Local superderivations on Cartan type Lie superalgebras

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Abstract. In this paper, we characterize the local superderivations on Cartan type Lie superalgebras over the complex field C. Furthermore, we prove that every local superderivation on Cartan type simple Lie superalgebras is a superderivation. As an application, using the results on local superderivations we characterize the 2-local superderivations on Cartan type Lie superalgebras. We prove that every 2-local superderivation on Cartan type Lie superalgebras is a superderivation.

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1. Introduction

As a natural generalization of Lie algebras, Lie superalgebras are closely related to many branches of mathematics. The classification of all finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero has been obtained by Kac [10], which consists of classical Lie superalgebras and Cartan type Lie superalgebras. Cartan type Lie superalgebras play an important role in the category of Lie superalgebras. Cartan type Lie superalgebras over C are subalgebras of the full superderivation algebras of the exterior superalgebras. The structural theory of these superalgebras has been playing a

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key role in the theory of Lie superalgebras. Derivations and generalized derivations are very important notions in the research of algebras and their generalizations.

The concept of local derivation was introduced in 1990 by Kadison [11], Larson and Sourour [13], and the authors studied local derivations of Banach algebra. In 2001 Johnson showed that every local derivation from a $C^*$-algebra $A$ into a Banach $A$-bimodule is a derivation [9]. Local derivations on the algebra $S(M, \tau)$ were studied deeply in paper [1].

In recent years, local derivations have aroused the interest of a great many authors, see [4, 8, 19]. The local derivations of Lie algebras have been sufficiently studied. In 2016, the local derivations of Lie algebras were proved by Ayupov and Kudaybergenov [2], and the authors proved that every local derivation of a finite dimensional semisimple Lie algebras over an algebraically closed field of characteristic zero is a derivation. In 2018, Ayupov and Kudaybergenov showed that in the class of solvable Lie algebras there exist two facts. One is that local derivation is different from any other derivation and the second is that there indeed exists a kind of algebras in which each local derivation is a derivation [3]. In 2017, Chen, Wang and Nan mainly studied local superderivations on basic classical Lie superalgebras, and the authors proved that every local superderivation on $A(1,1)$ is a superderivation [6].

In 2018, Chen and Wang studied local superderivations on Lie superalgebras $q(n)$, and the authors proved that every local superderivation on $q(n)$, $n > 3$, is a superderivation [5].

In this paper, we are interested in determining all local superderivations and 2-local superderivations on Cartan type Lie superalgebras over $C$. Let $L$ be a Cartan type Lie superalgebra over $C$. The main result in this paper is a complete characterization of the local superderivations on $L$:

$$\text{LDer}(L) = \text{Der}(L).$$

The paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we establish several lemmas, which will be used to characterize the local superderivations on Cartan type Lie superalgebras. In Section 4, we determine all local superderivations on Cartan type Lie superalgebras. In Section 5, as an application, using the results on local superderivations we determine all 2-local superderivations on Cartan type Lie superalgebras.

2. Preliminaries

Throughout $C$ is the field of complex numbers, $N$ the set of nonnegative integers and $Z_2 = \{0, 1\}$ the additive group of two elements. For a vector superspace $V = V_0 \oplus V_1$, we write $|x|$ for the parity of $x \in V_\alpha$, where $\alpha \in Z_2$. Once the symbol $|x|$ appears in this paper, it will imply that $x$ is a $Z_2$-homogeneous element. We also adopt the following notation: For a proposition $P$, put $\delta_P = 1$ if $P$ is true and $\delta_P = 0$ otherwise.

2.1. Lie superalgebras, superderivation

Let us recall some definitions relative to Lie superalgebras and superderivations [10].

**Definition 2.1.** A Lie superalgebra is a vector superspace $L = L_0 \oplus L_1$ with an even bilinear mapping $[,] : L \times L \rightarrow L$ satisfying the following axioms:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$$

for all $x, y, z \in L$. 

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Definition 2.2. We call a linear map $D : L \rightarrow L$ a superderivation of Lie superalgebra $L$ if it satisfies the following equation:

$$D([x, y]) = [D(x), y] + (-1)^{|D||x|}[x, D(y)]$$

for all $x, y \in L$.

Write $\text{Der}_0(L)$ (resp. $\text{Der}_1(L)$) for the set of all superderivations of $\mathbb{Z}_2$-homogeneous 0 (resp. 1) of $L$. Denote

$$\text{Der}(L) = \text{Der}_0(L) \oplus \text{Der}_1(L).$$

2.2. Local superderivation and 2-local superderivation

Let us recall some definitions relative to local superderivations and 2-local superderivations $\mathbb{L}$. Let $L$ be a Lie superalgebra.

Definition 2.3. Recall that a linear map $\phi : L \rightarrow L$ is called a local superderivation if for every $x \in L$, there exists a superderivation $D_x \in \text{Der}L$ (depending on $x$) such that $\phi(x) = D_x(x)$.

Definition 2.4. Recall that a linear map $\phi : L \rightarrow L$ is called a 2-local superderivation if for any two elements $x, y \in L$, there exists a superderivation $D_{x,y} \in \text{Der}L$ (depending on $x, y$) such that $\phi(x) = D_{x,y}(x)$ and $\phi(y) = D_{x,y}(y)$.

A local superderivation $\phi$ of $\mathbb{Z}_2$-homogeneous $\alpha$ of $L$ is a local superderivation such that $\phi(L_{\beta}) \subseteq L_{\alpha+\beta}$ for any $\beta \in \mathbb{Z}_2$. Write $\text{LDer}_0(L)$ (resp. $\text{LDer}_1(L)$) for the set of all super-biderivations of $\mathbb{Z}_2$-homogeneous 0 (resp. 1) of $L$. Denote

$$\text{LDer}(L) = \text{LDer}_0(L) \oplus \text{LDer}_1(L).$$

2.3. Cartan type Lie superalgebras

Let $n \geq 4$ be an integer and $\Lambda(n)$ be the exterior algebra in $n$ indeterminates $x_1, x_2, \ldots, x_n$ with $\mathbb{Z}_2$-grading structure given by $|x_i| = 1$. One may define a $\mathbb{Z}$-grading on $\Lambda(n)$ by letting $\deg x_i = 1$, where $1 \leq i \leq n$. Write $n = 2r$ or $n = 2r + 1$, where $r \in \mathbb{N}$. Put $\left\lceil \frac{n}{2} \right\rceil = r$.

Cartan type Lie superalgebras consist of four series of simple Lie superalgebras contained in the full superderivation algebras of $\Lambda(n)$:

$$W(n) = \left\{ \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n) \right\},$$

$$S(n) = \left\{ \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n), \sum_{i=1}^{n} \partial_i(f_i) = 0 \right\},$$

$$\tilde{S}(n) = \left\{ (1 - x_1x_2 \cdots x_n) \sum_{i=1}^{n} f_i \partial_i \mid f_i \in \Lambda(n), \sum_{i=1}^{n} \partial_i(f_i) = 0 \right\} \ (n \text{ is an even integer}),$$

$$H(n) = \left\{ D_H(f) \mid f \in \oplus_{i=0}^{n-1} \Lambda(n)_i \right\} \ (n > 4),$$

where

$$D_H(f) = (-1)^{|f|} \sum_{i=1}^{n} \partial_i(f)\partial_i.$$
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\[ i' = \begin{cases} 
  i + r, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
  i - r, & \text{if } \left\lceil \frac{n}{2} \right\rceil < i \leq 2 \left\lfloor \frac{n}{2} \right\rfloor, \\
  i, & \text{otherwise}. 
\end{cases} \]

One may define a \( \mathbb{Z} \)-grading on \( W(n) \) by letting \( \deg x_i = 1 = \deg \partial_i \), where \( 1 \leq i \leq n \). Thus \( W(n) \) becomes a \( \mathbb{Z} \)-graded Lie superalgebra of 1 depth: \( W(n) = \bigoplus_{j=0}^{\xi_w} W(n)_j \), where \( \xi_w = n - 1 \). Suppose \( L = S(n) \) or \( H(n) \). Then \( L \) is a \( \mathbb{Z} \)-graded subalgebra of \( W(n) \).

The \( \mathbb{Z} \)-grading is defined as follows: \( L = \bigoplus_{j=0}^{\xi_L} L_j \), where \( L_j = L \cap W(n)_j \) and

\[ \xi_L = \begin{cases} 
  n - 2, & \text{if } L = S(n), \\
  n - 3, & \text{if } H(n). 
\end{cases} \]

Put \( \xi_i = x_1 x_2 \cdots x_n \partial_i \), \( S(n)_{-1} = \text{span}_\mathbb{C} \{ \partial_i - \xi_i \mid 1 \leq i \leq n \} \), \( S(n)_i = S(n) \), for \( i > 1 \).

Then \( S(n) \) becomes a \( \mathbb{Z}_n \)-graded Lie superalgebra: \( S(n) = \bigoplus_{i=1}^{\xi_S} S(n)_i \), where \( \xi_S = n - 2 \). The 0-degree components of these superalgebras are classical Lie algebras:

\[ W(n)_0 \cong \mathfrak{sl}(n), \ S(n)_0 = \tilde{S}(n)_0 \cong \mathfrak{sl}(n), \ H(n)_0 \cong \mathfrak{so}(n). \]

Let \( L = \bigoplus_{i \in \mathbb{Z}} L_i \) be a \( \mathbb{Z} \)-graded Lie superalgebra, \( H_L \) be the standard Cartan subalgebra of \( L \), \( \theta \in H_L^* \) be the zero root, \( \Delta_L \) be the root system of \( L \). Let us describe the roots of Cartan type Lie superalgebras. If \( L = W(n) \), we choose the standard basis \( \{ \epsilon_1, \ldots, \epsilon_n \} \) in \( H_L^* \), and then

\[ \Delta_L = \{ \epsilon_{i_1} + \cdots + \epsilon_{i_k} - \epsilon_i \mid 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq i \leq n \}. \]

The root systems of \( S(n) \) and \( \tilde{S}(n) \) are obtained from the root system of \( W(n) \) by removing the roots \( \epsilon_1 + \cdots + \epsilon_n - \epsilon_i \), where \( 1 \leq i \leq n \). Finally if \( L = H(n) \), then

\[ \Delta_L = \left\{ \pm \epsilon_{i_1} \pm \cdots \pm \epsilon_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}. \]

3. General lemmas

In this section, let us establish several lemmas, which will be used to characterize the local superderivations on Cartan type Lie superalgebras. Put \( \mathcal{C} = \sum_{i=1}^n x_i \partial_i \) and \( \mathcal{H}(n) = \{ \mathcal{D}_H(f) \mid f \in \Lambda(n) \} \). By [18], we have the following lemma.

**Lemma 3.1.** Let \( L \) be a Cartan type Lie superalgebra. Then \( \text{Der} L = \mathfrak{ad} L' \), where

\[ L' \cong \begin{cases} 
  L, & \text{if } L = W(n), \tilde{S}(n), \\
  L \oplus \mathbb{C} \mathcal{C}, & \text{if } L = S(n), \\
  \tilde{H}(n) \oplus \mathbb{C} \mathcal{C}, & \text{if } L = H(n). 
\end{cases} \]

Let \( L \) be a Cartan type Lie superalgebra. By Lemma 3.1 and a simple computation, we have \( \Delta_{L'} = \Delta_L \) and the following lemma.

**Lemma 3.2.** Let \( L \) be a Cartan type Lie superalgebra. Then \( L' \) is transitive, that is \( a \in \bigoplus_{i \geq 0} L'_i \) and \( [a, L'_{-1}] = 0 \), then \( a = 0 \).
Suppose $L$ is a Cartan type Lie superalgebra. For $i \in \mathbb{Z}$ and $\alpha \in \Delta_L$, we put
\[ L\text{Der}(L)_{i,\alpha} = \{ \phi \in L\text{Der}(L) \mid \text{there exists } u_x^i \in L' \cap L'_\alpha \text{ such that } \phi(x) = [u_x^i, x] \text{ for all } x \in L \}. \]
Then we have the following lemma.

Lemma 3.3. Let $L$ be a Cartan type Lie superalgebra. Then the following conclusion hold:

1. If $L \neq \tilde{S}(n)$, then $L\text{Der}(L) = \bigoplus_{i \in \mathbb{Z}, \alpha \in \Delta_L} L\text{Der}(L)_{i,\alpha}$.

2. If $L = \tilde{S}(n)$, then $L\text{Der}(L) = \bigoplus_{i \in \mathbb{Z}, \alpha \in \Delta_L} L\text{Der}(L)_{i,\alpha}$.

Proof. (1) By Lemma 3.1 we have “$\supseteq$” part is complete. Next, we verify the “$\subseteq$” part. Let $\phi \in L\text{Der}(L)$. For each $x \in L$, by Lemma 3.1 there exists an element $u_x \in L'$ such that $\phi(x) = [u_x, x]$, where $u_x \in L'$. Since $L'$ has the $\mathbb{Z} \times \Delta_L$-grading, we can write
\[ u_x = \sum_{i \in \mathbb{Z}, \alpha \in \Delta_L} u_x^{i,\alpha}, \]
where $u_x^{i,\alpha} \in L'_{i,\alpha}$. For $i \in \mathbb{Z}$ and $\alpha \in \Delta_L$, we set
\[ \phi_{i,\alpha}(x) = [u_x^{i,\alpha}, x]. \]
A direct verification shows that $\phi_{i,\alpha} \in L\text{Der}(L)_{i,\alpha}$ and
\[ \sum_{i \in \mathbb{Z}, \alpha \in \Delta_L} \phi_{i,\alpha}(x) = \sum_{i \in \mathbb{Z}, \alpha \in \Delta_L} [u_x^{i,\alpha}, x] = [u_x, x] = \phi(x). \]

(2) A similar argument as for $L \neq \tilde{S}(n)$ works also for $L = \tilde{S}(n)$. \hfill \square

4. Local Superderivations of Cartan type Lie superalgebras

In this section we shall characterize the local superderivations on Cartan type Lie superalgebras. Let $L$ be a Cartan type Lie superalgebra and $\{h_1, \ldots, h_l\}$ be the standard basis of $H_L$. Set $h_0 = \sum_{i=1}^l t^i h_i$, where $t$ is a fixed algebraic number from $\mathbb{C}$ of degree bigger than $l$. Then we have the following propositions.

Proposition 4.1. Let $L$ be a Cartan type Lie superalgebra and $\phi \in L\text{Der}(L)$. If $\phi(h_0) = 0$, then $\phi|_{H_L} = 0$.

Proof. For any $h_i (i = 1, \ldots, l)$, there exists element
\[ u^i = \sum_{1 \leq k_1 < \cdots < k_j \leq n} \sum_{j=1}^n a_{j,k_1,\ldots,k_j} x_{k_1} \cdots x_{k_j} \partial_j \in L', \]
where $a_{j,k_1,\ldots,k_j} \in \mathbb{C}$ such that $\phi(h_i) = [u^i, h_i]$. Then
\[ \phi(h_i) = -\sum_{1 \leq k_1 < \cdots < k_j \leq n} \sum_{j=1}^n a_{j,k_1,\ldots,k_j} (\varepsilon_{k_1} + \cdots + \varepsilon_{k_j} - \varepsilon_j)(h_i) x_{k_1} \cdots x_{k_j} \partial_j, \]
If \( l \neq 1 \) or \( l = 1 \) and \( k_1 \neq j \), then exists \( 1 \leq k \leq n \) such that \( (\epsilon_{k_1} + \cdots + \epsilon_{k_1} - \epsilon_j)(h_k) \neq 0 \). Put

\[ h_{ik} = (\epsilon_{k_1} + \cdots + \epsilon_{k_1} - \epsilon_j)(h_k)h_i - (\epsilon_{k_1} + \cdots + \epsilon_{k_1} - \epsilon_j)(h_i)h_k. \]

Then there exists element

\[ u^{ik} = \sum_{1 \leq k_1 < \cdots < k_l \leq n} \sum_{j=1}^n a^{ik}_{j,k_1,\ldots,k_l} x_{k_1} \cdots x_{k_l} \partial_j \in L', \]

where \( a^{ik}_{j,k_1,\ldots,k_l} \in \mathbb{C} \) such that \( \phi(h_{ik}) = [u^{ik}, h_{ik}] \). Thus \( \phi(h_{ik}) \cap L_{\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j} = 0 \). On the other hand,

\[ \phi(h_{ik}) = (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_k)\phi(h_i) = (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)\phi(h_k). \]

Hence

\[ (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_k)(\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)a^{ik}_{j,k_1,\ldots,k_l} = (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)\phi(h_k). \]

that is

\[ (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)a^{ik}_{j,k_1,\ldots,k_l} = (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)a^{ik}_{j,k_1,\ldots,k_l}. \]

Then

\[ \phi(h_0) = \sum_{i=1}^l t^i \phi(h_i) = - \sum_{i=1}^l \sum_{1 \leq k_1 < \cdots < k_l \leq n} \sum_{j=1}^n t^i a^{ik}_{j,k_1,\ldots,k_l} (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i)x_{k_1} \cdots x_{k_l} \partial_j. \]

Since \( \phi(h_0) = 0 \), we have \( a^{ik}_{j,k_1,\ldots,k_l} \sum_{i=1}^l t^i (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i) = 0 \). Since \( t \) is a algebraic number from \( \mathbb{C} \) of degree bigger than \( l \) and \( (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i) \) are integers, then \( \sum_{i=1}^l t^i (\epsilon_{k_1} + \cdots + \epsilon_{k_l} - \epsilon_j)(h_i) \neq 0 \). Thus \( a^{ik}_{j,k_1,\ldots,k_l} = 0 \) and \( \phi(h_i) = 0 \). The proof is complete. \( \square \)

To apply Lemma \( \ref{lem:superderivation} \), we give the following proposition.

**Proposition 4.2.** Let \( L \) be a Cartan type Lie superalgebra, \( \phi \in \text{LDer}(L) \) and \( \phi \mid_{L - i \oplus H_L} = 0 \). Then the following conclusion hold:

1. If \( L \neq \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \alpha}(L) \), where \( i \in \mathbb{Z} \) and \( \alpha \in \Delta_{L'} \), then \( \phi \) is zero.
2. If \( L = \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \alpha}(L) \), where \( i \in \mathbb{Z}_n \) and \( \alpha \in \Delta_{L'} \), then \( \phi \) is zero.

**Proof.** For any \( x \in \oplus_{j \geq 0} L_j \), there exists an element \( u_x \in L_{i_{-o}} \cap L'_{\alpha} \) such that

\[ \phi(x) = \phi(x + \partial_1 + \cdots + \partial_n - \delta_{L_{\bar{S}}}(\xi_1 + \cdots + \xi_n)) = [u_x, x + \partial_1 + \cdots + \partial_n - \delta_{L_{\bar{S}}}(\xi_1 + \cdots + \xi_n)] \in \oplus_{j \geq 0} L_{i + j}. \]

Since

\[ [u_x, \partial_1 + \cdots + \partial_n - \delta_{L_{\bar{S}}}(\xi_1 + \cdots + \xi_n)] \in L_{i_{-1}} \]

and \( [u_x, x] \in \oplus_{j \geq 0} L_{i_{+j}} \), then

\[ [u_x, \partial_1 + \cdots + \partial_n - \delta_{L_{\bar{S}}}(\xi_1 + \cdots + \xi_n)] = 0. \]
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Because \( \partial_1 - \delta_{L,S} \xi_1, \ldots, \partial_n - \delta_{L,S} \xi_n \) belong to different root space, so \( [u_x, \partial_j - \delta_{L,S} \xi_j] = 0 \) for all \( 1 \leq j \leq n \). By Lemma [3.2] we have \( u_x \in L'_{-1} \). Note that every root space of \( L'_{-1} \) is one dimension. Then there is \( 1 \leq k \leq l \) and \( a_x \in \mathbb{C} \) such that \( u_x = a_x(\partial_k - \delta_{L,S} \xi_k) \). Take \( h \in H_L \) satisfied \((\partial_k - \delta_{L,S} \xi_k, h) \neq 0 \). By

\[
[a_x(\partial_k - \delta_{L,S} \xi_k), x] = \phi(x) = \phi(h + x) = [a_{h+x}(\partial_k - \delta_{L,S} \xi_k), h + x],
\]

we have \( a_{h+x}(\partial_k - \delta_{L,S} \xi_k), h \) \( = 0 \) for all \( x \in \oplus_{i \geq 1} L_i \) or \( x \in L_0 \cap L_\beta \), where \( \theta \neq \beta \in \Delta_L \). Then \( a_{h+x} = 0 \), that is \( \phi(x) = 0 \) for all \( x \in \oplus_{i \geq 1} L_i \) or \( x \in L_0 \cap L_\beta \), where \( \theta \neq \beta \in \Delta_L \). This together with \( \phi |_{L_{-1} \oplus H_L} = 0 \) implies that \( \phi = 0 \). The proof is complete. \( \square \)

By Proposition [4.2] we have the following local superderivations vanishing propositions.

**Proposition 4.3.** Let \( L \) be a Cartan type Lie superalgebra, \( \phi \in \text{LDer}(L) \) and \( \phi |_{H_L} = 0 \). Then the following conclusion hold:

1. If \( L \neq \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \times \alpha}(L) \), where \( i \in \mathbb{Z} \) and \( \theta \neq \alpha \in \Delta_L \), then \( \phi \) is zero.
2. If \( L = \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \times \alpha}(L) \), where \( i \in \mathbb{Z}_n \) and \( \theta \neq \alpha \in \Delta_L \), then \( \phi \) is zero.

**Proof.** Since \( \alpha \neq \theta \), there exists an element \( h \in H_L \) such that \( \alpha(h) \neq 0 \). Let \( 1 \leq k \leq n \). By Lemma [3.1] we know that there is \( u_k, v_k \in L'_\alpha \) such that

\[
\phi(\partial_k - \delta_{L,S} \xi_k) = [u_k, \partial_k - \delta_{L,S} \xi_k],
\]

\[
\phi(h + \partial_k - \delta_{L,S} \xi_k) = [v_k, h + \partial_k - \delta_{L,S} \xi_k].
\]

Since \( \phi(H_L) = 0 \),

\[
[u_k, \partial_k - \delta_{L,S} \xi_k] = \phi(\partial_k - \delta_{L,S} \xi_k) = \phi(h + \partial_k - \delta_{L,S} \xi_k) = [v_k, h + \partial_k - \delta_{L,S} \xi_k] = -\alpha(h)v_k + [v_k, \partial_k - \delta_{L,S} \xi_k].
\]

Since \( \alpha \neq \theta \), we have \( v_k = 0 \). Then

\[
\phi(\partial_k - \delta_{L,S} \xi_k) = \phi(h + \partial_k - \delta_{L,S} \xi_k) = [v_k, h + \partial_k - \delta_{L,S} \xi_k] = 0,
\]

that is \( \phi |_{L_{-1}} = 0 \). By Proposition [1.2] we have \( \phi = 0 \). \( \square \)

**Proposition 4.4.** Let \( L \) be a Cartan type Lie superalgebra and \( \phi \in \text{LDer}(L) \). Then the following conclusion hold:

1. If \( L \neq \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \times \theta}(L) \), where \( i \in \mathbb{Z} \), then \( \phi \) is a superderivation.
2. If \( L = \tilde{S}(n) \) and \( \phi \in \text{LDer}_{i \times \theta}(L) \), where \( i \in \mathbb{Z}_n \), then \( \phi \) is a superderivation.

**Proof.** Let \( 1 \leq k \leq n \). For \( \partial_k - \delta_{L,S} \xi_k \), there exists an element \( u_k \in L'_\theta \) such that

\[
\phi(\partial_k - \delta_{L,S} \xi_k) = [u_k, \partial_k - \delta_{L,S} \xi_k].
\]

For \( \partial_1 + \cdots + \partial_n - \delta_{L,S}(\xi_1 + \cdots + \xi_n) \), there exists an element \( u \in L'_\theta \) such that

\[
\phi(\partial_1 + \cdots + \partial_n - \delta_{L,S}(\xi_1 + \cdots + \xi_n)) = [u, \partial_1 + \cdots + \partial_n - \delta_{L,S}(\xi_1 + \cdots + \xi_n)].
\]
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Since $\phi$ is a linear map,
\[ [u, \partial_1 + \cdots + \partial_n - \delta_{L, \mathcal{S}}(\xi_1 + \cdots + \xi_n)] = [u_1, \partial_1 - \delta_{L, \mathcal{S}}\xi_1] + \cdots + [u_n, \partial_n - \delta_{L, \mathcal{S}}\xi_n]. \]
Then
\[ [u, \partial_k - \delta_{L, \mathcal{S}}\xi_k] = [u_1, \partial_k - \delta_{L, \mathcal{S}}\xi_k] + \cdots + [u_n, \partial_k - \delta_{L, \mathcal{S}}\xi_k] \]
for all $1 \leq k \leq n$. Put $\phi' = \phi - \text{ad}_u$. Then $\phi'(\partial_k - \delta_{L, \mathcal{S}}\xi_k) = 0$ for all $1 \leq k \leq n$, that is $\phi' |_{L-1} = 0$. Since $\phi \in \text{LDer}_{\times 1}(L)$, we have $\phi' |_{H_L} = 0$. By Proposition 4.2, we have $\phi' = 0$. Then $\phi = \text{ad}_u$ is a superderivation. The proof is complete.

By Propositions 4.1, 4.3 and 4.4, we have the following theorem.

**Theorem 4.5.** Let $L$ be a Cartan type Lie superalgebra. Then
\[ \text{LDer}(L) = \text{Der}(L). \]

**Proof.** Only the “$\subseteq$” part needs a verification. Let $\phi \in \text{LDer}(L)$. By Lemma 3.1, there exists an element $u \in L'$ such that $\phi(h_0) = [u, h_0]$. Put $\phi' = \phi - \text{ad}_u$. Then $\phi'(h_0) = 0$. By Proposition 4.1, we have $\phi' |_{H_L} = 0$. Propositions 4.3 and 4.4 together with Lemma 3.3 implies that $\phi' \in \text{Der}(L)$. Therefore, $\phi \in \text{Der}(L)$.

5. **Applications**

In this section, we will characterize the 2-local superderivation of Cartan type Lie superalgebras. In [17] P. Šemrl introduced the concept of 2-local derivations. Moreover, the author proved that every 2-local derivation on $B(H)$ is a derivation. Similarly, some authors started to describe 2-local derivation. In [12] S. Kim and J. Kim give a short proof of that every 2-local derivation on the algebra $M_n(\mathbb{C})$ is a derivation. A similar description for the finite-dimensional case appeared later in [13]. In the paper [14] 2-local derivations and automorphisms have been described on matrix algebras over finite-dimensional division rings. Later J. Zhang and H. Li [20] extended the above result for arbitrary symmetric digraph matrix algebras and construct an example of 2-local derivation which is not a derivation on the algebra of all upper triangular complex $2 \times 2$ matrices. In [7] Fošner introduced the concept of 2-local superderivations on the associative superalgebra and the authors proved that every 2-local superderivation on superalgebra $M_n(\mathbb{C})$ is a superderivation. In 2017, Chen, Wang and Nan mainly studied 2-local superderivations on basic classical Lie Superalgebras, the authors proved that every 2-local superderivations on basic classical Lie superalgebras except for $A(1, 1)$ over the complex number field $\mathbb{C}$ is a superderivation [18].

Using the results on local superderivations we have the following theorem.

**Theorem 5.1.** Let $L$ be a Cartan type Lie superalgebra. Then every 2-local superderivation of $L$ is a superderivation.

**Proof.** By the definition of 2-local superderivation, we know that every 2-local superderivation of $L$ is a local superderivation. Let $\phi$ is an 2-local superderivation of $L$. Then $\phi \in \text{LDer}(L)$. By Theorem 4.5 we have $\phi \in \text{Der}(L)$.
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