THE 2-DIMENSIONAL NONLINEAR SCHRÖDINGER-MAXWELL SYSTEM

ANTONIO AZZOLLINI AND MARCOS T.O. PIMENTA

ABSTRACT. In this paper we carry on the study of a system recently introduced by the first author as the planar version of the well known electrostatic Schrödinger-Maxwell equations.

In the positive potential case, we exhibit situations where the existence of solutions depends on the strength of the coupling, being this one modulated by a parameter.

We also present some results in the case of a sign-changing potential.

1. INTRODUCTION

In this paper we wish to perform a deeper study of the following Schrödinger-Maxwell problem

\[
\begin{align*}
\{ & -\Delta u + V(x)u - q\phi u + W'(u) = 0 \text{ in } \mathbb{R}^2, \\
\Delta \phi = u^2 & \text{ in } \mathbb{R}^2,
\end{align*}
\]

introduced in [1]. The origin of this system is purely physical. Indeed, the system \((P_0)\) is obtained by coupling the nonlinear Schrödinger equation with the Maxwell equations and it represents a model for describing the dynamics of a charged particle interacting with the electromagnetic field generated by itself.

Due to its important meaning, this system was widely studied in three-dimensions by many authors, since it was introduced, in its linear version, in the pioneering paper by Benci and Fortunato [6] as an eigenvalue problem in a bounded domain. In the last 25 years the literature has been enriched by a lot of papers dealing with several situations. In particular, we recall fundamental contributions from [5, 9, 10, 14, 16] in looking for existence and multiplicity results, specifically concerning radial solutions, for the problem

\[
\begin{align*}
\{ & -\Delta u + u - q\phi u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3, \\
\Delta \phi = u^2 & \text{ in } \mathbb{R}^3.
\end{align*}
\]

Abandoning the radial function constraint, in [3] it was proved the existence of a ground state solution for (1) when \(p \in (3, 6)\), and for a nonautonomous version of (1) obtained introducing a Rabinowitz type function \(V(x)\) as a coefficient of the linear term \(u\) when \(p \in (4, 6)\) (the range \(p \in (3, 4)\) was later successfully treated in [19]). Ranges of nonexistence depending on values of \(p\) and \(q\) are provided in [3, 11, 16]. We also recall [2] where system (1) was considered in presence of a general nonlinearity and [13, 17] where it was studied an equation which is equivalent to a zero mass version of (1).

In spite of the broad interest reserved to problem (1), up to our knowledge the literature on \((P_0)\) is definitely less rich.

Motivated by physical considerations on positiveness of energy density related with the rest mass (see [15] for more details), the problem \((P_0)\) was introduced in [1] for \(V(x) = 1 + |x|^\alpha\) and \(W(s) \geq 0\). In particular, in [1] it was studied the problem

\[\begin{align*}
\{ & -\Delta u + u - q\phi u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3, \\
\Delta \phi = u^2 & \text{ in } \mathbb{R}^3.
\end{align*}\]

\[
\begin{align*}
\{ & -\Delta u + V(x)u - q\phi u + W'(u) = 0 \text{ in } \mathbb{R}^2, \\
\Delta \phi = u^2 & \text{ in } \mathbb{R}^2,
\end{align*}
\]

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\[
(P_1) \quad \begin{cases}
-\Delta u + (1 + |x|^\alpha)u - q\phi u + |u|^{p-2}u = 0 \text{ in } \mathbb{R}^2, \\
\Delta \phi = u^2 \text{ in } \mathbb{R}^2,
\end{cases}
\]

and proved the following result

**Theorem 1.1.** Assume \(2 < p\). If it holds one or the other of these situations

- \(p > 4\) and \(\alpha \in (0, \tilde{p}]\) with \(\tilde{p} = \frac{2p-4}{p-4}\),
- \(2 < p \leq 4\) and \(\alpha > 0\),

then for any \(q > 0\) \((P_1)\) possesses infinitely many nontrivial solutions.

Even if such a multiplicity result was obtained for \(q = 1\), the proof can be repeated without any variation for an arbitrary \(q > 0\).

Looking at the proof of Theorem 1.1, we observe how strongly the choice of the exponent \(\alpha\) influences geometrical and compactness properties of the functional associated to the problem for \(p > 4\). At a first sight, difficulties in applying usual variational minimax arguments for \(p > 4\) and large values of \(\alpha\) could appear exquisitely technical, but it is not the case.

Indeed, as a first result, in Theorem 3.1 we use a new estimate to show that the existence of solutions to \((P_1)\) is, in fact, compromised when the impact of the coupling term \(\phi u\) is, roughly speaking, not strong enough.

In this sense, the main purpose of this paper is to study existence of solutions to \((P_1)\) assuming \(p > 4\), \(\alpha > \tilde{p}\), and letting \(q\) play the role of a parameter, or, when needed, of a variable of the problem, in order to modulate the effect of coupling.

An interesting feature coming out of our study is that, under suitable assumptions, we can find both radial and nonradial solutions of \((P_1)\).

As to the radial case, there are a lot of surprising analogies between \((P_1)\) under conditions \(p > 4\), \(\alpha > \tilde{p}\) and the three-dimensional Schrödinger-Maxwell system (1) studied by Ruiz [16] under assumption \(p \in (2, 3]\).

In particular, we use a new estimate (see Lemma 2.1) and exploit a variant of Strauss radial lemma to prove lower boundedness of the constrained functional and verify Palais - Smale condition. Choosing \(q > 0\) as large as we need to allow the functional to achieve negative levels, we have all the ingredients to show that minimum is attained for a nontrivial radial function and, at a later stage, that a multiplicity result analogous to that in [5] holds for a possibly larger \(q\).

As we will explain in Remark 3.6, the situation in the nonradial case is quite different, at least from a technical point of view. We point out that, differently from [8] where the better known and studied Schrödinger-Poisson system

\[
(P_0) \quad \begin{cases}
-\Delta u + u + \phi u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^2, \\
\Delta \phi = u^2 \text{ in } \mathbb{R}^2,
\end{cases}
\]

is treated, our strategy to find nonradial solutions does not rely only on a suitable change of the functional framework. Indeed we will introduce a different approach to \((P_0)\), transforming the problem in the following nonlinear eigenvalue type: look for a nonradial \(u \neq 0\) and \(q > 0\) such that

\[-\Delta u + (1 + |x|^\alpha)u + W'(u) = q(\log |x| \star u^2)u.\]

In Theorem 3.7 we are able to prove the existence of at least a nonradial solution to a class of problem including \((P_1)\) for \(p > 2\).

We would stress the importance of the role played by the dimension in solving \((P_1)\), as it falls in the category of problem like \((P_0)\) with **positive potential** (namely \(V(x)u^2 + W(u) \geq 0\)). Indeed, existence and multiplicity results we are going to provide in Theorems 3.5 and 3.7 are in contrast to what happens for the
analogous three-dimensional version of nonlinear Schrödinger-Maxwell system with positive potential. In [10, Proposition 1.2] it was proved a general result implying nonexistence of nontrivial solutions to \((P_1)\) in \(\mathbb{R}^3\).

Finally, in the last part of the paper we are interested in studying \((P_0)\) for other types of nonlinearities, as, for instance, those changing sign. As an example, we will treat the following model problem
\[
(P_2) \quad \begin{cases}
- \Delta u + (1 + |x|^\alpha)u - q\phi u - |u|^{p-2}u = 0 & \text{in } \mathbb{R}^2, \\
\Delta \phi = u^2 & \text{in } \mathbb{R}^2,
\end{cases}
\]
for \(\alpha > 0\) and \(p > 2\).

This situation, which appears at a first sight simpler to treat because of the formal analogy with \((1)\), presents indeed unexpected difficulties in proving boundedness of Palais-Smith sequences when \(p < 4\). Similar difficulties have been observed and overcome in [12] by suitable estimates used inside a contradiction argument. Unfortunately, Du and Weth idea of proof seems to be only partially workable in our situation. Indeed, adapting their estimates to our problem, we are able to get our goal only in the range \(2 < p < 3\), remaining the existence of solutions to \((P_2)\) an open problem for \(3 \leq p < 4\).

The paper is organized as follows: in Section 2 we provide some preliminaries to develop our arguments. Section 3 is devoted to the study of \((P_1)\). It is divided in three parts: in the first, we present a nonexistence result depending on estimate provided in Lemma 2.1 and the combination of a large value of \(\alpha\) and a small value of \(q\); in the second we look for existence of radial solutions, providing an existence and multiplicity result due to the choice of a sufficiently large \(q\); the third is related with the existence of a nonradial solution to a class of problems, including \((P_1)\). Finally, Section 4 is devoted to the discussion of problem \((P_2)\).

2. VARIATIONAL FRAMEWORK AND SOME TECHNICAL RESULTS

First of all note that, when considering \((P_1)\) and \((P_2)\), looking for the second equation of each one, one can see that it is quite natural to consider \(\phi\) as the newtonian potential of \(u^2\), i.e.,
\[
\phi(x) = (\Phi_2 \ast u^2)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x - y|) u^2(y) dy,
\]
where \(\Phi_2\) is the fundamental solution of the laplacian operator in \(\mathbb{R}^2\). Taking this into account, one can see that \((P_1)\) is equivalent to
\[
(P'_1) \quad - \Delta u + (1 + |x|^\alpha)u + q(\Phi_2 \ast u^2)u + |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^2
\]
and \((P_2)\), to
\[
(P'_2) \quad - \Delta u + (1 + |x|^\alpha)u + q(\Phi_2 \ast u^2)u - |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^2.
\]
In this section we set the variational background to deal with \((P'_1)\) and \((P'_2)\). First of all let us define
\[
X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 dx < +\infty \right\},
\]
edowed with the norm
\[
\|u\| = \left( \|\nabla u\|_2^2 + \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 dx \right)^{\frac{1}{2}}.
\]
Let us also denote
\[
\|u\|_* = \left( \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 dx \right)^{\frac{1}{2}}.
\]
in such a way that
\[ \|u\| = (\|\nabla u\|_2^2 + \|u\|_2^2)^{\frac{1}{2}}. \]

Note that, since \( \|u\|_{H^1(\mathbb{R}^2)} \leq \|u\| \) for all \( u \in X \), it follows that the following embeddings are continuous,
\[ X \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all } r \geq 2. \]

As in \([8, 18]\), let us define
\[ V_0(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)dxdy, \]
\[ V_1(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x-y|)u^2(x)u^2(y)dxdy, \]
\[ V_2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x-y|}\right)u^2(x)u^2(y)dxdy \]
\[ \text{and note that } V_0 = V_1 - V_2, \text{ where } V_1, V_2 \geq 0. \]
Moreover, as in \([1, \text{Section } 3.1]\), one can prove that there exists \( C_\alpha > 1 \) such that
\[ V_1(u) \leq \frac{C_\alpha}{\pi} \|u\|_2^2 \|u\|_*^2 \]
and, as in \([8, \text{Section } 2]\), there exists \( D > 0 \)
\[ V_2(u) \leq D\|u\|_4^\frac{4}{3}. \]

For \( u \in X \), we define
\[ I_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2 dx - \frac{q}{4} V_0(u) + \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx, \]
and
\[ J_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2 dx - \frac{q}{4} V_0(u) - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx. \]

By (8) and (9), we can see that \( I_\alpha \) and \( J_\alpha \) are well defined in \( X \). Also, as in \([8, \text{Lemma } 2.2]\), it follows that \( I_\alpha \) and \( J_\alpha \) are \( C^1 \) in this space and their critical points are solutions, respectively, of \((P'_1)\) and \((P'_2)\).

We start with the following fundamental estimate

**Lemma 2.1.** Let \( \alpha > 0 \), \( p > 2 \) and \( \beta > 2 \) such that the term \((\alpha, p, \beta)\) satisfies the following inequality \( \frac{\alpha}{(p-2)(\beta-1)} > 2 \), and \( \varepsilon > 0 \). Then there exists a positive constant \( C \) depending on \((\alpha, p, \beta, \varepsilon)\) such that for any \( u \in L^2(\mathbb{R}^2, (1 + |x|^\alpha) dx) \cap L^p(\mathbb{R}^2) \) we have
\[ \left( \int_{\mathbb{R}^2} \log(2 + |x|)u^2(x) \, dx \right)^2 \leq \frac{2\varepsilon}{\beta} \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2(x) \, dx + C\|u\|_p^{\frac{4(\beta-1)}{3}}. \]

**Proof.** Consider \( u \) as in the assumption and take \( \alpha, p, \beta \) satisfying the prescribed inequalities. Take \( \varepsilon > 0 \) and compute
\[ \int_{\mathbb{R}^2} \log(2 + |x|)u^2(x) \, dx = \int_{\mathbb{R}^2} \frac{\log(2 + |x|)}{u^2} \frac{2\alpha - 2}{\alpha} (1 + |x|^\alpha)^\frac{2}{\alpha} u^\frac{2}{\alpha} (x) \, dx \]
\[ \leq \left( \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2(x) \, dx \right)^\frac{2}{\alpha} \left( \int_{\mathbb{R}^2} \frac{\log(2 + |x|)}{\varepsilon(1 + |x|^\alpha)}u^2(x) \, dx \right)^\frac{\beta - 1}{\beta}. \]
As a consequence, applying the Young inequality, we obtain

\[
\left( \int_{\mathbb{R}^2} \log(2 + |x|)u^2(x) \, dx \right)^2 \\
\leq \frac{2\varepsilon}{\beta} \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2(x) \, dx + \frac{\beta - 2}{\beta \varepsilon^{\frac{\beta}{\alpha} - 2}} \left( \int_{\mathbb{R}^2} \log^{\frac{\beta}{\beta - 1}}(2 + |x|) \, dx \right)^{\frac{2(\beta - 1)}{(\beta - 2)p - 2}}
\]

We conclude observing that in the last line the integral converges. \(\square\)

Now, let us prove a technical result involving \(V_0\).

**Lemma 2.2.** \(V_0\) is weakly continuous in \(X\).

As a consequence, since \(\langle V_0(u), u \rangle = 4V_0(u)\) for all \(u \in X\), also the map

\(u \in X \mapsto \langle V_0(u), u \rangle \in \mathbb{R}\)

is weakly continuous.

**Proof.** Let \((u_n) \subset X\) and \(u \in X\) such that

\(u_n \rightharpoonup u\) in \(X\).

By the coerciveness of the potential \(x \mapsto (1 + |x|^\alpha)\), the embedding \(X \hookrightarrow L^r(\mathbb{R}^2)\) is compact, for all \(r \geq 2\). Then,

\(u_n \to u\) in \(L^r(\mathbb{R}^2)\), for all \(r \geq 2\).

Then

\[
|V_1(u_n) - V_1(u)| = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x - y|) (u_n^2(x) - u_n^2(y)) \, dxdy + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x - y|) (u_n^2(x) - u_n^2(y)) \, dxdy
\]

\[
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x|) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \, dxdy
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |y|) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \, dxdy
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x|) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \, dxdy
\]

\[
+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |y|) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \, dxdy.
\]  

(11)

Let us denote \(C_\alpha, C'_\alpha > 0\), such that

(12) \(\log(2 + r) \leq C_\alpha(1 + r^\alpha)\) and \(\log^2(2 + r) \leq C'_\alpha(1 + r^\alpha)\),
for all $r > 0$. Then, \((11), (12)\) and Hölder inequality imply that
\[
|V_1(u_n) - V_1(u)| \leq C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \ dx \ dy
\]
\[
+ \|u_n\|^2 \int_{\mathbb{R}^2} \log^2 (2 + |y|) |u_n(y) + u(y)|^2 \ dy \|u_n - u\|^2
\]
\[
+ C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u_n^2(x) |u_n(y) - u(y)| |u_n(y) + u(y)| \ dx \ dy
\]
\[
+ \|u_n\|^2 \int_{\mathbb{R}^2} \log^2 (2 + |y|) |u_n(y) + u(y)|^2 \ dy \|u_n - u\|^2
\]
\[
\leq C_\alpha \|u_n\|^2 \|u_n - u\|^2 + C'_\alpha \|u_n\|^2 \|u_n - u\|^2 + C''_\alpha \|u\|^2 \|u_n - u\|^2 + \|u\|^2
\]
\[
= o_n(1).
\]
Hence $V_1$ is weakly continuous. Moreover, by \([8, \text{Lemma 2.2}]\), $V_2$ is also weakly continuous. By these facts, it follows that
\[
V_0(u_n) \rightarrow V_0(u),
\]
as $n \rightarrow +\infty$. \hfill \qedsymbol

3. Existence and nonexistence results to \((P_1)\) for $p > 4$

3.1. A nonexistence result. We start this subsection by proving a nonexistence result to \((P_1)\), for $p > 4$. More specifically, we are going to prove the following result.

**Theorem 3.1.** Let $p > 4$ and $\alpha > \tilde{p}$. Then there exists a constant $\tilde{q} > 0$ such that if $q \in (0, \tilde{q})$, then \((P_1)\) (or equivalently \((P_1')\)) does not have any nontrivial solution.

**Proof.** Assume $u \in X$ is a solution of \((P_1')\). Then
\[
0 = \int_{\mathbb{R}^2} |\nabla u(x)|^2 \ dx + \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \ dx
\]
\[
- \frac{q}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x-y|) u^2(x) u^2(y) \ dx \ dy
\]
\[
+ \frac{q}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left(1 + \frac{|x-y|}{2} \right) u^2(x) u^2(y) \ dx \ dy + \int_{\mathbb{R}^2} |u(x)|^p \ dx
\]
\[
\geq \int_{\mathbb{R}^2} |\nabla u(x)|^2 \ dx + \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \ dx
\]
\[
- \frac{q}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x|) u^2(x) u^2(y) \ dx \ dy + \int_{\mathbb{R}^2} |u(x)|^p \ dx
\]
\[
\geq \int_{\mathbb{R}^2} |\nabla u(x)|^2 \ dx + \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \ dx
\]
\[
- \frac{q}{\pi \log 2} \left( \int_{\mathbb{R}^2} \log(2 + |x|) u^2(x) \ dx \right)^2 + \int_{\mathbb{R}^2} |u(x)|^p \ dx.
\]
By Lemma 2.1 applied for $\beta = \frac{2p-4}{p-1}$ and $\varepsilon = 1$ we can proceed as follows
\[
0 \geq \int_{\mathbb{R}^2} |\nabla u(x)|^2 \ dx + \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \ dx
\]
\[
- qC_1 \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \ dx - qC_2 \|u\|^p + \|u\|^p
\]
Choosing $\tilde{q} = \min \left( \frac{1}{C_1}, \frac{1}{C_2} \right)$, we easily conclude that $u = 0$. \hfill \qedsymbol
3.2. Existence of radial solutions. In this subsection, we are going to study the problem in the set of radial functions. Following the very original approach developed by Ruiz [16, Section 4], we again would like to emphasize the analogies arising between the three dimensional Schrödinger-Maxwell system perturbed by the nonlinear local term $-|u|^{p-2}u$, and this two dimensional model of the Schrödinger-Maxwell system, in presence of the reversed sign local nonlinearity $|u|^{p-2}u$.

We introduce the space

$$X_r = \{ u \in X; u(x) = u(|x|) \},$$

which, as is well known, is a natural constraint.

The following Strauss type radial Lemma will be a useful tool for subsequent estimates

**Lemma 3.2.** Let $\alpha > 0$. Then every $u \in X_r$ is almost everywhere equal to a continuous function $U \in \mathbb{R}^2 \setminus \{0\}$. Moreover, there exists a constant $C > 0$ uniform with respect to $u \in X_r$ such that

$$|u(x)| \leq \frac{C}{|x|^{\alpha+2}} \| u \|, \quad \text{for } |x| \geq 1.$$  

**Proof.** Since the first part is standard, we only prove the estimate.

Let $k \leq \frac{\alpha+2}{2}$ and consider $u$ a radial function in $C_0^\infty(\mathbb{R}^2)$. For any $r > 0$, we have that

$$\left| \frac{d}{dr} \left( r^k u^2(r) \right) \right| \leq k r^{k-1} u^2(r) + 2r^k |u(r)||u'(r)|$$

$$\leq k r^{k-1} u^2(r) + r^{2k-1} u^2(r) + r |u'(r)|^2.$$  

Now, fix $r \geq 1$ and integrate $-\frac{d}{ds} (s^k u^2(s))$ in the interval $[r, +\infty)$. We have

$$r^k u^2(r) \leq k \int_r^{+\infty} s^{k-2-\alpha} s^\alpha u^2(s) \, ds + \int_r^{+\infty} s^{2k-2-\alpha} s^\alpha u^2(s) \, ds + \frac{\| \nabla u \|_2^2}{2\pi}$$

$$\leq \frac{k}{\sqrt{2\pi}} r^{k-2-\alpha} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx + \frac{r^{2k-2-\alpha}}{\sqrt{2\pi}} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx + \frac{\| \nabla u \|_2^2}{2\pi}.$$  

The conclusion follows taking $k = \frac{\alpha+2}{2}$ and then by a density argument. \hfill $\square$

**Lemma 3.3.** For any $\alpha > \tilde{p}$ there exists $\tilde{q} > 0$ such that, for every $q \geq \tilde{q}$ we have $\inf_{u \in X_r} I_\alpha(u) \in (-\infty, 0)$.

**Proof.** First of all, let us consider $\tilde{q}$ large enough in such a way that $\inf_{u \in X_r} I_\alpha(u) < 0$ (such a constant $\tilde{q}$ exists by (21)). For $q \geq \tilde{q}$, let us show that

$$\inf_{u \in X_r} I_\alpha(u) \neq -\infty.$$  

Using Young’s and Hölder’s inequalities as in (10), we obtain, for $\beta > 2$ and $\varepsilon > 0$

$$\left( \int_{\mathbb{R}^2} \log (2 + |x|) u^2(x) \, dx \right)^2$$

$$\leq \frac{2\varepsilon}{\beta} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \, dx + \frac{\beta - 2}{\beta \varepsilon \pi^{\frac{1}{\beta}}} \left( \int_{\mathbb{R}^2} \log \frac{\beta}{\varepsilon} (2 + |x|) \, u^2(x) \, dx \right)^{\frac{2(\beta-1)}{\beta-2}}$$

$$\leq \frac{2\varepsilon}{\beta} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2(x) \, dx$$

$$+ \frac{\beta - 2}{\beta \varepsilon \pi^{\frac{1}{\beta}}} \left( \int_{\mathbb{R}^2} \log^2 (2 + |x|) \, dx \right)^{\frac{\beta}{\beta-2}} \int_{\mathbb{R}^2} |u|^{4(\beta-1)/\beta-2} (x) \, dx.$$
Assume that $\alpha > \beta$ in such a way that
\[
\int_{\mathbb{R}^2} \frac{\log^2(2 + |x|)}{(1 + |x|^\alpha)^{\frac{1}{\beta}}} \, dx < +\infty.
\]

Then in a similar way as in Theorem 3.1
\[
I_\alpha(u) \geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx - \frac{q}{4} V_0(u) + \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx + \frac{q}{4} V_2(u)
\]
\[
- \frac{q}{4 \pi \log^2 2} \left( \int_{\mathbb{R}^2} \log(2 + |x|) u^2(x) \, dx \right)^2 + \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left( \frac{1}{4} - \frac{q\varepsilon}{2 \pi \beta \log 2} \right) \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx
\]
\[
+ \frac{q}{4} V_2(u) + \int_{\mathbb{R}^2} \left( \frac{u^2}{4} - C_\varepsilon |u|^{\frac{4(\beta - 1)}{\beta - 2}} + \frac{1}{p} |u|^p \right) \, dx.
\]

Now, since the functional has to be coercive in its principal part, we choose $\varepsilon > 0$ such that $\frac{1}{4} - \frac{q\varepsilon}{2 \pi \beta \log 2} > \frac{1}{8}$. Moreover, since $\alpha > \tilde{p}$, we can choose $\beta$ sufficiently close to $\alpha$ in such a way $\frac{4(\beta - 1)}{\beta - 2} < p$. On the other hand, it is immediate to see that $4 < \frac{4(\beta - 1)}{\beta - 2}$.

Now we prove that the functional $H$ in $X_\varepsilon$ defined by
\[
H(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left( \frac{1}{4} - \frac{q\varepsilon}{2 \pi \beta \log 2} \right) \int_{\mathbb{R}^2} (1 + |x|^\alpha) u^2 \, dx
\]
\[
+ \frac{q}{4} V_2(u) + \int_{\mathbb{R}^2} \left( \frac{u^2}{4} - C_\varepsilon |u|^{\frac{4(\beta - 1)}{\beta - 2}} + \frac{1}{p} |u|^p \right) \, dx
\]
is bounded from below.

We define $f : [0, +\infty) \to \mathbb{R}$ such that
\[
f(s) = \frac{s^2}{4} - C_\varepsilon s^{\frac{4(\beta - 1)}{\beta - 2}} + \frac{1}{p} s^p
\]
and set $m = \min_{s \geq 0} f(s)$. Assume $m < 0$ (otherwise we are already done) and consider the open set $N = \{ s > 0 \mid f(s) < 0 \}$. A simple study of $f$ shows that $N$ is an open bounded interval $(\gamma, \delta)$ with $\gamma > 0$.

We proceed with the following lower estimate on $H$
\begin{align*}
(13) \quad H(u) &\geq \int_{\mathbb{R}^2} \frac{1}{4} |\nabla u|^2 \, dx + \frac{1}{8} (1 + |x|^\alpha) u^2 \, dx + \frac{q}{4} V_2(u) + \int_{\mathbb{R}^2} f(u) \, dx \\
&\geq \int_{\mathbb{R}^2} \frac{1}{4} |\nabla u|^2 \, dx + \frac{1}{8} (1 + |x|^\alpha) u^2 \, dx + \frac{q}{4} V_2(u) + \int_{\{x \in \mathbb{R}^2 \mid |u(x)| \in (\gamma, \delta)\}} f(u) \, dx \\
&\geq \int_{\mathbb{R}^2} \frac{1}{4} |\nabla u|^2 \, dx + \frac{1}{8} (1 + |x|^\alpha) u^2 \, dx + \frac{q}{4} V_2(u) + m |A_u|
\end{align*}
where $A_u = \{ x \in \mathbb{R}^2 \mid |u(x)| \in (\gamma, \delta) \}$ and $|A_u|$ denotes the Lebesgue measure of $A_u$.

Assume by contradiction that there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $X_\varepsilon$ such that $\lim_{n \to +\infty} H(u_n) = -\infty$. By this fact, and since we can assume that up to a subsequence $H(u_n) < 0$ for all $n \geq 0$, we have
1. $\lim_{n \to +\infty} |A_n| = +\infty$,
2. $\frac{1}{8} |u_n|^2 \leq -m |A_n|$, for all $n \geq 1$,
3. $\frac{q}{4} V_2(u_n) \leq -m |A_n|$, for all $n \geq 1$,
where $A_n = A_{u_n}$.
Take $n \geq 1$ and set $\rho_n = \sup_{x \in A_n} |x|$. Since $|A_n| > 0$, certainly $\rho_n > 0$ and, by Lemma 3.2, $\rho_n = \max_{x \in A_n} |x| \in \mathbb{R}$.

By 3., we have

$$|A_n| \geq -\frac{q}{4m}V_2(u_n) \geq -\frac{q}{8m\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x-y|}\right) u^2(x)u^2(y) \, dx \, dy$$

$$\geq -\frac{q}{8m\pi} \int_{A_n} \int_{A_n} \log \left(1 + \frac{2}{|x-y|}\right) u^2(x)u^2(y) \, dx \, dy$$

$$\geq -\frac{q^4}{8m\pi} \log \left(1 + \frac{1}{\rho_n}\right) |A_n|^2,$$

and then there exists $c_1 > 0$ such that

$$(14) \quad \log \left(1 + \frac{1}{\rho_n}\right) |A_n| \leq c_1.$$

Now, if $(\rho_n)_n$ is bounded from above, we easily get a contradiction comparing (14) with 1. Assume then that $\lim_n \rho_n = +\infty$. Define $x_n \in A_n$ such that $|x_n| = \rho_n$. By 2., Lemma 3.2 and since we can assume $|x_n| \geq 1$ for every $n \geq 1$,

$$\gamma \leq |u_n(x_n)| \leq \frac{C}{|x_n|} \|u_n\| \leq \frac{\sqrt{8mC}}{\rho_n^{\frac{\beta}{2}}} |A_n|^{\frac{1}{\gamma}}$$

and then there exists $c_2 > 0$ such that

$$(15) \quad \frac{\alpha+2}{\rho_n^{\frac{\beta}{2}}} \leq c_2 |A_n|.$$

Comparing (14) with (15) we have

$$\frac{\alpha+2}{\rho_n^{\frac{\beta}{2}}} \log \left(1 + \frac{1}{\rho_n}\right) \leq c_1 c_2$$

and, since $\alpha > 0$, this contradicts the fact that $\lim_n \rho_n = +\infty$. $\square$

**Lemma 3.4.** The functional $I_\alpha$ restricted to $X_r$ satisfies the Palais-Smale condition.

**Proof.** Assume that $(u_n)_n$ is a Palais-Smale sequence for the functional $I_\alpha|X_r$. In particular, we have that, for any $n$ large enough,

$$\langle I'_\alpha(u_n), u_n \rangle \leq \|I'_\alpha(u_n), u_n\| \leq \|n\|$$

that is

$$(16) \quad \|u_n\|^2 - qV_0(u_n) + \|u_n\|_p^p \leq \|u_n\|.$$

On the other hand, making computations similar to those in Lemma 3.3, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$(17) \quad \|u_n\|^2 - qV_0(u_n) + \|u_n\|_p^p$$

$$\geq \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \left(\frac{1}{2} - \frac{2q\varepsilon}{\pi\beta \log 2}\right) \int_{\mathbb{R}^2} (1 + |x|)u_n^2 \, dx$$

$$+ qV_2(u_n) + \int_{\mathbb{R}^2} \left(\frac{1}{2} u^2 - C_\varepsilon |u_n|^{\frac{4(\beta-1)}{3\beta-2}} + |u_n|^p\right) \, dx.$$

Comparing (16) and (17), for a suitable choice of $\varepsilon > 0$, we obtain

$$\frac{1}{4} \|u_n\|^2 + qV_2(u_n) + \int_{\mathbb{R}^2} \left(\frac{1}{2} u^2 - C_\varepsilon |u_n|^{\frac{4(\beta-1)}{3\beta-2}} + |u_n|^p\right) \, dx \leq \|u_n\|.$$
We claim that \( (\|u_n\|) \) is bounded. Otherwise, for any large value of \( n \), we would have
\[
\frac{1}{8} \|u_n\|^2 + q V_2(u_n) + \int_{\mathbb{R}^2} \left( \frac{1}{2} u^2 - C_2 |u_n|^{\frac{a(q-1)}{q-2}} + |u_n|^p \right) dx \leq \|u_n\| - \frac{1}{8} \|u_n\|^2 \leq 0
\]
which leads to a contradiction by the same arguments developed in the proof of Lemma 3.3 from inequality (13) on. Now, as usual, we extract a subsequence, relabeled in the same way, weakly convergent to \( \bar{u} \in X_r \). At this point we can conclude as usual, once we have observed that
\[
\|u_n\|^2 = q V_0(u_n) - \|u_n\|^p + o_n(1)
= q V_0(\bar{u}) - \|\bar{u}\|^p + o_n(1)
= \|\bar{u}\|^2 + o_n(1)
\]
where we have used
\[
\langle I'_n(u_n), u_n \rangle = o_n(1)
\]
in the first equality, compactness in the second and
\[
0 = \lim_n \langle I'_n(u_n), \bar{u} \rangle = \langle I'_n(\bar{u}), \bar{u} \rangle
\]
in the third.

**Theorem 3.5.** Let \( \alpha > \tilde{p} \) and \( q \geq \tilde{q} \). Then the system \( (P_1) \) has both a minimizer (at a negative level) and a mountain pass solution on the set of radial functions.

Moreover, there exists a nonincreasing sequence \( (q_n)_{n \geq 1} \) such that, for any \( n \geq 1 \), system \( (P_1) \) possesses at least \( n \) couples of radial solutions \( (\pm u_{k,q}) \), \( k = 1, \ldots, n \), such that \( I_\alpha(u_{k,q}) < 0 \) and \( n \) couples of radial solutions \( (\pm v_{k,q}) \), \( k = 1, \ldots, n \), such that \( I_\alpha(v_{k,q}) > 0 \), for any \( q \geq q_n \).

**Proof.** Using the Ekeland variational principle, the existence of a minimizer follows directly from Lemma 3.3 and Lemma 3.4 by a well known variant of Weierstrass Theorem.

To show that there exists a mountain pass solution, by Lemma 3.4, we only have to check the geometrical conditions of Ambrosetti-Rabinowitz mountain pass Theorem.

Observe that, if we take any \( u \in X_r \) such that \( \|u\| = \rho \) with \( \rho > 0 \) small enough, then by (3) and (8)
\[
I_\alpha(u) \geq \frac{1}{2} \|u\|^2 - C_1 \|u\|^2 \|u\|^2 + \frac{1}{p} \|u\|^p \\
\geq \frac{1}{2} \|u\|^2 - C_1 \|u\|^4 = \rho^2 \left( \frac{1}{2} - C_1 \rho^2 \right) \\
\geq \frac{\rho^2}{4}
\]
Moreover the second mountain pass geometric condition is trivially satisfied by the minimizer \( u_0 \).

In order to prove the second part of the theorem, we will use abstract results of [4], proceeding in a similar way as in [5, Theorem 3.1].

First we look for solutions at negative levels of the functional. Consider an increasing (in the sense of inclusion) sequence \( (X_n)_{n \geq 1} \) such that for any \( n \geq 1 : X_n \) is an \( n \)-dimensional subspace of \( X_r \). Set \( S = \{ u \in X_r : \|u\| = 1 \} \) and \( S'_n = X_n \cap S \) for any \( n \geq 1 \).
We define
\[
M_1 = \max_{u \in S_n} \frac{1}{2} \int_{R^2} (|\nabla u|^2 + (1 + |x|^a)u^2) \, dx, \quad m_2 = \min_{u \in S_n} \frac{1}{8\pi} \left( \int_{R^2} u^2 \, dx \right)^2,
\]
\[
M_3 = \max_{u \in S_n} \frac{1}{4} |V_0(u)|, \quad M_4 = \max_{u \in S_n} \frac{1}{p} \int_{R^2} |u|^p \, dx,
\]
and, for all \( u \in X_r \), set \( u_t = u(\cdot/t) \). Consider \( \bar{t} > 1 \) such that \( m_2 \log \bar{t} - M_3 > 0 \):
\[
I_\alpha(u_t) = \frac{1}{2} ||\nabla u||^2 + \frac{\bar{t}^2}{2} \int_{R^2} (1 + \bar{t}^a |x|^a)u^2 \, dx \\
- \frac{q_1(\bar{t})^4 \log \bar{t}}{8\pi} \left( \int_{R^2} u^2 \, dx \right)^2 - \frac{q_2(\bar{t})^4}{4} V_0(u) + \frac{\bar{t}^2}{p} \int_{R^2} |u|^p \, dx \\
\leq M_1 \bar{t}^{2+a} - q_2(m_2 \log \bar{t} - M_3) \bar{t}^4 + M_4 \bar{t}^2.
\]
We take \( q_2(n) > 0 \) such that\( M_1 \bar{t}^{2+a} - q_2(n) (m_2 \bar{t}^4 \log \bar{t} - M_3) \bar{t}^4 + M_4 \bar{t}^2 < 0 \) and define \( \tau : X_r \to X_r \) such that \( \tau(u) = u_{\bar{t}} \). Of course, \( \tau \) is continuous and odd and then by well known properties coming from index theory, we have \( \gamma(\tau(S_n)) \geq n \), where \( \gamma \) denotes the Krasnoselski genus.

Now take \( q \geq q_2(n) \). By (18) we deduce that, for any \( k = 1, \ldots, n \),
\[
d_k = \inf_{\gamma(\alpha) \leq k} \sup_{A \in \Sigma(X_r)} \{I_\alpha(u); u \in A\} < 0
\]
where \( \Sigma(X_r) \) represents the set of closed subsets of \( X_r \setminus \{0\} \) which are symmetric with respect to the origin. Since \( I_\alpha \) satisfies the Palais-Smale condition and \( I_\alpha \) is bounded from below, by [4, Corollary 2.24] every value \( d_k \) is critical, and there exists at least \( n \) couples of solutions, some of them possibly at the same level.

As for the (at least) \( n \) couples of solutions at positive levels, we just have to verify condition \( I_\gamma \) to apply [4, Theorem 2.23]. Consider \( n \geq 1 \) and \( u_1, \ldots, u_n \in X_r \) linearly independent. Define \( S^{n-1} = \{\sigma \in R^n; |\sigma| = 1\} \) and for any \( \sigma = (\sigma_1, \ldots, \sigma_n) \in S^{n-1} \) set
\[
u_\sigma = \sum_{i=1}^n \sigma_i u_i.
\]
Moreover, we define
\[
M_1 = \max_{\sigma \in S^{n-1}} \frac{1}{2} \int_{R^2} (|\nabla u_\sigma|^2 + (1 + |x|^a)u_\sigma^2) \, dx, \quad m_2 = \min_{\sigma \in S^{n-1}} \frac{1}{8\pi} \left( \int_{R^2} u_\sigma^2 \, dx \right)^2,
\]
\[
M_3 = \max_{\sigma \in S^{n-1}} \frac{1}{4} |V_0(u_\sigma)|, \quad M_4 = \max_{\sigma \in S^{n-1}} \frac{1}{p} \int_{R^2} |u_\sigma|^p \, dx,
\]
and \( u_{\sigma t} = u_\sigma(\cdot/t) \). Consider \( \bar{t} > 1 \) such that \( m_2 \log \bar{t} - M_3 > 0 \) and take \( q_2(n) > 0 \) such that
\[
I_\alpha(u_{\sigma t}) = ||\nabla u_\sigma||^2 + \frac{\bar{t}^2}{2} \int_{R^2} (1 + \bar{t}^a |x|^a)u_\sigma^2 \, dx \\
- \frac{q_2(n) \bar{t}^4 \log \bar{t}}{8\pi} \left( \int_{R^2} u_\sigma^2 \, dx \right)^2 - \frac{q_2(n) \bar{t}^4}{4} V_0(u_\sigma) + \frac{\bar{t}^2}{p} \int_{R^2} |u_\sigma|^p \, dx \\
\leq M_1 \bar{t}^{2+a} - q_2(n) (m_2 \log \bar{t} - M_3) \bar{t}^4 + M_4 \bar{t}^2 < 0.
\]
Observe that for any \( q \geq q_2(n) \), the set \( A\{u_\sigma; \sigma \in S^{n-1}\} \) verifies \( I_\gamma \) of [4, Theorem 2.23].

We conclude our proof taking \( q_n = \max\{q_2(1), q_2(n)\} \) for all \( n \geq 1 \). □

**Remark 3.6.** Observe that the minimizing argument, and in particular the proof of Lemma 3.3, can not be repeated excluding the property of radiality. In [17] it was showed that radiality assumption is not just technical in the three dimensional case, since the author
proved that, on the other hand, the unconstrained functional turns out to be unbounded from below. Unfortunately, differently from [17], we are not able to obtain an analogous result in our situation, so that the question about global boundedness and existence of a global minimizer remains an interesting open problem.

3.3. Existence of a nonradial solution by constrained minimization. It is worthy of note that the arguments previously used to prove the existence of solutions work by the assumption of radial symmetry on the functional framework, necessary to exploit Strauss estimate. To complete our study on problem \( (P_1) \), we are going to show the existence of a nonradial solution.

Unfortunately, since we will proceed by means of constrained minimization, the result we present is weaker than Theorem 3.5, due to the fact that we treat \( q \) as an unknown of the problem, coming in the form of a Lagrange multiplier.

First of all, in order to prevent our solution to be radial, we change our functional framework from \( X_r \) to

\[
\tilde{X} = \{ u \in X; u(-x_1, x_2) = -u(x_1, x_2) \text{ and } u(x_1, -x_2) = u(x_1, x_2), \forall (x_1, x_2) \in \mathbb{R}^2 \}.
\]

Such a space has been introduced in [12] in order to prove the existence of nonradial solutions to the planar Schrödinger-Poisson system. By the well known Palais principle of symmetric criticality, \( \tilde{X} \) is a natural constraint. We are going to prove our result in a more general setting. Indeed, assume that \( W: \mathbb{R} \to \mathbb{R} \) satisfies the following assumptions

\begin{itemize}
  \item [W1)] \( W = W(s) \in C^1(\mathbb{R}) \),
  \item [W2)] \( W \) is nonnegative,
  \item [W3)] there exist \( p > 2, C_1 \) and \( C_2 \) positive constants such that \( W(s) \leq C_1 s^2 + C_2 |s|^p, \)
\end{itemize}

**Theorem 3.7.** Assume W1, W2 and W3 and consider the problem

\[
(P_W) \quad \begin{cases}
-\Delta u + (1 + |x|)u - q\phi u + W'(u) = 0 & \text{in } \mathbb{R}^2, \\
\Delta \phi = u^2 & \text{in } \mathbb{R}^2,
\end{cases}
\]

Then for any \( \alpha > 0 \) there exists \( q > 0 \) such that \( (P_W) \) has a nonradial solution. Such a solution is odd with respect to the first variable and even with respect to the second.

**Proof.** Let us consider the functional

\[
G_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} (1 + |x|)u^2 \, dx + \int_{\mathbb{R}^2} W(u) \, dx
\]

constrained to the set

\[
H := \left\{ u \in \tilde{X}; V_0(u) = 1 \right\}.
\]

It is straightforward to see that \( G_\alpha \) is a \( C^1 \) functional in \( \tilde{X} \). Also, \( H \) is a \( C^1 \) manifold in \( X \), since for every \( u \in H \), by [12, Lemma 2.3], \( V \in C^1(X) \) and also \( \langle V'(u), u \rangle = 4V_0(u) = 4 \). Moreover, \( H \) is also a non empty set. Indeed, for \( u \in X \setminus \{0\} \), we consider \( u_t = u(\cdot/t) \) and compute

\[
V_0(u_t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)u_t^2(x)u_t^2(y) \, dxdy
= \frac{t^4}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)u^2(x)u^2(y) \, dxdy + \frac{t^4 \log t}{2\pi} \left( \int_{\mathbb{R}^2} u^2(x) \, dx \right)^2.
\]

Since

\[
\lim_{t \to 0} V_0(u_t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} V_0(u_t) = +\infty,
\]

\[
\lim_{t \to 0} V_0(u_t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} V_0(u_t) = +\infty,
\]
it follows that there exists $t_0 > 0$ such that $t_0 u \in H$.

Coerciveness of $G$ is straightforward to see, since

$$G_{\alpha}(u) \geq \frac{1}{2} \|u\|^2.$$  

In order to show that there exists $\overline{u} \in H$ such that

$$G_{\alpha}(\overline{u}) = \inf_{u \in H} G_{\alpha}(u),$$

let $(u_n) \subset H$ be such that

$$\lim_{n \to +\infty} G_{\alpha}(u_n) = \inf_{u \in H} G_{\alpha}(u).$$

By the coercivity of $G_{\alpha}$, it follows that $(u_n)$ is bounded in $\tilde{X}$. Then, there exists $\overline{u} \in \tilde{X}$ such that, up to a subsequence,

$$u_n \rightharpoonup \overline{u} \text{ in } \tilde{X}, \text{ as } n \to +\infty.$$  

From Lemma 2.2, $\overline{u} \in H$. Also, from the compactness of the embeddings $\tilde{X} \hookrightarrow L^r(\mathbb{R}^2)$, for all $r \geq 2$, it follows that

$$u_n \to \overline{u} \text{ in } L^r(\mathbb{R}^2), \text{ as } n \to +\infty.$$  

In particular,

$$\|u_n\|_p^p \to \|\overline{u}\|_p^p, \text{ as } n \to +\infty.$$  

Hence, it follows that

$$\inf_{u \in H} G_{\alpha}(u) \leq G_{\alpha}(\overline{u}) \leq \liminf_{n \to +\infty} \left( \frac{1}{2} \|u_n\|^2 + \frac{1}{p} \|u_n\|_p^p \right) = \liminf_{n \to +\infty} G_{\alpha}(u_n) = \inf_{u \in H} G_{\alpha}(u),$$

from where it follows that $\overline{u}$ is a minimizer of $G_{\alpha}$ on $H$.

Then, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$G'_{\alpha}(\overline{u}) = \lambda V'_0(\overline{u}).$$

In order to prove that $\lambda > 0$, we recall an argument used in [7, Page 327]. First observe that $\lambda \neq 0$. Indeed, if we assume $\lambda = 0$, then $\overline{u}$ would be a nontrivial solution of the equation

$$-\Delta u + (1 + |x|^\alpha)u + W'(u) = 0 \text{ in } \mathbb{R}^2,$$

However, by standard arguments, we could show that $\overline{u}$ satisfies the following Pohozaev identity

$$\|\overline{u}\|_2^2 + \left( \frac{2 + \alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha \overline{u}^2 \, dx + 2 \int_{\mathbb{R}^2} W(\overline{u}) \, dx = 0$$

which, by assumption $W2$, is satisfied only by $\overline{u} = 0$.

So, assume by contradiction that $\lambda < 0$. Then, since $\langle V'_0(\overline{u}), \overline{u} \rangle > 0$, by (26) certainly $\langle G'_{\alpha}(\overline{u}), \overline{u} \rangle < 0$. Choosing $\varepsilon > 0$ small enough, we would have

$$V_0((1 + \varepsilon)\overline{u}) = V_0(\overline{u}) + \varepsilon \langle V'_0(\overline{u}), \overline{u} \rangle + o(\varepsilon) > V_0(\overline{u})$$

$$G_{\alpha}((1 + \varepsilon)\overline{u}) = G_{\alpha}(\overline{u}) + \varepsilon \lambda \langle V'_0(\overline{u}), \overline{u} \rangle + o(\varepsilon) < G_{\alpha}(\overline{u})$$
So we deduce that there exists \( \bar{v} \in X \), with \( \bar{v} \neq 0 \), such that \( V_0(\bar{v}) > V_0(u) = 1 \) and \( G_\alpha(\bar{v}) < G_\alpha(u) = \inf_{u \in H} G_\alpha(u) \). This immediately leads to a contradiction, since there exists \( \sigma \in (0,1) \) such that \( \bar{v} = \bar{v}(\cdot/\sigma) \in H \) (see (20)), and

\[
G_\alpha(\bar{v}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \bar{v}|^2 \, dx + \frac{\sigma^2}{2} \int_{\mathbb{R}^2} (1 + |\sigma x|^\alpha) \bar{v}^2 \, dx + \sigma^2 \int_{\mathbb{R}^2} W(\bar{v}) \, dx < G_\alpha(u) < \inf_{u \in H} G_\alpha(u).
\]

Taking \( q = 4 \lambda \), we get a solution of \( (P_1) \).

\[\square\]

4. Existence of solution for \( (P_2) \)

In this section, we use variational methods to find solutions of \( (P_2) \) (or \( (P'_2) \), equivalently). Our final result is the following

**Theorem 4.1.** If \( \alpha > 0 \) and either \( 2 < p < 3 \) or \( p \geq 4 \), then \( (P_2) \) possesses infinitely many solutions.

In order to get it, let us show that the functional \( J_\alpha \) associated to \( (P'_2) \), satisfies the following geometrical properties

**Proposition 4.2.** Take \( r > \frac{\alpha-2}{2} \), \( w \in X \setminus \{0\} \) and for all \( t > 0 \) set \( w_t(\cdot) = t^r w(\cdot/t) \). Then there exist \( \rho, \beta, t > 0 \) such that

i) \( J_\alpha(u) \geq \beta \) for all \( u \in X \) such that \( \|u\| = \rho \);

ii) \( J_\alpha(w_t) < 0 \) and \( \|w_t\| > \rho \).

**Proof.** Note that, for any \( u \in X \), from (3) and (8)

\[
J_\alpha(u) = \frac{1}{2} \|u\|^2 - \frac{1}{4} V_1(u) + \frac{1}{4} V_2(u) - \frac{1}{p} \|u\|^p_p
\]

\[
\geq \frac{1}{2} \|u\|^2 - \frac{C_\alpha}{4\pi} \|u\|^2 \|u\|^2 - \frac{C}{p} \|u\|^p_p
\]

\[
\geq \frac{1}{2} \|u\|^2 - \frac{C_\alpha}{4\pi} \|u\|^4 - \frac{C}{p} \|u\|^p_p
\]

\[
= \|u\|^2 \left( \frac{1}{2} - \frac{C_\alpha}{4\pi} \|u\|^2 - \frac{C}{p} \|u\|^p_p \right)
\]

\[
\geq \rho^2 \left( \frac{1}{2} - \frac{C_\alpha}{4\pi} \rho^2 - \frac{C}{p} \rho^{p-2} \right)
\]

\[
= \beta.
\]

where \( \rho > 0 \) is such that \( \frac{1}{2} - \frac{C_\alpha}{4\pi} \rho^2 - \frac{C}{p} \rho^{p-2} > 0 \). This proves i).

As to ii), we compute

\[
J_\alpha(w_t) = \frac{t^{2r}}{2} \int_{\mathbb{R}^2} |\nabla w_t|^2 \, dx + \frac{t^{2r+2}}{2} \int_{\mathbb{R}^2} (1 + t^\alpha |x|^\alpha) w_t^2 \, dx
\]

\[
- \frac{t^{4r+4} \log t}{8\pi} \|w_t\|^4 - \frac{t^{4r+4}}{4} V_0(w) - \frac{t^{p+2}}{p} \|w_t\|^p_p.
\]

Then, by the choice of \( r \), it follows that \( J_\alpha(w_t) \to -\infty \) as \( t \to +\infty \). Moreover, since \( \|w_t\| \to +\infty \) as \( t \to +\infty \), we can choose \( t \) such that such that \( J_\alpha(w_t) < 0 \) and \( \|w_t\| > \rho \), which proves ii).

\[\square\]

Now take \( k \in \mathbb{N} \), \( k \geq 1 \) and consider the following constraint introduced by Du and Weth in [12] (see (iii) in Example 1.1.):

\[
X_k := \{ u \in X; A_j^1 \ast u = u, \forall j = 1, 2, \ldots, 2k \}
\]
where the operator $A^j : X \to X$ is such that
$$u \in X \mapsto (-1)^j (u \circ A^j)$$
and $A^j : \mathbb{R}^2 \to \mathbb{R}^2$ is such that for any $x \in \mathbb{R}^2 : A^j(x) = A^j \cdot x$ with
$$A^j = \begin{pmatrix} \cos \frac{j \pi}{k} & -\sin \frac{j \pi}{k} \\ \sin \frac{j \pi}{k} & \cos \frac{j \pi}{k} \end{pmatrix}.$$  
Of course, apart from the null functions, all functions in $X_k$ are sign changing. Moreover by arguments based on Palais symmetric criticality principle, it can be proved that $X_k$ is a natural constraint for $J_\alpha$. In what follows we are interested in finding a critical point of $J_\alpha |_{X_k}$ at the mountain pass level.

First observe that, since for every $w \in X_k \setminus \{0\}$ and $t > 0$ we have $w_t \in X_k$, the set
$$\Gamma_k = \{ \gamma \in C([0, 1], X_k) : \gamma(0) = 0 \text{ and } J_\alpha(\gamma(1)) < 0, \|\gamma(1)\| > \rho \}$$
is nonempty by ii) of Proposition 4.2. Then, by i) of Proposition 4.2, the number
$$c_{\alpha,k} = \inf_{\gamma \in \Gamma_k} \max_{0 \leq t \leq 1} J_\alpha(\gamma(t))$$
is well defined and larger than $\beta$.

**Proposition 4.3.** There exists $(u_n)_{n \in N} \subset X_k$ such that, as $n \to +\infty$,
$$J_\alpha(u_n) \to c_{\alpha,k},$$
$$J'_\alpha(u_n) \to 0$$
and
(27)  $$P_\alpha(u_n) \to 0,$$
where for an arbitrary $r \in \mathbb{R}$,
$$P_\alpha(u) = r \|\nabla u\|_2^2 + (r + 1)\|u\|_2^2 + \left( r + 1 + \frac{\alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha u^2 \, dx - \frac{q}{8\pi} \|u\|_2^2 - q(r + 1)V_0(u) - \frac{pr + 2}{p} \|u\|_p^p.$$

The proof follows by the same arguments as [1, Lemma 3.5].

Note that, by the last result, there exists $(u_n)_{n \in N} \subset X_k$ a Palais-Smale sequence for $J_\alpha$ at the level $c_{\alpha,k}$. On the other hand, in the next result, we prove that such a sequence is bounded in $X_k$.

**Proposition 4.4.** If $p \geq 4$, then any Palais-Smale sequence for $J_\alpha$ is bounded. If $2 < p < 3$, then the sequence $(u_n)_{n \in N}$ coming from Proposition 4.3 is bounded in $X_k$.

**Proof.** First of all, consider the case $p \geq 4$. Assume that $(v_n)_{n \in N}$ in $X$ is a Palais-Smale sequence for $J_\alpha$. Then, since $J'_\alpha(v_n) \to 0$, we have
$$o_n(1)\|v_n\| = \langle J'_\alpha(v_n), v_n \rangle$$
$$= \|\nabla v_n\|^2_2 + \|v_n\|^2_2 + \int_{\mathbb{R}^N} |x|^\alpha v_n^2 \, dx - qV_0(v_n) - \|v_n\|^p_p$$
$$=: N_\alpha(v_n).$$
Moreover, since $J_\alpha(v_n)$ is bounded, then there exists $M > 0$ such that
$$M + o_n(1)\|v_n\| \geq J_\alpha(v_n) - \frac{1}{4} N_\alpha(v_n)$$
$$= \frac{1}{4} \|\nabla v_n\|^2_2 + \frac{1}{4} \int_{\mathbb{R}^2} (1 + |x|^\alpha) v_n^2 \, dx + \left( \frac{1}{4} - \frac{1}{p} \right) \|v_n\|^p_p.$$  
This, in turn, implies that $(v_n)_{n \in N}$ is bounded in $X_k$. 

Now assume $2 < p < 3$ and consider $(u_n)_n$ as in Proposition 4.3. Applying (27) we obtain

\begin{equation}
(29) \quad c_{\alpha,k} + o_n(1) = J_\alpha(u_n) - \frac{1}{4(r + 1)} P_\alpha(u_n)
\end{equation}

\begin{equation*}
= \left( \frac{r + 2}{4(r + 1)} \right) \| \nabla u_n \|_2^2 + \frac{1}{4} \| u_n \|_2^2
\end{equation*}

\begin{equation*}
+ \left( \frac{2(r + 1) - \alpha}{4(r + 1)} \right) \int_{\mathbb{R}^2} |x|^\alpha u_n^2 \, dx
\end{equation*}

\begin{equation*}
+ \left( \frac{(p - 4)r - 2}{4p(r + 1)} \right) \| u_n \|_p^p + \frac{q}{32\pi(r + 1)} \| u_n \|_2^4.
\end{equation*}

If we take $r > \frac{4p - 2}{p - 4}$, there exist $C_i > 0$, $i \in \{1, \ldots, 5\}$, such that

\begin{equation}
(30) \quad c_{\alpha,k} + o_n(1) = C_1 \| \nabla u_n \|_2^2 + C_2 \| u_n \|_2^2 - C_3 \| u_n \|_p^p + C_4 \int_{\mathbb{R}^2} |x|^\alpha u_n^2 \, dx + C_5 \| u_n \|_2^4.
\end{equation}

Suppose by contradiction that, up to a subsequence, $\| \nabla u_n \|_2 \to +\infty$, as $n \to +\infty$. Let $t_n = \| \nabla u_n \|_2^{-\frac{1}{2}}$ and note that $\lim_{n \to +\infty} t_n = 0$. Define

\begin{equation*}
v_n(x) = t_n^2 u_n(t_n x),
\end{equation*}

in such a way that, for all $1 \leq q < +\infty$,

\begin{equation}
(31) \quad \| \nabla v_n \|_2^2 = t_n^4 \| \nabla u_n \|_2^2 = 1, \quad \| v_n \|_q^q = t_n^{2q - 2} \| u_n \|_q^q.
\end{equation}

Then, multiplying (30) by $t_n^4$, it follows that

\begin{equation}
(32) \quad c_{\alpha,k} t_n^4 + o(t_n^4) = C_1 t_n^4 \| \nabla u_n \|_2^2 + C_2 t_n^4 \| u_n \|_2^2
\end{equation}

\begin{equation*}
- C_3 t_n^4 \| u_n \|_p^p + C_4 t_n^4 \int_{\mathbb{R}^2} |x|^\alpha u_n^2 \, dx + C_5 t_n^4 \| u_n \|_2^4.
\end{equation*}

Moreover, by Gagliardo-Nirenberg’s inequality,

\begin{equation}
(33) \quad \| u_n \|_p^p \leq C \| u_n \|_2^2 \| \nabla u_n \|_2^{p - 2} = C t_n^{4 - 2p} \| u_n \|_2^2.
\end{equation}

Then,

\begin{equation}
(34) \quad t_n^4 \| u_n \|_p^p \leq C t_n^{6 - 2p} \| v_n \|_2^2.
\end{equation}

By (32) and (34), it follows that

\begin{equation*}
\tilde{C} t_n^4 \geq c_{\alpha,k} t_n^4 + o(t_n^4)
\end{equation*}

\begin{equation*}
\geq C_5 \| v_n \|_2^2 - C t_n^{6 - 2p} \| v_n \|_2^2.
\end{equation*}

Hence, we can see that there exists $\tilde{C} > 0$ such that, for $n$ sufficiently large,

\begin{equation}
(35) \quad \| v_n \|_2 \leq \tilde{C} t_n^{3 - p}.
\end{equation}

Moreover, from (32), (34) and (35), it follows that

\begin{equation}
(36) \quad t_n^4 \int_{\mathbb{R}^2} |x|^\alpha u_n^2 \, dx \leq c_{\alpha,k} t_n^4 + o(t_n^4) + C_5 t_n^4 \| u_n \|_p^p = o_n(1).
\end{equation}

Moreover, by (35),

\begin{equation}
(37) \quad t_n^4 V_0(u_n) = V_0(v_n) + \| v_n \|_2^4 \log(t_n) = V_0(v_n) + o_n(1).
\end{equation}
From Proposition 4.3 applying (27) for \( r = 0 \), it follows that

\[
\begin{align*}
\text{Proof of Theorem } & (41) + \text{Lemma } (41) \\
(38) & \qquad o(t_n^4) = t_n^4 P_{\alpha}(u_n) \\
& = t_n^4 \| u_n \|^2_2 + \left( \frac{2 + \alpha}{2} \right) t_n^4 \int_{\mathbb{R}^2} |x|^{\alpha} u_n^2 dx - q t_n^4 V_0(u_n) \\
& \quad - \frac{q}{8\pi} t_n^4 \| u_n \|^3_2 - \frac{2}{p} t_n^4 \| u_n \|^p_p \\
& = t_n^2 \| v_n \|^2_2 + \left( \frac{2 + \alpha}{2} \right) t_n^4 \int_{\mathbb{R}^2} |x|^{\alpha} u_n^2 dx - \frac{q}{8\pi} \| v_n \|^2_2 \\
& \quad - \frac{2}{p} t_n^4 \| u_n \|^p_p - q V_0(v_n) + o_n(1).
\end{align*}
\]

From (34) and (35), it follows that

\[
(39) \quad t_n^4 \| u_n \|^p_p \leq C \| v_n \|^2_2 \cdot t_n^6 - 2p = o_n(1).
\]

Hence, (36), (38) and (39) imply that

\[
(40) \quad V_0(v_n) = o_n(1).
\]

Then, multiplying (28) by \( t_n^4 \) and using (31), (36), (37), (39) and (40), we have that

\[
\begin{align*}
o(t_n^4) \| u_n \| & = t_n^4 \| \nabla u_n \|^2_2 + t_n^4 \| u_n \|^2_2 + t_n^4 \int_{\mathbb{R}^2} |x|^{\alpha} u_n^2 dx \\
& \quad - q t_n^4 V_0(u_n) - t_n^4 \| u_n \|^p_p \\
& = 1 + o_n(1),
\end{align*}
\]

which is a contradiction since, by (31) and (36),

\[
t_n^4 \| u_n \| = \left( t_n^4 + t_n^8 \int_{\mathbb{R}^2} |x|^{\alpha} u_n^2 dx \right)^{\frac{1}{2}} = o_n(1).
\]

Hence, \( \langle \nabla u_n \rangle \) is bounded in \( L^2(\mathbb{R}^2) \).

Moreover, by (30) and (33) and taking into account that there exists \( C_6 > 0 \) such that \( \| \nabla u_n \|^2_2 \leq C_6 \) for all \( n \in \mathbb{N} \), we have

\[
C_5 \| u_n \|^4_2 + (C_2 - C C_3 C_6^{p-2}) \| u_n \|^2_2 + C_4 \int_{\mathbb{R}^2} |x|^{\alpha} u_n^2 dx \leq C_{\alpha,k} + o_n(1).
\]

The last estimate, in turn, implies that \( \langle u_n \rangle_n \) is bounded in \( L^2(\mathbb{R}^N) \) and also in \( X_k \).

\[ \square \]

**Proof of Theorem 4.1.** Take \( \langle u_n \rangle_n \) as in Proposition 4.4. It follows that there exists \( u_k \in X_k \) such that

\[
u_n \rightarrow u_k, \quad \text{in } X_k.
\]

From the compact embeddings \( X_k \hookrightarrow L^r(\mathbb{R}^2) \) for \( r \geq 2 \), we have that

\[
(41) \quad u_n \rightarrow u_k, \quad \text{in } L^r(\mathbb{R}^2), \quad \text{for } r \geq 2.
\]

Hence, the same arguments as in Lemma 2.2 and (41) imply that \( u_k \) is a weak solution to \( (P_\alpha^1) \). Moreover, since \( \langle J_\alpha(u_n) \rangle = o_n(1) \) and \( \langle J_\alpha(u) \rangle = 0 \), by Lemma 2.2 it follows that

\[
\| u_n \|^2 = \frac{q}{4} \langle V_0(u_n) \rangle + \| u_n \|^p_p + o_n(1) = \frac{q}{4} \langle V_0(u_k) \rangle + \| u_k \|^p_p + o_n(1) = \| u_k \|^2 + o_n(1).
\]

Hence, from the last equality it follows that

\[
u_n \rightarrow u_k, \quad \text{in } X
\]

and then,

\[
J_\alpha(u_k) = \kappa_{\alpha,k}.
\]
Since $c_{\alpha,k} > 0$, we have that $u_k \in X_k$ is a nontrivial sign-changing solution of $(P^k_\alpha)$ at the mountain pass minimax level.

Now, in order to prove multiplicity, we can proceed exactly as in [12, Proof of Corollary 1.4.] to obtain a sequence of solutions $(u_{3h})_h$ such that, for all $h \geq 1$

- $u_{3h} \in X_{3h}$,
- $J_{\alpha}(u_{3h}) = c_{\alpha,3h}$,

being the sequence $(c_{\alpha,3h})_h$ of the mountain pass levels of $J_{\alpha}$ on $X_{3h}$ non-decreasing and unbounded from above. □

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Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata,
Via dell’Ateneo Lucano 10, I-85100 Potenza, Italy
Email address: antonio.azzollini@unibas.it

Departamento de Matemática e Computação, Universidade Estadual Paulista - Unesp,
CEP: 19060-900, Presidente Prudente - SP, Brazil
Email address: marcos.pimenta@unesp.br