite{RockafellarWets}. If $\varphi$ is a $C^1$-smooth function, then $\partial \varphi = \nabla \varphi$ and that the inclusion (1.5) reduces to the gradient system (1.1).

One of the generalized Newton algorithms, which is designed in this paper to solve the subgradient inclusion (1.5), is also based on the second-order subdifferential $\partial^2 \varphi$ with replacing (1.4) by

$$-v^k \in \partial^2 \varphi(x^k - \lambda v^k; v^k)(\lambda v^k + d^k) \text{ with } v^k := \frac{1}{\lambda}(x^k - \text{Prox}_{\lambda \varphi}(x^k)), \quad (1.6)$$

where $\text{Prox}_{\lambda \varphi}(x)$ stands for the \textit{proximal mapping} of $\varphi$ corresponding to a constructive choice of the parameter $\lambda > 0$. This form is shown to be closely related to the Newton-type algorithm developed by Mordukhovich and Sarabi \cite{MordukhovichSarabi}, in terms of Moreau envelopes with somewhat different choice of parameters, to find tilt-stable minimizers of prox-regular and subdifferentially continuous functions $\varphi$. As mentioned, the latter is not an ultimate framework of (1.5).

Here we develop a new approach to solvability of systems (1.6) with respect to the directions $d^k$ and to local superlinear convergence of iterates $x^k \to \bar{x}$ under certain metric regularity and subregularity properties of the subdifferential mapping $\partial \varphi$. In particular, all our assumptions hold if $\partial \varphi$ is semismooth* at the reference point and strongly metrically regular around it. As shown by Drusvyatskiy and Lewis \cite{DrusvyatskiyLewis}, for the class of prox-regular and subdifferentially continuous functions the latter property is equivalent to tilt stability of $\bar{x}$ required by Mordukhovich and Sarabi \cite{MordukhovichSarabi} \textit{provided that $\bar{x}$ is a local minimizer of $\varphi$}, which is not assumed in this paper.

We also extend our generalized Newton method to solve the following structured class of composite optimization problems given in the form

$$\text{minimize } \varphi(x) := f(x) + g(x) \text{ subject to } x \in \mathbb{R}^n, \quad (1.7)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is $C^2$-smooth while the regularizer $g : \mathbb{R}^n \to [\mathbb{R}]$ is prox-regular and subdifferentially continuous. Problems written in format (1.7) frequently arise in many applied areas such as machine learning, compressed sensing, image processing, etc. Since $g$ is generally extended-real-valued, the unconstrained format (1.7) encompasses problems of \textit{constrained optimization}. If, in particular, $g$ is the indicator function of a closed set, then (1.7) becomes an optimization problem with geometric constraints. In our algorithm to solve problems of type (1.7) we employ the machinery of \textit{forward-backward envelopes}.

The developed generalized Newton method for subgradient inclusions is finally applied to solving regularized least square problems that appear in practical models of statistics, machine learning, etc. For such problems, we compute the second-order subdifferential and the proximal mapping from (1.6) entirely in terms of the given data, derive explicit calculation formulas, and then provide solvability and local convergence results for our algorithms applied to problems of this class.

The rest of the paper is organized as follows. Section 2 presents and discusses those notions of variational analysis and generalized differentiation, which are broadly used in the formulations and proofs of the main results obtained below. Section 3 is devoted to \textit{solvability} of the \textit{generalized equations}

$$-\bar{v} \in D^*F(\bar{x}, \bar{v})(d) \text{ for } d \in \mathbb{R}^n, \quad (1.8)$$

where $D^*F(\bar{x}, \bar{v})(\cdot)$ is the \textit{coderivative} of a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that is associated with the limiting subdifferential $\partial \varphi$ as defined in Section 2. The framework of (1.8) encompasses all the versions (1.3), (1.4), and (1.6) of the Newton-type algorithms discussed above. Using well-developed calculus rules of limiting generalized differentiation allows us to prove that (1.8)
is solvable for $d$ if the mapping $F$ is strongly metrically subregular at the reference point $(\bar{x}, \bar{v})$. Furthermore, the strong metric regularity of $F$, in particular, ensures the solvability of (1.8) and the compactness of the (generalized) Newton directions therein around the point in question. The solvability results established in Section 3 for coderivative inclusions are then applied in Section 4 to solvability issues for generalized Newton systems of types (1.4) and (1.6) involving the second-order subdifferential $\partial^2 \varphi$. In this way we identify broad classes of functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ for which the required assumptions on $F = \partial \varphi$ are satisfied.

Section 5 presents a generalized Newton algorithm to solve the gradient equations (1.1) with functions $\varphi$ of class $C^{1,1}$ according to the iteration procedure (1.4). The main result of this section establishes a local superlinear convergence of iterates (1.4) to a designated solution $\bar{x}$ of (1.1) under the semismoothness* of the gradient mapping $\nabla \varphi$ at $\bar{x}$ and under merely its metric regularity around this point. Even in the case of tilt-stable local minimizers of $\varphi$, the obtained result improves the one from Mordukhovich and Sarabi [48]. We also compare the new algorithm with some other generalized Newton methods and show, in particular, that our algorithm is well-defined and exhibits a superlinear convergence of iterates when the semismooth Newton algorithm cannot be even constructed.

Section 6 and Section 7 are the culmination of the paper. They describe and justify new Newton-type algorithms to solve the subgradient inclusions (1.5), where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a prox-regular function, and where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is the cost function of the composite optimization problem (1.7), respectively. Note that both these extended-real-valued frameworks of $\varphi$ incorporates problems of constrained optimization for which inclusion (1.5) provides a necessary condition for local minimizers. The results obtained here justifies a constructive and well-defined algorithm, with a verifiable choice of the starting point, that superlinearly converges to the solution $\bar{x}$ of (1.5) under about the same assumptions as in Section 5, but being now addressed to $\partial \varphi$ instead of $\nabla \varphi$. In fact, the proofs of the main results in this section are based on the reduction to the $C^{1,1}$ case by using Moreau envelopes, forward-backward envelopes, and the machinery of variational analysis taken from Mordukhovich [44] and Rockafellar and Wets [57].

In Section 8 we present applications of the developed Newton-type algorithms to solving some nonsmooth, convex and nonconvex regularized least square problems, where we completely calculate all the algorithm parameters in terms of the given problem data.

The concluding Section 9 summarizes the major contributions of the paper and discusses some topics of the future research. Our notation is standard in variational analysis and optimization and can be found in the aforementioned books [44] and [57]. Recall that $B_r(\bar{x}) := \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq r\}$ stands for the closed ball with center $\bar{x}$ and radius $r > 0$.

## 2 Variational Analysis: Preliminaries and Discussions

Here we present the needed background material from variational analysis and generalized differentiation by following the books of Mordukhovich [43, 44] and Rockafellar and Wets [57].

Given a set $\Omega \subset \mathbb{R}^n$ with $\bar{z} \in \Omega$, the (Bouligand-Severi) tangent/contingent cone to $\Omega$ at $\bar{z}$ is

$$T_\Omega(\bar{z}) := \{w \in \mathbb{R}^n \mid \exists t_k \downarrow 0, w_k \rightarrow w \text{ as } k \rightarrow \infty \text{ with } \bar{z} + t_k w_k \in \Omega\}. \quad (2.1)$$

The (Fréchet) regular normal cone to $\Omega$ at $\bar{z} \in \Omega$ is defined by

$$\hat{N}_\Omega(\bar{z}) := \{v \in \mathbb{R}^n \mid \limsup_{\bar{z} \downarrow \bar{z}^+} \langle v, z - \bar{z} \rangle / \|z - \bar{z}\| \leq 0\}, \quad (2.2)$$

where the symbol $z \overset{\Omega}{\rightarrow} \bar{z}$ indicates that $z \rightarrow \bar{z}$ with $z \in \Omega$. It can be equivalently described via a duality correspondence with (2.1) by

$$\hat{N}_\Omega(\bar{z}) = T^*_\Omega(\bar{z}) := \{v \in \mathbb{R}^n \mid \langle v, w \rangle \leq 0 \text{ for all } w \in T_\Omega(\bar{z})\}. \quad (2.3)$$
The (Mordukhovich) \textit{limiting normal cones} to $\Omega$ at $\bar{x} \in \Omega$ is defined by

$$N_{\Omega}(\bar{z}) := \{v \in \mathbb{R}^s \mid \exists z_k \underset{\Omega}{\rightarrow} \bar{z}, v_k \rightarrow v \text{ as } k \rightarrow \infty \text{ with } v_k \in \hat{N}_{\Omega}(z_k)\}. \quad (2.4)$$

Note that the regular normal cone (2.2) is always convex, while the limiting normal cone (2.4) is often nonconvex (e.g., for the graph of $|x|$ at $\bar{z} = (0,0)$), and hence it cannot be obtained by the duality correspondence of type (2.3) from any tangential approximation of $\Omega$ at $\bar{z}$. Nevertheless, the normal cone (2.4), as well as the coderivative and subdifferential constructions for mappings and functions generated by it and described below, enjoy comprehensive \textit{calculus rules} that are based on \textit{variational/extremal principles} of variational analysis.

Given further a set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the \textit{graph}

$$\text{gph } F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\},$$

the \textit{graphical derivative} of $F$ at $(\bar{x}, \bar{y}) \in \text{gph } F$ is defined via (2.1) by

$$DF(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^m \mid (u,v) \in T_{\text{gph } F}(\bar{x}, \bar{y})\}, \quad u \in \mathbb{R}^n. \quad (2.5)$$

Consider also the \textit{domain}, \textit{kernel}, and \textit{range} of $F$ denoted, respectively, by

$$\text{dom } F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}, \quad \ker F := \{x \in \mathbb{R}^n \mid 0 \in F(x)\},$$

$$\text{rge } F := \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ with } y \in F(x)\}.$$ 

The inverse mapping of $F$ is the set-valued mapping $F^{-1}: \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$F^{-1}(y) := \{x \in \mathbb{R}^n \mid y \in F(x)\}, \quad y \in \mathbb{R}^m.$$ 

The coderivative constructions for $F$ at $(\bar{x}, \bar{y}) \in \text{gph } F$ are defined via the regular normal cone (2.2) and the limiting normal cone (2.4) to the graph of $F$ at this point. They are, respectively, the \textit{regular coderivative} and the \textit{limiting coderivative} of $F$ at $(\bar{x}, \bar{y})$ given by

$$\hat{D}^*F(\bar{x}, \bar{y})(v) := \{u \in \mathbb{R}^n \mid (u,-v) \in \hat{N}_{\text{gph } F}(\bar{x}, \bar{y})\}, \quad v \in \mathbb{R}^m, \quad (2.6)$$

$$D^*F(\bar{x}, \bar{y})(v) := \{u \in \mathbb{R}^n \mid (u,-v) \in N_{\text{gph } F}(\bar{x}, \bar{y})\}, \quad v \in \mathbb{R}^m. \quad (2.7)$$

In the case where $F(\bar{x})$ is the singleton $\{\bar{y}\}$, we omit $\bar{y}$ in the notation of (2.5)–(2.7). Note that if $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $C^1$-smooth around $\bar{x}$, then

$$DF(\bar{x}) = \nabla F(\bar{x}) \quad \text{and} \quad \hat{D}^*F(\bar{x}) = D^*F(\bar{x}) = \nabla F(\bar{x})^*,$$

where $\nabla F(\bar{x})^*$ is the adjoint/transpose matrix of the Jacobian $\nabla F(\bar{x})$.

Before considering the first- and second-order subdifferential constructions for extended-real-valued functions, which are employed in this paper and are closely related to the limiting normals and coderivatives, we formulate the \textit{metric regularity} and \textit{subregularity} properties of set-valued mappings that are highly recognized in variational analysis and optimization. These properties are frequently used below.

To proceed, recall that the \textit{distance function} associated with a set $\Omega \subset \mathbb{R}^s$ is

$$\text{dist}(x; \Omega) := \inf \{\|w-x\| \mid w \in \Omega\}, \quad x \in \mathbb{R}^s.$$ 

A mapping $\hat{F}: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a \textit{localization} of $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that we have

$$\text{gph } \hat{F} = \text{gph } F \cap (U \times V).$$
Definition 2.1 (metric regularity and subregularity of mappings). Let \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be a set-valued mapping, and let \((\bar{x}, \bar{y}) \in \text{gph } F\). We say that:

(i) \( F \) is metrically regular around \((\bar{x}, \bar{y})\) with modulus \( \mu > 0 \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) providing the estimate

\[
\text{dist}(x; F^{-1}(y)) \leq \mu \text{dist}(y; F(x)) \quad \text{for all } (x, y) \in U \times V.
\]

If in addition \( F^{-1} \) has a single-valued localization around \((\bar{y}, \bar{x})\), then \( F \) is strongly metrically regular around \((\bar{x}, \bar{y})\) with modulus \( \mu > 0 \).

(ii) \( F \) is metrically subregular at \((\bar{x}, \bar{y})\) with modulus \( \mu > 0 \) if there exists a neighborhood \( U \) of \( \bar{x} \) such that we have

\[
\text{dist}(x; F^{-1}(\bar{y})) \leq \mu \text{dist}(\bar{y}; F(x)) \quad \text{for all } x \in U.
\]

If \( F^{-1}(\bar{y}) \cap U = \{\bar{x}\} \), then \( F \) is strongly metrically subregular at \((\bar{x}, \bar{y})\) with modulus \( \mu > 0 \).

Observe that the metric regularity properties in Definition 2.1(i) are stable/robust with respect to small perturbations of the reference point \((\bar{x}, \bar{y})\), i.e., they are preserved in a neighborhood of the point \((\bar{x}, \bar{y})\). It is not always the case for metric subregularity. The next remark summarizes relationships between the above metric regularity/subregularity properties of mappings and presents generalized differential characterizations of the major ones that are broadly used in this paper.

Remark 2.2 (on metric regularity and subregularity). Observe the following:

(i) We obviously have that the strong metric regularity of a set-valued mapping implies that both its metric regularity and strong metric subregularity properties hold. Furthermore, a strongly metrically subregular mapping is metrically subregular at the corresponding point. However, metric regularity and strong metric subregularity are generally incomparable. For example, the mapping \( F(x) := |x| \) for all \( x \in \mathbb{R} \) is strongly metrically subregular at \((0, 0)\), but it is not metrically regular around this point. On the other hand, \( F : \mathbb{R} \rightrightarrows \mathbb{R} \) defined by \( F(x) := [x, \infty) \) is metrically regular around \((0, 0)\) while not being strongly metrically subregular at this point.

(ii) Simple examples show that a mapping may exhibit the strong metric subregularity property at some point while not being strongly metrically regular and even merely metrically regular around the reference point. Indeed, consider the simplest nonsmooth function \( F(x) := |x| \) with \((\bar{x}, \bar{y}) = (0, 0)\) discussed in (i). The relationships between metric subregularity, metric regularity, strong metric subregularity, and strong metric regularity are illustrated in the following diagram, where the arrow \( \longrightarrow \) reads “implies”.

\[
\begin{array}{ccc}
\text{metric regularity} & \longrightarrow & \text{strong metric regularity} \\
\text{metric subregularity} & \longrightarrow & \text{strong metric subregularity}
\end{array}
\]
(iii) A major advantage of the generalized differential constructions defined above is the possibility to get in their terms complete pointwise characterizations of the metric regularity and strong metric subregularity properties of general set-valued mappings. Namely, a mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, which graph is locally closed around $(\bar{x}, \bar{y}) \in \text{gph} F$, is metrically regular around this point if and only if we have the kernel coderivative condition

$$\ker D^*F(\bar{x}, \bar{y}) := \{ v \in \mathbb{R}^m \mid 0 \in D^*F(\bar{x}, \bar{y})(v) \} = \{0\}$$

established by Mordukhovich [42, Theorem 3.6] via his limiting coderivative (2.7) and then labeled as the Mordukhovich criterion in Rockafellar and Wets [57, Theorems 7.40 and 7.43]. Broad applications of this result are based on robustness and full calculus available for the limiting coderivative; see the books by Mordukhovich [43,44] and by Rockafellar and Wets [57] for more details and references.

A parallel characterization of the (nonrobust) strong metric subregularity property of a (locally) closed-graph mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \text{gph} F$ is given by $\ker DF(\bar{x}, \bar{y}) = \{0\}$ via the (nonrobust) graphical derivative (2.5) of $F$ at $(\bar{x}, \bar{y})$ and is known as the Levy-Rockafellar criterion; see the book by Dontchev and Rockafellar [13, Theorem 4E.1] with the references and discussions therein.

Finally in this section, we recall the limiting first-order and second-order subdifferential constructions for extended-real-valued functions that are used for describing the subgradient inclusions (1.5) and the Newton-type algorithms (1.4) and (1.6) to compute their solutions. Given an extended-real-valued function $\varphi: \mathbb{R}^n \to \overline{\mathbb{R}}$, consider its effective domain and epigraph defined by, respectively,

$$\text{dom } \varphi := \{ x \in \mathbb{R}^n \mid \varphi(x) < \infty \} \quad \text{and} \quad \text{epi } \varphi := \{ (x, \alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq \varphi(x) \}.$$

Then for a fixed point $\bar{x} \in \text{dom } \varphi$ we define the basic/limiting subdifferential and the singular/horizon subdifferential of $\varphi$ at $\bar{x}$ by, respectively,

$$\partial \varphi(\bar{x}) := \{ v \in \mathbb{R}^n \mid (v, -1) \in N_{\text{epi } \varphi}(\bar{x}, \varphi(\bar{x})) \},$$

$$\partial^\infty \varphi(\bar{x}) := \{ v \in \mathbb{R}^n \mid (v, 0) \in N_{\text{epi } \varphi}(\bar{x}, \varphi(\bar{x})) \}$$

via the limiting normal cone (2.4) to the epigraph of $\varphi$ at $(\bar{x}, \varphi(\bar{x}))$. For simplicity we use here the geometric definitions of the subdifferentials (2.9) and (2.10) while referring the reader to the aforementioned monographs on variational analysis for equivalent analytic representations. Recall that the basic subdifferential $\partial \varphi(\bar{x})$ reduces to the gradient $\{\nabla \varphi(\bar{x})\}$ if $\varphi$ is $C^1$-smooth around $\bar{x}$ (or merely strictly differentiable at this point), and that $\partial \varphi(\bar{x})$ is the subdifferential of convex analysis if $\varphi$ is convex. On the other hand, the singular subdifferential $\partial^\infty \varphi(\bar{x})$ of a lower semicontinuous (l.s.c.) function $\varphi$ reduces to $\{0\}$ if and only if $\varphi$ is locally Lipschitzian around $\bar{x}$. Both constructions (2.9) and (2.10) enjoy in parallel full subgradient calculi in very general settings. Let us also mention the scalarization formula

$$D^*F(\bar{x})(v) = \partial \langle v, F(\bar{x}) \rangle \quad \text{for all } v \in \mathbb{R}^n,$$

which holds whenever $F: \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitzian around $\bar{x}$.

Now we are ready to define the second-order subdifferential of $\varphi: \mathbb{R}^n \to \overline{\mathbb{R}}$ at $\bar{x} \in \text{dom } \varphi$ for $\bar{v} \in \partial \varphi(\bar{x})$ in the sense of Mordukhovich [41] as the mapping $\partial^2 \varphi(\bar{x}, \bar{v}): \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that

$$\partial^2 \varphi(\bar{x}, \bar{v})(u) := (D^* \partial \varphi)(\bar{x}, \bar{v})(u) \quad \text{for all } u \in \mathbb{R}^n,$$

i.e., by applying the coderivative (2.7) to the first-order subgradient mapping (2.9). This second-order subdifferential (or generalized Hessian) (2.12) appears in the Newton-type iterations (1.4)
and (1.6) for $C^{1,1}$ and prox-regular functions, respectively, which both go back to the classical Newton algorithm (1.2) for $C^2$-smooth functions due to the relationship
\[
\partial^2 \varphi(\bar{x})(u) = \{\nabla^2 \varphi(\bar{x})u\}
\]
whenever $u \in \mathbb{R}^n$ (2.13) in the $C^2$-smooth case. If $\varphi$ is of class $C^{1,1}$ around $\bar{x}$, then the computation of $\partial^2 \varphi(\bar{x})$ reduces to the computation of the limiting subdifferential (2.9) of the gradient mapping $\nabla \varphi$ by the scalarization formula (2.11). Besides the well-developed second-order calculus for (2.12), variational analysis achieves constructive computations of the second-order subdifferential, entirely in terms of the given problem data, for major classes of nonsmooth functions arising in important problems of constrained optimization, bilevel programming, optimal control, operations research, mechanics, economics, statistics, machine learning, etc. Among many other publications, we refer the reader to Colombo et al. [9], Ding et al. [11], Dontchev and Rockafellar [12], Henrion et al. [22, 24], Mordukhovich [43, 44], Mordukhovich and Outrata [46], Mordukhovich and Rockafellar [47], Outrata and Sun [50], Yao and Yen [64], and the bibliographies therein. The new computation of this type is provided in Section 8 for the practically important regularized least square problems arising in statistics and machine learning.

3 Solvability of Coderivative Inclusions

A crucial step in the design and justification of numerical algorithms is to establish their well-posedness, i.e., the solvability of the corresponding iterative systems. In the case of the classical Newton method to solve $\nabla \varphi(x) = 0$, we have the equation for $d \in \mathbb{R}^n$ written as
\[
- \nabla \varphi(x) = \nabla^2 \varphi(x)d,
\]
which is solvable if the Hessian matrix $\nabla^2 \varphi(x)$ is invertible. In the case of the generalized Newton algorithm for nonsmooth functions discussed in Section 1, we extend (3.1) in the following way. Given a subgradient $v \in \partial \varphi(x)$, consider the inclusion
\[
- v \in \partial^2 \varphi(x,v)(d)
\]
and find conditions ensuring the solvability of (3.2) with respect to $d \in \mathbb{R}^n$. Due to (2.12), the second-order inclusion (3.2) can be written as
\[
- v \in D^*F(x,v)(d)
\]
with $F := \partial \varphi$, where $D^*$ stands for the limiting coderivative (2.7).

The major goal of this section is to investigate the solvability of the coderivative inclusion (3.3) with respect to $d \in \mathbb{R}^n$. In the next section, we proceed with the study of solvability of the generalized Newton systems (3.2) and establish appropriate conditions on functions $\varphi$, which allow us to efficiently apply the solvability results obtained for (3.3) to the case of systems (3.2) of our main interest.

The first theorem here verifies the solvability of (3.3) for $d$ at any point $(\bar{x}, \bar{v}) \in \text{gph}\ F$ where the mapping $F$ is strongly metrically subregular. The proof of this result is based on major calculus rules for the limiting generalized differential constructions.

**Theorem 3.1 (solvability of coderivative inclusions).** Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping whose graph is locally closed around a given point $(\bar{x}, \bar{v}) \in \text{gph}\ F$. If $F$ is strongly metrically subregular at $(\bar{x}, \bar{v})$, then there exists $d \in \mathbb{R}^n$ satisfying the inclusion
\[
- \bar{v} \in D^*F(\bar{x}, \bar{v})(d).
\]
which implies that \( x \) have gph which implies in turn that a subsequence \( \{ x \} \) converging to \( v \) such that \( F^{-1}(\bar{v}) \cap U = \{ \bar{x} \} \) and

\[
\| x - \bar{x} \| \leq \ell \| v - \bar{v} \| \text{ for all } x \in F^{-1}(v) \cap U \text{ and } v \in V.
\]

(3.5)

Passing to appropriate subsets, we suppose for convenience that the sets \( U \) and \( V \) are closed and bounded. Due to the local closedness of \( F \) around \( (\bar{x}, \bar{v}) \), we can assume further without loss of generality that the set \( gphG \cap (U \times V) \) is closed. Consider further the set-valued mapping \( G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) given by

\[
G(v) := F^{-1}(v) \cap U \text{ for all } v \in \mathbb{R}^n
\]

and then define the marginal/optimal value function \( \mu: \mathbb{R}^n \rightarrow \overline{\mathbb{R}} \) and the corresponding argmin-imum mapping \( M: \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) by, respectively,

\[
\mu(v) := \inf \{ \varphi(v, x) \mid x \in G(v) \}
\]

and

\[
M(v) := \{ x \in G(v) \mid \varphi(v, x) = \mu(v) \}, \quad v \in \mathbb{R}^n,
\]

(3.6)

where \( \varphi(v, x) := \langle \bar{v}, x \rangle \) for all \((v, x) \in \mathbb{R}^n \times \mathbb{R}^n \). It is clear that \((\bar{v}, \bar{x}) \in gphM \). We intend to verify that the marginal function (3.6) is l.s.c. around \( \bar{v} \). To proceed, pick any \( v \in \text{int}V \) and check first that for \( \varepsilon > 0 \) there exists \( \delta > 0 \) ensuring the implication

\[
\| u - v \| \leq \delta \implies G(u) \subset G(v) + \varepsilon \mathbb{B}
\]

(3.7)

Suppose on the contrary that (3.7) does not hold and then find \( \varepsilon > 0 \) and a sequence \( \{ u_k \} \) converging to \( v \) such that \( G(u_k) \subset G(v) + \varepsilon \mathbb{B} \). This means that for each \( k \in \mathbb{N} \) there exists \( x_k \in G(u_k) \) such that \( x_k \notin G(v) + \varepsilon \mathbb{B} \). Since we have \( \{ x_k \} \subset U \) where \( U \) is compact, there exists a subsequence \( \{ x_{k_j} \} \) of \( \{ x_k \} \) converging to some \( x^* \) as \( j \rightarrow \infty \). It follows from the closedness of \( gphG \cap (U \times V) \) and the choice of \( x_{k_j} \in G(u_{k_j}) \) and \( u_{k_j} \rightarrow v \) with \( x_{k_j} \rightarrow x^* \) that \( x^* \in G(v) \), which implies that \( x_{k_j} \notin \mathbb{B}_\varepsilon(x^*) \) for any \( j \in \mathbb{N} \). This is a clear contradiction that verifies (3.7).

Let us now show that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\mu(u) \geq \mu(v) - \varepsilon \quad \text{whenever } \| u - v \| \leq \delta.
\]

(3.8)

Indeed, using (3.7) gives us \( \delta > 0 \) providing the implication

\[
\| u - v \| \leq \delta \implies G(u) \subset G(v) + \varepsilon \mathbb{B}.
\]

(3.9)

Pick \( u \in \mathbb{B}_\delta(v) \). By (3.9), for any \( x' \in G(u) \), there is \( x \in G(v) \) with \( x' - x \in \varepsilon/(\| \bar{v} \| + 1) \mathbb{B} \), and so

\[
\langle \bar{v}, x' \rangle - \langle \bar{v}, x \rangle = \langle \bar{v}, x' - x \rangle \geq -\| \bar{v} \| \cdot \| x' - x \| \geq -\varepsilon,
\]

which implies in turn that

\[
\langle \bar{v}, x' \rangle \geq \mu(v) - \varepsilon \quad \text{for all } x' \in G(u).
\]

Therefore, we arrive at the conditions

\[
\mu(u) = \inf_{x' \in G(u)} \langle \bar{v}, x' \rangle \geq \mu(v) - \varepsilon,
\]

and hence justifies (3.8), which means that the marginal function \( \mu \) is l.s.c. around \( \bar{v} \).

To show that the mapping \( M \) is locally bounded around \( \bar{v} \), observe that for each \( v \in V \) we have

\[
M(v) \subset G(v) = F^{-1}(v) \cap U \subset U,
\]
which tells us that the image set $M(V)$ is bounded, i.e., the mapping $M$ is locally bounded around $\bar{v}$. To evaluate now the limiting subdifferential (2.9) of the marginal function (3.6) by Theorem 4.1(ii) from Mordukhovich [44], it remains to check the qualification condition

$$\partial^\infty \varphi(\bar{v}, \bar{x}) \cap \{- N_{\text{gph} G}(\bar{v}, \bar{x})\} = \{0\}$$

therein, which automatically holds due to the Lipschitz continuity of the function $\varphi$ around $(\bar{x}, \bar{v})$. Since we have in (3.6) that $M(\bar{v}) = \{\bar{x}\}$, $\nabla_x \varphi(\bar{v}, \bar{x}) = 0$, and $\nabla_x \varphi(\bar{v}, \bar{x}) = \bar{v}$ in (3.6), it follows that

$$\partial \mu(\bar{v}) \subseteq \nabla_x \varphi(\bar{v}, \bar{x}) + D^* G(\bar{v}, \bar{x}) \nabla_x \varphi(\bar{v}, \bar{x}) = D^* G(\bar{v}, \bar{x})(\bar{v}). \quad (3.10)$$

To proceed further, consider the mapping $H(v) \equiv U$ on $\mathbb{R}^n$ and observe that $N_{\text{gph} H}(\bar{v}, \bar{x}) = \{0\}$ by $(\bar{v}, \bar{x}) \in \text{int}(\text{gph} H)$. Thus $N_{\text{gph} H}(\bar{v}, \bar{x}) = \{0\}$ and the qualification condition

$$N_{\text{gph} H}(\bar{v}, \bar{x}) \cap \{- N_{\text{gph} F^{-1}}(\bar{v}, \bar{x})\} = \{0\}$$

is satisfied. This allows us to apply the normal cone intersection rule from Mordukhovich [44, Theorem 2.16] and get the relationships

$$N_{\text{gph} G}(\bar{v}, \bar{x}) \subseteq N_{\text{gph} F^{-1}}(\bar{v}, \bar{x}) + N_{\text{gph} H}(\bar{v}, \bar{x}) = N_{\text{gph} F^{-1}}(\bar{v}, \bar{x}).$$

Combining the latter with (3.10) gives us the inclusions

$$\partial \mu(\bar{v}) \subseteq D^* G(\bar{v}, \bar{x})(\bar{v}) \subseteq D^* F^{-1}(\bar{v}, \bar{x})(\bar{v}). \quad (3.11)$$

Let us now show that $\partial \mu(\bar{v}) \neq \emptyset$. Recall from Mohammadi et al. [38, Proposition 2.1] that if a function $\psi: \mathbb{R}^n \to \mathbb{R}$ is l.s.c. around $\bar{v} \in \text{dom} \psi$ and satisfies the “lower calmness” property

$$\psi(v) \geq \psi(\bar{v}) - \eta \|v - \bar{v}\| \quad \text{for all} \quad v \in V \quad (3.12)$$

with some $\eta \geq 0$ and a neighborhood $V$ of $\bar{v}$, then $\partial \psi(\bar{v}) \neq \emptyset$. To establish (3.12) for $\psi := \mu$, we verify in what follows the fulfillment of the estimate

$$\mu(v) - \mu(\bar{v}) \geq -\ell \|v\| \cdot \|v - \bar{v}\| \quad \text{whenever} \quad v \in V. \quad (3.13)$$

Observe first that if $G(v) = \emptyset$ for some $v \in V$, then $\mu(v) = \infty$, and so estimate (3.13) is obviously satisfied. In the remaining case where $G(v) \neq \emptyset$ for a fixed vector $v \in V$, pick any $x \in G(v)$ and then check by (3.5) that

$$\langle \bar{v}, x \rangle - \langle \bar{v}, x \rangle \geq -\|\bar{v}\| \cdot \|x - \bar{x}\| \geq -\ell \|\bar{v}\| \cdot \|v - \bar{v}\|.$$

Indeed, the condition $M(\bar{v}) = \{\bar{x}\}$ clearly yields (3.13), and hence $\partial \mu(\bar{v}) \neq \emptyset$. It justifies by (3.11) the existence of $u \in \mathbb{R}^n$ satisfying $u \in D^* F^{-1}(\bar{v}, \bar{x})(\bar{v})$. To complete the proof of the theorem, recall that

$$z \in D^* F(\bar{x}, \bar{y})(u) \iff -u \in (D^* F^{-1})(\bar{y}, \bar{x})(-z), \quad (3.14)$$

which readily ensures the fulfillment of (3.4) and thus finishes the proof. \qed

The following example shows that the strong metric subregularity assumption of Theorem 3.1 cannot be replaced by metric subregularity of $F$ at $(\bar{x}, \bar{v})$, or even by metric regularity of $F$ around this point in order to guarantee the solvability of the coderivative inclusion (3.4).
Example 3.2 (insolvability of coderivative inclusions under metric regularity). Consider the set-valued mapping $F: \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := [0, 1] \text{ for all } x \in \mathbb{R}. \quad (3.15)$$

Then the graphical set $\text{gph} F = \mathbb{R} \times [0, 1]$ is convex, and we easily calculate the limiting normal cone to the graph of (3.15) as follows:

$$N_{\text{gph} F}(x,y) = \begin{cases} \{(0,0)\} & \text{if } y \in (0,1), \\ \{0\} \times \mathbb{R}_+ & \text{if } y = 1, \\ \{0\} \times \mathbb{R}_- & \text{if } y = 0. \end{cases}$$

Pick $(\bar{x}, \bar{v}) := (1, \frac{1}{2}) \in \text{gph} F$ and show that $F$ is metrically regular around $(\bar{x}, \bar{v})$. Indeed, taking $u \in \mathbb{R}^n$ with $0 \in D^*F(\bar{x}, \bar{v})(u)$, we readily have that

$$(0, -u) \in N_{\text{gph} F}(\bar{x}, \bar{v}) = \{(0,0)\},$$

and hence $u = 0$. It follows from the Mordukhovich criterion (2.8) that $F$ is metrically regular around $(\bar{x}, \bar{v})$. However, it is easy to see that there exists no $d \in \mathbb{R}$ which solves (3.4).

The next example shows that the strong metric subregularity, being a nonrobust property, does not ensure the robust solvability of the coderivative inclusion (3.3), i.e., its solvability in a neighborhood of the reference point. Given $(x,v) \in \text{gph} F$, consider the set of feasible solutions to inclusion (3.3) at this point that is defined by

$$\Gamma_F(x,v) := \{d \in \mathbb{R}^n \mid -v \in D^*F(x,v)(d)\}. \quad (3.16)$$

Example 3.3 (failure of robust solvability under strong metric subregularity). Consider the set-valued mapping $F: \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := \begin{cases} \{0\} \cup [1, \infty) & \text{if } x = 0, \\ [1, \infty) & \text{if } x \neq 0. \end{cases} \quad (3.17)$$

It is clear that $F$ is strongly metrically subregular at $(0,0)$ and that the graph of $F$ is closed around this point. Let us show that for any neighborhood $U \times V$ of the origin in $\mathbb{R}^2$ and any nonzero pair $(x,v) \in \text{gph} F \cap (U \times V)$ there exists no $d \in \mathbb{R}$ satisfying the coderivative inclusion (3.3). To proceed, observe that $\text{gph} F = (0,0) \cup \mathbb{R} \times [1,\infty)$ for $F$ from (3.17). Then we readily get that

$$N_{\text{gph} F}(x,v) = \begin{cases} \mathbb{R}^2 & \text{if } x = 0, \ v = 0, \\ \{(0,\lambda) \mid \lambda \leq 0\} & \text{if } v \geq 1, \end{cases}$$

which yields the limiting coderivative expression

$$D^*F(x,v)(u) = \begin{cases} \mathbb{R} & \text{if } (x,v) = (0,0), \\ \{0\} & \text{if } (x,v) \neq (0,0), \ u \geq 0, \\ \emptyset & \text{otherwise}. \end{cases}$$

It easily follows from this formula that there exists no $d \in \mathbb{R}$ satisfying (3.3) for any $(x,v) \in \text{gph} F$ except $(x,v) = (0,0)$. Note also that the set $\Gamma_F(0,0) = \mathbb{R}$ from (3.16) is not bounded.
Now we are ready to show that the replacement of the strong metric subregularity of $F$ at $(\bar{x}, \bar{v})$ in Theorem 3.1 by its robust counterpart, which is the strong metric regularity of $F$ around $(\bar{x}, \bar{v})$, leads us to robust solvability of the coderivative inclusion (3.3) and thus ensures the well-posedness of generalized Newton iterations in the algorithms designed below in this paper.

**Theorem 3.4 (robust solvability of coderivative inclusions).** Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping whose graph is closed around $(\bar{x}, \bar{v}) \in \text{gph} \, F$. If $F$ is strongly metrically regular around this point, then there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that for each $(x, v) \in \text{gph} \, F \cap (U \times V)$ there exists a direction $d \in \mathbb{R}^n$ satisfying the coderivative inclusion (3.3). Moreover, the set-valued mapping $\Gamma_F$ from (3.16) is compact-valued for all $(x, v) \in \text{gph} \, F \cap (U \times V)$.

**Proof.** Since $F$ is strongly metrically regular around $(\bar{x}, \bar{v})$, it follows from Definition 2.1(i) that the inverse mapping $F^{-1}: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ admits a single-valued localization $\vartheta: V \to U$ around $(\bar{v}, \bar{x})$, which is locally Lipschitzian around $\bar{v}$. This implies, together with the scalarization formula (2.11), that for each $(x, v) \in \text{gph} \, F \cap (U \times V)$ we have the representations

$$D^*F^{-1}(x)(v) = D^*\vartheta(v)(v) = \vartheta(v, \vartheta)(v). \quad (3.18)$$

It is well known (see, e.g., Mordukhovich [44, Theorem 1.22]) that the limiting subgradient set of a locally Lipschitzian function is nonempty and compact. Hence it follows from (3.18) that the set $D^*F^{-1}(x)(v) = \vartheta(v, \vartheta)$ is nonempty and compact in $\mathbb{R}^n$. Taking any $u \in D^*F^{-1}(x)(v) = \vartheta(v, \vartheta)$, we deduce from (3.14) that $-v \in D^*F(x, v)(d)$ with $d := -u$, which verifies the robust solvability of (3.3) around $(\bar{x}, \bar{v})$. The claimed compactness of (3.16) follows from the compactness of $\vartheta(v, \vartheta)(v)$. \hfill $\square$

Note that the strong metric regularity in Theorem 3.4 is just a sufficient condition for robust solvability of the coderivative inclusion (3.3) and its second-order subdifferential specifications studied in the next section. As we see below, the required robust solvability is exhibited even in the case of coderivative inclusions (1.3) generated by subdifferentially continuous functions $\varphi$ without the strong metric regularity assumption on $\partial \varphi$.

## 4 Solvability of Generalized Newton Systems

In this section we consider the second-order subdifferential inclusions (3.2) generated by extended-real-valued functions $\varphi: \mathbb{R}^n \to \overline{\mathbb{R}}$. Such systems appear in our generalized Newton algorithms, which were discussed in Section 1 and will be fully developed in what follows. As mentioned, the second-order subdifferential systems (3.2) are specifications of the coderivative ones (3.3) for $F := \partial \varphi$, while the subdifferential structure of $F$ creates strong opportunities to efficiently implement and improve the assumptions of Theorems 3.1 and 3.4 for important classes of functions $\varphi$ that are often encountered in finite-dimensional variational analysis and optimization.

There are two groups of assumptions in both Theorems 3.1 and 3.4: one on the closed graph of $F$, and the other on the strong metric subregularity and regularity properties. Let us start with the first one: when is the limiting subgradient mapping $\partial \varphi$ of (locally) closed graph?

It is well known and easily follows from definitions (2.4) and (2.9) that the limiting subgradient mapping $\partial \varphi: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is closed-graph around $(\bar{x}, \bar{v}) \in \text{gph} \, \partial \varphi$ if $\varphi$ is continuous around $\bar{x}$. However, this important and broad setting does not encompass functions that are locally extended-real-valued around $\bar{x}$, while such functions are the most interesting for applications, e.g., to constrained optimization. This is the reason for the following definition taken from Rockafellar and Wets [57, Definition 13.28].
**Definition 4.1 (subdifferentially continuous functions).** A function \( \varphi: \mathbb{R}^n \to \overline{\mathbb{R}} \) is subdifferentially continuous at \( \bar{x} \in \text{dom} \varphi \) for \( \bar{v} \in \partial \varphi(\bar{x}) \) if for any \((x_k, v_k) \to (\bar{x}, \bar{v})\) with \( v_k \in \partial \varphi(x_k) \) we have \( \varphi(x_k) \to \varphi(\bar{x}) \) as \( k \to \infty \). If this holds for all \( \bar{v} \in \partial \varphi(\bar{x}) \), then \( \varphi \) is subdifferentially continuous at \( \bar{x} \).

Note that \( \varphi \) is obviously subdifferentially continuous at any point \( \bar{x} \in \text{dom} \varphi \) where \( \varphi \) is continuous merely relative to its domain. It easily follows from the subdifferential construction of convex analysis that any convex extended-real-valued function is subdifferentially continuous at every \( \bar{x} \in \text{dom} \varphi \). As has been well recognized in variational analysis and optimization, the class of subdifferentially continuous functions is much broader and includes, in particular, *strongly amenable functions, lower-C^2 functions*, etc.; see Rockafellar and Wets [57].

The next theorem on solvability and robust solvability of generalized Newton systems is a direct consequence of Theorems 3.1, 3.4 and Definition 4.1.

**Theorem 4.2 (solvability and robust solvability of generalized Newton systems).** Let \( \varphi: \mathbb{R}^n \to \overline{\mathbb{R}} \) be subdifferentially continuous around some \( \bar{x} \in \text{dom} \varphi \). Then the following hold:

(i) Given \( \bar{v} \in \partial \varphi(\bar{x}) \), assume that the mapping \( \partial \varphi: \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is strongly metrically subregular at \((\bar{x}, \bar{v})\). Then there exists \( d \in \mathbb{R}^n \) satisfying the second-order subdifferential inclusion (3.2) for \((x,v) := (\bar{x}, \bar{v})\).

(ii) Given \( \bar{v} \in \partial \varphi(\bar{x}) \), assume that the subgradient mapping \( \partial \varphi \) is strongly metrically regular around \((\bar{x}, \bar{v})\). Then there is a neighborhood \( U \times V \) of \((\bar{x}, \bar{v})\) such that for each \((x,v) \in \text{gph} \partial \varphi \cap (U \times V)\) there exists a direction \( d \in \mathbb{R}^n \) satisfying the second-order subdifferential inclusion (3.2). Moreover, the set-valued mapping \( \Gamma_{\partial \varphi} \) from (3.16) is compact-valued for all \((x,v) \in \text{gph} \partial \varphi \cap (U \times V)\).

**Proof.** It is easy to check that the imposed subdifferential continuity assumption on \( \varphi \) ensures that the graph of \( \partial \varphi \) is locally closed around \((\bar{x}, \bar{v}) \in \text{gph} \partial \varphi \). Then the claimed assertions (i) and (ii) follow from Theorem 3.1 and Theorem 3.4, respectively.

Observe that Example 3.2, which demonstrates that the strong metric subregularity assumption on the mapping \( F \) in (3.3) cannot be replaced by the metric regularity and hence by the metric subregularity ones in the conclusion of Theorem 3.1, still works for Theorem 4.2(i) dealing with mappings \( F = \partial \varphi \) of the subdifferential type. Indeed, it is shown by Wang [63, Theorem 4.6] that there exists a Lipschitz continuous function \( \varphi: \mathbb{R} \to \mathbb{R} \) such that \( \partial \varphi(x) = [0,1] \) for all \( x \in \mathbb{R} \). Thus we get from Example 3.2 that the subgradient mapping \( \partial \varphi \) for this function is metrically regular around the point \((0,1/2) \in \text{gph} \partial \varphi \), while the second-order subdifferential inclusion (3.2) is not solvable for \( d \) at this point.

Let us further reveal the class of functions where the metric regularity of \( \partial \varphi \) can replace the strong metric subregularity assumption in the solvability result of Theorem 4.2(i). This issue is very appealing since metric regularity is a robust property, which is fully characterized—via the Mordukhovich criterion (2.8)—by the robust limiting coderivative (and hence by the second-order subdifferential (2.12) for the subgradient systems (1.5)) enjoying comprehensive calculus rules and computation formulas discussed above. To proceed in this direction, we significantly use the subdifferential structure of (1.5) with \( F = \partial \varphi \) in (3.3). The following class of functions was introduced by Poliquin and Rockafellar [53].

**Definition 4.3 (prox-regular functions).** A function \( \varphi: \mathbb{R}^n \to \overline{\mathbb{R}} \) is prox-regular at a point \( x \in \text{dom} \varphi \) for a subgradient \( \bar{v} \in \partial \varphi(\bar{x}) \) with modulus \( r > 0 \) if \( \varphi \) is l.s.c. around \( \bar{x} \) and there exists \( \varepsilon > 0 \) such that for all \( x,u \in B_{\varepsilon}(\bar{x}) \) with \( |\varphi(u) - \varphi(\bar{x})| < \varepsilon \) we have

\[
\varphi(x) \geq \varphi(u) + \langle v, x-u \rangle - \frac{r}{2}\|x-u\|^2 \quad \text{whenever} \quad v \in \partial \varphi(u) \cap B_{\varepsilon}(\bar{v}).
\]

(4.1)
If this holds for all \( \bar{v} \in \partial \varphi(\bar{x}) \), \( \varphi \) is said to be prox-regular at \( \bar{x} \in \text{dom} \varphi \).

In what follows we say that \( \varphi \) is continuously prox-regular at \( \bar{x} \) for \( \bar{v} \) (and just at \( \bar{x} \)) if it is simultaneously prox-regular and subdifferentially continuous according to Definitions 4.1 and 4.3. It is easy to see that if \( \varphi \) is continuously prox-regular at \( \bar{x} \) for \( \bar{v} \in \partial \varphi(\bar{x}) \), then the condition \( |\varphi(u) - \varphi(\bar{x})| < \varepsilon \) the definition of prox-regularity can be omitted. Furthermore, in this case the graph of \( \partial \varphi \) is locally closed around \((\bar{x}, \bar{v})\). As discussed in the book by Rockafellar and Wets [57], the class of continuously prox-regular functions is fairly broad containing, besides \( C^2 \)-smooth functions, also functions of class \( C^{1,1} \), convex l.s.c. functions, lower-\( C^2 \) functions, strongly amenable functions, etc. This class plays a central role in second-order variational analysis and its applications; see the books by Rockafellar and Wets [57] and by Mordukhovich [44] with the commentaries and references therein.

To establish the desired solvability theorem for the second-order subdifferential inclusions (3.2) with continuously prox-regular functions \( \varphi \), we need to recall yet another notion of generalized second-order differentiability taken from Rockafellar and Wets [57, Chapter 13].

Given \( \varphi: \mathbb{R}^n \to \mathbb{R} \) with \( \bar{x} \in \text{dom} \varphi \), consider the family of second-order finite differences

\[
\Delta^2 \varphi(\bar{x}, v)(u) := \frac{\varphi(\bar{x} + \tau u) - \varphi(\bar{x}) - \tau(v, u)}{\frac{1}{2} \tau^2}
\]

and define the second subderivative of \( \varphi \) at \( \bar{x} \) for \( v \in \mathbb{R}^n \) and \( w \in \mathbb{R}^n \) by

\[
d^2 \varphi(\bar{x}, v)(w) := \liminf_{\tau \downarrow 0} \Delta^2 \varphi(\bar{x}, v)(u).
\]

Then \( \varphi \) is said to be twice epi-differentiable at \( \bar{x} \) for \( v \) if for every \( w \in \mathbb{R}^n \) and every choice \( \tau_k \downarrow 0 \) there exists a sequence \( w^k \to w \) such that

\[
\frac{\varphi(\bar{x} + \tau_k w^k) - \varphi(\bar{x}) - \tau_k(v, w^k)}{\frac{1}{2} \tau_k^2} \to d^2 \varphi(\bar{x}, v)(w) \text{ as } k \to \infty.
\]

Twice epi-differentiability has been recognized as an important property in second-order variational analysis with numerous applications to optimization; see the aforementioned monograph by Rockafellar and Wets and the recent papers by Mohammadi et al. [38–40]. In particular, the latter papers develop a systematic approach to verify epi-differentiability via parabolic regularity, which is a major second-order property of extended-real-valued functions that goes far beyond the class of fully amenable functions investigated in Rockafellar and Wets [57].

Now we are ready to establish solvability of the second-order subdifferential inclusion (3.2) under merely metric regularity of the limiting subgradient mappings \( \partial \varphi \) for the class of continuously prox-regular and twice epi-differentiable functions.

**Theorem 4.4 (solvability of generalized Newton systems under metric regularity).** Let \( \varphi: \mathbb{R}^n \to \mathbb{R} \) be continuously prox-regular at \( \bar{x} \) for some \( \bar{v} \in \partial \varphi(\bar{x}) \). Suppose in addition that the subgradient mapping \( \partial \varphi \) is metrically regular around \((\bar{x}, \bar{v})\) and that one of two following properties holds:

(i) \( \varphi \) is a univariate function, i.e., \( n = 1 \).

(ii) \( \varphi \) is twice epi-differentiable at \( \bar{x} \) for \( \bar{v} \).

Then there exists \( d \in \mathbb{R}^n \) satisfying the second-order subdifferential system (3.2) at \((\bar{x}, \bar{v})\).

**Proof.** As mentioned above, the subdifferential graph \( \partial \varphi \) is locally closed around \((\bar{x}, \bar{v})\). Let us show now that the imposed assumptions ensure that \( \partial \varphi \) is strongly metrically subregular at
In the univariate case (i) inclusion (4.3) was proved by Rockafellar and Zagrodnik [58, Theorem 4.1], while the twice epi-differentiable case (ii) was done in the equivalent form in Theorem 1.1 of the latter paper; see also Rockafellar and Wets [57, Theorem 13.57]. Since the subdifferential mapping \( \partial \varphi \) is assumed to be metrically regular at \((\bar{x}, \bar{v})\), we get by using (4.3) and the Mordukhovich criterion (2.8) that

\[
0 \in (D \partial \varphi)(\bar{x}, \bar{v})(u) \implies 0 \in (D^* \partial \varphi)(\bar{x}, \bar{v})(u) \implies u = 0,
\]

which ensures by (4.2) that \( \partial \varphi \) is strongly metrically subregular at \((\bar{x}, \bar{v})\). Using finally the result of Theorem 4.2(i), we arrive at the claimed solvability and thus complete the proof. □

Note that Theorem 4.4 concerns solvability of the second-order subdifferential inclusion (3.2) at the chosen point \((\bar{x}, \bar{v}) \in \text{gph} \partial \varphi\). What about robust solvability of (3.2) around the reference point in the line of Theorem 4.2(ii)? This is discussed in the following remark.

**Remark 4.5 (robust solvability under metric regularity).** Theorem 4.2(ii) tells us that the strong metric regularity of \( \partial \varphi \) around \((\bar{x}, \bar{v})\) ensures the robust solvability of (3.2) around this point. But it has been recognized that the strong metric regularity of subgradient mappings \( \partial \varphi \) is equivalent to merely the metric regularity of them for major subclasses of continuously prox-regular functions \( \varphi : \mathbb{R}^n \to \mathbb{R} \) with the conjecture that it holds for the entire class of such functions at local minimizers of \( \varphi \); see Drusvyatskiy et al. [15, Conjecture 4.7]. This is largely discussed in the mentioned paper by Drusvyatskiy et al., and now we recall some results from that paper. Indeed, the equivalence clearly holds (and not only for local minimizers of \( \varphi \)) for \( \mathcal{C}^2 \)-smooth functions and for l.s.c. convex functions due the fundamental Kenderov theorem on maximal monotone operators [30]. The claimed equivalence is also valid for a broad class of functions given by \( \varphi(x) = \varphi_0(x) + \delta_{\Omega}(x) \), where \( \varphi_0 \) is a \( \mathcal{C}^2 \)-smooth function, and where \( \delta_{\Omega} \) is the indicator function of a polyhedral convex set; see Dontchev and Rockafellar [12]. Yet another large setting of such an equivalence is revealed in Drusvyatskiy et al. [15, Theorem 4.13] for continuously prox-regular functions \( \varphi \) with \( 0 \in \partial \varphi(\bar{x}) \) under the additional condition that the second-order subdifferential \( \partial^2 \varphi(\bar{x}, 0) \) is positive-semidefinite in the sense that

\[
\langle v, u \rangle \geq 0 \quad \text{for all} \quad v \in \partial^2 \varphi(\bar{x}, 0)(u), \quad u \neq 0.
\]

Note that the requirement that \( \bar{x} \) is a local minimizer of \( \varphi \) is essential for the validity of this conjecture even for twice epi-differentiable functions of class \( \mathcal{C}^{1,1} \) with piecewise linear and directionally differentiable gradients; see Example 5.2 in the next section.

## 5 Generalized Newton Method for \( \mathcal{C}^{1,1} \) Gradient Equations

In this section we propose and justify a generalized Newton algorithm to solve gradient systems of type (1.1), where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is a function of class \( \mathcal{C}^{1,1} \) around a given point \( \bar{x} \). To begin with, let us formulate the semismooth* property of set-valued mappings \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^m \) introduced
recently by Gfrerer and Outrata [19]. This property is used here for the justification of local superlinear convergence of our Newton-type algorithm to solve gradient equations (1.1) and then to solve subgradient inclusions (1.5) in the subsequent sections of the paper.

To formulate the semismooth* property of set-valued mappings, recall first the notion of the directional limiting normal cone to a set $\Omega \subset \mathbb{R}^n$ at $\bar{z} \in \Omega$ in the direction $d \in \mathbb{R}^n$ introduced by Ginchev and Mordukhovich [20] by implementing the limiting process

$$N_{\Omega}(\bar{z}; d) := \{ v \in \mathbb{R}^s \mid \exists t_k \downarrow 0, d_k \to d, v_k \to v \text{ with } v_k \in \tilde{N}_{\Omega}(\bar{z} + t_k d_k) \}. \quad (5.1)$$

It is obvious that (5.1) reduces to the limiting normal cone (2.4) for $d = 0$. Given a set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x}, \bar{y}) \in \text{gph} F$, the directional limiting coderivative of $F$ at $(\bar{x}, \bar{y})$ in the direction $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ is defined by Gfrerer [17] as

$$D^*F((\bar{x}, \bar{y}); (u, v))(v^*) := \{ u^* \in \mathbb{R}^n \mid (u^*, -v^*) \in N_{\text{gph} F(\bar{x}, \bar{y})}(u, v) \} \text{ for all } v^* \in \mathbb{R}^m$$

by using the directional normal cone (5.1) to the graph of $F$ at $(\bar{x}, \bar{y})$ in the direction $(u, v)$. The aforementioned semismooth* property of $F$ is now formulated as follows.

**Definition 5.1 (semismooth* property of set-valued mappings).** A mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is semismooth* at $(\bar{x}, \bar{y}) \in \text{gph} F$ if whenever $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ we have the equality

$$\langle u^*, u \rangle = \langle v^*, v \rangle \text{ for all } (v^*, u^*) \in \text{gph } D^*F((\bar{x}, \bar{y}); (u, v))$$

via the graph of the directional limiting coderivative of $F$ at $(\bar{x}, \bar{y})$ in all the directions $(u, v)$.

Semismooth* mappings were introduced and largely investigated in Gfrerer and Outrata [19], where this property is verified for any mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the graph represented as a union of finitely many closed and convex sets, for normal cone mappings generated by convex polyhedral sets. Other equivalent descriptions and properties of semismooth* mappings are given in Mordukhovich and Sarabi [48]. If $F: \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitzian around $\bar{x}$ and directionally differentiable at this point, then its semismooth* property reduces to the classical semismoothness. Although the standard semismooth property of locally Lipschitzian mappings involving directional derivatives has been conventionally used in the literature for the semismooth Newton method, some important results were obtained without the directional differentiability assumption; see, e.g., Kummer [33], Meng et al. [35], and Sun [59]. Such a relaxed semismooth property of single-valued locally Lipschitzian mappings is known as $G$-semismoothness. Note that, in contrast to $G$-semismoothness, the semismooth* property is defined for arbitrary set-valued mappings, and it is used for subgradient ones in this paper. But even for single-valued Lipschitzian mappings, the semismooth* definition based on coderivatives may have some advantages in comparison with the $G$-semismooth one due to perfect coderivative calculus rules.

Now we are ready to present and discuss the major assumptions used in the rest of the paper for the design and justification of our generalized Newton algorithms to solve the gradient and subgradient systems. The following assumptions are formulated for the general subgradient inclusions (1.5) at a reference point $\bar{x}$ satisfying (1.5).

**(H1)** Given a subgradient $\bar{v} \in \partial \varphi(\bar{x})$, the second-order subdifferential inclusion (3.2) is robustly solvable around $(\bar{x}, \bar{v})$, i.e., there is a neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ such that for every $(x, v) \in \text{gph } \partial \varphi \cap (U \times V)$ there exists a direction $d \in \mathbb{R}^n$ satisfying (3.2).

**(H2)** The subgradient mapping $\partial \varphi$ is metrically regular around $(\bar{x}, \bar{v})$. 


(H3) The subgradient mapping $\partial \varphi$ is semismooth* at $(\bar{x}, \bar{v})$.

Observe that in the case where $\varphi$ is of class $C^{1,1}$ around $\bar{x}$, we have $v = \nabla \varphi(x)$ and the second-order subdifferential system (3.2) is written as

$$-\nabla \varphi(x) \in \partial^2 \varphi(x)(d).$$

The robust solvability assumption (H1) has been discussed in Section 4 with presenting sufficient conditions for its fulfillment; see Theorems 4.2(ii), 4.4 and Remark 4.5. Note that the strong metric regularity of $\partial \varphi$ around $(\bar{x}, \bar{v})$ for subdifferentially continuous functions $\varphi$ ensures that both assumptions (H1) and (H2) are satisfied. However, this is just a sufficient condition for the validity of (H1) and (H2). The following example borrowed from Klatte and Kummer [32, Example BE.4], where it was constructed for different purposes, presents a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ of class $C^{1,1}$ (i.e., certainly being continuously prox-regular), which is twice epi-differentiable on the entire space $\mathbb{R}^2$ with the semismooth, metrically regular, but not strongly metrically regular gradient mapping $\nabla \varphi$ around the point in question. It is worth mentioning that the given example illustrates that assuming $\bar{x}$ to be a local minimizer of $\varphi$ is essential to the validity of Conjecture 4.7 from Drusvyatskiy et al. [15]; see Remark 4.5.

**Example 5.2 (all assumptions hold without strong metric regularity).** Let $z := (x, y) \in \mathbb{R}^2$ be written in the polar coordinates $(r, \theta)$ by

$$z = r(\cos \theta + i \sin \theta).$$

We now describe the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and its partial derivatives on the eight cones

$$C(k) := \left\{ z := (r \cos \theta, r \sin \theta) \mid \theta \in \left( (k - 1)\frac{\pi}{4}, k\frac{\pi}{4} \right], \ r \geq 0 \right\}, \quad k = 1, \ldots, 8.$$

The analytic expressions of $\varphi$, $\nabla_x \varphi$, and $\nabla_y \varphi$ are collected in the table:

| $k$ | $C(k)$ | $\varphi(z)$ | $\nabla_x \varphi(z)$ | $\nabla_y \varphi(z)$ |
|-----|--------|---------------|-----------------|-----------------|
| 1   | $C(1)$ | $y(y - x)$    | $-y$            | $2y - x$        |
| 2   | $C(2)$ | $x(y - x)$    | $-2x + y$       | $x$             |
| 3   | $C(3)$ | $x(y + x)$    | $2x + y$        | $x$             |
| 4   | $C(4)$ | $-y(y + x)$   | $-y$            | $-2y - x$       |
| 5   | $C(5)$ | $y(y - x)$    | $-y$            | $2y - x$        |
| 6   | $C(6)$ | $x(y - x)$    | $-2x + y$       | $x$             |
| 7   | $C(7)$ | $x(y + x)$    | $2x + y$        | $x$             |
| 8   | $C(8)$ | $-y(y + x)$   | $-y$            | $-2y - x$       |

The function $\varphi$ and its gradient have the following properties:

(a) The function $\varphi$ is of class $C^{1,1}$ with $\nabla \varphi$ being piecewise linear on $\mathbb{R}^2$. This is an obvious consequence of the definition. Thus $\varphi$ is continuously prox-regular on $\mathbb{R}^2$.

(b) The mapping $\nabla \varphi$ is metrically regular around $(0, 0)$. Indeed, it is observed by Klatte and Kummer [32, Example BE.4] that the inverse mapping $(\nabla \varphi)^{-1}$ is Lipschitz-like (pseudo-Lipschitz, Aubin) around $(\bar{x}, \bar{v})$ with $\bar{x} = (0, 0)$ and $\bar{v} = (0, 0)$. As well known (see, e.g., Mordukhovich [44, Theorem 3.2(ii)]), the latter property is equivalent to the metric regularity of the mapping $\nabla \varphi$ around $\bar{x}$.

(c) The mapping $\nabla \varphi$ is directionally differentiable on $\mathbb{R}^2$, which follows from Facchinei and
Pang [16, Lemma 4.6.1]. Hence \( \varphi \) is twice epi-differentiable by Rockafellar and Wets [57, Theorems 9.50(b), 13.40].

(d) The mapping \( \nabla \varphi \) is semismooth on \( \mathbb{R}^2 \) due to its piecewise linearity. This fact can be found, e.g., in Facchinei and Pang [16, Proposition 7.4.6] and Ulbrich [62, Proposition 2.26].

(e) The mapping \( \nabla \varphi \) is not strongly metrically regular around \((\bar{x}, \bar{v})\) with \( \bar{x} = (0,0) \) and \( \bar{v} = (0,0) \). To verify it, we proceed accordingly to Definition 2.1(i) and let \( \theta \) be an arbitrary localization of \((\nabla \varphi)^{-1}\) around \( \bar{v} \), i.e., such that

\[
\text{gph } \theta = \text{gph } (\nabla \varphi)^{-1} \cap (V \times U)
\]

for some neighborhoods \( V \) of \( \bar{v} \) and \( U \) of \( \bar{x} \). Find \( \varepsilon, \gamma > 0 \) with \( B_{\varepsilon}(\bar{x}) \subset U \) and \( B_{\gamma}(\bar{v}) \subset V \) and then pick \( t \in \mathbb{R} \) such that \( 0 < t < \min\{\gamma, \varepsilon/\sqrt{5}\} \). This shows that \((t, 0) \in B_{\gamma}(\bar{v}) \subset V \) with

\[
(0, t) = (r \cos \theta, r \sin \theta) \quad \text{for } r = t \quad \text{and } \quad \theta = \frac{\pi}{2},
\]

and thus \( \nabla \varphi(0,t) = (t, 0) \). Furthermore, we have

\[
(-2t, -t) = (r \cos \theta, r \sin \theta) \quad \text{for } r = \sqrt{5}t \quad \text{and } \quad \theta \in \left[\frac{5\pi}{4}, \frac{9\pi}{4}\right], \quad \cos \theta = -\frac{2}{\sqrt{5}}, \quad \sin \theta = \frac{1}{\sqrt{5}},
\]

which tells us that \( \nabla \varphi(-2t, -t) = (t, 0) \) and \((0, t), (-2t, -t) \in B_{\varepsilon}(\bar{x}) \subset U \). This yields

\[
((t, 0), (0, t)), ((t, 0), (-2t, -t)) \in \text{gph } (\nabla \varphi)^{-1} \cap \left( B_{\gamma}(\bar{v}) \times B_{\varepsilon}(\bar{x}) \right) \subset \text{gph } \theta.
\]

The latter means that there does not exist a localization \( \theta \) of \((\nabla \varphi)^{-1}\) around \((\bar{v}, \bar{x})\) which is single-valued, and hence the mapping \( \nabla \varphi \) is not strongly metrically regular around \((\bar{x}, \bar{v})\).

Remembering finally that the metric regularity is a robust property and therefore holds for all points in some neighborhood of \((0,0)\), we deduce from (a)–(d) and Theorem 4.4 that all the imposed assumptions (H1)–(H3) are satisfied without the fulfillment of the strong metric regularity of \( \nabla \varphi \) around \((0,0)\) as shown in (e). It is easy to see that \( \bar{x} = (0,0) \) is a stationary point of \( \varphi \) while not its local minimizer. Thus this example does not contradict the conjecture from Drusvyatskiy et al. [15] discussed in Remark 4.5.

Now we are ready to formulate a generalized Newton algorithm to solve the gradient equation

\[\nabla \varphi(x) = 0, \quad \text{labeled as (1.1) in Section 1, where } \varphi \text{ is of class } C^{1,1} \text{ around the reference point.} \]

This algorithm is based on the second-order subdifferential/generalized Hessian (2.12) of the function \( \varphi \) in question.

**Algorithm 5.3** *(Newton-type algorithm for \( C^{1,1} \) functions).* Do the following:

**Step 0:** Choose a starting point \( x^0 \in \mathbb{R}^n \) and set \( k = 0 \).

**Step 1:** If \( \nabla \varphi(x^k) = 0 \), stop the algorithm. Otherwise move to Step 2.

**Step 2:** Choose \( d^k \in \mathbb{R}^n \) satisfying

\[
- \nabla \varphi(x^k) \in \partial (d^k, \nabla \varphi)(x^k),
\]

**Step 3:** Set \( x^{k+1} \) given by

\[
x^{k+1} := x^k + d^k \quad \text{for all } k = 0, 1, \ldots
\]

**Step 4:** Increase \( k \) by 1 and go to Step 1.
The major step and novelty of Algorithm 5.3 is the generalized Newton system (5.3) expressed in terms of the limiting subdifferential of the scalarized gradient mapping. Due to the coderivative scalarization formula (2.11) and the second-order construction (2.12) we have
\[ \partial (d^k, \nabla \varphi)(x^k) = (D^*\nabla \varphi)(x^k)(d^k) = \partial^2 \varphi(x^k)(d^k), \]
i.e., the iteration system (5.3) agrees with the second-order subdifferential inclusion (3.2) whose solvability was discussed in Section 4; see Theorem 4.4. The computation of the second-order subdifferential for \( C^{1,1} \) functions reduces to that of the (first-order) limiting subdifferential of the classical gradient mapping; this significantly simplifies the numerical implementation. Note also that, according to definition (2.7), the direction \( d^k \) in (5.3) can be found from
\[ ( - \nabla \varphi(x^k), -d^k) \in N((x^k, \nabla \varphi(x^k)); \text{gph} \nabla \varphi). \]

The main goal for the rest of this section is to show that the metric regularity and the semismooth* properties of \( \nabla \varphi \) imposed in (H2) and (H3) ensure the convergence of iterates \( x^k \rightarrow \bar{x} \) with superlinear rate. To proceed in this way, we present the following three lemmas of their own interest. The first lemma gives us a necessary and sufficient condition for the metric regularity of continuous single-valued mappings \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) via their limiting coderivatives. Since \( F \) is single-valued, we are talking about its metric regularity around \( \bar{x} \) instead of \( (\bar{x}, F(\bar{x})) \).

**Lemma 5.4 (yet another characterization of metric regularity).** Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be continuous around \( \bar{x} \). Then it is metrically regular around this point if and only if there exists \( c > 0 \) and a neighborhood \( U \) of \( \bar{x} \) such that we have the estimate
\[ \|v\| \geq c \|u\| \quad \text{for all} \quad v \in D^*F(x)(u), \ x \in U, \quad \text{and} \quad u \in \mathbb{R}^m. \]  

**Proof.** If \( F \) is metrically regular and continuous around \( \bar{x} \), then it follows from the book by Mordukhovich [43, Theorem 1.54] that there are \( c > 0 \) and an (open) neighborhood \( U \) of \( \bar{x} \) with
\[ \|v\| \geq c \|u\| \quad \text{for all} \quad v \in \hat{D}^*F(x)(u), \ x \in U, \quad \text{and} \quad u \in \mathbb{R}^m \]  
in terms of the regular coderivative (2.6). Fix any \( x \in U, \ u \in \mathbb{R}^m \), and an element \( v \in D^*F(x)(u) \) from the limiting coderivative (2.7). The continuity of \( F \) ensures that the graphical set \( \text{gph} F \) is closed around \( (x, F(x)) \). Then using Corollary 2.36 from the aforementioned book gives us sequences \( x_k \rightarrow x, \ u_k \rightarrow u \), and \( v_k \rightarrow v \) as \( k \rightarrow \infty \) such that \( v_k \in \hat{D}^*F(x_k)(u_k) \), and thus \( x_k \in U \) for all \( k \in \mathbb{N} \) sufficiently large. It follows from (5.5) that
\[ \|v_k\| \geq c \|u_k\| \quad \text{for large} \quad k \in \mathbb{N}. \]

Letting \( k \rightarrow \infty \), we arrive at the estimate \( \|v\| \geq c \|u\| \). On the other hand, the fulfillment of (5.4) immediately yields the metric regularity of \( F \) around \( \bar{x} \) by the coderivative criterion (2.8). \( \square \)

The next lemma presents an equivalent description of semismoothness* for Lipschitzian gradient mappings. This result follows from the combination of Proposition 3.7 in Gfrerer and Outrata [19] and Theorem 13.52 in Rockafellar and Wets [57] due to the symmetry of generalized Hessian matrices; see Mordukhovich and Sarabi [48, Proposition 2.4] for more details.

**Lemma 5.5 (equivalent description of semismoothness*).** Let \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) be of class \( C^{1,1} \) around \( \bar{x} \). Then the gradient mapping \( \nabla \varphi \) is semismooth* at \( \bar{x} \) if and only if
\[ \nabla \varphi(x) - \nabla \varphi(\bar{x}) - \partial^2 \varphi(x)(x - \bar{x}) \subset o(\|x - \bar{x}\|), \]
which means that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \|\nabla \varphi(x) - \nabla \varphi(\bar{x}) - v\| \leq \varepsilon \|x - \bar{x}\| \quad \text{for all} \quad v \in \partial^2 \varphi(x)(x - \bar{x}) \quad \text{and} \quad x \in \mathbb{B}_\delta(\bar{x}). \]
The last lemma establishes a useful subadditivity property of the limiting coderivative of single-valued and locally Lipschitzian mappings.

**Lemma 5.6 (subadditivity of coderivatives).** If $F: \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitzian around $\bar{x}$, then we have the inclusion

$$D^*F(\bar{x})(u_1 + u_2) \subset D^*F(\bar{x})(u_1) + D^*F(\bar{x})(u_2) \quad \text{for all } u_1, u_2 \in \mathbb{R}^n.$$  \hspace{1cm} (5.6)

**Proof.** It follows from the scalarization formula (2.11) that

$$D^*F(\bar{x})(u_1 + u_2) = \partial(u_1 + u_2, F)(\bar{x}).$$

The subdifferential sum rule from Mordukhovich [43, Theorem 2.33] and the aforementioned scalarization formula ensure that

$$\partial(u_1 + u_2, F)(\bar{x}) \subset \partial(u_1, F)(\bar{x}) + \partial(u_2, F)(\bar{x}) = D^*F(\bar{x})(u_1) + D^*F(\bar{x})(u_2),$$

which therefore completes the proof of the lemma. \hfill \Box

The next theorem is the main result of this section that establishes the local superlinear convergence of Algorithm 5.3 under the imposed assumptions.

**Theorem 5.7 (local convergence of the Newton-type algorithm for $C^{1,1}$ functions).** Let $\bar{x}$ be a solution to (1.1) for which assumptions (H1)-(H3) are satisfied. Then there exists a neighborhood $U$ of $\bar{x}$ such that for all $x^0 \in U$ Algorithm 5.3 is well-defined and generates a sequence $\{x^k\}$ that converges $Q$-superlinearly to $\bar{x}$, i.e., we have

$$\lim_{k \to \infty} \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} = 0.$$

**Proof.** Define the set-valued mapping $G_\varphi: \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$G_\varphi(x, u) := -(D^*\nabla \varphi)(x)(-u) = -\partial^2 \varphi(x)(-u) \quad \text{for all } x, u \in \mathbb{R}^n.$$  

Assumption (H1) allows us to construct the sequence of iterates $\{x^k\}$ in Algorithm 5.3. Using (H2) and its characterization from Lemma 5.4, we find $c > 0$ and a neighborhood $U$ of $\bar{x}$ with

$$\|v\| \geq c\|u\| \quad \text{for all } v \in G_\varphi(x, u), x \in U, \text{ and } u \in \mathbb{R}^n.$$

Let us now verify the inclusion

$$\nabla \varphi(x) - \nabla \varphi(\bar{x}) + G_\varphi(x, u) \subset G_\varphi(x, x + u - \bar{x}) + o(x - \bar{x})B$$  \hspace{1cm} (5.7)

for the above vectors $x, u$. Indeed, taking any $v_1 \in G_\varphi(x, u)$, i.e., $-v_1 \in \partial^2 \varphi(x)(-u)$, and using the subadditivity inclusion from Lemma 5.6 lead us to

$$\partial^2 \varphi(x)(-u) \subset \partial^2 \varphi(x)(-x - u + \bar{x}) + \partial^2 \varphi(x)(x - \bar{x})$$

and thus ensure the existence of $v_2 \in \partial^2 \varphi(x)(-x - u + \bar{x})$ such that $-v_1 - v_2 \in \partial^2 \varphi(x)(x - \bar{x})$. The semismoothness* assumption (H3) and its equivalent description in Lemma 5.5 tell us that

$$\lim_{x \to \bar{x}} \frac{\|\nabla \varphi(x) - \nabla \varphi(\bar{x}) + v_1 + v_2\|}{\|x - \bar{x}\|} = 0,$$

which therefore verifies (5.7). All of this allows us to proceed similarly to the proof of Theorem 10.7 in Klatte and Kummer [32] and thus find a neighborhood $U$ of $\bar{x}$ such that, whenever the starting point $x^0 \in U$ is selected, Algorithm 5.3 generates a well-defined sequence of iterates $\{x^k\}$, which converges $Q$-superlinearly to $\bar{x}$. This therefore completes proof of the theorem. \hfill \Box
Assumption (H1) has been already discussed and obviously cannot be removed or relaxed; otherwise Algorithm 5.3 is not well-defined. Now we present two examples showing that assumptions (H2) and (H3) are essential for the convergence (not even talking about its $Q$-superlinear rate) of Algorithm 5.3. Let us start with the semismooth* assumption (H3).

**Example 5.8 (semismooth* property is essential for convergence).** Consider the Lipschitz continuous function on $\mathbb{R}$ given by

$$
\psi(x) := \begin{cases} 
  x^2 \sin \frac{1}{x} + 2x & \text{if } x \neq 0, \\
  0 & \text{if } x = 0,
\end{cases}
$$

which was used in Jiang et al. [26] to show that the semismooth Newton method for solving the equation $\psi(x) = 0$ fails to locally converge to $\bar{x} := 0$. Consider further the $C^{1,1}$ function

$$
\varphi(x) := \int_0^x \psi(t)dt, \quad x \in \mathbb{R},
$$

with $\nabla \varphi(x) = \psi(x)$ on $\mathbb{R}$, and hence $\nabla \varphi(\bar{x}) = 0$. As shown in Mordukhovich and Sarabi [48, Example 4.5], the mapping $\nabla \varphi$ is not semismooth* at $\bar{x}$ and iterations (5.3) constructed to compute tilt-stable local minimizers of $\varphi$ (see below) do not locally converge to $\bar{x}$.

The next example reveals that assumption (H2) cannot be improved by relaxing the metric regularity property to the metric subregularity or even to the strong metric subregularity of the gradient mapping $\nabla \varphi$ at $\bar{x}$ in order to keep the convergence of iterates in Algorithm 5.3.

**Example 5.9 (convergence failure under strong metric subregularity).** Consider the function $\varphi: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$
\varphi(x, y) := \frac{1}{2}x|x| + \frac{1}{2}y|y| \quad \text{for all } (x, y) \in \mathbb{R}^2.
$$

(5.8)

It is clear that the function $\varphi$ is of class $C^{1,1}$ around $\bar{z} := (0, 0)$ with $\nabla \varphi(x, y) = (|x|, |y|)$ for all $z := (x, y) \in \mathbb{R}^2$ and $\nabla \varphi(\bar{z}) = 0$. The simple computation tells us that

$$(D\nabla \varphi)(x, y)(u_1, u_2) = \begin{cases} 
  \{(|u_1|, |u_2|) \mid u_1, u_2 \in \mathbb{R}\} & \text{if } x = 0, y = 0, \\
  \{(u_1 \text{ sgn}(x), |u_2|) \mid u_1, u_2 \in \mathbb{R}\} & \text{if } x \neq 0, y = 0, \\
  \{(|u_1|, u_2 \text{ sgn}(y)) \mid u_1, u_2 \in \mathbb{R}\} & \text{if } x = 0, y \neq 0, \\
  \{(u_1 \text{ sgn}(x), u_2 \text{ sgn}(y)) \mid u_1, u_2 \in \mathbb{R}\} & \text{if } x \neq 0, y \neq 0.
\end{cases}
$$

It follows from the Levy-Rockafellar criterion that the mapping $\nabla \varphi$ is strongly metrically subregular at any point $(x, y) \in \text{gph} \nabla \varphi$. Furthermore, $\nabla \varphi$ is semismooth* at $\bar{z}$ since it is piecewise linear on $\mathbb{R}^2$, and thus assumption (H3) is satisfied. The fulfillment of (H1) is proved in Theorem 4.2. Let us now show that the sequence of iterates $\{z^k\}$ generated by Algorithm 5.3 does not converge to $\bar{z}$. Indeed, fix any $r > 0$ and pick an arbitrary starting point $z^0$ in the form $z^0 := (0, r) \in \mathbb{B}_r(\bar{z})$. To run the algorithm, we need to find $d^0 \in \mathbb{R}^2$ such that

$$
- \nabla \varphi(z^0) \in \partial^2 \varphi(z^0)(d^0).
$$

(5.9)

Using the second-order subdifferential (2.12) for the function $\varphi$ from (5.8) gives us

$$
\partial^2 \varphi(z^0)(u_1, u_2) = \begin{cases} 
  \{(\alpha u_1, u_2) \mid \alpha \in [-1, 1]\} & \text{if } u_1 \geq 0, u_2 \in \mathbb{R}, \\
  \{(\alpha u_1, u_2) \mid \alpha \in [-1, 1]\} & \text{if } u_1 < 0, u_2 \in \mathbb{R}.
\end{cases}
$$
This shows that the direction \( d^0 = (1, -r) \) satisfies inclusion (5.9). Put further \( z^1 := z^0 + d^0 = (1, 0) \) and find a direction \( d^1 \in \mathbb{R}^2 \) satisfying the inclusion
\[
-\nabla \varphi(z^1) \in \partial^2 \varphi(z^1)(d^1).
\]
(5.10)
Computing again the second-order subdifferential brings us to the expression
\[
\partial^2 \varphi(z^1)(u_1, u_2) = \begin{cases} 
\{(u_1, \alpha u_2) | \alpha \in [-1, 1]\} & \text{if } u_1 \in \mathbb{R}, u_2 \geq 0, \\
\{(u_1, \alpha u_2) | \alpha \in \{-1, 1\}\} & \text{if } u_1 \in \mathbb{R}, u_2 < 0
\end{cases}
\]
and then verifies that the direction \( d^1 = (-1, r) \) satisfies the inclusion in (5.10). Thus \( z^2 := z^1 + d^1 = (0, r) \). Continuing this process, we construct the sequence of iterates \( \{z^k\} \) such that \( z^{2k} = z^0 \) and \( z^{2k+1} = z^1 \) for all \( k \in \mathbb{N} \). It is obvious that \( \{z^k\} \) does not converge to \( \tilde{z} \).

There are several Newton-type methods to solve Lipschitzian equations \( f(x) = 0 \) that apply to gradient systems (1.1) with \( f(x) := \nabla \varphi(x) \), where \( \varphi \) is of class \( C^{1,1} \). Such methods are mainly based of various generalized directional derivatives and can be found, e.g., in Klatte and Kummer [32], Pang [51], Hoheisel et al. [25], Mordukhovich and Sarabi [48], and the references therein. It is beyond the scope of this paper to discuss their detailed relationships with Algorithm 5.3. However, let us compare the proposed algorithm with the semismooth Newton method to solve equation (1.1), which is based on the generalized Jacobian by Clarke [8] for the mapping \( f = \nabla \varphi \).

**Remark 5.10 (comparing Algorithm 5.3 with the semismooth Newton method and its variants).** In the setting of (1.1), the semismooth Newton method constructs the iterations
\[
x^{k+1} = x^k - (A^k)^{-1} \nabla \varphi(x^k), \quad k = 0, 1, \ldots,
\]
(5.11)
where for each \( k \) a nonsingular matrix \( A^k \) is taken from the generalized Jacobian of \( \nabla \varphi \) at \( x = x^k \), which is expressed for functions \( \varphi \) of class \( C^{1,1} \) around \( x \) via the convex hull \( A^k \in \text{co} \nabla^2 \varphi(x^k) \) of the limiting Hessian matrices (or Bouligand Jacobian) defined by
\[
\nabla^2 \varphi(x) := \left\{ \lim_{m \to \infty} \nabla^2 \varphi(u_m) \mid u_m \to x, u_m \in Q_{\varphi} \right\}, \quad x \in \mathbb{R}^n,
\]
(5.12)
where \( Q_{\varphi} \) stands for the set on which \( \varphi \) is twice differentiable. It follows from the classical Rademacher theorem that (5.12) is a nonempty compact in \( \mathbb{R}^{n \times n} \). Observe that
\[
\text{co} \left[ \partial \langle u, \nabla \varphi \rangle(x) \right] = \text{co} \partial^2 \varphi(x)(u) = \left\{ A^*u \mid A \in \text{co} \nabla^2 \varphi(x) \right\}, \quad u \in \mathbb{R}^n,
\]
which tells us that, in contrast to our algorithm (5.3), the semismooth method (5.11) employs the convex hull of the corresponding set. A serious disadvantage of (5.11) is that it requires the invertibility of all the matrices from \( \text{co} \nabla^2 \varphi(x^k) \); otherwise the semismooth Newton algorithm (5.11) is simply not well-defined. Observe that nothing like that is required to run our Algorithm 5.3. Indeed, the invertibility assumption is even more restrictive than the strong metric regularity of \( \nabla \varphi \) around \( \tilde{x} \), which is also not required in the imposed assumptions (H1)–(H3), that ensure the well-posedness and local superlinear convergence of Algorithm 5.3. To overcome this disadvantage of the semismooth Newton method, one of the possible ideas developed by Sun [59] is to take matrices from \( \nabla^2 \varphi(x) \) instead of \( \text{co} \nabla^2 \varphi(x) \), i.e., to construct the iterations as in (5.11) with \( A^k \in \nabla^2 \varphi(x^k) \). This method requires the nonsingularity of \( \nabla^2 \varphi(\tilde{x}) \), which is weaker than the nonsingularity of the Clarke generalized Jacobian. However, there are some drawbacks of this requirement in comparison with our assumption that \( \nabla \varphi \) is metrically regular around \( \tilde{x} \). The first observation to make is that, to the best of our knowledge, calculus rules for the Bouligand Jacobian is more limited in comparison with the extensive ones available for
coderivatives. Meanwhile, the metric regularity can be fully characterized by the Mordukhovich coderivative criterion for general set-valued mappings. Furthermore, the Bouligand-based semismooth Newton method clearly addresses just \( C^{1,1} \) functions, while Algorithm 5.3 in our approach can be seen as a bridge to develop a generalized Newton method for solving subgradient inclusions \( 0 \in \partial \varphi(x) \), where \( \varphi \) is a prox-regular function as in Section 6, and where \( \varphi = f + g \) with a \( C^2 \)-smooth function \( f \) and a prox-regular function \( g \) as in Section 7. Observe that our standing assumptions are formulated and used for set-valued mappings without any directional differentiability requirements or the like.

To conclude this remark, we present a specific one-dimensional example, where all the imposed assumptions (H1)–(H3) of Theorem 5.7 are satisfied while the directional differentiability is not required. Indeed, consider the cost function \( \varphi : [-1, 1] \to \mathbb{R} \) defined by

\[
\varphi(x) := \int_{-1}^{x} H(t)dt, \quad x \in [-1, 1],
\]

with the integrand \( H : [-1, 1] \to \mathbb{R} \) having the following properties:

(i) \( H \) is Lipschitz continuous and metrically regular around \( \bar{x} := 0 \) with \( H(\bar{x}) = 0 \).

(ii) \( H \) is monotone on \((-1, 1)\).

(iii) \( \|H(x) - H(\bar{x}) + v\| = o(\|x - \bar{x}\|) \) for all \( x \) near \( \bar{x} \), \( v \in DH(x)(\bar{x} - x) \).

(iv) \( H \) is not directionally differentiable at \( \bar{x} \).

An explicitly constructed function \( H \) of this type can be found in Hoheisel et al. [25, Example 4.11]. It is clear that \( \varphi \) is a differentiable convex function on \((-1, 1)\) by (ii) and the fact that \( \nabla \varphi(x) = H(x) \) for all \( x \in (-1, 1) \). Thus the metric regularity of the gradient mapping \( \nabla \varphi \) around \( \bar{x} \) is equivalent to its strong metric regularity around \( \bar{x} \) due to Aragón Artacho and Geoffroy [1, Proposition 3.8]. This verifies the assumptions (H1) and (H2) in our paper. By Proposition 2.4 from Mordukhovich and Sarabi [48] and property (iii), we conclude that \( \nabla \varphi \) is semismooth* at \( \bar{x} \). Meanwhile, property (iv) tells us that \( \nabla \varphi \) is not directionally differentiable at \( \bar{x} \).

Next we consider a particular case of the gradient stationary equation (1.1), where \( \bar{x} \) is a local minimizer of \( \varphi \). Moreover, our attention is paid to the remarkable class of local minimizers exhibiting the property of tilt stability introduced by Poliquin and Rockafellar [54]. In the case of tilt-stable minimizers, Algorithm 5.3 was developed by Mordukhovich and Sarabi [48]. The results established below improve those from the latter paper. First we recall the notion of tilt-stable local minimizers for the general case of extended-real-valued functions \( \varphi : \mathbb{R}^n \to \mathbb{R} \).

**Definition 5.11 (tilt-stable local minimizers).** Given \( \varphi : \mathbb{R}^n \to \mathbb{R} \), a point \( \bar{x} \in \text{dom} \varphi \) is a tilt-stable local minimizer of \( \varphi \) if there exists a number \( \gamma > 0 \) such that the mapping

\[
M_\gamma : v \mapsto \text{argmin}\{\varphi(x) - \langle v, x \rangle \mid x \in \mathbb{B}_\gamma(\bar{x})\}
\]

is single-valued and Lipschitz continuous on some neighborhood of \( 0 \in \mathbb{R}^n \) with \( M_\gamma(0) = \{\bar{x}\} \).

This notion was largely investigated and characterized in second-order variational analysis with many applications to constrained optimization. Besides the seminal paper by Poliquin and Rockafellar [54], we refer the reader to Chieu et al. [6], Drusvyatskiy and Lewis [14], Drusvyatskiy et al. [15], Gfrerer and Mordukhovich [18], Mordukhovich [44], Mordukhovich and Nghia [45], Mordukhovich and Rockafellar [47] and the bibliographies therein. Some of these characterizations are used in the following theorem.
Theorem 5.12 (Newton-type method for tilt-stable minimizers of $C^{1,1}$ functions). Let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be of class $C^{1,1}$ around a given point $\bar{x}$, which is a tilt-stable local minimizer of $\varphi$. Then there is a neighborhood $U$ of $\bar{x}$ such that the following assertions hold:

(i) For any $x \in U$ there exists a direction $d \in \mathbb{R}^n$ satisfying the inclusion

$$\nabla \varphi(x) \in \partial(d, \nabla \varphi)(x). \tag{5.14}$$

Furthermore, we have that $(\nabla \varphi(x), d) < 0$ whenever $x \neq \bar{x}$ and that for each $c \in (0, 1)$ there is $\delta > 0$ ensuring the fulfillment of the inequality

$$\varphi(x + td) \leq \varphi(x) + ct(\nabla \varphi(x), d) \text{ for all } t \in (0, \delta). \tag{5.15}$$

(ii) If in addition the gradient mapping $\nabla \varphi$ is semismooth* at $(\bar{x}, 0)$, then Algorithm 5.3 is well-defined for any starting point $x^0 \in U$ and generates a sequence $\{x^k\}$ that $Q$-superlinearly converges to $\bar{x}$, while the sequence of the function values $\{\varphi(x^k)\}$ $Q$-superlinearly converges to $\varphi(\bar{x})$, and the sequence of the gradient values $\{\nabla \varphi(x^k)\}$ $Q$-superlinearly converges to 0.

Proof. To verify (i), deduce from Drusvyatskiy and Lewis [14, Proposition 3.1] that the imposed tilt stability of the local minimizer $\bar{x}$ implies that the gradient mapping $\nabla \varphi$ is strongly metrically regular around $(\bar{x}, 0)$. Then Theorem 3.4 tells us that there exists a neighborhood $U_1$ of $\bar{x}$ such that for all $x \in U_1$ we can find a direction $d \in \mathbb{R}^n$ satisfying (5.14). Furthermore, it follows from Chieu et al. [7, Theorem 4.7] that there exists another neighborhood $U_2$ of $\bar{x}$ such that $\varphi$ is strongly convex on $U_2$, and we have

$$\langle z, u \rangle > 0 \text{ for all } z \in \partial^2 \varphi(x)(u) \text{ and } x \in U_2, u \neq 0. \tag{5.16}$$

Denote $U := U_1 \cap U_2$ and fix any $x \in U$ with $x \neq \bar{x}$, which gives us $d \in \mathbb{R}^n$ satisfying (5.14). To show that $d \neq 0$, assume the contrary and then get from (5.14) that $\nabla \varphi(x) = 0$. Hence it follows from the strong convexity of $\varphi$ that $x$ is a strict global minimizer of $\varphi$ on $U$, which clearly contradicts the tilt stability of $\bar{x}$ by Definition 5.11, and thus $d \neq 0$. Combining the latter with (5.14) and (5.16), we get $(\nabla \varphi(x), d) < 0$ and hence conclude that (5.15) holds by using, e.g., Lemma 2.19 from Izmailov and Solodov [27].

Next we verify assertion (ii). As follows from Theorem 4.2(ii), the imposed strong metric regularity of $\nabla \varphi$ around $\bar{x}$ ensures the fulfillment of assumption (H1) and of course (H2). The additional semismooth* assumption on $\nabla \varphi$ at $(\bar{x}, 0)$ is (H3) in this setting, and hence we deduce the well-posedness and local superlinear convergence of iterates $\{x^k\}$ in Algorithm 5.3 from Theorem 5.7. To show that the sequence $\{\varphi(x^k)\}$ converges superlinearly to $\varphi(\bar{x})$, we conclude by the $C^{1,1}$ property of $\varphi$ around $\bar{x}$ and Lemma A.11 from Izmailov and Solodov [27] that there exists $\ell > 0$ yielding

$$|\varphi(x^{k+1}) - \varphi(\bar{x})| \leq \frac{\ell}{2}||x^{k+1} - \bar{x}||^2 \text{ for sufficiently large } k \in \mathbb{N}. \tag{5.17}$$

Furthermore, the second-order growth condition that follows from tilt stability of $\bar{x}$ (see, e.g., Mordukhovich and Nghia [45, Theorem 3.2]) gives us $\kappa > 0$ such that

$$|\varphi(x^k) - \varphi(\bar{x})| \geq \varphi(x^k) - \varphi(\bar{x}) \geq \frac{1}{2\kappa}||x^k - \bar{x}||^2 \text{ for large } k \in \mathbb{N}. \tag{5.18}$$

Combining the two estimates above produces the inequality

$$\frac{|\varphi(x^{k+1}) - \varphi(\bar{x})|}{|\varphi(x^k) - \varphi(\bar{x})|} \leq \frac{\ell\kappa}{\kappa} \cdot \frac{||x^{k+1} - \bar{x}||}{||x^k - \bar{x}||}, \tag{5.19}$$
which deduces the claimed superlinear convergence of \( \{ \varphi(\bar{x}) \} \) from the one established for \( \{ x^k \} \).

To finish the proof, it remains to to show that the sequence \( \{ \nabla \varphi(x^k) \} \) superlinearly converges to \( 0 \). Indeed, the Lipschitz continuity of \( \nabla \varphi \) around \( \bar{x} \) gives us a constant \( \ell > 0 \) such that

\[
\| \nabla \varphi(x^{k+1}) \| = \| \nabla \varphi(x^{k+1}) - \nabla \varphi(\bar{x}) \| \leq \ell \| x^{k+1} - \bar{x} \| \quad \text{for large } k \in \mathbb{N}.
\]

The strong local monotonicity of \( \nabla \varphi \) around \((\bar{x}, 0)\) under the tilt stability of \( \bar{x} \) (see, e.g., Poliquin and Rockafellar [54, Theorem 1.3] for a more general result) tells us that there exists a constant \( \kappa > 0 \) with

\[
\langle \nabla \varphi(x^k) - \nabla \varphi(\bar{x}), x^k - \bar{x} \rangle \geq \kappa \| x^k - \bar{x} \|^2 \quad \text{for large } k \in \mathbb{N},
\]

and hence \( \| \nabla \varphi(x^k) \| \geq \kappa \| x^k - \bar{x} \| \) for such \( k \). Thus we arrive at the estimate

\[
\frac{\| \nabla \varphi(x^{k+1}) \|}{\| \nabla \varphi(x^k) \|} \leq \frac{\ell}{\kappa} \frac{\| x^{k+1} - \bar{x} \|}{\| x^k - \bar{x} \|},
\]

which verifies that the gradient sequence \( \{ \nabla \varphi(x^k) \} \) superlinearly converges to \( 0 \) as \( k \to \infty \) due to such a convergence of \( x^k \to \bar{x} \) obtained above. This completes the proof of the theorem. \( \square \)

To conclude, we compare Theorem 5.12 with the recent results from Mordukhovich and Sarabi [48].

**Remark 5.13 (comparison with known results under tilt stability).** The aforementioned paper [48] developed Algorithm 5.3, written in an equivalent form, for computing tilt-stable minimizers \( \bar{x} \) of \( \varphi \in C^{1,1} \). As follows from Drusvyatskiy and Lewis [14, Theorem 3.3] (see also Drusvyatskiy et al. [15, Proposition 4.5] for a more precise statement), that the tilt-stability of \( \bar{x} \) for \( \varphi \) is equivalent to the strong metric regularity of \( \nabla \varphi \) around \((\bar{x}, 0)\) provided that \( \bar{x} \) is a local minimizer of \( \varphi \). The latter requirement is essential as trivially illustrated by the function \( \varphi(x) := -x^2 \) on \( \mathbb{R} \), where \( \bar{x} = 0 \) is not a tilt-stable local minimizer while \( \nabla \varphi \) is strongly metrically regular around \((\bar{x}, 0)\). Note also that the solvability/well-posedness of Algorithm 5.3 holds under weaker assumptions than the strong metric regularity; see Section 4. The local superlinear convergence of \( \{ x^k \} \) to a tilt-stable minimizer \( \bar{x} \) in Theorem 5.12(ii) follows from Mordukhovich and Sarabi [48, Theorem 4.3] under the semismoothness* of \( \nabla \varphi \) at \((\bar{x}, 0)\). Besides the local superlinear convergence of \( \varphi(x^k) \to \varphi(\bar{x}) \) and \( \nabla \varphi(x^k) \to 0 \) in Theorem 5.12(ii), the new statements of Theorem 5.12(i) include the descent property \( \langle \nabla \varphi(x^k), d^k \rangle < 0 \) of the algorithm and the backtracking line search (5.15) at each iteration.

## 6 Generalized Newton Algorithm for Prox-Regular Functions

This section is devoted to the design and justification of a generalized Newton algorithm to solve subgradient inclusions of type (1.5), where \( \varphi: \mathbb{R}^n \to \mathbb{R} \) is a continuously prox-regular function. We have already considered this remarkable class of functions in Section 4 concerning solvability of the second-order subdifferential systems (3.2), which play a crucial role in the design of the generalized Newton algorithm to find a solution of (1.5) in this section. The approach developed here is to reduce the subgradient inclusion (1.5) to a gradient one of type (1.1) with the replacement of \( \varphi \) from the class of continuously prox-regular functions by its Moreau envelope, which is proved to be of class \( C^{1,1} \). This leads us to the well-defined and implementable generalized Newton algorithm expressed in terms of the second-order subdifferential of \( \varphi \) and the single-valued, monotone, and Lipschitz continuous proximal mapping associated with this function. We show that the proposed algorithm exhibits local superlinear convergence under the standing assumptions imposed and discussed above.

First we formulate the notions of Moreau envelopes and proximal mappings associated with extended-real-valued functions. Recall that \( \varphi: \mathbb{R}^n \to \mathbb{R} \) is proper if \( \text{dom} \varphi \neq \emptyset \).
Definition 6.1 (Moreau envelopes and proximal mappings). Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ be proper and l.s.c., and let $\lambda > 0$. The Moreau envelope $e_\lambda \varphi$ and the proximal mapping $\text{Prox}_\lambda \varphi$ are defined by

$$
e_\lambda \varphi(x) := \inf \left\{ \varphi(y) + \frac{1}{2\lambda} \|y - x\|^2 \mid y \in \mathbb{R}^n \right\},$$

(6.1)

$$\text{Prox}_\lambda \varphi(x) := \text{argmin} \left\{ \varphi(y) + \frac{1}{2\lambda} \|y - x\|^2 \mid y \in \mathbb{R}^n \right\}.$$  

(6.2)

If $\lambda = 1$, we use the notation $e \varphi(x)$ and $\text{Prox} \varphi(x)$ in (6.1) and (6.2), respectively.

Both Moreau envelopes and proximal mappings have been well recognized in variational analysis and optimization as efficient tools of regularization and approximation of nonsmooth functions. This has have done particularly for convex functions and more recently for continuously prox-regular functions; see Rockafellar and Wets [57] and the references therein. Proximal mappings and the like have been also used in numerical algorithms of the various types revolving around computing proximal points; see, e.g., the book by Beck [2] and the paper by Hare and Sagastizábal [21] among many other publications. In what follows we are going to use the proximal mapping (6.2) for designing a generalized Newton algorithm to solve subgradient inclusions (1.5) for continuously prox-regular functions with applications to regularized least square problems.

Here is our basic Newton-type algorithm to solve subgradient inclusions (1.5) generated by prox-regular functions $\varphi$. Note that this algorithm constructively describes the area of choosing a starting point that depends on the constant of prox-regularity. The subproblem of this algorithm at each iteration consists of finding a unique solution to the optimization problem in (6.2), which is a regularization of $\varphi$ by using quadratic penalties.

Algorithm 6.2 (Newton-type algorithm for subgradient inclusions). Let $r > 0$ be a constant of prox-regularity of $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ from (4.1).

**Step 0:** Pick any $\lambda \in (0, r^{-1})$, choose a starting point $x^0$ by

$$x^0 \in U_\lambda := \text{rge}(I + \lambda \partial \varphi),$$

(6.3)

and set $k := 0$.

**Step 1:** If $0 \in \partial \varphi(x^k)$, then stop. Otherwise compute

$$v^k := \frac{1}{\lambda} \left( x^k - \text{Prox}_\lambda \varphi(x^k) \right).$$

(6.4)

**Step 2:** Choose $d^k \in \mathbb{R}^n$ such that

$$-v^k \in \partial^2 \varphi(x^k - \lambda v^k, v^k)(\lambda v^k + d^k).$$

(6.5)

**Step 3:** Compute $x^{k+1}$ by

$$x^{k+1} := x^k + d^k, \quad k = 0, 1, \ldots.$$

**Step 4:** Increase $k$ by 1 and go to Step 1.

Note that Algorithm 6.2 does not include computing the Moreau envelope (6.1) while requiring to solve subproblem (6.4) built upon the proximal mapping (6.2). By definition of the second-order subdifferential (2.12) and the limiting coderivative (2.7), the implicit inclusion (6.5) for $d^k$ can be rewritten explicitly as

$$(-v^k, -\lambda v^k - d^k) \in N_{\text{gph} \partial \varphi}(x^k - \lambda v^k, v^k).$$

(6.6)
Observe that for convex functions \( \varphi : \mathbb{R}^n \to \mathbb{R} \) we can choose \( \lambda \) in Step 0 arbitrarily from \((0, \infty)\) with \( U_\lambda = \mathbb{R}^n \) in (6.3). This is due to a well-known result of convex analysis, which is reflected in the following lemma that plays a crucial role in the justification of Algorithm 6.2. Recall that \( I \) stands for the identity operator, and that \( \varphi \) is \textit{prox-bounded} if it is bounded from below by a quadratic function on \( \mathbb{R}^n \).

**Lemma 6.3 (Moreau envelopes and proximal mappings for prox-regular functions).** Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be prox-bounded on \( \mathbb{R}^n \) and continuously prox-regular at \( \bar{x} \) for \( \bar{v} \in \partial \varphi(\bar{x}) \) with modulus \( r > 0 \). Then the following assertions hold for all \( \lambda \in (0, r^{-1}) \), where the parameter \( \lambda \) can be chosen arbitrarily from \((0, \infty)\) with \( U_\lambda = \mathbb{R}^n \) if \( \varphi \) is convex:

(i) The Moreau envelope \( e_\lambda \varphi \) is of class \( C^{1,1} \) on \( U_\lambda \) taken from (6.3), which contains a neighborhood of \( \bar{x} + \lambda \bar{v} \). Furthermore, \( \bar{x} \) is a solution to (1.5) if and only if \( \nabla e_\lambda \varphi(\bar{x}) = 0 \).

(ii) The proximal mapping \( P_\lambda \varphi \) is single-valued, monotone, and Lipschitz continuous on \( U_\lambda \) and satisfies the condition \( P_\lambda \varphi(\bar{x} + \lambda \bar{v}) = \bar{x} \).

(iii) The gradient of \( e_\lambda \varphi \) is calculated by

\[
\nabla e_\lambda \varphi(x) = \frac{1}{\lambda} \left( x - \text{Prox}_\lambda \varphi(x) \right) = (\lambda I + \partial \varphi^{-1})^{-1}(x) \quad \text{for all} \quad x \in U_\lambda.
\]

**Proof.** Denote \( \varphi_0(x) := \varphi(x + \bar{x}) - \varphi(\bar{x}) - \langle \bar{v}, x \rangle \) and observe that \( \varphi_0 \) satisfies the assumptions from Poliquin and Rockafellar [53, Theorem 4.4]. This yields assertions (i) and (ii). The results for convex functions \( \varphi \) follow from [57, Theorem 2.26]. Assertion (iii) is taken from Poliquin and Rockafellar [53, Theorem 4.4]. \( \square \)

The next simple lemma is also needed in the proof of the main result of this section.

**Lemma 6.4 (second-order subdifferential graph).** In the setting of Lemma 6.3, for any \( \lambda \in (0, r^{-1}), x \in U_\lambda, \) and \( v = \nabla e_\lambda \varphi(x) \) we have the equivalence

\[
(v^*, x^*) \in \text{gph}(D^*\nabla e_\lambda \varphi)(x, v) \iff (v^* - \lambda x^*, x^*) \in \text{gph} \partial^2 \varphi(x - \lambda v, v).
\]

**Proof.** The relationships in (3.14) and (6.7) tell us that \( (v^*, x^*) \in \text{gph}(D^*\nabla e_\lambda \varphi)(x, v) \) if and only if

\[-v^* \in D^*(\lambda I + \partial \varphi^{-1})(v, x)(-x^*).\]

Elementary operations with the limiting coderivative yield

\[
D^*(\lambda I + \partial \varphi^{-1})(v, x)(-x^*) = -\lambda x^* + (D^*\partial \varphi^{-1})(v, x - \lambda v)(-x^*).
\]

This ensures the equivalence of (6.8) to the inclusion

\[
\lambda x^* - v^* \in (D^*\partial \varphi^{-1})(v, x - \lambda v)(-x^*),
\]

which implies by (3.14) that \( x^* \in \partial^2 \varphi(x - \lambda v, v)(v^* - \lambda x^*) \) and thus completes the proof. \( \square \)

**Remark 6.5 (on the iterative sequence generated by Algorithm 6.2).** Lemma 6.3 and Lemma 6.4 allow us to show that Algorithm 6.2 is a special case of the general scheme given in Algorithm 5.3. Indeed, we can equivalently rewrite the conditions in (6.4) and (6.5) as

\[
v^k = \nabla e_\lambda \varphi(x^k) \quad \text{and} \quad -\nabla e_\lambda \varphi(x^k) \in \partial^2 e_\lambda \varphi(x^k)(d^k) = \partial(d^k, \nabla e_\lambda \varphi)(x^k).
\]

Therefore, Algorithm 6.2 reduces to Algorithm 5.3 with \( \varphi := e_\lambda \varphi. \)
Now we proceed with the formulation and proof of the major result of this section on the well-posedness and superlinear convergence of the proposed algorithm for prox-regular functions.

**Theorem 6.6 (local superlinear convergence of Algorithm 6.2).** In addition to the standing assumption (H1)–(H3) with $\tilde{v} = 0$ therein, let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be prox-bounded on $\mathbb{R}^n$ and continuously prox-regular at $\bar{x}$ for $0 \in \partial \varphi(\bar{x})$ with constants $r, \varepsilon > 0$ from (4.1). Then there exists a neighborhood $U$ of $\bar{x}$ such that for all starting points $x^0 \in U$ we have that Algorithm 6.2 is well-defined and generates a sequence of iterates $\{x^k\}$, which converges superlinearly to the solution $\bar{x}$ of (1.5) as $k \to \infty$.

**Proof.** It follows from Remark 6.5 that solving the subgradient inclusion (1.5) for the class of prox-regular functions under consideration by Algorithm 6.2 is equivalent to solving the gradient system $\nabla e_\lambda \varphi(x) = 0$ with the $C^{1,1}$ function $e_\lambda \varphi$ under the indicated choice of parameters by Algorithm 5.3. To apply Algorithm 5.3 to the equation $\nabla e_\lambda \varphi(x) = 0$, we need to check that assumptions (H1)–(H3) imposed in this theorem on $\partial \varphi$ are equivalent to the corresponding assumptions on $\nabla e_\lambda \varphi$ imposed in Theorem 5.7.

Let us first verify that assumption (H1) of the theorem yields its fulfillment of its counterpart for $e_\lambda \varphi$. Since $U_\lambda$ from (6.3) contains a neighborhood of $\bar{x}$ by Lemma 6.3(i), it suffices to show that there is a neighborhood $\tilde{U}$ of $\bar{x}$ such that for each $x \in \tilde{U}$ there exists $d \in \mathbb{R}^n$ satisfying

$$-\nabla e_\lambda \varphi(x) \in (D^* \nabla e_\lambda \varphi)(x)(d).$$

Indeed, assumption (H1) gives us neighborhoods $U$ of $\bar{x}$ and $V$ of $\tilde{v} = 0$ for which inclusion (3.2) holds. Since $\nabla e_\lambda \varphi$ and $I - \nabla e_\lambda \varphi$ are continuous around $\bar{x}$ and since

$$((I - \lambda \nabla e_\lambda \varphi)(\bar{x}), \nabla e_\lambda \varphi(\bar{x})) = (\bar{x}, 0) \in (U \cap U_\lambda) \times V,$$

there exists a neighborhood $\tilde{U}$ of $\bar{x}$ such that $\tilde{U} \subset (U \cap U_\lambda)$ and that

$$(I - \lambda \nabla e_\lambda \varphi)(\tilde{U}) \times \nabla e_\lambda \varphi(\tilde{U}) \subset (U \cap U_\lambda) \times V.$$

Fix now any $x \in \tilde{U}$ and put $y := \nabla e_\lambda \varphi(x)$. Performing elementary transformations brings us to

$$D^*(\nabla e_\lambda \varphi)^{-1}(y, x)(y) = D^*(\lambda I + \partial \varphi^{-1})(y, x)(x) = \lambda y + (D^* \partial \varphi^{-1})(y, x - \lambda y)(y).$$

It follows from (3.2) that there exists $\bar{d} \in \mathbb{R}^n$ such that $-y \in (D^* \partial \varphi)(x - \lambda y, y)(\bar{d})$, which implies that $-\bar{d} \in (D^* \partial \varphi^{-1})(y, x - \lambda y)(y)$. Denoting $d := -\lambda y + \bar{d}$, we arrive at

$$-d \in D^*(\nabla e_\lambda \varphi)^{-1}(y, x)(y),$$

which tells us by (3.14) that $-\nabla e_\lambda \varphi(x) \in (D^* \nabla e_\lambda \varphi)(x)(d)$ and thus verifies (6.9).

Next we show that the metric regularity assumption (H2) on $\partial \varphi$ around $(\bar{x}, 0)$ is equivalent to the metric regularity of the Moreau envelope gradient $\nabla e_\lambda \varphi$. To proceed, pick any $\lambda \in (0, r^{-1})$ from Step 0 of the algorithm and then get by Lemma 6.4 that

$$0 \in (D^* \partial \varphi)(\bar{x}, 0)(u) \iff 0 \in (D^* \nabla e_\lambda \varphi)(\bar{x}, 0)(u).$$

Then the claimed equivalence between the metric regularity properties follows directly from the Mordukhovich coderivative criterion (2.8) applied to both mappings $\partial \varphi$ and $\nabla e_\lambda \varphi$.

The equivalence between the semismoothness* of $\partial \varphi$ at $(\bar{x}, 0)$ and of $\nabla e_\lambda \varphi$ around this point is verified in the proof of Theorem 6.2 in Mordukhovich and Sarabi [48]. Thus we can apply the above Theorem 5.7 to the gradient system $\nabla e_\lambda \varphi(x) = 0$ and complete the proof of this theorem by using Lemma 6.3. 

□
Next we present a consequence of Theorem 6.6 for \textit{tilt-stable local minimizers}.

**Corollary 6.7 (computing tilt-stable minimizers of prox-regular functions).** Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be continuous prox-regular at \( \bar{x} \) for \( \bar{v} := 0 \in \partial \varphi(\bar{x}) \) with \( r > 0 \) taken from (4.1), and let \( \bar{x} \) be a tilt-stable local minimizer of \( \varphi \). Assume that the subgradient mapping \( \partial \varphi \) is semismooth* at \((\bar{x}, \bar{v})\). Then whenever \( \lambda \in (0, r^{-1}) \) there exists a neighborhood \( U \) of \( \bar{x} \) such that Algorithm 6.2 is well-defined for all \( x^0 \in U \) and generates a sequence of iterates \( \{x^k\} \), which superlinearly converges to \( \bar{x} \) as \( k \to \infty \).

**Proof.** Observe that, in the case of minimizing \( \varphi \) under consideration, the prox-boundedness assumption on \( \varphi \) imposed in Theorem 6.6 can be dismissed without loss of generality. Indeed, we can always get this property by adding to \( \varphi \) the indicator function of some compact set containing a neighborhood of \( \bar{x} \), which makes \( \varphi \) to be prox-bounded without changing the local minimization. Furthermore, it follows from Drusvyatskiy and Lewis [14, Theorem 3.3] that the mapping \( \partial \varphi \) is strongly metrically regular around \((\bar{x}, \bar{v})\). Thus assumption (H2) holds automatically, while (H1) follows from Theorem 4.2(ii). The fulfillment of (H3) is assumed in this corollary, and so we deduce its conclusions from Theorem 6.6. \( \square \)

Note that the well-posedness and local superlinear convergence of the coderivative-based generalized Newton algorithm of computing tilt-stable local minimizers of the Moreau envelope \( e_{\lambda} \varphi \) under the semismoothness* assumption on \( \partial \varphi \) has been recently obtained by Mordukhovich and Sarabi [48, Theorem 6.2]. As follows from the discussions above, the latter algorithm is equivalent to Algorithm 6.2 in the case of tilt-stable local minimizers. However, the explicit form of Algorithm 6.2 seems to be more convenient for implementations, which is demonstrated below. Moreover, in Algorithm 6.2 we specify the area of starting points \( x^0 \) in Step 0 and also verify the choice of the prox-parameter \( \lambda \) ensuring the best performance of the proposed algorithm.

Let us illustrate Algorithm 6.2 by the following example of solving the subgradient inclusion (1.5) generated by a nonconvex, nonsmooth, and continuously prox-regular function \( \varphi : \mathbb{R} \to \mathbb{R} \). This function is taken from Mordukhovich and Outrata [46, Example 4.1] and relates to the modeling of equilibria.

**Example 6.8 (illustration of computing by Algorithm 6.2).** Consider the function
\[
\varphi(x) := |x| + \frac{1}{2}(\max\{x, 0\})^2 - \frac{1}{2}(\max\{0, -x\})^2 + \delta_{\Gamma}(x), \quad x \in \mathbb{R},
\]
where \( \delta_{\Gamma} \) is the indicator function of the set \( \Gamma := [-1, 1] \). We can clearly write \( \varphi \) in the form
\[
\varphi(x) = \vartheta(x) + \delta_{\Gamma}(x), \quad x \in \mathbb{R},
\]
where
\[
\vartheta(x) := \begin{cases} 
-x - \frac{1}{2}x^2 & \text{if } x \in [-1, 0], \\
+x + \frac{1}{2}x^2 & \text{if } x \in [0, 1].
\end{cases}
\]

Then we get by the direct calculations that
\[
\partial \varphi(x) = \begin{cases} 
(-\infty, 0] & \text{if } x = -1, \\
[-1, x) & \text{if } x \in (-1, 0), \\
[-1, 1] & \text{if } x = 0, \\
[1, x) & \text{if } x \in (0, 1), \\
[2, \infty) & \text{if } x = 1.
\end{cases}
\]
It is not hard to check that the mapping $\partial \varphi$ is strongly metrically regular around $(\bar{x},0)$ with $\bar{x} = 0$ and semismooth* at this point, and thus the assumptions of Theorem 6.6 hold. Moreover, we get

$$\varphi(x) \geq \varphi(u) + v(x - u) - \frac{1}{2} \|x - u\|^2 \quad \text{for all} \quad (u,v) \in \text{gph} \partial \varphi \cap (U \times \mathbb{R}), \quad x \in U$$

where $U = [-1,1]$, which implies that $\varphi$ is continuously prox-regular on $U$ with modulus $r = 1$. Choosing $\lambda := \frac{1}{2} \in (0,1)$ and $x^0 := \frac{1}{3} \in \text{rge}(I + \frac{1}{2} \partial \varphi)$, we run Algorithm 6.2 with

$$\text{Prox}_\lambda \varphi(x^0) = \arg\min_{y \in \mathbb{R}} \{ \varphi(y) + (y - x^0)^2 \} = 0.$$ 

Thus the vector $v^0$ in Step 1 of the algorithm is calculated by $v^0 = \frac{1}{\lambda}(x^0 - P_{\lambda \varphi}(x^0)) = \frac{2}{3}$. To find $d^0 \in \mathbb{R}$ in Step 2 of the algorithm, we have by (6.6) that

$$(−v^0, −\lambda v^0 − d^0) \in N_{\text{gph} \partial \varphi}(x^0 − \lambda v^0, v^0) = N_{\text{gph} \partial \varphi}(0, 2/3).$$

The second-order calculations in Mordukhovich and Outrata [41, Equation (4.7)] yield

$$N_{\text{gph} \partial \varphi}(0, 2/3) = \{ (\omega, z) \in \mathbb{R}^2 \mid z = 0 \}.$$

This tells us that $−\lambda v^0 − d^0 = 0$, and hence $d^0 = −\frac{1}{3}$. Letting $x^1 := x^0 + d^0 = 0$ by Step 3, we arrive at $0 \in \partial \varphi(x^1)$, and thus solve the subgradient inclusion (1.5) for $\varphi$ from (6.10).

We conclude this section with the brief discussion on some other Newton-type methods to solve set-valued inclusions involving the subgradient systems (1.5).

**Remark 6.9** (Comparison with other Newton-type methods to solve generalized equations). Let us make the following observations:

(i) Josephy [29] was the first to propose a Newton-type algorithm to solve generalized equations in the sense of Robinson [56] written in the form

$$0 \in f(x) + F(x), \quad (6.11)$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth single-valued mapping, and where $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued one, which originally was considered as the normal cone to a convex set while covering in this case classical variational inequalities and nonlinear complementarity problems. The Josephy-Newton method constructs the next iterate $x^{k+1}$ by solving the linearized generalized equation

$$0 \in f(x^k) + \nabla f(x^k)(x - x^k) + F(x), \quad (6.12)$$

with the same set-valued part $F(x)$ as in (6.11). Note that in this method we must have a nonzero function $f$ in (6.11); otherwise algorithm (6.12) stops at $x^1$. There are several results on the well-posedness and superlinear convergence of the Josephy-Newton method under appropriate assumptions; see, e.g., the books by Facchinei and Pang [16] and by Izmailov and Solodov [27] with the references therein. The major assumption for (6.11), which is the most related to our paper, is the strong metric regularity of $f + F$ around $(\bar{x},0)$ imposed in Dontchev and Rockafellar [13, Theorem 6C.1]. In the case of subgradient systems (1.5), we actually have $f = \nabla \varphi$ and $F = 0$ in (6.11), which gives us the strong metric regularity of $\nabla \varphi$. As discussed in Sections 4 and 5, the latter assumption is more restrictive that our standing ones imposed in Theorem 5.7. A similar strong metric regularity assumption is imposed in the semismooth Newton method of solving generalized equations (without changing $F$ in iterations) presented in Theorem 6F.1 of the aforementioned book by Dontchev and Rockafellar.
(ii) Gfrerer and Outrata [19] have recently introduced the new semismooth* Newton method to solve the generalized equation $0 \in F(x)$. In contrast to the Josephy-Newton and semismooth Newton methods for generalized equations, the new method approximates the set-valued part $F$ of (6.11) by using a certain linear structure inside the limiting coderivative graph under the nonsingularity of matrices from the mentioned linear structure. Local superlinear convergence of the proposed algorithm is proved there under the additional semismooth* assumptions on the mapping $F$.

(iii) Another Newton-type method to solve the inclusions $0 \in F(x)$, with the verification of well-posedness and local superlinear convergence, was developed by Dias and Smirnov [10] based on tangential approximations of the graph of $F$. The major assumption therein is the metric regularity of $F$ and the main tool of analysis is the Mordukhovich criterion (2.8). Observe that the well-posedness result of the suggested algorithm requires the Lipschitz continuity of $F$, which is rarely the case for subgradient mappings associated with nonsmooth functions.

7 Generalized Newton Algorithm in Composite Optimization

In this section we develop a generalized Newton algorithm to solve the subgradient inclusion $0 \in \partial \varphi(x)$ with the cost function $\varphi : \mathbb{R}^n \to \mathbb{R}$ given in the composite optimization form by

$$\text{minimize } \varphi(x) := f(x) + g(x) \quad \text{subject to } x \in \mathbb{R}^n,$$  

where the function $f : \mathbb{R}^n \to \mathbb{R}$ is $C^2$-smooth while the regularizer $g : \mathbb{R}^n \to \mathbb{R}$ is continuously prox-regular. Since the function $\varphi$ in (7.1) is also continuously prox-regular, we can solve this problem by using the generalized Newton algorithm developed in Section 6. However, one of the difficulties in implementing Algorithm 6.2 is to compute the proximal mapping of $\varphi$, which is the sum of two functions. This motivates us to design a generalized Newton algorithm to solve the subgradient inclusion $0 \in \partial \varphi(x)$ associated with the composite optimization problem (7.1) by calculating the proximal mapping only for the regularizer $g$ that is often easy to compute.

To proceed, we employ the construction of forward-backward envelopes introduced by Patri- nos and Bemporad [52] for convex composite functions and then largely used to design numerical methods in various composite frameworks of optimization; see, e.g., Themelis et al. [60] with the references therein. This leads us to developing a well-defined and implementable generalized Newton algorithm expressed in terms of the second-order subdifferential of the regularizer $g$ and the single-valued, monotone, and Lipschitz continuous proximal mapping associated with this function. We show that the proposed algorithm exhibits local superlinear convergence under the standing assumptions formulated in Section 5.

Definition 7.1 (forward-backward envelopes). Let $\varphi := f + g$, where $f : \mathbb{R}^n \to \mathbb{R}$ is $C^1$-smooth, and where $g : \mathbb{R}^n \to \mathbb{R}$ is a proper l.s.c. function. The forward-backward envelope (FBE) of $\varphi$ with the parameter value $\gamma > 0$ is given by

$$\varphi_\gamma(x) := \inf_{y \in \mathbb{R}^n} \left\{ f(x) + \langle \nabla f(x), y - x \rangle + g(y) + \frac{1}{2\gamma} \| y - x \|^2 \right\}.$$  

(7.2)

By definition (6.1) of the Moreau envelope, (7.2) is equivalent to

$$\varphi_\gamma(x) = f(x) - \frac{\gamma}{2} \| \nabla f(x) \|^2 + e_\gamma g(x - \gamma \nabla f(x)).$$  

(7.3)

Now we derive several properties of FBE in our setting, that are of independent interest while being instrumental to design the aforementioned generalized Newton algorithm for (7.1) and justify its well-definedness and superlinear convergence.

The first proposition verifies the $C^1$-smoothness of FBE with computing its gradient and verify the desired properties of the prox-gradient mapping associated with (7.1).
Proposition 7.2 (smoothness of FBE and related properties). Let \( f \) be of class \( C^2 \) around \( \bar{x} \) where \( 0 \in \partial \varphi(\bar{x}) \), and let \( g \) be prox-bounded and continuously prox-regular at \( \bar{x} \) for \(-\nabla f(\bar{x})\) with modulus \( r > 0 \). Then for all \( \gamma \in (0, r^{-1}) \) there exists a neighborhood \( U \) of \( \bar{x} \) on which the following properties hold:

(i) The mapping \( x \mapsto \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)) \) is single-valued and Lipschitz continuous.

(ii) The FBE \( \varphi_\gamma \) is continuously differentiable and its gradient mapping is computed by

\[
\nabla \varphi_\gamma(x) = \gamma^{-1}(I - \gamma \nabla^2 f(x))(x - \text{Prox}_{\gamma g}(x - \gamma \nabla f(x))).
\]

(7.4)

Moreover, if \( I - \gamma \nabla^2 f(x) \) is nonsingular for \( x \in U \), we have the equivalence

\[
\nabla \varphi_\gamma(x) = 0 \iff 0 \in \partial \varphi(x).
\]

(7.5)

If in addition the regularizer \( g \) is convex, then \( \gamma \) can be chosen in \((0, \infty)\).

Proof. By Lemma 6.3 and the continuous prox-regularity of \( g \), for each \( \gamma \in (0, r^{-1}) \) there is a neighborhood \( U_\gamma \) of \( \bar{x} - \gamma \nabla f(\bar{x}) \) such that the proximal mapping \( \text{Prox}_{\gamma g} \) is single-valued and Lipschitz continuous on \( U_\gamma \). Since \( f \) is \( C^2 \)-smooth around \( \bar{x} \), there is a neighborhood \( U \) of \( \bar{x} \) ensuring the Lipschitz continuity of the mapping \( x \mapsto x - \gamma \nabla f(x) \) on \( U \) and the fulfillment of the relationships

\[
x \in U \implies x - \gamma \nabla f(x) \in U_\gamma,
\]

which therefore verifies the single-valuedness of the prox-gradient mapping \( x \mapsto \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)) \) on \( U \). The Lipschitz continuity of this mapping follows from the Lipschitz continuity of the proximal mapping \( \text{Prox}_{\gamma g} \) on \( U_\gamma \) and of the mapping \( x \mapsto x - \gamma \nabla f(x) \) on \( U \). This verifies assertion (i).

The gradient calculation in (7.4) of assertion (ii) follows directly from (7.3) and Lemma 6.3(iii). It remains to verify implication (7.5) if \( I - \gamma \nabla^2 f(x) \) is nonsingular for \( x \in U \). Indeed, due to the nonsingular of \( I - \gamma \nabla^2 f(x) \), the gradient equation \( \nabla \varphi_\gamma(x) = 0 \) is equivalent to \( x = \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)) \). The latter is equivalent to the fact that \( x = (I + \gamma \partial g)^{-1}(x - \gamma \nabla f(x)) \) by Lemma 6.3. This means that \( 0 \in \nabla f(x) + \partial g(x) \). Using finally the limiting subdifferential sum rule with equalities from Mordukhovich [43, Proposition 1.107], we obtain the inclusion \( 0 \in \partial \varphi(x) \). In the case where \( g \) is convex, we proceed similarly to the above to justifying that \( \gamma \) can be arbitrary chosen from \((0, \infty)\), and thus we complete the proof of this proposition. \( \Box \)

Proposition 7.2 tells us that using the forward-backward envelope (7.2) makes it possible to pass from the nonsmooth composite problem (7.1) to the unconstrained one

\[
\text{minimize } \varphi_\gamma(x) \text{ subject to } x \in \mathbb{R}^n
\]

(7.6)

with a smooth cost function. Due to the explicit calculations of \( \varphi_\gamma \) in (7.3) and its gradient (7.4), we can extend Algorithm 5.3 to cover problem (7.1) by reducing it to (7.6). The implementation of this procedure requires revealing appropriate assumptions on \( \varphi \) in (7.1), which ensure the fulfillment of those for \( \varphi_\gamma \) and thus allow us to apply the results of Section 5 to the reduced problem (7.6).

Note that (7.6) is generally not a problem of \( C^{1,1} \) optimization, since Proposition 7.2 does not guarantee the Lipschitz continuity of \( \nabla \varphi_\gamma \). Nevertheless, in the case where \( f \) is quadratic, the \( C^{1,1} \) property of \( \varphi_\gamma \) follows from Proposition 7.2 by formula (7.4). From now on, we consider the problem

\[
\text{minimize } \varphi(x) := \frac{1}{2}\langle Ax, x \rangle + \langle b, x \rangle + \alpha + g(x), \quad x \in \mathbb{R}^n,
\]

(7.7)
where $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $b \in \mathbb{R}^{n}$, $\alpha \in \mathbb{R}$, and the regularizer $g : \mathbb{R}^{n} \to \mathbb{R}$ is continuously prox-regular at $\bar{x} \in \mathbb{R}^{n}$ for $-Az - b$ with $\bar{x}$ satisfying the stationary condition $0 \in \partial \varphi(\bar{x})$. Clearly, $(7.7)$ is a subclass of the optimization problem $(7.1)$ with $f(x) := \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + \alpha$.

Let us highlight that problems of type $(7.7)$ are important in their own right, while they also arise frequently as subproblems in various numerical algorithms including sequential quadratic programming methods (SQP) [4,27], proximal Newton methods [28,34,49], etc. Observe furthermore that optimization problems of this type often appear in practical models related, e.g., to machine learning and statistics.

Now we start the procedure of designing and justifying a locally convergent generalized Newton algorithm to solve the inclusion $0 \in \partial \varphi(x)$, where $\varphi$ is the cost function in the composite problem $(7.7)$, by applying the corresponding results for the $C^{1,1}$ gradient systems $\nabla \varphi_{\gamma}(x) = 0$ obtained in Section 5. The first step is to express the generalized Hessian of the FBE $\varphi_{\gamma}$ from $(7.6)$ in terms of the given data of $(7.7)$.

**Proposition 7.3 (calculating the generalized Hessian of FBE).** Let $\varphi = f + g$ be as in $(7.7)$ with a prox-bounded and continuously prox-regular $g : \mathbb{R}^{n} \to \mathbb{R}$, and let $\gamma \in (0, r^{-1})$ be such that the matrix $B := I - \gamma A$ is nonsingular, where $r > 0$ is a constant of prox-regularity of $g$.

Then there exists a neighborhood $U$ of $\bar{x}$ on which we have the second-order calculation formula

$$z \in \partial^{2} \varphi_{\gamma}(x)(w) \iff B^{-1}z - Aw \in \partial^{2} g \left( \text{Prox}_{\gamma g}(u), \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \right)(w - \gamma B^{-1}z)$$

(7.8)

for any $x \in U$, $w \in \mathbb{R}^{n}$, and $u := x - \gamma(Ax + b)$.

**Proof.** Using the Moreau envelope, define the function $h : \mathbb{R}^{n} \to \mathbb{R}$ by

$$h(x) := e_{\gamma}g(x - \gamma(Ax + b)) \quad \text{for all} \quad x \in \mathbb{R}^{n}.$$  

Due to Proposition 7.2, there exists a neighborhood $U$ of $\bar{x}$ such that $\varphi_{\gamma}$ is $C^{1}$-smooth and that the mapping $x \mapsto \text{Prox}_{\gamma g}(x - \gamma(Ax + b))$ is continuous on $U$, which implies that $h$ is continuous differentiable around $\bar{x}$ due to $(7.3)$. Take any $x \in U$ and put $u := x - \gamma(Ax + b)$. It follows from the standard chain rule that

$$\nabla h(x) = (I - \gamma A)^{n} \nabla e_{\gamma}g(u) = B \nabla e_{\gamma}g(u).$$

Using $(7.3)$ and the second-order sum rule from Mordukhovich [43, Proposition 1.121], for each $w \in \mathbb{R}^{n}$ we have

$$\partial^{2} \varphi_{\gamma}(x)(w) = (A - \gamma A^{*}A)w + \partial^{2}h(x)(w) = BAw + \partial^{2}h(x)(w).$$

(7.9)

Employing another second-order chain rule taken from [43, Theorem 1.127] yields

$$\partial^{2}h(x)(w) = B\partial^{2}e_{\gamma}g(u)(Bw) \quad \text{for all} \quad w \in \mathbb{R}^{n}.$$  

(7.10)

Combining $(7.9)$ and $(7.10)$ brings us to the formula

$$\partial^{2} \varphi_{\gamma}(x)(w) = BAw + B\partial^{2}e_{\gamma}g(u)(Bw)$$

valid for the same $w$. This verifies the equivalences

$$z \in \partial^{2} \varphi_{\gamma}(x)(w) \iff z - BAw \in \partial^{2}e_{\gamma}g(u)(Bw) \iff B^{-1}z - Aw \in \partial^{2}e_{\gamma}g(u)(Bw)$$

for all $w \in \mathbb{R}^{n}$. Using finally Lemma 6.4, we arrive at the inclusion

$$B^{-1}z - Aw \in \partial^{2} g \left( \text{Prox}_{\gamma g}(u), \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \right)(Bw - \gamma B^{-1}z + \gamma Aw)$$

and thus complete the proof of the proposition due to $Bw + \gamma Aw = w$.  \qed
The next proposition shows that the metric regularity and tilt stability of the original cost function \( \varphi \) in (7.7) is equivalent to the corresponding properties of its FBE \( \varphi_\gamma \) in (7.6).

**Proposition 7.4 (metric regularity and tilt-stability of FBE).** Let in the setting of Proposition 7.3 \( \bar{x} \) be such that \( 0 \in \partial \varphi(\bar{x}) \). Then we have the following assertions:

(i) \( \partial \varphi \) is metrically regular around \((\bar{x}, 0)\) if and only if \( \nabla \varphi_\gamma \) is metrically regular around \( \bar{x} \).

(ii) If \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi \), then \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi_\gamma \) provided that the matrix \( B = I - \gamma A \) is positive definite.

**Proof.** It follows from Proposition 7.3 that

\[
z \in \partial^2 \varphi_\gamma(\bar{x})(w) \iff B^{-1}z \in Aw + \partial^2 g \left( \text{Prox}_{\gamma g}(\bar{u}), \frac{1}{\gamma}(\bar{u} - \text{Prox}_{\gamma g}(\bar{u})) \right) (w - \gamma B^{-1}z) \tag{7.11}
\]

with \( \bar{u} \) defined therein. Proposition 7.2 gives us \( \nabla \varphi_\gamma(\bar{x}) = 0 \), which means that

\[
\gamma^{-1}(I - \gamma A)(x - \text{Prox}_{\gamma g}(x - \gamma \nabla f(x))) = 0
\]

and implies that \( \bar{x} = \text{Prox}_{\gamma g}(\bar{x} - \gamma \nabla f(\bar{x})) \) since \( B = I - \gamma A \) is nonsingular. This tells us that \( \bar{x} - \text{Prox}_{\gamma g}(\bar{u}) = 0 \), and thus (7.11) is equivalent to

\[
B^{-1}z \in Aw + \partial^2 g(\bar{x}, -A\bar{x} - b)(w - \gamma B^{-1}z)
= A(w - \gamma B^{-1}z) + \partial^2 g(\bar{x}, -A\bar{x} - b)(w - \gamma B^{-1}z) + \gamma AB^{-1}z. \tag{7.12}
\]

The second-order subdifferential sum rule from Mordukhovich [43, Proposition 1.121] yields

\[
A(w - \gamma B^{-1}z) + \partial^2 g(\bar{x}, -A\bar{x} - b)(w - \gamma B^{-1}z) = \partial^2 (f + g)(\bar{x}, 0)(w - \gamma B^{-1}z)
= \partial^2 \varphi(\bar{x}, 0)(w - \gamma B^{-1}z). \tag{7.13}
\]

Combining (7.11), (7.12), and (7.13), we arrive at the equivalence

\[
z \in \partial^2 \varphi_\gamma(\bar{x})(w) \iff B^{-1}z - \gamma AB^{-1}z \in \partial^2 \varphi(\bar{x}, 0)(w - \gamma B^{-1}z)
\iff z \in \partial^2 \varphi(\bar{x}, 0)(w - \gamma B^{-1}z). \tag{7.14}
\]

It follows from the coderivative criterion (2.2) and the equivalence in (7.14) that \( \partial \varphi \) is metrically regular around \((\bar{x}, 0)\) if and only if \( \nabla \varphi_\gamma \) is metrically regular around \( \bar{x} \), which justifies assertion (i).

To clarify assertion (ii), suppose that \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi \), which also implies that \( \partial \varphi \) is metrically regular around \((\bar{x}, 0)\). Let \( w \in \mathbb{R}^n \), \( w \neq 0 \) and \( z \in \partial^2 \varphi_\gamma(\bar{x})(w) \). By (7.14), we deduce that \( z \in \partial^2 \varphi(\bar{x}, 0)(w - \gamma B^{-1}z) \), which ensures that \( z \neq 0 \) by the coderivative criterion (2.2). It follows from the point-based second-order characterization of tilt-stability by Poliquin and Rockafellar [54, Theorem 1.3] that \( \langle z, w - \gamma B^{-1}z \rangle \geq 0 \), and thus

\[
\langle z, w \rangle \geq \gamma \langle B^{-1}z, z \rangle > 0. \tag{7.15}
\]

due to the positive definiteness of \( B^{-1} \), which tells us that \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi_\gamma \) due to [54, Theorem 1.3]. This completes the proof of the proposition.

We need another proposition before formulating our generalized Newton algorithm to solve the composite optimization problem (7.7) and justifying its superlinear convergence via the given data.
Proposition 7.5 (semismoothness\* of FBE derivatives). In the setting of Proposition 7.4, we claim that the semismooth\* property of \( \partial g \) at \((\bar{x}, \bar{v})\) with \( \bar{v} := -A\bar{x} - b \) yields the semismoothness\* of \( \nabla \varphi_\gamma \) at \( \bar{x} \).

**Proof.** Denote \( h_\gamma(x) := \text{Prox}_\gamma(x - \gamma(Ax + b)) \) on \( \mathbb{R}^n \) and get by Proposition 7.2 that
\[
\nabla \varphi_\gamma(x) = \gamma^{-1}(I - \gamma A)(x - h_\gamma(x)) = \gamma^{-1}Bx - \gamma^{-1}Bh_\gamma(x) \quad \text{for all} \quad x \in \mathbb{R}^n. 
\]
(7.16)

Since \( 0 \in \partial \varphi(\bar{x}) \), it follows from Proposition 7.2 that \( \nabla \varphi_\gamma(\bar{x}) = 0 \). Due to (7.16) and the nonsingularity of \( B = I - \gamma A \), we have \( \bar{x} = h_\gamma(\bar{x}) \). Furthermore, it follows from Lemma 6.3, we have
\[
\text{Prox}_\gamma(x) = (I + \gamma \partial g)^{-1}(x) \quad \text{for all} \quad x \nearrow \bar{x}.
\]

The semismoothness\* of \( \partial g \) at \((\bar{x}, \bar{v})\) yields this property for \( I + \gamma \partial g \) at \((\bar{x}, \bar{x} + \gamma \bar{v})\) due to Gfrerer and Outrata [19, Proposition 3.6], and hence \((I + \gamma \partial g)^{-1}\) is semismooth\* at \( \bar{x} + \gamma \bar{v} \) (see [19, p. 496]) meaning that \( \text{Prox}_\gamma \) is semismooth\* at \( \bar{x} - \gamma(A\bar{x} + b) \). It follows therefore from Khanh et al. [31, Lemma 7.3] that \( h_\gamma \) is semismooth\* at \( \bar{x} \). Employing finally (7.16) together with [19, Proposition 3.6] tells us that the gradient mapping \( \nabla \varphi_\gamma \) is semismooth\* at \( \bar{x} \). \( \square \)

The proposed generalized Newton algorithm to solve problem (7.7) is formulated as follows.

**Algorithm 7.6 (Newton-type algorithm in nonconvex composite optimization).** Let \( \varphi \) be taken from (7.7), and let \( r > 0 \) be a constant of prox-regularity of \( g: \mathbb{R}^n \rightarrow \mathbb{R} \).

**Step 0:** Pick any \( \gamma \in (0, r^{-1}) \) such that \( I - \gamma A \) is nonsingular, choose a starting point \( x^0 \), and set \( k := 0 \).

**Step 1:** If \( 0 \in \partial \varphi(x^k) \), then stop. Otherwise compute
\[
u^k := x^k - \gamma(Ax^k + b), \quad v^k := \text{Prox}_\gamma(u^k).
\]
(7.17)

**Step 2:** Choose \( d^k \in \mathbb{R}^n \) such that
\[
- \frac{1}{\gamma}(x^k - v^k) - Ad^k \in \partial^2 g\left(v^k, \frac{1}{\gamma}(u^k - v^k)\right)(x^k - v^k + d^k).
\]
(7.18)

**Step 3:** Compute \( x^{k+1} \) by
\[
x^{k+1} := x^k + d^k, \quad k = 0, 1, \ldots.
\]

**Step 4:** Increase \( k \) by 1 and go to Step 1.

**Remark 7.7 (on the iterative sequence generated by Algorithm 7.6).** Propositions 7.2 and 7.3 allow us to show that Algorithm 7.6 is a special case of the general scheme given in Algorithm 5.3. Indeed, by Proposition 7.2 and the constructions of \( \{u^k\} \) and \( \{v^k\} \) in Algorithm 7.6, we have
\[
\nabla \varphi_\gamma(x^k) = \frac{1}{\gamma}B(x^k - v^k), \quad \text{where} \quad B := I - \gamma A.
\]

Therefore, inclusion (7.18) can be equivalently rewritten as
\[
-B^{-1}\nabla \varphi_\gamma(x^k) - Ad^k \in \partial^2 g\left(\text{Prox}_\gamma(u^k), \frac{1}{\gamma}(u^k - \text{Prox}_\gamma(u^k))\right)(d^k + \gamma B^{-1}\nabla \varphi_\gamma(x^k)),
\]
which means that \( -\nabla \varphi_\gamma(x^k) \in \partial^2 \varphi_\gamma(x^k)(d^k) = \partial(d^k, \nabla \varphi_\gamma)(x^k) \) by using Proposition 7.3 and (2.11). Therefore, Algorithm 7.6 reduces to Algorithm 5.3 with \( \varphi := \varphi_\gamma \).
The next theorem justifies the existence and local superlinear convergence of iterates in Algorithm 7.6 under basically the standing assumptions of this paper. Note that we do not assume the convexity of the regularizer $g$ as in problems of convex composite optimization.

**Theorem 7.8. (local superlinear convergence of Algorithm 7.6).** Let $\varphi = f + g$ be as in (7.7), where the regularizer $g: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is continuously prox-regular at the stationary point $\bar{x}$ for $-A\bar{x} - b$. In addition to the standing assumption (H1)–(H3) with $\bar{v} = 0$, suppose that $g$ is prox-bounded on $\mathbb{R}^n$. Then there exists a neighborhood $U$ of $\bar{x}$ such that for any starting points $x^0 \in U$ we have that Algorithm 7.6 is well-defined and generates a sequence of iterates $\{x^k\}$, which converges $Q$-superlinearly to the solution $\bar{x}$ of the subgradient inclusion $0 \in \partial \varphi(x)$.

**Proof.** It follows from Proposition 7.2 that solving the gradient system $\nabla \varphi(x) = 0$ gives us a solution to the original subgradient inclusion $0 \in \partial \varphi(x)$ under the indicated choice of parameters. Furthermore, Remark 7.7 gives us an opportunity to reduce solving the inclusion $0 \in \partial \varphi(x)$ by Algorithm 7.6 to solving the equation $\nabla \varphi(x) = 0$ for the function $\varphi_\gamma$ of class $C^{1,1}$ by using Algorithm 5.3. To justify this reduction, we need to check that assumptions (H1)–(H3) imposed in this theorem on $\partial \varphi$ implies the fulfillment of the corresponding assumptions on $\nabla \varphi_\gamma$ imposed in Theorem 5.7.

To proceed, observe that Propositions 7.4 and 7.5 show that assumptions (H2) and (H3) of this theorem yield the fulfillment of their counterparts for $\varphi_\gamma$. Pick any $\gamma \in (0, r^{-1})$ such that $I - \gamma A$ is nonsingular. Since $f$ is a quadratic function, there exists a neighborhood $U_\gamma$ of $\bar{x}$ on which the prox-gradient mapping $x \mapsto \text{Prox}_{\gamma g}(x - \gamma \nabla f(x))$ is single-valued and Lipschitz continuous, while $\varphi_\gamma$ is of class $C^{1,1}$ on $U_\gamma$ by Proposition 7.2. Now it suffices to find a neighborhood $\tilde{U}$ of $\bar{x}$ such that for each $x \in \tilde{U}$ there exists $\tilde{d} \in \mathbb{R}^n$ satisfying the second-order inclusion

$$-\nabla \varphi_\gamma(x) \in \partial^2 \varphi_\gamma(x)(\tilde{d}). \quad (7.19)$$

Assumption (H1) tells us that there are neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{v} = 0$ such that for every $(x, v) \in \text{gph } \partial \varphi \cap (U \times V)$ there exists a direction $d \in \mathbb{R}^n$ satisfying (3.2). Consider the single-valued mapping $T: U \cap U_\gamma \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$T(x) := \left( \text{Prox}_{\gamma g}(u), A\text{Prox}_{\gamma g}(u) + b + \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \right) \quad \text{for all } x \in U \cap U_\gamma, \ u := x - \gamma(Ax + b).$$

Since $T$ is continuous with $T(\bar{x}) = (\bar{x}, 0)$, there exists a neighborhood $\tilde{U}$ of $\bar{x}$ such that $T(\tilde{U}) \subset (U \cap U_\gamma) \times V$. For any $x \in \tilde{U}$ with $u := x - \gamma(Ax + b)$ we have

$$\frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \in \partial g(\text{Prox}_{\gamma g}(u)). \quad (7.20)$$

Applying the subdifferential sum rule from Mordukhovich [43, Proposition 1.107] gives us

$$\partial \varphi(\text{Prox}_{\gamma g}(u)) = A\text{Prox}_{\gamma g}(u) + b + \partial g(\text{Prox}_{\gamma g}(u)). \quad (7.21)$$

Combining (7.20) and (7.21), we obtain the inclusion

$$\left( \text{Prox}_{\gamma g}(u), A\text{Prox}_{\gamma g}(u) + b + \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \right) \in \text{gph } \partial \varphi \cap (U \times V).$$

Therefore, there exists a direction $d \in \mathbb{R}^n$ satisfying

$$-A\text{Prox}_{\gamma g}(u) - b - \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \in \partial^2 \varphi\left( \text{Prox}_{\gamma g}(u), A\text{Prox}_{\gamma g}(u) + b + \frac{1}{\gamma}(u - \text{Prox}_{\gamma g}(u)) \right)(d). \quad (7.22)$$
Using the second-order subdifferential sum rule from the aforementioned book [43, Proposition 1.121] leads us to the generalized Hessian representation

\[ \partial^2 \varphi \left( \text{Prox}_{\gamma g}(u), A \text{Prox}_{\gamma g}(u) + b + \frac{1}{\gamma} u - \text{Prox}_{\gamma g}(u) \right)(d) = Ad + \partial^2 g \left( \text{Prox}_{\gamma g}(u), \frac{1}{\gamma} u - \text{Prox}_{\gamma g}(u) \right)(d). \]

Combining now the latter with (7.22) leads us to

\[ -A \text{Prox}_{\gamma g}(u) - b - \frac{1}{\gamma} (u - \text{Prox}_{\gamma g}(u)) - Ad \in \partial^2 g \left( \text{Prox}_{\gamma g}(u), \frac{1}{\gamma} (u - \text{Prox}_{\gamma g}(u)) \right)(d). \quad (7.23) \]

Moreover, we clearly have the equalities

\[ -A \text{Prox}_{\gamma g}(u) - b - \frac{1}{\gamma} (u - \text{Prox}_{\gamma g}(u)) = -A \text{Prox}_{\gamma g}(u) - b - \frac{1}{\gamma} (x - \gamma(Ax - b) - \text{Prox}_{\gamma g}(u)) = \gamma^{-1} (I - \gamma A)(\text{Prox}_{\gamma g}(u) - x) = -\nabla \varphi_{\gamma}(x). \quad (7.24) \]

Put \( B := I - \gamma A \) and \( \tilde{d} := d - \gamma B^{-1} \nabla \varphi_{\gamma}(x) \). Using (7.23) and (7.24), we deduce that

\[ -B^{-1} \nabla \varphi_{\gamma}(x) - Ad \in \partial^2 g \left( \text{Prox}_{\gamma g}(u), \frac{1}{\gamma} (u - \text{Prox}_{\gamma g}(u)) \right)(\tilde{d} + \gamma B^{-1} \nabla \varphi_{\gamma}(x)), \]

which verifies (7.19) due to Proposition 7.3. This allows us to apply Theorem 5.7 to the gradient equation \( \nabla \varphi_{\gamma}(x) = 0 \) and thus completes the proof of this theorem.

We have the following specification of Theorem 7.8 in the case of of tilt-stable minimizers.

**Corollary 7.9 (computing tilt-stable minimizers in nonconvex composite optimization).** Let \( \bar{x} \) be a tilt-stable local minimizer of the function \( \varphi \) in (7.7), where the regularizer \( g: \mathbb{R}^n \to \mathbb{R}^n \) is prox-bounded on \( \mathbb{R}^n \) and continuously prox-regular at \( \bar{x} \) for \( -A\bar{x} - b \), and where the mapping \( \partial g \) is semismooth* at \( (\bar{x}, -\nabla f(\bar{x})) \) with \( f(x) := \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + \alpha \). Then there exists a neighborhood \( U \) of \( \bar{x} \) such that Algorithm 7.6 is well-defined for all \( x^0 \in U \) and generates a sequence of iterates \( \{x^k\} \), which Q-superlinearly converges to the local tilt-stable minimizer \( \bar{x} \).

**Proof.** By the tilt stability of \( \bar{x} \), the assumptions of Theorem 7.8 are satisfied; see Remark 5.13. □

We conclude this section with two remarks. The first one concerns the nonsingularity assumption on the matrix \( I - \gamma A \) in Algorithm 7.6.

**Remark 7.10 (on nonsingularity of the shifted matrix in Algorithm 7.6).** Observe first that in this paper we have never imposed the positive-definiteness assumption on the generalized Hessian \( \partial^2 \varphi \) of the cost functions. Furthermore, we do not assume that the quadratic matrix \( A \) in the composite optimization problem (7.7) is nonsingular. However, the nonsingularity of the shifted matrix \( I - \gamma A \) for some \( \gamma > 0 \) is required in the very construction of Algorithm 7.6. Note that we can always find a number \( \gamma \) such that \( I - \gamma A \) is positive definite, and thus it is also nonsingular due to the symmetry of the matrix \( A \) and the openness of the set of positive definite matrices.

The second remark discusses the comparison between our Algorithm 7.6 to solve (7.7) and a version of the semismooth Newton method in composite optimization.
Remark 7.11 (comparison with the semismooth Newton method in composite optimization). The thesis by Milzarek [37] deals with the composite minimization problem (7.1), where $f$ is twice continuously differentiable (possibly nonconvex), while the regularizer $g$ is a proper l.s.c. and convex function. The developed approach employs the semismooth Newton method to solve the equation

$$F^\Lambda(x) := x - \text{Prox}_g^\Lambda(x - \Lambda^{-1}\nabla f(x)) = 0,$$

where $\Lambda$ is a symmetric, real, and positive-definite $n \times n$ matrix, and where

$$\text{Prox}_g^\Lambda(x) := \arg\min \left\{ g(y) + \frac{1}{2\lambda} \langle \Lambda(x - y), x - y \rangle \mid y \in \mathbb{R}^n \right\}.$$  

(7.25)

The local superlinear convergence results are obtained in the thesis under the assumptions that the proximity operator (7.25) is semismooth and the generalized Jacobian of $F^\Lambda(x)$ is nonsingular. Meanwhile, our method addresses solving the subgradient inclusion $0 \in \partial \varphi(x)$ with $\varphi$ taken from (7.7), where the regularizer $g$ is merely prox-regular under the metric regularity and semismooth* assumption on the subgradient mapping; see Remark 5.10 for the related discussions.

8 Applications to Regularized Least Square Problems

In this section we provide applications of our main Algorithm 7.6 to solving two practical models arising in statistics, machine learning, image processing, etc. The first model is known as the Lasso problem (or as the $\ell^1$-regularized least square optimization problem). It was formulated by Tibshirani [61] for statistical applications. The second model we label here as the $\ell^{1,2}$-regularized least square optimization problem. Both of these problems can be described in the form

$$\text{minimize} \quad \varphi(x) := \frac{1}{2} \|Ax - b\|_2^2 + g(x), \quad x \in \mathbb{R}^n,$$

(8.1)

where $A$ is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and the regularizer $g: \mathbb{R}^n \to \mathbb{R}$ is a nonsmooth function, which is convex in the first problem and nonconvex in the second one. Rewriting these problems in the composite optimization form (7.7), we’ll constructively express in what follows all the elements and assumptions of Algorithm 7.6 entirely in terms of the given data for both problems under consideration.

Let is start with the Lasso problem formulated as

$$\text{minimize} \quad \varphi(x) := \frac{1}{2} \|Ax - b\|_2^2 + \mu \|x\|_1, \quad x \in \mathbb{R}^n,$$

(8.2)

where $A$ is an $m \times n$ matrix, $\mu > 0$, $b \in \mathbb{R}^m$, $\| \cdot \|_2$ and $\| \cdot \|_1$ stand for the standard Euclidean and sum norms, respectively. Note that (8.2) is a subclass of (8.1) with $g(x) := \mu \|x\|_1$ and

$$f(x) := \frac{1}{2} \langle \tilde{A}x, x \rangle + \langle \tilde{b}, x \rangle + \tilde{\alpha} \quad \text{with} \quad \tilde{A} := A^*A, \quad \tilde{b} := -A^*b, \quad \text{and} \quad \tilde{\alpha} := \frac{1}{2}\|b\|^2,$$

(8.3)

where the matrix $\tilde{A} = A^*A$ is always symmetric and positive-semidefinite. The Lasso problem (8.2) is convex and always admits an optimal solution, which is fully characterized by the subgradient inclusion

$$0 \in \partial \varphi(x) \text{ with } \varphi \text{ from (8.2)}.$$

(8.4)

To efficiently apply Algorithm 7.6 to solving the subgradient system (8.4), we have to provide explicit calculations of $\partial \varphi$, $\partial^2 g$, and Prox$_\gamma g$ via the initial data of (8.2). Here they are.
Proposition 8.1 (explicit calculations for Lasso). For problem (8.2), the following hold:

\[
(\text{Prox}_g(x))_i = \begin{cases} 
  x_i - \mu \gamma & \text{if } x_i > \mu \gamma, \\
  0 & \text{if } -\mu \gamma \leq x_i \leq \mu \gamma, \\
  x_i + \mu \gamma & \text{if } x_i < -\mu \gamma.
\end{cases}
\] 

(8.5)

where the set-valued mapping \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is computed by

\[
F(x) = \left\{ v \in \mathbb{R}^n \mid v_j = \text{sgn}(x_j), x_j \neq 0, \quad v_j \in [-1, 1], \quad x_j = 0 \right\}.
\] 

(8.7)

For each \((x, y) \in \text{gph } \partial g\) and \(v = (v_1, \ldots, v_n) \in \mathbb{R}^n\), we have

\[
\partial^2 g(x, y)(v) = \left\{ w \in \mathbb{R}^n \mid \left( \frac{1}{\mu} w_i, -v_i \right) \in G \left( x_i, \frac{1}{\mu} y_i \right), \quad i = 1, \ldots, n \right\},
\]

(8.8)

where the mapping \( G : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2 \) is given by

\[
G(t, p) := \begin{cases} 
  \{0\} \times \mathbb{R} & \text{if } t \neq 0, \quad p \in \{-1, 1\}, \\
  \mathbb{R} \times \{0\} & \text{if } t = 0, \quad p \in (-1, 1), \\
  (\mathbb{R}_+ \times \mathbb{R}_-) \cup \{0\} \times \mathbb{R} \cup (\mathbb{R} \times \{0\}) & \text{if } t = 0, \quad p = -1, \\
  (\mathbb{R}_- \times \mathbb{R}_+) \cup \{0\} \times \mathbb{R} \cup (\mathbb{R} \times \{0\}) & \text{if } t = 0, \quad p = 1, \\
  \emptyset & \text{otherwise.}
\end{cases}
\]

(8.9)

**Proof.** The formula for the proximal mapping (8.5) follows from definition (6.2). Proceeding with the subdifferential calculation, we clearly get \( \partial (\| \cdot \|_1)(x) = F(x) \) with \( F(x) \) is computed by (8.7), and thus

\[
\partial g(x) = \left\{ v \in \mathbb{R}^n \mid v_j = \text{sgn}(x_j), x_j \neq 0, \quad v_j \in [-\mu, \mu], \quad x_j = 0 \right\}
\]

whenever \( x \in \mathbb{R}^n \).

(8.10)

Then (8.6) follows from (8.10) and the subdifferential sum rule of convex analysis.

It remains to verify the second-order subdifferential formula (8.8) for \( g \) at \((x, y) \in \text{gph } \partial g\). Considering the auxiliary function \( \psi(t) := |t| \) on \( \mathbb{R} \), observe that

\[
\text{gph } \partial \psi = \left( (\mathbb{R} \setminus \{0\}) \times \{-1, 1\} \right) \cup \{0\} \times [-1, 1], \quad \text{and } \|x\|_1 = \sum_{i=1}^n \psi(x_i), \quad x \in \mathbb{R}^n.
\]

Furthermore, we see that \( N_{\text{gph } \partial \psi} = G \) for the mapping \( G \) defined in (8.9). It now follows from Mordukhovich and Outrata [46, Theorem 4.3] that

\[
\partial^2 (\| \cdot \|_1)(x, \frac{1}{\mu} y)(v) = \left\{ u \in \mathbb{R}^n \mid (u_i, -v_i) \in N_{\text{gph } \partial \psi} \left( x_i, \frac{1}{\mu} y_i \right) \right\},
\]

which therefore justifies the fulfillment of (8.8) and thus completes the proof of the proposition.

\[\square\]

Note that by applying to (8.2) the second-order subdifferential sum rule from Mordukhovich [43, Theorem 1.21] and using (8.8), we easily arrive at the calculation of \( \partial^2 \varphi \), which is not needed to run Algorithm 7.6 for (8.2). However, to run our algorithm, we have to explicitly determine
the sequences \( \{v^k\} \) and \( \{d^k\} \) taken from (7.17) and (7.18), respectively. For the case of \( v^k \), it follows from (8.5) that

\[
(v^k)_i = \begin{cases} 
(u^k)_i - \mu \gamma & \text{if } (u^k)_i > \mu \gamma, \\
0 & \text{if } -\mu \gamma \leq (u^k)_i \leq \mu \gamma, \\
(u^k)_i + \mu \gamma & \text{if } (u^k)_i < -\mu \gamma.
\end{cases}
\]

To find \( d^k \), we substitute into (7.18) the calculation of \( \partial^2 g \) from (8.8), which brings us to the conditions

\[
\begin{cases} 
(-\frac{1}{2}(x^k - v^k) - A^* A d^k)_i = 0 & \text{if } (v^k)_i \neq 0, \\
(x^k - v^k + d^k)_i = 0 & \text{if } (v^k)_i = 0.
\end{cases}
\]

Thus \( d^k \) can be computed by solving the linear equation \( X^k d = v^k - x^k \), where

\[
(X^k)_i := \begin{cases} 
\gamma(A^* A)_i & \text{if } (v^k)_i \neq 0, \\
I_i & \text{if } (v^k)_i = 0.
\end{cases}
\]

(8.11)

where \( (X^k)_i \), \( (A^* A)_i \), \( I_i \) are the \( i \)-th rows of the matrices \( X^k \), \( A^* A \), and \( I \), respectively.

The above calculations allow us to apply Algorithm 7.6 to solving the Lasso problem (8.2).

**Theorem 8.2 (solving Lasso).** Let \( \bar{x} \) be an optimal solution to (8.2) such that \( \partial \varphi \) is metrically regular around \((\bar{x}, 0)\). Then all the assumptions of Theorem 7.8 are satisfied at \( \bar{x} \). Thus Algorithm 7.6, with the parameter \( \gamma > 0 \) for which \( I - \gamma A^* A \) is nonsingular and the data \( \partial \varphi \), \( \partial^2 g \), and \( \text{Prox}_{\gamma g} \) explicitly computed in terms of \((A, b, \mu)\) in Proposition 8.1, is well-defined around \( \bar{x} \), and the sequence of its iterates \( \{x^k\} \) locally \( Q \)-superlinearly converges to \( \bar{x} \). Furthermore, \( \bar{x} \) is a tilt-stable local minimizer of \( \varphi \).

**Proof.** Observe first that the subdifferential mapping \( \partial \varphi \) enjoys the property of strong metric regularity around \((\bar{x}, 0)\). This follows from the convexity of \( \varphi \) and the result of Aragón Artacho and Geoffroy [1, Proposition 3.8]. Furthermore, we get from the formula for the mapping \( F \) in Proposition 8.1 that the graph of \( F \) is the union of finitely many closed convex polyhedra, and hence \( F \) is semismooth* on its graph. Since the function \( x \mapsto A^* (Ax - b) \) is continuously differentiable on \( \mathbb{R}^n \), it follows from the subdifferential formula (8.6) and Proposition 3.6 from Gfrerer and Outrata [19] that the mapping \( \partial \varphi \) is semismooth* at every point of its graph. Taking into account the solvability results of Section 4, we see that all the assumptions of Theorem 7.8 hold for the Lasso problem (8.2). Therefore, the solvability and convergence conclusions of the theorem follow from Theorem 7.8, where the obtained the specification of \( \gamma \) is due to the convexity of \( g \). Finally, the tilt-stability of \( \bar{x} \) is a consequence of Proposition 3.1 from Drusvyatskiy and Lewis [14].

The next remark provides an essential clarification of our assumptions.

**Remark 8.3 (positive-definiteness and metric regularity).** Note that when the matrix \( A^* A \) is positive-definite, the metric regularity of \( \partial \varphi \) around the reference point in Theorem 8.2 is automatically satisfied. Observe that the metric regularity assumption is weaker than the positive-definiteness of \( A^* A \). A simple class of functions \( \varphi \) illustrating this is given by \( \varphi = f + |x| \). In particular, for \( f \equiv 0 \) we have

\[
\partial^2 \varphi(0, 0)(v) = \{ w \in \mathbb{R} \mid (w, -v) \in \mathbb{R} \times \{0\} \},
\]

and thus \( \ker \partial^2 \varphi(0, 0) = \{0\} \), which tells us by (2.2) that \( \partial \varphi \) is metrically regular around \((0, 0)\). Some sufficient conditions for the metric regularity of the subdifferential operator of the objective function (8.2) can be found in the recent preprint by Berk et al. [3].
In the remainder of this section, we develop applications of Algorithm 7.6 to solving the subgradient inclusion $0 \in \partial \varphi(x)$ corresponding to the stationary condition for the nonsmooth and nonconvex composite optimization problem formulated as follows:

$$
\text{minimize } \varphi(x) := \frac{1}{2} \|Ax - b\|_2^2 + \mu_1 \|x\|_1 - \mu_2 \|x\|_2^2, \quad x \in \mathbb{R}^n, \quad (8.12)
$$

where $A$ is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $\mu_1, \mu_2 > 0$. We label (8.12) as the $\ell^{1,2}$-regularized least square optimization problem. It is clear that (8.12) is a subclass of (8.1) whose cost function is represented in the composite form $\varphi := f + g$, where

$$
f(x) := \frac{1}{2} \langle \tilde{A}x, x \rangle + \langle \tilde{b}, x \rangle + \tilde{\alpha} \quad \text{and} \quad g(x) := \mu_1 \|x\|_1 - \mu_2 \|x\|_2^2 \quad (8.13)
$$

with $\tilde{A} := A^*A$, $\tilde{b} := -A^*b$, and $\tilde{\alpha} := \frac{1}{2} \|b\|_2^2$, and where the matrix $\tilde{A} = A^*A$ is symmetric and positive-semidefinite. Note that the regularizer $g$ is nonsmooth and nonconvex, while being prox-bounded and continuously prox-regular on $\mathbb{R}^n$ with modulus $2\mu_2$ due to the convexity of the quadratically shifted function $g + \mu_2 \|\cdot\|_2^2$. This allows us to implement Algorithm 7.6 to solving the subgradient system $0 \in \partial \varphi(x)$, where $\varphi$ is taken from (8.12). To proceed, we provide explicit calculations of all the algorithm ingredients in terms of the initial data of problem (8.12).

**Proposition 8.4 (precise algorithmic calculations for $\ell^{1,2}$-regularizer).** For the cost function $\varphi$ in (8.12), we have the subdifferential formula

$$
\partial \varphi(x) = A^*(Ax - b) + \mu_1 F(x) - 2\mu_2 x \quad \text{whenever } x \in \mathbb{R}^n, \quad (8.14)
$$

where the set-valued mapping $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is taken from (8.7). The proximal mapping (6.2) associated with the regularizer $g(x) := \mu_1 \|x\|_1 - \mu_2 \|x\|_2^2$ is computed by

$$
(\text{Prox}_{\gamma g}(x))_i = \begin{cases} 
  x_i - \gamma \frac{\mu_1}{1 - 2\gamma \mu_2} & \text{if } x_i > \mu_1 \gamma, \\
  0 & \text{if } -\mu_1 \gamma \leq x_i \leq \mu_1 \gamma, \\
  x_i + \gamma \frac{\mu_1}{1 - 2\gamma \mu_2} & \text{if } x_i < -\mu_1 \gamma
\end{cases} \quad (8.15)
$$

provided that $\gamma \in \left(0, \frac{1}{2\mu_2}\right)$. For each $(x, y) \in \text{gph} \partial g$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, the generalized Hessian of $g$ is computed by the formula

$$
\partial^2 g(x, y)(v) = \left\{ w \in \mathbb{R}^n \mid \left( \frac{1}{\mu_1} (w_i + 2\mu_2 v_i), -v_i \right) \in G \left( x_i, \frac{1}{\mu_1} (y_i + 2\mu_2 x_i) \right), \quad i = 1, \ldots, n \right\}, \quad (8.16)
$$

where the mapping $G: \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is taken from (8.9).

**Proof.** The calculation of $\text{Prox}_{\gamma g}(x)$ in (8.15) follows from the definition. Also, we can easily get

$$
\partial g(x) = \mu_1 F(x) - 2\mu_2 x \quad \text{whenever } x \in \mathbb{R}^n, \quad (8.17)
$$

where $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined in (8.7). Thus the subdifferential formula (8.14) is a consequence of (8.17) and the standard first-order subdifferential sum rule. Arguing similarly to the proof of Proposition 8.1, we verify the second-order calculation formula (8.16) and thus complete the proof. \qed
The obtained calculations allow us to compute parameters \( v^k \) and \( d^k \) in Algorithm 7.6 to solve the stationary inclusion \( 0 \in \partial \varphi(x) \) for problem (8.12). Indeed, it follows from (8.15) that

\[
(v^k)_i = \begin{cases}
\frac{(u^k)_i - \gamma \mu_1}{1 - 2\gamma \mu_2} & \text{if } (u^k)_i > \mu_1 \gamma, \\
0 & \text{if } -\mu_1 \gamma \leq (u^k)_i \leq \mu_1 \gamma, \\
\frac{(u^k)_i + \gamma \mu_1}{1 - 2\gamma \mu_2} & \text{if } (u^k)_i < -\mu_1 \gamma,
\end{cases}
\]

Using further the formulas in (8.16) and (8.17), we get the relationships

\[
\begin{align*}
&\left\{ \left( -\frac{1}{\gamma}(x^k - v^k) - A^* A d^k + 2\mu_2(x^k - v^k + d^k) \right) \right\}_i = 0 & \text{if } (v^k)_i \neq 0, \\
&(x^k - v^k + d^k)_i = 0 & \text{if } (v^k)_i = 0
\end{align*}
\]

for \( d^k \) and hence find these directions by solving the equation \( X^k d = (1 - 2\gamma \mu_2)(v^k - x^k) \), where

\[
(X^k)_i := \begin{cases}
\gamma (A^* A)_i - 2\gamma \mu_2 I_i & \text{if } (v^k)_i \neq 0, \\
(1 - 2\gamma \mu_2) I_i & \text{if } (v^k)_i = 0.
\end{cases}
\]

Employing the above calculations in the framework of Theorem 7.8, we arrive at the following efficient computational procedure to solve problem (8.12) with justifying its excellent performance.

**Theorem 8.5.** Let \( \bar{x} \) be such that \( 0 \in \partial \varphi(\bar{x}) \) with \( \varphi \) defined in (8.12), and let \( \partial \varphi \) be metrically regular around \( (\bar{x}, 0) \). Then all the assumptions of Theorem 7.8 are satisfied for Algorithm 7.6 with the parameter \( \gamma \in \left( 0, \frac{1}{\mu_2} \right) \) such that \( I - \gamma A^* A \) is nonsingular and the data \( \partial \varphi, \partial^2 g, \) and Prox\(_{\gamma g}\) explicitly computed in terms of \((A, b, \mu)\) by the formulas in Proposition 8.4. Thus Algorithm 7.6 is well-defined around \( \bar{x} \) for this problem, and the sequence of its iterates \( \{x^k\} \) \( Q \)-superlinearly converges to \( \bar{x} \).

**Proof.** According to Theorem 4.4, the second-order subdifferential inclusion (3.2) is robustly solvable around \((\bar{x}, 0)\) for \( \varphi \) from (8.12) provided that the mapping \( \partial \varphi \) is metrically regular around \((\bar{x}, 0)\) as assumed, and if \( \varphi \) and twice epi-differentiable at any points near \( \bar{x} \). Let us check the fulfillment of the latter property on the whole space \( \mathbb{R}^n \). Indeed, observe that \( \varphi(x) = \varphi_1 + \varphi_2 \), where

\[
\varphi_1(x) := \frac{1}{2} \|Ax - b\|^2 - \mu_2 \|x\|_2^2 \quad \text{and} \quad \varphi_2(x) := \mu_1 \|x\|_1.
\]

Since \( \varphi_2 \) is a proper, convex, and piecewise linear-quadratic function on \( \mathbb{R}^n \), it follows from Rockafellar and Wets [57, Proposition 13.9] that \( \varphi_2 \) is twice epi-differentiable on \( \mathbb{R}^n \). Furthermore, the function \( \varphi_1 \) also enjoys this property on \( \mathbb{R}^n \), since it is \( C^2 \)-smooth of \( \varphi_1 \); see Exercise 13.18 in the aforementioned book [57]. This clearly verifies the twice epi-differentiability of the sum function \( \varphi \).

It remains to check that the mapping \( \partial \varphi \) is semismooth\(^* \) at the reference point. But it can be done similarly to the proof of Theorem 8.2 due to the subdifferential expression in (8.14). \( \square \)

**9 Concluding Remarks**

This paper proposes and develops a generalized Newton method to solve gradient and subgradient systems by using second-order subdifferentials of \( C^{1,1} \) and extended-real-valued prox-regular functions, respectively, together with the appropriate tools of second-order variational analysis. The suggested algorithms for gradient and subgradient systems are comprehensively investigated.
with establishing verifiable conditions for their well-posedness/solvability and local superlinear convergence. Applications to solving important classes of nonsmooth regularized least square problems are given with complete computations of algorithm ingredients and assumption verifications.

The results of this paper concern far-going nonsmooth extensions of the basic Newton method for $C^2$-smooth functions. Besides further implementations of the obtained results and their applications to practical models, in our future research we intend to develop other, more advanced versions of nonsmooth second-order algorithms of the Newton and quasi-Newton types with establishing their local and global convergence as well as efficient specifications for broad classes of variational inequalities and constrained optimization problems.

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