MATRIX ALGEBRAS WITH A CERTAIN COMPRESSION PROPERTY II

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Abstract. A subalgebra $A$ of $\mathbb{M}_n(\mathbb{C})$ is said to be idempotent compressible if $EAE$ is an algebra for all idempotents $E \in \mathbb{M}_n(\mathbb{C})$. Likewise, $A$ is said to be projection compressible if $PAP$ is an algebra for all orthogonal projections $P \in \mathbb{M}_n(\mathbb{C})$. In this paper, a case-by-case analysis is used to classify the unital projection compressible subalgebras of $\mathbb{M}_n(\mathbb{C})$, $n \geq 4$, up to transposition and unitary equivalence. It is observed that every algebra shown to admit the projection compression property is, in fact, idempotent compressible. We therefore extend the findings of [3] in the setting of $\mathbb{M}_3(\mathbb{C})$, proving that the two notions of compressibility agree for all unital matrix algebras.

§1 Introduction

Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices with complex entries. In [3], the notions of idempotent compressibility and projection compressibility were defined for subalgebras of $\mathbb{M}_n$. In particular, a subalgebra $A$ of $\mathbb{M}_n$ was said to be idempotent compressible if the corner $EAE$ is an algebra for all idempotents $E \in \mathbb{M}_n$. Analogously, $A$ was said to be projection compressible if the corner $PAP$ is an algebra for all orthogonal projections $P \in \mathbb{M}_n$.

It is immediate from the definitions that every idempotent compressible subalgebra of $\mathbb{M}_n$ is also projection compressible, though the converse is much less clear. When $n \leq 2$, dimension considerations that every algebra is idempotent compressible—hence projection compressible—though this fact does not hold for $n \geq 3$. In [3], however, it was shown that every unital subalgebra of $\mathbb{M}_3$ with the projection compression property is in fact, idempotent compressible. Furthermore, a complete description of unital subalgebras of $\mathbb{M}_3$ that admit these properties was obtained up to transposition and similarity [3, Theorem 6.0.1].

The goal of this paper is to extend the results of [3] to higher dimensional settings. Specifically, we wish to obtain a classification of the unital subalgebras of $\mathbb{M}_n$, $n \geq 4$, that admit the projection compression property, and investigate whether or not this notion agrees with that of idempotent compressibility.

Several subalgebras of $\mathbb{M}_n$, $n \geq 4$, are known to exhibit the idempotent compression property. For example, if $A$ is the intersection of a left ideal and a right ideal, then $A$ is idempotent compressible [3, Corollary 2.0.11]. Algebras of this form are known as LR-algebras, and are exactly the algebras of the form $A = PM_nQ$ for some projections $P$ and $Q$ in $\mathbb{M}_n$ [3, Corollary 2.0.10].

The following example showcases three additional collections of algebras that exhibit the idempotent compression property.

Example 1.0.1. [3, Examples 3.1.1, 3.1.3, 3.1.6] Let $n \geq 4$ be an integer, and let $Q_1$, $Q_2$, and $Q_3$ be projections in $\mathbb{M}_n$ that sum to $I$. In what follows, all matrices are expressed with respect to the decomposition $\mathbb{C}^n = \text{ran}(Q_1) \oplus \text{ran}(Q_2) \oplus \text{ran}(Q_3)$.

(i) The algebra

$A = CQ_1 + CQ_3 + (Q_1 + Q_2)M_n(Q_2 + Q_3)$

$= \left\{ \begin{pmatrix} \alpha I & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & \beta I \end{pmatrix} : \alpha, \beta \in \mathbb{C}, M_{ij} \in \mathbb{M}_n \right\}$

is idempotent compressible.
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(iii) If \( \text{rank}(Q_1) = \text{rank}(Q_2) = 1 \), then the algebra

\[
\mathcal{A} = \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)M_nQ_3
\]

\[
= \begin{cases}
\alpha & 0 & M_{13} \\
0 & \beta & M_{23} \\
0 & 0 & \gamma I
\end{cases} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in Q_iM_nQ_j
\]

is idempotent compressible.

Our main result, Theorem 7.1.1, states that for every integer \( n \geq 4 \), the algebras from Example 1.0.1 together with the unitization of \( \mathcal{L}R \)-algebras described above, form an exhaustive list of unital projection compressible subalgebras of \( M_n \) up to transposition and similarity. Since each algebra in this collection is known to be idempotent compressible, it will follow that a unital matrix algebra is projection compressible if and only if it is idempotent compressible.

As in [3], a case-by-case analysis will be used to obtain the classification of unital projection compressible algebras described above. The requisite results from [7] concerning the structure theory for matrix algebras will be reintroduced in §2. In §3, we present a necessary condition for projection compressibility (Theorem 3.0.1) that imposes significant restrictions on the structure of a projection compressible algebra. As we shall see, the algebras that satisfy this condition can be grouped into three distinct types determined by their block upper triangular forms. The unital projection compressible algebras of each type will be classified up to transposition and similarity in sections §4-6, and ultimately up to transposition and unitary equivalence in §7.

§2 Preliminaries

We will begin by reintroducing the notation, definitions, and preliminary results from [3] surrounding idempotent and projection compressibility. Additionally, we will present some of the key results from [7] concerning the structure theory for matrix algebras.

Notation. Given vectors \( x, y \in \mathbb{C}^n \), define \( x \otimes y^* : \mathbb{C}^n \to \mathbb{C}^n \) to be the rank-one operator \( z \mapsto \langle z, y \rangle x \).

Observe that if \( \mathcal{A} \) is a subalgebra of \( M_n \) and \( E \in M_n \) is an idempotent, then \( EAE \) is always a linear space. Thus, \( EAE \) is an algebra if and only if it is multiplicatively closed. By dimension considerations, \( EAE \) be an algebra for all idempotents \( E \) of rank 1.

Definition 2.0.1. [3] Definition 2.0.2 Given a subset \( \mathcal{A} \) of \( M_n \), we define the transpose and anti-transpose of \( \mathcal{A} \) to be

\[
\mathcal{A}^T := \{ A^T : A \in \mathcal{A} \} \quad \text{and} \quad \mathcal{A}^{\alpha T} := \{ J A^T J : A \in \mathcal{A} \},
\]

respectively, where \( J \) denotes the anti-diagonal unitary matrix whose \( (i, j) \)-entry is \( \delta_{j,n-i+1} \). We say that two subalgebras \( \mathcal{A} \) and \( \mathcal{B} \) of \( M_n \) are transpose similar (resp. transpose equivalent) if \( \mathcal{B} \) is similar (resp. unitarily equivalent) to \( \mathcal{A} \) or \( \mathcal{A}^T \).

Since the set of idempotents in \( M_n \) is closed under transpose similarity, so too is the set of all idempotent compressible subalgebras of \( M_n \). Likewise, the set of projection compressible subalgebras of \( M_n \) is closed under transpose equivalence. From this it follows that for a given algebra \( \mathcal{A} \), either \( \mathcal{A} \), \( \mathcal{A}^T \), and \( \mathcal{A}^{\alpha T} \) are all idempotent (resp. projection) compressible, or none of them are.

Finally, if an algebra \( \mathcal{A} \) is idempotent (resp. projection) compressible, then so too is its unitization \( \tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}I \) [3] Proposition 2.0.6. The converse, however, is false.

The classification of unital projection compressible subalgebras of \( M_n \), \( n \geq 4 \), will require much of structure theory for matrix algebras applied in the analysis from [3]. Thus, it will be important to recall the following.
Definition 2.0.2. [7 Definition 9] A subalgebra $A$ of $M_n$ is said to have a reduced block upper triangular form with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 + \mathcal{V}_2 + \cdots + \mathcal{V}_m$ if

(i) when expressed as a matrix, every $A$ in $\mathcal{A}$ has the form

$$A = \begin{bmatrix}
    A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\
    0 & A_{22} & A_{23} & \cdots & A_{2m} \\
    0 & 0 & A_{33} & \cdots & A_{3m} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & A_{mm}
\end{bmatrix}$$

with respect to this decomposition, and

(ii) for each $i$, the subalgebra $A_{ii} := \{ A_{ii} : A \in \mathcal{A} \}$ is irreducible. That is, either $A_{ii} = \{0\}$ and $\dim \mathcal{V}_i = 1$, or $A_{ii} = M_{\dim \mathcal{V}_i}$.

An application of Burnside's Theorem [2] shows that every subalgebra $A$ admits a reduced block upper triangular form with respect to some orthogonal decomposition of $\mathbb{C}^n$. Moreover, one may verify that if $S$ is an invertible matrix that is block upper triangular with respect to the same decomposition as that of $A$, then $S^{-1}AS$ also has a reduced block upper triangular form with respect to this decomposition.

Theorem 2.0.3. [7 Corollary 14] If a subalgebra $A$ of $M_n$ has a reduced block upper triangular form with respect to a decomposition $\mathbb{C}^n = \mathcal{V}_1 + \mathcal{V}_2 + \cdots + \mathcal{V}_m$, then the set $\{1, 2, \ldots, m\}$ can be partitioned into disjoint subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that

(i) If $i \in \Gamma_s$ and $A_{ii} \neq \{0\}$, then there exists $G^{<i>}$ in $\mathcal{A}$ such that $G^{<i>}_{jj} = I_{\mathcal{V}_j}$ for all $j \in \Gamma_s$, and $G^{<i>}_{jj} = 0$ for all $j \notin \Gamma_s$.

(ii) If $i$ and $j$ belong to the same $\Gamma_s$, then $\dim \mathcal{V}_i = \dim \mathcal{V}_j$, and there is an invertible linear map $S_{ij} : \mathcal{V}_i \rightarrow \mathcal{V}_j$ such that

$$A_{ii} = S_{ij}^{-1}A_{jj}S_{ij}$$

for all $A \in \mathcal{A}$.

(iii) If $i$ and $j$ do not belong to the same $\Gamma_s$, then

$$\{(A_{ii}, A_{jj}) : A \in \mathcal{A}\} = \{A_{ii} : A \in \mathcal{A}\} \times \{A_{jj} : A \in \mathcal{A}\}.$$
If an algebra $\mathcal{A}$ in reduced block upper triangular form with respect to some decomposition of $\mathbb{C}^n$ is such that $\mathcal{A} = BD(\mathcal{A}) + \text{Rad}(\mathcal{A})$, we say that $\mathcal{A}$ is unhinged with respect to this decomposition.

We emphasize that the transformation of an algebra $\mathcal{A}$ into an unhinged reduced block upper triangular form described in Theorem 2.0.3 can be achieved via application of a block upper triangular similarity, but not, in general, via unitary equivalence. Additionally, we note that if $\mathcal{A}$ is in reduced block upper triangular form and $BD(\mathcal{A}) = CI$, then Theorem 2.0.3 implies that $\mathcal{A} = CI + \text{Rad}(\mathcal{A})$. Thus, $\mathcal{A}$ is unhinged with respect to any decomposition in which it admits a reduced block upper triangular form.

§3 A STRATEGY FOR CLASSIFICATION

In this section we will develop a strategy for characterizing the unital subalgebra of $\mathbb{M}_n$ that admit the projection compression property. By the comments preceding Theorem 2.0.3 we may assume that all algebras under consideration are expressed in reduced block upper triangular form with respect to some orthogonal direct sum decomposition of $\mathbb{C}^n$.

We begin by presenting a simple structural requirement for a unital subalgebra of $\mathbb{M}_n$, $n \geq 4$, to admit the projection compression property. This result and its corollaries impose substantial restrictions on the reduced block upper triangular form of a projection compressible algebra.

**Theorem 3.0.1.** Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a projection compressible subalgebra of $\mathbb{M}_n$. Suppose there exist mutually orthogonal projections $P_1$ and $P_2$ in $\mathbb{M}_n$ such that $\min(\text{rank}(P_1), \text{rank}(P_2)) \geq 2$ and $P_2AP_1 = \{0\}$. Then $P_1AP_1 = \mathbb{C}P_1$ or $P_2AP_2 = \mathbb{C}P_2$.

**Proof.** First assume that $\text{rank}(P_1) = \text{rank}(P_2) = 2$. By replacing $\mathcal{A}$ with the compression $(P_1 + P_2)\mathcal{A}(P_1 + P_2)$ if necessary, we may also assume that $P_1 + P_2 = I$.

Arguing by contradiction, suppose that $P_1AP_1 \notin \mathbb{C}P_1$ and $P_2AP_2 \notin \mathbb{C}P_2$. There then exists an operator $A \in \mathcal{A}$ such that $P_1AP_1 \notin \mathbb{C}P_1$ for each $i \in \{1, 2\}$. Indeed, choose operators $A_1, A_2 \in \mathcal{A}$ such that $P_1A_1P_1 \notin \mathbb{C}P_1$ and $P_2A_2P_2 \notin \mathbb{C}P_2$. If $P_2A_1P_2 \in \mathbb{C}P_2$ or $P_1A_2P_1 \notin \mathbb{C}P_1$, then $A_1$ or $A_2$ will satisfy the above requirements. Otherwise, $A := A_1 + A_2$ will suffice.

Thus, assume that $A \in \mathcal{A}$ has been chosen such that $P_1AP_1 \notin \mathbb{C}P_1$ and $P_2AP_2 \notin \mathbb{C}P_2$. For each $i \in \{1, 2\}$, choose an orthonormal basis $\{e_i^{(1)}, e_i^{(2)}\}$ for $\text{ran}(P_i)$ such that $P_iAP_i$ is not diagonal with respect to $B = \{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)}\}$. By permuting the basis vectors if necessary, we may assume that $\langle Ae_2^{(i)}, e_1^{(i)} \rangle \neq 0$ for each $i \in \{1, 2\}$.

Consider the matrix

$$Q := \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}$$

written with respect to $B$. It is straightforward to check that $\frac{1}{2}Q$ is a projection in $\mathbb{M}_4$ and every $B \in QAQ$ satisfies $\langle Be_2^{(1)}, e_1^{(1)} \rangle = 0$. With $A$ as above, however,

$$\langle (QAQ)^2e_2^{(1)}, e_1^{(2)} \rangle = 8\langle Ae_2^{(1)}, e_1^{(1)} \rangle \langle Ae_2^{(2)}, e_1^{(2)} \rangle \neq 0.$$  

Thus, $(QAQ)^2$ does not belong to $QAQ$, so $QAQ$ is not an algebra. This contradicts the assumption that $\mathcal{A}$ is projection compressible.

Now consider the general case in which each $P_i$ has rank at least 2. One may deduce from the above analysis that for some $i \in \{1, 2\}$, every rank-two subprojection $P \leq P_i$ is such that $PAP = \mathbb{C}P$. It then follows that $P_iAP_i = \mathbb{C}P_i$, as required. 

As we shall see in the coming analysis, Theorem 3.0.1 has significant implications for the classification of projection compressible algebras. Additionally, it highlights a major difference between the classification in this setting and that of $\mathbb{M}_3$. Since $\mathbb{M}_3$ cannot contain projections $P_1$ and $P_2$ as described in Theorem 3.0.1, this result may help to explain why there exist certain projection compressible subalgebras of $\mathbb{M}_3$ that do not admit analogues in higher dimensions (see Examples 3.2.1, 3.2.4, 3.2.7).

The following corollaries to Theorem 3.0.1 provide a more explicit description of the reduced block upper triangular forms that can exist for a unital projection compressible algebra.
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Corollary 3.0.2. Let \( n \geq 4 \) be an integer, and let \( \mathcal{A} \) be a unital subalgebra of \( M_n \). Suppose that there is an orthogonal decomposition \( \bigoplus_{i=1}^{m} V_i \) of \( \mathbb{C}^n \) with respect to which

(i) \( \mathcal{A} \) is reduced block upper triangular, and

(ii) there is an index \( k \in \{1, 2, \ldots, m\} \) such that if \( Q_1, Q_2, \) and \( Q_3 \) denote the orthogonal projections onto \( \bigoplus_{i<k} V_i, V_k, \) and \( \bigoplus_{i>k} V_i \) respectively, then

\[
(Q_1 + Q_2)A(Q_1 + Q_2) \neq \mathbb{C}(Q_1 + Q_2) \quad \text{and} \quad (Q_2 + Q_3)A(Q_2 + Q_3) \neq \mathbb{C}(Q_2 + Q_3).
\]

If \( \mathcal{A} \) is projection compressible, then \( k \) is unique. When this is the case, \( Q_1A\mathcal{Q}_1 = \mathbb{C}Q_1 \) and \( Q_3A\mathcal{Q}_3 = \mathbb{C}Q_3 \).

Proof. Assume that \( \mathcal{A} \) is projection compressible. Suppose to the contrary that there were two distinct indices \( k, k' \) together with corresponding projections \( Q_1', Q_2', Q_3' \) such that

\[
(Q_1' + Q_2')A(Q_1' + Q_2') \neq \mathbb{C}(Q_1' + Q_2') \quad \text{and} \quad (Q_2' + Q_3')A(Q_2' + Q_3') \neq \mathbb{C}(Q_2' + Q_3').
\]

Assume without loss of generality that \( k < k' \). The projections \( P_1 := Q_1 + Q_2 \) and \( P_2 := Q_2' + Q_3' \) then satisfy the hypotheses of Theorem 3.0.1 so \( P_1AP_1 = \mathbb{C}P_1 \) or \( P_2AP_2 = \mathbb{C}P_2 \). This is a contradiction.

The final claim follows immediately from the uniqueness of \( k \). Indeed, if \( Q_1A\mathcal{Q}_1 \neq \mathbb{C}Q_1 \), then \( k - 1 \) would be another such index. If instead \( Q_3A\mathcal{Q}_3 \neq \mathbb{C}Q_3 \), then one could derive a similar contradiction by considering the index \( k + 1 \).


The following special case of Corollary 3.0.2 describes the situation for algebras whose block-diagonal contains a block of size at least 2.

Corollary 3.0.3. Let \( n \geq 4 \) be an integer, and let \( \mathcal{A} \) be a unital subalgebra of \( M_n \). Suppose that there is a decomposition \( \bigoplus_{i=1}^{m} V_i \) of \( \mathbb{C}^n \) with respect to which

(i) \( \mathcal{A} \) is reduced block upper triangular, and

(ii) there is an index \( k \in \{1, 2, \ldots, m\} \) such that \( \dim V_k \geq 2 \).

If \( \mathcal{A} \) is projection compressible, then \( k \) is unique. When this is the case, if \( Q_1, Q_2, \) and \( Q_3 \) denote the orthogonal projections onto \( \bigoplus_{i<k} V_i, V_k, \) and \( \bigoplus_{i>k} V_i \) respectively, then

\[
Q_1A\mathcal{Q}_1 = \mathbb{C}Q_1, \quad Q_2A\mathcal{Q}_2 = \mathbb{C}Q_2, \quad \text{and} \quad Q_3A\mathcal{Q}_3 = \mathbb{C}Q_3.
\]

The results presented above provide a strategy for classifying the unital subalgebras of \( M_n \) that exhibit the projection compression property. Indeed, we may use Corollaries 3.0.2 and 3.0.3 to partition the unital subalgebras of \( M_n \) into the following three distinct types determined by their reduced block upper triangular forms:

Type I: \( \mathcal{A} \) has a reduced block upper triangular form with respect to an orthogonal decomposition of \( \mathbb{C}^n \) such that there does not exist an index \( k \) as in Corollary 3.0.2.

Type II: \( \mathcal{A} \) has a reduced block upper triangular form with respect to an orthogonal decomposition of \( \mathbb{C}^n \) such that \( BD(\mathcal{A}) \) contains a block of size at least 2 (i.e., there is an integer \( k \) as in Corollary 3.0.3).

Type III: For each orthogonal decomposition of \( \mathbb{C}^n \) with respect to which \( \mathcal{A} \) is reduced block upper triangular, every block in \( BD(\mathcal{A}) \) is \( 1 \times 1 \), and there is an integer \( k \) as in Corollary 3.0.2.

The unital projection compressible algebras of type I, II, and III will be analysed in §4, §5, and §6, respectively. In each case, a classification of these algebras will be obtained up to transpose similarity by examining the structure their semi-simple and radical parts.

§4 Algebras of Type I

In what follows, the term type I will be used to describe a unital subalgebra \( \mathcal{A} \) of \( M_n \), \( n \geq 4 \), that has a reduced block upper triangular form with respect to an orthogonal decomposition \( \bigoplus_{i=1}^{m} V_i \) of \( \mathbb{C}^n \), such that there does not exist an integer \( k \) as in Corollary 3.0.2. If \( \mathcal{A} \) is such an algebra, then it must be the case that \( \dim V_i = 1 \) for all \( i \) (i.e., \( m = n \)). For instance, the algebra from Example 1.0.1(i) is of type I if and only if \( Q_2 = 0 \); or \( \text{rank}(Q_2) = 1 \) and \( Q_1 = 0 \) for some \( i \in \{1, 3\} \).
The goal of this section is to determine which type I algebras possess the projection compression property. As we shall see, the type I algebras satisfying this condition are either unitizations of LR-algebras, or unitarily equivalent to the type I algebra from Example 1.0.1(i). In order to demonstrate this systematically, it will be useful to keep a record of the orthogonal decompositions of \( \mathbb{C}^n \) with respect to which \( \mathcal{A} \) satisfies the definition of type I.

**Definition 4.0.1.** If \( \mathcal{A} \) is an algebra of type I, let \( \mathcal{F}_I = \mathcal{F}_I(\mathcal{A}) \) denote the set of pairs \( \Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i) \), where

(i) \( \bigoplus_{i=1}^n \mathcal{V}_i \) is an orthogonal decomposition of \( \mathbb{C}^n \) with respect to which \( \mathcal{A} \) is reduced block upper triangular, and

(ii) \( d \) is an integer in \( \{1, 2, \ldots, n\} \) such that if \( Q_{1\Omega} \) denotes the orthogonal projection onto \( \bigoplus_{i=1}^d \mathcal{V}_i \), and \( Q_{2\Omega} \) denotes its complement \( I - Q_{1\Omega} \), then

\[
Q_{1\Omega}AQ_{1\Omega} = CQ_{1\Omega} \quad \text{and} \quad Q_{2\Omega}AQ_{2\Omega} = CQ_{2\Omega}.
\]

**Notation.** If \( \mathcal{A} \) is a type I algebra and \( \Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i) \) is a pair in \( \mathcal{F}_I(\mathcal{A}) \), the notation \( n_{1\Omega} = d \) and \( n_{2\Omega} = n - d \) will be used to refer to the ranks of \( Q_{1\Omega} \) and \( Q_{2\Omega} \), respectively.

Suppose that \( \mathcal{A} \) is a projection compressible algebra of type I and \( \Omega \) is a pair in \( \mathcal{F}_I(\mathcal{A}) \). In the language of §2, each corner \( Q_{1\Omega}AQ_{1\Omega} = CQ_{1\Omega} \) is a diagonal algebra comprised of mutually linked \( 1 \times 1 \) blocks. Note that the blocks in \( Q_{1\Omega}AQ_{1\Omega} \) may or may not be linked to those in \( Q_{2\Omega}AQ_{2\Omega} \). If these blocks are linked, we will say that the projections \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are **linked**. Otherwise, we will say that \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are **unlinked**. Note that the projections \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are linked for some pair in \( \Omega \in \mathcal{F}_I(\mathcal{A}) \) if and only if they are linked for every pair in \( \mathcal{F}_I(\mathcal{A}) \).

It will be important to distinguish between the type I algebras whose projections are linked and those whose projections are unlinked. The projection compressible type I algebras with unlinked projections will be classified in 4.1, while those with linked projections will be classified in 4.2. Before our analysis splits, however, let us examine one extreme case that will be relevant to the classification in either setting.

Observe that if \( \mathcal{A} \) is an algebra of type I and \( \mathcal{F}_I(\mathcal{A}) \) contains a pair \( \Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i) \) with \( d = n \), then \( \mathcal{A} = C1 \), and hence \( \mathcal{A} \) is idempotent compressible. If instead \( d = 1 \) or \( d = n - 1 \), then Proposition 10.3 indicates that \( \mathcal{A} \) is the unitization of an LR-algebra. The proof of this result relies on the following structure theorem for \( \mathbb{M}_n \)-modules, which will be applied frequently throughout our analysis. For reference, see [6] Theorem 3.3).

**Theorem 4.0.2.** Let \( n \) and \( p \) be positive integers.

(i) If \( S \subseteq \mathbb{M}_{n \times p} \) is a left \( \mathbb{M}_n \)-module, then there is a projection \( Q \in \mathbb{M}_p \) such that \( S = \mathbb{M}_{n \times p}Q \).

(ii) If \( S \subseteq \mathbb{M}_{p \times n} \) is a right \( \mathbb{M}_n \)-module, then there is a projection \( Q \in \mathbb{M}_p \) such that \( S = Q\mathbb{M}_{p \times n} \).

**Proposition 4.0.3.** Let \( \mathcal{A} \) be a type I subalgebra of \( \mathbb{M}_n \). If there is a pair \( \Omega = (d, \bigoplus_{i=1}^n \mathcal{V}_i) \) in \( \mathcal{F}_I(\mathcal{A}) \) with \( d = 1 \) or \( d = n - 1 \), then \( \mathcal{A} \) is the unitization of an LR-algebra, and hence \( \mathcal{A} \) is idempotent compressible.

**Proof.** Assume that \( \mathcal{F}_I(\mathcal{A}) \) contains a pair \( \Omega = (n-1, \bigoplus_{i=1}^n \mathcal{V}_i) \). By Theorem 2.0.3 there exists an invertible upper triangular matrix \( S \) such that \( \mathcal{A}_0 := S^{-1}AS \) is unhinged with respect to \( \bigoplus_{i=1}^n \mathcal{V}_i \). Thus, since the class of LR-algebras is invariant under similarity, it suffices to prove that \( \mathcal{A}_0 \) is the unitization of an LR-algebra.

Note that by Theorem 4.0.2 there is a subprojection \( Q'_1 \leq_Q 1_{\Omega} \) such that

\[
Q_{1\Omega}A_0Q_{2\Omega} = Q_{1\Omega}Rad(A_0)Q_{2\Omega} = Q'_1M_nQ_{2\Omega}.
\]

Thus, either \( \mathcal{V}_n \) is linked to the other \( \mathcal{V}_i \)'s, in which case \( A_0 = Q'_1M_nQ_{2\Omega} + CI \); or \( \mathcal{V}_n \) is not linked to the other \( \mathcal{V}_i \)'s, in which case \( A_0 = (Q'_1 + Q_{2\Omega})M_nQ_{2\Omega} + CI \). In either scenario, \( A_0 \) is the unitization of an LR-algebra.

Suppose instead that \( \mathcal{F}_I(\mathcal{A}) \) contains a pair whose first entry is 1. It follows that \( \mathcal{F}_I(A^{aT}) \) contains a pair whose first entry is \( n - 1 \). The above analysis then shows that \( A^{aT} \) is the unitization of an LR-algebra, and thus so too is \( \mathcal{A} \).
§4.1 Type I Algebras with Unlinked Projections. In this section we consider the type I algebras \( A \) for which the pairs \( \Omega = (\alpha, \beta) \) in \( \mathcal{F}(A) \) are such that \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are unlinked. In light of Proposition 4.0.3 and its preceding remarks, we may assume that \( 1 < d < n - 1 \) for all pairs \( \Omega \). Thus, if \( \Omega \) is any such pair, then \( \min(d, n - d) \geq 2 \). That is, the corresponding projections \( Q_{1\Omega} \) and \( Q_{2\Omega} \) have ranks \( n_{1\Omega} \geq 2 \) and \( n_{2\Omega} \geq 2 \), respectively.

It will be shown in Theorem 4.1.9 that every projection compressible type I algebra satisfying the above assumptions is unitarily equivalent to the type I algebra from Example 1.0.1(i). The majority of the work leading to this classification, however, occurs in Lemma 4.1.6. The proof of Lemma 4.1.6 itself relies on several intermediate results concerning the structure of the radical of a projection compressible type I algebra.

It should be noted that while Lemmas 4.1.1, 4.1.2, and 4.1.3 are presented here in the context of type I algebras with unlinked projections, these results are also applicable to type I algebras whose projections are linked.

**Lemma 4.1.1.** Let \( A \) be a projection compressible type I subalgebra of \( \mathbb{M}_n \), and suppose that \( \Omega = (d, \bigoplus_{i=1}^{n} \mathcal{V}_i) \) is a pair in \( \mathcal{F}(A) \) with \( 1 < d < n - 1 \). Suppose further that there are orthonormal bases \( \{ e_i^{(1)} \}_{i=1}^{n_{1\Omega}} \) for \( \text{ran}(Q_{1\Omega}) \) and \( \{ e_i^{(2)} \}_{i=1}^{n_{2\Omega}} \) for \( \text{ran}(Q_{2\Omega}) \), as well as indices \( i_0 \) and \( j_0 \) such that

\[
\langle \text{Re}_{j_0}, e_{i_0}^{(1)} \rangle = 0 \quad \text{for all} \quad R \in \text{Rad}(A).
\]

Then \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are linked, and either \( \langle \text{Re}_{j_0}, e_{k}^{(2)} \rangle = 0 \) for all \( k \in \{1, 2, \ldots, n_{1\Omega}\} \), or \( \langle \text{Re}_{k}, e_{i_0}^{(1)} \rangle = 0 \) for all \( k \in \{1, 2, \ldots, n_{2\Omega}\} \).

*Proof.* Suppose to the contrary that \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are unlinked. By considering a suitable principal compression of \( A \) to a subalgebra of \( \mathbb{M}_4 \), we may assume without loss of generality that \( d = n_{1\Omega} = n_{2\Omega} = 2 \). Furthermore, we may reorder the bases if necessary to assume that \( \langle \text{Re}_{k}, e_{i_0}^{(1)} \rangle = 0 \) for all \( R \in \text{Rad}(A) \).

Since \( A \) is similar to \( BD(A) + \text{Rad}(A) \) via an upper triangular similarity, there is a fixed matrix \( M \) in \( Q_{1\Omega}AQ_{2\Omega} \) such that with respect to the basis \( \{ e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)} \} \), every \( A \) in \( A \) has the form

\[
A = \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \beta & 0 \\
0 & \beta & 0 & \beta
\end{bmatrix} + (\beta - \alpha)M + R
\]

for some \( \alpha, \beta \in \mathbb{C} \) and \( R \in \text{Rad}(A) \).

For each \( i, j \in \{1, 2\} \) define \( m_{ij} = \langle Me_j^{(2)}, e_i^{(1)} \rangle \). Furthermore, for each \( k \in \mathbb{R} \) let \( P_k \) denote the matrix

\[
P_k := \begin{bmatrix}
k^2 + 1 & 0 & 0 & 0 \\
0 & k^2 & 0 & -k \\
0 & 0 & k^2 + 1 & 0 \\
0 & -k & 0 & 1
\end{bmatrix},
\]

so that \( \frac{1}{n^2 + 1}P_k \) is a projection in \( \mathbb{M}_4 \). By direct computation, one may verify that every element \( B = (b_{ij}) \) in \( P_kAP_k \) satisfies the equation

\[
(k^2 + 1)b_{23} - m_{21}k^2(b_{33} - b_{11}) = 0.
\]

If, however, \( A \) is as above with \( \alpha = 0, \beta = 1, \) and \( R = 0 \), then for \( (P_kAP_k)^2 = (c_{ij}) \), we have

\[
(k^2 + 1)c_{23} - m_{21}k^2(c_{33} - c_{11}) = m_{21}k^2(k^2 + 1)^2(1 - km_{22}).
\]

The fact that \( A \) is projection compressible implies that \( (P_kAP_k)^2 \) belongs to \( P_kAP_k \), and hence the right-hand side of the above expression must be 0 for all \( k \). We therefore deduce that \( m_{21} = \langle Me_1^{(2)}, e_1^{(1)} \rangle = 0 \).

It now follows that \( \langle Ae_1^{(2)}, e_1^{(1)} \rangle = 0 \) for all \( A \in A \). So with respect to the basis \( \{ e_1^{(1)}, e_2^{(1)}, e_2^{(2)}, e_2^{(2)} \} \) for \( \mathbb{C}^4 \), every \( A \in A \) may be expressed as

\[
A = \begin{bmatrix}
\alpha & (\beta - \alpha)m_{11} + r_{11} & 0 & (\beta - \alpha)m_{12} + r_{12} \\
\beta & 0 & 0 & \alpha \\
0 & 0 & \beta & (\beta - \alpha)m_{22} + r_{22} \\
0 & 0 & \beta & (\beta - \alpha)m_{22} + r_{22}
\end{bmatrix},
\]
for some $\alpha$, $\beta$, and $r_{ij}$ in $\mathbb{C}$. Since $\alpha$ and $\beta$ may be chosen arbitrarily, this contradicts Theorem 3.0.1. Thus, $Q_{1\Omega}$ and $Q_{2\Omega}$ must be linked.

For the final claim, observe that $BD(A) = CI$ as $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked. By the remarks following Theorem 2.0.5 we have that $A = CI + Rad(A)$, and hence

$$\langle A(e_{j_0}^{(2)}, e_{i_0}^{(1)}), A \rangle = 0 \text{ for all } A \in A.$$

Suppose to the contrary that there exist indices $k_1 \in \{1, 2, \ldots, n_{1\Omega}\} \setminus \{i_0\}$, $k_2 \in \{1, 2, \ldots, n_{2\Omega}\} \setminus \{j_0\}$ and operators $A_1, A_2 \in A$ such that $\langle A_1 e_{j_0}^{(2)}, e_{i_0}^{(1)} \rangle \neq 0$ and $\langle A_2 e_{k_2}^{(2)}, e_{i_0}^{(1)} \rangle \neq 0$. Let $P_1$ and $P_2$ denote the orthogonal projections onto $\text{span}\{e_{i_0}^{(1)}, e_{j_0}^{(2)}\}$ and $\text{span}\{e_{i_0}^{(1)}, e_{k_2}^{(2)}\}$, respectively. It is easy to see that $P_1 A P_1 \neq \mathbb{C} P_1$, $P_2 A P_2 \neq \mathbb{C} P_2$, and $P_2 A P_1 = \{0\}$. Thus, Theorem 3.0.1 indicates that $A$ is not projection compressible—a contradiction.

Lemma 4.1.2. Let $n \geq 4$ be an even integer, and let $A$ be a projection compressible subalgebra of $\mathbb{M}_n$. Let $Q_1$ be a projection in $\mathbb{M}_n$ of rank $n/2$ and define $Q_2 := I - Q_1$. If $E \in \mathbb{M}_n$ is a partial isometry satisfying $E^*E = Q_1$ and $EE^* = Q_2$, then the linear space

$$A_0 := \{Q_1 A Q_1 + E^* A Q_1 + Q_1 A E + E^* A E : A \in A\}$$

is an algebra.

Proof. The assumptions on $E$ imply that the operator $P := \frac{1}{2}(I + E + E^*)$ is a projection in $\mathbb{M}_n$, and hence $P A P$ is an algebra. One may verify that with respect to the decomposition $\mathbb{C}^n = \text{ran}(Q_1) \oplus \text{ran}(Q_2)$, we have

$$P A P = \left\{ \begin{bmatrix} X & X \\ X & X \end{bmatrix} : X \in A_0 \right\}.$$

It follows that for any $X$ and $Y$ in $A_0$,

$$\begin{bmatrix} X & Y \\ X & Y \end{bmatrix} \begin{bmatrix} X & Y \\ X & Y \end{bmatrix} = 2 \begin{bmatrix} X Y & X Y \\ X Y & X Y \end{bmatrix} \in P A P,$$

and hence $XY$ belongs to $A_0$ as well. Thus, $A_0$ is an algebra.

Lemma 4.1.3. Let $A$ be a type I subalgebra of $\mathbb{M}_4$. If $\text{Rad}(A)$ is 3-dimensional and $\mathcal{F}_1(A)$ contains a pair $\Omega = (\mathbb{B}_{d=1}^d V_i)$ with $d = 2$, then $A$ is not projection compressible.

Proof. Suppose that $\dim \text{Rad}(A) = 3$ and $\Omega$ is a pair in $\mathcal{F}_1(A)$ as described above. Write $A = S + \text{Rad}(A)$, where $S$ is similar to $BD(A)$ via a block upper triangular similarity. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, then $A = \{\alpha I : \alpha \in \mathbb{C}\} + \text{Rad}(A)$. If instead $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, there is a matrix $M \in Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega}$ such that

$$A = \{\alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha) M : \alpha, \beta \in \mathbb{C}\} + \text{Rad}(A).$$

Note that the only distinctions between the linked and unlinked settings are the presence of the matrix $M$ and the freedom to choose $\alpha$ and $\beta$ independently. In the arguments that follow, we treat the entries of $M$ as arbitrary constants (possibly zero), and make no attempt to choose independent values for $\alpha$ and $\beta$. Thus, these arguments are applicable to both cases.

For each $i \in \{1, 2\}$, let $\{e_{i}^{(1)}, e_{i}^{(2)}\}$ be an orthonormal basis for $\text{ran}(Q_{i\Omega})$. Since $\text{Rad}(A)$ is a 3-dimensional subspace of $Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega}$, there is a non-zero matrix $\Gamma \in Q_{1\Omega} \mathbb{M}_4 Q_{2\Omega}$ such that $\text{Tr}(\Gamma^* R) = 0$ for all $R$ in $\text{Rad}(A)$. By reordering the bases for $\text{ran}(Q_{1\Omega})$ and $\text{ran}(Q_{2\Omega})$ if necessary, we may assume that $\langle \Gamma e_{i}^{(2)}, e_{i}^{(1)} \rangle$ is non-zero. From this it follows that there exist $\gamma_{12}, \gamma_{21}, \gamma_{22} \in \mathbb{C}$ such that

$$\text{Rad}(A) = \left\{ \begin{bmatrix} 0 & 0 & \gamma_{12} r_{12} & \gamma_{21} r_{21} + \gamma_{22} r_{22} \\ 0 & 0 & r_{21} & r_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : r_{12}, r_{21}, r_{22} \in \mathbb{C} \right\}$$

with respect to the basis $\{e_{1}^{(1)}, e_{1}^{(2)}, e_{2}^{(1)}, e_{2}^{(2)}\}$ for $\mathbb{C}^4$. 
To see that \( \mathcal{A} \) is not projection compressible, consider the matrix

\[
P := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},
\]

and note that \( \frac{1}{2}P \) is a projection in \( \mathbb{M}_4 \). One may verify that every operator \( B = (b_{ij}) \) in \( PAP \) satisfies the equation

\[
b_{13} - 4\gamma_{22}b_{24} - 2\gamma_{21}b_{23} - 2\gamma_{12}b_{14} - (\gamma_{12}m_{12} + \gamma_{21}m_{21} - \gamma_{22}(1 - m_{22}) - m_{11})b_{11}
\]

\[
+ (\gamma_{12}m_{12} + \gamma_{21}m_{21} + \gamma_{22}(1 + m_{22}) - m_{11})b_{33} = 0,
\]

where for each \( i, j \in \{1, 2\} \), we define \( m_{ij} = \langle Me_j^{(2)}, e_i^{(1)} \rangle \). If, however, \( A \) is the element of \( \mathcal{A} \) obtained by setting \( \alpha = \beta = r_{12} = r_{21} = 1 \) and \( r_{22} = 0 \), then \( B := (PAP)^2 \) produces a value of 8 on the left-hand side of the above equation. Consequently, \( (PAP)^2 \) does not belong to \( PAP \), so \( PAP \) is not an algebra. \( \blacksquare \)

The following classical theorem from linear algebra will be applied in the proof of Lemma 4.1.6 and used extensively throughout §5. For reference, see [5, Theorem 2.6.3].

**Theorem 4.1.4 (Singular Value Decomposition).** Let \( n \) and \( p \) be positive integers, and let \( \mathcal{A} \) be a complex \( n \times p \) matrix.

(i) If \( n \leq p \), then there are unitaries \( U \in \mathbb{M}_n \) and \( V \in \mathbb{M}_p \), and a positive semi-definite diagonal matrix \( D \in \mathbb{M}_n \) such that

\[
U^*AV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.
\]

(ii) If \( n \geq p \), then there are unitaries \( U \in \mathbb{M}_n \) and \( V \in \mathbb{M}_p \), and a positive semi-definite diagonal matrix \( D \in \mathbb{M}_p \) such that

\[
U^*AV = \begin{bmatrix} D \\ 0 \end{bmatrix}.
\]

The principal application of Theorem 4.1.4 will be in simplifying the structure of the semi-simple part of an algebra \( \mathcal{A} \) in reduced block upper triangular form. Indeed, suppose that \( \mathcal{A} = \mathcal{S} \oplus \text{Rad}(\mathcal{A}) \) is a type I subalgebra of \( \mathbb{M}_n \) where \( \mathcal{S} \) is semi-simple. Let \( \Omega = (d, \bigoplus_{i=1}^n V_i) \) be a pair in \( \mathcal{F}_f(\mathcal{A}) \), and assume that the projections \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are unlinked. For each \( i \in \{1, 2\} \), let \( \{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{\Omega}}^{(i)}\} \) be an orthonormal basis for \( \text{ran}(Q_{i\Omega}) \). As a consequence of Theorem 2.0.5, there is a matrix \( M \in Q_{1\Omega}M_nQ_{2\Omega} \) such that

\[
\mathcal{S} = \{\alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha)M : \alpha, \beta \in \mathbb{C}\}.
\]

It then follows from Theorem 4.1.4 that there is a unitary \( U \in \mathbb{M}_n \) such that \( Q_{1\Omega}UQ_{2\Omega} = 0 \), \( Q_{2\Omega}UQ_{1\Omega} = 0 \), and \( \langle U^*MU_{e_i^{(2)}}, e_i^{(1)} \rangle = 0 \) whenever \( i \neq j \).

Finally, the proof of Lemma 4.1.6 will require the following result of Azoff concerning the minimum dimension of a transitive space of linear operators. Recall that a set \( \mathcal{L} \) of linear transformations from \( \mathbb{C}^n \) to \( \mathbb{C}^m \) is said to be transitive if for every non-zero \( x \in \mathbb{C}^n \) and arbitrary \( y \in \mathbb{C}^m \), there exists some \( L \in \mathcal{L} \) such that \( Lx = y \).

**Theorem 4.1.5.** [1, Proposition 4.7] If \( \mathcal{L} \) is a transitive space of linear transformations from \( \mathbb{C}^n \) to \( \mathbb{C}^m \), then the dimension of \( \mathcal{L} \) is at least \( m + n - 1 \).

We are now prepared to state and prove Lemma 4.1.6. This result indicates that under certain restrictive assumptions, a projection compressible type I algebra with unlinked projections is unitarily equivalent to the type I algebra from Example 1.0.1(i). Loosening these assumptions will require a refinement of Theorems 4.1.4 to specific classes of transitive spaces of operators.

**Lemma 4.1.6.** Let \( n \geq 4 \) be an even integer, and let \( \mathcal{A} \) be a projection compressible type I subalgebra of \( \mathbb{M}_n \). Suppose that \( \mathcal{F}_f(\mathcal{A}) \) contains a pair \( \Omega = (d, \bigoplus_{i=1}^n V_i) \) with \( d = n/2 \). If the projections \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are unlinked, then \( \mathcal{A} \) is unitarily equivalent to

\[
\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{2\Omega} + Q_{1\Omega}M_nQ_{2\Omega},
\]

the type I algebra from Example 1.0.1(i). Consequently, \( \mathcal{A} \) is idempotent compressible.
Proof. For each \( i \in \{1, 2\} \), let \( \{e_{1}^{(i)}, e_{2}^{(i)}, \ldots, e_{d}^{(i)}\} \) be an orthonormal basis for \( \text{ran}(Q_{1\Omega}) \). As a consequence of Theorem 2.0.5 there is a matrix \( M \) in \( Q_{1\Omega}M_{n}Q_{2\Omega} \) such that

\[
\mathcal{A} = \{ \alpha Q_{1\Omega} + \beta Q_{2\Omega} + (\beta - \alpha)M : \alpha, \beta \in \mathbb{C} \} + \text{Rad}(\mathcal{A}).
\]

In fact, one may assume by Theorem 4.1.4 and its subsequent remarks that there are constants \( m_{ij} \geq 0 \) such that \( \langle Me_{j}^{(2)}, e_{i}^{(1)} \rangle = \delta_{ij}m_{ij} \) for all \( i \) and \( j \).

Let \( E \in M_{n} \) denote the partial isometry satisfying \( Ee_{i}^{(1)} = e_{i}^{(2)} \) and \( Ee_{i}^{(2)} = 0 \) for all \( i \in \{1, 2, \ldots, d\} \). Since \( \mathcal{A} \) is projection compressible, Lemma 4.1.3 implies that

\[
\mathcal{A}_{0} := \{ (\alpha + \beta)Q_{1\Omega} + (\beta - \alpha)ME + RE : \alpha, \beta \in \mathbb{C}, R \in \text{Rad}(\mathcal{A}) \}
\]

is a subalgebra of \( Q_{1\Omega}M_{n}Q_{2\Omega} \). If this subalgebra were proper, then by Burnside’s theorem, we may change the orthonormal basis for \( \text{ran}(Q_{1\Omega}) \) if necessary to assume that \( \langle Ae_{1}^{(1)}, e_{d}^{(1)} \rangle = 0 \) for all \( A \in \mathcal{A}_{0} \). In this case, one may change the orthonormal basis for \( \text{ran}(Q_{2\Omega}) \) accordingly and assume that \( \langle Re_{1}^{(2)}, e_{d}^{(1)} \rangle = 0 \) for all \( R \in \text{Rad}(\mathcal{A}) \). Since \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are unlinked, an application of Lemma 4.1.1 demonstrates that \( \mathcal{A} \) lacks the projection compression property—a contradiction.

We may therefore assume that \( \mathcal{A}_{0} \) is equal to \( Q_{1\Omega}M_{n}Q_{2\Omega} \). This means that \( \text{Rad}(\mathcal{A})E \) can be enlarged to a \( d^{2} \)-dimensional space by adding

\[
\{ \alpha(Q_{1\Omega} - ME) + \beta(Q_{1\Omega} + ME) : \alpha, \beta \in \mathbb{C} \},
\]

the linear span of two diagonal matrices in \( Q_{1\Omega}M_{n}Q_{2\Omega} \). It follows that

\[
\dim \text{Rad}(\mathcal{A})E = \dim \text{Rad}(\mathcal{A}) \geq d^{2} - 2,
\]

and any entries in \( \text{Rad}(\mathcal{A})E \) that depend linearly on other entries must be located on the diagonal. Our goal is to show that \( \dim \text{Rad}(\mathcal{A}) = d^{2} \), and hence \( \text{Rad}(\mathcal{A}) = Q_{1\Omega}M_{n}Q_{2\Omega} \).

Let us begin by addressing the case in which \( n = 4 \), and hence \( d = 2 \). If \( \dim \text{Rad}(\mathcal{A}) \) is strictly less than \( d^{2} = 4 \), then \( \text{Rad}(\mathcal{A}) \) is 2- or 3-dimensional by the analysis above. If \( \dim \text{Rad}(\mathcal{A}) = 2 \), then by Theorem 4.1.6 \( \text{Rad}(\mathcal{A}) \) is not transitive as a space of linear maps from \( \text{ran}(Q_{2\Omega}) \) to \( \text{ran}(Q_{1\Omega}) \). In this case there exist unit vectors \( v \in \text{ran}(Q_{1\Omega}) \) and \( w \in \text{ran}(Q_{2\Omega}) \) such that \( Rw \in \mathbb{C}v \) for every \( R \in \text{Rad}(\mathcal{A}) \). Choose unit vectors \( v' \in \text{ran}(Q_{1\Omega}) \cap \text{ran}(\mathbb{C}v) \) and \( w' \in \text{ran}(Q_{2\Omega}) \cap \text{ran}(\mathbb{C}w) \), and replace the orthonormal bases for \( \text{ran}(Q_{1\Omega}) \) and \( \text{ran}(Q_{2\Omega}) \) with \( \{ v, v' \} \) and \( \{ w, w' \} \), respectively. Since

\[
\langle Rw, v' \rangle = \langle \lambda v, v' \rangle = 0 \quad \text{for all } R \in \text{Rad}(\mathcal{A}),
\]

\( \mathcal{A} \) lacks the projection compression property by Lemma 4.1.1—a contradiction. Using Lemma 4.1.3 one may also obtain a contradiction in the case that \( \dim \text{Rad}(\mathcal{A}) = 3 \).

Assume now that \( n > 4 \). By the above analysis, there are at most two entries from \( \text{Rad}(\mathcal{A})E \) which cannot be chosen arbitrarily, and these entries necessarily occur on the diagonal. By reordering the bases for \( \text{ran}(Q_{1\Omega}) \) and \( \text{ran}(Q_{2\Omega}) \), we may relocate the linearly dependent entries to the \((1, n - 1)\) and \((2, n)\) positions of \( \text{Rad}(\mathcal{A}) \), respectively. That is, we may assume that with respect to the decomposition

\[
\mathbb{C}^{n} = \vee \left\{ e_{1}^{(1)}, e_{2}^{(1)}, \ldots, e_{d-1}^{(1)} \right\} \oplus \vee \left\{ e_{1}^{(2)}, e_{2}^{(2)} \right\} \oplus \vee \left\{ e_{2}^{(2)}, e_{3}^{(2)}, \ldots, e_{d}^{(2)} \right\},
\]

we have

\[
\text{Rad}(\mathcal{A})E = \{ \alpha \vee \beta : \alpha, \beta \in \mathbb{C} \}. 
\]
each \( A \in \mathcal{A} \) can be represented by a matrix of the form

\[
A = \begin{bmatrix}
\alpha & 0 & t_{11} & t_{12} & \cdots & t_{1,d-2} & \gamma_1 & t_{1d} \\
\alpha & 0 & t_{21} & t_{22} & \cdots & t_{2,d-2} & t_{2,d-1} & \gamma_2 \\
\alpha & 0 & t_{31} & t_{32} & \cdots & t_{3,d-2} & t_{3,d-1} & t_{3d} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha & 0 & t_{d-1,1} & t_{d-1,2} & \cdots & t_{d-1,d-2} & t_{d-1,d-1} & t_{d-1,d} \\
\alpha & t_{d1} & t_{d2} & \cdots & t_{d,d-2} & t_{d,d-1} & t_{dd} \\
\beta & 0 & \cdots & 0 & 0 & 0 & 0 \\
\beta & \beta & \beta & \beta & \beta & \beta & \beta \\
\end{bmatrix},
\]

where \( \alpha, \beta, \) and \( t_{ij} \) can be chosen arbitrarily, and \( \gamma_1 \) and \( \gamma_2 \) may depend linearly on these entries. We will demonstrate that, in fact, \( \gamma_1 \) and \( \gamma_2 \) can be chosen arbitrarily and independently of the remaining terms.

Consider the matrix

\[
P = \begin{bmatrix}
2 & 2 & \cdots & 2 \\
2 & 1 & 1 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & \cdots & 2 \\
\end{bmatrix}
\]

written with respect to the decomposition above. Observe that \( \frac{1}{2}P \) is a projection in \( \mathbb{M}_n \). Direct computations show that with \( A \) as above, \( PAP \) is given by

\[
\begin{bmatrix}
4\alpha & 4\alpha & 4\alpha & \vdots & \vdots & \vdots & \vdots \\
2t_{11} & 2t_{11} & 4t_{12} & \cdots & 4t_{1,d-2} & 4\gamma_1 & 4t_{1d} \\
2t_{21} & 2t_{21} & 4t_{22} & \cdots & 4t_{2,d-2} & 4t_{2,d-1} & 4\gamma_2 \\
2t_{31} & 2t_{31} & 4t_{32} & \cdots & 4t_{3,d-2} & 4t_{3,d-1} & 4t_{3d} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2t_{d-1,1} & 2t_{d-1,1} & 4t_{d-1,2} & \cdots & 4t_{d-1,d-2} & 4t_{d-1,d-1} & 4t_{d-1,d} \\
\alpha + \beta + t_{d1} & \alpha + \beta + t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
\alpha + \beta + t_{d1} & \alpha + \beta + t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
\alpha & \alpha & \cdots & \alpha & 4\beta & 4\beta & 4\beta \\
\beta & \beta & \beta & \beta & \beta & \beta & \beta \\
\end{bmatrix}
\]

Hence, it suffices to prove that \( e_1^{(1)} \otimes e_{d-1}^{(2)*} \) and \( e_2^{(1)} \otimes e_d^{(2)*} \) belong to \( PAP \).

To see that this is the case, let \( A \) be as above with \( t_{11} = t_{d,d-1} = 1 \) and \( \alpha = \beta = t_{ij} = 0 \) for all other indices \( i \) and \( j \). It is straightforward to verify that

\[
(PAP)^2 = 8e_1^{(1)} \otimes e_{d-1}^{(2)*}.
\]
Consequently, $e^{(1)}_1 \otimes e^{(2)*}_{d-1}$ belongs to $PAP$, so $\gamma_1$ can indeed be chosen arbitrarily. By reordering the basis to interchange the positions of $\gamma_1$ and $\gamma_2$, one may repeat this process to show that $\gamma_2$ may be chosen arbitrarily as well.

Observe that the success of Lemma 4.1.6 relied heavily on the existence of the pair $\Omega = (d, \bigoplus_{i=1}^n V_i)$ with $d = n/2$. Indeed, without such a pair, one would be unable to directly apply Lemma 4.1.2 or Burnside’s Theorem to infer that $\dim \operatorname{Rad}(A) \geq d^2 - 2$.

Our final goal of this section is to generalize Lemma 4.1.6 to type I algebras $A$ that may not admit a pair $\Omega$ as describe above. We will accomplish this goal by applying Lemma 4.1.6 to study the structure of the radical of certain principal compressions of $A$. It will then follow from [4, Theorem 1.2]—an extension of Theorem 4.1.5—that $A$ is unitarily equivalent to the type I algebra from Example 1.0.1(i). In order to introduce this extension, we first present the following definition.

**Definition 4.1.7.** Let $L$ be a vector space of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, and let $k$ be a positive integer. We say that $L$ is $k$-transitive if for every choice of $k$ linearly independent vectors $x_1, x_2, \ldots, x_k$ in $\mathbb{C}^n$, and every choice of $k$ arbitrary vectors $y_1, y_2, \ldots, y_k$ in $\mathbb{C}^m$, there is an element $A \in L$ such that $Ax_i = y_i$ for all $i \in \{1, 2, \ldots, k\}$.

**Theorem 4.1.8.** [4, Theorem 1.2] If $L$ is a $k$-transitive space of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$, then the dimension of $L$ is at least $k(m + n - k)$.

We are now prepared to prove the classification in the general case of type I algebras with unlinked projections.

**Theorem 4.1.9.** Let $A$ be a projection compressible type I subalgebra of $M_n$, and let $\Omega = (d, \bigoplus_{i=1}^n V_i)$ be a pair in $\mathcal{F}_I(A)$ with $1 < d < n - 1$. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are unlinked, then $A$ is unitarily equivalent to $\mathbb{C}Q_{1\Omega} + \mathbb{C}Q_{2\Omega} + Q_{1\Omega}M_nQ_{2\Omega}$, the type I algebra from Example 1.0.1(i). Consequently, $A$ is idempotent compressible.

**Proof.** By replacing $A$ with $A^{\sigma T}$ if necessary, we may assume that $d \leq n - d$. That is, $n_{1\Omega} \leq n_{2\Omega}$. We will demonstrate that $\operatorname{Rad}(A)$ has dimension $d(n - d)$, and hence must be equal to $Q_{1\Omega}M_nQ_{2\Omega}$. Of course, it is clear that $\dim \operatorname{Rad}(A) \leq d(n - d)$.

Note that $\operatorname{Rad}(A)$ is $d$-transitive as a space of linear maps from $\operatorname{ran}(Q_{2\Omega})$ to $\operatorname{ran}(Q_{1\Omega})$. Indeed, let $S$ be a linearly independent $d$-element subset of $\operatorname{ran}(Q_{2\Omega})$, and let $Q_S$ denote the orthogonal projection onto the span of $S$. Since $Q_{1\Omega}$ and $Q_S$ are both of rank $d$, Lemma 4.1.6 implies that the radical of $A_0 := (Q_{1\Omega} + Q_S)A(Q_{1\Omega} + Q_S)$ is equal to $Q_{1\Omega}M_nQ_S$. As a result, the vectors in $S$ can be mapped anywhere in $\operatorname{ran}(Q_{1\Omega})$ by elements of $\operatorname{Rad}(A)$. We conclude that $\operatorname{Rad}(A)$ is $d$-transitive.

The proof ends with an application of Theorem 4.1.8 Since $\operatorname{Rad}(A)$ is a $d$-transitive subspace of $Q_{1\Omega}M_nQ_{2\Omega}$, we have that $\dim \operatorname{Rad}(A) \geq d(\dim (Q_{1\Omega} + Q_S) - d) = d(n - d)$. 

§4.2 Type I Algebras with Linked Projections. We now wish to describe the projection compressible type I algebras $A$ for which the pairs $\Omega = (d, \bigoplus_{i=1}^n V_i)$ in $\mathcal{F}_I(A)$ are such that $Q_{1\Omega}$ is linked to $Q_{2\Omega}$. An inductive argument in Theorem 4.1.2 will demonstrate that every such algebra is the unitization of an $\mathcal{L}\mathcal{R}$-algebra. The base case of this argument will require the following lemma.

**Lemma 4.2.1.** Let $A$ be a projection compressible type I subalgebra of $M_4$, and suppose that $\mathcal{F}_I(A)$ contains a pair $\Omega = (d, \bigoplus_{i=1}^4 V_i)$ with $d = 2$. If $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked, then there are projections $Q'_1 \leq Q_{1\Omega}$ and $Q'_2 \leq Q_{2\Omega}$ such that $\operatorname{Rad}(A) = Q'_1 M_2 Q'_2$. In this case $A$ is the unitization of an $\mathcal{L}\mathcal{R}$-algebra, so $A$ is idempotent compressible.

**Proof.** Let $\Omega$ be a pair in $\mathcal{F}_I(A)$ as above, and assume that $Q_{1\Omega}$ and $Q_{2\Omega}$ are linked. By the observations following Theorem 2.0.5, $A = CI + \operatorname{Rad}(A)$.
For each \( i \in \{1, 2\} \), let \( \left\{ e_1^{(i)}, e_2^{(i)} \right\} \) be a fixed orthonormal basis for \( \text{ran}(Q_{\Omega^1}) \). Furthermore, let \( E \in \mathbb{M}_n \) denote the partial isometry satisfying \( Ee_i^{(1)} = e_i^{(2)} \) and \( Ee_i^{(2)} = 0 \) for each \( i \in \{1, 2\} \). By Lemma 4.1.2

\[
\mathcal{A}_0 := \mathbb{C}Q_{\Omega^1} + \text{Rad}(\mathcal{A})E
\]

is a subalgebra of \( Q_{\Omega^1}\mathbb{M}_4 Q_{\Omega^1} \). If this subalgebra \( \mathcal{A}_0 \) is proper, then by Burnside’s Theorem, we may change the orthonormal basis for \( \text{ran}(Q_{\Omega^1}) \) if required and assume that \( \langle Ae_1^{(1)}, e_2^{(1)} \rangle = 0 \) for all \( A \in \mathcal{A}_0 \). In this case we may adjust the orthonormal basis for \( \text{ran}(Q_{2\Omega^1}) \) accordingly and assume that \( \langle Re_i^{(2)}, e_1^{(1)} \rangle = 0 \) for all \( R \in \text{Rad}(\mathcal{A}) \). Thus, by Lemma 4.1.1 either \( \langle Re_i^{(2)}, e_1^{(1)} \rangle = 0 \) for all \( R \in \text{Rad}(\mathcal{A}) \), or \( \langle Re_i^{(2)}, e_2^{(1)} \rangle = 0 \) for all \( R \in \text{Rad}(\mathcal{A}) \). The fact that \( \text{Rad}(\mathcal{A}) \) has the required form now follows from Theorem 1.0.2.

Suppose instead that \( \mathbb{C}Q_{\Omega^1} + \text{Rad}(\mathcal{A})E \) is equal to \( Q_{\Omega^1}\mathbb{M}_4 Q_{\Omega^1} \). It follows that \( \text{Rad}(\mathcal{A}) \) is at least 3-dimensional. If \( \dim \text{Rad}(\mathcal{A}) = 3 \), then \( \mathcal{A} \) is of the form described in Lemma 1.1.3 and hence \( \mathcal{A} \) is not projection compressible. We therefore have that \( \dim \text{Rad}(\mathcal{A}) = 4 \), so \( \text{Rad}(\mathcal{A}) = Q_{\Omega^1}\mathbb{M}_4 Q_{2\Omega^1} \).

**Theorem 4.2.2.** Let \( \mathcal{A} \) be a projection compressible type I subalgebra of \( \mathbb{M}_n \), and let \( \Omega = (d, \bigoplus_{i=1}^n V_i) \) be a pair in \( \mathcal{F}_1(\mathcal{A}) \). If \( Q_{\Omega^1} \) and \( Q_{2\Omega^1} \) are linked, then there are projections \( Q_1^1 \leq Q_{\Omega^1} \) and \( Q_2^1 \leq Q_{2\Omega^1} \) such that \( \text{Rad}(\mathcal{A}) = Q_1^1 \mathbb{M}_n Q_2^1 \). Thus, \( \mathcal{A} \) is the unitization of an LR-algebra, so \( \mathcal{A} \) is idempotent compressible.

**Proof.** We will proceed by induction on \( n \). By definition of a type I algebra, our base case occurs when \( n = 4 \). That is, let \( \mathcal{A} \) be a projection compressible type I subalgebra of \( \mathbb{M}_4 \), and suppose that \( \Omega = (d, \bigoplus_{i=1}^n V_i) \) is a pair in \( \mathcal{F}_1(\mathcal{A}) \) with \( Q_{\Omega^1} \) linked to \( Q_{2\Omega^1} \). If \( d = 1 \) or \( d = 3 \), then Proposition 1.0.3 guarantees that \( \text{Rad}(\mathcal{A}) \) admits the required form. If instead \( d = 2 \), then \( \mathcal{A} \) and \( \Omega \) are as in Lemma 4.2.1. Once again \( \text{Rad}(\mathcal{A}) \) is of the correct form.

Now fix an integer \( N \geq 5 \). Assume that for every positive integer \( n < N \), if \( \mathcal{A} \) is a projection compressible type I subalgebra of \( \mathbb{M}_n \) and \( \Omega \) is a pair in \( \mathcal{F}_1(\mathcal{A}) \) with \( Q_{\Omega^1} \) linked to \( Q_{2\Omega^1} \), then \( \text{Rad}(\mathcal{A}) = Q_1^1 \mathbb{M}_n Q_2^1 \) for some subprojections \( Q_1^1 \leq Q_{\Omega^1} \) and \( Q_2^1 \leq Q_{2\Omega^1} \). We claim that this is also the case for every such subalgebra \( \mathcal{A} \) of \( \mathbb{M}_N \) and pair \( \Omega \in \mathcal{F}_1(\mathcal{A}) \). Indeed, fix a subalgebra \( \mathcal{A} \) of \( \mathbb{M}_N \) and pair \( \Omega = (d, \bigoplus_{i=1}^n V_i) \) in \( \mathcal{F}_1(\mathcal{A}) \) as in the statement of the theorem. If \( d = 1 \) or \( d = N - 1 \), then Proposition 1.0.3 ensures that \( \text{Rad}(\mathcal{A}) \) is of the desired form. Thus, we will assume that \( 1 < d < N - 1 \). By replacing \( \mathcal{A} \) with \( \mathcal{A}^T \) if necessary, we will also assume that \( d < N - d \).

First consider the case that \( N \) is even and \( d = N - d = N/2 \). Fix orthonormal bases \( \left\{ e_1^{(1)}, e_2^{(1)}, \ldots, e_d^{(1)} \right\} \) and \( \left\{ e_1^{(2)}, e_2^{(2)}, \ldots, e_d^{(2)} \right\} \) for \( \text{ran}(Q_{\Omega^1}) \) and \( \text{ran}(Q_{2\Omega^1}) \), respectively. Let \( E \in \mathbb{M}_n \) denote the partial isometry satisfying \( Ee_i^{(1)} = e_i^{(2)} \) and \( Ee_i^{(2)} = 0 \) for each \( i \in \{1, 2, \ldots, d\} \). Arguing as in the proof of Lemma 4.1.6 either \( \mathbb{C}Q_{\Omega^1} + \text{Rad}(\mathcal{A})E \) is equal to \( Q_{\Omega^1}\mathbb{M}_N Q_{\Omega^1} \), or Burnside’s Theorem may be used to assume that

\[
\langle Re_i^{(2)}, e_1^{(1)} \rangle = 0 \quad \text{for all} \quad R \in \text{Rad}(\mathcal{A}).
\]

If the latter holds, then by Lemma 4.1.1 \( \text{Rad}(\mathcal{A}) \) contains a permanent row or column of zeros. In the case of a permanent row of zeros, consider the algebra \( \mathcal{A}_0 \) obtained by deleting this row and its corresponding column from \( \mathcal{A} \). We have that \( \mathcal{A}_0 \) is a projection compressible type I subalgebra of \( \mathbb{M}_{N-1} \), so \( \text{Rad}(\mathcal{A}_0) \) admits the required form by the inductive hypothesis. Upon reinserting the removed row and column, one can see that \( \text{Rad}(\mathcal{A}) \) is also of the required form. An analogous argument can be made in the case of a permanent column of zeros. We may therefore assume that \( \mathbb{C}Q_{\Omega^1} + \text{Rad}(\mathcal{A})E = Q_{\Omega^1}\mathbb{M}_N Q_{\Omega^1} \).

Since \( \text{Rad}(\mathcal{A})E \) can be enlarged to a \( d^2 \)-dimensional space by adding \( \mathbb{C}Q_{\Omega^1} \), \( \dim \text{Rad}(\mathcal{A}) \geq d^2 - 1 \). We claim that in fact, \( \dim \text{Rad}(\mathcal{A}) = d^2 \), and hence \( \text{Rad}(\mathcal{A}) = Q_{\Omega^1}\mathbb{M}_N Q_{2\Omega^1} \). To see this is the case, reorder the bases for \( \text{ran}(Q_{\Omega^1}) \) and \( \text{ran}(Q_{2\Omega^1}) \) if necessary to assume that with respect to the decomposition

\[
\mathbb{C}^N = \bigoplus \left\{ e_1^{(1)}, e_2^{(1)}, \ldots, e_d^{(1)} \right\} \oplus \bigoplus \left\{ e_1^{(2)}, e_2^{(2)} \right\} \oplus \bigoplus \left\{ e_2^{(2)}, e_3^{(2)}, \ldots, e_d^{(2)} \right\}.
\]
each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$
A = \begin{bmatrix}
\alpha & 0 & t_{11} & t_{12} & \cdots & t_{1,d-2} & \gamma & t_{1d} \\
\alpha & 0 & t_{21} & t_{22} & \cdots & t_{2,d-2} & t_{2,d-1} & t_{2d} \\
\alpha & 0 & t_{31} & t_{32} & \cdots & t_{3,d-2} & t_{3,d-1} & t_{3d} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\alpha & 0 & t_{d-1,1} & t_{d-1,2} & \cdots & t_{d-1,d-2} & t_{d-1,d-1} & t_{d-1,d} \\
\alpha & t_{d1} & t_{d2} & \cdots & t_{d,d-2} & t_{d,d-1} & t_{dd} & 0 \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha
\end{bmatrix}.
$$

Here, $\alpha$ and $t_{ij}$ are arbitrary values in $\mathbb{C}$, and $\gamma$ may depend linearly on these entries.

It will be shown that $\gamma$ is in fact, independent of the other terms. Indeed, let $P$ denote the matrix from the proof of Lemma 4.1.3 so that $\frac{1}{2}P$ is a projection in $\mathbb{M}_N$. Proceed now as in the proof of that lemma by noting that with $A$ as above, $PAP$ is given by

$$
\begin{bmatrix}
4\alpha & 0 & 2t_{11} & 2t_{11} & 4t_{12} & \cdots & 4t_{1,d-2} & 4\gamma & 4t_{1d} \\
4\alpha & 0 & 2t_{21} & 2t_{21} & 4t_{22} & \cdots & 4t_{2,d-2} & 4t_{2,d-1} & 4t_{2d} \\
4\alpha & 0 & 2t_{31} & 2t_{31} & 4t_{32} & \cdots & 4t_{3,d-2} & 4t_{3,d-1} & 4t_{3d} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
4\alpha & 0 & 2t_{d-1,1} & 2t_{d-1,1} & 4t_{d-1,2} & \cdots & 4t_{d-1,d-2} & 4t_{d-1,d-1} & 4t_{d-1,d} \\
4\alpha & 0 & 2t_{d1} & 2t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
4\alpha & 0 & 2t_{d1} & 2t_{d1} & 2t_{d2} & \cdots & 2t_{d,d-2} & 2t_{d,d-1} & 2t_{dd} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
4\alpha & 0 & 4\alpha & 4\alpha & \cdots & 4\alpha & 4\alpha & 4\alpha & 4\alpha
\end{bmatrix}.
$$

It therefore suffices to prove that $e_1^{(1)} \otimes e_{d-1}^{(2)*}$ belongs to $PAP$. But if $A$ denotes the particular element of $\mathcal{A}$ obtained by taking $t_{11} = t_{d,d-1} = 1$ and $\alpha = t_{ij} = 0$ for all other indices $i$ and $j$, then

$$(PAP)^2 = 8e_1^{(1)} \otimes e_{d-1}^{(2)*}.$$ 

Since $\mathcal{A}$ is projection compressible, this element belongs to $PAP$. We conclude that $\text{Rad}(\mathcal{A}) = Q_{1\Omega}^d \mathbb{M}_N Q_{2\Omega}$, and hence the proof of the $d = N - d$ case is complete.

Let us now turn to the case in which $d < N - d$. As above, let $\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{n_{1\Omega}}^{(1)}\}$ and $\{e_1^{(2)}, e_2^{(2)}, \ldots, e_{n_{2\Omega}}^{(2)}\}$ be fixed orthonormal bases for $\text{ran}(Q_{1\Omega})$ and $\text{ran}(Q_{2\Omega})$, respectively. For each linearly independent $d$-element subset $S$ of $\text{ran}(Q_{2\Omega})$, let $Q_S$ denote the orthogonal projection onto the span of $S$, and define $P_S := Q_{1\Omega} + Q_S$. Let $\mathcal{A}_S$ denote the compression $P_S \mathcal{A} P_S$, which we regard as a subalgebra of $\mathcal{C}_1 + Q_{1\Omega} \mathbb{M}_d Q_{2\Omega}$.

If each compression $\mathcal{A}_S$ is equal to $\mathcal{C}_1 + Q_{1\Omega} \mathbb{M}_d Q_{2\Omega}$, then $\text{Rad}(\mathcal{A})$ is a $d$-transitive space of linear maps from $\text{ran}(Q_{2\Omega})$ into $\text{ran}(Q_{1\Omega})$. In this case we may apply Theorem 4.1.8 to conclude that $\text{Rad}(\mathcal{A}) = Q_{1\Omega}^d \mathbb{M}_N Q_{2\Omega}$, as desired. Instead, suppose that one of the sets $S$ is such that the radical of $\mathcal{A}_S$ is properly contained in $Q_{1\Omega} \mathbb{M}_d Q_{2\Omega}$. For such an $S$, the inductive hypothesis gives rise to subprojections $Q'_S \leq Q_{1\Omega}$ and $Q'_S \leq Q_S$ such that

$$\text{Rad}(\mathcal{A}_S) = Q'_S \mathbb{M}_d Q'_S.$$ 

At least one of these subprojections must be proper.

If $Q'_S \neq Q_S$ or $Q'_1 = 0$, then there is an orthonormal basis for $\mathbb{C}^{2d}$ with respect to which $\text{Rad}(\mathcal{A}_S)$ has a permanent column of zeros. One may then extend this basis to an orthonormal basis for $\mathbb{C}^N$ with respect to which $\text{Rad}(\mathcal{A})$ also admits a permanent column of zeros. By deleting this column and its corresponding
row from $\mathcal{A}$, we obtain a projection compressible type I subalgebra of $M_{N-1}$. The inductive hypothesis then implies that the radical of this compression is of the desired form. Upon reinserting the column and row deleted from $\mathcal{A}$, it is easy to see that $Rad(\mathcal{A})$ is of the desired form as well.

On the other hand, if $Q_\delta = Q'_\delta$ and $Q'_i$ is a proper non-zero subprojection of $Q_i\Omega$, then it must be the case that $Rad(\mathcal{A}_k)$ has a permanent row of zeros, but not a permanent column of zeros. Thus, $Rad(\mathcal{A})$ has a permanent row of zeros by Lemma 4.1.1. By removing this row and its corresponding column from $\mathcal{A}$, we obtain a projection compressible type I subalgebra of $M_{N-1}$. The radical of this algebra is of the correct form by the inductive hypothesis, and hence so too is $Rad(\mathcal{A})$.

§5 Algebras of Type II

The term type II will be used to describe a unital subalgebra $\mathcal{A}$ of $\mathbb{M}_n$, $n \geq 4$, that has a reduced block upper triangular form with respect to an orthogonal decomposition $\bigoplus_{i=1}^{m} V_i$ of $\mathbb{C}^n$, such that $\dim V_k \geq 2$ for some $k$. For example, the algebra from Example 1.0.1(i) is of type II if and only if $\text{rank}(Q_2) \geq 2$. It follows from this definition that every type II algebra satisfies the assumptions of Corollary 3.0.3.

The purpose of this section is to classify the type II algebras that afford the projection compression property. It will be shown that every projection compressible algebra of type II is either the unitalization of an $\mathcal{L}$-algebra, or is unitarily equivalent to the type II algebra from Example 1.0.1(i).

As in the case of type I algebras, it will be helpful to keep a record of all orthogonal decompositions of $\mathbb{C}^n$ that satisfy the conditions of Corollary 3.0.3 for a given type II algebra $\mathcal{A}$. Thus, we make the following definition.

Definition 5.0.1. If $\mathcal{A}$ is an algebra of type II, let $\mathcal{F}_{II} = \mathcal{F}_{II}(\mathcal{A})$ denote the set of triples $\Omega = (d, k, \bigoplus_{i=1}^{m} V_i)$ that satisfy the following conditions:

(i) $\bigoplus_{i=1}^{m} V_i$ is an orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular;

(ii) $d$ and $k$ are integers such that $d \geq 2$, $k \in \{1, 2, \ldots, m\}$, and $\dim V_k = d$.

Notation. If $\mathcal{A}$ is an algebra of type II and $\Omega$ is a triple in $\mathcal{F}_{II}(\mathcal{A})$, let $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ denote the orthogonal projections onto $\bigoplus_{i<k} V_i$, $V_k$, and $\bigoplus_{i>k} V_i$, respectively. Furthermore, for each $i \in \{1, 2, 3\}$, let $n_{i\Omega}$ denote the rank of $Q_i\Omega$.

Observe that if $\mathcal{A}$ is a projection compressible type II subalgebra of $\mathbb{M}_n$ and $\mathcal{F}_{II}(\mathcal{A})$ contains a triple $\Omega = (d, \bigoplus_{i=1}^{m} V_i)$, then Corollary 3.0.3 implies that $Q_{2\Omega}AQ_{2\Omega} = Q_{2\Omega}M_nQ_{2\Omega}$ and $Q_{1\Omega}AQ_{1\Omega} = CQ_{1\Omega}$ for each $i \in \{1, 3\}$. In this case, $n_{1\Omega} = k - 1$, $n_{2\Omega} = d$, and $n_{3\Omega} = n - d - k + 1$.

We will begin by considering the extreme case of a type II algebra $\mathcal{A}$ such that $\mathcal{F}_{II}(\mathcal{A})$ contains a triple $\Omega = (d, k, \bigoplus_{i=1}^{m} V_i)$ with $k = 1$ or $k = m$. The projection compressible algebras of this form can be easily identified using Theorem 4.0.2.

Proposition 5.0.2. Let $\mathcal{A}$ be a projection compressible type II subalgebra of $\mathbb{M}_n$. If there exists a triple $\Omega = (d, k, \bigoplus_{i=1}^{m} V_i)$ in $\mathcal{F}_{II}(\mathcal{A})$ with $k = 1$ or $k = m$, then $\mathcal{A}$ is the unitalization of an $\mathcal{L}$-algebra. Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. Let $\Omega \in \mathcal{F}_{II}(\mathcal{A})$ be as in the statement above. By replacing $\mathcal{A}$ with $A^{eT}$ if necessary, we may assume that $k = m$. Furthermore, since any algebra similar to an $\mathcal{L}$-algebra is again an $\mathcal{L}$-algebra, we may assume that $\mathcal{A}$ is unihinged with respect to $\bigoplus_{i=1}^{m} V_i$.

Since $Rad(\mathcal{A})$ is a right $M_d$-module, Theorem 4.0.2 indicates that $Rad(\mathcal{A}) = Q_1M_nQ_2$ for some projection $Q'_1 \leq Q_1\Omega$. It follows that,

$$A = BD(\mathcal{A}) + Rad(\mathcal{A}) = (Q'_1 + Q_{2\Omega})M_nQ_{2\Omega} + CI,$$

and hence $\mathcal{A}$ is the unitalization of an $\mathcal{L}$-algebra.

By Proposition 5.0.2, it suffices to consider the type II algebras for which the triples $\Omega = (d, k, \bigoplus_{i=1}^{m} V_i)$ in $\mathcal{F}_{II}(\mathcal{A})$ are such that $1 < k < m$. For such an algebra $\mathcal{A}$ and triple $\Omega$, the projections $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ are all non-zero. In the language of Theorem 2.0.8 and the remarks that follow, the corners $Q_{1\Omega}AQ_{1\Omega}$ and $Q_{3\Omega}AQ_{3\Omega}$ are diagonal algebras, each comprised of mutually linked $1 \times 1$ blocks. Note that the blocks in
\( Q_{1\Omega}AQ_{1\Omega} \) may be linked to those in \( Q_{3\Omega}AQ_{3\Omega} \). If this is the case, we will say that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked. Otherwise, we will say that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are unlinked. In either case, dimension considerations imply that neither \( Q_{1\Omega} \) nor \( Q_{3\Omega} \) is linked to \( Q_{2\Omega} \). As in our analysis of type I algebras, it will be important to distinguish between these settings.

The following lemma concerns the independence of the blocks in the radical of an algebra in reduced block upper triangular form, and will play a key role in our study of type II algebras.

**Lemma 5.0.3.** Let \( n \) be a positive integer, and let \( A \) be a unital subalgebra of \( \mathbb{M}_n \) in reduced block upper triangular form with respect to a decomposition \( \bigoplus_{i=1}^n V_i \) of \( \mathbb{C}^n \). Suppose that there is an index \( 1 < k < m \), that is unlinked from all indices \( i \neq k \). Let \( Q_1, Q_2 \), and \( Q_3 \) denote the orthogonal projections onto \( \bigoplus_{i<k} V_i \), \( V_k \), and \( \bigoplus_{i>k} V_i \), respectively, and assume that \( Q_1 \operatorname{Rad}(A)Q_1 = Q_3 \operatorname{Rad}(A)Q_3 = \{0\} \).

(i) For every \( R \in \operatorname{Rad}(A) \), there are elements \( R' = Q_1RQ_2 \) and \( R'' = R''Q_3 \) in \( \operatorname{Rad}(A) \) such that

\[
R'Q_2 = Q_1RQ_2 \quad \text{and} \quad Q_2R'' = Q_2RQ_3.
\]

(ii) If there exist projections \( Q_1' \leq Q_1 \) and \( Q_3' \leq Q_3 \) such that

\[
Q_1 \operatorname{Rad}(A)Q_2 = Q_1'M_nQ_2, \quad Q_2 \operatorname{Rad}(A)Q_3 = Q_2'M_nQ_3,
\]

and

\[
Q_1 \operatorname{Rad}(A)Q_3 = Q_1' \operatorname{Rad}(A)Q_3',
\]

then

\[
\operatorname{Rad}(A) = Q_1'M_nQ_2 + Q_1'M_nQ_3' + Q_2'M_nQ_3'.
\]

**Proof.** For (i), let \( R \) belong to \( \operatorname{Rad}(A) \). Since \( V_k \) is unlinked from all other spaces \( V_i \), there is an element \( A \in A \) such that \( Q_1AQ_1 = Q_3AQ_3 = 0 \) and \( Q_2AQ_2 = Q_2 \). Thus, with respect to the decomposition \( \mathbb{C}^n = \operatorname{ran}(Q_1) \oplus \operatorname{ran}(Q_2) \oplus \operatorname{ran}(Q_3) \), \( A \) and \( R \) may be expressed as

\[
A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & I & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & 0 \end{bmatrix}
\]

for some \( A_{ij} \) and \( R_{ij} \). It is then easy to see that \( R' := RA \) and \( R'' := AR \) define elements of \( \operatorname{Rad}(A) \) that satisfy the requirements of (i).

For (ii), let \( M_1 \) and \( M_2 \) denote arbitrary elements of \( Q_1'M_nQ_2 \) and \( Q_2'M_nQ_3' \), respectively. By (i), there are elements \( S_1 \) and \( S_2 \) in \( Q_1'M_nQ_3 \) such that \( M_1 + S_1 \) and \( M_2 + S_2 \) belong to \( \operatorname{Rad}(A) \). Moreover, since \( Q_1 \operatorname{Rad}(A)Q_3 = Q_1' \operatorname{Rad}(A)Q_3' \), we have that \( S_1 \) and \( S_2 \) are contained in \( Q_1'M_nQ_3' \).

Observe that \( R := (M_1 + S_1)(M_2 + S_2) \) belongs to \( \operatorname{Rad}(A) \). With respect to the decomposition of \( \mathbb{C}^n \) described above, this element can be expressed as

\[
R = \begin{bmatrix} 0 & M_1 & S_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & S_2 \\ 0 & 0 & M_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & M_1M_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

But since \( M_1 \) and \( M_2 \) were arbitrary, this implies that \( Q_1'M_nQ_3' \subseteq \operatorname{Rad}(A) \). In particular, \( \operatorname{Rad}(A) \) contains \( S_1 \) and \( S_2 \). It then follows that \( M_1 \) and \( M_2 \) belong to \( \operatorname{Rad}(A) \) as well. We conclude that \( \operatorname{Rad}(A) \) contains \( Q_1'M_nQ_2 \) and \( Q_2'M_nQ_3' \), as \( M_1 \) and \( M_2 \) were arbitrary.

Notably, if \( A \) is a type II algebra for which the triples \( \Omega = (d, k, \bigoplus_{i=1}^m V_i) \) in \( \mathcal{F}_{II} \) are such that \( 1 < k < m \), then \( Q_{2\Omega} \) is necessarily unlinked from \( Q_{1\Omega} \) and \( Q_{3\Omega} \), and hence \( A \) satisfies the assumptions of Lemma 5.0.3.

### §5.1 Type II Algebras with Unlinked Projections

Let us first consider the type II algebra \( A \) for which the triples \( \Omega = (d, k, \bigoplus_{i=1}^m V_i) \) in \( \mathcal{F}_{II}(A) \) are such that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are unlinked. We aim to show that the only such algebras with the projection compression property are those that are unitarily equivalent to the type II algebra in Example 1.0.1(i). To accomplish this goal, we will first show in Lemma 5.1.1 that the result holds in the \( \mathbb{M}_4 \) setting. An extension to larger type II algebras will be made in Theorem 5.1.2 by applying Lemma 5.1.1 to their \( 4 \times 4 \) compressions.
Lemma 5.1.1. Let \( A \) be a projection compressible type II subalgebra of \( \mathbb{M}_4 \). Assume that \( \mathcal{F}_{1I}(A) \) contains a pair \( \Omega = (d, k, \bigoplus_{i=1}^3 V_i) \) such that \( d = k = 2 \). If \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are unlinked, then \( A \) is unitarily equivalent to \[ CQ_{1\Omega} + CQ_{3\Omega} + (Q_{1\Omega} + Q_{2\Omega})\mathbb{M}_4(Q_{2\Omega} + Q_{3\Omega}), \]
the type II algebra from Example 4.1.4; Consequently, \( A \) is idempotent compressible.

Proof. Suppose to the contrary that \( A \) is not unitarily equivalent to the algebra described above. Lemma 5.0.3 (ii) then implies that
\[
Q_{1\Omega}Rad(A)Q_{2\Omega} \neq Q_{1\Omega}\mathbb{M}_4Q_{2\Omega} \quad \text{or} \quad Q_{2\Omega}Rad(A)Q_{3\Omega} \neq Q_{2\Omega}\mathbb{M}_4Q_{3\Omega}.
\]
By replacing \( A \) with \( A^{rt} \) if necessary, we may assume that \( Q_{1\Omega}Rad(A)Q_{2\Omega} \neq Q_{1\Omega}\mathbb{M}_4Q_{2\Omega} \). Consequently, \( Q_{1\Omega}Rad(A)Q_{2\Omega} = \{0\} \) by Theorem 4.0.2.

An application of Theorem 2.0.5 provides a precise description of \( Q_{1\Omega}AQ_{2\Omega} \). Since \( A \) is similar to \( BD(A) + Rad(A) \) via a block upper triangular similarity, there is a fixed element \( T \in Q_{1\Omega}\mathbb{M}_4Q_{2\Omega} \) such that
\[ Q_{1\Omega}AQ_{2\Omega} = (Q_{1\Omega}AQ_{2\Omega})T - T(Q_{2\Omega}AQ_{2\Omega}) \] for every \( A \in \mathcal{A} \).

For each \( i \in \{1, 2, 3\} \), fix an orthonormal basis \( \{e^{(i)}_1, e^{(i)}_2, \ldots, e^{(i)}_{n_i}\} \) for \( \text{ran}(Q_{i\Omega}) \). To simplify matters, we may use Theorem 4.1.4 and the remarks that follow to assume that \( \langle Te^{(2)}_2, e^{(1)}_1 \rangle = 0 \). That is, with respect to the basis \( \{e^{(1)}_1, e^{(2)}_1, e^{(2)}_2, e^{(3)}_1\} \) for \( \mathbb{C}^4 \), each \( A \in \mathcal{A} \) may be expressed as
\[
A = \begin{bmatrix}
  a_{11} & a_{11}t - ta_{22} & -ta_{23} & a_{14} \\
  a_{22} & a_{23} & a_{24} & \\
  a_{32} & a_{33} & a_{34} & \\
  a_{44} & & & \\
\end{bmatrix},
\]
where \( a_{ij} \in \mathbb{C} \) and \( t := \langle Te^{(2)}_2, e^{(1)}_1 \rangle \). Here, the entries on the block-diagonal may be selected arbitrarily.

To reach a contradiction, consider the matrices
\[
P_0 := \begin{bmatrix}
  2 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 2 & 0 \\
  0 & 1 & 0 & 1 \\
\end{bmatrix}, \quad P_1 := \begin{bmatrix}
  1 & 0 & 0 & -1 \\
  0 & 2 & 0 & 0 \\
  0 & 0 & 2 & 0 \\
 -1 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \text{and} \quad P_2 := \begin{bmatrix}
  1 & 0 & 0 & 1 \\
  0 & 2 & 0 & 0 \\
  0 & 0 & 2 & 0 \\
  1 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Observe that for each \( i \), \( \frac{1}{2}P_i \) is a projection in \( \mathbb{M}_4 \). Through direct computation, one may verify that
\[
\langle Be^{(2)}_2, e^{(1)}_1 \rangle + 2t\langle Be^{(2)}_2, e^{(2)}_1 \rangle = 0 \text{ for all } B \in P_0AP_0.
\]

Yet with \( A \) as above and \( B_0 := (P_0AP_0)^2 \), we have
\[
\langle B_0e^{(2)}_2, e^{(1)}_1 \rangle + 2t\langle B_0e^{(2)}_2, e^{(2)}_1 \rangle = 8a_{23}a_{14} - t(a_{11} - a_{44} - a_{24}).
\]
It follows that \( a_{23} = 0 \) for all \( A \in \mathcal{A} \) or \( a_{14} = t(a_{11} - a_{44} - a_{24}) \) for all \( A \in \mathcal{A} \). Indeed, it is clear that every element of \( A \) must satisfy at least one of these equations. But if \( A \) contained an operator \( A_1 \) satisfying only the first equation and an operator \( A_2 \) satisfying only the second, then neither equation would hold for \( A_1 + A_2 \). Since \( a_{23} \) may be selected arbitrarily, it must be that \( a_{14} = t(a_{11} - a_{44} - a_{24}) \) for every \( A \in \mathcal{A} \).

One may now derive similar relations using \( P_1 \) and \( P_2 \). Indeed, it is straightforward to check that for \( j \in \{1, 2\} \), the equation
\[
t(P_jAP_je^{(2)}_2, e^{(1)}_1) + 2(P_jAP_je^{(2)}_2, e^{(2)}_1) = 0
\]
holds for every \( A \in \mathcal{A} \). Yet if \( A_0 \) denotes any element of \( \mathcal{A} \) of the above form satisfying \( a_{11} = a_{23} = 1 \) and \( a_{44} = 0 \), then for \( B_j := (P_jAP_j)^2 \),
\[
\left(t(B_1e^{(2)}_2, e^{(2)}_1) + 2(B_1e^{(2)}_2, e^{(1)}_1)\right) - \left(t(B_2e^{(2)}_2, e^{(2)}_1) + 2(B_2e^{(2)}_2, e^{(1)}_1)\right) = 16t^2.
\]
Since \(B_1\) and \(B_2\) belong to \(P_1A_1P_2\) and \(P_2A_2P_2\), respectively, we conclude that \(t = 0\). That is, \(Q_{1}\Omega A Q_{2}\Omega = \{0\}\). It follows that with respect to the basis \(\{e_1^{(2)}, e_2^{(2)}, e_1^{(3)}\}\) for \(\mathbb{C}^4\), each \(A \in \mathcal{A}\) may be written as

\[
A = \begin{bmatrix}
a_{22} & a_{23} & 0 & a_{24} \\
a_{32} & a_{33} & 0 & a_{34} \\
a_{11} & 0 & a_{14} \\
a_{44}
\end{bmatrix}
\]

for some \(a_{ij} \in \mathbb{C}\). Theorem 3.0.1 now demonstrates that \(\mathcal{A}\) is not projection compressible, as the entries in \(BD(\mathcal{A})\) may be chosen arbitrarily. This is a contradiction. \(\blacksquare\)

**Theorem 5.1.2.** Let \(\mathcal{A}\) be a projection compressible type II subalgebra of \(M_n\), and assume that there is a triple \(\Omega = (d, k, \bigoplus_{i=1}^n \mathcal{V}_i)\) in \(\mathcal{F}_{II}(\mathcal{A})\) with \(1 < k < m\). If \(Q_{1}\Omega\) and \(Q_{3}\Omega\) are unlinked, then \(\mathcal{A}\) is unitarily equivalent to

\[
\mathbb{C}Q_{1}\Omega + \mathbb{C}Q_{3}\Omega + (Q_{1}\Omega + Q_{2}\Omega)M_n(Q_{2}\Omega + Q_{3}\Omega),
\]

the type II algebra from Example 1.0.1(i). Consequently, \(\mathcal{A}\) is idempotent compressible.

Proof. Suppose to the contrary that \(\mathcal{A}\) is not unitarily equivalent to the algebra described above. As in the proof of the previous result, we may appeal to Lemma 5.0.3 (ii) and assume without loss of generality that \(Q_{1}\Omega Rad(\mathcal{A})Q_{2}\Omega \neq Q_{1}\Omega M_n Q_{2}\Omega\). Thus, Theorem 4.0.2 gives rise to a proper subprojection \(Q_1\) of \(Q_{1}\Omega\) satisfying

\[
Q_{1}\Omega Rad(\mathcal{A})Q_{2}\Omega = Q_1' M_n Q_{2}\Omega.
\]

Define \(Q''_1 := Q_{1}\Omega - Q_1\) and let \(\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{n\Omega}^{(1)}\}\) be an orthonormal basis for \(\text{ran}(Q_{1}\Omega)\) such that

\[
\text{ran}(Q''_1) = \bigvee \{e_1^{(1)}, e_2^{(1)}, \ldots, e_{\ell}^{(1)}\}
\]

for some integer \(1 \leq \ell \leq n\Omega\). Since \(\mathcal{A}\) is similar to \(BD(\mathcal{A}) + Rad(\mathcal{A})\) via a matrix that is block upper triangular with respect to \(\mathbb{C}^\ell = \text{ran}(Q_{1}\Omega) \oplus \text{ran}(Q_{2}\Omega) \oplus \text{ran}(Q_{3\Omega})\), there is an operator \(T \in Q''_1 M_n Q_{2\Omega}\) such that

\[
Q''_1 A Q_{2\Omega} = (Q''_1 A Q''_1) T - T (Q_{2\Omega} A Q_{2\Omega}) \quad \text{for all} \quad A \in \mathcal{A}.
\]

By Theorem 4.1.4 one may choose a suitable orthonormal basis \(\{e_1^{(2)}, e_2^{(2)}, \ldots, e_{n\Omega}^{(2)}\}\) for \(\text{ran}(Q_{2\Omega})\) and adjust the basis for \(\text{ran}(Q''_1)\) if necessary to impose additional structure on \(T\). Specifically, one may assume that \(\langle Te_j^{(2)}, e_i^{(1)}\rangle = 0\) whenever \(i \neq j\).

Let \(e_1^{(3)}\) be any non-zero vector in \(\text{ran}(Q_{3\Omega})\), and define \(\mathcal{B} = \{e_1^{(1)}, e_2^{(1)}, e_2^{(2)}, e_1^{(3)}\}\). Let \(P\) denote the orthogonal projection onto the span of \(\mathcal{B}\), and consider the compression \(\mathcal{A}_0 := P \mathcal{A} \). It is easy to see that \(\mathcal{A}_0\) is a projection compressible type II subalgebra of \(M_n\). Moreover, if

\[
\mathcal{W}_1 := \mathbb{C} e_1^{(1)}, \quad \mathcal{W}_2 := \bigvee \{e_1^{(2)}, e_2^{(2)}\}, \quad \text{and} \quad \mathcal{W}_3 := \mathbb{C} e_1^{(3)},
\]

then the triple \(\Omega' = (2, 2, \bigoplus_{i=1}^3 \mathcal{W}_i)\) belongs to \(\mathcal{F}_{II}(\mathcal{A}_0)\). Since \(Q_{1}\Omega'\) and \(Q_{3}\Omega'\) are unlinked, \(\mathcal{A}_0\) is among the class of algebras addressed in Lemma 5.1.1. With respect to the basis \(\mathcal{B}\) for \(\text{ran}(P)\), however, every element of \(\mathcal{A}_0\) may be expressed as a matrix of the form

\[
A = \begin{bmatrix}
a_{11} & a_{11} t - ta_{22} & -ta_{23} & a_{14} \\
a_{22} & a_{23} & a_{24} \\
a_{32} & a_{33} & a_{34} \\
a_{44}
\end{bmatrix},
\]

where \(t := \langle Te_2^{(2)}, e_1^{(1)}\rangle\). Since \(\mathcal{A}_0\) is not of the form prescribed by Lemma 5.1.1 it follows that \(\mathcal{A}_0\) is not projection compressible—a contradiction. \(\blacksquare\)
§5.2 Type II Algebras with Linked Projections. Consider now the type II algebras \( \mathcal{A} \) for which the triples \( \Omega = (d, k, \bigoplus_{i=1}^{n} V_i) \) in \( \mathcal{F}_{II}(\mathcal{A}) \) are such that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked. It will be shown in Theorem 5.2.2 that all projection compressible algebras of this form are unitizations of \( \mathcal{Lr} \) algebras. The proof of this result requires a careful analysis of the upper triangular blocks in the semi-simple part of the algebra. The following lemma is the crux of this analysis.

**Lemma 5.2.1.** Let \( \mathcal{A} \) be a projection compressible type II subalgebra of \( \mathcal{M}_4 \). Assume that \( \mathcal{F}_{II}(\mathcal{A}) \) contains a triple \( \Omega = (d, k, \bigoplus_{i=1}^{3} V_i) \) with \( d = k = 2 \), and such that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked.

(i) If there are a constant \( t \in \mathbb{C} \) and for each \( i \in \{1, 2, 3\} \), an orthonormal basis \( \{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{1\Omega}}^{(i)}\} \) for \( \text{ran}(Q_{1\Omega}) \) such that

\[
\langle Ae_1^{(1)}, e_1^{(1)} \rangle = t \left( \langle Ae_1^{(1)}, e_1^{(1)} \rangle - \langle Ae_2^{(1)}, e_1^{(1)} \rangle \right)
\]

\[
\langle Ae_2^{(1)}, e_1^{(1)} \rangle = -t \langle Ae_2^{(1)}, e_1^{(1)} \rangle
\]

for all \( A \in \mathcal{A} \), then \( \langle Ae_1^{(1)}, e_1^{(1)} \rangle = -t \langle Ae_1^{(1)}, e_1^{(1)} \rangle \) for all \( A \in \mathcal{A} \).

(ii) If there are a constant \( t \in \mathbb{C} \) and for each \( i \in \{1, 2, 3\} \), an orthonormal basis \( \{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{3\Omega}}^{(i)}\} \) for \( \text{ran}(Q_{3\Omega}) \) as described above. Then with respect to the basis \( \{e_1^{(1)}, e_2^{(1)}, e_2^{(2)}, e_3^{(1)}\} \) for \( \mathbb{C}^4 \), each \( A \in \mathcal{A} \) can be written as

\[
A = \begin{bmatrix}
a_{11} & t(a_{11} - a_{22}) & -ta_{23} & a_{14} \\
a_{22} & a_{23} & a_{24} & \\
a_{32} & a_{33} & a_{34} & \\
1 & 0 & 1 & a_{11}
\end{bmatrix}
\]

for some \( a_{ij} \in \mathbb{C} \). Since \( \mathcal{A} \) is in reduced block upper triangular form, the entries on the block-diagonal may be chosen arbitrarily.

Consider the matrix

\[
P := \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

and note that \( \frac{1}{2}P \) is a projection in \( \mathcal{M}_4 \). One may verify that

\[
t\langle Be_2^{(2)}, e_1^{(1)} \rangle + 2\langle Be_2^{(2)}, e_1^{(1)} \rangle = 0 \quad \text{for all } B \in PAP.
\]

But with \( A \) as above and \( B := (PAP)^2 \), we see that

\[
t\langle Be_2^{(2)}, e_1^{(2)} \rangle + 2\langle Be_2^{(2)}, e_1^{(1)} \rangle = -8ta_{23}(a_{14} + ta_{24}).
\]

The projection compressibility of \( \mathcal{A} \) implies that \( B \) belongs to \( PAP \). Consequently, \( ta_{23}(a_{14} + ta_{24}) = 0 \) for all \( A \in \mathcal{A} \).

If \( t \neq 0 \), then either \( a_{23} = 0 \) for all \( A \in \mathcal{A} \) or \( a_{14} = -ta_{24} \) for all \( A \in \mathcal{A} \). Indeed, it is clear that every operator in \( \mathcal{A} \) must satisfy at least one of these equation. If, however, \( \mathcal{A} \) contained an operator \( A_1 \) satisfying the first equation but not the second, as well as an operator \( A_2 \) satisfying the second but not the first, then neither equation would hold for \( A_1 + A_2 \). Finally, since \( a_{23} \) can be selected arbitrarily, we conclude that either \( t = 0 \) or \( a_{14} = -ta_{24} \) for all \( A \).
If the latter holds, then every \( A \in \mathcal{A} \) satisfies the equation \( \langle Ae_1^{(3)}, e_1^{(1)} \rangle = -t\langle Ae_1^{(3)}, e_1^{(2)} \rangle \), as required. If instead \( t = 0 \), then with respect to the basis \( \{e_1^{(2)}, e_2^{(2)}, e_1^{(1)}, e_1^{(3)}\} \) for \( \mathbb{C}^4 \), each \( A \in \mathcal{A} \) may be expressed as a matrix of the form

\[
A = \begin{bmatrix}
a_{22} & a_{23} & 0 & a_{24} \\
a_{32} & a_{33} & 0 & a_{34} \\
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{11} & a_{12} & a_{13} & a_{14}
\end{bmatrix}
\]

for some \( a_{ij} \in \mathbb{C} \). It follows from Theorem 3.0.1 that \( a_{14} = \langle Ae_1^{(3)}, e_1^{(1)} \rangle = 0 \) for all \( A \), and hence the equation \( \langle Ae_1^{(3)}, e_1^{(1)} \rangle = -t\langle Ae_1^{(3)}, e_1^{(2)} \rangle \) holds in this case as well.

In the context of (ii), note that every \( A \in \mathcal{A} \) satisfies the equation \( \langle Ae_1^{(3)}, e_1^{(1)} \rangle = t\langle Ae_1^{(2)}, e_1^{(1)} \rangle \) for all \( A \in \mathcal{A} \).

**Theorem 5.2.2.** Let \( \mathcal{A} \) be a projection compressible type II subalgebra of \( \mathbb{M}_n \), and let \( \Omega = (d, k, \bigoplus_{i=1}^m V_i) \) be a triple in \( F_{II}(\mathcal{A}) \). If \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked, then \( \mathcal{A} \) is the unitization of an \( \mathcal{L}R \)-algebra. Consequently, \( \mathcal{A} \) is idempotent compressible.

**Proof.** Let \( \Omega \) be as above, and assume that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked. Note that if \( k = 1 \) or \( k = m \), then \( \mathcal{A} \) is the unitization of an \( \mathcal{L}R \)-algebra by Proposition 5.0.2. Thus, we will assume that \( 1 < k < m \). In this case, Theorem 4.0.2 gives rise to subprojections \( Q' \leq Q_{1\Omega} \) and \( Q' \leq Q_{3\Omega} \) such that

\[
Q_{1\Omega}Rad(\mathcal{A})Q_{2\Omega} = Q'\mathbb{M}_n Q_{2\Omega} \quad \text{and} \quad Q_{2\Omega}Rad(\mathcal{A})Q_{3\Omega} = Q'\mathbb{M}_n Q_{3\Omega}.
\]

Our goal is to show that \( \mathcal{A} \) is similar to \( \mathcal{A}_0 := (Q' + Q_{2\Omega})\mathbb{M}_n (Q_{2\Omega} + Q_3') + CI \).

Since \( \mathcal{A}_0 \) is the unitization of an \( \mathcal{L}R \)-algebra, this will demonstrate that so too is \( \mathcal{A} \). We will accomplish this task by determining the structure of \( Q_{1\Omega}AQ_{3\Omega} \).

Define \( Q'' := Q_{1\Omega} - Q' \) and \( Q''' := Q_{3\Omega} - Q_3' \). For each \( i \in \{1, 2, 3\} \), let \( \{e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}\} \) be an orthonormal basis for \( \text{ran}(Q_{i\Omega}) \) such that if \( Q'' \neq 0 \), then

\[
\text{ran}(Q''') = \bigvee \{e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)}\}
\]

for some \( \ell_i \in \{1, 2, \ldots, n_{i\Omega}\} \). Since \( \mathcal{A} \) is similar to \( BD(\mathcal{A}) + \text{Rad}(\mathcal{A}) \) via a matrix that is block upper triangular with respect to \( \mathbb{C}^n = \text{ran}(Q_{1\Omega}) \oplus \text{ran}(Q_{2\Omega}) \oplus \text{ran}(Q_{3\Omega}) \), there are operators \( T_1 \in Q''\mathbb{M}_n Q_{2\Omega} \) and \( T_2 \in Q_{2\Omega} Q_{3\Omega}'' \) such that every \( A \in \mathcal{A} \) satisfies

\[
Q'_{1\Omega}AQ_{2\Omega} = (Q''_{1\Omega}AQ''_{2\Omega})T_1 - T_1(Q_{2\Omega}AQ_{2\Omega}) \quad \text{and} \quad Q_{2\Omega}AQ'' = (Q_{2\Omega}AQ_{2\Omega})T_2 - T_2(Q''_{3\Omega}AQ'').
\]
We will begin by using Lemma 5.2.1 to identify the structure of \( Q''_1AQ_{3Ω} \). Of course, there is little to be said when \( Q''_1 = 0 \), so assume for now that \( Q''_1 \neq 0 \). By Theorem 4.1.4 and its subsequent remarks, one may change the orthonormal bases for \( \text{ran}(Q''_1) \) and \( \text{ran}(Q_{2Ω}) \) if required and assume that
\[
t_{ij}^{(1)} := \langle T_1e_j^{(2)}, e_i^{(1)} \rangle = 0 \text{ for all } i \neq j.
\]

Let \( i \) and \( i' \) be arbitrary indices from \{1, 2, \ldots, \ell_1\} and \{1, 2, \ldots, n_{3Ω}\}, respectively. Define
\[
j = \begin{cases} 
    i & \text{if } i \leq n_{2Ω}, \\
    1 & \text{otherwise},
\end{cases}
\]
and fix an index \( j' \in \{1, 2, \ldots, n_{2Ω}\} \setminus \{j\} \). Let \( P \) denote the orthogonal projection onto the span of \( B := \{ e_i^{(1)}, e_j^{(2)}, e_j^{(2)}_j, e_i^{(3)} \} \), and consider the algebra \( PAP \). If \( i > n_{2Ω} \), then for each \( A \in A \), \( PAP \) may be expressed as a matrix of the form
\[
PAP = \begin{bmatrix}
    a_{11} & 0 & 0 & a_{14} \\
    a_{22} & a_{23} & a_{24} & a_{34} \\
    a_{32} & a_{33} & a_{34} & a_{11} \\
    a_{41} & a_{42} & a_{43} & a_{11}
\end{bmatrix}
\]
with respect to \( B \). In this case, \( PAP \) is an algebra of the form described in Lemma 5.2.1 (i) with \( t = 0 \). Thus, this result implies that
\[
a_{14} = \langle Ae_i^{(3)}, e_i^{(1)} \rangle = 0 \text{ for all } A \in A.
\]
Suppose instead that \( i \leq n_{2Ω} \). We then have that for each \( A \in A \), \( PAP \) can be written as a matrix of the form
\[
PAP = \begin{bmatrix}
    a_{11} & a_{11}t_{ii}^{(1)} - t_{ii}^{(1)} & a_{23} & a_{14} \\
    a_{22} & a_{23} & a_{24} & a_{34} \\
    a_{32} & a_{33} & a_{34} & a_{11} \\
    a_{41} & a_{42} & a_{43} & a_{11}
\end{bmatrix}
\]
with respect to \( B \). It follows that \( PAP \) is of the form described in Lemma 5.2.1 (i) with \( t = t_{ii}^{(1)} \), and hence
\[
a_{14} = \langle Ae_i^{(3)}, e_i^{(1)} \rangle = -t_{ii}^{(1)} \langle Ae_i^{(3)}, e_i^{(2)} \rangle \text{ for all } A \in A.
\]
Since our choice of indices was arbitrary, these conclusions hold for all \( i \in \{1, 2, \ldots, \ell_1\} \) and all \( i' \in \{1, 2, \ldots, n_{3Ω}\} \). Consequently,
\[
Q''_1AQ_{3Ω} = -T_1Q_{2Ω}AQ_{3Ω} \text{ for all } A \in A.
\]

We now wish to obtain information on the structure of \( Q_{1Ω}AQ''_3 \). As in the analysis above, it will be convenient to simplify the description of \( T_2 \) by choosing suitable bases for \( \text{ran}(Q_{2Ω}) \) and \( \text{ran}(Q_{3Ω}) \). Specifically, Theorem 4.1.4 gives rise to operators \( V \in Q_{2Ω}M_nQ_{2Ω} \), \( W \in Q''_3M_nQ''_3 \), and a unitary \( U \in M_n \) such that
\[
(Q_{1Ω} + Q'_{3})U(Q_{1Ω} + Q'_{3}) = Q_{1Ω} + Q'_{3},
\]
\[
(Q_{2Ω} + Q''_{3})U(Q_{2Ω} + Q''_{3}) = V + W,
\]
and
\[
\langle U^*T_2Ue_j^{(3)}, e_i^{(2)} \rangle = \langle V^*T_2We_j^{(3)}, e_i^{(2)} \rangle = 0 \text{ for all } i \neq j.
\]
By considering the algebra \( U^*AU \) and arguing as above, one may deduce that
\[
(Q_{1Ω}AQ''_3) = (Q_{1Ω}AQ_{2Ω})T_2 \text{ for all } A \in A.
\]

Our findings thus far indicate that with respect to the decomposition
\[
\mathbb{C}^n = \text{ran}(Q''_3) \oplus \text{ran}(Q'_{3}) \oplus \text{ran}(Q_{2Ω}) \oplus \text{ran}(Q''_{3}) \oplus \text{ran}(Q'_{3}),
\]
each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix}
a_{11}I & 0 & a_{11}T_1 - T_1M & -T_1(MT_2 - a_{11}T_2) & -T_1J_2 \\
a_{11}I & J_1 & J_1T_2 & A_{25} \\
M & MT_2 - a_{11}T_2 & J_2 \\
a_{11}I & 0 & 0 & 0 \\
a_{11}I & 0 & 0 & 0 & J_2 \\
0 & 0 & 0 & 0 & a_{11}I \\
0 & 0 & 0 & 0 & a_{11}I \\
\end{bmatrix}$$

for some $a_{11} \in \mathbb{C}$ and operators $M \in Q_{20}M_nQ_{20}$, $J_1 \in Q_1^I\operatorname{Rad}(A)Q_{20}$, $J_2 \in Q_{20}\operatorname{Rad}(A)Q_3'$, and $A_{25} \in Q_1^I\mathbb{M}_nQ_3'$. With this description in hand we are prepared to show that $A$ is similar to $A_0$, and hence is the unitization of an $LR$-algebra.

Consider the operator $S := I - T_1 - T_2$. This $S$ is invertible with $S^{-1} = I + T_1 + T_2 + T_1T_2$. Moreover, for each $A \in \mathcal{A}$ as above, we have that

$$S^{-1}AS = \begin{bmatrix}
a_{11}I & 0 & 0 & 0 & 0 \\
a_{11}I & J_1 & 0 & A_{25} & 0 \\
M & 0 & J_2 & 0 & 0 \\
a_{11}I & 0 & 0 & 0 & J_2 \\
a_{11}I & 0 & 0 & 0 & a_{11}I \\
0 & 0 & 0 & 0 & a_{11}I \\
\end{bmatrix}$$

From here it is easy to see that $S^{-1}AS$ is a type II algebra that has a reduced block upper triangular form with respect to the above decomposition. Moreover,

$$Q_{1\Omega}\operatorname{Rad}(S^{-1}AS)Q_{2\Omega} = Q_1^I\operatorname{Rad}(A)Q_{2\Omega} = Q_1^I\mathbb{M}_nQ_{2\Omega}$$

and

$$Q_{2\Omega}\operatorname{Rad}(S^{-1}AS)Q_{3\Omega} = Q_{2\Omega}\operatorname{Rad}(A)Q_3' = Q_{2\Omega}\mathbb{M}_nQ_3'.$$

Thus, Lemma 5.10.3 (ii) implies that

$$S^{-1}AS = (Q_1^I + Q_{2\Omega})\mathbb{M}_n(Q_{2\Omega} + Q_3') + CI = A_0,$$

as claimed. 

§6 Algebra of Type III

We now begin the final stage of our classification of unital projection compressible subalgebras of $\mathbb{M}_n$ when $n \geq 4$. The term type III will be used to describe a unital subalgebra $A$ of $\mathbb{M}_n$, $n \geq 4$, such that for every orthogonal decomposition $\bigoplus_{i=1}^{n} \mathcal{V}_i$ of $\mathbb{C}^n$ with respect to which $A$ is reduced block upper triangular, $\dim \mathcal{V}_i = 1$ for all $i$ (i.e., $m = n$), and there is an integer $k$ as in Corollary 3.0.2. It is obvious that such a $k$ must lie strictly between $1$ and $n$.

As in the preceding sections, it will be important to maintain a record of the integers $k$ and decompositions of $\mathbb{C}^n$ that satisfy the assumptions of Corollary 3.0.2 for a given type III algebra $A$.

**Definition 6.0.1.** If $\mathcal{A}$ is an algebra of type III, let $\mathcal{F}_{III}(\mathcal{A})$ denote the set of pairs $\Omega = (k, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ that satisfy the following conditions:

(i) $\bigoplus_{i=1}^{n} \mathcal{V}_i$ is an orthogonal decomposition of $\mathbb{C}^n$ with respect to which $\mathcal{A}$ is reduced block upper triangular;

(ii) $k$ is an integer in $\{2, \ldots, n-1\}$ such that if $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ denote the orthogonal projections onto $\bigoplus_{i<k} \mathcal{V}_i$, $\mathcal{V}_k$, and $\bigoplus_{i>k} \mathcal{V}_i$, respectively, then for each $i \in \{1, 3\}$,

$$(Q_{i\Omega} + Q_{2\Omega})\mathcal{A}(Q_{i\Omega} + Q_{2\Omega}) \neq \mathcal{C}(Q_{i\Omega} + Q_{2\Omega}).$$

**Notation.** If $\mathcal{A}$ is an algebra of type III and $\Omega = (k, \bigoplus_{i=1}^{n} \mathcal{V}_i)$ is a pair in $\mathcal{F}_{III}(\mathcal{A})$, let $n_{1\Omega} = k - 1$, $n_{2\Omega} = 1$, and $n_{3\Omega} = n - k$ denote the ranks of $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$, respectively. Note that since $n_{2\Omega} = 1$ and $n \geq 4$, we necessarily have max$(n_{1\Omega}, n_{3\Omega}) \geq 2$.

If $\mathcal{A}$ is a projection compressible algebra of type III with pair $\Omega \in \mathcal{F}_{III}(\mathcal{A})$, then $Q_{i\Omega}AQ_{j\Omega} = \mathbb{C}Q_{j\Omega}$ for each $i \in \{1, 2, 3\}$. Thus, each corner $Q_{i\Omega}AQ_{j\Omega}$ is a diagonal algebra comprised of mutually linked $1 \times 1$ blocks. Of course, the blocks in $Q_{i\Omega}AQ_{j\Omega}$ may or may not be linked to those in $Q_{j\Omega}AQ_{i\Omega}$. If there is linkage
between these blocks, we will say that the projections $Q_{\Omega}$ and $Q_{\bar{\Omega}}$ are linked; otherwise, we will say that they are unlinked.

Unlike in §5, it is now entirely possible that $Q_{2\Omega}$ is linked to $Q_{1\Omega}$ or $Q_{3\Omega}$. As the following result demonstrates, however, there do not exist projection compressible algebras of type III for which all projections $Q_{\Omega}$ are mutually linked.

**Proposition 6.0.2.** Let $\mathcal{A}$ be a projection compressible algebra of type III, and let $\Omega$ be a pair in $\mathcal{F}_{III}(\mathcal{A})$.

(i) If $Q_{2\Omega}$ is linked to $Q_{1\Omega}$, then $n_{1\Omega} = 1$ and $Q_{1\Omega} \cap \text{Rad}(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}M_nQ_{2\Omega}$.

(ii) If $Q_{2\Omega}$ is linked to $Q_{3\Omega}$, then $n_{3\Omega} = 1$ and $Q_{3\Omega} \cap \text{Rad}(\mathcal{A})Q_{2\Omega} = Q_{2\Omega}M_nQ_{3\Omega}$.

Consequently, $Q_{2\Omega}$ cannot be linked to both $Q_{1\Omega}$ and $Q_{3\Omega}$.

**Proof.** Clearly (ii) follows from (i) by replacing $\mathcal{A}$ with $\mathcal{A}^{\sigma T}$. Thus, it suffices to prove (i).

Suppose to the contrary that $n_{1\Omega} \geq 2$. For each $i \in \{1, 2, 3\}$, let $\{e_i^{(1)}, e_i^{(2)}, \ldots, e_i^{(n_{i\Omega})}\}$ be an orthonormal basis for $\text{ran}(Q_{i\Omega})$. For each index $j$ in $\{1, 2, \ldots, n_{3\Omega}\}$, let $P_j$ denote the orthogonal projection onto the span of $B_j := \{e_j^{(1)}, e_j^{(2)}, e_j^{(3)}\}$. Furthermore, define $P_j'$ to be the operator

$$P_j' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

acting on $\text{ran}(P_1)$ and written with respect to the basis $B_j$. It is clear that $\frac{1}{2}P_j'$ is a subprojection of $P_j$.

One may verify that every $B \in P_j'AP_j'$ satisfies the equation $\langle Be_j^{(2)}, e_1^{(2)} \rangle = \langle Be_j^{(1)}, e_2^{(1)} \rangle$. But if $A$ belongs to $\mathcal{A}$ and $C := (P_j'AP_j')^2$, then

$$\langle Ce_j^{(2)}, e_1^{(2)} \rangle - \langle Ce_j^{(1)}, e_2^{(1)} \rangle = 8\langle Ae_j^{(2)}, e_1^{(1)} \rangle \langle Ae_j^{(3)}, e_2^{(1)} \rangle.$$ 

Since $C$ is an element of $P_j'AP_j'$, the right-hand side of this equation must be zero. To obtain a contradiction, it therefore suffices to exhibit an element $A$ in $\mathcal{A}$ such that for some $j \in \{1, 2, \ldots, n_{3\Omega}\}$, both $\langle Ae_j^{(2)}, e_1^{(1)} \rangle$ and $\langle Ae_j^{(3)}, e_2^{(1)} \rangle$ are non-zero.

First suppose that the projections $Q_{1\Omega}$, $Q_{2\Omega}$, and $Q_{3\Omega}$ are mutually linked. By definition of $\Omega$ as a pair in $\mathcal{F}_{III}(\mathcal{A})$, there exist $i \in \{1, 2, \ldots, n_{1\Omega}\}$ and $j \in \{1, 2, \ldots, n_{3\Omega}\}$, as well as $A_1, A_2 \in \mathcal{A}$, such that $\langle A_1 e_i^{(1)}, e_1^{(1)} \rangle \neq 0$ and $\langle A_2 e_j^{(1)}, e_1^{(1)} \rangle \neq 0$. By reordering the basis for $\text{ran}(Q_{1\Omega})$ if necessary, we may assume that $i = 1$. If $\langle A_2 e_j^{(2)}, e_1^{(1)} \rangle \neq 0$ or $\langle A_1 e_j^{(3)}, e_1^{(2)} \rangle \neq 0$, then we obtain the required contradiction. Otherwise, $A := A_1 + A_2$ is such that $\langle A_2 e_j^{(2)}, e_1^{(1)} \rangle \neq 0$ and $\langle A_1 e_j^{(3)}, e_1^{(2)} \rangle \neq 0$, as desired.

Now suppose that $Q_{2\Omega}$ is unlinked from $Q_{1\Omega}$ and $Q_{3\Omega}$. By reordering the basis for $\text{ran}(Q_{1\Omega})$ if necessary, we may obtain an element $A_1 \in \mathcal{A}$ such that $\langle A_1 e_j^{(2)}, e_1^{(1)} \rangle \neq 0$. If there is an element $A_2 \in \mathcal{A}$ such that $\langle A_2 e_j^{(3)}, e_1^{(2)} \rangle \neq 0$ for some $j \in \{1, 2, \ldots, n_{3\Omega}\}$, then arguments similar to those in the linked case above provide the required contradiction. Of course, it is now entirely possible that no such $A_2$ exists, as $Q_{2\Omega}$ and $Q_{3\Omega}$ are unlinked. That is, it may be that $Q_{2\Omega} \mathcal{A} Q_{3\Omega} = \{0\}$. Assume that this is the case.

Let $\mathcal{B} = \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}, e_2^{(2)}\}$, and define $P$ to be the orthogonal projection onto the span of $\mathcal{B}$. Note that with respect to the basis $\mathcal{B}$ for $\text{ran}(P)$, each $A \in P\mathcal{A}P$ may be written as

$$A = \begin{bmatrix} \alpha & 0 & a_{13} & a_{14} \\ \alpha & a_{23} & a_{24} & 0 \\ \beta & 0 & \alpha \end{bmatrix},$$

for some $\alpha, \beta$, and $a_{ij} \in \mathbb{C}$. Consider the operator

$$P' = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$
acting on \( \text{ran}(P) \) and written with respect to \( \mathcal{B} \). It is easy to see that \( \frac{1}{2}P' \) is a subprojection of \( P \). Moreover, one may verify that every element \( B = (b_{ij}) \) in \( P' \mathcal{A} \mathcal{P}' \) satisfies the equation \( b_{33} + 2b_{31} - b_{43} - 2b_{41} - b_{22} = 0 \). But if \( A \) is as above and we define \( (P' \mathcal{A}P')^2 = (c_{ij}) \), then
\[
c_{33} + 2c_{31} - c_{43} - 2c_{41} - c_{22} = 27a_{14}(\beta - \alpha).
\]
Since \( \alpha \) and \( \beta \) may be chosen arbitrarily, it must be that \( a_{14} = \langle A \varepsilon_1^{(2)} \rangle, e_1^{(1)} \rangle = 0 \) for all \( A \). This is a contradiction, as \( \langle A \varepsilon_1^{(2)}, e_1^{(1)} \rangle \neq 0 \). We therefore conclude that \( n_{3\Omega} = 1 \).

Since \( Q_{3\Omega} \) and \( Q_{2\Omega} \) are linked, yet \( Q_{4\Omega + Q_{2\Omega}} A (Q_{1\Omega} + Q_{2\Omega}) \neq \mathcal{C}(Q_{1\Omega} + Q_{2\Omega}) \) by definition of \( \Omega \) as a pair in \( \mathcal{F}_{III}(A) \), it follows that \( Q_{1\Omega} \text{Rad}(A) Q_{2\Omega} \neq \{0\} \). Consequently, \( Q_{3\Omega} \text{Rad}(A) Q_{2\Omega} = Q_{1\Omega} \mathcal{M}_{n} Q_{2\Omega} \) as \( n_{1\Omega} = n_{3\Omega} = 1 \).

The final claim now follows from the fact that \( \max \{n_{1\Omega}, n_{3\Omega}\} \geq 2 \).

The above result indicates that if \( A \) is a projection compressible algebra of type III and \( \Omega \) is a pair in \( \mathcal{F}_{III}(A) \), then there is a projection \( Q_{4\Omega} \) that is unlinked from \( Q_{2\Omega} \). In the case that this \( Q_{4\Omega} \) is also unlinked from the remaining projection \( Q_{4\Omega} \), one can say more about the structure of \( A \).

**Proposition 6.0.3.** Let \( A \) be a projection compressible type III subalgebra of \( \mathcal{M}_{n} \), and let \( \Omega \) be a pair in \( \mathcal{F}_{III}(A) \).

(i) If \( Q_{3\Omega} \) is unlinked from \( Q_{1\Omega} \) and \( Q_{2\Omega} \), then either \( Q_{2\Omega} \text{Rad}(A) Q_{3\Omega} = Q_{2\Omega} \mathcal{M}_{n} Q_{3\Omega} \); or \( n_{3\Omega} = 1 \) and \( Q_{2\Omega} \text{Rad}(A) Q_{3\Omega} = \{0\} \).

(ii) If \( Q_{1\Omega} \) is unlinked from \( Q_{2\Omega} \) and \( Q_{3\Omega} \), then either \( Q_{1\Omega} \text{Rad}(A) Q_{2\Omega} = Q_{1\Omega} \mathcal{M}_{n} Q_{2\Omega} \); or \( n_{1\Omega} = 1 \) and \( Q_{1\Omega} \text{Rad}(A) Q_{2\Omega} = \{0\} \).

**Proof.** As in the previous proof it is easy that (ii) follows from (i) by replacing \( A \) with \( A^{\sigma T} \). Thus, it suffices to prove (i).

Assume that \( Q_{3\Omega} \) is unlinked from both \( Q_{1\Omega} \) and \( Q_{2\Omega} \). Suppose for the sake of contradiction that \( n_{3\Omega} \geq 2 \) and \( Q_{2\Omega} \text{Rad}(A) Q_{3\Omega} \neq Q_{2\Omega} \mathcal{M}_{n} Q_{3\Omega} \). For each \( i \in \{1, 2, 3\} \), let \( \{e_{i}^{(1)}, e_{i}^{(2)}, \ldots, e_{i}^{(3)}\} \) be an orthonormal basis for \( \text{ran}(Q_{4\Omega}) \), and assume that the basis for \( \text{ran}(Q_{3\Omega}) \) is chosen so that \( \langle Re_{1}^{(3)}, e_{1}^{(2)} \rangle = 0 \) for all \( R \in \text{Rad}(A) \).

Define \( \mathcal{B} = \{e_{1}^{(1)}, e_{1}^{(2)}, e_{1}^{(3)}, e_{2}^{(3)}\} \), let \( P \) denote the orthogonal projection onto the span of \( \mathcal{B} \), and consider the compression \( A_0 := \mathcal{P} \mathcal{A} \mathcal{P} \). As a consequence of Theorem 2.0.5, there is a constant \( t \in \mathbb{C} \) such that with respect to the basis \( B \) for \( \text{ran}(P) \), each \( A \) in \( A_0 \) admits a matrix of the form
\[
A = \begin{bmatrix}
\alpha & a_{12} & a_{13} & a_{14} \\
\beta & t(\beta - \gamma) & a_{24} & 0 \\
\gamma & 0 & 0 & 0 \\
\end{bmatrix}
\]
for some \( \alpha, \beta, \gamma, \) and \( a_{ij} \) in \( \mathbb{C} \). Note that in the case that \( Q_{1\Omega} \) and \( Q_{2\Omega} \) are linked, \( \alpha \) and \( \beta \) must coincide for each \( A \in A_0 \). In the case that they are unlinked, these values may be chosen independently. With this in mind, the following arguments are applicable to either setting.

Consider the matrices
\[
P_1 := \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
P_2 := \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 \\
\end{bmatrix},
\]
acting on \( \text{ran}(P) \) and written with respect to the basis \( \mathcal{B} \). It is easy to see that \( \frac{1}{2}P_1 \) and \( \frac{1}{2}P_2 \) are subprojections of \( P \). In addition, one may verify that every \( B \in P_1 A_0 P_1 \) satisfies the equation
\[
\langle Be_1^{(3)}, e_{1}^{(2)} \rangle - t\langle Be_1^{(2)}, e_{1}^{(2)} \rangle + t\langle Be_1^{(3)}, e_{1}^{(3)} \rangle = 0.
\]
Thus, if \( A \) belongs to \( A_0 \) and \( C := (P_1 A \mathcal{P}_1)^2 \), then
\[
\langle Ce_1^{(3)}, e_{1}^{(2)} \rangle - t\langle Ce_1^{(2)}, e_{1}^{(2)} \rangle + t\langle Ce_1^{(3)}, e_{1}^{(3)} \rangle = 8\langle Ae_1^{(3)}, e_{1}^{(2)} \rangle \left( \langle Ae_1^{(3)}, e_{1}^{(1)} \rangle - t\langle Ae_1^{(2)}, e_{1}^{(1)} \rangle \right)
\]
must be zero. It follows that \( \langle Ae_1^{(3)}, e_{1}^{(2)} \rangle = 0 \) for all \( A \in A_0 \), or \( \langle Ae_1^{(3)}, e_{1}^{(1)} \rangle = t\langle Ae_1^{(2)}, e_{1}^{(1)} \rangle \) for all \( A \in A_0 \). Indeed, it is clear that every member of \( A_0 \) must satisfy at least one of these equations. If, however, there
were elements $A_1$ and $A_2$ in $\mathcal{A}_0$ such that $\langle A_1 e_2^{(3)}, e_1^{(2)} \rangle \neq 0$ and $\langle A_2 e_1^{(3)}, e_1^{(1)} \rangle \neq t\langle A_2 e_1^{(2)}, e_1^{(1)} \rangle$, then neither equation would be satisfied by their sum.

If it were the case that $\langle A_2 e_1^{(3)}, e_1^{(2)} \rangle = 0$ for every $A \in \mathcal{A}_0$, then by viewing $\mathcal{A}_0$ as an algebra of matrices with respect to the reordered basis $\{e_1^{(1)}, e_2^{(3)}, e_1^{(2)}, e_1^{(3)}\}$ for $\text{ran}(P)$, $\mathcal{A}_0$ would be seen to lack the projection compression property by Theorem 3.0.1. This is clearly a contradiction, so it must be that

$$\langle A e_1^{(3)}, e_1^{(1)} \rangle = t\langle A e_1^{(2)}, e_1^{(1)} \rangle$$
for all $A$.

From here one may verify that every $B \in P_2\mathcal{A}_0 P_2$ satisfies the equation

$$2(B e_1^{(3)}, e_1^{(2)}) - t(B e_1^{(2)}, e_1^{(2)}) + t(B e_2^{(3)}, e_2^{(3)}) = 0.$$

In particular, if $A \in \mathcal{A}_0$ is as above, then this equation must also hold for $D := (P_2 A P_2)^2$. Since

$$2(D e_1^{(3)}, e_1^{(2)}) - t(D e_1^{(2)}, e_1^{(2)}) + t(D e_2^{(3)}, e_2^{(3)}) = 8t(\beta - \gamma)(\alpha - \gamma)$$
and $\gamma$ may be selected independently from $\alpha$ and $\beta$, we deduce that $t = 0$. It is now evident that every $A \in \mathcal{A}_0$ can be expressed as a matrix of the form

$$A = \begin{bmatrix}
\alpha & 0 & a_{12} & a_{14} \\
\gamma & 0 & 0 & a_{24} \\
\beta & a_{12} & a_{14} & \gamma
\end{bmatrix}$$
with respect to the basis $\{e_1^{(1)}, e_2^{(3)}, e_1^{(2)}, e_2^{(3)}\}$ for $\text{ran}(P)$. Thus, Theorem 3.0.1 provides the required contradiction.

It must therefore be the case that $Q_{2\Omega} \text{Rad}(A)Q_{3\Omega} = Q_{2\Omega} M_n Q_{3\Omega}$ or $n_{3\Omega} = 1$. Of course, in the event that $Q_{2\Omega} \text{Rad}(A)Q_{3\Omega} \neq Q_{2\Omega} M_n Q_{3\Omega}$ and hence $n_{3\Omega} = 1$, it follows immediately that $Q_{2\Omega} \text{Rad}(A)Q_{3\Omega} = \{0\}$. ■

The preceding propositions will be key ingredients in our treatment of projection compressible algebras of type III. Our analysis will proceed in the same spirit as those for algebras of types I or II. We will begin in §6.1 by classifying the projection compressible type III algebras for which the projections $Q_{\Omega}$ are mutually unlinked. In §6.2 we will classify the projection compressible type III algebras for which exactly two distinct projections $Q_{\Omega}$ and $Q_{\Omega'}$ are linked.

§6.1 Type III Algebras with Unlinked Projections. In this section we present a classification of the projection compressible type III algebras for which the pairs $\Omega$ in $\mathcal{F}_{III}$ are such that no two distinct projections $Q_{\Omega}$ and $Q_{\Omega'}$ are linked. Such algebras include the algebra from Example 1.0.1(i) when $Q_1 \neq 0$, $Q_3 \neq 0$ and $\dim Q_2 = 1$; and the algebra from Example 1.0.1(ii). As the following theorem demonstrates, every projection compressible type III algebra with mutually unlinked projections is either transpose equivalent to the former, or transposable similar to the latter.

**Theorem 6.1.1.** Let $\mathcal{A}$ be a projection compressible type III subalgebra of $M_n$. If there is a pair $\Omega$ in $\mathcal{F}_{III}(\mathcal{A})$ such that no two distinct projections $Q_{\Omega}$ and $Q_{\Omega'}$ are linked, then $\mathcal{A}$ is transpose equivalent to the type III algebra from Example 1.0.1(i) or transpose similar to the algebra from Example 1.0.1(ii). Consequently, $\mathcal{A}$ is idem- potent compressible.

**Proof.** Let $\Omega = (k, \bigoplus_{i=1}^n V_i)$ be a pair in $\mathcal{F}_{III}(\mathcal{A})$ as in the statement of the theorem. For each $i$ in $\{1, 2, 3\}$, fix an orthonormal basis $\{e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_{\Omega}}^{(i)}\}$ for $\text{ran}(Q_{\Omega})$.

Note that if $Q_{1\Omega} \text{Rad}(\mathcal{A})Q_{2\Omega} = Q_{1\Omega} M_n Q_{2\Omega}$ and $Q_{2\Omega} \text{Rad}(\mathcal{A})Q_{3\Omega} = Q_{2\Omega} M_n Q_{3\Omega}$, then by Lemma 5.0.3(ii),

$$\text{Rad}(\mathcal{A}) = Q_{1\Omega} M_n Q_{2\Omega} + Q_{1\Omega} M_n Q_{3\Omega} + Q_{2\Omega} M_n Q_{3\Omega}.$$ 

In this case, $\mathcal{A}$ is the type III algebra from Example 1.0.1(i) so $\mathcal{A}$ is idem- potent compressible. It therefore suffices to consider the case in which $Q_{1\Omega} \text{Rad}(\mathcal{A})Q_{2\Omega} \neq Q_{1\Omega} M_n Q_{2\Omega}$ or $Q_{2\Omega} \text{Rad}(\mathcal{A})Q_{3\Omega} \neq Q_{2\Omega} M_n Q_{3\Omega}$.

By replacing $\mathcal{A}$ with $\mathcal{A}^{*T}$ if necessary, we may assume without loss of generality that

$$Q_{2\Omega} \text{Rad}(\mathcal{A})Q_{3\Omega} \neq Q_{2\Omega} M_n Q_{3\Omega}.$$ 
It then follows from Proposition 6.0.3(i) that $n_{3\Omega} = 1$ and $Q_{2\Omega} \text{Rad}(\mathcal{A})Q_{3\Omega} = \{0\}$. Consequently, $n_{1\Omega} \geq 2$ and hence $Q_{1\Omega} \text{Rad}(\mathcal{A})Q_{2\Omega} = Q_{1\Omega} M_n Q_{2\Omega}$ by Proposition 6.0.3(ii).
The above observations imply that for every \( X \in Q_{10}M_nQ_{2\Omega} \), there exists an element \( Y_X \in Q_{10}M_nQ_{3\Omega} \) such that \( X + Y_X \in \text{Rad}(A) \). Additionally, as a consequence of Theorem 2.0.5, there is a constant \( t \in \mathbb{C} \) such that

\[
\langle Ae_1^{(3)}, e_1^{(2)} \rangle = t \left( \langle Ae_1^{(2)}, e_1^{(2)} \rangle - \langle Ae_1^{(3)}, e_1^{(3)} \rangle \right)
\]

for all \( A \in \mathcal{A} \).

It therefore suffices to prove that \( \text{Rad}(A) = Q_{10}M_n(Q_{2\Omega} + Q_{3\Omega}) \). Indeed, when this is the case, consider the operator \( S := I - te_1^{(2)} \otimes e_1^{(3)*} \in M_n \). One may verify that \( S \) is invertible with \( S^{-1} = I + te_1^{(2)} \otimes e_1^{(3)*} \), and \( S^{-1}AS \) is the anti-transpose of the type III algebra from Example 1.0.1(iii).

To this end, note that since \( Q_{1\Omega}, Q_{2\Omega} \), and \( Q_{3\Omega} \) are mutually unlinked, there is an element \( A_1 \in \mathcal{A} \) such that \( Q_{2\Omega}A_1Q_{3\Omega} = Q_{2\Omega} \) and \( Q_{1\Omega}A_1Q_{1\Omega} = Q_{3\Omega}A_1Q_{3\Omega} = 0 \). With respect to the direct sum decomposition \( \mathbb{C}^n = \text{ran}(Q_{1\Omega}) \oplus \text{ran}(Q_{2\Omega}) \oplus \text{ran}(Q_{3\Omega}) \), we may write

\[
A_1 = \begin{bmatrix}
0 & A_{12} & A_{13} \\
1 & t & 0
\end{bmatrix}
\]

for some \( A_{12} \in Q_{10}M_nQ_{2\Omega} \) and \( A_{13} \in Q_{10}M_nQ_{3\Omega} \). Thus, for any \( X \in Q_{10}AQ_{2\Omega} \), there exists \( Y_X \in Q_{10}AQ_{3\Omega} \) such that \( \text{Rad}(A) \) contains

\[
(X + Y_X)A_1 = \begin{bmatrix}
0 & X & Y_X \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & A_{12} & A_{13} \\
1 & t & 0
\end{bmatrix} = \begin{bmatrix}
0 & X & tX \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We conclude that \( \text{Rad}(A) = \mathcal{R}^{(1)} + \mathcal{R}^{(2)} \) where

\[
\mathcal{R}^{(1)} := \left\{ \begin{bmatrix}
0 & X & tX \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} : X \in M_{(k-1) \times 1} \right\},
\]

and \( \mathcal{R}^{(2)} := \text{Rad}(A) \cap Q_{10}M_nQ_{3\Omega} \).

We claim that \( \mathcal{R}^{(2)} \) must be equal to \( Q_{10}M_nQ_{3\Omega} \). Suppose to the contrary that this is not the case. By changing the orthonormal basis for ran(\(Q_{1\Omega}) \) if necessary, we may assume that

\[
\langle Ye_1^{(3)}, e_1^{(1)} \rangle = 0 \quad \text{for all} \quad Y \in \mathcal{R}^{(2)}.
\]

Consider the set \( \mathcal{B} = \{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\} \) and let \( P \) denote the orthogonal projection onto the span of \( \mathcal{B} \). Define \( \mathcal{A}_0 \) to be the compression \( PA \), and accordingly, define

\[
\mathcal{R}_0^{(1)} := PR^{(1)}P \quad \text{and} \quad \mathcal{R}_0^{(2)} := PR^{(2)}P.
\]

Since \( \mathcal{A}_0 = S + \text{Rad}(\mathcal{A}_0) \) where \( S \) is similar to \( BD(\mathcal{A}_0) \) via a block upper triangular similarity, there are constants \( u_1, u_2, v_1, v_2 \in \mathbb{C} \) such that each \( A \in \mathcal{A}_0 \) can be written as

\[
A = \begin{bmatrix}
\alpha & 0 & v_1(\alpha - \beta) & u_1(\alpha - \gamma) - tv_1(\beta - \gamma) \\
\alpha & \beta & u_2(\alpha - \gamma) - tv_2(\beta - \gamma) & t(\beta - \gamma)
\end{bmatrix} + \begin{bmatrix}
0 & 0 & x_1 & tx_1 \\
0 & 0 & x_2 & tx_2
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & y
\end{bmatrix}.
\]

where the above summands are expressed with respect to the basis \( \mathcal{B} \) for ran(\(P\)), and belong to \( S, \mathcal{R}_0^{(1)} \), and \( \mathcal{R}_0^{(2)} \), respectively. We will obtain a contradiction by showing that a certain compression of \( \mathcal{A}_0 \) violates Theorem 3.0.1. To accomplish this goal, it will first be necessary to prove that \( t = u_1 = 0 \).

With this in mind, consider the matrices

\[
P_1 := \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}, \quad P_2 := \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad P_3 := \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix},
\]

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Adding these equations, it becomes evident that their entries satisfy the equations

\[ 4b_{14}^{(1)} + 2(tv_1 - u_1 + 1)b_{34}^{(1)} - 2t^2b_{13}^{(1)} + t(tv_1 - u_1 - 1)b_{22}^{(1)} - t(tv_1 - u_1 + 1)e_{33}^{(1)} = 0, \]

and

\[ 4b_{14}^{(2)} + 2(tv_1 - u_1 - 1)b_{34}^{(2)} - 2t^2b_{13}^{(2)} + t(tv_1 - u_1 + 1)b_{22}^{(2)} - t(tv_1 - u_1 - 1)e_{33}^{(2)} = 0. \]

Let \( A_0 \) denote the element of \( A_0 \) obtained by setting \( \alpha = \beta = x_2 = y = 0 \) and \( \gamma = x_1 = 1 \). That is,

\[
A_0 = \begin{bmatrix}
0 & 0 & 1 & tv_1 - u_1 + t \\
0 & 0 & tv_2 - u_2 \\
0 & -t & 1
\end{bmatrix}.
\]

Since \( A \) is projection compressible, \( C_1 := (P_1A_0P_1)^2 \) must satisfy the first equation above, while \( C_2 := (P_2A_0P_2)^2 \) must satisfy the second. But with \( C_1 = (c_{ij}^{(1)}) \) and \( C_2 = (c_{ij}^{(2)}) \), we have

\[
4c_{14}^{(1)} + 2(tv_1 - u_1 + 1)c_{34}^{(1)} - 2t^2c_{13}^{(1)} + t(tv_1 - u_1 - 1)c_{22}^{(1)} - t(tv_1 - u_1 + 1)e_{33}^{(1)} = 8t^2(tv_1 - u_1 - 1), \] and

\[
4c_{14}^{(2)} + 2(tv_1 - u_1 - 1)c_{34}^{(2)} - 2t^2c_{13}^{(2)} + t(tv_1 - u_1 + 1)c_{22}^{(2)} - t(tv_1 - u_1 - 1)e_{33}^{(2)} = -8t^2(tv_1 - u_1 + 1).
\]

Adding these equations, it becomes evident that \( t = 0 \). Consequently, \( Q_{10}\mathcal{R}_0^{(1)}Q_{30} = \{0\} \).

We now prove that \( u_1 = 0 \). Let \( A'_0 \) denote the element of \( A_0 \) obtained by setting \( \alpha = \beta = x_2 = y = 0 \). That is,

\[
A'_0 = \begin{bmatrix}
1 & 0 & 1 & u_1 \\
1 & 0 & u_2 \\
1 & 0 & 0
\end{bmatrix}.
\]

Since any element \( B_3 = (b_{ij}^{(3)}) \) in \( P_3A_0P_3 \) satisfies the equation \( 2b_{14}^{(3)} - u_1(b_{22}^{(3)} - b_{44}^{(3)}) = 0 \), it must be the case that the element \( C_3 := (P_3A_0P_3)^2 \) satisfies this equation as well. But if \( C_3 = (c_{ij}^{(3)}) \), then \( 2c_{14}^{(3)} - u_1(c_{22}^{(3)} - c_{44}^{(3)}) = 8u_1 \). Therefore, \( u_1 = 0 \).

We deduce that every element in \( A_0 \) admits a matrix representation of the form

\[
\begin{bmatrix}
\alpha & u_2(\alpha - \gamma) + y & 0 & v_2(\alpha - \beta) + x_2 \\
\gamma & 0 & 0 & \alpha \gamma \\
0 & 0 & 0 & \beta
\end{bmatrix}
\]

with respect to the reordered basis \( \{e_2^{(1)}, e_1^{(1)}, e_1^{(2)}, e_2^{(2)}\} \) for \( \text{ran}(P) \). Since the values of \( \alpha, \beta, \) and \( \gamma \) can be selected arbitrarily, an application of Theorem 3.0.1 shows that \( A_0 \) is not projection compressible—a contradiction.

The arguments above demonstrate that \( \mathcal{R}^{(2)} = Q_{10}\mathcal{M}Q_{30} \). Thus, \( \text{Rad}(A) = Q_{10}\mathcal{M}Q_{30},Q_{30} + Q_{30} \), as required.

### 6.2 Type III Algebras with Linked Projections

Let us now consider the projection compressible type III algebras that admit pairs \( \Omega \in \mathcal{F}_{III} \) with distinct mutually linked projections. By Proposition 6.0.2 it cannot be the case that all three projections \( Q_{10}, Q_{20}, \) and \( Q_{30} \) are mutually linked.

We begin with the case in which there is a pair \( \Omega \in \mathcal{F}_{III} \) with \( Q_{20} \) linked to \( Q_{10} \) or \( Q_{30} \). One example of such an algebra is given by the type III algebra from Example 1.0.1(iii). The following theorem demonstrates that this algebra is in fact, the only example up to transpose equivalence.
Theorem 6.2.1. Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$. If there is a pair $\Omega$ in $\mathcal{F}_{III}(\mathcal{A})$ such that $Q_{2\Omega}$ is linked to $Q_{1\Omega}$ or $Q_{3\Omega}$, then $\mathcal{A}$ is transverse equivalent to the algebra from Example [1.0.1(iii)]. Consequently, $\mathcal{A}$ is idempotent compressible.

Proof. Let $\Omega$ be as in the statement of the theorem. By replacing $\mathcal{A}$ with $\mathcal{A}^T$ if necessary, we may assume without loss of generality that $Q_{1\Omega}$ is the projection that is linked to $Q_{2\Omega}$. In this case, Proposition 6.0.2(i) implies that $n_{1\Omega} = 1$ and $Q_{1\Omega} \text{Rad}(\mathcal{A})Q_{2\Omega} = Q_{1\Omega}M_nQ_{2\Omega}$. It follows that $n_{3\Omega} \geq 2$, and hence $Q_{3\Omega}$ is unlinked from $Q_{2\Omega}$ and $Q_{3\Omega}$ by Proposition 6.0.2(ii). Thus, $Q_{2\Omega} \text{Rad}(\mathcal{A})Q_{3\Omega} = Q_{2\Omega}M_nQ_{3\Omega}$ by Proposition 6.0.3.

Fix operators $T_1 \in Q_{1\Omega}M_nQ_{2\Omega}$ and $T_2 \in Q_{2\Omega}M_nQ_{3\Omega}$. By the observations above, there exist $R_1, R_2$ in $\text{Rad}(\mathcal{A})$ such that $Q_{1\Omega}R_1Q_{2\Omega} = T_1$ and $Q_{2\Omega}R_2Q_{3\Omega} = T_2$. With respect to the direct sum decomposition $\mathbb{C}^n = \text{ran}(Q_{1\Omega}) \oplus \text{ran}(Q_{2\Omega}) \oplus \text{ran}(Q_{3\Omega})$, we may write

$$R_1 = \begin{bmatrix} 0 & T_1 & R_{13}^{(1)} \\ 0 & 0 & R_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} 0 & R_{12}^{(2)} & R_{13}^{(2)} \\ 0 & 0 & T_2 \\ 0 & 0 & 0 \end{bmatrix}$$

for some operators $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$. From here it is easy to see that $R_1R_2 = T_1T_2 \in \text{Rad}(\mathcal{A})$. Since $T_1$ and $T_2$ were arbitrary, we conclude that $\text{Rad}(\mathcal{A})$ contains $Q_{1\Omega}M_nQ_{3\Omega}$.

It will now be shown that each block $Q_{1\Omega}\text{Rad}(\mathcal{A})Q_{3\Omega}$ exists independently in $\text{Rad}(\mathcal{A})$. First, write $\mathcal{A} = S + \text{Rad}(\mathcal{A})$ where $S$ is semi-simple. Since $Q_{1\Omega}Q_{3\Omega}$ and $Q_{2\Omega}Q_{3\Omega}$ are linked, $S$ is similar to $\mathbb{C}(Q_{1\Omega} + Q_{2\Omega}) + \mathbb{C}Q_{3\Omega}$ via an upper triangular similarity. From this it follows that $Q_{1\Omega}SQ_{2\Omega} = \{0\}$, and hence $S$ contains an element $A$ of the form

$$A = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & I \end{bmatrix}.$$ 

Using the fact that $Q_{1\Omega}M_nQ_{3\Omega} \subseteq \text{Rad}(\mathcal{A})$, we deduce that $T_2 = R_2A - Q_{1\Omega}R_2AQ_{3\Omega}$ belongs to $\text{Rad}(\mathcal{A})$. Since $T_2$ was arbitrary, $\text{Rad}(\mathcal{A})$ contains $Q_{2\Omega}M_nQ_{3\Omega}$. Consequently, $T_1 = R_1 - Q_{1\Omega}R_1Q_{3\Omega} - Q_{2\Omega}R_1Q_{3\Omega}$ belongs to $\text{Rad}(\mathcal{A})$. This proves that $\text{Rad}(\mathcal{A})$ contains $Q_{1\Omega}M_nQ_{2\Omega}$, and therefore

$$\text{Rad}(\mathcal{A}) = Q_{1\Omega}M_nQ_{2\Omega} + Q_{1\Omega}M_nQ_{3\Omega} + Q_{2\Omega}M_nQ_{3\Omega}.$$ 

We conclude that $\mathcal{A} = \mathbb{C}(Q_{1\Omega} + Q_{2\Omega}) + \mathbb{C}Q_{3\Omega} + \text{Rad}(\mathcal{A})$. Thus, $\mathcal{A}$ is the algebra from Example [1.0.1(iii)] as claimed. □

With the proof of Theorem 6.2.1 complete, we are left only to classify the projection compressible type III algebras such that $\mathcal{F}_{III}$ contains a pair $\Omega$ in which $Q_{1\Omega}$ and $Q_{3\Omega}$ linked, yet neither of these projections is linked to $Q_{2\Omega}$. It will be shown in Theorem 6.2.3 that such an algebra is necessarily the unitization of an $\mathcal{LR}$-algebra. Unsurprisingly, the proof of this result shares many similarities with that of Theorem 5.2.2.

The first step in this direction is the following adaptation of Lemma 5.2.1 to the type III setting.

Lemma 6.2.2. Let $\mathcal{A}$ be a projection compressible type III subalgebra of $\mathbb{M}_n$, and suppose that $\mathcal{F}_{III}(\mathcal{A})$ contains a pair $\Omega$ in which $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked. Then

$$\text{(i)} \quad \text{If there exist a constant } t \in \mathbb{C} \text{ and for each } i \in \{1, 2, 3\}, \text{ an orthonormal basis } \{e_{1(i)}^{(i)}, e_{2(i)}^{(i)}, \ldots, e_{n(i)}^{(i)}\} \text{ for } \text{ran}(Q_{1\Omega}) \text{ such that }$$

$$\langle Ae_{1(i)}^{(i)}, e_{1(i)}^{(i)} \rangle = t\left(\langle e_{1(i)}^{(i)}, e_{1(i)}^{(1)} \rangle - \langle Ae_{1}^{(1)}, e_{1(i)}^{(2)} \rangle\right) \quad \text{for all } A \in \mathcal{A},$$

then $\langle Ae_{1(i)}^{(i)}, e_{1(i)}^{(1)} \rangle = -t\langle Ae_{1(i)}^{(1)}, e_{1(i)}^{(2)} \rangle$ for every $A \in \mathcal{A}$.

$$\text{(ii)} \quad \text{If there exist a constant } t \in \mathbb{C} \text{ and for each } i \in \{1, 2, 3\}, \text{ an orthonormal basis } \{e_{1(i)}^{(i)}, e_{2(i)}^{(i)}, \ldots, e_{n(i)}^{(i)}\} \text{ for } \text{ran}(Q_{3\Omega}) \text{ such that }$$

$$\langle Ae_{1(i)}^{(i)}, e_{1(i)}^{(i)} \rangle = t\left(\langle Ae_{1(i)}^{(i)}, e_{1(i)}^{(2)} \rangle - \langle Ae_{1(i)}^{(3)}, e_{1(i)}^{(3)} \rangle\right) \quad \text{for all } A \in \mathcal{A},$$

then $\langle Ae_{1(i)}^{(3)}, e_{1(i)}^{(1)} \rangle = t\langle Ae_{1(i)}^{(2)}, e_{1(i)}^{(1)} \rangle$ for every $A \in \mathcal{A}$ and each $i \in \{1, 2\}$. 


Proof. First note that since $Q_{1\Omega}$ and $Q_{3\Omega}$ are linked, Proposition [6.0.2] implies that neither of these projections is linked to $Q_{2\Omega}$.

We begin by considering the situation of (i). With respect to the basis $B = \{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\}$ for $\mathbb{C}^4$, each $A$ in $\mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix}
\alpha & 0 & t(\alpha - \beta) & a_{14} \\
\alpha & a_{23} & a_{24} & a_{34} \\
\beta & 0 & a_{34} & a_{14} \\
\alpha & 0 & a_{14} & a_{14}
\end{bmatrix}$$

for some $\alpha$, $\beta$, and $a_{ij}$ in $\mathbb{C}$. Consider the matrix

$$P = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}.$$  

It is straightforward to check that $\frac{1}{2}P$ is a projection in $\mathbb{M}_4$ and every element $B = (b_{ij})$ in $PAP$ satisfies the equation $2b_{13} - t(b_{22} - b_{33}) = 0$. But if $A \in \mathcal{A}$ is as above, and $C = (c_{ij})$ denotes the operator $(PAP)^2$, then

$$2c_{13} - t(c_{22} - c_{33}) = 8(ta_{34} + a_{14})(\alpha - \beta).$$

Since $\mathcal{A}$ is projection compressible, $C$ belongs to $PAP$, and hence the right-hand side of this equation must be 0 for all $\mathcal{A}$. Since $\alpha$ and $\beta$ may be chosen arbitrarily, it follows that either $t = 0$ or $a_{14} = -ta_{34}$ for all $A$ in $\mathcal{A}$.

If $t = 0$, then each $A \in \mathcal{A}$ can be expressed as a matrix of the form

$$A = \begin{bmatrix}
\alpha & a_{23} & 0 & a_{24} \\
\beta & 0 & a_{34} & a_{14} \\
\alpha & 0 & a_{14} & a_{14}
\end{bmatrix}$$

with respect to the reordered basis $\{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\}$ for $\mathbb{C}^4$. In this case, Theorem [3.0.1] demonstrates that $a_{14} = \langle Ae_1^{(3)}, e_1^{(1)}\rangle = 0$ for all $A$. Thus, the equation $a_{14} = -ta_{34}$ holds in either case. That is, $\langle Ae_1^{(3)}, e_1^{(1)}\rangle = -t\langle Ae_1^{(3)}, e_1^{(2)}\rangle$ for all $A \in \mathcal{A}$.

We now turn our attention to the proof of (ii). In this setting, every $A$ in $\mathcal{A}$ admits a matrix representation of the form

$$A = \begin{bmatrix}
\alpha & 0 & a_{13} & a_{14} \\
\alpha & a_{23} & a_{24} & a_{24} \\
\beta & 0 & t(\beta - \alpha) & a_{14} \\
\alpha & 0 & a_{14} & a_{14}
\end{bmatrix}$$

with respect to the basis $B = \{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_1^{(3)}\}$. With $P$ as in (i), every element $B = (b_{ij})$ in $PAP$ satisfies the equation $2b_{34} - t(b_{33} - b_{22}) = 0$. It can be verified, however, that if $A$ is as above and $C := (PAP)^2 = (e_{ij})$, then

$$2c_{34} - t(c_{33} - c_{22}) = 8(ta_{13} - a_{14})(\alpha - \beta).$$

Once again, it follows that either $t = 0$ or $a_{14} = ta_{13}$ for all $A \in \mathcal{A}$.

Suppose first that $t = 0$. Let $P'$ denote the matrix

$$P' = \begin{bmatrix}
2 & 0 & -1 & -1 \\
0 & 3 & 0 & 0 \\
-1 & 0 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix},$$

and since $A \in \mathcal{A}$, it can be verified that $A = P'AP$ is as above, with $a_{14} = -(ta_{13})$. Thus, the equation $a_{14} = -ta_{34}$ holds for all $A \in \mathcal{A}$.
As in the proof of Theorem 5.2.2, we will show that
\[ \text{the structure of} \quad \text{Proof.} \]
\[ \text{idempotent compressible.} \quad \text{Let} \quad \text{Theorem 6.2.3.} \]
\[ \text{We may obtain information on the structure of} \quad \text{Finally, by switching the order of the first two vectors in} \quad \text{Thus, the proof is complete.} \]

**Theorem 6.2.3.** Let \( A \) be a projection compressible type \( III \) subalgebra of \( M_n \). If there is a pair \( \Omega \) in \( F_{II1}(A) \) such that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked, then \( A \) is the unitization of an \( LR \)-algebra. Consequently, \( A \) is idempotent compressible.

**Proof.** Let \( \Omega \) be a pair in \( F_{II1}(A) \) such that \( Q_{1\Omega} \) and \( Q_{3\Omega} \) are linked. By replacing \( A \) with \( A^{aT} \) if necessary, we will assume that \( n_{1\Omega} = \max\{n_{1\Omega}, n_{3\Omega}\} \geq 2 \). Note that by Proposition 6.0.2, neither of these projections is linked to \( Q_{2\Omega} \).

By Theorem 4.0.2 there are subprojections \( Q' \leq Q_{1\Omega} \) and \( Q' \leq Q_{3\Omega} \) such that
\[ Q_{1\Omega} \text{Rad}(A) Q_{2\Omega} = Q' M_n Q_{2\Omega} \quad \text{and} \quad Q_{2\Omega} \text{Rad}(A) Q_{3\Omega} = Q' M_n Q' \]
As in the proof of Theorem 5.2.2 we will show that \( A \) is similar to
\[ A_{0} := (Q' + Q_{2\Omega}) M_n (Q_{2\Omega} + Q') + CI, \]
and hence that \( A \) is the unitization of an \( LR \)-algebra. To show that this is the case, we must first determine the structure of \( Q_{1\Omega} A Q_{3\Omega} \).

Define projections \( Q'' := Q_{1\Omega} - Q' \) and \( Q' := Q_{3\Omega} - Q' \). For each \( i \in \{1, 3\} \), let \( \{e_{i}^{(1)}, e_{i}^{(2)}, \ldots, e_{n_{i}}^{(1)}\} \) be an orthonormal basis for \( \text{ran}(Q_{i\Omega}) \) such that if \( Q''_{i} \neq 0 \), then
\[ \text{ran}(Q''_{i}) = \vee \{e_{i}^{(1)}, e_{i}^{(2)}, \ldots, e_{i}^{(1)}\} \]
for some index \( i \in \{1, 2, \ldots, n_{i}\} \). Furthermore, let \( e_{1}^{(2)} \) be a unit vector in \( \text{ran}(Q_{2\Omega}) \). Since \( A \) is similar to \( BD(A) + \text{Rad}(A) \) via an upper triangular similarity, there are matrices \( T_{1} \in Q'_{1\Omega} M_{n} Q_{2\Omega} \) and \( T_{2} \in Q_{2\Omega} M_{n} Q'_{3\Omega} \) such that for each \( A \in A \),
\[ Q'_{1\Omega} A Q_{2\Omega} = (Q'_{1\Omega} A Q'_{1\Omega}) T_{1} - \left( (Q_{2\Omega} A Q_{2\Omega}) \right) \quad \text{and} \quad Q_{2\Omega} A Q'_{3\Omega} = (Q_{2\Omega} A Q_{2\Omega}) T_{2} - \left( (Q''_{3\Omega} A Q'_{3\Omega}) \right) \]

We may obtain information on the structure of \( Q'_{1\Omega} A Q_{3\Omega} \) by appealing to Lemma 6.2.2. Of course, there is little to be said when \( Q'_{1\Omega} = 0 \). If instead \( Q'_{1\Omega} \neq 0 \), fix arbitrary indices \( i \in \{1, 2, \ldots, i\} \), \( i' \in \{1, 2, \ldots, n_{1\Omega}\} \setminus \{i\} \), and \( j \in \{1, 2, \ldots, n_{3\Omega}\} \). Define \( B = \{e_{i}^{(1)}, e_{i'}^{(1)}, e_{1}^{(2)}, e_{j}^{(3)}\} \) and let \( P \) denote the orthogonal projection onto the span of \( B \). With respect to the basis \( B \) for \( \text{ran}(P) \), every member of \( P A P \) can be written as a matrix of the form
\[
\begin{bmatrix}
\alpha & 0 & i_{i}^{(1)} (\alpha - \beta) & a_{14} \\
0 & \alpha & a_{23} & a_{24} \\
i_{i'}^{(1)} (\alpha - \beta) & a_{23} & \beta & a_{34} \\
0 & a_{24} & a_{34} & \alpha
\end{bmatrix}
\]
where \( t_i^{(1)} := \langle T_1 e_1^{(2)}, e_i^{(1)} \rangle \). Thus, an application Lemma 6.2.2 (i) demonstrates that
\[
\langle A e_j^{(3)}, e_i^{(1)} \rangle = -t_i^{(1)} \langle A e_j^{(3)}, e_1^{(2)} \rangle \text{ for all } A \in \mathcal{A}.
\]

Since the indices \( i, i', j \) were selected arbitrarily, it follows that
\[
Q_i'' AQ_3'\Omega = -T_1 Q_2\Omega AQ_3'\Omega \text{ for all } A \in \mathcal{A}.
\]

A similar argument can be used to determine the structure of \( Q_1\Omega AQ_3'' \). Indeed, there is nothing to be said when \( Q_i'' = 0 \). If instead \( Q_i'' \neq 0 \), choose distinct indices \( i \) and \( i' \) in \( \{1, 2, \ldots, n_1\} \), and let \( j \in \{1, 2, \ldots, \ell_3 \} \) be arbitrary. Define \( \mathcal{C} = \{ e_i^{(1)}, e_i^{(2)}, e_j^{(3)} \} \), and let \( P' \) denote the orthogonal projection onto the span of \( \mathcal{C} \). The compression \( P' AP'' \) is an algebra of the form described in Lemma 6.2.2 (ii), and hence this result indicates that each \( A \in \mathcal{A} \) satisfies the equation
\[
\langle A e_j^{(3)}, e_i^{(1)} \rangle = t_j^{(2)} \langle A e_1^{(2)}, e_i^{(1)} \rangle,
\]
where \( t_j^{(2)} := \langle T_2 e_j^{(3)}, e_1^{(2)} \rangle \). Again, the fact that \( i, i', j \) were chosen arbitrarily implies that \( Q_1\Omega AQ_3'' = Q_1\Omega AQ_2\Omega T_2 \) for all \( A \in \mathcal{A} \).

Our findings thus far indicate that with respect to the decomposition
\[
\mathbb{C}^n = \text{ran}(Q_1'') \oplus \text{ran}(Q_1') \oplus \text{ran}(Q_2\Omega) \oplus \text{ran}(Q_3') \oplus \text{ran}(Q_3'),
\]
each \( A \) in \( \mathcal{A} \) can be expressed as a matrix of the form
\[
A = \begin{bmatrix}
\alpha I & 0 & (\alpha - \beta)T_1 & A_{14} & A_{15} \\
\alpha I & J_1 & A_{24} & A_{25} \\
\beta & (\beta - \alpha)T_2 & J_2 \\
\alpha I & 0 & & & \\
& & & \alpha I & \\
\end{bmatrix},
\]
for some \( \alpha, \beta \in \mathbb{C} \), \( J_1 \in Q_1' \text{Rad}(\mathcal{A})Q_2\Omega \), \( J_2 \in Q_2\Omega \text{Rad}(\mathcal{A})Q_3' \), and operators \( A_{ij} \) satisfying the equations
\[
\begin{bmatrix}
A_{14} & A_{15}
\end{bmatrix} = -T_1 \begin{bmatrix}
(\beta - \alpha)T_2 & J_2
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
A_{14} \\
A_{24}
\end{bmatrix} = \begin{bmatrix}
(\alpha - \beta)T_1 \\
J_1
\end{bmatrix} T_2.
\]

To see that \( \mathcal{A} \) is similar to \( \mathcal{A}_0 = (Q_1' + Q_2\Omega)^{\ell \times n} (Q_2\Omega + Q_3') + \mathbb{C} I \), and hence is the unitization of an \( \mathcal{LR} \)-algebra, consider the operator \( S := I - T_1 - T_2 \). This map is invertible with \( S^{-1} = I + T_1 + T_2 + T_1T_2 \). In addition, we have that for \( A \) as above,
\[
S^{-1} AS = \begin{bmatrix}
\alpha I & 0 & 0 & 0 & 0 \\
\alpha I & J_1 & 0 & A_{25} & \\
\beta & 0 & J_2 & & \\
\alpha I & 0 & & & \\
& & & \alpha I & \\
\end{bmatrix}.
\]

It is now apparent that \( S^{-1} AS \) is a type III algebra that admits a reduced block upper triangular form with respect to the above decomposition. Since
\[
Q_1\Omega \text{Rad}(S^{-1} AS)Q_2\Omega = Q_1' \text{Rad}(\mathcal{A})Q_2\Omega = Q_1' M_n Q_2\Omega \quad \text{and} \quad Q_2\Omega \text{Rad}(S^{-1} AS)Q_3\Omega = Q_2\Omega \text{Rad}(\mathcal{A})Q_3' = Q_2\Omega M_n Q_3',
\]
it follows from Lemma 5.13 (ii) that \( S^{-1} AS = (Q_1' + Q_2\Omega)^{\ell \times n} (Q_2\Omega + Q_3') + \mathbb{C} I = \mathcal{A}_0 \). \( \blacksquare \)
§7 Main Result and Applications

§7.1 The Main Result. The analysis carried out in the preceding sections provides a description of the unital projection compressible subalgebras of $M_n$, $n \geq 4$ up to transpose similarity. Since every such algebra was also seen to admit the idempotent compression property, it follows that the two notions of compressibility coincide for unital algebras in this setting. We therefore obtain the following theorem, the main result of this paper.

Theorem 7.1.1. Let $A$ be a unital subalgebra of $M_n$ for some integer $n \geq 4$. The following are equivalent.

(i) $A$ is projection compressible;
(ii) $A$ is idempotent compressible;
(iii) $A$ is the unitization of an $LR$-algebra, or $A$ is transpose similar to one of the algebras from Example 7.0.1.

Combining Theorem 7.1.1 and [3, Theorem 6.0.1], we conclude that the two notions of compressibility coincide for all unital algebras.

Theorem 7.1.2. A unital subalgebra $A$ of $M_n$, $n \geq 2$, is projection compressible if and only if it is idempotent compressible.

In light of Theorem 7.1.2, we make the following definition.

Definition 7.1.3. A unital subalgebra $A$ of $M_n$ is compressible if $A$ is projection compressible (equivalently, if $A$ is idempotent compressible).

It is worth noting that nearly all of the classification results from §4-6 describe the various unital compressible subalgebras of $M_n$ up to transpose equivalence, not just transpose similarity. Indeed, the only instance in which a description up to transpose equivalence was not achieved was in Theorem 6.1.1. There it was shown that a projection compressible type III algebra is either transpose equivalent to the type III algebra from Example 1.0.1(i) or transpose similar to the algebra from Example 1.0.1(ii).

The following proposition describes the similarity orbit of the algebra from Example 1.0.1(ii) up to unitary equivalence, thereby providing a characterization of the (unital) compressible subalgebras of $M_n$, $n \geq 4$, up to transpose equivalence.

Proposition 7.1.4. Let $n \geq 3$ be an integer, let $Q_1$ and $Q_2$ be mutually orthogonal rank-one projections in $M_n$, and define $Q_3 := I - Q_1 - Q_2$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that $e_1 \in \text{ran}(Q_1)$, $e_2 \in \text{ran}(Q_2)$, and $e_i \in \text{ran}(Q_3)$ for all $i \geq 3$. If

$$A := \mathbb{C}Q_1 + \mathbb{C}Q_2 + \mathbb{C}Q_3 + (Q_1 + Q_2)M_nQ_3$$

$$= \left\{ \begin{bmatrix} \alpha & 0 & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in \mathbb{C} \right\}$$

denotes the compressible algebra from Example 1.0.1(ii) and $B$ is an algebra that is similar to $A$, then there is some $t \in \mathbb{C}$ such that $B$ is unitarily equivalent to

$$A_t := \{ A + t(\langle Ae_1, e_1 \rangle - \langle Ae_2, e_2 \rangle) e_1 \otimes e_2^* : A \in A \}$$

$$= \left\{ \begin{bmatrix} \alpha & t(\alpha - \beta) & M_{13} \\ 0 & \beta & M_{23} \\ 0 & 0 & \gamma I \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C}, M_{ij} \in \mathbb{C} \right\}.$$

Proof. Suppose that $B = S^{-1}AS$ for some invertible $S \in M_n$. For all indices $i \in \{1, 2\}$ and $j \in \{3, 4, \ldots, n\}$, define $E_{ij} := e_i \otimes e_j^*$ and $E'_{ij} := S^{-1}E_{ij}S$. Furthermore, define $Q_i' := S^{-1}Q_iS$ for $i \in \{1, 2, 3\}$. Observe that

$$B = S^{-1}AS = \text{span} \{ Q_1', Q_2', Q_3', E'_{ij} : i \in \{1, 2\}, j \in \{3, 4, \ldots, n\} \}.$$

Let $\{f_1, f_2, \ldots, f_n\}$ be an orthonormal basis for $\mathbb{C}^n$ such that $f_1$ and $f_2$ belong to $\ker(Q_3')$. Let $P_1$, $P_3$, and $P_3$ denote the orthogonal projections onto $\mathbb{C}f_1$, $\mathbb{C}f_2$, and $\text{span} \{ f_i : i \geq 3 \} = \ker(Q_3')^\perp$, respectively. Since $P_3Q_3P_3 = P_3$ and $Q_3Q_3' = Q_3'Q_3 = 0$, we have that $Q_3' = (P_1 + P_2)Q_3$ and $Q_3' = (P_1 + P_2)Q_3'$. Note that since $Q_1'Q_2 = Q_2'Q_1' = 0$, we may adjust the first two basis vectors if necessary to assume that $Q_1'$ and $Q_2'$ are upper triangular with respect to $\{f_1, f_2, \ldots, f_n\}$, and $\langle Q_i'f_j, f_j \rangle = \delta_{ij}$ for $i, j \in \{1, 2\}$. Thus,
there are matrices $X_{ij}$, $Y_{ij}$, and $Z_{ij}$, and a constant $t \in \mathbb{C}$ such that with respect to the decomposition
\[ \mathbb{C}^n = \text{ran}(P_1) \oplus \text{ran}(P_2) \oplus \text{ran}(P_3), \]
\[ Q'_1 = \begin{bmatrix} 1 & t & X_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q'_2 = \begin{bmatrix} 0 & -t & Y_{13} \\ 0 & 1 & Y_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad Q'_3 = \begin{bmatrix} 0 & 0 & Z_{13} \\ 0 & 0 & Z_{13} \\ 0 & 0 & 1 \end{bmatrix}. \]

Finally, since $E'_{ij} = (Q'_1 + Q'_2)E_{ij}Q'_3$, we have that $E'_{ij} = (P_1 + P_2)E_{ij}P_3$ for all indices $i$ and $j$. Dimension considerations then imply that
\[ \text{span} \{ E'_{ij} : i \in \{1,2\}, j \in \{3,4,\ldots,n\} \} = (P_1 + P_2)M_n P_3, \]
and therefore
\[ B = \{ B + t((Bf_1, f_1) - (Bf_2, f_2)) : B \in \mathbb{C}P_1 + \mathbb{C}P_2 + \mathbb{C}P_3 + (P_1 + P_2)M_n P_3 \}. \]
We conclude that $\mathcal{A}_t = U^*BU$ where $U \in M_n$ is the unitary satisfying $Ue_i = f_i$. $\blacksquare$

**Corollary 7.1.5.** Let $n \geq 4$ be an integer, and let $\mathcal{A}$ be a unital subalgebra of $M_n$. The following are equivalent.

(i) $\mathcal{A}$ is compressible;

(ii) $\mathcal{A}$ is the unitization of an $\mathcal{LR}$-algebra, or $\mathcal{A}$ is transpose equivalent to the algebra from Example 1.0.1(i), the algebra from Example 1.0.1(iii), or the algebra $\mathcal{A}_t$ from Proposition 7.1.4.

**Remark 7.1.6.** The above result, together with Theorem 7.1.1 implies that if $\mathcal{A}$ is transpose similar to an algebra from Theorem 7.1.1 (iii), then $\mathcal{A}$ is transpose equivalent to an algebra from Corollary 7.1.5 (ii). Indeed, Proposition 7.1.4 makes this fact explicit for the algebra in Example 1.0.1(ii), while in [3] it was shown that the class of $\mathcal{LR}$-algebras is invariant under transpose similarity. Arguments akin to those in the proof of Proposition 7.1.4 can be used to show that any algebra transpose similar to the algebra from Example 1.0.1(ii) (resp. Example 1.0.1(iii)) is in fact, transpose equivalent to it.

§7.2 **Applications.** Here we investigate some of the applications of the classification of unital compressible algebras. It follows from Theorem 7.1.2 that the class of all such algebras is invariant under similarity and transposition. Using this fact, it is relatively straightforward to determine which unital semi-simple algebras admit the compression property.

**Corollary 7.2.1.** Let $n \geq 2$ be an integer, and let $\mathcal{A}$ be a unital, semi-simple subalgebra of $M_n$. The following are equivalent:

(i) $\mathcal{A}$ is compressible;

(ii) $\mathcal{A} = CI$ or $\mathcal{A}$ is similar to $M_k \oplus CI_{n-k}$ for some positive integer $k$.

**Proof.** Since $CI$ and $M_k \oplus CI_{n-k}$ are unitizations of $\mathcal{LR}$-algebras, it is obvious that (ii) implies (i). Assume now that (i) holds, so $\mathcal{A}$ is a unital, semi-simple subalgebra of $M_n$ that admits the compression property. Assume as well that $\mathcal{A}$ is in reduced block upper triangular form with respect to some orthogonal decomposition $\bigoplus_{i=1}^m V_i$ of $\mathbb{C}^n$. By Theorem 2.0.5 $\mathcal{A}$ is similar to $B \cong BD(A)$. It therefore suffices to prove that $B$ is similar to an algebra of the form prescribed in (ii).

If $n = 2$, then $B$ is equal to $CI$, $C \oplus C$, or $M_2$, and hence $B$ is of the desired form. If instead $n = 3$, then either $B$ is equal to $CI$ or $M_3$, or $B$ is unitarily equivalent to $C \oplus CI_2$ or $M_2 \oplus C$. Indeed, the only other block diagonal subalgebra of $M_3$ is the algebra of all $3 \times 3$ diagonal matrices. This algebra was shown to lack the compression property in [3, Theorem 5.2.6], and hence cannot be similar to $B$. Again we see that (ii) holds.

Suppose now that $n \geq 4$. By Theorem 3.0.1 there is at most one space $V_i$ of dimension $2$ or greater. If such a space exists, we may reindex the sum $\bigoplus_{i=1}^m V_i$ if necessary and assume that dim($V_1$) = $k \geq 2$. Theorem 3.0.1 then implies that $V_i$ is linked to $V_j$ for all $i, j \geq 2$, so $B = M_k \oplus CI_{n-k}$. If instead dim $V_i = 1$ for all $i$, then Theorem 3.0.1 indicates that with at most one exception, all spaces $V_i$ are mutually linked. Thus, $B$ is equal to $CI$ or is unitarily equivalent to $C \oplus CI_{n-1}$. $\blacksquare$

Theorem 7.1.1 can also be used to quickly identify the operators $T \in M_n$ such that Alg$(T, I)$—the unital algebra generated by $T$—is compressible.
Corollary 7.2.2. Let $n \geq 2$ be an integer, and let $T \in \mathbb{M}_n$. The following are equivalent:

(i) $\text{Alg}(T, I)$ is compressible;
(ii) $\text{Alg}(T, I)$ is the unitization of an $\mathcal{LR}$-algebra;
(iii) $T \in \text{span}\{I, R\}$ for some $R \in \mathbb{M}_n$ of rank 1.

Proof. It is clear that (ii) implies (i). To see that (i) implies (iii), assume that $\text{Alg}(T, I)$ is compressible. It follows that $\text{Alg}(S^{-1}TS, I) = S^{-1}\text{Alg}(T, I)S$ is compressible for all invertible $S \in \mathbb{M}_n$; hence we may assume that $T$ is in Jordan canonical form with respect to the standard basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$.

If $T$ has a Jordan block of size at least 3, then $\text{Alg}(T, I)$ admits a principal compression of the form
\[
\begin{pmatrix}
x & y & z \\
0 & x & y \\
0 & 0 & x \\
\end{pmatrix} : x, y, z \in \mathbb{C}.
\]
Since this algebra was shown to lack the compression property in [3, Theorem 5.2.4], it must be the case that each Jordan block of $T$ has size at most 2. Note as well that if two or more Jordan blocks of size 2 were present, then $\text{Alg}(T, I)$ would lack the compression property by Theorem 3.0.1. Consequently, $T$ has at most one Jordan block of size 2, and the remaining blocks have size 1.

If a Jordan block of size 2 occurs, then $T$ cannot have two or more distinct eigenvalues. Indeed, if $T$ had at least two distinct eigenvalues, then $\text{Alg}(T, I)$ would admit a principal compression that is unitarily equivalent to
\[
\begin{pmatrix}
x & y & 0 \\
0 & x & 0 \\
0 & 0 & z \\
\end{pmatrix} : x, y, z \in \mathbb{C}.
\]
By [3, Theorem 5.2.2], this algebra is not compressible—a contradiction. Thus, $T$ must be unitarily equivalent to $e_1 \otimes e_2^* + \alpha I$ for some $\alpha \in \mathbb{C}$. We conclude that $T = \alpha I + R$ for some $R \in \mathbb{M}_n$ of rank 1.

Suppose now that every Jordan block of $T$ is $1 \times 1$, so $T$ is diagonal. If $T$ had at least three distinct eigenvalues, then the algebra $\mathcal{D}$ of all $3 \times 3$ diagonal matrices could be obtained as a principal compression of $\text{Alg}(T, I)$. Since no algebra similar to $\mathcal{D}$ is projection compressible by [3, Theorem 5.2.6], this is not possible. Therefore, $T$ has at most two distinct eigenvalues. By Theorem 3.0.1 one of the eigenvalues must have multiplicity 1. We deduce that either $T$ has exactly one eigenvalue, and hence is a multiple of the identity, or $T$ has exactly two eigenvalues, and hence is a rank-one perturbation of a multiple of the identity. Thus, (iii) holds in this case as well.

Finally, we will show that (iii) implies (ii). Suppose that $T \in \text{span}\{I, R\}$ for some rank-one operator $R \in \mathbb{M}_n$. That is, $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$. If $\beta = 0$, then $\text{Alg}(T, I) = \mathbb{C}I$. Otherwise, $\beta R$ has rank 1, and hence $\text{Alg}(T, I) = \text{Alg}(\beta R) + \mathbb{C}I$ is the unitization of an $\mathcal{LR}$-algebra by [3, Proposition 2.0.12].

It is interesting to note that in the 3-dimensional case, the matrices of the form $\alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$ and $R \in \mathbb{M}_3$ of rank one are exactly those with two or more Jordan blocks corresponding to a common eigenvalue. Such matrices are said to be derogatory [5, Definition 1.4.4]. One may therefore view Corollary 7.2.2 as a higher-dimensional analogue of [3, Corollary 5.1.3].

Throughout this exposition we have devoted our attention almost exclusively to unital subalgebras of $\mathbb{M}_n$. Of course, it is reasonable to ask which non-unital algebras admit the projection or idempotent compression properties. In particular, it would be interesting to know whether or not the equivalence of these notions proven above in the unital case extends to non-unital algebras as well.

By [3, Proposition 2.0.6], if a subalgebra $\mathcal{A}$ of $\mathbb{M}_n$ admits the projection (resp. idempotent) compression property, then so too does its unitization. As a result, Theorem 7.1.1 offers considerable insight into the non-unital projection (resp. idempotent) compressible algebras that exist in $\mathbb{M}_n$. Specifically, this result indicates that if $\mathcal{A}$ is a projection compressible subalgebra of $\mathbb{M}_n$, then $\tilde{\mathcal{A}} = \mathcal{A} + CI$ is the unitization of an $\mathcal{LR}$-algebra, or is transpose similar to one of the unital algebras from Example 1.0.1. Using this information, one can quickly obtain a non-unital analogue of Corollary 7.2.2.

Corollary 7.2.3. Let $n \geq 3$ be an integer, and let $T \in \mathbb{M}_n$. The following are equivalent:

(i) $\text{Alg}(T)$ is projection compressible;
(ii) $\text{Alg}(T)$ is idempotent compressible;
(iii) $\text{Alg}(T)$ is an $\mathcal{LR}$-algebra, or the unitization thereof.
(iv) $T \in \text{span}\{I, R\}$ for some $R \in \mathbb{M}_n$ of rank 1, and 0 does not occur as an eigenvalue of $T$ with algebraic multiplicity 1.

Proof. It is clear that (iii) implies (ii), and (ii) implies (i).

To see that (i) implies (iv), note that if $\mathcal{A}(T)$ is projection compressible, then so too is $\mathcal{A}(T, I)$. By Corollary 7.2.2, there is a rank-one operator $R \in \mathbb{M}_n$ such that $T \in \text{span}\{I, R\}$. For the final claim, write $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$, and suppose to the contrary that $\lambda = 0$ is an eigenvalue of $T$ with algebraic multiplicity 1. Since $\text{rank}(R) = 1$, there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$ with respect to which

$$\beta R = \gamma_1 e_1 \otimes e_1^* + \gamma_2 e_2 \otimes e_2^*$$

for some constants $\gamma_1, \gamma_2 \in \mathbb{C}$. Thus, when expressed as a matrix with respect to this basis, $T$ is upper triangular with diagonal entries $\alpha + \gamma_1$ with multiplicity 1, and $\alpha$ with multiplicity $n - 1$. It must therefore be the case that $\alpha + \gamma_1 = 0$ and $\alpha \neq 0$.

Let $P$ denote the orthogonal projection onto $\text{span}\{e_1, e_2, e_3\}$ and define $T' := PT P$. With respect to the ordered basis $B = \{e_1, e_2, e_3\}$ for $\text{ran}(P)$,

$$T' = \begin{bmatrix} 0 & \gamma_2 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

Thus, since $\mathcal{A}(T) = \mathbb{C}T$ is projection compressible, $P'\mathcal{A}(T)P' = \mathbb{C}P'TP'$ is an algebra for all projections $P' \leq P$. Consider the matrix

$$P' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

written with respect to the basis $B$. It is easy to see that $\frac{1}{2}P'$ is a subprojection of $P$. Moreover, it is straightforward to show that $\langle Be_2, e_2 \rangle = 4\langle Be_1, e_1 \rangle$ for all $B \in P'\mathcal{A}(T)P'$. One may verify, however, that

$$\langle (P'TP')^2 e_2, e_2 \rangle - 4\langle (P'TP')^2 e_1, e_1 \rangle = 8\alpha^2 \neq 0,$$

and thus $(P'TP')^2 \notin P'\mathcal{A}(T)P'$. This is clearly a contradiction.

It remains to show that (iv) implies (iii). To this end, let $T$ and $R$ be as in (iv), and write $T = \alpha I + \beta R$ for some $\alpha, \beta \in \mathbb{C}$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$ with respect to which

$$\beta R = \gamma_1 e_1 \otimes e_1^* + \gamma_2 e_2 \otimes e_2^*$$

for some $\gamma_1, \gamma_2 \in \mathbb{C}$.

First suppose that $\alpha = 0$, so $\mathcal{A}(T) = \mathcal{A}(\beta R)$. If $\beta = 0$ then this algebra is trivial. Otherwise, $\mathcal{A}(T)$ is an $\mathcal{LR}$-algebra by [3 Proposition 2.0.12]. If instead $\alpha \neq 0$, then our assumptions on $T$ imply that $\alpha + \gamma_1 \neq 0$. Consequently,

$$I = \left( \frac{1}{\alpha} + \frac{1}{\alpha + \gamma_1} \right) T - \frac{1}{\alpha(\alpha + \gamma_1)} T^2 \in \mathcal{A}(T).$$

It follows that $\mathcal{A}(T) = \mathcal{A}(T, I)$, so $\mathcal{A}(T)$ is the unitization of an $\mathcal{LR}$-algebra by Corollary 7.2.2.

The notions of projection compressibility and idempotent compressibility can also be naturally extended to algebras of bounded linear operators acting on a Hilbert space $\mathcal{H}$ of arbitrary dimension. It would therefore be interesting to obtain analogues of the above results that apply in this setting.

One approach to understanding the structure of a projection (resp. idempotent) compressible operator algebra $\mathcal{A}$ would be to apply Theorem 7.1.1 to the unital compressions $PAP$, where $P$ is a projection (resp. idempotent) of finite rank. This technique may have its limits, however, as there could exist operator algebras $\mathcal{A}$ that lack the projection compression property, yet such that $PAP$ is an algebra for all finite-rank projections $P$. With this in mind, the most viable avenue for understanding the compression properties in this setting may be to first obtain an intrinsic explanation as to why these notions coincide for unital subalgebras of $\mathbb{M}_n$. 

\[ \square \]
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