SPECTRAL RIGIDITY AND SUBGROUPS OF FREE GROUPS

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Abstract. A subset \( \Sigma \subseteq F_N \) of the free group of rank \( N \) is called spectrally rigid if whenever trees \( T, T' \) in Culler-Vogtmann Outer Space are such that \( \| g \|_T = \| g \|_{T'} \) for every \( g \in \Sigma \), it follows that \( T = T' \). Results of Smillie, Vogtmann, Cohen, Lustig, and Steiner prove that (for \( N \geq 2 \)) no finite subset of \( F_N \) is spectrally rigid in \( F_N \). We prove that if \( \{ H_i \}_{i=1}^k \) is a finite collection of subgroups, each of infinite index, and \( g_i \in F_N \), then \( \bigcup_{i=1}^k g_i H_i \) is not spectrally rigid in \( F_N \). Taking \( H_i = 1 \), we recover the results about finite sets. We also prove that any coset of a nontrivial normal subgroup \( H \trianglelefteq F_N \) is spectrally rigid.

1. Introduction

The Culler-Vogtmann Outer Space (denoted \( \text{cv}_N \)) was introduced by Culler and Vogtmann [9] as a free group analog of Teichmuller Space. While the latter admits an action by the Teichmuller Modular Group, Outer Space admits an action by \( \text{Out}(F_N) \), the outer automorphism group of \( F_N \). The space \( \text{cv}_N \) has proven to be a key tool in studying \( \text{Out}(F_N) \).

Formally, \( \text{cv}_N \) is the space of all free, minimal, discrete, isometric actions of \( F_N \) on \( \mathbb{R} \)-trees, \( T \), considered up to \( F_N \)-equivariant isometry. One may think of a point in the space as a triple \( (\Gamma, \tau, l) \) where \( \Gamma \) is a finite graph equipped with both a marking isomorphism, \( \tau : F_N \to \pi_1(\Gamma) \), and a function \( l : E(\Gamma) \to \mathbb{R}_{>0} \) which assigns a length to each edge. Given such a triple, one sets \( T = \tilde{\Gamma} \) so that \( T \) is equipped with a covering space action of \( F_N \cong \pi_1(\Gamma) \) which is free, minimal, discrete, and isometric. One obtains \( (\Gamma, \tau, l) \) from \( T \) by considering the quotient graph \( \Gamma = F_N \setminus T \). To each \( T \) we associate a length function \( \| \cdot \|_T : F_N \to \mathbb{R}_{\geq 0} \) defined by \( \| g \|_T = \inf_{x \in T} \{ d(x, g x) \} \). Equivalently, \( \| g \|_T \) is the length of the cyclically reduced loop (in \( \Gamma \)) which represents the conjugacy class of \( \tau(g) \). The closure, \( \text{cv}_N \), of \( \text{cv}_N \) is known to consist of the so called very small actions of \( F_N \) on \( \mathbb{R} \)-trees [1, 8]. The fact that \( \| \cdot \|_T \) determines \( T \) [6] gives an injection \( \ell : \text{cv}_N \to (\mathbb{R}_{\geq 0})^{F_N} \); we call \( \ell(T) \) the marked length spectrum of \( T \). Let \( \ell = \ell|_{\text{cv}_N} \) and note that \( \ell \) is also injective. Morally speaking, a subset \( \Sigma \subseteq F_N \) is spectrally rigid (resp. strongly spectrally rigid) if one can replace the target of \( \ell \) (resp. \( \ell \)) by \( (\mathbb{R}_{\geq 0})^2 \) and still retain injectivity. The term was recently coined (in the free group setting) by Kapovich [15], though similarly spirited work dates back to the early 1980’s (see [7, 24]).

Definition 1.1 (Strongly Spectrally rigid). We say \( \Sigma \subseteq F_N \) is (strongly) spectrally rigid if whenever \( T_1, T_2 \in \text{cv}_N \) (resp. \( \text{cv}_N \)) are such that \( \| g \|_{T_1} = \| g \|_{T_2} \) for every \( g \in \Sigma \), then \( T_1 = T_2 \) in \( \text{cv}_N \) (resp \( \text{cv}_N \)).

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For \( N \geq 3 \) Smillie and Vogtmann [24] prove that no finite set is spectrally rigid in \( F_N \). More specifically, they show that given a finite set, \( \Sigma \), of conjugacy classes there exists a one parameter family of trees, \( T_t \in \mathcal{CV}_N \), whose translation length functions all agree on \( \Sigma \). For \( N = 2 \), Cohen, Lustig and Steiner [7] provide a similar argument as to why no finite set is spectrally rigid in \( F_N \). We prove the following.

**Theorem A.** Let \( N \geq 2 \). Let \( \{H_i\}_{i=1}^k \) be a finite collection of finitely generated subgroups \( H_i < F_N \). Let \( \{g_i\}_{i=1}^k \) be a finite collection of elements \( g_i \in F_N \). Let \( \mathcal{H} = \bigcup_{i=1}^k g_iH_i \). Then the following are equivalent.

1. For every \( i \), \( [F_N : H_i] = \infty \).
2. \( \mathcal{H} \subset F_N \) is not spectrally rigid in \( F_N \).

In the above theorem, we recover the results of Smillie, Vogtmann, Cohen, Lustig, and Steiner by setting \( H_i = \{1\} \). The proof by Smillie and Vogtmann uses the fact that in a finite set of conjugacy classes, there is a universal bound on the exponent with which a given primitive element may occur. Their argument, however, works for any set \( \Sigma \) which satisfies the following property: there exists a triple \((\mathcal{A}, a, M)\) with \( \mathcal{A} \) a free basis, \( a \in \mathcal{A} \), and \( M \geq 1 \), so that for any \( g \in \Sigma \), if \( a^k \) occurs as a subword of the cyclically reduced form (over \( \mathcal{A} \)) of \( g \), then \( |k| \leq M \) (see Definition 2.1 and Theorem 2.2). To prove Theorem A, we use *laminations associated to a fully irreducible automorphism*. Introduced by Bestvina, Feighn, and Handel in [2] these laminations are defined – for a given fully irreducible \( \varphi \in \text{Out}(F_N) \) – in terms of a train track representative \( f: \Gamma \to \Gamma \). Leaves of the lamination are “generated” by edgepaths of the form \( f^n(e) \) for any \( e \in E(\Gamma) \) and \( n \in \mathbb{N} \) (see [2, 12]). It is a theorem of Bestvina, Feighn, and Handel [2] that only finitely generated subgroups of finite index may “carry a leaf” of such a lamination. It follows that, for finitely generated infinite index subgroups, edgepaths of the form \( f^n(e) \) (for suitably large \( n \)) can not be “read” in \( H \). Thus if a primitive loop contains such an \( f^n(e) \), then it cannot be a subword of any \( h \in H \) (compare Definition 2.1).

Theorem A fails if we allow infinite unions. Indeed, consider \( \mathcal{H} = \bigcup_{g \in F_N} g\{1\} \). Then \( \mathcal{H} = F_N \) and so is spectrally rigid. Since translation length functions satisfy \( \|g^n\| = n\|g\| \), any finite index subgroup is spectrally rigid. More generally, if \( \pi: F_N \to G \) is a presentation homomorphism \( (G \cong F_N/\ker \pi) \) for a torsion group, \( G \), then \( \ker \pi \) is spectrally rigid. Note that this applies to (normal) subgroups of infinite index (in the case \( G \) is infinite), and these subgroups are necessarily infinitely generated. Thus the finite generation assumption in Theorem A is also essential.

Carette, Francaviglia, Kapovich, and Martino prove [10] that if \( H < \text{Aut}(F_N) \) is such that the image of \( H \) in \( \text{Out}(F_N) \) contains an infinite normal subgroup, then for any \( g \in F_N \) (so long as \( N \neq 2 \) or \( g \) not conjugate to a power of \([a, b]\) in \( F_2 = F(a, b) \)) the orbit \( Hg \) is spectrally rigid. This shows, for example, that the commutator subgroup, \([F_N, F_N]\), of \( F_N \) is spectrally rigid, as it contains the orbit \( \text{Aut}(F_N)[g, h] \) for a suitable commutator \([g, h]\). More generally, any nontrivial verbal or marginal subgroup is spectrally rigid since such groups are known to be characteristic [23, §2.3].

Motivated by the above result regarding characteristic subgroups, Ilya Kapovich asked the following question: Is it true that for any nontrivial normal subgroup \( H < F_N \), \( H \) is a spectrally rigid set? We prove the following.
Theorem B. Let $H \triangleleft F_N$ be a nontrivial normal subgroup. Then for any $g \in F_N$, the coset $gH$ is strongly spectrally rigid.

The proof uses geodesic currents on free groups (see [14, 17, 18, 20]). These are positive Radon measures on $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$ (where $\Delta$ is the diagonal) which are invariant under the diagonal action of $F_N$ and under the map which exchanges the coordinates. The space of all such currents is denoted $\text{Curr}(F_N)$. The spaces $\text{Curr}(F_N)$ and $\overline{\text{Curr}}_N$ are related by an intersection form $[17]$, $(\cdot, \cdot) : \overline{\text{Curr}}_N \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$. The main feature of the intersection form that we use is as follows: given a conjugacy class $g \in F_N$, there is a canonically associated current $\eta_g \in \text{Curr}(F_N)$ such that for any $T \in \overline{\text{Curr}}_N$, we have $(T, \eta_g) = \|g\|_T$. Now given any nontrivial $g, r \in F_N$, we show that the normal closure, ncl$(r)$, of $r$ contains a sequence of words $w_n$ such that large powers of $g$ exhaust $w_n$. Linearity of the intersection form then reveals that the associated counting currents converge to the counting current of $g$, and that if two trees $T, T'$ agree on ncl$(r)$ (and hence on the $w_n$), then they must agree on $g$. Since $g$ is arbitrary, we conclude $T = T'$. Our result then follows from the fact that any nontrivial normal subgroup contains ncl$(r)$ for some nontrivial $r$.

2. Preliminaries

The $N$-Rose, $R_N$, is the graph with one vertex and $N$ topological edges. If $p$ is an edgepath in a graph, $G$, we let $[p]$ denote the reduced edgepath homotopic (rel. endpoints) to $p$. If $p$ is a closed path, we let $[[p]]$ denote the cyclically reduced edgepath freely homotopic to $p$. If $g \in F_N$ and $A$ is a free basis, then $[g]_A$ (resp. $[[g]]_A$) denotes the freely reduced (resp. cyclically reduced) form of $g$ over the basis $A$. If $v, w$ are edgepaths in $G$, we write $v \equiv w$ to indicate that $v$ is a subpath of $w$. If $g, h \in F_N$ and a free basis, say $A$, is specified, then $g \equiv h$ means the edgepath $g$ occurs as a subpath of the edgepath $h$ in the graph $G = R_A$, the rose with petals labelled by the elements of $A$.

2.1. Property $W$. As mentioned in the introduction, the proof by Smillie and Vogtmann [24] that no finite set is spectrally rigid in $F_N$ (for $N \geq 3$) admits a generalization. Specifically, their argument proceeds verbatim so long as the subset $\Sigma \subset F_N$ satisfies the following property.

Definition 2.1 (Property $W$). Let $\Sigma \subset F_N$. We say $\Sigma$ has property $W$ if there exist a free basis $A$ of $F_N$, $a \in A$, and $M \geq 1$ so that for any $\sigma \in \Sigma$, if $a^k \equiv [[\sigma]]_A$ then $|k| \leq M$.

In [21] it is verified that the argument provided by Cohen, Lustig, and Steiner [7] (for $N = 2$) also works under the assumption of property $W$.

Theorem 2.2 (CLS, R, SV[7, 21, 24]). For $N \geq 2$, if a subset $\Sigma \subset F_N$ has Property $W$, then $\Sigma$ is not spectrally rigid.

2.2. Stallings Subgroup Graphs. Given a point $T = (\Gamma, \tau, l) \in \text{ev}_N$, and a finitely generated subgroup $H \triangleleft F_N$, we consider the (possibly infinite) cover of $\Gamma$ corresponding to the subgroup $\tau(H)$, denoted $(X^T_H)_\tau$. Its core, denoted $X^T_H$, is finite (since $H$ is finitely generated) and represents the conjugacy class of $\tau(H)$ in $\tau_1(\Gamma)$. Note that $X^T_H = (X^T_H)_\tau$ (i.e. $X^T_H$ is $E(\Gamma)$-regular) if and only if $[F_N : H] < \infty$ if and only if $X^T_H \to \Gamma$ is a covering. If $H$ is of infinite index, then one obtains $(X^T_H)_\tau$ from
2.3. Train Track Maps. By a graph map, we mean a function \( f : V(\Gamma) \cup E(\Gamma) \to V(\Gamma') \cup P(\Gamma') \) sending vertices to vertices and edges to edgpaths and for which the notions of incidence and inverse are preserved. If \( A \) is a free basis of \( F_N \) and \( \varphi \in \text{Out}(F_N) \), then we can always think of \( \varphi \) as a graph map \( R_N \to R_N \) by pairing \( E(R_N) \) with \( A \). By a marked graph we mean a pair \((\Gamma, \tau)\), where \( \Gamma \) is a graph and \( \tau : R_N \to \Gamma \) is a homotopy equivalence. In the event \( \Gamma = R_N \), we write \( R_N = R_A \), with \( A \) the free basis determined by the pairing \( E(R_N) \) with \( A \). Given \( \varphi \in \text{Out}(F_N) \), and a marked graph \((\Gamma, \tau)\) we say that a graph map \( f : \Gamma \to \Gamma \) is a topological representative of \( \varphi \) with respect to \( \tau \) if \( f \) is a homotopy equivalence and the outer automorphism determined by \( \tau \circ f \circ \tau^{-1} : R_N \to R_N \) is equal to \( \varphi : R_N \to R_N \).

**Definition 2.3** (Train track map). A graph map \( f : \Gamma \to \Gamma \) is called a train track map if for every \( e \in E(\Gamma) \), the edgpath \( f^n(e) \) is not a vertex and is reduced.

We say that \( \varphi \in \text{Out}(F_N) \) is reducible if there is a free factorization \( F_N = F^0 \star \cdots \star F^{l-1} \star H \), with \( l \geq 1 \) and \( 1 \leq \text{rk}(F^0) < N \), so that \( \varphi \) permutes the conjugacy classes of the \( F^i \); we allow \( H = 1 \). Otherwise, we say \( \varphi \) is irreducible.

**Theorem 2.4** ([Bestvina and Handel [3, Theorem 1.7]]). Every irreducible automorphism is topologically represented by a train track map.

If all positive powers of \( \varphi \) are irreducible, then we say that \( \varphi \) is irreducible with irreducible powers (iwip for short). Thus \( \varphi \) is fully irreducible if and only if for all \( k \geq 1 \), \( \varphi^k \) does not preserve the conjugacy class of a proper free factor. In what follows, we use train track maps \( f : \Gamma \to \Gamma \) representing iwip automorphisms in \( \text{Out}(F_N) \). The existence of such automorphisms – for every rank \( N \geq 2 \) – is well known (see, for example, [11, 22]).

2.4. Bestvina-Feighn-Handel Laminations. Fix a free basis \( A \) of \( F_N \). Let \( \partial F_N \) be the Gromov boundary of the word hyperbolic group, \( F_N \); that is, \( \partial F_N := \{ a_1 a_2 \cdots | a_i \in A^\pm, a_i^{-1} \neq a_{i+1} \} \). The double boundary of \( F_N \), denoted by \( \partial^2 F_N \), is defined by \( \partial^2 F_N := (\partial F_N \times \partial F_N) \setminus \Delta \), where \( \Delta \subset \partial F_N \times \partial F_N \) consists of those pairs \((\zeta_1, \zeta_2)\) for which \( \zeta_1 = \zeta_2 \). More generally, given a marked graph \((\Gamma, \tau)\), one defines \( \partial^2 \Gamma \) by considering instead one sided infinite reduced edgpaths in \( \Gamma \).

**Definition 2.5** (Bestvina-Feighn-Handel lamination). Let \( \varphi \in \text{Out}(F_N) \) be a fully irreducible automorphism equipped with a train track map \( f : \Gamma \to \Gamma \). The Bestvina-Feighn-Handel lamination associated to \( \varphi \in \text{Out}(F_N) \), denoted by \( L_{BFH}(\varphi, f, \Gamma) \), is the set of pairs \((\zeta_1, \zeta_2) \in \partial^2 \Gamma \) which have the following property: for every finite subpath \([z_1, z_2] \subset (\zeta_1, \zeta_2)\) there exists an \( e \in E(\Gamma) \) and an \( n \geq 1 \) so that \( f^n(e) \supseteq \pi([z_1, z_2]) \). Here \( \pi : \partial^2 \Gamma \to \Gamma \) is the “labelling” map which, on each coordinate, coincides with projection from the universal covering, \( \pi : \tilde{\Gamma} \to \Gamma \). Such a pair \((\zeta_1, \zeta_2)\) is called a leaf of the lamination.
If \( f : \Gamma \to \Gamma \) is a train track representing an iwip, then given any pair of edges \( e, e' \in E(\Gamma) \), there exists an \( n \geq 1 \) so that either \( f^n(e) \equiv e' \) or \( f^n(e^{-1}) \equiv e' \). In what follows, we will always assume that we have passed to a power of \( \varphi \) so that this is indeed the case. Definition 2.5 can be formulated (equivalently) in terms of a fixed edge \( e \in E(\Gamma) \). Furthermore, the leaves of the Bestvina-Feighn-Handel lamination satisfy a certain “recurrence” property, which we now formulate precisely.

**Definition 2.6** (Quasiperiodicity). A leaf \((\zeta_1, \zeta_2)\) of \(\mathcal{L}_{BFH}(\varphi, f, \Gamma)\) is said to be quasiperiodic if for every \( L > 0 \), there exists \( L' > L \) so that the following holds: if \([z_1, z_2]\), and \([w_1, w_2]\) are subpaths of \((\zeta_1, \zeta_2)\) for which \(|[z_1, z_2]| > L'\) and \(|[w_1, w_2]| < L\), then \(\pi([w_1, w_2]) \in \pi([z_1, z_2])\) (here \(\pi : \Gamma \to \Gamma\) is the labeling map).

**Proposition 2.7** [Bestvina, Feighn, and Handel [2, Proposition 1.8]]. Every leaf \((\zeta_1, \zeta_2)\) of \(\mathcal{L}_{BFH}(\varphi, f, \Gamma)\) is quasiperiodic.

**Definition 2.8** (Carrying a leaf). Let \( H \leq F_N \) be a finitely generated subgroup. Let \( \varphi \in \text{Out}(F_N) \) be a fully irreducible automorphism equipped with a train track map \( f : \Gamma \to \Gamma \) (the marking on \( \Gamma \) being \( \tau : R_N \to \Gamma \)). Let \( T = (\Gamma, \tau, l) \). Let \( X_H^T \) be the Stallings subgroup graph corresponding to the (conjugacy class of the) subgroup \( \tau(H) \leq \pi_1(\Gamma, \tau(\ast)) \). We say that (the conjugacy class of) \( H \) carries the leaf \((\zeta_1, \zeta_2)\) of \(\mathcal{L}_{BFH}(\Phi, f, \Gamma)\) if for every finite subpath \([z_1, z_2]\) of \((\zeta_1, \zeta_2)\), the map \(\pi([z_1, z_2]) : [z_1, z_2] \to \Gamma\) factors through \(X_H^T\) as a map \([z_1, z_2] \to X_H^T \to \Gamma\).

Of particular interest to us is the following proposition.

**Proposition 2.9** [Bestvina, Feighn, and Handel [2, Lemma 2.4]]. Let \( \varphi \in \text{Out}(F_N) \) be a fully irreducible automorphism equipped with a train track \( f : \Gamma \to \Gamma \). If a finitely generated subgroup \( H \leq F_N \) carries a leaf of \(\mathcal{L}_{BFH}(\varphi, f, \Gamma)\), then \( [F_N : H] < \infty \).

2.5. **Stability of Quasi-Geodesics.** We will also need some basic results about stability of quasi-geodesics.

**Proposition 2.10** [Bridson and Haefliger [5, III.H Theorem 1.7]]. For all \( \delta > 0, \lambda \geq 1, \epsilon \geq 0 \) there exists a constant \( R = R(\delta, \lambda, \epsilon) \) with the following property: If \( X \) is a \( \delta \)-hyperbolic geodesic space, \( e \) is a \((\lambda, \epsilon)\)-quasi-geodesic in \( X \) and \([p, q]\) is a geodesic segment joining the endpoints of \( e \), then the Hausdorff distance between \([p, q]\) and the image of \( e \) is less than \( R \).

**Proposition 2.11.** Let \( X, Y \) be \( \delta \)-hyperbolic geodesic metric spaces equipped with a \((\lambda, \epsilon)\)-quasi-isometry \( f : X \to Y \). Suppose \( A, B \) are collections of geodesic paths in \( X \) with the following property, there exists a constant \( C \) such that for any \( \alpha \in A \) and \( \beta \in B \), \( \text{diam}_X(\alpha \cap \beta) < C \). Let \( \alpha' = [f(\alpha)], \beta' = [f(\beta)] \) be geodesics in \( Y \) obtained by reducing the images \( f(\alpha), f(\beta) \), respectively. The there exists a constant \( C' \) such that for all \( \alpha \in A \), \( \beta \in B \), we have \( \text{diam}_Y(\alpha' \cap \beta') < C' \).

**Proof.** Let \( R = R(0, \lambda, \epsilon) \) be the constant afforded by Proposition 2.10. Let \( \{\alpha_i\}, \{\beta_i\} \subset X \) and \( \{\alpha'_i\}, \{\beta'_i\} \subset Y \) be arbitrary sequences of geodesics (with \( \alpha'_i, \beta'_i \) contained in the \( R \)-neighborhood of \( f(\alpha_i), f(\beta_i) \), respectively). Suppose without loss that \( d_X(\alpha_i, \alpha) \to \infty \) (the proposition is obvious otherwise). Since \( R \) is finite and since \( f \) is a quasi-isometry, we must have that \( d_X(\alpha_i, \alpha) \to \infty \). Similarly, the distance (in \( Y \)) between \( f(\alpha_i \cap \beta_i) \) and \( \alpha'_i \cap \beta'_i \) is uniformly bounded.
Let \( d = \text{diam}_Y(\alpha_i \cap \beta_i) \to \infty \). Then (by passing to a subsequence and reindexing) we may find a sequence of points \( a_i \in \alpha_i, b_i \in \beta_i \) with \( d_X(a_i, \alpha_i \cap \beta_i), d_X(b_i, \alpha_i \cap \beta_i) \to \infty \) for which \( d_Y(f(a_i), f(b_i)) \) is uniformly bounded. This contradicts the fact that \( f \) is a quasi-isometry.

**Corollary 2.12.** Let \((\Gamma, \tau)\) be a marked graph. Suppose \( z \) is a loop in \( G \) which represents a primitive element of \( \pi_1(\Gamma) \). Let \( A \) be an arbitrary collection of loops in \( G \). Suppose there exists a number \( M \) such that for all \( a \in A \) whenever \( z^k \equiv a \) it follows that \( |k| \leq M \). Then for any free basis \( B \) with \( z \in B \) and a number \( M' \) such that for all \( a \in A \) whenever \( z^k \equiv [a]_B \) it follows that \( |k| \leq M' \).

**Proof.** Let \( Z = \{z^k \mid k \in \mathbb{Z}\} \). Let \( f : \tilde{\Gamma} \to \text{Cay}(F_N, B) \) be a quasi-isometry with \( B \) any free basis containing \( z \). We may think of \( Z, A \) as defining a collection of geodesics in \( \tilde{\Gamma} \). The assumption on \( z \) affords a constant, \( C \), such that for any \( \zeta \in Z \) and any \( \alpha \in A \), we have \( \text{diam}_z(\zeta \cap \alpha) < C \). We now apply Proposition 2.11 and conclude that (in \( \text{Cay}(F_N, B) \)) there is a bound, \( C' \) on \( \text{diam}_{\text{Cay}(F_N, B)}(\zeta', \alpha') \). Thus there is a bound \( M' \) so that if \( z^k \equiv [a]_B \), then \( |k| \leq M' \).

**2.6. Geodesic Currents.** A geodesic current on \( F_N \) is a positive Radon measure (a Borel measure that is finite on compact sets) on \( \partial^2 F_N \) which is invariant under the diagonal action of \( F_N \) and under the map which interchanges the coordinates of \( \partial^2 F_N \). The space of all currents is denoted \( \text{Curr}(F_N) \). If \( \nu, \mu \in \text{Curr}(F_N) \) are (nontrivial) currents such that \( \nu = \lambda \mu \) for some \( \lambda \in \mathbb{R}^\times \) then we write \( [\nu] = [\mu] \), where \([\cdot]\) denotes the projective class of a given current. To each root free conjugacy class \( g \in F_N \) we associate a counting current, \( \eta_g \) as follows. Let \( A \) be a free basis of \( F_N \). Write \( g \) as a cyclically reduced word over \( A \). We can thus think of \( g \) as the label of a directed graph, \( \Gamma_g \), which is topologically homeomorphic to \( S^1 \) and has \( \|g\|_A \) edges (and hence \( \|g\|_A \) vertices), each edge labelled by the appropriate \( a_i \in A \). Now let \( v \) be any freely reduced word. By the number of occurrences of \( v \) or \( v^{-1} \) in \( g \), denoted \( n_g(v) \), we mean the number of vertices of \( \Gamma_g \) at which one may read the word \( v \) or \( v^{-1} \) along \( \Gamma_g \) (going ‘with’ the oriented edges) without leaving \( \Gamma_g \). Note that \( n_{a_i}(v^\pm) = n_{a_i^{-1}}(v^\mp) \) by definition. Now let \( \tilde{v} \) be a lift of \( v \) to the Cayley tree of \( F_N \) with respect to \( A \). Let \( \text{Cyl}(v) \subset \partial^2 F_N \) be the set of all \((\zeta_1, \zeta_2) \subset \partial^2 F_N \) such that the bi-infinite geodesic representing \((\zeta_1, \zeta_2) \) passes through \( \tilde{v} \). Then \( \eta_g(\text{Cyl}(v)) = n_g(v^\pm) \). If \( g = h^k \) for \( h \) a root free conjugacy class, we define \( n_g(v^\pm) = k n_h(v^\pm) \). Currents of the form \( \lambda \eta_g \) for \( \lambda > 0 \) and \( g \in F_N \) form a dense subset of \( \text{Curr}(F_N) \) \([13, \text{Corollary 3.5}]\). For more information, see \([14, 17, 18, 20]\). We will need the notion of an intersection form \((\cdot, \cdot) : \text{Cov}_N \times \text{Curr}(F_N) \to \mathbb{R} \).

**Proposition 2.13** (Kapovich and Lustig \([17, \text{Theorem A}]\)). There exists a map \((\cdot, \cdot) : \text{Cov}_N \times \text{Curr}(F_N) \to \mathbb{R} \) which is continuous, \( \text{Out}(F_N) \)-invariant, and linear with respect to the second argument. Furthermore for every \( g \in F_N \), and \( T \in \text{Cov}_N \) we have \((T, \eta_g) = \|g\|_T \).

3. PROOF OF THEOREM A

In what follows, whenever we speak of an iwip \( \varphi \in \text{Out}(F_N) \) equipped with train track map \( f : \Gamma \to \Gamma \) we assume that we have passed to a suitable power of \( \varphi \) so that for any topological edge \( e \) in \( \Gamma \), \( f(e) \) crosses each topological edge in \( \Gamma \). We will, however, continue to write \( \varphi, f \).
Proposition 3.1. Let $H \leq F_N$ be a finitely generated subgroup of infinite index. Let $\varphi \in \text{Out}(F_N)$ be fully irreducible with train track map $f : \Gamma \to \Gamma$ on the marked graph $(\Gamma, \tau)$. Let $T = (\Gamma, \tau, l)$ in $\text{cv}_N$. Let $X_H^T$ be the Stallings subgroup graph corresponding to the conjugacy class of $\tau(H)$ in $\pi_1(\Gamma)$. Then there exists a power $m$, such that for all $n \geq m$, and for any $e \in E(\Gamma)$, the edgepath $f^n(e)$ does not factor through $X_H^T$. Furthermore, given a basepoint $b$ in $(X_H^T)_e$, we may choose $m$ so that for all $n \geq m$, $f^n(e)$ does not factor through $(X_H^T)_b$.

Proof. By Proposition 2.9, $H$ cannot carry a leaf of $L_{BFH}(\varphi, f, \Gamma)$. Thus there is a finite subpath $[z_1, z_2]$ (of a leaf $(\zeta_1, \zeta_2) \subset \partial^2 \Gamma$) which does not factor through $X_H^T$. Let $L = [[z_1, z_2]]$. Since leaves are quasiperiodic, there exists $L'$ so that for any subpath $[w_1, w_2]$ of length at least $L'$, we have $[w_1, w_2] \not\equiv [z_1, z_2]$. Choose a basepoint $v \in \Gamma$ and let $b$ be the basepoint in $(X_H^T)_e$ so that the image of $\pi_1((X_H^T)_b)$ in $\pi_1(\Gamma, v)$ is equal to $\tau(H)$. Let $l$ be the length of the bridge $[X_H^T, b]$. Set $L'' = L' + 2l + 2 \text{diam}(X_H^T) + 1$. Now let $m \geq 1$ be so that the length of $f^n(e)$ is at least $L''$ for any $e \in E(\Gamma)$. Then for any $n \geq m$, $f^n(e) \equiv [z_1, z_2]$ for any $e \in E(\Gamma)$. Thus for any $n \geq m$, and for any $e \in E(\Gamma)$, we have that $f^n(e)$ does not factor through $X_H^T$, nor can it factor through $(X_H^T)_b$ since at least $2l + 1$ edges in $f^n(e)$ must lie outside of $X_H^T$.

Corollary 3.2. Let $\mathcal{H} = \cup H_i$ be a finite union of subgroups $H_i < F_N$ of infinite index. Then there exists a power $m$ such that for all $n \geq m$ and for any $e \in E(\Gamma)$ and for all $i$, the edgepath $f^n(e)$ does not factor through $(X_{H_i}^T)_b$. Furthermore, there exists a primitive element $z \in \pi_1(\Gamma)$ so that $z$ contains $f^n(e)$ and hence $z$ does not factor through $(X_{H_i}^T)_b$ for any $i$.

Proof. Fix a fully irreducible $\varphi \in \text{Out}(F_N)$ equipped with train track map $f : \Gamma \to \Gamma$. Fix an edge $e \in E(\Gamma)$. Now apply Proposition 3.1 to each $H_i$ and obtain a collection $\{m_i\}_{i=1}^k$ of exponents for which the following holds for each $i$: for all $n \geq m_i$, the edgepath $f^n(e)$ does not factor through $(X_{H_i}^T)_b$. Set $m = \max_i \{m_i\}$.

Then for all $n \geq m$ and for all $i$ we have that $f^n(e)$ does not factor through $(X_{H_i}^T)_b$.

Now let $a$ be any primitive loop in $\pi_1(\Gamma)$. Since $\varphi$ is fully irreducible, $z$ cannot be a periodic conjugacy class, and hence $l_\Gamma([\{f^k(a)\}]) \to \infty$. Thus we can find an edge $e'$ in suitable iterate, $f^k(a)$, of $a$ so that in all further iterates of $a$ by $f$ there is no cancellation in the images of $e'$. Thus for all $n \geq k + m + 1$, we have that $f^n(e)$ occurs in $f^n(a)$ and hence $f^n(a)$ cannot factor through $(X_{H_i}^T)_b$ for any $i$. Set $z = f^n(a)$ for some $n \geq k + m + 1$.

Proposition 3.3. Let $\mathcal{H} = \cup H_i$ be a finite union of subgroups $H_i < F_N$ of infinite index. Then there exists a primitive element $z$ and a free basis $B$ with $z \in B$ and a number $M$ so that for all $i$ and for all $h \in H_i$, if $z^k \in [h]_B$, then $|k| \leq M$.

Proof. Let $z$ be the primitive element afforded by Corollary 3.2. Let $Z = \{z^k \mid k \in \mathbb{Z}\}$. Let $B = \{\tau(h) \mid h \in H\}$. The previous proposition gives the following: there exists a constant $C$ such that for any $\zeta \in Z$ and for any $\beta \in B$ and for any choice of lifts $\tilde{\zeta}, \tilde{\beta}$ in $\tilde{\Gamma}$, the intersection $\text{diam}(\tilde{\zeta} \cap \tilde{\beta}) < C$. Let $f : \tilde{\Gamma} \to \text{Cay}(F_N, B)$ be a quasi-isometry to the Cayley graph of $F_N$ with respect to a basis $B$ with $z \in B$.

Now apply Proposition 2.10 using $Z, B$ and $f$. We conclude there exists a bound $M$ so that for any $h \in H$, if $z^k \in [h]_B$, then $|k| \leq M$. 

□
**Theorem 3.4.** Let $\mathcal{H} = \cup g_i H_i$ be a finite collection of cosets of finitely generated subgroups of infinite index, $H_i \leq F_N$. Then the set $\mathcal{H}$ has property $\mathcal{W}$.

**Proof.** Let $z, \mathcal{B}$ be the primitive element and free basis afforded by Proposition 3.3. Thus there exists an $M'$ such that for all $i$ and all $h \in H_i$, if $z^k \in [h_i]_g$, then $|k| \leq M'$. Since $\{g_i\}_{i=1}^n$ is finite, there is $M_g$ so that for all $i$, if $z^k \in [g_i]_g$, then $|k| \leq M_g$. Thus there is an $M$ so that for any $i$ and any $h_i \in H_i$, if $z^k \in ([g_i]_g)_g$, then $|k| \leq M$. Thus $H$ has property $\mathcal{W}$ with respect to $(z, \mathcal{B}, M)$. $\square$

Since $\|g^n\| = n\|g\|$ it is clear that any (coset of a) subgroup of finite index is spectrally rigid. We thus have the following.

**Theorem A.** Let $N \geq 2$. Let $\{H_i\}_{i=1}^k$ be a finite collection of finitely generated subgroups $H_i < F_N$. Let $\{g_i\}_{i=1}^k$ be a finite collection of elements $g_i \in F_N$. Let $\mathcal{H} = \cup_{i=1}^k g_i H_i$. Then the following are equivalent.

1. For every $i$, $[F_N : H_i] = \infty$.
2. $\mathcal{H} \subset F_N$ satisfies property $\mathcal{W}$.
3. $\mathcal{H} \subset F_N$ is not spectrally rigid in $F_N$.

**Proof.** Theorem 3.4 gives (1) implies (2). Theorem 2.2 gives (2) implies (3). Finally, if at least one subgroup is of finite index (i.e. not (1)), then $\mathcal{H}$ is spectrally rigid (i.e. not (3)). $\square$

4. PROOF OF THEOREM B

Recall that the space, $\text{Curr}(F_N)$, of geodesic currents consists of positive radon measures on $\partial^2 F_N$ which are $F_N$ and flip invariant. Also recall that $\text{Curr}(F_N)$ and $\mathcal{W}_N$ are related by an intersection form $\langle \cdot, \cdot \rangle$ for which $\langle \eta_g, T \rangle = \|g\|_T$ (see §2.6 and Proposition 2.13).

**Proposition 4.1** (Kapovich and Lustig [16, Lemma 2.10]). Let $A$ be a free basis of $F_N$. Then for any cyclically reduced words $w_n, w \in F_N$, we have

$$\lim_{n \to \infty} \frac{\eta_{w_n}}{\|w_n\|_A} = \frac{\eta_w}{\|w\|_A}$$

if and only if

**Proposition 4.2.** For any nontrivial $r \in F_N$, the set $\text{ncl}(r)$ is strongly spectrally rigid.

**Proof.** Let $u \in F_N$ be an arbitrary cyclically reduced word. We will show that $\|u\|_T = \|u\|_{r^\prime}$ from which it will follow that $T = T'$ (since $\| \cdot \|$ is constant on conjugacy classes). To that end, we first construct a sequence of cyclically reduced words $\{w_i\} \subset \text{ncl}(r)$ with the property that

$$[\eta_{w_i}] \to [\eta_u]$$

To that end, let

$$w_i = \alpha u^i \beta r \beta^{-1} u^{-i} \alpha^{-1} \gamma u^i \delta r \delta^{-1} u^{-i} \gamma^{-1}$$

Evidently $w_i$ is a product of conjugates of $r$, so is in $\text{ncl}(r)$. It is clear that we may choose $\alpha, \beta, \delta, \gamma$ so that $w_i$ is freely (and so cyclically, by inspection) reduced. Now let $v \in F_N$ be arbitrary. By Proposition 4.1 it is enough to show that

$$\lim_{i \to \infty} \frac{n_{w_i}(v^\pm)}{\|w_i\|_A} = \frac{n_u(v^\pm)}{\|u\|_A}$$
Note that there is a uniform bound on the number of occurrences of $v$ or $v^{-1}$ in $w_i$ which do not occur entirely within the $u^i$ or $u^{-i}$ subwords of $w_i$. Furthermore the difference between $\|w_i\|_A$ and $4\|u^i\|_A$ is uniformly bounded. We now have that

$$\lim_{i \to \infty} \frac{n_{w_i}(v^\pm)}{\|w_i\|_A} = \lim_{i \to \infty} \frac{2n_{u_i}(v^\pm) + 2n_{u_{i-1}}(v^\pm) + D_i}{4i\|u\|_A + C_i} = \lim_{i \to \infty} \frac{2in_{u}(v^\pm) + 2in_{u_{i-1}}(v^\pm) + D_i}{4i\|u\|_A + C_i} = \lim_{i \to \infty} \frac{4in_{u}(v^\pm) + D_i}{4i\|u\|_A + C_i} = \frac{n_{u}(v^\pm)}{\|u\|_A}$$

as desired.

Now suppose that $T, T' \in \mathcal{T}_N$ agree on $\text{ncl}(r)$. Since $[\eta_{w_i}] \to [\eta_u]$ there exists a sequence $\lambda_i$ such that

$$\lim_{i \to \infty} \lambda_i \eta_{w_i} = \eta_u$$

Then we have that

$$\|u\|_T = \langle T, \eta_u \rangle = \langle T, \lim_{i \to \infty} \lambda_i \eta_{w_i} \rangle = \lim_{i \to \infty} \lambda_i \langle T, \eta_{w_i} \rangle = \lim_{i \to \infty} \lambda_i \|w_i\|_T = \lim_{i \to \infty} \lambda_i \|w_i\|_{T'} = \lim_{i \to \infty} \lambda_i \langle T', \eta_{w_i} \rangle = \langle T', \lim_{i \to \infty} \lambda_i \eta_{w_i} \rangle = \langle T', \eta_u \rangle = \|u\|_{T'}$$

Recall that $u$ was an arbitrary conjugacy class. Since length functions are class functions, we conclude that $\| \cdot \|_T = \| \cdot \|_{T'}$ and so $T = T'$ in $\mathcal{T}_N$. \hfill $\Box$

**Corollary 4.3.** Let $r \in F_N$ be nontrivial and consider $g \text{ncl}(r)$. Then $g \text{ncl}(r)$ is strongly spectrally rigid.

**Proof.** In the proof of the above proposition, we may choose $\alpha, \beta, \gamma, \delta$ so that $gw_i$ is cyclically reduced. Then – with perhaps different constants $C_i, D_i$ – we obtain that $[\eta_{gw_i}] \to [\eta_u]$. We conclude as above that $g \text{ncl}(r)$ is strongly spectrally rigid. \hfill $\Box$

**Theorem B.** Let $H \triangleleft F_N$ be a nontrivial normal subgroup. Then for any $g \in F_N$, the coset $gH$ is strongly spectrally rigid.

**Proof.** Since $H$ is nontrivial and normal, $H \supset \text{ncl}(r)$ for some nontrivial $r$. By the above proposition, for any $g \in F_N$, $g \text{ncl}(r)$ is strongly spectrally rigid. Thus so is $gH$. \hfill $\Box$

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