Light-cone observations and cosmological models: implications for inhomogeneous models mimicking dark energy

Edward W. Kolb and Callum R. Lamb

Department of Astronomy and Astrophysics, Enrico Fermi Institute, and Kavli Institute for Cosmological Physics, the University of Chicago, Chicago, Illinois 60637-1433

Cosmological observables are used to construct cosmological models. Since cosmological observations are limited to the light cone, a fixed number of observables (even measured to arbitrary accuracy) may not uniquely determine a cosmological model without additional assumptions or considerations. A prescription for constructing a spherically symmetric, inhomogeneous cosmological model that exactly reproduces the luminosity-distance as a function of redshift and the light-cone mass density as a function of redshift of a ΛCDM model is employed to gain insight into how an inhomogeneous cosmological model might mimic dark energy models.

PACS numbers: 98.70.Cq

I. INTRODUCTION

In the last decade or so remarkable progress has been made in measuring cosmological parameters to unprecedented accuracy. Parameters such as the Hubble constant, \( H_0 = 72 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1} \) [1], the temperature of the cosmic background radiation (CBR), \( T_0 = 2.728 \pm 0.004 \text{ K} \) [2], and many other parameters, are now known to impressive precision. Of course, we don’t invest so much time and effort determining cosmological parameters because of an interest in numerology, but rather, because the parameters are necessary inputs for the task of constructing a standard cosmological model.

In turn, we do not construct cosmological models simply to make predictions to compare with observations, but rather, to guide us to, in the words of Einstein [3], “a deeper and more consistent comprehension of the physical and astronomical facts.”

The precision cosmological measurements have lead to the latest cosmological model, usually called the standard cosmological model, the concordance model, or simply ΛCDM, where Λ indicates the inclusion of Einstein’s cosmological constant (or more generally, dark energy), and CDM stands for cold dark matter. The model is but one possible realization of homogeneous and isotropic cosmological solutions to Einstein’s equations. (We will refer to homogeneous/isotropic models as Friedmann–Lemaître–Robertson–Walker (FLRW) models.) The question yet to be answered is whether today’s standard model will lead us to a deeper and more consistent comprehension of the physical and astronomical facts.

A troubling feature of the ΛCDM model is the present composition of the universe: 95% is dark! Of the dark components, roughly 25% of the total mass-energy density is dark matter, associated with galaxies, clusters of galaxies, and other bound structures. The bulk of the mass-energy of the universe, about 70% in the standard ΛCDM model, is attributed to dark energy, which is usually said to drive an accelerated expansion of the Universe.

The observations and phenomena that lead to the assumptions of dark matter and dark energy are real, but that does not necessarily imply that dark matter and dark energy as usually envisioned are real. Dark matter and dark energy are good names for the phenomena. But naming is not explaining!

In some sense, an even more remarkable feature of the standard cosmological model is that it seems capable of accounting for all cosmological observations; i.e., it seems to work! Here, we address the issue of whether the agreement with observations prejudices our judgment regarding the likelihood that the ΛCDM model is the final answer. In particular, we will discuss the possibility that dark energy, per se, does not exist.

The most important point regarding dark energy (and also, for that matter, the acceleration of the expansion of the universe), which is central to the motivation for this paper, is that all evidence for dark energy is indirect. The only effect of dark energy is on the expansion history of the Universe.

The expansion rate of the Universe is perhaps the most fundamental quantity in cosmology. In the standard cosmological model the expansion rate is determined from the 00-component of the Einstein equations, \( G_{00} = \kappa T_{00} \),
where $\kappa = 8\pi G$. In FLRW models, $G_{00} = 3H^2 + k/a^2$, where $H = \dot{a}/a$ is the expansion rate (here, $a$ is the Robertson-Walker scale factor and $k = \pm 1$ or 0, depending on the geometry). With the assumption of a perfect-fluid stress tensor, $T_{00} = \rho$, where $\rho$ is the mass density.

The stress tensor is assumed to consist of several components with different equations of state. For any component $i$, the equation of state for that component is defined as $w_i = p_i/\rho_i$. If $w_i$ is constant, the energy density in component $i$ evolves with redshift $z \equiv a_0/a - 1$ as $\rho_i(z) = \rho_i(0)(1 + z)^{3(1+w_i)}$, where $a_0$ is the present value of the scale factor. Then the expansion rate as a function of redshift can be expressed as

$$H^2(z) = H_0^2 \left\{ \Omega_k \exp \left[ \int_0^z \frac{dz_1}{1 + z_1} 3 \left[ 1 + \omega_{\Lambda}(z_1) \right] \right] + \Omega_k(1 + z)^2 + \Omega_M(1 + z)^3 + \Omega_R(1 + z)^4 \right\}. \tag{1}$$

Here, $\Omega_i$ is the ratio of the present energy density in component $i$ compared to the critical density $\rho_C = 3H_0^2/\kappa$. The subscript “$k$” indicates a spatial-curvature contribution ($w_k = -1/3$), “$M$” indicates a matter component ($w_M = 0$), “$R$” indicates a radiation component ($w_R = 1/3$), and “$\Lambda$” represents dark energy. In the event that the function $w_{\Lambda}(z)$ in the last term is independent of $z$, the last term in $H^2(z)$ becomes $\Omega_{\Lambda}(1 + z)^3(1 + w_{\Lambda})$. The equation of state for a cosmological constant is $w_{\Lambda} = -1$.

It appears that the expansion rate as a function of $z$ depends on four constants $\{\Omega_k, \Omega_M, \Omega_R, \Omega_{\Lambda}\}$ and one function $w_{\Lambda}(z)$. In reality, the situation is much more predictive. First, since $H^2(0) = H_0^2$, it follows that $\sum_i \Omega_i = 1$, removing one constant.

The value of $\Omega_k$ is well determined by WMAP to be small, $\Omega_k = -0.0026^{+0.0066}_{-0.0064}$ at the 68% confidence level [4], effectively removing yet another constant. The value of $\Omega_{\Lambda}$ is also well determined by CBR measurements ($5 \times 10^{-5}$) removing a further constant. One may then make the less secure assumption that $\Omega_M$ is “known,” or more profitably, use observational constraints to marginalize over the $\Omega_i$’s, leaving only the function $w_{\Lambda}(z)$ undetermined. The function must be determined through its effect on the expansion history of the Universe, which is manifest through various cosmological observables.

Many cosmological observables depend on the expansion history of the Universe. They often depend on the expansion history through the coordinate distance $r$ of a source of redshift $z$. Recall that the Robertson-Walker metric can be written in the form

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right], \tag{2}$$

where here $d\Omega$ is the angular differential and $r$ is the comoving “radial” coordinate. The coordinate radial of a source at redshift $z$ is determined by integrating the null geodesic equation ($ds^2 = 0$) to obtain

$$r(z) = \sin \left[ \frac{1}{\sinh \left( \int_0^z \frac{dz'}{H(z')} \right) } \right], \tag{3}$$

where $\sin, 1, \sinh$ obtains for $k = +1, 0, 1$, respectively.

Observables such as the luminosity distance, the angular-diameter distance, the volume element, and the age of the Universe, all depend on the time evolution of the expansion rate, hence on the properties of the stress tensor, hence on the dark energy fluid characterized by $w_{\Lambda}$.

At present, a constant $w_{\Lambda}(z)$ seems to fit the data, and furthermore $w_{\Lambda} = -1$, the value of $w_{\Lambda}$ for a cosmological constant, is not disallowed. It is also true that since there is no apparent reason to exclude vacuum energy from the stress tensor, there is no reason not to imagine it enters the determination of the expansion rate. The problem is that the magnitude of the expected vacuum energy is very many orders of magnitude larger than expected. If the cosmological constant is part of nature, then either some principle must be found to explain the much smaller-than-expected magnitude of $\Omega_{\Lambda}$, or one must resort to anthropic arguments.

This situation has led many to consider alternate explanations of the observations: modified gravity, new long-range forces, new ultra-light scalar fields, extra dimensions, branes, bulk, Lorentz-invariance violation, etc. Some of these approaches are, in many respects, more drastic than a small cosmological constant. For instance, theories of modified gravity typically do not respect the principle of general covariance. Since we should not abandon our principles without good cause, it is tempting to retreat to the position that since a cosmological constant is adequate to account for the data, just assume that it exists and call it a day?

We resist the retreat because our goal in cosmology is more than just explaining the observations. The ingredients of a cosmological model must be deeply grounded in fundamental physics. Dark matter, dark energy, modified gravity, mysterious new forces and particles, etc., unless part of an overarching model of nature, should not be part of a cosmological model. We may plant new ideas and features into cosmological models, but they will ultimately prove
barren unless grounded in fundamental physics. No matter how well a concordance model fits the data, its ingredients must not be discordant with fundamental physics.

In this paper we explore a less traveled path, and consider the possibility that the Friedmann equation is not an adequate description of our inhomogeneous Universe. In this approach there is no need for dark energy, so no need for new long-range forces, modifications of general relativity, new ultralight particles, or anthropic reasoning. The phenomenon of inhomogeneities mimicking dark energy is sometimes referred to as “backreaction.”

In this paper we follow the prescription of Mustapha, Hellaby, and Ellis, recently applied to $\Lambda$CDM models by Célerier, Bolejko, Krasinski, and Hellaby, and explore the implications of the fact that one can construct a spherically symmetric inhomogeneous model that exactly reproduces the redshift dependence of the luminosity distance and mass density of any given $\Lambda$CDM model. We show that in general one expects the cosmological solution thus obtained to have a “mixmaster” approach to the initial singularity, with the radial scale factor diverging and the angular scale factor vanishing. We then show that there is another cosmological solution with the same density and curvature functions as the singular model but with regular initial conditions where both scale factors vanish at the singularity, and that this regular solution well approximates another $\Lambda$CDM model. We then discuss the utility of different averaging prescriptions for understanding the results.

In the next section we briefly review spherically symmetric cosmological solutions. In Sect. II we review the construction of such an inhomogeneous model that reproduces key observational features of $\Lambda$CDM. In Sect. III we evolve the model off the light cone, paying special attention to its behavior near the bang. In Sect. IV we calculate some spatially averaged features of the model. Section V presents the conclusions and discusses their implications.

II. LEMAÎTRE–TOLMAN–BONDI MODELS

In this section we discuss Lemaître–Tolman–Bondi (LTB) models in general, and in the next section we discuss a particular LTB model that exactly reproduces select observational features of $\Lambda$CDM model.

LTB models are spherically symmetric cosmological solutions to the Einstein equations with a dust stress-energy tensor. The LTB model is based on the assumptions that the system has purely radial motion and the motion is geodesic without shell crossing (otherwise the pressure could not be neglected). The line element in the synchronous gauge can be written in the form

$$ds^2 = -dt^2 + \frac{R^2(r, t)}{1 + \beta(r)} dr^2 + R^2(r, t) d\Omega^2.$$  

Here, the prime superscript denotes $d/dr$. We will denote $d/dt$ by an overdot. The function $\beta(r)$ is an arbitrary function of $r$. The Robertson–Walker metric can be recovered if $R(r, t) \to a(t)r$ and $\beta(r) \to -kr^2$.

In spherically symmetric models, in general there are two expansion rates: an angular expansion rate, $H_\perp \equiv R(r, t)/R(t)$, and a radial expansion rate, $H_r \equiv \dot{R}(r, t)/R(r, t)$. With the dust equation of state, the Einstein equations may be expressed as

$$H_r^2(r, t) + 2\dot{H}_r(r, t)H_\perp(r, t) - \frac{\beta(r)}{R^2(r, t)} - \frac{\beta'(r)}{R(r, t)R'(r, t)} = \kappa \rho(r, t)$$

$$6 \frac{\dot{R}(r, t)}{R(r, t)} + 2H_\perp^2(r, t) - 2 \frac{\beta(r)}{R^2(r, t)} - 2H_r(r, t)H_\perp(r, t) + \frac{\beta'(r)}{R(r, t)R'(r, t)} = -\kappa \rho(r, t).$$

These represent the generalization of the Friedmann equation for a homogeneous/isotropic universe to a spherically symmetric inhomogeneous universe.

The equations may be manipulated to result in an easily integrable dynamical equation for $R(r, t)$:

$$\dot{R}(r, t) = \sqrt{\beta(r) + \frac{\alpha(r)}{R(r, t)}},$$

1 Similar formalism was developed by Chung and Romano to fit only the luminosity-distance, and by Yoo, Kai, and Nakao who simultaneously fit the luminosity distance and require no spatial variations of the age of the Universe.
where \( \alpha(r) \) is a function of \( r \) related to the density \( \rho(r,t) \) by

\[
\kappa \rho(r,t) = \frac{\alpha'(r)}{R^2(r,t)R'(r,t)}.
\]

Equation (7) can be differentiated to yield the dynamical equation for \( R'(r,t) \):

\[
\dot{R}'(r,t) = \frac{\beta'(r) + \alpha'(r)/R(r,t) - \alpha(r)R'(r,t)/R^2(r,t)}{2R(r,t)}.
\]

The two functions \( \alpha(r) \) and \( \beta(r) \) define the LTB model, playing roles analogous to the total energy density parameter \( \Omega_0 \) and the spatial curvature parameter \( \Omega_k \) of FLRW models. To see this, recall that the evolution of the scale factor and time for a hyperbolic dust Friedmann model evolves as (\( \eta \) is conformal time)

\[
a(t) = a_0 \frac{\Omega_0}{2\Omega_k} [\cosh \eta - 1], \quad t - t_{BB} = H_0^{-1} \frac{\Omega_0}{2\Omega_k^{3/2}} [\sinh \eta - \eta],
\]

where \( a_0 \) is the present value of the scale factor and \( t_{BB} \) is the time of the big bang (usually set to zero).

Since Friedmann models are homogeneous and isotropic, they are zero-dimensional models. The LTB models are spherically symmetric, so they are one-dimensional problems. So for LTB models the density parameter \( \Omega_0 \) is replaced by a density function \( \alpha(r) \) and the curvature parameter \( \Omega_k \) is replaced by a curvature function \( \beta(r) \). If \( \beta(r) > 0 \), which is the analog of hyperbolic Friedmann models, the evolution of the metric function \( R(r,t) \) can be described by

\[
R(r,t) = \frac{\alpha(r)}{2\beta(r)} [\cosh \eta(r) - 1]; \quad t - t_{BB}(r) = \frac{\alpha(r)}{2\beta^{3/2}(r)} [\sinh \eta(r) - \eta(r)],
\]

where now, in general, the bang time, \( t_{BB}(r) \) is a function of \( r \), as is the effective conformal time \( \eta(r) \).

The photon geodesic equation for the position of the photon as a function of time, \( \hat{t}(r) \), is found from the LTB metric:

\[
\frac{d\hat{t}}{dr} = -\frac{R'(r,\hat{t}(r))}{\sqrt{1 + \beta(r)}}.
\]

The redshift of the photon, \( z(r) \), is

\[
\frac{dz}{dr} = (1 + z) \frac{\dot{R}'(r,\hat{t}(r))}{\sqrt{1 + \beta(r)}}.
\]

Equations (11) and (12) are solved with initial conditions \( z(r = 0) = 0 \) and \( \hat{t}(r = 0) = 0 \). The luminosity distance is then simply

\[
\hat{d}_L(z) = (1 + z)^2 R(r,\hat{t}(r)).
\]

(We denote the luminosity distance by \( \hat{d}_L \) to emphasize that it is a light-cone observable.)

### III. RECONSTRUCTING AN LTB MODEL WITH \( \Lambda \text{CDM} \) OBSERVATIONAL FEATURES

In this section we will review the procedure and results that allow one to construct an LTB model that reproduces

1. the luminosity-distance–redshift relationship, \( \hat{d}_L(z) \), of a fiducial \( \Lambda \text{CDM} \) model, and
2. the light-cone matter density as a function of redshift, \( \hat{\rho}(z) \), of the fiducial \( \Lambda \text{CDM} \) model.

\footnote{The dimension of a model refers to the number of dynamically independent combinations of spatial dimensions that enter in physical quantities.}
The “fiducial” ΛCDM model may in principle be any FLRW model with a cosmological constant. In this paper we will assume for the fiducial model a spatially-flat Universe, so Ω_M + Ω_Λ = 1. Unless otherwise specified, we will also assume Ω_M = 0.7 (so Ω_M = 0.3). The assumption of spatial flatness will simplify our calculations. Neither that assumption, nor the exact value of Ω_Λ, is important in our considerations.

The reconstruction method we follow was discussed in general by Mustapha, Hellaby, and Ellis in 1998 [16] [MHE], and recently applied to ΛCDM models by Célérier, Bolejko, Krasinski, and Hellaby [CBKH], [17]. Related formalism was developed by Chung and Romano [18] and by Yoo, Kai, and Nakao [19].

Let us review the procedure. To follow the discussion, it is useful to refer to Fig. 1. Observers obtain information about the universe by means of photons, so our direct information about the Universe is limited to the light cone (in theory, not literally). What we say about the Universe off the light cone is only as trustworthy as the cosmological model used to generate the information. But as emphasized by MHE, light-cone observations (or at least the two light-cone observations we use) do not uniquely determine the cosmological model on the light-cone (i.e., both the ΛCDM and LTB models give the same results), so of course they do not uniquely determine the cosmological model off the light cone.

As in MHE, we take advantage of a coordinate freedom to simplify the calculation. The radial coordinate r has no physical significance; it can be rescaled in any convenient way. Here, we rescale r such that on the light cone

\[ \hat{R}' = H_0^{-1} \sqrt{1 + \beta(r)}. \]  

(15)

In order to find α(r) and β(r) we follow the procedure outlined by MHE and followed in CBKH. Here we take the view that we will construct an LTB model (i.e., find α(r) and β(r)) that agrees with observations of \( \hat{d}_L(z) \) and \( \hat{\rho}(z) \) only out to \( r \) corresponding to \( z = 2 \), and beyond that we have no useful information about \( \hat{d}_L(z) \) or \( \hat{\rho}(z) \) that would constrain α(r) and β(r). Perhaps information from cosmic microwave background (CMB) measurements could fill that niche, but since the purpose of this paper is to investigate constraints from the local Universe, we will not pursue constructing the model beyond a value of \( r \) corresponding to \( z = 2 \).

1. The first step is to use the assumption that the LTB luminosity-distance–redshift relationship matches that of the fiducial ΛCDM model. In spatially flat ΛCDM, the luminosity distance is given by

\[ \hat{d}_L(z) = (1 + z) \int_0^z \frac{dz_1}{H_{\Lambda \text{CDM}}(z_1)}. \]  

(16)

where \( H_{\Lambda \text{CDM}}(z) = H_0 \sqrt{\Omega_M(1 + z)^3 + \Omega_\Lambda} \). Therefore, using Eq. (13),

\[ \hat{R}(z) = \frac{H_0^{-1}}{1 + z} \int_0^z \frac{dz_1}{\sqrt{\Omega_M(1 + z_1)^3 + \Omega_\Lambda}}. \]  

(17)

2. The next step is to enforce the assumption that the observed mass density on the light cone in the LTB model as a function of \( z \), \( \hat{\rho}(z) \), matches the observed mass density as a function of \( z \) in the fiducial ΛCDM model. Using \( \hat{\rho}(z) d^3 V_{\Lambda \text{CDM}} = \rho_{M, \Lambda \text{CDM}} d^3 V_{\Lambda \text{CDM}} \), we find\(^3\)

\[ H_0^{-1} \hat{R}^2(z) \kappa \hat{\rho}(z) \frac{dr}{dz} = \Omega_M 3 H_0^3 \frac{H_0^2}{H_{\Lambda \text{CDM}}(z)} \left[ \int_0^z \frac{dz_1}{H_{\Lambda \text{CDM}}(z_1)} \right]^2, \]  

(18)

where we have used the fact that \( \rho_{M, \Lambda \text{CDM}}(z) = \Omega_M 3 H_0^2 \kappa^{-1}(1 + z)^3 \).

\(^3\) In MHE and CBKH, \( H_0^{-1} \hat{R}^2(z)\kappa \hat{\rho}_{\text{LTB}}(z) dr = \kappa m ndz \).
FIG. 1: This figure illustrates features and notation of the LTB model analyzed in this paper. The time coordinate is defined such that its value at the present time is $t_0$; $t_0$ is not the present age of the Universe. In LTB models the age of the universe is a function of $r$, since, in general, the bang time, $t_{BB}$, is a function of $r$. The age as a function of $r$ is given by $t_0 - t_{BB}(r)$.

The light cone (the past-directed null geodesic) for an observer at $r=0$ is denoted by the solid line. A "hat" indicates that a quantity is evaluated on the light cone, and hence is a function of only one variable: $\hat{f}(r) \equiv f(r, \hat{t}(r))$.

Hatted quantities are often written as functions of $z$, since that is the light-cone coordinate accessible to observations. The $r$ coordinate is scaled such that $\hat{t}(r)$ is given by $\hat{t}(r) = t_0 - H_0^{-1}r$.

The shaded area is our causal past.

3. We make use of the fact that

$$\frac{dz}{dr} = \left[ H_0 \frac{R(z)}{dz}(1 + z) \right]^{-1} \left[ 1 - \frac{1}{2} \int_0^z H^{-1}_0 \kappa \hat{\rho}(z) \hat{R}(z_1)(1 + z_1) \frac{dz_1}{dz} \right] = (1 + z) \frac{H_{\Lambda CDM}(z)}{H_0}, \quad (19)$$

where the last equality (not found in MHE or CBKH) obtains only for a spatially flat Friedmann model. The resulting $z(r)$ is shown in Fig. 2.

4. We then solve the differential equation from MHE for $\alpha(r)$:

$$\frac{d\alpha}{dr} = \frac{1}{2} H_0^{-1} \hat{R}^2(z) \kappa \hat{\rho}(z) \left[ \frac{1}{H_0 \hat{R}/dr} \left( 1 - \frac{\alpha}{\hat{R}} \right) + H_0 \frac{d\hat{R}}{dr} \right], \quad (20)$$

where $d\hat{R}/dr = d\hat{R}/dz \cdot dz/dr$. The initial condition is $\alpha(0) = 0$.

5. Finally, $\beta(r)$ is given by

$$\beta(r) = \left( \frac{d\alpha}{dr} \frac{1}{H_0^{-1} \hat{R}^2 \kappa \hat{\rho}} \right)^2 - 1. \quad (21)$$

The values of $\alpha$, $\beta$, $\alpha'$, and $\beta'$ obtained using this procedure are shown in Fig. 3. They agree with the results of CBKH.

---

4 Here we mention a subtlety in Eq. (20). The factor $d\hat{R}/dr$ vanishes at some value of $z$ (in the fiducial $\Lambda CDM$ model, at $z \sim 1.6$), but since $1 - \alpha/\hat{R}$ also vanishes at that point, the equation is regular. However, care must be taken when numerically integrating the equation.
We emphasize that for these values of $\alpha(r)$ and $\beta(r)$, the LTB model exactly reproduces the luminosity-distance–redshift relation and the mass-density–redshift relation of ΛCDM.

Once $\alpha(r)$ and $\beta(r)$ are determined, along with $\hat{R}$ and $\hat{R}'$, one may use Eq. (8) to solve for $R'(r,t)$ and either Eq. (9) or Eq. (10) to solve for $R(r,t)$ from the bang time, $t_{BB}(r)$, until today, $t_0$.

Again, we return to a point made earlier: We only have observational information on our light cone. This means that although one might speak of the density as a function of $r$ at a particular time, it is not accessible to observations. Nevertheless, it is instructive to examine $\rho(r,t)$ at fixed times. Such an example is the present value of $\rho$, $\rho(r,t_0)$. CBKH present it as a function of the metric function $R(r,t_0)$, but here, we present it in terms of a physical quantity,
FIG. 4: The present value of $\rho$, $\rho(t_0)$, as a function of the present proper distance. It is normalized to the value in the fiducial $\Lambda$CDM model, $\rho_{\Lambda$CDM}(t_0) = 3H_0^2\kappa^{-1}\Omega_M$.

namely $d_P(t_0)$, the present proper distance to $r$. The present proper distance is given by

$$H_0d_P(t_0) = \int_0^r dr_1 \frac{R'(r_1,t_0)}{\sqrt{1 + \beta(r_1)}}.$$ (22)

and is shown in Fig. 2 as a function of coordinate $r$.

Again, as noted by CBKH, while previous attempts to construct an LTB model to reproduce $d_L(z)$ of the fiducial $\Lambda$CDM model have resorted to a large local void \cite{5, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38}, here we find a very mild increase (less than 4%) in $\rho(r)$. (As already remarked, this increase is unobservable since only $\tilde{\rho}(z)$ is observable, and that exactly agrees with $\Lambda$CDM).

A point that will be developed further is worth introducing here. It has often been remarked that it is not possible for inhomogeneous cosmologies to reproduce $d_L(z)$ of $\Lambda$CDM models because large, unobserved, inhomogeneities would be necessary. This is usually stated as either 1) The large inhomogeneities would result in large, unobserved, peculiar velocities, or 2) One can always write the metric in the perturbed Newtonian form. These two arguments are often said to exclude the possibility that an LTB model can mimic dark energy observables.

However, the present consideration seems to provide (yet another) counterexample to that statement, since in the LTB model constructed here, $d_L(z)$ is exactly that of the fiducial $\Lambda$CDM model while $\tilde{\rho}(z)$ is also exactly that of the fiducial $\Lambda$CDM model.

In the next section we explore the time evolution of the model off of the light cone.

IV. FROM THE BANG TIME UNTIL TODAY

Let us first examine the evolution of $R(r,t)$ and $R'(r,t)$ for the model constructed in the previous section. Again, we only consider the evolution out to $r$ corresponding to $z = 2$. The values of $R(r,t)$ and $R'(r,t)$ for several values of $r$ (labeled by the corresponding values of $z$) are indicated in Fig. 5.

The striking feature is that for $r \neq 0$, as $t_{BB}(r)$ is approached, $R'(r,t) \to \infty$. (The solution at the origin at $r = 0$ is divergent.\footnote{To better appreciate why this feature is striking, recall that for an FLRW solution, $R'(r,t) \to a(t)$, where $a$ is the scale factor. One would expect the scale factor to approach zero at the bang time. Here, the effective radial scale factor $R'$ blows up as the angular scale factor $R$ vanishes.)

\footnote{To better appreciate why this feature is striking, recall that for an FLRW solution, $R'(r,t) \to a(t)$, where $a$ is the scale factor. One would expect the scale factor to approach zero at the bang time. Here, the effective radial scale factor $R'$ blows up as the angular scale factor $R$ vanishes.)}
FIG. 5: The evolution of $R(r, t)$ and $R'(r, t)$ with time for the model discussed in Sec. III. Labels on the curve correspond to values of $r$ corresponding to the indicated values of $z$. $R(0, t) = 0$ is not indicated. The age of the universe in the fiducial ΛCDM model is shown by the dotted vertical line.

the bang time is $R'(0, t_{BB}(r)) = 0.$) The behavior of the radial scale factor increasing as the angular scale factor decreasing is reminiscent of a type of Type I Bianchi model (the so-called Kasner solution).

To understand this behavior, consider the evolution of $R'$ at early times. It is most convenient to work with the parameter $\eta$ rather than time. From Eq. (10), $dt/d\eta = R(r, t)/\beta 1/2$, and using Eq. (8), we find

$$\frac{dR'}{d\eta} = \frac{\alpha'}{4\beta} \eta - \frac{R'}{\eta}. \quad (23)$$

Two small-$\eta$ solutions are possible. First, if one assumes that $R' \rightarrow 0$ as $\eta \rightarrow 0$, then the small-$\eta$ solution is

$$R' \rightarrow \frac{\alpha'}{12\beta} \eta^2, \quad (24)$$

which vanishes at the bang time ($\eta = 0$). We will refer to this as the regular solution.

The other solution to Eq. (23) obtains if as $\eta \rightarrow 0$, the second term on the rhs of the equation dominates. In this case, the solution is

$$R' \rightarrow A \frac{\eta}{\eta}, \quad (25)$$

where $A$ is an constant. We will refer to this as the singular solution.

The different behaviors near the origin can be understood by expressing $R'(r, t)$ as

$$R'(r, t) = \left(\frac{\alpha'}{\alpha} - \frac{\beta'}{\beta}\right) R(r, t) - \left(\frac{\alpha'}{\alpha} - \frac{3\beta'}{2\beta}\right) \left[t - t_{BB}(r)\right] \dot{R}(r, t) - t_{BB}'(r) \dot{R}(r, t). \quad (26)$$

At $t = t_{BB}(r)$, $R(r, t_{BB}(r)) = 0$, and Eq. (26) implies $R'(r, t_{BB}(r)) = -t_{BB}'(r) \dot{R}(r, t_{BB}(r))$. Since $\dot{R}(r, t_{BB}(r)) = +\infty$ and in our case $t_{BB}'(r) < 0$, we see that $R'(r, t_{BB}(r)) = +\infty$ as seen in Fig. 5. Since as $t \rightarrow 0$,

$$t \rightarrow \frac{\alpha(r)}{\beta^{3/2}(r)} \frac{\eta^3(r)}{12}; \quad R \rightarrow \frac{\alpha(r)}{\beta(r)} \eta^2; \quad \dot{R} \rightarrow \frac{2\beta^{3/2}}{\eta}, \quad (27)$$

the constant $A$ in Eq. (25) is identified to be $A = -t_{BB}'(r) 2\beta^{3/2}$. 


FIG. 6: The evolution of $R(r, t)$ and $R'(r, t)$ with time for $\alpha(r)$ and $\beta(r)$ from the model discussed in Sec. III, but with the condition that $t'_{BB}(r) = 0$. Labels on the curve correspond to values of $r$ corresponding to the indicated values of $z$.

The solution $R'(r, t_{BB}(r)) = 0$ only will occur if $t'_{BB}(r) = 0$, or in other words, if the big-bang surface is synchronous. In this case we can set $t_{BB} = 0$, and one can easily recover Eq. (24) from Eq. (26).

Different initial conditions, either Eq. (24) or (25), will lead to different evolutions of $R'(\eta)$ for the same values of $R(\eta)$. It is not difficult to see that the space spanned by singular solutions is larger than the space spanned by the regular solution because the constant $A$ (related to $t'_{BB}(r)$) of the singular solutions is undetermined. The regular initial condition leads to a unique value of $R'$ at some later time, whereas the singular initial condition can lead to any value of $R'$ at some later time, modulo the value obtained from the regular initial condition. Or as we have shown, the regular initial condition only occurs for a simultaneous bang surface.

There is no particular reason to believe that $\hat{R}'$ would correspond to the unique value obtained from the regular initial condition. This conforms to the expectation that unless there is some symmetry to prevent divergence, the generic approach to the singularity should not be regular.

At this point it should be pointed out that the small-$\eta$ behavior might be irrelevant for three reasons: 1) The LTB model is merely a toy model, not to be taken seriously. 2) The LTB model we consider has only a dust component to the stress tensor. Adding radiation may change the outcome, or imagining an early inflationary phase might also change the outcome. 3) Finally, perhaps it is premature to reject the singular solution without fully exploring why it is inappropriate, i.e., what is the observational basis for excluding it?

We eschew these issues and ask a modest question of how the observations that went into the construction of $\alpha(r)$ and $\beta(r)$ would be changed if we substituted the regular initial conditions. This is quite straightforward: we have $\alpha(r)$ and $\beta(r)$, all we have to do is choose initial conditions $R(r, 0) = R'(r, 0) = 0$, evolve the metric functions according to Eqs. (6) and (8), enforce $t'_{BB}(r) = 0$, and calculate the observables $\hat{d}_L(z)$ and $\hat{\rho}(z)$.

The results for $R$ and $R'$ following this procedure are illustrated in Fig. 6

Now consider the observational difference between the regular and singular solutions. In Fig. 7 we show the difference in $\hat{d}_L(z)$ for the two solutions. The regular solution is distinguishable from the singular solution, but the singular solution closely resembles another $\Lambda$CDM model, one with $\Omega_{\Lambda} = 0.5$ rather than $\Omega_{\Lambda} = 0.7$ of the fiducial model, at least out to redshift two. In fact, out to redshift unity, the difference is less than about 0.02 magnitudes.

Just as the luminosity distance for the regular cosmological solution differs from the fiducial $\Lambda$CDM model, the density does as well. This is illustrated in Fig. 8. Note that the reference $\Lambda$CDM model for this figure is the model with $\Omega_{\Lambda} = 0.5$, which well fits the $\hat{d}_L(z)$ for the regular LTB model. This figure well illustrates that the density along the light cone compared to a fiducial model can decrease as a function of $z$, while at the present time, it can increase as a function of $r$.

It is beyond the purpose of this paper to see whether one can fiddle with $\alpha(r)$ and $\beta(r)$ to find a regular solution that is a better fit to the fiducial $\Lambda$CDM model.
FIG. 7: The luminosity-distance vs. redshift for several cosmological models. The regular LTB model is the LTB model with $\alpha(r)$ and $\beta(r)$ from Sect. III but with regular initial conditions. Also shown for comparison is $\hat{d}_L(z)$ for a $\Lambda$CDM model with $\Omega_\Lambda = 0.5$ instead of the fiducial model of this paper ($\Omega_\Lambda = 0.7$). The singular LTB solution is constructed to give the exact $\hat{d}_L(z)$ as for the fiducial $\Lambda$CDM model. Finally, in keeping with tradition, the result for the Einstein-de Sitter model (spatially flat dust cosmology) is shown. Here $\Delta(m - M)$ is the difference in magnitudes between the indicated model and the open, empty cosmological model. ($M$ is a nuisance parameter related to the absolute magnitude of supernovae.)

FIG. 8: The density as a function of redshift on the light cone (solid curve) and at the present time (dashed curve) for the regular LTB model constructed in Sec. III. For the regular LTB model, the reference $\Lambda$CDM model has $\Omega_\Lambda = 0.5$.

The conclusion of this section is that if one attempts to construct an LTB model that agrees exactly with $\hat{d}_L(z)$ and $\hat{\rho}(z)$ of the fiducial $\Lambda$CDM model, the resulting model will have singular initial conditions for $R'$. We have argued that this is to be expected. We surmise that this will be true in general, and is not specific to our choice of the fiducial $\Lambda$CDM model. We also demonstrated that one can find a regular solution with the same values of $\alpha(r)$ and $\beta(r)$ that results in a $\hat{d}_L(z)$ and $\hat{\rho}(z)$ that approximates a $\Lambda$CDM model close to the fiducial model at late time.

In the next section we return to the singular models and examine whether they can be described in terms of averaged...
V. DESCRIPTION IN TERMS OF AN AVERAGING PROCEDURE

One proposed route to understanding the role of inhomogeneities in mimicking dark energy is by employing an averaging procedure. Our calculations allow us to study this proposal. First, let us review the averaging procedure. A fundamental quantity in the analysis is the velocity gradient tensor, which is defined as

$$\Theta_{ij} = u^i,_{;j} = \frac{1}{2} h^{ik} \dot{h}_{kj}, \quad (28)$$

where $h^{ij}$ is the spatial metric and $u^\mu$ is the fluid four velocity. The tensor $\Theta_{ij}$ represents the extrinsic curvature of the spatial hypersurfaces orthogonal to the fluid flow. It may be decomposed in terms of a trace term and a traceless tensor as

$$\Theta_{ij} = \Theta \delta_{ij} + \sigma_{ij}, \quad (29)$$

where $\Theta$ is the volume-expansion scalar. The traceless tensor $\sigma_{ij}$ is the shear.

The evolution equations for the expansion and the shear come from the space-space components of Einstein’s equations (see e.g., Ref. [40]). Combining the expansion evolution equation with the energy constraint gives the Raychaudhuri equation,

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + 2\sigma^2 + \frac{1}{2} \kappa \rho = 0. \quad (30)$$

From the Raychaudhuri equation it is straightforward to verify that local fluid elements cannot undergo accelerated expansion. (This point was emphasized by Hirata and Seljak [41].) But, of course, as now appreciated, that point is irrelevant. Locally the expansion does not accelerate, but it is incorrect to assume that acceleration does not occur when the fluid is coarse-grained over a finite domain. The reason is trivial: the time derivative of the average of $\Theta$ and the average of the time derivative of $\Theta$ are not the same because of the time dependence of the coarse-graining volume.

There has been a lot of work recently regarding the averaging procedure. First, we take the original proposal and average at a fixed time. We will then average over the light cone.

Let us denote the coarse-grained value of a quantity $\mathcal{F}$ by its average over a spatial domain $D$:

$$\langle \mathcal{F} \rangle_D = \frac{\int_D \sqrt{h} \mathcal{F} \, d^3x}{\int_D \sqrt{h} \, d^3x}. \quad (31)$$

Following the work of Buchert [42, 43], we define a dimensionless scale factor

$$a_D(t) \equiv \left( \frac{V_D}{V_{D0}} \right)^{1/3} ; \quad V_D = \int_D \sqrt{h} \, d^3x, \quad (32)$$

where $V_D$ is the volume of our coarse-graining domain (the subscript “0” denotes the present time). In the LTB model

$$\sqrt{h} = R^2(r, t_0)R'(r, t_0)/\sqrt{1 + \beta(r)}.$$

The coarse-grained Hubble rate $H_D$ will be

$$H_D = \frac{\dot{a}_D}{a_D} = \frac{1}{3} \langle \Theta \rangle_D. \quad (33)$$

The smoothing procedure leads to the evolution equations for the coarse-grained scale factor [42, 43]:

$$\frac{\ddot{a}_D}{a_D} = -\frac{\kappa}{6} (\rho_{\text{eff}} + 3p_{\text{eff}}), \quad (34)$$

$$\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{\kappa}{3} \rho_{\text{eff}}, \quad (35)$$

quantities.
where the effective energy density and pressure terms are \((^{3}\mathcal{R})\) is the spatial curvature

\[
\rho_{\text{eff}} = \langle \rho \rangle_D - \frac{Q_D}{2\kappa} - \frac{(^{3}\mathcal{R})_D}{2\kappa},
\]

\[
p_{\text{eff}} = -\frac{Q_D}{2\kappa} + \frac{(^{3}\mathcal{R})_D}{6\kappa},
\]

and we have introduced the kinematical backreaction

\[
Q_D = \frac{2}{3} \left( \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \right) - 2 \langle \sigma^2 \rangle_D.
\]

Finally, we can define an effective equation of state, \(w_{\text{eff}}\), to be \(w_{\text{eff}} = p_{\text{eff}}/\rho_{\text{eff}}\).

Now for LTB models,

\[
\Theta = \Gamma^k_{0k} = 2\frac{\dot{R}}{R} + \frac{\dot{R}'}{R'},
\]

\[
\sigma^2 = \frac{1}{2} \sum_k \left( \Gamma^k_{0k} \right)^2 - \frac{1}{6} \left( \sum_k \Gamma^k_{0k} \right)^2 = \frac{1}{3} \left( \frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right)
\]

\[
^{3}\mathcal{R} = -2\frac{\beta}{R^2} - 2\frac{\beta'}{R'R}.
\]

The various terms that enter the calculation of \(\rho_{\text{eff}}\) and \(p_{\text{eff}}\) are shown in Fig. 3 as a function of the proper distance that sets the scale of the averaging volume. Note that the shear is small, as is the kinematical back reaction since \(\langle \Theta^2 \rangle_D \approx \langle \Theta \rangle_D^2\). Shown in Fig. 10 is the effective value of \(w\), \(w_{\text{eff}} = p_{\text{eff}}/\rho_{\text{eff}}\). Note that if one models the fiducial ΛCDM model as a single fluid and calculates \(w\), the value today is \(w = -0.7\).

Another approach is light-cone averaging. Here, what is of interest isn’t the average on a (acausal) constant-time hypersurface, but rather the average on a hypersurface coinciding with the light cone. One way to approach this is just to substitute the “hatted” quantities in Eqs. (32-39). The resulting terms that enter the calculation of the averaged quantities are shown in Fig. 11 and the effective value of the equation of state parameter is shown in Fig. 12. Also shown in Fig. 12 is the equation of state parameter for the fiducial ΛCDM model if one would model the total energy (including matter and Λ) as a single fluid.

There are two things we learn from the exercise of this section. First, neither averaging on a constant-time hypersurface nor light-cone averaging is easy to connect with the observations corresponding to parameters of the ΛCDM model. However, for both averaging prescriptions the effective equation-of-state parameter is negative, indicating acceleration of some averaged scale factor. This may be a valuable check in more complicated models. It is interesting to note that the shear contribution to the evolution of the averaged quantities is insignificant in both averaging prescriptions. This seems to be a counterexample to naïve claims that if one constructs an inhomogeneous universe to mimic the observations in ΛCDM one would generate unacceptably large shear terms.

VI. DISCUSSION AND CONCLUSIONS

In the discussion of what we have learned in this investigation, it is useful again to refer the reader to Fig. 1 and to reiterate that all direct cosmological information we have is limited to the light cone, illustrated by the solid line in the figure. Any statement about the Universe off of the light cone is unsupported by direct observations, and hence requires a cosmological model for the evolution of the Universe.

An interesting lesson from the consideration of this paper is that if one is just restricted to the two cosmological observables employed in this paper, \(d_L(z)\) and \(\tilde{\rho}(z)\), it is impossible to determine uniquely a cosmological model \([16, 17]\). In our case, the fiducial ΛCDM model and the singular LTB model result in identical observations for \(d_L(z)\) and \(\tilde{\rho}(z)\). That result is valid even if one imagines perfect astronomical observations, since by construction the two cosmological observables are degenerate in the two models.

There are two ways to break the degeneracy, one observational and the other theoretical. One promising avenue to break the degeneracy with observational data might be to take advantage of the fact that in LTB models there are two different expansion rates, a radial expansion rate \(H_r\), and an angular expansion rate \(H_\perp\) (they were defined in Sect. II). Figure 13 shows the differences between the expansion rates in the singular model. Note that the fact that \(H_r = H_{\Lambda\text{CDM}}\) is guaranteed by the input requirements used to construct the model. Quartin and Amendola [44] have recently discussed using cosmic parallax and redshift drift to distinguish between void models and dark energy. It is possible to pile on additional observational constraints to break the degeneracy.
FIG. 9: The various terms that enter the calculation for $\rho_{\text{eff}}$ and $p_{\text{eff}}$ for the averaging at the present time as a function of proper distance that sets the scale of the averaging volume.

FIG. 10: The value of $w_{\text{eff}}$ as a function of proper distance that sets the scale of the averaging volume for the averaging at the present time.

While this can be done, it misses the important point that if one is restricted to light-cone observations, it is unclear whether any number of observations alone, even of arbitrary precision and with no systematic uncertainties, can uniquely determine a cosmological model, even if it is possible to rule out any LTB model. An FLRW model is a zero-dimensional (dimensions of space) model, and the LTB solution is a one-dimensional model. Intuition from other fields lead us to believe that the range of solutions in two-dimensional, or fully inhomogeneous three-dimensional models, will be richer. So it is unclear whether any number of cosmological observations, even to arbitrary accuracy, can uniquely specify a cosmological model.

This point was discussed by Kolb, Marra, and Matarrese (KMM) [45]. In that work they introduce the concept of cosmological background solutions. This concept will be useful in elucidating several points, so we briefly review some concepts from KMM.
A cosmological background solution is defined as a mean-field geometry of suitably averaged Einstein equations, in which the average expansion is described by a single scale factor. A cosmological background solution depends upon the spatial curvature and the mass-energy content. Associated with the mass-energy content is a stress-energy tensor that describes a fluid (or fluids) with equation(s) of state that satisfy local energy conditions.

KMM then define several types of background solutions; of relevance here is the Phenomenological Background Solution (PBS): a background solution that effectively describes the observations, that is, the data on the past light cone of an observer. The energy content and curvature of the PBS are not necessarily related to the spatial averages of the energy content and curvature of the observable universe. The equation of state of the stress-energy tensor of the PBS need not satisfy any of the local energy conditions of the mass-energy content of the universe.

In this nomenclature, one might imagine that the true background solution is the LTB solution, or more generally some inhomogeneous model, and the fiducial ΛCDM solution is merely a phenomenological background solution. It is unclear whether it is possible, in principle, to exclude this possibility just on the basis of light-cone observations.
Observations alone may not be able to determine the true cosmological model, but as remarked in the introduction, we expect the cosmological model to be part of our deeper comprehension of the physical and astronomical facts; it must fit into the larger framework of particle physics, general relativity, astrophysics, and cosmology.

As an example of larger considerations, consider the theory of structure formation. In the fiducial ΛCDM model, structure grows in a background that is close to homogeneous and isotropic, but with small seed perturbations initially generated by inflation. Even in this zero-dimensional model, structure formation is a complicated affair (at least in the nonlinear regime). Structure formation in LTB models is an even more complicated affair [23, 24]: each spherical shell evolves as an independent FLRW model. One would expect the evolution of structure formation in LTB models to differ from the fiducial ΛCDM model. However, since structure formation in a three-dimensional model is more complicated still, it is difficult to see how structure formation might be conclusive. The same argument might apply to other cosmological tests like the integrated Sachs–Wolfe effect.

Now, we return to the argument that purports to prove that inhomogeneous models cannot mimic the effects of dark energy (see, e.g., Refs. [46, 47, 48]). The argument might be paraphrased as follows: Perform an expansion of the FLRW metric to Newtonian order in potential and velocity but take into account the fully nonlinear density inhomogeneities. In the conformal Newtonian gauge, the metric will be of the form

\[ ds^2 = a^2(\eta)[-(1 + 2\phi) d\eta^2 + (1 - 2\phi) dx^2], \]  

(40)

where \( \eta \) is conformal time and \( \phi \simeq \psi \). The volume average of the equation then yields [48]

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \langle \rho \rangle (1 - 5W + 2K), \]  

(41)

where \( W \) and \( K \) are the Newtonian potential and kinetic energy (per unit mass),

\[ W = \frac{1}{2} \langle (1 + \delta)\phi \rangle, \quad K = \frac{1}{2} \langle (1 + \delta)v^2 \rangle. \]  

(42)

Here, \( \delta = \delta\rho/\langle \rho \rangle \) is the density perturbation in the matter rest frame and \( v \) is the peculiar velocity. The argument is that observationally both \( W \) and \( K \) are small, hence, inhomogeneities cannot mimic dark energy.

However the model constructed in Sec. III seems to belive this simple and compelling argument. If one interprets the observations of \( \tilde{\rho}(z) \) of the LTB model in terms of the fiducial ΛCDM phenomenological background solution, one would conclude that the perturbations are very small (viz. zero), hence \( W \) and \( K \) zero, and inhomogeneities could not mimic dark energy. But, at least for \( d_L(z) \) and \( \tilde{\rho}(z) \), inhomogeneities do exactly that!
We point out that the very notion of peculiar velocities requires a background solution (a norm) about which to define “peculiarity” [49]. Also, Enqvist, Mattsson, and Rigopoulos [50] explicitly demonstrate how spherically symmetric perturbations of a dust Universe can lead to acceleration while the perturbations of the gravitational potential remain small on all scales. Perhaps the general lesson is that there are highly non-Gaussian inhomogeneities in the late Universe, and the coherence of structures causes small deviations in observables to sum to a large deviation.

We can venture a different way to frame the argument: If the observed density fluctuations, $\delta \hat{\rho}(z)$, originated from the growth of perturbations in an FLRW cosmology, and the nonlinear perturbations on one scale did not affect the growth of structure or the evolution of the background solution on any other scale, and inhomogeneities do not show coherence or non-Gaussianity, and $\phi \simeq \psi$, then one can say that the interpretation of observations would yield a value of $W$ and $K$ that are too small to mimic dark energy.

This way of framing the argument makes manifest the fact that there are dynamical assumptions that enter, and it cannot simply be a matter that small peculiar velocities rule out the backreaction proposal. If any of the dynamical assumptions fail, the argument fails.

Another lesson that can be extracted from the exercise of this paper concerns using averaging procedures to quantify the effect of inhomogeneities on observables. This was the subject of the previous section. The conclusion was that while the two averaging prescriptions discussed give an indication that the LTB model would have observables that might be interpreted as arising from dark energy, the effective equation of state parameters do not resemble that of the fiducial $\Lambda$CDM model. This might be because the averaging approach is not very useful, or it might imply something quite troubling. Perhaps the equation of state parameter $w$, which has received so much attention, has a physical interpretation only in FLRW models, and in an inhomogeneous Universe, where there is no natural time slicing, it is an unphysical construct.

This paper has developed and discussed issues related to the interpretation of cosmological observables in constructing cosmological models and theories. Observations of $\delta_L(z)$ and $\bar{\rho}(z)$ do not, by and of themselves, prove that there is dark energy or that the Universe is accelerating in the usual sense; they do so only if one assumes that the Universe is homogeneous and isotropic and the dynamics of the Universe are governed by general relativity. This paper discussed aspects of the program to develop understanding of how an inhomogeneous Universe might mimic dark energy. While consideration of the results of this investigation might lead one to conclude that there are no arguments that conclusively exclude the backreaction proposal, we are no closer to developing an inhomogeneous cosmology alternative to dark energy.

One attempt to construct inhomogeneous cosmological models involves studying “Swiss-cheese” models comprised of low density voids embedded in an Einstein–de Sitter background [51, 52, 53, 54, 55, 56, 57, 58, 59]. If the void regions are packed together the regions between them form a network of high-density filamentary, or spaghetti-like, structures. Further refinements of this approach have included spherical mass concentrations referred to as “meatballs” [60]. A Swiss-cheese, meatball, and spaghetti cosmological model may include ingredients that are desirable on their own, or work well in combination with one other ingredient, but combining them all may lead to a model that is not easily digestible.

Whatever direction the study of inhomogeneous cosmologies might take, consideration of the results of this investigation should be important.

Acknowledgments

It is a pleasure to thank Marie-Nèölle Célerier, Valerio Marra, Sabino Matarrese, Albert Stebbins, and Dan Chung for useful discussions. This work was supported in part by the Department of Energy.

[1] W. L. Freedman et al., Ap. J. 553, 47 (2001).
[2] D. J. Fixen, et al., Ap. J. 473, 576 (1996).
[3] A. Einstein, in the foreword to Dialogue Concerning the Two Chief World Systems, translated by S. Drake (University of California Press, Berkeley, 1967, second edition).
[4] E. Komatsu et al., Ap. J. Suppl. 180, 330 (2009).
[5] M.-N. Célerièr, Astron. Astrophys. 353, 63 (2000).
[6] T. Buchert, JGRG 9, 306, (2000).
[7] D. J. Schwarz, “Accelerated expansion without dark energy,” arXiv:astro-ph/0209584.
[8] S. Rasanen, JCAP 0402, 003 (2004).
[9] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, Phys. Rev. D 71, 023524 (2005).
[10] S. Rasanen, JCAP 0411, 010 (2004).
