Reducing homological conjectures by $n$-recollements

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Abstract

$n$-recollements of triangulated categories and $n$-derived-simple algebras are introduced. The relations between the $n$-recollements of derived categories of algebras and the Cartan determinants, homological smoothness and Gorensteinness of algebras respectively are clarified. As applications, the Cartan determinant conjecture is reduced to 1-derived-simple algebras, and the Gorenstein symmetry conjecture is reduced to 2-derived-simple algebras.

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Keywords: $n$-recollement; $n$-derived-simple algebra; Cartan determinant; homologically smooth algebra; Gorenstein algebra.

1 Introduction

Throughout $k$ is a fixed field and all algebras are associative $k$-algebras with identity unless stated otherwise. Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [3], and play an important role in algebraic geometry and representation theory. Here, we focus on the recollements of derived categories of algebras which are the generalization of derived equivalences and provide a useful reduction technique for some homological properties such as the finiteness of global dimension [32, 21, 1], the finiteness of finitistic dimension [17, 9] and the finiteness of Hochschild dimension [15], some homological invariants such as $K$-theory [30, 35, 25, 29, 8, 1], Hochschild homology and cyclic homology [19] and Hochschild cohomology [15], and some homological conjectures such as the finitistic dimension conjecture [17, 9] and the Hochschild homology dimension conjecture [14].
In a recollement, two functors in the first layer always preserve compactness, i.e., send compact objects to compact ones, but other functors are not the case in general. If a recollement is perfect, i.e., two functors in the second layer also preserve compactness, then the Hochschild homologies, cyclic homologies and K-groups of the middle algebra are the direct sum of those of outer two algebras respectively \([19, 8, 1]\). Moreover, in this situation, the relations between recollements and the finitistic dimensions of algebras can be displayed very completely \([9]\). In order to clarify the relations between recollements and the homological smoothness and Gorensteinness of algebras respectively, we need even more layers of functors preserving compactness, which leads to the concept of \(n\)-recollement of triangulated categories inspired by that of ladder \([4]\), and further \(n\)-derived-simple algebra. In terms of \(n\)-recollements, the relations between recollements and the Cartan determinants, homological smoothness and Gorensteinness of algebras respectively are expressed as follows.

**Theorem I.** Let \(A, B\) and \(C\) be finite dimensional algebras, and \(D(\text{Mod}A)\) admit an \(n\)-recollement relative to \(D(\text{Mod}B)\) and \(D(\text{Mod}C)\) with \(n \geq 2\). Then \(\det C(A) = \det C(B) \cdot \det C(C)\).

**Theorem II.** Let \(A, B\) and \(C\) be algebras and \(D(\text{Mod}A)\) admit an \(n\)-recollement relative to \(D(\text{Mod}B)\) and \(D(\text{Mod}C)\).

1. \(n = 1\): if \(A\) is homologically smooth then so is \(B\);
2. \(n = 2\): if \(A\) is homologically smooth then so are \(B\) and \(C\);
3. \(n \geq 3\): \(A\) is homologically smooth if and only if so are \(B\) and \(C\).

**Theorem III.** Let \(A, B\) and \(C\) be finite dimensional algebras, and \(D(\text{Mod}A)\) admit an \(n\)-recollement relative to \(D(\text{Mod}B)\) and \(D(\text{Mod}C)\).

1. \(n = 3\): if \(A\) is Gorenstein then so are \(B\) and \(C\);
2. \(n \geq 4\): \(A\) is Gorenstein if and only if so are \(B\) and \(C\).

As applications of Theorem I and Theorem III, we will show that the Cartan determinant conjecture and the Gorenstein symmetry conjecture can be reduced to 1-derived-simple algebras and 2-derived-simple algebras respectively.

The paper is organized as follows: In section 2, we will introduce the concepts of \(n\)-recollement of triangulated categories and \(n\)-derived-simple algebra, and provide some typical examples and existence criteria of \(n\)-recollements of derived categories of algebras. In section 3, Theorem I is obtained and the Cartan determinant conjecture is reduced to 1-derived-simple algebras. In section 4, we will prove Theorem II. In section 5, The-
orem III is shown and the Gorenstein symmetry conjecture is reduced to 2-derived-simple algebras.

2 \( n \)-recollements and \( n \)-derived-simple algebras

In this section, we will introduce the concepts of \( n \)-recollement of triangulated categories and \( n \)-derived-simple algebra, and provide some examples and existence criteria of the \( n \)-recollements of derived categories of algebras. As we will see, the language of \( n \)-recollements is very convenient for us to observe the relations between recollements and certain homological properties, especially the Gorensteinness of algebras.

2.1 \( n \)-recollements of triangulated categories

**Definition 1.** (Beilinson-Bernstein-Deligne [3]) Let \( \mathcal{T}_1, \mathcal{T} \) and \( \mathcal{T}_2 \) be triangulated categories. A recollement of \( \mathcal{T} \) relative to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) is given by

\[
\begin{array}{ccc}
\mathcal{T}_1 & \xrightarrow{i^*} & \mathcal{T} \\
\downarrow{i_!} & & \downarrow{j^!} \\
\mathcal{T}_2 & \xleftarrow{j_*} & \mathcal{T} \\
\end{array}
\]

such that

- (R1) \((i^*, i_!), (i_!, i^!), (j^!, j_*)\) and \((j^*, j_!)\) are adjoint pairs of triangle functors;
- (R2) \(i_!, j_!\) and \(j_*\) are full embeddings;
- (R3) \(j^!i_* = 0\) (and thus also \(i^!j_! = 0\) and \(i^*j_! = 0\));
- (R4) for each \(X \in \mathcal{T}\), there are triangles

\[
\begin{align*}
j_!i^!X & \rightarrow X \rightarrow i_*i^*X \\
j^!j_*X & \rightarrow X \rightarrow j_*j^*X
\end{align*}
\]

where the arrows to and from \(X\) are the counits and the units of the adjoint pairs respectively.

**Definition 2.** Let \( \mathcal{T}_1, \mathcal{T} \) and \( \mathcal{T}_2 \) be triangulated categories, and \( n \) a positive integer. An \( n \)-recollement of \( \mathcal{T} \) relative to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) is given by \( n + 2 \) layers of triangle functors

\[
\begin{array}{ccc}
\mathcal{T}_1 & \xrightarrow{i^*} & \mathcal{T} \\
\downarrow{i_!} & & \downarrow{j^!} \\
\mathcal{T_2} & \xleftarrow{j_*} & \mathcal{T} \\
\downarrow{i^!} & & \downarrow{j_*} \\
\vdots & & \vdots \\
\mathcal{T}_1 & \xrightarrow{i^*} & \mathcal{T} \\
\downarrow{i_!} & & \downarrow{j^!} \\
\mathcal{T_2} & \xleftarrow{j_*} & \mathcal{T} \\
\downarrow{i^!} & & \downarrow{j_*} \\
\vdots & & \vdots
\end{array}
\]

such that every consecutive three layers form a recollement.
Obviously, a 1-recollement is nothing but a recollement. Moreover, if $\mathcal{T}$ admits an $n$-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$, then it must admit an $m$-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ for all $1 \leq m \leq n$ and a $p$-recollement relative to $\mathcal{T}_2$ and $\mathcal{T}_1$ for all $1 \leq p \leq n - 1$.

**Remark 1.** Let $\mathcal{T}$ be a skeletally small $k$-linear triangulated category with finite dimensional Hom-sets and split idempotents. If $\mathcal{T}$ has a Serre functor and admits a recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ then it admits an $n$-recollement relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ (resp. $\mathcal{T}_2$ and $\mathcal{T}_1$) for all $n \in \mathbb{Z}^+$ by [18, Theorem 7].

### 2.2 $n$-recollements of derived categories of algebras

Let $A$ be an algebra. Denote by $\text{Mod} A$ the category of right $A$-modules, and by $\text{mod} A$, $\text{Proj} A$, $\text{proj} A$ and $\text{inj} A$ its full subcategories consisting of all finitely generated modules, projective modules, finitely generated projective modules and finitely generated injective modules, respectively. For $* \in \{\text{nothing}, -, +, b\}$ and $\mathcal{A} = \text{Mod} A$ or any above subcategory of $\text{Mod} A$, denote by $K^*(\mathcal{A})$ (resp. $D^*(\mathcal{A})$) the homotopy category (resp. derived category) of cochain complexes of objects in $\mathcal{A}$ satisfying the corresponding boundedness condition. Up to isomorphism, the objects in $K^b(\text{proj} A)$ are precisely all the compact objects in $D(\text{Mod} A)$. For convenience, we do not distinguish $K^b(\text{proj} A)$ from the *perfect derived category* $D_{\text{per}}(A)$ of $A$, i.e., the full triangulated subcategory of $D A$ consisting of all compact objects, which will not cause any confusion. Moreover, we also do not distinguish $K^b(\text{inj} A), D^b(\text{Mod} A), D^b(\text{mod} A), D^+(\text{Mod} A)$ and $D^+(\text{Mod} A)$ from their essential images under the canonical full embeddings into $D(\text{Mod} A)$. Usually, we just write $D A$ instead of $D(\text{Mod} A)$.

In this paper, we focus on the $n$-recollements of derived categories of algebras, i.e., all three triangulated categories in an $n$-recollement are the derived categories of algebras. Clearly, in the $n$-recollement, the upper $n$ layers of functors have right adjoints preserving direct sums, thus they preserve compactness.

Now we provide some typical examples of $n$-recollements.

**Example 1.** (1) Stratifying ideals [11]. Let $A$ be an algebra, and $e$ an idempotent of $A$ such that $A e A$ is a *stratifying ideal*, i.e., $A e \otimes_L^L e A \cong A e A$ canonically. Then $D A$ admits a 1-recollement relative to $D(A/A e A)$ and $D(e A e)$.

(2) Triangular matrix algebras [1] Example 3.4. Let $B$ and $C$ be algebras,
$M$ a $C$-$B$-bimodule, and $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$. Then $\mathcal{D}A$ admits a 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Furthermore, $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ if $C M \in K^b(\text{proj}C^{\text{op}})$, and $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$ if $M_B \in K^b(\text{proj}B)$. What is more, $\mathcal{D}A$ admits a 4-recollement relative to $\mathcal{D}C$ and $\mathcal{D}B$ if $C M \in K^b(\text{proj}C^{\text{op}})$ and $M_B \in K^b(\text{proj}B)$. Note that the algebras $A, B$ and $C$ here need not be finite dimensional.

(3) Let $A$ be a finite dimensional algebra of finite global dimension and $\mathcal{D}A$ admit a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Then this recollement can be extended to an $n$-recollement for all $n \in \mathbb{Z}^+$ (Ref. [1, Proposition 3.3]).

(4) A derived equivalence induces a trivial $n$-recollement, i.e., an $n$-recollement whose left term or right term is zero, for all $n \in \mathbb{Z}^+$.

Usually we pay more attention to the $n$-recollements of derived categories of finite dimensional algebras. In this situation, we have some useful existence criteria of $n$-recollements.

**Lemma 1.** Let $A$ and $B$ be finite dimensional algebras, and the triangle functor $F : \mathcal{D}A \to \mathcal{D}B$ left adjoint to $G : \mathcal{D}B \to \mathcal{D}A$. Then:

1. $F$ restricts to $K^b(\text{proj})$ if and only if $G$ restricts to $D^b(\text{mod})$;
2. $F$ restricts to $D^b(\text{mod})$ if and only if $G$ restricts to $K^b(\text{inj})$.

**Proof.** (1) follows from [1, Lemma 2.7], and (2) is dual to (1). □

**Lemma 2.** Let $A$, $B$ and $C$ be finite dimensional algebras, and

$$
\begin{array}{ccc}
\mathcal{D}B & \xrightarrow{i^*} & \mathcal{D}A & \xleftarrow{j^!} & \mathcal{D}C \\
\xleftarrow{i^!} & & \xrightarrow{j^*} & & \xleftarrow{j^!} \\
\end{array}
$$

a recollement. Then the following statements hold:

1. $i^*$ and $j^!$ restrict to $K^b(\text{proj})$;
2. $i^!$ and $j^!$ restrict to $D^b(\text{mod})$;
3. $i^!$ and $j^*$ restrict to $K^b(\text{inj})$.

**Proof.** (1) is clear. (2) and (3) follow from Lemma [1] □

**Lemma 3.** Let $A$, $B$ and $C$ be finite dimensional algebras, and

$$
\begin{array}{ccc}
\mathcal{D}B & \xrightarrow{i^*} & \mathcal{D}A & \xleftarrow{j^!} & \mathcal{D}C \\
\xleftarrow{i^!} & & \xrightarrow{j^*} & & \xleftarrow{j^!} \\
\end{array}
$$

(R)
a recollement. Then the following statements are equivalent:

1. The recollement (R) can be extended one-step downwards;
2. $i_*$ or/and $j^!$ restricts to $K^b(\text{proj})$;
3. $i^! B \in K^b(\text{proj}A)$ or/and $j^!(A) \in K^b(\text{proj}C)$;
4. $i^!$ or/and $j_*$ restricts to $D^b(\text{mod})$;
5. The recollement (R) restricts to $D^-(\text{Mod})$.

Proof. (1) $\iff$ (2): It follows from \cite[Proposition 3.2 (a)]{1}.
(2) $\iff$ (2'): It follows from \cite[Lemma 2.5]{1}.
(2) $\iff$ (3): It follows from Lemma 1.
(4) $\iff$ (2'): It follows from \cite[Proposition 4.11]{1}.

Lemma 4. Let $A$, $B$ and $C$ be finite dimensional algebras, and

\[
\begin{array}{c}
 DB & \overset{i^*}{\longrightarrow} & DA & \overset{j^!}{\longrightarrow} & DC \\
 \overset{i_*}{\longleftarrow} & & \overset{j_*}{\longleftarrow}
\end{array}
\]

a recollement. Then the following statements are equivalent:

1. The recollement (R) can be extended one-step upwards;
2. $i_*$ or/and $j^!$ restricts to $K^b(\text{inj})$;
3. $i^!(DB) \in K^b(\text{inj}A)$ or/and $j^!(DA) \in K^b(\text{inj}C)$ where $D = \text{Hom}_k(-, k)$;
4. $i^!$ or/and $j_*$ restricts to $D^b(\text{mod})$;
5. The recollement (R) restricts to $D^+(\text{Mod})$.

Proof. This lemma is dual to Lemma 3.

Proposition 1. Let $A$, $B$ and $C$ be finite dimensional algebras. Then the following conditions are equivalent:

1. $DA$ admits a 2-recollement relative to $DB$ and $DC$;
2. $DA$ admits a recollement relative to $DB$ and $DC$ in which two functors in the second layer restrict to $K^b(\text{proj})$;
3. $DA$ admits a recollement relative to $DB$ and $DC$ in which two functors in the third layer restrict to $D^b(\text{mod})$;
4. $D^-(\text{Mod}A)$ admits a recollement relative to $D^-(\text{Mod}B)$ and $D^-(\text{Mod}C)$;
5. $DA$ admits a recollement relative to $DC$ and $DB$ in which two functors in the first layer restrict to $D^b(\text{mod})$;
6. $DA$ admits a recollement relative to $DC$ and $DB$ in which two functors in the second layer restrict to $K^b(\text{inj})$;
7. $D^+(\text{Mod}A)$ admits a recollement relative to $D^+(\text{Mod}C)$ and $D^+(\text{Mod}B)$.
Proposition 2. Let $A$, $B$ and $C$ be finite dimensional algebras. Then the following conditions are equivalent:

1. $\mathcal{D} A$ admits a $3$-recollement relative to $\mathcal{D} B$ and $\mathcal{D} C$;
2. $\mathcal{D} A$ admits a recollement relative to $\mathcal{D} B$ and $\mathcal{D} C$ in which all functors restrict to $\mathcal{K}^{b}(\text{proj})$;
3. $\mathcal{D}^{b}(\text{mod} A)$ admits a recollement relative to $\mathcal{D}^{b}(\text{mod} C)$ and $\mathcal{D}^{b}(\text{mod} B)$;
4. $\mathcal{D} A$ admits a recollement relative to $\mathcal{D} B$ and $\mathcal{D} C$ in which all functors restrict to $\mathcal{K}^{b}(\text{inj})$;
5. $\mathcal{D}^{b}(\text{Mod} A)$ admits a recollement relative to $\mathcal{D}^{b}(\text{Mod} C)$ and $\mathcal{D}^{b}(\text{Mod} B)$.

Proof. (1) $\iff$ (2) $\iff$ (3) $\iff$ (4): It follows from [1, Proposition 4.1] and Lemma 3.

(1) $\iff$ (5) $\iff$ (6) $\iff$ (7): Analogous to [1, Proposition 4.1], any $\mathcal{D}^{+}(\text{Mod})$-recollement can be lifted to a $\mathcal{D}(\text{Mod})$-recollement as well. Then it follows from Lemma 4. □

2.3 $n$-derived-simple algebras

For any recollement of derived categories of finite dimensional algebras, the Grothendieck group of the middle algebra is the direct sum of those of the outer two algebras [1, Proposition 6.5]. Thus the process of reducing homological properties, homological invariants and homological conjectures by recollements must terminate after finitely many steps. This leads to derived simple algebras, whose derived categories admit no nontrivial recollements any more. This definition dates from Wiedemann [32], where the author considered the stratifications of bounded derived categories. Later on, recollements of unbounded and bounded above derived categories attract considerable attention, and so do the corresponding derived simple algebras [1]. When we consider the stratifications along $n$-recollements, $n$-derived-simple algebras are defined naturally.

Definition 3. A finite dimensional algebra $A$ is said to be $n$-derived-simple if its derived category $\mathcal{D} A$ admits no nontrivial $n$-recollements.
Clearly, an $n$-derived-simple algebra must be indecomposable/connected. Note that 1-derived-simple algebras are just the $\mathcal{D}(\text{Mod})$-derived simple algebras. For finite dimensional algebras, by Proposition 1 and Proposition 2, 2 (resp. 3)-derived-simple algebras are exactly $\mathcal{D}^-(\text{Mod})$ (resp. $\mathcal{D}^b(\text{mod})$)-derived simple algebras in the sense of [1]. Moreover, $n$-derived-simple algebras must be $m$-derived simple for all $m \geq n$, and it is worth noting that for a finite dimensional algebra $A$ of finite global dimension, the $n$-derived-simplicity of $A$ does not depend on the choice of $n$.

Although it is difficult to find out all the $n$-derived-simple algebras, there are still some known examples.

**Example 2.** (1) Finite dimensional local algebras, blocks of finite group algebras and indecomposable representation-finite symmetric algebras are 1-derived-simple [32, 23];

(2) Some finite dimensional two-point algebras of finite global dimension are $n$-derived-simple for all $n \in \mathbb{Z}^+$ (Ref. [16, 24]);

(3) Indecomposable symmetric algebras are 2-derived-simple [23];

(4) There exist 2-derived-simple algebras which are not 1-derived-simple [1, Example 5.8], 3-derived-simple algebras which are not 2-derived-simple [1, Example 5.10], and 4-derived-simple algebras which are not 3-derived-simple [1, Example 4.13], respectively.

Let’s end this section by listing some known results on reducing homological conjectures via recollements. First, the finitistic dimension conjecture, which says that every finite dimensional algebra has finite finitistic dimension, was reduced to 3-derived-simple algebras by Happel [17]. Recently, Chen and Xi extended his result by reducing the finitistic dimension conjecture to 2-derived-simple algebras [9]. Second, it follows from [19, Proposition 2.9(b)] and [1, Proposition 2.14] that the Hochschild homology dimension conjecture, which states that the finite dimensional algebras of finite Hochschild homology dimension are of finite global dimension [14], can be reduced to 2-derived-simple algebras. Last but not least, both vanishing conjecture and dual vanishing conjecture can be reduced to 3-derived-simple algebras [34].

### 3 $n$-recollements and Cartan determinants

In this section, we will observe the relations between $n$-recollements and the Cartan determinants of algebras, and reduce the Cartan determinant conjecture to 1-derived-simple algebras.
Let \( \mathcal{E} \) be a skeletally small exact category, \( F \) the free abelian group generated by the isomorphism classes \([X]\) of objects \( X \) in \( \mathcal{E} \), and \( F_0 \) be the subgroup of \( F \) generated by \([X] - [Y] + [Z]\) for all conflations \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{E} \). The Grothendieck group \( K_0(\mathcal{E}) \) of \( \mathcal{E} \) is the factor group \( F/F_0 \). The Grothendieck group of a skeletally small triangulated category is defined similarly, just need replace conflations with triangles.

Let \( A \) be a finite dimensional algebra and \( \{P_1, \ldots, P_r\} \) a complete set of non-isomorphic indecomposable projective \( A \)-modules. Then their tops \( \{S_1, \ldots, S_r\} \) form a complete set of non-isomorphic simple \( A \)-modules (resp. \( A' \)-modules, \( A'' \)-modules). Then their tops \( \{S'_1, \ldots, S'_r\} \) (resp. \( \{S_1, \ldots, S_r\}, \{S'_1, \ldots, S''_r\} \)) form a complete set of non-isomorphic simple \( A' \)-modules (resp. \( A \)-modules, \( A'' \)-modules). By \cite[Theorem 1.1]{[1]} or \cite[Proposition 6.5]{[1]}, we have \( r' + r'' = r \).

Consider the triangles \( j\!j^\prime P_u \to P_u \to i^* i^! P_u \to \) for all \( 1 \leq u \leq r \). Since \( P_u \in K^b(\text{proj} A) \), we have \( j\!j^\prime P_u \in K^b(\text{proj} A'' \} = \text{tria}\{P_1, \ldots, P_r\} \subseteq \mathcal{D} A'' \).
and \( i^*P_u \in K^b(\text{proj} A') = \text{tria}\{P'_1, \ldots, P'_r\} \subseteq D A' \). Here, for a class \( \mathcal{X} \) of objects in a triangulated category \( T \), \( \text{tria}\mathcal{X} \) denotes the smallest full triangulated subcategory of \( T \) containing \( \mathcal{X} \). Furthermore, we have \( j_i j^*P_u \in \text{tria}\{j_i P''_1, \ldots, j_i P''_r\} \subseteq D A \) and \( i_* i^*P_u \in \text{tria}\{i_* P'_1, \ldots, i_* P'_r\} \subseteq D A \). Hence \( P_u \in \text{tria}\{i_* P'_1, \ldots, i_* P'_r, j_i P''_1, \ldots, j_i P''_r\} \subseteq D A \), and \( K^b(\text{proj} A) = \text{tria}\{P_1, \ldots, P_r\} = \text{tria}\{i_* P'_1, \ldots, i_* P'_r, j_i P''_1, \ldots, j_i P''_r\} \subseteq D A \). Therefore, \( \{[i_* P'_1], \ldots, [j_i P''_1], \ldots, [j_i P''_r]\} \) is a \( \mathbb{Z} \)-basis of \( K_0(K^b(\text{proj} A)) \).

Consider the triangles \( i_i i^!S_u \to S_u \to j_* j^!S_u \to \) for all \( 1 \leq u \leq r \). Since \( S_u \in D^b(\text{mod} A) \), we have \( i_i i^!S_u \in D^b(\text{mod} A') = \text{tria}\{S'_1, \ldots, S'_r\} \subseteq D A' \) and \( j_i j^!S_u \in D^b(\text{mod} A') = \text{tria}\{S''_1, \ldots, S''_r\} \subseteq D A'' \). Furthermore, we have \( i_i i^!S_u \in \text{tria}\{i_* S'_1, \ldots, i_* S'_r\} \subseteq D A \) and \( j_* j^!S_u \in \text{tria}\{j_* S''_1, \ldots, j_* S''_r\} \subseteq D A \). Hence \( S_u \in \text{tria}\{i_* S'_1, \ldots, i_* S'_r, j_* S''_1, \ldots, j_* S''_r\} \subseteq D A \) and \( D^b(\text{mod} A) = \text{tria}\{S_1, \ldots, S_r\} = \text{tria}\{i_* S'_1, \ldots, i_* S'_r, j_* S''_1, \ldots, j_* S''_r\} \subseteq D A \). Therefore, \( \{[i_* S'_1], \ldots, [j_* S''_1], \ldots, [j_* S''_r]\} \) is a \( \mathbb{Z} \)-basis of \( K_0(D^b(\text{mod} A)) \).

We have clearly the following commutative diagram

\[
\begin{array}{ccc}
K_0(K^b(\text{proj} A')) & \xrightarrow{i_*} & K_0(K^b(\text{proj} A)) \\
\downarrow{C_A} & & \downarrow{C_{A'}} \\
K_0(D^b(\text{mod} A')) & \xrightarrow{j^!} & K_0(D^b(\text{mod} A))
\end{array}
\]

where the horizontal maps are naturally induced by the functors \( i_* \) and \( j^! \). It is not difficult to see that the matrix of the Cartan map \( C_A \) of \( A \) under the \( \mathbb{Z} \)-basis \( \{[i_* P'_1], \ldots, [i_* P'_r], [j_i P''_1], \ldots, [j_i P''_r]\} \) of \( K_0(K^b(\text{proj} A)) \) and the \( \mathbb{Z} \)-basis \( \{[i_* S'_1], \ldots, [i_* S'_r], [j_* S''_1], \ldots, [j_* S''_r]\} \) of \( K_0(D^b(\text{mod} A)) \) is of the form

\[
\begin{bmatrix}
C(A') & * \\
0 & C(A'')
\end{bmatrix}
\]

Note that the matrix of the Cartan map \( C_A \) of \( A \) under the \( \mathbb{Z} \)-basis \( \{[P_1], \ldots, [P_r]\} \) of \( K_0(K^b(\text{proj} A)) \) and the \( \mathbb{Z} \)-basis \( \{[S_1], \ldots, [S_r]\} \) of \( K_0(D^b(\text{mod} A)) \) is just \( C(A) \).

Both \( \{[i_* P'_1], \ldots, [i_* P'_r], [j_i P''_1], \ldots, [j_i P''_r]\} \) and \( \{[P_1], \ldots, [P_r]\} \) are \( \mathbb{Z} \)-bases of \( K_0(K^b(\text{proj} A)) \), thus there exist invertible matrices \( U \) and \( V \) in \( M_r(\mathbb{Z}) \) such that

\[
([i_* P'_1], \ldots, [i_* P'_r], [j_i P''_1], \ldots, [j_i P''_r]) = ([P_1], \ldots, [P_r]) \cdot U
\]

and

\[
([P_1], \ldots, [P_r]) = ([i_* P'_1], \ldots, [i_* P'_r], [j_i P''_1], \ldots, [j_i P''_r]) \cdot V.
\]

Hence, \( UV = VU = I \), the identity matrix. Therefore, \( \det U = \det V = \pm 1 \).
Both \( \{[i_{*}S'_{1}], \ldots, [i_{*}S'_{r}], [j_{*}S''_{i'}], \ldots, [j_{*}S''_{r}] \} \) and \( \{[S_{1}], \ldots, [S_{r}] \} \) are \( \mathbb{Z} \)-bases of \( K_{0}(D^{b}(\text{mod} A)) \), thus there exist invertible matrices \( Q \) and \( R \) in \( M_{r}(\mathbb{Z}) \) such that

\[
([i_{*}S'_{1}], \ldots, [i_{*}S'_{r}], [j_{*}S''_{i'}], \ldots, [j_{*}S''_{r}]) = ([S_{1}], \ldots, [S_{r}]) \cdot Q
\]

and

\[
([S_{1}], \ldots, [S_{r}]) = ([i_{*}S'_{1}], \ldots, [i_{*}S'_{r}], [j_{*}S''_{i'}], \ldots, [j_{*}S''_{r}]) \cdot R.
\]

Hence, \( QR = RQ = I \), the identity matrix. Therefore, \( \det Q = \det R = \pm 1 \).

Combining the equalities above with \( ([P_{1}], \ldots, [P_{r}]) = ([S_{1}], \ldots, [S_{r}]) \cdot C(A) \) and \( ([i_{*}P'_{1}], \ldots, [i_{*}P'_{r}], [j_{*}P''_{i'}], \ldots, [j_{*}P''_{r}]) = ([i_{*}S'_{1}], \ldots, [i_{*}S'_{r}], [j_{*}S''_{i'}], \ldots, [j_{*}S''_{r}]) \), we have \( C(A) \cdot U = Q \cdot \begin{bmatrix} C(A') & * \\ 0 & C(A'') \end{bmatrix} \).

Furthermore, \( \det C(A) = \pm \det C(A') \cdot \det C(A'') \).

On the other hand, we can define a \( \mathbb{Z} \)-bilinear form

\[
\langle -, - \rangle : K_{0}(K^{b}(\text{proj} A)) \times K_{0}(K^{b}(\text{proj} A)) \to \mathbb{Z}
\]

by

\[
\langle [X], [Y] \rangle := \sum_{l \in \mathbb{Z}} (-1)^{l} \dim_{k}\text{Hom}_{K^{b}(\text{proj} A)}(X, Y[l])
\]

for all \( X, Y \in K^{b}(\text{proj} A) \).

Since \( i_{*} \) and \( j_{!} \) are full embeddings and \( j_{!}i_{*} = 0 \), we have

\[
\langle [i_{*}P'_{1}], [i_{*}P'_{r}] \rangle = \dim_{k}\text{Hom}_{A'}(P'_{u}, P'_{v}), \quad u, v = 1, \ldots, r';
\]

\[
\langle [j_{!}P''_{1}], [i_{*}P'_{r}] \rangle = 0, \quad u = 1, \ldots, r';
\]

\[
\langle [j_{!}P''_{1}], [j_{!}P''_{r}] \rangle = \dim_{k}\text{Hom}_{A''}(P''_{u}, P''_{v}), \quad u, v = 1, \ldots, r''.
\]

Thus the matrix of \( \langle -, - \rangle \) under the basis \( \{[i_{*}P'_{1}], \ldots, [i_{*}P'_{r}], [j_{*}P''_{i'}], \ldots, [j_{*}P''_{r}] \} \) is

\[
\begin{bmatrix}
D' \cdot C(A') & * \\
0 & D'' \cdot C(A'')
\end{bmatrix}
\]

where \( D' = \text{diag}\{c'_{1}, \ldots, c'_{r}\} \) with \( c'_{v} = \dim_{k}\text{End}_{A'}(S'_{v}) \) for all \( v = 1, \ldots, r' \) and \( D'' = \text{diag}\{c''_{1}, \ldots, c''_{r}\} \) with \( c''_{w} = \dim_{k}\text{End}_{A''}(S''_{w}) \) for all \( w = 1, \ldots, r'' \).

Let \( D = \text{diag}\{c_{1}, \ldots, c_{r}\} \) with \( c_{u} = \dim_{k}\text{End}_{A}(S_{u}) \) for all \( u = 1, \ldots, r \).

Note that the matrix of \( \langle -, - \rangle \) under the basis \( \{[P_{1}], \ldots, [P_{r}] \} \) is \( D \cdot C(A) \).

Thus \( D \cdot C(A) \) and

\[
\begin{bmatrix}
D' \cdot C(A') & * \\
0 & D'' \cdot C(A'')
\end{bmatrix}
\]

are the matrices of \( \langle -, - \rangle \) with respect to two different bases. Hence, there exists an invertible matrix \( T \in M_{r}(\mathbb{Z}) \) such that \( D \cdot C(A) = T^{\text{tr}} \cdot \begin{bmatrix} D' \cdot C(A') & * \\ 0 & D'' \cdot C(A'') \end{bmatrix} \cdot T \).

Therefore, \( \det C(A) = \frac{c'_{1} \cdots c'_{r} c''_{1} \cdots c''_{r}}{c_{1} \cdots c_{r}} \cdot (\det T)^{2} \cdot \det C(A') \cdot \det C(A'') \).

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If \( \det C(A) = 0 \) then \( \det C(A') \cdot \det C(A'') = 0 \). Thus \( \det C(A) = \det C(A') \cdot \det C(A'') \). If \( \det C(A) \neq 0 \) then \( \det C(A') \cdot \det C(A'') \neq 0 \). Thus we must have \( \frac{e_1 \cdots e_r}{e_1 \cdots e_r} \cdot (\det T)^2 = 1 \) but not \(-1\). Hence \( \det C(A) = \det C(A') \cdot \det C(A'') \).

Applying Theorem 1 to the trivial 2-recollement in Example 1 (4), we can obtain the following corollary which generalizes \([5, Proposition 1.5]\) to an arbitrary base field.

**Corollary 1.** Let \( A \) and \( B \) be derived equivalent finite dimensional algebras. Then \( \det C(A) = \det C(B) \).

Next we study the Cartan determinant conjecture. In 1954, Eilenberg showed that if \( A \) is a finite dimensional algebra of finite global dimension then \( \det C(A) = \pm 1 \) (Ref. \([12]\)). After that, the following conjecture was posed:

**Cartan determinant conjecture.** Let \( A \) be an artin algebra of finite global dimension. Then \( \det C(A) = 1 \).

The Cartan determinant conjecture remains open except for some special classes of algebras, such as the algebras of global dimension two \([37]\), the positively graded algebras \([33]\), the Cartan filtered algebras \([13]\), the left serial algebras \([7]\), the quasi-hereditary algebras \([5]\), and the artin algebras admitting a strongly adequate grading by an aperiodic commutative monoid \([28]\).

**Proposition 3.** Let \( A' \), \( A \) and \( A'' \) be finite dimensional algebras, and \( \mathcal{D}A \) admit a recollement relative to \( \mathcal{D}A' \) and \( \mathcal{D}A'' \). If both \( A' \) and \( A'' \) satisfy the Cartan determinant conjecture, then so does \( A \).

**Proof.** If \( A \) is of finite global dimension then so are \( A' \) and \( A'' \) by \([1, Proposition 2.14]\). Thus \( \det C(A') = \det C(A'') = 1 \) by the assumption and the recollement induces a 2-recollement, see Example 1 (3). By Theorem 1 we have \( \det C(A) = \det C(A') \cdot \det C(A'') = 1 \).

The following corollary implies that the Cartan determinant conjecture can be reduced to an arbitrary complete set of representatives of the derived equivalence classes of finite dimensional algebras.

**Corollary 2.** Let \( A \) and \( B \) be derived equivalent finite dimensional algebras. Then \( A \) satisfies the Cartan determinant conjecture if and only if so does \( B \).
Proof. It is enough to apply Proposition 3 to the trivial recollements, see Example 1 (4).

Applying Proposition 3, we can reduce the Cartan determinant conjecture to 1-derived-simple algebras.

Corollary 3. The Cartan determinant conjecture holds for all finite dimensional algebras if and only if it holds for all 1-derived-simple algebras.

Proof. For any finite dimensional algebra $A$, by [1, Proposition 6.5], $\mathcal{D}A$ admits a finite stratification of derived categories along recollements with 1-derived-simple factors. Then the corollary follows from Proposition 3.

Although Corollary 3 provides a reduction technique, the Cartan determinant conjecture seems far from being settled, because it is still a problem to deal with all the 1-derived-simple algebras of finite global dimension. Nonetheless, for the known examples described in Example 2 (1) and (2), the Cartan determinant conjecture holds true [16, 24].

Let’s end this section by pointing out that Theorem 1 can be applied to prove the $n$-derived-simplicity of certain algebras as well.

Remark 2. A finite dimensional two-point algebra $A$ with $\det C(A) \leq 0$ must be 2-derived-simple: Otherwise, $\mathcal{D}A$ admits a non-trivial 2-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Then both $B$ and $C$ are finite dimensional local algebras since $A$ has only two simple modules up to isomorphism. Therefore, $\det C(B) > 0$ and $\det C(C) > 0$. By Theorem 1, we get $\det C(A) > 0$. It is a contradiction. The examples of this kind of 2-derived-simple algebras include:

1. $1 \begin{array}{c} \alpha \\ \beta \end{array} 2$, $(\alpha \beta)^n = 0 = (\beta \alpha)^n, n \in \mathbb{Z}^+$;

2. $\gamma \bigcap 1 \begin{array}{c} \alpha \\ \beta \end{array} 2 \bigcup \delta$, $\alpha \beta = \beta \alpha = \gamma^2 = 2\delta = \gamma \alpha - \alpha \delta = \delta \beta - \beta \gamma = 0$;

3. Let $A$ be one of the algebras in (1) and (2), and $B$ an arbitrary finite dimensional local algebra. Then the tensor product algebra $A \otimes_k B$ is again 2-derived-simple by the same reason.

Remark 3. A representation-finite selfinjective two-point algebra $A$ with $\det C(A) \leq 0$ must be 1-derived-simple. Indeed, for a representation-finite selfinjective algebra $A$, if $\mathcal{D}A$ admits a recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$, this recollement must be perfect [23, Proposition 4.1]. Therefore, the 2-derived-simplicity of these algebras implies the 1-derived-simplicity. For example, the algebras in Remark 2 (1) are 1-derived-simple.

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4 n-recollements and homological smoothness

In this section, we will observe the relation between n-recollements and the homological smoothness of algebras.

Let $A$ be an algebra and $A^e := A^{op} \otimes_k A$ its enveloping algebra. The algebra $A$ is said to be smooth if the projective dimension of $A$ as an $A^e$-module is finite [31]. The algebra $A$ is said to be homologically smooth if $A$ is compact in $\mathcal{D}(A^e)$, i.e., $A$ is isomorphic in $\mathcal{D}(A^e)$ to an object in $K^b(\text{proj}A^e)$ (Ref. [22]). Clearly, all homologically smooth algebras are smooth. Moreover, if $A$ is a finite dimensional algebra then the concepts of smoothness and homological smoothness coincide. However, they are different in general. For example, the infinite Kronecker algebra is smooth but not homologically smooth [15, Remark 4].

Let $A$ and $B$ be two derived equivalent algebras. Then, by [27, Proposition 2.5], there is a triangle equivalence functor from $\mathcal{D}(A^e)$ to $\mathcal{D}(B^e)$ sending $A_{A^e}$ to $B_{B^e}$. Since the equivalence functor can restrict to $K^b(\text{Proj})$ and $K^b(\text{proj})$, both the smoothness and the homological smoothness of algebras are invariant under derived equivalences. Moreover, the relations between recollements and the smoothness of algebras have been clarified in [15]:

**Proposition 4.** (See [15, Theorem 3]) Let $A$, $B$ and $C$ be algebras, and $DA$ admit a recollement relative to $DB$ and $DC$. Then $A$ is smooth if and only if so are $B$ and $C$.

However, Proposition 4 is not correct for homological smoothness any more. Here is an example:

**Example 3.** (See [15, Remark 4]) Let $A$ be the infinite Kronecker algebra
\[
\begin{pmatrix}
k & 0 \\
k^{(n)} & k
\end{pmatrix}.
\]
Then by Example 1 (2), $DA$ admits a 2-recollement relative to $Dk$ and $Dk$, but $A$ is not homologically smooth.

Due to Example 3 even though $DA$ admits a 2-recollement relative to $DB$ and $DC$, the homological smoothness of $B$ and $C$ can not imply the homological smoothness of $A$. Nonetheless, we have the following theorem which is just Theorem II.

**Theorem 2.** Let $A$, $B$ and $C$ be algebras, and $DA$ admit an n-recollement relative to $DB$ and $DC$.

1. $n = 1$: if $A$ is homologically smooth then so is $B$;
2. $n = 2$: if $A$ is homologically smooth then so are $B$ and $C$;
3. $n \geq 3$: $A$ is homologically smooth if and only if so are $B$ and $C$. 

Proof. (1) See [20, Proposition 3.10 (c)].

(2) If $A$ is homologically smooth then $B$ is also homologically smooth by (1). Since $n = 2$, we have a recollement of $\mathcal{D}A$ relative to $\mathcal{D}C$ and $\mathcal{D}B$, and thus $C$ is also homologically smooth by (1) again.

(3) Assume $\mathcal{D}A$ admits a 3-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$. Consider the recollement formed by the middle three layers. By [15, Proposition 3], we may assume that it is of the form

$$
\begin{array}{ccc}
\mathcal{D}C & \xrightarrow{i^* = -\otimes_A^LY} & \mathcal{D}A & \xrightarrow{j^* = -\otimes_B^XY^*} & \mathcal{D}B \\
& \xrightarrow{i_* = -\otimes_B^LY} & & \xrightarrow{j_* = -\otimes_B^XY^*} &
\end{array}
$$

where $X \in \mathcal{D}(B^{\text{op}} \otimes A)$, $Y \in \mathcal{D}(A^{\text{op}} \otimes C)$, $X^* := \text{RHom}_{A}(X, A)$ and $Y^* := \text{RHom}_{C}(Y, C)$. Clearly, $A^Y$ and $Y^*_A$ are compact since the recollement can be extended one step upwards and one step downwards respectively [1, Proposition 3.2 and Lemma 2.8].

By [15, Theorem 1] and [15, Theorem 2], a recollement of derived categories of algebras induces those of tensor product algebras and opposite algebras respectively. Thus we have the following three recollements induced by the recollement $(R')$:

$$
\begin{array}{ccc}
\mathcal{D}(C^e) & \xrightarrow{F_1} & \mathcal{D}(A^e) & \xrightarrow{F_2} & \mathcal{D}(C^e) \\
& \xrightarrow{L_1} & & \xrightarrow{L_2} &
\end{array}
$$

where $L_1 = Y^* \otimes_A^L -$, $F_1 = Y \otimes_B^L -$, $L_2 = -\otimes_B^LY$, $F_2 = -\otimes_B^LY^*$, $L_3 = -\otimes_B^X -$, $F_3 = -\otimes_A^L X^*$, $L_4 = X^* \otimes_B^L -$ and $F_4 = X \otimes_A^L -$. Consider the canonical triangle

$$
L_3 F_3 A \rightarrow A \rightarrow F_2 L_2 A \rightarrow \text{ in } D(A^e),
$$

and note that $F_2 L_2 A = Y \otimes_B^L Y^* = F_2 F_1 C$, $L_3 F_3 A = X^* \otimes_B^L X = L_3 L_4 B$. Clearly, the functors $L_3$ and $L_4$ preserve compactness, so are $F_1$ and $F_2$ since $A^Y$ and $Y^*_A$ are compact. Applying these to the above triangle, we get $A \in K^b(\text{proj}A^e)$ whenever $B \in K^b(\text{proj}B^e)$ and $C \in K^b(\text{proj}C^e)$. Namely, the homological smoothness of $B$ and $C$ implies that of $A$. 

\qed
According to Example 3 and the statement followed, we see that in Theorem 2 (3), the requirement $n \geq 3$ is optimal.

Applying Theorem 2 to triangular matrix algebras, we get the following corollary which provides a construction of homologically smooth algebras.

**Corollary 4.** Let $B$ and $C$ be algebras, $M$ a $C$-$B$-bimodule, and $A := \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$.

1. If $A$ is homologically smooth, then so are $B$ and $C$;
2. If $B$ and $C$ are homologically smooth and $CM \in K^b(\text{proj} C^{\text{op}})$ or $MB \in K^b(\text{proj} B)$, then $A$ is also homologically smooth.

**Proof.** It follows from Example 1 (2) and Theorem 2.

---

**5 $n$-recollements and Gorensteinness**

In this section, we will observe the relations between $n$-recollements and the Gorensteinness of algebras, and reduce the Gorenstein symmetry conjecture to 2-derived-simple algebras.

A finite dimensional algebra $A$ is said to be Gorenstein if $\text{id} A < \infty$ and $\text{id} A^{\text{op}} < \infty$. Clearly, a finite dimensional algebra $A$ is Gorenstein if and only if $K^b(\text{proj} A) = K^b(\text{inj} A)$ as strictly full triangulated subcategories of $\mathcal{D}A$. Thus, the Gorensteinness of algebras is invariant under derived equivalences. It is natural to consider the relation between recollements and the Gorensteinness of algebras. In [26], Pan proved that the Gorensteinness of $A$ implies the Gorensteinness of $B$ and $C$ if there exists a recollement of $\mathcal{D}b(\text{mod} A)$ relative to $\mathcal{D}b(\text{mod} B)$ and $\mathcal{D}b(\text{mod} C)$. Now we complete it using the language of $n$-recollements. The following theorem is just Theorem III.

**Theorem 3.** Let $A$, $B$ and $C$ be finite dimensional algebras, and $\mathcal{D}A$ admit an $n$-recollement relative to $\mathcal{D}B$ and $\mathcal{D}C$.

1. $n = 3$: if $A$ is Gorenstein then so are $B$ and $C$;
2. $n \geq 4$: $A$ is Gorenstein if and only if so are $B$ and $C$.

**Proof.** (1) It follows from Proposition 2 that $\mathcal{D}b(\text{mod} A)$ admits a recollement relative to $\mathcal{D}b(\text{mod} C)$ and $\mathcal{D}b(\text{mod} B)$. Therefore, the statement follows from Pan [26]. Here we provide another proof. Consider the following recollement consisting of the middle three layers of functors of the 3-recollement:

$$
\mathcal{D}(C) \xrightarrow{i^*} \mathcal{D}(A) \xrightarrow{j^*} \mathcal{D}(B) .
$$
By Lemma \[i^*, i_*, j_!\] and \(j^!\) restrict to \(K^b(\text{proj})\), and \(i^*, i^!, j^!\) and \(j_*\) restrict to \(K^b(\text{inj})\). If \(A\) is Gorenstein then \(K^b(\text{proj}A) = K^b(\text{inj}A)\).

Note that \(DC := \text{Hom}_k(C, k)\). Thus \(DC \cong i^*i_*(DC) \subset i^*(K^b(\text{inj}A)) \subset K^b(\text{proj}A) \subset K^b(\text{inj}C)\). Thus, \(\text{pd}_C(DC) < \infty\), equivalently, \(\text{id}_{\text{op}C} \subset \infty\). On the other hand, \(C \cong i^!i_*C \subset i^!i_*K^b(\text{proj}C) \subset i^!K^b(\text{proj}A) = i^!K^b(\text{inj}A) \subset K^b(\text{inj}C)\). Thus, \(\text{id}_C \subset \infty\). Therefore, \(C\) is Gorenstein.

Similarly, \(DB \cong j^!j_*(DB) \subset j^!j_*(K^b(\text{inj}B)) \subset j^!K^b(\text{proj}B) \subset K^b(\text{proj}B)\). Thus, \(\text{pd}_B(DB) < \infty\), equivalently, \(\text{id}_{\text{op}B} < \infty\). On the other hand, \(B \cong j^!*j_!B \subset j^!*j_!K^b(\text{proj}B) \subset j^!*K^b(\text{proj}A) = j^!*K^b(\text{inj}A) \subset K^b(\text{inj}B)\). Thus, \(\text{id}_B \subset \infty\). Therefore, \(B\) is Gorenstein.

(2) Let \[
\begin{array}{c}
DB \\
\downarrow \\
DA \\
\downarrow \\
DC
\end{array}
\]

be a 4-recollement. By Lemma \[i^*, i_*, j_!\] and \(j^!\) restrict to both \(K^b(\text{proj})\) and \(K^b(\text{inj})\). If both \(B\) and \(C\) are Gorenstein, then \(K^b(\text{proj}B) = K^b(\text{inj}B)\) and \(K^b(\text{proj}C) = K^b(\text{inj}C)\).

Consider the triangle \(j^!j_!(DA) \rightarrow DA \rightarrow i^!i_*(DA) \rightarrow \). We have \(j^!j_!(DA) \subset j^!j_!K^b(\text{inj}A) \subset i^!*K^b(\text{proj}A) \subset K^b(\text{proj}A)\) and \(i^!*K^b(\text{proj}A) \subset i^!K^b(\text{proj}B) = i^!K^b(\text{inj}B) \subset K^b(\text{inj}B)\). Thus, \(DA \in K^b(\text{proj}A)\), i.e., \(\text{pd}_A(DA) < \infty\). Hence, \(\text{id}_{\text{op}A} < \infty\).

Similarly, consider the triangle \(j^!j_!A \rightarrow A \rightarrow i^!*A \rightarrow \). We have \(j^!j_!A \subset j^!j_!K^b(\text{proj}A) \subset j^!K^b(\text{inj}C) \subset K^b(\text{inj}A)\) and \(i^!*A \subset i^!*K^b(\text{proj}A) \subset i^!K^b(\text{proj}B) = i^!K^b(\text{inj}B) \subset K^b(\text{inj}B)\). Thus, \(A \in K^b(\text{inj}A)\), i.e., \(\text{id}_A < \infty\). Therefore, \(A\) is Gorenstein.

Applying Theorem 3 to triangular matrix algebras, we get the following corollaries, which imply the condition \(n \geq 4\) in Theorem 3 (2) is optimal.

**Corollary 5.** ([10, Theorem 3.3]) Let \(B\) and \(C\) be Gorenstein algebras, \(M\) a finite generated \(C\)-\(B\)-bimodule, and \(A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}\). Then \(A\) is Gorenstein if and only if \(\text{pd}_{\text{op}}M < \infty\) and \(\text{pd}_B M < \infty\).

**Proof.** Assume that \(A\) is Gorenstein. Set \(e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) and \(e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\).
By [1, Example 3.4], there is a 2-recollement of the form

\[ \begin{array}{c}
\mathcal{D}B \\
i^* \\
i^* \leftarrow \mathcal{D}A \\
\mathcal{D}A \rightarrow \mathcal{D}C \\
\end{array} \]

\[ j_i^* \leftarrow \leftarrow j_j^* = - \otimes_{B^e}^L e_1 A \]

It follows from Lemma 2 that \( i^* \) restricts to \( K^b (\text{inj}) \), and further restricts to \( K^b (\text{proj}) \) by the Gorensteinness of \( A \) and \( B \). Thus, \( i^* A = B \oplus M_B \in K^b (\text{proj} B) \). Hence, \( \text{pd}_B M < \infty \). Similarly, it follows from Lemma 2 that \( j^* \) restricts to \( K^b (\text{proj}) \), and further restricts to \( K^b (\text{inj}) \) by the Gorensteinness of \( A \) and \( C \). By Lemma 1, \( j_i^* \) restricts to \( D^b (\text{mod}) \). Since \( j_i = - \otimes_C^L e_2 A \), this is equivalent to \( C(e_2 A) \in K^b (\text{proj} C^\text{op}) \) (Ref. [1, Lemma 2.8]). Note that \( C(e_2 A) = C \oplus C M \), thus \( \text{pd}_{C^\text{op}} M < \infty \).

Conversely, assume that \( \text{pd}_{C^\text{op}} M < \infty \) and \( \text{pd}_B M < \infty \), then by Example 1 (2), the above 2-recollement can be extended one step upwards and one step downwards to a 4-recollement. Therefore, the Gorensteinness of \( B \) and \( C \) implies the Gorensteinness of \( A \) by Theorem 3.

**Corollary 6.** (\cite[Theorem 2.2 (iii)]{36}) Let \( B \) and \( C \) be finite dimensional algebras, \( M \) a finite generated \( C-B \)-bimodule with \( \text{pd}_{C^\text{op}} M < \infty \) and \( \text{pd}_B M < \infty \), and \( A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix} \). Then \( A \) is Gorenstein if and only if so are \( B \) and \( C \).

**Proof.** It follows from Example 1 (2) and Theorem 3.

Next we study the Gorenstein symmetry conjecture.

**Gorenstein symmetry conjecture.** Let \( A \) be an artin algebra. Then \( \text{id}_A A < \infty \) if and only if \( \text{id}_{A^\text{op}} A < \infty \).

This conjecture is listed in Auslander-Reiten-Smalø’s book \cite[p.410, Conjecture (13)]{2}, and it closely connects with other homological conjectures. For example, it is known that the finitistic dimension conjecture implies the Gorenstein symmetry conjecture. But so far all these conjectures are still open. As mentioned before, the finitistic dimension conjecture can be reduced to 2-derived-simple algebras. Now, let us utilize Theorem 8 to reduce the Gorenstein symmetry conjecture to 2-derived-simple algebras.

**Proposition 5.** Let \( A, B \) and \( C \) be finite dimensional algebras, and \( DA \) admit a 2-recollement relative to \( DB \) and \( DC \). If both \( B \) and \( C \) satisfy the Gorenstein symmetry conjecture, then so does \( A \).
Proof. Assume that

\[
\begin{array}{c}
\mathcal{D}B \xrightarrow{i^!} \mathcal{D}A \xrightarrow{j^!} \mathcal{D}C
\end{array}
\]

is a 2-recollement, and both \(B\) and \(C\) satisfy the Gorenstein symmetry conjecture.

If \(\text{id}_{A^{op}} A < \infty\), then \(K^b(\text{proj}A) \subseteq K^b(\text{inj}A)\). By Lemma 2, we have \(B \cong i^!i_*(K^b(\text{proj}B)) \subseteq i^!(K^b(\text{proj}A)) \subseteq i^!(K^b(\text{inj}A)) \subseteq K^b(\text{inj}B)\), i.e., \(\text{id}_B B < \infty\). Since \(B\) satisfies the Gorenstein symmetry conjecture, we obtain that \(B\) is Gorenstein. By Lemma 2 again, we have \(i^!A \in i^!(K^b(\text{proj}A)) \subseteq i^!(K^b(\text{inj}A)) \subseteq K^b(\text{inj}B) = K^b(\text{proj}B)\). Due to Lemma 3, \(i^!A \in K^b(\text{proj}B)\) implies that the 2-recollement \((R'')\) can be extended one step downwards. Therefore, we get a 2-recollement of \(\mathcal{D}A\) relative to \(\mathcal{D}C\) and \(\mathcal{D}B\). Analogous to the above proof, we obtain that \(C\) is Gorenstein and the 2-recollement \((R'')\) can be extended two steps downwards to a 4-recollement of \(\mathcal{D}A\) relative to \(\mathcal{D}B\) and \(\mathcal{D}C\). By Theorem 3, \(A\) is Gorenstein. Thus \(\text{id}_{A^{op}} A < \infty\).

The following corollary implies that the Gorenstein symmetry conjecture can be reduced to an arbitrary complete set of representatives of the derived equivalence classes of finite dimensional algebras.

**Corollary 7.** Let \(A\) and \(B\) be derived equivalent finite dimensional algebras. Then \(A\) satisfies the Gorenstein symmetry conjecture if and only if so does \(B\).

**Proof.** It is enough to apply Proposition 5 to the trivial 2-recollements, see Example 1 (4).

Applying Proposition 5, we can reduce the Gorenstein symmetry conjecture to 2-derived-simple algebras.
Corollary 8. The Gorenstein symmetry conjecture holds for all finite dimensional algebras if and only if it holds for all 2-derived-simple algebras.

Proof. For any finite dimensional algebra $A$, by [1, Proposition 6.5], $\mathcal{DA}$ admits a finite stratification of derived categories along 2-recollements with 2-derived-simple factors. Then the corollary follows from Proposition 5.

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