Effective theory of incompressible quantum Hall liquid crystals

Michael M. Fogler

Department of Physics, Massachusetts Institute of Technology,
77 Massachusetts Avenue, Cambridge, Massachusetts 02139

I propose an effective theory of zero-temperature phases of the quantum Hall stripes: a smectic phase where the stripes are static and a novel quantum nematic phase where the positional order is destroyed by quantum fluctuations. The nematic is viewed as a Bose condensate of dislocations whose interactions are mediated by a $U(1)$ gauge field. Collective mode spectrum and the dynamical structure factor in the two phases are calculated.

Stripe phases are extremely common in nature. Once studied mainly in the context of pattern formation and soft condensed-matter systems (convection rolls, ferrofluids, diblock co-polymers, etc. [1]), they are now recognized to be important in venues ranging from neutron stars [2] to neural networks of the brain [3]. The “hard” condensed-matter community was alerted to the relevance of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4]. The subject of stripes after their discovery in transitional metal oxides, especially, high-$T_c$ cuprates [4].

FIG. 1: (a) Smectic (b) Nematic. One dislocation is encircled.
density from the equilibrium value \( n_0 \), and \( U \) is the integral operator with kernel \( U(r) \), the electron-electron interaction potential. \( U \) is Coulombic at large \( r \) but is modified by many-body screening, exchange and correlation effects at short distances (see more details in [2]). The last term in Eq. (1) accounts for the dependence of the smectic period on \( n_0 \), with \( C = Y \partial \ln q_0 / \partial n_0 \). It vanishes at the half-filling due to electron-hole symmetry but is nonzero otherwise.

One may wonder why include “incompressible” background \( n \) in our low-energy Hamiltonian (1). The reason is the important role \( n \) plays in the dynamics of \( u \). Indeed, \( u \) determines the density fluctuations near the star of the soft mode, e.g., \( n_{q_0 + k} = \frac{1}{2} (\Psi_0^* e^{-i q_0 \cdot k}) \), while the density operators of \( q_0 \) projected onto a single Landau level are dynamically linked by the commutation relation [22],

\[
[n_q, n_{q'}] = 2i \sin(\frac{1}{2} l^2 q \wedge q') n_{q + q'},
\]

where \( l = \sqrt{\hbar/m \omega_c} \) is the magnetic length and \( \omega_c = eB/mc \) is the cyclotron frequency for the external magnetic field \( B \). The above commutation relation is exact but difficult to deal with. I achieve a simplification by replacing it with its expectation value expanded to the lowest order in \( q \). This way I obtain the following approximate commutation relation for use in our effective theory:

\[
[n_q, u_q] = (2\pi)^2 q_p \delta(q + q')
\]  

(2)

(all other commutators vanish in the \( q \to 0 \) limit). Introducing the canonical momentum \( p \) by \( \delta n = \partial_p \rho \), I unify Eqs. (1) and (2) into the effective action [19],

\[
A_{\text{eff}} = \int_0^\beta d\tau \int d^2r \left\{ -\frac{i}{\hbar} \hbar \partial_\tau u + \frac{Y}{2} (\partial_x u)^2 + \frac{K}{2} (\partial_y u)^2 + \frac{1}{2} (\partial_y p) U(\partial_y p) - C \partial_x u \partial_y p \right\}.
\]

This general form can be shown to pass two important tests. First, after a change of variables it reproduces [14] an effective action derived for a specific model of the quantum Hall smectic [13]. Second, the density structure factor, easily calculated for the Gaussian theory [3],

\[
S(q, \omega) \simeq \frac{\hbar}{m} \operatorname{Im} \frac{q^2 \omega^2}{Q(q)} (q / \omega_q)^2 - \omega^2 \omega_c^2 - i \omega \theta,
\]

(4)

coincides up to the Bose factor and dissipative terms with the finite temperature result of Ref. [22]. The notations used here are \( Q(q) = (Y q_x^2 + K q_y^2) / mn_0 \), \( Y = Y - C^2 / U \), and \( \omega_p(q) = \sqrt{n_0 U(q)^2 / m} \). The \( \omega / \omega_c \) depends on \( q \) consists of a single \( \delta \)-function at the frequency of the magneto phonon mode,

\[
\omega(q) = \frac{q q_e^2}{\hbar \omega_c} \sqrt{Q(q)}.
\]

Dislocations and duality.—2D smectics can exist only at zero temperature. At \( T > 0 \) thermal fluctuations of the stripes restore the translational symmetry, so that the highest possible degree of ordering is that of the nematic [20]. The actual crossover of the Hamiltonian from the short-distance smectic (1) to the long-distance nematic form is quite nontrivial. It is driven by thermally excited dislocations, which have an ability to screen the compressional stress [22]. It is natural to assume then that the smectic-nematic quantum phase transition is also driven by topological defects. Pictorially, the difference between the smectic and nematic can be represented as follows. The dislocations are viewed as lines in the \((2 + 1)\)D space. In the smectic phase, they form small closed loops (Fig. 2a) that depict virtual pair creation-annihilation events; in the nematic phase arbitrarily long dislocation worldlines exist and may entangle (Fig. 2b), similar to worldlines of particles in a Bose superfluid [25].

Now I will present a mathematical formalism supporting these qualitative considerations. It is analogous to the duality transformation introduced in Ref. [20].

Our first step is to incorporate the dislocations into the effective action (3). This is accomplished by factorizing the smectic order parameter, \( \Psi = \Psi_0 e^{i q_0 u} \times \Psi_D \), where \( u \) is a regular single-valued function and \( \Psi_D \) is the phase factor due to the dislocations. The derivatives of \( u \) in Eq. (3) should now be replaced by \( \partial_\mu u - (i / q_0) \Psi_D^* \partial_\mu \Psi_D \equiv (\partial_\mu u)_\text{tot} \), \( \mu = \tau, x, y \). Decoupling the quadratic part of the action with a Hubbard-Stratanovich field \( \sigma_\mu \), I get

\[
A = A_D + \int_0^\beta d\tau \int d^2r \left[ -i (\partial_\mu u)_\text{tot} \partial_\mu \sigma_\mu - H_\alpha \right],
\]

(6)

\[
H_\alpha = -i \frac{C}{Y \ell^2} \sigma_\tau \partial_\tau \sigma_\tau + \frac{\sigma^2_\tau}{2Y} + \sigma_y (-2K \partial^2_\theta)^{-1} \sigma_y
\]

\[
+ \frac{\hbar^4}{2} \partial_\theta \sigma_\tau \left( U - C^2 / Y \right) \partial_\theta \sigma_\tau,
\]

(7)

where \( h = 1, \sigma_\tau \equiv p / \ell^2 \), and \( A_D \) contains terms describing dislocation cores (see below). Integration over \( u \) gives the constraint \( \partial_\mu \sigma_\mu = 0 \), which I implement by means of an auxiliary \( U(1) \) gauge field \( a_\mu \),

\[
\sigma_\mu = \epsilon_{\mu \nu} \partial_\nu a_\lambda \equiv [\partial \times a]_\mu.
\]

(8)
This leads to the (dual) action of the form

$$A_{\text{dual}} = A_D + \int_0^\beta d\tau \int d^2 r (\ldots)$$

where $\Lambda = 2\pi/q_0$ and $j_\mu = (2\pi i)^{-1} t_{\mu\nu} \lambda \partial_r (\psi_D^* \partial_r \psi_D)$ is the dislocation 3-current. This current can be expressed in terms of the second-quantized dislocation field $\Phi$,

$$j_\mu = t_\mu \Phi^* (\ldots)$$

which is assumed to be bosonic. Other types of quantum statistics for dislocations are exotic alternatives, which are not investigated here.

Making standard assumptions about $A_D$, I arrive at

$$A_{\text{dual}} = \int_0^\beta d\tau \int d^2 r \left\{ \frac{t_{\mu\nu}}{2} (\ldots) + V(\Phi) + H_\alpha(\ldots) \right\},$$

The obtained dual theory, Eqs. (1), (3), and (11) describes $\Phi$ bosons interacting with each other and with the $U(1)$ gauge field $a_\mu$. The phenomenological parameters introduced above are as follows. Parameter $t_x \sim h^2/E_c$, where $E_c$ is the dislocation core energy. It was estimated within the Hartree-Fock approximation in Refs. [4] and [14]. Near $\nu \sim \nu_c$ where the interstripe separation $\Lambda$ approaches the quantum fluctuation scale $l$, $E_c$ is expected to be much smaller than the Hartree-Fock value, but any precise estimate is challenging. Parameter $t_x$ of dimension of energy $\times$ (length)$^2$ is the hopping matrix element for dislocation motion in the $x$-direction, i.e., dislocation glide. Such a glide requires quantum tunneling and is exponentially small unless $\Lambda \lesssim l$. Parameter $t_y$ describes the dislocation climb, which also originates from the dynamics on the microscopic length scales. One may recall that the climb requires mobile point defects. Although such defects are not among fundamental low-energy excitations of the theory, composite point defects, in the form of short stripe segments or dislocation pairs, do exist. They are most likely short lived because dislocations are not expected to form bound pairs in the nematic phase. Yet during their limited life span such composite defects can perfectly well assist the climb by micromotions of constituent electrons. In general, I am not aware of any fundamental principle that would protect $t_y = 0$ value, although it seems reasonable that $t_y \ll t_x$ in our case.

Yet another phenomenological variable is the potential $V(\Phi) = m_\phi |\Phi|^2 + r_\phi |\Phi|^4 + \ldots$, which accounts for a self-energy and a short-range interaction between the dislocations; the scales of $m_\phi$ and $r_\phi$ are set by $E_c$ and $E_c \Lambda^2$, respectively. Finally, $e_D$ is electric charge of the dislocation that couples to the external vector potential $\partial_{\tau_{\text{ext}}} = a_{\tau_{\text{ext}}} = 0, a_{\text{ext}} = Bx$. This coupling is introduced only for the sake of generality. Since I study electron liquid crystal phases derived from incompressible liquids, I expect dislocations to be electrically neutral, i.e., $e_D = 0$.

Another few comments are in order. The derived theory is meant to capture only the dynamics of neutral (dipolar) excitations of the system. The underlying incompressible state has its own dynamics characterized, most importantly, by the quantized Hall conductivity.

There is a very strong similarity between our theory and the statistical mechanics model of the 3D smectic-nematic transition studied by Toner and others [27].

Integrating out the gauge field from $A_{\text{dual}}$ leads to the model of dislocation lines interacting via an effective Biot-Savart potential (which in our case is short-range Coulombic $U$ [14]). One may argue [28] that the unlimited expansion of the dislocation loops occurs when the energy cost for creating a large loop of length $L$ is compensated by the “entropic” gain $L/l_0$, where the persistence length $l_0$ is determined by short-range physics. Despite the physical appeal of this argument, the gauge theory (1) is presumably better suited for a quantitative analysis (perhaps, along the lines of Ref. [27]).

**Phases and their collective modes.**— Let us now see how the smectic and nematic states are reproduced in the dual theory. As discussed above, the smectic phase corresponds to $\langle \Phi \rangle = 0$. In this case $A_{\text{dual}}$ reduces to $H_\alpha$, which is quadratic. The low-energy dynamics is that of a gas of noninteracting Goldstone bosons, which are the aforementioned magnetophons. It is a simple exercise to verify that their dispersion relation is given by Eq. (4).

In the nematic phase dislocations have condensed, $\langle \Phi \rangle \neq 0$. In conventional local $U(1)$ gauge theories, the appearance of such an order parameter triggers the Anderson-Higgs mechanism, eliminating the gapless Goldstone modes. This can not be the case here because the nematic state does break the continuous symmetry with respect to spatial rotations. The seeming paradox is resolved due to the peculiar feature of the present gauge theory: the nonlocality of the gauge-field strength term $H_\alpha$, see Eq. (1). By virtue of that, the condensation of $\Phi$ merely stiffens the collective mode, leaving it gapless. Neglecting terms proportional to $C$, I find that

$$\omega_1(q) = \left( \frac{m_x}{m_\tau} q_x^2 + m_x K q_y^2 \right)^{1/2}, \ m_\mu = t_\mu (\ldots)$$

As for the magnetophonon mode of the smectic [3], it indeed acquires a small gap $\sqrt{m_y Y}$ at $q = 0$. It anticrosses with the acoustic branch [12] near the point $\omega_1(q) \sim m_y Y$, and at larger $q$ becomes the lowest frequency collective mode with the dispersion relation

$$\omega_2(q) = \left[ \frac{q_x^2 q_y^2}{m^2 \omega_c^2} Y U(q) + m_y Y \right]^{1/2}$$

only slightly different from [3]. At such $q$ the structure
factor of the nematic has two sets of $\delta$-functional peaks, 
\[ S(\omega, \mathbf{q}) = \frac{\pi \hbar^2 q^2}{m \omega_c^2} \left[ \frac{K q^4}{m n_0} \delta(\omega^2 - \omega_1^2) + \frac{Y q^2}{m n_0} \delta(\omega^2 - \omega_2^2) \right], \]
which split between themselves the spectral weight of the single collective mode of the smectic. The presence of the two modes can be explained by the existence of two order parameters: a unit vector (more precisely, director) \( \mathbf{N} \) normal to the local stripe orientation and the complex wavefunction \( \Phi_0 \) of the dislocation condensate. Classical 2D nematics have two (overdamped) modes virtually for the same reason \( \square \).

In conclusion, let us address measurable properties of the novel quantum Hall states considered here. At low temperature both the parent incompressible state and its liquid crystal descendants will be insulating. If \( T \) is not too small, formation of stripe superstructures can be deduced from the anisotropic magnetoresistance \( \square \). On the other hand, the microwave absorption will be anisotropic even at low \( T \) and would enable one to further distinguish between the smectic and nematic phases: the nematic will show two dispersing collective modes while the smectic will produce a single one. To circumvent disorder pinning effects, such measurements should be done at high enough \( q \).

Acknowledgements—This work is supported by MIT Pappalardo Program in Physics. I wish to thank X.-G. Wen for useful discussions, and also A. Dorsey, L. Radzihovsky, and C. Wexler for valuable comments on the manuscript.

[1] M. Seul and D. Andelman, Science, 267, 476 (1995).
[2] S. Reddy, G. Bertsch, and M. Prakash, Phys. Lett. B 475, 1 (2000).
[3] D. B. Chklovskii and A. A. Koulakov, Physica A 284, 318 (2000).
[4] J. M. Tranquada et al., Nature 375, 561 (1995); S. Mori, C. H. Chen, and S.-W. Cheong, Nature 392, 473 (1998).
[5] A. A. Koulakov, M. M. Fogler, and B. I. Shklovskii, Phys. Rev. Lett. 76, 499 (1996); M. M. Fogler, A. A. Koulakov, and B. I. Shklovskii, Phys. Rev. B 54, 1853 (1996).
[6] R. Moessner and J. T. Chalker, Phys. Rev. B 54, 5006 (1996).
[7] M. M. Fogler and A. A. Koulakov, Phys. Rev. B 55, 9326 (1997); E. H. Rezayi, F. D. M. Haldane, and K. Yang, Phys. Rev. Lett. 83, 1219 (1999) N. Shibata and D. Yoshioka, cond-mat/0101401.
[8] V. W. Scarola, K. Park, J. K. Jain, Phys. Rev. B 62, R16259 (2001).
[9] M. P. Lilly, K. B. Cooper, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 82, 394 (1999); R. R. Du, D. C. Tsui, H. L. Störmer, L. N. Pfeiffer, and K. W. West, Solid State Commun. 109, 389 (1999).
[10] L. Balents, Europhys. Lett. 33, 291 (1996).
[11] E. Fradkin and S. A. Kivelson, Phys. Rev. B 59, 8065 (1999).
[12] H. Yi, H. A. Fertig, and R. Côté, Phys. Rev. Lett. (2001).
[13] A. H. MacDonald and M. P. A. Fisher, Phys. Rev. B 61, 5724 (2000).
[14] M. M. Fogler, in preparation.
[15] K. Musaelian and R. Joynt, J. Phys. Cond. Mat. 8, L105 (1996).
[16] A finite-size study by Rezayi et al. suggests that the transition from the stripe phase to a uniform state as a function of the interaction parameters can also occur via a first-order transition without the intermediate nematic phase.
[17] I am interested in a regime with very strong quantum fluctuations where the smectic order parameter is small and only the main harmonic of the soft mode is important.
[18] If the parent state is compressible, the additional low-energy degrees of freedom would generate an effective viscosity of Landau damping (or Caldeira-Leggett) type entering Eq. (4). Its consequences are left for future study.
[19] In principle, \( \delta n = -i \partial_p p + f(u) \) with some function \( f \). In the absence of dislocations the most relevant term is \( f = A \partial_p u \), which leads again to Eq. (5). If dislocations are allowed, the situation is more complicated, see later in the main text.
[20] P. G. de Gennes and J. Prost, The Physics of Liquid Crystals (Oxford University Press, New York, 1995).
[21] M. M. Fogler and V. M. Vinokur, Phys. Rev. Lett. 84, 5828 (2000).
[22] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B 33, 2481 (1986).
[23] J. Toner and D. R. Nelson, Phys. Rev. B 23, 316 (1981). For application of this theory to the quantum Hall stripes, see Refs. \[1\] and \[24\].
[24] C. Wexler and A. T. Dorsey, cond-mat/0009096.
[25] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).
[26] M. P. A. Fisher and D. H. Lee, Phys. Rev. B 39, 2756 (1989).
[27] J. Toner, Phys. Rev. B 26, 462 (1982); A. R. Day, T. C. Lubensky, and A. J. McKane, Phys. Rev. A 27, 1461 (1983).
[28] W. Helfrich, J. Phys. (Paris) 39, 1199 (1978).