Symmetry Protected Topological Charge in Symmetry Broken Phase: Spin-Chern, Spin-Valley-Chern and Mirror-Chern Numbers

Motohiko Ezawa
Department of Applied Physics, University of Tokyo, Hongo 7-3-1, 113-8656, Japan

The Chern number is a genuine topological number. On the other hand, a symmetry protected topological (SPT) charge is a topological number only when a symmetry exists. We propose a formula for the SPT charge as a derivative of the Chern number in terms of the Green function in such a way that it is valid and related to the associated Hall current even when the symmetry is broken. We estimate the amount of deviation from the quantized value as a function of the strength of the broken symmetry. We present two examples. First, we consider Dirac electrons with the spin-orbit coupling on honeycomb lattice, where the SPT charges are given by the spin-Chern, valley-Chern and spin-valley-Chern numbers. Though the spin-Chern charge is not quantized in the presence of the Rashba coupling, the deviation is estimated to be $10^{-7}$ in the case of silicene, a silicon cousin of graphene. Second, we analyze the effect of the mirror-symmetry breaking of the mirror-Chern number in a thin-film of topological crystalline insulator.

I. INTRODUCTION

There are two types of conserved charges, the Nöther charge and the topological charge. The Nöther charge conserves as a result of the equations of motion, while the topological charge conserves identically of any local perturbation. The third type of conserved charges may arise in the context of the symmetry protected topological (SPT) order. The SPT order generates a topological state only when some symmetry exists. The SPT charge is a topological charge as far as the symmetry is not broken.

Topological insulator is one of the most fascinating concepts found in this decade. It is characterized by topological numbers such as the Chern number and SPT numbers. The Chern number is a genuine topological charge, and indexes the quantum Hall state. Examples of SPT numbers are the $\mathbb{Z}_2$ index protected by the time-reversal symmetry (TRS), the spin-Chern number protected by the spin-rotation symmetry, and the mirror-Chern number protected by the mirror symmetry. There are other SPT numbers. Honeycomb systems are indexed by the valley-Chern and spin-valley-Chern numbers in addition to the spin-Chern number due to the valley-degree of freedom.

In this Letter we start with the Chern number $C$ expressed in terms of the single-particle Green function rather than the single-particle Hamiltonian. A merit is that we can calculate the Chern number even in the presence of interactions. We then propose to define the SPT charge $C_\Lambda$ corresponding to a symmetry $\Lambda$ of the Hamiltonian in such a way that the Hall conductivity $\sigma_{xy}^\Lambda$ is related to the SPT charge $C_\Lambda$ by the well-known formula,

$$\sigma_{xy}^\Lambda = \frac{e}{2\pi\hbar} C_\Lambda. \quad (1)$$

The SPT charge $C_\Lambda$ is expressed also in terms of the single-particle Green function. It is a topological number and quantized when the symmetry $\Lambda$ is unbroken. Our main result is that, even if $\Lambda$ is not a symmetry, the SPT charge $C_\Lambda$ is well defined and related to the Hall conductivity by this formula.

The SPT insulator may have symmetry protected gapless edge (surface) modes, indicating the topological nature of this order. However, the SPT charge can be continuously made zero through a continuous deformation of the Hamiltonian so as to break the symmetry. The gapless edge (surface) modes disappear as a result of this deformation though the gap of the bulk energy spectrum keeps open.

The SPT charge $C_\Lambda$ may be a continuous function of the strength $\xi$ of the symmetry-breaking coupling. It is then shown that $C_\Lambda = 1 - o(\xi^2)$. When the SPT charge is almost quantized ($|\xi| \ll 1$), the associated Hall current is almost quantized. The edge modes remains almost gapless.

We explicitly analyze Dirac electrons on honeycomb lattice in the presence of the spin-orbit (SO) coupling. They form a QSH insulator, which is an SPT state protected by TRS and by the spin $s_z$-symmetry. We consider a model where TRS has been broken by the antiferromagnetic (AF) order of $m_z$ in the perpendicular direction. We then break the $s_z$-symmetry by introducing the AF order in the in-plane component $m_x$. We calculate explicitly the spin-Chern charge $C_s$ as a function of these external parameters. We show that $C_s$ is continuously transformed from $C_s = 1$ to $C_s = 0$ by controlling them externally. We also analyze the effect of the Rashba coupling on the spin-Chern charge, and find $C_s = 1 - 5.9 \times 10^{-7}$ in the case of silicene, the silicon cousin of graphene. Consequently the spin-Chern charge is almost quantized.

We also analyze a thin film of topological crystalline insulator, which is protected by the mirror symmetry. The mirror-Chern number is half quantized. We introduce a mirror-symmetry breaking term and calculate the mirror-Chern charge. It becomes not quantized but the deviation is the second order in the breaking-term strength.

II. CHERN NUMBER

We analyze the two-dimensional system described by the Hamiltonian $H$ with a gapped energy spectrum. The Chern number is an integral of the Berry curvature over the first Brillouin zone.

$$C = (2\pi)^{-1} \int d^2k F(k), \quad (2)$$
where \( F(\mathbf{k}) = \partial_x a_y - \partial_y a_x \) and \( a_k = -i \sum_{\alpha} \langle \phi'^{\alpha}(\mathbf{k}) | \partial_k | \phi^{\alpha}(\mathbf{k}) \rangle \),

\[
a_k = -i \sum_{\alpha} \langle \phi'^{\alpha}(\mathbf{k}) | \partial_k | \phi^{\alpha}(\mathbf{k}) \rangle,
\]

with \( \phi^{\alpha}(\mathbf{k}) \) standing for the wave function of the ground state which is an insulator. There exists an alternative expression for the Chern number in terms of the Green function. With the use of the Matsubara Green function,

\[
G(\mathbf{k}) = [i \omega - H(\mathbf{k})]^{-1},
\]

with \( i \omega \) referring to the Matsubara frequency (\( \omega \): real), the Chern number is expressed as

\[
C = (2\pi)^{-2} \int d^2k \int_{-\infty}^{\infty} d\omega \Omega,
\]

with

\[
\Omega = \frac{1}{6} \varepsilon_{\mu \nu \rho} \text{Tr} \{ G \partial_\mu \Gamma G^{-1} \partial_\nu \Gamma G^{-1} \partial_\rho \Gamma G^{-1} \}.
\]

III. SYMMETRY PROTECTED CHERN NUMBER

As far as the symmetry \( \Lambda \) is not broken, the SPT charge \( C_\Lambda \) is a topological charge and quantized. An example is given by the spin \( s_z \)-symmetry with \( \Lambda = \sigma_z \), where the SPT charge is the spin-Chern number \( C_\Lambda \) and observable by measuring the Hall current,

\[
\sigma^{\Lambda}_{xy} = \frac{e}{2\pi \hbar} C_\Lambda.
\]

The main purpose of this work is to formulate the SPT charge \( C_\Lambda \) even when the symmetry \( \Lambda \) is broken in such a way that it is still related to the Hall current \( \sigma^{\Lambda}_{xy} \) by formula (7).

We start with the SPT current \( j_x \) defined by the symmetric product of the velocity \( v_y \) and the symmetry operator \( \Lambda \) as a natural extension of the spin current,

\[
j_x = \frac{1}{2} \{ v_x, \Lambda \},
\]

where \( v_i = \hbar^{-1} \partial_i H \). In the Kubo formalism the AC Hall conductivity of the SPT charge is given by the correlation function between the SPT current \( j_x \) and the current \( j_y = v_y \),

\[
\sigma^{\Lambda}_{xy}(\omega) = \varepsilon_{\mu \nu \rho} \frac{\varepsilon}{2\pi \hbar} K_\mu(\omega),
\]

\[
K_\mu(\omega) = \frac{\varepsilon_{\mu \nu \rho}}{\omega} \int \frac{d^2k}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega \times \text{Tr} [ j_\alpha G(\omega + \omega) j_\beta G(\omega) ].
\]

The DC Hall conductivity is given by \( \lim_{\omega \to 0} \sigma^{\Lambda}_{xy}(\omega) \). By making the Taylor expansion of \( G(\omega + \omega) \) in \( \omega \), and using \( \partial_i H = \partial_i H^{-1} \), it is straightforward to derive

\[
\lim_{\omega \to 0} K_\mu(\omega) = C_\Lambda,
\]

from which (7) follows, where

\[
C_\Lambda = (2\pi)^{-2} \int d^2k \int_{-\infty}^{\infty} d\omega \Omega(\Lambda),
\]

with

\[
\Omega(\Lambda) = \frac{1}{6} \varepsilon_{\mu \nu \rho} \text{Tr} \{ \Lambda G \partial_\mu \Gamma G^{-1} \partial_\nu \Gamma G^{-1} \partial_\rho \Gamma G^{-1} \}.
\]

In this derivation we have not assumed that \( \Lambda \) is a symmetry, that is, it can be that \( \Lambda H(\mathbf{k}) \neq H(\mathbf{k}) \Lambda \). Consequently, for any symmetry \( \Lambda \) for which the formula (8) makes sense, the SPT charge (12) is valid and related to the Hall current (7) even if the symmetry is broken.

For the sake of completeness let us prove that, when \( \Lambda \) is a symmetry, that is, \( \Lambda H(\mathbf{k}) = H(\mathbf{k}) \Lambda \), the SPT charge (12) is a conserved charge independent of the equations of motion and quantized. A proof is made by following Ref. 20. When the symmetry operator commutes with the Green function \( \Lambda G = G \Lambda \), \( \Omega(\Lambda) \) is simply given by

\[
\Omega(\Lambda) = \frac{1}{6} \varepsilon_{\mu \nu \rho} \text{Tr} \{ \Lambda G \partial_\mu \Gamma G^{-1} \partial_\nu \Gamma G^{-1} \partial_\rho \Gamma G^{-1} \}.
\]

We make a small modification \( \delta H \) of the Hamiltonian \( H \), which results in the modification \( \delta G \). By requiring that the perturbation does not break the symmetry, or \( \Lambda \delta G = \delta G \Lambda \), it is straightforward to show that

\[
\delta C_\Lambda = - (2\pi)^{-2} \int d^2k \int_{-\infty}^{\infty} d\omega \delta \Omega,
\]

with

\[
\delta \Omega(\Lambda) = \frac{1}{6} \varepsilon_{\mu \nu \rho} \text{Tr} \{ \Lambda \delta G \partial_\mu \Gamma G^{-1} \partial_\nu \Gamma G^{-1} \partial_\rho \Gamma G^{-1} \}.
\]

Since \( \delta C_\Lambda \) is an integral over a total derivative, \( C_\Lambda \) is a topological charge. It is said to be protected by the symmetry \( \Lambda \). Such a conserved charge is the SPT number.

When \( \Lambda \) is an element of a Lie algebra, it is possible to identify \( \Lambda \) with the diagonal element of the Pauli matrix \( \sigma^\Lambda \). Then the symmetry breaking coupling is given by \( \xi_x \sigma^\Lambda_x + \xi_y \sigma^\Lambda_y \) with external parameters \( \xi_x \) and \( \xi_y \). By controlling them we can continuously bring \( C_\Lambda \) to zero. Namely, the \( \Lambda \)-SPT insulator with \( C_\Lambda = 1 \) is continuously transformed into the \( \Lambda \)-trivial state with \( C_\Lambda = 0 \). Note that the \( \Lambda \)-trivial state may still be an SPT state with respect to another symmetry.

There exists an SPT charge \( C_\Lambda \) associated with each symmetry \( \Lambda \) of the unperturbed Hamiltonian density. We consider a maximum set \( \{ \Lambda_1, \Lambda_2, \ldots, \Lambda_N \} \) of these symmetries each of which is commutative with another, \( [\Lambda_i, \Lambda_j] = 0 \). It is interesting when the set may be identified with the Cartan subalgebra of a certain Lie algebra. We can identify \( \Lambda_i \) with the diagonal element of the Pauli matrix \( \sigma^\Lambda_i \). Then the symmetry breaking coupling is given by \( \xi_x \sigma^\Lambda_x + \xi_y \sigma^\Lambda_y \) with external
parameters $\xi'_s$ and $\xi'_v$. The $\Lambda_s$-SPT state is continuously transformed into the $\Lambda_s$-trivial state by controlling these parameters. In the next section we shall see an example in the honeycomb system where $\sigma_z$, $\eta_z$ and $\tau_z$ constitute the Cartan subalgebra of the SU(4) algebra with $\sigma_z$ and $\eta_z$ representing the spin and valley degrees of freedom.

In the rest of this paper we consider the system where we introduce a symmetry-breaking perturbation term to the symmetric Hamiltonian. When it contains a continuous parameter $\xi$, the Green function depends on $\xi$. We have $\partial \mathcal{C}_A(\xi) / \partial \xi \neq 0$. By integrating it we find that $\mathcal{C}_A(\xi)$ changes continuously from its quantized value as $\xi$ changes. We estimate $\mathcal{C}_A(\xi)$ for small symmetry-breaking perturbation. Since $\Lambda$ is a symmetry without the perturbation we have $\mathcal{C}'_A(0) = 0$, or

$$\mathcal{C}_A(\xi) = \mathcal{C}_A(0) + \frac{1}{2} \mathcal{C}''_A(0) \xi^2 + o(\xi^3). \quad (18)$$

As far as perturbations are small ($|\xi| \ll 1$) the SPT charge $\mathcal{C}_A$ is almost quantized.

We discuss the bulk-edge correspondence associated with the SPT charge. When $\Lambda$ is a good symmetry ($\xi = 0$), the SPT insulator may have symmetry-protected edge modes. However, as soon as the symmetry is broken ($\xi \neq 0$), these gapless edge modes disappear though the bulk gap keeps open. Nevertheless, as far as the symmetry breaking is small ($|\xi| \ll 1$), we can use the emergence of almost gapless edge modes as a signal of the SPT insulator. The edge-mode gap increases continuously from zero as $|\xi|$ increases. The SPT charge $\mathcal{C}_A(\xi)$ can be continuously transformed from the SPT phase ($\mathcal{C}_A = 1$) to the trivial phase ($\mathcal{C}_A = 0$). The edge-mode gap becomes larger than the bulk gap in the trivial phase. See Fig.1(a).

The emergence of gapless edge modes is a consequence of the reasoning that the SPT number can change its quantized value discontinuously across the edge only when the gap closes just as in the case of a genuine topological number. Accordingly, no gapless edge modes would appear since there is no need of gap closing provided the SPT number becomes ill-defined across the edge, that is, in vacuum. We have already encountered such a case for the valley-Chern and spin-valley-Chern numbers because the valley degree of freedom becomes ill-defined in vacuum. See also Section VI and Fig.1.

IV. DIRAC ELECTRONS ON HONEYCOMB LATTICE

After describing general features of SPT charges we now present an example to get an intuitive picture on them. Dirac electrons are ubiquitous in monolayer honeycomb systems: There are four types of them corresponding to the spin and valley degrees of freedom. It is intriguing that the SO coupling makes Dirac electrons massive and turns the systems into topological insulators. Silicene is a typical example. It is particularly interesting since we can control various topological phases externally by applying electric field, photo-irradiation, and exchange coupling.

The honeycomb lattice consists of two sublattices made of $A$ and $B$ sites. The states near the Fermi energy are $\pi$ orbitals residing near the $K$ and $K'$ points at opposite corners of the hexagonal Brillouin zone. The low-energy dynamics in the $K$ and $K'$ valleys is described by the Dirac theory. In what follows we use notations $s_z = \pm 1$ for spin $(\uparrow, \downarrow)$, $\tau_z = \pm 1$ for sublattice $(A, B)$ and $\eta_z = \pm 1$ for valley $(K, K')$. We also use the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ for the spin, the sublattice pseudospin and the valley pseudospin, respectively.

We analyze the Dirac Hamiltonian $H^{\nu}_{0}$ at the $K$ or $K'$ point as the unperturbed symmetric Hamiltonian. The first term represents electron hoppings with the Fermi velocity $v_F$. The second term describes the SO coupling with coupling $\lambda_{SO}$. The third term has been introduced to break TRS, whose physical meaning is the antiferromagnetic exchange magnetization in the $z$-direction.

There are three elements,

$$\Lambda_s = \sigma_z, \quad \Lambda_v = \eta_z, \quad \Lambda_{sv} = \sigma_z \eta_z, \quad (20)$$

that commute with the Hamiltonian $H^\nu_{0}$, forming the Cartan subalgebra of the SU(4) algebra. Accordingly we may introduce three SPT numbers $\mathcal{C}_s$, $\mathcal{C}_v$ and $\mathcal{C}_{sv}$. They are the spin-Chern numbers, the valley-Chern numbers, and the spin-valley-Chern numbers.

They are calculable by using the Hamiltonian (19) for $H$ and the operators (20) for $\Lambda$ in (12).

$$\mathcal{C} = \mathcal{C}_s^K + \mathcal{C}_v^K + \mathcal{C}_s^K + \mathcal{C}_v^K, \quad (21)$$

$$\mathcal{C}_s = \frac{1}{2}(\mathcal{C}_s^K + \mathcal{C}_v^K - \mathcal{C}_s^K - \mathcal{C}_v^K), \quad (22)$$

$$\mathcal{C}_v = \frac{1}{2}(\mathcal{C}_s^K - \mathcal{C}_v^K + \mathcal{C}_s^K + \mathcal{C}_v^K), \quad (23)$$

$$\mathcal{C}_{sv} = \frac{1}{2}(\mathcal{C}_s^K - \mathcal{C}_v^K - \mathcal{C}_s^K + \mathcal{C}_v^K), \quad (24)$$

where $\mathcal{C}_s^K = \frac{i}{2} \eta_z \text{sgn}(\Delta_{s_z})$ with $\Delta_{s_z}$ being the Dirac mass, $\Delta_{s_z} = \eta_z s_z \lambda_{SO} + m_z s_z$. There are two types of insulator phases in the model Hamiltonian (19). We find

$$(\mathcal{C}, \mathcal{C}_s, \mathcal{C}_v, \mathcal{C}_{sv}) = \left\{ \begin{array}{ll} (0, 1, 0, 0) & \text{for } |m_z| < \lambda_{SO} \\ (0, 0, 0, 1) & \text{for } |m_z| > \lambda_{SO} \end{array} \right.. \quad (25)$$

We call the $(0, 1, 0, 0)$ state a spin-Chern insulator for $m_z \neq 0$ since it has a nontrivial spin-Chern number: It is protected by the $s_z$-symmetry. Recall that there is no TRS when $m_z \neq 0$. The system undergoes a topological phase transition from the spin-Chern insulator ($\mathcal{C}_s = 1$) to the trivial insulator ($\mathcal{C}_s = 0$) as $m_z$ changes along the $m_z$ axis. The phase transition occurs at $m_z = \pm \lambda_{SO}$, where the gap closes. We remark that this trivial insulator is actually the spin-valley-Chern insulator ($\mathcal{C}_{sv} = 1$).

A comment is in order. TRS is recovered when we set $m_z = 0$. In this case the topological insulator is called the QSH insulator characterized by the $Z_2$ index. When there exist both TRS and the $s_z$-symmetry, the spin-Chern number is equal to the $Z_2$ index mod 2.
V. SECOND RASHBA TERM

We first study how these numbers are affected when the Hamiltonian (19) is deformed to break the $s_z$-symmetry. The second Rashba coupling exists in silicene as an intrinsic coupling[27,37]. It breaks the $s_z$-symmetry by mixing up and down spins on the next-nearest neighbor hopping sites. The additional term2 to the Dirac theory (19) is

$$H_{R2}^{m_z} = a\lambda R2\eta_z\tau_z (k_y\sigma_x - k_x\sigma_y), \quad (26)$$

with $a$ the lattice constant. It is straightforward to calculate the SPT charge (12) with $H^{m_z} = H_{R2}^{m_z} + H_{R2}^{AF}$ for the spin-Chern and spin-valley charges,

$$C_s = \sum_\eta \eta_z sgn(\eta_z\lambda_\text{SO} - m_z) \over [1 + (a_1a_2/hv_F)^2]$$

$$C_{sv} = \sum_\eta \eta_z sgn(\eta_z\lambda_\text{SO} - m_z) \over [1 + (a_1a_2/hv_F)^2]. \quad (27a)$$

This yields $C_s = 1 - 5.9 \times 10^{-7}$, where we have used $v_F = 5.5 \times 10^4$ m/s, $a = 3.86\AA$ and $\lambda R2 = 0.7$ meV as sample parameters of silicene. Surely $C_s$ does not yield a quantized number, but the deviation of the spin-Chern charge from 1 is negligibly small. Gapless edge modes must disappear when $m_z \neq 0$, but we do not recognize any discrepancy from zero within the accuracy of numerical calculation. The spin mixing can be neglected in practical purposes. We remark that gapless edge modes appear when $m_z = 0$ however large $\lambda R2$ may be, because they are protected by TRS.

VI. IN-PLANE AF ORDER

We next introduce the in-plane antiferromagnetic order $m_z$ to control the $s_z$-symmetry breaking externally[27,38-40] to the Dirac Hamiltonian (19). The additional term is

$$H_{AF}^{m_z} = m_z\sigma_x\tau_z. \quad (28)$$

We neglect the second Rashba coupling (26) since its effect is negligible. The Hamiltonian $H^{m_z} = H_{R2}^{m_z} + H_{AF}^{m_z}$ has the energy spectrum given by

$$E(k) = \pm \sqrt{(\hbar v_F^2 k^2 + (\eta_z\lambda_\text{SO} - m_z)^2 + m_z^2}, \quad (29)$$

with the band being $|E(0)|$. The system is an insulator except for two isolated points $(\eta_z\lambda_\text{SO}, 0)$ with $\eta_z = \pm 1$ in the $(m_z, m_x)$ phase diagram: See Fig.1(a). It is a spin-Chern insulator $(C_s = 1, C_{sv} = 0)$ in the region $(m_z, 0)$ with $|m_z| < \lambda_\text{SO}$, while it is a spin-valley-Chern insulator $(C_s = 0, C_{sv} = 1)$ for $|m_z| > \lambda_\text{SO}$.

We place a zigzag nanoribbon in vacuum: See Fig.1(a). We find that gapless edge modes emerge only in this spin-Chern insulator region[27]. This is because the spin-Chern number is well-defined but the spin-valley-Chern number is not for the vacuum[28]. The spin-Chern number cannot change its value from $C_s = 1$ to $C_s = 0$ across the edge without gap closing.

On the other hand, there is no need of the gap closing since the spin-valley-Chern number is not defined in the vacuum. We use (12) with $H^{m_z} = H_{R2}^{m_z} + H_{AF}^{m_z}$ to calculate the spin-Chern and spin-valley-Chern charges,

$$C_s = \sum_{\eta_z = \pm 1} \frac{\lambda_\text{SO} - \eta_z m_z}{2\sqrt{(\eta_z\lambda_\text{SO} - m_z)^2 + m_z^2}}, \quad (30a)$$

$$C_{sv} = \sum_{\eta_z = \pm 1} \frac{\eta_z\lambda_\text{SO} - m_z}{2\sqrt{(\eta_z\lambda_\text{SO} - m_z)^2 + m_z^2}}. \quad (30b)$$

They are quantized only when $m_z = 0$, yielding $C_s$. They are continuous function in all other region: See Fig.1(b) and (c). However, we may control parameters $m_z$ and $m_x$ so that the phase transition takes place without gap closing. Indeed, we can choose a path connecting the spin-Chern insulator $(C_s = 1, C_{sv} = 0)$ and the spin-valley-Chern insulator $(C_s = 0, C_{sv} = 1)$ along which the gap never closes, $|E(0)| \neq 0$, and the spin-Chern and spin-valley-Chern charges continuously change.

VII. FIRST RASHBA TERM

We also analyze the effect from the first Rashba coupling, which is a non-negligible ingredient of the Kane-Mele model.
The coupling mixes up and down spins on the nearest neighbor hopping sites. It yields the following term to the Dirac Hamiltonian \( H_{\text{R1}} \):

\[
H_{\text{R1}} = \frac{1}{2} \lambda_{\text{R1}} \sum_{\eta_z} (\eta_z \tau_x \sigma_y - \tau_y \sigma_x).
\]  

(31)

We calculate the SPT charge \( C \) for the spin-Chern and spin-valley-Chern charges with the use of the SO term and the first Rashba term,

\[
C_x = \frac{1}{2} \sum_{\eta_z} \eta_z \text{sgn}(\eta_z \lambda_{\text{SO}} - m_z) 4 - \xi^{-1} \text{arctanh} \xi,
\]

\[
C_{sv} = \frac{1}{2} \sum_{\eta_z} \eta_z \text{sgn}(\eta_z \lambda_{\text{SO}} - m_z) 4 - \xi^{-1} \text{arctanh} \xi,
\]

(32a)

(32b)

where \( \xi = |\lambda_{\text{R1}}/\lambda_{\text{SO}}| \).

**VIII. TOPOLOGICAL CRYSTALLINE INSULATOR THIN FILM**

We next study the topological crystalline insulator (TCI), which is a new type of topological insulator protected by the crystal group symmetry. It has been predicted that the TCI is realized in three-dimensional materials, SnTe and Pb\(_{1-x}\)Sn\(_{x}\)Se(Te). Gapless surface states have been observed in these materials. In particular, their surface system is characterized by the mirror-Chern number.

Dirac electrons emerge on the surface. We consider a thin film made of a TCI, where there are interferences between the two surfaces. The effective Hamiltonian for a TCI thin film is given by a two-dimensional model,

\[
H(k) = (v_x k_x \sigma_x - v_y k_y \sigma_y) \tau_x + m \tau_z,
\]

(33)

where \( \sigma_i \) is the Pauli matrix of spins and \( \tau_i \) is that of pseudospins representing two Dirac cones existing in the bulk TCI. Note that there are two Dirac cones in the bulk TCI, which are projected into one Dirac cone with the same momentum on the [001] TCI surface. The velocities \( v_x, v_y \) and the Dirac mass \( m \) are to be derived from microscopic parameters of surface states in the 3D TCI and their hybridization strengths.

This Hamiltonian has a mirror symmetry \( \Lambda_M H(k) \Lambda_M^{-1} = H(k) \) with

\[
\Lambda_M = -i \sigma_z \tau_z.
\]

(34)

The energy spectrum is given by

\[
E = \pm \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 + m^2}.
\]

(35)

The mirror-Chern number is calculated from \( C_M \) as

\[
C_M = \frac{1}{2} \text{sgn}(mv_x v_y).
\]

(36)

It has two phases characterized by \( C_M = \pm \frac{1}{2} \). The band is inverted and the system is topological for \( m < 0 \), while it is trivial for \( m > 0 \). A topological phase transition occurs at \( m = 0 \) with gap closing.

We break the mirror symmetry by introducing the term \( H' = \lambda_E \tau_y \), satisfying \( \Lambda_M H' \Lambda_M^{-1} = -H' \). The energy spectrum is modified as

\[
E = \pm \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2 + m^2 + \lambda_E^2}.
\]

(37)

The gap closes only at one point \( (m = 0, \lambda_E = 0) \). The mirror-Chern charge is calculated as

\[
C_M = \frac{1}{2} \frac{m}{\sqrt{m^2 + \lambda_E^2}} \text{sgn}(v_x v_y).
\]

(38)

It changes continuously from the topological insulator with \( C_M = -\frac{1}{2} (m < 0, \lambda_E = 0) \) to the topological insulator with \( C_M = \frac{1}{2} (m > 0, \lambda_E = 0) \) without gap closing by first switching on \( \lambda_E \), then changing \( m \) and finally switching off \( \lambda_E \). The mirror-Chern charge is no longer quantized in the presence of the mirror-symmetry breaking term, but its deviation is the second order in \( \lambda_E \).

**IX. DISCUSSIONS**

The SPT charge is a topological number and quantized when the associated symmetry is unbroken. We have presented its Green function representation valid even in the symmetry-broken phase. Our main result is that, even if \( \Lambda \) is not a symmetry, the SPT charge \( C \) is well defined and related to the Hall current formula (7). For instance, the spin current is observed as the difference between the up-spin and down-spin currents with respect to the z axis even if the spin is not a good symmetry. It is interesting that our formulas are valid even if the symmetry breaking is large. For instance, the spin-Chern charge \( C_M \) is valid even for any value of \( \lambda_{\text{R2}} \) or \( m_x \). The spin-Chern charge simply becomes zero when they are large enough. It implies that the average value of the \( s_z \) component becomes zero, which is physically reasonable.

We have elsewhere proposed possible topological devices with the use of edge channels carrying SPT charges in a honeycomb structure. Provided the edge-mode gap is far less than experimental resolution, we may use these almost gapless edge modes as carriers of currents. In principle, pure samples can be fabricated, where the effects of symmetry-breaking impurities must be made negligible. Then, SPT charges may well be treated as if they were topological numbers.

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1. Z.-C. Gu, X.-G. Wen, Phys. Rev. B 80, 155131 (2009).
2. A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B 83, 075102 (2011).
3. X. Chen, Z.-X. Liu, X.-G. Wen, Phys. Rev. B 84, 235141 (2011).
4. F. Pollmann, E. Berg, A. M. Turner, M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
5. Z.-C. Gu, X.-G. Wen, cond-mat/arXiv:1201.2648.
6. D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).
7. C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005); Phys. Rev. Lett. 95, 146802 (2005).
8. E. Prodan, Phys. Rev. B 80, 125327 (2009); New J. Phys. 12, 065003 (2010).
9. L. Sheng, D. N. Sheng, C. S. Ting, and F. D. M. Haldane, Phys. Rev. Lett. 95, 136602 (2005).
10. D. N. Sheng, Z. Y. Weng, L. Sheng and F. D. M. Haldane, Phys. Rev. Lett. 97, 036808 (2006).
11. Y. Yang, Z. Xu, L. Sheng, B. Wang, D.Y. Xing, and D. N. Sheng, Phys. Rev. Lett. 107, 066602 (2011).
12. Jeffrey C.Y. Teo, Liang Fu, C.L. Kane, Phys. Rev. B 78, 045426 (2008).
13. R. Takahashi and S. Murakami, Phys. Rev. Lett. 107, 166805 (2011).
14. F. Zhang, J. Jung, G. A. Fiete, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. 106, 156801 (2011).
15. M. Ezawa, Phys. Rev. B 87, 155415 (2013).
16. G. E. Volovik, The Universe in a Helium Droplet (Oxford University Press, New York, 2003).
17. Z. Wang, X.-L. Qi, and S.-C. Zhang, Phys. Rev. Lett. 105, 256803 (2010).
18. Z. Wang and S.-C. Zhang, Phys. Rev. X 2, 031008 (2012).
19. V. Gurarie, Phys. Rev. B 83, 085426 (2011).
20. L. Fu, Phys. Rev. Lett. 106, 106802 (2011).
21. T. H. Hsieh, H. Lin, J. Liu, W. Duan, A. Bansil, L. Fu, Nat. Com. 3, 982 (2012).
22. J. Liu, W. Duan, L. Fu, Phys. Rev. B, 88, 241303(R) (2013).
23. J. Liu, T. H. Hsieh, P. Wei, W. Duan, J. Moodera and L. Fu, Nat. Mat. 13, 178 (2014).
24. K. Ishikawa and T. Matsuyama, Z. Phys. C 33, 41 (1986), Nucl. Phys. B 280, 523 (1987).
25. J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. 92, 126603 (2004).
26. X.-L. Qi, T. Hughes, S.-C. Zhang, Phys. Rev. B 78 195424 (2008).
27. M. Ezawa, Y. Tanaka and N. Nagaosa, Sci. Rep. 3, 2790 (2013).
28. M. Ezawa, Phys. Rev. B 88, 161406(R) (2013).
29. C.-C. Liu, W. Feng, and Y. Yao, Phys. Rev. Lett. 107 (2011) 076802.
30. M. Ezawa, New J. Phys. 14 (2012) 033003.
31. M. Ezawa, Phys. Rev. Lett. 110, 026603 (2013).
32. M. Ezawa, Phys. Rev. Lett 109 (2012) 055502.
33. X. Li, T. Cao, Q. Niu, J. Shi, and J Feng, PNAS 2013 110 (10) 3738-3742 (2013).
34. Q.-F. Liang, L.-H. Wu, X. Hu, New J. Phys. 15 063031 (2013).
35. F. Zhang, A. H. MacDonald, and E. J. Mele, Proc. Natl. Acad. Sci. USA 110, 10546 (2013).
36. J. Li, A. F. Morpurgo, M. Büttiker, and I. Martin, Phys. Rev. B 82, 245404 (2010).
37. C.-C. Liu, H. Jiang, and Y. Yao, Phys. Rev. B, 84 (2011) 195430.
38. S. Rachel, K. L. Hur, Phys. Rev. B 82, 075106 (2010).
39. W. Wu, S. Rachel, W.-M. Liu, K. L. Hur, Phys. Rev. B 85, 205102 (2012).
40. J. Reuther, R. Thomale, S. Rachel, Phys. Rev. B 86, 155127 (2012).
41. Y. Tanaka, Z. Ren, T. Sato, K. Nakayama, S. Souma, T. Takahashi, K. Segawa, Y. Ando, Nature Physics 8, 800 (2012).
42. P. Dziawa, B. J. Kowalski, K. Dybko, R. Buczko, A. Szczersakow, M. Szot, Lusakowska, T. Balasubramanian, B. M. Wojek, M. H. Berntsen, O. Tjernberg and T. Story, Nature Materials 11, 1023 (2012).
43. S.-Y. Xu, C. Liu, N. Alidoust, M. Neupane, D. Qian, I. Belopolski, J.D. Denlinger, Y.J. Wang, H. Lin, L.A. Wray, G. Landolt, B. Slomski, J.H. Dil, A. Marcinkova, E. Morosan, Q. Gibson, R. Sankar, F.C. Chou, R.J. Cava, A. Bansil, M.Z. Hasan, Nature Communications 3 1192 (2012).
44. M. Ezawa, Appl. Phys. Lett. 102, 172103 (2013).