Slow passage through parametric resonance for a weakly nonlinear dispersive wave

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Abstract

A solution of the nonlinear Klein-Gordon equation perturbed by a parametric driver is studied. The frequency of the parametric perturbation varies slowly and passes through a resonant value. It yields a change in a solution. We obtain a connection formula for the asymptotic solution before and after the resonance.

1 Introduction

This work is devoted to the problem on a control of a nearly monochromatic weakly nonlinear dispersive wave with small amplitude in a strong nonlinear media. It is well-known that packets of nearly monochromatic waves propagate without changing of their shape when the envelope function of the packet is a soliton of the Nonlinear Schrödinger equation (NLSE). The solitary packets of waves would be more suitable for communication in optical fibers on a large distance if one can control the parameters of the envelope function for such packets. The wave packets with a soliton-like shape form for sufficiently large range of initial data. The control of the shape is possible by a parametric perturbation of the system.

Here we propose a new approach for controlling of parameters of the solitary packets. In our approach the wave packets are controlled due to a slowly passage through a local parametric resonance.

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In this paper we study the parametrically driven nonlinear Klein-Gordon equation. The frequency of the driver varies slowly. It is well-known that an envelope function of weak nonlinear wave is a solution of the NLSE [1]-[3]. In general case this is valid proposition for the parametrically driven Klein-Gordon equation. But the small driving force plays a central role in narrow layers where the frequency passes through the resonant value. As a result fast and slow variables appear which define the behaviors of the solution. In these layers the envelope function of a primary parametric resonance equation instead of the NLSE. In the slow variables the passage through the resonant layer looks as a jump of the envelope function. Our goal is a connection formula for this jump.

The change of the solution in the resonant layers for a local resonance (not parametric) was studied in [4]-[8]. The detail description of the phenomenon was presented in [9]. In that case the solution in the resonant layer is defined by the Fresnel integral. For parametric resonance the solution was studied in [10] for the Mathieu equation. They shown that the solution defines by parabolic cylinder functions. In our work we found that the parabolic cylinder equation also allows one to obtain the connection formulas for the small solutions of nonlinear Klein-Gordon equation.

In this work we use the singular perturbation theory and matching of different asymptotic expansions to obtain the connection formula. Our approach is based on the matching method of asymptotic expansions [11].

The structure of the paper is as follows. In Section 2 we formulate the result. The numerical simulations are inserted in Section 3. Section 4 contains the formal construction for the asymptotic solution out of the resonant layers. In Section 5 the asymptotic solution is constructed in the resonant layer. In Section 6 we match the main terms of the asymptotic expansions.

2 Statement of the problem and main result

We study the Klein-Gordon equation with a cubic nonlinearity

$$\begin{align*}
\partial_t^2 U - \partial_x^2 U + \left(1 + \varepsilon f \cos \left(\frac{S(\varepsilon^2 x, \varepsilon^2 t)}{\varepsilon^2}\right)\right) U + \gamma U^3 &= 0, \quad 0 < \varepsilon \ll 1.
\end{align*}$$

(1)

Here $\gamma$ and $f$ are constants. The phase function $S(y, z)$ and all derivatives of $S(y, z)$ are bounded.

Our goal is to obtain an asymptotic solution for (1). To formulate the result we use the following notations for slow variables:

$$x_j = \varepsilon^j x, \quad t_j = \varepsilon^j t, \quad j = 1, 2.$$
Let us define
\[ L[\chi] = [\partial_{x_2}\chi]^2 - [\partial_{x_2}\chi]^2 - 1 \]
and
\[ l(x_2, t_2) = L[\omega t_2 + k x_2 + S]. \]

The main result of the paper is formulated in the Theorem.

**Theorem 1** In the domain \( l < 0 \) the formal asymptotic solution of (1) has the form
\[ U(x, t, \varepsilon) \sim \varepsilon u_1(x_1, t_1, t_2) \exp\{i(kx + \omega t)\} + c.c.. \]

The amplitude \( u_1 = \Psi \exp\{-i\frac{f^2}{4\omega}G(x_2, t_2)\} \), where \( G \) is an antiderivative of
\[ g(x_2, t_2) = \left[ \frac{1}{L[\omega t_2 + k x_2 - S]} + \frac{1}{L[\omega t_2 + k x_2 + S]} \right], \]
with respect to \( t_2. \)

Function \( \Psi \) is determined by the nonlinear Schrödinger equation
\[ i\omega \partial_{t_2} \Psi - \partial_{x_1}^2 \Psi + 3\gamma |\Psi|^2 \Psi = 0, \quad \zeta = \omega x_1 + kt_1. \]

In the domain \( l > 0 \) the formal asymptotic solution has the same form
\[ U(x, t, \varepsilon) \sim \varepsilon v_1(x_1, t_1, t_2) \exp\{i(kx + \omega t)\} + c.c.. \]

The amplitude \( v_1 = \Psi \exp\{i\frac{f^2}{4\omega}G(x_2, t_2)\} \) and \( \Psi \) is determined by the nonlinear Schrödinger equation also and initial datum on the curve \( l = 0 \)
\[ \Psi(t_2, \zeta)|_{t=0} = e^{i\frac{f^2}{8} \psi} + e^{i\pi/4} e^{\frac{i\pi^2}{16}} e^{-i\frac{f^2}{8} \ln(2)} f\sqrt{\pi} \psi, \]
where \( \psi = \Psi(t_2, \zeta)|_{t=-0} \)

### 3 Numerical Simulation

Here we illustrate the analytical results which are formulated in Theorem 1. Let us consider (1) with \( \gamma = -1/3 \) and \( S(t_2) = t_2^2/2. \) We consider the solution when \( k = 0. \) It yields \( \omega = 1 \) and \( \zeta = x_1. \) The envelope function is a solution of the nonlinear Schrödinger equation:
\[ 2i\partial_{t_2} \Psi - \partial_{x_1}^2 \Psi - |\Psi|^2 \Psi = 0. \]
It is well-known that this equation has a soliton solution:

\[ \Psi(x_1, t_2) = \frac{\sqrt{2} \eta e^{i(kx_1+(\kappa^2-\eta^2)t_2/2)}}{\cosh(\eta(x_1 + \kappa t_2))}. \]

In this section we reduce the problem and take into account the envelope function as \( \eta = 1 \) and \( \kappa = 0 \). In this case the asymptotic solution before resonance has the following form:

\[ u \sim \varepsilon \sqrt{2} e^{it_2/2} e^{i(kx_1+\omega t - \frac{t^2}{8} \ln \left(\frac{t_2 + \varepsilon}{t_2 - \varepsilon} \right))}. \]

In this simplest case the resonant curves are \( t_2 = -2, t_2 = 0 \) and \( t_2 = 2 \). Let us consider \( \varepsilon = 0.1 \) then the resonant curves are \( t = -200, t = 0 \) and \( t = 200 \). Following figures show the behavior of the solution for (1). Initial data are represented as:

\[ u|_{t=-900} = \varepsilon \left( \frac{\sqrt{2} e^{i2t_2/2}}{\cosh(\varepsilon x)} e^{i\frac{t^2}{8} \ln \left(\frac{t_2 + \varepsilon}{t_2 - \varepsilon} \right))} \right)|_{t=-900}, \]

\[ \partial_t u|_{t=-900} = \varepsilon \partial_t \left( \frac{\sqrt{2} e^{i\varepsilon t_2/2}}{\cosh(\varepsilon x)} e^{i\left(t-\frac{t^2}{8} \ln \left(\frac{t_2 + \varepsilon}{t_2 - \varepsilon} \right) \right))} \right)|_{t=-900}. \]
Fig. 1. 3-Dimensional behaviour of numerical solution of (1)
4 The first external expansion

In this section the formal asymptotic solution is constructed in the domain before the resonance. This domain is defined by the condition $l < 0$. The asymptotic expansion has the form of WKB-type. The leading-order term of asymptotic expansion has the order of $\varepsilon$ and oscillates.

**Theorem 2** When $l < 0$ the formal asymptotic expansion for the solution of (1) modulo $O(\varepsilon^N)$ has the form

$$U = \sum_{n \geq 1}^{N} \varepsilon^n U_n, \quad N \in \mathbb{N}$$

(2)
where
\[ U_1 = \sum_{j=\pm 1} u_{1(j,0)}(t_1, x_1, t_2) \exp\{ij(kx_2 + \omega t_2)/\varepsilon^2\} \] (3)

and
\[ U_n = \sum_{(j,m) \in \Omega_n} u_{n,(j,m)}(t_1, x_1, t_2) \exp(i\chi_{j,m}(x_2, t_2)/\varepsilon^2). \]

Here \( \chi_{j,m} = j(kx_2 + \omega t_2) + mS \) is the phase of the oscillating mode and the set \( \Omega_n \) contains the pairs \((j, m)\) such that:
\[ \Omega_1 = \{ (\pm 1, 0) \}, \]
\[ \Omega_n = \{ (j_{n-1}, m_{n-1}) \pm 1, (j_{n-1}, m_{n-1}) \in \Omega_{n-1} \} \cup \{(j_{l_1} + j_{l_2} + j_{l_3}, m_{l_1} + m_{l_2} + m_{l_3}) : l_1 + l_2 + l_3 = n, (j_{l_1}, m_{l_1}) \in \Omega_l, l_q \in \mathbb{N} \}. \]

The coefficient \( u_{1(1,0)} \) is defined out of (12). The higher-order terms \( u_{n,(j,m)} \) are defined by linearized equation (18) as \((j, m) = (\pm 1, 0)\) and by algebraic equations (17) as \((j, m) \neq (\pm 1, 0)\).

This expansion is valid in the domains
\[ -\varepsilon^{-1} L[\chi_{j,m}] \gg 1, \quad \forall (j, m) \in \cup_{q=1}^N \Omega_q. \]

This theorem give the asymptotic solution for a set of the domains on the plane \((x_2, t_2)\). But this theorem does not give an answer about a connection between the asymptotic solutions for nearest-neighbour domains.

This theorem does not give the solution in the form asymptotic series for (1) in general. Because the resonant lines \( L[\chi_{j,m}] = 0 \) are dense as \((j, m) \in \mathbb{Z}^2\) and full series (2) does not asymptotical anywhere.

Let us consider an example as \( S \equiv t_2^2/2 \) and \( k = 0 \). The set \( \Omega_n \) contains a phases \( \chi_{j,m} \) as \( j = 2q + 1 \) and \( q, m = \pm 0, 1, 2, \ldots, |j| + |m| < n \). In this case \( \omega = 1 \) and the lines
\[ 1 - (2q + 1)^2 + m[-2(2q + 1)t_2 - mt_2^2] = 0 \]
are resonant. These lines
\[ t_2 = -\frac{2q + 1}{m} \pm \frac{1}{m} \]
are dance everywhere when \( N = \infty \). But for any segments of (2) the sets of resonant lines are finite. When \( N = 3 \) the set is \( t_2 = -2, t_2 = 0 \) and \( t_2 = 2 \).
4.1 Construction of pre-resonant solution

4.1.1 Derivation of equations

Let us substitute (2) in equation (1) and gather the terms of same order of \( \varepsilon \). As a result we obtain a recurrent sequence of the equations:

\[
\partial_t^2 U_n - \partial_x^2 U_n + U_n = -f \cos \left( \frac{S}{\varepsilon^2} \right) U_{n-1} - \gamma \sum_{j+l+m=n} U_j U_l U_m - 2\partial_t \partial_x U_{n-1} \\
+ 2\partial_x \partial_x U_{n-1} - \partial_x^2 U_{n-2} + \partial_x^2 U_{n-2} \\
- 2\partial_t \partial_t U_{n-2} - 2\partial_t \partial_x U_{n-3} - \partial_x^2 U_{n-4}.
\] (4)

Let us consider this equation at \( n = 1 \):

\[
\left( -\omega^2 + k^2 + 1 \right) u_{1,(1,0)} \exp(i(kx + \omega t)) + c.c. = 0.
\]

This equation defines the dispersion formula:

\[
\omega^2 = k^2 + 1.
\]

Equation at \( n = 2 \) is

\[
\sum_{(j,m) \in \Omega_2} \left[ \left( - (\partial_t \chi_{j,m})^2 + (\partial_x \chi_{j,m})^2 + 1 \right) u_{2,(j,m)} \exp(i\chi_{j,m}/\varepsilon^2) \right] = \\
- \left( 2i\omega \partial_t \chi_{1,(1,0)} + 2ik \partial_x \chi_{1,(1,0)} \right) \exp(i(kx + \omega t)) - \\
f \cos(S/\varepsilon^2 + \phi_0) u_{1,(1,0)} \exp(i(kx + \omega t)) + c.c.,
\] (5)

where \( \Omega_2 = (j, m), j = \pm 1, m = \pm 1 \).

We construct a bounded solution respect to fast time \( t \). We remove terms with modes \( \exp(i(kx + \omega t)) \) from the right-hand side of (5). It yields:

\[
\omega \partial_t u_{1,(1,0)} - k \partial_x u_{1,(1,0)} = 0.
\] (6)

Solution of this equation is an arbitrary function with respect to \( \zeta = kt_1 + \omega x_1 \).

The set of of \( \Omega_2 \) is defined by modes of right-hand side of (5). The amplitudes for these modes are given by following formulas:

\[
\left( - (\partial_t \chi_{j,m})^2 + (\partial_x \chi_{j,m})^2 + 1 \right) u_{2,(j,m)} = -\frac{1}{2} f u_{1,(1,0)},
\] (7)

where \( (j, m) \in \{(1, \pm 1)\} \) and complex conjugated equations for \( (j, m) \in \{(-1, \pm 1)\} \). Thus we define \( U_2 \).
It is easy to see that $U_2$ has singularity of first order on the curves:

$$L[(kx_2 + \omega t_2) + S] = 0$$
$$L[(kx_2 + \omega t_2) - S] = 0. \quad (8)$$

Equation as $\varepsilon^3$ looks like

$$\partial_t^2 U_3 - \partial_x^2 U_3 + U_3 = +2i\omega \partial_t u_{1,(1,0)} \exp(ikx_2 + \omega t_2) \]
$$+ \sum_{(j,m) \in \Omega} \left[ 2i\partial_t \chi_{j,m} \partial_t u_{2,(j,m)} - 
2i\partial_x \chi_{j,m} \partial_x u_{2,(j,m)} \right] \exp(i\chi_{j,m}/\varepsilon^2) + 
(\partial_t^2 u_{1,(1,0)} - \partial_x^2 u_{1,(1,0)}) \exp(i(kx_2 + \omega t_2)) + 
2i\omega \partial_t u_2(1,0) \exp(3i(kx_2 + \omega t_2)) + 
3|u_{1,(1,0)}|^2 u_{1,(1,0)} \exp(i(kx_2 + \omega t_2)) \right] + c.c. = 0. \quad (10)$$

This equation has a bounded solution with respect to $t$ if the right-hand side does not contain the terms with $\exp(\pm i(kx_2 + \omega t_2))$. Thus we obtain two equations. First of them defines the dependence of $u_{2,(\pm 1,0)}$ with respect to $x_1$ and $t_1$:

$$\omega \partial_t u_{2,(1,0)} - k \partial_x u_{2,(1,0)} = 0. \quad (11)$$

The second equation defines the dependence of $u_{1,(1,0)}$ with respect to $t_2$ and $\zeta$:

$$2i\omega \partial_t u_{1,(1,0)} - \partial_\zeta^2 u_{1,(1,0)} + 
3\gamma|u_{1,(1,0)}|^2 u_{1,(1,0)} = \frac{f^2}{4} g(x_2, t_2) u_{1,(1,0)},$$

where

$$g(x_2, t_2) = \left[ \frac{1}{L[\omega t_2 + kx_2 - S]} + \frac{1}{L[\omega t_2 + kx_2 + S]} \right]$$

Let us denote

$$u_{1(1,1)} = \Psi \exp \left\{-i \frac{f^2}{8\omega} G(x_2, t_2) \right\}, \quad (12)$$

where $G$ is an antiderivative of $g(x_2, t_2)$ with respect to $t_2$. Then the function $\Psi$ is determined by the nonlinear Schrödinger equation

$$2i\omega \partial_t \Psi - \partial_\zeta^2 \Psi + 3\gamma|\Psi|^2 \Psi = 0.$$

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The coefficients with \( \exp(i(k_2x + \omega t_2 \pm S)/\varepsilon^2) \) and their complex conjugations are defined by:

\[
    u_{3,(j,\pm 1)} = \frac{1}{L[\chi_{j,\pm 1}]} \left[ 2i\partial_{t_2}\chi_{j,\pm 1}\partial_{t_1}u_{2,(j,\pm 1)} - 2i\partial_{x_2}\chi\partial_{x_1}u_{2,(j,\pm 1)} \right], \quad j = \pm 1. \tag{13}
\]

The coefficients with \( \exp(i(k_2x + \omega t_2 \pm 2S)/\varepsilon^2) \) and their complex conjugations are defined by:

\[
    u_{3,(j,\pm 2)} = -\frac{1}{2L[\chi_{j,\pm 2}]} fu_{2,(j,\pm 1)}, \quad j = \pm 1 \tag{14}
\]

The following equations and their complex conjugations define the term with coefficient \( \exp(3(k_2x + \omega t_2)/\varepsilon^2) \) and \( \exp(-3(k_2x + \omega t_2)/\varepsilon^2) \) accordingly.

\[
    u_{3,(3,0)} = -\frac{1}{L[\chi_{3,0}]} \gamma u_1^3. \tag{15}
\]

As a result one obtains \( U_3 \) where

\[
    \Omega_3 = \{(1, \pm m); (-1, \pm m), m = 1, 2; (\pm 3, 0)\}.
\]

The third correction \( U_3 \) has the singularity of the second order with respect to \( L[(kx_2 + \omega t) \pm S] \) on the curves \([\mathcal{S}]\) and \([\mathcal{O}]\), and the singularity of the first order with respect to \( L[(kx_2 + \omega t) \pm 2S] \) on the curves:

\[
    L[(kx_2 + \omega t) + 2S] = 0, \quad L[(kx_2 + \omega t) - 2S] = 0. \tag{16}
\]

Let us consider equation \((\Omega)\) for \( n > 3 \). The right-hand side of \((\Omega)\) contains the phases \( \chi_{j,m} \), where \((j, m) \in \Omega_n \) and

\[
    \Omega_n = \Omega_{n-1} \cup \{(j_{n-1}, m_{n-1} \pm 1), (j_{n-1}, m_{n-1}) \in \Omega_{n-1}\} \cup \{(j_{l_1} + j_{l_2} + j_{l_3}, m_{l_1} + m_{l_2} + m_{l_3}), l_1 + l_2 + l_3 = n, (j_{l_1}, m_{l_1}) \in \Omega_{l_1}, l_1 \in \mathbb{N}\}.
\]

The amplitudes are defined by the following formula as \((j, m) \neq (\pm 1, 0)\):

\[
    u_{n,(j,m)} = \frac{1}{L[\chi_{j,m}]} \left( -i\partial_{t_2}\chi_{j,m}\partial_{t_1}u_{n-1,(j,m)} + \left[ 2i\partial_{t_2}\chi_{j,m}\partial_{t_1}u_{n-1,(j,m)} - 2i\partial_{x_2}\chi\partial_{x_1}u_{n-1,(j,m)} \right] - \frac{i}{2} f u_{n-1,(j,m-1)} - \frac{i}{2} f u_{n-1,(j,m+1)} - \sum_{q_1 + q_2 + q_3 = n} u_{q_1,(j_1,m_1)} u_{q_2,(j_2,m_2)} u_{q_3,(j_3,m_3)} \right). \tag{17}
\]
If \((j, m) \equiv (\pm 1, 0)\) then the functions \(u_{n-1,(\pm 1,0)} \equiv u_{n-1,(\pm 1,0)}(t_2, \zeta)\). Function \(u_{n,(\pm 1,0)}(t_2, \zeta)\) stay indefinite and \(u_{n-2,(\pm 1,0)}\) is a solution of the equation:

\[
2i\omega \partial_{t_2} u_{n-2,(\pm 1,0)} + \partial_\zeta^2 u_{n-2,(\pm 1,0)} + \sum_{\gamma} u_{q_1,(j_1,m_1)}u_{q_2,(j_2,m_2)}u_{q_3,(j_3,m_3)} = 0. \tag{18}
\]

The term \(U_n\) has the singularity on the curve \(L[\chi_{j,m}]\) of the order \(p = n-(|j|+|m|)+1\). It means that the series for the solution loses the asymptotic property in a neighborhood of this curve.

The series for the solution has an asymptotic property if

\[
\frac{\varepsilon^{n+1}U_{n+1}}{\varepsilon^n U_n} \ll 1.
\]

It yields the domain of validity for the constructed asymptotic solution:

\[
\varepsilon L[\chi_{j,m}] \ll 1.
\]

The Theorem 2 is proved.

5 Internal expansion

5.1 Expansion near primary resonant curve

The primary singularity is situated at the curves \(L[\chi_{1,\pm 1}] = 0\). It is easy to see that the \(n\)-th coefficients of expansion \(2\) has the singularity of the order \(n-1\) on these curves.

A typical local resonance generates new harmonics with

\[
k_1 = k \pm \partial_{x_2} S_{L[\chi_{1,\pm 1}]} = 0, \quad \text{as} \quad |k_1| \neq k
\]

and

\[
\omega_1 = \omega \pm \partial_{t_2} S_{L[\chi_{1,\pm 1}]} = 0.
\]

That case may be studied by the same approach as \(3\).

Here we consider a special case of the local resonance when

\[
k - \partial_{x_2} S = -k.
\]
This special case we call by the local parametric resonance. In this case a new harmonics are not generated in the leading-order term of the asymptotic expansion but the envelope function changes.

In this subsection we construct the formal asymptotic solution near \( L[\chi_{1,1}] = 0 \) and obtain an connection formulas for the nearest-neighbor solution (2).

**Theorem 3** In a neighbourhood of the parametric resonant curve the formal asymptotic expansion for the solution of (7) has the form

\[
U(x, t, \varepsilon) = \sum_{n=1}^{N} \varepsilon^n W_n(x_1, t_1, x_2, t_2, \varepsilon),
\]

(19)

where

\[
W_n = \sum_{j=-n+1}^{n} w_{n,j}(x_1, t_1) \exp(i\frac{(2j-1)S}{2\varepsilon^2}),
\]

Function \( w_{n,j} \) is determined from equations (21), (23) and (25). This expansion is valid when

\[
|\left(\partial_t^2 S\right)^2 - \left(\partial_x^2 S\right)^2 - 4| \ll \varepsilon^{-1}.
\]

5.2 Formal Construction

In this section we prove the Theorem 3. Substitute (19) into (1). Define:

\[
\varepsilon \lambda = -\frac{1}{4}(\partial_t^2 S)^2 + \frac{1}{4}(\partial_x^2 S)^2 + 1.
\]

(20)

Gather the terms of same order of \( \varepsilon \). In order of \( \varepsilon^2 \) it yields:

\[
\sum_{j=-1,j\neq0,1}^{2} \left[ -\frac{1}{4}\left( (2j-1)\partial_t^2 S \right)^2 + \frac{1}{4}\left( (2j-1)\partial_x^2 S \right)^2 + 1 \right] \times \]

\[
\frac{1}{2} w_{2,j} \exp\left(i\frac{(2j-1)S}{2\varepsilon^2}\right) = \frac{1}{2} w_{1,1,1} f \exp\left(i\frac{3S}{2\varepsilon^2}\right) + \frac{1}{2} w_{1,-1,1} \exp\left(-i\frac{3S}{2\varepsilon^2}\right) + \frac{f}{2} \left[ \frac{S}{w_{1,1,1}} \exp\left(i\frac{S}{2\varepsilon^2}\right) + \right]
\]

\[
\left( (i\partial_x^2 S - \partial_t^2 S) \partial_x \right) w_{1,1,1} + \lambda w_{1,1} + \frac{f}{2} \left[ \frac{S}{w_{1,-1,1}} \exp\left(-i\frac{S}{2\varepsilon^2}\right) + \right]
\]

\[
\left( (i\partial_x^2 S - \partial_t^2 S) \partial_x \right) w_{1,-1,1} + \lambda w_{1,-1} + \frac{f}{2} \left[ \frac{S}{w_{1,-1,1}} \exp\left(-i\frac{S}{2\varepsilon^2}\right) \right],
\]

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This formula contains linear independent exponents with respect to fast variables. Collect the coefficients of such exponents. As a result one obtains equations for $w_{1,\pm 1}$

$$i(\partial_{x_2}S\partial_{x_1} - \partial_{t_2}S\partial_{t_1})w_{1,\pm 1} + \lambda w_{1,\pm 1} + \frac{f}{2}w_{1,\pm 1} = 0, \quad (21)$$

and a formula for $w_{2,\pm 3}$:

$$w_{2,\pm 3} = \frac{1}{4} w_{1,\pm 1} f.$$

In order of $\varepsilon^n$ we obtain the following equation:

$$\sum_{j = -n + 1}^{n} \left[ -\frac{1}{4}((2j - 1)\partial_{t_2}S)^2 + \frac{1}{4}((2j - 1)\partial_{x_2}S)^2 + 1 \right] w_{n,j} \exp(i \frac{(2j - 1)S}{2\varepsilon^2}) =$$

$$[(i\partial_{x_2}S\partial_{x_1} - \partial_{t_2}S\partial_{t_1})w_{n-1,1} + \lambda w_{n-1,1} +$$

$$\frac{1}{2}\tilde{w}_{n-1,1}] \exp(i \frac{S}{2\varepsilon^2}) +$$

$$[(i\partial_{x_2}S\partial_{x_1} - \partial_{t_2}S\partial_{t_1})w_{n-1,-1} + \lambda w_{n-1,-1} +$$

$$\frac{1}{2}\tilde{w}_{n-1,-1}] \exp(-i \frac{S}{2\varepsilon^2}) +$$

$$- \sum_{j = -n + 1}^{n} \left( \partial_{t_2}^2 w_{n-4,2j-1} +$$

$$i(2j - 1)\partial_{t_2}S\partial_{t_2}w_{n-2,2j-1} +$$

$$\partial_{t_1}^2 w_{n-2,2j-1} - \partial_{x_2}^2 w_{n-2,2j-1} +$$

$$\frac{f}{2}w_{n-1,2j-3} + \frac{f}{2}w_{n-1,2j+1} +$$

$$\gamma \sum_{p+q+r = n} w_{p,j_p}w_{q,j_q}w_{r,j_r} \exp(i \frac{(2j - 1)S}{2\varepsilon^2}). \quad (22)$$

Collect coefficients with linear independent exponents with respect to fast variables. It yields:

$$i\partial_{t_2}S\partial_{t_1}w_{n,1} - i\partial_{x_2}S\partial_{x_1}w_{n,1} + \lambda w_{n,1} + \frac{1}{2}\tilde{w}_{n,1} = F_{n,1} \quad (23)$$

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where

\[
F_{n,1} = -i\partial_{t_2}S\partial_{t_2}w_{n-1,1} + i\partial_{x_2}S\partial_{x_2}w_{n-1,1} + \frac{1}{4}(\partial_{t_2}S)^2w_{n-1,1} - \frac{1}{4}(\partial_{x_2}S)^2w_{n-1,1} - \partial_{t_1}^2w_{n-1,1} + \partial_{x_1}^2W_{n-1,1} - 2\partial_{t_2}\partial_{t_1}w_{n-2,1} + 2\partial_{x_2}\partial_{x_1}w_{n-2,1} - \partial_{t_2}^2w_{n-3,1} + \partial_{x_2}^2w_{n-3,1} - \sum_{n_1 + n_2 + n_3 = n + 1, k_1 + k_2 + k_3 = 1, k_j \in \Omega_{n_j}, j = 1, 2, 3} w_{n_1,k_1}w_{n_2,k_2}w_{n_3,k_3}.
\]

(24)

The term \(w_{n,j}, j \neq 1\) is determined by algebraic equation

\[
w_{n,j} = \frac{\gamma}{(2j-1)^2 - 1} \left( -2i\partial_{t_2}S\partial_{t_2}w_{n-2,j} + 2i\partial_{x_2}S\partial_{x_2}w_{n-2,j} + (\partial_{t_2}S)^2w_{n-2,j} - (\partial_{x_2}S)^2w_{n-2,j} - \partial_{t_1}^2w_{n-2,j} + \partial_{x_1}^2w_{n-2,j} - 2\partial_{t_2}\partial_{t_1}w_{n-3,j} + 2\partial_{x_2}\partial_{x_1}w_{n-3,j} - \partial_{t_2}^2w_{n-4,j} + \partial_{x_2}^2w_{n-4,j} - \sum_{n_1 + n_2 + n_3 = n + 1, k_1 + k_2 + k_3 = j, k_j \in \Omega_{n_j}, j = 1, 2, 3} w_{n_1,k_1}w_{n_2,k_2}w_{n_3,k_3} \right).
\]

(25)

5.2.1 Characteristic variables

The function \(w_{n,1}\) satisfies equation (24). The solution is constructed by the method of characteristics. Define the characteristic variables \(\sigma, \xi\). We choose a point \((x_1^0, t_1^0)\) such that \(\partial_{x_2}l|_{(x_1^0, t_1^0)} \neq 0\) as an origin and denote by \(\sigma\) the variable along the characteristics for equation (21). We suppose \(\sigma = 0\) on the curve \(\lambda = 0\). The variable \(\xi\) mensurates the distance along the curve \(\lambda = 0\) from the point \((x_1^0, t_1^0)\). This point \((x_1^0, t_1^0)\) corresponds to \(\xi = 0\). The positive direction for parameter \(\xi\) coincides with the positive direction of \(x_2\) in the neighborhood of \((x_1^0, t_1^0)\).

The characteristic equations for (21) have a form

\[
\frac{dt_1}{d\sigma} = \partial_{t_2}S(\varepsilon x_1, \varepsilon t_1), \quad \frac{dx_1}{d\sigma} = -\partial_{x_2}S(\varepsilon x_1, \varepsilon t_1).
\]

(26)

The initial conditions for the equations are

\[
x_1|_{\sigma = 0} = x_1^0, \quad t_1|_{\sigma = 0} = t_1^0.
\]

(27)
Lemma 1  The Cauchy problem for characteristics has a solutions when $|\sigma| < c_1 \varepsilon^{-1}$, $c_1 = \text{const} > 0$.

Proof. The Cauchy problem \((26), (27)\) is equivalent to the system of the integral equations

\[
t_1 = t_1^0 + \int_0^\sigma \partial_2 S(\varepsilon x_1, \varepsilon t_1) d\zeta, \quad x_1 = x_1^0 - \int_0^\sigma \partial_2 S(\varepsilon x_1, \varepsilon t_1) d\zeta. \tag{28}
\]

Substituting $\tilde{t}_2 = (t_1 - t_1^0) \varepsilon$, $\tilde{x}_2 = (x_1 - x_1^0) \varepsilon$, we obtain

\[
\tilde{t}_2 = \int_0^\varepsilon \partial_2 S(\tilde{x}_2 - \varepsilon x_1^0, \tilde{t}_2 - \varepsilon t_1^0) d\zeta, \quad \tilde{x}_2 = -\int_0^\varepsilon \partial_2 S(\tilde{x}_2 - \varepsilon x_1^0, \tilde{t}_2 - \varepsilon t_1^0) d\zeta.
\]

The integrands are smooth and bounded functions on the plane $x_2, t_2$. There exists the constant $c_1 = \text{const} > 0$ such that the integral operator is a contraction operator when $\varepsilon |\sigma| < c_1$. Lemma 1 is proved.

It is convenient to use the following asymptotic formulas for the change of variables $(x_1, t_1) \rightarrow (\sigma, \xi)$.

Lemma 2  In the domain $|\sigma| \ll \varepsilon^{-1}$ the asymptotics as $\varepsilon \rightarrow 0$ of the solutions for Cauchy problem \((26), (27)\) have the form

\[
x_1(\sigma, \xi, \varepsilon) - x_1^0(\xi) = -\sigma \partial_2 S + \sum_{n=1}^N \varepsilon^n \sigma^{n+1} g_n(\varepsilon x_1, \varepsilon t_1) + O(\varepsilon^{N+1} \sigma^{N+2}), \tag{29}
\]

\[
t_1(\sigma, \xi, \varepsilon) - t_1^0(\xi) = \sigma \partial_2 S + \sum_{n=1}^N \varepsilon^n \sigma^{n+1} h_n(\varepsilon x_1, \varepsilon t_1) + O(\varepsilon^{N+1} \sigma^{N+2}), \tag{30}
\]

where

\[
g_n = -\left. \frac{d^n}{d\sigma^n}(\partial_2 S) \right|_{\sigma=0}, \quad h_n = \left. \frac{d^n}{d\sigma^n}(\partial_2 S) \right|_{\sigma=0}.
\]

The lemma proves by integration by parts of equations \((28)\).

The next proposition gives us the asymptotic formula which relates the variables $\sigma$ and $\lambda$ as $\sigma, \lambda \rightarrow \pm \infty$.

Lemma 3  Let be $\sigma \ll \varepsilon^{-1}$, then:

\[
\lambda = \varphi(\xi) \sigma + O(\varepsilon \sigma^2), \quad \sigma \rightarrow \infty, \quad \varphi(\xi) = \left. \frac{d\lambda}{d\sigma} \right|_{\sigma=0}. \tag{31}
\]

Proof. From formula \((20)\) we obtain the representation in the form

\[
\lambda = \sum_{j=1}^\infty \lambda_j(x_1, t_1, \varepsilon) \sigma^j \varepsilon^{j-1},
\]
where
\[ \lambda_j(x_1,t_1,\varepsilon) = \frac{1}{j!} \frac{d^j}{d\sigma^j} \lambda(x_1,t_1,\varepsilon)|_{\sigma=0}. \]

It yields
\[ \lambda = \frac{d\lambda}{d\sigma}|_{\sigma=0} + O(\varepsilon\sigma^2d^2\lambda/d\sigma^2). \]

Let be
\[ \left| \frac{d^2l}{d\sigma^2} \right| \geq \text{const}, \; \xi \in R. \]

The function \( d\lambda/d\sigma \) is not equal to zero
\[ \frac{d\lambda}{d\sigma} = \frac{1}{2} \left( -\partial_{x_2}\lambda\partial_{x_2}S + \partial_{t_2}\lambda\partial_{t_2}S \right) \neq 0. \]

Let us suppose \( d\lambda/d\sigma > 0 \). It yields
\[ \lambda = \varphi(\xi)\sigma + O(\varepsilon\sigma^2), \quad \varphi(\xi) = \frac{d\lambda}{d\sigma}|_{\sigma=0} \]

The lemma is proved.

5.2.2 Ordinary differential equations for local parametric resonance

The equation (21) may be considered as an ordinary differential equation along the characteristic.

\[ i\frac{dw_{1,1}}{d\sigma} + \lambda(\sigma,\xi,\varepsilon)w_{1,1} + \frac{f}{2}w_{1,1} = 0. \] (31)

Let us change variable: \( \kappa = \alpha\sigma, \; F = f/(2\alpha) \) where \( \alpha^2(\xi) = \varphi(\xi) \), then we obtain:
\[ i\frac{dw_{1,1}}{d\kappa} + \kappa w_{1,1} + Fw_{1,1} = 0. \] (32)

The same equation for \( w_{n,1} \) has the following form:
\[ i\frac{dw_{n,1}}{d\kappa} + \kappa w_{n,1} + Fw_{n,1} = \frac{1}{\alpha} F_{n,1} + \frac{1}{\alpha} \sum_{j+l=n-1} \lambda_{l+1} \left( \frac{\kappa}{\alpha} \right)^{l+1} w_{j,1}. \] (33)
A solution for the equation (32) has obtained in [10]:

\[ V(\kappa; C) = \frac{1}{2} (-1 - i)^{1-\frac{\kappa^2}{2}} F C D_z(e^{\frac{\kappa}{2}} \sqrt{2\kappa}) + \]
\[ \left( (1 - i)^{\frac{\kappa^2}{2}} C + 2(-1 - i)^{-1-\frac{\kappa^2}{2}} e^{\frac{\kappa^2}{2}} \sqrt{2\pi} F \Gamma(-i\frac{\kappa^2}{2}) C \right) D_{-z-1}(e^{\frac{\kappa}{2}} \sqrt{2\kappa}), \]

where \( D_z(y) \) is a parabolic cylinder function, \( z = i\frac{\kappa^2}{2} - 1 \) and \( C \) is an arbitrary complex constant.

Let us define

\[ w_{1,1} = V(\kappa, C). \quad (34) \]

Let us consider the nonhomogeneous equation:

\[ i \frac{dU}{d\kappa} + \kappa U + F\overline{U} = G(\kappa). \quad (35) \]

To solve this equation we use two linear independent solutions of homogeneous equation. These solutions are:

\[ V_1 = \partial_\alpha V(\kappa, C), \quad V_2 = \partial_\beta V(\kappa, C), \quad C = \alpha + i\beta. \]

Let us define the Wronskian of these solutions as:

\[ W = V_1\overline{V_2} - V_2\overline{V_1}. \]

It easy to see that \( W = \text{const} \neq 0. \)

The solution of (35) has a form:

\[ U = V_1 \int_0^\kappa (\overline{V_2} - V_2 \overline{G})(\chi) \frac{d\chi}{W} + \]
\[ V_2 \int_0^\kappa (\overline{V_1} - V_1 \overline{G})(\chi) \frac{d\chi}{W} + c_1 V_1 + c_2 V_2, \quad (36) \]

where \( c_1 \) and \( c_2 \) are real constants which are parameters of the solution.

The general formula (36) allows to us to solve the equations for any high-order term.

### 5.3 Asymptotics as \( \lambda \to \infty \) and domain of validity of the internal expansion

The domain of validity of the internal expansion is determined by the asymptotic behaviour of higher-order terms. In this section we show, that the \( n \)-th order term of the asymptotic solution grows as \( \lambda^{n-1} \) when \( \lambda \to \infty. \) This growth of higher-order terms allows us to determine the domain of validity for internal asymptotic expansion [19] as \( \lambda \to \infty. \)
5.3.1 Asymptotic behaviour of leading-order term

The leading-order term has the following behaviour as $\kappa \to -\infty$:

$$w_{1,1} = C_{1,1} e^{i\left(\frac{\kappa^2}{2} - \frac{a^2}{2} \ln(-\kappa)\right)} (1 + O(\kappa^{-1})), \quad C_{1,1} \in \mathbb{C} \quad (37)$$

and

$$w_{1,1} = \left( e^{\frac{a^2}{2}} C_{1,1} + \frac{(1 + i)e^{\frac{a^2}{4}} e^{i\frac{a^2}{4} \ln(2)} a \sqrt{\pi}}{\Gamma(1 - i \frac{a^2}{2})} C_{1,1} \right) e^{i\left(\frac{\kappa^2}{2} - \frac{a^2}{2} \ln(\kappa)\right)} (1 + O(\kappa^{-1})) \quad (38)$$

as $\kappa \to \infty$.

These formulas were obtained using the asymptotic behavior of solution (32).

5.3.2 Asymptotic behavior of higher-order terms

The $n$-th order term has the following asymptotic formula:

$$w_{1,n} = O(\kappa^{n-2}).$$

Then the internal expansion is valid as $|\kappa| \ll \varepsilon^{-1}$, or the same as $\varepsilon|\lambda| \ll 1$.

6 Matching

The domains of validity for the internal and external asymptotic expansions are intersected. It allows us to match these expansions and as a result to obtain an uniform asymptotic expansion that is valid in both domains.

To match the expansions one should reexpand the external asymptotic expansion in the terms of internal variables $\xi$ and $\kappa$ and equate the terms with the same order of $\varepsilon$. It yields the indefinite functions in the internal asymptotic expansion. Here we make this step for the main terms of the internal and external asymptotic expansions before the local resonant layer:

$$\varepsilon \left( w_{1,-1}(\xi, \kappa) \exp(iS/\varepsilon^2) + c.c. \right) \sim \varepsilon \left( \Psi(t_2, \zeta) \exp(-i\frac{f^2}{4\omega} G(x_2, t_2)) \exp(i(kx_2 + \omega t_2)/\varepsilon^2) + c.c. \right), \quad (39)$$

as $1 \ll -\kappa \ll \varepsilon^{-1}$, $(t_2, x_2) \in L[\chi_{1,1}] = 0$ and $\zeta = \omega x_2(t_2) + kt_2(t_2)$. This formula define the function $w_{1,-1}(\xi, \kappa)$ as $\kappa \to -\infty$. 

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The same formula for the matching is valid after the resonant layer:

\[
\varepsilon \left( \Psi(t_2, \zeta) \exp(-i \frac{f^2}{4\omega} G(x_2, t_2)) \exp(i(kx_2 + \omega t_2)/\varepsilon^2) + \text{c.c.} \right) \sim \\
\varepsilon \left( w_{1,-1}(\xi, \kappa) \exp(iS/\varepsilon^2) + \text{c.c.} \right).
\] (40)

as \(1 \ll \kappa \ll \varepsilon^{-1}, \ (t_2, x_2) \in L[\chi_{1,1}] = 0\) and \(\zeta = \omega x_1^0(\xi) + k t_1^0(\xi)\). This formula defines the function \(\Psi(t_2, \zeta)\) coming out of the resonant layer.

The matching gives a jump for the \(\Psi\) at the resonant line:

\[
\Psi(t_2, \zeta)|_{L[\chi_{1,1}]+0} = \left( e^{\frac{f^2\pi}{8}} \Psi(t_2, \zeta)|_{L[\chi_{1,1}]-0} + \right.\\
\left. \frac{(1 + i) e^{\frac{f^2\pi}{16}} e^{i \frac{f^2}{2} \ln(2)} f \sqrt{\pi}}{2\Gamma(1 - i \frac{f^2}{8})} \Psi(t_2, \zeta)|_{L[\chi_{1,1}]-0} \right)
\] (41)

6.1 Post-resonant expansion

The asymptotic structure of the post-resonant solution has the similar structure as the pre-resonant asymptotic solution. The main term of the solution is:

\[ U(x, t, \varepsilon) \sim \varepsilon v_1(x_1, t_1, t_2) \exp\{i(kx + \omega t)\} \]

The amplitude \(v_1 = \Psi \exp\left\{ i \frac{f^2}{4\omega} G(x_2, t_2) \right\}\) and \(\Psi\) is determined by nonlinear Schrödinger equation also and initial datum on the curve \(L[\chi_{1,1}] = 0\)

\[
\Psi(t_2, \zeta)|_{L[\chi_{1,1}]+0} = e^{\frac{f^2\pi}{8}} \Psi(t_2, \zeta)|_{L[\chi_{1,1}]-0} + \frac{(1 + i) e^{\frac{f^2\pi}{16}} e^{i \frac{f^2}{2} \ln(2)} f \sqrt{\pi}}{2\Gamma(1 - i \frac{f^2}{8})} \Psi(t_2, \zeta)|_{L[\chi_{1,1}]-0}.
\]

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