On simple ideal hyperbolic Coxeter polytopes

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Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space and let $P$ be a simple polytope in $\mathbb{H}^n$. $P$ is called a Coxeter polytope if all dihedral angles of $P$ are submultiples of $\pi$.

Hyperbolic Coxeter polytopes are not classified yet. Examples of compact Coxeter hyperbolic polytopes are known up to dimension $n \leq 8$ only, and examples of non-compact finite volume Coxeter polytopes are known up to dimension $n \leq 19$ [8], [9] and $n = 21$ [1]. It is also known that hyperbolic spaces of high dimension contain no finite volume Coxeter polytope. The estimate for the highest possible dimension of a finite volume Coxeter polytope is based on the following result of V. V. Nikulin.

Let $P$ be an $n$-dimensional simple polytope (where “simple” means that any $k$-dimensional face of $P$ belongs to exactly $n-k$ facets), and $\alpha_i$, $i = 0, 1, \ldots, n-1$, be the number of its $i$-dimensional faces ($i$-faces for short). For any face $f$ of $P$ denote by $\alpha_f$ the number of its $i$-faces. Denote by

$$\alpha^{(i)}_k = \frac{1}{\alpha_k} \sum_{\text{dim} f = k} \alpha^i_f$$

the average number of $i$-faces of a $k$-face of $P$.

Proposition 1 (Nikulin [4]). For any simple convex compact polytope $P$ in $\mathbb{R}^n$ for any $i < k \leq \lfloor n/2 \rfloor$ the following estimate holds:

$$\alpha^{(i)}_k < \left( \frac{n-i}{n-k} \right) \left( \binom{\lfloor n/2 \rfloor}{i} + \binom{(n+1)/2}{i} \right) \left( \binom{\lfloor n/2 \rfloor}{k} + \binom{(n+1)/2}{k} \right).$$

Using this estimate for 2-faces ($i = 0$ and $k = 2$) and the fact that any compact Coxeter polytope is simple, Vinberg [6] proved that no compact Coxeter polytope exists in $\mathbb{H}^n$ for $n > 29$.

In [3], Khovanskij proved that Nikulin’s estimate holds for edge-simple polytopes (a polytope is called edge-simple if any edge is the intersection of exactly $n-1$ facets). This was used by Prokhorov [2] when he proved that no Coxeter polytope of finite volume exists in $\mathbb{H}^n$ for $n \geq 996$.

A polytope $P$ is called ideal if all vertices of $P$ belong to the boundary of $\mathbb{H}^n$.

In this paper, we study simple ideal hyperbolic Coxeter polytopes. The main result is the following theorem.

Theorem 1. No simple ideal Coxeter polytope exists in $\mathbb{H}^n$ when $n > 8$. 
Section 1 contains basic definition and facts concerning Coxeter diagrams of spherical, Euclidean, and hyperbolic Coxeter polytopes. In Section 2, we study the combinatorics of Coxeter diagrams of simple ideal hyperbolic Coxeter polytopes. We show that if \( n > 5 \) then such a polytope has no triangular 2-faces and a few quadrilateral 2-faces. As shown in section 3 if \( n > 8 \) this contradicts Niculin’s estimate.

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1 Coxeter diagrams

It is convenient to describe Coxeter polytopes in terms of Coxeter diagrams.

A Coxeter diagram is one-dimensional simplicial complex with weighted edges, where weights are either of the type \( \cos \frac{\pi}{m} \) for some integer \( m \geq 3 \) or positive real numbers no less than one. We can suppress the weights but indicate the same information by labeling the edges of a Coxeter diagram in the following way: if the weight \( w_{ij} \) equals \( \cos \frac{\pi}{m} \), \( v_i \) and \( v_j \) are joined by an \((m - 2)\)-fold edge or a simple edge labeled by \( m \); if \( w_{ij} = 1 \), \( v_i \) and \( v_j \) are joined by a bold edge; if \( w_{ij} > 1 \), \( v_i \) and \( v_j \) are joined by a dotted edge labeled by its weight.

A subdiagram of a Coxeter diagram \( \Sigma \) is a subcomplex with the same weights as in \( \Sigma \).

Let \( \Sigma \) be a diagram with \( d \) nodes \( u_1, \ldots, u_d \). Define a symmetric \( d \times d \) matrix \( G(\Sigma) \) in the following way: \( g_{ii} = 1 \); if two nodes \( u_i \) and \( u_j \) are joined by an edge with weight \( w_{ij} \) then \( g_{ij} = -w_{ij} \); if two nodes \( u_i \) and \( u_j \) are not adjacent then \( g_{ij} = 0 \).

A Coxeter diagram \( \Sigma(P) \) of Coxeter polytope \( P \) is a Coxeter diagram whose matrix \( G(\Sigma) \) coincides with Gram matrix of \( P \). In other words, nodes of Coxeter diagram correspond to facets of \( P \). Two nodes are joined by either \((m - 2)\)-fold edge or \( m \)-labeled edge if the corresponding dihedral angle equals \( \frac{\pi}{m} \). If the corresponding facets are parallel the nodes are joined by a bold edge, and if they diverge then the nodes are joined by a dotted edge.

By the order of the diagram we mean the number of its nodes. By signature and rank of diagram \( \Sigma \) we mean the signature and the rank of the matrix \( G(\Sigma) \).

A Coxeter diagram \( \Sigma \) is called elliptic if the matrix \( G(\Sigma) \) is positive definite. A connected Coxeter diagram \( \Sigma \) is called parabolic if the matrix \( G(\Sigma) \) is degenerate, and any subdiagram of \( \Sigma \) is elliptic. Elliptic and connected parabolic diagrams are exactly Coxeter diagrams of spherical and Euclidean Coxeter simplices respectively, they were classified by Coxeter [2]. We represent the complete list of elliptic and connected parabolic diagrams in Table 1.

A non-connected diagram is called parabolic if it is a disjoint union of connected parabolic diagrams. A diagram is called indefinite if it contains at least one connected component that is neither elliptic nor parabolic.

Let \( F \) be a \( k \)-dimensional face of \( P \). Since \( P \) is simple, the face \( F \) belongs to exactly \( n - k \) facets \( f_1, \ldots, f_{n-k} \). Denote by \( v_1, \ldots, v_{n-k} \) the corresponding nodes of \( \Sigma(P) \).
Table 1: Connected elliptic and parabolic Coxeter diagrams are listed in the left and right columns respectively.

| Diagram | Description |
|---------|-------------|
| $A_n$ ($n \geq 1$) | [Diagram of $A_n$] |
| $\tilde{A}_1$ | [Diagram of $\tilde{A}_1$] |
| $\tilde{A}_n$ ($n \geq 2$) | [Diagram of $\tilde{A}_n$] |
| $B_n = C_n$ ($n \geq 2$) | [Diagram of $B_n = C_n$] |
| $\tilde{B}_n$ ($n \geq 3$) | [Diagram of $\tilde{B}_n$] |
| $\tilde{C}_n$ ($n \geq 2$) | [Diagram of $\tilde{C}_n$] |
| $D_n$ ($n \geq 4$) | [Diagram of $D_n$] |
| $\tilde{D}_n$ ($n \geq 4$) | [Diagram of $\tilde{D}_n$] |
| $G_{2}^{(m)}$ | [Diagram of $G_{2}^{(m)}$] |
| $\tilde{G}_2$ | [Diagram of $\tilde{G}_2$] |
| $F_4$ | [Diagram of $F_4$] |
| $\tilde{F}_4$ | [Diagram of $\tilde{F}_4$] |
| $E_6$ | [Diagram of $E_6$] |
| $\tilde{E}_6$ | [Diagram of $\tilde{E}_6$] |
| $E_7$ | [Diagram of $E_7$] |
| $\tilde{E}_7$ | [Diagram of $\tilde{E}_7$] |
| $E_8$ | [Diagram of $E_8$] |
| $\tilde{E}_8$ | [Diagram of $\tilde{E}_8$] |
| $H_3$ | [Diagram of $H_3$] |
| $H_4$ | [Diagram of $H_4$] |

$\Sigma_F$ be a subdiagram of $\Sigma(P)$ with nodes $v_1, \ldots, v_{n-k}$. We say that $\Sigma_F$ is the *diagram of the face* $F$. By *complete diagram of the face* $F$ we mean the minimal subdiagram of $\Sigma(P)$ containing the diagrams of all vertices of $F$.

The following properties of $\Sigma(P)$ and $\Sigma_F$ are proved in [7].

- [Cor. of Th. 2.1] the signature of $G(\Sigma(P))$ equals $(n, 1)$;
- [Cor. of Th. 3.1] if a $k$-face $F$ is not an ideal vertex of $P$ (i.e. $F$ is not a point at the boundary of $\mathbb{H}^n$), then $\Sigma_F$ is an elliptic diagram of rank $n - k$;
• [Cor. of Th. 3.2] if $F$ is an ideal vertex of $P$ then $\Sigma_F$ is a parabolic diagram of rank $n - 1$; if $F$ is a simple ideal vertex of $P$ (i.e. $F$ belongs to exactly $n$ facets) then $\Sigma_F$ is connected;

• [Cor. of Th. 3.1 and Th. 3.2] any elliptic subdiagram of $\Sigma(P)$ corresponds to a face of $P$; any parabolic subdiagram of $\Sigma(P)$ is a subdiagram of the diagram of exactly one ideal vertex of $P$.

For a simple ideal Coxeter polytope $P \subset \mathbb{H}^n$ this implies that

(i) Any two non-intersecting indefinite subdiagrams of $\Sigma(P)$ are joined in $\Sigma(P)$.

(ii) Any elliptic subdiagram of $\Sigma(P)$ contains at most $n - 1$ nodes.

(iii) Any parabolic subdiagram of $\Sigma(P)$ is connected and contains exactly $n$ nodes.

**Lemma 1.** A Coxeter diagram of a simple ideal Coxeter polytope in $\mathbb{H}^n$, $n > 3$, contains only simple edges, 2-fold edges and dotted edges.

**Proof.** It follows from Table 1 that any connected parabolic diagram containing at least three nodes contains neither bold edges nor edges of multiplicity $m > 2$. Thus, its enough to show that any non-dotted edge of $\Sigma(P)$ belongs to some connected parabolic subdiagram of order $n$. Indeed, such an edge (denote it by $uv$) together with its ends compose a rank 2 elliptic subdiagram. Hence, it is a diagram of some $(n - 2)$-face $F$ of $P$. The face $F$ has at least one vertex, and the diagram of this vertex is a connected parabolic subdiagram of $\Sigma(P)$ of order $n$ containing the diagram of $F$, i.e. containing the edge $uv$.

\[
\boxempty
\]

**Notation**

Let $F$ be a $k$-face of $P$ and let $f_1, \ldots, f_{n-k}$ be the facets of $P$ containing $F$. Let $v_1, \ldots, v_{n-k}$ be the corresponding nodes of $\Sigma(P)$. As above, we denote by $\Sigma_F$ the diagram of the face $F$, i.e. the subdiagram of $\Sigma(P)$ spanned by the nodes $v_1, \ldots, v_{n-k}$.

- We write $\Sigma_F = \langle v_1, \ldots, v_{n-k} \rangle$ and $\Sigma_F = \langle v_1, \Theta \rangle$, where $\Theta = \langle v_2, \ldots, v_{n-k} \rangle$.
- We denote by $\Sigma \setminus \{v_1, \ldots, v_m\}$ the subdiagram of $\Sigma$ spanned by all nodes of $\Sigma$ different from $v_1, \ldots, v_m$.
- For elliptic and parabolic diagrams we use standard notation (see Table 1). For example, we write $\Sigma_F = \tilde{A}_{n-1}$ if $F$ is an ideal vertex of the type $\tilde{A}_{n-1}$.

- Let $v$ and $u$ be two nodes of $\Sigma(P)$. We write

\[
[v, u] = 0 \text{ if } u \text{ and } v \text{ are not joined in } \Sigma(P);
\]

\[
[v, u] = 1 \text{ if } u \text{ and } v \text{ are joined by a simple edge};
\]

\[
[v, u] = 2 \text{ if } u \text{ and } v \text{ are joined by a 2-fold edge};
\]

\[
[v, u] = \infty \text{ if } u \text{ and } v \text{ are joined by a dotted edge}.
\]
2 Absence of triangular 2-faces and estimate for quadrilateral 2-faces.

Let $P$ be a simple ideal Coxeter polytope in $\mathbb{H}^n$ and let $V$ be a vertex of $P$. Since $P$ is simple, the vertex $V$ is contained in exactly $n$ edges $VV_i$, $i = 1, \ldots, n$. Denote by $v_i$ the node of $\Sigma_V$ such that $\Sigma_{V V_i} = \Sigma_V \setminus \{v_i\}$. Denote by $u_i$ the node of $\Sigma(P)$ such that $\Sigma_{V_i} = <u_i, \Sigma_{V V_i}>$. Clearly, the diagram $\Sigma_i = <u_i, \Sigma_V> = <v_i, \Sigma_{V V_i}> = <u_i, v_i, \Sigma_{V V_i}>$ is the complete diagram of the edge $VV_i$.

Notice that $\Sigma_i$ contains exactly two parabolic subdiagrams $\Sigma_V$ and $\Sigma_{V_i}$, therefore, it is possible to find the nodes $v_i$ and $u_i$ in $\Sigma_i$ by formulae $u_i = \Sigma_i \setminus \Sigma_V$, $v_i = \Sigma_i \setminus \Sigma_{V_i}$. We say that a complete diagram of the edge $\Sigma_i$ is elementary if there exists an automorphism of the diagram $\Sigma_i$ interchanging the nodes $u_i$ and $v_i$ and preserving the rest nodes. Otherwise we say that the complete diagram of the edge is non-elementary.

For any connected parabolic diagram $\Sigma_V$ it is not difficult to describe all possible complete diagrams of edges containing $\Sigma_V$. For example, suppose that $\Sigma_V = \tilde{A}_{n-1}$, $n \neq 3, 8, 9$. Then $\Sigma_{V V_i} = \Sigma_V \setminus v_i = A_{n-1}$. It is easy to see, that if $n \neq 3, 8, 9$ then $\tilde{A}_{n-1}$ is the only parabolic diagram with $n$ nodes containing a subdiagram $A_{n-1}$. Thus, $\Sigma_{V_i} = A_{n-1}$, and the diagram $<v_i, u_i, \Sigma_{V V_i}>$ is elementary. Furthermore, $[v_i, u_i] \neq 0$ and $[v_i, u_i] \neq 1$, otherwise $<v_i, u_i, \Sigma_{V V_i}>$ does not satisfy condition (iii). Hence, either $[v_i, u_i] = 2$ or $[v_i, u_i] = \infty$ (Lemma 1), and the diagram $<v_i, u_i, \Sigma_{V V_i}>$ is one of two diagrams shown in Fig 1.

Figure 1: Two possibilities for the complete diagram $<v_i, u_i, \Sigma_{V V_i}>$ of the edge $VV_i$, if $\Sigma_V = \tilde{A}_{n-1}$, $n \neq 3, 8, 9$. Both complete diagrams are elementary.

Similarly, one can list all possible diagrams $<u_i, v_i, \Sigma_{V V_i}>$ for any other type of $\Sigma_V$ (recall that $\Sigma_V$ is one of the diagrams shown in the right column of Table 1).

Lemma 2. Let $<u_i, v_i, \Sigma_{V V_i}>$ be a non-elementary complete diagram of the edge $VV_i$. If $n > 5$ and $\Sigma_{V V_i}$ is connected then $<u_i, v_i, \Sigma_{V V_i}>$ is one of the diagrams listed in Table 2.
Table 2: Non-elementary diagram of the edge \(<v_i, u_i, \Sigma_{VV_i}>\) such that \(n > 5\) and the diagram \(\Sigma_{VV_i}\) is connected. The waved edge connecting \(v_i\) and \(u_i\) means that \([v_i, u_i] \in \{0, 1, 2, \infty\}\) (for some of these values the conditions (i)–(iii) does not hold).

| \(\Sigma_{VV_i} = A_{n-1}\) | \(\Sigma_{VV_i} = B_{n-1}\) | \(\Sigma_{VV_i} = D_{n-1}\) |
|-------------------------------|-------------------------------|-------------------------------|
| ![Diagram A_{n-1}]            | ![Diagram B_{n-1}]            | ![Diagram D_{n-1}]            |
| \(\bar{A}_7 \leftrightarrow \bar{E}_7\) | \(\bar{C}_{n-1} \leftrightarrow \bar{B}_{n-1}\) | \(\bar{D}_{n-1} \leftrightarrow \bar{B}_{n-1}\) |
| \(\bar{A}_8 \leftrightarrow \bar{E}_8\) | ![Diagram C_{n-1} & D_{n-1}] | ![Diagram E_{n-1} & D_{n-1}] |
| \(\bar{E}_8 \leftrightarrow \bar{E}_8\) | ![Diagram E_{n-1} & D_{n-1}] | ![Diagram E_{n-1} & D_{n-1}] |

**Proof.** There are two ways to obtain an edge \(VV_i\) with a non-elementary complete diagram: either the diagrams of the vertices \(V\) and \(V_i\) are different or the diagrams are same but the nodes \(u_i\) and \(v_i\) are attached to the diagram \(\Sigma_{VV_i}\) in different ways. Since \(n > 5\) and the diagram \(\Sigma_{VV_i}\) is connected, the diagram \(\Sigma_{VV_i}\) should be of one of the types \(A_{n-1}, B_{n-1}, D_{n-1}\) and \(E_6, E_7, E_8\). Consider these cases.

1. Suppose that \(\Sigma_{VV_i} = A_{n-1}\). Then the diagrams \(<u_i, \Sigma_{VV_i}>\) and \(<v_i, \Sigma_{VV_i}>\) are parabolic diagrams of order \(n\) containing a subdiagram of the type \(A_{n-1}\). Hence, each of these diagrams is of one of the types \(\bar{A}_{n-1} (n \geq 6), \bar{E}_7 (n = 8)\), and \(\bar{E}_8 (n = 9)\).

Furthermore, the diagram \(A_{n-1}\) extends to \(A_{n-1}\) in a unique way, \(A_7\) extends to \(\bar{E}_7\) in a unique way, and \(A_8\) extends to \(\bar{E}_8\) in two different ways. Thus, if \(n \neq 8, 9\) the complete diagram of the edge \(VV_i\) is always elementary. If \(n = 8\) we obtain a unique non-elementary diagram (where the multiplicity of the edge \(u_i v_i\) may vary), denote this diagram by \(A_7 \leftrightarrow \bar{E}_7\). If \(n = 9\) we obtain three non-elementary diagrams (two diagrams of the type \(\bar{A}_8 \leftrightarrow \bar{E}_8\) and one of the type \(\bar{E}_8 \leftrightarrow \bar{E}_8\)), however, two diagrams of the type
\( \bar{A}_8 \leftrightarrow \bar{E}_8 \) coincide modulo the renumbering of the nodes. So, in case \( \Sigma_{V^i} = A_{n-1} \) we obtain three non-elementary diagrams, see the left column of Table 2.

2. Suppose that \( \Sigma_{V^i} = B_{n-1} \). Since \( n > 5 \), each of the diagrams \( <u_i, \Sigma_{V^i}> \) and \( <v_i, \Sigma_{V^i}> \) is of the type \( \bar{B}_{n-1} \) or \( \bar{C}_{n-1} \). The diagram \( B_{n-1} \) may be extended to each of these diagram in a unique way, and we obtain a unique non-elementary diagram \( \bar{C}_{n-1} \leftrightarrow \bar{B}_{n-1} \), see the middle column of Table 2.

3. Suppose that \( \Sigma_{V^i} = D_{n-1} \). Then each of the diagrams \( <u_i, \Sigma_{V^i}> \) and \( <v_i, \Sigma_{V^i}> \) is of one of the types \( \bar{B}_{n-1} \), \( \bar{D}_{n-1} \) \( (n \geq 6) \), and \( \bar{E}_8 \) \( (n = 9) \). Since \( n > 5 \), the diagram \( D_{n-1} \) extends to each of the diagrams \( \bar{B}_{n-1} \) and \( \bar{D}_{n-1} \) in a unique way. The diagram \( D_8 \) extends to the diagram \( \bar{E}_8 \) in two different ways, and we obtain four non-elementary diagrams \( <u_i, v_i, \Sigma_{V^i}> \) shown in the right column of Table 2.

4. Suppose that \( \Sigma_{V^i} = E_6, E_7 \) or \( E_8 \). Then each of the diagrams \( <u_i, \Sigma_{V^i}> \) \( <v_i, \Sigma_{V^i}> \) is of the types \( \bar{E}_6, \bar{E}_7 \) or \( \bar{E}_8 \) respectively. Since each of the diagrams \( E_k \) \( (k = 6, 7, 8) \) extends to the diagram \( \bar{E}_8 \) in a unique (modulo the renumbering of the nodes) way, the diagram \( <u_i, v_i, \Sigma_{V^i}> \) is elementary, and the lemma is proved.

The node \( v \) of the diagram \( \Sigma \) is called a leaf of the diagram \( \Sigma \), if \( v \) belongs to exactly one edge of \( \Sigma \).

**Lemma 3.** Let \( <v_i, u_i, \Sigma_{V^i}> \) be a complete diagram of the edge \( VV_i \). If \( n > 5 \) then \( [v_i, u_i] \neq 0 \) and \( [v_i, u_i] \neq 1 \).

**Proof.** Suppose that \( [v_i, u_i] = 0 \) or \( 1 \).

Assume that the complete diagram \( <v_i, u_i, \Sigma_{V^i}> \) of the edge \( VV_i \) is elementary, and consider two cases.

(a) Suppose that \( v_i \) is a leaf of \( <v_i, \Sigma_{V^i}> \). Denote by \( a \) the node of \( <v_i, \Sigma_{V^i}> \) joined with \( v_i \). Then the assumptions that the diagram \( <v_i, u_i, \Sigma_{V^i}> \) is elementary and that \( [v_i, u_i] \neq \infty \) imply that \( <v_i, u_i, \Sigma_{V^i} \setminus a> \) is an elliptic subdiagram of order \( n \), that contradicts condition (1).

(b) Suppose that \( v_i \) is not a leaf of \( <v_i, \Sigma_{V^i}> \). Then there are at least two nodes \( a_1 \) and \( a_2 \) in \( <v_i, \Sigma_{V^i}> \) joined with \( v_i \). Table 1 implies that one of the edges \( a_1v_i \) and \( a_2v_i \) is simple and another one is either simple or double. Since the diagram \( <v_i, u_i, \Sigma_{V^i}> \) is elementary and \( [v_i, u_i] = 0 \) or \( 1 \), we obtain that the diagram \( <v_i, u_i, a_1, a_2> \) contains a parabolic subdiagram of the type \( \bar{A}_2, \bar{C}_2 \) or \( \bar{A}_3 \), which is impossible by condition (3).

Now, suppose that the diagram \( <v_i, u_i, \Sigma_{V^i}> \) is not elementary. Suppose in addition that \( \Sigma_{V^i} \) is connected. Then by Lemma 2 the diagram \( <u_i, v_i, \Sigma_{V^i}> \) is one of the diagrams listed in Table 2. However, if \( [v_i, u_i] = 0 \) or \( 1 \), none of these diagrams satisfies conditions (1) and (3) simultaneously.
Therefore, the diagram $\Sigma_{VV_i}$ is not connected. Let $\Sigma_1$ and $\Sigma_2$ be some connected components of $\Sigma_{VV_i}$ (it follows from Table that $\Sigma_{VV_i}$ contains at most 3 connected components). Clearly, each of the nodes $v_i$ and $u_i$ is joined with each connected component by exactly one edge. Hence, the diagram $<\Sigma_1, \Sigma_2, v_i, u_i>$ contains a cycle $C$ including the nodes $v_i$ and $u_i$.

Suppose that the subdiagram $<\Sigma_1, \Sigma_2>$ contains no double edges. Then all edges of $<\Sigma_1, \Sigma_2, v_i, u_i>$ are simple, and the cycle $C$ is a parabolic diagram of the type $\tilde{A}_k$ containing the nodes $u_i$ and $v_i$. If $k < n − 1$ this is impossible by condition (iii), and the case $k = n − 1$ contradicts the assumption that $<v_i, u_i, \Sigma_{VV_i}>$ is the complete diagram of the edge $VV_i$.

Therefore, at least one of the diagrams $\Sigma_1$ and $\Sigma_2$ contains a double edge which is included in the cycle $C$, i.e. either $\Sigma_1$ or $\Sigma_2$ is a diagram $B_k$ for some $2 \leq k < n − 1$. We assume that $\Sigma_1 = B_k$ and denote by $t_1$ and $t_2$ the ends of the double edge in such a way that $t_1$ is a leaf of $\Sigma_1$. Since the edge $t_1t_2$ belongs to the cycle $C$, one of the nodes $u_i$ and $v_i$ (say, $u_i$) is joined with $t_1$, and another one ($v_i$) is not. If $k > 2$ then the nodes $t_1$ and $t_2$ are not leaves of the parabolic diagram $<u_i, \Sigma_{VV_i}>$, and hence, $<u_i, \Sigma_{VV_i}> = \tilde{F}_4$ (see Table), that contradicts the assumption that $n > 5$.

Thus, $\Sigma_1 = B_2 = t_1t_2$, and $u_i$ is joined with $t_1$, while $v_i$ is joined with $t_2$. It follows from the classification of parabolic diagrams that the edges $u_it_1$ and $v_it_2$ are simple. Consider two cases: $[u_i, v_i] = 1$ or $[u_i, v_i] = 0$.

- If $[u_i, v_i] = 1$ then the diagram $<u_i, v_i, \Sigma_2>$ also contains a cycle, and by the same reasoning as above we obtain $\Sigma_2 = B_2$. Since the diagram $\tilde{C}_{n−1}$ is the only connected parabolic linear diagram of order $n$ containing a subdiagram of the type $B_2 + B_2$, the diagram $\Sigma_{VV_i}$ contains no other connected components besides $\Sigma_1 = B_2$ and $\Sigma_2 = B_2$. Therefore, $n − 1 = 4$, which contradicts the assumption that $n > 5$.

- If $[u_i, v_i] = 0$ then $<u_i, v_i, \Sigma_1> = F_4$. Suppose that $u_i$ and $v_i$ are joined with one and the same node $x$ of $\Sigma_2$. Then $<u_i, v_i, \Sigma_1 \setminus x>$ is an elliptic diagram of order $n$, which is impossible by condition (ii). Hence, $u_i$ and $v_i$ are joined with distinct nodes $x_1$ and $x_2$ of $\Sigma_2$. If $u_i$ is joined with $x_1$ by a simple edge, then $<x_1, u_i, t_1, t_2, v_i> = \tilde{F}_4$ which contradicts either condition (iii) or the assumption $n > 5$. If $u_i$ is joined with $x_1$ by a double edge then $<x_1, u_i, t_1, t_2> = \tilde{C}_3$, which contradicts either condition (iii) or the assumption $n > 5$ again.

\[\square\]

**Lemma 4.** Let $P$ be a simple ideal Coxeter polytope in $\mathbb{H}^n$, $n > 5$. Then $P$ has no triangular 2-faces.

**Proof.** Suppose that $UVW$ is a triangular 2-face of $P$. Since $P$ is simple, the triangle $UVW$ is contained in exactly $n−2$ facets. There exists a unique facet containing the edge $VW$ and not containing the triangle $UVW$. Denote this facet by $\tilde{u}$. Similarly, determine
facets $\bar{v}$ and $\bar{w}$ as facets containing the edges $UW$ and $UV$ and not containing $UVW$. Denote by $u$, $v$ and $w$ the nodes of $\Sigma(P)$ corresponding to $\bar{u}$, $\bar{v}$ and $\bar{w}$ respectively. Notice that the diagram of $U$ coincides with $<v, w, \Sigma_{UVW}>$. Similarly, $\Sigma_V = <u, w, \Sigma_{UVW}>$ and $\Sigma_W = <u, v, \Sigma_{UVW}>$ (see Figure 2a). In particular, (iii) implies that all these diagrams are connected and parabolic. We also obtain that the diagram $\Sigma = <u, v, w, \Sigma_{UVW}>$ is the complete diagram of the triangular 2-face $UVW$ as well as the complete diagram of each of the edges $VW$, $UW$ and $UV$.

Consider the edge of $\Sigma$ joining $u$ and $v$. By Lemma 3, either $[u, v] = 2$ or $[u, v] = \infty$. Suppose that $[u, v] = \infty$, then $\Sigma_W = <u, v, \Sigma_{UVW}>$ contains a dotted edge in contradiction to the assumption that $\Sigma_W$ is parabolic. Thus, $[u, v] \neq \infty$, i.e. $[u, v] = 2$. Similarly, $[v, w] = 2$ and $[u, w] = 2$. Furthermore, since $\Sigma_W$ is a parabolic diagram of order $n > 5$, one of the nodes $u$ and $v$ of the double edge $uv$ is a leaf. Assume that $u$ is a leaf of $\Sigma_W$, i.e. $u$ is not joined with $\Sigma_{UVW}$. Then, evidently, $v$ is joined with $\Sigma_{UVW}$. Similarly, from the diagram $\Sigma_U = <v, w, \Sigma_{UVW}>$ we obtain that $w$ is not joined with $\Sigma_{UVW}$. Hence, the diagram $<u, w, \Sigma_{UVW}> = \Sigma_V$ is not connected in contradiction to condition (iii).

Notice, that an ideal Coxeter polytope in $\mathbb{H}^5$ may have a triangular 2-face. For example, the Coxeter diagram shown in Figure 3 determines a 5-dimensional ideal Coxeter simplex. All 2-faces of any simplex are triangles.

Figure 2: Notation for a triangle (a) and for a quadrilateral (b).

Figure 3: This diagram determines a 5-dimensional ideal Coxeter simplex.
Lemma 5. Let $V$ be a vertex of simple ideal Coxeter polytope $P$ in $\mathbb{H}^n$, $n > 9$. Then $V$ belongs to at most $n + 3$ quadrilateral 2-faces.

Proof. Let $Q$ be a quadrilateral 2-face with vertices $V, V_i, V_j$ and $V_{ij}$. Then $Q$ belongs to $n - 2$ facets, each edge of $Q$ belongs to $n - 1$ facets and each vertex belongs to $n$ facets. Denote by $\tilde{v}_i, \tilde{u}_i, \tilde{v}_j$ and $\tilde{u}_j$ the facets not containing $Q$ and containing the edges $VV_j, V_iV_{ij}, VV_i$ and $V_jV_{ij}$ respectively (see Figure 2b). Denote by $v_i, u_i, v_j$ and $u_j$ the nodes of $\Sigma(P)$ corresponding to the facets $\tilde{v}_i, \tilde{u}_i, \tilde{v}_j$ and $\tilde{u}_j$ respectively.

Then $\Sigma_V = \langle v_i, v_j, \Sigma_Q \rangle$, $\Sigma_{V_i} = \langle v_j, u_i, \Sigma_Q \rangle$, $\Sigma_{V_j} = \langle v_i, u_j, \Sigma_Q \rangle$, and $\Sigma_{V_{ij}} = \langle u_i, u_j, \Sigma_Q \rangle$. Thus, $\Sigma = \langle v_i, u_i, v_j, u_j, \Sigma_Q \rangle$ is the complete diagram of the face $Q$ (see Fig. 4 for the example of a complete diagram of a quadrilateral).

![Diagram](image-url)

Figure 4: Example of a quadrilateral $VV_iV_{ij}V_j$.

Suppose that $\Sigma_V = \tilde{A}_{n-1}$. Since $n > 8$, the diagram of each of the vertices $V_i, V_j, V_{ij}$ is of the type $\tilde{A}_{n-1}$. Consider the diagram $\langle v_i, u_i, \Sigma_{VV_i} \rangle$, i.e. the complete diagram of the edge $VV_i$. By Lemma 3, $[v_i, u_i] = \infty$ or 2 (compare with Fig. 1). Similarly, from the complete diagram of the edge $VV_j$, we obtain that $[v_j, u_j] = \infty$ or 2.

Suppose that the nodes $v_i$ and $v_j$ are not joined in $\Sigma_V$. Then the diagram $\Sigma_Q = \Sigma_V \setminus \{v_i, v_j\}$ is not connected. On the other hand, $\Sigma_{V_{ij}} \setminus \{u_i, u_j\} = \Sigma_Q$, and we obtain $[u_i, u_j] = 0$ (see Fig. 5). Since $n > 5$, at least one of the connected components $\Sigma_1$ and $\Sigma_2$ of the diagram $\Sigma_Q$ contains at least three nodes, and we may assume that $\Sigma_1 = A_k$, $k \geq 3$. Denote by $w_i$ and $w_j$ the leaves of the diagram $\Sigma_1$ joining $\Sigma_V$ with $v_i$ and $v_j$ respectively. Then $\Sigma(P)$ contains two unjoined indefinite subdiagrams $v_i u_i w_i$ and $v_j u_j w_j$, which is
impossible by condition (1). Therefore, the nodes \( v_i \) and \( v_j \) are joined in the diagram \( \Sigma_V \), that implies that each quadrilateral face containing the vertex \( V \) corresponds to a pair of neighboring nodes in \( \Sigma_V \). Hence, \( V \) belongs to at most \( n \) quadrilateral 2-faces.

Figure 5: \( v_i u_i w_i \) and \( v_j u_j w_j \) are unjoined indefinite subdiagrams. In this diagram \( k_i, k_j = 2 \) or \( \infty \).

From now on we assume that \( \Sigma_V \neq \overline{A}_{n-1} \). Since \( n > 9 \), \( \Sigma_V = \overline{B}_{n-1}, \overline{C}_{n-1} \) or \( \overline{D}_{n-1} \). Define a distance \( \rho(u, w) \) between two nodes \( u \) and \( w \) of connected graph as the number of edges in the shortest path connecting \( u \) and \( w \).

Let \( x \) be a leaf of \( \Sigma_V \). Denote by \( \Sigma_V^{(5)}(x) \) a connected subdiagram of \( \Sigma_V \) spanned by five nodes closest to the leaf \( x \) in \( \Sigma_V \) (i.e., if \( v_k \in \Sigma_V^{(5)}(x) \) and \( v_l \notin \Sigma_V^{(5)}(x) \) then \( \rho(x, v_k) \leq \rho(x, v_l) \)). Notice that if \( \Sigma_V = \overline{B}_{n-1}, \overline{C}_{n-1} \) or \( \overline{D}_{n-1} \) when \( n \geq 9 \), then diagram \( \Sigma_V^{(5)}(x) \) is well-defined for any leaf \( x \) of \( \Sigma_V \).

Denote by \( L(\Sigma_V) \) the set of leaves of \( \Sigma_V \). Define

\[
\Sigma_V^{(5)} \overset{\text{def}}{=} \bigcup_{x \in L(\Sigma_V)} \Sigma_V^{(5)}(x)
\]

(see Fig. 6 for the example). It is easy to see that if \( n > 10 \) then \( \Sigma_V^{(5)} \) consists of two connected components. If \( n = 10 \), \( \Sigma_V^{(5)} \) is connected. However, it contains two leaves \( x \) and \( y \) such that \( \Sigma_V = \Sigma_V^{(5)}(x) \cup \Sigma_V^{(5)}(y) \), and \( \Sigma_V^{(5)}(x) \cap \Sigma_V^{(5)}(y) = \emptyset \). In this case we say that the diagrams \( \Sigma_V^{(5)}(x) \) and \( \Sigma_V^{(5)}(y) \) are “components”, and use this notion in case \( n = 10 \) instead of the connected components in general case \( n > 10 \).

Figure 6: Subdiagram \( \Sigma_V^{(5)} \) for \( \Sigma_V = \overline{B}_{12} \).
Suppose that \( v_i \) and \( v_j \) do not belong to the same connected component of \( \Sigma^{(5)}_V \) (respectively, to a “component” for \( n = 10 \)). Suppose that \( v_i \) is not joined with \( v_j \). A direct check of the conditions (i)–(iii) for each of the possible diagrams shows that if \( \Sigma_Q \) is a diagram of a quadrilateral 2-face then the corresponding connected component (or the “component”) of the diagram \( \Sigma^{(5)}_V \) coincides (modulo interchanging of \( v_i \) and \( v_j \)) with one of the following diagrams:

\[
\begin{align*}
\text{Diagram 1} & : v_i \quad v_j \\
\text{Diagram 2} & : v_i \quad v_j \\
\text{Diagram 3} & : v_i \quad v_j \\
\end{align*}
\]

Therefore, a quadrilateral 2-face \( Q \) containing \( V \) is of one of the following types: either \( Q \) corresponds to a pair of joined nodes in \( \Sigma_V \) (there are \( n - 1 \) of such pairs) or to one of two pairs described above for each of the connected components (or “components” for \( n = 10 \)) of the diagram \( \Sigma^{(5)}_V \). Hence, \( V \) belongs to at most \( 2 + 2 + (n - 1) = n + 3 \) quadrilaterals.

\[\square\]

**Lemma 6.** Let \( V \) be a vertex of a simple ideal Coxeter polytope \( P \) in \( \mathbb{H}^9 \). Then \( V \) belongs to at most 15 quadrilateral 2-faces.

**Proof.** Let \( v_1, \ldots, v_9 \) be the nodes of \( \Sigma_V \). While proving Lemma 5 we estimated the number of pairs \((v_i, v_j)\) such that the 2-face corresponding to the diagram \( \Sigma_V \setminus \{v_i, v_j\} \) may be quadrilateral. In other words, we were looking for the pairs \( (v_i, v_j) \), such that the diagram \( \Sigma_V \) can be accompanied by some additional nodes \( x \) and \( y \) subject to the following two properties: 1) the diagram \( <x, y, \Sigma_V> \) satisfies to conditions (i)—(iii), 2) the diagrams \( <x, \Sigma_V \setminus v_i>, <y, \Sigma_V \setminus v_j> \) and \( <x, y, \Sigma_V \setminus \{v_i, v_j\}> \) are connected and parabolic. In this case \( <x, y, \Sigma_V> \) is a complete diagram of a quadrilateral 2-face with diagram \( \Sigma_V \setminus \{v_i, v_j\} \).

Clearly, this estimate of the number of quadrilaterals is rough. In particular, for \( n = 9 \) the result of this estimate worse than one claimed in the lemma. To prove the lemma we use another method leading to the better estimate, but using much more computations. We proceed by the following algorithm:

**Step 1.** We consider the cases \( \Sigma_V = \tilde{A}_8, \tilde{B}_8, \tilde{C}_8, \tilde{D}_8 \) and \( \tilde{E}_8 \) separately.

**Step 2.** We want to list all possibilities for complete diagrams of edges incident to \( V \). To do this, for each node \( v_i \) of \( \Sigma_V \) \((i = 1, \ldots, 9)\) we list all possible diagrams \( <u^k_i, \Sigma_V>, k = 1, \ldots, k_i \) satisfying conditions (i)—(iii) and such that \( <u^k_i, \Sigma_V \setminus v_i> \) is a parabolic diagram. Here \( k_i \) stays for the number of different complete diagrams of edges found for each of the nodes \( v_i \) of \( \Sigma_V \). A straightforward check shows that \( 1 \leq k_i \leq 8 \) for different nodes of the diagrams \( \tilde{A}_8, \tilde{B}_8, \tilde{C}_8, \tilde{D}_8 \) and \( \tilde{E}_8 \) (for instance, \( k_i = 8 \) for one of the nodes of \( \tilde{E}_8 \)).
Step 3. For each pair \( (u^k_i, \Sigma_V), (u^l_j, \Sigma_V) \) of complete diagrams of edges \((i \neq j)\) we check if it is possible to assign a weight to the edge \(u^k_i u^l_j\) in order to turn the diagram \(u^k_i, u^l_j, \Sigma_V\) into a complete diagram of a quadrilateral. In particular, this implies that the diagram \(u^k_i, u^l_j, \Sigma_V \backslash \{v_i, v_j\}\) is parabolic, and hence, the nodes \(u^k_i\) and \(u^l_j\) are either unjoined or joined by a simple or double edge. Each of the pairs \((u^k_i, \Sigma_V), (u^l_j, \Sigma_V)\) obtained we call a good pair of complete diagrams of edges.

Step 4. For each of the nodes \(v_i, 1 \leq i \leq 9\), we choose a complete diagram \(u^r_i, \Sigma_V\) of an edge, \(1 \leq r_i \leq k_i\). Then compute the total number of the good pairs \((u^r_i, \Sigma_V), (u^l_j, \Sigma_V)\) \((\text{where } i \neq j)\). At this step we should check rather huge number of cases \((\text{more than 15000 in case } \Sigma_V = \tilde{E}_8)\), therefore, this was done by a computer program.

The number obtained in this step we denote by \(M(r_1, \ldots, r_9)\). Denote by \(M(\Sigma_V)\) the maximal value of \(M(r_1, \ldots, r_9)\) on the 9-tuples \((r_1, \ldots, r_9)\), where \(1 \leq r_i \leq k_i\). Clearly, the number of quadrilateral 2-faces containing the vertex \(V\) is bounded by \(M(\Sigma_V)\). Notice also, that the estimate is still rough \((\text{for example, we do not check if the conditions (i)—(iii) are satisfied by subdiagrams containing more than } n + 2 \text{ nodes})\). The computation shows that

\[
\begin{align*}
M(\tilde{A}_8) &= 15, \\
M(\tilde{B}_8) &= 14, \\
M(\tilde{C}_8) &= 12, \\
M(\tilde{D}_8) &= 15, \\
M(\tilde{E}_8) &= 14.
\end{align*}
\]

Thus, for any type of \(\Sigma_V\) we obtain that \(V\) belongs to at most 15 quadrilateral 2-facets.

\[
\square
\]

3 Absence of simple ideal Coxeter polytopes in large dimensions.

Recall that \(\alpha_i\) denotes the number of \(i\)-faces of a polytope \(P\) and \(\alpha_k^{(i)}\) denotes the average number of \(i\)-faces of \(k\)-face of \(P\).

Lemma 7. Let \(P\) be an \(n\)-dimensional simple polytope and let \(l\) be the number of vertices of \(P\). Then

\[
\frac{l}{\alpha_2} = \frac{2}{n(n - 1)} \alpha_2^{(1)}. \quad (1)
\]
Proof. Denote by $m_i$ the number of $i$-angular 2-faces of $P$. Let us compute the total number $N$ of vertices of 2-faces. Clearly, $N = \sum_{i \geq 3} i \cdot m_i$. On the other hand, each pair of edges incident to one vertex of simple polytope determines a 2-face of the polytope. Thus, $N = \frac{n(n-1)}{2}$, and we obtain the following equality

$$\frac{n(n-1)}{2} = \sum_{i \geq 3} i \cdot m_i. \tag{2}$$

By definition,

$$\alpha_2^{(1)} = \frac{\sum_{i \geq 3} i \cdot m_i}{\alpha_2}. \tag{3}$$

Combining (2) and (3), we obtain

$$\frac{l}{\alpha_2} = \frac{2}{n(n-1)} \sum_{i \geq 3} i \cdot m_i = \frac{2}{n(n-1)} \alpha_2^{(1)}. \tag{6}$$

Proof of the theorem. We use the notation from Lemma 7. Recall, that $\alpha_2 = \sum_{i \geq 3} m_i$.

By Lemma 4, $m_3 = 0$. Using (3), we obtain

$$\alpha_2^{(1)} \geq \frac{1}{\alpha_2} (4m_4 + 5 \sum_{i \geq 5} m_i) = \frac{1}{\alpha_2} (5 \sum_{i \geq 4} m_i - m_4) = 5 - \frac{m_4}{\alpha_2}. \tag{4}$$

Consider Nikulin’s estimate for $\alpha_2^{(1)}$:

$$\alpha_2^{(1)} < \left( \frac{n-1}{n-2} \right) \left( \frac{\lfloor n/2 \rfloor}{\binom{n}{2}} \right) + \left( \frac{\lfloor (n+1)/2 \rfloor}{\binom{n}{2}} \right) = \frac{n-1 + \varepsilon}{n-2 + \varepsilon}, \tag{5}$$

where $\varepsilon = 0$ if $n$ is even and $\varepsilon = 1$ if $n$ is odd.

Combining (4) with (5), we obtain

$$5 - \frac{m_4}{\alpha_2} \leq \alpha_2^{(1)} < \frac{n-1 + \varepsilon}{n-2 + \varepsilon}. \tag{6}$$

Denote by $l$ the number of vertices of $P$. Denote by $N_4$ the total number of vertices of quadrilateral 2-faces. Clearly, $N_4 = 4m_4$. By Lemmas 5 and 6, each of $l$ vertices is incident to at most $n+6$ quadrilaterals. Thus, $N_4 \leq l(n+6)$ and we have $4m_4 \leq l(n+6)$. In view of (1) and (5), we have

$$\frac{m_4}{\alpha_2} \leq \frac{1}{4} \frac{l(n+6)}{\alpha_2} = \frac{n+6}{4} \frac{2}{n(n-1)} \alpha_2^{(1)} <$$
\[ \frac{n+6}{2n(n-1)} \frac{4(n-1+\varepsilon)}{(n-2+\varepsilon)} = \frac{n+6}{n(n-1)} \frac{(n-1+\varepsilon)}{(n-2+\varepsilon)}. \tag{7} \]

Combining (6) and (7), we obtain
\[ 5 - \frac{4(n-1+\varepsilon)}{(n-2+\varepsilon)} < \frac{m_4}{\alpha_2} < 2 \frac{n+6}{n(n-1)} \frac{(n-1+\varepsilon)}{(n-2+\varepsilon)}. \]

This implies
\[ (n-6+\varepsilon)n(n-1) < 2(n+6)(n-1+\varepsilon). \]

This is equivalent to \( n^2 - 8n - 12 < 0 \) if \( n \) is even and to \( n^2 - 8n - 7 < 0 \) if \( n \) is odd. The first inequality has no solutions for \( n \geq 10 \), and the second one has no solutions for \( n \geq 9 \). So, the theorem is proved.

\[ \square \]

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