Numerical simulations and theoretical modelling

Examples of the simulated reflectance are shown in Fig. S1, with and without strong exciton-photon coupling. Without exciton-photon coupling, the reflectance dips correspond to the cavity photon modes, which can be modelled by Eq. (2) of the main text, as shown by the dashed lines in Fig. S1a. In the strong coupling regime, the reflectance dips form 4 exciton-polariton branches, as shown in Fig. S1b. All branches can be modelled by Eq. (1) using the parameters of the cavity photon with the exciton-photon coupling strength $V$ as the fitting parameter.

Figure S1: Simulated reflectance using the 4×4 transfer matrix method. a “Empty” anisotropic microcavity with exciton-photon coupling turned off. Dashed lines correspond to the exciton (known a priori) and cavity photon energies (fitted using Eq. 2 of the main text). b Same as a but with the exciton-photon coupling turned on. Dashed lines are energies extracted from the model, Eq. 1 of the main text, of the main text using the exciton and cavity photon energies extracted in (a), and the exciton-photon coupling strength $V$ as the fit parameter.

We fitted Lorentzian profiles to the reflectance dips to extract the energy (peak centre) and linewidth (peak width), which are presented as data points in Fig. S2a, b and Fig. S2c, d for the weak and strong coupling regimes, respectively. Evidently, the energies of the lower polariton branches (see Fig. S2e) inherit the same behaviour from the cavity photon, but the crossing occurs at lower momenta for the polariton branches. However, the linewidth behaviour is different for the two cases. The linewidth of the TE (TM) polarised cavity photon increases (decreases) with momentum, resulting in the linewidths crossing along the $k_y$ direction (see Fig. S2b). This means the Fermi arcs form a single closed contour in momentum space (see dashed contour in Fig. 1G of the main text). For the polariton branches, the linewidths increase with momentum, regardless of the polarisation. This is because the excitonic fraction (which has a large linewidth) increases with momentum. As a result, the polariton linewidths cross along the same direction as the energies,
suggesting that the Fermi arcs form two closed loops in momentum space, as illustrated by the solid arcs in Fig. 1G of the main text.

To confirm that the model, Eqs. (1,2) of the main text, describe the non-Hermitian dispersion of exciton polaritons, we fit the eigenvalues of Eq. (1,2) to the extracted complex dispersion. We used Eq. (2) of the main text to fit to the complex cavity photon dispersion, shown as solid lines in Fig. S2a, b, resulting in the following cavity photon parameters: $\tilde{E}_c = -(2.306-4.58 \times 10^{-4}i)$ eV, $\tilde{\chi}_1 = (2.3 \times 10^{-3} - 1.45 \times 10^{-7}i)$ $\mu$m$^{-2}$ eV, $\tilde{\chi}_2 = (8.76 \times 10^{-7} - 8.59 \times 10^{-8}i)$ $\mu$m$^{-4}$ eV, $\tilde{\alpha} = (8 \times 10^{-3} - 4.28 \times 10^{-6}i)$ eV, $\tilde{\beta}_1 = (1.76 \times 10^{-4} - 6.4 \times 10^{-6}i)$ $\mu$m$^{-2}$ eV, $\tilde{\beta}_2 = (5.01 \times 10^{-7} - 5.33 \times 10^{-8}i)$ $\mu$m$^{-4}$ eV. The complex exciton energy is known a priori as an input to the simulation with a value of $\tilde{E}_x = (2.407 - 6.08 \times 10^{-3}i)$ eV. We then use these parameters to fit to the eigenvalues of Eq. (1) to the complex lower exciton-polariton dispersion as shown in Fig. S2b, c, resulting in an exciton-photon coupling strength of $V = 61$ meV. Note that the model starts to deviate from the exciton-polariton dispersion at higher momenta. This is due to the weak polarisation and momentum-dependent exciton-photon coupling strength, which is neglected in the model for simplicity.

For sufficiently large negative detuning of cavity photon with respect to the exciton energy, Eq. (2) of the main text can be used to describe the lower exciton-polariton branches. The $2 \times 2$ Hamiltonian
is easier to understand and can describe the energy crossings and topology of the exceptional points presented in this work. Moreover, it enables a simple theoretical model for the effective gauge field for exciton polaritons. The validity of the $2 \times 2$ approximation is demonstrated in Fig. S3, where the crossing and anti-crossing behaviour of the complex dispersion is qualitatively captured by the $2 \times 2$ Hamiltonian. The model greatly deviates from the simulations at higher momenta, such that the energy and linewidth crossings occur at different momenta compared to the simulations. Higher order terms can be added to the $\tilde{\chi}(k)$ and $\tilde{\beta}(k)$ terms to improve the model.

**Figure S3: Simulated and theoretical exciton-polariton complex dispersion.** (a) Energy and linewidth (FWHM) of the lower exciton-polariton modes (upper polariton not shown) extracted from the $4 \times 4$ transfer matrix simulation of an anisotropic microcavity with embedded excitons in the strong coupling regime. (b) Mean-subtracted energy and linewidth of the simulated data in (a). (c) Mean-subtracted energy and linewidth of the simple model (Eq. (1) of main text, after accounting for the losses), showing a similar behavior as that in (b).

**Supplementary experimental data**

The transfer matrix calculations and the theoretical model also capture the linewidth behaviour of the experiment shown in Fig. S4. Regardless of the direction, the linewidth increases with $k$ as the exciton fraction of polariton increases. However, the experimental linewidth increases more or less linearly with $k$ (see Fig. S4), compared to the near parabolic behaviour of the numerical simulation. This can arise from the inhomogeneous broadening of the exciton resonance, which is not accounted for in the simulations.
Figure S4: Experimentally measured linewidths. FWHM of the experimental data presented in Fig. 2c of the main text. The linewidths of both modes are split at $k = 0$ and increase with $k$ as the excitonic fraction of the exciton-polariton increases.

The measured energies and linewidths near the paired EPs in momentum space are shown in Fig. S5. The complex energies are sorted following the same method in Fig. 3 of the main text to highlight the crossings in energy and linewidth.

Figure S5: 2D complex spectra. Experimentally extracted (a) Energy and (b) linewidth surfaces in the vicinity of the paired exceptional points (pink dots).

The spin texture of the excitonic PL of CsPbBr$_3$ perovskite crystals in momentum space are shown in Fig. S6. This was measured directly in momentum space without spectral filtering. The same background spin texture is observed outside the paired EPs in Fig. 4C of the main text. Note that, as expected, this chirality does not depend on the orientation (by rotation) of the sample.
Figure S6: Weak inherent chirality of CsPbBr$_3$ perovskite crystals. $S_3$ texture in $k$-space of the energy-integrated photoluminescence of a bare perovskite crystal.

The extraction of the energy and linewidth is performed by fitting the spectrum at each point in momentum space. For regions away from the EPs, two peaks appear in the polarized spectrum, as shown in Fig. S7a, which can be fitted to a double Lorentzian function. Near the EPs, the two peaks overlap so only one peak appears in the measurements. These peaks can be distinguished using the pair of orthogonally linearly polarized measurements as shown in Fig. S7b, c. We then select the pair that produces the largest energy splitting.

Figure S7: Fitting of spectral data. (a) Spectral profile (blue dots) at a $k$ point away from energy crossings. The red line is the double Lorentzian function fit, and the shaded areas are the contributing Lorentzian functions. Crossed polarized spectral profiles at a $k$ point (b) near and (c) very close to an exceptional point. A single Lorentzian function (shaded region) is used to fit the polarized data (blue dots). Dashed vertical lines correspond to the peak energy of the upper panel.

Topology of the eigenstates of the non-Hermitian model

For completeness, we show here the topology of the eigenstates corresponding to the discussions related to Fig. 1H-L of the main text. In Fig. S8, we show changes of the lower eigenstate with
increasing real-valued $\sigma_z$-term perturbation along with the topology of the eigenenergies, defined the spectral phase. Similar to Fig. 1K-L of the main text, the EPs in a pair each other (see Fig. S8a, b) and annihilate leading to a trivial non-Hermitian topology, i.e. zero spectral winding, as shown in Fig. S8c.

The in-plane pseudospin angle of the lower eigenstate (shown in Fig. S8d-f) is the opposite of that of the upper eigenstate (see Fig. 1H-J). Indeed, at some finite $\sigma_z$-term perturbation, the pseudospin singularity migrates to one of the eigenstates. The upper eigenstate (Fig. 1I) retains the negative (clockwise) winding while the lower eigenstate retains the positive winding. The singularity is accompanied by high circular polarisation degree, as shown by the spin textures (or $S_3$) in Fig. S8g-l. The sign of the circular polarisation follows the winding direction of the in-plane pseudospin angle. Note that since the pseudospin singularities do not follow the EP in momentum space for finite $\sigma_z$-term perturbation, the EP becomes elliptically polarised.

The evolution of the windings of both eigenstates and eigenenergies as a function of a $\sigma_z$-perturbation are shown in Fig. S9a. For a vanishing $\sigma_z$-perturbation, both eigenstates share the same winding of $\pi$. However, for a nonzero $\sigma_z$-term, as soon as the EP moves away from the pseudospin singularity, the shared winding migrates to one of the eigenstates resulting in a winding of $2\pi$ for that eigenstate and zero for the other. The windings of $2\pi$ and 0 persist for any strength of $\sigma_z$-perturbation. The zero $\sigma_z$-term case is a transition point that demonstrates the instability of the $\pi$-winding of the eigenstates against $\sigma_z$-perturbation. This is reminiscent of the instability of Dirac points in the Hermitian case, where any $\sigma_z$-perturbation destroys the degeneracy and opens the gap. 8
In contrast, the $\pi$ winding of the eigenenergies is more robust, requiring a finite $\sigma_z$-perturbation to change the winding number, as shown by the solid curve in Fig. S9a. The EPs meet and annihilate at this finite $\sigma_z$-perturbation, consequently destroying the bulk Fermi arc. This signifies the non-Hermitian topological transition and is accompanied by the sudden disappearance of the $\pi$ jump of the spectral phase at the bulk Fermi arc, as shown in Fig. S9b. By contrast, the phase jump of the eigenstate at the bulk Fermi arc continuously decreases from $\pi$ for any nonzero $\sigma_z$-perturbation, which demonstrate the stark difference between the topology of the eigenstates and eigenenergies.

**Figure S9: Evolution with $\sigma_z$-perturbation.** a Numerically calculated winding of the eigenstates (dashed) and eigenenergies (solid) as a function of $\sigma_z$-perturbation. b Discontinuity jump of the in-plane pseudospin angle (dashed line) and spectral phase (solid) at the midpoint of the bulk Fermi arc as a function of $\sigma_z$-perturbation.

**Magnitude of circular polarisation degree near the EPs**

The order-of-magnitude reduction of the measured circular polarisation degree near the EPs can be explained by the close proximity of the exceptional points. The high circular polarisation near the EPs (due to the pseudospin singularities) has opposite handedness and finite extent in momentum space, as shown in the theoretical spin texture shown in Fig. 4B of the main text. Hence, the opposite spin textures due to the EPs tend to overlap and cancel each other. In the measurement, this will result in the reduction of the circular polarisation degree: the closer the pseudospin singularities in momentum space, the larger is the reduction in the measured circular polarisation.

We observe the same behaviour in the transfer matrix simulations, which simulates the experiment, as presented in Fig. S9. In the simulations, we varied the number of DBR pairs (from 4.5 to 6.5) and calculate the $S_3$ of reflectance at constant energy (near the EP energy). Increasing the number of DBR pairs decreases the linewidth and hence reducing the separation between the EPs. As clearly shown in Fig. S9, the maximum $|S_3|$ decreases with increasing number of DBR layers. This suggests that samples with very high Q-factor will bring the EPs very close to each other such that the EP pairs cannot be resolved in the experiment.
Figure S10: Simulated circular polarisation degree near the EPs. Simulated spin texture in momentum space at constant energy (near the EP energy) for an anisotropic microcavity with different number of DBR pairs: a 4.5 pairs, b 5.5 pairs, c 6.5 pairs.