LARGE-N EXPANSION AS SEMICLASSICAL APPROXIMATION TO THE THIRD-QUANTIZED THEORY

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Abstract

The semiclassical theory for the large-$N$ field models is developed from an unusual point of view. Analogously to the procedure of the second quantization in quantum mechanics, the functional Schrodinger large-$N$ equation is presented in a third-quantized form. The third-quantized creation and annihilation operators depend on the field $\varphi(x)$. If the coefficient of the $\varphi^4$-term is of order $1/N$ (this is a usual condition of applicability of the $1/N$-expansion), one can rescale the third-quantized operators in such a way that their commutator will be small, while the Heisenberg equations will not contain large or small parameters. This means that classical equation of motion is an equation on the functional $\Phi[\varphi(\cdot)]$. This equation being a nonlinear analog of the functional Schrodinger equation for the one-field theory is investigated. The exact solutions are constructed and the renormalization problem is analysed. We also perform a quantization procedure about found classical solutions. The corresponding semiclassical theory is a theory of a variable number of fields. The developed third-quantized semiclassical approach is applied to the problem of finding the large-$N$ spectrum. The results are compared with obtained by known methods. We show that not only known but also new energy levels can be found.

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1 Introduction

Investigation of quantum field theory as the number of fields tends to infinity is very important. For example, the number of colours of quarks $N_c$ seems to be the only possible large parameter in quantum chromodynamics, so that one can perform asymptotic expansions in a series of $1/N_c$ \[1, 2\].

Vector models are also suitable for the large-$N$ analysis. Expansions in a series of $1/N_f$ ($N_f$ - number of flavours of quarks) is widely used in QCD. Examples of applications of the $1/N$-expansion are:

- large-$N$ calculation of Green functions \[3\];
- evaluation of the effective action \[4\];
- investigations of the spontaneous symmetry breaking \[5, 6\];
- research of processes of particle creation and back reaction in the strong external field \[7\];
- investigation of the evolution of the pion condensate \[8\];
- evaluations of cross-sections of the processes like “1 particle” $\rightarrow$ “$n$ particles” \[9\].

There are various approaches to the $1/N$-expansion. Consider, for example, the simplest vector model

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a - \frac{m^2}{2} \varphi^a \varphi^a - \frac{\lambda}{4N} \varphi^a \varphi^b \varphi^b \quad (1)$$

($a, b = 1, ..., N, \varphi^1, ..., \varphi^N$ are scalar fields, we sum over repeated indices). Some of these approaches are the following.

1. One can extract the so-called bubble Feynman graphs and evaluate their contributions to the Green functions and scattering amplitudes \[3, 2\]; one can then show that contribution of other graphs can be neglected.

2. One can present physical quantities via functional integrals like

$$\int D\varphi e^{i\int dx \mathcal{L}}, \quad (2)$$

introduce the auxiliary field $\chi$ \[3, 10, 11\], insert the identity

$$1 = \int D\chi e^{\frac{i\lambda}{4N} \int dx (N\chi - \varphi^a \varphi^a)^2}$$

into the functional integral (2) and evaluate the Gaussian integral over $\varphi$ exactly. Then remaining integral over $\chi$ is to be evaluated semiclassically (by the saddle-point technique). It is the condition $N \rightarrow \infty$ that allows us to apply the semiclassical approach.

3. One can use the collective field approach \[12\]. The Schrodinger wave functional $\Psi[\varphi_1(\cdot), ..., \varphi_N(\cdot)]$ can be assumed to depend only on the collective field

$$\psi(x, y) = \sum_{a=1}^N \varphi^a(x) \varphi^a(y). \quad (3)$$

The ansatz

$$\Psi[\varphi_1(\cdot), ..., \varphi_N(\cdot)] = \Psi[\psi(\cdot, \cdot)] \quad (4)$$

satisfies the Schrodinger equation; the obtained relation for $\Psi[\psi(\cdot, \cdot)]$ can be treated semiclassically.

4. One can develop the following semiclassical approach \[13\]. Consider the operators

$$\hat{\psi}(x, y) = \sum_{a=1}^N \hat{\varphi}^a(x) \hat{\varphi}^a(y),$$

$$\hat{\xi}(x, y) = \sum_{a=1}^N \hat{\pi}^a(x) \hat{\varphi}^a(y),$$

$$\hat{\chi}(x, y) = \sum_{a=1}^N \hat{\varphi}^a(x) \hat{\varphi}^a(y),$$

$$\hat{T}(x, y) = \sum_{a=1}^N \hat{\varphi}^a(x) \hat{\varphi}^a(y).$$

These operators can be interpreted as the creation and destruction operators of the fields $\varphi^a$. The Schrodinger equation is then written as

$$\frac{\partial}{\partial t} \Psi[\psi(\cdot, \cdot)] = i \mathcal{H} \Psi[\psi(\cdot, \cdot)],$$

where $\mathcal{H}$ is the Hamiltonian of the system.
The system of Heisenberg equations for these operators is expressed via these operators only, so that one can use the argumentation like the Ehrenfest theorem in quantum mechanics. One can consider the average values of the right- and left-hand sides of the Heisenberg equations and obtain the classical equations.

5. One can consider the Heisenberg equations for the fields $\varphi^a$. Making use of the averaging procedure [14], one can obtain the large-N equations.

In this paper we present yet another approach to the $1/N$-expansion. We show that semiclassical investigation of the large-N theories may lead to classical equations of another type rather than to equations on “collective field” $\psi(x,y)$ (3) or $\varphi^a$.

We consider states of a more general form than $\Psi[\varphi^a]$. The state functional

$$\Psi[\varphi_1(\cdot), ..., \varphi_N(\cdot)] (5)$$

is considered to be symmetric with respect to transformations of fields $\varphi_1(x), ..., \varphi_N(x)$. We will also discuss the general, non-symmetric, case.

In quantum statistical mechanics the most suitable way to investigate the $N$-particle Schrödinger equation is the method of second quantization. The Hamiltonian operator is presented via the operators of creating and annihilating particles, $a^\pm(q)$, with given coordinate $q$. Making use of the canonical commutation relations, one can investigate different properties of the $N$-particle system.

One can hope that this conception can be applied also to the $N$-field system. The $N$-field Hamiltonian operator is to be presented via the operators $A^\pm[\varphi(\cdot)]$ which “create” or “annihilate” the field configuration $\varphi(\cdot)$. This procedure can be called as “third quantization” because quantum field is already second-quantized.

The notion of third quantization usually arise in quantum gravity and cosmology [13] when the processes (like wormhole transition) with variable number of universes are studied. Creation (annihilation) operators create (annihilate) the universe.

The notion of third quantization is also useful in string theory: splitting and joining of strings are interpreted in terms of creation and annihilation operators [10].

We see that the idea of third quantization can be applied also to ordinary field theories. We will show that this idea is very useful and allows us to construct new asymptotics in large-N field theories.

For example, one can investigate the processes with a large number $n$ of particles, $n \sim N$, by using the developed technique. It is known [11] that the usual $1/N$-expansion is applicable only for processes with a small number of particles.

We will consider the formalism of third quantization in more details in section 2. We will show that the coefficient of $\varphi^4$ in eq. (3) is of order $O(1/N)$ is a condition of applicability of semiclassical approximation to the third-quantized theory. The corresponding classical equations are unusual. Namely, the classical equations corresponding to quantum mechanics are ordinary differential equations for the classical trajectory $(p(t), q(t))$, $p(t)$ are momenta, $q(t)$ are coordinates. When one applies semiclassical approximation to the (second-quantized) field theory, one finds classical equations for the classical field $\varphi(x,t)$. In the third-quantized theory the classical equation is equation for the functional $\Phi[t, \varphi(\cdot)]$ depending on the field $\varphi(x)$ and time $t$. Although this equation resembles the ordinary quantum functional Schrödinger equation in the theory of one field, an essential feature of this equation is non-linearity. Another interesting feature of the equation on $\Phi[t, \varphi(\cdot)]$ is exact solvability for some interesting cases.

The conception of classical “master field” is often used in investigation of large-N systems. However, “master field” is usually considered as a function on a finite-dimensional space. For example, field $\psi(x,y)$ (3) or fields $\varphi^a(x)$ are usually considered as classical variables [13, 14, 17, 18]. We see that in our approach the role of “master field” is played by the functional $\Phi[t, \varphi(\cdot)]$ which can be treated as time-dependent vector of the one-field Fock space. Classical mechanics for $\psi(x,y)$ is in agreement with our dynamics for $\Phi$. Namely, the Gaussian ansatz for our equation for $\Phi$ leads to the equation on $\psi$ which was obtained.

The system of Heisenberg equations for these operators is expressed via these operators only, so that one can use the argumentation like the Ehrenfest theorem in quantum mechanics. One can consider the average values of the right- and left-hand sides of the Heisenberg equations and obtain the classical equations.
in refs.\cite{12}. However, our classical theory is much more rich than the mechanics for $\psi$ because one can consider other solutions for the equation for $\Phi$. The reason is that we are considering solutions to the $N$-field equation which are not $O(N)$-symmetric but symmetric with respect to transpositions. This class is much more wide than the class of solutions \cite{4}. This means that we are able not only to reproduce the known asymptotics but also to construct new asymptotic solutions to the large-$N$ quantum-field Schrödinger equation.

An interesting feature of the classical equations on $\Phi[\varphi(\cdot)]$ is that it can be interpreted as a Hamiltonian system corresponding to the flat phase space. On the other hand, the phase space of refs.\cite{13} is curved. There are a lot of well-developed semiclassical methods \cite{19, 20, 21, 22, 25} for flat phase space, so that all of them are applicable for the third-quantized system.

We will construct the asymptotics for the entire $N$-field wave functional \cite{5}. This approach gives more information than the averaging procedure of the Heisenberg field equations. The difference can be explained as follows. Consider the semiclassical approximation to the ordinary quantum mechanics. One can derive classical equations with the help of the Ehrenfest theorem. This result allows us to find the trajectory of propagation of the wave packet. To find the evolution of the shape of the wave packet, it is necessary to apply more complicated semiclassical methods such as the complex-WKB method \cite{20} which allows us to construct the approximation for the wave function rather than for average values of the semiclassical observables.

This paper is organized as follows. Section 2 deals with the simple quantum mechanical model, the $O(N)$-symmetric anharmonic oscillator. One can test different versions of the $1/N$-expansion on such a toy model. We consider the analogs of the collective field approach and of our approach. These methods are compared, and it is shown that our approach allows us to construct much more rich set of approximate solutions to the Schrödinger equation. In section 3 we introduce the notion of third quantization (as related to the large-$N$ system). We write down the third-quantized quantum Hamiltonian, as well as the Heisenberg equations of motion. The semiclassical approximation is applied and the problems of regularization and renormalization are considered. In section 4 we apply the developed method to the problem of investigation of the large-$N$ field theory in the finite volume. Section 5 contains concluding remarks. Appendices A and B deal with the brief review of the complex-WKB technique \cite{20, 38}.

2 \hspace{1em} \textbf{$O(N)$ - symmetric anharmonic oscillator: different approaches to $1/N$-expansion}

In this section we consider the simplest example of the large-$N$ system - the $O(N)$-symmetric anharmonic oscillator with quartic interaction. The wave function of this system depends on $N$ coordinates, $x_1, ..., x_N$, while the Hamiltonian is

\begin{equation}
H = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{ax_i^2}{2} \right) + \frac{g}{4N} \sum_{i,j=1}^{N} x_i^2 x_j^2.
\end{equation}

Different approaches to $1/N$-expansion can be illustrated by applying to such a simple system. Let us compare the third-quantized approach developed in this paper with other approaches.

2.1 \hspace{1em} \textbf{The collective field approach}

The collective field approach for the Hamiltonian \cite{6} which is analogous to the ansatz \cite{4} was developed in \cite{12}. The idea is to consider the $O(N)$-symmetric wave function depending only on $r = \sqrt{x_1^2 + ... + x_N^2}$,

\begin{equation}
\psi_N(x_1, ..., x_N) = \psi(\sqrt{x_1^2 + ... + x_N^2}).
\end{equation}
However, the full probability will not have the usual form $\int dr |\psi(r)|^2$, since the area of the surface of the sphere depends on its radius. This means that

$$\int dx_1...dx_N |\psi_N(x_1, ..., x_N)|^2 = C_N \int_0^\infty dr |\psi(r)|^2 r^{N-1},$$

where $C_N$ is an $r$-independent constant. Let us consider the quantity

$$\varphi(r) = \psi(r) r^{\frac{N-1}{2}}, \quad (7)$$

which plays the role of the probability amplitude that $r = \sqrt{x_1^2 + ... + x_N^2}$. Time evolution of the function (7) $\varphi^t(r)$ is specified by the equation derivable from eq.(6):

$$i \frac{\partial \varphi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \varphi}{\partial r^2} + \frac{a}{8r^2} \varphi + \frac{g}{4N} r^4 \varphi \quad (8)$$

The rescaling

$$r = \xi \sqrt{N}$$

transforms eq.(8) to the semiclassical form

$$i \frac{N}{\xi} \frac{\partial \varphi}{\partial t} = -\frac{1}{2N^2} \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{1}{8\xi^2} \left(1 - \frac{1}{N^2}\right) \varphi + \frac{a\xi^2}{2} \varphi + \frac{g\xi^4}{4} \varphi, \quad (9)$$

where the analog of the Planck constant is $1/N$. The condition $N \to \infty$ is the condition of applicability of semiclassical methods to eq.(9).

Eq.(9) specifies the motion of the particle in the potential

$$U = \frac{1}{8\xi^2} + \frac{a\xi^2}{2} + \frac{g\xi^4}{4}. \quad (10)$$

One can apply different semiclassical methods to eq.(9) which are described in Appendix A. Spectrum in the vicinity of the ground state can be found by using the oscillator approximation of refs.[21, 20, 23]:

$$\frac{E_n}{N} = U(\bar{\xi}) + \frac{1}{N} \sqrt{U''(\bar{\xi})} (n + 1/2) + O(1/N^2), \quad (11)$$

where $\bar{\xi}$ is the minimum of the potential (10).

### 2.2 The “second-quantized” approach

Let us develop for the $O(N)$ - symmetric system the approach based on the Fock space. Instead of the Schrodinger equation for $\psi_N(x_1, ..., x_N)$, we consider the evolution equation for the set of functions

$$(\psi_0, \psi_1(x_1), ..., \psi_N(x_1, ..., x_N), ...) \quad (12)$$

($\psi_N$ being symmetric). Introduce in the Fock space of sets (12) creation and annihilation operators as usual,

$$(a^- (x) \psi)_{n-1}(x_1, ..., x_{n-1}) = \sqrt{n} \psi_n(x_1, ..., x_n),$$

$$(a^+ (x) \psi)_n(x_1, ..., x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \delta(x - x_i) \psi_{n-1}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Since the operator

$$\sum_{i=1}^n x_i^2$$
is presented via the introduced operators as

\[
\int dx a^+(x)x^2 a^-(x),
\]

while for the operator \( \sum_{i=1}^n (-\partial^2/\partial x_i^2) \) one has

\[
\int dx a^+(x) \left(-\frac{\partial^2}{\partial x^2}\right) a^-(x),
\]

one can present the Schrödinger equation in the second-quantized form

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \hat{H} \psi
\]

where

\[
\hat{H} = \int dx a^+(x) \left(-\frac{1}{2 \hbar^2} + \frac{ax^2}{2}\right) a^-(x) + \frac{g\hbar}{4} \left( \int dx a^+(x)x^2 a^-(x) \right)^2,
\]

for \( \hbar = 1/N \). Since the operators \( a^\pm(x) \) obey canonical commutation relations

\[
[a^+(x), a^\pm(y)] = 0, \quad [a^-(x), a^+(y)] = \delta(x - y),
\]

one can use the following functional Schrödinger representation. The vacuum state of the Fock space (1,0,0,...) corresponds to the functional

\[
\Psi(\xi) = \exp \left(-\frac{1}{2\hbar} \int dx \xi^2(x)\right)
\]

while creation and annihilation operators are written as

\[
a^\pm(x) = \frac{\xi(x) \mp \hbar \delta/\delta\xi(x)}{\sqrt{2\hbar}}.
\]

In this representation eq. (13) takes the semiclassical form

\[
i\hbar \frac{\partial \psi[\xi]}{\partial t} = \mathcal{H} \left[ \xi, -i\hbar \frac{\delta}{\delta\xi} \right] \psi[\xi(\cdot)],
\]

where

\[
\mathcal{H}[\xi, \pi] = H \left[ \frac{\xi - i\pi}{\sqrt{2}}, \frac{\xi + i\pi}{\sqrt{2}} \right],
\]

\[
H[\varphi^*, \varphi] = \int dx \varphi^*(x) \left(-\frac{1}{2 \hbar^2} + \frac{ax^2}{2}\right) \varphi(x) + \frac{g\hbar}{4} \left( \int dx x^2|\varphi(x)|^2 \right)^2.
\]

An analog of the Planck constant is \( \hbar \). One can try to apply semiclassical methods to eq.(15). One finds that the corresponding classical equations

\[
\frac{d}{dt} \xi = \frac{\delta H}{\delta \pi}, \quad \frac{d}{dt} \pi = -\frac{\delta H}{\delta \xi}
\]

can be presented as

\[
i\hbar \frac{d}{dt} \varphi = \frac{\delta H}{\delta \varphi^*}, \quad \varphi = \frac{\xi + i\pi}{\sqrt{2}}.
\]
We see that classical trajectory in our approach is specified not by one coordinate and one momentum but by infinite set of coordinates and momenta, \((\xi(\cdot),\pi(\cdot))\), or, equivalently, by the complex function \(\varphi(t,x)\) obeying the following dynamical equation
\[
i\frac{\partial \varphi(t,x)}{\partial t} = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{ax^2}{2}\right)\varphi(t,x) + \frac{g}{2}x^2\varphi(t,x) \int dy y^2|\varphi(t,y)|^2. \tag{19}
\]

The reason for extending the phase space is the following. First, we consider not the “N-argument” Schrodinger equation but infinite sets of such equations for \(N = 0, \infty\). The average value of “number of arguments of wave function” analogously to the average number of particles in quantum statistics
\[
< \int dxa^+(x)a^-(x) > = \frac{1}{2\varepsilon} < \int dx \left(\xi(x) - \varepsilon \frac{\delta}{\delta \xi(x)}\right) \left(\xi(x) + \varepsilon \frac{\delta}{\delta \xi(x)}\right) >
\]
in the “semiclassical state” is approximately equal to
\[
\frac{1}{2\varepsilon} \int dx(\xi(x) - i\pi(x))(\xi(x) + i\pi(x)) = \frac{1}{\varepsilon} \int dx|\varphi(x)|^2.
\]

Because of the relation \(\varepsilon = 1/N\) one should impose on the function \(\varphi\) the normalization condition
\[
\int dx|\varphi(x)|^2 = 1. \tag{20}
\]

However, the extension of the state space to the Fock space \((12)\) is not the only reason for extension of the phase space. Another reason is that we consider not only \(O(N)\) - symmetric solutions to the Schrodinger equation for \(\psi_N\) but solutions of a more general form.

### 2.3 Correspondence between Hamiltonian systems

Let us show that the collective-field equations can be obtained as a partial case of our classical equation \((19)\). To investigate this problem, consider the substitution
\[
\varphi(t,x) = ce^{\frac{i\alpha}{\sqrt{2}}x^2} \tag{21}
\]
to eq.\((13)\) for \(t\)-dependent complex numbers \(c\) and \(\alpha\). Making use of the condition \((20)\), we find that the function \((21)\) obeys eq.\((19)\) if
\[
i\frac{d}{dt}c = -\frac{i\alpha}{2}, \tag{22}
\]
\[
\frac{d\alpha}{dt} + \alpha^2 + a + \frac{g}{2Im\alpha} = 0 \tag{23}
\]

However, eq.\((23)\) can be transformed to the form of the Hamiltonian system corresponding to the potential \((10)\). Let us extract real and imaginary parts of \(\alpha\),
\[
p = \frac{Re\alpha}{\sqrt{2Im\alpha}}, \quad x = \frac{1}{\sqrt{2Im\alpha}}.
\]

Eq.\((23)\) takes the form
\[
\frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\frac{\partial U}{\partial x}.
\]
We see that our approach contains the collective field approach as a partial case. However, one can consider not only the Gaussian ansatz \((21)\) for eq.\((19)\) but also other substitutions. We see that one can construct new asymptotic solutions for the large-N systems.
2.4 Asymptotic spectrum of the second-quantized Hamiltonian

Let us show that the oscillator approximation applied to the Hamiltonian system with the Hamiltonian equation (14) leads us to the semiclassical spectrum of energy which is much more rich than eq.(11).

There are many semiclassical methods. One of them - quantization of periodic trajectories - is the following [21]. One should find the classical periodic solution. Then one considers small variations around it and finds the “stability angles” [21] related with the distances between semiclassical energy levels. This approach is certainly applicable to the Hamiltonian system (16) since there are many periodic solutions to eq.(19) of the form

\[ \varphi(t, x) = \varphi(x)e^{-i\Omega t}. \]  

However, we will apply the much more simple approach - quantization of static solutions. To apply this approach, it is necessary [32] to consider the Hamiltonian

\[ \hat{H}_\Omega = \hat{H} - \Omega \int dx a^+(x)a^-(x) \]  

instead of the operator (14). Since the spectra of the N-particle parts of these Hamiltonians are related by shifting by \(\Omega N\), one can investigate the Hamiltonian (25) instead of (14).

The classical Hamiltonian function corresponding to the operator (25) has the form (16), where

\[ H_\Omega[\varphi^*, \varphi] = \int dx \varphi^*(x) \left( -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{ax^2}{2} - \Omega \right) \varphi(x) + \frac{g}{4} \left( \int dx \varphi(x)^2 \right)^2. \]  

Let us find the static solutions of the classical equation of motion (18) which takes the form of the stationary harmonic oscillator equation

\[ \left( -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega^2x^2}{2} - \Omega \right) \varphi(x) = 0, \]  

where the frequency \(\omega\) is expressed via the average value of \(x^2\):

\[ \omega^2 = a + g \int dx x^2 |\varphi(x)|^2. \]  

Note that the same equation (27) could be obtained by substitution of periodic solution (24) to the equation (19).

The solution to eq.(27) can be chosen as the \(K\)-th excited state of the harmonic oscillator:

\[ \varphi = \Psi_K = \frac{(A^+)^K}{\sqrt{K!}} \Psi_0, \]

where

\[ \Psi_0 = \text{const}e^{-\frac{\omega^2}{4}} \]

is the ground oscillator state, while

\[ A^+ = \sqrt{\frac{\omega}{2}}x - \frac{1}{\sqrt{2\omega}} \frac{\partial}{\partial x} \]

is the creation operator. The parameter \(\Omega\) is

\[ \Omega = \omega(K + 1/2). \]

The normalizing factor is determined by eq.(20). Since the operator of coordinate can be presented via creation and annihilation operators,

\[ x = \frac{1}{\sqrt{2\omega}}(A^+ + A^-) \]  

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one can find the average value

\[(\Psi_K, x^2 \Psi_K) = \frac{1}{\omega}(K + 1/2),\]

so that the equation on \(\omega\) can be presented as

\[\omega^2 = a + \frac{g}{\omega}(K + 1/2).\]

Let us quantize the constructed static solution. It is necessary to consider the small perturbations around it:

\[\xi(x) \rightarrow \xi(x) + \delta\xi(x,t), \quad \pi(x) \rightarrow \pi(x) + \delta\pi(x,t).\]

Then one should consider such variations that

\[\delta\xi(x,t) = \delta\xi(x)e^{i\beta t}, \quad \delta\pi(x,t) = \delta\pi(x)e^{i\beta t}.\]

This means that variations of classical variables \(\xi\) and \(\pi\) should be complex, although the classical variables are real. It is convenient to make a transformation:

\[\delta\varphi = \frac{\delta\xi + i\delta\pi}{\sqrt{2}}, \quad \delta\varphi^* = \frac{\delta\xi - i\delta\pi}{\sqrt{2}}.\]

The fact of complexity \(\delta\xi\) and \(\delta\pi\) means that the variations of \(\varphi\) and \(\varphi^*\) are not conjugated each other; they are independent. Thus, to investigate the small variations around static solution, one should consider the Hamiltonian system

\[i\frac{\partial\varphi}{\partial t} = \frac{\delta H}{\delta\varphi^*}, \quad -i\frac{\partial\varphi^*}{\partial t} = \frac{\delta H}{\delta\varphi},\]

consider the independent perturbations of \(\varphi\) and \(\varphi^*\):

\[\varphi \rightarrow \varphi + Fe^{i\beta t}, \quad \varphi^* \rightarrow \varphi^* + Ge^{i\beta t},\]

and find a spectrum of \(\beta\):

\[-\beta F = \frac{\delta^2 H}{\delta\varphi^* \delta\varphi} \frac{F}{F} + \frac{\delta^2 H}{\delta\varphi \delta\varphi^*} \frac{G}{G},\]

\[\beta G = \frac{\delta^2 H}{\delta\varphi \delta\varphi} \frac{F}{F} + \frac{\delta^2 H}{\delta\varphi \delta\varphi^*} \frac{G}{G},\]

which should be real if the static solution is stable. The more detailed derivation of the semiclassical spectrum is presented in Appendix B.

The variation system (30) has the following form for the Hamiltonian (26):

\[-\beta F(x) = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2 x^2}{2} - \Omega \right) F(x) + \frac{g}{2} x^2 \varphi^2(x) \int dy y^2 \varphi(y)(F + G)(y),\]

\[\beta G(x) = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2 x^2}{2} - \Omega \right) G(x) + \frac{g}{2} x^2 \varphi^2(x) \int dy y^2 \varphi(y)(F + G)(y),\]

The solutions to eq.(31) are presented as follows. First of all, let us try to substitute instead of \(F\) and \(G\) the \(n\)-th eigenfunction of the oscillator:

\[F_n^+ = 0, G_n^+ = \Psi_n,\]
\[ F^-_n = \Psi_n, G^-_n = 0. \] (33)

Since the matrix element \((\Psi_K, x^2\Psi_n) = 0\) if \(n - K \neq 0, \pm 2\), functions \((32), (33)\) obeys eq.(31) if

\[ \beta^+_n = \omega(n - K), \beta^-_n = \omega(K - n). \]

Let us find other solutions to eq.(31). First of all, there is a “zero-mode” solution corresponding to the invariance of the Hamiltonian system with respect to transformations \(\varphi \rightarrow \varphi e^{i\alpha}, \varphi^* \rightarrow \varphi^* e^{-i\alpha}:

\[ F = -\Psi_K, G = \Psi_K, \beta = 0. \]

Further, making use of eq.(29), one has

\[ x^2 \varphi = \frac{1}{2\omega} \left( \sqrt{(K + 1)(K + 2)}\Psi_{K+2} + (2K + 1)\Psi_K + \sqrt{K(K - 1)}\Psi_{K-2} \right) \]

Thus, we find the following two solutions to system (31):

\[ F = \sqrt{K(K - 1)}\Psi_{K+2}, G = -\sqrt{(K + 1)(K + 2)}\Psi_{K-2}, \beta = -2\omega; \]

\[ G = \sqrt{K(K - 1)}\Psi_{K+2}, F = -\sqrt{(K + 1)(K + 2)}\Psi_{K-2}, \beta = 2\omega \]

because for these cases \((x^2\Psi_K, F + G) = 0.\)

The non-trivial solution to eq.(31) is

\[ F = -\frac{\sqrt{(K + 1)(K + 2)}}{\beta + 2\omega} \Psi_{K+2} - \frac{\sqrt{K(K - 1)}}{\beta - 2\omega} \Psi_{K-2}, \]

\[ G = \frac{\sqrt{(K + 1)(K + 2)}}{\beta - 2\omega} \Psi_{K+2} + \frac{\sqrt{K(K - 1)}}{\beta + 2\omega} \Psi_{K-2}, \]

where

\[ \beta^2 = 4\omega^2 + \frac{g}{\omega} (2K + 1). \]

There is also a solution to the non-stationary variation system which is proportional to \(t:\)

\[ \delta \varphi^* - \delta \varphi = 2i\omega(2K + 1)t\Psi_K, \]

\[ \delta \varphi^* + \delta \varphi = -\sqrt{(K + 1)(K + 2)}\Psi_{K+2} + \sqrt{K(K - 1)}\Psi_{K-2} + (2 + \frac{8\omega^3}{(2K + 1)g})\Psi_K. \]

However, such linear instability of classical static solution is usual for quantum systems with zero-modes. There are several approaches to resolve the difficulty. It is shown in Appendix B that such an instability is not an obstacle for quantization.

The asymptotic energy spectrum for the quantum Hamiltonian \(H_\Omega\) is expressed via frequencies \(\beta:\)

\[ E^n_\Omega = NE_\Omega + \varepsilon_0 + \sum_m \beta_m n_m + O(1/N), \] (34)

where \(E^n_\Omega\) is a quantum energy, \(E_\Omega\) is a corresponding “classical” energy,

\[ E_\Omega = H_\Omega(\varphi^*, \varphi), \]

\(\varepsilon_0\) is a quantum correction, \(\beta_m\) are energy excitations. Non-negative numbers \(n_m\) are parameters specifying the energy levels.
It is important that not all values of \( \beta_m \) found above enters to eq.(34). When we quantize ordinary harmonic oscillator (or free quantum field), there are positive-frequency and negative-frequency solutions to the variation equations (for free fields they coincide with classical equations). However, one usually takes into account only positive frequencies to construct a spectrum.

It happens (see appendix B) that in our case we should take into account not positive values of \( \beta \) but such values of \( \beta \) that obey the following condition:

\[
(G, G) > (F, F).
\]

This means that the excitation energies entering to eq.(34) are:

\[
\beta_m = \omega(m - K), m \neq K, K \pm 2,
\]

\[
\beta_K = -2\omega,
\]

\[
\beta_{K+2} = +\sqrt{4\omega^2 + \frac{g}{\omega}(2K + 1)}.
\]

An asymptotic energy spectrum of the operator \( \hat{H} \) which corresponds to the periodic solution (24) has then the form

\[
E^n = NE_\Omega + \varepsilon_0 + \sum_{m \geq 0, m \neq K} \beta_m n_m + O(1/N),
\]

where classical energy is

\[
\mathcal{E} = H(\varphi^*, \varphi) = \omega(K + 1/2) - \frac{g}{4\omega^2}(K + 1/2)^2.
\]

### 2.5 Correspondence between asymptotic spectra

Let us investigate the correspondence between found asymptotic formulas for the spectrum, eqs.(11) and (35). Let us set in eq.(35) \( K = 0 \). In this case the frequency \( \omega \) obeys the equation

\[
\omega^2 = a + \frac{g}{2\omega}.
\]

Let us find the minimum of the potential (10). The corresponding equation for \( \overline{\xi} \) is

\[
-\frac{1}{4\overline{\xi}^3} + a\overline{\xi} + g\overline{\xi}^3 = 0.
\]

However, eqs.(36) and (37) coincides if one sets:

\[
\overline{\xi}^2 = \frac{1}{2\omega}.
\]

It is easy to see that \( \mathcal{E} = U(\overline{\xi}) \), while \( \beta_2 = \sqrt{U''(\overline{\xi})} \). Therefore, the asymptotic spectrum in the case \( K = 0 \) is

\[
E^n = NU(\overline{\xi}) + \varepsilon_0 + \sqrt{U''(\overline{\xi})}n_2 + \omega n_3 + ...\]

The result can be interpreted as follows. There are many perturbations around classical solution. However, the excitations with energy \( \beta_2 \) do not break the quantum \( O(N) \)-symmetry, while other excitations break this symmetry. This is a reason that they have not been discovered by the collective-field approach which allows us to find \( O(N) \)-symmetric solutions only. The method of second quantization allows us also to find \( O(N) \)-asymmetric solutions.

We can also notice that one can perform a quantization around solutions with \( K \neq 0 \). These solutions do not have form (21), so that the corresponding quantum states are \( O(N) \)-nonsymmetric. One can say that quantization of the \( K = 0 \)-solution is investigation of symmetric and nonsymmetric perturbations around \( O(N) \)-symmetric quantum state while quantization around other solutions correspond to research of the fluctuations around non-symmetric state. Thus, the second-quantized approach allows us to find new energy levels of the anharmonic oscillator.
3 Formalism of third quantization and semiclassical approximation

This section deals with the third-quantized formulation of the quantum-field theory (1). We present the Hamiltonian operator of this theory via the creation and annihilation operators $A^\pm(\varphi(\cdot))$. We are also going to consider the problem of applicability of the semiclassical concept to the third-quantized theory. The “classical” equations for $\Phi[\varphi(\cdot)]$ are to be derived.

3.1 Creation and annihilation operators

In the functional Schrödinger representation states of the $N$-field system are specified by (time-dependent) functionals $\Psi[\varphi_1(\cdot),\ldots,\varphi_N(\cdot)]$, while their evolution is described by the functional Schrödinger equation

$$i \frac{d\Psi_N}{dt} = H_N \Psi_N,$$  \hspace{1cm} (38)

where

$$H_N = \int dx \left( -\frac{1}{2} \frac{\delta^2}{\delta \varphi_a^\dagger(x) \delta \varphi_a(x)} + \frac{1}{2} \nabla \varphi_a(x) \nabla \varphi_a(x) + \frac{m^2}{2} \varphi_a(x) \varphi_a(x) \right) +$$

$$+ \frac{\varepsilon \lambda}{4} \int dx \varphi_a(x) \varphi_a(x) \varphi_b(x) \varphi_b(x),$$

where $\varepsilon$ is a coupling constant in the $N$-field theory, $\varepsilon = 1/N$.

The first step to construct the third-quantized formulation is to extend the Hilbert state space. Instead of the space of functionals $\Psi[\varphi_1(\cdot),\ldots,\varphi_N(\cdot)]$, let us consider the “extended state space” which is a Fock space of sets

$$\begin{pmatrix}
\Psi_0 \\
\Psi_1[\varphi_1(\cdot)] \\
\vdots \\
\Psi_N[\varphi_1(\cdot),\ldots,\varphi_N(\cdot)] \\
\vdots
\end{pmatrix}$$

(39)

where $\Psi_k$ is symmetric with respect to transpositions of $\varphi_i$ and $\varphi_j$. One could conclude that the physical meaning of $\Psi_0, \Psi_1,\ldots,\Psi_N,\ldots$ is the following: $\Psi_0$ is the probability amplitude that there are no fields, $\Psi_1$ is the amplitude that there is only one field etc. However, theories with uncertain set of matter fields are not of widely use, so that one should consider only such extended state vectors (39) that $\Psi_N \neq 0$ and $\Psi_k = 0, k \neq N$. However, considering the extended Schrödinger equation

$$i \frac{d\Psi_k}{dt} = H_k \Psi_k, k = 0, 1, \ldots$$

(40)

on the Fock vector (39) instead of eq. (38) will be shown to be very useful, since system of equations (40) is more suitable for the semiclassical analysis. One can construct asymptotic solution to the system (40) and then extract the $N$-th component of the extended state (39).

Eq.(40) can be presented via the creation and annihilation operators. These operators $A^\pm(\varphi(\cdot))$ transform the extended state (39) to the Fock vector

$$\begin{pmatrix}
(A^\pm[\varphi(\cdot)]\Psi)_0 \\
(A^\pm[\varphi(\cdot)]\Psi)_1(\varphi_1(\cdot)) \\
\vdots \\
(A^\pm[\varphi(\cdot)]\Psi)_N(\varphi_1(\cdot),\ldots,\varphi_N(\cdot)) \\
\vdots
\end{pmatrix},$$

(41)
where
\[
(A^+[\varphi(\cdot)]\Psi)_k[\varphi_1(\cdot), \ldots, \varphi_k(\cdot)] = \frac{1}{\sqrt{k}} \sum_{a=1}^{k} \delta(\varphi(\cdot) - \varphi_a(\cdot)) \times \\
\times \Psi_{k-1}[\varphi_1(\cdot), \ldots, \varphi_{a-1}(\cdot), \varphi_{a+1}(\cdot), \ldots, \varphi_k(\cdot)],
\]
(42)

\[
(A^-[\varphi(\cdot)]\Psi)_{k-1}[\varphi_1(\cdot), \ldots, \varphi_{k-1}(\cdot)] = \sqrt{k}\Psi_k[\varphi(\cdot), \varphi_1(\cdot), \ldots, \varphi_{k-1}(\cdot)].
\]

The functional $\delta$-function entering to eq. (42) can be formally defined via the functional integral over all spatial field configurations
\[
\int D\phi \delta(\varphi(\cdot) - \varphi(\cdot)) F[\varphi(\cdot)] = F[\varphi(\cdot)]
\]
for arbitrary functional $F$. The formal integration measure $D\phi$ enters also to the inner product,
\[
(\Psi, \Psi) = \sum_{k=0}^{\infty} \int D\varphi_1 \ldots D\varphi_k |\Psi[\varphi_1(\cdot), \ldots, \varphi_k(\cdot)]|^2.
\]

The introduced operators (42) obey usual canonical commutation relations
\[
[A^+[\varphi(\cdot)], A^-[\varphi(\cdot)]] = 0, [A^-[\varphi(\cdot)], A^+[\varphi(\cdot)]] = \delta(\varphi(\cdot) - \varphi(\cdot))
\]
(43)

Let us present different operators in the extended state space (39) through the operators (42). It follows from eq. (42) that the operator of “number of fields” which multiplies the $k$-th component of (39) by $k$ can be presented as
\[
\int D\varphi A^+[\varphi(\cdot)]A^-[\varphi(\cdot)].
\]
(44)

Analogously, the operator
\[
\sum_{a=1}^{k} \varphi_a(x)\varphi_a(y)
\]
can be written as
\[
\int D\varphi A^+[\varphi(\cdot)]A^+[\varphi(x)]A^-[\varphi(\cdot)].
\]
(45)

One has also the following relation
\[
\sum_{a=1}^{k} \frac{\delta^2}{\delta \varphi_a(x) \delta \varphi_a(y)} = \int D\varphi A^+[\varphi(\cdot)] \frac{\delta^2}{\delta \varphi(x) \delta \varphi(y)} A^-[\varphi(\cdot)]
\]
(46)

Making use of the obtained relations, we present the quantum Hamiltonian via the creation and annihilation operators
\[
H = \int D\varphi A^+[\varphi(\cdot)] \int dx \left( -\frac{1}{2} \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x)} + \frac{1}{2} \nabla \varphi(x) \nabla \varphi(x) + \frac{m^2}{2} \varphi(x) \varphi(x) \right) A^-[\varphi(\cdot)] + \\
+ \frac{\varepsilon \lambda}{4} \int D\varphi D\phi \int dx \varphi^2(x)\phi^2(x)A^+[\varphi(\cdot)]A^-[\varphi(\cdot)]A^+[\phi(\cdot)]A^-[\phi(\cdot)].
\]
(47)

The system of equations (41) can be presented in a simpler form
\[
i \frac{d\Psi}{dt} = H\Psi,
\]
(48)

where $\Psi$ is an extended vector (39), $H$ is the operator (47).
3.2 Perturbation theory or semiclassical approximation?

Let us consider the case of small coupling constant $\varepsilon$. At first sight, at small $\varepsilon$ the theory (47) can be approximated by the theory of free fields. One can use then the perturbation theory which can be shown to be equivalent to the ordinary, second-quantized, perturbation theory.

However, we know from the quantum field theory that there are no n-perturbative effects even at small values of coupling constant. Namely, the conditions of applicability of such non-perturbative methods as soliton quantization [21], quantization in the background of the strong external field [26] or instanton method [27, 21] is that the coupling constant should be small.

To investigate the problem of applicability of perturbative or non-perturbative approach, consider the Heisenberg equations for the operators

$$A^\pm_t[\varphi(\cdot)] = e^{iHt}A^\pm[\varphi(\cdot)]e^{-iHt},$$

which have the form

$$i\frac{d}{dt}A^\pm_t[\varphi(\cdot)] = \pm[A^\pm_t[\varphi(\cdot)], H]$$

and can be simplified by using the canonical commutation relations (43). The equation for the annihilation operator is

$$i\frac{d}{dt}A^-_t[\varphi(\cdot)] = \int d\mathbf{x}\left(-\frac{1}{2}\delta^2\delta\varphi(\mathbf{x})\delta\varphi(\mathbf{x}) + \frac{1}{2}\nabla\varphi(\mathbf{x})\nabla\varphi(\mathbf{x}) + \frac{m^2}{2}\varphi(\mathbf{x})\varphi(\mathbf{x})\right)A^-_t[\varphi(\cdot)] +$$

$$+\frac{\varepsilon\lambda}{2}\int D\phi\int d\mathbf{x}\varphi^2(\mathbf{x})\phi^2(\mathbf{x})A^+_t[\phi(\cdot)].A^-_t[\varphi(\cdot)]$$

(49)

the conjugated equation is the equation for the creation operator.

The usual procedure to derive the classical equations of motion is averaging of eq.(49). If the second term of the right-hand side of eq.(49) can be neglected, the perturbation theory is applicable. Otherwise, one should develop a non-perturbative approach. The difficulty may arise from the factor

$$\int D\phi A^+_t[\phi(\cdot)].\phi^2(\mathbf{x}).A^-_t[\phi(\cdot)]$$

which is of order $k$ when it is applied to the $k$-field component. Thus, we see that if the number of fields $N$ satisfies the condition

$$N << 1/\varepsilon$$

the perturbation theory is applicable, while if

$$N \sim 1/\varepsilon,$$

another approach is necessary.

3.3 Classical equations

3.3.1 Averaging the Heisenberg equations

One of the approaches to the semiclassical theory is the following. Let us rescale the creation and annihilation operators

$$\Phi^\pm[\varphi(\cdot)] = \sqrt{\varepsilon}A^\pm[\varphi(\cdot)].$$

The Heisenberg equation (49) will take the form

$$i\frac{d}{dt}\Phi^-_t(\varphi(\cdot)) = \int d\mathbf{x}\left(-\frac{1}{2}\delta^2\delta\varphi(\mathbf{x})\delta\varphi(\mathbf{x}) + \frac{1}{2}\nabla\varphi(\mathbf{x})\nabla\varphi(\mathbf{x}) + \frac{m^2}{2}\varphi(\mathbf{x})\varphi(\mathbf{x})\right)\Phi^-_t(\varphi(\cdot)) +$$
$$+ \frac{\lambda}{2} \int D\phi \int d\mathbf{x} \varphi^2(\mathbf{x}) \phi^2(\mathbf{x}) \Phi_t^+(\phi(\cdot)) \Phi_t^-(\phi(\cdot))$$

(50)

which does not contain the small parameter $\varepsilon$.

The coupling constant $\varepsilon$ appears, however, in the canonical commutation relations

$$[\Phi_t^-[\phi(\cdot)], \Phi_t^+[\phi(\cdot)]] = \varepsilon \delta(\phi(\cdot) - \varphi(\cdot)).$$

(51)

The fact that the small parameter arises in the commutator and does not arise in the dynamical equations resembles semiclassical quantum mechanics: the Heisenberg equations are regular in semiclassical limit (they transforms to Hamiltonian system), while the commutator between coordinate and momenta operator tends to zero.

Analogously, one can try to substitute the operators entering to eq.(50) by the $c$-number quantities,

$$\Phi_t^- \rightarrow \Phi_t, \quad \Phi_t^+ \rightarrow \Phi_t^*,$$

and obtain the classical equation

$$i \frac{d}{dt} \Phi_t[\varphi(\cdot)] = \int d\mathbf{x} \left( - \frac{1}{2} \delta \varphi(\mathbf{x}) \delta \varphi(\mathbf{x}) + \frac{1}{2} \nabla \varphi(\mathbf{x}) \nabla \varphi(\mathbf{x}) + \frac{m^2}{2} \varphi(\mathbf{x}) \varphi(\mathbf{x}) \right) \Phi_t[\varphi(\cdot)] +$$

$$+ \frac{\lambda}{2} \int D\phi \int d\mathbf{x} \varphi^2(\mathbf{x}) \phi^2(\mathbf{x}) \Phi_t^+(\phi(\cdot)) \Phi_t(\phi(\cdot)) \Phi_t(\varphi(\cdot))$$

(52)

Such substitution can be explained via the averaging procedure of eq.(50) analogously to the derivation of the Ehrenfest theorem in quantum mechanics. Since the commutator between operators $\Phi_t^\pm$ is small, the uncertainty of them can be made of order $O(\sqrt{\varepsilon})$. This means that for some states average value

$$< A(\Phi_t^+, \Phi_t^-) >$$

(53)

can be approximated up to $O(\sqrt{\varepsilon})$ by the quantity $A(\Phi_t^*, \Phi_t)$,

$$< A(\Phi_t^+, \Phi_t^-) > \rightarrow A(\Phi_t^*, \Phi_t).$$

(54)

This means that classical limit of eq.(50) is eq.(52).

The states obeying eq.(54) can be constructed as follows. Let $Y$ be $\varepsilon$-independent extended state (39). Consider the unitary operator

$$U_\Phi = \exp \left[ \frac{1}{\sqrt{\varepsilon}} \int D\phi (\Phi(\phi(\cdot)) A^+(\varphi(\cdot)) - \Phi^*(\varphi(\cdot)) A^-(\varphi(\cdot))) \right]$$

(55)

obeying the relations

$$U_\Phi^{-1} A^- U_\Phi = A^- + \frac{1}{\sqrt{\varepsilon}} \Phi$$
$$U_\Phi^{-1} A^+ U_\Phi = A^+ + \frac{1}{\sqrt{\varepsilon}} \Phi^*$$

(56)

which are corollaries of the canonical commutation relations (13). Consider the semiclassical state

$$U_\Phi Y.$$
The “semiclassical” state (57) is a Fock vector (39) with all non-zero components. One can try to extract from eq.(57) one of the component, for example, the \(N\)-th component. However, one should be careful because our calculations are not exact but approximate. If the \(N\)-th component of eq.(57) is exponentially small as \(\varepsilon \to 0\), it should be neglected, so that the approximate wave functional will vanish, and the constructed asymptotics will be trivial. For the simplest case (\(Y\) is a vacuum) the probability that there are \(N\) fields is given by the Poisson distribution with the maximum at

\[
N = \frac{1}{\varepsilon} \int D\phi |\Phi(\phi(\cdot))|^2.
\]

(58)

This relation can be derived also from eqs.(44) and (54). Eq.(58) means that semiclassical approximation allows us to investigate the large-\(N\) field theory if \(N \sim 1/\varepsilon\). Note that the case \(N \sim 1\) corresponds to \(\Phi \sim \sqrt{\varepsilon}\), classical equation (52) transforms to the equation for free fields, so that the perturbation theory is applicable.

### 3.3.2 BBGKY-like approach

Eq.(52) can be also derived by the BBGKY (Bogoliubov - Born - Green - Kirkwood - Yvon) approach (28), see also (29) from the \(N\)-field equation (38). Let us briefly discuss this technique.

Consider the \(k\)-field correlation functional

\[
R_k^t(\varphi_1(\cdot), ..., \varphi_k(\cdot); \phi_1(\cdot), ..., \phi_k(\cdot)) = \int D\varphi_{k+1}...D\varphi_N
\]

\[
\times \Psi_N^*[\varphi_1(\cdot), ..., \varphi_N(\cdot)]\Psi_N[\phi_1(\cdot), ..., \phi_k(\cdot), \varphi_{k+1}(\cdot), ..., \varphi_N(\cdot)]
\]

(59)

corresponding to the \(N\)-field wave functional \(\Psi_N\). The correlator (59) can be presented via the creation and annihilation operators

\[
\frac{(N-k)!}{N!} < A^+[\varphi_1(\cdot)]...A^+[\varphi_k(\cdot)]A^-[\phi_1(\cdot)]...A^-[\phi_k(\cdot)] >.
\]

It follows from eq.(19) that functionals (59) obey the BBGKY-like hierarchy of equations.

Suppose that the correlators (59) factorize as \(k = \text{const}, N \to \infty\)

\[
R_k^t \sim \Phi_t^*[\varphi_1(\cdot)]...\Phi_t^*[\varphi_k(\cdot)]\Phi_t[\phi_1(\cdot)]...\Phi_t[\phi_k(\cdot)].
\]

(60)

Substituting eq.(60) to the BBGKY-hierarchy, one can find that for some phase factor \(e^{i\gamma t}\) the functional \(\Phi_t e^{i\gamma t}\) obeys eq.(52).

### 3.3.3 Variational principles

The classical equation of motion (52) can be presented as a Hamiltonian system. Namely, this equation is derivable from the variational principle

\[
\int_0^t dt \left[ \int D\varphi \Phi_t^*[\varphi(\cdot)] \frac{d}{dt} \Phi_t[\varphi(\cdot)] - \int D\varphi \Phi_t^*[\varphi(\cdot)] \int dx \left( -\frac{1}{2} \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x)} + \frac{1}{2} \nabla \varphi(x) \nabla \varphi(x) + \frac{m^2}{2} \varphi(x) \varphi(x) \right) \Phi_t[\varphi(\cdot)] - \frac{\lambda}{4} \int D\varphi D\phi \int dx \varphi^2(x) \phi^2(x) |\Phi_t[\varphi(\cdot)]|^2 |\Phi_t[\phi(\cdot)]|^2 \right] \to \text{extr.}
\]

(61)

This variational principle is Hamiltonian. Namely, one can extract real and imaginary part

\[
\Phi = \frac{Q + iP}{\sqrt{2}}
\]
and notice that the principle \((61)\) is
\[
\int dt (P\dot{Q} - H) \to \text{extr}.
\]
Thus, classical equations of motion correspond to infinite-dimensional flat phase space rather than to curved space considered in ref. [13, 17].

We will see that the (time-dependent) Gaussian ansatz
\[
\Phi[\varphi(\cdot)] = c \exp \left( \frac{i}{2} \int dxdy \varphi(x)A(x,y)\varphi(y) \right) (62)
\]
satisfies eq.\((52)\), so that one can consider the subspace of the phase space which is associated with Gaussian states \((62)\). The corresponding variational principle for classical mechanics can be obtained by substitution eq.\((62)\) to eq.\((61)\). The obtained dynamics coincides with refs. [13, 17].

However, one can consider other solutions to eq.\((52)\), for example, the product of the polynomial in \(\varphi\) by the Gaussian exponent \((62)\). These solutions will not belong to the reduced curved phase space but will belong to the flat space \(\{\Phi[\varphi(\cdot)]\}\).

The variational principle \((61)\) can be interpreted as follows. One can consider the time-dependent variational principle [30, 31]
\[
\int_0^t dt (\psi_t^N, (i\frac{d}{dt} - H_N)\psi_t^N) (63)
\]
for the \(N\)-field equation and substitute the \(N\)-particle time-dependent test functional
\[
\Psi_t^N[\varphi_1(\cdot), ..., \varphi_N(\cdot)] = \Phi_t[\varphi_1(\cdot)]...\Phi_t[\varphi_N(\cdot)] (64)
\]
to eq.\((63)\). One obtains then the classical variational principle \((61)\).

Note that the Gaussian ansatz to the quantum variational principle instead of \((64)\) lead to the classical mechanics in the reducrd curved space.

Although the wave functional \((64)\) will be shown to be not the asymptotic solution to eq.\((38)\), eq.\((52)\) is correct and can be derived also by the substituting the asymptotics for the wave functional to eq.\((38)\). This will be done in section 4.

### 3.3.4 Operator formulation of classical mechanics

Let us discuss relation between eq.\((52)\) and Heisenberg approach [14]. Consider the “Heisenberg” field \(\hat{\varphi}^t(x)\) which can be defined as follows. Let \(U^t\) be a non-linear operator transferring initial condition for eq.\((52)\) to the solution to eq.\((52)\) at time \(t\):
\[
\Phi_t = U_t(\Phi_0).
\]

The Heisenberg operator is
\[
\hat{\varphi}^t(x) = U_t^{-1}\varphi(x)U_t
\]
such that
\[
(\Phi, \hat{\varphi}^t(x)\Phi) = (\Phi^t, \hat{\varphi}(x)\Phi^t). \quad (65)
\]
It follows from eqs.\((52)\) and \((63)\) that the Heisenberg field \(\hat{\varphi}^t(x)\) obeys the equation
\[
\partial_\mu \partial_\mu \hat{\varphi}(x) + m^2 \hat{\varphi}(x) + \lambda < \hat{\varphi}^2(x) > \hat{\varphi}(x) = 0 \quad (66)
\]
which was obtained in [14].

An interesting feature of this second-quantized theory which is classical for the \(N\)-field model is that the field \(\hat{\varphi}(x)\) is a non-linear operator in the one-field state space. We will discuss in the next sections the applications of the semiclassical approach to eq.\((38)\) and clarify the role of eq.\((52)\) in constructing asymptotic solutions to the \(N\)-field equation.
3.4 Regularization and renormalization

In the previous subsection we have derived the classical equation (52) on the functional $\Phi_t[\varphi(\cdot)]$. However, this derivation was formal. We have not taken into account the problem of divergences and renormalization in quantum field theory. Thus, eq.(52) is not well-defined, and it is necessary to investigate the problem of correct definition of the classical equations in details.

3.4.1 Regularization

There are many ways to regularize the quantum field theory. Relativistic-invariant regularizations are usually applied to evaluation of Feynman graphs, while lattice regularization is studied in non-perturbative approaches. Another way to regularize the theory is to substitute the field $\varphi(x)$ by the cutoffed field $\varphi_{\Lambda}(x)$:

$$\varphi_{\Lambda}(x) = \int d y A_{\Lambda}(x - y) \varphi(y),$$

(67)

where $A_{\Lambda}(x - y) \to \delta(x - y)$ as the parameter of the ultraviolet cutoff $\Lambda$ tends to infinity. Eq.(67) allows us to regularize the canonical commutation relations between field and momentum:

$$[\varphi_{\Lambda}(x), \pi(x)] = i A_{\Lambda}(x - y)$$

instead of

$$[\varphi(x), \pi(x)] = i \delta(x - y).$$

To perform infrared regularization, we consider the theory in the box with sizes $L \times L \times ... \times L$ with the periodic boundary conditions.

After regularization eq.(52) is written as

$$i \Phi_t[\varphi(\cdot)] = \int d x \left( \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi_{\Lambda})^2(x) + \frac{u_t(x)}{2} \varphi_{\Lambda}^2(x) \right) \Phi_t[\varphi(\cdot)],$$

(68)

where

$$u_t(x) = m^2 + \lambda(\Phi_t, \varphi_{\Lambda}^2(x) \Phi_t), (\Phi_t, \Phi_t) = 1.$$  

(69)

Let us investigate the system of equations (68),(69) and consider the divergences in this system.

3.4.2 Renormalization in the Schrodinger picture

Consider the problem of renormalizing the obtained equations. For the spatially homogeneous classical field, analogous problem was considered in [31, 33], while some examples for the inhomogeneous field case are presented in [34]. Another approach was considered in refs. [35, 36]. The idea is that one should impose the conditions not only on the counterterms entering to the Hamiltonian but also on the state vector. One cannot consider the state vectors which are regular as the cutoffs tends to infinity. One should specify how the state vector depends on the parameters of the cutoffs; this dependence is singular. Then one should check that these conditions are invariant under time evolutions.

Let us perform the renormalization procedure in such a way for Gaussian and non-Gaussian solutions to eq.(68).

3.4.3 Gaussian and non-Gaussian states

We are going to investigate eq.(68). First of all, suppose that the function $u_t(x)$ is known. We will require this function to be non-singular:

$$\lim_{\Lambda \to \infty} u_t(x) < \infty$$

(70)
because it enters to classical equations of motion for the model (68).

Consider the wave functional being a product of a polynomial by the Gaussian exponent:

$$\sum_n \int dx_1...dx_n f_n(x_1,...,x_n) \varphi(x_1)...\varphi(x_n) \exp \left[ \frac{i}{2} \int dx dy \varphi(x) R(x,y) \varphi(y) \right].$$  \hspace{1cm} (71)

To investigate the divergences in the solution of the Cauchy problem for eq.(68), it is convenient to notice that the operator

$$B^+[P_t,Q_t] = \int dx \left[ \varphi(x) P_t(x) - \frac{1}{i} \frac{\delta}{\delta \varphi(x)} Q_t(x) \right].$$ \hspace{1cm} (72)

commutes with the operator $i \frac{d}{dt} - H$ ($H$ is the operator entering to the right-hand side of eq.(68)) if

$$\dot{Q}_t(x) = P_t(x),$$ \hspace{1cm} (73)

$$\dot{P}_t(x) + \int dy dz A_{\Lambda}(x - y)(-\Delta_y + u_t(y)) A_{\Lambda}(y - z) Q_t(z) = 0.$$ \hspace{1cm} (76)

The initial condition (71) can be presented as a linear combination of the states like:

$$B^+[P_0^1,Q_0^1]...B^+[P_0^m,Q_0^m]\Phi,$$ \hspace{1cm} (74)

where $\Phi$ is a Gaussian state. The solution to eq.(68) which obeys the initial condition (74) is expressed via the solutions $(P_1^t, Q_1^t)$, ..., $(P_m^t, Q_m^t)$ to the system eq. (73),

$$B^+[P_1^t,Q_1^t]...B^+[P_m^t,Q_m^t]\Phi.$$ \hspace{1cm} (75)

Since eqs.(73) have regular limits as $\Lambda \to \infty$, the operators (72) are regular as $\Lambda \to \infty$. This means that singularities may arise in the Gaussian state $\Phi$ only.

Let us consider the Gaussian solution to eq.(68):

$$c_t \exp \left[ \frac{i}{2} \int dx dy \varphi(x) R_t(x,y) \varphi(y) \right].$$ \hspace{1cm} (75)

Substitute formula (75) to the Schrödinger equation (68). By $\hat{R}$ we denote the operator with kernel $R(x,y)$:

$$(\hat{R}f)(x) = \int dy R(x,y) f(y).$$

Analogously, by $\hat{A}$ we denote the operator with kernel $A_{\Lambda}(x,y)$. We find that the Gaussian quadratic form obeys the nonlinear equation:

$$\frac{d}{dt} \hat{R}_t + \hat{R}_t \hat{R}_t + \hat{A}(-\Delta + u_t) \hat{A} = 0,$$ \hspace{1cm} (76)

where $u_t$ is the operator of multiplication by $u_t(x)$. The prefactor $c_t$ satisfies the equation

$$\frac{d}{dt} (\ln c_t) = - \frac{1}{2} Tr \hat{R}_t.$$ \hspace{1cm} (77)
3.4.4 Operational calculus

To investigate singularities in eq. (76), it is convenient [23, 24] to consider the symbols of the operators, i.e. to present each operator via operators $x$ and $-i \frac{\partial}{\partial x}$, for example:

$$\hat{R}_t = R_t(\hat{x}, -i \frac{\partial}{\partial \hat{x}}).$$

(78)

Since the operators $x$ and $-i \frac{\partial}{\partial x}$ do not commute, it is important to specify ordering of the operators. Notation (78) means that momenta operators act first and multiplication operators act next, so that the operator (78) transforms the vector $f_p(x) = const e^{ipx}$ to the vector

$$(\hat{R}_t f_p)(x) = R_t(x, p)f_p(x)$$

(79)

Eq. (79) can be treated as a definition of the symbol of the operator $\hat{R}_t$.

Let us consider the product of the operators $\hat{B} = B(\hat{x}, -i \frac{\partial}{\partial \hat{x}})$ and $\hat{C} = C(\hat{x}, -i \frac{\partial}{\partial \hat{x}})$. It can be also presented via coordinate and momentum operators:

$$\hat{B}\hat{C} = (B * C)(\hat{x}, -i \frac{\partial}{\partial \hat{x}}).$$

It happens that the “product” $B * C$ can be presented as

$$(B * C)(x, k) = B(\hat{x}, k - i \frac{\partial}{\partial \hat{x}})C(x, k).$$

(80)

To prove formula (80), one can consider first the case

$$C(x, k) = e^{ipx}C(k).$$

(81)

In this case

$$\hat{C}f_k(x) = C(k)f_{k+p}(x),$$

$$\hat{B}\hat{C}f_k(x) = B(x, k + p)C(k)f_{k+p}(x) = B(x, k + p)e^{ipx}C(k)f_k(x),$$

so that

$$(B * C)(x, k) = B(x, k + p)C(x, k),$$

$$B(\hat{x}, k - i \frac{\partial}{\partial \hat{x}})C(x, k) = B(x, k + p)C(x, k)$$

Formula (80) is then proved for this partial case. To check eq. (80) for general case, notice that if eq. (80) is valid for operators $\hat{C} = \hat{C}_1$ and $\hat{C} = \hat{C}_2$, it is also valid for $C = \hat{C}_1 + \hat{C}_2$. But any function can be presented as a linear combination of functions (81). Thus, we justify eq. (80) for general case.

Eq. (76) can be considered as an equation for the symbol of the operator $R_t$:

$$\dot{R}_t(x, k) + (R_t * R_t)(x, k) + A_k * (k^2 + u_t(x))A_k = 0.$$  

(82)

We have taken into account that the operator $\hat{A}$ can be presented as $A(-i \frac{\partial}{\partial x})$, where $A(k) = A_k$ is a Fourier transformation of the function $A_\Lambda$. As $\Lambda \to \infty$, $A_k \to 1$. 

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3.4.5 Singularities of the Gaussian quadratic form

To investigate the ultraviolet singularities, it is important to consider the behaviour of the symbol of the operator $R_t$ as $k \to \infty$. It is convenient to expand eq. (80) into an asymptotic series:

\[(B \ast C)(x, k) = \sum_{l \geq 0} \frac{(-i)^l}{l!} \frac{\partial^l B(x, k)}{\partial k_{i_1} \cdots \partial k_{i_l}} \frac{\partial^l C(x, k)}{\partial x_{i_1} \cdots \partial x_{i_l}}.\]  

(83)

This is an expansion in $1/|k|$, because the ratio between the next order of the series and the previous one is $O(1/|k|)$. We see that in the leading order of $1/|k|$

\[(B \ast C)(x, k) \sim B(x, k)C(x, k).\]

Let us consider the behaviour of eq. (82) as $k \to \infty$. Since the third term of the left-hand side of eq. (82) is of order $|k|^2$, one can think that $R_t \sim |k|$ as $|k| \to \infty$. Thus, it is reasonable to extract the most singular term from $R_t$:

\[R_t(x, k) = iA_k \sqrt{k^2 + \mu^2} + r_t(x, k).\]  

(84)

Eq. (82) takes the form

\[\dot{r}_t(x, k) + 2iA_k \sqrt{k^2 + \mu^2} r_t(x, k) + A_k(u_t(x) - \mu^2)A_k = O(|k|^{-1}).\]

We see that the quantity $r_t(x, k)$ should be of order $|k|^{-1}$:

\[r_t(x, k) \sim \frac{iA_k(u_t(x) - \mu^2)}{2\sqrt{k^2 + \mu^2}} + O(|k|^{-2}).\]  

(85)

Analogously, one can find a solution to the equation (83) with the help of the iteration procedure up to $O(|k|^{-\alpha})$ for arbitrary $\alpha$.

Let us impose at the initial time moment the condition

\[R_t(x, k) = R_t^{sing}(x, k) + R_t^{reg}(x, k),\]  

(86)

where $R_t^{sing}$ is the obtained solution to eq. (79), while $R_t^{reg}$ is of order $O(|k|^{-\alpha})$. Then eq. (86) will be invariant under time evolution. One can therefore find all the singularities of the quadratic form entering to the Gaussian exponent.

3.4.6 Renormalization of mass and coupling constant

We are now going to check our main condition (70). It is then necessary to find the singularities of the average value of $\varphi^2_A(x)$. This matrix element can be written via the functional integral

\[< \varphi^2_A(x) > = \int D\varphi P[\varphi(\cdot)]\varphi^2_A(x) \exp[-(\varphi, \Im R \varphi)],\]  

(87)

where $P$ is a polynomial functional of $\varphi(\cdot)$ with smooth coefficient functions. Calculation of this integral is standard: it is presented as

\[P \int d\varphi \varphi^2_A(x-y)A_A(x-z) \frac{\delta^2}{\delta j(y) \delta j(z)} \int D\varphi e^{(j, \Im R \varphi)},\]

Since the remaining integral is easily evaluated, the latter expression can be simplified:

\[P \int d\varphi \varphi^2_A(x-y)A_A(x-z) \frac{\delta^2}{\delta j(y) \delta j(z)} e^{i(\varphi, \Im R \varphi)},\]  

(88)
The only singularity in this average value comes from the operator $\frac{\partial^2}{\partial \delta^2}$. This means that the matrix element consists of the regular and singular parts:

$$<\varphi_2^2(x)> = \frac{1}{2} <x|\hat{A}(Im\hat{R})^{-1}\hat{A}|x> + <\varphi_2^2(x)>_{reg}.$$ 

Since the symbol of the operator $R_t$ is

$$R_t(x, k) = iA_k \sqrt{k^2 + \mu^2} \left(1 + \frac{u_t(x) - \mu^2}{2(k^2 + \mu^2)} + O(|k|^{-3})\right),$$

the leading orders for the symbol of the operator $A(Im\hat{R})^{-1}\hat{A}$ have the following form:

$$(\hat{A}(Im\hat{R})^{-1}\hat{A})(x, k) = \frac{A_k}{\sqrt{k^2 + \mu^2}}(1 - \frac{u_t(x) - \mu^2}{2(k^2 + \mu^2)} + O(|k|^{-3})).$$

Since the matrix element of arbitrary operator $\hat{B}$ is expressed via its symbol as

$$<x|\hat{B}|x> = \sum_p |<x|p>|^2 B(x, p) = \frac{1}{L^d} \sum_p B(x, p),$$

the average value (87) is presented up to a finite quantity $<\varphi^2(x)>_{reg}$ in the following form:

$$<\varphi_2^2(x)> = \frac{1}{2L^d} \sum_k \frac{A_k}{\sqrt{k^2 + \mu^2}} - \frac{u_t(x) - \mu^2}{4L^d} \sum_k \frac{A_k}{(k^2 + \mu^2)^{3/2}} + <\varphi^2(x)>_{reg}. \quad (89)$$

Substituting eq.(89) to eq.(88), one obtains the following condition:

$$m^2 + \lambda \left[ \frac{1}{2L^d} \sum_k \frac{A_k}{\sqrt{k^2 + \mu^2}} - \frac{u_t(x) - \mu^2}{4L^d} \sum_k \frac{A_k}{(k^2 + \mu^2)^{3/2}} \right] + \lambda <\varphi^2(x)>_{reg} = u_t(x). \quad (90)$$

Let us investigate eq.(90). First of all, set

$$\mu^2 = m^2 + \frac{\lambda}{2L^d} \sum_k \frac{A_k}{\sqrt{k^2 + \mu^2}}. \quad (91)$$

Since the sum in the right-hand side of eq.(91) diverges as $\Lambda \to \infty$, one should choose the quantity $m^2$ (square of the bare mass) to be infinite to make $\mu^2$ finite. We will see in the following sections that the quantity $\mu$ is a physical mass of elementary particles, so that it must be finite.

Eq.(90) can be presented then as

$$\frac{u_t(x) - \mu^2}{\lambda} = <\varphi^2(x)>_{reg} - \frac{u_t(x) - \mu^2}{L^d} \sum_k \frac{A_k}{4(k^2 + \mu^2)^{3/2}}.$$

The sum over $k$ entering to this equation is infinite for the (3+1)-dimensional case. One should then choose the bare coupling constant $\lambda$ in such a way that

$$\frac{1}{\lambda} + \frac{1}{L^d} \sum_k \frac{A_k}{4(k^2 + \mu^2)^{3/2}} = \frac{1}{\lambda_R} < \infty. \quad (92)$$

The quantity $\lambda_R$ is a renormalized coupling constant.
3.5 Complex-WKB approach for the third-quantized systems

In subsection 3.3 we have found the classical equations with the help of conjecture (54) on the average values of the Heisenberg operators. Let us now check this conjecture and develop a systematic semiclassical theory. We will substitute an ansatz for the third-quantized state to the equation of motion. We show that the constructed state approximately satisfies the equation.

3.5.1 The complex-WKB ansatz

Let us consider the following vector of the third-quantized Fock space:

$$\Psi = \exp \left( \frac{i}{\varepsilon} S^t \right) U_{\Phi^t} Y^t, \quad (93)$$

where $S^t$ is a real number, $U_{\Phi^t}$ is the unitary operator (55), $Y^t$ is an $\varepsilon$-independent state. It happens that the third-quantized state (93) being a vector (39) approximately obeys the equation (48):

$$i\frac{d\Psi}{dt} = \frac{1}{\varepsilon} H(\sqrt{\varepsilon}A^+, \sqrt{\varepsilon}A^-) \Psi. \quad (94)$$

Namely, one can use the commutation relations (56), as well as the formula

$$U_{\Phi^t}^{-1} \frac{d}{dt} U_{\Phi^t} =$$

$$= \frac{1}{\sqrt{\varepsilon}} \int D\varphi (\Phi^t[\varphi(\cdot)]A^+[\varphi(\cdot)] - \Phi^t[\varphi(\cdot)]A^-[\varphi(\cdot)]) + \frac{1}{2\varepsilon} \int D\varphi (\Phi^t[\varphi(\cdot)]\Phi^t[\varphi(\cdot)] - \Phi^t[\varphi(\cdot)]\Phi^t[\varphi(\cdot)]).$$

Analogously to appendix B, one can make equal first the terms of order $O(1/\varepsilon)$ in eq.(94):

$$\frac{dS^t}{dt} = -\frac{i}{2\varepsilon} [(\dot{\Phi}^t, \Phi^t) - (\dot{\Phi}^t, \dot{\Phi}^t)] - H(\Phi^*, \Phi). \quad (95)$$

Then one can consider the terms of order $O(\varepsilon^{-1/2})$ and obtain the classical equation of motion (52). Finally, the remaining non-vanishing as $\varepsilon \to 0$ terms of order $O(1)$ give us the following equation on $Y^t$:

$$i\frac{dY^t}{dt} = H_2 Y^t, \quad (96)$$

where

$$H_2 = \int D\varphi A^+[\varphi(\cdot)] \int dx \left( -\frac{\delta^2}{2 \delta \varphi(x) \delta \varphi(x)} + \frac{1}{2} (\nabla \varphi)^2(x) + \frac{1}{2} \varphi^2(x)(m^2 + (\Phi^t, \varphi^2(x)\Phi^t)) \right) A^-[\varphi(\cdot)] +$$

$$+ \frac{\lambda}{4} \int dx \left[ \int D\varphi \varphi^2(\Phi^t[\varphi(\cdot)]A^-[\varphi(\cdot)] + \Phi[\varphi(\cdot)]A^+[\varphi(\cdot)]) \right]^2 \quad (97)$$

An interesting feature of eq.(97) is that it contains terms with two creation operators. This means that vacuum vector is not a solution of eq.(97). Therefore, the product (64) is not an asymptotic solution to the $N$-field equation: one cannot suppose the fields to be independent, it is necessary to take into account the correlatins between fields even in the leading order of $1/N$-approximation.
3.5.2 Fixing number of fields

The constructed asymptotics (93) is a vector of the third-quantized Fock space. All its components are non-zero. However, our purpose is to construct the asymptotic solutions to the $N$-field equation. This means that we should consider the $N$-th component of the vector (93). Consider the operator $P_N$ of projecting on the $N$-field subspace:

$$P_N(\Psi_0, \Psi_1, ..., \Psi_N, ...) = (0, 0, ..., 0, \Psi_N, 0, ...).$$

According to subsection 3.2, the vector

$$P_N U_\Phi Y$$

is not exponentially small if

$$\int D\varphi |\Phi[\varphi(\cdot)]|^2 = \varepsilon N.$$ \hfill (99)

The fact that we are interested in the $N$-th component of the vector (93) only means that expression (93) contains more information than necessary. This implies that two different vectors $Y_1$ and $Y_2$ may lead to equal $N$-field states $P_N U_\Phi Y_1$ and $P_N U_\Phi Y_2$.

Namely, let us use the identity:

$$P_N \left( \varepsilon \int D\varphi A^+ A^- - \varepsilon N \right) U_\Phi X = 0.$$ \hfill (100)

Applying the commutation relations (53), one obtains that

$$P_N U_\Phi \left( \int D\varphi |\Phi[\varphi(\cdot)]|^2 - \varepsilon N + \sqrt{\varepsilon} (a^+ + a^-) + \varepsilon \int D\varphi A^+ A^- \right) X = 0,$$

where

$$a^+ = \int D\varphi A^+ \Phi[\varphi(\cdot)],$$

$$a^- = \int D\varphi A^- \Phi^* [\varphi(\cdot)].$$

Taking into account eq.(100), one obtains that for

$$Y = (a^+ + a^-) X$$

the $N$-field state (93) is small as $\varepsilon \to 0$. This means that there is an invariance of the vector (93) under transformations

$$Y \to Y + (a^+ + a^-) X.$$

One can perform the “gauge-fixing” procedure by imposing on $Y$ the additional condition, for example,

$$a^- \tilde{Y} = 0.$$ \hfill (101)

Another approach is to consider the gauge-invariant generalized state vector

$$Z = \delta(a^+ + a^-) Y$$ \hfill (102)

instead of the Fock space vector $Y$. Eq.(102) is associated with the vector $\tilde{Y}$ by the relation

$$Z = e^{-\frac{1}{2} a^+ a^-} \tilde{Y}.$$ \hfill (103)

Since the operator $a^+ + a^-$ commutes with the operator $i \frac{d}{dt} - H_2$, we find that the generalized vector $Z^t$ obeys the evolution equation:

$$i \frac{dZ^t}{dt} = H_2 Z^t.$$ \hfill (104)
On the other hand, the vector \( \tilde{Y}^t \) does not obey eq.(104) since the operator \( a^- \) does not commute with \( i \frac{d}{dt} - H_2 \). However, the most suitable form of the \( N \)-particle wave functional uses the vector \( \tilde{Y} \). Since the vector \( U_n \tilde{Y} \) can be expressed via components of \( \tilde{Y} \) as follows,

\[
\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \tilde{Y}_n[\varphi_1(\cdot), \ldots, \varphi_n(\cdot)] A^+ \varphi_1(\cdot) \ldots A^+ \varphi_n(\cdot) D\varphi_1 \ldots D\varphi_n \sum_{K=0}^{\infty} \frac{e^{-\frac{N}{2}K}(a^+)^K}{K!}|0>,
\]

the expression for the \( N \)-th component of the asymptotics is:

\[
\Psi_N^t[\varphi_1(\cdot), \ldots, \varphi_n(\cdot)] = e^{iNS^t} \sum_{n=0}^{N} \frac{1}{\sqrt{N^n n!}} \sum_{1 \leq i_1 \neq \ldots \neq i_n \leq N} \tilde{Y}_n[\varphi_{i_1}(\cdot), \ldots, \varphi_{i_n}(\cdot)] \prod_{i \neq i_1 \ldots i_n} \Phi^t[\varphi_i(\cdot)]. \tag{105}
\]

Condition (101) means that the functional \( \tilde{Y}_n^t \) is orthogonal to \( \Phi^t \):

\[
\int D\varphi \Phi^t[\varphi_1(\cdot)] \tilde{Y}_n[\varphi_{i_1}(\cdot), \ldots, \varphi_{i_n}(\cdot)] = 0.
\]

Thus, eq.(103) has the following meaning. The \( n = 0 \)-term corresponds to probability amplitude that all the fields are in the state \( \Phi \), the \( n \)-th term specifies that amplitude that \( N - n \) fields has the wave functional \( \Phi \) etc.

### 3.5.3 Renormalization of the cosmological constant

We have seen in the previous subsection that the phase factor in the solution to the classical equation \( \Phi^t \) diverges. However, the quantity \( S^t \) also diverges. Let us show that these divergences cancel in a vector \( e^{iS^t} \Phi^t \).

First of all, it is necessary to modify the classical equation of motion, eq.(68), by adding a cosmological constant \( \mathcal{E} \) to the right-hand side:

\[
i\dot{\Phi}_t[\varphi(\cdot)] = \int dx \left( \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi_\lambda)^2(x) + \frac{u_t(x)}{2} \varphi_\lambda^2(x) + \mathcal{E} \right) \Phi_t[\varphi(\cdot)]. \tag{106}
\]

This corresponds to usual renormalization of the vacuum energy in the quantum field theory. Modification (106) means that the \( N \)-field Hamiltonian is modified by \( N\mathcal{E} \). Eq.(74) for the singular pre-exponential factor becomes the following:

\[
\frac{d}{dt}(\ln c_t) = -\frac{1}{2} Tr \hat{R}_t - i \int dx \mathcal{E}. \tag{107}
\]

It follows from eqs.(95) and (107) that the quantity

\[
a_t = c_t e^{iS_t}
\]

obeys the equation:

\[
\frac{d}{dt}(\ln a_t) = -\frac{1}{2} Tr \hat{R}_t - i \int dx \mathcal{E} + \frac{i \lambda}{4} \int dx < \varphi_\lambda(x)^2 > . \tag{108}
\]

Let us find the singularities of \( Tr \hat{R}_t \). It is necessary to expand the symbol of \( \hat{R}_t \) up to \( O(1/|\omega_k|^4) \), where \( \omega_k = \sqrt{k^2 + \mu^2} \),

\[
R_t(x, k) = i A_k \omega_k + r_1 + r_2 + r_3 + O(1/|\omega_k|^4), \tag{109}
\]

where \( r_m = O(1/|\omega_k|^m) \).

Substitute expansion (109) to eq.(76). Making use of eq.(83), we obtain the following relations. The terms of orders \( O(1), O(1/|\omega_k|) \) and \( O(1/|\omega_k|^2) \) are:

\[
2i A_k \omega_k r_1 + A_k \ast (u - \mu^2) A_k = 0,
\]

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\[ \dot{r}_1 + 2iA_k \omega_k r_2 - i \frac{\partial}{\partial k}(iA_k \omega_k) \frac{\partial r_1}{\partial x} = 0, \]
\[ \dot{r}_2 + 2iA_k \omega_k r_3 + r_2^2 - i \frac{\partial}{\partial k}(iA_k \omega_k) \frac{\partial r_2}{\partial x} - \frac{1}{2} \frac{\partial^2}{\partial k_m \partial k_n}(iA_k \omega_k) \frac{\partial^2 r_1}{\partial x_m \partial x_n} = 0. \]

Since trace of the operator is expressed via integral of its symbol over \( x \), while integral of the derivative vanish, one finds:

\[ Trr_2 = -\frac{d}{dt} Tr r_1. \]
\[ Trr_3 = \frac{d^2}{dt^2} Tr r_1 - Tr r_1 = 0. \]

We see that the singular part of \( Tr \hat{R}_t \) is presented as:

\[ (Tr \hat{R}_t)_{sing} = \int dx \frac{1}{L^d} \sum_k A_k \left( i\omega_k + \frac{i}{2} \frac{u_t(x) - \mu^2}{\omega_k} - \frac{i}{8} \frac{(u_t(x) - \mu^2)^2}{\omega_k^3} \right) + \frac{dF}{dt} \]

for some singular function \( F \) depending on \( u_t(x) \) and its derivatives. One can then extract singularities from the factor \( a_t \):

\[ a_t = e^{\int b_t}, \]

where \( b_t \) is regular. The fact that the prefactor in the Schrodinger representation is singular is usual for quantum field theory [35]. However, this singularity vanishes if one changes the representation [35].

Thus, we see that the cosmological constant should be chosen in order to make the quantity

\[ -\frac{1}{2L^d} \sum_k A_k \left( i\omega_k + \frac{i}{2} \frac{u_t(x) - \mu^2}{\omega_k} - \frac{i}{8} \frac{(u_t(x) - \mu^2)^2}{\omega_k^3} \right) + \frac{\lambda}{4} < \varphi^2_\lambda(x) > - \mathcal{E} = -\tilde{\mathcal{E}} \]

finite. Making use of the relation (69)

\[ < \varphi^2_\lambda(x) > = \frac{u_t(x) - \mu^2}{\lambda} + \frac{\mu^2 - m^2}{\lambda} \]

and eqs.(91), (92), we obtain that the quantity \( -\tilde{\mathcal{E}} \) consists of constant and regular parts:

\[ -\tilde{\mathcal{E}} = -\mathcal{E} + \frac{1}{4\lambda R}(u - \mu^2)^2 + \frac{1}{4\lambda}(\mu^2 - m^2)^2 - \frac{1}{2L^d} \sum_k A_k \omega_k. \]

If one sets the bare cosmological constant to be

\[ \mathcal{E} = \frac{1}{4\lambda}(\mu^2 - m^2)^2 = -\frac{1}{2L^d} \sum_k A_k \omega_k, \]

the divergences in the vacuum energy will be canceled.

### 3.6 Non-symmetric solutions to the large-N theory

The formalism of second quantization works in quantum mechanics only if we require the wave function to be symmetric or antisymmetric. Analogously, the bosonic third quantization procedure is applicable to the \( N \)-field theory only if we require the wave functional \( \Psi_N[\varphi_1(\cdot), ..., \varphi_N(\cdot)] \) to be symmetric with respect to transpositions of the fields. This means that third quantization procedure allows us to construct only symmetric approximate solutions to the \( N \)-field Schrodinger equation. One can notice that the functional (103) is symmetric.
However, one can be interested in more general solutions to the large-$N$ Schrodinger equation. To construct such solutions, one can consider the wave functional

$$\Psi[\chi_1(\cdot), \ldots, \chi_k(\cdot), \varphi_1(\cdot), \ldots, \varphi_{N-k}(\cdot)]$$

to be symmetric with respect to transpositions of the last $N-k$ arguments $\varphi_1, \ldots, \varphi_{N-k}$ only.

Let us perform now the procedure of third quantization only for arguments $\varphi$. This means that we consider the extended Fock space of sets

$$\left( \begin{array}{c}
\Psi_0[\chi_1(\cdot), \ldots, \chi_k(\cdot)] \\
\Psi_1[\chi_1(\cdot), \ldots, \chi_k(\cdot), \varphi_1(\cdot)] \\
\cdots \\
\Psi_{N-k}[\chi_1(\cdot), \ldots, \chi_k(\cdot), \varphi_1(\cdot), \ldots, \varphi_{N-k}(\cdot)] \\
\cdots
\end{array} \right)$$

(110)

The $(N-k)$-field Hamiltonian can be presented as a sum of Hamiltonians of the fields $\varphi, \chi$ and interaction term:

$$H_{N-k} = H^\varphi_{N-k} + H^{\varphi\chi} + H^\chi_{N-k},$$

where

$$H^\varphi_{N-k} = \sum_{a=1}^{N-k} \int dx \left( -\frac{1}{2} \frac{\partial^2}{\partial \varphi^a(x) \partial \varphi^a(x)} + \frac{1}{2} \nabla \varphi_a(x) \nabla \varphi_a(x) + \frac{m^2}{2} \varphi_a(x) \varphi_a(x) \right) +$$

$$+ \frac{\lambda \varepsilon}{4} \int dx \sum_{a,b=1}^{N-k} \varphi_a(x) \varphi_a(x) \varphi_b(x) \varphi_b(x)$$

is the $\varphi$-Hamiltonian,

$$H^\chi_{N-k} = \sum_{a=1}^k \int dx \left( -\frac{1}{2} \frac{\partial^2}{\partial \chi^a(x) \partial \chi^a(x)} + \frac{1}{2} \nabla \chi_a(x) \nabla \chi_a(x) + \frac{m^2}{2} \chi_a(x) \chi_a(x) \right) +$$

$$+ \frac{\lambda \varepsilon}{4} \int dx \sum_{a,b=1}^k \chi_a(x) \chi_a(x) \chi_b(x) \chi_b(x)$$

is the Hamiltonian of the $\chi$-field, while the interaction term is

$$H^{\varphi\chi} = \frac{\lambda \varepsilon}{4} \int dx \sum_{a=1}^k \chi_a(x) \chi_a(x) \sum_{a=1}^{N-k} \varphi_a(x) \varphi_a(x).$$

For the simplicity, the cutoffs are omitted. The third-quantized version of the Hamiltonian $H^\varphi_{N-k}$ was written in subsection 3.1, eq.(17). The Hamiltonian $H^\chi_{N-k}$ is not to be third-quantized. The interaction Hamiltonian is presented as

$$H^{\varphi\chi} = \frac{\lambda \varepsilon}{4} \int dx \sum_{a=1}^k \chi_a(x) \chi_a(x) \int D\varphi A^+[\varphi(\cdot)] \varphi_a(x) \varphi_a(x) A^- [\varphi(\cdot)].$$

One can construct the asymptotics as $N \to \infty$, $k = \text{const}$ for the obtained third-quantized equation by the complex-WKB technique considered in the previous subsection. The ansatz (13) is substituted to the Schrodinger equation, where $Y^t$ is a vector (11). Equation for $S^t$ and classical equation for $\Phi^t$ coincide with obtained in the previous subsection. Equation for $Y^t$ has the form:

$$i \frac{dY^t}{dt} = (H_2 + H_\lambda)Y^t,$$

(111)
where the operator $H_2$ is given by eq. (111), while the Hamiltonian $H_χ$ is

$$H_χ = \sum_{α=1}^{k} \int dx \left( -\frac{1}{2} \frac{δ^2}{δχ^α(x) δχ^α(x)} + \frac{1}{2} δχ(x) δχ(x) + \frac{m^2 + λ(ϕ^t, ϕ^2(x))}{2} χ(x) χ(x) \right). \quad (112)$$

We see that $χ$- and Fock degrees of freedom are split in eq. (111). Each of fields $χ$ is a free field interacting with the external potential $u_t(x)$ specified by the classical solution $φ^t$.

To fix number of fields, one can project the constructed asymptotics on the $(N-k)$-field subspace. All further derivations are analogous to the previous subsubsection.

### 3.7 Generalization to other models

We have considered an example of the large-$N$ theory, the $Φ^4$-theory. Let us briefly discuss other large-$N$ models and applications of the third-quantized approach.

#### 3.7.1 The $ϕϕχ$-model

Consider the quantum field theory of $N$ scalar fields $ϕ^1,...,ϕ^N$ and one field $χ$ with the Lagrangian:

$$L = \frac{1}{2} \partial_μϕ^a∂_μϕ^a - \frac{m^2}{2} ϕ^aϕ^a + \frac{1}{2} δχ δχ - \frac{M^2}{2} χχ - \frac{g}{√N} ϕ^aϕ^aχ.$$ Let us show that semiclassical third-quantized approach is applicable to this theory and find classical variables and classical equations for this model.

The Hamiltonian of this model can be expressed via creation and annihilation operators $A^±[ϕ(·)]$ in the third-quantized form:

$$H = \int DφA^+[ϕ(·)] dφ \left[ -\frac{1}{2} \frac{δ^2}{δφ(x) δφ(x)} + \frac{1}{2} (δϕ)^2(x) + \frac{m^2}{2} ϕ^2(x) + g√εϕ^2(x)χ(x) \right] A^−[ϕ(·)] +$$

$$+ \int dφ \left[ -\frac{1}{2} \frac{δ^2}{δχ(x) δχ(x)} + \frac{1}{2} (δχ)^2(x) + \frac{M^2}{2} χ^2(x) \right].$$

When one performs the rescaling of “quantum” variables,

$$A^±√ε = φ^±, √εχ = Y,$$

one finds that the Hamiltonian is proportional to $1/ε$, the commutator between operators $Φ^±[ϕ(·)]$ is small, and the coefficient of each differentiation operator is $ε$. This means that semiclassical methods can be applied to this model, while classical variables are the following: complex functional $Φ[ϕ(·)]$, classical field $Y(x)$ and canonically conjugated momentum $P(x)$.

The classical equations can be obtained by substituting the complex-WKB ansatz

$$e^{iS^t} U_{φ^t} \exp \left[ \frac{i}{√ε} \int dφ [P(x)χ(x) - Y(x)\frac{1}{i} \frac{δ}{δχ(x)}] \right] Y^t \quad (113)$$

to the Schrodinger equation. One will obtain the following classical dynamics:

$$i\frac{d}{dt} Φ_t[φ(·)] = \int dφ \left[ -\frac{1}{2} \frac{δ^2}{δφ(x) δφ(x)} + \frac{1}{2} (δϕ)^2(x) + \frac{m^2}{2} ϕ^2(x) + gφ^2(x)Y(x) \right] Φ_t[φ(·)],$$

$$Y = P, \quad -\dot{P} = -ΔY + M^2Y + g(Φ, φ(x)Φ).$$
3.7.2 Spontaneous symmetry breaking case

Let us consider the spontaneous symmetry broken large-N field theory,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a - \frac{\lambda}{4N} (\varphi^a \varphi^a - Nv^2)^2. \quad (114)$$

Since the Hamiltonian can be presented in a third-quantized form,

$$H = \int D\varphi A^+[\varphi(\cdot)] \int dx \left[ -\frac{1}{2} \delta^2 \left( \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x)} \right) + \frac{1}{2} (\nabla \varphi)^2(x) \right] A^-[\varphi(\cdot)] +$$

$$+ \frac{\varepsilon \lambda}{4} \int dx \left( \int D\varphi A^+[\varphi(\cdot)] \varphi^2(x) A^-[\varphi(\cdot)] - \frac{\rho^2}{\varepsilon} \right)^2,$$

our semiclassical approach is applicable to this case. However, the ansatz (93) seems to correspond to the spontaneous unbroken phase of the theory because fields $\varphi^1, ..., \varphi^N$ are not distinguished in the asymptotic formula.

To specify solutions corresponding to the broken phase of the theory, it is reasonable to denote $\varphi^N = \chi$ and consider the model

$$\mathcal{L} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a - \frac{\lambda}{4N} (\chi^2 + \varphi^a \varphi^a - Nv^2)^2. \quad (115)$$

One can then analyse this model analogously to the previous subsubsection; the classical variables will be the same: the complex functional $\Phi[\varphi(\cdot)]$, the field $Y$ and momentum $P$.

3.7.3 $O(N)$ - nonsymmetric theory

The method of semiclassical third quantization can be applied to the theories of a much more wide types than traditional large-N theories. For example, consider the $O(N)$-nonsymmetric $\Phi^4$-model

$$\mathcal{L} = \sum_{a=1}^{N} \left[ \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a - \frac{m^2}{2} \varphi^a \varphi^a - \frac{g}{4} (\varphi^a)^4 \right] - \frac{\lambda}{4N} (\sum_{a=1}^{N} \varphi^a \varphi^a)^2. \quad (116)$$

Since the third-quantized form of the hamiltonian depends on the small parameter according to eq.(94), one can apply the complex-WKB approach and obtain the following classical equation:

$$i \dot{\Phi}_t[\varphi(\cdot)] = \int dx \left( \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi)^2(x) + \frac{u_t(x)}{2} \varphi^2(x) + g \varphi^4(x) \right) \Phi_t[\varphi(\cdot)],$$

where

$$u_t(x) = m^2 + \lambda (\Phi_t, \varphi^2(x) \Phi_t), \quad (\Phi_t, \Phi_t) = 1.$$

Thus, we see that the classical theory for the large-N system (116) in the external self-consistent field. Although this model is not exactly solvable, one can investigate qualitative features of it and obtain an information about the model (116).

4 Energy spectrum of the large-N theory

In this section we find stationary asymptotic solutions of the large-N Schrodinger equation.
4.1 Approximate stationary solutions

In the previous section we have constructed the asymptotic solution (105) to the $N$-field equation. Let us find in what cases eq.(105) gives us a stationary state:

$$
Psi_N^t = e^{-iEt}Psi_N^0. \tag{117}
$$

First of all, notice that the solution $Phi^t$ to the classical equation should depend on $t$ as $e^{-i\Omega t}$:

$$
Phi^t = e^{-i\Omega t}Phi^0. \tag{118}
$$

Then, the $n$-th component of the third-quantized state $\tilde{Y}$ is also a periodic function:

$$
\tilde{Y}_n^t = e^{-i(n\Omega + \omega)t}\tilde{Y}_n^0. \tag{119}
$$

If the conditions (118) and (119) are satisfied, eq.(105) gives us a stationary $N$-field state, because

$$
dS/dt = i(\Phi^t, \dot{\Phi}^t) - H(\Phi^t, \Phi^t) = \Omega - H(\Phi^0, \Phi^0),
$$

so that the quantum energy is expressed as:

$$
E = NH(\Phi^*, \Phi) + \omega + O(1/N). \tag{120}
$$

Let us investigate the obtained conditions, (118) and (119). Condition (118) means that one should consider the stationary analog of eq.(118):

$$
\int dx \left[ \frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla \varphi \Lambda)^2(x) + \frac{u(x)}{2} \varphi^2 \Lambda(x) \right] \Phi = \Omega \Phi, \tag{121}
$$

$$
u(x) = m^2 + \lambda(\Phi, \varphi^2 \Lambda(x) \Phi).
$$

To analyze eq.(113), notice that it can be presented as:

$$
Z_n^t = e^{-i(n\Omega + \omega)t}Z_n^0, \tag{122}
$$

where the generalized third-quantized vector $Z$ obeying eq.(104) is related with $\tilde{Y}$ by eq.(103). Eq.(122) means that

$$(H_2Z)_n = (n\Omega + \omega)Z_n.
$$

Making use of the form of the operator of number of fields, one obtains:

$$
(H_2 - \Omega \int D\varphi A^+[\varphi(\cdot)]A^-[\varphi(\cdot)] Z = \omega Z. \tag{123}
$$

Eq.(122) implies the following relation:

$$
\int D\varphi \left( A^+[\varphi(\cdot)]\Phi[\varphi(\cdot)] + A^-[\varphi(\cdot)]\Phi^*[\varphi(\cdot)] \right) Z = 0. \tag{124}
$$

Thus, the problem of finding the large-$N$ spectrum is reduced to the problem of finding a solution to classical equation (111) and oscillator system (123), (124) consisting of the oscillator Schrodinger equation and anonstraint relation (124).
4.2 Ground state energy in the large-N theory

Let us consider the following solution to eq. (121). Suppose \( u(x) \) to be a constant:

\[ u(x) = \mu^2. \]

Choose the classical solution \( \Phi \) to be a vacuum state of the one-field system with mass \( \mu \). Thus, our main equation (121) takes the form:

\[ \mu^2 = m^2 + \lambda < 0|\varphi^2(x)|0>. \]

The field \( \varphi(x) \) can be expressed via creation and annihilation operators,

\[ \varphi(x) = \frac{1}{L^{d/2}} \sum_k \frac{\sqrt{A_k}}{\sqrt{2\omega_k}} [a_k^+ e^{-ikx} + a_k e^{ikx}], \]

(125)

where the factor \( \sqrt{A_k} \) arises because of regularization considered in eq. (121), while \( \omega_k = \sqrt{k^2 + \mu^2} \).

Therefore, one can evaluate the average value of \( \varphi^2(x) \):

\[ <0|\varphi^2(x)|0> = \frac{1}{L^d} \sum_k \frac{A_k}{2\omega_k}. \]

(126)

Thus, we obtain the relation (111) on the parameter \( \mu \) playing a role of mass of elementary excitations. Namely, non-symmetric excitations obey Schrodinger equation with the Hamiltonian (112). However, this Hamiltonian corresponds to the free field of the mass \( \mu \).

Let us find the leading order of the energy, \( NH(\Phi^*, \Phi) \) expressed via average values:

\[ H(\Phi^*, \Phi) = \int dx \left[ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi)^2(x) + \frac{m^2}{2} \varphi^2(x) \right] + \frac{\lambda}{4} <0|\varphi^2(x)|0 >^2 \]

(127)

making use of eqs. (121) and (126), one finds:

\[ E \sim N \left( \Omega - \frac{\lambda}{4L^d} \left( \sum_k \frac{A_k}{2\omega_k} \right)^2 \right) + O(1) \]

(128)

Since the vacuum energy is \( \frac{1}{2} \sum_k A_k \omega_k \), eqs. (128) for the ground-state energy density has the form:

\[ \frac{E}{NL^d} \rightarrow_{N \rightarrow \infty} \frac{1}{2L^d} \sum_k A_k \omega_k - \lambda \left( \frac{1}{4L^d} \sum_k \frac{A_k}{\omega_k} \right)^2 \]

(129)

One can investigate then the dependence of the vacuum energy density on the boundary conditions.

4.3 Structure of the ground state wave functional

Let us find the asymptotic ground state wave functional in the large-N field theory. It is sufficient to construct a solution to oscillator-type equation (123) which satisfies the constraint condition (124). We will look for this solution in a Gaussian form:

\[ Z = \exp \left[ \frac{1}{2} \int D\varphi D\phi A^+[\varphi(\cdot)]R[\varphi(\cdot), \phi(\cdot)]A^+[\phi(\cdot)] \right] |0>, \]

(130)

where \( |0> \) is a third-quantized vacuum containing no fields. The constraint (124) leads to the following condition on \( R \):

\[ \int R[\varphi(\cdot), \phi(\cdot)]\Phi^*[\phi(\cdot)]D\phi = -\Phi[\varphi(\cdot)], \]

(131)
while Schrödinger equation (123) gives us a nonlinear relation

\[
\sum_{i=1}^{2} \left( \int dx \left[ -\frac{1}{2} \delta \varphi_i(x) \frac{\delta^2}{\delta \varphi_i(x)} + \frac{1}{2} (\nabla \varphi_i)^2(x) + \frac{\mu^2}{2} \varphi_i^2(x) \right] - \Omega \right) R[\varphi_1(\cdot), \varphi_2(\cdot)] + \\
\frac{\lambda}{2} \int dx \prod_{i=1}^{2} \left( \varphi_i^2(x) \Phi[\varphi_i(\cdot)] + \int D\varphi_1 \varphi_1^2(x) \Phi^*[\varphi_i(\cdot)] R[\varphi_i(\cdot), \varphi_i(\cdot)] \right) = 0. \tag{132}
\]

It follows from eq. (103) that the third-quantized vector \( \tilde{Y} \) entering to eq. (107) has also the Gaussian form:

\[
\tilde{Y} = \exp \left[ \frac{1}{2} \int D\varphi D\phi A^+[\varphi(\cdot)] R[\varphi(\cdot), \phi(\cdot)] A^+[\phi(\cdot)] \right] |0>, \tag{133}
\]

where the two-field state \( M \) being equal to

\[
M[\varphi(\cdot), \phi(\cdot)] = R[\varphi(\cdot), \phi(\cdot)] + \Phi[\varphi(\cdot)] \Phi[\phi(\cdot)]
\]

is orthogonal to \( \Phi[\varphi(\cdot)] \) because of eq. (131). The ground state \( N \)-field wave functional is presented according to eq. (103) as

\[
\Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)] = \sum_{l=0}^{[N/2]} \frac{1}{(2N)^{[l/2]}} \sum_{1 \leq i_1 \neq \ldots \neq i_{2l} \leq N} M[\varphi_{i_1}(\cdot), \varphi_{i_2}(\cdot)] \ldots M[\varphi_{i_{2l-1}}(\cdot), \varphi_{i_{2l}}(\cdot)] \prod_{i \neq i_1 \ldots i_{2l}} \Phi[\varphi_i(\cdot)] \tag{134}
\]

We see that one should take into account the two-field correlations even in the leading order of the semiclassical approximation.

Let us find the two-field states \( M \) and \( R \) which should obey the following condition: the norm of the operator with kernel \( M[\varphi(\cdot), \phi(\cdot)] \) is lesser than 1. Otherwise, the vector (133) will not belong to the Fock space [37].

It is convenient to consider the following one-field functionals:

\[
\Phi_{k_1k_2} = \frac{1}{\sqrt{2}} a_{k_1}^+ a_{k_2}^+ \Phi[\varphi(\cdot)], \tag{135}
\]

where \( \Phi[\varphi(\cdot)] \) is a vacuum state functional corresponding to the field with mass \( \mu \). The operators \( a_{k}^+ \) are second-quantized creation operators being linear combinations of field and momentum operators acting in the space of one-field states.

We are looking for the solution to the nonlinear equation (132) in the following form:

\[
R[\varphi_1(\cdot), \varphi_2(\cdot)] = -\Phi[\varphi_1(\cdot)] \Phi[\varphi_2(\cdot)] + \sum_{k_1k_2p_1p_2} c_{k_1k_2p_1p_2} \Phi_{k_1k_2} |\varphi_1(\cdot)| \Phi_{p_1p_2} |\varphi_2(\cdot)| \tag{136}
\]

which automatically obeys the constraint (131).

Substitution (133) gives us the following equation on the coefficients \( c_{k_1k_2p_1p_2} \):

\[
\sum_{k_1k_2p_1p_2} (A_{k_1} \omega_{k_1} + A_{k_2} \omega_{k_2} + A_{p_1} \omega_{p_1} + A_{p_2} \omega_{p_2}) c_{k_1k_2p_1p_2} \Phi_{k_1k_2} |\varphi_1(\cdot)| \Phi_{p_1p_2} |\varphi_2(\cdot)| + \\
\frac{\lambda}{2} \int dx \left[ (\varphi_{1,\lambda}^2(x) - |0|\varphi_{\lambda}^2(x)|0> |\Phi[\varphi_1(\cdot)] + \sum_{k_1k_2} c_{k_1k_2k_1k_2} \Phi_{k_1k_2} |\varphi_1(\cdot)| <0|\varphi_{\lambda}^2(x) |\Phi_{k_1k_2} > > \\
\times [(\varphi_{2,\lambda}^2(x) - |0|\varphi_{\lambda}^2(x)|0> |\Phi[\varphi_2(\cdot)] + \sum_{p_1p_2} c_{p_1p_2p_1p_2} \Phi_{p_1p_2} |\varphi_2(\cdot)| <0|\varphi_{\lambda}^2(x) |\Phi_{p_1p_2} > > = 0 \tag{137}
\]

Notice that the state vector $\varphi_\lambda^2(\mathbf{x})|0 >$ contains only the vacuum and two-particle components, so that

$$(\varphi_{2,\lambda}^2(\mathbf{x}) - <0|\varphi_{\lambda}^2(\mathbf{x})|0>): \Phi = \sum_{k_1k_2} \Phi_{k_1k_2} < \Phi_{k_1k_2} | \varphi_\lambda^2(\mathbf{x})|0 >.$$ 

This means that the expression (136) satisfies eq.(132) if

$$(A_{k_1}\omega_{k_1} + A_{k_2}\omega_{k_2} + A_{p_1}\omega_{p_1} + A_{p_2}\omega_{p_2})c_{k_1k_2p_1p_2} + \frac{\lambda}{2} \int d\mathbf{x} \left( \Phi_{k_1k_2}|\varphi_\lambda^2(\mathbf{x})|0 > + \sum_{k_1'k_2'} c_{k_1k_2k_1'k_2'} < 0|\varphi_\lambda^2(\mathbf{x})|\Phi_{k_1'k_2'} > \right) \times \left( < \Phi_{p_1p_2}|\varphi_\lambda^2(\mathbf{x})|0 > + \sum_{p_1'p_2'} c_{p_1p_2p_1'p_2'} < 0|\varphi_\lambda^2(\mathbf{x})|\Phi_{p_1'p_2'} > \right) = 0$$  

(38)

The quantity $c_{k_1k_2p_1p_2}$ is a probability amplitude that there are two particles of one type with momenta $k_1, k_2$ and of another type with momenta $p_1, p_2$. However, the full momentum should be equal to zero in the ground state. This means that $c_{k_1k_2p_1p_2} \sim \delta_{k_1+k_2+p_1+p_2}$. Let us denote by $P$ and $-P$ the full momentum of the fields, so that

$$c_{k_1k_2p_1p_2} = \sum_P \alpha_k^P \delta_{k_1+k_2} \delta_{p_1+p_2}.$$ 

The matrix elements entering to eq. (138) are calculable:

$$< \Phi_{k_1k_2}|\varphi_\lambda^2(\mathbf{x})|0 > = \frac{1}{L_d} \frac{A_{k_1}A_{k_2}}{\sqrt{2\omega_{k_1}\omega_{k_2}}} e^{-i(k_1+k_2)x},$$

so that eq.(138) is simplified as:

$$(A_{k_1}\omega_{k_1} + A_{-k_1}\omega_{-k_1} + A_{p_1}\omega_{p_1} + A_{-p_1}\omega_{-p_1})\alpha_k^P + \frac{\lambda}{2L_d} \sum_{k_1'p_1'} \sqrt{A_{k_1'}A_{p_1'}} \sqrt{A_{k_1}A_{-p_1}} (\delta_{k_1k_1'} + \alpha_k^P \alpha_{k_1}^P) = 0$$

(39)

Thus, the equation on the functional $\mathcal{R}[\varphi(\cdot), \phi(\cdot)]$ has been reduced to the equation on the function $\alpha_{kp}^P$ of two desired variables $k, p$ at fixed $P$. To investigate eq. (139), let us denote by $\alpha^P$ the operator with kernel $\alpha_{kp}^P$, by $T^P$ we denote the operator of multiplication by

$$T^P = (A_{k}\omega_{k} + A_{-k}\omega_{-k}),$$

while $B^P$ will be the operator with the following matrix element:

$$B_{kp}^P = \frac{\lambda}{2L_d} \sqrt{A_{k}A_{p-k}} \sqrt{A_{p}A_{-p-k}}.$$ 

Eq.(139) takes the form:

$$T^P \alpha^P + \alpha^P T^P + (1 + \alpha^P) B^P (1 + \alpha^P) = 0.$$ 

The solution to this operator equation can be written as:

$$\alpha^P = (1 + M^P)^{-1}(1 - M^P),$$

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where
\[ M^P = (T^P)^{-1/2}[(T^P)^2 + 2(T^P)^{1/2} B^P (T^P)^{1/2}]^{-1/2}. \] (140)

We see that the solution to eq.(132) is constructed if the operator entering to eq.(140) is positively definite,
\[ [(T^P)^2 + 2(T^P)^{1/2} B^P (T^P)^{1/2}]^{-1/2} > 0. \] (141)

In this case, one can construct an approximate stationary solution to the large-N Schrödinger equation.

Usually, one is able to find semiclassical eigenfunctions around the static classical solution if it is stable due to small perturbations. We have seen that investigation of the nonlinear equation gives us an alternative way to investigate this problem. We have also obtained the criteria of stability of classical solution (118), eq.(141).

4.4 Perturbatively excited states

Let us consider now the elementary excitations around the vacuum solution. It is necessary to construct excited eigenvectors of eq.(123). To construct such a third-quantized vectors, let us find such an operator \( \Lambda^+ \) that
\[ [\Lambda^+, H_2 - \Omega \int D\varphi A^+[\varphi(\cdot)] A^-[\varphi(\cdot)]] = -\beta \Lambda^+ \] (142)
and
\[ [\Lambda^+, \int D\varphi (A^+[\varphi(\cdot)] \Phi[\varphi(\cdot)] + A^-[\varphi(\cdot)] \Phi^*[\varphi(\cdot)])] = 0. \] (143)

This operator will satisfy then the following properties. Let \( Z \) be a solution to eq.(123) which satisfies the condition (124). Eq.(143) implies that the vector \( \Lambda^+ Z \) will also obey the constraint (124). It follows from eq.(142) that the vector \( \Lambda^+ Z \) will be also an eigenstate of eq.(123) with eigenvalue \( \omega + \beta \).

Thus, the operator \( \Lambda^+ \) shifts the energy by \( \beta \).

Let us find such operators \( \Lambda^+ \). We are looking for them as follows:
\[ \Lambda^+(F, G) = i \int D\varphi (A^+[\varphi(\cdot)] G^*[\varphi(\cdot)] - A^-[\varphi(\cdot)] F^*[\varphi(\cdot)]) \] (144)
where \( G \) and \( F \) are functionals to be found. Since the Hamiltonian entering to the left-hand side of eq.(123) is quadratic, while operator \( \Lambda^+ \) linearly depends on creation and annihilation operators, the commutator will be also a linear combination of these operators. Thus, eq.(142) will imply the system of two equations:
\[ (\int d\mathbf{x} [\frac{1}{2} \pi^2(\mathbf{x}) + \frac{1}{2} (\nabla \varphi^2)^2(\mathbf{x}) + \frac{u(\mathbf{x})}{2} \varphi^2(\mathbf{x})] - \Omega) F[\varphi(\cdot)] + \frac{\lambda}{2} \int d\mathbf{x} \varphi^2(\mathbf{x}) \int D\phi \Phi^2(\mathbf{x})(F[\phi(\cdot)] \Phi^*[\phi(\cdot)] + G[\phi(\cdot)] \Phi^*[\phi(\cdot)]) = -\beta F[\varphi(\cdot)], \] (145)
\[ (\int d\mathbf{x} [\frac{1}{2} \pi^2(\mathbf{x}) + \frac{1}{2} (\nabla \varphi^2)^2(\mathbf{x}) + \frac{u(\mathbf{x})}{2} \varphi^2(\mathbf{x})] - \Omega) G[\varphi(\cdot)] + \frac{\lambda}{2} \int d\mathbf{x} \varphi^2(\mathbf{x}) \int D\phi \Phi^2(\mathbf{x})(F[\phi(\cdot)] \Phi^*[\phi(\cdot)] + G[\phi(\cdot)] \Phi^*[\phi(\cdot)]) = \beta G[\varphi(\cdot)]. \]

Thus, we see that spectrum of the variation system (145) coincides with the spectrum of differences between energy levels, because the operator \( \Lambda^+ \) increases the energy by \( \beta \).

An important feature is that one must check the property \( \Lambda^+ Z \neq 0 \). Otherwise, the operator \( \Lambda^+ \) will give us a trivial solution to eq.(123).
When one applies the operator \([144]\) to the Gaussian state \([30]\), one will obtain the vector

\[
\int D\varphi A^+[\varphi(\cdot)]Y[\varphi(\cdot)]Z,
\]

where

\[
Y[\varphi(\cdot)] = iG^*[\varphi(\cdot)] - i \int D\phi R[\varphi(\cdot), \phi(\cdot)]F^*[\phi(\cdot)].
\]

This functional is certainly non-zero if

\[
(G, G) > (F, F)
\]

because the norm of the operator \(\hat{R}\) with the kernel \(R[\varphi(\cdot), \phi(\cdot)]\) is not greater than 1. Thus, the solutions of the system \([145]\) which satisfy the condition \([147]\) will give us the spectrum of elementary excitations.

On the other hand, let us consider the solutions to eq. \([145]\) which do not obey eq. \([147]\). Such solutions can be obtained if one substitute to the system \([145]\) \(F^*\) instead of \(G\) and \(G^*\) instead of \(F\). The corresponding operator \(\Lambda^+[G^*, F^*]\) denoted as \(\Lambda^-[F, G]Z\) is conjugated to operator \([144]\). Consider the vector \(\Lambda^-[F, G]Z\). It has the form \([146]\) where

\[
Y[\varphi(\cdot)] = iF[\varphi(\cdot)] - i \int D\phi R[\varphi(\cdot), \phi(\cdot)]G[\phi(\cdot)].
\]

However, one of the definitions of the operator \(R\) with the kernel \(R[\varphi, \phi]\) is the following: \(RG = F\), see appendix B. This means that \(\Lambda^-[F, G]Z = 0\) so that we should take into account only those solutions to the variation system that obey eq. \([147]\).

Let us find these solutions. First of all, notice that the functionals

\[
F_{k_1\ldots k_n} = 0, G_{k_1\ldots k_n} = a_{k_1}\ldots a_{k_n}^+ \Phi^*, \beta = A_{k_1} \omega_{k_1} + \ldots + A_{k_n} \omega_{k_n},
\]

obey eqs. \([145]\) if \(n \neq 0, 2\). Namely, the vector \(\varphi_2^L(x)|0\rangle\) contains vacuum and two-particle components only, so that the matrix element \(\langle F + G|\varphi_2^L(x)|0\rangle\) entering to eq. \([147]\) vanishes.

One can interpret the quantity \(\beta\) as \(\Lambda \to \infty\)

\[
\omega_{k_1} + \ldots + \omega_{k_n}
\]

as an energy of the elementary excitation of the \(N\)-field system. Why elementary? One can be surprised that the theory \([149]\) is the energy of the system of \(n\) particles. However, one excitation \([148]\) with energy \([149]\) is not equivalent to \(n\) excitations with energies \(\omega_{k_1}, \ldots, \omega_{k_n}\). Excitation \([148]\) describes \(n\) particles of one type, while \(n\) excitations describe \(n\) particles of different types. The third-quantized language is of course very unusual.

The only non-trivial elementary excitation corresponds to two particles of one type. Let us look for the corresponding solution to the variation system as:

\[
G = g_0 \Phi^* + \sum_{k_1, k_2} g_{k_1 k_2}^* \Phi_{k_1 k_2}^*,
\]

\[
F = f_0 \Phi + \sum_{k_1, k_2} f_{k_1 k_2} \Phi_{k_1 k_2},
\]

where \(\Phi_{k_1 k_2}\) is defined by formula \([133]\). Eqs. \([145]\) imply the following nontrivial relations:

\[
(A_{k_1} \omega_{k_1} + A_{k_2} \omega_{k_2} + \beta) f_{k_1 k_2} + \frac{\lambda}{2L^d} \sum_{p_1 p_2} \frac{\sqrt{A_{k_1} A_{k_2}}}{\sqrt{2\omega_{k_1} \omega_{k_2}}} \frac{\sqrt{A_{p_1} A_{p_2}}}{\sqrt{2\omega_{p_1} \omega_{p_2}}} (f_{p_1 p_2} \delta_{p_1 + p_2, k_1 + k_2} + g_{p_1 p_2} \delta_{p_1 + p_2 + k_1 + k_2}) = 0,
\]

\[
(A_{k_1} \omega_{k_1} + A_{k_2} \omega_{k_2} - \beta) g_{k_1 k_2} + \frac{\lambda}{2L^d} \sum_{p_1 p_2} \frac{\sqrt{A_{k_1} A_{k_2}}}{\sqrt{2\omega_{k_1} \omega_{k_2}}} \frac{\sqrt{A_{p_1} A_{p_2}}}{\sqrt{2\omega_{p_1} \omega_{p_2}}} (g_{p_1 p_2} \delta_{p_1 + p_2, k_1 + k_2} + f_{p_1 p_2} \delta_{p_1 + p_2 + k_1 + k_2}) = 0.
\]
The next step to simplify the system is to consider the momentum of two particles to be equal to $P$: 

$$f_{k_1k_2} = f_{-k_1} \delta_{k_1+k_2+P}, \quad g_{k_1k_2} = g_{k_1} \delta_{k_1+k_2-P}.$$ 

The obtained system

$$(A_{k_1} \omega_{k_1} + A_{p-k_1} \omega_{p-k_1} + \beta) f_{k_1} + \frac{\lambda}{2L^d} \sum_{p_1 p_2} \sqrt{A_{k_1} A_{p-k_1}} \sqrt{A_{p_1} A_{p-p_1}} (f_{p_1} + g_{p_1}) = 0,$$

$$(A_{k_1} \omega_{k_1} + A_{p-k_1} \omega_{p-k_1} - \beta) g_{k_1} + \frac{\lambda}{2L^d} \sum_{p_1 p_2} \sqrt{A_{k_1} A_{p-k_1}} \sqrt{A_{p_1} A_{p-p_1}} (f_{p_1} + g_{p_1}) = 0$$

allows us to express the functions $f_k$ and $g_k$ via eigenvalue $\beta$ and unknown constant $a_P$:

$$f_k = -\frac{\lambda}{2L^d} \frac{A_{p} \sqrt{A_{k} A_{p-k}}}{\sqrt{2 \omega_{k} \omega_{p-k}}} \frac{1}{A_{k} \omega_{k} + A_{p-k} \omega_{p-k} + \beta}$$

$$g_k = \frac{\lambda}{2L^d} \frac{a_P \sqrt{A_{k} A_{p-k}}}{\sqrt{2 \omega_{k} \omega_{p-k}}} \frac{1}{-A_{k} \omega_{k} - A_{p-k} \omega_{p-k} + \beta}$$

which is determined from the condition

$$a_P = \sum_p \frac{f_p + g_p}{\sqrt{2 \omega_p \omega_{p-P}}} \sqrt{A_p A_{p-P}}$$

implying equation on $\beta$:

$$\frac{1}{\lambda} = \frac{1}{2L^d} \sum_p \frac{A_p A_{p-P}}{\omega_p \omega_{p-P} \beta^2 - (A_p \omega_p + A_{p-P} \omega_{p-P})^2}.$$

For $d + 1 = 4$, the left-hand and right-hand side of this equation diverge as $\Lambda \rightarrow \infty$. However, making us of the definition of the renormalized coupling constant, we obtain the regular relation:

$$\frac{1}{\lambda_R} = \frac{1}{2L^d} \sum_p \frac{1}{\omega_p \omega_{p-P} \beta^2 - (\omega_p + \omega_{p-P})^2} - \frac{1}{2 \omega_p^2}.$$  \hspace{1cm} (150)

Eq. (150) allows us to evaluate the energy of bound two-particle states and scattering amplitudes for these particles.

### 4.5 Non-perturbatively excited states

In the previous subsection we have considered the vacuum state of the large-$N$ theory and states with finite number of particles. However, one can also investigate other excited states which cannot be described by the perturbation theory of the previous subsection.

To understand the problem, consider the simple analogy. In ordinary quantum field theory at small values of the coupling constant one can apply the perturbation theory and obtain that excitations correspond to free particles of given mass in a leading order of perturbation theory. On the other hand, one can also apply the soliton quantization approach which is also applicable at small values of coupling constant and obtain non-perturbative series of states corresponding to the soliton.

In the large-$N$ theory, the constructed asymptotic energy levels are found provided that the number of particles is much lesser than $N$. If we consider the state consisting of $N$ excitations we should take into account the interaction between the excitations, so that another approach is necessary.
To consider such a non-perturbatively excited state, it is necessary to consider another solution to the classical equation \((\text{[121]}\)) rather than vacuum. For example, the solution
\[
a_k^+ |0> \]
\((|0>\) is the vacuum of the field system with the mass \(M\), \(a_k^+\) is a creation operator) to eq.\((\text{[121]}\)) will correspond to the quantum state of \(N\) particles of a different type with the momentum \(k\). One will also be able to find the corresponding Gaussian state, solutions to the variation system etc.

Let us suppose
\[
u(x) = M^2\]
and
\[
\Phi = c a_k^+ a_{kn}^+ |0> \tag{151}\]
where \(c\) is a normalizing factor. Analogously to section 4.2, one can evaluate the average value of \(\varphi^2_\lambda(x)\):
\[
(\Phi, \varphi^2_\lambda(x)\Phi) = \frac{1}{L^d} \sum_k \frac{A_k}{2\sqrt{k^2 + M^2}} + \frac{1}{L^d} \sum_{i=1}^n \frac{A_{ki}}{\sqrt{k_i^2 + M^2}}. \tag{152}\]

Eq. \((\text{[121]}\)) on the mass \(M\) is to be renormalized. Making use of the definition of the physical mass \(\mu\) of perturbative excitations, one obtains:
\[
\frac{M^2 - \mu^2}{\lambda} = \frac{1}{2L^d} \sum_k A_k \left(\frac{1}{\sqrt{k^2 + M^2}} - \frac{1}{\sqrt{k^2 + \mu^2}}\right) + \frac{1}{L^d} \sum_{i=1}^n \frac{A_{ki}}{\sqrt{k_i^2 + M^2}}. \]

For \(d = 1, 2\) the right-hand and left-hand sides of this equation are finite. For \(d=3\), it is necessary to perform the renormalization of the coupling constant:
\[
\frac{M^2 - \mu^2}{\lambda_R} = \frac{1}{2L^d} \sum_k A_k \left(\frac{1}{\sqrt{k^2 + M^2}} - \frac{1}{\sqrt{k^2 + \mu^2}} + \frac{M^2 - \mu^2}{2(k^2 + \mu^2)^{3/2}}\right) + \frac{1}{L^d} \sum_{i=1}^n \frac{A_{ki}}{\sqrt{k_i^2 + M^2}}. \]

The quantity \(M\) plays a role of a mass of small excitations around the background state corresponding to the classical functional \((\text{[151]}\)). It differs from \(\mu\).

To find the leading order of the energy of the quantum state, let us use eq.\((\text{[127]}\)). One obtains:
\[
\frac{E}{N} = \Omega - \frac{\lambda}{4} \int dx < \varphi^2_\lambda(x) >^2, \]
so that
\[
\frac{E}{N} = \frac{1}{2} \sum_k A_k \sqrt{k^2 + M^2} + \sum_{i=1}^n A_{ki} \sqrt{k_i^2 + M^2} - \frac{\lambda L^d}{4} \left(\frac{M^2 - \mu^2}{\lambda}\right)^2, \]
where we have used eq.\((\text{[121]}\)). The difference between this energy and ground state energy \((\text{[29]}\)) is
\[
\frac{E - E_0}{N} = \frac{1}{2} \sum_k A_k \left(\sqrt{k^2 + M^2} - \sqrt{k^2 + \mu^2}\right) + \sum_{i=1}^n A_{ki} \sqrt{k_i^2 + M^2} - \frac{\lambda L^d}{4\lambda} (M^2 - \mu^2)(M^2 + \mu^2 - 2m^2), \]
since
\[
M^2 + \mu^2 - 2m^2 = \frac{\lambda}{L^d} \left(\sum_k A_k \left(\frac{1}{\sqrt{k^2 + M^2}} + \frac{1}{\sqrt{k^2 + \mu^2}}\right) + \sum_{i=1}^n \frac{A_{ki}}{\sqrt{k_i^2 + M^2}}\right)\]

The energy of non-perturbative excitation is
\[
\frac{E - E_0}{N} = \frac{1}{2} \sum_k A_k \left(\sqrt{k^2 + M^2} - \sqrt{k^2 + \mu^2} - \frac{M^2 - \mu^2}{4} \left(\frac{1}{\sqrt{k^2 + M^2}} + \frac{1}{\sqrt{k^2 + \mu^2}}\right)\right).\]
It is interesting that for $d \geq 5$ the sum entering to eq. (153) is divergent. This corresponds to the fact that the $\phi^4$-theory in such dimensions is not renormalizable. Usually this feature of the theory is important in calculations of the higher orders of perturbation theory. We see that for the method of the large-$N$ expansion the feature of the renormalizability is very important even in the leading order.

One can also consider small excitations around the considered non-perturbative state. One should investigate then the variation system which has the form (145) where $\Phi$ is not the vacuum state functional but the functional (154). There will be trivial solutions to eq. (145) with $F = 0$, this is the case

$$<G^*|\varphi^2_\Lambda(x)|0> = 0.$$  

In this case the eigenvalue $\beta$ will coincide with one of the eigenvalues of the operator $H_M - \Omega$ ($H_M$ is the Hamiltonian of the field of mass $M$).

To find non-trivial values of $\beta$, denote by $F$ and $X$ the one-field states corresponding to the functionals $F[\varphi(\cdot)]$ and $G^*[\varphi(\cdot)]$. Eqs. (145) will take then the form:

$$(H_M - \Omega)F + \frac{\lambda}{2} \int d^d x \varphi^2_\Lambda(x) (<X|\varphi^2_\Lambda(x)|\Phi > + <\Phi|\varphi^2_\Lambda(x)|F>)\Phi = -\beta F \tag{154}$$

$$(H_M - \Omega)X + \frac{\lambda}{2} \int d^d x \varphi^2_\Lambda(x) (<\Phi|\varphi^2_\Lambda(x)|X > + <F|\varphi^2_\Lambda(x)|\Phi>)\Phi = \beta X$$

Let $K = k_1 + ... + k_n$. To simplify the system (154), consider the state $X$ to have momentum $P$ and $F$ to have momentum $2K - P$: for momentum operator $\hat{P} = \sum_k k \hat{a}_k^+ \hat{a}_k^-$ one has:

$$\hat{P}X = PX, \quad \hat{P}F = (2K - P)F.$$  

This condition implies that

$$<X|\varphi^2_\Lambda(x)|\Phi > = <X|e^{-ix_0^{-1} \hat{P}}e^{ix_0}\varphi^2_\Lambda(0)|\Phi > = e^{i(K - P)x} <X|\varphi^2_\Lambda(0)|\Phi >,$$

while

$$<\Phi|\varphi^2_\Lambda(x)|F > = e^{i(K - P)x} <\Phi|\varphi^2_\Lambda(0)|F >.$$  

Analogously, one can express the operator $\int d^d x e^{ikx} \varphi^2(x)$ via the projection operators $\Pi_Q$ on the subspace corresponding to a given momentum $Q$,

$$\int d^d x e^{i(K - P)x} \varphi^2_\Lambda(x)\Phi = L^d \Pi_{2K - P}\varphi^2_\Lambda(0)\Phi.$$  

Thus, one simplifies the variation system as

$$(H_M - \Omega + \beta)F + \frac{\lambda L^d a^*}{2} \Pi_{2K - P}\varphi^2_\Lambda(0)\Phi = 0,$$

$$(H_M - \Omega - \beta)X + \frac{\lambda L^d a^*}{2} \Pi_P\varphi^2_\Lambda(0)\Phi = 0,$$

$$a = <\Phi|\varphi^2_\Lambda(0)|X > + <F|\varphi^2_\Lambda(0)|\Phi >,$$

so that the non-trivial eigenvalues $\beta$ are expressed from the equation

$$\frac{1}{\lambda} = \frac{1}{L^d} <\Phi|\varphi^2_\Lambda(0)|((\beta - H_M + \Omega)^{-1}\Pi_P - (\beta + H_M - \Omega)^{-1}\Pi_{2K - P})\varphi^2_\Lambda(0)|\Phi >$$

analogously to the previous subsection.
The method of second quantization is very useful in quantum many-particle mechanics even in the case of a fixed number of particles. For example, this approach allows us to introduce a notion of quasiparticles which can be created or annihilated even if all the particles of the system are stable. The quasiparticle conception is very important for the condensed matter theory.

Analogously, we have seen that the notion of third quantization can be applied to the theory of $N$ fields and allows us to construct the approximate solutions to the $N$-field functional Schrodinger equation. These asymptotics are expressed via the solution to eqs. (52) and (97).

Although eq. (52) is a classical equation, it resembles a functional Schrodinger equation of quantum theory of one field rather than a classical field equation. Eq. (97) is an equation for the vector of the third-quantized Fock space vector. The corresponding Hamiltonian is expressed via the operators $A^\pm[\varphi(\cdot)]$ which can be called as operators of creation and annihilation of the “quasifield” since the analogous operators in quantum statistics create and annihilate quasiparticles. The quantum theory of fixed number of fields is reduced to the theory of variable number of fields (97). Since the Hamiltonian is quadratic, such a model is exactly-solvable by the Bogoliubov-like transformation. However, the coefficients of the transformation are functionals expressing via the variation system (143).

In quantum field theory it is also assumed that the set of elementary particles is fixed: there are electrons, muons etc. There were no attempts to construct a theory admitting existence of the electron field with some probability. However, we see that the theory (97) tells us that there are probability amplitudes that there are no fields, that there is only one type of particles, etc. Since the theory (97) has been shown to be equivalent to the ordinary $N$-field theory, the model (97) obeys all the properties of the field theory, for example, it is relativistic invariant etc.

The conception that particles can be created or annihilated was very important in constructing the quantum field theory. One can hope that the idea that number of fields can be variable will be also useful in developing further theories of everything.

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Appendix A. The complex-WKB semiclassical approach

In this appendix we briefly review the complex-WKB method which was developed in [20].

A.1. The WKB and complex-WKB ansatz

Semiclassical approximation is the powerful tool to construct asymptotic solutions to the quantum mechanical $d$-dimensional equations like

$$i\hbar\frac{\partial \psi}{\partial t} = H\left(x, -i\hbar \frac{\partial}{\partial x}\right)\psi$$

for the wave function $\psi^t(x)$ as $\hbar \to 0$. We have seen that equations like (155) arise in the large-$N$ field theory, while the small parameter $\hbar$ may be not related with the Planck constant $\hbar$: in the large-$N$ theory $\hbar = 1/N$.

The most famous semiclassical approach is the WKB-approach which allows us to construct semiclassical solutions to eq.(155) of the rapidly oscillating form

$$\varphi^t(x)e^{\frac{i}{\hbar}S^t(x)}$$

where $S$ is a real function. One can easily see that the ansatz (156) really satisfies eq.(155), obtain the Hamilton-Jacobi equation for $S$, the expansion for $\varphi$ etc.
However, there exists other semiclassical wave functions that also approximately obey eq.(155) as $h \to 0$. One more example of the semiclassical ansatz to eq.(155) is the complex-WKB ansatz \[ 17 \] 

$$
\psi^t(x) = \frac{1}{\hbar^{d/4}} e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} f^t\left(\frac{x-Q^t}{\sqrt{\hbar}}\right) 
$$

(157)

where $f^t(\xi)$ is a rapidly damping function of $\xi$, while $S^t$, $P^t$, $Q^t$ do not depend on $x$, $P^t$ and $Q^t$ are $d$-dimensional vectors. The normalizing factor $\hbar^{-d/4}$ is a corollary of the condition $(\psi, \psi) = O(1)$.

At fixed moment of time the wave function (157) is a wave packet with the width of order $O(\sqrt{\hbar})$. The average value of the coordinate is $Q^t$, average momentum is $P^t$, uncertainties of the coordinate and momentum are of order $\sqrt{\hbar}$, so that the uncertainty relation $\delta P \delta Q \sim \hbar$ is satisfied. Note that the WKB-wave function has the uncertainty of the coordinate and momentum of order $O(1)$. Thus, the wave function (157) really determines motion of the classical particle along the classical trajectory, while the WKB-function (156) does not determine a classical trajectory.

Let us show that the wave function (157) really approximately satisfies the evolution equation (155). First of all, notice that extraction of the multiplier $\psi$ from the wave function

$$
\psi^t(x) = e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} \chi^t(x)
$$

is equivalent to shifting the differential operators:

$$
-i\hbar \frac{\partial}{\partial x} e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} = e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} \left(P^t - i\hbar \frac{\partial}{\partial x}\right),
$$

(158)

$$
i\hbar \frac{\partial}{\partial t} e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} = e^{i \frac{S}{\hbar}} e^{i \frac{P^t(x-Q^t)}{\sqrt{\hbar}}} \left(i\hbar \frac{\partial}{\partial t} - \dot{S}^t - \dot{P}^t(x-Q^t) + P^t \dot{Q}^t\right).
$$

Identities (158) for operators can be justified by applying them to arbitrary test function. Eqs.(158) imply that

$$
(i\hbar \frac{\partial}{\partial t} - \dot{S}^t - \dot{P}^t(x-Q^t) + P^t \dot{Q}^t - H(x, P^t - i\hbar \frac{\partial}{\partial x})) \chi^t(x) = 0.
$$

(159)

Let us consider the limit $\hbar \to 0$ of the latter equation. At first sight, one should simply neglect all the terms containing $\hbar$, so that one would wonder why the function $S^t$ does not depend on $x$. However, one should take into account that the wave function $\chi^t(x)$ is a wave packet with width of order $\sqrt{\hbar}$, so that

$$
x \chi^t \simeq Q^t \chi^t.
$$

This means that one should not only set $\hbar = 0$ in eq.(154) in order to obtain a leading order in $\hbar$ but also set $x = Q^t$ in this equation, so that the leading semiclassical approximation gives us the following relation on $S^t$:

$$
\dot{S}^t = P^t \dot{Q}^t - H(Q^t, P^t),
$$

(160)

so that $S^t$ is the action on the trajectory of motion of the wave packet. To find next corrections, consider the substitution

$$
x - Q^t = \xi \sqrt{\hbar},
$$

so that the average number of the observable $\xi$ is of order $O(1)$. Using eq.(160), one finds that eq.(159) takes the form

$$
(i\hbar \frac{\partial}{\partial t} - \dot{P}^t \sqrt{\hbar} - i\sqrt{\hbar} \frac{\partial}{\partial \xi} \dot{Q}^t - H(Q^t + \sqrt{\hbar} \xi, P^t - i\hbar \frac{\partial}{\partial \xi}))(f^t(\xi)) = 0.
$$

(161)
Eq. (161) imply that \((P_t, Q_t)\) should obey the Hamiltonian system:

\[
\dot{Q} = \frac{\partial H}{\partial P}, \quad \dot{P} = -\frac{\partial H}{\partial Q},
\]

while the dynamical equation for \(f^t\) is the oscillator Schrödinger equation:

\[
(i\frac{\partial}{\partial t} - H_2)f^t = 0, \tag{163}
\]

where

\[
H_2 = \frac{1}{2} \xi \frac{\partial^2 H}{\partial Q \partial \xi} + \frac{1}{2} \xi \frac{\partial^2 H}{\partial P \partial \xi} + \frac{1}{2} i \xi \frac{\partial}{\partial \xi} \frac{\partial^2 H}{\partial P \partial P} \frac{1}{i \xi}. \tag{164}
\]

The arguments \((Q^t, P^t)\) of the classical Hamiltonian are omitted.

Not that for \(h\)-dependent Hamiltonian, \(H(Q, P) + \hbar H_1(Q, P)\), the classical equations (162) will not change, while the oscillator Hamiltonian (164) will contain the additional term, \(H_1(Q^t, P^t)\).

We see that the complex-WKB approach is a good method to justify the Ehrenfest theorem. Moreover, one can find not only the classical trajectory of motion of the wave packet but also the evolution of the shape of this wave packet. Let us show how the complex-WKB approach allows us to obtain the asymptotic spectrum of the Hamiltonian.

### A.2. Stationary complex-WKB solutions

Let us find “semiclassical” stationary states with the help of the complex-WKB approach. Since the wave function (157) should be stationary, the wave packet should not move, so that classical coordinate and momentum should be time-independent, \(P^t = \text{const}, Q^t = \text{const}\). The phase factor depends on \(t\) as

\[
S^t = \text{const} - H(Q, P)t.
\]

The wave function (157) depends on \(t\) as \(e^{-\frac{i}{\hbar}\varepsilon t}\) for

\[
\varepsilon = H(Q, P) + \hbar \varepsilon,
\]

if

\[
H_2 f = \varepsilon f. \tag{165}
\]

We see that the problem of finding approximate energy levels of an arbitrary Hamiltonian is reduced to the problem of finding exact spectrum of the oscillator Hamiltonian \(H_2\). We see that the oscillator approximation considered in [21] is really semiclassical.

The procedure of the diagonalization of the quadratic Hamiltonian (164) is standard [37]. One should consider \(d\) independent creation and annihilation operators

\[
A^-_k = p^{(k)}_\xi - q^{(k)}_{\xi} \frac{1}{i} \frac{\partial}{\partial \xi},
\]

\[
A^+_k = p^{(k)^*}_\xi - q^{(k)^*}_{\xi} \frac{1}{i} \frac{\partial}{\partial \xi}, \tag{166}
\]

obeying the canonical commutation relations

\[
[A^-_k, A^+_l] = \delta_{kl}, [A^-_k, A^-_l] = 0.
\]

which are equivalent to

\[
p^{(k)} q^{(l)^*} - q^{(k)} p^{(l)^*} = i\delta_{kl},
\]

40
The $d$-dimensional vectors $p^{(k)}$ and $q^{(k)}$ entering to creation and annihilation operators (166) are defined from the relations

$$[H_2, A^\pm_k] = \pm \omega_k A^\pm_k. \quad (167)$$

Since there are $2d$ linearly independent operators $A^\pm$, one can present $2d$ coordinate and momenta operators via creation and annihilation operators. Thus, eq.(167) means that the operator $H_2 - \sum_k \omega_k A^+_k A^-_k$ commutes with coordinate and momentum operators. Therefore, it is a c-number, so that

$$H_2 = \sum_k \omega_k A^+_k A^-_k + \varepsilon^{(0)}.$$ 

The “frequencies” $\omega_k$ are obtained from eqs.(167) being equivalent to the variation system to the classical equations:

$$-i\omega p = \frac{\partial^2 H}{\partial Q \partial P} p + \frac{\partial^2 H}{\partial Q \partial Q} q,$$

$$i\omega q = \frac{\partial^2 H}{\partial P \partial P} p + \frac{\partial^2 H}{\partial P \partial Q} q,$$

One can show [20] that the described procedure of diagonalization is well-defined if and only if the stationary solution $(P, Q)$ of the classical equations of motion is stable.

The wave function of the ground state is defined from the condition

$$A_1^- f = 0, ..., A_d^- f = 0.$$

The function $f$ has the Gaussian form

$$f = \exp(i \frac{1}{2} \xi \alpha \xi),$$

where

$$p^{(k)} = \alpha q^{(k)} \quad (168)$$

one can show that the matrix consisting of $d$ vectors $q^{(1)}, ..., q^{(d)}$ is reversible, so that eq. (168) determines the matrix $\alpha$ in a unique fashion. The ground state energy can be written as

$$\varepsilon^{(0)} = \frac{1}{2i} Tr(\frac{\partial^2 H}{\partial Q \partial P} \alpha) + H_1. \quad (169)$$

The excited states are found by applying creation operators to the ground state

$$(A^+_1)^{n_1}...(A^+_d)^{n_d} f_0. \quad (170)$$

and have energies

$$\varepsilon^{(0)} + \sum k \omega_k n_k. \quad (171)$$

We have shown that the stationary case of the complex-WKB approach leads to the oscillator approximation.

Appendix B. The complex-WKB method in Fock space

In this appendix we develop the complex-WKB method for the second-quantized equations. For more details, see [38].
B.1 The complex-WKB ansatz

Consider the Schrödinger equation

\[ i \frac{d\Psi}{dt} = \hat{H}\Psi \]  

for the time-dependent vector \( \Psi \) of the Fock space. It happens that semiclassical methods are applicable to eq.(172) if the Hamiltonian depends on the bosonic creation and annihilation operators and small parameter \( \varepsilon \) as follows:

\[ \hat{H} = \frac{1}{\varepsilon} H(\sqrt{\varepsilon}a^+, \sqrt{\varepsilon}a^-) \]  

(173)

For the case of a finite number of degrees of freedom one can consider the following representation for the creations and annihilation operators. The vacuum state corresponds to the wave function

\[ \Psi_0(\xi) = \exp(-\frac{1}{2\varepsilon} \sum_i x_i^2), \]  

(174)

while the operators \( a^\pm \) can be presented as

\[ a^\pm_i = \frac{x_i \mp \varepsilon \partial/\partial x_i}{\sqrt{2\varepsilon}}. \]  

(175)

Substituting eq.(175) to eq.(173), one finds that the Schrödinger equation (172) is taken to the semiclassical form:

\[ i\varepsilon \frac{\partial \psi}{\partial t} = H(\frac{x - \partial/\partial x}{\sqrt{2}}, \frac{x + \partial/\partial x}{\sqrt{2}})\psi, \]

while the analog of the Planck constant is \( \varepsilon, \varepsilon = \hbar \).

We see that one can apply the complex-WKB method to the Hamiltonian of the type (173). Note that N-field Hamiltonians considered in this paper depend on the small parameter analogously to (173).

To construct an analog of the wave packet (154), let us consider the partial case of the formula (154), the particle ith zero coordinate and momentum, with the wave function

\[ f(x/\sqrt{\varepsilon}). \]

It is easy to see that this quantum state corresponds to an \( \varepsilon \)-independent state vector of the Fock space. Really, such a vector can be expressed via creation operators and vacuum state (174). However, the wave function (174) depends on \( x/\sqrt{\varepsilon} \) only, while the creation operators are expressed via operators of multiplication by \( x/\sqrt{\varepsilon} \) and operator of differentiation with respect to \( x_i/\sqrt{\varepsilon} \). This means that \( \varepsilon \)-independent state describes a wave packet with \( P = 0, Q = 0 \).

To construct the Fock-space analog of the wave function (154) in general case, present it as

\[ \text{const}_{\varepsilon} e^{\frac{1}{\sqrt{\varepsilon}} \sum_k (\phi_k a^+_k - \phi_k^* a^-_k)} f(x/\sqrt{\varepsilon}). \]

Since operators \( x \) and \( \partial/\partial x \) are linear combinations of creation and annihilation operators, the exponent can be presented as

\[ \exp(\frac{1}{\sqrt{\varepsilon}} \sum_k (\phi_k a^+_k - \phi_k^* a^-_k)) \]

for some numbers \( \phi_i \). Thus, we see that the complex-WKB ansatz in the Fock space should depend on \( \varepsilon \) as follows:

\[ \Psi^t = U_{\phi^t, \psi_t} Y^t, \]  

(176)

where

\[ U_{\phi, \psi} = e^{i S} e^{\frac{1}{\sqrt{\varepsilon}} \sum_k (\phi_k a^+_k - \phi_k^* a^-_k)}. \]
This ansatz can be considered also in the infinite-dimensional case.

Analogously to appendix A, one can show that vector (176) approximately satisfies eq. (172). Namely, the following relations take place:

\begin{equation}
 a_k^+ U_{\varphi,s} = U_{\varphi,s}(a_k^+ + \frac{\varphi_k^*}{\sqrt{\varepsilon}})
 \end{equation}

(177)

\begin{equation}
 a_k^- U_{\varphi,s} = U_{\varphi,s}(a_k^- + \frac{\varphi_k}{\sqrt{\varepsilon}})
 \end{equation}

(178)

\[ U_{\varphi,s}^{-1} \frac{d}{dt} U_{\varphi,s} = (-\frac{1}{\varepsilon} \frac{dS^t}{dt} + \frac{i}{2\varepsilon} \sum_k (\varphi_k^* \frac{d\varphi_k}{dt} - \frac{d\varphi_k^*}{dt} \varphi_k) + \frac{i}{\sqrt{\varepsilon}} \sum_k (\varphi_k a_k^+ - \varphi_k^* a_k^-) \]

In the leading order $O(1/\varepsilon)$ we obtain the condition on \( S^t \)

\[ \dot{S} = \frac{i}{2} \sum_k (\varphi_k^* \frac{d\varphi_k}{dt} - \frac{d\varphi_k^*}{dt} \varphi_k) - H(\varphi^*, \varphi). \]

(179)

The next order $O(1/\sqrt{\varepsilon})$ leads us to classical equations for classical variables $\varphi_k^*$:

\[ i \frac{d\varphi_k}{dt} = \frac{\partial H}{\partial \varphi_k^*}, \]

(180)

while the terms of order $O(1)$ allow us to obtain the equation for the Fock space vector $Y^t$:

\[ i \frac{dY^t}{dt} = H_2 Y^t. \]

(181)

where

\[ H_2 = \sum_{k,l} \left[ \frac{1}{2} a_k^+ \frac{\partial^2 H}{\partial \varphi_k^* \partial \varphi_l} a_l^+ + a_k^+ \frac{\partial^2 H}{\partial \varphi_k \partial \varphi_l} a_l^+ + \frac{1}{2} a_k^- \frac{\partial^2 H}{\partial \varphi_k \partial \varphi_l} a_l^- \right] \]

(182)

**B.2 Asymptotic spectrum of the $N$-particle Hamiltonian**

Let us consider the Hamiltonians (173) that consists of terms with equal number of creation and annihilation operators, i.e. the Hamiltonians that are invariant under the global transformations, $a_k^\pm \rightarrow a_k^\pm e^{\pm i\alpha}$. This means that the number of particles is an integral of motion.

To find eigenstates of the $N$-particle component of the Hamiltonian, consider the $N$-th component of eq.(176) and find whether it is a stationary vector. For the simplicity, set $N = 1/\varepsilon$. Eqs.(177) implies that the average value of any observable $f(\varepsilon \sum_m a_m^+ a_m^-)$, where $f$ is an $\varepsilon$-independent function, is the following for the state $\Psi^t$,

\[ < f(\varepsilon \sum_m a_m^+ a_m^-) \rightarrow \varepsilon \rightarrow 0 f(\varepsilon \sum_m \varphi_m^* \varphi_m). \]

This means that the average value of $\varepsilon N$ is $\sum_m \varphi_m^* \varphi_m^-$. To make sure that the $N$-th component is not exponentially small, one should set

\[ \sum_m \varphi_m^* \varphi_m = 1. \]

(183)

Let the classical solution to eq.(180) be

\[ \varphi_k^t = \varphi_k e^{-i\Omega t}, \]

(184)
while the $n$-th component of $Y^t$ depend on $t$ as

$$Y_n^t = Y_ne^{-i\Omega t - i\omega t}. $$

(185)

It follows from the relation

$$e^{-i\Omega t} \sum_m a_m^+ a_m^* e^{i\Omega t} \sum_m a_m^+ a_m^* = a_k^+ e^{i\Omega t}$$

that in this case

$$U_{\psi^0,0} = e^{-i\Omega t} \sum_m a_m^+ a_m U_{\psi^0,0} e^{i\Omega t} \sum_m a_m^+ a_m.$$

Eq.(185) means that the vector $e^{i\Omega t} \sum_m a_m^+ a_m^* Y^t$ is time-independent. Therefore, the $N$-th component of the state $U_{\psi^0,0} Y^t$ behaves as $e^{-i(N\Omega + \omega)t}$. Eq.(179) implies $S = (\Omega - H(\varphi^*, \varphi))t$, so that the $N$-th component of the vector (176) depends on $t$ as

$$\psi_N^t = \psi_N^0 e^{-i(NH(\varphi^*, \varphi) + \omega)t}$$

(186)

and gives us an eigenvector of the $N$-particle Hamiltonian with the energy

$$E = NH(\varphi^*, \varphi) + \omega$$

up to $O(1/N)$.

The next observation is that if one chooses the vector $Y$ as

$$Y = \sum_m (a_m^+ \varphi_m + a_m^* \varphi_m + \sqrt{\varepsilon}\varphi_m^* \varphi_m)X,$$

the $N$-th component of the expression (176) will vanish. This is a straightforward corollary of canonical commutation relations (177) and the condition (183). Since the vector $Y$ is defined up to $O(\sqrt{\varepsilon})$, one can conclude that if the more weak condition

$$Y^t = e^{-i\Omega t} \sum_m a_m^+ a_m^* - i\omega t \sum_m (a_m^+ \varphi_m + a_m^* \varphi_m^*)X^t$$

(187)

for some time-dependent vector $X^t$ is satisfied instead of eq.(185), the $N$-particle wave function will be also stationary, so that eq.(186) will be valid up to $O(\sqrt{\varepsilon})$.

In order to avoid appearance of the Fock space vector $X^t$, it is convenient to consider the generalized state vector

$$Z^t = \delta(\sum_m (a_m^+ \varphi_m + a_m^* \varphi_m^*))Y^t.$$

(188)

The vector $Y^t$ is defined from the relation (188) not uniquely but up to the state vector of the form

$$\sum_m (a_m^+ \varphi_m + a_m^* \varphi_m^*)X^t.$$ However, different vectors $Y^t$ obeying eq.(188) determine the same $N$-particle state $P_N U_{\psi^0} Y$ up to $O(\sqrt{\varepsilon})$. This means that the vector $Z^t$ specifies the asymptotic formula for the $N$-particle state uniquely.

Eq.(187) implies that the vector $Z$ should obey the following stationary equation

$$H_2 Z = \omega Z$$

(189)

and the constraint equation:

$$(\sum_m (a_m^+ \varphi_m + a_m^* \varphi_m^*))Z^t = 0.$$ 

(190)

Thus, one should investigate the Schrödinger equation with the quadratic Hamiltonian for the constrained system. Analogously to subsection A.2, introduce new creation and annihilation operators being linear combinations of the operators $a^\pm$:

$$A_k^+ = \sum_l (G^{(k)_l} l^+ - F^{(k)_l} k^-),$$
\[ A_k^- = \sum_l (G_l^{(k)} a_l^- - F_l^{(k)} a_l^+). \]

Such operators obey eq. (167) if the variation system is satisfied:

\[ -\omega_k F_m^{(k)} = \sum_l \left( \frac{\partial^2 H}{\partial \varphi^*_m \partial \varphi_l} - \Omega \delta_{ml} \right) F_l^{(k)} + \sum_l \frac{\partial^2 H}{\partial \varphi_m \partial \varphi_l} G_l^{(k)}, \]

(192)

\[ \omega_k G_m^{(k)} = \sum_l \left( \frac{\partial^2 H}{\partial \varphi^*_m \partial \varphi^*_l} - \Omega \delta_{ml} \right) G_l^{(k)} + \sum_l \frac{\partial^2 H}{\partial \varphi_m \partial \varphi_l} F_l^{(k)}. \]

The ground state vector of eq. (189) is to be found from the conditions:

\[ A_k^- Z = 0. \]

(193)

Eqs. (193) and (190) determine the state uniquely if the set of vectors \( \varphi^*, G^{(1)}, G^{(2)}, \ldots \) is complete. The new vacuum vector will be of the Gaussian type

\[ Z = \exp \left( \frac{1}{2} \sum_{lm} a_l^+ R_{lm} a_m^+ \right) |0> \]

(194)

where the operator \( R \) entering to the quadratic form is defined from the relations:

\[ R \varphi^* = -\varphi, \quad RG^{(k)} = F^{(k)}. \]

The fact that the set \( G^{(1)}, G^{(2)}, \ldots \) may be not complete is important. This means that the non-stationary analog of eq. (192) may have one linearly growing at \( t \) solution. If we considered the more strong relation on \( Y \), eq. (185), we would be unable to construct an asymptotics in such a case.

One of the solutions to eq. (188) on the vector \( Y \) is the following:

\[ Y = \exp \left( \frac{1}{2} \sum_{lm} a_l^+ (R_{lm} + \varphi_l \varphi_m) a_m^+ \right) |0> \]

(195)

One should then ensure that the expression (195) really determines the Fock space vector, i.e. that \( \sum_{mn} |R_{mn}|^2 < \infty \) and \( ||M|| < 1 \), where the matrix of the operator \( M \) is \( M_{mn} = R_{mn} + \varphi_m \varphi_n \). Since \( MG^{(k)} = -F^{(k)} \), one should require that

\[ ||G^{(k)}|| > ||F^{(k)}||, \]

(196)

this rule allows us to select one of two frequencies, \( \omega \) or -\( \omega \).

Excited states are expressed by eq. (170), while their energies have the form (171). We see that the complex-WKB approach allows us to find asymptotic spectrum of the \( N \)-particle Hamiltonian which corresponds to periodic solutions (184) of eq. (180). For more details, see [39].

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