THE UNIQUENESS OF THE ENNEPER SURFACES AND
CHERN–RICCI FUNCTIONS ON MINIMAL SURFACES

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ABSTRACT. We construct the first and second Chern-Ricci functions on negatively curved minimal surfaces in $\mathbb{R}^3$ using Gauss curvature and angle functions, and establish that they become harmonic functions on the minimal surfaces. We prove that a minimal surface has constant first Chern-Ricci function if and only if it is Enneper’s surface. We explicitly determine the moduli space of minimal surfaces having constant second Chern-Ricci function, which contains catenoids, helicoids, and their associate families.

To my mother who always gives me courage

1. ENNEPER’S SURFACES AND OTHER MINIMAL SURFACES IN $\mathbb{R}^3$

What is the Enneper surface? The Enneper-Weierstrass representation \[9\] says that a simply connected minimal surface in $\mathbb{R}^3$ can be constructed by the conformal harmonic mapping

$$X(\zeta) = X(\zeta_0) + \left( \text{Re} \int_{\zeta_0}^{\zeta} \phi_1(\zeta) d\zeta, \text{Re} \int_{\zeta_0}^{\zeta} \phi_2(\zeta) d\zeta, \text{Re} \int_{\zeta_0}^{\zeta} \phi_3(\zeta) d\zeta \right),$$

where the holomorphic null curve $\phi(\zeta)$ is determined by the Weierstrass data $(G(\zeta), \Psi(\zeta) d\zeta)$:

$$\phi(\zeta) = (\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta)) = \left( \frac{1}{2} (1 - G^2) \Psi, \frac{i}{2} (1 + G^2) \Psi, G \Psi \right).$$

Taking the simplest data $(G(\zeta), \Psi(\zeta) d\zeta) = (\zeta, d\zeta)$, $\zeta = u + iv \in \mathbb{C}$ yields Enneper’s surface

$$X_{\text{enn}}(u, v) = \left( \frac{1}{2} \left( u - \frac{u^3}{3} + uv^2 \right), \frac{1}{2} \left( -v + \frac{v^3}{3} - u^2 v \right), \frac{1}{2} (u^2 - v^2) \right).$$

One can see the inner rotational symmetry of its induced metric $g_{\text{enn}} = \left( \frac{1+|\zeta|^2}{2} \right)^2 |d\zeta|^2$ and strictly negative Gauss curvature $K_{\text{enn}} = K_{g_{\text{enn}}} = -\left( \frac{2}{1+|\zeta|^2} \right)^4$. We observe that

1. the conformally changed metric $(-K_{\text{enn}})^{\frac{1}{2}} g_{\text{enn}} = |d\zeta|^2$ is flat, and that
2. $(-K_{\text{enn}}) g_{\text{enn}} = \left( \frac{2}{1+|\zeta|^2} \right)^2 |d\zeta|^2$ becomes the metric on the complex $\zeta$-plane induced by the stereographic projection of the unit sphere sitting in $\mathbb{R}^3$ with respect to the north pole.

Do these intrinsic properties uniquely determine Enneper’s surfaces among minimal surfaces? The answer is completely no. Ricci \[1, 3, 6, 7, 8\] showed that, if $\Sigma$ is a minimal surface in $\mathbb{R}^3$, on non-flat points, the metric $(-K_{\text{ricci}})^{\frac{1}{2}} g_{\Sigma}$ is flat. The Ricci condition guarantees that the metric $(-K_{\text{ricci}}) g_{\Sigma}$ has constant Gauss curvature 1. We see that Enneper’s surface becomes the simplest example illustrating Ricci conditions for negatively curved minimal surfaces in $\mathbb{R}^3$. 

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We shall present the coordinate-free characterization of the Enneper surface given by the patch (1.1). To achieve this goal, we discover a geometric identity on Enneper’s surface. Let’s recall

**Definition 1.1 (Angle function on the oriented surface in \( \mathbb{R}^3 \)).** Given a surface \( \Sigma \) in \( \mathbb{R}^3 \) oriented by the unit normal vector field \( N \) and a constant unit vector field \( V(p) = V \) in \( \mathbb{R}^3 \), we introduce the angle function \( N_V : \Sigma \to [-1, 1] \) by the formula

\[
N_V(p) := \langle N(p), V \rangle_{\mathbb{R}^3}, \quad p \in \Sigma,
\]

which is the normal component of the vector field \( V \).

We observe that the induced unit normal vector field \( N_{\text{Enn}} \) on the Enneper surface given by

\[
X = X_{\text{Enn}}(u, v) = \left( \frac{1}{2} \left( u - \frac{u^3}{3} + uv^2 \right), \frac{1}{2} \left( v + \frac{v^3}{3} - u^2v \right), \frac{1}{2} \left( u^2 - v^2 \right) \right)
\]

reads

\[
N_{\text{Enn}}(u, v) = \frac{1}{X_u \times X_v} X_u \times X_v = \frac{1}{1 + u^2 + v^2} \left( 2u, 2v, -1 + u^2 + v^2 \right).
\]

Taking the downward vector \( -e_3 = (0, 0, -1) \), we see that the following quantity is constant:

\[
(-K)^{-\frac{1}{2}} \left( 1 + N_{(-e_3)} \right) = \left( 1 + \frac{1}{1 + u^2 + v^2} \right), \quad \left( 1 + \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right) = 1.
\]

It turns out that the constancy of this quantity captures the geometric uniqueness of Enneper’s surfaces among minimal surfaces in \( \mathbb{R}^3 \), and that the geometric quantity \( (-K)^{-\frac{1}{2}} (1 + N_{(-e_3)}) \) motivates the first Chern-Ricci harmonic function on minimal surfaces.

**Theorem 1.2 (The first Chern-Ricci harmonic function and uniqueness of Enneper’s surfaces).** Let \( \Sigma \) denote a minimal surface immersed in \( \mathbb{R}^3 \) with the Gauss curvature \( K = K_{\mathbb{S}^2} \) \( < 0 \), unit normal vector field \( N \), and angle function \( N_V \in (-1, 1) \) for a constant unit vector field \( V \) in \( \mathbb{R}^3 \).

1. The first Chern-Ricci function \( \text{CR}^1_V := \ln \left( (-K)^{-\frac{1}{2}} (1 + N_V) \right) \) is harmonic on \( \Sigma : \)

\[
\Delta_{g_\Sigma} \ln \left( \frac{1 + N_V}{(-K)^{\frac{1}{2}}} \right) = 0.
\]

2. If the first Chern-Ricci function is constant, then it should be a part of the Enneper surface, up to isometries and homotheties in \( \mathbb{R}^3 \).

**Theorem 1.3 (Harmonicity of the second Chern-Ricci function and classification of minimal surfaces with constant second Chern-Ricci function).** Let \( \Sigma \) denote a minimal surface in \( \mathbb{R}^3 \) with the Gauss curvature \( K = K_{\mathbb{S}^2} \) \( < 0 \), unit normal vector field \( N \), and angle function \( N_V \in (-1, 1) \) for a constant unit vector field \( V \) in \( \mathbb{R}^3 \).
1. The second Chern-Ricci function \( CR_V := \ln \left( (-K)^{-\frac{1}{2}} \left( 1 - N^2 \right) \right) \) is harmonic on \( \Sigma \):
\[
\triangle_{g_\Sigma} \ln \left( \frac{1-N^2}{(-K)^{\frac{1}{2}}} \right) = 0.
\]

2. Any member of the moduli space of minimal surfaces of constant second Chern-Ricci function is a part of the minimal surface given by Weierstrass data \( \left( g(\zeta), \frac{1}{g'(\zeta)}d\zeta \right) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha d\zeta}) \) for some constants \( \alpha \in \mathbb{C} \setminus \{0\} \) and \( \zeta_0 \in \mathbb{C} \). The moduli space contains catenoids, helicoids, and their associate families.

Example 1.4 (The second Chern-Ricci function on helicoids). Taking the Weierstrass data \( (G(\zeta), \Psi(\zeta)d\zeta) = (e^\zeta, -ie^{-\zeta}d\zeta), \zeta = u + iv \in \mathbb{C} \) gives \( X(u, v) = (-\sinh u \cos v, \sinh u \sin v, \sinh v) \).

The second Chern-Ricci function with respect to the vector field \( V = e_3 = (0, 0, 1) \) is constant:
\[
CR_{e_3}^2 = \ln \left( (-K)^{-\frac{1}{2}} \left( 1 - N_{e_3}^2 \right) \right) = \ln \left( \frac{1+|e|^2}{2|e|} \right)^2 = \ln 1 = 0.
\]

Remark 1.5. It would be interesting to generalize the Chern-Ricci harmonic functions on minimal hypersurfaces in higher dimensional Euclidean space.

2. Construction of the first and second Chern-Ricci harmonic functions

It is a classical fact that the flat points of a minimal surface are isolated, unless itself is flat everywhere. We use the geometric identities due to Chern and Ricci to construct harmonic functions on negatively curved minimal surfaces.

Theorem 2.1 (Harmonicity of Chern-Ricci functions on minimal surfaces in \( \mathbb{R}^3 \)). Let \( \Sigma \) denote a minimal surface immersed in \( \mathbb{R}^3 \) with the Gauss curvature \( K = K_{g_\Sigma} \leq 0 \), unit normal vector field \( N \), and angle function \( N_V \in [-1, 1] \) with respect to a constant unit vector field \( V \) in \( \mathbb{R}^3 \).

1. When \(-1 < N_V \leq 1\), the first Chern-Ricci function \( CR_V^1 = \ln \left( \frac{1+N_V}{(-K)^{\frac{1}{2}}} \right) \) is harmonic on \( \Sigma \).

2. When \(-1 < N_V < 1\), the second Chern-Ricci function \( CR_V^2 = \ln \left( \frac{1-N^2}{(-K)^{\frac{1}{2}}} \right) \) is harmonic on \( \Sigma \).

Proof. The key idea is to combine two intriguing identities for Gauss curvature function on negatively curved minimal surfaces. On the one hand, in his simple proof of Bernstein’s Theorem for entire minimal graphs, Chern \((4\text{ Section 4})\) used the geometric identity
\[
K = \triangle_{g_\Sigma} \ln \left( 1 + N_V \right).
\]

On the other hand, Ricci \((1\text{ 3\text{ 6\text{ 7\text{ 8})\)) obtained the geometric identity
\[
4K = \triangle_{g_\Sigma} \ln \left( -N \right), \quad \text{or equivalently,} \quad -K = \triangle_{g_\Sigma} \ln \left( \frac{1}{(-K)^{\frac{1}{2}}} \right).
\]

Chern’s identity and Ricci’s identity imply the harmonicity of the first Chern-Ricci function:
\[
\triangle_{g_\Sigma} CR_V^1 = \triangle_{g_\Sigma} \ln \left( \frac{1+N_V}{(-K)^{\frac{1}{2}}} \right) = \triangle_{g_\Sigma} \ln \left( 1 + N_V \right) + \triangle_{g_\Sigma} \ln \left( \frac{1}{(-K)^{\frac{1}{2}}} \right) = 0.
\]

The identity \(-N_V = N(-V)\) and the linear combination of two first Chern-Ricci functions imply
\[
\triangle_{g_\Sigma} CR_V^2 = \triangle_{g_\Sigma} \ln \left( \frac{1-N^2}{(-K)^{\frac{1}{2}}} \right) = \triangle_{g_\Sigma} \ln \left( \frac{1+N}{(-K)^{\frac{1}{2}}} \right) + \triangle_{g_\Sigma} \ln \left( \frac{1+N(-V)}{(-K)^{\frac{1}{2}}} \right) = 0.
\]
3. CLASSIFICATIONS OF MINIMAL SURFACES WITH CONSTANT
CHERN-RICCI FUNCTIONS

Theorem 3.1 (Uniqueness of Enneper’s minimal surfaces). Let \( \Sigma \) be a minimal surface in \( \mathbb{R}^3 \) with the Gauss curvature \( K = K_{\mathbb{R}^3} < 0 \) and unit normal vector \( \mathbf{N} \). Suppose that there exists a constant unit vector field \( \mathbf{V} \) in \( \mathbb{R}^3 \) such that \(-1 < N_V \leq 1\) and that the first Chern-Ricci function \( CR_V = \ln \left( \frac{1 + N_V}{(-K)^{\frac{1}{2}}} \right) \) is constant. Then, the minimal surface \( \Sigma \) should be a part of Enneper’s surface.

Proof. For the simplicity, rotating the coordinate system in \( \mathbb{R}^3 \), we can take the normalization \( \mathbf{V} = -e_3 = (0, 0, -1) \). We assume that the first Chern-Ricci function

\[
\ln \left[ (-K)^{-\frac{1}{4}} \left( 1 + N_{(-e_3)} \right) \right] = C
\]

is constant. The key idea is to take the orthogonal lines of curvature on our minimal surface \( \Sigma \) in order to read the information in (3.1) in terms of the corresponding Gauss map. We first begin with an arbitrary local conformal coordinate \( w \) on \( \Sigma \) to find the conformal harmonic map:

\[
\mathbf{X} = \mathbf{X}(w) = \mathbf{X}(w_0) + \left( \Re \int_{w_0}^w \omega_1, \Re \int_{w_0}^w \omega_2, \Re \int_{w_0}^w \omega_3 \right),
\]

where the holomorphic 1-forms \((\omega_1, \omega_2, \omega_3)\) are given by the Weierstrass data \((G(w), \Psi(w)dw)\):

\[
(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} (1 - G(w)^2) \Psi(w) dw, \frac{i}{2} \left( 1 + G(w)^2 \right) \Psi(w) dw, G(w) \Psi(w) dw \right).
\]

We introduce the isothermic coordinate \( \zeta \) from the initial conformal coordinate \( w \) by the rule

\[
w \mapsto \zeta = \zeta_0 + \int_{w_0}^w \sqrt{G'(w)\Psi(w)} \, dw,
\]

and the mapping \( g(\zeta) := G(w) \). Then, the Enneper-Weierstrass representation becomes

\[
\mathbf{X} = \mathbf{X}(\zeta) = \mathbf{X}(\zeta_0) + \left( \Re \int_{\zeta_0}^\zeta \omega_1, \Re \int_{\zeta_0}^\zeta \omega_2, \Re \int_{\zeta_0}^\zeta \omega_3 \right),
\]

where the holomorphic 1-forms \((\omega_1, \omega_2, \omega_3)\) are given by the Weierstrass data \((g(\zeta), \frac{1}{g'(\zeta)} d\zeta)\):

\[
(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (g(\zeta))^2}{g'(\zeta)} \, d\zeta, \frac{i}{2} \cdot \frac{1 + (g(\zeta))^2}{g'(\zeta)} \, d\zeta, \frac{g(\zeta)}{g'(\zeta)} \, d\zeta \right).
\]

The conformal coordinates \((u, v) = (\Re \zeta, \Im \zeta)\) give us the orthogonal lines of curvature on \( \Sigma \):

\[
0 = \Im \left( -g'(\zeta) \cdot \frac{1}{g'(\zeta)} \, d\zeta \right) = \Im (-d\zeta^2) = -2 \, du \, dv.
\]

Using classical formulas (cf. Chapter 9), we compute Gauss curvature and angle function:

\[
K = -\left( \frac{2|g'(\zeta)|}{1 + |g(\zeta)|^2} \right)^4 \quad \text{and} \quad N_{(-e_3)} = \frac{1 - |g(\zeta)|^2}{1 + |g(\zeta)|^2}.
\]

Combining (3.1) and (3.4) gives

\[
C = \ln \left[ (-K)^{-\frac{1}{4}} \left( 1 + N_{(-e_3)} \right) \right] = \ln \left[ 1 + |g(\zeta)|^2 \right] + \frac{2}{2|g'(\zeta)|} \cdot \frac{2}{1 + |g(\zeta)|^2} = -\Re \left[ \log g'(\zeta) \right].
\]

By the holomorphicity of the Gauss map \( g(\zeta) \), we have \( g(\zeta) = \alpha (\zeta - \zeta_0) \). Plugging this into (3.3), the Enneper-Weierstrass representation (3.2) shows that \( \Sigma \) is Enneper’s surface. \(\square\)
Theorem 3.3 (Classification of minimal surfaces with constant second Chern-Ricci function). Let $\Sigma$ be a minimal surface in $\mathbb{R}^3$ with the Gauss curvature $K = K_{\mathbb{R}^3} < 0$ and unit normal vector $N$. Suppose that there exists a constant unit vector field $V$ in $\mathbb{R}^3$ such that $-1 < N_V < 1$ and that the second Chern-Ricci function $CR^2_V = \ln \left( (-K)^{-\frac{1}{2}} (1 - N_V^2) \right)$ is constant. Then, $\Sigma$ is a part of the minimal surface given by the Enneper-Weierstrass representation

\begin{equation}
X(\zeta) = X(\zeta_0) + \left( Re \int_{\zeta_0} \omega_1, \ Re \int_{\zeta_0} \omega_2, \ Re \int_{\zeta_0} \omega_3 \right)
\end{equation}

where the holomorphic 1-forms

\begin{equation}
(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (g(\zeta))^2}{g'(\zeta)} \, d\zeta, \ i \cdot \frac{1}{2} \cdot \frac{1 + (g(\zeta))^2}{g'(\zeta)} \, d\zeta, \ g(\zeta) \cdot \frac{g'(\zeta)}{g''(\zeta)} \, d\zeta \right).
\end{equation}

are given by the Weierstrass data \((g(\zeta), \frac{1}{g'(\zeta)} \, d\zeta) = (e^{\alpha(\zeta - \zeta_0)}, e^{-\alpha\zeta} \, d\zeta)\) for some constants $\alpha \in C - \{0\}$ and $\zeta_0 \in C$.

Proof. For the simplicity, rotating the coordinate system in $\mathbb{R}^3$, we can take the normalization $V = e_3 = (0, 0, 1)$. As in the proof of Theorem 3.1, taking the orthogonal lines of curvature coordinates $\zeta$ on the minimal surface under the normalization of Hopf differential $-d\zeta^2$, we can write the second Chern-Ricci function in terms of the corresponding Gauss map $g(\zeta)$:

\begin{equation}
CR^2_{e_3} = \ln \left( (-K)^{-\frac{1}{2}} (1 - N_{e_3}^2) \right) = \ln \left[ \left( \frac{1 + i g(\zeta)^2}{2 |g(\zeta)|} \right)^2 \left( \frac{2 |g(\zeta)|}{1 + i g(\zeta)^2} \right)^2 \right] = 2 \ln \left( |g(\zeta)| \right),
\end{equation}

or equivalently,

\begin{equation}
CR^2_{e_3} = -2 \ Re \left[ \log \frac{g'(\zeta)}{g(\zeta)} \right] = -2 \ Re \left[ \log \left( \frac{g(\zeta)}{g''(\zeta)} \right) \right].
\end{equation}

Since the function $CR^2_{e_3}$ is constant, by the holomorphicity of $g(\zeta)$, we have the Weierstrass data

\begin{equation}
\left( g(\zeta), \frac{1}{g'(\zeta)} \, d\zeta \right) = \left( e^{\alpha(z - \zeta_0)}, e^{-\alpha\zeta} \, d\zeta \right).
\end{equation}

Remark 3.4 (Examples of minimal surfaces with constant second Chern-Ricci function). The moduli space in Theorem 3.3 contains catenoids [2 Section 2.1.2] and helicoids [2 Section 2.1.4]. In fact, under the transformation $\zeta \mapsto z := e^{\alpha(\zeta - \zeta_0)}$, we obtain the Weierstrass data

\begin{equation}
\left( g(\zeta), \frac{1}{g'(\zeta)} \, d\zeta \right) = \left( e^{\alpha(z - \zeta_0)}, e^{-\alpha\zeta} \, d\zeta \right) = \left( z, e^{\alpha \zeta_0} \cdot \frac{dz}{z^2} \right),
\end{equation}

which recovers the associate family of helicoids.

APPENDIX. FOUR HOLOMORPHIC QUADRATIC DIFFERENTIALS ON MINIMAL SURFACES IN $\mathbb{R}^3$

We summarize definitions of holomorphic quadratic differentials on minimal surfaces in $\mathbb{R}^3$. Throughout this section, as in the proofs of Theorems 3.1 and 3.3, we use the orthogonal lines of curvature coordinates $\zeta$ on the negatively curved minimal surface $\Sigma$ with the normalization of Hopf’s holomorphic differential $\Omega = -\frac{1}{2}d\zeta^2$. The minimal surface $\Sigma$ is parameterized by

\begin{equation}
X(\zeta) = X(\zeta_0) + \left( Re \int_{\zeta_0} \omega_1, \ Re \int_{\zeta_0} \omega_2, \ Re \int_{\zeta_0} \omega_3 \right),
\end{equation}

where we have the holomorphic 1-forms $(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{2} \cdot \frac{1 - (g(\zeta))^2}{g'(\zeta)}, \ i \cdot \frac{1 + (g(\zeta))^2}{g'(\zeta)}, \ g(\zeta) \cdot \frac{g'(\zeta)}{g''(\zeta)} \right) d\zeta$. 
I. Bernstein-Mettler’s entropy differential [3]. Applying the variational structure of Ricci’s intrinsic condition induced from Hamilton’s entropy functional for Ricci flow, Bernstein and Mettler constructed the entropy differential. In [3, Corollary A.2], they observed that, if the minimal surface \( \Sigma \) is an Enneper surface, the conformally changed metric \((-K_{\text{can}})^{\frac{1}{2}} g_{\text{can}}\) recovers the positively curved cigar soliton, and also proved that this property characterizes Enneper surfaces among minimal surfaces in \(\mathbb{R}^3\). The formula in [3, Proposition 3.2] implies that the Schwarzian derivative of the Gauss map \( g(\zeta) \) realizes the holomorphic quadratic differential

\[
S_g(\zeta) \, \d\zeta^2 := \left[ \left( \frac{g''(\zeta)}{g'(\zeta)} \right)' - \frac{1}{2} \left( \frac{g''(\zeta)}{g'(\zeta)} \right)^2 \right] \, \d\zeta^2.
\]

We would like to add that the Schwarzian derivative of the gauss map realizes the squared complex curvature of the lifted holomorphic null curve from the minimal surface [5, Section 3].

II. Induced holomorphic quadratic differential from Chern-Ricci functions. By the identity (3.5), the holomorphicity of the corresponding Gauss map \( g(\zeta) \) also implies the harmonicity of the first Chern-Ricci function

\[
\text{CR}_{\{\text{e}_1\}} = \ln \left[ (-K)^{-\frac{1}{2}} (1 + N_{\{\text{e}_1\}}) \right] = -\text{Re} \left[ \log g'(\zeta) \right].
\]

The harmonic function \(-\text{CR}_{\{\text{e}_1\}}\) induces the holomorphic quadratic differential

\[
\Omega_1 = \left( \log g'(\zeta) \right)'' \, \d\zeta^2 = \left( \frac{g''(\zeta)}{g'(\zeta)} \right)' \, \d\zeta^2.
\]

By the identity (3.5), the holomorphicity of the Gauss map \( g(\zeta) \) also implies the harmonicity of the second Chern-Ricci function

\[
\text{CR}_{\text{e}_2} = \ln \left( (-K)^{-\frac{1}{2}} (1 - N_{\text{e}_2}^2) \right) = -2 \text{Re} \left[ \log \left( \log g'(\zeta) \right)' \right].
\]

The harmonic function \(-\frac{1}{2}\text{CR}_{\text{e}_2}\) induces the holomorphic quadratic differential

\[
\Omega_2 = \left( \log \left( \log g'(\zeta) \right)' \right)'' \, \d\zeta^2 = \left[ \left( \frac{g''(\zeta)}{g'(\zeta)} \right)' - \left( \frac{g'(\zeta)}{g'(\zeta)} \right)' \right] \, \d\zeta^2.
\]

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