On the existence of low regularity solutions to semilinear generalized Tricomi equations in mixed type domains

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Abstract

In [19-20], we have established the existence and singularity structures of low regularity solutions to the semilinear generalized Tricomi equations in the degenerate hyperbolic regions and to the higher order degenerate hyperbolic equations, respectively. In the present paper, we shall be concerned with the low regularity solution problem for the semilinear mixed type equation $\partial_t^2 u - t^{2l-1} \Delta u = f(t, x, u)$ with an initial data $u(0, x) = \varphi(x) \in H^s(\mathbb{R}^n) (0 \leq s < \frac{n}{2})$, where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, $l \in \mathbb{N}$, $f(t, x, u)$ is $C^1$ smooth in its arguments and has compact support with respect to the variable $x$. Under the assumption of the subcritical growth of $f(t, x, u)$ on $u$, we will show the existence and regularity of the considered solution in the mixed type domain $[-T_0, T_0] \times \mathbb{R}^n$ for some fixed constant $T_0 > 0$.

Keywords: Generalized Tricomi equation, mixed type equation, confluent hypergeometric function, modified Bessel functions, Calderón-Zygmund decomposition, multiplier

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67, 76N15

1 Introduction

In [19-20], we have established the existence and singularity structures of low regularity solutions to the semilinear generalized Tricomi equations in the hyperbolic regions and to the higher order degenerate hyperbolic equations, respectively. In the present paper, we have a further

\textsuperscript{*}Ruan Zhuoping and Yin Huicheng were supported by the NSFC (No. 10931007, No. 11025105), and by the Priority Academic Program Development of Jiangsu Higher Education Institutions. This research was started when Ruan Zhuoping and Yin Huicheng were visiting the Mathematical Institute of the University of Göttingen in February-March of 2013.
study on the existence and regularities of solutions to the following $n$-dimensional semilinear generalized Tricomi equation in the mixed type domain $\mathbb{R} \times \mathbb{R}^n$

$$
\begin{cases}
\partial_t^2 u - t^{2l-1} \Delta u = f(t, x, u), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0, x) = \varphi(x), & x \in \mathbb{R}^n,
\end{cases}
\quad (1.1)
$$

where $l \in \mathbb{N}$, $x = (x_1, ..., x_n)$, $n \geq 2$, $\Delta = \sum_{i=1}^{n} \partial_i^2$, $\varphi(x) \in H^s(\mathbb{R}^n)$ ($0 \leq s < \frac{n}{2}$), $f(t, x, u)$ is $C^1$ smooth in its arguments and has a compact support $E$ on the variable $x$. Moreover, for any $T > 0$, there exists $C_T > 0$ such that for $(t, x, u) \in [-T, T] \times E \times \mathbb{R}$,

$$
|f(t, x, u)| \leq C_T(1 + |u|^\mu) \quad \text{and} \quad |\partial_u f(t, x, u)| \leq C_T(1 + |u|)^{\max(\mu-1, 0)},
\quad (1.2)
$$

where $C_T > 0$ is a constant depending only on $T$, and the exponent $\mu \geq 0$ fulfills

$$
\mu < p_0 \equiv \frac{2n}{n - 2s}.
\quad (1.3)
$$

Here we point out that the number $p_0$ defined in (1.3) comes from the Sobolev imbedding formula $H^s(\mathbb{R}^n) \subset L^{p_0}(\mathbb{R}^n)$. Thus (1.2) and (1.3) mean that the nonlinearity $f$ in (1.1) admits a “subcritical” growth on the variable $u$. In addition, we shall illustrate that the scope of the exponent $\mu$ for solving the problem (1.1) is closely related to the number $Q_0 \equiv 1 + \frac{n(2l+1)}{2}$.

In the terminology of [16] and the references therein, $Q_0$ is called the homogeneous dimension corresponding to the degenerate elliptic operator $\partial_t^2 u - t^{2l-1} \Delta$ for $t \leq 0$. Our main result in this paper is:

**Theorem 1.1.** Under the conditions (1.2)-(1.3), there exists a constant $T_0 > 0$ such that the problem (1.1) has a unique solution $u(t, x) \in C([-T_0, T_0], L^{p_0}(\mathbb{R}^n))$ when $0 \leq \mu \leq 1$, or when $1 < \mu < p_0$ and $Q_0 \leq \frac{p_0}{\mu - 1}$.

**Remark 1.1.** It seems to be necessary that the restriction of $Q_0 \leq \frac{p_0}{\mu - 1}$ is posed when $1 < \mu < p_0$ in Theorem 1.1, otherwise, the standard iteration scheme for solving the problem (1.1) only works in finite steps or the solution $u \notin C([-T_0, T_0], L^{p_0}(\mathbb{R}^n))$ when $Q_0 > \frac{p_0}{\mu - 1}$.

One can see Remark 4.1 and the related explanations below (4.7) in §4, respectively.

**Remark 1.2.** For $l = 1$, (1.1) is the well-known semilinear Tricomi equation $\partial_t^2 u - t \Delta u = f(t, x, u)$. When an initial data $u(0, x, \varphi(x) \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ is given and the crucial assumption of $\text{supp} f \subset \{ t \geq 0 \}$ is posed (namely, $f \equiv 0$ holds in $t \leq 0$, which means that the related Tricomi equation is linear in the elliptic region $\{ t \leq 0 \}$), M. Beals in [2] shows that the problem (1.1) has a regular solution $u \in C([\infty, T], H^s(\mathbb{R}^n)) \cap C^1([\infty, T], H^{-\frac{1}{2}}(\mathbb{R}^n)) \cap C^2([\infty, T], H^{-\frac{1}{2}} (\mathbb{R}^n))$ for some constant $T > 0$. Here we point out that the key assumption of $\text{supp} f \subset \{ t \geq 0 \}$ in [2] has been removed as well as the local existence of low regularity solutions is established in our present paper.
Remark 1.3. If the initial data \( \varphi(x) \in L^\infty(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \) for \( n = 2, 3 \) and \( s \geq 0 \) is given, then we can remove all the assumptions in (1.2)-(1.3). In fact, in this case, from the proof procedure in §4, we can derive that the solution \( u(t, x) \) of (1.1) satisfies: \( u(t, x), \partial_t u(t, x) \in L^\infty([-T_0, 0] \times \mathbb{R}^n) \) for some fixed \( T_0 > 0 \). Based on this, together with the \( L^\infty_{loc}([0, +\infty) \times \mathbb{R}^n) \) property of the solution \( v(t, x) \) to the linear equation \( \partial_t^2 v - t^{2-1} \Delta v = 0 \) with \( (v(0, x), \partial_t v(0, x)) = (\varphi_0(x), \varphi_1(x)) \), where \( \varphi_0(x), \varphi_1(x) \in L^\infty(\mathbb{R}^n) \), we may show that (1.1) is also solvable in the degenerate hyperbolic region in the space \( L^\infty([0, T_1] \times \mathbb{R}^n) \) for some positive constant \( T_1 > 0 \). Combining these two cases, we know that (1.1) is locally solvable (see more detailed explanations in Remark 5.1 of §5 below).

Remark 1.4. If the nonlinearity \( f(t, x, u) \) in (1.1) increases sublinearly or linearly with respect to the variable \( u \), namely, \( 0 \leq \mu \leq 1 \) holds in (1.2), then by a minor modification on the proof of Theorem 1.1, we can obtain a global solution \( u \in C(\mathbb{R}, L^p_0(\mathbb{R}^n)) \) to (1.1).

Remark 1.5. In order to guarantee the existence and uniqueness of the solution \( u \) to (1.1), the assumption that \( f(t, x, u) \) is compactly supported on the variable \( x \) is required. Otherwise, in case \( f(t, x, u) \sim |u|^{p-1} u \) with \( p \geq 1 \), due to the existence of infinite eigenvalues for the Tricomi operator \( \partial_t^2 - t \Delta \) with \( t \in \mathbb{R} \), (1.1) can admit infinitely many solutions or have no solution (one can see the references [13], [15-16] and so on).

Remark 1.6. With respect to the existence, singularity structures, and singularity propagation theories of classical solutions to the semilinear generalized Tricomi equations in the degenerate hyperbolic regions/mixed type regions or to the higher order degenerate hyperbolic equations, so far there have been many interesting results (one can see [1-5], [7], [19-20], [25-26] and the references therein). For the linear Tricomi equation in the mixed type region, when the closed boundary value is given, the authors in [14] establish the existence and uniqueness of weak solutions. Here our focus in Theorem 1.1 is on the existence of the low regularity solution to the semilinear problem (1.1) in the mixed type region \( \mathbb{R} \times \mathbb{R}^n \).

Remark 1.7. For the linear Tricomi equation \( \partial_x^2 u - x \partial_y^2 u = 0 \) with an initial data \( u(0, y) = u_0(y) \), its solvability in the whole region \( \mathbb{R}^2 \) has become an important practical application in the continuous transonic gas dynamics of isentropic and irrotational flows. Indeed, once the solution \( u \) is found, one can seek out the corresponding de Laval nozzle walls through its streamlines and then the required position of the sonic curve (corresponding to the lone \( \{ x = 0 \} \) in the hodograph plane) in the de Laval nozzle can be determined (more introductions on this physical background can be found in [12] and so on).

For \( n = 1, l = 1 \) and \( f(t, x, u) \equiv 0 \), the equation in (1.1) becomes the classical Tricomi equation which arises in transonic gas dynamics and has been extensively investigated in bounded domain with suitable boundary conditions from various viewpoints (one can see the review paper [18] and the references therein). For \( l = 1 \) and \( n = 2 \), with respect to the equation \( \partial_t^2 u - t \Delta u = f(t, x, u) \) with an initial data \( u(0, x) \in H^s(\mathbb{R}^n) \) \( (s > \frac{n}{2}) \), under the crucial assumption of \( supp f \subset \{ t \geq 0 \} \), M. Beals in [2] established the local existence of the solution \( u(t, x) \in C((-\infty, T], H^s(\mathbb{R}^n)) \cap C^1((-\infty, T], H^{s-\frac{n}{2}}(\mathbb{R}^n)) \cap C^2((-\infty, T], H^{s-\frac{2n}{2}}(\mathbb{R}^n)) \) for some \( T > 0 \), moreover the \( H^s(\mathbb{R}^n) \) conormal regularity of \( u \) with respect to the characteristic
Theorem 1.1 in [2]. For more general nonlinear degenerate hyperbolic equations with discontinuous initial data, the authors in [19-20] obtained the local existence of low regularity solutions. In the present paper, we focus on the low regularity solution problem for the semilinear generalized Tricomi equation with an initial data \( u(0, x) \) in the mixed type region \( \mathbb{R} \times \mathbb{R}^n \).

We now comment on the proof of Theorem 1.1. In order to establish the existence and regularity of the solution to (1.1) with an initial data \( u(0, x) = \varphi(x) \), we first consider the linear equation \( \partial_t^2 v - t^{2l-1} \Delta v = 0 \) with \( v(0, x) = \varphi(x) \) in the domain \( \mathbb{R} \times \mathbb{R}^n \) and obtain \( v(t, x) \in C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-\frac{d+1}{2}}) \). Subsequently, we set \( w = u - v \) and get a second order nonlinear degenerate equation of \( w \) from (1.1) in the domain \( \{ t \leq 0 \} \). By utilizing some delicate harmonic analysis methods (e.g., Calderón-Zygmund decomposition, interpolation, multiplier, fractional integral, and so on) as in [8-9], we can establish some suitably weighted \( W^{2,p}(\mathbb{R}^+ \times \mathbb{R}^n) \) estimates on \( w \) and further obtain the solvability of \( w \) in \( \{ t \leq 0 \} \) by the fixed point principle. From this, one can get another initial data \( \partial_t u(0, x) \), which is necessary to solve (1.1) in the degenerate hyperbolic region \( \{ t \geq 0 \} \). Finally by using some techniques in degenerate hyperbolic equations (see [19-20], [25-26] and the references therein), we can establish the solvability and regularity of the solution \( u \) to (1.1) in the domain \( \{ t \geq 0 \} \).

Then a local solution \( u \) in the mixed type domain \( \mathbb{R} \times \mathbb{R}^n \) could be obtained by patching the two solutions got in the degenerate elliptic domain and the degenerate hyperbolic domain separately.

This paper is organized as follows. In §2, we will give some preliminary results and useful estimates on the solutions to the linear degenerate elliptic equation \( \partial_t^2 u + t^m \Delta u = t^m f(t, x) \) with \( u(0, x) = 0 \) and \( f(t, x) \in L^p(\mathbb{R}^+ \times \mathbb{R}^n) \), where \( m \in \mathbb{N} \) and \( t \geq 0 \). In §3, we establish more general weighted \( W^{2,p} \) estimates of the solutions to the equation \( \partial_t^2 u + t^m \Delta u = t^\nu f(t, x) \) in \( \{ t \geq 0 \} \) for \( 0 < \nu < m \). Here we emphasize that such weighted estimates also admit independent interests in the degenerate elliptic equations. Based on the results in §2 and §3, we can show the local existence and regularity of (1.1) in the degenerate elliptic region \( \{ t \leq 0 \} \) in §4. Moreover, we may obtain \( \partial_t u(0, x) \) from Theorem 4.1 in §4. Together with the initial data \( u(0, x) \), we can solve (1.1) locally in the hyperbolic region \( \{ t \geq 0 \} \) in §5, and subsequently complete the proof of Theorem 1.1 in §6.

## 2 Some preliminaries and \( W^{2,p} \) estimates for the inhomogeneous generalized Tricomi equation

In this section, we mainly study the \( W^{2,p} \) regularity of the solution \( u(t, x) \) to the linear generalized Tricomi equation \( \partial_t^2 u + t^m \Delta u = t^m g(t, x) \) with the boundary value \( u(0, x) = 0 \) for \( t \geq 0 \), \( m \in \mathbb{N} \) and \( g(t, x) \in L^p(\mathbb{R}^+ \times \mathbb{R}^n) \) \( (1 < p < \infty) \). To this end, we require to apply some harmonic analysis tools (e.g., Calderón-Zygmund decomposition, generalized Hörmander’s multiplier theorem and so on) and some properties of modified Bessel functions. The so-called modified Bessel function \( K_\nu(t) = \int_0^\infty e^{-t \cosh z} \cosh(\nu z) dz \) \((\nu \in \mathbb{R})\) is a solution to the equation \( \left( t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - (t^2 + \nu^2) \right) K_\nu(t) = 0 \) for \( t > 0 \), moreover, there holds that \( \lim_{t \to +\infty} K_\nu(t) = 0 \)
and $\lim_{t \to 0^+} t^{\frac{1}{2}} K_\nu(t) = C_\nu > 0$ (one can see more properties of $K_\nu(t)$ in [6], [23] and so on). As in [8-9], set $\lambda(t) = C \frac{1}{m+2} t^{\frac{1}{2}} K_{\frac{1}{m+2}} (\frac{2}{m+2} t^{\frac{m+2}{m+4}})$ for $t > 0$ and $m \in \mathbb{N}$. Then a direct verification yields

**Lemma 2.1.** For $t \geq 0$,  
(i) $\lambda(t)$ is a solution to the equation $u''(t) - t^m u(t) = 0$ with $u(0) = 1$ and $u(+\infty) = 0$.  
(ii) The equation $u''(t) - t^m u(t) = g(t)$ with $u(0) = 1$ and $u(+\infty) = 0$ has a solution  
$$u(t) = -\lambda(t) \int_0^\infty \left( \int_0^{\min(t,\sigma)} \frac{1}{\lambda^2(y)} dy \right) \lambda(\sigma) g(\sigma) d\sigma.$$  

We now cite the Lemma 2.1 of [8], which illustrates some basic properties of $\lambda(t)$.

**Lemma 2.2.**  
(i) $\lambda(t)$ and $-\lambda'(t)$ are decreasing.  
(ii) $\lambda(t) + |\lambda'(t)| \leq C_M (1 + t)^{-\lambda}$ holds for any $M \in \mathbb{N}$, where $C_M$ is a positive constant depending on $M$.  
(iii) $\frac{\lambda(a)}{\lambda(b)} \leq \left( \frac{a}{b} \right)^{\frac{2}{m+2}} \exp \left( \frac{2}{m+2} b^{\frac{m+2}{m+4}} - \frac{2}{m+2} a^{\frac{m+2}{m+4}} \right)$ when $a \geq b > 0$.  
(iv) There exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{|\lambda'(t)|}{\lambda(t)t^{m/2}}$ for all $t > 0$ and $\frac{|\lambda'(t)|}{\lambda(t)t^{m/2}} \leq C$ for $t \geq 1$ hold.

Next we prove the global existence and regularity of the solution to the linear generalized Tricomi equation with an initial data in the whole mixed-type domain $\mathbb{R} \times \mathbb{R}^n$.

**Lemma 2.3.** Consider the problem  
$$\begin{cases} 
\partial_t^2 v - t^{2l-1} \Delta v = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
v(0, x) = \psi(x), & x \in \mathbb{R}^n,
\end{cases}  \quad (2.5)$$

where $l \in \mathbb{N}$, $\psi(x) \in H^\gamma(\mathbb{R}^n)$ ($\gamma \in \mathbb{R}$), then (2.5) has a solution $v(t, x) \in C(\mathbb{R}, H^\gamma) \cap C^1(\mathbb{R}, H^{\gamma - \frac{2l-1}{2l+3}})$.

**Proof.** At first we study (2.5) in the elliptic region $\{ t \leq 0 \}$

$$\begin{cases} 
\partial_t^2 w - t^{2l-1} \Delta w = 0, & (t, x) \in (-\infty, 0] \times \mathbb{R}^n, \\
w(0, x) = \psi(x).
\end{cases}  \quad (2.6)$$

Taking Fourier transform with respect to the variable $x$ in (2.6) yields

$$\begin{cases} 
\partial_t^2 \hat{w}(t, \xi) + t^{2l-1} |\xi|^2 \hat{w}(t, \xi) = 0, & (t, \xi) \in (-\infty, 0] \times \mathbb{R}^n, \\
\hat{w}(0, \xi) = \hat{\psi}(\xi).
\end{cases}  \quad (2.7)$$
By Lemma 2.1.(i), the solution \( \hat{w}(t, \xi) \) to (2.7) can be expressed as
\[
\hat{w}(t, \xi) = \lambda(-ts) \hat{\psi}(\xi) \quad \text{with} \quad s = |\xi|^{\frac{2}{m+1}}.
\] (2.8)

Then it follows from (2.8) and Lemma 2.2.(ii) that
\[
\|w(t, \cdot)\|_{H^\gamma(\mathbb{R}^n)} = \|\lambda(-ts) < \xi >^\gamma \hat{\psi}(\xi)\|_{L^2(\mathbb{R}^n)}
\leq C \| < \xi >^\gamma \hat{\psi}(\xi)\|_{L^2(\mathbb{R}^n)} = C \|\hat{\psi}\|_{H^\gamma(\mathbb{R}^n)}
\] (2.9)
and
\[
\|\partial_t w(t, \cdot)\|_{H^{\gamma-rac{2}{m+1}}(\mathbb{R}^n)} = \|\lambda'(-ts)s < \xi >^\gamma \hat{\psi}(\xi)\|_{L^2(\mathbb{R}^n)}
\leq C \|s < \xi >^\gamma \hat{\psi}(\xi)\|_{L^2(\mathbb{R}^n)}
\leq C \|\hat{\psi}\|_{H^\gamma(\mathbb{R}^n)}.
\] (2.10)

Thus, we have from (2.9) and (2.10)
\[
w(t, x) \in C((\infty, 0], H^\gamma(\mathbb{R}^n)) \cap C^1((\infty, 0], H^{\gamma-rac{2}{m+1}}(\mathbb{R}^n)).
\] (2.11)

Next we consider the corresponding degenerate hyperbolic part of (2.5) in the region \( \{t \geq 0\} \)
\[
\begin{cases}
\partial_t^2 u - t^{2l-1}\Delta u = 0, & (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
u(0, x) = \psi(x), & \partial_t u(0, x) = \partial_t v(0, x),
\end{cases}
\] (2.12)
where \( \partial_t v(0, x) \in H^{\gamma-rac{2}{m+1}}(\mathbb{R}^n) \) comes from (2.11).

Upon applying Proposition 3.3 in [19], we arrive at
\[
u(t, x) \in C([0, +\infty), H^\gamma(\mathbb{R}^n)) \cap C^1([0, \infty), H^{\gamma-rac{2l+3}{2l+2}}(\mathbb{R}^n)).
\] (2.13)

Combining (2.11) with (2.13) yields that problem (2.5) has a solution \( v(t, x) \) satisfying
\[
v(t, x) \in C(\mathbb{R}, H^\gamma(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{\gamma-rac{2l+3}{2l+2}}(\mathbb{R}^n)),
\] therefore, we complete the proof of Lemma 2.3. \( \square \)

To get the solvability of the nonlinear problem (1.1), as the first step we intend to solve (1.1) in the degenerate elliptic region \( (-\infty, 0] \times \mathbb{R}^n \). To do so, we require to derive the weighted \( W^{2,p} \) estimate of the solution to the problem \( \partial_t^2 v + t^{m}\Delta v = t^\nu g(t, x) \) \( (0 \leq \nu \leq m) \) with \( v(0, x) = 0 \) in \( \{t \geq 0\} \) so that (1.1) can be solved by applying the Hardy’s inequality and the fixed point theorem (one can see the details in §4 below), where \( g(t, x) \in L^p(\mathbb{R}^{n+1}) \). In this section, we only treat the case of \( \nu = m \), where the \( W^{2,p} \) (not weighted \( W^{2,p} \)) estimate can be derived. Based on this, by the interpolation method we can obtain the weighted \( W^{2,p} \) estimates for the general \( \nu \) (see Theorem 3.1 and its proof in §3 below).
From the equation $\partial_t^2 w + t^m \Delta w = t^m g(t, x)$ with $w(0, x) = 0$, we have

$$
\begin{align*}
\begin{cases}
\partial_t^2 \hat{w}(t, \xi) - t^m |\xi|^2 \hat{w}(t, \xi) = t^m \hat{g}(t, \xi), & (t, \xi) \in [0, +\infty) \times \mathbb{R}^n, \\
\hat{w}(0, \xi) = 0.
\end{cases}
\end{align*}
$$

(2.14)

This, together with Lemma 2.1(ii), yields

$$
\hat{w}(t, \xi) = \int_0^{+\infty} \hat{T}(t, \sigma, \xi) \sigma^m \hat{g}(\sigma, \xi) d\sigma,
$$

(2.15)

where $\hat{T}(t, \sigma, \xi) = \int_0^{\min(t, \sigma)} \frac{\lambda(ts)\lambda(\sigma s)}{\lambda^2(ys)} dy$ and $s = |\xi|^{2/m+2}$. By (2.15), we have

$$
\Delta w = F_{\xi}^{-1}(|\xi|^2 \hat{u}(t, \xi)) = F_{\xi}^{-1}\left(\int_0^{+\infty} \hat{K}(t, \sigma, \xi) \hat{g}(\sigma, \xi) d\sigma\right),
$$

(2.16)

here $\hat{K}(t, \sigma, \xi) = |\xi|^2 \sigma^m \hat{T}(t, \sigma, \xi)$. We start to analyze the property of kernel $\hat{K}(t, \sigma, \xi)$ so that the $L^p(\mathbb{R}^+ \times \mathbb{R}^n)$ estimate of $\Delta w$ can be obtained. To this end, we require to apply a basic result on the $L^p$ boundedness for a class of integral operators, which is established in Theorem 1.1 of [8] (a generalized Hörmander multiplier theorem in Theorem 7.95 in [10]). Suppose that the temperate distribution $K(t, \sigma, x) \in S'(\mathbb{R}^+ \times \mathbb{R}^n)$ ($t \in \mathbb{R}^+$ is taken as a parameter) satisfies

(i) For each fixed $(t, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\hat{K}(t, \sigma, \xi) = F_{\xi}(K(t, \sigma, \cdot))(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$, and $\hat{K}$ is piecewise continuous in $(t, \sigma)$.

(ii) $\sup_{t, \xi} \int_0^{+\infty} |\hat{K}(t, \sigma, \xi)| d\sigma \leq C$.

(2.18)

(iii) $\sup_{\sigma, \xi} \int_0^{+\infty} |\hat{K}(t, \sigma, \xi)| d\sigma \leq C$.

(2.19)

(iv) Denote by $\Delta(a, b) = (a - b, a + b)$ and $C\Delta(a, 2^{q+1}b) = \mathbb{R}^+ \setminus \Delta(a, 2^{q+1}b)$ for some integer $q$ with $q > n$. For $h(\sigma) \in C^\infty(\mathbb{R}^+_0)$ and $r > 0$, there exists a constant $C_q > 0$ depending only on $q$ such that for all $|\alpha| \leq q$

$$
\sup_{\xi \leq |\xi| \leq 2r} r^{|\alpha|} \int_0^{+\infty} \left| \partial_\xi^\alpha \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C_q \int_0^{+\infty} |h(\sigma)| d\sigma,
$$

(2.20)

$$
\sup_{\xi \leq |\xi| \leq 2r} r^{|\alpha|} \int_{C\Delta(a, 2^{q+1}b)} \left| \partial_\xi^\alpha \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C_q D_1(a, b, r) \int_0^{+\infty} |h(\sigma)| d\sigma,
$$

(2.21)

and if $\int_0^{+\infty} h(\sigma) d\sigma = 0$, then

$$
\sup_{\xi \leq |\xi| \leq 2r} r^{|\alpha|} \int_0^{+\infty} \left| \partial_\xi^\alpha \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C_q D_2(a, b, r) \int |h(\sigma)| d\sigma,
$$

(2.22)

where $D_i(a, b, r) (i = 1, 2)$ are some positive functions of $a, b, r$. Then one has
Lemma 2.4. (See Theorem 1.1 of [8]) Let (2.17)-(2.22) be fulfilled. Assume that for each compactly supported \( f(t, x) \in L^1(\mathbb{R}^+ \times \mathbb{R}^n) \), and for each \( \delta > 0 \), there is a Calderón-Zygmund decomposition

\[
f = f_0 + \sum_{k=1}^{\infty} r_k,
\]

where \( \|f_0\|_{L^1} + \sum \|r_k\|_{L^1} \leq 3\|f\|_{L^1} \), and \( \sup |f_0| \leq C\delta \). In addition, for some disjoint cubes \( Q_k = \Delta_k(a_k, b_k) \times I_k(x_k, b_k) \), here \( \Delta_k \) (or \( I_k \)) is the interval (or cube) centered at \( a_k \) (or \( x_k \)) with side-length \( 2b_k \) (or \( 2b_k \)), assuming that

\[
r_k \text{ is supported in } Q_k, \quad \int_{Q_k} r_k dt dx = 0, \quad \text{and} \quad \delta \sum |Q_k| \leq \|f\|_{L^1}.
\]

Moreover, for any \( k \in \mathbb{N} \), we also suppose that

\[
\sum_{j \geq \lceil \log_2 b_k \rceil} D_1(a_k, b_k, 2^j) + \sum_{j \leq \lceil \log_2 b_k \rceil} D_2(a_k, b_k, 2^j) \leq C, \tag{2.23}
\]

where \( C > 0 \) is a generic positive constant independent of \( \delta, f, k, a_k, x_k, b_k \) and \( \tilde{b}_k \).

Then the operator \( \tilde{K} \) defined by

\[
(\tilde{K} f)(t, x) = \int_0^\infty K(t, \sigma, \cdot) * f(\sigma, \cdot) d\sigma
\]

is bounded on \( L^p(\mathbb{R}^+ \times \mathbb{R}^n) \) for all \( p \in (1, \infty) \).

Next we apply Lemma 2.4 to establish the \( L^p \) boundedness of \( \Delta w \) in (2.16). For this purpose, we require to verify that the kernel \( \hat{K}(t, \sigma, \xi) = |\xi|^2 \sigma^m \hat{T}(t, \sigma, \xi) \) in (2.16) satisfies (2.17)-(2.22). Here we point out that our analysis for \( \tilde{K}(t, \sigma, \xi) \) is much more delicate and involved than that for the kernel \( \hat{K}_0(t, \sigma, \xi) = |\xi|^2 \sigma^m \hat{T}(t, \sigma, \xi) \) in [8]. The main reason is: In the integrals (2.18)-(2.22), the variables \( t \) and \( x \) are the parameter variable and the integration variable respectively. This brings more troubles in treating the integrals (2.18)-(2.22) of \( \tilde{K}(t, \sigma, \xi) \) than in treating the corresponding integrals of \( \hat{K}_0(t, \sigma, \xi) \) due to the appearance of the integral variable factor \( \sigma^m \) in \( \hat{K}(t, \sigma, \xi) \).

Lemma 2.5. Let \( \hat{K}(t, \sigma, \xi) \) be defined in (2.16), then

(a) (2.18) and (2.19) hold.

(b) (2.20) holds for \( \alpha = 0 \), namely, for any \( h(\sigma) \in C_0^\infty(\mathbb{R}^+) \), one has

\[
\sup_{\frac{\pi}{2} \leq |\xi| \leq 2r} \int_0^{+\infty} \int_0^{+\infty} |\hat{K}(t, \sigma, \xi)h(\sigma)| dt \lesssim \int_0^{+\infty} |h(\sigma)| d\sigma. \tag{2.24}
\]

Proof. (a) Noting that

\[
\int_0^{+\infty} |\hat{K}(t, \sigma, \xi)| d\sigma = \int_0^{+\infty} |\xi|^2 \sigma^m \left( \int_0^{\min(t, \sigma)} \frac{\lambda(t)s\lambda(\sigma s)}{\lambda^2(ys)} ds \right) d\sigma.
\]
The boundedness of $L_i$ and $L_3$ can be obtained by applying (2.3). Indeed,

$$L_2 \leq (\sigma s)^{m+\frac{1}{2}} \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) \int_{1}^{\frac{\sigma s}{t}} y \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) dy \times \int_{t}^{+\infty} t^{-m} \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) dt(\frac{m+2}{2})$$

$$\leq (\sigma s)^{m+\frac{1}{2}} \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) \int_{1}^{\frac{\sigma s}{t}} y^{-1} \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) dy \times \int_{t}^{+\infty} t^{-m} \exp\left(-\frac{2}{m+2} (\sigma s)^{m+\frac{1}{2}}\right) dt(\frac{m+2}{2})$$

Next we treat each $L_i$ ($i = 1, 2, 3$) in (2.25). From (2.1) and (2.2), one has

$$L_1 \leq C \lambda(\sigma s)(\sigma s)^{m-\frac{1}{2}} \int_{0}^{+\infty} (1+t)^{-\frac{2}{2}} dt \leq C.$$  (2.26)
and

$$L_3 \leq (\sigma s)^{m+\frac{1}{2}} \exp\left(-\frac{2}{m+2}(\sigma s)^{m+\frac{1}{2}}\right) \int_0^{\sigma s} y^{-1} \exp\left(\frac{4}{m+2} y^{m+\frac{1}{2}}\right) dy$$

$$\times \int_y^{+\infty} t^{-\frac{1}{2}} \exp\left(-\frac{2}{m+2}(\sigma s)^{m+\frac{1}{2}}\right) d(t^{\frac{m+2}{2}})$$

$$\leq (\sigma s)^{m+\frac{1}{2}} \exp\left(-\frac{2}{m+2}(\sigma s)^{m+\frac{1}{2}}\right) \int_0^{\sigma s} y^{-m-\frac{1}{2}} \exp\left(\frac{2}{m+2} y^{m+\frac{1}{2}}\right) d(y^{m+\frac{1}{2}})$$

$$\leq (\sigma s)^{m+\frac{1}{2}} \left(\frac{\sigma s}{2}\right)^{-m-\frac{1}{2}} \leq C. \quad (2.28)$$

Substituting (2.26)-(2.28) into (2.25) yields (2.19). (b) It follows from (a) that

$$\int_0^{+\infty} \left|\int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq \int_0^{+\infty} |h(\sigma)| \left(\int_0^{+\infty} \hat{K}(t, \sigma, \xi) dt\right) d\sigma \leq C \int_0^{+\infty} |h(\sigma)| d\sigma.$$

Thus (2.20) holds for $\alpha = 0$. \qed

**Lemma 2.6.** Let $\hat{K}(t, \sigma, \xi)$ be defined in (2.16), then (2.21) holds for $\alpha = 0$, i.e.,

$$\sup_{\frac{1}{2} \leq |\xi| \leq 2r} \int_{C\Delta(a,2r+1)b} \left|\int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C_q P_1(ar^{\frac{2}{m+2}}, br^{\frac{2}{m+2}}) \int_0^{+\infty} |h(\sigma)| d\sigma, \quad (2.29)$$

where $h(\sigma) \in C_0^\infty(\mathbb{R}^+)$ is supported in $\Delta = (a-b, a+b)$, and $P_1(a,b) = \exp\left(-Ca^{m/2}b\right)$.

**Proof.** Obviously, it suffices to establish (2.29) for $q = 0$ if the constant $C_q$ can be shown to be independent of $q$. Notice that

$$C\Delta(a,2b) = \begin{cases} (a + 2b, +\infty), & \text{if } a \leq 2b \\ (a + 2b, +\infty) \cup (0, a - 2b), & \text{if } a > 2b \end{cases}$$

then in order to estimate the integral $\int_{C\Delta(a,2r+1)b}$ in (2.29), we require to deal with the integrals $\int_{a+2b}$ and $\int_{a-2b}$ separately.

It follows from a direct computation that

$$\int_{a+2b}^{+\infty} \left|\int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt$$

$$\leq \int_{a+2b}^{+\infty} dt \left(\int_{a-b}^{a+b} |h(\sigma)| |\xi|^2 \sigma^m d\sigma \left(\int_0^{\min(t,\sigma)} \lambda(ts)\lambda(s)\sigma^2(y) dy\right)\right)$$

$$= \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^m \lambda(\sigma s) d\sigma \left(\int_0^{\sigma s} dy \int_{(a+2b)s}^{+\infty} \lambda(t) dt\right)$$

$$\leq \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{m+1} d\sigma \left(\int_{(a+2b)s}^{+\infty} \frac{\lambda(t)}{\lambda(\sigma s)} dt\right) \quad \text{(by (2.1))}$$
where the positive constant $C$ is independent of $a$ and $b$.

If $a - 2b \geq 0$, then

$$\int_{0}^{a+b} dt \left| \int_{0}^{\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| \leq \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{m+1} \exp \left( \frac{2}{m+2} (\sigma s)^{m+2} \right) d\sigma \left( \int_{0}^{\frac{(a-2b)s}{2}} \lambda(t) dt \right) \left( \int_{0}^{t} dt \right) \lambda(t)$$

$$\leq \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{m} \lambda(\sigma s) d\sigma \left( \int_{0}^{\frac{(a-2b)s}{2}} \lambda(t) dt \right) \left( \int_{0}^{t} dt \right) \lambda(t)$$

$$\leq \int_{0}^{(a-2b)s} t dt \left( \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{m} \lambda(\sigma s) d\sigma \right) \left( \int_{0}^{t} dt \right) \lambda(t)$$

$$\leq C((a - 2b)s)^{\frac{3}{2}} \left((a + b)s\right)^{m+\frac{1}{2}} \exp \left( \frac{2}{m+2} \left((a - 2b)s\right)^{m+2} \right) \exp \left( - \frac{2}{m+2} \left((a + b)s\right)^{m+2} \right) \times \int_{0}^{\infty} |h(\sigma)| d\sigma.$$

From (2.30) and (2.31), we see that

$$\int_{C_{\Delta(a,2^{a+1}b)}} \int_{0}^{\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma dt \leq C \int_{0}^{\infty} |h(\sigma)| d\sigma,$$  \hspace{1cm} (2.32)

where $P_{1}(a, b) = \exp \left( - C a^{m/2} b \right)$. Thus, (2.29) obviously holds due to (2.32).

**Lemma 2.7.** Let $\hat{K}(t, \sigma, \xi)$ be defined in (2.16), then (2.22) holds for $\alpha = 0$, i.e.,

$$\sup_{\frac{1}{2} \leq |\xi| \leq 2r} \int_{0}^{\infty} \int_{0}^{\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma dt \leq C \int_{0}^{\infty} |h(\sigma)| d\sigma,$$  \hspace{1cm} (2.33)
where \( h(\sigma) \in C_0^\infty(\mathbb{R}^+) \), supph(\sigma) \subset \Delta = (a - b, a + b), \int_0^{+\infty} h(\sigma)d\sigma = 0, \text{ and } P_2(a, b) = ba^2 + b^2a^m + b^2a^\frac{m}{2}.

**Proof.** We will divide the proof of (2.33) into the following three parts:

\[
\left( \int_{a+b}^{+\infty} + \int_0^{a-b} + \int_{a-b}^{a+b} \right) \int_0^{+\infty} \hat{K}(t, \sigma, \xi)h(\sigma)d\sigma dt.
\]

**Part 1.** \( \int_{a+b}^{+\infty} \int_0^{+\infty} \hat{K}(t, \sigma, \xi)h(\sigma)d\sigma dt \leq Cbs(as)^{\frac{m}{2}} \int_0^{+\infty} |h(\sigma)|d\sigma \) holds

By a direct computation, we have

\[
\int_{a+b}^{+\infty} \int_0^{+\infty} \hat{K}(t, \sigma, \xi)h(\sigma)d\sigma dt
\]

\[
= \int_{a+b}^{+\infty} \int_{a-b}^{a+b} (\hat{K}(t, \sigma, \xi) - \hat{K}(t, a + b, \xi))h(\sigma)d\sigma dt \quad \text{(by } \int h(\sigma)d\sigma = 0)\]

\[
= \int_{a+b}^{+\infty} \lambda(t)dt \int_{a-b}^{a+b} \|h(\sigma)\|^2 a^m \lambda(\sigma) \int_0^\sigma \frac{dy}{\lambda^2(y)} - (a + b)^m \lambda((a + b)s) \int_{a+b}^{+\infty} \frac{dy}{\lambda^2(ys)} d\sigma
\]

\[
\leq \int_{(a+b)s}^{+\infty} \lambda(t)dt \int_{a-b}^{a+b} |h(\sigma)(\sigma^m - (a + b)s)^m| \lambda((a + b)s)d\sigma \int_{(a+b)s}^{+\infty} \frac{dy}{\lambda^2(y)}
\]

\[
= I_1 + I_2 + I_3.
\]

Next we treat each \( I_i \) (\( i = 1, 2, 3 \)) in (2.34). For \( I_1 \), one has

\[
I_1 = \int_{a-b}^{a+b} |h(\sigma)|\sigma^m \lambda(\sigma)s d\sigma \left( \int_{(a+b)s}^{+\infty} \frac{\lambda(\sigma)s - \lambda((a + b)s)}{\lambda^2(y)} dy \right) \int_{(a+b)s}^{+\infty} \frac{\lambda(t)}{\lambda(\sigma)s} dt
\]

\[
\leq Cbs \int_{a-b}^{a+b} |h(\sigma)|\sigma^m \lambda(\sigma)s d\sigma \left( \int_0^{+\infty} \frac{\sigma - \lambda'(y)}{\lambda^2(y)} dy \right) \int_{(a+b)s}^{+\infty} \left( \frac{t}{\sigma} \right)^{\frac{1}{2}}
\]

\[
\times \exp \left( \frac{2}{m + 2} (\sigma^m) - \frac{2}{m + 2} t^{m+2} \right) dt \quad \text{(by (2.1) and (2.3))}
\]

\[
=Cbs \int_{a-b}^{a+b} |h(\sigma)|\sigma^{m-\frac{1}{2}} \exp \left( \frac{2}{m + 2} (\sigma^{m+2}) - \frac{1}{m + 2} (a + b)^{m+2} \right) \left( 1 - \lambda(\sigma) \right) d\sigma
\]

\[
\times \int_{(a+b)s}^{+\infty} t^{\frac{1-m}{2}} \exp \left( - \frac{2}{m + 2} (a + b)^{m+2} \right) \left( 1 - \lambda(\sigma) \right) d\sigma
\]

\[
\leq Cbs ((a + b)s)^{\frac{1-m}{2}} \exp \left( - \frac{2}{m + 2} (a + b)^{m+2} \right) \int_{a-b}^{a+b} |h(\sigma)|\sigma^{m-\frac{1}{2}}
\]

\[
\times \exp \left( \frac{2}{m + 2} (\sigma^{m+2}) - \frac{1}{m + 2} (a + b)^{m+2} \right) \left( 1 - \lambda(\sigma) \right) d\sigma
\]
Similarly, applying Lemma 2.2 yields

\[
I_2 \leq C_b s \left( (a+b)s \right)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( \int_0^\infty \frac{\lambda(s)}{\lambda^2(y)}dy \right) \left( \int_{(a+b)s}^{+\infty} \frac{\lambda(t)}{\lambda(s)}dt \right)
\]

\[
\leq C_b s \left( (a+b)s \right)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( \int_0^{\sigma s} \frac{\lambda(s)}{\lambda(y)}dy \right) \left( \int_{(a+b)s}^{+\infty} \frac{\lambda(t)}{\lambda(s)}dt \right)
\]

\[
\leq C_b s \left( (a+b)s \right)^{m-1} \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( \int_0^{\sigma s} \frac{\lambda(s)}{\lambda(y)}dy \right) \left( \int_{(a+b)s}^{+\infty} \frac{\lambda(t)}{\lambda(s)}dt \right)
\]

\[
\times \left( \int_{(a+b)s}^{+\infty} \frac{t}{s} \exp\left( \frac{2}{m+2} (s) \frac{m+2}{2} - \frac{2}{m+2} t \frac{m+2}{2} \right) dt \right) \quad \text{(by (2.1) and (2.3))}
\]

\[
\leq C_b s \left( (a+b)s \right)^{-m} \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( \int_0^{\sigma s} \frac{\lambda(s)}{\lambda(y)}dy \right) \left( \int_{(a+b)s}^{+\infty} \frac{\lambda(t)}{\lambda(s)}dt \right)
\]

\[
\times \left( \int_{(a+b)s}^{+\infty} \exp\left( - \frac{2}{m+2} t \frac{m+2}{2} \right) dt \right)
\]

\[
\leq C_b s \left( (a+b)s \right)^{-m} \int_0^{+\infty} |h(\sigma)|d\sigma
\]

and

\[
I_3 = \left( (a+b)s \right)^{m} \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{a-b}^{a+b} \lambda^2((a+b)s) \lambda^2(y) \lambda((a+b)s) \lambda^2(y) \lambda((a+b)s) \lambda(t) \lambda((a+b)s) \lambda(t) dt
\]

\[
\leq C_b s \left( (a+b)s \right)^{m} \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{a-b}^{a+b} \lambda((a+b)s) \lambda^2(t) \lambda((a+b)s) \lambda(t) \lambda((a+b)s) \lambda(t) dt
\]

\[
\times \exp\left( \frac{2}{m+2} ((a+b)s)^{m+2} - \frac{2}{m+2} t^{m+2} \right) dt \quad \text{(by (2.3))}
\]

\[
= C_b s \left( (a+b)s \right)^{-m} \exp\left( \frac{2}{m+2} ((a+b)s)^{m+2} \right) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{a-b}^{a+b} \lambda((a+b)s) \lambda^2(t) \lambda((a+b)s) \lambda(t) \lambda((a+b)s) \lambda(t) dt
\]

\[
\times \exp\left( - \frac{2}{m+2} t^{m+2} \right) dt \left( t^{m+1} \right)
\]

\[
\leq C_b s \left( (a+b)s \right)^{-m} \int_0^{+\infty} |h(\sigma)|d\sigma.
\]

Substituting (2.35)-(2.37) into (2.34) yields the conclusion in Part 1.

**Part 2.** \( \int_0^{a-b} \int_0^{+\infty} \tilde{K}(t, \sigma, \xi) h(\sigma)d\sigma dt \leq C_b s \left( (a+b)s \right)^{-m} \int_0^{+\infty} |h(\sigma)|d\sigma \) holds
We only estimate $\equiv II_2$.

To estimate the term $II_1$ in (2.38), we will consider the following three cases

Case 1. $(a - b)s \geq 2$

One now has

$$II_1 \leq Cbs \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m| |\lambda'(\sigma s)| d\sigma \left( \int_0^1 \frac{dy}{\lambda^2(y)} + \int_1^{(a-b)s} \frac{dy}{\lambda^2(y)} \right)$$

$$+ \int_{(a-b)s}^{(a-b)s \frac{t}{2}} \lambda(t) dt \int_0^1 \frac{dy}{\lambda^2(y)} + \int_{(a-b)s \frac{t}{2}}^{(a-b)s} \lambda(t) dt \int_1 \frac{dy}{\lambda^2(y)} + \int_{(a-b)s}^{(a-b)s \frac{t}{2}} \lambda(t) dt \int_{(a-b)s \frac{t}{2}}^{(a-b)s} \frac{dy}{\lambda^2(y)}$$

$$\equiv II_1^{(1)} + II_1^{(2)} + II_1^{(3)} + II_1^{(4)} + II_1^{(5)}.$$ (2.39)

We only estimate $II_1^{(1)}$, $II_1^{(4)}$ and $II_1^{(5)}$ since the remaining terms can be treated similarly.

It follows from Lemma 2.2 that

$$II_1^{(1)} \leq C(bs) \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m| |\lambda'(\sigma s)| d\sigma \int_0^1 \frac{dy}{\lambda^2(t)} dt \quad \text{(by (2.1))}$$

$$\leq C\lambda(1)^{-1} bs \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m| |\lambda'(\sigma s)| d\sigma \quad \text{(by (2.1))}$$

$$\leq Cbs(as)^{\frac{m}{2}} \int_0^{+\infty} |h(\sigma)| d\sigma \quad \text{(by (2.2))}$$ (2.40)

and

$$II_1^{(4)} \leq Cbs \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m| |\lambda'(\sigma s)| d\sigma \int_{(a-b)s}^{(a-b)s \frac{t}{2}} \frac{dy}{\lambda^2(y)} dt \int_1 \frac{dy}{\lambda^2(y)}$$
\[ \leq \text{Cbs} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{\frac{3m}{2}} d\sigma \int_{\frac{(a-b)s}{2}}^{(a-b)s} \frac{4}{m+2} y^{\frac{m+2}{2}} - \frac{2}{m+2} t^{\frac{m+2}{2}} - \frac{2}{m+2} (\sigma s)^{\frac{m+2}{2}} dy \]

(by (2.4) since \((a-b)s \geq 2\) and by (2.3))

\[ \leq \text{Cbs} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{\frac{3m+1}{2}} e^{x p \left(-\frac{2}{m+2} (\sigma s)^{\frac{m+2}{2}}\right)} \]

\[ \times \exp \left( \frac{2}{m+2} \left( \frac{(a-b)s}{2} \right)^{\frac{m+2}{2}} \right) d\sigma \]

\[ \leq \text{Cbs}(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.41) \]

In addition, by (2.3) and (2.4), we arrive at

\[ II_1^{(5)} = \text{Cbs} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{m} \frac{\lambda'(\sigma s)}{\lambda(\sigma s)} d\sigma \int_{\frac{(a-b)s}{2}}^{(a-b)s} \frac{\lambda(\sigma s) \lambda(y)}{\lambda^2(t)} dt \]

\[ \leq \text{Cbs} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{\frac{3m+1}{2}} e^{x p \left(-\frac{2}{m+2} (\sigma s)^{\frac{m+2}{2}}\right)} d\sigma \]

\[ \times \int_{\frac{(a-b)s}{2}}^{(a-b)s} t^{-\frac{1}{2}} e^{x p \left(-\frac{2}{m+2} t^{\frac{m+2}{2}}\right)} d(t^{\frac{m+1}{2}}) \int_{\frac{(a-b)s}{2}}^{t} y^{-\frac{1}{2}} e^{x p \left(\frac{4}{m+2} y^{\frac{m+2}{2}}\right)} d(y^{\frac{m+2}{2}}) \]

\[ \leq \text{Cbs}(as)^{\frac{m}{2}} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^{\frac{m+1}{2}} e^{x p \left(-\frac{2}{m+2} (\sigma s)^{\frac{m+2}{2}}\right)} d\sigma \]

\[ \times \left( (a-b)s \right)^{-\frac{(m+1)}{2}} e^{x p \left(-\frac{2}{m+2} (a-b)s^{\frac{m+2}{2}}\right)} . \quad (2.42) \]

Notice that the function \( \eta(z) = z^{m+\frac{1}{2}} e^{x p \left(-\frac{2}{m+2} z^{\frac{m+2}{2}}\right)} \) is strictly decreasing for \( z \geq 2 \), then (2.42) can be dominated by

\[ \text{Cbs}(a+b)^{s} \int_{0}^{+\infty} |h(\sigma)| d\sigma \leq \text{Cbs}(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma. \]

Collecting all the analysis above yields

\[ II_1 \leq \text{Cbs}(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma \quad \text{for} \quad (a-b)s \geq 2. \quad (2.43) \]

**Case 2.** \((a-b)s \leq 1\)

In this case, we have

\[ II_1 \leq II_1^{(1)} \leq \text{Cbs}(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.44) \]
Case 3. \(1 \leq (a-b)s \leq 2\)

In this case, we have

\[II_1 \leq II_1^{(1)} + II_1^{(6)},\]

where \(II_1^{(6)} = Cbs \int_{a-b}^{a+b} |h(\sigma)|[(\sigma s)^m]\lambda'((\sigma s))d\sigma \int_{1}^{2} \lambda(t)dt \int_{0}^{t} \frac{dy}{\lambda^2(y)}\).

As in Case 2, one has \(II_1^{(1)} \leq Cbs(as)^\frac{m}{2} \int_{\mathbb{R}} |h(\sigma)|d\sigma\). In addition,

\[II_1^{(6)} \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|[(\sigma s)^m]\lambda'((\sigma s))d\sigma \int_{1}^{2} \lambda(t)dt \quad \text{(by (2.1))}\]

\[\leq Cbs(as)^\frac{m}{2} \int_{\mathbb{R}} |h(\sigma)|d\sigma.\]

Therefore,

\[II_1 \leq Cbs(as)^\frac{m}{2} \int_{0}^{+\infty} |h(\sigma)|d\sigma \quad \text{for } 1 \leq (a-b)s \leq 2. \quad (2.45)\]

Combining (2.43)-(2.45) yields

\[II_1 \leq Cbs(as)^\frac{m}{2} \int_{0}^{+\infty} |h(\sigma)|d\sigma. \quad (2.46)\]

Next we estimate \(II_2\) in (2.38), which will be divided into the following two cases.

**Case A.** \((a-b)s \geq 1\)

We have

\[II_2 \leq Cbs((a+b)s)^{m-1}\lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma\]

\[\times \left( \int_{0}^{1} \lambda(t)dt \int_{0}^{t} \frac{dy}{\lambda^2(y)} + \int_{1}^{(a-b)s} \lambda(t)dt \int_{1}^{t} \frac{dy}{\lambda^2(y)} + \int_{1}^{(a-b)s} \lambda(t)dt \int_{1}^{t} \frac{dy}{\lambda^2(y)} \right)\]

\[\equiv II_2^{(1)} + II_2^{(2)} + II_2^{(3)}. \quad (2.47)\]

Noting the function \(\lambda(t)\) is decreasing, then by (2.1)-(2.3)

\[II_2^{(1)} \leq Cbs\left(\frac{2}{m+2} \frac{t^{m+2}}{t^{m+2}} - \frac{2}{m+2} \left((a+b)s \right)^{m+2} \right) \int_{a-b}^{a+b} |h(\sigma)|d\sigma\]

\[\leq Cbs\left(\frac{2}{m+2} \frac{t^{m+2}}{t^{m+2}} - \frac{2}{m+2} \left((a+b)s \right)^{m+2} \right) \int_{a-b}^{a+b} |h(\sigma)|d\sigma\]

\[\times \int_{0}^{t} \frac{1}{t^{1/2}} \exp\left( - \frac{2}{m+2} t^{m+2} \right) dt\]

\[\leq Cbs(as)^\frac{m}{2} \int_{0}^{+\infty} |h(\sigma)|d\sigma. \quad (2.48)\]
and

\[
II_2^{(2)} \leq C \frac{bs}{\lambda^2(1)} ((a + b)s)^{m-1} \lambda((a + b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{1}^{+\infty} \lambda(t) dt
\]

\[
\leq Cbs((a + b)s)^{\frac{m}{2}} ((a + b)s)^{\frac{m-1}{2}} \lambda((a + b)s) \int_{0}^{+\infty} |h(\sigma)| d\sigma
\]

\[
\leq Cbs(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma. \tag{2.49}
\]

For the term \(II_2^{(3)}\), one has from (2.3)

\[
II_2^{(3)} = Cbs((a + b)s)^{m-1} \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{1}^{(a-b)s} dt \int_{1}^{t} \frac{\lambda(t) \lambda((a + b)s)}{\lambda^2(y)} dy
\]

\[
\leq Cbs\left((a + b)s\right)^{\frac{m-1}{2}} \exp\left(-\frac{2}{m+2}\left((a + b)s\right)^{\frac{m+2}{2}}\right) \int_{a-b}^{a+b} |h(\sigma)| d\sigma
\]

\[
\times \int_{1}^{(a-b)s} t^{1/2} \exp\left(\frac{2}{m+2} t^{m+2}\right) dt
\]

\[
\leq Cbs\left((a + b)s\right)^{\frac{m-1}{2}} \left((a + b)s\right)^{\frac{1}{2}} \int_{a-b}^{a+b} |h(\sigma)| d\sigma
\]

\[
\times \exp\left(\frac{2}{m+2}\left((a - b)s\right)^{\frac{m+2}{2}} - \frac{2}{m+2}\left((a + b)s\right)^{\frac{m+2}{2}}\right)
\]

\[
\leq Cbs(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma. \tag{2.50}
\]

From (2.48)-(2.50), we see that

\[
II_2 \leq Cbs(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma \quad \text{for} \quad (a - b)s \geq 1. \tag{2.51}
\]

**Case B.** \((a - b)s \leq 1\)

One easily obtains

\[
II_2 \leq II_2^{(1)} \leq Cbs(as)^{\frac{m}{2}} \int_{0}^{+\infty} |h(\sigma)| d\sigma \quad \text{for} \quad (a - b)s \leq 1. \tag{2.52}
\]

Therefore, it follows from (2.38), (2.46) and (2.51)-(2.52) that the conclusion in Part 2 holds.

**Part 3.** \[\int_{a-b}^{a+b} \left\{ \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right\} dt \leq C \left( bs(as)^{\frac{m}{2}} + (bs)^2(as)^m + (bs)^3(as)^{\frac{3m}{2}} \right) \int_{0}^{+\infty} |h(\sigma)| d\sigma \]

holds

We have

\[
\int_{a-b}^{a+b} \left\{ \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right\} dt
\]
\[
\begin{align*}
&= \int_{a-b}^{a+b} |\int_{a-b}^{a+b} h(\sigma)|^2 \left( \sigma^m \int_0^{\min(t,\sigma)} \frac{\lambda(ts)\lambda(\sigma s)}{\lambda^2(ys)} dy - (a+b)^m \int_0^t \frac{\lambda(ts)\lambda((a+b)s)}{\lambda^2(ys)} dy \right) d\sigma | dt \\
&\leq \int_{(a-b)s}^{(a+b)s} \lambda(t) dt \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m \left( \lambda(\sigma s) - \lambda((a+b)s) \right) d\sigma \left( \int_0^{(a-b)s} + \int_{(a-b)s}^{\min(t,\sigma)s} \frac{dy}{\lambda^2(y)} \right) \\
&\quad + \int_{(a-b)s}^{(a+b)s} \lambda(t) dt \int_{a-b}^{a+b} |h(\sigma)| \left( \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \right) \left( \lambda((a-b)s) \int_{(a-b)s}^{(a+b)s} dt \right) \\
&\quad + \int_{(a-b)s}^{(a+b)s} \lambda(t) dt \int_{a-b}^{a+b} |h(\sigma)| \left( \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \right) \left( \lambda((a+b)s) \int_{\min(t,\sigma)s}^{a+b} \frac{dy}{\lambda^2(y)} \right) \\
&= \equiv III_1^{(1)} + III_2^{(1)} + III_3 + III_4 + III_5. \quad (2.53)
\end{align*}
\]

We now treat each \( III_i \) in (2.53) as follows.

**Step 1. Estimate of \( III_1 \)**

By applying Lemma 2.2, one obtains

\[
\begin{align*}
III_1 &= \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(\sigma s) - \lambda((a+b)s)}{\lambda^2(y)} dy \int_{(a-b)s}^{(a+b)s} \lambda(t) dt \\
&\leq C \left( (a+b)s \right)^m \int_{a-b}^{a+b} |h(\sigma)| d\sigma \left( bs \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \right) \left( \lambda((a-b)s) \int_{(a-b)s}^{(a+b)s} dt \right) \\
&\leq C (bs)^2 (as)^m \left( 1 - \lambda((a-b)s) \right) \int_{0}^{+\infty} |h(\sigma)| d\sigma \\
&\leq C (bs)^2 (as)^m \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.54)
\end{align*}
\]

**Step 2. Estimate of \( III_2 \)**

\[
\begin{align*}
III_2 &\leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m |\lambda'(\sigma s)| d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(t) dt}{\lambda^2(y)} \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \\
&\quad + Cbs \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m |\lambda'(\sigma s)| d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(t) dt}{\lambda^2(y)} \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \\
&= \equiv III_2^{(1)} + III_2^{(2)}. \quad (2.55)
\end{align*}
\]

If \((a-b)s \geq 1\), then

\[
\begin{align*}
III_2^{(1)} &\leq C (bs)^2 \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m |\lambda'(\sigma s)| d\sigma \int_{(a-b)s}^{(a+b)s} \frac{1}{\lambda(y)} dy \quad \text{(by (2.1))} \\
&\leq C (bs)^3 \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m \frac{|\lambda'(\sigma s)|}{\lambda(\sigma s)} d\sigma \quad \text{(by (2.1))} \\
&\leq C (bs)^3 \int_{a-b}^{a+b} |h(\sigma)|(\sigma s)^m d\sigma \quad \text{(by (2.4))}
\end{align*}
\]
\[ \leq C(bs)^3(\alpha)^{\frac{3m}{2}} \int_0^{+\infty} |h(\sigma)|d\sigma \]  

(2.56)

and

\[ III_2^{(2)} \leq C(bs)^2 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(t)}{\lambda^2(\sigma s)} dt \quad \text{(by (2.1))} \]

\[ = C(bs)^2 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(t)}{\lambda(\sigma s)} dt \]

\[ \leq C(bs)^3 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^{\frac{3m}{2}} d\sigma \quad \text{(by (2.1) and (2.4))} \]

\[ \leq C(bs)^3(\alpha)^{\frac{3m}{2}} \int_0^{+\infty} |h(\sigma)|d\sigma. \]  

(2.57)

If \((a-b)s \leq 1 \leq (a+b)s\), then one has by (2.1)

\[ III_2^{(1)} \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \lambda(t)dt \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \]

\[ + Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \lambda(t)dt \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \]

\[ \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma + C(bs)^3 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m \frac{|\lambda'(\sigma)|}{\lambda(\sigma s)} d\sigma \]

\[ \leq C \left( bs(as)^{\frac{m}{2}} + (bs)^3(\alpha)^{\frac{3m}{2}} \right) \int_0^{+\infty} |h(\sigma)|d\sigma \]  

(2.58)

and

\[ III_2^{(2)} \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \lambda(t)dt \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \]

\[ + C(bs)^2 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \frac{\lambda(t)}{\lambda(\sigma s)} dt \]

\[ \leq C \frac{1}{\lambda^2(1)} (bs)^2 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m |\lambda'(\sigma)| \lambda(\sigma s) d\sigma + C(bs)^3 \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^{\frac{3m}{2}} d\sigma \]

\[ \leq C \left( (bs)^2(as)^{\frac{m}{2}} + (bs)^3(\alpha)^{\frac{3m}{2}} \right) \int_0^{+\infty} |h(\sigma)|d\sigma. \]  

(2.59)

If \((a+b)s \leq 1\), then

\[ III_2^{(1)} \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \lambda(t)dt \int_{(a-b)s}^{(a+b)s} \frac{dy}{\lambda^2(y)} \]

\[ \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|(|\sigma|)^m|\lambda'(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} \frac{t}{\lambda(t)} dt \quad \text{(by (2.1))} \]
\[ \leq C_{bs} (as)^{3m} \int_{0}^{+\infty} |h(\sigma)| d\sigma \quad (2.60) \]

and

\[ III_2^{(2)} \leq C_{bs} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^m |\lambda'(\sigma s)| d\sigma \int_{(a-b)s}^{(a+b)s} \lambda(t) dt \int_{(a-b)s}^{1} dy \lambda^2(y) \]

\[ \leq C \lambda^{-2}(1)(bs)^2 \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^m |\lambda'(\sigma s)| \lambda(\sigma s) d\sigma \quad \text{(by (2.1))} \]

\[ \leq C (bs)^2 (as)^m \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.61) \]

Combining (2.55)-(2.61) yields

\[ III_2 \leq C \left( bs(as)^{3m} + (bs)^2 (as)^m + (bs)^3 (as)^{3m} \right) \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.62) \]

**Step 3. Estimate of III_3**

A direct computation derives that from (2.3)

\[ III_3 \leq C \left( bs \lambda((a-b)s) \left( bs((a+b)s)^{m-1} \lambda((a+b)s) \right) \left( \frac{(a-b)s}{\lambda^2((a-b)s)} \right) \int_{0}^{+\infty} |h(\sigma)| d\sigma \right. \]

\[ \leq C (bs)^2 (as)^m \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (2.63) \]

**Step 4. Estimate of III_4**

We have

\[ III_4 \leq C_{bs} ((a+b)s)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{(a-b)s}^{\sigma s} \lambda(t) dt \int_{(a-b)s}^{t} \frac{dy}{\lambda^2(y)} \]

\[ + C_{bs} ((a+b)s)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{\sigma s}^{(a+b)s} \lambda(t) dt \int_{(a-b)s}^{\sigma s} \frac{dy}{\lambda^2(y)} \]

\[ \equiv III_4^{(1)} + III_4^{(2)}. \]

It follows from Lemma 2.2 and (2.1) that

\[ III_4^{(1)} \leq C_{bs} ((a+b)s)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{(a-b)s}^{\sigma s} \frac{bs}{\lambda(t)} dt \]

\[ \leq C (bs)^2 ((a+b)s)^{m-1} \int_{a-b}^{a+b} |h(\sigma)| (\sigma s)^m \frac{\lambda((a+b)s)}{\lambda(\sigma s)} d\sigma \]

\[ \leq C (bs)^2 (as)^m \int_{0}^{+\infty} |h(\sigma)| d\sigma \]
Lemma 2.5. Suppose that for

\[ H \text{ satisfies (2.20)-(2.22) for all } \]

Substituting (2.54) and (2.62)-(2.65) into (2.53) derivesthe conclusion in Part 3. Therefore,

Step 5. Estimate of \( III_5 \)

One has

\[ III_5 = \left( (a + b)s \right)^m \lambda((a + b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{s}^{(a+b)s} \lambda(t) dt \int_{s}^{t} \frac{d\sigma}{\lambda^2(y)} \]

\[ \leq \left( (a + b)s \right)^m \lambda((a + b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{s}^{(a+b)s} \frac{2bs}{\lambda(t)} dt \]

\[ \leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (2.65) \]

Substituting (2.54) and (2.62)-(2.65) into (2.53) derives the conclusion in Part 3. Therefore, collecting all the results in Part 1 - Part 3, we have

\[ \int_0^{+\infty} \left| \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq CP_2(as, bs) \int_0^{+\infty} |h(\sigma)| d\sigma, \quad (2.66) \]

where \( P_2(a, b) = b a^2 + b^2 a^m + b^3 a^m \). Hence Lemma 2.7 is proved.

Based on Lemma 2.5-Lemma 2.7, we will show that the function \( \hat{K}(t, \sigma, \xi) \) defined in (2.16) satisfies (2.20)-(2.22) for all \( |\alpha| \leq q \) with some integer \( q > n \).

Lemma 2.8. Let \( \hat{K}(t, \sigma, \xi) \) be defined in (2.16), then (2.20)-(2.22) hold.

Proof. We now prove (2.20) by induction method. Note that (2.20) holds for \( |\alpha| = 0 \) by Lemma 2.5. Suppose that for \( |\alpha| \leq j < q \)

\[ \int_0^{+\infty} \left| \partial^\alpha \left( \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right) \right| dt \leq C_q |\xi|^{-|\alpha|} \int |h(\sigma)| d\sigma \quad \text{for any } h \in L^1(\mathbb{R}^+) ; \]

\[ \text{(A}_j) \]

\[ \int_{C\Delta(a, 2^j+1b)} \left| \partial^\alpha \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C_q D_1(a, b, R) |\xi|^{-|\alpha|} \int |h(\sigma)| d\sigma \]

for any \( h \in L^1(\mathbb{R}^+) \) with \( \text{supp} h \subset \Delta(a, b) ; \quad (B_j) \)
and
\[ \int_0^{+\infty} \left| \partial^a_{\xi} \right| \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma dt \leq C_2 D_2(a, b, R) |\xi|^{-|\alpha|} \int |h(\sigma)| d\sigma \]
for any \( h \in L^1(\mathbb{R}^+) \) with \( \text{supp} h \subset \Delta(a, b) \) and \( \int h(\sigma) d\sigma = 0 \). \( (C_j) \)

At first, we show \( (A_{j+1}) \) holds. Denote \( U^{(j+1)} = \partial^{j+1}_{\xi} \hat{u}(t, \xi) \). Taking \( \partial^{j+1}_{\xi} \) on the two hand sides of the equation \( \partial^2_{\xi} \hat{u} - m|\xi|^2 \hat{u} = t^m h(t) \) yields
\[ \partial^2_{\xi} U^{(j+1)} - t^m |\xi|^2 U^{(j+1)} = t^m \left( C_1 \partial_{\xi}(|\xi|^2) U^{(j)} + C_2 \partial^2_{\xi}(|\xi|^2) U^{(j-1)} \right), \]
(2.67)
where \( C_1 \) and \( C_2 \) are some constants. It follows from (2.67) and \( (A_0) \) that
\[ \int_0^{+\infty} |\xi|^2 |U^{(j+1)}(t, \xi)| dt \leq C |\xi|^{-1} \int_0^{+\infty} |\xi|^2 |U^{(j)}(t, \xi)| dt + C |\xi|^{-2} \int_0^{+\infty} |\xi|^2 |U^{(j-1)}(t, \xi)| dt. \]
(2.68)
This, together with the induction assumption \( (A_j) \) and the definition of \( \hat{K}(t, \sigma, \xi) \), yields \( (A_{j+1}) \).

In addition, \( (B_{j+1}) \) or \( (C_{j+1}) \) can be directly obtained by using (2.68) repeatedly until \( j = 1 \) and combining Lemma 2.6 or Lemma 2.7. Therefore, the proof of Lemma 2.8 is completed. \( \square \)

Next we show that functions
\[ D_1(a, b, r) = P_1(ar^{m+2}, br^{m+2}), \quad D_2(a, b, r) = P_2(ar^{m+2}, br^{m+2}) \]
satisfy the estimate (2.23) for suitable \( a_k, b_k \) and \( \tilde{b}_k \), where \( P_1(a, b) \) and \( P_2(a, b) \) are defined in (2.29) and (2.33) respectively.

**Lemma 2.9.** For \( D_1(a, b, r) \) and \( D_2(a, b, r) \) defined above, the estimate (2.23) in Lemma 2.4 holds for \( a_k, b_k \) and \( \tilde{b}_k \) with \( b_k \sim a_k^{m/2} \).

**Proof.** For any \( f \in L^1(\mathbb{R}^+ \times \mathbb{R}^n) \), one has the standard Calderón-Zygmund decomposition (see Theorem 4 in Chapter 1 of [21]):
\[ f = g + \sum r_k \]
such that
\[ g(t, x) = \begin{cases} f(t, x), & \text{in } \mathbb{R}^{n+1}_+ \setminus \bigcup_k Q_k \\ \frac{1}{|Q_k|} \int_{Q_k} f(\theta, y) d\theta dy, & \text{in } Q_k \end{cases} \]
and
\[ r_k(t, x) = \left( f(t, x) - g(t, x) \right) \chi_{Q_k}, \]
where the cube \( Q_k = \Delta_k(a_k, \tilde{b}_k) \times I_k(x_k, b_k) \) with \( \frac{1}{C} \leq \frac{b_k}{a_k^{m/2} \tilde{b}_k} \leq C \) for some positive constant \( C > 1 \) independent of \( k \).
Set
\[ D_1(a_k, \tilde{b}_k, r) \equiv P_1(a_k r^{2/(m+2)}, \tilde{b}_k r^{2/(m+2)}) = \exp(-C a_k^\frac{2}{m+2} \tilde{b}_k r), \]
and
\[ D_2(a_k, \tilde{b}_k, r) \equiv a_k^\frac{m}{2} \tilde{b}_k r + a_k^\frac{m}{2} \tilde{b}_k^2 r^2 + a_k^\frac{m}{2} \tilde{b}_k^3 r^3, \]
then
\[ \sum_{2^jb_k \geq 1} D_1(a_k, \tilde{b}_k, 2^j) + \sum_{2^jb_k \leq 1} D_2(a_k, \tilde{b}_k, 2^j) \]
\[ \leq \sum_{2^jb_k \geq 1} \exp(-C 2^jb_k) + C \sum_{2^jb_k \leq 1} \left( 2^jb_k + (2^jb_k)^2 + (2^jb_k)^3 \right) \]
\[ \leq C \]
and Lemma 2.9 is proved. \( \square \)

Based on Lemma 2.4-Lemma 2.9, we now prove

**Theorem 2.10.** Consider the problem
\[ \begin{cases} 
\Delta w = m g(t, x), & (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
\Delta w = t^m, & (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
\partial_t w = 0, & (t, x) \in [0, +\infty) \times \mathbb{R}^n. 
\end{cases} \tag{2.69} \]

where \( m \in \mathbb{N} \), \( g \in L^p(\mathbb{R}^{n+1}) \) \( (1 < p < \infty) \), and \( \text{supp} g \subset \{(t, x) : 0 \leq t \leq M_0\} \) with \( M_0 > 0 \) being some fixed constant, then (2.69) has a unique solution \( w \in W^{2,p}([0, T] \times \mathbb{R}^n) \) for any \( T > 0 \), moreover \( w \) satisfies the following estimate
\[ \|\Delta w\|_{L^p(G_T)} \leq C_p,T \|g\|_{L^p(\mathbb{R}^{n+1})}, \tag{2.70} \]

where \( G_T = [0, T] \times \mathbb{R}^n \), and \( C_{p,T} \) is a generic positive constant depending on \( p \) and \( T \).

**Proof.** By (2.16), we know that the solution \( u \) to (2.69) satisfies
\[ \Delta w = \mathcal{F}_\xi^{-1}(|\xi|^2 \hat{u}(t, \xi)) = \mathcal{F}_\xi^{-1} \left( \int_0^{+\infty} \hat{K}(t, \sigma, \xi) \hat{g}(\sigma, \xi) d\sigma \right), \]
where \( \hat{K}(t, \sigma, \xi) = |\xi|^2 \sigma^m \hat{T}(t, \sigma, \xi) \). By Lemma 2.5-Lemma 2.9, one knows that \( \hat{K}(t, \sigma, \xi) \) satisfies all the requirements in Lemma 2.4. Hence we have from Lemma 2.4
\[ \|\Delta w\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|g\|_{L^p(\mathbb{R}^{n+1})}. \tag{2.71} \]
This, together with the equation (2.69), yields
\[ \| \partial_t^2 w \|_{L^p(G_T)} \leq C_p T^m \| g \|_{L^p(G_T)}. \]  
(2.72)

By (2.71)-(2.72) and the interpolation theory, we can obtain
\[ \sum_{j=1}^n \| \partial_{t,x_j}^2 w \|_{L^p(G_T)} \leq C_{p,T} \| g \|_{L^p(\mathbb{R}^{n+1})}. \]  
(2.73)

In addition, by (2.15) we have
\[ \hat{w}(t, \xi) = \lambda(t s) \int_0^t \frac{dy}{\lambda^2(ys)} \int_0^\infty \lambda(\sigma s) \sigma^m \hat{g}(\sigma, \xi) d\sigma, \]
which derives
\[ \partial_t \hat{w}(0, \xi) = \int_0^\infty \lambda(\sigma s) \sigma^m \hat{g}(\sigma, \xi) d\sigma = \int_0^{M_0} \lambda(\sigma s) \sigma^m \hat{g}(\sigma, \xi) d\sigma \]
and further
\[ \partial_t w(0, x) = \int_0^{M_0} \mathcal{F}_\xi^{-1}(\lambda(s)) \ast (\sigma^m g(\sigma, \cdot)) d\sigma. \]  
(2.74)

It is noted that
\[ \partial^\alpha_\xi (\lambda(-\sigma s)) = |\xi|^{-|\alpha|} \left( P_1(\sigma s, \omega)\lambda(-\sigma s) + P_2(\sigma s, \omega)\lambda'(-\sigma s) \right), \]  
(2.75)
where \( P_j(\theta, \omega) \) \((j = 1, 2)\) are the polynomials of \( \theta \), and smoothly depend on the variable \( \omega \in S^{n-1} \). From (2.75), we easily derive that for some integer \( q > n \)
\[ \sup_{t \in [-T, 0]} r^{2|\alpha|-n} \int_{\xi \leq |\xi| \leq 2r} |\partial^\alpha_\xi (\lambda(-t s))|^2 d\xi \leq C_q \quad \text{for all } r > 0 \text{ and } |\alpha| \leq q. \]  
(2.76)

Analogously, we have for some integer \( q > n \)
\[ \sup_{t \in [-T, 0]} r^{2|\alpha|-n} \int_{\xi \leq |\xi| \leq 2r} |\partial^\alpha_\xi (\lambda'(-t s))|^2 d\xi \leq C_q \quad \text{for all } r > 0 \text{ and } |\alpha| \leq q. \]  
(2.77)

Then it follows (2.76)-(2.77), Hörmander’s multiplier theorem (see Theorem 7.95 in [10]), Minkowski inequality and (2.74) that
\[ \| \partial_t w(0, x) \|_{L^p(\mathbb{R}^n)} \leq C_p M_0^{m+1-\frac{1}{p}} \| g \|_{L^p(\mathbb{R}^{n+1})}. \]  
(2.78)

This, together with (2.72), yields
\[ \| \partial_t w \|_{L^p(G_T)} \leq \| \partial_t w(0, \cdot) \|_{L^p(G_T)} + \left( \int_0^T \left\| \int_0^t \partial^2_t w(\tau, \cdot) d\tau \right\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}. \]
In addition, it follows from
\[ \text{In this section, for the later requirements of solving Theorem 1.1 in the degenerate elliptic region, we will establish the weighted} \]
\[ \text{estimates for (3.1) also admits independent interests in the study on the linear degenerate elliptic} \]
\[ \text{equations.} \]
\[ \text{By (2.73) and} (2.79) \text{that} \]
\[ \|w\|_{L^p(G_T, \mathbb{R}^{n+1})} \leq T\|\partial_t w\|_{L^p(G_T)} \leq C_p T^{1 + \frac{1}{p}} (M_0^{m+1 - \frac{1}{p}} + T^m) \|g\|_{L^p(\mathbb{R}^{n+1})}. \quad (2.79) \]
\[ \text{In addition, it follows from} w(0, x) = 0 \text{and (2.73) that} \]
\[ \sum_{j=1}^n \|\partial_j w\|_{L^p(G_T)} \leq C_{p,T} \|g\|_{L^p(\mathbb{R}^{n+1})}. \quad (2.81) \]
\[ \text{Combining (2.71)-(2.73) and (2.79)-(2.81) yield (2.70).} \]

3 Weighted $W^{2,p}$ estimates for the generalized Tricomi equations

In this section, for the later requirements of solving Theorem 1.1 in the degenerate elliptic region, we will establish the weighted $W^{2,p}$ estimates of the solution to the following problem
\[ \left\{ \begin{array}{l}
\partial_t^2 w + t^m \Delta_x w = t^\nu g(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^n, \\
w(0, x) = 0,
\end{array} \right. \quad (3.1) \]
where $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, and $0 \leq \nu \leq m$. Here we point out that showing the weighted $W^{2,p}$ estimates for (3.1) also admits independent interests in the study on the linear degenerate elliptic equations.

**Theorem 3.1.** Let $g \in L^p(\mathbb{R}^{n+1})$ with $1 < p < \infty$ and $\text{supp}(g) \subset \{(t, x) : 0 \leq t \leq M_0\}$ for some $M_0 > 0$, then for any fixed $T > 0$, there exists a generic constant $C_{p,T} > 0$ such that
\[ \|\partial_t^2 w\|_{L^p(G_T)} + \|t^{m-\nu} \Delta_x w\|_{L^p(G_T)} + \sum_{j=1}^n \|t^{\frac{m-\nu}{2}} \partial_{x_j}^2 w\|_{L^p(G_T)} \\
+ \|\partial_t w\|_{L^p(G_T)} + \sum_{j=1}^n \|t^{\frac{m-\nu}{2}} \partial_{x_j} w\|_{L^p(G_T)} + \|w\|_{L^p(G_T)} \leq C_{p,T} \|g\|_{L_p(\mathbb{R}^{n+1})}, \quad (3.2) \]
where $G_T = [0, T] \times \mathbb{R}^n$. In addition, we have further that $w, \partial_t w \in C([0, T], L^p(\mathbb{R}^n))$ and
\[ \|w(t, \cdot)\|_{L^p(\mathbb{R}^n)} + \|\partial_t w(t, \cdot)\|_{L^p(\mathbb{R}^n)} \leq C_{p,T} \|g\|_{L_p(\mathbb{R}^{n+1})}. \quad (3.3) \]
Moreover, if $1 < p < Q_\nu \equiv 1 + n \left( \frac{1}{2} - \frac{1}{\nu} \right)$, then $w \in C([0, T], L^p(\mathbb{R}^n))$, $\partial_t w \in L^p(G_T)$ with $\frac{1}{\overline{p}} = \frac{1}{p} - \frac{1}{Q_\nu}$, and
\[ \|w(t, \cdot)\|_{L^p(\mathbb{R}^n)} + \|\partial_t w\|_{L^p(G_T)} \leq C_{p,T} \|g\|_{L_p(\mathbb{R}^{n+1})}; \quad (3.4) \]
if \( p > Q_\nu \), then \( w \in C([0, T], L^\infty(\mathbb{R}^n)) \), \( \partial_t w \in L^\infty(G_T) \), and

\[
\|w(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|\partial_t w\|_{L^\infty(G_T)} \leq C_T \|g\|_{L^p(\mathbb{R}^n)}^{1/2};
\]

(3.5)

and if \( p = Q_\nu \), then \( w \in C([0, T], L^q(\mathbb{R}^n)) \), \( \partial_t w \in L^q(G_T) \) for any \( 1 < q < \infty \), and

\[
\|w(t, \cdot)\|_{L^q(\mathbb{R}^n)} + \|\partial_t w\|_{L^q(G_T)} \leq C_{q,T} \|g\|_{L^p(\mathbb{R}^n)}^{1/2};
\]

(3.6)

**Remark 3.1.** When \( \nu = 0 \) in (3.1), the weighted \( W^{2,p} \) or \( W^{1,p} \) estimates have been established under variable assumptions on the function \( g(t, x) \) and by different methods in [11], [21], and [24].

**Remark 3.2.** By [17] and the references therein, \( Q_\nu = 1 + (\frac{m-\nu}{2} + 1)n \) stands for the homogeneous dimension with respect to the operator \( \partial_t^2 + t^{m-\nu}\Delta \) for \( t \geq 0 \) and \( x \in \mathbb{R}^n \), and then it follows from the “Sobolev’s imbedding theorem” and (3.2) that (3.4)-(3.6) hold.

**Proof of Theorem 3.1.** Since \( w \) is a solution to (3.1), then we have from (2.14)-(2.15) that

\[
\hat{w}(t, \xi) = \int_0^{+\infty} \hat{T}(t, \sigma, \xi) \sigma^n \hat{g}(\sigma, \xi) d\sigma,
\]

where \( \hat{T}(t, \sigma, \xi) = \int_0^{\min(t,\sigma)} \frac{\lambda(ts)\lambda(\sigma s)}{\lambda^2(y s)} dy \) and \( s = |\xi|^{2/m+2} \). Thus,

\[
w(t, x) = \int_0^{+\infty} T(t, \sigma, \xi) * (\sigma^n g(\sigma, \cdot)) d\sigma,
\]

(3.7)

where \( T(t, \sigma, x) = \mathcal{F}^{-1}_\xi \hat{T}(t, \sigma, \xi) \). To show Theorem 3.1, as in Theorem 2.10, at first, we require to prove

\[
\|t^{m-\nu}\Delta w\|_{L^p(G_T)} \leq C \|g\|_{L^p},
\]

(3.8)

It is noted that

\[
t^{m-\nu}\Delta w = \mathcal{F}^{-1}_\xi (t^{m-\nu}|\xi|^2 \hat{w}(t, \xi)) = \mathcal{F}^{-1}_\xi \left( \int_0^{+\infty} \hat{K}(t, \sigma, \xi) \hat{g}(\sigma, \xi) d\sigma \right),
\]

where \( \hat{K}(t, \sigma, \xi) = |\xi|^2 t^{m-\nu} \sigma^n \hat{T}(t, \sigma, \xi) \). For notational convenience, we set \( \hat{K}_1(t, \sigma, \xi) = |\xi|^2 t^{m-\nu} \sigma^n \hat{T}(t, \sigma, \xi) \) and \( \hat{K}_2(t, \sigma, \xi) = |\xi|^2 t^{m} \hat{T}(t, \sigma, \xi) \).

With respect to \( \hat{K}_1(t, \sigma, \xi) \), its some crucial properties have been established in Lemma 2.5-Lemma 2.9 of §2. For \( \hat{K}_2(t, \sigma, \xi) \), the authors in [8] have shown that it satisfies (2.17)-(2.20) and (2.21)-(2.22) with the quantities

\[
D_1(a, b, r) = P_3(ar^{2/(m+2)}, br^{2/(m+2)}), \quad D_2(a, b, r) = P_4(ar^{2/(m+2)}, br^{2/(m+2)}),
\]

where

\[
P_3(a, b) = \begin{cases} 
  a & \text{if } a \leq 3, \\
  \exp(-Ca^{n/2}b) & \text{if } a > 3,
\end{cases}
\]

(3.9)
\[ P_4(a, b) = \begin{cases} a & \text{if } a \leq 3, \\ a^{m/2}b + a^m b^2 + ab^{2/m} & \text{if } a > 3. \end{cases} \] (3.10)

We now prove that \( \hat{K}(t, \sigma, \xi) \) satisfies (2.17)-(2.20) and (2.21)-(2.22) with suitable quantities \( D_1(a, b, r) \) and \( D_2(a, b, r) \). Notice that

\[
y^{m-\nu} \sigma^\nu \leq \begin{cases} y^m & \text{if } \sigma \leq y, \\ \sigma^m & \text{if } y \leq \sigma, \end{cases}
\]

then one has \( 0 \leq \hat{K}(t, \sigma, \xi) \leq \hat{K}_1(t, \sigma, \xi) + \hat{K}_2(t, \sigma, \xi) \). From this, we have

\[
\sup_{t, \xi} \int_0^{+\infty} |\hat{K}(t, \sigma, \xi)|d\sigma \leq \sup_{t, \xi} \int_0^{+\infty} |\hat{K}_1(t, \sigma, \xi)|d\sigma + \sup_{t, \xi} \int_0^{+\infty} |\hat{K}_2(t, \sigma, \xi)|d\sigma \leq C, \tag{3.11}
\]

\[
\sup_{\sigma, \xi} \int_0^{+\infty} |\hat{K}(t, \sigma, \xi)|d\sigma \leq \sup_{\sigma, \xi} \int_0^{+\infty} |\hat{K}_1(t, \sigma, \xi)|d\sigma + \sup_{\sigma, \xi} \int_0^{+\infty} |\hat{K}_2(t, \sigma, \xi)|d\sigma \leq C, \tag{3.12}
\]

\[
\int_0^{+\infty} \left| \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma)d\sigma \right| dt \leq \int_0^{+\infty} dt \int_0^{+\infty} \left( \hat{K}_1(t, \sigma, \xi) + \hat{K}_2(t, \sigma, \xi) \right) |h(\sigma)|d\sigma \\
\leq C \int_0^{+\infty} |h(\sigma)|d\sigma \tag{3.13}
\]

and

\[
\int_{C(\Delta(a, 2^{v+1}b))} \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma)d\sigma dt \\
\leq \int_{C(\Delta(a, 2^{v+1}b))} dt \int_0^{+\infty} \left( \hat{K}_1(t, \sigma, \xi) + \hat{K}_2(t, \sigma, \xi) \right) |h(\sigma)|d\sigma \\
\leq C \left( P_1(as, bs) + P_3(as, bs) \right) \int_0^{+\infty} |h(\sigma)|d\sigma, \tag{3.14}
\]

which means that \( \hat{K} \) satisfies the estimates (2.18)-(2.19), and (2.20)-(2.21) with \( \alpha = 0 \).

Next we verify the estimate (2.22) of \( \hat{K} \) when \( |\alpha| = 0 \). This procedure will be divided into the following three parts. From now on, we assume that the function \( h \in C_0^\infty(\mathbb{R}^+) \) with \( \int h(\sigma)d\sigma = 0 \) is supported in \( \Delta(a, b) \).

**Part 1.** \( \int_{a+b}^{+\infty} \left| \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma)d\sigma \right| dt \leq CP_5(as, bs) \int_0^{+\infty} |h(\sigma)|d\sigma \) holds for suitable function \( P_5(a, b) \)

It follows from a direct computation that

\[
\int_{a+b}^{+\infty} \left| \int_0^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma)d\sigma \right| dt
\]
Next we treat all the $I_i$ ($1 \leq i \leq 3$) in the distinct two cases.

**Case 1.** $as \geq 1$

In this case, we arrive at

$$I_1 \leq |X((a + b)s)| \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_0^{(a+b)s} \frac{\lambda(\sigma s) - \lambda((a + b)s)}{\lambda^2(y)} dy \quad \text{(by } t^m\lambda(t) = \lambda''(t))$$

$$\leq Cbs|X((a + b)s)| \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_0^{(a+b)s} -\frac{\lambda'(y)}{\lambda^2(y)} dy \quad \text{(by (2.1))}$$

$$= Cbs \frac{|X((a + b)s)|}{\lambda((a + b)s)} (1 - \lambda((a + b)s)) \int_{a-b}^{a+b} |h(\sigma)| d\sigma$$

$$\leq Cbs(as)^{\frac{m}{2}} \int_0^{+\infty} |h(\sigma)| d\sigma \quad (3.16)$$

and

$$I_2 \leq Cbs((a + b)s)^{\nu - 1} \lambda((a + b)s) \int_{a-b}^{a+b} |h(\sigma)| \frac{\sigma s}{\lambda^2(\sigma s)} d\sigma \int_{(a+b)s}^{+\infty} t^{m-\nu} \lambda(t) dt$$

$$\leq Cbs \int_{a-b}^{a+b} |h(\sigma)| \frac{\lambda((a + b)s)}{\lambda^2(\sigma s)} d\sigma \int_{(a+b)s}^{+\infty} t^{m-\nu} \lambda(t) dt$$

$$= Cbs \int_{a-b}^{a+b} |h(\sigma)| \frac{|\lambda'((a + b)s)|}{\lambda^2(\sigma s)} d\sigma$$

$$\leq Cbs \int_{a-b}^{a+b} |h(\sigma)| \frac{|\lambda'((a + b)s)| \lambda((a + b)s)}{\lambda^2((a + b)s)} d\sigma$$

$$\leq Cbs(as)^{\frac{m}{2}} \int_0^{+\infty} |h(\sigma)| d\sigma \quad (3.17)$$
and

\[ I_3 \leq C bs \frac{bs}{\lambda((a + b)s)} \int_{(a+b)s}^{+\infty} t^m \lambda(t) dt \int_0^{+\infty} |h(\sigma)| d\sigma \quad \text{(by (2.1))} \]

\[ = Cbs \frac{|\lambda'((a + b)s)|}{\lambda((a + b)s)} \int_0^{+\infty} |h(\sigma)| d\sigma \]

\[ \leq Cbs(as)^{\frac{m}{2}} \int_0^{+\infty} |h(\sigma)| d\sigma. \quad \text{(by (2.2))} \quad (3.18) \]

**Case 2.** \( as \leq 1 \)

Denote by

\[ H = \int_{a-b}^{a+b} |h(\sigma)| \lambda(\sigma s) d\sigma \int_0^{\sigma s} dy \lambda^2(y) \int_y^{+\infty} t^m \lambda(t) dt. \]

Then a direct computation yields

\[ H = \int_{a-b}^{a+b} |h(\sigma)| \lambda(\sigma s) d\sigma \int_0^{\sigma s} dy \frac{-\lambda'(y)}{\lambda^2(y)} \]

\[ = \int_{a-b}^{a+b} |h(\sigma)| \left( \lambda(0) - \lambda(\sigma s) \right) d\sigma \]

\[ \leq (a + b)s |h'(0)| \int_0^{+\infty} |h(\sigma)| d\sigma \quad \text{(by (2.1))} \]

\[ \leq Cas \int_0^{+\infty} |h(\sigma)| d\sigma. \]

Thus,

\[ I_1 \leq H \leq Cas \int_0^{+\infty} |h(\sigma)| d\sigma, \quad I_2 \leq H \leq Cas \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.19) \]

In addition,

\[ I_3 \leq \int_{a-b}^{a+b} |h(\sigma)| \lambda((a + b)s) d\sigma \int_0^{(a+b)s} dy \lambda^2(y) \int_y^{+\infty} t^m \lambda(t) dt \]

\[ = \int_{a-b}^{a+b} |h(\sigma)| \lambda(a + b)s d\sigma \int_0^{(a+b)s} dy \frac{-\lambda'(y)}{\lambda^2(y)} \]

\[ \leq Cas \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.20) \]

Substituting (3.16)-(3.20) into (3.15) yields

\[ \int_{a+b}^{+\infty} \left| \int_0^{+\infty} \hat{K}(t, \sigma) h(\sigma) d\sigma \right| dt \leq CP_5(as, bs) \int_0^{+\infty} |h(\sigma)| d\sigma, \quad (3.21) \]

where

\[ P_5(a, b) = \begin{cases} a & \text{if } a \leq 1 \\ ba^{m/2} & \text{if } a \geq 1 \end{cases} \quad (3.22) \]
Part 2. \( \int_{0}^{a-b} \left| \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C b s \alpha^{m/2} \int_{0}^{+\infty} |h(\sigma)| d\sigma \) holds

It follows from a direct computation that

\[
\int_{0}^{a-b} \left| \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt
= \int_{0}^{a-b} t^{m-\nu} \lambda(ts) dt \left| \int_{a-b}^{a+b} h(\sigma) \left( \sigma^{\nu} \int_{0}^{t} \frac{\lambda(\sigma s)}{\lambda^2(y)} dy - (a + b)\nu \int_{0}^{t} \frac{\lambda((a + b)s)}{\lambda^2(y)} dy \right) d\sigma \right|

\leq C b s \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s))| d\sigma \int_{a-b}^{a+b} t^{m-\nu} \lambda(ts) dt \int_{0}^{t} \frac{dy}{\lambda^2(y)}

\leq C b s \int_{a-b}^{a+b} |h(\sigma)| |(\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s))| d\sigma \int_{a-b}^{a+b} t^{m-\nu} \lambda(ts) dt \int_{0}^{t} \frac{dy}{\lambda^2(y)}

\leq C b s \int_{0}^{+\infty} |h(\sigma)| d\sigma. \quad (\text{as in (2.38)})

(3.23)

Part 3. \( \int_{a-b}^{a+b} \left| \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt \leq C (b s)^2 \alpha^m \int_{0}^{+\infty} |h(\sigma)| d\sigma \) holds

We have

\[
\int_{a-b}^{a+b} \left| \int_{0}^{+\infty} \hat{K}(t, \sigma, \xi) h(\sigma) d\sigma \right| dt
= \int_{a-b}^{a+b} \left| \int_{a-b}^{a+b} h(\sigma) |(\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s))| d\sigma \int_{0}^{t} \frac{\lambda(ts)\lambda((a + b)s)}{\lambda^2(y)} dy - (a + b)\nu \int_{0}^{t} \frac{\lambda(ts)\lambda((a + b)s)}{\lambda^2(y)} dy \right| d\sigma

\leq C b s \int_{a-b}^{a+b} \left| \int_{a-b}^{a+b} h(t) \left| (\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s)) \right| d\sigma \int_{a-b}^{a+b} \left| \int_{0}^{t} \frac{\lambda(ts)\lambda((a + b)s)}{\lambda^2(y)} dy \right| dt

\leq C b s \int_{a-b}^{a+b} \left| \int_{a-b}^{a+b} h(t) \left| (\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s)) \right| d\sigma \int_{a-b}^{a+b} \left| \int_{0}^{t} \frac{\lambda(ts)\lambda((a + b)s)}{\lambda^2(y)} dy \right| dt

\leq C b s \int_{a-b}^{a+b} |h(t)| |(\sigma s)^m (\lambda(\sigma s) - \lambda((a + b)s))| d\sigma \int_{a-b}^{a+b} \left| \int_{0}^{t} \frac{\lambda(ts)\lambda((a + b)s)}{\lambda^2(y)} dy \right| dt

(3.23)
\[ III_1 + III_2 + V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7, \]  
where \( III_1 \) and \( III_2 \) have been defined in (2.53), whose estimates have been established in (2.54) and (2.62) respectively. Next we treat each \( V_i \) \( (1 \leq i \leq 7) \). For the term \( V_1 \), we have

\[ V_1 \leq \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} t^m \lambda(t)dt \int_0^{(a-b)s} \frac{\lambda(\sigma s) - \lambda((a+b)s)}{\lambda^2(y)} dy \]

\[ \leq Cbs \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} t^m \lambda(t)dt \int_0^{(a-b)s} \frac{-\lambda'(y)}{\lambda^2(y)} dy \]

\[ \leq C(bs)^2 ((a+b)s)^m \lambda((a-b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( \lambda(\sigma s) - \frac{1}{\lambda((a-b)s)} \right) \quad \text{(by (2.1))} \]

\[ \leq C(bs)^2 (as)^m \int_0^{+\infty} |h(\sigma)|d\sigma. \]  

For the term \( V_2 \),

\[ V_2 = \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{(a-b)s}^{(a+b)s} t^m \lambda(t)dt \int_{(a-b)s}^{\sigma s} \frac{\lambda(\sigma s) - \lambda((a+b)s)}{\lambda^2(y)} dy \]

\[ \leq C(bs)^2 ((a+b)s)^m \int_{a-b}^{a+b} |h(\sigma)|d\sigma \lambda(\sigma s) \int_{(a-b)s}^{\sigma s} \frac{-\lambda'(y)}{\lambda^2(y)} dy \]

\[ = C(bs)^2 ((a+b)s)^m \int_{a-b}^{a+b} |h(\sigma)|d\sigma \lambda(\sigma s) \left( \frac{1}{\lambda(\sigma s)} - \frac{1}{\lambda((a-b)s)} \right) d\sigma \]

\[ \leq C(bs)^2 (as)^m \int_0^{+\infty} |h(\sigma)|d\sigma. \]  

For the term \( V_3 \),

\[ V_3 \leq Cbs ((a+b)s)^{\nu-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \left( bs \ (\sigma s)^{m-\nu} \lambda((a-b)s) \right) \left( \frac{(a-b)s}{\lambda^2((a-b)s)} \right) \]

\[ \leq C(bs)^2 ((a+b)s)^m \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \lambda((a-b)s) \]

\[ \leq C(bs)^2 (as)^m \int_0^{+\infty} |h(\sigma)|d\sigma. \]  

For the term \( V_4 \),

\[ V_4 \leq Cbs ((a+b)s)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{(a-b)s}^{\sigma s} \lambda(t)dt \int_{(a-b)s}^{\sigma s} \frac{dy}{\lambda^2(y)} \]

\[ \leq Cbs ((a+b)s)^{m-1} \lambda((a+b)s) \int_{a-b}^{a+b} |h(\sigma)|d\sigma \int_{(a-b)s}^{\sigma s} \frac{bs}{\lambda(t)} dt \]

\[ \leq C(bs)^2 ((a+b)s)^{m-1} \int_{a-b}^{a+b} \sigma s |h(\sigma)| \frac{\lambda((a+b)s)}{\lambda(\sigma s)} d\sigma \]
Finally, for the term $V_5$,

$$V_5 \leq C(bs)^2((a + b)s)^m \int_{a-b}^{a+b} |h(\sigma)| \frac{\lambda((a + b)s)\lambda(\sigma s)}{\lambda^2((a - b)s)} d\sigma \quad \text{(by (2.1))}$$

$$\leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.29)$$

For the term $V_6$,

$$V_6 \leq C(bs)^2((a + b)s)^{m-1} \int_{a-b}^{a+b} |h(\sigma)| \frac{\sigma s}{\lambda(\sigma s)} d\sigma \quad \text{(by (2.1))}$$

$$\leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.30)$$

Finally, for the term $V_7$,

$$V_7 \leq Cbs\lambda((a + b)s) \int_{a-b}^{a+b} |h(\sigma)| d\sigma \int_{\sigma s}^{(a+b)s} \frac{t^m}{\lambda(t)} dt$$

$$\leq C(bs)^2((a + b)s)^m \int_0^{+\infty} |h(\sigma)| d\sigma$$

$$\leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.31)$$

Collecting (3.24)-(3.31) yields

$$\int_{a-b}^{a+b} \int_0^{+\infty} \tilde{K}(t, \sigma, \xi) h(\sigma) d\sigma dt \leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma.$$

Thus, by Part 1-Part 3, we arrive at

$$\int_0^{+\infty} \int_0^{+\infty} \tilde{K}(t, \sigma, \xi) h(\sigma) d\sigma dt \leq CP_6(as, bs) \int_0^{+\infty} |h(\sigma)| d\sigma, \quad (3.32)$$

where

$$P_6(a, b) = P_5(a, b) + ba^{m/2} + b^2 a^m.$$ 

Therefore, $\tilde{K}$ satisfies the estimate (2.12) for $|\alpha| = 0$. Applying the similar arguments as done in Lemma 2.8, we can obtain that $\tilde{K}$ satisfies estimates (2.20)-(2.22) for any $|\alpha| \geq 0$ by induction method.

In addition, it is easy to know that (2.23) holds for suitably chosen $a_k, b_k$ and $\tilde{b}_k$ with $b_k \sim a_k^{m/2} b_k$ when we set $D_1(a, b, r) = P_1(ar^{2/(m+2)}, br^{2/(m+2)}) + P_3(ar^{2/(m+2)}, br^{2/(m+2)})$ and $D_2(a, b, r) = P_6(ar^{2/(m+2)}, br^{2/(m+2)})$. 

$$\leq C(bs)^2(as)^m \int_0^{+\infty} |h(\sigma)| d\sigma. \quad (3.28)$$
Hence, applying Lemma 2.4 we could get
\[ \| t^{m-\nu} \Delta w \|_{L^p(\mathbb{R}^n+1)} \leq C_p \| g \|_{L^p(\mathbb{R}^n+1)}, \] (3.33)
and then as arguing in Theorem 2.10, we have
\[ \| \partial_t^2 w \|_{L^p(\mathcal{G}_T)} \leq C_p T^{\nu} \| g \|_{L^p(\mathbb{R}^n+1)}, \]
\[ \sum_{j=1}^{n} \| t^{m-\nu} \partial_{x_j}^2 w \|_{L^p(\mathcal{G}_T)} \leq C_{p,T} \| g \|_{L^p}. \] (3.34)
Note that from (3.3)
\[ \hat{w}(t, \xi) = \lambda(t) \int_0^t dy \frac{\lambda^2(y)}{\lambda^2(s)} \int_{y}^{\infty} \lambda(s) \sigma^{\nu} \hat{g}(\sigma, \xi) d\sigma, \]
which derives
\[ \partial_t \hat{w}(0, \xi) = \int_0^{\infty} \lambda(s) \sigma^{\nu} \hat{g}(\sigma, \xi) d\sigma = \int_0^{\infty} \lambda(s) \sigma^{\nu} \hat{g}(\sigma, \xi) d\sigma. \]
Thus,
\[ \partial_t w(0, x) = \int_0^{\infty} \mathcal{F}^{-1}_{\xi}(\lambda(s)) * (\sigma^{\nu} g(\sigma, \cdot)) d\sigma, \]
and \[ \| \partial_t w(0, x) \|_{L^p(\mathbb{R}^n)} \leq C_p M_0^{\nu+1-\frac{1}{p}} \| g \|_{L^p(\mathbb{R}^n+1)} \] holds as in the proof of Theorem 2.10. Together with (3.34), this yields for \( 0 \leq t \leq T \)
\[ \| \partial_t w \|_{L^p(\mathcal{G}_T)} \leq C_p T^{\nu} (M_0^{\nu+1-\frac{1}{p}} + T^{\nu}) \| g \|_{L^p(\mathbb{R}^n+1)} \] (3.35)
and
\[ \| \partial_t w(t, \cdot) \|_{L^p(\mathbb{R}^n)} \leq \| \partial_t w(0, \cdot) \|_{L^p(\mathbb{R}^n)} + \int_0^t \| \partial_t^2 w(\tau, \cdot) \|_{L^p(\mathbb{R}^n)} d\tau \]
\[ \leq C_p M_0^{\nu+1-\frac{1}{p}} \| g \|_{L^p(\mathbb{R}^n+1)} + T^{1-\frac{1}{p}} \| \partial_t^2 w(\tau, x) \|_{L^p(\mathcal{G}_T)} \]
\[ \leq C_p (M_0^{\nu+1-\frac{1}{p}} + T^{\nu+1-\frac{1}{p}}) \| g \|_{L^p(\mathbb{R}^n+1)}. \] (3.36)
From (3.35)-(3.36) and \( w(0, x) = 0 \), one has
\[ \begin{cases} \| w(t, \cdot) \|_{L^p(\mathbb{R}^n)} \leq C_p T (M_0^{\nu+1-\frac{1}{p}} + T^{\nu+1-\frac{1}{p}}) \| g \|_{L^p(\mathbb{R}^n+1)}, \quad \text{(3.37)} \\
\| w \|_{L^p(\mathcal{G}_T)} \leq C_p T^{1+\frac{1}{p}} (M_0^{\nu+1-\frac{1}{p}} + T^{\nu}) \| g \|_{L^p(\mathbb{R}^n+1)}. \end{cases} \]
In addition, it follows from \( w(0, x) = 0 \), (3.34) and Hardy's inequality that
\[ \sum_{j=1}^{n} \| t^{m-\nu} \partial_j w \|_{L^p(\mathcal{G}_T)} \leq C_{p,T} \| g \|_{L^p(\mathbb{R}^n+1)}. \] (3.38)
Combining (3.33)-(3.38) yields (3.2)-(3.3). In addition, (3.4)-(3.6) can be derived as indicated in Remark 3.2.
4 Solvability of (1.1) in the degenerate elliptic region \( \{ t \leq 0 \} \)

In this section, we will establish the solvability and regularity of problem (1.1) in the degenerate elliptic region \( \{ t \leq 0 \} \) by applying the weighted \( W^{2,p} \) estimates given in Theorem 3.1. Our main result is:

**Theorem 4.1.** Under the assumptions (1.2)-(1.3), there exists a constant \( T_0 > 0 \) such that when \( 0 \leq \mu \leq 1 \), or when \( 1 < \mu < p_0 \) with \( Q_0 \leq \frac{p_0}{\mu - 1} \), the following degenerate elliptic equation

\[
\begin{aligned}
\partial_t^2 u - t^{2l-1} \Delta u &= f(t, x, u), \\
\qquad (t, x) &\in (-\infty, 0] \times \mathbb{R}^n,
\end{aligned}
\]

\( u(0, x) = \varphi(x) \in H^s(\mathbb{R}^n), \) \( (4.1) \)

has a unique local solution \( u \) satisfying

\[
\begin{aligned}
u(t, x) &\in C\left([-T_0, 0], L^{p_0}(\mathbb{R}^n)\right), \\
\partial_t u(t, x) &\in C\left([-T_0, 0], L^{p_1}(\mathbb{R}^n)\right),
\end{aligned}
\]

\( (4.2) \)

where \( 0 \leq s < \frac{n}{2} \), \( p_0 = \frac{2n}{n - 2s} \), and \( \frac{1}{p_1} = \frac{1}{2} - \frac{1}{n} \left( s - \frac{2}{2l + 1} \right) \).

**Proof.** Without loss of generality, we assume that \( \mu \neq 0 \). Let \( \bar{u}(t, x) \) be a solution to the following linear problem

\[
\begin{aligned}
\partial_t^2 \bar{u} - t^{2l-1} \Delta \bar{u} &= 0, \\
\qquad (t, x) &\in (-\infty, 0] \times \mathbb{R}^n, \\
\bar{u}(0, x) &= \varphi(x) \in H^s(\mathbb{R}^n), \\n\end{aligned}
\]

Then from (2.11), we have

\[
\bar{u}(t, x) \in C((-\infty, 0], H^s(\mathbb{R}^n)) \cap C^1((-\infty, 0], H^{s - \frac{2}{2l + 1}}(\mathbb{R}^n)),
\]

which, together with Sobolev’s embedding theorem, yields

\[
\bar{u}(t, x) \in C((-\infty, 0], L^{p_0}(\mathbb{R}^n)) \cap C^1((-\infty, 0], L^{p_1}(\mathbb{R}^n)).
\]

\( (4.3) \)

Choosing a smooth function \( \chi(t) \) such that \( \chi(t) \equiv 1 \) for \( t \geq -1 \) and \( \chi(t) \equiv 0 \) for \( t \leq -2 \). For suitably small fixed constant \( T_0 > 0 \), we set \( \chi_{T_0}(t) = \chi(\frac{t}{T_0}). \) Let \( \bar{u}_{T_0} \) be a solution to the linear equation

\[
\begin{aligned}
\partial_t^2 \bar{u}_{T_0} - t^{2l-1} \Delta \bar{u}_{T_0} &= \chi_{T_0}(t) f(t, x, \bar{u}), \\
\qquad (t, x) &\in (-\infty, 0] \times \mathbb{R}^n, \\
\bar{u}_{T_0}(0, x) &= 0.
\end{aligned}
\]

\( (4.4) \)

Due to the compact support property of \( f(t, x, \bar{u}) \) on the variable \( x \) and the regularity of \( \bar{u}(t, x) \) in (4.3), one has from (1.2) that \( \chi_{T_0}(t) f(t, x, \bar{u}) \in L^p(\mathbb{R}^{n+1}) \), here \( 1 < p \leq \frac{p_0}{\mu} \) and \( \mathbb{R}^{n+1} = \{(t, x) : t \leq 0, x \in \mathbb{R}^n\} \). Next we derive the regularity of \( \bar{u}_{T_0} \) by applying the weighted \( W^{2,p} \)
estimates in Theorem 3.1. Its regularity will depend on the relationship between \( \frac{p_0}{\mu} \) and \( Q_0 \). Precisely,

**Case A.** \( 1 < \frac{p_0}{\mu} < Q_0 \)

We have from (3.3) and (3.4) that for any \( 1 < p \leq \frac{p_0}{\mu} \)

\[
\bar{u}_{T_0} \in C\left([-T_0, 0], L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right), \quad \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^p(\mathbb{R}^n)\right) \cap L^2(G_{T_0}),
\]  
(4.5)

where \( \frac{1}{p_2} = \frac{1}{p_0} - \frac{1}{Q_0} \).

**Case B.** \( \frac{p_0}{\mu} > Q_0 \)

We have from (3.3) and (3.5) that for any \( 1 < p \leq \frac{p_0}{\mu} \)

\[
\bar{u}_{T_0} \in C\left([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\right), \quad \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^p(\mathbb{R}^n)\right) \cap L^\infty(G_{T_0}),
\]

which means that for any \( 1 < q < \infty \)

\[
\bar{u}_{T_0} \in C\left([-T_0, 0], L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\right), \quad \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^q(\mathbb{R}^n)\right) \cap L^\infty(G_{T_0}).
\]  
(4.6)

**Case C.** \( \frac{p_0}{\mu} = Q_0 \)

We have from (3.3) and (3.6) that for any \( 1 < p < \infty \)

\[
\bar{u}_{T_0}, \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^p(\mathbb{R}^n)\right),
\]

especially,

\[
\bar{u}_{T_0} \in C\left([-T_0, 0], L^{po}(\mathbb{R}^n)\right), \quad \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^{po}(\mathbb{R}^n)\right).
\]  
(4.7)

Notice that for \( 1 < \mu < p_0 \), if \( Q_0 \leq \frac{p_0}{\mu - 1} \), there hold \( p_2 \geq p_0 \) and \( \frac{p_0}{\mu} > p_1 \); however, if \( Q_0 > \frac{p_0}{\mu - 1} \), then \( \frac{p_0}{\mu} < p_2 < p_0 \) and thus we cannot get the \( L^{po} \) (only \( L^{p_2} \)) regularity of \( \bar{u}_{T_0} \) with respect to the variable \( x \) from (4.5), which will lead to the difficulty in closing the \( L^{po} \) regularity argument on the solution \( u \) to the nonlinear problem (4.1). Consequently, we will restrict our consideration on the cases for \( 0 \leq \mu \leq 1 \), or for \( 1 < \mu < p_0 \) with \( Q_0 \leq \frac{p_0}{\mu - 1} \).

Collecting (4.5)-(4.7) yields that when \( 0 \leq \mu \leq 1 \), or when \( 1 < \mu < p_0 \) with \( Q_0 \leq \frac{p_0}{\mu - 1} \),

\[
\bar{u}_{T_0} \in C\left([-T_0, 0], L^{po}(\mathbb{R}^n)\right), \quad \partial_t \bar{u}_{T_0} \in C\left([-T_0, 0], L^{po}(\mathbb{R}^n)\right).
\]  
(4.8)

In order to solve (4.1) in \( \{t \leq 0\} \), we only require to consider the following problem

\[
\begin{aligned}
&\partial_t^2 v_{T_0} + t^{2l - 1} \Delta v_{T_0} = \chi_{T_0}(t) \left(f(t, x, \bar{u} + \bar{u}_{T_0} + v_{T_0}) - f(t, x, \bar{u})\right), \quad (t, x) \in (-\infty, 0] \times \mathbb{R}^n, \\
v_{T_0}(0, x) = 0.
\end{aligned}
\]  
(4.9)
Indeed, if (4.9) is solved, then \( u_{T_0} = \bar{u} + \bar{u}_{T_0} + v_{T_0} \) is a solution to the following problem

\[
\begin{aligned}
\partial_t^2 u_{T_0} - t^{2l-1} \Delta u_{T_0} &= \chi_{T_0}(t)f(t, x, u_{T_0}), \quad (t, x) \in (-\infty, 0) \times \mathbb{R}^n, \\
u_{T_0}(0, x) &= u_0(x),
\end{aligned}
\]

namely, we solve the problem (4.1) for \((t, x) \in [-T_0, 0] \times \mathbb{R}^n\).

We will use the following iteration scheme to study the problem (4.9)

\[
\begin{aligned}
\partial_t^2 v_{T_0}^{k+1} - t^{2l-1} \Delta v_{T_0}^{k+1} &= \chi_{T_0}(t)(f(t, x, \bar{u} + \bar{u}_{T_0} + v_{T_0}^k) - f(t, x, \bar{u})), \quad (t, x) \in (-\infty, 0) \times \mathbb{R}^n, \\
v_{T_0}^{k+1}(0, x) &= 0,
\end{aligned}
\]

where \( v_{T_0}^0(t, x) \equiv 0 \).

For \( k = 0 \), \( v_{T_0}^1 \) is the solution to the following problem

\[
\begin{aligned}
\partial_t^2 v_{T_0}^1 - t^{2l-1} \Delta v_{T_0}^1 &= \chi_{T_0}(t)(f(t, x, \bar{u} + \bar{u}_{T_0} - f(t, x, \bar{u})), \quad (t, x) \in (-\infty, 0) \times \mathbb{R}^n, \\
v_{T_0}^1(0, x) &= 0.
\end{aligned}
\]

Denote

\[ I_0 \equiv \frac{1}{t} \chi_{T_0}(t)(f(t, x, \bar{u} + \bar{u}_{T_0}) - f(t, x, \bar{u})). \]

Since \( \bar{u}_{T_0}(t, x) = \int_0^t \partial_t \bar{u}_{T_0}(\tau, x) d\tau \), we have from the condition (1.2) that

\[
|I_0| \leq \begin{cases} 
C \chi_{T_0}(t) \frac{1}{t} \int_0^t |\partial_\tau \bar{u}_{T_0}(\tau, x)| d\tau, & \text{when } 0 \leq \mu \leq 1, \\
C \chi_{T_0}(t)(1 + |\bar{u}|^{\mu-1} + |\bar{u}_{T_0}|^{\mu-1}) \frac{1}{t} \int_0^t |\partial_\tau \bar{u}_{T_0}(\tau, x)| d\tau, & \text{when } 1 < \mu < p_0.
\end{cases}
\]

Now we analyze the regularity of \( I_0 \). First consider the case \( 1 < \mu < p_0 \) with \( 1 < \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \). In this case, \( \bar{u} \in L^{p_0}(G_{T_0}), \bar{u}_{T_0}, \partial_t \bar{u}_{T_0} \in L^{p_2}(G_{T_0}) \), which together with Hölder’s inequality and Hardy’s inequality derives that

\[ I_0 \in L^{p_2}(\mathbb{R}_{-}^{n+1}) + L^{p_2/\mu}(\mathbb{R}_{-}^{n+1}) + L^1(\mathbb{R}_{-}^{n+1}), \]

where and below \( \frac{1}{r_1} = \frac{\mu - 1}{p_0} + \frac{1}{p_2} = \frac{2\mu - 1}{p_0} - \frac{1}{Q_0} \).

Since \( I_0 \) has compact support with respect to variables \( t \) and \( x \), and in this case, \( \min\{p_2, \frac{p_2}{\mu}, r_1\} = r_1 \), we have that for any \( 1 < \gamma \leq r_1 \),

\[ I_0 \in L^\gamma(\mathbb{R}_{-}^{n+1}), \quad \text{when } 1 < \mu < p_0 \quad \text{and } 1 < \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1}. \]
Similarly, we also have that

$$I_0 \in \begin{cases} L^p(\mathbb{R}^n_{+1}), & \text{for any } 1 < p \leq p_2, \text{ if } 0 < \mu \leq 1 \text{ and } 1 < \frac{p_0}{\mu} < Q_0, \\ L^q(\mathbb{R}^n_{+1}) \cap L^\infty(\mathbb{R}^n_{+1}), & \text{for any } 1 < q < \infty, \text{ if } 0 < \mu \leq 1 \text{ and } \frac{p_0}{\mu} > Q_0, \\ L^\delta(\mathbb{R}^n_{+1}), & \text{for any } 1 < \delta \leq \frac{p_0}{\mu - 1}, \text{ if } 1 < \mu < p_0 \text{ and } \frac{p_0}{\mu} > Q_0, \\ L^\eta(\mathbb{R}^n_{+1}), & \text{for any } 1 < \eta \leq \frac{p_0}{\mu}, \text{ if } p_0 = Q_0. \end{cases}$$

Based on (4.13)-(4.14), we then have from (4.11) and Theorem 3.1 with $m = 2l - 1$ and $\nu = 1$ that

(i) For $0 < \mu \leq 1$,

$$v^1_{T_0} \in \begin{cases} C([-T_0, 0], L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) & \text{for } 1 < q < \infty, \text{ if } \frac{p_0}{\mu} > Q_0, \text{ or if } \frac{p_0}{\mu} < Q_0 \text{ and } p_2 > Q_1, \\ C([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\theta(\mathbb{R}^n)) & \text{for } 1 < p \leq p_2 \text{ and } \frac{1}{\theta} \equiv \frac{1}{p_2} - \frac{1}{Q_1}, \text{ if } \frac{p_0}{\mu} < Q_0 \text{ and } p_2 < Q_1, \end{cases}$$

and

$$\partial_t v^1_{T_0} \in \begin{cases} C([-T_0, 0], L^q(\mathbb{R}^n) \cap L^\infty(G_{T_0})) & \text{for } 1 < q < \infty, \text{ if } \frac{p_0}{\mu} > Q_0, \text{ or if } \frac{p_0}{\mu} < Q_0 \text{ and } p_2 > Q_1, \\ C([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\theta(G_{T_0})) & \text{for } 1 < p \leq p_2, \text{ if } \frac{p_0}{\mu} < Q_0 \text{ and } 1 < p_2 < Q_1. \end{cases}$$

(ii) For $1 < \mu < p_0$,

$$v^1_{T_0} \in \begin{cases} C([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) & \text{for } 1 < p < \infty, \text{ if } \frac{p_0}{\mu} > Q_0, \text{ or if } \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \text{ and } r_1 > Q_1, \\ C([-T_0, 0], L^\gamma(\mathbb{R}^n) \cap L^\theta(\mathbb{R}^n)) & \text{for } 1 < \gamma \leq r_1, \text{ if } \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \text{ and } r_1 < Q_1, \end{cases}$$

and

$$\partial_t v^1_{T_0} \in \begin{cases} C([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\infty(G_{T_0})) & \text{for } 1 < p < \infty, \text{ if } \frac{p_0}{\mu} > Q_0, \text{ or if } 1 < \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1}, \text{ and } r_1 > Q_1, \\ C([-T_0, 0], L^\gamma(\mathbb{R}^n) \cap L^\theta(G_{T_0})) & \text{for } 1 < \gamma \leq r_1, \text{ if } 1 < \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \text{ and } 1 < r_1 < Q_1. \end{cases}$$
Moreover, when \( \frac{p_0}{\mu} = Q_0 \), or when \( 0 < \mu \leq 1 \) with \( \frac{p_0}{\mu} < Q_0 \) and \( p_2 = Q_1 \), or when \( 1 < \mu < p_0 \) with \( \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \) and \( r_1 = Q_1 \), we have that

\[
v_{T_0}^1, \partial_t v_{T_0}^1 \in C([-T_0, 0], L^q(\mathbb{R}^n)) \quad \text{for any} \ 1 < q < \infty.
\]

Therefore, collecting all the above analysis in (i)-(ii) yields

\[
v_{T_0}^1 \in C\left((-\infty, 0], L^{p_0}(\mathbb{R}^n)\right), \quad \partial_t v_{T_0}^1 \in C\left((-\infty, 0], L^{p_1}(\mathbb{R}^n)\right),
\]

when \( 0 < \mu \leq 1 \), or when \( 1 < \mu < p_0 \) and \( Q_0 \leq \frac{p_0}{\mu - 1} \).

To study the problem (4.10), we denote by

\[
I_k \equiv \frac{1}{t} \chi_{T_0}(t) \left(f(t, x, \bar{u} + \bar{u}_{T_0} + v_{T_0}^k) - f(t, x, \bar{u})\right).
\]

We now derive the regularities of \( I_k \) under different restrictions on \( \mu, l \) and \( p_0 \) in (1.2)-(1.3), whose classifications are based on the distinguished regularities of \( v_{T_0} \) and \( \partial_t v_{T_0} \).

Define the set \( \mathcal{M}_1 \)

\[
\mathcal{M}_1 = \{ g(t, x) : g(t, x) \in C([-T_0, 0], L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)), \partial_t g(t, x) \in C([-T_0, 0], L^q(\mathbb{R}^n) \cap L^\infty(G_{T_0})), ||g||_1 + ||\partial_t g||_1 \leq 2 \},
\]

for any \( 1 < q < \infty \),

where

\[
||g||_1 \equiv ||g||_{C([-T_0, 0], L^q(\mathbb{R}^n))} + ||\partial_t g||_{C([-T_0, 0], L^\infty(\mathbb{R}^n))}
\]

and

\[
||\partial_t g||_1 \equiv ||\partial_t g||_{C([-T_0, 0], L^q(\mathbb{R}^n))} + ||\partial_{t} g||_{L^\infty(G_{T_0})}.
\]

In particular, for \( g \in \mathcal{M}_1 \), one has \( g \in C([-T_0, 0], L^{p_0}(\mathbb{R}^n)) \cap C^{1}([-T_0, 0], L^{p_1}(\mathbb{R}^n)) \).

**Case 1.** \( 1 < \mu < p_0 \) and \( Q_0 < \frac{p_0}{\mu} \)

In this case, for \( v_{T_0}^k \in \mathcal{M}_1 \), we have from (1.2) that

\[
|I_k| \leq C x_{T_0}(t)(1 + |\bar{u}|^{\mu - 1} + |\bar{u}_{T_0}|^{\mu - 1} + |v_{T_0}^k|^{\mu - 1})(\frac{1}{t} \int_0^t \int_0^t \frac{1}{t} \int_0^t |\partial_{t} v_{T_0}^k(\tau, x)|d\tau|d\tau|d\tau|d\tau|)
\]

which, together with Hardy’s inequality and the regularities of \( \bar{u}, \bar{u}_{T_0}, v_{T_0}^k, \partial_t \bar{u}_{T_0} \) and \( \partial_t v_{T_0}^k \) in \( G_{T_0} \) (that is, \( \bar{u} \in L^{p_0}(G_{T_0}) \) and \( \bar{u}_{T_0}, \partial_t \bar{u}_{T_0}, v_{T_0}^k, \partial_t v_{T_0}^k \in L^\infty(G_{T_0}) \)), yields

\[
I_k \in L^\infty(\mathbb{R}_n^{n+1}) + L^{\frac{p_0}{\mu - 1}}(\mathbb{R}_n^{n+1})
\]

Since \( I_k \) has the compact support with respect to the variables \( t \) and \( x \), we have

\[
I_k \in L^{q}(\mathbb{R}^n) \quad \text{for any} \ 1 < q < \infty.
\]
Noting that in this case, \( \frac{p_0}{\mu - 1} > \frac{p_0}{\mu} > Q_0 > Q_1 \) hold, then we have from (4.10), (3.5) and (3.36)-(3.37) that \( v^{k+1}_{T_0} \in \mathcal{M}_1 \) for small \( T_0 > 0 \).

**Case 2.** \( 1 < \mu < p_0, \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \) and \( r_1 > Q_1 \)

In this case, for any \( v^k_{T_0} \in \mathcal{M}_1 \),
\[
\bar{u} \in L^{p_0}(G_{T_0}), \quad \bar{u}_{T_0} \in L^{p_2}(G_{T_0}), \quad \partial_t \bar{u}_{T_0} \in L^{p_2}(G_{T_0}), \quad v^k_{T_0} \in L^\infty(G_{T_0}), \quad \partial_t v^k_{T_0} \in L^\infty(G_{T_0}),
\]
and then as in Case 1, we have in \( G_{T_0} \),
\[
I_k \in L^{p_2} + L^\infty + L^{r_1} + L^{p_2/\mu} + L^{p_2/(\mu - 1)} + L^{p_0/(\mu - 1)}.
\]
Noting
\[
\min\{p_2, r_1, \frac{p_2}{\mu}, \frac{p_2}{\mu - 1}, \frac{p_0}{\mu - 1}\} = r_1
\]
and \( I_k \) has the compact support on the variables \( t \) and \( x \), then
\[
I_k \in L^\gamma(\mathbb{R}^{n+1}) \quad \text{for any } 1 < \gamma \leq r_1.
\]
Due to our assumption in this case, \( r_1 > Q_1 \) holds. Thus it follows from (3.36)-(3.37) and (3.5) in Theorem 3.1, and (4.10) that \( v^{k+1}_{T_0} \in \mathcal{M}_1 \) for small \( T_0 > 0 \).

**Case 3.** \( 0 < \mu \leq 1, Q_0 > \frac{p_0}{\mu} \) and \( p_2 > Q_1 \)

In this case, for \( v^k_{T_0} \in \mathcal{M}_1 \), the condition (1.2) implies that
\[
|I_k| \leq C\chi_{T_0}(t) \left( \frac{1}{t} \int_0^t \partial_t \bar{u}_{T_0}(\tau, x) d\tau \right) + \frac{1}{t} \int_0^t \partial_t v^k_{T_0}(\tau, x) d\tau
\]
which yields
\[
I_k \in L^\infty(\mathbb{R}^{n+1}) + L^{p_2}(\mathbb{R}^{n+1}) \subset L^{p_2}(\mathbb{R}^{n+1})
\]
and then
\[
I_k \in L^p(\mathbb{R}^{n+1}) \quad \text{for any } 1 < p \leq p_2.
\]
By \( p_2 > Q_1 \) from our assumption, then estimates (3.5), (3.36)-(3.37) in Theorem 3.1 and equation (4.10) that \( v^{m+1}_{T_0} \in \mathcal{M}_1 \) for small \( T_0 > 0 \).

**Case 4.** \( 0 < \mu \leq 1 \) and \( Q_0 < \frac{p_0}{\mu} \)

For \( v^k_{T_0} \in \mathcal{M}_1 \), we have
\[
I_k \in L^p(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1}) \quad \text{for any } 1 < p < \infty,
\]
thus, as in Case 3, \( v^{k+1}_{T_0} \in \mathcal{M}_1 \) for small \( T_0 > 0 \).

In order to get suitable regularities of \( I_k \) for some other left cases of \( \mu, l \) and \( p_0 \), we now define the second set \( \mathcal{M}_2 \) as follows
\[
\mathcal{M}_2 = \{g(t, x) : g \in C([-T_0, 0], L^p(\mathbb{R}^n) \cap L^\theta(\mathbb{R}^n)), \partial_t g \in C([-T_0, 0], L^p(\mathbb{R}^n)) \cap L^\theta(G_{T_0}),
\]
\[ \|g\|_2 + \|\partial_t g\|_2 \leq 2 \]

where
\[ \|g\|_2 = \|g\|_{C([-T_0,0],L^p(\mathbb{R}^n))} + \|g\|_{C([-T_0,0],L^p(\mathbb{R}^n))} \]

and
\[ \|\partial_t g\|_2 = \|\partial_t g\|_{C([-T_0,0],L^p(\mathbb{R}^n))} + \|\partial_t g\|_{L^p(G_{T_0})} \]

here \( 1 < p \leq p_2, \frac{1}{\theta} = \frac{1}{p_2} - \frac{1}{Q_1} \). We now analyze the regularity of \( I_k \) when \( v^k_{T_0} \in M_2 \).

**Case 5.** \( 0 < \mu \leq 1, Q_0 > \frac{p_0}{\mu} \) and \( Q_1 > p_2 \)

For \( v^k_{T_0} \in M_2 \), we have from Hardy’s inequality and the regularity of \( \partial_t u_{T_0} \in L^{p_2}(G_{T_0}) \) and \( \partial_t v^k_{T_0} \in L^\theta(G_{T_0}) \) that
\[ I_k \in L^{p_2}(\mathbb{R}^n_{-1}) + L^\theta(\mathbb{R}^n_{-1}) \subset L^{p_2}(\mathbb{R}^n_{-1}) \]

and further
\[ I_k \in L^p(\mathbb{R}^n_{-1}) \quad \text{for any } 1 < p \leq p_2. \]

Due to \( 1 < p_2 < Q_1 \) from our assumption in this case, we have from the estimates (3.36)-(3.37) and (3.4) in Theorem 3.1 together with (4.10) that \( v^k_{T_0} \in M_2 \) for small \( T_0 > 0 \). Obviously, under the assumption in Case 5, if \( g \in M_2 \), then
\[ g \in C([-T_0,0],L^{p_0}(\mathbb{R}^n)) \cap C^1([-T_0,0],L^{p_1}(\mathbb{R}^n)). \]

As before, we require to define the third set \( M_3 \)
\[ M_3 = \{ g(t,x) : g \in C([-T_0,0],L^\gamma(\mathbb{R}^n) \cap L^\Theta(\mathbb{R}^n)), \partial_t g \in C([-T_0,0],L^\gamma(\mathbb{R}^n)) \cap L^\Theta(G_{T_0}), \|g\|_3 + \|\partial_t g\|_3 \leq 2 \}, \]

where
\[ \|g\|_3 = \|g\|_{C([-T_0,0],L^\gamma(\mathbb{R}^n))} + \|g\|_{C([-T_0,0],L^\Theta(\mathbb{R}^n))} \]

and
\[ \|\partial_t g\|_3 = \|\partial_t g\|_{C([-T_0,0],L^\gamma(\mathbb{R}^n))} + \|\partial_t g\|_{L^\Theta(G_{T_0})}. \]

here \( 1 < \gamma \leq r_1, \frac{1}{\Theta} = \frac{1}{r_1} - \frac{1}{Q_1} \). We now derive the regularity of \( I_k \) in the following case.

**Case 6.** \( 1 < \mu < p_0, \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu - 1} \) and \( Q_1 > r_1 \)

At this moment, we have \( \Theta > p_0 \) and \( r_1 > p_1 \). Hence, in Case 6, if \( g \in M_3 \), then
\[ g \in C([-T_0,0],L^{p_0}(\mathbb{R}^n)) \cap C^1([-T_0,0],L^{p_1}(\mathbb{R}^n)), \]

and for \( v^k_{T_0} \in M_3 \),
\[ I_k \in L^{p_2} + L^{\Theta} + L^{r_1} + L^{p_2/\mu} + L^{p_1} + L^{n_2} + L^{n_3} + L^{\Theta/\mu}, \]
complete the proof of Theorem 4.1
Therefore,
\[ T \]

Noting that
\[ 1 \]
where
\[ \min\{p_2, \Theta, \frac{p_0}{\mu}, \frac{\Theta}{\mu}\} \]
then we cannot use the standard iteration scheme (4.10) to derive the existence of the

\[ \text{Case 9 when} \]

As in Case 1-Case 9, in the related spaces
\[ C \]

If the assumptions of Remark 4.1.

Case 8.

By the analogous analysis as in the previous cases, we can obtain
\[ \text{Case 7.} \]

Finally, we treat the term \( I_k \) for the left cases of \( \mu, l \) and \( p_0 \). To this end, as before, we define the fourth set \( M_4 \)
\[ \mathcal{M}_4 = \{g(t, x) : g, \partial_t g \in C([-T_0, 0], L^{p_0}(\mathbb{R}^n)), ||g||_4 + ||\partial_t g||_4 \leq 2\}, \]

where \( ||g||_4 = ||g||_{C([-T_0, 0], L^{p_0}(\mathbb{R}^n))} \) and \( ||\partial_t g||_4 = ||\partial_t g||_{C([-T_0, 0], L^{p_0}(\mathbb{R}^n))} \). We now distinguish the following three cases:

**Case 7.** \( 0 < \mu < p_0 \) and \( Q_0 = \frac{p_0}{\mu} \)

**Case 8.** \( 1 < \mu < p_0, \frac{p_0}{\mu} < Q_0 \leq \frac{p_0}{\mu} \) and \( Q_1 = r_1 \)

**Case 9.** \( 0 < \mu \leq 1, Q_0 > \frac{p_0}{\mu} \) and \( Q_1 = p_2 \)

By the analogous analysis as in the previous cases, we can obtain \( v_{T_0}^{k+1} \in \mathcal{M}_4 \) for the Case 7 -Case 9 when \( T_0 > 0 \) is small.

Next we show the convergence of the sequence \( \{v_{T_0}^k\} \) in the corresponding spaces \( \mathcal{M}_i \) (1 \( i \leq 4 \)). Set \( \tilde{v}_{T_0}^{k+1} = v_{T_0}^{k+1} - v_{T_0}^k \), then it follows from (4.10) that
\[
\begin{cases}
\partial_t^2 \tilde{v}_{T_0}^{k+1} - t^{2\mu - 2} \Delta \tilde{v}_{T_0}^{k+1} = \chi_{T_0}(t) \left( f(t, x, \bar{u} + \bar{u}_{T_0} + v_{T_0}^k) - f(t, x, \bar{u} + \bar{u}_{T_0} + v_{T_0}^{k-1}) \right), \\
\tilde{v}_{T_0}^{k+1}(0, x) = 0.
\end{cases}
\]

As in Case 1-Case 9, in the related spaces \( \mathcal{M}_j \) (1 \( j \leq 4 \)), we can obtain by (3.35)-(3.37) separately
\[
||\tilde{v}_{T_0}^{k+1}||_j + ||\partial_t \tilde{v}_{T_0}^{k+1}||_j \leq C(T_0)||\tilde{v}_{T_0}^k||_j,
\]
where \( C(T_0) \leq \frac{1}{2} \) for small \( T_0 > 0 \). Therefore, one can derive from (4.16) that for small fixed \( T_0 > 0 \), there exists a function \( v_T(t, x) \) such that \( v_{T_0}^{k+1} \rightarrow v_T(t, x) \) in \( C([-T_0, 0], L^{p_0}(\mathbb{R}^n)) \cap C^1([-T_0, 0], L^{p_1}(\mathbb{R}^n)) \) and \( v_T \in C([-T_0, 0], L^{p_0}(\mathbb{R}^n)) \cap C^1([-T_0, 0], L^{p_1}(\mathbb{R}^n)) \) solves (4.9). Therefore, \( u = \bar{u} + \bar{u}_{T_0} + v_{T_0} \) solves the problem (4.1) for \( (t, x) \in [-T_0, 0] \times \mathbb{R}^n \), and we complete the proof of Theorem 4.1

**Remark 4.1.** If the assumptions of \( 0 \leq \mu \leq 1 \) or \( 1 < \mu < p_0 \) with \( Q_0 \leq \frac{p_0}{\mu - 1} \) are violated, then we cannot use the standard iteration scheme (4.10) to derive the existence of the
solution to the problem (4.9). Indeed, for example, when $1 < \mu < p_0$ with $Q_0 > \frac{p_0}{\mu - 1}$ and $Q_1 > \frac{p_1}{\mu - 1}$, if the iteration scheme (4.10) does work, then it follows from Theorem 3.1 that for $v_{T_0}^k \in L^{q_k} (\mathbb{R}^{n+1})$

$$v_{T_0}^{k+1} \in L^{q_{k+1}} (\mathbb{R}^{n+1}), \quad \text{where} \quad \frac{1}{q_{k+1}} = \frac{\mu}{q_k} - \frac{1}{Q_1}.$$

Noticing that one has for $k \geq 1$

$$\frac{1}{q_{k+1}} - \frac{1}{q_k} = \mu^{-1} \left( \frac{1}{q_2} - \frac{1}{q_1} \right) = \mu^{-1} \left( \frac{p_1}{p_1} - \frac{Q_1}{Q_1} \right),$$

thus, we have

$$\frac{1}{q_k} = \frac{\mu^k}{p_1} - \frac{\mu^k}{(\mu - 1)Q_1} = \mu^k \left( \frac{1}{p_1} - \frac{1}{(\mu - 1)Q_1} \right) + \frac{1}{(\mu - 1)Q_1} \geq \mu^k \left( \frac{1}{p_1} - \frac{1}{(\mu - 1)Q_1} \right) \to +\infty \quad \text{as} \quad m \to +\infty.$$

Therefore, it seems that it is difficult for us to derive the uniform lower bound of $q_k > 1$ for any $k \in \mathbb{N}$. This means that the standard iteration scheme (4.10) only works for finite steps.

5 Solvability of (1.1) in the degenerate hyperbolic region

$\{ t \geq 0 \}$

Based on Theorem 4.1 in §4, we consider the problem (1.1) in the degenerate hyperbolic region $\{ t \geq 0 \}$

$$\begin{cases}
\partial_t^2 w - t^{2l-1} \Delta w = f(t, x, w), & (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
w(0, x) = \varphi(x), \\
\partial_t w(0, x) = \psi(x),
\end{cases} \tag{5.1}$$

where $\varphi(x) \in H^s(\mathbb{R}^n) \subset L^{p_0}(\mathbb{R}^n)$, $0 \leq s < \frac{n}{2}$, and $\psi(x) \equiv \partial_t u(0, x) \in L^{p_1}(\mathbb{R}^n)$, here $\partial_t u(0, x)$ comes from Theorem 4.1 in the degenerate elliptic part $\{ t \leq 0 \}$ of the problem (1.1). Our main result in this section is

**Theorem 5.1.** Under the conditions (1.2)-(1.3), there exists a small constant $T_0 > 0$ such that the problem (5.1) has a unique local solution $w(t, x) \in C([0, T], L^{p_0}(\mathbb{R}^n))$ when $0 \leq \mu \leq 1$, or when $1 < \mu < p_0$ with $Q_0 \leq \frac{p_0}{\mu - 1}$.

**Proof.** We first consider the following linear problem

$$\begin{cases}
\partial_t^2 w_1 - t^{2l-1} \Delta w_1 = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
w_1(0, x) = \varphi(x), \\
\partial_t w_1(0, x) = \psi(x).
\end{cases} \tag{5.2}$$
By [26] or Lemma 2.2 in [19], we know that the problem (5.2) has a unique solution \( w_1 \), which can be expressed as follows

\[
w_1(t, x) = V_1(t, D_x)\varphi(x) + V_2(t, D_x)\psi(x),
\]

where the pseudo-differential operator \( V_j(t, D_x) \) has the symbol \( V_j(t, |\xi|) \) for \( j = 1, 2 \),

\[
\begin{align*}
V_1(t, |\xi|) &= e^{-\frac{z}{2}} \Phi\left( \frac{2l - 1}{2(2l + 1)} \right) \left( 1 + O\left( |\xi|^{-1}\right) \right), \\
V_2(t, |\xi|) &= t e^{-\frac{z}{2}} \Phi\left( \frac{2l + 3}{2(2l + 1)} \right) \left( 1 + O\left( |\xi|^{-1}\right) \right),
\end{align*}
\]

here the confluent hypergeometric functions \( \Phi\left( \frac{2l - 1}{2(2l + 1)} \right) \) and \( \Phi\left( \frac{2l + 3}{2(2l + 1)} \right) \) are analytic functions of \( z \) with \( z = \frac{4i}{2l + 1} t \frac{2l+1}{2l+1} |\xi| \). Moreover, for sufficiently large \( |z| \),

\[
|\Phi\left( \frac{2l - 1}{2(2l + 1)} \right) \left( 1 + O\left( |\xi|^{-1}\right) \right),
\]

and

\[
|\Phi\left( \frac{2l + 3}{2(2l + 1)} \right) \left( 1 + O\left( |\xi|^{-1}\right) \right)| 
\]

In addition, by Lemma 3.2 in [19] and Sobolev’s embedding theorem, one has

\[
\|V_1(t, D_x)\varphi(x)\|_{L^{p_0}(\mathbb{R}^n)} \leq C \|V_1(t, D_x)\varphi(x)\|_{H^{s}(\mathbb{R}^n)} \leq C \|\varphi\|_{H^{s}(\mathbb{R}^n)}
\]

and

\[
\|
\partial_t V_1(t, D_x)\varphi(x)\|_{L^{p_0}(\mathbb{R}^n)} \leq C \|
\partial_t V_1(t, D_x)\varphi(x)\|_{H^{s-\frac{2l+1}{2(2l+1)}}(\mathbb{R}^n)} \leq C \|\varphi\|_{H^{s}(\mathbb{R}^n)},
\]

where \( \frac{1}{q_0} = \frac{1}{2} - \frac{1}{n}\left( s - \frac{2l + 3}{2(2l + 1)} \right) \).

On the other hand, it follows from the analytic property of \( \Phi\left( \frac{2l + 3}{2(2l + 1)} \right) \) on the variable \( z \) and the estimate (5.5) that there exists a constant \( C > 0 \) such that

\[
|z|^{\frac{q_0}{p_0}} |V_2(t, |\xi|)| \leq C t
\]

and further

\[
|V_2(t, |\xi|)| \leq C |\xi|^{-\frac{2}{2l+1}}.
\]

It is noted that \( \psi(x) = \partial_t u(0, x) \in L^{p_1}(\mathbb{R}^n) \) and

\[
1 < p_1 < p_0 < +\infty, \quad \frac{1}{p_1} - \frac{1}{p_0} = \frac{2}{n(2l + 1)}.
\]
This, together with (5.8) and Hardy-Littlewood-Sobolev’s inequality, yields
\[
\|V_2(t, D_x)\psi(x)\|_{L^p_0(\mathbb{R}^n)} \leq C\|\psi\|_{L^p_1(\mathbb{R}^n)}.
\]
(5.9)

Therefore, by (5.3), (5.6) and (5.9), we obtain
\[
w_1(t, x) \in C([0, T], L^p_0(\mathbb{R}^n)).
\]
(5.10)

Set \( w(t, x) = w(t, x) - w_1(t, x) \), then it follows from (5.1) that
\[
\begin{aligned}
\partial^2_t v - t^{2l-1} \Delta v &= f(t, x, v + w_1), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
v(0, x) &= 0, \quad \partial_t v(0, x) = 0.
\end{aligned}
\]
(5.11)

For suitably chosen constant \( T > 0 \), we define a set \( \mathcal{M} \):
\[
\mathcal{M} = \{ v \in C([0, T], L^p_0(\mathbb{R}^n)) : \sup_{t \in [0, T]} \|v(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} \leq 2 \}.
\]

And then we define a mapping \( \mathcal{T} \) as follows
\[
\mathcal{T}(v)(t, x) = \int_0^t \left( V_2(t, D_x)V_1(\tau, D_x) - V_1(t, D_x)V_2(\tau, D_x) \right) f(\tau, x, v + w_1) d\tau.
\]
It is easy to verify that \( \mathcal{T}(v) \) satisfies
\[
\begin{aligned}
\left( \partial^2_t - t^{2l-1} \right) \mathcal{T}(v) &= f(t, x, v + w_1), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
\mathcal{T}(v)(0, x) &= 0, \\
\partial_t \mathcal{T}(v)(0, x) &= 0.
\end{aligned}
\]
(5.12)

By Theorem 3.1 in [26], we have that for any \( g \in L^p(\mathbb{R}^n) \) with \( 1 < p < \infty \)
\[
\|V_1(t, D_x)g(x)\|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}
\]
(5.13)
and
\[
\|V_2(t, D_x)g(x)\|_{L^p(\mathbb{R}^n)} \leq C t\|g\|_{L^p(\mathbb{R}^n)}.
\]
(5.14)

Next we show that the mapping \( \mathcal{T} \) maps \( \mathcal{M} \) into itself and the mapping \( \mathcal{T} \) is contractible for small \( T > 0 \). If so, then we can get the solvability of (5.11). For this purpose, we will distinguish two cases as follows.

**Case 1.** \( 0 \leq \mu \leq 1 \):

in this case, one has
\[
\|\mathcal{T}(v)(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} \leq C t \int_0^t \|f(\tau, v + w_1)\|_{L^p_0(\mathbb{R}^n)} d\tau \quad \text{(by (5.13) and (5.14))}
\]
\[
\leq C t \int_0^t \|f(\tau, v + w_1)\|_{L^{p_0/\mu}(\mathbb{R}^n)} d\tau \quad \text{(by Hölder’s inequality)}
\]
\[
\begin{align*}
&\leq Ct \int_0^t (1 + \|v(\tau, \cdot) + w_1(\tau, \cdot))\|_{L^p_0(\mathbb{R}^n)} d\tau \quad \text{(by (1.2))} \\
&\leq Ct^2 \left(1 + \sup_{t \in [0,T]} \|v(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} + \|w_1(t, \cdot)\|_{L^p_0(\mathbb{R}^n)}\right). \quad (5.15)
\end{align*}
\]

**Case 2.** \(1 < \mu < p_0\) and \(Q_0 \leq \frac{p_0}{\mu - 1}\):

in this case, one has \(p_1 \leq \frac{p_0}{\mu} < p_0\). Thus, we have

\[
\|\mathcal{T}(v)(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} \leq C \int_0^t \|f(\tau, v + w_1)\|_{L^{p_1}(\mathbb{R}^n)} d\tau \quad \text{(by (5.11) and (5.13))}
\]

\[
\leq C \int_0^t \|f(\tau, v + w_1)\|_{L^{p_0/n}(\mathbb{R}^n)} d\tau \quad \text{(by Hölder’s inequality)}
\]

\[
\leq C \int_0^t (1 + \|v(\tau, \cdot) + w_1(\tau, \cdot))\|_{L^p_0(\mathbb{R}^n)} d\tau \quad \text{(by (1.2))}
\]

\[
\leq Ct \left(1 + \sup_{t \in [0,T]} \|v(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} + \|w_1(t, \cdot)\|_{L^p_0(\mathbb{R}^n)}\right).
\]

From (5.6), (5.9) and (5.16), we see that for small \(T > 0\), \(\mathcal{T}(v)(t, x) \in C([0, T], L^p_0(\mathbb{R}^n))\) and

\[
\sup_{t \in [0,T]} \|\mathcal{T}(v)(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} \leq C_{p_0} T \left(1 + \sup_{t \in [0,T]} \|v(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} + \sup_{t \in [0,T]} \|w_1(t, \cdot)\|_{L^p_0(\mathbb{R}^n)}\right) \leq 2,
\]

which implies that \(\mathcal{T}\) maps \(\mathcal{M}\) into itself.

In addition, for any \(v_1, v_2 \in \mathcal{M}\), as argued in (5.16), one can obtain for small \(T > 0\)

\[
\|\mathcal{T}(v_1)(t, \cdot) - \mathcal{T}(v_2)(t, \cdot)\|_{L^p_0(\mathbb{R}^n)} \leq \frac{1}{2} \sup_{t \in [0,T]} \|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^p_0(\mathbb{R}^n)}.
\]

Therefore, by (5.17)-(5.18) and the contraction map principle, we complete the proof of Theorem 5.1. \(\square\)

**Remark 5.1.** If the initial data \(u(0, x) = \varphi_0(x) \in L^\infty(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)\) in (1.1) for \(n = 2, 3\) and \(s \geq 0\) is given, we can remove all the assumptions in (1.2)-(1.3). Indeed, from the proof of Theorem 4.1, we can derive that \(u(t, x), \partial_t u(t, x) \in L^\infty([-T_0, 0] \times \mathbb{R}^n)\) for some fixed constant \(T_0 > 0\) without the assumptions (1.2)-(1.3), which obviously means \((u(0, x), \partial_t u(0, x)) = (\varphi_0(x), \varphi_1(x)) \in L^\infty(\mathbb{R}^n)\). In addition, by [25], one has the following formula

\[
w(t, x) = 2^{2-2\gamma} \Gamma(2\gamma) \int_0^1 v_{\varphi_0}(\phi(t)s, x)(1 - s^2)^{\gamma-1}ds
\]

\[
+ 2^{2\gamma} \Gamma(2 - 2\gamma) \int_0^1 v_{\varphi_1}(\phi(t)s, x)(1 - s^2)^{-\gamma}ds
\]

where \(w(t, x)\) is a solution to the equation \(\partial_t^2 w - t^{2l-1}\Delta w = 0\) with \((w(0, x), \partial_t w(0, x)) = (\varphi_0(x), \varphi_1(x))\), where \(\gamma = \frac{2l - 1}{2(2l + 1)}, \phi(t) = \frac{2}{2l + 1} t^{\frac{2l+1}{2}}, \) and \(v_{\varphi}\) denotes the solution to
the wave equation \( \partial_t^2 v - \Delta v = 0 \) with \((v(0, x), \partial_t v(0, x)) = (\varphi(x), 0)\). It is well-known that \( v \in L^\infty([0, T] \times \mathbb{R}^n) \) holds for \( n = 2, 3 \) and any \( T > 0 \) when \( \varphi(x) \in L^\infty(\mathbb{R}^n) \). Based on this, one easily knows \( w(t, x) \in L^\infty([0, T] \times \mathbb{R}^n) \) \((n = 2, 3)\). Therefore, by a simpler proof than that in Theorem 5.1, we can locally solve the problem (1.1) with no the assumptions (1.2)-(1.3) when \( n = 2, 3 \), and further \( u(t, x) \in L^\infty([-T_0, T_0] \times \mathbb{R}^n) \) can be derived and (1.1) is solved.

6 Proof of Theorem 1.1.

In this section, based on Theorem 4.1 and Theorem 5.1, we will complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, by Theorem 4.1, we know that (1.1) is solvable in the degenerate elliptic region \((t, x) \in [-T_0, 0] \times \mathbb{R}^n\), moreover, the corresponding solution \( u(t, x) \) satisfies

\[
 u(t, x) \in C\left([-T_0, 0], L^{p_0}(\mathbb{R}^n)\right), \quad \partial_t u(t, x) \in C\left([-T_0, 0], L^{p_1}(\mathbb{R}^n)\right).
\]

On the other hand, by the initial data \((u(0, x), \partial_t u(0, x))\) given in (6.1), it follows from Theorem 5.1 that the problem (1.1) admits a unique solution in the degenerate hyperbolic region \((t, x) \in [0, T_0] \times \mathbb{R}^n\) for small constant \( T_0 > 0 \), which satisfies

\[
 u(t, x) \in C\left([0, T_0], L^{p_0}(\mathbb{R}^n)\right).
\]

Therefore, combining (6.1) with (6.2) yields Theorem 1.1.

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