Research Article

Dynamical Yang-Baxter Maps Associated with Homogeneous Pre-Systems

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Dedicated to Professor Susumu Okubo on the occasion of his eightieth birthday

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Abstract We construct dynamical Yang-Baxter maps, which are set-theoretical solutions to a version of the quantum dynamical Yang-Baxter equation, by means of homogeneous pre-systems, that is, ternary systems encoded in the reductive homogeneous space satisfying suitable conditions. Moreover, a characterization of these dynamical Yang-Baxter maps is presented.

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1 Introduction

The quantum dynamical Yang-Baxter equation (QDYBE for short) [9, 10], a generalization of the quantum Yang-Baxter equation (QYBE for short) [2, 3, 40, 41], has been studied extensively in recent years (see [7] and the references therein). Dynamical Yang-Baxter maps [31, 32, 34] are set-theoretical solutions to a version of the QDYBE.

Let $H$ and $X$ be nonempty sets with a map $(\cdot): H \times X \ni (\lambda, x) \mapsto \lambda \cdot x \in H$. A map $R(\lambda): X \times X \to X \times X \quad (\lambda \in H)$ is a dynamical Yang-Baxter map associated with $H$, $X$ and $(\cdot)$, if and only if, for every $\lambda \in H$, $R(\lambda)$ satisfies the following equation on $X \times X \times X$:

$$R_{23}(\lambda)R_{13}(\lambda \cdot X^{(2)})R_{12}(\lambda) = R_{12}(\lambda \cdot X^{(3)})R_{13}(\lambda)R_{23}(\lambda \cdot X^{(1)}). \quad (1.1)$$

Here $R_{12}(\lambda)$, $R_{12}(\lambda \cdot X^{(3)})$, $R_{23}(\lambda \cdot X^{(1)})$, and others are the maps from $X \times X \times X$ to itself defined as follows: for $u, v, w \in X$,

$$R_{12}(\lambda)(u, v, w) = (R(\lambda)(u, v), w),$$
$$R_{12}(\lambda \cdot X^{(3)})(u, v, w) = R_{12}(\lambda \cdot w)(u, v, w),$$
$$R_{23}(\lambda \cdot X^{(1)})(u, v, w) = (u, R(\lambda)(u \cdot w)).$$

Set-theoretical solutions to the QYBE [6], also known as Yang-Baxter maps [39], are dynamical Yang-Baxter maps constant for the parameter $\lambda$ of any set $H$; indeed, the dynamical Yang-Baxter map is a generalization of the set-theoretical solution to the QYBE.

This dynamical Yang-Baxter map yields a bialgebroid [4]. Every dynamical Yang-Baxter map with some conditions gives birth to an $(H, X)$-bialgebroid [35], a generalization of the quantum group [5, 11], through the Faddeev-Reshetikhin-Takhtajan construction [8].

It is worth pointing out that a ternary system (Definition 1(3)) can produce the dynamical Yang-Baxter map [32]. Each triple $(L, M, \pi)$ consisting of a left quasigroup $L = (L, \cdot)$ (Definition 1(1)), a ternary system $M$ satisfying (2.2) and (2.3), and a (set-theoretic) bijection $\pi : L \to M$ gives a dynamical Yang-Baxter map $R(\lambda)$ associated with $L, L$ and $(\cdot)$ (see Section 2 for more details).

Homogeneous systems [18, 19, 20, 21, 22, 23] are algebraic features of the reductive homogeneous space [24, 28] satisfying suitable conditions. Let $A$ be a group with its subgroup $K$. We assume that a subset $G$ of the group $A$ satisfies the following:

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(1) the group $A$ is uniquely factorized as $A = GK$,
(2) $G^{-1} = G$,
(3) $kGk^{-1} = G$, for all $k \in K$.

Let $p : A \to G$ denote the canonical projection with respect to the factorization $A = GK$, and $(\cdot)$ the binary operation on $G$ defined by $x \cdot y = p(xy)$ $(x, y \in G)$. Because the map $L_x : G \ni y \mapsto x \cdot y \in G$ is bijective, we define the ternary operation $\eta$ on $G$ by

$$\eta(x, y, z) = L_x \left( (L_x)^{-1}(y) \cdot (L_x)^{-1}(z) \right), \quad x, y, z \in G.$$ 

This ternary system $G = (G, \eta)$ is a homogeneous system [23, Proposition 1] (see also Definition 7), and every homogeneous system is constructed in such a way.

If $G$ is a connected and second countable $C^\infty$-manifold with a $C^\infty$-map $\eta : G \times G \times G \to G$, then the homogeneous system $G = (G, \eta)$ is isomorphic to a reductive homogeneous space $A/K$ for a connected Lie group $A$ with its closed subgroup $K$ [19, Theorem 1]. The homogeneous system is a ternary system, an algebraic structure, encoded in the reductive homogeneous space (for ternary systems in differential geometry and mathematical physics, see [13, 14, 15, 29]).

It is natural to relate this homogeneous system to the dynamical Yang-Baxter map through the ternary system.

The aim of this paper is to produce the dynamical Yang-Baxter maps by means of homogeneous pre-systems, which generalize the homogeneous systems. Furthermore, we characterize such dynamical Yang-Baxter maps.

The organization of this paper is as follows.

Section 2 contains a brief summary of the dynamical Yang-Baxter map. We focus on its construction by means of the ternary system. This construction yields a category $A$ concerning the ternary systems, which is equivalent to a category $D$ consisting of the dynamical Yang-Baxter maps.

Section 3 presents the notion of a homogeneous pre-system, together with examples.

In Section 4, our main results are stated and proved. Every homogeneous pre-system satisfying (4.1) can produce a dynamical Yang-Baxter map via the ternary system. More precisely, we construct a category $H$, isomorphic to the category $A$, by means of the homogeneous pre-systems with (4.1). Because the category $A$ is equivalent to the category $D$, each object of $H$ gives a dynamical Yang-Baxter map; in particular, we demonstrate dynamical Yang-Baxter maps provided by a certain left quasigroup and the examples in Section 3.

The last section, Section 5, deals with a relation between the homogeneous pre-system satisfying (4.1) and the left quasigroup with (5.1), which is due to the work in [32, Section 6]. We introduce a category $B$ concerning the left quasigroups satisfying (5.1) and an essentially surjective functor $J : B \to H$ to construct the dynamical Yang-Baxter maps by means of quasigroups of reflection [17, 27].

Our viewpoint sheds some light on the relation between geometry and the dynamical Yang-Baxter map.

2 Dynamical Yang-Baxter maps

In this section, we briefly summarize without proofs the relevant material in [32] on the construction of the dynamical Yang-Baxter map.

**Definition 1.** (1) $(L, \cdot)$ is a left quasigroup (resp. right quasigroup [38, Section I.4.3]), if and only if $L$ is a nonempty set, together with a binary operation $(\cdot)$ on $L$ having the property that, for all $u, w \in L$, there uniquely exists $v \in L$ such that $u \cdot v = w$ (resp. $v \cdot u = w$). For the simplicity, one uses the notation $uv$ instead of $u \cdot v$ ($u, v \in L$).

(2) A quasigroup $(Q, \cdot)$ is a left and right quasigroup (see [30, Definition I.1.1] and [38, Section I.2]).

(3) A ternary system $(M, \mu)$ is a pair of a nonempty set $M$ and a ternary operation $\mu : M \times M \times M \to M$.

By this definition, the left quasigroup $L = (L, \cdot)$ has another binary operation $\setminus_L$ called a left division [38, Section I.2.2]. For $u, w \in L$, we denote by $w \setminus_L u$ the unique element $v \in L$ satisfying $uw = w$,

$$w \setminus_L u = v \iff uw = w. \quad (2.1)$$

The binary operation on the quasigroup is not always associative.

**Example 2.** We define the binary operation $(\ast)$ on the set $Q = \{1, 2, 3, 4, 5\}$ of five elements by Table 1. Here $1 \ast 2 = 3$. This $Q = (Q, \ast)$ is a quasigroup, because each element in $Q$ appears once and only once in each row and in each column of Table 1 [30, Theorem I.1.3]. The binary operation $(\ast)$ is not associative, since $(1 \ast 2) \ast 3 \neq 1 \ast (2 \ast 3)$. This quasigroup $Q$ is due to Nobusawa [27, Section 6, type 1]. However, the order of the binary operation $(\ast)$ in Table 1 is reversed.
Each ternary system $M = (M, \mu)$ satisfying

\[
\mu(a, \mu(a, b, c)), \mu(a, b, c)) = \mu(a, \mu(b, c), d)) \quad \text{and} \quad \mu(\mu(a, b, c), d) = \mu(a, \mu(b, c), d).
\]

for any $a, b, c, d \in M$, can provide a dynamical Yang-Baxter map [32, Theorem 3.2]. Let $L = (L, \cdot)$ be a left quasigroup isomorphic to $M$ as sets, and $\pi : L \rightarrow M$ a (set-theoretic) bijection. For $\lambda, u \in L$, we define the maps

\[
\xi^{L,M,\pi}_\lambda(u) : L \rightarrow L \quad \text{and} \quad \eta^{L,M,\pi}_\lambda(u) : L \rightarrow L
\]

as follows: for $v \in L$,

\[
\xi^{L,M,\pi}_\lambda(u)(v) = \lambda(L^{\pi^{-1}}(\mu(\pi(\lambda u), \pi((\lambda u)v))))
\]

\[
\eta^{L,M,\pi}_\lambda(u)(v) = \lambda(\xi^{L,M,\pi}_\lambda(v)(u)) \cdot_L (\lambda(v)u).
\]

Let $R^{L,M,\pi}(\lambda) (\lambda \in L)$ denote the map from $L \times L$ to itself defined by

\[
R^{L,M,\pi}(\lambda)(u, v) = (\eta^{L,M,\pi}_\lambda(u)(v), \xi^{L,M,\pi}_\lambda(v)(u)), \quad u, v \in L.
\]

**Theorem 3.** The map $R^{L,M,\pi}(\lambda)$ (2.6) is a dynamical Yang-Baxter map (1.1) associated with $L, L$ and $\cdot$.

We now introduce two categories $A$ and $\mathcal{D}$ concerning a special class of the dynamical Yang-Baxter maps, which play a central role in this article.

The first task is to explain the category $A$ (cf. the category $A_2$ in [32, Section 6]). We follow the notation of [16, Chapter XI]. Let $L = (L, \cdot)$ be a left quasigroup, $M = (M, \mu)$ a ternary system satisfying (2.2) and

\[
\mu(a, b, c) = a, \quad \forall a, b \in M,
\]

\[
\mu(a, b, c, d) = \mu(a, b, c), \quad \forall a, b, c, d \in M,
\]

and $\pi : L \rightarrow M$ a bijection. The object of $A$ is, by definition, a triple $(L, M, \pi)$.

The morphism $f : (L, (M, \mu), \pi) \rightarrow (L', (M', \mu'), \pi')$ of $A$ is a homomorphism $f : L \rightarrow L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : M \rightarrow M'$ is a homomorphism of ternary systems; that is, the map $f : L \rightarrow L'$ satisfies

\[
f(a \cdot_L b) = f(a) \cdot_L f(b), \quad \forall a, b \in L,
\]

\[
h(\mu(a, b, c)) = \mu'(h(a), h(b), h(c)), \quad \forall a, b, c \in M.
\]

The identity $\text{id}_{L,M,\pi}$, the source $s(f)$ and the target $b(f)$ of a morphism $f : (L, M, \pi) \rightarrow (L', M', \pi')$, and the composition $g \circ f$ for morphisms $f : (L, M, \pi) \rightarrow (L', M', \pi')$ and $g : (L', M', \pi') \rightarrow (L'', M'', \pi'')$ are defined as follows: for an object $(L, M, \pi) \in A$,

\[
\text{id}_{L,M,\pi} = \text{id}_L,
\]

\[
s(f : (L, M, \pi) \rightarrow (L', M', \pi')) = (L, M, \pi),
\]

\[
b(f : (L, M, \pi) \rightarrow (L', M', \pi')) = (L', M', \pi'),
\]

the composition $g \circ f$ is the usual one of the maps $f : L \rightarrow L'$ and $g : L' \rightarrow L''$.

**Proposition 4.** $A$ is a category.

| * | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 2 | 4 |
| 2 | 5 | 2 | 4 | 1 | 3 |
| 3 | 4 | 1 | 3 | 5 | 2 |
| 4 | 3 | 5 | 2 | 4 | 1 |
| 5 | 2 | 4 | 1 | 3 | 5 |

**Table 1:** Binary operation ($*$) on $Q$. 
The next task is to describe the category $D$, which is exactly the category $D_2$ in [32, Section 6]. The object of this category $D$ is a pair $(L, R)$ of a left quasigroup $L = (L, ·)$ and a dynamical Yang-Baxter map $R(\lambda) : L \times L \rightarrow L \times L$ ($\lambda \in L$) satisfying

$$
\xi_\lambda(u)((\lambda u)\,\lambda) = \lambda \lambda, \quad \forall \lambda, u \in L, \\
(\lambda \xi_\lambda(u)(v))\eta_\lambda(v)(u) = (\lambda u)v, \quad \forall \lambda, u, v \in L, \\
(\lambda \xi_\lambda(u)(v)\xi_\lambda(u)(w))(\eta_\lambda(v)(w)) = \lambda \xi_\lambda(u)((\lambda u)((\lambda u)v)w), \quad \forall \lambda, u, v, w \in L.
$$

Here, $(\eta_\lambda(v)(u), \xi_\lambda(u)(v)) := R(\lambda(u, v)) (\lambda, u, v \in L)$.

The morphism $f : (L, R) \rightarrow (L', R')$ of $D$ is a homomorphism $f : L \rightarrow L'$ of left quasigroups satisfying

$$
R'(f(\lambda)) \circ f \times f = f \times f \circ R(\lambda), \quad \forall \lambda \in L.
$$

**Proposition 5.** $D$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $A$.

We can construct functors $S : A \rightarrow D$ and $T : D \rightarrow A$, which establish an equivalence of the categories $A$ and $D$ (cf. [32, Proposition 6.15]): for $(L, M, \pi) \in A$, set $S(L, M, \pi) = (L, R^{(L, M, \pi)})$. Here, $R^{(L, M, \pi)}(\lambda)$ is defined by (2.4), (2.5) and (2.6); for a morphism $f$ of $A$, write $S(f) = f$; for $(L, R) \in D$, $T(L, R)$ denotes the triple $(L, (M, \mu), \text{id}_L)$, where $M = L$ as sets and $\mu(a, b, c) = a \xi_\lambda(a) \lambda b \lambda c) (a, b, c \in M = L)$; for a morphism $f$ of $D$, set $T(f) = f$.

These functors $S$ and $T$ satisfy $ST = \text{id}_D$, and $\theta(L, M, \pi) := \text{id}_L ((L, M, \pi) \in A)$ gives a natural isomorphism $\theta : TS \rightarrow \text{id}_A$. Thus, the following theorem holds.

**Theorem 6.** $S : A \rightarrow D$ is an equivalence of categories.

### 3 Homogeneous pre-systems

This section is devoted to introducing homogeneous pre-systems.

**Definition 7.** (1) A ternary system $G = (G, \eta)$ (Definition 1(3)) is a homogeneous pre-system if and only if the ternary operation $\eta$ satisfies

$$
\eta(x, y, x) = y, \quad \forall x, y \in G, \\
\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)),
$$

for all $x, y, u, v, w \in G$.

(2) A homogeneous system $G = (G, \eta)$ [18] is a homogeneous pre-system satisfying

$$
\eta(x, y, x) = y, \quad \forall x, y \in G, \\
\eta(x, y, \eta(y, x, z)) = z, \quad \forall x, y, z \in G.
$$

We explain two examples in this section: one homogeneous pre-system and one homogeneous system, which imply dynamical Yang-Baxter maps in the next section.

Let $G$ be an abelian group. We define the ternary operation $\eta$ on $G$ by

$$
\eta(x, y, z) = x + y - z, \quad x, y, z \in G.
$$

A trivial verification shows that $G = (G, \eta)$ is a homogeneous pre-system, which is not always a homogeneous system because of (3.3) (cf. [18, Remark 4]).

Another example is a homogeneous system on an arbitrary group $G$ [18, Example in Section 1]. We define the ternary operation $\eta$ on the group $G$ by

$$
\eta(x, y, z) = yx^{-1}z, \quad x, y, z \in G.
$$

It is clear that this $G = (G, \eta)$ is a homogeneous system.

**Remark 8.** The homogeneous system $(G, \eta)$ (3.5) is equivalent to the notion of a torsor [25,33,36], also known as the principal homogeneous space, up to the choice of the unit element. Hence, the principal homogeneous space provides a homogeneous system.
4 A relation between dynamical Yang-Baxter maps and homogeneous pre-systems

In this section, we construct dynamical Yang-Baxter maps (2.6) by means of homogeneous pre-systems $G = (G, \eta)$ satisfying
\[
\eta(x, y, z) = \eta(w, \eta(y, y, w), z), \quad \forall x, y, z, w \in G. \tag{4.1}
\]
In fact, we present a category $H$ such that Baxter map.

Proposition 12. (In fact, we present a category $H$)

Proof.
It suffices to prove that $H$ is isomorphic to the category $A$ in Section 2, and, on account of Theorem 6, every object of $H$ consequently gives a dynamical Yang-Baxter map.

Let $L = (L, \cdot)$ be a left quasigroup, $G = (G, \eta)$ a homogeneous pre-system satisfying (4.1) and $\pi : L \to G$ a (set-theoretic) bijection. The object of $H$ is a triple $(L, G, \pi)$.

The morphism $f : (L, (G, \eta), \pi) \to (L', (G', \eta'), \pi')$ of $H$ is a homomorphism $f : L \to L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : G \to G'$ is a homomorphism of ternary systems; that is, the map $f : L \to L'$ satisfies (2.9) and
\[
h(\eta(x, y, z)) = \eta'(h(x), h(y), h(z)), \quad \forall x, y, z \in G.
\]

Proposition 9. $H$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $A$.

In order to prove that the category $H$ is isomorphic to the category $A$, we construct functors $F : A \to H$ and $F' : H \to A$.

We first introduce the functor $F : A \to H$. Let $(L, (M, \mu), \pi) \in A$. Define the ternary system $G = (G, \eta)$ by $G = M$ as sets and
\[
\eta(x, y, z) = \mu(y, x, z), \quad x, y, z \in G(= M). \tag{4.2}
\]

Proposition 10. $(L, G, \pi) \in H$.

Proof. We need only show that $G$ is a homogeneous pre-system satisfying (4.1).

An easy computation shows (3.1) and (4.1).

By virtue of (4.2), the left-hand side of (3.2) is $\mu(y, x, \mu(v, u, w))$, and, with the aid of (2.2), (2.7) and (2.8),
\[
\begin{align*}
\mu(y, x, \mu(v, u, w)) &= \mu(\mu(y, x, v), v, \mu(v, u, w)) \\
&= \mu(\mu(y, x, v), \mu(\mu(y, x, v), v, u), \mu(\mu(y, x, v), v, u), u, w) \\
&= \mu(\mu(y, x, v), \mu(y, x, u), \mu(y, x, w)),
\end{align*}
\]
which is the right-hand side of (3.2). This proves the proposition.

By setting $F(L, (M, \mu), \pi) = (L, G, \pi)$ and $F(f) = f$ for a morphism $f$ of $A$, the following proposition holds.

Proposition 11. $F : A \to H$ is a functor.

The next task is to construct a functor $F' : H \to A$. Let $(L, (G, \eta), \pi) \in H$. We define the ternary system $M_G = (M_G, \mu)$ by $M_G = G$ as sets and
\[
\mu(a, b, c) = \eta(b, a, c), \quad a, b, c \in M_G(= G). \tag{4.3}
\]

Proposition 12. $(L, M_G, \pi) \in A$.

Proof. It suffices to prove that $M_G$ satisfies (2.2), (2.7) and (2.8).

A trivial verification shows (2.7) and (2.8).

Due to (4.1) and (4.3), the left-hand side of (2.2) is
\[
\mu(a, \mu(a, b, c), \mu(\mu(a, b, c), c, d)) = \eta(\eta(b, a, c), a, \eta(b, a, d)).
\]

From (3.1) and (3.2),
\[
\eta(\eta(b, a, c), a, \eta(b, a, d)) = \eta(\eta(b, a, c), \eta(b, a, b), \eta(b, a, d)) = \eta(b, a, \eta(c, b, d)),
\]
which is exactly the right-hand side of (2.2).
By setting $F'(L, (G, \eta), \pi) = (L, M_G, \pi)$ and $F'(f) = f$ for a morphism $f$ of $\mathcal{H}$, the following proposition holds.

**Proposition 13.** $F' : \mathcal{H} \to A$ is a functor.

Since the functors $F$ and $F'$ satisfy $F'F = \text{id}_A$ and $FF' = \text{id}_\mathcal{H}$, the following theorem holds.

**Theorem 14.** The categories $A$ and $\mathcal{H}$ are isomorphic.

By taking account of Theorem 6, we have the following corollary.

**Corollary 15.** Each object of $\mathcal{H}$ provides a dynamical Yang-Baxter map.

The proof of the following proposition is straightforward.

**Proposition 16.** The ternary operations (3.4) and (3.5) satisfy (4.1).

As a consequence of Corollary 15 and Proposition 16, the homogeneous pre-system $G$ (3.4) and the homogeneous system $G$ (3.5) imply dynamical Yang-Baxter maps. Let $L = (G, \cdot)$ denote the left quasigroup whose binary operation $(\cdot)$ is defined by

$$u \cdot v = v, \quad u, v \in L,$$

(4.4)

and let $\pi : L(= G) \to G$ be the identity map on $G$. The corresponding dynamical Yang-Baxter maps are as follows: if $G$ is a homogeneous pre-system (3.4), then

$$R^{(L, M_G, \pi)}(\lambda)(u, v) = (v, \lambda + u - v), \quad \lambda, u, v \in L(= G),$$

and if $G$ is a homogeneous system (3.5), then

$$R^{(L, M_G, \pi)}(\lambda)(u, v) = (v, \lambda u^{-1}v), \quad \lambda, u, v \in L(= G).$$

5 A relation between homogeneous pre-systems and left quasigroups

Because of the work in [32, Section 6] and the fact that the categories $A$ and $\mathcal{H}$ are isomorphic, every homogeneous pre-system $G$ (Definition 7(1)) in the object $(L, G, \pi) \in \mathcal{H}$ is a left quasigroup (Definition 1(1)) whose binary operation gives the ternary operation of $G$. This last section demonstrates it by constructing a category $B$ concerning the left quasigroups with (5.1) and an essentially surjective functor $J : B \to \mathcal{H}$ (see [32, Proposition 6.17]). The functors $J : B \to \mathcal{H}, S : A \to D$ in Section 2, and $F' : \mathcal{H} \to A$ in Section 4, together with quasigroups of reflection [17, 27], provide examples of the dynamical Yang-Baxter map.

The first task is to introduce a category $B$. Let $L_1, L_2 = (L_2, \ast)$ be left quasigroups. We assume that the left quasigroup $L_2$ satisfies

$$(a \ast c) \backslash_{L_2} ((a \ast b) \ast c) = (a' \ast c) \backslash_{L_2} ((a' \ast b) \ast c), \quad \forall a, a', b, c \in L_2.
$$

(5.1)

Here the symbol $\backslash_{L_2}$ is the left division (2.1) of $L_2$. Let $\pi : L_1 \to L_2$ be a (set-theoretic) bijection. An object of $B$ is such a triple $(L_1, L_2, \pi)$.

A morphism $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ is a homomorphism $f : L_1 \to L_1'$ of left quasigroups such that $\pi' \circ f \circ \pi^{-1} : L_2 \to L_2'$ is also a homomorphism of left quasigroups.

**Proposition 17.** $B$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $A$.

The next task is to construct a functor $J : B \to \mathcal{H}$. Let $(L_1, (L_2, \ast), \pi) \in B$. We define the ternary system $G_{L_2} = (G_{L_2}, \eta_{L_2})$ by $G_{L_2} = L_2$ as sets and

$$\eta_{L_2}(x, y, z) = z \ast (x \backslash_{L_2} y), \quad x, y, z \in G_{L_2}(= L_2).$$

(5.2)

**Proposition 18.** $(L_1, G_{L_2}, \pi) \in \mathcal{H}$. 
Proof. It suffices to prove that $G_{L_2}$ is a homogeneous pre-system satisfying (4.1).

We give a proof only for (3.2) because the rest of the proof is straightforward. Let $x, y, u, v, w \in G(= L_2)$. From (5.2) we have
\[
\eta_{L_2}(x, y, \eta_{L_2}(u, v, w)) = (w \ast (u \setminus L_2 v)) \ast (x \setminus L_2 y)
\]
\[
= (w \ast (x \setminus L_2 y)) \ast \left( ((w \ast (x \setminus L_2 y)) \setminus L_2 ((w \ast (u \setminus L_2 v)) \ast (x \setminus L_2 y))) \right).
\]
(5.3)

With the aid of (5.1), the right-hand side of (5.3) is
\[
(w \ast (x \setminus L_2 y)) \ast \left( ((w \ast (x \setminus L_2 y)) \setminus L_2 ((w \ast (u \setminus L_2 v)) \ast (x \setminus L_2 y))) \right)
\]
\[
= (w \ast (x \setminus L_2 y)) \ast \left( ((u \ast (x \setminus L_2 y)) \setminus L_2 ((u \ast (u \setminus L_2 v)) \ast (x \setminus L_2 y))) \right)
\]
\[
= (w \ast (x \setminus L_2 y)) \ast \left( ((u \ast (x \setminus L_2 y)) \setminus L_2 (v \ast (x \setminus L_2 y))) \right),
\]
which is exactly $\eta_{L_2}(\eta_{L_2}(x, y, u), \eta_{L_2}(x, y, v), \eta_{L_2}(x, y, w))$. This is the desired conclusion.

Let $f : (L_1, L_2, \pi) \rightarrow (L'_1, L'_2, \pi')$ be a morphism of the category $\mathcal{B}$. The map $f : L_1 \rightarrow L'_1$ is a homomorphism of left quasigroups. Moreover, $h := \pi' \circ f \circ \pi^{-1} : L_2 \rightarrow L'_2$ is a homomorphism of ternary systems from $G_{L_2}$ to $G_{L_1}$, because $h$ is a homomorphism of left quasigroups. As a result, $f : (L_1, L_2, \pi) \rightarrow (L'_1, L'_2, \pi')$ is a morphism of the category $\mathcal{H}$.

We set $J(L_1, L_2, \pi) = (L_1, G_{L_2}, \pi)$ for $(L_1, L_2, \pi) \in \mathcal{B}$ and $J(f) = f$ for a morphism $f$ of $\mathcal{B}$.

Proposition 19. $J : \mathcal{B} \rightarrow \mathcal{H}$ is a functor.

This functor $J$ is essentially surjective. In fact, for any $(L, G, \pi) \in \mathcal{H}$, we can construct a left quasigroup $L_2$ such that $(L, L_2, \pi) \in \mathcal{B}$ and $J(L, L_2, \pi) = (L, G, \pi)$. We fix any element $\lambda_0 \in G$. Set $L_2 = G$ as sets and
\[
a \ast b = \eta(\lambda_0, b, a), \quad a, b \in L_2(= G).
\]
(5.4)

Due to (3.1) and (4.1), $L_2$ is a left quasigroup; its left division is defined by
\[
a \setminus_{L_2} c = \eta(a, c, \lambda_0), \quad a, c \in L_2.
\]
(5.5)

Proposition 20. $(L, L_2, \pi) \in \mathcal{B}$.

Proof. We need only show (5.1). Let $a, a', b, c \in L_2(= G)$. With the aid of (5.4) and (5.5) we have
\[
(a \ast c) \setminus_{L_2} ((a \ast b) \ast c) = \eta(\eta(\lambda_0, c, a), \eta(\lambda_0, c, \eta(\lambda_0, b, a)), \lambda_0).
\]
(5.6)

From (3.2) and (4.1),
\[
\eta(\lambda_0, c, \eta(\lambda_0, b, a)) = \eta(\lambda_0, c, \eta(\lambda_0, b, a')) = \eta(\eta(\lambda_0, c, a'), \eta(\lambda_0, c, \eta(\lambda_0, b, a')), \lambda_0).
\]
By taking into account (4.1) again, the right-hand side of (5.6) is
\[
\eta(\eta(\lambda_0, c, a), \eta(\lambda_0, c, \eta(\lambda_0, b, a)), \lambda_0) = \eta(\eta(\lambda_0, c, a), \eta(\lambda_0, c, \eta(\lambda_0, b, a')), \lambda_0),
\]
which is exactly the right-hand side of (5.1) by virtue of (5.4) and (5.5). This proves the proposition.

It is immediate that $J(L, L_2, \pi) = (L, G, \pi)$, and consequently, the following holds.

Proposition 21. The functor $J : \mathcal{B} \rightarrow \mathcal{H}$ is essentially surjective.

Corollary 22. The functor $S'J : \mathcal{B} \rightarrow \mathcal{D}$ is essentially surjective.

The final task of this section is to construct dynamical Yang-Baxter maps by means of the functor $S'J : \mathcal{B} \rightarrow \mathcal{D}$ and quasigroups of reflection; see [17, Section 1].

Definition 23. A pair $(G, *)$ of a nonempty set $G$ and a binary operation $(*)$ on $G$ is called a quasigroup of reflection if and only if $(G, *)$ is a left quasigroup (Definition 1(1)) satisfying
\[
x \ast x = x, \quad \forall x \in G,
\]
\[
(x \ast y) \ast y = x, \quad \forall x, y \in G,
\]
(5.7)
\[
(x \ast y) \ast z = (x \ast z) \ast (y \ast z), \quad \forall x, y, z \in G.
\]
(5.8)
It follows from (5.7) that \((G,+)\) is a quasigroup (Definition 1(2)).

**Remark 24.** (1) The above definition is slightly different from that in [17] (see also [26, II.1.1] and [27, Section 1]); the order of the binary operation \((+)\) on \(G\) is reversed.

(2) The identity (5.8) is called a right distributive law [30, Section V.2].

(3) The quasigroup of reflection gives an involutory quandle \([1,12,37]\) by reversing the order of the binary operation in Definition 23.

A straightforward computation shows that Nobusawa’s quasigroup \((Q,+)\) in Example 2 is a quasigroup of reflection.

Let \((G,+)\) be a quasigroup of reflection, and \(L\) a left quasigroup isomorphic to \(G\) as sets. We denote by \(\pi\) a set-theoretic bijection from \(L\) to \(G\). Because (5.8) immediately induces (5.1),

**Proposition 25.** \((L,\ G,\ \pi)\in B\).

The quasigroup \(G = (G,+)\) of reflection hence produces the dynamical Yang-Baxter map \(R(\lambda)\) defined by \((L, R) = SF'' J(L, G, \pi) \in D\).

For example, let \(L = (G,+)\) denote the left quasigroup (4.4) and \(\pi : L (=G) \rightarrow G\) the identity map on \(G\). The above dynamical Yang-Baxter map \(R(\lambda)\) induced by \((L, G, \pi)\in B\) is

\[
R(\lambda)(u, v) = (v, v \star (u \setminus_G \lambda)) \quad \lambda, u, v \in L (=G). \tag{5.9}
\]

For Nobusawa’s quasigroup \(Q = (Q,+)\), the corresponding dynamical Yang-Baxter map (5.9) is really dependent on the parameter \(\lambda\); in fact,

\[
R(1)(1, 1) = (1, 1), \quad R(2)(1, 1) = (1, 2).
\]

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