On Kippenhahn curves and higher-rank numerical ranges of some matrices *

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Abstract

The higher rank numerical ranges of generic matrices are described in terms of the components of their Kippenhahn curves. Cases of tridiagonal (in particular, reciprocal) 2-periodic matrices are treated in more detail.

1. Introduction

Let $M_n$ stand for the algebra of all $n$-by-$n$ matrices with the entries $a_{ij} \in \mathbb{C}$, $i, j = 1, \ldots, n$. We will identify $A \in M_n$ with a linear operator acting on $\mathbb{C}^n$, the latter being equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|x\| := \langle x, x \rangle^{1/2}$. The numerical range of $A$ is defined as

$$W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}, \quad (1.1)$$

see e.g. [10, Chapter 1] or more recent [7, Chapter 6] for the basic properties of $W(A)$, in particular its convexity and invariance under unitary similarities.

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In [6], this notion was generalized as follows: the rank-$k$ numerical range of $A$ is

$$\Lambda_k(A) = \{ \lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P \}. \tag{1.2}$$

Of course,

$$W(A) = \Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_n(A). \tag{1.3}$$

For $k > n/2$ the set $\Lambda_k(A)$ is empty or a singleton $\{\lambda_0\}$; in the latter case $\lambda_0$ is an eigenvalue of $A$ having geometric multiplicity at least $2k - n$ [6, Proposition 2.2]. In particular, $\Lambda_n(A) \neq \emptyset$ if and only if $A$ is a scalar multiple of the identity, and then all the sets in (1.3) coincide.

So, for $k = 1$ and $k > n/2$ the sets $\Lambda_k(A)$ are convex. Their convexity for intermediate values of $k$ was established in [16]. Shortly thereafter, in [15] it was shown that, moreover,

$$\Lambda_k(A) = \bigcap_{\theta \in [0,2\pi)} \{ \mu \in \mathbb{C} : \Re(e^{i\theta} \mu) \leq \lambda_k(\theta) \}, \tag{1.4}$$

where $\lambda_k(\theta)$ stands for the $k$-th largest (counting the multiplicities) eigenvalue of the matrix $\Re(e^{i\theta} A)$. As usual, for any $X \in \mathbb{M}_n$

$$\Re X = \frac{X + X^*}{2}, \quad \Im X = \frac{X - X^*}{2i}.$$  

When applied to normal matrices, (1.4) yields

$$\Lambda_k(N) = \cap \text{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}, \tag{1.5}$$

with the intersection taken over all $(n-k+1)$-tuples from the spectrum $\sigma(N)$ of a normal matrix $N$. This result is also from [15], confirming a conjecture from [6].

Our next observation is that the boundary lines

$$\ell_{\theta,k} = \{ \mu \in \mathbb{C} : \Re(e^{i\theta} \mu) = \lambda_k(\theta) \} \tag{1.6}$$

of the half-planes in the right hand side of (1.4), when taken for all $k = 1, \ldots, n$, form a family the envelope of which is the so called Kippenhahn curve $C(A)$ of the matrix $A$. It was shown in [13] (see also the English translation [14]) that $W(A) = \text{conv} \ C(A)$. From the discussion above it is
clear that, at least in principle, not only $W(A)$ but all the rank-$k$ numerical ranges of $A$ can be described in terms of $C(A)$.

Section 2 is devoted to generic matrices, for which $C(A)$ splits into $\lceil n/2 \rceil$ components, each solely responsible for the respective higher rank numerical range. These results are specified further in Section 3 for the case of tridiagonal 2-periodic matrices, when explicit formulas for $\lambda_k(\theta)$ are known. Finally, a particular case of reciprocal 2-periodic matrices is treated in Section 4.

2. Generic matrices

For $n = 2$, there are only two sets in the chain (1.3), both easily identifiable. If $n = 3$, the middle term is either a singleton or the empty set (since $2 > 3/2$). The next proposition allows to distinguish between the two possibilities.

**Proposition 1.** Let $A \in \mathbf{M}_3$. Then $\Lambda_2(A) \neq \emptyset$ if and only if $W(A)$ is an elliptical disk, possibly degenerating into a line segment.

**Proof.** Directly from the definition it follows that $\Lambda_2(A)$ is a singleton $\{\lambda\}$ if and only if $A$ is unitarily similar to $\begin{bmatrix} \lambda & 0 & x \\ 0 & \lambda & y \\ u & v & z \end{bmatrix}$. Applying another unitary similarity if needed, we may without loss of generality suppose that $u = 0$.

Case 1. $x = 0$. Then $A = (\lambda) \oplus B$, where $B = \begin{bmatrix} \lambda & y \\ v & z \end{bmatrix}$, and $W(A) = W(B)$ is either an elliptical disk or a line segment, depending on whether or not $B$ is normal.

Case 2. $x \neq 0$. Then $A$ is unitarily similar to the tridiagonal matrix $\begin{bmatrix} \lambda & x & 0 \\ 0 & z & v \\ 0 & y & \lambda \end{bmatrix}$ with (1, 2)- and (2, 1)-entries having distinct absolute values. According to [3, Lemma 8], $A$ is unitarily irreducible. On the other hand, its (1, 1)- and (3, 3)-entries coincide, which implies the ellipticity of $W(A)$ [4, Theorem 4.2].

Recall that a matrix $A \in \mathbf{M}_n$ is generic if $\lambda_1(\theta), \ldots, \lambda_n(\theta)$ are distinct for all $\theta$.

Normal matrices are not generic; for $n = 2$ the converse is also true. Hence, there is a direct relation with the shape of the numerical range: $A \in$
\( M_2 \) is generic if and only if \( W(A) \) is a non-degenerate elliptical disc. Already for \( n = 3 \), things get more subtle.

**Proposition 2.** Let \( A \in M_3 \). Then \( A \) is generic if and only if \( W(A) \):

(i) has an ovular shape, or

(ii) is an ellipse with no eigenvalues of \( A \) lying on its boundary.

Note that \( A \) is unitarily irreducible in case (i) while it may or may not be unitarily reducible (though not normal) in case (ii).

**Proof.** If \( A \) is unitarily irreducible, according to [12, Proposition 3.2] it is generic if and only if \( W(A) \) has no flat portions on the boundary. These are exactly ovular and elliptical shapes, as per Kippenhahn’s classification. Moreover, unitary irreducibility of \( A \) implies that its eigenvalues are not on the boundary.

Normal matrices are not generic, as was mentioned earlier. In the remaining case, \( W(A) \) is the convex hull of an ellipse \( E \) and a normal eigenvalue \( \lambda \) of \( A \). The matrix is generic if \( \lambda \) lies in the interior of \( E \), which falls under (ii), and non-generic otherwise. \( \square \)

Comparing Propositions [1] and [2] we see that for \( A \in M_3 \) non-empty and empty \( \Lambda_2(A) \) materialize both for generic and non-generic matrices.

**Example 1.** Let

\[
M_1 = \begin{bmatrix} 0 & -1/2 & 0 \\ 2 & 0 & -1/2 \\ 0 & 1/2 & \sqrt{2} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Figure 7 refers to the matrix \( M_1 \) and Figure 8 refers to the matrix \( M_2 \). Observe that \( W(M_1) \) is ovular, \( \Lambda_2(M_1) = \emptyset \), while \( W(M_2) \) is elliptical and \( \Lambda_2(M_2) = \{0\} \) is the eigenvalue of \( M_2 \) different from the foci \( \pm 3/2 \) of \( W(M_2) \).

Returning to generic matrices of arbitrary dimension \( n \), note that from their definition it immediately follows that

\[
\lambda_k(\theta) = -\lambda_{n-k+1}(\theta + \pi), \quad k = 1, \ldots, n. \quad (2.1)
\]

Since \( \lambda_{n-k+1}(\theta) > \lambda_k(\theta) \) for \( k > \lceil n/2 \rceil \), the half-planes corresponding to \( \theta \) and \( \theta + \pi \) in (1.4) are disjoint. Therefore, the rank-\( k \) numerical ranges of generic matrices \( A \) are empty for \( k > \lceil n/2 \rceil \). On the other hand, directly
from (1.4) we see that for generic matrices $A$ the inclusions in (1.3) are proper for $k = 1, \ldots, \lceil n/2 \rceil$; moreover, $\Lambda_{k+1}(A)$ lies in the interior of $\Lambda_k(A)$.

The structure of $C(A)$ and the related description of $\Lambda_k(A)$ for $k \leq \lceil n/2 \rceil$ are as follows.

**Theorem 3.** For a generic matrix $A \in \mathbb{M}_n$ its Kippenhahn curve $C(A)$ consists of the closed components

$$\gamma_k(A) = \{ \langle Az_k(\theta), z_k(\theta) \rangle : \theta \in [0, 2\pi] \}, \quad k = 1, \ldots, \lceil n/2 \rceil,$$

where $z_k(\theta)$ is the unit eigenvector associated with the eigenvalue $\lambda_k(\theta)$ of $\text{Re}(e^{i\theta}A)$. Respectively, the half-planes in the representation (1.4) of $\Lambda_k(A)$ are bounded by the family (1.6) of the tangent lines of $\gamma_k(A)$.

The first statement is a rewording (in different terms) of [11, Theorem
13], based in particular on (2.1); the second immediately follows from the first.

For \( n \) odd and \( k = \lceil \frac{n}{2} \rceil \) from (1.4), (2.1) it can be seen that in fact \( \Lambda_k(A) \) is the intersection of the tangent lines \( l_{\theta,k} \) to \( \gamma_k(A) \) defined by (1.6). This yields the following test for distinguishing between \( \Lambda_{\lceil \frac{n}{2} \rceil} \) being a singleton or the empty set.

**Corollary 1.** Let \( A \in \mathbf{M}_n \) be generic. If \( n \) is odd, then \( \Lambda_{\lceil \frac{n}{2} \rceil}(A) = \gamma_{\lceil \frac{n}{2} \rceil}(A) \) if \( \gamma_{\lceil \frac{n}{2} \rceil}(A) \) is a point, and \( \Lambda_{\lceil \frac{n}{2} \rceil}(A) = \emptyset \) otherwise.

Both cases are illustrated by Example 1.

Corollary 1 implies that for odd \( n \) the curve \( \gamma_{\lfloor \frac{n}{2} \rfloor}(A) \) cannot be convex unless it collapses to a single point. On the other hand, the outermost curve \( \gamma_1(A) \) of \( C(A) \) for a generic matrix \( A \) is always convex, and thus coincides with the boundary \( \partial W(A) \) of its numerical range. This means in particular that \( \partial W(A) \) does not have corners or flat portions. Other components of \( C(A) \) may exhibit cusps and swallowtails but no inflection points.

As can be seen from Fig. 1 cusps (but not swallowtails) materialize already when \( n = 3 \). The emergence of swallowtails will be demonstrated in Section 4; see Fig. 5–9.

Convexity of \( \gamma_1(A) \) implies that the subsequent components lie strictly inside of it. This, however, does not preclude \( \gamma_j(A) \) with \( j > 1 \) from intersecting, as soon as there are at least two of them (i.e., when \( n \geq 5 \) — see Fig. 3 in Section 3 for an example corresponding to \( n = 5 \)). Note that this is happening in spite of strict inclusions in (1.3).

### 3. Tridiagonal 2-periodic matrices

A matrix \( A \in \mathbf{M}_n \) is tridiagonal if \( a_{ij} = 0 \) whenever \(|i - j| > 1\). We will be making use of the well known (and easy to prove) recursive relation for the determinants \( \Delta_n \) of such matrices,

\[
\Delta_n = a_{nn} \Delta_{n-1} - a_{n-1,n} a_{n,n-1} \Delta_{n-2},
\]

implying in particular that \( \Delta_n \) is invariant under transpositions \( a_{i+1,i} \leftrightarrow a_{i,i+1} \) of its off-diagonal pairs.

Suppose now that these pairs are unbalanced, i.e.,

\[
|a_{i+1,i}| \neq |a_{i,i+1}| \quad \text{for } i = 1, \ldots, n - 1.
\]
Then hermitian matrices \( \text{Re}(e^{i\theta}A) \) will be \textit{proper} tridiagonal, i.e., their entries directly above and below the main diagonal will be non-zero. According to [3, Corollary 7], the eigenvalues of \( \text{Re}(e^{i\theta}A) \) are simple for all \( \theta \), thus implying the genericity of \( A \).

**Example 2.** Let

\[
M_3 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1/4 & 2 & 1/2 & 0 & 0 \\
0 & 1/4 & 0 & 3/4 & 0 \\
0 & 0 & 1/4 & -2 & 1 \\
0 & 0 & 0 & 1/4 & -1
\end{bmatrix}.
\]

This matrix is generic, since (3.2) holds. According to Corollary [1] \( \Lambda_3 = \emptyset \).

Figure 3: Kippenhahn curve of \( M_3 \). Notice that \( \gamma_2 \) intersects \( \gamma_3 \).

We will say that a tridiagonal matrix \( A \) is \textit{2-periodic} if so are the sequence of its diagonal entries and of its (non-ordered) off-diagonal pairs. For such matrices we will use the notation \( a_1, a_2 \) for the first two diagonal entries, and \( \{b_1, c_1\}, \{b_2, c_2\} \) for the first two (once again, non-ordered) pairs of the off-diagonal entries.

Along with \( A \), for any \( \theta \) the hermitian matrix \( \text{Re}(e^{i\theta}A) \) will be 2-periodic as well, with \( \alpha_j(\theta) =: \text{Re}(e^{i\theta}a_j) \) \((j = 1, 2)\) as the period of its main diagonals. Transposing their off-diagonal pairs as needed, we may arrange for the superdiagonal to also be 2-periodic, with

\[
\beta_j(\theta) =: (e^{i\theta}b_j + e^{-i\theta}c_j)/2, \quad j = 1, 2
\]

as the first two entries. According to (3.1), this rearrangement preserves the characteristic polynomial of \( \text{Re}(e^{i\theta}A) \). Therefore, explicit formulas from [9] can be used to compute \( \lambda_k(\theta) \) in our setting. The respective straightforward computation shows that

\[
\lambda_{k,n-k+1} = \frac{\alpha_1 + \alpha_2}{2} \pm \sqrt{\left( \frac{\alpha_1 - \alpha_2}{2} \right)^2 + |\beta_1|^2 + |\beta_2|^2 + 2|\beta_1\beta_2| Q_k}
\]
for \( k = 1, \ldots, m := \lfloor n/2 \rfloor \), while \( \lambda_{m+1} = \alpha_1 \) if \( n \) is odd.

Here \( Q_k = \cos \frac{k\pi}{m+1} \) if \( n \) is odd, and the \( k \)-th (in the decreasing order) root of the \( m \)-th degree polynomial \( q_m \) defined recursively via

\[
q_0 = 1, \quad q_1(\mu) = \mu + |\beta_2/\beta_1|, \quad q_{k+1}(\mu) = \mu q_k(\mu) - q_{k-1}(\mu) \quad \text{for} \quad k \geq 1 \quad (3.5)
\]

if \( n \) is even.

For odd \( n \), directly from the formula for \( \lambda_{m+1} \) we obtain

**Proposition 4.** Let \( A \in M_n \) be tridiagonal and 2-periodic. If \( n \) is odd, then \( \gamma_{\lfloor n/2 \rfloor}(A) = \{a_1\} \), the (1,1)-entry of \( A \).

According to Corollary \( \text{[1]} \) for such matrices \( \Lambda_{\lfloor n/2 \rfloor}(A) = \{a_1\} \). Also, by Proposition \( \text{[4]} \) a 2-periodic tridiagonal matrix \( A \in M_5 \) cannot have intersecting \( \gamma_2 \) and \( \gamma_3 \). For \( n = 6 \), however, this becomes a possibility; see Fig. \( \text{[8]} \) in Section \( \text{[4]} \).

The parameters \( Q_k \) are explicit and constant when \( n \) is odd, and implicit (and in general depending on \( \theta \)) if \( n \) is even. This makes consideration of even-sized matrices much harder. However, in the case

\[
\beta_1 c_2 = c_1 \beta_2 \quad (3.6)
\]

treated in \( \text{[1]} \), the ratio \( |\beta_2/\beta_1| \) is the same as \( |b_2/b_1| \) and thus \( \theta \)-independent. According to \( (3.5) \), \( Q_k \) then do not depend on \( \theta \) for even \( n \) as well. Formulas \( (3.4) \), with some additional nontrivial computations, provide an alternative approach to the complete description of rank-\( k \) numerical ranges of 2-periodic tridiagonal matrices satisfying \( (3.6) \). In agreement with \( \text{[1]} \), they all happen to be elliptical disks.

Condition \( (3.5) \) holds in particular for tridiagonal Toeplitz matrices. If in addition either the super- or the subdiagonal vanishes, then the dependence on \( \theta \) disappears in \( (3.4) \) altogether. In other words, \( \gamma_k \) are then concentric circles, and \( \Lambda_k(A) \) the respective circular disks. This covers the result on shift operators from \( \text{[8]} \).

**Example 3.** To illustrate other possible shapes of Kippenhahn curves for 2-periodic tridiagonal matrices, let \( M_4 \in M_7 \) have the zero main diagonal and \( b_1 = 3, b_2 = 6, c_1 = c_2 = 2 \).

See the next section for more specific examples.
4. Reciprocal matrices

Recall the notion of reciprocal matrices introduced in [2]. These are tridiagonal matrices with constant (without loss of generality, zero) main diagonal and the off diagonal pairs satisfying $a_{i+1,i}, a_{i,i+1} = 1$. Reciprocal matrices are of course proper tridiagonal. Denoting $|a_{j+1,j}|^2 + |a_{j,j+1}|^2 =: 2A_j$ we see that $A_j \geq 1$. Condition (3.2) for such matrices takes the form $A_j > 1$, $j = 1, \ldots, n - 1$.

A 2-periodic reciprocal matrix $A$ is completely characterized by its size $n$ and the values $a_1 := |a_{12}|, a_2 := |a_{23}|$ (alternatively, by $A_1$ and $A_2$). For $n \geq 4$ (the only interesting setting), $\text{Im } A$ has multiple eigenvalues if $A_1$ or $A_2$ is equal to one, and so conditions $A_1, A_2 > 1$ are not only sufficient but also necessary for $A$ to be generic.

Moreover, for reciprocal matrices (3.3) yields $|\beta_j| = \sqrt{(A_j + \tau)/2}$, where $\tau = \cos(2\theta)$. So, according to (3.4) $\lambda_{k,n-k+1}$ in this case are the square roots of

$$\zeta_k = \frac{1}{2}(A_1 + A_2 + 2\tau) + \sqrt{(A_1 + \tau)(A_2 + \tau)}Q_k, \quad j = 1, \ldots, m.$$ (4.1)

Observe that the right hand side of (4.1) is invariant under the substitutions $\theta \mapsto -\theta$ and $\theta \mapsto \theta + \pi$. Thus, we arrive at the following

**Corollary 2.** Let $A \in M_n$ be a 2-periodic reciprocal matrix. Then each component $\gamma_1, \ldots, \gamma_m$ of its Kippenhahn curve $C(A)$, and consequently its rank-$k$ numerical ranges $\Lambda_k(A)$ for $k = 1, \ldots, m$, are symmetric with respect to both horizontal and vertical coordinate axes. Also, $\gamma_{m+1} = \Lambda_{m+1} = \{0\}$ if $n$ is odd.
Furthermore, $\gamma_k$ is an ellipse if and only if $\zeta_k = x\tau + y$ with some constant $y > x > 0$. If $A_1 = A_2 := A$, this happens to be the case for all $k$, since then

$$\zeta_k = (A + \tau)(1 + Q_k),$$

with $Q_k$ constant (note that (3.6) holds in a trivial way). So, the rank-$k$ numerical ranges of such matrices are elliptical disks with the boundaries $\{\gamma_k\}_{k=1}^m$ forming a family of nested ellipses whose axes are coincident with the coordinate axes.

On the contrary, when $A_1 \neq A_2$ we have

**Theorem 5.** Let $A$ be a 2-periodic reciprocal matrix of odd size $n$ and $A_1 \neq A_2$. Then none of its rank-$k$ numerical ranges has an elliptical shape if $n = 1 \mod 4$. Otherwise, exactly one of them, namely $\Lambda_{(n+1)/4}(A)$, is an elliptical disk.

**Proof.** The first summand in the right hand side of (4.1) is of desired form. The second term, however, is such only if $Q_k = 0$. Since $Q_k = \cos \frac{k\pi}{m+1}$ for odd $n$, the result follows. \hfill $\Box$

Observe that for generic 4-by-4 matrices $\gamma_1$ and $\gamma_2$ (consequently, $A_1$ and $A_2$) are elliptical only simultaneously. Recall also that the numerical range of a reciprocal matrix $A \in \mathbf{M}_4$ is elliptical if and only if

$$A_2 = \phi A_1 - \phi^{-1} A_3 \text{ or } A_2 = \phi A_3 - \phi^{-1} A_1,$$

(4.2)

where $\phi$ is the golden ratio, and at least one of the inequalities $A_j \geq 1$ is strict [2, Theorem 7]. If $A$ in addition is 2-periodic, i.e. $A_1 = A_3$, then (4.2) implies $A_2 = A_1$. In other words, neither of rank-$k$ numerical ranges of such $A$ is elliptical, unless $A_1 = A_2$.

We suspect that this is the case for generic 2-periodic reciprocal matrices $A \in \mathbf{M}_n$ for all even $n > 2$, not just $n = 4$. Formulas (4.1) should be instrumental in proving this conjecture; the difficulty lies in the implicit nature of $Q_k$ for even values of $n$.

Kippenhahn curves of several reciprocal matrices are pictured below. The matrices are described by the triples $\{n, |a_1|, |a_2|\}$, or $\{n, A_1, A_2\}$. In Fig. 7, 8 and 10 the dotted curves are the best fitting ellipses to the components of $C(A)$ which look elliptical but in fact are not.
Figure 5: $n = 4, a_1 = 2, a_2 = 21/20$. The numerical range $\Lambda_1$ is bounded by the exterior component, while $\Lambda_2$ is bounded by the interior component with its swallowtails removed; $\Lambda_3 = \emptyset$.

Figure 6: $n = 5, a_1 = 2, a_2 = 21/20$. The picture is similar to Fig. 5 except that now $\Lambda_3 = \{0\}$. 

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Figure 7: $n = 6, A_1 = 1.25, A_2 = 1.5$. The components of $C(A)$ are nested, with $\gamma_1$ and $\gamma_2$ being convex and so coinciding with the boundaries of $\Lambda_1, \Lambda_2$, respectively. On the other hand, $\Lambda_3$ is bounded by the “middle portion” of $\gamma_3$.

Figure 8: $n = 6, A_1 = 1.05, A_2 = 1.62$. The component $\gamma_1$ and $\gamma_2$ are still convex. As opposed to Fig. 7, $\gamma_3$ is intersecting with $\gamma_2$.

Figure 9: $n = 7, A_1 = 1.05, A_2 = 1.62$. The picture is similar to Fig. 8 except that $\gamma_2$ is an exact ellipse, and there emerges $\gamma_4 = \{0\}$.
Figure 10: $n = 7, A_1 = 2, A_2 = 1.5$. The components $\gamma_j$ are convex for $j = 1, 2, 3$ and visually indistinguishable from ellipses, though only the middle one is a genuine ellipse.

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