The Infrared Behavior of One-Loop Gluon Amplitudes at Next-to-Next-to-Leading Order

Zvi Bern
Department of Physics and Astronomy
University of California at Los Angeles
Los Angeles, CA 90095-1547, USA

Vittorio Del Duca
Particle Physics Theory Group
Dept. of Physics and Astronomy
University of Edinburgh
Edinburgh EH9 3JZ, Scotland, UK

and

Carl R. Schmidt
Department of Physics and Astronomy
Michigan State University
East Lansing, MI 48824, USA

Abstract

For the case of $n$-jet production at next-to-next-to-leading order in the QCD coupling, in the infrared divergent corners of phase space where particles are collinear or soft, one must evaluate $(n + 1)$-parton final-state one-loop amplitudes through $O(\epsilon^2)$, where $\epsilon$ is the dimensional regularization parameter. For the case of gluons, we present to all orders in $\epsilon$ the required universal functions which describe the behavior of one-loop amplitudes in the soft and collinear regions of phase space. An explicit example is discussed for three-parton production in multi-Regge kinematics that has applications to the next-to-leading logarithmic corrections to the BFKL equation.

*On leave of absence from I.N.F.N., Sezione di Torino, Italy.
The quest to obtain ever increasing precision in perturbative QCD requires calculations to higher orders in the QCD coupling, $\alpha_s$. Over the years, significant effort has been expended on computing the next-to-leading order (NLO) contributions to multi-jet rates within perturbative QCD. An important next step in the endeavor to obtain higher precision would be the computation of next-to-next-to-leading-order (NNLO) contributions to multi-jet rates. As an example, although the NLO contributions to $e^+e^- \rightarrow 3$ jets have been computed for some time now [1, 2], the NNLO contributions have yet to be obtained. A calculation of these NNLO contributions would be needed to further reduce the theoretical uncertainty in the determination of $\alpha_s$ from event shape variables [3]. Considerable effort has also been expended in the computation of leading and next-to-leading logarithmic contributions to the BFKL equations for parton evolution at small $x$ [4, 5]. These two problems are, of course, connected since the logarithms that are resummed in the BFKL equations also appear at fixed orders of perturbation theory. Indeed, the one-loop Lipatov vertex may be extracted [6, 7] from one-loop five-gluon helicity amplitudes [8].

In order to compute a cross section at NNLO, three series of amplitudes are required in the squared matrix elements: a) tree-level, one-loop, and two-loop amplitudes for the production of $n$ particles; b) tree-level and one-loop amplitudes for the production of $n + 1$ particles; c) tree-level amplitudes for the production of $n + 2$ particles. For the case of NNLO $e^+e^- \rightarrow 3$ jets the five-parton final-state tree [8] amplitudes, as well as four-parton final-state one-loop amplitudes exist in both helicity [11] and squared matrix-element forms [11], but as we discuss below, in order to be used in NNLO computations additional terms enter because of infrared issues. For the required two-loop three-parton final-state amplitudes no computations exist, as yet. Indeed, no two-loop amplitude computations exist for cases containing more than a single kinematic variable, except in the special cases of maximal supersymmetry [12].

Besides these amplitudes it is also important to have a detailed understanding of the infrared singularities that arise from virtual loops and from unresolved real emission when the momenta of particles become either soft or collinear. (For hadronic initial states infrared divergences are also associated with initial-state parton distribution functions.) These infrared divergences show themselves as poles in the dimensional regularization parameter $\epsilon = (4 - D)/2$. By the Kinoshita-Lee-Nauenberg theorem [13] the infrared singularities must cancel for sufficiently inclusive physical quantities. However, it is only when the various contributions are combined that the infrared singularities cancel.

At NLO the structure of the infrared singularities has been extensively studied. The singularities occur in a universal way, i.e. independent of the particular particle production amplitude considered. Accordingly, soft singularities have been accounted for by universal soft functions [14, 15], and collinear singularities by universal splitting functions [16]. A detailed discussion of the infrared singularities at NLO for $e^+e^- \rightarrow$ jets may be found, for example, in ref. [17].
At NNLO the situation is less developed, although some work has already been performed to illuminate the structure of infrared divergences. In particular, in the squared tree-level amplitudes, any two of the \( n + 2 \) produced particles can be unresolved; accordingly the ensuing soft singularities, collinear singularities, and mixed collinear/soft singularities, have been accounted for by double-soft functions [15], double-splitting functions and soft-splitting functions [19], respectively. Furthermore, the universal structure of the coefficients of the \( 1/e^4 \), \( 1/e^3 \) and \( 1/e^2 \) poles has also been determined [20] for the two-loop virtual contributions for \( n \)-particle productions.

In this letter we shall discuss the \((n+1)\)-parton final-state case when the soft or collinear particles are gluons. In the interference term between a one-loop amplitude for the production of \( n + 1 \) particles and its tree-level counterpart any one of the produced particles can be unresolved in the final state; the phase-space integration gives at most an additional double pole in \( \epsilon \). Therefore the expansion in \( \epsilon \) of the interference term starts with a \( 1/e^4 \) pole, from mixed virtual/real infrared singularities, and in order to evaluate it to \( \mathcal{O}(\epsilon^0) \), the \((n+1)\)-parton one-loop amplitude needs to be evaluated to \( \mathcal{O}(\epsilon^2) \). (A similar need to evaluate one-loop amplitudes to higher orders in \( \epsilon \) has been previously noted in NNLO deep inelastic scattering [21] and in the NLL corrections to the BFKL equation [22].) For the case of NNLO corrections to \( e^+ e^- \rightarrow 3 \) jets, this would be a rather formidable task given the already non-trivial analytic structure of the one-loop \( e^+ e^- \rightarrow 4 \) partons helicity amplitudes presented in ref. [10] through \( \mathcal{O}(\epsilon^0) \).

A much more practical approach is to evaluate the amplitudes to higher order in \( \epsilon \) only in the infrared-divergent regions of phase-space. In the collinear and soft regions the amplitudes factorize into sums of products of \( n \)-parton final-state amplitudes multiplied by soft or collinear splitting functions. (The splitting functions in this letter are for amplitudes; the Altarelli-Parisi ones are roughly speaking the squares of these.) It is these soft or collinear splitting functions and the \( n \)-parton final-state one-loop amplitudes that must be evaluated to higher order in \( \epsilon \). This is a much simpler task than evaluating the full \((n+1)\)-parton final-state amplitudes beyond \( \mathcal{O}(\epsilon^0) \).

Here we focus on the issue of supplementing one-loop \((n+1)\)-parton final-state amplitudes that are known to \( \mathcal{O}(\epsilon^0) \) with higher order in \( \epsilon \) pieces in the soft and collinear regions of phase space. The one-loop splitting functions have been given through \( \mathcal{O}(\epsilon^0) \) [23, 24], and the one-loop soft functions through \( \mathcal{O}(\epsilon^0) \) may be extracted from the known four- [23] and five-parton [8, 26, 24] one-loop amplitudes. Below, we provide the one-loop gluon splitting and soft functions to all orders in \( \epsilon \), leaving the calculational details and a complete listing of the one-loop splitting and soft functions, including fermions, to a forthcoming paper.

As an example, we apply the framework outlined above to the one-loop amplitude for three-parton production in multi-Regge kinematics [27], for which the produced partons are strongly ordered in rapidity. In NNLO and in next-to-leading-logarithmic corrections
to two-jet scattering, this amplitude appears in an interference term multiplied by its tree-level counterpart. Because of the rapidity ordering, phase-space integration does not yield any collinear singularities; however, the gluon which is intermediate in rapidity can become soft. Accordingly, the one-loop amplitude must be determined to $O(\epsilon^0)$ plus the contribution with the soft intermediate gluon evaluated to $O(\epsilon)$ [22]. To determine the soft gluon contribution we use our all orders in $\epsilon$ determination of the soft functions together with previous all orders in $\epsilon$ determinations of the four-gluon amplitudes [28, 23, 29].

We first briefly review properties of $n$-gluon scattering amplitudes, since we use these below. The tree-level color decomposition is (see e.g. ref.[30] for details and normalizations)

$$M_n^{\text{tree}}(1, 2, \ldots n) = g^{(n-2)} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}) m_n^{\text{tree}}(\sigma(1), \sigma(2), \ldots, \sigma(n)),$$

(1)

where $S_n/Z_n$ is the set of all permutations, but with cyclic rotations removed. We have suppressed the dependence on the particle polarizations $\varepsilon_i$ and momenta $k_i$, but label each leg with the index $i$. The $T^{a_i}$ are fundamental representation matrices for the Yang-Mills gauge group $SU(N_c)$, normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$. The color decomposition of one-loop multi-gluon amplitudes with adjoint states circulating in the loop is [31]

$$M_n^{1\text{-loop}}(1, 2, \ldots n) = g^n \sum_{j=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n,j}} \text{Gr}_{n;j}(\sigma) m_{n;j}^{1\text{-loop}}(\sigma(1), \ldots, \sigma(n)),$$

(2)

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$, $\text{Gr}_{n;1}(1) \equiv N_c \text{Tr}(T^{a_1} \ldots T^{a_n})$, $\text{Gr}_{n;j}(1) = \text{Tr}(T^{a_1} \ldots T^{a_{j-1}}) \text{Tr}(T^{a_j} \ldots T^{a_n})$ for $j > 1$, and $S_{n;j}$ is the subset of permutations $S_n$ that leaves the trace structure $\text{Gr}_{n;j}$ invariant. It turns out that at one-loop the $m_{n;j>1}$ can be expressed in terms of $m_{n;1}^{1\text{-loop}}$ [32], so we need only discuss this case in this letter. The amplitudes with fundamental fermions in the loop contain only the $m_{n;1}^{1\text{-loop}}$ color structures and are scaled by a relative factor of $1/N_c$.

The behavior of color-ordered one-loop amplitudes as the momenta of two color adjacent legs becomes collinear, is [23, 24]

$$m_{n;1}^{1\text{-loop}} \xrightarrow{a \parallel b} \sum_{\lambda = \pm} \left\{ \text{Split}^{\text{tree}}_{\lambda}(a^{\lambda}, b^{\lambda}) m_{n-1;1}^{1\text{-loop}}(K^\lambda, \ldots) + \text{Split}^{1\text{-loop}}_{\lambda}(a^{\lambda}, b^{\lambda}) m_{n-1}^{\text{tree}}(K^\lambda, \ldots) \right\},$$

(3)

where $\lambda$ represents the helicity, $m_{n;1}^{1\text{-loop}}$ and $m_{n}^{\text{tree}}$ are color-decomposed one-loop and tree sub-amplitudes with a fixed ordering of legs and $a$ and $b$ are consecutive in the ordering, with $k_a = zK$ and $k_b = (1-z)K$. The splitting functions in eq. [3] have square-root singularities in the collinear limit. For the case of only gluons, the tree splitting functions splitting into a positive helicity gluon (with the convention that all particles are outgoing) is

$$\text{Split}^{\text{tree}}_{+}(a^+, b^+) = 0, \quad \text{Split}^{\text{tree}}_{-}(a^-, b^-) = \frac{-1}{\sqrt{z(1-z)}} \frac{1}{[a b]}.$$
\[ \text{Split}^{\text{tree}}_{+}(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)\langle a \ b \rangle}}, \quad \text{Split}^{\text{tree}}_{+}(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)\langle a \ b \rangle}}, \quad (4) \]

where the remaining ones may be obtained by parity. The spinor inner products \[ \langle i \ j \rangle = \langle i^- | j^+ \rangle \] and \[ [i \ j] = \langle i^+ | j^- \rangle, \] where \[ |i^\pm\rangle \] are massless Weyl spinors of momentum \( k_i \), labeled with the sign of the helicity. They are antisymmetric, with norm \( |\langle i \ j \rangle| = |[i \ j]| = \sqrt{s_{ij}} \), where \( s_{ij} = 2k_i \cdot k_j \).

The one-loop splitting functions are,

\[ \text{Split}^{\text{1-loop}}_{+}(a^-, b^-) = (G^f + G^n) \text{Split}^{\text{tree}}_{+}(a^-, b^-), \]
\[ \text{Split}^{\text{1-loop}}_{+}(a^+, b^\mp) = G^n \text{Split}^{\text{tree}}_{+}(a^+, b^\mp), \]
\[ \text{Split}^{\text{1-loop}}_{+}(a^+, b^+) = -G^f \frac{1}{\sqrt{z(1-z)\langle a \ b \rangle}^2}. \quad (5) \]

The function \( G^f \) arises from the ‘factorizing’ contributions and the function \( G^n \) arises from the ‘non-factorizing’ ones described in ref. [34] and are given through \( \mathcal{O}(\epsilon^0) \) by [23, 24]

\[ G^f = \frac{1}{48\pi^2} \left( 1 - \frac{N_f}{N_c} \right) z(1-z) + \mathcal{O}(\epsilon), \quad (6) \]
\[ G^n = c_T \left[ -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{z(1-z)(-s_{ab})} \right)^\epsilon + 2 \ln(z) \ln(1-z) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon), \]

with \( N_f \) the number of quark flavors and

\[ c_T = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \quad (7) \]

As at tree-level, the remaining splitting functions can be obtained by parity. The explicit values were obtained by taking the limit of five-point amplitudes; the universality of these functions for an arbitrary number of legs was proven in ref. [34]. A listing of one-loop splitting functions through \( \mathcal{O}(\epsilon^0) \) also involving fermions may be found in refs. [23, 24]. To \( \mathcal{O}(\epsilon^0) \), these splitting functions are independent of the regularization scheme parameter,

\[ \delta_R = \begin{cases} 
1 & \text{HV or CDR scheme}, \\
0 & \text{FDH or DR scheme}, 
\end{cases} \quad (8) \]

where CDR denotes the conventional dimensional regularization scheme, HV the ’t Hooft-Veltman scheme, DR the dimensional reduction scheme, and FDH the ‘four-dimensional helicity scheme. (For further discussions on scheme choices see refs. [25, 35].)

We have extended the above results for one-loop gluon splitting, as well as similar ones for soft functions, to all orders in \( \epsilon \) in several ways. The first way is by following the methods of ref. [34] and extending the discussion to include soft limits, but being careful to keep all contributions to higher order in \( \epsilon \). In this method the contributions are divided into the classes of ‘factorizing’ contributions, that may be obtained directly from one-loop
three-point Feynman diagram calculations and from ‘non-factorizing’ contributions, that are linked to the infrared-singular poles in $\epsilon$. An important ingredient in this construction is that the set of all possible loop integral functions that may enter into an amplitude are known functions to all orders in $\epsilon$. The method makes clear the universality of the splitting and soft functions since it does not rely on the computation of any particular amplitude.

As a second independent method for obtaining the values of the splitting and soft functions we have computed the amplitudes $gggH$ using the effective $ggH$ coupling due to a heavy fermion loop, again being careful to keep all higher order in $\epsilon$ contributions. (For a discussion of the calculation valid through $O(\epsilon^0)$ see ref. [37].) This is a convenient amplitude from which to extract the splitting and soft functions since it involves only four-point kinematics with one massive leg; the massive leg $H$ ensures that the $gggH$ amplitude has well defined limits when gluons are collinear or soft. As a third independent check we have also verified that the non-factorizing contributions obtained for the special case of $N = 4$ supersymmetric amplitudes agree with the above determinations. (The $N = 4$ case has no factorizing contributions and does not provide a check of these.) The $N = 4$ four- five- and six-point amplitudes have been given to all orders in $\epsilon$ in ref. [38] making it straightforward to extract the collinear and soft limits in terms of limits of loop integral functions.

From these calculations, our results for the all orders in $\epsilon$ contributions to the functions appearing in splitting functions are

$$G^f = \frac{2c_T}{(3-2\epsilon)(2-2\epsilon)(1-2\epsilon)} \left(1 - \epsilon \delta_R - \frac{N_f}{N_c} \right) \left(\frac{\mu^2}{-s_{ab}}\right)^\epsilon \omega(1-z),$$

$$G^a = c_T \left(\frac{\mu^2}{-s_{ab}}\right)^\epsilon \frac{1}{\epsilon^2} \left[-\left(\frac{1-z}{z}\right)^\epsilon \Gamma(1-\epsilon)\Gamma(1+\epsilon) + 2 \sum_{k=1,3,5,...} \epsilon^k \text{Li}_k\left(\frac{-z}{1-z}\right)\right],$$

where the polylogarithms are defined as

$$\text{Li}_1(z) = -\ln(1-z),$$

$$\text{Li}_k(z) = \int_0^z \frac{dt}{t} \text{Li}_{k-1}(t) \quad (k = 2, 3, \ldots) \right\} = \sum_{n=1}^\infty \frac{z^n}{n^k},$$

It is not difficult to verify that eq. (3) agrees with eq. (6) through $O(\epsilon^0)$. Although not obvious, the expression for $G^a$ in eq. (4) is symmetric in $z \leftrightarrow (1-z)$; indeed its expansion to $O(\epsilon^2)$ may be written as

$$G^a = c_T \left(\frac{\mu^2}{-s_{ab}}\right)^\epsilon \left\{[z(1-z)]^{-\epsilon} \left[-\frac{1}{\epsilon^2} + \ln z \ln(1-z) - \frac{\pi^2}{6}\right] + \ln z \ln(1-z) - 2\epsilon \left[\text{Li}_3(z) + \text{Li}_3(1-z) - \zeta(3)\right] + \epsilon^2 \left[-\frac{1}{6} \ln z \ln(1-z) \ln^2[z(1-z)] - \frac{2}{3} \ln^2 z \ln^2(1-z)\right]\right\}.$$
\[ + \frac{\pi^2}{3} \ln z \ln(1 - z) - \frac{7}{360} \pi^4 \] + \mathcal{O}(\epsilon^3). \]

The behavior of one-loop amplitudes in the soft limit is very similar to the above. As the momentum \( k \) of an external leg becomes soft the color-ordered one-loop amplitudes become

\[ m^{1\text{-loop}}_{n:1}(..., a, k^\pm, b, ...) |_{k \to 0} = \text{Soft}^{\text{tree}}(a, k^\pm, b) m^{1\text{-loop}}_{n-1:1}(..., a, b, ...) + \text{Soft}^{1\text{-loop}}(a, k^\pm, b) m^{\text{tree}}_{n-1}(..., a, b, ...) , \]

with the tree-level soft functions

\[ \text{Soft}^{\text{tree}}(a, k^+, b) = \frac{\langle a b \rangle \langle a k \rangle \langle k b \rangle}{\langle a k \rangle \langle k b \rangle} , \quad \text{Soft}^{\text{tree}}(a, k^-, b) = \frac{-[a b]}{[a k][k b]} . \]

Following analogous methods as for the collinear case, we have computed the one-loop gluon soft function to all orders of \( \epsilon \), with the result,

\[ \text{Soft}^{1\text{-loop}}(a, k^\pm, b) = -\text{Soft}^{\text{tree}}(a, k^\pm, b) c_T \frac{1}{\epsilon^2} \left( \frac{\mu^2(-s_{ab})}{(s_{ak})(s_{kb})} \right) \epsilon \frac{\pi \epsilon}{\sin(\pi \epsilon)} . \]

The soft function [14] does not depend on \( N_f \) or \( \delta_R \) and through \( \mathcal{O}(\epsilon^2) \) it is

\[ \text{Soft}^{1\text{-loop}}(a, k^\pm, b) = -\text{Soft}^{\text{tree}}(a, k^\pm, b) c_T \left( \frac{\mu^2(-s_{ab})}{(s_{ak})(s_{kb})} \right) \epsilon \left( \frac{1}{\epsilon^2} + \frac{\pi^2}{6} + \frac{7\pi^4}{360 \epsilon^2} \right) + \mathcal{O}(\epsilon^3) . \]

Through \( \mathcal{O}(\epsilon^0) \) this agrees with the results that may be extracted from four- [23] and five-parton [8, 26, 24] one-loop amplitudes that are known through \( \mathcal{O}(\epsilon^0) \).

We now apply these results for one-loop splitting [3] and soft [14] functions to the case of three-gluon production in multi-Regge kinematics. To do so, we also need the exact four-gluon one-loop amplitude through \( \mathcal{O}(\epsilon) \). In fact, this is known exactly to all orders of \( \epsilon \). From the string-inspired decomposition of the (unrenormalized) one-loop four-gluon sub-amplitude [24, 32], we write

\[ m^{1\text{-loop}}_{4:1} = A_4^g + \left( 4 - \frac{N_f}{N_c} \right) A_4^f + \left( 1 - \frac{N_f}{N_c} \right) A_4^s , \]

where \( A_4^g, -A_4^f, \) and \( A_4^s \) are the contributions from an \( N = 4 \) supersymmetric multiplet, an \( N = 1 \) chiral multiplet, and a complex scalar, respectively. We can also write

\[ A_4^x = c_T m^{\text{tree}}_4 V^x , \quad x = g, f, s , \]

with \( m^{\text{tree}}_4 \) the corresponding tree-level subamplitude. The functions \( V^f \) and \( V^s \) depend on the helicity configuration and can be extracted to all orders in \( \epsilon \) from ref. [24] by taking the massless limit. For configurations of type \( m^{1\text{-loop}}_{4:1}(1^-, 2^-, 3^+, 4^+) \) this yields,

\[ V^f = -D_2(s_{23}) - \epsilon \frac{s_{23}}{s_{13}} D_4^{P=6-2\epsilon} \]

\[ V^s = 2 \left( \frac{1 - \epsilon \frac{s_{23}}{s_{12}}}{s_{12}} D_2^{P=6-2\epsilon}(s_{23}) + \frac{s_{23}}{s_{12}} \epsilon(1 - \epsilon) D_4^{P=8-2\epsilon} \right) , \]
while for configurations of type $m^{1-\text{loop}}_{4,1}(1^-, 2^+, 3^-, 4^+)$ we have

$$V^\ell = \frac{s_{23}}{s_{13}} \tilde{I}_2(s_{12}) + \frac{s_{12}}{s_{13}} \tilde{I}_2(s_{23}) - \frac{s_{12}s_{23}}{s_{13}^2} (1 - \epsilon) \tilde{I}_4^{D=6-2\epsilon},$$

$$V^s = 2 \left[ -\frac{s_{12}s_{23}(s_{12} - s_{23})}{s_{13}^3} \epsilon \left( \tilde{I}_3^{D=6-2\epsilon}(s_{23}) - \tilde{I}_3^{D=6-2\epsilon}(s_{12}) \right) - \frac{s_{12}s_{23}}{s_{13}^2} (1 - \epsilon) \tilde{I}_4^{D=8-2\epsilon} + \frac{s_{12}s_{23}}{s_{13}^2} \epsilon(1 - \epsilon) \tilde{I}_4^{D=8-2\epsilon} \right],$$

with

$$\tilde{I}_2(s) = \left( \frac{\mu^2}{-s} \right)^\epsilon \frac{1}{\epsilon(1 - 2\epsilon)},$$

$$\tilde{I}_2^{D=6-2\epsilon}(s) = \left( \frac{\mu^2}{-s} \right)^\epsilon \frac{1}{2\epsilon(1 - 2\epsilon)(3 - 2\epsilon)},$$

$$\tilde{I}_3^{D=4-2\epsilon}(s) = -\left( \frac{\mu^2}{-s} \right)^\epsilon \frac{1}{\epsilon^2},$$

$$\tilde{I}_3^{D=6-2\epsilon}(s) = \left( \frac{\mu^2}{-s} \right)^\epsilon \frac{1}{2\epsilon(1 - 2\epsilon)(1 - \epsilon)},$$

$$\tilde{I}_4^{D=4-2\epsilon} = \frac{2}{\epsilon^2} \left[ \left( \frac{\mu^2}{s_{23}} \right)^\epsilon \text{$_2F_1$} \left( -\epsilon, -\epsilon; 1 - \epsilon; 1 + \frac{s_{23}}{s_{12}} \right) \right. + \left( \frac{\mu^2}{s_{12}} \right)^\epsilon \text{$_2F_1$} \left( -\epsilon, -\epsilon; 1 - \epsilon; 1 + \frac{s_{12}}{s_{23}} \right),$$

$$\tilde{I}_4^{D=6-2\epsilon} = -\frac{1}{2(1 - 2\epsilon)} \left[ 2 \tilde{I}_4^{D=6-2\epsilon}(s_{23}) + 2 \tilde{I}_3^{D=4-2\epsilon}(s_{12}) \right],$$

$$\tilde{I}_4^{D=8-2\epsilon} = -\frac{1}{2(3 - 2\epsilon)} \left[ \frac{s_{12}s_{23}}{s_{13}} \tilde{I}_4^{D=6-2\epsilon}(s_{23}) + 2 \frac{s_{23}}{s_{13}} \tilde{I}_3^{D=6-2\epsilon}(s_{23}) + \frac{s_{12}}{s_{13}} \tilde{I}_3^{D=6-2\epsilon}(s_{12}) \right].$$

The functions $\tilde{I}_n$ are scalar loop integrals in the indicated dimension scaled by prefactors so as to make them dimensionless. The function $V^g$ obtained from the $N = 4$ multiplet $[25, 23, 38]$ has the same functional form for either helicity configuration. To all orders in $\epsilon$ it is

$$V^g = -\tilde{I}_4^{D=4-2\epsilon} - \epsilon \delta_R V^s. \tag{21}$$

Any partial amplitude of the type $m^{1-\text{loop}}_{4,1}$ may then be obtained from sums of permutations of the $m^{1-\text{loop}}_{4,1}$.

We next need the dispersive part of this amplitude in the high-energy limit, $s \gg t$. The leading color orderings of the sub-amplitudes of type $m^{1-\text{loop}}_{4,1}$ are $[3] (A^-, A'^+, B'^+, B^-), (A^-, B'^+, B^-, A'^+), (A^-, A'^+, B^-, B'^+)$, and $(A^-, B^-, B'^+, A'^+)$ where we take the Mandelstam variables to be $s = s_{AB}, t = s_{BB}', \text{ and } u = s_{AB}'$. In eqs. (18) and (21) orderings
of type $m_{4:1}^{1\text{-loop}}(-,-,+,+)$ occur with $s_{12} \to s$ and $s_{23} \to t$, while in eqs. (19) and (21) orderings of type $m_{4:1}^{1\text{-loop}}(-,+,-,+)$ occur with $s_{12} \to u$ and $s_{23} \to t$ or $s_{12} \to t$ and $s_{23} \to u$. However, for the second helicity configuration, the functions in eqs. (19) and (21) are symmetric under the exchange of $s_{12}$ and $s_{23}$. Thus we can limit the analysis to one ordering for each type.

Using the usual prescription $\ln(t) = \ln(-t) - i\pi$ (for $t < 0$), we have

$$\text{Re} \left( \frac{\mu^2}{t} \right)^\epsilon = \left( \frac{\mu^2}{-t} \right)^\epsilon \cos(\pi\epsilon).$$  \hspace{1cm} (22)

In the high-energy limit, $s \gg -t$,

$$2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{t}{s_{12}}\right) = \Gamma(1-\epsilon)\Gamma(1+\epsilon) + O\left(\frac{t}{s_{12}}\right)$$

$$= \frac{\pi\epsilon}{\sin(\pi\epsilon)} + O\left(\frac{t}{s_{12}}\right),$$  \hspace{1cm} (23)

$$\left(\frac{\mu^2}{s_{12}}\right)^\epsilon 2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{s_{12}}{t}\right) = \left(\frac{\mu^2}{-t}\right)^\epsilon 2F_1\left(-\epsilon,1;1-\epsilon;1+\frac{t}{s_{12}}\right)$$

$$= -\left(\frac{\mu^2}{-t}\right)^\epsilon \epsilon \left[\psi(1) - \psi(-\epsilon) + \ln \frac{s_{12}}{-t}\right] + O\left(\frac{t}{s_{12}}\right).$$

with $s_{12} \to s$ or $s_{12} \to u$. Since $u = -s - t$, for either choice of $s_{12}$ the dispersive parts are the same to the required accuracy, and we can take $s_{12} \to s$ in eqs. (23). Substituting eq. (22) and (23) in eq. (20), and using the identity

$$-\frac{\pi}{\sin(\pi\epsilon)} = \psi(1+\epsilon) - \psi(-\epsilon),$$  \hspace{1cm} (24)

we obtain

$$\text{Disp} \tilde{f}_4^{D=4-2\epsilon} = -2\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left[\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1) + \ln \frac{s}{-t}\right] + O\left(\frac{t}{s}\right).$$  \hspace{1cm} (25)

In addition, in the high-energy limit both eq. (18) with $s_{12} \to s$ and $s_{23} \to t$, and eq. (19) with $s_{12} \to u$ and $s_{23} \to t$ reduce to

$$V^f = -\left(\frac{\mu^2}{-t}\right)^\epsilon \frac{1}{\epsilon(1-2\epsilon)} + O\left(\frac{t}{s}\right),$$  \hspace{1cm} (26)

$$V^s = \left(\frac{\mu^2}{-t}\right)^\epsilon \frac{1}{\epsilon(1-2\epsilon)(3-2\epsilon)} + O\left(\frac{t}{s}\right),$$

while the function $V^g$, eq. (21), is obtained from eq. (25) and (26).

Using eqs. (16), (17), (21), (23) and (24), and the fact that the proportionality factor between each tree-level subamplitude and its one-loop correction is the same for all color
orderings, we obtain the unrenormalized four-gluon one-loop amplitude in the high energy limit to all orders in $\epsilon$,

$$\text{Disp} M_4^{\text{1-loop}}(A^-, A'^+, B'^+, B^-) = M_4^{\text{tree}}(A^-, A'^+, B'^+, B^-) g^2 c_T \left( \frac{\mu^2}{-t} \right)^\epsilon \frac{1}{\epsilon(1-2\epsilon)}$$  \hspace{1cm} (27)

$$\times \left\{ N_c \left[ 2(1-2\epsilon) \left( \psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1) + \ln \frac{s}{-t} \right) + \frac{1-\delta_R \epsilon}{3-2\epsilon} - 4 \right] + \frac{2(1-\epsilon)}{3-2\epsilon} N_f \right\}.$$

Following the methods of ref. \[6\], we can then extract the unrenormalized one-loop correction to the helicity-conserving vertex, to NLL accuracy,

$$\frac{\text{Disp} C_{gg}^{g(1)}(-p_a,p_a')}{C_{gg}^{0}(0)} = c_T \left( \frac{\mu^2}{-t} \right)^\epsilon \frac{1}{\epsilon(1-2\epsilon)}$$  \hspace{1cm} (28)

$$\times \left\{ N_c \left[ (1-2\epsilon) \left[ \psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1) \right] + \frac{1-\delta_R \epsilon}{2(3-2\epsilon)} - 2 \right] + \frac{1-\epsilon}{3-2\epsilon} N_f \right\}.$$

Eq. (28) is valid to all orders in $\epsilon$ for $\delta_R = 0$ and 1; it agrees with the one-loop correction to the helicity-conserving vertex computed in ref. \[8\] to $\mathcal{O}(\epsilon^0)$ and with the one computed in ref. \[27, 40\] to all orders in $\epsilon$, for $\delta_R = 1$.

In an inclusive high-energy two-jet cross section at NNLO and in the NLL corrections to the BFKL kernel, the five-gluon one-loop amplitude is multiplied by the corresponding tree-level amplitude with the intermediate gluon $k$ integrated over its phase space. Assuming multi-Regge kinematics, the only infrared divergence that can arise is in the soft limit for the intermediate gluon, $k \to 0$. It gives a single pole in $\epsilon$. Thus, in order to generate correctly all the finite terms in the squared amplitude, the five-gluon one-loop amplitude must be computed exactly to $\mathcal{O}(\epsilon^0)$, and must be augmented by the $\mathcal{O}(\epsilon)$ corrections in the soft limit for the intermediate gluon. To achieve that, we need the dispersive part of eq. (12) in the physical region, $s_{ab} > 0$, $s_{ak} > 0$, $s_{kb} > 0$. (The other leading color orderings yield the same result.) Using eq. (14), (22), and the identity (24), we can write the dispersive part of the soft function to all orders in $\epsilon$ as,

$$\text{Disp Soft}^{\text{1-loop}}(a,k^\pm,b) = -\text{Soft}^{\text{tree}}(a,k^\pm,b) \frac{c_T}{\epsilon^2} \left( \frac{\mu^2}{s_{ab}} \right)^\epsilon \frac{s_{ab}}{s_{ak} s_{kb}} \left[ 1 + \epsilon \psi(1-\epsilon) - \epsilon \psi(1+\epsilon) \right].$$  \hspace{1cm} (29)

In addition, using the strong rapidity ordering,

$$y_a \gg y_b \gg y_b; \quad |k_{a\perp}| \simeq |k_{\perp}| \simeq |k_{b\perp}|,$$

the mass-shell condition for the intermediate gluon gives

$$s_{ab} = \frac{s_{ak} s_{kb}}{|k_{\perp}|^2}.$$  \hspace{1cm} (30)

Subsequently, eq. (29) becomes,

$$\text{Disp Soft}^{\text{1-loop}}(a,k^\pm,b) = -\text{Soft}^{\text{tree}}(a,k^\pm,b) \frac{c_T}{\epsilon^2} \left( \frac{\mu^2}{|k_{\perp}|^2} \right)^\epsilon \left[ 1 + \epsilon \psi(1-\epsilon) - \epsilon \psi(1+\epsilon) \right].$$  \hspace{1cm} (32)
The soft limit for the tree-level five-gluon sub-amplitudes,

\[ m_5^{\text{tree}}(A^-, A'^+, k^\pm, B'^+, B^-) = \text{Soft}^{\text{tree}}(A', k^\pm, B') m_4^{\text{tree}}(A^-, A'^+, B'^+, B^-), \quad (33) \]

holds for arbitrary kinematics. Thus, the unrenormalized five-gluon one-loop amplitude in the multi-Regge kinematics, and in the soft limit for the intermediate gluon and to all orders in \( \epsilon \), is obtained by using eq. (12), with the four-gluon one-loop amplitude (27), the loop soft function (32), and eq. (33), yielding

\[
\begin{align*}
\text{Disp} M_5^{1\text{-loop}}(A^-, A'^+, k^\pm, B'^+, B^-)|_{k \to 0} &= g^2 c_T M_5^{\text{tree}}(A^-, A'^+, k^\pm, B'^+, B^-)|_{k \to 0} \\
&\times \left\{ \left( \frac{\mu^2}{-t} \right)^\epsilon \left\{ N_c \left[ -\frac{4}{\epsilon^2} + \frac{2}{\epsilon} \left( \psi(1+\epsilon) - 2\psi(1-\epsilon) + \psi(1) + \ln \frac{s}{-t} \right) \right] \\
&+ \frac{1}{\epsilon(1-2\epsilon)} \left( \frac{1 - \delta_R \epsilon}{3 - 2\epsilon} - 4 \right) \right\} + \frac{2(1-\epsilon)}{\epsilon(1-2\epsilon)(3-2\epsilon)} N_f \right\} \\
&- N_c \left( \frac{\mu^2}{|k_\perp|^2} \right)^\epsilon \frac{1}{\epsilon^2} \left[ 1 + \epsilon \psi(1-\epsilon) - \epsilon \psi(1+\epsilon) \right].
\end{align*}
\]

To \( O(\epsilon) \), eq. (34) reads

\[
\begin{align*}
\text{Disp} M_5^{1\text{-loop}}(A^-, A'^+, k^\pm, B'^+, B^-)|_{k \to 0} &= g^2 c_T M_5^{\text{tree}}(A^-, A'^+, k^\pm, B'^+, B^-)|_{k \to 0} \\
&\times \left\{ \left( \frac{\mu^2}{-t} \right)^\epsilon \left\{ N_c \left[ -\frac{4}{\epsilon^2} + \frac{2}{\epsilon} \left( \ln \frac{s}{-t} + \pi^2 - 64 \frac{9}{27} + \frac{22}{9} \zeta(3) \right) \right] + \frac{10}{9} \left( 56 \frac{27}{\epsilon} \right) - N_c \left( \frac{\mu^2}{|k_\perp|^2} \right)^\epsilon \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{3} \right) \right\} + O(\epsilon^0),
\end{align*}
\]

with \( \beta_0 = (11N_c - 2N_f)/3 \). We have checked that eq. (35) agrees to \( O(\epsilon^0) \) with the five-gluon one-loop amplitude, with strong rapidity ordering and in the soft limit for the intermediate gluon [8, 7]. The above result may be used to verify the virtual next-to-leading log corrections to the Lipatov vertex for use in the BFKL equation as is done in ref. [3].

The same type of analysis may be applied more generally to the problem of NNLO QCD corrections. The one-loop gluon splitting and soft functions that we have presented here are valid to all orders in the dimensional regularization parameter, \( \epsilon \). This allows them to be used in NNLO calculations with infrared singular phase space where terms of up to two powers in \( \epsilon \) are necessary. Previous explicit determinations of the one-loop collinear splitting functions [23, 24] were not performed to the required order in \( \epsilon \). A systematic discussion of the soft and collinear splitting functions and further calculational details, including the case of external fermions, will be presented elsewhere. In particular, these functions can be used to aid in the computation of NNLO contributions to \( e^+ e^- \to 3 \) jets once all the matrix elements are available. However, much more remains to be done before this is realized.
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