A NOTE ON COMMUTATORS ON WEIGHTED MORREY SPACES ON SPACES OF HOMOGENEOUS TYPE

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Abstract: In this paper we study the boundedness and compactness characterizations of the commutator of Calderón–Zygmund operators $T$ on spaces of homogeneous type $(X, d, \mu)$ in the sense of Coifman and Weiss. More precisely, We show that the commutator $[b, T]$ is bounded on weighted Morrey space $L^{p,\kappa}(X)$ ($\kappa \in (0, 1), \omega \in A_p(X), 1 < p < \infty$) if and only if $b$ is in the BMO space. Moreover, the commutator $[b, T]$ is compact on weighted Morrey space $L^{p,\kappa}(X)$ ($\kappa \in (0, 1), \omega \in A_p(X), 1 < p < \infty$) if and only if $b$ is in the VMO space.

Keywords: commutator, compact operator, BMO space, VMO space, weighted Morrey space, space of homogeneous type

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1 Introduction

It is well-known that the boundedness and compactness of Calderón–Zygmund operator commutators on certain function spaces and their characterizations play an important role in various area, such as harmonic analysis, complex analysis, (nonlinear) PDE, etc. See for example [10, 9, 3, 19, 20, 13, 22, 18, 24, 25, 34] and the references therein. Recently, equivalent characterizations of the boundedness and the compactness of commutators were further extended to Morrey spaces over the Euclidean space by Di Fazio and Ragusa [16] and Chen et al. [5], and to weighted Morrey spaces by Komori and Shirai [27] for Calderón–Zygmund operator commutators and by Tao, Da. Yang and Do. Yang [31, 32] for the Cauchy integral and Buerling-Ahlfors transformation commutator, respectively. For more results on the boundedness of operators on Morrey spaces in different settings, we refer the reader to other studies [1, 15, 30, 17].

Thus, along this literature, it is natural to study the boundedness and compactness of Calderón–Zygmund operator commutators on weighted Morrey spaces in a more general setting: spaces of homogeneous type in the sense of Coifman and Weiss [8], as Yves Meyer remarked in his preface to [11], “One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.”

We say that $(X, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss if $d$ is a quasi-metric on $X$ and $\mu$ is a nonzero measure satisfying the doubling condition. A quasi-metric $d$ on a set $X$ is a function $d : X \times X \to [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) the quasi-triangle inequality: there is a constant $A_0 \in [1, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) \leq A_0[d(x, z) + d(z, y)].$$

(1.1)
Throughout this paper we assume that a nonzero measure $\mu$ satisfies the doubling condition if there is a constant $C_\mu$ such that for all $x \in X$ and $r > 0$, 
\[ \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty, \tag{1.2} \]
where $B(x, r)$ is the quasi-metric ball by $B(x, r) := \{ y \in X : d(x, y) < r \}$ for $x \in X$ and $r > 0$.

We point out that the doubling condition (1.2) implies that there exists a positive constant $n$ (the upper dimension of $\mu$) such that for all $x \in X$, $\lambda \geq 1$ and $r > 0$,
\[ \mu(B(x, \lambda r)) \leq C_\mu \lambda^n \mu(B(x, r)). \tag{1.3} \]

Throughout this paper we assume that $\mu(X) = \infty$ and that $\mu(\{x_0\}) = 0$ for every $x_0 \in X$.

We now recall the definition of Calderón–Zygmund operators on spaces of homogeneous type.

**Definition 1.1.** We say that $T$ is a Calderón–Zygmund operator on $(X, d, \mu)$ if $T$ is bounded on $L^2(X)$ and has an associated kernel $K(x, y)$ such that $T(f)(x) = \int_X K(x, y)f(y)d\mu(y)$ for any $x \notin \text{supp } f$, and $K(x, y)$ satisfies the following estimates: for all $x \neq y$,
\[ |K(x, y)| \leq \frac{C}{V(x, y)}, \tag{1.4} \]
and for $d(x, x') \leq (2A_0)^{-1}d(x, y)$,
\[ |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C}{V(x, y)} \beta \left( \frac{d(x, x')}{d(x, y)} \right), \tag{1.5} \]
where $V(x, y) = \mu(B(x, d(x, y)))$, $\beta : [0, 1] \rightarrow [0, \infty)$ is continuous, increasing, subadditive, and $\omega(0) = 0$. Throughout this paper we assume that $\beta(t) = t^{\sigma_0}$, for some $\sigma_0 > 0$.

Note that by the doubling condition we have that $V(x, y) \approx V(y, x)$. From [12] we assume for any Calderón–Zygmund operator $T$ as in Definition 1.1 with $\beta(t) \rightarrow 0$ as $t \rightarrow 0$, the following “non-degenerate” condition holds:

There exists positive constant $c_0$ and $\bar{A}$ such that for every $x \in X$ and $r > 0$, there exists $y \in B(x, \bar{A}r) \setminus B(x, r)$, satisfying
\[ |K(x, y)| \geq \frac{1}{c_0 \mu(B(x, r))}. \tag{1.6} \]

This condition gives a lower bound on the kernel and in $\mathbb{R}^n$ this “non degenerate” condition was first proposed in [22]. On stratified Lie groups, a similar condition of the Riesz transform kernel lower bound was verified in [13].

Let $T$ be a Calderón–Zygmund operator on $X$. Suppose $b \in L^1_{\text{loc}}(X)$ and $f \in L^p(X)$. Let $[b, T]$ be the commutator defined by
\[ [b, T]f(x) := b(x)T(f)(x) - T(bf)(x). \]

Let $p \in (1, \infty), \kappa \in (0, 1)$ and $\omega \in A_p(X)$. The weighted Morrey space $L^p_{\omega, \kappa}(X)$ is defined by
\[ L^p_{\omega, \kappa}(X) := \{ f \in L^p_{\text{loc}}(X) : \|f\|_{L^p_{\omega, \kappa}(X)} < \infty \} \]
Here
\[ \|f\|_{L^p_{\omega, \kappa}(X)} := \sup_B \left\{ \frac{1}{\omega(B)^\kappa} \int_B |f(x)|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}}. \]

Our main results are the following theorems.
Theorem 1.2. Let $p \in (1, \infty), \kappa \in (0, 1)$ and $\omega \in A_p(X)$. Suppose $b \in L^{1}_{\text{loc}}(X)$ and that $T$ is a Calderón–Zygmund operator as in Definition 1.1 and satisfies the non-degenerate condition (1.6). Then the commutator $[b, T]$ has the following boundedness characterization:

(i) If $b \in \text{BMO}(X)$, then $[b, T]$ is bounded on $L^{p, \kappa}_{\text{loc}}(X)$.

(ii) If $b$ is real valued and $[b, T]$ is bounded on $L^{p, \kappa}_{\text{loc}}(X)$, then $b \in \text{BMO}(X)$.

Theorem 1.3. Let $p \in (1, \infty), \kappa \in (0, 1)$ and $\omega \in A_p(X)$. Suppose $b \in L^{1}_{\text{loc}}(X)$ and that $T$ is a Calderón–Zygmund operator as in Definition 1.1 and satisfies the non-degenerate condition (1.6). Then the commutator $[b, T]$ has the following compactness characterization:

(i) If $b \in \text{VMO}(X)$, then $[b, T]$ is compact on $L^{p, \kappa}_{\text{loc}}(X)$.

(ii) If $b$ is real valued and $[b, T]$ is compact on $L^{p, \kappa}_{\text{loc}}(X)$, then $b \in \text{VMO}(X)$.

Throughout the paper, we denote by $C$ and $\tilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, we denote by $p'$ the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \leq Cg$ or $f \geq Cg$, we then write $f \preceq g$ or $f \succeq g$; and if $f \preceq g \preceq f$, we write $f \approx g$.

2 Preliminaries on Spaces of Homogeneous Type

Let $(X, d, \mu)$ be a space of homogeneous type as mentioned in Section 1. We now recall the BMO and VMO space.

Definition 2.1. A function $b \in L^{1}_{\text{loc}}(X)$ belongs to the BMO space $\text{BMO}(X)$ if

$$
\|b\|_{\text{BMO}(X)} := \sup_{B} M(b, B) := \sup_{B} \frac{1}{\mu(B)} \int_{B} |b(x) - b_{B}| \, d\mu(x) < \infty,
$$

where the sup is taken over all quasi-metric balls $B \subset X$ and

$$
b_{B} = \frac{1}{\mu(B)} \int_{B} b(y) \, d\mu(y).
$$

The following John-Nirenberg inequalities on spaces of homogeneous type comes from [26].

Lemma 2.2 ([26]). If $f \in \text{BMO}(X)$, then there exist positive constants $C_1$ and $C_2$ such that for every ball $B \subset X$ and every $\alpha > 0$, we have

$$
\mu(\{x \in B : |f(x) - f_{B}| > \alpha\}) \leq C_1 \lambda(B) \exp\left\{-\frac{C_2}{\|f\|_{\text{BMO}(X)}} \alpha\right\}.
$$

We recall the median value $\alpha_{B}(f)$ ([4]). For any real valued function $f \in L^{1}_{\text{loc}}(X)$ and $B \subset X$, let $\alpha_{B}(f)$ be a real number such that

$$
\inf_{c \in \mathbb{R}} \frac{1}{\mu(B)} \int_{B} |f(x) - c| \, d\mu(x)
$$

is attained. Moreover, it is known that $\alpha_{B}(f)$ satisfies that

$$
\mu(\{x \in B : f(x) > \alpha_{B}(f)\}) \leq \frac{\mu(B)}{2}
$$

(2.1)
\[ \mu(\{ x \in B : f(x) < \alpha_B(f) \}) \leq \frac{\mu(B)}{2}. \] 
(2.2)

And it is easy to see that for any ball \( B \subseteq X \),
\[ M(b, B) \approx \frac{1}{\mu(B)} \int_B |b(x) - \alpha_B(b)| d\mu(x), \]
(2.3)
where the implicit constants are independent of the function \( b \) and the ball \( B \).

By Lip(\( \beta \)), \( 0 < \beta < \infty \), we denote the set of all functions \( \phi(x) \) defined on \( X \) such that there exists a finite constant \( C \) satisfying
\[ |\phi(x) - \phi(y)| \leq Cd(x, y)^\beta \]
for every \( x \) and \( y \) in \( X \). \( \| \phi \|_\beta \) will stand for the least constant \( C \) satisfying the condition above.

By Lip\( _c(\beta) \), we denote the set of all Lip(\( \beta \)) functions with compact support on \( X \).

**Definition 2.3.** We define \( \text{VMO}(X) \) as the closure of the Lip\( _c(\beta) \) functions \( X \) under the norm of the BMO space.

We also need to establish the characterisation of \( \text{VMO}(X) \). We will give its proof in Appendix. For the Euclidean and the stratified Lie groups case one can refer to [33] and [4].

**Lemma 2.4.** Let \( f \in \text{BMO}(X) \). Then \( f \in \text{VMO}(X) \) if and only if \( f \) satisfies the following three conditions:

(i) \( \lim_{a \to 0} \sup_{r_B = a} M(f, B) = 0 \);

(ii) \( \lim_{a \to \infty} \sup_{r_B = a} M(f, B) = 0 \);

(iii) \( \lim_{r \to \infty} \sup_{B \subseteq X \setminus B(x_0, r)} M(f, B) = 0 \),
where \( r_B \) is the radius of the ball \( B \) and \( x_0 \) is a fixed point in \( X \).

To this end, we recall the definition of \( A_p \) weights.

**Definition 2.5.** Let \( \omega(x) \) be a nonnegative locally integrable function on \( X \). For \( 1 < p < \infty \), we say \( \omega \) is an \( A_p \) weight, written \( \omega \in A_p \), if
\[ [\omega]_{A_p} := \sup_B \left( \int_B \omega \right)^{1/(p-1)} \left( \int_B \left( \frac{1}{\omega} \right)^{1/(p-1)} \right)^{p-1} < \infty. \]

Here the suprema are taken over all balls \( B \subseteq X \). The quantity \([\omega]_{A_p}\) is called the \( A_p \) constant of \( \omega \). For \( p = 1 \), we say \( \omega \) is an \( A_1 \) weight, written \( \omega \in A_1 \), if \( M(\omega)(x) \leq \omega(x) \) for \( \mu \)-almost every \( x \in X \), and let \( A_{\infty} := \cup_{1 \leq p < \infty} A_p \) and we have \([\omega]_{A_{\infty}} := \sup_B \left( \int_B \omega \right) \exp \left( \int_B \log(\frac{1}{\omega}) \right) \) < \infty.

Next we note that for \( \omega \in A_p \) the measure \( \omega(x)d\mu(x) \) is a doubling measure on \( X \). To be more precise, we have that for all \( \lambda > 1 \) and all balls \( B \subseteq X \),
\[ \omega(\lambda B) \leq \lambda^{np}[\omega]_{A_p} \omega(B), \]
(2.4)
where \( n \) is the upper dimension of the measure \( \mu \), as in (1.3).

We also point out that for \( \omega \in A_\infty \), there exists \( \gamma > 0 \) such that for every ball \( B \),

\[
\mu\left( \{ x \in B : \omega(x) \geq \gamma \int_B \omega \} \right) \geq \frac{1}{2} \mu(B).
\]

And this implies that for every ball \( B \) and for all \( \delta \in (0,1) \),

\[
\int_B \omega \leq C \left( \int_B \omega^\delta \right)^{1/\delta}; \quad (2.5)
\]

see also [25].

By the definition of \( A_p \) weight and Hölder’s inequality, we can easily obtain the following standard properties.

**Lemma 2.6.** Let \( \omega \in A_p(X), \ p \geq 1 \). Then there exists constants \( \hat{C}_1, \hat{C}_2 > 0 \) and \( \sigma \in (0,1) \) such that the following holds

\[
\hat{C}_1 \left( \frac{\mu(E)}{\mu(B)} \right)^p \leq \frac{\omega(E)}{\omega(B)} \leq \hat{C}_2 \left( \frac{\mu(E)}{\mu(B)} \right)^\sigma
\]

for any measurable set \( E \) of a quasi metric ball \( B \).

According to [2, Theorem 5.5], we have the following result for BMO functions on \( X \).

**Lemma 2.7.** Let \( 0 < p < \infty, v \in A_\infty(X), f \in \text{BMO}(X) \). Then

\[
\|f\|_{\text{BMO}(X)} \approx \sup_{B \subset X} \left\{ \frac{1}{\omega(B)} \int_B |f(x) - f_{B,v}|^p v(x) d\mu(x) \right\}^{1/p},
\]

where \( f_{B,v} = \frac{1}{\omega(B)} \int_B f(y)v(y)d\mu(y) \).

## 3  Boundedness Characterization of Commutators

In this section, we will give the proof of Theorem 1.2.

### 3.1 Proof of Theorem 1.2(i).

In order to prove Theorem 1.2(i), we need the following lemma.

**Lemma 3.1** ([12]). Let \( b \in \text{BMO}(X) \) and \( T \) be Calderón–Zygmund operator on \((X,d,\mu)\) a Space of homogeneous type. If \( \kappa \in (0,1), 1 < p < \infty \) and \( \omega \in A_p(X) \), then \([b,T]\) is bounded on \( L^p_{\omega^\kappa}(X) \).

**Proof of Theorem 1.2(i).** Let \( 1 < p < \infty \). Then it suffices to show that

\[
\left( \frac{1}{\omega(B)} \right)^{\kappa} \int_B |[b,T](x)|^p \omega(x) d\mu(x) \right)^{1/p} \lesssim \|b\|_{\text{BMO}(X)}\|f\|_{L^p_{\omega^\kappa}(X)},
\]

holds for any ball \( B \).

Now we will fix a ball \( B = B(x_0,r) \) and then decompose \( f = f_{\chi_{2B}} + f_{\chi_{X\setminus 2B}} =: f_1 + f_2 \).

Then we have

\[
\frac{1}{\omega(B)^\kappa} \int_B |[b,T](x)|^p \omega(x) d\mu(x)
\]
\[
\lesssim \left( \frac{1}{\omega(B)^\kappa} \right)^\kappa \left( \left| [b, T] f_1(x) \right|^p \omega(x) \right) d\mu(x) + \frac{1}{\omega(B)^\kappa} \int_B \left| [b, T] f_2(x) \right|^p \omega(x) d\mu(x) \] 
\[
=: I + II.
\]

For the first term \(I\) here, we use Lemma 3.1 and we obtain
\[
\frac{1}{\omega(B)^\kappa} \int_B \left| [b, T] f_1(x) \right|^p \omega(x) d\mu(x) \leq \frac{1}{\omega(B)^\kappa} \int_X \left| [b, T] f_1(x) \right|^p \omega(x) d\mu(x)
\]
\[
\lesssim \|b\|_{\text{BMO}(X)}^p \omega(B)^\kappa \int_{2B} |f(x)|^p \omega(x) d\mu(x)
\]
\[
\lesssim \|b\|_{\text{BMO}(X)}^p \|f\|^p_{L^p_{\omega}(X)}.
\]

So we have
\[
\| [b, T] f_1 \|_{L^p_{\omega}(X)} \lesssim \|b\|_{\text{BMO}(X)} \|f\|^p_{L^p_{\omega}(X)}.
\]

Now for the second term \(II\), observe that for \(x \in B\), by (1.4), we have
\[
\left| [b, T] f_2(x) \right|^p \leq \left( \int_X \left| b(x) - b(y) \right| |K(x, y)||f_2(y)| d\mu(y) \right)^p
\]
\[
\lesssim \left( \int_{X \setminus 2B} \frac{|b(x) - b(y)|}{V(x, y)} |f(y)| d\mu(y) \right)^p
\]
\[
\lesssim \left( \int_{X \setminus 2B} \frac{|f(y)|}{V(x, y)} \left( |b(x) - b_{B, \omega}| + |b_{B, \omega} - b(y)| \right) d\mu(y) \right)^p
\]
\[
\lesssim \left( \int_{X \setminus 2B} \frac{|f(y)|}{V(x, y)} d\mu(y) \right)^p |b(x) - b_{B, \omega}|^p + \left( \int_{X \setminus 2B} \frac{|f(y)|}{V(x, y)} |b_{B, \omega} - b(y)| d\mu(y) \right)^p,
\]

where \(b_{B, \omega} = \frac{1}{\omega(B)} \int_B b(y) \omega(y) d\mu(y)\). Hence we have the following
\[
\frac{1}{\omega(B)^\kappa} \int_B \left| [b, T] f_2(x) \right|^p \omega(x) d\mu(x) \lesssim \frac{1}{\omega(B)^\kappa} \left( \int_{X \setminus 2B} \frac{|f(y)|}{V(x, y)} d\mu(y) \right)^p \int_B |b(x) - b_{B, \omega}|^p \omega(x) d\mu(x)
\]
\[
+ \left( \int_{X \setminus 2B} \frac{|f(y)|}{V(x, y)} |b_{B, \omega} - b(y)| d\mu(y) \right)^p \omega(B)^{1-\kappa}
\]
\[
=: III + IV.
\]

Note that \(\lim_{k \to \infty} \mu(2^k B) = \infty\). Then there exist \(j_k \in \mathbb{N}\) such that
\[
\mu(2^k B) \geq 2 \mu(B) \text{ and } \mu(2^{j_k+1} B) \geq 2 \mu(2^k B).
\]

For \(III\), using the Hölder inequality, and using Lemma 2.6 and Lemma 2.7, we get
\[
III \lesssim \|f\|^p_{L^p_{\omega}(X)} \frac{1}{\omega(B)^\kappa} \left( \sum_{k=0}^{\infty} \int_{2^{j_k+1} B \cap 2^{2k} B} \frac{|f(y)|}{V(x, y)} d\mu(y) \right)^p \int_B |b(x) - b_{B, \omega}|^p \omega(x) d\mu(x)
\]
\[
\lesssim \|f\|^p_{L^p_{\omega}(X)} \frac{1}{\omega(B)^\kappa} \left( \sum_{k=0}^{\infty} \frac{1}{\omega(2^{j_k+1} B)^{1-\kappa}} \right)^p \int_B |b(x) - b_{B, \omega}|^p \omega(x) d\mu(x)
\]
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Using Hölder’s inequality for the term IV, we get

\[
IV \lesssim \left( \sum_{k=0}^{\infty} \frac{1}{\mu(2k+1)B} \left( \int_{2^{k+1}B} |f(y)||b_{B,\omega} - b(y)|d\mu(y) \right)^p \right)^{1-\kappa}
\]

\[
\lesssim \left( \sum_{k=0}^{\infty} \frac{1}{\mu(2k+1)B} \left( \int_{2^{k+1}B} |f(y)|^p \omega(y) d\mu(y) \right)^{1-\frac{1}{p}} \right)^p \omega(B)^{1-\kappa}
\]

\[
\times \left( \int_{2^{k+1}B} |b_{B,\omega} - b(y)|^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}} \omega(B)^{1-\kappa}
\]

\[
\lesssim \left\| f \right\|_{L^p_{\mu} (X)} \left\{ \sum_{k=0}^{\infty} \frac{\omega(2^{k+1}B)^{\frac{1}{p'}}}{\mu(2k+1)B} \left( \int_{2^{k+1}B} |b_{B,\omega} - b(y)|^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}} \right\} \omega(B)^{1-\kappa}.
\]

Now observe that

\[
\left( \int_{2^{k+1}B} |b_{B,\omega} - b(y)|^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}} \leq \left( \int_{2^{k+1}B} \left( |b(y) - b_{2^{k+1}B,\omega^{1-p'}}| + |b_{2^{k+1}B,\omega^{1-p'}} - b_{B,\omega}| \right)^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}}
\]

\[
\leq \left( \int_{2^{k+1}B} \left( |b(y) - b_{2^{k+1}B,\omega^{1-p'}}| \right)^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}}
\]

\[
+ \left( \int_{2^{k+1}B} \left( |b_{2^{k+1}B,\omega^{1-p'}} - b_{B,\omega}| \right)^{p'} \omega(y)^{1-p'} d\mu(y) \right)^{\frac{1}{p'}}
\]

\[
=: V + VI.
\]

We have \( \omega^{1-p'} \in A_{p'}(X) \) since \( \omega \in A_p(X) \). So we obtain

\[
V \lesssim \|b\|_{\text{BMO}(X)} \omega^{1-p'} (2^{k+1}B)^{\frac{1}{p'}}.
\]

For VI, we have

\[
|b_{2^{k+1}B,\omega^{1-p'}} - b_{B,\omega}| \leq \left| b_{2^{k+1}B,\omega^{1-p'}} - b_{2^{k+1}B} \right| + \left| b_{2^{k+1}B} - b_B \right| + |b_B - b_{B,\omega}|
\]

\[
\lesssim \frac{1}{\omega^{1-p'}(2^{k+1}B)} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| \omega(y)^{1-p'} d\mu(y)
\]

\[
+ (k+1) \|b\|_{\text{BMO}(X)} + \frac{1}{\omega(B)} \int_B |b(y) - b_B| \omega(y) d\mu(y).
\]

As we have \( b \in \text{BMO}(X) \), by Lemma 2.2, there exists some constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for any ball \( B \) and \( \alpha > 0 \)

\[
\mu(\{x \in B : |b(x) - b_B| > \alpha\}) \leq C_1 \mu(B) e^{-\frac{C_2\alpha}{\|b\|_{\text{BMO}(X)}}}.
\]
Then using Lemma 2.6, we get
\[ \omega(\{x \in B : |b(x) - b_B| > \alpha\}) \leq C_1 \omega(B) e^{-\frac{C_{\sigma, \alpha}}{|b|_{\text{BMO}(X)}}} \]
for some \( \sigma \in (0, 1) \). Hence we have
\[
\int_B |b(y) - b_B| \omega(y) d\mu(y) = \int_0^\infty \omega(\{y \in B : |b(y) - b_B| > \alpha\}) d\alpha \\
\leq \omega(B) \int_0^\infty e^{-\frac{C_{\sigma, \alpha}}{|b|_{\text{BMO}(X)}}} d\alpha \\
\leq \omega(B) \|b\|_{\text{BMO}(X)}.
\]
Similarly, we also get
\[
\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| \omega(y)^{1-p'} d\mu(y) \right)^{1/p} \lesssim (k + 1) \|b\|_{\text{BMO}(X)} \omega^{1-p'}(2^{j+1}B)^{1/p'}.
\]
Together with Lemma 2.6, we have the following
\[
IV \lesssim \|f\|_{L^p_{\omega}(X)}^p \|b\|_{\text{BMO}(X)}^p \left[ \sum_{k=0}^\infty \frac{\omega(2^{j+1}B)^\sigma}{\mu(2^{j+1}B)} (k+1)^{-p'}(2^{j+1}B)^{1/p'} \right] \omega(B)^{1-\kappa} \]
\[
\lesssim \|f\|_{L^p_{\omega}(X)}^p \|b\|_{\text{BMO}(X)}^p \left[ \sum_{k=0}^\infty \frac{(k+1)^{1-\kappa}}{\omega(2^{j+1}B)^{1-\kappa}} \right] \]
\[
\lesssim \|f\|_{L^p_{\omega}(X)}^p \|b\|_{\text{BMO}(X)}^p \sum_{k=0}^\infty (k+1)2^{-\frac{(k+1)(1-\kappa)}{p}} \]
\[
\lesssim \|f\|_{L^p_{\omega}(X)}^p \|b\|_{\text{BMO}(X)}^p.
\]
Therefore we have
\[
\| [b, T] f \|_{L^p_{\omega}(X)} \lesssim \|f\|_{L^p_{\omega}(X)} \|b\|_{\text{BMO}(X)}.
\]
This completes the proof.

3.2 Proof of Theorem 1.2(ii).

We first recall another version of the homogeneous condition (formulated in [12]): there exist positive constants \( 3 \leq A_1 \leq A_2 \) such that for any ball \( B := B(x_0, r) \subset X \), there exist balls \( \tilde{B} := B(y_0, r) \) such that \( A_1 r \leq d(x_0, y_0) \leq A_2 r \), and for all \( (x, y) \in (B \times \tilde{B}) \), \( K(x, y) \) does not change sign and
\[
|K(x, y)| \gtrsim \frac{1}{\mu(B)}, \quad (3.1)
\]
If the kernel \( K(x, y) := K_1(x, y) + iK_2(x, y) \) is complex-valued, where \( i^2 = -1 \), then at least one of \( K_i \) satisfies (3.1).

Then we first point out that the homogeneous condition (1.6) implies (3.1).

Lemma 3.2 ([12]). Let \( T \) be the Calderón–Zygmund operator as in Definition 1.1 and satisfy the homogeneous condition as in (1.6). Then \( T \) satisfies (3.1).
Proof of Theorem 1.2(ii). To prove \( b \in \text{BMO}(X) \), it is sufficient to show for any ball \( B \subset X \), we have \( M(b, B) \lesssim 1 \). Let \( B = B(x_0, r) \) be a quasi metric ball in \( X \). Also let \( \tilde{B} := B(y_0, r) \subset X \) be the measurable set in (3.1). Following [12], we take

\[
E_1 := \{ x \in B : b(x) \geq \alpha_{\tilde{B}}(b) \} \quad E_2 := \{ x \in B : b(x) < \alpha_{\tilde{B}}(b) \};
\]

\[
F_1 \subset \{ y \in \tilde{B} : b(y) \leq \alpha_{\tilde{B}}(b) \} \quad F_2 \subset \{ y \in \tilde{B} : b(y) \geq \alpha_{\tilde{B}}(b) \},
\]

with \( \alpha_{\tilde{B}}(b) \) the median value of \( b \) over \( \tilde{B} \), such that \( \mu(F_1) = \mu(F_2) = \frac{1}{2} \mu(\tilde{B}) \) and \( F_1 \cap F_2 = \emptyset \). For any \((x, y) \in E_j \times F_j, j \in \{1, 2\}, \) we have

\[
|b(x) - b(y)| = |b(x) - \alpha_{\tilde{B}}(b)| + |\alpha_{\tilde{B}}(b) - b(y)| \geq |b(x) - \alpha_{\tilde{B}}(b)|.
\]

As \( b \) is a real valued, using Lemma 2.6, Hölder’s inequality, and using the boundedness of \([b,T] \) on \( L^{p,\kappa}(X) \) and (3.1), we get that

\[
M(b, B) \lesssim \frac{1}{\mu(B)} \int_B |b(x) - \alpha_{\tilde{B}}(b)| \, d\mu(x) \lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} |b(x) - \alpha_{\tilde{B}}(b)| \, d\mu(x)
\]
\[
\lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \int_{F_j} \frac{|b(x) - \alpha_{\tilde{B}}(b)|}{V(x,y)} \, d\mu(y) \, d\mu(x)
\]
\[
\lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \int_{F_j} \frac{|b(x) - b(y)|}{V(x,y)} \, d\mu(y) \, d\mu(x)
\]
\[
\lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \left| \int_{F_j} |b(x) - b(y)| K(x, y) \, d\mu(y) \right| \, d\mu(x)
\]
\[
\lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \left| [b, T] \chi_{F_j}(x) \right| \, d\mu(x)
\]
\[
\lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \left| [b, T] \chi_{F_j}(x) \right| \, d\mu(x) \lesssim \sum_{j=1}^2 \frac{1}{\mu(B)} \int_{E_j} \| [b, T] \chi_{F_j} \|_{L^{p,\kappa}(X)} \, d\mu(B) \lesssim \| [b, T] \|_{L^{p,\kappa}(X) \to L^{p,\kappa}(X)} \| \chi_{F_j} \|_{L^{p,\kappa}(X)} \| \omega(B) \|^{\frac{n-1}{p}}
\]
\[
\lesssim \| [b, T] \|_{L^{p,\kappa}(X) \to L^{p,\kappa}(X)} \| \omega(\tilde{B}) \|^{\frac{1-n}{p}} \| \omega(B) \|^{\frac{n-1}{p}}
\]
\[
\lesssim \| [b, T] \|_{L^{p,\kappa}(X) \to L^{p,\kappa}(X)} \cdot
\]

This completes the proof of Theorem 1.2(ii). \( \square \)

4 Compactness Characterization of the Commutator

Now we will prove Theorem 1.3.
4.1 Proof of Theorem 1.3(i).

Now we will give sufficient conditions for the subsets of weighted Morrey spaces to be relatively compact. We define a subset \( F \) of \( L^{p,\kappa}_\omega(X) \) to be totally bounded if the \( L^{p,\kappa}_\omega(X) \) closure of \( F \) is compact.

**Lemma 4.1.** Let \( p \in (1, \infty), \kappa \in (0, 1) \) and \( \omega \in A^p(X) \), then a subset \( F \) of \( L^{p,\kappa}_\omega(X) \) is totally bounded if the set \( F \) satisfies the following three conditions:

(i) \( F \) is bounded, namely,
\[ \sup_{f \in F} \|f\|_{L^{p,\kappa}_\omega(X)} < \infty; \]

(ii) \( F \) vanishes uniformly at infinity, namely, for any \( \epsilon \in (0, \infty) \), there exists some positive constant \( M \) such that, for any \( f \in F \),
\[ \|f \chi_{\{x \in X : d(x_0, x) > M\}}\|_{L^{p,\kappa}_\omega(X)} < \epsilon, \]
where \( x_0 \) is a fixed point in \( X \);

(iii) \( F \) is uniformly equicontinuous, namely,
\[ \lim_{r \to 0} \|f(x) - f_{B(x,r)}\|_{L^{p,\kappa}_\omega(X)} = 0 \]
uniformly for \( f \in F \).

The proof for the lemma above, follows from [29] using a small modification from Euclidean setting to space of homogeneous type, this only requires following properties of the underlying space: metric on space and doubling measure.

We will now show the boundedness of the maximal operator \( T^*_\eta \) of a family of smooth truncated operators \( \{T_\eta\}_{\eta \in (0, \infty)} \) as follows. For \( \eta \in (0, \infty) \), we take
\[ T_\eta f(x) := \int_X K_\eta(x, y) f(y) d\mu(y), \]
where the kernel \( K_\eta := K(x, y) \varphi\left(\frac{d(x, y)}{\eta}\right) \) with \( \varphi \in C^\infty(\mathbb{R}) \) and \( \varphi \) satisfies the following
\[ \varphi(t) = \begin{cases} \varphi(t) \equiv 0 & \text{if } t \in \left(-\infty, \frac{1}{2}\right) \\ \varphi(t) \in [0, 1], & \text{if } t \in \left[\frac{1}{2}, 1\right] \\ \varphi(t) \equiv 1, & \text{if } t \in (1, \infty). \end{cases} \]

Let
\[ [b, T_\eta] f(x) := \int_X [b(x) - b(y)] K_\eta(x, y) f(y) d\mu(y). \]

The maximal operator \( T^*_\eta \) is defined as below
\[ T^*_\eta f(x) := \sup_{\eta \in (0, \infty)} \left| \int_X K_\eta(x, y) f(y) d\mu(y) \right|. \]

Observe that the Hardy-Littlewood maximal Operator \( M \) is given as
\[ Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \]
for any \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \), here we take the supremum over all quasi-metric balls \( B \) of \( X \) that contain \( x \).

Then we have the following lemmas.
Lemma 4.2. There exists a positive constant $C$ such that we have, for any $b \in \text{Lip}(\beta)$, $0 < \beta < \infty$, $f \in L^1_{\text{loc}}(X)$ and $x \in X$

$$|[b, T_\eta] f(x) - [b, T] f(x)| \leq C \eta^\beta M f(x).$$

Proof. Let $f \in L^1_{\text{loc}}(X)$. Now for any $x \in X$, we get

$$|[b, T_\eta] f(x) - [b, T] f(x)|$$

$$= \left| \int_{d(x,y) < \eta} [b(x) - b(y)] K_\eta(x,y) f(y) d\mu(y) - \int_{d(x,y) < \eta} [b(x) - b(y)] K(x,y) f(y) d\mu(y) \right|$$

$$\lesssim \int_{d(x,y) < \eta} |b(x) - b(y)||K(x,y)||f(y)| d\mu(y).$$

Since $b \in \text{Lip}(\beta)$ and (1.4), we have that

$$\int_{d(x,y) < \eta} |b(x) - b(y)||K(x,y)||f(y)| d\mu(y)$$

$$\lesssim C \sum_{j=0}^\infty \int_{2^{j+1} \leq d(x,y) < 2^j} \frac{d(x,y)^2}{V(x,y)} |f(y)| d\mu(y)$$

$$\lesssim C \eta^\beta M f(x),$$

this completes the proof of the Lemma 4.2. 

Lemma 4.3. Let $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $\omega \in A_p(X)$. Then there exists a positive constant $C$ such that, for any $f \in L^p_{\omega} (X)$,

$$||T_*||_{L^p_{\omega}(X)} + ||Mf||_{L^p_{\omega}(X)} \leq C ||f||_{L^p_{\omega}(X)}.$$

Proof. To show the boundedness of $M$ on $L^p_{\omega}(X)$ one can refer to [2]. We will now only consider the boundedness of $T_*$. For any fixed quasi-metric ball $B \subset X$ and $f \in L^p_{\omega} (X)$, we write the following

$$f := f_1 + f_2 := f \chi_{2B} + f \chi_{X\backslash 2B}.$$ 

Note that $\lim_{k \to \infty} \mu(2^kB) = \infty$. Then there exist $j_k \in \mathbb{N}$ such that

$$\mu(2^kB) \geq 2\mu(B) \text{ and } \mu(2^{j_k+1}B) \geq 2\mu(2^kB).$$

Observe that $f_1 \in L^p_{\omega} (X)$. Then using the boundedness of $T_*$ on $L^p_{\omega} (X)$ (see, for example, [23, Theorem 1.1] ) and from the Hölder inequality, also using size and smoothness of Kernel, we have that

$$\left[ \int_B |T_*f(x)|^p \omega(x) d\mu(x) \right]^\frac{1}{p}$$

$$\lesssim \left[ \int_B |T_*f_1(x)|^p \omega(x) d\mu(x) \right]^\frac{1}{p} + \sum_{k=0}^\infty \left\{ \int_B \left[ \int_{2^{k+1}B \backslash 2kB} \frac{|f(y)|}{V(x,y)} d\mu(y) \right]^p \omega(x)d\mu(x) \right\}^\frac{1}{p}$$

$$\lesssim \left[ \int_{2B} |f(x)|^p \omega(x) d\mu(x) \right]^\frac{1}{p} + \sum_{k=0}^\infty \left[ \frac{\omega(B)}{\mu(2^kB)^p} \left( \int_{2^{k+1}B} |f(y)|\omega(y)\right)^\frac{p}{p-1} d\mu(y) \right]^\frac{1}{p}$$
in the fourth inequality above, we have used Lemma 2.6 for some $\sigma \in (0,1)$. This completes the proof of Lemma 4.3.

Proof of Theorem 1.3(i). When $b \in \text{VMO} (X)$, then for any $\epsilon \in (0, \infty)$, there exists $b(\epsilon) \in \text{Lip}_c (\beta), 0 < \beta < \infty$ such that we have $\| b - b(\epsilon) \|_{\text{BMO}(X)} < \epsilon$. Then, using the boundedness of the commutator $[b, T]$ on $L^{p,\kappa}_\omega (X)$, we obtain

$$\left\| [b, T] f - [b(\epsilon), T] f \right\|_{L^{p,\kappa}_\omega (X)} = \left\| [b - b(\epsilon), T] f \right\|_{L^{p,\kappa}_\omega (X)} \lesssim \| b - b(\epsilon) \|_{\text{BMO}(X)} \| f \|_{L^{p,\kappa}_\omega (X)} \lesssim \epsilon \| f \|_{L^{p,\kappa}_\omega (X)}.$$

Also using Lemmas 4.2 and 4.3, we have the following

$$\lim_{\eta \to 0} \| [b, T_\eta] - [b, T] \|_{L^{p,\kappa}_\omega (X) \to L^{p,\kappa}_\omega (X)} = 0.$$ 

It sufficient to show that, for any $b \in \text{Lip}_c (\beta), 0 < \beta < \infty$ and $\eta \in (0, \infty)$ small enough, $[b, T_\eta]$ is a compact operator on $L^{p,\kappa}_\omega (X)$, this is equivalent to showing that, for any bounded subset $F \subset L^{p,\kappa}_\omega (X)$, $[b, T_\eta] F$ is relatively compact. Which means, we need to show that $[b, T_\eta]$ satisfies the conditions (i) through (iii) of Lemma 4.1.

Observe by [27, Theorem 3.4] and using the fact that $b \in \text{BMO} (X)$, we have that $[b, T_\eta]$ is bounded on $L^{p,\kappa}_\omega (X)$ for the given $p \in (1, \infty), \kappa \in (0,1)$ and $\omega \in A_p (X)$, this shows that $[b, T_\eta] F$ satisfies condition (i) of Lemma 4.1.

Now, let $x_0$ be a fixed point in $X$. Since $b \in \text{Lip}_c (\beta)$, we can further assume that $\| b \|_{L^\infty} = 1$. Recall that there exists a positive constant $R_0$ such that supp $(b) \subset B (x_0, R_0)$. Let $M \in (10R_0, \infty)$. Thus, for any $y \in B (x_0, R_0)$ and $x \in X$ with $d(x_0, x) > M$, $d(y, x) \sim d(x_0, x)$. Then, for $x \in X$ with $d(x_0, x) > M$, by Hölder inequality and using that $V(x, y) \sim \mu(B(x_0, d(x_0, x)))$ we deduce that

$$\| [b, T_\eta] f(x) \| \leq \int_X |b(x) - b(y)||K_\eta (x, y)||f(y)||d\mu(y)$$
$$\leq \int_X |b(y)||K_\eta (x, y)||f(y)||d\mu(y)$$
$$\lesssim \int_{B(x_0, R_0)} \frac{|f(y)|}{V(x, y)} d\mu(y)$$
$$\lesssim \int_{B(x_0, R_0)} \frac{|f(y)|}{\mu(B(x_0, d(x_0, x)))} d\mu(y)$$
$$\lesssim \frac{1}{\mu(B(x_0, d(x_0, x)))} \left( \int_{B(x_0, R_0)} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}} \left( \int_{B(x_0, R_0)} |\omega(y)|^\frac{p}{p'-1} d\mu(y) \right)^{\frac{1}{p}-1}$$
Boundedness and compactness of commutators

\[
\begin{align*}
\mu(B(x_0, R_0)) & \geq \mu(B(x_0, d(x, x_0))) |\omega(B(x_0, R_0))|^{\frac{k}{n}} \|f\|_{L^p_c(X)}. \\
\text{From } \lim_{k \to \infty} \mu(B(x_0, kM)) & = \infty, \text{ we have that there exist } j_k \in \mathbb{N} \text{ such that} \\
\mu(B(x_0, 2^{j_k}M)) & \geq 2\mu(B(x_0, M)) \text{ and } \mu(B(x_0, 2^{j_k+1}M)) \geq 2\mu(B(x_0, 2^{j_k}M)).
\end{align*}
\]

Hence, for any fixed ball \( B := B(\bar{x}, \bar{r}) \subset X \), by Lemma 2.6, we get that

\[
\begin{align*}
\frac{1}{|\omega(B)|^k} \int_{B \cap \{x \in X : d(x, x_0) > M\}} |[b, T_\eta]f(x)| |\omega(x)| d\mu(x) & \\
\lesssim \mu(B(x_0, R_0))^p |\omega(B(x_0, R_0))|^{(k-1)} \|f\|_{L^p_c(X)} \sum_{k=0}^\infty \omega(B \cap \{x \in X : 2^{j_k}M < d(x, x_0) \leq 2^{j_k+1}M\}) \\
\lesssim \|f\|_{L^p_c(X)} \sum_{k=0}^\infty \frac{\mu(B(x_0, R_0))^{\frac{p}{\kappa}}}{\mu(B(x_0, 2^{j_k}M))^{\frac{p}{\kappa}}} \\
\lesssim \|f\|_{L^p_c(X)} \sum_{k=0}^\infty \frac{\mu(B(x_0, 2^{j_k+1}M))^{\frac{p}{\kappa}}}{\mu(B(x_0, 2^{j_k}M))^{\frac{p}{\kappa}}} \\
\lesssim \|f\|_{L^p_c(X)} \sum_{k=0}^\infty \frac{\mu(B(x_0, R_0))^{\frac{p}{\kappa}}}{\mu(B(x_0, M))^{\frac{p}{\kappa}}} \|f\|_{L^p_c(X)}.
\end{align*}
\]

Therefore the condition (ii) of Lemma 4.1 holds for \([b, T_\eta]f\) with large \( M \).

Now we will prove \([b, T_\eta]f\) also satisfies (iii) of Lemma 4.1. Let \( \eta \) be a fixed positive constant small enough and \( r < \frac{\eta}{8\kappa_0} \). Now, for any \( x \in X \), we have

\[
[b, T_\eta]f(x) - ([b, T_\eta]f)_{B(x, r)} = \frac{1}{\mu(B(x, r))} \int_{B(x, r)} [b, T_\eta]f(y) d\mu(y).
\]

Note that

\[
[b, T_\eta]f(x) - [b, T_\eta]f(y) = [b(x) - b(y)] \int_X K_\eta(x, z) f(z) d\mu(z) + \int_X [K_\eta(x, z) - K_\eta(y, z)] [b(y) - b(z)] f(z) d\mu(z)
\]

\[= L_1(x, y) + L_2(x, y).\]

As \( b \in \text{Lip}_c(\beta) \), it follows that, for any \( y \in B(x, r) \)

\[
|L_1(x, y)| = |b(x) - b(y)| \int_X K_\eta(x, z) f(z) d\mu(z) \lesssim r^\beta T_\eta(f)(x).
\]

To estimate \( L_2(x, y) \), we first recall that \( K_\eta(x, z) = 0, K_\eta(y, z) = 0 \) for any \( y \in B(x, r) \),
\( d(x, z) \leq \frac{\eta}{4\kappa_0} \) and \( r < \frac{\eta}{8\kappa_0} \). Using the definition of \( K_\eta \) we have that, for any \( y \in B(x, r) \),
\( d(x, z) > \frac{\eta}{4\kappa_0} \) and \( r < \frac{\eta}{8\kappa_0} \),

\[
|K_\eta(x, z) - K_\eta(y, z)| \lesssim \frac{1}{V(x, z)} d(x, y)^{\kappa_0} d(x, z)^{\kappa_0}.
\]
Hence this implies, for any \( y \in B(x, r) \)
\[
|L_2(x, y)| \lesssim \int_{d(x, z) > \frac{r}{2} \eta_0} \frac{|f(z)|}{V(x, z)} d(x, y)^{\sigma_0} d\mu(z) 
\lesssim \sum_{k=0}^{\infty} \frac{\rho^{\sigma_0}}{(2^k \eta_0)^{\sigma_0}} \mu(B(x, 2^k \eta_0 \Delta_0)) \int_{\frac{2^k \eta_0}{2^{k+1}} < d(x, z) \leq \frac{2^k \eta_0}{2^{k+1}}} |f(z)| d\mu(z) 
\lesssim \frac{\rho^{\sigma_0}}{\eta_0^{\sigma_0}} Mf(x).
\]

Using the estimates of \( L_1(x, y) \) and \( L_2(x, y) \), we have
\[
\left| [b, T_n] f(x) - (b, T_n) f(x, r) \right| \lesssim r^\beta T_{\nu}(f)(x) + \frac{\rho^{\sigma_0}}{\eta_0^{\sigma_0}} Mf(x).
\]

Then, using Lemma 4.3 and the boundedness of \( M \) on \( L^{p,\kappa}_0(X) \), we obtain
\[
||[b, T_n] f(x) - (b, T_n) f(x, r)||_{L^{p,\kappa}_0} \lesssim (r^\beta + \frac{\rho^{\sigma_0}}{\eta_0^{\sigma_0}})||f||_{L^{p,\kappa}_0}.
\]

Hence we observe that, \([b, T_n] F \) satisfies condition \((iii)\) of Lemma 4.1. So we have that, \([b, T_n] \) is a compact operator for any \( b \in \text{Lip}_{\nu}(\beta) \). This completes the proof of Theorem 1.3(i).

4.2 Proof of Theorem 1.3(ii).

Next, we establish a lemma for the upper and the lower bounds of integrals of \([b, T] f_j \) on certain balls \( B_j \) in \( X \) for any \( j \in \mathbb{N} \).

Lemma 4.4. Let \( p \in (1, \infty), \kappa \in (0, 1) \) and \( \omega \in A_p(X) \). Suppose that \( b \in \text{BMO}_p(X) \) is a real-valued function with \( ||b||_{\text{BMO}_p(X)} = 1 \) and there exists \( \gamma \in (0, \infty) \) and a sequence \( B_j \}_{j \in \mathbb{N}} := \{ B(x_j, r_j) \}_{j \in \mathbb{N}} \) of balls in \( X \), with \( \{ x_j \}_{j \in \mathbb{N}} \subset X \) and \( \{ r_j \}_{j \in \mathbb{N}} \subset (0, \infty) \) such that, for any \( j \in \mathbb{N} \)
\[
M(b, B_j) > \gamma.
\]

Then there exist real-valued functions \( \{ f_j \}_{j \in \mathbb{N}} \subset L^{p,\kappa}_0(X) \), positive constants \( \tilde{C}_0, \tilde{C}_1 \) and \( \tilde{C}_2 \) such that, for any \( j \in \mathbb{N} \) and integer \( k \geq K_0, ||f_j||_{L^{p,\kappa}_0(X)} \leq \tilde{C}_0, \)
\[
\int_{B^k_j} ||[b, T] f_j(x) ||^p \omega(x) d\mu(x) \geq \tilde{C}_1 \frac{\gamma^p \mu(B_j)^p}{\mu(A^k_2 B_j)^p} [\omega(B_j)]^{\kappa-1} \omega\left(A^k_2 B_j \right),
\]
where \( B^k_j := A_2^{k-1} B_j \) is the ball associates with \( A^{k-1}_2 B_j \) in (3.1) and
\[
\int_{A_2^{k+1} B_j \setminus A_2^k B_j} ||[b, T] f_j(x) ||^p \omega(x) d\mu(x) \leq \tilde{C}_2 \frac{\mu(B_j)^p}{\mu(A^k_2 B_j)^p} [\omega(B_j)]^{\kappa-1} \omega\left(A^k_2 B_j \right).
\]

Proof. For each \( j \in \mathbb{N} \), we define function \( f_j \) as follows:
\[
f^{(1)}_j := \chi_{B_{j,1}} - \chi_{B_{j,2}} := \chi_{\{ x \in B_j : \beta(x) > \alpha B_j(b) \}} - \chi_{\{ x \in B_j : \beta(x) < \alpha B_j(b) \}}, \quad f^{(2)}_j := a_j \chi_{B_j},
\]
and
\[
f_j := [\omega(B_j)]^{\frac{1}{p}} \left( f^{(1)}_j - f^{(2)}_j \right),
\]
where $B_j$ is as in the assumption of Lemma 4.4 and $a_j \in \mathbb{R}$ is a constant such that
\[
\int_X f_j(x) d\mu(x) = 0.
\tag{4.4}
\]
Then, using the definition of $a_j$, (2.1) and (2.2) we have $|a_j| \leq 1/2, \text{supp } (f_j) \subset B_j$ and, for any $x \in B_j$,
\[
f_j(x) (b(x) - \alpha_{B_j}(b)) \geq 0. \tag{4.5}
\]
Also, since $|a_j| \leq 1/2$, we obtain that, for any $x \in (B_{j,1} \cup B_{j,2})$,
\[
|f_j(x)| \sim [\omega (B_j)]^{\frac{p-1}{p}} \tag{4.6}
\]
and therefore
\[
\|f_j\|_{L^{p,\omega}_0(X)} \lesssim \sup_{B \subset X} \left\{ \frac{\omega (B \cap B_j)}{\omega (B)} \right\}^{\frac{1}{p}} [\omega (B_j)]^{\frac{k-1}{p}} \\
\lesssim \sup_{B \subset X} [\omega (B \cap B_j)]^{\frac{1-\alpha}{p}} [\omega (B_j)]^{\frac{k-1}{p}} \lesssim 1.
\]
Observe that, for any $k \in \mathbb{N}$, we get
\[
A_2^{k-1} B_j \subset (A_2 + 1) B_j^k \subset A_2^{k+1} B_j \tag{4.7}
\]
hence we have
\[
\omega (B_j^k) \sim \omega (A_2^k B_j) \tag{4.8}
\]
Observe that
\[
[b, T](f) = [b - \alpha_B(b)] T(f) - T([b - \alpha_B(b)] f). \tag{4.9}
\]
Using Kernel estimates, (4.4), (4.6) and the fact that $d(x, x_j) \sim d(x, \xi)$ for any $x \in B_j^k$ with integer $k \geq 2$ and $\xi \in B_j$, we have, for any $x \in B_j^k$,
\[
| [b(x) - \alpha_{B_j}(b)] T (f_j) (x) | = | b(x) - \alpha_{B_j}(b) | \left| \int_{B_j} [K(x, \xi) - K (x, x_j)] f_j(\xi) d\mu(\xi) \right| \tag{4.10}
\]
\[
\leq | b(x) - \alpha_{B_j}(b) | \int_{B_j} |K(x, \xi) - K (x, x_j)| | f_j(\xi) | d\mu(\xi) \tag{4.10}
\]
\[
\lesssim [\omega (B_j)]^{\frac{p-1}{p}} | b(x) - \alpha_{B_j}(b) | \int_{B_j} \frac{1}{V(x, x_j)} \left( \frac{d(x, \xi)}{d(x, x_j)} \right)^{\sigma_0} d\mu(\xi) \tag{4.10}
\]
\[
\lesssim \frac{[\omega (B_j)]^{\frac{p-1}{p}} \mu(B_j)}{A_2^{k\sigma_0}} [b(x) - \alpha_{B_j}(b)].
\]
As $\|b\|_{\text{BMO}(X)} = 1$ by John-Nirenberg inequality (c.f. [6]), for each $k \in \mathbb{N}$ and ball $B \subset X$, we have
\[
\int_{A_2^{k+1} B} |b(x) - \alpha_B(b)|^p d\mu(x) \lesssim \int_{A_2^{k+1} B} |b(x) - \alpha_{A_2^{k+1} B}(b)|^p d\mu(x) + \mu(A_2^{k+1} B) \left| \alpha_{A_2^{k+1} B}(b) - \alpha_B(b) \right|^p
\]
\[
\lesssim k^p \mu(A_2^k B), \tag{4.11}
\]
where the last inequality follows from the fact that
\[ |\alpha_{A_2^{k+1}B}(b) - \alpha_B(b)| \lesssim |\alpha_{A_2^{k+1}B}(b) - b_{A_2^{k+1}B}| + |b_{A_2^{k+1}B} - B| + |B - \alpha_B(b)| \lesssim k. \]
As \( \omega \in A_p(X) \), we observe that there exists \( \varepsilon \in (0, \infty) \) such that the reverse Hölder inequality
\[
\left( \frac{1}{\mu(B)} \int_B \omega(x)^{1+\varepsilon} d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \lesssim \frac{1}{\mu(B)} \int_B \omega(x) d\mu(x)
\]
holds for any ball \( B \subset X \). Then using the Hölder inequality, (4.11), (4.7) and (4.10) we can obtain a positive constant \( \tilde{C}_3 \) such that, for any \( k \in \mathbb{N} \)
\[
\int_{B_j^k} \left| \left| b(x) - \alpha_{B_j}(b) \right| T(f_j)(x) \right|^p \omega(x) d\mu(x) \tag{4.12}
\]
\[ \lesssim \left[ \frac{\omega(B_j)}{A_2^k} \right]^{\kappa-1} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p} \int_{A_2^{k+1}B_j} \left| b(x) - \alpha_{B_j}(b) \right|^p \omega(x) d\mu(x) \]
\[ \lesssim \left[ \frac{\omega(B_j)}{A_2^k} \right]^{\kappa-1} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p} \left( \frac{1}{\mu(A_2^{k+1}B_j)} \int_{A_2^{k+1}B_j} \left| b(x) - \alpha_{B_j}(b) \right|^{p(1+\varepsilon)} d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \]
\[ \times \left( \frac{1}{\mu(A_2^{k+1}B_j)} \int_{A_2^{k+1}B_j} \omega(x)^{1+\varepsilon} d\mu(x) \right)^{\frac{1}{1+\varepsilon}} \]
\[ \leq \tilde{C}_3 \frac{k^p}{A_2^k} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p} \left[ \omega(B_j) \right]^{\kappa-1} \omega \left( A_2^k B_j \right). \]
Using Lemma 4.1, (4.5), (4.6), (2.3), (4.1) and (1.6) for any \( x \in B_j^k \), we get that
\[
|T \left( \left[ b - \alpha_{B_j}(b) \right] f_j \right)(x)| = \left| \int_{B_{j,1} \cup B_{j,2}} K(x, \xi) \left[ b(\xi) - \alpha_{B_j}(b) \right] f_j(\xi) d\xi \right|
\]
\[ \lesssim \left| \int_{B_{j,1} \cup B_{j,2}} \frac{\left[ b(\xi) - \alpha_{B_j}(b) \right] f_j(\xi)}{\mu(B(x, d(x, \xi)))} d\mu(\xi) \right|
\]
\[ \lesssim \frac{1}{\mu(A_2^k B_j)} \left[ \omega(B_j) \right]^{\frac{1}{p-1}} \left| \int_{B_j} |b(\xi) - \alpha_{B_j}(b)| d\mu(\xi) \right|
\]
\[ \lesssim \frac{\gamma \mu(B_j)}{\mu(A_2^k B_j)} \left[ \omega(B_j) \right]^{\frac{1}{p-1}}. \]
Along with (4.8) we deduce that there exists a positive constant \( \tilde{C}_4 \) such that
\[
\int_{B_j^k} \left| T \left( \left[ b - \alpha_{B_j}(b) \right] f_j \right)(x) \right|^p \omega(x) d\mu(x) \geq \frac{\gamma^p \mu(B_j)^p}{\mu(A_2^k B_j)^p} \left[ \omega(B_j) \right]^{\kappa-1} \omega \left( B_j^k \right) \tag{4.13}
\]
\[ \geq \tilde{C}_4 \frac{\gamma^p \mu(B_j)^p}{\mu(A_2^k B_j)^p} \left[ \omega(B_j) \right]^{\kappa-1} \omega \left( A_2^k B_j \right). \]
Now let us take \( K_0 \in (0, \infty) \) large enough such that, for any integer \( k \geq K_0 \)
\[ \tilde{C}_4 \frac{\gamma^p}{2p-1} - \tilde{C}_3 \frac{k^p}{A_2^k} \geq \tilde{C}_4 \frac{\gamma^p}{2p}. \]
Using this and (4.9), (4.12) and (4.13), we get

\[
\int_{B_j^k} \|b, T[f_j(x)]^p \omega(x)\,d\mu(x) \\
\geq \frac{1}{2^{p-1}} \int_{B_j^k} |T\left([b - \alpha_{B_j}(b)]f_j(x)\right)|^p \omega(x)\,d\mu(x) - \int_{B_j^k} \left|\left[b(x) - \alpha_{B_j}(b)\right] T(f_j(x))\right|^p \omega(x)\,d\mu(x) \\
\geq \left(\tilde{C}_4 \frac{\gamma^p}{2^{p-1}} - \tilde{C}_5 \frac{k^p}{A_2^{k_0 p}}\right) \frac{\mu(B_j)^p}{\mu(A_2^{k_0} B_j)^p} [\omega(B_j)]^{\kappa-1} \omega(A_2^{k_0} B_j) \\
\geq \tilde{C}_4 \frac{\gamma^p}{2^{p}} \frac{\mu(B_j)^p}{\mu(A_2^{k_0} B_j)^p} [\omega(B_j)]^{\kappa-1} \omega(A_2^{k_0} B_j).
\]

This implies (4.2).

Also, since \(\text{supp} (f_j) \subset B_j\), by (4.6) and (2.3) and \(\|b\|_{\text{BMO}(X)} = 1\), we deduce that, for any \(x \in A_2^{k+1} B_j \setminus A_2^k B_j\)

\[
|T\left(\left[b - \alpha_{B_j}(b)\right]f_j(x)\right)| \lesssim [\omega(B_j)]^{\frac{1}{p}} \int_{B_j} \left|\frac{b(\xi) - \alpha_{B_j}(b)}{V(x, \xi)}\right| \,d\mu(\xi) \lesssim [\omega(B_j)]^{\frac{1}{p}} \frac{\mu(B_j)}{\mu(A_2^{k_0} B_j)}.
\]

Therefore, by (4.12) with \(B_j^k\) replaced by \(A_2^{k+1} B_j \setminus A_2^k B_j\), we can deduce that, for any integer \(k \geq K_0\)

\[
\int_{A_2^{k+1} B_j \setminus A_2^k B_j} \|b, T[f_j(x)]^p \omega(x)\,d\mu(x) \\
\lesssim \int_{A_2^{k+1} B_j \setminus A_2^k B_j} |T\left(\left[b - \alpha_{B_j}(b)\right]f_j(x)\right)|^p \omega(x)\,d\mu(x) \\
+ \int_{A_2^{k+1} B_j \setminus A_2^k B_j} \left|\left[b(x) - \alpha_{B_j}(b)\right] T(f_j(x))\right|^p \omega(x)\,d\mu(x) \\
\lesssim \frac{\mu(B_j)^p}{\mu(A_2^{k_0} B_j)^p} [\omega(B_j)]^{\kappa-1} \omega(A_2^{k_0} B_j) + \frac{k^p}{A_2^{k_0 p}} \frac{\mu(B_j)^p}{\mu(A_2^{k_0} B_j)^p} [\omega(B_j)]^{\kappa-1} \omega(A_2^{k_0} B_j) \\
\lesssim \frac{\mu(B_j)^p}{\mu(A_2^{k_0} B_j)^p} [\omega(B_j)]^{\kappa-1} \omega(A_2^{k_0} B_j).
\]

This completes the proof of Lemma 4.4. □

The following technical result are needed to handle the weighted estimate for showing the necessity of the compactness of the commutators.

**Lemma 4.5.** Let \(1 < p < \infty, 0 < \kappa < 1, \omega \in A_p(X), b \in \text{BMO}(X), \gamma, K_0 > 0, \{f_j\}_{j \in \mathbb{N}}\) and \(\{B_j\}_{j \in \mathbb{N}}\) be as given in Lemma 4.4. Now assume that \(\{B_j\}_{j \in \mathbb{N}} := \{B(x_j, r_j)\}_{j \in \mathbb{N}}\) also satisfies the following two conditions:

(i) \(\forall \ell, m \in \mathbb{N} \text{ and } \ell \neq m\)

\[
A_2 C_1 B_1 \cap A_2 C_1 B_m = \emptyset, \tag{4.14}
\]

where \(C_1 := A_2^{K_1} > C_2 := A_2^{K_0}\) for some \(K_1 \in \mathbb{N}\) large enough.
(ii) \( \{ r_j \}_{j \in \mathbb{N}} \) is either non-increasing or non-decreasing in \( j \), or there exist positive constants \( C_{\min} \) and \( C_{\max} \) such that, for any \( j \in \mathbb{N} \)

\[
C_{\min} \leq r_j \leq C_{\max}.
\]

Then there exists a positive constant \( C \) such that, for any \( j, m \in \mathbb{N} \)

\[
\| [b, T] f_j - [b, T] f_{j+m} \|_{L^p(X)} \geq C.
\]

**Proof.** Without loss of generality, we assume that \( \| b \|_{BMO(X)} = 1 \) and \( \{ r_j \}_{j \in \mathbb{N}} \) is non-increasing. Let \( \{ f_j \}_{j \in \mathbb{N}}, \tilde{C}_1, \tilde{C}_2 \) be as in Lemma 4.4 associated with \( \{ B_j \}_{j \in \mathbb{N}} \).

By (4.2), (4.8), Lemma 2.6 with \( \omega \in A_p(X) \), we observe that, for any \( j \in \mathbb{N} \),

\[
\begin{align*}
\left[ \int_{A_2^{K_0} B_j} |[b, T] f_j(x)|^p \omega(x) d\mu(x) \right]^{1/p} & \geq \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{-\kappa/p} \left\{ \int_{B_2^{K_0-1}} |[b, T] f_j(x)|^p \omega(x) d\mu(x) \right\}^{1/p} \\
& \geq \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{-\kappa/p} \left\{ \tilde{C}_1 \gamma^p \frac{\mu(B_j)^p}{\mu(A_2^{K_0-1} B_j)^p} \left[ \omega (B_j) \right]^{\kappa-1} \omega \left( A_2^{K_0-1} B_j \right) \right\}^{1/p} \\
& \geq \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{-\kappa/p} \left\{ \gamma^p \frac{\left[ \omega (B_j) \right]^\kappa}{A_2^{\kappa p (K_0-1)}} \right\}^{1/p} \\
& \geq C_3 \gamma A_2^{-n(k K_0 + K_0 - 1)} \left[ \omega (B_j) \right]^{-\kappa/p} \left[ \omega (B_j) \right]^{\kappa/p} \\
& = C_3 \gamma A_2^{-n(k K_0 + K_0 - 1)}
\end{align*}
\]

holds for a positive constant \( C_3 \) independent of \( \gamma \) and \( A_2 \). We also show that, for any \( j, m \in \mathbb{N} \),

\[
\begin{align*}
\left[ \int_{A_2^{K_0} B_j} |[b, T] f_{j+m}(x)|^p \omega(x) d\mu(x) \right]^{1/p} & \geq \left[ \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{-\kappa/p} \frac{1}{2} \right] \leq \frac{1}{2} C_3 \gamma A_2^{-n(k K_0 + K_0 - 1)}.
\end{align*}
\]

As \( \text{supp} \ (f_{j+m}) \subset B_{j+m} \), from (2.3), (4.6), (4.14) and \( \| b \|_{BMO(X)} = 1 \), it follows that, for any \( x \in A_2^{K_0} B_j \)

\[
|T \left( [b - \alpha_{B_{j+m}}(b)] f_{j+m} \right) (x)| \lesssim \left[ \omega (B_{j+m}) \right]^{\frac{1}{p}} \int_{B_{j+m}} |K(x, \xi) | |b(x) - \alpha_{B_{j+m}}(b)| \ d\mu(\xi)
\]

\[
\lesssim \left[ \omega (B_{j+m}) \right]^{\frac{1}{p}} \frac{\mu (B_{j+m})}{V(x_j, x_{j+m})}.
\]

So we have

\[
\begin{align*}
\left\{ \int_{A_2^{K_0} B_j} |T \left( [b - \alpha_{B_{j+m}}(b)] f_{j+m} \right) (x)|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}} & \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{\frac{\kappa-1}{p}} \\
& \lesssim \left[ \omega (B_{j+m}) \right]^{\frac{\kappa-1}{p}} \frac{\mu (B_{j+m})}{V(x_j, x_{j+m})} \left[ \omega \left( A_2^{K_0} B_j \right) \right]^{\frac{\kappa-1}{p}} .
\end{align*}
\]
Also, using (4.6) we deduce that, for any $x \in A^{K_0}_2 B_j$

$$|T(f_{j+m})(x)| \leq \int_{B_{j+m}} |K(x, \xi) - K(x, x_{j+m})| |f_{j+m}(\xi)| \, d\mu(\xi)$$

$$\lesssim [\omega(B_{j+m})]^{n-1/p} \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} r^{\sigma_0}_{j+m} \left( \int_{B_{j+m}} |(b(x) - \alpha_{B_{j+m}}(b)) T(f_{j+m})(x)|^p \omega(x) \, d\mu(x) \right)^{1/p} [\omega(A^{K_0}_2 B_j)]^{-\kappa/p}. \quad (4.18)$$

Hence, using (4.18) and the fact $\{r_j\}_{j \in \mathbb{N}}$ is non-increasing in $j$ and from Hölder and reverse Hölder inequalities we get that

$$\begin{align*}
\left\{ \int_{A^{K_0}_2 B_j} \left[ |b(x) - \alpha_{B_{j+m}}(b)| T(f_{j+m})(x) \right|^p \omega(x) \, d\mu(x) \right\}^{1/p} &\lesssim [\omega(B_{j+m})]^{n-1/p} \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} r^{\sigma_0}_{j+m} \left( \int_{B_{j+m}} |(b(x) - \alpha_{B_{j+m}}(b))| \omega(x) \, d\mu(x) \right)^{1/p} [\omega(A^{K_0}_2 B_j)]^{-\kappa/p} \\
&\times \left[ \int_{A^{K_0}_2 B_j} \left| (b(x) - \alpha_{B_{j+m}}(b)) \right|^p \omega(x) \, d\mu(x) \right]^{1/p} \\
&\lesssim [\omega(B_{j+m})]^{n-1/p} \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} r^{\sigma_0}_{j+m} \left( \int_{B_{j+m}} \omega(x) \, d\mu(x) \right)^{1/p} \left( \frac{d(x_j, x_{j+m})}{r_{j+m}} \right)^{\frac{\kappa}{\sigma_0}} \log \frac{d(x_j, x_{j+m})}{r_{j+m}}. 
\end{align*}$$

Observe that, for $C_1$ large enough, using (4.14) we know that $d(x_j, x_{j+m})$ is also large enough and so we have

$$\left( \frac{d(x_j, x_{j+m})}{r_{j+m}} \right)^{-\sigma_0} \log \frac{d(x_j, x_{j+m})}{r_{j+m}} \lesssim 1. \quad (4.19)$$

Using (4.17), (4.18) and (4.19), we obtain that

$$\begin{align*}
\left\{ \int_{A^{K_0}_2 B_j} \left[ |b, T| (f_{j+m})(x) \right|^p \omega(x) \, d\mu(x) \right\}^{1/p} &\lesssim [\omega(A^{K_0}_2 B_j)]^{-\kappa/p} \\
&\leq \left\{ \int_{A^{K_0}_2 B_j} \left[ |(b - \alpha_{B_{j+m}}(b)) f_{j+m}(x) \right|^p \omega(x) \, d\mu(x) \right\}^{1/p} [\omega(A^{K_0}_2 B_j)]^{-\kappa/p} \\
&\quad + \left\{ \int_{A^{K_0}_2 B_j} \left[ |b(x) - \alpha_{B_{j+m}}(b)| T(f_{j+m})(x) \right|^p \omega(x) \, d\mu(x) \right\}^{1/p} [\omega(A^{K_0}_2 B_j)]^{-\kappa/p} \\
&\lesssim [\omega(B_{j+m})]^{n-1/p} \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} \left[ \frac{\omega(B(x_j, d(x_j, x_{j+m})))}{\omega(B_{j+m})} \right]^{1/p} \\
&\lesssim [\omega(B_j)]^{n-1/p} \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} \left[ \frac{\omega(B(x_j, d(x_j, x_{j+m})))}{\omega(B_{j+m})} \right]^{1/p} \\
&\leq C^* \left[ \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} \right]^\kappa.
\end{align*}$$
Note that $\lim_{k \to \infty} \mu(A^k_2B_{j+m}) = \infty$. Then for $C_1$ large enough, we have

$$\mu(C_1B_{j+m}) \geq \left( \frac{2C'}{C_3\gamma A^*_2 - n(\kappa K_0 + K_0 - 1)} \right)^{\frac{1}{\kappa}} \mu(B_{j+m}).$$

This implies that $C' \left( \frac{\mu(B_{j+m})}{V(x_j, x_{j+m})} \right)^{\kappa} \leq C' \left( \frac{\mu(B_{j+m})}{\mu(C_1B_{j+m})} \right)^{\kappa} \leq \frac{1}{2} C_3 \gamma A^*_2 - n(\kappa K_0 + K_0 - 1)$. This gives the proof of (4.16). Using (1.6) and (4.16) we know that, for any $j, m \in \mathbb{N}$ and $C_1$ large enough

$$\left\{ \int_{A^*_2B_j} \|b, T\(f_j\)(x) - [b, T](f_{j+m})(x)\|^{\frac{p}{\kappa}} \omega(x) d\mu(x) \right\}^{1/p} \left[ \omega\left(A^*_2B_j \right) \right]^{-\frac{\kappa}{p}}$$

$$\geq \left\{ \int_{A^*_2B_j} \|b, T\(f_j\)(x)\|^{\frac{p}{\kappa}} \omega(x) d\mu(x) \right\}^{1/p} \left[ \omega\left(A^*_2B_j \right) \right]^{-\frac{\kappa}{p}}$$

$$- \left\{ \int_{A^*_2B_j} \|b, T\(f_{j+m}\)(x)\|^{\frac{p}{\kappa}} \omega(x) d\mu(x) \right\}^{1/p} \left[ \omega\left(A^*_2B_j \right) \right]^{-\frac{\kappa}{p}} \geq \frac{1}{2} C_3 \gamma A^*_2 - n(\kappa K_0 + K_0 - 1).$$

This completes the proof of Lemma 4.7.

**Proof of Theorem 1.3(ii).** Without loss of generality, we assume that $\|b\|_{\text{VMO}(X)} = 1$. To prove $b \in \text{VMO}(X)$, observe that $b \in \text{BMO}(X)$ is a real-valued function, we will use a contradiction argument via Lemmas 2.4, 4.4 and 4.5. Now note that, if $b \notin \text{VMO}(X)$, then $b$ does not satisfy at least one of (i) through (iii) of Lemma 2.4. We show that $[b, T]$ is not compact on $L^{p,\kappa}(X)$ in any of the following three cases.

**Case (i)** $b$ does not satisfy condition (i) Lemma 2.4. Hence there exist $\gamma \in (0, \infty)$ and a sequence

$$\left\{ B_j^{(1)} \right\}_{j \in \mathbb{N}} := \left\{ B(x_j^{(1)}, r_j^{(1)}) \right\}_{j \in \mathbb{N}}$$

of balls in $X$ satisfying (4.1) and that $r_j^{(1)} \to 0$ as $j \to \infty$. Let $x_0$ be a fixed point in $X$. We now consider the following two subcases.

**Subcase (i)** There exists a positive constant $M$ such that $0 \leq d(x_0, x_j^{(1)}) < M$ for all $x_j^{(1)}, j \in \mathbb{N}$. That is, $x_j^{(1)} \in B_0 := B(x_0, M), \forall j \in \mathbb{N}$. Let $\{f_j\}_{j \in \mathbb{N}}$ be associated with the sequence $B_j$ in Lemmas 4.4 and 4.5. Let $p_0 \in (1, p)$ be such that $\omega \in A^*_{p_0}(X)$ and $C_4 := A^*_2 > C_2 = A^*_{K_0}$ for $K_2 \in \mathbb{N}$ large enough so that

$$C_5 := \frac{\tilde{C}_1 \tilde{C}_2 \gamma p}{C_\mu} A^{nK_0} > \frac{\tilde{C}_2}{C_1} A^{K_2(p_0 - p)} > \frac{\tilde{C}_2}{C_1} A^{K_2(p_0 - p) - 1}.$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are as in Lemma 2.6. As we know $\left| x_j^{(1)} \right| \to 0$ as $j \to \infty$ and $\left\{ x_j^{(1)} \right\}_{j \in \mathbb{N}} \subset B_0$, we choose a subsequence $\left\{ B_{j_{\ell}}^{(1)} \right\}_{\ell \in \mathbb{N}}$ of $\left\{ B_j^{(1)} \right\}_{j \in \mathbb{N}}$ so that, for any $j \in \mathbb{N}$,

$$\frac{\mu\left( B_{j_{\ell}+1}^{(1)} \right)}{\mu\left( B_{j_{\ell}}^{(1)} \right)} < \frac{1}{C_4}$$

and $\omega\left( B_{j_{\ell}+1}^{(1)} \right) \leq \omega\left( B_{j_{\ell}}^{(1)} \right)$. (4.21)
Define for any fixed \( \ell, m \in \mathbb{N} \)

\[
\mathcal{J} := C_4 B_{j_\ell}^{(1)} \setminus C_2 B_{j_\ell}^{(1)}, \quad \mathcal{J}_1 := \mathcal{J} \setminus C_4 B_{j_{\ell+m}}^{(1)} \quad \text{and} \quad \mathcal{J}_2 := X \setminus C_4 B_{j_{\ell+m}}^{(1)}.
\]

Observe that

\[
\mathcal{J}_1 \subset \left( C_4 B_{j_\ell}^{(1)} \right) \cap \mathcal{J}_2 \quad \text{and} \quad \mathcal{J}_1 = \mathcal{J} \cap \mathcal{J}_2.
\]

Hence we have

\[
\left\{ \int_{C_4 B_{j_\ell}^{(1)}} \left[ [b,T] (f_{j_\ell})(x) - [b,T] (f_{j_{\ell+m}}) (x) \right]^p \omega(x) d\mu(x) \right\}^{1/p} \geq \left\{ \int_{\mathcal{J}_1} \left[ [b,T] (f_{j_\ell})(x) - [b,T] (f_{j_{\ell+m}}) (x) \right]^p \omega(x) d\mu(x) \right\}^{1/p} \geq \left\{ \int_{\mathcal{J}_2} \left[ [b,T] (f_{j_\ell})(x) \right]^p \omega(x) d\mu(x) \right\}^{1/p} - \left\{ \int_{\mathcal{J}_2} \left[ [b,T] (f_{j_{\ell+m}}) (x) \right]^p \omega(x) d\mu(x) \right\}^{1/p}
\]

\[
=: F_1 - F_2.
\]

First we consider the term \( F_1 \) and assume that \( E_{j_\ell} := \mathcal{J} \setminus \mathcal{J}_2 \neq \emptyset \). Then \( E_{j_\ell} \subset C_4 B_{j_{\ell+m}}^{(1)} \) by (4.21) we have

\[
\mu(E_{j_\ell}) \leq C_4^0 \mu \left( B_{j_{\ell+m}}^{(1)} \right) < \mu \left( B_{j_\ell}^{(1)} \right).
\]

Now take

\[
B_{j_{\ell,k}}^{(1)} := A_2^{k-1} B_{j_\ell}^{(1)},
\]

to be the ball associates with \( A_2^{k-1} B_{j_\ell}^{(1)} \) in (3.1). Now using (4.23), we get

\[
\mu \left( B_{j_{\ell,k}}^{(1)} \right) = \mu \left( A_2^{k-1} B_{j_\ell}^{(1)} \right) > \mu(E_{j_\ell}).
\]

Using this, we further know that there exist finite \( \left\{ B_{j_{\ell,k}}^{(1)} \right\}_{k=K_0}^{K_2-2} \) intersecting \( E_{j_\ell} \). Then, from (4.2) and Lemma 2.6, we deduce that

\[
F_1^p \geq \sum_{k=K_0, B_{j_{\ell,k}}^{(1)} \cap E_{j_\ell} = \emptyset}^{K_2-2} \mu \left( B_{j_{\ell,k}}^{(1)} \right)^p \mu \left( A_2^{k-1} B_{j_\ell}^{(1)} \right)^{\kappa - 1} \mu \left( A_2^{k} B_{j_\ell}^{(1)} \right)^{\kappa} \omega \left( A_2^{k} B_{j_\ell}^{(1)} \right) \quad \text{and} \quad F_2^p \geq \sum_{k=K_0, B_{j_{\ell,k}}^{(1)} \cap E_{j_\ell} = \emptyset}^{K_2-2} \tilde{C}_1 \gamma^p \mu \left( B_{j_{\ell,k}}^{(1)} \right)^p / \mu \left( A_2^{k} B_{j_\ell}^{(1)} \right)^{\kappa} \omega \left( A_2^{k} B_{j_\ell}^{(1)} \right) \omega \left( B_{j_\ell}^{(1)} \right)^{\kappa}.
\]
\[ \frac{\tilde{C}_1 \tilde{C}_2 \gamma^p}{C_\mu} A_2^{\mu_{K_0(n-\mu)}} \omega(B_{j\ell})^\kappa = C_5 \omega(B_{j\ell})^\kappa. \]

If \( E_j := J \setminus J_2 = \emptyset \), the inequality is still holds true.

Observe that \( \lim_{k \to \infty} \mu(A^k_{j_k} B_{j_k}) = \infty \). Then there exist \( j_k \in \mathbb{N} \) such that

\[ \mu(A^k_{j_k} B_{j_k}) \geq A^2_{K_2} \mu(A^k_{j_k} B_{j_k}) \text{ and } \mu(A^{k+1}_{j_k} B_{j_k}) \geq A^2_{K_2} \mu(A^{k+1}_{j_k} B_{j_k}). \]

Also, using the proof of (4.3), Lemma 4.4, (4.20) and (4.21), we obtain that

\[ F^p_2 \leq \sum_{k=0}^{\infty} \int A^{k+1}_{j_k} B_{j_k} \setminus A^{k}_{j_k} B_{j_k} \omega(x) d\mu(x) \]

\[ \leq \tilde{C}_2 \sum_{k=0}^{\infty} \mu(A^{k}_{j_k} B_{j_k})^p \omega(B_{j_k}) - \omega(A^{k}_{j_k} B_{j_k}) \]

\[ \leq \tilde{C}_2 \sum_{k=0}^{\infty} \frac{\mu(A^{k}_{j_k} B_{j_k})^p}{\mu(A^{k+1}_{j_k} B_{j_k})^p} \omega(B_{j_k}) - \omega(A^{k}_{j_k} B_{j_k}) \]

By (4.21), (4.22), (4.24) and (4.25) we deduce

\[ \left\{ \int C_{1} B_{j\ell}^{(1)} \omega(x) d\mu(x) \right\}^{1/p} \]

\[ \geq C_5^{1/p} \left\{ \omega(B_{j\ell})^{1/p} - \left( \frac{C_5}{\tilde{C}_1} \right)^{1/p} \omega(B_{j\ell})^{1/p} \right\} \geq C_5^{1/p} \omega(B_{j\ell}). \]

Hence we get, \( \{ [b, T] f_j \}_{j \in \mathbb{N}} \) is not relatively compact in \( L^{p,\kappa}_\omega(X) \), which implies that \( [b, T] \) is not compact on \( L^{p,\kappa}_\omega(X) \). So, \( b \) satisfies condition (i) of Lemma 2.4.

**Subcase** (ii) There exists a subsequence \( \{ B_{j\ell}^{(1)} \}_{\ell \in \mathbb{N}} := \{ B(x_{j\ell}^{(1)}, r_{j\ell}^{(1)}) \}_{\ell \in \mathbb{N}} \) of \( \{ B_{j\ell} \}_{j \in \mathbb{N}} \) such that \( d(x_0, x_{j\ell}^{(1)}) \to \infty \) as \( \ell \to \infty \). In this subcase, by \( \mu(B_{j\ell}^{(1)}) \to 0 \) as \( \ell \to \infty \), we take a mutually disjoint subsequence of \( \{ B_{j\ell}^{(1)} \}_{\ell \in \mathbb{N}} \) and denote by \( \{ B_{j\ell}^{(1)} \}_{\ell \in \mathbb{N}} \) satisfying (4.14) as well.

This, via Lemma 4.5 implies that \( [b, T] \) is not compact on \( L^{p,\kappa}_\omega(X) \), which is a contradiction to our assumption. Hence, \( b \) satisfies condition (i) of Lemma 2.4.

**Case** (ii) If \( b \) does not satisfy condition (ii) of Lemma 2.4. In this case, there exist \( \gamma \in (0, \infty) \) and a sequence \( \{ B_{j\ell}^{(2)} \}_{j \in \mathbb{N}} \) of balls in \( X \) satisfying (4.1) and that \( \{ r_{j\ell}^{(2)} \to \infty \) as \( j \to \infty \). We also consider the following two subcases as well.
Subcase (i) There exists an infinite subsequence \( \left\{ B_{j\ell}^{(2)} \right\}_{\ell \in \mathbb{N}} \) of \( \left\{ B_j^{(2)} \right\}_{j \in \mathbb{N}} \) and a point \( x_0 \in X \) such that, for any \( \ell \in \mathbb{N}, x_0 \in A_2 C_1 B_{j\ell}^{(2)} \). As \( |r_{B_j^{(2)}}| \to \infty \) as \( \ell \to \infty \), it follows that there exists a subsequence, denoted as earlier by \( \left\{ B_{j\ell}^{(2)} \right\}_{\ell \in \mathbb{N}} \), such that, for any \( \ell \in \mathbb{N} \)

\[
\frac{\mu(B_{j\ell}^{(2)})}{\mu(B_{j\ell+1}^{(2)})} < \frac{1}{C_4}.
\]  

(4.26)

Note that \( 2A_2 C_1 B_{j\ell}^{(2)} \subset 2A_2 C_1 B_{j\ell+1}^{(2)} \) for any \( j, \ell \in \mathbb{N} \) and hence

\[
\omega\left(2A_2 C_1 B_{j\ell+1}^{(2)}\right) \geq \omega\left(2A_2 C_1 B_{j\ell}^{(2)}\right), \quad M(b, 2A_2 C_1 B_{j\ell}) > \frac{\gamma}{8A_2 C_1^2}. \]  

(4.27)

Using similar method as that used in Subcase (i) of Case (i) and we redefine our sets in a reversed order. That is, for any fixed \( \ell, k \in \mathbb{N} \), we let

\[
\tilde{J} := 2A_2 C_4 C_1 B_{\ell+k}^{(2)} \setminus 2A_2 C_2 C_1 B_{\ell+k}^{(2)};
\]

\[
\tilde{J}_1 := \tilde{J} \setminus 2A_2 C_4 C_1 B_{\ell}^{(2)};
\]

\[
\tilde{J}_2 := X \setminus 2A_2 C_4 C_1 B_{\ell}^{(2)}.
\]

As in Case (i), by Lemma 4.4, (4.26) and (4.27), we deduce that the commutator \([b, T]\) is not compact on \( L^{\infty}_0(X) \). This contradiction gives that \( b \) satisfies condition (ii) of Lemma 4.4.

Subcase (ii) For any \( z \in X \) the number of \( \left\{ A_2 C_1 B_{j\ell}^{(2)} \right\}_{j \in \mathbb{N}} \) containing \( z \) is finite. In this subcase, for each square \( B_{j0}^{(2)} \in \left\{ B_j^{(2)} \right\}_{j \in \mathbb{N}} \), the number of \( \left\{ A_2 C_1 B_{j\ell}^{(2)} \right\}_{j \in \mathbb{N}} \) intersecting \( A_2 C_1 B_{j0}^{(2)} \) is finite. Then we take a mutually disjoint subsequence \( \left\{ B_{j\ell}^{(2)} \right\}_{\ell \in \mathbb{N}} \) satisfying (4.1) and (4.14). From Lemma 4.5, we can deduce that \([b, T]\) is not compact on \( L^{\infty}_0(X) \). Thus, \( b \) satisfies condition (ii) of Lemma 2.4.

Case (iii) Condition (iii) of Lemma 2.4 does not hold for \( b \). Then there exists \( \gamma > 0 \) such that for any \( r > 0 \), there exists \( B \subset X \setminus B(x_0, r) \) with \( M(b, B) > \gamma \). As in [4] for the \( \gamma \) above, there exists a sequence \( \left\{ B_{j}^{(3)} \right\}_{j} \) of balls such that for any \( j \),

\[
M(b, B_{j}^{(3)}) > \gamma, \quad (4.28)
\]

and for any \( i \neq m \),

\[
\gamma_1 B_{i}^{(3)} \cap \gamma_1 B_{m}^{(3)} = \emptyset, \quad (4.29)
\]

for sufficiently large \( \gamma_1 \) since, by Case (i) and (ii), \( \left\{ B_{j}^{(3)} \right\}_{j \in \mathbb{N}} \) satisfies the conditions (i) and (ii) of Lemma 2.4, it follows that there exist positive constants \( C_{\text{min}} \) and \( C_{\text{max}} \) such that

\[
C_{\text{min}} \leq r_j \leq C_{\text{max}}, \quad \forall j \in \mathbb{N}.
\]

Using this and Lemma 4.5 we deduce that, if \([b, T]\) is compact on \( L^{\infty}_0(X) \), then \( b \) also satisfies condition (iii) of Lemma 2.4. This completes the proof of Theorem 1.3(ii) and hence of Theorem 1.3.
5 Appendix: characterisation of VMO(X)

In this section, we provide the characterisation of VMO space on X by giving the proof of Lemma 2.4.

Proof of Lemma 2.4. In the following, for any integer m, we use $B^m$ to denote the ball $B(x_0, 2^m)$, where $x_0$ is a fixed point in X.

Necessary condition: Assume that $f \in \text{VMO}(X)$. If $f \in \text{Lip}_x(\beta)$, then (i)-(iii) hold. In fact, by the uniform continuity, $f$ satisfies (i). Since $f \in L^1(X)$, $f$ satisfies (ii). By the fact that $f$ is compactly supported, $f$ satisfies (iii). If $f \in \text{VMO}(X) \setminus \text{Lip}_x(\beta)$, by definition, for any given $\varepsilon > 0$, there exists $f_\varepsilon \in \text{Lip}_x(\beta)$ such that $\|f - f_\varepsilon\|_{\text{BMO}(X)} < \varepsilon$. Since $f_\varepsilon$ satisfies (i)-(iii), by the triangle inequality of BMO(X) norm, we can see (i)-(iii) hold for $f$.

Sufficient condition: In this proof for $j = 1, 2, \ldots, 8$, the value $\alpha_j$ is a positive constant depending only on $n$ and $\epsilon_i$ for $1 \leq i < j$. Assume that $f \in \text{BMO}(X)$ and satisfies (i)-(iii). To prove that $f \in \text{VMO}(X)$, it suffices to show that there exist positive constants $\alpha_1, \alpha_2$ such that, for any $\varepsilon > 0$, there exists $\phi_\varepsilon \in \text{BMO}(X)$ satisfying

$$\inf_{h \in \text{Lip}_x(\beta)} \|\phi_\varepsilon - h\|_{\text{BMO}(X)} < \alpha_1 \varepsilon,$$

(5.1)

and

$$\|\phi_\varepsilon - f\|_{\text{BMO}(X)} < \alpha_2 \varepsilon.$$

(5.2)

By (i), there exist $i_\varepsilon \in \mathbb{N}$ such that

$$\sup \{M(f, B) : r_B \leq 2^{-i_\varepsilon + 4}\} < \varepsilon.$$

(5.3)

By (iii), there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\sup \{M(f, B) : B \cap B^{j_\varepsilon} = \emptyset\} < \varepsilon.$$

(5.4)

We first establish a cover of $X$. Observe that

$$B^{j_\varepsilon} = B^{-i_\varepsilon} \bigcup \left( \bigcup_{\nu = 1}^{2^j - i_\varepsilon - 1} B \left( x_0, (\nu + 1)2^{-i_\varepsilon} \right) \right) \setminus B \left( x_0, 2^i \right) =: \bigcup_{\nu = 0}^{2^j - i_\varepsilon - 1} R^{j_\varepsilon}_{\nu, -i_\varepsilon}.$$

For $m > j_\varepsilon$,

$$B^m \setminus B^{m-1} = \bigcup_{\nu = 0}^{2^j + i_\varepsilon - 1 - 1} B \left( x_0, 2^{m-1} + (\nu + 1)2^{m-j_\varepsilon - i_\varepsilon} \right) \setminus B \left( x_0, 2^{m-1} + \nu 2^{m-j_\varepsilon - i_\varepsilon} \right) =: \bigcup_{\nu = 0}^{2^j + i_\varepsilon - 1 - 1} R^m_{\nu, m-j_\varepsilon - i_\varepsilon}.$$

For each $R^m_{\nu, m-j_\varepsilon - i_\varepsilon}$, $\nu = 1, 2, \ldots, 2^j + i_\varepsilon - 1$, let $B^{j_\varepsilon}_{\nu, -i_\varepsilon}$ be an open cover of $R^m_{\nu, m-j_\varepsilon - i_\varepsilon}$ consisting of open balls with radius $2^{-i_\varepsilon}$ and center on the sphere $S(x_0, (\nu + 1)2^{-i_\varepsilon})$. Let $B^m_{0, -i_\varepsilon} = \{B(x_0, 2^{-i_\varepsilon})\}$ and $B^m_{\nu, -i_\varepsilon}$ be the finite subcover of $B^{j_\varepsilon}_{\nu, -i_\varepsilon}$. Similarly, for each $m > j_\varepsilon$ and $\nu = 0, 1, \ldots, 2^j + i_\varepsilon - 1$ - 1, let $B^m_{\nu, m-j_\varepsilon - i_\varepsilon}$ be the finite cover of $R^m_{\nu, m-j_\varepsilon - i_\varepsilon}$ consisting of open balls with radius $2^{m-j_\varepsilon - i_\varepsilon}$ and center on the sphere $S(x_0, (2^{m-1} + (\nu + 2^{-1})2^{m-j_\varepsilon - i_\varepsilon})$. 

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We define $B_x$ as follows. If $x \in B^{2^j}$, then there is $\nu \in \{0, 1, \ldots, 2^{j+1} - 1\}$ such that $x \in R^j_{\nu, -i}$, let $B_x$ be a ball in $B^j_{\nu, -i}$ that contains $x$. If $x \in B^m \setminus B^{m-1}$, $m > j$, then there is $\nu \in \{0, 1, \ldots, 2^{j+1} - 1\}$ such that $x \in R^m_{\nu, m-j-1}$, let $B_x$ be a ball in $B^m_{\nu, m-j-1}$ that contains $x$. We can see that if $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$, then

$$\text{either } r_{B_x} \leq 2 r_{B_{x'}} \text{ or } r_{B_{x'}} \leq 2 r_{B_x}. \quad (5.5)$$

In fact, if $r_{B_x} > 2 r_{B_{x'}}$, then there is $m_0 \in \mathbb{N}$ such that $x \in B^{m_0+2} \setminus B^{m_0+1}$ and $x' \in B^{m_0}$, thus

$$d(x, x') \geq d(x_0, x) - d(x_0, x') \geq 2^{m_0+1} - 2^{m_0} > 2^{m_0+2-j} + 2^{m_0-j-i} = r_{B_x} + r_{B_{x'}},$$

which is contradict to the fact that $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$ (Without loss of generality, here we assume that $A_0 = 1$ in the quasi-triangle inequality. Otherwise, we just need to take $r_B = (\lceil 2A_0 \rceil + 1)m$ and make some modifications).

Now we define $\phi_\varepsilon$. By (ii), there exists $m_\varepsilon > j_\varepsilon$ large enough such that when $r_B > 2^{m_\varepsilon-j_\varepsilon}$, we have

$$M(f, B) < 2^{n(-j_\varepsilon-j_\varepsilon-1)} \varepsilon. \quad (5.6)$$

Define

$$\phi_\varepsilon(g) = \begin{cases} f_{B_x}, & \text{if } x \in B^{m_\varepsilon}, \\ f_{B^{m_\varepsilon}-B^{m_\varepsilon-1}}, & \text{if } x \in X \setminus B^{m_\varepsilon}. \end{cases}$$

We claim that there exists a positive constant $\alpha_3, \alpha_4$ such that if $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$ or $x, x' \in X \setminus B^{m_\varepsilon-1}$, then

$$|\phi_\varepsilon(x) - \phi_\varepsilon(x')| < \alpha_3 \varepsilon. \quad (5.7)$$

And if $2B_x \cap 2B_{x'} \neq \emptyset$, then for any $x_1, x_2 \in B_x, B_{x'}$, we have

$$|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)| < \alpha_4 \varepsilon. \quad (5.8)$$

Assume (5.7) and (5.8) at the moment, we now continue to prove the sufficiency of Lemma 2.4.

Now we show (5.1). Let $\tilde{h}_\varepsilon(x) := \phi_\varepsilon(x) - f_{B^{m_\varepsilon}-B^{m_\varepsilon-1}}$. By definition of $\phi_\varepsilon$, we can see that $\tilde{h}_\varepsilon(x) = 0$ for $x \in X \setminus B^{m_\varepsilon}$ and $\|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{\text{BMO}(X)} = 0$.

Observe that supp $(\tilde{h}_\varepsilon) \subset B^{m_\varepsilon}$ and there exists a function $h_\varepsilon \in C_c(X)$ such that for any $x \in X$, $|\tilde{h}_\varepsilon(x) - h_\varepsilon(x)| < \varepsilon$. Let $\eta(s)$ be an infinitely differentiable function defined on $[0, \infty)$ such that $0 \leq \eta(s) \leq 1, \eta(s) = 1$ for $0 \leq s \leq 1$ and $\eta(s) = 0$ for $s \geq 2$. And let

$$\rho(x, y, t) = \left( \int_X \eta(d(x, z)/t) d\mu(z) \right)^{-1} \eta(d(x, z)/t)$$

and

$$h^t_\varepsilon(x) = \int_X \rho(x, y, t) h_\varepsilon(y) d\mu(y).$$

Then by [28, Lemmas 3.15 and 3.23], $h^t_\varepsilon(x)$ approaches to $h_\varepsilon(x)$ uniformly for $x \in X$ as $t$ goes to 0 and $h^t_\varepsilon \in \text{Lip}_\beta$ for $\beta > 0$. Since

$$\|h^t_\varepsilon - \phi_\varepsilon\|_{\text{BMO}(X)} \leq \|h^t_\varepsilon - h_\varepsilon\|_{\text{BMO}(X)} + \|h_\varepsilon - \tilde{h}_\varepsilon\|_{\text{BMO}(X)} + \|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{\text{BMO}(X)} + \|h_\varepsilon - \phi_\varepsilon\|_{\text{BMO}(X)} \leq \|h^t_\varepsilon - h_\varepsilon\|_{\text{BMO}(X)} + 2 \varepsilon,$$
we can obtain (5.1) by letting $t$ go to 0 and by taking $\alpha_1 = 2$.

Now we show (5.2). To this end, we only need to prove that for any ball $B \subset X$,

$$M(f - \phi_x, B) < \alpha_2 \varepsilon.$$ 

We first prove that for every $B_x$ with $x \in B^m_e$,

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x') \leq \alpha_5 \varepsilon \mu(B_x). \quad (5.9)$$

In fact,

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x') = \int_{B_x \cap B^{m_e}} |f(x') - f_{B_x}| \, d\mu(x') + \int_{B_x \cap (X \setminus B^{m_e})} |f(x') - f_{B^{m_e} \setminus B^{m_e-1}}| \, d\mu(x').$$

When $x \in B(x_0, 2^{m_e - 2^j_e - j_e})$, then $B_x \subset B^{m_e}$, thus

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x') = \int_{B_x} |f(x') - f_{B_x}| \, d\mu(x') \leq \int_{B_x} |f(x') - f_{B_x}| \, d\mu(x') + \int_{B_x} |f_{B_x} - f_{B_x}| \, d\mu(x')$$

$$= \mu(B_x) M(f, B_x) + \int_{B_x} |f_{B_x} - f_{B_x}| \, d\mu(x').$$

Note that if $x' \in B_x$, then $B_x \cap B_{x'} \neq \emptyset$. Therefore, If $B_x \cap B^{j_e} = \emptyset$, by (5.4) and (5.7), we have

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x') < (\varepsilon + \alpha_3 \varepsilon) \mu(B_x).$$

If $B_x \cap B^{j_e} \neq \emptyset$, then $r_{B_x} \leq 2^{-j_e + 1}$, then by (5.3) and (5.7),

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x') < (\varepsilon + \alpha_3 \varepsilon) \mu(B_x).$$

When $x \in B^{m_e} \setminus B(x_0, 2^{m_e - 2^j_e - j_e})$, it is clear that $B_x \cap B^{j_e} = \emptyset$, then by (5.4), (5.6) and (5.7), we have

$$\int_{B_x} |f(x') - \phi_x(x')| \, d\mu(x')$$

$$\leq \int_{B_x \cap B^{m_e}} |f(x') - f_{B_x}| \, d\mu(x') + \int_{B_x \cap B^{m_e}} |f_{B_x} - f_{B_x}| \, d\mu(x')$$

$$+ \int_{B_x \cap (X \setminus B^{m_e})} |f(x') - f_{B^{m_e+1}}| \, d\mu(x') + \int_{B_x \cap (X \setminus B^{m_e})} |f_{B^{m_e+1}} - f_{B^{m_e} \setminus B^{m_e-1}}| \, d\mu(x')$$

$$\leq \mu(B_x) M(f, B_x) + \alpha_3 \varepsilon \mu(B_x) + \mu(B^{m_e+1}) M(f, B^{m_e+1}) + \mu(B^{m_e+1}) \mu(B_x) \frac{M(f, B^{m_e+1})}{\mu(B^{m_e+1} \setminus B^{m_e-1})}$$

$$< (C_1 \varepsilon + \alpha_3 \varepsilon) \mu(B_x).$$

Then (5.9) holds by taking $\alpha_5 = (C_1 + \alpha_3)$.

Let $B$ be an arbitrary ball in $X$, then $M(f - \phi_x, B) \leq M(f, B) + M(\phi_x, B)$. If $B \subset B^{m_e}$ and $\max\{|r_{B_x} : B_x \cap B \neq \emptyset\} > 8r_B$, then

$$\min\{|r_{B_x} : B_x \cap B \neq \emptyset\} > 2r_B. \quad (5.10)$$
In fact, assume that $r_{B_x} = \max\{r_{B_x} : B_x \cap B \neq \emptyset\}$ and $\hat{x} \in B^0 \setminus B^{i_0 - 1}$ for some $l_0 \in \mathbb{Z}$. Then $B \subset B^0 \cap \frac{3}{2} B_{\hat{x}}$. If $l_0 \leq j, x$, then (5.10) holds. If $l_0 > j, x$, then $r_{B_x} = 2^{l_0 - j, x - i_0}$, and

$$r_B \leq \frac{1}{8} r_{B_{\hat{x}}} = 2^{l_0 - j, x - i_0 - 3}.$$  

Since for any $x' \in \frac{3}{2} B_{\hat{x}}$,

$$d(x_0, x') \geq d(x_0, \hat{x}) - d(\hat{x}, x') \geq 2^{l_0 - 1} - \frac{3}{2} 2^{l_0 - j, x - i_0} > 2^{l_0 - 1} - 2^{l_0 - j, x - i_0 + 1},$$

we have

$$\text{dist}(x_0, \frac{3}{2} B_{\hat{x}}) := \inf_{x' \in \frac{3}{2} B_{\hat{x}}} d(x_0, x') \geq 2^{l_0 - 1} - 2^{l_0 - j, x - i_0 + 1}.$$

Thus $B \subset B^0 \setminus \frac{3}{2} B^{l_0 - 2}$. Therefore, if $B_x \cap B \neq \emptyset$, then $x \in B^0 \setminus B^{l_0 - 2}$, which implies that

$$r_{B_x} \geq 2^{l_0 - 2 - j, x - i_0} > 2 r_B.$$

From (5.10) we can see that if $B_x, \cap B \neq \emptyset$ and $B_x, \cap B \neq \emptyset$, then $2 B_x, \cap 2 B_x \neq \emptyset$. Then by (5.8), we can get

$$M(\phi_x, B) \leq \frac{1}{\mu(B)} \int_B \frac{1}{\mu(B)} \int_B |\phi_x(x) - \phi_x(x')| \, d\mu(x') \, d\mu(x)$$

$$= \frac{1}{\mu(B)^2} \sum_{i:B_x \cap B \neq \emptyset} \int_{B_x \cap B} \sum_{j:B_x \cap B \neq \emptyset} \int_{B_x \cap B} |\phi_x(x) - \phi_x(x')| \, d\mu(x') \, d\mu(x)$$

$$\leq \alpha_4 \varepsilon \frac{1}{\mu(B)^2} \left( \sum_{i:B_x \cap B \neq \emptyset} \mu(B_x \cap B) \right) \left( \sum_{i:B_x \cap B \neq \emptyset} \mu(B_x \cap B) \right) < \alpha_4 \alpha_6^2 \varepsilon.$$

Moreover, if $B \cap B^{j} \neq \emptyset$, then by (5.10), $r_B < 2^{-i_0}$, thus by (5.3), we have $M(f, B) < \varepsilon$. If $B \cap B^{j} = \emptyset$, then by (5.4), $M(f, B) < \varepsilon$. Consequently,

$$M(f - \phi_x, B) \leq M(f, B) + M(\phi_x, B) < (1 + \alpha_4 \alpha_6^2) \varepsilon.$$

If $B \subset B^{m_0}$ and $\max\{r_{B_x} : B_x \cap B \neq \emptyset\} \leq 8 r_B$, since the number of $B_x$ with $x \in B^{m_0}$ that covers $B$ is bounded by $\alpha_7$, by (5.9), we have

$$M(f - \phi_x, B) \leq \frac{2}{\mu(B)} \int_B |f(x) - \phi_x(x)| \, d\mu(x) \leq \frac{2}{\mu(B)} \sum_{i:B_x \cap B \neq \emptyset} \int_{B_x} |f(x) - \phi_x(x)| \, d\mu(x)$$

$$\leq \frac{2}{\mu(B)} \alpha_5 \varepsilon \sum_{i:B_x \cap B \neq \emptyset} \mu(B_x) \leq \frac{2}{\mu(B)} \alpha_5 \alpha_7 \varepsilon \mu(8B) \leq C_2 \alpha_5 \alpha_7 \varepsilon.$$

If $B \subset X \setminus B^{m_{i - 1}}$, then $B \cap B^{j} = \emptyset$, from (5.4) we can see $M(f, B) < \varepsilon$. By (5.7),

$$M(\phi_x, B) \leq \frac{1}{\mu(B)} \int_B \frac{1}{\mu(B)} \int_B |\phi_x(x) - \phi_x(x')| \, d\mu(x') \, d\mu(x) < \alpha_3 \varepsilon.$$

Therefore,

$$M(f - \phi_x, B) \leq M(f, B) + M(\phi_x, B) < (1 + \alpha_3) \varepsilon.$$
If $B \cap (X \setminus B^{m_e}) \neq \emptyset$ and $B \cap B^{m_e-1} \neq \emptyset$. Let $p_B$ be the smallest integer such that $B \subset B^{p_B}$, then $p_B > m_e$. If $p_B = m_e + 1$, then $r_B > \frac{1}{2}(2^{m_e} - 2^{m_e-1}) = 2^{m_e-2}$. If $p_B > m_e + 1$, then $r_B > \frac{1}{2}(2^{p_B-1} - 2^{m_e-1})$. Thus

$$\frac{\mu(B^{p_B})}{\mu(B)} \leq C_3.$$  

Therefore,

$$M(f - \phi, B) \leq \frac{1}{\mu(B)} \int_B |f(x) - \phi(x) - (f - \phi)_B| \, d\mu(x) + |(f - \phi)_B| B \leq 2 \frac{\mu(B^{p_B})}{\mu(B)} \int_{B^{p_B}} |f(x) - \phi(x) - (f - \phi)_B| \, d\mu(x) \leq C_3 \left( M(f, B^{p_B}) + M(\phi, B^{p_B}) \right) \leq C_3 (\varepsilon + M(\phi, B^{p_B})),$$

where the last inequality comes from (5.6). By definition,

$$M(\phi, B^{p_B}) \leq \frac{1}{\mu(B^{p_B})} \int_{B^{p_B}} |\phi(x) - (\phi)_B| B^{p_B} \setminus B^{m_e} \, d\mu(x) + |(\phi)_B B^{p_B} \setminus B^{m_e} - (\phi)_B| B^{p_B} \leq \frac{2}{\mu(B^{p_B})} \int_{B^{p_B}} |\phi(x) - (\phi)_B| B^{p_B} \setminus B^{m_e} \, d\mu(x).$$

By (5.4), (5.9) and the fact that $\phi(x) = f_{B^{m_e} \setminus B^{m_e-1}}$ if $x \in X \setminus B^{m_e}$, we have

$$\int_{B^{p_B}} |\phi(x) - (\phi)_B B^{p_B} \setminus B^{m_e} | \, d\mu(x) \leq \int_{B^{p_B}} \frac{1}{\mu(B^{p_B} \setminus B^{m_e})} \int_{B^{p_B} \setminus B^{m_e}} |\phi(x) - \phi(x')| \, d\mu(x') \, d\mu(x) = \int_{B^{m_e}} |\phi(x) - f_{B^{m_e} \setminus B^{m_e-1}}| \, d\mu(x) \leq \int_{B^{m_e}} |\phi(x) - f(x)| \, d\mu(x) + \int_{B^{m_e}} |f(x) - f_{B^{m_e} \setminus B^{m_e-1}}| \, d\mu(x) \leq \sum_{i : B_{z_i} \neq \emptyset, x_i \in B^{m_e}} |\phi(x) - f(x)| \, d\mu(x) + \left( \mu(B^{m_e}) + \frac{\mu(B^{m_e})^2}{\mu(B^{m_e} \setminus B^{m_e-1})} \right) M(f, B^{m_e}) \leq \alpha_5 \varepsilon \sum_{i : B_{z_i} \neq \emptyset, x_i \in B^{m_e}} \mu(B_{z_i}) + 3 \varepsilon \mu(B^{m_e}) < (\alpha_5 \alpha_8 + 3) \varepsilon \mu(B^{m_e}).$$

Therefore,

$$M(f - \phi, B) \leq C_3 \left( \varepsilon + M(\phi, B^{p_B}) \right) \leq C_3 \left( \varepsilon + 2 \frac{\mu(B^{m_e})}{\mu(B^{p_B})} (\alpha_5 \alpha_8 + 3) \varepsilon \right) < C_4 (\alpha_5 \alpha_8 + 3) \varepsilon.$$  

Then (5.2) holds by taking $\alpha_2 = \max\{1 + \alpha_4 \alpha_6^2, 1 + \alpha_3, C_4 (\alpha_5 \alpha_8 + 3)\}$. This finishes the proof of Lemma 2.4.

**Proof of (5.7):**

We first claim that

$$\sup \{ |f_{B_{z_i}} - f_{B_{z_i}}| : x, x' \in B^{m_e} \setminus B^{m_e-1} \} < C_5 \varepsilon.$$  

(5.11)

By (5.6), for any $x \in B^{m_e} \setminus B^{m_e-1}$, we have

$$|f_{B_{z_i}} - f_{B_{z_i}+1}| \leq \frac{\mu(B^{m_e+1})}{\mu(B_{z_i+1})} \frac{1}{\mu(B^{m_e+1})} \int_{B^{m_e+1}} |f(x') - f_{B_{z_i}+1}| \, d\mu(x').$$
Boundedness and compactness of commutators

Similarly, for any \( x' \in B_{m_e} \setminus B_{m_e - 1} \), \( |f_{B_{x'}} - f_{B_{m_e + 1}}| < C_5 \frac{\varepsilon}{2} \). Consequently, (5.11) holds.

For the case \( x, x' \in X \setminus B_{m_e} \), firstly, if \( x, x' \in X \setminus B_{m_e} \), then by definition

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| = 0.
\]

Secondly, if \( x, x' \in B_{m_e} \setminus B_{m_e - 1} \), then by (5.11), we have

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| < C_5 \varepsilon.
\]

Thirdly, without loss of generality, we may assume that \( x \in B_{m_e} \setminus B_{m_e - 1} \) and \( x' \in X \setminus B_{m_e} \), then by (5.6), we have

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| = |f_{B_{x}} - f_{B_{x'}}| \leq |f_{B_{x}} - f_{B_{m_e + 1}}| + |f_{B_{m_e + 1}} - f_{B_{x'}}| \\
\leq \frac{\mu(B_{m_e + 1})}{\mu(B_{x})} M(f, B_{m_e + 1}) + \frac{\mu(B_{m_e})}{\mu(B_{m_e - 1})} M(f, B_{m_e}) \\
\leq \left( \frac{\mu(B_{m_e + 1})}{\mu(B_{x})} + \frac{\mu(B_{m_e})}{\mu(B_{m_e - 1})} \right) M(f, B_{m_e}) \\
< C_7 \varepsilon.
\]

For the case \( \overline{B_x} \cap \overline{B_{x'}} \neq \emptyset \) and \( x, x' \in B_{m_e - 1} \), we may assume \( B_x \neq B_{x'} \) and \( r_{B_x} \leq r_{B_{x'}} \). By (5.5), \( B_{x'} \subset 5B_x \subset 15B_{x'} \). If \( x' \in B_{j+1} \), then by (5.3), we have

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| = |f_{B_{x}} - f_{B_{x'}}| \leq |f_{B_{x}} - f_{3B_{x'}}| + |f_{B_{x'}} - f_{3B_{x'}}| \\
\leq \left( \frac{\mu(3B_{x'})}{\mu(B_{x})} + \frac{\mu(3B_x)}{\mu(B_{x'})} \right) M(f, 3B_{x'}) \\
\leq C_7 \varepsilon.
\]

If \( x' \notin B_{j+1} \), then \( 3B_{x'} \cap B_{j+1} = \emptyset \), by (5.4), we have

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| \leq C_7 M(f, 3B_{x'}) \leq C_7 \varepsilon.
\]

Therefore, (5.7) holds by taking \( \alpha_3 = \max\{C_5, C_6, C_7\} \).

Proof of (5.8):

Since \( x_1 \in B_x \), \( x_2 \in B_{x'} \), we have \( B_{x_1} \cap B_{x_2} \neq \emptyset \) and \( B_{x_2} \cap B_{x'} \neq \emptyset \), by (5.7),

\[
|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)| \leq |\phi_\varepsilon(x_1) - \phi_\varepsilon(x)| + |\phi_\varepsilon(x) - \phi_\varepsilon(x_2)| + |\phi_\varepsilon(x_2) - \phi_\varepsilon(x_2)| \\
\leq 2\alpha_3 \varepsilon + |\phi_\varepsilon(x) - \phi_\varepsilon(x')|.
\]

We may assume \( B_x \neq B_{x'} \) and \( r_{B_x} \leq r_{B_{x'}} \). If \( x, x' \in X \setminus B_{m_e - 1} \), then (5.8) follows from (5.7). If \( x, x' \in B_{m_e - 1} \), when \( x' \in B_{j+1} \), then \( 2^{-j+2} \leq r_{B_x} \leq r_{B_{x'}} \leq 2^{-j+1} \), thus \( B_{x'} \subset 10B_x \subset 60B_{x'} \), by (5.3), we have

\[
|\phi_\varepsilon(x) - \phi_\varepsilon(x')| \leq |f_{B_x} - f_{6B_{x'}}| + |f_{B_{x'}} - f_{6B_{x'}}| = \left( \frac{\mu(6B_{x'})}{\mu(B_{x})} + \frac{\mu(6B_x)}{\mu(B_{x'})} \right) M(f, 6B_{x'}) \\
\leq C_9 \varepsilon.
\]
When \( x' \notin B^{j_{\varepsilon}+1} \), then there exist \( \tilde{m}_0 \in \mathbb{N} \) and \( \tilde{m}_0 \geq j_{\varepsilon} + 2 \) such that \( x' \in B^{\tilde{m}_0} \setminus B^{\tilde{m}_0-1} \). Since \( 2B_x \cap 2B_{x'} \neq \emptyset \), we have \( B_x \subset 6B_{x'} \). Note that \( 6B_{x'} \cap B^{\tilde{m}_0-2} = \emptyset \), (in fact, for any \( \tilde{x} \in 6B_{x'} \), \( d(x_0, \tilde{x}) \geq d(x_0, x') = d(x', \tilde{x}) \geq 2^{\tilde{m}_0-1} - 6 \cdot 2^{\tilde{m}_0-j_{\varepsilon} - i_{\varepsilon}} > 2^{\tilde{m}_0-2} \)), thus \( B_x \cap B^{\tilde{m}_0-2} = \emptyset \) and then \( \frac{1}{2} r_{B_{x'}} = 2^{\tilde{m}_0-1-j_{\varepsilon} - i_{\varepsilon}} \leq r_{B_x} \leq 2^{\tilde{m}_0-j_{\varepsilon} - i_{\varepsilon}} = r_{B_{x'}} \). Therefore, \( B_{x'} \subset 10B_x \). Then by (5.4), we have
\[
|\phi_{\varepsilon}(x) - \phi_{\varepsilon}(x')| \leq C_{\varepsilon} M(f, 6B_{x'}) < C_{\varepsilon} \varepsilon.
\]
If \( x \in B^{m_{s}-1} \) and \( x' \in X \setminus B^{m_{s}-1} \), since \( 2B_x \cap 2B_{x'} \neq \emptyset \), by the construction of \( B_x \) we can see that \( x \in B^{m_{s}-1} \setminus B^{m_{s}-2} \) and \( x' \in B^{m_{s}} \setminus B^{m_{s}-1} \). Thus, \( B_{x'} \subset 10B_x \subset 40B_{x'} \). Then by (5.6), we have
\[
|\phi_{\varepsilon}(x) - \phi_{\varepsilon}(x')| < C_{10} M(f, 4B_{x'}) < C_{10} \varepsilon.
\]
Taking \( \alpha_4 = C_{\varepsilon} + C_{10} + 2\alpha_3 \), then (5.8) holds. \( \square \)

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