Existence and asymptotic behavior of $C^1$ solutions to the multidimensional compressible Euler equations with damping

Daoyuan Fang†, Jiang Xu‡
Department of Mathematics, Zhejiang University, Hangzhou 310027, P.R.China

Abstract

In this paper, the existence and asymptotic behavior of $C^1$ solutions to the multidimensional compressible Euler equations with damping on the framework of Besov space are considered. We weaken the regularity requirement of the initial data, and improve the well-posedness results of SIDERIS-THOMASES-WANG (Comm.P.D.E. 28 (2003) 953). The global existence lies on a crucial a-priori estimate which is proved by the spectral localization method. The main analytic tools are the Littlewood-Paley decomposition and Bony’s para-product formula.

MSC: 35L65; 76N15

Keywords: Euler equations; damping; classical solutions; spectral localization

1 Introduction and main results

In this paper, we study the following Euler equation with damping for a perfect gas flow:

\[
\begin{aligned}
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) &= 0 \\
(n \mathbf{u})_t + \nabla \cdot (n \mathbf{u} \otimes \mathbf{u}) + \nabla p(n) &= -an \mathbf{u}
\end{aligned}
\] (1.1)

for $(t, x) \in [0, +\infty) \times \mathbb{R}^N$, $N \geq 1$, where $n$ and $\mathbf{u} = (u^1, u^2, \cdots, u^N)^T$ ($T$ represents transpose) denote the density, velocity for the gas respectively. $n \mathbf{u}$ stands for the momentum. The pressure $p$ satisfies the $\gamma$-law:

\[
p = p(n) = An^\gamma,
\] (1.2)

where the case $\gamma > 1$ corresponds to the isentropic gas and $\gamma = 1$ corresponds to the isothermal gas, $A$ is a positive constant. The positive constant $a$ is the damping coefficient. The system is supplemented with the initial data

\[
(n, \mathbf{u})(x, 0) = (n_0, \mathbf{u}_0)(x), \quad x \in \mathbb{R}^N.
\] (1.3)

The system (1.1) describes that the compressible gas flow passes a porous medium and the medium induces a friction force, proportional to the linear momentum in the opposite direction. It is hyperbolic with two characteristic speeds $\lambda = \mathbf{u} \pm \sqrt{p(n)}$. As a vacuum appears, it fails to be strict hyperbolic. Thus, the system involves three mechanisms: nonlinear convection, lower-order dissipation of damping and the resonance due to vacuum. After NISHIDA’s [11, 12] pioneer works for (1.1), many contributions have been made on the small smooth solutions and piecewise smooth Riemann solutions away from

---

*This work is supported by NSFC 10571158.
†E-mail: dyf@zju.edu.cn
‡E-mail: jiangxu_79@yahoo.com.cn
vacuum, we can cite [4, 5, 6, 15, 13] and their references. Among them, for the one dimensional case [4], the system can be written in the Lagrangian coordinates as follows:

\[
\begin{align*}
\begin{cases}
    v_t - u_x = 0, \\
    u_t + p(v)_x = -au,
\end{cases}
\end{align*}
\]  

(1.4)

where \( v = 1/n \) is the specific volume. It was shown that the system (1.4) was time asymptotically equivalent to the porous media equation. In [13], Sideris, Thomases & Wang explained that the damping only presented weak dissipation in 3D space: it could prevent the development of singularities if the initial data was small and smooth, furthermore, they obtained the decay of classical solutions to the constant background state in \( L^\infty \) at a rate of \((1 + t)^{-3(2/3)}\), but singularities was exhibited for large data under some assumption. However, the main open problems for (1.1) with vacuum are still far from well-known. One of them is to study the singular evolution of the vacuum interface. As the first step in this direction, Xu & Yang [16] proved that a local existence theorem on a perturbation of a planar wave solution for (1.1) under the assumption of physical vacuum boundary condition. For the large-time asymptotic behavior for the solutions with vacuum, recently, Huang & Pan et al. [7, 8] gave a complete answer to this problem. In fact, they showed that the \( L^\infty \) weak entropy solutions with vacuum for the Cauchy problem converged to the Barenblatt’s profile of the porous medium equation strongly in \( L^p \).

In this paper, we are concerned with lowering the regularity of the initial data in the generally multidimensional space. As in [13], we also consider a perturbation of the constant equilibrium state \((\bar{n}, 0)(\bar{n} > 0)\). First of all, we give a local existence result in Besov space \( B^2_{2, 1}(\mathbb{R}^N) \) (\( \sigma = 1 + \frac{N}{2} \)) for (1.1)-(1.3) away from vacuum.

**Theorem 1.1.** \((N \geq 1)\) Suppose that \((n_0 - \bar{n}, u_0) \in B^2_{2, 1}(\mathbb{R}^N)\) with \( n_0 > 0 \), then there exist a time \( T_0 > 0 \) and a unique solution \((n, u)\) of the system (1.1)-(1.3) such that

\[
(n, u) \in C^1([0, T_0] \times \mathbb{R}^N) \quad \text{with} \quad n > 0 \quad \text{for all} \quad t \in [0, T_0]
\]

and

\[
(n - \bar{n}, u) \in C([0, T_0], B^2_{2, 1}(\mathbb{R}^N)) \cap C^1([0, T_0], B^{\sigma - 1}_{2, 1}(\mathbb{R}^N)).
\]

**Remark 1.1.** The nonlinear pressure term makes computation more fuzzy in the spectral localization estimates due to commutators. To get around this, we introduce a new variable (sound speed) which transforms the nonlinear term into linear and double-linear terms in virtue of the ideas in [13]. In fact, the original system (1.1)-(1.3) transforms into a symmetric hyperbolic system (3.1)-(3.2) where we can obtain the effective a-priori estimates. Different from the local existence result in [13], Theorem 1.1 follows from Proposition 4.1, Remark 4.1 and Remark 3.1. The proof of Proposition 4.1 is organized as follows. First, we regularize the initial data of (3.1)-(3.2) and obtain approximative local solutions based on Kato’s results. Second, we find a uniform positive time \( T_0 \) such that the approximative solution sequence is uniform bounded in \( C([0, T_0]; B^2_{2, 1}(\mathbb{R}^N)) \cap C^1([0, T_0], B^{\sigma - 1}_{2, 1}(\mathbb{R}^N)) \). Finally, we utilize the compactness argument to pass the limit (For detail, see Proposition 4.1).

Under a smallness assumption, we establish the global existence of classical solutions in Besov space \( B^2_{2, 2}(\mathbb{R}^N) \) (\( \sigma = 1 + \frac{N}{2}, \varepsilon > 0 \)) for (1.1)-(1.3).

**Theorem 1.2.** \((N \geq 3)\) Suppose that \((n_0 - \bar{n}, u_0) \in B^2_{2, 2}(\mathbb{R}^N)\). There exists a positive constant \( \delta_0 \) depending only on \( A, \gamma, a \) and \( \bar{n} \) such that if

\[
\|(n - \bar{n}, u)(\cdot, 0)\|_{B^2_{2, 2}(\mathbb{R}^N)} + \|(n_t, u_t)(\cdot, 0)\|_{B^{\sigma - 1}_{2, 2}(\mathbb{R}^N)} \leq \delta_0,
\]

then there exists a unique global solution \((n, u)\) of the system (1.1)-(1.3) satisfying

\[
(n, u) \in C^1([0, \infty) \times \mathbb{R}^N)
\]

and

\[
(n - \bar{n}, u) \in C([0, \infty), B^{\sigma + \varepsilon}_{2, 2}(\mathbb{R}^N)) \cap C^1([0, \infty), B^{\sigma - 1 + \varepsilon}_{2, 2}(\mathbb{R}^N)).
\]
Moreover, we have the energy estimate
\[
\|(n - \tilde{n}, u)(\cdot, t)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 + \|(n_t, u_t)(\cdot, t)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 \\
+ \mu_0 \int_0^t \left(\|u(\cdot, \tau)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 + \|v_n, n_t, u_t(\cdot, \tau)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2\right) d\tau \\
\leq \|(n - \tilde{n}, u)(\cdot, 0)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 + \|(n_t, u_t)(\cdot, 0)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2, \quad t \geq 0,
\]
where the positive constant \(\mu_0\) depends only on \(A, \gamma, a\) and \(\tilde{n}\).

**Remark 1.2.** In fact, the smallness of \((n_u, u_t)(x, 0)\) can be derived by Eqs. (1.1) and the smallness of \((n - \tilde{n}, u)(x, 0)\). For the simplicity of the statement, we give the assumption (1.5) directly.

**Remark 1.3.** Theorem 1.2 follows from Proposition 5.1 and Remark 3.1. The proof of a crucial a-priori estimate (Proposition 5.2) is separated into the low frequency part (Lemma 5.5) and high frequency part (Lemma 5.6) elaborately. On each high frequency \((q \geq 0), \|\triangle_q n\|_{L^2} \) is equivalent to \(2^q \|\triangle_q m\|_{L^2}\) by Lemma 2.1, but it is not valid for low frequency \((q = -1)\). In [2], we knew that the Poisson potential remedied the estimate on \(\|\triangle_{-1} n\|_{L^2}\). Here, we can’t get any estimates on \(\|\triangle_{-1} n\|_{L^2}\), however, with the help of Hölder’s inequality and Gagliardo-Nirenberg-Sobolev inequality \((N > 2)\), we can get the estimates on \(\|\triangle_{-1} \nabla m\|_{L^2}\) (For detail, see (5.12), (5.18), (5.20), (5.24) and (5.26)). Hence, in order to ensure our functional space still imbedding into \(C^1\) space, we need to increase a little of regularity, furthermore, which leads to the global existence of classical solutions to (3.1)-(3.2).

In [13], the authors obtained the decay of classical solutions to the equilibrium state \((\bar{n}, 0)\) in \(L^\infty\) at a rate of \((1 + t)^{-\gamma/2}\). According to the energy estimate in Theorem 1.2, we also see the large-time asymptotic behavior of solutions in Besov space roughly.

**Corollary 1.1.** \((N \geq 3)\) Let \((n, u)\) be the solution in Theorem 1.2, we have \((\sigma = 1 + \frac{N}{2}, \varepsilon' < \varepsilon)\)
\[
\|n(\cdot, t) - \bar{n}\|_{B^{2-\frac{\sigma}{2}}_{p,2}(\mathbb{R}^N)} \to 0 \quad (p = \frac{2N}{N-2}), \quad \|u(\cdot, t)\|_{B^{2-\frac{\sigma}{2}}_{p,2'}(\mathbb{R}^N)} \to 0, \quad \text{as} \quad t \to +\infty.
\]

Different from the result in [13], the following theorem characterizes the exponential decay of the vorticity in Besov space \(B^{2-1}_{2,1}(\mathbb{R}^N)\).

**Theorem 1.3.** \((N=3)\) Let \((n, u)\) be the solution in Theorem 1.2. If
\[
\|(n - \bar{n}, u)(\cdot, 0)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 + \|(n_t, u_t)(\cdot, 0)\|_{B^{2-\frac{\sigma}{2}}_{2,2}(\mathbb{R}^N)}^2 \leq \delta_0^2,
\]
then the vorticity \(\omega = \nabla \times u\) decays exponentially in \(B^{2-1}_{2,1}(\mathbb{R}^N)\):
\[
\|\omega(\cdot, t)\|_{B^{2-1}_{2,1}(\mathbb{R}^N)} \leq \|\omega(\cdot, 0)\|_{B^{2-1}_{2,1}(\mathbb{R}^N)} \exp(-\mu_0 t), \quad t \geq 0,
\]
where \(\omega(x, 0) = \nabla \times u_0\), the positive constants \(\delta_0^2 = \min\{\delta_0, \frac{\mu_0^2}{\delta_0^2}\}\) and \(\mu_0^2\) depend only on \(A, \gamma, a\) and \(\bar{n}\) \((C_0\) a uniform constant given in (5.33)).

The paper is arranged as follows. In Section 2, we present some definitions and basic facts on the Littlewood-Paley decomposition and Bony’s para-product formula. In Section 3, we reformulate the system (1.1)-(1.3) in order to obtain the effective a-priori estimates by the spectral localization method. In Section 4, we are concerned with the local existence and uniqueness of classical solutions to (3.1)-(3.2) with general initial data. In the last section, we deduce a crucial a-priori estimate under a smallness assumption which is used to complete the proof of global existence. Finally, it is also shown that the vorticity decays to zero in time exponentially.

Throughout this paper, the symbol \(C\) denotes a harmless constant. All functional spaces will be considered in \(\mathbb{R}^N\), so we can omit the space dependence for simplicity.
\section{Littlewood-Paley analysis}

In this section, these definitions and basic facts can be found in Darchin’s mini-course [1].

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwarz class. $(\varphi, \chi)$ is a couple of smooth functions valued in $[0,1]$ such that $\varphi$ is supported in the shell $C(0, \frac{1}{2}, \frac{2}{3}) = \{\xi \in \mathbb{R}^N | \frac{1}{2} \leq |\xi| \leq \frac{2}{3}\}$, $\chi$ is supported in the ball $B(0, \frac{1}{2}) = \{\xi \in \mathbb{R}^N | |\xi| \leq \frac{1}{2}\}$ and

\[\chi(\xi) + \sum_{q=0}^{\infty} \varphi(2^{-q}\xi) = 1, \quad q \in \mathbb{Z}, \quad \xi \in \mathbb{R}^N.\]

For $f \in \mathcal{S}'$ (denote the set of temperate distributions which is the dual one of $\mathcal{S}$), we can define the nonhomogeneous dyadic blocks as follows:

\[\Delta_{-1} f := \chi(D)f = \tilde{h} * f \quad \text{with} \quad \tilde{h} = F^{-1}\chi; \]
\[\Delta_q f := \varphi(2^{-q}D)f = 2^{qN} \int h(2^{q}y)f(x-y)dy \quad \text{with} \quad h = F^{-1}\varphi, \quad \text{if} \quad q \geq 0;\]

where $\ast$, $F^{-1}$ represent the convolution operator and the inverse Fourier transform, respectively. The nonhomogeneous Littlewood-Paley decomposition is

\[f = \sum_{q \geq -1} \Delta_q f \quad \text{in} \quad \mathcal{S}'.\]

Define the low frequency cut-off by

\[S_q f := \sum_{p \leq q} \Delta_p f.\]

Of course, $S_0 f = \Delta_{-1} f$. The above Littlewood-Paley decomposition is almost orthogonal in $L^2$.

\begin{proposition}
For any $f \in \mathcal{S}'(\mathbb{R}^N)$ and $g \in \mathcal{S}'(\mathbb{R}^N)$, the following properties hold:

\[\Delta_p \Delta_q f \equiv 0 \quad \text{if} \quad |p - q| \geq 2;\]
\[\Delta_q (S_{p-1} f \Delta_p g) \equiv 0 \quad \text{if} \quad |p - q| \geq 5.\]

Besov space can be characterized in virtue of the Littlewood-Paley decomposition.

\begin{definition}
Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $1 \leq r < \infty$, the Besov spaces $B^s_{p,r}(\mathbb{R}^N)$ are defined by

\[f \in B^s_{p,r}(\mathbb{R}^N) \iff \left( \sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L^p})^r \right)^{\frac{1}{r}} < \infty\]

and $B^s_{p,\infty}(\mathbb{R}^N)$ are defined by

\[f \in B^s_{p,\infty}(\mathbb{R}^N) \iff \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L^p} < \infty.\]

In particular, $B^s_{2,2}(\mathbb{R}^N) \equiv H^s(\mathbb{R}^N)$.

\end{definition}

\begin{definition}
Let $f, g$ be two temperate distributions. The product $f \cdot g$ has the Bony’s decomposition:

\[f \cdot g = T_f g + T_g f + R(f, g),\]

where $T_f g$ is paraproduct of $g$ by $f$,

\[T_f g = \sum_{p \leq q - 2} \Delta_p f \Delta_q g = \sum_q S_{q-1} f \Delta_q v\]

and the remainder $R(f, g)$ is denoted by

\[R(f, g) = \sum_q \Delta_q f \tilde{\Delta}_q g \quad \text{with} \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.\]

\end{definition}
Remark 3.1. Then the system (1.1) is transformed into the following system for $C$: and denoting the sound speed at a background density $\bar{n}$ into the following system for Lemma 2.1. (Bernstein) Let $(0, R_1, R_2) \Rightarrow \sup \|\partial^\alpha f\|_{L^s} \leq C^{k+1} \lambda^{k+N}(A - B/2)\|f\|_{L^s};$

Sup $\mathcal{F}f \subset C(0, R_1, R_2) \Rightarrow C^{-k-1} \lambda^k\|f\|_{L^s} \leq \sup \|\partial^\alpha f\|_{L^s}\leq C^{k+1} \lambda^k\|f\|_{L^s}.$

Here, $\mathcal{F}$ represents the Fourier transform.

A result of compactness for Besov space:

Proposition 2.2. Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\phi \in C_c^\infty$, the map $f \mapsto \phi f$ is compact from $B_p^{s,r}(\mathbb{R}^N)$ to $B_{p,r}^s(\mathbb{R}^N)$.

Finally, we state a result of continuity for the composition which is used to end this section.

Proposition 2.3. Let $1 \leq p, r \leq \infty$, $I$ be open interval of $\mathbb{R}$. Let $s > 0$ and $n$ be the smallest integer such that $n \geq s$. Let $F : I \rightarrow \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{n,\infty}(I, \mathbb{R})$. Assume that $v \in B_p^s$, takes values in $J \subset I$. Then $F(v) \in B_{p,r}^s$, and there exists a constant $C$ depending only on $s, I, J$ and $N$ such that

\[ \|F(v)\|_{B_{p,r}^s} \leq C(1 + \|v\|_{L^s})^n \|F'\|_{W^{n,\infty}(I)} \|v\|_{B_{p,r}^s}. \]

3 Reformulation of the original system

In this section, we are going to reformulate (1.1)-(1.3) in order to get the effective a-priori estimates by the spectral localization method. For the isentropic case $(\gamma > 1)$: introducing the sound speed

\[ \psi(n) = \sqrt{\frac{\gamma}{\gamma-1}n}, \]

and denoting the sound speed at a background density $\bar{n}$ by $\bar{\psi} = \psi(\bar{n})$, as in [13], we define

\[ m = \frac{2}{\gamma-1}(\psi(n) - \bar{\psi}). \]

Then the system (1.1) is transformed into the following system for $C^1$ solutions:

\[
\begin{aligned}
m_t + \bar{\psi} \text{div} u &= -u \cdot \nabla m - \frac{2}{\gamma-1}m \text{div} u, \\
u_t + \bar{\psi} \nabla m + au &= -u \cdot \nabla u - \frac{2}{\gamma-1}m \nabla m.
\end{aligned}
\] (3.1)

The initial data (1.3) becomes

\[ (m, u)|_{t=0} = (m_0, u_0) \]

with

\[ m_0 = \frac{2}{\gamma-1}(\psi(n_0) - \bar{\psi}). \]

Remark 3.1. For any $T > 0$, $(n, u) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (1.1)-(1.2) with $n > 0$, then $(m, u) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (3.1)-(3.2) with $\frac{2}{\gamma-1}m + \bar{\psi} > 0$; Conversely, if $(m, u) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (3.1)-(3.2) with $\frac{2}{\gamma-1}m + \bar{\psi} > 0$, let $n = \psi^{-1}(\frac{2}{\gamma-1}m + \bar{\psi})$, then $(n, u) \in C^1([0, T] \times \mathbb{R}^N)$ solves the system (1.1)-(1.2) with $n > 0$.

For the isothermal case $(\gamma = 1)$: set $\bar{n} = \sqrt{\lambda}(\ln n - \ln \bar{n})$, then the system (1.1) can be transformed into the following system for $C^1$ solutions:

\[
\begin{aligned}
\bar{n}_t + \sqrt{\lambda} \text{div} u &= -u \cdot \nabla \bar{n}, \\
u_t + \sqrt{\lambda} \nabla \bar{n} + au &= -u \cdot \nabla u.
\end{aligned}
\] (3.3)

The initial data (1.3) becomes

\[ (\bar{n}, u)|_{t=0} = (\sqrt{\lambda}(\ln n_0 - \ln \bar{n}), u_0). \]

(3.4)
Remark 3.2. For any $T > 0$, $(n, u) \in C^1([0,T] \times \mathbb{R}^N)$ solves the system (1.1)-(1.2) with $n > 0$, then $(\tilde{n}, \tilde{u}) \in C^1([0,T] \times \mathbb{R}^N)$ solves the system (3.3)-(3.4); Conversely, if $(\tilde{n}, \tilde{u}) \in C^1([0,T] \times \mathbb{R}^N)$ solves the system (3.3)-(3.4), let $n = \tilde{n} \exp(A - \frac{2}{3} \tilde{n})$, then $(n, u) \in C^1([0,T] \times \mathbb{R}^N)$ solves the system (1.1)-(1.2) with $n > 0$.

In the subsequent sections, we study the system (3.1)-(3.2) and prove the main results in this paper only, (3.3)-(3.4) can be studied through the similar process.

4 Local existence

In this section, we shall first give the estimates of some commutators in virtue of Bony’s para-product formula and the Littlewood-Paley decomposition in Besov $B^s_{2,1}$ space. Second, using the regularized means and compactness argument, we complete the proof of local existence for (3.1)-(3.2).

Applying the operator $\triangle_q$ to (3.1) yields
\[
\begin{cases}
\partial_t \triangle_q m + (u \cdot \nabla) \triangle_q m = -\psi \triangle_q \text{div} u + [u, \triangle_q] \cdot \nabla m - \frac{2}{3} \triangle_q (m \text{div} u), \\
\partial_t \triangle_q u + (u \cdot \nabla) \triangle_q u + a \triangle_q u = -\psi \triangle_q (\nabla m) + [u, \triangle_q] \cdot \nabla u - \frac{2}{3} \triangle_q (m \nabla m),
\end{cases}
\]
where the commutor $[f, g] = fg - gf$.

Multiplying the first equation of Eqs.(4.1) by $\triangle_q m$ and the second one by $\triangle_q u$, adding the resulting equations together and integrating them in $\mathbb{R}^n$, we obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt}(\|\triangle_q m\|^2_{L^2} + \|\triangle_q u\|^2_{L^2}) &+ a\|\triangle_q u\|^2_{L^2} \\
= \frac{1}{2} \int_{\mathbb{R}^n} \text{div} u (|\triangle_q m|^2 + |\triangle_q u|^2) + \int_{\mathbb{R}^n} ([u, \triangle_q] \cdot \nabla m \triangle_q m + [u, \triangle_q] \cdot \nabla u \triangle_q u) \\
&- \frac{\gamma - 1}{2} \int_{\mathbb{R}^n} (\triangle_q (m \text{div} u) \triangle_q m + \triangle_q (m \nabla m) \triangle_q u).
\end{align*}
\]

Noticing that the bi-linear spectral localization term, we have
\[
\int_{\mathbb{R}^n} [\triangle_q (m \text{div} u) \triangle_q m + \triangle_q (m \nabla m) \triangle_q u] = -\int_{\mathbb{R}^n} \triangle_q m (\nabla m \cdot \triangle_q u) + \int_{\mathbb{R}^n} [\triangle_q, m] \nabla m \cdot \triangle_q u + \int_{\mathbb{R}^n} [\triangle_q, m] \text{div} u \triangle_q m.
\]

Here, we give the following lemma to estimate these commutators in $L^2$- norm.

Lemma 4.1. The following estimates hold for any $m, u \in B^s_{2,1}$:
\[
\begin{align*}
2^s \|\triangle_q m\|_{L^2} &\leq C c_q \|m\|_{B^s_{2,1}} \|u\|_{B^s_{2,1}}, \\
2^s \|\triangle_q \nabla m\|_{L^2} &\leq C c_q \|\nabla m\|_{L^\infty} \|m\|_{B^s_{2,1}}, \\
2^s \|\triangle_q \nabla u\|_{L^2} &\leq C c_q \|\nabla u\|_{L^\infty} \|u\|_{B^s_{2,1}}, \\
2^s \|\triangle_q \cdot \nabla u\|_{L^2} &\leq C c_q \|\nabla u\|_{L^\infty} \|u\|_{B^s_{2,1}},
\end{align*}
\]
where $C$ denotes a harmless constant, $c_q$ denotes a sequence such that $\|(c_q)\|_{\ell^1} \leq 1$.

Proof. We are going to show (4.4) holds only, others can be proved similarly. In order to obtain (4.4) in Besov space $B^s_{2,1}$, we have to split $m$ into low and high frequencies: $m = \triangle_{-\gamma} m + \tilde{m}$. Since there exists a radius $0 < R < \frac{3}{4}$ such that $\text{Supp} \mathcal{F} \tilde{m} \cap B(0, R) = \emptyset$, Lemma 2.1 implies
\[
\|\triangle_q \nabla \tilde{m}\|_{L^q} = 2^q \|\triangle_q \tilde{m}\|_{L^q}, \quad a \in [1, \infty], \quad q > -1.
\]
Taking advantage of Bony’s decomposition, we have

\[
[m, \triangle_q] \text{div} \mathbf{u} = [\tilde{m}, \triangle_q] \text{div} \mathbf{u} + [\triangle_{-1} m, \triangle_q] \text{div} \mathbf{u}
\]

\[
= \tilde{m} \triangle_q \text{div} \mathbf{u} - \triangle_q (\tilde{m} \text{div} \mathbf{u}) + [\triangle_{-1} m, \triangle_q] \text{div} \mathbf{u}
\]

\[
= T_{\tilde{m}} \triangle_q \text{div} \mathbf{u} + T_{\triangle_q \text{div} \mathbf{u}} \tilde{m} + R(\tilde{m}, \triangle_q \text{div} \mathbf{u})
\]

\[-\triangle_q (T_{\tilde{m}} \text{div} \mathbf{u} + T_{\triangle_q \text{div} \mathbf{u}} \tilde{m} + R(\tilde{m}, \triangle_q \text{div} \mathbf{u})) + [\triangle_{-1} m, \triangle_q] \text{div} \mathbf{u}.
\]

Then we can write \([m, \triangle_q] \text{div} \mathbf{u} = \sum_{i=1}^{6} F_i^q\) where

\[
F_1^q = T_{\tilde{m}} \triangle_q \partial_j u^j - \triangle_q T_{\tilde{m}} \partial_j u^j, \quad (\text{div} \mathbf{u} := \partial_j u^j)
\]

\[
F_2^q = T_{\triangle_q \partial_j u^j} \tilde{m},
\]

\[
F_3^q = -\triangle_q T_{\partial_j u^j} \tilde{m},
\]

\[
F_4^q \bigg|_{q' \leq q - 3} \sum_{|q' - q| \leq 4} h(y) \begin{vmatrix} S_{q' - 1} \tilde{m}(x) - S_{q' - 1} \tilde{m}(x - 2^{-q}y) \end{vmatrix} \partial_j \triangle_q u^j (x - 2^{-q}y) dy.
\]

Then, applying first order Taylor’s formula, Young’s inequality and (4.8), we get

\[
2^{-q} \| F_3^q \|_{L^2} \leq C \sum_{|q' - q| \leq 4} \| \nabla \tilde{m} \|_{L^\infty} 2^{(\sigma - 1)(q - q')} 2^{q'} \| \triangle_q u^j \|_{L^2}
\]

\[
\leq C c_{q1} \| \nabla m \|_{L^\infty} \| \mathbf{u} \|_{B_{2,1}^\sigma}, \quad c_{q1} = \sum_{|q - q'| \leq 4} 2^{q'} \| \triangle_q \mathbf{u} \|_{L^2}
\]

and

\[
2^{-q} \| F_4^q \|_{L^2} = 2^{-q} \sum_{q' \geq q - 3} \| \triangle_q \tilde{m} \|_{L^\infty} \| S_{q' - 1} \partial_j \triangle_q u^j \|_{L^2}
\]

\[
\leq 2^{-q} \sum_{q' \geq q - 3} \| \triangle_q \tilde{m} \|_{L^\infty} \| S_{q' - 1} \partial_j \triangle_q u^j \|_{L^2}
\]

\[
\leq C \sum_{q' \geq q - 3} 2^{-q'} \| \nabla m \|_{L^\infty} 2^{-q'} \| \triangle_q \mathbf{u} \|_{L^2}
\]

\[
\leq C c_{q2} \| \nabla m \|_{L^\infty} \| \mathbf{u} \|_{B_{2,1}^\sigma}, \quad c_{q2} = \frac{2^{-q} \| \triangle_q \mathbf{u} \|_{L^2}}{\| \mathbf{u} \|_{B_{2,1}^\sigma}}.
\]

The third part \(F_3^q\) is proceeded as follows:

\[
F_3^q = -\triangle_q T_{\partial_j u^j} \tilde{m}
\]

\[
= -\sum_{|q' - q| \leq 4} \triangle_q (S_{q' - 1} \partial_j u^j \triangle_q \tilde{m}),
\]
Taylor’s formula, Young’s inequality and (4.8), we have

\[
2^{\varrho q} \| F_q^3 \|_{L^2} \leq C \sum_{|q-q'| \leq 4} 2^{(q-q')\sigma} 2^{q'\sigma} \| S_{q'-1} \partial_j u^j \triangle_q \tilde{m} \|_{L^2}
\]

\[
\leq C \sum_{|q-q'| \leq 4} 2^{(q-q')\sigma} \| S_{q'-1} \partial_j u^j \|_{L^\infty} 2^{q'\sigma - 1} \| \triangle_q \nabla \tilde{m} \|_{L^2}
\]

\[
\leq C \eta_3 \| \nabla m \|_{B^{\sigma-1}_{2,1}} \| u \|_{B^{\sigma+1}_{2,1}}, \quad \eta_3 = \sum_{|q-q'| \leq 4} \frac{2^{q'\sigma - 1} \| \triangle_q \nabla \tilde{m} \|_{L^2}}{9 \| \nabla m \|_{B^{\sigma-1}_{2,1}}^2}.
\]

(Here, we use the imbedding $B^{\sigma-1}_{2,1} \hookrightarrow C_0$ (continuous bounded functions which decay to zero at infinity).)

By the definition 2.2, we have

\[
F_q^4 = \partial_j R(\tilde{m}, \triangle_q u^j) - \partial_j \triangle_q R(\tilde{m}, u^j) = \sum_{|q-q'| \leq 1} \partial_j (\triangle_{q'} \tilde{m} \triangle_{q'} \triangle_q u^j) - \partial_j \triangle_q R(\tilde{m}, u^j) = F_q^{4,1} + F_q^{4,2}.
\]

For the first term, using (4.8) only, we get

\[
2^{\varrho q} \| F_q^{4,1} \|_{L^2} \leq C \| \nabla m \|_{L^\infty} \sum_{|q-q'| \leq 1} 2^{(q-q')\sigma} 2^{q'\sigma} \| \triangle_q u^j \|_{L^2}
\]

\[
\leq C \eta_4(1) \| \nabla m \|_{L^\infty} \| u \|_{B^{\sigma}_{2,1}}, \quad \eta_4(1) = \sum_{|q-q'| \leq 1} \frac{2^{q'\sigma} \| \triangle_q u^j \|_{L^2}}{4 \| u \|_{B^{\sigma}_{2,1}}^2}.
\]

The second term is estimated as follows:

\[
2^{\varrho q} \| F_q^{4,2} \|_{L^2} \leq 2^{\varrho q} \| \partial_j \triangle_q R(\tilde{m}, u^j) \|_{L^2}
\]

\[
\leq C 2^{q(\sigma + 1)} \| \triangle_q R(\tilde{m}, u^j) \|_{L^2}
\]

\[
\leq C \eta_4(2) \| \nabla m \|_{B^{\sigma+1}_{2,1}} \| u \|_{B^{\sigma+1}_{2,1}}, \quad \eta_4(2) = \frac{2^{\varrho q} \| \triangle_q R(\tilde{m}, u^j) \|_{L^2}}{4 \| R(\tilde{m}, u^j) \|_{B^{\sigma+1}_{2,1}}^2}.
\]

(Here, we use the result of continuity for the remainder, see [1] Proposition 1.4.2. $\eta_4 := \eta_4(1) + \eta_4(2)$.)

For $F_q^5$, the same argument as $F_q^4$, we can obtain

\[
2^{\varrho q} \| F_q^5 \|_{L^2} \leq C \eta_5 \| \nabla m \|_{B^{\sigma-1}_{2,1}} \| u \|_{B^{\sigma}_{2,1}}, \quad \eta_5 = \left( \sum_{|q-q'| \leq 1} \frac{2^{q' \sigma} \| \triangle_q u^j \|_{L^2}}{4 \| u \|_{B^{\sigma}_{2,1}}} \right) + \frac{2^{\varrho q} \| \triangle_q R(\tilde{m}, u^j) \|_{L^2}}{4 \| R(\tilde{m}, u^j) \|_{B^{\sigma}_{2,1}}^2}.
\]

For $F_q^6 = \sum_{|q-q'| \leq 1} [\triangle_q (\triangle_{-1} m \partial_j \triangle_q u^j) - \triangle_{-1} m \triangle_q \triangle_{q'} \partial_j u^j]$ ($u^j = \sum_q \triangle_q u^j$), applying first order Taylor’s formula, Young’s inequality and (4.8), we have

\[
2^{\varrho q} \| F_q^6 \|_{L^2} \leq C \sum_{|q-q'| \leq 1} 2^{(q-q')\sigma - 1} \| \nabla \triangle_{-1} m \|_{L^\infty} 2^{q'\sigma} \| \triangle_q u^j \|_{L^2}
\]

\[
\leq C \eta_6 \| \nabla m \|_{L^\infty} \| u \|_{B^{\sigma}_{2,1}}, \quad \eta_6 = \sum_{|q-q'| \leq 1} \frac{2^{q'\sigma} \| \triangle_q u^j \|_{L^2}}{3 \| u \|_{B^{\sigma}_{2,1}}^2}.
\]

Adding above these inequalities together and choosing $\eta_q = \frac{1}{\varrho} \sum_{i=1}^{6} \eta_q$, we prove the estimate (4.4). \qed

Now, we give the local existence result of solutions to (3.1)-(3.2).
Proposition 4.1. Suppose that \((m_0, u_0) \in B^2_{2,1}\), then there exist a time \(T_0 > 0\) and a unique solution \((m, u)\) of (3.1)-(3.2) such that \((m, u) \in C^1([0, T_0] \times \mathbb{R}^N)\) and \((m, u) \in C([0, T_0], B^2_{2,1}) \cap C^1([0, T_0], B^{2-1}_{2,1})\).

Proof. (Existence)
Let \(U_0 = (m_0, u_0)^T \in B^2_{2,1}\). There exists a sequence \(\{U^k\} := \{(m^k_0, u^k_0)^T\} \in H^s(s > \sigma, \; s \in \mathbb{N})\) converging to \(U_0\) in \(B^2_{2,1}\) satisfying \(\|U^k_0\|_{B^2_{2,1}} \leq \|U_0\|_{B^2_{2,1}} + 1\). We define a sequence \(\{U^k\} = \{(m^k, u^k)^T\}\) solves the following equations:

\[
\begin{align*}
& m^k_t + \psi \text{div} u^k = -u^k \cdot \nabla m^k - \frac{2-1}{2}m^k \text{div} u^k \\
& u^k_t + \psi \nabla m^k + a u^k = -u^k \cdot \nabla u^k - \frac{2-1}{2}m^k \nabla m^k
\end{align*}
\]

(4.9)

with the initial data

\[
(m^k, u^k)|_{t=0} = \left(\frac{2}{\gamma-1}(\psi(n^0_\delta) - \tilde{\psi}), u^0_0\right).
\]

(4.10)

It is easy to see (4.9) is a symmetric hyperbolic system on \(G = \{U^k : -\infty < \frac{2-1}{2}m^k + \tilde{\psi} < \infty\}\), using Kato’s classical results in [2] or [10], we can get the following local existence result: there exist a time \(T_k > 0\) and a solution \(U^k\) of (4.9)-(4.10) such that

\[
U^k \in C^1([0, T_k] \times \mathbb{R}^N)
\]

and

\[
U^k \in C([0, T_k], H^s) \cap C^1([0, T_k], H^{s-1}).
\]

We define \([0, T^*_k] \subseteq [0, T_k]\) is the maximal interval of local existence for above solutions of (4.9)-(4.10). According to the discussion in [10], we have the blow-up criterion:

\[
T_k^* < \infty \quad \Leftrightarrow \quad \limsup_{t \to T_k^*} (\|U^k_t\|_{L^\infty} + \|\nabla U^k\|_{L^\infty}) = +\infty\quad \text{or}
\]

for any compact subset \(K \subseteq G\), \(U^k(x, t)\) escapes \(K\) as \(t \to T_k^*\).

Claim: For \( t \in [0, \min\{T^*_k, T_0\}) \), we have \(\|U^k(t)\|_{B^2_{2,1}} \leq 4\lambda_0\), where \(\lambda_0 = \|U_0\|_{B^2_{2,1}} + 1\), \(T_0 = 1/(2\tilde{C}\lambda_0)\) and \(\tilde{C}\) is a positive constant (independent of \(k\)) given in (4.12).

In fact, we have known

\[
\|U^k_0\|_{B^2_{2,1}} \leq \lambda_0.
\]

There exists a small \(\bar{T} \in (0, \min\{T^*_k, T_0\})\) (owing to \(U^k(t) \in C([0, T_k], H^s)\)) such that

\[
\sup_{t \in [0, \bar{T}]} \|U^k(t)\|_{B^2_{2,1}} \leq 4\lambda_0.
\]

(4.11)

We can assume that (4.11) holds on arbitrary interval \([0, T'] \subset [0, \min\{T^*_k, T_0\})\), then we shall show (4.19) holds.

Here, we don’t need to consider the effect of damping term. Therefore, by (4.2)-(4.4), Hölder’s inequality and Lemma 4.1, we can obtain

\[
\frac{d}{dt} (\|\triangle u^k\|_{L^2}) \leq \tilde{C} \{ \|\nabla u^k\|_{L^\infty}^2 \|\triangle m^k\|_{L^2} + \|\triangle m^k\|_{L^2} + \|\Delta u^k\|_{L^2} + \bar{c}_q \|\text{div} u^k\|_{L^\infty} \|m^k\|_{B^2_{2,1}} m^k \|B^2_{2,1}\| \}
\]

(4.12)

Taking (4.12) \(L^1\)-norm, we obtain the a-priori estimate of \(U^k\):

\[
\frac{d}{dt} \|U^k(t)\|_{B^2_{2,1}} \leq \tilde{C} \|U^k(t)\|_{B^2_{2,1}}^2, \quad t \in [0, T').
\]

(4.13)
Integrating (4.13) on variable $t$ to get
\[
\|U^k(t)\|_{B^q_{2,1}} \leq \|U^k_0\|_{B^q_{2,1}} + \tilde{C} \int_0^t \|U^k(\xi)\|_{B^q_{2,1}}^2 d\xi \\
\leq (\|U_0\|_{B^q_{2,1}} + 1) + \tilde{C} \int_0^t \|U^k(\xi)\|_{B^q_{2,1}}^2 d\xi, \quad t \in [0, T'].
\]  
(4.14)

Furthermore, we have
\[
\sup_{0 \leq \xi \leq t} \|U^k(\xi)\|_{B^q_{2,1}} \leq (\|U_0\|_{B^q_{2,1}} + 1) + \tilde{C} \int_0^t \sup_{0 \leq \xi' \leq \xi} \|U^k(\xi')\|_{B^q_{2,1}}^2 d\xi, \quad t \in [0, T'].
\]  
(4.15)

Set
\[
\lambda_1(t) \equiv (\|U_0\|_{B^q_{2,1}} + 1) + \tilde{C} \int_0^t \sup_{0 \leq \xi' \leq \xi} \|U^k(\xi')\|_{B^q_{2,1}}^2 d\xi.
\]

Then, we have
\[
\frac{d}{dt} \lambda_1 \leq \tilde{C} \lambda_1^2, \quad \lambda_1(0) = \|U_0\|_{B^q_{2,1}} + 1, \quad t \in [0, T'].
\]  
(4.16)

Let $\lambda(t)$ solves Riccati equation:
\[
\frac{d}{dt} \lambda = \tilde{C} \lambda^2, \quad \lambda(0) = \|U_0\|_{B^q_{2,1}} + 1.
\]  
(4.17)

The time $T_0 = 1/(2\tilde{C}\lambda_0)$ is less than the blow-up time for (4.17). Then by solving the differential inequality (4.16), we know $\lambda_1(t) \leq \lambda(t)$ for $t \in [0, T_0]$. Solving (4.17) yields
\[
\lambda_1(t) \leq \frac{\|U_0\|_{B^q_{2,1}} + 1}{1 - \tilde{C}t(\|U_0\|_{B^q_{2,1}} + 1)} = \lambda(t), \quad t \in [0, T_0].
\]  
(4.18)

Therefore, we see
\[
\sup_{t \in [0, T']} \|U^k(t)\|_{B^q_{2,1}} \leq 2\lambda_0, \quad [0, T'] \subset [0, \min\{T^*_k, T_0\}).
\]  
(4.19)

Combining with (4.11) and (4.19), by the continuum principle, we prove the claim immediately.

Furthermore, using Eqs. (4.9), we can conclude
\[
\|U^k(t)\|_{B^q_{2,1}'} \leq \lambda'_0, \quad t \in [0, \min\{T^*_k, T_0\}),
\]
where $\lambda'_0$ is a positive constant only depending on the initial data $U_0$. The blow-up criterion implies $0 < T_0 < T^*_k$, so we have $0 < T_0 \leq \inf_k T^*_k$.

That is, we find a positive time $T_0$ (only depending on the initial data $U_0$) such that the approximative solution sequence $\{U^k\}$ of (4.9)-(4.10) is uniform bounded in $C([0, T_0], B^q_{2,1}) \cap C^1([0, T_0], B^{q-1}_{2,1})$. Moreover, it weak-*converges (up to a subsequence) to some $U$ in $L^\infty([0, T_0], B^q_{2,1})$, in terms of the Banach-Alaoglu theorem (see [14] Remark 2 on p.180 in Triebel, 1983). Because $\{U^k\}$ is also uniform bounded in $C([0, T_0], B^{q-1}_{2,1})$ (it weak-*converges to $U$ in $L^\infty([0, T_0], B^{q-1}_{2,1})$), then $\{U^k\}$ is uniform bounded in $\text{Lip}(0, T_0, B^{q-1}_{2,1})$, hence uniform equicontinuous on $[0, T_0]$ valued in $B^{q-1}_{2,1}$. By Proposition 2.2, Ascoli-Arzelà theorem and Cantor diagonal process, we deduce that
\[
\phi U^k \to \phi U \quad \text{in} \quad C([0, T_0], B^{q-1}_{2,1}) \quad \text{as} \quad k \to \infty, \quad \text{for any} \quad \phi \in C_c^\infty(\mathbb{R}).
\]

The properties of strong convergence enable us to pass to the limit in (4.9)-(4.10). Indeed, $U$ is a solution of (3.1)-(3.2). Now, what remains is to check $U$ has the required regularity. First, we already have known $U \in C([0, T_0], B^{q-1}_{2,1})$, an interpolation argument insures $U \in C([0, T_0], B^{q'}_{2,1})$ for any $q' < q$. Furthermore, for any $q \in \mathbb{N}$, $S_q U \in C([0, T_0], B^{q}_{2,1})$. Combining with (4.12)(throw off the superscript $k$), we derive that $\{S_q U\}$ converges uniformly to $U$ on $[0, T_0]$ valued in $B^{q}_{2,1}$. This achieves to prove that $U \in C([0, T_0], B^{q}_{2,1})$. 

10
Moreover, using Eqs. (3.1), we see that $U_t \in C([0,T_0], B_{2,1}^{r_1-1})$, so $U(t, x) \in C^1([0,T_0] \times \mathbb{R}^N)$.

\[ \text{(Uniqueness)} \]

Let $\tilde{m} = m_1 - m_2, \tilde{u} = u_1 - u_2$, where $U_1 = (m_1, u_1)^T, U_2 = (m_2, u_2)^T$ are two solutions for the system (3.1)-(3.2) with the same initial data, respectively. Then $\tilde{U} = (\tilde{m}, \tilde{u})$ satisfies the following equations:

\[
\begin{aligned}
\tilde{m}_t + \tilde{u} \nabla \tilde{m} - \tilde{u} \nabla m_2 - \frac{\gamma - 1}{2} m_1 \tilde{u} \nabla u_2, \\
\tilde{u}_t + \tilde{u} \nabla \tilde{m} + a \tilde{u} = -u_1 \cdot \nabla u - u_2 \nabla m_1 - \frac{\gamma - 1}{2} \tilde{m} \tilde{u} \nabla m_2.
\end{aligned}
\]

(4.20)

Similar to the derivation of (4.13), we obtain the following estimate:

\[
\|\tilde{U}(t)\|_{B_{2,1}^{r_1}} \leq C \int_0^t \|\tilde{U}(\varsigma)\|_{B_{2,1}^{r_1}} (\|U_1(\varsigma)\|_{B_{2,1}^{r_1}} + \|U_2(\varsigma)\|_{B_{2,1}^{r_1}}) d\varsigma, \quad \text{for } t \in [0, T_0].
\]

(4.21)

By Gronwall’s inequality, we conclude $\tilde{U} \equiv 0$. \(\Box\)

Remark 4.1. If we add the assumption $\frac{\gamma - 1}{2} m_2 + \bar{\psi} > 0$ in Proposition 4.1, defining the flow map $X(t; s, x)$ of $u$ starting from $x \in \mathbb{R}^N$ at time $s \in [0, T_0]$ by

\[
\frac{dX}{dt} = u(t, X(t; s, x)), \quad X(t; s, x)|_{t=s} = x
\]

and using the first equation of Eqs. (3.1), we can get

\[
\frac{\gamma - 1}{2} m(s, x) + \bar{\psi} > 0 \quad \text{for } (s, x) \in [0, T_0] \times \mathbb{R}^N.
\]

5 A-priori estimates and global existence

In this section, we first give the proposition on the global existence of classical solutions to (3.1)-(3.2).

Proposition 5.1. \( (N \geq 3) \) Suppose that $U_0 \in B_{2,2}^{2+\varepsilon}$ \( (\varepsilon > 0) \). There exists a positive constant $\delta_2 < \frac{1}{2}\delta_1$ depending only on $A, \gamma, a$ and $\bar{u}$ such that if

\[
\|U(\cdot, 0)\|_{B_{2,2}^{2+\varepsilon}}^2 + \|U_t(\cdot, 0)\|_{B_{2,2}^{2+\varepsilon}}^2 \leq \delta_2,
\]

then there exists a unique global solution $U$ of (3.1)-(3.2) satisfying

\[
U \in C((0, \infty), B_{2,2}^{2+\varepsilon}) \cap C^1((0, \infty), B_{2,2}^{2+\varepsilon})
\]

and

\[
\|U(\cdot, t)\|_{B_{2,2}^{2+\varepsilon}}^2 + \|U_t(\cdot, t)\|_{B_{2,2}^{2+\varepsilon}}^2 \leq \mu_1 \int_0^t (\|u(\cdot, \tau)\|_{B_{2,2}^{2+\varepsilon}}^2 + \|\nabla m(\cdot, \tau)\|_{B_{2,2}^{2+\varepsilon}}^2 + \|U_t(\cdot, \tau)\|_{B_{2,2}^{2+\varepsilon}}^2) d\tau
\]

\[
\leq \|U(\cdot, 0)\|_{B_{2,2}^{2+\varepsilon}}^2 + \|U_t(\cdot, 0)\|_{B_{2,2}^{2+\varepsilon}}^2, \quad t \geq 0,
\]

(5.1)

where $\delta_1$ and $\mu_1$ are some positive constants given by Proposition 5.2, $U = (m, u)$ and $U_t = (m_t, u_t)$.

Remark 5.1. The energy estimate (5.1) implies the full solution $(m, u)$ does not decay to zero in time exponentially.

The proof of above proposition mainly depends on a crucial a-priori estimate (Proposition 5.2). To do this, we need the following lemmas.
Lemma 5.1. If \((m, u) \in C([0, T], B^{\sigma+\varepsilon}_{2, 2}) \cap C^1([0, T], B^{-1+\varepsilon}_{2, 2})\) is a solution of Eqs. (3.1) for any given \(T > 0\), then

\[
\frac{d}{dt} \left( \|\triangle m_t\|_{L^2}^2 + \|\triangle u_t\|_{L^2}^2 \right) + 2\nu \|\triangle u_t\|_{L^2}^2 
\leq 2\|u_t\|_{L^\infty} (\|\triangle \nabla m\|_{L^2} + \|\triangle \nabla u\|_{L^2} + \|\triangle \nabla \nabla m\|_{L^2} + \|\triangle \nabla \nabla u\|_{L^2}) 
+ \|\nabla u_t\|_{L^\infty} (\|\triangle m_t\|_{L^2} + \|\triangle u_t\|_{L^2}) 
+ (\gamma - 1) \|\nabla m\|_{L^\infty} (\|\triangle m_t\|_{L^2} + \|\triangle u_t\|_{L^2}) 
+ (\gamma - 1) \|\nabla u\|_{L^\infty} (\|\triangle m_t\|_{L^2} + \|\triangle u_t\|_{L^2}) 
+ (\gamma - 1) \|\nabla u\|_{L^\infty} (\|\triangle m_t\|_{L^2} + \|\triangle u_t\|_{L^2}).
\]

Proof. By differentiating the first two equations of Eqs. (3.1) with respect to the variable \(t\) once, integrating them over \(\mathbb{R}^N\) after multiplying \(\triangle m_t, \triangle u_t\), respectively, similar to the derivation of (4.12), we can obtain (5.2) directly. \(\square\)

Lemma 5.2. The following estimates hold for any \(m, u \in C([0, T], B^{\sigma+\varepsilon}_{2, 2}) \cap C^1([0, T], B^{-1+\varepsilon}_{2, 2})\)\((T > 0)\):

\[
2^\sigma (\gamma - 1) \|\nabla \nabla m\|_{L^2} \leq C \nu \|\nabla m\|_{B^{\sigma+\varepsilon}_{2, 2}},
\]

\[
2^\sigma (\gamma - 1) \|\nabla \nabla u\|_{L^2} \leq C \nu \|\nabla u\|_{B^{\sigma+\varepsilon}_{2, 2}},
\]

\[
2^\sigma (\gamma - 1) \|\nabla \nabla u\|_{L^2} \leq C \nu \|\nabla u\|_{B^{\sigma+\varepsilon}_{2, 2}},
\]

\[
2^\sigma (\gamma - 1) \|\nabla \nabla m\|_{L^2} \leq C \nu \|\nabla m\|_{B^{\sigma+\varepsilon}_{2, 2}},
\]

where \(C\) denotes a harmless constant, \(c_q\) denotes a sequence such that \(\|c_q\|_2^2 \leq 1\).

Remark 5.2. The proof is similar to that of Lemma 4.1, so we omit it here.

In order to establish the differential inequality (5.30), we still need some auxiliary estimates.

Lemma 5.3. If \((m, u) \in C([0, T], B^{\sigma+\varepsilon}_{2, 2}) \cap C^1([0, T], B^{-1+\varepsilon}_{2, 2})\) is a solution of Eqs. (3.1) for any given \(T > 0\), then

\[
\|\triangle m_t\|_{L^2}^2 \leq \left( \tilde{\nu} \tilde{C} 2^\sigma \|\nabla \nabla u\|_{L^2} + \|u\|_{L^\infty} \|\nabla \nabla \nabla m\|_{L^2} + \frac{\gamma - 1}{2} \|m\|_{L^\infty} \|\nabla \nabla u\|_{L^2} \right) \|\triangle m_t\|_{L^2},
\]

\[
\tilde{\nu} \|\nabla \nabla m\|_{L^2}^2 \leq \left( a \|\nabla \nabla m\|_{L^2} + \|\nabla \nabla u\|_{L^2} + \|u\|_{L^\infty} \|\nabla \nabla \nabla m\|_{L^2} + \|u, \nabla u\|_{L^2} \right) \|\nabla \nabla m\|_{L^2},
\]

where the uniform constant \(\tilde{C}\) is independent of \(A, \gamma, a\) and \(\tilde{n}\).
Proof. (1) Using the first equation of Eqs. (3.1), we have
\[
m_t = -(\psi \text{div} u + u \cdot \nabla m + \frac{\gamma - 1}{2} m \text{div} u). \tag{5.13}
\]
By applying the operator \(\triangle_q(q \geq -1)\) to (5.13), integrating it over \(\mathbb{R}^N\) after multiplying \(\triangle_q m_t\), we can get (5.11) only by Hölder’s inequality.

(2) Using the second equation of Eqs. (3.1), we get
\[
\bar{\psi} \nabla m = -(u_t + au + u \cdot \nabla u + \frac{\gamma - 1}{2} m \nabla m). \tag{5.14}
\]
By applying the operator \(\triangle_q(q \geq -1)\) to (5.14), integrating it over \(\mathbb{R}^N\) after multiplying \(\triangle_q \nabla m\), we can get (5.12) immediately.

By Lemma 2.1, we have
\[
\|\triangle_q \nabla m\|_{L^2} \approx 2^q \|\triangle_q m\|_{L^2} \quad (q \geq 0),
\]
however, we can’t get any estimates on \(\|\triangle_{-1} m\|_{L^2}\) according to (5.12), furthermore, we can’t obtain the total estimates on \(\|\triangle_q m\|_{L^2}\) \((q \geq -1)\), which is the essential difference with Euler-Poisson equation in [2]. That is why we need the functional space \(B^\sigma_{q,2} \) to deal with the global existence of classical solutions for (3.1)-(3.2). Hence, we must modify those estimates in Lemma 4.1 and divide them into the cases of the high and low frequency.

Lemma 5.4. The following estimates hold for any \(m, u \in B^\sigma_{q,2}\):
\[
2^{q(\sigma + \varepsilon)} \|u, \triangle_q \|_{\nabla u} \|_{L^2} \leq C \epsilon \|u\|_{B^\sigma_{q,2}} \|\nabla u\|_{B^{q-1+\varepsilon}_{q,2}}, \tag{5.15}
\]
\[
2^{q(\sigma + \varepsilon)} \|m, \triangle_q \|_{\nabla m} \|_{L^2} \leq C \epsilon \|m\|_{B^\sigma_{q,2}} \|\nabla m\|_{B^{q+1-\varepsilon}_{q,2}}, \tag{5.16}
\]
\[
2^{q(\sigma + \varepsilon)} \|u, \triangle_q \|_{\nabla m} \|_{L^2} \leq C \epsilon \|u\|_{B^\sigma_{q,2}} \|\nabla m\|_{B^{q-1+\varepsilon}_{q,2}}, \tag{5.17}
\]
\[
2^{-(\sigma + \varepsilon)} \|\nabla u\|_{L^2} \leq C \epsilon \|u\|_{B^\sigma_{q,2}} \|\nabla m\|_{B^{q-1+\varepsilon}_{q,2}}, \tag{5.18}
\]
\[
2^{q(\sigma + \varepsilon)} \|m, \triangle_q \|_{\nabla u} \|_{L^2} \leq C \epsilon \|m\|_{B^\sigma_{q,2}} \|\nabla m\|_{B^{q+1-\varepsilon}_{q,2}}, \tag{5.19}
\]
\[
2^{-(\sigma + \varepsilon)} \|\nabla u\|_{L^2} \leq C \epsilon \|m\|_{B^\sigma_{q,2}} \|\nabla m\|_{B^{q-1+\varepsilon}_{q,2}}, \tag{5.20}
\]
where \(C\) denotes a harmless constant, \(c_q(q \geq -1)\) denotes a sequence such that \(\|(c_q)\|_2^2 \leq 1\).

Remark 5.3. The proof is similar to that of Lemma 4.1, so we also omit it here.

Now, we give the crucial a-priori estimate in the following proposition.

Proposition 5.2. There exist two positive constants \(\delta_1\) and \(\mu_1\) depending only on \(A, \gamma, a\) and \(\bar{n}\) such that for any \(T > 0\), if
\[
\sup_{0 \leq t \leq T} \left(\|U(\cdot, t)\|_{B^\sigma_{q,2}}^2 + \|U_t(\cdot, t)\|_{B^{q-1+\varepsilon}_{q,2}}^2\right) \leq \delta_1, \tag{5.21}
\]
then
\[
\|U(\cdot, t)\|_{B^\sigma_{q,2}}^2 + \|U_t(\cdot, t)\|_{B^{q-1+\varepsilon}_{q,2}}^2 + \mu_1 \int_0^T \left(\|\nabla m(\cdot, \tau)\|_{B^\sigma_{q,2}}^2 + \|\nabla m(\cdot, \tau)\|_{B^{q-1+\varepsilon}_{q,2}}^2 + \|U_t(\cdot, \tau)\|_{B^{q-1+\varepsilon}_{q,2}}^2\right) d\tau 
\leq \|U(\cdot, 0)\|_{B^\sigma_{q,2}}^2 + \|U_t(\cdot, 0)\|_{B^{q-1+\varepsilon}_{q,2}}^2, \quad t \geq 0. \tag{5.22}
\]
**Proof.** From the a-priori assumption (5.21), we deduce
\[
\sup_{0 \leq t \leq T} \left( \|U(\cdot, t)\|_{L^\infty} + \|U(\cdot, t)\|_{L^\infty} + \|\nabla U(\cdot, t)\|_{L^\infty} + \|\nabla U(\cdot, t)\|_{L^\infty} + \|U_t(\cdot, t)\|_{L^\infty} \right) \leq C\delta_t^2 \quad (N > 2(5.23))
\]

In addition, in order to obtain the global existence of $C^1$ solutions of the original system (1.1)-(1.3), we can choose $0 < \delta_1 \leq \frac{\psi^2}{(\gamma - 1)^2 c^2}$, then
\[
\frac{\gamma - 1}{2} \cdot m(t, x) + \psi \geq \frac{\psi}{2} > 0, \quad (t, x) \in [0, T] \times \mathbb{R}^N.
\]

From (4.2)-(4.3), by Hölder’s inequity, we set
\[
I_{1,1} = \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + 2\|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + 2\|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2}
\]
and
\[
I_{1,2} = \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + 2\|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + 2\|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2}
\]

I_{2,q} denotes the right side of inequality (5.1) ($q \geq 0$).

For the proof of Proposition 5.2, we are going to divide it into the following two lemmas.

**Lemma 5.5.** ($q = -1$) There exists a positive constant $\mu_2$ depending only on $A, \gamma, a$ and $n$ such that the following estimate holds:
\[
\frac{d}{dt} \left( 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U\|_{L^2}^2 + 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U_t\|_{L^2}^2 \right) + \mu_2 \left( 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U\|_{L^2}^2 + 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U_t\|_{L^2}^2 \right)
\]
\[
\leq C \left( \|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^\frac{2}{m+2}} \|\nabla u\|_{L^\infty} \|\Delta_1 m\|_{L^2} \right)
\]
\[
+ \left( 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U\|_{L^2}^2 + 2^{-2(\sigma + \varepsilon)} \|\Delta_1 U_t\|_{L^2}^2 \right) \left( 2^{2q(\sigma + \varepsilon)} \|\Delta_q U\|_{L^2}^2 + \|\Delta_q U_t\|_{L^2}^2 \right)
\]
\[
+ c_2^2 \left( \|\nabla u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 \right) (q \geq -1).
\]

**Proof of Lemma 5.5.** Combining (4.2)-(4.3), Lemma 5.1 and 5.3, we have
\[
\frac{d}{dt} \left( 2^{-2(\|\Delta_1 m\|_{L^2}^2 + \|\Delta_1 u\|_{L^2}^2) + (\|\Delta_1 m_t\|_{L^2}^2 + \|\Delta_1 u_t\|_{L^2}^2) \right) + \beta_1 \psi \|\Delta_1 m\|_{L^2}^2 + 2\alpha \|\Delta_1 m\|_{L^2}^2 + \beta_2 \|\Delta_1 m\|_{L^2}^2 + 2\alpha \|\Delta_1 u\|_{L^2}^2
\]
\[
\leq 2^{-2} I_{1,1} + I_{2,1} + \beta_1 \left( \|\nabla u\|_{L^\infty} \|\Delta_1 U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 U_t\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 u_t\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2}
\]
\[
+ \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla U\|_{L^2}
\]
\[
+ \frac{\gamma - 1}{2} \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla m\|_{L^2} + \frac{\gamma - 1}{2} \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla \cdot m\|_{L^2} + \frac{\gamma - 1}{2} \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla m\|_{L^2} + \frac{\gamma - 1}{2} \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla m\|_{L^2} + \frac{\gamma - 1}{2} \|\nabla u\|_{L^\infty} \|\Delta_1 \nabla m\|_{L^2},
\]
where two positive constants $\beta_1, \beta_2$ satisfy
\[ \beta_1 \leq \min \left\{ \frac{\bar{\psi}}{8a}, \frac{\psi}{\psi_2 C^2} \right\} \text{ and } \beta_2 \leq \frac{a}{\psi_2 C^2}, \text{ respectively.} \]

We introduce them in order to eliminate quadratic terms in the right side of (5.27). First, we notice that there are no quadratic terms in $I_{1,-1}$ and $I_{2,-1}$. The first quadratic term can be handled directly by Young’s inequality:
\[
\beta_1 a \| \Delta_{-1} u \|_{L^2} \| \Delta_{-1} \nabla m \|_{L^2}
= \sqrt{\frac{\beta_1}{\bar{\psi}}} \| \Delta_{-1} u \|_{L^2} \cdot \sqrt{\beta_1 \bar{\psi}} \| \Delta_{-1} \nabla m \|_{L^2}
\leq \frac{\beta_1 a^2}{\bar{\psi}} \| \Delta_{-1} u \|_{L^2}^2 + \frac{1}{4} \beta_1 \bar{\psi} \| \Delta_{-1} \nabla m \|_{L^2}^2
\leq \frac{1}{8} a \| \Delta_{-1} u \|_{L^2}^2 + \frac{1}{4} \beta_1 \bar{\psi} \| \Delta_{-1} \nabla m \|_{L^2}^2.
\]

The remainder quadratic terms in the right side of (5.27) are estimated similarly as follows:
\[
\beta_1 \| \Delta_{-1} u \|_{L^2} \| \Delta_{-1} \nabla m \|_{L^2} \leq \frac{\beta_1}{\bar{\psi}} \| \Delta_{-1} u \|_{L^2}^2 + \frac{1}{4} \beta_1 \bar{\psi} \| \Delta_{-1} \nabla m \|_{L^2}^2
\leq a \| \Delta_{-1} u \|_{L^2}^2 + \frac{1}{4} \beta_1 \bar{\psi} \| \Delta_{-1} \nabla m \|_{L^2}^2; \\
\frac{1}{2} \beta_2 \bar{\psi} \tilde{C} \| \Delta_{-1} u \|_{L^2} \| \Delta_{-1} m \|_{L^2} \leq \frac{1}{8} \beta_2 \bar{\psi} \tilde{C} \| \Delta_{-1} u \|_{L^2}^2 + \frac{1}{2} \beta_2 \| \Delta_{-1} m \|_{L^2}^2,
\]

Then (5.27) becomes into
\[
\frac{d}{dt} \left( 2^{-2} (\| \Delta_{-1} m \|_{L^2}^2 + \| \Delta_{-1} \nabla u \|_{L^2}^2) \right) + \| \Delta_{-1} \nabla m \|_{L^2}^2 + a \| \Delta_{-1} \nabla u \|_{L^2}^2 + \frac{1}{2} \beta_1 \bar{\psi} \| \Delta_{-1} \nabla m \|_{L^2}^2 + \frac{1}{2} \beta_2 \| \Delta_{-1} m \|_{L^2}^2
\leq 2^{-2} I_{1,-1} + I_{2,-1} + \beta_1 \left( \| u \|_{L^\infty} \| \Delta_{-1} \nabla u \|_{L^2} + \| u, \Delta_{-1} \nabla u \|_{L^2} \right)
+ \frac{\gamma - 1}{2} \| m \|_{L^\infty} \| \Delta_{-1} \nabla m \|_{L^2} + \frac{\gamma - 1}{2} \| m, \Delta_{-1} \nabla m \|_{L^2} \| \Delta_{-1} \nabla m \|_{L^2}
+ \beta_2 \left( \| u \|_{L^\infty} \| \Delta_{-1} \nabla m \|_{L^2} + \| m \|_{L^\infty} \| \Delta_{-1} \nabla m \|_{L^2} \right) \| \Delta_{-1} m \|_{L^2}.
\]

Multiplying (5.28) by $2^{-2(\sigma-1+\varepsilon)}$ and combining Lemma 5.2, 5.4, we can get (5.26) with the aid of Gagliardo-Nirenberg-Sobolev inequality \( \| \Delta_{-1} m \|_{L^2} \leq C \| \Delta_{-1} \nabla m \|_{L^2} (N > 2) \) and Young’s inequality. \( \square \)

For the case of high frequency \( \tilde{n} \geq 0 \), we also have the following a-priori estimate in a similar way:

**Lemma 5.6.** \( \tilde{n} \geq 0 \) There exists a positive constant $\mu_3$ depending only on $A, \gamma, a$ and $\tilde{n}$ such that the following estimate holds:
\[
\frac{d}{dt} \left( 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} U \|_{L^2}^2 + 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} U_t \|_{L^2}^2 \right)
+ \mu_3 \left( 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} U \|_{L^2}^2 + 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} \nabla m \|_{L^2}^2 + \| \Delta_{q} U_t \|_{L^2}^2 \right)
\leq C \left( \| \nabla U \|_{L^\infty} + \| U \|_{B^2_{2,2} + 1} \right) \left( 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} \nabla m \|_{L^2}^2 + 2^{2q(\sigma-1+\varepsilon)} \| \Delta_{q} U_t \|_{L^2}^2 \\
+ c_q^2 \| U \|_{B^2_{2,2} + 1} + c_q^2 \| \nabla U \|_{B^2_{2,2} + 1} \right) + J_q,
\]

(5.29)
where $J_q$ is defined in Lemma 5.5.

Summing (5.29) on $q \in \mathbb{N} \cup \{0\}$ and adding (5.26) together, according to a priori assumption (5.23), we get the following differential inequality:

$$\frac{d}{dt} \left( \|U(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \right) + \mu_4 \left( \|u(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|\nabla m\|_{B^{2-1+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \right) \leq C \delta^2 \left( \|u(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|\nabla m\|_{B^{2-1+\varepsilon}_{2,2}}^2 + \|U(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \right), \quad (5.30)$$

where the constant $\mu_4$ depends only on $A, \gamma, a$ and $\bar{n}$. Furthermore, choosing $\delta_1 \leq \min \left\{ \frac{\mu_4}{C \delta^2}, \frac{\mu_4}{C \delta^2} \right\}$, we conclude the proof of Proposition 5.2 with $\mu_1 = \frac{\mu_4}{C \delta^2}$. \qed

**Proof of Proposition 5.1.** In fact, Proposition 4.1 also holds on the framework of functional space $B^{2+\varepsilon}_{2,2}$. From the assumption

$$\|U(\cdot, 0)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, 0)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \leq \delta_2,$$

we can determine a time $T_1 > 0$ ($T_1 < T_0$) such that

$$\|U(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \leq 2\delta_2, \quad \text{for all} \ t \in [0, T_1].$$

**Claim:** One can choose a positive constant $\delta_2$ satisfying $\delta_2 < \frac{1}{2} \delta_1$ such that

$$\|U(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 < \delta_1, \quad \text{for all} \ t \in [0, T_0]. \quad (5.31)$$

In fact, otherwise, we can assume that there exists a time $T_2$ ($T_1 < T_2 < T_0$) such that (5.31) is satisfied for all $t \in [0, T_2)$ and

$$\|U(\cdot, T_2)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, T_2)\|_{B^{2-1+\varepsilon}_{2,2}}^2 = \delta_1, \quad (5.32)$$

since we see (5.31) is satisfied as $t \rightarrow 0$ for such choice of $\delta_2$.

By Proposition 5.2, for all $t \in [0, T^k]$ ($T^k \rightarrow T_2$)

$$\|U(\cdot, t)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, t)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \leq \|U(\cdot, 0)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, 0)\|_{B^{2-1+\varepsilon}_{2,2}}^2.$$

In particular,

$$\|U(\cdot, T^k)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, T^k)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \leq \|U(\cdot, 0)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, 0)\|_{B^{2-1+\varepsilon}_{2,2}}^2.$$

By the continuity on $t \in [0, T_0]$, we get

$$\|U(\cdot, T_2)\|_{B^{2+\varepsilon}_{2,2}}^2 + \|U_t(\cdot, T_2)\|_{B^{2-1+\varepsilon}_{2,2}}^2 \leq \delta_2 < \delta_1,$$

which contradicts (5.32). Hence, (5.31) is true. From Proposition 4.1 and 5.2, we can prove Proposition 5.1 by using the standard bootstrap argument. \qed

By Besov imbedding property, $(m, u) \in C^1([0, \infty) \times \mathbb{R}^N)$ solves (3.1)-(3.2). The choice of $\delta_1$ is sufficient to ensure $\frac{2N}{N-2}m + \psi > 0$. According to Remark 3.1, we deduce that $(m, u) \in C^1([0, \infty) \times \mathbb{R}^N)$ solves (1.1)-(1.3) with $\bar{n} > 0$. Furthermore, we attain the main result (Theorem 1.2) in this paper.

In what follows, we state a direct consequence of Proposition 5.1.

**Corollary 5.1.** Let $(m, u)$ be the solution in Proposition 5.1, we have $(\sigma = 1 + \frac{N}{2}, \epsilon' < \epsilon)$

$$\|m(\cdot, t)\|_{B^{2-1+\epsilon'}_{p,2}} \rightarrow 0 \quad (p = \frac{2N}{N-2}), \quad \|u(\cdot, t)\|_{B^{2+\epsilon'}_{2,2}} \rightarrow 0, \quad \text{as} \ t \rightarrow +\infty.$$
Proof. Because of the similar argument, we show the former only. From the energy estimate in Proposition 5.1, we get
\[ \nabla m \in L^2 t B_2^{-1+\varepsilon}, \quad \nabla m_t \in L^2 t B_2^{-2+\varepsilon}. \]
Set
\[ H(t) = \|\nabla m(\cdot, t)\|^2_{B_2^{-2+\varepsilon}} \in L^1_t. \]
After an easy computation, we have
\[ \frac{d}{dt} H(t) \leq \|\nabla m(\cdot, t)\|^2_{B_2^{-2+\varepsilon}} + \|\nabla m_t(\cdot, t)\|^2_{B_2^{-2+\varepsilon}} \in L^1_t. \]
Hence, \( H(t) \to 0 \) as \( t \to +\infty \). Since \( m(t, x) \) is bounded in \( C([0, \infty), B_2^{-\varepsilon}) \), by interpolation argument, we can obtain \( (\varepsilon' < \varepsilon) \)
\[ \|\nabla m(\cdot, t)\|_{B_2^{-1+\varepsilon'}} \to 0, \quad \text{as} \quad t \to +\infty, \]
which completes the proof after using Gagliardo-Nirenberg-Sobolev inequality. \( \square \)

Finally, we show the exponential decay of the vorticity.

**Proof of Theorem 1.3.** When \( N = 3 \), the curl of the velocity equation in Eqs.(1.1) gives
\[ \partial_t \omega + a\omega + u \cdot \nabla \omega - \omega \cdot \nabla u = 0. \]
Hence,
\[ \frac{1}{2} \frac{d}{dt} \|\Delta q\omega\|_{L^2}^2 + a \|\Delta q\omega\|_{L^2}^2 \leq C_0 (\|\nabla u\|_{L^\infty} \|\Delta q\omega\|_{L^2} + \|\omega\|_{L^\infty} \|\Delta q\nabla u\|_{L^2} + c_q \|\nabla u\|_{B_2^{-1}} \|\omega\|_{B_2^{-1}}) \|\Delta q\omega\|_{L^2}, \quad (5.33) \]
Dividing (5.33) by \( \|\Delta q\omega\|_{L^2} \) and summing it on \( q \geq -1 (q \in \mathbb{Z}) \) after multiplying the factor \( 2^q(\sigma - 1) \), from Theorem 1.2, we have
\[ \frac{1}{2} \frac{d}{dt} \|\omega(\cdot, t)\|_{B_2^{-1}} + a \|\omega(\cdot, t)\|_{B_2^{-1}} \leq C_0 \|u(\cdot, t)\|_{B_2^{-1}} \|\omega(\cdot, t)\|_{B_2^{-1}} \]
\[ \leq C_0 \left( \|(n - \bar{n}, u)(\cdot, 0)\|_{B_2^{-1}}^2 + \|(n_t, u_t)(\cdot, 0)\|_{B_2^{-1}}^2 \right)^{\frac{1}{2}} \|\omega(\cdot, t)\|_{B_2^{-1}} \]
\[ \leq C_0 \min \{ \delta_0^2, \frac{a}{2C_0} \} \|\omega(\cdot, t)\|_{B_2^{-1}} \]
\[ \leq \frac{1}{2} a \|\omega(\cdot, t)\|_{B_2^{-1}}. \]
Therefore, we obtain the exponential decay of \( \|\omega(\cdot, t)\|_{B_2^{-1}} \) with \( \mu_0' = a \). \( \square \)

**References**

[1] R. Danchin, Fourier Analysis Methods for PDE’s (2005).

[2] D. Y. Fang, J. Xu and T. Zhang, Global exponential stability of classical solutions to the hydrodynamic model for semiconductors, to appear on Mathematical Models and Methods in Applied Sciences.

[3] L. Hsiao, Quasilinear hyperbolic systems and dissipative mechanisms, World Scientific, 1997.

[4] L. Hsiao and T. P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys. 143 (1992) 599–605.

[5] L. Hsiao and R. Pan, initial-boundary value problem for the system of compressible adiabatic flow through porous media, J. Differential Equations 159 (1999) 280–305.
[6] L. Hsiao and D. Serre, Global existence of solutions for the system of compressible adiabatic flow through porous media, SIAM J. Math. Anal. 27 (1996) 70–77.

[7] F. Huang, P. Marcati and R. Pan, Convergence rate for compressible Euler equations with damping and vacuum, Arch. Rational Mech. Anal. 176 (2005) 1–24.

[8] F. Huang and R. Pan, Convergence rate for compressible Euler equations with damping and vacuum, Arch. Rational Mech. Anal. 166 (2003) 359–376.

[9] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rational Mech. Anal. 58 (1975) 181–205.

[10] A. Majda, Compressible Fluid Flow and Conservation laws in Several Space Variables, Springer-Verlag, Berlin/New York, 1984.

[11] T. Nishida, Global solutions for an initial-boundary value problem of a quasilinear hyperbolic systems, Proc. Japan Acad. 44 (1968) 642–646.

[12] T. Nishida, Nonlinear hyperbolic equations and relates topics in fluid dynamics, Publ. Math. D’Orsay (1978) 46–53.

[13] T. Sideris, B. Thomases and D. H. Wang, Long time behavior of solutions to the 3D compressible Euler with damping, Comm. P. D. E. 28 (2003) 953–978.

[14] H. Triebel, Theory of function spaces, Birkhäuser, 1983.

[15] W. Wang and T. Yang, The pointwise estimates of solutions for Euler equations with damping in multi-dimensions, J. Differential Equations 173 (2001) 410–450.

[16] C. J. Xu and T. Yang, Local existence with physical vacuum boundary condition to Euler equations with damping, J. Differential Equations 210 (2005) 217–231.