Optimal Control of Double Integrator with Minimum Total Variation

C. Yalçın Kaya*

January 4, 2019

Abstract

We study the well-known minimum-energy control of double integrator, along with a simultaneous minimization of the total variation in the control variable. We derive optimality conditions and obtain an optimal solution for the combined problem. We study the problem from a multi-objective optimal control viewpoint, constructing the Pareto front. For a particular instance of the problem, we also derive an asymptotic optimal solution for the minimization of the total variation alone.

Key words: Optimal control, Minimum-energy control, Total variation, Multi-objective optimal control, Pareto front.

AMS subject classifications. Primary 49J15, 90C29 Secondary 49N05

1 Introduction

Double integrator is a mathematical model for a car in rectilinear motion on a flat and frictionless plane, as schematically illustrated in Figure 1. It also constitutes a model for analogous rotational-mechanical and electrical systems [14]. Because of its simplicity, optimal control of double integrator is studied virtually in every course of lectures on optimal control theory. Such a course is typically delivered in the final undergraduate year or at the postgraduate level in mathematics, as well as in the disciplines of economy and engineering. In the teaching of optimal control theory and its applications, although the minimum energy, minimum-effort and minimum-time control of double integrator are widely studied, minimization of total variation is not even considered, presumably because a maximum principle for the minimum-total-variation control does not exist as yet.

The double integrator model is so simple that a student can work out an analytical solution relatively easily for the problem of energy minimization, as a classroom exercise. Moreover, for the case of minimum-time control, where the control variable is bound-constrained, the optimal control can simply be shown to be bang–bang with at most one switching, i.e., the control variable switches from one bound to the other, and it does so at most once. The control structure can also be worked out easily in the case of minimum-effort control, where the $L^1$-norm of the control function is minimized. In summary, optimal control of double integrator yields simple but rich-enough examples for illustrations of some key aspects of the theory of optimal control [10].

Total variation of a function can be broadly described as the total vertical distance traversed by the graph of the function (a precise definition is to be given in Section 3.1). A small total

*School of Information Technology and Mathematical Sciences, University of South Australia, Mawson Lakes, S.A. 5095, Australia. E-mail: yalcin.kaya@unisa.edu.au.
variation in the control function is obviously desirable, as it would make the control system easier to design and implement, resulting in, for example, smaller or lighter motors for a robot or a spacecraft.

Although there is a lack of theory and results for the pure minimization of total variation, it is often imposed in addition to the minimization of another functional, for instance, energy or duration of time. This is done in the earlier works [9, 12, 13], where the optimal control problem is discretized directly by assuming piecewise-constant optimal control variables. This discretization simplifies the expression for the total variation in control; however, optimality conditions for the original (continuous-time) problem cannot be derived or verified, because of the discretization itself.

Total variation is widely used as a regularization term in more general optimization problems such as imaging and signal processing (see [3] and the references therein). It has also relatively recently been used as a regularization term for parameter estimation in linear quadratic control [7]. A bound on the total variation in the control is derived for minimum-time linear control problems in [11], although the total variation itself is not incorporated into the minimization problem.

In the present article, in addition to the minimization of energy, we consider the minimization of the total variation in the control variable of double integrator. In other words, we aim to study simultaneous minimization of energy and total variation, giving rise to multi-objective optimization and the study of the set of all trade-off/compromise solutions called the Pareto front. Optimal control problems involving total variation have not been studied yet from the view point of multi-objective optimal control.

In this paper, we use a tutorial approach. First, in Section 2, we introduce the double integrator model as well as the problem of energy minimization as an optimal control problem. This is a standard problem in optimal control; so, we derive the optimal solution without going into much of the details.

In Section 3, we define the total variation of a function and state the energy and total variation minimization problem, by appending the total variation in control as a weighted term to the energy functional. Next, we augment the state variable vector, so that the problem can be rewritten and posed as an optimal control problem in standard form. We derive optimality conditions, and discuss the problem as a multi-objective optimal control problem. We provide a video illustration of the multi-objective solutions on the Pareto front, so that an evolution of the solutions as the weight of total variation is varied can be animated and observed. Via asymptotic analysis, we derive an optimal solution for the pure total variation minimization problem. Such results on total variation do not exist in the literature.

Finally, in Section 4, we offer some concluding remarks and provide various relevant open problems for future work.
2 Minimum-energy Control

Consider the car as a point unit mass, moving on a frictionless plane ground in a fixed line of action, as shown in Figure 1. Let the position of the car at time \( t \) be given by \( y(t) \) and the velocity by \( \dot{y}(t) := (dy/dt)(t) \). By Newton’s second law of motion, \( \ddot{y}(t) = u(t) \), where \( u(t) \) is the summation of all the external forces applied on the car, in this case the force simply representing the acceleration and deceleration of the car. This differential equation model is referred to as the double integrator in system theory literature, since \( y(t) \) can be obtained by integrating \( u(t) \) twice.

Let \( x_1 := y \) and \( x_2 := \dot{y} \). The problem of minimizing the energy of the car, which starts at a position \( x_1(0) = s_0 \) with a velocity \( x_2(0) = v_0 \) and finishes at the final position \( x_1(1) = s_f \) with velocity \( x_2(1) = v_f \), within one unit of time, can be posed as follows:

\[
\begin{align*}
\text{(Pe)} & \quad \min \quad \frac{1}{2} \int_0^1 u^2(t) \, dt \\
& \quad \text{subject to} \quad \dot{x}_1(t) = x_2(t), \quad x_1(0) = s_0, \quad x_1(1) = s_f, \\
& \quad \dot{x}_2(t) = u(t), \quad x_2(0) = v_0, \quad x_2(1) = v_f.
\end{align*}
\]

Here, the functions \( x_1 \) and \( x_2 \) are referred to as the state variables and \( u \) the control variable. As a first step in writing the conditions of optimality for this optimization problem, define the Hamiltonian function \( H \) for Problem (Pe) in the usual way as

\[
H(x_1, x_2, u, \lambda_1, \lambda_2) := \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u,
\]

where \( \lambda(t) := (\lambda_1(t), \lambda_2(t)) \in \mathbb{R}^2 \) is the adjoint variable (or costate) vector such that (see [5])

\[
\dot{\lambda}_1 = -\partial H/\partial x_1 \quad \text{and} \quad \dot{\lambda}_2 = -\partial H/\partial x_2.
\]

Equations in (2) simply reduce to

\[
\lambda_1(t) = \bar{\lambda}_1 \quad \text{and} \quad \lambda_2(t) = -\bar{\lambda}_1 t - c,
\]

where \( \bar{\lambda}_1 \) and \( c \) are real constants. By calculus of variations, or the maximum principle with an unconstrained control variable (see [4] or [5]), if \( u \) is optimal, then

\[
\partial H/\partial u = 0, \quad \text{i.e.,} \quad u(t) = -\lambda_2(t) = \bar{\lambda}_1 t + c.
\]

Substituting \( u(t) \) in (4) into the differential equations and solving these equations by also utilizing the boundary conditions in Problem (Pe), one gets the analytical solution

\[
\begin{align*}
\lambda_1(t) &= \bar{\lambda}_1 t + c, \\
x_1(t) &= \frac{1}{6} \bar{\lambda}_1 t^3 + \frac{1}{2} c t^2 + v_0 t + s_0, \\
x_2(t) &= \frac{1}{2} \bar{\lambda}_1 t^2 + c t + v_0,
\end{align*}
\]

for all \( t \in [0, 1] \), where

\[
\bar{\lambda}_1 = -12 (s_f - s_0) + 6 (v_0 + v_f), \quad c = 6 (s_f - s_0) - 2 (2 v_0 + v_f).
\]

We note that the position variable \( x_1(t) \) of the car is a cubic polynomial of time. Therefore, the minimum-energy control solution, despite being so simple, constitutes a building block for the problem of finding a cubic spline interpolant passing through a given set of points.
3 Minimization of Total Variation

3.1 Total variation of a function

The total variation of a function $u : [t_0, t_f] \rightarrow \mathbb{R}$ is defined as

$$TV(u) := \sup_{N} \sum_{i=1}^{N} |u(t_i) - u(t_{i-1})|,$$  \hspace{1cm} (10)

where the supremum is taken over all partitions

$$t_0 < t_1 < \cdots < t_N = t_f$$  \hspace{1cm} (11)

of the interval $[t_0, t_f]$ (see [8]). Here, $N \in \{1, 2, 3, \ldots\}$ is arbitrary so is the choice of the values $t_1, \ldots, t_{N-1}$ in $[t_0, t_f]$ which, however, must satisfy (11). The function $u$ is said to be of bounded variation on $[t_0, t_f]$, if $TV(u)$ is finite. If $u$ is piecewise differentiable and continuous on $[t_0, t_f]$, then

$$TV(u) = \int_{t_0}^{t_f} |\dot{u}(t)| \, dt,$$  \hspace{1cm} (12)

where $\dot{u} := du/dt$. Practically speaking, $TV(u)$ as given in (12) represents the total distance traversed by the projection of the $u(t)$ vs. $t$ graph along the vertical $u(t)$ axis. Figure 2 illustrates this interpretation with $u(t) = \sin t$ over $[0, 3\pi/2]$, where clearly $TV(u) = 3$.

3.2 Minimum-total-variation control of double integrator

We consider optimal control problems where we aim to minimize the total variation in the control variables in addition to a general objective functional.

\[
(P_{tv}) \begin{cases} 
\min & \frac{1}{2} \int_0^1 u^2(t) \, dt + \alpha \, TV(u) \\
\text{subject to} & \dot{x}_1(t) = x_2(t), \quad x_1(0) = s_0, \quad x_1(1) = s_f, \\
& \dot{x}_2(t) = u(t), \quad x_2(0) = v_0, \quad x_2(1) = v_f,
\end{cases}
\]
where \( \alpha \geq 0 \) is referred to as the weight. Define the new control variable \( v(t) := \dot{u}(t) \) for a.e. \( t \in [0, t_f] \). Using (12), Problem (Ptv) can be reformulated as

\[
\begin{align*}
\text{(Paug)} \quad \min & \quad \frac{1}{2} \int_0^1 (u^2(t) \, dt + \alpha \, |v(t)|) \, dt \\
\text{subject to} & \quad \dot{x}_1(t) = x_2(t), \quad x_1(0) = s_0, \quad x_1(1) = s_f, \\
& \quad \dot{x}_2(t) = u(t), \quad x_2(0) = v_0, \quad x_2(1) = v_f, \\
& \quad \dot{u}(t) = v(t), \quad \text{for a.e. } t \in [0, t_f].
\end{align*}
\]

In this augmented form of the problem, \( u \) becomes a new state variable.

### 3.3 Optimality conditions

The Hamiltonian function for Problem (Paug) is given by

\[
H(x_1, x_2, u, v, \lambda_1, \lambda_2, \eta) := \frac{1}{2} u^2 + \alpha \, |v| + \lambda_1 x_2 + \lambda_2 u + \eta v,
\]

where \( \lambda(t) = (\lambda_1(t), \lambda_2(t)) \in \mathbb{R}^2 \) and \( \eta(t) \in \mathbb{R} \) are adjoint variables defined by (see [5])

\[
\lambda_1 := -\partial H/\partial x_1 = 0 \quad \text{and} \quad \lambda_2 := -\partial H/\partial x_2 = -\lambda_1, \\
\eta := -\partial H/\partial u = -u - \lambda_2, \quad \eta(0) = 0, \quad \eta(1) = 0,
\]

In other words,

\[
\lambda_1(t) = \bar{\lambda}_1, \quad \text{and} \quad \lambda_2(t) = -\bar{\lambda}_1 t - c, \\
\eta(t) = -u(t) + \bar{\lambda}_1 t + c, \quad \eta(0) = 0, \quad \eta(1) = 0.
\]

where \( \bar{\lambda}_1 \) and \( c \) are real constants. Note that, although the expressions in (14) are respectively the same as those in (2), the real constants \( \bar{\lambda}_1 \) and \( c \) in this case depend on the value of \( \alpha \) and so are different in general.

**Maximum Principle.** If \( v \) is an optimal control for Problem (Paug), then there exist continuously differentiable adjoint variables \( \lambda \) and \( \eta \), as defined in (14)–(15), such that \( (\lambda(t), \eta(t)) \neq 0 \) for all \( t \in [0, 1] \) and that

\[
v = \arg\min_{w \in \mathbb{R}} H(x_1, x_2, u, v, \lambda_1, \lambda_2, \eta) = \arg\min_{w \in \mathbb{R}} \alpha \, |w| + \eta \, w; \quad \text{for a.e. } t \in [0, 1],
\]

see e.g. [4,5]. Condition (18) implies that

\[
v(t) = \begin{cases} 
0, & \text{if } |\eta(t)| < \alpha, \\
\text{undetermined}, & \text{if } |\eta(t)| = \alpha,
\end{cases}
\]

for a.e. \( t \in [0, 1] \). Note that \( |\eta(t)| > \alpha \) is not allowed by the maximum principle, as otherwise one would get \( v(t) = -\infty \).

In view of (19), when \(-\alpha < \eta(t) < \alpha\), a.e. \( t \in [0, 1] \), the original control \( u(t) \) is (possibly piecewise) constant. What if \(|\eta(t)| \equiv \alpha\) over a subinterval of \([0, 1]\)? If so, then we refer to the optimal control in this subinterval as singular control, which we elaborate further next.

**Singular control.** If there exist \( s_1 \) and \( s_2 \) such that \(|\eta(t)| = \alpha\) for every \( t \in [s_1, s_2] \subset [0, t_f] \) (in fact, one has either \( \eta(t) = \alpha \) or \( \eta(t) = -\alpha \) for every \( t \in [s_1, s_2] \), because of the continuity of \( \eta \)), then the control variable \( v(t) \) for every \( t \in [s_1, s_2] \) is said to be singular. A candidate for a singular optimal control \( v(t) \) might be obtained by observing that, since \( \eta(t) \) is constant.
over \([s_1, s_2]\), one will have \(\dot{\eta}(t) = \ddot{\eta}(t) = 0\) for every \(t \in [s_1, s_2]\). By using (17), this observation yields
\[
\dot{\eta}(t) \equiv 0 = -u(t) - \lambda_2(t),
\]
i.e.,
\[
u(t) = \bar{\lambda}_1 t + c,
\]
and so
\[
v(t) = \bar{\lambda}_1,
\]
for all \(t \in [s_1, s_2]\).

**Optimal solution.** With the incorporation of the singular control, and by the continuity of the adjoint variable \(\eta\), (19) can be rewritten as
\[
v(t) = \begin{cases} 0, & \text{if } |\eta(t)| < \alpha, \\ \bar{\lambda}_1, & \text{if } |\eta(t)| = \alpha, \end{cases}
\]
for all \(t \in [0, 1]\). Note that \(v(t)\) in (20) is piecewise-constant and so \(u(t)\) is piecewise-linear and continuous in \(t\). Then, by (17), \(\eta(t)\) is continuous and piecewise-quadratic in \(t\). Note in particular that, differentiating both sides of the ODE in (17), using \(\ddot{u} = v\) and substituting (20), one gets
\[
\ddot{\eta}(t) = \begin{cases} \bar{\lambda}_1, & \text{if } |\eta(t)| < \alpha, \\ 0, & \text{if } |\eta(t)| = \alpha. \end{cases}
\]
The expression in (21) and the boundary conditions in (17) imply that there will be at most two junction points, \(0 < t_1 < t_2 < 1\), for \(\eta(t)\). Namely, either \(\eta(t) = \alpha\) or \(\eta(t) = -\alpha\), for \(t_1 < t < t_2\), and \(\eta(t)\) is quadratic in \(t\), for \(0 \leq t < t_1\) and \(t_2 < t \leq 1\), with the same constant second derivative \(\bar{\lambda}_1\). In other words,
\[
v(t) = \begin{cases} 0, & \text{if } 0 \leq t < t_1 \text{ or } t_2 \leq t \leq 1, \\ \bar{\lambda}_1, & \text{if } t_1 \leq t < t_2. \end{cases}
\]
Then from \(\ddot{u} = v\) and continuity of \(u\), one gets
\[
u(t) = \begin{cases} u_1, & \text{if } 0 \leq t < t_1, \\ u_1 + \bar{\lambda}_1 (t - t_1), & \text{if } t_1 \leq t < t_2, \\ u_3, & \text{if } t_2 \leq t \leq 1. \end{cases}
\]
where \(u_1\) and \(u_3\) are unknown constants. Subsequently, \(c = -\bar{\lambda}_1 t_1 + u_1\),
\[
\lambda_2(t) = \bar{\lambda}_1 (t_1 - t) - u_1,
\]
and
\[
\eta(t) = \begin{cases} \frac{1}{2} \bar{\lambda}_1 (t^2 - 2t_1 t), & \text{if } 0 \leq t < t_1, \\ \alpha \text{ or } -\alpha, & \text{if } t_1 \leq t < t_2, \\ \frac{1}{2} \bar{\lambda}_1 [t^2 + 2t_2 (1 - t) - 1], & \text{if } t_2 \leq t \leq 1. \end{cases}
\]
Note that \(\eta(t_1) = \eta(t_2)\) (both equal to \(\alpha\) or \(-\alpha\)), which, after some simple algebraic manipulations, yields
\[
t_1 = 1 - t_2.
\]

**Solution when \(\alpha = 0\).** This is the case when one has Problem (Pe)—minimizing only the energy. In this case, \(\eta(t) = 0\), and so \(v(t) = \bar{\lambda}_1\) and \(u(t) = \bar{\lambda}_1 t + c\), for all \(t \in [0, 1]\), the latter expression being the same as that in (4), resulting in the solution given in (5)–(9). In this case, clearly \(\text{TV}(u) = |12 \ (s_f - s_0) - 6 \ (v_0 + v_f)|\).
3.4 Multi-objective optimal control

Problem (Ptv), or equivalently Problem (Paug), concerns a simultaneous minimization of two objectives, which can simply be written as

\[
(P_{mo}) \quad \min_{u \in U} [\varphi_1(u), \varphi_2(u)],
\]

where

\[
\varphi_1(u) := \frac{1}{2} \int_0^1 u^2(t) \, dt \quad \text{and} \quad \varphi_2(u) := TV(u).
\]

Problem (Pmo) is referred to as a multi-objective, or vector, optimal control problem, with \(U\) representing the feasible, or admissible, set of all control functions satisfying the differential equation constraints and the boundary conditions—see [2] and the references therein. The set of all solutions of (26) is usually infinite, consisting of all trade-off, or Pareto, solutions. Broadly speaking, a Pareto solution is a solution where one cannot improve the value of one objective functional without making the other worse. The set of all solutions in the \(\varphi_1, \varphi_2\)-plane (or the value space) is referred to as the Pareto front of Problem (Pmo). An example of a Pareto front is given in Figure 3(a) (see details in Section 3.5).

For solving (26), a typical approach is to consider a scalarization of the vector objective and so reduce Problem (Pmo) to a single-objective optimal control problem. Note that \(\varphi_1\) and \(\varphi_2\) are convex and the constraint set represents linear differential equations and linear boundary conditions. Therefore we can use the weighted-sum scalarization (see [2]):

\[
(P_{s1}) \quad \min_{u \in U} \alpha_1 \varphi_1(u) + (1 - \alpha_1) \varphi_2(u),
\]

where \(\alpha_1 \in (0, 1]\). Since \(\alpha_1 \neq 0\), we can define \(\alpha := (1 - \alpha_1)/\alpha_1\) and write

\[
(P_{s2}) \quad \min_{u \in U} \varphi_1(u) + \alpha \varphi_2(u),
\]

with \(\alpha \in [0, \infty)\). Note that (29) is in the same form as Problem (Ptv).

In this case, the individual functionals in (27) can be calculated using (23) and (25), in terms of the unknown parameters \(t_1, u_1\) and \(u_3\), as follows.

\[
\varphi_1(u) = \frac{1}{2} \left[ (u_1^2 + u_3^2) t_1 + \frac{1}{3 \lambda_1} (u_3^3 - u_1^3) \right],
\]

\[
\varphi_2(u) = u_3 - u_1.
\]

3.5 Solution for a particular instance

We consider the particular instance when \(s_0 = 0, s_f = 0, v_0 = 1\) and \(v_f = 0\); i.e., the car with an initial unit velocity is required to come to rest in the same position where it started the motion. By using (23) and the initial conditions in Problem (Paug), one can obtain the following expressions for the state variables \(x_1(t)\) and \(x_2(t)\).

\[
x_1(t) = \begin{cases} 
\frac{1}{2} u_1 t^2 + t, & \text{if } 0 \leq t < t_1, \\
\frac{1}{3} \lambda_1 (t - t_1)^3 + \frac{1}{2} u_1 t^2 + t, & \text{if } t_1 \leq t < t_2, \\
\frac{1}{6} \lambda_1 (t_2 - t_1)^3 + \frac{1}{2} u_1 t_2^2 + t_2 + \frac{1}{2} u_3 (t - t_2)^2 + \left[ \frac{1}{2} \lambda_1 (t_2 - t_1)^2 + u_1 t_2 + 1 \right] (t - t_2), & \text{if } t_2 \leq t \leq 1;
\end{cases}
\]
Finally, writing out the terminal conditions $x_1(1) = 0$ and $x_2(1) = 0$ using (32) and (33), respectively, using (25), and carrying out lengthy manipulations, we obtain the following.

$$4t_1^3 - 3 \left(2 + \frac{1}{\alpha}\right) t_1^2 + 1 = 0 \quad (34)$$

$$\bar{\lambda}_1 = \frac{2\alpha}{t_1^2}, \quad (35)$$

$$u_1 = -\frac{\bar{\lambda}_1}{2}(1 - 2t_1) - 1, \quad (36)$$

$$u_3 = -(u_1 + 2). \quad (37)$$

All of the above relations are closed-form expressions except the cubic equation in (34), which also involves alternative signs in the coefficient of $t_1^2$. One can show that (34) will have a unique solution $t_1 \in [0, 1]$ with the coefficient $-3\left(2 + \frac{1}{\alpha}\right)$: Let $f_1(t_1) := 4t_1^3 - 3 \left(2 + \frac{1}{\alpha}\right) t_1^2 + 1$. Then for any $\alpha > 0$, $f_1(0) > 0$ and $f_1(1) < 0$. Moreover, $f_1'(t_1) := t_1 [12t_1^2 - 6 \left(2 + \frac{1}{\alpha}\right)] = t_1 [12 (t_1 - 1) - 6/\alpha] < 0$ for all $t_1 \in [0, 1]$ and $\alpha > 0$. Therefore $f_1(t_1)$ has a unique zero over $[0, 1]$, for any $\alpha > 0$.

On the other hand, the cubic equation in (34) does not have a solution $t_1 \in [0, 1]$ for every $\alpha > 0$, with the coefficient $-3\left(2 - \frac{1}{\alpha}\right)$: Let $f_2(t_1) := 4t_1^3 - 3 \left(2 - \frac{1}{\alpha}\right) t_1^2 + 1$. Then $f_2(t_1) = t_1^2 \left[4t_1^2 - 3 \left(2 - \frac{1}{\alpha}\right)\right] + 1 = 0$ if and only if $4t_1^2 - 3 \left(2 - \frac{1}{\alpha}\right) = -1 < 0$. With the requirement that $t_1 \in [0, 1]$, this inequality does not hold for every $\alpha > 0$, for example, for $\alpha < 1/2$. Therefore, in the rest of this paper, we will consider (34) with the coefficient $-3\left(2 + \frac{1}{\alpha}\right)$.

From the above paragraph, for certain values of $\alpha$, for example, for $\alpha > 2/3$, $f_2(t_1) = 0$ will have a solution. In fact, $f_2(t_1) = 0$ can have three real solutions in $[0, 1]$, which can easily be checked graphically. An analysis of these “auxiliary” solutions seems to be involved; therefore we leave them outside the scope of the current short paper.

**Solution when $\alpha = 0$.** The solution can be obtained directly, after substituting $s_0 = 0$, $s_f = 0$, $v_0 = 1$ and $v_f = 0$ into (5)–(9), as

$$u(t) = 6t - 4,$$

$$x_1(t) = t^3 - 2t^2 + t,$$

$$x_2(t) = 3t^2 - 4t + 1,$$

for $t \in [0, 1]$. In this case, clearly, $\text{TV}(u) = 6$.

**Multi-objective solution.** Figure 3 depicts the full Pareto front, as well as the optimal control variable for the parameter values $\alpha = 10^{-6}$, 0.05, 0.4 and $10^6$. In drawing the graphs, first, Equations (34)–(37) and (25) have been solved for the unknown parameters $t_1$, $t_2$, $\bar{\lambda}_1$, $u_1$, $u_3$, and $t_2$. Then $u(t)$, $\varphi_1(u)$ and $\varphi_2(u)$ have been computed as given in (23) and (30)–(31).

For a rather “continuous” range of values of $\alpha$, we have generated a movie, by using MATLAB, called `mintotalvar.avi` (submitted along with this manuscript), an instance of which for $\alpha = 0.589$ is shown in Figure 4. For a large number of values of $\alpha$, the movie
depicts/animates the Pareto front and the graphs of the control and state variables, as well as the graph of the adjoint variable $\eta(t)$ divided (or normalized) by $\alpha$. The graph of $\eta(t)/\alpha$ in the bottom-right corner reconfirms that $u(t)$ is constant when $|\eta(t)| \leq \alpha$ and $u(t)$ is linear in $t$ when $|\eta(t)| = \alpha$.

As expected, reduction in total control variation is obtained as the value of $\alpha$ is increased, with the trade-off that minimum energy is increased. Figure 3(b), and the movie, reveal that, as $\alpha$ gets larger, the control variable appears to become a piecewise-constant function, switching from the constant level $-3$ to the constant level $1$, resulting in $TV(u) = 4$. In the next subsection, by using an asymptotic analysis, we prove that, as $\alpha \to \infty$, $u$ indeed tends to a piecewise constant function.

3.5.1 Asymptotic solution for minimum total variation

As mentioned in Introduction, it is not possible to write down the necessary conditions of optimality for the minimization of the total variation in the control variable alone. Nevertheless, an analytic solution can still be obtained by studying the asymptotic behaviour of the solutions when $\alpha \to \infty$. In this case, Equation (34) becomes $4t_1^2 - 6t_1^2 + 1 = 0$, which means that $t_1 \to 1/2$. Then, by (25), $t_2 \to 1/2$. Moreover, from Equation (35), $\lambda_1 \to \infty$. However, these make the expression in (36) indeterminate. Therefore, we need to write the asymptotic expressions for the state variables (with $t_1 = t_2 = 1/2$), in order to proceed:

$$ x_1(t) = \begin{cases} \frac{1}{2} u_1 t^2 + t, & \text{if } 0 \leq t < 1/2, \\ \frac{1}{8} u_1 + \frac{1}{2} + \left(\frac{1}{2} u_1 + 1\right) \left(t - \frac{1}{2}\right) + \frac{1}{2} u_3 \left(t - \frac{1}{2}\right)^2, & \text{if } 1/2 \leq t \leq 1, \end{cases} $$

$$ x_2(t) = \begin{cases} u_1 t + 1, & \text{if } 0 \leq t < 1/2, \\ \frac{1}{2} u_1 + 1 + u_3 \left(t - \frac{1}{2}\right), & \text{if } 1/2 \leq t \leq 1. \end{cases} $$

Then the boundary conditions $x_1(1) = 0$ and $x_2(1) = 0$ give

$$ 3u_1 + u_3 = -8 \quad \text{and} \quad u_1 + u_3 = -2, $$
yielding the constant control values as $u_1 = -3$ and $u_3 = 1$. In other words, one gets an asymptotic expression for the control variable, as $\alpha \to \infty$, as

$$u(t) = \begin{cases} 
-3, & \text{if } 0 \leq t < 1/2, \\
1, & \text{if } 1/2 \leq t \leq 1.
\end{cases} \quad (40)$$

Clearly, as $\alpha \to \infty$, $\text{TV}(u) \to 4$. Finally, the asymptotic expressions for the state variables in (38) and (39) can be rewritten neatly as

$$x_1(t) = \begin{cases} 
-3t^2 + t, & \text{if } 0 \leq t < 1/2, \\
\frac{1}{2}(t-1)^2, & \text{if } 1/2 \leq t \leq 1,
\end{cases} \quad (41)$$

$$x_2(t) = \begin{cases} 
-3t + 1, & \text{if } 0 \leq t < 1/2, \\
t - 1, & \text{if } 1/2 \leq t \leq 1.
\end{cases} \quad (42)$$
4 Conclusion and Future Work

We have derived a solution to the optimal control problem of simultaneous minimization of energy and total variation in control for double integrator. The minimum-energy control problem which we have also considered is a special case of a general linear quadratic control problem. An approach similar to the one employed in the current paper can be employed for the more general linear quadratic control (or linear quadratic programming) problem where one is additionally concerned with the minimization of total variation, namely the problem

\[
\text{(LQPTV)} \begin{cases}
\min & \frac{1}{2} \int_0^1 [x(t)^T Q(t) x(t) + u(t)^T R(t) u(t)] \, dt + \alpha \text{TV}(u) \\
\text{subject to} & \dot{x}(t) = A(t) x(t) + B(t) u(t), \quad \text{for all } t \in [0,1], \\
& x(0) = x_0, \quad x(1) = x_f.
\end{cases}
\]

The time horizon in Problem (LQPTV) has been set to be \([0,1]\), but, without loss of generality, it can be taken to be any interval \([t_0, t_f]\), with \(t_0\) and \(t_f\) specified. The state variable vector \(x(t) \in \mathbb{R}^n\) and the control variable vector \(u(t) \in \mathbb{R}^m\). The time-varying matrices \(A : [0, 1] \to \mathbb{R}^{n \times n}\) and \(B : [0, 1] \to \mathbb{R}^{n \times m}\) are continuous, \(Q : [0, 1] \to \mathbb{R}^{n \times n}\) is symmetric positive definite and continuous in \(t\), and \(R : [0, 1] \to \mathbb{R}^{m \times m}\) is positive definite and continuous in \(t\). The initial and terminal states are specified as \(x_0\) and \(x_f\), respectively. Since there are more than just one control variable, i.e., \(u(t) = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m\), the total variation in (12) can be generalized for this case as

\[
\text{TV}(u) := \int_0^1 (|\dot{u}_1(t)| + \ldots + |\dot{u}_m(t)|) \, dt.
\]

It should be noted that the problem we have studied in the current paper fits into the above problem description (LQPTV) with \(n = 2, m = 1, Q = 0\) and \(R = 1\), and the appropriate constant system and control matrices \(A\) and \(B\).

The general linear quadratic problem is a convex problem, so the weighted-sum scalarization can still be used (see [2,6]) when it is combined with the minimization of total variation. However, for a generalization to nonconvex problems, a scalarization different from the weighted-sum scalarization needs to be considered. This requires some specialized numerical techniques in obtaining a solution—see [6] and the pertaining discussion therein for problems which also have constraints on the state and control variables.

It would be interesting to investigate the optimal control solutions associated with the auxiliary/multiple solutions of \(f_2(t_1) = 0\) that was discussed Section 3.5, in some future work.

References

[1] H. H. BAUSCHKE, R. S. BURACHIK, AND C. Y. KAYA, Constraint splitting and projection methods for optimal control of double integrator. ArXiv: 1804.03767v1, 2018. To appear in a Springer volume.

[2] H. BONNEL AND C. Y. KAYA, Optimization over the efficient set of multi-objective convex optimal control problems. J. Optim. Theory Appl., 147 (2010), 93–112.

[3] D. GONG, M. K. TAN, Q. F. SHI, A. VAN DEN HENGEN, AND Y. N. ZHANG, MPTV: Matching pursuit-based total variation minimization for image deconvolution. IEEE Trans. Image Proc., 28 (2019), 1851–1865.
[4] R. F. Hartl, S. P. Sethi, and R. G. Vickson, A survey of the maximum principles for optimal control problems with state constraints. SIAM Rev., 37 (1995), 181–218.

[5] M. R. Hestenes, Calculus of Variations and Optimal Control Theory. John Wiley & Sons, New York, 1966.

[6] C. Y. Kaya and H. Maurer, A numerical method for nonconvex multi-objective optimal control problems. Comput. Optim. Appl., 57 (3) (2014), 685–702.

[7] O. I. Kostyukova and M. A. Kurdina, Parametric identification problem with a regularizer in the form of the total variation of the control. Differential Equations (Differentsial’nye Uravneniya), 49 (2013), 1056–1068.

[8] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, Inc., U. S. A., 1978.

[9] R. Loxton, Q. Lin, and K. L. Teo, Minimizing control variation in nonlinear optimal control. Automatica, 49 (2013), 2652–2664.

[10] V. G. Rao and D. S. Bernstein, Naive control of the double integrator. ITEE Control Systems Magazine, October (2001), 86–97.

[11] D. B. Silin, Total variation of optimal control in linear systems. Translated from Matematicheskie Zametki, 31 (1982), 761–772.

[12] K. L. Teo and L. S. Jennings, Optimal control with a cost on changing control. Journal of Optimization Theory and Applications, 68 (1991), 33–357.

[13] Y. Wang, C. Yu, and K. L. Teo, A new computational strategy for optimal control problem with a cost on changing control. Numerical Algebra, Control and Optimization, 6 (2016), 339–364.

[14] P. E. Wellstead, Introduction to Physical System Modelling. Control Systems Principles, United Kingdom, 2000.