Photon & Axion Oscillation In a Magnetized Medium: A Covariant Treatment

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October 2008

Abstract
Pseudoscalar particles, with almost zero mass and very weak coupling to the visible matter, arise in many extensions of the standard model of particle physics. Their mixing with photons in the presence of an external magnetic field leads to many interesting astrophysical and cosmological consequences. This mixing depends on the medium properties, the momentum of the photon and the background magnetic field. Here we give a general treatment of pseudoscalar-photon oscillations in a background magnetic field, taking the Faraday term into account. We give predictions valid in all regimes, under the assumption that the frequency of the wave is much higher than the plasma frequency of the medium. At sufficiently high frequencies, the Faraday effect is negligible and we reproduce the standard pseudoscalar-photon mixing phenomenon. However at low frequencies, where Faraday effect is important, the mixing formulae are considerably modified. We explicitly compute the contribution due to the longitudinal mode of the photon and show that it is negligible.

1 Introduction
The standard model of particle physics suggests the existence of a very light, weakly coupled particle, called the axion. It arises as a pseudo-Goldstone boson of the broken Peccei Quinn (PQ) symmetry in a generalization of the standard model \cite{1–7}. Similar particles are also predicted by supergravity \cite{8} and superstrings theory \cite{9, 10}. Such a pseudoscalar particle has an effective coupling to two photons. As a consequence, in an external magnetic field, an axion can oscillate to a photon and vice versa \cite{11–18}. The mixing can change both the intensity and the state of polarization of the photons. Since the mixing is dependent on the frequency of the wave, it also leads to a change in the spectrum of electromagnetic radiation \cite{19–23}. This phenomenon has been used for laboratory and astrophysical searches for such particles, leading to stringent limits \cite{24–34}. Furthermore, the astrophysical and cosmological consequences of this mixing have been studied extensively in the literature \cite{33–44}. The mixing increases the transparency of the intergalactic and galactic medium to propagation of high energy photons due to the very weak coupling of pseudoscalars to visible matter \cite{45–47}.

Inside a medium, there exists an additional loop induced, axion-photon vertex, providing an extra contribution to the usual axion-photon coupling in vacuum. In some kinematical limit this contribution has medium, momentum and magnetic field dependence. In vacuum one of the two transverse modes of photon gets coupled to an axion. In a medium the photon acquires an extra degree of freedom (i.e., the
longitudinal mode) and as a consequence axion also couples to this additional degree of freedom. In a magnetized medium, the two transverse degrees of freedom are usually coupled due to the Faraday effect. Hence in the presence of axions the two transverse modes would also get coupled to the longitudinal degree of freedom.

In this paper we study the axion-photon evolution, taking the Faraday term into account. Our objectives in this study are two-folds. First we give a general treatment of pseudoscalar-photon mixing in a medium. This allows us to compute the next to leading order temperature dependent corrections. Next we give general solutions to the oscillation problem, taking into account the Faraday effect. At very large frequencies we expect that the Faraday effect would be negligible and the standard treatment of pseudoscalar-photon mixing would apply. However at low frequencies the standard pseudoscalar-photon mixing result may deviate significantly due to the presence of near degeneracies in the mixing matrix. In several laboratory and astrophysical situations this regime may be applicable and hence our treatment would be useful in such cases. In the present paper we restrict ourselves to providing a general treatment and do not address the issue of applications to laboratory experiments or to astrophysics.

1.1 Interactions and the Polarization Tensor in a Medium

The classical Lagrangian of a free electromagnetic field, including the gauge fixing term is given by,

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2\zeta} (\partial_\alpha \Lambda^{\alpha}(x))^2 + j^{\text{ext}} \cdot A .
\]  

(1.1)

In Eq. (1.1), \( F_{\mu\nu}(x) = (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) \), \( \zeta \) is the gauge fixing parameter, \( j^{\text{ext}} \) is the external current and \( A(x) \)'s are the vector potential. For the sake of simplicity, we use the Feynman gauge and set \( \zeta = 1 \).

As one takes quantum corrections into account, the quadratic part of the tree level Lagrangian gets modified because of quantum corrections coming from terms proportional to the vacuum polarization tensor \( \Pi_{\mu\nu} \). The resulting Lagrangian, in the momentum space, is

\[
\mathcal{L} = \frac{1}{2} \left[ -k^2 \tilde{g}_{\mu\nu} + \Pi_{\mu\nu}(k) \right] A^\mu(k) A^\nu(-k) - j^{\text{ext}} A^\mu(k) + \mathcal{L}_G .
\]  

(1.2)

In Eq. (1.2) above, \( \mathcal{L}_G \) corresponds to the gauge fixing term and \( \tilde{g}_{\mu\nu} = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \). In presence of a medium of finite density and temperature, the polarization tensor get corrections from the matter and temperature dependent parts. The resulting equation of motion for such a system can be written as,

\[
[ -k^2 \tilde{g}_{\mu\nu} + \Pi_{\mu\nu}(k) ] A^\nu(k) = j^{\text{ext}} ,
\]  

(1.3)

where we have not retained the pieces coming from the gauge fixing term (i.e. terms proportional to \( \frac{k_\mu k_\nu}{k^2} \)) since they can be shown to vanish. In a medium composed of electrons, the polarization tensor, \( \Pi_{\mu\nu}(k) \), can be written in terms of scalar form factors and tensors composed out of the four vectors available for the system, i.e., the medium four velocity \( u^\mu = (1, 0, 0, 0) \) and \( k^\mu = (\omega, \vec{k}) \). They are expressed as follows [48],

\[
\Pi_{\mu\nu}(k) = \Pi_T R_{\mu\nu} + \Pi_L Q_{\mu\nu} ,
\]

(1.4)

where

\[
\begin{align*}
Q_{\mu\nu} &= \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} , \\
R_{\mu\nu} &= \tilde{g}_{\mu\nu} - Q_{\mu\nu} ,
\end{align*}
\]

in absence of any external field. The vector \( \tilde{u}_\mu \) is given by \( \tilde{u}_\mu = \tilde{g}_{\mu\nu} u^\nu \). The scalar form factor \( \Pi_L(k) \), corresponding to the longitudinal degree of freedom is given by,

\[
\Pi_L(k) = -\frac{k^2}{|\vec{k}|^2} \Pi_{\mu\nu}(k) u^\mu u^\nu , \quad \text{where,} \quad u^\mu u^\nu \Pi_{\mu\nu}(k) = \omega_p^2 \left( \frac{|\vec{k}|^2}{\omega^2} + 3 \frac{|\vec{k}|^4 T}{\omega^4 m^2} \right)
\]  

(1.5)
Similarly the transverse form factor $\Pi_T$ is given by the following expression:

$$\Pi_T(k) = R^{\mu\nu} \Pi_{\mu\nu}(k)$$

and,

$$R^{\mu\nu} \Pi_{\mu\nu}(k) = \frac{\omega_p^2}{1 + \frac{|\vec{k}|^2}{\omega^2}}.$$  \hspace{1cm} (1.6)

In the expressions above $\omega_p$ denotes the plasma frequency. In the classical limit, to leading order in $\frac{T}{m}$, it is given by:

$$\omega_p = \sqrt{\frac{4\pi n_e m}{\hbar^2}} \left( 1 - \frac{5T^2}{2m} \right).$$  \hspace{1cm} (1.7)

In Eq. (1.7) $n_e$ is the number density of electrons.

We next express the form factors in terms of the dielectric constants. Denoting, $F_{\mu\nu} = -i(k_{\mu} A_{\nu} - k_{\nu} A_{\mu})$, so that $\vec{E} = i\omega \vec{A} - i\vec{k} \times \vec{A}$, one can further define longitudinal and transverse parts of the electric field as, $E_l = k(\vec{k} \cdot \vec{E})$, $E_t = \vec{E} - E_l$. Using the Kubo formula \cite{1} for linear response analysis, the induced current, $j^{ind}(k)$, can be written as,

$$j^{ind} = i\omega \left[ (1 - \epsilon_l) E_l + (1 - \epsilon_t) E_t \right]$$

where longitudinal dielectric function $\epsilon_l$ and transverse dielectric function $\epsilon_t$ are given as follows,

$$(1 - \epsilon_l) = \frac{\Pi_l}{\omega^2}$$

$$(1 - \epsilon_t) = \frac{\Pi_T}{\omega^2}.$$ \hspace{1cm} (1.9)

Using the relations given by Eq. (1.6) for $\Pi_T(k)$ and $\Pi_L(k)$, one can further express the dielectric functions in terms of $\omega$ and $\vec{k}$.

## 2 Inclusion of the Faraday Term

### 2.1 Exact Faraday Term

The contribution to the vacuum polarization tensor which is odd in $B$ has been estimated using real time finite temperature field theory in Refs. [49, 50]. It can be expressed as,

$$\Pi_{\chi\rho}(k) = 4ie^2 \int \frac{d^4 p}{(2\pi)^4} \eta_{-}(p) \int_{-\infty}^{\infty} ds \ e^{\Phi(p,s)} \int_{0}^{\infty} ds' e^{\Phi(p',s')} \left[ R_{\lambda\rho}^{(1)} + R_{\lambda\rho}^{(2a)} \right]$$

$$= 4ie^2 \varepsilon_{\chi\alpha\beta\gamma} k^\beta \int \frac{d^4 p}{(2\pi)^4} \eta_{-}(p) \int_{-\infty}^{\infty} ds \ e^{\Phi(p,s)} \int_{0}^{\infty} ds' e^{\Phi(p',s')} \times \left[ \frac{\vec{p}^\alpha}{\tan eBs + \vec{p}^\alpha} \tan eBs' - \frac{\tan eBs \ tan eBs'}{\tan eBs' + \tan eBs' + (p + p')} \right].$$ \hspace{1cm} (2.1)

This is exact to all orders in $eB$.

In Eq. (2.1) $\alpha_{\parallel}$ stands for 0 or the 3. Moreover appearance of $\alpha_{\parallel}$ and $\alpha_{\parallel}$ together in any product would mean: if $\alpha_{\parallel}$ takes the value 0, $\alpha_{\parallel}$ would take the value 3 and vice versa. Furthermore for thermal fermions, $\eta_F(p)$ is the distribution function,

$$\eta_F(p) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta).$$ \hspace{1cm} (2.2)

---

\(^{(1)}\) i.e. $j^{ind}_{\mu}(k) = -\Pi_{\mu\nu}(k) A^\nu(k)$. 

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Here, \( \Theta \) is the step function, which takes the value \(+1\) for positive values of its argument and vanishes for negative values of the argument, and \( f_F \) denotes the Fermi-Dirac distribution function,

\[
f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p\cdot u - \mu)} + 1}. \tag{2.3}
\]

Lastly \( \phi(p, s) \) in the exponential stands for,

\[
\Phi(p, s) = \frac{1}{8} \sum_{i=1}^{3} k_i^2 |K| \frac{k^i}{|K|} u^\beta = \Pi^p(k) P_{\mu\nu},
\]

with \( P_{\mu\nu} = i \epsilon_{\mu\nu\alpha\beta} k^\alpha |K| u^\beta \).

The scalar form factor associated with the Faraday rotation term is given by \[50\\]

\[
\Pi^p(k) = \frac{\omega B \omega_p^2}{\omega^2 - \omega_B^2}, \quad \text{where} \quad \omega_B = \frac{eB}{m}. \tag{2.7}
\]

### 2.2 Equation of Motion with Faraday Contribution

In presence of external magnetic field, Eq. \[13\\] is further modified to \[49, 50\\]

\[
\left[-k^2 \tilde{g}_{\mu\nu} + \Pi^p_{\mu\nu}(k) + \Pi^p_{\mu\nu}(k)\right] A^\nu(k) = j^\text{ext}_{\mu}. \tag{2.6}
\]

We define,

\[
|K| = \left[ \sum_{i=1}^{3} k_i^2 \right]^{\frac{1}{2}}.
\]

In the limit \( |K| \to 0 \), one can express,

\[
\Pi^p_{\mu\nu}(k) = \Pi^p(k) i \epsilon_{\mu\nu\alpha\beta} \frac{k^\alpha}{|K|} u^\beta = \Pi^p(k) P_{\mu\nu},
\]

with \( P_{\mu\nu} = i \epsilon_{\mu\nu\alpha\beta} k^\alpha |K| u^\beta \).

The limit \( |K| \to 0 \) should be taken in such a way that

\[
\mathcal{L}_{|K|\to0} \left( \frac{k^i}{|K|} \right) \to 1.
\]

The scalar form factor associated with the Faraday rotation term is given by \[50\\]

\[
\Pi^p(k) = \frac{\omega B \omega_p^2}{\omega^2 - \omega_B^2}, \quad \text{where} \quad \omega_B = \frac{eB}{m}. \tag{2.7}
\]

### 2.3 Axion Electrodynamics

In this subsection we obtain an expression for the effective Lagrangian for axion-electrodynamics in a magnetized medium. It is important to note here that the axion contribution to the effective Lagrangian in a magnetized medium does get modified due to the presence of medium and magnetic field. However all the modifications can be included by redefining the axion-photon coupling constant \[18\\]. The structure of the interaction Lagrangian remains identical to that at tree level. In light of this observation, we work with the tree level axion-photon Lagrangian in a magnetic field. In momentum space this effective Lagrangian is given by:

\[
\mathcal{L} = \frac{1}{2} \left[ -A_\mu k^2 \tilde{g}^{\mu\nu} A_\nu + A_\mu \Pi^{\mu\nu} A_\nu + i \frac{\tilde{F}^{\mu\nu} k_\mu A_\nu a}{2M_a} - a(k^2 - m^2)a \right]. \tag{2.8}
\]
In the expression above, $\tilde{F}^{\mu\nu}$ corresponds to the field strength tensor of the external field, $A_{\mu}$ is the vector potential for the photon, $a$ is the axion field, $1/M_a$ is the pseudoscalar-photon coupling and $\tilde{\Pi}^{\mu\nu}$ is the polarization tensor in matter along with the Faraday contribution,

$$\tilde{\Pi}^{\mu\nu}(k) = \Pi_T(k) R^{\mu\nu} + \Pi_L(k) Q^{\mu\nu}(k) + \Pi_p(k) P^{\mu\nu}. \quad (2.9)$$

The equations of motion for the Lagrangian given by Eq. (2.8) are the following:

$$\left(-k^2 \tilde{g}_{\alpha\nu} + \tilde{\Pi}_{\alpha\nu}(k)\right) A^\nu(k) = -i \frac{k^\mu \tilde{F}_{\mu\alpha} a}{2M_a} \quad (2.10)$$

$$(k^2 - m^2) a = i b^{(2)}_{\mu} A^\mu(k) \quad (2.11)$$

From now on we would denote $b^{(2)}_{\alpha} = k^\mu \tilde{F}_{\mu\alpha}$.

### 2.4 Expanding $A^\mu(k)$ in Orthogonal Basis.

In this subsection we construct a system of orthonormal basis vectors out of the vectors available to us. Recall that the available vectors we have at our disposal are, $u^\mu$ the 4 velocity of the medium and $k^\mu$ the external momentum of the photon. Using these two and $F^{\mu\nu}$ we can further construct two more vectors,

$$b^{(1)}_{\nu} = k_\mu F^{\mu\nu}, \quad (2.12)$$

and

$$b^{(2)}_{\nu} = k_\mu \tilde{F}^{\mu\nu}. \quad (2.13)$$

We also define,

$$\tilde{u}^\nu = \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) u_\mu. \quad (2.14)$$

We note that, the vectors $b^{(1)}$, $b^{(2)}$ and $\tilde{u}$ are all orthogonal to $k$. Now we can construct another vector, $I^\nu$, such that it is orthogonal to both $\tilde{u}$ and $b^{(2)}$,

$$I^\nu = \left( b^{(2)}_{\nu} - \frac{(\tilde{u}_{\mu} b^{(2)}_{\mu})}{\tilde{u}^2} \tilde{u}_{\nu} \right). \quad (2.15)$$

It is easy to verify that the vectors, $I^\nu$, $\tilde{u}^\nu$, $k^\nu$, $b^{(1)}_{\nu}$ are mutually orthogonal to each other. Therefore we can express the gauge potential as their linear combination,

$$A_\alpha(k) = A_1(k) N_1 b^{(1)}_{\alpha} + A_2(k) N_2 I_{\alpha} + A_L(k) N_L \tilde{u}_{\alpha} + k_{\alpha} N_{||} A_{||}(k). \quad (2.16)$$

The component, $A_{||}(k)$ can be set to zero, since it is associated with the gauge degrees of freedom. In Eq. (2.16) the $N_i$’s are normalization constants for the corresponding basis vectors. We can easily see that the normalization constants are given such to be,

$$N_1 = \frac{1}{\sqrt{-b^{(1)}_{\mu} b^{(1)}_{\mu}}} \quad (2.17)$$

$$N_2 = \frac{1}{\sqrt{-I_{\mu} I^\mu}} \quad (2.18)$$
\[ N_L = \frac{1}{\sqrt{-\tilde{u}_\mu \tilde{u}^\mu}} \]  

(2.19)

The negative sign under the square root is for maintenance of the reality of the normalization constants. These constants take the values,

\[ N_1 = \frac{1}{B_z K_\perp} \]
\[ N_2 = \frac{|\vec{K}|}{\omega K_\perp B_z} \]
\[ N_L = \frac{K}{|\vec{K}|} \]

3 Equation of Motion in terms of the Form Factors

The equation of motion for photon, Eq. (2.10), can be written in an expanded form,

\[ \left[ k^2 g_{\mu \nu} - \Pi_T(k) R_{\mu \nu} - \Pi_L(k) Q_{\mu \nu}(k) - \Pi_p(k) P_{\mu \nu} \right] A^\nu(k) = \frac{i b^{(2)} A}{2M_a} \]  

(3.1)

where we have used Eq. (2.9). In order to arrive at the equations of motions in terms of the form factors, one needs to substitute Eq. (2.16) in Eq. (3.1) and project out the different components of this equation. Multiplication from left by the normalized basis vectors, \( N_1 \times b^{(1)}_\mu, N_2 \times I_\mu \) and \( N_L \times \tilde{u}_\mu \) leads to the equations,

\[ - (k^2 - \Pi_T(k)) A_2(k) + i \Pi_p N_1 N_2 \left[ \epsilon_{\mu \perp, \nu \perp} b^{(1) \mu} \right] N_1 A_1(k) = \frac{i N_2 b^{(2)}_\mu P^\mu}{2M_a}, \]
\[ (k^2 - \Pi_T(k)) A_1(k) + i \Pi_p N_1 N_2 \left[ \epsilon_{\mu \perp, \nu \perp} b^{(1) \mu} \right] A_2(k) = 0, \]
\[ (k^2 - \Pi_L) A_L(k) = \frac{i N_L \left( b^{(2)}_\mu \tilde{u}^\mu \right) a}{2M_a} \]  

(3.2)

respectively. The equation of motion for the pseudoscalar field can be expressed as,

\[ \left[ \frac{\left( i b^{(2)}_\mu \right) P^\mu}{2M_a} N_2 A_2(k) + \frac{\left( i b^{(2)}_\mu \tilde{u}^\mu \right)}{2M_a} N_L A_L(k) \right] = (k^2 - m^2) A. \]  

(3.3)

This completes the closed set of equations involving the axion and the photon vector potential. It is easy to see that, if one sets the axion field to be equal to zero, one recovers the usual Maxwell Equations modified by the Faraday pieces.

4 Mixing Matrix

One can observe from Eqs. (3.2), (3.3) that, the coupling to axions mixes the longitudinal component \( A_L \) to the transverse components. So even if \( A_L \) is zero to begin with, the same can be generated through coupling through the pseudoscalar field \( a \). However the coupling is suppressed by the PQ symmetry breaking scale. One needs to study these equations carefully to determine the effect of pseudoscalar-photon coupling and magnetized medium on the Electro-Magnetic (EM) vector potentials.

We are interested in a quasi-monochromatic wave solution. We shall assume that the wave propagates in the \( z \) direction and express the solution in the form

\[ \phi_i(t, z) = e^{-i\omega t} \phi_i(0, z) \]  

(4.1)
where $\phi_i$ may represent the pseudoscalar field or any of the components of the electromagnetic wave. We work in the eikonal limit, where $\omega \approx k$ is the largest energy scale. We may now express Eqs. (8.2), (8.3) in real space in the matrix form

$$\left[(\omega^2 + \partial_z^2)I - M\right] \begin{pmatrix} A_1(k) \\ A_2(k) \\ A_L(k) \\ a(k) \end{pmatrix} = 0. \quad (4.2)$$

where $I$ is a $4 \times 4$ identity matrix and the mixing matrix,

$$M = \begin{pmatrix} \Pi_T & -i\Pi_p \epsilon_{\mu \nu \lambda} \delta_0 b(1)^{\mu} I^\nu & 0 & 0 \\ i\Pi_p \epsilon_{\mu \nu \lambda} \delta_0 b(1)^{\mu} I^\nu & +\Pi_T & 0 & 0 \\ 0 & 0 & \Pi_L & 0 \\ 0 & iN_2 b(2)^{\mu} \mu / 2M_a & 0 & iN_L b(2)^{\mu} \mu / 2M_a \end{pmatrix}. \quad (4.3)$$

In order to find out axion-photon oscillation, we need to diagonalize the matrix given in Eq. (4.3). Although it can be diagonalized exactly, the resulting formulae are too cumbersome to be directly useful. We make some simplifying assumptions to make the problem tractable. The longitudinal component has been shown to contribute negligibly in Ref. [51]. Hence at leading order we set $A_L(z) = 0$ and compute its contribution perturbatively.

Our final aim is to compute the Stokes parameters $I(z)$, $Q(z)$, $U(z)$, $V(z)$ at some distance $z$, given the values of these parameters at the origin $z = 0$. For this purpose we may compute the density matrix

$$\rho(z) = \begin{pmatrix} < A_1 A_1^* > & < A_1 A_2^* > & < A_1 a^* > \\ < A_2 A_1^* > & < A_2 A_2^* > & < A_2 a^* > \\ < a A_1^* > & < a A_2^* > & < aa^* > \end{pmatrix}. \quad (4.4)$$

where the angular brackets $<>$ represent ensemble averages. We work in the Lorentz gauge $\Box A = 0$ and hence from these matrix we can directly compute the coherency matrix

$$\tilde{\rho}(z) = \begin{pmatrix} < E_1 E_1^* > & < E_1 E_2^* > \\ < E_2 E_1^* > & < E_2 E_2^* > \end{pmatrix}. \quad (4.5)$$

where $E_i$ are the electric fields.

## 5 Solutions

We now solve the equations of motion, ignoring the longitudinal mode. We work in the low temperature limit and ignore the terms proportional to temperature. We may express the resulting mixing matrix $M$ as

$$M = \begin{pmatrix} A & iF & 0 \\ -iF & A & -iT \\ 0 & iT & B \end{pmatrix}, \quad (5.1)$$

where $A = \omega_p^2$, $B = m_a^2$, $F = \omega \omega_B \omega_p^2 \cos \theta / (\omega^2 - \omega_B^2)$, and $T = |\vec{B}| \omega \sin \theta / 2M_a$. Here $\theta$ is the angle between the background magnetic field and the direction of propagation. Throughout we shall assume that the background magnetic field is independent of space and time. The component transverse to the direction of propagation is taken to point along the ‘$y$’ or ‘$2$’ axis.

We denote the eigenvalues of $M$ by $\lambda_i$. We define an alternate matrix $\bar{M}$,

$$\bar{M} = (M - AI)/(B - A) = \begin{pmatrix} 0 & ix & 0 \\ -ix & 0 & -iy \\ iy & 0 & 1 \end{pmatrix}. \quad (5.2)$$
where $I$ is an identity matrix, $x = F/(B - A)$ and $y = T/(B - A)$. The eigenvalues $\tilde{\lambda}_i$ of $\tilde{M}$ are related to $\lambda_i$ by
\[
\lambda_i = \bar{\lambda}_i(B - A) + A.
\] (5.3)
The eigenvectors of $\tilde{M}$ (or $M$) may be expressed as
\[
|\bar{\lambda}_i\rangle = \frac{1}{D_i} \left( \begin{array}{c}
x(\bar{\lambda}_i - 1) \\
-i\bar{\lambda}_i(\bar{\lambda}_i - 1) \\
y\bar{\lambda}_i
\end{array} \right),
\] (5.4)
where $D_i^2 = 2x^2(\bar{\lambda}_i^2 - 1)^2 + y^2(2\bar{\lambda}_i^2 - \bar{\lambda}_i)$. The eigenvalues $\bar{\lambda}_i$ may be written as
\[
\bar{\lambda}_1 = 2\sqrt{-Q}\cos \left( \frac{\theta + 4\pi}{3} \right) + \frac{1}{3},
\]
\[
\bar{\lambda}_2 = 2\sqrt{-Q}\cos \left( \frac{\theta + 2\pi}{3} \right) + \frac{1}{3},
\]
\[
\bar{\lambda}_3 = 2\sqrt{-Q}\cos \left( \frac{\theta}{3} \right) + \frac{1}{3},
\] (5.5)
where
\[
\theta = \cos^{-1} \left( \frac{R}{\sqrt{-Q^3}} \right),
\]
\[
R = \frac{1}{54} (9y^2 - 18x^2 + 2),
\]
\[
Q = -\frac{x^2 + y^2}{3} - \frac{1}{9}.
\] (5.6)
We point out that for a cubic equation we expect three real eigenvalues if $Q^3 + R^2 < 0$, which must of course be valid in our case.

Using this we construct the unitary matrix $U$ to diagonalize $M$ and hence solve the propagation equation. The final result for the density matrix after propagating a distance $z$ from the initial point is
\[
\rho(z) = UPU^{\dagger} \rho(0)UP^{\dagger}U^{\dagger},
\] (5.7)
where $P$ is the propagation matrix,
\[
P = \begin{pmatrix}
\exp(i\bar{k}_1z) & 0 & 0 \\
0 & \exp(i\bar{k}_2z) & 0 \\
0 & 0 & \exp(i\bar{k}_3z)
\end{pmatrix},
\] (5.8)
and $\bar{k}_i = \sqrt{\omega^2 - \bar{\lambda}_i} \approx \omega - \frac{\lambda_i}{2\omega}$.

Although the problem is solvable exactly, we find that there exist parameter ranges where the exact formulas are not very useful for numerical calculations. Furthermore in order to get some insight into the solution it is useful to explore it analytically in some limiting cases. The problem is a little tricky due to near degeneracy of the two modes of the electromagnetic wave.

For astrophysical and cosmological applications we are usually interested in the case where both the parameters $T$ and $F$ are very small compared to $|A - B|$, i.e. $x << 1$ and $y << 1$. This is the case of weak magnetic field and pseudoscalar-photon coupling. In laboratory experiments the parameter $y$ may not be small compared to unity if the plasma density is taken to be very small. We postpone a detailed discussion of laboratory experiments to a separate publication. For now we discuss the astrophysically interesting limit of $x << 1$ and $y << 1$. 

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We expand the eigenvalues in powers of \( x \) and \( y \). This expansion is a little messy since we find that we need to include terms at least up to order \( y^4 \). This is because in the limit \( x \to 0 \), these terms contribute at leading order. The expression \( R/\sqrt{-Q^3} \to 1 \) in the limit \( x \to 1, y \to 1 \). Hence we may express
\[
R/\sqrt{-Q^3} = 1 - \alpha - \beta ,
\]
where
\[
\alpha = \frac{27x^2}{2} << 1, \\
\beta = \frac{27}{8}y^4 - \frac{27 \times 17}{8}x^4 - 54x^2y^2 << 1.
\]
The \( \cos^{-1}(1 - \alpha - \beta) \) function has a non-analytic structure in the neighbourhood of unity. Expanding in powers of \( x \) and \( y \) we find [52]
\[
\theta = \cos^{-1}(1 - \alpha - \beta) = \frac{\sqrt{\alpha + \beta}}{\sqrt{2}} \left[ 2 + \frac{\alpha + \beta}{6} + \frac{3(\alpha + \beta)^2}{80} + ... \right].
\]
The eigenvalues are approximately given by
\[
\tilde{\lambda}_1 = -\frac{1}{2}(x^2 + y^2) + \frac{2}{3\sqrt{6}}\sqrt{\alpha + \beta} + \frac{3}{8}(x^2 + y^2)^2 + \frac{1}{\sqrt{6}}(x^2 + y^2)\sqrt{\alpha + \beta} \\
+ \frac{1}{27}(\alpha + \beta) + \frac{\alpha}{18}(x^2 + y^2), \\
\tilde{\lambda}_2 = -\frac{1}{2}(x^2 + y^2) - \frac{2}{3\sqrt{6}}\sqrt{\alpha + \beta} + \frac{3}{8}(x^2 + y^2)^2 - \frac{1}{\sqrt{6}}(x^2 + y^2)\sqrt{\alpha + \beta} \\
+ \frac{1}{27}(\alpha + \beta) + \frac{\alpha}{18}(x^2 + y^2), \\
\tilde{\lambda}_3 = 1 + (x^2 + y^2) - \frac{2}{27}(\alpha + \beta) - \frac{3}{4}(x^2 + y^2)^2 - \frac{1}{9}(\alpha + \beta)(x^2 + y^2).
\]
Here we have displayed terms accurate to order \( x^2, y^4 \) and \( x^2y^2 \). We again emphasize that in the limit \( x \to 0 \), the dominant contribution comes from terms proportional to \( y^4 \). Hence we need expand up to order \( y^4 \). As we can see the expansion is quite cumbersome and does not yield a simple analytic result. However it is suitable for numerical calculations in the limit the parameters \( x \) and \( y \) are very small. We next consider some limiting cases in which a simple analytic result can be obtained.

### 5.1 Limit 1: \( T << F << |A - B| \)

This is the regime where the Faraday effect is dominant and hence relevant at low frequencies. We expand the eigenvalues and eigenvectors in powers of \( T \) and obtain results accurate to order \( T^2 \). This expansion may also be obtained directly by using perturbation theory. We may first exactly diagonalize the matrix \( M \) in the limit \( T = 0 \) and then compute leading order corrections in \( T \). The resulting eigenvalues are given by
\[
\lambda_1 = A - F + \frac{T^2}{2(A - F - B)} \\
\lambda_2 = A + F + \frac{T^2}{2(A + F - B)} \\
\lambda_3 = B - \frac{T^2}{2(A - F - B)} - \frac{T^2}{2(A + F - B)}
\]
with the corresponding eigenvectors
\[
|\lambda_1> = \frac{1}{\sqrt{2}} \left( i \left[ 1 - \frac{T^2}{4(A - F - B)^2} + \frac{T^2}{4F(A - F - B)} \right] \right)
\]
\[
|\lambda_2> = \frac{1}{\sqrt{2}} \left( i \left[ 1 - \frac{T^2}{4(A + F - B)^2} - \frac{T^2}{4F(A + F - B)} \right] \right)
\]
\[
|\lambda_3> = \frac{1}{\sqrt{2}} \left( i \left[ 1 - \frac{T^2}{4(A - F - B)^2} - \frac{T^2}{4(A + F - B)} \right] \right)
\]
\begin{align*}
|\lambda_2| &= \frac{1}{\sqrt{2}} \left( 1 - \frac{T^2}{4(A+F-B)^2} \right) \\
|\lambda_3| &= \frac{1}{\sqrt{2}} \left( i \left[ 1 - \frac{T^2}{4(A+F-B)^2} - \frac{T^2}{4F(A+F-B)} \right] \right)
\end{align*}

(5.14)

In the limit under consideration we may expand the denominators in powers of \(F/(A-B)\). Since \(F \gg T\), we compute the density matrix accurate to order \(T^2/[F(A-B)]\) and drop terms proportional to \(T^2/(A-B)^2\). We also assume that all correlators involving the pseudoscalar field are zero initially. The relevant elements of the density matrix are given by

\[
\rho_{11}(Z) = \frac{1}{2} \left[ (\rho_{11}(0) + \rho_{22}(0)) + \frac{\cos(\Delta_{12}Z)}{2} (\rho_{11}(0) - \rho_{22}(0)) \right] + c.c.
\]

\[
\rho_{22}(Z) = \frac{1}{2} \left[ (\rho_{11}(0) + \rho_{22}(0)) - \frac{\cos(\Delta_{12}Z)}{2} (\rho_{11}(0) - \rho_{22}(0)) \right] + c.c.
\]

\[
\rho_{12}(Z) = \frac{\sin(\Delta_{12}Z)}{2} \left[ (\rho_{11}(0) - \rho_{22}(0)) + \frac{iT}{4F(A-B)}(1 - \cos(\Delta_{12}Z))(\rho_{11}(0) + \rho_{22}(0)) \right] + c.c.
\]

where \(\Delta_{12} = (\lambda_1 - \lambda_2)/(2\omega)\). We find the usual Faraday rotation along with additional contributions proportional to \(T^2/[F(A-B)]\). If the initial beam is unpolarized or linearly polarized, pseudoscalar-photon will generate circular polarization. The dominant contribution arises if the initial beam has a non-zero real part of \(\rho_{12}\) or equivalently a non-zero Stokes parameter \(U\). In this case in the limit \(\Delta_{12}Z << 1\), the circular polarization generated is proportional to \(T^2Z/[\omega(A-B)]\). Hence this contribution is proportional to \(\omega\).

\subsection*{5.2 Limit 2: \(F/|A-B| << T^2/|A-B|^2 << 1\)}

We next consider another limiting case where the Faraday rotation effect is negligible and pseudoscalar-photon mixing dominates. This is the case where the leading order contribution leads to the standard pseudoscalar-photon mixing. In this limit we can also obtain the results by first diagonalizing the matrix \(M\) with \(F = 0\) and then obtaining leading order corrections in \(F\). From the exact formulas we find that this expansion is valid only in the range,

\[
\delta_F = \frac{x}{y^2} = \frac{F(A-B)}{T^2} << 1
\]

(5.16)

It is clear that the range over which the leading order results are valid is considerably limited due to the nonlinear dependence on \(T\) in this inequality.

The three eigenvalues, accurate to order \((F/T)^2\) are given by

\[
\lambda_1 = A - (A-B) \left( \frac{F}{T} \right)^2
\]

\[
\lambda_2 = A + \frac{T^2}{(A-B)} + (A-B) \left( \frac{F}{T} \right)^2
\]

\[
\lambda_3 = \frac{1}{2} \left( i \left[ 1 - \frac{T^2}{4(A+F-B)^2} - \frac{T^2}{4F(A+F-B)} \right] \right)
\]
\[ \lambda_3 = B - \frac{T^2}{(A-B)} \]  
(5.17)

We display the eigenvectors only to order \( \delta_F \) or up to terms linear in \( F \), since these give the dominant corrections. The corresponding eigenvectors are given by

\[
|\lambda_1 > = \begin{pmatrix} 1 \\ i\delta_F \\ 0 \end{pmatrix} \\
|\lambda_2 > = \frac{1}{D_T} \begin{pmatrix} \delta_F \\ -i \\ \delta_T \end{pmatrix} \\
|\lambda_3 > = \frac{1}{D_T} \begin{pmatrix} 0 \\ \delta_T \\ -i \end{pmatrix}
\]  
(5.18)

where \( D_T = \sqrt{1 + \delta_T^2}, \delta_T = T/(A-B) \). The density matrix elements, accurate to order \( \delta_F \) are

\[
\rho_{11}(Z) = \rho_{11}(0) + \left( i\rho_{12}(0)\delta_F \left[ 1 - e^{i(\lambda_1 - \lambda_2)Z/(2\omega)} \right] + c.c. \right)
\]

\[
\rho_{22}(Z) = \frac{\rho_{22}(0)}{D_T^2} \left[ 1 + \delta_T^2 e^{i(\lambda_2 - \lambda_3)Z/(2\omega)} \right]^2 + \left( i\rho_{12}(0)\delta_F \left[ e^{i(\lambda_2 - \lambda_1)Z/(2\omega)} - 1 \right] + c.c. \right)
\]

\[
\rho_{12}(Z) = -i\delta_F \rho_{11}(0) \left[ 1 - e^{i(\lambda_2 - \lambda_1)Z/(2\omega)} \right] + i\delta_F \rho_{22}(0) \left[ 1 - e^{i(\lambda_2 - \lambda_1)Z/(2\omega)} \right]
\]

\[
+ \frac{\rho_{12}(0)}{D_T^2} \left[ e^{i(\lambda_2 - \lambda_1)Z/(2\omega)} + \delta_T^2 e^{i(\lambda_3 - \lambda_1)Z/(2\omega)} \right]
\]  
(5.19)

5.3 Limit 3: \( F/|A-B| \sim T^2/|A-B|^2 << 1 \)

We finally consider the case where \( x \) and \( y^2 \) are of the same order but both are much smaller than unity. The eigenvalues \( \bar{\lambda}_i \) at leading order are

\[
\bar{\lambda}_1 = \sqrt{x^2 + y^4/4 - y^2/2} \\
\bar{\lambda}_2 = -\sqrt{x^2 + y^4/4 - y^2/2} \\
\bar{\lambda}_3 = 1 + y^2
\]  
(5.20)

with \( \lambda_i \) related to \( \bar{\lambda}_i \) by Eq. 5.3. The corresponding eigenvectors, accurate to order \( y \), are

\[
|\bar{\lambda}_1 > = \frac{1}{d_1} \begin{pmatrix} -1 \\ i\bar{\lambda}_1/x \\ y\bar{\lambda}_1/x \end{pmatrix} \\
|\bar{\lambda}_2 > = \frac{1}{d_2} \begin{pmatrix} -1 \\ i\bar{\lambda}_2/x \\ y\bar{\lambda}_2/x \end{pmatrix} \\
|\bar{\lambda}_3 > = \frac{1}{d_3} \begin{pmatrix} 0 \\ -iy \\ 1 \end{pmatrix}
\]  
(5.21)

where \( d_1 = \sqrt{1 + \bar{\lambda}_1^2/x^2} \), and \( d_2 = \sqrt{1 + \bar{\lambda}_2^2/x^2} \) and \( d_3 = 1 \). The density matrix elements in this case are given by

\[
\rho_{11}(Z) = \rho_{11}(0) \left( 1 - \frac{2}{d_1^2 d_2^2} [1 - \cos(\Delta_{12}Z)] \right) + \rho_{22}(0) \frac{2}{d_1^2 d_2^2} [1 - \cos(\Delta_{12}Z)]
\]
easily check that

\[ \rho_{22}(Z) = \rho_{11}(0) + \rho_{22}(0) (1 - \frac{2}{d_1^2 d_2^2} \{1 - \cos(\Delta_{12} Z)\}) \rho_{12}(0) + c.c. \]

\[ \rho_{12}(Z) = \frac{\rho_{11}(0)}{d_2^2} [\frac{1}{d_1^2} + \frac{1}{d_2^2} e^{i\Delta_{12} Z}] \rho_{12}(0) \]

where \( \Delta_{12} = (\lambda_1 - \lambda_2)/(2\omega) \). We point out that in the density matrix elements we have kept only the leading order terms. The terms of order \( y^2 \) have been dropped, unless they come multiplied by the distance factor \( Z \). This is reasonable since for very large \( Z \) values such terms may give significant contribution but are otherwise expected to be negligible compared to the terms we have kept.

### 6 Contribution of the Longitudinal Component

In the full mixing matrix \([43]\) we have so far ignored the longitudinal mixing terms, i.e., \( M_{34} \) and \( M_{43} \). We now determine the contribution of the longitudinal mode by treating these terms as perturbation. The matrix element \( M_{34} \) of the mixing matrix \([43]\) is given by,

\[ |M_{34}| = N_L h^{(2)}_\mu \bar{u}^\mu = \frac{\omega_p}{2M} |\vec{B}| \sin \theta \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{1/2} \]  

(6.1)

The full 4 × 4 matrix, given in Eq. \([43]\) may be expressed as,

\[ M = M_0 + M' = 
\begin{bmatrix}
A & iF & 0 & 0 \\
-iF & A & 0 & -iT \\
0 & 0 & \Pi_L & 0 \\
0 & iT & 0 & 0
\end{bmatrix} + 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & iL \\
0 & 0 & -iL & 0
\end{bmatrix} \]  

(6.2)

where \( L \approx -\omega_p |\vec{B}| \sin \theta/(2M) \) in the limit \( \omega >> \omega_p \). Here we treat \( M' \) as a perturbation. The contribution of the perturbation is small as long as

\[ \frac{L}{|\lambda_i - \lambda_j|} << 1, \]  

(6.3)

where \( \lambda_i \) are the unperturbed eigenvalues. In the limit of high frequencies, where the pseudoscalar-photon mixing effect may dominate, this difference is of order \( T^2/A \), ignoring the pseudoscalar mass. One can easily check that \( LA/T^2 << 1 \) for a wide range of parameters. In the low frequency limit, where Faraday effect dominates, the contribution of the longitudinal mode is also small. Hence we find that for a wide range of parameters we can treat the longitudinal mode perturbatively.
We next explicitly compute the modification of the eigenvectors and eigenvalues for the case discussed in section 5.3 above. Similar results apply in all cases. The eigenvectors and eigenvalues for the unperturbed matrix $M_0$ can be obtained from section 5.3. The unperturbed eigenvalues can be expressed as,

$$
\begin{align*}
\lambda_1^0 &= \frac{T^2}{2A} + A - \sqrt{F^2 + \frac{T^4}{4A^2}}, \\
\lambda_2^0 &= \frac{T^2}{2A} + A + \sqrt{F^2 + \frac{T^4}{4A^2}}, \\
\lambda_4^0 &= -\frac{T^2}{A},
\end{align*}
$$

where we have set $B = m_2^2 = 0$. Similarly, it follows that, the unperturbed eigenvectors,

$$
\begin{align*}
|\lambda_1^0 > &= \frac{1}{d_1} \begin{pmatrix} -1 \\ v_1 \\ 0 \\ w_1 \end{pmatrix} \\
|\lambda_2^0 > &= \frac{1}{d_2} \begin{pmatrix} -1 \\ v_2 \\ 0 \\ w_2 \end{pmatrix} \\
|\lambda_4^0 > &= \frac{1}{d_4} \begin{pmatrix} 0 \\ v_4 \\ 0 \\ w_4 \end{pmatrix}
\end{align*}
$$

where, $v_1 = i\bar{\lambda}_1/x$, $v_2 = i\bar{\lambda}_2/x$, $v_4 = -iy$, $w_1 = y\bar{\lambda}_1/x$, $w_2 = y\bar{\lambda}_2/x$, $w_4 = 1$, $d_4 = 1$ and $d_1$, $d_2$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are defined in section 5.3. We also have,

$$
\begin{align*}
\lambda_3^0 &= \Pi_L \\
|\lambda_3^0 > &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
$$

The leading order corrections to the eigenvalues are given by,

$$
M_j^j' = \left< \lambda_j^0 | M' | \lambda_j^0 \right>
$$

For all values for $j$, the leading order corrections to the eigenvalues due to the longitudinal perturbative piece vanish. The corrections to the eigenvectors are as follows,

$$
C_{jk} = \frac{\left< \lambda_j^0 | M' | \lambda_k^0 \right>}{(\lambda_k - \lambda_j)}
$$

We can verify that if $k \neq 3$ then the corrections to the eigenvectors are zero. Hence for $j = 1, 2, 4$, we have,

$$
|\lambda_j > = \sqrt{1 - |C_{j3}|^2} |\lambda_j^0 > + C_{j3} |\lambda_3^0 > .
$$

where,

$$
C_{j3} = \frac{-iLw_j}{d_j (\lambda_3 - \lambda_j)}
$$
Similarly, for $|\lambda_3|$ we can write,

$$
|\lambda_3| = \sqrt{1 - \sum_{j=\{1,2,4\}} |C_{3j}|^2} \sum_{j=\{1,2,4\}} C_{3j} |\lambda_j^0| >
$$

(6.11)

We find that for a wide range of parameters, $|C_{j3}| = |C_{3j}| < 1$. Hence we see that the contribution of the longitudinal part is small and can be neglected at leading order.

7 Conclusion

In this paper we have given a general treatment of pseudoscalar-photon mixing in a magnetized medium. We solve the resulting coupled wave equations in several different regimes with the assumption that the frequency of the electromagnetic wave is much larger than the plasma frequency. The problem is a little complicated due to the presence of near degeneracy in the two transverse modes of the electromagnetic wave. We find, as expected, that at very high frequencies, the Faraday effect gives negligible contribution and the standard treatment of pseudoscalar-photon mixing is applicable. However at low frequencies, Faraday effect cannot be neglected. We extend the pseudoscalar-photon oscillation formulas so that they are applicable in this regime also. We find that in this case the oscillation effect is considerably modified due to the presence of near degeneracy in the mixing matrix. These results may be useful in many laboratory experiments and in astrophysical observations.

8 Acknowledgement

We thank Amit Banerjee for collaborating in the initial stages of this work. S. Mandal thanks a Department of Science and Technology (DST) for financial support. P. Jain thanks John P. Ralston for pointing out potential problems that can arise in the phenomenon of pseudoscalar-photon mixing due to the presence of near degeneracies.

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