Exact relations between particle fluctuations and entanglement in Fermi gases

Pasquale Calabrese, Mihail Mintchev and Ettore Vicari

Dipartimento di Fisica dell’Università di Pisa and INFN - Pisa, Italy, EU

received 9 March 2012; accepted 26 March 2012
published online 12 April 2012

PACS 03.65.Ud – Entanglement and quantum nonlocality (e.g. EPR paradox, Bell’s inequalities, GHZ states, etc.)
PACS 05.30.Fk – Fermion systems and electron gas
PACS 03.67.Mn – Entanglement measures, witnesses, and other characterizations

Abstract – We derive exact relations between the Rényi entanglement entropies and the particle-number fluctuations of (connected and disjoint) spatial regions in systems of N non-interacting fermions in arbitrary dimension. We prove that the asymptotic large-N behavior of the entanglement entropies is proportional to the variance of the particle number. We also consider 1D Fermi gases with a localized impurity, where all particle cumulants contribute to the asymptotic large-N behavior of the entanglement entropies. The particle cumulant expansion turns out to be convergent for all integer-order Rényi entropies (except for the von Neumann entropy) and the first few cumulants provide already a good approximation. Since the particle cumulants are accessible to experiments, these relations may provide a measure of entanglement in these systems.

The nature of the quantum correlations of many-body systems, and in particular the entanglement phenomenon, are fundamental physical issues. They have attracted much theoretical interest in the last few decades, due to the impressive progress in the experimental activity in atomic physics, quantum optics and nanoscience, which has provided a great opportunity to investigate the interplay between quantum and statistical behaviors in particle systems. The great ability in the manipulation of cold atoms in optical lattice (see, e.g., ref. [1]) has allowed the realization of physical systems which are accurately described by theoretical models such as Hubbard and Bose-Hubbard models in different dimensions, achieving through experimental checks of the fundamental theoretical paradigm of condensed matter physics.

The quantum correlations arising in the ground state of quantum many-body systems can be characterized by the expectation values of the products of local operators, such as the particle density and one-particle operators, or by their integral over a space region A, such as the particle-number correlators within A,

$$\langle N_A^m \rangle_c = \int_A d^d x \prod_{i=1}^m n(x_i) \prod_{i=1}^m n(x_i)$$

where $n(x)$ is the particle-density operator and

$$N_A = \int_A d^d x n(x)$$

counts the number of particles in A. Quantum correlations are also characterized by the fundamental phenomenon of entanglement, which gives rise to non-trivial connections between different parts of extended quantum systems [2]. A widely accepted measure of entanglement is given by the Rényi entropies of the reduced density matrix $\rho_A$ of a subsystem A:

$$S_A^{(\alpha)} = \frac{1}{1-\alpha} \ln \text{Tr} \rho_A^\alpha$$

whose limit $\alpha \to 1$ provides the von Neumann (vN) entropy. Local correlations and bipartite entanglement entropies provide important and complementary information of the quantum features of many-body systems, of their ground states and of their unitary evolutions, because they probe different features of the quantum dynamics. However, the entanglement entropy is a highly nonlocal quantity which is difficult to measure. Designing an experimental protocol for its measurement represents a major challenge.

A recent interesting proposal considers the particle fluctuations as effective probes of many-body entanglement [3–6]. This is based on the result that, for
non-interacting fermions, one can write down a formal expansion of the entanglement entropies of bipartitions in terms of the even cumulants $V_A^{(2k)}$ of the particle-number distribution, which can be defined through a generator function as

$$V^{(m)} = (-i\partial_A)^m \ln(e^{i\lambda N_A})|_{\lambda=0}. \quad (4)$$

Indeed, the Rényi entropies can be written as [4,6]

$$S_A^{(a)} = \sum_{k=1}^{\infty} s_k^{(a)} V_A^{(2k)}, \quad (5)$$

$$s_k^{(a)} = (-1)^k (2\pi)^{2k} \frac{2\zeta(-2k, (1+a)/2)}{(a-1)a^{2k}(2k)!}, \quad (6)$$

where $\zeta$ is the generalized Riemann zeta function. In particular, for the lowest integer $a$ we have

$$S_A^{(1)} = \frac{\pi^2}{3} V_A^{(2)} + \frac{\pi^4}{45} V_A^{(4)} + \frac{2\pi^6}{945} V_A^{(6)} + \cdots, \quad (7)$$

$$S_A^{(2)} = \frac{2\pi^2}{4} V_A^{(2)} - \frac{\pi^4}{192} V_A^{(4)} + \frac{2\pi^6}{23040} V_A^{(6)} + \cdots. \quad (8)$$

The possibility of turning these expansions into an effective measure of entanglement depends on its convergence properties, which appear problematic due to the behavior of the coefficients $s_k^{(a)}$ with increasing $k$.

In this paper we investigate the relations between entanglement entropies and particle fluctuations, and in particular the convergence properties of the formal expansions (5)–(8). We will show that, in non-interacting fermion gases with $N$ particles in a finite volume of any dimension $d$, the expansion (5) gets effectively truncated in the large-$N$ limit, because the high cumulants $V_A^{(m)}$ with $n > 2$ are all suppressed relatively to the particle variance

$$V_A^{(2)} = \langle N_A^2 \rangle_c \equiv \langle N_A^2 \rangle - \langle N_A \rangle^2. \quad (9)$$

The leading $N^{(d-1)/d} \ln N$ asymptotic behavior of $S_A^{(a)}$ in eqs. (5), (7) and (8) arises from $V_A^{(2)}$ only, because the leading order of each cumulant $V_k^{(2)}$ with $k > 2$ vanishes for any subsystem $A$ (including disjoint ones) in any dimension. This implies the general asymptotic relation

$$\frac{S_A^{(a)}}{V_A^{(2)}} = \frac{(1+\alpha^{-1})\pi^2}{6} + o(1). \quad (10)$$

We will also consider the effect of a localized impurity in 1D fermion gases. In this case all particle-number cumulants contribute to the leading logarithmic behavior of the Rényi entropies. The expansion (5) turns out to be convergent for the integer Rényi entropies, so that a few particle cumulants provide a good estimate, with the exception of the $\nu N$ entropy which appears problematic in this respect. Relations like eq. (10) may provide an experimental measure of entanglement.

This letter is organized as follows. First we consider Fermi gases in any dimension, and prove eq. (10) for any subsystem $A$, by rigorously computing the asymptotic behaviors of all cumulants of the particle distribution. Then we focus on one-dimensional Fermi gases, for which more general results, including subleading terms, can be obtained for the asymptotic behaviors of the particle fluctuations and entanglement entropies of connected and disjoint subsystems. Finally, we extend our analysis to one-dimensional Fermi gases in the presence of a defect.

The leading behavior in arbitrary dimension. – We consider a system of $N$ non-interacting spinless fermions with discrete one-particle energy spectrum, which may arise from a finite volume or an external potential. The many-body ground state is obtained by filling the lowest $N$ one-particle energy levels $\phi_n(x)$. The cumulants $V_A^{(m)}$ of the particle-number distribution and the entanglement entropies of a subsystem $A$ can be written in terms of the overlap matrix $[7,8]$.

$$\mathcal{A}_{nm} = \int_A d^d x \phi_n^*(x)\phi_m(x), \quad n, m = 1, \ldots, N. \quad (11)$$

The eigenvalues $\alpha_i$ of $\mathcal{A}$ are real and limited, $\alpha_i \in (0, 1)$. The matrix

$$\tilde{\mathcal{A}} = \mathbb{I} - \mathcal{A} \quad (12)$$

is the overlap matrix of the complement of the region $A$.

The particle fluctuations within a region $A$ can be characterized by the cumulants $V_A^{(m)}$ of the particle distribution. The cumulant generator function for lattice free fermions has been derived in ref. [6] in terms of the two-point correlation function $C_A$ restricted to the subsystem $A$. From this, taking the continuum limit and using the fundamental property

$$\text{Tr} \, C_A^n = \text{Tr} \, \tilde{\mathcal{A}}^n, \quad \text{for any } n \in \mathbb{N}, \quad (13)$$

the cumulants generator function of the particle distribution can be expressed in terms of the overlap matrix $\mathcal{A}$ by replacing $C_A$ with $\mathcal{A}$, obtaining

$$V_A^{(m)} = (-i\partial_A)^m G(\lambda, \mathcal{A})|_{\lambda=0}. \quad (14)$$

$$G(\lambda, \mathcal{A}) = \text{Tr} \ln [1 + (e^{i\lambda} - 1) \mathcal{A}^\dagger]. \quad (15)$$

The even cumulants $V^{(2k)}$, which enter eq. (5), can be cast in the form

$$V_A^{(2k)} = \sum_{n=1}^{k} w_{k,n} \text{Tr} \, E^n, \quad (16)$$

where

$$E \equiv \mathcal{A}(1 - \mathbb{I}) = \mathbb{I} \tilde{\mathcal{A}}, \quad (17)$$

$$w_{k,n} = 2 \sum_{p=1}^{n} (-1)^{p+1} p^{2k} \frac{(2n-1)!}{(n-p)! (n+p)!}. \quad (18)$$

In particular,

$$V_A^{(2)} = \text{Tr} \, E, \quad V_A^{(4)} = \text{Tr} \, [E - 6E^2]. \quad (19)$$
The entanglement entropies are obtained as [7,8]

\[ S_A^{(\alpha)} = \frac{1}{1 - \alpha} \text{Tr} \ln[A^{\alpha} + (1 - A)^{\alpha}]. \]  

(20)

In particular, the \( \alpha = 2 \) entropy can be written as

\[ S_A^{(2)} = -\text{Tr} \ln(1 - 2E) = \sum_{k=1}^{\infty} \frac{\zeta_k}{k} \text{Tr} E^k. \]  

(21)

The eigenvalues of \( E \) satisfies \( c_i, \epsilon_i \in (0, 1/4) \), thus the series (21) is convergent for any \( N \), providing a systematic approximation scheme in terms of \( V_A^{(2k)} \) by inverting eq. (16).

In systems of non-interacting fermions and for arbitrary dimension \( d \), the entanglement entropy of connected bipartitions grows asymptotically as \( N^{(d-1)/d} \ln N \) [9]. The logarithm of the asymptotic behavior is related to the logarithmic area-law violation in lattice free fermions [10–16]. In homogeneous systems with periodic and open (hardwall) boundary conditions (PBC and OBC, respectively) the prefactor can be analytically computed [9] using the Widom conjecture [17]. This method applies, and is even more suited, to compute the large-\( N \) behavior of particle fluctuations with both PBC and OBC in any dimension. In fact, unlike the entanglement entropies, we deal with smooth functions, for which the Widom conjecture has been proved [13,15]. We apply this theorem to the overlap matrix of a subsystem \( A \) (with smooth boundaries \( \partial A \)) of a finite system of size \( L^d \), with PBC, which is [9]

\[ A_{nm} = L^{-d} \int_A d^d x e^{i2\pi(k_m - k_n) \cdot x/L}. \]  

(22)

with \( k \in \mathbb{Z}^d \) within the Fermi surface \( \partial \Gamma \). It allows us to derive the large-\( N \) behavior of \( \text{Tr} F(A) \), where \( F(z) \) is any function analytic in \( \{ z : |z| < 1 + \epsilon \} \) with \( F(0) = F(1) = 0 \), obtaining

\[
\text{Tr} F(A) = C(F) N^{(d-1)/d} \ln N + o(N^{(d-1)/d} \ln N),
\]

(23)

where we set \( L = 1 \), \( n_x \) and \( n_k \) are the normal vectors on \( \partial A \) and on \( \partial f \) which is the Fermi surface \( \partial \Gamma \) rescaled to enclose a unit volume, and

\[ I(F) = \int_0^1 dx \frac{F(z)}{z(z-1)}. \]  

(24)

Note that the function \( F \) enters only the integral \( I(F) \). The result (23) applies also to OBC.

The asymptotic large-\( N \) behavior of \( \text{Tr} E^k \) can be computed using eqs. (23) and (24). Indeed, it corresponds to the function

\[ F_k(z) = z^k (1-z)^k, \]  

(25)

thus

\[ I(F_k) = \frac{[(k-1)]!^2}{(2k-1)!}. \]  

(26)

Plugging the last result into eq. (23), we finally have

\[
\text{Tr} E^k = N^{1-1/d} \ln N \frac{1}{4d\pi^2(2k-1)!} \int_{\partial A} \int_{\partial f} dS_x dS_k |n_x \cdot n_k|. \]

(27)

The large-\( N \) leading behaviors of \( V_A^{(2k)} \) are obtained by inserting these asymptotic results into their expressions in terms of \( \text{Tr} E^k \), cf. eq. (16). For any spatial region \( A \) in any dimension \( d \), the variance \( V_A^{(2)} \) is

\[
V_A^{(2)} = N^{1-1/d} \ln N \frac{1}{4d\pi^2} \int_{\partial A} \int_{\partial f} dS_x dS_k |n_x \cdot n_k|, \]

(28)

while, very remarkably, this leading term cancels for higher cumulants. For odd cumulants \( V_A^{(2k+1)} \) the leading term vanishes, because they are odd under \( A \to \bar{A} \).

In the sum (5), the leading behavior of the Rényi entropies gets a finite contribution only from the variance and the resulting entropies agree the direct computation in ref. [9]. Taking the ratio, the asymptotic large-\( N \) relation (10) follows. The above calculations do not allow us to determine the behavior of the suppressed corrections in eq. (10). Finite-\( N \) calculations up to \( N = O(10^3) \) indicate that they are \( O(1/\ln N) \). This is also supported by the analytic calculations in 1D systems reported below.

Notice that eq. (10) relating particle fluctuations and entanglement entropies can also be obtained for lattice free fermions in the thermodynamic limit, exploiting the correspondence [8,9] between the overlap matrix \( A \) and the lattice two-point function \( C_{ij} \) where \( i, j \) are the lattice sites within the region \( A \). Indeed, analogous formulas for the particle-number cumulants hold by replacing \( A \) with \( \mathbb{C} \) and the lattice version of some of the above results has been already derived (as, e.g., in refs. [6,11,18]). Moreover, developing the results of refs. [19,20], analogous results can be also inferred for free fermion gases in external potential, such as a harmonic one which is usually present in experiments of cold atoms [1].

**One-dimensional Fermi gas.** – For 1D systems one may also consider an alternative computation based on the Fisher-Hartwig conjecture [21] and generalizations [22], similarly to what has been done for the entanglement entropies [8,23,24]. This exact approach allows us to calculate not only the leading term in \( V_A^{(m)} \) but also the leading and subleading corrections to the scaling.

Let us consider a 1D system of size \( L \) with PBC or OBC, and the interval \( A = [0, x] \) as subsystem. Setting \( L = 1 \), the corresponding overlap matrix is [7]

\[ A_{nm} = P_{nm}(x) \equiv \frac{\sin[\pi(n-m)x]}{\pi(n-m)} \]  

(PBC)  

(29)

and

\[ A_{nm} = O_{nm}(x) \equiv \frac{\sin[\pi(n-m)x]}{\pi(n-m)} - \frac{\sin[\pi(n+m)x]}{\pi(n+m)} \]  

(OBC).  

(30)

20003-p3
The asymptotic large-$N$ behavior of the entanglement entropies has been already reported in refs. [7,8]:
\[ S^{(\alpha)}(x) = \frac{1 + \alpha^{-1}}{6} \ln(N \sin \pi x) + b_\alpha + o(1) \] (31)
for PBC, and
\[ S^{(\alpha)}(x) = \frac{1 + \alpha^{-1}}{12} \ln(2N \sin \pi x) + \frac{b_\alpha}{2} + o(1) \] (32)
for OBC, where also the constants $b_\alpha$ and the $o(1)$ corrections are known. Fisher-Hartwig calculations can be generalized to the particle cumulants (details will be reported elsewhere), obtaining for PBC
\[ \text{Tr} E^k = \frac{1}{\pi^2} \left( \frac{(k-1)!!}{(2k-1)!} \right)^2 \ln(N \sin \pi x) \]
\[ + i \int_{-\infty}^\infty dz \frac{g_k(\tanh \pi x)}{\cosh^k(\pi x)} \ln \frac{\Gamma(1/2 + ix)}{\Gamma(1/2 - ix)} + o(1), \] (33)
with
\[ g_k(z) = k z [(1 - z^2)/4]^{k-1/2}. \] (34)
The leading terms agree with eq. (27), but eq. (33) also provides the $O(1)$ corrections. Then, using eq. (16), we obtain
\[ V^{(2)}(x) = \text{Tr} \Xi = \frac{1}{\pi^2} \ln(N \sin \pi x) + v_2 + o(1), \] (35)
\[ V^{(2k)}(x) = v_{2k} + o(1) \text{ for } k > 1, \] (36)
where
\[ v_2 = (1 + \gamma_E + \ln 2)/\pi^2, \]
\[ v_4 = -0.0185104, \quad v_6 = 0.00808937, \] (38)
etc. The odd cumulants are suppressed,
\[ V^{(2k+1)}(x) = o(1). \] (39)
For OBC we find
\[ V^{(m)}_{\text{OBC}}(N) = V^{(m)}_{\text{PBC}}(2N)/2 + o(1). \] (40)
Following ref. [24], the $o(1)$ subleading corrections can be systematically obtained.

**Asymptotic behaviors for disjoint subsystems.**

We now show that the asymptotic relation (10) holds also for disjoint subsystems, for which some result can be obtained. Let us consider a Fermi gas of size $L = 1$ with PBC. The overlap matrix of a generic disjoint subsystem
\[ [x_1, x_2] \cup [x_3, x_4], \quad 0 < x_1 < x_2 < x_3 < x_4 < 1, \] (41)
reads
\[ A_{nm} = e^{i\pi(n-m)(x_4-x_3)} \mathbb{P}_{nm}(x_4-x_3) + e^{i\pi(n-m)(x_2-x_1)} \mathbb{P}_{nm}(x_2-x_1), \] (42)
where the matrix $\mathbb{P}_{nm}$ is defined in eq. (29). For the particular subsystem
\[ B = [0,x] \cup [1/2,1/2+x], \quad 0 < x < 1/2, \] (43)
the overlap matrix takes the simple form
\[ A_{nm} = e^{i\pi(n-m)d}[1 + (-1)^{n-m}]\mathbb{P}_{nm}(x). \] (44)
Since we are interested in the traces of powers of $A_{nm}$, we may neglect the global phase. Then, exploiting the fact that all terms with different parity (odd $n-m$) vanish, we have for any integer $k$
\[ \text{Tr} A_B^k(N) = 2\text{Tr} A_A^k(N/2), \quad A = [0,2x], \] (45)
where $A_A(N/2)$ is the overlap matrix of the connected interval $A = [0,2x]$ with $N/2$ particles reported in eq. (29). This exact relation implies
\[ S_B^{(\alpha)}(N) = 2S_A^{(\alpha)}(N/2), \] (46)
\[ V_B^{(m)}(N) = 2V_A^{(m)}(N/2). \] (47)
Using the asymptotic behaviors (31), (35) and (36) for the r.h.s. of the above equations, we again recover the asymptotic relation (10). Moreover, we obtain the asymptotic behaviors of the entanglement entropies
\[ S_B^{(\alpha)} = \frac{1 + \alpha^{-1}}{3} (\ln N + \ln \sin 2\pi x) + b_\alpha + o(1), \] (48)
which agree with the free-fermion conformal field theory prediction [25]
\[ S^{(\alpha)}_{[x_1,x_2] \cup [x_3,x_4]} = \frac{1 + \alpha^{-1}}{6} \left[ \ln(4N^2) + \ln \frac{\sin(\pi x_{21}) \sin(\pi x_{43}) \sin(\pi x_{41}) \sin(\pi x_{32})}{\sin(\pi x_{31}) \sin(\pi x_{42})} \right] + b_\alpha + o(1), \] (49)
where $x_{ij} = x_i - x_j$ and $0 < x_1 < x_2 < x_3 < x_4 < 1$. While we explicitly reported the proof of eq. (10) for the particular choice of the intervals in eq. (43), the proof can be extended to subsystems with an arbitrary union of equal and equidistant intervals. We expect that the validity of eq. (10) extends to the most general case of disjoint subsystems.

**One-dimensional Fermi gas with an impurity.**

We now investigate how the above relations between entanglement entropy and particle fluctuations may change in the presence of localized interactions, such as those arising from local impurities. We consider 1D fermion gases with an impurity localized at the point separating the system in two equal parts of size $L = 1$, with hard-wall boundary conditions at their ends. The allowed scale invariant conditions at the vertex, describing the universal features arising from the presence of a defect, are fully encoded in the scattering matrix [26]
\[ S(T) = \frac{1}{1 + \epsilon^2} \begin{pmatrix} -1 + \epsilon^2 & 2 \epsilon \\ 2 \epsilon & 1 - \epsilon^2 \end{pmatrix}, \quad T = \frac{2\epsilon}{1 + \epsilon^2}, \] (50)
Table 1: In order to check the convergence of the expansions (5) for the coefficients $C_{S^a(\alpha)}$ of the large-$N$ logarithmic behavior $S^\alpha \approx C_{S^a(\alpha)} \ln N$ in the presence of a defect with $T = 1/2$, we report the sum $\sum_{k=1}^{K} C_{S^a(\alpha)}$ for the $\alpha = 1, 2, 3$ Rényi entropies, whose leading-log coefficients are 0.0570281, 0.0264623, 0.0203204, respectively. An analogous pattern of convergence is shown by the data at fixed $N$.

| $K$ | $nV(\alpha = 1)$ | $\alpha = 2$ | $\alpha = 3$ |
|-----|------------------|--------------|--------------|
| 1   | 0.0416667        | 0.0312500    | 0.0277778    |
| 2   | 0.0622283        | 0.0264309    | 0.0201623    |
| 3   | 0.0622283        | 0.0264309    | 0.0201623    |
| 4   | 0.0288468        | 0.0264614    | 0.0203150    |
| 5   | 0.0763883        | 0.0264262    | 0.0203256    |
| 6   | 0.703467         | 0.0264262    | 0.0203209    |
| 7   | 3.61878          | 0.0264623    | 0.0203202    |
| 8   | 3.47035          | 0.0264262    | 0.0203203    |
| 9   | 949.44           | 0.0264623    | 0.0203204    |
| 10  | 6860.34          | 0.0264623    | 0.0203204    |

where $T$ is the transmission coefficient. $T = 1$ corresponds to full transmission, i.e., no impurity, thus to the bipartition into two equal parts of a 1D system with OBC. The corresponding entropies and cumulants can be obtained by setting $x = 1/2$ in eqs. (32)–(36).

For $|T| < 1$ the ground-state entanglement entropies and particle fluctuations of one of the two edges can be again derived from its overlap matrix $A$. For even $N$ we have [7,27]

$$\mathcal{A}_{nm} = \frac{2\epsilon}{1 + \epsilon^2} \mathcal{O}_{nm}(1/2) \quad \text{for} \ n \neq m,$$

$$\mathcal{A}_{nn} = \frac{1}{1 + \epsilon^2} \quad \text{for odd} \ n, \quad \mathcal{A}_{nn} = \frac{\epsilon^2}{1 + \epsilon^2} \quad \text{for even} \ n,$$

where $\mathcal{O}_{nm}$ is defined in eq. (30). The symmetry $a_k \rightarrow 1 - a_k$ of the spectrum of $A$ implies that any odd observable with respect to $A \rightarrow 1 - A$, such as the odd particle cumulants, vanishes.

Using the result of ref. [27],

$$\text{Tr} \mathbb{E}^k(T) = T^{2k} \text{Tr} \mathbb{E}^k \ (T = 1),$$

the particle fluctuations and entanglement entropies can be computed using eqs. (14)–(21). In particular, for the particle variance we obtain

$$V^{(2)}(T) = \text{Tr} \mathbb{E}(T) = T^2 V^{(2)} \ (T = 1).$$

This shows that, like the homogeneous case $T = 1$, the particle variance grows as $\ln N$, but with a smaller coefficient $T^2/(2\pi^2)$. Analogous results for the large-$N$ behavior of higher cumulants follow and, unlike the homogeneous case, also the higher-order even particle cumulants grow logarithmically when $|T| < 1$

$$V^{(2k)} \approx C_{V^{(2k)}}(T) \ln N.$$
For interacting systems not conserving the particle number, the entanglement should be related to the more fundamental energy transport. Recent proposals [4,32,33] to measure the entanglement entropy in general 1D systems have considered protocols based on appropriate quantum quenches.

Finally, let us mention that the relations between particle fluctuations and entanglement entropies apply also to 1D Bose gases with short-ranged repulsive interactions. Indeed, in the limit of strong interaction (i.e. a gas of hard-core bosons), this can be mapped to free fermions, so that their particle-number cumulants and entanglement entropies coincide for connected regions. The hard-core Bose gas also describes the dilute limit of the finite-strength models [34], which implies that it also provides their infinite size limit at fixed $N$ [20] (with $O(L^{-1})$ corrections). Therefore, in this regime, the same asymptotic large-$N$ behavior is expected, and in particular eq. (10) remains valid.

***

We thank C. Flindt and I. Peschel for correspondence. PC research was supported by ERC under the Starting Grant No. 279391 EDEQS.

REFERENCES

[1] Bloch I., Dalibard J. and Zwerger W., Rev. Mod. Phys., 80 (2008) 885.
[2] Amico L., Fazio R., Osterloh A. and Vedral V., Rev. Mod. Phys., 80 (2008) 517; Eisert J., Cramer M. and Plenio M. B., Rev. Mod. Phys., 82 (2010) 277; Calabrese P., Cardy J. and Doyon B. (Editors), J. Phys. A, 42 (2009) 500301.
[3] Klich I., Refael G. and Silva A., Phys. Rev. A, 74 (2006) 032306.
[4] Klich I. and Levitov L., Phys. Rev. Lett., 102 (2009) 100502.
[5] Song H. F., Rachel S. and Le Hur K., Phys. Rev. B, 82 (2010) 012405.
[6] Song H. F., Flindt C., Rachel S., Klich I. and Le Hur K., Phys. Rev. B, 83 (2011) 161408; Song H. F., Rachel S., Flindt C., Klich I., Laplourence N. and Le Hur K., Phys. Rev. B, 85 (2012) 035409.
[7] Calabrese P., Mintchev M. and Vicari E., Phys. Rev. Lett., 107 (2011) 020601.
[8] Calabrese P., Mintchev M. and Vicari E., J. Stat. Mech. (2011) P09028.
[9] Calabrese P., Mintchev M. and Vicari E., EPL, 97 (2012) 20009.
[10] Wolf M. M., Phys. Rev. Lett., 96 (2006) 010404.
[11] Groev D. and Klich I., Phys. Rev. Lett., 96 (2006) 100503.
[12] Baez T., Chung M.-C. and Schollwöck U., Phys. Rev. A, 74 (2006) 022329.
[13] Helling R., Leschke H. and Spitzer W., Int. Math. Res. Not., 2011 (2011) 1451.
[14] Ding L., Bray-Ali N., Yu R. and Haas S., Phys. Rev. Lett., 100 (2008) 215701.
[15] Sobolev A., Funct. Anal. Appl., 44 (2010) 313; arXiv:1004.2576.
[16] Swingle B., Phys. Rev. Lett., 105 (2010) 050502.
[17] Widom H., Oper. Theory: Adv. Appl., 4 (1982) 477.
[18] Eisler V., Legeza Ž. and Rácz Z., J. Stat. Mech. (2006) P11013.
[19] Camposini M. and Vicari E., Phys. Rev. A, 81 (2010) 023606; 81 (2010) 063614; J. Stat. Mech. (2010) P08020.
[20] Camposini M. and Vicari E., Phys. Rev. A, 82 (2010) 063636.
[21] The Fisher-Hartwig conjecture (Fisher M. E. and Hartwig R. E., Adv. Chem. Phys., 15 (1968) 333) has been rigorously proven for the case at hand, see, e.g., Basor E. L. and Morrison K. E., Linear Algebra Appl., 202 (1994) 129.
[22] Deift P., Its A. and Krasovsky I., Ann. Math., 174 (2011) 1243.
[23] Jin B.-Q. and Korepin V. E., J. Stat. Phys., 116 (2004) 79.
[24] Fagotti M. and Calabrese P., J. Stat. Mech. (2011) P01017; Calabrese P., Camposini M., Essler F. and Nienhuis B., Phys. Rev. Lett., 104 (2010) 095701; Calabrese P. and Essler F. H. L., J. Stat. Mech. (2010) P08029.
[25] Calabrese P. and Cardy J., J. Stat. Mech. (2004) P06002; Ryu S. and Takayanagi T., Phys. Rev. Lett., 96 (2006) 181602; Casini H., Fosco C. D. and Huerta M., J. Stat. Mech. (2005) P05007; Calabrese P., Cardy J. and Tonni E., J. Stat. Mech. (2009) P11001; (2011) P01021.
[26] Bellazzini B. and Mintchev M., J. Phys. A, 39 (2006) 11101; Bellazzini B., Mintchev M. and Sorba P., J. Phys. A, 40 (2007) 2485; Bellazzini B., Calabrese P. and Mintchev M., Phys. Rev. B, 79 (2009) 085122.
[27] Calabrese P., Mintchev M. and Vicari E., J. Phys. A, 45 (2012) 105206.
[28] Eisler V. and Peschel I., Ann. Phys. (Berlin), 522 (2010) 679; Peschel I. and Eisler V., J. Phys. A: Math. Theor., 45 (2012) 155301; Peschel I., arXiv:1109.0159.
[29] Eisler V. and Garmon S., Phys. Rev. B, 82 (2010) 174202.
[30] Stenger J., Inouye S., Chikkatur A. P., Stamper-Kurn D. M., Pritchard D. E. and Ketterle W., Phys. Rev. Lett., 82 (1999) 4569; Fabbrini N., Huber S. D., Clément D., Fallani L., Fort C., Inguscio M. and Altman E., arXiv:1109.1241.
[31] Bakr W. S., Peng A., Tai M. E., Ma R., Simon J., Gillen J. I., Fölling S., Pollet L. and Greiner M., Science, 329 (2010) 547.
[32] Hsu B., Grossfeld E. and Fradkin E., Phys. Rev. B, 80 (2009) 235412.
[33] Cardy J., Phys. Rev. Lett., 106 (2011) 150404.
[34] Lieb E. H. and Liniger W., Phys. Rev., 130 (1963) 1605.