Density Analysis of Network Community Divisions

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We present a compact matrix formulation of the modularity, a commonly used quality measure for the community division in a network. Using this formulation we calculate the density of modularities, and we conclude that the general features of the modularity density are quite similar for different networks. From a simple model of the modularity we conclude that all connected networks must show similar shapes of their modularity densities. The general features of this density may give valuable information in the search for good optimization schemes of the modularity.

I. INTRODUCTION

The nodes of a network can be grouped into communities which are loosely defined as groups of nodes that are more “related” to each other in some fashion than they are related to the rest of the network. Such a community division can reveal important structures of the network. In a recent study, for instance, Wilkinson and Huberman introduced a method to create a network of gene co–occurrences from the literature and interpret its community structure as groups of genes related to each other by their function. Since some of the genes in these communities are not known to be related to the community’s function, this method possibly aids in identifying unknown relationships of this sort. Massen and Doye used a community analysis on a potential energy landscape to identify transition states of small Lennard–Jones cluster. Networks have been very successfully used also to simulate dynamics in various systems. By modeling a community structure of individuals using a contact network model, Meyers et al. predicted the dynamics of a SARS outbreak.

Many different approaches have been used to identify community structures in networks. To name a few more recent methods: vertex similarity, vertex degree gradient, resistor network, Potts Hamiltonian model, and an information–theoretic approach. The most popular methods appear to be ones based on the network modularity $Q$ introduced by Newman and coworkers. The advantage with the modularity $Q$ is that it is a well defined number that gives the quality of a particular community division in a network. It is bounded, $-1 \leq Q \leq 1$, and is larger for divisions that split the network into groups with many intra–edges and few inter–edges between the groups.

A number of different strategies have been proposed for finding the optimal community division based on the modularity. These methods can be broadly divided into two different classes. Path bound methods are agglomerative or divisive and either successively add or take away edges in the network so as to reduce the number of communities by merging existing communities (agglomerative) or to increase the number of communities by taking away edges and splitting existing communities (divisive). In both cases, the number of possible community divisions depend on the previous steps in the algorithm, or the particular path that was taken in the space of all possible community divisions. The resulting evolution of the community structure is commonly called a dendrogram.

The different methods in this class differ in the way they identify the edges to be removed or added. Examples are the shortest–path betweenness, random–path betweenness, or the greedy algorithm. All these methods have in common that they follow a dendrogram and attempt to identify the edges to be removed or added by optimizing the effect of modularity change. The number of communities is changed by at most 1 in each step and only information from the previous step is used. The quality of these methods is very sensitive to the strategy employed for identifying the critical edges.

Methods in the second class are not path bound and try to optimize $Q$ directly without regards to a dendrogram. Simulated annealing is a recent example of one of these techniques, but other techniques as for instance genetic algorithms are also possible.

Current results suggest that non path bound optimization strategies outperform dendrogram bound methods. However, the number of possible ways of dividing a network with $N$ nodes into $C$ communities is immense and given by the Stirling number of the second kind $S_N^C$. Due to the discrete nature of how nodes are assigned to communities, the modularity takes on a discrete set of values. The possibility exists that several divisions are of similar high quality, or that $Q$ is degenerate as a function of the community division. All of these properties of the modularity make it difficult to optimize the modularity.
In a non path bound way by using standard optimization techniques.

In this article we study the properties of the modularity $Q$ in a statistical sense. Our aim is to gain a deeper understanding of the complexity of network community divisions in general, and the modularity in particular. By gaining knowledge of the modularity we believe that it may be possible to find faster and more accurate community division algorithms. We introduce a matrix algebra formalism to define a connected community division and obtain the modularity. We calculate a community division density in the modularity–community space where we show how the values of $Q$ are distributed in terms of the number of communities in the division for several different networks.

This article is organized as follows: In section II we will describe the modularity and introduce a matrix representation of it. In section III we will present and discuss our results and in section IV we present our conclusions.

II. THEORY

A. Modularity and its Matrix Representation

A network can be represented by its corresponding adjacency matrix $A$. For a network with $N$ nodes, this matrix is of size $N \times N$ where the element $A_{ij}$ represents the edge between nodes $i$ and $j$. For unweighted networks, $A_{ij} = 1$ if the edge exists and 0 otherwise. For weighted networks, $A_{ij} = w_{ij}$, the weight associated with this edge, and in the case of an undirected network, $A$ is symmetric. If we do not include self–edges, the diagonal of $A$ is zero. The underlying network can be divided into $C$ communities, which amounts to labeling each node with one of $C$ community labels. A compact way of expressing a specific community division is through the community matrix $P$ which we define as a matrix of size $C \times N$, with elements $P_{ij}$ given by

$$P_{ij} = \begin{cases} 1 & \text{node } j \text{ is a member of community } i \\ 0 & \text{otherwise.} \end{cases}$$

(1)

Newman’s assortative mixing matrix $e$, can then be expressed as

$$e = \left( \sum_{ij} A_{ij} \right)^{-1} P A P^T,$$

(2)

where $P^T$ is the transpose of $P$. The modularity is given by

$$Q = \text{Tr}(e) - \sum_{ij} (e^2)_{ij}.$$

(3)

The larger the value of $Q$ the better the community division. The modularity has the property that it has an upper bound, $Q \leq 1 - 1/C$. This has to be regarded as a theoretical upper bound, however; in practice the upper bound is lower.

B. Statistical Analysis of the Modularity

The modularity can be interpreted as a function of the community matrix $P$. In the space of all possible community divisions, $Q(P)$ defines a rugged and complicated surface. In Fig. 1 we show several curves of $Q(P)$ vs. $C$ obtained by randomly choosing a path through this space along a dendrogram in the Zachary Karate Club network [18]. The path is chosen by starting with a diagonal $N \times N$ matrix $P$ so that all nodes are in their own community. Then, by summing two randomly selected rows in $P$ we merge two of the communities. We can easily check that the new community is connected by checking the assortative mixing matrix $e$. We can see in the figure that the qualities of the various community divisions depend strongly on the chosen paths and the success in the previous steps.

In the following we will try to gain a more general understanding of the structure of $Q(P)$ for different networks. We will not attempt to optimize the modularity but rather map its structure in the space of possible community divisions. Put in other words, we would like to know how many community matrices $P$ exist for a given number of communities $C$ in a given modularity interval between $Q$ and $Q + \delta Q$. If we start out with the completely split network, i.e. $C = N$, and sample $Q(P)$ along a random dendrogram until we reach $C = 1$ we will find that some values of $Q_C$ are more likely to be found this way than others. After sampling a large number of these random dendrograms we can analyze the result as a fre-
frequency distribution $f(Q)$ vs. $Q$ and $C$ and get a density of modularities, $N(Q)$. We expect the following: For a given number of communities, $C$, there will be a range of possible values of $Q_C$. We know from previous studies that it is difficult to find a division with a large modularity. This implies that it is unlikely for us to find a large value of $Q$ with our random dendrograms and likely that we find some average $Q$.

III. RESULTS

A. Examples of modularity densities

1. Real networks

In Fig. 1 we present plots of the modularity as a function of $C$ along random dendrograms in the Zachary Karate Club network [18]. The modularities in these examples are quite different. By performing a large set of such dendrogram walks where we save each modularity plot we obtain a statistical image of the modularity.

In Fig. 2 we show the modularity density for the Zachary Karate Club network where we have calculated 100,000 modularity plots from random dendrogram walks. We find that the modularity density is not uniformly distributed in the modularity–communities space and has a strong peak for large values of $C$. This peak decreases rapidly as the number of communities is decreased. For small values of $C$ the density is low and spread over a large range of modularities. By construction, the integral over modularity for constant number of communities is always the same. Consequently, if the peak is very low, the range over which the modularity density is spread out will be large and it thus seems most probable to find the maximum $Q$ within this regime.

We notice that although the density is very low in the case of a small number of communities the peak does not disappear and the top of the peak outlines a curved shape in the modularity–communities space. The position of this peak determines the most probable $Q(C)$ relation for the network.

In Fig. 3 we show the modularity density surface for the network of simultaneous appearance on stage for the characters in the Les Misérables musical [11, 19]. We find that the shape and general features are similar to the Zachary modularity density surface. The main differences are that the Les Misérables surface shows smaller regions of negative modularity and a more shallow curvature. In these two examples, we find that the general features of the modularity density are very similar. Therefore we will investigate the modularity density of a few artificial networks to see whether the same general features can be found.

2. Random networks

In Fig. 4 we show the modularity density for a random network with 34 nodes and 78 undirected edges, which is the same average degree $\langle k \rangle = 4.59$ as the Zachary Karate Club. The edges are randomly distributed however and the network does not exhibit any particular community structure and certainly not the same community structure as the Zachary network. The modularity density on the other hand does exhibit a very similar structure to what we found for the networks of Figs. 2 and 3. This indicates that to a large extent the structure of the modularity density is not associated with the particular community structure, but rather with the network itself.

In Fig. 5 we show the modularity density for a random network with 34 nodes and 78 undirected edges. Before distributing the edges, the nodes were randomly assigned to 2 communities. The edge distribution was random with the constraint that the probability $p_{in}$ of connecting two nodes within the same community was chosen to be 10 times larger than the probability $p_{out}$ (see Ref. [11] or
Sec. III B 1 for a more detailed description of the $p_{in}/p_{out}$ algorithm of connecting two nodes which are in different communities. The generated network is shown in Fig. 6 and it shows a clear community structure.

3. Fully connected network

Fig. 7 shows our results for a fully connected network of 34 nodes. The overall structure of the modularity density is markedly different from the other cases and we attribute this difference to the much higher level of connectedness of the nodes. In this case the modularity is always less than zero and the maximum value is at $C = 1$, where $Q = 0$. The fact that the modularity is negative is due to the fact that the number of inter-community edges is always larger than the number of intra-community edges for this network. The off-diagonal elements of the assortative mixing matrix $e$ are therefore always large and contribute strongly to the negative term in the expression for the modularity, eq. 3. The average degree in this fully connected network is $\langle k \rangle = 33$ as compared to $\langle k \rangle = 4.59$ for the Zachary network.

4. Symmetric network

In Fig. 8 we show the modularity density of a completely symmetric regular network consisting of 64 nodes arranged in an $8 \times 8$ grid with 4 edges per node and periodic boundary conditions. Intuitively, this network should not show any strong community structure. But as shown in the figure, there is a high modularity region around $C \approx 8$ communities.

B. Analysis

1. General networks

The appearance of the modularity density is very similar for different networks and in particular the shape of the most probable $Q(C)$ region, the “ridge” in the density surface plots, shows remarkable similarity between
FIG. 5: (Color online) Modularity density for a random network with 34 nodes and 78 edges. The nodes were randomly assigned to 2 communities and the edges were randomly distributed under the constraint that the probability of connecting two nodes within the same community is 10 times larger than the probability of connecting two nodes which are in different communities. On the x–axis is the number of communities, $C$.

FIG. 6: (Color online) Network structure of a random network with 34 nodes, $\langle k \rangle = 4.59$ and $p_{in}/p_{out} = 10$.

FIG. 7: (Color online) Modularity Density for a fully connected network with 34 nodes. On the x–axis is the number of communities, $C$.

different networks. In an effort to describe the general shape of the $Q(C)$ ridge we will employ a simple model of the modularity as a function of the number of communities. We start by observing that in order to maximize the modularity it is desirable to minimize the number of off–diagonal elements and evenly distribute the diagonal elements in the assortative mixing matrix $e$. In the case where the communities are completely disconnected, the maximum modularity is given by

$$Q = 1 - \frac{1}{C}, \quad (4)$$

where the first term is due to the trace of $e$ and the second is due to the $\sum_{ij} (e^2)_{ij}$ term of eq. 3. This formula is the upper theoretical limit of the modularity for any network but does not describe the general shape of the modularity density very well. In order to make a correction to this formula for networks that are connected we introduce the function $\delta(C)$ which represents the average number of edges connecting a community with the other communities. For simplicity we will only consider cases where the communities have equal size and the edges between the communities are distributed evenly. The $C \times C$
of community formation in the network. In addition, the values of the probabilities are chosen such that the average degree per node, \( \langle k \rangle \), can be controlled. Since we are interested in deriving a simple closed form expression for \( \delta(C) \) we approximate the probability of finding a particular pair of nodes by assuming that the network is empty and does not contain any edges. Our approximation will be good for sparsely connected networks and few communities, but will become progressively worse for \( C \to N \) and \( \langle k \rangle \to N-1 \). In a network without self-edges and \( C \) communities we therefore require that

\[
\left( \frac{N}{C} - 1 \right) p_{in} + N \left( 1 - \frac{1}{C} \right) p_{out} = \langle k \rangle .
\]

As a shorthand we introduce \( \lambda = p_{in}/p_{out} \geq 1 \), a freely tunable parameter. The first term in eq. (7) corresponds to the average number of edges per node connecting two nodes within the same community, \( \langle k_{in} \rangle \). The second term is the corresponding number of edges connecting two nodes in different communities, \( \langle k_{out} \rangle \). It is this second term that we need in order to estimate the parameter \( \delta(C) \). The parameter \( \delta(C) \) is given by

\[
\delta(C) = N \frac{\langle k_{out} \rangle}{\langle k \rangle} = N \frac{\langle k \rangle}{(1 - C/N) \lambda + (C - 1)}.
\]

The fully connected like network can easily be derived now by setting \( \lambda = 1 \), and

\[
\delta(C) = \langle k \rangle \frac{N^2}{N-1} \frac{C-1}{C^2}.
\]

We chose \( N = 34 \) and \( \langle k \rangle = 4.59 \) to model the Zachary Karate Club and our results for a range of different values of \( \lambda \) are shown in Fig. 4 in the upper and lower panels, respectively. As can be seen from the figure, the model reproduces the shape of the most likely modularity, the modularity density ridge, well for the networks shown in Figs. 2, 3, 6 and 7. In particular the general feature of a peak in modularity at a small value of \( C \) is reproduced solely by the introduction of the \( \delta(C) \) term that describes the average number of edges that connect communities. We observe that any connected network can be approximated by an appropriate choice of \( \delta(C) \). This suggests that any connected network will show a peak in the modularity density for small \( C \ll N \) just as our numerical results for real networks indicate. Note that the model is such that one needs to generate a new network for each value of \( C \) in order to keep \( \lambda \) constant. Consequently, the modularity plots in Fig. 4 are really to be understood as a collection of modularity values for different networks given the constraint that \( \lambda \) is fixed. It is interesting to note also that the fully connected network, shown in Fig. 4, is qualitatively reproduced by our model in the case of \( \lambda = 1 \) although \( \langle k \rangle \) is far from the fully connected value. In our model, \( \langle k \rangle \) represents only a scaling constant and

\[
e_{C} = \frac{1}{M} \begin{pmatrix}
\frac{M}{C} - \delta(C) & \delta(C)/C \\
\delta(C)/C & \frac{M}{C} - \delta(C)
\end{pmatrix}
\]

where \( M \) is the total number of directed edges in the network. The modularity in this case is given by

\[
Q = 1 - C \frac{\delta(C)}{M} - \frac{1}{C}.
\]

In order to make a reasonable estimate of \( \delta(C) \) we will revisit the \( p_{in}/p_{out} \)-model already used in section 3.2.2. The model, as introduced by Newman and Girvan [11], is used to generate networks with a predetermined community structure by randomly choosing pairs of nodes and connecting them based on the two probabilities \( p_{in} \) and \( p_{out} \). The ratio of the probabilities determines the extent

FIG. 8: (Color online) Modularity density for a regular network of 8 x 8 nodes on a two dimensional square grid with periodic boundary conditions. On the x-axis is the number of communities, \( C \).

assortative mixing matrix will then look like
does not alter the qualitative result. We can not expect to get quantitative agreement since the model is only accounting for the general behavior of the $C$–dependence of the modularity with fixed $\lambda$. We note that it is in principle possible to write down the $C$–dependence of $\lambda$ for a regular lattice but that has not been performed in the current study. However, the limitations of the model do not affect our general conclusion that the maximum in the modularity occurs for relatively small values of $C$ for any connected network.

IV. CONCLUSIONS

We have presented a matrix formalism to describe the modularity of a community division in a network. We have described the modularity for some well studied networks as well as some synthetic networks from a statistical point of view and introduced the concept of modularity density.

In conclusion we found that the modularity density is quite similar for different networks. Even random networks with no apparent community structure exhibit a remarkably similar modularity density. This suggests that most of the structure of the modularity density is independent of the network itself. We have introduced a simple model that describes the general shape of the modularity density based on the $p_{in}/p_{out}$ concept of Newman and Girvan [11] and concluded that any connected network must show a peak in the modularity density at a small number of communities compared to the size of the network.

The presence of a general shape indicates that it should be possible to develop global optimization strategies which work well for most networks. The maximum modularity curve of course depends on the particular network.

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