On the existence of distributional potentials

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Abstract
We present proofs for the existence of distributional potentials $F \in D'(\Omega)$ for distributional vector fields $G \in D'(\Omega)^n$, that is, $\text{grad} F = G$, where $\Omega$ is an open subset of $\mathbb{R}^n$. The hypothesis in these proofs is the compatibility condition $\partial_j G_k = \partial_k G_j$ for all $j, k \in \{1, \ldots, n\}$, if $\Omega$ is simply connected, and a stronger condition in the general case. A key tool in our treatment is the Bogovskiĭ formula, assigning vector fields $v \in D(\Omega)^n$ satisfying $\text{div} v = \varphi$ to functions $\varphi \in D(\Omega)$ with $\int \varphi(x) dx = 0$. The results are applied to properties of Hilbert spaces of functions occurring in the treatment of the Stokes operator and the Navier–Stokes equations.

KEYWORDS
Bogovskiĭ formula, distribution, deRham’s theorem, Poincaré’s lemma, stokes operator

MSC (2020)
46F10, 46E35

1 | INTRODUCTION

The most elementary version of Poincaré’s lemma is the statement that, given a $C^1$-vector field $v : \mathbb{R}^n \to \mathbb{K}^n$ satisfying $\partial_j v_k = \partial_k v_j$ for all $j, k = 1, \ldots, n$, there exists a potential $p \in C^2(\mathbb{R}^n)$ for $v$, that is, $\text{grad} p = v$. The main issue of this paper is to present proofs of the following two distributional versions of this kind of existence theorem.

Theorem 1.1 (“de Rham style”). Let $\Omega \subseteq \mathbb{R}^n$ be open and connected. Let $G = (G_1, \ldots, G_n) \in D'(\Omega)^n$, and suppose that

$$\langle G, \varphi \rangle = 0 \quad (\varphi \in D(\Omega)^n, \ \text{div} \ \varphi = 0).$$

Then there exists a distribution $F \in D'(\Omega)$ such that $\text{grad} F = G$. If $n \geq 2$ and $G$ has compact support, then $F$ can be chosen with compact support.

Theorem 1.2 (“Poincaré style”). Let $\Omega \subseteq \mathbb{R}^n$ be open and simply connected. Let $G \in D'(\Omega)^n$ be such that $\partial_j G_k = \partial_k G_j$ for all $j, k \in \{1, \ldots, n\}$. Then there exists $F \in D'(\Omega)$ such that $\text{grad} F = G$.

Proofs of these theorems have been provided by Mardare [10, Theorems 4.1 and 2.1]. In this paper, the author explains that “to this day, there is no proof, to the best knowledge of the author,” of Theorem 1.2 in the existing literature; and...
concerning Theorem 1.1, the proof given in [12, § 22, Theorem 17'] is said to require “an important prerequisite about chains and flows on differential manifolds.” For more motivation and background we refer to [10]. A proof of the local part of Theorem 1.2, on the basis of de Rham’s regularization, is given in [11, Corollary 3.6(i)].

The author was intrigued by these theorems as well as motivated by related problems connected with the Stokes operator, and this led to proofs of the theorems that are quite different from those presented in [10, 11]. Let us comment briefly on some differences between Mardare’s and our proofs of Theorem 1.2, which is Theorem 2.1 in [10]. Given $G \in D'(\Omega)^n$ as in Theorem 1.2, Mardare first carries out the local part of the proof on sets $\omega = \prod_{j=1}^n (a_j, b_j)$ by presenting—following Schwartz [13, chap. II, § 6]—a formula for the solution $F \in \mathcal{D}'(\omega)$. Our approach is to deal with the local part on bounded open sets that are star-shaped with respect to an open ball and to approximate $G$ by $C^\infty$-vector fields satisfying the compatibility conditions. In both proofs, the global part is achieved by a homotopy argument: Mardare just exploits the existence of local solutions, whereas our proof is arranged in a way to permit the application of the divergence theorem on the parameter set $[0,1]^2$ of the homotopy.

The proofs of Theorems 1.1 and 1.2 are the content of Sections 2 and 3, respectively. In the remaining part of the paper, we treat applications of the theorems stated above to Sobolev space versions of the theorems. In these results, the task is finding potentials in $L_2(\Omega)$ for suitable vector fields in $H^{-1}(\Omega)^n$. Here, the author was motivated, among others, by the recent paper [1].

## 2 PROOF OF DE RHAM’S THEOREM

In the proof of Theorem 1.1, we will need the Bogovskiĭ operator that we discuss next. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $\Omega$ star-shaped with respect to every point of an open ball $B(x_0, r) \subseteq \Omega$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)_+ := \{ \varphi \in C_c^\infty(\Omega); \varphi \geq 0 \}$ with $\text{spt} \varphi \subseteq B(x_0, r_0)$ and $\int \varphi(x) \, dx = 1$. For $\varphi \in D(\Omega)$, we define $B\varphi \in D(\Omega)^n$ by

$$B\varphi(x) := \int_{\Omega} \varphi(y)(x - y) \int_0^{\infty} \rho(y + r(x - y)) r^{n-1} \, dr \, dy \quad (x \in \Omega).$$

It is not too difficult to show that indeed $B\varphi$ belongs to $C^\infty(\Omega)^n$, and

$$\text{spt} \, B\varphi \subseteq \{ \lambda z_1 + (1 - \lambda) z_2; \, z_1 \in \text{spt} \varphi, \, z_2 \in \text{spt} \rho, \, 0 \leq \lambda \leq 1 \},$$

a compact subset of $\Omega$.

**Remark 2.1.** The linear mapping $B : D(\Omega) \rightarrow D(\Omega)^n$ is the **Bogovskiĭ operator**; we refer to [3] for its first appearance. It has the important property that $\int_\Omega \varphi(x) \, dx = 0$ implies that $\text{div} \, B\varphi = \varphi$; see Remark 3.3 below. This property will also be used in the less explicit version that for any $\varphi \in D(\Omega)$ with $\int \varphi(x) \, dx = 0$, there exists a vector field $\Phi \in D(\Omega)^n$ with the property $\text{div} \, \Phi = \varphi$. (The reader should be aware of the fact that the Bogovskiĭ operator depends on the function $\rho$; so the use of the definite article “the” might be somewhat misleading.)

**Lemma 2.2.** The operator $B : D(\Omega) \rightarrow D(\Omega)^n$ is continuous with respect to the standard topologies.

**Proof.** The function $B\varphi$ can be rewritten as

$$B\varphi(x) = \int_{z \in \mathbb{R}^n} \varphi(x - z) \frac{z}{|z|^n} \int_0^{\infty} \rho(x + s \frac{z}{|z|}) (s + |z|)^{n-1} \, ds \, dz$$

(where $\rho$ and $\varphi$ are considered as functions in $D(\mathbb{R}^n)$). This shows that derivatives of $B\varphi$ can be estimated by derivatives of $\varphi$ and $\rho$ of the same and lower order. This fact together with the support property (2.1) shows the assertion. \(\square\)

**Proof of Theorem 1.1.**

(1) In this part of the proof we suppose that $\Omega$ is star-shaped with respect to an open ball $B(x^0, r) \subseteq \Omega$. Let $\rho \in C_c^\infty(\mathbb{R}^n)_+$, $\text{spt} \rho \subseteq B(x^0, r)$ and $\int \rho(x) \, dx = 1$, and let $B : D(\Omega) \rightarrow D(\Omega)^n$ be the corresponding Bogovskiĭ operator. For $\varphi \in D(\Omega)$,
we define

\[ \langle F, \varphi \rangle := -(G, B \varphi). \]

Then Lemma 2.2 implies that \( F \in D'(\Omega) \).

In order to show that \( \text{grad} F = G \) let \( \varphi \in D(\Omega), \ j \in \{1, \ldots, n\} \). Then \( \int \delta_j \varphi(x) \, dx = 0 \), hence \( \text{div}(B \delta_j \varphi - \varphi e_j) = \delta_j \varphi - \delta_j \varphi = 0 \) (where \( e_j \) denotes the \( j \)th unit vector), and hypothesis (1.1) implies

\[ \langle G_j, \varphi \rangle = \langle G, \varphi e_j \rangle = \langle G, B \delta_j \varphi \rangle = -\langle F, \delta_j \varphi \rangle = \langle \delta_j F, \varphi \rangle. \]

This shows that \( G = \text{grad} F \).

(2) For the proof of the general case let \( (B(x^k, r_k))_{k \in N} \) be a countable covering of \( \Omega \) by open balls contained in \( \Omega \), with \( 1 \in N \subseteq \mathbb{N} \). To ease notation, we put \( \Omega_k := B(x^k, r_k) \) \( (k \in N) \). For each \( k \in N \), let \( \rho_k \in C^\infty_0(\mathbb{R}^n)_+ \) with \( \text{spt} \rho_k \subseteq \Omega_k \), \( \int \rho_k(x) \, dx = 1 \), and let \( B_k : D(\Omega_k) \to D(\Omega_k)^n \) be the corresponding Bogovski \u010ci operator.

For \( k \in N \) and some \( c_k \in \mathbb{K} \), we define \( F_k \in D'(\Omega_k) \) by

\[ \langle F_k, \varphi \rangle := c_k \int \varphi(x) \, dx - \langle G, B_k \varphi \rangle \quad (\varphi \in D(\Omega_k)). \]

Then part (1) shows that \( \text{grad} F_k = G \) on \( \Omega_k \). We are going to show that the constants \( c_k \) can be chosen such that the family \( (F_k)_{k \in N} \) of distributions is consistent.

For \( k \in N \), we observe that there exists a function \( \Phi_k \in D(\Omega)^n \) such that \( \text{div} \Phi_k = \rho_1 - \rho_k \); this can be seen by connecting \( \Omega_1 \) with \( \Omega_k \) by a finite chain of consecutively intersecting open balls and applying Remark 2.1 repeatedly.

With this function, we choose \( c_k \) in such a way that

\[ \langle F_k, \rho_k \rangle = \langle G, \Phi_k - B_k(\varphi - \rho_k) \rangle \quad (\varphi \in D(\Omega_k), \int \varphi(x) \, dx = 1). \]

In order to show that the family \( (F_k)_{k \in N} \) is consistent, let \( k, l \in N \) be such that \( \Omega_k \cap \Omega_l \neq \emptyset \), and let \( \varphi \in D(\Omega) \) with \( \text{spt} \varphi \subseteq \Omega_k \cap \Omega_l \), \( \int \varphi(x) \, dx = 1 \). Then

\[ \langle F_k, \varphi \rangle - \langle F_l, \varphi \rangle = \langle G, \Phi_k - B_k(\varphi - \rho_k) - \Phi_l + B_l(\varphi - \rho_l) \rangle = 0, \]

because the divergence of the function to which \( G \) is applied turns out to be \( (\rho_1 - \rho_k) - (\varphi - \rho_k) - (\rho_1 - \rho_l) + (\varphi - \rho_l) = 0 \). It follows that \( \langle F_k, \varphi \rangle = \langle F_l, \varphi \rangle \) for all \( \varphi \in D(\Omega_k \cap \Omega_l) \).

We conclude that the family \( (F_k)_{k \in N} \) composes to a distribution \( F \in D'(\Omega) \) satisfying \( \text{grad} F = G \).

(3) Now suppose that \( n \geq 2 \) and that \( G \) has compact support. There exists \( \psi \in D(\Omega) \) with \( \psi = 1 \) in a neighborhood of the support of \( G \). Then

\[ \langle \hat{G}, \varphi \rangle := \langle G, \psi \varphi \rangle \quad (\varphi \in D(\mathbb{R}^n)^n) \]

defines an extension of \( G \) to \( D(\mathbb{R}^n)^n \), satisfying \( \delta_j \hat{G}_k = \delta_k \hat{G}_j \) for all \( j, k \in \{1, \ldots, n\} \). According to part (2) above, there exists \( \hat{F} \in D'(\mathbb{R}^n) \) such that \( \text{grad} \hat{F} = \hat{G} \). Denote by \( \Omega_\infty \) the unbounded component of \( \mathbb{R}^n \setminus \text{spt} G \). Then \( \text{grad} \hat{F} = 0 \) on \( \Omega_\infty \); hence there exists \( c \in \mathbb{K} \) such that \( \hat{F} = c \) on \( \Omega_\infty \). Then the restriction of \( \hat{F} - c \) to \( D(\Omega) \) is as asserted. \( \square \)

3 PROOF OF THE DISTRIBUTIONAL VERSION OF POINCARÉ’S LEMMA

We start with a property that will be needed in the proof of Theorem 1.2.

**Lemma 3.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be open. Let \( v \in C^1(\Omega; \mathbb{K}^n) \) be a vector field satisfying

\[ \partial_j v_k = \partial_k v_j \quad (j, k = 1, \ldots, n). \]
Let $\gamma, \tilde{\gamma} : [0, 1] \rightarrow \Omega$ be $C^1$-paths with $\gamma(0) = \tilde{\gamma}(0) = x^0$, $\gamma'(0) = \tilde{\gamma}'(0) = 0$, $\gamma(1) = \tilde{\gamma}(1) = x^1$, $\gamma'(1) = \tilde{\gamma}'(1) = 0$. Suppose that $\Gamma : [0, 1]^2 \rightarrow \Omega$ is an FEP-homotopy between $\gamma$ and $\tilde{\gamma}$, that is, $\Gamma$ is continuous, $\Gamma(\cdot, 0) = \gamma$, $\Gamma(\cdot, 1) = \tilde{\gamma}$, $\Gamma(0, \cdot) = x^0$, $\Gamma(1, \cdot) = x^1$. (FEP stands for “fixed end points”). Then

$$\int_\gamma v := \int_0^1 v(\gamma(s)) \cdot \gamma'(s) ds = \int_0^1 v(\tilde{\gamma}(s)) \cdot \tilde{\gamma}'(s) ds = \int_{\tilde{\gamma}} v.$$

**Proof.**

(1) In this step, we suppose additionally that $\Gamma$ is twice continuously differentiable. We define the vector field $w : [0, 1]^2 \rightarrow \mathbb{R}^2$,

$$w(s, t) := \left( v(\Gamma(s, t)) \cdot \frac{\partial}{\partial t} \Gamma(s, t), -v(\Gamma(s, t)) \cdot \frac{\partial}{\partial s} \Gamma(s, t) \right).$$

Then, using the hypothesis on $v$, one easily obtains $\text{div} w = 0$; hence the divergence theorem yields

$$\int_{\partial [0,1]^2} w(s, t) \cdot \nu(s, t) d\sigma(s, t) = 0,$$

where $\nu$ is the outer unit normal and $\sigma$ the surface measure. The integrals over the lines with $s = 0$ and $s = 1$ vanish, because $\frac{\partial}{\partial t} \Gamma(s, t) = 0$ for $s = 0, s = 1$, and all $t \in [0, 1]$. Hence the remaining integrals yield

$$\int_0^1 v(\Gamma(s, 0)) \cdot \frac{\partial}{\partial s} \Gamma(s, 0) ds - \int_0^1 v(\Gamma(s, 1)) \cdot \frac{\partial}{\partial s} \Gamma(s, 1) ds = 0,$$

which is just the asserted equality.

(2) In order to apply step (1) by smoothing the given homotopy $\Gamma$, we first “contract and extend” $\Gamma$ as follows. We define the continuous function $\alpha : [-1/4, 5/4] \rightarrow [0, 1]$ by $\alpha|_{[-1/4, 1/4]} := 0, \alpha|_{[3/4, 5/4]} := 1$ and $\alpha|_{[1/4, 3/4]}$ affine linear, and then we put $\bar{\Gamma} : [-1/4, 5/4]^2 \rightarrow \Omega$, $\bar{\Gamma}(s, t) := \Gamma(\alpha(s), \alpha(t))$. It is easy to see that then

$$\int_{\bar{\Gamma}(,0)} v = \int_\gamma v \quad \text{and} \quad \int_{\bar{\Gamma}(,1)} v = \int_{\tilde{\gamma}} v. \quad (3.1)$$

Observe that

$$\bar{\Gamma}(s, \cdot)|_{[-1/4, 1/4]} = \bar{\Gamma}(s, 0), \quad \bar{\Gamma}(s, \cdot)|_{[3/4, 5/4]} = \bar{\Gamma}(s, 1) \quad (s \in [-1/4, 5/4]).$$

and

$$\bar{\Gamma}|_{[-1/4, 1/4] \times [-1/4, 1/4]} = x^0, \quad \bar{\Gamma}|_{[3/4, 5/4] \times [-1/4, 5/4]} = x^1.$$

Now let $(\rho_k)_{k \in \mathbb{N}}$ be a sequence in $C^\infty_c (\mathbb{R}_+)$, $\text{spt} \rho_k \subseteq [-1/k, 1/k]$, $\int \rho_k(x) dx = 1$ for all $k \in \mathbb{N}$, and put

$$\bar{\Gamma}_k(s, t) := \int_{-1/4}^{5/4} \int_{-1/4}^{5/4} \bar{\Gamma}(s', t') \rho_k(s - s') \rho_k(t - t') ds' dt' \quad (s, t \in [0, 1], k \geq 4).$$

The properties mentioned above imply that $\bar{\Gamma}_k(\cdot, j) = \bar{\Gamma}(\cdot, j) \ast \rho_k$ and $\bar{\Gamma}_k(j, \cdot) = \gamma(j)$ on $[0, 1]$ for $j = 0, 1$ and all $k \geq 4$. Using the uniform continuity of $\bar{\Gamma}$ one shows that there exists $k_0 \geq 4$ such that $\bar{\Gamma}_k([0, 1]^2) \subseteq \Omega$ for all $k \geq k_0$.

Having established these properties, we can apply part (1) of the proof to conclude that

$$\int_{\bar{\Gamma}_k(,0)} v = \int_{\bar{\Gamma}_k(,1)} v \quad (k \geq k_0). \quad (3.2)$$
From $\tilde{g}_{k}(\cdot, j) = \tilde{g}(\cdot, j) * \rho_k \to \tilde{g}(\cdot, j)$ in $C^1([0, 1]; \mathbb{R}^n)$ ($k \to \infty$), it follows that
\[
\lim_{k \to \infty} \int_{\Gamma_k(\cdot, j)} v = \int_{\Gamma(\cdot, j)} v \quad (j = 0, 1).
\] (3.3)

Combining (3.2), (3.3), and (3.1), we obtain the assertion of the lemma. $\square$

Proof of Theorem 1.2.

(1) In this part we prove the theorem for the case that $\Omega$ is bounded and star-shaped with respect to an open ball $B(x^0, r) \subseteq \Omega$. Let $\rho \in C_c^\infty(\Omega)$ with $\text{spt} \rho \subseteq B(x^0, r)$ and $\int \rho(x) \, dx = 1$, and let $B$ be the corresponding Bogovskiǐ operator. The transpose of $B : D(\Omega) \to D(\Omega)^n$ is the linear mapping $B^t : D'(\Omega)^n \to D'(\Omega)$, given by
\[
\langle B^tG, \varphi \rangle := \langle G, B\varphi \rangle \quad (\varphi \in D(\Omega), G \in D'(\Omega)^n),
\]
and $B^t$ is continuous with respect to the standard topologies.

The restriction of $-B^t$ to $C^\infty(\Omega; \mathbb{K}^n)$ is the operator $A : C^\infty(\Omega; \mathbb{K}^n) \to C^\infty(\Omega)$, given by
\[
Av(x) := \int \rho(y) \int_0^1 \nu(sy + (1 - s)x) \cdot (x - y) \, ds \, dy;
\] (3.4)
see Remark 3.3. It has the property that $\text{grad} A v = v$ for all $v \in C^\infty(\Omega; \mathbb{K}^n) := \{ u \in C^\infty(\Omega; \mathbb{K}^n); \partial_j u_k = \partial_k u_j \ (j, k = 1, \ldots, n) \}$.

Now let $G \in D'(\Omega)^n$ with $\partial_j G_k = \partial_k G_j$ for all $j, k \in \{1, \ldots, n\}$. We show that $G$ can be approximated by vector fields $v \in C^\infty(\Omega; \mathbb{K}^n)$. Without restriction, we assume that $x^0$ (the centre of the ball mentioned initially) is the origin. In a first step we introduce distributions $G^\lambda$ for $\lambda \in (1, \infty)$, defined on $\lambda \Omega$ by
\[
\langle G^\lambda, \varphi \rangle := \langle G, \varphi \left( \frac{1}{\lambda} \cdot \right) \rangle \quad (\varphi \in D(\lambda \Omega)^n).
\]
It is easy to see that $G^\lambda$ satisfies the compatibility condition and that for all $\varphi \in D(\Omega)^n$ one obtains $\langle G^\lambda, \varphi \rangle \to \langle G, \varphi \rangle$ as $\lambda \to 1$.

In order to approximate $G^\lambda$ from $C^\infty(\Omega; \mathbb{K}^n)$, we fix $\lambda > 1$ and observe that $\lambda \Omega$ is an open neighbourhood of the compact set $\overline{\Omega}$, due to the strict star-shapedness of $\Omega$. Let $(\rho_l)_{l \in \mathbb{N}}$ be a $\delta$-sequence in $C_c^\infty(\mathbb{R}^n)$. Then, for large $l$, the function
\[
x \mapsto v^{\lambda, l}(x) := \left\langle G^\lambda, \rho_l(\cdot - x) \right\rangle = \left( \left\langle G^\lambda_k, \rho_l(\cdot - x) \right\rangle \right)_{k = 1, \ldots, n}
\]
is defined on $\Omega$ and belongs to $C^\infty(\Omega; \mathbb{K}^n)$. For all $j, k = 1, \ldots, n$ one obtains
\[
\partial_{j, x} \langle G^\lambda_k, \rho_l(\cdot - x) \rangle = \langle G^\lambda_k, \partial_{j, x} \rho_l(\cdot - x) \rangle = -\langle G^\lambda_k, \partial_j \rho_l(\cdot - x) \rangle = \langle \partial_j G^\lambda_k, \rho_l(\cdot - x) \rangle = \langle \partial_k G^\lambda_j, \rho_l(\cdot - x) \rangle = \cdots = \partial_{k, x} \langle G^\lambda_j, \rho_l(\cdot - x) \rangle;
\]
hence $v^{\lambda, l} \in C^\infty(\Omega; \mathbb{K}^n)$.

In order to keep notation simple, we compute the convergence $v^{\lambda, l} \to G^\lambda$ in $D'(\Omega)^n$ componentwise (where we use the same symbol for the function $v^{\lambda, l}$ and for the regular distribution generated by this function). Thus, let $\varphi \in D(\Omega)$, $k \in \{1, \ldots, n\}$. Then
\[ \langle u^I_k, \varphi \rangle = \int u^I_k(x) \varphi(x) \, dx = \int \langle G^A_k, \rho(-x) \rangle \varphi(x) \, dx \]

\[ = \langle G^A_k, \int \rho(-x) \varphi(x) \, dx \rangle = \langle G^A_k, \rho \ast \varphi \rangle. \]

Now, \( \rho_l \ast \varphi \rightarrow \varphi \) in \( D(\Omega) \), hence \( u^I_k \rightarrow G^A_k \) in the weak topology \( \sigma(D(\Omega), D(\Omega)) \).

To complete this step, we first recall from above that

\[ -\nabla B_t v^I = \nabla A v^I = v^I. \]

Letting \( l \rightarrow \infty \), we obtain \( -\nabla B_t G^A = G^A \), and then letting \( \lambda \rightarrow 1 \), we finally get \( -\nabla B_t G = G \). This shows that \( F := -B_t G \) has the asserted property.

(2) For the general case, let \( (B(x^m, r_m))_{m \in N} \) be a countable covering of \( \Omega \) by open balls, with \( 1 \in N \subseteq \mathbb{N} \); for brevity we put \( \Omega_m := B(x^m, r_m) \) \( (m \in N) \). From part (1), we conclude that for each \( m \in N \) there exists \( F_m \in D'(\Omega_m) \) such that \( \nabla F_m = G \) on \( \Omega_m \).

Now it remains to show that there exist constants \( c_m \in \mathbb{K} \) such that \( (c_m + F_m)_{m \in N} \) is a consistent family of distributions. Without restriction, we assume that \( N = \{1, \ldots, m_0\} \) or \( N = \mathbb{N} \), and that \( \Omega_m \cap \bigcup_{k=1}^{m-1} \Omega_k \neq \emptyset \) for all \( m \in N \setminus \{1\} \). Put \( c_1 := 0 \).

Let \( m' \in N \setminus \{1\} \) and assume that \( c_2, \ldots, c_{m' - 1} \) are found such that the family \( (c_m + F_m)_{m=1}^{m'-1} \) is consistent; without restriction \( c_2 = \cdots = c_{m' - 1} = 0 \). Denote by \( \hat{F}_{m'} \in D'(\bigcup_{k=1}^{m'-1} \Omega_m) \) the composed distribution satisfying \( \hat{F}_{m'} = F_m \) on \( \Omega_m \) for all \( m = 1, \ldots, m' - 1 \). There exists \( m \in \{1, \ldots, m' - 1\} \) such that \( \Omega_m \cap \Omega_m' \neq \emptyset \), and \( \hat{F}_m = G \) on \( \Omega_m' \cap \Omega_m \) implies that there exists \( c_m \in \mathbb{K} \) such that \( c_m + F_m = F_m' \) on \( \Omega_m' \cap \Omega_m \). In order to make sure that \( c_m + F_m' \) is consistent with \( \hat{F}_{m'} \), we have to show that \( c_m + F_m' \) is consistent with \( \hat{F}_{m'} \) for all \( m \in \{1, \ldots, m' - 1\} \). Thus, let \( \hat{m} \in \{1, \ldots, m' - 1\} \setminus \{m\} \) be such that \( \Omega_m \cap \Omega_{\hat{m}} \neq \emptyset \). Then as before there exists \( \tilde{c}_m \in \mathbb{K} \) such that \( \tilde{c}_m + F_m' = \tilde{F}_{m'} \) on \( \Omega_{\hat{m}} \cap \Omega_m \). There exists a continuously differentiable path \( \gamma : [0,1] \rightarrow \Omega_m \cup \Omega_m' \) from \( x_m \) to \( x_{m'} \) satisfying \( \gamma'(0) = \gamma'(1) = 0 \), and there exists a continuously differentiable path \( \bar{\gamma} : [0,1] \rightarrow \bigcup_{k=1}^{m'-1} \Omega_k \) from \( x^m \) to \( x^{m'} \) satisfying \( \bar{\gamma}(0) = \bar{\gamma}(1) = 0 \). The simple connectedness of \( \Omega \) implies that there exists an FEP-homotopy \( \gamma : [0,1]^2 \rightarrow \Omega \) between \( \gamma \) and \( \bar{\gamma} \). As \( \Gamma([0,1]^2) \) is compact, there exists \( r > 0 \) such that \( \Gamma([0,1]^2) + B(0,2r) \subseteq \Omega \). Let \( \rho \in C_0^\infty(\mathbb{R}^n) \) with \( \text{spt} \rho \subseteq B(0,r) \), \( \int \rho(x) \, dx = 1 \), and put

\[ \nu(x) := \langle G, \rho \ast (\cdot - x) \rangle \quad (x \in \Gamma([0,1]^2) + B(0,r)). \]

Then \( \nu \in C_0^\infty(\Gamma([0,1]^2) + B(0,r); \mathbb{K}^n) \), and the hypothesis on \( G \) implies that \( \partial_j \nu_k = \partial_k \nu_j \) for all \( j, k = 1, \ldots, n \). Applying Lemma 3.1 with the open set \( \Gamma([0,1]^2) + B(0,r) \), we conclude that

\[ \int \gamma \nu = \int_{\gamma} \nu. \quad (3.5) \]

The distributions \( F_m, c_m + F_m' \) are consistent; call \( \hat{F} \in D'(\Omega_m \cup \Omega_{m'}) \) the combined distribution. Then \( \nabla \hat{F} = G \) on \( \Omega_m \cup \Omega_{m'} \), and

\[ \langle G, \rho(\cdot - y(s)) \rangle \cdot y'(s) = \langle \nabla \hat{F}, \rho(\cdot - y(s)) \rangle \cdot y'(s) \]

\[ = -\langle \hat{F}, \nabla \rho(\cdot - y(s)) \rangle \cdot y'(s) \quad (\text{by \( \partial_j \nu_k = \partial_k \nu_j \)}), \]

\[ \int_{\gamma} \nu = \int_{0}^{1} \langle G, \rho(\cdot - y(s)) \rangle \cdot y'(s) \, ds = \langle \hat{F}, \rho(\cdot - x^{m'}) \rangle - \langle \hat{F}, \rho(\cdot - x^{m}) \rangle \]

\[ = c_m + \langle F_{m'}, \rho(\cdot - x^{m'}) \rangle - \langle F_{m'}, \rho(\cdot - x^{m}) \rangle = c_m + \langle F_{m'}, \rho(\cdot - x^{m'}) \rangle - \langle \hat{F}_{m' - 1}, \rho(\cdot - x^{m}) \rangle. \]
By the same token,
\[
\int v = \int_0^1 \langle G, \rho(\cdot - \tilde{y}(s)) \rangle \cdot \tilde{y}'(s) \, ds = \int_0^{1/2} \langle G, \rho(\cdot - \tilde{y}(s)) \rangle \cdot \tilde{y}'(s) \, ds + \int_{1/2}^1 \langle G, \rho(\cdot - \tilde{y}(s)) \rangle \cdot \tilde{y}'(s) \, ds
\]
\[
= \langle \tilde{F}_{m'-1}, \rho(\cdot - x^{m}) \rangle - \langle \tilde{F}_{m'-1}, \rho(\cdot - x^{m'}) \rangle + (\tilde{c}_{m'} + F_{m'}, \rho(\cdot - x^{m'})) - (F_{m}, \rho(\cdot - x^{m})
\]
\[
= \tilde{c}_{m'} + \langle F_{m'}, \rho(\cdot - x^{m'}) \rangle - (\tilde{F}_{m'-1}, \rho(\cdot - x^{m})).
\]

From (3.5), we obtain \( c_{m'} = \tilde{c}_{m'} \), and this shows that the family \( (c_{m} + F_m)_{m \in \{1, \ldots, m'\}} \) is consistent; hence by induction, the constants \( c_{m} \) can be chosen such that the family \( (c_{m} + F_m)_{m \in \mathbb{N}} \) is consistent. □

Remark 3.2. We note that Theorem 1.2 implies classical versions of Poincaré’s lemma: If \( \Omega \subseteq \mathbb{R}^n \) is open and simply connected, and \( g \in \mathcal{C}^l(\Omega; \mathbb{K}^n) \) (for some \( l \in \mathbb{N}_0 \)) is a vector field satisfying
\[
\partial_j g_k = \partial_k g_j \quad (j,k = 1, \ldots, n),
\]
then there exists \( f \in \mathcal{C}^{l+1}(\Omega) \) with \( \text{grad} \ f = g \). Indeed, the distributional solution \( F \) of \( \text{grad} \ F = g \) obtained by Theorem 1.2 is automatically a regular distribution generated by a \( \mathcal{C}^{l+1} \)-function.

Remark 3.3. On the Bogovskiǐ operator \( B : \mathcal{C}_c^{\infty}(\Omega) \rightarrow \mathcal{C}_c^{\infty}(\Omega; \mathbb{K}^n) \) and the operator \( A : \mathcal{C}^{\infty}(\Omega; \mathbb{K}^n) \rightarrow \mathcal{C}^{\infty}(\Omega) \) from (3.4).

Let \( \Omega \) and \( \rho \) be as at the beginning of Section 2. It is a standard exercise of calculus that \( \text{grad} \ A v = v \) for all \( v \in \mathcal{C}_c^{\infty}(\Omega; \mathbb{K}^n) \). Substitution of variables yields
\[
\int (A v) \varphi \, dx = - \int v \cdot B \varphi \, dx \quad \text{for all} \quad v \in \mathcal{C}^{\infty}(\Omega; \mathbb{K}^n), \varphi \in \mathcal{C}_c^{\infty}(\Omega).
\]

Let \( \varphi, \psi \in \mathcal{C}_c^{\infty}(\Omega) \), \( \int \varphi(x) \, dx = 0 \). Note that \( \text{grad} \ \psi \in \mathcal{C}_c^{\infty}(\Omega; \mathbb{K}^n) \), therefore \( \text{grad}(\text{grad} \ \psi - \psi) = 0 \); hence \( \text{grad} \ \psi - \psi \) is constant, \( \int (\text{grad} \ \psi - \psi) \varphi \, dx = 0 \). This implies that
\[
\int \psi \, \text{div} \ B \varphi \, dx = - \int \text{grad} \ \psi \cdot B \varphi \, dx = \int (A \text{grad} \ \psi) \varphi \, dx = \int \psi \varphi \, dx.
\]
As this equality holds for all \( \psi \in \mathcal{C}_c^{\infty}(\Omega) \) one obtains \( \text{div} \ B \varphi = \varphi \).

4 | THE “COARSE” AND “SIMPLIFIED” VERSIONS OF DE RHAM’S THEOREM

In this section, we treat the existence of potentials in a Hilbert space context. Let \( \Omega \subseteq \mathbb{R}^n \) be a connected bounded open set. We define the Sobolev spaces
\[
H^1_{0,\sigma}(\Omega; \mathbb{K}^n) := \{ u \in H^1_0(\Omega; \mathbb{K}^n); \text{div} \ u = 0 \},
\]
\[
H^1_{\sigma,0}(\Omega; \mathbb{K}^n) := H^1_{0,\sigma}(\Omega; \mathbb{K}^n) = C^{\infty}_c(\Omega; \mathbb{K}^n) = H^1_{0,\sigma}(\Omega; \mathbb{K}^n),
\]
where \( C^{\infty}_c(\Omega; \mathbb{K}^n) := \{ \varphi \in C^{\infty}(\Omega; \mathbb{K}^n); \text{div} \ \varphi = 0 \} \). For a subspace \( V \subseteq H^1_0(\Omega; \mathbb{K}^n) \), we define the polar
\[
V^0 := \{ g \in H^{-1}(\Omega)^n; \langle g, \varphi \rangle_{H^{-1},H^1_0} = 0 (\varphi \in V) \},
\]
where \( H^{-1}(\Omega) \) is the anti-dual space of \( H^1_0(\Omega) \) in the Gelfand triple \( H^1_0(\Omega) \subseteq L_2(\Omega) \subseteq H^{-1}(\Omega) \).

We adopt the terminology in the title of this section from [1] (formerly used already in [8, Chap. I, § 2]), where it is defined that the “coarse version of de Rham’s theorem” holds (for \( \Omega \)) if
\[
\text{for all } g \in H^1_{0,\sigma}(\Omega; \mathbb{K}^n)^0 \text{ there exists } f \in L_2(\Omega) \text{ with } g = \text{grad} \ f, \quad (H_0)
\]
and the “simplified version of de Rham’s theorem” holds if
\[
\text{for all } g \in C^{\infty}_c(\Omega; \mathbb{K}^n)^0 \text{ there exists } f \in L_2(\Omega) \text{ with } g = \text{grad} \ f, \quad (H_c)
\]
Note that property \((H_c)\) could have been stated equivalently with \(H^1_{\sigma,0}(\Omega; \mathbb{K}^n)\) instead of \(C^\infty_{c,2}(\Omega; \mathbb{K}^n)\), because these spaces have the same polar. Also note that in both of these properties the existence of \(f\) could have been stated with the additional property that \(f \in L^2_0(\Omega) \coloneqq \{ g \in L^2(\Omega) : \int g(x) \, dx = 0 \} \).

From \(H^1_{\sigma,0}(\Omega; \mathbb{K}^n) \subseteq H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c\), it is clear that property \((H_c)\) implies \((H_0)\). It is shown in [1, Theorem 4.1] that for \(\Omega\) with Lipschitz boundary, “J. L. Lions’ lemma” together with \((H_0)\) implies \((H_c)\); see also Remark 4.6. We are going to show that \((H_c)\) is “hereditary” (Proposition 4.3), that \((H_0)\) holds and implies \((H_c)\) for strictly star-shaped sets (Remark 4.4) and that \((H_c)\) holds for sets with Lipschitz boundary (Corollary 4.5).

### Remark 4.1
Let \(\text{grad}\) denote the operator \(\text{grad} : L^2(\Omega) \to H^{-1}(\Omega)^n\), \(f \mapsto \text{grad} f\). Then \(\text{ran}(\text{grad}) \subseteq H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c \subseteq H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c\), and \((H_0)\) is equivalent to \(\text{ran}(\text{grad}) = H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c\), whereas \((H_c)\) is equivalent to \(\text{ran}(\text{grad}) = H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c\). This implies that \((H_c)\) is equivalent to \((H_0)\) together with \(H^1_{\sigma,0}(\Omega; \mathbb{K}^n) = H^1_{\sigma,0}(\Omega; \mathbb{K}^n)\). Note that the last equality is equivalent to the denseness of \(C^\infty_{c,2}(\Omega; \mathbb{K}^n)\) in \(H^1_{\sigma,0}(\Omega; \mathbb{K}^n)\).

We start our investigation of these properties by a proof of the equivalence of property \((H_c)\) and “J. L. Lions’ lemma”; see [1, Theorem 4.1].

### Theorem 4.2
Let \(\Omega \subseteq \mathbb{R}^n\) be a connected bounded open set. Then property \((H_c)\) for \(\Omega\) is equivalent to the property that for all \(F \in D'(\Omega)\) with \(\text{grad} F \in H^{-1}(\Omega)^n\) there exists \(f \in L^2(\Omega)\) with \(F = \text{grad} f\) (as distributions).

**Proof.** For the proof of the necessity let \(F \in D'(\Omega)\) such that \(\text{grad} F \in H^{-1}(\Omega)^n\). Then \(\text{grad} F \in C^\infty_2(\Omega; \mathbb{K}^n)^c\); hence \((H_c)\) implies that there exists \(f \in L^2(\Omega)\) with \(\text{grad} f = \text{grad} F\), and because \(\Omega\) is connected there exists \(c \in \mathbb{K}\) such that \(F\) is generated by the function \(f + c \in L^2(\Omega)\).

For the sufficiency let \(g \in C^\infty_2(\Omega; \mathbb{K}^n)^c\). Then Theorem 1.1 implies that there exists \(F \in D'(\Omega)\) such that \(\text{grad} F = g \in H^{-1}(\Omega; \mathbb{K}^n)\); hence there exists \(f \in L^2(\Omega)\) with \(F = f, g = \text{grad} F = \text{grad} f\).

The following proposition shows a hereditary property of \((H_c)\).

### Proposition 4.3
Let \(\Omega \subseteq \mathbb{R}^n\) be a connected bounded open set, and assume that there exists a (finite) covering \((\Omega_j)_{j=1,\ldots,m}\) of \(\Omega\) by connected open sets \(\Omega_j \subseteq \Omega\) with property \((H_c)\). Then property \((H_c)\) holds for \(\Omega\).

**Proof.** Let \(g \in H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^c\). Then clearly \(g^j := g|_{H^1_{\sigma,0}(\Omega_j; \mathbb{K}^n)} \in H^1_{\sigma,0}(\Omega_j; \mathbb{K}^n)^c\), and the hypothesis implies that for all \(j = 1, \ldots, m\) there exists \(f_j \in L^2(\Omega_j)\) such that \(g^j = \text{grad} f_j\). Note that each function \(f_j\) is only determined up to a constant, and we have to “glue together” suitable versions of these functions.

We apply Theorem 1.1 and obtain a distribution \(F \in D'(\Omega)\) such that \(g = \text{grad} F\). This implies that for each \(j \in \{1, \ldots, m\}\) one has \(F = \text{grad} f_j\) on \(\Omega_j\), and because \(\Omega_j\) is connected, there exists \(c_j \in \mathbb{K}\) such that \(F = f_j + c_j 1_{\Omega_j}\) on \(\Omega_j\). This implies that the family \((f_j + c_j 1_{\Omega_j})_{j=1,\ldots,m}\) of \(L^2\)-functions is consistent; hence there exists \(f \in L^2(\Omega)\) with \(f|_{\Omega_j} = f_j + c_j 1_{\Omega_j}\) for all \(j = 1, \ldots, m\), \(\text{grad} f = \text{grad} F = g\).

Next we turn to more concrete sufficient conditions for the validity of \((H_c)\). The fundamental source of these conditions is an important fact from [3, Lemma 1], stated in part (1) of the following remark.

### Remark 4.4
Let \(\Omega \subseteq \mathbb{R}^n\) be a bounded open set that is star-shaped with respect to an open ball \(B(x^0, r)\).

1. Let \(\rho \in C^\infty_c(\Omega)\), \(\text{spt} \rho \subseteq B(x^0, r), \int \rho(x) \, dx = 1\). Then the corresponding Bogovskii operator \(B : C^\infty_c(\Omega) \to C^\infty_c(\Omega; \mathbb{K}^n)\) possesses a (unique) bounded linear extension \(B : L^2(\Omega) \to H^1_0(\Omega; \mathbb{K}^n)\). (The paper [3] does not contain a proof of this fact, but rather the hint that for the proof one should use [5]; this hint has been executed in [4, Theorem 2.4], [7, Section III.3].) From \(\text{div} B\varphi = \varphi\) for \(\varphi \in C^\infty_c \cap L^2_0(\Omega)\) it follows that \(\text{div} B f = f\) for all \(f \in L^2_0(\Omega)\).

2. It is easy to see that the bounded operators \(\text{div} : H^1_0(\Omega; \mathbb{K}^n) \to L^2(\Omega)\) and \(\text{grad} : L^2(\Omega) \to H^{-1}(\Omega)^n\) are negative adjoints of each other, and therefore \(\text{ran}(\text{grad})^c = \ker(\text{div})\)—a well-established relation for operators on...
Hilbert spaces—implies \( \text{ran}(\text{grad}) = \ker(\text{div})^\circ \). As \( \text{ran}(\text{div}) = L^0_0(\Omega) \) is closed, the closed range theorem implies that \( \text{ran}(\text{grad}) \) is closed; hence \( \text{ran}(\text{grad}) = \ker(\text{div})^\circ = H^1_{0,\Sigma}(\Omega; \mathbb{K}^n)^\circ \). This shows that \((H_0)\) is satisfied.

(3) We now show that even \((H_c)\) is satisfied; this follows if we show that \( C^\infty_{c,\sigma}(\Omega; \mathbb{K}^n) \) is dense in \( H^1_{0,\sigma}(\Omega; \mathbb{K}^n) \). Without restriction \( x_0 = 0 \). Let \( \varphi \in H^1_{0,\sigma}(\Omega; \mathbb{K}^n) \). Then the extension of \( \varphi \) to \( \mathbb{R}^n \) by zero belongs to \( H^1_{0,\sigma}(\mathbb{R}^n; \mathbb{K}^n) \); we keep the notation \( \varphi \) for the extension. For \( \lambda \in (1, \infty) \) we note that \( \varphi(\lambda \cdot) \in H^1_{0,\sigma}(\Omega; \mathbb{K}^n) \) has compact support contained in \( \lambda^{-1} \Omega \subseteq \Omega \). Regularisation shows that \( \varphi(\lambda \cdot) \) is approximated by functions in \( C^\infty_{c,\sigma}(\Omega; \mathbb{K}^n) \); hence \( \varphi(\lambda \cdot) \in H^1_{\Sigma,0}(\Omega; \mathbb{K}^n) \).

Taking \( \lambda \to 1 \) we then obtain \( \varphi \in H^1_{\Sigma,0}(\Omega; \mathbb{K}^n) \). (This reasoning has also been used in [14, proof of Lemma 9, p. 31].)

**Corollary 4.5.** Let \( \Omega \subseteq \mathbb{R}^n \) be a connected bounded open set, and assume that there exists a covering \((\Omega_j)_{j=1,\ldots,m}\) of \( \Omega \) by open sets \( \Omega_j \subseteq \Omega \), where each \( \Omega_j \) is star-shaped with respect to an open ball. Then property \((H_c)\) holds for \( \Omega \).

These assertions apply to any connected bounded open set with Lipschitz boundary.

**Proof.** From Remark 4.4(3) we know that the sets \( \Omega_j \) have property \((H_c)\). Hence Proposition 4.3 implies the assertion.

If \( \Omega \) has Lipschitz boundary, then it is not difficult to see that for all \( x \in \overline{\Omega} \) there exists an open neighborhood \( U_x \) such that \( U_x \cap \Omega \) is star-shaped with respect to the points of a ball in \( U_x \cap \Omega \). This is obvious for \( x \in \Omega \), and for \( x \in \partial \Omega \) it results from the Lipschitz property of \( \partial \Omega \). The compactness of \( \overline{\Omega} \) implies that there exists a finite open covering \((\Omega_j)_{j=1,\ldots,m}\) of \( \Omega \) as required in the first part of the corollary. (See also [1, Theorem 2.1].)

**Remark 4.6.** In [1, Theorem 3.1] an “equivalence result” is established between various classical results, e.g. the (classical) J. L. Lions lemma, the J. Nečas inequality, property \((H_0)\) and the property that \( \text{div} : H^1_{0,\sigma}(\Omega; \mathbb{K}^n) \to L^0_0(\Omega) \) is surjective, for bounded open sets with Lipschitz boundary. Each of these results by itself is of a notoriously high technical level, and in [1, Section 2] several sources for the results are mentioned.

A treatment of part of these properties, for the case that \( \Omega \) has \( C_2 \)-boundary, can be found in [14, pp. 26–32]. Related problems are treated in a more general setting in [2]. For further information, we also refer to [6, 8, 15].

5 | A “WEAK VERSION” OF POINCARÉ’S LEMMA

In the previous section, we have seen how the distributional version of de Rham’s theorem can be applied in the \( L^2 \)-context. Here, we present an application of the distributional version of Poincaré’s lemma.

**Theorem 5.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a simply connected bounded open set satisfying property \((H_c)\). Let \( g \in H^{-1}(\Omega^n) \) be such that \( \partial_j g_k = \partial_k g_j \) for all \( j, k = 1, \ldots, n \) (with derivatives in the sense of distributions). Then there exists \( f \in L^2(\Omega) \) such that \( \text{grad} f = g \).

**Proof.** Theorem 1.2 implies that there exists \( F \in D'(\Omega) \) such that \( \text{grad} F = g \). This shows that for all \( \varphi \in C^\infty_{c,\sigma}(\Omega; \mathbb{K}^n) \) one obtains

\[
\langle g, \varphi \rangle_{H^{-1},H^1_0} = \langle \text{grad} F, \varphi \rangle = -\langle F, \text{div} \varphi \rangle = 0,
\]

that is, \( g \in C^\infty_{c,\sigma}(\Omega; \mathbb{K}^n)^\circ \). Now property \((H_c)\) implies that there exists \( f \in L^2(\Omega) \) such that \( \text{grad} f = g \). □

Recall from Corollary 4.5 that Theorem 5.1 applies to the case that \( \Omega \) is a simply connected bounded open set with Lipschitz boundary. For this case, proofs of Theorem 5.1 are contained in [9, proof of Theorem 2.1] and [1, Proof of Theorem 4.1], without the use of our Theorem 1.2.

ACKNOWLEDGMENTS

The author is grateful to M. Kunzinger (Vienna) for pointing out reference [11].
REFERENCES

[1] C. Amrouche, P. G. Ciarlet, and C. Mardare, On a lemma of Jacques-Louis Lions and its relation to other fundamental results, J. Math. Pures Appl. 104 (2015), 207–226.

[2] C. Amrouche and V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czechoslovak Math. J. 44 (1994), 109–140.

[3] M. E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Soviet Math. Dokl. 20 (1979), 1094–1098.

[4] W. Borchers and H. Sohr, On the equations rot v = g and div f with zero boundary conditions, Hokkaido Math. J. 19 (1990), 67–87.

[5] A. P. Calderón and A. Zygmund, On singular integrals, Amer. J. Math. 78 (1956), 289–309.

[6] P. G. Ciarlet, Linear and nonlinear functional analysis with applications, SIAM, Philadelphia, PA, 2013.

[7] G. P. Galdi, An introduction to the mathematical theory of the Navier–Stokes equations. Steady-state problems, 2nd ed., Springer, New York, 2011.

[8] V. Girault and P. A. Raviart, Finite element methods for Navier–Stokes equations, Springer, Berlin, 1986.

[9] S. Kesavan, On Poincaré’s and J.L. Lions’ lemmas, C. R. Math. Acad. Sci. Paris, Ser. I 340 (2005), 27–30.

[10] S. Mardare, On Poincaré and de Rham’s theorems, Rev. Roumaine Math. Pures Appl. 53 (2008), 523–541.

[11] J. E. Marsden, Generalized Hamiltonian mechanics, a mathematical exposition of non-smooth dynamical systems and classical Hamiltonian mechanics, Arch. Ration. Mech. Anal. 28 (1968), 323–361.

[12] G. de Rham, Differentiable manifolds, Springer, Berlin, 1984.

[13] L. Schwartz, Théorie des distributions, Hermann, Paris, 1966.

[14] L. Tartar, Topics in nonlinear analysis, Université de Paris Sud, Département de Mathématique, Orsay, 1978.

[15] R. Temam, Navier–Stokes equations, theory and numerical analysis, North-Holland, Amsterdam, 1977.

How to cite this article: J. Voigt, On the existence of distributional potentials, Math. Nachr. 296 (2023), 424–433.
https://doi.org/10.1002/mana.202100220