On the Supersymplectic Homogeneous Superspace
Underlying the OSp(1/2) Coherent States

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ABSTRACT

In this work we extend Onofri and Perelomov’s coherent states methods to the recently introduced $OSp(1/2)$ coherent states. These latter are shown to be parametrized by points of a supersymplectic supermanifold, namely the homogeneous superspace $OSp(1/2)/U(1)$, which is clearly identified with a supercoadjoint orbit of $OSp(1/2)$ by exhibiting the corresponding equivariant supermoment map. Moreover, this supermanifold is shown to be a nontrivial example of Rothstein’s supersymplectic supermanifolds. More precisely, we show that its supersymplectic structure is completely determined in terms of $SU(1,1)$-invariant (but unrelated) Kähler 2-form and Kähler metric on the unit disc. This result allows us to define the notions of a superKähler supermanifold and a superKähler superpotential, the geometric structure of the former being encoded into the latter.

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1. Introduction

In the past thirty years several Lie algebraic, group theoretic and geometric concepts have been successfully extended to the supersymmetric context, enriching these already established mathematical structures and giving birth to a new branch of mathematics called supermathematics. Numerous review articles and textbooks are available, see for instance [1-5]. For applications in physics we refer to [6].

Recently Balantekin et al. [7-8] (see also [9]) went a step further and introduced the so called supercoherent states (SCS), which extend in a natural way the well known Perelomov coherent states (CS) [10]. It seems then natural to pursue further this parallel with the non-super case, by addressing such questions as: what is the geometric structure underlying these SCS? In order to answer this question we will devote the first part of this work to the extension of Onofri’s analysis [11]. Let us briefly recall the main features of this analysis. Perelomov’s coherent states for a semi-simple Lie group $G$ are points of an orbit of a unitary irreducible representation (UIR) $U$ of $G$ in a Hilbert space $\mathcal{H}$ (more precisely in $\mathbf{CP}(\mathcal{H})$). Choosing an initial state in $\mathcal{H}$, called the fiducial state, the vectors of the corresponding $G$-orbit in $\mathcal{H}$ are parametrized by points of a homogeneous space $G/H$, where $H$ is the isotropy subgroup (up to a phase) of the fiducial state. Using this family of states (CS) Onofri showed that $G/H$ is a symplectic and even a Kähler manifold [11]. More precisely, he showed that a symplectic structure on $G/H$ can be explicitly obtained from the coherent states. From the quantization vs. classical limit point view, Onofri’s result identifies, through the CS, the classical limit of the quantum theory described by the UIR $(U, \mathcal{H})$ of $G$.

Here, we shall answer the above question in the case of the $OSp(1/2)$ CS constructed in [7]. In fact, we will show that the homogeneous superspace $OSp(1/2)/U(1)$ parametrizing these SCS is a supersymplectic supermanifold [1] [4]. Indeed, using an extension of Onofri’s analysis we will exhibit a supersymplectic structure on that homogeneous superspace. Moreover, using a supersymmetric extension of Berezin’s covariant symbols [12], we will be able to construct an equivariant supermoment map allowing us to show that $OSp(1/2)/U(1)$ is a supercoadjoint orbit of $OSp(1/2)$.

The general notion of a supersymplectic supermanifold has been considered by both Berezin [1] and Kostant [4] using different conventions. Since the second reference gives
a complete description of the subject, we will use its conventions. Recently, in extending Batchelor’s theorem [13] which allows the description of supermanifolds in terms of usual geometry, Rothstein [14] proved a similar result for the supersymplectic subcategory. In fact, according to Rothstein any supersymplectic supermanifold can be completely characterized by a set \((M, \omega, E, g, \nabla)\), where \((M, \omega)\) is a symplectic manifold and \(E\) is a vector bundle over \(M\) with a metric \(g\) and a \(g\)-compatible connection \(\nabla\). In the second part of this work, we will explicitly identify these five ingredients for the supersymplectic supermanifold underlying the \(OSp(1/2)\) CS, exhibiting thus a nontrivial example of Rothstein’s theorem. Moreover, from this example we will be able to define the new notion of a superKähler supermanifold.

The notion of a Kähler potential appears naturally in Onofri’s analysis. It constitutes the crucial step of that analysis, since it is defined in terms of the CS and it gives rise to the Kähler structure on the homogeneous space parametrizing these CS. Hence, it connects the quantum model to its classical limit. We shall show here that this notion extends also to the super-setting, allowing one to encode Rothstein’s theorem ingredients for a superKähler supermanifold into a superKähler superpotential.

Since \(OSp(1/2)\) is the simplest supersymmetric extension of \(SU(1,1)\) we shall start this letter in section 2 by rederiving Onofri’s analysis for the \(SU(1,1)\) CS. In section 3 we will extend it to \(OSp(1/2)\) basing our construction on the SCS obtained in [7]. We will exhibit the supersymplectic form and show that \(OSp(1/2)/U(1)\) is a supercoadjoint orbit of \(OSp(1/2)\). Then in section 4, we will show that what we have obtained is a non-trivial example of Rothstein’s supersymplectic supermanifolds. We will also define the notion of a superKähler supermanifold and show that the ingredients of Rothstein’s theorem can be read off from a superKähler superpotential. In section 5 we will show that our supersymplectic supermanifold satisfies the Darboux-Kostant theorem [4]. Finally, section 6 gathers a discussion, concluding remarks and prospects. Notice that throughout this work, \(OSp(1/2)\) stands for \(OSp(1/2, \mathbb{R})\).

2. \(SU(1,1)\) CS and the symplectic unit disc

Coherent states are special quantum states that have proven to be very useful in many areas of physics, especially in quantum optics (see [15] for references). They have also proven to be very useful in mathematical physics, mainly in connection with the quantization vs.
classical limit procedures [11] [12] [15]. As stressed in the introduction this last aspect will be our focus here. The group theoretic construction of coherent states [10] holds for Lie groups possessing square integrable representations. Hence for non-compact semisimple Lie groups such CS live within the discrete series representations. Let us consider here the simplest example of the \( SU(1, 1) \) CS.

The Lie algebra \( su(1, 1) \) of \( SU(1, 1) \) is described by the following commutation relations in the Cartan-Weyl basis,

\[
[K_0, K_{\pm}] = \pm K_{\pm} \quad \text{and} \quad [K_+, K_-] = -2K_0. \tag{2.1}
\]

The positive discrete series representations of \( SU(1, 1) \) are denoted \( D^k_+ \) for \( k \in \frac{1}{2}\mathbb{N} \) and \( k > \frac{1}{2} \). For given \( k \), the representation space is spanned by the set of states \( \{ |k, m\rangle, m = k, k + 1, k + 2, \ldots \} \). These basis elements are eigenstates of \( K_0 \) with eigenvalues \( k + m \). They result from the action of integer powers of \( K_+ \) on the “highest weight” state \( |k, k\rangle \) defined by

\[
K_0|k, k\rangle = k|k, k\rangle \quad \text{and} \quad K_-|k, k\rangle = 0. \tag{2.2}
\]

When considering \( |k, k\rangle \) as the fiducial state for the construction of \( SU(1, 1) \) CS, one can easily see that the latter are parametrized by points of the coset space \( SU(1, 1)/U(1) \). More precisely, here one uses the unit disc realization of \( SU(1, 1)/U(1) \), namely \( \mathcal{D}^{(1)} \cong SU(1, 1)/U(1) \), which appears in the following construction of the CS [10],

\[
|z\rangle \equiv D^k_+(g) \equiv (1 - |z|^2)^k e^{zK_+}|k, k\rangle. \tag{2.3}
\]

Note that \( \mathcal{D}^{(1)} = \{ z \in \mathbb{C} | |z|^2 < 1 \} \) and that \( g \) in (2.3) belongs to \( SU(1, 1) \) and projects down to \( z \in \mathcal{D}^{(1)} \) through the principal bundle projection \( SU(1, 1) \longrightarrow SU(1, 1)/U(1) \equiv \mathcal{D}^{(1)} \). Moreover, \((1 - |z|^2)^k \) in equation (2.3) ensures that \( |z\rangle \) is normalized to one. For short, the highest weight state \( |k, k\rangle \) will be, in the sequel, denoted \( |0\rangle \).

According to Onofri [11], the unit disc can be equipped with an \( SU(1, 1) \)-invariant symplectic (Kähler) 2-form that will make it into a classical phase space. This is realized in the following way. Define first the Kähler potential as the phase space function

\[
f_0(z, \bar{z}) \equiv \log |\langle 0|z\rangle|^{-2} \equiv \log |\langle 0|D^k_+(g)|0\rangle|^{-2} = -2k \log (1 - |z|^2). \tag{2.4}
\]
Then a two-form on $\mathcal{D}^{(1)}$ is obtained as follows,

$$\omega_0 \equiv -i\partial\bar{\partial}f_0(z, \bar{z}) = -2ik \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2}. \quad (2.5)$$

Clearly, $\omega_0$ is a closed and nondegenerate Kähler 2-form. In (2.5) $\partial \equiv dz \frac{\partial}{\partial z}$ and $\bar{\partial} \equiv d\bar{z} \frac{\partial}{\partial \bar{z}}$ such that the exterior derivative $d = \partial + \bar{\partial}$.

Having in hand the symplectic form we can now analyze the $SU(1, 1)$ action on $\mathcal{D}^{(1)}$. The classical observables corresponding to the generators of $SU(1, 1)$ given in (2.1) are nothing but the following Berezin covariant symbols [10] [12],

$$K_0(z, \bar{z}) \equiv \langle z|K_0|z \rangle = k \frac{(1 + |z|^2)}{(1 - |z|^2)}, \quad (2.6a)$$

$$K_+(z, \bar{z}) \equiv \langle z|K_+|z \rangle = 2k \frac{\bar{z}}{(1 - |z|^2)} \quad \text{and} \quad K_-(z, \bar{z}) \equiv \langle z|K_-|z \rangle = 2k \frac{z}{(1 - |z|^2)}. \quad (2.6b)$$

These functions generate a symplectic action of $SU(1, 1)$ on $\mathcal{D}^{(1)}$ through Hamiltonian vector fields which are obtained using the equation:

$$[X_H, \omega_0] = dH. \quad (2.7)$$

Here the phase space function $H$ uniquely defines a vector field $X_H$ through the procedure of contraction (or interior product) with the symplectic form $\omega_0$. A straightforward calculation gives:

$$X_{K_0} = iz \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial \bar{z}}, \quad X_{K_+} = i \frac{\partial}{\partial z} - iz^2 \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad X_{K_-} = iz^2 \frac{\partial}{\partial z} - i \frac{\partial}{\partial \bar{z}}. \quad (2.8)$$

Because of (2.7) one clearly sees that $\mathcal{L}_{X_H}\omega_0 = 0$ for $H = K_0, K_+$ or $K_-$, where $\mathcal{L}_X$ is the Lie derivative along $X$. This shows that the action of $SU(1, 1)$ on $\mathcal{D}^{(1)}$ is actually symplectic. Moreover, $\omega_0$ defines in $C^\infty(\mathcal{D}^{(1)})$ a Poisson bracket with respect to which the observables in (2.6) form a symplectic realization of $su(1, 1)$ in (2.1). Indeed, we have

$$\{K_0, K_{\pm}\} \equiv i[X_{K_0}]dK_{\pm} = \pm K_{\pm} \quad \text{and} \quad \{K_+, K_-\} \equiv i[X_{K_+}]dK_- = -2K_0. \quad (2.9)$$

Note that the factor $i$ appearing in the definition of the Poisson bracket is inherent to the present complex-analytic context.
From equation (2.6) one shows that $SU(1,1)/U(1)$ is a coadjoint orbit of $SU(1,1)$. In other words, using (2.6) one defines the equivariant momentum map,

$$K : \mathcal{D}^{(1)} \longrightarrow su^*(1,1) \equiv su(1,1)$$

$$z \mapsto (K_0(z, \bar{z}), K_1(z, \bar{z}), K_2(z, \bar{z})),$$  \hspace{1cm} (2.10)

where $K_+ = K_1 + iK_2$ and $K_- = K_1 - iK_2$. The three-vector on the right hand side of (2.10) spans a two dimensional surface in $\mathbb{R}^3$, the equation of which is given by the identity,

$$K_0^2(z, \bar{z}) - K_1^2(z, \bar{z}) - K_2^2(z, \bar{z}) = k^2.$$ \hspace{1cm} (2.11)

For $k > 0$ this is the upper component of the two-sheeted hyperbola in $\mathbb{R}^3$ which intersects the $K_0$ axis at the point $(k, 0, 0)$. The unit disc $\mathcal{D}^{(1)}$ is just the stereographic projection of the latter. Thus $(\mathcal{D}^{(1)}, \omega_0)$ is a coadjoint orbit of $SU(1,1)$.

### 3. OSp(1/2) CS and the supersymplectic superunit disc

Now let us show how the previous constructions extend to the supersymmetric world. We will consider for instance the $OSp(1/2)$ CS introduced in [7]. The $osp(1/2)$ Lie superalgebra has five supergenerators, $K_0, K_\pm$ and $F_\pm$; their commutation-anticommutation relations are as follows,

\begin{align*}
[K_0, K_\pm]_- &= \pm K_\pm \quad \text{and} \quad [K_+, K_-]_- = -2K_0, \quad (3.1a) \\
[K_0, F_\pm]_- &= \pm \frac{1}{2} F_\pm, \quad [K_\pm, F_\pm]_- = 0 \quad \text{and} \quad [K_\pm, F_\mp]_- = \mp F_\pm. \quad (3.1b) \\
[F_\pm, F_\mp]_+ &= K_\pm \quad \text{and} \quad [F_+, F_-]_+ = K_0. \quad (3.1c)
\end{align*}

The commutator and the anticommutator are respectively denoted $[\cdot, \cdot]_-$ and $[\cdot, \cdot]_+$. The use of the usual anticommutator $\{\cdot, \cdot\}$ is here avoided in order to prevent any confusion with the Poisson brackets.

It is quite remarkable that only the discrete series representations of $SU(1,1)$ extend to irreducible representations of $OSp(1/2)$ [16]. This observation encourages the study of the $OSp(1/2)$ CS as a natural extension of those of $SU(1,1)$. Each irreducible representation (IR) of $OSp(1/2)$ contains two IR of its subgroup $SU(1,1)$ (see (3.1a)). More precisely, each IR of $OSp(1/2)$ labelled by $\tau$, such that the Casimir operator $Q_2 \equiv \tau(\tau - \frac{1}{2})$, contains two discrete series representations of $SU(1,1)$, namely $D_+^{k=\tau}$ and $D_+^{k=\tau+\frac{1}{2}}$. The representation
space is spanned by the set of vectors \( \{|\tau, k = \tau, m\rangle, |\tau, k = \tau + \frac{1}{2}, m\rangle, m = k, k+1, \ldots \} \).

For more details concerning these representations we refer to [7] [16] and references therein.

The fiducial state is chosen as before to be the “highest weight” state \( |\tau, k = \tau, m = \tau\rangle \) defined through the equations:

\[
K_0 |\tau, k = \tau, m = \tau\rangle = \tau |\tau, k = \tau, m = \tau\rangle, \quad (3.2a)
K_- |\tau, k = \tau, m = \tau\rangle = 0 = F_- |\tau, k = \tau, m = \tau\rangle. \quad (3.2b)
\]

We will denote it \( |0\rangle \) as in section 2. The \( OSp(1/2) \) CS are then defined as follows [7],

\[
|z, \theta\rangle \equiv (1 - |z|^2)^\tau \left[ 1 - \frac{1}{2} \frac{\bar{\theta} \theta}{(1 - |z|^2)} \right]^\tau e^{zK_+ + \theta F_+} |0\rangle, \quad (3.3)
\]

here \( z \) is a complex number which clearly belongs to \( D^{(1)} \) and \( \theta \) is an odd coordinate, i.e. an anticommuting Grassmann number [1] [5]. Note that \( \bar{z} \) is the usual complex conjugate of \( z \) while \( \bar{\theta} \) is the so-called adjoint of \( \theta \) (denoted \( \theta^\# \) in [5]). This statement will be made clearer in section 5. Moreover, the expression in front of the exponential in (3.3) arises from a normalization procedure as in section 2. Because of (3.2) the pair \( (z, \theta) \) parametrizes a realization of the \( (2|2) \)-dimensional homogeneous superspace \( OSp(1/2)/U(1) \). This defines the superunit disc \( D^{(1|1)} \cong OSp(1/2)/U(1) \), the global geometric structure of which will be discussed in next section.

We are now ready to extend Onofri’s analysis to reveal the supersymplectic character of \( D^{(1|1)} \). As in section 2 we use the SCS in (3.3) to first define the following superpotential on \( D^{(1|1)} \),

\[
f(z, \bar{z}, \theta, \bar{\theta}) \equiv \log |\langle 0 |z, \theta\rangle|^{-2}
= -2\tau \log(1 - |z|^2) + \tau \frac{\bar{\theta} \theta}{(1 - |z|^2)}, \quad (3.4)
\]

from which we extract the even two-superform \( \omega \) on \( D^{(1|1)} \) given by,

\[
\omega \equiv -i\delta \bar{\delta} f(z, \bar{z}, \theta, \bar{\theta})
= -2i\tau \left[ 1 + \frac{1}{2} \frac{\theta \bar{\theta}}{(1 - |z|^2)^2} \right] \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2}
+ i\tau \frac{d\theta d\bar{\theta}}{(1 - |z|^2)^2}
+ i\tau \frac{d\bar{\theta} d\theta}{(1 - |z|^2)^2}
- i\tau \frac{\bar{z} d\bar{\theta} d\bar{z}}{(1 - |z|^2)^2}. \quad (3.5)
\]
In equation (3.5) \( \delta = dz \frac{\partial}{\partial z} + d\theta \frac{\partial}{\partial \theta} \) and \( \bar{\delta} = d\bar{z} \frac{\partial}{\partial \bar{z}} + d\bar{\theta} \frac{\partial}{\partial \bar{\theta}} \) such that the exterior superderivative \( d \equiv \delta + \bar{\delta} \). Note also that we use here Kostant’s conventions [4], according to which the exterior superalgebra on \( D^{(1|1)} \) has a \( \mathbb{Z}_+ \times \mathbb{Z}_2 \) bi-graded structure. The \( \mathbb{Z}_+ \) gradation is the usual gradation of exterior algebras while the \( \mathbb{Z}_2 \) one is the natural gradation of the Grassmann algebra. The commutation-anticommutation relations of superforms are governed by the following relation [4],

\[
\beta_1 \beta_2 = (-1)^{a_1 a_2 + b_1 b_2} \beta_2 \beta_1,
\]

where \( a_1 \) (\( b_1 \)) is the degree of the superform \( \beta_1 \) with respect to the \( \mathbb{Z}_+ \) (\( \mathbb{Z}_2 \)) gradation. For example, \( dz d\bar{z} = -d\bar{z} dz \) (this is clearly the usual wedge product), \( dz d\bar{\theta} = -d\bar{\theta} dz \) and \( d\theta d\bar{\theta} = d\bar{\theta} d\theta \). Using these conventions a straightforward calculation shows that \( \omega \) is closed, i.e. \( d\omega = 0 \). Hence, \( D^{(1|1)} \) is a supersymplectic supermanifold [1] [4] [14]. It is worth noting that different conventions based on another grading of the exterior superalgebra have been used by Berezin [1]. We will comment on this point in the concluding section.

The Berezin covariant symbols or the classical observables corresponding to the supergenerators of \( OSp(1/2) \) have already been evaluated in [7]. They are given by,

\[
K_0(z, \bar{z}, \theta, \bar{\theta}) \equiv \langle z, \theta | K_0 | z, \theta \rangle = \tau \left( \frac{1 + |z|^2}{1 - |z|^2} \right) \left[ 1 + \frac{1}{2} \left( \frac{\bar{\theta}}{1 - |z|^2} \right) \right],
\]

\[
K_+(z, \bar{z}, \theta, \bar{\theta}) \equiv \langle z, \theta | K_+ | z, \theta \rangle = 2\tau \left( \frac{z}{1 - |z|^2} \right) \left[ 1 + \frac{1}{2} \left( \frac{\bar{\theta}}{1 - |z|^2} \right) \right],
\]

\[
K_-(z, \bar{z}, \theta, \bar{\theta}) \equiv \langle z, \theta | K_- | z, \theta \rangle = 2\tau \left( \frac{\bar{z}}{1 - |z|^2} \right) \left[ 1 + \frac{1}{2} \left( \frac{\bar{\theta}}{1 - |z|^2} O(3, 2) \right) \right],
\]

\[
F_+(z, \bar{z}, \theta, \bar{\theta}) \equiv \langle z, \theta | F_+ | z, \theta \rangle = \tau \left( \frac{\bar{\theta} + \bar{z} \theta}{1 - |z|^2} \right),
\]

\[
F_-(z, \bar{z}, \theta, \bar{\theta}) \equiv \langle z, \theta | F_- | z, \theta \rangle = \tau \left( \frac{\theta + z \bar{\theta}}{1 - |z|^2} \right).
\]

Using equation (2.7), which applies equally well in the supersymplectic context [4], one can evaluate the Hamiltonian vector superfields corresponding to the observables given above. A lengthy but straightforward calculation gives:

\[
X_{K_0} = iz \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial \bar{z}} + \frac{i}{2} \theta \frac{\partial}{\partial \theta} - \frac{i}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}},
\]

\[
X_{K_+} = i \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial \bar{z}} - i\bar{\theta} \frac{\partial}{\partial \theta},
\]

\[
X_{K_-} = i \frac{\partial}{\partial \bar{z}} - i\bar{z} \frac{\partial}{\partial z} - i\theta \frac{\partial}{\partial \theta},
\]

\[
X_{F_+} = i \frac{\partial}{\partial \bar{z}} - i\bar{z} \frac{\partial}{\partial z} + i\bar{\theta} \frac{\partial}{\partial \theta},
\]

\[
X_{F_-} = i \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial \bar{z}} + i\theta \frac{\partial}{\partial \bar{\theta}}.
\]
\[ X_{K_-} = iz^2 \frac{\partial}{\partial z} - i \frac{\partial}{\partial \bar{z}} + iz\theta \frac{\partial}{\partial \theta}, \quad (3.8c) \]
\[ X_{F_+} = i \frac{\theta}{2} \frac{\partial}{\partial z} - \frac{i}{2} \bar{z}^\theta \frac{\partial}{\partial \bar{z}} - i \frac{\partial}{\partial \bar{\theta}} - iz \frac{\partial}{\partial \theta}, \quad (3.8d) \]
\[ X_{F_-} = i \frac{\bar{\theta}}{2} \frac{\partial}{\partial \bar{z}} - \frac{i}{2} \bar{z} \frac{\partial}{\partial \bar{z}} - iz \frac{\partial}{\partial \theta} - i \frac{\partial}{\partial \theta}. \quad (3.8e) \]

The same arguments which follow equation (2.6) still hold true here. For instance, \( OSp(1/2) \) acts, through the vector superfields found above, in a supersymplectic way on \( D^{(1|1)} \) and \( \omega \) defines a Poisson superbracket with respect to which the super observables in (3.7) form a supersymplectic realisation of \( osp(1/2) \). For example,
\[ \{K_0, F_\pm\} \equiv iX_{K_0} dF_\pm = \pm \frac{1}{2} F_\pm = -\{F_\pm, K_0\}, \quad (3.9a) \]
\[ \{F_+, F_-\} \equiv iX_{F_+} dF_- = K_0 = \{F_-, F_+\}. \quad (3.9b) \]

As in the non-super case the superJacobi identities are a direct consequence of the fact that \( \omega \) is closed [1] [4].

As in section 2, equations (3.7) define a supermoment map \( K^s \) as follows,
\[ K^s : D^{(1|1)} \to osp^*(1/2) \equiv osp(1/2) \]
\[ (z, \theta) \mapsto (K_0(z, \bar{z}, \theta, \bar{\theta}), K_i(z, \bar{z}, \theta, \bar{\theta}), F_i(z, \bar{z}, \theta, \bar{\theta})), \; i = 1, 2, \quad (3.10) \]
where \( K_+ = K_1 + iK_2, \; K_- = K_1 - iK_2, \; F_+ = F_1 + iF_2 \) and \( F_- = -i(F_1 - iF_2) \).
(Notice here the use of the notion of adjoint of a complex odd quantity, see section 5 for more details.) The \( (3|2) \)-vector on the right hand side of (3.10) spans a \( (2|2) \)-dimensional subsupermanifold in \( \mathbb{R}^{(3|2)} \), the equation of which is given by the identity,
\[ K_0^2(z, \bar{z}, \theta, \bar{\theta}) - K_1^2(z, \bar{z}, \theta, \bar{\theta}) - K_2^2(z, \bar{z}, \theta, \bar{\theta}) + 2F_1(z, \bar{z}, \theta, \bar{\theta})F_2(z, \bar{z}, \theta, \bar{\theta}) = \tau^2. \quad (3.11) \]
This is the equation of an \( OSp(1/2) \) supercoadjoint orbit. It is the supersymmetric extension of the upper component of the two-sheeted hyperboloid of section 2 (see eq. (2.11)). The superunit disc \( D^{(1|1)} \) can then be viewed as a superstereographic projection of the supercoadjoint orbit given in (3.11). Hence, \( OSp(1/2)/U(1) \cong (D^{(1|1)}, \omega) \) is a supercoadjoint orbit of \( OSp(1/2) \).

Finally, let us mention that all the equations derived in this section reduce to those of section 2 when one sets the odd parts to zero. This completely justifies our description of
the results of this section as supersymmetric extensions of those of section 2. In particular, the body of $\omega$ given in (3.5) is nothing but $\omega_0$ obtained in (2.5) with $k$ replaced by $\tau$.

4. Rothstein’s theorem and the superKähler character of the superunit disc

We want now to go a step further and show that $(D^{(1|1)}, \omega)$ fits within the general picture of a supersymplectic supermanifold recently depicted by Rothstein [14]. In order to make our statement understandable a brief survey of known results is mandatory. For more details see [1]-[4] and [17].

A $(p|q)$-dimensional supermanifold is a pair $(M, A_M)$ where $M$ is a $p$-dimensional manifold with structure sheaf $O_M$ and $A_M$ is a sheaf of supercommutative algebras ($A_M$ is called the superstructure sheaf of the supermanifold) such that: (a) $A_M/N$ is isomorphic to $O_M$, $N$ being the subsheaf of nilpotent elements of $A_M$ and (b) $E \equiv N/N^2$ is a locally free sheaf over $O_M$ such that $A_M$ is locally isomorphic to the exterior sheaf $\Lambda^\cdot E$. In other words (b) means that given an open cover $\{U_\alpha\}$ of $M$, the following map $\tau_\alpha$

$$\tau_\alpha : A_M(U_\alpha) \to O_M(U_\alpha) \otimes \Lambda^q \equiv \Lambda^\cdot E(U_\alpha) \quad (4.1)$$

is an isomorphism. Here $\Lambda^q$ is the exterior algebra on $R^q$ and $q$ is the odd dimension of $(M, A_M)$. (In the complex case $R$ should be replaced by $C$.) Local supercoordinates on $(M, A_M)$ are given by a set $(x^1, \ldots, x^p; \theta^1, \ldots, \theta^q)$ where $(x^1, \ldots, x^p)$ are local coordinates on $M$ and $(\theta^1, \ldots, \theta^q)$ form a basis of $E$ over $O_M$.

Batchelor’s theorem [13] states that the local isomorphism in (4.1) extends in a (non-canonical) way to a global one $A_M \cong \Lambda^\cdot E$. In other words, it shows that there exists a vector bundle $F$ over $M$ such that $A_M$ is (non-canonically) isomorphic to $\Gamma(M, \Lambda F)$, the sheaf of smooth sections of the exterior vector bundle $\Lambda F \to M$. More precisely this theorem applies only when $M$ is a $C^\infty$ or a real-analytic manifold. In these two cases the supermanifold is said to be split. When $M$ is a complex-analytic manifold the theorem may not hold true. This drawback shall not trouble us that much since it is sufficient for our purpose to use the fact that one can always associate a split supermanifold in the form $(M, \Gamma(M, \Lambda F))$ to any supermanifold $(M, A_M)$, for $M$ a $C^\infty$, a real-analytic, or a complex-analytic manifold. In fact in practice one always deals with supermanifolds of that form.
Rothstein’s theorem can be viewed as a supersymplectic extension of Batchelor’s theorem. A supersymplectic supermanifold is defined as a triple \((M, A_M, \omega)\), where \(\omega\) is a closed and non-degenerate even 2-superform on \((M, A_M)\). Rothstein’s theorem allows one to completely identify \(\omega\) in terms of a symplectic structure on \(M\) and extra structures in the vector bundle sector. More precisely, it states that to any supersymplectic supermanifold \((M, A_M, \omega)\) there corresponds a set \((M, \omega_0, E, g, \nabla)\), where \(\omega_0\) is a symplectic manifold, \(E\) is a vector bundle over \(M\) with metric \(g\) and \(g\)-compatible connection \(\nabla\), such that \(E\) is the sheaf of linear functionals on \(E\) and \(\omega\) is completely determined in terms of \((\omega_0, g, \nabla)\) as follows:

\[
\omega = \omega_0 + \frac{1}{2} g_{ab} R^b_{ijc} \theta^a \theta^c dx^i dx^j + g_{ab} D\theta^a D\theta^b. \tag{4.2}
\]

Here \(R^b_{ijc}\), for \(i, j \in \{1, \ldots, p = 2n\}\) and \(a, b \in \{1, \ldots, q\}\), are the components of the curvature of \(\nabla\), and \(D\) is an operator defined on \(\wedge E\) in terms of the components \(A^a_{ib}\) of \(\nabla\), namely

\[
D\theta^a \equiv d\theta^a + A^a_{ib} \theta^b dx^i \quad \text{and} \quad Dx^i = 0. \tag{4.3}
\]

In the \(C^\infty\) case this correspondence is one-to-one [14]. Subsequently, Rothstein’s data will refer to the set \((M, \omega_0, E, g, \nabla)\).

In what follows we will identify Rothstein’s data for the supersymplectic superunit disc \((D^{(1|1)}, \omega)\) obtained in the previous section. This will allow us to characterize \((D^{(1|1)}, \omega)\) using usual geometric structures. Clearly \(M\) in our case is nothing but the unit disc \(D^{(1)}\) and \(\omega_0\) is its \(SU(1,1)\)-invariant symplectic form given in (2.5) with \(k\) replaced by \(\tau\). (From now on \(\omega_0 \equiv \omega_0(k = \tau)\).) The complete identification of Rothstein’s data for \((D^{(1|1)}, \omega)\) is given in the following theorem.

**Theorem:** The \(OSp(1/2)\)-invariant supersymplectic structure of the superunit disc \(D^{(1|1)}\) is completely characterized by the set \(\left(D^{(1)}, \omega_0, T\!D^{(1)}, g \equiv \tau \frac{dz \otimes dz}{(1 - |z|^2)}, \nabla_g\right)\), where \(T\!D^{(1)}\) is the holomorphic tangent bundle of \(D^{(1)}\) and \(\nabla_g\) is the (unique) Hermitian connection associated to the (clearly) Hermitian and \(SU(1,1)\)-invariant metric \(g\).

**Proof:** The proof is omitted since it consists of a straightforward verification. Using the ingredients in the theorem one has to show that \(\omega\) in (3.5) is given by the formula (4.2). In doing so one must make a small change in (4.2), namely multiplying the terms at the right.
of $\omega_0$ by $i$. This is a natural change for the complex context. In fact, the formula (4.2) was originally derived in the real case. One can also easily verify that $g$ is $SU(1, 1)$-invariant, in other words that the vector fields $X_{K_0}$ and $X_{K_\pm}$ given in (2.8) are Killing vector fields for $g$, i.e. $\mathcal{L}_{X_{K_0}}(g) = 0 = \mathcal{L}_{X_{K_\pm}}(g)$. □

The metric $g$ above is in fact more than Hermitian. It is Kählerian. Indeed the associated 2-form $\omega_g \equiv -i\tau \frac{dz \wedge d\bar{z}}{(1 - |z|^2)}$ on $D^{(1)}$ is a Kähler form since $\omega_g$ is clearly closed. Hence, the $OSp(1/2)$-invariant supersymplectic structure $\omega$ on $D^{(1,1)}$ is completely determined given the two Kähler structures $\omega_0$ and $\omega_g$ on the unit disc $D^{(1)}$. This observation allows us to define the following general notion of a superKähler supermanifold.

**Definition:** A superKähler supermanifold $(M, \mathcal{A}_M, \omega)$ is a supersymplectic supermanifold, the supersymplectic structure of which is completely determined given two Kähler 2-forms $\omega_1$ and $\omega_2$ on $M$, such that Rothstein’s data is $(M, \omega_1, TM, g_{\omega_2}, \nabla_{g_{\omega_2}})$, where $g_{\omega_2}$ is the Kähler metric on $M$ associated to $\omega_2$ and $\nabla_{g_{\omega_2}}$ is the Hermitian connection associated to $g_{\omega_2}$.

Alternatively one can define the superKähler structure in terms of Kähler potentials $f_1$ and $f_2$, which give rise to $\omega_1$ and $\omega_2$, respectively. For $(D^{(1,1)}, \omega)$, $\omega_g$ can be obtained, as $\omega_0$ in (2.4)-(2.5), from a Kähler potential $f_g$. More precisely,

$$\omega_g = -i\partial\bar{\partial}f_g, \quad \text{where} \quad f_g \equiv \tau Li_2(|z|^2) \quad (4.4a)$$

and $Li_n(x)$ is the polylogarithmic function defined by the series

$$Li_n(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^n}, \quad 0 \leq x \leq 1. \quad (4.4b)$$

$Lii(x)$ is the so-called dilogarithmic function. Moreover, notice that the Kähler potential $f_0$, which gives rise to $\omega_0$ in (2.5), is nothing but $2\tau Li_1(|z|^2)$. Hence, one can completely characterize $(D^{(1,1)}, \omega)$ by the triplet $(D^{(1)}, f_0 = 2\tau Li_1(|z|^2), f_g = \tau Li_2(|z|^2))$. It is worth noting that a Kähler form does not specify uniquely a Kähler potential. Clearly, two Kähler potentials which differ only by a holomorphic or/and antiholomorphic function produce the same Kähler form.

In order to complete the present picture let us address the question of the existence of a superKähler potential. We have a good candidate for this, namely the superfunction
\( f(z, \bar{z}, \theta, \bar{\theta}) \) on \( \mathcal{D}^{(1|1)} \) which is at the origin of \( \omega \) in (3.4)-(3.5). This defines the notion of a superKähler potential. It can actually be related to Rothstein’s data. Indeed, noting that \( g_{zz} = \frac{\tau}{2} (1 - |z|^2)^{-1} = g_{\bar{z}\bar{z}} \), one can rewrite (3.4) in the following way:

\[
f(z, \bar{z}, \theta, \bar{\theta}) = f_0(z, \bar{z}) + 2g_{\bar{z}\bar{\theta}} \frac{\partial z}{\partial \bar{z}} f_g(z, \bar{z}) \bar{\theta}.
\] (4.5)

Or equivalently, using (4.4a),

\[
f(z, \bar{z}, \theta, \bar{\theta}) = f_0(z, \bar{z}) + \partial_z \partial_{\bar{z}} f_g(z, \bar{z}) \bar{\theta}.
\] (4.6)

From this observation one can state the following Lemma.

**Lemma:** Rothstein’s data for the superKähler superunit disc \( \mathcal{D}^{(1|1)} \) can be read off directly from the superKähler potential \( f \).

**Proof:** It consists of direct computations based on the observation made above. \( \square \)

On the other hand one can construct a superKähler potential for \( \omega \) in (3.5) given Rothstein’s data as in the definition above. These results generalize in a straightforward way to any superKähler supermanifold.

Let us close this section by a final remark. As previously stated, Rothstein’s theorem allows the description of supersymplectic supermanifolds in usual geometric terms. It seems that one can realize the invariance of supersymplectic forms under supergroups in terms of the invariance of the associated Rothstein’s data under usual groups within the even part of the supergroup in question. Actually, notice in our case, the \( OSp(1/2) \)-invariance of \( \omega \) in (3.5) versus the \( SU(1,1) \)-invariance of both \( \omega_0 \) and \( g \). This interesting question as well as those we have gathered in section 6 will be addressed in a forthcoming publication.

5. Darboux-Kostant coordinates

Darboux-Kostant’s theorem [4] is the supersymmetric generalization of Darboux’s theorem. Given a \((2n|q)\)-dimensional supersymplectic supermanifold \((M, A_M, \omega)\), it states that for any open neighbourhood \( U \) of some point \( m \) in \( M \) there exists a set \((q^1, \ldots, q^n, p_1, \ldots, p_n; \xi^1, \ldots, \xi^q)\) of local coordinates on \( \wedge \mathcal{E}(U) \) such that \( \omega \) on \( U \) can be written in the following form,

\[
\omega|_U \equiv \tilde{\omega} = \sum_{i=1}^n dp_i \wedge dq^i + \sum_{a=1}^q \frac{\epsilon_a}{2} (d\xi^a)^2, \quad \epsilon_a = \pm 1.
\] (5.1)
This reflects the fact that any supersymplectic supermanifold is locally a supersymplectic vector superspace. The coordinates in (5.1) are called Darboux-Kostant coordinates. Our aim in this section is to show that $(\mathcal{D}^{(1|1)}, \omega)$ satisfies Darboux-Kostant’s theorem. To this end, we use the following contraction-inspired trick. (The full contraction $OSp(1/2) \rightarrow H(2|2)$, $H(2|2)$ being the super Heisenberg group of the extended plane $\mathbb{R}^{(2|2)}$, as well as other contractions, will be addressed elsewhere.) Instead of investigating locally $\mathcal{D}^{(1|1)}$, we introduce a length scale $r$ with respect to which the coordinates $z$ and $\theta$ become length-like quantities, then we take the limit $r \rightarrow \infty$. In other words, this procedure amounts to rescaling $z$ in order to make the unit disc $\mathcal{D}(1)$ into a disc of radius $\sqrt{2}r$. In the $r \rightarrow \infty$ limit, this unit disc becomes the whole plane. For consistency, one needs to rescale $\theta$ also, such that in the same limit $\mathcal{D}^{(1|1)}$ becomes $\mathbb{R}^{(2|2)}$. Hence, the rescaled coordinates become in the limit $r \rightarrow \infty$ the Darboux-Kostant coordinates on $\mathcal{D}^{(1|1)}$. Explicitly, let $z'$ and $\theta'$ be the rescaled coordinates such that,

\[
z' = \sqrt{2} rz \quad \text{and} \quad \theta' = r \theta.
\]

Take also $\tau = r^2$. Rewriting $\omega$ in terms of $z'$ and $\theta'$ and taking the limit $r \rightarrow \infty$ we find,

\[
\tilde{\omega} \equiv \lim_{r \rightarrow \infty} \omega(z', \theta', \tau = r^2) = -idz' \wedge d\bar{z}' + id\theta' d\bar{\theta}'.
\]

Clearly this is a closed nondegenerate even 2-superform on $\mathbb{R}^{(2|2)}$. One needs now to rewrite (5.3) in terms of “real” coordinates. Let us define $q$, $p$, $\xi^1$ and $\xi^2$ as follows,

\[
z' = \frac{1}{\sqrt{2}}(q + ip), \quad \bar{z}' = \frac{1}{\sqrt{2}}(q - ip),
\]

and

\[
\theta' = \frac{1}{\sqrt{2}}(\xi^1 + i\xi^2), \quad \bar{\theta}' = -\frac{i}{\sqrt{2}}(\xi^1 - i\xi^2).
\]

Then,

\[
\tilde{\omega} = dp \wedge dq + \frac{1}{2}(d\xi^1)^2 + \frac{1}{2}(d\xi^2)^2;
\]

and is of the form (5.1). Finally, notice that in (5.4b) we have used the (unusual) notion of the adjoint of a complex Grassmann number discussed in [5]. There it is denoted by a sharp ($\theta^\#$) instead of a bar ($\bar{\theta}$). Hence, an odd Grassmann number is real when $\theta^\# \equiv \bar{\theta} = -i\theta$. This is to be compared with the notion of Hermitian conjugate of an odd
quantum operator obtained by Rothstein as a by product of his quantization of a simple example of a supersymplectic supermanifold [14]. The notion of reality we have used here is clearly different from that in [4]. On the other hand we could have considered the usual notion of reality and of complex conjugate, namely $\xi^1 + i\xi^2 \equiv \xi^1 - i\xi^2$. In doing so, one needs to use another form for (5.1). More precisely, $\epsilon_a = \pm i$ (see for instance [18]).

6. Discussion, conclusions and prospects

Here we discuss some of the results obtained in this work and describe their possible generalizations. We also briefly state other results.

.\textbf{OSp}(N/2). First let us mention that $D^{(1|1)} \cong OSp(1/2)/U(1)$ is not the only possible supersymmetric extension of the unit disc $D^{(1)} \cong SU(1,1)/U(1)$. In fact in the same way $SU(1,1)$ admits several supersymmetric extensions in the form of the orthosymplectic supergroups $OSp(N/2)$, the unit disc possesses several susy extensions which are homogeneous superspaces of the previous supergroups (we use here the fact that $SU(1,1)$ is isomorphic to $Sp(2,\mathbb{R})$). More precisely, simple arguments show that $OSp(N/2)/O(N) \times U(1) \cong D^{(1|N)}$ is a $(2|2N)$-dimensional supermanifold that supersymmetrically extends $D^{(1)}$. The knowledge of $OSp(N/2)$ CS will make the explicit identification of $D^{(1|N)}$ easy. As a consequence notice that we should have called $D^{(1|1)}$ the $N = 1$ superunit disc. The $N = 2$ case will be considered in a forthcoming publication. This will provide us with a supersymplectic supermanifold beyond the superKähler setup.

.\textbf{OSp}(N/2n). Clearly, superKähler homogeneous superspaces of $OSp(N/2)$ occur only when $N = 1$. This is the superunit disc $D^{(1|1)}$. A natural question then arises: when is a homogeneous superspace of $OSp(N/2n)$ superKähler? A purely dimensional argument based on the known classification of Kähler homogeneous bounded domains of type II (see [10] and references therein), which are the homogeneous spaces $Sp(2n,\mathbb{R})/U(n) \cong D^{(n^2+n)}$, helps one to conjecture the following: $OSp(N/2n)/O(N) \times U(n) \cong D^{(n^2+n|Nn)}$ is superKähler when $n = 2N - 2$. In other words, $D^{(2N^2-N|2N^2-N)} \cong OSp(N/4N-2)/O(N) \times U(2N-1)$ is a $(2N(2N-1)|2N(2N-1))$-dimensional superKähler homogeneous superspace of $OSp(N/4N-2)$ for $N \in \mathbb{N}$. This superKähler series starts with the $N = 1$ superunit disc $D^{(1|1)}$; for $N = 2$ we obtain the $(12|12)$-dimensional bounded superdomain $D^{(6|6)}$; etc.
Quantizations. Because of the $OSp(1/2)$-invariance of its supersymplectic structure, the superunit disc $D^{(1|1)}$ constitutes the perfect arena for investigating or testing supersymmetric extensions of the known quantization procedures. The simplest one to consider in the present context, is clearly the CS-based one, namely the Berezin quantization [12]. Extension of geometric quantization should also be tested in the present situation. With respect to this, another result of Rothstein [14] is of great help in achieving the prequantization. It states that the expression at the right of $\omega_0$ in (4.2) is an exact superform, i.e. $\omega = \omega_0 + d\alpha$, $\alpha$ being an even 1-superform. This allows one to prequantize $(M, A_M, \omega)$ whenever a prequantum map exists for $(M, \omega_0)$. This is actually the case for $(D^{(1)}, \omega_0)$. The prequantization of $(D^{(1|1)}, \omega)$ is then within reach. We will report on this elsewhere. Note that the (super)prequantization program is discussed in [4] and applied in [14] to a simple example. For completeness one should also consider $\ast$-quantization.

Berezin’s conventions. In the present work we have used Kostant’s conventions [4] (see eq.(3.6)). There exist other conventions, namely Berezin’s conventions. A variant of these latter is used in BRST formalism [18]. Berezin’s conventions are based on the following two points: (a) a simple grading of the superalgebra of exterior superforms which is induced from the Grassmann $Z_2$ parity (the Grassmann parity of an even (odd) 0-form being 0 (1)), and (b) the odd character of the exterior superderivative $d$. This last point means that when acting on a $p$-superform of Grassmann parity $\epsilon$, $d$ produces a $p+1$-superform of parity $\epsilon + 1$. The Leibniz rule is as follows $d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^{\omega_1}\omega_1 (d\omega_2)$. Moreover, if $r^A$ are coordinates of the supermanifold of parity $\epsilon_A$, we have [1]: $r^A r^B = (-1)^{\epsilon_A \epsilon_B} r^B r^A$, $r^A dr^B = (-1)^{\epsilon_A (\epsilon_B + 1)} dr^B r^A$ and $dr^A dr^B = (-1)^{(\epsilon_A + 1)(\epsilon_B + 1)} dr^B dr^A$. For example $dz d\bar{z} = -d\bar{z} dz$, $dz d\bar{\theta} = d\bar{\theta} dz$ and $d\theta d\bar{\theta} = d\bar{\theta} d\theta$. Using these conventions we can reproduce all the computations of section 3. The results are different, since Berezin and Kostant’s exterior superalgebras are not isomorphic [3]. The closed two-superform turns out to be:

$$\omega = -2i\tau \left[ 1 + \frac{1}{2} \bar{\theta} \frac{(1 + |z|^2)}{(1 - |z|^2)} \right] \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} - i\tau \frac{d\theta d\bar{\theta}}{(1 - |z|^2)^2} + i\tau \frac{\theta \bar{z}}{(1 - |z|^2)^2} d\theta d\bar{\theta} + i\tau \frac{z \bar{\theta}}{(1 - |z|^2)^2} dz d\bar{\theta}. \quad (6.1)$$

However, equation (2.7) is not consistent with Berezin’s conventions. In order to be able to show that the observables given in (3.7) form a supersymplectic realization of $osp(1/2)$,
as was the case within Kostant’s conventions in (3.9), one needs to modify equation (2.7) in the following way:

\[ X_H \equiv \begin{cases} 
  dH & \text{when } H \text{ is even, i.e. } \epsilon_H = 0 \\
  -dH & \text{when } H \text{ is odd, i.e. } \epsilon_H = 1 
\end{cases} \quad (6.2) \]

Using this equation one shows that the Hamiltonian vector superfields associated with the observables in (3.7) are exactly those in (3.8). Finally, equation (3.9) still holds true.

**SuperMeasure.** One of the most important properties of coherent states is the so-called resolution of identity [10]. It reflects the fact that CS constitute an overcomplete basis of the Hilbert space. The explicit form of this property involves an integration over phase space. A measure on that space is then needed. After lengthy computations based on a generalization of Bogoliubov transformations, which can only be used when one considers the harmonic oscillator representation of $OSp(1/2)$, Balantekin et al. were able to produce such a measure on $D^{(1|1)}$ [7]. Using Berezin’s notion of a density [1][3], our geometric approach allows us to determine this measure in a straightforward way. Indeed, up to a multiplicative constant, this measure is given by the formula,

\[ d\mu(z, \bar{z}, \theta, \bar{\theta}) \propto \sqrt{s\text{det} \| \omega_{AB} \|} \frac{dz \, d\bar{z}}{2\pi} \frac{d\theta \, d\bar{\theta}}{1 + \frac{1}{2} (1 - |z|^2)}, \quad (6.3) \]

Here $s\text{det} \| \omega_{AB} \|$ is the superdeterminant or Berezenian of the supermatrix $\| \omega_{AB} \|$, for $\omega$ given in (3.5) and $\omega = dr^A \omega_{AB} dr^B$. A direct computation gives,

\[ d\mu(z, \bar{z}, \theta, \bar{\theta}) \propto \frac{1}{\pi} \frac{1}{(1 - |z|^2)} \left[ 1 + \frac{1}{2} \frac{\bar{\theta} \theta}{(1 - |z|^2)} \right] dz \, d\bar{z} \, d\theta \, d\bar{\theta}, \quad (6.4) \]

which is the measure obtained in [7]. Starting with $\omega$ in (6.1) and repeating the above computations one obtains the same measure on $D^{(1|1)}$, except from the differences in the commutation-anticommutation relations of the 1-superforms.

**Integral curves.** The integration of the flows of the Hamiltonian vector superfields given in (3.8) amounts to solving a system of nonlinear ordinary superdifferential equations. These are super-Riccati equations, the resolution of which has been fully studied in [19].
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