SIMULATION OF PROBABILISTIC SEQUENTIAL SYSTEMS

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Abstract. In this paper we introduce the idea of probability in the definition of Sequential Dynamical Systems, thus obtaining a new concept, Probabilistic Sequential System. The introduction of a probabilistic structure on Sequential Dynamical Systems is an important and interesting problem. The notion of homomorphism of our new model, is a natural extension of homomorphism of sequential dynamical systems introduced and developed by Laubenbacher and Pareigeis in several papers. Our model, give the possibility to describe the dynamic of the systems using Markov chains and all the advantage of stochastic theory. The notion of simulation is introduced using the concept of homomorphisms, as usual. Several examples of homomorphisms, subsystems and simulations are given.

Introduction

Genetic Regulatory Networks had been modeled using discrete and continuous mathematical models. An important contribution to the simulation science is the theory of sequential dynamical systems (SDS) [1, 2, 3, 10, 11]. In these paper, the authors developed a new theory about the sequential aspect of the entities in a dynamical systems. In particular Laubenbacher and Pareigeis created an elegant mathematical background of the SDS, and with it solve several aspects of the theory and applications.

Probabilistic Boolean Networks (PBN) had been recently introduced [13, 14, 15], to model regulatory gene networks. The PBN are a generalization of the widely used Boolean Network model (BN) proposed by Kauffman (1969) [7, 8]. While the PBN eliminate one of the main limitations of the BN model, namely its inherent determinism, they do not provide the framework for considering the sequential behavior of the genes in the network, behavior observed by the biologists.

Here, we introduce the probabilistic structure in SDS, using for each vertices of the support graph a set of local functions, and more than one schedule in the sequence of the local function selected to form the update functions, obtaining a new concept: probabilistic sequential dynamical system (PSS). The notion of simulation of a PSS is introduced in Section 4 using the concept of homomorphism of PSS; and we prove
that the category of SDS is a full subcategory of the of the category PSS. Several examples of homomorphisms, subsystems and simulations are given.

On the other hand, Deterministic Sequential Dynamical Systems have been studied for the last few years. The introduction of a probabilistic structure on Sequential Dynamical Systems is an important and interesting problem. Our approach take into account, a number of issues that have already been recognized and solved for Sequential Dynamical Systems.

1. Sequential Dynamical Systems and SDS-homomorphism

This section is an introduction with the definitions and results of Sequential Dynamical System introduced by Laubenbacher and Pareigis. Here we use SDS over a finite field. In this paper, we denote the finite field $GF(p^r)$ by $K$, where $p$ is a prime number.

1.1. Sequential Dynamical Systems over finite fields. A Sequential Dynamical System (SDS) over a finite field $F = (\Gamma, (f_i)_{i=1}^n, \alpha)$ consists of

1. A finite graph $\Gamma = (V_\Gamma, E_\Gamma)$ with $V_\Gamma = \{1, \ldots, n\}$ vertices, and a set of edges $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$.
2. A family of local functions $f_i : K^n \to K^n$, that is $f_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, f_i, x_{i+1}, \ldots, x_n)$ where $f_i(x_1, \ldots, x_n)$ depends only of those variables which are connected to $i$ in $\Gamma$.
3. A permutation $\alpha = (\alpha(1) \ldots \alpha(n))$ in the set of vertices $V_\Gamma$, called an update schedule (i.e. a bijective map $\alpha : V_\Gamma \to V_\Gamma$).

The global update function of the SDS is $f = f_{\alpha(1)} \circ \ldots \circ f_{\alpha(n)}$. The function $f$ defines the dynamical behavior of the SDS and determines a finite directed graph with vertex set $K^n$ and directed edges $(x, f(x))$, for all $x \in K^n$, called the State Space of $F$.

1.2. Homomorphisms of SDS. The definition of homomorphism between two SDS uses the fact that the vertices $V_\Gamma = \{1, \ldots, n\}$ of an SDS and the states $K^n$ together with their evaluation map

$$K^n \times V_\Gamma \ni (x, a) \mapsto < x, a > = : x_a \in K_a$$

form a contravariant setup, so that morphisms between such structures should be defined contravariantly, i.e. by a pair of certain maps $(\phi : \Gamma \to \Delta, h_\phi : K^m \to K^n)$ or by a pair $(\phi : \Delta \to \Gamma, h_\phi : K^n \to K^m)$, with the graph $\Delta$ having $m$ vertices. Here we use a notation slightly different the one using in [11].

Let $F = (\Gamma, (f_i : K^n \to K^n), \alpha)$ and $G = (\Delta, (g_i : K^m \to K^m), \beta)$ be two SDS. Let $\phi : \Delta \to \Gamma$ be a digraph morphism, and

$$(\phi_b : K_{\phi(b)} \to K_b, \forall b \in \Delta),$$
be a family of maps in the category of $\textbf{Set}$. The map $h_\phi$ is an adjoint map, because is defined as follows: consider the pairing

$$K^n \times V_\Gamma \ni (x, a) \mapsto < x, a > := x_a \in K_a;$$

and similarly

$$K^m \times V_\Delta \ni (x, b) \mapsto < x, b > := x_b \in K_b.$$ 

The induced adjoint map is $< h_\phi(x), b > = \hat{\phi}_b(< x, \phi(b) >) = \hat{\phi}_b(x_{\phi(b)}).$ Then, $\phi,$ and $(\hat{\phi}_b)$ induce the adjoint map $h_\phi: K^n \rightarrow K^m$ defined as follows:

$$(D1.a) \quad \left( \prod_{\beta_j \in \phi^{-1}(\alpha_i)} \prod_{j=1}^{\phi^{-1}(\alpha_i)} (g_{\beta_j}) \right) \circ h_\phi = h_\phi \circ f_{\alpha_i},$$

where $\prod_{\beta_j \in \phi^{-1}(\alpha_i)} \prod_{j=1}^{\phi^{-1}(\alpha_i)} (g_{\beta_j}) = g_{\beta_i} \circ g_{\beta_{i+1}} \circ \cdots \circ g_{\beta_s}$ and $\{\beta_i, \beta_{i+1}, \ldots, \beta_s\} = \phi^{-1}(\alpha_i).$

If $\phi^{-1}(\alpha_i) = \emptyset,$ then $Id_{K^m} \circ h_\phi = h_\phi \circ f_{\alpha_i},$ and the commutative diagram is the following.

$$(D1.b) \quad \begin{array}{ccc}
K^n & \xrightarrow{f_{\alpha_i}} & K^n \\
| h_\phi & \downarrow & | h_\phi \\
K^m & \xrightarrow{Id_{K^m}} & K^m
\end{array}$$

For examples and properties see [11]. It is clear that the above diagrams implies the following one

$$(D2) \quad \begin{array}{ccc}
K^n & \xrightarrow{f_{\alpha_i} \circ \cdots \circ f_{\alpha_m}} & K^n \\
| h_\phi & \downarrow & | h_\phi \\
K^m & \xrightarrow{g_{\beta_1} \circ \cdots \circ g_{\beta_m}} & K^m
\end{array}$$

2. Probabilistic Sequential Dynamical Systems

The following definition give us the possibility to have several update functions acting in a sequential manner with assigned probabilities. All these, permit us to study the dynamic of these systems using Markov chains and other probability tools. We will use the acronym PSS (or SDS) for plural as well as singular instances.

**Definition 2.1.** A Probabilistic Sequential dynamical System (PSS) $D_S = (\Gamma, \{F_i\}_{i=1}^m, (\alpha_j)_{j=1}^n, C)$
over \( \mathcal{K} \) consists of:

1. a finite graph \( \Gamma = (V_\Gamma, E_\Gamma) \) with \( n \) vertices;
2. a set of local functions \( F_i = \{ f_{ik} : \mathcal{K}^n \rightarrow \mathcal{K}^n \} \) for each vertex \( i \) of \( \Gamma \) (i.e. a bijection map \( \sim : V_\Gamma \rightarrow \{ F_i \} \) for definition of local function see [1.1.2]);
3. a family of \( m \) permutations \( \alpha_j = (\alpha_j(1) \ldots \alpha_j(n)) \) in the set of vertices \( V_\Gamma \);
4. and a set \( C = \{ c_1, \ldots, c_s \} \), of selection probabilities.

We select one function in each set \( F_i \), that is one for each vertices \( i \) of \( \Gamma \), and a permutation \( \alpha \), with the order in which the vertices \( i \) are selected (similarly SDS), so there are \( m \) possibly different update functions \( f = f_{\alpha(1)k_1(1)} \circ \ldots \circ f_{\alpha(n)k_\alpha(n)} \), where \( m = m! \times \ell(1) \times \ldots \times \ell(n) \). The probabilities are assigned to the update functions, so there exists a subset \( S = \{ f_1, \ldots, f_s \} \) of update functions such that \( c_k = p(f_k) \), \( 1 \leq k \leq s \).

The State Space of \( \mathcal{D}_S \) is a digraph whose vertices are the elements of \( \mathcal{K}^n \) and there are an arrow going from \( x_1 \) to \( x_2 \) if there exists a functions \( f \), such that \( x_2 = f(x_1) \). For each one of the selected functions in \( S \) we have an SDS inside the PSS, so the state space of the function \( f \) is a subdigraph of the state space of the PSS, so, the State Space of \( \mathcal{D}_S \) is a superposition of all inside SDS. When we take the whole set of update functions generated by the data, we will say that we have the full PSS. We denote by \( \overline{S} \) the complement of set \( S \) in the set of all update functions, and we will call the PSS \( \mathcal{D}_{\overline{S}} \) building by the same data but taking \( \overline{S} \) as a set of update functions, one complement of the PSS \( \mathcal{D}_S \). All the complements have the same set of function but they can use different set of probabilities.

The probability of the arrow going from \( x_1 \) to \( x_2 \) for example, is the sum of the probabilities of all functions \( f \), such that \( x_2 = f(x_1) \). The PSS represents a generalization of the SDS: a SDS is a PSS for which every set of local functions has one element, and there is only one permutation in the family of permutations. The update functions of the PSS have probabilities and the state space of the PSS (or high level digraph) is described by a transition matrix, and the dynamic is described by a Markov Chain.

2.1. Example. Let

\[ \mathcal{D} = (\Gamma; F_1, F_2, F_3; \alpha^1, \alpha^2; F_i, (C(f_i))_{i=1}^{\delta}) \],

be the following PSS:

1 \[ \xrightarrow{} \] 3
2 \[ \xleftarrow{} \] 1

(1) The graph: \( \Gamma \)

(2) If \( x = (x_1, x_2, x_3) \in \{0,1\}^3 \), then the sets of local functions are the following:

\[ F_1 = \{ f_{11}(x) = (1, x_2, x_3), f_{12}(x) = (x_1 + 1, x_2, x_3) \} \]
\[ F_2 = \{ f_{21}(x) = (x_1, x_1x_2, x_3) \} \]
\[ F_3 = \{ f_{31}(x) = (x_1, x_2, x_1x_2), f_{32}(x) = (x_1, x_2, x_1x_2 + x_3) \} \]
In the definition of a homomorphism of PSS we establish condi-

The transition matrix of the system. The transition matrix of

The update functions are the following:

We obtain the following table of functions, and we select all of them for \( D \) because the probabilities given by \( C \).

The update functions are the following:

(4) The probabilities that we assign are: \( p_1 = C(f_1) = .18; p_2 = C(f_2) = .12; p_3 = C(f_3) = .18; p_4 = C(f_4) = .12; p_5 = C(f_5) = .12; p_6 = C(f_6) = .08; p_7 = c(f_7) = .12; p_8 = c(f_8) = .08 \). In order to study the state space, it is convenient to determine the transition matrix of the system. The transition matrix of \( D \) is \( T \),

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0.6 \\
0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.3 & 0.3 \\
0.4 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 \\
0.2 & 0.2 & 0 & 0 & 0.3 & 0.3 & 0 & 0 \\
0.24 & 0 & 0 & 0.16 & 0 & 0 & 0 & 0.6 \\
0.12 & 0.12 & 0.08 & 0.08 & 0 & 0 & 0.3 & 0.3
\end{pmatrix}.
\]

2.1.1. Example. We notice that there are several PSS that we can construct with the same initial data of functions and permutations, but with different set of probabilities, that is, sub-PSS of \( D \) see \textbf{3.1}. For example if \( S = \{ f_1, f_2, f_3, f_4 \} \) and \( C = \{ d(f_1) = .355, d(f_2) = .211, d(f_3) = .12, d(f_4) = .314 \} \), then

\[
B = (\Gamma; (F_1, F_2, F_3, \alpha_1, \alpha_2; S; C)).
\]

3. Homomorphisms of Probabilistic Sequential Systems

One objective in this section is to show that: homomorphism of PSS is a natural extension of homomorphism of SDS. We remark, in the introductory section \textbf{4}, the definition of SDS. In the definition of a homomorphism of PSS we establish conditions to connect the support graphs, the State Space and the assigned probabilities.

3.1. Definition of homomorphisms of PSS. Let \( D_1 = (\Gamma, (F_i)_{i=1}^{\Gamma}; (\alpha^j)_j, S_1, C_1) \) and \( D_2 = (\Delta, (G_i)_{i=1}^{\Delta}; (\beta^k)_k, S_2, C_2) \) be two PSS over a finite field \( K \).

A homomorphism from \( D_1 \) to \( D_2 \) is a pair of functions \( (\phi, h_\phi) \) where:

\[
(3.1) \phi : \Delta \to \Gamma \text{ is a graph morphism, and} \quad (\hat{\phi}_b : K_{\phi(b)} \to K_b \forall b \in \Delta),
\]
is a family of maps in the category of Set. They induce the adjoint function $h_\phi$, see (1.2) (I).

**(3.1.2)** The induced adjoint map $h_\phi : \mathcal{K}^n \to \mathcal{K}^m$ is a map such that for all update function $f$ in $S_1$ there exists an update function $g \in S_2$ such that $(\phi, h_\phi)$ is an SDS-homomorphism from $(\Gamma, (f : \mathcal{K}^n \to \mathcal{K}^n), \alpha^j)$ to $(\Delta, (g : \mathcal{K}^m \to \mathcal{K}^m), \beta^k)$.
That is, the diagrams (D1), and (D2) commute for all $f$ and the selected $g$.

**(3.1.3)** *(The $\epsilon$-condition)* For a fixed real number $0 \leq \epsilon < 1$, the map $h_\phi$ satisfies the following:

$$\max_{u,v} |c_{f_j}(u, v) - d_{g_j}(h_\phi(u), h_\phi(v))| \leq \epsilon$$

for all $f$ in $S_1$, and its selected $g$ in $S_2$ by condition (3.1.2), and $u, v \in \mathcal{K}^n$,
We will say that an homomorphism \((\phi, h_\phi)\) from \(D_1\) to \(D_2\) is an isomorphism if \(\phi\) and \(h_\phi\) are bijective functions, and \(d(h_\phi(u), h_\phi(g(u))) = c(u, f(u))\) for all \(u\) in \(K^n\), and all \(f\). We denote it by \(D_1 \cong D_2\).

3.2. Theorem. In the definition of homomorphism the condition (3.1.3) is a consequence of the condition (3.1.1), and (3.1.2).

Proof. Suppose \((\phi, h_\phi)\) satisfies (3.1.1), (3.1.2), and \(\max_{u,v}|c_{f_1}(u, v) - d_{g_1}(h_\phi(u), h_\phi(v))| \geq 1\), then for some \((u, v)\) we have one of the following cases

(Case 1) \(c_{f_1}(u, v) = 1\) and \(d_{g_1}(h_\phi(u), h_\phi(v)) = 0\), it is impossible by (3.1.2).

(Case 2) \(d_{g_1}(h_\phi(u), h_\phi(v)) = 1\), and \(c_{f_1}(u, v) = 0\), it is impossible because there exists at least other element \(w \in K^n\) such that \(c_{f_1}(u, w) \neq 0\), and so on.

Therefore the third condition (3.1.3) holds, and always \(\epsilon\) exists. \(\Box\)

3.3. Theorem. If \((\phi, h_\phi) : D_1 \rightarrow D_2\) is a PSS-homomorphism, and \(T_i\) denote the transition matrix of \(D_i\), and the entry \((u, v)\) of \(T_1\) is \(p(u, v)\) then:

\[
\lim_{m \to \infty} |(T_1)_u^m - (T_2)_u^m| = 0,
\]

for all possible \(u\) and \(v\) in \(K^n\).

Proof. The \(\epsilon\)-condition (3.3.3) asserts that for a fixed real number \(0 \leq \epsilon < 1\), the map \(h_\phi\) satisfies the following:

\[
\max_{u,v}|c_{f_1}(u, v) - d_{g_1}(\phi(u), \phi(v))| \leq \epsilon
\]

for all \(f\) in \(S_1\), and its selected \(g\) in \(S_2\) by condition (3.1.2), and \(u, v \in K^n\).

If we have a function \(f_i\) going from \(u\) to \(v = f_i(u)\) in \(K^n\), then there exists a function \(g_j\) going from \(h_\phi(u)\) to \(h_\phi(v)\), so \(g_j(h_\phi(u)) = h_\phi(f_i(u))\). Then for \(m = 2\), and by the Chapman-Kolmogorov equation [16, 9], we have the following:

\[
|c_{f_i}(u, f_i^2(u)) - d_{g_j}(h_\phi(u), g_j^2(h_\phi(u)))| = |c_{f_i}(u, f_i(u))c_{f_i}(f_i(u), f_i^2(u)) - d_{g_j}(h_\phi(u), g_j(h_\phi(u)))d_{g_j}(g_j(h_\phi(u)), g_j^2(h_\phi(u)))| = |c_{f_i}(u, f_i(u))c_{f_i}(f_i(u), f_i^2(u)) - d_{g_j}(h_\phi(u), h_\phi(f_i(u)))d_{g_j}(h_\phi(f_i(u)), h_\phi(g_j^2(u)))| \leq 0.
\]

By condition (2) in definition of homomorphism. Then we proved that \(|c_{f_i}(u, f_i^2(u)) - d_{g_j}(\phi(u), g_j^2(\phi(u)))| \leq 2\epsilon\). Using mathematical induction over \(m\), we can conclude that

\[
\max_{u,v}|(c_{f_i}(u, f_i^m(u)) - d_{g_j}(\phi(u), g_j^m(\phi(u)))| \leq m\epsilon
\]

Let \(S_o\) be the set of function in \(S_2\) associated to a function in \(S_1\). If \(u, v \in K^n\), and we denote by \(p_2(u, v) = \sum_{f_i} c_{f_i}(u, v)\) and \(p_2(h_\phi(u), h_\phi(v)) = \sum_{g_j} d_{g_j}(h_\phi(u), h_\phi(v))\), then condition (3.3.3) implies that \(|p_2(u, v) - p_2(h_\phi(u), h_\phi(v))| \leq 2k\epsilon + \delta^2\), where \(k\) is the maximum number of functions \(f_i\) going from one state to another in the state space \(K^n\). Denoting \(\delta^2 = \sum_{g_j \in S_o} d_{g_j}(h_\phi(u), h_\phi(v))\) we know that \(\delta^2 < 1\). So, if \(T_i\)
denote the transition matrix of $D_i$, and the entry $(u, v)$ of $(T_i)^2$ is $p_2(u, v)$ then the condition (3.3.3) implies that:

$$\max_{u, v} |(T_1)^2_{u,v} - (T_2)^2_{h\phi(u), h\phi(v)}| \leq 2k\epsilon + \delta^2$$

for all possible $u$ and $v$ in $K^n$.

Using equations (3.3.1), and (3.3.2), and denoting $\delta^m = \sum g_i g_{S_{\phi}} [d_{g_i}(h\phi(u), h\phi(v))]^m$, we conclude that

$$\max_{u, v} |(T_1)^m_{u,v} - (T_2)^m_{h\phi(u), h\phi(v)}| \leq mk\epsilon + \delta^m$$

for all possible $u$ and $v$ in $K^n$.

So, for all real number $0 < \epsilon' < 1$ there exists $m \in \mathbb{N}$, such that,

$$|T_1^{m'}_{u,v} - T_2^{m'}_{h\phi(u), h\phi(v)}| < \epsilon'$$

for all natural number $m' > m$, and for all possible $u, v \in K^n$.

In fact, using notation of equation (3.3.2), we have $\epsilon^m_{\phi} \ll \epsilon^m_{\phi}$, and this implies $(m'k)\epsilon^m_{\phi} < (mk)\epsilon^m_{\phi}$. Similarly $\delta^m_{\phi} < \delta^m_{\phi}$, So, selecting $m$ such that $(mk)\epsilon^m_{\phi} + \delta^m_{\phi} < \epsilon'$, we obtain

$$|(T_1)^m_{u,v} - (T_2)^m_{h\phi(u), h\phi(v)}| \leq (m'k)\epsilon^m_{\phi} + \delta^m_{\phi} < (mk)\epsilon^m_{\phi} + \delta^m_{\phi} < \epsilon'$$

where $k$ is the maximum number of functions going from one state to another in the state space of the power $m'$ of the functions. Therefore

$$\lim_{m \to \infty} |(T_1)^m_{u,v} - (T_2)^m_{h\phi(u), h\phi(v)}| = 0,$$

for all possible $u$ and $v$ in $K^n$, and the theorem holds.

3.4. Definition of $\epsilon$-equivalent. If $\phi$ and $h\phi$ are bijective functions, and the condition (3.1.3) holds, but the probabilities are not equal, we will say that $D_1$, and $D_2$ are $\epsilon$-equivalent, and we write $D_1 \simeq_{\epsilon} D_2$. So, $D_1$, and $D_2$ are $\epsilon$-equivalent if there exist $(h\phi, \phi)$, and $((h_{\phi}^{-1}, \phi^{-1})$, such that for all $f \in S_1$ and $g \in S_{\phi}$, we have $f = h_{\phi}^{-1} \circ g \circ h\phi$.

3.5. Image of $h\phi$. Consider $d(g)$ defined as follows: if $S_{\phi} = \{g_1, \ldots, g_s\}$ is the set of functions in $S_2$ selected in (3.1.2) for the map $h\phi$ then the new probability of $g_i$ is

$$d(g_i) = \frac{d(g_i)}{\sum_{g_i \in S_{\phi}} d(g_i)}.$$

The set of functions $S_{\phi}$ in $G$, together with the new probabilities defined above, form a new PSS, that we will call Image of $h\phi = Im(h\phi)$. So, the graph $\Delta$, the update functions $S_{\phi}$ determine de local functions associate to each vertex of $\Delta$, similarly the permutations using by these functions, and finally the new probabilities assigned.

3.6. Sub Probabilistic Sequential System. We will say an injective monomorphism is a PSS-homomorphism such that $\phi$ is surjective and the set of functions $\phi_i$, for all $i$ are injective functions, and so $h\phi_i$. Therefore, we will say that a PSS $G_X$ is sub Probabilistic Sequential System of $F_S$ if there exists an injective monomorphism from $G_X$ to $F_S$. 
3.7. Definitions.

(3.7.1) Let \( \mathcal{F} = (\Gamma, (F_i)_{i=1}^n, (\alpha^j)_{j \in J}, C) \) be a PSS. The pair of functions \( \mathcal{I} = (id_{\Gamma}, id_{\mathcal{K}^n}) \) is the identity homomorphism, and it is an example of an isomorphism.

(3.7.2) An homomorphism \((\phi_1, h_{\phi_1})\) of PSS is an injective monomorphism if \(\phi\) is surjective and \(h_{\phi_1}\) is injective, for example see (3.3). Similarly we will say that an homomorphism is a surjective epimorphism if \(\phi\) is injective and \(h_{\phi_1}\) is surjective, for a complete description of the properties of this class of monomorphism and epimorphism see Section 7 in [11].

3.8. Simulation of PSS. We consider that the PSS \( \mathcal{G} \) is simulated by \( \mathcal{F} \) if there exist a injective monomorphisms \((\phi_1, h_{\phi_1}) : \mathcal{F} \to \mathcal{G}\) or a surjective epimorphism \((\phi_2, h_{\phi_2}) : \mathcal{G} \to \mathcal{F}\).

4. Examples of Simulation and Homomorphisms of PSS.

In this section we give several examples of PSS-homomorphism, and simulation. In the second example we show how the condition (3.1.2) is verified under the supposition that a function \(\phi\) is defined. So, we have two examples in (4.2), one with \(\phi\) the natural inclusion, and the second with \(\phi\) the only possibility of a surjective map. In the last example we have a complete description of two PSS, where we have only one permutation and two or less functions for each vertices in the graph. In particular this homomorphism is an injective monomorphism, so is an example of simulation too.

(4.1) For the PSS, in the examples 2.1 and 2.1.1 we now define the natural inclusion \((Id_{\Gamma}, \iota) : B \to D\). It is clear that the inclusion satisfies the two first condition to be a homomorphism, and the third one is a simple consequence of the Theorem 3.2. In fact

\[
T_B = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.566 & 0.434 & 0 & 0 \\
0 & 0 & 0 & 1 & 0.525 & 0 & 0 & 0.475 \\
0 & 0 & 0 & 0 & 0.211 & 0.314 & 0.12 & 0.355 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.566 & 0.434 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0.08 & 0 & 0 & 0.434 & 0.566
\end{pmatrix}
\]

and \(|(T_B)_{u,v} - (T)_{\iota(u),\iota(v)}| < 0.4\). Here \(D\) is simulated by \(B\).

(4.2) Consider the two graphs below

\[
\begin{array}{c|c|c}
\Gamma & \begin{array}{c}
2 \\
1
\end{array} & \begin{array}{c}
3 \\
4
\end{array} \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\Delta & \begin{array}{c}
2 \\
1
\end{array} & \begin{array}{c}
3 \\
4
\end{array} \\
\hline
\end{array}
\]

Suppose that the functions associated to the vertices are the families \(\{f_1, f_2, f_3, f_4\}\) for \(\Gamma\) and \(\{g_1, g_2, g_3\}\) for \(\Delta\). The permutations are \(\alpha_1 = (4 3 2 1)\), \(\alpha_2 = (4 1 3 2)\) and \(\beta_1 = (3 2 1)\), \(\beta_2 = (1 3 2)\), so, \(S = \{f = f_4 \circ f_3 \circ f_2 \circ f_1; f = f_4 \circ f_1 \circ f_3 \circ f_2\}\), and \(X = \{g = g_3 \circ g_2 \circ g_1; g = g_1 \circ g_3 \circ g_2\}\). Then, we have constructed two PSS,
each one with two permutations and only one function associated to each vertex in the graph; denoted by:

\[ D_S = (\Gamma; f_1, f_2, f_3, f_4; \alpha^1, \alpha^2; S; C) \] and \[ B_X = (\Delta; g_1, g_2, g_3; \beta^1, \beta^2; X; D). \]

**Case (a)** We assume that there exists a homomorphism \((\phi, h_\phi)\) from \(D_S\) to \(B_X\), with the graph morphism \(\phi: \Delta \to \Gamma\) is given by \(\phi(1) = 1\), \(\phi(2) = 2\), \(\phi(3) = 3\). Suppose the functions

\[
(\hat{\phi}_b: K_{\phi(b)} \to K_b, \forall b \in \Delta),
\]

are giving, and the adjoint function

\[ h_\phi: Z_p^4 \to Z_p^3, \ h_\phi(x_1, x_2, x_3, x_4) = (\hat{\phi}_1(x_1), \hat{\phi}_2(x_2), \hat{\phi}_3(x_3)) \]

is defined too. If \((h_\phi, \phi)\) is an homomorphism, which satisfies the definition 3.11, then the following diagrams commute:

\[
\begin{array}{c}
Z_p^4 \xrightarrow{f_1} Z_p^4 \xrightarrow{f_3} Z_p^4 \xrightarrow{f_2} Z_p^4 \xrightarrow{f_1} Z_p^4 \xrightarrow{f} Z_p^4 \\
h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow \ h_\phi \downarrow, \ h_\phi \downarrow \ h_\phi \downarrow \\
Z_p^3 \xrightarrow{1d} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_2} Z_p^3 \xrightarrow{g_1} Z_p^3 \xrightarrow{g} Z_p^3 \\
\end{array}
\]

\[
\begin{array}{c}
Z_p^4 \xrightarrow{f_4} Z_p^4 \xrightarrow{f_3} Z_p^4 \xrightarrow{f_2} Z_p^4 \xrightarrow{f_1} Z_p^4 \xrightarrow{f} Z_p^4 \\
h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow \ h_\phi \downarrow, \ h_\phi \downarrow \ h_\phi \downarrow \\
Z_p^3 \xrightarrow{1d} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_2} Z_p^3 \xrightarrow{g_1} Z_p^3 \xrightarrow{g} Z_p^3 \\
\end{array}
\]

**Case (b)** Consider now the map \(\phi: \Gamma \to \Delta\), defined by \(\phi(1) = 1\), \(\phi(2) = 2\), \(\phi(3) = 3\), and \(\phi(4) = 1\). If there exists an homomorphism \((\phi, h_\phi) : B_X \to D_S\) that satisfies 3.112, then

\[
\begin{array}{c}
Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g} Z_p^3 \\
h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow \ h_\phi \downarrow, \ h_\phi \downarrow \ h_\phi \downarrow \\
Z_p^4 \xrightarrow{f_4f_3} Z_p^4 \xrightarrow{f_2} Z_p^4 \xrightarrow{f_1} Z_p^4 \xrightarrow{f} Z_p^4 \\
\end{array}
\]

\[
\begin{array}{c}
Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g_3} Z_p^3 \xrightarrow{g} Z_p^3 \\
h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow h_\phi \downarrow \ h_\phi \downarrow, \ h_\phi \downarrow \ h_\phi \downarrow \\
Z_p^4 \xrightarrow{f_4f_3} Z_p^4 \xrightarrow{f_2} Z_p^4 \xrightarrow{f_1} Z_p^4 \xrightarrow{f} Z_p^4 \\
\end{array}
\]

(4.3) We now construct a PSS-homomorphism from \(F_S = (\Gamma, (F_i)_3, \beta, S, C)\) to \(G_X = (\Delta, (G_i)_4, \alpha, X, D)\), with the property that \(\phi\) is surjective and the functions \(\phi_i\) are injective, that we call a injective monomorphism. The PSS \(F_S\) has a support graph \(\Gamma\) with three vertices, and the PSS \(G_X\) has a support graph \(\Delta\) with four vertices

\[
\begin{array}{c|c|c|c}
1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

The homomorphism \((h_\phi, \phi) : F_S \to G_X\), has the contravariant graph morphism \(\phi: \Delta \to \Gamma\), defined by the arrows of graphs, as follows \(\phi(1) = 1\), \(\phi(2) = \phi(3) = 2\),
and $\phi(4) = 3$. The family of functions $\hat{\phi}_i : K_{\phi(i)} \rightarrow K_{(i)}$, $\hat{\phi}(x_1) = x_1$; $\hat{\phi}_2(x_2) = x_2$; $\hat{\phi}_3(x_3) = x_3$; $\hat{\phi}_4(x_4) = x_4$. The adjoint function is $h_\phi : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^4$, with $h_\phi(x_1,x_2,x_3) = (\hat{\phi}(x_1), \hat{\phi}_2(x_2), \hat{\phi}_3(x_3), \hat{\phi}_4(x_4)) = (x_1,x_2,x_2,x_3)$. The first condition in the definition \[\text{3.1}\] holds.

The PSS $\mathcal{F}_S = (\Gamma; (F_i)_5; \beta; S; C)$, with data of functions $F_1 = \{ f_{11} = \text{Id}, f_{12}(x_1, x_2, x_3) = (1, x_2, x_3) \}$, $F_2 = \{ f_{21}(x_1, x_2, x_3) = (x_1, x_2, x_3) \}$, and $F_3 = \{ f_{31}(x_1, x_2, x_3) = (x_1, x_2, x_2 x_3) \}$, one permutation or schedule $\beta = (1 2 3)$; and probabilities $C = \{ c(f) = .5168, c(\hat{g}) = .4832 \}$, so $S = \{ f, g \}$; where the two update functions are $f = f_{11} \circ f_{21} \circ f_{31}$, $\hat{f}(x_1, x_2, x_3) = (x_1, x_2, x_2 x_3)$; $\hat{g} = f_{12} \circ f_{21} \circ f_{31}$, $\hat{g}(x_1, x_2, x_3) = (1, x_2, x_2 x_3)$.

The PSS $\mathcal{G}_X = (\Delta; (G_i)_4; \alpha; X; D)$ is a PSS, with the following data: the families of functions: $G_1 = \{ g_{11}, g_{12} \}$; $G_2 = \{ g_{21}, g_{22} \}$, $G_3 = \{ g_{31}, g_{32} \}$; and $G_4 = \{ g_4 \}$. One schedule $\alpha = (1 2 3 4)$, the eight possible update functions, and its probabilities $D = \{ d(g) = 0, d(f) = 0, d(\hat{f}) = 0, d(\hat{g}) = 0, d(\hat{f}) = .00252, d(\hat{g}) = .08321, d(\hat{f}) = .51428, d(\hat{g}) = .39999 \}$ whose determine $X = \{ f, \hat{g}, f, \hat{g} \}$ probabilities, are the following:

1) $g(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{21} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, x_1 x_2, x_2 x_3)$,
2) $f(x_1, x_2, x_3, x_4) = (g_{12} \circ g_{21} \circ g_{32} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, x_2, x_2 x_3)$,
3) $f(x_1, x_2, x_3, x_4) = (g_{12} \circ g_{21} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$,
4) $\hat{g}(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{22} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$,
5) $\hat{g}(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{22} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$,
6) $\hat{g}(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{22} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$,
7) $\hat{g}(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{22} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$,
8) $\hat{g}(x_1, x_2, x_3, x_4) = (g_{11} \circ g_{22} \circ g_{31} \circ g_4)(x_1, x_2, x_3, x_4) = (1, 1, 1, x_2 x_3)$.

We claim $(\phi, h_\phi) : \mathcal{F}_S \rightarrow \mathcal{G}_X$ is a homomorphism. We will prove that the following diagrams commute.

\[\begin{array}{ccc}
\mathbb{Z}_2^3 & \xrightarrow{\vec{f}} & \mathbb{Z}_2^3 \\
\downarrow h_\phi & & \downarrow h_\phi \\
\mathbb{Z}_2^4 & \xrightarrow{\vec{f}} & \mathbb{Z}_2^4 \\
\end{array}\]

In fact,

$(h_\phi \circ \vec{f})(x_1, x_2, x_3) = h_\phi(x_1, x_2, x_2 x_3) = (x_1, x_2, x_2 x_3) = (\vec{f} \circ h_\phi)(x_1, x_2, x_3)$.

On the other hand,

$(h_\phi \circ \vec{g})(x_1, x_2, x_3) = h(1, x_2, x_2 x_3) = (1, x_2, x_2 x_3) = (\vec{g} \circ h_\phi)(x_1, x_2, x_3)$.

We verify the composition of functions as follows
Figure 2. State Spaces of \( F \) and \( F' \). Transition Matrices: \( T_1 \), and \( T_{h\phi} \).

\[
\begin{align*}
Z_p^3 &\xrightarrow{f_{31}} Z_p^3 & Z_p^3 &\xrightarrow{f_{21}} Z_p^3 & Z_p^3 &\xrightarrow{f_{12}} Z_p^3 & Z_p^3 &\xrightarrow{f_{11}} Z_p^3 \\
&\downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} \\
Z_p^4 &\xrightarrow{g_1} Z_p^4 & g_1 &\circ g_2 & g_1 &\circ g_2 & g_1 &\circ g_2 \\
&\downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} & \downarrow h_{\phi} \\
Z_p^3 &\xrightarrow{f_{31}} Z_p^3 & Z_p^3 &\xrightarrow{f_{21}} Z_p^3 & Z_p^3 &\xrightarrow{f_{12}} Z_p^3 & Z_p^3 &\xrightarrow{f_{11}} Z_p^3 \\
\end{align*}
\]

\((h_{\phi} \circ f_{31})(x_1, x_2, x_3) = (x_1, x_2, x_2, x_2) = (g_4 \circ h_{\phi})(x_1, x_2, x_3),\)

\((h_{\phi} \circ f_{21})(x_1, x_2, x_3) = (x_1, x_2, x_2, x_3) = ((g_21 \circ g_{32}) \circ h_{\phi})(x_1, x_2, x_3),\)

\((h_{\phi} \circ f_{12})(x_1, x_2, x_3) = (1, x_2, x_3) = (g_{11} \circ h_{\phi})(x_1, x_2, x_3)\)

Similarly we check the condition for \( f \), and \( f' \).

The third condition holds, because the initial \( \epsilon \leq .009 \), because \( |T_1 - T_{h\phi}| < .009 \).

In fact:

\[
T_1 = \begin{pmatrix}
.5168 & 0 & 0 & 0 \\
.5168 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & .5168 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and

\[
T_{h\phi} = \begin{pmatrix}
.5168 & 0 & 0 & 0 \\
.5168 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & .51428 & 0 \\
.00252 & 0 & 0 & 0 \\
.00252 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
To describe the connection between the state spaces, see Figure 2.

5. The category PSS

In this section, we prove that the PSS with the homomorphisms form a category, that we denote by PSS. In Theorem 5.3, we prove that the category of Sequential Dynamical Systems SDS is a full subcategory of PSS.

Theorem and Definition 5.1. Let

\[ \mathcal{H}_1 = (\phi_1, h_{\phi_1}) : \mathcal{F} \to \mathcal{G} \text{ and } \mathcal{H}_2 = (\phi_2, h_{\phi_2}) : \mathcal{G} \to \mathcal{L} \]

be two PSS-homomorphisms. Then the composition \( \mathcal{H} = \mathcal{H}_2 \circ \mathcal{H}_1 : \mathcal{F} \to \mathcal{L} \) is defined as follows: \( \mathcal{H} = (\phi_1 \circ \phi_2, h_{\phi_2} \circ h_{\phi_1}) \).

The map \( \mathcal{H} : \mathcal{F} \to \mathcal{L} \) is a PSS-homomorphism.

Proof. The composite \( \phi = \phi_2 \circ \phi_1 : \Lambda \to \Gamma \) of two graph morphisms is obviously again a graph morphism. The composite \( h_{\phi} = h_{\phi_2} \circ h_{\phi_1} \) is again a digraph morphism which satisfies the conditions (3.1.1), and (3.1.2). In fact, using the Proposition and Definition 2.7 in [11], we can check these condition and conclude that the third condition holds, too by Theorem 2.1.1. So, \( (\phi, h_{\phi}) \) is again a PSS-homomorphism. \( \square \)

Theorem 5.2. The Probability Sequential Systems together with the homomorphisms of PSS form the category PSS.

Proof. Easily follows from Definition and Theorem 5.1. \( \square \)

Theorem 5.3. The SDS together with the morphisms defined in 3.1 form a full subcategory of the category PSS.

Proof. It is trivial. \( \square \)

In paper [11], the authors proved that the category SDS has finite product, similarly the category PSS has finite product too, so the following theorem for two PSS holds [12]

Theorem 5.4. Let \( \mathcal{D}_S \) and \( \mathcal{G}_X \) be two PSS over the finite field \( \mathcal{K} \). For all \( \epsilon \)-homomorphisms \( (\phi_1, h_{\phi_1}) : \mathcal{L} \to \mathcal{D}_S \) and \( (\phi_2, h_{\phi_2}) : \mathcal{L} \to \mathcal{G}_X \), then there exists a morphism \( (\phi, h_{\phi}) : \mathcal{L} \to \mathcal{D}_S \times \mathcal{G}_X \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{D}_S \times \mathcal{G}_X & \xrightarrow{\pi_1} & \mathcal{D}_S \\
\downarrow \phi \downarrow h_{\phi} & & \downarrow \phi \downarrow h_{\phi} \\
\mathcal{L} & \xrightarrow{\phi_1 \times \phi_2} & \mathcal{D}_S \end{array}
\]

\[
\begin{array}{ccc}
\mathcal{D}_S \times \mathcal{G}_X & \xrightarrow{\pi_2} & \mathcal{G}_X \\
\downarrow \phi \downarrow h_{\phi} & & \downarrow \phi \downarrow h_{\phi} \\
\mathcal{L} & \xrightarrow{\phi_2 \times \phi_2} & \mathcal{G}_X
\end{array}
\]

\[\mathcal{D}_S \times \mathcal{G}_X \]
5.1. The functor \( T : \text{PSS} \to \text{Ab} \). Let \( D = (\Gamma, (F_i)_{i=1}^n, (\alpha_j)_j, S, C) \) be a PSS over the finite field \( \mathcal{K} \). Let \( F = \{ f : \mathcal{K}^n \to \mathcal{K}^n \} = S \cup S \) be the set of all update functions that we can construct with the local functions and the permutation in \( \mathcal{F}_S \). Let us consider the free abelian group generated by \( F \), that we denote by \( \langle F \rangle \). We can notice that we are working over a finite field with characteristic the prime number \( p \), so \( p f = 0 \) for all \( f \in S \). We will take the quotients of these groups by \( p \), that is \( \langle F \rangle / \langle p F \rangle = \langle S \rangle / \langle p S \rangle \). Denoting \( \langle A \rangle / \langle p A \rangle = \langle A \rangle_p \), for an abelian group \( A \), we rewrite the above relation by \( \langle F \rangle_p = \langle S \rangle_p \). The functor \( T \) gives the possibility to work with PSS using group theory, for example, because \( \langle F \rangle_p = \langle S \rangle_p \), and there exists a covariant functor from the category \( \text{PSS} \) to the category of small abelian groups with morphism of such, \( \text{Ab} \), defined as follows.

\[
T : \text{PSS} \to \text{Ab}
\]

1. The object function is defined by \( T(F_S) = \langle S \rangle_p \).
2. The arrow function which assigns to each homomorphism \( (\phi, h_{\phi}) : (\mathcal{F}_S) \to (\mathcal{G}_X) \) in the category \( \text{PSS} \) an homomorphism of abelian groups \( T(\phi, h_{\phi}) = H_{\phi} : \langle S \rangle_p \to \langle X \rangle_p \) which is defined in a natural way, because \( h_{\phi} \circ (\sum_{f \in S} af) = (\sum_{g \in S_{\phi}} ag) \circ h_{\phi} \), where \( a \in \mathbb{Z}_p \), and \( S_{\phi} \subseteq X \), then

\[
H_{\phi}(\sum_{f \in S} af) = (\sum_{g \in X} ag).
\]

\( T \) is a functor, in fact,

1. \( T(1_{F_S}) = 1_{\langle S \rangle_p} \).
2. \( T((\phi_2, h_{\phi_2}) \circ (\phi_1, h_{\phi_1})) = T((\phi_2, h_{\phi_2})) \circ T((\phi_1, h_{\phi_1})) \)

We assign probabilities to the set \( S \) in some way, and we consider that all possible different assignments are \( \epsilon \)-isomorphic.

**Definition 5.5 (Complement of a PSS).** A complement of the PSS \( \mathcal{D}_S \) is the PSS \( \mathcal{D}_{\overline{S}} \), and all of the complement are \( \epsilon \)-equivalent, so we can select a distribution of probabilities for the complement having in account particular applications.

We use the definition of complement in order to define a decomposition of a PSS in two sub PSS, only looking the set of functions. One of the mean problem in modeling dynamical systems is the computational aspect of the number of functions and the calculation of steady states in the state space, in particular the reduction of number of functions is one of the most important problem to solve for determine which part of the network state space could be simplify.

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