A note on boundary differentiability of solutions of nondivergence elliptic equations with unbounded drift

Yongpan Huang

School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China

Abstract

Boundary differentiability is shown for solutions of nondivergence elliptic equations with unbounded drift.

Keywords: Boundary differentiability, Elliptic equations

1. Introduction

In this paper, we will study the boundary differentiability of strong solutions of elliptic equation with unbounded lower order coefficients. Suppose that \( u \in W^{2,\infty}_{loc}(Q^+_1) \cap C(Q^+_1) \) satisfies

\[
\begin{aligned}
Lu &:= -a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x) \quad \text{in} \quad Q^+_1; \\
u(x) &= 0 \quad \text{on} \quad T_1.
\end{aligned}
\]

We use the summation convention over repeated indices and the notations \( D_i := \frac{\partial}{\partial x_i} \) and \( D_{ij} := D_iD_j \).

We assume that \( a_{ij}, b_i \) and \( f \) are measurable functions on \( Q^+_1 \), \( b = (b_1, b_2, ..., b_n) \), the matrix \( (a_{ij}(x))_{n \times n} \) is symmetric and satisfies the uniformly elliptic condition

\[
\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e.} \ x \in Q^+_1, \tag{1.2}
\]

with a constant \( \lambda \in (0, 1] \) and \( b, f \in L^n(Q^+_1) \). Throughout the paper, we denote \( W(\Omega) := W^{2,\infty}_{loc}(\Omega) \cap C(\Omega) \) while \( \Omega \) is a bounded domain in \( \mathbb{R}^n \).

As for the boundary regularity of nondivergence elliptic equations: If the drift term \( |b| \) is bounded, Krylov [8] showed that the solution is \( C^{1,\alpha} \) along the boundary if \( \partial \Omega \) is \( C^{1,1} \); Lieberman [9] gave a more general estimates; Wang [10] proved a similar pointwise result as in [8] by an iteration method that will be adopted in this paper; Li and Wang in [11] showed the boundary differentiability of solutions of elliptic equations on convex domains. If \( |b| \) is unbounded, Ladyzhenskaya and Ural’tseva [5] proved boundary \( C^{1,\alpha} \) estimate of elliptic and parabolic inequalities on \( W^{2,q} \) domain with \( b \in L^q, \Phi \in L^q, q > n \) and nonlinear term \( \mu_i|Du|^2 \); Safonov [15] obtained the the Hopf-Oleinik lemma for elliptic equations and gave the counterexample which indicated that the Dini condition on \( b_n \) can not be removed for our theorem; Nazarov

\[\text{The author was supported by NSFC 11401460 and CSC 201506285016.}\]

\[\text{Corresponding author}\]

\[\text{Email address: huangyongpan@gmail.com (Yongpan Huang)}\]
proved the Hopf-Oleinik Lemma and boundary gradient estimate under minimal restrictions on lower-order coefficients; In [12] the boundary differentiability is shown for strong solution of nondivergence elliptic equation \(|b|\) and \(f\) satisfying Dini condition. Since the Hopf Oleinik Lemma and boundary Lipschitz Estimate [13] hold for solution of (1.1) only need \(b_n\) satisfies the Dini condition, it is natural conjecture that whether the boundary differentiability of solutions at 0 is true while \(b_n\) satisfying Dini condition at 0. In the following, we will show that the result is correct. Some related results concerning Dini continuity can be found in [3, 6, 7, 14].

The following Alexandroff-Bakelman-Pucci maximum principle and Harnack inequality are our main tools.

Theorem 1.1. (4, 5) Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\), and let \(u\) be a function in \(W(\Omega)\) such that \(Lu \leq f\) in \(\Omega\). Suppose that the matrix \((a_{ij}(x))_{n\times n}\) is symmetric and satisfies the uniformly elliptic condition (1.2), and \(b, f \in L^n(\Omega)\). Then

\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + N \operatorname{diam}(\Omega) \cdot e^{N||b||_{C^{0,1}}||f||_{L^n(\Omega)}},
\]

where \(N\) is a positive constant depending only on \(n\) and \(\lambda\).

Theorem 1.2. (Harnack Inequality) Let \(u\) be a nonnegative function in \(W(B_0)\), \(Lu = f\) in \(B_0\) and \(b, f \in L^n(B_0)\). There exists a positive constant \(C_0\) depending only on \(\lambda\) and \(n\), such that if \(||b||_{L^n(B_0)} \leq C_0\), then

\[
\sup_{B_1} u \leq C(\inf_{B_1} u + ||f||_{L^n(B_0)}),
\]

where \(C\) is constant depending only on \(\lambda\) and \(n\).

Theorem 1.2 follows from the the proof in [15] clearly. The most important thing is that the quantity \(||b||_{L^n}\) is scaling invariant(see Remark 1.4 in [15]) and the Harnack constant is invariant in the iteration procedure.

Notations.

\(|\vec{e}|\) \(\equiv\) the standard basis of \(\mathbb{R}^n\).

\(|x| := \sqrt{\sum_{i=1}^{n} x_i^2}\), \(\) the Euclidean norm of \(x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\).

\(B_r := \{x \in \mathbb{R}^n : |x| < r\}\).

\(T_r := \{x' \in \mathbb{R}^{n-1} : |x'| < r\}\).

\(Q_r := T_r \times (-r, r)\).

\(Q^*_r := T_r \times (0, r)\).

\(||f||_{L^n(\Omega)} := \left(\int_{\Omega} |f(x)|^n dx\right)^{1/n}\).

\(W(\Omega) := W^{2,1}_{lo}(\Omega) \cap C(\Omega)\).

Theorem 1.3. Assume that (1) \(u \in W(Q^*_1), u|_{T_1} = 0, Lu = f\) in \(Q^*_1\); (2) \(f \in L^n(Q^*_1)\) and \(\int_0^1 \frac{||f||_{L^n(Q^*_r)}}{r} dr < \infty\); (3) \(b \in L^n(Q^*_1)\) and \(\int_0^1 \frac{||b||_{C^{0,1}(Q^*_r)}}{r} dr < \infty\). Then \(u\) is differentiable at 0.
2. Proof of Theorem

By standard normalization, it is enough for us to prove the following Theorem 2.1 instead of proving Theorem 1.3.

**Theorem 2.1.** Assume that

1. \( u \in W(Q_1^+) \), \( \|u\|_{L^1(Q_1^+)} \leq 1 \), \( Lu = f \) in \( Q_1^+ \), and \( u|_{T_1} = 0 \);
2. \( f \in L^p(Q_1^+) \) with \( \|f\|_{L^p(Q_1^+)} \leq 1 \) and \( \int_0^1 \frac{\|b_n\|_{L^p(Q_1^+)} dr}{r} \leq 1 \);
3. \( b \) and \( b_n \) satisfy

\[
\|b\|_{L^1(Q_1^+)} \leq \min \left[ \frac{\delta}{4A_3(\frac{2M}{r} + 1)}, 1, \epsilon_0 \right] \quad \text{and} \quad \int_0^1 \frac{\|b_n\|_{L^1(Q_1^+)} dr}{r} \leq \min \left[ 1, \frac{\delta \ln 1}{16MA_3} \right],
\]

(2.1)

where \( \epsilon_0 \) is the constant in Theorem 2.2 and \( \delta, M, \mu, A_2 \) and \( A_3 \) are constants in Lemma 2.2.

Then \( u \) is differentiable at 0.

**Lemma 2.2.** There exist positive constants \( \delta(<1), \mu(<1), M, A_1, A_2 \) and \( A_3 \) depending only on \( \lambda \) and \( n \). If

\[
\begin{align*}
k x_n - B & \leq u(x) \leq K x_n + B \quad \text{in} \quad Q_1^+, \\
\end{align*}
\]

(2.2)

for some constants \( k, K \) and \( B(\geq 0) \) with \( k \leq K \), then there exist constants \( \tilde{k} \) and \( \tilde{K} \) such that for \( x \in Q_1^+ \),

\[
\begin{align*}
\tilde{k} x_n - A_1\|f\|_{L^1(Q_1^+)} - A_2(K - k + B)\|b\|_{L^1(Q_1^+)} - A_3(|k| + |k|)\|b_n\|_{L^1(Q_1^+)} \\
\leq u(x) - \tilde{K} x_n + A_1\|f\|_{L^1(Q_1^+)} + A_2(K - k + B)\|b\|_{L^1(Q_1^+)} + A_3(|k| + |k|)\|b_n\|_{L^1(Q_1^+)},
\end{align*}
\]

(2.3)

where either

\[
\tilde{k} = k - 2MB + \mu(K - k) \quad \text{and} \quad \tilde{K} = K + 2MB,
\]

(2.4)

or

\[
\tilde{k} = k - 2MB \quad \text{and} \quad \tilde{K} = K + 2MB - \mu(K - k).
\]

(2.5)

Obviously, we have \( \tilde{k} \leq \tilde{K} \).

**Proof of Lemma 2.2.** We prove the following claim first.

**Claim.** There exist positive constants \( M, \tilde{\delta} \) and \( C_1 \) depending only on \( \lambda \) and \( n \), such that for any \( x \in Q_1^+ \),

\[
\begin{align*}
(k - 2MB)x_n - C_1(|k|\|b_n\|_{L^1(Q_1^+)} + B\|b\|_{L^1(Q_1^+)} + \|f\|_{L^1(Q_1^+)}) \\
\leq u(x) - (K + 2MB)x_n - C_1(|k|\|b_n\|_{L^1(Q_1^+)} + B\|b\|_{L^1(Q_1^+)} + \|f\|_{L^1(Q_1^+)}).
\end{align*}
\]

(2.6)

**Proof.** Let \( M \geq \sqrt{n - 1}(1 + \frac{2\sqrt{n - 1}}{\lambda}) \) and \( \epsilon(>0) \) be small enough, such that

\[
4 - (1 + \epsilon)(2 + \epsilon)(M - 1)^e \geq 0.
\]

(2.7)

Let

\[
\tilde{\delta} = \frac{1}{M} \left( \frac{1}{3 \sqrt{n - 1}} \right), \quad \delta = \frac{\tilde{\delta}}{2M}
\]

(2.8)
By (2.13) and (2.16), the claim follows clearly.

The barrier function $\tilde{\psi}(x)$ is $C^2$ and satisfies the following conditions:

$\begin{align*}
(1) \tilde{\psi}(x) & \geq 1 \text{ on } \{x \in \mathbb{R}^n : |x'| \leq 1, x_n = \delta\}; \\
(2) \tilde{\psi}(x) & \geq 0 \text{ on } \{x \in \mathbb{R}^n : |x'| \leq 1, x_n = 0\}; \\
(3) \tilde{\psi}(x) & \geq 1 \text{ on } \{x \in \mathbb{R}^n : |x'| = 1, 0 \leq x_n \leq \delta\}; \\
(4) -u''(x)D_j\tilde{\psi}(x) & \geq 0 \text{ a.e. in } \{x \in \mathbb{R}^n : |x'| < 1, \quad 0 < x_n < \delta\}; \\
(5) \tilde{\psi}(x) & \leq \frac{2\lambda}{\delta} \text{ in } Q^*_\delta.
\end{align*}$

It follows that

$\begin{align*}
& L(kx_n - B\tilde{\psi}(x) - u(x)) \leq b_D(kx_n - B\tilde{\psi}(x)) - f(x) \text{ in } \tilde{Q}; \\
& kx_n - B\tilde{\psi}(x) - u(x) \leq 0 \text{ on } \partial \tilde{Q},
\end{align*}$

where $\tilde{Q} = T_1 \times (0, \delta)$.

According to the Alexandroff-Bakelman-Pucci maximum principle, we have

$\begin{align*}
kx_n - B\tilde{\psi}(x) - u(x) & \leq C_1(k||b||_{L^1(Q_1')} + B||b||_{L^1(Q_1')} + ||f||_{L^1(Q_1')}) \text{ in } \tilde{Q},
\end{align*}$

where $C_1$ is a constant depending only on $\lambda$ and $n$.

By (2.10)(5) (i.e. $\tilde{\psi}(x) \leq \frac{2\lambda}{\delta} = 2Mx_n$ in $Q^*_\delta$), we have

$\begin{align*}
(k - 2MB)x_n - C_1(k||b||_{L^1(Q_1')} + B||b||_{L^1(Q_1')} + ||f||_{L^1(Q_1')}) & \leq u(x) \text{ in } Q^*_\delta.
\end{align*}$

As in (2.11), we also have

$\begin{align*}
& L(u(x) - Kx_n - B\tilde{\psi}(x)) \leq -b_D(Kx_n + B\tilde{\psi}(x)) + f(x) \text{ in } \tilde{Q}; \\
& u(x) - Kx_n - B\tilde{\psi}(x) \leq 0 \text{ on } \partial \tilde{Q}.
\end{align*}$

According to the Alexandroff-Bakelman-Pucci maximum principle, we have

$\begin{align*}
u(x) - Kx_n - B\tilde{\psi}(x) & \leq C_1(||k||_{L^1(Q_1')} + B||b||_{L^1(Q_1')} + ||f||_{L^1(Q_1')}) \text{ in } \tilde{Q},
\end{align*}$

where $C_1$ is a constant depending only on $\lambda$ and $n$. Combining (2.15) and (2.10)(5), we get

$\begin{align*}
u(x) & \leq (K + 2MB)x_n + C_1(||k||_{L^1(Q_1')} + B||b||_{L^1(Q_1')} + ||f||_{L^1(Q_1')}) \text{ in } Q_\delta.
\end{align*}$

By (2.13) and (2.16), the claim follows clearly.

Let $\Gamma = [\delta c_n + T_M]$. Next, we will show (2.3) according to two cases: $u(\delta c_n) \geq \frac{1}{2}(K + k)\delta$ and $u(\delta c_n) < \frac{1}{2}(K + k)\delta$, corresponding to which (2.4) and (2.5) will hold respectively. Since the proofs of these two cases are similar, we will only show the proof for the case: $u(\delta c_n) \geq \frac{1}{2}(K + k)\delta$. 

4
Let \( v(x) = u(x) - (k - 2MB)x_n + C_1(\|b\|_{L^2(Q^n)} + B\|b\|_{L^2(Q^n)} + \|f\|_{L^2(Q^n)}) \). Then
\[
\nu(\delta x_n) \geq \left( \frac{K - k}{2} + 2MB \right) \delta + C_1(\|b\|_{L^2(Q^n)} + B\|b\|_{L^2(Q^n)} + \|f\|_{L^2(Q^n)}).
\] (2.17)

Since \( \nu(x) \geq 0 \) for \( x \in Q^n \), from (2.17) and the Harnack inequality, it follows that
\[
\sup_{\Gamma} \nu(x) \leq C_2(\inf_{\Gamma} \nu(x) + |k|\|b\|_{L^2(Q^n)} + B\|b\|_{L^2(Q^n)} + \|f\|_{L^2(Q^n)}),
\] (2.18)

where \( C_2(\gg 1) \) is a constant depending only on \( \lambda \) and \( n \). Combining (2.17),(2.18) and \( \nu(x) \geq 0 \), we have
\[
\inf_{\Gamma} \nu(x) \geq \left( \frac{1}{C_2} \left( \frac{K - k}{2} + 2MB \right) \right) + \left( \frac{C_1}{C_2} - 1 \right)(|k|\|b\|_{L^2(Q^n)} + B\|b\|_{L^2(Q^n)} + \|f\|_{L^2(Q^n)}) \quad \Rightarrow a. \quad (2.19)
\]

Let
\[
\psi(x) = \frac{1}{2} \left( \frac{x_n}{\delta} \right)^2 - \frac{\delta^2}{4(n-1)} \sum_{i=1}^{n-1} \left( \frac{|x_i|}{\delta} - 1 \right)^2 + \epsilon,
\] (2.20)

where \( \epsilon \) satisfies (2.7).

The barrier function \( \psi(x) \) is \( C^2 \) and satisfies the following conditions:
\[
\begin{align*}
(1) \psi(x) &\leq 1 \text{ on } \{ x \in \mathbb{R}^n : |x'| \leq M\delta, x_n = \delta \}; \\
(2) \psi(x) &\leq 0 \text{ on } \{ x \in \mathbb{R}^n : |x'| \leq M\delta, x_n = 0 \}; \\
(3) \psi(x) &\leq 0 \text{ on } \{ x \in \mathbb{R}^n : |x'| = M\delta, 0 < x_n < \delta \}; \\
(4) - a_{ij}(x)D_{ij}\psi(x) &\leq 0 \text{ a.e. in } \{ x \in \mathbb{R}^n : |x'| < M\delta, 0 < x_n < \delta \}; \\
(5) \psi(x) &\geq \frac{n}{2\delta} \text{ in } Q^n_*.
\end{align*}
\] (2.21)

It follows that
\[
\begin{cases}
L(a\psi(x) - \nu(x)) \leq b_i D_i(a\psi(x) + (k - 2MB)x_n) - f(x) &\text{in } \tilde{Q}^c, \\
a\psi(x) - \nu(x) \leq 0 &\text{on } \partial \tilde{Q}^c,
\end{cases}
\] (2.22)

where \( \tilde{Q} = T_{M\delta} \times (0, \delta) \).

According to the Alexandroff-Bakelman-Pucci maximum principle,
\[
a\psi(x) - \nu(x) \leq C_3(\lambda - k + B)\|b\|_{L^2(Q^n)} + C_4|k|\|b\|_{L^2(Q^n)} + C_5\|f\|_{L^2(Q^n)} \quad \text{in } \tilde{Q}^c,
\] (2.23)

where \( C_3, C_4, C_5 \) are constants depending only on \( \lambda \) and \( n \), and we have used \( K - k \geq 0 \).

From (2.21)(5), it follows that for each \( x \in Q^n_* \),
\[
a\psi(x) \geq \frac{a}{2\delta} x_n
\geq \frac{(K - k)\delta}{2\delta}
\geq \frac{K - k}{4C_2} x_n - |k|\|b\|_{L^2(Q^n)} - B\|b\|_{L^2(Q^n)} - \|f\|_{L^2(Q^n)}. \quad (2.24)
\]

5
Combining (2.23) and (2.24), we have that for each $x \in Q_0^+$,

$$u(x) \geq a\psi(x) + (k - 2MB)x_n - (C_1 + C_4)(|k||b||L^\infty(Q_0^+)| + B||b||L^\infty(Q_0^+))$$

$$- C_3(k - B)||b||L^\infty(Q_0^+) - (C_1 + C_3)||f||L^\infty(Q_0^+)$$

$$\geq \left(k - 2MB + \frac{1}{4C_2}(K - k)\right)x_n - (C_1 + C_3 + 1)(|k||b||L^\infty(Q_0^+))$$

$$- (C_1 + C_3 + C_4 + 1)(k - k + B)||b||L^\infty(Q_0^+) - (C_1 + C_3 + 1)||f||L^\infty(Q_0^+).$$

(2.25)

Let

$$\mu = \frac{1}{4C_2}, \quad A_1 = C_1 + C_5 + 1, \quad A_2 = C_1 + C_3 + C_4 + 1 \quad \text{and} \quad A_3 = C_1 + C_4 + 1.$$

(2.26)

Combining (2.16),(2.25) and (2.26), we have that (2.3) and (2.4) hold. □

By induction, the following lemma is a direct consequence of Lemma 2.2.

**Lemma 2.3.** There exist sequences $(k_m)_{m=0}^\infty$ and $(K_m)_{m=0}^\infty$ and nonnegative sequence $(B_m)_{m=0}^\infty$ with $k_0 = K_0 = 0$, $B_0 = 1$, and for $m = 0, 1, 2, \ldots$,

$$B_{m+1} = A_1\delta^m||f||L^\infty(Q_{m+1}^+) + A_2\delta^m(K_m - k_m + \frac{B_m}{\delta^m})||b||L^\infty(Q_{m+1}^+) + A_3\delta^m(|K_m| + |k_m|)||b||L^\infty(Q_{m+1}^+),$$

and

$$k_{m+1} = k_m - 2MB_m\frac{B_m}{\delta^m} + \mu(K_m - k_m) \quad \text{and} \quad K_{m+1} = K_m + 2MB_m\frac{B_m}{\delta^m},$$

or

$$k_{m+1} = k_m - 2MB_m\frac{B_m}{\delta^m} \quad \text{and} \quad K_{m+1} = K_m + 2MB_m\frac{B_m}{\delta^m} - \mu(K_m - k_m),$$

such that

$$k_m x_n - B_m \leq u(x) \leq k_m x_n + B_m \text{ in } Q_0^+, \quad (2.27)$$

where $\delta$, $\mu$, $M$, $A_1$ and $A_2$ are positive constants given by Lemma 2.2.

Now we present the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $(B_m)_{m=0}^\infty$, $(k_m)_{m=0}^\infty$ and $(K_m)_{m=0}^\infty$ be defined by Lemma 2.3. We will show the proof by the following three claims.

**Claim 1.** $\sum_{m=0}^\infty B_m\delta^m$ is convergent.

**Proof.** Firstly, notice that we take $K_0 = k_0 = 0$ and $B_0 = 1$, then by induction, we have $K_m \geq k_m$ for all $m \geq 0$.

For $m \geq 0$, we define $S_m = \sum_{i=0}^m B_i\delta^i$. For any $m \geq 0$, since

$$K_{m+1} \leq K_m + 2MB_m\frac{B_m}{\delta^m} \text{ and } K_0 = 0,$$

we have

$$K_{m+1} \leq 2MS_m \text{ for any } m \geq 0.$$
Similarly, we have

\[ k_{m+1} \geq -2MS_m \text{ for any } m \geq 0. \]

Hence,

\[ |K_{m+1}| + |k_{m+1}| \leq 4MS_m \text{ for any } m \geq 0. \] (2.28)

Now we consider the term \( K_m - k_m \). By Lemma 2.3, for any \( m \geq 0 \),

\[ K_{m+1} - k_{m+1} \leq (1 - \mu)(K_m - k_m) + 4MB_m. \]

Since \( K_0 = k_0 = 0 \), by iteration, we have that for any \( m \geq 0 \),

\[ K_{m+1} - k_{m+1} \leq \sum_{i=0}^{m} \frac{4MB_i}{\delta^i}(1 - \mu)^{m-i}. \] (2.29)

It follows that for \( m \geq 1 \),

\[ \sum_{j=0}^{m}(K_j - k_j) \leq \sum_{j=1}^{m-1} \sum_{i=0}^{j-1} \frac{4MB_i}{\delta^i}(1 - \mu)^{j-i-1} = \sum_{j=1}^{m-1} \sum_{i=0}^{j-1} \frac{4MB_i}{\delta^i}(1 - \mu)^{j-i}. \]

By changing the order of summation, we have

\[ \sum_{j=0}^{m-1} \sum_{i=0}^{j} \frac{4MB_i}{\delta^i}(1 - \mu)^{j-i} = \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} \frac{4MB_i}{\delta^i}(1 - \mu)^{j-i}. \]

By

\[ \sum_{j=i}^{\infty}(1 - \mu)^{j-i} = \frac{1}{\mu}, \]

we have that for \( m \geq 1 \),

\[ \sum_{j=0}^{m}(K_j - k_j) \leq \frac{4M}{\mu} \sum_{i=0}^{m-1} \frac{B_i}{\delta^i} = \frac{4M}{\mu}S_{m-1}. \] (2.30)

Since

\[ \frac{B_{i+1}}{2} = \frac{A_1}{\delta} ||f||_{L^1(\mathcal{Q}^i_j)} + \frac{A_2}{\delta} (K_i - k_i + \frac{B_i}{\delta^i})||b||_{L^\infty(\mathcal{Q}^i_j)} + \frac{A_3}{\delta} (|K_i| + |k_i|)||b||_{L^\infty(\mathcal{Q}^i_j)}, \]

for any \( i \geq 1 \), combining the above identity with (2.28), we obtain

\[ \frac{B_{i+1}}{2} \leq \frac{A_1}{\delta} ||f||_{L^1(\mathcal{Q}^i_j)} + \frac{A_2}{\delta} (K_i - k_i + \frac{B_i}{\delta^i})||b||_{L^\infty(\mathcal{Q}^i_j)} + \frac{A_3}{\delta} (|K_i| + |k_i|)||b||_{L^\infty(\mathcal{Q}^i_j)} + \frac{4MA_3}{\delta} S_{i-1}||b||_{L^\infty(\mathcal{Q}^i_j)}. \] (2.31)

It follows from (2.30) and (2.31) that for any \( m \geq 1 \),

\[ \sum_{i=1}^{m} \frac{B_{i+1}}{2} \leq \sum_{i=1}^{m} \frac{A_1}{\delta} ||f||_{L^1(\mathcal{Q}^i_j)} + \frac{A_2}{\delta} (\frac{4M}{\mu} + 1)S_{m+1}||b||_{L^\infty(\mathcal{Q}^i_j)} + \frac{4MA_3}{\delta} S_{m+1}||b||_{L^\infty(\mathcal{Q}^i_j)}. \]

7
Since
\[ \sum_{i=1}^{\infty} \frac{4MA_3}{\delta} \|b_n\|_{L^2(Q_{i-1})} \leq \frac{4MA_3}{\delta} \log \frac{1}{\delta} \int_0^1 \frac{\|b_n\|_{L^2(Q_1)}}{r} dr \leq \frac{1}{4}, \]
\[ \frac{A_2}{\delta} \left( \frac{4M}{\mu} + 1 \right) \|b\|_{L^2(Q_1)} \leq \frac{1}{4}, \]
and
\[ \sum_{i=1}^{\infty} \|f\|_{L^2(Q_i)} \leq \frac{1}{\ln \frac{1}{\delta}} \int_0^1 \frac{\|f\|_{L^2(Q_1)}}{r} dr, \]
it follows that
\[ S_{m+1} - S_1 = \sum_{i=1}^{m} B_{m+1} \delta^i \leq \frac{A_1}{\delta} \ln \frac{1}{\delta} \int_0^1 \frac{\|f\|_{L^2(Q_1)}}{r} dr + \frac{1}{2} S_{m+1}. \]

Therefore for all \( m \geq 1, \)
\[ S_{m+1} \leq \frac{2A_1}{\delta} \ln \frac{1}{\delta} \int_0^1 \frac{\|f\|_{L^2(Q_1)}}{r} dr + 2S_{m+1} \leq \frac{4A_1}{\delta} \ln \frac{1}{\delta} + 2S_{m+1} \leq \frac{4A_1}{\delta} \ln \frac{1}{\delta} + \frac{2(A_1 + A_2)}{\delta} + 2, \]
where we used \( \|f\|_{L^2(Q_1)} \leq 1, \) \( \int_0^1 \frac{\|f\|_{L^2(Q_1)}}{r} dr \leq 1 \) and \( \|b\|_{L^2(Q_1)} \leq 1. \) Then \( \{S_m\}_{m=0}^{\infty} \) is a uniformly bounded sequence. It follows that \( \sum_{m=0}^{\infty} B_{m+1} \delta^i \) is convergent. This completes the proof of Claim 1. \( \square \)

**Claim 2.**
\[ \lim_{m \to \infty} K_m = \lim_{m \to \infty} k_m = \theta. \]

**Proof.** It follows from Claim 1 that \( \{K_m\}_{m=0}^{\infty} \) and \( \{k_m\}_{m=0}^{\infty} \) are uniformly bounded. Since
\[ K_{m+1} - K_m \leq 2MB_{m+1} \delta^m = 2MS_m - 2MS_{m-1} \text{ for } m \geq 1, \]
we obtain
\[ K_{m+1} - 2MS_m \leq K_m - 2MS_{m-1} \text{ for } m \geq 1. \]
It follows that \( \{K_m - 2MS_{m-1}\}_{m=1}^{\infty} \) is a bounded nonincreasing sequence and \( \lim_{m \to \infty} (K_m - 2MS_{m-1}) \) exists. Hence \( \lim_{m \to \infty} K_m \) exists. Let \( \lim_{m \to \infty} K_m = \theta. \)

Since
\[ \sum_{j=0}^{\infty} (K_j - k_j) \leq 2M \sum_{i=0}^{\infty} B_i \delta^i < +\infty, \quad \forall \ m \geq 1, \]
we have \( \sum_{j=0}^{\infty} (K_j - k_j) \) is convergent. It follows that \( \lim_{m \to \infty} (K_m - k_m) = 0. \) Hence
\[ \lim_{m \to \infty} K_m = \lim_{m \to \infty} k_m = \theta. \]
This completes the proof of Claim 2. \( \square \)
Claim 3. Let $\theta$ be given by Claim 2. Then for each $m = 0, 1, 2, \ldots$, there exist $C_m$ such that
\[
\lim_{m \to +\infty} C_m = 0
\]
and that $|u(x) - \theta x_n| \leq C_m \delta^m$ for any $x \in Q^+_m$.

Proof. For any $m \geq 0$ and any $x \in Q^+_m$, we have
\[
|u(x) - \theta x_n| \leq (|K_m - \theta| + |k_m - \theta||x_n|) + \frac{B_m}{\delta^m} \leq (|K_m - \theta| + |k_m - \theta| + \frac{B_m}{\delta^m}) \delta^m.
\]
Let $C_m = |K_m - \theta| + |k_m - \theta| + \frac{B_m}{\delta^m}$. It follows that for any $m \geq 0$ and any $x \in Q^+_m$,
\[
|u(x) - \theta x_n| \leq C_m \delta^m,
\]
and
\[
\lim_{m \to +\infty} C_m = 0.
\]
This completes the proof of Claim 3.

By Claim 3, we deduce that $u(x)$ is differentiable at 0 with derivative $\theta \delta x_n$. This completes the proof of Theorem 2.1.

References

[1] D.E. Apushkinskaya, A.I. Nazarov; Boundary estimate for the first-order derivatives of a solution to a nondivergent parabolic equation with composite right-hand side and coefficients of lower-order derivatives, Journal of Mathematical Sciences, 77(4) (1995), 3257-3276.
[2] D.E. Apushkinskaya, A. I. Nazarov; A counterexample to the Hopf-Oleinik lemma (elliptic case), Analysis & PDE, 9(2) (2016), 439–458.
[3] C. Burch; The Dini condition and regularity of weak solutions of elliptic equations, Journal of Differential Equations, 30 (1970), 308–323.
[4] D. Gilbarg, N.S. Trudinger; Elliptic partial differential equations of second order, 2nd ed., Springer-Verlag, Berlin, (1983).
[5] O.A. Ladyzhenskaya, N.N. Uraltseva; Estimates on the boundary of a domain for the first derivatives of functions satisfying an elliptic or parabolic inequality, Trudy Mat. Inst.Steklov, 179 (1988), 102–125 (in Russian). English transl. in Proc. Steklov Inst. Math., 179 (1989), 109–135.
[6] J. Kovats; Fully nonlinear elliptic equations and the Dini condition, Comm. Partial Differential Equations, 22 (1997), 1911–1927.
[7] J. Kovats; Dini–Campanato spaces and applications to nonlinear elliptic equations, Electron. J. Differential Equations, 37 (1999), 1–20.
[8] N.V. Krylov; Boundedly inhomogeneous elliptic and parabolic equations in a domain, Izvestia Akad. Nauk. SSSR, 47 (1983), 75–108 (in Russian). English translation in Math. USSR Izv., 22 (1984), 67–97.
[9] G.M. Lieberman; The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data, Comm. in Partial Differential Equations, 11 (1986), 167–229.
[10] D.Li, L.Wang; Boundary differentiability of solutions of elliptic equations on convex domains, Manuscripta Mathematica, 121 (2006), 137–156.
[11] D.Li, L.Wang; Elliptic equations on convex domains with nonhomogeneous Dirichlet boundary conditions, Journal of Differential Equations, 246 (2009), 1723–1743.
[12] Y. Huang, Q. Zhai, S. Zhou; Boundary regularity for strong solution of nondivergence elliptic equations with unbounded drift, Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 39, 1–16.
[13] A.I. Nazarov; A centennial of Zaremba-Hopf-Oleinik Lemma, SIAM Journal of mathematical analysis, SIAM J. Math. Anal., 44 (2012), 437–453.
[14] M.V. Safonov; Boundary estimates for positive solutions to second order elliptic equations, [arXiv:0810.0522v2 [math.AP]].
[15] M.V. Safonov; Non-divergence Elliptic Equations of Second Order with Unbounded Drift, Nonlinear partial differential equations and related topics, 211–232, Amer. Math. Soc. Transl. Ser. 2, 229, Amer. Math. Soc., Providence, RI, (2010).
[16] L.Wang; On the regularity theory of fully nonlinear parabolic equations. II, Comm. Pure. Appl. Math., 45 (1992), 141–178.