Enhanced Charge and Spin Currents in the One-Dimensional Disordered Mesoscopic Hubbard Ring

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Abstract

We consider a one-dimensional mesoscopic Hubbard ring with and without disorder and compute charge and spin stiffness as a measure of the permanent currents. For finite disorder we identify critical disorder strength beyond which the charge currents in a system with repulsive interactions are larger than those for a free system. The spin currents in the disordered repulsive Hubbard model are enhanced only for small $U$, where the magnetic state of the system corresponds to a charge density wave pinned to the impurities. For large $U$, the state of the system corresponds to localized isolated spins and the spin currents are found to be suppressed. For the attractive Hubbard model we find that the charge currents are always suppressed compared to the free system at all length scales.

72.15Rn, 73.20Mf
I. INTRODUCTION

Over the last few years various experiments have measured the magnetic response of small, moderately disordered ensembles of quasi one-dimensional (1D) rings. These experiments confirm the existence of a persistent current which had been predicted already much earlier. However, the experimental value for the persistent current is two to three orders of magnitude larger than the theoretical predictions based on calculations in a disordered but non-interacting electron gas. It is thus commonly believed that the discrepancy could probably be resolved by accurately including the electron-electron interactions in these calculations.

Studying the full interacting electron problem with disorder is in general difficult. However, in one dimension, powerful analytical tools, such as Bethe Ansatz and Bosonization, give us a handle to treat at least the on-site part of the interaction exactly and the effect of disorder can then be studied within a perturbative renormalization group (RG) approach. Using these methods, Giamarchi and Shastry have shown that a repulsive interaction enhances the value of the persistent current in a mesoscopic Hubbard ring. This result has been confirmed by recent independent numerical and analytical studies treating the Hubbard interaction in first order perturbation theory.

In this paper we study the persistent currents in the 1D mesoscopic Hubbard ring both with and without disorder along the lines of Ref. Due to the absence of Galilean invariance, we note that the persistent currents exhibit a strong reduction as a function of both filling and interaction strength already for the clean ring. Thus it is a priori not clear that this initial reduction can be compensated completely by disorder as to give a net enhancement of the current. Therefore, we use the exact bosonization parameters, obtained from the Bethe Ansatz solution for the clean ring, as initial values for the solution of the RG equations. We remark that Ref. uses only perturbative starting values for the RG equations and normalizes the current w.r.t. non-interacting values. Our approach allows us to show that for arbitrarily weak disorder there is no enhancement in the currents. Fur-
thermore, we can identify critical disorder strengths beyond which the persistent current in the presence of both interactions and disorder is indeed *larger* than the current for the disordered but non interacting system.

We also study the effect of disorder on the *spin* currents in the Hubbard ring. We emphasize that even in the presence of Galilean invariance the spin currents can renormalize non-trivially under interactions. Physically a non-vanishing spin current implies that the system has long ranged spin correlations and it is thus interesting to see the effect of disorder on the spin currents.

The plan of the paper is as follows: In section II we define charge and spin stiffness for a 1D mesoscopic system and relate them to the corresponding currents. We briefly discuss the range of validity of these relations. In section III we first review the Bethe Ansatz (BA) solution for both the repulsive and the attractive clean Hubbard model. We then go on to compute the stiffnesses and their conjugate compressibilities by iterating the BA equations for a mesoscopic system. Next, we recall the bosonized description of the Hubbard model and show how to relate stiffness and compressibility to the new parameters of the bosonized Hamiltonian. In section IV, we then include the effects of disorder into the bosonized model by a RG calculation. These RG equations are then integrated numerically and we can identify a crossover from a phase with enhanced spin currents to a phase in which spin currents are suppressed as the on-site interaction is increased relative to the initial disorder strength. We discuss these results in section V.

II. PERSISTENT CURRENTS AND STIFFNESS OF A 1D MESOSCOPIC SYSTEM

For a finite system on a ring of length $L$, the response of the ground state energy to a finite Aharanov-Bohm (AB) flux is a measure of the persistent current at $T = 0$. The current is given as $J = L \partial E_0(\Phi)/\partial \Phi_{\Phi=0}$, where $E_0(\Phi)$ is the ground state energy of the full interacting system in the presence of the AB flux $\Phi$. The energy shift of the ground state
can be written as \( E_0(\Phi) - E_0(0) \equiv D\Phi^2/L + O(\Phi^4) \), where \( D \equiv (L/2)\partial^2 E_0(\Phi)/\partial\Phi^2|_{\Phi=0} \) is called the stiffness constant. \( D \) provides an operational definition for the persistent current for small values of the flux, given as \( J = 2D\Phi \). Higher order terms are important when the energy shift is comparable to the mean energy level spacing in the spectrum of the many-body system. In the case of a finite 1D metallic system, the gaps are \( O(1/L) \) and so non-quadratic corrections occur when \( \Phi \) is \( O(1) \). Level crossings would occur and perturbation theory would break down for \( \Phi \) of order \( \pi \).

In a system with up and down spins, the AB flux for each species can be treated as independent parameters \( \Phi_{\uparrow} \) and \( \Phi_{\downarrow} \). With this freedom, there are two stiffnesses that can be calculated and hence two currents. In the case when \( \Phi_{\uparrow} = \Phi_{\downarrow} = \Phi_c \), both species are coupled to the same flux and the shift in the energy gives the charge stiffness \( D_c \). In a Galilean invariant system this flux couples only to the center of mass coordinate and hence the persistent charge current will not depend on the interaction between the particles. However, for lattice models such as the Hubbard model, the center of mass momentum is conserved only up to the reciprocal lattice vector and we can observe a non-trivial dependence of the charge current on the interaction. In the sector of zero magnetization \( (S_z = 0) \), the second choice is to take \( \Phi_{\uparrow} = -\Phi_{\downarrow} = \Phi_s/2 \). In this case the two species are driven in opposite directions through each other which leads to a non-trivial dependence of the resulting spin current on the interactions between opposite spins even for a Galilean invariant system. The energy shift in the presence of \( \Phi_s \) defines the spin stiffness \( D_s \). Charge and spin currents are then given as \( J_r = 2D_r\Phi_r \) and

\[
D_r = \frac{L}{2} \left. \frac{\partial^2 E_0(\Phi_r)}{\partial\Phi_r^2} \right|_{\Phi_r=0}, \quad r = c, s. \tag{1}
\]

We note that \( D_c \) is related to the d.c. part of the conductivity \( \sigma(\omega) = D_{\text{Drude}}\delta(\omega) + \sigma_{\text{reg}} \) by \( D_{\text{Drude}} \equiv 2\pi D_c \).

III. THE 1D HUBBARD MODEL WITHOUT DISORDER
A. The repulsive interaction regime

1. Charge and spin stiffness and the conjugate compressibilities

The repulsive Hubbard model on a ring of size $L$ threaded by a spin dependent flux $\Phi_\sigma$ is described by the Hamiltonian

$$H = -t \sum_{i=1}^{L} \left( e^{i\Phi_\sigma/L} c_{i+1,\sigma}^\dagger c_{i,\sigma} + h.c. \right) + U \sum_{i=1}^{L} n_{i,\uparrow} n_{i,\downarrow}, \quad U > 0,$$

where $c_{i,\sigma}^\dagger$ and $c_{i,\sigma}$ create and annihilate fermions on site $i$ with spin $\sigma$ and $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$ is the density operator. We define our energy scales by setting $t = 1$ in the following. Note that this is in fact equivalent to using the Fermi velocity of the non interacting system $v_f$ as defining energy scales, as $v_f = 2t \sin[\pi n/2]$, where $n$ is the density.

The addition of a flux $\Phi_\sigma$ is compatible with integrability and the Bethe Ansatz equations for a chain with total number of particles $N = N_\uparrow + N_\downarrow$ and $M = N_\downarrow$ number of down spin particles are given as:

$$L k_n = 2\pi I_n + \Phi_\uparrow + 2 \sum_{j=1}^{M} \arctan \left[ 4(\Lambda_j - \sin k_n)/U \right], \quad (3a)$$

$$2 \sum_{n=1}^{N} \arctan \left[ 4(\Lambda_j - \sin k_n)/U \right] = 2\pi J_j + \Phi_\downarrow - \Phi_\uparrow + 2 \sum_{i,j} \arctan \left[ 2(\Lambda_j - \Lambda_i)/U \right]. \quad (3b)$$

The quantum numbers $I_n$ and $J_j$ label different states and statistics. For the fermionic ground state they are taken to be $\Delta I_n = M/2 \mod 1$ and $\Delta J_j = (N - M + 1)/2 \mod 1$. The energy of the system in a state corresponding to a solution of Eq. (3) is equal to $E_0(\Phi_\sigma) = -2 \sum \cos k_n$.

In order to study the currents in the ring one should in principle distinguish between cases of odd or even numbers of particles: If both $N_\uparrow$ and $N_\downarrow$ are odd, the number of right and left moving particles for each of the species are equal, the ground state is non-degenerate and the energy is minimum in zero external flux. For any other combination, i.e. when either $N_\uparrow$ or $N_\downarrow$ or both are even, the number of right and left moving particles are not the same due to the $k = 0$ state and hence the energy is not necessarily a minimum at zero charge or...
spin flux but rather at flux values of ±π. For the sake of simplicity, we shall only consider the
case of zero magnetization with $N$ even and $M = N/2$ odd in our numerical computations.

In Figs. 1 and 2, we show $D_c$ and $D_s$ as obtained from an iteration of the BA equations
for a ring of size $L = 100$ and various fillings $n = N/L$ and values of the interaction strength
$U$. Both figures show that there exists a reduction in both stiffnesses with increasing $U$ for
a given filling. The magnitude of the reduction for $D_c$ is considerably enhanced as the filling
is increased towards the half-filled case.

The other physical quantities that will be needed in the next section to compute the
parameters arising in the bosonized Hubbard Hamiltonian are the charge compressibility and
the spin susceptibility. The usual thermodynamic expression for the charge compressibility
is given as $\chi_c^{-1} = (n^2/L)\partial^2 E_0/\partial n^2|_{\Phi=0}$, where $n = N/L$ is the density of particles. For
a mesoscopic system, the derivatives should be replaced by finite differences. In order to
keep the magnetization fixed, the charge compressibility is then computed by adding both
a spin up and a spin down particle to the ground state configuration; the explicit form of
the derivative is

$$\chi_c^{-1} = 2 \times \frac{N^2}{4L} [E_0(N + 2) + E_0(N - 2) - 2E_0(N)]. \quad (4)$$

The leading factor of two ensures that a pair of particles is introduced.

Shastry and Sutherland\textsuperscript{14} have shown that the bulk spin susceptibility $\chi_s$ in the Hubbard
model is related to the spin stiffness by $D_s\chi_s = 1/2\pi^2n^2$ due to a remarkable property of the
BA equations (3). Thus in the thermodynamic limit we only need to compute either spin
stiffness or susceptibility. However, for a mesoscopic system, this relation does not hold due
to an avoided level crossing at $\Phi_s = 2\pi$.\textsuperscript{17} We remark that this situation is very much as for
the Heisenberg-Ising model in the momentum $\pi$ sector.\textsuperscript{18} Following Ref. 14, we then write
for the spin susceptibility

$$\chi_s^{-1} = \frac{N^2}{2L} [E_0(N, M = N/2 - 1) - E_0(N, N/2)]. \quad (5)$$

Note that in the $M = N/2 - 1$ sector the ground state has the quantum numbers $I_n = -(N + 1)/2 + n$ for $1 \leq n \leq N$ and $J_j = -(N/2 + 1)/2 + j$ for $1 \leq j \leq M$. 
The behavior of both \( \chi_c \) and \( \chi_s \) has been reported previously in the thermodynamic limit. Our mesoscopic results are qualitatively the same and differ at the most by 15\% from their thermodynamic values. Thus we refrain from including the corresponding figures here. However, we emphasize that these small deviations will be quite important in the following sections.

### 2. Boson representation of the repulsive Hubbard model

Away from half filling, the low-energy and large distance behavior of a one-dimensional fermion system with spin-independent interactions is described by the Hamiltonian:

\[
H = H_c + H_s + \frac{2g_{1\perp}}{(2\pi\alpha)^2} \int dx \cos (\sqrt{8} \phi_s(x)),
\]

where

\[
H_r = \frac{1}{2\pi} \int dx \left[ (v_r K_r)(\pi \Pi_r)^2 + \left( \frac{v_r}{K_r} \right) (\partial_x \phi_r)^2 \right]
\]

for \( r = c, s \). This Hamiltonian describes the most general 1D Hamiltonian with spin conserving interactions, provided that the proper values for \( K_r \) and \( v_r \) are used. The \( c \) and \( s \) parts of the Hamiltonian describe the charge and spin degrees of freedom of the system respectively. The operator \( \Pi_r \) is the momentum density conjugate to \( \phi_r \) and these operators obey Bose-like commutation relations: \([\Pi_r(x), \phi_r(x')] = -i\delta_{r,\mu}\delta(x - x')\). \( \alpha \) is a short-range cutoff parameter of the order of the lattice constant. The \( g_{1\perp} \) term represents scattering between opposite spins with a momentum transfer close to \( 2k_f \). The umklapp scattering transferring two particles from \(-k_f\) to \( k_f \), involves a momentum transfer \( 4k_f = 2\pi \) which in the half filled band \( (k_f = \pi/2) \) corresponds to a reciprocal lattice vector. Away from the half-filled case this term does not contribute and therefore has not been included in the Hamiltonian. The case \( g_{1\perp} = 0 \) describes independent long-wavelength oscillations of the charge and spin densities with linear dispersion relation \( \omega_r(k) = v_r|k| \). For \( g_{1\perp} \neq 0 \) the cosine term has to be treated perturbatively.
For the Hubbard Hamiltonian (2) the values for $K_c$ and $v_c$ are given by the following relations,

\begin{align}
K_c &= \pi \frac{n}{2} \sqrt{\frac{2D_c}{\chi_c^{-1}}}, \\
v_c &= \frac{2}{n} \sqrt{\frac{D_c \chi_c^{-1}}{2}}. 
\end{align}

Here $D_c$ and $\chi_c$ are defined as in Eq. (1) and (4), respectively, and $n = N/L$ is the density of particles. For $g_{1\perp} = 0$, we can similarly write

\begin{align}
K_s &= 2 \times \pi \frac{n}{2} \sqrt{\frac{2D_s}{\chi_s^{-1}}}, \\
v_s &= 2 \times \frac{2}{n} \sqrt{\frac{D_s \chi_s^{-1}}{2}},
\end{align}

with $D_s$ and $\chi_s$ defined as in Eq. (1) and (5) and the leading factor of 2 is due to our definition of the spin flux $\Phi_s$. However, $D_s$ and $\chi_s$ have been calculated from the exact solution of the Hubbard model, e.g., including the $2k_f$ scattering terms represented by the $g_{1\perp}$ term in the bosonized description of Eq. (6). Therefore, the identifications of Eq. (9) are not exact and only valid perturbatively in $g_{1\perp}$. Since for the Hubbard model $g_{1\perp} = U$, this means that Eq. (9) is valid only for small $U$.

Let us briefly elaborate on this point: For a macroscopic system $g_{1\perp}$ renormalizes to zero. The spin stiffness in this limit is given as $D_s = \frac{1}{4\pi} v_s K_s^*$ where $K_s^* = 1$ is the renormalized value of $K_s$. However, for a finite system, one generates additional terms of $O(g_{1\perp})$ coming from the term proportional to $g_{1\perp}$. We therefore define the spin stiffness as $D_s = \frac{1}{4\pi} v_s K_s$ with $v_s$ and $K_s$ given as in Eq. (9). When $K_s = 1$, we find the simple relation between the spin stiffness and the spin susceptibility as in last section. However, for mesoscopic systems deviation of $K_s$ from its macroscopic fixed point value occurs as a consequence of the avoided level crossing mentioned earlier. We find that $K_s$ varies from its non interacting value $K_s = 1$ up to $K_s \sim 1.15$ for $U = 20$ in a ring of size $L = 100$. 

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B. The attractive interaction regime

The attractive Hubbard model is again described by the Hamiltonian of Eq. (2), but now $U < 0$. The attraction gives rise to Cooper-like bound pairs of spin up and spin down particles. These pairs scatter off each other without diffraction. Hence for $N$ even and zero magnetization, we can write down effective BA equations for $M = N/2$ pairs.$^6$ Let $\alpha(k_i) = 2\arcsin \left( k_i + i\frac{U}{4} \right)$. Then the BA equations for the pairs are given as,

$$L \text{Re}\{\alpha(k_i)/2\} = 2\pi J_i + 2\Phi_p + \sum_{j=1}^{M} \arctan \left[ \frac{2}{U} (k_j - k_i) \right], \quad (10)$$

where $\Phi_p = \Phi^\uparrow = \Phi^\downarrow$ is the flux acting on the Cooper pair. The factor of two in front of $\Phi_p$ arises due to the charge of the Cooper pair. The quantum numbers for the ground state are $\Delta J_i = (M + 1)/2 \mod 1$, and the corresponding ground state energy is given as $E_0(\Phi_p) = -4 \sum_{i=1}^{M} \cos[\text{Re}\{\alpha(k_i)/2\}] \cosh[\text{Im}\{\alpha(k_i)/2\}]$. There exists a gap in the spin excitation spectrum equal to the binding energy of a pair and only the charge sector remains gapless. Hence only the charge sector in the bosonized Hamiltonian is retained and the only relevant parameters in (6) are $K_c$ and $v_c$. In order to make the reference to the Cooper pairs explicit, we shall rename these parameters $K_p$ and $v_p$.

The pair stiffness is defined as before $D_p = (L/2) \frac{\partial^2 E_0(\Phi_p)}{\partial \Phi_p^2}$ and the pair compressibility is computed by changing the ground state configuration by adding a Cooper pair. The thermodynamic expression for the pair compressibility is $\chi_p^{-1} = (m^2/L) \frac{\partial^2 E_0(M)}{\partial m^2}$, where $m$ is the density of Cooper pairs. The explicit form of the derivative for a mesoscopic system is $\chi_p^{-1} = (M^2/L) [E_0(M+1) + E_0(M-1) - 2E_0(M)]$. We can now compute $K_p$ and $v_p$ from the ground state energies. The relations are

$$K_p = \pi m \sqrt{\frac{2D_p}{\chi_p}}, \quad (11a)$$

$$v_p = \frac{1}{m} \sqrt{\frac{D_p \chi_p^{-1}}{2}}. \quad (11b)$$

The error involved in using the Cooper Pair representation is of $O(\exp[-2UL])$. From Fig. 1 we see that down to $|U| = 0.01$ there is very good agreement of the pair stiffness in
the attractive regime with the charge stiffness found in the repulsive regime. This justifies our use of the effective BA equations even for such small $U$. The general features of the pair stiffness are the same as that seen for the charge stiffness. One sees a reduction in the pair stiffness as the attraction is increased. Note that for small $|U|$ and density $n$, the value of $D_c$ is always larger or equal to $D_p$. Here the charge currents in the repulsive case are larger than in the attractive case. For densities close to half-filling and $U > 1$, this behavior is no longer true as $4k_f$ scattering terms become dominant in the repulsive regime.

IV. EFFECT OF DISORDER

In the limit of weak disorder, the interaction between the particles and disorder can be parameterized by two uncorrelated Gaussian random fields $\eta$ and $\xi$. These two fields describe the forward and backward scattering by the impurities. The forward scattering term can be treated exactly in one dimension and is found not to contribute to the conductivity. However, the effect of backward scattering is very important and leads to localization in the non-interacting limit. $\xi$ and $\xi^*$ correspond to the part of the random potential which has Fourier components close to $2k_f$. Higher Fourier components are less effective and do not correspond to low energy processes. A notable exception is the $4k_f$ term which we will have reason to discuss in the last section. In terms of the boson variables the impurity coupling to the particle density is given as

$$H_\Delta = \int dx \, \xi(x) \, e^{i\sqrt{2}\phi_\sigma(x)} \cos(\sqrt{2}\phi_\sigma) + h.c., \quad (12)$$

where $\xi$ is a Gaussian with $\langle \xi(x)\xi^*(x') \rangle = \Delta_\xi \delta(x - x')$. Contrary to the free case the charge and spin degrees of freedom are no longer independent but are coupled through the random potential.

Generally both impurity backscattering and the interaction term $g_{1\perp}$ give rise to divergent terms in a perturbation calculation. Hence, a perturbative approach in the disorder $\Delta_\xi$ and in the interaction $g_{1\perp}$ is used to generate the renormalization group equations under a change of the length scale $\alpha \to e'\alpha$, where $\alpha$ is the lattice spacing. The equations are
\[
\frac{dK_c(l)}{dl} = -\frac{1}{2} \left[ K_c^2 \frac{v_c}{v_s} \right] \Delta(l),
\]
(13a)

\[
\frac{dK_s(l)}{dl} = -\frac{1}{2} \left[ \Delta(l) + y(l)^2 \right] K_s^2,
\]
(13b)

\[
\frac{dv_c(l)}{dl} = -\frac{v_c^2 K_c}{2v_s} \Delta(l),
\]
(13c)

\[
\frac{dv_s(l)}{dl} = -\frac{v_s K_s}{2} \Delta(l),
\]
(13d)

\[
\frac{dy(l)}{dl} = [2 - 2K_s(l)] y(l) - \Delta(l),
\]
(13e)

\[
\frac{d\Delta(l)}{dl} = [3 - K_c(l) - K_s(l) - y(l)] \Delta(l),
\]
(13f)

with the dimensionless quantities \(\Delta\) and \(y\) defined as: \(\Delta = (2\Delta\alpha/\pi v_s^2) [v_s/v_c]^K_c\) and \(y = g_{1\perp}/\pi v_s\). The development of the RG equations is limited to first order in \(\Delta\alpha\) and to second order in \(g_{1\perp}\) and hence the renormalizations of \(K_r\) and \(v_r\) can be neglected on the right hand side of the first three equations.

For a mesoscopic system of size \(L\), the infra-red cutoff is expected to be given by \(L \sim \epsilon'\alpha\), or equivalently, \(l \sim \ln L/\alpha\). Thus we may calculate charge and spin stiffness at finite size by using the values of \(K_c(l), K_s(l), u_c(l)\) and \(c_s(l)\) and the formulas of section III.

The RG equations for finite disorder have to be numerically integrated taking appropriate initial conditions at \(l = 0\). Using perturbative values as initial data for the RG equations, e.g., \(v_s = v_f\) and \(K_s = 1 + y/2\), we see that we get an increasing value for the spin stiffness \(D_s = \frac{1}{4\pi} v_s K_s\) with increasing repulsion. This does not agree with the exact Bethe Ansatz solution for the spin stiffness as shown in Fig. 2 which clearly shows a reducing stiffness. Therefore, we shall use the clean Hubbard model parameters that were computed in section III and \(g_{1\perp} = U\). In the following sections we discuss the results of the integration primarily for \(n = 0.3\). The results for fillings corresponding to \(M = 10, 22, 42, 50, 62, 70, 82, 90\) and \(96\) are qualitatively the same.
A. The repulsive case

For the repulsive case we find that $\Delta$ always flows to $\infty$ as shown in Fig. 3. Assuming that there is no other fixed point at intermediate coupling, this whole region can then be identified with the localized phase.\textsuperscript{27,28} The magnetic properties of the system will then depend on the renormalized value of $y$:

(i) Increasing the repulsive interaction away from $U = 0$, we find that $y$ flows to $-\infty$ as shown in Fig. 4. The physical state corresponds to a non-magnetic system of localized pairs of spins which is equivalent to a charge density wave pinned by the impurities (PCDW).\textsuperscript{27,28} In Fig. 3 we see that the disorder scales less rapidly to infinity as the interaction strength is increased. This implies that pinning of the charge density wave is harder in the presence of a repulsive interaction. Thus we expect the interacting charge currents to be enhanced by the disorder. However, as shown in section \ref{section11} there is an initial reduction in the interacting charge currents due to the presence of the lattice. In Fig. 5 we show the renormalization of the charge stiffness for a fixed filling $n = 0.3$ and disorder strength for various values of the interaction strength. We see that there is a crossover between the non-interacting and interacting stiffnesses for finite $l_c$, or, equivalently, finite disorder $\Delta_\xi(l_c, U)$ and we find enhancement only for disorder values larger than $\Delta_\xi(l_c, U)$. The crossover regions for a fixed value of the disorder as a function of $n$ and $U$ are shown in Fig. 6. Previous studies of the disordered Hubbard ring\textsuperscript{11,12} treat the Hubbard interaction in first order perturbation theory for finite disorder and thus cannot identify this crossover.

In Fig. 4 we further note that $y$ scales less rapidly to $-\infty$ as the interaction strength is increased. As noted earlier, the physical state corresponds to a system of localized spins and hence larger the $U$ value lesser the localization. We therefore find that the spin currents are less pinned by the impurity as the repulsion is enhanced. In Fig. 7 we again find due to the reduced pinning effect there exists a critical disorder strength $\Delta_\xi(l_s, U)$ where there is a crossover between the non-interacting and interacting spin currents. $l_s$ here is the length at which the crossover occurs and $\Delta_\xi(l_s, U)$ is the value of the disorder at $l_s$ for a given $U$.\textsuperscript{12}
Note that we always find $l_c < l_s$ ($\Delta_\xi(l_c, U) < \Delta_\xi(l_s, U)$). We remark that the actual value of $\Delta_\xi(l_s, U)$ is independent of particle density $n$. However, we expect a strong dependence on the magnetization. As we have restricted our study to the sector of zero magnetization, we do not observe this later dependence here.

(ii) In Fig. 4, we see that further increase of $U$ results in $y$ flowing to $+\infty$. Hence there is a strong repulsion of up and down spins, and the particles start to localize as isolated spins on randomly distributed sites. The magnetic properties have been identified earlier as being typical of a random antiferromagnet (RAF)\cite{27,28}.

The charge currents in the RAF phase will continue to be enhanced. In Fig. 5, we see that, e.g., the charge stiffness for $n = 0.3$ and $U$ values of 0.8 and 1.0 is still above the non-interacting stiffness. Fig. 4 clearly shows that these values of $U$ already belong to the RAF fixed point $y^* = \infty$.

The strong repulsion of up and down spins, however, gives rise to a drastic fall in the spin currents as seen again for $n = 0.3$ in Fig. 4. We thus observe no enhancements in the spin currents in the RAF phase. This may be used to distinguish the two different phases of the system for repulsive $U$. In Fig. 8 the region above the line represents the RAF state where no enhancement of the spin currents is found, whereas the region below the line corresponds to the PCDW state. The reader should compare Fig. 8 with Fig. 6. The maxima of each curve in Fig. 8 corresponds to the transition point between PCDW and RAF phases as in Fig. 8. We note that for values of $n$ and $U$ such that the fixed point of the system belongs to the RAF phase, already small disorder values will localize the spins and thus lead to an enhancement of the charge stiffness.

B. The attractive case

For negative $U$ there exists a gap in the spin excitation spectrum and only the charge sector remains gapless. The RG Eqs. (13) can be reduced by taking into account only the charge (i.e. pair) excitations,\cite{27,28}.
\[
\frac{dK_p(l)}{dl} = -\frac{1}{2} K_p^2 \Delta(l), \quad (14a)
\]
\[
\frac{dv_p(l)}{dl} = -v_p^2 K_p \Delta(l), \quad (14b)
\]
\[
\frac{d\Delta(l)}{dl} = [3 - K_p(l)] \Delta(l), \quad (14c)
\]

with \(\Delta(l) = (2C_s \Delta \alpha / \pi v_p^2)\) and \(C_s\) a constant of order unity. These equations have been studied previously and we find in agreement with Ref. 10: From Eq. (14c) we see that as \(K_p < 3\), the disorder will always scale to large values thereby always leading to localization of the particles. This implies for the second equation that \(v_p \to 0\). The stiffness therefore always reduces in the presence of disorder. As the attraction is increased the disorder scales faster to infinity and since in the attractive Hubbard model the ground state contains strong charge density fluctuations they get easily pinned to the disorder. Due to the large reduction in the bare stiffness and an increased pinning effect to the disorder we therefore see no enhancement in the persistent currents as the attraction is increased.

V. CONCLUSIONS

We have studied in this work the behavior of charge and spin stiffness constants of a 1D disordered mesoscopic Hubbard ring. For the attractive Hubbard model without disorder, we find in agreement with previous studies\(^\text{[10]}\) that the charge stiffness gets reduced with increasing \(|U|\) and is always smaller than the corresponding charge stiffness for the repulsive regime at densities not close to half-filling. For finite disorder, we too find no enhancements in the stiffness for finite \(|U|\) over the non interacting currents.

For the repulsive Hubbard model without disorder, we again see that the charge and spin stiffnesses get reduced as \(U\) is increased. Additionally, we observe a very strong reduction as we approach the half-filled situation for finite \(U\). However, for finite disorder, the situation is very different from the attractive case: For small \(U\), the inclusion of disorder drives the system into a localized state with enhancement of both charge and spin stiffnesses. However, for a mesoscopic system, enhancement is only observed for large enough values of
the disorder, or, equivalently, large enough system sizes. The physical state in this phase corresponds to a non-magnetic system of localized pairs of spins which is equivalent to a charge density wave pinned by the impurities (PCDW).\textsuperscript{27, 28} As the interaction is increased the tendency towards pinning is found to reduce as the disorder scales less rapidly towards $\infty$.

For larger $U$, the effective interaction between up and down spins becomes repulsive and we identify the physical state as corresponding to isolated spins localized on randomly distributed sites (RAF).\textsuperscript{27, 28} This strong repulsion between unlike spins reduces the spin current drastically and we find no enhancements in the spin current. However, the charge currents in the RAF phase are still enhanced as compared to the non interacting current.

We emphasize that the transition from PCDW state to RAF state at intermediate values of the interaction strength $U$ can be identified by studying the different behavior of $D_s$ in the two phases.

In this study, we have neglected Umklapp and $4k_F$ scattering terms both in the bosonized formulation of the Hubbard Hamiltonian and in the RG treatment of the disorder. Thus processes close to half-filling and $U \gg 1$ are presumably not included in our study. In order to at least qualitatively see what happens for large $U$, we have studied a 4 site Hubbard ring by exact diagonalization. Then due to the finite value of $l_c$, we need to go to large values of the disorder strength $\Delta_c > 5$ in order to observe the crossover in $D_c$. However, we still do find a crossover even for $U = 100$.

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FIGURES

FIG. 1. The charge $D_c$ and pair $D_p$ stiffness for the Hubbard ring of size $L = 100$, as a function of the interaction strength for different values of the filling $N = 2M = 10, 22, 30, 42, 50, 62, 70, 82, 90, 96$ from bottom to top.

FIG. 2. The spin stiffness $D_s$ for the Hubbard ring of size $L = 100$, as a function of the interaction strength for different values of the filling $N = 2M = 10, 22, 30, 42, 50, 62, 70, 82, 90, 96$ from bottom to top. Note that for large $N$ and $U$, it becomes increasingly difficult to distinguish different $D_s$ curves due to limited numerical resolution.

FIG. 3. Values of the disorder $\Delta$ as a function of the system size $l \sim \ln L/\alpha$ obtained by numerically integrating Eq. 13 for different values of the interaction strength $U = 0, 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 1.2, 1.4, 1.5, 1.6, 1.8, 2.0, 4., 6., 8., 10., 15$ and 20 (small $U$ is represented by small dashes) for a fixed density $n = 0.3$. The initial value for the disorder is fixed to $\Delta/v_f = 5 \times 10^{-4}$. The disorder scales less rapidly to infinity as the interaction strength is increased.

FIG. 4. Values of $y$ as a function of the system size $l \sim \ln L/\alpha$. Density and initial disorder strength are as in Fig. 3. $U$ varies from 0 to 20. Increasing $U$ is indicated by increasing dash length. $y$ scales less rapidly to $-\infty$ as the interaction strength is increased from 0 to 0.6. Starting at $U = 0.8$, $y$ then scales to $+\infty$.

FIG. 5. Values of the charge stiffness $D_c$ as a function of the system size $l \sim \ln L/\alpha$. Density and initial disorder strength are as in Fig. 3. For clarity, $U$ is only shown for values equal to 0.0, 0.1, 0.2, 0.4, 0.6, 0.8 and 1. Increasing $U$ is indicated by increasing dash length. There exists a crossover region beyond which enhancement of the currents is observed. Note that the lines for $U = 0.8$ and $U = 1$ correspond to the RAF phase and are well above the $U = 0$ value.
FIG. 6. We plot the disorder strength $\Delta \xi (l_c) = (\pi v_s^2 \Delta /2\alpha)\left[v_c /v_s\right]^{K_c}$ at which the enhancement of interacting over non interacting $D_c$ occurs as function of interaction strength $U$. Different curves correspond to different fillings, i.e., $N = 2M = 10, 22, 30, 42, 50, 62, 70, 82, 90, 96$ from bottom to top. Note that we have only limited resolution in $U$ corresponding to the possible values of the interaction strength as in Fig. 3.

FIG. 7. Values of the spin stiffness $D_s$ as a function of the system size $l \sim \ln L/\alpha$. Density and initial disorder strength are as in Fig. 3. $U$ varies from 0 to 20. Increasing $U$ is indicated by increasing dash length. There exists a crossover region beyond which enhancement of the currents is observed for $0 \leq U \leq 0.6$. After the PCDW-RAF transition between $0.6 < U < 0.8$, $D_s$ rapidly scales to zero.

FIG. 8. Using the qualitatively different behavior of $D_s$, we identify the boundary between the PCDW and RAF phases. The data points represent the largest values of $U$ for a given filling at which we still see a crossover in $D_s$ and are thus part of the PCDW phase. Note that we have only a limited resolution in $U$ corresponding to the possible values of the interaction strength as in Fig. 3.