ON THE ASYMPTOTIC BEHAVIOUR OF THE FRACTIONAL SOBOLEV SEMINORMS IN METRIC MEASURE SPACES: ASYMPTOTIC VOLUME RATIO, VOLUME ENTROPY AND RIGIDITY

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Abstract. We study the asymptotic behaviour of suitably defined seminorms in general metric measure spaces. As particular cases we provide new and shorter proofs of the Maz’ya-Shaposhnikova’s theorem \cite{28} with general mollifiers as well as of Ludwig’s result \cite{25} concerning anisotropic Sobolev seminorms. Our result provides new spaces satisfying an asymptotic formula involving the asymptotic volume ratio (and the volume entropy), non-trivial examples fit our setting includes Carnot groups and metric measure spaces with synthetic lower Ricci curvature bound. Moreover, we prove two rigidity theorems which are new even in the smooth setting.

1. Introduction

About 20 years ago, Bourgain, Brezis and Mironescu \cite{9} on one hand and Maz’ya and Shaposhnikova \cite{28} on the other, revealed that fractional \((s, p)\)-seminorms can be used to recover the \(L^p\)-norm and the \(W^{1, p}\)-seminorms when \(s \to 0^+\) and \(s \to 1^-\) respectively. More precisely, for any \(s \in (0, 1), N \in \mathbb{N}\) and \(p \geq 1\), the fractional Sobolev space \(W^{s, p}(\mathbb{R}^N)\) is defined as the set of \(L^p(\mathbb{R}^N)\) functions with finite seminorm
\[
\|f\|_{W^{s, p}} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}} \, d\mathcal{L}^N(x) \, d\mathcal{L}^N(y) \right)^{\frac{1}{p}}
\]
where \(\mathcal{L}^N\) denotes the Lebesgue measure on \(\mathbb{R}^N\). The following important asymptotic formulas have been proved in \cite{9} and \cite{28} respectively: for any \(f \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^N) \cap W^{1, p}(\mathbb{R}^N)\), it holds
\[
\lim_{s \uparrow 1} (1 - s)\|f\|_{W^{s, p}}^p = K\|\nabla f\|_{L^p(\mathbb{R}^N)}^p \tag{BBM}
\]
and
\[
\lim_{s \downarrow 0} s\|f\|_{W^{s, p}}^p = L\|f\|_{L^p(\mathbb{R}^N)}^p \tag{MS}
\]
where \(K = K_{p,N}\) and \(L = L_{p,N}\) are positive constants depending only on \(p\) and \(N\).

Both formulas (BBM) and (MS) have been widely studied, cited in hundreds of papers in the fields of analysis, probability theory and geometry, and generalized to many different situations, see for example \cite{1,11,18,19,21–23,25,26,29,30,32–34} and the references therein. Therefore, we believe it is an interesting problem to find more non-trivial examples satisfying such asymptotic formulas, and learn more about the structural constants \(K\) and \(L\) appearing in the formulas.

In the present paper we study (MS) in the setting of general metric measure space and general mollifiers. Following a completely different proof than the one proposed...
in [28] we prove that (MS) holds in great generality and that the mollifier \( \overline{\frac{\delta}{|x-y|^{n+sp}}} \) can be replaced by more general ones (Theorem 2.2). As a consequence we provide a new proof of [28], as well as of [25] without using Blaschke-Petkantschin formula.

We also prove a surprising link between the Maz’ya-Shaposhnikova’s formula and the asymptotic volume ratio of the metric measure space (Lemma 2.5 and Theorem 2.7). Using this very precise and geometric information we provide several interesting examples of spaces satisfying a Maz’ya-Shaposhnikova’s type formula. In addition, we find a new dimension-free asymptotic formula involving volume entropy (Theorem 2.10). Finally, applying our result to the setting of metric measure spaces with synthetic curvature-dimension condition à la Lott-Sturm-Villani, we obtain two sharp estimates on the corresponding structural constants, and prove two rigidity theorems (Theorem 2.16 and Theorem 2.13). To our knowledge, these results are new even in the smooth setting.

Plan of the paper.

In subsection 2.1 we prove our main result, i.e. Theorem 2.2. In the next subsection 2.2, we apply our main theorem with two specific mollifiers and we prove Lemma 2.5, moreover we provide several examples of metric measure spaces where Theorem 2.2 and Lemma 2.5 hold. Finally, in subsection 2.3 we prove two rigidity results Theorem 2.16 and Theorem 2.13.

2. Main results

2.1. General theory. In this paper, a metric measure space is a triple \((X, d, m)\) where \((X, d)\) is a complete and separable metric space, \(m\) is a locally finite non-negative Borel measure on \(X\) with full support.

Let \((\rho_n)_{n \in \mathbb{N}}\) be a sequence of mollifiers i.e. each

\[ \rho_n : (X \times X) \setminus \{(x, y) \in X \times X \mid x = y\} \to (0, \infty) \]

is measurable and it satisfies the following set of assumptions:

**Assumption 2.1.**

1) There is a sequence of functions \((\tilde{\rho}_n)_{n \in \mathbb{N}}\) with \(\tilde{\rho}_n : (0, \infty) \to (0, \infty)\) continuous such that

\[ \rho_n(x, y) = \tilde{\rho}_n(d(x, y)) \]

for all \(x, y \in X, x \neq y\).

Moreover, there exists \(n_0 \in \mathbb{N}\) such that \(\tilde{\rho}_n(r)\) is non-increasing in \(r\) for any \(n > n_0\), there exists \(r_1 > 0\) such that \((\tilde{\rho}_n)_{n \in \mathbb{N}}\) is non-increasing in \(n \in \mathbb{N}\) for any \(r \in (r_1, \infty)\), and

\[ \lim_{n \to \infty} \tilde{\rho}_n(r) = 0, \quad \forall r \in (r_1, \infty). \]  

2) There exists \(r_0 > 0\) such that for any \(n, m \in \mathbb{N}\) with \(n > m\) the function

\[ (r_0, +\infty) \ni r \mapsto \frac{\tilde{\rho}_n(r)}{\tilde{\rho}_m(r)} \]

is non-decreasing.

3) There is a structural constant \(L \in [0, \infty)\) such that the following limit holds

\[ \lim_{\delta \to \infty} \lim_{n \to \infty} \int_{B_{\delta}(x)} \rho_n(x, y) \, dm(y) = L, \quad \forall x \in X \]

where \(B_{\delta}(x)\) denotes the complement of the open ball centred at \(x\) with radius \(\delta > 0\).
Let $p > 1$, $n \in \mathbb{N}$ and $(\rho_n)_{n \in \mathbb{N}}$ be a family of mollifiers. We define the $W(n, p)$ as the set of functions in $L^p(X, m)$ with finite (semi)norm $\mathcal{E}_n(\cdot)$, defined by

$$\mathcal{E}_n(u) := \left( \int_X \int_X |u(x) - u(y)|^p \rho_n(x, y) \, dm(x) \, dm(y) \right)^{\frac{1}{p}}.$$ 

We are now ready to prove a generalized Maz’ya-Shaposhnikova’s asymptotic formula under the Assumption 2.1.

**Theorem 2.2** (Generalized Maz’ya-Shaposhnikova’s formula). Given $p > 1$ and a metric measure space $(X, d, m)$. Let $(\rho_n)_{n \in \mathbb{N}}$ be a family of mollifiers. For any $u \in \bigcup_{n \in \mathbb{N}} W(n, p)$, there exists the limit

$$\lim_{n \to \infty} \mathcal{E}_n^p(u) = 2L\|u\|_{L^p}^p$$

where $L$ is the constant given in (2.2).

**Proof.** Fix $x_0 \in X$, for any $R \gg r_0$ ($r_0 > 0$ is as in Assumption 2.1-2) , we have the following decomposition of $X \times X$:

$$\begin{align*}
A & := \{(x, y) : d(x, y) \leq R\} \\
B & := \{(x, y) : d(x, y) > R\} \cap \{(x, y) : d(y, x_0) > 2d(x, x_0)\} \\
B' & := \{(x, y) : d(x, y) > R\} \cap \{(x, y) : d(y, x_0) < \frac{7}{8}d(x, x_0)\} \\
C & := \{(x, y) : d(x, y) > R\} \cap \{(x, y) : d(x, x_0) \leq d(y, x_0) \leq 2d(x, x_0)\} \\
C' & := \{(x, y) : d(x, y) > R\} \cap \{(x, y) : \frac{7}{8}d(x, x_0) \leq d(y, x_0) < 2d(x, x_0)\}
\end{align*}$$

According to this decomposition, and by symmetry (w.r.t. $x, y$), we divide $\mathcal{E}_n^p(u)$ into the following three parts

$$\begin{align*}
\mathcal{E}_n^p(u) &= \int_A |u(x) - u(y)|^p \rho_n(x, y) \, dm(x) \, dm(y) \\
& + 2 \int_B |u(x) - u(y)|^p \rho_n(x, y) \, dm(x) \, dm(y) \\
& + \int_{C \cup C'} |u(x) - u(y)|^p \rho_n(x, y) \, dm(x) \, dm(y).
\end{align*}$$

By hypothesis, there is $n_0 \in \mathbb{N}$ such that $u \in W(n_0, p)$. Concerning the term $I$, for any $n > n_0$, by Assumption 2.1, it holds

$$\begin{align*}
I &= \int_X \left( \int_{B_R(y)} |u(x) - u(y)|^p \rho_{n_0}(x, y) \frac{\rho_n(x, y)}{\rho_{n_0}(x, y)} \, dm(x) \right) \, dm(y) \\
& \leq \int_X \left( \int_{B_R(y)} |u(x) - u(y)|^p \rho_{n_0}(x, y) \frac{\tilde{\rho}_n(R)}{\tilde{\rho}_{n_0}(R)} \, dm(x) \right) \, dm(y) \\
& \leq \mathcal{E}_n^{p_0}(u) \frac{\tilde{\rho}_n(R)}{\tilde{\rho}_{n_0}(R)}.
\end{align*}$$

By Assumption 2.1-1) we get

$$\lim_{n \to +\infty} I(R, n) = 0.$$
On the other hand, for any $R > 0$, we have

$$B_1(R, n) := \int_B |u(y)|^p \rho_n(x, y) \, dm(x) \, dm(y)$$

$$= \int_X |u(y)|^p \left( \int_{\{x \mid d(x,y) > R, d(y,x_0) > 2d(x,x_0)\}} \rho_n(x, y) \, dm(x) \right) \, dm(y)$$

$$\leq \int_X |u(y)|^p \left( \int_{\{x \mid \frac{1}{2}d(y,x_0) \geq d(x,y) > \frac{1}{2}d(y,x_0) + R\}} \rho_n(x, y) \, dm(x) \right) \, dm(y)$$

$$= \int_X |u(y)|^p \left( \int_{\{x \mid d(x,y) > \frac{1}{2}d(y,x_0) + R\}} \rho_n(x, y) \, dm(x) \right) \, dm(y) - \int_{\{x \mid d(x,y) > \frac{1}{2}d(y,x_0) + R\}} \rho_n(x, y) \, dm(x) \, dm(y).$$

By Assumption 2.1-1) and 3), and the monotone convergence theorem, we have

$$\lim_{R \to \infty} \lim_{n \to \infty} \int_X |u(y)|^p \left( \int_{\{x \mid d(x,y) > \frac{1}{2}d(y,x_0) + R\}} \rho_n(x, y) \, dm(x) \right) \, dm(y) = L \|u\|_{L^p}^p.$$

Hence

$$\lim_{R \to \infty} \lim_{n \to \infty} B_1(R, n) = 0. \tag{2.5}$$

Notice that for any $x, y, x_0 \in X$, by triangle inequality

$$B'_{4d(x,x_0)}(x) \subset \{y \in X : d(y,x_0) > 2d(x,x_0)\} \subset B^0_{6d(x,x_0)}(x).$$

So we also have

$$B_2(R, n) := \int_B |u(x)|^p \rho_n(x, y) \, dm(x) \, dm(y)$$

$$= \int_X |u(x)|^p \left( \int_{\{y \mid d(x,y) > R, d(y,x_0) > 2d(x,x_0)\}} \rho_n(x, y) \, dm(y) \right) \, dm(x)$$

$$\geq \int_X |u(x)|^p \left( \int_{B'_{4d(x,x_0)}(x) \setminus R(x)} \rho_n(x, y) \, dm(y) \right) \, dm(x)$$

and

$$B_2(R, n) \leq \int_X |u(x)|^p \left( \int_{B^0_{6d(x,x_0)}(x) \setminus R(x)} \rho_n(x, y) \, dm(y) \right) \, dm(x).$$

By Assumption 2.1-1) and 3), and monotone convergence theorem, we obtain

$$\lim_{R \to \infty} \lim_{n \to \infty} B_2(R, n) = L \|u\|_{L^p}^p. \tag{2.6}$$
By triangle inequality we have
\[
(B_2)_{\frac{1}{p}} - (B_1)_{\frac{1}{p}} \leq II_{\frac{1}{p}} = \left( \int_B |u(x) - u(y)|^p \rho_n(x,y) \, dm(x) \, dm(y) \right)^{\frac{1}{p}} 
\leq (B_2)_{\frac{1}{p}} + (B_1)_{\frac{1}{p}}.
\]

Combining with (2.5) and (2.6), we get
\[(2.7) \lim_{R \to \infty} \lim_{n \to \infty} II(R, n) = L\|u\|_{L^p}.\]

We observe that
\[
C = \{(x,y) \in X \times X \mid d(x,y) > R\} \cap \{(x,y) \in X \times X \mid d(x,x_0) \leq d(y,x_0) \leq 2d(x,x_0)\}.
\]

By triangle inequality
\[(2.8) C \subset \{(x,y) \in X \times X \mid d(x,y) > R, \, d(x,x_0) > R/3\}\]
and
\[(2.9) C \subset \{(x,y) \in X \times X \mid d(x,y) > R, \, d(y,x_0) > R/2\}\].

Thus by (2.8) and (2.9) we can estimate
\[
III \leq 2 \int_C |u(x) - u(y)|^p \rho_n(x,y) \, dm(x) \, dm(y)
\leq 2^{p-1} \left( \int_C |u(x)|^p \rho_n(x,y) \, dm(x) \, dm(y) + \int_C |u(y)|^p \rho_n(x,y) \, dm(x) \, dm(y) \right)
\leq 2^{p-1} \int_{d(x,x_0) > \frac{R}{3}} |u(x)|^p \left( \int_{d(x,y) > R} \rho_n(x,y) \, dm(y) \right) \, dm(x)
+ 2^{p-1} \int_{d(y,x_0) > \frac{R}{2}} |u(y)|^p \left( \int_{d(x,y) > R} \rho_n(x,y) \, dm(x) \right) \, dm(y)
\leq 2^p \int_{d(x,x_0) > \frac{R}{3}} |u(x)|^p \left( \int_{d(x,y) > R} \rho_n(x,y) \, dm(y) \right) \, dm(x).
\]

So by Assumption 2.1 and the monotone convergence theorem
\[
\lim_{n \to \infty} III(R, n) \leq 2^p \int_{d(x,x_0) > \frac{R}{3}} |u(x)|^p \lim_{n \to \infty} \left( \int_{d(x,y) > R} \rho_n(x,y) \, dm(y) \right) \, dm(x).
\]
and
\[
\lim_{R \to \infty} \lim_{n \to \infty} III(R, n) = 0.
\]

The conclusion follows combining (2.4) and (2.7) obtained above.

\[\square\]
2.2. Applications and Examples. In this section we will apply our main theorem to some important spaces and to some specific mollifiers. Doing so we not only extend Maz’ya-Shaposhnikova’s theorem on Euclidean spaces (and Ludwig’s theorem on finite dimensional Banach spaces), but we also show that the key point in the asymptotic formula is the so-called ‘asymptotic volume ratio’ which describe the growth of the volume of a geodesic ball at infinity.

**Definition 2.3.** A metric measure space \((X,d,m)\) admits the asymptotic volume ratio at \(x_0 \in X\) with some \(N \in (0, +\infty)\), denoted by \(\text{AVR}(X,d,m)(x_0,N)\), provided

\[
\text{AVR}(X,d,m)(x_0,N) := \lim_{r \to +\infty} \frac{m(B_r(x_0))}{r^N} \in [0, +\infty].
\]

**Lemma 2.4.** Let \((X,d,m)\) be a metric measure space admitting finite asymptotic volume ratio \(\text{AVR}(X,d,m)(x_0,N)\) at \(x_0 \in X\) with some \(N > 0\). Then \((X,d,m)\) admits the asymptotic volume ratio at any \(x_1 \in X\) with the same constant \(N\) and

\[
\text{AVR}(X,d,m)(x_1,N) = \text{AVR}(X,d,m)(x_0,N).
\]

Furthermore, if \(\text{AVR}(X,d,m)(x_0,N) \in (0, +\infty)\), then \(N\) is unique.

If \(\text{AVR}(X,d,m)(x_0,N) = 0\) for some \(N > 0\), then \(\text{AVR}(X,d,m)(x_0,N') = 0\) for any \(N' \geq N\).

**Proof.** The uniqueness of \(N\) is obvious, we will just show that \(\text{AVR}(X,d,m)(x_0,N)\) is independent of the choice of \(x_0\). Let \(x_1 \in X\) with \(x_1 \neq x_0\). Using the triangle inequality it is easy to see that

\[
\frac{m(B_r(x_1))}{r^N} \leq \frac{m(B_{r+d(x_0,x_1)}(x_0))}{r^N}.
\]

So

\[
\lim_{r \to +\infty} \frac{m(B_r(x_1))}{r^N} \leq \lim_{r \to +\infty} \frac{m(B_{r+d(x_0,x_1)}(x_0))}{r^N} = \lim_{r \to +\infty} \frac{m(B_{r+d(x_0,x_1)}(x_0))}{(r + d(x_0,x_1))^N} \frac{(r + d(x_0,x_1))^N}{r^N} = \text{AVR}(X,d,m)(x_0,N).
\]

Similarly, we have

\[
\lim_{r \to +\infty} \frac{m(B_r(x_1))}{r^N} \geq \lim_{r \to +\infty} \frac{m(B_{r-d(x_0,x_1)}(x_0))}{r^N} = \lim_{r \to +\infty} \frac{m(B_{r-d(x_0,x_1)}(x_0))}{(r - d(x_0,x_1))^N} \frac{(r - d(x_0,x_1))^N}{r^N} = \text{AVR}(X,d,m)(x_0,N)
\]

and the conclusion follows.

Thanks to the previous result, if there exists \(x_0 \in X\) and \(N > 0\) such that \(\text{AVR}(X,d,m)(x_0,N) \in (0, +\infty)\) then we can write \(\text{AVR}(X,d,m)\) without any further specification. Moreover, if there is no risk of confusion we will just write \(\text{AVR}\) thus omitting also the dependence on the metric measure space. In case \(\text{AVR} > 0\), we say that \((X,d,m)\) has Euclidean-volume growth.
Lemma 2.5. Let \((X, d, m)\) be a metric measure space admitting finite asymptotic volume ratio, i.e. \(\text{AVR}(X, d, m) \in [0, +\infty)\), \(p > 1\) and let \((a_n)_{n \in \mathbb{N}}\) be a non-increasing sequence of strictly positive numbers converging to 0. For each \(n \in \mathbb{N}\) we define \(\rho_n : (X \times X) \setminus \{(x, y) \in X \times X \mid x = y\} \to (0, \infty)\) by

\[
\rho_n(x, y) = \frac{a_n}{d(x, y)^{N+a_np}},
\]

where \(N \in \mathbb{N}\) is as in Definition 2.3. Then \((\rho_n)_{n \in \mathbb{N}}\) satisfies Assumption 2.1. In particular,

\[
L = \frac{N}{p} \text{AVR}(X, d, m).
\]

Proof. Clearly \(\rho_n(x, y) = \tilde{\rho}_n(d(x, y))\) where \(\tilde{\rho}_n : (0, \infty) \to (0, \infty)\) is defined by

\[
(2.11) \quad \tilde{\rho}_n(r) = \frac{a_n}{r^{N+a_n p}}.
\]

Then \(\tilde{\rho}_n(r)\) is decreasing in \(r\) for every \(n \in \mathbb{N}\).

Fix \(r > 1\). We define \(\varphi : (0, \infty) \to (0, \infty)\) by \(\varphi(x) := \frac{x}{r^{N+x_p}}\). It can be seen that

\[
\varphi'(x) = \frac{r^{N+xp} - xpr^{N+xp} \ln r}{r^{2N+2xp}} > 0
\]

for sufficiently small \(x\). So \(\tilde{\rho}_n(r)\) is decreasing in \(n\) (for large \(n\)).

Given \(n > m\), by assumption \(a_m \geq a_n\), so

\[
\frac{\tilde{\rho}_n(r)}{\tilde{\rho}_m(r)} = \frac{a_m}{a_m} = (a_m - a_n)p
\]

which is increasing for \(r > 0\).

For simplicity, we assume that the asymptotic volume ratio \(\text{AVR}\) is positive, the case for \(\text{AVR} = 0\) can be proved in a similar way. For any \(\epsilon > 0\), there is \(\delta_0 > 0\) such that

\[
\text{AVR}(1-\epsilon)r^N \leq m(B_r(x)) \leq \text{AVR}(1+\epsilon)r^N, \quad \forall r \geq \delta_0.
\]

For simplicity, we can also write

\[
m(B_r(x)) = \text{AVR}(1 + O(\epsilon))r^N.
\]

Let \(\delta > \delta_0\). For any \(n \in \mathbb{N}\) we define the function \(\tilde{\rho}_{n, \delta} : X \times X \to \mathbb{R}\) by

\[
\tilde{\rho}_{n, \delta}(x, y) := \begin{cases} \rho_n(x, y) & d(x, y) > \delta, \\ \tilde{\rho}_n(\delta) & 0 \leq d(x, y) \leq \delta. \end{cases}
\]
Then by the Cavalieri’s formula (cf. [4, Chapter 6]) we can write
\[
\int_{B_\delta(x)} \rho_n(x, y) \, dm(y) + \tilde{\rho}_n(\delta) m(B_\delta(x))
\]
= \int_{x} \tilde{\rho}_n(\delta, x, y) \, dm(y)

by Cavalieri’s formula
= \int_{0}^{\tilde{\rho}_n(\delta)} m(B_{\tilde{\rho}_n^{-1}(r)}(x)) \, dr
= \int_{0}^{\tilde{\rho}_n(\delta)} \text{AVR}(1 + O(\epsilon))(\tilde{\rho}_n^{-1}(r))^N \, dr

let \( t = \tilde{\rho}_n^{-1}(r) \) = \( (1 + O(\epsilon)) \text{AVR} \int_{+\infty}^{t} t^{N} \tilde{\rho}_n^{p}(t) \, dt \)

by integration by part
= \( (1 + O(\epsilon)) \text{AVR} \left( \tilde{\rho}_n(\delta) \delta^{N} - \int_{+\infty}^{\delta} Nt^{N-1} \tilde{\rho}_n(t) \, dt \right) \)
= \( (1 + O(\epsilon)) \text{AVR} \left( \tilde{\rho}_n(\delta) \delta^{N} - \int_{+\infty}^{\delta} N \frac{a_n}{t^{1+a_p}} \, dt \right) \)
= \( (1 + O(\epsilon)) \text{AVR} \left( \tilde{\rho}_n(\delta) \delta^{N} + \frac{N}{p} \delta^{-a_p} \right) \)

Therefore, for \( \delta > \delta_0 \),
\[
\int_{B_\delta(x)} \rho_n(x, y) \, dm(y) = O(\epsilon) \text{AVR} \frac{a_n}{\delta^{a_p}} + (1 + O(\epsilon)) \text{AVR} \frac{N}{p} \delta^{-a_p}.
\]

Thus
\[
\lim_{\delta \to +\infty} \lim_{n \to \infty} \int_{B_\delta(x)} \rho_n(x, y) \, dm(y) = \frac{N}{p} \text{AVR}, \quad \forall x \in X
\]

which is the thesis. \( \square \)

**Remark 2.6.** The function \( r \to m(B_r(x)) \) is increasing, so it is differentiable except for a set of Lebesgue measure 0. In many situations, for example when the space satisfies some generalized Bishop-Gromov comparison theorem (cf. [36, Theorem 2.3] and Theorem 2.12), we have a stronger asymptotic formula:
\[
\lim_{r \to +\infty} \frac{d}{dr} m(B_r(x)) = \text{AVR}.
\]

Furthermore, when \((X, d)\) is geodesic, it is not hard to prove that
\[
\frac{d}{dr} m(B_r(x)) = m^+(\partial B_r(x)),
\]
where \( m^+ \) is the canonical Minkowski content defined by
\[
(2.12) \quad m^+(E) := \lim_{\epsilon \to 0} \frac{m(E^{\epsilon}) - m(E)}{\epsilon}
\]
where \( E \) is a Borel set and \( E^{\epsilon} := \{ x \in X : \exists y \in E \text{ such that } d(x, y) < \epsilon \} \) is the \( \epsilon \)-neighbourhood of \( E \) with respect to the metric \( d \).
Combining Theorem 2.2 and Lemma 2.5 we immediately get the following theorem.

**Theorem 2.7.** Let \((X, d, m)\) be a metric measure space admits finite asymptotic volume ratio \(\text{AVR}(X, d, m)\). For any \(p > 1\) and \(s \in (0, 1)\), define \(W^{s,p}(X, d, m)\) as the functions in \(L^p(X, d, m)\) with finite (semi)-norm

\[
\left( \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{N+sp}} \, dm(x) \, dm(y) \right)^{\frac{1}{p}}
\]

where \(N\) is the number given in Definition 2.3.

For any \(u \in \bigcup_{0 < \tau < 1} W^{\tau,p}(X, d, m)\), there exists the limit

\[
\lim_{\varepsilon \downarrow 0} s \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{N+sp}} \, dm(x) \, dm(y) = \frac{2N}{p} \text{AVR} \|u\|_{L^p}^p.
\]

In the study of metric (Riemannian) geometry, there are some important spaces where the asymptotic volume ratio \(\text{AVR} = +\infty\). In these cases, we can consider instead the volume entropy, which is an important concept in both Riemannian geometry (cf. [7]) and dynamical system (cf. [27]).

**Definition 2.8.** A metric measure space \((X, d, m)\) admits the volume entropy at \(x_0 \in X\), denoted by \(h(X, d, m)(x_0)\), provided

\[
h(X, d, m)(x_0) := \lim_{r \to +\infty} \frac{\ln m(B_r(x_0))}{r} \in [0, \infty].
\]

Proceeding as in Lemma 2.4, it is not hard to prove the following result (cf. [12] and the references therein).

**Lemma 2.9.** Let \((X, d, m)\) be a metric measure space admitting finite volume entropy \(h(X, d, m)(x_0)\) at \(x_0 \in X\). Then \((X, d, m)\) admits finite volume entropy at any \(x_1 \in X\) and

\[
h(X, d, m)(x_0) = h(X, d, m)(x_1).
\]

**Theorem 2.10.** Let \((X, d, m)\) be a metric measure space with \(h = h(X, d, m) \in (0, +\infty)\). Then for any \(p > 1\) and \(u \in L^p(X)\) such that \(\int_X \int_X \frac{|u(x) - u(y)|^p}{e^{(h+s)d(x,y)}} \, dm(x) \, dm(y) < +\infty\) for some \(\tau > 0\), there exists the limit

\[
\lim_{\varepsilon \downarrow 0} s \int_X \int_X \frac{|u(x) - u(y)|^p}{e^{(h+s)d(x,y)}} \, dm(x) \, dm(y) = 2h \|u\|_{L^p}^p.
\]

**Proof.** Similar to Lemma 2.5, we can check that the mollifiers \(\tilde{\rho}(s) := \frac{s}{e^{(h+s)p}}\) satisfy Assumption 2.1, and

\[
\lim_{\delta \to 0} \lim_{s \to 0} \int_{B_{\delta}(x)} \frac{s}{e^{(h+s)d(x,y)}} \, dm(y) = h, \quad \forall x \in X
\]

Then the assertion follows from Theorem 2.2. \(\square\)

**Example 2.11.** The range of applicability of our main theorem is pretty wide. We list below some relevant examples.
1) **Euclidean spaces**: it is known that $\text{AVR}(\mathbb{R}^N, |\cdot|, \mathcal{L}^N) = \omega_N = \frac{|S^{N-1}|}{N}$ where $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2} + 1\right)}$ denotes the volume of an $N$-dimensional unit ball and $|S^{N-1}|$ denotes its surface area. By Theorem 2.7 we get Maz’ya-Shaposhnikova’s original result [28, Theorem 3]:

$$\lim_{s \downarrow 0} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, d\mathcal{L}^N(x) \, d\mathcal{L}^N(y) = \frac{2|S^{N-1}|}{p} \|u\|_{L^p(\mathbb{R}^N)}^p.$$ 

2) **Finite dimensional Banach spaces**: let $(\mathbb{R}^N, \|\cdot\|, \mathcal{L}^N)$ be an $N$-dimensional Banach space. Denote its unit ball by $K$ (which is a convex body). Applying Theorem 2.7, we get Ludwig’s result [25, Theorem 2] for anisotropic fractional Sobolev norms:

$$\lim_{s \downarrow 0} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{\|x - y\|^{N+sp}} \, d\mathcal{L}^N(x) \, d\mathcal{L}^N(y) = \frac{2N}{p} |K| \|u\|_{L^p(\mathbb{R}^N)}^p,$$

where $|K|$ denotes the volume of $K$.

3) **Riemannian manifolds**: Let $(M^N, g)$ be a complete Riemannian manifold of dimension $N$ with $\text{Ric} \geq 0$. Let $d$ be the distance determined by $g$, and let $\mathcal{L}^n$ be the volume element determined by $g$. Since $\text{Ric} \geq 0$ then, by the classical Bishop-Gromov Volume Comparison Theorem, the asymptotic volume ratio exists so (2.13) holds. If $h(M, d, \mathcal{L}^n) \in (0, +\infty)$, then (2.15) holds.

4) **Carnot groups**: Let $G = (\mathbb{R}^n, \cdot)$ be a Carnot group of step $s$ endowed with the Carnot-Carathéodory distance $d_{cc}$ and the Lebesgue measure $\mathcal{L}^n$ (we address the reader to [8] for all the relevant definitions). It is well known that $\mathcal{L}^n(B(x, r)) = r^n \mathcal{L}^n(B(0, 1))$ where $N \in \mathbb{N}$ is the so called homogeneous dimension of $G$. It is then clear that $\text{AVR} = \mathcal{L}^n(B(0, 1)) > 0$ and (2.13) holds.

5) **MCP($0, N$) spaces**: Let $(X, d, \mathcal{L}^n)$ be a metric measure space satisfying the so-called Measure Contraction Property MCP($0, N$), a property introduced independently by Ohta [31] and Sturm [36] as synthetic curvature bound of metric measure spaces. By Generalized Bishop–Gromov volume growth inequality (cf. [36, Theorem 2.3]), the limit

$$\text{AVR} = \lim_{r \to +\infty} \frac{m(B_r(x))}{r^N}$$

exists finite (it can be 0) and does not depend on the point $x \in X$.

It was shown by Juillet [20] that the $n$-dimensional Heisenberg group $\text{Hess}^n$, which is the simplest example of a non-trivial sub-Riemannian manifold, equipped with the Carnot-Carathéodory metric and the Lebesgue measure, satisfies MCP($0, N$) for $N = 2n + 3$. Recently, interpolation inequalities à la Cordero-Erausquin–McCann–Schmuckenschläger [13] have been obtained, under suitable modifications, by Balogh, Kristály and Sipos [5] for the Heisenberg group and by Barilari and Rizzi [6] in the general ideal sub-Riemannian setting. As a consequence, more examples of spaces verifying MCP have been found, e.g. generalized $H$-type groups, the Grushin plane and Sasakian structures (see [6] for more details).

2.3. **Rigidity results.** In this subsection we apply Theorems 2.7 and Theorems 2.10 to get rigidity results for metric measure spaces satisfying the RCD($K, N$) condition. It is well-known that RCD($K, N$) space are obtained by adding a Riemannian
structure (called infinitesimally Hilbertian) to a metric measure space satisfying the CD\( (K, N)\) condition à la Lott-Sturm-Villani [24, 35, 36]. They have been introduced in [3] by Ambrosio-Gigli-Savaré (when \( N = \infty \)) and by Gigli in [16, 17] (treating RCD\( ^* (K, N) \) and infinitesimally Hilbert spaces).

Important examples of spaces satisfying RCD\( (K, N) \) and RCD\( (K, \infty) \) conditions include: weighted Riemannian manifolds satisfying Bakry-Émery condition, measured-Gromov Hausdorff limits of Riemannian manifolds with Ric \( \geq K \) (cf. [24, 35]), Alexandrov spaces with curvature \( \geq K \) (cf. [37]). We refer the reader to the ICM proceeding [2] by Ambrosio for an overview of this topic.

**Theorem 2.12.** Let \((X, d, m)\) be a RCD\( (0, N)\) metric measure space with \( N \in (1, +\infty) \). Then

\[
\lim_{s \downarrow 0} s \int_X \int_X |u(x) - u(y)|^p d^{N+sp}(x, y) \, dm(x) \, dm(y) \leq \frac{2N\omega_N}{p} \|u\|^p_{L^p}.
\]

for any \( u \in \cup_{r \in (0, 1)} W^{7,p} \), where \( \omega_N \) denotes the volume of an \( N \)-dimensional unit ball.

If the equality in (2.16) is attained by a non-zero function \( u \), then \( N \in \mathbb{N} \) and \((X, d, m)\) is isometric to a metric cone over an RCD\( (N - 2, N - 1) \) space.

**Proof.** Firstly, by Generalized Bishop–Gromov volume growth inequality (cf. [36, Theorem 2.3]) we know

\[
\text{AVR}(X, d, m) = \lim_{r \to \infty} m\left(\frac{B_r(x_0)}{r^N}\right) \leq \frac{m\left(\frac{B_R(x_0)}{R^N}\right)}{\frac{R}{r^N}} \leq \omega_N, \quad \forall R > 0.
\]

So (2.16) follows from Theorem 2.7.

If the equality in (2.16) is attained by a non-zero function \( u \), by Theorem 2.7 and (2.17) we can see that

\[
\frac{m\left(\frac{B_r(x_0)}{r^N}\right)}{\frac{R}{r^N}} = \omega_N, \quad \forall R > 0.
\]

By [14, Theorem 1.1] we know \((X, d, m)\) is isometric to a metric cone over an RCD\( (N - 2, N - 1) \) space. Furthermore, by [10] and [15] we know \( N \) is an integer. \(\square\)

**Theorem 2.13.** Let \((X, d, m)\) be a non-compact RCD\( -(N - 1), N)\) metric measure space with \( N \in (1, +\infty) \). Then

\[
\lim_{s \downarrow 0} s \int_X \int_X \frac{|u(x) - u(y)|^p}{e^{(h+s)d(x,y)}} \, dm(x) \, dm(y) \leq 2(N - 1)\|u\|^p_{L^p}
\]

for any \( u \in \text{Lip}_b(X, d) \) with bounded support.

If the equality in (2.18) is attained by a non-zero function \( u \), then \((X, d, m)\) is isometric to a warped product space \( \mathbb{R} \times_{e^h} X' \), where \( X' \) is an RCD\( (0, N) \) space.

**Proof.** By [12, Corollary 3.2] we know \( h(X, d, m) \leq N - 1 \). So (2.18) follows from Theorem 2.10.

If the equality in (2.18) is attained by a non-zero function \( u \), by Theorem 2.10 and the inequality \( h(X, d, m) \leq N - 1 \) we have \( h(X, d, m) = N - 1 \). Then by [12, Theorem 1.2], \((X, d, m)\) is isometric to a warped product space \( \mathbb{R} \times_{e^h} X' \), where \( X' \) is an RCD\( (0, N) \) space. \(\square\)
References

[1] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková. Fractional Orlicz-Sobolev embeddings. *J. Math. Pures Appl. (9)*, 149:216–253, 2021.

[2] L. Ambrosio. Calculus, heat flow and curvature-dimension bounds in metric measure spaces. *Proceedings of the ICM 2018*, 2018.

[3] L. Ambrosio, N. Gigli, and G. Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.*, 163:1405–1490, 2014.

[4] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*. Oxford University Press, 2004.

[5] Z. M. Balogh, A. Kristály, and K. Sipos. Geometric inequalities on Heisenberg groups. *Calc. Var. Partial Differential Equations*, 57(2):Art. 61, 41, 2018.

[6] D. Barilari and L. Rizzi. Sub-riemannian interpolation inequalities. *Invent. math.*, 2018. doi:10.1007/s00222-018-0840-y.

[7] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.

[8] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Oxford University Press, 2004.

[9] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In *Optimal control and partial differential equations*, pages 439–455. IOS, Amsterdam, 2001.

[10] E. Brué and D. Semola. Constancy of the dimension for RCD($K, N$) spaces via regularity of Lagrangian flows. *Comm. Pure Appl. Math.*, 73(6):1141–1204, 2020.

[11] M. Capolli, A. Maione, A. M. Salort, and E. Vecchi. Asymptotic behaviours in fractional Orlicz-Sobolev spaces on Carnot groups. *J. Geom. Anal.*, 31(3):3196–3229, 2021.

[12] C. Connell, X. Dai, J. Núñez-Zimbón, R. Perales, P. Suárez-Serrato, and G. Wei. Maximal volume entropy rigidity for RCD$^*(−(N − 1), N)$ spaces. *J. Lond. Math. Soc.*, Jun 2021.

[13] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math.*, 146(2):219–257, 2001.

[14] N. Gigli. The splitting theorem in non-smooth context. Preprint, arXiv:1302.5555., 2013.

[15] N. Gigli. On the differential structure of metric measure spaces and applications. *Mem. Amer. Math. Soc.*, 236(1113):vi+91, 2015.

[16] W. Görny. Bourgain-Brezis-Mironescu approach in metric spaces with euclidean tangents. Preprint, arXiv:2004.08448, 2020.

[17] B. Han and A. Pinamonti. On the asymptotic behaviour of the fractional Sobolev $s$-seminorm as $s \uparrow 1$: Bourgain-Brezis-Mironescu’s theorem revisited. Preprint, arXiv: 2108.06996, 2021.

[18] N. Juillet. Geometric inequalities and generalized Ricci bounds in the Heisenberg group. *Int. Math. Res. Not. IMRN*, (13):2347–2373, 2009.

[19] G. E. Karadžov, M. Milman, and J. Xiao. Limits of higher-order Besov spaces and sharp reiteration theorems. *J. Funct. Anal.*, 221(2):323–339, 2005.

[20] A. Kreuml and O. Mordhorst. Fractional Sobolev norms and BV functions on manifolds. *Nonlinear Anal.*, 187:450–466, 2019.

[21] N. Lam, A. Maalaoui, and A. Pinamonti. Characterizations of anisotropic high order Sobolev spaces. *Asymptot. Anal.*, 113(4):239–260, 2019.

[22] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[23] M. Ludwig. Anisotropic fractional Sobolev norms. *Adv. Math.*, 252:150–157, 2014.

[24] A. Manning. Topological entropy for geodesic flows. *Ann. of Math. (2)*, 110(3):567–573, 1979.

[25] V. Maz’ya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[26] M. Milman. Notes on limits of Sobolev spaces and the continuity of interpolation scales. *Trans. Amer. Math. Soc.*, 357(9):3425–3442, 2005.
[30] H.-M. Nguyen, A. Pinamonti, M. Squassina, and E. Vecchi. Some characterizations of magnetic Sobolev spaces. *Complex Var. Elliptic Equ.*, 65(7):1104–1114, 2020.

[31] S.-i. Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.

[32] A. Pinamonti, M. Squassina, and E. Vecchi. The Maz’ya-Shaposhnikova limit in the magnetic setting. *J. Math. Anal. Appl.*, 449(2):1152–1159, 2017.

[33] A. Pinamonti, M. Squassina, and E. Vecchi. Magnetic BV-functions and the Bourgain-Brezis-Mironescu formula. *Adv. Calc. Var.*, 12(3):225–252, 2019.

[34] A. C. Ponce. A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations*, 19(3):229–255, 2004.

[35] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[36] K.-T. Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.

[37] H.-C. Zhang and X.-P. Zhu. Ricci curvature on Alexandrov spaces and rigidity theorems. *Comm. Anal. Geom.*, 18(3):503–553, 2010.

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