Characterizing singularities of a surface in Lie sphere geometry
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Abstract
The conditions for a cuspidal edge, swallowtail and other fundamental singularities are given in the context of Lie sphere geometry. We then use these conditions to study the Lie sphere transformations of a surface.

1 Introduction

Lie sphere geometry is the study of spheres in spaceforms and their tangential contact. It was first developed by Sophus Lie [19] using his hexaspherical coordinate model. This model has been utilized by Blaschke [3] to study applicable surfaces in Lie sphere geometry, and the recent interest in integrable systems has revived this approach (see, for example, [7, 8, 9, 22, 24]). Spacelike linear Weingarten surfaces with singularities in spaceforms can be characterized in this setting ([4, 5]), and that is one motivation for the present work, in which we investigate relationships between Lie sphere geometry and the singularity theory of surfaces. In particular, we obtain the following:

- In Theorem 3.5 we characterize certain corank 1 singularities arising from non-umbilic points of Legendre immersions in Lie sphere geometry, those singularities being cuspidal edges, swallowtails, cuspidal lips, cuspidal beaks, cuspidal butterflies, and more degenerate singularities of certain types.

- In Theorem 3.8 we characterize certain corank 2 singularities arising from umbilic points of Legendre immersions, those being $D^\pm_4$ singularities.

- We classify these singularities by certain differential properties, and in Theorem 4.1 and Theorem 4.3 we show how Lie sphere transformations preserve these classes.

For simplicity of exposition we shall only consider singularities of surfaces in Euclidean 3-space, however, by employing analogous arguments as those used in this paper one can show that our results hold in any 3-dimensional Riemannian spaceform.

2 Preliminaries on Lie sphere geometry

In this section we will explain the notions and terminologies from Lie sphere geometry that will be used in this paper.

Let $\mathbb{R}^{4,2}$ be a 6-dimensional vector space equipped with the inner product $(\cdot, \cdot) = (- + + + + -)$. Let

$$L^5 = \{X \in \mathbb{R}^{4,2} \mid (X, X) = 0\}$$

denote the lightcone of $\mathbb{R}^{4,2}$.

Choosing nonzero $p \in \mathbb{R}^{4,2}$ and $q \in \mathbb{R}^{4,2}$ with $p \perp q$ and $p$ non-null, let us define the following 3-dimensional quadrics

$$M = \{X \in L^5 \mid (X, p) = 0, (X, q) = -1\} \subset \mathbb{R}^{4,2},$$

$$N = \{X \in L^5 \mid (X, q) = 0, (X, p) = -1\} \subset \mathbb{R}^{4,2}. \quad (2.1)$$
We call $p$ the point sphere complex and $q$ the spaceform vector of $M$. If $p$ is timelike (respectively, spacelike), then $M$ is isometric to a 3-dimensional Riemannian (respectively, Lorentzian) spaceform with constant sectional curvature $\kappa = -(q,q)$. For example, if $(p,p) = -1$ and $q$ is null, then $M$ is isometric to a Euclidean space $\mathbb{R}^3$. Cecil [6, Section 2.3] made this identification explicit via the isometry

$$\xi \in \mathbb{R}^3 \mapsto \frac{1}{2} \left( \frac{1 + \langle \xi, \xi \rangle}{2}, \frac{1 - \langle \xi, \xi \rangle}{2}, \xi_0 \right) \in M,$$

where $\langle \ , \ \rangle$ denotes the inner product of $\mathbb{R}^3$. In this case $M$ is determined by $p = t(0,0,0,0,1)$ and $q = t(1,-1,0,0,0)$. Points in the projective light cone $\mathbb{P}(L^5)$ of $\mathbb{R}^{4,2}$ correspond to spheres in spaceforms in the following way. Let $s \in \mathbb{P}(L^5)$, then we let

$$S := M \cap s^\perp.$$  

If $s \not\perp p$, i.e., $(\sigma,p) \neq 0$ for any $\sigma \in L^5$ such that $s = \text{span}\{\sigma\}$, then the set of points determined by $s$ is a metric sphere or a plane in $M$. Otherwise $S$ is a point in $M$. For example, suppose we are using the identification of $M$ with $\mathbb{R}^3$ in (2.2), where $p = t(0,0,0,0,1)$ and $q = t(1,-1,0,0,0)$. Write $\sigma \in L^5$ such that $s = \text{span}\{\sigma\}$ as $\sigma = t(a,b,\zeta,c)$, where $a,b,c \in \mathbb{R}$ and $\zeta \in \mathbb{R}^3$. Then $\gamma = t\left(\frac{1 + \langle \xi, \xi \rangle}{2}, \frac{1 - \langle \xi, \xi \rangle}{2}, \xi_0 \right) \in S$ if and only if

$$0 = (\gamma, \sigma) = -a \frac{1 + \langle \xi, \xi \rangle}{2} + b \frac{1 - \langle \xi, \xi \rangle}{2} + \langle \zeta, \xi \rangle = \frac{1}{2}(b-a) - \frac{\langle \xi, \xi \rangle}{2}(b+a) + \langle \zeta, \xi \rangle. \quad (2.3)$$

Furthermore, since $s \in \mathbb{P}(L^5)$,

$$0 = -a^2 + b^2 + \langle \zeta, \zeta \rangle - c^2 = (a+b)(b-a) + \langle \zeta, \zeta \rangle - c^2. \quad (2.4)$$

Now if $s \not\perp q$, i.e., $(\sigma,q) = a+b \neq 0$, then we may scale $\sigma$ so that $a+b = 1$. Then (2.3) and (2.4) imply that $\gamma \in S$ if and only if

$$\langle \xi - \zeta, \xi - \zeta \rangle = c^2.$$  

Hence, $S$ is a sphere of radius $|c|$ with center $\zeta$. If $s \perp q$, i.e., $(\sigma,q) = a+b = 0$, then we may scale $\sigma$ so that $c^2 = 1$. Then (2.4) implies that $\zeta \in \mathbb{R}^3$ has unit length and one has that (2.3) is equivalent to

$$\langle \zeta, \xi \rangle = \frac{1}{2}(b-a).$$  

Hence, $S$ defines a plane with unit normal $\zeta$. For a more in depth explanation, see [6, Section 2.3].

Now suppose that $s_1, s_2 \in \mathbb{P}(L^5)$ and let $S_i := M \cap s_i^\perp$ for $i \in \{1,2\}$ denote the corresponding spheres in $M$. Then $S_1$ and $S_2$ are in oriented tangential contact if and only if $s_1 \perp s_2$. The maximal dimension of null subspaces of $\mathbb{R}^{4,2}$ is 2. Let $Z$ denotes the Grassmannian of these null 2-dimensional subspaces. Alternatively, we can think of $Z$ as the space of lines in $\mathbb{P}(L^5)$. Then given $P \in Z$, any two elements $s_1, s_2 \in P$ satisfy $s_1 \perp s_2$ and thus $P$ corresponds to a pencil of spheres in oriented tangential contact in any spaceform $M$. Thus $P$ is referred to as a contact element. We will see that $Z$ is a contact manifold in the next section.
2.1 Legendre immersions

In this section we shall recall from (Section 4.1) how the notion of Legendre immersion from the perspective of Lie sphere geometry coincides with its more well known analogue using the unit tangent bundle $T_1S^3$ of $S^3$. To achieve this, let $p = \ell(0, 0, 0, 0, 0, 1)$ and $q = \ell(1, 0, 0, 0, 0, 0)$. Then we have that both $M$ and $N$ are isometric to $S^3$ by the maps

$$\zeta \in S^3 \mapsto \tilde{\zeta} = \ell(1, \zeta, 0) \in M, \quad \xi \in S^3 \mapsto \tilde{\xi} = \ell(0, \xi, 1) \in N.$$ 

Then a pair $(\zeta, \xi) \in S^3 \times S^3$ takes values in $T_1S^3 = \{((\zeta, \xi) \in S^3 \times S^3 | \langle \zeta, \xi \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product of $R^4$, if and only if $\text{span}\{\tilde{\zeta}, \tilde{\xi}\}$ takes values in $Z$. Thus we obtain a bijective map $C : T_1S^3 \to Z$, and we derive a differentiable structure on $Z$ from $C$. Since $T_1S^3$ has a standard contact structure, we also derive a contact structure on $Z$ by $C$.

Let $\mathcal{F}$ be a map from a 2-dimensional manifold $\Sigma$ to $Z$. Since $\mathcal{F}(x)$ is a 2-dimensional null subspace of $R^{4,2}$ for each $x \in \Sigma$, we can think of $\mathcal{F}$ as a rank 2 null subbundle of the trivial bundle $\Sigma \times R^{4,2}$ over $\Sigma$. Any independent pair of sections $\sigma_1, \sigma_2 \in \Gamma\mathcal{F}$, such that $\mathcal{F} = \text{span}\{\sigma_1, \sigma_2\}$, will be called a generator of $\mathcal{F}$. Let $F := \mathcal{F} \cap M$ and $T := \mathcal{F} \cap N$. Then according to our identifications we have $F = \ell(1, f, 0)$ and $T = \ell(0, t, 1)$, for some maps $f, t : \Sigma \to S^3$. Now $C^{-1} \circ \mathcal{F} = (f, t) : \Sigma \to T_1S^3$ is said to be a Legendre immersion if the map is an immersion and $(f, t)$ is isotropic for the contact structure on $T_1S^3$, i.e., $(df, dt) = 0$. It is then clear that the isotropy condition is equivalent to $(dF, T) = 0$. One can then easily deduce that this holds if and only if for any generators $\sigma_1, \sigma_2 \in \Gamma\mathcal{F}$

$$(d\sigma_1, \sigma_2) = 0.$$ 

It remains to characterize the condition that $C^{-1} \circ \mathcal{F}$ is an immersion:

**Lemma 2.1.** The map $C^{-1} \circ \mathcal{F} : \Sigma \to Z$ is an immersion if and only if for all $x \in \Sigma$ and $X \in T_x\Sigma$,

$$d_X\sigma \in \mathcal{F}(x) \quad \text{for all sections } \sigma \in \Gamma\mathcal{F} \text{ implies } X = 0,$$ 

where $d : \Gamma T\Sigma \times \Gamma(\Sigma \times R^{4,2}) \to \Gamma(\Sigma \times R^{4,2})$ is the flat connection of $\Sigma \times R^{4,2}$.

**Proof.** Since $\mathcal{F} = \text{span}\{F, T\}$, we may write any $\sigma \in \Gamma\mathcal{F}$ as

$$\sigma = \alpha \begin{pmatrix} 1 \\ f \\ 0 \\ t \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

for some choice of coefficient functions $\alpha, \beta$. Now for any $x \in \Sigma$ and $X \in T_x\Sigma$,

$$d_X\sigma = d_X\alpha \begin{pmatrix} 1 \\ f(x) \\ 0 \\ 0 \end{pmatrix} + \alpha(x) \begin{pmatrix} 0 \\ d_Xf \\ 0 \\ 0 \end{pmatrix} + d_X\beta \begin{pmatrix} 0 \\ 0 \\ t(x) \\ 1 \end{pmatrix} + \beta(x) \begin{pmatrix} 0 \\ 0 \\ d_Xt \ contribution to other sections in flat connection. \end{pmatrix}. \end{pmatrix}$$

Therefore, $d_X\sigma \in \mathcal{F}(x)$ if and only if $\alpha(x)d_Xf + \beta(x)d_Xt = 0$. This holds for arbitrary $\sigma$, i.e., for arbitrary $\alpha$ and $\beta$, if and only if $d_Xf = d_Xt = 0$. On the other hand the map $C^{-1} \circ \mathcal{F} = (f, t)$ is an immersion if and only if $\ker df \cap \ker dt = \{0\}$. 

Namely, $d_Xf = d_Xt = 0$ implies $X = 0$. \qed
Motivated by this result, one says that $F : \Sigma \to \mathcal{Z}$ is a Legendre immersion (in the context of Lie sphere geometry) if the following two conditions hold

1. for any sections $\sigma_1, \sigma_2 \in \Gamma F$, $(d\sigma_1, \sigma_2) = 0$,

2. for all $x \in \Sigma$ and $X \in T_x \Sigma$, if $d_X \sigma \in F(x)$ for all sections $\sigma \in \Gamma F$, then $X = 0$.

We have already seen in this subsection how one can identify $F$ with maps into the spaceform $S^3$. However, suppose now that we have a general point sphere complex $p$ and a spaceform vector $q$ defining quadrics $M$ and $N$ as in (2.1). Assume that

$$\det B \neq 0, \quad B := \begin{pmatrix} (\sigma_1, p) & (\sigma_2, p) \\ (\sigma_1, q) & (\sigma_2, q) \end{pmatrix}$$

for a generator $\{\sigma_1, \sigma_2\}$ of $F$. Then by (2.6), we can obtain a map $F : \Sigma \to M$ as an intersection of $F$ with $M$, and a map $T : \Sigma \to N$ as an intersection of $F$ with $N$ as follows:

$$F(x) = a(x)\sigma_1(x) + b(x)\sigma_2(x), \quad T(y) = c(x)\sigma_1(x) + d(x)\sigma_2(x), \quad \left( \begin{array}{c} a(x) \\ b(x) \end{array} \right) \left( \begin{array}{c} c(x) \\ d(x) \end{array} \right) = B^{-1} \left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right)$$

By [6, Theorem 4.2], $F$ can be interpreted as a projection of $F$ to $M$ by a Legendre fibration. Thus $F$ is a front in the sense of [1]. One then has that, at each point $x \in \Sigma$, $F(x)$ corresponds to the pencil of oriented spheres tangent to the surface $F$ at $x$. Furthermore, if one consider the sphere defined by $\text{span}\{T(x)\}$, this corresponds to the unique totally geodesic “plane” in the sphere pencil $F(x)$. Thus, for a Euclidean projection, this simply corresponds to the tangent plane of the surface $F$ at $x$. Therefore, we shall refer to $T$ as the tangent plane congruence of $F$.

Using the specific projection of Cecil [6, Section 2.3] to Euclidean 3-space $\mathbb{R}^3$, where $p = \iota(0, 0, 0, 0, 1)$ and $q = \iota(1, -1, 0, 0, 0)$, $F$ and $T$ have the form

$$F = \left( \begin{array}{c} 1 \right), \quad T = \iota(\iota(f, t), -\iota(f, t), t, 1),$$

where $f = (f_1, f_2, f_3)$, $t = (t_1, t_2, t_3)$ and $\langle a, b \rangle = a_1b_1 + a_2b_2 + a_3b_3$ ($a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$). We then obtain a smooth map $f = (f_1, f_2, f_3) : \Sigma \to \mathbb{R}^3$ and its unit normal vector $t = (t_1, t_2, t_3)$. Although $F$ is a Legendre immersion, $f$ may have singularities. Such a projection is a front, and the study of singularities of fronts has a long history, see [1,2] for example. It is known that generic singularities of fronts are cuspidal edges and swallowtails. Usefull criteria for them are given in [17], and using these criteria, singularities of surfaces which have special curvature properties are characterized by their geometric properties. (See [10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 23, 28, 31] for example.) It is natural to ask whether the properties of singularities of the front $f$ can be characterized using the Lie-geometric properties of $F$. Thus the purpose of this note is to study singularities of $f$ in the context of Lie sphere geometry. As an application, we shall study singularities of Lie sphere transformations of a regular surface in $\mathbb{R}^3$, as noted in the introduction. Since Lie sphere transformations include the transformations to parallel surfaces, this generalizes the results in [11].
2.2 Curvature sphere congruences

Suppose that we have a 2-dimensional manifold $\Sigma$ and a smooth map $s : \Sigma \to \mathbb{P}(L^5)$. Then at each point $x \in \Sigma$, $s(x)$ corresponds to a sphere in any given spaceform. Therefore, $s$ defines a 2-dimensional family of spheres in such spaceforms, i.e., a sphere congruence. Alternatively, we may think of such a map as a rank 1 null subbundle of the trivial bundle $\Sigma \times \mathbb{R}^{12}$. Now, given a Legendre immersion $\mathcal{F} : \Sigma \to \mathcal{Z}$, one may consider the rank 1 subbundles $s \leq \mathcal{F}$. These correspond to sphere congruences enveloped by the surface defined by $\mathcal{F}$ in any spaceform.

At a point $x \in \Sigma$, we say that a 1-dimensional subspace $s(x)$ of $\mathcal{F}(x)$ is a curvature sphere if there exists a subspace $T_x \subset T_x \Sigma$ such that

$$\text{for any } \sigma \in \Gamma \mathcal{F} \text{ satisfying } \sigma(x) \in s(x), \text{ and for any } X_x \in T_x, \; d_{X_x} \sigma \in \mathcal{F}(x). \quad (2.9)$$

If $s(x)$ is a curvature sphere, then take a maximal subspace $T_s(x)$ such that $(2.9)$ holds, and call $T_s(x)$ curvature space of $s(x)$. Cecil [6, Corollary 4.9] showed that at each point $x \in \Sigma$ there are at most two distinct curvature spheres. Umbilic points are exactly the points where there is only one curvature sphere. In that case $T_s(x) = T_x \Sigma$. In the case that $\mathcal{F}$ is umbilic-free, the curvature spheres form two rank one subbundles $s_1$ and $s_2$ of $\mathcal{F}$ (called curvature sphere congruences) with respective rank one curvature subbundles $T_1 := \bigcup_{x \in \Sigma} T_{s_1(x)}$ and $T_2 := \bigcup_{x \in \Sigma} T_{s_2(x)}$ of the tangent bundle $T \Sigma$. Furthermore, $\mathcal{F} = s_1 \oplus s_2$ and $T \Sigma = T_1 \oplus T_2$.

Suppose now that we have a point sphere complex $p$ and a spaceform vector $q$ defining $M$ and $N$ as in (2.1). Let $F = \mathcal{F} \cap M$ denote the spaceform projection and $T = \mathcal{F} \cap N$ denote the tangent plane congruence of $\mathcal{F}$. Where $F$ is immersed, let $\kappa_1$ and $\kappa_2$ denote the principal curvatures of $F$. Then Cecil [6, Chapter 4] showed that $T + \kappa_i F$ are lifts of the curvature sphere congruences $s_i$ of $\mathcal{F}$. This can be deduced by using Rodrigues equations in the spaceform $M$. For example, consider the lifts $F$ and $T$ of an umbilic-free surface $f : \Sigma \to \mathbb{R}^3$ in Euclidean space with unit normal $t : \Sigma \to S^2$ given in (2.8). For $i \in \{1, 2\}$ let $\partial_i$ be a principal curvature direction in $T \Sigma$ corresponding to $\kappa_i$. Then by Rodrigues equations $\partial_i t + \kappa_i \partial_i f = 0$. One can then check that this implies $\partial_i T + \kappa_i \partial_i F = 0$. Hence,

$$\partial_i (T + \kappa_i F) = \partial_i T + \partial_i \kappa_i F + \kappa_i \partial_i F = \partial_i \kappa_i F \in \Gamma \mathcal{F}.$$

Thus, one has that $s_i = \text{span} \{T + \kappa_i F\}$ and $T_i = \text{span} \{\partial_i\}$.

It is clear from the lifts $T + \kappa_i F$ that umbilic points of $F$, i.e., points where the principal curvatures coincide, are exactly the umbilic points of $\mathcal{F}$, i.e., the points where $\mathcal{F}$ has only one curvature sphere.

It should come as no surprise that the sphere $S_i := M \cap s_i^\perp$ corresponding to $s_i$ coincides with the classical notion of a curvature sphere, i.e., a sphere tangent to the surface $F$ with radius $\kappa_i^{-1}$. Therefore, for any point $x \in \Sigma$, $S_i(x)$ has second order contact with $F$ in the principal curvature direction $T_i(x)$.

2.3 Möbius and Lie sphere transformations

Let $\hat{M}$ denote $M$ union its (possible empty) ideal boundary $\partial M$. Möbius transformations of $M$ are the diffeomorphisms from $\hat{M}$ to $\hat{M}$ that map spheres to spheres, and they
are represented by pseudo-orthogonal transformations of $\mathbb{R}^{4,2}$ (i.e. transformations that preserve $(,)$, and $O(4,2)$ is a double cover of them) that fix $p$. Möbius transformations of $M$ preserve the conformal structure, so will preserve the conformal structure of any surface inside $M$ as well. Furthermore, Möbius transformations will preserve contact orders of any spheres tangent to the surface, and so will preserve the principal curvature spheres. As a direct consequence, an umbilic point of the surface will remain an umbilic point after the Möbius transformation is applied.

Lie sphere transformations are the transformations of spaceforms that map spheres to spheres and preserve the oriented contact of spheres, and, like for Möbius transformations, they are represented by matrices in $O(4,2)$, and the group of Lie sphere transformations is now isomorphic to all of $O(4,2)/\{\pm I\}$. Möbius transformations are of course a special case of this. The objects preserved by general Lie sphere transformations are oriented spheres, but not point spheres (unlike Möbius transformations). However, contact elements are mapped to contact elements, so restricting to point spheres within the contact elements gives maps taking points to points. From this latter point of view, all Lie sphere transformations of surfaces in $M$ are generated (by composition) from Möbius transformations and parallel surface transformations in various spaceforms (see [6], Theorem 3.18). Like for Möbius transformations, curvature spheres are preserved under Lie sphere transformations.

Rephrasing the above statements about Lie sphere transformations more precisely, given a Legendre immersion $F : \Sigma \to \mathcal{Z}$, we have that $AF$ is a Legendre immersion, for any $A \in O(4,2)$. Furthermore, if $s(x)$ is a curvature sphere of $F$ at $x$ with curvature space $T_{s(x)}$, then $As(x)$ is a curvature sphere of $AF$ at $x$ with curvature space $T_{As(x)} = T_{s(x)}$. With $p, q$ as chosen just before (2.8), let $f = (f_1, f_2, f_3) : \Sigma \to \mathbb{R}^3$ be the regular surface in $\mathbb{R}^3$ with unit normal $t : \Sigma \to S^2$ given by $F$. Continue on from (2.6) and (2.7), assuming that for a generator $\{\sigma_1, \sigma_2\}$ of $F$ (we could try for $\sigma_1 = F, \sigma_2 = T : \Sigma \to \mathbb{R}^{4,2}$ as defined in (2.8), for example) the matrix $B_A = \begin{pmatrix} (A\sigma_1, p) & (A\sigma_2, p) \\ (A\sigma_1, q) & (A\sigma_2, q) \end{pmatrix}$ is regular,

$$B_A = \begin{pmatrix} (A\sigma_1, p) & (A\sigma_2, p) \\ (A\sigma_1, q) & (A\sigma_2, q) \end{pmatrix}$$

then

$$\hat{F} = a A\sigma_1 + b A\sigma_2,$$

$$\hat{T} = c A\sigma_1 + d A\sigma_2,$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = B_A^{-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(2.10)

satisfy $(\hat{F}, p) = 0$, $(\hat{F}, q) = -1$, $(\hat{T}, p) = -1$ and $(\hat{T}, q) = 1$. Thus $\hat{F}, \hat{T}$ have the form

$$\hat{F} = \hat{t} \left( \frac{1}{2} \left(1 + \langle \hat{f}, \hat{\hat{f}} \rangle \right), \frac{1}{2} \left(1 - \langle \hat{f}, \hat{\hat{f}} \rangle \right), \hat{f}, 0 \right), \quad \hat{T} = \hat{t} \left( \langle \hat{f}, \hat{\hat{t}} \rangle, -\langle \hat{f}, \hat{\hat{t}} \rangle, \hat{t}, 1 \right)$$

(2.11)

Hence we can project $AF = \{A\sigma_1, A\sigma_2\}$ to $M$ (isometric to $\mathbb{R}^3$) and $N$, and we obtain $\hat{f} : \Sigma \to \mathbb{R}^3$ and $\hat{t} : \Sigma \to S^2$. We call $\hat{f}$ the Lie sphere transformation of $f$ by $A$. The unit normal of $\hat{f}$ is $\hat{t}$. 6
3 Conditions for singularities

3.1 Criteria for singularities

Let $\Sigma \subset \mathbb{R}^2$ be an open domain and $f : \Sigma \to \mathbb{R}^3$ a frontal, meaning there exists a unit normal vector field $t : \Sigma \to S^2$. The function $\det(f_u, f_v, t)$ is called the signed area density function with respect to the local coordinate system $(\Sigma; u, v)$ of $\Sigma$, where $f_\ast = \partial f/\partial \ast$ for $\ast = u, v$. If the map $(f, t) : \Sigma \to \mathbb{R}^3 \times S^2$ is an immersion, $f$ is called a front. Let $S(f)$ be the set of singular points of $f$. Take a point $x \in S(f)$ such that rank $df_x = 1$. Then there exists a non-zero vector field $X$ on a small neighborhood $V$ of $x$ such that $df_y(X) = 0$ holds for $y \in S(f) \cap V$. We call $X$ a null vector field with respect to $f$.

Let $g_1, g_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two map-germs at the origin. These map-germs are $A$-equivalent if there exist diffeomorphism-germs $\Xi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\Xi_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying $g_2 \circ \Xi = \Xi_t \circ g_1$. A map-germ at the origin $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a cuspidal edge, swallowtail, cuspidal beaks, cuspidal lips, cuspidal butterfly or $D_4^\pm$ singularity, respectively, if it is $A$-equivalent to the map germ

- $(u, v) \mapsto (u, v^2, v^3)$ (cuspidal edge),
- $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ (swallowtail),
- $(u, v) \mapsto (u, 2v^3 - u^2v, 3v^4 - u^2v^2)$ (cuspidal beaks),
- $(u, v) \mapsto (u, 2v^3 + u^2v, 3v^4 + u^2v^2)$ (cuspidal lips),
- $(u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2)$ (cuspidal butterfly),
- $(u, v) \mapsto (2uv, \pm u^2 + 3v^2, \pm 2u^2v + 2v^3)$ ($D_4^\pm$ singularity)

at the origin.

We have the following well-known characterizations of cuspidal edges and swallowtails [17, Proposition 1.3], see also [27, Corollary 2.5]. Also there are characterizations of cuspidal lips, cuspidal beaks and cuspidal butterflies, see [15, Theorem A.1] and [14, Theorem 8.2]. Two function-germs, or map-germs, are proportional if they coincide up to non-zero scalar functional multiplication.

**Lemma 3.1.** Let $f$ be a front and $x$ a singular point for which $df_x$ is rank 1. Let $\lambda$ be a function which is proportional to the signed area density function, and $X$ a null vector field in a neighborhood of $x$. Then $f$ at $x$ is a

1. cuspidal edge if and only if $df_x \lambda \neq 0$ at $x$,
2. swallowtail if and only if $df_x \lambda = 0$, $d_X df_x \lambda \neq 0$ and $d\lambda \neq 0$ at $x$.
3. cuspidal beaks if and only if $d\lambda = 0$, $d_X df_x \lambda \neq 0$ and $\det \text{Hess} \lambda < 0$ at $x$,
4. cuspidal lips if and only if $d\lambda = 0$ and $\det \text{Hess} \lambda > 0$ at $x$,
5. cuspidal butterfly if and only if $d\lambda = 0$, $d_X df_x \lambda = 0$, $d_X d_X d_X \lambda \neq 0$ and $d\lambda \neq 0$ at $x$.

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We shall now define three types of singularity. The motivation for defining these types will become clear in Section 4, where we apply Lemma 3.1 to the study of Lie sphere transformations of surfaces. Let \( f \) be a front and \( x \) a singular point such that \( \text{rank } df = 1 \). Let \( \lambda \) be a function which is proportional to the signed area density function, and let \( X \) be a null vector field in a neighborhood of \( x \). Then \( x \) is called a type 1 singularity if \( dX \lambda \neq 0 \). Furthermore, \( x \) is called a type 2 singularity if \( dX \lambda = 0 \) and \( dXdX \lambda \neq 0 \) at \( x \), and is called a type 3 singularity if \( dX \lambda = dXdX \lambda = 0 \) and \( dXdX \lambda \neq 0 \) at \( x \). By the non-degeneracy, one can show that these definitions do not depend on the choice of \( \lambda \) and \( X \). By Lemma 3.1, if \( x \) is a type 1 singularity, then \( x \) is a cuspidal edge. If \( x \) is a non-degenerate type 2 (respectively, type 3) singularity, then \( x \) is a swallowtail (respectively, cuspidal butterfly). Cuspidal beaks and cuspidal lips are examples of type 2 degenerate singularities.

In the case that \( \text{rank } df = 0 \), there is the following characterization for \( D_4^+ \) singularities:

**Lemma 3.2.** (20) Let \( f \) be a front with unit normal \( t \) and let \( \lambda \) be a function which is proportional to the signed area density function. A singular point \( x \) is a \( D_4^+ \) (respectively, \( D_4^- \)) singularity if and only if the following two conditions hold:

1. \( \text{rank } df = 0 \).
2. \( \det \text{Hess } \lambda < 0 \) (respectively, \( \det \text{Hess } \lambda > 0 \)) at \( x \).

Lemma 3.1 and Lemma 3.2 show how certain types of singularities are determined by any function proportional to the signed area density function. We will now derive such a function in the context of Lie sphere geometry. Using the lifts of (2.8), for any linearly independent vector fields \( X,Y \in \Gamma T\Sigma \), the determinant \( \det(dXF,dYF,T,F,q,p) \) is equal to the determinant of

\[
\begin{pmatrix}
\langle X,f \rangle & \langle Y,f \rangle & \langle t,f \rangle & \frac{1}{2}(1+\langle f,f \rangle) & 1 & 0 \\
-\langle X,f \rangle & -\langle Y,f \rangle & -\langle t,f \rangle & \frac{1}{2}(1-\langle f,f \rangle) & -1 & 0 \\
x \cdot f & y \cdot f & t \cdot f & 0 & 0 & 1
\end{pmatrix}.
\]

This is the same as

\[
- \det(dXf,dYf,t) \det \begin{pmatrix} \frac{1}{2}(1+\langle f,f \rangle) & 1 & 0 \\ \frac{1}{2}(1-\langle f,f \rangle) & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(dXf,dYf,t).
\]

Hence, the signed area density function can be taken to be

\[
\lambda = \det(dXF,dYF,T,F,q,p).
\]

Now given two linearly independent sections \( \sigma, \tilde{\sigma} \in \Gamma F \), we may write

\[
\sigma = -((\sigma,q)F-(\sigma,p)T) \quad \text{and} \quad \tilde{\sigma} = -((\tilde{\sigma},q)F-(\tilde{\sigma},p)T).
\]

Thus,

\[
F = \frac{1}{\Delta}((\sigma,p)\tilde{\sigma}-(\tilde{\sigma},p)\sigma).
\]
where $\Delta := (\tilde{\sigma}, p)(\sigma, q) - (\sigma, p)(\tilde{\sigma}, q)$. Hence,

$$dF = \frac{1}{\Delta} ((\sigma, p) d\tilde{\sigma} - (\tilde{\sigma}, p) d\sigma) \mod \Omega^1(\mathcal{F})$$

and the signed area density function is proportional to

$$\det \left( (\sigma, p)d_X\tilde{\sigma} - (\tilde{\sigma}, p)d_X\sigma, (\sigma, p)d_Y\tilde{\sigma} - (\tilde{\sigma}, p)d_Y\sigma, T, F, q, p \right). \quad (3.1)$$

### 3.2 Calculations at a non-umbilic point

Let $\mathcal{F}$ be a Legendre immersion, and $F$ as in (2.8). In this section, we assume that $x$ is a non-umbilic point of $f$. Then, as stated in the introduction, on a neighborhood of $x$ there are two distinct curvature sphere congruences $s_1$ and $s_2$ with curvature subbundles $T_1$ and $T_2$, respectively. We may then choose $\tilde{\sigma} = \sigma_1$ and $\sigma = \sigma_2$ in (3.1) for any non-zero lifts $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$. Furthermore, we take $X \in \Gamma T_1$ and $Y \in \Gamma T_2$. Then since $d_X\sigma_1, d_Y\sigma_2 \in \Gamma \mathcal{F}$ holds, (3.1) becomes

$$- (\sigma_1, p)(\sigma_2, p) \det(d_X\sigma_2, d_Y\sigma_1, T, F, q, p). \quad (3.2)$$

This gives rise to the following lemma:

**Lemma 3.3.** The signed area density function $\lambda$ of $f$ as in (2.8) is proportional to $(\sigma_1, p)(\sigma_2, p)$ for any choice of non-zero lifts of the curvature spheres $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$.

**Proof.** Following (3.2), it is sufficient to show that $\{d_X\sigma_2, d_Y\sigma_1, T, F, q, p\}$ is a basis for $\Sigma \times \mathbb{R}^1$. Since $\mathcal{F}$ is an isotropic map, we have that $d_X\sigma_2, d_Y\sigma_1 \in \Gamma \mathcal{F}^\perp$. Now if there exist functions $\lambda$ and $\mu$ such that $\lambda d_X\sigma_2 + \mu d_Y\sigma_1 \in \Gamma \mathcal{F}$, then $d_{\lambda X + \mu Y}(\sigma_2 + \sigma_1) \in \Gamma \mathcal{F}$. However, this implies that $\lambda = \mu = 0$ as otherwise $\sigma_2 + \sigma_1$ would span a curvature sphere congruence, contradicting that there are exactly two curvature sphere congruences $s_1$ and $s_2$. Thus, $\text{span}\{d_X\sigma_2, d_Y\sigma_1\} \oplus \mathcal{F}$ is a rank 4 subbundle of $\mathcal{F}^\perp$, i.e.,

$$\text{span}\{d_X\sigma_2, d_Y\sigma_1, T, F\} = \mathcal{F}^\perp.$$

By (2.6), we have that $\text{span}\{q, p\} \cap \mathcal{F}^\perp = \{0\}$ and the result follows. \qed

It is clear from Lemma 3.3 that if $\lambda(x) = 0$ at a point $x \in \Sigma$, then $s_1(x) \perp p$ or $s_2(x) \perp p$. Note that since $p$ is timelike, we cannot have that both $s_1(x) \perp p$ and $s_2(x) \perp p$. Assume, without loss of generality, that $s_1(x) \perp p$. Then by inertia, there exists an open neighborhood $V$ of $x$ such that $s_2(y) \not\perp p$, for all $y \in V$. Therefore, $y \in S(f) \cap V$ if and only if $s_1(y) \perp p$. This is equivalent to $F(y) \in s_1(y)$. In such an instance, since $d_X\sigma_1 \in \mathcal{F}(y)$ for all $X \in \Gamma T_1$ and $\sigma_1 \in \Gamma s_1$, we have that $d_X f = 0$. Hence $d_{X_1} F = 0$ and any $X \in \Gamma T_1$ locally yields a null vector field with respect to $f$. We conclude that:

**Lemma 3.4.** The point $x$ is a singular point of $f$ if and only if $s_1(x) \perp p$ or $s_2(x) \perp p$. In which case, any $X \in \Gamma T_1$ or $Y \in \Gamma T_2$, respectively, locally yields a null vector field for $f$.

**Conditions for singularities.** We will now state and prove our main theorem:
Theorem 3.5. Let \( x \in \Sigma \) be a non-umbilic point of \( \mathcal{F} \) and let \( p \) be a point sphere complex. Suppose that \( s_1(x) \) is perpendicular to \( p \) (and thus \( \mathcal{F} \) projects to a singular point in any spaceform with point sphere complex \( p \)). Let \( \sigma_1 \in \Gamma s_1 \) be any non-zero lift of \( s_1 \), and let \( X \in \Gamma T_1 \). Then

1. \( d_X \sigma_1 \notin s_1(x) \) if and only if \( \mathcal{F} \) projects to a cuspidal edge at \( x \),

2. \( d_X \sigma_1 \in s_1(x), (d_X d_X \sigma_1)_x \notin s_1(x) \) if and only if \( \mathcal{F} \) projects to a type 2 singularity at \( x \). Moreover, under this condition,

   - \( d\sigma_1 \not\perp p \) at \( x \), if and only if \( \mathcal{F} \) projects to a swallowtail at \( x \),
   - \( d\sigma_1 \perp p \), and \( \det \text{Hess}(\sigma_1, p) > 0 \) (respectively, \( \det \text{Hess}(\sigma_1, p) < 0 \)) at \( x \) if and only if \( \mathcal{F} \) projects to a cuspidal lips (respectively, cuspidal beaks) at \( x \),

3. \( d_X \sigma_1, (d_X d_X \sigma_1)_x \in s_1(x), (d_X d_X d_X \sigma_1)_x \notin s_1(x) \) if and only if \( \mathcal{F} \) projects to a type 3 singularity \( x \). Moreover, under this condition, \( d\sigma_1 \not\perp p \) at \( x \) if and only if \( \mathcal{F} \) projects to a cuspidal butterfly at \( x \).

Proof. In this proof we will prove a series of facts equating conditions on \( \lambda \) to conditions involving \( s_1 \) and the point sphere complex \( p \). It is then straightforward to complete this proof by applying these facts to Lemma 3.1.

Let \( \sigma_2 \) be a non-zero lift of the curvature sphere \( s_2 \). Since \( \sigma_1 \) is a lift of the curvature sphere congruence \( s_1 \) and \( X \in \Gamma T_1 \), we have that

\[
d_X \sigma_1 = \alpha \sigma_1 + \beta \sigma_2, \tag{3.3}
\]

for some smooth functions \( \alpha \) and \( \beta \). We prove this theorem by utilizing the result of Lemma 3.3 that we may replace the signed area density function in Lemma 3.1 by

\[
\lambda = (\sigma_1, p)(\sigma_2, p),
\]

for any non-zero lift of \( \sigma_2 \in \Gamma s_2 \). By the Leibniz rule, we then have for any \( V \in \Gamma T \Sigma \),

\[
d_V \lambda = (d_V \sigma_1, p)(\sigma_2, p) + (\sigma_1, p)(d_V \sigma_2, p). \tag{3.4}
\]

Since we assumed that \( s_1(x) \) is perpendicular to \( p \), we then have that

\[
d_V \sigma_1 = (d_V \sigma_1, p)(\sigma_2(x), p).
\]

By the assumption in (2.6) we have that \( (\sigma_2(x), p) \neq 0 \), thus \( d_V \lambda = 0 \) if and only if \( (d_V \sigma_1, p) = 0 \). Hence,

\[
d\lambda = 0 \text{ at } x \text{ if and only if } (d\sigma_1, p) = 0 \text{ at } x. \tag{3.5}
\]

If we replace \( V \) with \( X \) then by (3.3), \( (d_X \sigma_1, p) = 0 \) if and only if \( \beta(x) = 0 \). Therefore, we have shown that

\[
d_X \lambda = 0 \text{ at } x \text{ if and only if } d_X \sigma_1 \in s_1(x). \tag{3.6}
\]

Differentiating (3.4) with respect to \( W \in \Gamma T \Sigma \), we have that \( d_W d_V \lambda \) is given by

\[
(d_W d_V \sigma_1, p)(\sigma_2, p) + (d_V \sigma_1, p)(d_W \sigma_2, p) + (d_W \sigma_1, p)(d_V \sigma_2, p) + (\sigma_1, p)(d_W d_V \sigma_2, p). \tag{3.7}
\]
Lemma 3.6. The map $\Gamma$ is isotropic.

Proof. Firstly, let $Y, Z \in \Gamma T \Sigma$. Then, using the Leibniz rule and that $\mathcal{F}$ is isotropic,

$$(d_Y d_Z \sigma, \sigma) = (d_Z d_Y \sigma, \sigma) = (d_Y d_Y d_Z \sigma, \sigma).$$

Since $\sigma(x) \in s(x)$, $(d\sigma)_x \in T_x \Sigma \otimes \mathcal{F}(x)$ and thus $(d_Z \sigma, d_Y \sigma) = 0$ at $x$. Hence,

$$((d_Y d_Z \sigma)_x, \sigma(x)) = (\sigma(x), (d_Y d_Z \sigma)_x) = 0.$$  

3.3 Calculations at an umbilic point

Suppose that $x$ is an umbilic point of $\mathcal{F}$, i.e., there exists $s(x)$ in $\mathcal{F}(x)$ such that for any section $\sigma \in \Gamma \mathcal{F}$ with $\sigma(x) \in s(x)$ we have that

$$(d\sigma)_x \in T_x \Sigma \otimes \mathcal{F}(x).$$

Define

$$\mathcal{C}(X, Y, Z)(\sigma, \tilde{\sigma}) := (d_X d_Y d_Z \tilde{\sigma}, \sigma),$$

for $X, Y, Z \in \Gamma T \Sigma$ and $\sigma, \tilde{\sigma} \in \Gamma \mathcal{F}$ such that $\sigma(x) \in s(x)$.

Lemma 3.6. The map $\mathcal{C}$ is tensorial at $x$ and is symmetric in $X, Y, Z$ at $x$. Furthermore, $\mathcal{C}_\sigma(s(x), s(x)) = 0$. Hence, we may identify $\mathcal{C}_\sigma$ as an element of $\mathcal{S}^3(T^* \Sigma) \otimes s(x) \otimes f(x)/s(x)$.

Proof. Firstly, let $Y, Z \in \Gamma T \Sigma$. Then, using the Leibniz rule and that $\mathcal{F}$ is isotropic,

$$-(d_Z \sigma, d_Y \sigma) = (d_Y d_Z \sigma, \sigma) = (d_Z d_Y \sigma, \sigma) = (d_Y d_Z \sigma, \sigma).$$

Since $\sigma(x) \in s(x)$, $(d\sigma)_x \in T_x \Sigma \otimes \mathcal{F}(x)$ and thus $(d_Z \sigma, d_Y \sigma) = 0$ at $x$. Hence,

$$((d_Y d_Z \sigma)_x, \sigma(x)) = (\sigma(x), (d_Y d_Z \sigma)_x) = 0.$$  

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In particular, since \( \hat{\sigma} \) is arbitrary, we have that \((d_\mathcal{Y}d_\mathcal{Z}\sigma)_x \in (\mathcal{F}(\Sigma))^\perp \). Using the Leibniz rule and the fact that \( \mathcal{F} \) is isotropic one can show that

\[
(d_Xd_\mathcal{Y}d_\mathcal{Z}\sigma, \sigma) = -(d_Xd_\mathcal{Y}\sigma, d_\mathcal{Z}\sigma) - (d_\mathcal{Y}d_\mathcal{Z}\sigma, d_X\sigma) - (d_\mathcal{Y}d_\mathcal{Z}\sigma, d_X\sigma).
\] (3.13)

Therefore, since \((d\sigma)_x \in T_x\Sigma \otimes \mathcal{F}(\Sigma)\),

\[
((d_Xd_\mathcal{Y}d_\mathcal{Z}\sigma)_x, \sigma(x)) = 0.
\] (3.14)

Now let us assume that \( \sigma \) and \( \hat{\sigma} \) are everywhere linearly independent. Then an arbitrary section \( \hat{\sigma} \in \Gamma\mathcal{F} \) may be written as \( \hat{\sigma} = \alpha\hat{\sigma} + \beta\sigma \) for some smooth functions \( \alpha \) and \( \beta \). By applying (3.12) and (3.14), we have that

\[
((d_Xd_\mathcal{Y}d_\mathcal{Z}\hat{\sigma})_x, \sigma(x)) = \alpha(x)((d_Xd_\mathcal{Y}d_\mathcal{Z}\hat{\sigma})_x, \sigma(x)).
\]

Hence, the value of \( \mathcal{C} \) at \( x \) only depends on \( \sigma(x) \) and \( \hat{\sigma}(x) \mod s(x) \).

The tensorality of \( \mathcal{C} \) in \( X, Y, Z, F \) at \( x \) follows from the linearity of the connection \( d \) and (3.12). The symmetry of \( \mathcal{C} \) at \( x \) follows from (3.12) and the flatness of the trivial connection \( d \).

Suppose that \( c \in S^3(T_x^o\Sigma) \) is a cubic form. Then in terms of a basis \( X, Y \in T_x\Sigma \), its discriminant is given by

\[
(c_{X,X,X}c_{X,Y,Y} - c_{X,Y,X}^2)(c_{Y,Y,Y}c_{X,Y,X} - c_{X,Y,Y}^2) - (c_{X,X,X}c_{Y,Y,Y} - c_{X,Y,X}c_{X,Y,Y})^2.
\]

Now since \( s(x) \otimes f(x)/s(x) \) is a line bundle, \( C_x \) may be viewed as a conformal class of cubic forms. Since this is a conformal class, the sign of the discriminant of any two non-zero elements coincides. We will refer to this as the sign of the discriminant of \( C_x \).

Now suppose we are given a point sphere complex \( p \) and spaceform vector \( q \). Let \( F : \Sigma \to M \) denote the spaceform projection and \( T : \Sigma \to N \) denote the tangent plane congruence of \( \mathcal{F} \). If \( f \) is immersed at \( x \) with principal curvature \( \kappa = \kappa_1(x) = \kappa_2(x) \in \mathbb{R} \), then Rodrigues’ equations we have that \( \sigma := T + \kappa F \) satisfies \((d\sigma)_x \in T_x\Sigma \otimes \mathcal{F}(\Sigma) \). Let \( \hat{\sigma} := F \). Then in terms of this choice of \( \sigma \) and \( \hat{\sigma} \) we have that

\[
C_x(X, Y, Z)(T + \kappa F, F) = (d_Xd_\mathcal{Y}d_\mathcal{Z}F, T + \kappa F) = (d_Xd_\mathcal{Y}d_\mathcal{Z}F, T) + \kappa(d_Xd_\mathcal{Y}d_\mathcal{Z}F, T)
\]

\[
= (d_Xd_\mathcal{Y}d_\mathcal{Z}F, T) - \kappa((d_Xd_\mathcal{Y}F, d_\mathcal{Z}F) + (d_Xd_\mathcal{Y}d_\mathcal{Z}d_\mathcal{Y}F, d_XF) + (d_\mathcal{Y}d_\mathcal{Z}F, d_XF))
\]

by (3.13), noticing that \( \kappa \) is a constant. This coincides with the cubic form in Porteous [25]. Hence, we obtain the following result:

**Proposition 3.7.** Suppose that the sign of the discriminant of \( C_x \) is positive (respectively, negative). Then, when the spaceform projection immerses, \( \mathcal{F} \) projects to a elliptic (hyperbolic) umbilic at \( x \).

We will now examine the case that the spaceform projection does not immerse at \( x \). Assume that \( x \in \Sigma \) is an umbilic point of \( \mathcal{F} \). Recall from (3.1) that given two linearly independent sections \( \sigma, \hat{\sigma} \in \Gamma\mathcal{F}, \) we may assume that the signed area density function is proportional to

\[
\det \left( (\sigma, p)d_X\hat{\sigma} - (\hat{\sigma}, p)d_X\sigma, (\sigma, p)d_\mathcal{Y}\hat{\sigma} - (\hat{\sigma}, p)d_\mathcal{Y}\sigma, T, F, q, p \right)
\]
for any linearly independent $X$ and $Y$ in $T\Sigma$. Now we may choose $\sigma$ and $\tilde{\sigma}$ such that $\sigma(x) \in s(x)$, $\tilde{\sigma}$ lies nowhere in a curvature sphere and
\[ d\sigma = d\tilde{\sigma} \circ S, \]
where $S \in \Gamma \text{End}(T\Sigma)$. Now since $\tilde{\sigma}$ lies nowhere in a curvature sphere, we have that if $Z \in T_x \Sigma$ such that $d_2\tilde{\sigma} \in \mathcal{F}(x)$, then $Z = 0$. On the other hand, since $\sigma(x) \in s(x)$, we have that $(d\sigma)_x \in T_x \Sigma \otimes \mathcal{F}(x)$. Therefore we must have that $S(x) = 0$. Writing
\[ S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]
in terms of the basis $X, Y$ of $T\Sigma$, we may rewrite (3.1) as
\[
\det \left( \left( (\sigma, p) - \alpha(\tilde{\sigma}, p) \right) d_\chi \tilde{\sigma} - \beta(\tilde{\sigma}, p) d_\gamma \tilde{\sigma}, \left( (\sigma, p) - \delta(\tilde{\sigma}, p) \right) d_\chi \tilde{\sigma} - \gamma(\tilde{\sigma}, p) d_\chi \tilde{\sigma}, T, F, q, p \right) = \left( (\sigma, p) - \alpha(\tilde{\sigma}, p) \right) \left( (\sigma, p) - \delta(\tilde{\sigma}, p) \right) - \beta \gamma(\tilde{\sigma}, p)^2 \det(d_\chi \tilde{\sigma}, d_\gamma \tilde{\sigma}, T, F, q, p).
\]
Since $\sigma$ is nowhere a curvature sphere, similar arguments as in the proof of Lemma 3.3 show that $\{d_\chi \sigma, d_\gamma \sigma, T, F, q, p\}$ is a basis for $\Sigma \times \mathbb{R}^{4,2}$. Thus (3.1) is proportional to
\[ \lambda = \left( (\sigma, p) - \alpha(\tilde{\sigma}, p) \right) \left( (\sigma, p) - \delta(\tilde{\sigma}, p) \right) - \beta \gamma(\tilde{\sigma}, p)^2. \]
At the umbilic point $x, S(x) = 0$, we have that $\lambda(x) = (\sigma(x), p)^2$. Hence, $\lambda(x) = 0$ if and only if $s(x) \perp p$. So let us now assume that $s(x) \perp p$. Then
\[
\begin{align*}
\chi \lambda &= \left( \beta(d_\chi \tilde{\sigma}, p) - (d_\chi \alpha)(\tilde{\sigma}, p) \right) \left( (\sigma, p) - \delta(\tilde{\sigma}, p) \right) \\
&\quad + \left( (\sigma, p) - \alpha(\tilde{\sigma}, p) \right) \left( \beta(d_\gamma \tilde{\sigma}, p) + (\alpha - \delta)(d_\chi \tilde{\sigma}, p) - (d_\chi \delta)(\tilde{\sigma}, p) \right) \\
&\quad - d_\chi(\beta \gamma)(\tilde{\sigma}, p)^2 - 2 \beta \gamma(d_\chi \tilde{\sigma}, p)(\tilde{\sigma}, p), \\
\gamma \lambda &= \left( \gamma(d_\gamma \tilde{\sigma}, p) + (\delta - \alpha)(d_\gamma \tilde{\sigma}, p) - (d_\gamma \alpha)(\tilde{\sigma}, p) \right) \left( (\sigma, p) - \delta(\tilde{\sigma}, p) \right) \\
&\quad + \left( (\sigma, p) - \alpha(\tilde{\sigma}, p) \right) \left( \gamma(d_\chi \tilde{\sigma}, p) - (d_\chi \delta)(\tilde{\sigma}, p) \right) \\
&\quad - d_\gamma(\beta \gamma)(\tilde{\sigma}, p)^2 - 2 \beta \gamma(d_\gamma \tilde{\sigma}, p)(\tilde{\sigma}, p).
\end{align*}
\]
Using that $S(x) = 0$ and $(\sigma, p) = 0$ it is then clear that $(d\lambda)_x = 0$. We shall now consider the Hessian of $\lambda$ at $x$.
\[
\begin{align*}
(d_\chi d_\chi \lambda)_x &= 2 \left( (d_\gamma \alpha)(d_\chi \delta) - (d_\chi \alpha)(d_\chi \gamma) \right) (\tilde{\sigma}(x), p)^2 \\
(d_\chi d_\gamma \lambda)_x &= \left( (d_\chi \alpha)(d_\gamma \delta) + (d_\gamma \alpha)(d_\chi \delta) - (d_\chi \beta)(d_\gamma \gamma) - (d_\gamma \gamma)(d_\chi \beta) \right) (\tilde{\sigma}(x), p)^2 \\
(d_\gamma d_\chi \lambda)_x &= 2 \left( (d_\gamma \alpha)(d_\chi \delta) - (d_\chi \beta)(d_\gamma \gamma) \right) (\tilde{\sigma}(x), p)^2 \\
(d_\gamma d_\gamma \lambda)_x &= 2 \left( (d_\chi \alpha)(d_\gamma \delta) + (d_\gamma \alpha)(d_\chi \delta) - (d_\chi \beta)(d_\chi \gamma) - (d_\chi \gamma)(d_\chi \beta) \right) (\tilde{\sigma}(x), p)^2.
\end{align*}
\]
On the other hand, using by these special lifts $\sigma$ and $\tilde{\sigma}$ one can compute the discriminant of $C_x(\sigma, \tilde{\sigma})$ to be a positive scalar multiple of $(\det \text{Hess} \lambda)_x$. Therefore we have arrived at the following theorem:

**Theorem 3.8.** Let $x$ be an umbilic point of $\mathcal{F}$ such that $\lambda(x) = 0$. Then $f$ has a $D^+_4$ singularity (respectively, $D^-_4$ singularity) at $x$ if and only if the discriminant of $C_x$ is negative (respectively, positive).

### 4 Singularity of Lie sphere transformations

Let $\mathcal{F} : \Sigma \to \mathcal{Z}$ be a Legendre immersion, and let $A \in O(4, 2)$. Let $f : \Sigma \to \mathbb{R}^3$ be the projection of $\mathcal{F}$ to $\mathbb{R}^3$ in the manner of (2.7) and (2.8), and assume that $f$ is immersed. We described Lie sphere transformations $\tilde{f} : \Sigma \to \mathbb{R}^3$ of $f$ in Section 2, (2.10) and (2.11), and we shall now study singularities appearing on such transformations.
4.1 Non-umbilic points

Let \( x \) be a non-umbilic point of \( f \) and assume that \( \hat{\lambda}(x) = 0 \), where \( \hat{\lambda} \) denotes the signed area density function of \( \hat{f} \). Let \( \kappa_1 \) and \( \kappa_2 \) be the principal curvatures of \( f \). Then by Theorem 3.5, we have the following:

**Theorem 4.1.** Let \((u,v)\) be curvature line coordinates for \( f \) and assume that \( \hat{f}_u = 0 \) at \( x \) (thus \( x \) is a rank 1 singular point). Then

1. \( \kappa_{1,u} \neq 0 \) at \( x \) if and only if \( \hat{f} \) has a type 1 singularity at \( x \),
2. \( \kappa_{1,u} = 0 \) and \( \kappa_{1,uu} \neq 0 \) at \( x \) if and only if \( \hat{f} \) has a type 2 singularity at \( x \),
3. \( \kappa_{1,u} = \kappa_{1,uu} = 0 \) and \( \kappa_{1,uuu} \neq 0 \) at \( x \) if and only if \( \hat{f} \) has a type 3 singularity at \( x \).

**Proof.** The theorem follows by noting that \( T + \kappa_1 F \) is a lift of the curvature sphere \( s_1 \) of \( F \). Thus, \( AT + \kappa_1 AF \) is a lift of the curvature sphere \( As_1 \) of \( AF \). The result then follows by applying Theorem 3.5 to this lift. \( \square \)

We remark that in [11] and [25] similar conditions were derived for singularities of parallel surfaces, however in that case the singularities are precisely determined, allowed for by the fact that parallel transformations are a strict subgroup of the full group of Lie sphere transformations. Note that when taking parallel transformations of a regular surface there are at most two parallel surfaces with singularities at a given point. However, in the case of Lie sphere transformations there are infinitely many such surfaces.

4.1.1 Varying subclasses using Lie sphere transformations

In Theorem 4.1 we showed that the type of singularities appearing on a Lie sphere transformation is determined by the initial surface \( f \). Using Theorem 3.5 we will now show that we can explicitly construct Lie sphere transformations of the initial surface that give each possible subclass of the determined class.

To start with, we would obviously like our Lie sphere transformation \( \hat{f} \) to have a singularity at \( x \). To simplify matters we would also like the \( u \) derivative of \( \hat{f} \) to vanish at \( x \). To achieve this we choose a unit timelike vector \( \hat{p} \in R^{4,2} \) such that \( s_1(x) \perp \hat{p} \). There are many of these to choose from since \( s_1(x) \perp s_1(x) \) has signature \((3,1)\). Now since \( \hat{p} \) is a unit timelike vector there exists (many) \( A \in O(4,2) \) such that \( A\hat{p} = p \). If we then consider the Legendre immersion \( AF \), we have by the orthogonality of \( A \) that \( As_1(x) \perp A\hat{p} = p \), and thus \( f \) satisfies \( f_u = 0 \) at \( x \).

Immediately by Theorem 4.1 we see that if our initial surface satisfies \( \kappa_{1,u} \neq 0 \) at \( x \) then our Lie sphere transformation \( \hat{f} \) has a cuspidal edge at \( x \).

For type 2 and 3 singularities, one can see from Theorem 3.5 that we must impose some additional constraints on \( \hat{p} \). If our initial surface satisfies \( \kappa_{1,u} = 0 \) at \( x \) then by considering the lift \( \sigma_1 = T + \kappa_1 F \), it is clear that \( (\sigma_{1,u})_x \in s_1(x) \). We then have that \( W := \text{span}\{\sigma_1(x), d\sigma_1(T_x\Sigma)\} \perp s_1(x) \) has dimension 3 and signature \((2,1)\). Thus we may choose a unit timelike vector \( \hat{p} \), satisfying

\[
\hat{p} \mod s_1(x) \in W. \quad (4.1)
\]
This time by choosing $A \in O(4, 2)$ such that $A\hat{p} = p$, we have that $A\mathcal{F}$ satisfies $A_{s_1}(x) \perp p$ and, by (4.1),
\[ (d(A\sigma_1)_x, p) = (Ad(\sigma_1)_x, A\hat{p}) = (d(\sigma_1)_x, \hat{p}) = 0, \]
and so
\[ d(A\sigma_1)_x \perp p. \quad (4.2) \]
Now if we are in the case that $\kappa_{1,u} = 0$ and $\kappa_{1,uu} \neq 0$ at $x$, then by Theorem 4.1 we know that $\hat{f}$ has a type 2 singularity at $x$. Moreover, (4.2) implies that $\hat{f}$ has a degenerate singularity at $x$, and by Theorem 3.5 we can see that $\hat{f}$ has either cuspidal lips, cuspidal beaks or some other less well known type 2 degenerate singularity at $x$. In order to specify exactly which one of these we have, let $\hat{p}^\xi := \hat{p} + \xi \sigma_1$, for some $\xi \in \mathbb{R}$. Then $\hat{p}^\xi$ is again a unit timelike vector whose quotient lies in $W$ and, furthermore, one can check that
\[ \text{Hess}(\sigma_1, \hat{p}^\xi)_x = \text{Hess}(\sigma_1, \hat{p})_x + \xi c, \]
for some non-zero constant $c$. Therefore, by varying $\xi$, any sign of $\text{Hess}(\sigma_1, \hat{p}^\xi)_x$ can be achieved. Now by letting $A^\xi \in O(4, 2)$ such that $A^\xi \hat{p} = p$, we have by the orthogonality of $A^\xi$ that $\text{Hess}(A^\xi \sigma_1)_x = \text{Hess}(\sigma_1, \hat{p}^\xi)_x$. Then, by Theorem 3.5 by varying $\xi$ we change between cuspidal beaks, cuspidal lips and a less familiar type 2 degenerate singularity at $x$.

In the case that $\kappa_{1,u} = 0$, $\kappa_{1,uu} = 0$ and $\kappa_{1,uuu} \neq 0$ at $x$, we have by Theorem 4.1 that $\hat{f}$ projects to a type 3 singularity at $x$, and by Theorem 3.5 we have that $\hat{f}$ has a type 3 degenerate singularity at $x$.

On the other hand, in the case that $\kappa_{1,u} = 0$, $\kappa_{1,uu} \neq 0$ (respectively, $\kappa_{1,u} = \kappa_{1,uu} = 0$ and $\kappa_{1,uuu} \neq 0$) at $x$ if we wish to obtain a swallowtail (respectively, cuspidal butterfly) at $x$, the additional constraint on $\hat{p}$ we impose is that the quotient $\hat{p}$ mod $s_1(x) \not\in W$.

Now by letting $A \in O(4, 2)$ such that $A\hat{p} = p$, we have that $A\mathcal{F}$ satisfies $A_{s_1}(x) \perp p$ and $d(A\sigma_1)_x \not\perp p$. Therefore, by Theorem 3.5 $\hat{f}$ has a swallowtail (respectively, cuspidal butterfly) at $x$.

To illustrate the freedom given by Lie sphere transformations, we give the following example:

**Example 4.2.** The surface in $\mathbb{R}^3$ given by
\[ f(u, v) = (u, u^2, v) \]
satisfies $k_{1,u} = 0$ and $\kappa_{1,uu} \neq 0$ at $(u, v) = (0, 0)$. Let $\hat{f}$ be a Lie sphere transformation of $f$ determined by $A \in O(4, 2)$. Then by Theorem 4.1 if $\hat{f}_u = 0$ at $(0, 0)$ then $\hat{f}$ must have a type 2 singularity at $(0, 0)$. For example, $\hat{f}$ is a swallowtail at $(0, 0)$ if we use the Lie sphere transformation
\[
A = \begin{pmatrix}
\frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{-3}{2} & 1 \\
1 & 0 & 0 & 0 & -1 & 1 \\
\end{pmatrix}.
\]
Whereas, if we use, with $\chi = \sqrt{1 + 2\xi^2}$,

$$A^\xi = \begin{pmatrix} -\chi^{-1}(\frac{1}{\sqrt{2}} + \xi) & 0 & 0 & 0 & \chi^{-1}(\frac{1}{\sqrt{2}} - \xi) \\ \chi^{-1}\xi(-1 + \sqrt{2}\xi) & -\frac{\chi}{\sqrt{2}} & 0 & -\frac{\chi}{\sqrt{2}} & -\chi^{-1}\xi(1 + \sqrt{2}\xi) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{\chi}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} - \xi & \xi & 0 & \xi & 0 \\ \chi^{-1}\xi(1 + \sqrt{2}\xi) & -\frac{\chi}{\sqrt{2}} & 0 & -\frac{\chi}{\sqrt{2}} & -\chi^{-1}\xi(1 + \sqrt{2}\xi) \end{pmatrix},$$

we will have a cuspidal beaks (respectively, cuspidal lips) at $(0,0)$ if $\xi < 1/(2\sqrt{2})$ ($\xi > 1/(2\sqrt{2})$). When $\xi = 1/(2\sqrt{2})$, we obtain a less familiar type 2 degenerate singularity. Figure 4.1 shows how the singularity changes type as $\xi$ varies.

Figure 4.1: From left to right, cuspidal beaks ($\xi < 1/(2\sqrt{2})$), other type 2 degenerate singularity ($\xi = 1/(2\sqrt{2})$) and cuspidal lips ($\xi > 1/(2\sqrt{2})$), as in Example 4.2

4.2 Umbilic case

Now suppose that $x$ is an umbilic point of $\mathcal{F}$. Then for any Lie sphere transformation $A \in O(4,2)$, $x$ is an umbilic point of $A\mathcal{F}$. Let $\sigma, \tilde{\sigma} \in \Gamma\mathcal{F}$ such that $\sigma(x) \in s(x)$ and $\tilde{\sigma}(x) \notin s(x)$. Then, if we let $\sigma^A := A\sigma$ and $\tilde{\sigma}^A := A\tilde{\sigma}$, we have that $\sigma^A(x) \in As(x)$ and $\tilde{\sigma}^A(x) \notin As(x)$. We may then compute the cubic form $C^A_x(\sigma^A, \tilde{\sigma}^A)$ for $A\mathcal{F}$ in terms of these lifts, but since $A$ is constant and orthogonal it is clear that $C^A_x(\sigma^A, \tilde{\sigma}^A) = C_x(\sigma, \tilde{\sigma})$. By applying this to the spaceform projections $f : \Sigma \to \mathbb{R}^3$ and $\tilde{f} : \Sigma \to \mathbb{R}^3$, we obtain the following theorem as a corollary of Proposition 3.7 and Theorem 3.8:

**Theorem 4.3.** Suppose that $x$ is an elliptic (respectively, hyperbolic) umbilic point of $f$. If $\tilde{f}$ is immersed at $x$ then $x$ is an elliptic (respectively, hyperbolic) umbilic point of $\tilde{f}$. Otherwise, $\tilde{f}$ has a $D^-_4$ (respectively, $D^+_4$ singularity) at $x$.

This result extends the result of [11] for parallel transformations to the group of Lie sphere transformations.
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