SHARP BOHR RADIUS CONSTANTS FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract. The Bohr radius for a class $G$ consisting of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is the largest $r^*$ such that every function $f$ in the class $G$ satisfies the inequality

$$d \left( \sum_{n=0}^{\infty} |a_n z^n|, |f(0)| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial f(D))$$

for all $|z| = r \leq r^*$, where $d$ is the Euclidean distance. In this paper, our aim is to determine the Bohr radius for the classes of analytic functions $f$ satisfying differential subordination relations $zf'(z)/f(z) \prec h(z)$ and $f(z) + \beta zf'(z) + \gamma z^2 f''(z) \prec h(z)$, where $h$ is the Janowski function. Analogous results are obtained for the classes of $\alpha$-convex functions and typically real functions, respectively. All obtained results are sharp.

1. Introduction

In recent years, the Bohr radius problems attracted the attention of several researchers in various direction in geometric function theory. The Bohr inequality has emerged as an active area of research after Dixon [12] used it to disprove a conjecture in Banach algebra. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$ and $H(D, \Omega)$ denote the class of analytic functions mapping unit disc into a domain $\Omega$. Let $A$ denote the class of analytic functions in $D$ normalized by $f(0) = 0 = f'(0) - 1$. Let $S$ denote the subclass of $A$ consisting of univalent functions. For two analytic functions $f$ and $g$ in $D$, the function $f$ is said to be subordinate to $g$, written $f(z) \prec g(z)$, if there is an analytic map $w : D \to D$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. In particular, if the function $g$ is univalent in $D$, then $f$ is subordinate to $g$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$. Let $\phi$ be the analytic function with positive real part in $D$ that map the unit disc $D$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$. For such functions, Ma and Minda [21] introduced the following classes:

$$ST(\phi) := \left\{ f \in A ; \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \quad \text{and} \quad CV(\phi) := \left\{ f \in A ; 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}.$$

On taking $\phi(z) = (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$, the class $ST(\phi)$ reduces to the familiar class consisting of Janowski starlike functions [17], denoted by $ST[A, B]$. The special case $A = 1 - 2\alpha$ and $B = -1$, that is, $\phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$ yield the classes $ST(\alpha)$ and $CV(\alpha)$ of starlike and convex functions of order $\alpha$, respectively. In particular, $\alpha = 0$, that is, $A = 1$ and $B = 1$ leads to the usual classes $ST$ and $CV$ of starlike and convex functions, respectively. For $A = 1$ and $B = (1 - M)/M$, $M > 1/2$, we

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obtain the class $ST(M)$ introduced by Janowski [16]. In 1939, Robertson [28] introduced a very well known class $ST(\beta) := ST[\beta, -\beta], 0 < \beta \leq 1$. Also, $ST(\beta) := ST[\beta, 0]$ leads to a class which was introduced by MacGregor [22].

In 1914, Bohr [10] discovered that if a power series of an analytic function converges in the unit disc and its sum has a modulus less than one, then the sum of the absolute values of its terms is again less than one in the disc $|z| \leq 1/6$. Wiener, Riesz and Schur independently proved that the Bohr’s result holds in the disc $|z| \leq 1/3$ and the number $1/3$ is best possible. For the class of functions $f \in H(\mathbb{D}, \mathbb{D})$, the number $1/3$ is commonly called the Bohr radius, while the inequality $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ is known as the Bohr inequality. Later on, various proof of Bohr’s inequality were given in [24–26]. Using the Euclidean distance $d$, the Bohr inequality for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is written as

$$d \left( \sum_{n=0}^{\infty} |a_n z^n|, |a_0| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |f(0)| = d(f(0), \partial \mathbb{D}),$$

where $\partial \mathbb{D}$ is the boundary of the disc $\mathbb{D}$. For any domain $\Omega$ and all functions $f \in H(\mathbb{D}, \Omega)$, the Bohr radius is the largest radius $r^* > 0$ such that

$$d \left( \sum_{n=0}^{\infty} |a_n z^n|, |f(0)| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial \Omega),$$

for $|z| = r \leq r^*$. Let the class $G$ consisting of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which map disc $\mathbb{D}$ into a domain $\Omega$. Then Bohr radius for the class $G$ satisfies the inequality

$$d \left( \sum_{n=0}^{\infty} |a_n z^n|, |f(0)| \right) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(f(0), \partial f(\mathbb{D}))$$

for all $|z| = r \leq r^*$. In this case, the class $G$ is said to satisfy a Bohr phenomenon.

Ali et al. [4] obtained the Bohr radius for wedge domain $W_\alpha = \{ w : |\arg w| < \pi \alpha/2 \}, 1 \leq \alpha \leq 2$ and also determined the upper and lower bounds on the Bohr radius for odd analytic functions. Further, several different improved versions of the classical Bohr inequality were given in [19]. Alkhaleefah et al. [7] established the classical Bohr inequality for the class of quasiconformal mappings, while the classical Bohr inequality in the Poincaré disc model of the hyperbolic plane was extended in [5]. Bohr radii for the classes of convex univalent functions of order $\alpha$, close-to-convex functions and functions with positive real part were obtained by authors [6]. Powered Bohr radius for the class of analytic functions mapping the unit disc onto itself was studied in [18]. For more details, see [1].

In this paper, Section 2 provides the Bohr inequality for the class of Janowski starlike functions. In Section 3, the Bohr radius problem is determined for the class of analytic functions $f$ satisfying second order differential subordination relation $f(z) + \beta z f'(z) + \gamma z^2 f''(z) < h(z)$, where $h$ is a Janowski function. Section 4 yields the Bohr radius for the class of alpha-convex functions. In the last section, we compute the Bohr radius for the class of typically-real functions.

2. JANOWSKI STARLIKE FUNCTIONS

In the present section, we obtain the Bohr’s radius for the class of Janowski starlike functions. To prove our results we need the following lemmas.
Letting $p = 1$ and $\alpha = 0$ in [8, Theorem 3, p. 738], we get following result for Janowski starlike functions.

**Lemma 2.1.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST[A, B]$, then

\[ |a_n| \leq \prod_{k=0}^{n-2} \frac{|(B - A) + B k|}{k + 1}, \quad (n \geq 2) \]

and these bounds are sharp.

**Lemma 2.2.** [17, Theorem 4, p. 315] If the function $f \in ST[A, B]$, then for $|z| = r$; $(0 \leq r < 1)$,

\[ l_{(-A,-B)}(r) \leq |f(re^{i\theta})| \leq l_{(A,B)}(r) \]

where $l_{(A,B)} : \mathbb{D} \to \mathbb{C}$ is given by

\[ l_{(A,B)}(z) = \begin{cases} z(1 + Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0. \end{cases} \]

The result is sharp.

**Theorem 2.3.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $ST[A, B]$. Then

\[ |z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D})) \]

for $|z| < r^*$, where $r^* \in (0, 1]$ is the root of equations

\[ r + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|(B - A) + B k|}{k + 1} r^n - (1 - B)^{(A-B)/B} = 0, \quad \text{if} \quad B \neq 0 \]

\[ re^{A r} - e^{-A} = 0, \quad \text{if} \quad B = 0. \]

The number $r^*$ is the Bohr radius for the class $ST[A, B]$ which is best possible.

**Proof.** Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $ST[A, B]$, then using inequality (2.2), the distance between boundary and the origin of the function $f$ is given by

\[ d(0, \partial f(\mathbb{D})) = \inf_{\xi \in \partial f(\mathbb{D})} |f(\xi)| = l_{(-A,-B)}(1). \]

Note that the given $r^*$ satisfies the following equations:

For $B \neq 0$,

\[ r + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|(B - A) + B k|}{k + 1} r^n = (1 - B)^{(A-B)/B} \]

and $B = 0$,

\[ re^{A r} = e^{-A}. \]

Using (2.1), (2.4) and the fact that $A > 0$ for $B = 0$, we have

\[ |z| + \sum_{n=2}^{\infty} |a_n z^n| \leq r + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|(B - A) + B k|}{k + 1} r^n \]

\[ = \begin{cases} r + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|(B - A) + B k|}{k + 1} r^n, & B \neq 0; \\ re^{A r}, & B = 0. \end{cases} \]
for \( r \leq r^\ast \). To show the sharpness of the Bohr radius \( r^\ast \), consider the function \( l_{(A,B)} \) so that for \( |z| = r^\ast \), we have

\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| = r^\ast + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|B - A| + Bk}{k + 1} (r^\ast)^n
\]

\[
= \begin{cases} 
  r^\ast + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{|B - A| + Bk}{k + 1} (r^\ast)^n, & B \neq 0 \\
  r^\ast e^\beta r^\ast, & B = 0
\end{cases}
\]

\[
= l_{(-A,-B)}(1) = d(0, \partial f(\mathbb{D})).
\]

This completes the sharpness.

Bhowmik and Das [9, Theorem 3, p. 1093] found the Bohr radius for \( S^\ast(\alpha) \) where \( \alpha \in [0, 1/2] \). For \( 0 \leq \alpha < 1 \), \( A = 1 - 2\alpha \) and \( B = -1 \), Theorem 2.3 gives sharp Bohr radius for the class of starlike functions of order \( \alpha \).

**Corollary 2.4.** [3, Remark 3, p. 7] If \( 0 \leq \alpha < 1 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST(\alpha) \), then

\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))
\]

for \( |z| < r^\ast \), where \( r^\ast \in (0, 1) \) is the root of equation \((1 - r)^{2(1 - \alpha)} - 2^{2(1 - \alpha)} r \geq 0\).

**Remark 2.1.** In particular, for \( \alpha = 0 \), Corollary 2.4 yields the sharp Bohr radius for the class of starlike functions which is \( 3 - 2\sqrt{2} \).

Putting \( A = \beta \) and \( B = -\beta \), where \( 0 < \beta \leq 1 \) in Theorem 2.3, we get the sharp Bohr radius for the class \( ST(\beta) \).

**Corollary 2.5.** If \( 0 < \beta \leq 1 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST(\beta) \), then

\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))
\]

for \( |z| < r^\ast \), where \( r^\ast \in (0, 1) \) is given as

\[
r^\ast = -\frac{1 - 4\beta - \beta^2 + (1 + \beta)\sqrt{1 + \beta(6 + \beta)}}{2\beta^2}.
\]

If \( 0 < \beta \leq 1 \), \( A = \beta \) and \( B = 0 \), then Theorem 2.3 yields sharp Bohr radius for the class \( ST(\beta) \).

**Corollary 2.6.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST(\beta) \), then \( |z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D})) \) for \( |z| < r^\ast \), where \( r^\ast \in (0, 1) \) is the root of equation \( re^{\beta r} - e^{\beta} = 0 \).

Letting \( A = 1 \) and \( B = (1 - M)/M \) where \( M > 1/2 \), Theorem 2.3 provides following result for the class \( ST(M) \).
Corollary 2.7. If \( M > 1/2 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST(M) \), then
\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))
\]
for \( |z| < r^* \), where \( r^* \in (0, 1] \) is the root of equation
\[
r + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \left( \frac{(1 - 2M) + (1 - M)k}{M(k+1)} \right) r^n - \left( 2 - \frac{1}{M} \right)^{(1-2M)/(1+M)} = 0.
\]

3. Second Order Differential Subordination Associated With Janowski Functions

For \( \beta \geq \gamma \geq 0 \), we consider the class \( R(\beta, \gamma, h) \) which is defined by making use of subordination as
\[
R(\beta, \gamma, h) = \{ f \in A : f(z) + \beta z f'(z) + \gamma z^2 f''(z) \prec h(z), z \in \mathbb{D} \}
\]
where \( h \) is a Janowski function. The class \( R(\beta, \gamma, h) \) can be seen as an extension to the class
\[
R(\beta, h) = \{ f \in A : f'(z) + \beta z f''(z) \prec h(z), z \in \mathbb{D} \}.
\]
Many variations of this class have been studied by various authors [13, 31, 32].

For two analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), the Hadamard product (or convolution) is the function \( f \ast g \), defined by
\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.
\]

Consider the function \( \phi_\lambda \), defined by
\[
\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - z t^\lambda} = \sum_{n=0}^{\infty} \frac{z^n}{1 + \lambda n}.
\]
For \( \text{Re} \lambda \geq 0 \), the function \( \phi_\lambda \) is convex in \( \mathbb{D} \) [30, Theorem 5, p.113].

For \( \beta \geq \gamma \geq 0 \), let \( \nu + \mu = \beta - \gamma \) and \( \mu \nu = \gamma \) and
\[
q(z) = \int_0^1 \int_0^1 h(z t^\mu s^\nu) dt ds = (\phi_\nu \ast \phi_\mu) \ast h(z).
\]

Since \( \phi_\nu \ast \phi_\mu \) is a convex function and \( h \in ST[A, B] \) so it follows from [21, Theorem 5, p.167] that \( q \in ST[A, B] \). The following theorem gives the sharp Bohr radius for the class \( R(\beta, \gamma, h) \).

Theorem 3.1. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\beta, \gamma, h) \) and \( h \) be a Janowski starlike function. Then
\[
\sum_{n=1}^{\infty} |a_n z^n| \leq d(h(0), \partial h(\mathbb{D}))
\]
for all $|z| \leq r^*$, where $r^* \in (0, 1]$ is root of the equations

$$r \quad \frac{1}{1 + (\nu + \mu) + \nu \mu} + \sum_{n=2}^{\infty} \frac{n^2 |(B - A) + Bk|}{k + 1} \leq l_{(-A, -B)}(1),$$

where

$$l_{(-A, -B)}(1) = \begin{cases} (1 - B) e^{-\lambda}, & B \neq 0; \hfill \vspace{1cm} \\
B = 0 \hfill \vspace{1cm} \
\end{cases}$$

The result is sharp.

**Proof.** Let $F(z) = f(z) + \beta zf'(z) + \gamma z^2 f''(z) < h(z)$. It is noted that

$$F(z) = \sum_{n=0}^{\infty} (1 + \beta n + \gamma n(n-1))a_n z^n < h(z), \quad a_1 = 1.$$  

Consider

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} |1 + \beta n + \gamma n(n-1)|a_n z^n = \frac{F(z) - F(0)}{h'(0)} < \frac{h(z) - h(0)}{h'(0)} = H(z).$$

Since $h \in ST[A, B]$, it follows that $H \in ST[A, B]$. Thus, Lemma 2.1 gives

$$\left| \frac{1 + \beta n + \gamma n(n-1)}{h'(0)} \right| |a_n| \leq \prod_{k=0}^{n-2} \frac{|(B - A) + Bk|}{k + 1}$$

for each $n \geq 2$. In view of the above inequality, we have

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \frac{|h'(0)|}{1 + (\nu + \mu) + \mu \nu} r + \sum_{n=2}^{\infty} \frac{|h'(0)| \prod_{k=0}^{n-2} \frac{|(B - A) + Bk|}{k + 1}}{1 + (\nu + \mu) n + \mu \nu n^2} r^n.$$  

Since $H \in ST[A, B]$, using (2.3) and (2.2) the following inequality holds

$$l_{(-A, -B)}(r) \leq |H(re^{i\theta})| \leq l_{(A, B)}(r), \quad 0 < r \leq 1.$$  

So that

$$d(0, \partial H(\mathbb{D})) \geq l_{(-A, -B)}(1)$$

which gives

$$d(h(0), \partial h(\mathbb{D})) = \inf_{\xi \in \partial h(\mathbb{D})} |h(\xi) - h(0)| \geq |h'(0)| l_{(-A, -B)}(1).$$

Using (3.2) and (3.4) we obtain

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \frac{d(h(0), \partial h(\mathbb{D}))}{l_{(-A, -B)}(1)} \left(\frac{1}{1 + (\nu + \mu) + \mu \nu} r + \sum_{n=2}^{\infty} \frac{\prod_{k=0}^{n-2} \frac{|(B - A) + Bk|}{k + 1}}{1 + (\nu + \mu) n + \mu \nu n^2} r^n \right).$$

Thus, the Bohr radius $r^*$ is the smallest positive root of the equation

$$r \quad \frac{1}{1 + (\nu + \mu) + \mu \nu} + \sum_{n=2}^{\infty} \frac{\prod_{k=0}^{n-2} \frac{|(B - A) + Bk|}{k + 1}}{1 + (\nu + \mu) n + \mu \nu n^2} r^n = l_{(-A, -B)}(1).$$
For sharpness, consider the function
\[ f(z) = q(z) = (\phi_\nu \ast \phi_\mu) * h(z) \]
as defined in (3.1), where
\[ h(z) = l_{(A,B)}(z) = \begin{cases} z(1 + Bz)^{\frac{\nu}{\nu - \mu}}, & B \neq 0; \\ z^A, & B = 0. \end{cases} \]
Also \( f(z) \in \mathcal{R}(\beta, \gamma, h). \) This gives
\[ f(z) = \frac{z}{1 + (\nu + \mu) + \nu \mu} + \sum_{n=2}^{\infty} \frac{n-2}{1 + (\nu + \mu)n + \nu \mu n^2} z^n. \]

For \( |z| = r^* \),
\[ \sum_{n=1}^{\infty} |a_n z^n| = \left| h'(0) \right| \left( \frac{r^*}{1 + (\nu + \mu) + \nu \mu} + \sum_{n=2}^{\infty} \frac{n-2}{1 + (\nu + \mu)n + \nu \mu n^2} (r^*)^n \right) \]
\[ = \left| h'(0) \right| l_{(-A,-B)}(1) \]
\[ = d(h(0), \partial h(D)). \]

Thus the result is sharp.

For \( 0 \leq \alpha < 1, A = 1 - 2\alpha \) and \( B = -1 \), Theorem 3.1 reduces to the following result.

**Corollary 3.2.** [15, Theorem 3.3, p.7] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{R}(\beta, \gamma, h) \) and \( h \) be starlike of order \( \alpha \). Then
\[ \sum_{n=1}^{\infty} |a_n z^n| \leq d(h(0), \partial h(D)) \]
for all \( |z| \leq r^* \), where \( r^* \in (0, 1] \) is the smallest positive root of the equation
\[ r \left( \frac{1}{1 + (\nu + \mu) + \nu \mu} + \sum_{n=2}^{\infty} \frac{n-1}{1 + (\nu + \mu)n + \nu \mu n^2} \right) = \frac{1}{2^{2(1-\alpha)}.} \]

The result is sharp.

For \( A = 1 \) and \( B = -1 \), Theorem 3.1 yields the following result.

**Corollary 3.3.** [2, Theorem 3.3, p.131] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{R}(\beta, \gamma, h) \) and \( h \) be starlike. Then
\[ \sum_{n=1}^{\infty} |a_n z^n| \leq d(h(0), \partial h(D)) \]
for all \( |z| \leq r^* \), where \( r^* \in (0, 1] \) is the smallest positive root of the equation
\[ \frac{n}{1 + (\nu + \mu)n + \nu \mu n^2} = \frac{1}{4}. \]

The result is sharp.
4. Alpha-Convex Functions

In 1969, Mocanu \[23\] introduced the class of $\alpha$-convex functions. For $\alpha \in \mathbb{R}$, a normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be $\alpha$-convex in $D$ (or $\alpha$-convex) if the following conditions hold

$$\frac{f(z)}{z} : f'(z) \neq 0,$$

and

$$\text{Re} \left[ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \frac{zf'(z)}{f(z)} \right] \geq 0,$$

for all $z \in D$. The set of all such functions is denoted by $\alpha - \text{CV}$. For $\alpha = 0$, $\alpha - \text{CV}$ is the class of starlike functions and for $\alpha = 1$, $\alpha - \text{CV}$ is the class of convex functions.

In this section, we shall obtain the Bohr radius for the class of $\alpha$-convex functions. In order to obtain the Bohr radius for the class $\alpha - \text{CV}$, we need the following lemmas.

**Lemma 4.1.** [14, Theorem 7, p. 146] If $\alpha > 0$, and $f(z)$ is $\alpha$-convex, then

$$k(-r, \alpha) \leq |f(z)| \leq k(r, \alpha), \quad z = re^{i\theta},$$

where

$$k(z, \alpha) = \left( \frac{1}{\alpha} \int_{0}^{z} \frac{\xi^{\frac{1}{\alpha} - 1} d\xi}{(1 - \xi)^{\frac{1}{\alpha}}} \right)^{\alpha}. \quad (4.1)$$

The inequalities are sharp for each $\alpha > 0$ and each $r \in (0, 1)$.

**Lemma 4.2.** [20, Theorem 2, p. 208] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \alpha - \text{CV}$ and let $S(n)$ be the set of all $n$-tuples $(x_1, x_2, \cdots, x_n)$ of non-negative integers for which $\sum_{i=1}^{n} ix_i = n$ and for each $n$-tuple define $q$ by $\sum_{i=1}^{n} x_i = q$. If

$$\Upsilon(\alpha, q) = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - q)$$

with $\Upsilon(\alpha, 0) = \alpha$, then for $n = 1, 2, \cdots$

$$|a_{n+1}| \leq \sum_{(x_1, x_2, \cdots, x_n) \in S(n)} \frac{\Upsilon(\alpha, q-1)c_1^{x_1}c_2^{x_2} \cdots c_n^{x_n}}{x_1! x_2! \cdots x_n!}, \quad (4.2)$$

where summation is taken over all $n$-tuples in $S(n)$ and

$$c_n = \frac{1}{n! \alpha^n (1 + n\alpha)} \prod_{k=0}^{n-1} (2 + k\alpha).$$

The result is sharp.

**Theorem 4.3.** Let $\alpha > 0$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \alpha - \text{CV}$. Then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(D)).$$
for \(|z| \leq r^*\), where \(r^* \in (0, 1]\) is the positive root of the equation

\[
r + \sum_{n=2}^{\infty} \left( \sum_{\text{c_{n-1}}} \frac{\mathcal{Y}(\alpha, q-1)c_1 x_2 \cdots c_{n-1}}{x_1 \cdots x_{n-1}} \right) r^n = k(-1, \alpha)
\]

and where summation is as in (4.2). The result is sharp.

**Proof.** Let \(f \in \alpha - CV\). By Lemma 4.1, the growth inequality for the function \(f\) is given by

\[
k(-r, \alpha) \leq |f(re^{i\theta})| \leq k(r, \alpha),
\]

where \(k\) is the function as defined in (4.1). This immediately shows that

(4.3)

\[
d(0, \partial f(\mathbb{D})) \geq k(-1, \alpha).
\]

It is given that the Bohr radius \(r^*\) is the root of the equation

\[
r + \sum_{n=2}^{\infty} \left( \sum_{\text{c_{n-1}}} \frac{\mathcal{Y}(\alpha, q-1)c_1 x_2 \cdots c_{n-1}}{x_1 \cdots x_{n-1}} \right) r^n = k(-1, \alpha).
\]

For \(0 \leq r \leq r^*\), it is readily seen that \(k(r, \alpha) \leq k(-1, \alpha)\). Using (4.2) and (4.3), we have

\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq r + \sum_{n=2}^{\infty} \left( \sum_{\text{c_{n-1}}} \frac{\mathcal{Y}(\alpha, q-1)c_1 x_2 \cdots c_{n-1}}{x_1 \cdots x_{n-1}} \right) r^n \leq k(-1, \alpha) \leq d(0, \partial f(\mathbb{D}))
\]

for \(|z| = r \leq r^*\). In order to prove the sharpness, consider the function

\[
f(z) = k(z, \alpha) = \left[ \frac{1}{\alpha} \int_{0}^{\frac{\pi}{\alpha}} \frac{d\xi}{(1 - \xi)^{\frac{1}{\alpha}}} \right] z.
\]

For \(|z| = r^*\), we obtain

\[
|z| + \sum_{n=2}^{\infty} |a_n z^n| = r^* + \sum_{n=2}^{\infty} \left( \sum_{\text{c_{n-1}}} \frac{\mathcal{Y}(\alpha, q-1)c_1 x_2 \cdots c_{n-1}}{x_1 \cdots x_{n-1}} \right) (r^*)^n = k(-1, \alpha) = d(0, \partial f(\mathbb{D})).
\]

Thus the result is sharp.

5. **Typically Real Functions**

The class of typically real functions was introduced by Rogosinski [29]. An analytic function \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) which satisfies the condition \(\text{sign(Im } f(z)) = \text{sign(Im } z)\) for non real \(z \in \mathbb{D}\) is said to be typically real in \(\mathbb{D}\). The class of such functions is denoted by \(\mathcal{T}R\). In the present section, we obtain the Bohr radius for the class of typically real functions.

**Lemma 5.1.** [14, Theorem 3, p. 185] If \(f(z)\) is in \(\mathcal{T}R\) and \(z = re^{i\theta} \in \mathbb{D}\), then the coefficients satisfy the inequality

(5.1) \[m_n \leq a_n \leq n\]

where \(m_n = \min(\sin n\theta / \sin \theta)\) for each \(n\). The inequality is sharp for each \(n\).
Lemma 5.2. [27, Thm 1, p. 136] Let the function $f$ be in $\mathcal{T}\mathcal{R}$ and $z = re^{i\theta} \in \mathbb{D}$. Then

$$|f(z)| \geq \begin{cases} \left| \frac{z}{(1 + z)^2} \right|, & \text{if } \text{Re}\left(\frac{1 + z^2}{z}\right) \geq 2; \\ \left| \frac{r(1 - r^2)}{|1 - z^2|^2} \sin \theta \right|, & \text{if } \text{Re}\left(\frac{1 + z^2}{z}\right) \leq 2; \\ \left| \frac{z}{(1 - z)^2} \right|, & \text{if } \text{Re}\left(\frac{1 + z^2}{z}\right) \leq -2. \end{cases}$$

(5.2)

The result is sharp.

The next theorem gives the sharp Bohr radius for the class $\mathcal{T}\mathcal{R}$.

Theorem 5.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{T}\mathcal{R}$ and $z = re^{i\theta} \in \mathbb{D}$. Then

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

for $|z| \leq 3 - 2\sqrt{2} \approx 0.171573$. The result is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}\mathcal{R}$. By Lemma 5.2 and the proof of Theorem 2 (see [11]), it follows that the distance between the origin and the boundary of $f(\mathbb{D})$ satisfies the inequality

$$|f(z)| \geq \frac{1}{4}, \quad z \in \mathbb{D}.$$  

(5.3)

Using the inequality (5.3), we have

$$d(0, \partial f(\mathbb{D})) = \inf_{\xi \in \partial f(\mathbb{D})} |f(\xi)| \geq \frac{1}{4}.$$ 

(5.4)

In view of inequalities (5.1) and (5.4), we have

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq r + \sum_{n=2}^{\infty} nr^n = \frac{r}{(1 - r)^2} \leq \frac{1}{4} \leq d(0, \partial f(\mathbb{D}))$$

if $r < 3 - 2\sqrt{2}$. In order to prove the sharpness, for $-\pi < t \leq \pi$, consider the function $l_t : \mathbb{D} \to \mathbb{C}$ defined by

$$l_t(z) = \frac{z}{(1 - 2z \cos t + z^2)}.$$ 

For $|z| = r^*$ and $t = 0$, we obtain

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| = r^* + \sum_{n=2}^{\infty} n(r^*)^n = \frac{r^*}{(1 - r^*)^2} = \frac{1}{4} = d(0, \partial f(\mathbb{D})).$$

Thus the result is sharp.
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