Finite element modeling of creep of three-layered shallow shells

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Abstract. In the article the problems of calculation of three-layered shallow shells are considered with the help of the finite element method taking the creep into account. We offer rectangular in plane finite elements of shallow shells. The influence of the curvature of the shells on the change in displacements in time is investigated. A comparison is made with the LIRA-SAPR software package.

1. Introduction

Three-layer structures with lightweight filler are widely used in various industries, including civil and industrial construction, aircraft construction, shipbuilding, etc. With the same flexural rigidity, the weight of the three-layer structures is much smaller than that of single-layer structures. The middle layer of three-layer structures is often made of porous polymers, usually polyurethane foams. For such materials, in addition to the elastic properties, a pronounced viscoelasticity is characteristic. This dictates the need to apply to the calculation of three-layer constructions of the apparatus of the creep theory. In works [1-6] the questions of rheological calculation of three-layer beams are considered. Calculation of three-layer plates and shells is a more complicated task and is mainly performed in the elastic stage [7-9]. In papers [10-11] the questions of taking into account the instant nonlinearity of the filler in the problem of axisymmetric bending of circular plates are investigated. In article [12] a linear rheological calculation of reinforced concrete shell is performed using the theory of three-layer structures. In [13-15], flat finite elements of three-layer plates and shells are considered. In this paper, we consider a curvilinear element of a shallow three-layer shell

2. The derivation of the resolving equations

We obtain the resolving equations for a rectangular in plane finite element of the three-layered shallow shell, shown in Fig. 1. This element has 5 degrees of freedom in the node: the displacements of the skin in the xOy plane $u_i^x$, $u_i^y$, $v_i^x$, $v_i^y$ and also the deflection $w_i$. The derivation of the equations is feasible taking into account the orthotropy of the carrier layers, their different thicknesses, and the temperature effects, and we also take into account the possibility of creeping of the shells and filler.

The displacement field of the finite element is written as:

\[ \begin{align*}
&u(x, y, t) = \sum_{i=1}^{n} N_i(u_i^x + u_i^y + v_i^x + v_i^y + w_i(t)) \\
&v(x, y, t) = \sum_{i=1}^{n} N_i(w_i(t))
\end{align*} \]
\[ \{U\} = \begin{bmatrix} \{\rho_1\} \\ \{\rho_2\} \\ \{\rho_3\} \\ \{\rho_4\} \end{bmatrix}, \]

where \( \{\rho\} = \{u^+, v^+, u^-, v^-\} \).

For displacements, the following approximation is adopted:

\[ u^{(-)} = N_1 u_1^{(-)} + N_2 u_2^{(-)} + N_3 u_3^{(-)} + N_4 u_4^{(-)}; \]
\[ v^{(-)} = N_1 v_1^{(-)} + N_2 v_2^{(-)} + N_3 v_3^{(-)} + N_4 v_4^{(-)}; \]
\[ w = N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4, \]

where \( N_1, N_2, N_3, N_4 \) – functions of the form, defined by the formulas:

\[ N_1 = \frac{1}{ab} \left( \frac{a}{2} - x \right) \left( \frac{b}{2} - y \right); \]
\[ N_2 = \frac{1}{ab} \left( \frac{a}{2} + x \right) \left( \frac{b}{2} - y \right); \]
\[ N_3 = \frac{1}{ab} \left( \frac{a}{2} + x \right) \left( \frac{b}{2} + y \right); \]
\[ N_4 = \frac{1}{ab} \left( \frac{a}{2} - x \right) \left( \frac{b}{2} + y \right). \]

**Figure 1.** Rectangular in plane finite element of a three-layered shallow shell.

The deformations of the carrier layers in accordance with the theory of shallow shells are written in the form:

\[ \varepsilon_x^{(-)} = \frac{\partial u^{(-)}}{\partial x} + k_x w; \quad \varepsilon_y^{(-)} = \frac{\partial v^{(-)}}{\partial y} + k_y w; \quad \gamma_{xy}^{(-)} = \frac{\partial u^{(-)}}{\partial y} + \frac{\partial v^{(-)}}{\partial x}, \]

where \( k_x \) and \( k_y \) – principal curvatures (we assume that the \( x \) and \( y \) axes coincide with the directions of the principal curvatures).

For the displacements of the filler, we take the linear distribution over the thickness:

\[ u^m = \frac{u^+ + u^-}{2} + \frac{u^+ - u^-}{h} z; \quad v^m = \frac{v^+ + v^-}{2} + \frac{v^+ - v^-}{h} z. \]

Shear deformations of the filler are determined as follows:

\[ \gamma_{xz}^m = \frac{\partial u^m}{\partial z} + \frac{\partial w}{\partial x} = \frac{u^+ - u^-}{h} + \frac{\partial w}{\partial x}; \quad \gamma_{yz}^m = \frac{\partial v^m}{\partial z} + \frac{\partial w}{\partial y} = \frac{v^+ - v^-}{h} + \frac{\partial w}{\partial y}. \]

Substituting (2) into (4) and (6), we obtain:

\[ \{\varepsilon\} = [B]\{U\}, \]
where \( \{ \varepsilon \} = \{ \varepsilon_1^e, \varepsilon_2^e, \gamma_{xy}^e, \varepsilon_1^m, \varepsilon_2^m, \gamma_{xy}^m \}^T \).

The relationship between deformations and stresses for the carrier layers is written as:

\[
\varepsilon_i^{(+)} = \frac{1}{E_i^{(-)}} \left( \sigma_i^{(+)} - \nu_{i,1}^{(-)} \sigma_1^{(+)} \right) + \varepsilon_i^{(-)}, \quad \varepsilon_i^{(-)} = \frac{1}{E_i^{(-)}} \left( \sigma_i^{(-)} - \nu_{i,1}^{(-)} \sigma_1^{(-)} \right) + \varepsilon_i^{(-)},
\]

\[
\gamma_{xy}^{(+)} = \frac{\tau_{xy}^{(+)}}{G_i^{(-)}},
\]

where \( \varepsilon_i^{(-)}, \sigma_i^{(-)} \) – forced deformations representing the sum of temperature deformations and creep strains:

\[
\varepsilon_i^{(-)} = \varepsilon_i^{(-)} + \alpha_i \Delta T^{(-)}; \quad \varepsilon_i^{(-)} = \varepsilon_i^{(-)} + \alpha_i \Delta T^{(-)}; \quad \gamma_{xy}^{(-)} = \gamma_{xy}^{(-)},
\]

where \( \alpha_i, \alpha_j \) – coefficients of temperature expansion, \( \varepsilon_i^{(+)}, \varepsilon_i^{(-)}, \gamma_{xy}^{(+)} \) – creep strains.

The potential deformation energy is written as:

\[
W = \frac{1}{2} \int_{A} \left( \delta^{+} \left[ \sigma_i^{+} \varepsilon_i^{+} + \sigma_y^{+} \varepsilon_y^{+} + \tau_{xy}^{+} \gamma_{xy}^{+} \right] + \delta^{-} \left[ \sigma_i^{-} \varepsilon_i^{-} + \sigma_y^{-} \varepsilon_y^{-} + \tau_{xy}^{-} \gamma_{xy}^{-} \right] + \delta^{+} \left[ \sigma_i^{+} \varepsilon_i^{+} + \sigma_y^{+} \varepsilon_y^{+} + \tau_{xy}^{+} \gamma_{xy}^{+} \right] + \right.
\]

\[
+ \int_{-h/2}^{h/2} \tau_{xy}^{m} \gamma_{xy}^{m} dz + \int_{-h/2}^{h/2} \tau_{xy}^{m} \gamma_{xy}^{m} dz) dA,
\]

where the indices \( \text{el} \) correspond to elastic deformations representing the difference between total and forced deformations.

The change in the shear modulus of the filler over the thickness of the plate due to the temperature effect is taken into account by introducing the value of the averaged shear modulus \( \overline{G}_m \) and the average creep strains \( \overline{\gamma}_{xy}^{+} \) and \( \overline{\gamma}_{xy}^{-} \):

\[
\overline{G}_m = \frac{1}{h} \int_{-h/2}^{h/2} G_m(z) dz; \quad \overline{\gamma}_{xy}^{+} = \frac{1}{G_m} \int_{-h/2}^{h/2} G_m(z) y_{xy}^{+} dz; \quad \overline{\gamma}_{xy}^{-} = \frac{1}{G_m} \int_{-h/2}^{h/2} G_m(z) y_{xy}^{-} dz.
\]

The potential strain energy will take the form:

\[
W = \frac{1}{2} \int_{A} \left\{ N \right\}^T \left( \{ \varepsilon \} - \{ \varepsilon_f \} \right) dA,
\]

where \( \left\{ N \right\} = \left[ \sigma_x^{+} \delta_{x}^{+} \sigma_y^{+} \delta_{y}^{+} \tau_{xy}^{-} \delta_{xy}^{+} \sigma_x^{-} \delta_{x}^{-} \sigma_y^{-} \delta_{y}^{-} \tau_{xy}^{+} \delta_{xy}^{-} \overline{\tau}_{xy}^{m} \overline{\tau}_{xy}^{m} \right] - \text{vector of internal forces}, \]

\[
\{ \varepsilon_f \} = \left\{ \varepsilon_{x,f}^{+} \varepsilon_{y,f}^{+} \gamma_{xy,f}^{+} \varepsilon_{x,f}^{-} \varepsilon_{y,f}^{-} \gamma_{xy,f}^{-} \right\}^T - \text{vector of forced deformations}.
\]

Internal forces are related to deformations as follows:

\[
\{ N \} = [D] \left( \{ \varepsilon \} - \{ \varepsilon_f \} \right),
\]

where \([D]\) is the block matrix of elastic constants, having the form:

\[
[D] = \begin{bmatrix}
[D_+^T] \\
[D_-^T] \\
[D_m]
\end{bmatrix}.
\]
where

\[
[D^{(-)}] = \frac{1}{1-V_1^{(-)}V_2^{(-)}} \begin{bmatrix}
E_1^{(-)} & E_1^{(-)}V_2^{(-)} & 0 \\
E_2^{(-)}V_1^{(-)} & E_2^{(-)} & 0 \\
0 & 0 & G^{(-)}(1-V_1^{(-)}V_2^{(-)})
\end{bmatrix};
\]

\[
[D_m] = \bar{\sigma}_m h \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

After substituting (13) in (12) and further (7) into (12) with the following minimization of the Lagrangian functional with respect to the nodal displacements, we obtain a system of linear algebraic equations:

\[
[K]\{U\} = \{F\} + \{F_f\},
\]

where \([K]\) is the stiffness matrix, \(\{F_f\}\) is the vector of fictitious nodal forces due to forced deformations.

\[
[K] = \int [B]^T [D] [B] dA; \quad \{F_f\} = \int [B]^T dA [D] \{\epsilon_f\}.
\]

3. Statement and solution of the problem

A three-layered shallow shell was calculated, the surface of which is an elliptical paraboloid (Figure 2). The equation of the shell surface has the form:

\[
z = f \left[ \frac{f_x}{f} \left( 2 \frac{x}{a} - 1 \right)^2 + \frac{f_y}{f} \left( 2 \frac{y}{b} - 1 \right)^2 - 1 \right].
\]

It was assumed in the calculations that along the contour the shell is connected to diaphragms absolutely rigid in their plane and flexible from it. The calculation was carried out for a rectangular shell with dimensions \(a = b = 3\) m, with a total thickness \(h = 8\) cm. The carrier layers of the shell were taken as steel with a thickness of 1 mm. As the creep law, the Maxwell-Thompson equation was used:

\[
nG \frac{\partial \gamma}{\partial t} + H \gamma = n \frac{\partial \tau}{\partial t} + \tau,
\]

where \(G\) and \(H\) are respectively the instantaneous and long-term shear moduli, \(n\) is the relaxation time, \(\tau\) is the tangential stress, and \(\gamma\) is the total shear deformation. Calculation was performed at \(\kappa = 46.6\) MPa \(\cdot\) day, \(H = 3.17\) MPa, \(G = 4.85\) MPa.

The load was assumed to be constant: \(q = 2\) kPa, and the shell curvature was varied by changing the lift amount \(f\).

4. Results and discussion

The growth curves of the largest deflection relative to the original deflection with different values of the ratio \(f/a\) are shown in Figure 3. As a result of the calculation, it was found that with increasing curvature, the creep effect decreases, and for shells of large curvature, the creep of the aggregate has no effect on the amount of deflection.

To confirm the reliability of the results, a calculation was performed in the LIRA-SAPR 2013 program at \(f/a = 1/15\). The value of \(H\) was substituted for the long shear modulus of the filler. The carrier layers were modeled by plane shell finite elements, and the middle layer by a single row of bulk prismatic elements. The support contour was framed from plates, the thickness of which was equal to the thickness of the bearing layers. The calculation scheme was generated as a text file using a program written in the Matlab package. Then this text file was imported into the LIRA SAPR.

The isopoles of vertical displacements are shown in Figure 4. The greatest value of displacements, obtained in the PC LIRA, was 0.652 mm. When solving with the three-layered elements \(w_{\text{max}} \mid_{t \to \infty} = 0.635\) mm, which differs from the result in the LIRA SAPR by 2.6%.
5. Conclusions

As a result of calculation it is established that for three-layered shells of great curvature, the creep of the aggregate does not have a noticeable effect on the deflection. The coincidence of the results obtained by the author with the solution in the LIRA-SAPR program complex indicates their reliability.

6. References

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