SOME REDUCTIONS OF RANK 2 AND GENERA 2 AND 3
HITCHIN SYSTEMS

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Abstract. Certain reductions of the rank 2, genera 2 and 3 Hitchin systems
are considered, which are shown to give an integrable system of 2, resp. 3, in-
teracting points on the line. It is shown that the reduced systems are particular
cases of a certain universal integrable system related to the Lagrange inter-
polation polynomial. Admissibility of the reduction is proved using computer
technique. The corresponding codes are given in the text.

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1. Introduction

General procedures of finding the algebraic-geometric solutions, and the action-
angle coordinates for Hitchin systems are proposed in [3]. For the rank 2 genus 2
systems the problem had been considered earlier in [1] based on the analogy with
the Neumann system, and the results of [2]. However, if we ask what a dynamical
system corresponds to a given Hitchin Hamiltonian, say, in the Tyurin parametriza-
tion, we will find out that it is a problem even write down the corresponding equa-
tions because the expressions for the Hamiltonians consist of thousands of symbols.

In the present work we consider reductions of Hitchin systems, i.e. their restric-
tions to invariant subvarieties of a positive codimension. On this way we obtain
some manageable rank 2, genera 2 and 3 systems, certain relations for their dy-
namical variables, and some particular solutions to the original systems.

It turns out to be that on the genus $g$ curve the reduced system is integrable and
coincides with the universal integrable system related to the Lagrange interpolation
polynomial of degree $g - 1$.

The paper is organized as follows. In Section 2 we describe the reduction of the
Hitchin systems of rank 2, genera 2 and 3, prove it to be an admissible reduction,
observe that the reduced systems are completely integrable and find some their
particular solutions (which are solutions to the original Hitchin system as well). The

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2. Reduction, genera 2 and 3

Let \( \Sigma \) be a genus \( g \) hyperelliptic curve given by the equation \( y^2 = P_{2g+1}(x) \) where \( P_{2g+1} \) is a degree \( 2g + 1 \) polynomial. Consider a Lax operator with the spectral parameter on this curve, taking values in the full linear algebra \( \mathfrak{gl}(2) \), given in the Tyurin parametrization \( [3] \) (see \([6]–[11]\) for the further developments):

\[
L(x, y) = \sum_{i=0}^{g-1} L_i x^i + \sum_{s=1}^{2g} \alpha_s \beta_s^T y + b_s, \quad \frac{1}{x - a_s},
\]

where \((a_s, b_s) \in \Sigma, \alpha_s, \beta_s \in \mathbb{C}^2, \beta_s^T \alpha_s = 0, L(a_s, b_s)\alpha_s = \kappa_s \alpha_s, \kappa_s \in \mathbb{C} \) for any \( s = 1, \ldots, 2g \) (the upper \( T \) denotes transposing here). Observe that there arise no singularity in the above eigenvalue conditions due to the assumption \( \beta_s^T \alpha_s = 0 \), and also because the points \( a_s \) \((s = 1, \ldots, 2g)\) are assumed to be different. By fixing a gauge, we set the matrix formed by the vectors \( \alpha_{2g-1}, \alpha_{2g} \) to the unit \( 2 \times 2 \) matrix. Below, \( g = 2, 3 \).

A rank 2 Hitchin system on \( \Sigma \) is given by the canonical symplectic structure \( \omega = \sum_{s=1}^{2g} \alpha_s^T \wedge \beta_s + \sum_{s=1}^{2g} a_s \wedge \kappa_s \), and the Hamiltonians of the form

\[
H_{k,m} = \text{res}_{z=0} z^{m-1} \text{tr} L(z)^k \text{dx}(z)/y(z),
\]

where \( z = \frac{1}{x^2} \) is a local parameter in the neighborhood of the point \( x = \infty \), and \( k, m \in \mathbb{Z} \).

In the case \( g = 2 \) we consider the canonical equations corresponding to the Hamiltonian \( H_{2,2} \). Assuming \( p_1 = 0 \) where \( y^2 = x^5 + p_1 x^4 + \ldots + p_5 \) is the equation of the curve we state

**Proposition 2.1.** For a genus 2 curve \( \Sigma \) the reduction \( \alpha_{11} = \alpha_{22} = 0, \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0 \) is admissible (compatible with the system), and the corresponding reduced system has the form

\[
\begin{align*}
\dot{a}_1 &= -\frac{2(a_1 \kappa_4 + a_4 (-2 \kappa_1 + \kappa_4))}{(a_1 - a_4)^2}, & \dot{\kappa}_1 &= \frac{\kappa_1 - \kappa_4}{a_1 - a_4} \dot{a}_1, \\
\dot{a}_4 &= -\frac{2(a_4 \kappa_1 + a_1 (\kappa_1 - 2 \kappa_4))}{(a_1 - a_4)^2}, & \dot{\kappa}_4 &= \frac{\kappa_1 - \kappa_4}{a_1 - a_4} \dot{a}_4, \\
\dot{a}_2 &= -\frac{2(a_2 \kappa_3 + a_3 (-2 \kappa_2 + \kappa_3))}{(a_2 - a_3)^2}, & \dot{\kappa}_2 &= \frac{\kappa_2 - \kappa_3}{a_2 - a_3} \dot{a}_2, \\
\dot{a}_3 &= -\frac{2(a_3 \kappa_2 + a_2 (\kappa_2 - 2 \kappa_3))}{(a_2 - a_3)^2}, & \dot{\kappa}_3 &= \frac{\kappa_2 - \kappa_3}{a_2 - a_3} \dot{a}_3, \\
\dot{\alpha}_{12} &= \alpha_{12} \frac{b_2 + b_3}{a_2 - a_3} \dot{a}_3, & \dot{\alpha}_{21} &= \alpha_{21} \frac{b_1 + b_4}{a_1 - a_4} \dot{a}_4.
\end{align*}
\]
The system of the first $8$ equations (for the system of points) is Hamiltonian with the Hamiltonian

$$H_{2,2}^{(r)} = 2 (\kappa_1 - \kappa_4) (a_4 \kappa_1 - a_1 \kappa_4) / (a_1 - a_4)^2$$
$$+ 2 (\kappa_2 - \kappa_3) (a_3 \kappa_2 - a_2 \kappa_3) / (a_2 - a_3)^2$$

The proof consists of a direct calculation using the program "Wolfram Mathematica": it turns out to be that the derivatives of the corresponding variables by virtue of the system vanish after plugging $\alpha_{11} = \alpha_{22} = 0$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$. The programs for this and other calculations are given below (in Section 4 for genus 2, and in Section 5 for genus 3).

The obtained system of equations splits into the two consistent subsystems: one of them for unknowns $a_1$, $a_3$, $\kappa_1$, $\kappa_4$, $\alpha_{21}$, and another for unknowns $a_2$, $a_3$, $\kappa_2$, $\kappa_3$, $\alpha_{12}$. Both can be resolved similarly. For example, we have for the first one $\kappa_1 - \kappa_4 = \frac{a_1 - a_4}{a_1 - a_4}$, which implies $\kappa_1 = c_1 a_1 + c_2$, $\kappa_4 = c_1 a_4 + c_3$ ($c_1$, $c_2$, $c_3$ are constants). For $c_1 = c_2 = c_3$ the arising equations for $a_1$, $a_4$ can be completely integrated: $a_1 + a_4 + a_2^2 = ct + c$, $2a_4 + \ln(a_1 + a_4) + c = 0$, where $c$, $\varphi$, $\varphi'$ are constants.

Next, the equation for $\alpha_{21}$ can be transformed to the form $a \ln \alpha_{21} = \frac{h_1 + h_4}{a_1 - a_4} da_4$. The right hand side can be considered as a function of only $a_4$, which is, moreover, already known, and we have $\alpha_{21} = \exp \int \frac{h_1 + h_4}{a_1 - a_4} da_4$.

Besides the reduction $\alpha_{11} = \alpha_{22} = 0$ there is another one $\alpha_{12} = \alpha_{21} = 0$ which descends to the previous by means of permutation of the points $a_1$ and $a_2$.

For a genus 3 curve

$$y^2 = x^7 + p_1 x^6 + p_2 x^5 + p_3 x^4 + p_4 x^3 + p_5 x^2 + p_6 x + p_7$$
we fix a gauge by requiring that the vectors $\alpha_3$ and $\alpha_6$ form the unit matrix. Then for $p_1 = p_2 = p_3 = 0$ the reduced system splits to the two (independent) triples of interacting points, and the equations for the parameters $\alpha_{ij}$. For example for the Hamiltonian $H_{2,4}$ under the reduction

$$\alpha_{11} = 0, \alpha_{22} = 0, \alpha_{13} = 0, \alpha_{24} = 0,$$
$$\beta_2 = 0 \ (s = 1, \ldots, 6)$$
we obtain for the points $a_1$, $a_3$, $a_6$ (for which the first coordinates of the corresponding vectors $\alpha_s$ are equal to zero) a Hamiltonian system with the Hamiltonian

$$H_{2,4}^{(r)} = \frac{(a_6^2 (-\kappa_1 + \kappa_3) + a_3^2 (\kappa_1 - \kappa_6) + a_1^2 (-3 \kappa_3 + \kappa_6))^2}{(a_1 - a_3)^2 (a_1 - a_6)^2 (a_3 - a_6)^2} +$$
$$+ \frac{2 (a_6 (\kappa_3 - \kappa_1) + a_3 (\kappa_1 - \kappa_6) + a_1 (\kappa_6 - \kappa_3))}{(a_1 - a_3) (a_1 - a_6) (a_3 - a_6)} \times$$
$$\times \frac{a_6^2 (a_1 \kappa_3 - a_3 \kappa_1) + a_3^2 (a_6 \kappa_1 - a_1 \kappa_6) + a_1^2 (a_3 \kappa_6 - a_6 \kappa_3)}{(a_1 - a_3) (a_1 - a_6) (a_3 - a_6)}$$

The following relations between $\dot{\kappa}_i$ and $\dot{a}_i$ hold:

$$\dot{\kappa}_1 = \frac{(a_1 - a_6)^2 (\kappa_1 - \kappa_3) - (a_1 - a_3)^2 (\kappa_1 - \kappa_6)}{(a_1 - a_3) (a_1 - a_6) (a_3 - a_6)} \dot{a}_1$$
$$\dot{\kappa}_3 = \frac{(a_3 - a_6)^2 (\kappa_1 - \kappa_3) + (a_1 - a_3)^2 (\kappa_3 - \kappa_6)}{(a_1 - a_3) (a_1 - a_6) (a_3 - a_6)} \dot{a}_3$$
$$\dot{\kappa}_6 = \frac{(a_6 - a_1)^2 (\kappa_3 - \kappa_6) - (a_6 - a_3)^2 (\kappa_1 - \kappa_6)}{(a_1 - a_3) (a_1 - a_6) (a_3 - a_6)} \dot{a}_6$$
Also we have:

\[
(2.3) \quad \dot{a}_1 + \dot{a}_3 + \dot{a}_6 = \frac{2(a_6(-\kappa_1 + \kappa_3) + a_3(\kappa_1 - \kappa_6) + a_1(-\kappa_3 + \kappa_6))}{(a_1 - a_3)(a_1 - a_6)(a_3 - a_6)}.
\]

The relations (2.2) are proven below (see Lemma 3.3 and after that). As for (2.3), it still is only a result of computation.

Under assumption \( \kappa_1 = \kappa_3 = \kappa_6 = K (= \text{const}) \) we obtain the following equations:

\[
\begin{align*}
\dot{a}_1 &= \frac{2K}{(a_1 - a_3)(a_1 - a_6)} \\
\dot{a}_3 &= -\frac{2K}{(a_1 - a_3)(a_3 - a_6)} \\
\dot{a}_6 &= \frac{2K}{(a_1 - a_6)(a_3 - a_6)}
\end{align*}
\]

and

\[
\begin{align*}
\dot{a}_{21} &= \frac{2K(b_1 + b_6)}{(a_1 - a_6)^2(a_3 - a_6)} - \frac{2K(b_1 + b_3)}{(a_1 - a_3)^2(a_3 - a_6)} \\
\dot{a}_{23} &= \frac{2K(b_3 + b_6)}{(a_1 - a_6)(a_3 - a_6)^2} - \frac{2K(b_1 + b_3)}{(a_1 - a_3)^2(a_1 - a_6)},
\end{align*}
\]

in particular it follows that \( \dot{a}_1 + \dot{a}_3 + \dot{a}_6 = 0 \). The variables in the system of equations on \( a_k \) do separate: setting

\[
a_1 + a_3 + a_6 = C_1
\]

we obtain

\[
(\dot{a}_1 - \dot{a}_3)(a_1 - a_3) = -3(\dot{a}_1 + \dot{a}_3)(a_1 + a_3 - 2C_1/3)
\]

which is resolved in the form:

\[
\begin{align*}
a_1^3 + f_1C_1a_1^2 + f_2C_1^2a_1 &= -2Kt + C_2, \\
a_3 &= e^{\pm 2\pi i/3}a_1 + f_0C_1,
\end{align*}
\]

where \( f_0, f_1, f_2 \in \mathbb{C} \) are known coefficients (the explicit expressions for them are omitted here), \( C_1, C_2 \) are the integration constants. In particular, for \( C_1 = 0 \) we obtain

\[
a_1 = \sqrt{-2Kt + C_2}, \quad a_3 = e^{\pm 2\pi i/3}a_1, \quad a_1 + a_3 + a_6 = 0.
\]

The variables \( \alpha_i \) can be expressed via quadratures, similar to the genus 2 case.

**Proposition 2.2.** The reduced systems are completely integrable. The corresponding full system of integrals in involution for \( g = 2 \) (and the system of the points \( \{a_1, a_4\} \) ) is as follows:

\[
F_0 = \frac{a_1\kappa_4 - a_4\kappa_1}{a_1 - a_4}, \quad F_1 = \frac{\kappa_1 - \kappa_4}{a_1 - a_4},
\]

and for \( g = 3 \) as follows:

\[
\begin{align*}
F_0 &= \frac{a_1a_6(a_6 - a_1)\kappa_3 + a_3^2(a_6\kappa_1 - a_1\kappa_6) + a_3(a_3^2\kappa_6 - a_6^2\kappa_1)}{(a_1 - a_3)(a_1 - a_6)(a_3 - a_6)} \\
F_1 &= \frac{a_6^2(-\kappa_1 + \kappa_3) + a_3^2(\kappa_1 - \kappa_6) + a_3^2(-\kappa_3 + \kappa_6)}{(a_1 - a_3)(a_1 - a_6)(a_3 - a_6)} \\
F_2 &= \frac{a_6(-\kappa_1 + \kappa_3) + a_3(\kappa_1 - \kappa_6) + a_1(-\kappa_3 + \kappa_6)}{(a_1 - a_3)(a_1 - a_6)(a_3 - a_6)}
\end{align*}
\]
Proof. It is proven in the next section that the integrals $F_0, F_1$ for $g = 2$, and the integrals $F_0, F_1, F_2$ for $g = 3$, are in involution with respect to the canonical Poisson bracket (and obviously independent). The proposition follows now from the fact that $H^{(r)}_{2,2} = 2F_0F_1$ for $g = 2$ (by $H^{(r)}_{2,2}$ we mean here the part of the Hamiltonian (2.1) related to the points $a_1, a_4$), and $H^{(r)}_{2,4} = F_1^2 + 2F_0F_2$ for $g = 3$. Hence $H^{(r)}_{2,2}$ (resp., $H^{(r)}_{2,4}$) is in involution with the basis integrals. □

Remark. It looks like the above vanishing assumptions on the coefficients $p_i$ of the equation of the curve do not affect the reduced system. At least, for $g = 2$ we have $H^{(r)}_{2,2} = 2(F_0 + p_1 F_1)F_1$ for an arbitrary $p_1$, i.e. the basis integrals are the same.

The above listed integrals are related to the universal integrable system introduced in Section 3 (Proposition 3.1). We conclude the section with the following conjecture.

The space where a half of the vectors $\alpha_s$ is proportional to $(1, 0)^T$, the remainder to the $(0, 1)^T$, and $\beta_s = 0$ for every $s$, is an invariant subspace of the system. The reduced system splits into two independent Hamiltonian completely integrable systems. Each of two reduced systems is given by plugging the corresponding points to the integrals of the universal integrable system of Section 3. The above obtained results prove the conjecture for $g = 2, 3$.

3. Integrable system related to the Lagrange interpolation polynomial

Let $F(x) = \sum_{i=0}^{n-1} F_i x^i$ be a polynomial taking values $\kappa_1, \ldots, \kappa_n$ at the different points $a_1, \ldots, a_n$ (called nodes in the context of interpolation). Consider its coefficients as functions of the sets of nodes and values:

$$F_i = F_i(a_1, \ldots, a_n, \kappa_1, \ldots, \kappa_n), \quad i = 0, \ldots, n - 1.$$

By canonical Poisson bracket we mean here the Poisson bracket in $\mathbb{C}^{2n}$ equipped with coordinates $a_1, \ldots, a_n, \kappa_1, \ldots, \kappa_n$, defined by means the following relations:

$$\{a_r, \kappa_s\} = \delta_{rs}, \quad \{a_r, a_s\} = \{\kappa_r, \kappa_s\} = 0$$

for all pairs $r, s$, where $\delta_{rs}$ is the Kronecker symbol. The following statement is the main goal of this section.

Proposition 3.1. The functions $F_i$ commute with respect to the canonical Poisson bracket.

The proof immediately follows from the following lemma:

Lemma 3.2. There exist rational functions $M_k = M_k(a_1, \ldots, a_n, \kappa_1, \ldots, \kappa_n)$ such that for any $i = 0, \ldots, n - 1$

$$\frac{\partial F_i}{\partial a_k} = M_k \frac{\partial F_i}{\partial \kappa_k}, \quad k = 1, \ldots, n$$

independently of $i$.

Indeed, by the lemma we obtain

$$\frac{\partial F_i}{\partial a_k} \frac{\partial F_j}{\partial \kappa_k} = M_k \frac{\partial F_i}{\partial \kappa_k} \frac{\partial F_j}{\partial a_k},$$

hence

$$\frac{\partial F_i}{\partial a_k} \frac{\partial F_j}{\partial \kappa_k} - \frac{\partial F_j}{\partial a_k} \frac{\partial F_i}{\partial \kappa_k} = 0,$$
for all \( k = 1, \ldots, n \), and
\[
\{ F_i, F_j \} = \sum_{k=1}^{n} \left( \frac{\partial F_i}{\partial a_k} \frac{\partial F_j}{\partial \kappa_k} - \frac{\partial F_j}{\partial a_k} \frac{\partial F_i}{\partial \kappa_k} \right) = 0.
\]

**Proof of Lemma 3.2.** It is obviously sufficient to prove the lemma for \( k = 1 \).

Represent \( F \) in the form of Lagrange interpolation polynomial [4]:
\[
(3.1) \quad F(x) = \sum_{s=1}^{n} \kappa_s f_s(x)
\]
where
\[
f_s(x) = \frac{(x - a_1) \ldots (x - a_{s-1})(x - a_{s+1}) \ldots (x - a_n)}{(a_s - a_1) \ldots (a_s - a_{s-1})(a_s - a_{s+1}) \ldots (a_s - a_n)}.
\]
In particular,
\[
(3.2) \quad f_1(x) = \frac{(x - a_2) \ldots (x - a_n)}{(a_1 - a_2) \ldots (a_1 - a_n)},
\]
hence
\[
(3.3) \quad \frac{\partial}{\partial a_1} f_1(x) = (x - a_2) \ldots (x - a_n) \frac{1}{(a_1 - a_2) \ldots (a_1 - a_n)}.
\]

For \( s > 1 \) we have
\[
\frac{\partial}{\partial a_1} f_s(x) = \frac{(x - a_2) \ldots (x - a_{s-1})(x - a_{s+1}) \ldots (x - a_n)}{(a_s - a_2) \ldots (a_s - a_{s-1})(a_s - a_{s+1}) \ldots (a_s - a_n)} \cdot \frac{\partial}{\partial a_1} \frac{x - a_1}{a_s - a_1}.
\]
Since
\[
\frac{\partial}{\partial a_1} \frac{x - a_1}{a_s - a_1} = \frac{x - a_s}{(a_s - a_1)^2},
\]
the gap in the nominator of the previous expression will get filled up, and we obtain
\[
\frac{\partial}{\partial a_1} f_s(x) = \frac{(x - a_2) \ldots (x - a_n)}{(a_s - a_1)^2(a_s - a_2) \ldots (a_s - a_{s-1})(a_s - a_{s+1}) \ldots (a_s - a_n)}.
\]
We conclude that
\[
(3.2) \quad \frac{\partial}{\partial a_1} F(x) = M'_1(x - a_2) \ldots (x - a_n)
\]
where
\[
M'_1 = \frac{\partial}{\partial a_1} \frac{\kappa_1}{(a_1 - a_2) \ldots (a_1 - a_n)} + \sum_{s=2}^{n} \frac{\kappa_s}{(a_s - a_1)^2(a_s - a_2) \ldots (a_s - a_{s-1})(a_s - a_{s+1}) \ldots (a_s - a_n)}.
\]
Further on, by (3.1) we obviously have
\[
(3.3) \quad \frac{\partial}{\partial \kappa_1} F(x) = f_1(x) = \frac{(x - a_2) \ldots (x - a_n)}{(a_1 - a_2) \ldots (a_1 - a_n)}.
\]
Comparing (3.2) and (3.3), and setting \( M_1 = M'_1(a_1 - a_2) \ldots (a_1 - a_n) \), we conclude that
\[
\frac{\partial}{\partial a_1} F(x) = M_1 \frac{\partial}{\partial \kappa_1} F(x).
\]
The corresponding relation for the coefficients of the polynomials gives
\[
\frac{\partial F_i}{\partial a_1} = M_1 \frac{\partial F_i}{\partial \kappa_1}.
\]
as required.

The proof of the Proposition 3.1 is complete. Observe that Lemma 3.2 can be generalized in the following way.

**Lemma 3.3.** Let $H$ be an arbitrary polynomial in $F_0, F_1, \ldots, F_{n-1}$. Then

$$\frac{\partial H}{\partial a_k} = M_k \frac{\partial H}{\partial \kappa_k}, \quad k = 1, \ldots, n.$$ 

**Proof.**

$$\frac{\partial H}{\partial a_k} = \sum_{i=0}^{n-1} \frac{\partial H}{\partial F_i} \frac{\partial F_i}{\partial a_k} = \sum_{i=0}^{n-1} \frac{\partial H}{\partial F_i} \left( M_k \frac{\partial F_i}{\partial \kappa_k} \right) = M_k \frac{\partial H}{\partial \kappa_k}.$$ 

□

Lemma 3.3 is the source of relations of type (2.2). Indeed, by virtue of the canonical system of equations with the Hamiltonian $H$ we have $\dot{a}_k = \frac{\partial H}{\partial \kappa_k}$, $\dot{\kappa}_k = -\frac{\partial H}{\partial a_k}$, and Lemma 3.3 reads as

$$\dot{\kappa}_k = -M_k \dot{a}_k.$$

4. Genus 2. Codes

(* As the first step we calculate the Lax operator: *)

$$A_0 := \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \quad A_1 := \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \quad \alpha_1 := \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \quad \alpha_2 := \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}, \quad \alpha_3 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_4 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \beta_1 := \begin{pmatrix} \beta_{11} \\ \beta_{21} \end{pmatrix}, \quad \beta_2 := \begin{pmatrix} \beta_{12} \\ \beta_{22} \end{pmatrix}, \quad \beta_3 := \begin{pmatrix} 0 \\ \beta_{23} \end{pmatrix}, \quad \beta_4 := \begin{pmatrix} \beta_{14} \\ 0 \end{pmatrix}$$

$L := A_0 + A_1 x + \alpha_1 . Transpose [\beta_1] * ((y + b_1) / (x - a_1)) + \alpha_2 . Transpose [\beta_2] * ((y + b_2) / (x - a_2)) + \alpha_3 . Transpose [\beta_3] * ((y + b_3) / (x - a_3)) + \alpha_4 . Transpose [\beta_4] * ((y + b_4) / (x - a_4))$

(* $L$ is the Lax operator *)

**MatrixForm[L] (This results in the expression for L. Further on we form the eigenvalue conditions *)

$$\begin{pmatrix} l_{11} + \frac{l_{11} l_{12}}{z^2} & \frac{(y+b_1)\alpha_{11} \beta_{11}}{z^2 - a_1} + \frac{(y+b_2)\alpha_{12} \beta_{12}}{z^2 - a_2} \\ l_{21} + \frac{l_{12} l_{21}}{z^2} & \frac{(y+b_1)\alpha_{21} \beta_{21}}{z^2 - a_1} + \frac{(y+b_2)\alpha_{22} \beta_{22}}{z^2 - a_2} + \frac{(y+b_3)\beta_{23}}{z^2 - a_3} \end{pmatrix}$$

**MatrixForm[La] (These 2 commands calculate the left hand *)

**MatrixForm[La2] (sides of the subsequent system of equations *)

sol =

Solve[
\[
\left\{ l_{11} + a_3 l_{111} + \frac{(b_3 + b_4)a_{11} l_{11}}{a_3 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_3 - a_2} + \frac{(b_3 + b_4)a_{23} l_{12}}{a_3 - a_2} + \frac{(b_3 + b_4)a_{34} l_{12}}{a_3 - a_2} \right\} = \kappa_3 \beta_{11} l_{121} + a_3 l_{21} + \alpha_{11} \left( l_{11} + a_4 l_{111} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) + \alpha_{21} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) = \alpha_{11} \kappa_1 \beta_{11} + \alpha_{21} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) = \alpha_{12} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) = \alpha_{22} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) = \alpha_{22} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right) = \alpha_{22} \left( l_{12} + a_4 l_{112} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} + \frac{(b_3 + b_4)a_{12} l_{12}}{a_1 - a_2} \right)
\]

\[z := 1/z^2\] (* local parametrization at infinity *)

\[y = (1/z^5) \sqrt{1 + p_1 z^2 + p_2 z^4 + p_3 z^6 + p_4 z^8 + p_5 z^{10}}\]

(*Calculating the Hamiltonians as the coefficients of the expansion of T at infinity : *)

\[T = \text{Series}[\text{Tr}[\mathcal{L} L], \{z, 0, -2\}]\]

\[\text{(-1/z^3)Series[1/y, \{z, 0, 9\}] (*dx/y up to the factor dz*)}\]

\[\text{H2} = -\text{Residue} \left[ \left( 1/z \right) \star T \star \left( -z^2 + \frac{2 b_4}{2} + \left( -\frac{3 b_2^2}{2} + \frac{4 b_2}{2} \right) z \right), \{z, 0\} \right] (*with the differential z^\{ -1 \} dx/y*)\]

\[
\{(l_{11}, l_{12}, l_{21}, l_{111}, l_{112}, l_{121}, l_{122}) \} = \{l_{11}, l_{12}, l_{21}, l_{22}, l_{111}, l_{112}, l_{121}, l_{122}\} /. \text{sol}
\]

\[\text{H2} (*by this line we obtain a final expression for H2 taking account of replacements *)\]

(* The commands below correspond to the Hamiltonian equations with the Hamiltonian H2. The assumptions correspond to the assumptions of the reduction. The zero results prove that assumptions of the reduction are preserved along the trajectories. Da11 and so on denote the time derivatives of the corresponding variables. *)

\[\text{DA11 = Series}[\text{D}[\text{H2}, \beta_{11}] / \{\alpha_{11} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0\}, \{\alpha_{22}, 0, 0\}] / \{\alpha_{22} \to 0\}\]

0

\[\text{DB11 = Series}[\text{D}[\text{H2}, \alpha_{11}] / \{\alpha_{11} \to 0, \beta_{11} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0\}, \{\alpha_{22}, 0, 0\}] / \{\alpha_{22} \to 0\}\]

0

\[\text{DB21 = Series}[\text{D}[\text{H2}, \alpha_{21}] / \{\alpha_{11} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0\}, \{\alpha_{22}, 0, 0\}] / \{\alpha_{22} \to 0\}\]

0
\[ D[\beta_{12}] = -\text{Series}[D[H_2, \alpha_{12}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[\beta_{22}] = -\text{Series}[D[H_2, \alpha_{22}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[\beta_{13}] = -\text{Series}[D[H_2, \alpha_{13}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[\beta_{23}] = -\text{Series}[D[H_2, \alpha_{23}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[\beta_{14}] = -\text{Series}[D[H_2, \alpha_{14}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[\beta_{24}] = -\text{Series}[D[H_2, \alpha_{24}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[a_{22}] = \text{Series}[D[H_2, \beta_{22}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

0

\[ D[a_{12}] = \text{Series}[D[H_2, \beta_{12}]]/. \{ \alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0 \}, \{ \alpha_{22}, 0 \}]/. \{ \alpha_{22} \rightarrow 0 \} \]

\[ 2(b_2 + b_3)(a_2 a_12 + a_3 a_12 - 2a_2 a_12) \]

\[ (a_2 - a_3)^4 \]
Da21 = Series[D[H2, β21]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(β1 + β2 + a1 + a2 + a3 + 2a1a2)
\((a1 - a2)^2\)

Da1 = Series[D[H2, κ4]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(−2a3κ1 + a1κ4 + a4κ4)
\((a1 - a4)^2\)

Da2 = Series[D[H2, κ3]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(−2a3κ1 + a1κ4 + a4κ4)
\((a2 - a3)^2\)

Da3 = Series[D[H2, κ3]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(a2κ2 + a3κ2 - 2a2κ3)
\((a2 - a3)^2\)

Da4 = Series[D[H2, κ3]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(α1κ1 + a4κ4 - 2a1κ4)
\((a1 - a4)^2\)

Dn1 = Series[-D[H2, a4]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(κ1 - κ4)(−2a4κ1 + a1κ4 + a4κ4)
\((a1 - a4)^2\)

Dn2 = Series[-D[H2, a2]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(a2κ2 - κ3)(−2a2κ2 + a2κ3 + a3κ3)
\((a2 - a3)^2\)

Dn3 = Series[-D[H2, a3]/. {α11 → 0, β11 → 0, β21 → 0, β12 → 0, β22 → 0, β13 → 0, β23 → 0, β14 → 0, β24 → 0}, {α22, 0, 0}]/. {α22 → 0}

2(κ2 - κ3)(a2κ2 + a3κ2 - 2a2κ3)
\((a2 - a3)^2\)
$D\alpha_4 = \text{Series}[-D[H2, a_4]/\{\alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0\}, \{\alpha_{22}, 0, 0\}]/\{\alpha_{22} \rightarrow 0\}$

$H2_r = \text{Simplify}[/\text{Series}[H2]/\{\alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0\}, \{\alpha_{22}, 0, 0\}]/\{\alpha_{22} \rightarrow 0\}$

$-\frac{2(\kappa_1 - \kappa_4)(a_1\kappa_1 + a_4\kappa_4 - 2a_1\kappa_4)}{(a_3 - a_4)^5}$

$H2_r = \text{Simplify}[/\text{Series}[H2]/\{\alpha_{11} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0\}, \{\alpha_{22}, 0, 0\}]/\{\alpha_{22} \rightarrow 0\}$

$-\frac{1}{(a_2 - a_3)^2(a_1 - a_4^2)2(a_2^2(\kappa_1 - \kappa_4)(a_4\kappa_1 - a_1\kappa_4) + a_2(a_1 - a_4)^2(\kappa_2 - \kappa_3)\kappa_3 - 2a_3(\kappa_1 - \kappa_4)(a_4\kappa_1 - a_1\kappa_4)) + a_3((a_1 - a_4)^2\kappa_2 - \kappa_3) + a_3(\kappa_1 - \kappa_4)(a_4\kappa_1 - a_1\kappa_4))}$

(*It is equal to $2(\kappa_1 - \kappa_4)(a_4\kappa_1 - a_1\kappa_4)/(a_1 - a_4)^2 + 2(\kappa_2 - \kappa_3)(a_3\kappa_2 - a_2\kappa_3)/(a_2 - a_3)^2$ as in Proposition 1 *)

5. Genus 3. Codes

$$A_0 := \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \quad A_1 := \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \quad A_2 := \begin{pmatrix} l_{211} & l_{212} \\ l_{221} & l_{222} \end{pmatrix}, \quad A_3 := \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad A_4 := \begin{pmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{pmatrix}, \quad A_5 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_6 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(* $L$ is the Lax operator: *)

$L := A_0 + A_1x + A_2x^2 + \alpha_1.\text{Transpose}[\beta_1] * ((y + b_1)/(x - a_1)) + \alpha_2.\text{Transpose}[\beta_2] * ((y + b_2)/(x - a_2)) + \alpha_3.\text{Transpose}[\beta_3] * ((y + b_3)/(x - a_3)) + \alpha_4.\text{Transpose}[\beta_4] * ((y + b_4)/(x - a_4)) + \alpha_5.\text{Transpose}[\beta_5] * ((y + b_5)/(x - a_5)) + \alpha_6.\text{Transpose}[\beta_6] * ((y + b_6)/(x - a_6))$

(*Calculating the Hamiltonian : *)

$x := 1/z^2$

$y = (1/z^7)\text{Sqrt}[1 + p_1 z^2 + p_2 z^4 + p_3 z^6 + p_4 z^8 + p_5 z^{10} + p_6 z^{12} + p_7 z^{14}]$

(* at the infinity $1/y \sim z^7$, $z^{-1}dx/y \sim z^3dz$, hence $\text{Tr}[L,L]$ should be calculated up to $z^{-4}$ *)

$T = \text{Series}[/\text{Tr}[L,L], \{z, 0, -4\}]$

(*$\text{Tr}[L,L] \sim z^{-10}, z^{-1}\text{Tr}[L,L] \sim z^{-11}$, hence $dx/y$ should be calculated up to $z^{10}$ *)
\((-1/z^3)\) Series \([1/y, \{z, 0, 13\}]\) (* \(dx/y\) up to the factor \(dz\) *)

(*For \(p_1 = p_2 = p_3 = 0\) the Hamiltonian at \(z^{-5}\) gives trivial equations.*)

(* The following Hamiltonian is the Hamiltonian at \(z^{-4}\) (for \(p_1 = p_2 = p_3 = 0\)) : *)

\[
H_2 = -\text{Residue}[(1/z) \cdot T \cdot (-z^4 + \frac{\kappa_1}{z} + (-\frac{3\beta_1}{z} + \frac{\kappa_2}{z})z^8 + \frac{1}{16}(5p_1^2 - 12p_1p_2 + 8p_3)z^{10}), \{z, 0\}]
\]

(* with the differential \(z^{-1}dx/y\). Below \(p_1 = p_2 = p_3 = 0\) *)

(* First part of the equations out of eigenvalue conditions
(to generate the left hand sides of those equations one makes use of commands of the type \(\text{MatrixForm}[L\alpha_1]\)
like it has been done for genus = 2) : *)

\[
\text{sol} = \text{Solve}\{\{l_{11} + a_1l_{111} + a_5^2l_{211} + (b_1 + b_3)a_{11}\beta_{12} + (b_1 + b_5)a_{12}\beta_{13} + (b_1 + b_5)a_{13}\beta_{12} + (b_1 + b_5)a_{14}\beta_{14}\} == \kappa_1\alpha_{11} \& \& \ldots
\]

\[
l_{22} + a_5l_{121} + a_5^2l_{221} + (b_1 + b_5)a_{21}\beta_{22} + (b_1 + b_5)a_{22}\beta_{23} + (b_1 + b_5)a_{23}\beta_{22} + (b_1 + b_5)a_{24}\beta_{24} == \kappa_2\alpha_{12} \& \& \ldots
\]

\[
l_{23}(l_{11} + a_1l_{111} + a_5^2l_{211} + (b_1 + b_5)a_{11}\beta_{21} + (b_1 + b_5)a_{12}\beta_{23} + (b_1 + b_5)a_{13}\beta_{22} + (b_1 + b_5)a_{14}\beta_{24} == \kappa_3\alpha_{13} \& \& \ldots
\]

\[
l_{24}(l_{11} + a_1l_{111} + a_5^2l_{211} + (b_1 + b_5)a_{11}\beta_{21} + (b_1 + b_5)a_{12}\beta_{23} + (b_1 + b_5)a_{13}\beta_{22} + (b_1 + b_5)a_{14}\beta_{24} == \kappa_4\alpha_{14},
\]

\[
\{\{l_{11}, l_{12}, l_{111}, l_{211}, l_{211}, l_{212}\}\} = \{l_{11}, l_{12}, l_{111}, l_{121}, l_{211}, l_{212}\} / . \text{sol}
\]

(* Second part of the equations out of eigenvalue conditions: *)

\[
\text{sol} = \text{Solve}\{\{l_{21} + a_5l_{211} + a_5^2l_{221} + (b_1 + b_5)a_{21}\beta_{21} + (b_1 + b_5)a_{22}\beta_{22} + (b_1 + b_5)a_{23}\beta_{23} + (b_1 + b_5)a_{24}\beta_{24} == 0 \& \& \ldots
\]

\[
l_{22} + a_5l_{121} + a_5^2l_{221} + (b_1 + b_5)a_{21}\beta_{22} + (b_1 + b_5)a_{22}\beta_{23} + (b_1 + b_5)a_{23}\beta_{22} + (b_1 + b_5)a_{24}\beta_{24} == \kappa_5\alpha_{22} \& \& \ldots
\]

\[
\text{.sol}
\]
\( \alpha_{13}(l_{21} + a_3 l_{121} + a_3^2 l_{221} + \frac{(b_1+b_2) a_3 \beta_{11}}{-a_1-a_3} + \frac{(b_2+b_3) a_3 \beta_{21}}{-a_2-a_3} + \frac{(b_3+b_4) a_2 \beta_{14}}{-a_3-a_4} + \frac{(b_4+b_5) a_2 \beta_{16}}{-a_3-a_4}) +
\)
\( \alpha_{23}(l_{22} + a_3 l_{122} + a_3^2 l_{222} + \frac{(b_1+b_2) a_3 \beta_{11}}{-a_1-a_3} + \frac{(b_2+b_3) a_3 \beta_{22}}{-a_2-a_3} + \frac{(b_3+b_4) a_3 \beta_{24}}{-a_3-a_4} + \frac{(b_4+b_5) a_3 \beta_{16}}{-a_4-a_6}) +
\)
\( \alpha_{14}(l_{21} + a_4 l_{121} + a_4^2 l_{221} + \frac{(b_1+b_2) a_4 \beta_{11}}{-a_1-a_4} + \frac{(b_2+b_3) a_4 \beta_{22}}{-a_2-a_4} + \frac{(b_3+b_4) a_4 \beta_{24}}{-a_3-a_4} + \frac{(b_4+b_5) a_4 \beta_{26}}{-a_4-a_6}) +
\)
\( \alpha_{24}(l_{22} + a_4 l_{122} + a_4^2 l_{222} + \frac{(b_1+b_2) a_4 \beta_{11}}{-a_1-a_4} + \frac{(b_2+b_3) a_4 \beta_{22}}{-a_2-a_4} + \frac{(b_3+b_4) a_4 \beta_{24}}{-a_3-a_4} + \frac{(b_4+b_5) a_4 \beta_{26}}{-a_4-a_6}) = \kappa_3 \alpha_{23} + \kappa_4 \alpha_{24} + \kappa_6 \alpha_{14}.
\)

\[
\{(l_{21}, l_{22}, l_{121}, l_{122}, l_{221}, l_{222})\} = \{l_{21}, l_{22}, l_{121}, l_{122}, l_{221}, l_{222}\}/.\text{sol}
\]

(* It is recommended to collect and keep the results of the foregoing replacements for \(l_{ij}\),

\(l_{ijk}\) in a separate file because solution of the system takes quite a lot time *)

H2 (*calculating the Hamiltonian taking account of the replacements for \(l_{ij}, l_{ijk}\) *)

(* Next we check admissibility of the reduction: *)

DA11 = Simplify[Series[D[H2, \(\beta_{11}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0

DA22 = Simplify[Series[D[H2, \(\beta_{22}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0

DA13 = Simplify[Series[D[H2, \(\beta_{13}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0

DA24 = Simplify[Series[D[H2, \(\beta_{24}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0

DB11 = Simplify[Series[D[H2, \(\alpha_{11}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0

DB21 = Simplify[Series[D[H2, \(\alpha_{21}\)]]/. \{\(\alpha_{11} \to 0, \alpha_{13} \to 0, \alpha_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}, \{\alpha_{22}, 0, 0\}]/. \{\alpha_{22} \to 0\}]

0
Some reductions
(* Next we find the (right parts of the) reduced equations (from the full Hamiltonian) for the remainder of the variables, calculate the reduced Hamiltonian and check that the equations of the reduced system are the same as the equations from the reduced Hamiltonian *)

\[ \text{DA12} = \text{Simplify[Simplify[Simplify[Series} [] \] \right]}. \{ \alpha_{11} \rightarrow 0, \alpha_{13} \rightarrow 0, \alpha_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0}, \{ \alpha_{22}, 0, 0] \}. \{ \alpha_{22} \rightarrow 0] \}

\[ \frac{1}{(a_2-a_4)^3(a_2-a_3)(a_4-a_5)}2\alpha_{12}(-a_2\frac{1}{2}(b_2 + b_4) + a_1\frac{1}{4}(b_2 + b_5) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1(b_2 + b_5)) - 2a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{4}(b_2 + b_5))(a_5(k_2 - k_4) + a_2(k_4 - k_5) + a_4(-k_2 + k_5)) + (a_3\frac{1}{2}(b_2 + b_4) + a_2\frac{1}{2}(b_2 - b_5) - a_1\frac{1}{2}(b_2 + b_5) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1(b_2 + b_5)) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{2}(b_2 + b_5)) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{2}(b_2 - b_5) + a_2\frac{1}{2}(b_2 + b_5)) - 2a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{2}(b_2 + b_5))(a_2a_5(-a_2 + a_5)k_4 + a_2\frac{1}{2}(a_5k_2 - a_2k_5) + a_4(-a_5^2k_2 + a_2^2k_5)) \]

\[ \text{DA14} = \text{Simplify[Simplify[Simplify[Series} [] \] \right]}. \{ \alpha_{11} \rightarrow 0, \alpha_{13} \rightarrow 0, \alpha_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0}, \{ \alpha_{22}, 0, 0] \}. \{ \alpha_{22} \rightarrow 0] \}

\[ \frac{1}{(a_2-a_4)^3(a_2-a_3)(a_4-a_5)}2\alpha_{14}(-a_4\frac{1}{2}(b_2 + b_4) - a_2\frac{1}{2}(b_4 + b_5) + a_3\frac{1}{4}(b_2 + b_4) - a_2(b_4 + b_5)) - 2a_4\frac{1}{2}(b_2 + b_4) - a_2\frac{1}{2}(b_4 + b_5))(a_5(-k_2 + k_4) + a_4(k_2 - k_5) + a_2(-k_4 + k_5)) - (a_3\frac{1}{2}(b_2 + b_4) + a_2\frac{1}{2}(b_2 - b_5) - a_1\frac{1}{2}(b_2 + b_5) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1(b_2 + b_5)) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{2}(b_2 + b_5)) + a_2\frac{1}{2}(-a_5(b_2 + b_4) + a_1\frac{1}{2}(b_2 - b_5) + a_2\frac{1}{2}(b_2 + b_5)) - 2a_4\frac{1}{2}(b_2 + b_4) - a_2(b_4 + b_5))(a_2a_5(-a_2 + a_5)k_4 + a_2\frac{1}{2}(a_5k_2 - a_2k_5) + a_4(-a_5^2k_2 + a_2^2k_5)) \]

\[ \text{Da1} = \text{Simplify[Simplify[Series} [] \] \right]}. \{ \alpha_{11} \rightarrow 0, \alpha_{13} \rightarrow 0, \alpha_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0}, \{ \alpha_{22}, 0, 0] \}. \{ \alpha_{22} \rightarrow 0] \}

\[ \frac{1}{(a_1-a_3)^3(a_1-a_2)(a_3-a_5)}2(a_1\frac{1}{2}(k_1 - k_6) + a_2\frac{1}{2}(a_6(3k_1 - 2k_6) - a_1k_5) + a_6(a_1a_0k_3 + a_6(-k_1 + k_5)) + a_1\frac{1}{2}(-2k_3 + k_6)) + a_3(a_0\frac{1}{2}(-3k_1 + 2k_6) - a_1\frac{1}{2}(k_3 - 2k_6) + a_1a_6(-k_3 + k_6)) \]

\[ \text{Da2} = \text{Simplify[Series[Series} [] \] \right]}. \{ \alpha_{11} \rightarrow 0, \alpha_{13} \rightarrow 0, \alpha_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0}, \{ \alpha_{22}, 0, 0] \}. \{ \alpha_{22} \rightarrow 0] \}

\[ \frac{1}{(a_2-a_4)^3(a_2-a_3)(a_4-a_5)}2(a_1\frac{1}{2}(k_2 - k_5) + a_2\frac{1}{2}(a_5(3k_2 - 2k_5) - a_2k_3) + a_5(a_2a_5k_4 + a_2\frac{1}{2}(-k_2 + k_4) + a_2\frac{1}{2}(-2k_4 + k_5)) + a_4(a_2\frac{1}{2}(-3k_2 + 2k_5) - a_2\frac{1}{2}(k_4 - 2k_5) + a_2a_5(-k_4 + k_5)) \]

\[ \text{Da3} = \text{Simplify[Series[Series} [] \] \right]}. \{ \alpha_{11} \rightarrow 0, \alpha_{13} \rightarrow 0, \alpha_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0}, \{ \alpha_{22}, 0, 0] \}. \{ \alpha_{22} \rightarrow 0] \} (**

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\[
\begin{align*}
\frac{1}{(a_1-a_2)^2(a_1-a_3)(a_3-a_4)} & \cdot 2(a_1^3(k_4 - k_6) + a_1^2(a_5(3k_4 - 2k_6) - a_3k_6) + a_5(a_3a_5k_3 + a_3^2(k_1 - k_3) + a_3^2(-2k_4 + k_5)) + a_1(a_5(2k_1 - 3k_3) - a_3^2(k_1 - 2k_6) + a_3a_6(-k_1 + k_6))) \\
\end{align*}
\]

**Da4** = Simplify[Series[D[H2, k4], \{a11 \rightarrow 0, a13 \rightarrow 0, a24 \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}]. \{a22 \rightarrow 0\}]

\[
\begin{align*}
\frac{1}{(a_2-a_4)^2(a_2-a_3)(a_4-a_5)} & \cdot 2(a_2^2(a_5k_4 + a_5(2k_4 - 3k_5)) - a_2(a_5a_5k_2 + a_2^2(-2k_2 + k_4) + a_2^2(k_2 - k_5) + a_2^2(-2k_2 + 3k_5)) + a_2(a_5(k_2 - 2k_4) + a_5a_5(k_2 - k_4) + a_2^2(-2k_2 + 3k_5))) \\
\end{align*}
\]

**Da5** = Simplify[Series[D[H2, k5], \{a11 \rightarrow 0, a13 \rightarrow 0, a24 \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}]. \{a22 \rightarrow 0\}]

\[
\begin{align*}
\frac{1}{(a_1-a_3)(a_1-a_6)(a_3-a_4)} & \cdot 2(a_1^2(a_6k_3 + a_3(2k_3 - 3k_6)) - a_3(a_3a_6k_1 + a_6^2(-2k_1 + k_3) + a_1^2(k_1 - k_6)) + a_1^2(k_1 - k_6) + a_1(a_6^2(k_1 - 2k_3) + a_3a_6(k_1 - k_3) + a_6^2(-2k_1 + 3k_6))) \\
\end{align*}
\]

**Simplify[Da1 + Da3 + Da6]**

\[
\frac{2(a_6(-k_1+k_2)a_3(k_1-k_6)+a_1(-k_3+k_4))}{(a_1-a_3)(a_1-a_6)(a_3-a_4)}
\]

(*The dependence of a_s on b_s has been switched on at calculating the Dk4. It does not affect the result: *)

**Dx1** = Simplify[Series[-D[H2], \{b_1 \rightarrow f[a_1]\}, a_1\}]. \{a11 \rightarrow 0, a13 \rightarrow 0, a24 \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}]. \{a22 \rightarrow 0\}]

\[
\begin{align*}
\frac{1}{(a_1-a_3)(a_1-a_6)(a_3-a_4)} & \cdot 2(-a_6^2(k_1 - k_3)^2 + a_3a_6^2(-3k_1^2 + 5k_1k_3 - 2k_2^2) + a_3^2a_6^2(k_1 - k_3)(4k_1 - 3k_2) + a_3^2a_6^2(4k_1 - 3k_3)(k_1 - k_6) - a_3^2(k_1 - k_6)^2 + a_6^4(a_3k_3 - 2k_6) + a_6(2k_3 - k_6))(k_3 - k_6) - 3a_1^2(a_3 - a_6)(a_3(-k_1 + k_3) + a_3(k_1 - k_6)(a_6a_6 + a_3^2k_3 + a_3^2(-3k_1^2 + 5k_1k_6 - 2k_2^2) + a_1(a_6^2(2k_2^2 - 3k_1k_3 + k_3^2) + a_3^2a_6(4k_1 + 3k_3 - 5k_6)(k_1 - k_6) + a_3a_6^2(k_1 - k_3)(4k_1 - 3k_3 + 3k_6) + a_3^2(2k_1^2 - 3k_1k_6 + k_6^2) + a_3^2a_6(-4k_1^2 - 2k_2k_6 + 6k_1(k_1 + k_6)) - a_1^2(a_3a_6(k_2^2 - 6k_3k_6 + k_6^2) + 2k_1(k_3 + k_6)) + a_6^2(k_3k_3 - 3k_6) + a_1(-4k_3 + 2k_6)) + a_3^2(2k_1(k_3 - 2k_6) + a_6(-3k_3k_6 + 3k_6)))]
\end{align*}
\]
Dx2 = \text{Simplify[Series[-D[H2], \{b_2 \rightarrow f[a_2]\}, a_2]/. \{a_{11} \rightarrow 0, a_{13} \rightarrow 0, a_{24} \rightarrow 0, \beta_{11} \rightarrow 0, \beta_{21} \rightarrow 0, \\
\beta_{12} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{13} \rightarrow 0, \beta_{23} \rightarrow 0, \beta_{14} \rightarrow 0, \beta_{24} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}, \{a_{22}, 0, 0\}]}/. \{a_{22} \rightarrow 0\]
\[ a_4a_2^2(\kappa_2^2 + \kappa_2(-6\kappa_4 + 2\kappa_5 + \kappa_4(\kappa_1 + 2\kappa_5))) + a_2^2(\kappa_4(3\kappa_4 - 4\kappa_5)(-\kappa_2 + \kappa_5) + 3a_4a_5^2(\kappa_2^2 + \\
\kappa_2\kappa_5 - 2\kappa_4\kappa_5) + a_2^2(\kappa_4(5\kappa_4 - 4\kappa_5) + \kappa_2(-3\kappa_4 + 2\kappa_5)) + 3a_4a_5^2(\kappa_2(2\kappa_4 - 3\kappa_5) + \kappa_5(-3\kappa_4 + \\
4\kappa_5)) + a_4(-a_2^2(2\kappa_2^2 - 3\kappa_2\kappa_4 + \kappa_4^2) + a_2^2(\kappa_2 - \kappa_5)^2 + 3a_4a_5^2(\kappa_2 - \kappa_5)^2 - a_2^2a_5(\kappa_2^2 - 3\kappa_2\kappa_5 + \\
2\kappa_5^2) + a_4a_5^2(5\kappa_2^2 + 2\kappa_4\kappa_5 - \kappa_2(3\kappa_4 + 4\kappa_5)))) \]

\[
\text{Dx6} = \text{Simplify}\left[\text{Series}\left[-D[H2r, f[a_6]], a_6\right], \{a_{11} \to 0, a_{13} \to 0, a_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0, \{a_{22}, 0, 0\}\right], \{a_{22} \to 0\}\right] 
\]

\[
\text{H2r} = \text{Simplify}\left[\text{Cancel}\left[H2r, \{a_{11} \to 0, a_{13} \to 0, a_{24} \to 0, \beta_{11} \to 0, \beta_{21} \to 0, \beta_{12} \to 0, \beta_{22} \to 0, \beta_{13} \to 0, \beta_{23} \to 0, \beta_{14} \to 0, \beta_{24} \to 0, \beta_{25} \to 0, \beta_{16} \to 0\}\right], \{a_{22} \to 0\}\right] 
\]

\[
\text{Da6r} = \text{Simplify}\left[D[H2r, \kappa_6]\right] 
\]

\[
\frac{1}{(a_1 - a_3)^2(a_1 - a_6)^2(a_3 - a_6)^2} \left( a_2^2a_6a_5(\kappa_6 + 3\kappa_2 - 3\kappa_3) + a_6^2(3\kappa_2 - 3\kappa_3) + a_2^2(\kappa_2 - \kappa_5)^2 + 3a_4a_5^2(\kappa_2 - \kappa_5)^2 - a_2^2a_5(\kappa_2^2 - 3\kappa_2\kappa_5 + 2\kappa_5^2) + a_4a_5^2(5\kappa_2^2 + 2\kappa_4\kappa_5 - \kappa_2(3\kappa_4 + 4\kappa_5))) \right) 
\]

\[
\left( a_2^2(\kappa_2^2 + \kappa_2(-6\kappa_4 + 2\kappa_5 + \kappa_4(\kappa_1 + 2\kappa_5))) + a_2^2(\kappa_4(3\kappa_4 - 4\kappa_5)(-\kappa_2 + \kappa_5) + 3a_4a_5^2(\kappa_2^2 + \\
\kappa_2\kappa_5 - 2\kappa_4\kappa_5) + a_2^2(\kappa_4(5\kappa_4 - 4\kappa_5) + \kappa_2(-3\kappa_4 + 2\kappa_5)) + 3a_4a_5^2(\kappa_2(2\kappa_4 - 3\kappa_5) + \kappa_5(-3\kappa_4 + \\
4\kappa_5)) + a_4(-a_2^2(2\kappa_2^2 - 3\kappa_2\kappa_4 + \kappa_4^2) + a_2^2(\kappa_2 - \kappa_5)^2 + 3a_4a_5^2(\kappa_2 - \kappa_5)^2 - a_2^2a_5(\kappa_2^2 - 3\kappa_2\kappa_5 + \\
2\kappa_5^2) + a_4a_5^2(5\kappa_2^2 + 2\kappa_4\kappa_5 - \kappa_2(3\kappa_4 + 4\kappa_5)))) \right) 
\]

\[
\text{Da6r} = \text{Da6r} 
\]

\[
\text{True} 
\]

\[
\text{We use the notation } A_{ij} \text{ for } \alpha_{ij} * 
\]
DA21 = Simplify[Series[D[H2, $\beta_{21}$]]/. \{a_{11} \rightarrow 0, a_{13} \rightarrow 0, a_{24} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{23} \rightarrow 0, \\
\beta_{14} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}/. \{a_{22} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{24} \rightarrow 0\}]/. \{k_3 \rightarrow k_1, \kappa_6 \rightarrow k_1\}

\[
\frac{1}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)}2a_{21}(a_0^2(b_1 + b_3) + a_1^2(b_3 - b_6) - a_2^2(b_1 + b_6) + 2a_1(-a_0(b_1 + b_3) + a_3(b_1 + b_6)))(a_{23}^2(a_1k_1 - a_6k_1) + a_3(-a_2^2k_1 + a_2^2k_1) + a_1(a_1 - a_6)a_6k_3) + a_1(a_0^2(b_1 + b_3) - a_3(b_1 + b_6)) + a_1(-a_0^2(b_1 + b_3) + a_2^2(b_1 + b_6)))(a_6k_1 - k_3) + a_1(-k_1 + k_3) + a_3(b_3(b_6) + a_1^2(b_3 - b_6) - a_3^2(b_1 + b_3) + a_1(-a_6(b_1 + b_3)) + a_3(b_1 + b_6)) + a_1(-a_0^2(b_1 + b_3) + a_2^2(b_1 + b_6)))(a_6k_1 - k_3) + a_1(-k_1 + k_3))
\]

DA21 = Simplify[DA21/. \{k_1 \rightarrow K, \kappa_3 \rightarrow K, \kappa_6 \rightarrow K\}] (* we will regard to K as to a constant *)

\[
\frac{2K(a_0^2(b_1 + b_3) + a_1^2(b_3 - b_6) - a_2^2(b_1 + b_6))}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)}a_{21}
\]

(* after a hand made simplification we obtain: *)

DA21 := \[
\frac{2K(a_0^2(b_1 + b_3) + a_1^2(b_3 - b_6) - a_2^2(b_1 + b_6))}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)}a_{21} - \frac{2K(a_0^2(b_1 + b_3) - a_2^2(b_1 + b_6))}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)}
\]

(* Finally DA21 := *)

DA23 = Simplify[Series[D[H2, $\beta_{23}$]]/. \{a_{11} \rightarrow 0, a_{13} \rightarrow 0, a_{24} \rightarrow 0, \beta_{21} \rightarrow 0, \beta_{12} \rightarrow 0, \beta_{23} \rightarrow 0, \\
\beta_{14} \rightarrow 0, \beta_{25} \rightarrow 0, \beta_{16} \rightarrow 0\}/. \{a_{22} \rightarrow 0, \beta_{22} \rightarrow 0, \beta_{24} \rightarrow 0\}]

\[
\frac{1}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)}2a_{23}(a_0^2(b_1 + b_3) + a_1^2(b_1 + b_6) + a_2^2(a_6(b_1 + b_3) - a_1(b_3 + b_6)) - 2a_1(a_0^2(b_1 + b_3) - a_1^2(b_1 + b_6)))(a_0(-k_1 + k_3) + a_3(k_1 - k_6) + a_1(-k_3 + k_6)) + (a_2^2(b_1 + b_3) + a_3^2(b_1 - b_6) - a_1^2(b_3 + b_6))(a_0^2(k_1 - k_3) + a_3^2(k_1 - k_6) + a_1^2(-k_3 + k_6)) + (a_0^2(b_1 + b_3) + a_3^2(b_1 - b_6) - a_1^2(b_3 + b_6) - 2a_1(a_0(b_1 + b_3) - a_1(b_3 + b_6)))(a_0(-k_1 + k_3) + a_3(k_1 - k_6) + a_1(-k_3 + k_6))]
\]

(* Finally DA23 := *)

\[
\frac{2K(a_0^2(b_1 + b_3) + a_1^2(b_3 - b_6) - a_2^2(b_1 + b_6))}{(a_1-a_3)^2(a_1-a_6)^2(a_3-a_6)^2(b_1+b_3)} = \frac{2K}{(a_1-a_3)(a_1-a_6)(a_3-a_6)} - \frac{2K(a_0^2(b_1 + b_3) - a_2^2(b_1 + b_6))}{(a_1-a_3)(a_1-a_6)(a_3-a_6)^2(b_1+b_3)} (*
\]

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