ON THE CARDINALITY OF UNIQUE RANGE SETS WITH WEIGHT ONE

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Abstract. Two meromorphic functions \(f\) and \(g\) are said to share the set \(S \subseteq \mathbb{C} \cup \{\infty\}\) with weight \(l \in \mathbb{N} \cup \{0\} \cup \{\infty\}\), if \(E_f(S, l) = E_g(S, l)\) where
\[
E_f(S, l) = \bigcup_{a \in S} \{(z, t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},
\]
where \(t = p\) if \(p \leq l\) and \(t = p + 1\) if \(p > l\).

In this paper, we improve and supplement the result of L. W. Liao and C. C. Yang (On the cardinality of the unique range sets for meromorphic and entire functions, Indian J. Pure appl. Math., 31 (2000), no. 4, 431-440) by showing that there exist a finite set \(S\) with cardinality \(\geq 13\) such that \(E_f(S, 1) = E_g(S, 1)\) implies \(f \equiv g\).

1. Introduction and Definitions

By \(\mathbb{C}\) and \(\mathbb{N}\), we mean the set of complex numbers and set of natural numbers respectively. By meromorphic function, we mean an analytic function defined on \(\mathbb{C}\) except possibly at isolated singularities, each of which is a pole.

The tool we used in this paper is Nevanlinna Theory. For the standard notations of the Nevanlinna theory, one can go through the Hayman’s Monograph (\([8]\)).

It will be convenient to let \(E\) denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function \(h(z)\) we denote by \(S(r, h)\) any quantity satisfying
\[
S(r, h) = o(T(r, h)) \quad (r \rightarrow \infty, r \notin E).
\]

Suppose \(f\) and \(g\) be two non-constant meromorphic functions and \(a \in \mathbb{C}\). We say that \(f\) and \(g\) share the value \(a\)-CM (counting multiplicities), provided that \(f - a\) and \(g - a\) have the same zeros with the same multiplicities. Similarly, we say that \(f\) and \(g\) share the value \(a\)-IM (ignoring multiplicities), provided that \(f - a\) and \(g - a\) have the same set of zeros, where the multiplicities are not taken into account. Moreover, we say that \(f\) and \(g\) share \(\infty\)-CM (resp. IM), if \(1/f\) and \(1/g\) share 0-CM (resp. IM).

In the course of studying the factorization of meromorphic functions, F. Gross (\([6]\)) first generalized the idea of value sharing by introducing the concept of a unique range set. Before going to the details of the paper, we first recall the definition of set sharing.

Definition 1.1. (\([15]\)) For a non-constant meromorphic function \(f\) and any set \(S \subseteq \mathbb{C} \cup \{\infty\}\), we define
\[
E_f(S) = \bigcup_{a \in S} \{(z, p) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},
\]
\[
\overline{E}_f(S) = \bigcup_{a \in S} \{(z, 1) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\}.
\]

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If \( E_f(S) = E_g(S) \) (resp. \( \overline{E_f(S)} = \overline{E_g(S)} \)), then it is said that \( f \) and \( g \) share the set \( S \) counting multiplicities or in short CM (resp. ignoring multiplicities or in short IM).

Thus, if \( S \) is singleton, then it coincides with the usual definition of value sharing.

In 1977, F. Gross ([6]) proposed the following problem which has later became popular as “Gross Problem”. The problem was as follows:

**Question A.** Does there exist a finite set \( S \) such that any two non-constant entire functions \( f \) and \( g \) sharing the set \( S \) must be \( f = g \)?

In 1982, F. Gross and C. C. Yang ([7]) gave the affirmative answer to the above question as follows:

**Theorem A.** ([7]) Let \( S = \{ z \in \mathbb{C} : e^z + z = 0 \} \). If two entire functions \( f, g \) satisfy \( E_f(S) = E_g(S) \), then \( f \equiv g \).

In that paper ([7]), they first introduced the terminology unique range set for entire function (in short URSE). Later the analogous definition for meromorphic functions was also introduced in literature.

**Definition 1.2.** ([15]) Let \( S \subset \mathbb{C} \cup \{ \infty \} \) and \( f \) and \( g \) be two non-constant meromorphic (resp. entire) functions. If \( E_f(S) = E_g(S) \) implies \( f \equiv g \), then \( S \) is called a unique range set for meromorphic (resp. entire) functions or in brief URSM (resp. URSE).

In 1997, H. X. Yi ([18]) introduced the analogous definition for reduced unique range sets.

**Definition 1.3.** ([18]) A set \( S \subset \mathbb{C} \cup \{ \infty \} \) is said to be a unique range set for meromorphic (resp. entire) functions in ignoring multiplicity, in short URSM-IM (resp. URSE-IM) or a reduced unique range set for meromorphic (resp. entire) functions, in short RURSM (resp. RURSE) if \( E_f(S) = E_g(S) \) implies \( f \equiv g \) for any pair of non-constant meromorphic (resp. entire) functions.

During the last few years the notion of unique as well as reduced unique range sets have been generating an increasing interest among the researchers. For the literature, one can also go through the research monograph written by C. C. Yang and H. X. Yi ([15]).

Next we recall the following notations:

\[
\lambda_M = \inf \{ \sharp(S) \mid S \text{ is an URSM} \} \quad \text{and} \quad \lambda_E = \inf \{ \sharp(S) \mid S \text{ is an URSE} \},
\]

where \( \sharp(S) \) is the cardinality of the set \( S \).

In 1996, P. Li ([11]) showed that \( \lambda_E \geq 5 \) whereas C. C. Yang and H. X Yi ([15]) established that \( \lambda_M \geq 6 \). Also, G. Frank and M. Reinders ([3]) estimated that \( \lambda_M \leq 11 \). And for entire functions corresponding estimation is \( \lambda_E \leq 7 \). Till date these estimations are the best.

In course of time, researchers are also paying their attention to find the lowest cardinality of URSM-IM as well as URSE-IM. In 1997, H. X. Yi ([18]) gave the existence of URSM-IM with 19 elements. After one year, in 1998, H. X. Yi ([19]) further improved his result ([18]) and obtained URSM-IM of 17 elements. Also in 1998, M. L. Fang and H. Guo ([2]) and in 1999, S. Bartels ([1]) independently gave the existence of URSM-IM with 17 elements. In connection to our discussions, the following question is natural:

**Question 1.1.** Can one further reduced the lower bound of the unique range sets by relaxing the sharing notations?
As an attempt to reduce the cardinalities of unique range sets, L. W. Liao and C. C. Yang (13) introduced the following notation:

**Definition 1.4.** Let \( f \) be a non-constant meromorphic function and \( S \subset \mathbb{C} \cup \{\infty\} \). We define

\[
E_1(S, f) = \bigcup_{a \in S} E_1(a, f),
\]

where \( E_1(a, f) \) is the set of all simple \( a \)-points of \( f \).

For a positive integers \( n \geq 3 \) and \( c \neq 0,1 \), we shall denote by \( P(z) \), the Frank-Reinders polynomial (3) as:

\[
P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c,
\]

(1.1)

Clearly the restrictions on \( c \) implies that \( P(z) \) has only simple zeros. Using the methods of Frank-Reinders (3), in connection to the Question 1.1, L. W. Liao and C. C. Yang (13) proved following Theorem:

**Theorem B.** (13) Suppose that \( n(\geq 1) \) be a positive integer. Further suppose that \( S = \{z : P(z) = 0\} \) where the polynomial \( P(z) \) of degree \( n \) defined by (1.1). Let \( f \) and \( g \) be two non-constant meromorphic functions satisfying \( E_1(S, f) = E_1(S, g) \). If \( n \geq 15 \), then \( f \equiv g \).

The motivation of this paper is to improve and supplement Theorem B utilizing the method of Frank-Reinders (3). Before going to state our main result, we recall some well known definitions which will be useful for the proof of the main result of this paper.

**Definition 1.5.** (10) For a non-constant meromorphic function \( f \) and any set \( S \subset \mathbb{C} \cup \{\infty\}, l \in \mathbb{N} \cup \{0\} \cup \{\infty\} \), we define

\[
E_f(S, l) = \bigcup_{a \in S} \{(z, t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},
\]

where \( t = p \) if \( p \leq l \) and \( t = p + 1 \) if \( p > l \).

Two meromorphic functions \( f \) and \( g \) are said to share the set \( S \) with weight \( l \), if \( E_f(S, l) = E_g(S, l) \).

Clearly \( E_f(S) = E_f(S, \infty) \) and \( \overline{E}_f(S) = E_f(S, 0) \).

**Definition 1.6.** Suppose \( a \in \mathbb{C} \cup \{\infty\} \) and \( m \in \mathbb{N} \).

i) We denote by \( N(r, a; f | = 1) \), the counting function of simple \( a \)-points of \( f \),

ii) by \( N(r, a; f | \leq m) \) (resp. \( \bar{N}(r, a; f | \geq m) \)), we denote the counting function of those \( a \)-points of \( f \) whose multiplicities are not greater (resp. less) than \( m \) where each \( a \)-point is counted according to its multiplicity.

Similarly, one can define \( \underline{N}(r, a; f | \leq m) \) and \( \overline{N}(r, a; f | \geq m) \) as the reduced counting function of \( N(r, a; f | \leq m) \) and \( N(r, a; f | \geq m) \) respectively.

Analogously \( N(r, a; f | < m) \), \( N(r, a; f | > m) \), \( \overline{N}(r, a; f | < m) \) and \( \overline{N}(r, a; f | > m) \) are defined.

**Definition 1.7.** Suppose that \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share \((a,0)\). Further suppose that \( z_0 \) be an \( a \)-point of \( f \) with multiplicity \( p \), an \( a \)-point of \( g \) with multiplicity \( q \).

i) We denote by \( \overline{N}_L(r, a; f) \), the reduced counting function of those \( a \)-points of \( f \) and \( g \) where \( p > q \),
ii) by $N^1_E(r, a; f)$, the counting function of those $a$-points of $f$ and $g$ where $p = q = 1$.

iii) by $N^2_E(r, a; f)$, the reduced counting function of those $a$-points of $f$ and $g$ where $p = q \geq 2$.

Similarly, we can define $\overline{N}_L(r, a; f)$, $N^1_E(r, a; g)$, $N^2_E(r, a; g)$.

When $f$ and $g$ share $(a, m)$, $m \geq 1$, then $N^1_E(r, a; f) = N(r, a; f \mid = 1)$.

**Definition 1.8.** We denote by $N(r, a; f \mid = k)$, the reduced counting function of those $a$-points of $f$ whose multiplicities is exactly $k$, where $k \geq 2$ is an integer.

**Definition 1.9.** Let $f$, $g$ share a value $a$ IM. We denote by $N^\ast(r, a; f, g)$, the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$. Clearly

$N^\ast(r, a; f, g) \equiv N^\ast(r, a; g, f)$ and $N^\ast(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. **Main Results**

**Theorem 2.1.** Suppose that $n \geq 1$ be a positive integer. Further suppose that $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ of degree $n$ defined by (1.1). Let $f$ and $g$ be two non-constant meromorphic functions satisfying $E_f(S, 1) = E_g(S, 1)$. If $n \geq 13$, then $f \equiv g$.

**Corollary 2.1.** Suppose that $n \geq 1$ be a positive integer. Further suppose that $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ of degree $n$ defined by (1.1). Let $f$ and $g$ be two non-constant entire functions satisfying $E_f(S, 1) = E_g(S, 1)$. If $n \geq 8$, then $f \equiv g$.

3. **Lemmas**

We define for any two non-constant meromorphic functions $f$ and $g$

$$Q(z) = \frac{P(z) + c}{c}, \quad F = Q(f), \quad G = Q(g).$$

Henceforth, we shall denote by $H$ the following three functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1} \right).$$

**Lemma 3.1.** (14) Let $f$ be a non-constant meromorphic function and let

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$ Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$. 
Lemma 3.2. ([15]) For a non-constant meromorphic function \( f \),
\[
T \left( r, \frac{1}{f} \right) = T(r, f) + O(1),
\]
where \( O(1) \) is a bounded quantity depending on \( a \).

Lemma 3.3. ([15]) For a non-constant meromorphic function \( f \) and for a complex number \( a \in \mathbb{C} \cup \{\infty\} \)
\[
T \left( r, \frac{1}{f-a} \right) = T(r, f) + O(1),
\]
where \( O(1) \) is a bounded quantity depending on \( a \).

Lemma 3.4. ([15], page no. 23) Suppose that \( f \) is a non-constant meromorphic function in the complex plane and \( a_1, a_2, \ldots, a_q \) are \( q \geq 3 \) distinct values in \( \mathbb{C} \cup \{\infty\} \). Then
\[
(q-2)T(r, f) < \sum_{j=1}^{q} N(r, a_j; f) + S(r, f)
\]
where \( S(r, f) \) is a quantity such that \( \frac{S(r, f)}{T(r, f)} \rightarrow 0 \) as \( r \rightarrow +\infty \) out side of a set \( E \) in \( (0, \infty) \) with finite linear measure.

A polynomial \( P(z) \) over \( \mathbb{C} \), is called a uniqueness polynomial for meromorphic (resp. entire) functions, if for any two non-constant meromorphic (resp. entire) functions \( f \) and \( g \), \( P(f) \equiv P(g) \) implies \( f \equiv g \).

In 2000, H. Fujimoto ([4]) first discovered a special property of a polynomial, which was later termed as critical injection property. A polynomial \( P(z) \) is said to satisfy critical injection property if \( P(\alpha) \neq P(\beta) \) for any two distinct zeros \( \alpha, \beta \) of the derivative \( P'(z) \).

Let \( P(z) \) be a monic polynomial without multiple zero whose derivatives has mutually distinct \( k \)-zeros given by \( d_1, d_2, \ldots, d_k \) with multiplicities \( q_1, q_2, \ldots, q_k \) respectively. The following theorem of Fujimoto helps us to find many uniqueness polynomials.

Lemma 3.5. ([5]) Suppose that \( P(z) \) satisfy critical injection property. Then \( P(z) \) will be a uniqueness polynomial if and only if
\[
\sum_{1\leq l<m\leq k} q_l q_m > \sum_{l=1}^{k} q_l.
\]
In particular, the above inequality is always satisfied whenever \( k \geq 4 \). When \( k = 3 \) and \( \max\{q_1, q_2, q_3\} \geq 2 \) or when \( k = 2 \), \( \min\{q_1, q_2\} \geq 2 \) and \( q_1 + q_2 \geq 5 \), then also the above inequality holds.

4. Proof of the theorem

Proof of Theorem 2.1. By the assumption, it is clear that \( F \) and \( G \) shares (1,1). Now we consider two cases.

Case-1 First we assume that \( H \neq 0 \). Then by simple calculations, we have
\[
N(r, \infty; H) \leq N(r, 0; |F| \geq 2) + N(r, 0; |G| \geq 2) + N(r, \infty; F) + N(r, \infty; G) + N(r, 1; F, G) + N_0(r, 0; F') + N_0(r, 0; G'),
\]
where $\mathcal{N}_0(r, 0; F')$ is the reduced counting function of zeros of $F'$ which is not zeros of $F(F - 1)$. Thus

\begin{align}
(4.1) \quad N(r, \infty; H) & \leq \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 1; f) \\
& + \mathcal{N}(r, 1; g) + \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \\
& + \mathcal{N}_*(r, 1; F, G) + \mathcal{N}_*(r, 0; f') + \mathcal{N}_*(r, 0; g'), \nonumber
\end{align}

where $\mathcal{N}_*(r, 0; f')$ is the reduced counting function of zeros of $f'$ which is not zeros of $f(f - 1)$ and $(F - 1)$. We denote by $\mathcal{N}_*(r, 1; F, G)$ the reduced counting function of those 1-points of $F$ whose multiplicities differ from the multiplicities of the corresponding 1-points of $G$. Clearly

\begin{align}
(4.2) \quad \mathcal{N}(r, 1; F) = 1 = \mathcal{N}(r, 1; G) = 1 \leq N(r, \infty; H), \nonumber
\end{align}

where $\mathcal{N}(r, 1; F) = 1$ the counting function of those simple 1-points of $F$ which are also simple 1-points of $G$.

Now using the Second Fundamental Theorem, equations (4.1) and (4.2), we get

\begin{align}
(4.3) \quad (n + 1)(T(r, f) + T(r, g)) \\
& \leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) \\
& + \mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g) + \mathcal{N}(r, 1; F) + \mathcal{N}(r, 1; G) \\
& - N_*(r, 0; f') - N_*(r, 0; g') + S(r, f) + S(r, g) \\
& \leq 2\{\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) \\
& + \mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g)\} + \{\mathcal{N}(r, 1; F) + \mathcal{N}(r, 1; G) \\
& - \mathcal{N}(r, 1; F = 1)\} + \mathcal{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \nonumber
\end{align}

Again

\begin{align}
(4.4) \quad \mathcal{N}(r, 1; F) + \mathcal{N}(r, 1; G) - \mathcal{N}(r, 1; F) = 1 = \mathcal{N}_*(r, 1; F, G) \\
& \leq \frac{1}{2}\{N(r, 1; F) + N(r, 1; G)\} + \frac{1}{2}\{\mathcal{N}(r, 1; F) \geq 2\} + \mathcal{N}(r, 1, G \geq 2)\} \\
& \leq \frac{n}{2}(T(r, f) + T(r, g)) + \frac{1}{2}\{N(r, 0; f' \neq f) + N(r, 0; g' \neq g)\}. \nonumber
\end{align}

Using (4.3) and (4.4), we get

\begin{align}
(4.5) \quad \left(\frac{n}{2} - 3\right)(T(r, f) + T(r, g)) \\
& \leq 2\{\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g)\} + \frac{1}{2}\left\{N\left(r, 0; \frac{f'}{f}\right) + N\left(r, 0; \frac{g'}{g}\right)\right\} \\
& + S(r, f) + S(r, g) \\
& \leq 2\{\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g)\} + \frac{1}{2}\left\{N\left(r, \infty; \frac{f'}{f}\right) + N\left(r, \infty; \frac{g'}{g}\right)\right\} \\
& + S(r, f) + S(r, g) \\
& \leq \frac{5}{2}\{\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g)\} + \frac{1}{2}\{\mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g)\} + S(r, f) + S(r, g). \nonumber
\end{align}

That is,

\begin{align}
(4.6) \quad (n - 7)(T(r, f) + T(r, g)) \\
& \leq 5\{\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g)\} + S(r, f) + S(r, g), \nonumber
\end{align}

which is a contradiction if $n \geq 13$ (resp. 8) for meromorphic (resp. entire) case.
Case-2 Next we assume that $H \equiv 0$. Now on integration, two times, we have

$$F \equiv \frac{AG + B}{CG + D},$$

where $A, B, C, D$ are constant satisfying $AD - BC \neq 0$.

Thus applying Lemma (3.1) in equation (4.7), we get

$$T(r, f) = T(r, g) + O(1).$$

Next we consider the following two cases:

**Subcase-2.1** First assume $AC \neq 0$. Then equation (4.7) can be written as

$$F - \frac{A}{C} = \frac{BC - AD}{C(CG + D)}.$$

Thus

$$N\left(r, \frac{A}{C}; F\right) = N(r, \infty; G).$$

Now applying the Second Fundamental Theorem and equation (4.8), we get

$$nT(r, f) + O(1) = T(r, F)$$

$$\leq N(r, \infty; F) + N(r, 0; F) + N\left(r, \frac{A}{C}; F\right) + S(r, F)$$

$$\leq N(r, \infty; f) + N(r, 0; f) + 2T(r, f) + N(r, \infty; g) + S(r, f),$$

which is impossible as $n > 5$.

**Subcase-2.2** Thus we consider $AC = 0$. Since $AD - BC \neq 0$, so $A = C = 0$ never occur, thus the following two subcases are obvious:

**Subsubcase-2.2.1** First assume that $A = 0$ and $C \neq 0$. Then obviously $B \neq 0$, hence equation (4.7) can be written as

$$F = \frac{1}{\gamma G + \delta},$$

where $\gamma = \frac{C}{D}$ and $\delta = \frac{B}{D}$. Now we assume that there exist no $z_0$ such that $F(z_0) = 1$. Then applying the Second Fundamental Theorem and equation (4.8), we get

$$T(r, F)$$

$$\leq N(r, \infty; F) + N(r, 0; F) + N(r, 1; F) + S(r, F)$$

$$\leq N(r, \infty; f) + N(r, 0; f) + 2T(r, f) + S(r, f)$$

$$\leq \frac{4}{n}T(r, F) + S(r, F),$$

which is impossible as $n > 4$.

Thus there exist at least one $z_0$ such that $F(z_0) = 1$. Since $F$ and $G$ share 1, hence $\gamma + \delta = 1$ with $\gamma \neq 0$. Thus equation (4.10) becomes,

$$F = \frac{1}{\gamma G + 1 - \gamma},$$
and hence
\[ N(r, 0; G + \frac{1}{\gamma}) = N(r, \infty; F). \]

If \( \gamma \neq 1 \), then the Second Fundamental Theorem and equation (4.8), imply
\[
T(r, G) \\
\leq N(r, \infty; G) + N(r, 0; G) + N(r, 0; G + \frac{1}{\gamma}) + S(r, G) \\
\leq N(r, \infty; g) + N(r, 0; g) + 2T(r, g) + N(r, \infty; f) + S(r, g) \\
\leq \frac{5}{n} T(r, F) + S(r, F),
\]
which is again impossible as \( n > 5 \). Hence \( \gamma = 1 \), i.e., \( FG \equiv 1 \).

Then
\[
f^{n-2} \prod_{i=1}^{2} (f - \gamma_i) g^{n-2} \prod_{i=1}^{2} (g - \gamma_i) \equiv \frac{4c^2}{(n-1)^2(n-2)^2},
\]
where \( \gamma_i \) (i=1,2) are the roots of the equation \( z^2 - \frac{2n}{n-1} z + \frac{n}{n-2} = 0 \).

Let \( z_0 \) be a \( \gamma_i \)-point of \( f \) of order \( p \). Then \( z_0 \) must be a pole of \( g \), (say of order \( q \)). Then \( p = nq \geq n \). So
\[
N(r, \gamma_i; f) \leq \frac{1}{n} N(r, \gamma_i; f).
\]
Again, let \( z_0 \) be a zero of \( f \) of order \( t \). Then \( z_0 \) must be a pole of \( g \), (say of order \( s \)). Then \( (n-2)t = ns \). Thus \( t > s \). Now \( 2s = (n-2)(t-s) \geq (n-2) \). Consequently \( (n-2)t = ns \) gives \( t \geq \frac{n}{2} \). So
\[
N(r, 0; f) \leq \frac{2}{n} N(r, 0; f).
\]

Similar calculations are valid for \( g \) also. Thus
\[
N(r, \infty; f) \leq N(r, 0; g) + \sum_{i=0}^{2} N(r, \gamma_i; g) \\
\leq \frac{2}{n} N(r, 0; g) + 1 \sum_{i=0}^{2} N(r, \gamma_i; g) \\
\leq \frac{4}{n} T(r, g) + O(1).
\]
Next we apply the Second Fundamental Theorem for the identity (4.11) and we get
\[
2T(r, f) \\
\leq N(r, \infty; f) + N(r, 0; f) + \sum_{i=0}^{2} N(r, \gamma_i; f) + S(r, f) \\
\leq \frac{4}{n} T(r, f) + \frac{2}{n} T(r, f) + \frac{2}{n} T(r, f) + S(r, f),
\]
which is a contradiction as \( n \geq 5 \).
Subsubcase 2.2.2 Thus we consider $A \neq 0$ and $C = 0$. Then obviously $D \neq 0$ and equation (4.7) can be written as

$$F = \lambda G + \mu,$$

where $\lambda = \frac{A}{D}$ and $\mu = \frac{B}{D}$.

Then obviously there exist atleast one $z_0$ such that $F(z_0) = 1$, otherwise we arrived at a contradiction like previous subcase. Thus $\lambda + \mu = 1$ with $\lambda \neq 0$. Hence

$$\mathcal{N}(r, 0; G + 1 - \frac{\lambda}{\lambda}) = \mathcal{N}(r, 0; F).$$

If $\lambda \neq 1$, then using the Second Fundamental Theorem and equation (4.8), we get

$$T(r, G) \leq \mathcal{N}(r, \infty; G) + \mathcal{N}(r, 0; G) + \mathcal{N}(r, 0; G + 1 - \frac{\lambda}{\lambda}) + S(r, G)$$

$$\leq \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; g) + 2T(r, g) + \mathcal{N}(r, 0; f) + 2T(r, f) + S(r, g)$$

$$\leq \frac{7}{n}T(r, G) + S(r, G),$$

which is a contradiction as $n > 7$. Thus $\lambda = 1$, i.e., $F \equiv G$, i.e., $P(f) \equiv P(g)$.

Since $P'(z) = \frac{n(n-1)(n-2)}{2}z^{n-3}(z-1)^2$ and $P(0) \neq P(1)$, So $P(z)$ satisfy critical injection property. Thus in view of Lemma 3.5 $P(z)$ is a uniqueness polynomial. Hence $f \equiv g$. This completes the proof of the theorem. \hfill \Box

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