New q-Rung Orthopair Hesitant Fuzzy Decision Making Based on Linear Programming and TOPSIS

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ABSTRACT A new multiple attribute decision making method based on the q-rung orthopair hesitant fuzzy sets has been developed. The evaluation values are given as q-rung orthopair hesitant fuzzy values. Then some weighted similarity functions are defined. A linear programming model is proposed to derive attribute weights based on the similarity functions for the case of partly known attribute weight information and a formula is given to determine attribute weight based on similarity function and the Lagrange function for completely unknown attribute weights. Finally, TOPSIS method is used to rank alternatives. The application of the proposed approach is explored by the application of purchase self-service book sterilizer problem. Some comparisons are also conducted to demonstrate advantages of the proposed method.

INDEX TERMS Multiple attribute decision making, q-rung orthopair hesitant fuzzy set, linear programming model, TOPSIS, aggregation operator.

I. INTRODUCTION

A-s extension of intuitionistic fuzzy set and Pythagorean fuzzy set, q-rung orthopair fuzzy set [1] is more powerful and flexible since the sum of the $q$th power of the membership and the $q$th power of the nonmembership is not more than 1. Q-rung orthopair fuzzy set has gotten extensive research and application [2]–[6]. Some aggregation operators have been developed [7]–[27]. Liu and Wang developed the q-rung orthopair fuzzy weighted averaging operator in [7] and developed some weighted generalized Maclaurin symmetric mean operators in [9]. Darko and Liang [10] defined some q-rung orthopair fuzzy Hamacher aggregation operators. Some Muirhead mean q-rung orthopair fuzzy aggregation operators are studied in [11]. Wang et al. [12] developed some q-rung orthopair hesitant fuzzy power generalized Heronian mean operator by using the power average operator and generalized Heronian mean operator. Yang and Pang [13] defined some q-rung orthopair fuzzy partitioned Bonferroni mean operators. Yang and Pang [14] proposed new q-rung orthopair fuzzy Bonferroni Mean Dombi operators. Xing et al. [15] defined a new class of q-rung orthopair fuzzy point weighted aggregation operators. Du [16] defined some q-rung orthopair fuzzy weighted power mean operators. Wang et al. [21] defined a family of q-rung orthopair fuzzy Muirhead mean operators. Liu and Wang [22] developed q-rung orthopair fuzzy Archimedean Bonferroni Mean operator. Ju et al. [23] defined some interval-valued q-rung orthopair weighted averaging operators. Wei et al. [24] proposed the generalized q-rung orthopair fuzzy Heronian mean operator. Wang et al. [25] presented q-rung orthopair fuzzy Hamy mean operators. Some q-rung orthopair fuzzy distance measures, entropy measures, correlation measures are studied [28]–[36]. Rajkumar [28] defined the q-rung orthopair fuzzy divergence and entropy measures. Some new information measures are developed for q-rung orthopair fuzzy sets [29]. Liu et al. [30] defined some q-rung orthopair fuzzy cosine similarity measures and distance measures. Peng et al. [31] developed q-rung orthopair fuzzy weighted distance-based method. Du [34] defined some Minkowski-type distances for q-rung orthopair values. Some multiple attribute decision making methods have been presented [37]–[40]. Banerjee et al. [36] developed a q-rung orthopair fuzzy method based on the multiple criteria hierarchy process method and QUALIFLEX method. Liu and Huang [37] defined a new correlation measure based...
on linguistic scale function and developed a consensus reaching process for probabilistic linguistic q-rung orthopair fuzzy TOPSIS method. Pinar and Boran [38] used q-rung orthopair fuzzy TOPSIS and q-rung orthopair fuzzy ELECTRE to select the best supplier. Gao et al. [39] developed q-rung interval-valued orthopair fuzzy VIKOR model. Hussain et al. [40] presented a q-rung orthopair fuzzy rough set model based on TOPSIS. Q-rung orthopair fuzzy sets are further been extended to accommodate linguistic fuzzy values [41]–[43], interval values [39], etc. Q-rung orthopair hesitant fuzzy set [44], [45] is the extension of q-rung orthopair fuzzy set by combing it with hesitant fuzzy set. Since q-rung orthopair hesitant fuzzy set has been developed recently, there are few studies focusing on it. Wang et al. [12] developed some power generalised Heronian mean operators in q-rung orthopair hesitant fuzzy environment. Wang et al. [46] extended the TOPSIS to accommodate q-rung orthopair hesitant fuzzy values. Comparing with tools to model uncertain information, q-rung orthopair hesitant fuzzy set has larger feasible region, which can be used to solve problems that can’t be solved by other fuzzy sets. Different fuzzy sets can be got by taking different parameters in q-rung orthopair hesitant fuzzy set. Intuitionistic fuzzy set and Pythagorean fuzzy set are all special cases of the q-rung orthopair hesitant fuzzy set. In fact, decision makers can take parameter according to the risk attitude and real need of the problems. Since q-rung orthopair hesitant fuzzy set is more flexible and powerful, the evaluation values are given as q-rung orthopair hesitant fuzzy values in this paper.

There are many methods based on mathematical programming models to determine attribute weights including least squares, maximum entropy, linear programming models, etc. The linear programming technique for multidimensional analysis of preference (LINMAP) is an important compromising model based on linear programming in multiple attribute decision making field, which was initiated by Srinivasan and Shocker [47]. LINMAP method has been extended to accommodate different fuzzy values including intuitionistic fuzzy values [48], [49], interval-valued intuitionistic fuzzy values [50], hesitant fuzzy values [48], [49], linguistic hesitant fuzzy values [54], trapezoidal fuzzy values [55], [56], probabilistic linguistic values [57], interval type-2 fuzzy values [58], Pythagorean fuzzy values [59], [60], probabilistic interval-valued intuitionistic hesitant values [61]. The idea of LINMAP is used in determining attribute weights [62]–[67]. Yang et al. [62] used linear programming model to determine weight vector in linguistic hesitant intuitionistic fuzzy environment. Yang et al. [63] calculated attribute weights by using a linear programming model and closeness coefficients in intuitionistic fuzzy setting. Qin et al. [64] determined attributes weight vector in interval type-2 fuzzy group decision making problems by a linear programming model. Liu et al. [65] calculated combined attribute weights in linguistic intuitionistic fuzzy environments by a linear programming model. Song et al. [66] established a two-stage optimization model to determine attribute weights based on the extended LINMAP method. Wang et al. [67] derived attribute weights based on a linear programming model under intuitionistic fuzzy environment. The attribute weights determined by linear programming models have the following characteristics: attribute weights are more objective and reasonable, the linear programming model can be solved easily. In this paper, the attribute weights are determined by using the linear programming model based on the similarity function. TOPSIS is an important multiple attribute decision making method, which is studied and applied widely [68]–[73]. In TOPSIS, optimal alternative is determined based on the idea of close to the positive idea solution and far from the negative solution at the same time. TOPSIS is used to get compromise solution and easy to calculate. TOPSIS is used to rank alternatives in this paper to avoid complication computation.

Summarizing above discussion, we present a new multiple attribute decision making method by extending linear programming and TOPSIS method to q-rung orthopair hesitant fuzzy environments. The contributions of the paper are as following. (1) The evaluation values are given as q-rung orthopair hesitant values, which can model flexible and complicated decision information. (2) The weighted similarity functions between q-rung orthopair hesitant fuzzy sets have been defined based on generalized distance and cosine similarity measure. (3) New attribute weights determined method is proposed based on the weighted similarity function and a linear programming model. The weights got by the new method are more reasonable. (4) New multiple attribute decision making method is developed to rank the alternatives based on a linear programming model and TOPSIS in q-rung orthopair hesitant environment. The new method can be used to solve complicated decision problem with low computation complexity.

In order to do so, the rest of paper is organized as follows. Some basic concepts about q-rung orthopair fuzzy set, hesitant fuzzy set are reviewed in Section 2. Several weighted similarity function for q-rung orthopair hesitant fuzzy set is also defined. We also study the properties of the weighted similarity function. A new algorithm is given based on the weighted similarity function and idea of TOPSIS presented in Section 3. We give the methods to determine attribute weights for the cases of partly known and completely unknown attribute weights. Numerical example is presented in Section 4 and some comparisons are also conducted. Conclusions are given in Section 5.

II. PRELIMINARIES

Definition 1 [11]: Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a fixed set. A q-rung orthopair fuzzy set (q-ROFS) \( A \) on \( X \) can be represented as

\[
A = \{ x_i, \mu_A(x_i), \nu_A(x_i) | x_i \in X \},
\]

where \( \mu_A(x_i) : X \rightarrow [0, 1] \) is the degree of membership and \( \nu_A(x_i) : X \rightarrow [0, 1] \) is the degree of non-membership of \( x_i \in X \) to the set \( A \), respectively. For each \( x_i \in X \), it satisfies
the following condition $0 \leq (\mu_A(x_i))^q + (\nu_A(x_i))^q \leq 1$, $q \geq 1$.

For simplicity, $\mu_A(x_i), \nu_A(x_i)$ is called the q-rough orthopair fuzzy number (q-ROFN), denoted by $\mu_A, \nu_A$.

Definition 2 [44]: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set. A q-rough orthopair hesitant fuzzy set (q-ROHFS) on $X$ can be represented as

$$Q = \{< x_i, h_Q(x_i), g_Q(x_i) > | x_i \in X\},$$

where $h_Q(x_i)$ and $g_Q(x_i)$ are the sets of all the possible degrees of membership and the degrees of nonmembership, $\mu \in h_Q(x_i)$ and $\nu \in g_Q(x_i)$ and they satisfy conditions $0 \leq \mu, \nu \leq 1$, $0 \leq (\mu^q + (\nu^q)^q \leq 1$, $\mu^q = \max_{\nu \in g_Q(x_i)}(\mu)$, $\nu^q = \max_{\nu \in g_Q(x_i)}(\nu)$. For convenience, the $h_Q(x_i), g_Q(x_i)$ is called a q-rough orthopair hesitant fuzzy number (q-ROHFN).

Definition 3 [44]: Let $Q = < h_Q, g_Q >$, $Q_1 = < h_{Q_1}, g_{Q_1} >$ and $Q_2 = < h_{Q_2}, g_{Q_2} >$ be three q-ROHFNs, $\lambda > 0, q \geq 1$. The operational laws of q-ROHFNs are described as follows:

1. $Q_1 \oplus Q_2 = \bigcup_{Q_1 \epsilon h_Q, \nu \in g_Q} \{Q_1, Q_2\}, \{\nu Q_1, g_Q\}$;
2. $Q_1 \ominus Q_2 = \{Q_1, Q_2\} \ominus \{\nu Q_1, g_Q\}$;
3. $Q = \bigcup_{Q_1 \epsilon h_Q, \nu \in g_Q} \{Q_1, Q_2\}, \{\nu Q_1, g_Q\}$;
4. $Q^q = \bigcup_{Q_1 \epsilon h_Q, \nu \in g_Q} \{Q_1, Q_2\}, \{\nu Q_1, g_Q\}$

Theorem 1 [44]: Let $Q = < h_Q, g_Q >$, $Q_1 = < h_{Q_1}, g_{Q_1} >$ and $Q_2 = < h_{Q_2}, g_{Q_2} >$ be three q-ROHFSs, $\lambda > 0, q \geq 1$. Then

1. $(Q_1 \oplus Q_2 = Q_2 \oplus Q_1)\text{;}$
2. $(Q_1 \ominus Q_2 = Q_2 \ominus Q_1)\text{;}$
3. $\lambda (Q_1 \oplus Q_2) = \lambda Q_1 \oplus \lambda Q_2$;
4. $(Q_1 \ominus Q_2)^q = Q_1^q \ominus Q_2^q$.

Definition 4 [44]: Let $Q = < h_Q, g_Q > = < \mu_Q, \nu_Q > \ldots, \mu_{\theta(h_Q)}, \nu_{\theta(g_Q)} >$ be a q-ROHFN, where $l(h_Q), l(g_Q)$ are the numbers of the membership and the nonmembership in $h_Q, g_Q$, respectively. Then the score function $S(Q)$ can be defined as

$$S(Q) = \frac{1}{l(h_Q)} \sum_{\mu \epsilon h_Q} \mu_q^q - \frac{1}{l(g_Q)} \sum_{\nu \epsilon g_Q} \nu_q^q.$$

The accuracy function is defined as

$$A(Q) = \frac{1}{l(h_Q)} \sum_{\mu \epsilon h_Q} \mu_q^q + \frac{1}{l(g_Q)} \sum_{\nu \epsilon g_Q} \nu_q^q.$$

The comparison method of the q-ROHFNs based on the score function and accuracy function can be defined as follows [44].

Definition 5: Let $Q_1 = < h_{Q_1}, g_{Q_1} >$ and $Q_2 = < h_{Q_2}, g_{Q_2} >$ be two q-ROHFNs. Then

1. If $S(Q_1) > S(Q_2)$, then $Q_1 > Q_2$.
2. If $S(Q_1) = S(Q_2)$, then

if $A(Q_1) > A(Q_2)$, then $Q_1 > Q_2$.
if $A(Q_1) = A(Q_2)$, then $Q_1 \sim Q_2$.

q-ROHFNs should be extended if they have different numbers of the membership and nonmembership according to the risk attitude of decision makers. The smallest membership and the largest nonmembership can be added for risk-averse attitude of decision maker. The average value of the membership and the average value of nonmembership can be added for the case of risk-neutral attitude of decision maker. The largest membership and the smallest nonmembership can be added for risk-seeking attitude of decision maker.

Definition 6: Let $Q_1 = < h_{Q_1}, g_{Q_1} > = < \mu_{Q_1}, \nu_{Q_1} >, \ldots, \mu_{Q_{l(h_Q)}}, \nu_{Q_{l(g_Q)}} >$, $Q_2 = < h_{Q_2}, g_{Q_2} > = < \mu_{Q_2}, \nu_{Q_2} >, \ldots, \mu_{Q_{l(h_Q)}}, \nu_{Q_{l(g_Q)}} >$ be two extended q-ROHFSs, where $l(h_Q)$, $l(g_Q)$ are the numbers of the membership and nonmembership in $h_Q, g_Q$, respectively. Then the weighted similarity based on the generalized distance measure of the two q-ROHFSs $\tilde{S}(Q_1, Q_2)$ can be defined as follows:

$$\tilde{S}(Q_1, Q_2) = \sum_{i=1}^{n} w_i \left( \left( 1 - \frac{1}{l(h_Q)} \sum_{j=1}^{l(h_Q)} |(\mu_{Q_1}^{[j]}(x_i))^q \right) - \frac{1}{l(g_Q)} \sum_{j=1}^{l(g_Q)} |(\nu_{Q_1}^{[j]}(x_i))^q \right) \right) \left( 1 - \frac{1}{l(g_Q)} \sum_{j=1}^{l(g_Q)} |(\nu_{Q_2}^{[j]}(x_i))^q \right) \right) \right),$$

where $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$.

The weighted similarity based on the Hamming distance measure $\tilde{S}_1(Q_1, Q_2)$ of the two q-ROHFSs can be defined as

$$\tilde{S}_1(Q_1, Q_2) = \sum_{i=1}^{n} w_i \left( \left( 1 - \frac{1}{l(h_Q)} \sum_{j=1}^{l(h_Q)} |(\mu_{Q_1}^{[j]}(x_i))^q \right) - (\mu_{Q_2}^{[j]}(x_i))^q \right) + \frac{1}{l(g_Q)} \sum_{j=1}^{l(g_Q)} |(\nu_{Q_1}^{[j]}(x_i))^q \right) - (\nu_{Q_2}^{[j]}(x_i))^q \right) \right).$$

The weighted similarity based on the Euclidean distance measure $\tilde{S}_2(Q_1, Q_2)$ of the two q-ROHFSs can be defined as

$$\tilde{S}_2(Q_1, Q_2) = \sum_{i=1}^{n} w_i \left( \left( 1 - \frac{1}{l(h_Q)} \sum_{j=1}^{l(h_Q)} |(\mu_{Q_1}^{[j]}(x_i))^q \right) - (\mu_{Q_2}^{[j]}(x_i))^q \right) + \frac{1}{l(g_Q)} \sum_{j=1}^{l(g_Q)} |(\nu_{Q_1}^{[j]}(x_i))^q \right) - (\nu_{Q_2}^{[j]}(x_i))^q \right) \right)^{1/2).}.$$
Example 1: Let $X = \{x_1, x_2\}$, $Q_1' = \{< x_1, [0.65, 0.75, 0.80], [0.30, 0.40]>, < x_2, [0.60, 0.70], [0.40, 0.50] >\}$, $Q_2' = \{< x_1, [0.60, 0.70], [0.35, 0.45]>, < x_2, [0.70, 0.80], [0.45, 0.55] >\}$, $\nu = (w_1, w_2) = (0.35, 0.65)$, $q = 3$. Assume decision makers are risk averse, the smallest membership and the largest nonmembership are added. The extended q-ROHFs can be got as $Q_1 = \{< x_1, [0.65, 0.75, 0.80], [0.30, 0.40]>, < x_2, [0.60, 0.70], [0.40, 0.50] >\}$. $Q_2 = \{< x_1, [0.60, 0.70], [0.35, 0.45]>, < x_2, [0.70, 0.80], [0.45, 0.55] >\}$. Then $\hat{S}_1(Q_1, Q_2) = 0.9278$, $\hat{S}_2(Q_1, Q_2) = 0.9023$, $\hat{S}_3(Q_1, Q_2) = 0.8853$.

The similarity measure $\hat{S}(Q_1, Q_2)$ between two q-ROHFs $Q_1$ and $Q_2$ satisfies the following properties:

1. $0 \leq \hat{S}(Q_1, Q_2) \leq 1$.
2. $\hat{S}(Q_1, Q_2) = 1$ if and only if $Q_1 = Q_2$.
3. $\hat{S}(Q_1, Q_2) = \hat{S}(Q_2, Q_1)$.

$\hat{S}(Q_1, Q_2)$ can be proved to satisfy the above conditions.

Proof: (1) Since $0 \leq (\mu_{Q_1}(x_i))^{\theta} \leq 1$, $0 \leq (\lambda_{Q_2}(x_i))^{\eta} \leq 1$, then $0 \leq (\mu_{Q_2}(x_i))^{\theta} \leq (\mu_{Q_1}(x_i))^{\theta} \leq 1$, $0 \leq (\lambda_{Q_1}(x_i))^{\eta} \leq (\lambda_{Q_2}(x_i))^{\eta} \leq 1$, $0 \leq (\nu_{Q_1}(x_i))^{\theta} \leq (\nu_{Q_2}(x_i))^{\theta} \leq 1$, $0 \leq (\lambda_{Q_1}(x_i))^{\eta} \leq (\lambda_{Q_2}(x_i))^{\eta} \leq 1$, $0 \leq (\nu_{Q_1}(x_i))^{\theta} \leq (\nu_{Q_2}(x_i))^{\theta} \leq 1$. Since $\sum_{i=1}^{n} w_i = 1$, and

$$
0 \leq \sum_{i=1}^{n} w_i (1 - \left(\frac{1}{2} \left(\frac{1}{l(h_i)} \sum_{j=1}^{l(h_i)} (\mu_{Q_1}(x_i))^{\theta} \right) + \left(\frac{1}{l(g_i)} \sum_{j=1}^{l(g_i)} (\nu_{Q_1}(x_i))^{\theta} \right) \right) ^{1/\lambda}) \leq 1.
$$

Hence $0 \leq \hat{S}(Q_1, Q_2) \leq 1$.

(2) If $Q_1 = Q_2$, $\mu_{Q_1}(x_i) = \mu_{Q_2}(x_i)$ and $\nu_{Q_1}(x_i) = \nu_{Q_2}(x_i)$. Then $|\mu_{Q_1}(x_i) - \mu_{Q_2}(x_i)| = 0$, $|\nu_{Q_1}(x_i) - \nu_{Q_2}(x_i)| = 0$,

$$
\left(\frac{1}{2} \left(\frac{1}{l(h_i)} \sum_{j=1}^{l(h_i)} (|\mu_{Q_1}(x_i)|^{\theta} - (\mu_{Q_2}(x_i))^{\theta}) \right) + \left(\frac{1}{l(g_i)} \sum_{j=1}^{l(g_i)} (|\nu_{Q_1}(x_i)|^{\theta} - (\nu_{Q_2}(x_i))^{\theta}) \right) \right) ^{1/\lambda} = 0
$$

and $\hat{S}(Q_1, Q_2) = 1$. If $\hat{S}(Q_1, Q_2) = 1$, then

$$
\sum_{i=1}^{n} w_i \left(\frac{1}{2} \left(\frac{1}{l(h_i)} \sum_{j=1}^{l(h_i)} (|\mu_{Q_1}(x_i)|^{\theta} - (\mu_{Q_2}(x_i))^{\theta}) \right) + \left(\frac{1}{l(g_i)} \sum_{j=1}^{l(g_i)} (|\nu_{Q_1}(x_i)|^{\theta} - (\nu_{Q_2}(x_i))^{\theta}) \right) \right) ^{1/\lambda} = 0.
$$

Hence $\hat{S}(Q_1, Q_2) = 1$. where $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.

Example 2: $Q_1, Q_2$, $q$ and $w$ are the same as that in Example 1. If $\hat{S}_c(Q_1, Q_2)$ is used to calculate similarity as eq.(8), we can get ($\sum_{i=1}^{3} (\mu_{Q_1}(x_1) \mu_{Q_2}(x_1))^{q} + (\sum_{i=1}^{3} (\nu_{Q_1}(x_1) \nu_{Q_2}(x_1))^{q}) = 0.3330$, ($\sum_{i=1}^{3} (\mu_{Q_1}(x_2) \mu_{Q_2}(x_2))^{q} + (\sum_{i=1}^{3} (\nu_{Q_1}(x_2) \nu_{Q_2}(x_2))^{q}) = 0.3320$, ($\sum_{i=1}^{3} (\mu_{Q_1}(x_3) \mu_{Q_2}(x_3))^{q} + (\sum_{i=1}^{3} (\nu_{Q_1}(x_3) \nu_{Q_2}(x_3))^{q}) = 0.7214$, ($\sum_{i=1}^{3} (\mu_{Q_1}(x_4) \mu_{Q_2}(x_4))^{q} + (\sum_{i=1}^{3} (\nu_{Q_1}(x_4) \nu_{Q_2}(x_4))^{q}) = 0.4702$, ($\sum_{i=1}^{3} (\mu_{Q_1}(x_5) \mu_{Q_2}(x_5))^{q} + (\sum_{i=1}^{3} (\nu_{Q_1}(x_5) \nu_{Q_2}(x_5))^{q}) = 0.6800$. Then $\hat{S}_c(Q_1, Q_2) = 0.9864$. 

\[ \hat{S}_c(Q_1, Q_2) = \frac{\sum_{i=1}^{n} w_i \left(1 - \left(\frac{1}{2} \left(\frac{1}{l(h_i)} \sum_{j=1}^{l(h_i)} (|\mu_{Q_1}(x_i)|^{q}) + (\sum_{j=1}^{l(g_i)} (|\nu_{Q_1}(x_i)|^{q})) \right) \right) \right) ^{1/\lambda}}{\left(\sum_{i=1}^{n} w_i \left(\frac{1}{2} \left(\frac{1}{l(h_i)} \sum_{j=1}^{l(h_i)} (|\mu_{Q_1}(x_i)|^{q}) + (\sum_{j=1}^{l(g_i)} (|\nu_{Q_1}(x_i)|^{q})) \right) \right) \right) ^{1/\lambda}}. \]
\[ \hat{S}_c(Q_1, Q_2) \text{ also be proved to satisfy the above properties.} \]

(1). Since \( 0 \leq (\mu^{(i)}_{Q_1}(x_i))^q \leq 1, 0 \leq (\mu^{(i)}_{Q_2}(x_i))^q \leq 1, \)
then \( 0 \leq (\mu^{(i)}_{Q_1}(x_i))^q(\mu^{(i)}_{Q_2}(x_i))^q \leq 1. 0 \leq (v^{(i)}_{Q_1}(x_i))^q \leq 1, \)
\( 0 \leq (v^{(i)}_{Q_2}(x_i))^q(\nu^{(i)}_{Q_2}(x_i))^q \leq 1. \)

By using the cosine similarity measure in [74] \( C(A, B) = \sum_{i=1}^{n} \frac{\mu_1(x_i)\mu_2(x_i) + v_1(x_i)v_2(x_i)}{\sqrt{\mu_1^2(x_i) + \mu_2^2(x_i)}} \cdot \sqrt{v_1^2(x_i) + v_2^2(x_i)} , \)
we get

\[
0 \leq \sum_{i=1}^{n} w_i\left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i)\mu^{(i)}_{Q_2}(x_i))^q \right) \\
+ \left( \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_1}(x_i)v^{(i)}_{Q_2}(x_i))^q \right) \bigg/ \left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i))^2q + \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_2}(x_i))^2q \bigg)^{1/2} .
\]

(2). If \( Q_1 = Q_2, \) that is \( \mu^{(i)}_{Q_1}(x_i) = \mu^{(i)}_{Q_2}(x_i), v^{(i)}_{Q_1}(x_i) = v^{(i)}_{Q_2}(x_i) \) for all \( i, j \)
and

\[
\hat{S}_c(Q_1, Q_2) = \sum_{i=1}^{n} w_i\left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i))^{2q} + \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_2}(x_i))^2q \right)^{1/2} \\
= \sum_{i=1}^{n} w_i\left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i)\mu^{(i)}_{Q_2}(x_i))^q \right) \\
+ \left( \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_1}(x_i)v^{(i)}_{Q_2}(x_i))^q \right) \bigg/ \left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i))^2q + \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_2}(x_i))^2q \bigg)^{1/2} .
\]

(3).

\[
\hat{S}_c(Q_1, Q_2) = \sum_{i=1}^{n} w_i\left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i)\mu^{(i)}_{Q_2}(x_i))^q \right) \\
+ \left( \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_1}(x_i)v^{(i)}_{Q_2}(x_i))^q \right) \bigg/ \left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i))^2q + \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_2}(x_i))^2q \bigg)^{1/2} .
\]

Definition 8: Let \( Q_1 = \langle h_{Q_1}(x_i), g_{Q_1}(x_i) \rangle \gg \langle \mu^{(i)}_{Q_1}(x_i), \mu^{(i)}_{Q_1}(x_i) \rangle, \langle \nu^{(i)}_{Q_1}(x_i), \nu^{(i)}_{Q_1}(x_i) \rangle \rangle, Q_2 = \langle h_{Q_2}(x_i), g_{Q_2}(x_i) \rangle \gg \langle \mu^{(i)}_{Q_2}(x_i), \mu^{(i)}_{Q_2}(x_i) \rangle, \langle \nu^{(i)}_{Q_2}(x_i), \nu^{(i)}_{Q_2}(x_i) \rangle \rangle \) be two extended q-ROHFSs, where \( l(h_{Q_1}), l(g_{Q_1}) \) are the numbers of the membership and nonmembership in \( h_{Q_1}(x_i), g_{Q_1}(x_i), \) respectively. Then the weighted hybrid similarity \( \hat{S}_{w}(Q_1, Q_2) \) based on the cosine similarity measure \( \hat{S}_c(Q_1, Q_2) \) and general distance measure of the two q-ROHFSs \( \hat{S}_c(Q_1, Q_2) \) can be defined as follows:

\[
\hat{S}_{w}(Q_1, Q_2) = \frac{1}{2} \hat{S}_c(Q_1, Q_2) + \hat{S}_c(Q_1, Q_2) \\
= \frac{1}{2} \left( \sum_{i=1}^{n} w_i\left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i)\mu^{(i)}_{Q_2}(x_i))^q \right) \\
+ \left( \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_1}(x_i)v^{(i)}_{Q_2}(x_i))^q \right) \bigg/ \left( \sum_{j=1}^{l(h_{Q_1})} (\mu^{(i)}_{Q_1}(x_i))^2q + \sum_{j=1}^{l(h_{Q_2})} (v^{(i)}_{Q_2}(x_i))^2q \bigg)^{1/2} .
\right)
\]
The similarity \( \hat{S}_{ad}(Q_1, Q_2) \) can be proved to satisfy the three properties above easily by using the properties of \( \hat{S}_a(Q_1, Q_2) \) and \( \hat{S}_c(Q_1, Q_2) \), which are omitted here.

### III. A NEW MADM METHOD BASED ON Q-ROHFSs, THE LINEAR PROGRAMMING AND EXTENDED TOPSIS

Let \( A = \{A_1, A_2, \ldots, A_m\} \) be the set of alternatives and \( C = \{C_1, C_2, \ldots, C_n\} \) be the set of attributes. Let \( J_1 \) be the set of benefit attributes and \( J_2 \) be the set of cost attributes, \( J_1 \cap J_2 = \varnothing \) and \( J_1 \cup J_2 = C \). Let \( \tilde{D}' = (\tilde{\theta}_{ij})_{m \times n} \) be the q-rung orthopair hesitant fuzzy decision matrix given by decision makers, where \( \tilde{\theta}_{ij} = (\hat{h}_{ij}, \hat{g}_{ij}, \hat{g}'_{ij}, g''_{ij}) \) are the sets of memberships and nonmemberships, respectively. \( w_j \) is the weight of attribute \( C_j \) with \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 \). Extend decision matrix according to risk attitude of the decision makers and the extended decision matrix is \( \tilde{D} = (\tilde{\theta}_{ij})_{m \times n} \).

The proposed method is as follows.

In decision making process, there are cases that attribute weights are partly known. Generally, partly known attribute weight information can be expressed as: (1) \( \{w_i \geq w_j\} \), (2) \( \{w_i - w_j \geq e_i(e_i > 0)\} \), (3) \( \{w_i \geq \alpha w_j\}, 0 \leq \alpha_i \leq 1 \), (4) \( \{\beta_j \leq w_i \leq \beta_i + e_j\}, 0 \leq \beta_j < \beta_i + e_j \leq 1 \), (5) \( \{w_i - w_j \geq w_i - w_j\}, i \neq j \neq k \neq l \). The partly known attribute weights can be represented as \( \hat{H} \). The weights can be calculated according to the principle that reasonable weights should make the similarity \( \hat{S}(\hat{H}, \hat{\theta}^+) \) of \( A_i (i = 1, 2, \ldots, n) \) to the q-rung orthopair fuzzy positive ideal solution (PIS) \( \hat{\theta}^+ \) as large as possible and make the similarity \( \hat{S}(\hat{H}, \hat{\theta}^-) \) of \( A_i (i = 1, 2, \ldots, n) \) to the q-rung orthopair fuzzy negative ideal solution (NIS) \( \hat{\theta}^- \) as small as possible at the same time. Hence the following linear programming model (M-1) can be set up to determine attribute weights.

**Algorithm**

**Step 1.** Decision makers evaluate alternatives with respect to attributes with q-rung orthopair fuzzy evaluation values. The q-rung orthopair hesitant fuzzy evaluation decision matrix is got as \( \tilde{D}' = (\tilde{\theta}_{ij})_{m \times n} \) since decision makers can refuse to give any evaluation values if they are not familiar with the attributes.

**Step 2.** Extend decision matrix \( \tilde{D}' = (\tilde{\theta}_{ij})_{m \times n} \) according to method proposed in Section II. The extended decision matrix is represented as \( \tilde{D} = (\tilde{\theta}_{ij})_{m \times n} \), \( \tilde{\theta}_{ij} = (\hat{h}_{ij}, \hat{g}_{ij}) \).

**Step 3.** Determine the q-rung orthopair hesitant fuzzy positive ideal solution (q-ROHPIS) \( \hat{\theta}^+ \) and the q-rung orthopair hesitant fuzzy negative ideal solution (q-ROFHNIS) \( \hat{\theta}^- \) as follows:

\[
\hat{\theta}^+ = (\hat{\theta}^+_1, \hat{\theta}^+_2, \ldots, \hat{\theta}^+_n) = (\langle h_{1}, g_{1}^{*} \rangle, \langle h_{2}, g_{2}^{*} \rangle, \ldots, \langle h_{n}, g_{n}^{*} \rangle)
\]

\[
= (\langle \max_{ij} \tilde{\theta}_{ij} \in J_1 \rangle, (\min_{ij} \tilde{\theta}_{ij} \in J_2 \rangle)
\]

\[
\hat{\theta}^- = (\hat{\theta}^-_1, \hat{\theta}^-_2, \ldots, \hat{\theta}^-_n)
\]

\[
= (\langle h_{1}, g_{1}^{*} \rangle, \langle h_{2}, g_{2}^{*} \rangle, \ldots, \langle h_{n}, g_{n}^{*} \rangle)
\]

\[
= (\langle \min_{ij} \tilde{\theta}_{ij} \in J_1 \rangle, (\max_{ij} \tilde{\theta}_{ij} \in J_2 \rangle)
\]

**Step 4.** To solve (M-2), we construct the Lagrange function \( L(W, \lambda) \) as follows.

\[
L(W, \lambda) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_j \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+) + \frac{\lambda}{2} \sum_{j=1}^{n} (w_j - 1).
\]

Differentiate \( L(W, \lambda) \) with respect to \( w_j \) and \( \lambda \) and set these partial derivatives \( L_{w_j} \) and \( L_{\lambda} \) equal to zero to get

\[
\begin{align*}
\frac{\partial L}{\partial w_j} &= \sum_{i=1}^{m} \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+) + \lambda w_j = 0, \\
\frac{\partial L}{\partial \lambda} &= \frac{1}{2} \left( \sum_{j=1}^{n} w_j^2 - 1 \right) = 0.
\end{align*}
\]

Then we get the following equation

\[
w_j = \frac{\sum_{i=1}^{m} \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+)}{\sqrt{\sum_{j=1}^{n} (\sum_{i=1}^{m} \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+))^2}}.
\]

Then we normalize the weights to get

\[
w_j = \frac{\sum_{i=1}^{m} \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+)}{\sum_{j=1}^{n} \sum_{i=1}^{m} \hat{S}(\hat{\theta}_{ij}, \hat{\theta}^+) + \lambda/2}.
\]
Step 4. Calculate the similarity degree \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) \) between evaluation values \( \bar{\theta}_i \) and \( \bar{\theta}_j^+ \) as defined as
\[
\hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) = 1 - \left( \frac{1}{2} \left( \frac{1}{l(h_C)} \sum_{k=1}^{l(h_C)} |(\mu_{\bar{\theta}_j}^{\sigma(k)}) - (\mu_{\bar{\theta}_j^+}^{\sigma(k)})|^\lambda \right) + \frac{1}{l(g_C)} \sum_{k=1}^{l(g_C)} |(v_{\bar{\theta}_j}^{\sigma(k)}) - (v_{\bar{\theta}_j^+}^{\sigma(k)})|^\lambda \right)^{1/\lambda}.
\]
Then the weighted similarity \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) \) between evaluation values of alternative \( A_i \) and NIS \( \bar{\theta}_j^- \) is defined as
\[
\hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) = \sum_{j=1}^{n} w_j \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) = \frac{1}{n} \sum_{j=1}^{n} w_j \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-)
\]
where \( \bar{\theta}_i = (\bar{\theta}_{i1}, \bar{\theta}_{i2}, \ldots, \bar{\theta}_{im}) \), \( \bar{\theta}_j^- = (\bar{\theta}_{j1}, \bar{\theta}_{j2}, \ldots, \bar{\theta}_{jm}) \), \( \bar{\theta}_j^+ = (\bar{\theta}_{j1}^+, \bar{\theta}_{j2}^+, \ldots, \bar{\theta}_{jm}^+) \).

Step 5. Calculate the similarity degree \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) \) between evaluation values \( \bar{\theta}_j^- \) and \( \bar{\theta}_j^- \) as defined as
\[
\hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) = 1 - \left( \frac{1}{2} \left( \frac{1}{l(h_C)} \sum_{k=1}^{l(h_C)} |(\mu_{\bar{\theta}_j}^{\sigma(k)}) - (\mu_{\bar{\theta}_j}^{\sigma(k)})|^\lambda \right) + \frac{1}{l(g_C)} \sum_{k=1}^{l(g_C)} |(v_{\bar{\theta}_j}^{\sigma(k)}) - (v_{\bar{\theta}_j}^{\sigma(k)})|^\lambda \right)^{1/\lambda}.
\]

Step 6. For partly known attribute weights, we can determine the attribute weights by using the model (M-2). For completely unknown attribute weights, we can determine the attribute weight by using Eq. (13).

Step 7. Calculate the similarity degree \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) \) based on Eq. (17).

Step 8. Calculate the similarity degree \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) \) based on Eq. (19).

Step 9. Calculate the relative closeness \( RC_i \) of alternative \( A_i \) with respect to \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) \) and \( \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-) \) as follows:
\[
RC_i = \frac{\hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) + \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-)}{\hat{S}(\bar{\theta}_i, \bar{\theta}_j^+) + \hat{S}(\bar{\theta}_i, \bar{\theta}_j^-)},
\]
where \( RC_i \in [0, 1], i = 1, 2, \ldots, m \). Rank alternatives \( A_i (i = 1, 2, \ldots, n) \) according to the ranking of the \( RC_i (i = 1, 2, \ldots, n) \) and select the alternative with the largest \( RC_i \).

IV. NUMERICAL EXAMPLE

In this section, a case study concerning problem of purchase self-service book sterilizer problem is given to illustrate the new proposed method.

A. NUMERICAL EXAMPLE

Books in library are read by different readers. Books are easy to be contaminated in the process of reading and viruses and bacteria can pass in different readers. Bookworm bacteria and other bacteria are also produced when books are stored in library. In order to ensure the health of readers and extend life of books, library is planed to buy several self-service book sterilizers. After pre-valuation, there are still five suppliers left: A1—Longdian electric equipment co., Ltd, A2—Funuo Technology Co., Ltd, A3—Aidixun Technology Co., Ltd, A4—Runlian Technology Development Co., Ltd, A5—HuaTong Information Technology Co., Ltd. Four attributes are considered as: C1—price, C2—size, C3—efficiency, C4—service. Several experts from library and university have been invited to make decision. Alternatives are ranked by using the proposed method.

Step 1. The decision makers evaluate alternatives by the q-rung orthopair hesitant fuzzy evaluation values and the decision matrix is formed as \( D' = (\bar{\theta}_{ij})_{5 \times 4} \) in Table 1.

Step 2. Assume decision makers are risk-averse, the decision matrix \( D' \) is extended accordingly. The smallest membership and the largest non-membership are added. The extended is given as \( D = (\bar{\theta}_{ij})_{5 \times 4} \) in Table 2.

Step 3. Since the attributes are all benefit attributes, the q-rung orthopair hesitant fuzzy positive ideal solution \( \bar{\theta}^+ \) and the q-rung orthopair hesitant fuzzy negative ideal solution \( \bar{\theta}^- \) can be determined as \( \bar{\theta}^+ = (\bar{\theta}_{i1}^+, \bar{\theta}_{i2}^+, \bar{\theta}_{i3}^+, \bar{\theta}_{i4}^+) = (\bar{\theta}, \bar{\theta}, \bar{\theta}, \bar{\theta}) \).

\[
\begin{array}{cccc}
\bar{\theta}_1 & \bar{\theta}_2 & \bar{\theta}_3 & \bar{\theta}_4 \\
0.6 & 0.7 & 0.8 & 0.9 \\
0.5 & 0.6 & 0.7 & 0.8 \\
0.4 & 0.5 & 0.6 & 0.7 \\
0.3 & 0.4 & 0.5 & 0.6 \\
0.2 & 0.3 & 0.4 & 0.5 \\
\end{array}
\]
Step 7. The weighted similarity $\hat{S}(\hat{\theta}_i, \hat{\theta}^+) \text{ can be calculated by using eq.(17) to get}$
$$
\hat{S}(\hat{\theta}_1, \hat{\theta}^+) = 0.8310, \hat{S}(\hat{\theta}_2, \hat{\theta}^+) = 0.8263, \hat{S}(\hat{\theta}_3, \hat{\theta}^+) = 0.8783, \hat{S}(\hat{\theta}_4, \hat{\theta}^+) = 0.8678, \hat{S}(\hat{\theta}_5, \hat{\theta}^+) = 0.8415.
$$
Step 8. The weighted similarity $\hat{S}(\hat{\theta}_i, \hat{\theta}^-)$ can be calculated by using eq.(19) to get
$$
\hat{S}(\hat{\theta}_1, \hat{\theta}^-) = 0.8787, \hat{S}(\hat{\theta}_2, \hat{\theta}^-) = 0.8834, \hat{S}(\hat{\theta}_3, \hat{\theta}^-) = 0.8314, \hat{S}(\hat{\theta}_4, \hat{\theta}^-) = 0.8419, \hat{S}(\hat{\theta}_5, \hat{\theta}^-) = 0.8682.
$$
Step 9. The relative closeness $RC_I$ of alternative $A_i$ can be calculated as
$$
RC_1 = 0.4861, \quad RC_2 = 0.4833, \quad RC_3 = 0.5137, \quad RC_4 = 0.5076, \quad RC_5 = 0.4922.
$$
Step 10. The alternatives can be ranked according to the ranking of $R_i$ to get
$$
A_3 > A_4 > A_5 > A_1 > A_2.
$$
Then the optimal alternative is $A_3$.

### B. THE SENSITIVITY ANALYSIS

If other $q, \lambda$ are used in the similarity function, the final results are shown in Table 3. We can see that the optimal alternative are the same in most cases except for $q = 2, \lambda = 4$ and $q = 3, \lambda = 4$. The ranking results are slightly different for different $\lambda, q$. With the increasing of $\lambda, q$, evaluation values with large similarity and weight play more and more important role in decision making process. If the attribute weights are unknown completely, the first five steps are the same as above. Then we calculate the attribute weight vector by using eq. (13) to get $W = (0.2424, 0.2573, 0.2555, 0.2448)$. Then $S(\hat{\theta}_i, \hat{\theta}^+)$ can be calculated as $\hat{S}(\hat{\theta}_1, \hat{\theta}^+) = 0.8257, \hat{S}(\hat{\theta}_2, \hat{\theta}^+) = 0.8351, \hat{S}(\hat{\theta}_3, \hat{\theta}^+) = 0.8661, \hat{S}(\hat{\theta}_4, \hat{\theta}^+) = 0.8699, \hat{S}(\hat{\theta}_5, \hat{\theta}^+) = 0.8498$. Similarly, $\hat{S}(\hat{\theta}_i, \hat{\theta}^-)$ can be calculated as $\hat{S}(\hat{\theta}_1, \hat{\theta}^-) = 0.8831, \hat{S}(\hat{\theta}_2, \hat{\theta}^-) = 0.8737, \hat{S}(\hat{\theta}_3, \hat{\theta}^-) = 0.8426, \hat{S}(\hat{\theta}_4, \hat{\theta}^-) = 0.8419, \hat{S}(\hat{\theta}_5, \hat{\theta}^-) = 0.8590$. The relative closeness $RC_I$ can be calculated as $RC_1 = 0.4832, RC_2 = 0.4887, RC_3 = 0.5069, RC_4 = 0.5073, RC_5 = 0.4973$. The alternatives can be ranked accordingly as $A_4 > A_3 > A_5 > A_2 > A_1$. The optimal alternative is $A_4$. We can see that the ranking results are slightly different in the two methods. The reason for this is that weight vectors are differently in the two methods due to the reason that they are determined according to different principles. If other $q, \lambda$ are used in the similarity function in
this case, we can get the results shown in Table 4. The optimal alternative is \( A_4 \) in most cases except for \( \lambda = 3, q = 3 \). In this case, the difference of the attribute weights is very small. Hence the difference of similarity plays more important role in decision making process.

### C. COMPARISON ANALYSIS

We can also compare the proposed method with some other methods including the method based on TOPSIS and the methods based on aggregation operators.

If we rank alternatives by using the TOPSIS method, we should first determine the \( q \)-rungh orthopair hesitant fuzzy positive ideal solution as \( \tilde{\theta}^+ = (\tilde{\theta}_1^+, \tilde{\theta}_2^+, \tilde{\theta}_3^+, \tilde{\theta}_4^+) = (\langle, <, <, <, 0.3, 0.8, 0.1, 0.2 > \rangle, \langle, <, <, <, 0.7, 0.8, 0.3, 0.5 > \rangle, \langle, <, 0.3, 0.7, 0.2, 0.3 > \rangle ) \) and the \( q \)-rungh orthopair hesitant fuzzy negative ideal solution \( \tilde{\theta}^- = (\tilde{\theta}_1^-, \tilde{\theta}_2^-, \tilde{\theta}_3^-, \tilde{\theta}_4^-) = (\langle, <, <, <, 0.3, 0.5, 0.7, 0.8 > \rangle, \langle, <, <, <, 0.3, 0.5, 0.4, 0.5 > \rangle, \langle, <, 0.3, 0.4, 0.6, 0.8 > \rangle ) \). Calculate the distance of each evaluation value \( \tilde{\theta}_j \) to \( \tilde{\theta}^+ \) and \( \tilde{\theta}^- \). The distance measure is defined as

\[
\tilde{d} (\tilde{\theta}_j, \tilde{\theta}^+) = \frac{1}{2} \left( \frac{1}{L_j} \sum_{k=1}^{L_j} |\mu_{j\theta(k)} - \mu_{\tilde{\theta}^+(k)}| + \frac{1}{K_j} \sum_{i=1}^{K_j} |\nu_{j\theta(i)} - \nu_{\tilde{\theta}^+(i)}| \right) \quad \text{and} \quad \tilde{d} (\tilde{\theta}_j, \tilde{\theta}^-) = \frac{1}{2} \left( \frac{1}{L_j} \sum_{k=1}^{L_j} |\mu_{j\theta(k)} - \mu_{\tilde{\theta}^-(k)}| + \frac{1}{K_j} \sum_{i=1}^{K_j} |\nu_{j\theta(i)} - \nu_{\tilde{\theta}^- (i)}| \right).
\]

We still take the weight vector as \((0.2000, 0.1750, 0.3500, 0.2250)\). Then the weighted distance of \( A_1 \) to \( \tilde{\theta}^+ \) and \( \tilde{\theta}^- \) can be defined as

\[
\tilde{d} (\tilde{\theta}_j, \tilde{\theta}^+) = \sum_{j=1}^{n} w_j \tilde{d} (\tilde{\theta}_j, \tilde{\theta}^+) \quad \text{and} \quad \tilde{d} (\tilde{\theta}_j, \tilde{\theta}^-) = \sum_{j=1}^{n} w_j \tilde{d} (\tilde{\theta}_j, \tilde{\theta}^-).
\]

By using the above equation, we can get an order \( \tilde{d} (\tilde{\theta}_1, \tilde{\theta}^+) = 0.2358, \tilde{d} (\tilde{\theta}_2, \tilde{\theta}^+) = 0.2143, \tilde{d} (\tilde{\theta}_3, \tilde{\theta}^+) = 0.1830, \tilde{d} (\tilde{\theta}_4, \tilde{\theta}^+) = 0.1383, \tilde{d} (\tilde{\theta}_1, \tilde{\theta}^-) = 0.1590, \tilde{d} (\tilde{\theta}_2, \tilde{\theta}^-) = 0.3192, \tilde{d} (\tilde{\theta}_3, \tilde{\theta}^-) = 0.5659, \tilde{d} (\tilde{\theta}_4, \tilde{\theta}^-) = 0.5702, \tilde{d} (\tilde{\theta}_1, \tilde{\theta}^-) = 0.4178. \) The relative closeness coefficients can be calculated as

\[
CC_1 = \frac{\tilde{d} (\tilde{\theta}^+, \tilde{\theta}_1)}{\tilde{d} (\tilde{\theta}^+, \tilde{\theta}_1)} = (i = 1, 2, \ldots, 5).
\]

The results are calculated as \( CC_1 = 0.4027, CC_2 = 0.3192, CC_3 = 0.5659, CC_4 = 0.5702, CC_5 = 0.4178 \). Then alternatives can be ranked as \( A_4 > A_3 > A_5 > A_4 > A_2 \). If the \( q \)-rungh orthopair hesitant fuzzy positive averaging (q-ROHFPA) operator is used to aggregate evaluation values, we can get \( \tilde{\theta}_1 = \langle 0.5852, 0.6067, 0.6465, 0.6638, 0.6025, 0.6227, 0.6607, 0.6768 \rangle \), \( \tilde{\theta}_2 = \langle 0.5937, 0.6055 \rangle \), \( \tilde{\theta}_3 = \langle 0.6244, 0.6695, 0.6781, 0.7153, 0.6922, 0.7273, 0.7241, 0.7636 \rangle \), \( \tilde{\theta}_4 = \langle 0.6350, 0.6589, 0.6924, 0.7015, 0.6895, 0.3031, 0.3481, 0.3223, 0.3703, 0.3352, 0.3850, 0.3064, 0.3520, 0.3186, 0.3660, 0.3389, 0.3893, 0.3524, 0.4048 \rangle \), \( \tilde{\theta}_5 = \langle 0.6526, 0.6626, 0.3126, 0.3529, 0.3494, 0.3945 \rangle \). Where \( \tilde{\theta}_i = \langle h_{\tilde{\theta}_i}, g_{\tilde{\theta}_i} \rangle = \langle \mu_{\tilde{\theta}_i}, \nu_{\tilde{\theta}_i} \rangle \), \( \mu_{\tilde{\theta}_i} = \mu_{\tilde{\theta}_i}^{\tilde{\theta}_i}, \nu_{\tilde{\theta}_i} = \nu_{\tilde{\theta}_i}^{\tilde{\theta}_i} \) and weight vector is also taken as \( (0.2000, 0.1750, 0.3500, 0.2250) \). Then we can rank alternatives as \( A_3 > A_5 > A_4 > A_2 > A_1 \).

If the \( q \)-rungh orthopair hesitant fuzzy weighted geometric averaging (q-ROHFWGA) operator is used to aggregate evaluation values, we can get \( \tilde{\theta}_1 = \langle 0.5462, 0.5611, 0.5723, 0.5879, 0.5664, 0.5819, 0.5935, 0.6098 \rangle \). Here we can rank alternatives as \( A_3 > A_5 > A_4 > A_2 > A_1 \).

Then we can rank alternatives as \( A_3 > A_5 > A_4 > A_2 > A_1 \). The q-ROHFGA operator is defined as

\[
\text{q-ROHFGA}(\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n) = \bigcup_{\mu_{i} \in h_{\tilde{\theta}_i}} \bigcup_{v_{i} \in v_{\tilde{\theta}_i}} ((1 - \frac{1}{(\sum_{j=1}^{n} w_j (1 - \mu_{j\theta}^{\tilde{\theta}_i})^{1/y})^{1/y}})^{1/y}) > .
\]

where \( \tilde{\theta}_i = \langle h_{\tilde{\theta}_i}, g_{\tilde{\theta}_i} \rangle = \langle \mu_{\tilde{\theta}_i}, \nu_{\tilde{\theta}_i} \rangle \). Here \( q, y \) can take different values. The weight vector is also taken as \( (0.2000, 0.1750, 0.3500, 0.2250) \). We only give the ranking results for different \( q, y \) in Table 5 for space limit. The concrete computation process are omitted here for space limit. From the results, we can see that in most cases the ranking is \( A_3 > A_5 > A_4 > A_5 > A_1 > A_2 \), which is the same as that of proposed method in partly known attribute weight situation. The optimal alternative is \( A_3 \) except for \( q = 4, 5 \) and \( q = 2, 3 \). \( A_3 \) becomes the sub-optimal alternative in the cases of \( q = 2, 4 = 4 \) and \( q = 3, 4 = 4 \). \( A_5 \) becomes the sub-optimal alternative in the cases of \( y = 1, q = 2, q = 3 \).
TABLE 3. Ranking results of different $q$, $\lambda$ for partly known attribute weights.

| $q$ | $\lambda$ | $S(\theta_1)$ | $S(\theta_2)$ | $S(\theta_3)$ | $S(\theta_4)$ | $S(\theta_5)$ | Ordering | Optimal alternative |
|-----|-----------|---------------|---------------|---------------|---------------|---------------|----------|---------------------|
| 2   | 1         | 0.4861        | 0.4833        | 0.5137        | 0.5076        | 0.4972        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 3   | 1         | 0.4871        | 0.4861        | 0.5123        | 0.5088        | 0.4931        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 4   | 1         | 0.4881        | 0.4885        | 0.5094        | 0.5080        | 0.4940        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 5   | 1         | 0.4892        | 0.4904        | 0.5069        | 0.5066        | 0.4948        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 2   | 2         | 0.4662        | 0.4646        | 0.5288        | 0.5178        | 0.4793        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 3   | 2         | 0.4713        | 0.4718        | 0.5260        | 0.5180        | 0.4830        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 4   | 2         | 0.4748        | 0.4779        | 0.5192        | 0.5149        | 0.4866        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 5   | 2         | 0.4781        | 0.4827        | 0.5134        | 0.5117        | 0.4895        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 2   | 3         | 0.4601        | 0.4607        | 0.5246        | 0.5104        | 0.4727        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 3   | 3         | 0.4699        | 0.4667        | 0.5263        | 0.5083        | 0.4766        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 4   | 3         | 0.4747        | 0.4780        | 0.5184        | 0.5104        | 0.4853        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 5   | 3         | 0.4789        | 0.4838        | 0.5128        | 0.5085        | 0.4896        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 2   | 4         | 0.4526        | 0.4519        | 0.5289        | 0.5298        | 0.4805        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 3   | 4         | 0.4654        | 0.4965        | 0.5240        | 0.5269        | 0.4892        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 4   | 4         | 0.4725        | 0.5003        | 0.5271        | 0.5247        | 0.4973        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 5   | 4         | 0.4775        | 0.5023        | 0.5217        | 0.5209        | 0.5013        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 2   | 5         | 0.4539        | 0.4673        | 0.5193        | 0.5040        | 0.4595        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 3   | 5         | 0.4685        | 0.4765        | 0.5231        | 0.5062        | 0.4721        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 4   | 5         | 0.4762        | 0.4842        | 0.5177        | 0.5065        | 0.4826        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |
| 5   | 5         | 0.4810        | 0.4893        | 0.5124        | 0.5059        | 0.4890        | $A_3 > A_4 > A_5 > A_1 > A_2$ | $A_3$    |

and $\gamma = 5$, $q = 2$, $q = 3$. The q-ROHFDA operator is more flexible comparing with some other aggregation operators.

Wang et al. [46] extended TOPSIS to q-rung orthopair hesitant fuzzy environment, in which attribute weights determined based on firefly algorithm and entropy. Some entropy formulas are given based on the similarity measures first. Attribute weights are determined from evaluation values by using the firefly algorithm and entropy formula. Then the alternatives are ranked by using the revised TOPSIS method considering the risk preference of decision makers. While in this paper, some similarity measures are defined based on the cosine similarity measure, which are different from [46] and different weight attribute situations are considered including the attribute weights are partly known and completely unknown. Linear programming method is used for the case of partly known attribute weight information. Nonlinear programming model is set up for the case of completely unknown attribute weight information. Finally, alternatives are ranked by using the TOPSIS method to avoid too much computation.

The main differences of the proposed method from the existing method are summarized in Table 6. The main advantages and superiorities of the proposed methods are as follows. First, the evaluation values are given as q-rung orthopair hesitant Fuzzy values, which are more accurate since decision makers can refuse to give evaluation values if they are not familiar with attributes or alternatives. Second, the proposed method can deal with problems with partly known attribute weight information or completely unknown attribute weight information or completely unknown attribute weight information or completely unknown attribute weight information or completely unknown attribute weight information or completely unknown attribute weight information.
TABLE 5. The aggregated results by using q-ROHFDWA operator for different $q$, $\gamma$.

| $q$ | $\gamma$ | Aggregation Results |
|-----|-----------|---------------------|
| 2   | 1         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 2   | 2         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 2   | 3         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 2   | 4         | $A_4 > A_3 > A_5 > A_1 > A_2$ |
| 2   | 5         | $A_4 > A_5 > A_3 > A_1 > A_2$ |

| $q$ | $\gamma$ | Aggregation Results |
|-----|-----------|---------------------|
| 3   | 1         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 3   | 2         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 3   | 3         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 3   | 4         | $A_4 > A_3 > A_5 > A_1 > A_2$ |
| 3   | 5         | $A_4 > A_5 > A_3 > A_1 > A_2$ |

| $q$ | $\gamma$ | Aggregation Results |
|-----|-----------|---------------------|
| 4   | 1         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 4   | 2         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 4   | 3         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 4   | 4         | $A_4 > A_3 > A_5 > A_1 > A_2$ |
| 4   | 5         | $A_4 > A_5 > A_3 > A_1 > A_2$ |

| $q$ | $\gamma$ | Aggregation Results |
|-----|-----------|---------------------|
| 5   | 1         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 5   | 2         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 5   | 3         | $A_3 > A_5 > A_4 > A_2 > A_1$ |
| 5   | 4         | $A_4 > A_3 > A_5 > A_1 > A_2$ |
| 5   | 5         | $A_4 > A_5 > A_3 > A_1 > A_2$ |

TABLE 6. Comparison of the proposed method with other methods.

| Methods          | Information given by q-rung fuzzy number | Whether information modeled by hesitant fuzzy set | Whether attribute weight vector is determined objectively | Whether different attribute weight situations are considered |
|------------------|-----------------------------------------|--------------------------------------------------|------------------------------------------------------|----------------------------------------------------------|
| Liu and Wang [22] | Yes                                     | No                                               | No                                                   | No                                                       |
| Wei et al. [24]  | Yes                                     | No                                               | No                                                   | No                                                       |
| Liao et al. [57] | No                                      | No                                               | Yes                                                  | No                                                       |
| Wang et al. [46] | Yes                                     | Yes                                              | Yes                                                  | No                                                       |
| Yang et al. [75] | No                                      | No                                               | No                                                   | No                                                       |
| Our proposed method | Yes                                   | Yes                                              | Yes                                                  | Yes                                                      |

information. Third, attribute weights determined by using the similarity function, linear programming method and nonlinear programming method are more reasonable comparing with the methods that determined by decision makers subjectively. Fourth, TOPSIS method is used to rank alternatives can reduce computational complexity.

V. CONCLUSION

In this paper, we develop a q-rung orthopair hesitant fuzzy multiple attribute decision making method based on the linear programming model and idea of TOPSIS. We first give the methods to determine attribute weights for situations of partly known attribute weight and completely unknown attribute weight, respectively. Then we give a new q-rung orthopair hesitant fuzzy multiple attribute decision making algorithm based on the above method. The new algorithm has the following advantages: (1) Evaluation values are given as hesitant q-rung orthopair hesitant fuzzy values. Comparing with other tool to model fuzzy information, q-rung orthopair hesitant fuzzy value is more accurate and flexible. Decision makers can refuse to give evaluation values if they are not familiar with the attributes. Hence more accurate decision results can be got. (2) The attribute weights determined by using the linear programming model (M-2) or eq. (11) based on the similarity function are more objective. (3) The new method can be used to solve decision-making problems with q-rung orthopair hesitant fuzzy values, q-rung orthopair fuzzy values, Pythagorean hesitant fuzzy values, Pythagorean fuzzy values, etc. Hence, the proposed method has more wider application. Finally, the new algorithm is presented to solve new campus site selection problem.

In the future, we will apply our proposed our method to other fields including the evaluation of metro construction project, global supplier selection, emergency decision making problem, energy policy evaluation, etc. We will extend the proposed method to accommodate some other special features such as different types of the evaluation values, correlation of the attributes, incomplete preference information, etc. In addition, other methods to determine attribute weights also worth to further study.

CONFICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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