Varieties of contextuality based on complexion, quantum probabilities, and structural non-embeddability

Karl Svozil

Institute for Theoretical Physics, TU Wien, Wiedner Hauptstrasse 8-10/136, 1040 Vienna, Austria
(Dated: September 1, 2021)

Contextuality can either be synthetically defined in terms of outcome conditionality on the measurement conditions, or in terms of non-classical probability distributions. Another logico-algebraic “strong” form of contextuality characterizes collections of quantum observables that have no faithfully embedding into (extended) Boolean algebras. Any of these forms indicate a classical in- or underdetermination that can be termed “value indefinite”, and formalized by partial functions of theoretical computer sciences.

PACS numbers: 03.65.Ca, 02.50.-r, 02.10.-v, 03.65.Aa, 03.67.Ac, 03.65.Ud
Keywords: Quantum contextuality, Gleason theorem, Kochen-Specker theorem, Born rule, quantum logic, probability distributions

1. TYPES OF QUANTUM CONTEXTUALITY

Early synthetic conceptions of contextuality emerged from insights into the entangled complexion [1] of physical properties retrieved from quantum measurements. As expressed by Bohr [2]: “the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.” This yields a “conditionality of phenomena” [3, 4] relative to a “complex of conditions under which the measurement is performed” [5, 6].

In this line of thought, observable phenomena appear not as isolated properties of the object, but as signals from the object-measurement apparatus composite. (Entanglement may even extend to the observer [7, 8].) Indeed, if entanglement is involved, these signals are about the relational properties of the combined entangled system. It makes no sense to refer to a well-defined property of the individual object alone [1, 9]. Therefore, one should be careful interpreting a statement such as Bell’s observation [10] that “the result of an observation may reasonably depend… on the complete disposition of the apparatus”. In general there cannot be a strong, that is, one-to-one, non-stochastic correspondence, association, or translation between relevant (counterfactual) well-defined individual properties of the constituents (one imagined [11] as “object”) of an entangled quantized system on the one hand, and the signal resulting from observation of this entangled state on the other hand. Entanglement does not allow such an association, simply because the constituents of an entangled quantum state have no well-defined individual states.

Subsequent attempts to specify and quantify contextuality have presumed that individual objective properties of nevertheless exist even for quantized systems, and that these properties do not depend on any kind of disposition of the measurement apparatus; in particular not on some compatible observables that are measured simultaneously. The former assumption of the general existence of counterfactual properties or observables can be called omni-realism. The latter assumption is usually referred to as non-contextuality.

Historical attempts to prove contextuality have assumed both omni-existence and non-contextuality (thereby disregarding earlier synthetic concepts of contextuality by entanglement), and have concentrated on the differences between classical and quantum predictions. Thereby, omni-existence is usually implicitly taken for granted. Thus any empirically (falsifiable) discrepancies between classical and quantum predictions are interpreted to signify non-contextuality.

Most commonly, experimental violations of Boole-Bell-type inequalities are identified with quantum contextuality. Other empirical signatures of quantum contextuality are the experimental violations of ad hoc configurations whose classical interpretation (i) either merely assume the omni-existence of unrestricted classical non-contextual value assignments that do not depend on the complete disposition of the apparatus [12]; (ii) or, on preselected input, predict classical functional output that is violated in quantized systems [13].

Theoretical arguments against omni-existent non-contextual value assignments consider finite configurations of observables forming intertwining contexts that have no consistent classical value assignments [14]. This is not the place for a historic review but the literature indicates that what is now known as the Kochen-Specker theorem [15, Theorem 1] has been discussed [16, 17] as a direct consequence of Gleason’s theorem [18].

The variety of contextual signifiers has resulted in a great semantic spread of notions of contextuality that threatens to obscure subtle differences concerning the quality of non-classicality. Because the same collection of observables, taken from quasi-classical or quantum experimental configurations alike, may still allow very different types of probability distributions. The resulting differences in the prediction may be taken as signatures for contextuality. But this is incomparable to configurations of observables that do not, by any classical means, support such probability distributions, either because the existing classical value assignments that do not depend on the complete disposition of the apparatus are too scarce to resolve the logico-algebraic structure of observables at hand, or because this structure does not allow any such classical value assignment at all.

In what follows I, therefore, suggest refining the notion of contextuality by differentiating between two cases, depending
on whether the collection of observables

(i) violates some constraints on classical probabilities but still allows a faithful embedding into an extended Boolean subalgebra, or

(ii) does not, by any classical means, allow any faithful embedding into some extended Boolean subalgebra.

For the sake of this analysis note that maximal collections of (finitely many) mutually co-measurable observables can be “wrapped up” into contexts (or blocks or maximal observables) which, from a probabilistic and structural point of view, intrinsically behave classically. The respective probability distributions are in accord with Kolmogorov’s axioms. In particular, the probabilities of mutually exclusive events add up [18].

Quantum mechanics, as well as partition logics from a generalized urn or finite automata models featuring complementarity, allow two or more distinct contexts which, for more than two mutually exclusive outcomes per context, may intertwine in some observable(s) [19]. The remaining non-intertwining observables, taken from different contexts, exhibit complementarity. (One convenient and compact graphical representation depicts contexts as smooth lines, and mutually exclusive elementary observables by points on these lines.)

As has been mentioned earlier, despite exhibiting complementarity, a collection of contexts may still allow some quasi-classical interpretation in terms of extreme cases. Such extreme cases can be formalized by dispersionless two-valued states \( v(x) \in \{0, 1\} \) (or, logically interpreted, “false” and “true”) that are binary functions of the respective observables \( x \in X \) forming the contexts which are additive \( v(x \lor y) = v(x) + v(y) \) for mutually exclusive observables \( x \land y = \emptyset \), and add up to 1 for all mutually exclusive observables within any context.

However, if the observables on which the aforementioned collections of contexts are based are quantum, then there is no guarantee that “sufficiently many” classical extreme cases corresponding to dispersionless two-valued \( \{0, 1\} \)-states exist. Indeed, there exist configurations with no such classical extreme cases, or ones that cannot support “sufficiently many” classical extreme cases to allow embeddings preserving the respective logico-algebraic structure.

Already Kochen and Specker discussed this issue and presented a demarcation criterion [15, Theorem 0] that utilizes the (in)separability of the underlying binary elementary propositions by classical extreme cases: A set of observables \( X \) forming a collection of contexts is faithfully embeddable into an extended Boolean algebra if and only if these observables in \( X \) contained in the respective contexts support or allow a separating set of two-valued states \( V = \{v_1, \ldots, v_n\} \) such that, for any two observables \( x, y \in X \), there exists some \( v_i \in V \) for which \( v_i(x) \neq v_i(y) \). (In its extreme form the collection of observables \( X \) support no two-valued measure, and \( V = \emptyset \).)

If the observables in \( X \) support a separating set of two-valued states \( V \) then signifiers facilitates three tasks:

(i) It yields all classical probability distributions in the form of a convex combination \( P(x) = \sum_{i=1}^n \lambda_i v_i(x) \), such that with \( \sum_{i=1}^n \lambda_i = 1 \) and \( \lambda_j > 0 \) for all \( j \in \{1, \ldots, n\} \).

(ii) The values of the extreme cases formalized by dispersionless two-valued states on, say, \( k \) observables can be arranged in \( k \)-tuples. These tuples can be interpreted as extreme points or vertices of a compact convex subset of \( \mathbb{R}^k \), a convex polytope, that has an equivalent representation in terms of its hull; that is, as a set of half-spaces which are (in)equalities. In the quantum physical realm, these inequalities are often referred to as Boole-Bell-type inequalities [20, 21].

(iii) A complete set of \( n \) two-valued states allows a representation as a partition logic that explicitly represents a classical embedding into an extended Boolean algebra \( 2^n \) [22].

Therefore, as disclosed earlier, it is suggested to adopt the demarcation criterion of embeddability for a refined definition of contextuality: One could speak of strong contextuality if no classical representation of the respective observables, and also no classical probability distribution exists.

The weaker probabilistic contextuality, while featuring complementarity (because more than one context is involved), allows all kinds of classically embeddable collections of observables, as well as classical probability distributions—if only the probability distribution in some way differs from global classical Kolmogorovian probabilities that are not restricted to local contexts. At least one should state that, for such latter scenarios, the observables still allow classical probabilities, but those probabilities are not realized by the “probabilistic contextual” systems at hand.

Why should one make such a distinction? Because if there do not exist separating sets of two-valued states and thus no faithful classical embeddability then the respective structures do not any longer support classical probability distributions. This is quite different from any non-classical probability distribution on an otherwise “classical” structure of observables.

II. EXAMPLES OF PROBABILISTIC CONTEXTUALITY

In what follows several example configurations of probabilistic contextuality will be enumerated. To avoid unnecessary redundancies they are mentioned with a reference to the concrete computation. Often two-valued states will be rewritten in terms of the expectation values of dichotomic outcomes \( E \in \{-1, 1\} \) by affine transformations—a multiplication followed by a subtraction—from extreme cases encoded by two-valued states \( v \in \{0, 1\} \) or, conversely, \( v = \frac{1}{2}(E + 1) \). In quantum mechanics and Hilbert spaces of dimension greater than one, \( E \) generalizes to a unitary Householder transformation \( E_x = 1 - 2|x\rangle\langle x| \), where \( |x\rangle \) is a unit vector. The resulting eigensystem of \( E_x \) has eigenvalues \( \pm 1 \):

\(-1: |x\rangle \) is an eigenvector of \( E_x \) with eigenvalue \(-1\).
+1: The remaining \( n - 1 \) mutually orthogonal eigenvectors span the \( n - 1 \) dimensional orthogonal subspace of \([x]\). Every vector in that subspace has eigenvalue \(+1\). (For \( n > 2 \) the spectrum is degenerate.)

For any context represented by some orthonormal basis \( B = \{ |e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle \} \), \( E_{e_1}E_{e_2} \cdots E_{e_n} = -1 \).

All such approaches take the complementary observables from quantum mechanics and force a classical interpretation—in terms of classical extreme cases formalized by two-valued states or expectations of binary observables—upon them. There exist classical models of complementarity, for instance Moore automata \([23]\) or generalized urn models \([24]\), or more generally, partition logics \([19, 22]\).

A. Boole-Bell type signatures of contextuality by hull computations

As mentioned earlier the hull computation of the convex polytope formed by vertices that represent the encoded classical extreme cases of a given selection of observables yields inequalities that are identified with optimal Boole-Bell type inequalities \([20, 21]\). Violations of these inequalities by quantum probabilities are interpreted as signifying contextuality relative to the assumptions discussed earlier, in particular, omni-existence and non-contextuality.

For the sake of concrete examples of historic configurations used for Boole-Bell type inequalities consider a configurations with

(i) isolated contexts with no common observable, such as

(i. SZ) Suppes-Zanotti inequalities from three isolated contexts \([25]\);

(i.CHSH) CHSH inequalities from four isolated contexts \([25]\);

(i.TPB) two-party Bell inequalities from finitely many isolated contexts \([26]\);

(ii) intertwining contexts from three- or higher dimensional Hilbert spaces with common observables, such as

(ii.SB) for the Specker bug configuration \([27]\) that serves as graph theoretic true-implies-false gadget \([28]\);

(ii.KCBS) inequalities from five cyclically connected contexts \([29]\). (The Bub-Stairs inequality \([30]\) on the same configuration is ad hoc and does not follow from a hull computation but from a classical probability assessment.)

B. Functional signatures of contextuality

There exist configurations of observables that, interpreted classically, serve all kinds of (logical) functions. (Graph theoretically they are gadgets.) Usually, they have input and output terminals which, functionally interpreted, serve as arguments and functional values. Two historic configurations realize either true-implies-false \([13, 27]\) or true-implies-true functional relations \([15, 31]\): if a particular state is preselected on the input terminal then classical value assignments (implementing omni-existence and non-contextuality) enforce a particular dependent value assignment—either false or true, respectively—on the output terminal.

Violations of these dependencies by quantum probabilities are interpreted as signifying contextuality relative to the assumptions discussed earlier, in particular, omni-existence and non-contextuality.

For the sake of examples take (extensions of) the Specker bug \([13, 27]\), or the examples in Refs. \([32, 33]\) which use finite sets of quantum observables in three-dimensional Hilbert space, as well as Hardy type configurations \([15, 31]\) which use finite sets of quantum observables in four- or higher-dimensional \([13]\) Hilbert space. All of these gadgets have a classical interpretation in terms of partition logic, finite automaton models or generalized urn models. Their respective logico-algebraic structure can be faithfully embedded into extended Boolean algebras; for instance, \(2^n\), by identification with the union of elements of a partition obtained from analyzing a complete set of \(n\) two-valued states \([22]\).

C. Ad hoc signatures of contextuality

We just mention without detail that there exist other ad hoc methods to obtain finite collections observables whose classical interpretation, in particular, omni-existence and non-contextuality, yield estimates and predictions that are violated by experimentally testable quantum predictions \([12]\).

III. EXAMPLES OF STRONG CONTEXTUALITY

Even though there still may exist “many” two-valued states associated with classical value assignments, these assignments may not be able to resolve the structure of quantum observables supporting them. There exist escalations of the “smallness” of the set of two-valued states in terms of inseparability, non-unitality, or non-existence that will be briefly reviewed next. All these instances go beyond classical embeddability (preserving the logico-algebraic structure of the associated observables) as they do not satisfy Kochen and Specker’s demarcation criterion \([15, \text{Theorem } 0]\) for separability.

A. Nonseparability of classical value assignments

As mentioned earlier there exist finite sets of observables that do not allow separation by classical extreme cases. That is, in such circumstances, no classical value assignment exists that is capable to differentiate between, or separating, the individual constituents of some pair of distinct quantum observables.

For the sake of examples of contextuality based on inseparability, take Kochen and Specker’s combo \([15, \text{Graph } \Gamma_3]\)
of intertwining true-implies-true gadgets which contains two pairs of observables that cannot be classically separated. A respective four-dimensional example inspired by Hardy’s non-local configuration can be found in Figure 5 of Reference [31].

Other explicit experimentally testable cases of inseparability can be found in a configuration depicted in Figure 2 of Ref. [34], as well as in Figure 24.2c analyzed in Table 24.1 of Ref. [35], based on a configuration introduced in Figure 2 of Ref. [36].

B. Non-unitality of classical value assignments

Another, even stronger (because it includes and extends inseparability) form of logical contextuality are collections of observables with a unital set of two-valued states: if interpreted classically such structures enforce the non-occurrence (and occurrence) of certain observables.

Two explicit experimentally testable cases of inseparability are the same as mentioned earlier in a configuration depicted in Figure 2 of Ref. [34], as well as in Figure 24.2c analyzed in Table 24.1 of Ref. [35], based on a configuration introduced in Figure 2 of Ref. [36].

C. Nonexistence of classical value assignments

The most extreme form of strong contextuality occurs if the respective structure of observables allows no classical interpretation whatsoever. This result had already been announced by Specker in 1960 [14], and is nowadays called the Kochen-Specker theorem [15, Graph Γ2]. It has been perceived [16, 17] as a direct consequence of Gleason’s theorem [18].

One may, in a certain sense and relative to the mathematical means employed, extend these results by proving that there exist finite configurations of observables that do not allow any classical value definite existence beyond a single extreme case, and the (continuity of) contexts containing this extreme case. Proofs relative to global and total classical value assignments are in Refs. [37, 38]. Similar results are obtained with weaker assumptions allowing partial value assignments in Refs. [36, 39, 40]

IV. DISCUSSION

Let me add some pertinent issues: First, I would like to raise some concerns about a method of “collapsing” probabilities, also called the “support” of a standard probabilistic Hardy model [41], such that any non-zero probability is mapped onto one, which serves as the basis of various “logic” arguments. This “logical” method works well for particular gadgets such as true-implies-false gadgets because they are symmetrical with respect to their “terminals”: true-implies-false gadgets also work “the other way round” by inverting the terminal observables, but this is no longer the case for true-implies-true gadgets. In general by effectively “collapsing” all two-valued states, one might lose important relational information.

Second, to pronounce that contextuality “supplies the magic for quantum computation” suggests that contextuality might be some computational capacity or resource that raises high expectations based on a rather vague notion. One may even argue that contextuality might indicate restrictions rather than extensions of classical computations.

A. Omni-realism

Another, more serious concern, is the use of the term “contextual” as a denomination. Because, as has been noted earlier, this term may implicitly suggest, without direct empirical evidence, a form of omni-realism.

A generalized Jaynes’ principle is called “plausible reasoning”: one should not introduce unnecessary epistemic bias, superficial information, and individual ontologic projections into empirical evidence but rather stick to the “knowable” facts. In Jaynes’ words [42, Section 10.11, p. 331], “the onus is always on the user . . . that the full extent of his ignorance is also properly represented”.

Therefore, it might be more appropriate to talk about “quantum indeterminacy” as Pitowsky did [37, 38], and to allow partial functions and value indefiniteness. Partial functions have been first conceptualized [43] in theoretical computer science to cope with and formalize computability; in particular, with the recursive unsolvability of the halting problem. They are a staple of the theory of recursive functions and indicate capacities that go beyond certain (consistent formal) limits of operational expressibility. It has thus might me more appropriate to use the terms “partial functions” and “value indefinite” instead of “contextuality” [36, 39, 40, 44].

I believe that because of Pitowsky’s principle of indeterminism [37, 38] and newer theorems allowing partial functions as value assignments [36], the “message” of the quantum is straightforward: quantum systems are defined in their frame of preparation, and undefined in directions not perpendicular or collinear to it. Insistence in the assumption of omni-definiteness yields to “contextual” as well as stochastic value assignments. The respective measurement outcomes do not directly reflect any objective property prior to measurement, but on the complexion of the entire entangled state including the measurement apparatus.

B. Is contextuality haunted?

It should be kept in mind that any experimentally verifiable contextuality remains indirect (or “haunted”) and not direct, as quantum mechanics predicts the absence of direct verifications [45–48]; if say, two contexts \( \{a, b, c\} \) and \( \{a, d, e\} \) intertwining at observable \( a \) are considered, quantum mechanics is not ambiguous about the outcome corresponding to \( a \), regardless of the context measured. This can be directly experimentally tested on a single quantized particle, or on entangled particle pairs. A remaining “haunted” context-dependence might
manifest itself in a hidden and uncontrollable outcome dependence of the remaining complementary observable pairs \( \{b, c\} \) and \( \{d, e\} \). In four or higher dimensions this applies also to all quantum observables common to different contexts. For instance, for a configuration \( \{a, b, c, d\} \) and \( \{a, b, e, f\} \), quantum mechanics predicts that the observables \( a \) and \( b \) are non-contextual, whereas \( \{c, d\} \) and \( \{e, f\} \) might show (hidden and haunted) outcome dependence.

C. Historic aspects regarding the importance of embeddability for Specker

Let me add some afterthoughts on Kochen and Specker’s demarcation criterion [15, Theorem 0] for structure-preserving (non-)embeddability mentioned earlier as a decisive benchmark or measure for strong contextuality or value indefiniteness. I consider it not entirely unreasonable to speculate that Specker meant the lack of embeddability by announcing [14]: “An elementary geometrical argument shows that such an assignment is impossible and that therefore (aside from the exceptions noted above) no consistent prediction concerning a quantum mechanical system is possible.” Indeed, the ‘Comment’ section of the respective article in Specker’s ‘Selecta’ [49, p. 385] explicitly notes (the references are updated to match the current ones used here): “The impossibility to embed the lattice of subspaces of \( \mathbb{R}^3 \) into a Boolean algebra, mentioned at the end of [14], is proved in [15] (theorem 1 and subsequent remarks).”

In summary,

(i) The supposition of a well-defined physical operationalization of the properties associated with quantum observables, and, in particular, omni-existence lies at the basis of current so-called empirical tests of contextuality. This does not take into account the entanglement between the object under observation and the measurement apparatus. However, such a conception of measurement entails that the constituents of the entangled object-apparatus state are in no definite individual state.

(ii) As long as the explicit functional context-dependence of quantum observables common to different contexts is directly tested it is absent in quantized systems. Therefore, it might be “haunted”, as such dependence may only occur indirectly, and without direct experimental testability.

(iii) It might be prudent to differentiate between the logico-algebraic structure formed by the observables via complementarity and the probability distributions such logics can or cannot support.

(iv) Kochen and Specker gave a “demarcation criterion” for non-embeddability in terms of (in)separability: if the set of two-valued states on the logic can discriminate between every pair of atomic observables (aka elementary propositions in the sense of Birkhoff and von Neumann) then it can support classical models (and also quantum ones if there exist faithful orthogonal representations). If no extreme case or two-valued state can separate between two observables, then embeddability in some extended Boolean algebra breaks down. Such situations can be termed strong contextuality.

(v) Current empirical corroborations of contextuality associated with Boole-Bell-type inequalities, as long as they are based on hull computations of classical extreme cases on suitable ensembles of quantum observables with separating sets of two-valued states encoding these extreme cases, are merely about probabilistic contextuality. The same is true for separable ensembles of quantum observables with functional relations on endpoints.

(vi) In the spirit of theoretical computer science and of Itamar Pitowsky I suggest [35] to call these properties “partial value assignments”, “value indefinite”, or “indeterminate”.

ACKNOWLEDGMENTS

This research was funded in whole, or in part, by the Austrian Science Fund (FWF), Project No. I 4579-N. For the purpose of open access, the author has applied a CC BY public copyright licence to any Author Accepted Manuscript version arising from this submission.

I thank Adán Cabello for sharing his thoughts and questions on the time of origin of what is nowadays called [50] Kochen-Specker theorem.

All misconceptions and errors are mine.

The author declares no conflict of interest.

[1] E. Schrödinger, Die gegenwärtige Situation in der Quantenmechanik, Naturwissenschaften 23, 823 (1935).
[2] N. Bohr, Discussion with Einstein on epistemological problems in atomic physics, in Albert Einstein: Philosopher-Scientist, edited by P. A. Schilpp (The Library of Living Philosophers, Evanston, Ill., 1949) pp. 200–241.
[3] A. Khrennikov, Bohr against Bell: complementarity versus nonlocality, Open Physics 15, 734 (2017).
[4] G. Jaeger, Quantum contextuality in the copenhagen approach, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 377, 20190025 (2019).
[5] A. Khrennikov, Interpretations of Probability, 2nd ed. (Walter de Gruyter, Berlin, New York, 2009).
[6] A. Khrennikov, Contextual Approach to Quantum Formalism, Fundamental Theories of Physics, Vol. 160 (Springer Science + Business Media B.V., 2009).
[7] F. London and E. Bauer, La theorie de l’observation en mécanique quantique; No. 775 of Actualités scientifiques et in-
A list of references is provided, including titles, authors, and publication details for various works in the field of quantum mechanics and related areas. The references cover a range of topics, from foundational principles to specific applications and developments in the field.
[48] R. B. Griffiths, Quantum measurements and contextuality, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 377, 20190033 (2019), arXiv:1902.05633.

[49] E. Specker, Selecta (Birkhäuser Verlag, Basel, 1990).

[50] S. M. Stigler, Stigler’s law of eponymy, Transactions of the New York Academy of Sciences 39, 147 (1980), in “Science and social structure: a Festschrift for Robert K. Merton”, ed. by Thomas F. Gieryn, reprinted in [52].

[51] N. Zierler and M. Schlessinger, Boolean embeddings of orthomodular sets and quantum logic, in The Logico-Algebraic Approach to Quantum Mechanics: Volume I: Historical Evolution, edited by C. A. Hooker (Springer Netherlands, Dordrecht, 1975) pp. 247–262.

[52] S. M. Stigler, Statistics on the Table. The History of Statistical Concepts and Methods (Harvard University Press, Cambridge, MA, USA and London, England, 1999, 2002).