BRS symmetry for Yang-Mills theory with exact renormalization group

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Abstract

In the exact renormalization group (RG) flow in the infrared cutoff \( \Lambda \) one needs boundary conditions. In a previous paper on \( SU(2) \) Yang-Mills theory we proposed to use the nine physical relevant couplings of the effective action as boundary conditions at the physical point \( \Lambda = 0 \) (these couplings are defined at some non-vanishing subtraction point \( \mu \neq 0 \)). In this paper we show perturbatively that it is possible to appropriately fix these couplings in such a way that the full set of Slavnov-Taylor (ST) identities are satisfied. Three couplings are given by the vector and ghost wave function normalization and the three vector coupling at the subtraction point; three of the remaining six are vanishing (e.g. the vector mass) and the others are expressed by irrelevant vertices evaluated at the subtraction point. We follow the method used by Becchi to prove ST identities in the RG framework. There the boundary conditions are given at a non-physical point \( \Lambda = \Lambda' \neq 0 \), so that one avoids the need of a non-vanishing subtraction point.

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1 Introduction

Renormalization group (RG) formulation \[1\] provides the most physical way to deal with the ultraviolet (UV) singularities. Following a suggestion of Polchinski \[2\] (see also \[3\]-\[5\]) the exact RG formulation has been used recently to give self-contained and simple perturbative rederivations of many properties such as renormalizability \[2\]-\[5\], infrared finiteness of massless theories \[1\], \[4\], operator product expansion \[10\], decoupling theorem \[11\]. Some approximations have also been attempted within this context \[12\].

Its application has been recently extended to gauge and chiral gauge theories \[4\], \[13\]-\[17\]. The challenging problem here is the general fact that local gauge symmetry typically conflicts with the presence of a momentum cutoff. For the $SU(2)$ Yang-Mills theory it is shown in ref. \[5\] and in the present paper that, in spite of the explicit breaking of gauge symmetry, the Slavnov-Taylor (ST) identities can be implemented perturbatively by appropriately fixing the boundary conditions on the RG flow. Thus the RG formulation provides an alternative to dimensional regularization \[18\] of gauge theories.

To describe the situation we briefly recall the exact RG formulation for gauge theories. Consider the Wilsonian effective action $S_{\text{eff}}[\phi, \gamma; \Lambda]$ with $\phi$ and $\gamma$ the fields and the BRS sources and $\Lambda$ an infrared (IR) cutoff. This functional is obtained by path integration over the fields with frequencies above $\Lambda$ and up to an UV cutoff $\Lambda_0$. At $\Lambda = 0$ and $\Lambda_0 \to \infty$ one has performed the full path integral and therefore from the functional $S_{\text{eff}}[\phi, \gamma; \Lambda = 0]$ we can obtain the physical Green functions or the one particle irreducible vertices. Actually the vertices of $S_{\text{eff}}[\phi, \gamma; \Lambda = 0]$ are the amputated connected Green functions. We never write in the functionals the UV cutoff $\Lambda_0$ since we always understand $\Lambda_0 \to \infty$, which is possible due to perturbative renormalizability. The functional $S_{\text{eff}}[\phi, \gamma; \Lambda]$ satisfies an evolution equation in the IR cutoff, which is obtained by observing that the path integral over all frequencies can be done first by integrating over the frequencies above $\Lambda$ and then below $\Lambda$. The fact that the result does not depend on $\Lambda$ gives the mentioned evolution equation.

The exact RG equation is non-perturbative but it can be solved perturbatively once the boundary conditions are given at some value of $\Lambda$. An important point concerning the boundary conditions is the distinction between relevant and irrelevant parts of the functional $S_{\text{eff}}[\phi, \gamma; \Lambda]$ (and of any other functional such as the physical effective action $\Gamma[\phi, \gamma]$). As usual the vertices with negative mass dimension are called “irrelevant” and contribute to $S_{\text{eff,irr}}[\phi, \gamma; \Lambda]$. The remaining part $S_{\text{eff,rel}}[\phi, \gamma; \Lambda]$ contains only a finite number of parameters with non-negative dimension (relevant parameters) which, for $\Lambda \neq 0$, can be defined for instance as the first coefficients of the Taylor expansion of $S_{\text{eff}}[\phi, \gamma; \Lambda]$ around vanishing momenta.

The UV value $\Lambda = \Lambda_0$ is the obvious value where to fix the boundary conditions for $S_{\text{eff,irr}}[\phi, \gamma; \Lambda]$. For $\Lambda_0 \to \infty$ one expects that all vertices with negative dimension vanish thus one sets $S_{\text{eff,irr}}[\phi, \gamma; \Lambda = \Lambda_0] \to 0$ for $\Lambda_0 \to \infty$.

Finally the key problem is to fix the boundary conditions for the finite number of relevant parameters in $S_{\text{eff,rel}}[\phi, \gamma; \Lambda]$. Here is where one implements in the theory both the physical parameters (such as masses, couplings, wave function normalizations) and the
symmetry, *i.e.* ST identities.

In ref. [13] we considered the $SU(2)$ Yang-Mills theory which has nine relevant parameters. We proposed to fix these nine parameters at the physical point $\Lambda = 0$. Since the theory involves zero mass fields, at $\Lambda = 0$ one should define these parameters as the values of some vertices at non-vanishing subtraction points. The reason for selecting $\Lambda = 0$ is that $S_{\text{eff}}[\phi, \gamma; \Lambda = 0]$ is related via Legendre transform to the effective action $\Gamma[\phi, \gamma]$. Therefore some of these parameters are fixed at the physical values of the couplings, masses and wave function normalizations. We proposed to obtain the remaining parameters by imposing some of the ST identities and showed that this procedure can be implemented perturbatively. Therefore by construction some of the ST identities are satisfied, thus the fundamental question is whether the effective action $\Gamma[\phi, \gamma]$ computed by this procedure does indeed satisfy the full set of ST identities. In the paper of ref. [13] we were able to prove this only partially at one loop level.

Recently Becchi [5] considered the exact RG formulation of the $SU(2)$ Yang-Mills theory and imposed the boundary conditions for the relevant parameters in $S_{\text{eff},\text{rel}}[\phi, \gamma; \Lambda]$ at a non-physical point $\Lambda = \Lambda' \neq 0$ so that the relevant parameters can be defined by expanding the vertices around vanishing momenta. In this way the relevant parameters are not directly related to physical couplings in the effective action $\Gamma[\phi, \gamma]$ but the analysis of relevant parts of ST identities becomes easy. By using algebraic methods he was able to prove that the full set of ST identities can indeed be satisfied perturbatively. He identified a functional $\Delta_{\text{eff}}[\phi, \gamma; \Lambda]$ which gives the “defect” to ST identities and then showed that one can obtain perturbatively the nine relevant parameters in $S_{\text{eff},\text{rel}}[\phi, \gamma; \Lambda]$ by solving the fine tuning equation

$$\Delta_{\text{eff},\text{rel}}[\phi, \gamma; \Lambda] = 0$$

for the relevant part of the ST defect.

In the present paper we apply the same method to the case in which the boundary conditions are given at the physical point $\Lambda = 0$. In this case the operation of extracting the relevant parameters become rather complex since one has to use non-vanishing subtraction points. We are able to generalize the Becchi’s proof to this case and we explicitly prove that the procedure suggested in our previous paper gives indeed an effective action which satisfies perturbatively ST identities. Moreover we explicitly express the solution of the fine tuning equation by giving in terms of physical vertices those relevant parameters which are not fixed by the physical couplings and wave function normalizations.

This paper is organized as follows. In section 2 we recall the RG formulation for the $SU(2)$ Yang-Mills theory given in ref. [13]. We introduce the Wilsonian effective action $S_{\text{eff}}[\phi, \gamma; \Lambda]$, define its relevant and irrelevant parts, discuss the rôle of the boundary conditions and the local symmetry. In section 3 we study the operator which gives the violation to the ST identities and show that the problem of implementing the symmetry can be reduced to the solution of a finite number of equations, the *fine tuning equations*. Section 4 contains the explicit solution of these fine tuning equations and in section 5 we rephrase this solution in the usual algebraic language. Section 6 contains some conclusions.
2 Renormalization group flow for $SU(2)$ Yang-Mills theory

The fields and corresponding sources for $SU(2)$ Yang-Mills theory are

$$\phi = \{ A_\mu^a, c^a, \bar{c}^a \}, \quad j = \{ j_\mu^a, \bar{x}^a - \frac{1}{g} \partial_\mu u_\mu^a, -x^a \}, \quad \gamma = \{ u_\mu^a, v^a \},$$

where $u_\mu^a$ and $v^a$ are the sources associated to the BRS variation of $A_\mu^a$ and $c^a$ respectively. The generating functional, in the Feynman gauge, is

$$Z[j, \gamma] = e^{i W[j, \gamma]} = \int D\phi \exp i\{-\frac{1}{2} (\phi p^2 \phi)_{0\Lambda_0} + (j \phi)_{0\Lambda_0} + S_{\text{int}}[\phi, \gamma; \Lambda_0]\}, \quad (1)$$

where the path integral is regularized by assuming a UV cutoff $\Lambda_0$. In general we introduce the cutoff scalar products between fields and sources

$$\frac{1}{2} (\phi p^2 \phi)_{\Lambda\Lambda_0} \equiv \int_p p^2 K^{-1}_{\Lambda\Lambda_0}(p) \left\{ \frac{1}{2} A_\mu^a(-p) A_\nu^a(p) - \bar{c}^a(-p) c^a(p) \right\}, \quad \int_p \equiv \int \frac{d^4 p}{(2\pi)^4}$$

$$(j \phi)_{\Lambda\Lambda_0} \equiv \int_p K^{-1}_{\Lambda\Lambda_0}(p) \left\{ j_\mu^a(-p) A_\mu^a(p) + [\bar{x}^a(-p) - \frac{i}{g} p_\mu u_\mu^a(-p)] c^a(p) + \bar{c}^a(-p) x^a(p) \right\},$$

where $K_{\Lambda\Lambda_0}(p)$ is the cutoff function which is one for $\Lambda^2 < p^2 < \Lambda_0^2$ and rapidly vanishes outside.

The functional $S_{\text{int}}$ is the UV action involving monomials in the fields, BRS sources and their derivatives which have dimension not larger than four and are Lorentz and $SU(2)$ scalars. There are nine of these independent monomials

$$S_{\text{int}}[\phi, \gamma; \Lambda_0] = \int d^4 x \left\{ \frac{1}{2} A_\mu^a \left[ g_{\mu\nu} (\sigma_1^B + \sigma_2^B \partial^2) + \sigma_3^B \partial_\mu \partial_\nu \right] A_\nu^a + \sigma_4^B w_\mu \cdot \partial_\nu c + \sigma_5^B (\partial_\mu A_\mu) \cdot A_\nu + \sigma_6^B (A_\mu \wedge A_\nu) \cdot (A_\mu \wedge A_\nu) + \sigma_7^B (A_\mu \cdot A_\nu) (A_\mu \cdot A_\nu) + \sigma_8^B w_\mu \cdot c \wedge A_\mu + \sigma_9^B v \cdot c \wedge c \right\},$$

where $w_\mu^a = u_\mu^a + g \partial_\mu \bar{c}^a$ and we have introduced the usual scalar and external $SU(2)$ product. The nine couplings $\sigma_i^B$ depend on $\Lambda_0$ and have non-negative dimension (relevant parameters). In order to obtain the physical theory one has to show that the values of the $\sigma_i^B$ can be fixed in such a way that:

(1) the $\Lambda_0 \to \infty$ limit can be taken by fixing the physical parameters such as the masses, coupling $g$ and wave function normalization constant at a subtraction point $\mu$. Perturbative renormalizability ensures that this can be done \cite{2, 3, 13}.

(2) in the $\Lambda_0 \to \infty$ limit the Slavnov-Taylor identities must be satisfied. This is the crucial point to be discussed in this paper.

According to Wilson one integrates over the fields with frequencies $\Lambda^2 < p^2 < \Lambda_0^2$ and obtains (see appendix A)

$$e^{i W[j, \gamma]} = N[j, \gamma; \Lambda] \int D\phi \exp i\{-\frac{1}{2} (\phi p^2 \phi)_{0\Lambda} + (j \phi)_{0\Lambda} + S_{\text{eff}}[\phi, \gamma; \Lambda]\}, \quad (2)$$

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where \( N[j, \gamma; \Lambda] \) contributes to the quadratic part of \( W[j, \gamma] \)

\[
\log N[j, \gamma; \Lambda] = -i \int_p \frac{1}{p^2} \frac{K_{\Lambda\Lambda_0}(p)}{K_{0\Lambda_0}(p)} \left\{ \frac{1}{2} j^\mu_a(-p) j^\mu_a(p) - [\chi^a(-p) + \frac{i}{g} p^\mu u^a_\mu(-p)] \chi^a(p) \right\}.
\]

The functional \( S_{\text{eff}} \) is the Wilsonian effective action and is obtained by integrating the fields \( \phi' \) with the higher modes \( (p^2 > \Lambda^2) \)

\[
\exp i\{\frac{1}{2}(\phi p^2 \phi)_{\Lambda \Lambda_0} + S_{\text{eff}}[\phi, \gamma; \Lambda]\} = \int \mathcal{D}\phi' \exp i\{-\frac{1}{2}(\phi' p^2 \phi')_{\Lambda \Lambda_0} + (j' \phi')_{\Lambda \Lambda_0} + S_{\text{int}}[\phi', \gamma; \Lambda_0]\},
\]

where the source is

\[
j'(p) = p^2 \left\{ A^a_\mu(p), -c^a(p), c^a(p) \right\}.
\]

By comparing with the physical generating functional \( W[j, \gamma] \) in (3) we see that

\[
W[j', \gamma; \Lambda] = \frac{1}{2}(\phi p^2 \phi)_{\Lambda \Lambda_0} + S_{\text{eff}}[\phi, \gamma; \Lambda],
\]

as a functional of \( j' \), generates the connected Green functions in which the internal propagators have frequencies in the range \( \Lambda^2 < p^2 < \Lambda_0^2 \). The cutoff function in the external propagators is cancelled by the inverse cutoff so that the source \( j' \) can be taken for all values of \( p \). At \( \Lambda \to 0 \) and \( \Lambda_0 \to \infty \), this functional becomes the physical generating functional

\[
W[j', \gamma] = W[j', \gamma; \Lambda = 0] = \frac{1}{2}(\phi p^2 \phi)_{0\Lambda_0} + S_{\text{eff}}[\phi, \gamma; \Lambda = 0].
\]

Given the relation in (3) between \( j' \) and \( \phi \) we have that the vertices of \( S_{\text{eff}}[\phi, \gamma; \Lambda = 0] \) are the amputated connected Green functions.

Taking into account that the variable \( \Lambda \) enters as a cutoff in the internal propagators of connected Green functions, one derives the exact RG equation

\[
\Lambda \partial_\Lambda \frac{S_{\text{eff}}}{\hbar} = (2\pi)^8 \hbar \int_p \frac{1}{p^2} \Lambda \partial_\Lambda K_{0\Lambda}(p) e^{-iS_{\text{eff}}/\hbar} \left\{ \right. \frac{\delta}{\delta A^a_\mu(-p)} \frac{\delta}{\delta A^a_\mu(p)} + \frac{\delta}{\delta c^a(-p)} \frac{\delta}{\delta c^a(p)} \left. \right\} e^{iS_{\text{eff}}/\hbar}.
\]

This equation is non-perturbative but it can be solved perturbatively once the boundary conditions are given at some \( \Lambda \). We have included \( \hbar \) in (3) to show explicitly how the loop expansion is generated. In the r.h.s. there are two contributions. One is quadratic in the action and has a coefficient with the same power of \( \hbar \) as the l.h.s. thus involving Green functions with lower or equal loop order. The second, linear in the action, has a coefficient with an additional factor \( \hbar \) and therefore involves Green functions with a lower loop. Thus, starting at zero loop with \( S_{\text{eff}}^{(\ell=0)} \) given by the value of Green functions at the boundary conditions, eq. (3) allows one to obtain \( S_{\text{eff}}^{(\ell)} \) at all loops. The fact that one of the contributions in the r.h.s. in (3) is at the same loop order as the l.h.s. makes less transparent and less efficient the generation of the loop expansion. Actually, as shown in ref. [4, 5], the loop expansion is more easily generated if one integrates the RG flow not for the Wilsonian action \( S_{\text{eff}} \) but for a cutoff effective action \( \Gamma[\phi, \gamma; \Lambda] \) given by the usual Legendre transform of the cutoff functional \( W[j, \gamma; \Lambda] \) introduced in (3)

\[
\Gamma[\phi, \gamma; \Lambda] = W[j, \gamma; \Lambda] - W[0, 0; \Lambda] - \int_p \left\{ j^\mu_a(-p) A^a_\mu(p) + \bar{\chi}^a(-p) c^a(p) + c^a(-p) \chi^a(p) \right\}.
\]
This functional of the “classical” fields $\phi$ generates cutoff vertex functions in which the internal propagators have frequencies in the range $\Lambda^2 < p^2 < \Lambda_0$. From (7) we have indeed that at $\Lambda = 0$

$$\Gamma[\phi, \gamma; \Lambda = 0] = \Gamma[\phi, \gamma; \Lambda = 0]$$

is the physical effective action. One can convert (8) into the RG equation for $\Gamma[\phi, \gamma; \Lambda]$ which has the following form

$$\Lambda \partial_\Lambda \Gamma[\phi, \gamma; \Lambda] = \bar{h} I_R[\phi, \gamma; \Lambda],$$

where the functional $I_R[\phi, \gamma; \Lambda]$ is given in ref. [13] and depends non-linearly on the vertices of $\Gamma[\phi, \gamma; \Lambda]$. Then in the r.h.s. of (10) there are vertices at a loop order lower than the l.h.s., so that by solving iteratively this equation one automatically generates the loop expansion.

We now discuss the crucial point of the boundary conditions which provide the starting point for the loop expansion.

### 2.1 Boundary conditions: physical parameters and symmetry

There is an obvious value where the boundary conditions should be given which is the UV value $\Lambda = \Lambda_0$. Here one finds from (9)

$$S_{\text{eff}}[\phi, \gamma; \Lambda = \Lambda_0] = S_{\text{int}},$$

thus $S_{\text{eff}}$ becomes local and depends only on the nine relevant couplings $\sigma_i^R$ with non-negative dimension. Since $\Lambda_0$ is the only surviving mass parameter as $\Lambda_0 \to \infty$ one expects that, for $\Lambda_0 \to \infty$, all vertices with negative dimension vanish. For this reason vertices with negative dimension are called “irrelevant” and their contributions to the Wilsonian action will be denoted by $S_{\text{eff,irr}}[\phi, \gamma; \Lambda]$.

The remaining part $S_{\text{eff,rel}}[\phi, \gamma; \sigma_i(\Lambda)]$ is a local functional which contains nine relevant parameters $\sigma_i(\Lambda)$ with non-negative dimension. The form of $S_{\text{eff,rel}}[\phi, \gamma; \sigma_i(\Lambda)]$ is the same as $S_{\text{int}}$ in which the UV couplings $\sigma_i^R$ are replaced by $\sigma_i(\Lambda)$. Also for these parameters we need to fix the boundary conditions at some $\Lambda = \Lambda'$. The precise definition of $\sigma_i(\Lambda)$ in terms of $S_{\text{eff}}$ depends crucially on whether one assumes boundary conditions at $\Lambda = \Lambda' \neq 0$ (e.g. $\Lambda' = \Lambda_0$) or at the physical point $\Lambda' = 0$. In the first case the definition of $\sigma_i(\Lambda)$ is simple since one does not need to impose a subtraction point and can consider the Taylor expansion of the Green functions around vanishing momenta. Therefore one defines

$$S_{\text{eff,rel}}[\phi, \gamma; \sigma_i(\Lambda)] = T_4^{(0)} S_{\text{eff}}[\phi, \gamma; \Lambda], \quad \Lambda \neq 0,$$

where $T_4^{(0)}$ is the Taylor expansion around vanishing momenta truncated to the terms with coefficients with non-negative dimension (see for instance [3]).

Taking advantage of the fact that $S_{\text{eff}}[\phi, \gamma; \Lambda = 0]$ and $\Gamma[\phi, \gamma; \Lambda = 0]$ are physical functionals (see (7) and (9)), we have suggested in ref. [13] to fix the relevant parameters $\sigma_i(\Lambda)$ at the physical point $\Lambda = 0$ so that $\sigma_i(\Lambda = 0)$ are physical couplings, such as for
instance the mass, the wave function normalizations and the coupling constant $g$ at a subtraction point $\mu$. At $\Lambda = 0$ the definition of the functional $S_{\text{eff,rel}}[\phi, \gamma; \sigma_i(\Lambda = 0)]$ requires the introduction of a finite subtraction point $\mu$. We then define

$$S_{\text{eff,rel}}[\phi, \gamma; \sigma_i(\Lambda)] = T^{(\mu)}_4 S_{\text{eff}}[\phi, \gamma; \Lambda],$$

where the operator $T^{(\mu)}_4$ is given in appendix B for a scalar theory. The extension to the YM theory is simple in principle and $T^{(\mu)}_4 \Gamma[\phi, \gamma; \Lambda]$ is defined completely in appendix B.

For the boundary conditions for the RG equation (11) of the cutoff effective action $\Gamma[\phi, \gamma; \Lambda]$ we follow the same procedure:

1. we assume that the irrelevant parts of $\Gamma[\phi, \gamma; \Lambda = \Lambda_0]$ vanish

$$\Gamma_{\text{irr}}[\phi, \gamma; \Lambda = \Lambda_0] = \left[1 - T^{(\mu)}_4\right] \Gamma[\phi, \gamma; \Lambda = \Lambda_0] = 0,$$

2. we fix the remaining nine relevant parameters at the physical point $\Lambda = 0$. It is useful to rearrange the various monomials and use the following parametrization for the relevant part of the physical effective action $\Gamma[\phi, \gamma] = \Gamma[\phi, \gamma; \Lambda = 0]$

$$\Gamma_{\text{rel}}[\phi, \gamma] = \int d^4x \left\{ -z_1 \frac{1}{4} F_{\mu\nu} \cdot F_{\mu\nu} + z_2 \left[ \frac{1}{g} w_{\mu} \cdot D_{\mu} c - \frac{1}{2} v \cdot c \wedge c \right] + g(z_3 - 1) (\partial_{\nu} A_\mu) \cdot A_\nu \wedge A_\mu - \frac{1}{2} (\partial_{\nu} A_\mu) \cdot (\partial_{\nu} A_\mu) \right\} + \tilde{\Gamma}_{\text{rel}}[\phi, \gamma; \rho_i],$$

(11)

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g A_{\mu} \wedge A_{\nu}$, $D_{\mu} c = \partial_{\mu} c + g A_{\mu} \wedge c$ and

$$\tilde{\Gamma}_{\text{rel}}[\phi, \gamma; \rho_i] \equiv \int d^4x \left\{ \rho_1 \frac{1}{2} A_\mu \cdot A_\mu + \rho_2 \frac{1}{2} (\partial_{\mu} A_\mu) \cdot (\partial_{\nu} A_\mu) + \rho_3 w_{\mu} \cdot c \wedge A_\mu + \frac{1}{2} \rho_4 v \cdot c \wedge c + \frac{g^2}{4} \rho_5 (A_\mu \wedge A_{\nu}) \cdot (A_\mu \wedge A_{\nu}) + \frac{g^2}{4} \rho_6 (A_\mu \cdot A_{\nu}) (A_\mu \cdot A_{\nu}) \right\}.$$

(12)

In (11) we have singled out two BRS invariant monomials, the gauge fixing contribution and an additional three vector coupling monomial.

We now discuss the values $z_i$ and $\rho_i$ that one has to fix in order to have the physical couplings and the local $SU(2)$ gauge symmetry. Some of these couplings can be fixed to give the wave function normalizations and the three vector coupling $g$ at a subtraction point $\mu$. This can be implemented by assuming $z_1 = z_2 = z_3 = 1$ so that the first part of $\Gamma_{\text{rel}}$ in (11) is just the BRS action in the Feynman gauge $S_{\text{BRS}}[\phi, \gamma]$ and we have

$$\Gamma_{\text{rel}}[\phi, \gamma] = S_{\text{BRS}} + \hbar \tilde{\Gamma}_{\text{rel}}[\phi, \gamma; \rho_i].$$

(13)

Notice that the choice of boundary conditions at $\Lambda = 0$ for the relevant couplings provides a conspicuous advantage with respect to the case in which one takes $\Lambda = \Lambda' \neq 0$. We have in (13) that the relevant part of the effective action already contains the BRS invariant classical action which is the starting point of perturbation theory. We have also explicitly written the $\hbar$ dependence which shows that the other six couplings $\rho_i$ vanish at zero loop.

In ref. [13] we proposed to fix the $\rho_i$'s from the ST identities for the physical effective action $\Gamma[\phi, \gamma]$ given by

$$\Delta \Gamma[\phi, \gamma] \equiv S_F, \quad \Gamma'[\phi, \gamma] = 0,$$

(14)
where
\[ \Gamma'[\phi, \gamma] = \Gamma[\phi, \gamma] + \frac{1}{2} \int d^4x (\partial_\mu A_\mu)^2 \]
and \( S_\Gamma' \) is the usual Slavnov operator \[19\]
\[ S_{\Gamma'} = \int_p \left\{ \frac{\delta \Gamma'}{\delta u^a_\mu(-p)} \frac{\delta}{\delta A^a_\mu(p)} + \frac{\delta \Gamma'}{\delta A^a_\mu(p)} \frac{\delta}{\delta u^a_\mu(-p)} + \frac{\delta \Gamma'}{\delta v^a(p)} \frac{\delta}{\delta c^a(p)} + \frac{\delta \Gamma'}{\delta c^a(p)} \frac{\delta}{\delta v^a(-p)} \right\}. \] (15)

The six couplings \( \rho_i \) are involved in all vertices of the functional \( \Delta \Gamma[\phi, \gamma] \). In ref. [13] we obtained the various \( \rho_i \) from the simplest ST identities and found that these couplings are given by some irrelevant part of vertices evaluated at the subtraction point \( \mu \). We showed that this procedure can be implemented perturbatively. This is due to the fact that, as indicated in (13), the couplings \( \rho_i \) vanish at zero loop. At higher loops some of the couplings \( \rho_i \) are different from zero. The mass of the vector which is given by \( \rho_1 \), as expected, remains zero at all loops. In that paper we were not able to show that this procedure ensures that the ST identities are actually satisfied for all vertices. In the next section we will show that this is the case.

3 Gauge symmetry and fine tuning equation at \( \Lambda = 0 \)

The proof that gauge symmetry can be implemented perturbatively in the RG formulation has been given by Becchi in [3] in the case in which the boundary conditions for the nine parameters \( \sigma_i(\Lambda) \) are given at some non-physical point \( \Lambda = \Lambda' \neq 0 \). These parameters are defined by applying the operator \( T_4^{(0)} \) to the Wilsonian effective action. The perturbative analysis of the ST identities is simplified by the fact that \( T_4^{(0)} \) commutes with the ST functional derivative at zero loop.

Here we follow the same procedure but in the case in which one fixes the boundary conditions at the physical point \( \Lambda = 0 \). The price for working directly at \( \Lambda = 0 \) is that the relevant couplings are defined by the operator \( T_4^{(\mu)} \) which does not commute with the zero loop ST operator. As a result the analysis of the ST identities seems more complex. We are able to show that actually the analysis is easy and one can write a simple fine tuning equation which allows one to determine the six relevant couplings \( \rho_i \).

Following Becchi we consider the generalized BRS transformation
\[ \delta A^a_\mu(p) = -\frac{i}{g} p_\mu \eta c^a(p) + K_{0\Lambda}(p) \eta \frac{\delta S_{\text{eff}}}{\delta u^a_\mu(-p)}, \]
\[ \delta c^a(p) = K_{0\Lambda}(p) \eta \frac{\delta S_{\text{eff}}}{\delta v^a(-p)}, \]
\[ \delta \bar{c}^a(p) = \frac{i}{g} \eta p_\mu A^a_\mu(p), \]
with \( \eta \) a Grassmann parameter. Applying this change of variables to the functional integral (2), one deduces the following identity
\[ S Z[j, \gamma] = N[j, \gamma; \Lambda] \int \mathcal{D}\phi \ e^{i\left\{ -\frac{1}{2} (\partial \phi)^2 \phi + (j \phi)_{\alpha \lambda} + S_{\text{eff}}[\phi, \gamma; \Lambda] \right\}} \Delta[\phi, \gamma; \Lambda], \]
where

$$S \equiv \int_p \left( j_\mu^a(p) \frac{\delta}{\delta u_\mu^a(p)} - \chi^a(p) \frac{\delta}{\delta v^a(p)} - i \frac{g}{g} p_\mu \chi^a(p) \frac{\delta}{\delta j_\mu^a(p)} \right)$$

is the usual ST operator obtained from the variation of the source term \((j\phi)_{0\Lambda}\). Notice that the cutoff functions cancel. The functional \(\Delta\) arises from the variation of the other terms in the exponent and from the Jacobian. It can be divided into two parts, a linear and a quadratic one in the derivatives

$$\Delta = \Delta_1 + \Delta_2,$$

with

$$\Delta_1 = i \int_p \left\{ p^2 A_\mu^a(p) \frac{\delta}{\delta u_\mu^a(p)} + \frac{i}{g} p_\mu \bar{c}^a(p) \frac{\delta}{\delta A_\mu^a(p)} + \frac{i}{g} p_\mu u_\mu^a(p) \frac{\delta}{\delta v^a(p)} - \frac{i}{g} p_\mu \bar{c}^a(p) \frac{\delta}{\delta \bar{c}^a(p)} \right\} S_{\text{eff}}$$

and

$$\Delta_2 = \int_p K_{0\lambda}(p) e^{-i S_{\text{eff}}} \left\{ \frac{\delta}{\delta A_\mu^a(p)} \frac{\delta}{\delta u_\mu^a(-p)} + \frac{\delta}{\delta c^a(p)} \frac{\delta}{\delta v^a(-p)} \right\} e^{i S_{\text{eff}}}.$$

Since \(K_{0\lambda}(p)\) vanishes for \(\Lambda = 0\) we have that at the physical point \(\Delta[\phi, \gamma; \Lambda = 0] = \Delta_1[\phi, \gamma; \Lambda = 0]\).

We have to prove that one can select the six relevant couplings \(\rho_i\) in such a way that

$$\Delta[\phi, \gamma; \Lambda] = 0,$$

so that the ST identities

$$S Z[j, \gamma] = 0$$

are satisfied. Actually we will show that the functional equation (17) can be reduced to a finite number of relations, the fine tuning equations.

The RG flow for the functional \(\Delta\) is given by the following linear evolution equation

$$\Lambda \partial_\Lambda \Delta[\phi, \gamma; \Lambda] = M \cdot \Delta[\phi, \gamma; \Lambda] \equiv M[\Delta; \Lambda],$$

where the linear operator \(M\) depends on \(S_{\text{eff}}\)

$$M = -(2\pi)^8 \int_p \frac{1}{p^2} \Lambda \partial_\Lambda K_{0\lambda}(p) \left\{ \left( \frac{\delta S_{\text{eff}}}{\delta A_\mu^a(p)} - \frac{i h}{2} \frac{\delta}{\delta A_\mu^a(p)} \right) \frac{\delta}{\delta A_\mu^a(-p)} ight. + \left( \frac{\delta S_{\text{eff}}}{\delta c^a(p)} - \frac{i h}{2} \frac{\delta}{\delta c^a(p)} \right) \frac{\delta}{\delta c^a(-p)} - c \leftrightarrow \bar{c} \right\}.$$

The RG equation (18) requires boundary conditions. As before we discuss the boundary conditions for the relevant part of \(\Delta[\phi, \gamma; \Lambda]\) at \(\Lambda = 0\) and for the irrelevant part at the UV point \(\Lambda = \Lambda_0\).

The definition of the relevant part of \(\Delta[\phi, \gamma; \Lambda]\) at \(\Lambda = 0\) requires a subtraction point. This functional has dimension 1 and we have

$$\Delta_{\text{rel}}[\phi, \gamma; \delta_i] = T_{5}^{(\mu)} \Delta[\phi, \gamma; \Lambda = 0],$$

(19)
where $T^{(\mu)}_S$ is again defined as in appendix B. There are 11 relevant parameters $\delta_i$ (see appendix C) which are the coefficients of the 11 monomials in the fields, sources and momenta of dimension not greater than five and with ghost number one. Recall that at $\Lambda = 0$ the expression of $\Delta$ simplifies since the non-linear part $\Delta_2$ vanishes due to the cutoff function.

We can easily prove that the irrelevant part of the functional $\Delta$ vanishes at $\Lambda = \Lambda_0$ provided that the UV cutoff is sent to infinity

$$\Delta_{\text{irr}}[\phi, \gamma; \Lambda = \Lambda_0] \to 0, \quad \text{for} \quad \Lambda_0 \to \infty. \quad (20)$$

First observe that $\Delta^{(1)}[\Lambda = \Lambda_0]$ contains only relevant parts. This is due to the fact that at $\Lambda = \Lambda_0$ we have $S_{\text{eff}}[\Lambda = \Lambda_0] = S_{\text{int}}$ which is local. For $\Delta^{(2)}[\Lambda = \Lambda_0]$ we have that the functional in the integrand is local apart from the $p$-dependence of the cutoff function $K_{0\Lambda}(p)$ which gives an irrelevant contribution to $\Delta^{(2)}[\Lambda = \Lambda_0]$. However, for $\Lambda_0 \to \infty$ we have $K_{0\infty}(p) = 1$ so that $\Delta^{(2)}[\Lambda = \Lambda_0 \to \infty]$ becomes local.

Using the boundary condition (22) and (21) we can integrate the RG equation for $\Delta$ and obtain

$$\Delta[\phi, \gamma; \Lambda] = \Delta_{\text{rel}}[\phi, \gamma; \delta_i] + \int_0^{\Lambda} \frac{d\lambda}{\Lambda} M_{\text{rel}}[\Delta; \lambda] - \int_{\Lambda}^{\infty} \frac{d\lambda}{\Lambda} M_{\text{irr}}[\Delta; \lambda]. \quad (21)$$

We now prove the following two perturbative theorems for the functional $\Delta^{(\ell)}[\phi, \gamma; \Lambda]$ at loop $\ell$ and for the corresponding vertices $\Delta^{(\ell)}_n(...; \Lambda)$ where $n$ denotes the number of fields or sources and the dots denote momenta, internal and Lorentz indices.

**Theorem 1:** if $\Delta^{(\ell)}[\phi, \gamma; \Lambda] = 0$ for all loop $\ell' < \ell$, and if $\Delta^{(\ell)}_n(...; \Lambda) = 0$ for $n' < n$, then

$$\Delta^{(\ell)}_n(...; \Lambda) = \Delta^{(\ell)}_{\text{rel},n}(...; \Lambda = 0).$$

**Proof:** from the hypothesis the functional $M[\Delta; \Lambda]$ at loop $\ell$ is given by

$$M^{(\ell)}[\Delta; \Lambda] = -(2\pi)^8 \int \frac{dp}{p^2} \Lambda \delta A_{\mu}(p)\partial_{\Lambda} \delta A_{\mu}(-p) + \frac{\delta S^{(0)}_{\text{eff}}}{\delta c^a(p)} \frac{\delta}{\delta \bar{c}^a(-p)} - c \leftrightarrow \bar{c} \Delta^{(\ell)}[\phi, \gamma; \Lambda],\quad (22)$$

where $S^{(0)}_{\text{eff}}$ is the tree Wilsonian action in (3). Since the starting point of the perturbative expansion (see (13)) is $\Gamma^{(0)}[\phi, \gamma; \Lambda] = S_{\text{BRS}}[\phi, \gamma]$, the vertices of $S^{(0)}_{\text{eff}}[\phi, \gamma; \Lambda]$ are the amputated connected Green functions at tree level. Moreover $S^{(0)}_{\text{eff}}[\phi, \gamma; \Lambda]$ does not contain contributions quadratic in the fields, they are all included in $\frac{1}{2}(\phi^2 \phi \phi)$, see (3). Consider now the RG equations (21) for a vertex $\Delta^{(\ell)}_n(...; \Lambda)$. From (22) we have that $M^{(\ell)}[\Delta; \Lambda]$ involves only vertices $\Delta^{(\ell)}_n(...; \Lambda)$ with $n' < n$. They vanish according to the hypothesis thus only the first term in r.h.s. of (21) is left and the theorem is proved. An obvious consequence of this is the second theorem.

**Theorem 2:** if $\Delta^{(\ell)}[\phi, \gamma; \Lambda] = 0$ for all loop $\ell' < \ell$ and if at loop $\ell$ the relevant part of this functional vanishes at $\Lambda = 0$, namely if

$$\Delta_{\text{rel}}[\phi, \gamma; \delta^{(\ell)}_i] = 0, \quad (23)$$
then $\Delta^{(\ell)}[\phi, \gamma; \Lambda] = 0$.

The fine tuning condition (23) will be used to fix the relevant couplings $\rho_i$ of the effective action $\Gamma[\phi, \gamma]$. However the functional $\Delta[\phi, \gamma; \Lambda]$ defined in (16) involves vertices of the Wilsonian action $S_{\text{eff}}[\phi, \gamma; \Lambda]$. It is convenient to express the fine tuning condition at $\Lambda = 0$ in terms of a functional which involves directly the vertices of $\Gamma[\phi, \gamma]$. This can be simply done by taking the Legendre transform of $\Delta[\phi, \gamma; \Lambda]$ at $\Lambda = 0$. In this way the fine tuning equation (23) is equivalent to

$$
\Delta_{\Gamma, \text{rel}}[\phi, \gamma; \delta^{(\ell)}_i] = T^{(\ell)}_5 \Delta^{(\ell)}[\phi, \gamma] = 0, \quad \Delta[\phi, \gamma] \equiv S_{\Gamma, \Gamma'}[\phi, \gamma], \quad (24)
$$

where $\Delta_{\Gamma, \text{rel}}$ has the same decomposition as $\Delta_{\text{rel}}$ given in appendix C. Then both theorems 1 and 2 can be phrased for the functional $\Delta_{\Gamma}[\phi, \gamma]$.

From now on we consider $\Lambda = 0$ and we show perturbatively the equivalence of (23) and (24). By using the inductive assumption $\Delta^{(\ell')} = 0$ for $\ell' < \ell$, one has the following results at loop $\ell$:

(1) For the two field vertex one has $\Delta_{\Gamma}^{(\ell)(A_c)} = \Delta^{(\ell)(A_c)}$, then the fine tuning condition $\Delta_{\text{rel}}^{(\ell)(A_c)} = 0$ is equivalent to $\Delta_{\Gamma, \text{rel}}^{(\ell)(A_c)} = 0$. Imposing this condition the full vertex $\Delta_{\Gamma}^{(\ell)(A_c)}$ vanishes.

(2) For vertices with three fields one has $\Delta_{\Gamma}^{(\ell)(AAc)} = \Delta^{(\ell)(AAc)}$ and $\Delta_{\Gamma}^{(\ell)(wcc)} = \Delta^{(\ell)(wcc)}$, since the one particle reducible terms vanish due to (1). Thus the fine tuning conditions $\Delta_{\text{rel}}^{(\ell)(AAc)} = 0$ and $\Delta_{\Gamma, \text{rel}}^{(\ell)(wcc)} = 0$ are equivalent to $\Delta_{\Gamma, \text{rel}}^{(\ell)(AAc)} = 0$ and $\Delta_{\Gamma, \text{rel}}^{(\ell)(wcc)} = 0$, respectively. With these conditions these two vertices vanish.

(3) Similarly for the vertices with four fields one has $\Delta_{\Gamma}^{(\ell)(3Ac)} = \Delta^{(\ell)(3Ac)}$ and $\Delta_{\Gamma}^{(\ell)(wAcc)} = \Delta^{(\ell)(wAcc)}$ and the remaining fine tuning conditions become $\Delta_{\Gamma, \text{rel}}^{(\ell)(3Ac)} = 0$ and $\Delta_{\Gamma, \text{rel}}^{(\ell)(wAcc)} = 0$ and the two functions $\Delta_{\Gamma}^{(\ell)(3Ac)}$ and $\Delta_{\Gamma}^{(\ell)(wAcc)}$ vanish.

(4) By increasing the number of fields one has that the two functionals $\Delta^{(\ell)}$ and $\Delta_{\Gamma}^{(\ell)}$ are equal.

In the next section we prove by induction on the number of loops that it is possible to fix the six couplings $\rho_i^{(\ell)}$ in such a way that $\delta^{(\ell)}_i = 0$, so that the fine tuning equation is solved and from theorems 1 and 2 the ST identities hold perturbatively.

## 4 Perturbative solution of the fine tuning equation

The starting point of the proof by induction over the number of loops is $\Delta_{\Gamma}^{(\ell=0)}[\phi, \gamma] = 0$ which is valid since $\Gamma^{(\ell=0)} = S_{\text{BRS}}$. We suppose that the fine tuning equation is solved at $\ell' < \ell$ so that $\Delta_{\Gamma}^{(\ell')}[\phi, \gamma] = 0$. We want to show that it is possible to fix the six couplings $\rho_i^{(\ell)}$ in such a way that $\Delta_{\Gamma}^{(\ell)}[\phi, \gamma] = 0$. Notice that there are 6 parameters $\rho_i^{(\ell)}$ and 11 equations $\delta^{(\ell)}_i = 0$. Therefore the solvability of these equations requires that there must be 5 relations connecting the various $\delta^{(\ell)}_i$. They are provided by the so-called consistency conditions, a set
of equations which must be identically satisfied by the functional $\Delta \Gamma$ due to its definition (14). From the anticommuting character of $S\Gamma$, we have $S\Gamma \Delta \Gamma = S^{2}\Gamma$, $\Gamma' = 0$, where $S\Gamma$ is defined in (15). By using again the inductive hypothesis $\Delta(\ell') = 0$ for $\ell' < \ell$, the consistency condition simplifies and at loop $\ell$ one finds

$$S\Gamma_{o} \Delta(\ell) = 0,$$

where $S\Gamma_{o}$ is defined in (13) with $\Gamma$ given by $S_{BRS}$. In order to obtain from this equation the desired relations connecting the various $\delta_{i}(\ell)$, we have to extract its relevant part by applying the operator $T_{6}(\mu)$. Notice that this operator mixes relevant and irrelevant parts of $\Delta \Gamma$. This in general makes some of the consistency conditions complicated due to the presence of irrelevant contributions. We analyse in detail the fine tuning equations and consistency conditions for the various vertices of $\Delta \Gamma$. We start form the vertices with two and three fields and sources. We then analyse the remaining vertices. In the following all vertices will be considered at loop $\ell$.

1) The vertices with two and three fields are obtained from the definition of $\Delta \Gamma$ in (14)

$$\Delta_{\Gamma,\mu}(p) = \Gamma_{\mu}(p),$$

$$\Delta_{\Gamma,\mu}(p, q, k) = \Gamma_{\mu}(p) \Gamma_{\mu}(k) - \Gamma_{\mu}(p) \Gamma_{\mu}(k),$$

$$\Delta_{\Gamma,\mu}(p, q, k) = \Gamma_{\mu}(p) \Gamma_{\mu}(k) + \Gamma_{\mu}(p) \Gamma_{\mu}(k).$$

The relevant parameters of these vertices are the five $\delta_{i}$ with $i = 1, 2, 3, 4, 5$ (see appendix C) while the relevant couplings of the $\Gamma$ vertices are the four $\rho_{i}$ with $i = 1, 2, 3, 4$ (see (14) and appendix B). Then we need one relation among these $\delta_{i}$. Indeed if we consider the consistency condition (25) for the vertex of the $A-C-C$ fields and take its relevant part, we obtain

$$i g \delta_{2} = \delta_{3} + \delta_{4}.$$

It is a nice fact that this relation involves only relevant parameters. This is not a general fact since the operator $T_{6}(\mu)$ applied to $S\Gamma \Delta \Gamma$ typically extracts irrelevant vertices of $\Delta \Gamma$ evaluated at the subtraction point.

We are now able to satisfy the fine tuning equation for the vertices in (26)-(28). From (26) we obtain

$$\delta_{1} = - \frac{i}{g} \rho_{1}, \quad \delta_{2} = - \frac{i}{g} \rho_{2},$$

and we have $\delta_{1} = \delta_{2} = 0$ by fixing $\rho_{1} = \rho_{2} = 0$.

From (27) we obtain

$$\delta_{3} = \rho_{3},$$

and we have $\delta_{3} = \delta_{4} = 0$ by fixing $\rho_{3} = 0$.

Finally from (28) we obtain

$$\delta_{5} = - \frac{i}{g} \left\{ \rho_{4} + \left[ \frac{2}{p^{2}} p_{\mu} k_{\nu} \tilde{\Gamma}_{\mu\nu}(p, q, k) \right]_{3SP} \right\},$$

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and we have $\delta_5 = 0$ by fixing

$$\rho_4 = - \left[ \frac{2}{p^2} P^\mu_k P^\nu_k \Gamma^{(wCA)}(p, q, k) \right](32)$$

From theorem 2 of the previous section we have that, since the relevant parts of the vertices in (26)-(28) vanish, the complete vertices vanish. Moreover from theorem 1 of previous section the four field vertices of $\Delta_\Gamma$ could be only relevant.

2) We come now to analyse the vertices of $\Delta_\Gamma$ involving four fields. Consider first the vertex $\Delta^{(wCA)\alpha\beta\gamma\delta}(p, q, k, h)$, which contains the relevant parameter $\delta_6$ (see appendix C). From the definition of $\Delta_\Gamma[\phi, \gamma]$ in (14) one easily sees that this vertex involves only couplings $\rho_i$ already fixed thus one should be able to show that the fine tuning condition $\delta_6 = 0$ is satisfied automatically. Indeed there is a consistency condition which, after having imposed (30), (31) and (32), reduces to $\delta_6 = 0$. This is due to theorem 1 which implies that irrelevant parts of the vertices of $\Delta_\Gamma$ with four fields vanish.

The last vertex of $\Delta_\Gamma$ we have to consider is

$$\Delta^{(3Ac)\alpha\beta\gamma\delta}(p, q, k, h) = \Gamma^{(wC)}(h) \Gamma^{(4A)\alpha\beta\gamma\delta}(p, q, k, h)$$

$$+ \Gamma^{(AA)}(p) \Gamma^{(wCA)\alpha\beta\gamma\delta}(p, h, q, k) + \Gamma^{(AA)}(q) \Gamma^{(wCA)\alpha\beta\gamma\delta}(q, h, p, k)$$

$$+ \Gamma^{(AA)}(k) \Gamma^{(wCA)\alpha\beta\gamma\delta}(h, p, q, k) + \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\epsilon\delta\nu\rho} \Gamma^{(wC)}(q + k, h, p) \Gamma^{(AA)}(p + h, q, k)$$

$$+ \epsilon^{\epsilon\delta\nu\rho} \epsilon^{\alpha\beta\gamma\delta} \Gamma^{(wC)}(q + h, p, k) \Gamma^{(AA)}(q + h, k, p)$$

(33)

This vertex contains the five relevant parameters $\delta_i$ with $i = 7, \ldots, 11$ while the vertices involved here contain only the two relevant couplings not yet fixed, $\rho_5$ and $\rho_6$. The consistency condition (27) gives the three relations

$$\delta_7 = \delta_9 = \delta_{11}, \quad \delta_8 = \delta_{10}.$$ (34)

Notice that these relations involve only relevant parameters. As stated before this is a consequence of the fact that we have already set to zero the vertices of $\Delta_\Gamma$ with two and three fields and, according to theorem 1, the vertices with four fields do not have any irrelevant parts.

From (14) we have that it is possible to have $\delta_i = 0$, $i = 7, \ldots, 11$, simply setting to zero two independent $\delta_i$ (e.g. $\delta_7$ and $\delta_8$) by fixing $\rho_5$ and $\rho_6$. The easiest way of obtaining $\delta_7$ and $\delta_8$ is to contract in two different ways the vertex in eq. (33) by external momenta and summing over internal indices. Since this vertex does not have any irrelevant part, these contractions are proportional to two different linear combinations of $\delta_7$ and $\delta_8$. Using for simplicity the symmetric point we find

$$\frac{p^\mu q^\nu k^\rho}{p^4} \Delta^{(3Ac)\alpha\beta\gamma\delta}(p, q, k, h)\big|_{4SP} = -\frac{15}{9} (2\delta_7 + \delta_8)$$

$$= -\frac{10}{3} ig \rho_6 - \frac{i}{g} \left[ \frac{p^\mu q^\nu k^\rho}{p^4} \Gamma^{(4A)\alpha\beta\gamma\delta}(p, q, k, h) \right)_{4SP}$$

and

$$\frac{p^\mu q^\nu k^\rho}{p^4} \Delta^{(3Ac)\alpha\beta\gamma\delta}(p, q, k, h)\big|_{4SP} = -\frac{1}{3} (2\delta_7 - 7\delta_8)$$
\[= ig\left(\frac{16}{3} \rho_5 - \frac{2}{3} \rho_6\right) - \left[\frac{ip_\mu q_\nu h_\rho h_\sigma}{g p^4} \tilde{\Gamma}^{(AA)aabb}(p, q, k, h)\right.\]
\[+ \frac{p_\mu q_\nu}{p^2} (h_\rho + \frac{1}{3} k_\rho) \Gamma^{(wcc)Aaabb}(k, h, p, q) + 12 \frac{p_\mu q_\nu h_\rho}{p^4} (h_\rho + \frac{1}{3} k_\rho) \Gamma^{(wcc)A}(q + k, h, p) \Gamma^{(3A)}(p + h, q, k)\left.\right]_{4SP}.\]

The fine tuning equations \(\delta_7 = 0\) and \(\delta_8 = 0\) give

\[\rho_6 = -\frac{3}{10g^2} \left[\frac{p_\mu q_\nu k_\rho h_\sigma}{p^4} \tilde{\Gamma}^{(AA)aabb}(p, q, k, h)\right]_{4SP} \tag{35}\]

\[\rho_5 = \frac{1}{8} \rho_6 - \frac{3i}{16g^2} \left[\frac{ip_\mu q_\nu h_\rho h_\sigma}{p^4} \tilde{\Gamma}^{(AA)aabb}(p, q, k, h) + \frac{p_\mu q_\nu}{p^2} (h_\rho + \frac{1}{3} k_\rho) \Gamma^{(wcc)Aaabb}(k, h, p, q)\right.\]
\[+ 12 \frac{p_\mu q_\nu h_\rho}{p^4} \Gamma^{(wcc)A}(q + k, h, p) \Gamma^{(3A)}(p + h, q, k)\left.\right]_{4SP}. \tag{36}\]

This completes the proof that we can set to zero the 11 parameters \(\delta_i\) by fixing, at every loop, the 6 couplings \(\rho_i\) according to (30), (31), (32), (36) and (35).

### 5 Algebraic formulation

We want to connect the perturbative solution we have discussed with the algebraic formulation given in ref. \[4\]. This is based on the fact that if \(\Delta_{\Gamma,\text{rel}}[\phi, \gamma; \delta_i]\) can be written in the following form

\[\Delta_{\Gamma,\text{rel}}[\phi, \gamma; \delta_i] = S_{\Gamma^{(0)}} \Gamma_{\text{rel}}[\phi, \gamma; \tilde{\rho}_i], \tag{37}\]

then it is possible to find a perturbative solution of the fine tuning equation by appropriately fixing the \(\rho_i\). This property is discussed in \[4\] and recalled in appendix D. Eq. (37) is valid provided the 11 parameters in \(\Delta_{\Gamma,\text{rel}}\) fulfil the following five relations

\[ig\delta_2 = \delta_3 + \delta_4, \tag{38}\]
\[\delta_6 = -ig\delta_5, \quad \delta_7 = \delta_9 = \delta_{11}, \quad \delta_8 = \delta_{10}. \tag{39}\]

In our case, in which we have a non-vanishing subtraction point \(\mu \neq 0\), the first relation (38) holds (see (29)), while the relations (39) are not valid in general since there are contributions from irrelevant parts.

Still also for \(\mu \neq 0\) we can use the above theorem as follows. First observe that the relation (38) holds also for \(\mu \neq 0\) and involves only vertices with two and three fields, i.e. \(\Delta_{\Gamma}^{(Ac)}\), \(\Delta_{\Gamma}^{(AAc)}\) and \(\Delta_{\Gamma}^{(wcc)}\). Therefore the form (37) is valid for these vertices and from appendix D we deduce that we can solve the fine tuning equations \(\delta_i = 0\) for the 5 relevant parameters with \(i = 1, \ldots, 5\). This is obtained by fixing the four couplings \(\rho_1, \rho_2, \rho_3\) and \(\rho_4\), which enter in the vertices of \(\Gamma\) with two and three fields. From the previous section the solution is \(\rho_1 = \rho_2 = \rho_3 = 0\) and \(\rho_4\) given in (32).

\[2\] In ref. \[3\] some contributions to (36) were missing.
After solving these five fine tuning conditions, we have from theorem 2 that the complete vertices $\Delta^{(A)}_\Gamma$, $\Delta^{(AA)}_\Gamma$ and $\Delta^{(w+o)}_\Gamma$ vanish and from theorem 1 that the vertices with four fields are only relevant. Therefore the relations in (33) are now valid and all vertices of $\Delta_{\Gamma,\text{rel}}[\phi, \gamma; \delta_i]$ can be expressed as in (37). From appendix D we can perturbatively solve the remaining fine tuning equations and from previous section the solution is $\rho_5$ in (36) and $\rho_6$ in (35).

6 Conclusions

In this paper we have fixed the renormalization conditions for the $SU(2)$ Yang-Mills theory, i.e. the nine couplings ($z_i$ and $\rho_i$, see (11)), which enter in the relevant part of the effective action $\Gamma[\phi, \gamma]$ as boundary conditions at $\Lambda = 0$ for the exact RG flow. Three of these couplings are fixed to give, at the subtraction point $\mu \neq 0$, the vector and ghost wave functions and the three vector coupling ($z_1 = z_2 = z_3 = 1$). In this way one fixes a contribution of the effective action to be the BRS classical action (for instance in the Feynman gauge, see (13)). The other six couplings $\rho_i$ in (12) are absent at tree level. They are at our disposal in order to implement the gauge symmetry for the physical effective action, i.e. the ST identities $\Delta_\Gamma[\phi, \gamma] = 0$ in (14). The fact that the three couplings $z_i = 1$ are not affected by loop corrections is one of the advantages of working at the physical point $\Lambda = 0$ rather than at $\Lambda = \Lambda' \neq 0$. (We shall recall later the difficulties arising from the non-vanishing subtraction point). We have shown perturbatively the following two results.

The first result (see section 3) is that it is possible to satisfy the ST identities $\Delta_\Gamma[\phi, \gamma] = 0$ if one is able to solve the fine tuning equations $\Delta_{\Gamma,\text{rel}}[\phi, \gamma; \delta_i] = 0$. It is a consequence of the RG equation (18) (see theorem 1 and 2) and of the fact that the starting point of the loop expansion is the classical BRS action which satisfies the ST identities $\Delta_\Gamma^{(\ell=0)} = 0$.

The second result (see sections 4 and 5) is that it is possible to solve (perturbatively) the eleven fine tuning equations $\delta_i = 0$ by fixing the six $\rho_i$ couplings. This is possible since, due to the consistency condition (25), there are only six independent $\delta_i$. Moreover we have constructed the solution and found that the relevant part of the effective action is

$$\Gamma_{\text{rel}}[\phi, \gamma; \rho_i] = S_{\text{BRS}} + \hbar \int d^4x \left\{ \frac{\rho_6}{2} v \cdot c \wedge c + \frac{g^2}{4} \rho_5 (A_\mu \wedge A_\nu) \cdot (A_\mu \wedge A_\nu) + \frac{g^2}{4} \rho_6 (A_\mu \cdot A_\nu) (A_\mu \cdot A_\nu) \right\}.$$ 

The only non-vanishing couplings $\rho_4$, $\rho_5$ and $\rho_6$ are given in (32), (36) and (35), respectively, in terms of irrelevant vertices of $\Gamma$ evaluated at the subtraction point. This form allows one to deduce the perturbative expansion since irrelevant vertices at loop $\ell$ involve relevant couplings at lower loops $\ell' < \ell$.

We have followed the method used by Becchi [5] in which the boundary conditions are taken at the non-physical point $\Lambda = \Lambda' \neq 0$. In that case one can define the relevant parameters by expanding around vanishing momenta thus avoiding irrelevant contributions in the consistency conditions.

Since we work at $\Lambda = 0$ we have needed non-vanishing subtraction points and introduced the operator $T_k^{(u)}$ which defines the relevant parts of the various functionals. This makes
the analysis of the ST identities more difficult since the operator $T_k^{(\mu)}$ mixes relevant and irrelevant parts when applied to the product of two functionals. As discussed in section 5 the fine tuning equation $\Delta_{\Gamma}[\phi, \gamma; \delta_i] = 0$ can be perturbatively solved provided that the local functional $\Delta_{\Gamma,\text{rel}}$ can be parametrized as in (37). This is valid if the relations (38) and (39) hold. Actually the relations (39) are not valid in general if $\mu \neq 0$ due to the presence of the irrelevant contributions generated by the $T_6^{(\mu)}$ operator. However we have shown that one can use the parametrization in (37) by proceeding in two steps. First one uses the fact that eq. (38) is valid also for $\mu \neq 0$. This allows one to solve the fine tuning equations for the vertices of $\Delta_{\Gamma}$ with two and three fields. Once these equations are solved the remaining relations (39) hold since the irrelevant contributions vanish. In this way one can use (37) and solve the remaining fine tuning equations.

The method is general. As shown in ref. [17] it can be applied for instance to $SU(2)$ gauge theory with fermions. The application to the case of chiral gauge theory without anomalies should be also possible along the same lines.

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Appendix A

The generating functional (1) is equivalent to

\[
N[j, \gamma; \Lambda] \int D\phi D\phi_1 \exp i\left\{-\frac{1}{2}\left(\phi p^2\phi\right)_{0\Lambda} - \frac{1}{2}\left(p_1\phi_1 p^2\phi_1\right)_{\Lambda\Lambda_0} + (j\phi)_{0\Lambda} + S_{\text{int}}[\phi + \phi_1, \gamma; \Lambda_0]\right\},
\]

where

\[
K_{0\Lambda_0}(p) = K_{0\Lambda}(p) + K_{\Lambda\Lambda_0}(p)
\]

and the coefficient \(N\) is given in eq. (3). This can be easily seen by making in (40) the change of variables \(\phi_1 = \phi' - \phi\), which gives

\[
N[j, \gamma; \Lambda] \int D\phi' \exp i\left\{-\frac{1}{2}\left(\phi' p^2\phi'\right)_{\Lambda\Lambda_0} + S_{\text{int}}[\phi', \gamma; \Lambda_0]\right\}
\times \int D\phi \exp i\left\{-\frac{1}{2}\left(\phi p^2\phi\right)_{0\Lambda} - \frac{1}{2}\left(p\phi p^2\phi\right)_{\Lambda\Lambda_0} + (j_1\phi)_{\Lambda\Lambda_0} + (j\phi)_{0\Lambda}\right\},
\]

where the source \(j_1(p)\) is

\[
j_1(p) = p^2 (A^\alpha_\mu(p), -c^\alpha(p), c^\alpha(p)).
\]

By performing the integral over the field \(\phi\), which is gaussian, one obtains (1). On the other hand by integrating over the field \(\phi'\) in (41) and using (4) one obtains (2).

Appendix B

We explicitly give the form of the operator \(T_4^{(\mu)}\), which extracts the relevant part of a functional of a multicomponent massless scalar field \(\psi_i\). Due to the masslessness of the field, the Taylor expansion around vanishing momenta is affected by infrared singularities when we consider the four dimensional terms in the fields and momenta. Thus one is forced to introduce a non-vanishing subtraction point. For the two field components this point is assumed at \(p^2 = \mu^2\), while for the \(N\) field components it is assumed at the symmetric point \(NSP\) defined by

\[
\bar{p}_i\bar{p}_j = \frac{\mu^2}{N - 1} (N\delta_{ij} - 1), \quad N = 3, 4, \ldots.
\]

In order to have a more compact notation, we also introduce the Fourier transform \(\tilde{\psi}_i\) of the field \(\psi_i\). The form of \(T_4^{(\mu)}\) is then

\[
T_4^{(\mu)} F[\psi] \equiv F[0] + \int d^4 x \psi_i(x) \left\{\frac{\delta F}{\delta \psi_i(0)}\right\}_{\psi=0} + \frac{1}{2} \psi_j(x) \left\{\frac{\delta F}{\delta \psi_j(0)}\right\}_{\psi=0} + \frac{1}{6} \psi_j(x) \psi_k(x) \left\{\frac{\delta F}{\delta \psi_j(0) \delta \psi_k(0)}\right\}_{\psi=0}
\]

\[
+ \frac{i}{2} \partial_\mu \psi_j(x) \left[\frac{\partial}{\partial p_\mu} \frac{\delta F}{\delta \psi_j(0)}\right]_{p^2=\mu^2, \psi=0}
\]

\[
- \frac{1}{2} \partial_\mu \partial_\nu \psi_j(x) \left[\frac{\partial}{\partial p_\mu \partial p_\nu} \frac{\delta^2 F}{\delta \psi_j(0)}\right]_{p^2=\mu^2, \psi=0}
\]

\[
+ \frac{i}{6} \psi_j(x) \partial_\mu \psi_k(x) \left[\frac{\partial^2}{\partial p_\mu \partial p_\nu} \frac{\delta^3 F}{\delta \psi_j(0) \delta \psi_k(0)}\right]_{p_i=3SP, \psi=0}
\]

\[
+ \frac{1}{24} \psi_j(x) \psi_k(x) \psi_l(x) \left[\frac{\delta^4 F}{\delta \psi_j(p_1) \delta \psi_k(p_2) \delta \psi_l(p_3) \delta \psi_i(0)}\right]_{p_i=4SP, \psi=0}.
\]
In the following we explicitly give the relevant part of the effective action $\Gamma[\phi, \gamma]$ for the SU(2) Yang-Mills theory. We first isolate the vertices of $\Gamma[\phi, \gamma]$ which contain the relevant couplings

$$\Gamma[\phi, \gamma] = \frac{1}{2} \int p \mu \nu \rho \sigma (p) A_{\mu}^{a}(p) A_{\nu}^{a}(-p) A_{\rho}^{a}(p) + \frac{1}{3!} \epsilon^{a b c} \int \int \int \Gamma_{\mu \nu \rho \sigma}^{(3A)}(p, q, r, s) A_{\mu}^{a}(p) A_{\nu}^{b}(q) A_{\rho}^{c}(r)$$

$$+ \frac{1}{4!} \int \int \int \Gamma_{\mu \nu \rho \sigma}^{(4A)}(p, q, r, s) A_{\mu}^{a}(p) A_{\nu}^{b}(q) A_{\rho}^{c}(r) A_{\sigma}^{d}(s)$$

$$+ \int \int \int \int \Gamma_{\mu \nu \rho \sigma}^{(w c)}(p, q, r) u_{\mu}^{a}(p) c_{\nu}^{b}(q) c^{c}(r) + \ldots ,$$

where $r = -p - q$, $h = -p - q - k$ and the dots stand for all the remaining terms which contain only irrelevant vertices, since they are coefficients of monomials in the fields and sources with dimension higher than four.

The six vertices in (42) contain 9 relevant couplings which are defined as follows:

$$\Gamma_{\mu \nu}^{(3A)}(p) = g_{\mu \nu} [\sigma_{\mu A} + p_{\mu} (1 + \sigma_{\alpha}) + \Sigma_{\mu}(p)] + t_{\mu \nu}(p) [\sigma_{A} + \Sigma_{T}(p)] ,$$

$$\Gamma_{\mu \nu \rho \sigma}^{(3A)}(p, q, r) = [(p - q) g_{\mu \rho} + (q - r) g_{\nu \sigma} + (r - p) g_{\rho \sigma}] [\sigma_{A} + \Sigma_{T}(p)] ,$$

$$\Gamma_{\mu \nu \rho \sigma}^{(4A)}(p, q, r, s) = t_{\mu \nu \rho \sigma}^{(4A)}[\sigma_{A} + \Sigma_{T}(p)] = t_{\mu \nu \rho \sigma}^{(4A)}[\sigma_{A} + \Sigma_{T}(p)] ,$$

$$\Gamma_{\mu \nu \rho \sigma}^{(w c)}(p, q, r) = g_{\mu \nu} [\sigma_{A} + \Sigma_{T}(p)] + t_{\mu \nu \rho \sigma}^{(w c)}[\sigma_{A} + \Sigma_{T}(p)] ,$$

where $t_{\mu \nu}(p) = p_{\mu} g_{\nu \rho} - p_{\nu} g_{\mu \rho} ,

$$t_{\mu \nu \rho \sigma}^{(4A)} = (\epsilon^{a b c d} \epsilon^{e a b c} - \epsilon^{a b c d} \epsilon^{e a b c}) g_{\mu_{1} \mu_{2}} g_{\mu_{3} \mu_{4}} + (2 \leftrightarrow 3) + (3 \leftrightarrow 4)$$

is the four vector SU(2) structure appearing in the BRS action and

$$t_{\mu \nu \rho \sigma}^{(w c)} = (\delta^{a b} \delta^{c d} + \delta^{a c} \delta^{b d} + \delta^{a d} \delta^{b c}) (g_{\mu \nu} g_{\rho \sigma} + g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) .$$

The relevant couplings are defined by the conditions

$$\Sigma_{L}(0) = 0 , \quad \frac{\partial \Sigma_{L}(p)}{\partial p^{2}}|_{p^{2} = \mu^{2}} = 0 , \quad \Sigma_{T}(p)|_{p^{2} = \mu^{2}} = 0 ,$$

$$\Sigma_{3}^{(3A)}(p, q, r)|_{3SP} = 0 , \quad \Sigma_{1}^{(4A)}(p, q, k, h)|_{4SP} = 0 , \quad \Sigma_{2}^{(4A)}(p, q, k, h)|_{4SP} = 0 ,$$

From these conditions we can factorize in the vertices $\Sigma_{i}$ a dimensional function of $p$. Thus $\Sigma_{i}$ are “irrelevant” and contribute to the irrelevant part of the functional $\Gamma[\phi, \gamma]$. Similarly

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3 Notice that in order to follow the general rule of extracting the relevant part of a functional given by the operator $T_{4}^{(m)}$, the couplings $\sigma_{A}$ and $\sigma_{w c}$ are defined in a slightly different way with respect to ref. [13].
the vertices \( \tilde{\Gamma}_i \) are irrelevant since their Lorentz structure is (partially in the case of \( \tilde{\Gamma}^{(4A)} \)) given by external momenta.

We recall that the ghost propagator and the \( \bar{c}-c-A \) vertex are given in terms of the vertices \( \Gamma^{(wc)}_\mu \) and \( \Gamma^{(wcA)}_{\mu\nu} \) by

\[
\Gamma^{(\bar{c}c)}(p) = p^2 + ip^2[\sigma_{wc} + \Sigma^{(wc)}(p)], \quad \Gamma^{(\bar{c}cA)}_{\nu}(p, q, r) = -igp_\mu \Gamma^{(wcA)}_{\mu\nu}(p, q, r).
\]

In subsect. 2.1 in order to better identify the physical couplings in the effective action and study the gauge symmetry we have made a different definition of the relevant part of the effective action (see eq. (11)), introducing the couplings \( z_i \) and \( \rho_i \). The relation of these couplings with the \( \sigma_i \) is the following

\[
z_1 = 1 - \sigma_A - \sigma_\alpha, \quad z_2 = 1 + i\sigma_{wc}, \quad z_3 = 2 - \frac{i}{g}\sigma_{3A} - \sigma_A - \sigma_\alpha, \quad \rho_1 = \sigma_{mA}, \quad \rho_2 = \sigma_\alpha, \quad \rho_3 = 1 + i\sigma_{wc} + \sigma_{wcA},
\]

\[
\rho_4 = \sigma_{vcc} + i\sigma_{wc}, \quad \rho_5 = 1 - \sigma_A - \sigma_\alpha + \frac{1}{g^2}(\sigma_{4A} + \frac{1}{2}\sigma_{4A}'), \quad \rho_6 = \frac{3}{2g^2}\sigma_{4A}'.
\]

In particular when the physical boundary conditions \( z_1 = z_2 = z_3 = 1 \) are imposed one finds

\[
\sigma_\alpha + \sigma_A = 0, \quad \sigma_{wc} = 0, \quad \sigma_{3A} = -ig, \quad \sigma_{mA} = \rho_1, \quad \sigma_\alpha = \rho_2, \quad \sigma_{wcA} = -1 + \rho_3, \quad \sigma_{vcc} = \rho_4, \quad \sigma_{4A} = g^2(-1 + \rho_5 - \frac{1}{3}\rho_6), \quad \sigma_{4A}' = \frac{2}{3}g^2\rho_6,
\]

which show that these boundary conditions fix the vector and ghost wave function normalization and the three vector coupling. The other couplings in \( \Gamma[\phi, \gamma] \) are given in terms of the \( \rho_i \) and determined by the symmetry as in section 4.

**Appendix C**

We now extract the relevant part of the most general one dimensional functional of fields and sources with ghost number one. We call this generic functional \( \Delta \). First of all we isolate the vertices of \( \Delta \) which contain the relevant couplings

\[
\Delta = \int_p \Delta^{(Ac)}_{\mu}(p)A^a_\mu(-p)c^a(p) + \frac{1}{2}e^{abc} \int_p \int_q \Delta^{(AAc)}_{\mu\nu}(p, q, r)A^a_\mu(p)A^b_\nu(q)c^c(r) + \frac{1}{2}e^{abc} \int_p \int_q \Delta^{(wc)}_{\mu}(p, q)u^a_\mu(p)c^b(q)c^c(r) + \frac{1}{2}e^{abcd} \int_p \int_q \int_k \Delta^{(wAc)}_{\mu}(p, q, k, h)u^a_\mu(p)A^b_\nu(q)c^c(k)c^d(h) + \frac{1}{6}e^{abcd} \int_p \int_q \int_k \Delta^{(3Ac)}_{\mu\nu}(p, q, k, h)A^a_\mu(p)A^b_\nu(q)A^c_\rho(k)c^d(h) + \ldots,
\]

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where \( r = -p - q, \ h = -p - q - k \) and the dots stand for the remaining terms which are all irrelevant. Then we define the 11 relevant parameters as follows

\[
\begin{align*}
\Delta^{(Ac)}(p) &= p_\mu [\delta_1 + p^2 \delta_2 + \Delta^{(Ac)}(p)], \\
\Delta^{(AAc)}(p, q, r) &= g_{\mu\nu}(p^2 - q^2)[\delta_3 + \Delta^{(AAc)}(p, q, r)] \\
&+ (p_\mu p_\nu - q_\mu q_\nu)[\delta_4 + \Delta^{(AAc)}(p, q, r)] + \Delta^{(AAc)}(p, q, r), \\
\Delta^{(wcc)}(p, q, r) &= p_\mu [\delta_5 + \Delta^{(wcc)}(p, q, r)], \\
\Delta^{(wAcc)\alpha\beta\gamma\delta}(p, q, k, h) &= g_{\mu\nu}(\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_6 + \Delta^{(wAcc)}(p, q, k, h)] + \Delta^{(wAcc)\alpha\beta\gamma\delta}(p, q, k, h), \\
\Delta^{(3Ac)\alpha\beta\gamma\delta}(p, q, k, h) &= (\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_7 + \Delta^{(3Ac)}(p, q, k, h)] \\
&+ (\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_8 + \Delta^{(3Ac)}(p, q, k, h)] \\
&+ (\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_9 + \Delta^{(3Ac)}(p, q, k, h)] \\
&+ (\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_10 + \Delta^{(3Ac)}(p, q, k, h)] \\
&+ (\delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta})[\delta_11 + \Delta^{(3Ac)}(p, q, k, h)] + \Delta^{(3Ac)\alpha\beta\gamma\delta}(p, q, k, h),
\end{align*}
\]

where the dots stand for permutation over the gluon momenta and Lorentz and colour indices. The conditions defining the relevant parameters are

\[
\begin{align*}
\Delta^{(Ac)}(0) &= \frac{\partial}{\partial p^2} \Delta^{(Ac)}(p)|_{p^2 = \mu^2} = 0, \\
\Delta^{(AAc)}|_{3SP} &= 0, \\
\Delta^{(wcc)|3SP} &= 0, \\
\Delta^{(3Ac)|4SP} &= 0, \\
\Delta^{(wAcc)|4SP} &= 0.
\end{align*}
\]

Due these conditions one can isolate in these vertices a dimensional function of \( p \), thus they are irrelevant. Similarly the vertices \( \Delta \) have the Lorentz indices carried by momenta in a different way with respect to their relevant parts and are irrelevant.

**Appendix D**

In this appendix we prove that the fine tuning equations (24) can be solved if \( \Delta_{\Gamma,rel} \) satisfies

\[
\Delta_{\Gamma,rel}[\phi, \gamma; \delta_i] = S_{\Gamma^{(0)}} \Gamma_{rel}[\phi, \gamma; \bar{\rho}_i].
\]

An explicit calculation shows that eq. (43) implies the relations (38) and (39) among the \( \delta_i \). The parameters \( \bar{\rho}_i \) are given by the following functions of \( \delta_i \)

\[
\begin{align*}
\delta_1 &= -i \bar{\rho}_1, \quad \delta_2 = -i \bar{\rho}_2, \quad \delta_3 = \bar{\rho}_5, \\
\delta_5 &= -i (\bar{\rho}_6 - \bar{\rho}_5), \quad \delta_7 = ig(\bar{\rho}_5 - \bar{\rho}_3 + \bar{\rho}_4), \quad \delta_8 = -2ig(\bar{\rho}_5 - \bar{\rho}_3).
\end{align*}
\]

Now we show that it is possible to select the six parameters \( \rho_i^{(\ell)} \) in \( \Gamma_{rel}[\phi, \gamma; \rho_i^{(\ell)}] \) in such a way that six fine tuning equations, \( \delta_i^{(\ell)} = 0 \), are satisfied.

From (44) we have

\[
\begin{align*}
\Delta^{(\ell)}[\phi, \gamma] &= 2 S_{\Gamma^{(0)}} \Gamma^{(\ell)} + \sum_{k=1}^{\ell-1} S_{\Gamma^{(k)}} \Gamma^{(\ell-k)}.
\end{align*}
\]
By applying $T_5^{(μ)}$, we obtain the relevant part

$$\Delta_{\text{rel}}[\phi, \gamma; δ_5^{(ℓ)}] = 2 S_{Γ^{(0)}} Γ'_{\text{rel}}[\phi, \gamma; ρ_i^{(ℓ)}] + Ω^{(ℓ)}[\phi, \gamma],$$

where

$$Ω^{(ℓ)}[\phi, \gamma] = T_5^{(μ)} \sum_{k=1}^{ℓ-1} S_{Γ^{(k)}} Γ_{(ℓ-k)}^{(μ)} + 2 \left( T_5^{(μ)} S_{Γ^{(0)}} - S_{Γ^{(0)}} T_4^{(μ)} \right) Γ^{(ℓ)}. \quad (45)$$

The crucial observation now is that $Ω^{(ℓ)}$ does not depend on the relevant parameters $ρ_i^{(ℓ)}$. This is obvious, since the product of two relevant vertices is a relevant vertex. This implies

$$T_5^{(μ)} S_{Γ^{(0)}} T_4^{(μ)} = S_{Γ^{(0)}} T_4^{(μ)}. \quad (46)$$

As a consequence $(T_5^{(μ)} S_{Γ^{(0)}} - S_{Γ^{(0)}} T_4^{(μ)}) Γ'_{\text{rel}} = 0$ and $Ω^{(ℓ)}$ does not receive contribution from the couplings $ρ_i^{(ℓ)}$. Thus the last term in (45) involves only $Γ^\irr_{ℓ}[ϕ, γ]$ which is given in terms of $ρ_i^{(ℓ')} = \bar{ρ}_i^{(ℓ)}$ at lower loops $ℓ' < ℓ$.

Eq. (43) implies that $Ω^{(ℓ)}$ must be of the form

$$Ω^{(ℓ)} = S_{Γ^{(0)}} Γ'_{\text{rel}}[ϕ, γ; ρ_i^{(ℓ')}], \quad ρ_i^{(ℓ')} = ρ_i^{(ℓ)} - 2ρ_i^{(ℓ')}.$$

As previously observed, $ρ_i^{(ℓ')}$ are known from the calculation of $ρ_i^{(ℓ')}$ at loop $ℓ' < ℓ$. Therefore, by fixing

$$ρ_i^{(ℓ')} = -\frac{1}{2} ρ_i^{(ℓ')}, \quad (46)$$

we have $\bar{ρ}_i = 0$ and from (44) we have the final result $Δ_{Γ}^{(ℓ)}[ϕ, γ] = 0$. 

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