THE CAPACITY FOR THE LINEAR TIME-ININVARIANT GAUSSIAN RELAY CHANNEL

Youngchul Sung† and Cheulsoon Kim

ABSTRACT

In this paper, the Gaussian relay channel with linear time-invariant relay filtering is considered. Based on spectral theory for stationary processes, the maximum achievable rate for this subclass of linear Gaussian relay operation is obtained in finite-letter characterization. The maximum rate can be achieved by dividing the overall frequency band into at most eight subbands and by making the relay behave as an instantaneous amplify-and-forward relay at each subband. Numerical results are provided to evaluate the performance of LTI relaying.

Index Terms—Linear Gaussian relay channel, linear time-invariant filtering, Toeplitz distribution theorem, maximum achievable rate

I. INTRODUCTION

The relay channel problem is one of the classical problems in information theory, and still the capacity of this three node network is not exactly known. However, many ingenious coding strategies including decode-and-forward, compress-and-forward, etc. beyond the simple instantaneous amplify-and-forward (IAF) scheme have been developed [1, 2]. Recently, El Gamal et al. proposed a more advanced linear scheme for relay channels based on linear processing at the relay to compromise the complexity and performance between the complicated coding strategies and IAF [3], and showed that the scheme could perform well in certain cases by giving an example. Although the capacity for frequency-division linear relaying was obtained in their work, the general linear relay case was not explored fully, and for the capacity of the general linear relay channel is not still available; the general linear problem becomes a sequence of non-convex optimization problems and seemingly intractable [3] except the simple case of one-tap IAF [4]. To circumvent such difficulty, in [5] we considered more tractable and practical linear time-invariant (LTI) relaying, and proposed an efficient joint design algorithm for source and relay filters for general inter-symbol interference (ISI) relay channels. However, a performance bound for the LTI relaying was not obtained. In this paper, we provide the maximum achievable rate of LTI relaying in finite-letter characterization, based on the technique in [3] and results from spectral theory [6, 7]. The obtained result provides new insights into the structure and performance of optimal relay processing.

Notations: We will make use of standard notational conventions. Vectors and matrices are written in boldface with matrices in capitals. All vectors are column vectors. For a scalar a, a* denotes its complex conjugate. For a matrix A, AT, AH and tr(A) indicate the transpose, Hermitian transpose and trace of A, respectively. I is the identity matrix for the identity of size n (the subscript is omitted when unnecessary). The notation x ~ N(μ, Σ) means that x is Gaussian distributed with mean vector μ and covariance matrix Σ. E{·} denotes the expectation. R and C are the sets of reals and complex numbers, respectively. τ = √π.

2. SYSTEM MODEL AND BACKGROUND

We consider the general additive white Gaussian noise (AWGN) relay channel in Fig. 1. Here, x is the transmitted symbol at the source; x and yf are the transmitted and received symbols at the relay, respectively; and yd is the received symbol at the destination. We assume that the channel coefficients from the source to the destination, from the source to the relay and from the relay to the destination are 1, a and b, respectively. Then, the received signals at the relay and destination at the i-th symbol time are given by

\[ y_r[i] = ax_s[i] + w_r[i], \quad \text{and} \quad y_d[i] = bx_r[i] + w_d[i], \]

respectively, where \( w_r[i] \) and \( w_d[i] \) are independent and both are from \( N(0, \sigma^2) \). The source and relay have maximum available power \( P \) and \( γP \), respectively, for some \( γ > 0 \).

Here, we introduce the Toeplitz distribution theorem for our later development.

Theorem 1 [2, 8] Let \( \{r_k^Y := E\{y_n y_{n-k}^\ast\}\} \) be an absolutely summable autocovariance sequence of a stationary process \( \{y_n\} \); let \( \Sigma_k^Y = [r_{ji}^Y]_{i,j=1}^{\infty} \) be its Toeplitz covariance matrix; let \( f^S(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} r_k^Y e^{-ik\omega} \) be the spectrum of \( \{y_n\} \); and let \( \{\zeta^{(n)}_i\} \) be the eigenvalues of \( \Sigma_k^Y \). Then,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(\zeta^{(n)}_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(f^S(\omega)) d\omega \]

for any continuous function g(.)

3. LINEAR TIME-ININVARIANT RELAYING

3.1. General LTI relaying

The general (possibly noncausal) linear processing at the relay is given by

\[ x_r[i] = \sum_j h_{ij} y_r[i-j], \]

for arbitrary linear combination coefficients \( h_{ij} \). However, such linear processing requires time-varying filtering at the relay and is not readily realizable. Thus, in this paper we restrict ourselves to the case of LTI filtering at the relay. In this case, the relay output is given by

\[ x_r[i] = \sum_j h_j y_r[i-j], \]

where \( \cdots, h_{-1}, h_0, h_1, h_2, \cdots \) is the (possibly noncausal) LTI impulse response of the relay filter which is assumed to be stable, i.e., \( \sum_{j=-\infty}^{\infty} |h_j| < \infty \). Thus, the frequency response \( H(\omega) \) of the relay filter is well defined as \( H(\omega) = (1/2\pi) \sum_{j=-\infty}^{\infty} h_j e^{-ij\omega} \). Note that the frequency response
$H(\omega)$ is complex in general since $\{h_i\}$ is arbitrary except being stable. 2 can be written in vector form as

$$x_n^r = H_n y_n^r,$$

where

$$x_n^r = \begin{bmatrix} x_1 \, x_2 \cdots \, x_n \end{bmatrix}^T,$$

$$y_n^r = \begin{bmatrix} y_1 \, y_2 \cdots \, y_n \end{bmatrix}^T,$$

and

$$H_n = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_0 & h_1 & \cdots & h_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_1 & \cdots & h_0 \end{bmatrix}.$$

With the LTI filtering relay, the overall channel from the source to the destination becomes a Gaussian ISI channel, and stationary Gaussian input distribution is sufficient to achieve the capacity (3) pp.407-430). Thus, we assume stationary Gaussian input distribution hereafter:

$$x_n = \begin{bmatrix} x_1 \, x_2 \cdots \, x_n \end{bmatrix}^T \sim \mathcal{N}(0, \Sigma_n^x),$$

where $\Sigma_n^x$ is Hermitian and Toeplitz by the stationarity of the input process.

Then, the power constraints for the source and relay are respectively given by

$$\text{tr}(\Sigma_n^x) \leq n P, \quad \text{and}$$

$$\mathbb{E}\{\text{tr}(H_n, y_n^r (H_n y_n^r)^H)\} = \text{tr}(H_n (\sigma^2 + \sigma^2 I) H_n^H) \leq n\gamma P.$$

The received signal vector at the destination is given by

$$y_n^d = x_n^r + \lambda_0 w_n^d + w_n^d = \begin{bmatrix} (1 + \lambda_0 b H_n) x_n^r + \lambda_0 w_n^r + w_n^d \end{bmatrix},$$

where $y_n^d = [y_n^1, \cdots, y_n^n]^T$ and $w_n^d \sim \mathcal{N}(0,\Sigma^d)$ for $m = r, d$. The transmission rate in this case is given by

$$f^d(\omega) = 1 + \frac{|1 + a b H(\omega)|^2}{\sigma^2(1 + b^2 |H(\omega)|^2)} f^s(\omega),$$

and

$$f^s(\omega) = (a^2 f^r(\omega) + \sigma^2) |H(\omega)|^2.$$
channel coefficient, and hence the optimal source power allocation is \( \mu_i = \frac{\theta_{0n}(P - \epsilon'_n)}{n_0} \) for \( i = 1, \ldots, n_0 \). For global optimality the Karush-Kuhn-Tucker (KKT) condition should be satisfied for the remaining variables \( \{\mu_i, \lambda_i, i = n_0 + 1, \ldots, n\} \). For the problem \((15)\) the Lagrangian and KKT condition are respectively given by

\[
\mathcal{L} = \frac{1}{2n} \sum_{i=1}^{n} \log \left( 1 + \frac{\mu_i}{\sigma^2} \frac{|1 + ab\lambda_i|^2}{1 + b^2|\lambda_i|^2} \right) - \epsilon_n \leq \frac{1}{n} \mathcal{I}(x^n, y^n) \leq \frac{1}{2n} \sum_{i=1}^{n} \log \left( 1 + \frac{\mu_i}{\sigma^2} \frac{|1 + ab\lambda_i|^2}{1 + b^2|\lambda_i|^2} \right) + \epsilon_n
\]

subject to \( \tau_j, \theta_j \geq 0 \), the mode combination constraint \( \sum_{j=1}^{49} \tau_j = 1 \), the power distribution constraint \( \sum_{j=1}^{49} \tau_j |\lambda_j|^2 (a^2 \theta_j P/\tau_j + \sigma^2) = \gamma \). Here, \( \tau = [\tau_0, \tau_1, \ldots, \tau_{49}] \in \mathbb{R}^{49}, \theta = [\theta_0, \theta_1, \ldots, \theta_{49}] \in \mathbb{R}^{49}, \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{49}] \in \mathbb{C}^{49}, \) and \( C(x) = 1/2 \log(1 + x) \).

**Proof:** Substitute \((20)\) into \((15)\), and take limit as \( n \to \infty \). Then, we have \( \epsilon_n, \epsilon'_n, \epsilon''_n \to 0 \), \( \lim_{n \to \infty} \tau_j \to \tau_j \), and the limit of \((15)\) is \((21)\). (Converse) The achievable rate cannot be larger than \((21)\) because the maximum number of modes except mode 0 is 49 by Beozou’s theorem. (Achievability) Suppose that we have obtained \( \{\tau_j, \theta_j, \lambda_j\} \) from the optimization \((21)\). Shortly, we will see that the above rate can be obtained by partitioning the overall frequency band into 50 subbands and by using IAF with gain \( \lambda_j \) at subband \( j \). This can be accomplished by using a filter bank of 50 ideal band-pass filters (one for each subband and gain \( \lambda_j \) for subband \( j \)). The impulse response of this filter bank is the sum of the inverse DFTs of the frequency responses of the subband filters, and is stable.

**Remark 1**
(i) When the number of solutions to \((19)\) is less than 49, \((21)\) is still valid. Solving \((21)\) will yield the same result as solving a possible further-reduced optimization problem in this case. This is like that solving the size \( n \) problem \((12)\) directly should yield the same result as solving the reduced-size problem with the cost \((20)\) when the number of solutions is exactly 49. \((21)\) has already finite-letter characterization, but the number of the required modes can be reduced further by considering the structure of the optimization \((21)\) See Corollary 7.

(ii) Since the bins here are frequency bins, a mode is a frequency subband.

(iii) Since causal and stable LTI filters are contained in the set of the considered stable and possibly noncausal filters, \((21)\) is an upper bound on the capacity of the causal LTI Gaussian relay channel.

**Corollary 1** The capacity for the linear Gaussian relay channel with possibly noncausal LTI relay is given by

\[
C_{LTI}(P, \gamma P) = \max_{\tau, \theta, \lambda} C \left( \frac{\theta_0 P}{\tau_0 \sigma^2} \right) + \sum_{j=1}^{49} C \left( \frac{\theta_j P}{\tau_j \sigma^2} \right) \left( 1 + \frac{|a b \lambda_j|^2}{1 + b^2 |\lambda_j|^2} \right)
\]

for real \( a \) and \( b \), subject to \( \tau_j, \theta_j \geq 0 \), \( \sum_{j=0}^{49} \tau_j = 1 \), \( \sum_{j=1}^{49} \tau_j |\lambda_j|^2 (a^2 \theta_j P/\tau_j + \sigma^2) = \gamma \). Here, \( \tau = [\tau_0, \tau_1, \ldots, \tau_{49}] \in \mathbb{R}^{49}, \theta = [\theta_0, \theta_1, \ldots, \theta_{49}] \in \mathbb{R}^{49}, \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{49}] \in \mathbb{R}^{49}, \) and \( \mathcal{C}(x) = 1/2 \log(1 + x) \).

**Proof:** To maximize the argument, \( |1 + a b \lambda_j|^2 / (1 + b^2 |\lambda_j|^2) \) in \((20)\) \( \lambda_j \) should be adjusted with the complex conjugate of \( a b \) under the same magnitude. Hence, optimal \( \lambda_j \) is real, and we can perform the optimization only over real \( \lambda_j \) without loss of optimality. The same procedure as before can be performed except that \( \{\lambda_j\} \) are now real and that \( \mathcal{D}/\mathcal{D}\lambda_j \) is the ordinary real derivative. In this case, \( \lambda_j \) is a solution of a fixed 7th order univariate polynomial equation, \( \sum_{k=0}^{49} c_k x_k^n = 0 (c_0 \neq 0) \), regardless of \( i \). So, we only need at most seven real \( \lambda_j \)’s. (In the case that a and b are complex, still the phase of optimal \( \lambda_j \) is fixed and only the magnitude is a single real variable. Thus, we have the same result of at most seven different solutions.)

**Theorem 2** The capacity for the linear Gaussian relay channel with possibly noncausal LTI relay is given by

\[
C_{LTI}(P, \gamma P) = \max_{\tau, \theta, \lambda} C \left( \frac{\theta_0 P}{\tau_0 \sigma^2} \right) + \sum_{j=1}^{49} C \left( \frac{\theta_j P}{\tau_j \sigma^2} \right) \left( 1 + \frac{|a b \lambda_j|^2}{1 + b^2 |\lambda_j|^2} \right)
\]
include ideal low-pass filters, raised-cosine type filters, linear-phase filters with symmetric coefficients, etc.

In [3], El Gamal et al. obtained the capacity formula for the frequency-division (FD) linear Gaussian relay channel, given by

\[ C^{FD-L}(P, \gamma P) = \max_{\tau^{fd}, \theta^{jd}} \frac{\sum_{j=1}^{4} \tau^{jd}_j \eta_j}{\sigma^2 (1 + b^2 \eta_j^2)} \]

where \( \tau^{fd} = [\tau_0^{fd}, \ldots, \tau_4^{fd}], \theta^{jd} = [\theta_0^{jd}, \ldots, \theta_4^{jd}], \eta_j = [\eta_1, \ldots, \eta_4] \), subject to \( \tau_j^{fd}, \theta_j^{jd}, \eta_j \geq 0, \sum_{j=0}^{4} \tau_j^{fd} = \sum_{j=0}^{4} \theta_j^{jd} = 1 \), and \( \sum_{j=1}^{4} \tau_j^{fd} \eta_j (a^2 \sigma_j^2 P/\tau_j^{fd} + \sigma^2) = \gamma P \). One simple difference of the LTI relay from the FD relay is the maximum number of subbands (or modes) required to achieve the capacity. A more important difference lies in the difference in the operation at each frequency subband. In the LTI relay case, the effective signal-to-noise ratio (SNR) at subband \( j \) in [22] is given by

\[ \frac{P}{\sigma^2} \frac{(1 + ab\lambda_j)^2}{1 + b^2 \lambda_j^2} \]

This is exactly the same SNR of the relay channel equipped with IAF with gain \( \lambda_j \). (24) is easily obtained by considering that the signals along the two paths in Fig. 4 are added before reaching the destination.) Thus, Corollary [4] states that a capacity-achieving strategy is to divide the overall frequency band into at most eight subbands and to make the relay behave as an IAF relay with gain \( \lambda_j \) at subband \( j \). In the FD relay, on the other hand, the effective SNR in [22] is given by

\[ \frac{P}{\sigma^2} \frac{(1 + a^2 \eta_j^2)}{1 + b^2 \eta_j^2} \]

for subband \( j \). Here, let us consider the following data model:

\[ \begin{bmatrix} y_{d,1} \\ y_{d,2} \end{bmatrix} = \begin{bmatrix} a \lambda_j \\ 1 \end{bmatrix} x + \begin{bmatrix} b \lambda_j w_i + w_{d,1} \\ w_{d,2} \end{bmatrix} \]

where \( x \sim N(0, P) \) and \( w_{d,1}, w_{d,2}, w_i \sim i.i.d. N(0, \sigma^2) \). Note that the data model [22] corresponds to the FD relay channel in which the relay is IAF with gain \( \lambda_j \). The SNR after optimal matched filtering for the received signal in [22] is given by

\[ \frac{P}{\sigma^2} \frac{(1 + a^2 \lambda_j^2)}{1 + b^2 \lambda_j^2} \]

which is exactly the same as [22] with substitution \( \eta_j = \lambda_j^2 \). Hence, [22] states that a capacity-achieving strategy in the linear FD relay is to divide the overall frequency band into a finite number of subbands and to use IAF at each subband! Surprisingly, infinite frequency segmentation is not required. The optimality of this infinite frequency segmentation comes from the fact that the channel is flat-fading and thus each term in the Lagrangian \( L \) has the same form. In the ISI channel case, the frequency-domain channel coefficients \( a \) and \( b \) depend on the bin index \( i \). (We should use \( a_i \) and \( b_i \) instead of \( a \) and \( b \).) Hence, the solution \( (\mu_i, \lambda_i) \) to \( \partial L/\partial \mu_i = 0 \) and \( \partial L/\partial \lambda_i = 0 \) can be different for all \( i \in \{1, \ldots, n\} \). Thus, in the ISI case, the optimality of finite frequency segmentation is not guaranteed any more, and the capacity has infinite-letter characterization.

4. NUMERICAL RESULTS

We now provide some numerical results. [22] was evaluated by using a commercial optimization tool. ([33] and [34]) resulted in the same value.) Fig. 6 shows the rates of several schemes. Since the performance of other schemes is available in [3], we only considered the unlimited look-ahead cut-set bound, IAF and LTI relaying. Fig. 6(a) show the performance in the case of \( a = 1, b = 2 \) and \( \gamma = 1 \). In this case, it is known that the IAF already performs well and achieves the capacity when \( P \geq 1/3 \). [4] The LTI relaying improves the performance over the IAF at the very low SNR values, but the gain is not significant. Fig. 6(b) show the performance in the case of \( a = 2, b = 1 \) and \( \gamma = 1 \) in which the IAF has noticeable performance degradation from the cut-set bound. Even in this case, the gain by general LTI filtering over the IAF is not so significant. Thus, IAF seems quite sufficient for the general single-input single-output (SISO) flat-fading relay channel when linear filtering is considered for the relay function.

5. CONCLUSION

We have considered the LTI Gaussian relay channel. By using the Toeplitz distribution theorem and the technique in [3], we have obtained the capacity for LTI relaying in finite-letter characterization, and have shown that the capacity can be achieved by dividing the overall frequency band into at most eight subbands and by using IAF with possibly different gain in each subband. Thus, an optimal LTI relay can easily be implemented by using a filter bank.
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