Higher Dimensional Particle Model in Pure Lovelock Gravity

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In this paper, based on the thin-shell formalism, we introduce a classical model for particles in the framework of \( n+1 \)-dimensional \( \left[ \frac{n}{2} \right] \)-order pure Lovelock gravity (pLG). In particular we construct a spherically symmetric particle of radius \( a \) whose inside is a flat minkowski spacetime while its outside is charged pLG solution. Knowing that in \( n+1 \)-dimensional spherically symmetric Einstein (R-gravity) such a particle model cannot be constructed, provides the main motivation for this study.

I. INTRODUCTION

We introduce particle model to a rather interesting area of modified theory of gravity which is called pure Lovelock gravity (pLG). This terminology was used for the first time by Kastor and Mann [1] and has been developed by Cai, et al. [2, 3], Dadhich, et al. [4–13] and others [14–18]. We recall from the original Lovelock theory that the action of the \( n+1 \)-dimensional \( \left[ \frac{n}{2} \right] \)-order Lovelock gravity is given by [19]

\[
I_{LG} = \int d^{n+1}x \sqrt{-g} \left( \alpha_0 + \sum_{i=0}^{\left[ \frac{n}{2} \right]} \alpha_i \mathcal{L}_i + \mathcal{L}_{\text{matter}} \right)
\]  
(1)

in which \( \left[ \frac{n}{2} \right] \) stands for the integral part of the \( \frac{n}{2} \), \( \alpha_i \) are the \( i^{th} \)-order Lovelock parameters, and the Euler densities of a 2\( i \)-dimensional manifold are given by

\[
\mathcal{L}_i = \frac{1}{2^i} \delta_{\alpha_1 \beta_1 \ldots \alpha_i \beta_i} \prod_{s=1}^{i} R_{\mu_1 \nu_1 \ldots \mu_i \nu_i} \alpha_i \beta_i,
\]

(2)

where the generalized Kronecker delta \( \delta \) is defined as the antisymmetric product

\[
\delta_{\alpha_1 \beta_1 \ldots \alpha_i \beta_i} = i! \delta_{[\alpha_1 \beta_1 \ldots \alpha_i \beta_i]} \delta_{[\mu_1 \nu_1 \ldots \mu_i \nu_i]}.
\]

(3)

On the other hand, in pure LG the action is expressed as

\[
I_{\text{pure LG}} = \int d^{n+1}x \sqrt{-g} (\alpha_0 + \alpha_p \mathcal{L}_p + \mathcal{L}_{\text{matter}})
\]

(4)

in which \( \alpha_0 \) represents the cosmological constant and \( 1 \leq p \leq \left[ \frac{n}{2} \right] \). For instance, the Einstein R-gravity has \( p = 1 \), \( \alpha_1 = 1 \) and \( \mathcal{L}_1 = R \) which is the particular case of the pure LG applicable in all dimensions. For \( p = 2 \), one finds the pure Gauss-Bonnet (GB) gravity applicable in \( n+1 \geq 5 \). Finally, in this paper, \( p = 3 \) represents pure TOLG which is valid for \( n+1 \geq 7 \). Unlike the particle model in TOLG, herein we give a general formalism in the three different pure Lovelock theories mentioned above, i.e., the pure Einstein, GB and TOLG. It will be shown that in specific case where the inner and outer spacetimes of the chosen thin-shell boundary admit identical Lovelock parameters i.e., \( \alpha^+_p = \alpha^-_p \), the junction conditions result in the same in terms of the metric functions and their first derivatives, irrespective of the order of the Lovelock term. These junction conditions are simply the continuity of the bulk’s metric function and its first derivatives across the thin-shell which is the surface of the particle. These are the continuity of the first and second fundamental forms of the surface.

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II. PARTICLE MODEL IN PURE LOVELOCK GRAVITY

A. Pure Einstein gravity

In this section we construct an $n + 1$-dimensional chargeless particle model in pure Einstein $R$-gravity. We assume a static timelike spherical shell of radius $r = a$ such that its interior and exterior spacetimes are $f_-(r_-)$ and $f_+(r_+)$, respectively. The standard Israel junction conditions [20, 21] imply that

$$\sigma = -\frac{n-1}{a} \left( \sqrt{f_+} - \sqrt{f_-} \right) \Sigma$$  \hspace{1cm} (5)

and

$$p = \frac{1}{2} \left( \frac{f'_+}{\sqrt{f_+}} - \frac{f'_-}{\sqrt{f_-}} \right) - \frac{n-2}{n-1} \sigma.$$  \hspace{1cm} (6)

For a particle one expects $\sigma = 0$ and $p = 0$ simultaneously which result in the bulk’s metric function and its first derivative to be continuous across the surface of the particle, i.e. $(f_+ = f_-) \Sigma$ and $(f'_+ = f'_-) \Sigma$. In other words, continuity of the first and second fundamental forms across the thin-shell correspond to having the metric function and its first derivative continuous on the surface of the particle. Hence, we are very much restricted in choosing the spacetimes inside and outside of the particle. For instance, a Minkowski inner flat and an outer Schwarzschild spacetimes do not satisfy the two conditions. Also, an inner Minkowski flat and an outer Reissner-Nordström (RN) can not be matched, whereas a cloud of strings of the form

$$f_- = 1 - \frac{2\mu}{(n-1)r^{n-3}}$$  \hspace{1cm} (7)

can be matched to the RN spacetime with the metric function

$$f_+ = 1 - \frac{2m}{r^{n-2}} + \frac{q^2}{r^{2(n-2)}}.$$  \hspace{1cm} (8)

Herein, $\mu$ is an integration constant associated with the cloud of strings, $m$ is a mass parameter (proportional to the physical mass) and $q$ is the charge parameter (proportional to the physical total charge). Applying $(f_+ = f_-) \Sigma$ and $(f'_+ = f'_-) \Sigma$ yields

$$\frac{m}{q^2} = \frac{n-1}{2a^{n-2}},$$  \hspace{1cm} (9)

and

$$\frac{\mu}{q^2} = \frac{(n-1)(n-2)}{2a^{n-1}}.$$  \hspace{1cm} (10)

Specifically, in 4-dimensional spacetime ($n = 3$) one finds $a = \frac{q^2}{m}$ and $\mu = \left( \frac{m}{q} \right)^2$. Furthermore, in 4-dimensional spacetime the Newton’s potential inside the shell becomes a constant, i.e. $f_- = 1 + 2\Phi_- = 1 - \mu$, which leads to $\Phi_- = -\frac{\mu}{2}$. In addition, the exterior potential of the particle is given by $f_+ = 1 + 2\Phi_+ = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$, and explicitly $\Phi_+ = -\frac{m}{r} + \frac{q^2}{2r^2}$. On the surface, we trivially have $\Phi_- = \Phi_+ = -\frac{\mu}{2}$, which is in agreement with our understanding of Newtonian potential inside and outside a spherical shell in classical mechanics. We should admit, however, that although the constant Newtonian potential seems a better choice than a flat Minkowski for the particle’s interior, such a spacetime is singular due to the singular nature of the energy-momentum tensor for the string cloud

$$T^\nu_{\mu} = \text{diag} \left( \frac{\mu}{r^2}, \frac{\mu}{r^2}, 0, 0 \right).$$  \hspace{1cm} (11)

In higher-dimensional particle models without singularity at the center, one may think of a(n) (Anti-)de Sitter spacetime with the cosmological constant $\Lambda = \pm \frac{n(n-1)}{2\ell^2}$ as the interior of the particle in the form

$$f_- = 1 - \frac{r^2}{\ell^2},$$  \hspace{1cm} (12)
and an RN spacetime as the exterior of the particle, given in Eq. (8). Upon matching the two metrics, we obtain

\[
\frac{m}{q^2} = \frac{n - 1}{na^{n-2}},
\]

and

\[
q^2 \ell^2 = \frac{n}{(n - 2)} a^{2(n-1)}.
\]

Again in the specific 4-dimensional case \((n = 3)\), we find \(a = \frac{2}{3} \left( \frac{q^2}{m} \right)\) and \(\ell^2 = \frac{16}{27} q^6 m^4\). By reversing these expressions, we obtain the mass and the charge of a particle in terms of the geometric parameters of the theory. We must add that, in all cases the particle’s radius \(a\) should be greater than the radius of the event horizon of the exterior metric and smaller than the possible cosmological horizon of the interior metric. These are necessary to avoid any horizon or singularity inside or outside of the particle. For instance, again in four dimensions, in order to avoid any horizon within outer spacetime one has to assume \(a > \frac{2}{3} q\).

As the final example for this section, let us consider a global monopole for the interior spacetime with the metric function

\[
f_- = 1 - 2\eta
\]

in an arbitrary dimension, and \(f_+\) given by Eq. (8) for the exterior spacetime. The junction conditions impose \((f_+ = f_-)\Sigma\) and \((f'_+ = f'_-)\Sigma\) which in turn reveal the radius as

\[
a = \left( \frac{q^2}{m} \right) \frac{1}{n-2}.
\]

Note that, the energy-momentum tensor of the global monopole spacetime which is given by

\[
T^\mu_\nu = -\frac{\eta}{r^2} (n-2) \text{diag}(n-1, n-1, n-3, \ldots, n-3),
\]

implies that the spacetime is singular at \(r = 0\). In \(3 + 1\)-dimensions, however, the spacetime reduces to the cloud of strings of the previous example with \(\rho = -p_r = \eta/r^2\), and \(p_\theta = p_\phi = 0\). In higher dimensions, if one desires to keep the potential within the interior spacetime to be constant, the only candidate spacetime is the global monopole. Unfortunately, such a spacetime admits undesirable singularity at \(r = 0\).

### B. Pure Gauss-Bonnet gravity

In [2] the pure LG with cosmological constant has been investigated. In pure GB gravity it amounts to

\[
I_{GB} = \int d^{n+1}x \sqrt{-g} (\alpha_0 + \alpha_2 \mathcal{L}_{GB}), \quad n \geq 4
\]

in which \(\alpha_0\) is the cosmological constant, \(\alpha_2\) is the GB free parameter and

\[
\mathcal{L}_{GB} = R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} - 4 R_{\mu\nu} R_{\mu\nu} + R^2
\]

is the GB Lagrangian. In [2] a spherically symmetric solution with the line element

\[
ds^2 = -f(r) \, dt^2 + \frac{1}{f(r)} \, dr^2 + r^2 d\Omega_{n-1}^2,
\]

is considered such that the metric function is found to be

\[
f(r) = 1 \pm \frac{r^2}{\sqrt{\alpha_2}} \sqrt{\frac{2M}{(n-1) \Sigma_{n-1} r^{n-1}}} + \frac{1}{\ell^2}.
\]
Here $M$ is the mass of the solution, $\Sigma_{n-1} = \frac{2\pi n}{r(\frac{1}{2})}$ is the volume of $(n-1)$-sphere [18], $\tilde{\alpha}_2$ is given by

$$\tilde{\alpha}_2 = (n - 2)(n - 3)\alpha_2,$$  \hspace{1cm} (22)

and

$$\frac{1}{\ell^2} = -\frac{\alpha_0}{n(n-1)}$$  \hspace{1cm} (23)

is related to the cosmological constant $\alpha_0$. In particular, for $n + 1 = 7$ one finds

$$f(r) = 1 \pm \frac{\ell^2}{\sqrt{\tilde{\alpha}_2}} \sqrt{\frac{2M}{5\pi^{n/2}}} + \frac{1}{\ell^2},$$  \hspace{1cm} (24)

in which the parameters, upon choosing the $(+)$ sign are connected by $\frac{1}{\ell^2} = -\frac{\alpha_0}{30}$ and $\tilde{\alpha}_2 = 12\alpha_2$. Note that an interesting property of the 7-dimensional pure GB theory without cosmological constant is that, its potential from Eq. (24), gives the same fall-off as in the 4-dimensional Einstein gravity [5] i.e.,

$$\Phi = \pm \frac{m}{r},$$  \hspace{1cm} (25)

where $m = \frac{1}{2\sqrt{\alpha_0}} \sqrt{\frac{2M}{5\pi^{n/2}}}$. 

Next, let’s consider a particle of radius $r = a$ whose inner and outer spacetimes are the solutions in pure GB. The generalized Israel junction condition must be applied at the surface of the particle where the two incomplete spacetimes are glued. Based on the generalized Israel junction conditions, without assuming $dt_+ = dt_-$, one simply finds the induced metric on the shell to be given by

$$ds^2 = -d\tau^2 + a^2d\Omega^2_{n-1}.$$  \hspace{1cm} (26)

The surface energy momentum tensor $S^a = (-\sigma, \ p, ..., p)$ is expressed as

$$-S^a_b = 2[\alpha_2 (3J^a_b - J^a)_{n-1}]_+$$  \hspace{1cm} (27)

in which $J^a_b$ and $J$ are defined as

$$J^a_b = diag \left( -\frac{2!}{3} \left\{ \sum_{s=0}^{2} \frac{(-1)^s}{n} \binom{n}{s} \left[ sK^s + (n-s)K^{n-s} \right] \left( K^{n-1}K^{n-s} \right) \right\} \right)$$  \hspace{1cm} (28)

and

$$J = J^a_a$$  \hspace{1cm} (29)

respectively. Here, $K^r_r$ and $K^b_b = K^{a_1}_b = K^{a_2}_b = ... = K^{a_{n-1}}_b$ are the components of extrinsic curvature tensor associated with the time and angular coordinates of the timelike hyperplane (surface of the particle). An explicit calculation reveals

$$\sigma = \frac{2(n-1)}{3a^3} [\tilde{\alpha}_2 \sqrt{f(3-f^\prime)}]_+$$  \hspace{1cm} (30)

and

$$p = \frac{1}{a^2} \left[ \frac{\tilde{\alpha}_2 (f^2 - 1) f''}{\sqrt{f}} \right]_+ + \frac{n-4}{n-1} \alpha.$$  \hspace{1cm} (31)

Having considered specific metric functions $f_{\pm}$, and also $\tilde{\alpha}_2^-$ and $\tilde{\alpha}_2^+$ for interior and exterior spacetimes, respectively, the conditions $\sigma = 0$ and $p = 0$ yield

$$\tilde{\alpha}_2^- \sqrt{f_+ (f_+ - 3)} - \tilde{\alpha}_2^+ \sqrt{f_- (f_- - 3)} = 0$$  \hspace{1cm} (32)
and
\[ \tilde{\alpha}_2^+ \sqrt{f_- f'_+ (f_+ - 1)} - \tilde{\alpha}_2^- \sqrt{f_+ f'_- (f_- - 1)} = 0. \] (33)

For \( \tilde{\alpha}_2^+ = \tilde{\alpha}_2^- \), one may consider the trivial solution of Eq. (32) as \( (f_+ = f_-)_\Sigma \), and consequently, Eq. (33) yields \( (f'_+ = f'_-)_\Sigma \). For \( \tilde{\alpha}_2^+ \neq \tilde{\alpha}_2^- \), Eq. (32) implies

\[ \tilde{\alpha}_2^- = \frac{\sqrt{f_+ (f_+ - 3)}}{\sqrt{f_- (f_- - 3)}} \tilde{\alpha}_2^+ \] (34)

which upon inserting into Eq. (33) we obtain

\[ (f_- - 3) (f_+ - 1) f_- f'_+ + (f_+ - 3) (f_- - 1) f_+ f'_- = 0. \] (35)

This is the only constraint on the metric functions and their first derivatives to be satisfied on \( \Sigma \). As an example, let us consider \( f_+ = 1 \) and the positive branch of the solution in Eq. (21) as \( f_+ \). Considering \( \tilde{\alpha}_2^+ \neq \tilde{\alpha}_2^- \), from Eq. (35) and Eq. (34) we obtain

\[ M = 2 \frac{(n-1) \Sigma_{n-1} a^n}{\ell^2 (n-4)} \] (36)

and

\[ \tilde{\alpha}_2^- = \frac{\tilde{\alpha}_2^+}{2} (2 - \lambda_2) \sqrt{1 + \lambda_2} \] (37)

in which

\[ \lambda_2 = \frac{a^2}{\tilde{\alpha}_2^+} \sqrt{\frac{n}{(n-4) \ell^2}}. \] (38)

To complete the arguments, let us add that, in the case where \( \tilde{\alpha}_2^- = \tilde{\alpha}_2^+ \), the junction conditions simply reduce to the one in \( R^- \)–gravity, i.e., the continuity of the bulk’s metric function and its first derivative.

C. Pure third-order Lovelock gravity

Similar to the previous section, in this section we shall give a particle model in pure TOLG. Following [2, 3] the corresponding action is given by

\[ I_{\text{pure TOLG}} = \int dx^{n+1} \sqrt{-g} \left( \alpha_0 + \alpha_3 \mathcal{L}_3 \right), \quad n \geq 6 \] (39)

in which \( \alpha_0 \) stands for the cosmological constant, \( \alpha_3 \) is the third order Lovelock parameter and

\[ \mathcal{L}_3 = 2R^\kappa_{\lambda \rho} R_{\mu \sigma \nu} R^{\mu \nu} R_{\kappa \lambda} + 8R^{\mu \nu} R_{\kappa \lambda} R^{\kappa \lambda} + 24R^{\mu \lambda \nu \rho} R_{\mu \nu \lambda \rho} R_\kappa + 3RR_{\kappa \lambda \rho} R_{\mu \kappa \lambda \rho} + 24R^{\kappa \lambda \mu \nu} R_{\kappa \lambda \mu \nu} + 16R^{\mu \nu} R_{\mu \nu} R_\kappa - 12RR^{\mu \nu} R_{\mu \nu} + R^3 \] (40)

is the third order Lovelock Lagrangian. In [2], after supplementing the Lagrangian with a Maxwell term, an \( n + 1 \)-dimensional static spherically symmetric solution with electric charge is found whose line element is given by Eq. (21) with the metric function written as

\[ f(r) = 1 + \frac{r^2}{|\tilde{\alpha}_3|^{1/3}} \sqrt{\frac{2M}{(n-1) \Sigma_{n-1} r^n} - \frac{q^2}{r^{2(n-1)}}} + \frac{1}{\ell^2}. \] (41)

Therein, the integration constants \( M \) and \( q \) are the usual mass and electric charge parameters, \( \ell^2 > 0 \) represents the cosmological constant and \( \tilde{\alpha}_3 = -|\tilde{\alpha}_3| < 0 \). This solution contains two theory parameters i.e., \( \tilde{\alpha}_3 \) and \( \ell^2 \) and two solution parameters which are the integration constants i.e., \( M \) and \( q \).
Next, we consider a spherically symmetric particle that consists of inner and outer spacetimes as solutions of pure TOLG, such as the one given in (41). The associated junction conditions extracted from [22–25] are expressed as

\[ - S_b^a = [3\alpha_3 (5P_b^a - P\delta_b^a + L_{GB} (K_b^a + K\delta_b^a))]^+, \]

in which \( S_b^a \) and \( K_b^a \) are as before while \( K = K_a^a \) with

\[ P_b^a = \text{diag} \left( \frac{4!}{5} \left( \sum_{s=0}^4 \frac{(-1)^s}{n_s} \right) [sK_\tau^+ + (n-s)K_\beta^a] \right) \]

and \( P = P^a_a \). Within explicit calculations we obtain

\[ \sigma = -\frac{(n-1)}{5a^5} \left[ \tilde{\alpha}_3 \sqrt{f} (3f^2 - 10f + 15) \right]^+ \]

and

\[ p = \frac{3}{2a^3} \left[ \frac{\tilde{\alpha}_3 (1 - f)^2 f'}{\sqrt{f}} \right]^+ - \frac{n - 6}{n - 1} \sigma. \]

As of the boundary conditions on \( \Sigma \), the surface of the particle, the first fundamental form \( h_{ab} \) and the second fundamental form \( K_{ab} \) should be continuous. Technically these are equivalent to \( \sigma = 0 \) and \( p = 0 \) simultaneously. Similar to the pure GB case, there are two distinct possibilities: i) \( \tilde{\alpha}_3^+ = \tilde{\alpha}_3^- \) which might trivially imply \( f_+ = f_- \) \( \Sigma \) and so \( (f'_+ = f'_-)_\Sigma \), and ii) \( \tilde{\alpha}_3^* \neq \tilde{\alpha}_3^- \) which results in the following relation between \( \tilde{\alpha}_3^+ \) and \( \tilde{\alpha}_3^- \)

\[ \tilde{\alpha}_3^- = \frac{\sqrt{f_+} (3f_+^2 - 10f_+ + 15)}{\sqrt{f_-} (3f_-^2 - 10f_- + 15)} \tilde{\alpha}_3^+. \]

and a constraint condition on the metric functions

\[ f_+ f'_+ (1 - f_-)^2 (3f_+^2 - 10f_+ + 15) - f_- f'_+ (1 - f_+)^2 (3f_-^2 - 10f_- + 15) = 0. \]

Therefore, for \( \tilde{\alpha}_3^* \neq \tilde{\alpha}_3^- \), every considered solution for the inner and outer spacetimes must satisfy this condition on \( \Sigma \). Moreover with \( \tilde{\alpha}_3^* = \tilde{\alpha}_3^- \) the junction conditions reduce to the one in \( \text{R} \)-gravity and also in pure GB gravity with \( \tilde{\alpha}_3^* = \tilde{\alpha}_3^- \).

As an example, since we are interested to construct a singularity free particle model, let us set \( f_- = 1 \) (and consequently \( f'_- = 0 \)). The first condition, i.e. \( \sigma = 0 \), implies

\[ \tilde{\alpha}_3^- = \frac{\tilde{\alpha}_3^+}{8} \sqrt{f_+} (3f_+^2 - 10f_+ + 15). \]

Considering the second condition, i.e. \( p = 0 \), we find

\[ f'_+ (1 - f_+)^2 = 0 \]

which admits two possibilities, either \( f_+ = 1 \) on the surface which consequently implies \( \tilde{\alpha}_3^+ = \tilde{\alpha}_3^- \) or \( f'_+ = 0 \). Upon considering \( f_+ \) to be the general charged solution given in (41) without the cosmological constant, the latter equation, i.e., \( f'_+ = 0 \) admits the relation

\[ \frac{M}{q^2} = \frac{(n-1)(n-4)\Sigma_{n-1}}{(n-6)a^{n-2}} \]

between the mass and the charge of the particle in terms of its radius and

\[ \tilde{\alpha}_3^- = \frac{\tilde{\alpha}_3^+}{8} \sqrt{1 + \lambda_3 (3\lambda_3^2 - 4\lambda_3 + 8)} \]

in which

\[ \lambda_3 = a^2 \left( \frac{(n-2)q^2}{|\tilde{\alpha}_3^+| (n-6)a^{2(n-1)}} \right)^{1/3}. \]

In 7-dimensional spacetime where \( n = 6 \), however, such a particle model fails to work and one needs to consider additional theory parameters such as a cosmological constant. In short, having the third-order parameters involved in this specific case, provides us a particle model with mass and charge which could not be made in \( \text{R} \)-gravity. This shows the rich structure of the LG in any form in constructing particle models. Once more, we add that to avoid nonphysical particle models, the spacetime of the particle including inside, on, and outside the shell have to be singularity- and horizon-free. These are the additional conditions to be imposed on the radius of any physical particle.
We employed the cut and paste method to construct classical particle model in $n+1$–dimensional spherically symmetric pLG. In general $n+1$–dimensional particle model with an arbitrarily high order of the pLG when the Lovelock parameters of the outer and inner spacetimes are identical then irrespective of the dimensions and order of pLG the junction conditions reduce to the continuity of the metric function and its first derivative on the surface of the particle. On the other hands, with different Lovelock parameters the scenario changes dramatically such that depending on what order of Lovelock gravity is considered one obtains two distinct conditions relating the metric functions as well as the Lovelock parameters. In pGB gravity and pTOLG we have explicitly found these relations. Although in $n+1$–dimensional $R$–gravity a particle model with flat inside and Schwarzschild or RN outside are not possible (for $n = 3$ see [26]), in both pGB and pTOLG such a massive and charged model of particle with flat inside are feasible.

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