Research Article

Inclusion Relations between $\alpha$-Modulation Spaces and Triebel–Lizorkin Spaces

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In this paper, we obtain conditions of the inclusion relations between $\alpha$-modulation spaces and Triebel–Lizorkin spaces.

1. Introduction

The modulation space $M_{p,q}^\alpha$ was first introduced by Feichtinger [1] in 1983 by the short-time Fourier transform. Modulation space has a close relationship with the topics of time-frequency analysis (see [2]), and it has been regarded as a proper space for the study of partial differential equations (see [3–5]).

The $\alpha$-modulation space is introduced by Gröbner [6] to link Besov and modulation spaces by the parameter $0 \leq \alpha \leq 1$. One can find some basic properties about $\alpha$-modulation spaces in [7, 8]. Among many features of the $\alpha$-modulation spaces, an interesting subject is the inclusion between $\alpha$-modulation and function spaces, which has been concerned by many authors to this topic, see [8–11]. As applications, $\alpha$-modulation spaces are quite recently applied to the field of partial differential equations. In [12], Misiolek and Yoned proved locally ill-posedness of the Euler equations in the frame of $\alpha$-modulation spaces. Furthermore, Han and Wang [13] proved a global well-posedness for the nonlinear Schrödinger equation on $\alpha$-modulation spaces, and also in [14] studied the Cauchy problem for the derivative nonlinear Schrödinger equation on $\alpha$-modulation spaces.

Remark 1. Modulation spaces are special $\alpha$-modulation spaces in the case $\alpha = 0$, so our theorems also works well in the special case $\alpha = 0$.

In this research, we are interested in studying the inclusion relations between $\alpha$-modulation spaces $M_{p,q}^{\alpha}$ and Triebel–Lizorkin spaces $F_{p,q}$ for $p \leq 1$, which greatly improve and extend the results for the inclusion relations between local Hardy spaces and $\alpha$-modulation spaces obtained by Kato in [10].

2. Preliminaries

The notation $X \lesssim Y$ denotes the statement that $X \lesssim CY$ with a positive constant $C$ that may depend on $n, \alpha, p, q, s, r$. The notation $X \sim Y$ means the statement $X \leq CY$ and the notation $X = Y$ stands for $X = CY$. For a multi-index $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, we denote $|k|_{\infty} := \max_{i=1,2,\ldots,n}|k_i|$, $|k| = |k_1| + \cdots + |k_n|$ and $k \gtrsim (1 + |k|^2)^{1/2}$.

Let $S := S(\mathbb{R}^n)$ be the Schwartz space and $S' := S'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in S(\mathbb{R}^n)$ by

$$
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} \, d\xi.
$$

We give some definitions and properties of sequences.

Definition 2. Let $0 < p, q < \infty$, $s, r \in \mathbb{R}, \alpha \in [0, 1)$. Let $\{a_k\}_{k \in \mathbb{Z}^n}$ be a sequence, we denote its $\ell_p^\alpha$ (quasi-) norm

$$
\|\{a_k\}\|_{\ell_p^\alpha} = \begin{cases} 
\left( \sum_{k \in \mathbb{Z}^n} |a_k|^p \langle k \rangle^{q(1-\alpha)} \right)^{1/p} & 0 < p < \infty, \\
\sup_{k \in \mathbb{Z}^n} |a_k| \langle k \rangle^{s(1-\alpha)} & p = \infty,
\end{cases}
$$
and let \( e_p^{a,0} \) be the (quasi-) Banach space of sequences whose \( e_p^{a,1} \) (quasi-) norm is finite.
Let \( \{ b_j \}_{j \in \mathbb{N}} \) be a sequence, we denote its \( e_p^{a,1} \) (quasi-) norm
\[
\left\| \{ b_j \} \right\|_{e_p^{a,1}} = \left( \sum_{j \in \mathbb{N}} |b_j|^p \right)^{1/p} ; \quad 0 < p < \infty ,
\]
and let \( e_p^{a,0} \) be the (quasi-) Banach space of sequences whose \( e_p^{a,1} \) (quasi-) norm is finite.
Let \( \{ c_k \}_{k \in \mathbb{Z}^n} \) be a sequence, we denote its \( e_p^{a,0} \) (quasi-) norm
\[
\left\| \{ c_k \} \right\|_{e_p^{a,0}} = \left( \sum_{k \in \mathbb{Z}^n} |c_k|^p \right)^{1/p} ; \quad p = \infty ,
\]
and let \( e_p^{a,0} \) be the (quasi-) Banach space of sequences whose \( e_p^{a,1} \) (quasi-) norm is finite.

We recall some embedding lemmas about sequences defined above.

**Lemma 3** (sharpness of embedding, for uniform decomposition). Suppose \( 0 < p, q \leq \infty , s_1, s_2 \in \mathbb{R} \). Then
\[
e_p^{a,1} \subsetneq e_q^{a,1},
\]
holds if and only if
\[
s_1 \leq s_2 , \quad \frac{1}{q} \leq \frac{1}{p} + \frac{1}{n} ,
\]
or
\[
s_1 = s_2 , \quad p = q .
\]

**Lemma 4** (sharpness of embedding, for dyadic decomposition). Suppose \( 0 < p, q \leq \infty , s_1, s_2 \in \mathbb{R} \). Then.
\[
e_p^{a,1} \subsetneq e_q^{a,1},
\]
holds if and only if
\[
s_2 < s_1 , \quad \frac{1}{q} < \frac{1}{p}.
\]

**Lemma 5** (sharpness of embedding, for \( \alpha \)-decomposition). Suppose \( 0 < p, q \leq \infty , s_1, s_2 \in \mathbb{R} , \alpha \in [0, 1) \). Then
\[
e_p^{a,0} \subsetneq e_q^{a,0},
\]
holds if and only if
\[
\frac{1 - \alpha}{q} + \frac{s_2}{n} < \frac{1 - \alpha}{p} + \frac{s_1}{n} , \quad \frac{1}{q} \leq \frac{1}{p}.
\]
Let \( Q^n \) be the collection of all cubes \( Q_{v,k} \) in \( \mathbb{R}^n \) with sides parallel to the axes, centered at \( 2^{-v}k \), and with side length \( 2^{-v} \), where \( k \in \mathbb{Z}^n \) and \( v \in \mathbb{N}_p \).

Let \( Q \) be a cube in \( \mathbb{R}^n \) and \( m > 0 \), then \( mQ \) is the cube in \( \mathbb{R}^n \) concentric with \( Q \) and with side length \( m \) times the side length of \( Q \). We write \((v,k) < (v',k')\) if \( v \geq v' \) and

\[
Q_{v,k} \subset 2Q_{v',k'} \quad \text{with} \quad Q_{v,k} = 2Q_{v,k'} \subset Q^n.
\]

Let \( a \in \mathbb{R} \), then \( a_e = \max(a,0) \) and \([a]\) stands for the largest integer less than or equal to \( a \).

**Definition 7 (see [16])**. Let \( s \in \mathbb{R}, 0 < p \leq 1 < r \leq \infty \). Let \( K \) and \( L \) be integers with

\[
K \geq (\lceil s \rceil + 1)s, \quad \text{and} \quad L \geq \max\left\{\left\lfloor n\left(\frac{1}{p} - 1\right) - s\right\rfloor, -1\right\}.
\]

(1) The (complex-valued) function \( f(x) \) is called a \( s \)-atom if \( \supp a \subset 5Q \), for some \( Q = Q_{v,k} \in \mathbb{R}^n \) and

\[
|D^\alpha f(x)| \leq 1 \quad \text{for} \quad |\alpha| \leq K.
\]

(2) Let \( Q = Q_{v,k} \in \mathbb{R}^n \). The (complex-valued) function \( f(x) \) is called a \((Q,s,p,r)-\)atom if (20) is satisfied,

\[
|D^\alpha f(x)| \leq |Q|^{-1/r + (s/n) - (|\alpha|/n)} \quad \text{for} \quad |\alpha| \leq K,
\]

and

\[
\int_{\mathbb{R}^n} x^\beta f(x) dx = 0 \quad \text{for} \quad |\beta| \leq L.
\]

(3) The distribution \( g \in S' \) is called an \((s,p,r)-\)atom if

\[
g = \sum_{\mu \in E(s,k)} d_{\mu} a_{\mu}(x) \quad \text{(convergence in} \mathcal{F}_{p,r}')
\]

for some \( v \in \mathbb{N}_p \) and \( k \in \mathbb{Z}^n \), where \( a_{\mu}(x) \) is a \((Q_{v,k},s,p,r)-\)atom and \( d_{\mu} \) are complex numbers with

\[
\left( \sum_{\mu \in E(s,k)} |d_{\mu}| \right)^{1/p} \leq |Q_{v,k}|^{(1/p) - (1/q)}
\]

with usual modification if \( q = \infty \).

**Lemma 8** (see [16]). Let \( s \in \mathbb{R}, 0 < p \leq 1 < r \leq \infty \). Let \( K \) and \( L \) be fixed integers satisfying (19). Then \( f \in S' \) is an element of \( \mathcal{F}_{p,r}' \) if and only if it can be represented as

\[
f = \sum_{j=1}^{\infty} (\mu_j a_j + \lambda_j g_j) \quad \text{(convergence in} S')
\]

where \( a_j \) are \( s \)-atoms, \( g_j \) are \((s,p,r)-\)atoms, \( \mu_j \) and \( \lambda_j \) are complex numbers with

\[
\left( \sum_{j=1}^{\infty} |\mu_j|^p + |\lambda_j|^p \right)^{1/p} \leq \|f\|_{p,r}'.
\]

We also give the following lemma for inclusion relations between Besov and \( \alpha \)-modulation spaces [8].

**Lemma 9**. Let \( 0 < p, q \leq \infty \), and \( s \in \mathbb{R} \). Then the following tow statement are true:

1. \( M_{p,q}^\alpha \subset M_{p,q}^\alpha \quad \text{if and only if} \quad s > 0 \vee \left[ n(\alpha - 1) \left( \frac{1}{p} - \frac{1}{q} \right) \right] \vee \left[ n(\alpha - 1) \left( 1 - \frac{1}{p} \right) \right].
\]

2. \( B_{p,q} \subset M_{p,q}^\alpha \quad \text{if and only if} \quad s \leq 0 \wedge \left[ n(\alpha - 1) \left( \frac{1}{p} - \frac{1}{q} \right) \right] \wedge \left[ n(\alpha - 1) \left( 1 - \frac{1}{p} \right) \right] \).

**Lemma 10** (Young’s inequality).

(1) \( 0 < p \leq 1, R > 0, \supp f, \supp g \subseteq B(x,R) \subseteq \mathbb{R}^n \).

We have

\[
\|f \ast g\|_L_s \leq CR_{R^p}(1/p - 1)\|f\|_L_s \|g\|_L_s.
\]

for all \( f, g \in S(\mathbb{R}^n) \) and \( R > 0 \), where \( C \) independent of \( x \in \mathbb{R}^n \).

(2) \( 1 \leq p, q, r \leq \infty \) satisfy \( 1 + (1/q) = (1/p) + (1/r) \).

Then we have

\[
\|f \ast g\|_L_s \leq \|f\|_L_s \|g\|_L_s.
\]

The following Bernstein multiplier theorem will be used in our proof.

**Lemma 11** (Bernstein multiplier theorem). Let \( 0 < p \leq 1, \partial^p f \in L_2 \) for \( |\partial| \leq |n(1 - \alpha)((1/p) - (1/2))| + 1 \).

Then,

\[
\left\| \partial^p f \right\|_{L^p} \leq \sum_{|\partial| \leq |n(1 - \alpha)((1/p) - (1/2))|+1} \left\| \partial^p f \right\|_{L^p}.
\]

3. Main Results

Now, we state our main results as follows.

**Theorem 12**. Let \( 0 < p \leq 1, 0 < q, r \leq \infty \), \( s \in \mathbb{R} \), and \( 0 \leq \alpha < 1 \). Then, \( M_{p,q}^\alpha(\mathbb{R}^n) \subset F_{p,q}^\alpha(\mathbb{R}^n) \) holds if and only if either of the following conditions is satisfied.

1. \( p \geq q, s \geq 0, 1 \leq \frac{1}{p} \leq \frac{1}{q} \).
2. \( p < q, s > n(1 - \alpha)\left( \frac{1}{p} - \frac{1}{q} \right) \).

**Theorem 13**. Let \( 0 < p \leq 1, 0 < q, r \leq \infty \), \( s \in \mathbb{R} \), and \( 0 \leq \alpha < 1 \). Then, \( F_{p,q}^\alpha(\mathbb{R}^n) \subset M_{p,q}^\alpha(\mathbb{R}^n) \) holds if and only if either of the following conditions is satisfied.

1. \( p > q, s < -n(1 - \alpha)\left( \frac{1}{p} + \frac{1}{q} - 1 \right) \).
2. \( p \leq q, s \leq -n(1 - \alpha)\left( \frac{1}{p} + \frac{1}{q} - 1 \right) \).

We prove the following two propositions used for the proof of the Theorem 12.
Proposition 14. Let \( 0 < p < \infty \), \( 0 < q, r \leq \infty \), \( s \in \mathbb{R} \), and \( 0 \leq \alpha < 1 \). Then we have

\begin{align*}
(1) \quad & M_{p,q}^{\alpha} \subset F_{p,r} \Rightarrow \mathcal{E}_{q}^{\alpha(1-\alpha)} \subset \mathcal{E}_{r}^{\alpha}, \\
(2) \quad & F_{p,r} \subset M_{\alpha}^{p,q} \Rightarrow \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{r}^{\alpha(1-\alpha)}. \quad \Box
\end{align*}

Proof. Take \( f \) to be a nonzero smooth function whose Fourier transform has small support, such that \( \hat{f}_{k} \in \mathcal{F}_{p} \) if \( \mathbf{k} \neq \mathbf{m} \), where we denote \( \mathbf{f}_{k} = \hat{f}(k) \). We denote \( \mathbf{f}_{k} = \hat{f}(k) \). Denote

\[ E = \sum_{k \in \mathbb{Z}} a_{k} f_{k}. \]

For a truncated (only finite non-zero items) non-negative sequence \( \{ a_{k} \}_{k \in \mathbb{Z}} \), we have

\[ \|E\|_{F_{p,r}} = \left( \sum_{k \in \mathbb{Z}} |a_{k}|^{r} \right)^{1/r} \sim \left( \sum_{k \in \mathbb{Z}} |a_{k}|^{\alpha} \right)^{1/r}. \]

On the other hand, we use Proposition 15 and Lemma 3 to deduce that \( E \) is a smooth function whose Fourier sequence \( \{ a_{k} \}_{k \in \mathbb{Z}} \) non-negative sequence \( \{ a_{k} \}_{k \in \mathbb{Z}} \), where \( N \) is some large integer.

By the definition of \( \alpha \)-modulation space \( M_{p,q}^{\alpha} \), we have

\begin{align*}
\|H_{N}\|_{M_{p,q}^{\alpha}} &= \left( \sum_{k \in \mathbb{Z}} \|H_{N}^{k} \|_{L_{p}}^{q} \right)^{1/q} \\
&= \left( \sum_{k \in \mathbb{Z}} \|c_{k} f_{k}^{N} \|_{L_{p}}^{q} \right)^{1/q} \\
&= \left( \sum_{k \in \mathbb{Z}} \|c_{k}^{q} f_{k}^{q} \|_{L_{p}}^{q} \right)^{1/q} \\
&= \left( \sum_{k \in \mathbb{Z}} \|c_{k} f_{k} \|_{L_{p}}^{q} \right)^{1/q} \\
&= \|c_{k} f_{k} \|_{L_{p}}^{q}. \quad \Box
\end{align*}

Thus, we obtain \( \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{r}^{\alpha} \) if \( M_{p,q}^{\alpha} \subset F_{p,r} \).

On the other hand, we use Proposition 15 and Lemma 5 to deduce that \( \mathcal{E}_{r}^{\alpha} \subset \mathcal{E}_{q}^{\alpha} \), which implies \( (1/r) \leq (1/q) \).

For \( q \geq p_{0} \), using Proposition 14 and Lemma 3 to deduce \( \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{p_{0}}^{\alpha} \), which implies \( s > n(1-\alpha)/(1-(1/q)). \)

Sufficiency. For \( q \geq p \). We have \( (1/r) \leq (1/q) \), and then \( M_{p,q}^{\alpha} \subset B_{p,q} \). Using Lemma 9, we obtain \( M_{p,q}^{\alpha} \subset B_{p,q} \). Thus we deduce that \( M_{p,q}^{\alpha} \subset B_{p,q} \subset F_{p,q} \subset F_{p,r} \).

Thus, we obtain \( \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{r}^{\alpha} \) if \( M_{p,q}^{\alpha} \subset F_{p,r} \).

Proposition 15. Let \( 0 < p < \infty \), \( 0 < q, r \leq \infty \), \( s \in \mathbb{R} \) and \( 0 \leq \alpha < 1 \). Then we have

\begin{align*}
M_{p,q}^{\alpha} \subset F_{p,r} \Rightarrow \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{r}^{\alpha}, \\
M_{p,q}^{\alpha} \subset F_{p,r} \Rightarrow \mathcal{E}_{q}^{\alpha} \subset \mathcal{E}_{r}^{\alpha}. \quad \Box
\end{align*}
Proof. Let \( g \) be a nonzero Schwartz function whose Fourier transform has compact support in \( \{ \xi : 3/4 \leq |\xi| \leq 4/3 \} \), satisfying \( g(\xi) = 1 \) on \( \{ \xi : 7/8 \leq |\xi| \leq 8/7 \} \). Set \( \tilde{g}(\xi) = \tilde{g}(\xi/2^j) \). By the definition of \( \Delta_j \), we have \( \Delta_j g_j = g_j \) for \( j \geq 0 \), and \( \Delta_j g_j = 0 \) if \( j \neq i \). Denote
\[
A_j = \left\{ k \in \mathbb{Z}^n : g_j^N = F^{−1} \eta^N_k \right\}, \quad A_j = \left\{ k \in \mathbb{Z}^n : g_j^N \neq 0 \right\},
\]
we have \( |A_j| \sim |A_j| \sim 2^{2(j−1)} \) for \( j \geq N \), where \( N \) is a sufficiently large number. We define
\[
G_N = \sum_{j \geq N} b_j^N, \quad g_j^N(x) = g_j(x − jN),
\]
for a truncated (only finite nonzero items) nonnegative sequence \( \{b_j^N\} \).

We first prove that the inclusion \( M^{fa}_{p,q} \subset F_{p,s} \) implies \( \varepsilon_j^{\rho(1−\alpha)/q+\alpha(1−(1/p))}<1 \subset \varepsilon_{p}^{\rho(1−(1/p))}<1 \). By the definition of \( \alpha \)-modulation space, we obtain that
\[
\|G_N\|_{M^{fa}_{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| b_k g_N \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| b_k^N \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{j \geq N} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| F^{−1} \eta^N_k \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho}. \]

Hence,
\[
\|G_N\|_{M^{fa}_{p,q}} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho}. \]

Thus, if \( M^{fa}_{p,q} \subset F_{p,s} \), we obtain the desired inclusion
\[
\varepsilon_j^{\rho(1−\alpha)/q+\alpha(1−(1/p))}<1 \subset \varepsilon_{p}^{\rho(1−(1/p))}<1. \]

Next we prove that the inclusion \( F_{p,s} \subset M^{fa}_{p,q} \) implies \( \varepsilon_p^{\rho(1−(1/p))} \subset \varepsilon_{p}^{\rho(1−(1/p))} \). By the definition of \( \alpha \)-modulation space, we obtain that
\[
\|G_N\|_{M^{fa}_{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| b_k g_N \|_{L_p}^\rho \right)^{1/\rho} \geq \left( \sum_{j \geq N} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| b_k^N \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{j \geq N} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| \Delta_j g_j \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{j \geq N} \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\rho(1−\alpha)} \| \Delta_j g_j \|_{L_p}^\rho \right)^{1/\rho} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho}. \]

On the other hand, we turn to the estimate of \( \|G_N\|_{F_{p,s}} \), using the orthogonality of \( \{g_j^N\} \) as \( N \to \infty \), we obtain
\[
\|G_N\|_{F_{p,s}} = \left( \sum_{j \geq N} \| \Delta_j g_j^N \|_{L_p}^{\rho} \right)^{1/\rho} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho} \leq \left( \sum_{j \geq N} \| b_j^N \|_{L_p}^{\rho} \right)^{1/\rho}. \]

Hence,
\[
\lim_{N \to \infty} \|G_N\|_{F_{p,s}} = \|b_j\|_{L_p}^{\rho(1−(1/p))}. \]

Thus, if \( F_{p,s} \subset M^{fa}_{p,q} \), we obtain the desired inclusion
\[
\varepsilon_p^{\rho(1−(1/p))} \subset \varepsilon_{p}^{\rho(1−(1/p))}. \]

Proposition 17. Let \( 0 < p \leq 1 \). We have the following inclusion relation:
\[
F_{p,\alpha}^{(2/p)−1} \subset M^{fa}_{p,q}. \]
Proof. We first verify
\[ \|a\|_{M_{\nu}^{\alpha}} \leq 1, \quad (53) \]
for any \( n(1 - \alpha)((2/p) - 1) \)-atom \( a \). Tack \( a \) to be an \( n(1 - \alpha) \)
\( ((2/p) - 1) \)-atom as in Definition 7 (with \( s = n(1 - \alpha)((2/p) - 1) \)). Observing that \( K \geq n(1 - \alpha)((2/p) - 1) \) \( + 1 \geq n(1 - \alpha) \)
\( ((1/p) - (1/2)) \) \( + 1 \), we have
\[ |\tilde{\partial}^\beta a| \leq 1, \quad (54) \]
for \( |\delta| \leq n(1 - \alpha)((1/p) - (1/2)) \) \( + 1 \). By the Bernstein multiplier theorem, we have the following estimate of \( a \):
\[ \|a\|_{M_{\nu}^{\alpha}} \sim \|F^{-1} a\|_{L_2} \leq \sum_{\{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) \} + 1} \|\tilde{\partial}^\beta a\|_{L_2} \leq 1. \quad (55) \]
Next, we turn to the estimate of an \( (s, p, \infty) \)-atom for \( F_{p,\infty}^{(n-1)\alpha}(2/p-1) \). By Definition 7, an \( (s, p, \infty) \)-atom \( g \) can be represented by
\[ g = \sum_{(\mu,j) \in (k,v)} d_{\mu,j}a_{\mu,j}(x) \quad \text{convergence in } F_{p,\infty}^{(n-1)\alpha}(2/p-1), \quad (56) \]
for some \( k \in \mathbb{Z}^n \) and \( v \in \mathbb{N}_p \), where \( a_{\mu,j} \) are \( (\mu,j, s, p, \infty) \)-atoms and \( d_{\mu,j} \) are complex numbers with
\[ \sup_{(\mu,j) \in (k,v)} |d_{\mu,j}| \leq |Q_{\mu,j}|^{-1/p}, \quad (57) \]
for a fixed \( \tau \leq v \), we denote
\[ g_{\tau} = \sum_{(\tau,j) \in (k,v)} d_{\mu,j}a_{\mu,j}(x). \quad (58) \]
Then, \( g \) can be represented by
\[ g = \sum_{\tau \leq v} g_{\tau} \quad \text{convergence in } F_{p,\infty}^{(n-1)\alpha}(2/p-1), \quad (59) \]
We now concentrate on the estimate of \( g_{\tau} \). By Definition 7, we have,
\[ |\tilde{\partial}^\beta g_{\tau}| \leq \sum_{(\tau,j) \in (k,v)} d_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \leq \sum_{(\tau,j) \in (k,v)} d_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \leq |Q_{\mu,j}|^{-1/p} |\tilde{\partial}^\beta a_{\mu,j}(x)|, \quad (60) \]
for \( |\delta| \leq n(1 - \alpha)((1/p) - (1/2)) \) \( + 1 \). Recalling \( \text{supp } a_{\mu,j} \subset 5Q_{\mu,j} \), we use (60) and the almost orthogonality of \( a_{\mu,j} \) to deduce that
\[ |\tilde{\partial}^\beta g_{\tau}| \leq \sum_{(\tau,j) \in (k,v)} d_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \leq \sum_{(\tau,j) \in (k,v)} d_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \leq |Q_{\mu,j}|^{-1/p} |\tilde{\partial}^\beta a_{\mu,j}(x)|, \quad (61) \]
for all \( |\delta| \leq n(1 - \alpha)((1/p) - (1/2)) \) \( + 1 \). By the Bernstein multiplier theorem, we deduce that
\[ \|g_{\tau}\|_{M_{\nu}^{\alpha}} \sim \|F^{-1} g_{\tau}\|_{L_2} \leq \sum_{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) + 1} \|\tilde{\partial}^\beta g_{\tau}\|_{L_2} \leq \sum_{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) + 1} |Q_{\mu,j}|^{(1/p)-1} |\tilde{\partial}^\beta a_{\mu,j}(x)|. \quad (62) \]
By a dilation argument, we have
\[ \|g_{\tau}\|_{M_{\nu}^{\alpha}} \sim \|F^{-1} g_{\tau}\|_{L_2} \leq \sum_{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) + 1} |Q_{\mu,j}|^{(1/p)-1} |\tilde{\partial}^\beta a_{\mu,j}(x)|. \quad (63) \]
Thus,
\[ \|g_{\tau}\|_{M_{\nu}^{\alpha}} \leq \sum_{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) + 1} \|\tilde{\partial}^\beta g_{\tau}\|_{L_2} \leq \sum_{|\delta| \leq n(1 - \alpha)((1/p) - (1/2)) + 1} |Q_{\mu,j}|^{(1/p)-1} |\tilde{\partial}^\beta a_{\mu,j}(x)| \sim 1. \quad (64) \]
By Lemma 8 we have
\[ \|f\|_{M_{\nu}^{\alpha}} \leq \sum_{j=1}^{\infty} \left( \sum_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \right)^{1/p} \leq \sum_{j=1}^{\infty} \left( \sum_{\mu,j} |\tilde{\partial}^\beta a_{\mu,j}(x)| \right)^{1/p} \leq \sum_{j=1}^{\infty} \left( \sum_{\mu,j} \right)^{1/p} \leq \|f\|_{F_{p,\infty}^{(n-1)\alpha}(2/p-1)}, \quad (65) \]
which is the desired conclusion. \( \square \)

Proof of Theorem 13. We divide this proof into two parts.

Sufficiency. For \( p \geq q \), by Lemma 9, we obtain \( B_{p,\infty} \subset M_{p,\infty}^{(n-1)\alpha}(1/(1/p)) \). Using \( F_{p,\infty} \subset B_{p,\infty} \) we deduce that
\[ F_{p,\infty} \subset M_{p,\infty}^{(n-1)\alpha}(1/(1/p)) \alpha. \quad (66) \]
In addition, we have \( F_{p,\infty} \subset M_{p,\infty} \) by Proposition 17. By potential lifting, we obtain
\[ F_{p,\infty} \subset M_{p,\infty}^{(n-1)\alpha}(1/(1/p)) \alpha. \quad (67) \]
Thus, the desired conclusion can be deduced by a standard interpolation argument between (66) and (67).

For \( p < q \), recalling \( F_{p,\infty} \subset M_{p,\infty}^{(n-1)\alpha}(1/(2/p)) \alpha \) obtained in Proposition 17, we deduce that
\[ F_{p,\infty} \subset M_{p,\infty}^{(n-1)\alpha}(1/(2/p)) \alpha \subset M_{p,q}^{(n-1)\alpha}(1/(1/p) - (1/q)) - \varepsilon \alpha, \quad (68) \]
for any \( \varepsilon > 0, r \in (0, \infty) \).
Necessity. We use Proposition 16 to deduce inclusion relation $c_0^{(\alpha)}(1-(\alpha)/p)) \subset c_0^{(\alpha)}((\alpha)-1)$. Then, Lemma 4 yields that $s \leq -n(1-\alpha)((1/p)+(1/q)-1)$ for $p \leq q$, while the inequality is strict for $p > q$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] H. G. Feichtinger, “Modulation spaces on locally compact Abelian group,” in Proceedings of the International Conference on Wavelet and their Applications, pp. 99–140, University of Vienna, New Delhi Allied Publishers, India, 1983.
[2] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.
[3] M. Ruzhansky, M. Sugimoto, and B. Wang, “Modulation spaces and nonlinear evolution equations,” in Evolution Equations of Hyperbolic and Schrödinger Type. Progress in Mathematics, vol. 301, pp. 267–283, Springer, 2012.
[4] B. Wang and H. Hudzik, “The global cauchy problem for the NLS and NLKG with small rough data,” Journal of Differential Equations, vol. 232, no. 1, pp. 36–73, 2007.
[5] B. Wang and C. Huang, “Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations,” Journal of Differential Equations, vol. 239, no. 1, pp. 213–250, 2007.
[6] P. Gröbner, “Banachräume glatter Funktionen und Zerlegungsmethoden,” University of Vienna, 1992, Doctoral Thesis.
[7] W. Guo, D. Fan, H. Wu, and G. Zhao, “Sharpness of complex interpolation on $\alpha$-modulation spaces,” Journal of Fourier Analysis and Applications, vol. 22, no. 2, pp. 427–461, 2016.
[8] J. Han and B. Wang, “$\alpha$-modulation spaces (I) scaling, embedding and algebraic properties,” Journal of the Mathematical Society of Japan, vol. 66, no. 4, pp. 1315–1373, 2014.
[9] W. Guo, D. Fan, H. Wu, and G. Zhao, “Full characterization of inclusion relations between $\alpha$-modulation spaces,” Science China Mathematics, vol. 61, no. 7, pp. 1243–1272, 2018.
[10] T. Kato, “On modulation spaces and their applications to dispersive equations,” Graduate School of Mathematics, Nagoya University, 2016, Doctoral Thesis.
[11] J. Toft and P. Wahlberg, “Embeddings of $\alpha$-modulation spaces,” Pliska Studia Mathematica Bulgarica, vol. 21, pp. 25–46, 2012.
[12] G. Misiolek and T. Yoneda, “Loss of continuity of the solution map for the Euler equation in $\alpha$-modulation and Hölder spaces,” 2014, https://arxiv.org/abs/1412.4619.
[13] J. Han and B. Wang, “$\alpha$-modulation spaces and the cauchy problem for nonlinear Schrödinger equations,” in RIMS Kôkyûroku Bessatsu, vol. B49, pp. 119–130, Research Institute for Mathematical Sciences (RIMS), Kyoto, 2014.
[14] J. Han and B. Wang, ”$\alpha$-modulation spaces (II) derivative NLS,” Journal of Differential Equations, vol. 267, no. 6, pp. 3646–3692, 2019.
[15] M. Fornasier, “Banach frames for $\alpha$-modulation spaces,” Applied and Computational Harmonic Analysis, vol. 22, no. 2, pp. 157–175, 2007.
[16] H. Triebel, “Monographs in Mathematics,” Theory of function spaces. II, vol. 84, Birkhauser Verlag, Basel, 1992.