Extension of log pluricanonical forms from subvarieties

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Abstract

In this paper, I prove a very general extension theorem for log pluricanonical systems. The strategy and the techniques used here are the same as those in [Ts3, Ts6, Ts7, Ts8]. The main application of this extension theorem is (together with Kawamata’s subadjunction theorem ([K5])) to give an optimal subadjunction theorem which relates the positivities of canonical bundle of the ambient projective manifold and that of the (maximal) center of log canonical singularities. This is an extension of the corresponding result in [Ts7], where I dealt with log pluricanonical systems of general type. This subadjunction theorem indicates an approach to solve the abundance conjecture for canonical divisors (or log canonical divisors) in terms of the induction in dimension.

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1 Introduction

In this paper, I present a proof of the extension theorem of log pluricanonical forms announced in [Ts3, Ts5]. The special case of the extension theorem has already been proven and used in [Ts3, Ts5] (cf. [Ts5, Theorems 2.24,2.25]). Although the scheme of the proof is very similar to that of [Ts3], it requires a lot more estimates of Bergman kernels and technicalities.

1.1 Abundance conjecture

The main motivation to prove such an extension theorem is to investigate the pluri (log) canonical systems on a projective varieties. Since the finite generation of canonical rings has been settled very recently ([B-C-H-M]), the most outstanding conjecture in this direction is the following conjecture.
Conjecture 1.1 (Abundance conjecture) Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Then $K_X$ is abundant, i.e.,

\[ \Kod(X) = \nu(X) \]

holds, where $\Kod(X)$ denotes the Kodaira dimension of $X$ and $\nu(X)$ denotes the numerical Kodaira dimension of $X$ (cf. Definition 3.1). \(\square\)

Let us explain the geometric meaning of Conjecture 1.1. If $\nu(X) = -\infty$, it is clear that $\Kod(X) = -\infty$, since $\Kod(X) \leq \nu(X)$ (cf. Definition 3.1) always holds. Hence in this case, Conjecture 1.1 is trivial. Next let $X$ be a smooth projective variety with $\nu(X) \geq 0$. Then $K_X$ is pseudoeffective. Suppose that $\Kod(X) = \nu(X)$ holds. In this case for a sufficiently large $m$, the rational map associated with $|mK_X|$ gives a rational fibration (called the Iitaka fibration)

\[ \Phi := \Phi_{|mK_X|} : X \dasharrow Y \subseteq \mathbb{P}^{N_m} \quad (N_m := \dim |mK_X|) \]

with $\dim Y = \nu(X)$. Then a general fiber $F$ of $\Phi$ is a smooth projective variety such that

\[ \Kod(F) = \nu(F) = 0. \]

Let $h$ be an AZD of minimal singularities on $K_X$ (cf. Definition 2.17). $(K_X, h)$ is considered to be the maximal positive part of $K_X$. Then we see that the curvature current $\Theta_h$ of $h$ has no absolutely continuous part on $F$ or equivalently $(K_X, h)$ is numerically trivial on $F$ and $(K_X, h) \cdot C$ is strictly positive for every irreducible curve $C \subset X$ such that $\Phi^*(C) \neq 0$, i.e., $\Phi$ is a numerically trivial fibration of $(K_X, h)$. This means that the fibration $\Phi : X \dasharrow Y$ extracts all the positivity of $(K_X, h)$. In this sense, Conjecture 1.1 asserts that the Iitaka fibration extracts all the positivity of $K_X$. This is the meaning of Conjecture 1.1.

So far Conjecture 1.1 has been proven only in the case of $\dim X \leq 3$. The main reason why we cannot proceed beyond the case of $\dim X = 3$ is that we do not know how to construct sections of pluricanonical bundles without bigness. Actually the proof of the abundance conjecture in the case of surfaces depends on the classification of projective surfaces. And the proof of Conjecture 1.1 in the case of projective 3-folds (M2, M3, K4, K-M-M), the key ingredient of the proof is the clever use of Miyaoka-Yau type inequality (M1). But unfortunately this method works well only for the case of 3-folds, because the method depends on the speciality of the Riemann-Roch theorem in dimension 3. So these partial affirmative solutions of Conjecture 1.1 do not lead us any further. Hence it will be desirable to find a systematic method to study pluricanonical systems which works in all dimensions.

Let us consider what is needed to solve Conjecture 1.1. For simplicity, first we shall assume that $X$ is already minimal. Then the Conjecture 1.1 is equivalent to the stable base point freeness of $K_X$ by a theorem of Kawamata (K2). Thus the abundance conjecture can be viewed as a base point freeness theorem.

To prove the base point freeness of some linear systems, especially the case of adjoint line bundles, the well known method is to produce log canonical center and go by induction in dimension using Kawamata-Viehweg vanishing theorem. To solve Conjecture 1.1 we shall consider a similar approach. Let $X$ be a smooth projective variety with pseudoeffective canonical bundle and let $h$ be an AZD of $K_X$ with minimal singularities. The strategy is as follows.
1. Find a LC center (log canonical center), say $S$ for $\alpha K_X$ for some $\alpha > 0$. Here the LC center means a little bit broader sense, i.e., LC center here means the cosupport of the multiplier ideal sheaf with respect to a singular hermitian metric of $\alpha K_X$ with semipositive curvature current. We shall assume that $S$ is smooth and is a maximal LC center (This can be assured by taking an embedded resolution of the LC center).

2. Use subadjunction theorem due to Kawamata ([K5]) to compare the canonical divisor of the center and $K_X$.

3. Lift the pluricanonical system of the center (possibly twisted by some fixed ample line bundle $B$ on $X$) to a sub linear system of the pluricanonical system of $X$ (possibly twisted by the ample line bundle $B$).

If this strategy works, we can see the positivity of $(K_X, h)|_S$ dominates that of $K_S$. If we have already known the abundance of $K_S$ and the equality $\nu(S) = \nu(K_X|_S, h|_S)$ holds, then we see that the positivity of $(K_X, h)|_S$ can be extracted in terms of the Iitaka fibration of $S$. Moreover if we are able to construct the LC center as above through every point of $X$ (this is expected when $\nu(X) \geq 1$), then we may see that for a general such $S$ above have pseudoeffective $K_S$ and the equality $\nu(S) = \nu(K_X|_S, h|_S)$ holds. As a consequence, we may extract the positivity of $(K_X, h)$ in terms of a family of LC centers. And it is not difficult to prove Conjecture 1.1 in this case, if we have already proven Conjecture 1.1 for every smooth projective variety of dimension $< \dim X$. Hence this strategy can be viewed as an approach to Conjecture 1.1 by the induction in dimension.

This strategy has been first considered by the author in the series of papers [Ts5, Ts6, Ts7]. But in these papers, the varieties are assumed to be of general type and in this case the LC center $S$ is also of general type. Hence what we compare is the volume of $K_S$ and that of $(K_X, h)|_S$ and there are essentially no analytic difficulties. In fact [H-M, Ta] have interpreted [Ts7] in terms of algebro geometric languages.

But in contrast to the case of general type, the case of non general type is much harder in analysis. The reason is that there is no room of positivity to approximate AZD of $K_X$ by a sequence of pseudoeffective singular hermitian metrics with algebraic singularities. This phenomena (loss of positivity) was first observed in the work of Demailly ([D]) on the regularization of closed positive currents.

Hence in the case of non general type, we need to provide effective estimates to obtain the desired singular hermitian metrics as was observed in [Ts3, S1]. In this case the nefness should be replaced by the semipositivity of curvature. This is the crucial point to apply the $L^2$-extension theorem which is considered to be the substitute of Kawamata-Viehweg vanishing theorem ([K1]) in the case of varieties of non general type. We summerize the comparison of the cases of general type and of non general type in the following table.
The purpose of this article is to implement the steps 2 and 3 of the strategy above. At this moment I do not know how to produce LC centers.

1.2 Main results

Let us explain the main results in this article. Let \( X \) be a smooth projective variety and let \((L, h_L)\) be a singular hermitian line bundle on \( X \) such that \( \Theta h_L \geq 0 \) on \( X \). We assume that \( h_L \) is lowersemicontinuous. Throughout this paper we shall assume that all the singular hermitian metrics are lowersemicontinuous.

Let \( m_0 \) be a positive integer. Let \( \sigma_0 \in \Gamma(X, \mathcal{O}_X(m_0L) \otimes \mathcal{I}_\infty(h_{L}^{m_0})) \) be a bounded global section (cf. Section 2.1 for the definition of \( \mathcal{I}_\infty(h_{L}^{m_0}) \)). Let \( \alpha \) be a positive rational number \( \leq 1 \) and let \( S \) be an irreducible component of the center of LC(log canonical) singularity but not KLT (Kawamata log terminal) and \((X, (\alpha - \epsilon)(\sigma_0))\) is KLT on the generic point of \( S \) for every \( 0 < \epsilon << 1 \). We set

\[
\Psi_S = \alpha \cdot \log h_{L}^{m_0}(\sigma_0, \sigma_0).
\]

Suppose that \( S \) is smooth for simplicity. Let \( dV \) be a \( C^\infty \) volume form on \( X \).

In this situation we may define a (possibly singular) measure \( dV[\Psi_S] \) on \( S \) as the residue as follows. Let \( f : Y \rightarrow X \) be a log resolution of \((X, \alpha(\sigma_0))\).

Then we may define the singular volume form \( f^*dV[f^*\Psi_S] \) on the divisorial components of \( f^{-1}(S) \) (the volume form is identically 0 on the components with discrepancy \( > 1 \)) by taking residue along \( f^{-1}(S) \). The singular volume form \( dV[\Psi_S] \) is defined as the fibre integral of \( f^*dV[f^*\Psi_S] \) (the actual integration takes place only on the components with discrepancy \( -1 \)), i.e., \( dV[\Psi_S] \) is the residue volume form of general codimension.

Let \( dV_S \) be a \( C^\infty \) volume form on \( S \) and let \( \varphi \) be the function on \( S \) defined by

\[
\varphi := \log \frac{dV_S}{dV[\Psi_S]}
\]

\( (dV[\Psi_S]) \) may be singular on a subvariety of \( S \), also it may be totally singular on \( S \). The following is the main theorem in this article.

**Theorem 1.2** (cf. [Ts5, Theorem 5.1]) Let \( X, S, \Psi_S \) be as above. Suppose that \( S \) is smooth. Let \( d \) be a positive integer such that \( d > \alpha m_0 \). We assume that \((K_X + dL, e^{-\varphi} \cdot (dV^{-1} \cdot h_L^d) \mid S)\) is weakly pseudoeffective (cf. Definition 2.7) and let \( h_S \) be an AZD of \((K_X + dL, e^{-\varphi} \cdot (dV^{-1} \cdot h_L^d) \mid S)\).

Then every element of \( H^0(S, \mathcal{O}_S(m(K_X + dL)) \otimes \mathcal{I}(e^{-\varphi} \cdot h_L^d \mid S \cdot h_S^{m-1})) \)

extends to an element of \( H^0(X, \mathcal{O}_X(m(K_X + dL)) \otimes \mathcal{I}(h_L^d)). \)

\(^1\)Here we have used the convention that \( \Theta h_L = \sqrt{-1} \partial \bar{\partial} \log h_L \). The advantage of this convention is that \( \Theta h_L \) is always a real current.
In particular every element of 
\[ H^0(S, \mathcal{O}_S(m(K_X + dL)) \otimes \mathcal{I}(e^{-\varphi} \cdot h_L^d |_S) \cdot \mathcal{I}_\infty(e^{-(m-1)\varphi} \cdot h_L^{m-1} |_S)) \]
extends to an element of 
\[ H^0(X, \mathcal{O}_X(m(K_X + dL)) \otimes \mathcal{I}(h_L^d \cdot h_0^{m-1})) \]
where \( h \) is an AZD of \( K_X + dL \) of minimal singularities. \( \square \)

As we mentioned as above the smoothness assumption on \( S \) is just to make the statement simpler. And it may be worthwhile to note that the weight function \( \varphi \) is not necessary when \( dV[\Psi_S] \) is locally \( L^1 \) on \( S \) and \( h_L \) is bounded on \( S \) (see the proof in Section 7).

Theorem 1.2 follows from Theorem 1.3 below by using a limiting process (cf. Section 9).

**Theorem 1.3** Let \( X,S,\Psi_S \) be as above. Suppose that \( S \) is smooth. Let \((E,h_E)\) be a pseudoeffective singular hermitian line bundle on \( X \) (cf. Definition 2.4).

Let \( d \) be a positive integer such that \( d > \alpha m_0 \). Let \( m \) be a positive integer. We assume that 
\[ (K_X + dL + \frac{1}{m}E |_S, e^{-\varphi} \cdot (dV^{-1} \cdot h_L \cdot \frac{h_E}{m}) |_S) \]
is weakly pseudoeffective. Let \( h_S \) be an AZD of \((K_X + dL + \frac{1}{m}E |_S, e^{-\varphi} \cdot (dV^{-1} \cdot h_L \cdot \frac{h_E}{m}) |_S)\). Suppose that \( h_S \) is normal (cf. Definition 2.17).

\[
\dim H^0(S, \mathcal{O}_S(A + m\ell(K_X + dL + \frac{1}{m}E)) |_S) \otimes \mathcal{I}(h_S^{m\ell})) = O(\ell^\nu)
\]
holds for every ample line bundle \( A \) on \( X \), where \( \nu \) denotes the numerical Kodaira dimension \( \nu_{num}(K_X + L + \frac{1}{m}E |_S, h_S) \) of \((K_X + L + \frac{1}{m}E |_S, h_S)\) (cf. Definition 3.4).

Then every element of 
\[ H^0(S, \mathcal{O}_S((m+1)(K_X + dL) + E) \otimes \mathcal{I}(e^{-\varphi} \cdot h_L^d |_S \cdot h_S^m)) \]
extends to an element of 
\[ H^0(X, \mathcal{O}_X((m+1)(K_X + dL) + E) \otimes \mathcal{I}(h_L^d \cdot h_0^m)), \]
where \( h_0^\frac{1}{m} \) is an AZD of \( K_X + dL + \frac{1}{m}E \) of minimal singularities. \( \square \)

As an application of Theorem 1.3 we obtain the following theorem.

**Theorem 1.4** Let \( X \) be a smooth projective variety and let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \) numerically equivalent to \( \alpha K_X \) for some \( \alpha > 0 \). Let \( h \) be the supercanonical AZD of \( K_X \) (cf. Definition 2.4) or any AZD of minimal singularities (cf. Definition 2.17). Let \( S \) be a maximal log canonical center of \((X,D)\) in the sense that \( S \) cannot be a proper subvariety of the log canonical center of \((X,D)\). We assume that for every rational number \( 0 < \epsilon < 1 \), \((X,(1-\epsilon)D)\) is log terminal at the generic point of \( S \). Suppose that \( S \) is smooth and \( K_S \) is pseudoeffective.
Then for every ample line bundle $A$ and every sufficiently ample line bundle $B$ and every positive integer $m$, there exists an injection
\[ H^0(S, \mathcal{O}_S(mK_S + A)) \hookrightarrow \text{Image}(H^0(X, \mathcal{O}_X(m(1+d)K_X + A + B) \otimes \mathcal{I}(h^{m-1})) \to H^0(S, \mathcal{O}_S(m(1+d)K_X + A + B))). \]
In particular
\[ \nu(K_S) \leq \nu_{\text{asym}}(K_X|_S, h|_S) \]
holds (for the definitions of $\nu(K_S)$ and $\nu_{\text{asym}}(K_X|_S, h|_S)$ see Definitions 3.1 and 3.5 below). □

**Remark 1.5** The smoothness assumption of $S$ is not essential in Theorem 1.4. In fact we just need to take an embedded resolution. □

The following theorem is a variant of Theorem 1.4.

**Theorem 1.6** Let $X$ be a smooth projective variety and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ numerically equivalent to $\alpha K_X$ for some $\alpha > 0$. Let $h$ be the supercanonical AZD of $K_X$ ([Ts9]) or any AZD of minimal singularities (cf. Definition 2.17). Let $S$ be a maximal log canonical center of $(X, D)$ in the sense that $S$ cannot be a proper subvariety of the log canonical center of $(X, D)$. We assume that for every rational number $0 < \epsilon < 1$, $(X, (1-\epsilon)D)$ is log terminal at the generic point of $S$. Suppose that $S$ is smooth and $K_S$ is pseudoeffective.

Then for every sufficiently ample line bundle $B$ and every positive integer $m$, there exists an injection
\[ H^0(S, \mathcal{O}_S(mK_S)) \hookrightarrow \text{Image}(H^0(X, \mathcal{O}_X(m(1+d)K_X + B) \otimes \mathcal{I}(h^{m-1})) \to H^0(S, \mathcal{O}_S(m(1+d)K_X + B))). \]
□

**Remark 1.7** One can deduce Theorem 1.6 from Theorem 1.4 by taking $A$ sufficiently ample in Theorem 1.4. But the proof in Section 9 below implies that we just need to take $B$ with a continuous metric with semipositive curvature such that there exists an injection
\[ H^0(S, \mathcal{O}_S(mK_S)) \hookrightarrow H^0(S, \mathcal{O}_S(m(1+d)K_X + B) \otimes \mathcal{I}(h^{m-1}|_S)) \]
exists for all $m \geq 1$. Hence the ampleness of $B$ is somewhat irrelevant in Theorem 1.6. □

Theorem 1.4 implies that the positivity of $K_S$ is dominated by the positivity of $K_X$ (up to a constant multiple). Theorem 1.6 is obtained by a similar argument as in [Ts5, Ts6, Ts7] using Theorem 7.1 below and Kawamata's semipositivity theorem ([K5, Theorem 2], see Theorem 8.1 below).

In Theorem 1.4 the presence of $h$ is crucial, even if $K_X$ is assumed to be nef. Because in this case, the abundance of $K_X$ implies that we may replace $(K_X, h)$ by $K_X$ in Theorem 1.4. Hence this is somewhat opposite.

We note that for a minimal algebraic variety $X$, the abundance of $K_X$ implies that $K_X$ is numerically trivial on the fiber of the Iitaka fibration and the pluricanonical system comes from an ample Kawamata log terminal divisor on the base of the Iitaka fibration ([F-M]). Hence Theorems 1.4 is a supporting
evidence of Conjecture 1.1. In fact, assuming the existence of minimal models for projective varieties with pseudoeffective canonical bundles and the abundance conjecture, we may easily deduce Theorem 1.4 by using the $L^2$-extension theorem (Theorems 2.25 and 2.26).

The organization of this article is as follows. In Section 2, I collect basic tools to prove Theorem 1.2. This section is mainly for algebraists. In Section 3, I define the numerical Kodaira dimension and the asymptotic Kodaira dimension of a pseudoeffective singular hermitian line bundle on a smooth projective variety and study the relation between the two dimensions. In Section 4, I relate the Monge-Ampère measure of the curvature current of a pseudoeffective singular hermitian line bundle to the asymptotic expansion of Bergman kernels. This leads us to define the local volume of a pseudoeffective singular hermitian line bundles and a natural and very interesting conjecture for the relation between the asymptotics of Bergman kernels and the Monge-Ampère mass. This direction should be studied in near future. In Section 5, I prove an analogue of Kodaira’s lemma for big pseudoeffective singular hermitian line bundles. This lemma is used in the next section. In Section 6, I prove the dynamical construction of an AZD for adjoint type singular hermitian line bundles. The idea of the proof is similar to the one in [Ts6]. But it requires a little bit more complication. In Section 7, I prove Theorem 7.1 by using the dynamical construction of an AZD in Section 6. In Section 8, I prove Theorem 1.4 by using Kawamata’s semipositivity theorem (Theorem 8.1) and Theorem 1.2. In Section 9, I prove Theorem 1.2 combining results in the previous sections.

2 Preliminaries

In this section we collect the basic tools. They are standard except Definitions 2.6, 2.8 and 2.21.

2.1 Singular hermitian metrics

In this subsection $L$ will denote a holomorphic line bundle on a complex manifold $X$.

**Definition 2.1** A singular hermitian metric $h$ on $L$ is given by

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{loc}(X)$ is an arbitrary function on $X$. We call $\varphi$ a weight function of $h$. □

The curvature current $\Theta_h$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\partial \bar{\partial}$ is taken in the sense of a current and we have used the convention that $\Theta_{h_0} = \sqrt{-1} \partial \bar{\partial} \log h_0$. The $L^2$ sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\mathcal{L}^2(L, h)(U) := \{ \sigma \in \Gamma(U, \mathcal{O}_X(L)) \mid h(\sigma, \sigma) \in L^1_{loc}(U) \}.$$
where $U$ runs over the open subsets of $X$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$L^2(L, h) = \mathcal{O}_X(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of $(L, h)$. If we write $h$ as

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(X)$ is the weight function, we see that

$$\mathcal{I}(h) = L^2(\mathcal{O}_X, e^{-\varphi})$$

holds. For $\varphi \in L^1_{\text{loc}}(X)$ we define the multiplier ideal sheaf of $\varphi$ by

$$\mathcal{I}(\varphi) := L^2(\mathcal{O}_X, e^{-\varphi}).$$

Similarly for $1 \leq p \leq +\infty$, we define

$$L^p(L, h)(U) := \{ \sigma \in \Gamma(U, \mathcal{O}_X(L)) \mid h(\sigma, \sigma) \in L^{p/2}_{\text{loc}}(U) \},$$

where $U$ runs over the open subsets of $X$. In this case there exists an ideal sheaf $\mathcal{I}_p(h)$ such that

$$L^p(L, h) = \mathcal{O}_X(L) \otimes \mathcal{I}_p(h)$$

holds. We call $\mathcal{I}_p(h)$ the **$L^p$ multiplier ideal sheaf** of $(L, h)$.

**Remark 2.2** It is known that $\mathcal{I}(h)$ is coherent when $\Theta_h$ is locally bounded from below by a $C^\infty$ form ([N]). But it is not clear whether $\mathcal{I}_p(h)$ is coherent under the same condition for $p \neq 2$. □

**Example 2.3** Let $\sigma \in \Gamma(X, \mathcal{O}_X(L))$ be the global section. Then

$$h := \frac{1}{|\sigma|^2} = \frac{h_0}{h_0(\sigma, \sigma)}$$

is a singular hermitian metric on $L$, where $h_0$ is an arbitrary $C^\infty$ hermitian metric on $L$ (the right hand side is obviously independent of $h_0$). The curvature $\Theta_h$ is given by

$$\Theta_h = 2\pi(\sigma),$$

where $(\sigma)$ denotes the current of integration over the divisor of $\sigma$. □

First we define the pseudoeffectivity of a singular hermitian line bundle.

**Definition 2.4** A line bundle $L$ on a complex manifold is said to be **pseudoeffective**, if there exists a singular hermitian metric $h$ on $L$ such that the curvature current $\Theta_h$ is a closed positive current. A singular hermitian line bundle $(L, h_L)$ is said to be **pseudoeffective**, if the curvature current $\Theta_{h_L}$ is a closed positive current. □

An important class of singular hermitian metrics with semipositive curvature current is algebraic singular hermitian metrics.
Definition 2.5 Let \( h \) be a singular hermitian metric on \( L \). We say that \( h \) is algebraic, if there exists a positive integer \( m_0 \) and global holomorphic sections \( \sigma_0, \cdots, \sigma_N \) of \( m_0L \) such that

\[
h = \left( \sum_{i=0}^{N} |\sigma_i|^2 \right)^{-\frac{1}{m_0}}
\]

holds. □

Definition 2.5 is naturally generalized to the case of \( \mathbb{Q} \)-line bundles in an obvious way.

Also the following weaker version of pseudoeffectivity is also important.

Definition 2.6 Let \( (L, h_L) \) be a singular hermitian line bundle on a smooth projective variety \( X \). \( (L, h_L) \) is said to be weakly pseudoeffective, if there exists an ample line bundle \( A \) on \( X \) such that

\[
H^0(X, \mathcal{O}_X(mL + A) \otimes \mathcal{I}(h_L^m)) \neq 0
\]

holds for every \( m \geq 0 \). A line bundle on a smooth projective variety is said to be pseudoeffective there exists an ample line bundle \( A \) on \( X \) such that

\[
H^0(X, \mathcal{O}_X(mL + A)) \neq 0
\]

holds for every \( m \geq 0 \). □

Remark 2.7 Let \( (L, h_L) \) is a pseudoeffective singular hermitian line bundle on a smooth projective variety \( X \). Then \( (L, h_L) \) is weakly pseudoeffective. This follows from an easy application of Hörmander’s \( L^2 \)-estimate. □

The following definition is useful in this article.

Definition 2.8 Let \( (L, h_L) \) be a pseudoeffective singular hermitian line bundle on a smooth projective variety \( X \). \( (L, h_L) \) is said to be normal, if the set

\[
E := \{ x \in X \mid n(\Theta_{h_L}, x) > 0 \}
\]

is contained in a proper analytic set of \( X \), where \( n(\Theta_{h_L}, x) \) denotes the Lelong number of the closed positive current \( \Theta_{h_L} \) at \( x \)\(^2\). □

Remark 2.9 By the fundamental theorem of Siu (\cite{Si}), \( E \) is at most a countable union of subvarieties of \( X \). □

2.2 Analytic Zariski decompositions (AZD)

In this subsection we shall introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle a pseudoeffective line bundles like nef line bundles.

Definition 2.10 Let \( X \) be a compact complex manifold and let \( L \) be a holomorphic line bundle on \( X \). A singular hermitian metric \( h \) on \( L \) is said to be an analytic Zariski decomposition, if the followings hold.

\footnote{Usually I use \( \nu \) instead of \( n \). But in this article, I use \( n \) not to confuse with the numerical Kodaira dimension or the asymptotic Kodaira dimension.}
1. $\Theta_h$ is a closed positive current,

2. for every $m \geq 0$, the natural inclusion

$$H^0(X, \mathcal{O}_X(mL) \otimes I(h^m)) \to H^0(X, \mathcal{O}_X(mL))$$

is an isomorphism. □

**Remark 2.11** If an AZD exists on a line bundle $L$ on a smooth projective variety $X$, $L$ is pseudoeffective by the condition 1 above. □

**Theorem 2.12** ([Ts1, Ts2]) Let $L$ be a big line bundle on a smooth projective variety $X$. Then $L$ has an AZD. □

As for the existence of AZD for general pseudoeffective line bundles, now we have the following theorem.

**Theorem 2.13** ([D-P-S, Theorem 1.5]) Let $X$ be a smooth projective variety and let $L$ be a pseudoeffective line bundle on $X$. Then $L$ has an AZD. □

Although the proof is in [D-P-S], we shall give a proof here, because we shall use it afterwards.

Let $h_0$ be a fixed $C^\infty$ hermitian metric on $L$. Let $E$ be the set of singular hermitian metric on $L$ defined by

$$E = \{ h; h : lowersemicontinuous singular hermitian metric on L, \Theta_h \text{ is positive, } h \geq h_0 \}.$$  

Since $L$ is pseudoeffective, $E$ is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0 (i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

**Theorem 2.14** ([L, p.26, Theorem 5]) Let $\{ \varphi_t \}_{t \in T}$ be a family of plurisubharmonic functions on a domain $\Omega$ which is uniformly bounded from above on every compact subset of $\Omega$. Then $\psi = \sup_{t \in T} \varphi_t$ has a maximum uppersemicontinuous majorant $\psi^*$ which is plurisubharmonic. We call $\psi^*$ the uppersemicontinuous envelope of $\psi$. □

**Remark 2.15** In the above theorem the equality $\psi = \psi^*$ holds outside of a set of measure 0 (cf. [L, p.29]). □

In this paper, we shall call the uppersemicontinuous envelope (resp. lower semicontinuous envelope) by the upper envelope (resp. the lower envelope) for simplicity. By Theorem 2.14 we see that $h_L$ is also a singular hermitian metric on $L$ with $\Theta_h \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$
for some $m$ (otherwise the second condition in Definition 2.3 is empty). We note that
\[
\log |\sigma|_h^m
\]
gives the weight of a singular hermitian metric on $L$ with curvature $2\pi m^{-1}(\sigma)$, where $(\sigma)$ is the current of integration along the zero set of $\sigma$. By the construction we see that there exists a positive constant $c$ such that
\[
\frac{h_0}{|\sigma|_h^m} \geq c \cdot h_L
\]
holds. Hence
\[
\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m))
\]
holds. Hence in particular
\[
\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))
\]
holds. This means that $h_L$ is an AZD of $L$. □

Remark 2.16 By the above proof we have that for the AZD $h_L$ constructed as above,
\[
H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m)) \simeq H^0(X, \mathcal{O}_X(mL))
\]
holds for every $m$. □

It is easy to see that the multiplier ideal sheaves of $h_L^m$ ($m \geq 1$) constructed in the proof of Theorem 2.2 are independent of the choice of the $C^\infty$ hermitian metric $h_0$. The AZD constructed as in the proof of Theorem 2.13 has minimal singularity in the following sense.

Definition 2.17 Let $L$ be a pseudoeffective line bundle on a smooth projective variety $X$. An AZD $h$ on $L$ is said to be an AZD of minimal singularities, if for any AZD $h'$ on $L$, there exists a positive constant $C$ such that
\[
h \leq C \cdot h'
\]
holds. □

Remark 2.18 In [Ts9], I have constructed a canonical AZD (the supercanonical AZD) of the canonical bundle of a projective variety with pseudoeffective canonical bundle. The supercanonical AZD is completely determined by the complex structure. In the previous papers, I have called an AZD of minimal singularities a canonical AZD. Since this may cause a confusion, I have changed the name. □

The following proposition is trivial but important.

Proposition 2.19 Let $h$ be an AZD of a line bundle $L$ on a compact complex manifold $X$. Suppose that there exists a positive integer $m_0$ such that $|m_0L| \neq \emptyset$, then $h$ is normal. □

Proof. Suppose that there exists a positive integer $m_0$ such that $|m_0L| \neq \emptyset$. Let $x \in M$ be a point such that $n(\Theta_h, x) > 0$. Then $\mathcal{I}(h^m)_x \neq \mathcal{O}_{M,x}$ for every $m >> 0$, by the classical theorem of Bombieri ([Bo]). By Definition 2.10, this implies that
\[
\{x \in X \mid n(\Theta_h, x) > 0\} \subseteq \text{Supp} \mathcal{I}|m_0L|
\]
holds. □
2.3 AZD for weakly pseudoeffective singular hermitian line bundles

Similarly as Theorem 2.13, we obtain the following theorem.

**Theorem 2.20**  
Let \((L, h_0)\) be a singular hermitian line bundle on a smooth projective variety \(X\). Suppose that \((L, h_0)\) is weakly pseudoeffective. Then
\[
E(L, h_0) := \{ \varphi \in L^1_{\text{loc}}(X) \mid \varphi \leq 0, \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0 \}
\]
is nonempty and if we define the function \(\varphi_P \in L^1_{\text{loc}}(X)\) by
\[
\varphi_P(x) := \sup \{ \varphi(x) \mid \varphi \in E(L, h_0) \} \ (x \in X).
\]
Then \(h := e^{-\varphi_P} \cdot h_0\) is a singular hermitian metric on \(L\) such that
1. \(\Theta_h \geq 0\).
2. \(H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h^m)) \simeq H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h^m_0))\) holds for every \(m \geq 0\). \(\square\)

**Proof of Theorem 2.20.**  
Since \((L, h_0)\) is pseudoeffective, there exists an ample line bundle \(A\) on \(X\) such that
\[
H^0(X, \mathcal{O}_X(mL + A) \otimes \mathcal{I}(h^m_0)) \neq 0
\]
holds for every \(m \geq 0\). Let \(h_A\) be a \(C^\infty\) hermitian metric on \(A\) with strictly positive curvature and \(dV\) be a \(C^\infty\) volume form on \(X\). Let
\[
K_m := K(mL + A, h^m_0 \cdot h_A, dV)
\]
be the (diagonal part) of the Bergman kernel of \(mL + A\) with respect to the inner product
\[
(\sigma, \sigma') := \int_X \sigma \cdot \sigma' \cdot h^m_0 \cdot h_A \cdot dV
\]
on \(H^0(X, \mathcal{O}_X(mL + A) \otimes \mathcal{I}(h^m_0))\).

Let \(x \in X\) be an arbitrary point and let \((U, z_1, \ldots, z_n)\) be a coordinate neighbourhood centered at \(x\) such that \(U\) is biholomorphic to the open unit ball \(B(O, 1)\) in \(\mathbb{C}^n\) centered at the origin via the coordinate. Taking \(U\) to be sufficiently small, we may and do assume that there exist holomorphic frames \(e_A, L\) of \(A\) and \(L\) on \(U\) respectively. Then with respect to these frames, we may express \(h_A, h_L\) as
\[
h_A = e^{-\varphi_A}, h_L = e^{-\varphi_L}
\]
respectively in terms of plurisubharmonic functions \(\varphi_A, \varphi_L\) on \(U\). By the extremal property of Bergman kernels, we see that \(K_m(x) (x \in X)\) is expressed as
\[
K_m(x) = \sup \{ |\sigma(x)|^2 \mid \sigma \in \Gamma(X, \mathcal{O}_X(A + mL)), \int_X |\sigma|^2 \cdot h_A \cdot h^m_L \cdot dV = 1 \}.
\]

Let \(\sigma_0 \in \Gamma(X, \mathcal{O}_X(A + mL))\) with
\[
\int_X |\sigma_0|^2 \cdot h_A \cdot h^m_L \cdot dV = 1
\]
and \( |\sigma_0(x)|^2 = K_m(x) \). Let us write \( \sigma_0 = f \cdot e_A \cdot e_{\mathbb{T}^n} \) on \( U \) by using a holomorphic function \( f \) on \( U \). By the submeanvalue property of plurisubharmonic functions, we have that

\[
|f(O)|^2 \leq \frac{1}{\text{vol}(B(O, \varepsilon))} \int_{B(O, \varepsilon)} |f|^2 \, d\mu \\
\leq \left( \sup_{B(O, \varepsilon)} e^{\varphi_A \cdot e_{\mathbb{T}^n}} \right) \cdot \left( \frac{1}{\text{vol}(B(O, \varepsilon))} \int_{B(O, \varepsilon)} |f|^2 e^{-\varphi_A \cdot e_{\mathbb{T}^n}} \, dV \right) \cdot \left( \sup_{B(O, \varepsilon)} d\mu \right)
\]

hold, where \( d\mu \) is the standard Lebesgue measure on \( \mathbb{C}^n \). Hence there exists a positive constant \( C_\varepsilon \) independent of \( m \)

\[
K_m(x) \leq C_\varepsilon \cdot \sup_{w \in B(O, \varepsilon)} (h_A^{-1 \cdot h_{L}^{-m}})(w) \cdot dV
\]

holds. Hence taking the \( m \)-th roots of the both sides, letting \( m \) tend to infinity we and letting \( \varepsilon \) tend to 0, we see that

\[
K_\infty = \limsup_{m \to \infty} K(mL + A, h_0^m \cdot h_A, dV)\]

exists and satisfies the inequality

\[
K_\infty \leq h_0^{-1}
\]

holds almost everywhere on \( X \). Hence if we set

\[
h_\infty := \text{the lower envelope of } K_\infty^{-1}
\]

is an element of \( E(L, h_0) \). Hence \( E(L, h_0) \) is nonempty. The rest of the proof is the same as the one of Theorem 2.20

**Definition 2.21** Let \((L, h_0)\) be a singular hermitian line bundle on a complex manifold \( X \). A singular hermitian metric \( h \) on \( L \) is said to be an analytic Zariski decomposition (AZD) of \((L, h_0)\), if the followings hold:

1. \( \Theta_h \geq 0 \).
2. \( H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h^m)) \simeq H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_0^m)) \) holds for every \( m \geq 0 \), where

\[
\mathcal{I}_\infty(h^m) := \cap_{p \geq 1} \mathcal{I}(h^m).
\]

**Remark 2.22** This definition is slightly different from that in [Ts7]. See Remark 2.16 for the reason why we use \( L^\infty \) multiplier ideal sheaves instead of the usual multiplier ideal sheaves.

**Remark 2.23** In Theorem 2.20, \( E(L, h_0) \) is nonempty, if there exists a positive integer \( m_0 \) and \( \sigma \in H^0(X, \mathcal{O}_X(m_0L) \otimes \mathcal{I}_\infty(h_0^{m_0})) \) such that \( h_0^{m_0}(\sigma, \sigma) \leq 1 \). In this case

\[
\varphi := \frac{1}{m_0} \log h_0^{m_0}(\sigma, \sigma)
\]

belongs to \( E(L, h_0) \).

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About the normality of an AZD of a singular hermitian line bundle, we have the following proposition.

**Proposition 2.24** Let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on a smooth projective variety \(X\) and \(F\) be a line bundle on \(X\). Let \(h_F\) be a \(C^\infty\) hermitian metric on \(F\).

Assume that \((L + F, h_L \cdot h_F)\) is weakly pseudoeffective and let \(h\) be an AZD of \((L, h_L)\).

If \(h\) is normal and

\[
H^0(X, \mathcal{O}_X(m_0(L + F)) \otimes \mathcal{I}_\infty(h^{\infty}_L)) \neq 0
\]

, then \(h\) is also normal. \(\Box\)

2.4 \(L^2\)-extension theorem

The \(L^2\)-extension theorem is our crucial tool to investigate multi adjoint bundles in this article.

**Theorem 2.25** ([O-T, p.200, Theorem]) Let \(X\) be a Stein manifold of dimension \(n\), \(\psi\) a plurisubharmonic function on \(X\) and \(s\) a holomorphic function on \(X\) such that \(ds \neq 0\) on every branch of \(s^{-1}(0)\). We put \(Y := s^{-1}(0)\) and \(Y_0 := \{x \in Y; ds(x) \neq 0\}\). Let \(g\) be a holomorphic \((n-1)\)-form on \(Y_0\) with

\[
c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g} < \infty,
\]

where \(c_k = (-1)^{\frac{k(n-k)}{2}}(\sqrt{-1})^k\). Then there exists a holomorphic \(n\)-form \(G\) on \(X\) such that

\[
G(x) = g(x) \wedge ds(x)
\]
on \(Y_0\) and

\[
c_n \int_X e^{-\psi}(1 + |s|^2)^{-2} G \wedge \bar{G} \leq 1620\pi c_n \int_{Y_0} e^{-\psi} g \wedge \bar{g}.
\]

\(\Box\)

For the extension from an arbitrary dimensional submanifold, T. Ohsawa extended Theorem 2.25 in the following way.

Let \(X\) be a complex manifold of dimension \(n\) and let \(S\) be a closed complex submanifold of \(X\). Then we consider a class of continuous function \(\Psi : X \to [\psi, 0)\) such that

1. \(\Psi^{-1}(\psi) \supset S\),

2. if \(S\) is \(k\)-dimensional around a point \(x\), there exists a local coordinate \((z_1, \ldots, z_n)\) on a neighbourhood of \(x\) such that \(z_{k+1} = \cdots = z_n = 0\) on \(S \cap U\) and

\[
\sup_{U \setminus S} |\Psi(z) - (n-k) \log \sum_{j=k+1}^n |z_j|^2| < \infty.
\]
The set of such functions $\Psi$ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_X|\Psi|$ on $S$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_k} f \, d\mu \geq \limsup_{t \to -\infty} \frac{2(n-k)}{\nu_{2n-2k-1}} \int_X f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} \, dV_X$$

for any nonnegative continuous function $f$ with $\text{Supp} f \subset X$. Here $S_k$ denotes the $k$-dimensional component of $S$, $v_m$ denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$ and $\chi_{R(\Psi,t)}$ denotes the characteristic function of the set

$$R(\Psi,t) = \{ x \in M \mid -t - 1 < \Psi(x) < -t \}.$$

Let $X$ be a complex manifold and let $(E,h_E)$ be a holomorphic hermitian vector bundle over $X$. Given a positive measure $d\mu_X$ on $X$, we shall denote $A^2(X,E,h_E,d\mu_X)$ the space of $L^2$ holomorphic sections of $E$ over $X$ with respect to $h_E$ and $d\mu_X$. Let $S$ be a closed complex submanifold of $X$ and let $d\mu_S$ be a positive measure on $S$. The measured submanifold $(S,d\mu_S)$ is said to be a set of interpolation for $(E,h_E,d\mu_X)$, or for the space $A^2(X,E,h_E,d\mu_X)$, if there exists a bounded linear operator

$$I : A^2(S,E|_S,h_E,d\mu_S) \longrightarrow A^2(X,E,h_E,d\mu_X)$$

such that $I(f)|_S = f$ for any $f \in A^2(S,E|_S,h_E,d\mu_S)$. $I$ is called an interpolation operator. The following theorem is crucial.

**Theorem 2.26 ([\textbf{D}] Theorem 4)** Let $X$ be a complex manifold with a continuous volume form $dV_X$, let $E$ be a holomorphic vector bundle over $X$ with $C^\infty$ fiber metric $h_E$, let $S$ be a closed complex submanifold of $X$, let $\Psi \in \sharp(S)$ and let $K_X$ be the canonical bundle of $X$. Then $(S,dV_X(\Psi))$ is a set of interpolation for $(E \otimes K_X,h_E \otimes (dV_X)^{-1},dV_X)$, if the followings are satisfied.

1. There exists a closed set $F \subset X$ such that

   (a) $F$ is locally negligible with respect to $L^2$-holomorphic functions, i.e., for any local coordinate neighborhood $U \subset M$ and any $L^2$-holomorphic function $f$ on $U \setminus X$, there exists a holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f} | U \setminus F = f$.

   (b) $M \setminus F$ is a Stein manifold which intersects with every component of $S$.

2. $\Theta_{h_E} \geq 0$ in the sense of Nakano,

3. $\Psi \in \sharp(S) \cap C^\infty(X \setminus S),$

4. $e^{-(1+\epsilon)\Psi} \cdot h_E$ has semipositive curvature in the sense of Nakano for every $\epsilon \in [0,\delta]$ for some $\delta > 0$.

Under these conditions, there exists a constant $C$ and an interpolation operator from $A^2(S,E \otimes K_X|_S,h \otimes (dV_X)^{-1}|_S,dV_X(\Psi))$ to $A^2(X,E \otimes K_X,h \otimes (dV_X)^{-1},dV_X)$ whose norm does not exceed $C\delta^{-3/2}$. If $\Psi$ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$. □
The above theorem can be generalized to the case that \((E, h_E)\) is a singular hermitian line bundle with semipositive curvature current (we call such a singular hermitian line bundle \((E, h_E)\) a pseudoeffective singular hermitian line bundle) as was remarked in [O].

**Lemma 2.27** Let \(X, S, \Psi, dV_X, dV_X[\Psi], (E, h_E)\) be as in Theorem 2.26 Let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on \(X\). Then \(S\) is a set of interpolation for \((K_X \otimes E \otimes L, dV_X^{-1} \otimes h_E \otimes h_L)\). □

Later we shall use the more general residue volume form \(dV[\Psi]\) as is introduced in Section 1.2. Since the proof of Theorem 2.26 in [O] works without any change, we shall also apply Theorem 2.26 also for this generalized residue volume form \(dV[\Psi]\), too.

### 3 Numerical Kodaira dimension of pseudoeffective singular hermitian line bundles

Let \(X\) be a smooth projective variety and let \(L\) be a line bundle on \(X\).

**Definition 3.1 ([Nak])** The \(L\)-dimension \(\text{Kod}(L)\) is defined by
\[
\text{Kod}(L) := \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(mL))}{\log m}.
\]

The numerical dimension \(\nu(L)\) of \(L\) is defined by
\[
\nu(L) := \sup_A \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(A + mL))}{\log m},
\]
where \(A\) runs all the ample line bundles on \(X\). We often denote \(\text{Kod}(K_X)\) by \(\text{Kod}(X)\) and call it the Kodaira dimension of \(X\). And we often denote \(\nu(K_X)\) by \(\nu(X)\) and call it the numerical Kodaira dimension of \(X\). □

It is trivial to see that \(\text{Kod}(L) \leq \nu(L)\) holds. And \(\text{Kod}(L)\), \(\nu(L)\) are either \(-\infty\) or integers between 0 and \(\dim X\).

The purpose of this section is to define a similar numerical Kodaira dimension for a pseudoeffective singular hermitian line bundle on a smooth projective variety and study the relation between a numerical property of the singular hermitian line bundle and the asymptotic property of the powers of the singular hermitian line bundle twisted by a sufficiently positive line bundle.

#### 3.1 Intersection theory for pseudoeffective singular hermitian line bundles

To introduce the notion of the numerical Kodaira dimension of a pseudoeffective singular hermitian line bundle on a smooth projective variety, first we define the intersection number.

**Definition 3.2 ([Ts1])** Let \((L, h_L)\) be a weakly pseudoeffective singular hermitian line bundle on a smooth projective \(n\)-fold \(X\). The intersection number \((L, h_L)^n \cdot X\) defined by
\[
(L, h_L)^n \cdot X := n! \cdot \limsup_{m \to \infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)).
\]
\((L, h_L)\) is said to be big, if \((L, h_L)^n \cdot X\) is positive. For a \(r\)-dimensional subvariety \(V\) in \(X\) such that \(h_L | V\) is not identically \(+\infty\), we define

\[
(L, h_L)^r \cdot V := r! \cdot \limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}, \mathcal{O}_\tilde{V}(m\mu^* L) \otimes \mathcal{I}(\mu^* (h_L | V)^m)),
\]

where \(\mu : \tilde{V} \to V\) is a resolution of singularities. \((L, h_L)\) is said to be big on \(V\), if \((L, h_L)^r \cdot V > 0 (r = \dim V)\) holds. □

The well definedness of the intersection number is verified as follows.

**Proposition 3.3** The definition of \((L, h_L)^r \cdot V\) is independent of the choice of the resolution \(\pi : \tilde{V} \to V\). □

**Proof.** Let \(\mu : \tilde{V} \to V\) be an resolution and let \(\mu' : \tilde{V}'\) be another resolution factors through \(\mu\), i.e., there exists a morphism \(\phi : \tilde{V}' \to \tilde{V}\) such that \(\mu' = \mu \circ \phi\). Then

\[
\phi_*(\mathcal{O}_{\tilde{V}}(K_{\tilde{V}}) \otimes \mathcal{I}((\mu')^*(h | V)^m))) = \mathcal{O}_V(K_V) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

holds. Hence

\[
H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(K_{\tilde{V}} + m\mu^* L) \otimes \mathcal{I}(\mu^* (h | V)^m)) = H^0(\tilde{V'}, \mathcal{O}_{\tilde{V}'}(K_{\tilde{V}}') + m(\mu')^* L) \otimes \mathcal{I}((\mu')^*(h | V)^m))
\]

holds for every \(m \geq 1\). On the other hand

\[
\limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L + A - B) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

holds, since

\[
\limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L + A - B) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

\[
= \limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

holds for any very ample divisors \(A\) and \(B\) on \(\tilde{V}\). In fact this can be verified by using the exact sequence

\[
0 \to H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L + A)) \otimes \mathcal{I}(\mu^* (h | V)^m)) \to H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L + A - B)) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

\[
\to H^0(B, \mathcal{O}_{\tilde{V}}(m\mu^* L + A))
\]

, etc. And similar equality holds on \(\tilde{V}'\). Hence using the equality \[\square\], we conclude that

\[
\limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(m\mu^* L) \otimes \mathcal{I}(\mu^* (h | V)^m))
\]

\[
= \limsup_{m \to \infty} m^{-r} \dim H^0(\tilde{V}', \mathcal{O}_{\tilde{V}'}(m\mu^* L) \otimes \mathcal{I}((\mu')^*(h | V)^m))
\]

holds. This completes the proof of Proposition 3.3 □
3.2 The numerical Kodaira dimension and the asymptotic Kodaira dimension of a pseudoeffective singular hermitian line bundle

We shall define the numerical Kodaira dimension and the asymptotic Kodaira dimension for pseudoeffective singular hermitian line bundles on smooth projective varieties. These invariants play the essential roles in this article.

**Definition 3.4** $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on a projective manifold $X$. We set
\[ \nu_{\text{num}}(L, h_L) := \sup \left\{ \dim V \mid V \text{ is a subvariety of } X \text{ such that } h_L|_V \text{ is well defined and } (L, h_L)^{\dim V} V > 0 \right\}. \]

We call $\nu_{\text{num}}(L, h_L)$ the numerical Kodaira dimension of $(L, h_L)$. □

The following invariant is more analytic in nature.

**Definition 3.5** The asymptotic Kodaira dimension $\nu(L, h_L)$ of $(L, h_L)$ is defined by
\[ \nu_{\text{asym}}(L, h_L) = \sup_A \limsup_{m \to \infty} \frac{\log h^0(X, \mathcal{O}_X(A + mL) \otimes \mathcal{I}(h_m^P))}{\log m}, \]
where $A$ runs all the ample line bundles on $X$. □

The following example shows that $\nu_{\text{asym}}$ is not necessarily an integer.

**Example 3.6** Let $T$ be a closed positive $(1,1)$ current on $\mathbb{P}^1$
\[ T = \sum_{i=1}^{\infty} \sum_{j=1}^{3^n+1} \frac{1}{4^n} P_{ij} \]
where $\{P_{ij}\}$ are distinct points on $\mathbb{P}^1$. Then there exists a singular hermitian metric $h$ on $\mathcal{O}(1)$ such that $\Theta_h = 2\pi T$. Then we see that
\[ \nu_{\text{asym}}(\mathcal{O}(1), h) = \frac{\log 3}{\log 4} \]
and
\[ \nu_{\text{num}}(\mathcal{O}(1), h) = 0. \]
This implies that $\nu_{\text{num}} \neq \nu_{\text{asym}}$ in general. □

3.3 Seshadri constant for a pseudoeffective singular hermitian line bundle

In this subsection we shall give a criterion of the bigness of a pseudoeffective singular hermitian line bundles on smooth projective varieties.
Definition 3.7 Let $X$ be a smooth projective variety and let $H$ be an ample divisor on $X$. Let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on $X$. Let $x$ be a point on $X$. We set
\[ \epsilon((L, h_L), H, x) = \inf_C \frac{(L, h_L) \cdot C}{H \cdot C}, \]
where $C$ runs all the irreducible curves in $X$ passing through $x$. We call $\epsilon((L, h_L), H, x)$ the Seshadri constant of $(L, h_L)$ at $x$ with respect to $H$. □

Theorem 3.8 Let $X$ be a smooth projective variety and let $H$ be an ample divisor on $X$. Let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on $X$.

Then there exists at most countable union of proper subvarieties $F$ such that if there exists a point $x_0 \in X - F$ such that
\[ \epsilon((L, h_L), x_0) > 0 \]
holds, then $(L, h_L)$ is big. □

Proof of Theorem 3.8 We say that $C$ is a strongly movable curve, if
\[ C = \mu_* (\tilde{A}_1 \cap \cdots \cap \tilde{A}_{n-1}) \]
for some very ample divisors $\tilde{A}_j$ on $\tilde{X}$, where $\mu: \tilde{X} \rightarrow X$ is a modification. The strongly movable cone $SME(X)$ of $X$ is the cone of curves generated by all the strongly movable curves on $X$. Let $S$ be the family of strongly movable curves on $X$. We set
\[ U := \{ x \in X \mid \text{for every irreducible component of } C, \text{there exists an irreducible member } C \text{ belonging to the component and passing through } x. \} \]
Then we see that there exists at most a countable union of proper subvarieties $F$ of $X$ such that $U = X - F$.

Suppose that there exists a point $x_0 \in U$ such that $\epsilon_0 := \epsilon((L, h_L), H, x_0) > 0$. Let $dV$ be a $C^\infty$ volume form on $X$. Let $m$ be a positive integer such that $m > 2/\epsilon_0$ and let
\[ H^0(X, \mathcal{O}_X(mL + H) \otimes \mathcal{I}(h_{L_m}^{m})) \]
us consider the inner product
\[ (\sigma, \sigma') := \int_X \sigma \cdot \bar{\sigma} \cdot h_L^m \cdot h_H \cdot dV. \]
Let $K_m := K(X, mL + H, h_L^m \cdot h_H, dV)$ be the diagonal part of the Bergman kernel with respect to the above inner product. We set
\[ h_m := (K_m)^{-1/\mu}. \]
Then $h_m$ is a singular hermitian metric on $L + \frac{1}{m}H$ with algebraic singularities.

We note that for every strongly movable curve $C$ passing through $x_0$
\[ (L + \frac{1}{m}H, h_m | C) \cdot C \geq \frac{\epsilon_0}{2} H \cdot C \]
20
holds. Since the pseudoeffective cone of $X$ is the dual of $SME(X)$ as in [B-D-P-P], by the definition of $U$, we see that

$$(L + \frac{1}{m} H, h_m) - \frac{\epsilon_0}{2} H$$

is pseudoeffective. Letting $m$ tend to infinity, by Lemma 5.4, we see that

$$(L - \frac{\epsilon_0}{2} H, h_L, h_H)$$

is pseudoeffective (cf. Definition 2.4). Hence $(L, h_L)$ should be big. This completes the proof of Theorem 3.8.

\[\square\]

3.4 Non big pseudoeffective singular hermitian line bundles

In this subsection, we shall prove the following vanishing theorem.

**Proposition 3.9** Let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on a smooth projective variety $X$. Suppose that $(L, h_L)$ is not big and one of the followings holds.

1. $L$ is not big.
2. $L$ is normal (cf. Definition 2.8).

Then there exists a very ample divisor $H$ on $X$ such that

$$H^0(X, \mathcal{O}_X(mL - H) \otimes I(h^m_L)) \neq 0$$

holds for every $m \geq 1$. \[\square\]

**Proof of Proposition 3.9** Suppose that there exists a very ample divisor $H$ on $X$ such that

$$H^0(X, \mathcal{O}_X(mL - H) \otimes I(h^m_L)) \neq 0$$

holds for some $m \geq 1$. Then since $H$ is very ample, we see that

$$| H^0(X, \mathcal{O}_X(mL) \otimes I(h^m_L)) |$$

gives a birational rational map from $X$ into a projective space. Hence if $L$ is not big, then for every very ample divisor $H$,

$$H^0(X, \mathcal{O}_X(mL - H) \otimes I(h^m_L)) = 0$$

holds for every $m \geq 1$.

Next we shall assume that $(L, h_L)$ is normal. If we take $H$ very general, we see that

$$I(h^m_L |_H) = I(h^m_L) |_H$$

holds for every $m \geq 1$. This is possible by the following lemma.
Lemma 3.10 There exists a smooth member $H' \in |H|$, such that

$$\mathcal{I}(h^m_L) \otimes \mathcal{O}_{H'} = \mathcal{I}(h^m_L |_{H'})$$

holds for every $m \geq 1$. □

Proof of Lemma 3.10. Let $A$ be a sufficiently ample line bundle such that $\mathcal{O}_X(A + mL) \otimes \mathcal{I}(h^m_L)$ is globally generated for all $m \geq 1$. Let $\{\sigma_j^{(m)}\}_{j=1}^{N_m}$ be a (complete) basis of $H^0(X, \mathcal{O}_X(A + mL) \otimes \mathcal{I}(h^m_L))$. We consider the subset

$$U := \{ F \in |H| ; F \text{ is smooth, } \int_F |\sigma_j^{(m)}|^2 \cdot h^m_L \cdot h_A \cdot dV_F < +\infty \text{ for every } m \text{ and } 1 \leq j \leq N_m \}$$

of $|H|$, where $dV_F$ denotes the volume form on $F$ induced by the Kähler form $\omega$. We claim that such $U$ is the complement of at most a countable union of proper subvarieties of $|H|$. Let us fix a positive integer $m$. Since $\mathcal{I}(h^m_L)$ is a coherent sheaf ([N]), we see that

$$U_m := \{ F \in |H| ; F \text{ is smooth, } \int_F |\sigma_j^{(m)}|^2 \cdot h^m_L \cdot h_A \cdot dV_F < +\infty \text{ for } 1 \leq j \leq N_m \}$$

is a Zariski open subset of $|H|$. In fact this can be verified as follows. Let $\Lambda$ be a pencil contained in $|H|$ which contains a smooth member. Then by Fubini’s theorem, we see that a general member of $\Lambda$ is contained in $U_m$, unless for every general member $F$ of $\Lambda$ the set

$$\{ x \in F | h_A \cdot h^m_L \cdot |\sigma_j^{(m)}|^2 \notin L_{1,\text{loc}}(F, x) \text{ for some } 1 \leq j \leq N_m \}$$

is contained in the base locus of $\Lambda$. Hence we see that $U_m$ is Zariski dense in $|H|$. Then since $U = \cap_{m=1}^\infty U_m$, we complete the proof of Lemma 3.10. □.

Let us continue the proof of Proposition 3.9. Since $(L, h_L)$ is not big, by Theorem 3.8 we see that for a very general $x \in X$, $\epsilon((L, h_L), H, x) = 0$ holds.

Lemma 3.11 There exists at most a countable union of subvarieties $F$ such that for every $x \in X - F$ and $\delta > 0$, there exists an irreducible curve $C$ on $X$ passing through $x$ such that

1. $C$ is smooth at $x$,
2. $(L, h_L) \cdot C \leq \delta \cdot (H \cdot C)$.

□

By the $L^2$-extension theorem (Theorems 2.25 and 2.26) the intersection number $(L, h_L) \cdot C_t$ is lower semicontinuous with respect to the countable Zariski topology on $\Delta^{n-1}$. Then since the Hilbert scheme (of $X$) has only countably many irreducible components, by Lemma 3.11 we have the following lemma.

Lemma 3.12 For every $x \in X - F$ and $\delta > 0$ there exist a $(n-1)$-dimensional family of irreducible curves $\{C_t\}_{t \in \Delta^{n-1}}$ parametrized $\Delta^{n-1}$ and a coordinate neighbourhood $(U, z_1, \ldots, z_n)$ of $x$ such that
1. \((L, h_L) \cdot C_t \leq \delta (H \cdot C_t)\) holds for every \(t \in \Delta^{n-1}, @\)

2. \(z_1(x) = \cdots = z_n(x) = 0,\)

3. \(U\) is biholomorphic to \(\Delta^n\) via the coordinate \((z_1, \cdots, z_n)\).

4. For every \(t \in \Delta^{n-1},\)
   \[C_t \cap U = \{ p \in U \mid (z_1(p), \cdots, z_n(p)) = t \}\]
   holds for every \(t \in \Delta^{n-1}.\)

\[\square\]

By the assumption \(E\) is contained in a proper analytic subset of \(X, \) say \(V.\)
Taking \(H\) to be sufficiently ample, we may and do assume that \(\frac{1}{2} H\) is Cartier
and \(V\) is contained in a member \(H_0 \in |\frac{1}{2} H |.\) Let us take \(\delta < 1/2m\) in Lemma 3.12
Then since
   \[\deg \mathcal{O}_{C_t}(mL - H) \otimes \mathcal{I}(h^m_L |C_t) < -\frac{1}{2m} \cdot m \cdot (H \cdot C) + \frac{1}{2} H \cdot C < 0\]
hold (because \(V\) is contained in a member \(H_0 \in |\frac{1}{2} H |), we see that
\[H^0(C_t, \mathcal{O}_{C_t}(mL - H) \otimes \mathcal{I}(h^m_L |C_t)) = 0\]
holds for every \(t \in \Delta^{n-1}.\) By Fubini’s theorem and Lemma 3.12,
\[H^0(X, \mathcal{O}_X(mL - H) \otimes \mathcal{I}(h^m_L)) = 0\]
holds. We note that \(H\) can be taken independent of \(m.\) This completes the
proof of Proposition 3.9. \[\square\]

3.5 Relation between \(\nu_{\text{num}}\) and \(\nu_{\text{asym}}\)
In Example 3.6, we have seen that \(\nu_{\text{num}}\) and \(\nu_{\text{asym}}\) are different in general.
But in this subsection we shall prove that \(\nu_{\text{num}} = \nu_{\text{asym}}\) holds under a mild
condition.

**Theorem 3.13** Let \(X\) be a smooth projective variety and let \((L, h_L)\) be a pseudo-
effective singular hermitian line bundle on \(X.\)
Then
   \[\nu_{\text{num}}(L, h_L) \leq \nu_{\text{asym}}(L, h_L)\]
holds. Moreover if \(h_L\) is normal (cf. Definition 2.8), for every ample line bundle
\(A\) on \(X,
\[\dim H^0(X, \mathcal{O}_X(A + mL) \otimes \mathcal{I}(h^m_L)) = O(m^{\nu_{\text{num}}(L, h_L)}).\]
holds. In particular
\[\nu_{\text{num}}(L, h_L) = \nu_{\text{asym}}(L, h_L)\]
holds. \[\square\]
Remark 3.14 If $h_L$ is an AZD of $L$ with $Kod(L) \geq 0$, then the set

$$E := \{ x \in X \mid n(\Theta_{h_L}, x) > 0 \}$$

is contained in the stable base locus of $L$. Hence in this case $h_L$ is normal. □

Proof of Theorem 3.13 We denote $\nu_{\text{num}}(L, h_L)$ by $\nu$. Let $V$ be a $\nu$ dimensional subvariety such that $h_L |_V$ is well defined and $(L|_V, h_L | V)$ is big. Let $f : Y \to X$ be an embedded resolution of $V$. Then replacing $(L, h_L)$ by $f^*(L, h_L)$ we may assume that $V$ is smooth from the beginning. Let $A$ be a sufficiently ample line bundle such that every element of $H^0(Y, O_Y(A + mL) \otimes I(h^{m}_L | V))$ extends to an element $H^0(X, O_X(A + mL) \otimes I(h^{m}_L))$. Then we have that

$$\nu_{\text{num}}(L, h_L) \leq \limsup_{m \to \infty} \frac{\log \dim H^0(X, O_X(A + mL) \otimes I(h^{m}_L))}{\log m}$$

holds. Hence we have the inequality

$$\nu_{\text{num}}(L, h_L) \leq \nu_{\text{asym}}(L, h_L).$$

Next suppose that $(L, h_L)$ be a normal pseudoeffective singular hermitian line bundle on a smooth projective variety $X$. We shall prove

$$\nu_{\text{num}}(L, h_L) \geq \limsup_{m \to \infty} \frac{\log \dim H^0(X, O_X(A + mL) \otimes I(h^{m}_L))}{\log m}$$

holds by induction on $n = \dim X$.

If $n = 1$ and $(L, h_L)$ is not big, then $(L, h_L)$ is numerically trivial. Hence $\Theta_{h_L}$ has no absolutely continuous part. Since $h_L$ is normal,

$$\Theta_{h_L} = \sum a_i P_i$$

for some effective $\mathbb{R}$-divisor $\sum a_i P_i$ on $X$. Hence $\nu_{\text{num}}(L, h_L) = \nu_{\text{asym}}(L, h_L) = 0$ holds in this case.

Suppose that

$$\nu_{\text{num}}(F, h_F) = \limsup_{m \to \infty} \frac{\log \dim H^0(Y, O_Y(A + mF) \otimes I(h^{m}_F))}{\log m}$$

holds for every normal pseudoeffective singular hermitian line bundle $(F, h_F)$ and every sufficiently ample line bundle $A$ on a smooth projective variety $Y$ of dimension $\leq n - 1$.

Let $X$ be a smooth projective variety of dimension $n$ and let $(L, h_L)$ be a normal pseudoeffective line bundle on $X$. If $(L, h_L)$ is big, there is nothing to prove. We shall assume that $(L, h_L)$ is not big, i.e.,

$$\limsup_{m \to \infty} \frac{m^{-n} \dim H^0(X, O_X(mL) \otimes I(h^{m}_L))}{m} = 0$$

holds. Let $H$ be a sufficiently ample very ample smooth divisor on $X$. Then by Proposition 3.9

$$H^0(X, O_X(mL - H) \otimes I(h^{m}_L)) = 0$$

holds. □
holds for every \( m \geq 0 \). Let \( G \) be a smooth member of \( |2H| \). If we take \( G \) properly, by the Lemma 3.10 we may assume that
\[
\mathcal{I}(h^m_L) \otimes \mathcal{O}_G = \mathcal{I}(h^m_L | G)
\]
holds for every \( m \geq 1 \). Let us consider the exact sequence
\[
0 \to H^0(X, \mathcal{O}_X(mL - H) \otimes \mathcal{I}(h^m_L)) \to H^0(X, \mathcal{O}_X(mL + H) \otimes \mathcal{I}(h^m_L)) \to H^0(G, \mathcal{O}_G(mL + H) \otimes \mathcal{I}(h^m_L)).
\]
Then by (3) we have that
\[
\dim H^0(X, \mathcal{O}_X(H + mL) \otimes \mathcal{I}(h^m_L)) \leq \dim H^0(G, \mathcal{O}_G(mL + H) \otimes \mathcal{I}(h^m_L | G)).
\]
Since \( \mathcal{I}(h^m_L | G) = \mathcal{I}(h^m_L) | G \) holds for every \( m \geq 0 \) by the choice of \( G \), we see that
\[
\limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(H + mL) \otimes \mathcal{I}(h^m_L))}{\log m} \leq \limsup_{m \to \infty} \frac{\log \dim H^0(G, \mathcal{O}_G(mL + H) \otimes \mathcal{I}(h^m_L | G))}{\log m}.
\]
holds. By the induction assumption
\[
\limsup_{m \to \infty} \frac{\log \dim H^0(G, \mathcal{O}_G(mL + H) \otimes \mathcal{I}(h^m_L))}{\log m} = \nu_{num}(L | G, h_L | G)
\]
holds. By the assumption, if we take \( H \) sufficiently ample, we see that
\[
\nu_{num}(L, h_L) = \nu_{num}(L | G, h_L | G)
\]
holds. Hence combining (2) and (7), we see that
\[
\nu_{num}(L, h_L) \geq \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(H + mL) \otimes \mathcal{I}(h^m_L))}{\log m}
\]
holds.
Combining (2) and (7), we see that
\[
\nu_{num}(L, h_L) = \nu_{asym}(L, h_L)
\]
holds.

By using the construction as above inductively, we find a sequence of members \( G = G_1, G_2, \cdots, G_n \in |2H| \) such that
\begin{enumerate}
\item \( G_i \) intersects \( G_1 \cap \cdots \cap G_{i-1} \) transversally. We set \( X_i := G_1 \cap \cdots \cap G_i \) for \( 1 \leq i \leq n - \nu \), \( X_0 := X \).
\item \( h_L | X_i \) is well defined for every \( 1 \leq i \leq n - \nu \).
\item \( \mathcal{I}(h^m_L | X_i) = \mathcal{I}(h^m_L | X_i) \) for every \( 1 \leq i \leq n - \nu \).
\item \( \dim H^0(X_i, \mathcal{O}_{X_i}(H + mL) \otimes \mathcal{I}(h^m_L)) \leq \dim H^0(X_{i+1}, \mathcal{O}_{X_{i+1}}(H + mL) \otimes \mathcal{I}(h^m_L)) \)
\end{enumerate}
holds for every \( m \geq 1 \) and \( 0 \leq i \leq n - \nu - 1 \).
5. \((L, h_L) \mid_{X_{n-\nu}}\) is big.

Then we see that
\[
\dim H^0(X, \mathcal{O}_X(H + mL) \otimes \mathcal{I}(h_L^m)) \leq \dim H^0(X_{n-\nu}, \mathcal{O}_{X_{n-\nu}}(H + mL) \otimes \mathcal{I}(h_L^m \mid_{X_{n-\nu}}))
\]
holds. Since
\[
\dim H^0(X_{n-\nu}, \mathcal{O}_{X_{n-\nu}}(H + mL) \otimes \mathcal{I}(h_L^m \mid_{X_{n-\nu}})) = O(m^\nu),
\]
this completes the proof of Theorem 3.13. \(\square\)

4 Asymptotic expansion of Bergman kernels of pseudoeffective line bundles

The purpose of this section is to extend the asymptotic expansions of Bergman kernels associated with positive line bundles on a projective manifold to the case of pseudoeffective singular hermitian line bundles.

4.1 Local measure associated with quasiplurisubharmonic functions

The content of this subsection is taken from [G-Z]. Let \(u, v\) be bounded plurisubharmonic functions on some domain \(D\) in \(\mathbb{C}^n\). Then
\[
1\{u > v\}[dd^c u]^n = 1\{u > v\}[dd^c \max(u, v)]^n
\]
holds in weak sense of measure in \(D\) ([B-T]), where
\[
d^c := \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial)
\]
and the operation \([dd^c \cdot]^n\) is the wedge product of closed positive (1,1) current with bounded potential defined in [B-T]. Let \((X, \omega)\) be a compact Kähler manifold and we set
\[
PSH(X, \omega) = \{\varphi \in L^1_{loc}(X) \mid \text{uppersemicontinuous function on } X \text{ such that } \omega \varphi := \omega + dd^c \varphi \text{ is a closed semipositive (1,1) current on } X\}.
\]
We call \(PSH(X, \omega)\) the set of \(\omega\) plurisubharmonic functions. Let \(\varphi \in PSH(X, \omega)\) be a \(\omega\) plurisubharmonic function on \(X\).

The purpose of this subsection is to define the Monge-Ampère measure associated with the closed positive current \(\omega + dd^c \varphi\).

We set \(\varphi_j := \max(\varphi, -j) \in PSH(X, \omega)\). We call the sequence \(\{\varphi_j\}\) the canonical approximation of \(\varphi\) by bounded \(\omega\)-plurisubharmonic functions. This is a decreasing sequence and by (8)
\[
1\{\varphi_j > k\}[\omega + dd^c \varphi_j]^n = 1\{\varphi > -k\}[\omega + dd^c \max(\varphi_j, -k)]^n
\]
holds. If \(j > k\) holds, then \(\{\varphi_j > -k\} = \{\varphi > -k\}\) and \(\max(\varphi_j, -k) = \varphi_k\) hold. Hence
\[
j \geq k \Rightarrow 1\{\varphi > -j\}[\omega + dd^c \varphi_j]^n \geq 1\{\varphi > -k\}[\omega + dd^c \varphi_k]^n
\]
holds. Since \( \{ \varphi > -k \} \subseteq \{ \varphi > -j \} \) holds, we have

\[
j \geq k \Rightarrow \mathbf{1}_{\{ \varphi > -j \}}[\omega + dd^c \varphi_j]^n \geq 1_{\{ \varphi > -k \}}[\omega + dd^c \varphi_k]^n.
\]

We set

\[
d\mu_\varphi := \lim_{j \to \infty} \mathbf{1}_{\{ \varphi > -j \}}[\omega + dd^c \varphi_j]^n.
\]

This is a positive Borel measure which is precisely the non-pluripolar part of \( (\omega + dd^c \varphi)^n \).

In general the total mass

\[
\int_X d\mu_\varphi
\]

can take any value in \([0, \int_X \omega^n]\). This phenomena is caused by the escape of the measure toward the pluripolar set of \( \varphi \).

### 4.2 Local volume of pseudoeffective singular hermitian line bundles

Let \( X \) be a projective manifold of dimension \( n \) and let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on \( X \).

**Definition 4.1** In the above notation, we define the number \( \mu(L, h_L) \) by

\[
\mu(L, h_L) := n! \lim_{m \to \infty} \operatorname{sup} m^{-n} h^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))
\]

is called the volume of \((L, h_L)\). \( \square \)

**Definition 4.2** Let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on a projective manifold \( X \). \((L, h_L)\) is said to be big, if the volume \( \mu(L, h_L) \) is positive. \( \square \)

In this subsection we shall define the local version of \( \mu(L, h_L) \).

Let \( h_0 \) be a \( C^\infty \) hermitian metric on \( L \) and let \( \varphi \in L^1(X) \) be the weight function of \( h_L \) with respect to \( h_0 \), i.e., \( \varphi \) is a function such that

\[
h_L = e^{-\varphi} \cdot h_0
\]

holds. Let \( A \) be an ample line bundle on \( X \) such that \( A - L \) is very ample. Let \( \sigma \) be a nontrivial global holomorphic section of \( A - L \) and let \( h_{A-L} \) be a \( C^\infty \) hermitian metric on \( A - L \) such that \( h_A := h_0 \cdot h_{A-L} \) is a \( C^\infty \) hermitian metric on \( A \) with strictly positive curvature on \( X \). Then

\[
h_L \cdot \frac{1}{|\sigma|^2} = (h_0 \cdot h_{A-L}) \cdot e^{-(\varphi + \log h_{A-L}(\sigma, \sigma))}
\]

is a singular hermitian metric on \( A \). We set

\[
\psi := \varphi + \log h_{A-L}(\sigma, \sigma)
\]

and

\[
\omega = \frac{1}{2\pi} \Theta_{h_A}.
\]
Then
\[ \omega_\psi = \omega + dd^c \psi \]
is a closed positive current on \( X \) and
\[ \omega_\psi = \frac{1}{2\pi} \Theta_{h_L} + (\sigma), \]
where \((\sigma)\) denotes the current of integration over the divisor of \( \sigma \). In particular, \( \psi \) is a \( \omega \)-plurisubharmonic function on \( X \). Then by the result in Section 4.1, we may define
\[ d\mu(L, h_L) := d\mu_\psi. \]
It is easy to see that \( d\mu(L, h_L) \) is independent of the choice of \( A, h_A, \) etc.

**Definition 4.3** Let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on a projective manifold \( X \). We call the measure \( d\mu(L, h_L) \) the local volume of \((L, h_L)\). □

### 4.3 Asymptotic expansion of Bergman kernels

Let \( X \) be a smooth projective variety and let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on \( X \).

Let \( A \) be a sufficiently ample line bundle on \( X \) so that
\[ O_X(K_X + A + mL) \otimes I(h^m_L) \]
is globally generated for every \( m \geq 1 \). Let \( h_A \) be a \( C^\infty \) hermitian metric on \( A \) with strictly positive curvature. Let
\[ K(X, K_X + A + mL, h_A \cdot h^m_L) \]
be the Bergman kernel of \( K_X + A + mL \) with respect to the inner product
\[ (\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X \sigma \wedge \overline{\sigma'} \cdot h_A \cdot h^m_L. \]
The reason why we need \( A \) here is that to kill the higher cohomology of \( O_X(K_X + A + mL) \otimes I(h^m_L) \) and to localize the estimate below.

Now we are interested in the asymptotics of the volume form
\[ h_A \cdot h^m_L \cdot K(X, K_X + A + mL, h_A \cdot h^m_L) \]
as \( m \) tends to infinity.

We may also consider the local version of the above Bergman kernel. To consider the local version and to localize the estimate is quite crucial here.

Let \( x \) be a point on \( X \) and let \( U \) be a coordinate neighbourhood of \( x \) such that \( U \) is biholomorphic to a ball. Let
\[ K(U, K_X + A + mL, h_A \cdot h^m_L) \]
be the Bergman kernel of \( K_X + A + mL \mid_U \) with respect to the inner product
\[ (\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_U \sigma \wedge \overline{\sigma'} \cdot h_A \cdot h^m_L. \]
Then it is obvious that
\[ K(U, K_X + A + mL, h_A \cdot h_L^m) \geq K(X, K_X + A + mL, h_A \cdot h_L^m) \]
On the other hand if we replace \( A \) by its high multiple, if necessary, we see tat there exists a positive constant \( C \) such that
\[ K(U, K_X + A + mL, h_A \cdot h_L^m)(x) \leq C \cdot h_A \cdot K(X, K_X + 2A + mL, h_A^2 \cdot h_L^m)(x) \]
holds for every \( x \in B(O, 1/2) \) and every positive integer \( m \). This estimate immediately follows from the estremal property of the Bergman kernels, i.e.,
\[ K(U, K_X + A + mL, h_A \cdot h_L^m)(x) = \sup\{ |\sigma|^2(x) \mid (\sqrt{-1})^{n^2} \int_U \sigma \wedge \bar{\sigma} \cdot h_A \cdot h_L^m = 1 \} \]
and similar equality for \( K(X, K_X + A + mL, h_A \cdot h_L^m) \) and \( \partial \) operators.

In this way, thanks to the presence of \( A \), we may localize the estimate of the asymptotics of the global Bergman kernels in terms of that of local Bergman kernels.

The local asymptotics of \( h_A \cdot h_L^m \cdot K(U, K_X + A + mL, h_A \cdot h_L^m) \) (hence also the global asymptotics) can be explored by the following well known theorem, if \( h_L \) is \( C_\infty \).

**Theorem 4.4** (\( \mathcal{O}, \mathcal{P}, \mathcal{Z} \)) Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^n \) and let \( \varphi \) be a \( C_\infty \) plurisubharmonic function on \( \Omega \). For a positive integer \( m \), let \( K(K_\Omega, e^{-m\varphi}) \) be the Bergman kernel of \( K_\Omega \) with respect to the inner product
\[ (f, g) := (\sqrt{-1})^{n^2} \int_{\Omega} e^{-m\varphi} \cdot f \wedge \bar{g}. \]
Then
\[ e^{-m\varphi} \cdot K(K_\Omega, e^{-m\varphi}) = \frac{(dd^c\varphi)^n}{n!} m^n + O(m^{n-1}) \]
holds. □.

**Theorem 4.4** can be viewed as a (weak version of) local Riemann-Roch theorem.

**Definition 4.5** Let \( (L, h_L) \) be a pseudoeffective singular hermitian line bundle on a projective manifold \( X \) of dimension \( n \). Let \( A \) be a sufficiently ample line bundle on \( X \). We set
\[ d\mu^+(L, h_L) := n! \cdot \limsup_{m \to \infty} m^{-n} \cdot h_A \cdot h_L^m \cdot K(X, K_X + A + mL, h_A \cdot h_L^m)(z). \]
d\( \mu^+(L, h_L) \) is said to be the upper local volume of \( (L, h_L) \). Similarly we set
\[ d\mu^-(L, h_L) := n! \cdot \liminf_{m \to \infty} m^{-n} \cdot h_A \cdot h_L^m \cdot K(X, K_X + A + mL, h_A \cdot h_L^m)(z). \]
d\( \mu^-(L, h_L) \) is said to be the lower local volume of \( (L, h_L) \). □

Since
\[ \int_X h_A \cdot h_L^m \cdot K(X, K_X + A + mL, h_A \cdot h_L^m) = \dim H^0(X, \mathcal{O}_X(K_X + A + mL) \otimes \mathcal{I}(h_A \cdot h_L^m)) \]
holds, by Lebesgue-Fatou’s lemma, we have that
\[ \int_X d\mu^-(L, h_L) \leq \mu(L, h_L) \leq \int_X d\mu^+(L, h_L) \]
hold.
4.4 A general conjecture for the asymptotic expansion of Bergman kernels

Let $X$ be a smooth projective variety of dimension $n$ and let $(L, h_L)$ be a pseudoeffective singular hermitian line bundle on $X$. Let $A$ be a sufficiently ample line bundle on $X$ and let $h_A$ be a $C^\infty$ hermitian metric on $A$ with strictly positive curvature.

**Conjecture 4.6**

1. $d\mu^+(L, h_L) = d\mu^-(L, h_L)$ holds (cf. Definition 4.3). In particular
   \[ n! \cdot \lim_{m \to \infty} m^{-n} \cdot h_A \cdot h_L^m \cdot K(X, K_X + A + mL, h_A \cdot h_L^m) \exists \text{ on } X. \]

2. $n! \cdot \lim_{m \to \infty} m^{-n} \cdot h_A \cdot h_L^m \cdot K(X, K_X + A + mL, h_A \cdot h_L^m) = d\mu(L, h_L),$

where $d\mu(L, h_L)$ denotes the local volume defined in Definition 4.3.

Conjecture 4.6 is true for the case that $h_L$ is smooth by [13, Z]. Also by essentially the same proof, it is easy to verify that Conjecture 4.6 is true for the case that $h_L$ has algebraic singularities. But in general, Conjecture 4.6 seems to be very difficult to prove, since the asymptotic expansion of Bergman kernels breaks down, if the hermitian metric is not $C^\infty$. Of course one may approximate $h_L$ by a singular hermitian metric with algebraic singularities ([D]). But the difficulty arises when we take the limit, i.e., the lower order term of the asymptotic expansion may blow up.

In the next subsection, we shall prove a lower estimate for the asymptotic expansion of Bergman kernels, when the curvature is strictly positive.

4.5 Asymptotic expansion of Bergman kernels of singular hermitian line bundles with strictly positive curvature currents

In this subsection, we shall estimate the asymptotic expansion of Bergman kernels of singular hermitian line bundles with strictly positive curvature. We prove that the coefficient of the top term of the expansion at a point is strictly positive when the curvature is bounded from below by a strictly positive form locally around the point.

**Theorem 4.7** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ contained in the unit open ball $B(O, 1)$ centered at the origin with radius 1 and let $\varphi$ be a plurisubharmonic function on $\Omega$ such that

\[ dd^c \varphi \geq -dd^c \log(1 - \|z\|^2) \]

holds on $\Omega$, where $\|z\|^2 = \sum_{i=1}^n |z_i|^2$. Then there exists a positive constant $C$ independent of $\varphi$ such that for every positive integer $m$

\[ e^{-m\varphi} \cdot K(\Omega, e^{-m\varphi}) \geq C \cdot m^n \cdot |dz_1 \wedge \cdots \wedge dz_n|^2 \]

holds on $\Omega$. □
Proof. Since $B(O,1)$ is homogeneous, we may assume that $\Omega$ contains the origin $O \in \mathbb{C}^n$ and it is is enough to prove that

$$e^{-m\varphi} \cdot K(\Omega, e^{-m\varphi})(O) \geq C \cdot m^n \cdot |dz_1 \wedge \cdots \wedge dz_n|^2$$

holds at the origin $O$. Let $\rho \in C_0^\infty(\mathbb{C}^n)$ be a function such that

1. $0 \leq \rho \leq 1$ hold on $\mathbb{C}^n$.
2. $\text{Supp } \rho \subset B(O, 1/2)$ holds.
3. $|d\rho| < 3$, where the norm is taken with respect to the standard Euclidean metric on $\mathbb{C}^n$.

Since $dd^c \log(1-\|z\|^2)$ is a Kähler form invariant under $\text{Aut}(B(O,1))$, for every sufficiently large $m$, there exists a positive constant $c > 1$ depending only on $\rho$ such that

$$m \cdot dd^c \log(1-\|z\|^2) + dd^c(\rho(c \sqrt{m} z) \cdot \frac{1}{\|z\|^{2n}}) \geq \frac{m}{2} \cdot dd^c \log(1-\|z\|^2) \quad (9)$$

holds on $B(O,1)$. Hence by the standard $L^2$-estimates, we have the following lemma.

**Lemma 4.8** There exists a positive constant $c_0$ such that for every sufficiently large $m \geq 1$,

$$K(\Omega, e^{-m\varphi})(O) \geq c_0 \cdot K(B(O, \frac{c}{\sqrt{m}}), e^{-m\varphi})(O)$$

holds. □

**Proof of Lemma 4.8**. Let $f$ be a holomorphic $(n, 0)$ form on $B(O, \frac{c}{\sqrt{m}})$ such that

$$|f|^2 (O) = K(B(O, \frac{c}{\sqrt{m}}), e^{-m\varphi})(O)$$

and

$$\int_{B(O, \frac{c}{\sqrt{m}})} |f|^2 \cdot e^{-m\varphi} d\mu = 1$$

holds, i.e., $f$ is a peak section at $O$ with respect to the metric $e^{-m\varphi}$.

Let us fix a complete Kähler metric $g$ on $\Omega$ and let $\omega_g$ denote the Kähler form associated with $g$. Let $\gamma(z)$ be the minimum eigenvalue of $dd^c \log(1-\|z\|^2)$ with respect to $g(z)$. Then

$$dd^c \log(1-\|z\|^2) \geq \gamma(z) \omega_g(z)$$

holds. Then by the $L^2$-estimates and [9], we have that there exists a $(n, 0)$ form $u$ on $\Omega$ such that

$$\bar{\partial} u = \bar{\partial}(\rho(c \sqrt{m} z) f(z))$$

and

$$\int_{\Omega} |u|^2 \cdot e^{-m\varphi} \cdot e^{\rho(c \sqrt{m} z)} \frac{1}{\|z\|^{2n}} \leq \frac{2}{m} \int_{\Omega} \frac{1}{\gamma(z)} |\bar{\partial}(\rho(c \sqrt{m} z) f(z))|^2 \cdot e^{\rho(c \sqrt{m} z)} \frac{1}{\|z\|^{2n}} d\mu_g \quad (10)$$

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hold, where $|\bar{\partial}(\rho(z)f(z))|$ denotes the norm with respect to $g$ and $d\mu_g$ denotes the volume form with respect to $g$. Since there exists a positive constant $C_1$ independent of $m$ such that
\[
\int_{\Omega} \frac{1}{\gamma(z)} |\bar{\partial}(\rho(z)f(z))|^2 \cdot e^{\rho(z)} \cdot \frac{1}{\|z\|^{2n}} d\mu_g \leq C_1 \cdot m
\]
holds for every sufficiently large $m$, by (10), we see that there exists a positive constant $C_2$ independent of $m$ such that
\[
\int_{\Omega} |u|^2 \cdot e^{-m\varphi} \cdot e^{\rho(z)} \cdot \frac{1}{\|z\|^{2n}} \leq C_2.
\]
Since $u(O) = 0$ holds by the construction, we see that
\[
\rho(z)f(z) - u
\]
is a holomorphic extension of $f(O)$ at $O$ and there exists a positive constant $C_3$ independent of $m$ such that
\[
\int_{\Omega} |\rho(z)f(z) - u|^2 \cdot e^{-m\varphi} \leq C_3
\]
holds. Hence by the extremal property of Bergman kernels, we see that
\[
K(\Omega, e^{-m\varphi})(O) \geq \frac{1}{C_3} \cdot K(B(O, \frac{c}{\sqrt{m}}), e^{-m\varphi})(O)
\]
holds. This completes the proof of Lemma 4.8. □

On the other hand, by the $L^2$-extension theorem, we see that there exists a positive constant $C_0$ independent of $m$ such that for every sufficiently large $m$,
\[
K(B(O, \frac{c}{\sqrt{m}}), e^{-m\varphi})(O) \geq C_0 \cdot e^{m\varphi(O)} \cdot m^n \cdot |dz_1 \wedge \cdots \wedge dz_n|^2
\]
holds. Hence combining the above inequality and Lemma 4.8 we see that
\[
K(\Omega, e^{-m\varphi})(O) \geq c_0 \cdot C_0 \cdot e^{m\varphi(O)} \cdot m^n \cdot |dz_1 \wedge \cdots \wedge dz_n|^2
\]
holds. This completes the proof of Theorem 4.7. □

The global version of Theorem 4.7 is as follows.

**Theorem 4.9** Let $X$ be a smooth projective variety of dimension $n$ and let $(L, h_L)$ be a singular hermitian line bundle on $X$. Let $\omega$ be a $C^\infty$ Kähler form on $X$. Suppose that there exists a positive constant $\varepsilon$ such that
\[
\Theta_{h_L} \geq \varepsilon \cdot \omega
\]
holds on $X$. Then there exists a positive constant $C$ such that for every sufficiently large positive integer $m$,
\[
h_L^m \cdot K(X, K_X + mL, h_L^m) \geq C \cdot m^n \cdot \omega^n
\]
holds on $X$. □

The proof of Theorem 4.9 follows from the local version (Theorem 4.7) by the localization principle (cf. Section 4.3).
5 Kodaira’s lemma for big pseudoeffective singular hermitian line bundles

In this section we shall prove an analogue of Kodaira’s lemma for big pseudoeffective singular hermitian line bundles. Kodaira’s lemma has been extensively used in algebraic geometry (cf. [K-O], [K1]). To prove Theorem 1.2 we need an analogue of Kodaira’s lemma for big pseudoeffective singular hermitian line bundles.

Although I can state the new version as a lemma, it will be useful to state the singular hermitian version as a theorem. Because I believe that the new version will be also fundamental in complex geometry.

5.1 Statement of the theorem

First we shall state the original Kodaira’s lemma.

Theorem 5.1 ([K-O, Appendix], [K1, Lemma]) Let $X$ be a smooth projective variety and let $D$ be a big divisor on $X$. Then there exists an effective $\mathbb{Q}$-divisor $E$ such that $D - E$ is an ample $\mathbb{Q}$-divisor. □

The analogue for the case of big pseudoeffective singular hermitian line bundles is stated as follows.

Theorem 5.2 Let $X$ be a projective manifold and let $(L, h_L)$ be a big pseudoeffective singular hermitian line bundle. Then there exists a singular hermitian metric $h_L^+$ on $L$ such that

1. $\Theta_{h_L^+}$ is strictly positive everywhere on $X$,
2. $h_L^+ \geq h_L$ holds on $X$.

□

Let us explain the relation between Theorems 5.1 and 5.2. Let $D, E$ be as in Theorem 5.1. Let us identify divisors with line bundles. Theorem 5.1 says that there exists a $C^\infty$ hermitian metrics $h_D, h_E$ on $D, E$ respectively (the notion of hermitian metrics naturally extends to the case of $\mathbb{Q}$-line bundles) such that the curvature of $h_D \cdot h_E^{-1}$ is strictly positive. Let $\sigma_E$ be a multivalued holomorphic section of $E$ with divisor $E$ such that $h_E(\sigma_E, \sigma_E) \leq 1$ on $X$. Then

$$h_D^+ := \frac{h_D}{h_E(\sigma_E, \sigma_E)}$$

is a singular hermitian metric on $D$ such that

1. $\Theta_{h_D^+}$ is strictly positive everywhere on $X$.
2. $h_D \leq h_D^+$ holds on $X$.

In this way Theorem 5.2 can be viewed as an analogue of the usual Kodaira’s lemma to the case of big pseudoeffective singular hermitian line bundles.
5.2 Proof of Theorem 5.2

The proof of Theorem 5.2 presented here is not very much different from the original proof of Kodaira’s lemma (cf. [K1] or [K-O, Appendix]). But it requires estimates of Bergman kernels and additional care for the multiplier ideal sheaves.

Let \( X \) be a smooth projective variety of dimension \( n \) and let \((L, h_L)\) be a big pseudoeffective singular hermitian line bundle on \( X \). Let \( \omega \) be a Kähler form on \( X \) and let \( dV \) be the associated volume form on \( X \). Let \( H \) be a smooth very ample divisor on \( X \). The following lemma is a singular hermitian version of the theorem in [Tr].

**Lemma 5.3** There exists a positive integer \( m_0 \) such that \( m_0(L, h_L) - H \) is big, i.e.,

\[
\limsup_{\ell \to \infty} \ell^{-n} \dim H^0(X, \mathcal{O}_X(\ell(m_0L - H) \otimes I(h_{h_L}^{m_0\ell}))) > 0
\]

holds. \( \square \)

**Proof of Lemma 5.3** Replacing \( H \) by a suitable member of \(|H|\), by Lemma 3.10, we may assume that

\[
I(h_{h_L}^{m_0\ell} | H) = I(h_{h_L}^{m_0\ell} | H)
\]

holds for every \( m \geq 1 \). Let us consider the exact sequence

\[
0 \to H^0(X, \mathcal{O}_X(mL - H) \otimes I(h_{h_L}^m)) \to H^0(X, \mathcal{O}_X(mL) \otimes I(h_{h_L}^m)) \to H^0(H, \mathcal{O}_H(mL) \otimes I(h_{h_L}^m | H)).
\]

Then since \( \mu(L, h_L) > 0 \) and

\[
\dim H^0(H, \mathcal{O}_H(mL) \otimes I(h_{h_L}^m | H)) = O(m^{-1})
\]

we see that for every sufficiently large \( m \),

\[
H^0(X, \mathcal{O}_X(mL - H) \otimes I(h_{h_L}^m)) \neq 0
\]

holds.

To prove Lemma 5.3 we need to refine the above argument a little bit. Let \( m_0 \) be a positive integer such that

\[
m_0 > n \cdot \frac{(L, h_L)^{n-1} \cdot H}{(L, h_L)^n}
\]

holds. For very general \( H_1^{(\ell)}, \ldots, H_{\ell}^{(\ell)} \in |H| \), by Lemma 3.10, replacing \( m \) by \( m_0 \ell \) and \( H \) by \( \ell H \), we have the exact sequence

\[
0 \to H^0(X, \mathcal{O}_X(\ell(m_0L - H) \otimes I(h_{h_L}^{m_0\ell}))) \to H^0(X, \mathcal{O}_X(m_0\ell L) \otimes I(h_{h_L}^{m_0\ell})).
\]

\[
\to \otimes_{i=1}^{\ell} H^0(H_i^{(\ell)}, \mathcal{O}_{H_i}(m_0\ell L) \otimes I(h_{h_L}^{m_0\ell} | H_i)).
\]

We note that \( \{H_i^{(\ell)}\}_{i=1}^{\ell} \) are chosen for each \( \ell \). If we take \( \{H_i^{(\ell)}\}_{i=1}^{\ell} \) very general, we may assume that

\[
\dim H^0(H_i^{(\ell)}, \mathcal{O}_{H_i}(mL) \otimes I(h_{h_L}^m | H_i))
\]
is independent of $1 \leq i \leq \ell$ for every $m$. This implies that

$$\limsup_{\ell \to \infty} \ell^{-n} \cdot \dim H^0(X, \mathcal{O}_X(\ell(m_0L - H)) \otimes \mathcal{I}(h_L^{m_0\ell}))$$

$$\geq \frac{1}{n!} (L, h_L)^n \cdot m_0^n - \frac{1}{(n-1)!} ((L, h_L)^{n-1}, H) \cdot m_0^{n-1}$$

holds. By (11), we see that

$$\frac{1}{n!} (L, h_L)^n \cdot m_0^n - \frac{1}{(n-1)!} ((L, h_L)^{n-1}, H) m_0^{n-1}$$

is positive. This completes the proof of Lemma 5.3. $\blacksquare$

Let $A$ be a sufficiently ample line bundle on $X$ and let $h_A$ be a $C^\infty$ hermitian metric such that the curvature of $h_A$ is everywhere strictly positive on $X$. Here the meaning of “sufficiently ample” will be specified later. Let $m$ be a positive integer. Let us consider the inner product $(\sigma, \sigma') := \int_X h_A \cdot h_L^m \cdot \sigma \cdot \bar{\sigma}' dV$ on $H^0(X, \mathcal{O}_X(A + mL) \otimes \mathcal{I}(h_L^m))$ and let $K_m$ be the associated (diagonal part of) Bergman kernel. Let us consider the subspace $H^0(X, \mathcal{O}_X(A + \ell(m_0L - H)) \otimes \mathcal{I}(h_L^{m_0 \ell})) \subset H^0(X, \mathcal{O}_X(A + m_0L) \otimes \mathcal{I}(h_L^{m_0 \ell}))$ as a Hilbert subspace and let $K_{m_0 \ell}^+$ denotes the associated Bergman kernel with respect to the restriction of the inner product on $H^0(X, \mathcal{O}_X(A + \ell(m_0L - H)) \otimes \mathcal{I}(h_L^{m_0 \ell}))$ to the subspace $H^0(X, \mathcal{O}_X(A + \ell(m_0L - H)) \otimes \mathcal{I}(h_L^{m_0 \ell}))$. Then by definition, we have the trivial inequality:

$$K_{m_0 \ell}^+ \leq K_{m_0 \ell}$$

(12)

holds on $X$ for every $\ell \geq 1$.

The next lemma follows from the same argument as in [D].

Lemma 5.4 ([D]) If $A$ is sufficiently ample,

$$h_L := \text{the lower envelope of } (\limsup_{m \to \infty} \sqrt{K_m})^{-1}.$$

holds. $\blacksquare$

Proof of Lemma 5.4. The proof has been given in [D]. But for the completeness, we shall reproduce the proof here.

By the $L^2$-extension theorem (Theorem 2.25 or Theorem 2.26), if $A$ is sufficiently ample, there exists a positive constant $C_0$ such that

$$K_m \geq C_0 \cdot h_A^{-1} \cdot h_L^{-m}$$

(13)

holds on $X$ for every $m$.

On the other hand, let $x \in X$ be an arbitrary point and let $(U, z_1, \ldots, z_n)$ be a coordinate neighbourhood centered at $x$ such that $U$ is biholomorphic to the
open unit ball $B(O,1)$ in $\mathbb{C}^n$ centered at the origin via the coordinate. Taking $U$ to be sufficiently small, we may and do assume that $e_{A,L}$ be the holomorphic frame of $A$ and $L$ on $U$ respectively. Then with respect to these frame, we may express $h_A, h_L$ as

$$h_A = e^{-\varphi_A}, h_L = e^{-\varphi_L}$$

respectively in terms of plurisubharmonic functions $\varphi_A, \varphi_L$ on $U$. By the extremal property of Bergman kernels, we see that

$$K_m(x) = \sup\{ |\sigma(x)|^2 | \sigma \in \Gamma(X, O_X(A + mL)) \},$$

\[ \int_X |\sigma|^2 \cdot h_A \cdot h_L^m \cdot dV = 1 \}

Let $\sigma_0 \in \Gamma(X, O_X(A + mL))$ with

$$\int_X |\sigma_0|^2 \cdot h_A \cdot h_L^m \cdot dV = 1$$

and $|\sigma_0(x)|^2 = K_m(x)$. Let us write $\sigma_0 = f \cdot e_A \cdot e_L^0$ on $U$ by using a holomorphic function $f$ on $U$. By the submeanvalue property of plurisubharmonic functions, we have that

$$|f(O)|^2 \leq \frac{1}{\text{vol}(B(O,\varepsilon))} \int_{B(O,\varepsilon)} |f|^2 \, d\mu$$

$$\leq \left( \sup_{B(O,\varepsilon)} e^{\varphi_A} \cdot e^{m\varphi_L} \right) \cdot \left( \frac{1}{\text{vol}(B(O,\varepsilon))} \int_{B(\varepsilon)} |f|^2 \cdot e^{-\varphi_A} \cdot e^{-m\varphi_L} \, dV \right) \cdot \left( \sup_{B(O,\varepsilon)} \frac{d\mu}{dV} \right)$$

hold, where $d\mu$ is the standard Lebesgue measure on $\mathbb{C}^n$. Hence there exists a positive constant $C_\varepsilon$ independent of $m$

$$K_m(x) \leq C_\varepsilon \cdot \sup_{w \in B(O,\varepsilon)} (h_A^{-1} \cdot h_L^{-m})(w) \cdot dV$$

(14)

holds. By (13) and (14), we see that

$$C_\varepsilon^{\frac{1}{2}} (h_A^{-\frac{1}{2}} h_L^{-1}) \leq \sqrt{K_m(x)} \leq C_\varepsilon^{\frac{1}{2}} \cdot \left( \sup_{w \in B(O,\varepsilon)} (h_A^{-1} \cdot h_L^{-m})(w) \cdot dV \right)^{\frac{1}{2}}$$

holds. Hence letting $m$ tend to infinity and then letting $\varepsilon$ tend to 0, we have the desired equality. $\square$

We note that

$$\int_X h_A \cdot h_L^m \cdot K_m \cdot dV = \dim H^0(X, O_X(A + mL) \otimes \mathcal{I}(h_L^m))$$

and

$$\int_X h_A \cdot h_L^{m\ell} \cdot K_{m\ell}^+ \cdot dV = \dim H^0(X, O_X(A + \ell(m_0L - H)) \otimes \mathcal{I}(h_L^{m\ell}))$$

hold. Hence by Lemma 5.3

$$\limsup_{\ell \to \infty} (m_0\ell)^{-n} \cdot \int_X h_L^{m\ell} \cdot K_{m\ell}^+ \cdot dV > 0$$

(15)
holds. Then by Fatou’s lemma, we see that
\[ \int_X \limsup_{\ell \to \infty} \frac{h_A \cdot h_{m_0 \ell} \cdot K^+_{m_0 \ell}}{(m_0 \ell)^n} \geq \limsup_{\ell \to \infty} \int h_A \cdot h_{m_0 \ell} \cdot K^+_{m_0 \ell} > 0 \]
hold. In particular
\[ \limsup_{\ell \to \infty} \frac{h_A \cdot h_{m_0 \ell} \cdot K^+_{m_0 \ell}}{(m_0 \ell)^n} \]
is not identically 0. This implies that
\[ \limsup_{\ell \to \infty} m_0 \sqrt{K^+_{m_0 \ell}} \]
is not identically 0 and by Lemma 5.4 and (12), it is finite. Let \( h_H \) be a \( C^\infty \) hermitian metric on \( H \) with strictly positive curvature and let \( \tau \) be a global holomorphic section of \( O_X(H) \) with divisor \( H \) such that \( h_H(\tau, \tau) \leq 1 \) holds on \( X \). We set
\[ h^+_L := (\limsup_{\ell \to \infty} m_0 \sqrt{K^+_{m_0 \ell}})^{-1} \cdot h_H(\tau, \tau). \]
Then \( h^+_L \) is a singular hermitian metric on \( L \), since
\[ (\limsup_{\ell \to \infty} m_0 \sqrt{K^+_{m_0 \ell}})^{-1} \cdot | \tau |^2 \]
can be viewed as a singular hermitian metric on \( L - H \) with semipositive curvature current. By the construction it is clear that the curvature current of \( h^+_L \) is bigger than or equal to the curvature of \( h_H \). In particular the curvature current of \( h^+_L \) is strictly positive. And by the construction
\[ h_L \leq h^+_L \]
holds on \( X \). This completes the proof of Theorem 5.2. \( \square \)

6 Dynamical construction of an AZD

This section is almost completely the same as [Ts1, Section 3]. The only difference is that we consider the adjoint line bundles instead of canonical bundles.

6.1 Sub extension problem of a singular hermitian metric with semipositive curvature

Let \( X \) be a smooth projective variety and let \( S \) be a smooth subvariety of \( X \). Let \( E \) be a line bundle on \( X \). Suppose that there exists a singular hermitian metric \( h_{E,S} \) on \( E |_S \) such that \( \Theta h_{E,S} \geq 0 \) holds on \( S \). We shall consider the following sub extension problem of singular hermitian metrics.

**Problem 6.1** Let \( X, S, E, h_{E,S} \) be as above. Construct a singular hermitian metric \( h_E \) on \( E \) such that

1. \( h_E |_S \leq h_{E,S} \) holds on \( S \).
2. \( \Theta h_E \geq 0 \) holds on \( X \).

\(^3\)At this moment, there is a possibility that it is identically +\( \infty \).
Of course in general such a sub extension $h_E$ does not exist. But under certain conditions, sub extension $h_E$ exists. Since singular hermitian metrics are real object, it is not easy to extend them directly.

The sub extension strategy used in this article is as follows.

1. Take a sufficiently ample line bundle $A$ on $X$.

2. Approximate $h_{E,S}$ by a sequence of singular hermitian metrics $\{h_{m,S}\}$, where $h_{m,S}$ is an algebraic singular hermitian metric (cf. Definition 2.5) on $\frac{1}{m}A_{S}+L_{S}$ which is of the form

$$h_{m,S} = \left( \sum_{i=0}^{N(m)} |\tau_{i}|^{2} \right)^{-\frac{1}{m}},$$

where $\tau_{i} \in H^{0}(S, O_{S}(A + mL))$.

3. Find a holomorphic extension $\tilde{\tau}_{i} \in H^{0}(X, O_{X}(A + mL))$ of $\tau_{i}$ for every $i = 0, \cdots, N(m)$.

4. Define an algebraic singular hermitian metric $\tilde{h}_{m}$ on $\frac{1}{m}A + L$ by

$$\tilde{h}_{m} := \left( \sum_{i=0}^{N(m)} |\tilde{\tau}_{i}|^{2} \right)^{-\frac{1}{m}}.$$

5. Prove the existence of

$$h_{E} := \text{the lower envelope of } \lim_{m \to \infty} \tilde{h}_{m}$$

as a nontrivial singular hermitian metric on $E$.

In the above strategy one cannot expect that the equality $h_{E}|_{S} = h_{E,S}$ holds on $S$, since we have taken the lower envelope. But in most applications, sub extension is enough.

### 6.2 Dynamical construction of singular hermitian metrics

To implement the sub extension strategy in Section 6.1, first task is to approximate the given singular hermitian metric by a sequence of algebraic singular hermitian metrics. Here we shall consider the case that the given hermitian metric is an AZD.

Instead of approximating the singular hermitian metric, we shall construct another AZD which is a limit of the algebraic singular hermitian metrics.

Let $X$ be a smooth projective variety and let $K_{X}$ be the canonical line bundle of $X$. Let $n$ denote the dimension of $X$. Let $(L,h_{L})$ be a pseudoeffective singular hermitian line bundle on $X$. Let $dV$ be a $C^{\infty}$ volume form on $X$. Suppose that $(K_{X} + L, dV^{-1} \cdot h_{L})$ is weakly pseudoeffective (cf. Definition 2.6). Then

$$E(K_{X} + L, dV^{-1} \cdot h_{L}) := \{ \varphi \in L^{1}_{\text{loc}}(X) \mid \varphi \leq 0, \Theta_{dV^{-1} \cdot h_{L}} + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0 \}$$

is nonempty (cf. the proof of Theorem 2.20). Then $(K_{X} + L, dV^{-1} \cdot h_{L})$ admits an AZD $h$ as in Theorem 2.20, i.e., the following holds:
1. $\Theta_h \geq 0$,

2. $H^0(X, O_X(m(K_X + L)) \otimes I_{\infty}(h^m)) \simeq H^0(X, O_X(m(K_X + L)) \otimes I_{\infty}(h^m))$

holds for every $m \geq 0$.

We assume that for every ample line bundle $B$ on $X$

$$\dim H^0(X, O_X(B + m(K_X + L)) \otimes I(h^m)) = O(m^\nu)$$

holds, where $\nu$ denotes the numerical Kodaira dimension $\nu_{num}(K_X + L, h)$ of $(K_X + L, h)$ (cf. Definition 3.4). We note that if $h$ is normal, this follows from Theorem 3.13.

Let $A$ be a sufficiently ample line bundle on $X$ such that for every pseudo-effective singular hermitian line bundle $(F, h_F)$,

$$O_X(A + F) \otimes I(h_F)$$

and

$$O_X(K_X + A + F) \otimes I(h_F)$$

are globally generated. This is possible by the $L^2$-estimate of $\bar{\partial}$ operator (cf. [S1, p. 667, Proposition 1]).

Let $h_A$ be a $C^\infty$ hermitian metric on $A$ with strictly positive curvature. Let $\{\sigma^{(1)}_0, \cdots, \sigma^{(1)}_{N(1)}\}$ be a complete orthonormal basis of $H^0(X, O_X(K_X + L + A) \otimes I(h_L))$ with respect to the inner product

$$(\sigma, \tau) := (\sqrt{-1})^n \int_X \sigma \wedge \bar{\tau} \cdot h \cdot h_A$$

We set

$$K_1 := \sum_{i=0}^{N(0)} |\sigma^{(1)}_i|^2.$$ 

We define the singular hermitian metric $h_1$ on $K_X + L + A$ by

$$h_1 := K_1^{-1}.$$ 

By taking a complete orthonormal basis of $H^0(X, O_X(2(K_X + L) + A) \otimes I(h_L, h))$ with respect to the inner product

$$(\sigma, \tau) := (\sqrt{-1})^2 \int_X \sigma \wedge \bar{\tau} \cdot h_1,$$

we define $K_2$ and the singular hermitian metric

$$h_2 := K_2^{-1}.$$ 

Suppose that we have already constructed $\{h_1, \cdots, h_{m-1}\}$. We set

$$V_m := H^0(X, O_X(m(K_X + L) + A) \otimes I(h_L, h^{m-1}))$$

By taking a complete orthonormal basis of $V_m$ with respect to the inner product

$$(\sigma, \tau) := (\sqrt{-1})^n \int_X \sigma \wedge \bar{\tau} \cdot h_L \cdot h_{m-1},$$
we define $K_m$ and the singular hermitian metric

$$h_m := K_m^{-1}$$

in the same manner. We note that for every $x \in X$ and $m$,

$$h_m^{-1}(x) = K_m = \sup \{ |\sigma|^2 (x); \sigma \in V_m, \int_X h_{m-1} \cdot |\sigma|^2 = 1 \}$$

holds by definition (cf. [Kl, p. 46, Proposition 1.4.16]).

We set

$$\nu := \limsup_{m \to \infty} \frac{\log \dim H^0(X, O_X(m(K_X + L) + A) \otimes I(h^m))}{\log m},$$

i.e., $\nu$ is the asymptotic Kodaira dimension of $(K_X + L, dV^{-1} \cdot h)$ (cf. Definition 3.5) and is an integer between 0 and $n = \dim X$ by Theorem 3.13. The following theorem is the main result in this section.

**Theorem 6.2** (cf. [Ts6]) Let $X, (L, h_L), \{K_m\}_{m=1}^\infty$ and $\{h_m\}_{m=1}^\infty$ be as above. Then

$$K_\infty := \text{the upper envelope of } \limsup_{m \to \infty} \sqrt[\nu]{(m!)^{-\nu} K_m}$$

exists and

$$h_\infty := 1/K_\infty$$

is an AZD of $(K_X + L, dV^{-1} \cdot h_L)$, where $dV$ is an arbitrary $C^\infty$ volume form on $X$.

**6.3 Proof of Theorem 6.2**

By the assumption there exists a positive constant $C$ such that

$$h^0(X, O_X(m(K_X + L) + A) \otimes I(h^m)) \leq C \cdot m^\nu$$

holds for every $m \geq 1$.

Let us fix a Kähler form $\omega$ on $X$. Let $dV$ be the volume form on $X$ with respect to $\omega$ and let $h_{L,0}$ be a $C^\infty$ hermitian metric on $L$. The following estimate is an easy consequence of the submeanvalue property of plurisubharmonic functions.

**Lemma 6.3** There exists a positive constant $\tilde{C}$ such that for every $m \geq 1$,

$$K_m \leq \tilde{C}^m \cdot (m!)^{\nu} \cdot (dV)^m \cdot h_A^{-1} \cdot h_{L,0}^{-m}$$

holds.

**Proof.** Let $p \in X$ be an arbitrary point. Let $(U, z_1, \ldots, z_n)$ be a local coordinate around $p$ such that

1. $z_1(p) = \cdots = z_n(p) = 0$,

2. $U$ is biholomorphic to the open unit polydisk in $\mathbb{C}^n$ with center $O \in \mathbb{C}^n$ by the coordinate, 

In such a case, it may be appropriate to write $K_X + (L, h_L)$ instead of $(K_X + L, dV^{-1} \cdot h_L)$.
3. \( z_1, \ldots, z_n \) are holomorphic on a neighbourhood of the closure of \( U \),

4. there exists a holomorphic frame \( e \) of \( A \) on the closure of \( U \).

Taking \( U \) sufficiently small we may assume that there exist holomorphic frames \( e_L \) of \( L \) and \( e_A \) of \( A \) on \( U \) respectively. We set

\[
\Omega := (-1)^{n(n-1)/2} (\sqrt{-1})^n dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\]

For every \( m \geq 0 \), we set

\[
B_m := \sup_{x \in U} \frac{K_m}{\Omega^m \cdot |e_L|^{2m} \cdot |e_A|^2}(x).
\]

We note that for any \( x \in X \)

\[
K_m(x) = \sup\{| \phi \|^2(x) ; \phi \in V_m, \int_X h_{m-1} | \phi \|^2 = 1 \}
\]

holds. Let \( \phi_0 \) be the element of \( V_m \) such that

\[
\int_X h_{m-1} | \phi_0 \|^2 = 1.
\]

Then there exists a holomorphic function \( f \) on \( U \) such that

\[
\phi_0 |_U = f \cdot (dz_1 \wedge \cdots dz_n)^m \cdot e_L^m \cdot e_A
\]

holds. Then

\[
\int_U | \phi_0 \|^2 \cdot h_A \cdot h_L^m \cdot \Omega^{-(m-1)} = \int_U | f \|^2 \cdot h_L(e_L, e_L) \cdot h_A(e_A, e_A) \cdot \Omega
\]

holds. On the other hand by the definition of \( B_{m-1} \) we see that

\[
\int_U h_A | \phi_0 \|^2 \Omega^{-(m-1)} \leq B_{m-1} \int_U h_{m-1} | \phi_0 \|^2 \leq B_{m-1}
\]

hold. Combining above inequalities we have that

\[
\int_U | f \|^2 h_A(e,e) \Omega \leq B_{m-1}
\]

holds. Let \( 0 < \delta << 1 \) be a sufficiently small number. Let \( U_\delta \) be the inverse image of

\[
\{ (y_1, \ldots, y_n) \in \mathbb{C}^n ; | y_i | < 1 - \delta \}
\]

by the coordinate \((z_1, \ldots, z_n)\).

Then by the subharmonicity of \( | f \|^2 \), there exists a positive constant \( C_\delta \) independent of \( m \) such that

\[
| f(x) |^2 \leq C_\delta \cdot B_{m-1}
\]

holds for every \( x \in U_\delta \). Then we have that

\[
K_m(x) \leq C_\delta \cdot B_{m-1} \cdot | e_A |^2 \cdot | e_L |^m \cdot \Omega^m(x)
\]
holds for every \( x \in U_{\delta} \). Summing up the estimates for the orthonormal basis, moving \( p \), by the compactness of \( X \) we see that there exists a positive constant \( \tilde{C} \) such that
\[
K_m \leq \tilde{C}^m \cdot (ml)^\nu \cdot (dV)^m \cdot h_A^{-1} \cdot h_{L,0}^{-m}
\]
holds on \( X \). This completes the proof of Lemma 6.3. □

Let \( V \) be a \( \nu \) dimensional nonsingular subvariety such that
1. \( V \) is not contained in the pluripolar set of \( h \).
2. \( (K_X + L, h) \mid V \) is big, i.e.,
\[
\limsup_{m \to \infty} m^{-\nu} \cdot h^0(V, \mathcal{O}_V(m(K_X + L)) \otimes \mathcal{I}(h_m \mid V)) > 0
\]
holds.

Since \( (K_X + L \mid V, h \mid V) \) is big, by Theorem 5.2, there exists a singular hermitian metric \( h_V \) on \( K_X + L \mid V \) such that
1. \( \Theta h_V \) is strictly positive everywhere on \( V \),
2. \( h \mid V \leq h_V \) holds on \( V \).

Suppose that for some \( m \geq 2 \)
\[
K_{m-1}(x) \geq C_{m-1} \cdot h_A^{-1} \cdot h_V^{-m}
\]
holds on \( x \in V \). Let us estimate \( K_m(x) \) from below at the point \( x \).

**Lemma 6.4** Let \( \varepsilon \) be a positive number. There exists a positive constant \( C(\varepsilon) \) depending only on \( \varepsilon \) and \( V \) such that
\[
K_m(x) \geq C(\varepsilon)^m \cdot (ml)^\nu \cdot h_A^{-1} \cdot (h_V^\varepsilon \cdot h^{(1-\varepsilon)})^{-m}
\]
holds at \( x \).

**Proof.** We shall give a proof only for the case \( \varepsilon = 1 \). The general case follows from the proof below in the same manner. We note that
\[
K_m(x) = \sup \{ |\sigma|^2(x) ; \sigma \in \Gamma(X, \mathcal{O}_{X}(A + m(K_X + L))), \int_X h_{m-1} \cdot |\sigma|^2 = 1 \}.
\]
holds. Then by the asymptotic expansion of Bergman kernels (Theorem 4.7), we can extend any element \( \sigma_x \) of the fiber \( (m(K_X + L) + A)_x \) of the line bundle of \( m(K_X + L) + A \) to an element
\[
\sigma_V \in H^0(U \cap V, \mathcal{O}_V(m(K_X + L) + A) \otimes \mathcal{I}(h_L \cdot h_V^{-1}))
\]
with
\[
\int_{U \cap V} h_V^m \cdot h_A(\sigma_V, \sigma_V) \cdot dz_1 \wedge \cdots \wedge dz_\nu \leq C \cdot m^{-\nu} \cdot h_V^m \cdot h_A(\sigma_x, \sigma_x)
\]

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where \( C \) is a positive constant depending only on \( x \) and \( V \). Since \( h \mid V \leq h_V \) holds on \( V \), this implies that

\[
\int_{U \cap V} h^m \cdot h_A(\sigma_V, \sigma_V) \mid dz_1 \wedge \cdots \wedge dz_\nu \mid^2 \leq C \cdot m^{-\nu} \cdot h_V^m \cdot h_A(\sigma_x, \sigma_x)
\]

holds. Then we can extend \( \sigma_V \) to \( \sigma_U \in H^0(U, \mathcal{O}_X(m(K_X + L) + A) \otimes \mathcal{I}(h^{m-1})) \)

by the \( L^2 \)-extension theorem (Theorems [O-T] and 2.26) so that

\[
(\sqrt{-1})^2 \int_U \sigma_U \wedge \bar{\partial}_U \cdot h^{m-1} \cdot h_A \leq C_U \cdot C \cdot m^{-\nu} \cdot h_V^m \cdot h_A(\sigma_x, \sigma_x)
\]

where \( C_U \) is a positive constant depending only on \( U \). Let \( \rho \) be a \( C^\infty \) function such that

1. \( \text{Supp } \rho \subset U \),
2. \( 0 \leq \rho \leq 1 \) on \( X \),
3. \( \rho \equiv 1 \) on some neighbourhood \( W \) of \( x \).

Taking \( A \) sufficiently ample, we may assume that

\[
\sqrt{-1} \bar{\partial} \bar{\partial}(n \rho \log \sum_{i=1}^n |z_i|^2) + \Theta_{h_A}
\]

is strictly positive on \( X \). Then we may solve the \( \bar{\partial} \)-equation

\[
\bar{\partial} u = \bar{\partial}(\rho \cdot \sigma_U)
\]

with

\[
\int_X \exp(-(n + 1)\rho \cdot \log \sum_{i=1}^n |z_i|^2) \cdot |u|^2 \cdot h_A \cdot h^{m-1} \leq C'_U \cdot \int_X \exp(-(n + 1)\rho) \cdot \log \sum_{i=1}^n |z_i|^2 \cdot |\bar{\partial}(\rho \cdot \sigma_U)|^2 \, dV
\]

holds, where \( |\bar{\partial}(\rho \cdot \sigma_U)|^2 \) denotes the norm with respect to \( h_A \cdot h^{m-1} \) and \( \omega \) and \( C'_U \) is a positive constant depending only on the supremum of the norm of \( \bar{\partial} \rho \) with respect to \( \omega \). This implies that

\[
u(x) = 0
\]

and there exists a positive constant \( C \) independent of \( (L, h_L) \) and \( \sigma_x \) such that

\[
\int_X h_A \cdot h_L \cdot |u|^2 \leq C \cdot (dV^{-1} \cdot h_A \cdot h_L)(\sigma_x, \sigma_x)
\]

holds. Then

\[
\sigma := \rho \cdot \sigma_U - u \in H^0(X, \mathcal{O}_X(m(K_X + L) + A) \otimes \mathcal{I}(h^{m-1}))
\]
is an extension of \( \sigma_x \) such that

\[
(\sqrt{-1})^{n^2} \int_X \sigma \wedge \bar{\sigma} \cdot h^{m-1} \cdot h_A \leq C' \cdot C_U \cdot C \cdot m^{-\nu} \cdot (h^\nu_V \cdot h_A)(\sigma_x, \sigma_x)
\]

where \( C' \) is a positive constant depending \( V \) and \( U \). Hence by the extremal property of Bergman kernels, we obtain that

\[
K_m(x) \geq (C' \cdot C_U \cdot C)^{-1} \cdot m^{\nu} \cdot C_{m-1} \cdot h_A^{-1} \cdot (h_V) -m
\]

holds. We note that for every \( 0 < \varepsilon < 1 \), \( h^\varepsilon_V \cdot h^{1-\varepsilon}_V \) is a singular hermitian metric with strictly positive curvature on \( V \). Then replacing \( h_V \) by \( h^\varepsilon_V \cdot h^{1-\varepsilon}_V \) we obtain that there exists a positive constant \( C(\varepsilon) \) depending on \( V \) and \( \varepsilon \) such that

\[
K_m(x) \geq C(\varepsilon) \cdot m^{\nu} \cdot C_{m-1} \cdot h_A^{-1} \cdot (h^\varepsilon_V \cdot h^{(1-\varepsilon)}_V) -m
\]

holds. Hence summing up the estimates for \( m \), we get

\[
K_m(x) \geq C(\varepsilon)^m \cdot (m!)^{\nu} \cdot h_A^{-1} \cdot (h^\varepsilon_V \cdot h^{(1-\varepsilon)}_V) -m
\]

at \( x \). The estimate is valid on a neighbourhood of \( x \) in \( V \) and moving \( x \) on \( V \), we may assume that the above estimate is valid on \( V \).

Hence on \( V \), there exists a positive constant \( C_V \) such that

\[
K_m(y) \geq C_V(\varepsilon)^m \cdot (m!)^{\nu} \cdot h_A^{-1} \cdot (h^\varepsilon_V \cdot h^{(1-\varepsilon)}_V) -m
\]

holds for every \( y \in V \).

Combining Lemmas 6.3 and 6.4 we have that the limit:

\[
\limsup_{m \to \infty} \sqrt[n]{(m!)^{\nu} K_m}
\]

exists and nonzero on \( V \). Moving \( V \), the limit exists on the whole \( X \). We set

\[
h_\infty := \text{the lower envelope of } \left( \limsup_{m \to \infty} \sqrt[n]{(m!)^{\nu} K_m} \right)^{-1}
\]

Let us prove \( h_\infty \) is an AZD of \( (K_X + L, dV^{-1} \cdot h_L) \). Let \( x \in X \) be an arbitrary point. Then there exists a family \( \{V_t\} \) of smooth projective subvariety of dimension \( \nu \) in \( X \) and a local coordinate \((U, z_1, \cdots, z_n)\) such that

1. \((z_1(x), \cdots, z_n(x)) = O, \)
2. \( U \) is biholomorphic to \( \Delta^n \) via \((z_1, \cdots, z_n), \)
3. \( V_t \cap U = \{p \in U \mid (z_{\nu+1}(p), \cdots, z_n(p)) = t\}. \)
4. \((K_X + L, h) \mid V_t \) is big for every \( t \in \Delta^{n-\nu}. \)

Let

\[
\phi : V \longrightarrow \Delta^{n-\nu}
\]

be the above family. By the proof of Theorem 5.2 we may take a singular hermitian metric \( h_V \) on \( \phi^*(K_X + L) \) such that

1. \( h_V \mid V_t \) is a singular hermitian metric with strictly positive curvature.
2. There exists a positive constant $C$ such that
\[ h_{V_t} \leq C \cdot h_{V_t} |_{V_t} \]
holds on $V_t$ for every $t \in \Delta^{a-\nu}$.

Since for every $y \in U$, there exists a positive constant $C(\varepsilon)$ such that
\[ K_m(y) \geq C(\varepsilon)^m \cdot (m!)^\nu \cdot h_{A}^{-1} \cdot (h_{V_t} \cdot h^{(1-\varepsilon)})^{-m} \]
holds for every $m \geq 1$ by Lemma 6.4. Then since $\limsup_{m \to \infty} \sqrt{m!}^{-\nu} K_m$ exist, letting $m$ tend to infinity, see that
\[ h_{\infty} |_{V_t} \leq C(\varepsilon)^{-1} \cdot h_{V_t} \cdot h^{(1-\varepsilon)} \]
holds. Let $\sigma \in \Gamma(U, \mathcal{O}_X(m(K_X + L)) \otimes \mathcal{I}_\infty(h^m))$ be any element. Then by the above inequality, we see that
\[ \int_U |\sigma|^p \cdot (h_{\infty})^p dV \leq C(\varepsilon)^{-p} \cdot \int_U |\sigma|^p \cdot (h_{V_t} \cdot h^{(1-\varepsilon)})^p dV \]
holds. Hence letting $\varepsilon$ tend to 0 and shrinking $U$, if necessary, we see that
\[ \sigma \in L^p_{\text{loc}}(m(K_X + L), h^p_{\infty}) \]
holds for every $p \geq 1$. This implies that
\[ \sigma \in \Gamma(U, \mathcal{O}_X(m(K_X + L)) \otimes \mathcal{I}_\infty(h^m_{\infty})) \]
holds. Hence we see that $h_{\infty}$ an AZD of $(K_X + L, dV^{-1} \cdot h_L)$ (cf. Definition 2.21). This completes the proof of Theorem 6.2 \(\Box\)

7 Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3 together with the following theorem which is a little bit weaker than Theorem 1.2.

**Theorem 7.1** ([Ts5, Theorem 5.1]) Let $X, S, \Psi_S$ be as in Section 1.2. Suppose that $S$ is smooth. Let $d$ be a positive integer such that $d > a \nu_0$. We assume that $(K_X + L, e^{-d} \cdot dV^{-1} \cdot h_L |_S)$ is weakly pseudoeffective (cf. Definition 2.7) and let $h_S$ be an AZD of $(K_X + L, e^{-d} \cdot dV^{-1} \cdot h_L |_S)$. Suppose that $h_S$ is normal (cf. Definition 2.8) or for every ample line bundle $A$ on $X$
\[ \dim H^0(S, \mathcal{O}_S(A + m(K_X + L) |_S) \otimes \mathcal{I}(h^m_S)) = O(m^\nu) \]
holds, where $\nu$ denotes the numerical Kodaira dimension $\nu_{\text{num}}(K_X + L |_S, h_S)$ of $(K_X + L |_S, h_S)$ (cf. Definition 3.4).

Then every element of
\[ H^0(S, \mathcal{O}_S(m(K_X + dL) \otimes \mathcal{I}(e^{-d} \cdot h^d_L \cdot h^m_L)) \]
extends to an element of
\[ H^0(X, \mathcal{O}_X(m(K_X + dL) \otimes \mathcal{I}(h^d_L \cdot h^m_L))) \],
where $h_0$ is an AZD of $K_X + dL$ with minimal singularities. In particular every element of

$$H^0(S, \mathcal{O}_S(m(K_X + dL)) \otimes \mathcal{I}(e^{-\varphi} \cdot h_L^d) \cdot \mathcal{I}_\infty(h_L^{m-1} |_{S} \cdot e^{-(m-1) \epsilon}))$$

extends to an element of

$$H^0(X, \mathcal{O}_X(m(K_X + dL)) \otimes \mathcal{I}(h_0^m)).$$

Remark 7.2 The normality condition on $h_S$ is more restrictive than the condition

$$\nu_{\text{num}}(K_X + L | S, h_S) = \nu_{\text{asymp}}(K_X + L | S, h_S)$$

(see Theorem 3.13). But practically, the normality of $h_S$ is much easier to verify than the equality of $\nu_{\text{num}}$ and $\nu_{\text{asymp}}$. For example if $h_L | S$ is normal and $K_X | S$ is $\mathbb{Q}$-effective or admits a normal singular hermitian metric with semipositive curvature current, then $h_S$ is normal. □

7.1 Setup

Let $X$ be a smooth projective variety and let $(L, h_L)$ be a singular hermitian line bundle on $X$ such that $\Theta_{h_L} \geq 0$ on $X$. As we mentioned in Section 1.2, we assume that $h_L$ is lowersemicontinuous. This is a technical assumption so that a local potential of the curvature current of $h_L$ is plurisubharmonic.

Let $m_0$ be a positive integer and let $\sigma \in \Gamma(X, \mathcal{O}_X(m_0L) \otimes \mathcal{I}(h_L^{m_0}))$ be a global section. Let $\alpha$ be a positive rational number $\leq 1$ and let $S$ be an irreducible subvariety of $X$ such that $(X, \alpha(\sigma))$ is LC(log canonical) but not KLT(Kawamata log terminal) on the generic point of $S$ and $(X, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of $S$ for every $0 < \epsilon << 1$. We set

$$\Psi_S = \alpha \log h_L(\sigma, \sigma).$$

Suppose that $S$ is smooth for simplicity (if $S$ is not smooth, we just need to take an embedded resolution to apply Theorem 7.1). We shall assume that $S$ is not contained in the singular locus of $h_L$, where the singular locus of $h_L$ means the set of points where $h_L$ is $+\infty$. Let $dV$ be a $C^\infty$ volume form on $X$.

Then we may define a (possibly singular) measure $dV[\Psi_S]$ on $S$ as in the introduction. Let $dV_S$ be a $C^\infty$ volume form on $S$ and let $\varphi$ be the function on $S$ defined by

$$\varphi := \log \frac{dV_S}{dV[\Psi_S]}$$

($dV[\Psi_S]$ may be singular on a subvariety of $S$, also it may be totally singular on $S$).

7.2 Dynamical construction of singular hermitian metrics with successive extensions

Let us start the proof of Theorem 7.1. Let $d$ be a positive integer such that $d > \alpha m_0$. Replacing $L$ by $dL$, we may and do assume that $d = 1$ from the
beginning. By the assumption, there exists an AZD $h_S$ of $(K_X + L \mid S, (dV^{-1} \cdot h_L) \mid S \cdot e^{-\varphi})$. We shall define sequences of the hermitian metrics $\{h_m\}$ on $A + m(K_X + L) \mid S$ and $\{\tilde{h}_m\}(m \geq 1)$ on $A + m(K_X + L)$ inductively as follows. Let $h_A$ be a $C^\infty$ hermitian metric on $A$. Let 

$$\{\sigma_0^{(1)}, \cdots, \sigma_{N_1}^{(1)}\}$$

be an orthonormal basis of $H^0(S, \mathcal{O}_S(A + (K_X + L)) \otimes \mathcal{I}(h_S))$ with respect to the inner product :

$$(\sigma, \tau) = \int_S \sigma \cdot \bar{\tau} \cdot (h_A \cdot h_L \cdot dV^{-1}) dV [\Psi_S](\sigma, \tau \in H^0(S, \mathcal{O}_S(A + (K_X + L)) \otimes \mathcal{I}(h_S))).$$

We set

$$K_1 = \sum_{i=0}^{N_1} |\sigma_i^{(1)}|^2$$

and set

$$h_1 = 1/K_1.$$ 

Then $h_1$ is a singular hermitian metric on $A + (K_X + L) \mid S$. By the choice of $A$, we see that $\mathcal{O}_S(A + (K_X + L)) \otimes \mathcal{I}(h_S)$ is globally genereated on $S$. Hence we see that

$$h_1 \leq O(h_A \cdot h_S)$$

holds, where this means that $h_1 \cdot (h_A \cdot h_S)^{-1}$ is bounded from above on $S$. Then by the $L^2$-extension theorem (Theorem 2.26), each $\sigma_i^{(1)}$ extends to a section 

$$\tilde{\sigma}_i^{(1)} \in H^0(X, \mathcal{O}_X(A + (K_X + L)))$$

such that

$$\|\sigma_i^{(1)}\|^2 = \left( \int_X |\sigma_i^{(1)}|^2 \cdot (h_A \cdot h_L \cdot dV^{-1}) \cdot dV \right)^{\frac{1}{2}}$$

satisfies the inequality

$$\|\sigma_i^{(1)}\| \leq C,$$

where $C$ is the positive constant as in Theorem 2.26. And we set

$$\tilde{K}_1 = \sum_{i=0}^{N_1} |\sigma_i^{(1)}|^2$$

and

$$\tilde{h}_1 := 1/\tilde{K}_1.$$ 

We note that $\tilde{K}_1$ depends on the choice of the orthonormal basis.

Suppose that we have already constructed $\{h_i\}_{i=1}^m$ and $\{\tilde{h}_i\}_{i=1}^m$ and

$$h_i = O(h_A \cdot \tilde{h}_S)$$

holds for every $i = 1, \cdots, m$.

Let 

$$\{\sigma_0^{(m+1)}, \cdots, \sigma_{N_{m+1}}^{(m+1)}\}$$

...
be an orthonormal basis of $H^0(S, \mathcal{O}_S(A + (m + 1)(K_X + L)) \otimes \mathcal{I} (h_S^{m+1}))$ with respect to the inner product:

$$(\sigma, \tau) = \int_S \sigma \cdot \overline\tau \cdot (h_m \cdot h_L \cdot dV^{-1}) dV \Psi_S.$$ 

Here we note that by the assumption $h_m = O(h_A \cdot h_S^m)$, this inner product is well defined. By the choice of $A$, we see that $\mathcal{O}_S(A + (m + 1)(K_X + L)) \otimes \mathcal{I} (h_A h_S^{m+1})$ is globally generated. Hence

$$h_{m+1} = O(h_A \cdot h_S^{m+1})$$

holds.

We define

$$K_{m+1} := \sum_{i=0}^{N_{m+1}} |\sigma_i^{(m+1)}|^2$$

and

$$h_{m+1} := 1/K_{m+1}.$$ 

Then $h_{m+1}$ is a singular hermitian metric on $A + (m + 1)(K_X + L)$ with semi-positive curvature current.

Then by the $L^2$-extension theorem (Theorem 2.26), each $\sigma_i^{(m+1)}$ extends to a section

$$\tilde{\sigma}_i^{(m+1)} \in H^0(X, \mathcal{O}_X(A + (K_X + L)))$$

such that

$$\|\tilde{\sigma}_i^{(m+1)}\| = \left( \int_X |\tilde{\sigma}_i^{(m+1)}|^2 \cdot (h_L \cdot dV^{-1} \cdot \tilde{h}_m) \cdot dV \right)^{1/2}$$

satisfies the inequality

$$\|\tilde{\sigma}_i^{(m+1)}\| \leq C,$$

where $C$ is the positive constant independent of $m$ as in Theorem 2.26. And we set

$$\tilde{K}_{m+1} := \sum_{i=0}^{N_{m+1}} |\tilde{\sigma}_i^{(m+1)}|^2.$$ 

and

$$\tilde{h}_{m+1} := 1/\tilde{K}_{m+1}.$$ 

In this way we construct the sequences $\{K_m\}_{m=1}^{\infty}$, $\{\tilde{K}_m\}_{m=1}^{\infty}$, $\{h_m\}_{m=1}^{\infty}$ and $\{\tilde{h}_m\}_{m=1}^{\infty}$. Next we shall discuss the (normalized) convergence of these sequences. Let $\nu$ be the asymptotic Kodaira dimension (cf. Definition 3.5) of $(K_X + L \mid S, h_S)$, i.e.,

$$\nu := \limsup_{m \to \infty} \frac{\log \dim H^0(S, \mathcal{O}_S(A + m(K_X + L)) \otimes \mathcal{I} (h_S^m))}{\log m}.$$ 

$\nu$ is a nonnegative integer between 0 and $\dim S$ by Theorem 3.13.
Lemma 7.3

\[ h_\infty := \text{the lower envelope of } \liminf_{m \to \infty} \sqrt[\nu]{(m!)^{\nu} \cdot h_m} \]

exists as a singular hermitian metric on \( K_X + L |_S \) and is an AZD of \( (K_X + L |_S, dV^{-1} \cdot h_L \cdot e^{-\varphi}) \) on \( S \). And

\[ \tilde{h}_\infty := \text{the lower envelope of } \liminf_{m \to \infty} \sqrt[\nu]{(m!)^{\nu} \cdot \tilde{h}_m} \]

exists as a singular hermitian metric on \( K_X + L \) with semipositive curvature current and

\[ h_\infty \geq \tilde{h}_\infty |_S \]

holds on \( S \). \( \square \)

**Proof of Lemma 7.3** To prove Lemma 7.3 we shall estimate \( K_m \) from above and below and \( \tilde{K}_m \) from above.

The estimate for \( \{K_m\} \) is identical as the one in the proof of Theorem 6.2 in the last section. Hence we obtain that

\[ h_\infty := \text{the lower envelope of } \liminf_{m \to \infty} \sqrt[\nu]{(m!)^{\nu} \cdot h_m} \]

exists and is an AZD of \( (K_X + L |_S, dV^{-1} \cdot h_L \cdot e^{-\varphi}) \) on \( S \).

Let us fix \( h_{L,0} \) be a \( C^\infty \) hermitian metric on \( L \). By the same proof as that of Lemma 6.3 we obtain that there exists a positive constant \( C_+ \) such that

\[ \tilde{K}_m \leq (C_+)^{m} \cdot (m!)^{\nu} \cdot h_A^{-1} \cdot (dV)^m \cdot h_{L,0}^{-m} \]

holds on \( X \) for every \( m \geq 0 \). Hence by this estimate, we see that

\[ \tilde{K}_\infty := \text{the upper envelope of } \limsup_{m \to \infty} \sqrt[\nu]{\tilde{K}_m} \]

exists on \( X \) and is an extension of \( K_\infty \) by the construction. In particular \( \tilde{K}_\infty \) is not identically zero on \( X \). Hence

\[ h_\infty := \frac{1}{\tilde{K}_\infty} \]

is a well defined singular hermitian metric on \( K_X + L \). And

\[ h_\infty \geq \tilde{h}_\infty |_S \]

holds by the construction. This completes the proof of Lemma 7.3. \( \square \).

Let us complete the proof of Theorem 7.1. By Lemma 7.3 and Theorem 2.26 we see that every element of

\[ H^0(S, O_S(m(K_X + L)) \otimes I(e^{-\varphi} \cdot h_L |_S \cdot h_S^{m-1})) \]

extends to an element of

\[ H^0(X, O_X(m(K_X + L) \otimes I(h_L \cdot h_\infty^{m-1}))). \]
Let $h_0$ be an AZD of $K_X + L$ of minimal singularities. Since $h_\infty$ has semipositive curvature in the sense of current, we see that there exists a positive constant $C_\infty$ such that

$$h_0 \leq C_\infty \cdot h_\infty,$$

holds. Hence we see that every element of

$$H^0(S, \mathcal{O}_S(m(K_X + L)) \otimes \mathcal{I}(e^{-\varphi} \cdot h_L |_S \cdot h_S^{m-1}))$$

extends to an element of

$$H^0(X, \mathcal{O}_X(m(K_X + L) \otimes \mathcal{I}(h_L \cdot h_0^{m-1})).$$

This completes the proof of Theorem 7.1. □

The proof of Theorem 1.3 is very similar. Hence we shall indicate the necessary change. The method of the proof is an extension of an AZD of $(K_X + L + \frac{1}{m} E |_S, dV^{-1} \cdot h_L \cdot h_E^\frac{1}{m})$. Here we need to perform the dynamical construction of AZD on the fractional singular hermitian line bundle. The necessary change is that we need to tensorize $(E, h_E)$ at every $m$ step instead of tensorize $(L, h_L)$ at every step as above (see Section 9.1 below, where one can see the concrete construction). □

8 Proof of Theorem 1.4

In this section we shall prove Theorem 1.4. The method of the proof is parallel to that of [Ts5, Ts6] except the use of Theorems 7.1 and 1.3.

8.1 Positivity result

In [K4], Y. Kawamata proved the following important theorem.

**Theorem 8.1** ([K4, p.894, Theorem 2]) Let $f : X \rightarrow B$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum P_j$ and $Q = \sum \ell Q_\ell$ be normal crossing divisors on $X$ and $B$ respectively, such that $f^{-1}(Q) \subset P$ and $f$ is smooth over $B \setminus Q$. Let $D = \sum d_j P_j$ be a $Q$-divisor on $X$, where $d_j$ may be positive, zero or negative, which satisfies the following conditions:

1. $D = D^h + D^v$ such that $f : \text{Supp}(D^h) \rightarrow B$ is surjective and smooth over $B \setminus Q$, and $f(\text{Supp}(D^v)) \subset Q$. An irreducible component of $D^h$ (resp. $D^v$) is called horizontal (resp. vertical).

2. $d_j < 1$ for all $j$.

3. The natural homomorphism $\mathcal{O}_B \rightarrow f_* \mathcal{O}_X([-D])$ is surjective at the generic point of $B$.

4. $K_X + D \sim_Q f^*(K_B + L)$ for some $Q$-divisor $L$ on $B$.  

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Let
\[ f^*Q_\ell = \sum_j w_{\ell j} P_j \]
\[ \bar{d}_j := \frac{d_j + w_{\ell j} - 1}{w_{\ell j}} \text{ if } f(P_j) = Q_\ell \]
\[ \delta_\ell := \max\{\bar{d}_j; f(P_j) = Q_\ell\} \]
\[ \Delta := \sum_\ell \delta_\ell Q_\ell \]
\[ M := L - \Delta. \]

Then \( X \) is nef. \( \square \)

**Remark 8.2** In Theorem 8.1, the condition: \( d_j < 1 \) is irrelevant for every \( D_j \) with \( f(D_j) \subset Q \) by a trivial reason. In fact in this case, if we replace \( D \) by \( D' := D - \alpha f^*Q \) and replace \( L \) by \( L' := L - \alpha Q \) for a sufficiently large positive rational number \( \alpha \), \( D' = \sum d'_j D_j \) satisfies the condition: \( d'_j < 1 \) for all \( j \). \( \square \)

The meaning of the divisor \( \Delta \) may be difficult to understand. So I would like to give an geometric interpretation of \( \Delta \). Let \( X, P, Q, D, B, \Delta \) be as above. Let \( dV \) be a \( C^\infty \) volume form on \( X \). Let \( \sigma_j \) be a global section of \( O_X(P_j) \) with divisor \( P_j \). Let \( \| \sigma_j \| \) denote the hermitian norm of \( \sigma_j \) with respect to a \( C^\infty \) hermitian metric on \( O_X(P_j) \) respectively. Let us consider the singular volume form
\[ \Omega := \prod_j \| \sigma_j \|^{2d_j} \]
on \( X \). Then by taking the fiber integral of \( \Omega \) with respect to \( f : X \rightarrow B \), we obtain a singular volume form \( \int_{X/B} \Omega \) on \( B \), where the fiber integral \( \int_{X/B} \Omega \) is defined by the property that for any open set \( U \) in \( B \),
\[ \int_U \left( \int_{X/B} \Omega \right) = \int_{f^{-1}(U)} \Omega \]
holds. We note that the condition 2 in Theorem 8.1 assures that \( \int_{X/B} \Omega \) is continuous on a nonempty Zariski open subset of \( B \). Also by the condition 4 in Theorem 8.1, computing the differential \( df \), we see that \( K_X + D \) is numerically \( f \)-trivial and \( (\int_{X/B} \Omega)^{-1} \) is a \( C^0 \)-hermitian metric on the \( Q \)-line bundle \( K_B + \Delta \). Thus the divisor \( \Delta \) corresponds exactly to singularities (poles and degenerations) of the singular volume form \( \int_{X/B} \Omega \) on \( B \).

### 8.2 Proof of Theorem 1.4

Let \( \sigma \) be a multivalued holomorphic section of \( D \) with divisor \( D \). Let \( h \) be the supercanonical AZD ([Ts9]) of \( K_X \) or any AZD constructed in the proof of Theorem 2.13. Let \( dV \) be a \( C^\infty \) volume form on \( X \) and let
\[ \Psi := \log h^\alpha(\sigma, \sigma) \]
Let \( dV[\Psi] \) be the residue volume form on \( S \) defined by
\[ dV[\Psi] = \text{Res}_S(e^{-\Psi}dV). \]
Let 
\[ \pi : Y \longrightarrow X \]
be a log resolution of \((X, D)\). Then by the assumption there exist irreducible components of \(\pi^{-1}(S)\) with discrepancy \(-1\) which dominates \(S\). We divide the proof into the following two cases.

1. There exists a unique irreducible component of \(\pi^{-1}(S)\) with discrepancy \(-1\) which dominates \(S\).

2. There exist several irreducible components of \(\pi^{-1}(S)\) with discrepancy \(-1\) which dominate \(S\).

In the first case, the residue volume form \(dV[\Psi_S]\) is not identically \(+\infty\) on \(S\).

In the second case the residue volume form \(dV[\Psi_S]\) is identically \(+\infty\) on \(S\). We shall reduce this case to the first case above by a minor modification.

First we shall consider the first case. Let 
\[ \varpi : F \longrightarrow S \]
be the restriction of \(\pi\) to \(F\). We shall write 
\[ \pi^*(K_X + D) = K_Y + F + E, \]
where \(\text{Supp}(F + E)\) is a divisor with normal crossings and \(F\) and \(E\) have no common divisorial component. We set \(G := E \mid_F\). Then we have that there exists a \(\mathbb{Q}\)-divisor \(L\) on \(S\).

\[ K_F + G = \varpi^*(K_S + L). \]  
(16)

Now we shall apply Theorem 8.1 and obtain that 
\[ L - \Delta \]
is nef, where \(\Delta\) is the \(\mathbb{Q}\)-divisor on \(S\) defined as in Theorem 8.1. We note that \(\Delta\) is effective in this case, since \(S\) is smooth. Let \(\sigma_{\Delta}\) be a multivalued holomorphic section of \(\Delta\) on \(S\) with divisor \(\Delta\) and let \(h_{\Delta}\) be a \(C^\infty\) hermitian metric on \(\Delta\). Let \(dV_S\) be a \(C^\infty\) volume form on \(S\). Then by the definition of \(dV[\Psi]\), there exists a positive constant \(C\) such that 
\[ dV[\Psi] \leq C \cdot \frac{dV^{-\alpha} \cdot h^{-\alpha}}{h_{\Delta}(\sigma_{\Delta}, \sigma_{\Delta})} \cdot dV_S \]
holds on \(S\). We set 
\[ \varphi = \log \frac{dV_S}{dV[\Psi]}. \]
Then by the above inequality, we have that 
\[ e^{-\varphi} \cdot h^{\alpha} \leq C \cdot dV^{-\alpha} \cdot \frac{1}{h_{\Delta}(\sigma_{\Delta}, \sigma_{\Delta})} \]  
(17)
holds.

Let \(d\) be a positive integer greater than \(\alpha\). By (16), (17) and the facts that \(L - \Delta\) is nef and \(K_S\) is pseudoeffective, we see that \((1 + d)K_X \mid_S\) admits a singular hermitian metric with semipositive curvature current which dominates \(e^{-\varphi} \cdot (dV \mid_S)^{-1} \cdot h^d\). Let \(h_A\) be a \(C^\infty\) hermitian metric on \(A\) with strictly positive curvature. Then \(((1 + d)K_X + \frac{m}{d} A \mid_S, e^{-\varphi} \cdot (dV \mid_S)^{-1} \cdot h^d \cdot h_A^m)\) is big and admits an AZD \(h_{S,m}\). By Theorem 1.3 we have the following lemma.
Lemma 8.3 Let $A$ be an ample line bundle on $X$. For every positive integer $m$, every element of

$$H^0(S, \mathcal{O}_S((m + 1)(1 + d)K_X + A) \otimes \mathcal{I}(e^{-\varphi} \cdot h^d \cdot h_{S,m}^m))$$

extends to an element of

$$H^0(X, \mathcal{O}_X((m + 1)(1 + d)K_X + A) \otimes \mathcal{I}(h^{(1+d)(m+1)})).$$

On the other hand by (17), we have that there exists an inclusion

$$H^0(S, \mathcal{O}_S((m + 1)(1 + d)K_X + A) \otimes \mathcal{I}(h^{(m+1)(d-\alpha)} \cdot h_\Delta(\sigma_\Delta, \sigma_\Delta)^{-1})) \hookrightarrow$$

$$H^0(S, \mathcal{O}_S((m + 1)(1 + d)K_X + A) \otimes \mathcal{I}(e^{-\varphi} \cdot h^d \cdot h_{S,m}^m))$$

for every $m \geq 1$. Since $L - \Delta$ is nef and

$$\pi^*(K_X + D) |_{F \sim Q} \cong \pi^*(K_S + L)$$

holds, if we take a sufficiently ample line bundle $B$ on $X$, there exists an inclusion

$$H^0(S, \mathcal{O}_S((m + 1)K_S + A)) \hookrightarrow$$

$$H^0(S, \mathcal{O}_S((m+1)(1+d)K_X + A+B) \otimes \mathcal{I}(h^{(m+1)(d-\alpha)} \cdot h_\Delta(\sigma_\Delta, \sigma_\Delta)^{-1}))$$

for every $m \geq 1$. Hence by (17), we see that there exists a natural inclusion

$$H^0(S, \mathcal{O}_S((m+1)K_S+A)) \hookrightarrow H^0(S, \mathcal{O}_S((m+1)(1+d)K_X + A+B) \otimes \mathcal{I}(e^{-\varphi} \cdot h^d \cdot h_{S,m}^m)).$$

By the above inclusion and Lemma 8.3 we obtain an injection

$$H^0(S, \mathcal{O}_S(mK_S + A)) \hookrightarrow$$

$$\text{Image}\{H^0(X, \mathcal{O}_X(m(1 + d)K_X + A + B)) \rightarrow H^0(S, \mathcal{O}_S(m(1 + d)K_X + A + B))\}$$

for every $m \geq 1$.

Next we shall consider the second case, i.e., there are several divisorial component of discrepancy $-1$ of $\pi^*(K_X + D)$ which dominates $S$. Let $A$ be an ample divisor on $X$ as above. We may assume that $\text{Supp} \, R$ contains all the component of $\pi^*D$. Let us $(X, D)$ by $(X, (1 + \delta(\varepsilon))(D - \varepsilon R))$ where $1 + \delta(\varepsilon)$ is the log canonical threshold of $D - \varepsilon R$ along $S$. Then perturbing the coefficients of $R$, if necessary, we may assume that there exists a unique irreducible divisor with discrepancy $-1$ over $S$. Let $L$ be a sufficiently positive integer such that $\ell A'$ is a very ample Cartier divisor and let $\{\tau_0, \cdots, \tau_N\}$ a basis of $H^0(X, \pi_* \mathcal{O}_Y(\ell A'))$. Then we modify $\Psi$ as

$$\Psi_m := \frac{1}{1 + \delta(\varepsilon/\ell m)} \{\log h^\alpha(\sigma, \sigma) + \frac{\varepsilon}{\ell m} \log(\sum_{i=0}^{N} h_{A'}^\ell(\tau_i, \tau_i))\},$$

where the parameter $\varepsilon$ is a positive number less than 1. Then by the choice of $R$, we see that $dV[\Psi_m]$ is not identically $+\infty$ on $S$. As Lemma 8.3 for every ample line bundle $A$ on $X$, we can extend every element of

$$H^0(S, \mathcal{O}_S((m + 1)(1 + d)K_X + A) \otimes \mathcal{I}(e^{-\varphi} \cdot h^d \cdot h_{S,m}^m))$$

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to an element of

\[ H^0(X, \mathcal{O}_X(m(1+d)K_X + A) \otimes \mathcal{I}(h^m)) \]

for every \( m \geq 1 \), where \( h_{S,m} \) is an AZD of

\[ ((1 + d)K_X + \frac{1}{\ell}A, e^{-\varphi_m} \cdot (dV|_S)^{-1} \cdot h_d \cdot h_A^\perp) \]

Then replacing \( \varphi \) by \( \varphi_m \) and tracing the proof of the first case, if we take a sufficiently ample line bundle \( B \) on \( X \) and take \( \varepsilon \) sufficiently small, again we obtain an injection

\[ H^0(S, \mathcal{O}_S(mK_S + A) \hookrightarrow \text{Image}\{ H^0(X, \mathcal{O}_X(m(1+d)K_X + A + B) \otimes \mathcal{I}(h^m)) \} \]

for every \( m \geq 1 \). Hence we obtain that

\[ \nu((K_X, h)|_S) \geq \nu(K_S) \]

holds. This completes the proof of Theorem 1.4. \( \square \)

9 Proof of Theorems 1.2 and 1.6

Let \( M, S, (L, h_L), \Psi_S, dV, \varphi \) be as in Theorem 1.2. By the assumption

\( (K_X + L|_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L|_S) \) is weakly pseudoeffective. We set

\[ n := \dim S. \]

9.1 Dynamical construction with a parameter

Let \( A \) be an ample line bundle on \( X \) and let \( h_A \) be a \( C^\infty \) hermitian metric on \( A \) with strictly positive curvature. Then we see that \( (K_X + L + \frac{1}{\ell}A|_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L \cdot h_A^\perp|_S) \) is big for every \( \ell > 0 \). The main idea of the proof of Theorem 1.2 is to extend an AZD of \( (K_X + L + \frac{1}{\ell}A|_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L \cdot h_A^\perp|_S) \) to a singular hermitian metric of \( K_X + L + \frac{1}{\ell}A \) of semipositive curvature current with uniform estimates with respect to the parameter \( \ell \). And prove the normalized convergence as \( \ell \) tends to infinity.

Let \( h_S \) be an AZD of \( (K_X + L|_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L|_S) \) with minimal singularities. Then as in Section 7.2, we construct an AZD on \( (K_X + L + \frac{1}{\ell}A|_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L|_S) \) by using the dynamical system of Bergman kernels as

\[ K_{\ell,1} := \sum_i |\sigma_i^{(\ell,1)}|^2, \]

where \( \{ \sigma_0^{(\ell,1)}, \ldots, \sigma_N^{(\ell,1)} \} \) is an orthonormal basis of

\[ H^0(S, \mathcal{O}_S(K_X + L + A|_S) \otimes \mathcal{I}(h_S)) \]

with respect to the inner product:

\[ (\sigma, \sigma') := \int_S \sigma \cdot \sigma' \cdot dV^{-1} \cdot h_L \cdot h_A \cdot dV[\Psi_S]. \]

And we define

\[ h_{\ell,1} := \frac{1}{K_{\ell,1}}. \]
Suppose that we have already defined the singular hermitian metrics \( \{ h_{\ell,1}, \cdots, h_{\ell,m-1} \} \), where \( h_{\ell,j} (0 \leq j \leq m - 1) \) is a singular hermitian metric on \( j(K_X + L) + (1 + [j/\ell])A \) respectively. Then we define \( K_{\ell,m} \) and \( h_{\ell,m} \) by

\[
K_{\ell,m} := \sum_{i=0}^{N(\ell,m)} |\sigma_i^{(\ell,m)}|^2
\]

where \( \{ \sigma_0^{(\ell,m)}, \cdots, \sigma_{N(\ell,m)}^{(\ell,m)} \} \) is an orthonormal basis of

\[
H^0(S, \mathcal{O}_S(m(K_X + L) + (1 + [m/\ell])A |_S) \otimes \mathcal{I}(h_S^m))
\]

with respect to the inner product

\[
(\sigma, \sigma') := \int_S \sigma \cdot \bar{\sigma}' \cdot dV^{-1} \cdot h_L^{-1} \cdot h_A^{(\lceil m/\ell \rceil - \lceil m-1/\ell \rceil)} \cdot h_{\ell,m-1} \cdot dV[\Psi_S].
\]

and we define the singular hermitian metric on \( m(K_X + L) + (1 + [m/\ell])A |_S \) by

\[
h_{\ell,m} := \frac{1}{K_{\ell,m}}.
\]

In this way we construct the dynamical system of Bergman kernels \( \{ K_{\ell,m} \} \) and the dynamical system of singular hermitian metrics \( \{ h_{\ell,m} \} \) respectively with the parameter \( \ell \).

### 9.2 Upper estimate

We set

\[
d_{\ell,m} = \dim H^0(S, \mathcal{O}_S(m(K_X + L) + (1 + [m/\ell])A) \otimes \mathcal{I}(h_S^m)).
\]

We shall fix a \( C^\infty \) hermitian metric \( h_{L,0} \) on \( L \). In view of the proof of Lemma 6.2, we see that there exists a positive constant \( C \) independent of \( \ell \) and \( m \) such that

\[
K_{\ell,m+1} \leq C \cdot d_{\ell,m} \cdot K_{\ell,m} \cdot dV \cdot h_{L,0}^{-1} \cdot h_A^{-\lfloor \frac{m+1}{\ell} \rfloor - \lceil \frac{m}{\ell} \rceil}
\]

holds. Then summing up the estimates, we have that

\[
K_{\ell,m} \leq C_0 \cdot C^{m-1} \cdot \left( \prod_{j=1}^{m-1} d_{\ell,j} \right) \cdot h_A^{-\lfloor 1 + \frac{m}{\ell} \rfloor} \cdot dV^m \cdot h_{L,0}^m, \tag{18}
\]

where \( C_0 \) is a positive constant such that

\[
K_{\ell,1} \leq C_0 \cdot h_A^{-1} \cdot dV \cdot h_{L,0}^{-1}
\]

holds on \( S \). We note that by the definition of \( (K_X + L + \frac{1}{\ell}A |_S, h_S \cdot h_A^\frac{1}{\ell})^n \) (see Definition 3.2),

\[
\limsup_{m \to \infty} \left( \frac{1}{m!} \cdot \prod_{j=1}^{m} d_{\ell,j} \right)^\frac{1}{m} \leq \frac{1}{m!} (K_X + L + \frac{1}{\ell}A |_S, h_S \cdot h_A^\frac{1}{\ell})^n \tag{19}
\]

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holds. Hence combining (18) and (19), we see that

\[ K_{\ell,\infty} := \limsup_{m \to \infty} \frac{\sqrt{(m!)^{-n} K_{\ell,m}}}{n!} \leq \frac{C}{n!} \left( (K_X + L + \frac{1}{\ell} A |_S, h_S \cdot h_A^\frac{1}{n})^n \cdot dV^{-1} \cdot h_{L,0}^{-1} \cdot h_A^\frac{1}{n} \right) \]

hold.

### 9.3 Lower estimate

Now we shall estimate \( K_{\ell,m} \) from below. Let \( \nu \) denotes the numerical Kodaira dimension of \((K_X + L |_S, h_S)\). Let \( V \) be a very general smooth complete intersection of \((n - \nu)\) members of \(| A |_S\) such that \((K_X + L |_S, h_S)^\nu \cdot V > 0\).

Then as in the proof of Lemma 6.4 in Section 6.3 we see that

\[ \limsup_{\ell \to \infty} \ell^{n-\nu} \cdot K_{\ell,\infty} \neq 0 \] (21)

holds. Here the factor \( \ell^{n-\nu} \) appears by the decay of the curvature of \( \Theta_{h_S} + \ell^{-1} \Theta_{h_A} \) along \( V \) in the normal direction of \( V \). Let \( h_V \) be a singular hermitian metric on \( K_X + L |_V \) such that

1. \( h_V \geq h_S|_V \) holds on \( V \).
2. \( \Theta_{h_V} \) is strictly positive everywhere on \( V \).

Then by repeating the proof of Lemma 6.4 we see that there exists a positive constant \( C_\varepsilon \) depending only on \( 0 < \varepsilon < 1 \) such that

\[ \limsup_{\ell \to \infty} \ell^{n-\nu} \cdot K_{\ell,\infty} \geq C_\varepsilon \cdot (h_V^\varepsilon \cdot h_S |_V^{1-\varepsilon})^{-1} \] (22)

holds on \( V \).

### 9.4 Completion of the proofs of Theorems 1.2 and 1.6

Let us set

\[ K_{S,\infty} := \limsup_{\ell \to \infty} \left( \{(K_X + L + \frac{1}{\ell} A, h_S \cdot h_A^\frac{1}{n})^n \}^{-1} \cdot (K_{\ell,\infty}) \right) \cdot (K_{\ell,\infty}) \]

Then since there exists a positive constant \( c_0 \) such that

\[ (K_X + L + \frac{1}{\ell} A, h_S \cdot h_A^\frac{1}{n})^n \geq c_0 \cdot \left( \frac{1}{\ell^{n-\nu}} \right) \]

holds by the assumption, by (20) and (21), we see that \( K_{S,\infty} \) exists on \( S \) and nonzero and by (22), moving \( V \),

\[ h_{S,\infty} := \text{the lower envelope of} \ \frac{1}{K_{S,\infty}} \]

is an AZD of \((K_X + L |_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L)\).
Repeating the argument in Section 7.2, we may extend $K_{\ell,\infty}$ to $\tilde{K}_{\ell,\infty}$ on $X$. Then we set

$$\tilde{K}_{S,\infty} := \limsup_{\ell \to \infty} \left( \{(K_X + L + \frac{1}{\ell} A \mid_S, h_s \cdot h_A^{\frac{1}{\ell}})^n \}^{-1} \cdot (\tilde{K}_{\ell,\infty}) \right)$$

and

$$\tilde{h}_{S,\infty} := \text{the lower envelope of } \frac{1}{\tilde{K}_{S,\infty}}$$

exists. By the construction, it is clear that $\Theta_{h_{\cdot,\infty}}$ is closed semipositive in the sense of current. Then we see that

$$\tilde{h}_{S,\infty} \mid_S \leq h_{S,\infty}$$

and $\tilde{h}_{S,\infty} \mid_S$ is an AZD of $(K_X + L \mid_S, e^{-\varphi} \cdot dV^{-1} \cdot h_L)$.

Then by Theorem 2.26, we may extend every element of

$$H^0(S, \mathcal{O}_S(m(K_X + L)) \otimes I(e^{-\varphi} \cdot h_L \cdot h_{S,\infty}^{m-1}))$$

to an element of

$$H^0(X, \mathcal{O}_X(m(K_X + L)) \otimes I(h_L \cdot \tilde{h}_{S,\infty}^{m-1})).$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.6 The proof of Theorem 1.6 follows from the combination of the proof of Theorem 1.4 and the above argument of eliminating $A$. □

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