Spatial Optical Solitons due to Multistep Cascading

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We introduce a novel class of parametric optical solitons supported simultaneously by two second-order nonlinear cascading processes, second-harmonic generation and sum-frequency mixing. We obtain, analytically and numerically, the solutions for three-wave spatial solitons and show that the presence of an additional cascading mechanism can change dramatically the properties and stability of two-wave quadratic solitary waves.

As is known, optical cascaded nonlinearities due to parametric wave mixing can lead to a large nonlinear phase shift and spatial solitary waves, resembling those for a Kerr medium. However, solitary waves supported by cascaded nonlinearities demonstrate much richer dynamics due to nonintegrability of governing nonlinear equations and, unlike solitons of the Kerr nonlinearity, the quadratic solitons can become unstable in a certain narrow region of their parameters.

In this Letter we introduce a novel class of parametric spatial solitons supported simultaneously by two nonlinear quadratic (or $\chi^{(2)}$) optical processes: second-harmonic generation (SHG) and sum-frequency mixing (SFM). As has been recently shown by Koynov and Saltiel for continuous waves, under the condition that the two wave-mixing processes are nearly phase matched, the presence of multistep cascading leads to a four fold reduction of the input intensity required to achieve a large nonlinear phase shift. Here, we demonstrate that the multistep cascading can lead to a new type of parametric solitons. Introducing a third wave generated via a SFM process, we find that it can alter both the general properties and stability of the two-wave $\chi^{(2)}$ spatial solitons. Moreover, we reveal the existence of a new type of the so-called quasi-soliton, that appear for a negative mismatch of the SFM process.

To introduce the model of multistep cascading, we consider the fundamental beam with frequency $\omega$ entering a noncentrosymmetric nonlinear medium with a $\chi^{(2)}$ response. As a first step, the second-harmonic wave with frequency $2\omega$ is generated via the SHG process. As a second step, we expect the generation of higher order harmonics due to SFM, for example, a third harmonic ($\omega + 2\omega = 3\omega$) or even fourth harmonic ($2\omega + 2\omega = 4\omega$). When both such processes are nearly phase matched, they can lead, via down-conversion, to a large nonlinear phase shift of the fundamental wave. Additionally, as we demonstrate in this paper, the multistep cascading can support a novel type of three-wave spatial solitary waves in a diffractive $\chi^{(2)}$ nonlinear medium, multistep cascading solitons.

We start our analysis with the reduced amplitude equations derived in the slowly varying envelope approximation with the assumption of zero absorption of all interacting waves (see, e.g., Ref. [3]). Introducing the effect of diffraction in a slab waveguide geometry, we obtain

\begin{equation}
\begin{align*}
2ik_1\frac{\partial A_1}{\partial z} + \frac{\partial^2 A_1}{\partial x^2} + \chi_1 A_3 A_2^* e^{-i\Delta k_3 z} + \chi_2 A_2 A_1^* e^{-i\Delta k_2 z} &= 0, \\
4ik_2\frac{\partial A_2}{\partial z} + \frac{\partial^2 A_2}{\partial x^2} + \chi_4 A_3 A_1^* e^{-i\Delta k_3 z} + \chi_5 A_1 A_2^* e^{i\Delta k_2 z} &= 0, \\
6ik_3\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial x^2} + \chi_3 A_2 A_1 e^{i\Delta k_3 z} &= 0,
\end{align*}
\end{equation}

where $\chi_{1,2} = 2k_1\sigma_{1,2}$, $\chi_3 = 6k_1\sigma_3$, and $\chi_{4,5} = 4k_1\sigma_{4,5}$, and the nonlinear coupling coefficients $\sigma_k$ are proportional to the elements of the second-order susceptibility tensor which we assume to satisfy the following relations (no dispersion), $\sigma_3 = 3\sigma_1$, $\sigma_2 = \sigma_5$, and $\sigma_4 = 2\sigma_1$.

In Eqs. (1), $A_1, A_2$, and $A_3$ are the complex electric field envelopes of the fundamental harmonic (FH), second harmonic (SH), and third harmonic (TH), respectively, $\Delta k_2 = 2k_1 - k_2$ is the wavevector mismatch for the SHG process, and $\Delta k_3 = k_1 + k_2 - k_3$ is the wavevector mismatch for the SFM process. The subscripts ‘1’ denote the FH wave, the subscripts ‘2’ denote the SH wave, and the subscripts ‘3’, the TH wave. Following the technique earlier employed in Refs. [2, 3], we look for stationary solutions of Eq. (1) and introduce the normalised envelope $w(z,x)$, $v(z,x)$, and $u(z,x)$ according to the relations,
\[ A_1 = \frac{\sqrt{2\beta k_1}}{\sqrt{\lambda_2}} e^{i\beta z}, \quad A_2 = \frac{2\beta k_1}{\lambda_2} e^{2i\beta z + \Delta k z} v, \quad A_3 = \frac{\sqrt{2\beta k_1}}{\chi_1 \sqrt{\lambda_5}} e^{3i\beta z + \Delta k z} u, \] (2)

where \( \Delta k \equiv \Delta k_2 + \Delta k_3 \). Renormalising the variables as \( z \to z/\beta \) and \( x \to x/\sqrt{2\beta k_1} \), we finally obtain a system of coupled equations,

\[
\begin{align*}
\frac{i}{\chi} \frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} - w + w^* v + v^* u &= 0, \\
2i \frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2} w^2 + w^* u &= 0, \\
3i \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} - \alpha u + \chi w v &= 0,
\end{align*}
\] (3)

where \( \alpha = 2(2\beta + \Delta k_2)/\beta \) and \( \alpha_1 = 3(3\beta + \Delta k)/\beta \) are two dimensionless parameters that characterise the nonlinear phase matching between the parametrically interacting waves. Dimensionless material parameter \( \chi \equiv \chi_1 \chi_3/\lambda_2^2 = 9(\sigma_1/\sigma_2)^2 \) depends on the type of phase matching, and it can take different values of order of one. For example, when both SHG and SFM are due to quasi-phase matching (QPM), we have \( \sigma_1 = (2/\pi m)(\pi/\Lambda_1 n_1)^2 \chi_3^{(2)} [\omega - (4j)\omega; -(3j)\omega] \), where \( j = 1, 2 \). Then, for the first-order (\( m = 1 \)) QPM processes (see, e.g., Ref. [5]), we have \( \sigma_1 = \sigma_2 \), and therefore \( \chi = 9 \). When SFM is due to the third-order QPM process (see, e.g., Ref. [7]), we should take \( \sigma_1 = \sigma_2/3 \), and therefore \( \chi = 1 \). At last, when SFM is the fifth-order QPM process, we have \( \sigma_1 = \sigma_2/5 \) and \( \chi = 9/25 \).

Dimensionless equations (3) present a fundamental model for three-wave multistep cascading solitons in the absence of walk-off. Additionally to the type I SHG solitons (see, e.g., Refs [5]), the multistep cascading solitons involve the phase-matched SFM interaction (\( \omega + 2\omega = 3\omega \)) that generates a third harmonic wave. If this latter process is not phase-matched, we should consider \( \alpha_1 \) as a large parameter, and then look for solutions of Eq. (3) in the form of an asymptotic series in \( \alpha_1 \). Substituting \( w = w_0 + \varepsilon w_1 + \ldots, \quad v = v_0 + \varepsilon v_1 + \ldots \) and \( u = \varepsilon u_1 \), we find \( u_1 \approx \chi w v \), and the system (3) reduces to a model of competing nonlinearities,

\[
\begin{align*}
\frac{i}{\chi} \frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial x^2} - w + w^* v + \varepsilon \chi |v|^2 w &= 0, \\
2i \frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2} w^2 + \varepsilon \chi |w|^2 v &= 0.
\end{align*}
\] (4)

In the limit \( \varepsilon \to 0 \), Eqs. (4) coincide with the model of two-wave solitons due to the type I SHG earlier analysed in Refs [5].

For smaller \( \alpha_1 \), the system (3) cannot be reduced to Eq. (4), and its two-parameter family of localised solutions consists of three mutually coupled waves. It is interesting to note that, similar to the case of nondegenerate three-wave mixing (3), Eqs. (3) possess an exact solution. To find it, we make a substitution \( w = w_0 \sech^2(\eta x), \quad v = v_0 \sech^2(\eta x) \) and \( u = u_0 \sech^2(\eta x) \), and obtain unknown parameters from the following algebraic equations

\[
\begin{align*}
w_0^2 &= \frac{9v_0}{3 + 4\chi v_0}, \quad 4\chi v_0^2 + 6v_0 = 9, \quad u_0 = \frac{2}{3} \chi w_0 v_0, \quad (5)
\end{align*}
\]

valid for \( \eta = \frac{1}{2} \) and \( \alpha = \alpha_1 = 1 \). Equations (3) have two solutions corresponding to positive and negative values of the amplitude. This indicates a possibility of multivalued solutions, even within the class of exact solutions.

In a general case, three-wave solitons of Eqs. (3) can be found only numerically. Figures 1(a) and 1(b) present two examples of solitary waves for different sets of the mismatch parameters \( \alpha \) and \( \alpha_1 \). When \( \alpha_1 \gg 1 \) [see Fig. 1(a)], which corresponds to an unmatched SFM process, the amplitude of the third harmonic is small, and it vanishes for \( \alpha_1 \to \infty \) according to the asymptotic solution of Eq. (3) discussed above.

To summarise different types of three-wave solitary waves, in Fig. 2 we plot the dependence of the total soliton power defined as

\[
P = \int_{-\infty}^{+\infty} dx \left( |w|^2 + 4|v|^2 + \frac{9}{\chi} |w|^2 \right),
\] (6)

on the mismatch parameter \( \alpha_1 \), for fixed \( \alpha = 1 \). It is clearly seen that for some values of \( \alpha_1 \) (including the exact solution at \( \alpha_1 = 1 \) shown by two filled circles), there exist two different branches of three-wave solitary waves, and only...
one of those branches approaches, for large values of $\alpha_1$, a family of two-wave solitons of the cascading limit (Fig. 2, dashed). The slope of the branches changes from negative (for small $\alpha_1$) to positive (for large $\alpha_1$), indicating a possible change of the soliton stability. However, the soliton stability should be defined in terms of physical parameters, and in the case of two-parameter solitons as we have here, the stability threshold is determined by a certain integral determinant condition, similar to that first derived for the three-wave mixing problem [9].

Ratios of the maximum amplitudes of the soliton components for the three-wave solitons of the lower branch in the model [8] are presented in Fig. 3, where the upper dashed curve is the asymptotic limit of two-wave solitons for $\alpha_1 \to \infty$. Soliton solutions of the second (upper) branch in Fig. 2 correspond to large values of the total power and they have been verified numerically to be unstable.

The analysis of the asymptotics for Eqs. (3) suggests that localised solutions should not occur for $\alpha_1 < 0$. However, we reveal the existence of an extended class of very robust localised solutions which we classify as ‘quasi-solitons’ [10], solitary waves with small-amplitude oscillating tails. In principle, such solitons are known in one-component models (see, e.g., Ref. [11]) but here the nonvanishing tails appear only due to a resonance with the third-harmonic field [see Fig. 4(a)]. Such solitons are expected to be weakly unstable, and this is indeed demonstrated in Fig. 4(b) for rather long propagation distances.

Existence of quasi-solitons for any value of negative phase-matching with a higher-order harmonic field indicates that all two-wave quadratic solitons can become unstable due to an additional SFM process. This is confirmed in Figs. 4(c,d) where we present the results of numerical simulations of the dynamics of an initially launched two-wave soliton for two cases, positive and negative phase-matching of a SFM process. For $\alpha_1 > 0$ [see Fig. 4(c)], a very small harmonic ($v_{\text{max}} \approx 0.1$) is generated and the initial two-component beam converges to a three-wave soliton. In contrast, for $\alpha_1 < 0$ [see Fig. 4(d)], the input beam decays rapidly into radiation and diffracting harmonic fields.

In conclusion, we have investigated, analytically and numerically, multistep cascading and nonlinear beam propagation in a diffractive optical medium and introduced a novel type of three-wave parametric spatial optical solitons, multistep cascading solitons. The detailed analysis of the soliton stability, the effect of walk-off, higher-dimensional and spatio-temporal effects are possible directions of the future research.

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FIG. 1. Examples of three-wave solitary waves of the model (3) for (a) \( \alpha = 0.05, \alpha_1 = 5 \), and (b) \( \alpha = 5, \alpha_1 = 0.35 \).

FIG. 2. Two branches of multistep cascading solitons shown as the total soliton power \( P \) vs. the parameter \( \alpha_1 \) for \( \alpha = 1 \) and \( \chi = 1 \). Filled circles show the analytical solutions. The lower branch approaches a family of two-wave quadratic solitons (for \( \alpha_1 \to \infty \)) shown by a dashed line.
FIG. 3. Peak intensity ratios for the family of three-wave solitary waves ($\chi = 1$) at $\alpha_1 = -10$ (solid), $\alpha_1 = 10$ (doted). Upper dashed curve shows the asymptotic limit of large $\alpha_1$ corresponding to the two-wave quadratic solitons.
FIG. 4. Examples of (a) a three-wave quasi-soliton ($\alpha = 1$, $\alpha_1 = -5$) and (b) its instability-induced long-term evolution shown for the peak intensities ($\alpha = 1$, $\alpha_1 = -10$). (c,d) Evolution of the fundamental harmonic of a two-wave soliton owing to an unseeded unmatched SFM process with $\alpha_1 = +8$ and $\alpha_1 = -8$, respectively.