Padé-Improvement of QCD Running Coupling Constants, Running Masses, Higgs Decay Rates, and Scalar Channel Sum Rules

V. Elias and T. G. Steele
Department of Physics and Engineering Physics
University of Saskatchewan
Saskatoon, Saskatchewan S7N 5C6
Canada
and
F. Chishtie, R. Migneron and K. Sprague
Department of Applied Mathematics
The University of Western Ontario
London, Ontario N6A 5B7
Canada

March 27, 2022

PACS: 11.10.Hi, 11.55.Hx, 12.38.Aw, 14.80.Bn

Abstract

We discuss Padé-improvement of known four-loop order results based upon an asymptotic three-parameter error formula for Padé-approximants. We derive an explicit formula estimating the next-order coefficient $R_4$ from the previous coefficients in a series $1 + R_1 x + R_2 x^2 + R_3 x^3$. We show that such an estimate is within 0.18% of the known five-loop order term in the $O(1)$ $\beta$-function, and within 10% of the known five-loop term in the $O(1)$ anomalous mass-dimension function $\gamma_m(g)$. We apply the same formula to generate a $[2|2]$ Padé-summation of the QCD $\beta$-function and anomalous mass dimension in order to demonstrate both the relative insensitivity of the evolution of $\alpha_s(\mu)$ and the running quark masses to higher order corrections, as well as a somewhat increased compatibility of the present empirical range for $\alpha_s(m_t)$ with the range anticipated via evolution from the present empirical range for $\alpha_s(M_Z)$. For $3 \leq n_f \leq 6$ we demonstrate that positive zeros of any $[2|2]$ Padé-summation estimate of the all-orders $\beta$-function which incorporates known two-, three-, and four-loop contributions necessarily correspond to ultraviolet fixed points, regardless of the unknown five-loop term. Padé-improvement of higher-order perturbative expressions is presented for the decay rates of the Higgs into two gluons and into a $b\bar{b}$ pair, and is used to show the relative insensitivity of these rates to higher order effects. However, Padé-improvement of the purely-perturbative component of scalar/pseudoscalar current correlation functions is indicative of large theoretical uncertainties in QCD sum rules for these channels, particularly if the continuum-threshold parameter $s_0$ is near 1 GeV$^2$.

*Permanent Address: Department of Applied Mathematics, The University of Western Ontario, London, Ontario N6A 5B7, CANADA
1. Introduction

A recent body of work [1-3] has demonstrated how higher order terms in a number of field-theoretical perturbative series can be estimated by Padé-approximant techniques.

Of particular interest are applications to QCD quantities, particularly $\beta$- and $\gamma$-functions now known to four-loop order in $\alpha_s$ [4-6]. Padé-approximant methods have already addressed the $n_f$ (flavour-number-) dependence of higher order terms in the QCD $\beta$- and $\gamma$-functions [2,3]. Near cancellations between coefficients of successive powers of $n_f$, however, can lead to large uncertainty in the estimated overall size of such higher-loop contributions – small errors in fitted coefficients of $n_f^k$ have been seen to lead to much larger errors in the aggregate (now known) four-loop contribution to the $\beta$-function [2].

In the present paper, our focus will be on using Padé-approximant methods to estimate the magnitude of higher-order corrections to quantities already calculated to three- and four-loop order in QCD. We assess the theoretical uncertainty of such calculations by seeing how closely they coincide with their own Padé-improvements, as well as whether successive orders of perturbation theory exhibit convergence toward Padé-summation estimates of the full perturbative series.

The present paper is phenomenologically oriented, specifically aimed at developing Padé-improved estimates of what are hoped to be computationally and/or experimentally accessible quantities. The particular items of interest considered are the $O(N)$ $\beta$-function and anomalous mass dimension (Section 2), the running QCD coupling constant (Section 3), the running quark masses (Section 4), the Higgs-boson decay rates into two gluons and into a $b \bar{b}$ pair (Section 5), and the purely perturbative content of QCD-sum rules based upon scalar and pseudoscalar current correlation functions (Section 6). In Section 3, we also discuss some general implications of Padé-summation as an approximation to all orders of perturbation theory. In particular, we analyze the fixed point structure of the most general [2 | 2] Padé-summation estimates of the full content of QCD $\beta$-functions for $n_f = \{3, 4, 5, 6\}$ whose Maclaurin expansions coincide with presently known perturbative contributions.

In the section that immediately follows, we discuss how Padé-approximant methods can be used to estimate higher-order corrections to perturbative series in which the leading four terms are known. Although this methodology appears in other work [3], the presentation leading to eq. (2.12) [which has not appeared elsewhere] will, it is hoped, be of some value to those unfamiliar with Padé-approximant methods. We also demonstrate the reasonable agreement of predictions based on (2.12) with now-known five-loop terms in $O(N)$ $\beta$- and $\gamma$-functions.

2. The APAP Algorithm: Padé-Improvement for Pedestrians

We consider the general problem of developing a Padé-improvement of the series

$$S \equiv 1 + R_1 x + R_2 x^2 + R_3 x^3 + ..., \quad (2.1)$$

where $\{R_1, R_2, R_3\}$ are known and $\{R_4, R_5, ...\}$ are not known, through use of the asymptotic error formula for estimating $R_{N+M+1} x^{N+M+1}$ via the Padé approximant

$$S_{[N|M]} = \frac{1 + a_1 x + a_2 x^2 + ... a_N x^N}{1 + b_1 x + b_2 x^2 + ... + b_M x^M} = 1 + R_1 x + R_2 x^2 + R_3 x^3 + ... + R_{N+M+1} x^{N+M+1} + ... \quad (2.2)$$
Let $R_{N+M+1}^{Padé}$ be the prediction one would obtain from the $[N|M]$ Padé Approximant, and let $R_{N+M+1}$ be the true value of the coefficient. The structure of the asymptotic error formula is given by [2]

$$\frac{R_{N+M+1}^{Padé} - R_{N+M+1}}{R_{N+M+1}} = -\frac{M!A^M}{[N + M + aM + b]^M}$$

(2.3)

with numbers $\{A, a, b\}$ (independent of $N, M$) to be determined.

This error formula simplifies considerably if $M$ is always chosen to be 1 [3]: the right hand side of (C) becomes $-A/[N + 1 + (a + b)]$, with only the two numbers $\{A, a + b\}$ to be determined. These can be determined explicitly for the series $S$ in (2.1) given knowledge of the three coefficients $\{R_1, R_2, R_3\}$.

Given knowledge of $R_1$ only, the $[0|1]$ Padé approximant

$$S_{[0|1]} = \frac{1}{1 + b_1x} = 1 - b_1x + b_1^2x^2... = 1 + R_1x + R_2^{Padé}x^2$$

(2.4)

predicts $R_2^{Padé} = R_1^2$. Consequently, we see from the asymptotic error formula (2.3) that

$$\frac{R_2^{Padé} - R_2}{R_2} = \frac{R_2^2 - R_2}{R_2} = \frac{-A}{1 + (a + b)} \equiv \delta_2$$

(2.5)

Given knowledge of only $R_1$ and $R_2$, the $[1|1]$ Padé approximant

$$S_{[1|1]} = \frac{1 + a_1x}{1 + b_1x} = 1 + (a_1 - b_1)x + b_1(a_1 - a_1)x^2 + b_1^2(a_1 - b_1)x^3 + ...$$

$$= 1 + R_1x + R_2x^2 + R_3^{Padé}x^3...$$

(2.6)

predicts $R_3^{Padé} = R_2^2/R_1$. The asymptotic error formula (2.3) for this case implies that

$$\frac{R_2^2 / R_1 - R_3}{R_3} = \frac{-A}{2 + (a + b)} \equiv \delta_3$$

(2.7)

Given knowledge of $\{R_1, R_2, R_3\}$ in the series (2.1), the relative errors $\delta_2$ and $\delta_3$ are specified completely by the left-hand sides of (2.5) and (2.7). These two equations may be regarded as two equations in the two unknowns $A$ and $(a + b)$ characterizing the asymptotic error formula (2.3) when $M = 1$. The solution to these two equations is

$$A = [1/\delta_2 - 1/\delta_3]^{-1},$$

(2.8)

$$(a + b) = \frac{\delta_2 - 2\delta_3}{\delta_3 - \delta_2}.$$  

(2.9)

This information is sufficient to generate an asymptotic Padé approximant (APAP) estimate [2,3] of the unknown coefficient $R_4$ in the series (2.1). Consider the $[2|1]$ Padé approximant

$$S_{[2|1]} = \frac{1 + a_1x + a_2x^2}{1 + b_1x}$$

$$= 1 + (a_1 - b_1)x + [a_2 - b_1(a_1 - b_1)]x^2$$

$$+ [-b_1[a_2 - b_1(a_1 - b_1)]]x^3$$

$$+ [b_1^2[a_2 - b_1(a_1 - b_1)]]x^4 + ...$$

$$= 1 + R_1x + R_2x^2 + R_3x^3 + R_4^{Padé}x^4$$

(2.10)
The three known values of \( \{R_1, R_2, R_3\} \) completely determine the three parameters \( \{a_1, a_2, b_1\} \) characterizing \( S_{[2]} \). We see from (2.10) that \( R_4^{\text{padé}} = R_2^3/R_2 \). However, the asymptotic error formula (2.3) suggests that a more accurate estimate of the true value \( R_4 \) differs from \( R_4^{\text{padé}} \) by a predictable relative error:

\[
\frac{R_3^2/R_2 - R_4}{R_4} = \delta_4 = \frac{-A}{3 + (a + b)},
\]

in which case we find from (2.11), (2.8) and (2.9) that

\[
R_4 = \frac{R_3^2/R_2}{1 + \delta_4} = \frac{R_3^2(\delta_2 - 2\delta_1)}{R_2(\delta_2 - 2\delta_1 - \delta_1\delta_2)} = \frac{R_3^2(R_3^2 + R_1R_2R_3 - 2R_1^3R_3)}{R_2(2R_2^3 - R_1^3R_3 - R_1^2R_2^2)}. \tag{2.12}
\]

As an example, we test the applicability of (2.12) by comparing its prediction to the known \( O(g^6) \) coefficient of the \( O(1) \) \( \beta \)-function [7]:

\[
\beta^{(1)}(g) = 1.5g^2\left[1 - (17/9)g + 10.8499g^2 - 90.5353g^3 + 949.523g^4 + O(g^5)\right]. \tag{2.13}
\]

Identifying \( R_1 = -17/9, R_2 = 10.8499 \) and \( R_3 = -90.5353 \), we find from (2.12) that \( R_4 \) = 947.8 in startlingly close agreement to the next \((g^4)\) term within (2.13). Although APAP improvement has also been applied elsewhere [2] to \( O(g^6) \) terms in \( O(N) \) \( \beta \)-functions, the result obtained here relies on direct and explicit use of the full asymptotic error formula (2.3). A comparison of (2.12) predictions and exact values of \( \beta_4 \), the \( O(N) \) \( \beta \)-function coefficient of \( g^6 \), is presented in Table I. We emphasize that these predictions are not obtained by a fitting of the \( N \)-dependence or any knowledge of the \( N^2 \)-dependence of \( \beta^{(4)} \), as is the case in Table 3 of ref. [2]; for comparative purposes, the predictions of ref. [2] are also listed in Table I.

We can use (2.12) to predict the known \( R_4 \) coefficient within the \( O(1) \) theory’s anomalous mass dimension [7], as well:

\[
\gamma_m(g) = (g/2)\left[1 - 0.8333g + 3.500g^2 - 19.96g^3 + 150.8g^4 + O(g^5)\right]. \tag{2.14}
\]

Eq. (2.12) predicts \( R_4 = 135.1 \), a result only 10% off the 150.8 value given in (2.14). Table I shows that predicted \( R_4 \) coefficients for \( \gamma_m(g) \) within \( O(2), O(3), \) and \( O(4) \) also remain within 20% of their true values, as given in [7]. These results provide a reasonable basis for applying (2.12) [and its concomitant asymptotic error formula (2.3)] to the \( \beta \)- and \( \gamma \)-functions of QCD, as we will do in Sections 3 and 4.
arbitrarily truncating the series after the $R_4 x^4$ term. Given known values of $\{R_1, R_2, R_3\}$ and using the APAP estimate (2.12) for $R_4$, the approximant $S_{[2]}$ of the infinite series $S$ is fully determined:

\[
S \rightarrow S_{[2]} = \frac{1 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2},
\]

(2.15)

\[
b_1 = \frac{R_1 R_4 - R_2 R_3}{R_2^2 - R_1 R_3},
\]

(2.16)

\[
b_2 = \frac{R_3^2 - R_2 R_4}{R_2^2 - R_1 R_3},
\]

(2.17)

\[
a_1 = R_1 + b_1,
\]

(2.18)

\[
a_2 = R_2 + b_1 R_1 + b_2.
\]

(2.19)

Eq. (2.15), as determined from (2.12) and (2.16-19), constitutes the procedure we denote as “Padé-summation” of the series $S$ in (2.1).

3. Pade-Improvement of the QCD Coupling

The QCD minimal subtraction (MS or $\overline{\text{MS}}$) renormalization-group functions $\beta(x)$ and $\gamma(x)$ are now known to 4-loop order [4,5,6]. Prior work involving Padé-improvement methods has attempted to predict the flavour-dependence of these functions to 4- and 5-loop order [2,3]. $\beta_3$ and $\beta_4$, the 4- and 5-loop order corrections to the $\beta$-function, are respectively third- and fourth-degree polynomials in $n_f$, and Padé methods have already shown some success in predicting the polynomial coefficients now known for $\beta_3$ [2]. However, an accurate determination of the polynomial coefficients within $\beta_3$ is not reflected in the accuracy with which $\beta_3$ is itself determined. Thus the overall Padé-driven estimate of $\beta_3$ presented in [2] for $n_f = 3$ [$\beta_3 = (7.6 \pm 0.1) \cdot 10^3/256 = 30 \pm 1$, using normalization conventions appropriate for (3.1) below] is substantially below the true value ($\beta_3 = 47.23$), even after allowances are made for claimed uncertainties arising from quadratic Casimir contributions [3]. It should also be noted that this estimate, once disentangled from a fitting procedure aimed at ascertaining the explicit $n_f$ dependence of $\beta_3$, follows from a simplified version of the asymptotic error formula (2.3), in which the denominator $(N + M + aM + b)^M$ is taken to be $N^M$ [2]. Indeed, it is difficult to tell at this stage whether the discrepancy between Padé-estimates of $\beta_3$ and the true value arises primarily from quadratic-Casimir contributions not occurring in lower orders, or alternatively, arises from the error involved in simplifying the asymptotic error formula in order to make an estimate of $\beta_3$ possible. Without such a simplification, the error formula (2.3) has (in principle) three arbitrary constants $(A,a,b)$ instead of one $(A)$.

Consequently, in this section we will predict $\beta_4$ directly using the APAP-algorithm (2.12) following from the full asymptotic error formula (2.3). We will not prejudice these predictions of $\beta_4$ by attempting a fit of the polynomial dependence on $n_f$, nor will we attempt to disentangle quadratic-and-higher Casimir contributions from $\beta_3$ and $\beta_4$. Rather, we will make distinct predictions of $\beta_4$ via (2.12) for $n_f = 3, 4, 5, 6$. The validity of such an approach, particularly the possibility that the full asymptotic error formula is inclusive of higher-order Casimir contributions to $\beta(x)$, would be best established by comparison to an exact calculation of $\beta_4$, when available.

We define the $\beta$-function as in [4]:
\[ \mu^2 \frac{d}{d\mu^2} x = -x^2 \sum_{i=0}^{\infty} \beta_i x^i \]
\[ = -\beta_0 x^2 \sum_{i=0}^{\infty} R_i x^i, \quad (3.1a) \]
with \( x \equiv \alpha_s(\mu)/\pi \) and
\[ R_i \equiv \beta_i/\beta_0, \quad (3.1b) \]
Known values of \( \beta_0 - \beta_3 \) are as follows [4,5]:
\[ \beta_0 = (11 - 2n_f/3)/4, \quad (3.2a) \]
\[ \beta_1 = (102 - 38n_f/3)/16, \quad (3.2b) \]
\[ \beta_2 = (2857/2 - 5033n_f/18 + 325n_f^2/54)/64, \quad (3.2c) \]
\[ \beta_3 = 114.23033 - 27.133944n_f + 1.5823791n_f^2 \]
\[ + 5.85669582 \cdot 10^{-3}n_f^3. \quad (3.2d) \]
We use (3.2), (3.1b), and (2.12) to predict the following values for \( \beta_4 \):
\[ n_f = 3 : \quad R_4 = -849.74, \quad \beta_4 = -1911.9; \quad (3.3) \]
\[ n_f = 4 : \quad R_4 = 40.203, \quad \beta_4 = 83.7563; \quad (3.4) \]
\[ n_f = 5 : \quad R_4 = 70.203, \quad \beta_4 = 134.56; \quad (3.5) \]
\[ n_f = 6 : \quad R_4 = -239.22, \quad \beta_4 = -418.64. \quad (3.6) \]
The large negative values for \( n_f = 3 \) and \( n_f = 6 \) reflect the near cancellation of the factor \((2R_2^3 - R_3^2R_3 - R_1^2R_2^2)\) in the denominator of (2.12). Since a change of sign can easily occur if this cancellation is over- or under-estimated, the safest interpretation of (3.3) and (3.6) is to predict a relatively large magnitude for \( \beta_4 \), with the sign uncertain.

Corresponding results from Table III of reference [3] with quadratic Casimir contributions and a 1/1024 normalization factor appropriate to (3.1a) included are \( \beta_4 = \{278(n_f = 3), 202(n_f = 4), 165(n_f = 5), 166(n_f = 6)\} \). As stated earlier, these latter results are based upon a fit to the polynomial coefficients of \( n_f^k \) in \( \beta_4 \), with near cancellation of very large opposite-sign coefficients for the \( k = 0 \) and \( k = 1 \) terms. These latter results appear most consistent with those obtained above when \( n_f = 5 \).

To get a feeling of the magnitude of these Padé estimates of 5-loop effects, we generate the \([2|2]\) approximant (2.15) from the known values for \( R_1, R_2, \) and \( R_3 \) and our estimate of \( R_4 \), and incorporate this approximant directly into the \( \beta \)-function \( [\mu^2 d\beta/\mu^2 = \beta(x)] \):
\[ n_f = 3 : \quad \beta(x) = -\frac{9x^2}{4} \left[ \frac{1 + 94.383x - 75.605x^2}{1 + 92.606x - 244.71x^2} \right], \quad (3.7a) \]
\[
\beta(x) = \begin{cases} 
\frac{-25x^2}{12} \left[ 1 - 5.8963x - 4.0110x^2 \right] & \text{for } n_f = 4, \\
\frac{23x^2}{12} \left[ 1 - 5.9761x - 6.9861x^2 \right] & \text{for } n_f = 5,
\end{cases}
\]

We can use these Padé approximations to the full β-function to evolve \(\alpha_s(\mu) = \pi x(\mu)\) down to \(\mu = 1\) GeV from an initial condition \(\alpha_s(M_z) = 0.118\) [8] through use of the following (4- and 5-) flavour threshold matching conditions with \(m(\mu_{th}) = \mu_{th}\) [9]:

\[
x^{(n_f-1)}(\mu_{th}) = x^{(n_f)}(\mu_{th}) \left[ 1 + 0.1528 \left( x^{(n_f)}(\mu_{th}) \right)^2 + \{0.9721 - 0.0847(n_f - 1)\} \left( x^{(n_f)}(\mu_{th}) \right)^3 \right] .
\]

In Figure 1 we display the evolution of \(\alpha_s(\mu)\) to low energies through use of two-loop, three-loop, four-loop, and the Padé-improved four-loop β-functions of (3.7). All of these curves are generated from the initial condition \(\alpha_s(M_z) = 0.118\), with \(\mu_{th} = 4.3\) GeV (i.e. \(m_b(m_b) = 4.3\) GeV) identified as the \(n_f = 5\) flavour threshold, and with \(\mu_{th} = 1.3\) GeV identified as the \(n_f = 4\) flavour threshold. Eq. (3.8) is utilized in full in both the four-loop and Padé improved calculations; it is utilized to \(\mathcal{O}(x^2)\) to generate flavour-threshold initial conditions in the three-loop calculation, and the matching condition is trivial for the two-loop calculation. It is evident from the figure that curves from successive orders of the β-function appear to converge from below to that generated via (3.7), the Padé-summation approximating all orders. The gaps between curves of successive order clearly narrow as the order increases. Figure 1 shows that the Padé-summation leads to a curve for \(\alpha_s(\mu)\) that exceeds the unimproved four-loop curve by less than 1%. Such a difference is inconsequential compared to the estimated uncertainties in \(\alpha_s(M_z)\) and \(\mu_{th}\) for four- and five-flavour thresholds. Both the four-loop and the Padé-improved curve are indicative of benchmark values \(\alpha_s(1\text{GeV}) = 0.48\) and \(\alpha_s(m_{\tau}) = 0.32\). However, if the flavour thresholds and \(\alpha_s(M_z)\) are assigned their accepted [8] lower-bound values \([\alpha_s(M_z) = 0.115, \mu_{th}^{(n_f=5)} = 4.1\text{ GeV}, \mu_{th}^{(n_f=4)} = 1.0\text{ GeV}]\), the values we obtain at \(\mu = 1\) GeV and \(\mu = m_{\tau}\) from either four-loop or Padé-summation β-functions are \(\alpha_s(1\text{GeV}) = 0.41\) and \(\alpha_s(m_{\tau}) = 0.29\). Corresponding upper-bound values \([\alpha_s(M_z) = 0.121, \mu_{th}^{(n_f=5)} = 4.5\text{ GeV}, \mu_{th}^{(n_f=4)} = 1.6\text{ GeV}]\) lead to \(\alpha_s(1\text{GeV}) = 0.57\) and \(\alpha_s(m_{\tau}) = 0.35\). In view of these (much-) larger-than-1% uncertainties in low-energy values, the best possible test at present of Padé-improvement would be nonempirical, i.e. a comparison to an explicit 5-loop calculation of the β-function.

We note, however, that higher-order effects do appear to increase the overlap between the present (somewhat large) empirical range for \(\alpha_s(m_{\tau})\) \((0.370 \pm 0.033\) [8]) and the range predicted via evolution down from the present empirical range for \(\alpha_s(M_z)\) \((0.118 \pm 0.003\) [8]). Taking into account the present uncertainty in the five-flavour threshold \((\mu_{th} = 4.3 \pm 0.3\) GeV [8]) and incorporating the matching condition (3.8) for \(\alpha_s(\mu)\) below and above \(\mu_{th}\), we find the following predicted ranges for \(\alpha_s(m_{\tau})\) for two-loop, three-loop, four-loop and Padé-summation β-functions:

\[
\begin{align*}
0.2910 & \leq \alpha_s(m_{\tau}) \leq 0.3391, \\
[\alpha_s(m_{\tau})]_{LO} & = 0.3137,
\end{align*}
\]
3–Loop:
\[0.2944 \leq \alpha_s(m_\tau) \leq 0.3451,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3182,\]  
\[0.2944 \leq \alpha_s(m_\tau) \leq 0.3451,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3182,\]  
\[0.2957 \leq \alpha_s(m_\tau) \leq 0.3477,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3200,\]  
\[0.2963 \leq \alpha_s(m_\tau) \leq 0.3489,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3208.\]  

4–Loop:
\[0.2957 \leq \alpha_s(m_\tau) \leq 0.3477,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3200,\]  
\[0.2963 \leq \alpha_s(m_\tau) \leq 0.3489,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3208.\]  

Padé-Improved:
\[0.2963 \leq \alpha_s(m_\tau) \leq 0.3489,\]  
\[\left[\alpha_s(m_\tau)\right]_{cv} = 0.3208.\]  

The ranges listed above progressively overlap the low end of the present experimental range \(0.337 \leq \alpha_s(m_\tau) \leq 0.403\). The central values (\(cv\)) displayed above are evolved down from \(\alpha_s(M_z) = 0.118\) with a five-flavour threshold at \(\mu_{th} = 4.3\) GeV. The lower bounds evolve from \(\alpha_s(M_z) = 0.115\) with \(\mu_{th} = 4.0\) GeV, and the upper bounds evolve from \(\alpha_s(M_z) = 0.121\) with \(\mu_{th} = 4.6\) GeV.

There are also some unexpected theoretical consequences arising from estimating the summation of the full \(\beta\)-function series through use of an appropriately chosen Padé approximant [1]. The most general [2] approximant must yield \(1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4\) as the first five terms of its Maclaurin expansion. For optimal generality, we require that \(R_1, R_2,\) and \(R_3\) be given by (3.1b) and (3.2), but allow \(R_4\) to be arbitrary. We then find that
\[
\beta(x) = -\beta_0 x^2 \left[1 + a_1 x + a_2 x^2 \right] \left[1 + b_1 x + b_2 x^2 \right],
\]
with \(\{a_1, a_2, b_1, b_2\}\) linear in \(R_4\) as follows:

\(n_f = 3:\)
\[
a_1 = 7.1945 - 0.10261 R_4, \tag{3.14a}
\]
\[
a_2 = -11.329 + 0.075643 R_4, \tag{3.14b}
\]
\[
b_1 = 5.4168 - 0.10261 R_4, \tag{3.14c}
\]
\[
b_2 = -25.430 + 0.25806 R_4; \tag{3.14d}
\]

\(n_f = 4:\)
\[
a_1 = 4.8401 - 0.11068 R_4, \tag{3.15a}
\]
\[ a_2 = -8.1842 + 0.10836R_4, \]  
\[ b_1 = 3.3001 - 0.11068R_4, \]  
\[ b_2 = -16.314 + 0.21904R_4 \]  
(3.15b)

\[ n_f = 5: \]
\[ a_1 = 2.6793 - 0.12329R_4, \]  
\[ a_2 = -6.19671 - 0.011245R_4, \]  
\[ b_1 = 1.4184 - 0.12329R_4, \]  
\[ b_2 = -9.4599 + 0.14421R_4; \]  
(3.16a)

\[ b_1 = -0.31772 - 0.18424R_4, \]  
\[ b_2 = -6.0423 - 0.057574R_4. \]  
(3.16d)

\[ b_2 = -9.4599 + 0.14421R_4 \]  
(3.17d)

\[ n_f = 6: \]
\[ a_1 = 0.61085 - 0.18424R_4, \]  
\[ a_2 = -6.6275 - 0.24181R_4, \]  
\[ b_1 = -0.31772 - 0.18424R_4, \]  
\[ b_2 = -6.0423 - 0.057574R_4. \]  
(3.17d)

For all \( n_f \) values listed above, the first positive zero of \( 1 + a_1 x + a_2 x^2 \) in (3.13) is found to be above the first positive zero of \( 1 + b_1 x + b_2 x^2, \) regardless of the choice for \( R_4. \) Consequently, the smallest positive zero of \( \beta(x), \) if given by (3.13), is necessarily an ultraviolet fixed point and not an infrared fixed point, an inescapable result of the denominator sign change for \( x \) between 0 and the first positive zero of (3.13). Moreover, for those values of \( R_4 \) for which a second positive zero of \( 1 + a_1 x + a_2 x^2 \) is possible, we find from (3.14-17) that a second positive zero of \( 1 + b_1 x + b_2 x^2 \) will also occur at some value of \( x \) between the two positive zeros of \( 1 + a_1 x + a_2 x^2. \) This ensures that neither positive zero of \( 1 + a_1 x + a_2 x^2 \) corresponds to an infrared fixed point.

In Figure 2, a schematic diagram is presented showing different branches for the evolution of \( x(\mu) \) anticipated from a \( \beta \)-function (3.13) with the above-described alternation of positive denominator and numerator zeros, with the smallest positive zero occurring for the denominator. Zeros of \( 1 + b_1 x + b_2 x^2 \) represent values of \( x \) for which \( x(\mu) \) has infinite slope. Zeros of \( 1 + a_1 x + a_2 x^2 \) are fixed-point values of \( x. \) As is evident from the figure, all such fixed points are necessarily ultraviolet, and the infrared region is inaccessible for values of \( \mu \) less than those corresponding to zeros of the denominator. Such behaviour suggests...

1) ...the possible existence of a strong phase of QCD at short-distances, reflective of a nonzero ultraviolet fixed point, and
2) ... the inapplicability of a perturbative theory of quarks and gluons to the infrared region, specifically the region excluded from the domain of the first positive branch of $x(\mu)$ [Fig 2].

The former statement above may have ramifications for scenarios of dynamical electroweak symmetry breaking that usually involve a distinct technicolour group. The latter statement parallels old infrared slavery ideas, except that the inapplicability of perturbation theory to low energies is not seen to follow from the coupling constant growing infinite (or nonperturbatively large), as in infrared slavery, but from an explicit decoupling of the infrared region from the ultraviolet by virtue of $\beta$-function singularities alternating with $\beta$-function zeros (Fig. 2). We have verified that this alternation occurs in the most general $[2|2]$ Padé-summation of the $\beta$-function even when $n_f = 0$.

4. Padé-Improvement of the Running Mass

The running mass $m(\mu)$ satisfies the differential equation

$$\frac{dm}{dx} = m \frac{\gamma(x)}{\beta(x)}$$  \hspace{1cm} (4.1)

where $x \equiv \alpha_s(\mu)/\pi$, as in the previous section, and where $\beta(x) (\equiv \mu^2 dx/d\mu^2)$ is given by (3.1a) and (3.2). The QCD MS anomalous mass dimension function $\gamma(x)$ has been calculated to four-loop order [4,6]:

$$\gamma(x) = -x[1 + \sum_{i=1}^3 \gamma_i x^i]$$  \hspace{1cm} (4.2a)

$$\gamma_1 = 4.20833 - 0.138889n_f$$  \hspace{1cm} (4.2b)

$$\gamma_2 = 19.5156 - 2.28412n_f - 0.0270062n_f^2$$  \hspace{1cm} (4.2c)

$$\gamma_3 = 98.9434 - 19.1075n_f + 0.276163n_f^2 - 0.00579322n_f^3.$$  \hspace{1cm} (4.2d)

Padé improvement of the square-bracketed expression within (4.2a) is straightforward via the methods of Section 2 by identifying $R_1, R_2$ and $R_3$ in (2.1) with $\gamma_1, \gamma_2$ and $\gamma_3$. Using (2.12) we obtain the following APAP estimates for $\gamma_4$:

$$n_f = 3 : \gamma_4 = 162.987$$  \hspace{1cm} (4.3)

$$n_f = 4 : \gamma_4 = 75.2349$$  \hspace{1cm} (4.4)

$$n_f = 5 : \gamma_4 = 12.5550$$  \hspace{1cm} (4.5)

$$n_f = 6 : \gamma_4 = 12.1820$$  \hspace{1cm} (4.6)

One can obtain a solution for $m[x(\mu)]$ which includes the $O(x^4)$ Padé improvement of $\beta(x)$ and $\gamma(x)$ by expressing $\gamma(x)/\beta(x)$ as a Maclaurin series in $x$, using APAP estimates(3.3-6) for $\beta_4$ and (4.3-6) for $\gamma_4$. By truncating this series after $x^4$, the differential equation (4.1) can be approximated by

$$\frac{x \frac{dm}{dx}}{m} = \left[\beta_0^{-1} + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4\right],$$  \hspace{1cm} (4.7)
with $\beta_0$ given by (3.2a), and with $d_i$ given as follows:

\[
\begin{align*}
n_f = 3 : & \quad d_1 = 0.895063, \ d_2 = 1.94172, \ d_3 = 2.88956, \ d_4 = 417.493; \\
n_f = 4 : & \quad d_1 = 1.014131, \ d_2 = 1.74994, \ d_3 = 0.0880435, \ d_4 = -3.93256; \\
n_f = 5 : & \quad d_1 = 1.17549, \ d_2 = 0.809817, \ d_3 = -1.05016, \ d_4 = -10.0138; \\
n_f = 6 : & \quad d_1 = 1.39796, \ d_2 = 1.63266, \ d_3 = -6.84005, \ d_4 = 142.769; \\
\end{align*}
\]

The values for $d_1, d_2$ and $d_3$ are exactly determined by four-loop calculations of $\beta(x)$ and $\gamma(x)$. The value of $d_4$ is underlined to emphasize that it is determined from APAP estimates. The large values for $d_4$ when $n_f = 3$ and $n_f = 6$ reflect correspondingly large values for $\beta_4$ that are discussed in the previous section.

The solution to (4.7) can be expressed in terms of $x(\mu)$ evaluated at two different values of $\mu$: $x(\mu_1) \equiv x_1, x(\mu_2) \equiv x_2$, where $x(\mu)$ is the running coupling whose evaluation is discussed in the previous section. This solution to (4.7) is [4]

\[
m(x_2) = m(x_1)c(x_2)/c(x_1),
\]

where

\[
c(x) = x^{1/\beta_0} \left\{ 1 + d_1 x + \left[ (d_1^2 + d_2)/2 \right] x^2 \\
+ \left[ (d_1^3 + 2d_3 + 3d_1d_2)/6 \right] x^3 \\
+ \left[ (d_1^4 + 3d_2^2 + 6d_4 + 6d_1^2d_2 + 8d_1d_3)/24 \right] x^4 \right\}.
\]

Coefficients of $x, x^2,$ and $x^3$ are determined in full by known coefficients in the four-loop $\beta$ and $\gamma$ functions. The $x^4$ term is the lowest-degree term sensitive to Padé driven estimates of $\beta_4$ and $\gamma_4$. We find the following set of expressions for $c(x)$ from the $d_i$ in (4.8-11):

\[
\begin{align*}
n_f = 3 : & \quad c(x) = x^{4/9} \left[ 1 + 0.895063x + 1.37143x^2 \\
& + 1.95168x^3 + 106.122x^4 \right] \\
\end{align*}
\]

\[
\begin{align*}
n_f = 4 : & \quad c(x) = x^{12/25} \left[ 1 + 1.01413x + 1.38920x^2 \\
& + 1.09052x^3 - 0.0765827x^4 \right], \\
\end{align*}
\]

\[
\begin{align*}
n_f = 5 : & \quad c(x) = x^{12/25} \left[ 1 + 1.17549x + 1.50071x^2 \\
& + 0.172486x^3 - 10.2813x^4 \right], \\
\end{align*}
\]
\[ n_f = 6 : \quad c(x) = x^{4/7} \left[ 1 + 1.39796x + 1.79347x^2 \right. \]
\[ \left. - 0.683486x^3 + 33.7949x^4 \right]. \quad (4.16) \]

These same expressions are obtained to \( \mathcal{O}(x^3) \) in ref. [4]; the effects of Padé-improvement reside entirely in the \( x^4 \) terms.

It is important to recognise that these results ultimately derive from applying the asymptotic error formula (2.3) to the perturbative field-theoretical calculation of \( \beta(x) \) and \( \gamma(x) \), as argued in [2] and [3]. As in the previous section, the results (4.3-6) differ from those one would obtain using the fits of ref. [3] to the coefficients of \( n_f^i \) within \( \gamma_4 \), particularly as \( \gamma_4 \) so extracted involves the near cancellation of large terms from successive values of \( k \): from Table X of ref. [3] one finds for \( n_f = 5 \) that \( \gamma_4 = 530 - (143) \cdot 5 + (6.67) \cdot 5^2 + (0.037) \cdot 5^3 - (8.54 \cdot 10^{-5}) \cdot 5^4 = -13.7 \). Small variations in these Padé-estimated coefficients can easily lead to positive values comparable to (4.5).

It is also important to note that the application of the APAP algorithm at the field-theoretical level — i.e., to \( \beta(x) \) and \( \gamma(x) \) — is not equivalent to applying it to “perturbative” expressions which are obtained by integrating over these functions. One could question, for example, whether the \( x^4 \) terms appearing in (4.13-16) might be obtainable by direct application of the APAP algorithm (2.12) to lower-degree terms in \( x \). If we apply (2.12) directly to (4.13-16) using the explicit coefficients of \( x, x^2 \) and \( x^3 \) in order to estimate the coefficients of \( x^4 \), the \( x^4 \)-coefficients we obtain are very different from those listed. Instead we obtain respectively for \( n_f = \{3, 4, 5, 6\} : 2.683x^4, 0.7426x^4, 0.01839x^4, \) and \( 0.2850x^4 \). This discrepancy is indicative of the inapplicability of the error formula (2.3) to the series in (4.12b), assuming (2.3) is applicable to the perturbative field-theoretical series (3.1a) and (4.2a). Such applicability is suggested by the predictions of five-loop terms for \( \mathcal{O}(N) \) \( \beta \)- and \( \gamma \)-functions already noted in Section 2.

Figure 3a-d display the relative impact on running quark masses of higher order corrections both augmented and unaugmented by Padé-improvement. Given an initial value \( m_b(4.3 \text{ GeV}) = 4.3 \text{ GeV} \) [8], we evolve \( m_b(\mu) \) up to \( \mu = 175 \text{ GeV} \). Fig 3a indicates the evolution obtained via (4.1) from three-loop \( \beta \)- and \( \gamma \)-functions — i.e. from truncation of the series (3.1a) and (4.2a) after \( i = 2 \). Fig. 3b displays the relative effects of higher-order corrections augmented and unaugmented by Padé-improvement. We first consider the unaugmented 4-loop case. The upper curve in Fig 3b is the ratio of \( m_b \) obtained from 4-loop \( \beta \)- and \( \gamma \)-functions to \( m_b \) obtained from 3-loop \( \beta \)- and \( \gamma \)-functions (Fig 3a). As \( \mu \) increases from 4 GeV to 175 GeV, the relative decrease in \( m_b(\mu) \) from use of four-loop information is seen to be less than 0.1%.

Padé-improvement of four-loop results is displayed in the lower curve of Fig 3b. This curve is the ratio of a fully Padé-improved estimate of \( m_b(\mu) \) to the three-loop calculation of \( m_b(\mu) \) displayed in Fig 3a. The full Padé-improvement is obtained via an APAP-algorithm determination of \( \beta_4 \) and \( \gamma_4 \), which is then utilized to construct Padé-summation [2][2] approximants as estimates of the aggregate effect of all higher-order terms in \( \beta \) and \( \gamma \). The [2][2] approximant for \( \beta(x) \) is given by (3.7c); the [2][2] approximant within

\[ \gamma(x) = -x \left[ 1 + 1.19485x + 1.02765x^2 \right. \]
\[ \left. \left/ \left( 1 - 2.31904x + 1.75664x^2 \right) \right. \right] \quad (4.17) \]

is obtained via (2.15-19) using the \( n_f = 5 \) values of \( \gamma_1, \gamma_2 \), and \( \gamma_3 \) as well as the APAP-algorithm value for \( \gamma_4 \) given in (4.5). The full [2][2] approximants are then used to evaluate \( c(x) \) in (4.12a): \( c(x) = \exp\left[ \int \frac{\gamma(x)}{\beta(x)} \right] dx \). It is evident from Fig 3b that Padé-improvement does not significantly alter \( m_b(\mu) \) beyond a correction comparable to five-loop expectations; the relative change from such Padé
improvement is respectively within 0.1% (Fig 3b) and 0.01% of the unimproved three- and four-loop results.

Fig. 3c displays the corresponding evolution of the charmed quark mass $m_c(\mu)$ for $1.3 \text{ GeV} \leq \mu \leq 20 \text{ GeV}$, as obtained from $\beta$- and $\gamma$-functions evaluated to two-, three-, and four-loop order as well as from Padé-improvement of the four-loop $\beta$- and $\gamma$-functions, both below and above the five-flavour threshold. The initial value is taken to be $m_c(1.3 \text{ GeV}) = 1.3 \text{ GeV}$ for all four curves, and the five-flavour threshold is assumed to occur at 4.3 GeV [8]. The “Padé” curve is obtained by direct substitution of appropriate $[2|2]$ Padé-summation $\beta$- and $\gamma$-functions into (4.1). At the five-flavour threshold, we utilize the threshold-matching constraint [9]

$$m_q^{(n_f)}(\mu_{th}) = m_q^{(n_f-1)}(\mu_{th}) \left[ 1 + 0.2060 \left( \alpha_s^{(n_f)}(\mu_{th})/\pi \right)^2 \right]^{-1}$$

with $n_f = 5$ and $\mu_{th} = 4.3 \text{ GeV}$ to generate the above-threshold initial condition for $m_c$. Thus, the above-threshold portion of the Padé curve in Fig. 3c is obtained from this initial condition via substitution of (3.7c) and (4.17) into the differential equation (4.1). The below threshold portion of the Padé curve is obtained from the initial condition $m_c(1.3 \text{ GeV}) = 1.3 \text{ GeV}$ via substitution of the Padé-summation four-flavour $\beta$-function (3.7b) and four-flavour $\gamma$-function

$$\gamma(x) = -x \left[ \frac{1 - 0.4541x - 1.3454x^2}{1 - 4.1069x + 3.7090x^2} \right]$$

into the differential equation (4.1). The constraint (4.18) is also utilized to generate the four-loop curve in Fig. 3c, and [when taken to order-$\left( \alpha_s^{(n_f)}(\mu_{th})/\pi \right)^2$] to generate the three-loop curve. As in Fig. 1, curves of successive order appear to converge (at least qualitatively) to the Padé estimate, which is almost indistinguishable from the four-loop curve.

Padé-improvement effects are somewhat larger for light quarks (u, d, s). The evolution of light quarks from an initial value (normalized to unity) at $\mu = 1 \text{ GeV}$ is displayed in Fig. 3d. This latter set of curves is obtained via utilization of (4.18) at both $n_f = 4$ and $n_f = 5$ flavour thresholds, which are respectively taken to be at 1.3 GeV and 4.3 GeV [8]. Below the 1.3 GeV four-flavour threshold, the Padé curve is generated via (4.1) using the $n_f = 3$ Padé-summation $\gamma$-function

$$\gamma(x) = -x \left[ \frac{1 - 1.2373x - 1.8485x^2}{1 - 5.0289x + 4.7993x^2} \right]$$

and the $n_f = 3$ $\beta$-function (3.7a). Between the four- and five-flavour thresholds we utilize (4.19) and (3.7b), and above the five-flavour threshold, we utilize (3.17) and (3.7c) as before. At $\mu = 5 \text{ GeV}$, Fig. 3d shows that there is a 1% difference between running masses obtained via unimproved and Padé-improved four-loop $\beta$- and $\gamma$-functions. Once again, however, the distance between curves of successive order decreases as the order increases, giving the appearance of convergence towards the Padé-improved curve.

5. Application to Higgs Decays

Although the Higgs particle has yet to be directly observed, expressions for its decay into either two-gluons [10] or a $b\bar{b}$ pair [11] have been worked out with precision in perturbation theory. In much the
same way, knowledge of the $Z^0$ decay widths, whose precise values are important for bounds on standard-model parameters, preceded the discovery of the $Z^0$ itself. In this section, we apply Padé-improvement to the decay processes $H \rightarrow$ two gluons and $H \rightarrow b\bar{b}$, and examine whether such improvement leads to detectable changes from the calculated rates obtained without such Padé improvement.

### 5.1 Higgs $\rightarrow$ Two Gluons

The decay rate $H \rightarrow gg$ has been calculated to three-loop order in perturbation theory [10]:

$$
\Gamma(H \rightarrow gg) = \frac{G_F M_H^3 x_H^2}{36\pi\sqrt{2}} \times [1 + 17.9167 x_H] + \left(156.808 - 5.70833 \ln(m_b^2/M_H^2)\right)x_H^2 + \mathcal{O}(x_H^3),
$$

(5.1)

where $x_H = x(M_H) = \alpha_s(M_H)/\pi$, and where $M_H$ is assumed to be less than $m_t$. Padé-improvement can enter this expression both in the actual value of $\alpha_s(M_H)$ evolving from a Padé-improved $\beta$-function, as well as in a Padé-driven estimate of the $\mathcal{O}(x_H^3)$ contribution to the square bracketed expression in (5.1). One cannot apply the APAP-algorithm of Section 2 to estimate this term because for a given value of $M_H$, only the coefficient of $x$ ($R_1 = 17.9167$) and $x^2$ [$R_2 = 156.808 - 5.70833 \ln(m_b^2/M_H^2)$] are known; the coefficient $R_3$ of $x^3$ is not known. One way to estimate $R_3$ is to express $1+R_1x+R_2x^2$ as a [1\,1] Padé approximant, which upon expansion yields $R_3 = R_3^2/R_1$. A refinement on this estimate that is actually utilized to approximate $\beta_3$ in ref. [2] is to assume in the asymptotic error formula (2.3) that $a+b$ is small compared to $N+1$. One can then argue from (2.5) and (2.7) that $\beta_3 \simeq \beta_2/2$, where $\beta_2 = (R_1^2 - R_2)/R_2$ is determined in full by $R_1$ and $R_2$. Eq. (2.7) can then be rearranged to yield the following estimate of $R_3$:

$$
R_3 = \frac{R_3^2/R_1}{1 + \beta_3} = \frac{2R_3^2}{R_3^2/R_1 + R_1 R_2},
$$

(5.2)

For different $M_H$ values we plot in Fig. 4 the ratio of the Padé-improved $H \rightarrow gg$ rate to the rate obtained directly from (5.1), using the three-loop $\beta$ function to obtain $x_H$ from the initial condition $x(M_Z) = 0.118/\pi$ [8]. The Padé-improvement of the $H \rightarrow gg$ rate is obtained for a given choice of $M_H$ first by evolving $x_H$ from the same initial condition via the $[2\,2]$ approximant (3.7c) for the $\beta$-function, then by using (5.2) to estimate the $\mathcal{O}(x_H^3)$ term in (5.1), and finally by replacing the now-known cubic $1 + R_1 x_H + R_2 x_H^2 + R_3 x_H^3$ in (5.1) with its appropriate $[2\,1]$ Padé summation:

$$
1 + R_1 x + R_2 x^2 + R_3 x^3 \rightarrow \frac{1 + (R_1 - R_3/R_3)x + (R_2 - R_3 R_1/R_2)x^2}{1 - (R_3/R_2)x}.
$$

(5.3)

Figure 4 shows that such Padé improvement yields a 2.5% - 3% increase in the $H \rightarrow gg$ rate, with very little sensitivity to the Higgs mass. Such improvement is best understood to be an estimate of (unknown) higher order corrections to (5.1) that should be eventually testable against both experimental and future higher-order calculations of the $H \rightarrow gg$ rate.

### 5.2 Higgs $\rightarrow b\bar{b}$

The decay rate $H \rightarrow b\bar{b}$ has been calculated to four-loop order in perturbation theory [11]:

$$
\Gamma(H \rightarrow b\bar{b}) = \left[3G_F M_H m_b^2/(4\pi\sqrt{2})\right]
$$

14
\[ \times \left\{ \left[ 1 + (17/3)x_H + 29.1467x_H^2 + 41.7581x_H^3 \right] - (6m_b^2(M_H)/M_H^2)[1 + (20/3)x_H + 14.62x_H^2] \right\} \]  
(5.4)

Full Padé-improvement of this expression for a given value of \( M_H \) (with \( M_H < m_t \)) entails...

1) ...determination of \( x(M_H) \) through use of (3.7), the \([2|2]\) Padé-summation of the \( \beta \)-function, to evolve \( x(\mu) \) from an appropriate initial condition; e.g. \( \alpha_s(M_z) = 0.118 \) [8];

2) ...determination of \( m_b(M_H) \) through substitution into (4.1) of (3.7c) and (4.17), the \([2|2]\) Padé summations for \( \beta(x) \) and \( \gamma(x) \), so as to evolve \( m_b(\mu) \) from an appropriate initial condition; e.g. \( m_b(4.3\text{GeV}) = 4.3 \text{ GeV} \) [8];

3) ...Padé-improvement and \([2|2]\) Padé summation of the cubic expression in (5.4),

\[ 1 + (17/3)x + 29.1467x^2 + 41.7581x^3 \to \frac{[1 + 4.30262x + 21.0641x^2]}{[1 - 1.36405x - 0.352971x^2]}, \]  
(5.5)

where (2.12) is used to generate an estimate of the \( x^4 \) coefficient \([67.2472]\), and where (2.15 - 19) are used to generate the \([2|2]\) approximant in (5.5), and

4) ...Padé-improvement and \([2|1]\) Padé summation of the quadratic expression in (5.4),

\[ 1 + (20/3)x + 14.62x^2 \to \frac{[1 + 5.581x + 7.382x^2]}{[1 - 1.086x]}, \]  
(5.6)

where (5.2) is used to generate an estimate of the \( x^3 \) coefficient \([15.87]\), and where (5.3) is used to generate the \([2|1]\) approximant in (5.6).

The relative size of all these corrections, referenced to the rate calculated to the next-to-highest-known [three-loop] order in perturbation theory, is displayed in Fig. 5. The top curve is the ratio \([\Gamma(H \to bb)]_4\text{-loop}/[\Gamma(H \to bb)]_3\text{-loop} \), as a function of \( M_H \). The 4-loop rate is obtained directly from (5.4), with \( m_b(M_H) \) and \( x_H(= x(M_H)) \) obtained from \( \beta \)- and \( \gamma \)-functions that are truncated to zero after their \( \beta_3 \) and \( \gamma_3 \) contributions. The 3-loop rate is obtained by truncating off the highest order terms in (5.4) — specifically the \( O(x^3) \) term on the left-hand side of (5.5) and the \( O(x^2) \) term on the left-hand side of (5.6) — and by obtaining \( x_H \) and \( m_b(M_H) \) from \( \beta \)- and \( \gamma \)-functions that are truncated to zero after \( \beta_2 \) and \( \gamma_2 \) contributions. The top curve shows a change of 0.02% to 0.09% in going from three to four-loop order.

The bottom curve compares the ratio of \( \Gamma(H \to bb) \), obtained by full Padé improvement of \([\Gamma(H \to bb)] \), as described above, to \([\Gamma(H \to bb)]_3\text{-loop} \). It is evident from the figure that the two ratios are within 0.0001 of each other for all values of \( M_H \) below \( m_t \). In other words, Padé-improvement reduces the 4-loop result for \( \Gamma(H \to bb) \) by at most 0.01%. The effect that is seen seems to derive wholly from the Padé improvement of \( m_b(\mu) \), as is evident from comparison of Fig 5 to Fig 3b. Of course, such close agreement between (5.4) and its fully Padé-improved version suggests that the expression (5.4) is more than adequate for future comparison to experiment. Thus the purpose of the analysis presented here is really to demonstrate the robustness of (5.4) against Padé estimates of higher order corrections.

The small size of corrections past even the 3-loop order is partly consequence of the small size of \( x(M_H) \) characterizing Higgs decay rates. Padé corrections are of much more interest when the magnitude
of $x$ is larger, suggesting their usefulness in assessing the perturbative content of low-energy QCD – i.e. QCD sum rules. In the section that follows we will address how Padé-improvement can be utilized to estimate substantial higher-order corrections to sum rules relevant to scalar- and pseudoscalar-meson static properties.

6. Perturbative Content of Scalar/Pseudoscalar QCD Sum Rules

The resonance content of finite-energy $F_k$ and Laplace $R_k$ QCD sum rules [12,13] is obtained from integrals over the imaginary part of current correlation functions $\Pi(s, \mu^2)$ in the subcontinuum region ($s < s_0$):

$$F_k(s_0) = \frac{1}{\pi} \int_0^{s_0} Im\Pi(s, s_0)s^k ds,$$

$$R_k(\tau, s_0) = \frac{1}{\pi} \int_0^{s_0} Im\Pi(s, 1/\tau)s^k e^{-s\tau} ds.\quad (6.2)$$

We consider here the purely perturbative content of the correlation function for scalar currents, which is presently known to 4-loop order [11]:

$$\frac{1}{\pi} Im\Pi(s, \mu^2) = \frac{3s}{8\pi^2} \left\{ 1 + \left( \frac{\alpha_s(\mu)}{\pi} \right) \left[ a_0 + a_1 ln \left( \frac{\mu^2}{s} \right) \right] + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ b_0 + b_1 ln \left( \frac{\mu^2}{s} \right) + b_2 \left( ln \left( \frac{\mu^2}{s} \right) \right)^2 \right] + \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \left[ c_0 + c_1 ln \left( \frac{\mu^2}{s} \right) + c_2 \left( ln \left( \frac{\mu^2}{s} \right) \right)^2 + c_3 \left( ln \left( \frac{\mu^2}{s} \right) \right)^3 \right] + \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 R_4 + ... \right\}.\quad (6.3)$$

The problem we will address in this section is the computation of $R_4$, which is necessary for the determination of $O(\alpha_s^4)$ contributions to $F_k$ and $R_k$. For three flavours Chetyrkin [11] has found that

$$a_0 = 17/3, \quad a_1 = 2, \quad (6.4a)$$

$$b_0 = 31.8640, \quad b_1 = 31.6667, \quad b_2 = 17/4, \quad (6.4b)$$

$$c_0 = 89.1564, \quad c_1 = 297.596, \quad c_2 = 229/2, \quad c_3 = 9.20833. \quad (6.4c)$$

If we define $w \equiv s/\mu^2$, we can estimate $R_4[w]$ directly by substituting

$$R_1[w] = a_0 - a_1 ln w \quad (6.5a)$$

$$R_2[w] = b_0 - b_1 ln w + b_2(ln w)^2 \quad (6.5b)$$
\[ R_3[w] = c_0 - c_1 \ln w + c_2(\ln w)^2 - c_3(\ln w)^3 \]  
(6.5c)

directly into (2.12). This enables one to determine explicitly \( \mathcal{O}(\alpha_s^4) \) corrections to the sum-rules (6.1) and (6.2), even though \( R_4[w] \) determined in this way is manifestly not a fourth-order polynomial in \( \ln(w) \). In particular, we easily find the \( \mathcal{O}(\alpha_s^4) \) contribution to \( F_k(s_0) \) to be

\[
\Delta F_0(s_0) = \frac{3s_0^2}{16\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \int_0^1 2R_4[w] \, dw
= \frac{3s_0^2}{16\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 (2059.4),
\]

(6.6a)

\[
\Delta F_1(s_0) = \frac{s_0^3}{8\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \int_0^1 3R_4[w] \, w \, dw
= \frac{s_0^3}{8\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 (1158.4),
\]

(6.6b)

\[
\Delta F_2(s_0) = \frac{3s_0^4}{32\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \int_0^1 4R_4[w] \, w^2 \, dw
= \frac{3s_0^4}{32\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 (833.47).
\]

(6.6c)

The integrals in (6.6) have been evaluated numerically. This approach, however, ignores the known structural dependence of \( R_4 \) on the variable \( w \),

\[ R_4[w] = d_0 - d_1 \ln w + d_2(\ln w)^2 - d_3(\ln w)^3 + d_4(\ln w)^4, \]

(6.7)

which may be important when one integrates over the \( w \)-variable, as in (6.6). The \( \mathcal{O}(\alpha_s^4) \) corrections to the first three finite energy sum rules are easily determined in terms of the constants \( d_i \) by substitution of (6.7) into the integrand of (6.1):

\[
\Delta F_0(s_0) = \frac{3s_0^2}{16\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \left( d_0 + \frac{d_1}{2} + \frac{d_2}{2} + \frac{3d_3}{4} + \frac{3d_4}{2} \right),
\]

(6.8)

\[
\Delta F_1(s_0) = \frac{s_0^3}{8\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \left( d_0 + \frac{d_1}{3} + \frac{2d_2}{9} + \frac{2d_3}{9} + \frac{8d_4}{27} \right),
\]

(6.9)

\[
\Delta F_2(s_0) = \frac{3s_0^4}{32\pi^2} \left( \frac{\alpha_s(s_0^{1/2})}{\pi} \right)^4 \left( d_0 + \frac{d_1}{4} + \frac{d_2}{8} + \frac{3d_3}{32} + \frac{3d_4}{32} \right),
\]

(6.10)
We can use the Padé algorithm (2.12) to estimate the coefficients \(d_i\). To do so, we let \(R_1\), \(R_2\), and \(R_3\) be given by (6.5) for five representative values of \(w\) between zero and one: \(w = \{1, e^{-1/4}, e^{-1/2}, e^{-1}, e^{-2}\}\). When \(w = 1\), corresponding to \(s = s_0\) in the finite-energy sum rule integrand (6.1), we see from (6.5) that \(R_1 = a_0\), \(R_2 = b_0\), \(R_3 = c_0\). Using the APAP-algorithm (2.12), we find that \(R_4[1] = 251.442 = d_0\).

When \(w = e^{-1/4}\) (\(s = 0.779s_0\)), we find from (6.5) that \(R_1 = 37/6\), \(R_2 = 40.0463\), \(R_3 = 170.8554\). Using (2.12) and (6.7), respectively, we then find that

\[
R_4[e^{-1/4}] = 699.398 = d_0 + d_1/4 + d_2/16 + d_3/64 + d_4/256. 
\]  

(6.11)

Similarly we find the following results when \(w = e^{-1/2}\) (\(s = 0.606s_0\)), \(w = e^{-1}\) (\(s = 0.368s_0\)), and \(w = e^{-2}\) (\(s = 0.135s_0\)):

\[
R_4[e^{-1/2}] = 1389.82 = d_0 + d_1/2 + d_2/4 + d_3/8 + d_4/16, 
\]  

(6.12)

\[
R_4[e^{-1}] = 3652.36 = d_0 + d_1 + d_2 + d_3 + d_4, 
\]  

(6.13)

\[
R_4[e^{-2}] = 12804.9 = d_0 + 2d_1 + 4d_2 + 8d_3 + 16d_4. 
\]  

(6.14)

We solve the four linear equations (6.11-14) for the four unknowns \(d_1, d_2, d_3, d_4\) using the value already obtained for \(d_0(= 251.422)\), and we obtain \(d_1 = 1357.84\), \(d_2 = 1634.53\), \(d_3 = 404.630\), \(d_4 = 3.9097\). Substitution of these numbers into (6.8-10) yields results remarkably close to those obtained in (6.6). These results are listed in the underlined highest-order terms given below for the Padé-improved perturbative content of the first three finite energy sum rules \((x \equiv \alpha_s(s_0^{1/2})/\pi)\):

\[
\mathcal{F}_0(s_0) = \frac{3s_0^2}{16\pi^2} \left[ 1 + 6.66667x + 49.8223x^2 
\right. 
+ \left. 302.110x^3 + 2057.0x^4 \right], 
\]  

(6.15a)

\[
\mathcal{F}_1(s_0) = \frac{s_0^3}{8\pi^2} \left[ 1 + 6.33333x + 43.3640x^2 
\right. 
+ \left. 215.846x^3 + 1158.4x^4 \right], 
\]  

(6.15b)

\[
\mathcal{F}_2(s_0) = \frac{3s_0^4}{32\pi^2} \left[ 1 + 6.16667x + 40.3119x^2 
\right. 
+ \left. 178.731x^3 + 833.52x^4 \right]. 
\]  

(6.15c)

Padé corrections to the Laplace sum rules (6.2) are not listed, as they are complicated by the occurrence of two scale variables \((s_0\) and \(\tau\)). However, such corrections are straightforward to obtain via integration of the \(O(\alpha_s^4)\) term of (6.3), which we have already obtained via APAP estimates of \(d_0\)–4:

\[
R_4 = 251.44 + 1357.8 \ln \left( \frac{\mu^2}{s} \right) + 1634.5 \left( \ln \left( \frac{\mu^2}{s} \right) \right)^2 
+ 404.63 \left( \ln \left( \frac{\mu^2}{s} \right) \right)^3 + 3.9097 \left( \ln \left( \frac{\mu^2}{s} \right) \right)^4. 
\]  

(6.16)
It is worth noting that the underlined $R_4$ terms in $(6.15)$ are not very different from those one would obtain using either the $R_2^2/R_2$ estimate suggested by a $[2|1]$ approximant, as discussed following $(2.10)$, or the APAP algorithm $(2.12)$ applied directly to the known (non-underlined) terms of $(6.15)$. The increase in the size of coefficients with increasing powers of $x$ suggests the utility of a $[2|2]$ Padé summation for these three expressions as an improvement over truncating off what may be substantial $O(x^3)$ corrections to $(6.15)$. By applying $(2.15-19)$ to the three equations $(6.15)$, we obtain the following $[2|2]$ approximants to the full perturbative content of the first three finite-energy sum rules:

\begin{align}
\mathcal{F}_0(s_0) &= \frac{3s_0^2}{16\pi^2} \left[ \frac{1 + 3.8073x + 6.8124x^2}{1 - 2.8594x - 23.947x^2} \right], \\
\mathcal{F}_1(s_0) &= \frac{s_0^3}{8\pi^2} \left[ \frac{1 + 2.3917x + 11.308x^2}{1 - 3.9416x - 7.0930x^2} \right], \\
\mathcal{F}_2(s_0) &= \frac{3s_0^4}{32\pi^2} \left[ \frac{1 + 2.2174x + 12.791x^2}{1 - 3.9492x - 3.1670x^2} \right].
\end{align}

Both $(6.15a)$ and $(6.17a)$ are indicative of a need for $s_0^{1/2}$ to be substantially larger than $1$ GeV for finite-energy sum rules to be useful in the scalar and pseudoscalar channels. If $s_0^{1/2} = 1$ GeV, we see from Fig. 1 that $x(1\text{GeV}) = 0.153$. For this value of $x$, each successive term in the square brackets of $(6.15a)$ is approximately unity, indicative of nonconvergence. This is reflected by a near-vanishing of the denominator of $(6.17a)$, implying a divergent result for the summation of the full perturbative series. If $s_0 = 3.24$ GeV$^2$, we see from Fig. 1 that $x(s_0^{1/2}) \approx 0.10$. The truncated series $(6.15a)$ is then seen to yield a value that is only $87\%$ of that obtained via the Padé summation $(6.17a)$, indicative of the magnitude of the higher order terms missing from $(6.15a)$. Note that a choice for $s_0$ near or somewhat above $3$ GeV$^2$ is suggested by Laplace sum-rule fits in both the pseudoscalar [14] and scalar [15] resonance channels.

The finite energy sum rule $\mathcal{F}_0(s_0)$ provides an example of how it is not enough just to have precise higher-order results. Even though $(6.15a)$ includes 4-loop effects as well as an APAP-algorithm estimate of 5-loop effects, the five terms listed demonstrate only sluggish convergence for a realistic choice of $s_0$. There is found to be enough of a difference between the truncated series $(6.15a)$ and its Padé-summation $(6.17a)$ to suggest the advisability of using the latter.

### 7. Summary

Using a Padé-motivated algorithm $(2.12)$, we have estimated in Section 3 the five-loop contributions to the $\beta$-function for $n_f = \{3, 4, 5, 6\}$, and we have compared the evolution of $\alpha_s(\mu)$ from $\mu = M_z$ obtained from two-loop, three-loop, four-loop, and Padé-summation estimates of the full $\beta$-function. Low energy values of $\alpha_s$ obtained from the four-loop $\beta$-functions with quoted flavour thresholds and appropriate threshold matching conditions are within $1\%$ of those obtained from the Padé-summation $\beta$-functions, a small effect compared to the much larger sensitivity of $\alpha_s(1$ GeV$)$ and $\alpha_s(m_{\tau})$ to present uncertainties in $\alpha_s(M_z)$ and $c$- and $b$-quark flavour thresholds.

We concluded Section 3 by extracting the most general set of $[2|2]$ Padé-summation estimates of 3, 4, 5, and 6 flavour QCD $\beta$-functions whose Maclaurin expansions yield known four-loop results for their first four terms. For positive values of $\alpha_s$, these Padé-summation estimates of the $\beta$-function were
shown to alternate denominator and numerator zeros, regardless of the size of the (presently unknown) five-loop term serving as a free parameter in the approximants. Such alternation necessarily implies that all positive numerator zeros represent ultraviolet fixed points, behaviour which, if applicable to the true \( \beta \)-function, would decouple the (suitably defined) infrared region from perturbative QCD.

In Section 4, we applied Padé-improvement methods to the running quark mass by estimating five-loop contributions to the \( \gamma \)-functions for 3, 4, 5, and 6 quark flavours. We then extracted an estimate for the \( O(x^4) \) contribution to closed-form expressions for \( m_q[\alpha_s(\mu)/\pi] \) that had been obtained earlier [4,6] to \( O(x^3) \). We compared the evolution of 3-loop, 4-loop, and Padé-summation estimates of \( m_b(\mu) \), once again finding very little relative difference (0.01\%) between Padé-summation and four-loop determinations of \( m_b(\mu) \) over the range \( \mu < m_t \), given identical five-flavour-threshold initial conditions. Corresponding agreement was still seen to occur at the 1\% level for light quarks.

In Section 5 we applied the results of the previous two sections to higher-loop calculations of the Higgs decay ratio \( H \to gg \) and \( H \to b\bar{b} \), rates which are sensitive to running couplings and running masses, as well as higher-loop corrections that are polynomial in \( \alpha_s(m_H) \). The calculated three-loop \( H \to gg \) rate is shown to be within 3\% of our Padé-improvement estimate, given identical choices for \( M_H, \alpha(M_z), \) and \( m_b \)-threshold initial conditions. Similarly, the calculated four-loop \( H \to b\bar{b} \) rate is seen to differ from full Padé-improvement by at most 0.01\%.

All the results summarized up until this point are indicative of close agreement between known perturbation theory and Padé-approximant improvements intended to take into account higher-order effects. Consequently, the theoretical uncertainties associated with the truncation of any such calculations at the three-or-four-loop order are shown to be small. In Section 6, we considered quantities known to four-loop order for which this is not the case, the purely-perturbative content of QCD sum rules in scalar/pseudoscalar-resonance channels. We constructed a Padé-algorithm estimate of the purely-perturbative \( O(\alpha_s^4) \) contribution to the imaginary part of scalar/pseudoscalar correlation functions, and we obtained [2] Padé-summation estimates of the all-orders content of the first three finite energy sum rules. We found the overall convergence of the primary sum rule to be doubtful for values of the QCD continuum threshold near \( s_0 = 1 \text{ GeV}^2 \). Even for \( s_0 \) above 3 GeV\(^2\), we found a greater-than-10\% discrepancy between Padé-summation and four-loop-order contributions to this sum rule, suggesting the existence of substantial theoretical uncertainties from higher-than-four-loop contributions.

**Acknowledgements**

We are grateful for Mark Samuel’s participation in the early stages of this work, prior to his sudden passing last November. His enthusiasm, energy, and sharp intelligence will be missed by all who had the opportunity to work with him. We are also grateful to A.L. Kataev for a seminal discussion on higher-order terms in sum rules, K.G. Chetyrkin for useful correspondence on both the running quark mass and scalar correlation functions, and to the Natural Sciences and Engineering Research Council of Canada for research support. VE is grateful for the hospitality of the Department of Physics and Engineering Physics at the University of Saskatchewan.
References:

1. M. A. Samuel, G. Li, and E. Steinfelds, Phys. Rev. D 48, 869 (1993) and Phys. Rev. E 51, 3911 (1995); M. A. Samuel, J. Ellis, and M. Karliner, Phys. Rev. Lett. 74, 4380 (1995).
2. J. Ellis, M. Karliner, and M. A. Samuel, Phys. Lett. B 400, 176 (1997).
3. J. Ellis, I. Jack, D. R. T. Jones, M. Karliner, and M. A. Samuel, Phys. Rev. D 57, 2665 (1998).
4. K. G. Chetyrkin, Phys. Lett. B 404, 161 (1997).
5. T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B 400, 379 (1997).
6. T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B 405, 323 (1997).
7. H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin, and S. A. Larin, Phys. Lett. B 272, 39 (1991) and B 319, 545 (1993) Erratum.
8. R. M. Barnett et al. [Particle Data Group], Phys. Rev. D 54, 1 (1996).
9. K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Max-Planck-Institut Preprint MPI/PhT/97-041 [hep-ph/9708258 v2].
10. K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Phys. Rev. Lett. 79, 353 (1997).
11. K. G. Chetyrkin, Phys. Lett. B 390, 309 (1997).
12. W. Hubschmid and S. Mallik, Nucl. Phys. B 193, 368 (1981).
13. M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 385 and 448 (1979).
14. V. Elias, A. H. Fariborz, M. A. Samuel, Fang Shi, and T. G. Steele, Phys. Lett. B 412, 131 (1997); T. G. Steele, J. C. Breckenridge, M. Benmerrouche, V. Elias, and A. H. Fariborz, Nucl. Phys. A 624, 517 (1997).
15. V. Elias, A. H. Fariborz, Fang Shi, and T. G. Steele, Nucl. Phys. A 633, 279 (1998).
Table 1: Comparison of $O(N)$ $\beta$-function and $\gamma$-function coefficients obtained via Padé estimates to those obtained via exact calculation. The “$\beta_4$ via (2.12)” column displays estimates obtained via the APAP algorithm (2.12), based on $\beta_0-3$ listed in [7]. The “$\beta_4^{\text{true}}$” column displays the results for $\beta_4$ obtained in [7] by explicit calculation. The “$\beta_4$ via [2]” column lists the Padé estimates given in Table 3 of ref. [2], involving knowledge of the $N^4$ dependence and a fit of the overall $N$-dependence of $\beta_4$, as well as a simplified asymptotic error formula. The “($\gamma_4/\gamma_0$) via (2.12)” column displays estimates of $\gamma_4/\gamma_0$ in the anomalous mass dimension $\gamma$-function obtained via the APAP algorithm (2.12), based on prior coefficients listed in [7]. The final column, “($\gamma_4/\gamma_0)^{\text{true}}$” are coefficients explicitly calculated in [7].

Figure Captions:

Figure 1: A comparison of the evolution of $x(\mu) \equiv \alpha_s(\mu)/\pi$ for $m_t \geq \mu \geq 1$ GeV obtained from the evolution of two-loop (2L), three-loop (3L), four-loop (4L), and [2|2] Padé-summation $\beta$-functions. All curves are generated from the initial condition $\alpha_s(M_Z) = 0.118$, with five- and four-flavour threshold matchings occurring at 4.3 GeV and 1.3 GeV, respectively.

Figure 2: Schematic behaviour of $x(\mu)$ obtained from a [2|2] Padé-summation estimate of the $\beta$-function whose positive numerator and denominator zeros alternate. The denominator zeros are denoted by $x_{d1}$ and $x_{d2}$, the numerator zeros are denoted by $x_{n1}$ and $x_{n2}$, and the alternation of zeros is consistent with the smallest positive zero being a zero of the denominator: $0 < x_{d1} < x_{n1} < x_{d2} < x_{n2}$. The value $\mu_{d1}$ is defined such that $x(\mu_{d1}) = x_{d1}$; values of $\mu < \mu_{d1}$ are outside the domain of $x(\mu)$.

Figure 3a: Evolution of the running mass $m_b(\mu)$ obtained from three-loop (3L) $\beta$- and $\gamma$-functions from the initial condition $m_b(4.3 \text{ GeV}) = 4.3 \text{ GeV}$.

Figure 3b: A comparison of the ratio of $m_b(\mu)$ obtained from four-loop (4L) and [2|2] Padé-summation $\beta$- and $\gamma$-functions to $m_b(\mu)$ obtained from three-loop (3L) $\beta$- and $\gamma$-functions, as shown in Fig. 3a, given the initial condition $m_b(4.3 \text{ GeV}) = 4.3 \text{ GeV}$.

Figure 3c: A comparison of the evolution of $m_c(\mu)$ obtained from evolution of two-loop (2L), three-loop (3L) four-loop (4L) and [2|2] Padé-summation $\beta$- and $\gamma$-functions.

Figure 3d: Masses of light (u, d, s) quarks from 2-loop, 3-loop, 4-loop, and [2|2] Padé-summation $\beta$- and $\gamma$-functions. Masses at $\mu = 1$ GeV are normalized to unity.

Figure 4: A comparison of the ratio of the Higgs decay rate into two gluons obtained via Padé-improvement discussed in the text to the three-loop (3L) expression for the same rate.

Figure 5: A comparison of the ratio of the Higgs decay rate into a $b\bar{b}$ pair obtained to four-loop (4L) order and through subsequent Padé-improvement (as described in the text) to the same rate obtained to three-loop (3L) order.
Fig. 3b

\[
\frac{m_b}{m_b^{(3L)}} \quad \frac{4L}{3L} \quad \text{Padé} / 3L
\]

\[\mu (\text{GeV})\]
Fig. 5

\[ \frac{\Gamma_{bb}}{\Gamma_{bb}^{(3L)}} \]

\[ M_H (\text{GeV}) \]

Curves labeled:
- 4L / 3L
- Padé / 3L