Partial covering arrays for data hiding and quantization*  

Vladimir N. Potapov  

email: vpotapov@math.nsc.ru;  

Abstract  
We consider the problem of finding a set (partial covering array) $S$ of vertices of the Boolean $n$-cube having cardinality $2^n - k$ and intersecting with maximum number of $k$-dimensional faces. We prove that the ratio between the numbers of the $k$-faces containing elements of $S$ to $k$-faces is less than $1 - \frac{1+o(1)}{\sqrt{2\pi k}}$ as $n \to \infty$ for sufficiently large $k$. The solution of the problem in the class of linear codes is found. Connections between this problem, cryptography and an efficiency of quantization are discussed.  

Keywords: linear code, covering array, data hiding, wiretap channel, quantization, wet paper stegoscheme.

1 Introduction  
Let $F_2^n$ be the set of binary vectors of length $n$ (hypercube). We consider some problems relating to information transmission. The first problem is the message transmission over wiretap channel [4]. Consider the following situation. An adversary can intercept $n-k$ bits (in random positions) of an $n$-bit message. The encoder is to be designed to minimize the adversary’s information about the initial data. A general approach for solving this problem is to split hypercube into $C_1, \ldots, C_{2k}$ sets, for example, by syndromes of some linear code, and to encode $k$-bit data $x$ by a random $n$-bit word from $C_x$. Let $\Gamma$ be $k$-dimensional face ($k$-face) defined by intercepted $n-k$ bits. Then the adversary is forced to choose between all $x$ such that $C_x \cap \Gamma \neq \emptyset$. The encoder needs a partition such that each $k$-face of the hypercube intersects with as many as possible elements of partition. In other words each $C_x$ must intersect as many $k$-faces as possible.  

One of a well-known stegoscheme is based on coding theory. Encoder changes one or more bits of the initial message in order to the resulting word has a special syndrome. This syndrome is a hiding message. It is assumed that the changes of initial message are not perceptible. However, data obtained by modern methods of coding images contain control bits of different kind that cannot be changed. So called wet paper stegoscheme divides the coordinates into wet coordinates that can be used for hiding information and dry coordinates that cannot be altered [7]. However alternation of different wet coordinates corresponds to hugely different effects. Places of the least significant bits depend of image. Consider this issue in detail.

The second problem is an efficient quantization of real data. It is the important stage for lossy compression of images or speeches. From the nature of the things a part of data values is on the edges of the quantization intervals. The last bit of such value is the least significant one for the quality of quantization. Thus useful stegoscheme should provide possibility of alternation of different bits for embedding the same message. We conclude that the set of words generated by stegoscheme must intersect as many $k$-faces as possible where $k$ is the maximum number of

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alternating bits. A special method for choosing this least significant bits is used for data hiding in image and video [6].

It is possible to utilize this redundancy for data compression [2]. Consider an \( n \)-tuple consisted of the last bits of quantized values. Suppose that each \( n \)-tuple contains \( k' \) negligible bits. Let \( C \) be some code (array of codewords) with cardinality \( 2^{n-k} \) and let for each \( n \)-tuple there exists a codeword such that this \( n \)-tuple and the codeword differ only in negligible bits. We will transmit the codeword (rather its number) instead of the initial \( n \)-tuple. So, we will truncate \( k' \) bits of the \( n \)-bit message.

Consider a mathematical formulation of the problem. We would like to construct a minimal code such that for any binary \( n \)-tuple \( v \) and for each set of \( k \) positions (that contains the least significant bits) there exists a codeword \( u \) such that \( v \) and \( u \) can differ from each other only in these \( k \) positions. I.e., the \( k \)-face defined by these positions contains a codeword. We need either to find such code (covering array) with the least cardinality or to fix a cardinality of the code and to maximize the number of the fit positions. Note that this problem is different from a construction of covering codes. A code \( C \) is called \( k \)-covering code if for every \( n \)-tuple there exists a codeword which is different in certain \( k \) positions. But we going to maximize the number of the fit \( k \)-sets.

A perfect solution for the problems written above would be a set of codewords containing only one element of each \( k \)-face of \( F^2_n \). Such sets have cardinalities \( 2^{n-k} \) and are called MDS codes. For the \( q \)-ary hypercubes (\( q \) is a prime power), there are MDS codes with several code distances. If the cardinality of an MDS code is \( q^{n-k} \) then the code distance is equal to \( k+1 \). But in the Boolean hypercube, there exist only two nonequivalent MDS codes: the parity check code \((k = 1)\) and the pair of antipodal vectors \((k = n-1)\), for example, \( \overline{0} \) and \( \overline{1} \) [1]. Consequently, we need to find approximate solutions.

A subset \( T \) of the hypercube is called a binary covering array \( CA([|T|], n, n-k) \) with strength \( n-k \) if for each \( v \in F^2_n \) and for any \( k \) positions there exists \( u \in T \) such that \( v \) and \( u \) can differ only in these \( k \) positions. A survey of constructions and bounds for cardinalities of covering arrays can be found in [3]. At this moment, exact bounds are obtained only for small \( n \) or for \( k=1, 2, n-2, n-1 \) and an arbitrary \( n \). If \( n > k + 1 \), the cardinality of minimum covering array \( F(n, k) \) is greater than \( 2^{n-k-2}(2 + \log(k+2)) \) and \( F(n, k) \leq 2^n/(k+1) \). Moreover, it is known that \( F(n, k) \approx 2^{n-k} \log n \) as \( n \to \infty \) and \( n-k \) is fixed. With the exception of the parity check code mentioned above, linear codes are not useful as binary covering arrays because any other proper linear code does not intersect with some \( \lceil \frac{n}{3} \rceil \)-faces.

We will consider a bit different mathematical problem: to construct a partial covering array \( S_k \subset F^2_n \), \( |S_k| = 2^{n-k} \) with the following property. The number of \( k \)-faces containing elements of \( S_k \) is as large as possible. In practice it is convenient to encode elements of sets of such cardinality. Apparently for the first time partial covering arrays or ”covering array with budget constraints” is considered in [9]. In [8] an existence of partial covering arrays with some parameters is established by probabilistic methods. Denote by \( \nu_k(S_k) \) the ratio between the number of \( k \)-faces that contain elements of \( S_k \) and the number \( \binom{n}{k} 2^{n-k} \) of all \( k \)-faces. In Section 2, we prove that 

\[
\max \nu_k(S_k) \approx 1 - \frac{1 + o(1)}{\sqrt{2\pi k}} \quad \text{as} \quad n \to \infty
\]

for sufficiently large \( k \).

For application in information transmission a device or algorithmic function that performs a quantization must be cost-effective. For example, linear codes are easily implemented. In Section 3, we find precise value of \( \max \nu_k(S_k) \) for linear sets as \( n = 2^k - 1 \) and construct corresponding \( S_k \).

Another approach to study suitability of linear codes for similar tasks was developed in [5].

In the beginning, we consider a random subset \( T \) of the hypercube with cardinality \( 2^{n-k} \). We suppose that the elements of \( T \) are selected independently. Since the probability of any \( k \)-face \( \Gamma \)
equals $1/2^{n-k}$, the probability $\Pr(T \cap \Gamma = \emptyset)$ equals $(1 - \frac{1}{2^{n-k}})^{2^{n-k}}$. Since $(1 - \frac{1}{2^{n-k}})^{2^{n-k}} \to 1/e$ as $n \to \infty$, the following proposition is true.

**Proposition 1** \( \lim_{n \to \infty} E\nu_k(T) = 1 - 1/e \), where \( T \subset E^n \) is a random set, \( |T| = 2^{n-k} \), and \( k \) is fixed.

## 2 Upper bound for partial covering arrays

In this section we will use definitions and methods from \[ \square \]. Consider the vector space \( \mathbb{V} \) of real-valued functions on \( F_2^n \) with the scalar product \( (f, g) = \frac{1}{2^n} \sum_{x \in F_2^n} f(x)g(x) \). For every \( z \in F_2^n \) define a character \( \phi_z(x) = (-1)^{\langle x, z \rangle} \), where \( \langle x, z \rangle = x_1z_1 + \ldots + x_nz_n \). Here all arithmetic operations are performed on real numbers. As is generally known, the characters of the group form an orthonormal basis of \( \mathbb{V} \).

Let \( M \) be the adjacency matrix of the hypercube \( F_2^n \). This means that \( Mf(x) = \sum_{y:d(x,y)=1} f(y) \), where \( d(x, y) \) is the Hamming distance. It is well known that the characters are eigenvectors of \( M \). Indeed, we have

\[
M\phi_z(x) = (n - 2\text{wt}(z))\phi_z(x),
\]

where \( \text{wt}(z) \) is the number of nonzero coordinates of \( z \).

Let \( M_r \) be the adjacency matrix of the distance \( r \), i.e. \( M_r(x, y) = 1 \) iff \( d(x, y) = r \). Obviously, this matrix generates Bose — Mesner algebra. We have the equations

\[
M_rM = (n - r + 1)M_{r-1} + (r + 1)M_{r+1},
\]

\[
M^r = a^r_0M_0 + a_1^rM_1 + \ldots + a^r_nM_n,
\]

where

\[
a^r_1 = ia^r_{i-1} + (n - i)a^r_{i+1},
\]

\[
a^r_{i-1} = a^r_{i+1} = 0.
\]

The following properties of coefficients \( a^r_i \) are not difficult to prove by induction using \( \square \).

1. \( a^r_i = r! \).
2. \( a^r_i = 0 \) if \( i > r \).
3. \( a^r_i = 0 \) if \( r \) and \( i \) have different parity.
4. Consider \( a^r_i \) as a function of the variable \( n \). Then \( a^r_i(n) \) is a polynomial of degree \( (r - i)/2 \) as \( r \) and \( i \) have the same parity. Moreover, \( a^r_i(n) = C(r, r - i)n^{(r-i)/2} + p_{r,i}(n) \) where \( C(r, s) = \binom{r+s}{s} \) are entries of Catalan’s triangle and \( p_{r,i} \) is some polynomial of the degree at most \( r+i - 1 \).

Notice that \( a^r_i \) is the number of the paths of the length \( r \) in the hypercube such that distance between the origin and the end of the path is equal to \( i \). It is known that \( a^r_i = (\cosh n^{-i}(x)\sinh^{r-i}(x))|_{x=0} \)

\[
a^r_i = \frac{1}{r!} \sum_{k=0}^n (n - 2k)^r P_k(i, n) \]

where

\[
P_k(r, n) = \sum_{j=0}^k (-1)^j \binom{r}{j} \binom{n-r}{k-j} \]

are the \( k \)th Kravtchouk polynomials of the variable \( r \).

Define \( \nu_k(n) = \max_{S \subset F_2^n} \nu_k(S) \) where \( S \subset F_2^n \), \( |S| = 2^{n-k} \).

**Theorem 1** Let \( r \leq k \) be even. Then \( \nu_k(n) \leq 1 - \frac{1+o(1)}{2^{n-k}} \binom{k}{r} \) as \( n \to \infty \).
The number of monomials of the degree $S$ points from $n$ elements is decompositions by row only.

Since a pair of points at the distance $r$, The characteristic function $\chi$

Proposition 2

3 Linear codes

The results of this section were announced in \[2\].

**Proposition 2** Let $C$ be a linear code and $\Gamma$ be a $k$-face. Then it is possible two cases

1) $|C \cap (x + \Gamma)| = 0$ or $2^k$ for each $x \in F_2^n$,

2) $|C \cap (x + \Gamma)| = 1$ for each $x \in F_2^n$.

Proof. System of linear equations over $GF(2)$ has 1 or 0 and $2^k$ solutions for different right-hand side vectors. \(\square\)

Denote by $\mu_k(S)$ the number of $k$-faces containing only one element of a linear code $\mathcal{S}$ (this element is $0$). It is clear that $\nu_k(S) \geq \frac{2^{-k}M(S)}{|S|}$.

Let $H = \{h_{ij}\}$ be a binary $k \times n$ matrix. Consider the linear code $C = \{x \in F_2^n | Hx = 0\}$.

The characteristic function $\chi^C$ of $C$ is equal to $\prod_i (1 + \bigoplus_j h_{ij}x_j)$.

**Proposition 3** The number of monomials of the degree $k$ in $\chi^C$ is equal to $\mu_k(C)$.

It follows from the properties of Möbius transform of Boolean function.

Let $H[x]$ be a matrix with columns $h_jx_j$ where $h_j$ is a column of $H$, $j = 1, \ldots, n$. For example, if $H_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ then $H_2[x] = \begin{pmatrix} 0 & x_2 & x_3 \\ 0 & x_1 & 0 \end{pmatrix}$.

The permanent of the matrix $H$ (square or rectangular $k \times n$, $k \leq n$) is the sum of all products $h_{1j_1} \ldots h_{kj_k}$ where $j_1, \ldots, j_k$ are mutually different. There we use addition by modulo $2$. For example, $\text{per}_2(H_2[x]) = x_1x_2 + x_2x_3 + x_1x_3$.

By the definition we can obtain the following properties of $\text{per}_2(H[x])$.

1. If we add a row of $H[x]$ to another row of $H[x]$ then $\text{per}_2(H[x])$ does not change.

2. Laplace expansion for determinants is true for the permanents of rectangular matrices (decompositions by row only).

3. If two columns $h_{n-1}$ and $h_n$ of $H$ coincide, then $\text{per}_2(H[x_1, \ldots, x_n])$ is equal to
per₂(H'[x₁, \ldots, x_{n-2}, x_{n-1} + x_n]) where H' is H without the last column.

These properties are proved analogously to the property of the determinant because the permanent and the determinant coincide modulo 2.

Let Hₖ be the matrix k × (2ᵏ - 1) such that the jth row of Hₖ is the binary representation bₖ(j) of j. Determine real functions fₖ by the equations fₖ(x₁, \ldots, xₙ) = per₂(Hₖ[x₁, \ldots, xₙ]), where n = 2ᵏ - 1.

**Proposition 4** fₖ(x₁, \ldots, xₙ) = \sum_{i₁, \ldots, iₖ \in I} x_{i₁} \cdots x_{iₖ},
where I = \{i₁, \ldots, iₖ\} and vectors bₖ(i₁), \ldots, bₖ(iₖ) are linearly independent over GF(2).

Let H be a binary k × n matrix having aⱼ columns bₖ(j) for j = 1 \ldots 2ᵏ - 1. Consider the linear code C = \{x \in F²ⁿ | Hx = 0\}. Applying properties of permanents we can prove

**Proposition 5** μₖ(C) = fₖ(a₁, \ldots, a₂ᴺ⁻₁).

**Proof.** By Proposition 3 and the third property of the permanent

μₖ(C) = per₂(H[x₁, \ldots, xₙ]|_{x₁=1} = per₂(Hₖ[\underbrace{x_{i₁} + \ldots, x_{iₖ}}_{a₁}, \ldots, \underbrace{x_{iₖ} + \ldots}_{a₂ᴺ⁻₁}]|_{x₁=1} = fₖ(a₁, \ldots, a₂ᴺ⁻₁).

Then we can conclude that the original problem is equivalent to finding maximum of fₖ where the sum of the arguments is fixed. Consider the set

Tₖ(s) = \{(x₁, \ldots, xₙ)|x_j ≥ 0, \sum_{j=1}^{n} x_j = s\},

where n = 2ᵏ - 1.

**Theorem 2** \max_{x \in Tₖ(s)} fₖ(x) = fₖ(s/n, \ldots, s/n).

**Proof.** By induction on k. For k = 2 the statement of the theorem can be verified by the standard methods of analysis. Consider occurrences of the variable x₁ in the polynomial fₖ(x₁, \ldots, xₙ). By the second and the third properties of the permanent we get

per₂(Hₖ⁺¹[x₁, \ldots, x_{2n+1}]) = x₁per₂(Hₖ[(x₁i¹, \ldots, x_{2n}i^{2n+1})] + h(x₂, \ldots, x_{2n+1}),

where a function h does not depend on the variable x₁, bₖ(i₁²j) and bₖ(i₁²j⁺¹) differ only in the first position.

By the first property of the permanent all variables of fₖ(x₁, \ldots, xₙ) are equivalent and each monomial has degree k. Hence we have the equation

fₖ⁺¹(x₁, \ldots, x_{2n+1}) = \frac{1}{k + 1} \sum_{j=1}^{2n+1} x_j fₖ(x_{j¹}, \ldots, x_{j²n⁻¹} + x_{j²n}),

where \{x_{n, j}\} is the set of all variables without j.
By induction, we find \( f_{k+1}(x_1, \ldots, x_{2n+1}) \leq \frac{1}{k+1} \sum_{j=1}^{2n+1} x_j f_k((s-x_j)/n, \ldots, (s-x_j)/n) \). Determine \( y_j = s - x_j \). Obviously we have \( \sum_{j=1}^{2n+1} y_j = 2ns \). Then we obtain an inequality
\[
f_{k+1}(x_1, \ldots, x_{2n+1}) \leq \frac{1}{k+1} \sum_{j=1}^{2n+1} (s - y_j) \left( \frac{y_j}{n} \right)^k.
\]

By the method of Lagrange multipliers, it is not complicated to prove that the function
\[
g(y_1, \ldots, y_{2n+1}) = \sum_{j=1}^{2n+1} (s - y_j)y_j^k
\]
has maximum in the interior point \( y_1 = \ldots = y_{2n+1} = \frac{2ns}{2n+1} \) if \( \sum_{j=1}^{2n+1} y_j = 2ns, 0 \leq y_j \leq s \). In the edge points (where \( y_i = 0 \) for some \( i, i = 1, \ldots, 2n + 1 \), this function isn’t positive. Induction step is proved. \( \triangle \)

**Corollary 1** For fixed \( k \) and \( n = m(2^k - 1) \) the maximum value of \( \mu_k(C_{k,m}) \) corresponds to the code \( C_{k,m} \) with the check matrix \( b_k(j) \) for all \( j = 1, \ldots, 2^k - 1 \). If \( m = 1 \) then \( C_{k,1} \) is the Hamming code.

For example, \( H = (111 \ldots 1) \); \( H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \); \( H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \).

By linear algebra it is routine to prove the following proposition.

**Proposition 6** Let \( n = 2^k - 1, k \geq 2 \) then \( \mu_k(C_{k,1}) = (2^k - 1)(2^k - 2)(2^k - 4) \cdots (2^k - 2^{k-1})/k! \) and
\[
\nu_k(C_{k,1}) = \frac{1}{k! 2^k \begin{pmatrix} k \end{pmatrix}} \sum_{t=1}^{k-1} (2^k - 1)(2^k - 2)(2^k - 4) \cdots (2^k - 2^t)(2^t - 1)^2t^{t-1}.
\]

It is possible to calculate that \( \nu_2(C_{2,1}) = 1, \nu_3(C_{3,1}) = 9/10, \nu_4(C_{4,1}) = 10/13 \).

Define \( \nu'_k(C) = \mu_k(C)/\binom{k}{2} \). Obviously, \( \nu'_k(C) \) is a lower bound of \( \nu_k(C) \). Then \( \nu'_k(C_{k,1}) = (2^k - 1)(2^k - 2)(2^k - 4) \cdots (2^k - 2^{k-1})/(2^k - 1)(2^k - 2)(2^k - 3) \cdots (2^k - k) \) and \( \lim_{k \to \infty} \nu'_k(C_{k,1}) \approx 0.2888 \).

**Corollary 2** \( \lim_{s \to \infty} \nu'_k(C_{k,s}) = (2^k - 1)(2^k - 2)(2^k - 4) \cdots (2^k - 2^{k-1})/(2^k - 1)^k \).

4 Conclusion

As we can see from Proposition 6 the best (for our task) linear code is not better than a random set as \( k \) is large. But if \( k \) is a small integer than the best linear code is tight to the best unrestricted partial covering array. Problems to find the best unrestricted set or to find an asymptotic of its cardinality are open.

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