To unify azimuthally traveling-wave and standing-wave structured light by ray-wave duality

Zhaoyang Wang1,3, Yijie Shen1,*, Qiang Liu2,3 and Xing Fu2,3,*

1 Optoelectronics Research Centre, University of Southampton, Southampton SO17 1BJ, United Kingdom
2 Key Laboratory of Photonic Control Technology (Tsinghua University), Ministry of Education, Beijing 100084, People’s Republic of China
3 State Key Laboratory of Precision Measurement Technology and Instruments, Department of Precision Instrument, Tsinghua University, Beijing 100084, People’s Republic of China

E-mail: y.shen@soton.ac.uk and fuxing@mail.tsinghua.edu.cn

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Abstract
Structured light, with its ability to tailor diverse degrees of freedom (DoFs) of light has recently attracted increasing attention, especially as geometric beams that can open multiple DoFs by the ray-wave duality. Here, a generalized ray-wave beam family is proposed to unify the azimuthally traveling-wave (TW) and standing-wave (SW) structured light, by introducing a ray-split/fusion state as a tunable spatial mode to continuously transform the TW and SW states of light. We derive an elegant closed-form expression by utilizing frequency-degenerate Ince–Gaussian eigenmodes to construct a new coherent state of structured light to precisely parameterize the ray-split/fusion effect, pushing structured light control in higher dimensions. We also experimentally generate these modes by dynamic control of a digital hologram system, revealing their potential applications in optical manipulation and communication.

Keywords: structured light, ray-wave beams, traveling-wave, standing-wave

(Some figures may appear in colour only in the online journal)

1. Introduction
Structured light, with the ability to arbitrarily tailor its amplitude, phase and polarization profiles [1], has attracted great attention due to its wide applications such as optical tweezers, quantum entanglement, and communication [2–5]. Multifarious beams as solutions of paraxial wave equation (PWE) were identified in a structured light family, such as Hermite–Gaussian (HG) modes, Laguerre–Gaussian (LG) modes, Ince–Gaussian (IG) modes and helical-Ince–Gaussian (HIG) modes [6], as well as various superposed states of eigen-modes including hybrid HG modes [7], vortex lattices [8], and SU(2) geometric beams [9, 10]. The unified representation of spatial mode evolution for prior beams has raised enthusiasm. For example, the generalized Hermite-Laguerre–Gaussian (HLG) modes realize the continuously tunable spatial mode evolution between LG modes and HG modes [11, 12], IG modes interpret the transition from even and odd LG modes to HG modes [6], and a singularity hybrid evolution model further accommodates the HLG and HIG modes together [13]. Recently, a SU(2) Poincaré sphere model was proposed to universally reveal the topology of orbital angular momentum (OAM) eigenmodes and coherent state beams [14]. Nevertheless, a unified representation to describe more generalized structured modes still requires further exploration.
As another important category of structured light, the standing-wave (SW) beams family has attracted wide attention. Generally, SW refers to a superposition of two traveling-waves (TWs) with opposite longitudinally propagating directions as shown in figure 1(a). Additionally, a superposition of two TWs with opposite azimuthally propagating directions, as shown in figure 1(b), could form an interesting azimuthally SW. As for structured light, SW beams refer to a superposition of two TW beams with opposite azimuthally topological charges [15], as shown in figure 2(a), where LG modes with topological charge |ℓ| = 3 are selected as an example. It should be noted that all of the so-called SW beams in this work refer to azimuthally SW beams, composed by two TW beams with opposite azimuthally topological charges.

A SW has distinct characteristics and intriguing patterns. For instance, a complex SW beam composed by vector vortex TW beams was generated to simulate entangled beating and applied in optical machining [4]. This concept can also be introduced in exotic ray-wave beams, as shown in figure 2(b). Ray-wave beam is a kind of exotic structured light constructed by coherent state [9, 16]. The coherent state is a specific quantum state whose behavior most closely resembles the classical state, where the quantum probability wave-packet can be coupled with classical movement, i.e., so-called ray-wave duality [17, 18]. The classic light analogy of quantum coherent state could be described by both spatial wave packet and ray representation, which could be called ray-wave beams [2, 9, 19]. The TW and SW ray-wave beams unveil the ray-like structures of light for propagation in free-space [20, 21] and oscillation in resonator [9, 22], respectively. Notably, such a ray-wave geometric beam has recently extended the frontier of modern physics such as non-diffraction effect [23], topological phase [24], and quantum–classical informatics [25]. However, the tunable spatial mode evolution of TW and SW ray-wave beams has never been studied in either theory or experiment, and it is intriguing and significant to break the TW-SW boundary in ray-wave beams for unveiling more generally spatial mode evolution and potential applications.

In this work, a unified representation for spatial mode evolution of TW and SW ray-wave beams is proposed as we construct the novel ray-wave beams with ray-splitting/fusion structure that have not been observed before. We analyze the continuously tunable spatial mode evolution among the eigenmodes of PWE including IG modes, HLG modes and HIG modes, etc (section 2.1). Then, based on the unified spatial mode evolution of these eigenmodes, we propose novel ray-wave beams with ray-splitting/fusion structure, unifying the spatial mode evolution of TW and SW ray-wave beams (section 2.2). In addition, we experimentally generate these complex beams by the digital holography method with digital micromirror devices (section 3) [26, 27]. The simulated and experimental results intuitively demonstrate the unified spatial mode evolution of TW and SW ray-wave beams. Our work realizes the continuously tunable spatial mode evolution between TW and SW ray-wave beams, further extending the topological states of ray-wave beams family and pushing structured light control into higher dimensions.

2. Theoretical model

2.1. TW-SW unified eigenmodes

2.1.1. SW LG modes converting into HG modes via IG modes. The PWE has various analytic solutions in different coordinates. In cylindrical coordinates, the eigenmodes are TW and SW LG modes. The TW LG modes labeled as LGρ,ℓ,θ(z) can be expressed as [28]:

\[
LG_{\rho,\ell,\theta}(r,\theta,z) = \sqrt{\frac{2\rho^{|\ell|}}{\pi (\rho + |\ell|)!}} \frac{1}{w(z)} r^{|\ell|} e^{-r^2/2} L_{\rho}^{|\ell|} (\rho^2) e^{i\theta} \times \exp [i\kappa_{\rho,m,n}z - i (2\rho + |\ell| + 1) \theta(z)]
\] (1)
where $L^i_\nu(z)$ is a generalized Laguerre polynomial with radial and azimuthal indices $\rho$ and $\ell$, $l$ is the longitudinal index, $\tilde{r} = \sqrt{\tilde{r}} = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$, $k_{m,l} = 2\pi w_0/\ell_\nu$, $\omega_{m,l} = (n + m + l)\omega_0 + (l + 1/2)\omega_z$, $\omega_0$ and $\omega_z$ are transverse and longitudinal mode frequency steps, $c$ is the light speed, $\tilde{z} = z + \frac{(x^2 + y^2)z}{2(z^2 + \tilde{r}^2)}$, $\tilde{\vartheta} = \tan^{-1}(z/\tilde{r})$ is the Gouy phase, $w(z) = w_0 \sqrt{1 + (z/\tilde{r})^2}$ is the Gaussian beam waist parameter, $w_0 = \sqrt{2\pi k_0/\pi}$ is the fundamental mode radius at the waist, $\tilde{r}$ is the Rayleigh range, $\lambda$ is the wavelength of light. The expression of the TW LG mode in equation (1) contains azimuthal phase dependence $e^{i\theta}$. According to the Euler formula $\cos(i\theta) = (e^{i\theta} + e^{-i\theta})/2$, $\sin(i\theta) = (e^{i\theta} - e^{-i\theta})/2i$, it can be known that $\cos(i\theta)$ and $\sin(i\theta)$ are equivalent to the superposition of azimuthal phase dependence with opposite azimuthal directions. Thus, LG modes which azimuthal components taken as $\cos(i\theta)$ and $\sin(i\theta)$ are called SW LG modes, also called even and odd LG modes labeled as $LG^{e}_{p,l,l}$ and $LG_{o,p,l,l}$. The relation between SW LG modes and TW LG modes can be expressed as $LG_{p,l,l}^{e} = (LG_{p,l,l}^{+} + LG_{p,l,l}^{-})/\sqrt{2}$ and $LG_{o,p,l,l} = (LG_{p,l,l}^{+} - LG_{p,l,l}^{-})/\sqrt{2}$, respectively.

The eigenmodes in the elliptic coordinates are the Ince-Gaussian (IG) modes [28]:

$$IG_{n,m,l}((x,y,z)|e) = C_{G} \psi_{m,l}^{\nu}(\xi,\eta) \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \times \exp \left[ i k_{m,l} \tilde{z} - i (l + 1) \theta \tilde{z} \right], 0 \leq v \leq u$$

$$IG_{n,m,l}^o((x,y,z)|e) = S_{G} \psi_{m,l}^{\nu}(\xi,\eta) \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \times \exp \left[ i k_{m,l} \tilde{z} - i (l + 1) \theta \tilde{z} \right], 1 \leq v \leq u$$

(2)

where $IG_{n,m,l}^e$ and $IG_{n,m,l}^o$ represent the even and odd IG modes, $u = m + n + 13$, $k_{m,l} = k_{m,l} \epsilon_\omega w_0/\ell_\nu$, $\omega_{m,l} = (u + 1)\omega_0 + (l + 1/2)\omega_z$, $\epsilon$ is ellipticity, $C_{G}$ and $S_{G}$ are the normalization coefficients, $C_{\nu}(\epsilon)$ and $S_{\nu}(\epsilon)$ are even and odd Ince polynomials, $(\xi,\eta)$ is elliptic coordinates.

In Cartesian coordinates, the solutions of PWE are HG modes [29]:

$$HG_{n,m,l}((x,y,z)|e) = \frac{1}{\sqrt{n^2 + m^2 + 1}} \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \times H_{\nu}(\tilde{r}) H_{\nu}(\tilde{\vartheta}) \exp \left[ i k_{m,l} \tilde{z} - i (n + m + 1) \theta \tilde{z} \right] \times \exp \left[ i k_{m,l} \tilde{z} - i (l + 1) \theta \tilde{z} \right]$$

(3)

where $H_{\nu}(\cdot)$ is the Hermite polynomial of nth order.

The mapping relation between elliptic coordinates $(\xi,\eta)$ and Cartesian coordinates $(x,y,z)$ is [13] $x = w(z) \sqrt{e/2} \cos \xi \cos \vartheta$, $y = w(z) \sqrt{e/2} \sin \xi \sin \vartheta$, where $\xi \in [0, \infty)$, $\eta \in [0, \pi/2]$. When $e \to \infty$ or 0, elliptic coordinates $(\xi,\eta)$ tend to Cartesian coordinates $(x,y,z)$ or cylindrical coordinates $(r,\theta,z)$, respectively. From the coordinates mapping relation, it can be inferred that HG modes and SW LG modes can be served as limiting IG modes. HG modes correspond to IG modes with $e \to \infty$ as $HG_{n,m,l} = (-i)^m IG_{n+m,l}^e (e \to \infty)$ for $(-i)^m = 1$, $HG_{n,m,l} = (-i)^m IG_{n+m,l}^o (e \to \infty)$ for $(-i)^m \neq 1$ [13]. And SW LG modes correspond to IG modes with $e \to 0$ as $LG_{p,l,l}^e = (-i)^m IG_{2m+l+l}^e (e \to 0)$, $LG_{o,p,l,l} = (-i)^m + 1 IG_{2m+l+l}^o (e \to 0)$ [13]. Thus, IG modes act as the intermediate state modes between HG modes and SW LG modes via the eccentricity $e$. Therefore, the spatial mode evolution from SW LG modes to HG modes by IG modes can be uniformly expressed as [13]:

$$\psi_{m,l}^{\nu}(x,y,z|e) = \begin{cases} (-i)^m IG_{m+n+1,l}^e (x,y,z|e), & (-1)^m = 1 \\ (-i)^m IG_{m+n+1,l}^o (x,y,z|e), & (-1)^m \neq 1 \end{cases} \quad (4)$$

where $(n,m)$ are transverse indices of HG modes, $IG_{\nu}^{\pm,i}$ represents even/odd IG modes, $(-i)^m$ is a global phase factor to make phase distribution equal [13]. The mode $\psi_{m,l}^{\nu}$ tends to HG$_{n,m}$ mode when $e \to \infty$. When $e \to 0$, $\psi_{m,l}^{\nu}$ tends to LG$_{o,p,l}$ mode with $\rho = m/2$, $\ell = n$ for $(1)^m = 1$ or to LG$_{e,p,l}$ mode with $\rho = (m - n)/2$, $\ell = n + 1$ for $(-1)^m \neq 0$. The mode $\psi_{n,m,l}$ defined in equation (4) unifies the spatial mode evolution from SW LG mode to HG mode with $e$ evolving from $0$ to $\infty$, as shown in figures (a1)–(a4) and (b1)–(b4).

2.1.2. HG modes converting into TW LG modes via HLG modes. The spatial mode evolution from HG modes to TW LG modes could be expressed analytically by exploiting HG modes as [11, 13, 30]:

$$\psi_{n,m,l}^{HLG}((x,y,z) | \alpha) = \frac{1}{\sqrt{2^m n! m!}} \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \times H_{n,m,l}(\tilde{r}) \exp \left[ i k_{m,l} \tilde{z} - i (n + m + 1) \theta \tilde{z} \right] \times \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right]$$

(5)

$$\psi_{n,m,l}^{HLG}(r,\alpha) = \frac{\partial^m \varphi^{\nu}}{\partial s^{m} \partial \alpha^{s}} G(r,s,\alpha)|_{s=0}$$

$$G(r,s,\alpha) = \exp[-(U^* s)^T (U s) + 2\sqrt{2\pi} (U^* s)^T r]$$

(6)

where $\psi_{n,m,l}^{HLG}(\cdot)$ is the Hermite–Laguerre polynomial, $G(r,s,\alpha)$ is the generating function, $r = (x,y)^T$, $s = (x,s)^T$, $U = \begin{pmatrix} \cos(\alpha) & i \sin(\alpha) \\ i \sin(\alpha) & \cos(\alpha) \end{pmatrix}$, $\alpha$ ranges from $-\pi/4$ to $\pi/4$. The mode $\psi_{n,m,l}^{HLG}$ tends to HG$_{n,m,l}$ mode for $\alpha = 0$, and $\psi_{n,m,l}^{HLG}$ tends to TW LG$_{p,l,l}(x,y,z)$ with $\rho = \min(m,n)$, $\ell = \pm \sqrt{n-m}$ for $\alpha = \pm \pi/4$, respectively. The mode $\psi_{n,m,l}^{HLG}$ defined in equation (6) unifies the spatial mode evolution from HG modes to TW LG modes with $\alpha$ increasing from 0 to $\pi/4$, as shown in figures (a4)–(a7) and (b4)–(b7).

2.1.3. TW LG modes converting into hybrid HG modes via HIG modes. The TW LG modes can be decomposed into a superposition of two SW LG modes with a phase shift of $\pi/2$. And the SW LG modes correspond to IG modes with $e \to 0$. Thus, TW LG modes can also be seen as a superposition of the odd and even IG modes with $e \to 0$ and a phase shift of $\pi/2$. In fact, such linear superposition of odd and even IG modes are essentially HIG modes. The HIG mode, as shown in figure 3(a), can be constructed based on odd and even IG modes with a phase shift $\pi/2$ as [6]:

$$HIG_{p,l,l}^{\pm}(x,y,z|e) = IG_{p,l,l}^{e}(x,y,z|e) \pm i IG_{p,l,l}^{o}(x,y,z|e) \quad (7)$$
where HG modes with $\varepsilon \to 0$ are equivalent to TW LG modes, and HG modes with $\varepsilon \to \infty$ correspond to a superposition of two HG modes with a phase shift of $\pi/2$. Such a linear combination of two HG modes could have non-zero OAM, which could be called hybrid HG modes [7]. In our work, the limiting case of HG modes ($\varepsilon \to \infty$) could be seen as a kind of hybrid HG modes, which could be explicitly expressed as $H_{n,m,l}^h = H_{n,m,l}^{\varepsilon} + iH_{n,m,l}^{\varepsilon}$ for $(-1)^m = 1$, and $H_{n,m,l}^h = H_{n,m,l}^{\varepsilon} + iH_{n,m,l}^{\varepsilon}$ for $(-1)^m \neq 1$, where $\rho = \min(n,m), \ell = n - m$. In addition, the properties of hybrid HG modes are briefly introduced, which are distinguished from the pure HG modes [7, 29, 31]. The mode $HG_{n+1,m}^h$ can form $n^2 + (n + 1)^2$ vortex arrays arranged in square as shown in figure 3(b). The corresponding normalized OAM (orbital angular momentum) can be derived as $J_\rho/z = n + 1 [7, 31]$. If $\ell = 2\rho + 1$ for $(-1)^m = 1$, the hybrid HG modes can form $(2\rho)^2 + (2\rho + 1)^2$ vortex arrays with normalized OAM $2\rho + 1$. If $\ell = 2\rho$ for $(-1)^m \neq 1$, the hybrid HG modes can form $(2\rho)^2 + (2\rho + 1)^2$ vortex arrays with normalized OAM $2\rho$.

Therefore, the spatial mode evolution can be extended from TW LG modes to hybrid HG modes as:

$$
\psi_{n,m,l}^{HIG}(x,y,z|\varepsilon) = (-1)^mH_{2\rho+1,|\ell|,|\ell|}^2(x,y,z|\varepsilon)
$$

where $\rho = \min(n,m), \ell = n - m$, symbol + for $n > m$ and symbol - for $n < m$. The mode $\psi_{n,m,l}^{HIG}$ tends to TW LG modes for $\varepsilon = 0$ and to hybrid HG modes $H_{|\ell|,|\ell|}^2 + iH_{|\ell|,-2\rho+1,|\ell|}^2$ for $\varepsilon \to \infty$. The mode $\psi_{n,m,l}^{HIG}(x,y,z|\varepsilon)$ unifies the spatial mode evolution from TW LG to hybrid HG modes with $\varepsilon$ evolving from 0 to $\infty$, as shown in figures 4(a7)–(a10) and (b7)–(b10). So far, the generalized eigenmode family $\{\psi_{n,m,l}^{IG,HLG,HLG}^\varepsilon\}$ realizes the spatial mode evolution of SW LG $\leftrightarrow$ TW LG modes (as shown in figures 4(a1)–(a7) and (b1)–(b7)) and even extends to hybrid HG modes (as shown in figures 4(a7)–(a10) and (b7)–(b10)).

### 2.2. TW-SW unified ray-wave geometric beams

#### 2.2.1. Spatial wave packet of ray-wave beams

In the above sections, we have proposed the generalized eigenmode family $\{\psi_{n,m,l}^{IG,HLG,HLG}^\varepsilon\}$ to unify the spatial mode evolution from SW LG modes to hybrid HG modes, which allows us to explore more complex ray-wave beams as on-demand superposed spatial wave packet of generalized eigenmodes. Hereinafter, we will demonstrate a family of TW-SW unified ray-wave beams with ray-splitting/fusion structure and merging properties that have not been observed before. We exploit the formation of SU(2) coherent state to construct such ray-wave beams [9, 14, 32] as:

$$
\Phi_{n,m,l}(x,y,z) = \frac{1}{2^{N/2}} \sum_{K=0}^{N} |K\rangle \langle K| e^{iK\phi} \psi_{n,m,l}^{\varepsilon=0}(x,y,z),
$$

where $\phi$ is coherent phase, $(p, q, s)$ are three integers related to frequency coupling among transverse and longitudinal modes $Q = p + q, s = -p, (P, Q)$ are a pair of coprime integers for fulfilling frequency-degenerate condition [9]. $\Phi_{n,m,l}^{\varepsilon=0} = \{\psi_{n,m,l}^{IG,HLG,HLG}^\varepsilon\}$. The corresponding spatial mode evolution of spatial wave packet for $(n, m) = (10,0)$ and $(p, q) = (4, 0)$ are shown in figures 4(d1)–(d10) and (e1)–(e10), where the spatial wave packets are located on multiple rays as shown in figures 4(c1)–(e10). Therefore these beams could be called as multi-path ray-wave beams. The evolution from planar multi-path ray-wave beams (figures 4(d4) and (e4)) to TW multi-path ray-wave beams (figures 4(d7) and (e7)) with oscillating and propagating ray paths was studied in [20, 22, 27]. But we create new ray-wave beams as the transitional state between SW multi-path ray-wave beams and planar multi-path ray-wave beams, as shown in figures 4(d1)–(d4) and (e1)–(e4), where rays of a planar ray-wave beam gradually split into two fold and distribute into circularly oscillating rays of the SW ray-wave beam. Thus, the spatial mode evolution of TW and SW ray-wave beams has been unified as shown in figures 4(d1)–(e7) and (e1)–(e7). Furthermore, we could utilize the mode $\psi_{n,m,l}^{HIG}$ to extend more exotic transformation from planar multi-path ray-wave beams into a pseudo-planar multi-path ray-wave beam as shown in figures 4(d7)–(d10) and (e7)–(e10), where the pseudo-planar multi-path ray-wave beam (figures 4(d10) and (e10)) is a superposition of a series of hybrid HG modes under frequency-degenerate condition. Since its spatial wave packet is similar to planar multi-path ray-wave beams, we call such a superposition of a series of hybrid HG modes a pseudo-planar multi-path ray-wave beam to distinguish them. In addition, the extended spatial mode evolution shown in figures 4(d8)–(d10) and (e8)–(e10) could be called pseudo- multi-path ray-wave beams to distinguish them from the spatial mode evolution shown in figures 4(d4)–(d6) and (e4)–(e6).
Figure 4. The continuously tunable spatial mode evolution of eigenmodes from SW LG modes to hybrid HG modes (a), (b), multi-path ray-wave beams \((n, m) = (10, 0)\) (c)–(e), and multi-axis ray-wave beams \((n, m) = (10, 3)\) (f)–(h). Panels (a1)–(a10), (d1)–(d10), and (g1)–(g10) show the corresponding spatial wave packets. Panels (c1)–(c10) and (f1)–(f10) show the corresponding rays for multi-path and multi-axis ray-wave beams, respectively. \(z\) ranges from 0 to 2\(z_R\). Panels (b1)–(b10), (e1)–(e10), and (h1)–(h10) show the corresponding transverse intensity and phase distributions at \(z = 0\) plane. The subplots in red box exhibit the newly proposed ray-wave beams with ray-splitting/fusion structure. The subplots in blue box exhibit the extended spatial mode evolution, i.e. pseudo- multi-axis/path ray-wave beams. (Colormap: darkness to brightness means 0 to 1 for intensity and \(-\pi\) to \(\pi\) for phase).

We also explore the higher-order formation of such coherent state beams, as the results for \((n, m) = (10, 3)\) and \((p, q) = (4, 0)\) demonstrated in figures 4(f1)–(f10), (g1)–(g10) and (h1)–(h10), where the light on each ray changes into higher-order eigenmode formation, namely the multi-axis ray-wave beams [33]. Here, we markedly generalize such multi-axis ray-wave beams that the planar multi-axis ray-wave beams (figures 4(g4) and (h4)) can evolve into SW multi-axis ray-wave beam (figures 4(g1)–(g4) and (h1)–(h4)) and further extend TW multi-axis ray-wave beam (figures 4(g7) and (h7)) to pseudo-planar multi-axis ray-wave beam (figures 4(g7)–(g10) and (h7)–(h10)). The newly proposed ray-wave beams
(in red and blue boxes of figure 4) largely enrich the structured light family and inspire the tailoring of more exotic structured light.

2.2.2. Ray-splitting/fusion structure. Ray-wave duality [19] is an intrinsic property of the SU(2) coherent state, where the corresponding classical light, SU(2) geometric beams, would be localized on a cluster of periodic rays [9]. The newly proposed ray-wave beams are constructed by SU(2) coherent state and also have ray-wave duality. Therefore, we could exploit the cluster of rays to characterize our newly proposed ray-wave beams. The collection \( \{x_i^b, y_i^b, z_i^b\} \) represents the rays of planar multi-path/axis ray-wave beams, which can be expressed as [34, 35]:

\[
\begin{align*}
x_i^b(z) &= \sqrt{N_i}w(z)\cos[\theta_s z + \phi_x \pm \theta_G(z)] \\
y_i^b(z) &= \sqrt{N_i}w(z)\cos[\theta_s z + \phi_y \pm \theta_G(z)],
\end{align*}
\]

(10)

where \( (N_x, N_y) \) are transverse indices proportional to \((n, m)\) determining transverse scales in \(x\)- and \(y\)-directions, \(\phi_x\) and \(\phi_y\) are phase factors related to coherent state phase \(\phi_c\), and \(\pm\) represents forward and backward propagation, respectively. For the rays of planar multi-path ray-wave beam shown in figure 4(c4), \(\theta_s = (2\pi s)/(P/Q)\) (\(s = 0, 1, 2, \ldots, Q - 1\)), \(\phi_c = \phi/Q\) [36] and \(N = 0\) thus \(\phi\) and \(\theta_s\) is not considered. For the rays of planar multi-axis ray-wave beam shown in figure 4(f4), we select \(N_x > N_y \neq 0\), \(\theta_{s,z} = (2\pi s)/(P/Q)\) and \(\theta_{z,s} = (2\pi s)/(P/Q)(1 + pM_1/M_2)\), where \(s = 0, 1, 2, \ldots, M_2 Q - 1\), \((M_1, M_2)\) is a pair of coprime integers and \(M_1 = 1\) selected usually [34, 35].

By SU(2) transformation, a cluster of rays could be constructed to characterize evolution from planar to TW multi-path/axis ray-wave beams as:

\[
\begin{align*}
x_i^b(z) &= \left[ e^{-i\pi/4}\cos(\alpha) \quad e^{-i\pi/4}\sin(\alpha) \right] \left[ x_i^s(z) \right] \\
y_i^b(z) &= \left[ e^{i\pi/4}\sin(\alpha) \quad e^{i\pi/4}\cos(\alpha) \right] \left[ y_i^s(z) \right],
\end{align*}
\]

(11)

where the cluster \( \{x_i^s(z), y_i^s(z), z|\alpha\} \) (figures 4(c4)–(c7) and (f4)–(f7)) is coupled with the spatial wave packet of equation (9) with \(\Psi_{n,m,l}^{\text{eigen}} = \psi_{n,m,l}^{\text{HLG}}\), (figures 4(d4)–(d7) and (g4)–(g7)). The cluster \( \{x_i^s(z), y_i^s(z), z|\alpha\} \) would evolve from planar to circularly distributing rays with \(\alpha\) increasing from 0 to \(\pm\pi/4\), corresponding to the spatial mode evolution from planar to TW ray-wave beams.

Based on the cluster of rays of planar-TW ray-wave beams \( \{x_i^b(z), y_i^b(z), z|\alpha\} \), we can further study the ray-wave structure of newly proposed ray-wave beams as shown in red and blue boxes of figure 4. SW ray-wave beams can be decomposed into two TW ray-wave beams, which reveals that the rays of SW ray-wave beams can be interpreted as a superposition of two clusters of rays of TW ray-wave beams as:

\[
\begin{align*}
\{x_i^b(z), y_i^b(z), z\}^{\text{SW}} &= \{x_i^b(z), y_i^b(z), z|\alpha = \pi/4\} \\
+ \{x_i^b(z), y_i^b(z), z|\alpha = -\pi/4\},
\end{align*}
\]

(12)

where \( \{x_i^b(z), y_i^b(z), z|\alpha = \pi/4\} \) are clusters of rays coupled with spatial wave packet of equation (9) with \(\Psi_{n,m,l}^{\text{eigen}} = \psi_{n,m,l}^{\text{HLG}}\) \((\alpha = \pm\pi/4)\) in two opposite azimuthal directions. Furthermore, the rays for characterizing the evolution from SW to planar ray-wave beams can be constructed by a superposition of two clusters of planar-TW ray-wave beams with opposite azimuthally directions as:

\[
\begin{align*}
\{x_i^b(z), y_i^b(z), z|\alpha\}^{\text{IG}} &= \{x_i^b(z), y_i^b(z), z|\alpha(\epsilon)\} + \{x_i^b(z), y_i^b(z), z|\alpha(\epsilon)\},
\end{align*}
\]

(13)

where \(\alpha(\epsilon)\) is still the parameter in equation (11) that is just related to \(\epsilon\), which decreases from \(\pi/4\) to 0 with \(\epsilon\) increasing from 0 to \(\infty\), as shown in figure 9. The cluster \( \{x_i^b(z), y_i^b(z), z|\alpha\}^{\text{IG}} \) is coupled with spatial wave packet of equation (9) with \(\Psi_{n,m,l}^{\text{eigen}} = \psi_{n,m,l}^{\text{IG}}\). The cluster \( \{x_i^b(z), y_i^b(z), z|\alpha\}^{\text{IG}} \) would evolve from circular to planar distributing rays with \(\epsilon\) increasing from 0 to \(\infty\), corresponding to the spatial mode evolution from SW to planar ray-wave beams. During this process, the number of rays doubles, which demonstrates that our newly proposed ray-wave beams have an exotic ray-wave duality, ray-splitting/fusion structure as shown in the red box of figure 4. Figure 5 illustrates this exotic ray-splitting/fusion structure clearly where; (a) the parameter

![Figure 5.](image-url)
\[\alpha(x) = \pi/4 \text{ for SW multi-path ray-wave beam}, \text{(b), (c)} \alpha(x) = \pi/8 \text{ and } \pi/16 \text{ for multi-path ray-wave beams with elliptically distributing rays, and (d)} \alpha(x) = 0 \text{ for planar multi-path ray-wave beams, respectively. The relation between } \alpha(x) \text{ and } \varepsilon \text{ is } \varepsilon \text{ increasing from 0 to } \pi/4 \text{ with } \alpha(x) \text{ decreasing from } \pi/4 \text{ to 0, which is discussed in detail in the appendix.}

The cluster of rays for extended ray-wave beams (shown in blue box of figure 4) can also be constructed based on the cluster of planar-TW ray-wave beams. Pseudo-planar ray-wave beams (corresponding to equation (9) with } \psi_{\text{HIG}} = \psi_{n,m,l} (\varepsilon \rightarrow \infty) \text{ can be decomposed into two planar ray-wave beams with slightly different } (N_x, N_y), \text{ which reveals that the rays of pseudo-planar ray-wave beams can be interpreted as a superposition as:}

\begin{align*}
\{x^b(z), y^b(z), z|\alpha = \infty\}^{\text{HIG}} &= \{x^b_{1,i}(z), y^b_{1,i}(z), z|\alpha = 0\} \\
+ \{x^b_{2,i}(z), y^b_{2,i}(z), z|\alpha = 0\},
\end{align*}

where \(\{x^b_{1,i}(z), y^b_{1,i}(z), z|\alpha = 0\} \) is clusters coupled with spatial wave packet of equation (9) with } \psi_{\text{HIG}} = \psi_{n,m,l} (\alpha = 0). \text{ Furthermore, the rays for characterizing the evolution from TW to pseudo-planar ray-wave beams can be constructed by a superposition of two clusters of planar-TW ray-wave beams with slightly different } (N_x, N_y) \text{ as:}

\begin{align*}
\{x^b(z), y^b(z), z|\varepsilon\}^{\text{HIG}} &= \{x^b_{1,i}(z), y^b_{1,i}(z), z|\alpha(\varepsilon)\} \\
+ \{x^b_{2,i}(z), y^b_{2,i}(z), z|\alpha(\varepsilon)\},
\end{align*}

where \(\{x^b_{1,i}(z), y^b_{1,i}(z), z\} \) is the cluster of rays with slight different } \psi_{\text{HIG}} = \psi_{n,m,l}(\varepsilon), \text{ as shown in the blue box of figure 4.

A unified representation for TW and SW structured light is proposed here, which incorporates TW and SW formations of eigenmodes and ray-wave beams into a continuously tunable spatial mode evolution. We analyze the continuous spatial mode evolution among typical free-space eigenmodes (IG, HLG and HIG modes). Then, we create the new ray-wave beams with ray-split/fusion structure (marked with red box in figure 4) to realize the continuously tunable spatial mode evolution of SW-TW ray-wave beams and extend the unified spatial mode evolution from the TW to pseudo-ray-wave beams (marked with blue box in figure 4). In the next section, we generate the continuously tunable spatial mode evolution among generalized eigenmodes and ray-wave beams in experiment.

3. Experiment

We exploit a digital micromirror (DMD) device to experimentally generate these complex modes by the digital holography method [26, 27]. The DMD is a micro-electromechanical system device with millions of tiny switchable mirrors [26, 27, 28]. The DMD used in this experiment is a DLP Light Crafter 6500 produced by Texas Instruments, 1920 \times 1080 pixels. The experimental setup of digital dynamic tailoring light based on DMD is shown in figure 6. The beam emitted by the source is collimated and expanded by lens L1 and L2. Then the collimated beam enters the DMD plane and is modulated by the mask loaded on DMD. The outgoing beam is focused by lens L3 and filtered by the aperture (AP) at the Fourier plane. The filtering process to extract +1th-order diffraction component are shown in the bottom local enlargement of figure 6, where the target structured light exists in the +1th-order diffraction component and only three diffraction orders are displayed to illustrate the filtering process. Finally, the target structured light is imaged by lens L4, with its intensity collected by CCD. By changing corresponding DMD masks, we can easily generate a series of target structured light. The focuses of four lenses L1, L2, L3 and L4 are noted as } \begin{align*} f_1, f_2, f_3 \text{ and } f_4 \text{, respectively. As for the placement of experimental elements, the distance between L1 and L2 is } f_1 + f_2. } \begin{align*} \text{The AP is placed in the focus plane of L3, so the distance between L3 and the AP is } f_3. } \begin{align*} \text{The lenses L3 and L4 compose 4f system, thus } f_3 = f_4 \text{ and the distance between L3 and L4 is } 2f_3. \text{ The distances between the other components are adjustable over a wide range.}

The mask (essentially computer generated binary hologram) loaded on the DMD carries the amplitude and phase information of the target structured light. In order to encode the phase and amplitude of the target structured light with binary amplitude DMD, the mathematical expression of mask is [38, 39]:

\begin{align*}
\text{Hol} = \frac{1}{2} + \frac{1}{2} \text{sgn}(\cos(\phi_0(x,y) + \phi(x,y)) - \sin^{-1}(A(x,y)))
\end{align*}

(16)
where \( A(x, y) \) and \( \phi(x, y) \) are target structured light’s amplitude and phase, \( \phi_g \) is a linear phase tilting factor that determines the period and angle of grating, \( \text{sgn}(x) = 1, 0, -1 \) for \( x > 0, = 0, < 0 \) respectively. \( \phi_g(x, y) = 2\pi(x/\Delta x + y/\Delta y), \Delta x = \Delta y = 35 \) are used in generating masks in this work. The value of \( Hol = 0.5 \) is set to \( Hol = 0 \), since the masks are binary in nature. The value of \( N = 8 \) is used in the experimentally generated ray-wave beams. The mask and experiment result of SW multi-path ray-wave beams are shown in the top section of figure 6, where the zoom shows local details of the mask. Experimental results of TW-SW unified structured light are shown in figure 7, where rows from top to bottom are the patterns of generalized eigenmodes, multi-path ray-wave beams and multi-axis ray-wave beams, respectively, recorded at \( z = 0 \) plane, corresponding to the simulated results in figure 4. In addition, for demonstrating the longitudinal variance upon propagation, we captured the experimental patterns of SW multi-path/axis ray-wave beams at multiple transverse planes as shown in figure 8. Our newly proposed SW ray-wave beams exhibits distinct spatial distribution that the number of spots/rings changes upon propagation, which is distinguished with beams of pattern-invariant upon propagation and offers a possible way to longitudinally manipulate particles.
Figure 9. The relation between $\alpha(\epsilon)$ and $\epsilon$. (a) The phase distributions of IG modes with various values of $\epsilon$. (b) Various values of $\epsilon$ correspond to the ratios $b/a$ of length of ellipse long and short half-axis.

4. Discussion

We construct a generalized spatial mode evolution model, which realizes the continuously tunable spatial mode evolution between SW and TW structured light and contains ray-split/fusion process. For optical manipulation, the continuous spatial mode evolution could be exploited to dynamically manipulate particles. The ray-wave duality means this new beam could manipulate multiple particles in various locations. Also, the ray-split/fusion property means this beam could be used to realize sorting or concentration of particles. For optical communication, more degrees of freedom (DoFs) means larger communication capacity. Our newly proposed model introduces a new controllable ray-split/fusion DoF, which is controlled by parameter $\epsilon$ and $\alpha$, extending the potential applications of structured light in high-capacity optical communications.

Notably, our theoretical framework reveals the potential to be further extended to unveil more generalized structured light. For instance, it can be applied to more complex ray-wave beams such as ray-wave Lissajous and trochooidal wave packets [27, 40]. We can also study its general structure in astigmatic and vectorial optical fields. We can exploit the non-diffraction eigenmodes in various orthogonal coordinates [41], such as Bessel, Mathieu, Airy modes [42], to unify their TW and SW formations. Additionally, we can also consider other coherent states to extend more complex structured modes, e.g. SU(1,1) coherent state [43] and hybrid coherent state [36]. The TW-SW unification also acts as a new mechanism to extend structured light, so as to enable novel applications. The ray-splitting/fusion structure provides a new DoF to create multipartite classical entangled state [25], which can be employed in high-speed optical encryption and communication [5]. The multi-singularity and complex OAM evolution of new structured light is also needed for advanced optical tweezers and trapping [3]. Therefore, our method offers a practical way to universally tailor a wide variety of structured light and the general spatial modes, which can enrich both high-dimensional ray-wave beams and distributions of multiple phase singularities, enabling various cutting-edge applications such as multi-particle trapping, high-capacity communications, precise imaging and metrology [1, 2].

5. Conclusion

In summary, we propose a new model to unify the TW and SW ray-wave beams. It generalizes the family of ray-wave beams based on the continuously tunable spatial mode evolution of generalized eigenmodes (IG, HLG, and HIG modes), extending new ray-wave structures in a ray-splitting/fusion structure. The generalized theoretical framework has strong extensibility and applicability to construct more complex beams and to study OAM with multi-singularities, which could inspire the exploration of more novel structured light forms with their advanced applications.

Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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Conflict of interest

The authors declare that they have no conflict of interest.

Appendix. The parametric relation in the ray representation

The relation between $\alpha(\epsilon)$ and $\epsilon$ could be obtained numerically. The details about how to obtain the relation between $\alpha(\epsilon)$ and $\epsilon$ are as follows:

Step 1: The relation between $\alpha$ and $b/a$ (the ratio of length of long and short half-axis of the ellipse where the rays are located on). For the cluster $\{x_1^\alpha(z), y_1^\alpha(z), z|\alpha\}$ with $N_x \neq 0$ and $N_y = 0$, it can be derived as $x_1^\alpha(z) = \ldots$
\\[ \psi_0 \to \sqrt{\text{ORCID iD}} \]

\[ \psi \]

\[ \mathrm{LG}_{n,l}^\pm, \mathrm{LG}_{n,l}^{\text{even/odd}}, \mathrm{IG}_{n,l}^{\alpha}, H_{n,m,l}^{\alpha}, \Theta_{n,m,l}, H_{n,m,l}^{\text{hybrid}}, \mathrm{IG}^{\alpha}, \mathrm{IG}^{\text{SW}}, H_{n,m,l}^{\text{TW}}, \Theta_{n,m,l}^{\text{TW}}, W_{n,m,l}^{\text{ray}}, W_{n,m,l}^{\text{ray SW}}, W_{n,m,l}^{\text{ray SW IG}}, W_{n,m,l}^{\text{ray TW}}, W_{n,m,l}^{\text{ray TW IG}}, W_{n,m,l}^{\text{ray TW pseudo-planar}} \]

\[ \sqrt{N} \cos(\alpha) w(z) \cos[\theta_{1,0} \pm \theta_{2,0} (z) - \pi/4] \] and

\[ \psi_i(z) = \sqrt{N} \sin(\alpha) w(z) \cos[\theta_{1,0} \pm \theta_{2,0} (z) + \pi/4], \]

which would locate on an ellipse with the ratios of length of long and short half-axis as \( b/a = \tan(\alpha) \).

Step 2: The relation between \( \varepsilon \) and \( b/a \). The jump points of phase distribution of IG modes could form an ellipse as shown in figure 9(a), where the ellipses formed by jump points of phase distribution are labelled by red curves for various values of \( \varepsilon \), and values of \( b/a \) are also labelled on the bottom of subplots.

Step 3: The relation between \( \varepsilon \) and \( \alpha(\varepsilon) \). By using \( \alpha = \tan^{-1}(b/a) \), the relation between \( \alpha(\varepsilon) \) and \( \varepsilon \) can be numerically obtained, as shown in figure 9(b), where \( \varepsilon \) increases from 0 to \( \infty \) while \( \alpha(\varepsilon) \) decreases from \( \pi/4 \) to 0.

In addition, we summarize the notations and abbreviations into a table for a clear illustration (see table 1).

**ORCID iD**

Yijie Shen [https://orcid.org/0000-0002-6700-9902](https://orcid.org/0000-0002-6700-9902)

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