Simulation of viscoelastic flow past circular airfoil by using the modified LS-STAG immersed boundary method

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1. Introduction

Many fluids encountered in various technical applications do not describe by the Newton’s viscous friction law, i.e. they are non-Newtonian fluids [1]. The such liquids viscosity may depend on the rate-of-strain tensor second invariant and the flow rate. However, some non-Newtonian fluids also exhibit elastic behavior, i.e. they combine the solid and liquid characteristics.

The most common viscoelastic fluids [2] include various polymer solutions. Polymer solutions are used as an additive to reduce resistance or to stabilize the interface between immiscible liquids. Also, polymers are the main thickeners in the production of consumer goods, such as paints and toothpastes. The polymer solutions addition allows to obtain materials that have high viscosity at low shear rates and low viscosity at high shear rates, therefore, it is convenient to use them.

In some industrial processes, such as injection molding, the viscoelastic polymers flow occurs in domains with complex geometries. To optimize such processes, it is necessary to simulate
the polymers flow to determine the product shape with sufficient accuracy. For this reason, numerical methods are being actively developed for solving such problems. But the most severe issue is the breakdown of the numerical algorithms for highly elastic flows: the so-called high Weissenberg number problem (HWNP). The Weissenberg number critical value that is reached is highly dependent on the discretization method employed. It is also observed that mesh refinement does not help raise the threshold of critical value, and one observes the paradox that the finer the mesh is, the lower is the Weissenberg number critical value [3].

Immersed boundary methods [4] are suitable for numerical simulation in domains with complex geometry, since they do not require a coincidence of cell edges and boundaries of the computational domain, and allow to solve problems when domain shape is irregular or it changes significantly in the simulation process due to body motion. At the same time, it is important to ensure high accuracy in cells which are cut by the boundary, the so-called cut-cells. The LS-STAG immersed boundary method [5] allows to do this for Newtonian fluids. The immersed boundary is represented with the level-set function [6]. The LS-STAG-discretization in the cut-cells has the ability to preserve the five-point Cartesian structure of the stencil, resulting in a highly computationally efficient method. To date, the LS-STAG-discretization of the 2D Navier — Stokes equations for a viscous incompressible flow has been constructed [5], the LS-STAG method modifications for coupled hydroelastic problems [7, 8] and for Smagorinsky, Spalart — Allmaras, $k-\varepsilon$, $k-\omega$ and $k-\omega$ SST turbulence models within RANS, LES and DES approaches [9] have been developed. The LS-STAG method modification for viscoelastic flows has been developed for the case of non-moving immersed boundaries. The obtained method is implemented in the developed software package [10]. It is verified on various test problems.

The purpose of this paper is to verify the developed modification of the LS-STAG immersed boundary method for viscoelastic flows described by rate-type models. The test problem about viscoelastic Oldroyd-B flow past a circular airfoil is considered.

2. Test problem statement
To verify the modified LS-STAG immersed boundary method, we consider the following 2D test problem. The flow around a fixed circular airfoil assumed to be viscoelastic. The airfoil is rigid. Its diameter is equal $D$. We denote airfoils boundary as $K$. The airfoil radius $R = D/2$ is chosen as a base length as in [11]. The flow is considered at rectangular computational domain $\Omega = [0; 20D] \times [0; 2D]$ (see figure 1). The airfoil center is the same as the computational domain center. Test problem statement in dimensionless variables is the following:
Figure 2. Staggered arrangement of the variables on the LS-STAG mesh.

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \frac{1}{\text{Re}} \Delta \mathbf{v} &= \nabla \cdot \boldsymbol{\tau}^e, \\
\mathbf{v}(x, y, 0) &= \mathbf{v}_0(x, y), \\
\tau^{e}(x, y, 0) &= \tau^{e}_0(x, y), \\
(x, y) &\in \Omega, \\
\mathbf{v}|_{\Gamma_1} &= \mathbf{V}_{\text{inflow}} = V_{\infty} y (2D - y) e_x, \\
\mathbf{v}|_{\Gamma_3} &= \mathbf{v}|_{\Gamma_4} = 0, \\
\frac{\partial \mathbf{v}}{\partial n}|_{\Gamma_4} &= 0, \\
\tau^{e}\big|_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} &= \tau^{e, bc}(x, y, t), \\
\tau^{e, bc} &= 2\lambda \nu_e \left( \frac{\partial u}{\partial y} \right)^2, \\
\tau^{e, bc} &= 0, \\
\tau^{e, bc} &= \nu_e \frac{\partial u}{\partial y}, \\
\frac{\partial \tau^{e}}{\partial n}\big|_{\Gamma_4} &= 0, \\
\frac{\partial \tau^{e}}{\partial n}|_{\Gamma_4} &= 0, \\
\frac{\partial p}{\partial n}|_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} &= 0.
\end{align*}
\]

Here \( t \) is the dimensionless time; \( x, y \) is the dimensionless coordinates; \( p \) is the dimensionless pressure; \( \text{Re} \) is the Reynolds number; \( n \) is the external normal; \( \mathbf{v} = \mathbf{v}(x, y, t) = \mathbf{v} \cdot e_x + \mathbf{v} \cdot e_y \) is the dimensionless velocity; \( \lambda \) is the dimensionless relaxation time; \( \lambda_r \) is the dimensionless retardation time; \( \nu_e \) is the dimensionless polymer viscosity; \( \tilde{D} \) is the time derivative differential operator (it can be a particular derivative or one of the convective derivatives depending on the non-Newtonian model); \( \tau^e \) is the extra-stress tensor [12]; \( S = \frac{1}{2} \left( \nabla \mathbf{v} + [\nabla \mathbf{v}]^T \right) \) is the rate-of-strain tensor. The dimensionless fluid density \( \rho \) is equal 1. The considered fluid can be described by one of the following rate type models depending on \( \lambda_r \) and operator \( \tilde{D} \) type: Maxwell, Maxwell-A or Upper Convective Maxwell model [13], Jeffreys model [14], Oldroyd-B or Oldroyd-A model [15], Johnson — Segalman model [16]. In this paper the computational results for Oldroyd-B fluid will be presented.

**Main ideas of the modified LS-STAG method.** The Cartesian mesh with cells \( \Omega_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j) \) is introduced in the rectangular computational domain \( \Omega \). It is a control volume for continuity equation and transport equation for the normal extra-stresses. It is the base mesh. It is denoted that \( \Gamma_{i,j} \) is the face of \( \Omega_{i,j} \) and \( \mathbf{x}^{e}_{i,j} = (x^e_i, y^e_j) \) is the center of this cell (see figure 2).

Pressure, normal stresses and normal extra-stresses are computed in the center of \( \Omega_{i,j} \). Unknown components \( u_{i,j} \) and \( v_{i,j} \) of velocity vector \( \mathbf{v} \) are computed in the middle of fluid parts of the cell faces. These points are the centers of control volumes \( \Omega^u_{i,j} = (x^u_{i-1}, x^u_i) \times (y_{j-1}, y_j) \) (x-mesh) and \( \Omega^v_{i,j} = (x_{i-1}, x_i) \times (y^v_{j-1}, y^v_j) \) (y-mesh) with faces \( \Gamma^u_{i,j} \) and \( \Gamma^v_{i,j} \) and squares \( M^x_{ij} \) and \( M^y_{ij} \), respectively (see figure 2).
Figure 3. Location of the variables discretization points in the case of generic cells on the LS-STAG mesh: north-west triangular cell (at left), north trapezoidal cell (in center) and north-west pentagonal cell (at right).

In case of the LS-STAG method usage for viscoelastic fluids the fourth mesh (xy-mesh) with cells $\Omega_{xy}^{i,j} = (x_i^e, x_{i+1}^e) \times (y_j^{e}, y_{j+1}^{e})$ is needed. It is a control volume for transport equation for the shear extra-stresses. The faces of these cells are $\Gamma_{xy}^{i,j}$ (see figure 2) and their areas are $M_{xy}^{i,j}$ [9].

The level-set function $\varphi = \varphi(\mathbf{r}) = \varphi(x,y)$ [6] is used for immersed boundary $\Gamma_{ib}$ description [5]:

$$
\left\{
\begin{array}{ll}
\varphi(\mathbf{r}) < 0, & \mathbf{r} \in \Omega_d^f = \Omega \setminus (\Omega_{ib} \cup \Gamma_{ib}), \\
\varphi(\mathbf{r}) = 0, & \mathbf{r} \in \Gamma_{ib}, \\
\varphi(\mathbf{r}) > 0, & \mathbf{r} \in \Omega_{ib}.
\end{array}
\right.
$$

(2)

In the considered test problem the level-set function for a circular airfoil with the center at point with coordinates $(x_C, y_C)$ can be defined by the following analytical formula:

$$
\varphi(x,y) = R - \sqrt{(x-x_C)^2 + (y-y_C)^2}.
$$

(3)

The boundary $\Gamma_{ib}$ is represented by a line segment on the cut-cell $\Omega_{i,j}$. Location of this segment endpoints is defined by a linear interpolation of the variable $\varphi_{i,j} = \varphi(x_i,y_j)$. The cell-face fraction ratios $\vartheta_{n,i,j}$ and $\vartheta_{p,i,j}$ are introduced [5]. They take values in interval $[0,1]$ and represent the fluid parts of the east and north faces of $\Gamma_{i,j}$, respectively. The cell-face fraction ratios are defined by a one-dimensional linear interpolations of functions $\varphi(x,y)$ in interval $[y_{j-1}, y_j]$ and $\varphi(x,y)$ in interval $[x_{i-1}, x_i]$:

$$
\vartheta_{n,i,j} = \frac{\min(\varphi_{i,j-1}, \varphi_{i,j})}{\min(\varphi_{i,j-1}, \varphi_{i,j}) - \max(\varphi_{i,j-1}, \varphi_{i,j})}, \quad \vartheta_{p,i,j} = \frac{\min(\varphi_{i-1,j}, \varphi_{i,j})}{\min(\varphi_{i-1,j}, \varphi_{i,j}) - \max(\varphi_{i-1,j}, \varphi_{i,j})}.
$$

(4)

In 2D case, the cut-cells can be classified into trapezoidal, triangular and pentagonal cells. Examples of each type cut-cells are shown on figure 3. Location of the variables discretization points depends on the type cell.

The time integration of the differential algebraic system arising after spatial LS-STAG-discretization is performed using a method based on the first-order predictor-corrector scheme. This method consists of two steps.

Predictor step leads to discrete analogues of the Helmholtz equation for velocities prediction $\tilde{U}$ in time point $t_{n+1} = (n + 1)\Delta t$:

$$
\frac{M(\tilde{U}^{n+1})}{\Delta t} + C[U^n]U^n + S^{ib,c,n} - DT(P^n - T_{norm}^{e,n}) - DR^{e,n} - \frac{1}{Re}(K\tilde{U} + S^{ib,n}) = 0.
$$

(5)

Here $P^n$ is a vector with $p^n_{i,j}$ components; $T_{norm}^{e,n}$ is a vector whose components are the normal extra-stress values $\tau^{e,n}_{xx,i,j}$ and $\tau^{e,n}_{yy,i,j}$; $T_{xy}^{e,n}$ is a vector with $\tau^{e,n}_{xy,i,j}$ components; $U^n$ is a vector whose
components are the velocity values $u^v_{i,j}$ and $v^v_{i,j}$; $S^{ab,c,\nu}$ and $S^{ab,\nu}$ are the source terms arising from the boundary conditions; $M$ is the diagonal matrix whose elements are areas of $\Omega_{i,j}$ and $\Omega_{i,j}^{x,y}$ cells; $C[U]$ and $K$ are matrices arising from the LS-STAG-discretization of convective and diffusive flows respectively; $-D^T$ is the matrix defining the gradient operator discrete analogue; $D^T$ is the matrix defining the divergence operator discrete analogue on the $xy$-mesh. To construct $D^T$, we apply the Ostrogradsky — Gauss formula:

$$\int_{\Omega_{i,j}^{x,y}} \nabla \cdot \tau_{xy}^e \, dV = \int_{\Gamma_{i,j}^{x,y}} \tau_{xy}^e e_y \cdot n \, dS, \quad \int_{\Omega_{i,j}^{x,y}} \nabla \cdot \tau_{xy}^e \, dV = \int_{\Gamma_{i,j}^{x,y}} \tau_{xy}^e e_x \cdot n \, dS. \tag{6}$$

Discrete analogues of these integrals can be computed by the following formulae for all cell types:

$$\int_{\Gamma_{i,j}^{x,y}} \tau_{xy}^e e_y \cdot n \, dS \approx \frac{1}{2} (\varphi_{i,j}^v \Delta x_i + \varphi_{i+1,j}^v \Delta x_{i+1}) \tau_{xy}^e_{i,j} - \frac{1}{2} (\varphi_{i,j-1}^v \Delta x_i + \varphi_{i+1,j-1}^v \Delta x_{i+1}) \tau_{xy}^e_{i+1,j-1}, \tag{7}$$

$$\int_{\Gamma_{i,j}^{x,y}} \tau_{xy}^e e_x \cdot n \, dS \approx \frac{1}{2} (\varphi_{i,j}^v \Delta y_j + \varphi_{i+1,j}^v \Delta y_{j+1}) \tau_{xy}^e_{i,j} - \frac{1}{2} (\varphi_{i-1,j}^v \Delta y_j + \varphi_{i+1,j+1}^v \Delta y_{j+1}) \tau_{xy}^e_{i+1,j+1}. \tag{8}$$

So, $D^T$ consists of $D^T_x$ and $D^T_y$ blocks, which are built according to the following patterns:

$$i = 1, N_x - 1 : \quad \left\{ \begin{array}{l}
D^T_x, p(i,j) = \frac{1}{2} (\varphi_{i,j}^v \Delta x_i + \varphi_{i+1,j}^v \Delta x_{i+1}), \quad j = 1, N_y - 1; \\
D^T_x, s(i,j) = -\frac{1}{2} (\varphi_{i,j-1}^v \Delta x_i + \varphi_{i+1,j-1}^v \Delta x_{i+1}), \quad j = 2, N_y; \\
\end{array} \right. \tag{9}$$

$$j = 1, N_y - 1 : \quad \left\{ \begin{array}{l}
D^T_y, p(i,j) = \frac{1}{2} (\varphi_{i,j}^v \Delta y_j + \varphi_{i,j+1}^v \Delta y_{j+1}), \quad i = 1, N_x - 1; \\
D^T_y, w(i,j) = -\frac{1}{2} (\varphi_{i-1,j}^v \Delta y_j + \varphi_{i-1,j+1}^v \Delta y_{j+1}), \quad i = 2, N_x. \\
\end{array} \right. \tag{10}$$

Here $N_x$ and $N_y$ is the number of the base mesh cells along the $Ox$ and $Oy$ axis, respectively. The following notation is used for matrix elements: the element with the index $P$ is on the diagonal in the row of the corresponding cell with the number $(i,j)$, the element with the index of $W$ is in the column with the number of the control volume neighboring this cell from the west, $N$ is in the column with the number of the control volume neighboring this cell from the north, etc.

Corrector step leads to the following discrete analogue of Poisson equation for pressure function $\Phi = \Delta t (P^{n+1} - P^n)$:

$$A \Phi = D \bar{U}^{\theta b,n+1}, \quad A = -DM^{-1}D^T. \tag{11}$$

Then flow variables at the time point $t_{n+1}$ are computed by the following formulæ:

$$U^{n+1} = \bar{U} + M^{-1}D^T \Phi, P^{n+1} = P^n + (\Phi/\Delta t). \tag{12}$$

After this, new values of extra-stresses $T^{e,n+1}$ are computed according by the following way:

$$M_e T^{e,n+1} + \frac{M_e (T^{e,n+1} - T^{e,n})}{\Delta t} = 2\nu_e M_e S^{n+1} + \frac{2\nu_e \lambda_e M_e (S^{n+1} - S^n)}{\Delta t} + CD(\bar{U}^{n+1}, X^n). \tag{13}$$

Here $M_e$ is the diagonal matrix whose elements are areas of $\Omega_{i,j}$ and $\Omega_{i,j}^{x,y}$ cells; $S$ is a vector whose components are the rate-of-strain tensor components at the corresponding points; $CD(\bar{U}^{n+1}, X^n)$ is a discrete analog of convective derivative $X^n$ excluding time derivative; $X^n = 2\nu_e \lambda_e S^{n+1} - \lambda T^{e,n}$. The convective derivative can be upper (Oldroyd) derivative, lower (Cotter — Rivlin) derivative or rotational (Yauman or Yauman — Zaremba — Noll) derivative [10] depending on the used viscoelastic fluid model.
3. Numerical experiments.
When solving the problem (1), we assume that the base length is equal to \( R = 1 \) and the base velocity is equal to \( V_\infty = 1 \). Then the Weissenberg number is equal to

\[
\text{We} = \frac{\lambda \cdot V_\infty}{R} = \lambda,
\]

(14)

and the Reynolds number is defined as follows:

\[
\text{Re} = \frac{V_\infty \cdot R}{\nu} = \frac{1}{\nu} \frac{1}{\nu_s + \nu_e} = \frac{\beta}{\nu_s}.
\]

(15)

Here \( \nu_s \) is the solvent viscosity; \( \beta \) is the solvent viscosity ratio. So, the polymer viscosity can be defined by the following formula:

\[
\nu_e = \frac{1 - \beta}{\text{Re}}.
\]

(16)

And the retardation time can be calculated by the following way:

\[
\lambda_r = \beta \cdot \text{We}.
\]

(17)

As in experiment [17], we have chosen \( \beta = 0.73 \).

Computations were carried out on two non-uniform meshes (see table 1). A mesh size along the \( Oy \) axis did not change and it was equal to \( h \). A mesh size along the \( Ox \) axis also was equal to \( h \) on the segment \([x_C - 1.5D; x_C + 1.5D]\). Here \( x_C = 10D \) is the airfoil center abscissa. A mesh size along the \( Ox \) was equal to \( 2h \) in the rest of the computational domain. Thus, an uniform mesh block with a \( h \) step was used in the airfoil vicinity.

As in study [17] dimensionless stationary aerodynamic drag coefficient \( C_D \) was calculated by the following formula:

\[
C_D = \frac{F_D}{\nu \cdot V_\infty \cdot R}.
\]

(18)

Here \( F_D \) is a drag force. For the case of the LS-STAG-discretization usage, its calculation is described in detail in [5, 9]. Comparisons of time averaged drag coefficient \( C_D \) with experimental data [17] are shown on figures 4 and 5. Values computed on \( M1 \) and \( M2 \) meshes differ by less than 0.2% in all computations. For this reason only values computed on \( M2 \) mesh are shown on figures 4 and 5. They are indicated by black circles.

Data obtained for a Newtonian fluid at low Reynolds numbers \( \text{Re} < 0.55 \) is shown on figure 4. The drag coefficient value should not depend on the Reynolds number [17] at \( C_D \) computation by the formula (18). However, as the experiment [17] authors note, the drag measurements are less accurate at low Reynolds numbers \( \text{Re} < 0.2 \), therefore, the results differ by about 10% in this Reynolds numbers range. Experimental \( C_D \) mean is about 76.61 [17] at \( 0.2 < \text{Re} < 0.55 \). Wherein computed at this Reynolds numbers range on \( M2 \) mesh \( C_D \) mean value is about 76.82. Thus, for the case of Newtonian fluid the computational results are in good agreement with the experimental data [17] at low Reynolds number for the LS-STAG method.

### Table 1. Salient properties of the meshes used for the model problem.

| Mesh | Time step \( \Delta t \) | \( h \) | Number of cells | Number of solid cells | Number of cut-cells |
|------|------------------------|------|----------------|---------------------|-------------------|
| \( M1 \) | 0.050 | 0.2 | 2300 | 60 | 28 |
| \( M2 \) | 0.025 | 0.1 | 4600 | 276 | 68 |
Table 2. Comparison of dimensionless vortex wake length with experimental data [17].

| Re  | We  | Experiment [17] | Computation on M1 | Computation on M2 |
|-----|-----|-----------------|-------------------|-------------------|
| 0.230 | 0.000 | 3.670          | 3.400              | 3.700             |
| 0.068 | 1.420 | 6.396          | 6.200              | 6.300             |
| 0.171 | 2.720 | 8.996          | 8.600              | 8.700             |

Viscoelastic flow simulations showed that the greatest differences between viscoelastic and Newtonian fluid occur in the wake past an airfoil: the vortex wake length increases with the Weisenberg number $We$. The comparison of the simulated vortex wake length with the experimental data [17] is presented in Table 2. It can be seen that the difference between the results obtained on different meshes reaches 10% in contrast to the $C_D$ values. Nevertheless, the results are close to the experimental data on both meshes.

The drag coefficient $C_D$ value also increases with the Weissenberg number (see figure 5). The difference between the computational results and the experimental data reaches 3% at $We < 2.5$ and this difference does not exceed 4% at higher values.

![Figure 4](image1.png)

**Figure 4.** Comparison of time averaged drag coefficient $C_D$ with experimental data (solid lines) for Newtonian fluid.

![Figure 5](image2.png)

**Figure 5.** Comparison of time averaged drag coefficient $C_D$ with experimental data (solid lines) for viscoelastic fluid.

4. Conclusions.

The numerical LS-STAG immersed boundary method modification is presented. This modification allows to simulate viscoelastic flows described by linear and quasilinear rate-type models (Maxwell model, Jeffrey model, Johnson — Sigelman model, Maxwell-A model, Oldroyd-B model, Oldroyd-A model, Upper Convective Maxwell model). In particular, formulae for differential types of convected time derivatives (Oldroyd, Cotter — Rivlin, Jaumann — Zaremba — Noll derivatives) the LS-STAG discretization is obtained. Main ideas for the extra-stresses simulation by the developed numerical method are presented.

The numerical algorithm developed in present study is implemented in a software written in C++. As an example, the test problem about viscoelastic Oldroyd-B flow past a circular airfoil is considered. The problem was solved by using the developed modification of the LS-STAG method. Computational experiments were carried out at Weissenberg number in the range from 0 to 4 on sufficiently coarse meshes (no more than 20 cells corresponded to profile diameter).
The obtained numerical results are used to verify robustness and computational efficiency of the developed LS-STAG method modification. The computed values of the drag coefficients and the wake length are in good agreement with the experimental data.

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