LOCAL TRANSLATIONS ASSOCIATED TO SPECTRAL SETS

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Abstract. In connection to the Fuglede conjecture, we study groups of local translations associated to spectral sets, i.e., measurable sets in $\mathbb{R}$ or $\mathbb{Z}$ that have an orthogonal basis of exponential functions. We investigate the connections between the groups of local translations on $\mathbb{Z}$ and on $\mathbb{R}$ and present some examples for low cardinality. We present some relations between the group of local translations and tilings.

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1. Introduction

In his study of commuting self-adjoint extensions of partial differential operators, Fuglede proposed the following conjecture [Fug74]:

**Conjecture 1.1.** Denote by $e_{\lambda}(x) = e^{2\pi i \lambda \cdot x}$, $\lambda, x \in \mathbb{R}^d$. Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^d$ of finite positive measure. There exists a set $\Lambda$ such that $\{e_{\lambda} : \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$ if and only if $\Omega$ tiles $\mathbb{R}^d$ by translations.

**Definition 1.2.** Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^d$ of finite, positive measure. We say that $\Omega$ is a spectral set if there exists a set $\Lambda$ in $\mathbb{R}^d$ such that $\{e_{\lambda} : \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$. In this case, $\Lambda$ is called a spectrum for $\Omega$.

We say that $\Omega$ tiles $\mathbb{R}^d$ by translations if there exists a set $\mathcal{T}$ in $\mathbb{R}^d$ such that the sets $\Omega + t$, $t \in \mathcal{T}$ form a partition of $\mathbb{R}^d$, up to measure zero.

Terrence Tao [Tao04] has proved that spectral-tile implication in the Fuglede conjecture is false in dimensions $d \geq 5$ and later both directions were disproved in dimensions $d \geq 3$, see [KM06a]. At the moment the conjecture is still open in both directions for dimensions 1 and 2. In this paper, we will only focus on dimension $d = 1$.

Recent investigations have shown that the Fuglede can be reduced to analogous statements in $\mathbb{Z}$, see [DL13].

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Definition 1.3. Let $A$ be a finite subset of $\mathbb{Z}$, $|A| = N$. We say that $A$ is \textit{spectral} if there exists a set $\Gamma$ in $\mathbb{R}$ such that $\{e_\gamma : \gamma \in \Gamma\}$ is an orthogonal basis in $l^2(A)$, equivalently, the matrix
\begin{equation}
\frac{1}{\sqrt{N}}(e^{2\pi i \gamma a})_{\gamma \in \Gamma, a \in A}
\end{equation}
is unitary. This matrix is called the \textit{Hadamard matrix associated to the pair $(A, \Gamma)$}.

We say that $A$ \textit{tiles} $\mathbb{Z}$ \textit{by translations} if there exists a set $\mathcal{T}$ in $\mathbb{Z}$ such that the sets $A + t$, $t \in \mathcal{T}$ forms a partition of $\mathbb{Z}$.

The Fuglede conjecture for $\mathbb{Z}$ can be formulated as follows

Conjecture 1.4. A finite subset $A$ of $\mathbb{Z}$ is spectral if and only if it tiles $\mathbb{Z}$ by translations.

As shown in [DL13] the tile-spectral implications for $\mathbb{R}$ and $\mathbb{Z}$ are equivalent:

Theorem 1.5. Every bounded tile in $\mathbb{R}$ is spectral if and only if every tile in $\mathbb{Z}$ is spectral.

It is not clear if the spectral-tile implications for $\mathbb{R}$ and $\mathbb{Z}$ are equivalent. It is known that, if every spectral set in $\mathbb{R}$ is a tile, then every spectral set in $\mathbb{Z}$ is a tile in $\mathbb{Z}$. It was shown in [DL13], that the reverse also holds under some extra assumptions.

Theorem 1.6. Suppose every bounded spectral set $\Omega$ in $\mathbb{R}$, of Lebesgue measure $|\Omega| = 1$, has a rational spectrum, $\Lambda \subset \mathbb{Q}$. Then the spectral-tile implications for $\mathbb{R}$ and for $\mathbb{Z}$ are equivalent, i.e., every bounded spectral set in $\mathbb{R}$ is a tile if and only if every spectral set in $\mathbb{Z}$ is a tile.

All known examples of spectral sets of Lebesgue measure 1 have a rational spectrum. There is another, stronger variation of the spectral-tile implication in $\mathbb{Z}$ which is equivalent to the spectral-tile implication in $\mathbb{R}$.

Definition 1.7. We say that a set $\Lambda$ has \textit{period} $p$ if $\Lambda + p = \Lambda$. The smallest positive $p$ with this property is called the \textit{minimal period}.

Theorem 1.8. [DL13] The following statements are equivalent

(i) Every bounded spectral set in $\mathbb{R}$ is a tile.

(ii) For every finite union of intervals $\Omega = A + [0, 1]$ with $A \subset \mathbb{Z}$, if $\Lambda$ is a spectrum of $\Omega$ with minimal period $\frac{1}{N}$, then $\Omega$ tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset N\mathbb{Z}$.

We should note here that for a finite set $A$ in $\mathbb{Z}$, the set $A + [0, 1]$ is spectral in $\mathbb{R}$ if and only if $A$ is spectral in $\mathbb{Z}$. Also, if $\Gamma$ is a spectrum for $A$ then $\Lambda := \Gamma + \mathbb{Z}$ is a spectrum of $A + [0, 1]$, therefore $\Lambda$ has period 1 and the minimal period will have to be of the form $\frac{1}{N}$. See [DL13] for details.

The spectral property of a set, in either $\mathbb{R}$ or $\mathbb{Z}$, can be characterized by the existence of a certain unitary group of local translations. We will describe this in the next section and present some properties of these groups. In Theorem 2.11 we give a characterization of spectral sets in $\mathbb{Z}$ in terms of the existence of groups of local translations or of a local translation matrix. In Proposition 2.14 we establish a formula that connects the local translation matrix for a spectral set $A$ in $\mathbb{Z}$ and the group of local translations for the spectral set $A + [0, 1]$ in $\mathbb{R}$. Proposition 2.16 shows how one can look for tiling sets for the spectral set $A$ using the local translation matrix. Proposition 2.18 shows that the rationality of the spectrum is characterized by the periodicity of the group of local translations.
2. Local translations

In this section we introduce the unitary groups of local translations associated to spectral sets. These are one-parameter groups of unitary operators on $L^2(\Omega)$, for subsets $\Omega$ of $\mathbb{R}$, or on $l^2(A)$, for subsets of $\mathbb{Z}$, which act as translations on $\Omega$ or $A$ whenever such translations are possible. The existence of such groups was already noticed earlier by Fuglede [Fug74] and [Ped87]. They were further studied in [DJ12b]. The idea is the following: the existence of an orthonormal basis \( \{e_\lambda : \lambda \in \Lambda\} \) allows the construction of the Fourier transform from $L^2(\Omega)$ to $l^2(\Lambda)$. On $l^2(\Lambda)$ one has the unitary group of modulation operators, i.e., multiplication by $e^{it\lambda}$, or the diagonal matrix with entries $e^{2\pi i \lambda t}$, $\lambda \in \Lambda$. Conjugating via the Fourier transform we obtain the unitary group of local translations. For further information on local translations and connections to self-adjoint extensions and scattering theory, see [JPT12c].

**Definition 2.1.** Let $\Omega$ be a bounded Borel subset of $\mathbb{R}$. A unitary group of local translations on $\Omega$ is a strongly continuous one parameter unitary group $U(t)$ on $L^2(\Omega)$ with the property that for any $f \in L^2(\Omega)$ and any $t \in \mathbb{R}$,

\[
(U(t)f)(x) = f(x + t) \quad \text{for a.e. } x \in \Omega \cap (\Omega - t).
\]

If $\Omega$ is spectral with spectrum $\Lambda$, we define the Fourier transform $\mathcal{F} : L^2(\Omega) \to l^2(\Lambda)$

\[
\mathcal{F}f = \left( \frac{1}{\sqrt{|\Omega|}} e_\lambda \right)_{\lambda \in \Lambda}, \quad (f \in L^2(\Omega)).
\]

We define the unitary group of local translations associated to $\Lambda$ by

\[
U_\Lambda(t) = \mathcal{F}^{-1} \hat{U}_\Lambda(t) \mathcal{F} \quad \text{where} \quad \hat{U}_\Lambda(t)(a_\lambda)_{\lambda \in \Lambda} = (e^{2\pi i t \lambda} a_\lambda)_{\lambda \in \Lambda}: \quad ((a_\lambda) \in l^2(\Lambda)).
\]

**Proposition 2.2.** With the notations in Definition 2.1

\[
U_\Lambda(t)e_\lambda = e_\lambda(t)e_\lambda, \quad (t \in \mathbb{R}, \lambda \in \Lambda)
\]

Thus the functions $e_\lambda$ are the eigenvectors for the operators $U(t)$ corresponding to the eigenvalues $e^{2\pi i t \lambda}$ with multiplicity one.

**Proof.** Clearly $\mathcal{F} e_\lambda = \sqrt{|\Omega|} \delta_\lambda$ for all $\lambda \in \Lambda$. The rest follows from a simple computation. \qed

**Theorem 2.3.** Let $\Omega$ be a bounded Borel subset of $\mathbb{R}$. Assume that $\Omega$ is spectral with spectrum $\Lambda$. Let $U_\Lambda$ be the associated unitary group as in Definition 2.1. Then $U := U_\Lambda$ is a unitary group of local translations.

In the particular case when $\Omega$ is a finite union of intervals the converse also holds:

**Theorem 2.4.** [DJ12b] The set $\Omega = \bigcup_{i=1}^n (\alpha_i, \beta_i)$ is spectral if and only if there exists a strongly continuous one parameter unitary group $(U(t))_{t \in \mathbb{R}}$ on $L^2(\Omega)$ with the property that, for all $t \in \mathbb{R}$ and $f \in L^2(\Omega)$:

\[
(U(t)f)(x) = f(x + t), \quad \text{for almost every } x \in \Omega \cap (\Omega - t).
\]

Moreover, given the unitary group $U$, the spectrum $\Lambda$ of the self-adjoint infinitesimal generator $D$ of the group $U(t) = e^{2\pi i t D}$ (as in Stone’s theorem), is a spectrum for $\Omega$.

**Remark 2.5.** As shown in [DJ12b], and actually even in the motivation of Fuglede [Fug74] for his studies of spectral sets, the self-adjoint operator $D$ appearing in Theorem 2.4 are self-adjoint extensions of the differential operator $\frac{1}{2\pi i} \frac{d}{dx}$ on $L^2(\Omega)$. 
Example 2.6. The simplest example of a spectral set is $\Omega = [0, 1]$ with spectrum $\Lambda = \mathbb{Z}$. In this case, the group of local translations is

$$(U(t)f)(x) = f((x + t) \mod 1), \quad (f \in L^2[0, 1], x, t \in [0, 1]).$$

This can be checked by verifying that $U(t)e_n = e_n(t)e_n$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

Proposition 2.7. Let $\Omega = \bigcup_{i=1}^{n} (\alpha_i, \beta_i)$ be a spectral set with spectrum $\Lambda$. Let $E$ be a Lebesgue measurable subset of $\Omega$ and $t_0 \in \mathbb{R}$ such that $E + t_0 \subset \Omega$. Then

$$(2.6) \quad U(-t_0)\chi_E = \chi_{E+t_0}.$$

Proof. Since $E + t_0$ is contained in $\Omega \cap (\Omega + t_0)$, by Theorem 2.4 we have that, for almost every $x \in E + t_0$, $(U(-t_0)\chi_E)(x) = \chi_E(x-t_0) = \chi_{E+t_0}(x)$. Thus, if $g := U(-t_0)\chi_E$, then $g(x) = \chi_{E+t_0}(x)$ for a.e., $x \in E + t_0$. On the other hand, since $U(-t_0)$ is unitary, we have that $\|g\|_{L^2}^2 = \|\chi_E\|_{L^2}^2 = \mu(E)$. But

$$\mu(E) = \|g\|_{L^2}^2 = \int_{E+t_0} |g(x)|^2 \, dx + \int_{\Omega \setminus (E+t_0)} |g(x)|^2 \, dx = \mu(E + t_0) = \int_{\Omega \setminus (E+t_0)} |g(x)|^2 \, dx,$$

so $g(x) = 0$ for a.e. $x \in \Omega \setminus (E + t_0)$.

Next, we focus on spectral subsets of $\mathbb{Z}$ and define the one-parameter unitary group of local translations in an analogous way. As we will see, in this case, the parameter can be restricted from $\mathbb{R}$ to $\mathbb{Z}$ and thus the unitary group of local translations is determined by a local translation unitary matrix.

Definition 2.8. Let $A$ be a finite subset of $\mathbb{Z}$. A group of local translations on $A$ is a continuous one-parameter unitary group $U(t)$, $t \in \mathbb{R}$ on $l^2(A)$ with the property that

$$(2.7) \quad U(a - a')\delta_a = \delta_{a'}, \quad (a, a' \in A)$$

A unitary matrix $B$ on $l^2(A)$ is called a local translation matrix if

$$(2.8) \quad B^{a-a'}\delta_a = \delta_{a'}, \quad (a, a' \in A)$$

If $A$ is a spectral subset of $\mathbb{Z}$ with spectrum $\Gamma$ and $|A| = N$, we define the Fourier transform from $l^2(A)$ to $l^2(\Gamma)$ by the matrix:

$$(2.9) \quad \mathcal{F} = \frac{1}{\sqrt{N}} \left( e^{-2\pi i \lambda a} \right)_{\lambda \in \Gamma, a \in A}.$$

Let $D_\Gamma(t)$ be the diagonal matrix with entries $e^{2\pi i \lambda t}$, $\lambda \in \Gamma$. We define the group of local translations on $A$ associated to $\Gamma$ by

$$(2.10) \quad U_\Gamma(t) := \mathcal{F}^* D_\Gamma(t) \mathcal{F}, \quad (t \in \mathbb{R})$$

The local translation matrix associated to $\Gamma$ is $B = U_\Gamma(1)$.

Proposition 2.9. With the notations as in Definition 2.8,

$$(2.11) \quad U_\Gamma(t)e_\lambda = e_\lambda(t)e_\lambda, \quad Be_\lambda = e^{2\pi i \lambda}e_\lambda, \quad (\lambda \in \Gamma)$$

Thus the vectors $e_\lambda$ in $l^2(A)$ are the eigenvectors of $B$ corresponding to the eigenvalues $e^{2\pi i \lambda}$ of multiplicity one.
The matrix entries of \( U_\Gamma(t) \) are
\[
(2.12) \quad U_\Gamma(t)_{a a'} = \frac{1}{N} \sum_{\lambda \in \Gamma} e^{2\pi i (a - a' + t) \lambda}, \quad B_{a a'} = \frac{1}{N} \sum_{\lambda \in \Gamma} e^{2\pi i (a - a' + 1) \lambda}, \quad (a, a' \in A, t \in \mathbb{R}).
\]

Proof. We have \( F e_\lambda = \sqrt{N} \delta_\lambda \), for all \( \lambda \in \Gamma \). The rest follows from an easy computation. \( \square \)

**Theorem 2.10.** Let \( A \) be a spectral subset of \( \mathbb{Z} \) with spectrum \( \Gamma \) and let \( U_\Gamma \) be the unitary group associated to \( \Gamma \) as in Definition 2.8. Then \( U_\Gamma \) is a group of local translations on \( \Gamma \), i.e., equation (2.7) is satisfied. Also \( B := U_\Gamma(1) \) is a local translation matrix.

Proof. We have \( F \delta_a = (e^{-2\pi i \lambda a})_{\lambda \in \Gamma} \). Then \( D\Gamma(a - a') F \delta_a = (e^{-2\pi i \lambda a'})_{\lambda \in \Gamma} = F \delta_{a'} \). Hence \( U_\Gamma(a - a') \delta_a = \delta_{a'} \). \( \square \)

The converse holds also in the case of subsets of \( \mathbb{Z} \), i.e., the existence of a group of local translations, or of a local translation matrix guarantees that \( A \) is spectral.

**Theorem 2.11.** Let \( A \) be a finite subset of \( \mathbb{Z} \). The following statements are equivalent:

(i) \( A \) is spectral.

(ii) There exists a unitary group of local translations \( U(t) \), \( t \in \mathbb{R} \), on \( A \).

(iii) There exists a local translation matrix \( B \) on \( A \).

The correspondence from (i) to (ii) is given by \( U = U_\Gamma \) where \( \Gamma \) is a spectrum for \( A \). The correspondence from (ii) to (iii) is given by \( B = U(1) \). The correspondence from (iii) to (i) is given by: if \( \{ e^{2\pi i \lambda} : \lambda \in \Gamma \} \) is the spectrum of \( B \) then \( \Gamma \) is a spectrum for \( A \).

Proof. The implications (i)\( \Rightarrow \) (ii)\( \Rightarrow \) (iii) were proved above. We focus on (iii)\( \Rightarrow \) (i). Let \( \{ e^{2\pi i \lambda} : \lambda \in \Gamma \} \) be the spectrum of the unitary matrix \( B \), the eigenvalues repeated according to multiplicity and let \( \{ v_\lambda : \lambda \in \Gamma \} \) be an orthonormal basis of corresponding eigenvectors. Let \( P_\lambda \) be the orthogonal projection onto \( v_\lambda \). Then
\[
B^m = \sum_{\lambda \in \Gamma} e^{2\pi i m \lambda} P_\lambda, \quad (m \in \mathbb{Z}).
\]

We have, from (2.8),
\[
\delta_{a'} = \sum_\lambda e^{2\pi i \lambda (a - a')} P_\lambda \delta_a
\]
so \( P_\lambda \delta_{a'} = e^{2\pi i \lambda (a - a')} P_\lambda \delta_a \) for all \( \lambda \in \Gamma \) which implies that \( e^{2\pi i a \lambda} P_\lambda \delta_a \) does not depend on \( a \), so it is equal to \( c(\lambda) v_\lambda \) for some \( c(\lambda) \in \mathbb{C} \). Then \( P_\lambda \delta_a = e^{-2\pi i a \lambda} c(\lambda) v_\lambda \) for all \( \lambda \in \Gamma \) and
\[
\delta_a = \sum_\lambda c(\lambda) e^{-2\pi i a \lambda} v_\lambda,
\]
so
\[
\sum_\lambda c(\lambda) e^{-2\pi i a \lambda} v_\lambda(a') = \delta_{aa'}, \quad (a, a' \in A)
\]
Consider the matrices \( S = (c(\lambda)e^{-2\pi i a \lambda})_{a \in A, \lambda \in \Gamma} \) and \( T = (v_\lambda(a))_{\lambda \in \Gamma, a \in A} \). The previous equation implies that \( ST = I \) and since \( T \) is unitary, we get that \( S \) is also. But then the columns have unit norm so
\[
1 = \sum_{a \in A} |c(\lambda)|^2
\]
and this implies that $|c(\lambda)| = \frac{1}{\sqrt{N}}$. The fact that the rows are orthonormal means that

$$\frac{1}{N} \sum_{\lambda \in \Gamma} e^{2\pi i (a - a') \lambda} = \delta_{aa'}.$$ 

But this means, first, that all the $\lambda$’s are distinct and that $\Gamma$ is a spectrum for $A$. \hfill \Box

**Remark 2.12.** Given the group of local translations $U(t)$, $t \in \mathbb{R}$, the local translation matrix is given by $B = U(1)$. Conversely, given the local translation matrix $B$, this defines $U$ on $\mathbb{Z}$ in a unique way $U(n) = B^n$, $n \in \mathbb{Z}$. However, there are many ways to interpolate this to obtain a local translation group depending on the real parameter $t$. One can pick some choices for $\Gamma$ such that \{e^{2\pi i \lambda} : \lambda \in \Gamma\} is the spectrum of $B$. Then consider the spectral decomposition

$$B = \sum_{\lambda \in \Gamma} e^{2\pi i \lambda} P_{\lambda}.$$ 

Define

$$U(t) = \sum_{\lambda \in \Gamma} e^{2\pi i \lambda t} P_{\lambda}, \quad (t \in \mathbb{R}).$$ 

Note that $U(t)$ depends on the choice of $\Gamma$. Any two such choices $\Gamma$, $\Gamma'$ are congruent modulo $\mathbb{Z}$, and therefore the corresponding groups $U_{\Gamma}(t)$ and $U_{\Gamma'}(t)$ coincide for $t \in \mathbb{Z}$.

**Example 2.13.** The simplest example of a spectral set in $\mathbb{Z}$ is $A = \{0, 1, \ldots, N-1\}$ with spectrum $\Gamma = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$. The local translation matrix associated to $\Gamma$ is the permutation matrix:

$$B = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix}.$$

To see this, it is enough to check that, for $k = 0, \ldots, N-1$,

$$B \begin{pmatrix} e^{\frac{2\pi i}{N} (0)} \\ \vdots \\ e^{\frac{2\pi i}{N} (N-1)} \end{pmatrix} = e^{2\pi i \frac{k}{N}} \begin{pmatrix} e^{\frac{2\pi i}{N} (0)} \\ \vdots \\ e^{\frac{2\pi i}{N} (N-1)} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i k}{N} (1)} \\ \vdots \\ e^{\frac{2\pi i k}{N} (N-1)} \end{pmatrix}.$$ 

In the next proposition we link the two concepts for $\mathbb{Z}$ and for $\mathbb{R}$: if $A$ is a spectral set in $\mathbb{Z}$, with spectrum $\Gamma$, then $\Omega = A + [0, 1]$ is a spectral set in $\mathbb{R}$ with spectrum $\Lambda = \Gamma + \mathbb{Z}$. The local group of local translations for $\Omega$ and $\Lambda$ can be expressed in terms of the local translation matrix associated to $A$ and $\Gamma$.

**Proposition 2.14.** Let $A = \{a_0, \ldots, a_{N-1}\}$ be a spectral set in $\mathbb{Z}$, with spectrum $\Gamma = \{\lambda_0, \ldots, \lambda_{N-1}\}$. Then the set $\Omega = A + [0, 1]$ is spectral in $\mathbb{R}$ with spectrum $\Lambda = \Gamma + [0, 1]$. Define the matrix of the Fourier transform from $l^2(A)$ to $l^2(\Gamma)$:

\[ F = F_{A, \Gamma} = \frac{1}{\sqrt{N}} \left( e^{-2\pi i \lambda_j a_k} \right)_{j,k=0}^{N-1}, \] 

(2.13)
and let $D_G$ be the $N \times N$ diagonal matrix with entries $e^{2\pi i \lambda_j}$, $j = 0, \ldots, N - 1$.

\begin{equation}
B = \mathcal{F}^* D_G \mathcal{F}
\end{equation}

The group of local translations $(U\Lambda(t))_{t \in \mathbb{R}}$ associated to the spectrum $\Lambda$ of $\Omega$ is given by

\begin{equation}
(U\Lambda(t)f)(x + a_0) = B^{[x+t]}(f\{x+t\} + a_0), \quad (f \in L^2(\Omega), x \in [0, 1], t \in \mathbb{R}).
\end{equation}

where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ represent the integer and the fractional parts respectively.

Proof. The fact that $\Lambda$ is a spectrum for $\Omega$ can be found, for example, in [KM06b]. The formula for $U\Lambda$ appears in a slightly different form in [DIT21], but we can check it here directly in a different way: it is enough to prove that

\begin{equation}
U\Lambda(t)e_\lambda = e_\lambda(t)e_\lambda \text{ for all } \lambda \in \Lambda
\end{equation}

Thus, we have to plug in $f = e_{\lambda_i+n}$ in the right hand side of (2.15), with $\lambda_i \in \Gamma$, $n \in \mathbb{Z}$. For the computation, we will use the following relation:

\begin{equation}
\mathcal{F} \frac{1}{\sqrt{N}} \begin{pmatrix}
e^{2\pi i \lambda_i a_0} \\
\vdots \\
e^{2\pi i \lambda_i a_{N-1}}
\end{pmatrix} = \delta_i.
\end{equation}

Indeed, we have, for $j = 0, \ldots, N - 1$,

$$
\frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i \lambda_j a_k} e^{2\pi i \lambda_i a_k} = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i (\lambda_i - \lambda_j) a_k} = \delta_{ij},
$$

because $\Gamma$ is a spectrum for $A$.

Then, for $m \in \mathbb{Z}$,

\begin{equation}
B^m \begin{pmatrix} e^{2\pi i \lambda_i a_0} \\
\vdots \\
e^{2\pi i \lambda_i a_{N-1}}
\end{pmatrix} = e^{2\pi i \lambda_i m} \begin{pmatrix} e^{2\pi i \lambda_i a_0} \\
\vdots \\
e^{2\pi i \lambda_i a_{N-1}}
\end{pmatrix}
\end{equation}

We have, for $x \in [0, 1]$ and $t \in \mathbb{R}$:

\begin{align*}
B^{[x+t]} \begin{pmatrix} e^{2\pi i(x+t)(\lambda_i + n)} \\
\vdots \\
e^{2\pi i(x+t)(\lambda_i + n - 1)}
\end{pmatrix} &= e^{2\pi i(x+t)(\lambda_i + n)} B^{[x+t]} \begin{pmatrix} e^{2\pi i a_0 \lambda_i} \\
\vdots \\
e^{2\pi i a_{N-1} \lambda_i}
\end{pmatrix} \\
&= e^{2\pi i(x+t)(\lambda_i + n)} e^{2\pi i [x+t] \lambda_i} \begin{pmatrix} e^{2\pi i a_0 \lambda_i} \\
\vdots \\
e^{2\pi i a_{N-1} \lambda_i}
\end{pmatrix} \\
&= e^{2\pi i(\lambda_i + n)(x+t)} \begin{pmatrix} e^{2\pi i a_0 \lambda_i} \\
\vdots \\
e^{2\pi i a_{N-1} \lambda_i}
\end{pmatrix} = e^{2\pi i(\lambda_i + n)(x+t)} \begin{pmatrix} e^{2\pi i(x+a_0)(\lambda_i + n)} \\
\vdots \\
e^{2\pi i(x+a_{N-1})(\lambda_i + n)}
\end{pmatrix}.
\end{align*}
This proves \(2.16\). \(\square\)

In the following we present some connections between the local matrix \(B\) and possible tilings for the set \(A\). We define a set \(\Theta_B\) as the set of powers of the matrix \(B\) which have a canonical vector as a column, with 1 not on the diagonal.

**Definition 2.15.** Let \(A\) be a spectral subset of \(\mathbb{Z}\) with spectrum \(\Gamma\). Let \(B\) be the associated local translation matrix. Define

\[
(2.19) \quad \Theta_B := \{ m \in \mathbb{Z} : B^m \text{ has a column } a \text{ equal to the canonical vector } \delta_{a'} \text{ for some } a \neq a' \}
\]

\[
= \{ m \in \mathbb{Z} : B^m \delta_a = \delta_{a'} \text{ for some } a \neq a' \text{ in } A \}.
\]

**Proposition 2.16.** Let \(A\) be a spectral subset of \(\mathbb{Z}\) with spectrum \(\Gamma\), \(|A| = N\). Assume \(0 \in \Gamma\). Assume in addition that the smallest lattice that contains \(\Gamma\) is \(\mathbb{Z}d\) for some mutually prime integers \(r,d \geq 1\). For a subset \(\mathcal{T}\) of \(\mathbb{Z}\) the following statements are equivalent:

(i) \(\mathcal{T} \oplus A = \mathbb{Z}d\), in the sense that \(\mathcal{T} \oplus A\) is a complete set of representatives modulo \(d\) and every element \(x\) in \(\mathcal{T} + A\) can be represented in a unique way as \(x = t + a\) with \(t \in \mathcal{T}\) and \(a \in A\). In this case \(A\) tiles \(\mathbb{Z}\) by \(\mathcal{T} \oplus d\mathbb{Z}\).

(ii) \((\mathcal{T} - \mathcal{T}) \cap \Theta_B = \{0\}\) and \(|\mathcal{T}||A| = d\).

**Proof.** First, we present \(\Theta_B\) in a more explicit form. By Proposition 2.9 we have

\[
(B^m)_{aa'} = \frac{1}{N} \sum_{\lambda \in \Gamma} e^{2\pi i (a-a'+m)\lambda}.
\]

Since we want \((B^m)_{aa'}\) to be 1 for some \(a \neq a'\), we must have equality in the triangle inequality

\[
|(B^m)_{aa'}| \leq \frac{1}{N} \sum_{\lambda \in \Gamma} 1 = 1,
\]

and since \(0 \in \Gamma\) this implies that \(e^{2\pi i (a-a'+m)\lambda} = 1\) so \((a-a'+m)\lambda \in \mathbb{Z}\) for all \(\lambda \in \Gamma\). Since the smallest lattice that contains \(\Gamma\) is \(\mathbb{Z}d\), we obtain that \((a-a'+m)\frac{\lambda}{d} \in \mathbb{Z}\) which means that \(m \equiv a' - a \mod d\). The converse also holds: if \(a' \equiv a + m \mod d\) then \(B^m\) has a 1 on position \(aa'\). Thus

\[
(2.20) \quad \Theta_B = \{ m \in \mathbb{Z} : m \equiv a' - a \mod d \text{ for some } a \neq a' \in A \}
\]

(i)\(\Rightarrow\)(ii). Suppose there exists \(t \neq t'\) in \(\mathcal{T}\) such that \(t - t' \in \Theta_B\). Then there exist \(a \neq a'\) in \(A\) such that \(a' - a \equiv t - t' \mod d\). Then \(a' + t' \equiv a + t \mod d\), a contradiction. Also if \(A \oplus \mathcal{T} = \mathbb{Z}d\), then \(|A||\mathcal{T}| = d\).

(ii)\(\Rightarrow\)(i). It is enough to prove that \((A - A) \cap (\mathcal{T} - \mathcal{T}) = \{0\} \mod d\), because this implies that the map from \(A \times \mathcal{T}\) to \((A + T) \mod d\), \((a,t) \mapsto a + t \mod d\) is injective, and the condition \(|A||\mathcal{T}| = d\) implies that it has to be bijective.

Suppose not. Then there exist \(a \neq a'\) in \(A\) \(t \neq t'\) in \(\mathcal{T}\) such that \(a + t \equiv a' + t' \mod d\). Then \(t - t' \equiv a' - a \mod d\). Therefore \(t - t' \in \Theta_B\) which contradicts the hypothesis. \(\square\)
Corollary 2.17. If the local translation matrix \( B \) is
\[
B = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix},
\]
then \( A \) tiles \( \mathbb{Z} \) by \( N \mathbb{Z} \).

Proof. We have \( \Theta_B = \{1, \ldots, N-1\} + N \mathbb{Z} \) so one can take \( T = \{0\} \) in Proposition 2.16.

Another piece of information that is contained in the local translation matrix is the rationality of the spectrum:

Proposition 2.18. Let \( A \) be a spectral set in \( \mathbb{Z} \) with spectrum \( \Gamma \) and local translation matrix \( B \). Let \( d \in \mathbb{Z}, \ d \geq 1 \). Then \( \Gamma \subset \frac{1}{d} \mathbb{Z} \) if and only if \( B^d = I \). The spectrum \( \Gamma \) is rational if and only if the group of local translations \( U_\Gamma \) has an integer period, i.e., there exists \( p \in \mathbb{Z}, \ p \geq 1 \) such that \( U_\Gamma(t + p) = U_\Gamma(t), \ t \in \mathbb{R} \).

Proof. If \( \Gamma \subset \frac{1}{d} \mathbb{Z} \), using equation (2.10), with \( t = d \), we have that \( D_\Gamma(d) = I \) so \( B^d = I \). Conversely, if \( B^d = I \) then \( D_\Gamma(d) = I \) and therefore \( d\lambda \in \mathbb{Z} \) for all \( \lambda \in \Gamma \). The second statement follows from the first.

3. Examples

In this section we study the local translation groups associated to spectral sets of low cardinality \( N = 2, 3, 4, 5 \). Such sets were described in [DH12]. We recall here the results:

Definition 3.1. The standard \( N \times N \) Hadamard matrix is
\[
\frac{1}{\sqrt{N}} \left( e^{2\pi i jk/N} \right)_{j,k=0}^{N-1}.
\]
We say that a \( N \times N \) matrix is equivalent to the standard Hadamard matrix if it can be obtained from it by permutations of rows and columns.

Let \( A \) and \( L \) be two subsets of \( \mathbb{Z} \) and \( R \in \mathbb{Z}, \ R \geq 1 \). We say that \((A, L)\) is a Hadamard pair with scaling factor \( R \) if \( \frac{1}{R} L \) is a spectrum for \( A \).

Theorem 3.2. Let \( A \subset \mathbb{Z} \) have \( N \) elements and spectrum \( \Gamma \). Assume \( 0 \) is in \( A \) and \( \Gamma \). Suppose the Hadamard matrix associated to \((A, \Gamma)\) is equivalent to the standard \( N \) by \( N \) Hadamard matrix. Then \( A \) has the form \( A = dA_0 \) where \( d \) is an integer and \( A_0 \) is a complete set of residues modulo \( N \) with \( \gcd(A_0) = 1 \). In this case any such spectrum \( \Gamma \) has the form \( \Gamma = \frac{1}{R} fL_0 \) where \( f \) and \( R \) are integers, \( L_0 \) is a complete set of residues modulo \( N \) with greatest common divisor one, and \( R = NS \) where \( S \) divides \( df \) and \( \frac{df}{S} \) is mutually prime with \( N \). The converse also holds.

Since for \( N = 2, 3, 5 \) our Hadamard matrices are equivalent to the standard one (see [Haa97, TZ06]) the next corollary follows:

Corollary 3.3. A set \( A \subset \mathbb{Z} \) with \( |A| = N = 2, 3, \) or \( 5 \), where \( 0 \in A \) is spectral if and only if \( A = N^k A_0 \) where \( k \) is a positive integer and \( A_0 \) is a complete set of residues modulo \( N \).
For cardinality $N = 4$ the situation is more complex:

**Theorem 3.4.** Let $A$ be spectral with spectrum $\Gamma$ and size $N = 4$. Assume $0$ is in both sets. Then there exists a set of integers $L$, containing $0$, and an integer scaling factor $R$ so that $\Gamma = \frac{1}{R} L$.

$(A, L)$ is a Hadamard pair (each containing $0$) of integers of size $N = 4$, with scaling factor $R$, if and only if $R = 2^{C + M + a + 1} d$, $A = 2^C \{0, 2^a c_1, c_2, c_2 + 2^a c_3\}$, and $L = 2^M \{0, n_1, n_1 + 2^a n_2, 2^a n_3\}$, where $c_i$ and $n_i$ are all odd, $a$ is a positive integer, $C$ and $M$ are non-negative integers, and $d$ divides $c_1 n_1$, $c_3 n_1$, $n_2 c_2$, and $n_3 c_3$, where $c$ is the greatest common divisor of the $c_k$'s and similarly for $n$.

The next proposition helps us simplify our study:

**Proposition 3.5.** Let $A$ be a spectral set in $\mathbb{Z}$ with spectrum $\Gamma$, local translation group $U_\Gamma$ and local translation matrix $B$. Let $d \in \mathbb{Z}, d \geq 1$. Then $dA$ is spectral with spectrum $\frac{1}{d} \Gamma$. The local translation group $U_{\frac{1}{d} \Gamma}$ and the local translation matrix $B_{\frac{1}{d} \Gamma}$ are related to the corresponding ones for $A$ and $\Gamma$ by

$$U_{\frac{1}{d} \Gamma}(t) = U_\Gamma\left(\frac{t}{d}\right), \quad (t \in \mathbb{R}), \quad B_{\frac{1}{d} \Gamma} = B_\Gamma.$$

**Proof.** Everything follows from (2.10) by a simple calculation. □

**N=2.** We can take $A = 2^c \{0, a_0\}$, with $c \in \mathbb{Z}$, $c \geq 0$ and $a_0$ odd, and $\Gamma = \{0, \gamma_1 = \frac{g}{2^{a_0}}\}$ with $g$ odd. The matrix of the Fourier transform is

$$\mathcal{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By equation (2.10), we can compute the local translation matrix $B$ and the local translation group $U_\Gamma$:

$$B = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i \gamma_1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (3.3)$$

Here $\gamma_1$ depends on the non-zero element of $A$, called $a_1$. Let $a_1 = 2^c a_0$, where $a_0$ is odd. Then $\gamma_1 = \frac{g}{2^{a_0}}$, where $g$ is odd. Multiplying, we obtain

$$B = \frac{1}{2} \begin{pmatrix} 1 + e^{\pi i \gamma_1} & 1 - e^{\pi i \gamma_1} \\ 1 - e^{\pi i \gamma_1} & 1 + e^{\pi i \gamma_1} \end{pmatrix}. \quad (3.4)$$

We also have

$$U_\Gamma(t) = \frac{1}{2} \begin{pmatrix} 1 + e^{t \pi i \gamma_1} & 1 - e^{t \pi i \gamma_1} \\ 1 - e^{t \pi i \gamma_1} & 1 + e^{t \pi i \gamma_1} \end{pmatrix}. \quad (3.5)$$

Note that when $c = 0$,

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

In this case $\Theta_B = \mathbb{Z}$. 

We compute the group of local translations:

\[
U_\Gamma(t) = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 \\
\omega & \bar{\omega} & \omega \\
\bar{\omega} & \omega & \bar{\omega}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
e^{2\pi i \gamma_1 t} & 0 & e^{2\pi i \gamma_2 t} \\
0 & e^{2\pi i \gamma_2 t} & 0
\end{pmatrix} \begin{pmatrix}
1 & 1 & 1 \\
\bar{\omega} & \omega & \bar{\omega}
\end{pmatrix}.
\]

Multiplying, we obtain

\[
U_\Gamma(t) = \frac{1}{3} \begin{pmatrix}
1 + e^{2\pi i \gamma_1 t} + e^{2\pi i \gamma_2 t} \\
1 + \omega e^{2\pi i \gamma_1 t} + \bar{\omega} e^{2\pi i \gamma_2 t} \\
1 + \bar{\omega} e^{2\pi i \gamma_1 t} + \omega e^{2\pi i \gamma_2 t}
\end{pmatrix} \begin{pmatrix}
1 + \omega e^{2\pi i \gamma_1 t} + \bar{\omega} e^{2\pi i \gamma_2 t} \\
1 + e^{2\pi i \gamma_1 t} + e^{2\pi i \gamma_2 t} \\
1 + e^{2\pi i \gamma_1 t} + e^{2\pi i \gamma_2 t}
\end{pmatrix} \begin{pmatrix}
1 + e^{2\pi i \gamma_1 t} + e^{2\pi i \gamma_2 t} \\
1 + \omega e^{2\pi i \gamma_1 t} + \bar{\omega} e^{2\pi i \gamma_2 t} \\
1 + \bar{\omega} e^{2\pi i \gamma_1 t} + \omega e^{2\pi i \gamma_2 t}
\end{pmatrix}.
\]

Note that, when \( c = 0 \),

\[
B = U_\Gamma(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

In this case \( \Theta_B = \mathbb{Z} \).

**N=4.** We take a simple case to obtain some nice symmetry, so we will ignore, after some rescaling, the common factor. So take \( A = \{0, 2^a c_1, c_2, c_3 + 2^a c_3\} \), \( \Gamma = \frac{1}{2\pi i} \{0, 2^a n_1, n_2, n_2 + 2^a n_3\} \) as in Theorem 3.4. The matrix of the Fourier transform is

\[
(3.7) \quad \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -\rho & \rho \\
1 & -1 & -\rho & \rho
\end{pmatrix},
\]

where \( \rho = \exp \left( -\frac{\pi i 2\gamma_2}{2\gamma_1} \right) \). We compute integers powers of the spectral matrix \( B \):

\[
(3.8) \quad B^k = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -\rho & \rho \\
1 & -1 & -\rho & \rho
\end{pmatrix}^* \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (-1)^k & 0 & 0 \\
0 & 0 & z^k & 0 \\
0 & 0 & 0 & (-z)^k
\end{pmatrix} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -\rho & \rho \\
1 & -1 & -\rho & \rho
\end{pmatrix},
\]

where \( z = \exp \left( \frac{\pi i n_2}{2\gamma} \right) \).

We obtain for odd \( k \),

\[
(3.9) \quad B^k = \frac{1}{2} \begin{pmatrix}
0 & 0 & 1 + z^k \rho & 1 - z^k \rho \\
0 & 0 & 1 - z^k \rho & 1 + z^k \rho \\
1 + z^k \rho & 1 - z^k \rho & 0 & 0 \\
1 - z^k \rho & 1 + z^k \rho & 0 & 0
\end{pmatrix}.
\]
We obtain for even $k$,

\begin{equation}
B^k = \frac{1}{2} \begin{pmatrix}
1 + z^k & 1 - z^k & 0 & 0 \\
1 - z^k & 1 + z^k & 0 & 0 \\
0 & 0 & 1 + z^k & 1 - z^k \\
0 & 0 & 1 - z^k & 1 + z^k
\end{pmatrix}.
\end{equation}

We compute $\Theta_B$. We have $k \in \Theta_B$ if and only if one of the following situations occurs: $z^k = -1$, $z^k \rho = \pm 1$, $z^k \bar{\rho} = \pm 1$. This means that $\frac{k n_2}{2} = 2m + 1$ or $\frac{k n_2}{2} \pm \frac{c n_2}{2} = m$ for some $m \in \mathbb{Z}$. Since $n_2$ is odd, this implies that

$$\Theta_B = \{2^a (2m + 1) : m \in \mathbb{Z}\} \cup \{2^a m \pm c_2 : m \in \mathbb{Z}\}.$$

Then, one can easily see that $\mathcal{T} := \{0, 2, 4, \ldots, 2^a - 2\}$ satisfies the conditions in Proposition 2.16, and therefore $A \oplus \mathcal{T} = \mathbb{Z}_{2^{a+1}}$.

**Example 3.6.** Let $A = \{0, 1, 4, 5\}$ and $\Gamma = \frac{1}{8} \{0, 1, 4, 5\}$. We illustrate how the group of local translations acts on an indicator function.

![Figure 1](image-url)
The indicator function $f$ is the one in the first picture, for $t = 0$. We use negative values for $t$ to move the function to the right. We show here the absolute value of $U(t)f$. Note that for $t \approx -1, -4$ and $-5$, since the the interval $[0, 1]$ is moved into the intervals $[1, 2], [4, 5] [5, 6]$, which are contained in $A + [0, 1]$, as predicted by the theory, e.g., Proposition 2.7, the group $U(t)$ really acts a simple translation.

For $t \approx 2$, the interval $[0, 1] + 2$ is no longer contained in $A + [0, 1]$. The local translation $U(-2)$ splits the indicator function into 2 pieces, supported on $[0, 1]$ and $[4, 5] [5, 6]$. Similarly for $t \approx -3, -6, -7$.

Since $\Gamma$ is contained in $\frac{1}{8}\mathbb{Z}$, the group of local translations has period 8. We see this in the last picture $U(-8)f = f$.

**N=5.** For simplicity, by rescaling we can ignore the common factors in $A$ and $\Gamma$ so we take $A = \{0, a_1, a_2, a_3, a_4\}$ with $a_j \equiv j \mod 5$ and $\Gamma = \frac{1}{5}\{0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with $\gamma_j \equiv j \mod 5$. Then the matrix of the Fourier transform is

$$
F = \frac{1}{\sqrt{5}} \left( e^{2\pi i jk/5} \right)_{j,k=0}^4.
$$

The local translation matrix is

$$
B = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

In this case $\Theta_B = \mathbb{Z}$.

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