Dynamical decoupling leads to improved scaling in noisy quantum metrology

Pavel Sekatski¹, Michalis Skotiniotis¹,² and Wolfgang Dür¹

¹ Institut für Theoretische Physik, Universität Innsbruck, Technikerstr. 21a, A-6020 Innsbruck, Austria
² Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

E-mail: pavel.sekatski@uibk.ac.at

Keywords: quantum metrology, dynamical decoupling, Heisenberg scaling

Abstract
We consider the usage of dynamical decoupling in quantum metrology, where the joint evolution of system plus environment is described by a Hamiltonian. We show that by ultra-fast unitary control operations acting locally only on system qubits, noise can be eliminated while the desired evolution is only reduced by at most a constant factor, leading to Heisenberg scaling. We identify all kinds of noise where such an approach is applicable. Only noise that is generated by the Hamiltonian to be estimated itself cannot be altered. However, even for such parallel noise, one can achieve an improved scaling as compared to the standard quantum limit for any local noise by means of symmetrization. Our results are also applicable in other schemes based on dynamical decoupling, e.g. the generation of high-fidelity entangling gates.

1. Introduction
Quantum mechanics offers the promise to significantly enhance the precision of estimating unknown parameters as compared to any classical approach [1–3]. Such high-precision measurements are of central importance in physics and other areas of science, and include possible applications in frequency standards [4–6], atomic clocks [7–9], or gravitational wave detectors [10, 11]. Whereas the quantum advantage can survive in the presence of some particular noise types [12, 13], it can not be maintained in the presence of a general incoherent noise processes [5, 14–23].

Identifying schemes that realize this advantage in practice are of high theoretical and practical relevance. It was shown that quantum error correction [23–26], dynamical decoupling [27, 28] or limited control over the environment [29, 30] allow for an improved scaling for certain, specific noise processes.

Here we consider the usage of dynamical decoupling techniques [31–34] for quantum metrology. These techniques have been shown to be applicable in the context of storage or for the realization of quantum gates [35, 36], and are nowadays widely used in various experimental settings [37–45]. We show that with the help of ultra-fast control operations that act locally on the system qubits, one can effectively decouple the system from its environment and fully protect it against decoherence effects, while at the same time maintaining its sensing capabilities and its quantum advantage in metrology. We identify all noise processes that can be eliminated, and discuss additional techniques that allow one to obtain a super-classical scaling even in situations where these methods are not applicable. Our systematic investigation of which kinds of noise are correctable is also applicable in different contexts, e.g. for the realization of high-fidelity quantum gates using dynamical decoupling techniques [35, 36].

System and environment interact via a coherent evolution governed by some Hamiltonian $H_{SE}$. In addition, the sensing system is effected by a Hamiltonian $H_S$ that includes an unknown parameter that should be estimated. We assume that a coherent description is appropriate at all times, and that the evolution can be interrupted by fast control operations. This is similar to what is done in dynamical decoupling [31, 32, 35, 36, 46], with the requirement that the desired evolution is not cancelled. In addition we do not
enforce the evolution to be left invariant, but allow it to be altered in a controlled way. In practice, these
intermediate pulses will not be infinitely fast, and only noise up to a certain frequency can be treated and
eliminated this way. However, in what follows we will assume that the accessible time is much faster than any
other timescale in the problem.

Our main findings are summarized below:

(1) We show that for any local system–environment interaction of rank one or two one can completely eliminate
the noise process at the cost of reducing the sensing capabilities of the system by a constant factor leading to
Heisenberg scaling in precision. Only interactions that are of rank three—and hence necessarily contain a
component that is generated by the system Hamiltonian that should be estimated—cannot be fully
corrected.

(2) Still one can achieve that only this parallel noise part remains, and, even in this case one can still maintain a
quantum advantage. In particular, for any local noise process, we show that one can achieve a super-standard
quantum scaling in precision by preparing N qubit probes in the Greenberger–Horne–Zeilinger (GHZ) state
and randomizing the qubits via fast, intermediate swap gates.

(3) For correlated noise processes we provide a general lower bound on the achievable precision, applicable even
if one assumes perfect quantum control and auxiliary resources, demonstrating that one is in general
restricted to the standard quantum limit (SQL). Even in this case, one may still improve the precision by a
constant factor, though the effect depends on the details of the system–environment interactions and the type
of fluctuations.

(4) Finally we demonstrate that that our results are applicable in the design of high-fidelity quantum gates for
fault-tolerant quantum computation.

2. Background

Quantum metrology is the science of optimally measuring and estimating an unknown parameter such as the
frequency in an atomic clock or the strength of a magnetic
field. In a typical metrology scenario, a system of N
probes evolves for time t under the Hamiltonian

\[ H_S = \omega \sum \sigma_j^{(3)}, \]

where \( \omega \) is the parameter to be estimated, and is
subsequently measured. This process is repeated v times and \( \omega \) is estimated from the measurement statistics. The
Cramér–Rao bound provides a limitation on the estimation precision \( \tilde{\omega} \geq \frac{1}{v \sqrt{\mathcal{F}_{\text{SQL}}}} \), which is attainable for large
enough number of repetitions v. Here \( \mathcal{F} \) is the quantum Fisher information (QFI) [47]. For classical strategies
(separable probe states) the ultimate precision is given by the SQL, \( \mathcal{F} \propto N \). Using entangled states, one can
achieve the so-called Heisenberg limit, \( \mathcal{F} \propto N^2 \), i.e., a quadratic improvement as compared to any classical
strategy [1–3].

However, in practice the system is not isolated but also interacts with its environment. In general, this leads
to a noisy evolution, where it was shown that in case of incoherent noise described by a master equation, the
quantum advantage is limited to a constant factor rather than a different scaling [15–22]. The way to describe
noise processes crucially depends on the timescales of the problem. Here we assume that we have ultra-fast
access to the system, and can interject the evolution by control operations, faster than the relaxation time of the
environment. In this case it is appropriate to describe the dynamics of system plus environment in a coherent
way, i.e., by means of an overall Hamiltonian that governs the unitary evolution. As we will show shortly, the fact
that system and environment evolve coherently grants us with additional freedom that allows to maintain
Heisenberg scaling even in the presence of uncontrolled interaction with some environment.

3. General setting

We begin by first considering the case of a single qubit. The most general evolution of a single qubit plus
environment is described by the Hamiltonian

\[ H = H_S + H_{\text{SE}} \equiv \omega \sigma_3 \otimes 1 + \sum_{j=0}^{3} \epsilon_j \sigma_j \otimes A_j, \]

where \( H_S = \omega \sigma_3 \otimes 1 \) describes the evolution of the system, \( H_{\text{SE}} \) describes the system-environment interaction
with \( A_j \) arbitrary environment operators and \( \epsilon_j \) the coupling strengths. Here and throughout the remainder of
this work \( \sigma_j, j = \{1, 2, 3\} \) denote the Pauli matrices, \( \sigma \) denotes the vector of Pauli matrices and \( \sigma_n = n \cdot \sigma \).

In the case where we have N probe systems, the exact form of the Hamiltonian governing their coherent
evolution with the environment depends on whether the N probes couple to individual environments, or to a
common environment. In the former case the Hamiltonian is given by

\[ H = \sum_{i=1}^{N} \left( \omega \sigma_i^{(a)} \otimes 1 + \sum_{j=0}^{3} \alpha_j \sigma_j^{(a)} \otimes A_j^{(a)} \right), \]

where \( A_j^{(a)} \) act on different Hilbert spaces. If the N probes couple to a common environment the \( A_j^{(a)} \) act on the same Hilbert space and might even be equal for all \( a \) in the extreme case.

4. Decoupling strategy

The most general dynamical decoupling strategy consists of applying an arbitrarily fast time sequence of unitary gates \( \{ u_i\}_{i=0}^{n} \) applied at times \( \{ t_0, t_1, \ldots, t_n \} \), i.e., intersecting the system evolution with fast pulses. If this is done fast enough, one can use a first order Trotter expansion to describe the effective evolution as being generated by the Hamiltonian

\[ H_{\text{eff}} = \mathcal{E}(H) = \sum_{i=1}^{n} p_i HUU_i^\dagger \]

modulo an irrelevant final unitary, where we redefine the gates as \( U_1 = u_0 \), \( U_{i+1} = U_i u_i \) and the probabilities are obtained from the time sequence \( p_i = \frac{1}{t_i - t_{i-1}} \). This is similar to what is done in optimal control theory or bang-bang control [31, 32], with the requirement that the desired evolution \( H_3 \) is not completely eliminated. For the simple case of orthogonal \( \sigma_\eta \) noise, fast intermediate \( \sigma_\eta \) operations suffice (see [28]).

The applicability of dynamical decoupling techniques is determined by the timescales of the problem. We are considering a situation where control operations can be performed infinitely fast. This is a standard assumption in optimal control theory [31, 32], and justifies the expansion in equation (3) leading to the effective Hamiltonian. The requirements on the applicability of the first order Taylor expansion, i.e., the requirement on the frequency of intermediate control operations, depend on the spectral radius of the Hamiltonian (and thus the number of systems \( N \)). In addition, in order to write the effective Hamiltonian as a coherent sum of terms for different times it is important that the environment operators and the coupling terms do not change. This provides the condition that control operations need to be much faster than all relevant relaxation timescales of the environment. We would like to point out, though, that methods in dynamical decoupling techniques have been developed where this stringent assumption of infinite fast control, or in turn a restriction of results to noise up to a certain frequency for any given control rate, can be significantly softened [48].

5. Single qubit

In the single qubit case the parametrization of completely positive trace preserving (CPTP) maps in equation (3) spans exactly all the unital maps, i.e. such that \( \mathcal{E}(I) = I \) [49], and any unital map can be constructed with the set of four gates \( \{ \sigma_i \}_{i=0}^{3} \). As all single qubit CPTP maps corresponds to affine transformations on the Bloch sphere, a unital CPTP map is uniquely specified by a real matrix \( A = RDR \), where \( R, D \in SO(3) \), \( D = \text{diag}(\eta_0, \eta_1, \eta_2) \) (the values \( \eta_i \) have to satisfy several constraints which are, however, not important in our context), and the action of such map on any Pauli matrix is given by \( \mathcal{E}(\sigma_\eta) = A n \cdot \sigma \). The second rotation \( R \) corresponds to an inconsequential change of basis, so we assume \( R = R^\dagger \).

Accordingly, a noise term is completely eliminated by the dynamical decoupling strategy if and only if it belongs to the kernel of the corresponding matrix \( DR \); if a noise term belongs to the kernel it is eliminated by the action of \( DR \), while if it does not at least a part of it remains after the application of the dynamical decoupling sequence. On the other hand \( D = 0 \) since some part of the system evolution has to survive. Therefore to identify all the noises that can be removed it is sufficient to consider rank one projectors \( A = I \eta \) on a general direction \( r = (r_1, r_2, r_3)^T \). It follows that the action of the corresponding map \( \mathcal{E} = \pi_r \) on Pauli matrices is given by \( \pi_r(\sigma_\eta) = (r \cdot n) \sigma_\eta \). Notice that this remains true even if one allows for auxiliary systems and intermediate unitary operations in equation (3) to act on the enlarged system. Even when embedded in a larger vector space (spanned by operators which could also act on auxiliary qubits), the support of the evolution and of the noise operators is still on a subspace that has the same dimension as the initial space. Hence, the purely geometrical argument regarding which type of vectors can be eliminated is still valid even when using higher-dimensional vectors that form a subspace of the same dimension as the initial space. Finally note that a unital map \( \pi_r \) can be easily implemented by applying \( U = \sigma_r \) at regular intervals, so that the effective Hamiltonian is \( H_{\text{eff}} = \pi_r(H) = (H + \sigma_r H \sigma_r)/2 \).
Now we identify all the noises that can be removed by dynamical decoupling. The noise term $1 \otimes A_0$ in equation (1) only acts on the environment and cannot be altered by control operations performed on the system. In what follows we will ignore this term. The action of the decoupling strategy $\pi_3$ on the rest of the Hamiltonian in equation (1) leads to

$$H_{\text{eff}} = \omega \, \sigma_3 \otimes 1 + \sigma_\tau \otimes \sum_{j=1}^{3} c_j A_j.$$  \hfill (4)

Consequently the noise can be effectively decoupled if and only if $\exists \alpha_1, \alpha_2 \in \mathbb{R}$ such that $c_j A_j = \sum_{j=1}^{3} \alpha_j c_j A_j$, in which case the effective system evolution is slowed down by the factor $\tau_3 = (1 + \alpha_1^2 + \alpha_2^2)^{-1/2}$. In the case of bounded operators $A_j$ the Hamiltonian equation (1) can be put in the standard form

$$H = \omega \, \sigma_3 \otimes 1 + \sum_{j=1}^{3} b_j \sigma_n \otimes B_j,$$  \hfill (5)

where $\text{tr}B_j B_k = \delta_{jk} \{ n_1, n_2, n_3 \}$ is an orthonormal frame and $b_1 \geq b_2 \geq b_3$ are the ordered Schmidt coefficients, see supplementary material. This allows a more intuitive geometrical picture of dynamical decoupling. For any rank one or two noise, i.e. $b_1 = 0$, we can choose $r = n_1 \times n_2$, orthogonal to the plane where noise acts. The fact that the desired evolution $\sigma_3 \otimes 1$ is not completely canceled requires that $r \cdot z = 0$, which is equivalent to $z \notin \text{span} \{ n_1, n_2 \}$. In this case, the noise is completely eliminated by dynamical decoupling and, as above, the system evolution $H_3 = \tau_3 \omega \, \sigma_3 \otimes 1$ is slowed down by a factor $\tau_3 = z \cdot (n_1 \times n_2)$. The decrease of the coupling strength leads to a constant reduction of the achievable accuracy by $(\tau_3)^2$ but, as all noise is completely eliminated, we still obtain Heisenberg scaling precision. Notice that noise that is perpendicular to the system Hamiltonian, i.e., any combination of $\sigma_1$ and $\sigma_2$ noise, can be eliminated without altering the evolution, i.e. $\tau_3 = 1$. This is done by using fast intermediate $\sigma_3$ pulses.

6. Single qubit and full rank noise

From the above geometrical argument, it follows that one cannot eliminate rank three noise as such noise spans the whole three-dimensional space. To see this note that the effect of the decoupling procedure on any rank-three noise model is to project both the system Hamiltonian and noise onto direction $r$, so one obtains an effective Hamiltonian equation (4) for system plus environment which is unitarily equivalent to $\sigma_3$ evolution and parallel noise, with $\sum_{j=1}^{3} r_j c_j A_j = \sum_{j=1}^{3} (n_1 \cdot r) b_j B_j$ for the standard form. As noise generated by the same operator as the system Hamiltonian cannot be eliminated without eliminating the system evolution as well, the best one can hope for in this case is to reduce any rank-three noise to noise parallel to the system evolution.

The choice of direction $r$ in equation (4) determines both the effective coupling strength of the system Hamiltonian $\tau_3$ and the strength of the noise. One can then choose $r$ to optimize the ratio between the modified coupling strength $\tau_3$ and the variance of noise fluctuations after projection. Notice, however, that while this shows that in general we can reduce the problem to parallel noise it is not clear if this reduction to parallel noise does in fact correspond to the optimal strategy. Later on we will show what this optimal ratio is for the case of local Gaussian noise.

7. $N$ qubits

We now turn to the case of $N$ two level systems. Consider first the case where each qubit encounters an independent environment which corresponds to a local noise process. The total Hamiltonian describing the evolution of all $N$ qubits plus environment is given by equation (2), with the system evolution $H_S = \omega \sum_j \sigma_j^0 \otimes 1 + \omega_{S3}$ and the system–environment interaction $H_{SE} = \sum_{a=1}^{N} \sum_{j=1}^{3} c_j^{(a)} a_j^{(a)} \otimes A_j^{(a)}$. One can use the above dynamical decoupling strategy independently on each of the systems so that the results of the previous section directly apply. For each qubit, noise of rank one or two can be eliminated, while full rank noise can be reduced to parallel noise $H_{SE} = \sum_a \dot{z}^{(a)} \sigma_\tau^{(a)} \otimes \tilde{A}^{(a)}$ with $\tilde{z}^{(a)} \tilde{A}^{(a)} = \sum_j c_j^{(a)} a_j^{(a)} A_j^{(a)}$ and $H_3 = \omega \sum_a r_3^{(a)} \sigma_\tau^{(a)}$.

In addition, one can randomize the system particles by means of fast intermediate permutations. Such permutations leave $H_S$ unchanged, but project out all asymmetric noise terms onto their symmetric contribution. For single-body noise terms (where each term only acts non-trivially on one of the qubits) such symmetrization is achieved by applying a circular shift at regular time intervals, and each shift can be realized by a sequence of $N-1$ two-qubit swap gates acting on all pairs of neighboring qubits one after the other. This follows from the fact that after $N$ circular shifts the qubits are back to their initial position, and each qubit was...

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3 For this to hold one requires perfect knowledge of $\tau_3$. This means perfect knowledge of the direction of the sensing Hamiltonian $\sigma_3$ and perfect control of the decoupling pulses. This is a standard assumption in metrology.
coupled to each local environment for the same duration. Hence, the remaining effective noise term is given by

\[ H_{SE} = \tilde{\epsilon}_3 S_3 \otimes \tilde{A}, \quad \tilde{A} = \frac{1}{\tilde{\epsilon}_3} \sum_{a=1}^{N} \tilde{\epsilon}^{(a)} \tilde{A}^{(a)}, \quad \tilde{\epsilon}_3 = \frac{1}{N} \sum_{a=1}^{N} \tilde{\epsilon}^{(a)}. \]  

(6)

Notice that in general \( \tilde{A} \) depends on the individual coupling strengths \( \tilde{\epsilon}^{(a)} \) unless all \( \tilde{A}^{(a)} \) are identical. As we show later symmetrization of all system qubits can, in the presence of independent couplings or fluctuating coupling strengths, help boost precision to super-classical scaling. We remark that if the noise has no symmetric contributions then \( \tilde{\epsilon}_3 = 0 \) and even locally full rank noise can be eliminated by symmetrization.

We now consider the case where the \( N \) qubits couple to a common environment, which may possess both temporal and spatial correlations, i.e., the operators \( \tilde{A}^{(a)} \) in equation (2) are arbitrary. Suppose first that the system–environment interactions are such that each system qubit interacts individually with the environment. Then a strategy similar to the single-qubit case can be applied, where one eliminates all noise except the one generated by the (symmetrized) system Hamiltonian itself. By way of example consider the following local decoupling strategy where one applies fast local \( \sigma_z^{(a)} \) on each of the qubits. This allows to eliminate all noise terms including \( \sigma_x^{(a)}, \sigma_y^{(a)} \) without altering \( H_3 \), and together with fast random permutations reduces all noise to one generated by the system Hamiltonian itself, see equation (6). The only difference as compared to the case of independent environments treated above is that now the operators \( \tilde{A}^{(a)} \) may act on the same environment. In general a more involved decoupling strategy requiring non-local operations may be needed in order to partially or fully remove the noise. However, it is not clear if all noise operators except those parallel to \( H_3 \) can be completely removed in this case as not all unital maps can be expressed as convex combinations of unitary operations [50]. Moreover, whatever the dynamical decoupling procedure, the condition that \( H_3 \) has a non-zero overlap with the kernel of the unital map must hold in order to be able to estimate \( \omega \).

One may also consider noise arising from terms in the Hamiltonian that act on several systems simultaneously (not to be confused with correlated noise terms where different systems couple to the same environment). From a physical standpoint such many-body noise processes are less important as they usually correspond to higher order processes. Nevertheless, these many-body processes can be eliminated by means of dynamical decoupling, and for any quasi-local noise process one still recovers Heisenberg scaling in the absence of noise generated by \( S_3 \), see supplementary material.

8. Parallel noise

Hitherto, we have seen how to eliminate all kinds of noise, except noise generated by \( S_3 \). The latter is indistinguishable from the desired evolution, and cannot be eliminated. However, we will now show that even such parallel noise does not automatically imply the SQL. In fact, the scaling of the QFI depends on the particular situation considered. For instance, for local uncorrelated noise a super-SQL scaling \( O(N^{3/2}) \) of the QFI can be achieved, as we show later in this section.

Consider the effect of the system plus environment evolution described by equation (6) on the system alone. Tracing out the environment in the eigenbasis \( \{ |\ell\rangle \} \) of \( \tilde{A}_3 \) one can always represent the noise as

\[ E(\rho) = \int p(\tilde{\epsilon}_3) f(\ell) e^{-i\omega \ell S_3} \rho e^{i\omega \ell S_3} d\ell d\tilde{\epsilon}_3, \]

(7)

where \( p(\tilde{\epsilon}_3) \) corresponds to fluctuations of the interaction strength between experimental runs, and \( f(\ell) = \langle \ell | \rho \ell \rangle \) depends on the initial state of the environment (if \( \tilde{A}_3 \) has a discrete spectrum then the integral over \( \ell \) is replaced by a sum).

The effect of the system–environment coupling, when the environment is not in an eigenstate of \( \tilde{A}_3 \), is similar to a fluctuating interaction strength. Both cases lead to an ensemble average over evolutions governed by the same Hamiltonian as \( H_3 \) with a random prefix. This averaging ultimately limits the achievable accuracy, as it corresponds to fluctuations of the parameter \( \omega \) to be estimated. However, the resulting scaling depends on details, such as the spectrum of environment and whether these fluctuations are correlated or uncorrelated.

The worst case is when the interaction strength, \( \tilde{\epsilon}_3 \), is fixed but unknown. This type of noise leads to a systematic error on the estimated value of \( \omega \) and there is no way to decrease the error below a certain value set by the initial knowledge of the interaction strength and the state of the environment (except the trivial case where the environment is in the zero eigenstate of \( \tilde{A}_3 \)).

Next consider the case where the mean interaction strength \( \tilde{\epsilon}_3 \) is known but fluctuates around the mean value between experimental runs following a smooth distribution \( p(\tilde{\epsilon}_3) \) and \( \tilde{A}_3 = 1 \). This is equivalent to the case of a fixed \( \tilde{\epsilon}_3 \) but a continuous spectrum of \( \tilde{A}_3 \) with smooth \( f(\ell) \). We show in the supplementary material that for any \( p(\tilde{\epsilon}_3) \) the optimal QFI per unit time is upper bounded by
\[ \frac{\mathcal{F}}{t} \leq N \sqrt{\mathcal{F}_d(p(\xi))}, \]  

(8)

where \( \mathcal{F}_d(p(\xi)) \) remains finite for every smooth noise distribution \( p(\xi) \) enforcing the SQL in this case. For a normal distribution \( p(\xi) \) with width \( \sigma \) the bound takes the simple form \( \mathcal{F}/t \leq N/\sigma \), whereas a strategy utilizing an \( N \)-qubit GHZ state \( \frac{1}{\sqrt{N}}(|0\rangle^\otimes N + |1\rangle^\otimes N) \) gives a maximal QFI per unit time \( \mathcal{F}/t \approx 0.43 N/\sigma \) for the optimal choice of \( t \).

Next consider local parallel noise, where each \( \xi^{(1)} \) in equation (6) is an independent and normally distributed random variable with width \( \sigma \). After randomly permuting the probes one finds that \( \xi \) is also a normally distributed random variable whose width is reduced \( \bar{\sigma} = \frac{\sigma}{\sqrt{N}} \). Consequently, preparing the probes in the GHZ state yields a super-SQL scaling

\[ \frac{\mathcal{F}_{\text{GHZ}}}{t_{\text{opt}}} = \frac{N^{3/2}}{\sqrt{2\pi \bar{\sigma}}}, \]

(9)

where \( t_{\text{opt}} = 1/\sqrt{N\bar{\sigma}^2} \). The Cramér–Rao bound \( \delta \omega \geq (\nu \mathcal{F})^{-1/2} = (T \mathcal{F}/t_{\text{opt}})^{-1/2} \) is attainable for large total running time \( T = t_{\text{opt}} \). Hence symmetrization of the probes allows one to significantly reduce the overall effects of noise (as already noted in [51, 52]), and restore super-SQL scaling.

Finally, in the case where \( \xi \) is fixed and \( A_3 \) has a discrete spectrum, the effective noise distribution \( f(\xi) \) is discrete and \( \mathcal{F}_d(p(\xi)) \) is unbounded. Making the bound of equation (8) trivial. No general statements can be made with regards to the optimal QFI per unit time in this case. For example, if \( A_3 \) has an equally gapped spectrum with gap \( \Delta \), then at time \( t = \frac{\Delta}{2\bar{\sigma}} \), the noise completely cancels. This final example, though artificial, demonstrates that one cannot provide general statements on achievable scaling without specifying further details of the type of fluctuations, interaction, spectrum, and initial state of the environment.

9. Summary and outlook

We have shown that when an overall Hamiltonian description of the system plus environment is appropriate, ultra fast control allow one to alleviate a large class of noise processes, and recover Heisenberg scaling. We remark that the dynamical procedure outlined here are also applicable in other contexts, e.g. for the design of ultrafast control allow one to alleviate a large class of noise processes, and recover Heisenberg scaling. We have shown that when an overall Hamiltonian description of the system plus environment is appropriate, interaction, spectrum, and initial state of the environment.

Our results are in stark contrast to situations where a master equation description of the system-environment interaction is required. There, it has been shown that with the help of auxiliary systems and fast error correction only rank one Pauli noise processes can be eliminated, while even full quantum control including ultrafast pulses and quantum error correction do not allow one to go beyond SQL scaling [23]. Hence, our results provide a big promise for practical applications of quantum metrology in various contexts, opening the way towards ultra-sensitive devices with widespread potential application in all branches of science.

Finally an interesting question is how our technique can be applied beyond the single-parameter metrology scenario studied here. In general, in multi-parameter estimation might the sensing Hamiltonian might contain all three Pauli operators, so the the local dynamical decoupling techniques we described in this paper can not be directly applied (since they would cancel parts of the sensing Hamiltonian). However remark that it is always possible to sense each of the generators separately by dividing the probes in groups and applying the local decoupling strategy on each group separately; in a particular case this strategy was shown to achieve the optimal scaling [53]. Moreover note that the symmetrization technique of section 8 can be applied without modification to any multi-parameter scenario.

Acknowledgments

We thank J Kołodyński for useful discussions. This work was supported by the Austrian Science Fund (FWF): P24273-N16, P28000-N27 and the Swiss National Science Foundation grant P2GEP2_151964.
Appendix A. Standard form of the Hamiltonian

Consider a Hamiltonian of the form

\[ H = \sum_{j=1}^{3} c_j \sigma_j \otimes A_j. \]  

(A.1)

For bounded operators \( \tilde{C}_j = c_j A_j \) we define the overlap matrix

\[ \tilde{O}_k = \text{tr}\tilde{C}_i \tilde{C}_k, \]  

(A.2)

which is real and symmetric \( \tilde{O} = \tilde{O}^T \) (as imposed by the hermiticity of the Hamiltonian \( \tilde{C}_j = \tilde{C}_j^\dagger \)). Expressing the Pauli operators in a rotated frame \( \sigma_i = \tilde{R} \sigma_i \tilde{R}^\dagger \) allows one to rewrite the Hamiltonian as

\[ H = \sum_{k=1}^{3} \sigma_k \otimes \sum_j \tilde{R}_j \tilde{C}_j = \sum_{k=1}^{3} \sigma_k \otimes C_k, \]  

(A.3)

with \( C_k = \sum_{j=1}^{3} \tilde{R}_j \tilde{C}_k \). Accordingly the overlap matrix for the operators \( O_k = \text{tr}C_i C_k \) is given by

\[ O = R^T \tilde{O} R. \]  

(A.4)

Choosing the rotation that diagonalizes the symmetric matrix \( \tilde{O} = R \text{ diag} (\lambda_1, \lambda_2, \lambda_3) R^T \) leads to

\[ \text{tr}C_i C_k = \delta_{ik} \lambda_i. \]  

(A.5)

Which also shows that \( \lambda_i \geq 0 \), being the trace of the square of an Hermitian operator. Finally, denoting \( B_j = \frac{1}{\sqrt{\lambda_j}} C_j \) and \( b_j = \sqrt{\lambda_j} \) allows one to rewrite the Hamiltonian as

\[ H = \sum_{j=1}^{3} b_j \sigma_j \otimes B_j, \]  

(A.6)

with \( \text{tr}B_i B_k = \delta_{ik} \).

Appendix B. Correlated noise

We now consider correlated noise processes where several systems are affected jointly. In general, the system-environment Hamiltonian of a \( N \)-qubit system is given by

\[ H_{SE} = \sum_j c_j^{(a)} \sigma_j^{(a)} \otimes A_j^{(a)}, \]

where \( T_j^{(a)} = \sigma_j^{(a)} \otimes \sigma_j^{(a)} \ldots \sigma_j^{(a)} \) denotes a tensor product of Pauli operators. Using fast intermediate \( \sigma_j \) operations on all qubits allows one to eliminate all terms with \( \sigma_j \) acting on at least one system qubit. We are then left with a Hamiltonian where \( \lambda_j \in \{0, 3\} \) and noise is solely diagonal. In case of localized noise, i.e., where there is a certain spatial structure and only qubits that are spatially close are jointly affected by noise, one can use fast intermediate \( \sigma_j \) operations acting sparsely to eliminate noise terms of range \( k \). For instance, performing such an action on every second qubit eliminates all nearest neighbor two-qubit noise terms in a 1D setting (acting on every \( k \)th qubit eliminates confined \( k \)-body noise terms). However, this also eliminates the desired evolution for half of the particles (every \( k \)th particle), and these particles no longer contribute to the sensing process. As long as the number of systems to be decoupled is given by \( \alpha N \) with \( \alpha \) being some constant—which is the case for any finite range \( k \) noise operators—we still obtain Heisenberg scaling \( O(\alpha^2 N^2) \).

Appendix C. Parallel noise upper-bound QFI

In this section we derive a limitation on the maximally achievable QFI in presence of the parallel noise, i.e., noise that is described by the same generator as the Hamiltonian \( H \) that governs the evolution. Such parallel noise results in the channel

\[ \mathcal{E}(\rho) = \int p(\lambda) e^{-i\lambda H} \rho e^{i\lambda H} d\lambda, \]  

(C.1)

where \( \lambda \) is a random variable and \( p(\lambda) \) is a probability distribution with standard deviation \( \sigma \) characterizing the strength of the noise. As already mentioned such type of noise cannot be ameliorated using error correction as the operator generating it is identical to the Hamiltonian generating the desired evolution. The noise process in equation (C.1) can be viewed as describing classical noise applied directly on the estimated parameter \( \omega \), i.e., in every run of the experiment the observed parameter fluctuates by an amount \( \lambda \), with \( \lambda \) being a random variable with corresponding probability distribution \( p(\lambda) \).
Recall that the QFI of a state $\rho$ is given by
\[
\mathcal{F}(\rho(\theta)) = 8 \left( 1 - F(\rho(\theta), \rho(\theta + d\theta)) \right),
\] (C.2)
where $\rho(\theta) = e^{-i\theta\mathcal{H}} \rho e^{i\theta\mathcal{H}}$ and $F(\rho, \sigma) = \text{tr}\left(\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}\right)$ is the Uhlmann fidelity. So one can access the QFI in the presence of parallel noise (equation (C.1)) through the Uhlmann fidelity
\[
F(\mathcal{E}(\rho(t\omega)), \mathcal{E}(\rho(t\omega + t\d\omega))) = \int p(\lambda)rho(t\omega + t\lambda)d\lambda, \int p(\lambda - d\omega)rho(t\omega + t\lambda)d\lambda.
\] (C.3)

The Uhlmann fidelity is strongly concave
\[
F\left(\sum_i \rho_i, \sum_i \sigma_i \right) \geq \sum_i \sqrt{\rho_i \sigma_i} F(\rho_i, \sigma_i),
\] (C.4)
which implies that equation (C.3) is lower bounded by the fidelity of the probability distributions $p(\lambda)$ and $p(\lambda + d\omega)$
\[
F(\mathcal{E}(\rho(t\omega)), \mathcal{E}(\rho(t\omega + t\d\omega))) \geq \int \sqrt{p(\lambda)p(\lambda - d\omega)}d\lambda F(\rho(t\omega + t\lambda), \rho(t\omega + t\lambda)),
\] (C.5)
\[
= F_d(p(\lambda), p(\lambda - d\omega)).
\] (C.6)

Consequently the QFI in the presence of parallel noise is bounded by the classical Fisher information of the probability distribution characterizing the noise
\[
\mathcal{F}(\mathcal{E}(\rho)) \leq \mathcal{F}_d(p(\lambda)) = \int \frac{(p'(\lambda))^2}{p(\lambda)} d\lambda.
\] (C.7)

On the other hand we know than the QFI in the noisy case is lower than the noiseless QFI (attained by the GHZ state), therefore $\mathcal{F}(\mathcal{E}(\xi)) \leq t^2N^2$. Combining the two bounds one gets for the QFI per unit time
\[
\frac{\mathcal{F}(\mathcal{E}(\xi))}{t} \leq \min\left( t^2N^2, \frac{\mathcal{F}_d(p(\lambda))}{t} \right).
\] (C.8)

It remains to find the time $t$ that maximizes the rhs. Trivially the maximum is attained when $tN^2 = \frac{\mathcal{F}_d(p(\lambda))}{t}$, which yields
\[
\frac{\mathcal{F}(\mathcal{E}(\xi))}{t} \leq N\sqrt{\mathcal{F}_d(p(\lambda))}.
\] (C.9)

For any smooth distribution $p(\lambda)$ the classical Fisher information $\mathcal{F}_d(p(\lambda))$ is finite, and therefore SQL scaling for the QFI per unit time is enforced. In particular for a Gaussian noise with $p(\lambda) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\lambda^2/2\sigma^2}$ this bound implies
\[
\frac{\mathcal{F}(\mathcal{E}(\xi))}{t} \leq \frac{N}{\sigma}.
\] (C.10)

While for a simple strategy with GHZ states and the optimal choice of the time a straightforward calculation gives $\mathcal{F}/t \approx 0.429 N/\sigma$, which is roughly half of the bound above.

**Appendix D. Two-qubit Hamiltonians and gate implementation**

Consider the following Hamiltonian acting on two qubits plus an unspecified environment
\[
H = \sum_{i,j} \sigma_i \otimes \sigma_i \otimes 1 + \sum_i (\sigma_i \otimes 1 \otimes A_i + 1 \otimes \sigma_i \otimes B_i) + \sum_{i,j} \sigma_i \otimes \sigma_j \otimes C_{ij}.
\] (D.1)

In the equation above the first term corresponds to the useful two-qubit interaction, the second term couples one of the qubits to the environment locally (though the environment can be global if e.g. $A_i = B_i$), and the last term corresponds to non-local noise. Since we are interested in the implementation of two-qubit gates we consider a general dynamical decoupling sequence which only contains local pulse, i.e., unitaries that act on one of the qubits locally. Analogously to equation (3) a general sequence of this form modifies the Hamiltonian as

\[
H \rightarrow H + \sum_{i,j} \sigma_i \otimes \sigma_j \otimes C_{ij} - \text{perturbation term}
\]
\[
H_{\text{eff}} = \sum_{i=1}^{n} P_i U_i^A U_i^B H U_{i+1}^{B\dagger} U_{i+1}^A. \tag{D.2}
\]

Note that in this context \(H\) is the most general two-qubit plus environment Hamiltonian, as the missing Hamiltonian of the environment \(I \otimes I \otimes E\) does not affect the capabilities of dynamical decoupling protocol, while local Hamiltonians of the qubits \(H_q \otimes I \otimes I\) and \(I \otimes H_q \otimes I\) can be effectively removed by continually applying a field that compensates for their effect.

Just as before, our goal is to identify all the Hamiltonians \(H\) for which all the noise terms can be decoupled while keeping some part of the interaction between the qubits. If this is the case one can perform any two qubit gate in a noiseless fashion by tuning the interaction time, which allows for universal quantum computation. It has been shown in \([32]\) that by applying a correlated depolarizing sequence of pulses all local noise terms \(H_{\text{AE}} + H_{\text{Ec}}\) can be decoupled while keeping one interaction term, say \(\sigma_i \otimes \sigma_j\) intact.

In fact dynamical decoupling also allows to deal with non-local noise terms. By applying the gates \(\{U_i^A \otimes U_i^B\} = \{I \otimes I, I \otimes \sigma_2, \sigma_2 \otimes I, I \otimes \sigma_3, \sigma_3 \otimes I, I \otimes \sigma_1, \sigma_1 \otimes I, \sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_1, \sigma_3 \otimes \sigma_1, \sigma_1 \otimes \sigma_3, \sigma_3 \otimes \sigma_2, \sigma_2 \otimes \sigma_3\}\) with equal probability in equation (D.2) one cancels all the terms containing \(\sigma_i \otimes \sigma_j\) in the Hamiltonian except \(\sigma_2 \otimes \sigma_3\). Obviously, by an adopted sequence one can also project out all the terms except a product of two Paulis in two arbitrary directions \(\sigma_a \otimes \sigma_b\), with \(a = (a_1, a_2, a_3)^T\) and \(b = (b_1, b_2, b_3)^T\). Consequently, such a strategy allows to recover a noiseless interaction iff there exist two vectors \(a\) and \(b\) such that

\[
\sum_{ij} a_i b_i g_{ij} = 0 \tag{D.3}
\]

while \(\sum_{i} a_i b_i g_{ii} \neq 0\). In other words, at least one of the environment operators in the locally rotated frame is zero

\[
0 = \tilde{C}_{ij} = \sum_{ij} O_0^A O_0^B C_{ij}, \tag{D.4}
\]

where \(O^A\) and \(O^B\) are \(3 \times 3\) orthogonal matrices. This means that noises up to rank 15 can in principle be corrected.

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