Certified Robustness via Randomized Smoothing over Multiplicative Parameters

Nikita Muravev  
Lomonosov Moscow State University  
Huawei Moscow Research Center  
muravev.nikita@huawei.com

Aleksandr Petiushko  
Lomonosov Moscow State University  
Huawei Moscow Research Center  
petyushko.alexander1@huawei.com

Abstract

We propose a novel approach of randomized smoothing over multiplicative parameters. Using this method we construct certifiably robust classifiers with respect to a gamma-correction perturbation and compare the result with classifiers obtained via Gaussian smoothing. To the best of our knowledge it is the first work concerning certified robustness against the multiplicative gamma-correction transformation.

1 Introduction

It is well known [2, 22] that modern classification models are vulnerable to adversarial attacks. There are many attempts [3, 10, 14, 17, 18] to construct empirically robust classifiers, but all of them are proven to be vulnerable to some new more powerful attacks [1, 4, 23]. As a result, a lot of certification methods are introduced [9, 19, 20, 21, 24] that provide provably correct robustness certificates. Among this variety of certification methods the randomized smoothing remains one of the most effective and feasible approaches which is easy to implement and to parallel. Randomized smoothing can be applied to data and classifiers of any nature since it does not use any information about the internal structure of the classifier. It is also easily scalable to large networks unlike other methods. All these benefits lead to the fact that the randomized smoothing approach is currently one of the most popular and powerful solutions to robustness certification tasks.

Consequently, there are a lot of works devoted to the randomized smoothing. While most of them concern a setting of robustness within $l_p$- or $l_{\infty}$-balls, some generalize the approach to cover parameterized transformations [9]. These parameters usually add up to each other when composing transformations, which allows to smooth classifiers over them. Yet there are some other types of perturbations such that their parameters do not add up but multiply to each other (e.g. a volume change of audio signals or a gamma-correction of images). Though it is possible to reduce this case to the additive one by a logarithm (like it is done in [8]), we find that alternative distributions may result in better certificates. Thus we propose a new approach of smoothing directly over multiplicative parameters. We prove the theoretical guarantees for classifiers smoothed with Rayleigh distribution and compare them with those obtained via Gaussian smoothing for the gamma-correction perturbation.

The main contributions of this work are:

- Direct generalization of the randomized smoothing method to the case of multiplicative parameters.
- Practical comparison of the proposed method and the Gaussian smoothing for the gamma-correction perturbation, which shows superiority of our method for some factors.
- Construction of the first certifiably robust classifiers for the gamma-correction perturbation.
2 Background

The randomized smoothing is a method of replacement of an original classifier \( f \) with its “smoothed” version \( g(x) = \arg \max_c \mathbb{P}_\beta(f \circ \psi(x) = c) \) that returns the class the base classifier \( f \) is most likely to return when an instance is perturbed by some randomly distributed composable transformation \( \psi \). A right choice of the distribution \( \beta \) allows to predict a robustness certificate for a given input \( x \) for this smoothed classifier \( g \) using its confidence in its own answer. Here a robustness certificate is a range of transformations such that they do not change the predicted class \( g(x) \) when applied to the given input \( x \). One expects the new (smoothed) classifier to be reasonably good if the base one is sufficiently robust.

This technique has only two major drawbacks. The first one is that the transformation we want out classifier to be robust against has to be composable. It is a strong constraint since even theoretically composable transformations lose this property in practice due to rounding and interpolation errors. The other one is that generally we are unable to calculate the smoothed classifier \( g \) directly and instead have to use approximations. Since we cannot calculate the actual prediction of \( g \), we use the Monte-Carlo approximation algorithm with \( n \) samples to obtain the result with an arbitrary high level of confidence. The introduced parameter \( n \) controls the trade-off between the approximation accuracy and the evaluation time on inference. Though it is possible to smooth a default undefended base classifier \( f \), experiments show that the best results can be achieved by smoothing of already (empirically) robust base classifier \( f \). Thus during training of the base classifier \( f \) we use the same type of augmentation we expect the smoothed classifier \( g \) to be robust against. For a more detailed description of the randomized smoothing method see \([6, 16, 15]\).

3 Generalization of smoothing for multiplicative parameters

Let us extend the notion of composable transformations in case of multiplicative parameters.

Definition 3.1. A parameterized map \( \psi_\delta : X \to X, \ \delta \in \mathcal{B} \subset \mathbb{R}^n \) is called multiplicatively composable if

\[
(\psi_\delta \circ \psi_\theta)(x) = \psi_{(\delta \cdot \theta)}(x), \ \forall x \in X, \ \forall \delta, \theta \in \mathcal{B},
\]

where \( \delta \cdot \theta \) means the element-wise multiplication of vectors.

Usually multiplicative parameters are positive, thus one needs probability distributions with positive supports for randomized smoothing (it is not the case for Gaussian distribution). We propose a Rayleigh distribution for these needs.

Definition 3.2. A random variable \( \zeta \) has a Rayleigh distribution with the scale parameter \( \sigma > 0 \) (\( \zeta \sim \text{Rayleigh}(\sigma) \)) if its probability density function (PDF) has a form

\[
p_\zeta(z) = \sigma^{-2} z e^{-z^2/(2\sigma^2)}, \ z \geq 0.
\]

For classifiers smoothed with the Rayleigh distribution one obtains the following robustness guarantee:

Theorem 3.3. Let \( x \in \mathbb{R}^m, \ f : \mathbb{R}^m \to Y \) be a classifier, \( \psi_\beta : \mathbb{R}^m \to \mathbb{R}^m \) be a multiplicatively composable transformation for \( \beta \sim \text{Rayleigh}(\sigma) \) and \( g(x) = \arg \max_c \mathbb{P}_\beta(f \circ \psi_\beta(x) = c) \). Denote

\[
p_A = \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_A)
\]

\[
p_B = \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_B).
\]

If

\[
p_A \geq p_A > p_B \geq p_B,
\]

then \( g \circ \psi_\gamma(x) = c_A \) for all \( \gamma \) satisfying \( \gamma_1 < \gamma < \gamma_2 \), where \( \gamma_1, \gamma_2 \) are the only solutions of the following equations

\[
F(\gamma_1^{-1} F^{-1}(p_A)) + F(\gamma_1^{-1} F^{-1}(1 - p_A)) = 1,
\]

\[
F(\gamma_2^{-1} F^{-1}(p_A)) + F(\gamma_2^{-1} F^{-1}(1 - p_A)) = 1,
\]

and \( F(z) = 1 - e^{-z^2/(2\sigma^2)} \) is the CDF of \( \beta \).

A proof is similar to the one provided in \([8]\), though it is also possible to obtain the same result using the constrained adversarial certification framework \([25]\). Both variants are given in the supplementary materials. Notice that the certificate bounds \( \gamma_1, \gamma_2 \) can be easily found.
numerically (e.g. by a simple bisection method). But it is also possible to use a trivial estimation $p_B = 1 - p_A$ for the probability of the top two class. In that case the equations can be solved analytically:

$$
\gamma_1 = \frac{F^{-1}(1 - p_A)}{F^{-1}(1/2)}, \quad \gamma_2 = \frac{F^{-1}(p_A)}{F^{-1}(1/2)}.
$$

The only parameter to be chosen for the Rayleigh distribution is the scale $\sigma$. It seems reasonable to choose it in such a way that the median of a random value equals one (value multiplied by 1 stays the same). Thus hereafter the scale is equal to $1 / \sqrt{2 \ln(2)}$.

The above theorem can be generalized in case of transformations with multiple multiplicative parameters.

**Theorem 3.4.** Let $x \in \mathbb{R}^m$, $f : \mathbb{R}^m \to Y$ be a classifier, $\psi_\beta : \mathbb{R}^m \to \mathbb{R}^m$, $\beta = (\beta_1, \ldots, \beta_n)^T$ be a multiplicatively composable transformation for independent and identically distributed random variables $\beta_i \sim \text{Rayleigh}(\sigma)$ and $g(x) = \arg \max_c \mathbb{P}_\beta(f \circ \psi_\beta(x) = c)$. Denote

$$
p_A = \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_A)
$$

$$
p_B = \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_B).
$$

If

$$
p_A \geq p_B > p_B \geq p_B,
$$

then $g \circ \psi_\gamma(x) = c_A$ for all $\gamma \in \Omega$, where $\Omega$ is a region defined by the following inequality

$$
\mathbb{P}((\gamma_1^2 - 1) \beta_1^2 + \ldots + (\gamma_n^2 - 1) \beta_n^2 \leq r) > \mathbb{P}((\gamma_1^2 - 1) \beta_1^2 + \ldots + (\gamma_n^2 - 1) \beta_n^2 \geq \theta),
$$

where $r$ and $\theta$ are the only solutions of the following equations

$$
\mathbb{P}((1 - \gamma_1^{-2}) \beta_1^2 + \ldots + (1 - \gamma_n^{-2}) \beta_n^2 \leq r) = p_A,
$$

$$
\mathbb{P}((1 - \gamma_1^{-2}) \beta_1^2 + \ldots + (1 - \gamma_n^{-2}) \beta_n^2 \geq \theta) = p_B.
$$

A proof is analogous to the one for a single parameter case and can be found in the supplementary materials.

Usually we cannot evaluate the probabilities $p_A, p_B$. Thus we use the Clopper-Pearson [5] bounds $p_B \leq P_B < p_A \leq P_A$ that can be calculated with an arbitrary high confidence probability $1 - \alpha$. In case $p_A \leq 1/2$ the classifier $g$ abstains from answering (i.e. returns the “abstain” answer).

We present $\gamma_1, \gamma_2$ values for some $p_A, p_B$ in Table 1 to show what certificates can possibly be achieved.

### 4 Experiments

There are a lot of multiplicatively composable transformations. We choose the gamma-correction of images for our experiments due to its simplicity. If we consider an input image in the RGB-format as a tensor $x$ with entries in $[0, 1]$, then the gamma-correction $G_\gamma$ with the gamma factor $\gamma$ simply raises $x$ to the power $\gamma$ in the element-wise manner: $G_\gamma(x) = x^\gamma$. For a human eye it looks like change of brightness (Fig. 1). Obviously, this transformation is multiplicatively composable:

$$
G_\beta \circ G_\gamma(x) = (x^\gamma)^\beta = x^{\gamma \beta} = G_{\gamma \beta}(x).
$$

But it should be noticed that in reality a gamma-corrected image is likely to be converted to some image format with colour channel of limited width afterwards. In that case some information is lost and the resulting transformation is no longer composable. Thus we have two settings: 1) “idealized” — when the colour channel is so wide that the conversion error is negligible in comparison with the rounding error of a machine; 2) “realistic” — when the conversion error is significant and cannot be ignored.

| $p_A$ | $p_B$ | $\gamma_1$ | $\gamma_2$ |
|-------|-------|------------|------------|
| 0.6   | 0.4   | 0.86       | 1.15       |
| 0.6   | 0.2   | 0.71       | 1.33       |
| 0.7   | 0.3   | 0.72       | 1.32       |
| 0.7   | 0.1   | 0.54       | 1.56       |
| 0.8   | 0.2   | 0.57       | 1.52       |
| 0.9   | 0.1   | 0.39       | 1.82       |
| 0.999 | 0.001 | 0.12       | 2.58       |
| 0.999 | 0.001 | 0.04       | 3.16       |

Table 1: Calculated robustness certificates $(\gamma_1, \gamma_2)$ for the top two class probabilities $p_A, p_B$. 
4.1 Idealized setting

As it is mentioned before the multiplicative parameters $\beta, \gamma$ can be converted into the additive ones $a = \log_c \gamma$, $b = \log_c \beta$ via logarithm:

$$\gamma \cdot \beta = c^{\log_c \gamma} \cdot c^{\log_c \beta} = c^a \cdot c^b = c^{a+b},$$  \hfill (10)

where $c$ is some fixed base. Thus one can smooth over these new parameters with the standard Gaussian distribution to obtain certificates on the original ones. The certified accuracy for a factor $\gamma$ is the proportion of correctly classified test images whose certificates contain the value $\gamma$. We present the results for ResNet-20 on CIFAR-10 \cite{cifar10} for some bases $c$ and deviations $\sigma$ in Fig.2. All certificates are obtained with the mistake probability $\alpha = 0.001$. Our study shows that there are no base $c$ and standard deviation $\sigma$ such that the Gaussian smoothing over $a = \log_c \gamma$ consistently outperforms the Rayleigh smoothing over $\gamma$, where $\gamma$ is the gamma-correction factor. But it must be noticed that the Gaussian smoothing is more flexible since one can vary both the base and the deviation.

We also compare the theoretical certificates for the smoothed classifier with empirical ones for: the same classifier (real smoothed classifier), the base architecture without augmentations on training (undefended classifier) and the base architecture with augmentations on training (augmented classifier) (Fig.3). These empirical certificates are obtained via evaluations of classifiers’ predictions on perturbed images. For every image $x$ starting from gamma-factor $\gamma_0 = 1$ we evaluate classifier’s predictions on $G_{\gamma_k}(x)$ for every $\gamma_k = \gamma_{k-1} + \epsilon$ with a step $\epsilon = 0.01$. If the classifier predicts a wrong label on the input $G_{\gamma_k}(x)$ for some $\gamma_k$, we assume that it is robust only in the interval $[1, \gamma_{k-1}]$ and stop the certification procedure. The same procedure is used to find the left end of a certification interval. We use $\epsilon = 0.1$ searching for the right ends of certificate intervals for some classifiers (you can spot them by the less smooth right parts of corresponding graphs) due to computational complexity of this process. The resulting certificate can be over optimistic but still it provides a good approximation of the actual robustness. One can see that the real accuracy is bounded.
from below by the theoretically guaranteed one, which is expected. We also observe that the smoothed classifier has almost the same actual robustness as the just augmented one. Thus it can be concluded that smoothing provides theoretical certificates at a cost of extra time on inference but does not significantly impact the actual robustness of the augmented base classifier.

The randomized smoothing method is easily scalable to deep models and large datasets. We repeat these experiments for ResNet-50 on ImageNet [7] (Fig.4), but do not approximate actual robustness due to computational complexity of this process.

4.2 Realistic setting

Our study shows that if we convert gamma-corrected images to some format with 8 bits per colour channel (RGB true colour), then the conversion error is usually too big to be ignored. Thus we need to make the base classifier $f$ robust in some $l_2$-ball with a radius larger then the conversion error for the most images. For these needs Fischer et al. [8] propose to initially smooth the base classifier with the additive Gaussian noise to make it robust in $l_2$-norm. This new classifier can then be used as a base one for smoothing over transformation parameters. Thus instead of $n$ samples for the Monte-Carlo simulation algorithm we use $n_\varepsilon$ samples to simulate the smoothing in $l_2$-norm and $n_\gamma$ samples to simulate the smoothing over the gamma-correction. Since we need $n_\varepsilon$ samples for every gamma-correction sample, the resulting number of samples is equal to $n_\varepsilon \cdot n_\gamma$.

There are two types of guarantees that can be obtained by this method: 1) a distributional guarantee, when the conversion error is pre-calculated on the training dataset; 2) an individual guarantee, when we calculate the conversion error for each input at inference time.

For a given input image $x$ and the gamma-correction transformation $G$ we define

$$\varepsilon(\beta, \gamma, x) := G_{\beta} \circ G_{\gamma}(x) - G_{\beta, \gamma}(x)$$  \hspace{1cm} (11)

as the conversion error for gamma-factors $\beta, \gamma$. In the idealized setting we assume $\varepsilon(\beta, \gamma, x) = 0$ for all inputs. But now we need to estimate this value with some upper bound $E$.

In case of distributional guarantees the conversion error can be estimated with $E$ such that

$$q_E = \mathbb{P}_{x \sim D, \beta \sim Rayleigh} \left( \max_{\gamma \in \Gamma} \| \varepsilon(\beta, \gamma, x) \|_2 \leq E \right),$$  \hspace{1cm} (12)

where $D$ is the data distribution, $\Gamma$ is a fixed interval of possible attacks and $q_E$ is a desirable error rate.

![Figure 3: Comparison of theoretically predicted certificates to empirically obtained ones for ResNet-20 on CIFAR-10 for the gamma-correction perturbation.](image1)

![Figure 4: Comparison of theoretically predicted certificates to empirically obtained ones for ResNet-50 on ImageNet for the gamma-correction perturbation.](image2)
The conversion errors \( E \) are estimated for intervals \( \Gamma \) of gamma-factors \([0.86, 1.15]\) and \([0.71, 1.33]\) with the error rate \( q_E = 0.9 \) as 0.18 and 0.22 respectively. These intervals are chosen randomly but in such a way that they could be certificate intervals obtained with the Rayleigh smoothing for some input images, i.e. there exist such \( p_A, p_B \) that the corresponding certificates obtained via Theorem 3.3 equal to \([0.86, 1.15]\) or \([0.71, 1.33]\). Then we smooth the base classifier with the Gaussian noise with a deviation \( \sigma = 0.25 \), that is found experimentally to deliver the best results for our base models and datasets.

The resulting mistake probability \( \rho \) of that smoothed classifier on a ball \( B_\Gamma(x) \) for a given \( x \) can be estimated as the sum of mistake probabilities on all steps:

\[
\rho \leq \alpha + 1 - q_E + \alpha_E,
\]

where \( 1 - \alpha \) is the confidence of the base classifier, \( q_E \) is the probabilistic guarantee for \( E \) and \( 1 - \alpha_E \) is the confidence with which \( E \) is obtained.

This smoothed classifier is then smoothed with the Rayleigh distributed gamma-correction. For the certification procedure the probabilities \( p_A, p_B \) are adjusted by \( \rho \):

\[
p_A' = p_A - \rho, \quad p_B' = p_B + \rho
\]

such that we take into consideration the mistake probability of the base classifier we have estimated previously. In order to preserve the correctness of conversion error estimations the obtained certificates are clipped to the selected attack intervals \( \Gamma \).

We provide distributional guarantees for smoothed classifiers on CIFAR-10 for different numbers of samples \( n_e, n_\gamma \), mistake probabilities \( \alpha \) and attack intervals \( \Gamma \) (Fig. 5). A significant drop in accuracy can be seen when we switch from the idealized setting to the realistic one. It happens because we are not able to preserve the same number of samples for smoothing over the gamma parameter (due to the computational complexity) as well as because our base classifier has to be robust in \( l_2 \)-norm.

One can see that the smaller attack interval allows the lower estimation \( E \) (which is expected) and thus the higher accuracy can be obtained. It should also be noticed that a greater mistake probability \( \alpha \) increases the accuracy due to fewer "abstain" answers by both smoothed classifiers, but at the same time a greater \( \alpha \) decreases the resulting confidence with which certificates are obtained.

We combine the achieved certification results in one table (2).

| Dataset    | Setting    | \( \gamma_1 \) | \( \gamma_2 \) | \( n_e \) | \( n_\gamma \) | \( \alpha \) | Acc  |
|------------|------------|----------------|---------------|--------|--------------|--------|------|
| CIFAR-10   | Idealized  | 0.154          | 2.44          | -      | 1000         | 0.001  | 0.81 |
| CIFAR-10   | Realistic  | 0.71           | 1.33          | 50     | 40           | 0.001  | 0.13 |
| CIFAR-10   | Realistic  | 0.71           | 1.33          | 50     | 40           | 0.01   | 0.2  |
| CIFAR-10   | Realistic  | 0.86           | 1.15          | 50     | 20           | 0.001  | 0.18 |
| CIFAR-10   | Realistic  | 0.86           | 1.15          | 50     | 40           | 0.01   | 0.33 |
| ImageNet   | Idealized  | 0.1            | 2.68          | -      | 1000         | 0.001  | 0.63 |

Table 2: Certification results for smoothed classifiers. Here “Acc” means the portion of images that were correctly classified and whose certificates contain an attack interval \( \Gamma = (\gamma_1, \gamma_2) \).
4.3 Experiments with scaling

The other type of transformations we experiment with is an image scaling. By scaling $S_r(x)$ we mean a change of image $x$ resolution with factor $r$ followed by an interpolation to the original size. For a human eye it looks like a loss of clarity (Fig. 6). The resulting interpolation error is a perturbation we want our classifier to be robust against. It is clear that in the idealized setting scale factors multiply to each other when we scale an image several times and in that case the interpolation error does not exist. Thus this transformation is multiplicatively composable. But in reality we face non-zero interpolation errors that cannot be ignored. Though it is possible to handle this problem via double smoothing like it is done with the gamma-correction, we find that the average interpolation error on CIFAR-10 images is too great to be handled via Gaussian smoothing with satisfactory accuracy. Fischer et al. [8] encounter the same problem dealing with rotations. They propose several techniques to reduce interpolation errors (preprocessing with vignetting and low-pass filter, noise partitioning). Some of these techniques can be applied to our case but we find that the problem persists even for high-resolution images (e.g. ImageNet) that is not the case for rotations.

We report results obtained with the single Rayleigh smoothing (without initial Gaussian smoothing in $l_2$-norm) of ResNet-20 on CIFAR-10 for the scaling perturbation (Fig. 7). One can see that the graph for theoretically predicted certificates does not lie below the graph for empirically obtained ones. It means that we cannot ignore interpolation errors because they significantly impact the base classifier’s predictions. Therefore the obtained classifier is not certifiably robust. Still these graphs show that smoothed classifiers can be considered as the empirical defences even when transformations do not truly compose.

It should be noticed that downscaling presents a greater challenge than upscaling since the latter is theoretically invertible and results in smaller interpolation errors. Therefore it seems reasonable to use non-symmetric (with respect to multiplication) distributions for smoothing like the Rayleigh one, which has a distinct bias to-
wards smaller values rather than bigger ones. One can expect such distributions to provide better certification results than the standard Gaussian and other symmetric distributions for perturbations whose symmetric parameter values do not cause the same level of distortion.

5 Further research

Recently, there are some works revealing the limitations of the standard Gaussian smoothing \cite{4,24,25}, especially in case of multidimensional perturbations with $l_p$-ball certificates for large $p$. It seems reasonable to search for alternative smoothing distributions not only among those suitable for additive parameters but also among distributions of multiplicative parameters. Indeed, it is interesting to find a good family of distributions for smoothing over multiplicative parameters, so that one could vary the distribution like in additive case with the Gaussian smoothing. There is a chance that it will allow to avoid the pitfalls and limitations of the Gaussian smoothing and achieve better results in the robustness certification tasks.

Also this technique can be applied to other types of perturbations and even to other types of data (audio signals, video, etc.). Therefore further experiments can include construction of certifiably robust audio and video classifiers based on the Rayleigh smoothing and comparison of them with those obtained via other methods.

6 Conclusion

This work proposes a novel approach of randomized smoothing directly over multiplicative parameters. We provide the experimental comparison of this method with the standard Gaussian smoothing for gamma-correction transformations and show that it cannot be outperformed by the latter. As a result, the first certifiably robust image classifiers in the idealized and realistic settings are constructed for the gamma-correction perturbation. The method is also implemented for scaling transformations but with much worse results due to huge interpolation errors.

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7 Appendix

7.1 Proofs

Here we prove the main theorems that we stated in the work.

**Theorem 7.1.** Let \( x \in \mathbb{R}^m, f : \mathbb{R}^m \to Y \) be a classifier, \( \psi_\beta : \mathbb{R}^m \to \mathbb{R}^m \) be a multiplicatively composable transformation for \( \beta \sim Rayleigh(\sigma) \) and \( g(x) = \arg \max_c \mathbb{P}_\beta(f \circ \psi_\beta(x) = c) \). Denote

\[
\begin{align*}
    p_A &= \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_A), \\
    p_B &= \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_B).
\end{align*}
\]

If

\[
p_A \geq p_A > p_B \geq p_B,
\]

then \( g \circ \psi_\gamma(x) = c_A \) for all \( \gamma \) satisfying \( \gamma_1 < \gamma < \gamma_2 \), where \( \gamma_1, \gamma_2 \) are the only solutions of the following equations

\[
\begin{align*}
    F(\gamma_1^{-1} F^{-1}(p_B)) + F(\gamma_1^{-1} F^{-1}(1 - p_A)) &= 1, \\
    F(\gamma_2^{-1} F^{-1}(p_A)) + F(\gamma_2^{-1} F^{-1}(1 - p_B)) &= 1, \\
    F(z) &= 1 - e^{-z^2/(2\sigma^2)} \text{ is the CDF of } \beta.
\end{align*}
\]

The proof is analogous to the one in [8]. We try to preserve its structure and used abbreviations.

**Proof.** Let us show that

\[
\begin{align*}
    \mathbb{P}_\beta(f \circ \psi_\beta_\gamma(x) = c_A) \overset{(1)}{=} \mathbb{P}_\beta(\beta \gamma \in A) \overset{(2)}{>} \mathbb{P}_\beta(\beta \gamma \in B) \overset{(3)}{=} \mathbb{P}_\beta(f \circ \psi_\beta_\gamma(x) = c_B)
\end{align*}
\]

for all \( \gamma_1 < \gamma < \gamma_2 \), where

\[
A = \{ z \mid z \leq F^{-1}(p_A) \}, \quad B = \{ z \mid z \geq F^{-1}(1 - p_B) \}.
\]

Then the required statement for \( \gamma \geq 1 \) follows directly from these inequalities.

1) Assume \( p_\zeta(z) \) is the PDF of random variable \( \zeta \). Then

\[
\begin{align*}
    &\mathbb{P}(f \circ \psi_\beta_\gamma(x) = c_A) - \mathbb{P}(\beta \gamma \in A) \\
    &= \int_{\mathbb{R}} [f \circ \psi_\zeta(x) = c_A] p_{\beta \gamma}(z) dz - \int_A p_{\beta \gamma}(z) dz \\
    &= \left( \int_{\mathbb{R} \setminus A} [f \circ \psi_\zeta(x) = c_A] p_{\beta \gamma}(z) dz + \int_A [f \circ \psi_\zeta(x) = c_A] p_{\beta \gamma}(z) dz \right) \\
    &\quad - \left( \int_A [f \circ \psi_\zeta(x) = c_A] p_{\beta \gamma}(z) dz + \int_A [f \circ \psi_\zeta(x) \neq c_A] p_{\beta \gamma}(z) dz \right)
\end{align*}
\]
\[
\int_{\mathbb{R} \setminus A} [f \circ \psi_z(x) = c_A] p_{\beta \gamma}(z) \, dz - \int_{A} [f \circ \psi_z(x) \neq c_A] p_{\beta}(z) \, dz
\]

**Lemma 7.2**

\[
\frac{p_{\beta \gamma}(z)}{p_{\beta}(z)} = \gamma^{-2} \cdot \exp \left( -\frac{z^2}{2\sigma^2 \gamma^2} + \frac{z^2}{2\sigma^2} \right) = \gamma^{-2} \cdot \exp \left( \frac{z^2(\gamma^2 - 1)}{2\sigma^2 \gamma^2} \right) \leq t
\]

\(\iff\)

\[
\frac{z^2(\gamma^2 - 1)}{2\sigma^2 \gamma^2} \leq \ln(t\gamma^2)
\]

\(\iff\)

\[
(\gamma^2 - 1)z^2 \leq 2\sigma^2 \gamma^2 \ln(t\gamma^2)
\]

\(\iff\)

\[
z \leq \frac{\sigma \gamma \sqrt{2(\gamma^2 - 1) \ln(t\gamma^2)}}{\sqrt{\gamma^2 - 1}}.
\]
What is the lowest $t$ such that $\frac{p_A(z)}{p_B(z)} \leq t$ for all $z \in A$?

Because of $A = \{z \mid z \leq F^{-1}(p_A)\}$ we know that $z \leq F^{-1}(p_A)$. Does there exist such a threshold $t$ that both upper bounds coincide? Yes, namely

$$t = \gamma^{-2} \exp \left( \frac{(\gamma^2 - 1)(F^{-1}(p_A))^2}{2\sigma^2 \gamma^2} \right).$$

The theorem can be generalized to the case of transformations with multiple multiplicative parameters.

**Theorem 7.3.** Let $x \in \mathbb{R}^m$, $f : \mathbb{R}^m \to Y$ be a classifier, $\psi_\beta : \mathbb{R}^m \to \mathbb{R}^m$, $\beta = (\beta_1, \ldots, \beta_n)^T$ be a multiplicatively composable transformation for independent random variables $\beta_i \sim \text{Rayleigh}(\sigma)$ and $g(x) = \arg \max_c \mathbb{P}_\beta(f \circ \psi_\beta(x) = c)$ Denote

$$p_A = \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_A),$$

$$p_B = \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_B).$$

If

$$p_A \geq p_B > p_B \geq p_B,$$

then $g \circ \psi_\gamma(x) = c_A$ for all $\gamma \in \Omega$, where $\Omega$ is a region defined by the following inequality

$$\mathbb{P}((\gamma_1^2 - 1)\beta_1^2 + \ldots + (\gamma_n^2 - 1)\beta_n^2 \leq r) > \mathbb{P}((\gamma_1^2 - 1)\beta_1^2 + \ldots + (\gamma_n^2 - 1)\beta_n^2 \geq \theta),$$

where $r$ and $\theta$ are the only solutions of the following equations

$$\mathbb{P}((1 - \gamma_1^{-2})\beta_1^2 + \ldots + (1 - \gamma_n^{-2})\beta_n^2 \leq r) = \frac{p_A}{2},$$

$$\mathbb{P}((1 - \gamma_1^{-2})\beta_1^2 + \ldots + (1 - \gamma_n^{-2})\beta_n^2 \geq \theta) = \frac{p_B}{2}.$$ 

**Proof.** Let us show that

$$\mathbb{P}_\beta(f \circ \psi_\beta(x) = c_A) \geq \mathbb{P}_\beta(\beta_\gamma \in A) \geq \mathbb{P}_\beta(\beta_\gamma \in B) \geq \mathbb{P}_\beta(f \circ \psi_\beta(x) = c_B)$$

for all $\gamma \in \Omega$, where

$$A = \{(z_1, \ldots, z_n) \mid (1 - \gamma_1^{-2})z_1^2 + \ldots + (1 - \gamma_n^{-2})z_n^2 \leq r\},$$

$$B = \{(z_1, \ldots, z_n) \mid (1 - \gamma_1^{-2})z_1^2 + \ldots + (1 - \gamma_n^{-2})z_n^2 \geq \theta\}.$$ 

Then the required statement follows directly from these inequalities.

1) Assume $p_\zeta(z)$ is the PDF of a random variable $\zeta$. Then

$$\mathbb{P}(f \circ \psi_\beta(x) = c_A) - \mathbb{P}(\beta_\gamma \in A) = \int_{\mathbb{R}^n} [f \circ \psi_\beta(x) = c_A]p_\beta_\gamma(z)dz - \int_A p_\beta_\gamma(z)dz$$

$$= \left( \int_{\mathbb{R}^n \setminus A} [f \circ \psi_\beta(x) = c_A]p_\beta_\gamma(z)dz + \int_A [f \circ \psi_\beta(x) = c_A]p_\beta_\gamma(z)dz \right)$$
Proof.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$.

1) The inequality

$$
\int_{\mathbb{R}^n \setminus A} [f \circ \psi_z(x) = c_A] p_{\gamma}(z) dz = \int_{\mathbb{R}^n \setminus A} [f \circ \psi_z(x) \neq c_A] p_{\gamma}(z) dz
$$

2) The inequality

$$
P((\gamma_1^2 - 1) \beta_1^2 + \ldots + (\gamma_n^2 - 1) \beta_n^2 \leq r) = P(\beta \gamma \in A)$$

holds for all $\gamma \in \Omega$ by the definitions of $A$, $B$ and $\Omega$.

3) The proof is analogous to the proof for (1).

Lemma 7.4. There exists $t > 0$ such that $p_{\beta,\gamma}(z) \leq t \cdot p_{\beta}(z)$ for all $z \in A$. And further $p_{\beta,\gamma}(z) > t \cdot p_{\beta}(z)$ for all $a \in \mathbb{R}^n \setminus A$.

Proof.

$$
\frac{p_{\beta,\gamma}(z)}{p_{\beta}(z)} = \frac{p_{\beta_1,\gamma_1}(z_1) \cdot \ldots \cdot p_{\beta_n,\gamma_n}(z_n)}{p_{\beta_1}(z_1) \cdot \ldots \cdot p_{\beta_n}(z_n)}
$$

$$
= (\gamma_1 \ldots \gamma_n)^{-2} \cdot \exp\left(-\frac{z_1^2}{2\sigma_1^2} + \frac{z_1^2}{2\sigma_1^2} \ldots \cdot \exp\left(-\frac{z_n^2}{2\sigma_n^2} + \frac{z_n^2}{2\sigma_n^2}\right)\right)
$$

$$
= (\gamma_1 \ldots \gamma_n)^{-2} \cdot \exp\left(\frac{z_1^2}{2\sigma_1^2} \ldots + \frac{z_n^2}{2\sigma_n^2} - \frac{z_1^2}{2\sigma_1^2} \ldots + \frac{z_n^2}{2\sigma_n^2}\right) \leq t
$$

$$
\Rightarrow z_1^2(1 - \gamma_1^{-2}) + \ldots + z_n^2(1 - \gamma_n^{-2}) \leq 2\sigma^2 ln(t(\gamma_1 \ldots \gamma_n)^2).
$$

What is the lowest $t$ such that $\frac{p_{\beta,\gamma}(z)}{p_{\beta}(z)} \leq t$ for all $z \in A$?

Because of $A = \{(z_1, \ldots, z_n) \mid (1 - \gamma_1^{-2})z_1^2 + \ldots + (1 - \gamma_n^{-2})z_n^2 \leq r\}$ we know that $(1 - \gamma_1^{-2})z_1^2 + \ldots + (1 - \gamma_n^{-2})z_n^2 \leq r$.

Does there exist such a threshold $t$ that both upper bounds coincide? Yes, namely

$$
t = (\gamma_1 \ldots \gamma_n)^{-2} \exp\left(\frac{r}{2\sigma^2}\right).
$$

We have found the region $\Omega$ in which the classifier is robust. But how to compute this region in practice? Note, that the $\gamma = (1, \ldots, 1)$ point does not necessary lie in $\Omega$. But of course we can add it to $\Omega$ as the classifier has the same outputs on it. Then it can be shown that this region is bounded. Thus the region defining inequality can be solved numerically.
7.2 Alternative proofs

The above theorems can be proven via a slightly modified functional optimization based framework \[25\]. Let \( f^\# : \mathbb{R}^m \to \{0, 1\} \) be a binary classifier, \( \mathcal{F} \) be a function class that is known to include \( f^\# \), \( \psi : \mathbb{R}^m \to \mathbb{R}^m \) be a multiplicative composable transformation with a scalar parameter \( z \) and \( f^\#_{\pi_0}(x_0) := \mathbb{E}_{z \sim \pi_0}[f^\#(\psi(z)(x_0))] \) be the smoothed classifier over \( \psi \) with the probability distribution \( \pi_0 \). Assume \( f^\#_{\pi_0}(x_0) > \frac{1}{2} \). Then we want to certify that \( f^\#_{\pi_0}(\psi(z)(x_0)) > \frac{1}{2} \) for every \( \delta \in \mathcal{B} = [\gamma_1, \gamma_2] \). The constrained adversarial certification framework yields the following lower bound:

\[
\min_{\delta \in \mathcal{B}} f^\#_{\pi_0}(\psi(z)(x_0)) \geq \min_{f \in \mathcal{F}} \min_{\delta \in \mathcal{B}} \{ f_{\pi_0}(\psi(z)(x_0)) \ s.t. \ f_{\pi_0}(x_0) = f^\#_{\pi_0}(x_0) \}
\geq \min_{f \in \mathcal{F}} \min_{\delta \in \mathcal{B}} \max_{\lambda \in \mathbb{R}} \{ f_{\pi_0}(\psi(z)(x_0)) - \lambda (f_{\pi_0}(x_0) - f^\#_{\pi_0}(x_0)) \}
\geq \max_{\lambda \in \mathbb{R}} \min_{f \in \mathcal{F}} \min_{\delta \in \mathcal{B}} \{ f_{\pi_0}(\psi(z)(x_0)) - \lambda (f_{\pi_0}(x_0) - f^\#_{\pi_0}(x_0)) \}
= \max_{\lambda \in \mathbb{R}} \left\{ \lambda f^\#_{\pi_0}(x_0) - \max_{\delta \in \mathcal{B}} \mathbb{D}_\mathcal{F}(\lambda \pi_0 \| \pi_\delta) \right\},
\]

where \( \pi_\delta \) is the distribution of \( \psi(z) \) when \( z \sim \pi_0 \) and we define the discrepancy term

\[
\mathbb{D}_\mathcal{F}(\lambda \pi_0 \| \pi_\delta) := \max_{f \in \mathcal{F}} \{ \lambda f_{\pi_0}(x_0) - f_{\pi_\delta}(x_0) \}.
\]

If \( \mathcal{F} = \{ f : f(x) \in [0, 1], \ x \in \mathbb{R}^m \} \), then

\[
\mathbb{D}_\mathcal{F}(\lambda \pi_0 \| \pi_\delta) = \int (\lambda \pi_0(z) - \pi_\delta(z))_+ dz,
\]

where \( (t)_+ = \max\{0, t\} \).

With strong duality we have

\[
\max_{\lambda \in \mathbb{R}} \left\{ \lambda f^\#_{\pi_0}(x_0) - \max_{\delta \in \mathcal{B}} \mathbb{D}_\mathcal{F}(\lambda \pi_0 \| \pi_\delta) \right\} \geq \min_{\delta \in \mathcal{B}} \max_{\lambda \geq 0} \left\{ \lambda f^\#_{\pi_0}(x_0) - \int (\lambda \pi_0(z) - \pi_\delta(z))_+ dz \right\}
\]

So, all we need to do is to find such an interval \( \mathcal{B} = [\gamma_1, \gamma_2] \), that

\[
\min_{\delta \in \mathcal{B}} \max_{\lambda \geq 0} \left\{ \lambda f^\#_{\pi_0}(x_0) - \int (\lambda \pi_0(z) - \pi_\delta(z))_+ dz \right\} > \frac{1}{2}.
\]

In our case \( f^\#_{\pi_0}(x_0) \) is the probability \( p_A \) of top 1 class \( A \) which we estimate with \( p_A \), \( \pi_0 \) has the Rayleigh distribution and

\[
\pi_0(z) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}}, \quad \pi_\delta(z) = \frac{z}{\sigma^2\delta^2} e^{-\frac{z^2}{2\sigma^2\delta^2}}.
\]

Let

\[
C_\lambda := \{ z : \lambda \pi_0(z) \geq \pi_\delta(z) \} = \{ z : \lambda \delta^2 e^{-\frac{z^2}{2\sigma^2 \delta^2}} \geq e^{-\frac{z^2}{2\sigma^2}} \}
= \{ z : \frac{-z^2}{2\sigma^2} + \ln \lambda \delta^2 \geq \frac{-z^2}{2\sigma^2} \} = \{ z : z^2(1 - \delta^2) \geq -2\sigma^2 \delta^2 \ln \lambda \delta^2 \}.
\]

First, we find the lower bound \( \gamma_1 \). For this purpose we consider \( \mathcal{B} = [\gamma_1, 1] \) which implies \( \delta \leq 1 \). In this case

\[
C_\lambda = \{ z : z^2 \geq \frac{-2\sigma^2 \delta^2 \ln \lambda \delta^2}{1 - \delta^2} \}.
\]
max \lambda \geq 0 \left\{ f_{\pi_0}(x_0) - \int (\lambda \pi_0(z) - \pi_\delta(z))_+ dz \right\} \geq \max \lambda \geq 0 \left\{ p_{\lambda A} - \int_{C_\lambda} (\lambda \pi_0(z) - \pi_\delta(z)) dz \right\}

= \max \lambda \geq 0 \left\{ p_{\lambda A} - \lambda p \left( \pi_\delta^2 \geq \frac{-2 \sigma^2 \delta^2 \ln \lambda \delta^2}{1 - \delta^2} \right) + p \left( \pi_\delta^2 \geq \frac{-2 \sigma^2 \ln \lambda \delta^2}{1 - \delta^2} \right) \right\}

= \max_{0 \leq \lambda \leq \delta - 2} \left\{ p_{\lambda A} - \lambda p \left( \pi_\delta^2 \geq \frac{-2 \sigma^2 \delta^2 \ln \lambda \delta^2}{1 - \delta^2} \right) + p \left( \pi_\delta^2 \geq \frac{-2 \sigma^2 \ln \lambda \delta^2}{1 - \delta^2} \right) \right\}

= \max_{0 \leq \lambda \leq \delta - 2} \left\{ p_{\lambda A} - \lambda \left( \frac{1}{F(\sqrt{-2 \sigma^2 \delta^2 \ln \lambda \delta^2})} + \frac{1}{F(\sqrt{-2 \sigma^2 \ln \lambda \delta^2})} \right) \right\},

where \( F(\cdot) \) is the CDF of the Rayleigh distribution with the scale parameter \( \sigma \).

Let

\[ G(\delta, \lambda) := p_{\lambda A} - \int (\lambda \pi_0(z) - \pi_\delta(z))_+ dz. \]

For any \( \delta \in B \) \( G \) is concave with respect to \( \lambda \). So for any \( \delta \) \( G \) has the only maximum located at \( \lambda_\delta \) which is the solution of \( \frac{\partial G(\delta, \lambda)}{\partial \lambda} = 0 \). Simple calculations show that

\[ \lambda_\delta = \frac{1}{\delta^2 p_{\lambda A}^{1-\frac{1}{\delta^2}}} . \]

Thus

\[ \min_{\delta \in B} \max_{\lambda \geq 0} G(\delta, \lambda) = \min_{\delta \in B} G(\delta, \lambda_\delta) \]

\[ = \min_{\delta \in B} \left\{ p_{\lambda_\delta A} - \lambda_\delta \left( \frac{1}{F(\sqrt{-2 \sigma^2 \delta^2 \ln \lambda_\delta \delta^2})} + \frac{1}{F(\sqrt{-2 \sigma^2 \ln \lambda_\delta \delta^2})} \right) \right\} \]

\[ = \min_{\delta \in [\gamma_1, 1]} \left\{ \frac{p_{\lambda A}}{\delta^2} = \frac{p_{\lambda A}}{1 - p_{\lambda A}} > \frac{1}{2} \iff \gamma_1 > \left( \log \frac{p_{\lambda A}}{1 - p_{\lambda A}} \right)^{-\frac{1}{2}} = \frac{1 - p_{\lambda A}}{F^{-1}(\frac{1}{2})} \right\} \]

which coincides with the result obtained by the previous method when we use a trivial bound on probability of top 2 class \( p_B = 1 - p_{\lambda A} \).

The upper bound \( \gamma_2 \) can be calculated in the same way.

The result can be generalized to multi-class case as it was shown in [25].