Labyrinth Chaos is not Hamiltonian but still has a Vector Potential

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Abstract

We provide here a comprehensive proof that the so called Labyrinth chaos systems, a member of the Thomas-Rössler (TR) class of systems do not admit a Hamiltonian; yet they admit a vector potential. The proof starts from the general case of TR systems, which are in general non conservative and we show that this is also true for the conservative (volume preserving) case known as ‘Labyrith chaos’ [2,3]. To our knowledge, this is the first instance reported where a conservative chaotic system does not, in principle, admit a Hamiltonian symplectic structure. Still, a vector potential is readily admissible and thus, constructed.

Keywords: Labyrinth Chaos, Non-Hamiltonian Dynamics, Conservative Dynamics, Vector Potential

1 Introduction

The investigations on Thomas-Rössler (TR) class of systems started with the work of René Thomas and Otto Rössler [1,2,3] while they were examining the role of feed-back circuits and their related logical (Boolean) structure, especially, for a necessary condition for the appearance of chaos. The biological relevance of this approach has been, ever since, a very active and fruitful area of research [5,7,8,6,10] since feedback circuits provide a framework of understanding basic dynamical features such as multistationarity, homeostasis and
memory. Moreover this method and classes of systems extend to the study of the fundamentals of emergence of complex behaviours from simple circuit structures in complex systems at large [9, 10]. The main results of this approach can be summarized in three general statements: (i) a positive circuit is necessary to display multiple stable states (ii) a negative circuit is necessary to have robust sustained oscillations (iii) a necessary condition for chaos is the presence of both a positive and a negative circuit. In particular a class of systems that they named “Labyrinth chaos” and the “Arabesques” [2, 4, 13] drew attention as examples of “elegant chaos”. This signifies a type of minimal, generic, models that highlight interesting peculiarities of chaotic behaviour [11].

The TR class of systems exhibit the whole repertoire of dynamics, periodicity, multistationarity, coexisting attractors, bifurcation scenarii etc. The TR systems with complex symmetric attractors, termed “Labyrinth chaos” [1, 2, 3] possess a peculiar special case of a chaotic state that occurs without any attractors and where the trajectories wonder via a fractional Brownian motion and in addition they produce very interesting symbolic dynamics [12]. Moreover, recently it was reported that coupled arrays of Labyrinth Chaotic systems are able to produce normative chimera-like states [13].

The class of systems can be expressed as a set of $N$ cyclically coupled ordinary differential equations, with real variables $X_i$ evolving in time $t \in \mathbb{R}$, and $i = 1, 2, 3\ldots N$, comprehensively written as:

$$\frac{dX_i}{dt} = -bX_i + F(X_{i+1}) \mod N$$

(1)

The dissipation parameter, $b$ serves as the only bifurcation parameter and, over the years, a variety of forms for the function $F(\cdot)$ have been studied [11, 2, 4, 13]. This class of systems has a great repertoire of behaviour ranging, as expected, from trivial fixed points to simple periodicity, complicated periodicity and chaos with coexisting strange attractors with their associated bifurcation scenarii, see also Fig. [1].
Figure 1: Typical trajectories of the 3-dimensional TR-system of Eq. (3), projected on the \((x, y)\)-plane with \((z = 0)\) as the bifurcation parameter \(b\) varies and after transient time, \(t = 400\).

Originally, drawing from its biologically inspired background, the function \(F(\cdot)\) has been considered and has been serving as a model of threshold-type of functions processes in its dynamics. Yet, \(F(\cdot)\) can be also periodic \[2\] and indeed this is where the novel dynamical behaviour has been observed when the dissipative term, \(-bX_i\), vanishes i.e when \(b = 0\). It is by now known that when \(F(\cdot) = \sin(\cdot)\) the necessary condition (iii) for chaos is satisfied, yet, not any attractors are present! The system possesses a countably infinite set of stationary points arranged in a \(N\)-dimensional lattice with its points where \(\sin(X_i) = 0\). These are the infinite fixed points of the system; and they are all unstable, since \(\left[\frac{\partial \sin(X_i)}{\partial X_i}\right](X_i = 2k\pi) = -\left[\cos(X_i)\right](X_i = 2k\pi) = -1\) for \(k \in \mathbb{Z}\). It is also notable that for \(b = 0\) the system conserves the phase space volume and it is tempting to be considered Hamiltonian, but as it turns out and we show here, it is not! Notice also is due to the fact that for \(b = 0\) and \(F(\cdot) = \sin(\cdot)\) the system is time-reversible \((t \rightarrow -t)\) and preserves parity \(\{X_i\} \rightarrow \{-X_i\}\) while, depending on the value of its dimensionality \(N\), a plethora of symmetries for this lattice is present.

Of course the simplest case of Labyrinth Chaos is when \(N = 3\), i.e. it comprises of the minimal set of three differential equations necessary for chaos. So, we shall now turn, and throughout the remaining of the paper, to this case as our representative system for Labyrinth Chaos.

In summary, this “particularly simple and mathematically elegant example of chaos” \[2,11\], it is cyclical symmetric with respect to \(\{x, y, z\}\); the parameter \(b\), solely governs the damping and the attractor(s) dimension in the familiar the range \(0 \neq b < 3\); and in its limit value \(b = 0\) we have conservative but not
Hamiltonian chaos. This impossibility of Hamiltonization we discuss in the last section along with the significance of this result, possible generalizations and implications with connection to other interestingly similar dynamical systems.

2 Labyrinth Chaos cannot be Hamiltonianized

In this section, we are going to show that the Thomas-Rössler system known as Labyrinth chaos, for \( N = 3 \), does not admit a Hamiltonian in the sense defined in [14]. In this approach, the starting point is an autonomous dynamical system

\[
\nabla H^T f_c(X) = 0
\]

\[\dot{X} = f(X)\] where the “dot” denotes the time-derivative, \( X \) an \( n \)-dimensional vector field and \( f \) a smooth function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) and the \( \nabla H^T \) being the transposed of the vector \( \nabla H \). The velocity vector field \( f(X) \) can be written as a sum of two vector fields, a conservative part, let say \( f_c(X) \) and a dissipative part, \( f_d(X) \) (see [15]). To find the Hamiltonian \( H(X) \) as an energy associated to the dynamical system, the procedure consists on drawing the function \( H(X) \) out of the partial differential equation (2).

The function \( H(X) \), solution of (2), can be considered as a candidate for the generalized Hamiltonian of the conservative part of the dynamical system. This “candidate” can be retained as the Hamiltonian \( H(X) \) if it can be written in the form \( \dot{X} = J(X)\nabla H \) where \( J(X) \) is a skew symmetric matrix satisfying the Jacobi’s closure condition [16]. If such a function \( H(X) \) exist, then it represents the energy associated to the dynamical system \( \dot{X} = f(X) \). This energy is dissipated due to the dissipative component \( f_d(X) \) of the velocity vector field according to the equation, \( \dot{H} = \nabla H^T f_d(x) \). We are going to show that it is not possible to find a function \( H(X) \), solution of the partial differential equation (1) for the following dynamical system, the so called Labyrinth chaos.

\[
\begin{align*}
\frac{dx}{dt} &= -bx + \sin(y) \\
\frac{dy}{dt} &= -by + \sin(z) \\
\frac{dz}{dt} &= -bz + \sin(x)
\end{align*}
\] (3)

At our knowledge, this is the first dynamical system exhibiting such behavior. The conservative part \( f_c(X) \) and the dissipative part \( f_d(X) \) of the system (3) are

\[
\begin{align*}
f_c(X) &= \begin{pmatrix} \sin(y) \\ \sin(z) \\ \sin(x) \end{pmatrix} ; \quad f_d(X) &= \begin{pmatrix} -bx \\ -by \\ -bz \end{pmatrix}
\end{align*}
\] (4)

The relation (2) together with \( f_c(X) \) yields

\[
\frac{\partial H}{\partial x} \sin(y) + \frac{\partial H}{\partial y} \sin(z) + \frac{\partial H}{\partial z} \sin(x) = 0.
\] (5)
First suppose that \( \frac{\partial H}{\partial x} \neq 0, \frac{\partial H}{\partial y} \neq 0, \frac{\partial H}{\partial z} \neq 0 \). It is obvious that for the relation (5) to be true, \( \frac{\partial H}{\partial x} \) must contain sin(x) and/or sin(y), \( \frac{\partial H}{\partial y} \) must contain sin(y) and/or sin(x) and \( \frac{\partial H}{\partial z} \), must contain sin(y) and/or sin(z). Different options are the possible for \( \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \) and \( \frac{\partial H}{\partial z} \).

Here we write two of them; The first option, as one of the simplest possible options, is

\[
\frac{\partial H}{\partial x} = g(x, y, z) \sin(z) \quad (6a) \\
\frac{\partial H}{\partial y} = -g(x, y, z) \sin(y) - h(x, y, z) \sin(x) \quad (6b) \\
\frac{\partial H}{\partial z} = h(x, y, z) \sin(z) \quad (6c)
\]

where \( g \) and \( h \) are smooth functions of \( R^3 \) such that \( g \) does not explicitly contain \( \sin(z) \) nor \( \sin(y) \), and \( h \) does not explicitly contain \( \sin(x) \) nor \( \sin(y) \). A second possible option is

\[
\frac{\partial H}{\partial x} = g_1(x, y, z) \sin(z) - g_2(x, y, z) \sin(x) \quad (7a) \\
\frac{\partial H}{\partial y} = -g_1(x, y, z) \sin(y) - h_1(x, y, z) \sin(x) \quad (7b) \\
\frac{\partial H}{\partial z} = h_1(x, y, z) \sin(z) + g_1(x, y, z) \sin(y) \quad (7c)
\]

where \( g_1, g_2 \) and \( h_1 \) are smooth functions of \( R^3 \) such that \( g_1 \) does not explicitly contain \( \sin(z) \) nor \( \sin(y) \), \( g_2 \) does not explicitly contain \( \sin(x) \) nor \( \sin(y) \) and \( h_1 \) does not explicitly contain \( \sin(x) \) nor \( \sin(z) \).

Let us first consider the proposal (6). The integration of the relation (6a) with respect to \( x \) yields

\[
H(x, y, z) = \sin(z) \int g(x, y, z) dx + A(y, z) \quad (8)
\]

where \( A(y, z) \) is the “constant” of integration. Computing \( \frac{\partial H}{\partial y} \) through (8) and compare it to (6b) yields

\[
\frac{\partial A(y, z)}{\partial y} = -\sin(z) \int \frac{\partial g(x, y, z)}{\partial y} dx - g(x, y, z) \sin(y) - h(x, y, z) \sin(x) \quad (9)
\]

On the left side of (9), there is no dependence on \( x \) while on the right side, even if we suppose that \( g \) and \( h \) depend only on \( y \) and \( z \), we can not wipe off the dependence on \( x \) of the third term in the right hand side of (9) through the \( \sin(x) \). There are two possibilities to overcome this problem. The first is to let the function \( h = 0 \), but in this case according to (6b), \( \frac{\partial H}{\partial z} = 0 \) which contradicts our hypothesis. The second possibility in order to overcome the problem would be to let \( \frac{\partial A}{\partial z} = 0 \). Then, the derivative of (9) with respect to \( x \) yields

\[
\frac{\partial q}{\partial y} \sin(z) + \frac{\partial q}{\partial x} \sin(y) + \frac{\partial}{\partial x} [h \sin(x)] = 0. \quad (10)
\]
Following the same procedure, (5b) and (5c) yield

$$\frac{\partial h}{\partial y} \sin(z) + \frac{\partial g}{\partial z} \sin(y) + \frac{\partial h}{\partial z} \sin(x) = 0 \tag{11}$$

But we have seen that $g$ does not contain $\sin(z)$ nor $\sin(y)$ and $h$ does not contain $\sin(x)$ nor $\sin(z)$. So then, the nullity of (10) and (11) become impossible. It is not difficult to see that the second option (7) or any other option for $\partial H/\partial x$, $\partial H/\partial y$, $\partial H/\partial z$ consistent with the relation (5) will end up to the same type of contradictions. The only remaining case is when one of the partial derivatives $\partial H/\partial x$, $\partial H/\partial y$, $\partial H/\partial z$ is null. Let for instance $\partial H/\partial x = 0$. The relation (4) becomes then

$$\frac{\partial H}{\partial y} \sin(z) + \frac{\partial H}{\partial z} \sin(x) = 0 \tag{12}$$

But having assumed $\partial H/\partial x = 0$ means that $H$ does not depend on $x$ and enters in contradiction with (12) since again the dependence on $x$ through the $\sin(x)$ can not be wiped out. Hence, we can conclude that this is not possible to find any function $H(x, y, z)$ as a solution of the partial differential equation (2) and consequently the system (3) is not Hamiltonian in the sense described here.

It worth remarking that in spite the fact that the system (3) is not Hamiltonian, it does have a vector potential. Indeed, it is easy to see that $\nabla f_c = 0$. Thus, there exist a field $F(F_1, F_2, F_3)$, called the vector potential [17], such that $\nabla \times F = f_c$ yielding to the following system

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \sin(y) \tag{13a}$$
$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \sin(z) \tag{13b}$$
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \sin(x) \tag{13c}$$

As we know, vector potential is not unique. To find a simple solution, we can let $F_3 = 0$ and straight forward calculations yields to

$$F(-\cos(z), -z \sin(y) - \cos(x), 0). \tag{14}$$

Remind that for a conservative system, a vector potential is related to the flow of the field vector $f_c$ through the Stokes’ theorem.

3 Discussion

The question which arises here, is how one could explain that a conservative flow, having naturally a vector potential does not have a Hamiltonian representing its energy function. This is an open question to which we are going to bring some
hints. As mentioned previously, for the Hamiltonian of \( f_c(X) \), the conservative part of the field of the dynamical system \( f_c \), represents the energy function of the conservative flow of the system. The system \( f_c \) can be seen as a correlated system of oscillators, by adopting a mechanical point of view one can tell that its ‘energy’ will be the sum of its kinetic and potential energies. The kinetic energy can be directly calculated from the system \( f_c \). For the potential energy, one can easily draw out the acceleration field, proportional to a force, from the system \( f_c \).

\[
\begin{align*}
\frac{d^2 x}{dt^2} &= \sin(z) \cos(y) \\
\frac{d^2 y}{dt^2} &= \sin(x) \cos(z) \\
\frac{d^2 z}{dt^2} &= \sin(y) \cos(x)
\end{align*}
\]

(15a)

(15b)

(15c)

The system being conservative, the potential as the opposite of the path-integral of the force is path-independent. So, to calculate this vector potential, one can choose any path. Let us calculate this potential from the point \((0, 0, 0)\) to the point \((x, y, z)\) following three straight lines from \((0, 0, 0)\) to \((x, 0, 0)\), then from \((x, 0, 0)\) to \((x, y, 0)\) and finally from \((x, y, 0)\) to \((x, y, z)\). This yields to the potential \(U(x, y, z)\)

\[
U(x, y, z) = -y \sin(x) - z \sin(y) \cos(z).
\]

(16)

But then, we should have

\[
\left( -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \right) = \left( \frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2} \right)
\]

which is obviously not the case. To explain this contradiction, one can imagine that in spite the fact that the system is conservative, a part of its energy although being “conserved” within the system, is somehow “consumed” to maintain the correlation of oscillators. The mechanism of this correlation leads to an “internal dissipation” and the correct formalism to define a proper Hamiltonian for this system as well as the algebraic structure of such phenomena have to be discovered and clarified and will be the subject of forthcoming studies.

### 4 Conclusion & Outlook

Labyrinth Chaos in its simplicity and elegance naturally suggests the possibility of its Hamiltonization, as it is known to happen for other nonlinear cyclically coupled systems for example the Lotka-Voltera and its many variants types \[20\], the famous Arnold-Beltrami-Childress (“1:1:1 ABC”) model \[18, 19\] etc. But as we have seen the symmetries afforded by Labyrinth Chaos and the special structure of its solely ‘kinetic’ part makes it to break the rules of such a known Hamiltonian construction \[15, 16, 14\].
There is also a class of systems that are conservative in the sense of \(2\) and non-Hamiltonian due to specific considerations pertaining to the construction of ‘Hoover thermostats’ in classical statistical mechanics as developed in \(21\) \(22\). Although this theory provides a framework for analyzing non-Hamiltonian systems and determining the precise phase-space distribution generated by the equations of motion assuming ergodicity, again, the kinetic, dissipative and potential parts here are specifically designed to express a balance that makes the phase space incompressible. Still, its connection to chaotic dynamics has not been reported or elucidated, to our knowledge.

The above places the Labyrinth chaos at a class of its own. Following the preceding discussion there are three points, among others, to ponder for future investigations:

(a) The discrepancy, as far as a Hamiltonian structure is concerned, between the right hand side of \(1\) when \(F(\cdot) = \sin(\cdot)\) and its approximation via a polynomial (Taylor) series expansion that gives rise to Arabesque-like systems, especially with \(b = 0\) \(3\) \(4\).

(b) The differences in symmetry considerations and kinetic versus potential terms of \(1\) \(2\) and the \((1:1:1)\)ABC model \(18\) \(19\) where combinations of only ‘\(\sin(\cdot)\)’ versus ‘\(\sin(\cdot)\) and \(\cos(\cdot)\)’ terms appear in their right hand sides, respectively.

(c) The role of the vector potential as ‘carrier of information’ \(23\) \(24\) \(17\) is intriguing to investigate. Especially in connection to arrays of Labyrinth chaos systems \(13\) where Kuramoto-type phase locking resulting in chimera-like states has been observed recently.

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