Backreaction of accreting matter onto a black hole in the Eddington–Finkelstein coordinates

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Abstract

We study backreaction of accreting matter onto a spherically symmetric black hole in a perturbative way, when accretion is in a quasi-steady state. General expressions for corrections to the metric coefficients are found in the Eddington–Finkelstein coordinates. It is shown that near the horizon of a black hole, independently of the form of the energy–momentum tensor, the leading corrections to the metric are of the Vaidya form. The relation to other solutions is discussed and particular examples are presented.

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1. Introduction

The study of spherically symmetric black hole accretion has a long story. In Newtonian gravity, first the problem of matter accretion onto compact objects has been formulated in a self-similar manner by Bondi [1]. Later, Michel investigated the steady-state spherically symmetric flow of a test gas onto a Schwarzschild black hole in the framework of general relativity [2]. Accretion of a perfect fluid with a general equation of state onto a Schwarzschild black hole has been investigated in [3, 4], and a similar analysis for a charged black hole has been done in [5]. In [6], an exact analytic solution of non-spherical accretion of a stiff fluid onto a rotating black hole was derived, and a generalization of this analysis to the case of a rotating charged black hole was done in [7]. Other studies of spherically symmetric accretion of different types of fluids onto black holes and wormholes were undertaken in a number of works [8]. See also an excellent review on black holes in FRW universe [10].

Usually in the analytic study of the problem of accretion, an important simplification is made—backreaction of an accreting matter is neglected. In practice, this assumption means that the accreting matter is very light. It is, however, of a great interest to investigate backreaction effects on the metric of a black hole. To solve this problem in a full way is notoriously difficult, and only a few analytic solutions are known, where backreaction is taken into account.
In particular, the celebrated Lemaître–Tolman solution describes a spherically symmetric collapsing dust [11]. Another well-known solution is the Vaidya solution [12], which may describe a black hole surrounded by radially moving infalling photons. Generalizations of the Vaidya solution to include more general energy–momentum tensor are possible, see e.g. [13].

In this paper, we take a different approach to attack the problem of backreaction of accreting matter. Instead of looking for particular exact solutions, we implement a perturbative scheme: for an accreting matter with general energy–momentum tensor, we find corrections to the metric due to backreaction. The scheme goes as follows. Zero approximation is a solution for an accreting matter in the test fluid approximation, i.e. when backreaction of accreting matter is neglected. The first correction to the metric is found from the Einstein equations, where the source term is computed using zeroth-order solution for matter. Although the perturbative approach is not designed to find exact solutions, neither to take into account strong backreaction effects, it allows us to study the problem in general manner, without relying on a specific choice of the energy–momentum tensor. We also note that a similar approach was implemented in [14] for calculating backreaction effects using the Schwarzschild (static) coordinates. Here, instead, we use ingoing Eddington–Finkelstein coordinates, in which case we avoid problems related to the singularity of the coordinate system at the horizon of a black hole.

The paper is organized as follows. In section 2.1, the general perturbation scheme is described; in section 2.2, corrections to the metric are found for the case of accretion onto a Schwarzschild black hole according to the scheme, and in section 2.3, those for the case of a Reissner–Nordström black hole. In section 2.4, the results are applied to accretion of a perfect fluid with an arbitrary equation of state. The shift of the apparent horizon is discussed in section 2.5. In section 3, we compare corrections found with the use of the perturbative approach to known exact solutions. It is emphasized that near the horizon, corrections to the metric are of the Vaidya form, independently of the energy–momentum tensor of the accreting matter. Section 4 contains particular examples: in section 4.1, corrections are computed for accretion of a stiff fluid while in section 4.2, those for accretion of a canonical scalar field and a Galileon field. We conclude in section 5.

2. Corrections to the metric

2.1. Perturbation scheme

In the general case of a black hole accretion, the full solution to the Einstein equations are not known. Indeed, there are only a few special cases of the energy–momentum tensor for which the exact solutions were found, e.g. the Lemaître–Tolman [11] and the Vaidya [12] solutions. Here we proceed with the perturbative approach to the problem. We find corrections to the metric due to accretion of matter with a general energy–momentum tensor, \( T_{\mu\nu} \), in the (almost) steady-state regime of accretion. In this regime, the energy–momentum tensor is a function of the radial coordinate only. The solution for the energy–momentum tensor \( T_{\mu\nu}(r) \) is assumed to be found in the test fluid approximation, i.e. without taking into account backreaction. Then, using \( T_{\mu\nu}(r) \) at zeroth-order approximation, we can find first-order corrections to the metric due to the backreaction of matter.

More precisely, our approach is as follows. The full set of equations consist of the Einstein equations

\[
G_{\mu\nu}[g_{\mu\nu}] = 8\pi T_{\mu\nu}[g_{\mu\nu}, \phi].
\]
and the equations of motion for matter
\[ \mathcal{E}[\phi, g_{\mu\nu}] = 0, \]  
where by \( \phi \) we denote collectively the degrees of freedom for the accreting matter. For example, in the case of accretion of a single scalar field, \( \phi \) is a scalar field itself. The equations of motion for matter (2) can be obtained from the Bianchi identities; therefore, they are not independent, but in practice it is convenient to keep them.

A zeroth-order approximation is found by neglecting backreaction of the matter in equation (1). Put differently, the solution in this approximation is the vacuum solution for the metric, \( g_{\mu\nu}^{(0)} = g_{\mu\nu}^{\text{vac}} \), so that
\[ G_{\mu\nu}^{(0)} = 0. \] (3)

In the same approximation, the solution for the matter field(s), \( \phi(0) \), is found from
\[ \mathcal{E}[\phi(0), g_{\mu\nu}^{(0)}] = 0. \] (4)

Solutions at this zeroth-order approximation has been used in the literature extensively.

A step beyond the test-fluid approximation is to find corrections to the metric due to backreaction of matter. In a first-order approximation, we can substitute \( \phi(0) \) and \( g_{\mu\nu}^{(0)} \) on the rhs of (1), so that the Einstein equations take the form
\[ G_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} = 8\pi T_{\mu\nu}^{(0)} \] (5)

Assuming that \( g_{\mu\nu}^{(1)} \) is small with respect to \( g_{\mu\nu}^{(0)} \), one can linearize the above equation to obtain a system of linear partial differential equations. We will implement this approach below.

### 2.2. Schwarzschild black hole

The general spherically symmetric line element can be written in the following form (see, e.g., [15, 16]):
\[ ds^2 = e^{\nu(V,r)} \left[ dV^2 - \frac{2}{r} \frac{dV}{r} dr - r^2 d\Omega \right], \] (6)

where \( \nu(V, r) \) and \( \lambda(V, r) \) are two arbitrary functions. This is a coordinate frame of the Eddington–Finkelstein (EF) type, related to the infalling null (photon) geodesics. The metric (6) is similar to the one introduced by Bondi et al [17] using ‘radiation coordinates’. The vacuum Schwarzschild solution is recovered by setting \( \lambda = 0 \) and \( e^{\nu(V, r)} = 1 - 2M_0/r \), where \( M_0 \) is the mass of the black hole, so that
\[ ds^2_{\text{vac}} = \left( 1 - \frac{2M_0}{r} \right) dV^2 - 2 dV dr - r^2 d\Omega. \] (7)

Note that in our conventions equation (7) is the solution for the metric at zeroth-order approximation.

Analogously, instead of the metric coefficient \( \nu(V, r) \) we will use a function \( M(V, r) \), defined as follows:
\[ e^{\nu(V,r)} \equiv 1 - \frac{2M(V, r)}{r}, \] (8)

so that at zeroth-order approximation, \( M(V, r) = M_0 = \text{const} \).

Plugging (8) and (6) into the perturbed Einstein equations (5), we obtain the following system of equations (other components of (5) are trivial):
\[ 8\pi T_0^0 = -e^\nu \left( \frac{1}{r^2} + \frac{\nu'}{r} + \frac{1}{r^2} \right) \] (9)
\[ 8\pi T_0^1 = e^\nu \frac{\nu'}{r}, \] (10)
\[ 8\pi T_0^0 = \frac{2(e^{-\lambda})'}{r}, \quad (11) \]

\[ 8\pi T_0^1 = -e^\nu \left( \frac{1}{r^2} + \frac{v'}{r} \right) + \frac{1}{r^2} - \frac{2\lambda'}{r} e^\nu, \quad (12) \]

\[ 8\pi T_2^2 = 8\pi T_3^3 = -e^\nu \left( \frac{\lambda'' + v''}{2} \right) - e^{-\lambda} \dot{\lambda}' - e^\nu \left( \frac{\lambda'^2}{2} + \frac{\lambda' + v'}{r} + \frac{3}{2} \lambda' v' \right), \quad (13) \]

where dots stand for $\frac{\partial}{\partial V}$ and primes denote $\frac{\partial}{\partial r}$. Note that according to the general scheme, see equation (5), the lhs of equations (9)–(13) contain components of the energy–momentum tensor taken at zeroth-order approximation, i.e. on solutions of the matter field(s) when backreaction is neglected. Not all components of the above system of equation are independent—using the Bianchi identities it can be shown that equation (13) is a combination of equations (9)–(12). Thus, we will not take equation (13) into account below.

Substituting (8) into (9) and (10), we find

\[ M' = 4\pi T_0^0 r^2, \quad (14) \]

and

\[ M = A, \quad (15) \]

correspondingly, where in the last equation we introduced a notation

\[ A \equiv -4\pi T_0^1 r^2 \quad (16) \]

for the total flux of energy crossing some radius $r$. The rhs of equations (14) and (15) are taken at zeroth-order approximation. The components of the energy–momentum tensor are time independent in the given approximation. The flux $A$ does not depend on $r$ as well, i.e. $A = \text{const}$. For each particular case of black hole accretion, the value of $A$ is calculated by solving the equations of motion in a fixed metric (normally by finding integrals of motion). Additional physical requirements can be used to fix solutions. For example, in the case of a perfect fluid, the flux $A$ is fixed at the critical point [2, 3].

Integrating equations (14) and (15), we find

\[ M(V, r) = C_0 + AV + 4\pi \int_{r_0}^r T_0^0(r)r^2 dr, \quad (17) \]

where $C_0$ is a constant of integration and $r_0$ is the horizon locus. Similarly, integrating equation (11), in the limit $\lambda \ll 1$, one obtains

\[ \lambda(r) = -4\pi \int_{r_0}^r T_1^0 r dr + s(V), \quad (18) \]

where $s(V)$ is an arbitrary function. Note that integration of equation (12) gives the same result for $\lambda$, equation (18). In equation (18), the function $s(V)$ can be set to zero by redefinition of the time coordinate, $e^{(V)} dV \rightarrow dV$. The constant $C_0$, in turn, can be identified with $M_0$ by an appropriate shift $V \rightarrow V + \text{const}$. Therefore, equations (17) and (18) can be rewritten as follows:

\[ M(V, r) = M_0 + AV + 4\pi \int_{r_0}^r T_0^0(r)r^2 dr, \quad (19) \]

\[ \lambda(r) = -4\pi \int_{r_0}^r T_1^0 r dr. \quad (20) \]

The above equations are the central result of this work.
If the components of the energy–momentum tensor are slowly varying functions of the radial coordinate (which is a reasonable assumption for non-pathological matters), then from (19) and (20) one can find the expressions for the corrections of the metric in the vicinity of the black hole horizon

$$M(V, r) = M_0 + AV + 4\pi r_0^2(r - r_0)T_0^0 \bigg|_{r = r_0},$$  \hspace{1cm} (21)$$

$$\lambda(r) = -4\pi r_0(r - r_0)T_1^0 \bigg|_{r = r_0}. \hspace{1cm} (22)$$

According to our perturbative scheme, the above equations (19) and (20) are only valid if the corrections are small. Thus, we require

$$|AV| \ll M_0, \quad \left| 4\pi \int_{r_0}^{r} T_0^0(r) r^2 \, dr \right| \ll M_0, \quad \left| 4\pi \int_{r_0}^{r} T_1^0 r \, dr \right| \ll 1. \hspace{1cm} (23)$$

The first of the inequalities in (23) is simply a condition that the total flux of the infalling matter is smaller than the bare black hole mass. Similarly, the second condition says that the total mass of the matter inside a radius \( r \) is smaller than the bare mass of the black hole. The last condition in (23) has no obvious interpretation and it reflects the fact that the correction to the non-diagonal part of the metric due to the presence of infalling matter must not be too big.

### 2.3. Reissner–Nordström black hole

We can generalize our results to the case of the Reissner–Nordström black hole. Accordingly, instead of (8) we now write the metric coefficient \( \nu \) in the following form:

$$e^{\nu(V, r)} = 1 - \frac{2M(V, r)}{r} + \frac{Q^2}{r^2}, \hspace{1cm} (24)$$

where \( Q \) is a charge of a black hole. We assume that the accreting fluid carries no electric charge, otherwise the total charge of the black hole needs to be modified as well. Substituting (24) into (9)–(12) and taking into account a contribution of the electromagnetic field in the total energy–momentum tensor,

$$T_0^0 \rightarrow T_0^0 + \frac{Q^2}{8\pi r^4}, \quad T_1^1 \rightarrow T_1^1 + \frac{Q^2}{8\pi r^4},$$

$$T_2^2 \rightarrow T_2^2 - \frac{Q^2}{8\pi r^4}, \quad T_3^3 \rightarrow T_3^3 - \frac{Q^2}{8\pi r^4}, \hspace{1cm} (25)$$

after lengthy but straightforward calculations one can check that the expressions for the metric coefficients \( M(V, r) \) and \( \lambda(V, r) \) are the same as in the case of the Schwarzschild black hole, so that the results are given by equations (19) and (20), where \( r_0 \) has now the meaning of the event horizon of a bare charged black hole.

### 2.4. Accretion of a perfect fluid

In this section, we consider accretion of a perfect fluid onto a black hole. The stress tensor of a perfect fluid is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \hspace{1cm} (26)$$

where \( \rho \) is the energy density and \( p \) is the pressure of the fluid in the comoving coordinates, and \( u^\mu \) is a 4-velocity of the fluid. In the EF coordinates, the 4-velocity is given by

$$u^\mu = \left( \frac{1}{\sqrt{f_0 + u^2}}, -u, 0, 0 \right), \hspace{1cm} (27)$$
where \( u \equiv |dr/ds| > 0 \) is the absolute value of the radial component of 4-velocity in the static coordinates and \( f_0 \equiv 1 - 2M_0/r \). It is easy to check that all the components of \( u^\mu \) and \( u_{\mu} \) are non-divergent at the horizon. The relevant components of the energy–momentum tensor are

\[
T^0_0 = \rho \sqrt{f_0 + u^2} - pu \sqrt{f_0 + u^2} + u , \quad T^1_0 = -\frac{\rho + p}{\sqrt{f_0 + u^2}}. \tag{28}
\]

In the vicinity of the horizon, \( f_0 \rightarrow 0 \), we find

\[
T^0_0 \rightarrow \frac{1}{2}(\rho - p), \quad T^1_0 \rightarrow -\frac{\rho + p}{4u^2}. \tag{29}
\]

Thus, from (21) and (22), we can write for the corrections

\[
M(V, r) \approx M_0 + A V + 2\pi r_0^2 (\rho - p)(r - r_0), \tag{30}
\]

\[
\lambda(r) \approx \pi r_0 \frac{\rho + p}{u^2} (r - r_0). \tag{31}
\]

The above expressions are valid for any perfect fluid in the vicinity of the horizon of a black hole. Note that the energy flux onto the black hole is given by (16), with \( T^{1}_0 = -(\rho + p)u \sqrt{f_0 + u^2} \) for a perfect fluid; see [3]. Therefore, as can be seen from (30), the accretion of phantom energy, \( \rho + p < 0 \), decreases the black hole mass. Thus, we confirm that the backreaction does not affect the results, which were reported previously in [3], where the zeroth-order approximation has been used.

Note that expression (30) implies, in particular, that light observers, having different \( V = \text{const} \), see the black hole with the different masses \( M \) (at different stages of accretion).

### 2.5. Shift of the apparent horizon

The position of the apparent horizon depends on the choice of a coordinate system. For the metric (6), it can be shown that the location of the apparent horizon, \( r_h \), can be found from the following equation [18]:

\[
e^{\nu(V, r)} = 0. \tag{32}
\]

The above expression can be obtained by requiring that for a radial outgoing photon \( dr/dV = 0 \). Indeed, from \( ds^2 = 0 \), one can find the two radial light geodesics

\[
dV = 0, \quad \frac{dr}{dV} e^{-\nu} = \frac{1}{2}. \tag{33}
\]

The apparent horizon satisfies the condition that the photons do not cross the \( r = \text{const} \) surfaces in the growing \( r \)-direction. It gives \( dr/dV = 0 \), and from (33) one obtains \( e^{\nu+\lambda} = 0 \) and hence \( e^{\nu(V, r)} = 0 \). The later is due to regularity of \( \lambda \). Now, for a Schwarzschild black hole from (32) and (19), we find

\[
M_0 + A V + 4\pi \int_{r_0}^{r_h} T^0_0(r)r^2 dr = \frac{r_h}{2}, \tag{34}
\]

which is an implicit equation for \( r_h \). It is not difficult to check that for small shifts of the apparent horizon the last term on the lhs of (34) can be neglected, since it is of the next order in the expansion, so that we find

\[
r_h \approx 2M_0 + 2AV. \tag{35}
\]

Thus, the leading term in the apparent horizon shift only depends on the total flux \( A \) and is independent on other components of the energy–momentum tensor.
Similarly, for the shift of the apparent horizon in the case of a charged black hole one has, instead of (34),

$$M_0 + AV + 4\pi \int_{r_h}^{r_0} T_0^0(r)r^2 dr = \frac{r_h}{2} + \frac{Q^2}{2r_h}.$$  

(36)

Again, neglecting the last term on the lhs of the above equation, we obtain, for the case of the Reissner–Nordström black hole

$$r_h \approx M_0 + AV + \sqrt{M_0^2 - Q^2 + 2M_0AV}.$$  

(37)

Note that for the extreme black hole, $M_0 = Q$, the existence of the apparent horizon depends on the sign of the flux, $A$. For phantom accretion, the apparent horizon ceases to exist for positive $V$, which means that our perturbation scheme breaks down for this particular case. The reason for the breaking down of our scheme is that accretion of phantom decreases the black hole mass; therefore, any amount of phantom matter converts the extreme black hole to a naked singularity, thus going beyond our quasi-steady-state approximation. On the other hand, if normal (non-phantom) matter accretes, then (37) can be perfectly applied. It is worth to mention that in the static Schwarzschild coordinates the test fluid approximation is violated for accretion of any type of matter (phantom and non-phantom) onto the extreme black hole; see [14].

3. Relation to other solutions

Having found the general form of the corrections to the metric due to backreaction, we can now compare our results to other solutions. In particular, it is interesting to see how our result is related to known full analytic solutions.

Let us first consider the case when matter is the cosmological constant term. In this case, it is easy to verify that $T_0^0 = \rho_\Lambda$, $T_{1}^0 = 0$ and $A = 0$, where $\rho_\Lambda$ is the energy density of the vacuum term. The general expressions (19) and (20) give then

$$M(V, r) = M_0 + \frac{4\pi}{3} \rho_\Lambda (r^3 - r_0^3), \quad \lambda = 0,$$  

(38)

which, after the redefinition $M_0 \rightarrow M_0 + 4\pi \rho_\Lambda r_0^3 / 3$, can be identified with the Schwarzschild–de Sitter solution.

As a second example, we consider the case of radially infalling photons, so that the energy–momentum tensor is of the form

$$T_{\mu}^\nu = -\frac{M}{4\pi r^2} k_\mu k^\nu.$$  

(39)

The only non-zero component of the energy–momentum tensor which contributes to the corrections of the metric (19) and (20) is $T_{1}^0$ corresponding to a non-zero total flux, $A \neq 0$, while $T_0^0 = T_{1}^0 = 0$. Thus,

$$M(V, r) = M_0 + AV, \quad \lambda = 0,$$  

(40)

which corresponds to the Vaidya solution [12] for the constant flux, $A = \text{const}$.

It is also interesting to observe that in the vicinity of the horizon of a Schwarzschild black hole, independently of the form of the energy–momentum tensor of the accreting fluid, the corrections to the metric are of the Vaidya form. Indeed, for $r = 2M_0$, from (19) and (20) we infer that $M(V, r = r_0) = AV, \lambda = 0$, which coincides with (40). This means, in particular, that the first correction to the motion of free particles in the vicinity of the horizon does not depend on the energy–momentum tensor of the accreting fluid: it depends on the total flux $\mathcal{A}$ only. One should note, however, that although the metric itself is asymptotically Vaidya
near the horizon, the derivatives of the metric are not the same (not even asymptotically) as in the Vaidya solution. Said differently, the Riemann tensor depends (even near the black hole horizon) on the type of the accreting matter.

4. Examples

4.1. Stiff fluid

At zeroth-order approximation, the steady-state accretion in the Schwarzschild coordinates corresponds to the steady-state solution in the EF coordinates. Therefore, we obtain the same integrals of motion (as a function of \( r \) only), the same expression for the critical points, and finally the same profile of density and velocity as in both coordinate systems. Thus, we can take results of [3], derived in the static coordinate system and apply them here. In particular, for the linear equation of state \( p = \rho - \rho_0 \), one has

\[
\rho = \rho_0 + \left( \rho_\infty - \rho_0 \right) \left( 1 + \frac{\rho_0}{r} \right) \left( 1 + \frac{\rho_0}{r^2} \right),
\]

where \( \rho_\infty \) is the energy density at infinity. The corresponding profile for the velocity \( u = u(r) \) can be calculated from the energy flux conservation equation. In the case of a stiff perfect fluid (\( \rho_0 = 0 \)), we find, see [3], \( \rho u^2 = r_0 \rho_\infty / r^4 \). The relevant components of the energy–momentum tensor in this case are given by

\[
T_{00} = \rho_\infty \left( 1 + \frac{\rho_0}{r} \right) \left( 1 - \frac{r_0}{r^2} \right), \quad T_{11} = -2 \rho_\infty \left( 1 + \frac{\rho_0}{r} \right)^2,
\]

and the flux,

\[
A = 8\pi \rho_\infty r_0^2.
\]

Therefore, the corrections in the vicinity of the horizon take the form

\[
M = M_0 + 8\pi \rho_\infty r_0^2 V + 4\pi \rho_\infty r_0^3 \left( \frac{x^3}{3} + \frac{x^2}{2} - x^2 + \frac{x}{6} - \log x \right),
\]

\[
\lambda = 4\pi \rho_\infty r_0^2 (x^2 + 4x - 5 + 2\log x),
\]

where we have introduced a short-hand notation \( x \equiv r/r_0 \).

4.2. Scalar field

Here we consider accretion of a scalar field and find corrections to the metric. The boundary conditions at the infinity are posed by the cosmological evolution

\[
\phi|_\infty = \dot{\phi}_c.
\]

Let us first consider the standard kinetic term (which in the hydrodynamical description corresponds to a stiff perfect fluid), so that the action is given by

\[
S_{\text{can}} = \int d^4x \sqrt{-g} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right).
\]

Solving the Klein–Gordon equation, \( \Box \phi = 0 \), in the metric of the Schwarzschild black hole, one can deduce that the stationary non-singular solution for \( \phi \) is given by

\[
\phi = \dot{\phi}_c \left( V - r - r_0 \log \left( \frac{r}{r_0} \right) \right).
\]

Taking into account that the energy–momentum tensor for a canonical scalar field is

\[
T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu},
\]
one can easily calculate the relevant components of the energy–momentum tensor on the solution (48):

\[ T_{0}^{0} = \frac{\dot{\phi}^2}{2} \left(1 - \frac{r_0^2}{r^2}\right) \left(1 + \frac{r_0}{r}\right), \quad T_{1}^{0} = -\frac{\dot{\phi}^2}{2} \left(1 + \frac{r_0}{r}\right)^2 \tag{50} \]

and

\[ \mathcal{A} = 4\pi \dot{\phi}^2 r_0^2. \tag{51} \]

Substituting the above expressions to (19) and (20), one can see that the corrections to the metric are given by (44) and (45) after identification \( \dot{\phi}^2 \rightarrow 2\rho_{\infty} \), as one could expect.

It is worth mentioning phantom accretion in the context of backreaction effects. One of the examples for a phantom, which was considered in [3], consisted of a scalar field with the ‘wrong’ sign in the action. In terms of a perfect fluid, this simply means that the density and the pressure of the effective perfect fluid become negative. This, in particular, implies from (44) that the black hole mass decreases during accretion of a phantom field. This result takes into account backreaction of the accreting field and it confirms the conclusion of [3], where the backreaction was neglected.

Now let us consider accretion of a Galileon scalar field. The general Galileon model consists of five non-trivial terms: the first one is a potential of the form \( \text{const} \times \phi^4 \), the second one is the canonical scalar field (which we already considered above) and three other nonlinear terms. Here we consider accretion of the Galileon with the action

\[ S_{\text{gal}} = -\int d^4x\sqrt{-g}(\nabla \phi \nabla \phi - \Gamma^\alpha_{\mu\nu} \nabla_\alpha \phi \nabla_\mu \phi \nabla_\nu \phi). \tag{52} \]

The stress tensor for this action is

\[ T_{\mu\nu}^{\text{gal}} = -2\phi_{,\mu} \phi_{,\nu} \nabla \phi + 2\phi_{,\alpha} \nabla_\alpha (\nabla \phi)^2 - g_{\mu\nu} \phi_{,\alpha} \nabla_\alpha (\nabla \phi)^2. \tag{53} \]

The solution for steady-state spherically symmetric accretion onto a black hole was found in [9]:

\[ \phi = \dot{\phi}_c \left(V - r + 2\sqrt{r_0r} - 2\sqrt{r_0} \log \left(\frac{r}{r_0} + 1\right)\right). \tag{54} \]

Substituting (54) into (53) and (16), one finds for the Galileon

\[ T_{0}^{0} = \frac{\dot{\phi}^2_c}{3} \frac{3}{x(1 + \sqrt{x})}, \quad T_{1}^{0} = -\frac{\dot{\phi}^2_c}{3} \frac{3}{\sqrt{x}(1 + \sqrt{x})^2}, \quad \mathcal{A} = 12\pi \dot{\phi}^2_c r_0^2. \tag{55} \]

Thus, the corrections to the metric are

\[ M = M_0 + 12\pi \dot{\phi}^2_c r_0^2 V + 4\pi \dot{\phi}^2_c r_0^3 (6(\sqrt{x} - 1) - 3(x - 1) + 2(x^{3/2} - 1) - 6\log(\sqrt{x} + 1)), \tag{56} \]

\[ \lambda = 12\pi \dot{\phi}^2_c r_0^2 \left(\frac{2}{1 + \sqrt{x}} + 2\sqrt{x} - 4\log(1 + \sqrt{x}) - 1 + 4\log 2\right). \tag{57} \]

5. Conclusion

In the studies of accretion of matter onto black holes, there are basically two analytic approaches. One is to neglect backreaction of an accreting matter onto the metric of the black hole—which is reasonable when the matter is light—and to study dynamics of matter on a fixed background metric. Another way is to try to find solutions for the full system of the Einstein and matter equations. There are only a few such solutions known; the reason is that it is in general very difficult to find solutions to the full nonlinear system of equations.
In this work, we studied the backreaction of an accreting fluid on the black hole metric in a perturbative way. Accretion is spherically symmetric and in quasi-steady state. In this approximation we assume that the accretion flow is small and matter is light, so we could use a perturbation scheme. At zeroth-order approximation the accreting matter does not backreact, so this step is equivalent to the test fluid approximation, equations (3) and (4). Then, the first-order corrections to the metric are found from the perturbed Einstein equations, in which the source term is given by the accreting matter at zeroth-order approximation, equation (5). In particular, for the case of spherically symmetric accretion onto a Schwarzschild black hole, the resulting equations for the correction are written down in (9)–(13). The solutions for the corrections to the metric coefficients are found in general form, equations (19) and (20); this is the central result of our paper. In a similar manner, the general perturbation scheme is applied to the Reissner–Nordström black hole, and we have found the same form for the correction of the metric, equations (19) and (20).

We compared our perturbative results with known analytic solutions. In particular, we verified that our scheme gives correct results in the case of the cosmological vacuum term (up to a redefinition of the central mass), as well as for the radially infalling photons; in this case, we reproduced the Vaidya solution.

An interesting consequence of the general result is universality of the corrections to the metric in the vicinity of the horizon of a black hole. We found that if the accretion rate is nonzero, then independently on the energy–momentum tensor of an accreting fluid, the leading corrections to the metric are of the Vaidya type, equation (40).

We applied the obtained results for calculation of correction to the metric for a generic perfect fluids and also for particular examples: accretion of a stiff perfect fluid (equations (44) and (45)) as well as accretion of a canonical scalar field and the Galileon scalar field (equations (56) and (57)).

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