A new exactly solvable two-dimensional quantum model not amenable to separation of variables

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Abstract

The supersymmetric intertwining relations with second-order supercharges allow us to investigate a new two-dimensional model which is not amenable to standard separation of variables. The corresponding potential being the two-dimensional generalization of a well-known one-dimensional Pöschl–Teller model is proven to be exactly solvable for an arbitrary integer value of the parameter \(p\). All its bound state energy eigenvalues are found analytically, and the algorithm for analytical calculation of all wavefunctions is given. The shape invariance of the model and its integrability are of essential importance to obtain these results.

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1. Introduction

The beautiful idea of supersymmetry (SUSY) was first introduced [1] and developed in quantum field theory and elementary particle theory in the 1970s. During these years, SUSY became one of the most popular and promising branches of modern high-energy physics [2].

SUSY was also studied in the simplest toy model of (0+1) quantum field theory (i.e. in nonrelativistic quantum mechanics) in order to clarify some delicate problems of spontaneous supersymmetry breaking [3]. Very soon, this by-product of supersymmetrical quantum field theory became a new independent tool to study many problems in quantum mechanics itself [4]. In particular, the notions of SUSY intertwining relations [4, 5] and of shape invariance [6] provided both new methods to derive some old results and to obtain new interesting results. As an example, all previously known one-dimensional exactly solvable potentials were reproduced as potentials obeying the shape invariance [7]. In turn, SUSY intertwining relations were successfully used [8–11] in two-dimensional quantum mechanics to obtain a
variety of partially (quasi-exactly) solvable\(^3\) models which are not amenable to conventional separation of variables [13]. Until recently, the latter method was the sole practical tool to solve analytically two-dimensional (and higher dimensional) quantum problems.

Thus, supersymmetrical approach can be considered as a new method [9, 10] to solve (at least, partially) some two-dimensional Schrödinger equations. The procedure can be called SUSY separation of variables. It is realizable for models where the equation for zero modes of second-order supercharges allows separation of variables [10]. More than that, after separation of variables, one-dimensional equations must also be exactly solvable. Another procedure of SUSY separation of variables works if one of the partner Hamiltonians does allow standard separation of variables due to special choice of parameters. Then, SUSY intertwining relations may allow one to obtain eigenfunctions of the second partner Hamiltonian, which is not amenable to standard separation of variables. This approach was used successfully [14] for the two-dimensional generalization of the Morse potential with integer or half-integer values of the parameter.

This paper provides a new exactly solvable two-dimensional model with potential depending on three parameters, one of which has to be an integer. To solve the problem, it will be necessary to explore essentially both main ingredients of SUSY quantum mechanics: SUSY intertwining relations and shape invariance. Schematically, to solve the Schrödinger equation with the potential \(V(\vec{x}; A, B, p)\) depending on the parameters \(A, B, p\), three steps will be followed. First, to find such an exclusive value of the parameter (actually, \(p = 1\)) that the initial Hamiltonian \(H(p = 1)\) does allow conventional separation of variables. Second, using SUSY intertwining relations and shape invariance, to build eigenfunctions for Hamiltonians \(H(p), \ p = 2, 3, \ldots\), which are not already amenable to separation of variables. And finally, to prove that all constructed wavefunctions are normalizable and that no extra levels exist.

The structure of the paper is as follows. In section 2, the model and its main properties are formulated, and the scheme of investigation is reviewed. The separation of variables for the first Hamiltonian \(H(p = 1)\) is performed and some delicate properties of the potential are discussed in section 3. The zero modes of the supercharge are built in section 4, and they are used for the construction of wavefunctions in section 5. Normalizability of wavefunctions is studied in section 6, where the absence of any other bound states was proven. A few examples of wavefunctions for low values of the parameter \(p\) are given in section 7. Conclusions include the comparison of obtained results with the limiting case which is explicitly solvable. Rather cumbersome calculation of coefficients necessary for wavefunctions and the spectrum are presented in the appendix.

2. Formulation of the model and the general scheme

We consider the intertwining relations of the form

\[
Q^- H = \tilde{H} Q^-; \quad H Q^+ = Q^+ \tilde{H},
\]

where \(H\) and \(\tilde{H}\) are two-dimensional Hamiltonians of the Schrödinger type:

\[
H = -\left(\partial_1^2 + \partial_2^2\right) + V(x_1, x_2), \quad \tilde{H} = -\left(\partial_1^2 + \partial_2^2\right) + \tilde{V}(x_1, x_2),
\]

and the intertwining operators \(Q^\pm\) are second-order differential operators. A number of models of this kind were investigated in [8–11]. In [14], it was proven that one of these—the

\(^3\) By definition, partial (quasi-exact) solvability of the model means that a part of its energy spectrum and corresponding wavefunctions are known. Such models take up an intermediate place between exactly solvable ones and models with unknown spectra [12].
generalized two-dimensional Morse—possesses exact solvability. Here we shall show that another model [11] involved in intertwining relations (1) is also exactly solvable. It reads

\[ H(p) = -\partial^2_1 - \partial^2_2 - 2p(p - 1)(\cosh^2(x_+) + \cosh^2(x_-)) + k_1(\sinh^2(x_2) - \cosh^2(x_1)) + k_2(\cosh^2(x_2) - \sinh^2(x_1)), \]  

(3)

\[ \tilde{H}(p) = -\partial^2_1 - \partial^2_2 - 2p(p + 1)(\cosh^2(x_+) + \cosh^2(x_-)) + k_1(\sinh^2(x_2) - \cosh^2(x_1)) + k_2(\cosh^2(x_2) - \sinh^2(x_1)), \]  

(4)

\[ Q^± = \partial^2_1 - \partial^2_2 \pm 2p(\tanh(x_+) + \tanh(x_-)) \partial_1 \pm 2p(\tanh(x_-) - \tanh(x_+)) \partial_1 \pm 4p^2 \tanh(x_+) \tanh(x_-) + k_1(\sinh^2(x_2) + \cosh^2(x_1)) + k_2(\cosh^2(x_2) + \sinh^2(x_1)), \]  

(5)

where \( p \) and \( k_{1,2} \) are real parameters, so far arbitrary, and \( x_\pm = x_1 \pm x_2 \). The Hamiltonians in (3), (4) can be represented in the form

\[ H_{p,T}(x_1) + H_{p,T}(x_2) + f(x_1, x_2), \]  

(6)

where \( H_{p,T}(x) \) are well-known one-dimensional Pöschl–Teller Hamiltonians, and \( f(x_1, x_2) \) is the specific term mixing the variables \( x_1 \) and \( x_2 \) in potentials. Due to this expansion, the potentials \( V(x_1, x_2), \tilde{V}(x_1, x_2) \) may be considered [11] as a two-dimensional generalization of the Pöschl–Teller potential. These models are shape-invariant [6] with respect to the parameter \( p \):

\[ H(p + 1) = \tilde{H}(p). \]  

(7)

Properties (1) and (7) will be essential for the proof of exact solvability of the model for positive integer values of \( p \).

We remark that, by construction, these models are integrable since from intertwining relations (1) it follows that

\[ [H, R] = 0, \quad [\tilde{H}, \tilde{R}] = 0; \quad R = Q^+ Q^- , \quad \tilde{R} = Q^- Q^+ , \]  

(8)

with symmetry operators of fourth order in momenta.

The general scheme to determine the spectrum of the model (3) could be adopted from [14], where the full spectrum of the two-dimensional generalization of the Morse potential [9, 10] was found. In the present context, the plan for construction could be as follows: we start with \( H(p = 1) \) and find all normalizable solutions \( \Psi(\vec{x}; p = 1) \) for the corresponding Schrödinger equation (section 3) as far as it is amenable to separation of variables. Then, by means of intertwining relations (1), we find eigenfunctions \( \tilde{\Psi}(\vec{x}; p = 1) \) of \( \tilde{H}(1) \). In general, they might be of two types [9]: some of them are inherited from \( H(1) \) as: \( \tilde{\Psi}(\vec{x}; 1) = Q^-(1)\Psi(\vec{x}; 1) \), and others are zero modes of the intertwining operator \( Q^+(1)\tilde{\Psi}(\vec{x}; 1) = 0 \). In such a way we obtain all eigenfunctions of \( H(2) \). Due to the shape invariance (7) of the model, \( \Psi(\vec{x}; p + 1) = \tilde{\Psi}(\vec{x}; p) \), and therefore, we have calculated already \( \Psi(\vec{x}; 2) \). Following this strategy step by step, we expect to find the eigenfunctions and eigenvalues for the Hamiltonians \( \tilde{H}(p) = H(p+1) \) with arbitrary integer values \( p = 1, 2, \ldots \).

At each step, the full variety of eigenfunctions of \( H(p + 1) \) will belong to one of two classes: (1) each normalizable wavefunction \( \Psi(\vec{x}; p) \) leads to the normalizable wavefunction \( \Psi(\vec{x}; p + 1) = \Psi(\vec{x}; p) = Q^-(p)\Psi(\vec{x}; p) \); (2) the same Hamiltonian \( H(p + 1) \) also has some number of extra normalizable functions which are specific linear combinations of zero modes \( \Omega(\vec{x}; p) \) of the operator \( Q^+(p) \). We shall see below that this plan has to be modified suitably for the case of our present model, but the main ideas will be analogous to that of [9, 10, 14].
3. Separation of variables for $H(p = 1)$

For the Hamiltonian $H(1)$ the standard procedure of separation of variables in Cartesian coordinates can be applied. Looking for solutions of the Schrödinger equation $H(1)\psi(\vec{x}; 1) = E\psi(\vec{x}; 1)$ in the form $\Psi(\vec{x}; 1) = \eta(x_1)\rho(x_2)$, one obtains two one-dimensional equations for unknown functions $\rho$, $\eta$:

\begin{align*}
-\eta''(x_1) - \left(\frac{k_1}{\cosh^2 x_1} + \frac{k_2}{\sinh^2 x_1}\right)\eta(x_1) &= \varepsilon\eta(x_1), \\
-\rho''(x_2) + \left(\frac{k_2}{\cosh^2 x_2} + \frac{k_1}{\sinh^2 x_2}\right)\rho(x_2) &= \tilde{\varepsilon}\rho(x_2),
\end{align*}

where prime denotes the derivative of the function with respect to its argument, $\varepsilon + \tilde{\varepsilon} = E$ is the energy value for $H(1)$, and both $\varepsilon$ and $\tilde{\varepsilon}$ must be negative for the discrete part of the spectrum.

Thus, we have to consider solutions of one-dimensional Schrödinger equations (9) and (10) with Pöschl–Teller potentials $V_{P-T}(x)$. It is convenient to replace the parameters $k_1, k_2$ by $A, B$ according to $k_1 \equiv B(B - 1)$; $k_2 \equiv -A(A - 1)$. Avoiding the case of fall onto the center [15], we shall restrict ourselves with reasonably attracting singularity in $V_{P-T}(x)$, $V_{P-T}(x)$ with coefficients $k_1 \in (-1/4, 0)$, $k_2 \in (0, 1/4)$, i.e. it is sufficient to take $A, B \in (0, 1/2)$. The substitution $\eta(x_1) = \sinh^2(x_1)\cosh^2(x_1)\rho(x_1)$, and the subsequent change of variable $x_1$ to $z = -\sinh^2(x_1)$, turns (9) into the hypergeometric equation for the function $F(z)$:

\[ z(1 - z)\frac{d^2 F(z)}{dz^2} + \left(A + \frac{1}{2} - (A + B + 1)z\right)\frac{dF(z)}{dz} + \left(-\frac{1}{4} (A + B)^2 - \frac{1}{4} \varepsilon\right)F(z) = 0. \]

The pair of independent solutions for the given value of $\varepsilon$ reads (see 2.3.1(1) in [16])

\begin{align*}
\eta^{(1)}_{\varepsilon}(x) &= \sinh^2(x)\cosh^2(x)\, _2F_1\left(\frac{A + B + \sqrt{-\varepsilon}}{2}, \frac{A + B - \sqrt{-\varepsilon}}{2}; A + \frac{1}{2}; -\sinh^2(x)\right), \\
\eta^{(2)}_{\varepsilon}(x) &= \sinh^{1-A}(x)\cosh^2(x)\, _2F_1\left(\frac{1 - A + B + \sqrt{-\varepsilon}}{2}, \frac{1 - A + B - \sqrt{-\varepsilon}}{2}; \frac{3}{2} - A; -\sinh^2(x)\right).
\end{align*}

The similarity of expressions (11) and (12) reflects the obvious symmetry of potential in (9) under $A \rightarrow (1 - A)$. The potential under consideration also obeys the similar symmetry under $B \rightarrow (1 - B)$. But the corresponding independent solutions are related to solutions (11) and (12) according to relations between hypergeometric functions (see 2.1.4(23) in [16]). The formulas analogous to (11) and (12) also hold for $\rho^{(1,2)}_{\varepsilon}(x_2)$, but with the necessary changes $A \rightarrow B, B \rightarrow A$ and $\varepsilon \rightarrow \tilde{\varepsilon}$.

To provide the normalizability of $\Psi(\vec{x}; 1)$, both $\rho$ and $\eta$ must be normalizable. To analyze the possible bound states of $H(p = 1)$ it will be sufficient to consider the asymptotic behavior for large $|x_1|, |x_2|$.

The general solution $\eta_{\varepsilon}(x)$ is a linear combination

\[ \eta_{\varepsilon}(x) = \alpha_1\eta^{(1)}_{\varepsilon}(x) + \alpha_2\eta^{(2)}_{\varepsilon}(x) \]

with arbitrary constants $\alpha_1, \alpha_2$. The asymptotic behavior of the analytic continuation of hypergeometric functions for large $z = -\sinh^2 x$ (see 2.10(2) and 2.10(5) in [16]) reads

\[ _2F_1(a, b; c; z) = B_1(a, b, c)[(-z)^{-a} + O((-z)^{-a-1})] + B_2(a, b, c)[(-z)^{-b} + O((-z)^{-b-1})], \]

(14)
where the constants $B_{1,2}$ are expressed in terms of Gamma functions:

$$B_1(a, b, c) = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)}, \quad B_2(a, b, c) = \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)}.$$  

Substitution of (14) into (11), (12) and (13) gives asymptotically two groups of terms in $\eta_\varepsilon(x)$: proportional to $(-z)^{-\sqrt{1/2}}(1 + O(z^{-1}))$ and proportional to $(-z)^{\sqrt{1/2}}(1 + O(z^{-1}))$, correspondingly. In order to forbid the growing term in wavefunction, one has to require the coefficient to vanish:

$$\alpha_1 B_2 \left( \frac{A + B - \sqrt{\varepsilon}}{2}, \frac{A + B - \sqrt{\varepsilon}}{2}, A + \frac{1}{2} \right) + \alpha_2 B_2 \left( \frac{1 - A + B + \sqrt{\varepsilon}}{2}, \frac{1 - A + B - \sqrt{\varepsilon}}{2}, \frac{3}{2} - A \right) = 0. $$

In general, there are two options to fulfill this requirement:

$$\alpha_2 = 0; \quad B_2 \left( \frac{A + B + \sqrt{\varepsilon}}{2}, \frac{A + B + \sqrt{\varepsilon}}{2}, A + \frac{1}{2} \right) = 0; $$

$$\alpha_1 = 0; \quad B_2 \left( \frac{1 - A + B + \sqrt{\varepsilon}}{2}, \frac{1 - A + B + \sqrt{\varepsilon}}{2}, \frac{3}{2} - A \right) = 0. $$

In the case of arbitrary $A, B$, these conditions can be achieved by means of suitable choices of energy values $\varepsilon$, due to Gamma functions in denominators of coefficient $B_2$: arguments $a$ or $(c - b)$ of these Gamma functions must be equal $-n$ with $n = 0, 1, \ldots$. Just this condition might give the energies of bound states. However, one can check easily that in our present case of the parameters $A, B \in (0, 1/2)$, these conditions cannot be fulfilled for positive values of $\sqrt{-\varepsilon}$.

Thus, we are not able to kill the growing terms in the asymptotic of $\eta_\varepsilon$ and $\rho_\varepsilon$. Therefore, the Hamiltonian $H(p = 1)$ has no bound states because of asymptotic behavior at large $|x|$, and the first source for the construction of eigenfunctions of $\tilde{H}(1)$: $\Psi(\tilde{x}; 1) = Q^-(1)\Psi(\tilde{x}; 1)$ does not work. We stress that this statement depends crucially on a chosen region for values of $A, B$, which in its turn was dictated by conditions on both potentials $V_1$ and $V_2$, simultaneously. Taken separately, these Hamiltonians would have bound states, but for different values of $A, B$.

We also note that the conclusion above does not depend on the behavior of solutions at the singular point $x = 0$. Nevertheless, we will discuss the $x \to 0$ asymptotic of $\eta_\varepsilon^{(1,2)}$ here, since it will be necessary for the analysis in subsequent sections. The point is that (in contrast to the standard situation of nonsingular potentials) both solutions (11) and (12) have a zero limit at the origin $x \to 0$ for $A \in (0, 1/2)$. Namely, their behavior is $\eta_\varepsilon^{(1)} \sim x^A$ and $\eta_\varepsilon^{(2)} \sim x^{1-A}$. This is the typical situation of the so-called limit circle kind (see [17], appendix to section 10.1), which was widely discussed in the literature in the context of the one-dimensional potential (the so-called Calogero potential) $g/x^2$ on the semiaxis or on the whole axis (e.g. see [18–20]). In such a case, there is continuous freedom in choosing (among many opportunities) some ‘good kind of behavior’ for wavefunctions. The resulting spectrum of the model depends on this choice, thereby defining the kind of its quantization. The preferences are usually motivated by physical arguments [18, 19, 21]. In the case of the Hamiltonian $H(p = 1)$ this problem is of little importance due to the asymptotic behavior of solutions at infinity discussed above. One more remark concerns the extension of solutions to the negative semiaxis: it is reasonable to choose an odd way, taking $\eta(-|x|) = -\eta(|x|)$. This choice provides continuity of the derivative $\eta'(x)$ at the origin.

We have to remark that from a mathematical point of view, both a one-dimensional Schrödinger operator with the Pöschl–Teller potential in (9) and (10) and a two-dimensional
operator in (3) and (4) produce a rather nontrivial problem. It is possible to check that both (in the two-dimensional case, due to Green’s identity of vector calculus) are symmetric operators, but in a strictly mathematical approach, they are unbounded and not self-adjoint for the conventional choice of smooth functions with a compact support (dense in $L^2$) as a domain $D(H_{\alpha^{-}\tau}(x))$. Similar to the analysis given in [19] for the case $V = \alpha/x^2$, the self-adjoint extension of $H_{\alpha^{-}\tau}(x)$ also includes the functions from $L^2$ with specific asymptotic at the singular point $x = 0$. For details, we refer readers to [18–20] and references therein, where one will also find the description of some paradoxes induced by a too naive approach to singular potentials of $\alpha/x^2$ type.

4. Construction of zero modes of $Q^+$

As mentioned above, the second possible source of eigenfunctions for $\tilde{H}(p)$ are the zero modes of the operator $Q^+(p)$. It is well known [9, 10] that the subspace of zero modes of $Q^+(p)$ is invariant under the action of $\tilde{H}(p)$. This means that if $\tilde{\omega}_i(\vec{x}; p)$ is the zero mode of $Q^+(p)$, i.e. $Q^+(p)\tilde{\omega}_i(\vec{x}; p) = 0$, then due to intertwining relations (1)

$$\tilde{H}(p)\tilde{\omega}_i(\vec{x}; p) = \sum_{k=0}^{N} C_{ki}\tilde{\omega}_k(\vec{x}; p)$$

(15)

(where $C_{ki}$ is the $(N + 1) \times (N + 1)$ matrix with complex elements). If the matrix $C_{ki}$ can be diagonalized by some matrix $B$:

$$BC = \Lambda B; \quad \Lambda = \text{diag}(E_0, E_1, \ldots, E_N),$$

(16)

the functions

$$\tilde{\psi}_i(\vec{x}; p) = \sum_{k=0}^{N} B_{ki}\tilde{\omega}_k(\vec{x}; p)$$

(17)

are the eigenfunctions of $\tilde{H}(p)$: $\tilde{H}(p)\tilde{\psi}_i(\vec{x}; p) = E_i\tilde{\psi}_i(\vec{x}; p)$.

At first, one needs to calculate $\tilde{\omega}_i(\vec{x}; p)$. For this purpose, it is useful to perform the similarity transformation, which will help to separate variables:

$$q^+(p) = e^{p\chi(\vec{x})} Q^+(p) e^{-p\chi(\vec{x})}, \quad \tilde{\omega}_i = e^{-p\chi(\vec{x})}\tilde{\omega}_i,$$

(18)

where

$$\chi = \ln(\cosh(x_1)\cosh(x_2)).$$

After that, the problem $Q^+(p)\tilde{\omega}_i(\vec{x}; p) = 0$ becomes

$$q^+(p)\tilde{\omega}_i(\vec{x}; p) = 0,$$

(19)

where $q^+$ reads

$$q^+ = \partial_1^2 - \partial_2^2 + k_1(\sinh^{-2}(x_2) + \cosh^{-2}(x_1)) + k_2(\cosh^{-2}(x_2) + \sinh^{-2}(x_1)).$$

(20)

The choice of the function $\chi$ provides that (19) is amenable to separation of variables in Cartesian coordinates. The two-dimensional equation (19) is equivalent to the pair of one-dimensional ones if one takes $\tilde{\omega}_i = \eta_i(x_1)\rho_i(x_2)$, and it appears that they are exactly equations (9) and (10), but with $\varepsilon = \tilde{\varepsilon}$. The solutions can be written as linear combinations of

$$\tilde{\omega}_i = \eta_i(x_1)\rho_i(x_2),$$

(21)

where $\eta_i$ (and analogously $\rho_i$) must be built from solutions (11) and (12). Of course, we are interested only in normalizable zero modes $\Omega(\vec{x}; p)$ of the two-dimensional operator $Q^+(p)$.
but the normalizability condition, in comparison with section 3, is essentially less restrictive now:

\[
\int |\tilde{\Omega}(\tilde{x}; p)|^2 \, d^2 x = \int e^{-2p\chi(\tilde{x})}|\eta(x_1)|^2 |\rho(x_2)|^2 \, d^2 x = \int (\cosh(x_+)\cosh(x_-))^{-2p} |\eta(x_1)|^2 |\rho(x_2)|^2 \, d^2 x < \infty. \tag{22}
\]

The factor \( \exp(-2p\chi(\tilde{x})) \) in (22) is exponentially decreasing at infinity in all directions on the plane, and it is able to compensate even growing functions \( \tilde{a}_k \). Due to asymptotic equivalence \( \cosh x \sim \sinh x \) at infinity, the asymptotical behavior of the integrand of (22) can be represented as

\[
\tilde{\Omega}(\tilde{x}; p)^2 \sim (\cosh x_+ \cosh x_-)^{-2p} (\cosh x_+ - \cosh x_-)^{2\sqrt{-\varepsilon}}.
\]

Therefore, the functions \( \tilde{\Omega} \) are normalizable for arbitrary values of \( p \) and \( \varepsilon \), satisfying \( \varepsilon > -p^2 \). This fact has to be taken into account in calculation of the spectrum of \( H(p + 1) \) (see section 5).

But at first, we must define the variety of functions \( \tilde{\Omega}_k \), which may be used for the construction of actual wavefunctions. In this context, the functions \( \eta_1(x_1) \) and \( \rho_2(x_2) \) are the auxiliary objects for the construction of zero modes \( \tilde{\Omega} \) according to (18) and (21). Therefore, all four possible combinations can be used, in general. The first of them \( \tilde{\Omega}_e^{(1)}(\tilde{x}) = \exp(-p\chi(\tilde{x}))(A, B) \rho^{(1)}(x_2) \) is

\[
\tilde{\Omega}_e^{(1)}(\tilde{x}) = \pm (\cosh(x_+)\cosh(x_-))^{-p} \cosh^2(|x_1|) \cosh^2(|x_2|) \sinh^4(|x_2|) \cosh^2(|x_1|) \sinh^2(|x_2|) \cosh^2(|x_1|)
\]

\[
\cdot \cosh^2(|x_2|) \frac{\partial}{\partial t} F_1 \left(a_\varepsilon, b_\varepsilon; A + \frac{1}{2}; z_1\right) \cdot F_1 \left(a_\varepsilon, b_\varepsilon; B + \frac{1}{2}; z_2\right) - \varepsilon.
\]

where

\[
a_\varepsilon = \frac{A + B - \sqrt{-\varepsilon}}{2}; \quad b_\varepsilon = \frac{A + B + \sqrt{-\varepsilon}}{2}; \quad z_1 = -\sinh^2 x_1; \quad z_2 = -\sinh^2 x_2.
\]

and the sign \( \pm \) depends on a quarter of a plane \((x_1, x_2)\), according to the choice at the end of section 3. Other zero modes \( \tilde{\Omega}_e^{(2)}(\tilde{x}), \tilde{\Omega}_e^{(3)}(\tilde{x}), \tilde{\Omega}_e^{(4)}(\tilde{x}) \) are obtained from \( \tilde{\Omega}_e^{(1)}(\tilde{x}) \) by means of substitutions of pairs of parameters \((1 - A, B)\) for \( \tilde{\Omega}_e^{(2)}(\tilde{x})\), of \((A, 1 - B)\) for \( \tilde{\Omega}_e^{(3)}(\tilde{x})\) and \((1 - A, 1 - B)\) for \( \tilde{\Omega}_e^{(4)}(\tilde{x})\), instead of \((A, B)\) in \( \tilde{\Omega}_e^{(1)}(\tilde{x})\).

5. Eigenfunctions of the Hamiltonian \( H \)

In this section we look for linear combinations of zero modes of \( Q^+(p) \), which are simultaneously the eigenfunctions of the Hamiltonian \( \tilde{H}(p) = H(p + 1) \) in (4). Being interested in the discrete energy spectrum \( \tilde{E}_n \), we suppose that the corresponding wavefunctions are built from the finite number of zero modes \( \tilde{\Omega}_e^{(\gamma)}(\tilde{x}; p), \gamma = 1, 2, 3, 4 \), with the parameters \( a_\varepsilon \equiv a_{\gamma\varepsilon} \) in (24), numbering by discrete values \( k = 0, 1, 2, \ldots, N^{(\gamma)} \), so that the constants \( a_k \) are ordered as \( a_0 > a_1 > \cdots > a_N^{(\gamma)} \). We also suppose that four kinds of such wavefunctions exist: each is built from the corresponding zero modes \( \tilde{\Omega}_e^{(\gamma)} \) with a fixed value of \( \gamma \) (the value of \( \gamma \) defines the behavior at the origin). According to (15),

\[
\tilde{H}(p)\tilde{\Omega}_e^{(\gamma)}(\tilde{x}) = \sum_{i=0}^{N} C_i^{(\gamma)} \tilde{\Omega}_e^{(\gamma)}(\tilde{x}), \tag{25}
\]

where \( C_i^{(\gamma)} \) are constants, and \( N \) also depend on \( \gamma = 1, 2, 3, 4 \).
Performing with $\tilde{H}(p)$ the similarity transformation analogous to that with $Q^*(p)$ in (18), one obtains

$$h(p) = e^{p\epsilon(s)} \tilde{H}(p) e^{-p\epsilon(s)} = -a_2 - a_3 + D - \frac{k_1}{\cosh^2 x_1} - \frac{k_2}{\sinh^2 x_1}$$

$$+ \frac{k_2}{\cosh x_2} + \frac{k_1}{\sinh x_2} - 4p^2,$$

where the mixing operator $D$ is defined as

$$D = \frac{2p}{\cosh x_1 \cosh x_2} (\sinh(2x_1)\partial_1 + \sinh(2x_2)\partial_2).$$

(26)

Then, exploring (9), (10), the action of $\tilde{h}(p)$ on $\tilde{\omega}_{k_i}(p)$ (see its definition in (18) and (21)) can be expressed as

$$\tilde{h}(p)\tilde{\omega}_{k_i}(p) = 2(2p(A + B) + \epsilon_k - 2p^2)\tilde{\omega}_{k_i} + \sinh^4(x_1) \sinh^4(x_2)$$

$$\cdot \cosh^2(x_2) \tilde{D}(z F_1(a_k, b_k; A + 1/2; -\sinh^2 x_1) z F_1(a_k, b_k; B + 1/2; -\sinh^2 x_2)),$$

and (25) takes the form

$$2(2p(A + B) + \epsilon_k - 2p^2) z F_1(a_k, b_k; A + 1/2; z_1) z F_1(a_k, b_k; B + 1/2; z_2)$$

$$+ \tilde{D}(z F_1(a_k, b_k; A + 1/2; z_1) z F_1(a_k, b_k; B + 1/2; z_2))$$

$$= \sum_{i=0}^{N} C_{k_i}^{(1)} z F_1(a_i, b_i; A + 1/2; z_1) z F_1(a_i, b_i; B + 1/2; z_2).$$

(27)

After straightforward calculations, (25) can be rewritten as

$$2(2p(A + B) + \epsilon_k - 2p^2) z F_1(a_k, b_k; c_1; z_1) \cdot z F_1(a_k, b_k; c_2; z_2)$$

$$+ \frac{8a_k b_k p}{1 - z_1 - z_2} \left( \frac{z_1(1 - z_1)}{c_1} z F_1(a_k + 1, b_k + 1; c_1 + 1; z_1) \cdot z F_1(a_k, b_k; c_2; z_2) \right.$$

$$+ \frac{z_2(1 - z_2)}{c_2} \cdot z F_1(a_k + 1, b_k + 1; c_2 + 1; z_2) \cdot z F_1(a_k, b_k; c_1; z_1) \bigg)$$

$$= \sum_{i=0}^{N} C_{k_i}^{(1)} \cdot z F_1(a_i, b_i; c_1; z_1) \cdot z F_1(a_i, b_i; c_2; z_2).$$

(28)

where

$$c_1 = A + \frac{1}{2}, \quad c_2 = B + \frac{1}{2}, \quad b_k \equiv b_{k_i} = \frac{A + B + \sqrt{-\epsilon_k}}{2}.$$

For $z_2 = 0$ (27) reads

$$2(2p(A + B) - 2p^2 + \epsilon_k) z F_1(a_k, b_k; c_1; z_1) + \frac{8a_k b_k p}{c_1} z_1 \cdot z F_1(a_k + 1, b_k + 1; c_1 + 1; z_1)$$

$$= \sum_{i=0}^{N} C_{k_i}^{(1)} \cdot z F_1(a_i, b_i; c_1; z_1).$$

(29)

In the $z_1 \to -\infty$ limit, the largest power on the lhs of (29) is $(-z_1)^{-a_k}$. Therefore,

$$C_{k,i}^{(1)} = 0 \quad \text{for} \quad i > k,$$

i.e. equation (29) takes the form

$$2(2p(A + B) - 2p^2 + \epsilon_k) \cdot z F_1(a_k, b_k; c_1; z_1) + \frac{8a_k b_k p}{c_1} z_1 \cdot z F_1(a_k + 1, b_k + 1; c_1 + 1; z_1)$$

$$= \sum_{i=0}^{k} C_{k_i}^{(1)} \cdot z F_1(a_i, b_i; c_1; z_1).$$

(30)
Further, for particular values \( z_1 = 0 \) and \( k = 0 \), it gives
\[
C^{(1)}_{k0} = 2(2p(A + B) - 2p^2 + \varepsilon_0).
\]
(32)
Substitution of (32) back into (31) with arbitrary \( z_1 \) and \( k = 0 \) leads to \( a_0 \cdot b_0 = 0 \). Since all \( b_k \) are positive, the only opportunity is \( a_0 = 0 \). Comparing next powers in equation (31) for \( z_1 \to -\infty \), we obtain
\[
a_k + 1 = a_{k-1}.
\]
Together with \( a_0 = 0 \), this relation uniquely defines all values of \( a_k \):
\[
a_k = -k, \quad k = 0, 1, \ldots, N^{(1)}.
\]
(33)
In turn, the comparison of coefficients of \((-z_1)^{-n_k}\) in equation (31) gives values of elements \( C^{(1)}_{kk} \). Due to (30), the matrix \( C^{(1)}_{kk} \) is triangular. Its diagonal elements \( C^{(1)}_{kk} \) coincide with elements of the diagonal matrix \( \Lambda \) in (16), and therefore, \( C^{(1)}_{kk} \) gives a part of the eigenvalues of discrete energy spectrum of the Hamiltonians \( \tilde{H}(p) = H(p + 1) \):
\[
\tilde{E}^{(1)}_k(p) = E^{(1)}_k(p + 1) = -2((A + B + 2k - p)^2 + p^2), \quad k = 0, 1, \ldots, N^{(1)}.
\]
(34)
It is clear from (34) that the lowest energy state corresponds to the maximal \( k \), i.e. to \( k = N^{(1)} \), which can be defined from conditions of normalizability of \( \tilde{\Omega}^{(1)}_{k_0}(\tilde{x}) \). These conditions were formulated in section 4, and they can be rewritten now as
\[
k^{(1)} < \frac{1}{2}(p - A - B), \quad N^{(1)} = \left[ \frac{1}{2}(p - A - B) \right],
\]
(35)
where \([c]\) means the integer part of \( c \).

Analogously one can construct three other kinds of energy levels \( \tilde{E}^{(\nu)}_k(p) \) of \( \tilde{H}(p) = H(p + 1) \) by replacing everywhere above \( (A, B) \) by \((1 - A, B), (A, 1 - B) \) or \((1 - A, 1 - B) \), correspondingly. The result is as follows:
\[
\begin{align*}
\tilde{E}^{(2)}_k(p) &= E^{(2)}_k(p + 1) = -2((1 - A + B + 2k - p)^2 + p^2), \quad k^{(2)} = 0, 1, \ldots, N^{(2)}; \\
\tilde{E}^{(3)}_k(p) &= E^{(3)}_k(p + 1) = -2((1 + A - B + 2k - p)^2 + p^2), \quad k^{(3)} = 0, 1, \ldots, N^{(3)}; \\
\tilde{E}^{(4)}_k(p) &= E^{(4)}_k(p + 1) = -2((2 - A - B + 2k - p)^2 + p^2), \quad k^{(4)} = 0, 1, \ldots, N^{(4)}.
\end{align*}
\]
(36) - (38)
where
\[
\begin{align*}
k^{(2)} &< \frac{1}{2}(p - 1 + A - B), \\
k^{(3)} &< \frac{1}{2}(p - 1 - A + B), \\
k^{(4)} &< \frac{1}{2}(p - 2 + A + B).
\end{align*}
\]
(39) - (41)
The energy spectra and the corresponding wavefunctions for several lowest values of \( p \) are given in section 7.

According to (17), the eigenfunctions of \( H(p + 1) \):
\[
\Psi^{(\nu)}_k(\tilde{x}; p + 1) = \tilde{\Psi}^{(\nu)}_k(\tilde{x}; p) = \sum_{i=0}^{N} B^{(\nu)}_{ki} \tilde{\Xi}^{(\nu)}_i(\tilde{x}; p), \quad \nu = 1, 2, 3, 4,
\]
(42)
Thus, each Hamiltonian $H(p+1)$ (for $N^{(γ)} > 0$) possesses $N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)} + 4$ bound states $Ψ^{(γ)}_k(x; p+1)$ with energy levels $E^{(γ)}_k(p+1), k = 0, 1, \ldots, N^{(γ)}$, which are absent in the spectrum of $H(p)$. As we know, due to SUSY intertwining relations (1) (see also section 2), each of these wavefunctions produce the tower of extra eigenfunctions for higher Hamiltonians $H(p+n+1), n = 1, 2, \ldots$, with the same energy values $E^{(γ)}_k(p+1)$. These wavefunctions $Ψ^{(γ)}_k(x; p + n + 1)$ are built by the action of $n$ operators $Q^-(p+m), m = 1, 2, \ldots, n$:

$$Ψ^{(γ)}_k(x; p + n + 1) = Ψ^{(γ)}_k(x; p + n) = Q^-(p+n)Q^-(p+n-1)\ldots Q^-(p+1)Ψ^{(γ)}_k(x; p+1), \quad (43)$$

and their indices indicate the number $k$ among $N^{(γ)}$ bound states of the original Hamiltonian $H(p+1)$, and the number $n$ of one after another acting operators $Q^-$. 

6. Normalizability of the wavefunctions

According to results of previous sections, the Hamiltonian $H(p+1) = \tilde{H}(p)$ has two classes of bound state wavefunctions. The second (in terminology of section 2) $Ψ^{(γ)}_k(x; p+1), k = 0, 1, \ldots, N^{(γ)}$, is produced by normalizable zero modes of $Q^+$ via their suitable linear combinations. The first class is obtained from the eigenfunctions of lower Hamiltonians by means of operators $Q^-$. In the notations introduced above, they are

$$Ψ^{(γ)}_{m,(p-n)}(x; p+1) = Q^- (p)Q^- (p-1)\ldots Q^- (n+1)Ψ^{(γ)}_m(x; n+1), \quad (44)$$

with energy values $E^{(γ)}_m(n+1)$ (see (34)). The restrictions for values of $m$ depend on $γ$:

$$m < (n-A-B)/2, \quad γ = 1; \quad m < (n-1-A-B)/2, \quad γ = 2; \quad m < (n-1-A+B)/2, \quad γ = 3; \quad m < (n-2+A+B), \quad γ = 4.$$

This is an appropriate point to remark that the situation of general position corresponds to the simple spectrum of $H(p+1)$, which consists of levels $E^{(γ)}_k(p+1)$ (see (34)–(38)) for $Ψ^{(γ)}_k(p+1)$ and levels $E^{(γ)}_m(n+1)$ for the states (44). But nothing prohibits the possible occasional degeneracy of the spectrum for some specific values of parameters. Indeed, this situation is nongeneric: an occasional degeneracy of levels may occur only for some single values of the parameters $A, B$. In such a case, the degeneracy can be removed easily by an arbitrary small variation of $A, B$.

The normalizability of functions of the second class is obvious by construction, but this property for the wavefunctions (44) will be proven now. The Hamiltonian $\tilde{H}(p)$ has the symmetry operator $R(p) = Q^-(p)Q^+(p)$ (see (8)), and in turn, $H(p+1)$ has its own symmetry operator $R(p+1) = Q^+(p+1)Q^- (p+1)$. As far as these Hamiltonians coincide (shape invariance) $H(p+1) = \tilde{H}(p)$, the corresponding symmetry operators must coincide as well, but up to the function of the Hamiltonian. Indeed, by straightforward calculation one obtains the relation

$$R(p+1) - \tilde{R}(p) = 8(p+1)(\tilde{H}(p) + 2(2p^2 + 2p+1)), \quad (45)$$

which will help to analyze the normalizability.

The norm of the arbitrary wavefunction $Ψ^{(γ)}_{m,(p-n)}(x; p+1)$ (44) can be written as

$$\|Ψ^{(γ)}_{m,(p-n)}(x; p+1)\|^2 = |Ψ^{(γ)}_m(x; n+1)|^2 Q^+(n+1)Q^+(n+2)\ldots Q^+(p)Q^- (p)Q^- (p-1)\ldots Q^- (n+2)Q^- (n+1)Ψ^{(γ)}_m(x; n+1)|.$$
To simplify it, one may explore equation (45) and its consequence:

\[
(Q^+(n + 1)Q^-(n + 1) - Q^-(n)Q^+(n))Q^+(n)Q^- (n - 1) \ldots Q^- (m) = Q^- (n)Q^- (n - 1) \ldots Q^- (m) \Gamma_{mn},
\]

where \( \Gamma_{mn} \) is the function of the Hamiltonian:

\[
\Gamma_{mn} = 8(2n + 1)(H(m) + 2(2n^2 + 2n + 1)).
\]

The following relation can be derived by induction:

\[
Q^+(n + 1)Q^+(n + 2) \ldots Q^+(p)Q^- (p)Q^- (p - 1) \ldots Q^- (n + 1)
= R(n + 1)(R(n + 1) + \Gamma_{n+1,n+1}) \cdot (R(n + 1) + \Gamma_{n+1,n+1} + \Gamma_{n+1,n+2}) \ldots (R(n + 1) + \Gamma_{n+1,n+1} + \Gamma_{n+1,n+2} + \ldots + \Gamma_{n+1,p-1}),
\]

and finally, one obtains that norms of the wavefunctions \( \Psi^{(p)}_{m,(p-n)}(\vec{x}; p + 1) \) of the second class for \( H(p + 1) \) are proportional to the norms of wavefunctions \( \Psi^{(p)}_{m,(p-n)}(\vec{x}; n + 1) \) of the first class for the Hamiltonian \( H(n + 1) \):

\[
\left\| \Psi^{(p)}_{m,(p-n)}(\vec{x}; p + 1) \right\|^2 = 64(2n + 1)(E^{(p)}_{m}(n + 1) + 2n^2 + 2n + 1)
\cdot \prod_{q=n+1}^{p-1} (\sqrt{(q + 1)^2 - n^2}E^{(p)}_{m}(n + 1) + 2((q + 1)^2 + n^2)) \left\| \Psi^{(p)}_{m,(p-n)}(\vec{x}; n + 1) \right\|^2.
\]

It is easy to check explicitly that the coefficient of proportionality is positive. Hence, as far as the initial state \( \Psi^{(p)}_{m,(p-n)}(n + 1) \) is normalizable by the construction, any wavefunction \( \Psi^{(p)}_{m,(p-n)}(p + 1) \) is also normalizable.

One more statement is necessary to prove in order to be sure that the full variety of eigenfunctions for \( H(p + 1) \) was constructed above. Namely, we must prove that no additional normalizable wavefunctions exist besides those in (42) and (43). Starting from the lowest Hamiltonians, let us suppose that \( H(2) \) has such an additional eigenfunction \( \Phi(2) \), which differs from the linear combination of zero modes of \( Q^+(1) \). Then, it follows from the intertwining relations that \( Q^+(1) \Phi(2) \) must satisfy the Schrödinger equation with the Hamiltonian \( H(1) \). One may check that the supercharges \( Q^\pm(p) \) do not change the normalizability either at infinity or at the coordinate axes \( x_1, x_2 \): detailed analysis is presented in the next few paragraphs of this section. As we already know from section 3, the Hamiltonian \( H(1) \) has no bound states at all, and therefore, our supposition was wrong. Let us suppose now that the first Hamiltonian possessing such an additional state \( \Phi(p + 1) \) is \( H(p + 1) \), while all previous Hamiltonians \( H(p), H(p - 1), \ldots, H(2) \) have bound states of the forms (42) and (43) only. Then, due to intertwining relations, \( Q^+(p) \Phi(p + 1) \) is the eigenfunction \( \Psi(p) \) of \( H(p) \), and therefore, coincides either with \( \Psi_1(p) \) or with \( \Psi_{11}(p) \), by our assumption. For simplicity, we do not consider here the case of possible degeneracy of levels of \( H(p) \) (the conclusion will be the same in this case). Acting by \( Q^+(p) \) onto \( \Phi(p + 1) \) and using relation (45), one obtains by straightforward calculations that for both options, \( \Psi(p) \) is proportional to \( Q^+(n_1,1)(p + 1) \) with some suitable \( n \). Therefore, the wavefunction \( \Phi(p + 1) \) coincides with \( \Psi_{11}(n_1,1)(p + 1) \) up to zero modes of \( Q^+(p) \):

\[
\Phi(\vec{x}; p + 1) = c_1(p + 1)\Psi_{11}(\vec{x}; p + 1) + c_2(p + 1)\Psi_{2}(\vec{x}; p + 1),
\]

where \( c_{1,2}(p + 1) \) are constants.

Thus, the problem is reduced to the question: whether the operators \( Q^\pm(p) \) are able to change the normalizability of functions. If they are not, no additional normalizable eigenfunctions of \( H(p + 1) \) exist. It is evident from the explicit expressions (5) of \( Q^\pm(p) \) and
from taking into account the exponential decreasing of $\Psi$ at infinity that $Q^{\pm}$ cannot violate integrability of $|\Psi|^2$ at $\pm \infty$.

A more difficult problem arises in the neighborhood of $x_1 \to 0$ and/or $x_2 \to 0$. The part of $Q^{\pm}$ linear in derivatives coincides with the operator $\hat{D}$, defined in (26). In the limit $x_2 \to 0$, $x_1 \neq 0$, it is

$$\hat{D} \sim 4p(\tanh x_1)\partial_1 - 4p(1 - \tanh^2 x_1)x_2\partial_2,$$

i.e. it does not change the asymptotic behavior of the function. An analogous conclusion is true in the limit $x_2 \to 0$, $x_1 \neq 0$. To analyze the limit when both $x_{1,2} \to 0$, it is convenient to use the polar coordinates $x_+ = R \cos \varphi$, $x_- = R \sin \varphi$. Asymptotically, $\hat{D}$ for $R \to 0$ is

$$\hat{D} \sim 4p(\sin(2\varphi))R\partial_R + \cos(2\varphi)\partial_\varphi,$$

i.e. in this limit $\hat{D}$ cannot change the behavior of function as well.

Coming back to the operators $Q^{\pm}$, the only terms which could in principle change the behavior of function at $x_1 \to 0$ and/or $x_2 \to 0$ are

$$Q^{\pm}(p) \sim -\left(-\partial_1^2 - k_2 \sinh^{-2}(x_1)\right) + \left(-\partial_2^2 + k_1 \sinh^{-2}(x_2)\right).$$

Comparing these terms with (3) and (4), we observe the same parts (although with different signs) in asymptotical expressions:

$$H(p) \sim -\left(-\partial_1^2 - k_2 \sinh^{-2}(x_1)\right) + \left(-\partial_2^2 + k_1 \sinh^{-2}(x_2)\right).$$

Since $\Psi(\vec{x}; p)$ are eigenfunctions of $H(p)$, $Q^{\pm}(p)$ are not able to change the behavior of $\Psi$, and the absence of any additional wavefunctions besides that of (42) and (43) types was thus proven.

7. Examples

The explicit expressions for matrix elements $B_{\alpha}$ from (16), which are necessary to build the eigenfunctions (42) of $H(p + 1)$, seem difficult to present in a general form. Nevertheless, the problem can be solved straightforwardly for low values of $p$. By means of separation of variables, we demonstrated in section 3 that the Hamiltonian $H(1)$ has no bound states.

The next Hamiltonian $H(2)$ (corresponding to $p = 1$ in formulas above) has two bound states: one bound state with $k^{(1)} = 0$, due to inequality (35) with $p = 1$, and the second with $k^{(2)} = 0$ or $k^{(3)} = 0$, due to inequalities (39) and (40), depending on the positivity of $(A - B)$ or $(B - A)$. Of course, these bound states are of the second class, i.e. are built from the zero modes:

$$\Psi^{(1)}_0(\vec{x}; 2) \sim \tilde{\Omega}^{(1)}_0(\vec{x}; 1) = \pm(\cosh(x_+) \cosh(x_-))^{-1} \cdot \sinh^A(|x_1|) \cosh^B(|x_1|) \sinh^B(|x_2|) \cosh^A(|x_2|)$$

with energy $E^{(1)}_0(2) = -2((A + B - 1)^2 + 1)$, and (for $A > B$)

$$\Psi^{(2)}_0(\vec{x}; 2) \sim \tilde{\Omega}^{(2)}_0(\vec{x}; 1) = \pm(\cosh(x_+) \cosh(x_-))^{-1} \cdot \sinh^{1-A}(|x_1|) \cosh^B(|x_1|) \sinh^B(|x_2|) \cosh^{1-A}(|x_2|)$$

with the energy $E^{(2)}_0(2) = -2((B - A)^2 + 1)$. No bound states of the first class exist in this case.

For the next value $p = 2$, i.e. for the Hamiltonian $H(3)$, four bound states are of the second class being built by the zero modes

$$\Psi^{(\gamma)}_0(\vec{x}; 3) \sim \tilde{\Omega}^{(\gamma)}_0(\vec{x}; 2): \quad \gamma = 1, 2, 3, 4$$

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with the energies

\[ E^{(1)}_0(3) = -2((A + B - 2)^2 + 1), \quad E^{(2)}_0(3) = -2((B - A - 1)^2 + 1), \]
\[ E^{(3)}_0(3) = -2((A - B - 1)^2 + 1), \quad E^{(4)}_0(3) = -2((A + B)^2 + 1). \]

But in this case, two wavefunctions of the first class can also be built from (46) and (47) by the procedure (43):

\[ \Psi^{(\gamma)}_0(\vec{x}; 3) \sim Q^{-2}(2)\Psi^{(\gamma)}_0(\vec{x}; 2), \quad \gamma = 1, 2, \quad A > B. \] (48)

Their energies \( E^{(\gamma)}_0(3) \) coincide with \( E^{(\gamma)}_0(2) \), \( \gamma = 1, 2 \), above.

The Hamiltonian \( H(p + 1) \) with \( p = 3 \) for \( A > B \) has six states which are built from bound states of \( H(3) \) by means of the operator \( Q^{-2}(4) \). They have the same energies \( E^{(1)}_0(3), E^{(2)}_0(3), E^{(3)}_0(3), E^{(4)}_0(3) \) and \( E^{(1)}_0(2), E^{(2)}_0(2) \). As for the second-class bound states, six such states exist: \( k^{(1)} = 0, 1; k^{(2)} = 0, 1; k^{(3)} = k^{(4)} = 0 \). This set includes two wavefunctions coinciding with \( \tilde{\Omega}^{(1)}_0(\vec{x}; 4); \tilde{\Omega}^{(2)}_0(\vec{x}; 4); \tilde{\Omega}^{(3)}_0(\vec{x}; 4) \), and four other wavefunctions have to be built as linear combinations of pairs of zero modes \( \tilde{\Omega}^{(1)}_0(\vec{x}; 4), \tilde{\Omega}^{(2)}_0(\vec{x}; 4) \) and \( \tilde{\Omega}^{(3)}_0(\vec{x}; 4), \tilde{\Omega}^{(4)}_0(\vec{x}; 4) \), since \( N^{(1)} = N^{(2)} = 1 \) for \( p = 4 \). The matrix elements of the \( 2 \times 2 \) triangular matrix \( C_{kl}^{(i)} \) are defined by (A.7) and (A.8):

\[ C_{00}^{(1)} = E^{(1)}_0(4) = -2((A + B - 3)^2 + 9); \quad C_{11}^{(1)} = E^{(1)}_1(4) = -2((A + B - 1)^2 + 9); \]
\[ C_{10}^{(1)} = -24. \]

These elements are necessary to determine the coefficients \( B_{kl}^{(i)} \) for (42). From equation (16), one can find that for \( C_{00}^{(1)} \neq C_{11}^{(1)} \), as in the present case, the matrix \( B_{kl}^{(i)} \) is also triangular \( B_{00}^{(1)} = 0 \). Moreover, while

\[ B_{10}^{(1)} = \frac{C_{10}^{(1)}}{C_{11}^{(1)} - C_{00}^{(1)}} B_{11}^{(1)}, \]

the last coefficient \( B_{00}^{(1)} \) is arbitrary. This fact is not discouraging, since the linear combination (42) for \( \Psi^{(1)}_0(4) \) includes only one term (with arbitrary \( B_{00}^{(1)} \)). Thus, the value of \( B_{00}^{(1)} \) is fixed by the unity norm of \( \Psi^{(1)}_0(4) \). The second linear combination (42) for \( \Psi^{(2)}_0(4) \) includes both \( B_{10}^{(1)} \) and \( B_{11}^{(1)} \), which are proportional to each other. The absolute values of these coefficients will also be fixed by normalization of the wavefunction. Analogous calculations can be easily repeated for \( \gamma = 2 \).

8. Conclusions

An exhaustive procedure of analytical solution of two-dimensional generalization of the Pöschl–Teller model with integer values of the parameter \( p \) was presented above. Being based on SUSY intertwining relations and shape invariance of the model, the procedure replaces the standard method of separation of variables which is not applicable here, and it can be considered as a special—SUSY—separation of variables.

In order to confirm obtained results for \( H(p + 1) \), it is useful to compare them with the limiting case which possesses the direct solution by means of separation of variables. Indeed, if the parameters \( A, B \), which originally belong to the interval \((0, 1/2)\), are chosen on the limit of range \( A, B \to 0 \), the procedure above (starting from section 3) does not work. But due to conventional separation of variables, the Hamiltonian \( H(p + 1) \) from (3) is reduced
(up to a trivial multiplier 2) to a sum of two one-dimensional Hamiltonians with well-known
reflectionless potentials in variables \(x_\pm\):

\[
h(p + 1)(x) = -\partial^2 - \frac{p(p + 1)}{\cosh^2 x}.
\]

The spectra of these Hamiltonians are well known: for \(p \in [L, L + 1)\) they have exactly
\(L\) bound states. To compare the properties of spectra with that in section 7, one has to
explore the original inequalities (35), (39)–(41) and (34), (36)–(38) where \(A\) and \(B\) are written
explicitly. The point is that some of the bound states described in section 7 disappear in the
limit \(A, B \to 0\).

Thus, \(H(p + 1)\) with \(p = 1\) has one bound state with energy \(E_0^{(1)}(2) = -4\) in the limiting
case, since only (35) can be satisfied with \(k^{(1)} = 0\). Taking into account the multiplier 2
mentioned above, this value of energy coincides with the double value of eigenvalue of
\(h(2)\).

For \(p = 2\), three bound states of the second class for \(H(p + 1)\) exist in the limiting case,
with \(k^{(1)} = k^{(2)} = k^{(3)} = 0\) and energies \(E_0^{(1)}(3) = -16\), \(E_0^{(2)}(3) = E_0^{(3)}(3) = -10\), and one
bound state of the first class with energy \(E_0^{(1)}(2) = -4\). The one-dimensional Hamiltonian
\(h(3)\) has two bound states, leading just to four possible bilinear combinations in the standard
separation of variables.

For \(p = 3\) with \(A > B\) the limiting Hamiltonian \(H(4)\) has nine bound states: five of
second class \(k^{(1)} = 0, 1; k^{(2)} = k^{(3)} = k^{(4)} = 0\), and additionally four bound states of first
class inherited from four bound states of \(H(3)\). As it should be, this number coincides with
\(3 \times 3 = 9\) possible combinations of one-dimensional wavefunctions.

Finally, it is necessary to note that the results of this paper might have several implications.
First, in realizing new pure analytical methods of analysis of two-dimensional quantum
mechanics, which are currently few in number. Second, in using these new methods for
quantum design in different applications, such as quantum dots, modern nanodevices and
some cosmological models. Third, successful use of the one-dimensional P"oschl–Teller
potential for the description of interaction in diatomic molecules indicates use of its two-
dimensional analogs in quantum chemistry. Fourth, the complete solvability of the present
model presents the opportunity to check the validity of different approximate schemes in
many-particle quantum physics.

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Appendix. Calculation of the coefficients \(C_{ki}\)

We will calculate the coefficients \(C_{k\gamma}^{(1)}\), but \(C_{k\gamma}^{(\nu)}\) with \(\nu = 2, 3, 4\) can be easily obtained by
replacing in formulas below \(A \to (1 - A)\), etc. The calculation will be started from equation
(27), where parameters are defined according to (24), (33) and (28). It is useful to express the
hypergeometric functions (with \(a \equiv -k\)) in terms of Jacobi polynomials:

\[
\begin{align*}
2F_1(a_k, b_k; c_1; z_1) &= \frac{k!}{(\alpha + 1)_k} P_k^{(\alpha, \beta)}(y_1), \\
2F_1(a_k + 1, b_k + 1; c_1 + 1; z_1) &= \frac{(k - 1)!}{(\alpha + 2)_k} P_{k-1}^{(\alpha+1, \beta+1)}(y_1),
\end{align*}
\]
Due to this relation, (A.1) from [22], we obtain
\[ P_k^{(\beta,\alpha)}(y_2), \]
where Pochhammer symbols \((\Gamma)_k\) are defined as \((\Gamma)_k \equiv \Gamma(\Gamma + 1) \cdots (\Gamma + k - 1)\), and new variables and new parameters will be more suitable:
\[ y_{1,2} \equiv 1 - 2z_{1,2}, \quad \alpha \equiv A - \frac{1}{2}, \quad \beta \equiv B - \frac{1}{2}, \]
and now
\[ b_k = 1 + \alpha + \beta + k, \quad c_1 = 1 + \alpha, \quad c_2 = 1 + \beta, \quad \sqrt{-c_k} = 1 + \alpha + \beta + 2k. \]
Taking into account that \((\alpha + 1)(\alpha + 2)\) \(\equiv (\alpha + 1)_k\), the lhs of (27) becomes
\[ M \equiv \frac{(k!)^2}{(\alpha + 1)_k(\beta + 1)_k} \left\{ 2(2p(1 + \alpha + \beta) - (1 + \alpha + \beta + 2k)^2 - 2p^2) P_k^{(\alpha,\beta)}(y_1) P_k^{(\beta,\alpha)}(y_2) \right. \]
\[ - \frac{4p(1 + \alpha + \beta + k)}{y_1 + y_2} \left( (1 - y_1^2) P_k^{(\alpha + 1,\beta + 1)}(y_1) P_k^{(\beta,\alpha)}(y_2) \right. \]
\[ + \left( 1 - y_2^2 \right) P_k^{(\beta + 1,\alpha + 1)}(y_2) P_k^{(\alpha,\beta)}(y_1) \left. \right\}. \tag{A.1} \]

Let us use the relation for Jacobi polynomials 22.17.15 from [22] with \(n\) replaced by \(n + 1\) and \(\beta\) by \(\beta - 1\). Multiplying it by \((1 + x)\) and using the relation 22.7.16 from [22], one obtains
\[ \left( n + \frac{\alpha + \beta + 1}{2} \right) (1 - x^2) P_{n-1}^{(\alpha + 1,\beta + 1)}(x) = \frac{2(n + \alpha)(n + \beta)}{2n + \alpha + \beta} P_{n-1}^{(\alpha,\beta)}(x) \]
\[ - \frac{2n(n + 1)}{2(n + 1) + \alpha + \beta} P_n^{(\alpha,\beta)}(x) + 2 \left( \frac{2n + 1}{2n + \alpha + \beta} - \frac{n + 1 + \beta}{2(n + 1) + \alpha + \beta} \right) P_n^{(\alpha,\beta)}(x). \tag{A.2} \]

Let us write the same relation but for \(x \mapsto y\) and \(\alpha \leftrightarrow \beta\), and add it to the initial equation (A.1). Then, rewriting \(P_{n+1}\) as a combination of \(P_{n-1}\) and \(P_n\) (according to 22.7.1 from [22]), we obtain
\[ (1 - x^2) P_{n-1}^{(\alpha + 1,\beta + 1)}(x) P_n^{(\beta,\alpha)}(y) + (1 - y^2) P_{n-1}^{(\beta + 1,\alpha + 1)}(y) P_n^{(\alpha,\beta)}(x) \]
\[ = \frac{2}{n + 1 + \alpha + \beta} \left( \frac{2(n + \alpha)(n + \beta)}{2n + \alpha + \beta} P_{n-1}^{(\alpha,\beta)}(x) P_n^{(\beta,\alpha)}(y) + P_{n-1}^{(\beta,\alpha)}(y) P_n^{(\alpha,\beta)}(x) \right) \]
\[ - n(x + y) P_n^{(\alpha,\beta)}(x) P_n^{(\beta,\alpha)}(y). \]

Due to this relation, (A.1) takes the form
\[ M = \frac{(k!)^2}{(\alpha + 1)_k(\beta + 1)_k} \left\{ -2((1 + \alpha + \beta + 2k - p)^2 + p^2) P_k^{(\alpha,\beta)}(y_1) P_k^{(\beta,\alpha)}(y_2) \right. \]
\[ - \frac{16p(k + \alpha)(k + \beta)}{2k + \alpha + \beta} \left( \frac{1}{y_1 + y_2} \left( P_{k-1}^{(\alpha,\beta)}(y_1) P_k^{(\beta,\alpha)}(y_2) + P_{k-1}^{(\beta,\alpha)}(y_2) P_k^{(\alpha,\beta)}(y_1) \right) \right. \left. \right\}. \tag{A.3} \]

The rhs of (A.3) includes the combination
\[ \Phi_{k-1,k} \equiv P_{k-1}^{(\alpha,\beta)}(y_1) P_k^{(\beta,\alpha)}(y_2) + P_k^{(\alpha,\beta)}(y_2) P_{k-1}^{(\beta,\alpha)}(y_1), \]
which (by means of the recurrent formula 22.7.1 from [22]) satisfies
\[ \Phi_{k-1,k} = (y_1 + y_2) \frac{a_{3(k-1)}}{a_{1(k-1)}} P_{k-1}^{(\alpha,\beta)}(y_1) P_{k-1}^{(\beta,\alpha)}(y_2) - \frac{a_{4(k-1)}}{a_{1(k-1)}} \Phi_{k-1,k-2}. \tag{A.4} \]
where the following definitions were introduced:

\[ a_{1n} = 2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta), \]
\[ a_{2n} = (2n + \alpha + \beta + 1)(\alpha^2 - \beta^2), \]
\[ a_{3n} = (2n + \alpha + \beta)(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2), \]
\[ a_{4n} = 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2). \]

Since

\[ \Phi_{0,1} = (y_1 + y_2) \frac{a_{30}}{a_{10}} P_0^{(\alpha, \beta)}(y_1) P_0^{(\beta, \alpha)}(y_2), \quad (A.5) \]

equations (A.4) and (A.5) give

\[ \Phi_{k-1,k} = (y_1 + y_2) \left( \frac{a_{3(k-1)}}{a_{1(k-1)}} \frac{a_{4(k-1)}}{a_{1(k-1)}} P_{k-1}^{(\alpha, \beta)}(y_1) P_{k-1}^{(\beta, \alpha)}(y_2) \right) - \frac{a_{3(k-2)}}{a_{1(k-2)}} \frac{a_{4(k-2)}}{a_{1(k-2)}} P_{k-2}^{(\alpha, \beta)}(y_1) P_{k-2}^{(\beta, \alpha)}(y_2) \]
\[ + \frac{a_{3(k-3)}}{a_{1(k-3)}} \frac{a_{4(k-3)}}{a_{1(k-2)}} P_{k-3}^{(\alpha, \beta)}(y_1) P_{k-3}^{(\beta, \alpha)}(y_2) \]
\[ + \cdots (-1)^{k-1} \frac{a_{31}}{a_{11}} \frac{a_{41}}{a_{11}} P_1^{(\alpha, \beta)}(y_1) P_1^{(\beta, \alpha)}(y_2). \quad (A.6) \]

The coefficients

\[ b_{ki} \equiv (-1)^{k-i} \frac{a_{3i}}{a_{1i}} \frac{a_{4i}}{a_{1i}} P_{i}^{(\alpha, \beta)}(y_1) P_{i}^{(\beta, \alpha)}(y_2). \]

allow us to write (A.6) more compactly:

\[ \Phi_{k-1,k} = (y_1 + y_2) \sum_{i=0}^{k-1} b_{ki} P_{i}^{(\alpha, \beta)}(y_1) P_{i}^{(\beta, \alpha)}(y_2). \]

Finally, substituting it into (A.1), we obtain from (27) the general expressions for the desired coefficients \( C_{ki}^{(1)} \):

\[ C_{kk}^{(1)} = -2((1 + \alpha + \beta + 2k - p)^2 + p^2), \quad (A.7) \]
\[ C_{k,i<k}^{(1)} = \frac{16(k!)^2 p(k + \alpha)(1 + \beta)_k(1 + \beta)_i}{(l!)^2(1 + \alpha)_k(1 + \beta)_l(2k + \alpha + \beta)} b_{ki}. \quad (A.8) \]

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