Two Piggybacking Codes with Flexible Sub-Packetization to Achieve Lower Repair Bandwidth

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Abstract

As a special class of array codes, \((n, k, m)\) piggybacking codes are MDS codes (i.e., any \(k\) out of \(n\) nodes can retrieve all data symbols) that can achieve low repair bandwidth for single-node failure with low sub-packetization \(m\). In this paper, we propose two new piggybacking codes that have lower repair bandwidth than the existing piggybacking codes given the same parameters. Our first piggybacking codes can support flexible sub-packetization \(m\) with \(2 \leq m \leq n - k\), where \(n - k > 3\). We show that our first piggybacking codes have lower repair bandwidth for any single-node failure than the existing piggybacking codes when \(n - k = 8, 9\), \(m = 6\) and \(30 \leq k \leq 100\). Moreover, we propose second piggybacking codes such that the sub-packetization is a multiple of the number of parity nodes (i.e., \((n - k)|m\)), by jointly designing the piggyback function for data node repair and transformation function for parity node repair. We show that the proposed second piggybacking codes have lowest repair bandwidth for any single-node failure among all the existing piggybacking codes for the evaluated parameters \(k/n = 0.75, 0.8, 0.9\) and \(n - k \geq 4\).

Index Terms

Piggybacking codes, repair bandwidth, sub-packetization, single-node failure, transformation.

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I. INTRODUCTION

Maximum distance separable (MDS) array codes are widely employed in the modern distributed storage systems because they provide the maximum data reliability for a given level of storage overhead. An \((n, k, m)\) MDS array code encodes \(km\) data symbols into \(nm\) coded symbols that are equally stored in \(n\) nodes, where each node stores \(m\) symbols. We call the number of symbols stored in each node as the sub-packetization level. The \((n, k, m)\) MDS array codes satisfy the MDS property that is any \(k\) out of \(n\) nodes can retrieve all \(km\) data symbols. The codes are referred to as systematic codes if the \(km\) data symbols are included in the first \(k\) nodes of obtained \(n\) nodes. Reed-Solomon (RS) codes [2] are typical MDS array codes with \(m = 1\). In this paper, we consider systematic MDS array codes that contain \(k\) data nodes which store the \(km\) data symbols and \(r = n - k\) parity nodes which store the \(rm\) parity symbols.

In modern distributed storage systems, node failures are common and single-node failures occur most frequently among all failures [3], [4]. It is important to repair the failed node with the repair bandwidth defined as the total amount of symbols downloaded from other surviving nodes as small as possible [5]. Recently many constructions of \((n, k, m)\) MDS array codes to achieve the minimum repair bandwidth \(\frac{dn}{d-k+1}\) from \(d\) surviving nodes have been proposed in [6]–[11], all the high-code-rate (i.e., \(k/n > 0.5\)) MDS array codes with the minimum repair bandwidth need an exponential sub-packetization level in parameters \(n\) and \(k\) [12]. It is practical important to design high-code-rate MDS array codes with repair bandwidth as small as possible, for a given small sub-packetization level. HashTag Erasure Codes (HTEC) [13] is a high-code-rate that have efficient repair method for data nodes with sub-packetization level \(2 \leq m \leq r\lceil\frac{k}{r}\rceil\), however no efficient repair method for parity nodes and the required field size should be large enough to keep the MDS property.

Piggybacking codes which were first proposed by Rashmi et al. in [14] are an important class of MDS array codes that have both low repair bandwidth for single-node failure and low sub-packetization level. The central idea of piggybacking codes is creating \(m\) instances of RS codes as base codes and designing ingenious piggyback function (i.e., a linear combination of some selected symbols in some instances) which will be added to other instances. Many follow-up piggybacking codes [15]–[20] have been proposed to reduce the repair bandwidth.

In this paper, we present two constructions of piggybacking codes that have lower repair bandwidth than the existing piggybacking codes for the same parameters. We summarize the
main contributions as follows.

1) First, we propose first piggybacking codes for \( r \geq 4 \) and \( m \leq r \). We show that our first piggybacking codes have lower repair bandwidth for any single-node failure than all the existing piggybacking codes for the evaluated parameters \( r = 8, 9, m = 6 \) and \( k = 30, 31, \ldots, 100 \).

2) Second, we propose second piggybacking codes by jointly designing piggyback function for data nodes repair and transformation function for parity nodes repair, where the sub-packetization level is a multiple of the number of parity nodes. The proposed second piggybacking codes have the lowest repair bandwidth for any single-node failure among all the existing piggybacking codes for the evaluated parameters \( \frac{k_n}{n} = 0.75, 0.8, 0.9 \) and \( r \geq 4 \).

Note that a parallel work [20] also designs piggybacking codes to obtain low repair bandwidth for \( m \leq r \). The differences of codes [20] and our first piggybacking codes are as follows. We design the piggyback function by considering the repair bandwidth reduction for both data nodes and parity nodes. While in [20], the piggyback functions for data nodes and parity nodes are respectively designed. Because of the above difference, our codes have a slightly lower repair bandwidth than that of codes in [20]. Please refer to Section VI-A for the comparison.

The main differences between our first piggybacking codes and codes in our conference version [1] are of two-folds. First, our first piggybacking codes can support flexible sub-packetization \( m \), i.e., \( 2 \leq m \leq r \), while codes in [1] only suitable for \( m = r \). Second, the piggyback structure of our first piggybacking codes can be jointly designed with the proposed transformation function, while not for codes in [1].

Our second piggybacking codes is partially inspired by the generic transformation in [8]. The difference is that new MDS array codes with exponential sub-packetization level and optimal repair for any single-node failure can be obtained in [8] by recursively applying the transformation for MDS codes, while we use the transformation idea to design the transformation function for parity nodes in order to reduce the repair bandwidth in the meanwhile keeping the low repair bandwidth for data nodes. Note that it is not natural to obtain repair bandwidth reduction when we design piggyback functions for data nodes and transformation functions for parity nodes, since both piggyback function and transformation function are added in the same parity symbol. We need to carefully design the two functions to achieve lower repair bandwidth. Moreover, we can’t design transformation functions for parity nodes of piggybacking codes with invertible transformation functions [1], [14], [16], because the invertible transformation structure will be
destroyed if the transformation idea is employed for parity nodes.

The rest of the paper is organized as follows. Section II gives the construction for the first piggybacking codes. Section III presents the repair method for the first piggybacking codes. Section IV gives the construction for the second piggybacking codes. Section V presents the repair method for the second piggybacking codes. Section VI evaluates the repair bandwidth for the proposed piggybacking codes and the existing related piggybacking codes. Section VII concludes the paper.

II. CONSTRUCTION OF THE FIRST PIGGYBACKING CODES

Our first piggybacking codes can be represented by an \( n \times m \) array, where the \( m \) symbols in each row are stored in a node and \( m \leq r = n - k \). We label the index of the \( n \) rows in the array from 1 to \( n \) and the index of the \( m \) columns from 1 to \( m \). Let \( \{ a_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,k})^T \}_{i=1}^m \) be \( m \) columns of the \( k \times m \) data symbols and \( (a_{i,1}, a_{i,2}, \ldots, a_{i,k}, f_1(a_i), \ldots, f_r(a_i))^T \) be codeword \( i \) of the \( (n, k) \) MDS codes over \( \mathbb{F}_q \), where \( f_j(a_i) \) is the parity symbol \( j \) in codeword \( i \), \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, r \).

We divide \( n \) nodes into \( L \) disjoint subsets \( \Phi_1, \Phi_2, \ldots, \Phi_L \), where \( 1 \leq L < m \). Each of the first \( n - \left\lfloor \frac{n}{L} \right\rfloor L \) subsets has size \( \left\lfloor \frac{n}{L} \right\rfloor \) and each of the last \( \left( \left\lfloor \frac{n}{L} \right\rfloor + 1 \right) L - n \) subsets has size \( \left\lceil \frac{n}{L} \right\rceil \), i.e., \( |\Phi_i| = \left\lfloor \frac{n}{L} \right\rfloor \) for \( i = 1, 2, \ldots, n - \left\lfloor \frac{n}{L} \right\rfloor L \) and \( |\Phi_i| = \left\lceil \frac{n}{L} \right\rceil \) for \( i = n - \left\lfloor \frac{n}{L} \right\rfloor L + 1, \ldots, L \). We can check that

\[
\left\lceil \frac{n}{L} \right\rceil (n - \left\lfloor \frac{n}{L} \right\rfloor L) + \left\lfloor \frac{n}{L} \right\rfloor ((\left\lfloor \frac{n}{L} \right\rfloor + 1)L - n) = n,
\]

i.e., the \( n \) nodes \( \{1, 2, \ldots, n\} \) are partitioned by the \( L \) disjoint subsets. In this paper, we consider high-code-rate and suppose that the \( r \) parity nodes are in the subset \( \Phi_L \), i.e.,

\[
\left\lfloor \frac{n}{L} \right\rfloor \geq r.
\]

For example, when \( k = 6 \), \( r = 5 \) and \( L = 2 \), the \( n = 11 \) nodes are divided into \( L = 2 \) subsets \( \Phi_1 = \{1, 2, 3, 4, 5, 6\} \) and \( \Phi_2 = \{7, 8, 9, 10, 11\} \).

Recall that \( |\Phi_i| \geq \left\lfloor \frac{n}{L} \right\rfloor \geq r \geq m \) and \( \Phi_i \) contains \( |\Phi_i| \) nodes, where \( i = 1, 2, \ldots, L \). For \( i = 1, 2, \ldots, L \), define the first \( m - i \) symbols of each node in \( \Phi_i \) as Protect Symbols (PS) which will be added to some parity symbols as piggyback function. In the following, we present a method of designing the \((r-1)L\) piggyback functions that are added to the \((r-1)L\) parity symbols such that the number of PS used in computing each piggyback function as average as possible.
The total number of PS in $\Phi_i$ is $p_i = |\Phi_i|(m-i)$. For $i = 1, 2, \ldots, L$, denote the PS in column $j$ with $j = 1, 2, \ldots, m-i$ in row $(\sum_{\alpha=1}^{i-1} |\Phi_\alpha|) + \ell$ with $\ell = 1, 2, \ldots, |\Phi_i|$ as $t_{i,(\ell-1)(m-i)+j}$. For example, when $i = 1$, we have

$$(t_{1,1}, t_{1,2}, \ldots, t_{1,m-1}) = (a_{1,1}, a_{2,1}, \ldots, a_{m-1,1}),$$

$$(t_{1,m}, t_{1,m+1}, \ldots, t_{1,p_1}) = (a_{1,2}, a_{2,2}, \ldots, a_{m-1,|\Phi_i|}).$$

For $1 \leq \alpha \leq r-1$ and $1 \leq \beta < L$, let $s_{\alpha,\beta} = 1 + \lfloor \frac{p_\beta-\alpha}{r-1} \rfloor$ and define the piggyback function $g(\alpha, \beta)$ as,

$$g(\alpha, \beta) = \sum_{\ell=1}^{s_{\alpha,\beta}} t_{\beta,((\ell-1)(r-1)+\alpha}.$$ (1)

For $1 \leq \alpha \leq r-1$ and $\beta = L$, we define the piggyback function $g(\alpha, \beta)$ as,

$$g(\alpha, \beta) = \sum_{\ell=1}^{s_{\alpha,\beta}} t_{\beta,((\ell-1)(r-1)+\alpha} - ((m-L)r \mod (r-1))$$

$$+ \sum_{x=1}^{m-L} \sum_{y=1}^{m-L} f_x(a_y)(I[x+y = \alpha + 1] + I[x+y - (r-1) = \alpha + 1]),$$

where $I[\cdot]$ is characteristic function (i.e., $I[A] = 1$ if $A$ is true, otherwise $I[A] = 0$) and

$$s_{\alpha,L} = 1 + \left\lfloor \frac{|\Phi_L| - r)(m-L) - \alpha + (m-L)r \mod (r-1)}{r-1} \right\rfloor.$$ (2)

Notice that we let $t_{L,\ell} = 0$ if $\ell \leq 0$ in Eq. (2). We add the piggyback function $g(\alpha, \beta)$ to the symbol in row $\alpha + k + 1$ and column $m + 1 - \beta$.

We denote the above designed piggybacking codes as $C_1(n, k, m, L)$. Fig. 1 shows the construction structure of $C_1(n, k, m, L)$. When $m = r$, the piggyback structure of our $C_1(n, k, m = r, L)$ is quite similar the conference version $[1]$. One difference is that invertible transformation is used in codes $[1]$, while not in our $C_1(n, k, m = r, L)$. This is why we can jointly design piggyback function and transformation function for repair bandwidth reduction (please refer to Section $[V]$ for details), however the jointly design is not suitable for codes $[1]$.
**Example 1.** Consider the example of $C_1(11, 6, 4, 2)$, which is shown in Fig. 2. We divide the $n = 11$ nodes into $L = 2$ disjoint subsets, $\Phi_1 = \{1, 2, 3, 4, 5, 6\}$ and $\Phi_2 = \{7, 8, 9, 10, 11\}$. By Eq. (1) and Eq. (2), we have

\[

g(1, 1) = a_{1,1} + a_{2,2} + a_{3,3} + a_{1,5} + a_{2,6}, \\
g(2, 1) = a_{2,1} + a_{3,2} + a_{1,4} + a_{2,5} + a_{3,6}, \\
g(3, 1) = a_{3,1} + a_{1,3} + a_{2,4} + a_{3,5}, \\
g(4, 1) = a_{1,2} + a_{2,3} + a_{3,4} + a_{1,6}, \\
g(1, 2) = f_1(a_1) + f_4(a_2) + f_5(a_1), \\
g(2, 2) = f_1(a_2) + f_2(a_1) + f_5(a_2), \\
g(3, 2) = f_2(a_2) + f_3(a_1), \\
g(4, 2) = f_3(a_2) + f_4(a_1).
\]

According to the above definition of the piggyback function, we can easily know that any two data symbols in the same row are not used to compute one piggyback function because of $m \leq r$. In the next lemma, we show that this is also true for parity nodes.

**Lemma 2.** When $r \geq 4$, we have,

(i) Any two parity symbols in the same row are not used to compute the same piggyback function.

(ii) Any parity symbol in a row used in computing a piggyback function is not in the same row of the piggyback function.
Proof. Since $|\Phi_i| \geq r \geq m$, the piggyback functions which are computed from parity symbols are $g(\alpha, L)$ with $1 \leq \alpha \leq r - 1$.

Consider the first claim. Suppose that two parity symbols $f_x(a_{y_1})$ and $f_x(a_{y_2})$ that are in the same row which are used to compute the piggyback function $g(\alpha, L)$, where $x \in \{1, 2, \ldots, r\}$ and $1 \leq y_1 < y_2 \leq m - L$, then

$$x + y_1 = \alpha + 1,$$
$$x + y_2 - r + 1 = \alpha + 1.$$ 

We have that $y_2 - y_1 = r - 1$, which contradicts with $r \geq m$ and $L \geq 1$.

Consider the second claim. Suppose that the parity symbol $f_x(a_y)$ is used to compute the piggyback function $g(\alpha, \beta)$ in the same row, we have $x = \alpha + 1$, where $x \in \{1, 2, \ldots, r\}$, $y \in \{1, 2, \ldots, m - L\}$, $x + y = \alpha + 1$ or $x + y - (r - 1) = \alpha + 1$. We can obtain that $y = 0$ or $y = r - 1$, which contradicts with $1 \leq y \leq m - L$, $r \geq m$ and $L \geq 1$. \hfill $\square$

By Lemma 2, we can repair any single-node failure by employing the piggyback functions to reduce repair bandwidth and we present the repair method in the next section.

III. REPAIR METHOD OF CODES $C_1(n, k, m, L)$

In this section, we present the repair method for any single-node failure of the proposed code $C_1(n, k, m, L)$ and show the repair bandwidth.
A. Repair Method for Data Nodes

Suppose that node $t \in \Phi_i$ fails, where $t \in \{1, 2, \ldots, k\}$ and $i \in \{1, 2, \ldots, L\}$, the repair method is as follows.

1) We download $ki$ symbols in the last $i$ columns of the first $k + 1$ rows except row $t$ to recover the symbols $a_{m-i+1,t}, \ldots, a_{m,t}$ and $f_x(a_i)$ for $x = 2, 3, \ldots, r$.

2) We download the parity symbols of which the corresponding piggyback functions containing symbols of node $t$ and symbols contained by these piggyback functions except symbols of node $t$, together with $f_x(a_i)$ for $x = 2, 3, \ldots, r$, we can recover $a_{1,t}, a_{2,t}, \ldots, a_{m-i,t}$.

Consider the code $C_1(11, 6, 4, 2)$ in Example 1, we have $L = 2$ disjoint subsets $\Phi_1 = \{1, 2, 3, 4, 5, 6\}$ and $\Phi_2 = \{7, 8, 9, 10, 11\}$.

Suppose that node 1 fails, we can recover the five symbols

$$a_{4,1}, f_2(a_4), f_3(a_4), f_4(a_4), f_5(a_4)$$

by downloading the following six symbols

$$a_{4,2}, a_{4,3}, a_{4,4}, a_{4,5}, a_{4,6}, f_1(a_4).$$

Note that the erased three symbols $a_{1,1}$, $a_{2,1}$ and $a_{3,1}$ are used in computing the three piggyback functions $g(1, 1)$, $g(2, 1)$ and $g(3, 1)$, respectively. We download the three parity symbols

$$f_2(a_4) + g(1, 1), f_3(a_4) + g(2, 1), f_4(a_4) + g(3, 1)$$

and the following symbols

$$a_{2,2}, a_{3,3}, a_{1,5}, a_{2,6}, a_{3,2}, a_{1,2}, a_{2,5}, a_{3,6}, a_{1,3}, a_{2,4}, a_{3,5}$$

that are used in computing three piggyback functions $g(1, 1)$, $g(2, 1)$ and $g(3, 1)$, together with $f_2(a_4), f_3(a_4), f_4(a_4)$, to recover the three symbols $a_{1,1}, a_{2,1}$ and $a_{3,1}$. The repair bandwidth of node 1 is 20 symbols.

We can repair each of the other data nodes similarly and we can calculate that the repair bandwidth of each node in $\{2, 5, 6\}$ is 20 symbols, the repair bandwidth of each node in $\{3, 4\}$ is 19 symbols.
B. Repair Method for Parity Nodes

Suppose that node $t$ fails, where $t \in \{k+1, k+2, \ldots, k+r\}$ and $t \in \Phi_L$, the repair method is given as follows.

1) We download $kL$ symbols in the last $L$ columns of the first $k$ rows to recover $f_{t-k}(a_j)$ with $j = m-L+1, m-L+2, \ldots, m$ and $f_x(a_{m-L+1})$ with $x = 1, 2, \ldots, t-k-1, t-k+1, \ldots, r$.

2) Note that the $m-L$ erased symbols $f_{t-k}(a_1), f_{t-k}(a_2), \ldots, f_{t-k}(a_{m-L})$ are used in computing the $m-L$ piggyback functions $g((t-k-1+s) \mod (r-1), L)$ with $s = 1, 2, \ldots, m-L$. We can recover $m-L$ symbols $f_{t-k}(a_1), f_{t-k}(a_2), \ldots, f_{t-k}(a_{m-L})$ by downloading the $m-L$ symbols $f_{(t-k+s) \mod (r-1)}(a_{m-L+1}) + g((t-k-1+s) \mod (r-1), L)$ with $s = 1, 2, \ldots, m-L$ and the symbols used in computing the piggyback functions $g((t-k-1+s) \mod (r-1), L)$ with $s = 1, 2, \ldots, m-L$.

3) We can recover the last $L$ symbols in node $t$ by downloading the symbols used in computing the $L$ piggyback functions added in node $t$, together with the computed symbols in the first step.

Continue the code in Example 1. Suppose that parity 1 fails. According to the above repair method, in the first step, we can compute the four symbols $f_1(a_3), f_1(a_4), f_2(a_3), f_3(a_3)$ by downloading the following 12 symbols

$$a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, a_{3,5}, a_{3,6},$$
$$a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}, a_{4,5}, a_{4,6}.$$

By the second step, we can recover the erased two symbols $f_1(a_1), f_1(a_2)$ by downloading

$$f_1(a_2), f_5(a_1), f_2(a_1), f_5(a_1), f_2(a_3) + g(1, 2), f_3(a_3) + g(2, 2),$$

since

$$g(1, 2) = f_1(a_1) + f_4(a_2) + f_5(a_1),$$
$$g(2, 2) = f_1(a_2) + f_2(a_1) + f_5(a_2).$$

There is no piggyback function added in the parity 1 and we have recovered the last two symbols in parity 1, the third step is not necessary in repairing parity 1.
Suppose that parity 2 fails. By the first step, we can compute the four symbols \( f_2(a_3), f_2(a_4), f_3(a_3), f_4(a_3) \) by downloading the following 12 symbols

\[
a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, a_{3,5}, a_{3,6},
\]

\[
a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}, a_{4,5}, a_{4,6}.
\]

By the second step, we recover two symbols \( f_2(a_1), f_2(a_2) \) by downloading the following symbols

\[
f_1(a_2), f_5(a_2), f_3(a_1), f_3(a_3) + g(2, 2), f_4(a_3) + g(3, 2).
\]

By the third step, we can recover the two symbols \( f_2(a_3) + g(1, 2), f_2(a_4) + g(1, 1) \) by downloading the symbols

\[
a_{1,1}, a_{2,2}, a_{1,5}, a_{2,6}, f_2(a_2), f_3(a_1),
\]

together with \( a_{3,3} \) and \( f_2(a_3), f_2(a_4) \).

We can calculate that the repair bandwidth of parity 2 is 23 symbols. Similarly, we can show that the repair bandwidth of each node in parity \( \{3, 5\} \) is 24 symbols and the repair bandwidth of parity 4 is 23 symbols.

C. Average Repair Bandwidth Ratio of \( C_1(n, k, m, L) \)

In the following, we analyse the repair bandwidth of our codes \( C_1(n, k, m, L) \). We define the average repair bandwidth ratio of all nodes as the ratio of the average repair bandwidth of \( n \) nodes to the number of data symbols \( kr \).

**Lemma 3.** If \( L \) is a factor of \( n \), the lower bound \( \gamma_{\text{all}}^{\text{min}} \) and the upper bound \( \gamma_{\text{all}}^{\text{max}} \) of the average repair bandwidth ratio of all nodes of \( C_1(n, k, m, L) \) is

\[
\gamma_{\text{all}}^{\text{min}} = \frac{L+1}{2m} + \frac{(k+r)(m^2-m(L+1)+(L+1)(2L+1))}{Lmk(r-1)} + \frac{m-L}{km},
\]

\[
\gamma_{\text{all}}^{\text{max}} = \gamma_{\text{all}}^{\text{min}} + \frac{L(r-1)^2}{4(k+r)mk}.
\]

**Proof.** Denote the number of symbols used in computing the piggyback function \( g(\alpha, r - m + \beta) \) as \( n_{\alpha, \beta} \), where \( \alpha = 1, 2, \ldots, r - 1 \) and \( \beta = 1, 2, \ldots, L \). According to the definition of the piggyback function in Eq. (1) and Eq. (2), we can know that \( (n_{\alpha_1, \beta} - n_{\alpha_2, \beta})^2 \in \{0, 1\} \) for \( \alpha_1 \neq \alpha_2 \in \{1, 2, \ldots, r - 1\} \) and \( \beta = 1, 2, \ldots, L \).

According to the repair methods in Section III-A and III-B, we need to download \( \sum_{\beta=1}^{L} |\Phi_{\beta}| k_{\beta} \) symbols in repairing each of the \( n \) nodes in first step, download \( \sum_{\beta=1}^{L} \sum_{\alpha=1}^{r-1} n_{\alpha, \beta}^2 \) symbols in
repairing each of the $n$ nodes in the second step and download $(m - L)(k + r)$ symbols in repairing each of the $r$ parity nodes in the third step in total. Therefore, the average repair bandwidth ratio for all nodes $\gamma^{all}$ is

$$\gamma^{all} = \frac{\sum_{\beta=1}^{L} (|\Phi_\beta| k \beta + \sum_{\alpha=1}^{r-1} n_{\alpha,\beta}^2) + (m - L)(k + r)}{(k + r)mk}.$$ 

Since the equation

$$\left(\sum_{\alpha=1}^{r-1} n_{\alpha,\beta}\right)^2 + \sum_{\alpha_1 \neq \alpha_2 \in \{1,2,...,r-1\}} (n_{\alpha_1,\beta} - n_{\alpha_2,\beta})^2 = (r - 1) \sum_{\alpha=1}^{r-1} n_{\alpha,\beta}^2$$

holds, we can obtain that

$$\sum_{\alpha=1}^{r-1} n_{\alpha,\beta}^2 = \frac{(\sum_{\alpha=1}^{r-1} n_{\alpha,\beta})^2 + \sum_{\alpha_1 \neq \alpha_2 \in \{1,2,...,r-1\}} (n_{\alpha_1,\beta} - n_{\alpha_2,\beta})^2}{r - 1}$$

for $\beta = 1, 2, \ldots, L$.

Let $v_\beta = |\Phi_\beta|(m - \beta) - \left[\frac{|\Phi_\beta|(m-\beta)}{r-1}\right](r - 1)$, where $\beta = 1, 2, \ldots, L$. Because of $\sum_{\alpha=1}^{r-1} n_{\alpha,\beta} = |\Phi_\beta|(m - \beta)$ and $\sum_{\alpha_1 \neq \alpha_2 \in \{1,2,...,r-1\}} (n_{\alpha_1,\beta} - n_{\alpha_2,\beta})^2 = \beta(r - 1 - \beta)$, we can calculate that

$$\sum_{\alpha=1}^{r-1} n_{\alpha,\beta}^2 = \frac{(\sum_{\alpha=1}^{r-1} n_{\alpha,\beta})^2 + \beta(r - 1 - \beta)}{r - 1}.$$ 

Therefore, we can get $0 \leq v_\beta(r - 1 - \beta) \leq \left(\frac{r - 1}{2}\right)^2$ and further obtain the lower bound and upper bound in the lemma.

By Lemma 3, we can know $|\gamma_{\alpha\min}^{all} - \gamma^{all}| \leq |\gamma_{\alpha\max}^{all} - \gamma_{\alpha\min}^{all}| = \frac{L(r-1)^2}{4(4k+r)mk}$. When $k >> r$, we have $\frac{L(r-1)^2}{4(4k+r)mk} \rightarrow 0$. Therefore, we have $\gamma^{all} = \gamma_{\alpha\min}^{all}$ when $k >> r$.

**Lemma 4.** When $L$ is a factor of $n$ and $k >> r$, the minimum value of the average repair bandwidth ratio $\gamma^{all}$ of $C_1(n, k, m, L)$ is achieved when $L = \sqrt{\frac{6m^2 - 6m + 1}{3r - 1}}$.

**Proof.** When $k >> r$, we have $\gamma^{all} = \gamma_{\alpha\min}^{all}$. Then we can get

$$\gamma^{all} = \frac{L + 1}{2m} + \frac{m^2 L - mL^2 - mL + \frac{1}{2}L^3 + \frac{1}{2}L^2 + \frac{L}{6}}{L^2 m(r - 1)}.$$ 

We can calculate that

$$\frac{\partial \gamma^{all}}{\partial L} = \frac{1}{2m} + \frac{1}{3} \frac{m^2 - m + \frac{1}{2}}{L^2} \frac{1}{m(r - 1)} = 0,$$

and further obtain

$$L = \sqrt{\frac{6m^2 - 6m + 1}{3r - 1}}.$$
If \( L > \sqrt{\frac{6m^2-6m+1}{3r-1}} \), then \( \frac{\partial \gamma_{\text{all}}}{\partial L} > 0 \); if \( L < \sqrt{\frac{6m^2-6m+1}{3r-1}} \), then \( \frac{\partial \gamma_{\text{all}}}{\partial L} < 0 \); if \( L = \sqrt{\frac{6m^2-6m+1}{3r-1}} \), then \( \frac{\partial \gamma_{\text{all}}}{\partial L} = 0 \). Therefore, when \( L = \sqrt{\frac{6m^2-6m+1}{3r-1}} \), \( \gamma_{\text{all}} \) achieves the minimum value. \( \square \)

Since \( L \) is a positive integer, we take \( L = \lceil \sqrt{\frac{6m^2-6m+1}{3r-1}} \rceil \) or \( L = \lfloor \sqrt{\frac{6m^2-6m+1}{3r-1}} \rfloor \) to achieve the minimum repair bandwidth.

### IV. The Second Piggybacking Codes

In this section, we present construction of the second piggybacking codes that have lower repair bandwidth than all the existing piggybacking codes.

The second piggybacking codes is an \( n \times m \) array, where \( m = sr = s(n-k) \) and \( 2 \leq s \leq r \). We label the index of the \( n \) rows from 1 to \( n \) and the index of the \( m \) columns from 1 to \( m \). The first \( k \) nodes are data nodes that store data symbols and the last \( r \) nodes are parity nodes that store parity symbols. We divide the \( k \) data nodes into \( L \) disjoint subsets \( \Phi_1, \Phi_2, \ldots, \Phi_L \), where \( 1 \leq L < s \).

Each of the first \( k - \lceil \frac{k}{L} \rceil L \) subsets has size \( \lceil \frac{k}{L} \rceil \) and each of the last \( (\lceil \frac{k}{L} \rceil + 1)L - k \) subsets has size \( \lfloor \frac{k}{L} \rfloor \), i.e., \( |\Phi_i| = \lceil \frac{k}{L} \rceil \) for \( i = 1, 2, \ldots, k - \lfloor \frac{k}{L} \rfloor L \) and \( |\Phi_i| = \lfloor \frac{k}{L} \rfloor \) for \( i = k - \lfloor \frac{k}{L} \rfloor L + 1, \ldots, L \).

We divide the \( sr \) columns into \( s \) disjoint subsets \( W_\ell = \{(l-1)r+1, (l-1)r+2, \ldots, lr\} \), where \( \ell = 1, 2, \ldots, s \). The construction of the second piggybacking codes is as follows.

1. We create \( m = sr \) instances of \((n, k)\) MDS code. Let \((a_{i,1}^{(1)}, a_{i,2}^{(1)}, \ldots, a_{i,k}^{(1)}, f_1(a_{i,1}^{(1)}), \ldots, f_r(a_{i,k}^{(1)}))^T\) be an instance of \((n, k)\) MDS code over \( \mathbb{F}_q \) which is in column \((\ell-1)r+i\) of the \( n \times sr \) array, where \( a_i^{(\ell)} = (a_{i,1}^{(\ell)}, a_{i,2}^{(\ell)}, \ldots, a_{i,k}^{(\ell)})^T \) are data symbols, \( \ell = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, r \).

2. We take \((i-1)\) cyclic-shift for each column of the \( r \) parity symbols \((f_1(a_{i,1}^{(\ell)}), f_2(a_{i,1}^{(\ell)}), \ldots, f_r(a_{i,1}^{(\ell)}))^T\) to obtain

\[
(f_1((i-1) \mod r) + 1(a_{i}^{(\ell)}), f_1((2-i) \mod r) + 1(a_{i}^{(\ell)}), \ldots, f_r(a_{i,1}^{(\ell)}), f_1(a_{i,1}^{(\ell)}), f_2(a_{i,1}^{(\ell)}), \ldots, f_{r+1-i}(a_{i,1}^{(\ell)}))^T,
\]

where \( \ell = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, r \).

3. For \( i = 1, 2, \ldots, L \), we define the first \((s-i)r\) symbols of the \( |\Phi_i| \) nodes in \( \Phi_i \) as Protect Symbols (PS). We can calculate that the total number of PS in \( \Phi_i \) is \( p_i = |\Phi_i|(s-i)r \) and the total number of PS in all \( k \) data nodes are \( \sum_{i=1}^{L} p_i = \sum_{i=1}^{L} |\Phi_i|(s-i)r \). For \( i = 1, 2, \ldots, L \), denote the PS in column \( j \) with \( j = 1, 2, \ldots, (s-i)r \) in row \( \sum_{i=1}^{L} |\Phi_i| + \ell \) with \( \ell = 1, 2, \ldots, |\Phi_i| \) as \( t_{i,(\ell-1)(s-i)r+j} \). For example, when \( i = 1 \), we have

\[
(t_{1,1}, t_{1,2}, \ldots, t_{1,(s-1)r}) = (a_{1,1}^{(1)}, a_{2,1}^{(1)}, \ldots, a_{r,1}^{(s-1)}),
\]

\[
(t_{1,(s-1)r+1}, \ldots, t_{1,p_1}) = (a_{1,2}^{(1)}, \ldots, a_{r,|\Phi_1|}^{(s-1)}).
\]
We define \((r - 1)rL\) piggyback functions which will be added to the \((r - 1)rL\) parity symbols in the last \(L\) subsets \(W_i\) with \(i = s - L + 1, s - L + 2, \ldots, s\) such that the number of PS used in computing each piggyback function as average as possible. For \(i = 1, 2, \ldots, L\) and \(j = 1, 2, \ldots, r(r - 1)\), we define the piggyback function \(g(j, s - i + 1)\) as

\[
g(j, s - i + 1) = \sum_{\ell=1}^{s_{i,j}} t_{i,(\ell-1)(r-1)r+j},
\]

where \(s_{i,j} = 1 + \lfloor \frac{p_i}{r-1} \rfloor\) and the \(r(r - 1)L\) piggyback functions are added to the \(r(r - 1)L\) parity symbols in the last \(L\) subsets \(W_\ell\) with \(\ell = s - L + 1, s - L + 2, \ldots, s\), which is shown in Fig. 3. For notational convenience, denote the symbol in column \((\ell - 1)r + i\) and row \(j\) of the obtained \(n \times sr\) array as \(f'_{x-k}(a^{(\ell)}_i)\), where \(j = k + 1, k + 2, \ldots, k + r\), \(\ell = 1, 2, \ldots, s\) and \(i = 1, 2, \ldots, r\).

| Parity 1 | \(f_1(a_i^{(\ell)})\) | \(f_1(a_2^{(\ell)}) + g(1, s - \ell + 1)\) | \(\ldots\) | \(f_{s-1}(a_{i-1}^{(\ell)}) + g(r - 2, s - \ell + 1)\) | \(f_1(a_i^{(\ell)}) + g(r - 1, s - \ell + 1)\) |
|---|---|---|---|---|---|
| Parity 2 | \(f_2(a_i^{(\ell)}) + g(r, s - \ell + 1)\) | \(f_1(a_2^{(\ell)})\) | \(\ldots\) | \(f_1(a_{i-1}^{(\ell)})\) | \(f_2(a_i^{(\ell)}) + g(2r - 2, s - \ell + 1)\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |
| Parity \(r - 1\) | \(f_{r-1}(a_i^{(\ell)}) + g(r^2 - 3r + 3, s - \ell + 1)\) | \(f_{r-2}(a_i^{(\ell)}) + g(r^2 - 3r + 4, s - \ell + 1)\) | \(\ldots\) | \(f_{r-1}(a_i^{(\ell)})\) | \(f_{r-1}(a_i^{(\ell)}) + g(r^2 - 2r + 1, s - \ell + 1)\) |
| Parity \(r\) | \(f_r(a_i^{(\ell)}) + g(r^2 - 2r + 2, s - \ell + 1)\) | \(f_{r-1}(a_i^{(\ell)}) + g(r^2 - 2r + 3, s - \ell + 1)\) | \(\ldots\) | \(f_1(a_i^{(\ell)})\) | \(f_1(a_i^{(\ell)}) + g(r^2 - 2r + r, s - \ell + 1)\) |

Fig. 3: Piggyback functions added to the symbols in \(W_\ell\), \(\ell = s - L + 1, s - L + 2, \ldots, s\).

4) We replace the symbol \(f_x'(a_i^{(\ell)})\) in column \((\ell - 1)r + i\) and row \(x+k\) by the transformation function \(f_x''(a_i^{(\ell)})\) as follows,

\[
f_x''(a_i^{(\ell)}) = \begin{cases} 
    f_x'(a_i^{(\ell)}) + f_x'(a_x^{(\ell)}) & \text{if } x < i, \\
    \theta_{x,i} \cdot f_x'(a_x^{(\ell)}) + f_x'(a_i^{(\ell)}) & \text{if } x > i, \\
    f_x'(a_i^{(\ell)}) & \text{if } x = i,
\end{cases}
\]

where \(\theta_{x,i} \in \mathbb{F}_q \setminus \{0,1\}\) such that \(\theta_{x,i} - 1\) is invertible, \(x = 1, 2, \ldots, r\), \(\ell = 1, 2, \ldots, s\) and \(j = 1, 2, \ldots, r\).

We denote the above construction of our second piggybacking codes as \(C_2(n, k, m, L)\). In the construction of \(C_2(n, k, m, L)\), we design the piggyback functions in step three to repair the single-node failure of data nodes and employ the transformation functions designed in step four to repair the single-node failure of parity nodes. The piggyback structure of \(C_2(n, k, m, L)\) is
similar to that of our first piggybacking codes $C_1(n, k, m, L)$. The difference is that the piggyback functions of $C_2(n, k, m, L)$ are designed for data node repair, while the piggyback functions of $C_1(n, k, m, L)$ are designed for the repair of both data node and parity node.

Given that $x > i$, we have $f''_x(a_i^{(r)}) = \theta_{x,i} \cdot f'_x(a_i^{(r)}) + f''_x(a_i^{(r)})$ and $f''_x(a_i^{(r)}) = f'_x(a_{x,i}^{(r)}) + f''_x(a_i^{(r)})$. It is easy to check that we can compute $f'_x(a_i^{(r)})$ and $f''_x(a_i^{(r)})$ from $f''_x(a_i^{(r)})$ and $f''_x(a_i^{(r)})$. We can also compute $f''_x(a_i^{(r)})$ from any two out of the three symbols

$f'_x(a_i^{(r)}), f'_x(a_i^{(r)}), f''_x(a_i^{(r)})$.

Example 5. Consider the example of $(n, k, m, L) = (12, 8, 16, 2)$, the $k = 8$ data nodes are divided into two subsets $\Phi_1 = \{1, 2, 3, 4\}$ and $\Phi_2 = \{5, 6, 7, 8\}$, and the $m = 16$ columns are divided into four subsets $W_\ell = \{4(\ell - 1) + 1, 4(\ell - 1) + 2, 4(\ell - 1) + 3, 4(\ell - 1) + 4\}$ with $\ell = 1, 2, 3, 4$. According to Eq. (3), the piggyback functions are defined as follows.

$g(i, 1) = a_{1,i}^{(1)} + a_{i,2}^{(1)} + a_{i,3}^{(1)} + a_{i,4}^{(1)}$ for $i = 1, 2, 3, 4$,

$g(4 + i, 1) = a_{1,i}^{(2)} + a_{i,2}^{(2)} + a_{i,3}^{(2)} + a_{i,4}^{(2)}$ for $i = 1, 2, 3, 4$,

$g(8 + i, 1) = a_{1,i}^{(3)} + a_{i,2}^{(3)} + a_{i,3}^{(3)} + a_{i,4}^{(3)}$ for $i = 1, 2, 3, 4$,

$g(i, 2) = a_{i,5}^{(1)} + a_{i,6}^{(1)} + a_{i,7}^{(1)}$ for $i = 1, 2, 3, 4$,

$g(4 + i, 2) = a_{i,5}^{(2)} + a_{i,7}^{(2)}$ for $i = 1, 2, 3, 4$,

$g(8 + i, 2) = a_{i,6}^{(1)} + a_{i,7}^{(2)}$ for $i = 1, 2, 3, 4$.

The transformation functions are given in Eq. (4) with $x = 1, 2, 3, 4$, $i = 1, 2, 3, 4$ and $\ell = 1, 2, 3, 4$, where the symbol $f'_x(a_i^{(r)})$ is as follows,

$\left( f'_1(a_1^{(r)}), f'_2(a_2^{(r)}) \right) = \left( f_1(a_1^{(r)}), f_1(a_2^{(r)}) \right)$,

$\left( f'_3(a_3^{(r)}), f'_4(a_4^{(r)}) \right) = \left( f_1(a_3^{(r)}), f_1(a_4^{(r)}) \right)$,

$\left( f'_1(a_2^{(r)}), f'_1(a_3^{(r)}) \right) = \left( f_4(a_2^{(r)}) + g(1, 5 - \ell), f_4(a_2^{(r)}) + g(2, 5 - \ell) \right)$,

$\left( f'_1(a_1^{(r)}), f'_2(a_1^{(r)}) \right) = \left( f_4(a_2^{(r)}) + g(3, 5 - \ell), f_2(a_1^{(r)}) + g(4, 5 - \ell) \right)$,

$\left( f'_2(a_3^{(r)}), f'_2(a_4^{(r)}) \right) = \left( f_4(a_3^{(r)}) + g(5, 5 - \ell), f_3(a_1^{(r)}) + g(6, 5 - \ell) \right)$,

$\left( f'_3(a_1^{(r)}), f'_3(a_2^{(r)}) \right) = \left( f_3(a_1^{(r)}) + g(7, 5 - \ell), f_2(a_2^{(r)}) + g(8, 5 - \ell) \right)$,

$\left( f'_3(a_4^{(r)}), f'_4(a_1^{(r)}) \right) = \left( f_4(a_4^{(r)}) + g(9, 5 - \ell), f_4(a_1^{(r)}) + g(10, 5 - \ell) \right)$,

$\left( f'_4(a_2^{(r)}), f'_3(a_3^{(r)}) \right) = \left( f_3(a_2^{(r)}) + g(11, 5 - \ell), f_2(a_3^{(r)}) + g(12, 5 - \ell) \right)$,
where \( \ell = 1, 2, 3, 4 \) and \( g(\cdot, x) = 0 \) for \( x \geq 3 \).

The code \( C_2(12, 8, 16, 2) \) is shown in Fig. 4.

| Node 1 | \( a_{1,1}^{(1)} \) | \( a_{1,2}^{(1)} \) | \( a_{1,3}^{(1)} \) | \( a_{1,4}^{(1)} \) | \( a_{1,5}^{(1)} \) | \( a_{1,6}^{(1)} \) | \( a_{1,7}^{(1)} \) | \( a_{1,8}^{(1)} \) | \( a_{1,9}^{(1)} \) | \( a_{1,10}^{(1)} \) | \( a_{1,11}^{(1)} \) |
|--------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Node 2 | \( a_{2,1}^{(1)} \) | \( a_{2,2}^{(1)} \) | \( a_{2,3}^{(1)} \) | \( a_{2,4}^{(1)} \) | \( a_{2,5}^{(1)} \) | \( a_{2,6}^{(1)} \) | \( a_{2,7}^{(1)} \) | \( a_{2,8}^{(1)} \) | \( a_{2,9}^{(1)} \) | \( a_{2,10}^{(1)} \) | \( a_{2,11}^{(1)} \) |
| Parity 1 | \( f_1(a_1^{(1)}) \) | \( f_1(a_2^{(1)}) \) | \( f_1(a_3^{(1)}) \) | \( f_1(a_4^{(1)}) \) | \( f_1(a_5^{(1)}) \) | \( f_1(a_6^{(1)}) \) | \( f_1(a_7^{(1)}) \) | \( f_1(a_8^{(1)}) \) | \( f_1(a_9^{(1)}) \) | \( f_1(a_{10}^{(1)}) \) | \( f_1(a_{11}^{(1)}) \) |
| Parity 2 | \( \theta_1 f_1(a_1^{(1)}) + f_2(a_1^{(1)}) \) | \( \theta_1 f_1(a_2^{(1)}) + f_2(a_2^{(1)}) \) | \( \theta_1 f_1(a_3^{(1)}) + f_2(a_3^{(1)}) \) | \( \theta_1 f_1(a_4^{(1)}) + f_2(a_4^{(1)}) \) | \( \theta_1 f_1(a_5^{(1)}) + f_2(a_5^{(1)}) \) | \( \theta_1 f_1(a_6^{(1)}) + f_2(a_6^{(1)}) \) | \( \theta_1 f_1(a_7^{(1)}) + f_2(a_7^{(1)}) \) | \( \theta_1 f_1(a_8^{(1)}) + f_2(a_8^{(1)}) \) | \( \theta_1 f_1(a_9^{(1)}) + f_2(a_9^{(1)}) \) | \( \theta_1 f_1(a_{10}^{(1)}) + f_2(a_{10}^{(1)}) \) | \( \theta_1 f_1(a_{11}^{(1)}) + f_2(a_{11}^{(1)}) \) |
| Parity 3 | \( \theta_2 f_1(a_1^{(1)}) + f_2(a_1^{(1)}) \) | \( \theta_2 f_1(a_2^{(1)}) + f_2(a_2^{(1)}) \) | \( \theta_2 f_1(a_3^{(1)}) + f_2(a_3^{(1)}) \) | \( \theta_2 f_1(a_4^{(1)}) + f_2(a_4^{(1)}) \) | \( \theta_2 f_1(a_5^{(1)}) + f_2(a_5^{(1)}) \) | \( \theta_2 f_1(a_6^{(1)}) + f_2(a_6^{(1)}) \) | \( \theta_2 f_1(a_7^{(1)}) + f_2(a_7^{(1)}) \) | \( \theta_2 f_1(a_8^{(1)}) + f_2(a_8^{(1)}) \) | \( \theta_2 f_1(a_9^{(1)}) + f_2(a_9^{(1)}) \) | \( \theta_2 f_1(a_{10}^{(1)}) + f_2(a_{10}^{(1)}) \) | \( \theta_2 f_1(a_{11}^{(1)}) + f_2(a_{11}^{(1)}) \) |
| Parity 4 | \( \theta_3 f_1(a_1^{(1)}) + f_2(a_1^{(1)}) \) | \( \theta_3 f_1(a_2^{(1)}) + f_2(a_2^{(1)}) \) | \( \theta_3 f_1(a_3^{(1)}) + f_2(a_3^{(1)}) \) | \( \theta_3 f_1(a_4^{(1)}) + f_2(a_4^{(1)}) \) | \( \theta_3 f_1(a_5^{(1)}) + f_2(a_5^{(1)}) \) | \( \theta_3 f_1(a_6^{(1)}) + f_2(a_6^{(1)}) \) | \( \theta_3 f_1(a_7^{(1)}) + f_2(a_7^{(1)}) \) | \( \theta_3 f_1(a_8^{(1)}) + f_2(a_8^{(1)}) \) | \( \theta_3 f_1(a_9^{(1)}) + f_2(a_9^{(1)}) \) | \( \theta_3 f_1(a_{10}^{(1)}) + f_2(a_{10}^{(1)}) \) | \( \theta_3 f_1(a_{11}^{(1)}) + f_2(a_{11}^{(1)}) \) |

Fig. 4: Code \( C_2(12, 8, 16, 2) \).

V. REPAIR METHOD OF \( C_2(n, k, m, L) \)

In this section, we present repair method for any single-node erasure of \( C_2(n, k, m, L) \).

A. Repair Method for Data Node

The repair method for data node of \( C_2(n, k, m, L) \) is quite similar to that of \( C_1(n, k, m, L) \), because the two codes have the same structure of piggyback function. The difference is that the piggyback function of \( C_1(n, k, m, L) \) is computed from both data symbols and parity symbols, while the piggyback function of \( C_2(n, k, m, L) \) is computed from only data symbols. Suppose that data node \( t \in \Phi_u \) fails, where \( t \in \{1, 2, \ldots, k\} \) and \( u \in \{1, 2, \ldots, L\} \), the repair method is as follows.

1) Recall that there is one parity symbol in each column which is not added by piggyback function. We can recover the last \( ur \) erased symbols in node \( t \) by downloading \( (k - 1)ur \) data symbols in the last \( ur \) columns in the first \( k \) rows except row \( t \) and downloading the \( ur \) parity symbols in the last \( ur \) columns of without attaching piggyback function.

2) We download the parity symbols of which the corresponding piggyback functions are computed from the symbols in node \( t \), together with \( f_x(a_i^{(\ell)}) \) with \( x = 1, 2, \ldots, r \), \( i = 1, 2, \ldots, r \) and \( \ell = s - u + 1, \ldots, s \), to recover the first \( (s - u)r \) symbols in node \( t \).
Continue the code in Example 5. Suppose that node \( t = 1 \) fails, we have \( u = 1 \) and we can recover the last four symbols \( a_{i,1}^{(4)} \) with \( i = 1, 2, 3, 4 \) in node 1 and the 16 parity symbols \( f_x(a_i^{(4)}) \) with \( x = 1, 2, 3, 4 \) and \( i = 1, 2, 3, 4 \), by downloading the 28 symbols \( a_{i,j}^{(4)} \) with \( j = 2, 3, \ldots, 8 \) and \( i = 1, 2, 3, 4 \), and the four parity symbols
\[
f_1(a_1^{(4)}), f_1(a_2^{(4)}), f_1(a_3^{(4)}), f_1(a_4^{(4)}).
\]

Then we download the following 12 parity symbols
\[
\begin{align*}
&f_1'(a_2^{(4)}) + f_2'(a_1^{(4)}), f_1'(a_3^{(4)}) + f_3'(a_1^{(4)}), \\
f_2'(a_3^{(4)}) + f_3'(a_2^{(4)}), f_2'(a_4^{(4)}) + f_4'(a_2^{(4)}), \\
f_3'(a_3^{(4)}) + f_3'(a_1^{(4)}), f_1'(a_4^{(4)}) + f_4'(a_1^{(4)}), \\
&\theta_{4,1} \cdot f_1'(a_1^{(4)}) + f_4'(a_1^{(4)}), \theta_{2,1} \cdot f_1'(a_2^{(4)}) + f_2'(a_1^{(4)}), \\
&\theta_{3,1} \cdot f_1'(a_3^{(4)}) + f_3'(a_1^{(4)}), \theta_{3,2} \cdot f_2'(a_3^{(4)}) + f_2'(a_3^{(4)}), \\
&\theta_{4,2} \cdot f_2'(a_4^{(4)}) + f_4'(a_2^{(4)}), \theta_{4,3} \cdot f_3'(a_4^{(4)}) + f_4'(a_3^{(4)}),
\end{align*}
\]

and the 36 data symbols \( a_{i,j}^{(\ell)} \) with \( \ell = 1, 2, 3 \), \( i = 1, 2, 3, 4 \) and \( j = 2, 3, 4 \) to recover the first 12 data symbols in node 1. The repair bandwidth of node 1 is 80 symbols. Similarly, we can calculate that the repair bandwidth of each node in \{2, 3, 4\} is 80 symbols, the repair bandwidth of each node in \{5, 8\} is 90 symbols, and the repair bandwidth of each node in \{6, 7\} is 86 symbols.

\section*{B. Repair Process for Parity Node}

Suppose that node \( t \) fails, where \( t \in \{k + 1, k + 2, \ldots, k + r\} \). The repair method of node \( t \) is as follow.

1) We compute the \( sr \) symbols \( f_x(a_{\ell-k}^{(\ell)}) \) with \( \ell = 1, 2, \ldots, s \) and \( x = 1, 2, \ldots, r \) by downloading the \( sk \) symbols in the first \( k \) rows in columns \((\ell - 1)r + t - k\) with \( \ell = 1, 2, \ldots, s \).

2) We recover the erased \( sr \) symbols \( f_x''(a_{\ell-k}^{(\ell)}) \) with \( \ell = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, r \) by downloading the data symbols which are used in computing the piggyback functions located in the \( r \) rows \( k + 1, k + 2, \ldots, k + r \) in the \( L \) columns \((\ell - 1)r + t - k\) with \( \ell = s - L + 1, s - L + 2, \ldots, s \) and downloading the \( s(r - 1) \) symbols \( f_x''(a_{\ell-k}^{(\ell)}) \) with \( \ell = 1, 2, \ldots, s \) and \( x = 1, 2, \ldots, t - k - 1, t - k + 1, \ldots, r \).
Continue the code in Example 5. Suppose that node \( t = k + 1 \) fails, we download 32 data symbols \( a_{1,j}^{(\ell)} \) with \( \ell = 1, 2, 3, 4 \) and \( j = 1, 2, \ldots, 8 \) to calculate \( f_x(a_{1}^{(\ell)}) \) for \( x = 1, 2, 3, 4 \) and \( \ell = 1, 2, 3, 4 \). Then we download the following 20 data symbols
\[
\begin{align*}
d_{4,1}^{(1)}, d_{4,2}^{(1)}, d_{4,3}^{(1)}, d_{4,4}^{(1)}, d_{3,1}^{(2)}, d_{3,2}^{(2)}, d_{3,3}^{(2)}, d_{3,4}^{(2)}, \\
d_{2,1}^{(3)}, d_{2,2}^{(3)}, d_{2,3}^{(3)}, d_{2,4}^{(3)}, d_{1,1}^{(4)}, d_{1,2}^{(4)}, d_{1,3}^{(4)}, d_{1,4}^{(4)}.
\end{align*}
\]
to calculate the piggyback functions
\[g(4, 1), g(7, 1), g(10, 1), g(4, 2), g(7, 2), g(10, 2).\]
Together with \( f_x(a_{1}^{(\ell)}) \) for \( x = 1, 2, 3, 4 \) and \( \ell = 1, 2, 3, 4 \), and the above 6 piggyback functions, we can recover all the symbols in node \( t \) by downloading the following symbols.
\[
\begin{align*}
f'_1(a_2^{(3)}) + f'_2(a_1^{(3)}), f'_1(a_3^{(3)}) + f'_2(a_1^{(3)}), \\
f'_1(a_4^{(3)}) + f'_4(a_1^{(3)}), f'_1(a_2^{(4)}) + f'_2(a_1^{(4)}), \\
f'_1(a_3^{(4)}) + f'_3(a_1^{(4)}), f'_1(a_4^{(4)}) + f'_4(a_1^{(4)}).
\end{align*}
\]

C. Repair Bandwidth of \( C_2(n, k, m, L) \)

When \( L \) is a factor of \( k \), with similar proof in Lemma 6, we can show the bounds of repair bandwidth of data nodes of \( C_2(n, k, m, L) \) in the next lemma and we omit the proof.

Lemma 6. If \( L \) is a factor of \( k \), the lower bound \( \gamma_{min}^{sys} \) and the upper bound \( \gamma_{max}^{sys} \) of the average repair ratio of data nodes of \( C_2(n, k, m, L) \) is
\[
\gamma_{min}^{sys} = \frac{L+1}{2a} + \frac{s^2 - s(L+1) + \frac{(L+1)(2L+1)}{8}}{s(L-1)} + \frac{(r-1)(L-3)}{2sr}.
\]
\[
\gamma_{max}^{sys} = \gamma_{min}^{all} + \frac{L(r-1)}{4sk^2}.
\]

By Lemma 6, we have \( |\gamma_{sys} - \gamma_{min}^{sys}| \leq |\gamma_{sys}^{sys} - \gamma_{min}^{sys}| = \frac{L(r-1)}{4sk^2} \). When \( k \to \infty \), we have \( \gamma_{sys} = \gamma_{min}^{sys} \). Therefore, we do not distinguish between \( \gamma_{sys} \) and \( \gamma_{min}^{sys} \) in the rest of the paper.

Similar to the proof of Lemma 6, we can also show that the minimum value of average repair bandwidth ratio \( \gamma_{sys} \) is achieved when \( L = \sqrt{\frac{6s^2 - 6s + 1}{3r - 1}} \).

Lemma 7. If \( L \) is a factor of \( k \), the average repair bandwidth ratio of parity nodes for codes \( C_2(n, k, m, L) \) is
\[
\gamma_{parity} = \frac{2}{r} + \frac{1}{k} - \frac{1}{kr} - \frac{L + 1}{2sr}.
\]
Proof. According to the repair methods in Section V-B, we need to download \( \sum_{i=1}^{r} s k \) symbols in repairing each of the \( r \) parity nodes in the first step. In the second step, we need to download \( \sum_{i=1}^{r} s(r - 1) + 2 \sum_{i=1}^{L} |\Phi_i| r(s - i) \) in repairing each of the \( r \) parity nodes. Therefore, we can calculate that

\[
\gamma_{\text{parity}} = \frac{\sum_{i=1}^{r} s(k + r - 1) + \sum_{i=1}^{L} |\Phi_i| r(s - i)}{skr^2}.
\]

Because \( L \) is a factor of \( k \), we have \( |\Phi_i| = \frac{k}{L} \) for \( i = 1, 2, \ldots, L \), and further obtain that

\[
\gamma_{\text{parity}} = \frac{2}{r} + \frac{1}{k} - \frac{1}{kr} - \frac{L + 1}{2sr}.
\]

\( \square \)

VI. COMPARISON

In this section, we evaluate the average repair bandwidth of all \( n \) nodes for our two codes and the existing piggybacking codes with low repair bandwidth.

A. Piggybacking Codes with \( m < r \)

Fig. 5: The average repair bandwidth ratio of all nodes for the proposed codes \( C_1 \), REPB codes [15], codes \( C_3 \) [20] and codes \( C \) [21], where \( r = 8, 9 \), \( m = 6 \) and \( k = 30, 31, \ldots, 100 \).

First, we evaluate \( C_1(n, k, m, L) \) and the existing piggybacking codes [15], [20], [21] such that the sub-packetization is no larger than \( r \). Denote the codes in [20] as \( C_3(n, k, m) \), the MDS codes (the first codes) in [21] as \( C \), the codes in [15] as REPB.
Fig. 5 shows the evaluations for $r = 8, 9, m = 6$ and $k = 30, 31, \ldots, 100$. Note that the codes in our conference paper [1] can only support the parameter $m = r$ and do not draw the points for codes [1] in Fig. 5. The results in Fig. 5 demonstrate that our codes $C_1(n, k, m, L)$ have lower repair bandwidth than all existing piggybacking codes when $m < r$ for all the evaluated parameters.

**B. Piggybacking Codes with $m \geq r$**

Fig. 6: The average repair bandwidth ratio of all nodes for the proposed codes $C_2$, OOP codes [16], codes $C_4$ [1] and codes $C_5$ [19], where $r = 4, 5, 6, 7, \ldots, 20$, code rate $\frac{k}{n} = 0.75, 0.8, 0.9$ and $m = 6r, r^2$.

In the following, we evaluate $C_2(n, k, m, L)$ and the existing piggybacking codes with $m \geq r$.

We label the codes in [1] as $C_4$, the codes in [19] as $C_5$. Fig. 6 shows the average repair bandwidth ratio of all nodes for our codes $C_2(n, k, m, L)$ and the existing piggybacking codes, including OOP, $C_4$ and $C_5$ when $r = 4, 5, \ldots, 20$, $\frac{k}{n} = 0.75, 0.8, 0.9$ and $m = 6r, r^2$. The results show that our codes $C_2(n, k, m, L)$ have lower repair bandwidth than the other codes for all the evaluated parameters. The essential reason of lower repair bandwidth of our $C_2(n, k, m, L)$ is that we jointly design the piggyback function for data node repair and the transformation function for parity node repair.
VII. Conclusion

In this paper, we design two classes of piggybacking codes with flexible sub-packetization level. The first piggybacking codes $C_1$ have the lower repair bandwidth for single-node failure than all existing piggybacking codes with $m \leq r$ for all the evaluated parameters. Our second piggybacking codes can support sub-packetization $m = sr$ with $2 \leq s \leq r$ that have the lowest average repair bandwidth for all nodes among all existing piggybacking codes for the evaluated parameters. The piggybacking codes constructions by jointly designing piggyback functions and transformation functions to support more larger sub-packetization, say $m = sr$ with $s > r$, to further reduce repair bandwidth is one of our future work.

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