Markovian master equations, often called Liouvillians or Lindbladians, are used to describe decay and decoherence of a quantum system induced by that system’s environment. While a natural environment is detrimental to fragile quantum properties, an engineered environment can drive the system toward exotic phases of matter or toward subspaces protected from noise. These cases often require the Lindbladian to have more than one steady state, and such Lindbladians are dissipative analogues of Hamiltonians with multiple ground states. This thesis studies Lindbladian extensions of topics commonplace in degenerate Hamiltonian systems, providing examples and historical context along the way.

An important property of Lindbladians is their behavior in the limit of infinite time, and the first part of this work focuses on deriving a formula for the asymptotic projection — the map corresponding to infinite-time Lindbladian evolution. This formula is applied to determine the dependence of a system’s steady state on its initial state, to determine the extent to which decay affects a system’s linear or adiabatic response, and to determine geometrical structures (holonomy, curvature, and metric) associated with adiabatically deformed steady-state subspaces. Using the asymptotic projection to partition the physical system into a subspace free from nonunitary effects and that subspace’s complement (and making a few other minor assumptions), a Dyson series is derived to all orders in an arbitrary perturbation. The terms in the Dyson series up to second order in the perturbation are shown to reproduce quantum Zeno dynamics and the effective operator formalism.
Lindbladians with multiple steady states: theory and applications

A Dissertation
Presented to the Faculty of the Graduate School of Yale University in Candidacy for the Degree of Doctor of Philosophy

by
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May 2017
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ACRONYMS

Op(H) space of operators that act on a Hilbert space H

As(H) Asymptotic subspace — the subspace of Op(H) for which evolution under a given Lindbladian is unitary

DFS Decoherence-free subspace — specific type of As(H)

NS Noiseless subspace — specific type of As(H) that factorizes into a tensor product of a DFS and a unique auxiliary mixed state

QGT Quantum geometric tensor — a quantity providing a metric on and encoding the geometric phase properties of a subspace

LIST OF THEOREMS

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For my parents
Tatyana & Thomas
and my wife
Olga
ACKNOWLEDGMENTS

First I would like to thank my adviser, my wife, and my parents, all of whom created the environment and provided the opportunities for a work such as this to be possible. Their contributions however were on different timescales, but they are all nevertheless important to the final cumulative outcome. Leading by example, my parents always made me believe that whatever abilities I have would be sufficient for “success” as long as they are complemented with hard work. My wife has always supported my professional life by listening to me when I needed to talk and letting me devote as much time to work as I wanted. (Luckily, work hasn’t amounted to all my time, so I continue to be happily married!) Of course, a proper upbringing and a supportive better half would be for nothing if one is in a stressful relationship with their adviser. Luckily, it has been the opposite: Liang has been supportive, thoughtful, patient, and kind and has demonstrated what it takes to be a good physicist. As I developed my own tastes, he let me pursue other directions while at the same time letting me know what he thought was important.

I also acknowledge the support of my committee of collaborators. Despite all of them being extremely busy, I know that I can always count on Profs. Devoret, Girvin, Glazman, and Schoelkopf to lend an ear and help sort out scientific and even personal problems. Their combination of accomplishments and humility is truly inspiring. Among my other collaborators, I want to thank especially Martin Fraas and Barry Bradlyn for carefully checking a large portion of this work and helping overcome a negative referee report.

Several other faculty, both at Yale and elsewhere, have contributed to the supportive scientific environment I have enjoyed so far: Lorenza Viola (whom I want to also acknowledge for being an external reader for this work), Zaki Leghtas, Mazyar Mirrahimi, Barbara Terhal, Alexey Gorshkov, Nick Read, Steve Flammia, Francesco Ticozzi, Marlan Scully, Dariusz Chruscinski, Oscar Viyuela, Kirill Velizhanin, and Frank Harris. Members of my adviser’s group have provided a welcoming, active, and collaborative learning atmosphere: Stefan Krastanov, Sreraman Muralidharan, Arpit Dua, Kyungjoo Noh, Chao Shen, Linshu Li, Chang-ling Zou, Chi Shu, Marios Michael, and Jianming Wen. Throughout the years, I was also fortunate to have fruitful discussions with Richard Brierley, Matti Silveri, Alex Petrescu, Aris Alexandradinata, Andrey Gromov, Judith Höller, Phil Reinhold, Reinier Heeres, Volodymyr Sivak, Ion Garate, Zaheen Sadeq, and many of the members of Qlab and RSL.

Last and definitely not least, I want to thank my long-time friends at Yale — Brian Tennyson, Jukka Väyrynen, Staff Sheehan, Zlatko Minev, Kevin Yaung, and Alexei Surguchev — for being there during all these years.
“Give me a Hamiltonian and I will move the world.”

– Leonid I. Glazman

INTRODUCTION AND MOTIVATION

1.1 THE FIELD OF STUDY

1.1.1 Applied quantum physics

We are often taught in school that physics consists of the never-ending interplay between theorist and experimentalist. Theorists propose models for experimentalists to verify, while unexpected experimental results cause theorists to adjust their models. While this type of science is currently happening in high-energy areas of physics (such as dark matter detection), other areas have become more crystallized and their theories more verified. One such area can be called low-energy or applied quantum physics. In this broad field, the theory — quantum mechanics — is used to describe almost all processes and is not generally questioned. In other words, enough evidence has been gathered showing that the vast majority of processes on nanometer length scales and at nanoKelvin temperatures are most effectively and to a high degree of precision described by quantum mechanics. Instead of testing theories, some primary aims of this field are to notice interesting quantum mechanical effects, to demonstrate (or realize) these effects using various currently available quantum technologies, and to develop devices based on these effects which have potential applications to the “real world”. In the past, examples of such devices include the laser, the transistor, and nuclear magnetic resonance imagers. Although these devices can be described using mostly classical physics [113, 266], it is difficult to do so without resorting to the discretized nature of quantum energy levels and even more difficult to claim that the rise of quantum mechanics did not directly contribute to their development.

Currently, a primary potential application of this field is the development of a quantum computer [117, 311] — a device that has been theoretically proven to perform certain computational tasks significantly faster than any ordinary computer. By “significantly”, we mean that the time taken by a quantum computer to perform a task is a couple of days while a classical computer would take at least until the sun burns out. Since one of the tasks — integer factorization — can be used to crack an often-used encryption scheme used to communicate securely over the internet and since quantum technologies are improving very quickly, quantum computation continues to steadily gain attention from the non-scientific community.
Another important task that “quantum computers can do in their sleep” \cite{117} is \textit{quantum simulation} — the tailoring of one quantum system, in this case the quantum computer, to simulate another, less well-understood quantum system. Therefore, the development of a quantum computer should in principle lead to a qualitatively improved understanding of quantum processes in other fields such as chemistry or biology.

A closely related field is \textit{quantum metrology and sensing} — the use of quantum devices to perform precise measurements and microscopy which are not possible with “classical devices”. This field’s associated quantum technologies have promising applications ranging from table-top measurements of nature’s fundamental constants to imaging of biological systems.

Developments in applied quantum physics should also potentially allow one to better understand and even synthesize exotic \textit{quantum phases of matter}. “For a large collection of similar particles, a \textit{phase} is a region in some parameter space in which the thermal equilibrium states possess some properties in common that can be distinguished from those in other phases” \cite{239}. Naturally, a quantum phase of matter is one which is most efficiently described by quantum mechanics. More such phases have been predicted and theoretically studied than realized in the laboratory, and one of the goals of this field is to bridge this gap.

Another primary application of this field is \textit{quantum communication} — using the properties of quantum mechanics to communicate in such a way that any eavesdropper is readily noticed as soon as they try to intercept any communication. Such secure quantum communication channels not only have obvious applications in society, but also have lead to the development of the field of \textit{quantum information theory} — a merging of applied quantum physics and information theory which analyzes, among other things, how much information can securely go through quantum communication channels \cite{301}.

While the applications are clear, it is currently unclear which technology will \textit{actually} build the first practical (i.e., scalable!) quantum computer. Consequently, this lack of a precise focus on one technology has stimulated a broad and thrilling theoretical investigation into all areas of quantum mechanics which are even remotely useful in building quantum devices. The area that characterizes this thesis is \textit{open quantum systems} — the study of quantum systems which are in contact with a larger environment or reservoir.

\subsection{Open quantum systems}

Introductory physics courses devote much time to studying systems which are isolated from their environments. Examples include an object falling without air resistance or, more informally, a “spherical cow in vacuum”. Besides being necessary for a complete understanding of the systems in question, environmental effects can also steer the systems in favorable directions. For example, including air resistance in the calculation of a falling object leads to the understanding that a parachute can prevent said object from falling too quickly.

Environmental effects are even more pronounced in quantum systems. On the one hand, environment or \textit{quantum reservoir engineering} is poised to synthesize longer-lasting quantum memories, faster quantum computers, and exotic and previously inaccessible quantum phases of matter. On the other hand, commonplace environmental effects are known to destroy delicate quantum states, preventing us from observing them in our everyday lives. One aspect of the field of open quantum systems deals with understanding the limitations and possibilities of using the environment of a quantum system to further the aforementioned goals of applied quantum physics.
While ordinary or closed quantum system dynamics is generated by a Hamiltonian, open quantum system dynamics is generally not. We now derive the most general form of evolution of a system coupled to an environment (see, e.g., [67], Sec. 3.2.1). Assuming that the dynamics of the universe is generated by a Hamiltonian, the reduced dynamics of any open quantum system \( S \) coupled to an environment \( E \) can be derived starting from the joint Hamiltonian equation of motion (in units of \( \hbar = 1 \))

\[
\frac{d\rho_{SE}}{dt} = -i \{ H_{SE}, \rho_{SE} \},
\]

where \( \rho_{SE} \) is the quantum-mechanical density matrix of \( S \) and \( E \), \( H_{SE} \) is the Hamiltonian governing the dynamics of \( S \) and \( E \), and \( \{ A, B \} \equiv AB - BA \) is the commutator of \( A \) and \( B \). The reduced density matrix of the system at time \( t \) is then

\[
\rho(t) \equiv \text{Tr}_E \{ \rho_{SE}(t) \} = \text{Tr}_E \left\{ e^{-iH_{SE}t} \rho_{SE}(0) e^{iH_{SE}t} \right\},
\]

where \( \text{Tr}_E \{ A \} \equiv \sum_\ell \langle \ell | A | \ell \rangle \) is a tracing out of the environment and \( \{ | \ell \rangle \}_\ell \) is a basis of states for the environment. Closed-system evolution preserves the purity of quantum states, i.e., \( \rho(t) \) can be written as a rank-one projection\(^1\) onto some state \( |\psi(t)\rangle \) \( \{ \rho(t) = |\psi(t)\rangle \langle \psi(t)| \} \), assuming that \( \rho(0) = |\psi(0)\rangle \langle \psi(0)| \). In the open scenario, the system and the environment may become entangled under \( H_{SE} \) and the resulting initially pure reduced density matrix may become mixed (i.e., not pure). Therefore, throughout this thesis, we will always denote a quantum state by its density matrix, which happens to also be an operator on the Hilbert space.

For simplicity, let us now make the assumption that the initial state factorizes. In other words, \( \rho_{SE}(0) = \rho_{in} \otimes |0\rangle \langle 0| \), where \( \rho_{in} \) is an arbitrary initial state of the system and \( |0\rangle = |\ell = 0\rangle \).\(^2\)

Explicitly writing out \( \text{Tr}_E \), we can massage \( \rho(t) \) into the alternative form

\[
\rho(t) = \sum_\ell E^{\ell\dagger}_t(t) \rho_{in} E^\ell_t(t),
\]

where the Kraus operators \( E^\ell_t(t) \equiv \langle \ell | e^{-iH_{SE}t} | 0 \rangle \) operate only on the system Hilbert space. This Kraus map \([164]\), quantum channel, or completely positive trace-preserving (CPTP) map is the most general map from density matrices to density matrices that respects the laws of quantum mechanics, namely \([227\text{, Ch. }3]\),

1. it preserves the trace of \( \rho \): \( \text{Tr} \{ \rho(t) \} = 1 \) for all \( t \),
2. it preserves positivity: \( \rho(t) \geq 0 \) for all \( t \), and
3. it preserves positivity when acting on a part of a larger system.

The unitarity of \( e^{-iH_{SE}t} \) and completeness of \( \{ |\ell\rangle \}_\ell \) can be used to derive the constraint

\[
\sum_\ell E^{\ell\dagger}_t(t) E^\ell_t(t) = I,
\]

where \( I \) is the identity on the system. This constraint is equivalent to Property 1 above. Equation (1.3) can arguably be taken as the starting point of the entire field of open quantum systems.

\(^1\) We remind the reader that the rank of a diagonalizable matrix is the number of its not necessarily distinct nonzero eigenvalues.

\(^2\) The following derivation is easily extendable to an environment in an arbitrary initial state, \( \rho_E = \sum_\ell c_\ell |\ell \rangle \langle \ell| \), as long as the joint initial state is factorizable, \( \rho_{SE}(0) = \rho_{in} \otimes \rho_E \).
and there are many types of such quantum channels [307]. In the next Section, we simplify it to derive the Lindbladian — the most restrictive form of open-system evolution but also the simplest non-trivial extension of Hamiltonian-based quantum mechanics.

1.2 WHAT IS A LINDBLADIAN?

Let us study the time evolution of the system density matrix \( \rho (t) \) over an infinitesimal time increment \( dt \), following Ch. 3 of Ref. [227]. We assume that the time evolution over only this increment takes the same form as eq. (1.3), namely,

\[
\rho (t + dt) = \sum_{\ell} E^{\ell} (dt) \rho (t) E^{\ell \dagger} (dt) \equiv \mathcal{E}_{dt} [\rho (t)] .
\]  

This assumption implies that the behavior of \( \rho (t + dt) \) depends only on \( \rho (t) \) and not any previous times \( \tau < t \). Such a statement is commonly known as the Markov approximation, and it is necessary to obtain a Lindbladian from the more general form (1.3). We proceed to approximate \( \mathcal{E}_{dt} \) and construct a bona fide linear differential equation for \( \rho \), but first let us sketch what this will accomplish. Namely, we show that evolution due to any quantum channel \( \mathcal{E}_\ell \) is generated by a Lindbladian when \( t \) is “small”:

\[
\mathcal{E}_{dt} = \mathcal{I} + dt \mathcal{L} + \cdots \quad \text{and} \quad \mathcal{L} \equiv \lim_{dt \to 0} \frac{\mathcal{E}_{dt} - \mathcal{I}}{dt} ,
\]  

where \( \mathcal{I} \) is the identity channel, \( \mathcal{I} (\rho) = \rho \), and \( \mathcal{L} \) is a Lindbladian. To determine the precise form of \( \mathcal{L} \), we expand \( E^{\ell} (dt) \) and keep only the terms up to order \( O(dt) \). If evolution of \( \rho \) is governed by a Hamiltonian \( H \) only, then \( E^{0} = I - iHdt \) and all other \( E^{\ell>0} = 0 \), yielding the von Neumann equation \( \frac{d\rho}{dt} = -i[H, \rho] \) analogous to eq. (1.1). However, in this more general case, we include an order \( O(\sqrt{dt}) \) dependence of \( E^{\ell} \), which contributes another piece of order \( O(dt) \) since \( E^{\ell} \) acts on both sides of \( \rho (t) \) simultaneously. Without loss of generality, let us write

\[
E^{0} (dt) \sim I + (-iH + V) dt \quad \text{and} \quad E^{\ell>0} (dt) \sim \sqrt{\kappa_{\ell} dt} F^{\ell} ,
\]  

where \( \kappa_{\ell} \) are real nonzero rates and \( V \) is a to-be-determined Hermitian operator. To determine \( V \), we plug these leading-order forms into eqs. (1.4-1.5) and keep only terms up to \( O(dt) \). Since we want eq. (1.4) to be satisfied to this order, we require \( V = -\frac{1}{2} \sum_{\ell>0} F^{\ell \dagger} F^{\ell} \). Plugging this into eq. (1.5) and dividing both sides by \( dt \) yields the \textit{Lindbladian}

\[
\frac{d\rho}{dt} = \mathcal{L} (\rho) \equiv -i[H, \rho] + \frac{1}{2} \sum_{\ell>0} \kappa_{\ell} \left( 2F^{\ell} \rho F^{\ell \dagger} - F^{\ell \dagger} F^{\ell} \rho - \rho F^{\ell \dagger} F^{\ell} \right) ,
\]  

with Hamiltonian \( H \), \textit{jump operators} \( F^{\ell} \), and rates \( \kappa_{\ell} > 0 \). The \textit{recycling, sandwich,} or \textit{jump term} \( F^{\ell} \cdot F^{\ell \dagger} \) acts non-trivially on the state from both sides simultaneously and is the reason one cannot reduce the above equation to one involving only a ket-state \( |\psi\rangle \). The remaining terms \( F^{\ell \dagger} F^{\ell} \) can combined with \( H \) to form the non-Hermitian operator

\[
K \equiv H - \frac{i}{2} \sum_{\ell>0} \kappa_{\ell} F^{\ell \dagger} F^{\ell} .
\]
This operator generates the deterministic or no-jump part of the evolution and its contribution can be expressed as a modified von Neumann equation using the redefined commutator $[K, \rho]^\dagger \equiv K\rho - \rho K^\dagger$. Together, the recycling and deterministic terms conspire to make the evolution of $\rho$ preserve properties 1-3 listed in the previous Section, unlike systems whose time evolution is generated by a non-Hermitian operator alone. In particular, using the cyclic property of the trace, it is straightforward to show that $\text{Tr}\{\rho\}$ remains conserved throughout the evolution: $\text{Tr}\{d\rho/dt\} = 0$. In this way, the jump term compensates for the decay of probability caused by the deterministic term.

Solving eq. (1.8) yields the system density matrix for all times,

$$\rho(t) \equiv e^{t\mathcal{L}}(\rho_{\text{in}}),$$

where $e^{t\mathcal{L}}$ is a formal power series in $t\mathcal{L}$ and $\rho_{\text{in}}$ is the initial state of the system. The exponential can be done directly by re-expressing the $N \times N$ density matrix as an $N^2 \times 1$ vector (assuming the system Hilbert space is $N$-dimensional), which in turn allows one to re-express $\mathcal{L}$ as an $N^2 \times N^2$ matrix acting from the left on the vector version of $\rho$ (see Sec. 1.6 for further details). However, there is another method to solve the above equation for initially pure $\rho_{\text{in}} = |\psi\rangle\langle\psi|$ that allows one to work with $N \times N$ matrices instead of $N^2 \times N^2$ ones. This method is based on an average over multiple instances (or quantum trajectories) of a procedure applied to the initial ket-state $|\psi\rangle$ [224]. During a trajectory of a typical version of this unraveling procedure, the system is acted on by one instance of the jump term (and then renormalized) at a discrete set of certain randomly generated times $\{\tau_n\}$ and otherwise evolved deterministically under $K$ during the increments of time between neighboring $\tau_n$. More specifically, at time $t = 0$ and assuming one jump operator, a random number $r \in [0,1]$ is generated and the system evolves under $K$ until its norm decreases to $r$, i.e., until a time $\tau_1$ at which $|e^{\tau_1 K}|\psi\rangle|^2 = r$. Then, the evolved state is acted on by the jump term and renormalized, resulting in the state $|\psi_1\rangle = Fe^{\tau_1 K}|\psi\rangle / |F e^{\tau_1 K}|\psi\rangle|$. The same procedure is then repeated with $|\psi_1\rangle$ until the desired time of evolution $t$ is reached. It turns out that, in the limit of an infinite number of such trajectories, the average over the final states of the trajectories is exactly $\rho(t)$ from eq. (1.10). Unraveling is not only useful from a numerical standpoint, but it also has a physical analogy to the evolution of a continuously measured quantum state conditioned on the results of the measurement. For example, if the measured quantity is a photon detector and the sole jump operator represents photon loss, clicks of the detector are associated with applications of the jump term to the density matrix while periods of time for which the detector is quiet are associated with evolution under $K$.

1.3 A BRIEF HISTORICAL CONTEXT

Lindbladians (1.8) are also called Louivillians, Lindblad master equations, Lindblad-Kossakowski differential equations, or GKS-L equations. Those names mostly stem from two nearly simultaneous papers, one by Gorini, Kossakowski, and Sudarshan [125] that derives the equation for finite-dimensional systems and the other by Lindblad [177] for the infinite-dimensional case. Variations of eq. (1.8) were written down earlier [48, 169], although complete positivity of $\mathcal{L}$ was not proven (see [92] for a history). Note also a later chain of interest from the high-energy com-

---

3 Systems where the evolution is governed by $K$ only are also sometimes called “open quantum systems” [248] despite not preserving Properties 1-3 listed above.
Figure 1.1: Sketch of some of the ways in which Lindbladians ($\mathcal{L}$) fit into the field of quantum physics and of connections between them and other subfields. Details and references are provided in Sec. 1.3.

munity (e.g., [58]), started by Ref. [36]. Equation (1.8) is most general form for time-independent Hamiltonian and jump operators, but time-dependent extensions are also possible [91].

The heuristic derivation of $\mathcal{L}$ in the previous section has not covered all of the conditions on a system and reservoir for which Lindbladian evolution captures the dynamics of the system. The standard treatment [12] covers three common types of derivations of $\mathcal{L}$ starting from $H_{SE}$: the weak coupling limit, the low density approximation, and the singular coupling limit. Each of these relies on specific physical assumptions regarding, e.g., correlation functions of the environment. For example, in the weak coupling limit derivation, one typically assumes that (a) correlations of the system with the environment develop slowly, (b) excitations of the environment caused by system decay quickly, and (c) terms which are fast-oscillating when compared to the system timescale of interest can be neglected. These three approximations are called Born, Markov (as already discussed above), and rotating wave, respectively. The weak coupling limit is common in quantum optical scenarios (e.g., Ch. 1 of Ref. [80]), and there are comparisons to other models [157] and analyses studying its validity in said scenarios [65, 246] as well as in heat or electron transport [111, 245, 299] and many-body systems [124, 133].

Lindbladians lie at the nexus of a multitude of topics in applied quantum physics and beyond, some of which are listed in Fig. 1.1. We conclude this Section by commenting about each of them in approximately chronological order, noting that this is by no means a complete survey.

1. Studies of general Lindbladians arose in mathematical physics under the umbrella of quantum dynamical semigroups [137, 161] — continuous, one-parameter families of dynamical maps satisfying the (homogeneous) semigroup property $T_t T_s = T_{t+s}$ (clearly satisfied when $T_t = e^{t\mathcal{L}}$). Early work (e.g., [13, 126, 273]) often focused on Lindbladians with a unique
steady state, on systems stabilizing states in thermal equilibrium, and on systems satisfying detailed balance. Unbounded generators, a topic outside the scope of this thesis, have been rigorously studied in this field (e.g., [103, 267]). Overviews are provided in the books [12, 244, 276] and an important relatively recent work is Ref. [45].

2. Physicists’ interest in Lindbladians initiated in quantum optics, where the approximations needed to derive the Lindblad form are often well-justified. Expositions of this literature are provided in the books [80, 120], interesting methods to tackle such problems are given in Refs. [112, 155], and some specific older references are discussed in the beginning of Sec. 3.3. The term “reservoir engineering” was coined in this context [226].

3. At the turn of the millennium, researchers began to realize the potential for Lindbladians to stabilize subspaces for quantum information and error correction [316]. In other words, Lindbladians with multiple steady states are the focus of this field. Their steady-state subspaces can be decoherence-free subspaces [175], noiseless subsystems [156], and, most generally, information-preserving structures [60, 61]. These structures are covered thoroughly in this thesis and have been realized in various quantum technologies, including quantum optics [64, 166], liquid-state NMR [63, 118, 292], trapped ions [40, 41, 152, 235, 247, 256], and circuit QED [168, 286]. Such Lindbladians can also model an autonomous version of quantum error-correction [6, 39, 219], where syndromes are measured and corrections applied in continuous fashion using L’s jump operators. Holonomic quantum computation [317] with such information-preserving structures and its associated literature are discussed in the beginning of Ch. 5.

4. Some researchers began to take a look at Lindbladians from the mathematical perspective of control/systems theory with the goal of controllability [15, 16] and state/subspace stabilization [106, 107, 193, 258], with earlier discussions focused on more general quantum channels [182]. We discuss some of this literature in Sec. 3.4. Researchers also applied the well-developed theory of continuous monitoring and feedback to state stabilization, at first for the simplest case of a two-level system [296, 305] and later more generally (e.g., [280]); see the books [37, 304] for an introduction and further literature. The effect of a generic measurement on a system can also be described by a Lindbladian, collapsing the system to a convex combination of pointer states in a quantum-to-classical transition [324]; see Sec. 3.1 for such a Lindbladian.

5. Physicists started thinking about using many-body Lindbladians to stabilize non-equilibrium (i.e., not in the form of a Gibbs ensemble) steady states (NESS). Most of the initial theory [104, 119, 163] was motivated by cold atom physics, but another early work [84] focused on optical cavities. Besides state generation, researchers found they could engineer phase transitions [228, 232] and encode the outcome of a quantum computation into the steady state [288]. These early works initiated the burgeoning field of driven-dissipative open systems — an extension of the study of Hamiltonian-stabilized phases and phase transitions to Lindbladians; we discuss some of this literature in Sec. 3.4. The reader should consult Refs. [207, 208] for more thorough reviews of driven-dissipative systems.

6. A series of works [25, 26, 29] thoroughly investigated Lindbladian generalizations of geometric/quantized adiabatic response. We investigate these and related efforts in Chs. 4-5
and provide a review and generalization of the quantum geometric tensor (QGT), a related geometric quantity [56, 233], in Ch. 6. These works rely on a rigorous formulation of the adiabatic theorem for Lindbladians, which is reviewed in Ch. 5.

7. A simple “classical” open system with loss and gain is one where time evolution is generated by a “non-Hermitian Hamiltonian,” which is similar to the deterministic term \( K \) (1.9). However, the dependence of the eigenvalues and eigenvectors of \( \mathcal{L} \) on \( K \), and therefore the connection between Lindbladians and systems with loss and gain, is not completely understood. We show in Sec. 2.1.3 that, for certain types of \( \mathcal{L} \), a subspace of the system undergoes exactly the evolution generated by \( K \) and the recycling term merely takes one out of that subspace. There also exist methods to extend a given \( K \) into full Lindblad form [58, 261]. Such methods, and Lindbladians in general, may be useful as phenomenological models of resonance decay [122].

1.4 WHICH LINDBLADIANs ARE THE FOCUS OF THIS WORK?

Initial states \( \rho_{in} \) undergoing Lindbladian evolution (1.10) evolve into infinite-time or asymptotic states \( \rho_{\infty} \) for sufficiently long times [45],

\[
\rho_{in} \xrightarrow{t \to \infty} \rho_{\infty} \equiv \lim_{t \to \infty} e^{t\mathcal{L}} (\rho_{in}) = e^{-iH_{\infty}t} \mathcal{P}_{\infty} (\rho_{in}) e^{iH_{\infty}t}.
\]  

(1.11)

The non-unitary effect of Lindbladian time evolution is encapsulated in the asymptotic projection superoperator \( \mathcal{P}_{\infty} \) (with \( \mathcal{P}_{\infty}^2 = \mathcal{P}_{\infty} \)). The extra Hamiltonian \( H_{\infty} \) quantifies any residual unitary evolution, which persists for all time and, of course, does not cause any further decoherence. The various asymptotic states \( \rho_{\infty} \) are elements of an asymptotic subspace \( \text{As}(H) \) — a subspace of the space of operators \( \text{Op}(H) \) acting on the system Hilbert space \( H \),

\[
\text{As}(H) \equiv \mathcal{P}_{\infty} \text{Op}(H).
\]  

(1.12)

The asymptotic subspace attracts all initial states, is free from the non-unitary effects of \( \mathcal{L} \), and any remaining time evolution within \( \text{As}(H) \) is exclusively unitary. The asymptotic subspace can thus be thought of as a Hamiltonian-evolving subspace embedded in a larger Lindbladian-evolving space. If \( \text{As}(H) \) has no residual unitary evolution, then \( H_{\infty} \) is zero and all \( \rho_{\infty} \) are stationary or steady.

The results presented in this thesis, with the notable exception of Ch. 7, are novel for Lindbladians which (a) admit multiple steady states and (b) cause one (or more) state population to decay. We clarify these notions in this Section, introducing convenient notation along the way.

1.4.1 Multiple steady states

When \( \text{dim As}(H) = 1 \), only one asymptotic state exists and all \( \rho_{in} \) converge to it. This is what happens generically, i.e., if one were to pick a Lindbladian at random. Therefore, for the same reason as Hamiltonians with degenerate ground states, the set of Lindbladians with multiple steady states is “small” (i.e., of measure zero) compared to the set of all \( \mathcal{L} \). In general however, \( \text{As}(H) \) may be multi-dimensional and, in that case, the resulting asymptotic state will depend
on the initial condition $\rho_{in}$. Such Lindbladians are interesting to study (once again) for similar reasons as Hamiltonians with degenerate ground states, although one most likely needs a properly engineered environment to synthesize such Lindbladians. Subspace stabilization is also the default scenario from the systems theory perspective \cite{280, 281, 283}; this work however focuses on asymptotics and response instead of stabilizability of particular states or subspaces.

On one hand, $\text{As}(\mathcal{H})$ which can support quantum information are promising candidates for storing, preserving, and manipulating such information, particularly when their states can be engineered to possess favorable features (e.g., topological protection \cite{38, 105}). With many experimental efforts (see Sec. 1.3) aimed at engineering environments admitting nontrivial asymptotic subspaces, it is important to gain a comprehensive understanding of any differences between the properties of these subspaces and analogous subspaces of Hamiltonian systems (e.g., subspaces spanned by degenerate energy eigenstates).

On the other hand, response properties of $\text{As}(\mathcal{H})$ which do not necessarily support quantum information can help model experimental probes into driven-dissipative open systems. Due to, for example, symmetry \cite{8, 71} or topology \cite{105}, the asymptotic subspace can be degenerate yet not support a qubit. For example, an $\text{As}(\mathcal{H})$ spanned by two orthogonal pure state projections \( \Psi_0 = |\psi_0\rangle\langle\psi_0| \) and \( \Psi_1 = |\psi_1\rangle\langle\psi_1| \) (see Sec. 3.1) only consists of density matrices which are their convex superpositions, \( \rho_{\infty} = c\Psi_0 + (1 - c)\Psi_1 \) with \( 0 \leq c \leq 1 \), so no off-diagonal coherences between \( \Psi_0 \) and \( \Psi_1 \) are present. For these and similar cases, standard thermodynamical concepts \cite{13, 99, 141, 275} (see Ref. \cite{180} for a review) may not apply and steady states may no longer be thermal or even full-rank. The work here is directly tailored to such systems, i.e., those possessing one or more non-equilibrium steady states whose rank is less than \( \dim \mathcal{H} \). The rank constraint implies the presence of population decay, which is the remaining feature that we now address.

### 1.4.2 Presence of population decay

Unlike Hamiltonians, Lindbladians have the capacity to model decay. As a result, Lindbladians are often used to describe commonplace non-Hamiltonian processes (e.g., cooling to a ground state). We define the presence of population decay as the disappearance of at least one population component of all possible $\rho_{in}$ in the infinite-time limit. In other words, there exist one or more states $|\psi\rangle$ such that

\[
\langle \psi | e^{t\mathcal{L}} (|\psi\rangle\langle \psi|) | \psi \rangle \mathop\longrightarrow\limits^{t\to\infty} 0.
\]

Before proceeding, it is important to make a clear distinction between the decaying and non-decaying parts of the \( N \)-dimensional system Hilbert space $\mathcal{H}$. Let us group all non-decaying parts of $\text{Op}(\mathcal{H})$, the space of operators on $\mathcal{H}$, into the upper left corner [of the matrix representation of $\text{Op}(\mathcal{H})$] and denote them by the “upper-left” block \( \square \). Thereby, any completely decaying parts will be in the complementary \( \square \) block, and coherences between the two will be in the “off-diagonal” blocks \( \square \). We can discuss such a decomposition in the familiar language of NMR \cite{113}: the \( \square \) block consists of a degenerate ground state subspace \( \{|\psi_k\rangle\}_{k=0}^{d-1} \) immune to nonunitary effects, the \( \square \) block contains the set of populations decaying with rates commonly known as $1/T_1$, and the \( \square \) block is the set of coherences dephasing with rate $1/T_2$. While we have implicitly assumed only one pair of decay times $T_{1,2}$ for simplicity, in general every population in \( \square \) and coherence in \( \square \) has its own decay time.
For the NMR case above, \( \text{As}(H) = \mathbb{B} \) forms a \( d^2 \)-dimensional \textit{decoherence-free subspace} (DFS) \cite{175}. More generally, there can be further dephasing \textit{within} \( \mathbb{B} \) without population decay. While we postpone the discussion of the (many!) types of \( \text{As}(H) \) until Ch. 2, let us briefly mention one illustrative example. In the NMR case, \( \text{As}(H) \) is spanned by \( \{ |\psi_k\rangle\langle \psi_l| \}_{k,l=0}^{d-1} \). If we add a dephasing process which makes all coherences \( |\psi_k\rangle\langle \psi_{l\neq k}| \) between the degenerate ground states decay, \( \text{As}(H) \) will then reduce to the \( d \)-dimensional subspace spanned by only the state populations \( \{ |\psi_k\rangle\langle \psi_k| \}_{k=0}^{d-1} \). This is a case of \( \text{As}(H) \subseteq \mathbb{B} \), but \( \text{As}(H) \subseteq \mathbb{B} \) in general. An example of \( \text{As}(H) \) which includes dephasing is shown in the gray region in Fig. 1.2(a).

Let us now formally define the superoperator projections on the blocks. The subspaces \( \mathbb{B}, \mathbb{B}_0, \mathbb{B}_1, \mathbb{B}_2 \) and \( \mathbb{B}^0 \) are just four blocks (or corners) making up a matrix, so a nonspecialist reader may simply visualize them without focusing too much on their technical definitions in this paragraph. Let \( P \) be the orthogonal operator projection \( (P = P^2 = P^\dagger) \) on and only on the non-decaying subspace of \( H \) (also, the maximal invariant subspace). This projection is uniquely defined by the following conditions:

\[
\rho_\infty = P\rho_\infty P \quad \forall \quad \rho_\infty \in \text{As}(H),
\]

\[
\text{Tr}\{P\} = \max_{\rho_\infty}\{\text{rank}(\rho_\infty)\}. \tag{1.14}
\]

The first condition makes sure that \( P \) projects onto all non-decaying subspaces while the second guarantees that \( P \) does not project onto any decaying subspace. Naturally, the orthogonal projection onto the maximal decaying subspace of \( H \) is \( Q \equiv I - P \) (with \( PQ = QP = 0 \)). Therefore, population decay (1.13) occurs for any part of \( \rho_\infty \) that is in \( \mathbb{B}^0 \):

\[
Q\rho(t)Q \to 0 \quad \text{as} \quad t \to \infty. \tag{1.15}
\]
The projection \( P \) is also the projection on the range or support\(^4\) of \( \mathcal{P}_m(I) \). We define the four-corners projections acting on \( A \in \text{Op}(H) \) as follows:

\[
\begin{align*}
A_\square & \equiv \mathcal{P}_\square(A) \equiv PAP \\
A_\triangledown & \equiv \mathcal{P}_\triangledown(A) \equiv PAQ \\
A_\blacklozenge & \equiv \mathcal{P}_\blacklozenge(A) \equiv QAP \\
A_\blacklozenge & \equiv \mathcal{P}_\blacklozenge(A) \equiv QAQ.
\end{align*}
\]

(1.16)

By our convention, taking the conjugate transpose of the upper-right part places it in the lower-left subspace (projection acts before adjoint): \( A^\dagger_\square \equiv (A_\square)^\dagger = (A^\dagger_\square)_\square \). The superoperators \( \mathcal{P}_\square \) (with \( \square \in \{ \square, \triangledown, \blacklozenge, \blacklozenge \} \)) are projections \( (\mathcal{P}_\square = \mathcal{P}_\square^2) \) which partition the identity \( I \) on \( \text{Op}(H) \), \( \mathcal{P}_\square + \mathcal{P}_\triangledown + \mathcal{P}_\blacklozenge + \mathcal{P}_\blacklozenge = I \),

analogous to \( P + Q = I \). They conveniently add, e.g.,

\[
\mathcal{P}_\square = \mathcal{P}_\square + \mathcal{P}_\triangledown \quad \text{and} \quad \mathcal{P}_\blacklozenge = \mathcal{P}_\blacklozenge + \mathcal{P}_\blacklozenge.
\]

(1.18)

The subspace \( \square \equiv \mathcal{P}_\square \text{Op}(H) \) consists of all coherences between \( PH \) and \( QH \), and the “diagonal” subspace \( \blacklozenge \equiv \mathcal{P}_\blacklozenge \text{Op}(H) \) consists of all operators which do not contain any such coherences. While most of the cases we focus on have \( \square \neq 0 \), for completeness we mention two cases which do not contain decaying subspaces. The four-corners projections are different from those of the Nakajima-Zwanzig method \([203, 325]\) and are instead somewhat related to the Feshbach projection method \([116, 248]\) (see Sec. 2.1.3).

**Hamiltonian case:** If \( L = -i[H, \cdot] \) for some Hamiltonian, any state written in terms of the \( N \) eigenstate projections \( |E_k\rangle\langle E_k| \) of \( H (H|E_kangle = E_k|E_k\rangle) \) is a steady state. Therefore, there is no decaying subspace in Hamiltonian evolution \( (P = I) \).

**Unique state case (full-rank):** In the case of a one-dimensional \( \text{As}(H) \), \( P \) is the projection on the rank of the unique steady state \( \rho_\infty \equiv \varrho \). If the state’s spectral decomposition is \( \varrho = \sum_{k=0}^{d_\varrho-1} \lambda_k |\psi_k\rangle\langle \psi_k| \) (with \( d_\varrho \) being the number of nonzero eigenvalues of \( \varrho \)), then \( P = \sum_{k=0}^{d_\varrho-1} |\psi_k\rangle\langle \psi_k| \). If all \( N \) eigenvalues are nonzero, then \( \varrho \) is full-rank (e.g., in a Gibbs state) and there is no decaying subspace \( (P = I) \).

1.5 **Summary and reading guide**

Nontrivial decaying subspaces \( \square \) are ubiquitous in actively researched quantum information schemes (e.g., \([194, 218, 240]\)). For example, consider driven two-photon absorption — a bosonic Lindbladian with jump operator

\[
F = a^2 - \alpha^2,
\]

(1.19)

\(^4\) The **support** (kernel) of an operator \( A \) is the set of all vectors \( |x\rangle \) which are not mapped (are mapped) to zero under \( A \). The **range** of \( A \) is the set of all vectors \( |y\rangle \) which are mapped to under action of \( A \) on some other vector \( |x\rangle \): \( A|x\rangle = |y\rangle \). If \( A|y\rangle \notin \ker(A) \) if and only if \( |y\rangle \in \text{ran}(A) \), then the range and the support of \( A \) are equal (up to the zero vector). Whenever we use the word support, we will be describing operators for which this is true.
bosonic lowering operator $a ([a, a^\dagger] = I)$, and real non-negative parameter $\alpha$. The steady states of such a Lindbladian are the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, since both are annihilated by $F$. (In this Section, we always consider the $\alpha \gg 1$ limit, meaning that the overlap between the two states is negligible.) In this pseudo double-well system (recently realized experimentally [168, 286]), the asymptotic subspace $\text{As}(H) = \mathbb{H}$ is spanned by the outer products $|\alpha\rangle\langle a|$, $|\alpha\rangle\langle -\alpha|$, $|-\alpha\rangle\langle -\alpha|$, and $|-\alpha\rangle\langle a|$, and the projection $P$ (1.14) generating the four-corners decomposition (in the large $\alpha$ limit) is
\[
P = |\alpha\rangle\langle a| + |-\alpha\rangle\langle -\alpha| \ .
\] (1.20)
All states orthogonal to $|\pm\alpha\rangle$ constitute QH and the decaying subspace $\mathbb{F}$ is spanned by outer products of those states. Similarly, the coherences $\mathbb{F}$ are spanned by all states of the form $|\pm\alpha\rangle\langle \psi|$ and $|\psi\rangle\langle \pm\alpha|$, where $\langle \pm\alpha| \psi \rangle = 0$. For simplicity, we phrase the concepts tackled in this work in terms of questions about this particular example. While many of the questions we consider have already been answered for this type of $\text{As}(H)$, it is simpler to first state them in this context and detail the extensions to the various types of $\text{As}(H)$ later. We summarize the remaining chapters below, explicitly mention all collaborators directly related to this thesis, and end with a brief reading guide.

**Chapter Two** contains an application of the four-corners decomposition to Lindbladians and a derivation of when Lindbladians admit a decaying subspace $\mathbb{F}$ (Thm. 2). In process, we present various types of $L$, making contact with quantum channel simulation and non-Hermitian systems. It turns out that, while not all quantum channels can be expressed as $e^{itL}$ for any finite $t$ (since $L$ generates only Markovian channels), all channels can be embedded in some asymptotic projection $P_\infty = \lim_{t \to \infty} e^{itL}$ (assuming $H_\infty \neq 0$).

Building on previous results [8, 140], we then proceed to derive an analytical formula for the asymptotic projection $P_\infty$ from eq. (1.11) in terms of conserved quantities of $L$ (Thms. 3-4). For driven two-photon absorption and other examples where $\text{As}(H) = \mathbb{F}$, the formula states
\[
\rho_\infty = P_\infty (\rho_\text{in}) = P_\mathbb{F} (\rho_\text{in}) - P_\mathbb{F} L P_\mathbb{F} L^{-1} P_\mathbb{F} (\rho_\text{in}) .
\] (1.21)
One can see that there are two terms. The first ($P_\mathbb{F}$) term states the obvious: if one starts in $\text{As}(H)$, then nothing happens for all time. The second term shows how an initial state in $\mathbb{F}$ is transferred to an asymptotic state in $\mathbb{H}$. Since there are no other terms, it turns out that $\rho_\infty$ does not depend on any initial coherences in $\mathbb{F}$. We summarize the ramifications of a more general version of this formula in terms of the no-leak (LP1) and the clean-leak (LP2) properties. We then overview the various types of $\text{As}(H)$, summarize the results derived in Chs. 4-5 for each $\text{As}(H)$ type, and build notation for the various types that is used to derive said results.

We conclude the chapter in Sec. 2.6 with some statements about the relation between conserved quantities and symmetries of Lindbladians. Symmetries of a Hamiltonian $H$ are powerful tools since they can be used to block-diagonalize $H$,
\[
H = \bigoplus_k H_k ,
\] (1.22)
such that all states in a given block $H_k$ have the same eigenvalue of the symmetry operator. Each block can then be further diagonalized by finding its eigenstates $\{|E_k\rangle\}$, and each projection $|E_k\rangle\langle E_k|$ is a steady state $([H, |E_k\rangle\langle E_k|] = 0)$. So if each block is $d_k$ dimensional, then
$H_k$ admits at least $d_k$ steady states. Turning to Lindbladians, one can also use symmetries on the superoperator level to block-diagonalize $L$,

$$L = \bigoplus_{\kappa} O_\kappa,$$

where each superoperator $O_\kappa$ is not generally in Lindblad form. Therefore, $O_\kappa$ may not admit any steady states at all! We will learn in the correspondence from Thm. 3 that there are as many steady-state basis elements as there are conserved quantities. Thus, if $O_\kappa$ does not admit any steady states, it also does not admit any conserved quantities. This is a key difference between Hamiltonians and Lindbladians that breaks the usual duality between symmetries and conserved quantities known as Noether’s theorem. We explore this idea in Sec. 2.6.

This chapter is technical and heavily based on Albert and Jiang [8] and Albert, Bradlyn, Fraas, and Jiang [10], although some parts have been expanded and further clarified. The connection to non-Hermitian systems in Sec. 2.1.3 is new.

**Chapter Three** reviews examples of conserved quantities from few-qubit systems and undriven ($\alpha = 0$) two-photon absorption. We begin with the simplest possible example — a single two-level system admitting a two-dimensional $\text{As}(H)$ and expound on its relation to more complicated systems. We continue with a two-qubit example, studying (among other things) the effect of residual unitary evolution on $\rho_\infty$ [$H_\infty \neq 0$ in eq. (1.21)]. In turns out that in general $\mathcal{R}_\infty$ depends on $H_\infty$, leading to additional dephasing of $\rho_\infty$ caused by a “misalignment” of the driving inside $\text{As}(H)$ with the decay coming from outside. We conclude with a many-body example which allows stabilization of the ground-state subspace of any frustration-free Hamiltonian (Thm. 5), making contact with and reviewing earlier work on state stabilization.

This chapter is non-technical and is a collection of single-body examples from Refs. [8, 10]. The many-body example in Sec. 3.4 is synthesized from Refs. [281, 282] (see also [283]).

**Chapter Four** studies Lindbladian perturbation theory. We begin with the effect of Hamiltonian perturbations on $\rho_\infty$. One can show numerically [194] that applying a perturbation of the type

$$V = \epsilon (a + a^\dagger)$$

(1.24)

to the two-photon absorption Lindbladian generates, to linear order, motion within $\text{As}(H)$ due to the effective Hamiltonian $V_{\text{eff}} \equiv PVP$. Related results [314, 315] also show that Hamiltonian perturbations and perturbations to the jump operators of $L$ generate unitary evolution within some $\text{As}(H)$ to linear order. Do these results hold in general? In Sec. 4.1.1, we apply our formula for $\mathcal{P}$ to prove that such perturbations induce unitary evolution within all $\text{As}(H)$ to linear order. This result also holds for perturbations to the jump operators, $F \rightarrow F + f$, extending the capabilities of environment-assisted quantum computation and quantum Zeno dynamics ([22, 115, 255, 268]; see also [18, 47]).

Extending Ref. [212], we determine the energy scale governing leakage out of $\text{As}(H)$ due to Hamiltonian perturbations, jump operator perturbations, and adiabatic evolution (the latter is shown in the next chapter). Contrary to popular belief, this scale is not always the dissipative gap of $L$ — the nonzero eigenvalue with the smallest real part [see eq. (1.32)]. On the contrary, this leakage scale is the dissipative gap of $\mathcal{P} L \mathcal{P} \equiv L_{\text{eff}}$. The derivation
is given in Sec. 4.1.2 for Hamiltonian/jump operator perturbations and in Sec. 5.3 for nonadiabatic corrections.

More generally, Ch. 4 contains an application of the four-corners partition to the Kubo formula, splitting the formula into a part within As(H) which closely corresponds to the ordinary Hamiltonian-based formula and parts which cause leakage out of As(H) and contain non-unitary effects. Theorem 6 provides an all-order Dyson expansion for cases where \( \dim \text{As}(H) \geq 1 \), given a slowly ramping-up perturbation and assuming the initial state is already in As(H). An important distinction from most previous work is that we do not assume anything about \( L \) or its steady states (their number, detailed balance, thermodynamic equilibrium, etc.), making this analysis applicable to thermodynamic and quantum computational systems alike. It turns out that the number of terms to each order in this expansion is equal to a Catalan number. We finish the chapter by making contact with dark states [163], geometric response [25], and the effective operator formalism [241].

All first-order perturbation theory results are from Ref. [10]. The exact Dyson series for all higher orders and connections to previous work in Sec. 4.2 are new and will be studied further in a future publication [7].

**Chapter Five** studies the geometric “phase” acquired by \( \rho_\infty \) after cyclic adiabatic deformations of \( L \). Adiabatically changing the value of \( \alpha \) of a coherent state \( |\alpha\rangle \) in our two-photon absorption example over a closed path produces a Berry phase (more generally, a holonomy) proportional to the area enclosed by the path [87]. However, does this result still hold when the coherent state is part of an As(H) of an open system? Can Lindbladians induce any additional undesirable effects in the adiabatic limit for the various types of As(H)? We extend previous results [11, 25, 83, 212, 252] to show in Sec. 5.3 that cyclic Lindbladian-based [26] adiabatic evolution of states in As(H) is always unitary. This result extends the capabilities of holonomic quantum computation [214, 313, 317] via reservoir engineering. This chapter also contains an application of the four-corners partition of the leading nonadiabatic corrections to adiabatic evolution.

This chapter is a reshuffled version of the results from Ref. [10].

**Chapter Six** introduces a Lindbladian version of the quantum geometric tensor (Quantum geometric tensor — a quantity providing a metric on and encoding the geometric phase properties of a subspace (QGT)) [56, 233] which encodes both the curvature associated with the aforementioned adiabatic deformations and a metric associated with distances between adiabatically connected steady states. We also construct other geometric tensors and discuss why these are not always relevant to adiabatic deformations.

This chapter is a reshuffled version of the results from Ref. [10].

**Chapter Seven** applies what the results derived from the previous chapters to the driven two-photon absorption system. We discuss in detail the steady states for all parameters \( \alpha \) and derive the system’s conserved quantities. Using said conserved quantities, we find out \( \rho_\infty \) for various initial states. We then study the leading-order effect of the aforementioned Hamiltonian perturbation (1.24) and two types of noise — dephasing and loss. The perturbation theory formalism of Ch. 4 reveals that the system’s steady states are resilient to dephasing noise. We then apply the adiabatic results of Ch. 5 and review a way to induce holonomic quantum computation within As(H) by adiabatically varying the state parameters \( |\pm\alpha\rangle \).
1.6 A technical introduction

1.6.1 The playground and its features

This subsection contains a standard introduction into superoperators and double-ket notation [86, 113, 162, 199]. Lindbladians operate on the space of (linear) operators on $H$, or $\text{Op}(H) \equiv H \otimes H^*$ (also known as Liouville, von Neumann, or Hilbert-Schmidt space). This space is also a Hilbert space when endowed with the Hilbert-Schmidt inner product and Frobenius norm (for $N \equiv \dim H < \infty$). An operator $A$ in quantum mechanics is thus both in the space of operators acting on ordinary states and in the space of vectors acted on by superoperators. We denote the two respective cases as $A|\psi\rangle$ and $O|A\rangle$ (for $|\psi\rangle \in H$ and for a superoperator $O$). Strictly speaking, $|\rho\rangle$ is an $N^2$-by-1 vector and $\rho$ is an $N$-by-$N$ matrix, and superoperators acting on the vector version of $\rho$ are constructed according to the conversion rule

$$A\rho B \leftrightarrow (A \otimes B^T)|\rho\rangle,$$

where $B^T$ is the transpose of $B$. This extra transpose is necessary because the bra (row) part of the outer products making up $\rho$ is flipped when $\rho$ is written as a (column) vector, $|\psi\rangle\langle\phi| \rightarrow |\psi\rangle(\langle\phi|)^*$ (see Sec. 2.1.4.5 of [113] for details). The double-ket notation differentiates between superoperators acting on the matrix or vector versions of operators: $O|A\rangle$ and $|O(A)\rangle$ are written in vector form while $O(A)$ is a matrix.
Figure 1.3: A plot of a spectrum of an example \( \mathcal{L} \) with 21 eigenvalues \( \Lambda \) in the complex plane.

For \( A, B \in \text{Op}(H) \), the Hilbert-Schmidt inner product and Frobenius norm are respectively

\[
\langle\langle A|B \rangle\rangle \equiv \text{Tr}\{A^\dagger B\} \quad \text{and} \quad \|A\| \equiv \sqrt{\langle\langle A|A \rangle\rangle}.
\] (1.26)

The inner product allows one to define an adjoint operation \( \dagger \) which complements the adjoint operation \( \dagger \) on matrices in \( \text{Op}(H) \):

\[
\langle\langle A|\mathcal{O}(B) \rangle\rangle = \langle\langle A|\mathcal{O}|B \rangle\rangle = \langle\langle \mathcal{O}^\dagger(A)|B \rangle\rangle.
\] (1.27)

Writing \( \mathcal{O} \) as an \( N^2 \)-by-\( N^2 \) matrix, \( \mathcal{O}^\dagger \) is just the conjugate transpose of that matrix. For example, if \( \mathcal{O}(\cdot) = A \cdot B \), then one can use eq. (1.26) to verify that

\[
\mathcal{O}^\dagger(\cdot) = A^\dagger \cdot B^\dagger.
\] (1.28)

Similar to the Hamiltonian description of quantum mechanics, \( \mathcal{O} \) is Hermitian if \( \mathcal{O}^\dagger = \mathcal{O} \). For example, all projections \( \mathcal{P}_\Lambda \) from eq. (1.16) are Hermitian.

### 1.6.2 More on Lindbladians

The form of the Lindbladian (1.8) is not unique due to the following “gauge” transformation (for complex \( g_\ell \)),

\[
H \rightarrow H - \frac{i}{2} \sum_\ell \kappa_\ell (g_\ell^* F_\ell - g_\ell F_\ell^\dagger)
\]

\[
F_\ell \rightarrow F_\ell + g_\ell I,
\] (1.29)

that allows parts of the Hamiltonian to be included in the jump operators (and vice versa) while keeping \( \mathcal{L} (1.8) \) invariant. Note that there exists a unique “gauge” in which \( F_\ell \) are traceless ([125], Thm. 2.2). The Lindbladian is also invariant under unitary transformations on the jumps: for any unitary matrix \( u_{\ell\ell'} \),

\[
\sqrt{\kappa_\ell} F_\ell \rightarrow \sum_{\ell'} u_{\ell\ell'} \sqrt{\kappa_{\ell'}} F_{\ell'}.
\] (1.30)
It is easy to determine how an observable $A \in \text{Op}(H)$ evolves (in the Heisenberg picture) using the definition of the adjoint \((1.27)\) and cyclic permutations under the trace:

$$L^\dagger(A) = -\mathcal{H}(A) + \frac{1}{2} \sum_{\ell} \kappa_{\ell} \left( 2F^{\ell\dagger}AF^\ell - \left\{ F^{\ell\dagger}F^\ell, A \right\} \right). \quad (1.31)$$

The superoperator $\mathcal{H}(\cdot) \equiv -i[H, \cdot]$ corresponding to the Hamiltonian (more precisely, the adjoint representation of $H$) is therefore anti-Hermitian because we have absorbed the “$i$” in its definition.

The norm of a wavefunction corresponds to the trace of $\rho$ ($\langle\langle I|\rho\rangle\rangle = \langle\langle I|\rho\rangle\rangle$); we have already seen in Sec. 1.2 that it is preserved under both Hamiltonian and Lindbladian evolution. It is easy to check that the exponential of any superoperator of the above form preserves both trace \([\langle\langle I|L|\rho\rangle\rangle = 0 \text{ with } I \text{ the identity operator}]\) and Hermiticity \([L(A^\dagger) = [L(A)]^\dagger]\) as can be verified from eq. \((1.8)\). However, the norm/purity of $\rho$ ($\langle\langle |\rho\rangle\rangle = \text{Tr}\{\rho^2\}$) is not always preserved under Lindbladian evolution.

While $\mathcal{L}$ may not be diagonalizable, one can still obtain information about the dynamics by observing its eigenvalues (see Fig. 1.3). All eigenvalues $\Lambda$ lie on the non-positive plane and non-real eigenvalues exist in complex conjugate pairs (hence the symmetry under complex conjugation). The dots with zero real part in the Figure represent $\Lambda_1 \in \mathbb{R}$, whose eigenstates have pure imaginary eigenvalues ($\Lambda = i\Delta$ for real $\Delta$) and thus survive in the infinite-time limit. The value

$$\Lambda_{\text{dg}} \equiv \min_{\text{Re}\Lambda \neq 0} |\text{Re}\Lambda|$$

is the dissipative/dissipation/damping/relaxation gap (also, asymptotic decay rate \([151]\)) – the slowest non-zero rate of convergence toward $\text{As}(H)$.

One can eigendecompose $\mathcal{L}$ to obtain, in principle, the evolution for all time. Let us first assume that $\mathcal{L}$ is diagonalizable with eigenvalues $\Lambda$, an additional index $\mu$ which labels any degeneracies for each $\Lambda$, right eigenmatrices $R_{\Lambda\mu}$ ($\langle\langle L|R_{\Lambda\mu}\rangle\rangle = \Lambda|R_{\Lambda\mu}\rangle\rangle$), and left eingematrices $L^{\Lambda\mu}$ ($\langle\langle L|L^{\Lambda\mu}\rangle\rangle = \Lambda|L^{\Lambda\mu}\rangle\rangle$). Then, the evolution superoperator \((1.10)\) can be written as

$$e^{tL}\rho_{in} = \sum_{\Lambda,\mu} e^{\Lambda t}|R_{\Lambda\mu}\rangle\rangle \langle\langle L^{\Lambda\mu}|\rho_{in}\rangle\rangle.$$ \quad (1.33)

If $\mathcal{L}$ is not diagonalizable, there exists at least one Jordan block of $\mathcal{L}$ (in Jordan normal form) which has only one eigenmatrix, with the remaining matrix basis elements in the support of the block making up the block’s generalized eigenmatrices (e.g., \([236]\), Sec. 10.2). Exponentiating such an $\mathcal{L}$ brings about extra powers of $t$ in front of the exponent $e^{\Lambda t}$ above as well as off-diagonal elements of the form $|R_{\Lambda\mu}\rangle\rangle \langle\langle L^{\Lambda\nu\neq\mu}|$. For example, if $\Lambda$ has a two-dimensional Jordan block with right eigenmatrix $|R_{A0}\rangle\rangle$ ($\langle\langle L|R_{A0}\rangle\rangle = \Lambda|R_{A0}\rangle\rangle$) and generalized right eigenmatrix $|R_{A1}\rangle\rangle$ ($\langle\langle L|R_{A1}\rangle\rangle = |R_{A0}\rangle\rangle + \Lambda|R_{A1}\rangle\rangle$), then $e^{t\mathcal{L}}$ on that block is

$$e^{\Lambda t} \left( |R_{A0}\rangle\rangle \langle\langle L^{A0}| + t|R_{A1}\rangle\rangle \langle\langle L^{A0}| + |R_{A1}\rangle\rangle \langle\langle L^{A1}| \right) = e^{\Lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \quad (1.34)$$

Let us partition the Jordan normal form of $\mathcal{L}$ into blocks that are either diagonal or have an upper diagonal of ones. Since there could exist blocks associated with a particular $\Lambda$ which contain both
diagonal and off-diagonal sub-blocks, the sum over $\Lambda$ has to include each sub-block separately. Generalizing eq. (1.33), the full expansion of $e^{tL}$ is then

$$e^{tL}\rho_{in}\rangle = \sum_{\Lambda,\mu} e^{\Lambda t} |R_{\Lambda,\mu}\rangle \sum_{v \geq \mu} \frac{(\delta_{\Lambda 0})^{v-\mu}}{(v-\mu)!} \langle L^\Lambda v|\rho_{in}\rangle ,$$  

(1.35)

with $\mu, v \in \{0, 1, \cdots\}$ indexing either the degeneracy of the eigenspace of each $\Lambda$ if the Jordan block associated with $\Lambda$ is diagonal ($\delta_{\Lambda 0} = 0$; here $0^0 = 1$) or indexing the generalized eigenmatrices of $\Lambda$’s Jordan block if the block is not diagonal ($\delta_{\Lambda 1} = 1$).

Equations (1.34-1.35) immediately reveal that all Jordan blocks with pure imaginary eigenvalues $\Lambda = i\Delta$ are diagonal ([258], Sec. 5; [45], Thm. 18; [306], Prop. 6.2). By contradiction, if one assumes that $L$ is not diagonalizable in the subspace of the Jordan normal form with diagonals of zero real part, then exponentiating those Jordan blocks causes the dynamics to diverge as $t \to \infty$.

1.6.3 Double-bra/ket basis for steady states

We now bring in intuition from Hamiltonian-based quantum mechanics by building bases for $\text{Op}(H)$ from those for $H$. Given any orthonormal basis $\{|\phi_k\rangle\}_{k=0}^{N-1}$ for $H$, one can construct the corresponding orthonormal (under the trace) outer product basis for $\text{Op}(H)$,

$$\{|\Phi_{kl}\rangle\}_{k,l=0}^{N-1} = |\phi_k\rangle \langle \phi_l| ,$$

(1.36)

The analogy with quantum mechanics is that the matrices $\Phi_{kl} \leftrightarrow |\Phi_{kl}\rangle$ and $\Phi_{kl}^\dagger \leftrightarrow \langle\Phi_{kl}|$ are vectors in the vector space $\text{Op}(H)$ and superoperators $\mathcal{O}$ are linear operators on those vectors.
Furthermore, one can save an index and use properly normalized Hermitian matrices $\Gamma^\dagger_\kappa = \Gamma_\kappa$ to form an orthonormal basis $\{ |\Gamma_\kappa\rangle \}_{\kappa=0}^{N^2-1}$:

$$\langle \Gamma_\kappa | \Gamma_\lambda \rangle \equiv \text{Tr} \{ \Gamma^\dagger_\kappa \Gamma_\lambda \} = \text{Tr} \{ \Gamma_\kappa \Gamma_\lambda \} = \delta_{\kappa\lambda} . \quad (1.37)$$

Each $\Gamma_\kappa$ consists of Hermitian linear superpositions of the outer products $|\phi_k\rangle \langle \phi_l|$ and is not a density matrix. For example, an orthonormal Hermitian matrix basis for $\text{Op}(\mathcal{H})$ with $\mathcal{H}$ two-dimensional consists of the identity matrix and the three Pauli matrices, all normalized by $1/\sqrt{2}$.

An example for $N = 3$ is the set of properly normalized Gell-Mann matrices.

It is easy to see that the coefficients in the expansion of any Hermitian operator in such a matrix basis are real. For example, the coefficients $c_\kappa$ in the expansion of a density matrix, $|\rho\rangle = \sum_{\kappa=0}^{N^2-1} c_\kappa |\Gamma_\kappa\rangle$ with $c_\kappa = \langle \Gamma_\kappa | \rho \rangle$, are clearly real and represent the components of a generalized Bloch/coherence vector [12, 258].

Furthermore, defining

$$\mathcal{O}_{\kappa\lambda} \equiv \langle \langle \Gamma_\kappa | \mathcal{O} | \Gamma_\lambda \rangle \rangle \equiv \text{Tr} \{ \Gamma^\dagger_\kappa \mathcal{O} (\Gamma_\lambda) \} \quad (1.39)$$

for any superoperator $\mathcal{O}$, one can write

$$\mathcal{O} = \sum_{\kappa, \lambda=0}^{N^2-1} \mathcal{O}_{\kappa\lambda} |\Gamma_\kappa\rangle \langle \Gamma_\lambda| . \quad (1.40)$$

There are many physical $\mathcal{O}$ for which the “matrix” elements $\mathcal{O}_{\kappa\lambda}$ are real. For example, we define the superoperator equivalent of a Hamiltonian $H$ acting on a state $\rho$ as $\mathcal{H}(\rho) \equiv -i[H, \rho]$ [so that if $H$ generates time evolution, $\partial_t \rho = \mathcal{H}(\rho)$]. For this case, it is easy to show that matrix elements $\mathcal{H}_{\kappa\lambda}$ are real using cyclic permutations under the trace and Hermiticity of the $\Gamma$’s:

$$\mathcal{H}^\star_{\kappa\lambda} = \langle \langle \Gamma_\kappa | \mathcal{H} | \Gamma_\lambda \rangle \rangle^\star = i \text{Tr} \{ \Gamma^\dagger_\lambda [H, \Gamma_\kappa] \} = -i \text{Tr} \{ \Gamma_\kappa [H, \Gamma_\lambda] \} = \langle \langle \Gamma_\kappa | \mathcal{H} | \Gamma_\lambda \rangle \rangle = \mathcal{H}_{\kappa\lambda} . \quad (1.41)$$

This calculation easily extends to all Hermiticity-preserving $\mathcal{O}$, i.e., superoperators such that $\mathcal{O}(A^\dagger) = [\mathcal{O}(A)]^\dagger$ for all operators $A$.

Given a Lindbladian, one can provide necessary and sufficient conditions under which it generates Hamiltonian time evolution. This early key result in open quantum systems can be used to determine whether a perturbation generates unitary evolution.

**Theorem 1** (When Lindbladians generate unitary evolution [161]). The matrix $\mathcal{L}_{\kappa\lambda} = \langle \langle \Gamma_\kappa | \mathcal{L} | \Gamma_\lambda \rangle \rangle$ is real. Moreover,

$$\mathcal{L}_{\lambda\kappa} = -\mathcal{L}_{\kappa\lambda} \iff \mathcal{L} = -i [H, \cdot ] \text{ with Hamiltonian } H. \quad (1.42)$$

**Proof.** To prove reality, use the definition of the adjoint of $\mathcal{L}$, Hermiticity of $\Gamma_\kappa$, and cyclicity under the trace:

$$\mathcal{L}^\star_{\kappa\lambda} = \langle \langle \Gamma_\lambda | \mathcal{L}^\dagger | \Gamma_\kappa \rangle \rangle = \langle \langle \mathcal{L} (\Gamma_\lambda) | \Gamma_\kappa \rangle \rangle = \langle \langle \Gamma_\kappa | \mathcal{L} | \Gamma_\lambda \rangle \rangle = \mathcal{L}_{\kappa\lambda} . \quad (1.43)$$

**Proof.** To prove reality, use the definition of the adjoint of $\mathcal{L}$, Hermiticity of $\Gamma_\kappa$, and cyclicity under the trace:
Assume $L$ generates unitary evolution. Then there exists a Hamiltonian $H$ such that $L|\Gamma_\kappa\rangle\rangle = -i[[H, \Gamma_\kappa]]\langle\langle$ and $L$ is antisymmetric:

$$L_{\lambda\kappa} = -i\text{Tr}\{\Gamma_\lambda[H, \Gamma_\kappa]\} = i\text{Tr}\{\Gamma_\kappa[H, \Gamma_\lambda]\} = -L_{\kappa\lambda}. \quad (1.44)$$

Assume $L_{\kappa\lambda}$ is antisymmetric, so $L^\dagger = -L$. Then the dynamical semigroup $\{e^{tL}; t \geq 0\}$ is isometric (norm-preserving): let $t \geq 0$ and $|A\rangle\rangle \in \text{Op}(H)$ and observe that

$$\langle\langle e^{tL}(A)|e^{tL}(A)\rangle\rangle = \langle\langle A|e^{-iL}e^{iL}|A\rangle\rangle = \langle\langle A|A\rangle\rangle. \quad (1.45)$$

Since it is clearly invertible, $e^{tL} : \text{Op}(H) \to \text{Op}(H)$ is a surjective map. All surjective isometric one-parameter dynamical semigroups can be expressed as $e^{tL}(\rho) = U_t \rho U_t^\dagger$ with $U_t$ belonging to a one-parameter unitary group $\{U_t; t \in \mathbb{R}\}$ acting on $H$ ([161], Thm. 6). Therefore, there exists a Hamiltonian $H$ such that $U_t = e^{-iHt}$ and $L(\rho) = -i[H, \rho]$. \qed

\footnote{An alternative way to prove this part is to observe that all eigenvalues of $L$ lie on the imaginary axis and use Thm. 18-3 in [45].}
“[...] the orientation in the literature on semigroups is towards the proof of rigorous mathematical results and hence the connections to quantum optics applications are somewhat indirect.”

– Howard J. Carmichael

THE ASYMPTOTIC PROJECTION AND CONSERVED QUANTITIES

2.1 FOUR-CORNERS PARTITION OF LINDBLADIANS, WITH EXAMPLES

From the previous chapter, we learned that the four-corners projections (1.16) partition every operator $A \in \text{Op}(H)$ into four independent parts. Combining this notation with the vectorized or double-ket notation for matrices in $\text{Op}(H)$ (see Sec. 1.6), we can express any $A$ as a vector whose components are the respective parts. The following are therefore equivalent,

$$A = \begin{pmatrix} A_{\square} & A_{\blacksquare} \\ A_{\blacksquare} & A_{\blacklozenge} \end{pmatrix} \iff |A\rangle = \begin{bmatrix} |A_{\square}\rangle \\ |A_{\blacksquare}\rangle \\ |A_{\blacklozenge}\rangle \end{bmatrix},$$

and $A_{\blacksquare} = A_{\square} + A_{\blacklozenge}$. With $A$ written as a block vector, superoperators can now be represented as 3-by-3 block matrices acting on said vector. Note that we use square-brackets for partitioning superoperators and parentheses for operators in $\text{Op}(H)$ [as in Fig. 1.2 and eq. (2.1)]. We will do so with the Lindbladian $\mathcal{L}$ (1.8). Recall that

$$\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum_{\ell} \kappa_{\ell} \left( 2F^{\ell} \rho F^{\ell\dagger} - F^{\ell\dagger} F^{\ell} \rho - \rho F^{\ell\dagger} F^{\ell} \right)$$

with Hamiltonian $H$, jump operators $F^{\ell} \in \text{Op}(H)$, and positive rates $\kappa_{\ell}$. By writing $\mathcal{L} = \mathcal{I} \mathcal{L} \mathcal{I}$ using eqs. (1.17) and (1.18), we find that

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{\square} & \mathcal{P}_{\square} \mathcal{L} \mathcal{P}_{\square} & \mathcal{P}_{\blacksquare} \mathcal{L} \mathcal{P}_{\blacksquare} \\ 0 & \mathcal{L}_{\blacksquare} & \mathcal{P}_{\blacklozenge} \mathcal{L} \mathcal{P}_{\blacklozenge} \\ 0 & 0 & \mathcal{L}_{\blacklozenge} \end{bmatrix},$$

where $\mathcal{L}_{\square} \equiv \mathcal{P}_{\square} \mathcal{L} \mathcal{P}_{\square}$. Note that $\mathcal{L}_{\square}$ is a bona fide Lindbladian governing evolution within $\square$. The reason for the zeros in the first column is the inability of $\mathcal{L}$ to take anything out of $\square$ (stemming
from the definition of the four-corners projections). This turns out to be sufficient for \( \mathcal{P}_S \mathcal{L} \mathcal{P}_S \) to also be zero, leading to the block upper-triangular form above. These constraints on \( \mathcal{L} \) translate to well-known constraints on the Hamiltonian and jump operators as follows.

**Theorem 2** (When Lindbladians generate decay [10, 45, 262, 280]). Let \( \{P, Q\} \) be projections on \( \mathcal{H} \) and \( \{\mathcal{P}_S, \mathcal{P}_N, \mathcal{P}_N', \mathcal{P}_S'\} \) be their corresponding projections on \( \mathcal{O}(\mathcal{H}) \). Then

\[
F_\ell^S = 0 \text{ for all } \ell \quad (2.4)
\]

\[
H_S = -\frac{i}{2} \sum_\ell \kappa_\ell F_\ell^S F_\ell^S \quad (2.5)
\]

**Proof.** By definition (1.14), \( \mathcal{S} \) is the smallest subspace of \( \mathcal{O}(\mathcal{H}) \) containing all asymptotic states. Therefore, all states evolving under \( \mathcal{L} \) converge to states in \( \mathcal{S} \) as \( t \to \infty \) ([45], Thm. 2-1). This implies invariance, i.e., states \( \rho_\mathcal{S} = \mathcal{P}_S(\rho) \) remain there under application of \( \mathcal{L} \):

\[
\mathcal{L}(\rho_\mathcal{S}) = \mathcal{L}\mathcal{P}_S(\rho) = \mathcal{P}_S \mathcal{L}\mathcal{P}_S(\rho) \quad (2.6)
\]

Applying \( \mathcal{P}_S \), we get \( \mathcal{P}_S \mathcal{L}\mathcal{P}_S(\rho) = \sum_\ell \kappa_\ell F_\ell^S F_\ell^S = 0 \) since the projections are mutually orthogonal. Taking the trace, \( \langle \langle I \rangle \rangle \mathcal{P}_S \mathcal{L}\mathcal{P}_S(\rho) = \sum_\ell \kappa_\ell \text{Tr}\{\rho F_\ell^S F_\ell^S\} = 0 \). If \( \rho \) is a full rank density matrix (\( \text{rank}\{\rho\} = \text{Tr}\{P\} \)), then each summand above is non-negative (since \( \kappa_\ell > 0 \) and \( F_\ell^S F_\ell^S \) are positive semidefinite). Thus the only way for the above to hold for all \( \rho \) is for \( F_\ell^S F_\ell^S = 0 \) for all \( \ell \), which implies that \( F_\ell^S = 0 \). Applying \( \mathcal{P}_S \) to eq. (2.6) and simplifying using \( F_\ell^S = 0 \) gives

\[
\mathcal{P}_S \mathcal{L}\mathcal{P}_S(\rho) = P\rho \left( iH_S - \frac{1}{2} \sum_\ell \kappa_\ell F_\ell^S F_\ell^S \right) = 0 \quad (2.7)
\]

implying the condition on \( H_S \).

The constraints on \( H_S \) and \( F_\ell^S \) (due to Hermiticity, \( H_S = H_S^\dagger \)) leave only their complements as degrees of freedom. The four-corners decomposition provides simple expressions for the surviving matrix elements of \( \mathcal{L} \) (2.3) in terms of \( H_S, F_\ell^S \); these are shown below. From eq. (2.5), one can see that \( H_S \) in general depends on the nonzero jump operator rates \( \kappa_\ell \), demonstrating an intricate cancellation of Lindbladian effects via a Hamiltonian term. However, \( P \) is independent of \( \kappa_\ell \) in many physically relevant (\( \kappa \)-robust [280]) Lindbladians, and in those cases \( H_S = 0 \) and either \( F_\ell^S = 0 \) or \( F_\ell^S = 0 \) for each \( \ell \).

We now list all of the matrix elements of \( \mathcal{L} \) (2.3) and mention important special cases of DFS type, making contact with previous works and applications. Based on conditions (1.14) and after
Four-Corners Partition of Lindbladians, with Examples

Simplifications due to Thm. 2, the non-zero elements of eq. (2.3) acting on a Hermitian matrix $\rho = \rho_\square + \rho_\perp + \rho_\downarrow$ are

$$L_\square(\rho) = -i [H_\square, \rho_\square] + \frac{1}{2} \sum_\ell \kappa_\ell \left( 2F_\ell \rho_\square F_\ell^\dagger - F_\ell^\dagger F_\ell \rho_\square - \rho_\square (F_\ell^\dagger F_\ell) \right)$$  \hspace{1cm} (2.8a)

$$L_\perp(\rho) = -i (H_\perp \rho_\perp - \rho_\perp H_\perp) + \frac{1}{2} \sum_\ell \kappa_\ell \left[ 2F_\ell \rho_\perp F_\ell^\dagger - F_\ell^\dagger F_\ell \rho_\perp - \rho_\perp (F_\ell^\dagger F_\ell) \right]$$  \hspace{1cm} (2.8b)

$$L_\downarrow(\rho) = [L_\square(\rho)]^\dagger \hspace{1cm} (2.8c)$$

$$L_\square(\rho) = -i [H_\square, \rho_\square] + \frac{1}{2} \sum_\ell \kappa_\ell \left[ 2F_\ell^\dagger \rho_\square F_\ell - (F_\ell^\dagger F_\ell) \rho_\square - \rho_\square (F_\ell^\dagger F_\ell) \right]$$  \hspace{1cm} (2.8d)

$$P_\square \mathcal{L} P_\square(\rho) = \sum_\ell \kappa_\ell \left( F_\ell^\dagger \rho_\square F_\ell - \rho_\square (F_\ell^\dagger F_\ell) \right) + H.c. \hspace{1cm} (2.8e)$$

$$P_\perp \mathcal{L} P_\perp(\rho) = \sum_\ell \kappa_\ell \left( F_\ell^\dagger \rho_\perp F_\ell - (F_\ell^\dagger F_\ell) \rho_\perp \right) + H.c. \hspace{1cm} (2.8f)$$

$$P_\downarrow \mathcal{L} P_\downarrow(\rho) = \sum_\ell \kappa_\ell F_\ell^\dagger \rho_\downarrow F_\ell^\dagger. \hspace{1cm} (2.8g)$$

Note that $(F_\ell^\dagger F_\ell) = F_\ell^\dagger F_\ell + F_\ell^\dagger F_\ell$, that $P_\square \mathcal{L} P_\square = P_\perp \mathcal{L} P_\perp = 0$, i.e., evolution of coherences decouples under this decomposition, and that the transfer term $P_\square \mathcal{L} P_\square$ is in the form of a quantum channel.

### 2.1.1 DFS case

Recall from Sec. 1.4.2 that now $\text{As}(H) = \square$ and $P = \sum_{k=0}^{d-1} |\psi_k\rangle\langle\psi_k|$ is the projection on the DFS states $\{|\psi_k\rangle\}_{k=0}^{d-1}$. In the case of a non-steady DFS, any residual evolution within $\square$ is exclusively unitary and generated by the Hamiltonian superoperator $\mathcal{H}_\infty \equiv L_\square$. The jump operators in $L_\square$ (2.8a) must then act trivially, and the most general condition for them (for some complex constants $a_\ell$) is

$$F_\ell = a_\ell P. \hspace{1cm} (2.9)$$

This implies that $P_\square \mathcal{L} P_\square$ (2.8e) is zero and the partition (2.3) becomes

$$\mathcal{L} = \begin{bmatrix} \mathcal{H}_\infty & 0 & P_\perp \mathcal{L} P_\perp \\ 0 & \mathcal{L}_\perp & P_\downarrow \mathcal{L} P_\downarrow \\ 0 & 0 & \mathcal{L}_\downarrow \end{bmatrix}. \hspace{1cm} (2.10)$$

This extra zero will prove important when we examine the Kubo formula for this case in Ch. 4.
2.1.2 Semisimple DFS case

A simplified (semisimple [175]) DFS case can be obtained by setting \( F_\ell = 0 \), i.e., \( a_\ell = 0 \). Then, all DFS states are annihilated by the jumps,

\[
F_\ell |\psi_k\rangle = 0.
\]

If \(|\psi_k\rangle\) are also eigenstates of \( H \), they are called dark states [163]. We can also set \( H = 0 \), meaning that the DFS is stationary (\( H_\infty = 0 \)). Stationarity leads to only one new zero matrix element of \( \mathcal{L} \),

\[
\mathcal{L} = \begin{bmatrix}
0 & 0 & P_{\square} & \mathcal{L}_{\square} \\
0 & \mathcal{L}_{\square} & P_{\square} & \mathcal{L}_{\square} \\
0 & 0 & \mathcal{L}_{\square} & \mathcal{L}_{\square}
\end{bmatrix}.
\]

(2.12)

However, the \( \square \) part now decays deterministically:

\[
\mathcal{L}_{\square}(\rho) = -\frac{1}{2} \sum_\ell \kappa_\ell \{\rho_{\square}, (F_\ell^\dagger F_\ell)_{\square}\} \equiv -\{\rho_{\square}, H_{\text{edg}}\},
\]

(2.13)

where the ground states of the decoherence [148] or parent [135] Hamiltonian \( H_{\text{edg}} \) are exactly the DFS states. We see that the excitation gap \( \Delta_{\text{edg}} \) (where edg stands for effective dissipative gap) of this Hamiltonian is relevant in Lindbladian first-order perturbation theory in Sec. 4.3.1. All examples considered in Chs. 7-8 are of this type.

2.1.3 Non-Hermitian Hamiltonian systems

Now let us keep \( H \neq 0 \) and further simplify the jumps to

\[
F = F_{\square}.
\]

(2.14)

Having \( F_{\square} = 0 \) leads to significant simplification of the matrix elements of \( \mathcal{L} \) for the DFS case (2.10). Now, \( P_{\square} \mathcal{L} P_{\square} = 0 \) and evolution within \( \square \) has no jump terms and is governed solely by the deterministic part (1.9) of \( \mathcal{L} \),

\[
K = K_{\square} \equiv H_{\square} - \frac{1}{2} \sum_\ell \kappa_\ell F_\ell^\dagger F_\ell \quad \text{with superoperator} \quad K(\rho) \equiv -i(K\rho - \rho K^\dagger).
\]

(2.15)

Since \( K_{\square} = H_{\square} \), evolution within \( \square \) is unitary and governed by the Hamiltonian piece \( H_{\square} \). The remaining \( \square \) parts evolve under \( K_{\square} \) which includes the positive definite \( (\sum_\ell \kappa_\ell F_\ell^\dagger F_\ell > 0 \text{ on } \square) \) contribution due to the jumps and thereby guarantees decay of everything initially in \( \square \). Due to trace preservation, the transfer term (2.8g) \( P_{\square} \mathcal{L} P_{\square} \neq 0 \) makes sure that populations in \( \square \) are
transferred into $\mathcal{H}$ in the limit of infinite time, making up the only non-deterministic part of the evolution. The full decomposition is then

$$
\mathcal{L} = \begin{bmatrix}
K_{\mathcal{H}} & 0 & \mathcal{P}_{\mathcal{H}} L \mathcal{P}_{\mathcal{H}} \\
0 & K_{\mathcal{H}} & 0 \\
0 & 0 & K_{\mathcal{H}}
\end{bmatrix}.
$$

(2.16)

Thus, in the $\mathcal{H}$ sector, the evolution is generated by the non-Hermitian Hamiltonian $K$. In a related work [78], a non-Hermitian Hamiltonian governs evolution in $\mathcal{H}$ but not in $\mathcal{F}$.

It is useful here to remark on the relation between the four-corners projections and the Feshbach projection method [116, 248]. The former splits the Hilbert space into decaying and non-decaying parts while the latter splits it into subspaces spanned by bound and scattering states. Connections between the two become clearer when one looks at the effect of perturbations which excite states from $\mathcal{H}$ into $\mathcal{F}$ on $\mathcal{L}$ of the type (2.16). We calculate leading-order corrections to evolution within $\mathcal{H}$ due to these effects in Sec. 4.3.6, showing that they are identical to those derived by the effective operator formalism [241] — an extension of the Feshbach method to Lindbladians.

### 2.1.4 Quantum channel simulation

In this most simplified example, $\operatorname{As}(H)$ is a steady DFS ($H_\infty = H = 0$) of dimension $d$ and there is a decaying subspace $\mathcal{F}$ of dimension $N - d$. We assume that $F = F_{\mathcal{H}}$, that all rates $\kappa_\ell$ of the Lindbladian are equal to one rate $\kappa_{\text{eff}}$, and that

$$
\sum_\ell \mathcal{P}_{\mathcal{H}}^{\dagger} F_\ell \mathcal{P}_{\mathcal{H}} = \sum_\ell \mathcal{P}_{\mathcal{F}}^{\dagger} F_\ell \mathcal{P}_{\mathcal{F}} = Q.
$$

(2.17)

The last condition guarantees that the transfer term (2.8g), $\mathcal{P}_{\mathcal{H}} L \mathcal{P}_{\mathcal{F}} \equiv \kappa_{\text{eff}} \mathcal{E}$, is proportional to a bona fide quantum channel $\mathcal{E}$ from a $N - d$-dimensional input space $\mathcal{F}$ to a $d$-dimensional output space $\mathcal{H}$. The condition (2.17) also makes sure that $\mathcal{F}$ decays uniformly:

$$
\mathcal{L} = \kappa_{\text{eff}} \begin{bmatrix}
0 & 0 & \mathcal{E} \\
0 & -\frac{1}{2} \mathcal{P}_{\mathcal{F}} & 0 \\
0 & 0 & -\mathcal{P}_{\mathcal{H}}
\end{bmatrix}.
$$

(2.18)

The rate $\kappa_{\text{eff}}$ is the inverse of the uniform relaxation time $T_1$ for $\mathcal{F}$. Notice how the uniform coherence relaxation time $T_2 = \frac{1}{2} T_1$.

The example is constructed such that the piece of the asymptotic projection $\mathcal{P}_{\mathcal{F}} \mathcal{P}_{\mathcal{H}}$ (1.11) taking states from $\mathcal{F}$ into $\mathcal{H}$ is exactly the channel $\mathcal{E}$. While we derive the equation for $\mathcal{P}$ below, we apply that result (Thm. 4) here since the simplifications are trivial:

$$
\mathcal{P}_{\mathcal{F}} \mathcal{P}_{\mathcal{H}} = - (\mathcal{P}_{\mathcal{H}} L \mathcal{P}_{\mathcal{F}}) \mathcal{L}^{-1} = \frac{1}{\kappa_{\text{eff}}} \mathcal{P}_{\mathcal{H}} L \mathcal{P}_{\mathcal{F}} = \mathcal{E}.
$$

(2.19)

In other words, all quantum channels can be embedded in some $\mathcal{P}_{\mathcal{F}} = \lim_{t \to \infty} e^{t\mathcal{L}}$. Note that this scheme is different from constructing a Lindbladian $\mathcal{L} \equiv \mathcal{E} - \mathcal{I}$ out of a channel $\mathcal{E}$, shown in eq. (3.42). Here, the channel $\mathcal{E}$ does not act on all of $\operatorname{Op}(H)$ because the channel’s input and
output spaces are different subspaces of \( \text{Op}(H) \). It is impossible to simulate a more general channel that maps all of \( \text{Op}(H) \) to itself using only a Lindbladian [307].

The immediate application of this construction is quantum error correction [277], where \( \mathcal{P} \) is the subspace into which a protected state (initially in \( \mathcal{P} \)) is taken after application of an error channel. The channel \( \mathcal{P} = \mathcal{P}_\mathcal{P} \) then recovers the information, and the above construction shows that all such recovery channels can be implemented to arbitrary accuracy by turning on Lindbladian evolution for a sufficiently long period of time.

### 2.2 The Asymptotic Projection

Armed with the partition of \( \mathcal{L} \) from eq. (2.3), here we study cases where \( \text{As}(H) \) contains unitarily evolving states \( [H_\infty \neq 0 \text{ from eq. (1.11)]} \) and formally introduce the asymptotic projection \( \mathcal{P}_\infty \). The basis for \( \text{As}(H) \) consists of right eigenmatrices of \( \mathcal{L} \) with pure imaginary eigenvalues. Recalling eq. (1.35), we can expand \( |\rho_\infty\rangle \rangle \) in such a basis since exponentials \( (e^{it\Lambda}) \) of any eigenvalues \( \Lambda \) with negative real parts decay to zero for large \( t \). We call the non-decaying eigenmatrices right asymptotic eigenmatrices \( |\Psi_\Delta\rangle \rangle \) or \( \text{As}(H) \) basis elements with purely imaginary eigenvalue \( \Lambda = i\Delta \) (used here as an index) and degeneracy index \( \mu \) (that depends on \( \Delta \in \mathbb{R} \)). By definition, \( |\Psi_\Delta\rangle \rangle \in \mathfrak{P} \) and the eigenvalue equation is

\[
\mathcal{L}|\Psi_\Delta\rangle \rangle = i\Delta|\Psi_\Delta\rangle \rangle .
\]

(2.20)

Since \( \mathcal{L} \) is not always diagonalizable, any degeneracy may induce a non-trivial Jordan block structure for a given \( \Delta \) (see Sec. 1.6.2). However, we show below that all Jordan blocks corresponding to asymptotic eigenmatrices are diagonal. Therefore, there exists a dual set of conserved quantities or left asymptotic eigenmatrices \( \langle\langle J^{\Delta\mu} \rangle\rangle \) such that

\[
\langle\langle J^{\Delta\mu} \rangle\rangle \mathcal{L} = i\Delta\langle\langle J^{\Delta\mu} \rangle\rangle \iff \mathcal{L}^\dagger (J^{\Delta\mu}) = -i\Delta(J^{\Delta\mu}) .
\]

(2.21)

We state and prove this duality in the following correspondence.

**Theorem 3** (Conserved quantity – steady state correspondence [8]). Let \( \{|\Psi_\Delta\rangle \rangle\}_{\Delta,\mu} \) be an orthonormal basis for \( \text{As}(H) \subseteq \text{Op}(H) \) of dimension \( D \), i.e., \( \langle\langle \Psi_\Delta\rangle \rangle |\Psi_\Theta\rangle \rangle = \delta_{\Delta\Theta} \delta_{\mu\nu} \). Then, for all \( \rho_\in \in \text{Op}(H) \), the asymptotic state \( \rho_\in \) is expressible as

\[
|\rho_\in(t)\rangle \rangle = \sum_{\Delta,\mu} c_{\Delta\mu} e^{i\Delta t} |\Psi_\Delta\rangle \rangle
\]

(2.22)

and there exist \( D \) conserved quantities \( \{J^{\Delta\mu}\}_{\Delta,\mu} \) such that

\[
c_{\Delta\mu} \equiv \langle\langle J^{\Delta\mu} |\rho_\in\rangle \rangle = \text{Tr}\{ (J^{\Delta\mu})^\dagger \rho_\in \}
\]

(2.23)

\[
\langle\langle J^{\Delta\mu} |\Psi_\Theta\rangle \rangle = \delta_{\Delta\Theta} \delta_{\mu\nu} .
\]

(2.24)

**Proof.** The matrix form of \( \mathcal{L} \) can be put into Jordan normal form \( \mathcal{C} \) via a non-unitary similarity transformation \( \mathcal{Z} \),

\[
\mathcal{L} = \mathcal{Z}^{-1} \mathcal{C} \mathcal{Z} .
\]

(2.25)

We have already seen from eq. (1.34) that all Jordan blocks with pure imaginary eigenvalues \( i\Delta \) must be diagonal in order for the rules of quantum mechanics to not be violated as \( t \to \infty \).
Therefore, the respective transformed left and right eigenmatrices, $|\Psi_{\Delta \mu}\rangle = Z|\Psi_{\Delta \mu}\rangle$ and $\langle J^\Delta \mu| = \langle J^\Delta \mu| Z^{-1}$, are linearly independent and orthogonal to all other basis matrices of $\text{Op}(\mathcal{H})$. Thus they can be made to be biorthogonal [186], $\langle J^\Delta \mu|\Psi_{\theta \nu}\rangle = \delta_{\theta \mu} \delta_{\mu \nu}$, by a procedure similar to the Gram-Schmidt process. It is clear that once the transformed vectors are biorthogonal, the original ones are also, satisfying eq. (2.24). Taking the expectation value of $J^\Delta \mu$ with respect to $\rho_\infty$ yields
\[
\langle J^\Delta \mu| \rho(t) \rangle = \langle J^\Delta \mu| e^{t\mathcal{L}} |\rho_\infty\rangle = e^{i\Delta t} \langle J^\Delta \mu| \rho_\infty\rangle \equiv c_{\Delta \mu} e^{i\Delta t}.
\] (2.26)
Since $J^\Delta \mu$ are dual to $\Psi_{\Delta \mu}$ in the sense described above, eq. (2.22) holds.

The coefficients $c_{\Delta \mu}$ determine the footprint that $\rho_\infty$ leaves on $\rho_\infty$, implying that $\rho_\infty$ does not depend on dynamics at any intermediate times. In general, any part of $|\rho_\infty\rangle$ not in the kernel of $\mathcal{P}_\infty$ imprints on the asymptotic state since, by definition, that part overlaps with some $J^\Delta \mu$. Equation (2.26) tells us that the $J^\Delta \mu$ are either conserved in time (when $\Delta = 0$) or oscillating indefinitely. While the term non-decaying quantity is thus more accurate, we use the term conserved quantity to describe all $J^\Delta \mu$ since they are are bona fide conserved quantities in the rotating frame of $\mathcal{H}_\infty$ (as we now show). Since it was shown that evolution of asymptotic states is exclusively unitary ([45], Thm. 2), it must be that the eigenvalue set $\{\Delta\}$ is that of a Hamiltonian superoperator, which we define to be $\mathcal{H}_\infty \equiv -i[\mathcal{H}_\infty, \cdot]$. In other words, we use the set $\{\Delta\}$ to construct a Hamiltonian $\mathcal{H}_\infty \in \mathfrak{g}$ (defined up to a constant energy shift) such that each $\Delta$ is a difference of the energies of $\mathcal{H}_\infty$ and $|\Psi_{\Delta \mu}\rangle$ are eigenmatrices of $\mathcal{H}_\infty$.1 Because of this, the eigenmatrices $\{\Psi, J\}$ must come in complex conjugate pairs: $\Psi_{-\Delta \mu} = \Psi_{\Delta \mu}^\dagger$ (which obstructs us from constructing a Hermitian basis for $\{\Psi_{\Delta \neq 0, \mu}\}$) and same for $J^\Delta \mu$. The explicit form of $\mathcal{H}_\infty$ depends on the structure of $\text{As}(\mathcal{H})$.

Combining $\mathcal{P}_\infty$ with the definition of $\mathcal{H}_\infty$ allows us to rearrange eq. (2.22) into
\[
|\rho_\infty(t)\rangle \equiv \lim_{t \to \infty} e^{t\mathcal{L}} |\rho_\infty\rangle = e^{i\mathcal{H}_\infty} \sum_{\Delta \mu} |\Psi_{\Delta \mu}\rangle \langle J^\Delta \mu| \rho_\infty\rangle = e^{i\mathcal{H}_\infty} \mathcal{P}_\infty |\rho_\infty\rangle,
\] (2.27)
where the outer products are used to express the asymptotic projection
\[
\mathcal{P}_\infty \equiv \sum_{\Delta \mu} |\Psi_{\Delta \mu}\rangle \langle J^\Delta \mu|.
\] (2.28)
This is indeed a projection ($\mathcal{P}_\infty^2 = \mathcal{P}_\infty$) due to eq. (2.24). This projection is onto the peripheral spectrum of $e^{t\mathcal{L}}$ — all eigenvalues of $e^{t\mathcal{L}}$ (for $t > 0$) whose modulus is one. As a result, one can apply standard formulas to express it in two other ways,
\[
\mathcal{P}_\infty = \lim_{T \to \infty} \frac{1}{T} \sum_{\Delta} \int_0^T dt e^{t(\mathcal{L} - i\Delta)} = -\frac{1}{2\pi i} \int_{\Gamma} dz (\mathcal{L} - z)^{-1}.
\] (2.29)
The first (ergodic) way is to take a proper limit such that the projections on all eigenspaces associated with $i\Delta$ remain [[306], eq. (6.15)]. The second way defines $\mathcal{P}_\infty$ in terms of Riesz projections on the relevant eigenspaces: $\Gamma$ is the contour in the complex plane which encircles all $i\Delta$ and no other points in the spectrum of $\mathcal{L}$ [[150], Ch. 1, eq. (5.23)]. While both ways exist and are

1 The eigenvalues $i\Delta$ can be extracted from $\mathcal{H}_\infty$ as shown in eq. (2.40). Note that $\mathcal{H}_\infty$ shares the same eigenvalues as $\mathcal{P}_\infty \mathcal{L} \mathcal{P}_\infty$, but $\mathcal{H}_\infty \neq \mathcal{P}_\infty \mathcal{L} \mathcal{P}_\infty$ because the latter is not anti-Hermitian.
equal to each other since we assume a finite-dimensional $\text{Op}(H)$, they are also valid for some infinite-dimensional examples (see Chs. 7-8).

### 2.3 Analytical Formula for Conserved Quantities

We proceed to determine $J^\Delta$ by plugging in the four-corners partition of $\mathcal{L}$ (2.3) into the eigenvalue equation (2.21). The block upper-triangular structure of $\mathcal{L}$ readily implies that $|J^\Delta|$ are left eigenmatrices of $\mathcal{L}$:

$$\langle \langle J^\Delta | \mathcal{L} = i\Delta \langle \langle J^\Delta | .$$  \hspace{1cm} (2.30)

Writing out the conditions on the remaining components $|J^\Delta|$ yields an analytic expression for $|J^\Delta|$. We state this formula below, noting that $[\mathcal{L}, \mathcal{P}] = 0$. Plugging that result into eq. (2.28) and setting $\Delta = 0$ yields the formula for $\mathcal{P}$ (1.21) for the case when $H_\infty = 0$. We finish this Chapter by going through the ramifications of the formula for $\mathcal{P}$ for the various types of $\text{As}(H)$, introducing notation used throughout the rest of the thesis, and comparing our conserved quantities to those of Hamiltonian-based systems. In Ch. 3, we apply the decomposition of $\mathcal{P}$ into conserved quantities $J^\Delta$ to study examples of footprints left on $\rho_\infty$ by $\rho_{\text{in}}$.

**Theorem 4** (Analytical expression for conserved quantities [10]). The left eigenmatrices of $\mathcal{L}$ corresponding to pure imaginary eigenvalues $i\Delta$ are

$$\langle \langle J^\Delta | = \langle \langle J^\Delta | \left( \begin{array}{ccc} \mathcal{P} & \mathcal{L} & \mathcal{P} \\ \mathcal{L}^\dagger & -i\Delta \mathcal{P} & \mathcal{L}^\dagger \\ \mathcal{P} & \mathcal{L}^\dagger & \mathcal{P} \end{array} \right) \right) \longleftrightarrow J^\Delta = J^\Delta - \mathcal{P} \mathcal{L}^\dagger \mathcal{P} + \mathcal{L}^\dagger (i^\Delta),$$  \hspace{1cm} (2.31)

where $\langle \langle J^\Delta |$ are left eigenmatrices of $\mathcal{L}$.

**Proof.** For a left eigenmatrix $\langle \langle J^\Delta |$ with eigenvalue $i\Delta$, $\mathcal{L}^\dagger |J^\Delta| = -i\Delta |J^\Delta|$. Now partition this eigenvalue equation using the projections $\{\mathcal{P}, \mathcal{P}, \mathcal{P}\}$. Taking the adjoint of the partitioned $\mathcal{L}$ from eq. (2.3) results in

$$\mathcal{L}^\dagger |J^\Delta| = \left[ \begin{array}{ccc} \mathcal{L}^\dagger & 0 & 0 \\ \mathcal{P} & \mathcal{L}^\dagger \mathcal{P} & \mathcal{L}^\dagger \\ \mathcal{P} & \mathcal{L}^\dagger \mathcal{P} & \mathcal{L}^\dagger \mathcal{P} & \mathcal{L}^\dagger \mathcal{P} \end{array} \right] \left[ \begin{array}{c} |J^\Delta| \\ |J^\Delta| \\ |J^\Delta| \end{array} \right].$$  \hspace{1cm} (2.32)

The eigenvalue equation is then equivalent to the following three conditions:

$$-i\Delta |J^\Delta| = \mathcal{L}^\dagger (i^\Delta)$$  \hspace{1cm} (2.33a)

$$-i\Delta |J^\Delta| = \mathcal{P} \mathcal{L}^\dagger \mathcal{P} + \mathcal{L}^\dagger (i^\Delta)$$  \hspace{1cm} (2.33b)

$$-i\Delta |J^\Delta| = \mathcal{P} \mathcal{L}^\dagger \mathcal{P} + \mathcal{L}^\dagger (i^\Delta) + \mathcal{P} \mathcal{L}^\dagger \mathcal{P} + \mathcal{L}^\dagger (i^\Delta).$$  \hspace{1cm} (2.33c)

We now examine them in order.

---

2 There exist related formulas for the parts of $\mathcal{P} \mathcal{P}$ corresponding to fixed points of discrete-time quantum channels in Lemma 5.8 of Ref. [61] and Prop. 7 of Ref. [93] and of Markov chains in Thm. 3.3 of Ref. [209].
1. Condition (2.33a) implies that \([F^\ell_\mu, J^\Delta_\mu] = 0\) for all \(\ell, \Delta, \mu\). To show this, we use the *dissipation function* \(\mathcal{J}\) associated with \(L_\mu\) [177]. For some \(A \in \mathbb{K}\),

\[
\mathcal{J}(A) = L_0(A^\dagger A) - L_0(A)A - A^\dagger L_0(A) = \sum_\ell \kappa_\ell [F^\ell_\mu, A]^\dagger [F^\ell_\mu, A].
\] (2.34)

Using (2.33a) and remembering that \(J^\Delta_\mu = J^{-\Delta_\mu}\), the two expressions for \(\mathcal{J}(J^\Delta_\mu)\) imply that

\[
L_0(J^\Delta_\mu J^\Delta_\mu^\dagger) = \sum_\ell \kappa_\ell [F^\ell_\mu, J^\Delta_\mu]^\dagger [F^\ell_\mu, J^\Delta_\mu].
\] (2.35)

We now take the trace using the full rank steady-state density matrix

\[
|\rho_\infty\rangle = \mathcal{P}_\mu |\rho_\infty\rangle \equiv \sum_\mu c_\mu |\Psi_0_\mu\rangle.
\] (2.36)

Such an asymptotic state is simply that from eq. (2.22) with \(c_\Delta = \delta_\Delta_0 c_\mu\) and \(c_\mu \neq 0\). It is full rank because it is a linear superposition of projections on eigenstates of \(H_\infty\) and such projections provide a basis for all diagonal matrices of \(\mathbb{K}\). Taking the trace of the left-hand side of eq. (2.35) yields

\[
\langle \rho_\infty | L_0(J^\Delta_\mu J^\Delta_\mu^\dagger) | \rho_\infty \rangle = \langle L_0(\rho_\infty) | J^\Delta_\mu J^\Delta_\mu \rangle = 0,
\] (2.37)

implying that the trace of the right-hand side is zero:

\[
\sum_\ell \kappa_\ell \text{Tr}\{\rho_\infty[F^\ell_\mu, J^\Delta_\mu]^\dagger [F^\ell_\mu, J^\Delta_\mu]\} = 0.
\] (2.38)

Each summand above is non-negative (since \(\kappa_\ell > 0\), the commutator products are positive semidefinite, and \(\rho_\infty\) is positive definite). Thus the only way for the above to hold is for \([F^\ell_\mu, J^\Delta_\mu]\) to commute for all \(\ell, \Delta, \mu\). If we once again remember that \(J^\Delta_\mu = J^{-\Delta_\mu}\) and that the eigenvalues come in pairs \(\pm \Delta\), then

\[
[F^\ell_\mu, J^\Delta_\mu] = [F^\ell_\mu, J^{-\Delta_\mu}] = 0.
\] (2.39)

With the jump operators out of the picture, condition (2.33a) now becomes

\[
- i[H_\infty, J^\Delta_\mu] = i\Delta^\Delta_\mu,
\] (2.40)

confirming that \(\Delta\) are differences of eigenvalues of a Hamiltonian.

2. Now consider condition (2.33b). The first term on the right-hand side can be obtained from eq. (2.8e) and is as follows:

\[
\mathcal{P}_\mu L_\mu \mathcal{P}_\mu (J^\Delta_\mu) = \sum_\ell \kappa_\ell (F^{\ell+}_{\mu} F^{\ell+}_{\mu} - J^{\Delta_\mu}_{\mu} F^{\ell+}_{\mu} F^{\ell_\mu}_{\mu} + \sum_\ell \kappa_\ell (F^{\ell+}_{\mu} F^{\ell+}_{\mu} F^{\ell_\mu}_{\mu} - F^{\ell+}_{\mu} F^{\ell+}_{\mu} F^{\ell_\mu}_{\mu}).
\] (2.41)

---

3 This part is essentially the Lindblad version of a similar statement for quantum channels ([61], Lemma 5.2). Another way to prove this is to apply “well-known” algebra decomposition theorems (e.g., [156], Thm. 5).
2.4 NO-LEAK AND CLEAN-LEAK PROPERTIES

Armed with the analytical formula for $J^{\Delta \mu}$ from Thm. 4, we can now construct the most general $P$. We state our result for the $H_\infty = 0$ case and outline the non-trivial consequences of $H_\infty \neq 0$ in Sec. 3.2.3. We can split $P$ as follows:

$$P = P_\Psi P + P_\Psi P = P_\Psi (I - LL^{-1}),$$

(2.44)

where $P_\Psi$ is the minimal projection that further projects $\mathfrak{H}$ onto As(H). The form of $P_\Psi$, the asymptotic projection of $L_\mathfrak{H}$ (which does not admit a decaying subspace), depends on the details of As(H) and is already known [45, 61]. Our work therefore extends previous Lindbladian results to cases when a decaying subspace is present. Of course, any pair $\{L_\mathfrak{H}, P_\Psi\}$ can be extended (via Thm. 2) to the pair $\{L, P\}$ that admits an arbitrarily large decaying subspace $\mathfrak{H}$. The first $(P_\Psi P_\Psi)$ terms states that if one starts in $\mathfrak{H}$, then one is simple projected into As(H) $\subseteq \mathfrak{H}$. The second term $(P_\Psi P_\Psi)$ shows that an initial state in $\mathfrak{H}$ is transferred to an asymptotic state in $\mathfrak{H}$ via application of the projected inverse $L_\mathfrak{H}^{-1}$. Since the projection $P_\Psi$ is not present in the above formula, we can immediately read off that no coherences between the non-decaying subspace and its counterpart are preserved (see Sec. 7.3). Thus, the above formula allows us to determine which parts of $\rho_{in}$ are preserved in the large-time limit. The DFS case (when $P_\Psi = P_\mathfrak{H}$) was addressed in Sec. 1.5.

Since superoperator perturbation theory requires spectral projections such as $P_{\Psi}$, the above formula is also useful in determining how states that are already in As(H) respond to perturbations. We now switch gears and sketch the effect of small perturbations $\mathcal{O}$ on a state $\rho_{in}$ already in As(H) in order to lay the groundwork for Chs. 4-6. The perturbations of interest are either Hamiltonian perturbations $V \equiv -i[V, \cdot]$ (with Hamiltonian $V$ and small parameter $\epsilon$) or derivatives $\partial_{\alpha} \equiv \partial / \partial x_{\alpha}$.
(with parameters $x_a$ and adiabatic evolution time $T$) of the now parameter-dependent $\rho_\infty(x_a)$ and $L(x_a)$:

$$O \in \{ \epsilon V, \frac{1}{T} \partial_a \}.$$  \hfill (2.45)

In Chs. 4-5, we show that both of these can be used to induce unitary operations on As(H). The latter determine adiabatic connection(s) and thus help with defining parallel transport (i.e., adiabatic evolution) of As(H). We show in Ch. 4 that this analysis holds for jump operator perturbations $F^\ell \to F^\ell + f^\ell$ as well, but omit discussing those perturbations for now to keep things simple. Within first order for the case of perturbation theory ($\epsilon \to 0$) and approaching the adiabatic limit for the case of parallel transport ($T \to \infty$), two relevant perturbative processes after the action of $O$ on an asymptotic state are (A) subsequent projection onto As(H) and (B) leakage out of As(H) via the perturbation and $L^{-1}$:

$$\rho_\infty \to P_\square O(\rho_\infty) - L^{-1}O(\rho_\infty).$$  \hfill (2.46)

We study these terms here and show later that they occur both in the Kubo formula and in adiabatic response.

We first observe that $O$ is limited in its effect on $\rho_\infty$. Acting with $O$ once does not connect $\square$ with $\square$ because $O$ does not act non-trivially on $\rho_\infty$ from both sides simultaneously. This no-leak property can be understood if one observes that Hamiltonian superoperator perturbations $V$ act nontrivially on $\rho_\infty$ only from one side at a time due to their commutator form. Likewise, derivatives $\partial_a$ act nontrivially on either the “ket” or “bra” parts of all basis elements used to write $\rho_\infty$ due to the product rule. Therefore, acting with $O$ once only connects $\square$ to itself and nearest-neighbor squares $\square$ and does not cause “transitions” into $\square$.

$$O(\rho_\infty) = P_\square O(\rho_\infty),$$  \hfill (LP1)

where $P_\square \equiv I - P_\square$. Moreover, despite two actions of $O$ connecting $\square$ to $\square$ (LP1) still provides some insight into second-order effects (see Sec. 4.3.5).

The no-leak property is important in determining the energy scale governing leakage out of As(H). Let us apply this property to the second term in eq. (2.46):

$$L^{-1}O(\rho_\infty) = L^{-1}P_\square O(\rho_\infty) = L^{-1}O(\rho_\infty),$$  \hfill (2.47)

where $L^{-1}_\square \equiv (P_\square LP_\square)^{-1}$ with $\square$ being any block(s). Note that the last step in eq. (2.47) also uses $P_\square LP_\square = 0$ from eq. (2.3). Since the restriction to studying $L$ on $\square$ in linear response has previously gone unnoticed, it is conventionally believed that the leakage energy scale is determined by the dissipative gap $\Delta_{dg}$ (1.32) — the nonzero eigenvalue of $L$ with smallest real part. As shown in eq. (2.47), that energy scale is actually governed by the effective dissipative gap $\Delta_{edg} \geq \Delta_{dg}$ — the nonzero eigenvalue of $L_\square$ with smallest real part. In Hamiltonian systems ($L = -i[H, \cdot]$), a special case of the no-leak property states that the energy denominator in the first-order perturbative correction to the $k^{\text{th}}$ eigenstate of $H$ contains only energy differences involving the energy $E_k$ of that eigenstate (and not, e.g., $E_{k-1} - E_{k+1}$).
We now project $O(\rho_\infty)$ back to $\text{As}(H)$ to examine the first term in eq. (2.46). Applying $P_\infty$ to eq. (LP1) and using $P_\infty P_\Psi = 0$ from eq. (2.44) removes two more squares:

$$P_\infty O(\rho_\infty) = (P_\infty P_\Psi)(P_\Psi O)(\rho_\infty) = P_\infty P_\Psi O(\rho_\infty) = P_\Psi O P_\Psi(\rho_\infty). \quad (\text{LP2})$$

The clean-leak property shows that any leakage of the perturbed $\rho_\infty$ into $\Psi$ does not contribute to the first-order effect of $O$ within $\text{As}(H)$. Essentially, the clean-leak property makes $\text{As}(H)$ resistant to the non-unitary effects of Lindbladian evolution and allows for a closer analogue between $\text{As}(H)$ and subspaces of unitary systems. The clean-leak property simplifies calculations of both Hamiltonian perturbations and adiabatic/Berry connections. It can be used to show that $P_\Psi$ (instead of $P_\infty$) fully governs adiabatic evolution, so a natural Lindbladian generalization of the quantum geometric tensor (QGT) is

$$Q_{\alpha\beta} \equiv P_\Psi \partial_\alpha P_\Psi \partial_\beta P_\Psi P_\Psi. \quad (2.48)$$

In Ch. 6, we show that the part of the QGT anti-symmetric in $\alpha, \beta$ corresponds to the adiabatic curvature $\mathcal{F}_{\alpha\beta}$ (determined from the Berry connections) and, for most relevant $\text{As}(H)$, derive a metric $\mathcal{M}_{\alpha\beta}$ on the parameter space from the part of the QGT symmetric in $\alpha, \beta$. 

---

**Figure 2.1**: Sketches of various types of $\text{As}(H)$ (gray) embedded in a 25-dimensional space of matrices $\text{Op}(H)$ (corresponding to a five-dimensional Hilbert space $H$). The number of (non-overlapping) gray squares counts the dimension of $\text{As}(H)$ for each type. The first two admit no decaying subspace and correspond to (a) unitary evolution and (b) evolution due to a dephasing Lindbladian (see Sec. 3.1), where all coherences are absent in the $\infty$-time limit. The last three admit a decaying subspace and correspond to Lindbladian evolution admitting (c) a unique pure steady state, (d) a unique mixed steady state of rank two, (e) a qubit decoherence-free subspace, and (f) a qubit noiseless subsystem with a two-dimensional auxiliary subspace.
2.5 Summary of Applications to Various \( \text{As(H)} \)

2.5.1 Hamiltonian case

As a sanity check, let us review how the structures we are studying (trivially) simplify when \( L = -i[H, \cdot] \) [Fig. 2.1(a)]. Then, there is no decaying subspace \( \mathcal{P} = I \) and there is no extra dephasing either \( \text{As(H)} = \text{Op(H)} \). Literally everything is preserved in the long-time limit, so \( \mathcal{P}_\infty = I \).

2.5.2 Unique state case

The most common case occurs when \( L \) admits only one steady state [Fig. 2.1(c,d)], i.e., there exists a unique state \( \varrho \in \text{As(H)} \) such that

\[
\mathcal{L}(\varrho) = 0.
\]

(2.49)

Recall that, according to the duality from Thm. 3, there is a unique conserved quantity whose expectation value is preserved in the infinite-time limit. Due to trace preservation of \( e^{tL} \), that quantity is just the identity \( I, \mathcal{L}^\dagger(I) = 0 \), and the only quantity preserved is the trace of the initial state. In the double-ket notation, the asymptotic projection can be written as \( \mathcal{P}_\infty = |\varrho\rangle\langle\varrho| \). In the case when \( \varrho \) is not full-rank, \( \varrho \in \mathfrak{M} \) and there is a decaying subspace \( \mathfrak{P} \). Then, the projection \( P \) on the support of \( \varrho \) is the conserved quantity of \( \mathcal{L}_{\mathfrak{P}} \) and the minimal projection is \( \mathcal{P}_\Psi = |\varrho\rangle\langle\Psi| \).

This case is relatively trivial when examining its perturbation theory: since \( \text{As(H)} \) is one-dimensional, there is nowhere to move within \( \text{As(H)} \). Indeed, it is easy to show that for any trace-preserving perturbation \( \mathcal{O} \),

\[
\mathcal{P}_\infty \mathcal{O} = |\varrho\rangle\langle I| \mathcal{O} = |\varrho\rangle\langle I| \mathcal{O}^\dagger(I) = 0.
\]

(2.50)

The unique case does nevertheless admit a nontrivial QGT and corresponding metric,

\[
\mathcal{M}_{\alpha\beta} = \text{Tr}\left\{ \partial_{(\alpha} P \partial_{\beta)} \varrho \right\},
\]

(2.51)

where \( A_{(\alpha}B_{\beta)} = A_{\alpha}B_{\beta} + A_{\beta}B_{\alpha} \). This metric is distinct from the Hilbert-Schmidt metric \( \text{Tr}\{\partial_{(\alpha} \varrho \partial_{\beta)} \varrho \} \) for mixed \( \varrho \) and is nonzero only when \( \varrho \) is not full-rank. For pure steady states, both metrics reduce to the Fubini-Study metric \([233]\).

2.5.3 DFS case

Recall that the simplest multi-dimensional \( \text{As(H)} \) which stores quantum information is a decoherence-free subspace or DFS [Fig. 2.1(e)]. A \( d^2 \)-dimensional DFS block

\[
\text{As(H)} = \mathfrak{M}
\]

(2.52)
is spanned by matrices \{ |\psi_k\rangle\langle\psi_l| \}_{k,l=0}^{d-1}, where \{ |\psi_k\rangle \}_{k=0}^{d-1} is a basis for a subspace of the \( d \leq N \)-dimensional system space. The decaying block \( \mathcal{L} \) is then spanned by \{ |\psi_k\rangle\langle\psi_l| \}_{k,l=0}^{N-1}. Evolution of the DFS under \( L \) is exclusively unitary,

\[
\partial_t (|\psi_k\rangle\langle\psi_l|) = \mathcal{L}(|\psi_k\rangle\langle\psi_l|) = -i [H_\infty, |\psi_k\rangle\langle\psi_l|],
\]

(2.53)

where \( H_\infty = H_\oplus \) is the asymptotic Hamiltonian and \( k,l \leq d-1 \).

Since the entire upper-left block is preserved for a DFS,

\[
\mathcal{P}_\oplus (\rho_{in}) = \mathcal{P}_\oplus (\rho_{in}) = P \rho_{in} P.
\]

(2.54)

We can thus deduce from (LP2) that the effect of Hamiltonian perturbations \( V \) within \( \text{As}(H) \) is \( V_\oplus = P V P \) — the Hamiltonian projected onto the DFS (see Sec. 4.1.1). Likewise, if \( O = \partial_\alpha \) then the Lindbladian adiabatic connection can be shown to reduce to \( \partial_\alpha P \cdot P \), the adiabatic connection of the DFS (see Sec. 5.3.1). Naturally, the QGT and its corresponding metric also reduces to that of the DFS states. In other words, all such results are the same regardless of whether the states form a DFS of a Lindbladian or a degenerate subspace of a Hamiltonian. We apply this powerful formula to coherent state quantum information processing schemes in Chs. 7-8. Applications of this formula to waveguide QED quantum computation schemes can be found in Ref. [218]. We note that this extends a previous related result (Ref. [314], footnote [23]).

Since all states in \( \mathfrak{H} \) are asymptotic, the steady-state basis elements and conserved quantities of \( \mathcal{L}_\oplus = \mathcal{H}_\infty = -i [H_\infty, \cdot] \) are equal. In the notation of Thm. 4, \( J^{\Delta_H} = \Psi^{\Delta_H} \). The structure of the piece \( J^{\Delta_H} \) can be determined by Thm. 4 and depends on the \( \mathcal{L}_\oplus \).

2.5.4 NS case

This important case is a combination of the DFS and unique steady-state cases [Fig. 2.1(f)]. In this case, the non-decaying portion of the system Hilbert space (\( PH \)) factors into a \( d \)-dimensional subspace \( H_{dfs} \) spanned by DFS states and a \( d_{ax} \)-dimensional auxiliary (also, gauge) subspace \( H_{ax} \), which is the range of some unique steady state \( \varrho_{ax} \),

\[
d_{ax} = \dim(H_{ax}) = \text{rank}(\varrho_{ax}).
\]

(2.55)

This combination of a DFS tensored with the auxiliary state \( \varrho_{ax} \) is called a noiseless subsystem (NS) [156]. For one NS block, \( H \) and \( \text{Op}(H) \) decompose as

\[
\begin{align*}
H &= PH \oplus QH = (H_{dfs} \otimes H_{ax}) \oplus QH, \\
\text{Op}(H) &= \mathfrak{H} \oplus \mathfrak{P} = [\text{Op}(H_{dfs}) \otimes \text{Op}(H_{ax})] \oplus \mathfrak{P}.
\end{align*}
\]

(2.56a)

(2.56b)

An NS block is possible if \( \mathcal{L} \) respects this decomposition and does not cause any decoherence within the DFS part. The Lindbladian in \( \mathfrak{H} \) is then

\[
\mathcal{L}_\oplus = H_{dfs} \otimes \mathcal{P}_{ax} + \mathcal{P}_{dfs} \otimes \mathcal{L}_{ax} = H_\infty + \mathcal{P}_{dfs} \otimes \mathcal{L}_{ax},
\]

(2.57)

where \( \mathcal{P}_{dfs} (\mathcal{P}_{ax}) \) is the superoperator identity on the DFS (auxiliary) space. The auxiliary state \( \varrho_{ax} \) is the unique steady state of the Lindbladian \( \mathcal{L}_{ax} \).
2.5 Summary of Applications to Various As(H)

Note that the auxiliary factor becomes trivial when $e_{ax}$ is a pure state ($d_{ax} = 1$), reducing the NS to a DFS. This means that the NS case is distinct from the DFS case only when $e_{ax}$ is mixed ($d_{ax} \neq 1$). Similarly, if the dimension of the DFS $d^2 = d = 1$, the NS reduces to the unique steady-state case. The NS case thus encapsulates both the DFS and unique state cases.

The DFS basis elements $|\psi_k\rangle\langle\psi_l|$ from eq. (2.53) generalize to $|\psi_k\rangle\langle\psi_l| \otimes e_{ax}$. Let us now focus on a stationary NS ($H_\infty = 0$) and construct a basis that will be used throughout the rest of the thesis. Denote the respective As($H$) basis elements and conserved quantities as $|\Psi_\mu\rangle\rangle \equiv |\Psi_{\Lambda=0,\mu}\rangle$ and $|J^\mu\rangle\rangle \equiv |J^{\Lambda=0,\mu}\rangle$. Since As($H$) is stationary, we can construct a Hermitian matrix basis for both As($H$) and the corresponding conserved quantities that uses one index and is orthonormal (see Sec. 1.6). For the DFS part of the NS, we define the Hermitian matrix basis $\{|\Psi_{dfs}\rangle\}_{\mu=0}^{d^2-1}$. In this new notation, the basis elements for one NS block are then

$$|\Psi_\mu\rangle = \frac{1}{n_{ax}} \left( |\Psi_{dfs}\rangle \otimes |e_{ax}\rangle \right) 0 \right) . \tag{2.58}$$

We have normalized the states using the auxiliary state norm (purity)

$$n_{ax} \equiv \sqrt{\langle\langle e_{ax}|e_{ax}\rangle\rangle} = \sqrt{\text{Tr}\{e_{ax}^2\}} \tag{2.59}$$

to ensure that $\langle\langle \Psi_\mu|\Psi_\nu\rangle\rangle = \delta_{\mu\nu}$. Since an NS block is a combination of the unique and DFS cases, the conserved quantities of $L$ (i.e., of $L_{dfs}$) are direct products of the DFS and auxiliary conserved quantities [45, 61]. The unique auxiliary conserved quantity is $P_{ax}$, the identity on the auxiliary subspace $H_{ax}$. Combining this with the result above and multiplying by $n_{ax}$ so that $\Psi_\mu$ and $J^\mu$ are biorthogonal [see eq. (2.24)], we obtain

$$\langle\langle J^\mu = n_{ax} \left( \langle\langle \Psi_{dfs}| \otimes \langle\langle P_{ax}| \right) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) . \tag{2.60}$$

We use the NS block basis of the above form throughout the thesis. The two projections are then

$$P_\infty = \sum_\mu |\Psi_\mu\rangle\langle J^\mu| = P_\Psi - P_\Psi L L_{dfs}^{-1} \tag{2.61a}$$
$$P_\Psi = \sum_\mu |\Psi_\mu\rangle\langle J^\mu| = P_{dfs} \otimes |e_{ax}\rangle\langle P_{ax}| , \tag{2.61b}$$

where the superoperator projection onto the DFS is explicitly

$$P_{dfs}(\cdot) = \sum_\mu |\Psi_{dfs}\rangle\langle \cdot|_{dfs} \langle\langle \Psi_{dfs}| \otimes |P_{dfs}| \cdot , \tag{2.62}$$

Therefore, states in $\mathcal{B}$ are not perfectly preserved, but are instead partially traced over the auxiliary subspace:

$$P_\Psi (\rho_{in}) = \text{Tr}_{ax} \{ P \rho_{in} P \} \otimes e_{ax} , \tag{2.63}$$

where $P = P_{dfs} \otimes P_{ax}$ and $P_{dfs}$ ($P_{ax}$) is the identity on $H_{dfs}$ ($H_{ax}$).
For this case, the effect of perturbations $V$ on $\text{As}(H)$ is more subtle due to the auxiliary factor, but the induced time evolution on the DFS is still unitary. The effective DFS Hamiltonian can be extracted from $P_\Psi V P_\Psi$ (see Sec. 4.1.1) and is

$$W = \text{Tr}_{\text{ax}} \{ q_{\text{ax}} V \}.$$  \hfill (2.64)

Similarly, if we define generators of motion $G_\alpha$ in the $x_\alpha$-direction in parameter space (i.e., such that $\partial_\alpha \rho_\infty = -i [G_\alpha, \rho_\infty]$; see Sec. 5.3.1), then the corresponding holonomy (Berry phase) after a closed path is the path-ordered integral of the various DFS adiabatic connections

$$\tilde{A}_{\text{dfs}}^\alpha = \text{Tr}_{\text{ax}} \{ q_{\text{ax}} (G_\alpha) \}.$$  \hfill (2.65)

In both cases, the effect of the perturbation on the DFS part depends on $q_{\text{ax}}$, meaning that $q_{\text{ax}}$ can be used to modulate both Hamiltonian-based or holonomic quantum gates. Making contact with adiabatic evolution, Zanardi and Campos Venuti showed that first-order Hamiltonian evolution within $\text{As}(H)$ can be thought of as a holonomy. The results (2.64-2.65) further develop a connection between holonomies and first-order perturbative effects within $\text{As}(H)$ ([315], Prop. 1) by showing that, for both processes, evolution is generated by the same type of effective Hamiltonian ($W$ and $\tilde{A}_{\text{dfs}}^\alpha$, respectively).

The QGT for this case is rather complicated due to the $q_{\text{ax}}$-assisted adiabatic evolution. However, we devote Ch. 6 to showing that the QGT does endow us with a metric on the parameter space for one NS block.

### 2.5.5 Multi-block case

The noiseless subsystem is the most general form of one block of asymptotic states of $L$, and the most general $\text{As}(H)$ is a direct sum of such NS blocks [45, 102, 280] (also called “superselection sectors” [216]) [see Fig. 1.2(a)] with corresponding minimal projection $P_\Psi$. This important result applies to all quantum channels [46, 60, 61, 79, 178] and stems from a well-known algebra decomposition theorem (see [306] for a technical introduction). More technically, $\text{As}(H)$ is a matrix or von Neumann algebra, but with each block $\kappa$ distorted [61, 146] (see also Corr. 6.7 in Ref. [306]) by its corresponding fixed auxiliary steady state $q_{\text{ax}}^{(\kappa)}$:

$$\text{As}(H) = \bigoplus_{\kappa} \text{Op} \left( H_{\text{dfs}}^{(\kappa)} \right) \otimes q_{\text{ax}}^{(\kappa)}.$$  \hfill (2.66)

Of course, the above reduces to the NS case when there is only one block. Throughout the text, we explicitly calculate properties of one NS block and sketch any straightforward generalizations to the multi-block case. To reiterate, the subtleties of $\text{As}(H) \subseteq \mathcal{M}$ are independent of the presence of a decaying subspace $\mathcal{P}$.

If there are two NS blocks (characterized by projections $P_{\text{dfs}}^{(\kappa)} \otimes P_{\text{ax}}^{(\kappa)}$ with $\kappa \in \{1, 2\}$) and no decaying subspace, then the conserved quantities $J^{\kappa \mu} = \Psi_{\kappa \mu}^{\text{dfs}} \otimes P_{\text{ax}}^{(\kappa)}$ do not have presence in the subspace of coherences between the blocks. The blocks in which $J^{\mu} \neq 0$ are shaded in gray in Fig. 1.2(b), dual to $\Psi_{\mu}$ in Fig. 1.2(a).

Both eqs. (2.64) and (2.65) extend straightforwardly to the multi-block case, provided that the blocks maintain their shape during adiabatic evolution. We make the same connection between
ordinary and adiabatic perturbations to jump operators of \( \mathcal{L} \); the latter were first studied in Avron et al. [25]. If we add in jump operator perturbations \( F \to F + f \), then the generalization of the effective Hamiltonian \( V_{\mathfrak{E}} \) (2.64) is derived in Sec. 4.1.3 to be

\[
X_{\mathfrak{E}} \equiv V_{\mathfrak{E}} + \frac{i}{2} \kappa \left( F^{\dagger} f - f^{\dagger} F \right)
\]  

(2.67)

projected onto all of the NS blocks. This quantity has previously been introduced ([25], Thm. 5) as the operator resulting from joint adiabatic variation of the Hamiltonian and jump operators of \( \mathcal{L} \). It is thus not surprising that the effect of perturbations to the Hamiltonian and jump operators on \( \rho_\infty \) is \( X \) projected onto \( \text{As}(\mathcal{H}) \).

### 2.6 Relation of Conserved Quantities to Symmetries

For \( H_\infty = 0 \), all \( J^{A\mu} = J^{A=0,\mu} \) are conserved quantities, and a natural question is whether they always commute with the Hamiltonian \( H \) and the jump operators \( F^\ell \). It turns out that they do not always commute [8, 45], and so various generalizations of Noether’s theorem have to be considered [25, 127].

#### 2.6.1 A Noether’s theorem for Lindbladians?

**Hamiltonian case**

In a unitary system, an (explicitly time-independent) observable \( J = J^\dagger \) is a conserved quantity (i.e. constant of motion) if and only if it commutes with the Hamiltonian (e.g., angular momentum of the hydrogen atom). In the spirit of Noether’s theorem, one can then generate a continuous symmetry \( U = \exp(i\phi J) \) (for real \( \phi \)) that leaves the Hamiltonian invariant. There is thus the following set of equivalent statements for continuous symmetries in unitary evolution (with one-sided arrows depicting an “if-then” statement, two-sided arrows depicting “iff,” and the dot being total time derivative):

\[
[J, H] = 0 \quad \Leftarrow \quad \overset{\dddot{}}{J} = 0 \quad \iff \quad U^\dagger HU = H
\]  

(2.68)

We call the above triple iff relationship *Noether’s theorem*.

**Lindbladian case**

A conserved quantity in Lindbladian systems is one where \( \mathcal{L}^\dagger(J) = 0 \) (2.21). Let us define the superoperator corresponding to \( J \),

\[
\mathcal{J}(\cdot) \equiv +i[J, \cdot], \text{ and its corresponding unitary } U = \exp(\phi \mathcal{J}).
\]  

(2.69)

Just like with Hamiltonians \( H \) and their superoperators \( \mathcal{H} \),

\[
\mathcal{U}^\dagger(F^\ell) \equiv U^\dagger F^\ell U.
\]  

(2.70)
We consider only superoperators $\mathcal{U}$ which can be written in terms of a $J$ on the operator level. Using this notation, one produces an analogous set of statements for $\mathcal{L}$:

\[
\begin{align*}
[J, H] &= [J, F^\ell] = 0 \quad \forall \ell \\
\dot{J} &= \mathcal{L}^\dagger(J) = 0 \\
\mathcal{U}^\dagger \mathcal{L} \mathcal{U} &= \mathcal{L}
\end{align*}
\] (2.71)

In comparison to the original theorem (2.68), four arrows are lost! The two arrows emanating from $\mathcal{U}^\dagger \mathcal{L} \mathcal{U} = \mathcal{L}$ are lost because there exist operators which leave $\mathcal{L}$ invariant but are neither conserved nor commute with everything. A simple example of a symmetry that neither commutes with $F^\ell$ nor is conserved is $J = a^\dagger a \equiv \hat{n}$ for a bosonic $\mathcal{L}$ with the lowering operator $F = a, H = 0$; we discuss this case in detail in Sec. 2.6.2. The loss of the two arrows emanating from $\dot{J} = \mathcal{L}^\dagger(J) = 0$ is due to the decaying subspace, which guarantees that conserved quantities $J$ generally contain the piece $J|_\mathcal{U}$. However, it turns out that certain conditions can restore these arrows: (1) these arrows are restored in $\mathcal{E}$ and (2) they are restored for any $J$ which squares to the identity. We discuss these cases below, noting that the above theorem was motivated by Ref. [45].

**Noether’s theorem partially restored in non-decaying subspace**

The only restriction on the Lindbladian $\mathcal{L}|_\mathcal{E}$ is that it contains a full-rank steady state. It turns out that this restriction is sufficient to partially restore Noether’s theorem. Recall that each conserved quantity $J^\mu = J|_\mathcal{E} + J|_\mathcal{M}$. Using the dissipation function from the proof of Thm. 4, one can show that $[F^\ell, J^\mu] = 0$ for all $\ell$ and $\mu$. This restores some of the arrows and produces

\[
\begin{align*}
[J|_\mathcal{E}, H] &= [J|_\mathcal{E}, F^\ell] = 0 \quad \forall \ell \\
\dot{J}|_\mathcal{E} &= \mathcal{L}^\dagger(J|_\mathcal{E}) = 0 \\
\mathcal{U}^\dagger \mathcal{L} \mathcal{U} &= \mathcal{L}
\end{align*}
\] (2.72)

a partially restored Noether’s theorem that holds in $\mathcal{E}$. While being in $\mathcal{E}$ is thus a sufficient condition for the above to hold, it is not a necessary one (see Sec. 3.2.1 for an example).

**Parity & discrete rotations**

Having omitted discrete symmetries ($U$ where $\phi$ takes discrete values), we expound on parity since it is the simplest discrete symmetry and it is a good starting point for the further examples in Chs. 3 and 7. Equation (2.71) shows that if one can find a non-trivial operator that commutes with everything in $\mathcal{L}$, then one is lucky to have found both a symmetry and a conserved quantity. It turns out that systems with parity conservation necessarily have such an operator and parity can be thought of as a symmetry almost in the unitary sense of eq. (2.68):

\[
\begin{align*}
[P, H] &= [P, F^\ell] = 0 \quad \forall \ell \\
\dot{P} &= \mathcal{L}^\dagger(P) = 0 \\
[P, \mathcal{L}] &= 0
\end{align*}
\] (2.73)
In the above, $\mathcal{P}(F^t) = PF^tP$. The proof is simple. Assuming $L^\dagger(P) = 0$, it is possible to construct conserved positive- and negative-parity projections $\Pi_{\pm} = \frac{1}{2}(I \pm P)$, which in turn must commute with all operators in $L$ ([45], Lemma 7). Therefore, $P$ must commute with everything as well.

In general, any set of $d$ conserved projection operators partitions $\text{Op}(H)$ into $d^2$ subspaces which evolve independently under $L$ ([45], Thm. 3), with at least $d$ of the subspaces having their own steady state. In each of the $d$ subspaces, the steady state basis element is a population [e.g., $|n\rangle \langle n|$ with $n \in \{0, 1, \ldots, d - 1\}$; see Fig. 2.1(b)] and an $\text{As}(H)$ such as this corresponds to a classical $dit$ (see Sec. 3.1). However, such a symmetry is neither necessary nor sufficient for the existence of a DFS, i.e., for cases where all $d^2$ subspaces each have one steady-state basis element. Regarding the necessary condition, we study examples with symmetries which do not admit a DFS in Sec. 3.3 and there exist other examples which do not [71]. Regarding sufficiency, steady-state coherences $\Psi_{\mu\nu} \neq \mu$ can exist with or without a discrete symmetry (e.g., Sec. 5.3 in Ref. [45]). Both of these cases are demonstrated pictorially via the two types of $\rho_{in}$ below:

$$
\left( \begin{array}{c}
\Psi_{00} & \Psi_{01} \\
\Psi_{10} & \Psi_{11}
\end{array} \right) ,
\left( \begin{array}{c}
\Psi_{00} & \Psi_{01} \\
\Psi_{10} & \Psi_{11}
\end{array} \right) .
$$

(2.74)

In the above list, arrows represent parts of the space which converge to $\Psi_{\mu\nu} \in \text{As}(H)$, which form a qubit DFS. The left example symbolizes a system with no parity symmetry. In the right example of a system with parity symmetry, the full space is “cut-up” into four independent subspaces, each of which converges to a steady state/coherence.

### 2.6.2 Symmetries

We now partially extend the parity discussion in the previous Subsection to more general symmetries. As mentioned in eq. (2.71), a continuous symmetry $U$ is a unitary operator whose corresponding superoperator $U = e^{i\mathcal{J}}$ is such that $U^\dagger L U = L$, or equivalently $[\mathcal{J}, L] = 0$. It is therefore easy to see that both $U$ and $U^\dagger$ are symmetries of both $L$ and $L^\dagger$. To state in a different way, $U$ commutes with time-evolution generated by $L$,

$$
e^{\mathcal{L}t}U^\dagger(\rho_{in}) = e^{\mathcal{L}t}(U^\dagger \rho_{in}U) = U^\dagger e^{\mathcal{L}t}(\rho_{in})U = U^\dagger e^{\mathcal{L}t}(\rho_{in})$$

(2.75)

for any $\rho_{in} \in \text{Op}(H)$. Examples of symmetries include any $U$ such that $UHU^\dagger = H$ and $U^\dagger F U^\dagger = e^{i\Phi} F$ [38] or any permutations among the jump operators $F^i$ that leave $L$ invariant [71]. Note that the former case provides an example of a symmetry whose generator doesn’t commute with $F^i$. The Lindbladian can be block-diagonalized by $U$ (with each block corresponding to an eigenvalue of $U$) in the same way that a Hamiltonian can be block-diagonalized by $U$ (with each block corresponding to an eigenvalue of $U$). Symmetries can thus significantly reduce computational cost, with the additional complication that the blocks of $L$ may not be further diagonalizable. However, symmetries by themselves do not determine the dimension of $\text{As}(H)$ because some blocks may contain only decaying subspaces and no steady states. Diagonal parts of $\rho_{in}$ will always be in blocks with steady states since the trace is preserved. For a unitary $U$ such that $[U, H] = [U, F^i] = 0$, $\dim\{\text{As}(H)\}$ will be at least as much as the number of distinct
eigenvalues of $U$ ([71], Thm. A.1). One can see this by decomposing $U$ into a superposition of projections on its eigenspaces and applying eq. (2.73) to each projection. However, such a result once again does not say anything about whether $A(s|H)$ will be a quantum memory or a classical one.

An example of a symmetry is invariance of the zero-temperature cavity — $\mathcal{L}$ with $F=a$ and $H=0$ — under bosonic rotations $R_\phi \equiv e^{i\phi \hat{n}}$ (with $\hat{n} = a^\dagger a$). This is an example of a continuous symmetry which does not stem from a conserved quantity in $\text{Op}(H)$. Instead, this symmetry stems from the generator $\mathcal{N}$ of the corresponding $R_\phi \equiv e^{i\phi \mathcal{N}}$, which commutes with $\mathcal{L}$. The generator acts as $\mathcal{N}(\rho_{\text{in}}) = \hat{n}\rho_{\text{in}} - \rho_{\text{in}}\hat{n}$ and its commutation with $\mathcal{L}$ can be easily checked. The block diagonalization of $\mathcal{L}$ stemming from this symmetry corresponds to equations of motion for matrix elements $\langle n | \rho_{\text{in}} | m \rangle$ with $m - n = r$ being decoupled from those with $m - n \neq r$ ([120], eq. (6.1.6)). This will be used to calculate conserved quantities in Sec. 3.3. In this way, symmetries can help compartmentalize evolution of both states and operators.

Any symmetry of $\mathcal{L}$ leaves the asymptotic subspace invariant,

$$U^\dagger \mathcal{L} U = \mathcal{L} \quad \Rightarrow \quad U^\dagger \mathcal{P}_\infty U = \mathcal{P}_\infty. \quad (2.76)$$

This is because one can create a basis for $A(s|H)$ which consists of eigenstates of $U$. Symmetries can thus classify [38] unique steady states and/or constrain their properties [225, 245]. For the example of the previous paragraph, the vacuum $|0\rangle\langle 0|$ is rotationally invariant under $R_\phi$. When $A(s|H)$ is not one dimensional, symmetries will rotate $A(s|H)$ into itself and so can act nontrivially on any given state $\rho_\infty \in A(s|H)$. Symmetries can thus be used to perform unitary rotations on the steady-state subspace. We briefly mention the existence of anti-commuting symmetries such as chiral [38] or parity-time [230, 231] for dissipative dynamics. These can reveal symmetries in the spectrum of $\mathcal{L}$ and $\mathcal{L}^\dagger$ [231], similar to the spectrum of a chirally-symmetric Hamiltonian being symmetric around zero. A brute-force approach of finding all symmetries of $\mathcal{L}$ is to find the null space of the commutator super-superoperator $[\mathcal{L}, \cdot ]$ ([257], Appx. A; see also [320]).
“Never do a calculation until you know the answer.”

– Steven M. Girvin

EXAMPLES

Here, we present examples of $\mathcal{L}$ which do not have a unique steady state. The first three Sections focus on calculating conserved quantities of single-body systems such as qubits and oscillators \cite{8, 10}. The last section takes the systems theory perspective and reviews standard techniques for stabilizing ground state subspaces of many-body frustration-free Hamiltonians.

3.1 SINGLE-QUBIT DEPHASING

In this example, $\text{As}(H)$ is two-dimensional: all steady states can be written as convex combinations of two orthogonal basis states $\Psi_0$ and $\Psi_1$. In other words, such an $\text{As}(H)$ stores one classical bit (i.e., one probability’s worth of information). Such an $\text{As}(H)$ can also be thought of as a one-dimensional simplex \cite{187} and is the simplest version of an information-preserving structure \cite{61}. In the many-body case, such a system is called bistable \cite{171}. We also initially assume that there is no decaying subspace ($P = I$).

Consider one qubit undergoing dephasing on two of the three axes of its Bloch sphere, thereby stabilizing the Bloch vector onto the third axis. In this case, there is one jump operator $F = Z$ (with $X, Y, Z$ the usual Pauli matrices) and no Hamiltonian. The master equation simplifies to the Poisson semigroup generator (i.e., a Lindbladian with a unitary jump operator \cite{177})

$$
\mathcal{L}(\rho) = Z \rho Z - \frac{1}{2} \{ I, \rho \} = Z \rho Z - \rho = -\frac{1}{2} [Z, [Z, \rho]].
$$

(3.1)

Picking the eigenbasis of $Z$, $Z|\mu\rangle = (-)^{\mu}|\mu\rangle$ with $\mu = 0, 1$, one can see that the states $\Psi_\mu = |\mu\rangle \langle \mu|$ will be steady but the coherence $|0\rangle \langle 1|$ will not survive. The steady-state density matrix is then

$$
\rho_\infty = \lim_{t \to \infty} e^{t\mathcal{L}}(\rho_{\text{in}}) = \mathcal{P}_\infty(\rho_\infty) = c_0|0\rangle \langle 0| + c_1|1\rangle \langle 1|.
$$

(3.2)

Naturally, one expects the system to record the initial $Z$-component of $\rho_{\text{in}}$. One can see that $\mathcal{L}^\dagger = \mathcal{L}$ since the jump operator is Hermitian, so the conserved quantities $J^\mu = \Psi_\mu$. Letting $c_Z = \text{Tr}\{Z \rho_{\text{in}}\}$, one indeed determines that the $Z$-component is preserved and $c_\mu = \frac{1}{2} [1 + (-)^{\mu} c_Z]$.

This example can be straightforwardly generalized to an $N$-dimensional system whose $\text{As}(H)$ is spanned by all diagonal populations $|\mu\rangle \langle \mu|$ (for $\mu = 0, 1, \cdots, N-1$) and where there is still
no decaying subspace. Lindbladians with such $\text{As}(\mathcal{H})$ include *dephasing Lindbladians* [29] [see Fig. 2.1(b)]. These systems can be used to model what happens in a measurement [298], with $\Psi_\mu$ interpreted as *pointer states* of the system [324].

One can also extend this qubit example to include a decaying subspace $\mathcal{S}$. While the structure of $\text{As}(\mathcal{H})$ would remain the same, $c_\mu$ would additionally store information about the populations (and possibly the coherences) of states initially in $\mathcal{S}$. For example, consider adding a third level $|2\rangle$ to the Hilbert space and adding another jump operator $F' = |0\rangle\langle 2|$ which decays that new level to $|0\rangle$. Then, the conserved quantity associated with $|0\rangle$ gains an additional term which "catches" the decayed population: $J^0 = \Psi_0 + |2\rangle\langle 2|$.

Finally, this example can be extended to cases where the two (or more) $\Psi_\mu$ are mixed states. One can imagine such a case in an thermal equilibrium system with a parity symmetry: $\rho_\infty = c_0 \Psi_0 + c_1 \Psi_1$, where $\Psi_\mu$ are the unique steady states in each parity sector. We direct the interested reader to further examples of such systems in spin chains [71, 136, 191, 195, 196], fermionic systems ([229], Example 5.2), and quantum transport in models of energy harvesting [188].

### 3.2 Two-Qubit Dissipation

In this example, $\text{As}(\mathcal{H})$ is initially a one-qubit DFS with $H_\infty = 0$ and there is a two-dimensional decaying subspace $\mathcal{S}$. This example is taken from recent experimental work that stabilizes Bell states using trapped ions [41] and is closely related to stabilizer generators of qubit codes [254]. We study this case in detail by adding different Hamiltonians and jump operators [8, 10] and seeing how the structure of $\text{As}(\mathcal{H})$ and the conserved quantities changes.

#### 3.2.1 Clean case

Let $\text{Op}(\mathcal{H})$ be the space of matrices acting on the Hilbert space $\mathcal{H}$ of two qubits. Let $\mathcal{L}$ have one jump operator ($c$ in Box 1 of [41])

$$F = \frac{1}{2} (I - Z_1 Z_2) X_2,$$

where the subscript labels the qubit. Intuitively, $\text{As}(\mathcal{H})$ is equivalent to the space spanned by $|\psi_k\rangle\langle \psi_l|$, where $k, l \in \{0, 1\}$ and the Bell states $|\psi_k\rangle \equiv \frac{1}{\sqrt{2}} ([01] + (-)^k [10])$. While we can operate using the original Pauli matrices, let us instead call the other two Bell states $|\psi_k^\perp\rangle \equiv \frac{1}{\sqrt{2}} ([00] + (-)^k [11])$ and re-write the jump using this basis:

$$F = \sum_{k=0}^1 |\psi_k\rangle\langle \psi_k^\perp| = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = F_{\mathcal{S}}.$$
This allows us to conform to the structure, which is delineated using the two lines in the matrix above: the first two Bell states \( \{ |\psi_0\rangle, |\psi_1\rangle \} \) form the DFS and the latter two \( \{ |\psi_0^+\rangle, |\psi_1^+\rangle \} \) decay into the DFS. Obviously, \( \text{As}(H) \) is spanned by

\[
\Psi_{kl} = |\psi_k\rangle \langle \psi_l|.
\]  

(3.5)

The conserved quantities can be determined by the formula in Thm. 4:

\[
J^{kl} = |\psi_k\rangle \langle \psi_l| + |\psi_k^\perp\rangle \langle \psi_l^\perp|.
\]  

(3.6)

One would think that since there is a decaying subspace, the conserved quantities do not commute with \( F \) (see Sec. 2.6.1). However, the presence of \( \boxed{\text{structure}} \) is not a sufficient condition and we do in fact have \([J^{kl}, F] = 0\) because there is no extra decoherence in \( \boxed{\text{structure}} \) (\( F_{\text{DFS}} = 0 \) and because the decay of states is one-to-one (\(|\psi_k^\perp\rangle \rightarrow |\psi_k\rangle \) for \( k = 0, 1 \)). A few sanity checks: the steady diagonal state basis elements add up to the projection onto the DFS, \( P = \Psi_{00} + \Psi_{11}, \) and the diagonal conserved quantities correspondingly add up the identity, \( I = J^{00} + J^{11}. \) Both \( J^{kl} \) and \( \Psi_{kl} \) form the Lie algebra \( u(2). \) The steady state \( \rho_\infty \in \text{As}(H) \) for initial state \( \rho_{in} \in \text{Op}(H) \) can be expressed as

\[
\rho_\infty = \sum_{k,l=0}^1 \text{Tr}\{J^{kl} \rho_{in}\} |\psi_k\rangle \langle \psi_l| = P \rho_{in} P + \sum_{k,l=0}^1 \langle \psi_k^\perp| \rho_{in} |\psi_l^\perp\rangle |\psi_k\rangle \langle \psi_l|.
\]  

(3.7)

Notice that \( \Pi = J^{00} - J^{11} \) is a parity operator, meaning that the analysis from Sec. 2.6.1 holds and we can partition the evolution into invariant blocks. Everything in \( L \) commutes with \( \Pi, \) so \( \text{Op}(H) \) can be partitioned into four blocks of matrices that are built out of the two subspaces of \( H \) of positive and negative parity. Each block (indexed by \( k,l \)) is of the form \( \{ |\psi_k\rangle \langle \psi_l|, |\psi_k\rangle \langle \psi_l^\perp|, |\psi_k^\perp\rangle \langle \psi_l|, |\psi_k^\perp\rangle \langle \psi_l^\perp| \}, \) and there are four of them since \( k,l \in \{0,1\}. \) Each block has its own steady state basis element \( \Psi_{kl} \) and conserved quantity \( J^{kl} \) and evolves independently from the other blocks.

### 3.2.2 Coherence suppressed by a jump

Continuing from the previous DFS example, let us add a term to \( F_{\boxed{\text{structure}}} \):

\[
F = \sum_{k=0}^1 |\psi_k\rangle \langle \psi_k^\perp| + \alpha \sum_{k=0}^1 (-1)^k |\psi_k^\perp\rangle \langle \psi_k^\perp| = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \end{pmatrix},
\]  

(3.8)

where the \( \alpha \)-dependent term \( F_{\boxed{\text{structure}}} \) now dephases the non-DFS Bloch vector (with \( \alpha \in \mathbb{R} \)). The steady-state basis elements are still \( \Psi_{kl} = |\psi_k\rangle \langle \psi_l| \) since \( F_{\boxed{\text{structure}}} = 0. \) To determine the corresponding \( J^{kl} \), we use eq. (2.31). Acting on \( \Psi_{kl} \) with \( L^{\dagger} \) (1.31) and then the adjoint of \( L^{-1} \) (2.8d) yields the corresponding conserved quantities

\[
J^{kl} = |\psi_k\rangle \langle \psi_l| + \frac{|\psi_k^\perp\rangle \langle \psi_l^\perp|}{1 + 2\alpha^2 (1 - \delta_{kl})}.
\]  

(3.9)
The only non-trivial feature of the steady state is due to $F$ and $L^{-1}$. Namely, an initial nonzero coherence $\langle \psi_0^+ | \rho_{in} | \psi_1^+ \rangle$ leads necessarily to a mixed steady state due to coherence suppression of order $O(\alpha^{-2})$.

### 3.2.3 Coherence suppressed by a Hamiltonian ($H_\infty \neq 0$)

A similar coherence suppression can be achieved by adding the Hamiltonian

$$H = \frac{1}{2} \beta (|\psi_0\rangle \langle \psi_0| - |\psi_1\rangle \langle \psi_1|) = H_{\mathfrak{B}}$$

(with $\beta \in \mathbb{R}$) to the original jump $F$ from eq. (3.4). Now the DFS is non-stationary (with $H_\infty = H$) and the off-diagonal DFS elements $\Psi_{k \neq l}$ rotate. Abusing notation by omitting the corresponding eigenvalue $\Delta = \beta$, the left asymptotic eigenmatrices become

$$J_{kl} = |\psi_k\rangle \langle \psi_l| + \frac{|\psi_k^\perp\rangle \langle \psi_l^\perp|}{1 + i \beta (-)^{(1 - \delta_{kl})}}.$$  \hfill (3.11)

Despite the fact that $F_{\mathfrak{B}} = 0$, the inverse $(L - i \Delta)_{\mathfrak{B}}$ from Thm. 4 still inflicts damage to the initial state due to $H_\infty$ (for nonzero $\beta$), but now the coherence suppression is of order $O(\beta^{-1})$.

The coherence suppression shown above is due to $\mathfrak{B}$ rotating while states from $\mathfrak{B}$ flow into it. It turns out that one can cancel that suppression by also rotating $\mathfrak{B}$ in the same direction. If we add another term to the Hamiltonian,

$$H' = \frac{1}{2} \alpha (|\psi_0^\perp\rangle \langle \psi_0^\perp| - |\psi_1^\perp\rangle \langle \psi_1^\perp|) = H'_{\mathfrak{B}},$$

then, once again due to the inverse piece $(L - i \Delta)_{\mathfrak{B}}$ in determining $J_{kl}$,

$$J_{01} = J_{00} = |\psi_0\rangle \langle \psi_1| + \frac{|\psi_0^\perp\rangle \langle \psi_1^\perp|}{1 + i (\alpha - \beta)}.$$  \hfill (3.13)

Setting $\alpha = \beta$ means that the coherence suppression can be canceled by a proper rotation in the decaying subspace $\mathfrak{B}$. This effect will be studied in a future work.

### 3.2.4 Driven case

As a final example, let us take this DFS case and convert it into an NS by changing the original jump from eq. (3.4) to (assuming $\gamma \in \mathbb{R}$)

$$F = \sum_{k=0}^{1} |\psi_k\rangle \langle \psi_k^\perp| - \gamma I.$$  \hfill (3.14)

Note that we could have equivalently added the Hamiltonian

$$H = -i \gamma \sum_{k=0}^{1} |\psi_k\rangle \langle \psi_k^\perp| + H.c.$$  \hfill (3.15)
due to the “gauge” transformation (1.29). This driving expands the DFS into a qubit NS [Fig. 2.1(f)], absorbing the decaying subspace $\mathfrak{G}$. Since the new jump still commutes with everything the original ($\gamma = 0$) jump commuted with, we still have a parity symmetry and therefore can study each individual block of states $\{ |\psi_k\rangle \langle \psi_l|, |\psi_k^+\rangle \langle \psi_l|, |\psi_k^-\rangle \langle \psi_l| \}$. Now, the steady state basis element in each block is

$$
\Psi_{kl} = \frac{(1 + \gamma^2) |\psi_k\rangle \langle \psi_l| + \gamma |\psi_k\rangle \langle \psi_l^+| + \gamma |\psi_k^+\rangle \langle \psi_l| + \gamma^2 |\psi_k^+\rangle \langle \psi_l^+|}{\sqrt{1 + 4\gamma^2 + 2\gamma^4}},
$$

where we are adding $\gamma$-dependent factors so that $\langle \Psi_{kl} | \Psi_{pq} \rangle = \delta_{kp} \delta_{lq}$. The previous conserved quantities are carried over since they commute with $F$ for all values of $\gamma$, but now we have to add extra factors in order to make sure $\langle \Psi_{kl} | \Psi_{pq} \rangle = \delta_{kp} \delta_{lq}$:

$$
J^{kl} = \frac{\sqrt{1 + 4\gamma^2 + 2\gamma^4}}{1 + 2\gamma^2} \left( |\psi_k\rangle \langle \psi_l| + |\psi_k^+\rangle \langle \psi_l^+| \right).
$$

Let us now organize $\text{Op}(H)$ in a different way in order to reveal the NS structure and its associated quantities, introduced in Sec. 2.5. Both the $\Psi'$s and $J$'s can be put into the following factored form:

$$
\Psi_{kl} = |k\rangle \langle l| \otimes \frac{\Theta_{ax}}{n_{ax}} \quad \text{and} \quad J^{kl} = |k\rangle \langle l| \otimes n_{ax} \rho_{ax},
$$

where the auxiliary steady state, identity, and square-root of purity ($n_{ax} \equiv \sqrt{\text{Tr}(\rho_{ax}^2)}$) are

$$
\Theta_{ax} = \frac{1}{1 + 2\gamma^2} \begin{pmatrix} 1 + \gamma^2 & \gamma \\ \gamma & \gamma^2 \end{pmatrix}, \quad \rho_{ax} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n_{ax} = \frac{\sqrt{1 + 4\gamma^2 + 2\gamma^4}}{1 + 2\gamma^2}
$$

and $|k\rangle \langle l|$ is the basis for the DFS part of the NS. Naturally, the NS reduces to a DFS as $\gamma \to 0$, whereas the dissipation and driving balance out and produce a maximally mixed $\Theta_{ax}$ when $\gamma \to \infty$.

### 3.3 Many-Photon Absorption

In this example, $\text{As}(H)$ is initially a DFS with $H_0 = 0$ and there is an infinite-dimensional decaying subspace $\mathfrak{G}$. This family of examples includes single-mode two-photon [3, 123, 183, 269, 270] and $d > 2$-photon [154, 293, 323] absorption. Their dynamics has been analytically solved for all time in the aforementioned references, but studying As(H) does not require that tedious algebra. These and related quantum optical systems (see Ref. [109] for a brief review of the older literature) have been recently gaining interest from the quantum information (see Ch. 7) and optomechanics [62, 210] communities.

#### 3.3.1 Two-Photon Case

Consider bosonic systems with jump operator $F = a^2$ with $[a, a^+] = I$ and no Hamiltonian. While this system is infinite, one can successfully analyze it for finite energy using a large finite Fock space spanned by $\{|n\rangle\}_{n=0}^N$ (where $N \gg 1$) [258]. Here, As(H) has the same qubit DFS structure
as in Example 3.2, with basis \( \Psi_{kl} = |k\rangle \langle l| \) in Fock space (with \( k, l = 0, 1 \)), as well as a similar parity symmetry. The diagonal conserved quantities \( j^{kk} \) correspond to projectors on the even and odd subspaces respectively:

\[
    j^{kk} \equiv \sum_{n=0}^{\infty} |2n + k\rangle \langle 2n + k| \equiv \Pi_k,
\]

and we can once again build a parity operator \( j^{00} - j^{11} \) that commutes with \( F \). Due to this symmetry, \( \text{Op}(H) \) is once again split into four independent subspaces: the four blocks \( \{|2n + k\rangle \langle 2m + l|\}_{n,m=0}^{\infty} \) (labeled by \( k, l \in \{0, 1\} \)) evolve independently of each other.

The conserved quantity for the off-diagonal subspace,

\[
    j^{01} = \frac{(\hat{n} - 1)!}{\hat{n}!!} \Pi_0 \hat{a},
\]

where \((m + 2)!! \equiv (m + 2) \cdot m!!\) is the double factorial [128], does not commute with the jump operator \( F \). This conserved quantity \( (3.21) \), derived first in Ref. [270], has provided the primary motivation for the initial portions of this body of work. One can obtain this quantity by first using the parity symmetry to isolate the subspace where it exists and then solving the equation \( \mathcal{L}^\dagger(j) = 0 \) in that subspace. Due to the parity structure, we know that \( j^{01} \) is off-diagonal in the sense that \( j^{01} = \Pi_0 j^{01} \Pi_1 \). Furthermore, since \( \langle j^{01} | \Psi_{01} \rangle = 1 \), \( j^{01} \) has to overlap with its corresponding steady-state coherence \( \Psi_{01} = |0\rangle \langle 1| \). With those two constraints and symmetry of \( \mathcal{L} \) under \( V = e^{i \phi \hat{n}} \) (see Sec. 2.6.2), \( j^{01} \) must consist only of elements from the sector \( \{|2n\rangle \langle 2n + 1|\}_{n=0}^{\infty} \). Assuming a solution of the form \( j^{01} = j(\hat{n}) \Pi_0 \hat{a} \) and plugging into \( \mathcal{L}^\dagger(j^{01}) = 0 \) yields a recursion relation for \( j(\hat{n}) \), whose solution is eq. (3.21).

Physically, \( j^{01} \) represents how the environment distinguishes components of \( \rho_m \). It partially preserves information only from elements \( \{|2n\rangle \langle 2n + 1|\}_{n=0}^{\infty} \) since, in that case, the same number of photon pairs is lost in relaxing to \( |0\rangle \langle 1| \). In all other even-odd basis sectors, e.g., \( \{|2n\rangle \langle 2n - 1|\}_{n=0}^{\infty} \), different numbers of photon pairs are lost (\( n \) vs. \( n - 1 \) pairs for the example). While \( j^{01} \) tells us exactly which sector contributes to the asymptotic state, it does not tell us how coherences decay as photons are lost. To determine intermediate-time behavior, one should consider the eigenvalues of the other eigenmatrices of \( \mathcal{L} \) in each sector [see eq. (1.35)]. In fact, not all information is preserved even in the \( \{|2n\rangle \langle 2n + 1|\}_{n=0}^{\infty} \) sector! The remaining left eigenmatrices \( \{L^{01,m}\}_{m=0}^{\infty} \) in that sector are of the same form as \( j^{01} \), namely \( L^{01,m} = j^{01,m}(\hat{n}) \Pi_0 \hat{a} \), but with the double factorials generalized to operators whose nonzero entries are

\[
    \langle 2n|L^{01,m}(\hat{n})|2n\rangle = \frac{1}{\sqrt{2m + 1}} \prod_{k=m+1}^{n} \frac{k(2k - 1)}{2(k^2 - m^2)}
\]

and with the convention that the sum \( \prod_{k=j}^{l} f(k) = 1 \) when \( j = l + 1 \) and zero when \( j > l + 1 \) [323]. Their corresponding eigenvalues are \( \lambda^{01,m} = -4m^2 \) and one can see that \( L^{01,0} = j^{01} \). The smallest Fock state in the support of \( j^{01,m}(\hat{n}) \) is \( |2m\rangle \), implying that the coherences \( |2m\rangle \langle 2m + 1| \) decay no faster than \( e^{-4m^2 t} \).
3.3.2 \(d\)-photon case

Let us now generalize the \(d = 2\) case to all \(d > 0\) and consider the jump operator \(F = a^d\). Note that the \(d = 1\) case is simply single-photon loss, which has the vacuum Fock state as its unique steady state. For the general case, let

\[
\Pi_k = \sum_{n=0}^{\infty} |dn + k\rangle \langle dn + k| = \frac{1}{d} \sum_{l=0}^{d-1} e^{\frac{2\pi i}{d} (\hat{n} - k)l}
\]

(3.23)

be \(d\) different projections with \(k, l \in \{0, 1, \cdots, d - 1\}\). Noting the cyclic relationship among projection operators,

\[
\Pi_k a = a \Pi_{(k+1) \mod d} = \Pi_k a \Pi_{(k+1) \mod d},
\]

(3.24)

one can see that \([\Pi_k, a^d] = 0\). According to Sec. 2.6.1, the Fock space is then partitioned into \(d^2\) subspaces, each evolving independently. We can thus write

\[
\rho_\infty = \sum_{k,l=0}^{d-1} c_{kl} |k\rangle \langle l|
\]

(3.25)

with \(c_{kl} = \text{Tr} \{J_{kl}^\dagger \rho_\infty\}\). Extending the recipe of the \(d = 2\) case, there are \(d^2\) conserved quantities

\[
J_{kl} = \frac{j_{kl} (\hat{n})}{\sqrt{(l)_l}} \Pi_k a^{l-k},
\]

(3.26)

where the square-root is to satisfy the biorthogonality condition (2.24), \(J_{kl} = J_{kl}^\dagger\)

\[
|dn + k\rangle \langle dn + k| = \prod_{p=0}^{n-1} \frac{2 (dp + l)_{l-k}}{(dp + l)_l + (dp + l + d)_l}. \quad \text{(3.27)}
\]

(for all \(n \in \mathbb{N}\)) and zero elsewhere, and the falling factorial \((x)_n = x(x-1)\cdots(x-n+1)\). Since \((x)_0 = 1\), the diagonal conserved quantities simplify to \(J_{kk} = \Pi_k\). Since \(\sum_{k=0}^{d-1} J_{kk} = I\), only \(d^2 - 1\) quantities are independent. The off-diagonal quantity simplifies to eq. (3.21) for \(d = 2\) and only the identity remains for \(d = 1\). The \(J_{kl}\) are reducible into a direct sum of \(u(d)\) Lie algebras. In other words, \(\sum_{k,l=0}^{d-1} J_{kl}\) forms an infinite block-diagonal matrix with blocks of length \(d\), diagonal entries of \(I\), and off-diagonal entries depending on \(j_{kl}(\hat{n})\).

3.3.3 Steady state for an initial coherent state

As an example calculation, we determine \(\rho_\infty\) when \(\rho_{in} = |\beta\rangle \langle \beta|\), a coherent state \(a |\beta\rangle = |\beta\rangle |\beta\rangle\) with \(\beta \in \mathbb{C}\). Note that only the piece of \(\rho_{in}\) that initially lives in a given subspace, \(\Pi_k \rho_{in} \Pi_k\), contributes to the corresponding \(c_{kl}\) in \(\rho_\infty\). Since a coherent state fills the entire Fock space, all subspaces evolve non-trivially and equilibrate to

\[
c_{kl} = \frac{\beta^{l-k} e^{-|\beta|^2}}{\sqrt{(l)_l}} \sum_{n=0}^{\infty} \frac{j_{kl}(dn + k)}{(dn + k)!} (|\beta|^2)^{dn + k}. \quad \text{(3.28)}
\]
Since the factors \( P_n = j_{kl}(dn + k)/(dn + k)! \) are polynomials in \( n \), \( c_{kl} \) are generalized hypergeometric functions whose arguments are be roots of \( P_{n+1}/P_n \) [222]. The diagonal elements simplify if instead we express \( \Pi_k \) using the right-hand side of eq. (3.23),

\[
c_{kk} = \frac{1}{d} \sum_{l=0}^{d-1} e^{-i\frac{2\pi}{d} kl} \exp \left[ \beta^2 \left( e^{i\frac{2\pi}{d} l} - 1 \right) \right]. \tag{3.29}
\]

In the large \( |\beta| \) limit, \( c_{kk} \to 1/d \), distributing populations equally among the diagonal steady states. For \( k \neq l \) in this limit, \( c_{kl} \) converges to a constant times \( e^{-i\theta(k-l)} \), thus storing the phase \( \theta \equiv \arg(\beta) \) of the initial coherent state for any \( d \). Taking a look at specific cases, for \( d = 1 \), eq. (3.28) is just \( c_{00} = 1 \). For \( d = 2 \), expressing in the \( |k\rangle \langle l| \) basis,

\[
\rho_{\infty} = \left( \frac{1}{2}(1 + e^{-2|\beta|^2}) \right) \beta^* e^{-|\beta|^2} I_0(|\beta|^2) \left( \frac{1}{2}(1 - e^{-2|\beta|^2}) \right) \text{ c.c.}, \tag{3.30}
\]

where \( I_0 \) is the modified Bessel function of the first kind [201]. In the large \( |\beta| \) limit, \( c_{01} \to e^{-i\theta}/\sqrt{2\pi} \). We consider a generalized version of this case in Ch. 7.

### 3.4 Ground State Subspaces of Frustration-Free Hamiltonians

This final case is discussed primarily to present “many-body” Lindbladian examples and discuss some related work.¹ The focus here is not on conserved quantities, but on a recipe for jumps that guarantees stabilization of the ground state subspace of any given frustration-free Hamiltonian. The presentation is standalone for convenience, and these results can be reconstructed from, e.g., Thm. 1 of Ref. [281] and Corr. 1 of Ref. [282].

These many-body examples feature a uniformly factorizable Hilbert space \( H = H_0^\otimes M \) (a term borrowed from [216]), where \( H_0 \) is the Hilbert space of some “local” site (such as a spin) and \( M \) is the number of such sites. A Hamiltonian \( H \) is frustration-free if all ground states of \( H \) are also ground states of all individual terms used to construct \( H \). The ground states of \( H \) thus form the \( \text{As}(H) = \bigoplus \) of the Lindbladian constructed out of said jumps, making this a DFS case. To make things concrete, we state the following theorem and provide an expository proof.

**Theorem 5** (Stabilizing frustration-free Hamiltonian ground states [281, 282, 288]). Let the Hilbert space \( H = H_0^\otimes M \), \( H = \sum_k H_k \), where \( H_k \) is a Hamiltonian acting nontrivially on a subset of the \( M \) sites, and assume that \( H \) is frustration-free. Then, for each \( H_k \) there exist jump operators \( \{F^{k,\ell}_l\}_l \) such that \( F^{k,\ell}_l \) act nontrivially on the same subset of sites as \( H_k \) and the ground states of \( H \) form the DFS (\( \bigoplus \)) of the Lindbladian constructed out of \( \{F^{k,\ell}\}_k,l \). Moreover, the jump operators satisfy

\[
F^{k,\ell}_l = F^{k,\ell}_L \tag{3.31}
\]

\[
\sum_{k,\ell} F^{k,\ell}_L F^{k,\ell}_L > 0. \tag{3.32}
\]

**Proof.** The strategy is two-fold: first construct \( \{F^{k,\ell}_l\}_l \) whose Lindbladian \( \mathcal{L}_k \) stabilizes the ground states of \( H_k \) and then show that the full Lindbladian \( \mathcal{L} = \sum_k \mathcal{L}_k \) stabilizes only the ground states of \( H \).

¹ However, this is by no means an attempt to review the disconnected literature.
We can shift $H_k$ by a constant such that $H_k \geq 0$ for all $k$, meaning that all ground states of $H_k$ have eigenvalue zero. For each $k$, define the projection $Q_k$ to be on the excited states (also, range or support; see footnote 4 in Ch. 1) of $H_k$: $H_k = Q_k H_k Q_k$. Define $P_k = I - Q_k$ to be the projection on the kernel of $H_k$, i.e., $H_k P_k = P_k H_k P_k = 0$. The strategy is to create jumps $F^{k,\ell}$ which take us from the range to the kernel. If the range and kernel are the same dimension, we can do this by just having one jump $F^k$ which is the isometry $(F^k) \cdot F^k = Q_k$ acting on states in ran$Q_k$ and taking them to ker$Q_k$. If the range is bigger than the kernel, then we can have isometries $F^{k,\ell}$ between distinct and non-overlapping subspaces of the range and the kernel. In other words, if $\dim \ker Q_k = d$ and $\dim \ran Q_k = D$, we can have $[D/d]$ jumps $\{F^{k,\ell}\}_{\ell=1}^{[D/d]}$ with $d$ ones on a diagonal in the upper right sector such that together they form the matrix

$$
\begin{pmatrix}
0 & F^{k,1} & F^{k,2} & \cdots & F^{k,[D/d]} \\
0 & 0 & \cdots & & \\
\vdots & & & & \\
0 & & & &
\end{pmatrix},
$$

where the upper left block is the $d$-dimensional ker$Q_k$ and lower right block is $D$-dimensional ran$Q_k$. One can verify by simple block matrix multiplication that the Hamiltonian formed by our jumps,

$$K^k = \frac{1}{2} \sum_{\ell} F^{k,\ell} \cdot F^{k,\ell} = \begin{pmatrix} 0 & 0 \\
0 & \frac{1}{2} \sum_{\ell} F^{k,\ell} \cdot F^{k,\ell} > 0 \end{pmatrix},$$

is positive definite in the block corresponding to ran$Q_k$. Recall from Sec. 1.2 that one can think of evolution due to $L_k$ (constructed out of $\{F^{k,\ell}\}_{\ell}$) as being composed of a deterministic part, the anticommutator generated by $-K^k$, and a recycling part, generated by applications of the recycling term $F^{k,\ell} \cdot F^{k,\ell}$. Due to the structure of $F^{k,\ell}$, the recycling term always takes states out of ran$Q_k$ and maps them into ker$Q_k$. Due to $K^k > 0$, the deterministic part always decays anything in ran$Q_k$. To show this, one can consider the change in population on ran$Q_k$,

$$\Tr\{Q_k \dot{\rho}\} = \Tr\{Q_k L_k(\rho)\} = -2\Tr\{K\rho\} < 0,$$

where we have used the structure of $F^{k,\ell}$ and the fact that $K > 0$ on ran$Q_k$. A similar calculation, now using the four-corners (¶) decomposition, is now going to be done for the full Lindbladian $L = \sum_k L_k$.

Each dissipator $L_k$ drives states to the ground states subspace of its corresponding $H_k$ according to the above procedure. The full generator $L = \sum_k L_k$ should then drive states into ¶ — the intersection of the ground state spaces of all $H_k$, i.e., the ground state subspace of $H$. To prove this, we show that all states initially in ¶ decay to zero in ¶. The change in population in ¶ is

$$\Tr\{\dot{\rho}\} = \langle \rho | 1 | \mathcal{P}_{\Phi} L | \rho \rangle,$$

where $\mathcal{P}_{\Phi}$ is the superoperator projection onto ¶. We need to examine how the $F^{k,\ell}$’s decompose under the new block structure. Since ¶ is their joint kernel, all jumps annihilate states in ¶ ($F^{k,\ell}_{\Phi} = 0$) and no jump can take states out ($F^{k,\ell}_{\Phi} = 0$). Therefore, $F^{k,\ell} = F^{k,\ell}_{\Phi}$.
Since there is no Hamiltonian, this implies that the jumps in $L$ satisfy Thm. 2, meaning that $L$ cannot take states in $\mathfrak{m}$ to $\mathfrak{n}$. Applying this yields
\[
\text{Tr}\{\dot{\rho}\} = \langle I | \mathcal{P}_\mathfrak{m} L | \rho \rangle = \langle I | \mathcal{P}_\mathfrak{m} \mathcal{L} \mathcal{P}_\mathfrak{m} | \rho \rangle = \langle I | L \mathcal{P}_\mathfrak{m} | \rho \rangle .
\] (3.36)

The relevant piece $L_\mathfrak{m}$ (2.8d) is
\[
L_\mathfrak{m}(\rho_\mathfrak{m}) = \sum_{k,\ell} f^{k,\ell}_{\mathfrak{m}} f^{k,\ell \dagger}_{\mathfrak{m}} - \sum_k \left\{ k^k \rho_\mathfrak{m} \right\} .
\] (3.37)

The Hamiltonian $K^k$ consists of two pieces,
\[
K^k = \frac{1}{2} \sum_\ell \left( F^{k,\ell \dagger} F^{k,\ell} \right)_\mathfrak{m} = \frac{1}{2} \sum_\ell F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m} + F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m} .
\] (3.38)

The $F_\mathfrak{m}$ piece of $K^k$ conspires with the recycling term $F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m}$ and creates a bona fide Lindbladian with jump operator $F_\mathfrak{m}$. Since Lindbladians are trace-preserving, the $F_\mathfrak{m}$ pieces do not contribute to $\text{Tr}\{\dot{\rho}\}$. The $F_\mathfrak{m}$ has no corresponding recycling term, so that part is not of Lindblad form. The $F_\mathfrak{m}$ parts instead give us a potential decrease in trace:
\[
\text{Tr}\{\dot{\rho}\} = -\sum_{k,\ell} \text{Tr} \left\{ F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m} \rho_\mathfrak{m} \right\} .
\] (3.39)

Since $F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m} \geq 0$ and $\rho_\mathfrak{m} \geq 0$, each term in the above sum is $\leq 0$. We show that it is $< 0$, meaning that everything in $\mathfrak{m}$ decays. Since $\mathfrak{m}$ is the intersection of the kernels of all $F^{k,\ell}$ and since we are in $\mathfrak{n}$, there exists at least one $F^{j,m}_\mathfrak{m}$ for which $\text{Tr} \left\{ F^{j,m \dagger} F^{j,m}_\mathfrak{m} \rho_\mathfrak{m} \right\} > 0$. This provides a lower bound on the decay,
\[
\text{Tr}\{\dot{\rho}\} \leq -\text{Tr} \left\{ F^{j,m \dagger} F^{j,m}_\mathfrak{m} \rho_\mathfrak{m} \right\} < 0 ,
\] (3.40)

ensuring that all states initially in $\mathfrak{m}$ decay into $\mathfrak{n}$. Since the above is true for all $\rho_\mathfrak{m}$, $\sum_{k,\ell} F^{k,\ell \dagger}_\mathfrak{m} F^{k,\ell}_\mathfrak{m} > 0$ on $\mathfrak{m}$ and all steady states of $L$ are in $\mathfrak{m}$. □

The above procedure has been used in several specific cases throughout the literature, for example in obtaining stabilizer quantum error-correcting code states \[101, 163, 200\] or ground states of the AKLT model \[163, 321\]. However, the above recipe for $F^{k,\ell}$ is not unique. Instead of having each jump be an isometry from part of the range to the kernel of $H_{k,\ell}$, forming the shape in eq. (3.33), one can instead have $F^k$ (for each $k$) act like a ladder operator. In other words,
\[
F^k = \begin{pmatrix}
0 & f^{k,1} & 0 & \cdots & 0 \\
0 & f^{k,2} & 0 & \vdots \\
0 & 0 & \ddots & 0 \\
0 & \cdots & 0 & f^{k,\lceil D/d \rceil} \\
0 & \cdots & 0 & 0
\end{pmatrix} ,
\] (3.41)
where the upper left block is the $d$-dimensional $\ker Q_k$ and lower right block is the $D$-dimensional $\text{ran} Q_k$. To ensure that $f^{k,p}$ transfers all states in the block below it into the block to the left, we need to have $f^{k,p} f^{k,p+1} > 0$ when restricted to the block below $f^{k,p}$. If done this way, only one jump per each $k$ is sufficient. We note that the conserved quantities of $L = \sum_k L_k$ are complicated since $L_k \neq 0$ for both recipes, but they can nevertheless be determined by Thm. 4. One nice example of these types of jumps was used to stabilize the ground states of the Kitaev Majorana wire Hamiltonian [105] (see also [38]). Conversely, there exists an algorithm [279] (see also [310]) which, given a subspace, tries to decompose a jump operator into a structure similar to the above in order to check whether all states converge to failing if the $f$ in the lower right corner is not positive definite.

Of course, other stabilization schemes exist besides those described above [e.g., [288], eq. (6)]. A quite elegant family of schemes is based on the idea that, given a quantum channel $E$, the Lindbladian

$$L = E - I$$

(3.42)

is one whose semigroup $e^{tL}$ has the same fixed points as that of $E$ [217, 307]. Therefore, given a cleverly chosen $E$ which stabilizes some desirable states, the asymptotic projection $P_\infty = \lim_{t \to \infty} e^{tL}$ generated by the above $L$ will also stabilize those states. In another work, it is shown that only one jump operator is required to stabilize any state [284]. Note that Hamiltonian-based feedback control can also be used to make sure that the states of interest are stabilized ([281], Thm. 2). Since all local gapped Hamiltonians can be with approximated with ones that are frustration-free [132], the above recipes allow for stabilization of states close to any phase of matter that can be generated by such Hamiltonians. Extensions to stabilization of mixed states using frustration-free Lindbladians can be found in Ref. [146].

The bad news regarding all of these preparation schemes is that, for “exotic” states such as 2D topological phases and assuming some notion of locality for the jumps, the speed of convergence (i.e., inverse of the dissipative gap $\Delta_{dg}$) increases with the length scale $L$ associated with the system size. For example, an optimal toric code stabilizer [101] has gap $\Delta_{dg} = O(1/L)$, meaning that the system has arbitrarily small excitations above the steady state in the thermodynamic limit.
“I don’t always integrate, but when I do, I integrate by parts.”

– Nicholas Read

TIME-DEPENDENT PERTURBATION THEORY

In this chapter, we apply the four-corners decomposition to the first-order terms in ordinary time-dependent perturbation theory [10]. In Sec. 4.1, we determine that the first-order correction within As(H) is of Hamiltonian form and the energy scale of the first-order term causing leakage out of As(H) is governed by the dissipative gap of $L_\mathfrak{b}$. We extend these conclusions to jump operator perturbations $F^\ell \to F^\ell + f^\ell$. In Sec. 4.2, we determine the full Dyson expansion to all orders exactly, given a perturbation which slowly ramps up to a constant and an initial state that is a steady state of the unperturbed $L$. We conclude in Sec. 4.3 by making contact with previously studied topics: dark states, geometric linear response, the Dyson series for the case of an unperturbed $L$ with a unique steady state, quantum Zeno dynamics, and the effective operator formalism.

4.1 DECOMPOSING THE KUBO FORMULA

Let us assume that time evolution is governed by a time-independent Lindbladian $L$ and that the system is perturbed as

$$L \to L + g(t)O,$$

where the perturbation superoperator $O$ is multiplied by a slowly ramping up time-dependent factor $g(t)$ from time $-\infty$ to a time $t$. The Lindbladian-based Kubo formula [33, 34, 53, 54, 76, 141, 199, 264, 290, 297] is derived analogously to the Hamiltonian formula, i.e., it is a leading-order Dyson expansion of the full evolution.\(^1\) The main difference is that the derivation is performed in the superoperator formalism. However, the superoperator formalism lends a natural interpretation of the terms in the superoperator Dyson series. As a result, we use the intuitiveness of the terms to justify the expansion, omitting the quite standard technical modifications needed to obtain them.

The first term in such a series acts on a state $\rho$ as [290]

$$\mathcal{T}(1)^{(1)}|\rho\rangle = \int_0^t d\tau g(\tau) e^{(t-\tau)L} O e^{\tau L} |\rho\rangle.$$  \hspace{1cm} (4.2)

\(^1\) We note that there exists an adiabatic derivation as well [88], which is not addressed here.
We remind the reader that we use vectorized notation for matrices and the Hilbert-Schmidt inner product \( \langle A | \rho (t) \rangle \equiv \text{Tr} \{ A^\dagger \rho (t) \} \) (see Ch. 1.6). This term offers an intuitive interpretation if one thinks of the system as evolving from the right side of the expression to the left. Reading the integrand from right to left, the initial state \( \rho \) evolves under the unperturbed Lindbladian \( \mathcal{L} \) to time \( \tau \), is perturbed by \( \mathcal{O} \), and then evolves under \( \mathcal{L} \) from \( \tau \) to \( t \). The integral represents a sum over all possible acting times \( \tau \) of the perturbation. Applying an observable \( \langle A \rangle \) from the left is equivalent to evaluating said observable at time \( t \). If we now also make the assumption that we are in an initially steady state \( \rho = \rho_\infty \), the right-most exponential \( e^{i\mathcal{L} \tau} \) is removed since \( \mathcal{L} | \rho_\infty \rangle \rangle = 0 \). Since we do not have any evolution until the time of the perturbation with such an assumption, we can extend the initial time from 0 to \( -\infty \). These manipulations then produce the Kubo formula \[ 165 \] 
\[ \langle\langle A | T_t^{(1)} | \rho_\infty \rangle \rangle = \int_{-\infty}^{t} d\tau g (\tau) \langle\langle A | e^{i(t-\tau)\mathcal{L} \mathcal{O}} | \rho_\infty \rangle \rangle . \] (4.3)

We proceed to apply the four-corners decomposition to this formula. However, before doing so, let us show that this is indeed the original Kubo formula.

**Hamiltonian case** 
Let us set \( \mathcal{L} = \mathcal{H} = -i[H, \cdot] \), \( \mathcal{O} = -i[V, \cdot] \) for a Hamiltonian \( V \), and massage eq. (4.3) into standard form. For that, define \( \mathcal{O}(t) \equiv e^{i\mathcal{H} t} \mathcal{O} e^{-i\mathcal{H} t} = e^{-i\mathcal{H} t} (\mathcal{O}) \) and recall that \([H, \rho_\infty]\) = 0 since \( \rho_\infty \) is generically a superposition of projections on eigenstates of \( H \). We can then commute \( e^{i\mathcal{H} t} \) with \( \rho_\infty \) and cyclically permute under the trace to obtain 
\[ \langle\langle A | T_t^{(1)} | \rho_\infty \rangle \rangle = \frac{1}{i} \int_{-\infty}^{t} d\tau g (\tau) \text{Tr} \{ [A (t-\tau), V] \rho_\infty \} , \] (4.4)

recovering the usual time-ordered commutator expression.

The perturbations considered here are Hamiltonian and jump operator perturbations of \( \mathcal{L} \) (2.2), respectively

\[ H \rightarrow H + g (t) V \]  
(4.5a)  
\[ F^\ell \rightarrow F^\ell + g (t) f^\ell \]  
(4.5b)

[for \( V, f^\ell \in \text{Op}(H) \) and \( V^\dagger = V \)]. It will be shown that both generate unitary evolution within all As(H) and leakage caused by both does not take states into \( \mathbb{H} \). We first handle the Hamiltonian case first for simplicity,

\[ \mathcal{O} = -i [V, \cdot] \equiv V, \]  
(4.6)

returning to the jump case in Sec. 4.1.3.

We now use four-corners projections \( P_{\mathbb{H}} \) to partition eq. (4.3). Due to the no-leak property (LP), we have \( P_{\mathbb{H}} \mathcal{V} P_{\mathbb{H}} = 0 \). Remembering that the Lindbladian is block upper-triangular in the four-corners partition [see eq. (2.3)], it follows that \( e^{t\mathcal{L}} \) is also block upper-triangular. We do not
make any assumptions on the observable: $A = A_\mathcal{A} + A_\mathcal{B} + A_\mathcal{C}$. Further decomposing the first term using the asymptotic projection $P_\infty$ from eq. (2.61a) and its complement $Q_\infty \equiv I - P_\infty$ yields

$$
\langle\langle A\mid T_1^{(1)}\mid \rho_\infty \rangle\rangle = \int_{-\infty}^{t} d\tau g(\tau) \langle\langle A_\mathcal{B}\mid e^{(t-\tau)H_\infty}P_\infty V\mid \rho_\infty \rangle\rangle \tag{4.7A}
$$

$$
+ \int_{-\infty}^{t} d\tau g(\tau) \langle\langle A_\mathcal{B}\mid e^{(t-\tau)\mathcal{L}Q_\infty P_\infty V}\mid \rho_\infty \rangle\rangle \tag{4.7B}
$$

$$
+ \int_{-\infty}^{t} d\tau g(\tau) \langle\langle A_\mathcal{C}\mid e^{(t-\tau)\mathcal{L}P_\mathcal{B}V}\mid \rho_\infty \rangle\rangle. \tag{4.7C}
$$

The terms differ by which parts of $V$ perturb $\rho_\infty$ and also which parts of $A$ “capture” the evolved result. The three relevant parts of $A$ correspond to the three labels in Fig. 4.1. One can readily see that $A_\mathcal{B}$ is irrelevant to this order due to (LP1). The term (4.7A) consists of perturbing and evolving within the asymptotic subspace $\mathcal{A}$, shaded gray in the figure. The effect of the perturbation within $\text{As}(H)$ is $P_\infty V P_\infty$ (shown in Sec. 4.1.1 to be of Hamiltonian form), and $H_\infty$ is the part of the unperturbed $\mathcal{L}$ that generates unitary evolution within $\text{As}(H)$. The term (4.7A) therefore most closely resembles the traditional Hamiltonian-based Kubo formula. The remaining two terms quantify leakage out of $\text{As}(H)$ and contain non-Hamiltonian contributions. The term (4.7B) consists of perturbing into regions $\mathcal{B}$ and $\mathcal{C}$ in Fig. 4.1, but then evolving under $P_\mathcal{B}e^{\mathcal{L}P_\mathcal{B}}$ strictly into region $\mathcal{B}$ (since $P_\mathcal{B}e^{\mathcal{L}Q_\infty} = 0$). The term (4.7C) consists of perturbing into region $\mathcal{C}$ and remaining there after evolution due to $P_\mathcal{B}e^{\mathcal{L}P_\mathcal{B}}$. This term is eliminated if $A_\mathcal{B} = 0$, i.e., if the observable is strictly in $\mathcal{B}$.

**DFS Case**

Recall that in this case $\mathcal{B}$ is a DFS ($P_\mathcal{B}P_\mathcal{B} = P_\mathcal{B}$), and we do not assume it is stationary ($H_\infty \neq 0$). From eq. (2.10), we can see that $\mathcal{L}$ cannot take any coherences in $\mathcal{B}$ back into the DFS.
Therefore, the interference term (4.7B) is eliminated and the response formula reduces to

\[
\langle \langle A | T_1^{(1)} | \rho_\infty \rangle \rangle = \int_{-\infty}^{t} d\tau g(\tau) \langle \langle A | e^{(t-\tau)H_\infty} P_0 V P_0 | \rho_\infty \rangle \rangle + \int_{-\infty}^{t} d\tau g(\tau) \langle \langle A | e^{(t-\tau)\mathcal{L} P_0 | V | \rho_\infty \rangle \rangle. \quad (4.8A)
\]

If furthermore \( A_{\mathcal{O}} = 0 \), there are no interference terms coming from outside of the DFS and the Lindbladian linear response reduces to the purely Hamiltonian-based term (4.8A). Such a simplification can also be achieved when \( V_{\mathcal{O}} = 0 \), which implies that the Hamiltonian does not take \( \rho_\infty \) out of the DFS to begin with (\( P_0 V P_0 = 0 \)).

For the rest of this chapter, we set \( H_\infty = 0 \) and have the time-dependent part \( g(t) \) of our perturbation ramp up to a constant at \( t = 0 \): \[
g(t) = \lim_{\eta \to 0} e^{\eta t \Theta(-t)} = \begin{cases} \lim_{\eta \to 0} e^{\eta t} & t < 0 \\ 1 & t \geq 0 \end{cases}, \quad (4.9)
\]

where \( \Theta(t) \) is the Heaviside step function. As a result, the integrals in the Kubo formula can be performed exactly (see Sec. 4.2.1), simplifying the formula to\(^2\)

\[
\langle \langle A | T_1^{(1)} | \rho_\infty \rangle \rangle = \left( t + \frac{1}{\eta} \right) \langle \langle A | P_0 V P_0 | \rho_\infty \rangle \rangle - \langle \langle A | \mathcal{L}^{-1} V | \rho_\infty \rangle \rangle. \quad (4.10)
\]

This version, which is true for more general perturbations in Lindblad form \( (\mathcal{V} \to \mathcal{O}) \), invites an analogy with degenerate perturbation theory, in which \( P_0 V P_0 \) is the superoperator analogue of the perturbation projected onto the subspace of interest and \( \mathcal{L}^{-1} \mathcal{V} \) is an analogue of the term governing corrections to the wavefunction and including the famous “energy denominator”. The \( 1/\eta \) factor, an “infinity”, is the (unfortunate) consequence of the perturbation acting on the steady-state subspace for an infinite amount of time during the time interval \( (-\infty, 0] \) and in the \( \eta \to 0 \) limit.\(^3\) While this choice of \( g(t) \) creates this uncomfortable, but explainable, infinity within \( \text{As}(H) \), it allows us to write the leakage term strictly in terms of

\[
\mathcal{L}^{-1} = Q_\infty \mathcal{L}^{-1} Q_\infty \equiv -\int_{-\infty}^{t} d\tau g(\tau) e^{(t-\tau)\mathcal{L} Q_\infty} = -\int_{0}^{\infty} d\tau e^{\tau \mathcal{L} Q_\infty}. \quad (4.11)
\]

This pseudo-inverse \( (\mathcal{L}^{-1} \mathcal{L} = \mathcal{L} \mathcal{L}^{-1} = Q_\infty) \) is also the inverse of all invertible parts in the Jordan normal form of \( \mathcal{L} \) ([314], Appx. D). In the context of finite matrices, it is called the Drazin pseudoinverse \([ [49], \text{eq. (41)} \) for \( k = 1 \). In the Hamiltonian context, this is the familiar Green’s

\(^2\) We also note that one can have a sudden perturbation \( g(t) = \Theta(t) \) [314]. In that case, one does not obtain the \( 1/\eta \) term, but the leakage term now contains a “ringdown” contribution due to the sudden onset of the perturbation: \( \mathcal{L}^{-1} \mathcal{V} \to (e^{\mathcal{L} t} - 1) \mathcal{L}^{-1} \mathcal{V} \). None of the results in Sec. 4.1 depend on which \( g(t) \) one picks, but the all-order Dyson series in Thm. 6 relies on using a slowly ramping up perturbation in order to avoid the ringdown terms.

\(^3\) Note that, when evaluating the response in frequency space, a more careful treatment of \( \eta \) may be necessary [66].
function in Op(H) (i.e., Liouville space [199]). In the context of linear operators, this is simply the resolvent of \( L \) at \( z = 0 \), i.e.,

\[
L^{-1} = -\frac{1}{2\pi i} \oint \frac{dz}{z} (L - z)^{-1},
\]

where \( \Gamma \) is the contour which encircles zero and no other points in the spectrum of \( L \) [see [25], eq. (70) or [150], Ch. 3, eq. (6.23)]. Note that \( L^{-1} \) is not the Moore-Penrose pseudoinverse; while \( L^{-1} \) inverts the Jordan normal form of \( L \), the Moore-Penrose inverse inverts the diagonal matrix in the singular-value decomposition of \( L \). While \( L^{-1} \) appears naturally in the above formulation, the Moore-Penrose inverse can be used to study time-independent Lindbladian perturbation theory [172]. Since both pseudoinverses are basically identical for diagonalizable \( L \), we anticipate that differences between the formalisms (if any) should arise only in those parts of \( L \) which are not diagonalizable.

In the next Subsections, we use the no-leak and clean-leak properties to determine that evolution within \( \text{As}(H) \) is of Hamiltonian form and to quantify the leakage scale of the second term in eq. (4.10).

### 4.1.1 Evolution within \( \text{As}(H) \)

Let us focus on the term \( \mathcal{P}_\infty V \mathcal{P}_\infty \) from eq. (4.10), which quantifies the effect of the perturbation within \( \text{As}(H) \). A swift application of the no-leak and clean-leak properties (LP1-LP2) allows us to substitute \( \mathcal{P}_\Psi = \mathcal{P}_\infty \mathcal{P}_\infty \) for \( \mathcal{P}_\infty \). Recalling that \( \mathcal{P}_\Psi V \mathcal{P}_\Psi = 0 \) and that \( \mathcal{P}_\Psi V \mathcal{P}_\Psi \) is strictly acting on states \( \rho_\infty \in \text{As}(H) \) yields

\[
\mathcal{P}_\Psi V \mathcal{P}_\Psi = \mathcal{P}_\Psi V \mathcal{P}_\Psi = \mathcal{P}_\Psi V \mathcal{P}_\Psi. \tag{4.13}
\]

It turns out that this first-order effect of the perturbation within \( \text{As}(H) \) is always of Hamiltonian form, for some effective Hamiltonian that we determine now.

**DFS case**  Here, we can immediately read off the effective Hamiltonian. Since \( \mathcal{P}_\Psi = \mathcal{P}_\Phi \) for the DFS case,

\[
\mathcal{P}_\Psi V \mathcal{P}_\Psi = -i [V_{\Phi}, \cdot] \tag{4.14}
\]

with \( V_{\Phi} \) the perturbation projected onto the DFS.

**NS case**  In this case, we have to use the formula for \( \mathcal{P}_\Psi \) from eq. (2.61b), re-stated below:

\[
\mathcal{P}_\Psi = \mathcal{P}_\text{dfs} \otimes |q_{\text{ax}}\rangle \langle P_{\text{ax}}|, \tag{4.15}
\]

with \( \mathcal{P}_\text{dfs} (\cdot) = P_{\text{dfs}} \cdot P_{\text{dfs}} \) being the superoperator projection on the DFS part, \( P_{\text{ax}} \) being the operator projection on the auxiliary part, and \( P = \mathcal{P}_\text{dfs} \otimes P_{\text{ax}} \). Direct multiplication yields

\[
\mathcal{P}_\Psi V \mathcal{P}_\Psi = \langle P_{\text{ax}} | V | q_{\text{ax}} \rangle \otimes |q_{\text{ax}}\rangle \langle P_{\text{ax}}|, \tag{4.16}
\]

where the evolution within the auxiliary part is trivial and evolution within the DFS part is generated by the effective DFS Hamiltonian \( W \):

\[
\langle P_{\text{ax}} | V | q_{\text{ax}} \rangle = -i [\text{Tr}_{\text{ax}} (q_{\text{ax}} V_{\Phi}), \cdot] \equiv -i [W, \cdot]. \tag{4.17}
\]
To better reveal the effect of $\varrho_{ax}$, it is worthwhile to express $V_{\mathfrak{B}}$ as a sum of tensor products of various DFS and auxiliary Hamiltonians: $V_{\mathfrak{B}} = \sum_i V^i \otimes V_{ax}^i$. The effective Hamiltonian then becomes

$$W = \sum_i \text{Tr}_{ax}\{\varrho_{ax} V_{ax}^i\} V^i.$$  \hfill (4.18)

In words, $P_{\mathfrak{B}} V P_{\mathfrak{B}}$ is a linear combination of Hamiltonian perturbations $V^i$ on the DFS, with each perturbation weighted by the expectation value of the corresponding auxiliary operator $V_{ax}^i$ in the state $\varrho_{ax}$.

### 4.1.2 Leakage out of As(H)

Now we can apply the clean-leak property ($LP_2$) to narrow down those eigenvalues of $\mathcal{L}$ which are relevant in characterizing the scale of the leakage term $\mathcal{L}^{-1} V$ from the simplified Kubo formula (4.10). By definition (4.11), $\mathcal{L}^{-1}$ has the same block upper-triangular structure as $\mathcal{L}$ from eq. (2.3). This fact conspires with $P_{\mathfrak{B}} V P_{\mathfrak{B}} = 0$ to allow us to ignore $L_{\mathfrak{B}}$ and write

$$L^{-1} V |\rho_{\infty}\rangle\rangle = L^{-1}_{\mathfrak{B}} V |\rho_{\infty}\rangle\rangle.$$  \hfill (4.19)

Therefore, the relevant gap is the nonzero eigenvalue of $L_{\mathfrak{B}}$ with the smallest absolute value. However, we now show how the spectrum of $L_{\mathfrak{B}}$ is actually contained in the spectrum of $L_{\mathfrak{B}} + L_{H}$. Recalling the block upper-triangular structure of $\mathcal{L}$ from eq. (2.3), one can establish that its eigenvalues must consist of eigenvalues of $L_{\mathfrak{B}}$, $L_{\mathfrak{H}}$, and $L_{\mathfrak{H}}$. However, evolution of the two coherence blocks is decoupled, $L_{\mathfrak{B}} = L_{\mathfrak{B}} + L_{\mathfrak{H}}$ (see Sec. 2.1), and eigenvalues of $L_{\mathfrak{B}}$ come in pairs. Therefore, one can then define the effective dissipative gap $\Delta_{\text{edg}}$ to be the nonzero eigenvalue of $L_{\mathfrak{B}} + L_{\mathfrak{H}}$ with the smallest absolute value. If we want leakage to be suppressed, we want $\Delta_{\text{edg}}$ to be as large as possible.

**DFS case**  
Assume that we have a DFS case: all of $\mathfrak{B}$ evolves unitarily, so $L_{\mathfrak{B}} = H_{\infty}$ does not have a dissipative gap. In that case, we can omit $\mathfrak{B}$ from eq. (4.19) and simplify it to

$$L^{-1} V |\rho_{\infty}\rangle\rangle = L^{-1}_{\mathfrak{B}} V |\rho_{\infty}\rangle\rangle.$$  \hfill (4.20)

Therefore, the effective dissipative gap $\Delta_{\text{edg}}$ is just the dissipative gap of $L_{\mathfrak{B}}$.

### 4.1.3 Jump operator perturbations

Having covered Hamiltonian perturbations, let us return to jump operator perturbations of the Lindbladian (2.2). Recall from eq. (4.5b) that

$$F \rightarrow F + g(t) \, f$$  \hfill (4.21)

with $g(t)$ a ramping function and $f \in \text{Op}(H)$, not necessarily Hermitian. It was first shown in Ref. [315] that such perturbations actually induce unitary evolution on NS blocks of those Lindbladians which do not possess a nontrivial decaying space ($P = I$). Here we extend this interesting result to cases where $P \neq I$, thereby covering all $\mathcal{L}$. Namely, just like Hamiltonian
perturbations $\mathcal{V}$, jump operator perturbations induce unitary evolution within $\mathcal{A}_\Sigma(H)$ and the leakage scale associated with them is still $\Delta_{\text{edg}}$.

Returning to eq. (4.6), the action of the perturbation to first order in $g$ is

$$\mathcal{O}(\rho) \equiv \mathcal{Y}(\rho) = \kappa \left( F \rho f^\dagger + H.c. - \frac{1}{2} \left\{ f^\dagger F + F^\dagger f, \rho \right\} \right),$$  \hspace{1cm} (4.22)

where $\kappa$ is the rate corresponding to the jump operator $F$ (we ignore the index $\ell$ for clarity). We hope to invoke the clean-leak property (LP2) once again, but the first term on the right-hand side of the above acts simultaneously and non-trivially on both sides of $\rho$. There is thus a possibility that one can reach $\rho$ when acting with $\mathcal{Y}$ on a steady state. However, the condition $F = 0$ from Thm. 2 implies that $\mathcal{P}_\infty(F \rho f^\dagger)$ is zero for all $f$, so one can still substitute $\mathcal{P}_\Psi$ for $\mathcal{P}_\infty$:

$$\mathcal{P}_\infty \mathcal{Y} \mathcal{P}_\infty |\rho_\infty\rangle = \mathcal{P}_\Psi \mathcal{Y} \mathcal{P}_\Psi |\rho_\infty\rangle.$$  \hspace{1cm} (4.23)

Furthermore, the fact that $\mathcal{P} \mathcal{Y} \mathcal{P}_\infty = 0$ allows us to ignore $\mathcal{P}_\infty$ in determining the leakage energy scale associated with these jump operator perturbations. We finish with calculating the corresponding effective Hamiltonian for the most general cases.

**NS case** Having eliminated the influence of the decaying subspace we can now repeat the calculation done for Hamiltonian perturbations using the NS projection (4.15), yielding

$$\mathcal{P}_\Psi \mathcal{Y} \mathcal{P}_\Psi = \langle \langle p_{\alpha x}|\mathcal{Y}|q_{\alpha x}\rangle \otimes |q_{\alpha x}\rangle \langle p_{\alpha x}|.$$  \hspace{1cm} (4.24)

After some algebra, the DFS part reduces to Hamiltonian form [315]: $\langle \langle p_{\alpha x}|\mathcal{Y}|q_{\alpha x}\rangle = -i[Y, \cdot]$ where

$$Y \equiv \frac{i}{2} \kappa \text{Tr}_{\alpha x} \left\{ q_{\alpha x} \left( F^\dagger f^\dagger f - f^\dagger F f \right) \right\}.$$  \hspace{1cm} (4.25)

**Multi-block case** We now sketch the calculation of both Hamiltonian and jump operator perturbations, $\mathcal{O} = \mathcal{V} + \mathcal{Y}$, for the most general case of housing multiple NS blocks. Once again, we can get rid of the decaying subspace and substitute $\mathcal{P}_\Psi$ for $\mathcal{P}_\infty$. In addition, since $\mathcal{P}_\Psi$ does not have any presence except within the (gray) NS blocks of $\mathcal{H}$ [see Fig. 1.2], $\mathcal{P}_\Psi$ does not project onto any coherences between the NS blocks. The contributing part of $\mathcal{P}_\infty \mathcal{O} \mathcal{P}_\infty$ thus consists of the Hamiltonian and jump operator perturbations projected to each NS block. Combining the effective Hamiltonians arising from $\mathcal{V}$ and $\mathcal{Y}$ [respectively eqs. (4.17) and (4.25)], the effective evolution within the DFS part of each NS block (indexed by $\alpha$) is generated by the Hamiltonian

$$X^{(\alpha)} \equiv \text{Tr}_{\alpha x}^{(\alpha)} \left\{ q_{\alpha x}^{(\alpha)} \left( V_{\alpha} + i\kappa (F^\dagger f^\dagger f - f^\dagger F f) \right) \right\}.$$  \hspace{1cm} (4.26)

The unprojected Hamiltonian $X \equiv V + \frac{i}{2} \kappa (F^\dagger f - f^\dagger F)$ is exactly the operator resulting from joint variation of the Hamiltonian and jump operators of $\mathcal{L}$ ([25], Thm. 5).
4.2 EXACT ALL-ORDER DYSON EXPANSION

Let us now return to general Lindbladian perturbations $\mathcal{O}$ and study how higher-order terms in the Dyson series are also naturally interpreted from right to left. For example, the second-order term acts on a states in $\text{As}(\mathcal{H})$ as \[ T_2^{(2)} \rho_0 \equiv \int_{-\infty}^{t} g(\tau_2) d\tau_2 e^{(t-\tau_2)\mathcal{L}} \mathcal{O} \int_{-\infty}^{\tau_2} g(\tau_1) d\tau_1 e^{(\tau_2-\tau_1)\mathcal{L}} \mathcal{O} \rho_0, \] (4.27)

and one can see that it is a sum over all possible pairs of times $\tau_1 \leq \tau_2$ at which the perturbation can be applied. The $\rho_0$ on the left means that the initial state is necessarily in $\text{As}(\mathcal{H})$, and we study the full Dyson series with this restriction from now on. The full time-ordered ($T$) evolution operator is expanded as

\[ T e^{\int_{t=\infty}^{t} dt (\mathcal{L}+g(\tau)\mathcal{O})} \rho_0 = \sum_{N=0}^{\infty} T_t^{(N)} \rho_0 = \rho_0 + T_t^{(1)} \rho_0 + T_t^{(2)} \rho_0 + \cdots, \] (4.28)

where we have already seen the first two terms $T_t^{(1)}$ (4.2) and $T_t^{(2)}$ (4.27). The $N$th order term $T_t^{(N)} \rho_0$ consists of $N$ applications of the perturbation $g(\tau)\mathcal{O}$ at times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$ with evolution generated by the unperturbed term $\mathcal{L}$ between those times. Let us define the operator which acts with the perturbation $g(\tau)\mathcal{O}$ at time $\tau_n$, evolves with the unperturbed $\mathcal{L}$ from $\tau_n$ to $\tau_m$, and sums up over all possible $\tau_n$:

\[ S_{m,n} \equiv \int_{-\infty}^{\tau_m} g(\tau_n) d\tau_n e^{(\tau_m-\tau_n)\mathcal{L}} \mathcal{O}. \] (4.29)

Also, let $\tau_t \equiv t$ so that $S_{t,n}$ is an integral over $\tau_n$ from $-\infty$ to $t$. Then, the $N$th order term can be expressed as a convolution of $S$’s,

\[ T_t^{(N)} = S_{t,N-1} S_{N-1,N-2} \cdots S_{3,2} S_{2,1}. \] (4.30)

By convolution, we mean that $S_{n,n-1}$ is a function of the variable $\tau_n$ which is integrated out by $S_{n+1,n}$. This way, the $T$’s can be defined recursively:

\[ T_t^{(N+1)} = S_{t,N+1} T_t^{(N)}. \] (4.31)

We return to the case of $g(\tau)$ slowly ramping up to a constant, as in eq. (4.9), and continue analyzing terms for $N > 1$. While the $1/\eta$ infinity is ever-present in the entire expansion, eq. (4.9) allows us to compute all of the integrals in the series (4.28) exactly. We state the result first and prove it in the next Subsection.
Theorem 6 (Exact all-order Dyson expansion). The $N$th order term in the Dyson series (4.28), given a slowly ramping up perturbation (4.9) and an initial state in $\text{As}(H)$, is

$$T^{(N)}_t = \left(1 + \frac{\partial_t}{\eta}\right) \sum_{M=0}^{N} \frac{t^{N-M}}{(N-M)!} \sum_{\lambda \in \text{Cat}^N_M} \prod_{n=1}^{N} \mathcal{X}(\lambda_n),$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ is a sequence of $N$ nonnegative integers, $\text{Cat}^N_M$ is a particular set of such sequences,

$$\text{Cat}^N_M = \left\{ \lambda \text{ such that } \sum_{n=1}^{l} \lambda_{n+1-n} \leq \min \{l, M\} \quad 1 \leq l < N \right\},$$

and the operator that is put into the product for each element $\lambda_n$ is

$$\mathcal{X}(\lambda_n) = \begin{cases} -\mathcal{L}^{-\lambda_n} \mathcal{O} & \lambda_n > 0 \\ \mathcal{P}_0 \mathcal{O} & \lambda_n = 0 \end{cases}. \quad (4.34)$$

The pseudoinverse $\mathcal{L}^{-1}$ is defined in eq. (4.11) and it is implied that $\mathcal{L}^{-1}$ and its powers act on the range of $\mathcal{L}$, i.e., $\mathcal{L}^{-1} = \mathcal{L}^{-1} Q_\omega$. The sequences and their corresponding terms are presented in Tab. 4.1 up to $N = 4$. Recall that $(1 + \frac{1}{\eta} \partial_t) t^{N-M} / (N-M)!$ prepends and $\mathcal{P}_0$ appends each term.

The sequences readily lend themselves to a diagrammatic interpretation. Consider all paths on a two-dimensional grid from a point $(N, N)$ down to the point $(0, M)$ where one can only perform only the following types of steps: one step to the left ($\mathcal{O}$), no steps at all ($\mathcal{P}_0$), or $\lambda_n$ steps down ($-\mathcal{L}^{-\lambda_n}$ for $\lambda_n > 0$). Using these rules for the steps allows one to associate each sequence $\lambda \in \text{Cat}^N_M$ (4.33) with its own path from $(N, N)$ to $(0, M)$. It should be clear that each path contains exactly $N$ steps to the left (i.e., $N$ instances of $\mathcal{O}$) and $M$ steps down ($\sum_{n} \lambda_n = M$). Such a diagrammatic interpretation, along with its close connection to well-established perturbative methods [59, 70, 153], will be considered in a future publication [7].

Counting the sequences $\lambda \in \text{Cat}^N_M$ and the corresponding set of all sequences required to construct $T^{(N)}$,

$$\text{Cat}^N = \bigcup_{M=0}^{N} \text{Cat}^N_M,$$

reveals a quite interesting connection to the Catalan numbers — a sequence of numbers which has 215 different combinatorial interpretations [274]! The number of possible sequences $\lambda$ of length $N$ that sum up to $M$ is

$$|\text{Cat}^N_M| = \frac{(N+M)! (N+1-M)}{M! (N+1)!} \equiv C(N, M),$$

where $C(N, M)$ is an entry in Catalan’s triangle [160] and we define $C(N) \equiv C(N, N)$. As a result of the properties of Catalan’s triangle (see Tab. 4.2), the total number of terms for order $N$ is the $N + 1$st Catalan number:

$$\sum_{M=0}^{N} C(N, M) = \frac{(2N+2)!}{(N+1)! (N+2)!} = C(N+1). \quad (4.37)$$
| \(N\) | \(M\) | \(\text{Cat}_M^{N}\) | Term | \(N\) | \(M\) | \(\text{Cat}_M^{N}\) | Term | \(N\) | \(M\) | \(\text{Cat}_M^{N}\) | Term |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | \(\mathcal{P}_0\) | 0 | 0000 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 1020 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 1 | 1 | - \(L^{-1}_0\) | 0001 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 1200 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 0 | 00 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0010 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 2001 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 2 | 1 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0100 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 2100 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 1 | 01 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 1000 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3000 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 2 | 10 | - \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0011 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 1111 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 2 | 11 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | 0101 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 1120 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 2 | 20 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | 1010 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 1201 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 001 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 2000 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 1210 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 1 | 100 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | 0200 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 2011 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 110 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | 2010 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 2101 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 020 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | 3000 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 2110 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-2}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 011 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 1300 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3100 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 111 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0110 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3200 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 101 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 1110 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3300 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 201 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 1120 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3400 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 210 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0200 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3500 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |
| 3 | 300 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | 0210 | \(\mathcal{P}_0\) | \(\mathcal{P}_0\) | \(L^{-1}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) | 3600 | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-1}_{\mathcal{OP}}\) | \(L^{-3}_{\mathcal{OP}}\) | \(\mathcal{OP}_0\) |

Table 4.1: List of sequences \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)\) and their corresponding terms for the Dyson series in eq. (4.28) up to \(N = 4\). Recall that \((1 + \frac{r}{N})^N - M / (M - N)!\) prepends and \(\mathcal{P}\) appends each term.
The first few entries in Catalan’s triangle are reproduced in Tab. 4.2. Thus, the number of terms in the \( N \)th order piece \( T^{(N)} \) scales as

\[
C(N+1) \sim \frac{4^{N+1}}{(N+1)^{3/2} \sqrt{\pi}},
\]

indicating an exponentially increasing number of terms.

**4.2.1 Proof**

Since \( T^{(N)} \)'s can be defined recursively, we prove eq. (4.28) by induction on \( N \).

**Base Case** This corresponds to \( N = 1 \). Explicitly,

\[
T^{(1)}_t \mathcal{P}_\infty = \int_{-\infty}^{t} g(\tau_1) d\tau_1 e^{(t-\tau_1)L} \mathcal{O} \mathcal{P}_\infty. \tag{4.39}
\]

Inserting the decomposition \( \mathcal{I} = \mathcal{P}_\infty + \mathcal{Q}_\infty \) to the left of \( \mathcal{O} \) and simplifying yields

\[
T^{(1)}_t \mathcal{P}_\infty = \int_{-\infty}^{t} g(\tau_1) d\tau_1 \mathcal{P}_\infty \mathcal{O} \mathcal{P}_\infty + \int_{-\infty}^{t} d\tau_1 e^{(t-\tau_1)L} \mathcal{Q}_\infty \mathcal{O} \mathcal{P}_\infty, \tag{4.40}
\]

where the \( \eta \to 0 \) limit can readily be taken in the second integral. Performing both integrals and omitting \( \mathcal{Q}_\infty \) for conciseness yields

\[
T^{(1)}_t \mathcal{P}_\infty = \left[ \left( t + \frac{1}{\eta} \right) \mathcal{P}_\infty \mathcal{O} + L^{-1} \mathcal{O} \right] \mathcal{P}_\infty = \left( 1 + \frac{\partial}{\eta} \right) \left( t \mathcal{P}_\infty \mathcal{O} + L^{-1} \mathcal{O} \right) \mathcal{P}_\infty. \tag{4.41}
\]

For \( N = 1 \), the two sequences \( \lambda \) which satisfy the rules from eq. (4.33) are \( \lambda = (0) \) and \( \lambda = (1) \). These respectively correspond to the two terms above. The corresponding elements in the second line of Catalan’s triangle are \( C(1,0) = 1 \) and \( C(1,1) = 1 \), which sum up to the Catalan number \( C(2,2) = 2 \).
The integrals are those over $\tau \in (0, t]$. The above is zero when evaluated at $T_1 = -\infty$ due to the exponential $e^{\partial_T}$. It produces the “infinity” $(-)^M/\eta^{M+1}$ at $T_2 = 0$. The only remaining integrals are those over $[0, t]$, which are trivially evaluated. The two infinities coming from the integrals of $\tau_{N+1}^{-M}$ and $\tau_{N+1}^{-M-1}$ over $t \in (-\infty, 0]$ cancel due to having different signs, and, after simplification, all of this yields

$$\int_{-\infty}^{t} g(\tau_{N+1})d\tau_{N+1} \left(1 + \frac{\partial_{\tau_{N+1}}}{\eta}\right) \frac{\tau_{N+1}^{-M}}{(N-M)!} = \left(1 + \frac{\partial_{\tau}}{\eta}\right) \frac{t^{N+1-M}}{(N+1-M)!}.$$  

Now let us apply eq. (4.44) to the integrals in eq. (4.43b). This procedure is significantly simplified by observing that

$$\int d\tau e^{\partial_T} \frac{\tau^M}{M!} = (-)^M e^{\partial_T} \sum_{K=0}^{M} \frac{(-\tau)^K}{K!} \left(\frac{\tau}{M+1-K}\right).$$
When performing integrals of $e^{\tau \mathcal{L}}$, we can heuristically pretend $\mathcal{L}$ is a scalar since we are working only on its range. Performing these manipulations and simplifying signs yields

$$\int_{-\infty}^{t} d\tau_{N+1} \left( 1 + \frac{\partial \tau_{N+1}}{\partial \eta} \right) \frac{\tau_{N+1}^{N-M} e^{(t-\tau_{N+1})\mathcal{L}}}{(N-M)!} \mathcal{L} Q_{\infty} = \left( 1 + \frac{\partial \mathcal{L}}{\partial \eta} \right) \frac{\tau_{N+1}^{N-M} t^{K}}{K!} \left[ -\mathcal{L}^{-(N+1-M-K)} \right] Q_{\infty}.$$  

(4.47)

Plugging eqs. (4.45) and (4.47) and into $T_i^{(N+1)}$ (4.43a) yields

$$T_i^{(N+1)} = \left( 1 + \frac{\partial i}{\partial \eta} \right) \sum_{M=0}^{N+1-M} \frac{t^{K}}{(N+1-M)!} \mathcal{X} (N+1-M-K) \sum_{\lambda \in \text{Cat}_{M}^{N}} \prod_{n=2}^{N+1} \mathcal{X} (\lambda_n),$$

(4.48)

where we are now using the rule for $\mathcal{X}$ from eq. (4.34) and have also shifted the labeling of the sequences $\lambda$ by one ($\lambda_n \rightarrow \lambda_{n+1}$) for later convenience. Now, let us switch the order of the sums, sum over $K$ backwards ($\sum_{k=0}^{N+1} a_k = \sum_{k=0}^{N+1} a_{N+1-k}$), and rename indices as $M \leftrightarrow K$, yielding

$$T_i^{(N+1)} = \left( 1 + \frac{\partial i}{\partial \eta} \right) \sum_{M=0}^{N+1} \frac{t^{(N+1-M)}}{(N+1-M)!} M^{M-\delta_{M,N+1}} \mathcal{X} (M-K) \sum_{\lambda \in \text{Cat}_{M}^{N}} \prod_{n=2}^{N+1} \mathcal{X} (\lambda_n).$$

(4.49)

The Kronecker delta $\delta_{M,N+1}$ is there because the sum over $K$ for $M = N + 1$ has only $N + 1$ (and not $N + 2$) terms. For each $M$, we proceed to rewrite the sum over $K$ in terms of new sequences $\lambda'$ and show that those sequences are elements of $\text{Cat}_{M}^{N+1}$.

- According to eq. (4.33), the elements in each sequence $\lambda \in \text{Cat}_{M}^{N}$ sum to $K$. Therefore, if we prepend these sequences with $\lambda_1 = M - K$, we obtain new sequences

  $$\lambda' = (\lambda_1, \lambda_2, \cdots, \lambda_{N+1})$$

  (4.50)

whose elements sum to $M$. Since all elements $\lambda_{n \geq 2}$ satisfy eq. (4.33) and since $0 \leq \lambda_1 \leq N + 1$, the new sequences satisfy

$$\sum_{n=1}^{l} \lambda_{N+2-n} \begin{cases} \leq \min \{l, M\} & 1 \leq l < N + 1 \\ = M & l = N + 1 \end{cases}.$$  

(4.51)

Therefore, for each $M$, all of the new sequences $\lambda' \in \text{Cat}_{M}^{N+1}$.

- According to eq. (4.36), the total number of the old sequences $\lambda$ is $|\text{Cat}_{K}^{N}| = C (N, K)$ for each $K$. Therefore, the total number of new sequences $\lambda'$ for each $M$ is

$$\sum_{K=0}^{M-\delta_{M,N+1}} C (N, K) = C (N + 1, M - \delta_{M,N+1}),$$

(4.52)

where the right-hand side is true because of a property of Catalan’s triangle, namely, each term is a sum of entries above and to the left [160]. Therefore, the total number of sequences $\lambda'$ for each $M$ is $C (N + 1, M) = |\text{Cat}_{M}^{N+1}|$ [for this formula, we do not need the Kronecker
Using once again properties of Catalan numbers, the total number of terms for \( N + 1 \) is

\[
\sum_{M=0}^{N+1} C(N+1,M) = C(N+2) = |\text{Cat}^{N+1}|.
\]

Having shown that, for each \( M \), the sum over \( K \) can be rearranged as a sum over terms corresponding to all of the sequences in \( \text{Cat}_{N+1}^M \), we can rewrite eq. (4.49) as

\[
{T}^{(N+1)}_t = \left( 1 + \frac{\partial_t}{\eta} \right) \sum_{M=0}^{N+1} \frac{t(N+1-M)}{(N+1-M)!} \sum_{\lambda' \in \text{Cat}_{N+1}^M} \prod_{n=1}^{N+1} \lambda'(\lambda_n),
\]

thereby completing the inductive step and the proof. \( \square \)

4.3 RELATION TO PREVIOUS WORK

We now mention six connections of the above general derivations to previous works studying more specific cases. The first two deal with first-order perturbation theory while the last five make contact with higher-order effects.

4.3.1 Decoherence Hamiltonian and dark states

Focusing on the DFS case, we have \( \mathcal{L}^{-1}\mathcal{V}|\rho_\infty\rangle = \mathcal{L}^{-1}\mathcal{V}|\rho_\infty\rangle \) (4.20) and the leakage rate \( \Delta_{\text{edg}} \) is the dissipative gap of \( \mathcal{L} \). However, we can show something more with a few minor assumptions. Moreover, we show that for the semisimple DFS cases of Sec. 2.1.2, the dissipative gap of \( \mathcal{L} \) is the excitation gap of a related Hamiltonian. We assume that \( \mathcal{L} \) (1.8) can be written without a Hamiltonian part,

\[
\mathcal{L}(\rho) = \frac{1}{2} \sum_{\ell} \kappa_\ell (2F^\ell \rho F^{\ell\dagger} - F^{\ell\dagger} F^\ell \rho - \rho F^{\ell\dagger} F^\ell),
\]

and that DFS states are annihilated by the jump operators, \( F^\ell |\psi_k\rangle = 0 \) (if \( |\psi_k\rangle \) are also eigenstates of \( H \), they are called dark states [163]). This implies that \( F^\ell = P F^\ell P = 0 \) (with \( P = \sum_{k=0}^{d-1} |\psi_k\rangle \langle \psi_k| \)). We now determine \( \Delta_{\text{edg}} \) for such systems. Borrowing from Sec. 2.1 and using the above assumptions,

\[
\mathcal{L}(\rho) = -\frac{1}{2} \sum_{\ell} \kappa_\ell P \rho (F^{\ell\dagger} F^\ell \big|_{\mathfrak{A}} (2.13)).
\]

From this, we can extract the decoherence [148] or parent [135] Hamiltonian

\[
H_{\text{edg}} \equiv \frac{1}{2} \sum_{\ell} \kappa_\ell F^{\ell\dagger} F^\ell,
\]

The (zero-energy) ground states of \( H_{\text{edg}} \) are exactly the DFS states \( |\psi_k\rangle \) [135, 148] and the excitation gap of \( H_{\text{edg}} \) is \( \Delta_{\text{edg}} \). We come back to this case in Ch. 7.
4.3.2 Geometric linear response

We can avoid having to calculate the Green’s function \( \mathcal{L}^{-1} \) in the Kubo formula (4.10) by cleverly choosing an observable to measure. In what is essentially a linear response version of the adiabatic response calculation of Ref. [25], Thm. 9, let us define the flux of an operator \( X \)

\[
A \equiv \dot{X} = \mathcal{L}^\dagger(X). \tag{4.57}
\]

If we measure said flux \( A \), a simple manipulation of the Kubo formula (4.10) yields

\[
\langle\langle A|T_t^{(1)}|\rho_\infty\rangle\rangle = \langle\langle \mathcal{L}^\dagger(X)|[(t + \frac{1}{\eta})\mathcal{P}_\infty - \mathcal{L}^{-1}\mathcal{V}]|\rho_\infty\rangle\rangle \tag{4.58a}
\]

\[
= \langle\langle X|\mathcal{L}[(t + \frac{1}{\eta})\mathcal{P}_\infty - \mathcal{L}^{-1}\mathcal{V}]\mathcal{V}|\rho_\infty\rangle\rangle \tag{4.58b}
\]

\[
= -\langle\langle X|\mathcal{L}\mathcal{L}^{-1}\mathcal{V}|\rho_\infty\rangle\rangle \tag{4.58c}
\]

\[
= -\langle\langle X|\mathcal{Q}_\infty\mathcal{V}|\rho_\infty\rangle\rangle. \tag{4.58d}
\]

One can see that \( \mathcal{L} \) is not present in the above result. Therefore, if one perturbs with a current \( \mathcal{V} = i\partial_\alpha \) (i.e., \( -i[\mathcal{V}, \rho] = \partial_\alpha \rho \) for all \( \rho \)) and measures the flux of \( X = i\partial_\beta \) (for some parameters \( \alpha, \beta \)), then one obtains a Berry curvature in what can be called geometric linear response. This is a generalization of the geometric linear response of Hamiltonian systems ([129], Appx. C of Ref. [66]) to Lindbladians. A more detailed linear response calculation can be found in Sec. IV .A.1 of Ref. [10], complementing the earlier adiabatic response calculation in Sec. 7 of [25].

4.3.3 Dyson series for unique state case

Assume that the steady state is unique, so \( \mathcal{P}_\infty = |\varrho\rangle\langle\varrho| \). Then, assuming a trace-preserving perturbation, \( \mathcal{P}_\infty \mathcal{O} = |\varrho\rangle\langle\varrho| \langle\mathcal{O}^\dagger(1)| = 0 \) (2.50), all sequences \( \lambda \) with \( \lambda_n = 0 \) for some \( n \) in the Dyson expansion in Thm. 6 vanish. The only sequences that remain are of the form \( \lambda = (1, 1, \cdots, 1) \) and the \( N \)th order term reduces to

\[
\mathcal{T}_t^{(N)} = (-\mathcal{L}^{-1}\mathcal{O})^N. \tag{4.59}
\]

This matches the time-independent perturbation theory calculation from Ref. [172]. Contrary to the exponentially increasing number of terms when there is more than one steady state, the number of terms in the \( N \)th order Dyson series term for an unperturbed \( \mathcal{L} \) with a unique steady state is just... one!

4.3.4 Quantum Zeno dynamics

Recall that the power of \( t \) prepending each term is \( N - M \), meaning that the highest power of \( t \) always prepends the sole \( M = 0 \) term

\[
\frac{t^N}{N!} (\mathcal{P}_\infty \mathcal{O} \mathcal{P}_\infty)^N, \tag{4.60}
\]
associated with the sequence \( \lambda = (0, 0, \cdots, 0) \). Therefore, if we rescale our perturbation as \( \mathcal{O} \rightarrow \frac{1}{T} \mathcal{O} \) and evolve to time \( T \gg 1 \), then, for each \( N \), the term (4.60) is dominant and of order \( O(1) \) while the remaining \( M > 0 \) terms are of order \( O(1/T^M) \). Since this dominant term acts within \( \text{As}(H) \) and does not cause any leakage out of \( \text{As}(H) \), it is often said to generate quantum Zeno dynamics ([22, 115, 255]; see also [18, 47]). This effect has already been mentioned in Appx. D of Ref. [314] and derived to within first order using related methods [31, 218]. Since we have shown in Sec. 4.1.1 that \( \mathcal{R}_N \mathcal{P} \) is unitary for \( \mathcal{O} \) being a Hamiltonian (\( V \)) and/or jump operator (\( Y \)) perturbation, we can see that Zeno dynamics is always unitary for all perturbations of those forms for any order \( N \) in the \( T \rightarrow \infty \) limit.

4.3.5 Second-order terms

Omitting the \( 1/\eta \) infinity, the \( N = 2 \) terms are explicitly

\[
\mathcal{T}^{(2)}_t = \frac{(t \mathcal{P} \mathcal{O} \mathcal{P})^2}{2!} - t((\mathcal{P}_\infty \mathcal{O} \mathcal{L}^{-1} + \mathcal{L}^{-1} \mathcal{O} \mathcal{P}_\infty) \mathcal{O} \mathcal{P}_\infty + \mathcal{L}^{-1} (\mathcal{O} \mathcal{L}^{-1} - \mathcal{L}^{-1} \mathcal{O} \mathcal{P}_\infty) \mathcal{O} \mathcal{P}_\infty). \tag{4.61}
\]

Being the first nonunitary correction to \( \text{As}(H) \), the term

\[
\mathcal{L}_{\text{eff}} \equiv -\mathcal{P}_\infty \mathcal{O} \mathcal{L}^{-1} \mathcal{O} \mathcal{P}_\infty \tag{4.62}
\]

is relevant in a variety of many-body contexts [71, 73, 119, 170, 191, 195, 196, 260, 322] and quantum optical scenarios (see next Subsection). This term can also generate universal Lindbladian evolution on \( \text{As}(H) \) using the following clever Zeno-like scheme [319]. Assume that we have a rescaled Hamiltonian perturbation \( \mathcal{O} = \frac{1}{\sqrt{T}} \mathcal{V} \) and that \( \mathcal{P} \mathcal{V} \mathcal{P} = 0 \). Then, we can see that only \( \mathcal{L}_{\text{eff}} \) and \( \frac{1}{\sqrt{T}} \mathcal{L}^{-1} \mathcal{V} \mathcal{L}^{-1} \mathcal{V} \mathcal{P} \) remain, but the former is dominant in the \( T \rightarrow \infty \) limit. The first order leakage term \( \frac{1}{\sqrt{T}} \mathcal{L}^{-1} \mathcal{V} \mathcal{P} \) is then the dominant correction, being of order \( O(1/\sqrt{T}) \). We note that, in general, this term does not have to be in Lindblad form [319] and therefore can generate continuous-time but non-Markovian evolution.

Just like for \( N = 1 \), we can continue to apply the four-corners decomposition \( \mathcal{F} \) to the \( N \geq 2 \) terms. For the case of Hamiltonian (\( \mathcal{O} = \mathcal{V} \)) and/or jump operator (\( \mathcal{O} = \mathcal{Y} \)) perturbations, the piece \( \mathcal{L}_{\mathcal{F}} \) is also not relevant in \( \mathcal{L}_{\text{eff}} \). Since \( \mathcal{P}_{\mathcal{F}} \mathcal{O} \mathcal{P}_{\mathcal{F}} = 0 \) (see Sec. 4.1.1), one has

\[
\mathcal{L}_{\text{eff}} = -\mathcal{P}_{\mathcal{F}} \mathcal{O} \mathcal{L}^{-1} \mathcal{O} \mathcal{P}_{\mathcal{F}}. \tag{4.63}
\]

However, we cannot replace the remaining \( \mathcal{P}_{\mathcal{F}} \) with \( \mathcal{P}_{\mathcal{Y}} \) since two actions of \( \mathcal{V} \) can take the state from \( \mathcal{F} \) to \( \mathcal{F} \). Moreover, this is not always true when the perturbation \( \mathcal{O} \) is of general Lindblad form.

4.3.6 Effective operator formalism

Let us return to the type of \( \mathcal{L} \) studied in Sec. 2.1.3 and assume that \( F = F_{\mathcal{F}} \), \( H = H_{\mathcal{F}} \), and the perturbation \( \mathcal{O} = \mathcal{V} = -i[V, \cdot] \) is Hamiltonian with \( \mathcal{V} = V_{\mathcal{F}} \). Then, \( \mathcal{L}_{\text{eff}} \) (4.62) is exactly that
from the effective operator formalism of Ref. [241] (with $H_0 = 0$ for simplicity). Recall that now $L = \mathcal{K}$, where $\mathcal{K}(\rho) \equiv -i[K\rho - \rho K^\dagger]$ and the “non-Hermitian Hamiltonian”

$$K = K_0 \equiv H_0 - \frac{i}{2} \sum_\ell F_\ell^+ F_\ell$$

(4.64)

(with $K_0 > 0$ on $\mathfrak{H}$). We show that $L_{\text{eff}}$ is a Lindbladian,

$$L_{\text{eff}}(\rho) = -i[H_{\text{eff}}, \rho] + F_{\text{eff}}^\dagger F_{\text{eff}} - \frac{1}{2} \{F_{\text{eff}}^\dagger F_{\text{eff}}, \rho\},$$

(4.65)

with Hamiltonian $H_{\text{eff}} = -\frac{1}{2} V(K^{-1} + K^{-1\dagger}) V$ and jump operators $F_{\text{eff}}^\dagger = F^\dagger K^{-1} V_{\mathfrak{H}}$.

Recall that $P_{\mathfrak{H}}$ splits into two terms, and that, for this DFS case, the terms simplify to

$$P_{\mathfrak{H}} = P_{\mathfrak{F}} + P_{\mathfrak{H}}\mathcal{L}_{\mathfrak{H}}^{-1} = P_{\mathfrak{F}} - P_{\mathfrak{H}}\mathcal{L}_{\mathfrak{H}}^{-1} K_{\mathfrak{H}}^{-1}. \quad (4.66)$$

Given right and left eigenstates $|n\rangle$ and $\langle \tilde{m}|$ of $K$ and their respective eigenvalues $\lambda_n$ and $\lambda_m$,

$$\mathcal{K}_{\mathfrak{H}}^{-1}(|n\rangle \langle \tilde{m}|) = i \frac{|n\rangle \langle \tilde{m}|}{\lambda_n - \lambda_m^*}. \quad (4.67)$$

Using the formula for $P_{\mathfrak{H}}$, recalling (4.20), and remembering that $L_{\text{eff}}$ acts on $\rho_{\mathfrak{H}} \in \mathfrak{H}$ yields

$$L_{\text{eff}} = \left(-P_{\mathfrak{F}} + P_{\mathfrak{H}}\mathcal{L}_{\mathfrak{H}}^{-1} \right) V\mathcal{K}_{\mathfrak{H}}^{-1} V^\dagger P_{\mathfrak{H}}. \quad (4.68)$$

Let us first calculate the piece $V\mathcal{K}_{\mathfrak{H}}^{-1} V^\dagger P_{\mathfrak{H}}$ above:

$$V\mathcal{K}_{\mathfrak{H}}^{-1} V^\dagger P_{\mathfrak{H}}(\rho_{\mathfrak{H}}) = -i V\mathcal{K}_{\mathfrak{H}}^{-1}(V_{\mathfrak{F}} \rho_{\mathfrak{F}} - \rho_{\mathfrak{F}} V_{\mathfrak{F}}) \quad (4.69a)$$

$$= V(K^{-1} V_{\mathfrak{F}} \rho_{\mathfrak{F}} + \rho_{\mathfrak{F}} V_{\mathfrak{F}} K^{-1\dagger}) \quad (4.69b)$$

$$= -i(V_{\mathfrak{F}} \rho_{\mathfrak{F}} V_{\mathfrak{F}} K^{-1\dagger} - K^{-1} V_{\mathfrak{F}} \rho_{\mathfrak{F}} V_{\mathfrak{F}} + V_{\mathfrak{F}} K^{-1} \rho_{\mathfrak{F}} V_{\mathfrak{F}} - \rho_{\mathfrak{F}} V_{\mathfrak{F}} K^{-1\dagger} V_{\mathfrak{F}}). \quad (4.69c)$$

Here, $\mathcal{K}_{\mathfrak{H}}^{-1}$ can be expressed in terms of $K^{-1}$ by using eq. (4.67) and observing that either the bra or ket in each outer product of operators in $\mathfrak{H}$ is in the kernel of $K$. We now examine the first and second pair of terms to obtain the recycling and deterministic terms in $L_{\text{eff}}$.

- The first two terms in eq. (4.69c) involve $V_{\mathfrak{F}} \rho_{\mathfrak{F}} V_{\mathfrak{F}} \in \mathfrak{F}$ and are the terms seen by the second term of $L_{\text{eff}}$ (4.68). By decomposing $V_{\mathfrak{F}} \rho_{\mathfrak{F}} V_{\mathfrak{F}}$ into outer products $|n\rangle \langle \tilde{m}| \in \mathfrak{H}$ of eigenstates of $K$, we can simply study each outer product. For each $|n\rangle \langle \tilde{m}|$,

$$\mathcal{K}_{\mathfrak{H}}^{-1}(|n\rangle \langle \tilde{m}| K^{-1\dagger} - K^{-1} |n\rangle \langle \tilde{m}|) = \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_m} \right) \mathcal{K}_{\mathfrak{H}}^{-1}(|n\rangle \langle \tilde{m}|) \quad (4.70a)$$

$$= \frac{\lambda_n - \lambda_m^*}{\lambda_n \lambda_m^*} i |n\rangle \langle \tilde{m}| = i \frac{|n\rangle \langle \tilde{m}|}{\lambda_n \lambda_m^*} \quad (4.70b)$$

$$= i K^{-1} |n\rangle \langle \tilde{m}| K^{-1\dagger}. \quad (4.70c)$$

---

4 This assumes that $K$ is diagonalizable, although we are confident this treatment can be extended.
Plugging this in and using $\mathcal{P}_L \mathcal{L} \mathcal{P}_L (\cdot) = \sum_{\ell} F^\ell \cdot F^{\ell \dagger}$ (2.8g) yields the second term in $\mathcal{L}_{\text{eff}}$,

$$\mathcal{P}_L \mathcal{L} K^{-1}_L V K^{-1}_L V \mathcal{P}_L (\rho_\infty) = \sum_{\ell} F^\ell K^{-1}_L V \rho_\infty V K^{-1}_L F^{\ell \dagger} \equiv \sum_{\ell} F_{\text{eff}} \rho_\infty F_{\text{eff}}^{\dagger}. \quad (4.71)$$

The second two terms in eq. (4.69c) involve $V K^{-1}_L V \rho_\infty - H.c. \in \mathfrak{I}$ and are the terms seen by the second term of $\mathcal{L}_{\text{eff}}$ (4.68). Here, it is useful to decompose $K^{-1} = K^+_L + K^-_L$, where $K^\pm = \frac{1}{2} (K^{-1} \pm K^{-1 \dagger})$. The $K^+_L$ term produces an effective Hamiltonian $H_{\text{eff}}$ part of $V K^{-1}_L V \mathfrak{I}$ while the $K^-_L$ term can be combined with eq. (4.64) to relate $V K^{-1}_L V \mathfrak{I}$ to the anti-commutator piece consisting of $\sum_\ell F_{\text{eff}}^{\dagger} F_{\text{eff}}$:

$$-i V K^{-1}_L V \mathfrak{I} = -i V \frac{1}{2} (K^{-1} - K^{-1 \dagger}) V \mathfrak{I} = -i \frac{1}{2} V K^{-1 \dagger} (K - K^+) K^{-1} V \mathfrak{I} = -i \frac{1}{2} \sum_\ell F_{\text{eff}}^{\dagger} F_{\text{eff}}.$$

Thus, we have constructed both the jump and deterministic terms of eq. (4.65) and concisely linked the effective operator formalism to ordinary second-order perturbation theory.
“In general one may expect such effects whenever an isolated system is considered as being divided into two interacting parts, each slaved to a different aspect of the other. The systems considered [...] might be regarded as a special case, in which the coupling is with ‘the rest of the Universe’ (including us as observers). The only role of the rest of the Universe is to provide a Hamiltonian with slowly-varying parameters, thus forcing the system to evolve adiabatically with phase continuation governed by the time-dependent Schrödinger equation.”

– Michael V. Berry

5

ADIABATIC PERTURBATION THEORY

We now apply the four-corners decomposition to adiabatic perturbation theory. The leading order term governs adiabatic evolution within As(H) while all other terms are non-adiabatic corrections. We show that for a cyclic adiabatic deformation of steady As(H), the holonomy is unitary [10]. We also determine that the energy scale governing non-adiabatic corrections is once again governed by the effective dissipative gap $\Delta_{\text{edg}}$. We begin by reviewing the adiabatic/Berry connection for a Hamiltonian system in Sec. 5.1 and the DFS case in Sec. 5.2. We then continue to do full adiabatic perturbation theory for the Lindbladian case in Sec. 5.3.

The adiabatic limit has been generalized to Lindbladians [1, 25, 26, 77, 147, 259] and all orders of corrections to adiabatic evolution have been derived (e.g., [26], Thm. 6). By “the adiabatic limit”, we mean that dominated by the steady states of $\mathcal{L}$. Another adiabatic limit exists which is dominated by eigenstates of the Hamiltonian part of $\mathcal{L}$ [98, 221, 278], which we do not address here. Unlike adiabatic evolution of “non-Hermitian Hamiltonian” systems, Lindbladian adiabatic evolution always obeys the rules of quantum mechanics (i.e., is completely-positive and trace-preserving). In this work, we do not make the adiabatic approximation to Hamiltonians (and later to Lindbladians [251]) since it is not sufficient for the adiabatic theorem to hold. In the adiabatic approximation, one assumes that certain (seemingly reasonable) quantitative requirements on eigenstates and their derivatives w.r.t. parameters are sufficient for the system to be approximately adiabatic. However, those conditions have been shown to be insufficient (see [213] and refs. therein), so here we work strictly in the adiabatic limit and do not assume any of the conditions of the adiabatic approximation. We assume that As(H) is steady ($H_\infty = 0$), but
5.1 Hamiltonian Case

First, let us review two important consequences of the (Hamiltonian) quantum mechanical adiabatic theorem. Namely, adiabatic evolution can be thought of as either (1) being generated by an effective operator \([H_{\text{eff}}]\) or (2) generating transport of vectors in parameter space, leading to Abelian \([5, 55, 215, 291]\) or non-Abelian \([300]\) holonomies. We loosely follow the excellent expositions in Ch. 2.1.2 of Ref. [90] and Sec. 9 of Ref. [27]. We conclude with a summary of four different ways (\text{Hol}_1-\text{Hol}_4) of writing holonomies for the non-degenerate case and outline the generalizations done in the following Sections.

Let \(|\psi_0^{(t)}\rangle\) be the instantaneous unique (up to a phase) zero-energy ground state of a Hamiltonian \(H(t)\). We assume that the ground state is separated from all other eigenstates of \(H(t)\) by a nonzero excitation gap for all times of interest. Let us also rescale time \((s = t/T)\) such that the exact state \(|\psi(s)\rangle\) evolves according to

\[
\frac{1}{T} \partial_s |\psi(s)\rangle = -iH(s)|\psi(s)\rangle.
\]  

(5.1)

The adiabatic theorem states that \(|\psi(s)\rangle\) (with \(|\psi(0)\rangle = |\psi_0^{(s=0)}\rangle\)) remains an instantaneous eigenstate of \(H(s)\) (up to a phase \(\theta\)) in the limit as \(T \to \infty\), with corrections of order \(O(1/T)\). Let \(P_0^{(s)} = |\psi_0^{(s)}\rangle \langle \psi_0^{(s)}|\) be the projection onto the instantaneous ground state. In the adiabatic limit,

\[
|\psi(s)\rangle = e^{i\theta(s)}|\psi_0^{(s)}\rangle
\]  

and

(5.2)

and the initial projection \(P_0^{(0)}\) evolves into

\[
P_0^{(s)} = U_{\text{ad}}(s)P_0^{(0)}U_{\text{ad}}^+(s)
\]  

(5.3)

(with \(U_{\text{ad}}\) generating purely adiabatic evolution). The adiabatic evolution operator \(U_{\text{ad}}\) is determined by the Kato equation

\[
\partial_s U_{\text{ad}} = -iKU_{\text{ad}},
\]  

(5.4)
with so-called Kato Hamiltonian \([149]\) \((\hat{P}_0 \equiv \partial_s P_0)\)

\[ K = i[\hat{P}_0, P_0]. \]  

(5.5)

Such an adiabatic operator \(U_{\text{ad}}\) can be shown to satisfy eq. (5.3) (see [90], Prop. 2.1.1) using

\[
P_0\dot{P}_0P_0 = Q_0\dot{P}_0Q_0 = 0.
\]

(5.6)

The \(P_0\dot{P}_0P_0 = 0\) is a key consequence of the idempotence of projections while \(Q_0\dot{P}_0Q_0 = 0\) is obtained by application of the no-leak property (LP1); both are used throughout the text. The conventional adiabatic evolution operator is then a product of exponentials of \(-iK\) ordered along the path \(s' \in [0,s]\) (with path-ordering denoted by \(\mathcal{P}\)):

\[
U_{\text{ad}}(s) = \mathcal{P}\exp\left(\int_0^s [\hat{P}_0, P_0]ds'\right).
\]

(5.7)

Due to the intertwining property (5.3), \(U_{\text{ad}}(s)\) simultaneously transfers states in \(P_0^{(0)}\) to \(P_0^{(s)}\) and states in \(Q_0^{(0)}\) to \(Q_0^{(s)}\) (with \(Q_0 \equiv I - P_0\)) without mixing the two subspaces during the evolution. The term \(\hat{P}_0P_0\) in eq. (5.5) is responsible for generating the adiabatic evolution of \(P_0\) while the term \(P_0\dot{P}_0\) generates adiabatic evolution of \(Q_0\). To see this, observe that the adiabatically evolving state \(|\psi(s)\rangle = U_{\text{ad}}(s)|\psi^{(s=0)}\rangle \in P_0^{(s)}\) obeys the Schrödinger equation

\[
\partial_s|\psi(s)\rangle = [\hat{P}_0, P_0]|\psi(s)\rangle.
\]

(5.8)

Applying property (5.6), the second term in the commutator can be removed without changing the evolution. Since we are interested only in adiabatic evolution of the zero-eigenvalue subspace \(P_0\) (and not its complement), we can simplify \(U_{\text{ad}}\) by removing the second term in the Kato Hamiltonian. This results in the adiabatic equation

\[
\partial_s|\psi(s)\rangle = \hat{P}_0P_0|\psi(s)\rangle
\]

(5.9)

and effective adiabatic evolution operator

\[
U^{(s)} = \mathcal{P}\exp\left(\int_0^s \hat{P}_0P_0ds'\right).
\]

(5.10)

We now assume that \(s\) parameterizes a path in the parameter space \(M\) of some external time-dependent parameters of \(H(s)\). For simplicity, we assume that \(M\) is simply-connected.\(^1\) By writing \(P_0\) and \(\dot{P}_0\) in terms of \(|\psi_0\rangle\) and explicitly differentiating, the adiabatic Schrödinger equation (5.9) becomes

\[
\partial_s|\psi\rangle = (I - P_0)\partial_s|\psi\rangle.
\]

(5.11)

This implies a parallel transport condition

\[
0 = P_0\partial_s|\psi\rangle = \langle\psi|\partial_s\psi\rangle|\psi\rangle,
\]

(5.12)

---

\(^1\) If \(M\) is not simply connected (i.e., has holes), then the Berry phase may contain “topological” contributions. Such effects are responsible for anyonic statistics (e.g., [237], Sec. 1.B) and can produce Berry phases even for a one-dimensional parameter space [312].
which describes how to move the state vector from one point in M to another. The particular condition resulting from adiabatic evolution eliminates any first-order deviation from the unit overlap between nearby adiabatically evolving states [271]:

$$\langle \psi(s + \delta s)|\psi(s) \rangle = 1 + O(\delta s^2).$$

(5.13)

Therefore, we have shown two interpretations stemming from the adiabatic theorem. The first is that adiabatic evolution of $|\psi(s)\rangle$ (with $|\psi(0)\rangle = |\psi(0^{(s=0)})\rangle$) is generated (in the ordinary quantum mechanical sense) by the $P_0P_0$ piece of the Kato Hamiltonian $K$. The second is that adiabatic evolution realizes parallel transport of $|\psi(s)\rangle$ along a curve in parameter space. As we show now, either framework can be used to determine the adiabatically evolved state and the resulting Berry phase.

We now define a coordinate basis \( \{x_\alpha\} \) for the parameter space M. In other words,

$$\partial_t = \frac{1}{T} \partial_s = \frac{1}{T} \sum_\alpha x_\alpha \partial_\alpha,$$

(5.14)

where \( \partial_s \) is the derivative along the path, \( \partial_\alpha \equiv \partial/\partial x_\alpha \) are derivatives in various directions in parameter space, and \( x_\alpha \equiv \frac{dx_\alpha}{ds} \) are (dimensionless) parameter velocities. Combining eqs. (5.2) and (5.14) with the parallel transport condition (5.12) gives

$$0 = P_0 \partial_s |\psi\rangle = i \sum_\alpha x_\alpha (\partial_\alpha \theta - A_{\alpha,00}) |\psi\rangle,$$

(5.15)

where the adiabatic/Berry connection \( A_{\alpha,00} = i \langle \psi_0 | \partial_\alpha \psi_0 \rangle \) is a vector/gauge potential in parameter space. The reason we can think of \( A_{\alpha,00} \) as a gauge potential is because it transforms as one under gauge transformations $|\psi_0\rangle \to e^{i\theta} |\psi_0\rangle$ where \( \theta \in \mathbb{R} \):

$$A_{\alpha,00} \to A_{\alpha,00} - \partial_\alpha \theta.$$

(5.16)

These structures arise because the adiabatic theorem has furnished for us a vector bundle over the parameter-space manifold M [27, 271]. More formally, given the trivial bundle $M \times H$ (where at each point in M we have a copy of the full Hilbert space H), the projection $P_0$ defines a (possibly nontrivial) sub-bundle of $M \times H$ (in this case, a line bundle, since $P_0$ is rank one). The trivial bundle has a covariant derivative $\nabla_\alpha \equiv \partial_\alpha$ with an associated connection that can be taken to vanish. The Berry connection $A_{\alpha,00}$ is then simply the connection associated with the covariant derivative $P_0 \nabla_\alpha$ induced on the sub-bundle defined by $P_0$.

The Berry connection describes what happens to the initial state vector as it is parallel transported. It may happen that the vector does not return to itself after transport around a closed path in parameter space (due to e.g., curvature or non-simple connectedness of M). Given an initial condition \( \theta(0) = 0 \), the parallel transport condition (5.15) uniquely determines how \( \theta \) changes during adiabatic traversal of a path C parameterized by \( s \in [0,1] \), i.e., from a point \( x_\alpha(s=0) \in M \) to \( x_\alpha^{(1)} \). For a closed path \( (x_\alpha^{(1)} = x_\alpha^{(0)}) \) and assuming \( A_{\alpha,00} \) is defined uniquely for the whole path [237], the state transforms as $|\psi(0)\rangle \to B|\psi(0)\rangle$ with resulting gauge-invariant holonomy (here, Berry phase)

$$B \equiv \exp \left( i \sum_\alpha \int_C A_{\alpha,00} dx_\alpha \right).$$

(Hol1)
Alternatively, we can use (5.14) and the Schrödinger equation (5.9): $|\psi(0)\rangle \rightarrow U|\psi(0)\rangle$ with holonomy

$$U \equiv P \exp \left( \sum_{\alpha} \oint_{C} \partial_{\alpha} p_{0} d x_{\alpha} \right).$$  \hspace{1cm} \text{(Hol2)}$$

Since the geometric and Kato Hamiltonian formulations of adiabatic evolution are equivalent, eqs. (Hol1-Hol2) offer two ways to get to the same answer. They reveal two representations of the Berry connection and holonomy: the coordinate representation $\{iA_{s,00}, B\}$, which determines evolution of $\theta$ from eq. (5.2), and the operator representation $\{\partial_{\alpha} p_{0} p_{0}, U\}$, which determines evolution of $|\psi_{0}\rangle$ [Prop. 1.2 of [23], eq. (5) of [29]]. Despite the latter being a path-ordered product of matrices, it simplifies to the Berry phase in the case of closed paths.

For completeness, we also state an alternative form for each holonomy representation (Hol1-Hol2). If there are two or more parameters, the coordinate representation can be expressed in terms of the (here, Abelian) Berry curvature $F_{\alpha\beta,00} \equiv \partial_{\alpha} A_{\beta,00} - \partial_{\beta} A_{\alpha,00}$ using Stokes’ theorem:

$$B = \exp \left( \frac{i}{2} \sum_{\alpha,\beta} \int_{S} F_{\alpha\beta,00} d x_{\alpha} d x_{\beta} \right),$$ \hspace{1cm} \text{(Hol3)}

where $S$ is a surface whose boundary is the contour $C$. The operator representation can also be written as a product of the path-dependent projections $P_{0}$:

$$U = P \prod_{s \in C} P_{0}^{(s)},$$ \hspace{1cm} \text{(Hol4)}

where $P \prod$ denotes a continuous product ordered from right to left along the path $C$ [eq. (47) of [26], Prop. 1 of [315]]. This form of the holonomy should be reminiscent of the Pancharatnam phase [90, 215] and, more generally, of a dynamical quantum Zeno effect ([22, 115, 255]; see also [18, 47]).

In the following Sections, we generalize both representations to multi-dimensional subspaces in operator form, superoperator form, and to Lindblad NS blocks, so (in that order) the relevant quantities generalize to

$$P_{0} \rightarrow \{P_{dfs}, P_{dfs}, P_{\infty}\}$$

$$A \rightarrow \{A_{dfs}, A_{dfs}, A\}$$

$$B \rightarrow \{B_{dfs}, B_{dfs}, B\}$$

$$U \rightarrow \{U_{dfs}, U_{dfs}, U\}.$$ 

5.2 DFS case

We now generalize the above to a degenerate $d$-dimensional instantaneous ground state eigenspace spanned by orthonormal basis states $\{|\psi_{k}^{(s)}\rangle\}_{k=0}^{d-1}$ of a Hamiltonian $H(s)$. Due to the clean-leak condition (LP2), this analysis also holds if that same eigenspace comprises the DFS of a Lindblad-
This connection transforms as a gauge potential under transformations \( \text{Hol}_\mu \). The holonomy (here, Wilson loop) is given by the matrix unitary and Lindblad systems. To do this, we upgrade the state basis acting on the vector of coefficients \( c_k \) as

\[
\rho_{df}^{(s)}(\rho) = \sum_{\mu=0}^{d-1} |\psi_{\mu}^{dfs}(s)\rangle \langle \psi_{\mu}^{dfs}(s)| \rho \rangle = P_{dfs}^{(s)} \rho P_{dfs}^{(s)}
\]

where \( \psi_{\mu}^{dfs} \in \text{span}\{|\psi_k\rangle\} \) is a Hermitian matrix basis for the DFS, \( P_{dfs}^{+} = P_{dfs} \), \( P_{dfs}^{+}(\rho) = P_{dfs}^{+} \rho P_{dfs}^{+} = P_{dfs} \), and \( \rho \in \text{Op}(\mathcal{H}) \) in order to make contact with the next Section, where such a set is the DFS part of an NS block. For now however, adiabatic evolution of \( \{|\psi_k\rangle\} \) occurs under the ordinary Hamiltonian Schrödinger eq. (5.1).

Adiabatic evolution generalizes straightforwardly from the previous Section by letting \( P_{dfs}^{(s)} \rightarrow P_{dfs}^{(s)} \). The resulting Wilczek-Zee adiabatic connection [300] becomes a Hermitian matrix (for each \( \alpha \)) with elements

\[
A_{\alpha,kl}^{dfs} = i \langle \psi_k | \partial_\alpha \psi_l \rangle.
\]

This connection transforms as a gauge potential under transformations \( |\psi_k\rangle \rightarrow |\psi_l\rangle R_{lk} \), where \( R \in U(d) \) is a unitary rotation of the DFS states:

\[
A_{\alpha}^{dfs} \rightarrow R^{\dagger} A_{\alpha}^{dfs} R + iR^{\dagger} \partial_\alpha R.
\]

The holonomy (here, Wilson loop) is given by the matrix

\[
B^{dfs} = P \exp \left( i \sum_{\alpha} \oint_C A_{\alpha}^{dfs} dx_\alpha \right)
\]

acting on the vector of coefficients \( c_k \). Generalizing eq. (Hol1), \( |\psi\rangle \) transforms under the holonomy as

\[
|\psi(0)\rangle = \sum_{k=0}^{d-1} c_k |\psi_k^{(0)}\rangle \rightarrow \sum_{k,l=0}^{d-1} B_{kl}^{dfs} c_l |\psi_k^{(0)}\rangle.
\]

We now express the above structures in superoperator form in order to bridge the gap between unitary and Lindblad systems. To do this, we upgrade the state basis \( \{|\psi_k\rangle\}_{k=0}^{d-1} \) to the matrix basis \( \{|\psi_{\mu}^{dfs}\rangle\}_{\mu=0}^{d-1} \). The adiabatic Schrödinger equation can equivalently be expressed in operator and superoperator form using the superoperator projection from eq. (5.17):

\[
\partial_\alpha \rho = \langle \partial_\alpha | \psi \rangle + |\psi\rangle \langle \partial_\alpha | \psi \rangle = P_{dfs}^{+} \rho P_{dfs}^{+} \rho P_{dfs}^{+} = P_{dfs}^{+} \rho P_{dfs}^{+} \rho P_{dfs}^{+}.
\]

The superoperator version of the coordinate form of the DFS connection is then

\[
A_{\alpha,\mu

\text{dian. For notational convenience, we indicate the s-dependence as a superscript. We denote the respective operator and superoperator projections as}

\[
P_{dfs}^{(s)} = \sum_{k=0}^{d-1} |\psi_k^{(s)}\rangle \langle \psi_k^{(s)} |
\]

\[
P_{dfs}^{(s)}(\rho) = \sum_{\mu=0}^{d-1} |\psi_{\mu}^{dfs}(s)\rangle \langle \psi_{\mu}^{dfs}(s)| \rho \rangle = P_{dfs}^{(s)} \rho P_{dfs}^{(s)}
\]

\[
\text{where } \psi_{\mu}^{dfs} \in \text{span}\{|\psi_k\rangle\} \text{ is a Hermitian matrix basis for the DFS, } P_{dfs}^{+} = P_{dfs} \text{, } P_{dfs}^{+}(\rho) = P_{dfs}^{+} \rho P_{dfs}^{+} = P_{dfs} \text{, and } \rho \in \text{Op}(\mathcal{H}) \text{ in order to make contact with the next Section, where such a set is the DFS part of an NS block. For now however, adiabatic evolution of } \{|\psi_k\rangle\} \text{ occurs under the ordinary Hamiltonian Schrödinger eq. (5.1).}

\text{Adiabatic evolution generalizes straightforwardly from the previous Section by letting } P_{dfs}^{(s)} \rightarrow P_{dfs}^{(s)} \text{. The resulting Wilczek-Zee adiabatic connection [300] becomes a Hermitian matrix (for each } \alpha \text{) with elements}

\[
A_{\alpha,kl}^{dfs} = i \langle \psi_k | \partial_\alpha \psi_l \rangle.
\]

\text{This connection transforms as a gauge potential under transformations } |\psi_k\rangle \rightarrow |\psi_l\rangle R_{lk} \text{, where } R \in U(d) \text{ is a unitary rotation of the DFS states:}

\[
A_{\alpha}^{dfs} \rightarrow R^{\dagger} A_{\alpha}^{dfs} R + iR^{\dagger} \partial_\alpha R.
\]

\text{The holonomy (here, Wilson loop) is given by the matrix}

\[
B^{dfs} = P \exp \left( i \sum_{\alpha} \oint_C A_{\alpha}^{dfs} dx_\alpha \right)
\]

\text{acting on the vector of coefficients } c_k \text{. Generalizing eq. (Hol1), } |\psi\rangle \text{ transforms under the holonomy as}

\[
|\psi(0)\rangle = \sum_{k=0}^{d-1} c_k |\psi_k^{(0)}\rangle \rightarrow \sum_{k,l=0}^{d-1} B_{kl}^{dfs} c_l |\psi_k^{(0)}\rangle.
\]

\text{We now express the above structures in superoperator form in order to bridge the gap between unitary and Lindblad systems. To do this, we upgrade the state basis } \{|\psi_k\rangle\}_{k=0}^{d-1} \text{ to the matrix basis } \{|\psi_{\mu}^{dfs}\rangle\}_{\mu=0}^{d-1}. \text{ The adiabatic Schrödinger equation can equivalently be expressed in operator and superoperator form using the superoperator projection from eq. (5.17):}

\[
\partial_\alpha \rho = \langle \partial_\alpha | \psi \rangle + |\psi\rangle \langle \partial_\alpha | \psi \rangle = P_{dfs}^{+} \rho P_{dfs}^{+} \rho P_{dfs}^{+} = P_{dfs}^{+} \rho P_{dfs}^{+} \rho P_{dfs}^{+}.
\]

\text{since } \partial_\alpha | \psi \rangle = P_{dfs}^{+} \rho P_{dfs}^{+} | \psi \rangle, \quad P_{dfs}^{+}(\rho) = P_{dfs}^{+} \rho P_{dfs}^{+} + P_{dfs}^{+} \rho P_{dfs}^{+}, \quad \text{and } P_{dfs} = \partial_\alpha P_{dfs}. \text{ The operator representation of the holonomy is then the path-ordered product of exponentials of the generator } P_{dfs}^{+} \rho P_{dfs}^{+}. 

\text{The superoperator version of the coordinate form of the DFS connection is then}

\[
A_{\alpha,\mu}^{dfs} = \langle \langle \psi_{\mu}^{dfs} | \partial_\alpha \psi_{\mu}^{dfs} \rangle \rangle = \text{Tr} \left\{ \psi_{\mu}^{dfs} \partial_\alpha \psi_{\mu}^{dfs} \right\}.
\]
Sticking with the convention that $\Psi_0^{\text{dfs}} \equiv \frac{1}{\sqrt{T}} \hat{P}_{\text{dfs}}$ is the only traceful element and using property (5.6), $\Psi_0^{\text{dfs}} \partial_\alpha \Psi_0^{\text{dfs}} = 0$ and we can see that $A_{n,\mu}^{\text{dfs}} = A_{n,0 \mu}^{\text{dfs}} = 0$ for all $\mu$. Thus, $A_{n}^{\text{dfs}}$ consists of a direct sum of zero with a $(d^2 - 1)$-dimensional anti-symmetric matrix acting on the Bloch vector components $\{|\Psi_\mu^{\text{dfs}} \rangle\_{\mu \neq 0}\}$. Since the latter is anti-symmetric, the holonomy is unitary. Formally, letting $\text{Op}(H)^*$ be the space of traceless $d$-dimensional Hermitian matrices, $\mathcal{P}_{\text{dfs}}$ defines a sub-bundle of the trivial bundle $M \times \text{Op}(H)^*$ and $\mathcal{A}_n^{\text{dfs}}$ is the connection associated with the covariant derivative $\mathcal{P}_{\text{dfs}} \partial_\alpha$ induced on that sub-bundle.

### 5.3 Lindbladian Case

Throughout this entire Section, we assume that $\text{As}(H)$ is steady ($H_\alpha = 0$). Recall that a system evolves in a rescaled time $s = t/T \in [0,1]$ according to a time-dependent Lindbladian $\mathcal{L}(s)$, where the end time $T$ is infinite in the adiabatic limit. For all $s$, we define a continuous and differentiable family of asymptotic projections

$$\mathcal{P}_\infty^{(s)} = \sum_{\mu} |\Psi_\mu^{(s)}\rangle \langle \Psi_\mu^{(s)}|,$$

steady-state basis elements $\Psi_\mu^{(s)}$ (such that $\mathcal{L}(s) |\Psi_\mu^{(s)}\rangle = 0$), and conserved quantities $J_\mu^{(s)}$ (such that $\langle \Psi_\mu^{(s)} | J_\mu^{(s)} | \Psi_\mu^{(s)}\rangle = 0$). Each projection therefore is associated with its own instantaneous asymptotic subspace, $\mathcal{P}_\infty^{(s)} \text{Op}(H)$. The dimension of the instantaneous subspaces (i.e., the rank of $\mathcal{P}_\infty^{(s)}$) is assumed to stay constant during this evolution. In other words, the zero eigenvalue of $\mathcal{L}(s)$ is isolated from all other eigenvalues at all points $s$ by the dissipative gap $\Delta_{\text{dg}}$ [analogous to the excitation gap in Hamiltonian systems; see eq. (1.32)]. We once again assume that $s \in [0,1]$ parameterizes a path in a space of control parameters $M$, whose coordinate basis is $\{x_\alpha\}$, and use the parameterization from the Hamiltonian case in eq. (5.14):

$$\partial_t = \frac{1}{T} \partial_s = \frac{1}{T} \sum_\alpha x_\alpha \partial_\alpha,$$

where $\partial_s$ is the derivative along the path, $\partial_\alpha \equiv \partial/\partial x_\alpha$ are derivatives in various directions in parameter space, and $x_\alpha \equiv \frac{\partial s}{\partial x_\alpha}$ are (unitless) parameter velocities.

Following Ref. [26], starting with an initially steady state $|\rho(0)\rangle \in \text{As}(H)$, adiabatic perturbation theory is an expansion of the equation of motion

$$\frac{1}{T} \partial_s |\rho(s)\rangle = \mathcal{L}(s) |\rho(s)\rangle$$

in a series in $1/T$. Each term in the expansion is further divided using the decomposition $\mathcal{I} = \mathcal{P}_\infty + \mathcal{Q}_\infty$ into terms inside and outside the instantaneous $\text{As}(H)$. This allows one to derive both the adiabatic limit (when $T \to \infty$) and all corrections. The $O(1/T)$ expansion for the final state from Thm. 6 of Ref. [26] reads

$$|\rho(s)\rangle = U^{(s0)}(0) |\rho(0)\rangle$$

$$+ \frac{1}{T} \mathcal{L}^{-1}(s) \mathcal{P}^{(s)} \mathcal{U}^{(s0)} |\rho(0)\rangle$$

$$+ \frac{1}{T} \int_0^s dr U^{(sr)} \mathcal{P} \mathcal{L}^{-1} \mathcal{P} \mathcal{U}^{(r0)} |\rho(0)\rangle,$$

(5.27)
where all quantities in curly brackets are functions of \( r \), \( \mathcal{P}_\alpha \equiv \partial_s \mathcal{P}_\alpha \), \( Q_{\mathcal{P}} \equiv I - \mathcal{P}_\alpha \), and \( \mathcal{L}^{-1} \) is the instantaneous inverse (4.11). The superoperator

\[
\mathcal{U}^{(s,s')} = \mathbb{P} \exp \left( \int_{s'}^s \mathcal{P}_\alpha(r) \mathcal{P}_\alpha(r) dr \right) \tag{5.28}
\]

parallel transports states in \( \mathcal{P}_\alpha^{(s')} \text{Op}(\mathcal{H}) \) to states in \( \mathcal{P}_\alpha^{(s)} \text{Op}(\mathcal{H}) \) and is a path-ordered product of exponentials of the adiabatic connection \( \mathcal{P}_\alpha \mathcal{P}_\alpha \), the generator of Lindbladian adiabatic evolution.

Like the Kubo formula, all terms can be interpreted when read from right to left. The first term in eq. (5.27) represents adiabatic evolution of \( \text{As} (\mathcal{H}) \), the (second) leakage term quantifies leakage of \( |\rho(0)\rangle \) out of \( \text{As} (\mathcal{H}) \), and the (last) tunneling term represents interference coming back into \( \text{As} (\mathcal{H}) \) from outside. This term is a continuous sum of adiabatically evolved steady states which are perturbed by \( \mathcal{P}_\alpha \mathcal{L}^{-1} \mathcal{P}_\alpha \) at all points \( r \in [0, s] \) during evolution. Due to its dependence on the spectrum of \( \mathcal{L} \), this term needs to be minimized to determine the optimal adiabatic path through \( \text{As} (\mathcal{H}) \) [24]. Notice also the similarity between the leakage term and the leakage term (4.19) of the Kubo formula. Motivated by this, we proceed to apply the four-corners decomposition to all three terms.

### 5.3.1 Evolution within \( \text{As} (\mathcal{H}) \)

Let us now assume a closed path \( [\mathcal{L}(s) = \mathcal{L} (0)] \). In the adiabatic limit [according to eq. (5.27)], an initial steady state evolves in closed path \( C \) as

\[
|\rho(0)\rangle \rightarrow \mathcal{U} |\rho(0)\rangle \tag{5.29}
\]

and acquires a holonomy

\[
\mathcal{U} \equiv \mathcal{U}^{(3,0)} = \mathbb{P} \exp \left( \oint_C \mathcal{P}_\alpha \mathcal{P}_\alpha ds \right) . \tag{5.30}
\]

This is operator version of the holonomy and connection \( (\mathcal{P}_\alpha \mathcal{P}_\alpha) \) since the above expression acts on the steady-state basis elements \( \Psi_{\mu}^{(s=0)} \) used to express the initial steady state

\[
|\rho(0)\rangle = \sum_\mu c_\mu |\Psi_{\mu}^{(0)}\rangle . \tag{5.31}
\]

Let us now study the coordinate representation of the holonomy. This can be done by a straightforward generalization of the Hamiltonian analysis of Sec. 5.2 to Lindbladians [26, 252], which produces a parallel transport condition

\[
\mathcal{P}_\alpha \partial_s |\rho\rangle = 0 \tag{5.32}
\]

characterizing the Lindbladian adiabatic limit. After expressing \( \partial_s \) in terms of the various \( \partial_{\alpha} \)'s (5.25), this condition provides an equation of motion for the coordinate vector \( c_\mu \) from eq. (5.31). Solving this equation yields the coordinate representation of the holonomy

\[
\mathcal{B} = \mathbb{P} \exp \left( - \sum_\alpha \oint_C A_\alpha dx_\alpha \right) \tag{5.33}
\]
and corresponding adiabatic connection

$$A_{\alpha,\mu\nu} \equiv \langle J^\mu | \partial_\alpha \Psi^\nu \rangle .$$  \hfill (5.34)

Note that $A_\alpha$ is a real matrix since $\{ J^\mu, \Psi^\nu \}$ are Hermitian. The connection transforms as a gauge potential under $| \Psi^\mu \rangle \to | \Psi^\nu \rangle R_{\nu\mu}$ and $\langle J^\nu | \to R^{-1}_{\mu\nu} \langle J^\mu |$ for any $R \in GL[\dim \text{As}(H), \mathbb{R}]$:

$$A_\alpha \to R^{-1} A_\alpha R + R^{-1} \partial_\alpha R .$$  \hfill (5.35)

Upon evolution in the closed path, the density matrix transforms as

$$| \rho(0) \rangle = \sum_{\mu} c_\mu | \Psi^{(0)}_\mu \rangle \to \sum_{\mu, \nu} B_{\mu\nu} c_\nu | \Psi^{(0)}_\mu \rangle ,$$  \hfill (5.36)

equivalent to the operator representation (5.29). We study both representations below, showing that the holonomy is unitary for all $\text{As}(H)$.

First, let us remove the decaying subspace from both representations of the connection by applying the clean-leak property (LP2). Simplifying $A_\alpha$ turns out to be similar to calculating the effective Hamiltonian perturbation $W$ within $\text{As}(H)$ in Sec. 4.1.1. By (LP2),

$$A_{\alpha,\mu\nu} \equiv \langle J^\mu | \partial_\alpha \rho \rangle = \langle J^\mu | \partial_\alpha \Psi^\nu \rangle = \text{Tr} \{ \partial_\alpha \rho \} .$$  \hfill (5.37)

For the operator representation, one first applies (LP2) to the parallel transport condition (5.32):

$$0 = \mathcal{P}_\Psi | \partial_\nu \rho \rangle = \mathcal{P}_\Psi | \partial_\nu \rho \rangle .$$  \hfill (5.38)

Then, one uses this condition to obtain an equation of motion for $\rho$:

$$| \partial_\nu \rho \rangle = (\mathcal{I} - \mathcal{P}_\Psi) | \partial_\nu \rho \rangle = \mathcal{P}_\Psi \mathcal{P}_\Psi \rho \rangle .$$  \hfill (5.39)

The last equality above can be checked by expressing both sides in terms of the steady-state basis elements $\Psi^{(0)}_\mu$ and conserved quantities $J^\mu$. For a closed path, the solution to this equation of motion is then the same holonomy, but now with the minimal projection $\mathcal{P}_\Psi$ instead of the asymptotic projection $\mathcal{P}_\infty$:

$$\mathcal{U} = \mathcal{P} \exp \left( \oint_C \mathcal{P}_\Psi \mathcal{P}_\Psi ds \right) .$$  \hfill (5.40)

The holonomy $\mathcal{U}$ thus does not depend on the piece $\mathcal{P}_\Psi \mathcal{P}_\infty$ associated with the decaying subspace.

**Unique state case**

Now the only conserved quantity is the identity $J = I$, so it is easy to show that

$$A_\alpha = \langle I | \partial_\alpha \rho \rangle = \text{Tr} \{ \partial_\alpha \rho \} = 0 .$$  \hfill (5.41)

The unique steady state can never acquire a Berry phase. This may clash with the reader’s memories from introductory quantum mechanics, where a nonzero Berry phase was calculated for a Hamiltonian with a unique ground state. Such a phase is undetectable since it is an overall phase of the ground-state wavefunction. This phase disappears when the state is written as
density matrix. Since the Lindbladian formalism deals with density matrices directly, one never encounters such an overall phase. This should not be confused with interferometry experiments used to detect Berry phases. Such experiments implicitly assume that the adiabatically evolving subspace is more than one-dimensional. In that case, a phase gained by one basis component and not the others is then not an overall phase, but a relative (and thus observable) phase.

**NS case**

For this case, the NS factors into a DFS and an auxiliary part for each \( s \in [0,1] \). The DFS part is mapped into a reference DFS space spanned by a (parameter-independent Hermitian matrix) basis \( \{|\Psi_\mu^{\text{dfs}}\rangle\}_{\mu=0}^{d-1} \). We let \( S(s) \) (with \( S(\rho) \equiv S\rho S^\dagger \)) be the unitary operator which simultaneously maps the instantaneous basis elements \( |\Psi_\mu^{(s)}\rangle \) into the reference DFS basis and diagonalizes \( q_\alpha^{(s)} \).

Similarly, this \( S(s) \) factors the instantaneous conserved quantities \( \langle \rho_\alpha \rangle \) into a DFS part and the identity \( p_\alpha^{(s)} \) on the auxiliary space. Therefore, we define the family of instantaneous minimal projections as

\[
P_\Psi^{(s)} = S(s) \left( \overline{P}_\text{dfs} \otimes |q_\alpha^{(s)}\rangle \langle p_\alpha^{(s)}| \right) S^\dagger(s),
\]

where \( \overline{P}_\text{dfs}(\cdot) = \sum_{\mu=0}^{d-1} |\Psi_\mu^{\text{dfs}}\rangle \langle \Psi_\mu^{\text{dfs}}| \equiv \overline{P}_\text{dfs} \cdot P_\text{dfs} \) is the superoperator projection onto the \( x_\alpha \)-independent DFS reference basis. The generators of motion

\[
G_\alpha \equiv iS^\dagger \partial \alpha S \quad \text{and} \quad \mathcal{G}_\alpha \equiv -i[G_\alpha, \cdot]
\]

can mix up the DFS with the auxiliary part, generating novel dissipation-assisted adiabatic dynamics.

We note that \( q_\alpha^{(s)} \) (and therefore \( p_\alpha^{(s)} \)) can change rank \( (d_\alpha^{(s)}) \), provided that \( P_\Psi^{(s)} \) remains differentiable. For example, one can imagine \( q_\alpha^{(s)} \) to be a thermal state associated with some Hamiltonian on \( H_\alpha \) whose rank jumps from one to \( d_\alpha \) as the temperature is turned up from zero. This implies that \( p^{(s)} \) and thus \( P^{(s)} \) can change rank also. However, such deformations do not change the dimension \( d^2 \) of the steady-state subspace and thus do not close the dissipative gap. To account for such deformations in the one NS block case, the path can be partitioned into segments of constant rank \( \{P\} \) and the connection calculation below can be applied to each segment.

Simplifying eq. (5.37) by invoking the reference basis structure of \( \{J, \Psi\} \) from eq. (5.42) gives

\[
\mathcal{A}_\alpha = \mathcal{A}_\alpha^{\text{dfs}} + \mathcal{A}_\alpha^{\text{ax}} = -i[\mathcal{A}_\alpha^{\text{dfs}}, \cdot] \otimes |q_\alpha^{(s)}\rangle \langle p_\alpha^{(s)}| + \mathcal{A}_\alpha^{\text{ax}},
\]

where the DFS effective Hamiltonian is \([212]\)

\[
\mathcal{A}_\alpha^{\text{dfs}} \equiv \text{Tr}_\alpha \left\{ \left( \overline{P}_\text{dfs} \otimes q_\alpha^{(s)} \right) G_\alpha \right\}
\]

and the second term is the \( n_\alpha \)-dependent constant

\[
\mathcal{A}_\alpha^{\text{ax}} = -\partial \alpha \ln n^{(s)}_\alpha \delta_{\mu \nu}.
\]

\[\text{Note that in general } |\Psi_\mu^{\text{dfs}}\rangle \neq |\Psi_\mu^{\text{dfs}}(s = 0)\rangle \text{ since } s \text{ parameterizes a particular path in } M \text{ while } \{|\Psi_\mu^{\text{dfs}}\rangle\} \text{ is fixed.}\]
The first term clearly leaves the auxiliary part invariant and generates unitary evolution within the DFS part of the NS. We can thus see that DFS holonomies can be influenced by $\varepsilon^{(s)}_{\alpha x}$. We will see that the second term’s only role is to preserve the trace for open paths.

Sticking with the convention that $\Psi^{(s)}_0$ is traceful and the traceless $\Psi^{(s)}_\mu \neq 0$ carry the DFS Bloch vector, we notice that $A_\alpha$ transforms as a gauge potential under orthogonal Bloch vector rotations $R \in SO(d^2 - 1)$:

$$|\Psi_{\mu \neq 0}\rangle \rightarrow |\Psi_{\nu \neq 0}\rangle R_{\nu \mu} \quad \text{and} \quad |\Psi^{(s)}_B\rangle \rightarrow |\Psi^{(s)}_B\rangle R_{\nu \mu}.$$  

(5.47)

In addition, one has the freedom to internally rotate $\varepsilon^{(s)}_{ax}$ without mixing $\Psi_{\mu}^{-}$ with $\Psi_{\nu \neq \mu}^{-}$. Under such a transformation $S_{ax}$,

$$|\Psi_{\mu}^{-}\rangle \rightarrow S_{ax} |\Psi_{\mu}^{-}\rangle = S_{ax} |\Psi^{dfs}_{\mu} \otimes \frac{R_{ax}\varepsilon^{dfs}_{ax} R^+_{ax}}{n_{ax}}\rangle,$$

(5.48)

for some $R_{ax} \in U(d_{ax})$ and the connection transforms as an Abelian gauge potential:

$$A_{\alpha,\mu \nu} \rightarrow A_{\alpha,\mu \nu} + \langle \langle |\Psi^{(s)}_B\rangle S^+_{ax}\partial_{\alpha} S_{ax}|\Psi_{\nu}^{-}\rangle.$$

(5.49)

Plugging in eq. (5.44) into the Lindblad holonomy (5.33), we can see that $A_{ax}^{dfs}$ is proportional to the identity matrix (of the space of coefficients $c_{\mu}$) and thus can be factored out. Therefore,

$$B = \exp \left( \sum_{\alpha} \int_{\gamma} \partial_{\alpha} \ln n_{ax} d x_{ax} \right) B^{dfs},$$

(5.50)

where $B^{dfs}$ is the unitary $\varepsilon_{ax}^{-}$-influenced holonomy associated with $A^{dfs}$. The first term in the above product for an open path $s \in [0, 1]$ is simply $n_{ax}^{(1)} / n_{ax}^{(0)}$, providing the proper re-scaling of the coefficients $c_{\mu}$ to preserve the trace of $|\rho(0)\rangle\rangle$. For a closed path, this term vanishes (since $n_{ax}$ is real and positive) and $B = B^{dfs}$. Thus, we have shown that the holonomy after a closed-loop traversal of one NS block is unitary.

**Multi-block case**

The generalization to multiple NS blocks is straightforward: the reference basis now consists of multiple blocks. Recall that $J^{\mu}$ do not have presence in the off-diagonal parts neighboring the NS blocks [Fig. 1.2(b)] and that the only NS block that $\partial_{\alpha} \Psi_{\mu}$ has presence in is that of $\Psi_{\mu}$. Therefore, each NS block is imparted with its own unitary holonomy.

**5.3.2 Adiabatic curvature**

The adiabatic connection $A_{a}$ (5.34) can be used to define an adiabatic curvature defined on the parameter space induced by the steady states. For simply-connected parameter spaces $M$ (see footnote 1), the adiabatic curvature can be shown to generate the corresponding holonomy. More precisely, the Ambrose-Singer theorem ([202], Thm. 10.4) implies that the holonomy for an infinitesimal closed path $C$ with basepoint $x_{a}^{(0)}$ is the adiabatic curvature at $x_{a}^{(0)}$. One can

3 For open paths, $B$ is related to non-cyclic geometric phases in other dissipative systems [e.g., [272], eq. (47)] and non-Hermitian systems [289].
Alternatively use a generalization of Stokes’ theorem to non-Abelian connections \cite{21} to express the holonomy in terms of a “surface-ordered” integral of the corresponding adiabatic curvature, generalizing the Abelian case (\text{Hol}_\text{3}). Letting $\partial_{[\alpha}A_{\beta]} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$, the curvature is

$$F_{\alpha\beta,\mu\nu} = \partial_{[\alpha}A_{\beta]},_{\mu\nu} + [A_{\alpha},A_{\beta}]]_{\mu\nu}.$$ \hspace{1cm} (5.51)

Using the NS adiabatic connection (5.44) and remembering that $\partial_{\alpha}A_{\beta}$ is symmetric in $\alpha, \beta$, the adiabatic curvature for one NS block,

$$F_{\alpha\beta,\mu\nu} = \partial_{[\alpha}\tilde{A}_{\beta}],_{\mu\nu} + [\tilde{A}_{\alpha},\tilde{A}_{\beta}]]_{\mu\nu},$$ \hspace{1cm} (5.52)

is just the curvature associated with the connection $\tilde{A}^\text{dfs}$.  

5.3.3 Leakage out of the asymptotic subspace

We now return to the adiabatic response formula (5.27) to apply the four-corners decomposition to the $O(1/T)$ non-adiabatic corrections. By definition (4.11), $L^{-1}$ has the same block upper-triangular structure as $L$ from eq. (2.3). The derivative of the asymptotic projection has partition

$$\mathcal{P}_\infty = \begin{bmatrix} (\mathcal{P}_\Psi)_\square & \mathcal{P}_\text{E} \mathcal{P}_\Psi \mathcal{P}_\text{E} & \mathcal{P}_\text{E} \mathcal{P}_\Psi \mathcal{P}_\text{E} \\ \mathcal{P}_\text{E} \mathcal{P}_\Psi \mathcal{P}_\text{E} & 0 & \mathcal{P}_\text{E} \mathcal{P}_\Psi \mathcal{P}_\text{E} \\ 0 & 0 & 0 \end{bmatrix}. \hspace{1cm} (5.53)$$

One can interpret $\mathcal{P}_\infty$ as a perturbation, analogous to $V$ from Ch. 4, and observe from the above partition that $\mathcal{P}_\infty$ does not connect block diagonal spaces: $\mathcal{P}_\text{N} \mathcal{P}_\text{N} \mathcal{P}_\text{N} = 0$. In addition, whenever $\mathcal{P}_\infty^{(r)}$ acts on a parallel transported state living in $\mathcal{P}_\text{N} \text{Op}(H)$, only the first column in the above partition $(\mathcal{P}_\Psi \mathcal{P}_\text{E})$ is relevant. These observations result in $L^{-1} \rightarrow L^{-1}$ and the replacement of two factors of $\mathcal{P}_\infty$ with $\mathcal{P}_\Psi$ in eq. (5.27). Interestingly, we cannot replace the remaining $\mathcal{P}_\infty$ since $\mathcal{P}_\text{E} \mathcal{P}_\Psi \mathcal{P}_\text{E}$ contains contributions from $|\Psi^\nu\rangle \langle \partial_s^{(1)} \mathcal{P}_\infty \mathcal{P}_\Psi| \Psi^\nu\rangle$.

$$|\rho(s)\rangle = \mathcal{U}^{(s,0)}|\rho(0)\rangle + \frac{1}{T} L^{-1}(s) \mathcal{P}_\Psi^{(s)} \mathcal{U}^{(s,0)}|\rho(0)\rangle$$

$$+ \frac{1}{T} \int_0^s d\tau \mathcal{U}^{(s,\tau)} \{ \mathcal{P}_\infty L^{-1} \mathcal{P}_\Psi \}^{(\tau)} \mathcal{U}^{(\tau,0)}|\rho(0)\rangle.$$ \hspace{1cm} (5.54)

Using the results of Sec. 4.1.2, the energy scale governing the leading-order non-adiabatic corrections is once again the effective dissipative gap $\Delta_{\text{edg}}$ — the nonzero eigenvalue of $L_\square + L_\text{E}$ with the smallest real part. A similar result is shown for the leakage term in the Supplement of Ref. [212]. In addition, the tunneling term, which is similar to the second-order perturbative correction $\mathcal{P}_\Psi \mathcal{V} L^{-1} \mathcal{V} \mathcal{P}_\infty$ discussed in Sec. 4.1.2, does not contain contributions from $L_\square$.  

| Hamiltonian/DFS systems: operator notation | Hamiltonian/DFS systems: superoperator notation | Lindbladians: one NS block |
|------------------------------------------|-----------------------------------------------|-----------------------------|
| State basis | $|\psi_k\rangle = \text{DFS states}$ | $\Psi^\text{dfs}_\mu = (\Psi^\text{dfs}_\mu)^+ \in \text{span}\{ |\psi_k\rangle \langle \psi_1| \}$ |
| $P^\text{dfs} = \sum_{k=0}^{d-1} |\psi_k\rangle \langle \psi_k|$ | $\mathcal{P}^\text{dfs} = \sum_{\mu=0}^{d^2-1} |\Psi^\text{dfs}_\mu\rangle \langle \Psi^\text{dfs}_\mu| \langle \Psi^\text{dfs}\rangle = |\Psi^\text{dfs}_\mu\rangle \otimes |\chi_{\text{ax}}\rangle_{n_{\text{ax}}}$ |
| Connection | $A^\text{dfs}_{a,k,l} = i \langle \psi_k| \partial_a \psi_l \rangle$ | $A^\text{dfs}_{a,\mu\nu} = \langle \Psi^\text{dfs}_\mu| \partial_\alpha \Psi^\text{dfs}_\nu \rangle$ |
| Curvature | $F^\text{dfs}_{a,\beta} = \partial_{[a} A^\text{dfs}_{\beta]} - i [A^\text{dfs}_{a}, A^\text{dfs}_{\beta}]$ | $\mathcal{F}^\text{dfs}_{a,\mu\nu} = \partial_{[a} \mathcal{A}_{\beta]} + [A^\text{dfs}_{a}, A^\text{dfs}_{\beta}]$ |
| QGT | $Q^\text{dfs}_{a,k,l} = \langle \psi_k| \partial_a P^\text{dfs} \partial_\beta P^\text{dfs} |\psi_1\rangle - i \partial_a A^\text{dfs}_{\beta,k,l} - (A^\text{dfs}_{a} A^\text{dfs}_{\beta})_{k,l}$ | $Q^\text{dfs}_{a,k,l,\mu\nu} = \langle \Psi^\text{dfs}_\mu| \partial_{[a} P^\text{dfs} \partial_\beta P^\text{dfs} |\Psi^\text{dfs}_\nu\rangle = \partial_{[a} A^\text{dfs}_{\beta,\mu\nu} + (A^\text{dfs}_{a} A^\text{dfs}_{\beta})_{\mu\nu}$ |
| Metric tensor | $M^\text{dfs}_{a,\beta} = \text{Tr}\{ P^\text{dfs} \partial_{(a} P^\text{dfs} \partial_{\beta)} P^\text{dfs} \}$ | $M^\text{dfs}_{a,\beta} = \text{Tr}\{ P^\text{dfs} \partial_{(a} P^\text{dfs} \partial_{\beta)} P^\text{dfs} \}$ |

Table 5.1: Summary of quantities defined in Chs. 5 and 6.
“The main added value of the paper is that of providing results contained in 2-3 papers in a single one.”

– Anonymous Referee

QUANTUM GEOMETRIC TENSOR

Here, we introduce the Lindbladian QGT $Q$ and explicitly calculate it for the unique state and NS block cases [10]. The anti-symmetric part of the QGT is equal to the curvature $F$ generated by the connection $A$ (see Sec. 5.3.2). We show here that the symmetric part of the QGT produces a generalized metric tensor $M$ for parameter spaces associated with Lindbladian steady-state subspaces. We first review the Hamiltonian QGT for a single state in Sec. 6.1 and then extend to the DFS case in Sec. 6.2. The Lindbladian QGT is calculated in Sec. 6.3. We introduce other geometric quantities in Sec. 6.4, including an alternative geometric tensor $Q^{\text{alt}}$ whose curvature is different from the adiabatic curvature, but whose metric appears in the Lindbladian adiabatic path length. Most of the relevant quantities for the Hamiltonian, DFS, and Lindbladian cases are summarized in Table 5.1.

The original geometric quantity, later called the QGT by Berry [56], is introduced for Hamiltonian systems in Ref. [233]. This quantity encodes both a metric for measuring distances [19] and the adiabatic curvature. The QGT is experimentally probeable (e.g., via current noise measurements [205]). The Berry curvature can be obtained from adiabatic transport in Hamiltonian [28, 238, 309] and Lindbladian [25, 29] systems and even ordinary linear response (see Sec. 4.3.2). Singularities and scaling behavior of the metric are in correspondence with quantum phase transitions [75, 158, 318]. Conversely, flatness of the metric and curvature may be used to quantify stability of a given phase [44, 108, 139, 250], a topic of particular interest due to its applications in engineering exotic topological phases. Regarding generalization of the QGT to Lindbladians, to our knowledge there has been no introduction of a tensor including both the adiabatic curvature and a metric associated with $\text{As(H)}$. However, Refs. [35, 189] did apply various known metrics to study distinguishability within families of Gaussian fermionic and spin-chain steady states, respectively.

6.1 Hamiltonian Case

First let us review the non-degenerate Hamiltonian case before generalizing to degenerate Hamiltonians in operator/superoperator form. We recommend Refs. [27, 159] for detailed expositions.
Continuing from Sec. 5.1, we begin with an instantaneous zero-energy state $|\psi_0\rangle$ and projection $P_0 = |\psi_0\rangle\langle\psi_0|$ which are functions of a vector of control parameters $\{x_a\}$. The distance between the projections $P_0^{(s)}$ and $P_0^{(s+\delta)}$ along a path parameterized by $s \in [0,1]$ (with parameter vectors $x_a^{(s)}$ at each $s$) is governed by the QGT

$$Q_{a\beta,00} = \langle\psi_0|\partial_\alpha P_0 \partial_\beta P_0 |\psi_0\rangle = \langle\partial_\alpha |\psi_0\rangle (I - P_0) |\partial_\beta \psi_0\rangle.$$  

(6.1a)

The second form can be obtained from the former by explicit differentiation of $P_0$ and $\partial_\alpha P_0 \partial_\beta P_0 = (\partial_\alpha P_0) (\partial_\beta P_0)$ by convention. The $I - P_0$ term makes $Q_{a\beta,00}$ invariant upon the gauge transformations $|\psi_0\rangle \rightarrow e^{i\delta} |\psi_0\rangle$. The tensor can be split into symmetric and anti-symmetric parts,

$$2Q_{a\beta,00} = M_{a\beta,00} - iF_{a\beta,00},$$

(6.2)

which coincide with its real and imaginary parts. The anti-symmetric part is none other than the adiabatic/Berry curvature from eq. (Hol3). The symmetric part is the quantum Fubini-Study metric tensor [233]

$$M_{a\beta,00} = \text{Tr}\{P_0 \partial_\alpha P_0 \partial_\beta P_0\} = \text{Tr}\{\partial_\alpha P_0 \partial_\beta P_0\},$$

(6.3)

where $A_{(aB_\beta)} = A_a B_\beta + A_\beta B_a$ and the latter form can be obtained using $P_0 \partial_\alpha P_0 P_0 = 0$. This quantity is manifestly symmetric in $\alpha, \beta$ and real; it is also non-negative when evaluated in parameter space (see [242], Appx. D).

6.2 DFS Case

For degenerate Hamiltonian systems [185, 242] and in the DFS case, the QGT $Q^{dfs}$ is a tensor in both parameter $(a, \beta)$ and state $(k, l)$ indices and can be written as

$$Q^{dfs}_{a\beta,kl} = \langle\psi_k|\partial_\alpha P_{dfs} \partial_\beta P_{dfs} |\psi_l\rangle = \langle\partial_\alpha |\psi_k\rangle (I - P_{dfs}) |\partial_\beta \psi_l\rangle,$$

(6.4a)

(6.4b)

where $P_{dfs} = \sum_{k=0}^{d-1} |\psi_k\rangle\langle\psi_k|$ is the projection onto the degenerate zero eigenspace of $H(s)$. Since projections are invariant under changes of basis of their constituents, it is easy to see that $Q^{dfs}_{a\beta} \rightarrow R^\dagger Q^{dfs}_{a\beta} R$ under DFS changes of basis $|\psi_k\rangle \rightarrow |\psi_l\rangle R_{lk}$ for $R \in U(d)$. Notice that the QGT in eq. (6.4b) consists of overlaps between states outside of the zero eigenspace. For our applications, we write the QGT in a third way such that it consists of overlaps within the zero eigenspace only:

$$Q^{dfs}_{a\beta,kl} = -i\partial_\alpha A^{dfs}_{a\beta,kl} - (A^{dfs}_{a\beta} A^{dfs}_{a\beta})_{kl} - \langle\psi_k|\partial_\alpha \partial_\beta \psi_l\rangle,$$

(6.4c)

where $A^{dfs}_{a}$ is the DFS Berry connection and we used

$$0 = \partial_\beta \langle\psi_k|\psi_l\rangle = \langle\partial_\beta \psi_k|\psi_l\rangle + \langle\psi_k|\partial_\beta \psi_l\rangle$$

$$\partial_\alpha \langle\psi_k|\partial_\beta \psi_l\rangle = \langle\partial_\alpha \psi_k|\partial_\beta \psi_l\rangle + \langle\psi_k|\partial_\alpha \partial_\beta \psi_l\rangle.$$  

(6.5)
The Berry curvature is the part of the QGT anti-symmetric in $\alpha, \beta$ (here, also the imaginary part of the QGT): $F_{a\beta}^{dfs} = iQ_{a\beta}^{dfs}$. From (6.4c) we easily recover the proper form of the DFS Berry curvature listed in Table 5.1.

The symmetric part of the QGT appears in the infinitesimal distance between nearby parallel transported rays (i.e., states of arbitrary phase) $\psi(s)$ and $\psi(s + \delta s)$ in the degenerate subspace:

$$\langle \partial_s \psi | \partial_s \psi \rangle = \langle \partial_s \psi | (I - P_{dfs}) | \partial_s \psi \rangle ,$$  \hspace{1cm} (6.6)

where we used the parallel transport condition $P_{dfs} | \partial_s \psi \rangle = 0$. Expanding $\partial_s$ into parameter derivatives using eq. (5.14) and writing out $|\psi\rangle = \sum_{k=0}^{d-1} c_k |\psi_k\rangle$ yields

$$\langle \partial_s \psi | \partial_s \psi \rangle = \frac{1}{2} \sum_{a,\beta} \sum_{k,l=0}^{d-1} Q_{(a\beta),kl}^{dfs} x_a x_\beta c^*_k c_l .$$  \hspace{1cm} (6.7)

The corresponding Fubini-Study metric on the parameter space $M$ is $Q_{(a\beta)}^{dfs}$ traced over the degenerate subspace:

$$M_{a\beta}^{dfs} \equiv \sum_{k=0}^{d-1} Q_{(a\beta),kk}^{dfs} = \langle P_{dfs} | \partial_a P_{dfs} \partial_\beta P_{dfs} \rangle .$$  \hspace{1cm} (6.8)

All of this reasoning easily extends to the superoperator formalism ($|\psi_k\rangle \rightarrow |\psi^{dfs}_\mu\rangle$). The superoperator QGT corresponding to $Q^{dfs}$ can be written as

$$Q_{a\beta,\mu\nu}^{dfs} = \langle \langle \psi^{dfs}_\mu | \partial_a P_{dfs} \partial_\beta P_{dfs} | \psi^{dfs}_\nu \rangle \rangle$$  \hspace{1cm} (6.9)$$= \partial_a A_{\beta,\mu\nu}^{dfs} + (A_{a,\mu\nu}^{dfs} - \langle \langle \psi^{dfs}_\mu | \partial_a \partial_\beta \psi^{dfs}_\nu \rangle \rangle ,$$

where $A_{a,\mu\nu}^{dfs}$ is the adiabatic connection (5.23). The QGT is a real matrix (since $A_{a,\mu\nu}^{dfs}$ is real) and consists of parts symmetric ($Q_{(a\beta)}^{dfs}$) and antisymmetric ($Q_{[a\beta]}^{dfs}$) in $a, \beta$. Observing the second line of (6.9), it should be easy to see that the Berry curvature $F_{a\beta}^{dfs} = Q_{[a\beta]}^{dfs}$. The symmetric part of the superoperator QGT appears in the infinitesimal Hilbert-Schmidt distance ([50], Sec. 14.3) between nearby parallel transported DFS states $\rho(s)$ and $\rho(s + \delta s)$:

$$\langle \langle \partial_s \rho | \partial_s \rho \rangle \rangle = \langle \langle \partial_s \rho | (I - P_{dfs}) | \partial_s \rho \rangle \rangle ,$$  \hspace{1cm} (6.10)

where we used the parallel transport condition $P_{dfs} | \partial_s \rho \rangle = 0$. Similar manipulations as with the operator QGT, including the expansion $|\rho\rangle = \sum_{\mu=0}^{d^2-1} c_\mu |\psi^{dfs}_\mu\rangle$, yield

$$\langle \langle \partial_s \rho_\mu | \partial_s \rho_\mu \rangle \rangle = \frac{1}{2} \sum_{a,\beta} \sum_{\mu,\nu=0}^{d^2-1} Q_{(a\beta),\mu\nu}^{dfs} x_a x_\beta c_\mu c_\nu .$$  \hspace{1cm} (6.11)

The corresponding superoperator metric

$$M_{a\beta}^{dfs} \equiv \text{Tr} \{ P_{dfs} \partial_a P_{dfs} \partial_\beta P_{dfs} \} .$$  \hspace{1cm} (6.12)
6.3 Lindbladian Case

Now let us turn to the Lindbladian QGT and show that its symmetric part produces a generalized metric tensor for parameter spaces associated with Lindbladian steady-state subspaces. In Ch. 6, we showed using the operator representation of the adiabatic connection and the conditions (LP1-LP2) that the minimal projection \( P_\Psi = R P_\Psi \) (and not \( P_\xi \)) generates adiabatic evolution within \( \text{As}(H) \). Following this, we define

\[
Q_{\alpha\beta} \equiv P_\Psi \partial_\alpha P_\Psi \partial_\beta P_\Psi P_\Psi
\]

(6.14)

to be the associated QGT. While \( P_\Psi = \sum_\mu |\Psi_\mu\rangle \langle |\Psi_\mu\rangle \) is not always Hermitian due to \( J^\mu \Phi \neq \Psi_\mu \) (e.g., in the NS case), we show that the QGT nevertheless remains a meaningful geometric quantity. Looking at the matrix elements of \( Q_{\alpha\beta} \) and explicitly plugging in the instantaneous \( P_\Psi \) (5.42) yields the following three forms:

\[
Q_{\alpha\beta,\mu\nu} = \langle \langle J^\mu_\Phi | \partial_\alpha P_\Psi \partial_\beta P_\Psi | \Psi_\nu \rangle \rangle \quad (6.15a)
\]

\[
= \langle \langle \partial_\alpha J^\mu_\Phi | (I - P_\Psi) | \partial_\beta \Psi_\nu \rangle \rangle \quad (6.15b)
\]

\[
= \partial_\alpha \Lambda_\beta,\mu\nu + (\Lambda_\alpha \Lambda_\beta)_\mu\nu - \langle \langle J^\mu_\Phi | \partial_\alpha \partial_\beta \Psi_\nu \rangle \rangle , \quad (6.15c)
\]

with \( \Lambda_\alpha \) the Lindblad adiabatic connection (5.34). Since \( \Lambda_\alpha,\mu\nu \) are real and \( \{ J^\mu_\Phi, \Psi_\nu \} \) are Hermitian, the matrix elements are all real. From its second form, one easily deduces that the QGT transforms as \( Q_{\alpha\beta} \rightarrow R^{-1} Q_{\alpha\beta} R \) for any basis transformation \( R \in GL[\dim \text{As}(H), \mathbb{R}] \) [see eq. (5.35)]. Each matrix \( Q_{\alpha\beta} \) consists of parts symmetric \( (Q_{(\alpha\beta)} \rangle \rangle \) and antisymmetric \( (Q_{[\alpha\beta]} \rangle \rangle ) \) in \( \alpha, \beta \). From the third form, it is evident that its anti-symmetric part is exactly the adiabatic curvature \( F_{\alpha\beta} \) from eq. (5.51) (cf. [25], Prop. 13). The rest of this Section is devoted to calculating the symmetric part and its corresponding metric on \( M \), which is defined as the trace of the symmetric part of the QGT,

\[
\mathcal{M}_{\alpha\beta} \equiv \text{Tr} \{ P_\Psi \partial_\alpha P_\Psi \partial_\beta P_\Psi \} = \sum_{\mu=0}^{d^2-1} Q_{(\alpha\beta),\mu\mu} . \quad (6.16)
\]

Before proving that this is a metric for some of the relevant cases, let us first reveal how such a structure corresponds to an infinitesimal distance between adiabatically connected Lindbladian steady states by adapting results from non-Hermitian Hamiltonian systems [68, 69, 204]. The zero eigenspace of \( L_\Phi \) is diagonalized by right and left eigenmatrices \( |\Psi_\mu\rangle \) and \( \langle |\Psi_\mu\rangle \) respectively. In accordance with this duality between \( \Psi \) and \( J^\mu_\Phi \), we introduce an associated operator \( |\rho_\mu\rangle \rangle \) [68, 69],

\[
|\rho_\mu\rangle \rangle = \sum_{\mu=0}^{d^2-1} c_\mu |\Psi_\mu\rangle \leftrightarrow |\rho_\mu\rangle \rangle \equiv \sum_{\mu=0}^{d^2-1} c_\mu |J^\mu_\Phi \rangle \rangle , \quad (6.17)
\]
to every steady-state subspace operator $|\rho_\infty\rangle\rangle$. This allows us to define a modified inner product $\langle\langle A|B\rangle\rangle$ for matrices $A$ and $B$ living in the steady-state subspace. Since $\Psi_{\mu}$ and $J_{\mu}^\nu$ are biorthogonal ($\langle\langle J_{\mu}^\nu|\Psi_{\nu}\rangle\rangle = \delta_{\mu\nu}$), this inner product is surprisingly equivalent to the Hilbert-Schmidt inner product $\langle\langle A|B\rangle\rangle$. However, the infinitesimal distance is not the same:

$$\langle\langle \partial_s \hat{\rho}_\infty | \partial_s \rho_\infty \rangle\rangle \neq \langle\langle \partial_s \rho_\infty | \partial_s \rho_\infty \rangle\rangle.$$  

(6.18)

The symmetric part $Q_{(a\beta)}$ shows up in precisely this modified infinitesimal distance. Using eq. (6.17), the parallel transport condition (5.32), and parameterizing $\partial_s$ in terms of the $\partial_a$’s (5.25) yields

$$\langle\langle \partial_s \hat{\rho}_\infty | \partial_s \rho_\infty \rangle\rangle = \frac{1}{2} \sum_{\alpha\beta, \mu, \nu} d^2 Q_{(a\beta)\mu\nu} x_{a\beta} c_{\mu\nu},$$

(6.19)

as evidenced by the second form (6.15b) of the Lindblad QGT. Tracing the symmetric part over the steady-state subspace gives the metric $M_{a\beta}$.

6.3.1 Unique state case

Here things simplify significantly, yet the obtained metric turns out to be novel nonetheless. The asymptotic projection is $P_\Psi = |q\rangle\rangle \langle\langle P|$ and a straightforward calculation using eq. (6.15b) yields

$$M_{a\beta} = \langle\langle (\partial_a P)|\partial_\beta q\rangle\rangle.$$

(6.20)

Using the eigendecomposition $q = \sum_{k=0}^{d_s-1} \lambda_k |\psi_k\rangle \langle\psi_k|$, $M_{a\beta} = 2 \sum_{k=0}^{d_s-1} \lambda_k \langle\langle (\partial_a |\psi_k\rangle|Q|\partial_\beta |\psi_k\rangle\rangle$, (6.21)

where $Q = I - P$ and $(\partial_a |\psi_k\rangle|Q|\partial_\beta |\psi_k\rangle\rangle$ is the Fubini-Study metric corresponding to the eigenstate $|\psi_k\rangle$. In words, $M_{a\beta}$ is the sum of the eigenstate Fubini-Study metrics weighted by their respective eigenvalues/populations. If $q$ is pure, then it is clear that $M_{a\beta}$ reduces to the Fubini-Study metric. Finally, if $q$ is full rank, then $P = I$ and $M_{a\beta} = 0$. This means that the metric is non-zero only for those $q$ which are not full rank.

6.3.2 NS case

Recall from eq. (5.42) that adiabatic evolution on the NS is parameterized by the instantaneous minimal projections

$$P_{\Psi}^{(s)} = S(s) \left( P_{dfs}^{(s)} \otimes |\rho_\infty^{(s)}\rangle\rangle \langle\langle P_{dfs}^{(s)}| \right) S^\dagger(s),$$

(6.22)

where $P_{dfs}(\cdot) = \sum_{\mu=0}^{d_s-1} |\Psi_{\mu}^{dfs}\rangle\rangle \langle\langle \Psi_{\mu}^{dfs}|$ is the superoperator projection onto the $x_s$-independent DFS reference basis. We remind the reader (see Sec. 5.3.1) that the only assumption of such a parameterization is that the state $|\rho_\infty^{(s)}\rangle$ is unitarily equivalent (via unitary $S$) to a tensor product of a DFS state and auxiliary part for all points $s \in [0, 1]$ in the path.
We can simplify $M_{\alpha\beta}$ and show that it is indeed a metric (more technically, a semi-metric). In the reference basis decomposition of $P_{\Psi}$ from eq. (6.22), the operators $G_\alpha \equiv iS^\dagger \partial_\alpha S$ (with $S(s)|\rho\rangle \equiv |Ss^\dagger\rangle$) generate motion in parameter space. After significant simplification, one can express $M_{\alpha\beta}$ in terms of these generators:

$$M_{\alpha\beta} = M_{\alpha\beta}^{(1)} + M_{\alpha\beta}^{(2)}$$

$$M_{\alpha\beta}^{(1)} = 2d\langle \langle P_{\text{dfs}} \otimes \varrho_{\text{ax}} | G_\alpha (I - P_{\text{dfs}} \otimes P_{\text{ax}}) G_\beta \rangle \rangle$$

$$M_{\alpha\beta}^{(2)} = 2d\langle \langle G_\alpha | P_{\text{dfs}}^* \otimes O_{\text{ax}} | G_\beta \rangle \rangle$$

with projection $P_{\text{dfs}}^*$ consisting of only traceless DFS generators (we set $\varpi_{dfs} = \frac{1}{\sqrt{d}} P_{dfs}$),

$$P_{dfs}^* = \sum_{\mu=1}^{d^2-1} |\varpi_{dfs}^{-1} \rangle \langle \varpi_{dfs}^{-1}| = P_{dfs} - |\varpi_{dfs}^0 \rangle \langle \varpi_{dfs}^0|,$$  

(6.24)

and auxiliary superoperator defined (for all auxiliary operators $A$) as $O_{\text{ax}}(A) \equiv (A - \langle \langle \varrho_{\text{ax}} | A \rangle \rangle) \varrho_{\text{ax}}$.

The quantity $M_{\alpha\beta}$ is clearly real and symmetric in $\alpha, \beta$, so to show that it is a metric, we need to prove positivity ($w_\alpha M_{\alpha\beta} w_\beta \geq 0$, with sum over $\alpha, \beta$ implied, for all vectors $w$ in the tangent space $T_M(x)$ at a point $x \in M$ [202]). Since $\varrho_{ax}$ is positive definite, one can show that the first term in (6.23)

$$w_\alpha M_{\alpha\beta}^{(1)} w_\beta = 4d\langle \langle O | O \rangle \rangle \geq 0$$

(6.25)

with $O = (I - P_{dfs} \otimes P_{ax})(G_\alpha w_\alpha)(P_{dfs} \otimes \varrho_{dfs}^* \varrho_{dfs})$. For the second term $M_{\alpha\beta}^{(2)}$, we can see that $P_{dfs}^*$ is positive semidefinite since it is a projection. We show that $O_{ax}$ is positive semidefinite by utilizing yet another inner product associated with open systems [13]. First note that

$$\langle \langle A | O_{ax} | A \rangle \rangle = \text{Tr}\{\varrho_{ax} A^\dagger A\} - |\text{Tr}\{\varrho_{ax} A\}|^2.$$  

(6.26)

Since $\varrho_{ax}$ is full-rank, $\langle \langle A | B \rangle \rangle_{\varrho_{ax}} \equiv \text{Tr}\{\varrho_{ax} A^\dagger B\}$ is a valid inner product [13] and $\langle \langle A | O_{ax} | A \rangle \rangle \geq 0$ is merely a statement of the Cauchy–Schwarz inequality associated with this inner product. For Hermitian $A$, (6.26) reduces to the variance of $\langle \langle A | \varrho_{ax} \rangle \rangle$.

Roughly speaking, the first term $M_{\alpha\beta}^{(1)}$ describes how much the DFS and auxiliary parts mix and the second term $M_{\alpha\beta}^{(2)}$ describes how much they leave the $P$ block while moving in parameter space. For the DFS case, $M_{\alpha\beta}^{(2)} = 0$ (due to $O_{ax} = 0$ for that case) and the metric reduces to the standard DFS metric covered in Sec. 6.2. For the unique state case, $M_{\alpha\beta}^{(2)}$ is also zero (due to $P_{dfs}^*$ not containing any traceful DFS elements and thus reducing to zero when $P_{dfs} = 1$). The mixing term $M_{\alpha\beta}^{(2)}$ is thus of course nonzero only in the NS block case.

### 6.4 Other Geometric Tensors

In the previous Section, we showed that the anti-symmetric part of the QGT

$$Q = P_{\Psi} \partial P_{\Psi} \partial P_{\Psi} P_{\Psi}$$

(6.27)
corresponds to the curvature $\mathcal{F}$ associated with the adiabatic connection $A$ from Ch. 6. We thus postulate that this QGT and its corresponding symmetric part should be relevant in determining distances between adiabatically connected Lindbladian steady states. However, the story does not end there as there are two more tensorial quantities that can be defined using the steady-state subspace. The first is an extension of the Fubini-Study metric to non/pseudo-Hermitian Hamiltonians [68, 69, 197, 198] (different from [204]) that can also be generalized to Lindblad systems; we do not further comment on it here. The second is the alternative geometric tensor

$$Q_{\text{alt}} = \mathcal{P}_\Psi^\dagger \frac{\partial}{\partial \Psi^\dagger} \mathcal{P}_\Psi, \quad (6.28)$$

which is different from the QGT due to $\mathcal{P}_\Psi$ not being Hermitian. We show that $Q_{\text{alt}}$ appears in a bound on the adiabatic path length for Lindbladian systems, which has traditionally been used to determine the shortest possible distance between states in a parameter space $M$. Here we introduce the adiabatic path length, generalize it to Lindbladians, and comment on $Q_{\text{alt}}$.

6.4.1 Hamiltonian case

The adiabatic path length for Hamiltonian systems quantifies the distance between two adiabatically connected states $|\psi_0^{(s=0)}\rangle$ and $|\psi_0^{(1)}\rangle$. The adiabatic evolution operator (derived in Sec. 5.1) for an arbitrary path $s \in [0, 1]$ and for initial zero-energy state $|\psi_0^{(0)}\rangle$ is

$$U^{(1)} = \mathcal{P} \exp \left( \int_0^1 \mathcal{P}_0 \mathcal{P} ds \right). \quad (6.29)$$

Consider the Frobenius norm $\| U^{(1)} \|$ of $U^{(1)}$. By expanding the definition of the path-ordered exponential, one can show that $\| U^{(1)} \| \leq \exp(L_0)$ with path length

$$L_0 = \int_0^1 \| \mathcal{P}_0 \mathcal{P} \| ds. \quad (6.30)$$

Remembering that $\| A \| = \sqrt{\text{Tr}(A^\dagger A)}$ and writing $\partial_s$ in terms of parameter derivatives, we see that the Fubini-Study metric appears in the path length:

$$\| \mathcal{P}_0 \mathcal{P} \|^2 = \frac{1}{2} \sum_{\alpha, \beta} M_{\alpha,\beta,00} \dot{x}_\alpha \dot{x}_\beta. \quad (6.31)$$

Therefore, the shortest path between states in Hilbert space projects to a geodesic in parameter space satisfying the Euler-Lagrange equations associated with the metric $M_{\alpha,\beta,00}$ and minimizing the path length (e.g., [202], eq. (7.58)) (with sum implied)

$$L_0 = \int_0^1 \sqrt{\frac{1}{2} G_{\alpha,\beta,00} \dot{x}_\alpha \dot{x}_\beta} ds. \quad (6.32)$$

In Hamiltonian systems, the adiabatic path length appears in bounds on corrections to adiabatic evolution ([142], Thm. 3; see also [242]). This path length is also applicable when one wants to simulate adiabatic evolution in a much shorter time (counter-diabatic/superadiabatic dynamics...
6.4 OTHER GEOMETRIC TENSORS

[100, 176, 287] or shortcuts to adiabaticity [57, 285]) by explicitly engineering the Kato Hamiltonian $i[P_0, P_0]$ from eq. (5.5).

6.4.2 Lindbladian case

The tensor $Q_{\alpha \beta}^{alt}$ arises in the computation of the corresponding Lindbladian adiabatic path length

$$L \equiv \int_0^1 \| \mathcal{P}_\Psi \mathcal{P}_\Psi \| ds ,$$

(6.33)

where the superoperator norm of $\mathcal{P}_\Psi \mathcal{P}_\Psi$ is the analogue of the operator Frobenius norm from eq. (1.26): $\| O \| \equiv \sqrt{\text{Tr} \{ O^\dagger O \} }$ where $O$ is a superoperator. This path length provides an upper bound on the norm of the Lindblad adiabatic evolution superoperator (5.30)

$$\mathcal{U}^{(1,0)} = \mathcal{P} \exp \left( \int_0^1 \mathcal{P}_\Psi \mathcal{P}_\Psi ds \right).$$

(6.34)

Using properties of norms and assuming one NS block, it is straightforward to show that

$$\| \mathcal{U}^{(1,0)} \| \leq \exp(L) \quad \text{with} \quad L = \int_0^1 \sqrt{ \frac{1}{2} d_{ax} M_{\alpha \beta}^{alt} \dot{x}_\alpha \dot{x}_\beta } ds$$

(6.35)

(with sum over $\alpha, \beta$ implied). The metric governing this path length turns out to be

$$M_{\alpha \beta}^{alt} = \langle\langle \rho_{ax} | \rho_{ax} \rangle \rangle \sum_{\alpha, \beta} Q_{(\alpha \beta), \mu \mu}^{alt}.$$  

(6.36)

For a unique steady state $\rho$, this alternative metric reduces to the Hilbert-Schmidt metric

$$M_{\alpha \beta}^{alt} = \langle\langle \partial_{(\alpha} \rho | \partial_{\beta)} \rho \rangle \rangle .$$

(6.37)

Note the subtle difference between this metric and the QGT metric $M_{\alpha \beta} = \langle\langle \partial_{(\alpha} P | \partial_{\beta)} \rho \rangle \rangle.$

This difference is precisely due to the absence of $\rho$ in the left eigenmatrices $J_{[\alpha]}. For the QGT metric, $\rho$ is never in the same trace twice while for the alternative metric, the presence of $P_\Psi^\dagger$ yields such terms. We note that for a pure steady state $\rho = P$ (with $P$ being rank one), both metric tensors reduce to the Fubini-Study metric.

Another notable example is the DFS case ($\rho_{ax} = 1$). In that case, $J_{[\alpha]} = \Psi_{\mu}$ — the QGT and alternative tensor become equal ($Q^{alt} = Q$). It is therefore the presence of $\rho_{ax}$ that allows for two different metrics $M_{\alpha \beta}$ and $M_{\alpha \beta}^{alt}.$ However, for the NS case, the “alternative” curvature $Q_{\alpha \beta}^{alt}$ does not reduce to the adiabatic curvature $F_{\alpha \beta, \mu \nu}$ associated with the connection $A_a$ (unlike the QGT curvature). How this subtle difference between $Q_{\alpha \beta}$ and $Q_{\alpha \beta}^{alt}$ for the NS and unique steady state cases is relevant in determining distances between adiabatic steady states of Lindbladians should be a subject of future investigation.
“At the first of the 1960’s Rochester Coherence Conferences, I suggested that a license be required for use of the word ‘photon’, and offered to give such a license to properly qualified people. My records show that nobody working in Rochester, and very few other people elsewhere, ever took out a license to use the word ‘photon’.”

– Willis E. Lamb

APPLICATION: DRIVEN TWO-PHOTON ABSORPTION

This chapter consists of a detailed investigation of a Hamiltonian-driven version of the two-photon absorption process from Sec. 3.3.1 ([130]; [236], Sec. 13.2.2). This is also the same case we discussed in the overview of results in Sec. 1.5. Variants of this case are also manifest in the degenerate parametric oscillator ([308]; see also [81], eq. 12.10), a laser-driven trapped ion ([226], Fig. 2d; see also [85, 121]), nano-mechanical systems [294], and superconducting qubit systems [11, 30, 42, 95, 114, 168, 192, 194, 286] (where this case is colloquially known as the “two-cat pump”).

7.1 THE LINDBLADIAN AND ITS STEADY STATES

Consider a Lindbladian with a single jump operator

\[ F = a^2 - \alpha^2 = (a - \alpha)(a + \alpha), \]

where \( \alpha \in \mathbb{R} \), \([a, a^\dagger] = I \) and \( \hat{n} \equiv a^\dagger a \). Due to the “gauge” symmetry (1.29), this is equivalent to adding a squeezing Hamiltonian \( H = -i\alpha^2(a^2 - a^4) \) to a Lindbladian with the undriven two-photon absorption jump operator \( F = a^2 \). Unlike the driven case of the two-qubit example from Sec. 3.2.4, in which driving takes the DFS into an NS, here the undriven DFS remains a DFS for all driving parameters. Recall from Sec. 3.3.1 that, for \( \alpha = 0 \), \( \text{As}(H) \) is a qubit and consists of Fock states \(|k\rangle, k \in \{0, 1\} \) (since \( F \) annihilates both). We have also already mentioned in Ch. 1 that, for large enough \( \alpha \), \( \text{As}(H) \) remarkably retains its qubit form, which this time is spanned by superpositions of coherent states \(|\pm \alpha\rangle\). Here, a treatment is given which is valid for all \( \alpha \). One
may have noticed that both states $|\pm \alpha\rangle$ go to $|0\rangle$ in the $\alpha \to 0$ limit and do not reproduce the $\alpha = 0$ steady state basis. This issue is resolved by introducing the cat state basis \[ |\alpha\rangle \quad \text{(large $\alpha$)} \]

with normalization $\pi_k \equiv \frac{1}{2}[1 + (-)^k \exp(-2\alpha^2)]$. As $\alpha \to 0$, cat states approach Fock states while for $\alpha \to \infty$, the cat states (exponentially) quickly become “macroscopic” superpositions of $|\pm\alpha\rangle$. This Lindbladian thus has only two distinct parameter regimes: one in which coherent states come together ($\alpha \approx 1$) and one in which they are well-separated ($\alpha \gg 1$, or more practically $\alpha \geq 2$). Eq. (7.2) shows that (for large enough $\alpha$) cat states and coherent states become conjugate $z$- and $x$-bases respectively, forming the As(H) qubit (see Fig. 7.1). Using projections $\Pi_k = \sum_{n=0}^{\infty} |2n+k\rangle \langle 2n+k|$ (3.20), cat states can be concisely written as projected (and normalized) coherent states:

\[ |\alpha\rangle \equiv \frac{\Pi_k |\alpha\rangle}{\sqrt{\langle \alpha | \Pi_k |\alpha\rangle}} \quad \text{with normalization} \quad \pi_k \equiv \langle \alpha | \Pi_k |\alpha\rangle = \frac{1 + (-)^k e^{-2\alpha^2}}{2}. \] (7.3)

The projections are orthogonal: $\Pi_k \Pi_l = \delta_{kl}^{\text{mod}2} \Pi_k$, where $\delta_{qp}^{\text{mod}2} = 1$ whenever $q = p \text{ mod } 2$. Action of lowering or raising operators switches subspaces [see eq. (3.24)], implying that

\[ a \Pi_k = \Pi_{k+1 \text{mod}2}. \] (7.4)

The cat state label $k \in \{0, 1\}$ corresponds to the respective $\pm 1$ eigenspace of the parity operator $(-)^a = \Pi_0 - \Pi_1$. This parity operator commutes with $F$ for all $\alpha$, so Op(H) is split into four blocks,

\[ \{ |2n+k\rangle \langle 2m+l| \}_{n,m=0}^{\infty} \quad \text{(labeled by } k, l \in \{0, 1\}\text{)}, \] (7.5)

which evolve independently of each other. The outer product $\Psi_{kl} \equiv |k\rangle \langle l| \Pi_k \Pi_l$ is the unique steady-state basis element in the respective block $\{ |2n+k\rangle \langle 2m+l| \}_{n,m=0}^{\infty}$, and together the basis elements $\{ \Psi_{kl} \}_{k,l=0}^1$ span As(H) = \[.\] Outer products of all states orthogonal to $|\alpha\rangle$ span the decaying subspace \[.\]
The cat state basis, unlike the coherent state basis, is orthonormal for all values of $\alpha$ and simplifies most of the calculations done here, with all of the complexity coming from the normalization factors $\pi_k$. For example, using eqs. (7.3-7.4), orthogonality of projections, and the property of coherent states $a|\alpha\rangle = \alpha|\alpha\rangle$, the cat states have average occupation number

$$
\langle k_n|\hat{a}|k_\alpha\rangle = \frac{\langle \alpha|\Pi_k a^\dagger \Pi_{k+1}|\alpha\rangle}{\pi_k} = \alpha^2 \frac{\pi_{k+1}}{\pi_k} = \begin{cases} 
 k + O(\alpha^4) & \alpha \to 0 \\
 \alpha^2 + O(\alpha^2 e^{-2\alpha^2}) & \alpha \to \infty 
\end{cases}.
$$

(7.6)

This is sensible since Fock states have distinct average occupation numbers while coherent states with the same magnitude $\alpha$ have the same average occupation number.

### 7.2 Conserved Quantities

We now search for the four conserved quantities corresponding to $\Psi_{kl}$. By the correspondence from Thm. 3, there exist four $\{J^{kl}\}_{k,l=0}^1$ such that

$$
\mathcal{L}^\dagger (J^{kl}) = F^\dagger J^{kl} F - \frac{1}{2} \{F^\dagger F, J^{kl}\} = 0.
$$

(7.7)

Since parity symmetry is preserved, the diagonal $k = l$ conserved quantities remain the same as for the $\alpha = 0$ case: $J^{kk} = \Pi_k$. One can use Thm. 4 to determine the remaining conserved quantity $J^{01}$. However, since $F_{\alpha} \neq 0$, inverting $\mathcal{L}_{\alpha}$ is non-trivial. Fortunately, this inversion can be avoided and we can use the $\alpha = 0$ conserved quantity (3.26; now renamed to $J^{01,\alpha=0}$) to determine $J^{01}$. To do so, we apply $\mathcal{L}^\dagger$ to $J^{01,0}$, which yields nonzero terms only from the $\alpha$-dependent part of $\mathcal{L}^\dagger$ (since $J^{01,0}$ is conserved under the $\alpha$-independent part). These nonzero terms, which we call $J^{01,\alpha=\pm1}$, can in turn be plugged into $\mathcal{L}^\dagger$ themselves. Such recursive steps produce a pattern: the quantities $J^{01,q}$ (labeled by $q \in \mathbb{Z}$) turn out to be

$$
J^{01,q} = \begin{cases} 
\frac{(q-1)!!}{(n+2q)!!} \Pi_0 a^{2q+1} & q \geq 0 \\
\Pi_0 a^{2q+1} \frac{q!!}{(n+2|q|)!!} & q < 0 
\end{cases}
$$

(7.8)

and the equation of motion they satisfy is

$$
\mathcal{L}^\dagger (J^{01,q}) = \frac{1}{2} (2q + 1) \left[ a^2 \left( J^{01,q-1} - J^{01,q+1} \right) - 2q J^{01,q} \right].
$$

(7.9)

Recall that we are looking for a conserved quantity $J^{01}$ such that $\mathcal{L}^\dagger (J^{01}) = 0$. Since $J^{01} \to J^{01,\alpha=0}$ for $\alpha \to 0$ and since the set $\{J^{01,q}\}_{q \in \mathbb{Z}}$ is closed under application of $\mathcal{L}^\dagger$, $J^{01}$ for any $\alpha$ must be constructed out of the $J^{01,q}$’s:

$$
J^{01} \propto \sum_{q \in \mathbb{Z}} a_q J^{01,q},
$$

(7.10)

with some coefficients $a_q$. Determining these coefficients becomes easy when one notices that the equations of motion for $J^{01,q}$ mimic the recurrence relation

$$
a^2 \left[ I_{q-1} (\alpha^2) - I_{q+1} (\alpha^2) \right] + 2q I_q (\alpha^2) = 0
$$

(7.11)
satisfied by the modified Bessel functions of the first kind $I_q$ \[201\]. Taking care of the factor of $2q + 1$ in eq. (7.9) and an extra $q$-dependent sign yields

$$a_q = \frac{(-)^q I_q (\alpha^2)}{2q + 1}.$$  

(7.12)

Now all that is left is to biorthogonalize the $J_{01}^{\alpha}$ with its corresponding As(H) basis element $\Psi_{01}$, i.e., make sure that $\langle 0_\alpha | J_{01}^{\alpha\dagger} | 1_\alpha \rangle = 1$. Explicitly calculating

$$\langle \langle J_{01}^{\alpha} | \Psi_{01} \rangle \rangle = \langle 0_\alpha | J_{01}^{\alpha\dagger} | 1_\alpha \rangle = \sqrt{\frac{2\alpha^2}{\sinh 2\alpha^2}} I_q (\alpha^2)$$  

and using eq. (5.8.6.2) from [234],

$$\sum_{q \in \mathbb{Z}} \frac{(-)^q}{2q + 1} I_q (\alpha^2) I_q (\alpha^2) = \frac{\sinh 2\alpha^2}{2\alpha^2},$$  

(7.14)

we obtain the properly normalized conserved quantity

$$J_{01} = \sqrt{\frac{2\alpha^2}{\sinh 2\alpha^2}} \sum_{q \in \mathbb{Z}} \frac{(-)^q}{2q + 1} I_q (\alpha^2) J_{01,q}.$$  

(7.15)

One can check that $L^{\alpha \dagger} (J_{01}) = 0$ as follows. First, use linearity and the equation of motion (7.9) for $J_{01,q}$. Then, observe that each $J_{01,q}$ is supported on a different set of Fock state outer products $\{|2n\rangle \langle 2n + 1|\}_{n=0}^\infty$ for $q \geq 0$ and $\{|2n + 2\rangle \langle 2n + 1|\}_{n=0}^\infty$ for $q < 0$. This means that the coefficient in front of each $J_{01,q}$ must be zero for $J_{01}$ to be conserved. Rearranging the three infinite sums (coming from $J_{01,q}$, $J_{01,q+1}$, and $J_{01,q-1}$) in order to obtain that coefficient yields exactly the Bessel function recursion relation above.

### 7.3 State Initialization

We now determine the asymptotic state

$$\rho_\infty = \sum_{k,l=0}^1 c_{kl} \Psi^{kl} = \sum_{k,l=0}^1 c_{kl} |k_\alpha \rangle \langle l_\alpha|$$  

(7.16)

starting from an initial coherent state $\rho_{\text{in}} = |\beta \rangle \langle \beta|$. By the correspondence from Thm. 3, we know that $c_{kl} = \langle \langle J^{kl} | \rho_{\text{in}} \rangle \rangle$.

#### 7.3.1 Steady state for an initial fixed-parity state

Due to the decoupling of the blocks $\{|2n+k\rangle \langle 2m+l|\}_{n,m=0}^\infty$ (7.5), any state which starts exclusively in one of the blocks evolves within that block into the block’s fixed point $|k_\alpha \rangle \langle l_\alpha|$. Therefore, if we start in any state of fixed parity $k \in \{0,1\}$ (i.e., $(-)^{\hat{\alpha}^k} = (-)^k$ for that state), we necessarily converge to the pure asymptotic state $\rho_\infty = |k_\alpha \rangle \langle k_\alpha|$. This holds true for mixed fixed-parity initial
states as well, which is an example of the environment (which induces this Lindbladian) taking entropy out of the system.

7.3.2 Steady state for an initial coherent state

Now let \( \rho_{in} = |\beta\rangle \langle \beta | \) for some \( \beta \in \mathbb{C} \). The diagonal quantities \( c_{kk} = \langle J_{kk} | \rho_{in} \rangle \) have already been determined in eq. (3.29). The tricky part is the off-diagonal quantity, which simplifies to

\[
c_{01} = \langle \beta | J^{01} | \beta \rangle = \frac{i \alpha \beta^* e^{-|\beta|^2}}{\sqrt{2 \sinh 2 \alpha^2}} \int_{\phi=0}^{\pi} d\phi e^{-i\phi} I_0 \left( |\alpha|^2 - |\beta|^2 e^{2i\phi} \right) \tag{7.17}
\]

To derive this, we first apply eq. (7.15) to obtain the sum

\[
c_{01} = \frac{\sqrt{2\alpha} \beta^* e^{-|\beta|^2}}{\sqrt{\sinh 2 \alpha^2}} \sum_{q \in \mathbb{Z}} \frac{(-1)^q}{2q + 1} I_q (\alpha^2) I_q (|\beta|^2) e^{-i2q\theta}, \tag{7.18}
\]

where \( \theta = \arg \beta \). This sum is convergent because the sum without the \( 2q + 1 \) term is an addition theorem for \( I_q \) [eq. (5.8.7.2) from [234]]. To put the above into integral form, we use the identity (derivable from the addition theorem)

\[
I_q (\alpha^2) I_q (|\beta|^2) = \frac{1}{2\pi^2} \int_{\phi=0}^{2\pi} d\phi e^{iq(\phi + \pi)} I_0 \left( |\alpha|^2 - |\beta|^2 e^{i\phi} \right) \tag{7.19}
\]

Plugging in the above identity into eq. (7.18), interchanging the sum and integral (possible because of convergence), evaluating the sum (which is a simple Fourier series), and performing a change of variables yields the integral formula for \( c_{01} \).

Using eq. (5.8.1.15) from [234], one can calculate limits for large \( |\beta| \) along the real and imaginary axes in phase space of \( \beta \):

\[
\lim_{\beta \to \infty} c_{01} = \frac{1}{2} \frac{\text{erf}(\sqrt{2} \alpha)}{\sqrt{1 - e^{-4\alpha^2}}} \xrightarrow{\alpha \to \infty} \frac{1}{2} \quad \text{and} \quad \lim_{\beta \to \infty} c_{01} = -\frac{i}{2} \frac{\text{erfi}(\sqrt{2} \alpha)}{e^{4\alpha^2} - 1} \xrightarrow{\alpha \to \infty} 0, \tag{7.20}
\]

where \( \text{erf} \) and \( \text{erfi} \) are the error function and imaginary error function, respectively. Recalling that \( c_{kk} \to \frac{1}{2} \) in both limits (see Sec. 3.3.3), we see that \( \rho_{in} \) becomes pure when \( \beta \) is real and large and that \( \rho_{in} \) becomes maximally mixed when \( \beta \) is pure imaginary and large. To study the remaining sectors of \( \beta \) phase space, we numerically calculate the purity for a lattice of \( \beta \)'s in Fig. 7.2 for \( \alpha \) being 0.001, \( \frac{1}{2} \), 1, and 5. The rightmost panel shows the behavior for large \( \alpha \), showing that initial states \( \beta \) near the respective steady states \( |\pm \alpha\rangle \) converge to pure states. In fact, one can show that those pure states are exactly \( |\pm \alpha\rangle \). In other words, the two-photon system is similar to a classical double-well system in the combined large \( \alpha, \beta \) regime. However, starting in the state \( \propto |\beta\rangle + |-\beta\rangle \) for any \( \beta \) guarantees a pure asymptotic state by the symmetry arguments of the previous Subsection. Therefore, while a “classical” initial state \( |i\alpha\rangle \) results in a maximally mixed asymptotic state (in the large \( \alpha \) limit), the \( |-i\alpha\rangle \) component in an initial cat state \( \propto |i\alpha\rangle + |-i\alpha\rangle \) cancels that effect and results in a pure asymptotic state!
7.4 Ordinary Perturbation Theory

Figure 7.2: Purity of the asymptotic state of the driven two-photon absorption Lindbladian. The four panels correspond to $\alpha$ being 0.001, $\frac{1}{2}$, 1, and 5, respectively. For each panel, a point $\beta$ in phase space corresponds to the purity ($\text{Tr} \{\rho_\infty^2\}$) of the asymptotic state $\rho_\infty$ (7.16) given an initial coherent state $|\beta\rangle$. Besides $\rho_\infty$ being pure away from the vertical axis when $\alpha \gg 1$, one can observe that $\rho_\infty$ is also pure for initial states near the center of phase space. Indeed, starting in the vacuum state ($\beta = 0$), a fixed-parity state, the system is driven to the pure Schrödinger cat state $|0\alpha\rangle$ (7.3).

7.3.3 Steady state for an initial cat state

Now let us briefly consider an initial state proportional to $|\alpha\rangle + e^{i\theta}|\beta\rangle$ with $\beta \neq -\alpha$, i.e., a cat state in which one component is already in $\text{As}(H)$. For simplicity, let us consider the large $\alpha$ limit, meaning that all we say is true up to exponentially small corrections due to the overlap between coherent states. For this case, it is useful to consider the four-corners decomposition, in which $\text{As}(H) = \mathbb{R}$ is the cat-state subspace with projection

$$P = |0\alpha\rangle\langle 0\alpha| + |1\alpha\rangle\langle 1\alpha| \sim |\alpha\rangle\langle \alpha| + |\beta\rangle\langle \beta|.$$

(7.21)

We have seen above that initial states $|\beta\rangle$ which are much closer to $|\alpha\rangle$ than they are to $|-\alpha\rangle$ (i.e., $|\beta + \alpha| \gg |\beta - \alpha|$) converges to $|\alpha\rangle$. The same can be said of $|-\alpha\rangle$, the other “well” in this (approximately) double-well system. Let us assume that $\beta$ is much closer to $-\alpha$ so that both components in the initial cat state do not converge to the same well. However, we keep in mind that, in this approximation, $\langle \beta | \alpha \rangle \approx 0$, so $|\beta\rangle$ is still outside of both wells. This is a case in which $\rho_\infty$ contains components in all four corners of $\text{Op}(H)$:

$$\rho_\infty = \frac{1}{2} \begin{pmatrix} \rho_{\text{in}} & \rho_{\text{in}}^* \\ \rho_{\text{in}}^* & \rho_{\text{in}} \end{pmatrix} = \begin{pmatrix} |\alpha\rangle \langle \alpha| & e^{i\theta} |\alpha\rangle \langle \beta| \\ e^{-i\theta} |\beta\rangle \langle \alpha| & |\beta\rangle \langle \beta| \end{pmatrix}.$$

(7.22)

Due to Thm. 4, we know that the asymptotic projection $\mathcal{P}_\infty = \mathcal{P} \mathcal{P}^\dagger$ meaning that coherences $\rho_{\text{in}}$ are not preserved in the infinite-time limit. In the language of conserved quantities, $J_{\alpha\beta} = 0$. This means that $\theta$ is not imprinted on $\rho_\infty$. Moreover, since the component in $\mathbb{R}$ converges to a different location in $\mathbb{I}$ than the component already in $\mathbb{R}$, we necessarily have a mixed asymptotic state ($\rho_\infty = \frac{1}{2} \mathbb{P}$).

7.4 Ordinary Perturbation Theory

Let us now apply the first-order perturbation theory developed in Ch. 4 to study the behavior of the cat-state $\text{As}(H)$ under both Hamiltonian and Lindbladian perturbations. We learn how to in-
duce induce unitary evolution within $A_2(H)$ using Hamiltonians and that the driven two-photon absorption Lindbladian suppresses the effect of some (but not all) noise. In process, we apply Thm. 4, which greatly simplifies the calculations. All studied effects are verified numerically in Ref. [194], and the Hamiltonian perturbation calculations here offer another way to get to the same answers.

7.4.1 A Hamiltonian-based gate

Recall from eq. (4.10) that first-order response of a state in $A_2(H)$ due to a slowly ramped-up perturbation $O$ is

$$T_1^{(1)}|\rho_\infty\rangle = i\mathcal{P}_\infty O\mathcal{P}_\infty|\rho_\infty\rangle - \mathcal{L}^{-1}O|\rho_\infty\rangle,$$

(7.23)

where we have omitted the “infinity” which occurs within $A_2(H)$ (due to the slow ramp-up of the perturbation) since it doesn’t affect our conclusions (see footnote 2 in Ch. 4). Recall also from Sec. 4.3 and the References therein that if we rescale the perturbation as $V \rightarrow \frac{1}{\alpha^2} V$ and evolve to a time $t = T$, then the “Zeno term” $P V P$ is order $O(1)$ and dominates the $O(1/\tau)$ leakage term as $T \rightarrow \infty$. We use this effect to induce a Hamiltonian-based gate on $A_2(H)$.

EQUATION WITHIN $A_2(H)$ According to Sec. 4.1.1, $P_\infty V P_\infty$ is of Hamiltonian form and, for DFS cases, reduces to

$$P_\infty V P_\infty = -i[V^2, \cdot] \equiv -i[P V P, \cdot],$$

(7.24)

where $P$ is the projection onto the cat subspace. In other words, while $V$ can in general drive states in $A_2(H) = \mathbb{F}$ out of $A_2(H)$, $V_{\mathbb{F}} \neq 0$, that part of $V$ does not contribute to within first order in the perturbation. Consider the perturbative Hamiltonian

$$V = i\beta \left( a^\dagger e^{-i\theta} - a e^{i\theta} \right)$$

(7.25)

with $\beta \in \mathbb{R}$ and an angle $\theta = [0, 2\pi)$. After writing out both projections in $V_{\mathbb{F}} = P V P$ in terms of cat states, we need to calculate the matrix elements of $V$ in $A_2(H)$, i.e., $\langle k_\alpha | V | l_\alpha \rangle$. A calculation similar to the one from eq. (7.6) yields

$$\langle k_\alpha | V | l_\alpha \rangle = i\delta_{l,k+1}^{\text{mod}2} \alpha \beta \left( \sqrt{\frac{\pi k}{\pi k+1}} e^{-i\theta} - e^{i\theta} \sqrt{\frac{\pi k+1}{\pi k}} \right).$$

(7.26)

To make sense of these matrix elements, we consider the small and large $\alpha$ limits. Plugging the expansions from eq. (7.6) into $V_{\mathbb{F}}$ yields the two cases

$$V_{\mathbb{F}} \sim \begin{cases} 
  i\alpha \beta e^{-i\theta}|0\rangle\langle 1| + H.c. & \alpha \rightarrow 0 \\
  2\alpha \beta \sin \theta (|0_\alpha\rangle\langle 1_\alpha| + H.c.) & \alpha \rightarrow \infty 
\end{cases}.$$ 

(7.27)

For small $\alpha$, $V_{\mathbb{F}}$ is a rotation on $A_2(H)$, which is now spanned by outer products of Fock states $|k\rangle$ ($k \in \{0, 1\}$), and the axis of the rotation is determined by $\theta$. For large $\alpha$, $V_{\mathbb{F}}$ is also a rotation, but its axis is fixed to be the $x$-axis of the cat-qubit Bloch sphere from Fig. 7.1 and only its strength is dependent on $\theta$. For maximum effect in this limit, $\theta$ needs to be $\pi/2$, which translates to driving perpendicular to the horizontal line connecting $\alpha$ and $-\alpha$ in the phase space of the
oscillator ([194], Fig. 3). Graphically, such a gate shifts the fringes in the Wigner function of a cat state and produces the same effect as the holonomic gate from Fig. 7.5a. This gate was realized experimentally in Ref. [286].

**Leakage out of As(H)**  Let us now return to the leakage term $L^{-1}V$. Since we have a DFS case, we can apply the example from Sec. 4.3.1. The leakage caused by a Hamiltonian perturbation is then restricted to be in $L$, $L(\rho) = -\{(H_{edg})_{\rho}, \rho\}$, and the decoherence Hamiltonian is

$$H_{edg} \equiv \frac{1}{2} F^\dagger F^\ell = \frac{1}{2} \left[ \hat{n}(\hat{n} - 1) - \alpha^2 (a^2 + a^\dagger^2) + \alpha^4 \right].$$

(7.28)

The ground states of $H_{edg}$ are exactly the cat states $|k\alpha\rangle$, meaning that this Hamiltonian provides another way to stabilize such states [pur2016, 94]. Moreover, $\Delta_{edg}$ is the excitation gap of $H_{edg}$. It turns out that, for large enough $\alpha$, $\Delta_{edg}$ is larger than the dissipative gap $\Delta_{dg}$ of the full $L$, thereby providing another layer of protection against leakage besides the $T \to \infty$ Zeno limit. The excitation gap of $H_{edg}$ ($\Delta_{edg}$) is plotted in Fig. 7.3 vs. $\alpha$, along with $\Delta_{dg}$ and the eigenvalue of $L$ with smallest real part. One can see that for $\alpha > 1.5$, the dissipative gap of $L$ is smaller and does not coincide with the energy scale governing leakage.

### 7.4.2 Passive protection against dephasing noise

Let us now take a look at a perturbation of Lindblad form and consider the $\mathcal{P}_c \mathcal{O} \mathcal{P}_c$ (7.23) for

$$\mathcal{O}(\rho) = \frac{1}{2} \kappa (2\hat{n}\rho - \{\hat{n}^2, \rho\}) .$$

(7.29)

The above Lindbladian occurs when there are fluctuations in the frequency parameter of the oscillator and is called the dephasing error channel or, more colloquially, *dephasing noise*. The term $\mathcal{P}_c \mathcal{O} \mathcal{P}_c$ governs evolution within As(H), which no longer has to be unitary since $\mathcal{O}$ is not in Hamiltonian form. Since we cannot reduce this term to an operator like we did in the previous
Subsection, we have to consider the full superoperator and decompose the asymptotic projection in terms of steady states and conserved quantities,

$$\mathcal{P}_0 = \sum_{k,l=0}^{1} \langle \Psi_{kl} \rangle \langle \mathcal{F}^k \rangle.$$  \hspace{1cm} (7.30)

That way, $\mathcal{P}_0 \mathcal{O} \mathcal{P}_0$ is determined by the 16 matrix elements of $\mathcal{O}$ within $\text{As}(\mathcal{H})$, $\langle \mathcal{F}^k | \mathcal{O} | \mathcal{F}_q \rangle$ for $k,l,p,q \in \{0,1\}$. Luckily, dephasing noise preserves parity, so it also does not couple the blocks $\{ | 2n + k \rangle \langle 2m + l | \}^\infty_{n,m=0} \mathcal{O}$ \hspace{1cm} (7.5). Therefore, we can immediately say that $\langle \mathcal{F}^k | \mathcal{O} | \mathcal{F}_q \rangle \propto \delta_{kp} \delta_{lq}$. Moving $\mathcal{O}$ to act on the $J$’s and using $\mathcal{F}^k = \mathcal{F}_k$ (same for $\mathcal{F}^k$) and $\mathcal{O}^\dagger = \mathcal{O}$, we can instead consider how $\mathcal{O}$ acts on the conserved quantities:

$$\langle \mathcal{F}^k | \mathcal{O} | \mathcal{F}_k \rangle = \text{Tr} \{ \mathcal{F}^{k\dagger} \mathcal{O} \mathcal{F}_k \} = \text{Tr} \{ \mathcal{O}^{\dagger} \mathcal{F}^k \mathcal{F}_k \} = \langle \mathcal{F}^k | \mathcal{O} | \mathcal{F}^k \rangle.$$ \hspace{1cm} (7.31)

Since $\mathcal{O}(e^{i\pi \hat{\mathcal{H}}}) = 0$, the diagonal conserved quantities $\mathcal{F}^k$ remain conserved. Moreover, since $\mathcal{F}^0 = \mathcal{F}_{0\dagger}$ and $\mathcal{O}(\mathcal{J}^\dagger) = [\mathcal{O}(\mathcal{J})]^\dagger$ for any $\mathcal{O}$ in Lindblad form, we need only to determine the effect of $\mathcal{O}$ on $\mathcal{F}^0$. Recall that $\mathcal{F}^0$ \hspace{1cm} (7.15) is a superposition of $\mathcal{F}^{0,q}$’s for $q \in \mathbb{Z}$, which in turn are composed of superpositions of $\{ | 2n \rangle | 2n + 2q + 1 \rangle \}^\infty_{n=0}$ for $q \geq 0$ and $\{ | 2n + 2|q| \rangle | 2n + 1 \rangle \}^\infty_{n=0}$ for $q < 0$. Applying $\mathcal{O}$ to each $\mathcal{F}^{0,q}$ and simplifying yields the simple equation

$$\langle \mathcal{F}^{0,q} | \mathcal{O} | \mathcal{F}^{0,q} \rangle = -\frac{1}{2} \kappa (2q + 1)^2 \langle \mathcal{F}^{0,q} \rangle,$$ \hspace{1cm} (7.32)

signaling that $\mathcal{F}^{0,q}$ are actually eigenstates of $\mathcal{O}$. One of the $2q + 1$ terms cancels the $2q + 1$ in the denominator of the sum used to write $\mathcal{F}^0$ in terms of $\mathcal{F}^{0,q}$’s. The matrix element is then

$$\langle \mathcal{F}^{0,q} | \mathcal{O} | \mathcal{F}^{0,q} \rangle = \sqrt{\frac{2\alpha^2}{\sinh 2\alpha^2}} \sum_{q \in \mathbb{Z}} \frac{(-)^q}{2q + 1} I_q (\alpha^2) \langle \mathcal{F}^{0,q} | \mathcal{O} | \mathcal{F}^{0,q} \rangle$$ \hspace{1cm} (7.33a)

$$= -\frac{1}{2} \kappa \sqrt{\frac{2\alpha^2}{\sinh 2\alpha^2}} \sum_{q \in \mathbb{Z}} (-)^q (2q + 1) I_q (\alpha^2) \langle \mathcal{F}^{0,q} | \mathcal{O} | \mathcal{F}^{0,q} \rangle$$ \hspace{1cm} (7.33b)

$$= -\frac{1}{2} \kappa \frac{2\alpha^2}{\sinh 2\alpha^2} \sum_{q \in \mathbb{Z}} (-)^q (2q + 1) I_q (\alpha^2) I_q (\alpha^2)$$ \hspace{1cm} (7.33c)

$$= -\frac{\kappa \alpha^2}{\sinh 2\alpha^2},$$ \hspace{1cm} (7.33d)

where in the last two steps we used eq. \hspace{1cm} (7.13) and eq. \hspace{1cm} (5.8.7.2) from Ref. \hspace{1cm} [234], respectively. In the Zeno limit discussed above, the quantity $c_{01} = \langle \mathcal{F}^{01} | \mathcal{O} | \mathcal{F}^{01} \rangle$ representing the coherence of the cat qubit decays exponentially at the rate $\langle \mathcal{F}^{01} | \mathcal{O} | \mathcal{F}^{01} \rangle$. For small $\alpha$, the rate reduces to the usual dephasing rate $\kappa / 2$ induced on the Fock state outer product $| 0 \rangle \langle 1 |$ by $\mathcal{O}$. However, for large $\alpha$, the rate itself is exponentially suppressed for large $\alpha$ since $\langle \mathcal{F}^{01} | \mathcal{O} | \mathcal{F}^{01} \rangle \sim -2\kappa^2 e^{-2\alpha^2}$.

We have numerically confirmed that eq. \hspace{1cm} (7.33d) is indeed the first-order correction to $\text{As}(\mathcal{H})$ due to $\mathcal{O}$. In Fig. \hspace{1cm} 7.4, versus $\alpha$ and for various $\kappa$, we plot the dissipative gap of $\mathcal{L} + \mathcal{O}$ restricted to the block spanned by $\{ | 2n \rangle | 2m + 1 \rangle \}^\infty_{n,m=0}$. For small values of $\kappa$, the numerical result approaches our above analytical estimate. In fact, a similar trend holds for large values of $\kappa$, indicating that higher-order terms (see Sec. \hspace{1cm} 4.2) should scale in a similar fashion. Since the Lindbladian itself
is preventing dephasing noise from acting within As(H), we can say that the cat pump passively protects (in the sense of Ref. [277]) from this error process.

It is illuminating to determine the degree to which the protection from dephasing noise is coming from the driven two-photon Lindbladian $L$. To do so, we can split the perturbation into two terms,

$$L = \mathcal{P}_\infty O \mathcal{P}_\infty = \mathcal{P}_\infty^0 O \mathcal{P}_\infty^0 + \mathcal{P}_\infty^0 \mathcal{P}_\infty O \mathcal{P}_\infty$$  

and calculate the first term (which would be the only term if $L$ had been Hermitian). The second term is purely a dissipative effect and is due to $L$ not being Hermitian and therefore not having the same left and right eigenmatrices ($J_{kl} = \Psi_{kl}^{\dagger} + J_{kl} \neq \Psi_{kl}$; see Thm. 4). The first term has matrix elements $\langle \langle \Psi_{kl} | O | \Psi_{kl} \rangle \rangle$, which are easily evaluated using techniques from eq. (7.6):

$$\langle \langle \Psi_{kl} | O | \Psi_{kl} \rangle \rangle = -\frac{1}{2} \kappa \alpha^2 \left( \frac{\pi_{k+1}}{\pi_k} + \frac{\pi_{l+1}}{\pi_l} - 2 \alpha^2 \left[ \frac{\pi_{k+1} \pi_{l+1}}{\pi_k \pi_l} - 1 \right] \right) \sim \begin{cases} -\frac{1}{2} \kappa (k - l)^2 & \alpha \to 0 \\ -\kappa \alpha^2 & \alpha \to \infty \end{cases}$$

Therefore, for large $\alpha$, the piece $\mathcal{P}_\infty^0 O \mathcal{P}_\infty \sim -\kappa \alpha^2 \mathcal{P}_\infty$ is not trace-preserving. This shows the importance of using proper Lindbladian perturbation theory instead of merely projecting perturbations onto $\mathcal{H}$.

It is worth noting that the leakage term $L^{-1}O$ from eq. (7.23) dephases the cat-state basis elements that comprise the cat qubit, reducing the purity of the full density matrix $T_{\rho}^{(1)}(\rho_{\infty})$ (7.23). In phase space, this effect translates to a slight smearing of the two Gaussian peaks that represent each cat state. However, since this effect is due to leakage outside of As(H), the quantum information that is stored within As(H) (and represented by $c_{kl} = \langle \langle f_{kl} | \rho_{\infty} \rangle \rangle$) is not affected.
7.4.3 Decoherence under single-photon loss

While the cat pump is resilient to dephasing noise, it is unfortunately incapable of protecting the quantum information in $\mathbb{W}$ from the most common type of error in photonic systems — amplitude damping,

$$\mathcal{O}(\rho) = \frac{1}{2}\kappa \left( 2a^\dagger a \rho - \{\hat{n},\rho\} \right). \quad (7.36)$$

Let us show how the qubit in $\mathbb{W}$ breaks down under this type of error process by calculating $\mathcal{P}_\mathbb{W} \mathcal{O} \mathcal{P}_\mathbb{W}$. We are interested in large $\alpha$, so we work in the coherent state basis $|\pm\alpha\rangle$ (meaning that everything below is accurate up to exponential corrections due to the overlap between the two states). Luckily, the recycling term $a \cdot a^\dagger$ keeps us in $\mathbb{W}$ since $a|p\alpha\rangle = p\alpha|p\alpha\rangle$ with $p = \pm 1$. In addition, the anti-commutator term acts only from one side at a time and so does not take $\mathbb{W}$ into $\mathbb{W}$ by (LP1). Therefore, $\mathcal{P}_\mathbb{W} \mathcal{O} \mathcal{P}_\mathbb{W} = 0$ and we luckily only need to calculate $\mathcal{P}_\mathbb{W} \mathcal{O} \mathcal{P}_\mathbb{W}$. Since $|\pm\alpha\rangle$ are approximately orthogonal, $\mathcal{P}_\mathbb{W} \mathcal{O} \mathcal{P}_\mathbb{W}$ is diagonal in the coherent state basis. Letting $p, q \in \{\pm 1\}$, a straightforward calculation in the large $\alpha$ limit yields

$$\langle p\alpha|\mathcal{O}|q\alpha\rangle \langle q\alpha| \sim -\kappa \alpha^2 (1 - pq). \quad (7.37)$$

This perturbative result shows that the coherence $|\alpha\rangle\langle -\alpha|$ decays as $-2\kappa \alpha^2 t$, in agreement with the small $\kappa$ limit of the exact decay rate $-2\alpha^2 (1 - e^{-\kappa t})$ ([155], below eq. (9.11)).

7.5 Holonomic quantum control

Here, we apply the ideas learned in Ch. 5 to study holonomies on the cat-state As(H). Recall that a slow (i.e., adiabatic) variation of the parameters of a system in a closed loop returns the system to its initial state, up to an operation (called a holonomy) which is due to curvature and/or non-simple-connectedness of the parameter space. Such holonomies can be used to perform quantum gates, either on the ground states of a Hamiltonian or the As(H) of a Lindbladian, in a process called holonomic quantum computation [214, 313, 317]. We show how to perform such computation on the qubit spanned by the cat states $|k_\alpha\rangle$, $k \in \{0, 1\}$.

In order to perform one of the gates, we need to introduce another parameter into the previous Lindbladian, so from now we consider a slightly more general $\mathcal{L}$ with jump operator

$$F = (a - a_0)(a - a_1), \quad (7.38)$$

where $a_0, a_1 \in \{0, 1\}$, depend on time. This jump operator stabilizes a two-dimensional As(H) for all values of $a_0$ and $a_1$. To see this, observe that this new jump operator can be obtained by defining $\alpha_\pm = \frac{1}{2}(a_0 \pm a_1)$ and conjugating $a^2 - a_\pm^2$ with the displacement operator $D_\alpha$ (which acts on the vacuum state as $D_\alpha|0\rangle = |\alpha\rangle$):

$$D_{\alpha_+}(a^2 - a_\pm^2) D_{\alpha+}^\dagger = (a - a_\pm)^2 - a_\pm^2 = (a - a_0)(a - a_1) = F. \quad (7.39)$$

Since the two jumps are related by a unitary conjugation, all spectral properties of $\mathcal{L}$ with the original jump, including the existence of a dissipative gap, hold for this case as well. Most importantly, the two steady states are the displaced cat states $D_{a_+}|k_{a_+}\rangle$. While we can use this exact form of the steady states in the following calculations, we instead use the coherent state
basis and work in the $|\alpha_0 - \alpha_1| \gg 1$ limit for some of the time in order to simplify calculations. In this limit, $\text{As}(H)$ is spanned by the two coherent states $|\alpha_l\rangle$, $l \in \{0, 1\}$.

The positions of the cat-qubit’s two states $|\alpha_l(t)\rangle$ in phase space are now each controlled by a tunable parameter. We let $\alpha_0(0) = -\alpha_1(0) \equiv \alpha$, meaning that the states $|\pm\alpha\rangle$ are the starting point of parameter space evolution and the qubit defined by them (for large enough $\alpha$) is shown in Fig. 7.1. We now introduce two different gates for this cat-qubit, the loop gate and the collision gate, which together allow us to universally control said qubit. We work in the adiabatic limit, meaning that the time $T$ used to perform the parameter path is taken to infinity. Before proceeding, we want to briefly mention that the leading-order $O(1/T)$ non-adiabatic correction in the adiabatic perturbation theory expansion from Sec. 5.3.3 that causes leakage out of $\text{As}(H)$ is still governed by the dissipative gap $\Delta_{\text{edg}}$ of $\mathcal{L}_{-1}$ and not that of $\mathcal{L}_{-1}^{-1}$. This is identical to the effect of the leakage term $\mathcal{L}_{-1}^{-1} \mathcal{V} = \mathcal{L}_{-1}^{-1} \mathcal{V}$ in the ordinary perturbation theory calculations studied in Sec. 7.4, given a Hamiltonian perturbation $\mathcal{V}$.

### 7.5.1 Loop gate

The loop gate involves an adiabatic variation of $\alpha_1(t)$ through a closed path in phase space (see Fig. 7.5b). The state $|\alpha_1(t)\rangle$ follows the path and, as long as the path is well separated from $|\alpha_0(t)\rangle = |\alpha\rangle$, picks up a Berry phase of $\theta = 2A$, with $A$ being the area enclosed by the path [87]. It should be clear that initializing the qubit in $|\alpha\rangle$ produces only an irrelevant overall phase

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**Figure 7.5:** (a) Wigner function sketch of the state before (top) and after (bottom) a loop gate acting on $|\alpha\rangle$, depicting the path of $|\alpha_1(t)\rangle$ during the gate (blue) and a shift in the fringes between $|\pm\alpha\rangle$. (b) Phase space diagram for the loop gate; $X = \frac{1}{2}(a + a^\dagger)$ and $P = -\frac{i}{2}(a - a^\dagger)$. The parameter $\alpha_1(t)$ is varied along a closed path (blue) of area $A$, after which the state $|\alpha\rangle$ gains a phase $\theta = 2A$ relative to $|\alpha\rangle$. (c) Effective Bloch sphere of the $|\pm\alpha\rangle$ qubit depicting the rotation caused by the $d = 2$ loop gate. Black arrow depicts initial state while red arrow is the state after application of the gate. The dotted blue arrow does not represent the path traveled since the states leave the logical space $|\pm\alpha\rangle$ during the gate. (d-f) Analogous descriptions of the collision gate, which consists of reducing $\alpha$ to 0, driving back to $\alpha \exp(i\phi)$, and rotating back to $\alpha$. 

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upon application of the gate. However, once the qubit is initialized in a superposition of the two coherent states with coefficients \( c_\pm \), the gate imparts a relative phase:

\[
c_+ |\alpha\rangle + c_- |\!\!\!-\alpha\rangle \rightarrow c_+ |\alpha\rangle + c_- e^{i\theta} |\!\!\!-\alpha\rangle .
\]  

(7.40)

Recalling that \( |\alpha\rangle \) lies on the \( x \)-axis of the cat-qubit Bloch sphere from Fig. 7.1, this gate can be thought of as a rotation around that axis (depicted blue in Fig. 7.5c). Similarly, adiabatically traversing a closed and isolated path with the other state parameter \(|\alpha_0(t)\rangle\) induces a phase on \(|\alpha\rangle\).

### 7.5.2 Collision gate

For this gate, we utilize the small \( \alpha \) regime to perform rotations around the Bloch sphere \( z \)-axis (Fig. 7.5f), which effectively induce a collision and population transfer between \(|\alpha\rangle\) and \(|\!\!\!-\alpha\rangle\). The procedure hinges on the following observation: applying a bosonic rotation \( R_\phi \equiv \exp(i\phi \hat{n}) \) to well-separated coherent or cat state superpositions does not induce state-dependent phases while applying \( R_\phi \) to Fock state superpositions does. Only one tunable parameter \( \alpha_0(t) = -\alpha_1(t) \) is necessary here, so \( F = \sqrt{k} |\alpha|^2 - \alpha_0(t)^2 \) with \(|\alpha_0(0)\rangle = \alpha \). The collision gate consists of reducing \( \alpha \) to \( 0 \), driving back to \( \alpha \exp(i\phi) \), and rotating back to \( \alpha \) (Fig. 7.5e). Recall from Sec. 5.2 that, for the DFS case, the holonomy corresponding to \( A_\lambda(\phi) \) is generated by \( \partial_\lambda PP \) (for some parameter \( \lambda \)). Using eq. (Hol4), we can represent this holonomy as a path-ordered product of projections \( P_\alpha = |\alpha\rangle \langle \alpha| + |\!\!\!-\alpha\rangle \langle \!\!\!-\alpha| \) onto the DFS. Therefore, the part of the path which corresponds to the nonunitary driving from \( 0 \) to \( \alpha \exp(i\phi) \) can be approximated as \( S_\phi \equiv P_{\alpha e^{i\phi}} \cdots P_{\frac{\lambda}{M}\alpha e^{i\phi}} P_{\frac{\lambda}{M}\alpha e^{i\phi}} \) for integer \( M \gg 1 \). Similarly, the part which “deflates the cat” from \( \alpha \) to \( 0 \) is approximately \( S_0^\dagger \).

Combining these with the rotation for the last segment of the path, the full “pizza-slice” path is represented by \( R_\phi^\dagger S_\phi S_0^\dagger \). Since

\[
R_\phi^\dagger S_\phi R_\phi^\dagger = R_\phi^\dagger (R_\phi S_0 R_\phi^\dagger) S_0^\dagger = S_0 R_\phi^\dagger S_\phi^\dagger ,
\]  

(7.41)

the collision gate is equivalent to reducing \( \alpha \), applying \( R_\phi^\dagger \) on the steady-state Fock basis \(| k \rangle = |k_{\alpha=0}\rangle \), and driving back to \( \alpha \). The net result is thus a relative phase between the states \(| k_{\alpha}\rangle\):

\[
c_0 |0_{\alpha}\rangle + c_1 |1_{\alpha}\rangle \rightarrow c_0 |0_{\alpha}\rangle + c_1 e^{-i\phi} |1_{\alpha}\rangle .
\]  

(7.42)

In the coherent state basis, this translates to a coherent population transfer between \(|\pm\alpha\rangle\).

Both gates can also be understood in terms of Berry connections of the cat states,

\[
A_{kl}^\lambda \equiv i \langle k_{\alpha}| \partial_\lambda I_{\alpha} \rangle ,
\]  

(7.43)

where \( \lambda \) is a parameter that is varied during the path. (Recall from Ch. 5 that \( A^\lambda \) is the coordinate representation of the connection \( \partial_\lambda PP \).) For example, the collision gate arises from changes in the magnitude and phase of the coherent state parameter \( \alpha e^{i\phi} \). Therefore, \( \lambda \in \{|\alpha|, \arg \alpha \equiv \phi\} \) and a simple calculation reveals

\[
A_{kl}^{[\alpha]} = 0 \quad \text{and} \quad A_{kl}^\phi = -\delta_{kl} \langle k_{\alpha}| \hat{n} |k_{\alpha}\rangle .
\]  

(7.44)
Recalling the small and large $\alpha$ limits of the average occupation number from eq. (7.6), this confirms that the effective operation induced by the collision gate is indeed caused by the rotation induced on the Fock states at $\alpha = 0$. 
“For today’s electrical engineers worrying about [Moore’s Law], quantum mechanics is a bug, but the hope is that we can turn it into a feature.”

– Robert J. Schoelkopf

APPLICATION: SINGLE- AND MULTI-MODE CAT CODES

We now proceed to state a series of extensions of the cat-state As(H) stabilized by the “two-cat pump” of the previous chapter. The single-mode generalizations are called cat codes [9, 11, 94, 167, 194] — quantum memories for coherent-state quantum information processing [131, 144] which store information in superpositions of well-separated coherent states evenly distributed around the origin of phase space. Here, we review the single-mode generalizations and introduce an M-mode generalization of cat-codes, making contact with the Lindbladians necessary to generate these codes. We note that the states we consider have been studied in a quantum optical context for $M = 2$ [89, 181] and $M = 3$ [17].

8.1 single-mode cat codes

In the previous chapter, we studied features of the Lindbladian generated by the jump operator $F = a^2 - a^2$, which stabilized an As(H) consisting of cat states $|k\alpha\rangle$, $k \in \{0, 1\}$ (7.3). Through the lens of quantum information, this As(H) is part of a quantum code — a subspace that is used to store an arbitrary quantum state in order to prevent its quantum information from decohering or changing without notice. While we saw that such an As(H) is passively protected from dephasing noise, it is not protected from amplitude damping. We now double the size of this As(H) in order to accommodate (and thus protect from) the effects of amplitude damping.

Notice that the amplitude damping process (7.36) is generated by the jump operator $a$, which decreases the occupation number of a state by one, thereby flipping the occupation number parity. If we had some way of storing information in a subspace of fixed (say, even) occupation number parity which then could “jump” into an error subspace of odd parity after being acted upon by $a$, then we could in principle track such a jump and prevent the quantum information from decohering. This is similar to more traditional multi-qubit stabilizer codes [206], which provide a large enough number of error subspaces for a code such that the quantum information can, after undergoing an error, “jump” from the code subspace into an error subspace without overlapping with itself and decohering. However, here we gain an additional advantage: we do not have to correct the error and can simply track which subspace our quantum information is in. In order
to allow for the tracking of loss events \( a \), all we have to do is to make sure we have both an even- and an odd-parity subspace in our \( \text{As}(H) \), each of which are large enough to store a qubit. This can be achieved by “doubling” the jump operator to

\[
F = a^4 - \alpha^4. \tag{8.1}
\]

For large enough \( \alpha \), the four coherent states \( | \alpha^k \rangle \), \( k \in \{0, 1, 2, 3\} \), form the \( \text{As}(H) \) of the Lindbladian generated by this jump operator. However, we once again would like to build an orthonormal basis valid for all \( \alpha \) whose states are eigenstates of the parity operator \((-)^\hat{\Pi} \). Redefining projections \( \Pi_k = \sum_{n=0}^{\infty} \langle 4n + k | 4n + k \rangle \), the “four-cat” state basis consists of

\[
| k_\alpha \rangle \equiv \frac{\Pi_k | \alpha \rangle}{\sqrt{\langle \alpha | \Pi_k | \alpha \rangle}} \quad \text{with normalization} \quad \pi_k \equiv \langle \alpha | \Pi_k | \alpha \rangle. \tag{8.2}
\]

Similar to the two-cat pump states, these become Fock states \( | k \rangle \) for small \( \alpha \) and equal superpositions of coherent states \( \{| \alpha^k \rangle\}_{k=0}^{3} \) for large \( \alpha \). The states \( | 0_\alpha \rangle, | 2_\alpha \rangle \) are even parity states — \((-)^\hat{\Pi} | 0_\alpha \rangle = | 0_\alpha \rangle \) and same for \( | 2_\alpha \rangle \) — while the states \( | 1_\alpha \rangle, | 3_\alpha \rangle \) are odd parity. Instead of using the entire four-dimensional space to store a qudit, we can use the even parity subspace as the (qubit) code subspace and the odd parity subspace as the error subspace. That way, it is possible to track loss events as they happen. Such tracking has been experimentally realized in Ref. \([211]\).

The new “four-cat pump” Lindbladian generated by \( F \) from eq. (8.1) enjoys many of the same features as the two-cat pump. A Hamiltonian-based gate can be performed on the quantum information in either the even or odd-parity subspace using the Hamiltonian \( V = a^2 + H.c. \), just like \( V = a + H.c. \) performed a gate on the two-dimensional \( \text{As}(H) \) of the two-cat pump in Sec. 7.4. The new jump operator is also of the type \( F = F_{\text{d}} \), meaning that the effective dissipative gap \( \Delta_{\text{edg}} \) is the excitation gap of \( \frac{1}{2} F^+ F \). While an analytic representation for the 16 conserved quantities of this case has not yet been found, it has been numerically determined that dephasing noise is also suppressed (\([194]\], Fig. A1b). Holonomic quantum control can be performed on the entire \( \text{As}(H) \) or only on its even/odd parity blocks \([11]\).

We can continue along this line of reasoning and consider having \( d - 1 \) error subspaces in order to track up to \( d - 1 \) loss events (\([253]\), Supplementary Material; \([52, 174]\)). Such a scheme is realized by the jump operator

\[
F = a^{2d} - \alpha^{2d}. \tag{8.3}
\]

This operator annihilates the (unnormalized) states

\[
| k \rangle = e^{-\frac{1}{2} a^2} \sum_{n=0}^{\infty} \frac{\alpha^{2dn+k}}{\sqrt{(2dn+k)!}} | 2dn+k \rangle, \tag{8.4}
\]

where \( k \in \{0, 1, \cdots, 2d - 1\} \) and the \( 2d \) projections from eq. (3.23) are

\[
\Pi_k = \sum_{n=0}^{\infty} | 2dn+k \rangle \langle 2dn+k | = \frac{1}{2d} \sum_{l=0}^{2d-1} e^{i \pi \hat{n}(n-k)l}. \tag{8.5}
\]

These states are eigenstates of the discrete rotation operator \( e^{i \hat{\Pi} \hat{n}} \), just like the two-cat states are eigenstates of the parity operator \( e^{i \pi \hat{n}} \). The power of \( a \) is \( 2d \) (instead of \( d \)) because this provides enough room for (a two-dimensional) code subspace and the \( d - 1 \) (two-dimensional)
error subspaces that are required to track up to \( d - 1 \) loss events. If one wants to store a quDit’s worth of information in the code subspace while still protecting from \( d - 1 \) loss events, then one can consider the jump operator \( F = a^{Dd} - a^{dDd} \).

The above schemes can be extended even further by considering jump operators which are polynomials in \( a \),

\[
F = \prod_{k=0}^{d-1} (a - a_k)
\]

for some complex \( a_k \). Assuming that each \( a_k \) is well-separated from the others in phase space, the kernel of the jump operator is spanned by the \( d \) coherent states \( \{ |\alpha_k\rangle \}_{k=0}^{d-1} \). Holonomic gates on such \( \text{As}(\mathcal{H}) \) have been considered in Ref. [11]. However, this general case (for \( d > 2 \)) is more complicated to work with because, unlike the \( d = 2 \) case (see Sec. 7.5), it is not unitarily related to \( a^d - a^d \) for some \( a \).

8.2 TWO-MODE CAT CODES

Let us now consider a generalization of single-mode cat-codes to two modes [9] using the pair-coherent/Barut-Girardello states [2, 4, 43]. Recall that, in the single mode case, our codes were eigenstates of powers of the lowering operator \( a \). In this case, our code states are eigenstates of powers of \( ab \), where \( b \) is the lowering operator for another oscillator. Naturally, \( [b, b^\dagger] = 1 \) and \( \hat{m} \equiv b^\dagger b \). Recall also that cat states were built by projecting (using \( \Pi_k \)) a coherent state \( |\alpha\rangle \) onto eigenspaces of the rotation operator \( e^{i\frac{2\pi}{\hbar} \hat{m}} \). We perform the same trick with \( ab \). However, while \( a \) has only one type of eigenstate \( (|\alpha\rangle) \), \( ab \) has a countable infinity of types, each of which is indexed by a continuous parameter! Therefore, a careful analysis of the eigenstates of \( ab \) is required before we consider its higher powers.

An easy way to determine the eigenstates of \( a \) is to simply plug in a general state \( |\psi\rangle = \sum_{n=0}^\infty c_n |n\rangle \) into the eigenvalue relation

\[
a|\psi\rangle = \alpha |\psi\rangle
\]

and solve for the coefficients \( c_n = \frac{\alpha^n}{\sqrt{n!}} \). If we do this procedure with a two mode state \( |\psi\rangle = \sum_{n,m=0}^\infty c_{n,m} |n,m\rangle \), then we have two indices to consider. Instead, we can first use symmetries to restrict what types of states \( |\psi\rangle \) we can plug in and thereby avoid having to deal with two indices. Notice that \( [ab, \hat{m} - \hat{n}] = 0 \), meaning that \( ab \) preserves the occupation number difference

\[
\hat{\Lambda} \equiv \hat{m} - \hat{n}.
\]

Therefore, any eigenstates of \( ab \) are also eigenstates of \( \hat{\Lambda} \). The latter can be organized into subspaces of the same eigenvalue \( \Lambda \in \mathbb{Z} \), namely \( \{ |n, n + \Lambda\rangle \}_{n=0}^\infty \) for \( \Lambda \geq 0 \) and \( \{ |n + \Lambda, n\rangle \}_{n=0}^\infty \) for \( \Lambda < 0 \). That way, \( \Lambda |\psi\rangle = \Lambda |\psi\rangle \) for any state \( |\psi\rangle \) lying in a subspace of fixed \( \Lambda \).

For convenience, we can introduce the exchange operator

\[
X \equiv \exp \left[ i \frac{\pi}{2} (a^\dagger - b^\dagger)(a - b) \right], \quad \text{which acts as} \quad X |n,m\rangle = |m,n\rangle,
\]

and write the subspaces for all negative \( \Lambda \) as \( \{ X |n, n + \Lambda\rangle \}_{n=0}^\infty \). From now on, we assume that \( \Lambda \geq 0 \), remembering that an application of \( X \) yields the corresponding results for \( \Lambda < 0 \).
The projections onto each of the subspaces are

\[ P_\Delta \equiv \sum_{n=0}^{\infty} |n, n + \Delta\rangle \langle n, n + \Delta| = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i(n-n-\Delta)\theta} \]  

(8.10)

(for \( \Delta \geq 0 \) and \( XP_{|\Delta|}X \) for \( \Delta < 0 \)). Since \( ab \) is block diagonal in the decomposition of subspaces of fixed \( \hat{\Delta} \), we have to only consider general states in each subspace:

\[ ab|\psi_\Delta\rangle \equiv ab \sum_{n=0}^{\infty} c_n |n, n + \Delta\rangle = \gamma^2 |\psi_\Delta\rangle, \]  

with eigenvalue \( \gamma^2 \). Solving this equation by acting with \( ab \) on each Fock state yields the solution \( c_n = \frac{\gamma^{2n+D}}{\sqrt{n!(n+\Delta)!}} \), and the resulting normalized state is the pair-coherent state

\[ |\gamma_\Delta\rangle = \frac{1}{\sqrt{I_\Delta(2|\gamma|^2)}} \sum_{n=0}^{\infty} \frac{\gamma^{2n+\Delta}}{\sqrt{n!(n+\Delta)!}} |n, n + \Delta\rangle, \]  

(8.12)

where \( I_\Delta \) is the modified Bessel function of the first kind \([201]\). These well-known states satisfy several of the properties of ordinary coherent states: they are eigenstates of a lowering operator \( (ab) \) and they are overcomplete (on each subspace of fixed \( \Delta \)). However, they are not generated by a displacement-like operator: \( \exp(\gamma^*ab - H.c.)|0,0\rangle \) does not produce \( |\gamma_\Delta\rangle \) but instead produces what is known as a two-mode squeezed state \([scully]\). Nevertheless, such states can be conveniently related to a two-mode coherent state \( |\gamma, \gamma\rangle \) via the projections (8.10):

\[ |\gamma_\Delta\rangle = \frac{P_\Delta|\gamma, \gamma\rangle}{\sqrt{\langle \gamma, \gamma | P_\Delta |\gamma, \gamma\rangle}}. \]  

(8.13)

Having introduced all of the eigenstates of \( ab \), we can now further apply projections \( \Pi_k \) of the type discussed in the previous Section in order to present two-mode cat codes. Notice that, for the single-mode case in eq. (8.4), applying \( \Pi_k \) to a coherent state \( |\alpha\rangle \) is equivalent to having the index \( n \) of the sum over Fock states of \( |\alpha\rangle \) transform as \( n \rightarrow 2dn + k \). Here, we observe a similar pattern, but this time for the index \( n \) in the sum of the pair coherent state (8.12). Let us introduce projections onto eigenspaces of the joint rotation \( e^{i\frac{2\pi}{4d}(\hat{n}+\hat{m})} \) (where there is an extra factor of 2 compared to the single-mode case, \( e^{i\frac{2\pi}{2d}\hat{n}} \), corresponding to there being two modes),

\[ \Pi_k \equiv \frac{1}{4d} \sum_{l=0}^{4d-1} e^{i\frac{2\pi}{4d}(\hat{n}+\hat{m}-k)^l} = \sum_{n,m=0}^{\infty} |n, m\rangle \langle n, m| \delta_{n+m,k}^{4d}, \]  

(8.14)

where \( \delta_{n+m,k}^{4d} = 1 \) whenever \( n + m = k \) modulo \( 4d \). Notice that \([\Pi_k, P_\Delta] = 0 \) since they are both functions of \( \hat{n}, \hat{m} \) only and that \( \Pi_k |\alpha\rangle = a\Pi_{(k+1)\mod 4d} |\alpha\rangle \) (and same for \( b \)). This implies that

\[ \Pi_k(ab)^{2d} = (ab)^{2d} \Pi_{k+2d}b^{2d} = (ab)^{2d} \Pi_{k+4d} = (ab)^{2d} \Pi_k, \]  

(8.15)

meaning that any eigenstates of \((ab)^{2d}\) are also those of \( \Pi_k \).
Applying $\Pi_{2k+\Delta}$ to $|\gamma\rangle$ produces the two-mode cat code state

$$|k_{\gamma,\Delta}\rangle = \frac{\Pi_{2k+\Delta} P_\Delta |\gamma, \gamma\rangle}{\sqrt{\tau_{k,\Delta}}}$$

(8.16)

with normalization $\tau_{k,\Delta} \equiv \langle \gamma, \gamma | \Pi_{2k+\Delta} P_\Delta |\gamma, \gamma\rangle$ and $k \in \{0, 1, \cdots, 2d - 1\}$. Fixing $\gamma$ to be real, the Fock state representation of these states is

$$|k_{\gamma,\Delta}\rangle = \frac{e^{-\gamma^2}}{\sqrt{\tau_{k,\Delta}}} \sum_{n=0}^{\infty} \frac{\gamma^{2dn+k+\frac{1}{2}\Delta}}{(2dn+k)! (2dn+k+\Delta)!} |2dn+k, 2dn+k+\Delta\rangle.$$  

(8.17)

To show this, first observe that $P_\Delta |\gamma, \gamma\rangle$ consists of Fock states from the subspace $\{|n, n+\Delta\rangle\}_{n=0}^{\infty}$. Then, notice that $\Pi_{2k+\Delta}$ projects those Fock states further onto the subspace for which the total occupation number

$$2n + \Delta = 2k + \Delta \mod 4d.$$  

(8.18)

This implies that $n = k$ modulo $2d$. Therefore, for a given $k$, the subspace of the states $\{|n, n+\Delta\rangle\}_{n=0}^{\infty}$ that is preserved under $\Pi_{2k+\Delta}$ is $\{|2dn+k, 2dn+k+\Delta\rangle\}_{n=0}^{\infty}$.

One can check that $|k_{\gamma,\Delta}\rangle$, $k \in \{0, 1, \cdots, 2d - 1\}$, are eigenstates of $(ab)^{2d}$, meaning that the jump operator used to stabilize an $\text{As}(H)$ consisting of them is

$$F = (ab)^{2d} - \gamma^{2d}.$$  

(8.19)

As with the single-mode cat codes, we have verified numerically that this Lindbladian suppresses dephasing noise in both modes for $d = 1$. In addition, this $\text{As}(H)$ can store a qubit (say, in the $\Delta = k = 0$ subspace) that can be protected from arbitrary single-mode loss events $a^n$ and $b^m$ for $n, m \in \{1, 2, \cdots, \infty\}$ as well as joint events $(ab)^p$ for $p \leq d - 1$. This protection can be understood by studying how these loss events interact with the projections $P_\Delta$ and $\Pi_{2k+\Delta}$.

1. Single-mode loss events $a^n$ and $b^m$ shift the value of $\Delta$:

$$\begin{pmatrix} a \\ b \end{pmatrix} P_\Delta = \begin{pmatrix} P_{\Delta+1}a \\ P_{\Delta-1}b \end{pmatrix}. $$

(8.20)

Since the value of $\Delta$ is shifted in different directions depending on which mode incurred the losses, it is possible to track those events by continuously monitoring the occupation number difference $\hat{\Delta} = \hat{n} - \hat{\bar{n}}$. Since the eigenvalues of $\hat{\Delta}$ are integers, an arbitrary amount of single-mode loss events can be detected. Note that single-mode events also do not commute with $\Pi_{2k+\Delta}$, meaning that the error subspace to which the qubit jumps to after such events have different values of both $\hat{n} + \hat{\bar{n}}$ (modulo $4d$) and $\hat{\Delta}$.

2. For each $\Delta$, there are $2d$ states of fixed photon number difference. A joint loss event $ab$ commutes with $P_\Delta$ but not with $\Pi_k$, shifting $k \rightarrow k - 1$:

$$\Pi_{2k+\Delta} ab = ab \Pi_{2(k-1)+\Delta}. $$

(8.21)
Since the eigenspace of $\hat{\Lambda}$ doesn’t change upon these errors, the syndrome set is different from that associated with the single-mode events above. Since there are $d - 1$ error subspaces for each $\Delta$, it is possible to track up to $d - 1$ such joint loss events.

Some of the Hamiltonian-based and holonomic gates discussed in the previous chapter can also be extended to these cases. For example, for $d = 1$, the Hamiltonian $V = ab + H.c.$ performs a gate between the two states $|k,\gamma,\Delta\rangle$, $k \in \{0, 1\}$, for each $\Delta$. A holonomic gate which consists of the path $\gamma \to 0 \to \gamma e^{i\phi} \to \gamma$, a generalization of the single-mode collision gate from Sec. 7.5, induces a similar effect (again for each $\Delta$).

8.3 M-mode cat codes

The two-mode generalization above can be naturally extended to $M$ modes, whose corresponding operators are labeled $\{a_m, a^+_m, \hat{n}_m\}$ with $m \in \{1, 2, \cdots, M\}$. Such codes for $M \geq 3$ gain the additional advantage of being able to correct for higher-weight products of losses or for photon losses and gains at the same time (see Ref. [9]). One way to characterize their code states is to use to a vector of $M - 1$ occupation number differences between neighboring modes, $\vec{\Delta} = \langle \hat{n}_2 - \hat{n}_1, \hat{n}_3 - \hat{n}_2, \cdots, \hat{n}_M - \hat{n}_{M-1} \rangle$.

Projections onto subspaces of fixed differences $\vec{\Delta} = \langle \Delta_1, \Delta_2, \cdots, \Delta_{M-1} \rangle$ are

$$P_{\vec{\Delta}} = \sum_{n=0}^{\infty} \prod_{m=1}^{M} \left| n + \sum_{p=1}^{m-1} \Delta_p \right| \left( n + \sum_{p=1}^{m-1} \Delta_p \right)^{\Delta_m}$$

for $\Delta_p \geq 0$ and projections on the total occupation number $\hat{N} \equiv \sum_{m=1}^{M} \hat{n}_m$ generalize straightforwardly from eq. (8.14):

$$\Pi_k = \frac{1}{2dM} \sum_{l=0}^{2dM-1} e^{i\pi M (\hat{N} - k)l}.$$  

As an example, we write down only the $\vec{\Delta} = \vec{0}$ and $d = 1$ states, which are permutation symmetric. For $k \in \{0, 1\}$,

$$|k,\gamma,\vec{0}\rangle \equiv \Pi_k P_0 \cdot |\gamma\rangle^{\otimes M} \propto \sum_{n=0}^{\infty} \gamma^{M(2n+k)} \prod_{m=1}^{M} |2n+k\rangle \sqrt{(2n+k)!},$$

with normalization $\pi_{k,\vec{0}} \equiv \langle \gamma |^{\otimes M} \cdot \Pi_k P_0 \cdot |\gamma\rangle^{\otimes M}$. The jump operator which annihilates these states is

$$F = \left( \prod_{m=1}^{M} a_m \right)^2 - \gamma^{2M}.$$  

Once again, the analysis of the previous chapter is extendable to these codes.
“Rather than working on No-Go theorems, I prefer to do Lego experiments.”

– Michel H. Devoret

OUTLOOK

This work is concerned with Lindbladians which admit more than one steady state. The motivation for studying such Lindbladians is two-fold. First, using a set of techniques typically characterized as quantum reservoir engineering, such Lindbladians can be used to stabilize exotic phases of matter (corresponding to possibly degenerate ground states), quantum entanglement (for quantum communication or metrology), or desirable subspaces (for quantum information processing). Second, such Lindbladians can be used for autonomous/passive quantum error correction [277], suppressing the effect of errors on a steady-state subspace containing quantum information and/or driving any leaked quantum information back into said subspace after an error. We have reviewed and made contact with previous work, detailed relevant manuscripts to which the author of this thesis contributed, and presented previously unpublished results. A summary can be found in Sec. 1.5.

One item that is anticipated to gain further application is the all-order Dyson series for slowly ramping-up perturbations in Ch. 4. Due to no restrictions on the unperturbed Lindbladian, the number and type of its steady states, and the type of perturbation, that analysis is just about as general as one could hope for while still adhering to the laws of quantum mechanics. While we focus on response to Hamiltonian perturbations within first-order and evolution within the adiabatic limit, it would be of interest to further study other Lindbladian perturbations [76] and their corresponding higher-order effects. While several elements of this study consider asymptotic subspaces consisting of only one block of steady states, it is not unreasonable to imagine that the aforementioned second-order and/or non-adiabatic effects could produce transfer of information between two or more blocks. Recently developed diagrammatic series aimed for determining perturbed steady states [173] (see also [172]) may benefit from the four-corners decomposition (whenever the unperturbed steady state is not full-rank). The all-order Dyson series should provide insight into reservoir engineering theory and experiments in which the “good” dissipation is stronger than any “bad” noise. Finally, while we have applied the general result of Thm. 4 to ordinary (time-dependent) and adiabatic perturbation theory, future work could include an application to singular perturbation theory and adiabatic elimination techniques [31, 32, 193] or quasi-degenerate perturbation theory ([303], Appx. B).

A glaring item that is missing from this work, with the notable exception of the frustration-free Lindbladians of Sec. 3.4, is an application to many-body open systems with non-equilibrium steady...
states (NESS). This is not necessarily exclusive to this work, as many concepts are only recently being extended to NESS. Examples include topological order [72, 134, 243], Thouless pumping [179], and spontaneous symmetry breaking [302] to name a few. It would also be of interest to determine how the effective dissipative gap scales with system size vs. the true dissipative gap in many-body systems [73, 135, 302, 322]. Other many-body concepts have yet to be extended to Lindbladians with multiple NESS. Matrix product methods determining the steady state of a Lindbladian [97] and current applications of the Keldysh formalism to Lindbladians [265] do not tackle degenerate cases. Stability of NESS studied for the unique state case [96] should also be extendable.

Given a quantum channel $E$, there exists a recipe (see Sec. 2.1.4 and Ref. [10]) for a Lindbladian $L$ whose time evolution in the infinite-time limit contains one action of $E$. Such an embedding may prove useful in autonomous error correction and experimental quantum channel simulation. It would be of interest to study the applicability of this recipe in the broader context of previous efforts on channel simulation, both theoretical [20, 138, 182, 263, 295] and experimental [184, 190].

The metric stemming from the QGT will be examined in future work, particularly to see whether it reveals information about bounds on convergence rates [74, 145, 223, 249]. It remains to be seen whether the scaling behavior of the metric is correlated with phase stability [44, 108, 139, 250] and phase transitions [75, 158, 318] for NESS. We do not derive a QGT or metric for the case of multiple NS blocks, so taking into account any potential interaction of the blocks during adiabatic evolution remains an open problem.

It has recently been postulated [187] that Lindbladian meta-stable states also possess the same structure as the steady states. This may mean that our results regarding conserved quantities (which are dual to the steady states) also apply to the pseudo-conserved quantities (dual to the meta-stable states).

Lastly, the properties of Lindbladian eigenmatrices should be extendable to memory-kernel dynamics [143] and can be extended to eigenmatrices of more general quantum channels [46, 60, 61, 79]. Statements similar to Thm. 4 exist for fixed points of quantum channels [61, 93] and their extension to rotating points will be a subject of future work. These results may also be useful in determining properties of asymptotic algebras of observables [14, 103] and properties of quantum jump trajectories when the Lindbladian is “unraveled” [51, 304].
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