Asymptotic Solutions of the Kinetic Equation of the Radiation Propagation, Asymptotic Approximation of the $n$-th Order and the Improved Boundary Conditions

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Abstract

In the paper, asymptotic solutions of the kinetic equation of radiation propagation are constructed for two extreme cases: optically thick and optically thin media; in calculations of radiation propagation in optically thick media it is suggested to use asymptotic approximation of the $n$-th order. A formal solution has been obtained for the kinetic equation of radiation propagation along the line in the form of infinite series, for the optically thick medium, when the infinite series are certainly convergent, this formal solution is similar to the constructed asymptotic solution of the kinetic equation of radiation propagation. From the asymptotic solution of the kinetic equation of radiation propagation for optically thick media, improved boundary conditions (for inner boundaries and outer boundaries with vacuum), essential for practical application to calculations of radiation propagation, are derived.

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I. INTRODUCTION

Currently, in calculations of radiation propagation (gas-dynamic calculations with radiation), 3 basic approximations are used: kinetic approximation (numerical solution of the kinetic equation for radiation propagation), diffusion approximation and radiation heat conduction approximation; see, for instance, [1, 2]. The kinetic approximation, physically most accurate, is much more time-consuming in the calculations of radiation propagation. Besides, natural difference approximation of the first order derivatives in the kinetic equation of radiation propagation leads to non-monotonous difference schemes. Therefore, the majority of calculations of radiation propagation are still performed in the radiation heat conduction approximation and diffusion approximation. For these approximations, formulation of boundary conditions (for radiation) at the boundary of two substances (see below) represents a certain issue.

Below, in section II, asymptotic solutions of kinetic equation for radiation propagation will be constructed in two extreme cases: optically thick and optically thin media. From the asymptotic solution of the kinetic equation of radiation propagation for optically thick medium, we obtain a system of equations of asymptotic approximation of the n-th order, which is proposed to be used for calculations of radiation propagation in optically thick media instead of diffusion approximation and radiation heat conduction approximation. In calculating the propagation of radiation in optically not very thick media, it is proposed to use (minimally necessary) limitations on radiation mean free paths for all considered approximations (see [3] and the references, which are available there, concerning approximations with heat flux limitations).

In section III, the formal solution of the kinetic equation for radiation propagation along the line will be obtained in the form of infinite series (analogous infinite series is obtained in [2] due to somewhat different method, see equation (2.88) in [2]); for optically thick medium, when the infinite series are convergent, this formal solution is similar to the asymptotic solution of the kinetic equation of radiation propagation for optically thick medium, constructed in section II.

In section IV, from the asymptotic solution of the kinetic equation of radiation propagation for optically thick medium, improved boundary conditions (for inner and outer boundaries with vacuum) will be obtained. They are proposed to use in calculations of ra-
radiation propagation in the asymptotic approximation of the \( n \)-th order and in the diffusion approximation. Similar improved boundary conditions can also be used in the radiation heat conduction approximation. In the expressions for the improved boundary conditions, the limitations on radiation mean free paths are introduced, slightly more accurate, than those discussed in the section II.

II. ASYMPTOTIC SOLUTIONS OF KINETIC EQUATION OF RADIATION PROPAGATION

The approximated (light scattering is neglected, \ldots) kinetic equation of radiation propagation represents a first order differential equation in particular derivatives for time and direction depending on spatial coordinates, spectral intensity of radiation \( I_\nu(t, r, \Omega) \); see, for instance, equation (2.28) in [1]:

\[
\left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_\nu = \kappa'_\nu (I_{\nu P} - I_\nu),
\]

where \( I_\nu(t, r, \Omega) \, d\nu \, d\Omega \) represents a radiant energy in the spectral interval \( d\nu \), transported by light quanta having the motion direction within the element of solid angle \( d\Omega \) about the unit vector \( \Omega \) per unit time across a unit area located at point \( r \) perpendicular to the direction of \( \Omega \); \( I_{\nu P} \) is the spectral intensity of equilibrium radiation in the state of thermodynamic equilibrium determined by Planck formula:

\[
I_{\nu P} \equiv B_\nu \equiv \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1};
\]

\( \kappa'_\nu \equiv 1/l'_\nu \) is the absorption factor reduced by induced emission (see, for example, [1, Chapter II, § 5])

\[
\kappa'_\nu \equiv \kappa_\nu \left(1 - e^{-\frac{h\nu}{kT}}\right),
\]

\( \kappa_\nu I_\nu \, d\nu \, d\Omega \) is equal to the energy of light quanta with the frequency from \( \nu \) to \( \nu + d\nu \), having the motion direction within the element of solid angle \( d\Omega \) about the unit vector \( \Omega \), absorbed by a substance in a unit volume at unit time; \( l'_\nu \) is the mean free path of photons with frequency \( \nu \); \( c \) is the light velocity in vacuum; \( h \) is the Planck constant; \( \nu \) is the light quanta rate; \( k \) is Boltzmann constant; \( T \) is the substance temperature counted on the thermodynamic temperature scale.
Let us construct an asymptotic solution of equation (1) for two extreme cases: \( l'_\nu / L \rightarrow 0 \) (optically thick medium) and \( L / l'_\nu \rightarrow 0 \) (optically thin medium), where \( L \) is a characteristic distance of variations in radiation intensity. In order not to change the form of the physical equation, it is convenient not to specify explicitly a series expansion parameter (though in cases under consideration, it is very easy to specify a series expansion parameter, it is sufficient to go over to dimensionless independent variables in equation (1): \( \{c t / L, \ r / L\} \), based on powers of which, when it tends to zero, an asymptotic expansion with variable coefficients ([4], Chapter V, § 2, Section 5) is constructed, but to introduce it formally into the physical equation just as a smallness indicator of appropriate terms of the physical equation – compare with [5].

For an optically thick medium, let us introduce a series expansion (small) parameter \( \epsilon \) into equation (1) in the following way:

\[
\epsilon l'_\nu \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_\nu = (I_{\nu P} - I_\nu),
\]

(4)

(5)

Let us write down the asymptotic expansion of spectral radiation intensity in the form of the formal series of \( \epsilon \)-power successive approximations

\[
I_\nu = \epsilon^0 I_{\nu}^{(0)} + \epsilon I_{\nu}^{(1)} + \epsilon^2 I_{\nu}^{(2)} + \cdots,
\]

(6)

Let us substitute this power series into equation (4) and equate the variable coefficients at like powers \( \epsilon \). As the result, we will obtain the following system of equations of the method of successive approximations:

\[
I_{\nu}^{(0)} = I_{\nu P},
\]

(7a)

\[
I_{\nu}^{(1)} = -l'_\nu \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_{\nu P},
\]

(7b)

\[ \vdots \]

\[
I_{\nu}^{(n)} = -l'_\nu \left( \frac{\partial}{\partial t} + \Omega \cdot \nabla \right) I_{\nu}^{(n-1)}.
\]

(7c)

From (6) and (7), according to general definitions of spectral radiant energy density

\[
U_{\nu} \overset{\text{def}}{=} \frac{1}{c} \int_{(4\pi)} I_{\nu} d\Omega
\]

(8)
and spectral radiant energy flux

\[ S_\nu \overset{\text{def}}{=} \int_{(4\pi)} I_\nu \Omega \, d\Omega \]  

we obtain the asymptotic expansion of spectral radiant energy density

\[ U_\nu = \varepsilon^0 U^{(0)}_\nu + \varepsilon U^{(1)}_\nu + \varepsilon^2 U^{(2)}_\nu + \cdots \]  

where, in particular,

\[ U^{(0)}_\nu = U_{\nu P} = \frac{4\pi}{c} I_{\nu P} \]  

is equilibrium spectral radiant energy density,

\[ U^{(1)}_\nu = -t'_\nu \frac{\partial U_{\nu P}}{c \partial t}, \]  

\[ U^{(2)}_\nu = t'_\nu \frac{\partial}{c \partial t} \left( t'_\nu \frac{\partial U_{\nu P}}{c \partial t} \right) + \frac{1}{3} t'_\nu \nabla \cdot \left( t'_\nu \nabla U_{\nu P} \right), \]  

and the asymptotic expansion of spectral radiant energy flux is

\[ S_\nu = \varepsilon^0 S^{(0)}_\nu + \varepsilon S^{(1)}_\nu + \varepsilon^2 S^{(2)}_\nu + \cdots, \]  

where

\[ S^{(0)}_\nu = 0, \]  

\[ S^{(1)}_\nu = -\frac{4\pi}{3} t'_\nu \nabla I_{\nu P} = -\frac{c}{3} t'_\nu \nabla U^{(0)}_\nu, \]  

\[ S^{(2)}_\nu = \frac{4\pi}{3} t'_\nu \nabla \left( t'_\nu \frac{\partial I_{\nu P}}{c \partial t} \right) + \frac{4\pi}{3} t'_\nu \frac{\partial}{c \partial t} \left( t'_\nu \nabla I_{\nu P} \right) \]  

\[ = -\frac{c}{3} t'_\nu \nabla U^{(1)}_\nu - t'_\nu \frac{\partial S^{(1)}_\nu}{c \partial t}. \]  

Equations (11), (14) form equations system of asymptotic approximation of n-th order, that can be used in calculations of radiation propagation (gas-dynamic calculations with radiation) rather than considered below equations of diffusion approximation and radiation heat conduction approximation.

The n-th order asymptotic approximation can be used in calculations of radiation propagation, only if the series (9), (11), (14) converge. The formal convergence of these series
can be achieved due to a certain limitation on radiation mean free path, for example, if \( \tilde{I}'_\nu \) is used instead of \( I'_\nu \):

\[
\tilde{I}'_\nu = \min \left\{ I'_\nu, L'_\nu \right\},
\]

where

\[
L'_\nu = K_\nu \frac{I_{\nu P}}{|\nabla I_{\nu P}| + |\partial I_{\nu P}/c\partial t|} = K_\nu \frac{U_{\nu P}}{|\nabla U_{\nu P}| + |\partial U_{\nu P}/c\partial t|},
\]

and the coefficients \( K_\nu \), depending, generally speaking, on the conditions of the problem

\[
K_\nu \lesssim 1,
\]

compare with equation (17c). The limitation (18) can be considered as the minimum necessary limitation for obtaining reliable results in radiation propagation calculations. It makes sense to monitor the use of the limitation (18) in calculations and, if possible, to recalculate questionable problems in the kinetic approximation.

The first equation of diffusion approximation is derived (exactly) from the kinetic equation of radiation propagation (1) due to equation (1) integration over angles:

\[
\frac{\partial U_\nu}{\partial t} + \text{div} S_\nu = cK'_\nu (U_{\nu P} - U_\nu).
\]

The second equation of diffusion approximation can be derived from equation (16) by adding the terms of higher infinitesimal order of asymptotic expansions of spectral radiant energy density (10) and spectral radiant energy flux (14) into equation (16):

\[
S_\nu = -\frac{c}{3} l'_\nu \nabla U_\nu.
\]

Because of the presence of the second term on the right-hand side of the equation (17), the second equation of the diffusion approximation (and the diffusion approximation as a whole) has the first order of precision. Equations of (one-temperature) diffusion approximation (21), (22) represent the system of two first order equations in particular derivatives for two unknowns depending on radiation frequency (but not on the direction) of functions of spatial coordinates and time, i.e., spectral radiant energy density and spectral radiant energy flux. For optically thick media, the diffusion approximation is not more precise than the asymptotic first order approximation, but the first order asymptotic approximation is
much simpler. In calculations of radiation propagation in optically not very thick media the limitation on radiation mean free path \((18)\) can be used.

In the approximation of radiation heat conduction, the frequency-integral radiant energy flux is defined by an analytical formula (see, for example, [1, Chapter II, § 12]):

\[
S = -\frac{l'_R c}{3} \nabla U_P = -\frac{16 l'_R \sigma T^3}{3} \nabla T;
\]

(23)

in this equation

\[
l'_R = \int_0^\infty \frac{\nu}{\kappa} d\nu = \frac{1}{9} \int_0^\infty \frac{\kappa}{\rho} d\nu \frac{dI}{d\nu} d\nu
\]

(24)

is the Rosseland mean free path;

\[
U_P = \int_0^\infty U_\nu d\nu = \frac{4 \sigma T^4}{c}
\]

(25)

is the equilibrium density of radiation energy; \(\sigma\) is the Stephen-Boltzmann constant. Equilibrium radiant energy density enters in the gas-dynamic equation of energy transfer via the equation of state of substance with radiation.

Equation (23) corresponds to the second equation of diffusion approximation (22) and describes well radiation propagation in optically thick media but is unsuitable for description of radiation propagation in optically thin media. Application of geometrical radiation mean free paths to reduce the value of radiant energy flux, calculated due to formula (23), abnormally high in optically thin media, is ill-founded. One more critical drawback of the radiation heat conduction approximation is associated with the vertically cut-off profile of the heat wave produced in this approximation (see, for example, [1, Chapter X, § 3]); as the result, one can calculate infinitely high-value radiant energy flux at the front of heat wave by using formula (23) and refining the calculation mesh. For optically thick media, the radiation heat conduction approximation has the zeroth order of precision, because only equilibrium radiant energy density enters in the gas-dynamic equation of energy transfer

\[
U_P = U^{(0)}.
\]

(26)

In calculations of radiation propagation in optically not very thick media the limitation on radiation mean free path, analogous (18), can be used

\[
l'_R = \min \left\{ l'_R, L'_R \right\},
\]

(27)
where
\[ L_R' = K_R \frac{I_P}{|\nabla I_P| + |\partial I_P/c\partial t|} = K_R \frac{U_P}{|\nabla U_P| + |\partial U_P/c\partial t|}, \] (28)
\[ I_P = \int_0^\infty I_{\nu P} d\nu = \frac{\sigma T^4}{\pi}, \] (29)
– intensity of equilibrium radiation, and coefficient
\[ K_R \lesssim 1. \] (30)

For the optically thin medium let us introduce a series expansion parameter \( \xi \) into equation (1)
\[ l'_\nu \left( \frac{\partial}{c\partial t} + \Omega \cdot \nabla \right) I_\nu = \xi (I_{\nu P} - I_\nu), \] (31)
writing down the asymptotic \( \xi \) power expansion of radiation intensity in the form of a formal
series of successive approximations
\[ I_\nu = \xi^0 I_\nu^{(0)} + \xi I_\nu^{(1)} + \xi^2 I_\nu^{(2)} + \cdots, \] (33)
substituting this power series into equation (31) and equating variable coefficients at like
powers \( \xi \), we obtain the following system of equations of the method of successive approximations:
\[ l'_\nu \left( \frac{\partial}{c\partial t} + \Omega \cdot \nabla \right) I_\nu^{(0)} = 0, \] (34a)
\[ l'_\nu \left( \frac{\partial}{c\partial t} + \Omega \cdot \nabla \right) I_\nu^{(1)} = (I_{\nu P} - I_\nu^{(0)}), \] (34b)
\[ \vdots \]
\[ l'_\nu \left( \frac{\partial}{c\partial t} + \Omega \cdot \nabla \right) I_\nu^{(n)} = -I_\nu^{(n-1)}. \] (34c)

From (33) and (34), taking into account (8) and (9), we obtain the system of coupling
equations of asymptotic expansion of spectral radiant energy density
\[ U_\nu = \xi^0 U_\nu^{(0)} + \xi U_\nu^{(1)} + \xi^2 U_\nu^{(2)} + \cdots \] (35)
and spectral radiant energy flux

\[ S_\nu = \xi^0 S_\nu^{(0)} + \xi^1 S_\nu^{(1)} + \xi^2 S_\nu^{(2)} + \cdots , \quad (36) \]

which can be written in the form of:

\[
\begin{align*}
\frac{\partial U_\nu^{(0)}}{\partial t} + \text{div} S_\nu^{(0)} &= 0 , \\
\frac{\partial U_\nu^{(1)}}{\partial t} + \text{div} S_\nu^{(1)} &= c\kappa'_\nu \left( U_{\nu\nu} - U_\nu^{(0)} \right) , \\
&\vdots \\
\frac{\partial U_\nu^{(n)}}{\partial t} + \text{div} S_\nu^{(n)} &= -c\kappa'_\nu U_\nu^{(n-1)} .
\end{align*} \quad (37) \]

Unfortunately, the system of equations (37) yields nothing interesting. By summing the equations of system (37), we will just obtain the first equation of diffusion approximation (21).

### III. FORMAL SOLUTION OF THE KINETIC EQUATION OF RADIATION PROPAGATION

Let us select in space spherical system of coordinates \((r, \theta, \varphi)\) with the origin at the edge of a physical body, where radiation propagates, direct the polar axis, for definiteness, along the outer normal to the body surface.

Since the scattering of radiation leading to a change in the direction of photon motion is absent in the equation (1), considering the radiation propagation along the line, defined by the polar angle \(\theta\) and the azimuth angle \(\varphi\), we can write the formal solution of the equation (1) for the angles \(\pi/2 < \theta \leq \pi\) in the form:

\[ I_\nu (t, r, \theta, \varphi) = \int_0^r (\kappa'_\nu I_{\nu\nu})_{\nu', t - \frac{r-r_0}{c}} \exp \left[ -\int_{r'}^r (\kappa'_\nu)_{\nu', t - \frac{r-r_0}{c}} dr'' \right] dr' + \\
+ I_{\nu0} (0, t - \frac{r}{c}) \exp \left[ -\int_0^{r'} (\kappa'_\nu)_{\nu', t - \frac{r-r_0}{c}} dr'' \right] , \quad (38) \]

compare it, for instance, with equation (2.33) in [1]. \(I_{\nu0}\) in the solution (38) is an arbitrary integration constant corresponding to the intensity of radiation entering the body at point \(r = 0\), at time \(t - \frac{r}{c}\), in the direction specified by angles \((\theta, \varphi)\); the angular dependence in the right part of the solution (38) is not explicitly shown. According to the first summand in the solution (38), the radiation at point \(r\) of the explored line is formed by photons, generated
at points $r'$ of the segment $[0, r]$ at the earlier time instances, among which only the portion
\[
\exp \left[ - \int_{r'}^{r} (\kappa'_{\nu})_{r''', t- \frac{\kappa''_{\nu}}{c}} dr'' \right]
\]
reaches the point $r$. The solution (38) may be verified by its direct substitution into equation (1).

Similarly, for the polar angles $0 \leq \theta \leq \pi /2$, defining the direction of photon motion [the position of the photons on the line is defined by angles $(\pi - \theta, \pi + \varphi)$], the formal solution of the equation (1) can be written in the form:

\[
I_{\nu}(t, r, \theta, \varphi) = \int_{r}^{\infty} (\kappa'_{\nu} I_{\nu} P)_{r', t- \frac{\kappa''_{\nu}}{c}} \exp \left[ - \int_{r}^{r'} (\kappa'_{\nu})_{r'', t- \frac{\kappa''_{\nu}}{c}} dr'' \right] dr'.
\]  (39)

In compliance with the solution (38), (39) only photons, generated at the distance not exceeding several radiation mean free paths, contribute to the radiation at point $r$. Light covers this distance during the $\sim l'_{\nu}/c$ time period, which is much shorter than the characteristic variation time of substance parameters (temperature, density, ...), therefore the radiation field almost always can be considered quasi-steady, i.e., corresponding to instantaneous distribution of substance temperature and density, that is one can neglect the time lag in the formula (39) and use a simpler expression for formal solution of equation (1) instead of (39):

\[
I_{\nu}(t, r, \theta, \varphi) = \int_{r}^{\infty} \kappa'_{\nu} I_{\nu} P \exp \left[ - \int_{r}^{r'} \kappa'_{\nu} dr'' \right] dr'.
\]  (40)

or just

\[
I_{\nu}(t, r, \theta, \varphi) = \int_{r}^{\infty} \kappa'_{\nu} I_{\nu} P \exp \left[ - \int_{r}^{r'} \kappa'_{\nu} dr'' \right] dr'.
\]  (41)

Let us explore an important case when the medium in which the radiation propagates occupies an infinite semi-space $z \leq 0$ limited by a plane surface $z = 0$; let us direct the polar axis along the outer normal to the surface $z = 0$. Let us assume that it is a plane symmetry, i.e. the substance parameters actually depend only on the coordinate

\[
z = r \cos (\pi - \theta) = -r \cos \theta
\]  (42)

and time $t$. In this case we obtain from the equation (11), that the radiation intensity close to the body surface is

\[
I_{\nu}(t, 0, \theta, \varphi) = \int_{0}^{\infty} \kappa'_{\nu} I_{\nu} P \exp \left[ - \int_{0}^{r'} \kappa'_{\nu} dr'' \right] dr' = \int_{0}^{\infty} I_{\nu} P e^{-\gamma} d\tau,
\]  (43)
Integrating the latter integral in (43) by parts, we obtain the following expression for the radiation intensity in the vicinity of the body surface

\[ I_\nu(t, 0, \theta, \phi) = \int_0^\infty I_\nu P e^{-\tau} d\tau = I_\nu P + \sum_{i=1}^n \left. \frac{\partial^i I_\nu P}{\partial \tau^i} \right|_{\tau=0} + \int_0^\infty \frac{\partial^{n+1} I_\nu P}{\partial \tau^{n+1}} e^{-\tau} d\tau. \] (45)

The spatial derivative in the equation (7c) for the angles \(0 \leq \theta \leq \pi/2\) can be written as a derivative along the line, specified by angles \((\pi - \theta, \pi + \phi)\), taking into account (44):

\[ \bar{I}_\nu^{(n)}(t, 0, \theta, \phi) = \left. -l'\nu \left( \frac{\partial}{\partial c} \frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right) \bar{I}_\nu^{(n-1)} \right|_{\tau=0} = \left. \left( -l'\nu \frac{\partial}{\partial c} \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \bar{I}_\nu^{(n-1)} \right|_{\tau=0}. \] (46)

Correspondingly, the asymptotic expansion along the line, specified by angles \((\pi - \theta, \pi + \phi)\), can be written in the form

\[ \bar{I}_\nu(t, 0, \theta, \phi) = I_\nu P + \sum_{i=1}^n \left( -l'\nu \frac{\partial}{\partial c} \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \cdots \left( -l'\nu \frac{\partial}{\partial c} \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) I_\nu P \left|_{\tau=0} \right. \] (47)

(the smallness indicator \(\varepsilon\) is omitted). Therefore series (45) are similar to asymptotic expansion (7) and should converge at least for optically thick media. Below we will use asymptotic expansion (47), as more general, to derive improved boundary conditions.

IV. IMPROVED BOUNDARY CONDITIONS

A few words about the need to vary boundary conditions (in terms of heat conduction) at the boundary of media with different optical properties. The continuity of radiant energy density at the boundary of two media and the radiant energy flux component normal to the boundary follow from the condition of radiation intensity continuity:

\[ U_1 = U_2, \quad S_{n_1} = S_{n_2}; \] (48)

discontinuity of radiant energy density can lead to flux infinity since \(S \sim \nabla U\) and the discontinuity in the radiant energy flux contradicts the energy conservation law. Within the
framework of radiation heat conduction approximation it is difficult to reconcile the condition

\[ T_1 = T_2 , \]  

(49)

following from radiant energy density continuity, see equation (25), with the condition

\[ l'_\text{R} T_i^3 \nabla T_1 = l'_\text{R} T_i^3 \nabla T_2 , \]  

(50)

following from the equation (23). The above also relates to the diffusion approximation.

Proceeding to derivation of improved boundary conditions, let us assume, that the boundary between two substances can be locally considered plane – this is close approximation, if the mean free path of photons is much less than the radius of curvature of the boundary

\[ \frac{l'_\nu}{R} \ll 1 . \]  

(51)

We will also assume, that the substances totally fill their semi-spaces plane-symmetrically. From the equation (47), taking into account equations (42), (44), for the spectral intensity of radiation emitted from the substance surface occupying the left semi-space \( z \leq 0 \), we have:

\[
I^{-}_\nu (t, 0, \theta, \varphi) \simeq I^{-}_\nu \text{P} - \cos \theta \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial z} \right) \bigg|_{z=-0} - \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial t} \right) \bigg|_{z=-0} \\
+ \sum_{i=2}^{n} (-1)^i \left( \frac{1}{\cos \theta} \right)^i \left( l'_\nu \frac{\partial}{\partial z} \right)^{i-1} \left( \frac{\partial}{\partial t} \right) \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial z} \right) \bigg|_{z=-0} \\
+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} (-1)^i \left( \frac{1}{\cos \theta} \right)^j \left( l'_\nu \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial t} \right) \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial z} \right) \bigg|_{z=-0} \\
+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} (-1)^i \left( \frac{1}{\cos \theta} \right)^j \left( l'_\nu \frac{\partial}{\partial z} \right) \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial t} \right) \bigg|_{z=-0} \\
+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{1}{\cos \theta} \right)^j \left( l'_\nu \frac{\partial}{\partial t} \right) \left( l'_\nu \frac{\partial I^{-}_\nu \text{P}}{\partial z} \right) \bigg|_{z=-0} .
\]  

(52)

All summands in the right part of equation (52) relate to the boundary \( z = -0 \).

Integrating (52) over angles for the (right) semisphere, subject to (8) and (11), we obtain the expression for the spectral density of radiation in the left substance in the vicinity of
the surface

\[ U_\nu^- \simeq \frac{1}{c} \int_{(2\pi)} I_\nu^- (t, 0, \theta, \varphi) d\Omega = \frac{1}{2} U_{\nu P}^- - \frac{1}{4} \left( \nu' \frac{\partial U_{\nu P}^-}{\partial t} \right) \bigg|_{z=0} - \frac{1}{2} \left( \nu' \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \frac{(-1)^i}{i+1} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial z} \right) \left( \nu' \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i}{j+1} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial t} \right) \bigg|_{z=0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i}{j+1} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \frac{(-1)^i}{i+1} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]  \hspace{1cm} (53)

All the summands in the right part of equation (53) relate to the boundary \( z = -0 \).

Multiplying (52) by \( \cos \theta \) and integrating over angles for the (right) semisphere, taking into account (9), we find, that the projection to the axis \( z \) of the spectral radiant energy flux from the surface of the left substance, is

\[ S_\nu^- \simeq \int_{(2\pi)} I_\nu^- (t, 0, \theta, \varphi) \cos \theta d\Omega = \frac{c}{4} U_{\nu P}^- - \frac{c}{6} \left( \nu' \frac{\partial U_{\nu P}^-}{\partial t} \right) \bigg|_{z=0} - \frac{c}{4} \left( \nu' \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{c}{2} \sum_{i=2}^{n} \frac{(-1)^i}{i+2} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial z} \right) \left( \nu' \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i}{j+2} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial t} \right) \bigg|_{z=0} \]

\[ + \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i}{j+2} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]

\[ + \frac{c}{4} \sum_{i=2}^{n} \frac{(-1)^i}{i+1} \left( \nu' \frac{\partial}{\partial t} \right) \cdots \left( \nu' \frac{\partial}{\partial c} \frac{\partial U_{\nu P}^-}{\partial z} \right) \bigg|_{z=0} \]  \hspace{1cm} (54)

All summands in the right part of equation (53) relate to the boundary \( z = -0 \).

Comparing expressions (53), (54) and confining ourselves in the right parts of (53), (54)
only to the first summands, we obtain, in particular, the relation

\[ S_{\nu}^{-} \approx \frac{c}{2} U_{\nu}^{-} , \]  

(55)

which is often used as the boundary condition at the boundary between a substance and vacuum in the diffusion approximation, but condition (54) is more accurate, it takes into account non-uniformity of substance characteristics in the direction, perpendicular to the substance surface.

For the substance, occupying the right semi-space \( z \geq 0 \), upon integration over angles for the left semi-sphere the expressions similar to expressions (52), (53) and (54) are derived:

\[ I_{\nu}^{+} (t, 0, \theta, \varphi) \approx I_{\nu}^{+} - \cos \theta \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{\partial z} \right) \bigg|_{z=+0} - \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial t} \right) \bigg|_{z=+0} \]

\[ + \sum_{i=2}^{n} \frac{(-1)^i \cos \theta}{l'_{\nu}} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial z} \right) \prod_{i-1} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial t} \right) \bigg|_{z=+0} \]

\[ + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i \cos \theta}{l'_{\nu}} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial z} \right) \prod_{i-1} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial t} \right) \bigg|_{z=+0} \]

\[ + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^i \cos \theta}{l'_{\nu}} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial z} \right) \prod_{i-1} \left( \frac{l'_{\nu} + \partial I_{\nu}^{+}}{c \partial t} \right) \bigg|_{z=+0} , \]

(56)
\[ U_\nu^+ \simeq \frac{1}{c} \int_{(2\pi)} I_\nu^+ (t, 0, \theta, \varphi) \, d\Omega = \frac{1}{2} U_{\nu P}^+ + \frac{1}{4} \left( i^{\nu} \frac{\partial U_{\nu P}^+}{\partial z} \right) \bigg|_{z=+0} - \frac{1}{2} \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \frac{1}{i+1} \left( i^{\nu} \frac{\partial}{\partial z} \right) \cdots \left( i^{\nu} \frac{\partial}{\partial z} \right) \left( i^{\nu} \frac{\partial U_{\nu P}^+}{\partial z} \right) \bigg|_{z=+0} \]

\[ + \frac{1}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^{i+j}}{j+1} \left( i^{\nu} \frac{\partial}{\partial z} \right) \cdots \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \]

\[ - \frac{1}{2} \sum_{i=2}^{n} \left( i^{\nu} \frac{\partial}{c\partial t} \right) \cdots \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \] \hspace{1cm} (57)

\[ S_\nu^+ \simeq \int_{(2\pi)} I_\nu^+ (t, 0, \theta, \varphi) \cos \theta \, d\Omega = -\frac{c}{4} U_{\nu P}^+ - \frac{c}{6} \left( i^{\nu} \frac{\partial U_{\nu P}^+}{\partial z} \right) \bigg|_{z=+0} + \frac{c}{4} \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \]

\[ - \frac{c}{2} \sum_{i=2}^{n} \frac{1}{i+2} \left( i^{\nu} \frac{\partial}{\partial z} \right) \cdots \left( i^{\nu} \frac{\partial}{\partial z} \right) \left( i^{\nu} \frac{\partial U_{\nu P}^+}{\partial z} \right) \bigg|_{z=+0} \]

\[ - \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^{i+j}}{j+2} \left( i^{\nu} \frac{\partial}{\partial z} \right) \cdots \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \]

\[ - \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^{i+j}}{j+2} \left( i^{\nu} \frac{\partial}{c\partial t} \right) \cdots \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \]

\[ - \frac{c}{4} \sum_{i=2}^{n} (-1)^{i} \left( i^{\nu} \frac{\partial}{c\partial t} \right) \cdots \left( i^{\nu} \frac{\partial}{c\partial t} \right) \left( i^{\nu} \frac{\partial U_{\nu P}^+}{c\partial t} \right) \bigg|_{z=+0} \] \hspace{1cm} (58)

All the summands in the right parts of equations (56)-(58) relate to the boundary \( z = +0 \).

Expressions (53), (54) and (57), (58) can be used in calculations of radiation propagation in optically not very thick media, only if the series (52)–(54) and (56)–(58) converge. The formal convergence of these series can be achieved due to, as in section IIII, a limitation on radiation mean free path, if \( \tilde{U}_\nu^+ \) is used instead of \( U_\nu^+ \):

\[ \tilde{U}_\nu^+ = \min \{ U_\nu^+ , L_\nu^+ \} \] \hspace{1cm} (59)

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where
\[
L_{\nu}^I = K_{\nu}^I \frac{I_{\nu}^I}{\nabla I_{\nu}^I} + \frac{\partial I_{\nu}^I}{\partial t c}\quad = K_{\nu}^U \frac{U_{\nu}^I}{\nabla U_{\nu}^I} + \frac{\partial U_{\nu}^I}{\partial t c},
\]
and the coefficients \(K_{\nu}\), depending, generally speaking, on the conditions of the problem
\[
K_{\nu}^I \lesssim 1.
\]

So, at the outer boundary of the substance with vacuum in the asymptotic approximation of \(n\)-th order and in the spectral diffusion approximation with increased to \(n\)-order precision (see suggestion 2 in the conclusion below), it is proposed to use following expressions for normal components of the spectral radiant energy flux as boundary conditions, cf. with equations (54) and (58):
\[
\tilde{S}_{\nu}^- = \frac{c}{4} \frac{\partial U_{\nu}^-}{\partial z} \bigg|_{z=-0} - \frac{c}{4} \frac{\partial U_{\nu}^-}{\partial t c}\bigg|_{z=-0} + \frac{c}{2} \sum_{i=2}^{n} \left( \frac{-1}{i+2} \right) \prod_{j=1}^{i-1} \left( \frac{\partial U_{\nu}^-}{\partial z} \right) \bigg|_{z=-0}
\]
\[
+ \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{-1}{j+2} \right) \prod_{k=1}^{j-1} \left( \frac{\partial U_{\nu}^-}{\partial z} \right) \bigg|_{z=-0}
\]
\[
+ \frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left( \frac{-1}{j+2} \right) \prod_{k=1}^{j-1} \left( \frac{\partial U_{\nu}^-}{\partial t c} \right) \bigg|_{z=-0}
\]
\[
+ \frac{c}{4} \sum_{i=2}^{n} \left( \frac{-1}{i+2} \right) \prod_{j=1}^{i-1} \left( \frac{\partial U_{\nu}^-}{\partial z} \right) \bigg|_{z=-0},
\]
(62)
\[
\tilde{S}_n^+ = -\frac{c}{4} U_{vP}^+ - \frac{c}{6} \left( \tilde{i}_n^- \frac{\partial U_{vP}^+}{\partial z} \right) \bigg|_{z=+0} + \frac{c}{4} \left( \tilde{i}_n^- \frac{\partial U_{vP}^+}{\partial c^\prime t} \right) \bigg|_{z=+0}
\]

\[
-\frac{c}{2} \sum_{i=2}^{n} \frac{1}{i + 2} \left( \tilde{i}_n^- + \frac{\partial}{\partial z} \right) \cdots \left( \tilde{i}_n^- + \frac{\partial}{\partial z} \right) \left( \tilde{i}_n^+ \frac{\partial U_{vP}^+}{\partial z} \right) \bigg|_{z=+0}
\]

\[
-\frac{c}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{(-1)^{i+j}}{j + 2} \left( \tilde{i}_n^- + \frac{\partial}{\partial t} \right) \cdots \left( \tilde{i}_n^- + \frac{\partial}{\partial t} \right) \left( \tilde{i}_n^+ \frac{\partial U_{vP}^+}{\partial z} \right) \bigg|_{z=+0}
\]

\[
-\frac{c}{2} \sum_{i=2}^{n} \frac{(-1)^i}{i} \left( \tilde{i}_n^- + \frac{\partial}{\partial t} \right) \cdots \left( \tilde{i}_n^- + \frac{\partial}{\partial t} \right) \left( \tilde{i}_n^+ \frac{\partial U_{vP}^+}{\partial z} \right) \bigg|_{z=+0}, \quad (63)
\]

the summands in the right parts of equations (62), (63) are calculated at the substance boundary at points \( z = -0 \) and \( z = +0 \), respectively.

The projection of the total spectral radiant energy flux onto the normal to the boundary of two substances is equal to the algebraic sum of \( \tilde{S}_n^- \) and \( \tilde{S}_n^+ \) flux projections:

\[
\tilde{S}_n^b = \tilde{S}_n^- + \tilde{S}_n^+ = \frac{c}{4} \left( U_{vP}^- - U_{vP}^+ \right) - \frac{c}{6} \left( \tilde{i}_n^- \frac{\partial U_{vP}^-}{\partial z} \right) \bigg|_{z=-0} + \frac{c}{4} \left( \tilde{i}_n^+ \frac{\partial U_{vP}^+}{\partial z} \right) \bigg|_{z=+0} + \cdots \quad (64)
\]

It is proposed to use expression (64) at the inner boundaries in the asymptotic approximation of \( n \)-th order and in the spectral diffusion approximation with increased to \( n \)-order precision. If to equate \( U_{vP}^- = U_{vP}^+ \) and partial time derivatives in (64), then the opportunity to correctly calculate radiation propagation in problems with initial data discontinuity will be lost.

At the outer boundaries with vacuum for normal components of the integral (in terms of radiation frequency) radiant energy flux in the approximation of radiation heat conduction we propose to use expressions

\[
\tilde{S}^- = \frac{c}{4} U_P^- \bigg|_{z=-0} - \frac{c}{6} \left( \tilde{i}_R^- \frac{\partial U_P^-}{\partial z} \right) \bigg|_{z=-0} - \frac{c}{4} \left( \tilde{i}_R^- \frac{\partial U_P^-}{\partial c^\prime t} \right) \bigg|_{z=-0}, \quad (65)
\]

\[
\tilde{S}^+ = -\frac{c}{4} U_P^+ \bigg|_{z=+0} + \frac{c}{6} \left( \tilde{i}_R^+ \frac{\partial U_P^+}{\partial z} \right) \bigg|_{z=+0} + \frac{c}{4} \left( \tilde{i}_R^+ \frac{\partial U_P^+}{\partial c^\prime t} \right) \bigg|_{z=+0}, \quad (66)
\]

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corresponding to (62) and (63), with limitations similar to (59)

\[ \tilde{l}^2 = \min \left\{ I^R_l, L^R_l \right\}, \]  

(67)

where

\[ L^R_l = K^R_l \frac{\nabla I^R_l}{\left| \nabla I^R_l \right|} + \frac{\partial I^R_l}{\partial c \partial t} = K^R_l \frac{U^R_p}{\left| \nabla U^R_p \right|} + \frac{\partial U^R_p}{\partial c \partial t}, \]  

(68)

coefficients \( K^R_l \)

\[ K^R_l \lesssim 1. \]  

(69)

We propose to use an expression, corresponding to (64), for the normal component of the integral (in terms of radiation frequency) radiant energy flux

\[ \tilde{S} = \tilde{S}^- + \tilde{S}^+ = \frac{c}{4} (U_p^- - U^+_p) - \frac{c}{6} \left[ \left( \tilde{l}_R^- \frac{\partial U_p^-}{\partial z} \right) \bigg|_{z=-0} + \left( \tilde{l}_R^+ \frac{\partial U_p^+}{\partial z} \right) \bigg|_{z=+0} \right] \]

\[- \frac{c}{4} \left( \left( \tilde{l}_R^- \frac{\partial U_p^-}{\partial z} \right) \bigg|_{z=-0} - \frac{c}{4} \left( \tilde{l}_R^+ \frac{\partial U_p^+}{\partial z} \right) \bigg|_{z=+0} \right), \]  

(70)

with limitations (67) at inner boundaries in the approximation of radiation heat conduction. As well as for the spectral diffusion approximation, if to equate \( U_p^- = U^+_p \) and partial time derivatives in (70), then the opportunity to correctly calculate radiation propagation in problems with initial data discontinuity will be lost.

V. CONCLUSION

The asymptotic approximation of \( n \)-th order, received from the asymptotic solution of the kinetic equation of radiation propagation, may be used as follows:

1. to replace diffusion approximation or radiation heat conduction approximation, for example, with asymptotic approximation of the second order;

2. to increase diffusion approximation precision to the \( n \)-order, having used for spectral radiant energy flux the expression (14);

3. to increase radiation heat conduction approximation precision to the first order, having added the radiant energy density of the first order in the energy transfer equation

\[ U^{(1)} = - \frac{\tilde{l}_R^- \partial U_p^-}{\partial c \partial t}. \]
From the constructed asymptotic solution of the kinetic equation of radiation propagation and series expansion of the formal solution of the kinetic equation of radiation propagation, it follows the complete proof of coefficient 1/3 in the second equation of diffusion approximation.

The improved boundary conditions, obtained from the asymptotic solution of the kinetic equation of radiation propagation for optically thick medium, can be used instead of existing boundary conditions (at outer and inner boundaries) in calculations of radiation propagation in the asymptotic approximation of the $n$-order, in the multigroup diffusion approximation and in the still widely-used approximation of radiation heat conduction.

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