Indefinite Linear-Quadratic Optimal Control of Mean-Field Stochastic Differential Equation With Jump Diffusion: An Equivalent Cost Functional Method

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Abstract—In this article, we consider a linear-quadratic optimal control problem of mean-field stochastic differential equation with jump diffusion, which is also called as an mean-field linear-quadratic problem with jump diffusion (MF-LQJ) problem. Here, cost functional is allowed to be indefinite. We use an equivalent cost functional method to deal with the MF-LQJ problem with indefinite weighting matrices. Some equivalent cost functionals enable us to establish a bridge between indefinite and positive-definite MF-LQJ problems. With such a bridge, solvabilities of stochastic Hamiltonian system and Riccati equations are further characterized. Optimal control of the indefinite MF-LQJ problem is represented as a state feedback via solutions of Riccati equations. As a by-product, the method provides a new way to prove the existence and uniqueness of solution to mean-field forward-backward stochastic differential equation with jump diffusion, where existing methods in literatures do not work. Some examples are provided to illustrate our results.

Index Terms—Equivalent cost functional, existence and uniqueness of solution to mean-field forward-backward stochastic differential equation with jump diffusion (MF-FBSDEJ), indefinite MF-LQJ problem, Riccati equation, stochastic Hamiltonian system.

I. INTRODUCTION

In recent years, there is an increasing interest in mean-field control theory in mathematics, engineering, and finance. Comparing with classical stochastic optimal control, a new feature of this problem is that both objective functional and system dynamics involve state and control as well as their expected values. There are rich literatures on mean-field type optimal control problems, including deriving necessary conditions for optimality (Buckdahn et al. [5], He et al. [14], and Wang et al. [33]) and investigating dynamic programming principle (Bensoussan et al. [3], Lauriere and Pironneau [16], Pham and Wei [25], [26]). Linear quadratic (LQ) problems of mean-field type have also been considered. Yong [36] systematically studied an LQ problem of mean-field stochastic differential equation (MF-SDE). Elliott et al. [12] dealt with an LQ problem of MF-SDE with discrete time setting. Barreiro-Gomez et al. [1], [2] investigated LQ mean-field type games.

It is well known that jump diffusion processes play an increasing role in describing stochastic dynamical systems, due to its wide applications in financial, economic and engineering problems. For example, a geometric Brownian motion is usually used to model stock price, but it cannot reflect discontinuous characteristics, which may be induced by large fluctuations. There are various literatures on related topic. The interested readers may refer to Haadem et al. [13], Øksendal and Sulem [24], and Shen et al. [27] for more information. Haadem et al. [13] obtained a maximum principle for jump diffusion process with infinite horizon and dealt an optimal portfolio selection problem. Shen et al. [27] investigated stochastic maximum principle of mean-field jump diffusion process with delay and applied their results to a bicriteria mean-variance portfolio selection problem. Øksendal and Sulem [24] systematically discussed optimal control, optimal stopping, and impulse control of jump diffusion processes.

In this article, a kind of indefinite LQ problem of mean-field type jump diffusion process is investigated. Indefinite LQ problems, first studied by Chen et al. [9], have received considerable attention. Chen et al. [10] employed the method of completion of squares to study an indefinite stochastic LQ problem. Huang and Yu [13] and Yu [37] proposed an equivalent cost functional method to deal with stochastic LQ problems with indefinite weighting matrices. Ni et al. [22], [23] considered indefinite LQ problems of discrete-time MF-SDE for an infinite horizon and a finite horizon, respectively. Sun [28] and Li et al. [20] concerned open-loop and closed-loop solvabilities for LQ problems of MF-SDE, respectively. Li et al. [18] studied an indefinite
LQ problem of MF-SDE by introducing a relax compensator. Li et al. [17], Mukaidani et al. [21], Wang and Zhang [31], Wang et al. [32] investigated indefinite LQ mean-field games and social optimal control problems for various models. Indefinite mean-field linear-quadratic problem with jump diffusion (MF-LQJ) problems, which are natural generalizations of those in [29] and [30], have been not yet completely studied. Tang and Meng [30] investigated a definite MF-LQJ problem in finite horizon and derived two Riccati equations by decoupling optimality system. It was shown that under Assumption (S) given in Section III with $S_t \equiv 0$, $S_t \equiv 0$, these two Riccati equations are uniquely solvable and a feedback representation for optimal control is obtained. We point out that Assumption (S) is exactly the definite condition when we study an MF-LQJ problem. The MF-LQJ problem reduces to an NC-LQJ problem if $S_t \equiv 0$, $S_t \equiv 0$, where “NC” is the capital initials for “no cross.” Tang et al. [29] focused on open-loop and closed-loop solvabilities of an MF-LQJ problem, which extended results in [20] and [28]. However, the solvabilities of related Riccati equations without Assumption (S) have not been specified.

Inspired by [15] and [37], we use an equivalent cost functional method to deal with an indefinite MF-LQJ problem. As a preliminary result, we discuss a definite MF-LQJ problem. As mentioned above, the results obtained in [30] are not applicable for solving this MF-LQJ problem. We introduce an invertible linear transformation, which links the MF-LQJ problem with the corresponding NC-LQJ problem. Combining the results in existing literatures with this linear transformation, we obtain an optimal control of the MF-LQJ problem under Assumption (S). Then, we introduce two auxiliary variables to construct equivalent cost functionals. The original MF-LQJ problem with indefinite control weighting matrices is transformed into an MF-LQJ problem under Assumption (S). In a word, we can investigate the indefinite MF-LQJ problem by using this method.

Our article distinguishes from existing literatures in the following aspects.

1) An indefinite MF-LQJ problem is discussed in this article, which generalized the results in [18], [29], and [30]. The considered model could characterize more general problems and the jump diffusion item is important in some controlled dynamics systems. As we will see in Example 5.1, there is no equivalent cost functional satisfying Assumption (S) if jump diffusion item disappears, which implies that we cannot construct an optimal control directly in terms of Riccati equations. We further discussed the existence and uniqueness of solutions to the corresponding stochastic Hamiltonian system and Riccati equations without Assumption (S), which have not been considered in [29] and [30].

2) Compared with the derivation of optimal control for the definite NC-LQJ problem in [30], we derive a feedback control of “Problem MF” under Assumption (S) through a simple calculation. Actually, we introduce an invertible linear transformation, which links MF-LQJ problem with the corresponding NC-LQJ problem.

3) Our results provide an alternative and effective way to obtain the solvability of an MF-FBSDEJ, which does not satisfy classical conditions in existing literatures. In fact, when we consider two equivalent cost functionals with the same control system, we can get the equivalence by an invertible linear transformation between the corresponding stochastic Hamiltonian systems. We point out that the equivalence is existed in a family of stochastic Hamiltonian systems. Therefore, we can prove the solvability of a more general MF-FBSDEJ. Moreover, sometimes an MF-FBSDEJ may coincide with the stochastic Hamiltonian system of an MF-LQJ problem, which implies that the solvability of MF-FBSDEJ is actually the solvability of corresponding stochastic Hamiltonian system. Thus, in order to obtain the unique solvability of an MF-FBSDEJ, we need only to find an equivalent cost functional satisfying Assumption (S) of the related MF-LQJ problem.

4) Relying on the equivalent cost functional method, we can investigate the solvability of indefinite Riccati equations by virtue of the solvability of positive definite Riccati equations. In fact, the original MF-LQJ problem with indefinite control weighting matrices can be transformed into a definite MF-LQJ problem by looking for a simpler and more flexible equivalent cost functional. And there exists an equivalent relation between the corresponding Riccati equations. Similarly, we can get solvabilities of Riccati equations with indefinite conditions.

The rest of this article is organized as follows. In Section II, we formulate an MF-LQJ problem and give some assumptions throughout this article. Section III aims to study the MF-LQJ problem under Assumption (S). We reduce a general MF-LQJ problem to an NC-LQJ problem via an invertible linear transformation. In Section IV, we present our main results. We use the equivalent cost functional method to study an MF-LQJ problem with indefinite weighting matrices. Section V gives several illustrative examples. Finally, Section VI concludes this article.

II. PROBLEM FORMULATION

Let $\mathbb{R}^{n \times m}$ be an Euclidean space of all $n \times m$ real matrices with inner product $\langle \cdot, \cdot \rangle$ being given by $\langle M, N \rangle \rightarrow tr(M^T N)$, where the superscript $\top$ denotes the transpose of vectors or matrices. The induced norm is given by $|M| = \sqrt{tr(M^T M)}$. In particular, we denote by $S^n$ the set of all $n \times n$ symmetric matrices. We mean by an $n \times n$ matrix $N \geq 0$ that $N$ is a nonnegative matrix. Let $T > 0$ be a fixed time horizon and $(\Omega, F, F, \mathbb{P})$ be a complete filtered probability space. The filtration $\mathbb{F} \equiv \{ F_t \}_{t \geq 0}$ is generated by the following two mutually independent processes, augmented by all the $\mathbb{P}$-null sets: A standard 1-D Brownian motion $W_t$ and a Poisson random measure $N(dt, d\theta)$ on $\mathbb{R}^+ \times \Theta$, where $\Theta \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty set, with compensator $\tilde{N}(dt, d\theta) = \nu(d\theta)dt$, such that $\tilde{N}([0, t], \cdot) = N([0, t], A) - \tilde{N}([0, t], \cdot)$ is a martingale for all $A \in \mathcal{B}(\Theta)$ satisfying $\nu(A) < \infty$. $\mathcal{B}(\Theta)$ is the Borel $\sigma$-field generated by $\Theta$. Here, $\nu(d\theta)$ is a $\sigma$-finite measure on $(\Theta, \mathcal{B}(\Theta))$ satisfying $\int_{\Theta} (1 \wedge \theta^2) \nu(d\theta) < \infty$, which is called the characteristic measure. Then, $\tilde{N}(dt, d\theta) = \nu(d\theta)dt$.
\(N(dt, d\theta) - \nu(d\theta)dt\) is the compensated Poisson random measure. For any Euclidean space \(M\), we introduce the following spaces: \(L_2^p(M) = \{\xi : \Omega \rightarrow M | \xi \in \mathcal{F}_0\text{-measurable random vector, } \mathbb{E}[|\xi|^2] < \infty\}; L_\infty^p(0, T; M) = \{u : [0, T] \rightarrow M|u is a bounded function\}; \(L_2^p(0, T; M) = \{u : [0, T] \times \Omega \rightarrow M | u is an \mathcal{F}_0\text{-adapted stochastic process, } \mathbb{E}[\int_0^T |u|^2 dt] < \infty\}; S_2^p(0, T; M) = \{u : [0, T] \times \Omega \rightarrow M | u is an \mathcal{F}_0\text{-adapted càdlàg process in } L_2^p(0, T; M)\text{ such that } \mathbb{E}[\sup_{t \in [0, T]} |u|^2] < \infty\}; \(L_\infty^p(M) = \{r : [0, T] \times \Theta \rightarrow M | r is a deterministic function, } \sup_{t \in [0, T]} |r(t, \theta)|^2 dt \rightarrow 0\); \(L_{2, \nu'}(0, T; M) = \{r : [0, T] \times \Theta \rightarrow M | r is an \mathcal{F}_0\text{-predictable stochastic process } \mathbb{E}\left[\int_0^T |r(t, \theta)|^2 \nu(d\theta)dt\right] < \infty\).

Consider a controlled linear MF-SDE

\[
\begin{align*}
\dot{d}X_t = & \left\{ A_tX_t + \tilde{A}_t\mathbb{E}[X_t] + B_tu_t + \tilde{B}_t\mathbb{E}[u_t] \right\} dt \\
&+ \left\{ C_tX_t + \tilde{C}_t\mathbb{E}[X_t] + D_tu_t + \tilde{D}_t\mathbb{E}[u_t] \right\} dW_t \\
&+ \int_{\mathcal{F}_0} \left\{ E_{t, \theta}X_{t-} + \tilde{E}_{t, \theta}\mathbb{E}[X_{t-}] + F_{t, \theta}u_t \right\} d\Theta_t
\end{align*}
\]

where \(A_t, \tilde{A}_t, B_t, \tilde{B}_t, C_t, \tilde{C}_t, D_t, \tilde{D}_t, E_{t, \theta}, \tilde{E}_{t, \theta}, F_{t, \theta}, \tilde{F}_{t, \theta}\) are given matrix valued deterministic functions. In the above equation, \(u\), valued in \(\mathbb{R}^m\), is a control process and \(X\), valued in \(\mathbb{R}^n\), is the corresponding state process. In this article, an admissible control \(u\) is defined as a predictable process such that \(u \in L_2^p(0, T; \mathbb{R}^m)\). The set of all admissible control processes is denoted by \(U(0, T)\). We introduce a cost functional

\[
J[u] = \frac{1}{2} \mathbb{E}\left\{ \langle GX_T, X_T \rangle + \langle G\mathbb{E}[X_T], \mathbb{E}[X_T] \rangle \right\}
+ \int_0^T \left\{ \langle Q_{t, \theta} X_t + \tilde{Q}_{t, \theta} \mathbb{E}[X_t], X_t \rangle + \tilde{Q}_{t, \theta} \mathbb{E}[X_t], u_t \rangle \right\} dt
+ \int_0^T \left\{ \langle \tilde{R}_{t, \theta} X_t, \mathbb{E}[u_t] \rangle + \tilde{R}_{t, \theta} \mathbb{E}[u_t], u_t \rangle \right\} dt
\]

(2)

where \(G, \tilde{G}\) are symmetric matrices and \(Q_{t, \theta}, \tilde{Q}_{t, \theta}, R_{t, \theta}, \tilde{R}_{t, \theta}\) are deterministic matrix-valued functions with \(Q_t = Q_t^\top, R_t = R_t^\top, \tilde{Q}_t = \tilde{Q}_t^\top, \tilde{R}_t = \tilde{R}_t^\top\). Our MF-LQJ problem is stated as follows.

**Problem MF:** For any \(\xi \in L_2^p(\mathbb{R}^n)\), find a \(u^* \in U(0, T)\) such that

\[
J[u^*] = \inf_{u \in U(0, T)} J[u].
\]

Any \(u^* \in U(0, T)\) satisfying (3) is called an optimal control of Problem MF, and the corresponding state process \(X^* = X(\xi, u^*)\) is called an optimal state process. \((X^*, u^*)\) is called an optimal pair.

The following assumptions will be in force throughout this article.

**H1:** The coefficients of state equation satisfy

\[
\begin{align*}
A, \tilde{A}, C, \tilde{C} & \in L_\infty^p(0, T; \mathbb{R}^{n \times n}) , \\
B, \tilde{B}, D, \tilde{D} & \in L_\infty^p(0, T; \mathbb{R}^{n \times m}) , \\
E, \tilde{E} & \in L_2^p(\mathbb{R}^{n \times n}), \quad F, \tilde{F} \in L_2^p(\mathbb{R}^{n \times m}).
\end{align*}
\]

**H2:** The weighting matrices in cost functional satisfy

\[
\begin{align*}
Q, \tilde{Q} & \in L_\infty^p(0, T; \mathbb{R}^n) , \\
R, \tilde{R} & \in L_\infty^p(0, T; \mathbb{R}^m) , \\
S, \tilde{S} & \in L_\infty^p(0, T; \mathbb{R}^{n \times m}), \quad G, \tilde{G} \in \mathbb{R}^n.
\end{align*}
\]

We can show that under (H1), for any \(u \in U(0, T)\), (1) admits a unique solution \(X = X(\xi, u) \in S_2^p(0, T; \mathbb{R}^n)\) (see Lemma I.2 in Tang and Meng [30]). For simplicity of notations, we define \(\Delta = \Delta + \Delta\) with \(\Delta = A, B, C, D, E, F, Q, S, R, G\).

**III. MF-LQJ Problem under Standard Conditions**

In this section, we aim at studying Problem MF under standard conditions. We introduce an invertible linear transformation, which links MF-LQJ problem with the corresponding NC-LQJ problem.

**Assumption (S):** There exists \(\alpha_0 > 0\), such that

\[
\begin{align*}
R_t, \tilde{R}_t & \geq \alpha_0 I, Q_t - S_tR_t^{-1}S_t^\top \geq 0, \\
Q_t - \tilde{S}_t\tilde{R}_t^{-1}\tilde{S}_t^\top & \geq 0, \quad t \in [0, T], \quad G, \tilde{G} \geq 0.
\end{align*}
\]

We introduce a stochastic Hamiltonian system related to Problem MF

\[
\begin{align*}
\dot{d}X_t = & \left\{ A_tX_t + \tilde{A}_t\mathbb{E}[X_t] + B_t\mathbb{E}[u_t] + \tilde{B}_t\mathbb{E}[u_t] \right\} dt \\
&+ \left\{ C_tX_t + \tilde{C}_t\mathbb{E}[X_t] + D_t\mathbb{E}[u_t] + \tilde{D}_t\mathbb{E}[u_t] \right\} dW_t \\
&+ \int_{\mathcal{F}_0} \left\{ E_{t, \theta}X_{t-} + \tilde{E}_{t, \theta}\mathbb{E}[X_{t-}] + F_{t, \theta}\mathbb{E}[u_t] \right\} d\Theta_t
\end{align*}
\]

\[
\begin{align*}
\dot{d}Y_t = & - \left\{ A_t^\top Y_t - \tilde{A}_t^\top\mathbb{E}[Y_t] + C_t^\top Z_t + \tilde{C}_t^\top\mathbb{E}[Z_t] \right\} dt \\
&+ \int_{\mathcal{F}_0} \left\{ E_{t, \theta}^\top r_{t, \theta} + \tilde{E}_{t, \theta}^\top\mathbb{E}[r_{t, \theta}] \right\} \nu(d\theta) dt
\end{align*}
\]

Using the method in [30], we decouple the above stochastic Hamiltonian system and derive Riccati equations associated with Problem MF

\[
\begin{align*}
\dot{P}_t + P_tA + A_t^\top P_t + C_t^\top P_tC_t + \int_{\mathcal{F}_0} E_{t, \theta}^\top P_t E_{t, \theta}\nu(d\theta) + Q_t - L_t\Sigma_t^{-1}L_t^\top = 0, \\
P_T = G,
\end{align*}
\]

\[
\begin{align*}
\dot{\Pi}_t + \Pi_tA + A_t^\top \Pi_t + C_t^\top P_tC_t + \int_{\mathcal{F}_0} E_{t, \theta}^\top P_t E_{t, \theta}\nu(d\theta) + Q_t - L_t\Sigma_t^{-1}L_t^\top = 0, \\
\Pi_T = \tilde{G},
\end{align*}
\]

where

\[
\begin{align*}
L_t = & \tilde{S}_t + P_tB_t + C_t^\top P_tD_t + \int_{\mathcal{F}_0} E_{t, \theta}^\top P_t F_{t, \theta}\nu(d\theta), \\
\Sigma_t = & R_t + D_t^\top P_t D_t + \int_{\mathcal{F}_0} F_{t, \theta}^\top P_t F_{t, \theta}\nu(d\theta).
\end{align*}
\]
\[ L_t = \bar{S}_t + \Pi_t \bar{B}_t + \bar{C}_t^T P_t \bar{D}_t + \int_\Theta \bar{E}_{t,\theta}^T P_t \bar{F}_{t,\theta} \nu(d\theta), \]
\[ \bar{\Sigma}_t = \bar{R}_t + \bar{D}_t^T P_t \bar{D}_t + \int_\Theta \bar{F}_{t,\theta}^T P_t \bar{F}_{t,\theta} \nu(d\theta). \]

Note that (4) is a coupled MF-FBSDEJ, where the coupling comes from the last relation (which is essentially the maximum condition in the usual Pontryagin type maximum principle). Different from an NC-LQI problem, there are additional items 2 \{X_t, S_t u_t\} and 2 \{E[X_t], \bar{S}_t E[u_t]\} in cost functional (2). Next, we want to reduce Problem MF to an NC-LQI problem. For this, we introduce a controlled system

\[
\begin{aligned}
d\bar{X}_t &= (A_{1t} \bar{X}_t + B_t \bar{u}_t + \bar{A}_{1t} E[X_t] + \bar{B}_t E[u_t]) dt \\
&\quad + (C_{1t} \bar{X}_t + D_t \bar{u}_t + C_{1t} E[X_t] + D_t E[u_t]) dW_t \\
&\quad + \int_\Theta (E_{t,\theta}^T \bar{X}_t + F_{t,\theta} \bar{u}_t) \bar{N}(d\theta, d\theta), \\
\bar{X}_0 &= \xi
\end{aligned}
\]

and a cost functional

\[
J[\bar{\nu}] = \frac{1}{2} \mathbb{E} \left\{ \langle G \bar{X}_T, \bar{X}_T \rangle + \langle GE[X_T], E[X_T] \rangle \right\}
\]
\[
+ \int_0^T \left\{ \left( \begin{array}{c}
Q_{1t} & 0_{n \times m} \\
0_{m \times n} & R_t
\end{array} \right) \left( \begin{array}{c}
\bar{X}_t \\
\bar{u}_t
\end{array} \right) \right\} dt
\]
\[
+ \int_0^T \left\{ \left( \begin{array}{c}
Q_{1t} & 0_{n \times m} \\
0_{m \times n} & R_t
\end{array} \right) \left( \begin{array}{c}
E[X_t] \\
E[u_t]
\end{array} \right) \right\} dt \right\}
\]

where

\[
A_{1t} = A_t - B_t R_t^{-1} S_t^T, \quad \bar{A}_{1t} = \bar{A}_t - \bar{B}_t \bar{R}_t^{-1} \bar{S}_t^T - A_{1t},
\]
\[
C_{1t} = C_t - D_t R_t^{-1} S_t^T, \quad \bar{C}_{1t} = \bar{C}_t - \bar{D}_t \bar{R}_t^{-1} \bar{S}_t^T - C_{1t},
\]
\[
E_{t,\theta,\theta} = E_{t,\theta} - F_{t,\theta} R_t^{-1} S_t^T,
\]
\[
\bar{E}_{t,\theta,\theta} = \bar{E}_{t,\theta} - \bar{F}_{t,\theta} \bar{R}_t^{-1} \bar{S}_t^T - E_{1t,\theta},
\]
\[
Q_{1t} = Q_t - S_t R_t^{-1} S_t^T, \quad \bar{Q}_{1t} = \bar{Q}_t - \bar{S}_t \bar{R}_t^{-1} \bar{S}_t^T - Q_{1t}.
\]

The corresponding NC-LQI problem is stated as follows.

**Problem NC**: For any \( \xi \in L_2^2(\mathbb{R}^n) \), find a \( \bar{\nu}^* \in \mathcal{U}[0, T] \) such that

\[ J[\bar{\nu}^*] = \inf_{\bar{\nu} \in \mathcal{U}[0, T]} J[\bar{\nu}]. \]

Similar to Problem MF, we write the stochastic Hamiltonian system and Riccati equations corresponding to Problem NC

\[
\begin{aligned}
d\bar{X}_t &= \left\{ A_{1t} \bar{X}_t + \bar{A}_{1t} E[X_t] + B_t \bar{u}_t + \bar{B}_t E[u_t] \right\} dt \\
&\quad + \left\{ C_{1t} \bar{X}_t + \bar{C}_{1t} E[X_t] + D_t \bar{u}_t + D_t E[u_t] \right\} dW_t \\
&\quad + \int_\Theta \left\{ E_{t,\theta}^T \bar{X}_t + F_{t,\theta} \bar{u}_t \right\} \bar{N}(d\theta, d\theta), \\
\bar{X}_0 &= \xi
\end{aligned}
\]

\[
\begin{aligned}
d\bar{Y}_t &= \left\{ A_{1t} \bar{Y}_t + \bar{A}_{1t} E[Y_t] + C_{1t} \bar{Z}_t + C_{1t}^T \bar{Z}_t \right\} dt \\
&\quad + \int_\Theta \left\{ E_{t,\theta}^T \bar{Y}_t + F_{t,\theta} \bar{Z}_t \right\} \bar{N}(d\theta, d\theta), \\
\bar{X}_0 &= \xi
\end{aligned}
\]

**Lemma 3.1**: Let Assumption (S) hold. For any two pairs \((X, u)\) and \((\bar{X}, \bar{u})\), we introduce a linear transformation

\[
\begin{aligned}
(X - E[X]) &= \left( \begin{array}{c}
I_{n \times n} \\
0_{n \times m}
\end{array} \right) \left( \begin{array}{c}
\bar{X} - E[\bar{X}] \\
\bar{u} - E[\bar{u}]
\end{array} \right), \\
Y &= \left( \begin{array}{c}
0_{n \times n} \\
R^{-1} S^T
\end{array} \right) \left( \begin{array}{c}
\bar{X} - E[\bar{X}] \\
\bar{u} - E[\bar{u}]
\end{array} \right),
\end{aligned}
\]

Then, the following two statements are equivalent:

1. \((X, u)\) is an admissible (optimal) control of Problem MF.
2. \((\bar{X}, \bar{u})\) is an admissible (optimal) control of Problem NC.

Moreover, we have \(J[\bar{u}] = J[\bar{\nu}]\).

**Proof**: It follows from (13) and (14) that

\[
\begin{aligned}
\bar{X} - E[\bar{X}] &= \left( \begin{array}{c}
I_{n \times n} \\
0_{n \times m}
\end{array} \right) \left( \begin{array}{c}
X - E[X] \\
u - E[u]
\end{array} \right), \\
\bar{X} - E[\bar{X}] &= \left( \begin{array}{c}
I_{n \times n} \\
0_{n \times m}
\end{array} \right) \left( \begin{array}{c}
X - E[X] \\
\bar{u} - E[\bar{u}]
\end{array} \right),
\end{aligned}
\]
Then, linear transformation (13) with (14) is invertible. Through direct calculations, it is easy to verify statement 1) is equivalent to statement 2), and thus \( J[u] = J[\tilde{u}] \).

The above lemma tells us that there exists some equivalent relationship between Problem MF and Problem NC. We now analyze the relationship in terms of stochastic Hamiltonian system and Riccati equations, respectively.

**Lemma 3.2:** Under Assumption (S), \((\bar{X}^*, \bar{u}^*, Y, Z, r)\) is the solution of (10), if and only if \((X^*, u^*, Y, Z, r)\) is the solution of (4).

**Proof:** According to Lemma 1, it is not difficult to draw the conclusion.

**Lemma 3.3:** Under Assumption (S), we have
1) Riccati equations (11) and (5) are the same.
2) Riccati equations (12) and (6) are the same.

**Proof:** For simplicity of notations, we denote

\[
G(A, B, C, D, E, F; Q, R, S; P) = \begin{cases}
G(A_1, B, C_1, D_1, E_1, F_1; Q_1, R_0; P_0)
\end{cases}
\]

It is easy to prove \( G(A, B, C, D, E, F; Q, R, S; P) = G(A_1, B, C_1, D_1, E_1, F_1; Q_1, R_0; P_0) \). It is easy to see that \( \Sigma_t^{-1}L_t^1 - \Sigma_t^{-1}L_t^{1*} \). Thus, we calculate

\[
G(A, B, C, D, E, F; Q, R, S; P)
\]

Consequently, we complete the proof.

We cite a result in [30], which plays a role in deriving an optimal control of Problem MF under standard conditions.

**Lemma 3.4:** If (H1)–(H2) and Assumption (S) hold, then Riccati equations (11)–(12) admit unique solutions \( P \geq 0, \Pi \geq 0 \), respectively. Further, the optimal pair \((\hat{X}^*, \hat{u}^*)\) of Problem MF satisfies

\[
\begin{align*}
\hat{X}^*_t &= \left\{ \begin{array}{ll}
0_{n \times n} \\
-\Sigma_t^{-1}L_t^1(\bar{X}^*_t - E[\bar{X}^*_t]) - \Sigma_t^{-1}L_t^{1*}E[\bar{X}^*_t] \\
d\bar{X}^*_t &= \left\{ \begin{array}{ll}
0_{n \times n} \\
A_t\bar{X}^*_t + A_t\bar{E}[\bar{X}^*_t] + B_t\bar{u}^*_t + B_tE[\bar{u}^*_t] \\
+ C_t\bar{X}^*_t + C_tE[\bar{X}^*_t] + D_t\bar{u}^*_t + D_tE[\bar{u}^*_t] \\
+ E_t\bar{X}^*_t + E_tE[\bar{X}^*_t] \\
+ F_t\bar{u}^*_t + F_tE[\bar{u}^*_t] \\
\{ \end{array} \right. \\
X^*_0 &= \xi.
\end{align*}
\]

Defining
\[
\begin{align*}
\hat{Y}^*_t &= P_t(\hat{X}^*_t - E[\hat{X}^*_t]) + \Pi_tE[\hat{X}^*_t], \\
\hat{Z}^*_t &= \begin{cases}
0_{n \times n} \\
-\Sigma_t^{-1}L_t^1(\bar{X}^*_t - E[\bar{X}^*_t]) \\
+ P_tC_t - P_tD_t\Sigma_t^{-1}L_t^{1*}E[\bar{X}^*_t] \end{cases} \text{ and } \begin{cases}
\hat{r}^*_t \equiv 0_{n \times n} \\
-\Sigma_t^{-1}L_t^1(\bar{X}^*_t - E[\bar{X}^*_t]) \end{cases}
\end{align*}
\]

the five-tuple \((\hat{X}^*, \hat{u}^*, \hat{Y}^*, \hat{Z}^*, \hat{r}^*)\) is the unique solution to MF-FBSDE (10). Moreover

\[
\inf_{\hat{u} \in L^2[0, T]} J[\hat{u}] = \frac{1}{2}(P_0(0,0) - E[\xi, \xi] + E[\xi, \xi])
\]

Using the above lemmas, we obtain a main result of this section.

**Theorem 3.1:** If (H1)–(H2) and Assumption (S) hold, then Riccati equations (5) and (6) admit unique solutions \( P \geq 0, \Pi \geq 0 \), respectively. Further, the optimal pair \((X^*, u^*)\) of Problem MF satisfies

\[
\begin{align*}
\begin{cases}
\hat{X}^*_t &= \left\{ \begin{array}{ll}
0_{n \times n} \\
A_tX_t^* + A_tE[X_t^*] + B_tu_t^* + B_tE[u_t^*] \\
+ C_tX_t^* + C_tE[X_t^*] + D_tu_t^* + D_tE[u_t^*] \\
+ E_tX_t^* + E_tE[X_t^*] \\
+ F_tu_t^* + F_tE[u_t^*] \\
\{ \end{array} \right. \\
x_0^* &= \xi.
\end{cases}
\end{align*}
\]

Defining

\[
\begin{align*}
\hat{Y}^*_t &= P_t(X_t^* - E[X_t^*]) + \Pi_tE[X_t^*], \\
\hat{Z}^*_t &= \begin{cases}
0_{n \times n} \\
-\Sigma_t^{-1}L_t^1(X_t^* - E[X_t^*]) \end{cases} \text{ and } \begin{cases}
\hat{r}^*_t \equiv 0_{n \times n} \\
-\Sigma_t^{-1}L_t^1(X_t^* - E[X_t^*]) \end{cases}
\end{align*}
\]

the five-tuple \((X^*, u^*, Y^*, Z^*, r^*)\) is the unique solution to MF-FBSDE (4). Moreover

\[
\inf_{u \in L^2[0, T]} J[u] = \frac{1}{2}(P_0(0,0) - E[\xi, \xi] + E[\xi, \xi])
\]

**Proof:** We only need to prove the five-tuple \((X^*, u^*, Y^*, Z^*, r^*)\) is the unique solution to MF-FBSDE (4). By the linear transformation introduced in Lemma 1 and the representation of optimal control \(\hat{u}^*\) for Problem NC in Lemma 4, we get

\[
\begin{align*}
u_t^* &= u_t^* - E[u_t^*] + E[u_t^*] \\
&= -R_t^{-1}S_t^1(\hat{X}_t^* - E[\hat{X}_t^*]) + \hat{u}_t^* + E[\hat{u}_t^*] \\
&= -R_t^{-1}S_t^1E[\hat{X}_t^*] + E[\hat{u}_t^*] \\
&= -R_t^{-1}S_t^1(X_t^* - E[X_t^*]) - \Sigma_t^{-1}L_t^1(X_t^* - E[X_t^*]) \\
&= -R_t^{-1}S_t^1E[X_t^*] - \Sigma_t^{-1}L_t^1E[X_t^*] \\
&= -\Sigma_t^{-1}L_t^1(X_t^* - E[X_t^*]) - \Sigma_t^{-1}L_t^1E[X_t^*].
\end{align*}
\]
Using Lemma 3.2 and the representation of $\tilde{Y}^*, \tilde{Z}^*, \tilde{r}^*$ in Lemma 3.4, we have
\[
Y_t^* = \tilde{Y}_t^* = P_t(\tilde{X}_t^* - \mathbb{E}[\tilde{X}_t^*]) + \Pi_t \mathbb{E}[\tilde{X}_t^*]
\]
\[
= P_t(X_t^* - \mathbb{E}[X_t^*]) + \Pi_t \mathbb{E}[X_t^*],
\]
\[
Z_t^* = \tilde{Z}_t^* = [P_tC_t - P_tD_t \Sigma_t^{-1} L_t^*] (\tilde{X}_t^* - \mathbb{E}[\tilde{X}_t^*])
\]
\[
+ [P_t \tilde{C}_t - P_t \tilde{D}_t \Sigma_t^{-1} \tilde{L}_t^*] \mathbb{E}[X_t^*]
\]
\[
= \{P_t (C_t - D_t R_t^{-1} S_t^T) - P_tD_t \Sigma_t^{-1} L_t^* - R_t^{-1} S_t^T\}
\]
\[
\cdot (X_t^* - \mathbb{E}[X_t^*]) + \{P_t (\tilde{C}_t - \tilde{D}_t \tilde{R}_t^{-1} \tilde{S}_t^T)
\]
\[
- P_t \tilde{D}_t \Sigma_t^{-1} L_t^* - R_t^{-1} \tilde{S}_t^T\}
\]
\[
\cdot (\tilde{X}_t^* - \mathbb{E}[\tilde{X}_t^*])
\]
\[
+ [P_t \tilde{C}_t - P_t \tilde{D}_t \Sigma_t^{-1} \tilde{L}_t^*] \mathbb{E}[X_t^*].
\]

We can also obtain that $r_t^*$ satisfies (15) similarly. Then, the proof is completed.

IV. INDEFINITE MF-LQJ PROBLEM

In the case that Assumption (S) does not hold true, it is possible that Problem MF is well-posed and an optimal pair exists. In this section, we will apply the equivalent cost functional method to deal with Problem MF without Assumption (S).

Definition 4.1: For a given controlled system, if there exist two cost functionals $J$ and $\tilde{J}$ satisfying: For any admissible controls $u^1$ and $u^2$, $J[u^1] < J[u^2]$ if and only if $\tilde{J}[u^1] < \tilde{J}[u^2]$, then we say $J$ is equivalent to $\tilde{J}$.

Remark 4.1: The following two statements are equivalent.

1) Cost functional $J$ is equivalent to $\tilde{J}$.

2) For any admissible controls $u^1$ and $u^2$.
   a) $J[u^1] < J[u^2]$ if and only if $\tilde{J}[u^1] < \tilde{J}[u^2]$.
   b) $J[u^1] = J[u^2]$ if and only if $\tilde{J}[u^1] = \tilde{J}[u^2]$.
   c) $J[u^1] > J[u^2]$ if and only if $\tilde{J}[u^1] > \tilde{J}[u^2]$.

We denote
\[
\Phi = \{\varphi \in L^\infty (0, T; \mathbb{S}^n) | \varphi \text{ is a deterministic continuous differential function}\}.
\]

Inspired by Theorem 3.1, for any $H, K \in \Phi$, we introduce a family of cost functionals
\[
J^{HK}[u] = J[u] - \frac{1}{2} \mathbb{E} \left[ \langle H_0 (\xi - \mathbb{E}[\xi]), \xi - \mathbb{E}[\xi] \rangle + \langle K_0 \mathbb{E}[\xi], \mathbb{E}[\xi] \rangle \right]
\]
where $H_0$ and $K_0$ are the initial values of $H_t$ and $K_t$, respectively. Since $J^{HK}[u]$ and $J[u]$ differ by only a constant $\frac{1}{2} \mathbb{E}[\langle H_0 (\xi - \mathbb{E}[\xi]), \xi - \mathbb{E}[\xi] \rangle + \langle K_0 \mathbb{E}[\xi], \mathbb{E}[\xi] \rangle]$, they are equivalent. In other words, we get a family of equivalent cost functionals $\{J^{HK}[u]\}$, which includes the original cost functional $J[u]$ when $H = 0, K = 0$. For the sake of convenience, given any $H, K \in \Phi$, we call Problem MF “Problem $J^{HK}$” if $J[u]$ is replaced by $J^{HK}[u]$. According to Definition 1, we see that Problem $J^{HK}$ and Problem MF share a common optimal control. Applying Itô formula to $\langle H_t (X_t - \mathbb{E}[X_t]), X_t - \mathbb{E}[X_t] + K_t \mathbb{E}[X_t], \mathbb{E}[X_t] \rangle$ on the interval $[0, T]$, we derive
\[
J^{HK}[u]
\]
\[
= \frac{1}{2} \mathbb{E} \left( \langle G^{HK} X_T, X_T \rangle + \langle G^{HK} \mathbb{E}[X_T], \mathbb{E}[X_T] \rangle \right)
\]
\[
+ \int_0^T \langle \left( \begin{array}{c} Q_t^{HK} \\ S_t^{HK} \end{array} \right) \left( \begin{array}{c} X_t \\ u_t \end{array} \right), \left( \begin{array}{c} X_t \\ u_t \end{array} \right) \rangle dt
\]
\[
+ \int_0^T \langle \left( \begin{array}{c} \tilde{Q}_t^{HK} \\ \tilde{S}_t^{HK} \end{array} \right) \left( \begin{array}{c} \mathbb{E}[X_t] \\ \mathbb{E}[u_t] \end{array} \right), \left( \begin{array}{c} \mathbb{E}[X_t] \\ \mathbb{E}[u_t] \end{array} \right) \rangle dt \right)
\]
where
\[
Q_t^{HK} = Q_t + \tilde{H}_t + H_t A_t + A_t^T H_t + C_t^T H_t C_t
\]
\[
+ \int_\Theta \tilde{E}_t,0 H_t \tilde{E}_t,0 \nu(d\theta),
\]
\[
S_t^{HK} = S_t + H_t B_t + C_t^T H_t D_t + \int_\Theta \tilde{E}_t,0 H_t F_t,0 \nu(d\theta),
\]
\[
R_t^{HK} = R_t + D_t^T H_t D_t + \int_\Theta F_t,0 H_t F_t,0 \nu(d\theta),
\]
\[
G^{HK} = G - H_T,
\]
\[
\tilde{Q}_t^{HK} = \tilde{Q}_t^{HK} - Q_t^{HK}, \tilde{S}_t^{HK} = \tilde{S}_t^{HK} - S_t^{HK}, \tilde{R}_t^{HK} = \tilde{R}_t^{HK} - R_t^{HK}, \tilde{G}^{HK} = \tilde{G}^{HK} - G^{HK}
\]
with
\[
\tilde{Q}_t^{HK} = \tilde{Q}_t + \tilde{K}_t + K_t \tilde{A}_t + \tilde{A}_t^T K_t + \tilde{C}_t^T H_t \tilde{C}_t
\]
\[
+ \int_\Theta \tilde{E}_t,0 H_t \tilde{E}_t,0 \nu(d\theta),
\]
\[
\tilde{S}_t^{HK} = \tilde{S}_t + K_t \tilde{B}_t + \tilde{C}_t^T H_t \tilde{D}_t + \int_\Theta \tilde{E}_t,0 H_t \tilde{F}_t,0 \nu(d\theta),
\]
\[
\tilde{R}_t^{HK} = \tilde{R}_t + \tilde{D}_t^T H_t \tilde{D}_t + \int_\Theta \tilde{F}_t,0 H_t \tilde{F}_t,0 \nu(d\theta),
\]
\[
\tilde{G}^{HK} = \tilde{G} - K_T.
\]

Different from the original cost functional $J[u]$, the weighting matrices of cost functional $J^{HK}[u]$ are $(Q_t^{HK}, S_t^{HK}, R_t^{HK}, G^{HK})$, $(\tilde{Q}_t^{HK}, \tilde{S}_t^{HK}, \tilde{R}_t^{HK}, \tilde{G}^{HK})$. With this observation, we may transform an indefinite LQ control problem into a definite LQ control problem.
We write down stochastic Hamiltonian systems corresponding to Problem $J^{HK}$
\[
\begin{align*}
\dot{X}_t^{HK} &= \left( A_t X_t^{HK} + \tilde{A}_t \mathbb{E}[X_t^{HK}] + B_t \mathbb{E}[u_t^{HK}] \right) dt \\
&\quad + \left( C_t X_t^{HK} + \tilde{C}_t \mathbb{E}[X_t^{HK}] + D_t u_t^{HK} + \tilde{D}_t \mathbb{E}[u_t^{HK}] \right) dW_t \\
&\quad + \int_\Theta E_{t,\theta} X_t^{HK} + \tilde{E}_{t,\theta} \mathbb{E}[X_t^{HK}] d\hat{N}(dt, d\theta) \\
&\quad + F_{t,\theta} u_t^{HK} + \tilde{F}_{t,\theta} \mathbb{E}[u_t^{HK}] d\hat{N}(dt, d\theta), \\
\dot{Y}_t^{HK} &= \left( A_t Y_t^{HK} + \tilde{A}_t \mathbb{E}[Y_t^{HK}] + C_t^T Z_t^{HK} + \tilde{C}_t \mathbb{E}[Z_t^{HK}] \right) dt \\
&\quad + \left( \int_\Theta E_{t,\theta} Y_t^{HK} + \tilde{E}_{t,\theta} \mathbb{E}[Y_t^{HK}] d\hat{N}(dt, d\theta) \right) + Q^{HK} X_t^{HK} + \tilde{Q}^{HK} \mathbb{E}[X_t^{HK}] \\
&\quad + \left( \int_\Theta E_{t,\theta} u_t^{HK} + \tilde{E}_{t,\theta} \mathbb{E}[u_t^{HK}] d\hat{N}(dt, d\theta) \right) + P^{HK} u_t^{HK} + \tilde{P}^{HK} \mathbb{E}[u_t^{HK}] = 0. 
\end{align*}
\]
We note that (4) coincides with (17) while $H \equiv 0, K \equiv 0$. The following lemma shows that there exists an equivalent relationship among stochastic Hamiltonian systems (17).

**Lemma 4.1:** For all $H, K \in \Phi$, the existence and uniqueness of solutions to stochastic Hamiltonian systems (17) are equivalent.

**Proof:** We only need to prove that for all $H, K \in \Phi$, the existence and uniqueness of solutions to stochastic Hamiltonian systems (17) are equivalent to (4). For any given $H, K \in \Phi$, if $(X_t^{HK}, u_t^{HK}, Y_t^{HK}, Z_t^{HK}, r_t^{HK})$ is a solution of (17), we define
\[
\begin{align*}
X_t &= X_t^{HK}, \\
u_t &= u_t^{HK}, \\
Y_t &= Y_t^{HK} + H_t \left( X_t^{HK} - \mathbb{E}[X_t^{HK}] \right) + K_t \mathbb{E}[X_t^{HK}], \\
Z_t &= Z_t^{HK} + H_t \left( C_t X_t^{HK} + \tilde{C}_t \mathbb{E}[X_t^{HK}] + D_t u_t^{HK} + \tilde{D}_t \mathbb{E}[u_t^{HK}] \right), \\
r_{t,\theta} &= r_{t,\theta}^{HK} + H_t \left( E_{t,\theta} X_t^{HK} + \tilde{E}_{t,\theta} \mathbb{E}[X_t^{HK}] + F_{t,\theta} u_t^{HK} + \tilde{F}_{t,\theta} \mathbb{E}[u_t^{HK}] \right). 
\end{align*}
\]
It is obviously that $(X_t, u_t)$ satisfies the forward stochastic differential equation of (4). By Itô formula, we obtain
\[
\begin{align*}
\int_\Theta E_{t,\theta} r_{t,\theta}^{HK} \nu(d\theta) + \int_\Theta \tilde{E}_{t,\theta} r_{t,\theta}^{HK} \tilde{\nu}(d\theta) + \int_\Theta \tilde{E}_{t,\theta} \mathbb{E}[r_{t,\theta}^{HK}] \tilde{\nu}(d\theta) + \int_\Theta E_{t,\theta} \mathbb{E}[r_{t,\theta}^{HK}] \nu(d\theta) + Q^{HK} X_t^{HK} + \tilde{Q}^{HK} \mathbb{E}[X_t^{HK}] + S^{HK} u_t^{HK} + \tilde{S}^{HK} \mathbb{E}[u_t^{HK}]. 
\end{align*}
\]
Then, the conclusion follows immediately. 

We write down Riccati equations related to Problem $J^{HK}$:
\[
\begin{align*}
\dot{\Pi}_t^{HK} + \Pi_t^{HK} A_t + A_t^T \Pi_t^{HK} + C_t^T P_t^{HK} C_t \\
&\quad + \int_\Theta E_{t,\theta} \Pi_t^{HK} E_{t,\theta} \nu(d\theta) + Q^{HK} - L_t^{HK} \Sigma_t^{HK} - L_t^{HK} \Sigma_t^{HK} = 0, \\
\dot{\Pi}_t^{HK} + \Pi_t^{HK} A_t + A_t^T \Pi_t^{HK} + C_t^T P_t^{HK} C_t \\
&\quad + \int_\Theta E_{t,\theta} \Pi_t^{HK} E_{t,\theta} \nu(d\theta) + Q^{HK} - L_t^{HK} \Sigma_t^{HK} - L_t^{HK} \Sigma_t^{HK} = 0. 
\end{align*}
\]
where
\[
\begin{align*}
L_t^{HK} &= S_t^{HK} + P_t^{HK} B_t + C_t^T P_t^{HK} D_t \\
&\quad + \int_\Theta E_{t,\theta} P_t^{HK} E_{t,\theta} \nu(d\theta), \\
\tilde{L}_t^{HK} &= S_t^{HK} + \Pi_t^{HK} \tilde{B}_t + \tilde{C}_t^T P_t^{HK} \tilde{D}_t \\
&\quad + \int_\Theta \tilde{E}_{t,\theta} P_t^{HK} \tilde{E}_{t,\theta} \nu(d\theta), \\
\Sigma_t^{HK} &= R_t^{HK} + D_t^T P_t^{HK} D_t + \int_\Theta \tilde{F}_{t,\theta} P_t^{HK} \tilde{F}_{t,\theta} \nu(d\theta), \\
\tilde{\Sigma}_t^{HK} &= \tilde{R}_t^{HK} + \tilde{D}_t^T P_t^{HK} \tilde{D}_t + \int_\Theta \tilde{F}_{t,\theta} P_t^{HK} \tilde{F}_{t,\theta} \nu(d\theta). 
\end{align*}
\]

**Lemma 4.2:** For any $H, K \in \Phi$, the existence and uniqueness of solutions to Riccati equations associated with Problem $J^{HK}$ are equivalent.

**Proof:** We only need to prove that for all $H, K \in \Phi$, the existence and uniqueness of solutions to Riccati equations (18) and (19) are equivalent to Riccati equations (5) and (6). For any given $H, K \in \Phi$, if $\Pi^{HK}, \Pi^{HK}$ are solutions of Riccati
equations (18) and (19), respectively, then we define
\[ P_t = P_t^{HK} + H_t, \quad \Pi_t = \Pi_t^{HK} + K_t. \]
Through a straightforward calculation, we know that \( P, \Pi \) are solutions of Riccati equations (5) and (6), respectively. Thus, we complete the proof.

We now give two theorems which are our main results in this section.

**Theorem 4.1:** If there exist \( \bar{H}, \bar{K} \in \Phi \) such that the cost functional \( J^{HK} \) satisfies Assumption (S), then Riccati equations (5) and (6) admit unique solutions \( P, \Pi \), such that \( \Sigma_t \geq \alpha_1 I, \Sigma_t \geq \alpha_1 I \) for some \( \alpha_1 > 0 \), respectively. Further, the optimal pair \((X^*, u^*)\) of Problem MF satisfies
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\dot{P}_t = -\Sigma_t^{-1}L_t^\top (X_t^* - \mathbb{E}[X_t^*]) & - \Sigma_t^{-1}L_t^\top \mathbb{E}[X_t^*], \\
\dot{d}_t = \left( A_t X_t^* + \bar{A}_t \mathbb{E}[X_t^*] + B_t u_t^* + \bar{B}_t \mathbb{E}[u_t^*] \right) dt \\
& + \left( C_t X_t^* + \bar{C}_t \mathbb{E}[X_t^*] + D_t u_t^* + D_t \mathbb{E}[u_t^*] \right) dW_t \\
& + \int_\Theta \left\{ E_{t,\theta} X_t^* + \bar{E}_{t,\theta} \mathbb{E}[X_t^*] \\
& + F_{t,\theta} u_t^* + \bar{F}_{t,\theta} \mathbb{E}[u_t^*] \right\} \tilde{N}(dt,d\theta), \\
X_0^* = \xi, \\
\end{array} \right. \\
\end{aligned}
\]
for some \( \alpha_1 > 0 \). Using Theorem 1, further we know that the optimal pair \((X^{HK}, u^{HK})\) of Problem \( J^{HK} \) satisfies
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\dot{u}_t^{HK} = -\Sigma_t^{HK}^{-1}L_t^{HK}\top (X_t^{HK} - \mathbb{E}[X_t^{HK}]) & - \Sigma_t^{HK}^{-1}(L_t^{HK}\top)^\top \mathbb{E}[X_t^{HK}], \\
\dot{d}_t = \left( A_t X_t^{HK} + \bar{A}_t \mathbb{E}[X_t^{HK}] \\
& + B_t u_t^{HK} + \bar{B}_t \mathbb{E}[u_t^{HK}] \right) dt \\
& + \left( C_t X_t^{HK} + \bar{C}_t \mathbb{E}[X_t^{HK}] \\
& + D_t u_t^{HK} + D_t \mathbb{E}[u_t^{HK}] \right) dW_t \\
& + \int_\Theta \left\{ E_{t,\theta} X_t^{HK} + \bar{E}_{t,\theta} \mathbb{E}[X_t^{HK}] \\
& + F_{t,\theta} u_t^{HK} + \bar{F}_{t,\theta} \mathbb{E}[u_t^{HK}] \right\} \tilde{N}(dt,d\theta), \\
X_0^{HK} = \xi, \\
\end{array} \right. \\
\end{aligned}
\]
the five-tuple \((X^*, u^*, Y^*, Z^*, r^*)\) is the unique solution to MF-FBSDEJ (4). Moreover
\[
\inf_{u \in U[0,T]} J[u] = \frac{1}{2} \mathbb{E}[(P_t(\xi - \mathbb{E}[\xi])), \xi - \mathbb{E}[\xi]) + (\Pi_t \mathbb{E}[\xi], \mathbb{E}[\xi])].
\]

**Proof:** We now consider Problem \( J^{HK} \). For \( \bar{H}, \bar{K} \in \Phi \), according to Theorem 1, Riccati equations (18) and (19) associated with Problem \( J^{HK} \) admit unique solutions \( P^{HK} \geq 0, \Pi^{HK} \geq 0 \), respectively. By Lemma 6, Riccati equations (5) and (6) admit unique solutions
\[
P_t = P_t^{HK} + \bar{H}_t, \quad \Pi_t = \Pi_t^{HK} + \bar{K}_t, \text{ respectively.}
\]
Then, we have
\[
\Sigma_t = R_t + D_t^\top P_t D_t + \int_\Theta F_{t,\theta}^\top P_t F_{t,\theta} \nu(d\theta)
\]
and
\[
\tilde{\Sigma}_t = \tilde{R}_t + \tilde{D}_t^\top \tilde{P}_t \tilde{D}_t + \int_\Theta \tilde{F}_{t,\theta}^\top \tilde{P}_t \tilde{F}_{t,\theta} \nu(d\theta)
\]
which imply that
\[
\Sigma_t \geq \alpha_1 I, \tilde{\Sigma}_t \geq \alpha_1 I
\]
\[ + \frac{1}{2} \mathbb{E}[\langle \dot{H}_0(\xi - \mathbb{E}[\xi]), \xi - \mathbb{E}[\xi] \rangle + \langle \dot{K}_0 \mathbb{E}[\xi], \mathbb{E}[\xi] \rangle] \]
\[ = \frac{1}{2} \mathbb{E}[\langle P_0(\xi - \mathbb{E}[\xi]), \xi - \mathbb{E}[\xi] \rangle + \langle \Pi_0 \mathbb{E}[\xi], \mathbb{E}[\xi] \rangle]. \]

The proof is completed.

**Theorem 4.2**: If Riccati equations (5) and (6) admit unique solutions \( P, \Pi, \) respectively, and \( \Sigma_t \geq \alpha_1 I, \Sigma_\theta \geq \alpha_1 I \) for some \( \alpha_1 > 0, \) then there exist \( H, K \in \Phi \) such that cost functional \( J^{R,H} \) satisfies Assumption (S).

**Proof**: We consider the equivalent cost functional \( J^{P \Pi}[u]. \) It is easy to verify
\[
Q_t^{P \Pi} = Q_t + \dot{P}_t + P_t A_1 + A_1^\top P_t + C_1^\top P_t C_t + \int_\Theta E_{t,\theta}^\top P_t E_{t,\theta} \nu(d\theta),
\]
\[
S_t^{P \Pi} = S_t + P_t B_1 + C_1^\top P_t D_t + \int_\Theta E_{t,\theta}^\top P_t F_{t,\theta} \nu(d\theta),
\]
\[
R_t^{P \Pi} = R_t + D_1^\top P_t D_t + \int_\Theta F_{t,\theta}^\top P_t F_{t,\theta} \nu(d\theta),
\]
\[
G^{P \Pi} = G - P_T,
\]
\[
\overline{Q}_t^{P \Pi} = \overline{Q}_t + \dot{\Pi}_t + \Pi_t \overline{A}_t + \overline{A}_t^\top \Pi_t + \overline{C}_t^\top P_t \overline{C}_t + \int_\Theta \overline{E}_{t,\theta}^\top P_t \overline{E}_{t,\theta} \nu(d\theta),
\]
\[
\overline{S}_t^{P \Pi} = \overline{S}_t + \Pi_t \overline{B}_t + \overline{C}_t^\top P_t \overline{D}_t + \int_\Theta \overline{E}_{t,\theta}^\top P_t \overline{F}_{t,\theta} \nu(d\theta),
\]
\[
\overline{R}_t^{P \Pi} = \overline{R}_t + \overline{D}_t^\top P_t \overline{D}_t + \int_\Theta \overline{F}_{t,\theta}^\top P_t \overline{F}_{t,\theta} \nu(d\theta),
\]
\[
\overline{G}^{P \Pi} = \overline{G} - \Pi_T,
\]
and \((Q_t^{P \Pi}, S_t^{P \Pi}, R_t^{P \Pi}, G^{P \Pi}), (\overline{Q}_t^{P \Pi}, \overline{S}_t^{P \Pi}, \overline{R}_t^{P \Pi}, \overline{G}^{P \Pi})\) satisfy Assumption (S).

**V. Examples**

In this section, we present four illustrative examples, where Assumption (S) does not hold true for original optimal control problems. Example 5.1 shows that an optimal control exists even though Assumption (S) does not hold true. In Example 5.2, it is difficult to prove the existence and uniqueness of solutions to related Riccati equations. We use the equivalent cost functional method to construct an MF-LQJ problem which satisfies Assumption (S) first, and then we obtain an optimal control of the original stochastic control problem via solutions of Riccati equations. We also give the existence and uniqueness of solutions to a family of MF-FBSDEJs as a by-product of our results. With the in-depth study of Example 5.2, we apply our results to prove the existence and uniqueness of solution to an MF-FBSDEJ in Example 5.3, where existing methods in literatures do not work. In Example 5.4, we apply our results to solve an asset-liability management problem and give some numerical solutions.

**Example 5.1**: Consider a 1-D controlled MF-SDEJ
\[
\begin{aligned}
dx_t &= (X_t - \mathbb{E}[X_t] + u_t + \mathbb{E}[u_t]) dt + (2u_t - \mathbb{E}[u_t])dW_t \\
&+ \int_{[1, +\infty)} e^{-\theta} \mathbb{E}[u_t] \tilde{N}(dt, d\theta), \quad t \in [0, T],
\end{aligned}
\]
with a cost functional
\[
J[u] = \frac{1}{2} \mathbb{E}\left[ 2|X_T|^2 - |\mathbb{E}[X_T]|^2 \\
+ \int_0^T \left( -3|X_t|^2 + 3|\mathbb{E}[X_t]|^2 - 4|u_t|^2 + 2|\mathbb{E}[u_t]|^2 \right) dt \right].
\]

With the data, Assumption (S) does not hold. We write down the stochastic Hamiltonian system
\[
\begin{aligned}
dX_t &= (X_t - \mathbb{E}[X_t] + u_t + \mathbb{E}[u_t]) dt \\
&+ (2u_t - \mathbb{E}[u_t])dW_t + \int_{[1, +\infty)} e^{-\theta} \mathbb{E}[u_t] \tilde{N}(dt, d\theta),
\end{aligned}
\]
\[
dY_t = - \left( Y_t - \mathbb{E}[Y_t] - 3X_t + 3\mathbb{E}[X_t] \right) dt + Z_t dW_t \\
&+ \int_{[1, +\infty)} e^{-\theta} \mathbb{E}[r_t, \theta] \nu(d\theta, d\theta),
\]
\[
X_0 = x, \quad Y_T = 2X_T - \mathbb{E}[X_T], \quad -4u_t + 2\mathbb{E}[u_t] + Y_t + \mathbb{E}[Y_t] + 2Z_t - \mathbb{E}[Z_t] \\
&+ \int_{[1, +\infty)} e^{-\theta} \mathbb{E}[r_t, \theta] \nu(d\theta) = 0.
\]

The corresponding Riccati equations are
\[
\begin{aligned}
\dot{P}_t + 2P_t - 3 - \frac{P_t^2}{P_T - 4} &= 0, \\
P_T &= 2, \\
\dot{\Pi}_t - \frac{4\Pi_t^2}{2\Pi_T + 3} &= 0, \\
\Pi_T &= 1
\end{aligned}
\]
where \( \delta = \int_{[1, +\infty)} e^{-\theta} \nu(d\theta) > 0. \) Solving them, we get
\[
P_t = 2, \quad \Pi_t = \frac{\delta}{2T - 2t + \delta}.
\]

Note that
\[
-4 + 4P_t = 4, \quad -2 + P_t + \delta P_t = 2\delta.
\]

It follows from Theorem 4.2 that the equivalent cost functional \( J^{P \Pi}[u] \) satisfies Assumption (S). According to Theorem 4.1, the optimal pair \((X, u)\) satisfies
\[
\begin{aligned}
u_t &= -\frac{4}{3} (X_t - \mathbb{E}[X_t]) - \frac{2}{27 - 2t + 3} \mathbb{E}[X_t], \\
x_t &= \frac{4}{3} (X_t - \mathbb{E}[X_t]) + \frac{2}{27 - 2t + 3} \mathbb{E}[X_t] + \frac{2}{27 - 2t + 3} \mathbb{E}[X_t], \\
\int_{[1, +\infty)} e^{-\theta} \mathbb{E}[u_t] \tilde{N}(dt, d\theta),
\end{aligned}
\]
\[
\begin{aligned}
x_0 &= x, \\
Y_t &= 2(X_t - \mathbb{E}[X_t]) + \frac{4}{27 - 2t + 3} \mathbb{E}[X_t], \\
Z_t &= -2(X_t - \mathbb{E}[X_t]) - \frac{2}{27 - 2t + 3} \mathbb{E}[X_t], \\
r_t, \theta &= -\frac{2\mathbb{E}[u_t]}{27 - 2t + 3} \mathbb{E}[X_t].
\end{aligned}
\]

Defining
the five-tuple \((X, u, Y, Z, r)\) is a solution to MF-FBSDE (22). Moreover
\[
\inf_{u \in \mathcal{U}(0, T)} J[u] = \frac{\delta}{2(2T + \delta)} x^2, \quad \forall x \in \mathbb{R}.
\]

Example 5.2: Consider a 1-D controlled MF-SDEJ
\[
\begin{align*}
\frac{dX_t}{dt} &= \left(2X_t - E[X_t] + u_t \right) \, dt + 2u_t \, dW_t \\
&\quad + \int_{\Theta} \left(E_{t, \theta} X_{t-} + \tilde{E}_{t, \theta} E[X_{t-}]\right) \, dN_t \, (dt, d\theta), \\
X_0 &= x
\end{align*}
\]
with a cost functional
\[
J[u] = \frac{1}{2} \mathbb{E} \left\{ \alpha X_T^2 - (\alpha + 1) \mathbb{E}[X_T]^2 \right. \\
&\left. + \int_0^T \left(4E[X_t]^2 + 4X_tE[u] + R_t u_t^2 \right) \, dt \right\}
\]
where \(\alpha > \frac{1}{2}(T + 1)^2, R_t = (t + 1)^3 - 2(t + 1)^2, \tilde{R}_t = 1 - (t + 1)^3\).

Clearly, Assumption (S) does not hold. Now, we introduce an equivalent cost functional \(J^{H_0, K_0}[u]\) satisfying Assumption (S).

Recalling (16), we have
\[
\begin{align*}
Q_t^{HK} &= \tilde{H}_t + 4H_t + \delta_{1t} H_t, \\
R_t^{HK} &= (t + 1)^3 - 2(t + 1)^2 + 4H_t, \quad G_t^{HK} = \alpha - H_t, \\
\tilde{Q}_t^{HK} &= 4 + \tilde{K}_t + 2K_t + \delta_{2t} H_t, \quad \tilde{S}_t^{HK} = 2 + K_t, \\
\tilde{R}_t^{HK} &= 1 - 2(t + 1) - 4H_t, \quad \tilde{G}_t^{HK} = -1 - K_t, \quad \forall H, K \in \Phi
\end{align*}
\]
where \(\delta_{1t} = \int_\Theta \mathbb{E} [E_{t, \theta}^2 \nu(d\theta)] > 0, \delta_{2t} = \int_\Theta \mathbb{E} [\tilde{E}_{t, \theta}^2 \nu(d\theta)] > 0\). In particular, if we define \(H_{0t} = \frac{1}{2}(t + 1)^2, K_{0t} = \frac{1}{1 + (t + 1)^2} - 2\), then
\[
\begin{align*}
Q_t^{H_0, K_0} &= (t + 1)^2 + \delta_{1t} (t + 1)^2, \\
S_t^{H_0, K_0} &= \frac{1}{2} (t + 1)^2, \\
R_t^{H_0, K_0} &= (t + 1)^3 - 2(t + 1)^2 + 4H_t, \quad G_t^{H_0, K_0} = \alpha - \frac{1}{2}(T + 1)^2, \\
\tilde{Q}_t^{H_0, K_0} &= \frac{1}{1 + (T - t)^2} + \frac{2}{1 + (T - t)} + \tilde{\delta}_{2t} (t + 1)^2, \\
\tilde{S}_t^{H_0, K_0} &= \frac{1}{1 + (T - t)} \\
\tilde{R}_t^{H_0, K_0} &= 1, \quad \tilde{G}_t^{H_0, K_0} = 0.
\end{align*}
\]
It is easy to see that Assumption (S) holds true for \(J^{H_0, K_0}[u]\).

Then, Theorem 2 and Lemma 5 imply that for any \(H, K \in \Phi\), the MF-FBSDEJ
\[
\begin{align*}
\frac{dX_t^{HK}}{dt} &= \left(2X_t^{HK} - \mathbb{E}[X_t^{HK}] + u_t^{HK} \right) \, dt + 2u_t^{HK} \, dW_t \\
&\quad + \int_{\Theta} \left(E_{t, \theta} X_{t-}^{HK} + \tilde{E}_{t, \theta} \mathbb{E}[X_{t-}] \right) \, dN_t \, (dt, d\theta), \\
\frac{dY_t^{HK}}{dt} &= -\left(2Y_t^{HK} - \mathbb{E}[Y_t^{HK}] \right) \\
&\quad + \int_{\Theta} \left(\nu_{t, \theta}^{RHK} \mathbb{E}
+ \tilde{E}_{t, \theta} \mathbb{E}[X_{t-}] \right) \nu(d\theta) + \left[\tilde{Q}_t^{HK} X_t^{HK} + \tilde{S}_t^{HK} \tilde{u}_t^{HK} + \tilde{R}_t^{HK} \tilde{u}_t^{HK} \mathbb{E}[u_t^{HK}] \right] \, dt \\
&\quad + \tilde{Z}_t^{HK} \, dW_t + \int_{\Theta} \tilde{r}_{t, \theta} \tilde{N} \, (dt, d\theta), \\
X_0 &= x, \quad Y_T = \alpha X_T - (\alpha + 1) \mathbb{E}[X_T], \\
Y_t + 2Z_t + \tilde{R}_t u_t + \tilde{R}_t \tilde{u}_t &= 0.
\end{align*}
\]
has a unique solution. Further, Theorem 2 implies that Riccati equations
\[
\begin{align*}
\frac{\hat{P}_t + 4P_t + \delta_{1t} P_t - \frac{P_t^2}{(t + 1)^4 + 4(1 + t)^2}}{\gamma_0} &= \alpha, \\
P_T &= \alpha
\end{align*}
\]
and
\[
\begin{align*}
\frac{\Pi_t + 2\Pi_t + \delta_{2t} P_t - \frac{(\Pi_t + 2)^2}{1 - 2(1 + t)^2 + 4t^2}}{\gamma_0} &= 0, \\
\Pi_T &= -1
\end{align*}
\]
admit unique solutions \(P, \Pi\), respectively. And the optimal pair \((X^*, u^*)\) satisfies
\[
\begin{align*}
u_t^* &= \frac{P_t^*}{(1 + t + 2)} \mathbb{E}[X_t^*] \, \mathbb{E}[X_t^*] - \mathbb{E}[X_t^*], \\
&\quad - \frac{\Pi_t + 2}{1 - 2(1 + t)^2 + 4t^2} \mathbb{E}[X_t^*], \\
\frac{dX_t^*}{dt} &= \left(2X_t^* - \mathbb{E}[X_t^*] + u_t^* \right) \, dt + 2u_t^* \, dW_t \\
&\quad + \int_{\Theta} \left(E_{t, \theta} X_{t-}^* + \tilde{E}_{t, \theta} \mathbb{E}[X_{t-}] \right) \, dN_t \, (dt, d\theta), \\
X_0^* &= x.
\end{align*}
\]
We remark that the well-posedness of MF-FBSDE, i.e., the jump diffusion item in MF-FBSDEJ disappears, has been well studied (see [4], [6], [7], and [8]). In detail, Bensoussan [4] derived the existence and uniqueness of solution to MF-FBSDE under a monotonicity condition. Carmona and Delarue [6] obtained the solvability of MF-FBSDE by a compactness argument and the Schauder fixed point theorem under a bounded condition. Carmona and Delarue [8] took advantage of the convexity of the Hamiltonian to apply the continuation method, and proved the existence and uniqueness of solution to MF-FBSDE. Carmona and Delarue [7] derived the solvability results by using an approximation procedure under some convexity condition. Moreover, Li and Min [19] investigated the existence and uniqueness of solution to MF-FBSDEJ under a monotonicity condition, which extended the results in previous literatures. Different from the works above, our equivalent method provides an alternative way to solve MF-FBSDEJ. Specially, the original stochastic Hamiltonian system is of form
\[
\begin{align*}
\frac{dX_t}{dt} &= \left(2X_t - \mathbb{E}[X_t] + u_t \right) \, dt + 2u_t \, dW_t \\
&\quad + \int_{\Theta} \left(E_{t, \theta} X_{t-} + \tilde{E}_{t, \theta} \mathbb{E}[X_{t-}] \right) \, dN_t \, (dt, d\theta), \\
\frac{dY_t}{dt} &= \mathbb{E}[Y_t] - 2Y_t - \int_{\Theta} \left(\nu_{t, \theta}^R \mathbb{E}[X_{t-}] + \tilde{E}_{t, \theta} \mathbb{E}[X_{t-}] \right) \nu(d\theta) \\
&\quad - 4 \mathbb{E}[X_t] - 2 \mathbb{E}[u_t] \, dt + Z_t \, dW_t + \int_{\Theta} r_{t, \theta} \tilde{N} \, (dt, d\theta), \\
X_0 &= x, \quad Y_T = \alpha X_T - (\alpha + 1) \mathbb{E}[X_T], \\
Y_t + 2Z_t + R_t u_t + R_t \mathbb{E}[u_t] &= 0.
\end{align*}
\]
Since $R_t = (t+1)^3 - 2(t+1)^2$, $\bar{R}_t = 1 - 2(t+1)^2$, we cannot derive an expression of the optimal control process $u$ from the last equation in (25). The monotonicity condition in [4] and [19] and the bounded condition in [6] fail. Moreover, Carmona and Delarue [7], [8] assumed that cost functional satisfies some convex conditions, which are not true in our setting, thus the methods in [7] and [8] fail. We emphasize that our method is also effective in proving the solvability of MF-FBSDEJ with a slightly general and complicated form. The following example provides a better understanding on this issue.

Example 5.3: Consider an MF-FBSDEJ

\[
\begin{aligned}
dX_t &= (2X_t + E[X_t] + Y_t + Z_t) dt + (Y_t + Z_t) dW_t + \int_0^T \bar{E}_{t,0} E[X_{t-}] N(dt, d\theta), \\
dY_t &= (X_t - E[X_t] - 2Y_t - E[Y_t]) dt + Z_t dW_t + \int_0^T \bar{E}_{t,0} E[r_{t,\theta}] \nu(d\theta) dt + Z_t dW_t, \\
X_0 &= x, \\
Y_T &= 2X_T - E[X_T], \\
Z_t &= a_t, \\
X_0 &= x
\end{aligned}
\]

(26)

where $X, Y, Z, r$ are 1-D stochastic processes. We claim that (26) does not satisfy the conditions in [4], [6], and [19]. Indeed, we have

\[
\begin{aligned}
E\left[(X_{1T} - X_{2T})^2 \right] &\geq E[(X_{1T} - X_{2T}^2)] \\
\end{aligned}
\]

and

\[
\begin{aligned}
E\left[(Y_{1t} - Y_{2t}) \right] &\geq E[(Y_{1t} - Y_{2t}^2)] \\
\end{aligned}
\]

Different from (25), we can derive an explicit expression of the optimal control process $u$ from the last equation in (28). In fact, it is easy to see that stochastic Hamiltonian system (28) is exactly MF-FBSDEJ (26) with $u_t = Y_t + Z_t$ Note that the cost functional does not satisfy the convex conditions in [7] and [8].

For the above MF-LQJ problem, it is clear that Assumption (S) does not hold. Now, we introduce an equivalent cost functional $J^{H_0,K_0}[u]$ satisfying Assumption (S). Recalling (16), we have

\[
\begin{aligned}
Q_t^{HK} &= -1 + \bar{H}_t + 4H_t, \\
R_t^{HK} &= -1 + H_t, \\
\bar{Q}_t^{HK} &= \bar{K}_t + 6\bar{K}_t + \delta_t H_t, \\
\bar{R}_t^{HK} &= \delta_t H_t, \\
\bar{Q}_t^{HK} &= 7, \\
\bar{R}_t^{HK} &= 2, \\
\bar{Q}_t^{HK} &= 6 + 2\delta_t, \\
\bar{R}_t^{HK} &= 1, \\
\bar{Q}_t^{HK} &= 1, \\
\bar{R}_t^{HK} &= 0.
\end{aligned}
\]

With the data, Assumption (S) holds true for $J^{H_0,K_0}[u]$. Then, it follows from Theorem 2 that MF-FBSDEJ (28) admits a unique solution $(X, u, Y, Z, r)$, and $(X, Y, Z, r)$ is exactly the solution of (26).

Example 5.4: Consider a financial market consisting of a bond and a stock, in which two assets are trading continuously within the time horizon $[0, T]$. The dynamics of the bond price process $S_{1t}$ is governed by

\[
\begin{aligned}
ds_{1t} &= r_s S_{1t} dt, \\
S_{10} &= s_1
\end{aligned}
\]

where $r_s$ is the interest rate of the bond. The dynamics of the stock price process $S_{2t}$ is governed by

\[
\begin{aligned}
ds_{2t} &= \mu_s S_{2t} dt + \sigma_t S_{2t} dW_t, \\
S_{20} &= s_2
\end{aligned}
\]
where $\mu_t$ and $\sigma_t$ are the appreciation rate and the volatility coefficient of the stock, respectively. For simplicity, we assume that the coefficients $\mu_t \geq r_t > 0$, $\sigma_t$ and $\frac{1}{\sigma_t}$ are bounded and deterministic functions.

We assume that the trading of shares takes place continuously in a self-financing fashion and there are no transaction costs. We denote by $N_t$ the asset of an investor and by $u_t$ the amount allocated in the stock share at time $t$. Clearly, the amount invested in the risk-free asset is $N_t - u_t$. Without liability, the asset of the investor $N_t$ evolves as

$$dN_t = [r_t N_t + (\mu_t - r_t) u_t] dt + \sigma_t u_t dW_t, \quad N_0 = n_0.$$  \hspace{1cm} (29)

The investor’s accumulative liability at time $t$ is denoted by $\Upsilon_t$. Chiu and Li [11] and Wei and Wang [35] described the liability process by a geometric Brownian motion. In fact, it is possible that the control strategy and the mean of asset of the investor can influence the liability process, due to the complexity of the financial market and the risk aversion behavior of the investor. Such an example can be found in Wang et al. [34], where the liability process depends on a control strategy (e.g., capital injection or withdrawal) of the firm. Along this line, we proceed to improve the liability process here. Suppose that the dynamics of $\Upsilon_t$ satisfies

$$d\Upsilon_t = \left( a_t \Upsilon_t + c_1 \mathbb{E}[\Upsilon_t] \right) dt + b_t \Upsilon_t dW_t, \quad \Upsilon_0 = \Upsilon_0,$$  \hspace{1cm} (30)

where $a_t$ is the appreciation rate of the liability and $b_t$ is the corresponding volatility which satisfies the nondegeneracy condition. $a_t$, $b_t$, and $c_1$ are deterministic continuous functions on $[0, T]$. Taking the liability into consideration, the SDE for the net wealth of the investor at time $t$, denote by $I_t$, is obtained by subtracting (30) from (29)

$$dI_t = \left( r_t I_t + (r_t - a_t) \Upsilon_t + (\mu_t - r_t) u_t - c_1 \mathbb{E}[\Upsilon_t] \right) dt + \left( \sigma_t u_t - b_t \Upsilon_t \right) dW_t, \quad I_0 = n_0 - \Upsilon_0.$$  \hspace{1cm} (31)

**Definition 5.1:** An $\mathbb{R}$-valued portfolio strategy $u$ is called admissible, if $u$ is $\mathcal{F}$-adapted and $\mathbb{E}[\int_0^T u_t^2 dt] < \infty$. The set of all admissible portfolio strategies is denoted by $\mathcal{U}_{ad}$. For any $u \in \mathcal{U}_{ad}$, (31) admits a unique solution $\left( \Upsilon, I \right) \in S^2_{\mathbb{R}}(0, T; \mathbb{R}^2)$. We introduce a performance functional of the investor, which is in the form of

$$J[u] = \mathbb{E} \left\{ \int_0^T \left[ \Upsilon_t^2 + (I_t - \mathbb{E}[I_t])^2 \right] dt + (I_T - \mathbb{E}[I_T])^2 \right\}.$$  \hspace{1cm} (32)

Now, we pose an asset-liability management problem as follows:

**Problem AL:** Find a portfolio strategy $u^* \in \mathcal{U}_{ad}$ such that

$$J[u^*] = \inf_{u \in \mathcal{U}_{ad}} J[u].$$  \hspace{1cm} (33)

The problem implies that the investor aims to minimize the risk of net wealth and liability over the whole time horizon, simultaneously.

It is easy to see that Problem AL is a special case of Problem MF. Denoting $X_t = (\Upsilon_t, I_t)^T$, and $\xi = (\Upsilon_T, I_T - \mathbb{E}[I_T])^T$, we have

$$A_t = \begin{pmatrix} a_t & 0 \\ r_t - a_t & r_t \end{pmatrix}, \quad \bar{A}_t = \begin{pmatrix} c_t & c_t \\ -c_t & -c_t \end{pmatrix},$$

$$B_t = \begin{pmatrix} 0 & 0 \\ \mu_t - r_t \end{pmatrix}, \quad C_t = \begin{pmatrix} b_t \\ -b_t \end{pmatrix}, \quad D_t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad G_t = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{G}_t = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\bar{R}_t = 0, \quad S = \bar{S} = 0.$$  \hspace{1cm} (34)

Clearly, Assumption (S) does not hold. Define $H_t = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_t \end{pmatrix}$, $K_t = 0_{2 \times 2}$, where $\lambda_t$ is the solution of

$$\dot{\lambda}_t = \left( \frac{\mu_t - r_t}{\sigma_t} \right)^2 - 2r_t \lambda_t + \left[ r_t - a_t + \frac{b_t (u_t - r_t)}{\sigma_t} \right]^2 \lambda_t^2,$$

$$\lambda_T = \frac{1}{T}.$$  \hspace{1cm} (35)

Recalling (16), we have

$$Q_t^{HK} = \begin{pmatrix} 1 + b_t^2 \lambda_t & (r_t - a_t) \lambda_t \\ (r_t - a_t) \lambda_t & 1 + 2r_t \lambda_t + \lambda_t^2 \end{pmatrix},$$

$$S_t^{HK} = \begin{pmatrix} -b_t \sigma_t \lambda_t \\ (\mu_t - r_t) \lambda_t \end{pmatrix}, \quad R_t^{HK} = \lambda_t \sigma_t^2,$$

$$G_t^{HK} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \bar{Q}_t^{HK} = \begin{pmatrix} 1 + b_t^2 \lambda_t \\ 0 \end{pmatrix}, \quad \bar{G}_t^{HK} = \lambda_t \sigma_t^2, \quad \bar{R}_t^{HK} = 0.$$

It is easy to see that Assumption (S) holds true for $J^{HK}[u]$. According to Theorem 2, we know that the following Riccati equations have unique solutions:

$$\begin{align*}
\tilde{P}_t + P_t A_t + A_t^\top B_t + C_t^\top P_t C_t + Q_t & - (P_t B_t + C_t^\top P_t D_t) \Sigma_t^{-1} (B_t^\top P_t + D_t^\top P_t C_t) = 0, \\
P_t & = G_t,
\end{align*}$$

$$\begin{align*}
\tilde{P}_t + \Pi_t \tilde{A}_t + \tilde{A}_t^\top \Pi_t + C_t^\top P_t C_t + \tilde{Q}_t & - [\Pi_t B_t + C_t^\top P_t D_t] \Sigma_t^{-1} (B_t^\top \Pi_t + D_t^\top P_t C_t) = 0, \\
\Pi_t & = \bar{G}_t
\end{align*}$$

where

$$\Sigma_t = D_t^\top P_t D_t.$$  \hspace{1cm} (36)

An optimal portfolio strategy $u^*$ is given by

$$u^*_t = - \Sigma_t^{-1} (B_t^\top P_t + D_t^\top P_t C_t) (X_t - \mathbb{E}[X_t])$$

$$- \Sigma_t^{-1} (B_t^\top \Pi_t + D_t^\top P_t C_t) \mathbb{E}[X_t].$$  \hspace{1cm} (37)
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