A HOMOLOGY VALUED INVARIANT FOR TRIVALENT FATGRAPH SPINES

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Abstract. We introduce an invariant for trivalent fatgraph spines of a once bordered surface, which takes values in the first homology of the surface. This invariant is the secondary object coming from two 1-cocycles on the dual fatgraph complex, one introduced by Morita and Penner in 2008, and the other by Penner, Turaev, and the author in 2013. We present an explicit formula for this invariant and investigate its properties. We also show that the mod 2 reduction of the invariant is the difference of naturally defined two spin structures on the surface.

1. Introduction

Let $\Sigma_{g,1}$ be a once bordered $C^\infty$-surface of genus $g > 0$, and let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$ relative to the boundary. It is known that the Teichmüller space $\mathcal{T}(\Sigma_{g,1})$ of $\Sigma_{g,1}$ has an $\mathcal{M}_{g,1}$-equivariant ideal simplicial decomposition [24]. Taking its dual, one obtains a contractible CW complex $\hat{\mathcal{G}}(\Sigma_{g,1})$ on which $\mathcal{M}_{g,1}$ acts freely and properly discontinuously. This CW complex is called the dual fatgraph complex of $\Sigma_{g,1}$, since its cells are indexed by fatgraph spines of $\Sigma_{g,1}$, which are graphs embedded in the surface satisfying some conditions. Each 0-cell of $\hat{\mathcal{G}}(\Sigma_{g,1})$ corresponds to a trivalent fatgraph spine, and by contracting non-loop edges we obtain higher dimensional cells. In particular, each oriented 1-cell of $\hat{\mathcal{G}}(\Sigma_{g,1})$ corresponds to a flip (or a Whitehead move) between trivalent fatgraph spines of $\Sigma_{g,1}$.

This combinatorial structure of the Teichmüller space has a number of applications to the cohomology of the mapping class group and the moduli space of Riemann surfaces. See e.g., [7] [8] [9] [23] [15].

Recently, mainly motivated by the theory of the Johnson homomorphisms [11] [12] [20], several authors considered 1-cocycles on $\hat{\mathcal{G}}(\Sigma_{g,1})$ with coefficients in various $\mathcal{M}_{g,1}$-modules. In 2008, Morita and Penner [21] first gave such a 1-cocycle $j \in Z^1(\hat{\mathcal{G}}(\Sigma_{g,1}); \Lambda^3 H)$, where $\Lambda^3 H$ is the third exterior product of the first homology group $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. (In fact, they worked with a once punctured surface, but their construction works for $\Sigma_{g,1}$ as well.) Being a 1-cocycle on $\hat{\mathcal{G}}(\Sigma_{g,1})$, the cocycle $j$ associates an element of $\Lambda^3 H$ to each flip. Fixing a trivalent fatgraph spine of $\Sigma_{g,1}$, one obtains from $j$ a twisted 1-cocycle on $\mathcal{M}_{g,1}$. Morita and Penner proved that its cohomology class in $H^1(\mathcal{M}_{g,1}; \Lambda^3 H)$ is six times the extended first Johnson homomorphism $\tilde{k}$.
Similar constructions are also considered by Bene, Kawazumi and Penner for the second and higher Johnson homomorphisms, by Massuyeau for Morita’s refinement of the higher Johnson homomorphisms, and by Kuno, Penner and Turaev for the Earle class $k \in H^1(\mathcal{M}_{g,1}; H)$.

We emphasize that these cocycles on $\hat{\mathbb{G}}(\Sigma_{g,1})$ are all explicit and simple. In this way, the Johnson homomorphisms and related objects extend canonically to the Ptolemy groupoid, the combinatorial fundamental path groupoid of $\hat{\mathbb{G}}(\Sigma_{g,1})$.

It is interesting that there are many ways of constructing cocycle representatives for the cohomology classes such as $\tilde{k}$ and $k$, and that each construction reflects its own viewpoint for studying the mapping class group. It can happen that two cocycles constructed differently give the same cohomology class. In such a case, it is quite natural to compare these cocycles and to expect a secondary object behind there.

In this paper, we compare the Morita-Penner cocycle $j$ and the cocycle $m \in Z^1(\hat{\mathbb{G}}(\Sigma_{g,1}); H)$ which is related to $k$ and considered in [10]. Contracting the coefficients by using the intersection pairing on $H$, one has a natural homomorphism

$$C: Z^1(\hat{\mathbb{G}}(\Sigma_{g,1}); \Lambda^3 H) \to Z^1(\hat{\mathbb{G}}(\Sigma_{g,1}); H).$$

Let $j' = C \circ j$. It turns out that there is an $\mathcal{M}_{g,1}$-equivariant 0-cochain $\xi \in C^0(\hat{\mathbb{G}}(\Sigma_{g,1}); H)$ such that $2j' - m = \delta \xi$ (Proposition 3.1). The 0-cochain $\xi$ associates an element $\xi_G \in H$ to each trivalent fatgraph spine $G \subset \Sigma_{g,1}$.

We will study the secondary object $\xi_G$ as an $H$-valued invariant for trivalent fatgraph spines $G \subset \Sigma_{g,1}$. First of all, Theorem 3.4 gives an explicit formula for $\xi_G$. Based on this formula, we show in Theorem 5.2 that $\xi_G$ is non-trivial. At the present moment, we do not have a full understanding of the topological meaning of the invariant $\xi_G$. In Theorem 6.7 we give a partial result in this direction by relating the mod 2 reduction of $\xi_G$ to naturally defined two spin structures on $\Sigma_{g,1}$.

This paper is organized as follows. In §2, we first review the dual fatgraph complex and in particular describe its 2-skeleton. Then we recall the 1-cocycles $j$ from [21] and $m$ from [10]. Also, we correct an error in [10] about the evaluation of $m$. In §3, we show the existence and uniqueness of $\xi$, and then present an explicit formula for $\xi_G$ (Theorem 3.4). In §4, we show a certain gluing formula for $\xi_G$, and then the behavior of $\xi_G$ under a special kind of flip. The latter result makes it possible to define $\xi_G$ for a trivalent fatgraph spine $G$ of a punctured surface. In §5, we first prove the non-triviality of $\xi_G$. Then we discuss the primitivity of $\xi_G$ and show some partial results. In §6, we first show that given a trivalent fatgraph spine $G \subset \Sigma_{g,1}$, one can associate two spin structures on $\Sigma_{g,1}$. Then we prove that their difference coincides with the mod 2 reduction of $\xi_G$. In Appendix A, we consider another spin structure coming from a naturally defined non-singular vector field on $\Sigma_{g,1}$.

The author would like to thank Gwénaël Massuyeau for communicating to him the construction of the vector field $\mathcal{X}_G$ in Appendix A, Robert Penner for helpful remarks on a description of spin structures on $\Sigma_{g,1}$ in §6, and
We fix some notation about graphs. By a graph we mean a finite CW complex of dimension one. For a graph $G$, we denote by $V(G)$ the set of vertices of $G$, by $E(G)$ the set of edges of $G$, and by $E^\text{ori}(G)$ the set of oriented edges of $G$. For $v \in V(G)$, we denote by $E^\text{ori}(G)_v$ the set of oriented edges toward $v$. The number of elements of $E^\text{ori}(G)_v$ is called the valency of $v$. For $e \in E^\text{ori}(G)$, we denote by $\tilde{e} \in E^\text{ori}(G)$ the edge $e$ with reversed orientation. A fatgraph is a graph $G$ endowed with a cyclic ordering to $E^\text{ori}(G)_v$ about each $v \in V(G)$.

Let $\Sigma_{g,1}$ be a compact connected oriented $C^\infty$-surface of genus $g \geq 0$ with one boundary component. We fix two distinct points $p$ and $q$ on the boundary $\partial \Sigma_{g,1}$.

**Definition 2.1.** An embedding $\iota : G \hookrightarrow \Sigma_{g,1}$ of a fatgraph $G$ into $\Sigma_{g,1}$ is called a fatgraph spine of $\Sigma_{g,1}$ if the following conditions are satisfied.

1. The map $\iota$ is a homotopy equivalence.
2. For any $v \in V(G)$, the cyclic ordering given to $E^\text{ori}(G)_v$ is compatible with the orientation of $\Sigma_{g,1}$.
3. We have $\iota(G) \cap \partial \Sigma_{g,1} = \{p\}$ and $\iota^{-1}(p)$ is a unique univalent vertex of $G$. The other vertices have valencies greater than 2.

A unique edge connected to $\iota^{-1}(p)$ is called the tail of $G$. We consider fatgraph spines up to isotopies relative to $\partial \Sigma_{g,1}$. If there is no danger of confusion, we identify $G$ with $\iota(G)$, and write $G$ instead of $\iota : G \hookrightarrow \Sigma_{g,1}$. We denote by $V^\text{int}(G)$ the set of non-univalent vertices of $G$. We say that $G$ is trivalent if the valency of any non-univalent vertex of $G$ is 3.

Fatgraph spines appear naturally in the combinatorial description of the Teichmüller space of a punctured or bordered surface. This was first shown for punctured surfaces by Harer-Mumford [8] and Thurston from the holomorphic point of view based on a work by Strebel [28], and by Penner [22] and Bowditch-Epstein [5] from the point of view of hyperbolic geometry.

In this paper, we work mainly with the once bordered surface $\Sigma_{g,1}$. For definiteness, let us define the Teichmüller space $T(\Sigma_{g,1})$ as the space of Riemannian metric on $\Sigma_{g,1}$ of constant Gaussian curvature $-1$ with geodesic boundary, modulo pull-back of the metric by self-diffeomorphisms of $\Sigma_{g,1}$ fixing $q$ which are isotopic to the identity relative to $q$. Let $M_{g,1}$ be the mapping class group of $\Sigma_{g,1}$ relative to $\partial \Sigma_{g,1}$. Namely, $M_{g,1}$ is the group of self-diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary $\partial \Sigma_{g,1}$ pointwise, modulo isotopies fixing $\partial \Sigma_{g,1}$ pointwise. Note that $M_{g,1}$ is identified with the group of connected components of the group of self-diffeomorphisms of $\Sigma_{g,1}$ fixing $q$. Then pull-back of the metric induces an action of $M_{g,1}$ on $T(\Sigma_{g,1})$. This action is known to be free and properly discontinuous.

**Theorem 2.2** (Penner [22]). There is an $M_{g,1}$-equivariant ideal simplicial decomposition of $T(\Sigma_{g,1})$ with the following properties.

- Each simplex corresponds to a fatgraph spine of $\Sigma_{g,1}$.
Let \( \hat{G}(\Sigma_{g,1}) \) be the dual of this ideal simplicial decomposition. This is an honest CW complex of dimension \( 4g - 2 \). We call \( \hat{G}(\Sigma_{g,1}) \) the dual fatgraph complex of \( \Sigma_{g,1} \). Note that there is a natural cellular action of the mapping class group \( \mathcal{M}_{g,1} \) on \( \hat{G}(\Sigma_{g,1}) \). In fact, there is an \( \mathcal{M}_{g,1} \)-equivariant deformation retract of \( T(\Sigma_{g,1}) \) onto \( \hat{G}(\Sigma_{g,1}) \). See \[25\].

The 2-skeleton of \( \hat{G}(\Sigma_{g,1}) \) is described as follows.

- Each 0-cell corresponds to a trivalent fatgraph spine of \( \Sigma_{g,1} \).
- Each 1-cell corresponds to a fatgraph spine \( G \), where \( G \) has a unique 4-valent vertex and the other non-univalent vertices have valency 3.
- Each oriented 1-cell corresponds to a flip (or a Whitehead move) between trivalent fatgraph spines. Here, if \( e \) is a non-tail edge of a trivalent fatgraph spine, collapsing \( e \) and expanding the resulting 4-valent vertex to the unique distinct direction, one produces another trivalent fatgraph spine. We call this move a flip along \( e \), and denote it by \( W_e \). See Figure 1. If \( G' \) is obtained from \( G \) by a flip \( W = W_e \), we write it as \( G \xrightarrow{W} G' \). There is a natural bijection from \( E(G) \) to \( E(G') \) which restricts to an obvious identification of \( E(G) \setminus \{e\} \) with \( E(G') \setminus \{e'\} \). For this reason, we often use the same letter for edges of \( G \) and \( G' \) corresponding to each other by this bijection.
- Each 2-cell corresponds to a fatgraph spine \( G \), where either \( G \) has a unique 5-valent vertex and the other non-univalent vertices have valency 3, or \( G \) has two 4-valent vertices and the other non-univalent vertices have valency 3.

Let \( G \) and \( G' \) be trivalent fatgraph spines. Since \( \hat{G}(\Sigma_{g,1}) \) is connected, there is a finite sequence of flips

\[
G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} G_2 \cdots \xrightarrow{W_m} G_m = G'
\]

from \( G \) to \( G' \). This sequence is not uniquely determined, but any two such sequences are related to each other by the following three types of relations among flips.

1. Involutivity relation: \( W_e' \circ W_e = 1 \) in the notation of Figure 1.
2. Commutativity relation: \( W_{e_1} \circ W_{e_2} = W_{e_2} \circ W_{e_1} \) if \( e_1 \) and \( e_2 \) share no vertices.
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Figure 2. Pentagon relation

(3) Pentagon relation: $W_{f_4} \circ W_{g_3} \circ W_{f_2} \circ W_{g_1} \circ W_f = 1$ in the notation of Figure 2.

Here, we read composition of flips from right to left. The relations (2) and (3) come from the boundaries of 2-cells of $\hat{G}(\Sigma_{g,1})$.

There is a construction of twisted 1-cocycles on the mapping class group using the fatgraph complex appeared first in [21]. Let $M$ be a (left) $M_{g,1}$-module. By definition, a cellular 1-cochain $c$ on $\hat{G}(\Sigma_{g,1})$ with values in $M$ is an assignment of an element of $M$ to each flip $W$ satisfying $c(W_e) = -c(W_{e'})$ for any pair of flips $W_e$ and $W_{e'}$ as in Figure 1. Such a $c$ is a 1-cocycle if it satisfies the commutative equation

$$c(W_{e_1}) + c(W_{e_2}) = c(W_{e_2}) + c(W_{e_1}),$$

for any 5-tuple of flips as in Figure 2.

Now we assume that $c$ is a 1-cocycle and is $M_{g,1}$-equivariant in the sense that $\varphi \cdot c(W) = c(\varphi W)$ for any flip $W$ and $\varphi \in M_{g,1}$. Fix a trivalent fatgraph spine $G$. For $\varphi \in M_{g,1}$, taking a sequence of flips

$$G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} G_2 \rightarrow \cdots \xrightarrow{W_m} G_m = \varphi(G)$$

from $G$ to $\varphi(G)$, we set

$$c_G(\varphi) := \sum_{i=1}^{m} c(W_i) \in M.$$
Since \( c \) is a 1-cocycle, this value does not depend on the choice of the sequence. The map \( c_G: \mathcal{M}_{g,1} \to M \) is a twisted 1-cocycle. In fact, for \( \varphi, \psi \in \mathcal{M}_{g,1} \), take a sequence of flips from \( G \) to \( \varphi(G) \), and one from \( G \) to \( \psi(G) \). Then the first sequence followed by application of \( \varphi \) to the second is a sequence of flips from \( G \) to \( \varphi \psi(G) \). Since \( c \) is \( \mathcal{M}_{g,1} \)-equivariant, we obtain the cocycle condition

\[
c_G(\varphi \psi) = c_G(\varphi) + \varphi \cdot c_G(\psi).
\]

It is easy to see that the cohomology class \([c_G] \in H^1(\mathcal{M}_{g,1}; M)\) does not depend on the choice of \( G \).

Here we record an elementary fact which will be used later.

**Lemma 2.3.** Let \( M \) be an \( \mathcal{M}_{g,1} \)-module and suppose that \( c \) is an \( \mathcal{M}_{g,1} \)-equivariant cellular 1-cochain on \( \mathcal{G}(\Sigma_{g,1}) \) with values in \( M \). Then for any trivalent fatgraph spine \( G \) and any \( \varphi, \psi \in \mathcal{M}_{g,1} \), we have

\[
c_G(\psi) + c_{\psi(G)}(\varphi) = c_G(\varphi) + \varphi \cdot c_G(\psi).
\]

**Proof.** Consider a sequence of flips from \( G \) to \( \psi(G) \) and one from \( \psi(G) \) to \( \varphi \psi(G) \). The composition of these sequences is a sequence from \( G \) to \( \varphi \psi(G) \), and thus we obtain \( c_G(\varphi \psi) = c_G(\psi) + c_{\psi(G)}(\varphi) \). On the other hand, by the cocycle condition for \( c_G \), we have \( c_G(\varphi \psi) = c_G(\varphi) + \varphi \cdot c_G(\psi) \). \( \square \)

We denote by \( H = H_1(\Sigma_{g,1}; \mathbb{Z}) \) the first integral homology group of \( \Sigma_{g,1} \). Before giving examples of \( \mathcal{M}_{g,1} \)-equivariant cellular 1-cochains on \( \mathcal{G}(\Sigma_{g,1}) \), we recall from \([21]\) homology markings for edges of fatgraph spines. Let \( G \) be a (not necessarily trivalent) fatgraph spine of \( \Sigma_{g,1} \). For \( e \in E^{ori}(G) \), there is an oriented simple loop \( \bar{e} \) on \( \Sigma_{g,1} \) satisfying the following two conditions.

- The loop \( \bar{e} \) intersects \( G \) once transversely at the middle point of \( e \),
- The ordered pair of the velocity vectors of \( \bar{e} \) and \( e \) at their intersection is compatible with the orientation of \( \Sigma_{g,1} \).

Since the surface obtained from \( \Sigma_{g,1} \) by cutting along \( G \) is a disk, the homotopy class of such an \( \bar{e} \) is unique. We define \( \mu(e) \in H \) to be the homology class of \( \bar{e} \) and call it the *homology marking* of \( e \). The map \( \mu: E^{ori}(G) \to H \) has the following properties.

1. For any \( e \in E^{ori}(G) \), we have \( \mu(\bar{e}) = -\mu(e) \).
2. The set \( \{\mu(e)\}_{e \in E^{ori}(G)} \) generates \( H \).
3. For any \( v \in V(G) \), we have
   \[
   \sum_{e \in E^{ori}(G)} \mu(e) = 0.
   \]

For example, in the notation of the left part of Figure [4], where we orient edges \( a, b, c, d \) as indicated, we have \( \mu(a) + \mu(b) + \mu(c) + \mu(d) = 0 \).

In what follows, we consider \( \mathcal{M}_{g,1} \)-modules such as \( H \) and its third exterior product \( \Lambda^3 H \). There is a twisted cohomology class \( \hat{k} \in H^1(\mathcal{M}_{g,1}; \frac{1}{2} \Lambda^3 H) \) called the *extended first Johnson homomorphism* \([19]\). This cohomology class has a fundamental importance in the study of the cohomology of the mapping class group. See \([14]\).
Theorem 2.4 (Morita-Penner [21]). Keep the notation in Figure 1. For the flip $W_e$, set

$$j(W_e) = \mu(a) \wedge \mu(b) \wedge \mu(c) \in \Lambda^3 H.$$ 

Then $j$ is an $\mathcal{M}_{g,1}$-equivariant 1-cocycle on $\hat{\mathcal{G}}(\Sigma_{g,1})$, and we have

$$[j_G] = 6\tilde{k}.$$

Using the intersection pairing $(\cdot \cdot)$ on the homology, we define an $\text{Sp}(H)$-equivariant map

$$C : \Lambda^3 H \to H, \quad x \wedge y \wedge z \mapsto (x \cdot y)z + (y \cdot z)x + (z \cdot x)y$$

called the contraction. Morita [18] showed that if $g \geq 2$, the twisted cohomology group $H^1(\mathcal{M}_{g,1}; H)$ is infinite cyclic. As is remarked in [19], the element $k := C(2\tilde{k})$ is a generator of this cohomology group. Since Earle [6] first gave a cocycle representative for $k$, we call $k$ the Earle class. See [13].

Theorem 2.5 (Kuno-Penner-Turaev [16]). Keep the notation in Figure 1. For the flip $W_e$, set

$$m(W_e) = \mu(a) + \mu(c) \in H.$$ 

Then $m$ is an $\mathcal{M}_{g,1}$-equivariant 1-cocycle on $\hat{\mathcal{G}}(\Sigma_{g,1})$, and we have

$$[m_G] = 6k.$$ 

Here we correct an error in [16]. Let $\varphi_{BP} = \varphi$ be the torus BP map in [16] Fig.3, which was first considered in [21]. In [16] Lemma 1, it was asserted that $m(\varphi_{BP}) = 4a$, but this is not true. More precisely, in the proof of the lemma, we computed the contribution of the second Dehn twist (5 flips) as $-4a$, but this should be corrected as $4a$.

Lemma 2.6 (correction of [16] Lemma 1). Let $\varphi_{BP}$ be the torus BP map as above. Then we have

$$m(\varphi_{BP}) = 12\mu(a).$$

In [16] Theorem 6, it is asserted that $[m_G] = -2k$, but this should be corrected as in Theorem 2.5 above.

3. A secondary invariant

We consider the cocycle $j' = C \circ j$. For the flip $W_e$ in the notation of Figure 1 we have

$$j'(W_e) = (a \cdot b)\mu(c) + (b \cdot c)\mu(a) + (c \cdot a)\mu(b) \in H.$$ 

Here and throughout the paper, to simplify the notation we write e.g., $(a \cdot b)$ instead of $(\mu(a) \cdot \mu(b))$. By Theorems 2.4 and 2.5 for any trivalent fatgraph spine $G$, we have $2[j'_G] = [m_G] = 6k$. Therefore, there exists an element $\xi_G \in H$ such that $2j'_G - m_G = \delta \xi_G$. Here the symbol $\delta$ in the right hand side means the coboundary map in the standard cochain complex of $\mathcal{M}_{g,1}$ with coefficients in $H$. Explicitly, we have $(\delta \xi_G)(\varphi) = \varphi \cdot \xi_G - \xi_G$ for any $\varphi \in \mathcal{M}_{g,1}$. Such a $\xi_G$ is unique since only 0 is $\mathcal{M}_{g,1}$-invariant in $H$. We regard the collection $\xi = \{\xi_G\}_G$ as a cellular 0-cochain of $\hat{\mathcal{G}}(\Sigma_{g,1})$ with coefficients in $H$. 
Proposition 3.1. \(\begin{align*}
(1) \text{The 0-cochain } \xi \text{ is } M_{g,1}\text{-equivariant in the sense that } 
\xi_{\psi(G)} &= \psi \cdot \xi_G \text{ for any } \psi \in M_{g,1} \text{ and any trivalent fatgraph spine } G.
\end{align*}\)

(2) We have \(2j' - m = \delta \xi\). Namely, for any flip \(G \xrightarrow{W} G'\), we have \(\xi_G' - \xi_G = 2j'(W) - m(W)\).

Moreover, these two properties characterize \(\xi\).

Proof. (1) For simplicity we write \(s = 2j' - m\). Take \(\varphi \in M_{g,1}\). Using \(s_G(\varphi) = \delta \xi_G(\varphi) = \varphi \cdot \xi_G - \xi_G\), etc., we compute from Lemma 2.3 that
\[
\begin{align*}
\psi_{\psi(G)}(\varphi) &= s_G(\varphi) + \varphi \cdot s_G(\psi) - s_G(\psi) \\
&= \varphi \cdot \xi_G - \xi_G + \varphi \cdot (\psi \cdot \xi_G - \xi_G) - (\psi \cdot \xi_G - \xi_G) \\
&= \varphi \cdot (\psi \cdot \xi_G) - \psi \cdot \xi_G \\
&= \delta(\psi \cdot \xi_G)(\varphi).
\end{align*}
\]
This proves \(s_{\psi(G)} = \delta(\psi \cdot \xi_G)\). By the uniqueness of \(\xi_{\psi(G)}\), it follows that \(\xi_{\psi(G)} = \psi \cdot \xi_G\).

(2) can be proved analogously, and so we omit the detail.

Finally, suppose that \(\xi^0\) is an \(M_{g,1}\)-equivariant 0-cochain satisfying \(2j' - m = \delta \xi^0\). Then \(\xi - \xi^0\) is an \(M_{g,1}\)-equivariant 0-cocycle. This shows that \(\eta := \xi(G) - \xi^0(G) \in H\) is independent of \(G\) and \(\varphi \cdot \eta = \eta\) for any \(\varphi \in M_{g,1}\).

Therefore \(\eta\) must be zero and \(\xi^0 = \xi\). \(\square\)

Let \(G\) be a trivalent fatgraph spine of \(\Sigma_{g,1}\). We present an explicit formula for \(\xi_G\). To begin with, we introduce a total ordering for \(E^{ori}(G)\). Note that if we cut \(\Sigma_{g,1}\) along \(G\), we obtain an oriented closed disk \(D_G\).

Definition 3.2. \(\begin{align*}
(1) \text{For } e, e' \in E^{ori}(G), \text{ we say } e \prec e' \text{ if the edge } e \text{ occurs first when we go along the boundary of } D_G \text{ from } p \text{ by clockwise manner.}
\end{align*}\)

(2) Let \(e \in E^{ori}(G)\). We say that \(e\) has the preferred orientation (or \(e\) is preferably oriented) if \(e \prec \bar{e}\).

Note that any unoriented edge of \(G\) has the unique preferred orientation.

Let \(v \in V^{int}(G)\). We name three elements of \(E^{ori}_v(G)\) as \(e_1, e_2,\) and \(e_3\), so that

(1) \(e_1 \prec e_2\) and \(e_1 \prec e_3\), and

(2) the edge \(e_2\) is next to \(e_1\) in the cyclic ordering given to \(E^{ori}_v(G)\).

There are two possibilities for the ordering of \(e_i\) and its inverse \(\bar{e}_i\), namely,
\[
\begin{align*}
e_1 &< e_2 < e_3 \prec e_3 \prec e_1, \\
or \\
e_1 &< \bar{e}_2 < e_3 \prec e_1 < e_2 < e_3.
\end{align*}
\]

The vertex \(v\) is called of type 1 if the former case happens, and is called of type 2 otherwise. Figure 3 is an illustration of the situation.

We can count the number of vertices of type 1 and that of type 2.

Proposition 3.3. For any trivalent fatgraph spine \(G\) of \(\Sigma_{g,1}\), the number of trivalent vertices of type 1 is \(2g - 1\), and that of type 2 is \(2g\).
Proof. Let $V_1$ and $V_2$ be the numbers of trivalent vertices of type 1 and that of type 2, respectively. Since the number of trivalent vertices of $G$ is $4g - 1$, we have $V_1 + V_2 = 4g - 1$. We observe that if a trivalent vertex $v$ is of type $i$ ($i = 1, 2$), the number of preferably oriented edges toward $v$ is $i$. Thus $V_1 + 2V_2$ is equal to the number of edges of $G$, i.e., $6g - 1$. Hence we obtain $V_1 = 2g - 1$ and $V_2 = 2g$. □

We set

$$
\begin{cases}
    e_v = e_2 	ext{ and } f_v = e_3 & \text{if } v \text{ is of type 1}, \\
    e_v = e_1 	ext{ and } f_v = e_3 & \text{if } v \text{ is of type 2}.
\end{cases}
$$

Theorem 3.4. We have

$$
\xi_G = \sum_v (\mu(e_v) - \mu(f_v)),
$$

where the sum is taken over all trivalent vertices of $G$.

Proof. We set $\xi_G^0 = \sum_v (\mu(e_v) - \mu(f_v))$ and consider the collection $\xi_G^0 = \{\xi_G^0\}_G$. Clearly, $\xi_G^0$ is $M_{g,1}$-equivariant. By Proposition 5.1, it is sufficient to prove $2j' - m = \delta \xi_G^0$.

Take the notation as in Figure 3. For example, assume that $a < c < b < d$. For simplicity, we write $e$ instead of $\mu(e)$ for $e \in E_{ori}(G)$. Then we can see from the left part of Figure 4 that $(a \cdot b) = (c \cdot a) = 0$ and $(b \cdot c) = 1$, and so $j'(W_e) = a$. Thus $2j'(W_e) - m(W_e) = 2a - (a + c) = a - c$. On the other hand, we can compute from the right part of Figure 4 that

$$
\xi_G^0 - \xi_G^0 = (e_{v_1}' - f_{v_1}') + (e_{v_2}' - f_{v_2}') - (e_{v_1} - f_{v_1}) - (e_{v_2} - f_{v_2})
= (a + d - c) + (b + c - d) - (b - (c + d)) - (c - (a + b))
= 2a + b + d = 2a + b + (-a - b - c) = a - c.
$$

We can compute similarly for other cases as well, and we obtain $2j'(W_e) - m(W_e) = \xi_G^0 - \xi_G^0$. (There are essentially 6 cases to consider; in each case
Figure 4. The case where $a \prec c \prec b \prec d$

Figure 5. Situations near $e$

I: $a \prec b \prec c \prec d$

II: $a \prec b \prec d \prec c$

III: $a \prec c \prec b \prec d$

IV: $a \prec c \prec d \prec b$

V: $a \prec d \prec b \prec c$

VI: $a \prec d \prec c \prec b$

In Figure 5, there are two possibilities depending on whether $G$ corresponds to the left pictures or to the right pictures. The latter case reduces to the former case by changing the role of $G$ and $G'$.) Hence $2j' - m = \delta \xi^0$, as required. □
\begin{example}
Let $G$ be the fatgraph as shown in Figure 6. We name edges as in the figure and give them the preferred orientation. For $1 \leq i \leq g$ and $1 \leq j \leq 3$, let $e^j_i \in V^{\text{int}}(G)$ be the start point of $e^j_i$. For $1 \leq i \leq g - 1$, let $e^j_i \in V^{\text{int}}(G)$ be the end point of $e^j_i$.

Since $v^1_1$ is of type 2, its contribution is $\mu(e^4_1) - \mu(e^3_1) + \mu(e^2_1) - \mu(e^1_1)$. Since $v^2_1$ is of type 1, its contribution is $\mu(e^4_1) - \mu(e^3_1) + \mu(e^2_1) - \mu(e^1_1)$. Since $v^3_1$ is of type 1, its contribution is $\mu(e^4_1) - \mu(e^3_1) + \mu(e^2_1) - \mu(e^1_1)$. Here we understand that $e^g_1 = e^g_1$. Since $v^1_1$ is of type 2, its contribution is $\mu(e^4_1) - \mu(e^3_1) + \mu(e^2_1) - \mu(e^1_1)$.

Moreover, we have $\mu(e^3_1) = 0$, $\mu(e^2_1) + \mu(e^4_1) = \mu(e^4_1)$, and $\mu(e^4_1) = \mu(e^3_1) = -\mu(e^1_1)$. Using these relations, we obtain

$$\xi_G = \mu(e^1_1) + \sum_{i=1}^{g-1} 2\mu(e^4_i).$$

\end{example}

4. Elementary properties

In this section, we record two elementary properties of $\xi_G$.

We first show a certain gluing formula. Let $g$ and $g'$ be positive integers, and suppose that we have two trivalent fatgraph spines $\iota: G \hookrightarrow \Sigma_{g,1}$ and $\iota': G' \hookrightarrow \Sigma_{g',1}$. Fix $e \in E^{\text{ori}}(G)$. Plugging the tail of $G'$ in the right side of $e$, one produces a new fatgraph spine of $\Sigma_{g+g',1}$. A precise construction is as follows. Let $v_e$ be the middle point of $e$.

1. Take a small closed disk $D_e$ in $\Sigma_{g,1}$ such that $\text{Int}(D_e) \cap G = \emptyset$, the boundary $\partial D_e$ intersects $G$ once at $v_e$, and the center of $D_e$ is on the right side of $e$ with respect to the orientation of $e$.

2. Glue $\Sigma_{g,1} \setminus \text{Int}(D_e)$ with $\Sigma_{g',1}$ along the boundaries $\partial D_e$ and $\partial \Sigma_{g',1}$ so that the univalent vertex of $G'$ is identified with $v_e$.

3. Let $G''$ be the union of the images of $G$ and $G'$ in the result of gluing.

The glued surface is diffeomorphic to $\Sigma_{g+g',1}$. We consider $G''$ as a trivalent fatgraph spine of $\Sigma_{g+g',1}$ by dividing $e$ into two edges sharing the newly created trivalent vertex $v_e$. These two edges receive their orientation from $e$. We name them as $e_1, e_2 \in E^{\text{ori}}(G'')$ so that $v_e$ is the end point of $e_1$. The edges $e_1$ and $e_2$ have the same homology marking as $e$.

A schematic figure of this construction is Figure 7. We call $G''$ the gluing of $G$ and $G'$ at $e$. Note that the inclusions $\Sigma_{g,1} \setminus \text{Int}(D_e) \hookrightarrow \Sigma_{g+g',1}$ and $\Sigma_{g',1} \hookrightarrow \Sigma_{g+g',1}$ induce a direct sum decomposition

$$H_1(\Sigma_{g+g',1}; \mathbb{Z}) \cong H_1(\Sigma_{g,1}; \mathbb{Z}) \oplus H_1(\Sigma_{g',1}; \mathbb{Z}).$$

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{The fatgraph in Example 3.5.}
\end{figure}
Proposition 4.1 (gluing formula). Let $G''$ be the gluing of $G$ and $G'$ at $e$, as above. Then we have

$$\xi_{G''} = \xi_G + \mu(e) + \xi_{G'}.$$ 

Proof. We have a natural identification $V^\text{int}(G'') \cong V^\text{int}(G) \sqcup \{v_e\} \sqcup V^\text{int}(G')$. Observe that this identification respects the type of vertices. With the direct sum decomposition (4.1) in mind, we see that $V^\text{int}(G)$ and $V^\text{int}(G')$ contribute to $\xi_{G''}$ as $\xi_G$ and $\xi_{G'}$, respectively.

We compute the contribution from $v_e$. Let $t' \in E^\text{ori}_{v_e}(G'')$ be an edge coming from the tail of $G'$. The homology marking of $t'$ is trivial. If $e$ has the preferred orientation, we see that the contribution is $\mu(t') - \mu(\bar{e}_2) = \mu(e)$. Otherwise, the contribution is $\mu(e_1) - \mu(t') = \mu(e)$. This completes the proof. □

We next show a formula describing how $\xi_G$ changes under a special kind of flip. Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. We use the following notation.

- We denote by $t$ the tail of $G$, and give it the preferred orientation.
- $e_1 \in E^\text{ori}(G)$ is the oriented edge next to $t$ in the total ordering given to $E^\text{ori}(G)$.
- $v_1$ and $v_2$ are the start and end points of $e_1$, respectively.
- $b, c \in E^\text{ori}_{v_2}(G)$ are the edges such that $e_1, b,$ and $c$ are in this order in the cyclic ordering given to $E^\text{ori}_{v_2}(G)$.

The situation is illustrated in Figure 8. We call the flip along (the unoriented edge underlying) $e_1$ the tail slide to $G$.

Proposition 4.2 (tail slide formula). Let $G'$ be the result of the tail slide to $G$. Then we have

$$\xi_{G'} = \xi_G + \mu(c).$$

Proof. We work with Figure 8. Suppose $b < c$ in $E^\text{ori}(G)$. For simplicity, we write $e$ instead of $\mu(e)$ for $e \in E^\text{ori}(G)$. Then we compute

$$\xi_{G'} - \xi_G = (e_{v_1}' - f_{v_2}') + (e_{v_2}' - f_{v_1}') - (e_{v_1} - f_{v_2}) - (e_{v_2} - f_{v_1})$$

$$= (b - (-b)) + (c - (-b - c)) - (b + c - (-b - c)) - (b - c)$$

$$= c.$$

The case where $c < b$ can be computed similarly. □

As an application of Proposition 4.2, we can extend the definition of our invariant to trivalent fatgraph spines of a once punctured surface. Let $\Sigma^1_g$ be a surface obtained from $\Sigma_{g,1}$ by gluing a once punctured disk along the
boundaries. We regard $\Sigma_{g,1}$ as a subset of $\Sigma_g^1$. By definition, a fatgraph spine of $\Sigma_g^1$ is an embedding $\iota: G \hookrightarrow \Sigma_g^1$ of a fatgraph $G$ into $\Sigma_g^1$ satisfying the first two conditions in Definition 2.1 (with $\Sigma_{g,1}$ replaced by $\Sigma_g^1$), and the condition that all vertices have valency greater than 2.

Let $G$ be a trivalent fatgraph spine of $\Sigma_g^1$. By a suitable isotopy, we arrange that $G \subset \Sigma_{g,1}$. Let $e \in E^{\text{ori}}(G)$. Take a simple arc $\ell$ on $\Sigma_{g,1}$ starting from $p$, reaching $v_e$ from the right, and disjoint from $G \setminus \{v_e\}$. We say that such an arc $\ell$ is admissible for $e$. Regarding $v_e$ as a newly created trivalent vertex, we can consider the union $\bar{G}(e,\ell) = G \cup \ell$ as a trivalent fatgraph spine of $\Sigma_{g,1}$. The arc $\ell$ becomes the tail of $\bar{G}(e,\ell)$.

**Corollary 4.3.** Keep the notation as above. Then the element

$$\xi_{\bar{G}(e,\ell)} - \mu(e)$$

does not depend on the choice of $e$ and $\ell$. In particular, for a trivalent fatgraph spine $G \subset \Sigma_g^1$, we can define $\xi_G \in H = H_1(\Sigma_{g,1}; \mathbb{Z}) \cong H_1(\Sigma_g^1; \mathbb{Z})$ as

$$\xi_G := \xi_{\bar{G}(e,\ell)} - \mu(e).$$

**Proof.** Let $\ell'$ be another admissible arc for $e$. Then $\ell'$ is isotopic to the concatenation of some power of a simple based loop parallel to $\partial \Sigma_{g,1}$ and $\ell$. This implies that $\bar{G}(e,\ell')$ is obtained from $\bar{G}(e,\ell)$ by application of some power of the Dehn twist along $\partial \Sigma_{g,1}$. Since the Dehn twist along $\partial \Sigma_{g,1}$ acts on $H$ trivially, we have $\xi_{\bar{G}(e,\ell)} = \xi_{\bar{G}(e,\ell')}$. Hence $\xi_{\bar{G}(e,\ell)} - \mu(e)$ does not depend on the choice of $\ell$.

Now, we can give a cyclic ordering to the set $E^{\text{ori}}(G)$ by a way similar to the case where $G \subset \Sigma_{g,1}$ as in Definition 5.2. Suppose that $e, e' \in E^{\text{ori}}(G)$ are consecutive in this cyclic ordering. Fix an admissible arc $\ell$ for $e$. Let $v_0$ be the vertex of $G$ shared by $e$ and $e'$, and let $c \in E^{\text{ori}}(G)$ be an edge other than $e$ and $e'$. We denote by $e_0$ an unoriented edge of $\bar{G}(e,\ell)$ with end points $v_e$ and $v_0$.

Let $\bar{G}'$ be the result of flip along $e_0$. Then $\bar{G}'$ can be identified with $\bar{G}(e',\ell')$, where $\ell'$ corresponds to the tail of $\bar{G}'$. By Proposition 1.2 we have $\xi_{\bar{G}(e',\ell')} = \xi_{\bar{G}(e,\ell)} + \mu(c)$. Since $\mu(c) + \mu(e) = \mu(e')$, we obtain $\xi_{\bar{G}(e',\ell')} - \mu(e') = \xi_{\bar{G}(e,\ell)} - \mu(e)$. This shows that $\xi_{\bar{G}(e,\ell)} - \mu(e)$ does not depend on the choice of $e$, either.

\[ \Box \]

5. Non-triviality and primitivity

In this section, we first prove that the invariant $\xi_G$ is non-trivial. Then we consider the primitivity of $\xi_G$ and present some partial results.
Consider the mod 2 reduction of $\xi_G$:
$$\xi_G^2 := \xi_G \otimes (1 \mod 2) \in H \otimes \mathbb{Z}_2 \cong H_1(\Sigma_{g,1}; \mathbb{Z}_2).$$
Hereafter, $\equiv$ stands for an equality in $H \otimes \mathbb{Z}_2$. Since $\mu(e) = -\mu(e) \equiv \mu(e) \in H \otimes \mathbb{Z}_2$ for any $e \in E^{\text{ori}}(G)$, the homology marking $\mu$ induces a well-defined map $\mu^2: E(G) \to H \otimes \mathbb{Z}_2$. We call $\mu^2$ the mod 2 homology marking.

**Proposition 5.1.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then we have
$$\xi_G^2 = \sum_{e \in E(G)} \mu^2(e).$$

**Proof.** Let $v \in V^{\text{int}}(G)$. We work with Figure 3 and count preferably oriented edges toward $v$. By abuse of notation, we use the same letter for an oriented edge and its underlying unoriented edge. If $v$ is of type 1, only $e_1$ has the preferred orientation. Since $\mu(e_1) + \mu(e_2) + \mu(e_3) = 0$, we have
$$\mu(e_v) - \mu(f_v) = \mu(e_1) - \mu(e_2) - \mu(e_3) \equiv \mu(e_1).$$
If $v$ is of type 2, $e_1$ and $e_3$ have the preferred orientation and $e_2$ does not. Then we have
$$\mu(e_v) - \mu(f_v) = \mu(e_1) - \mu(e_3) \equiv \mu(e_1) + \mu(e_3).$$
Therefore, we have
$$\xi_G^2 = \sum_{v \in V^{\text{int}}(G)} \left( \text{the sum of the mod 2 homology markings} \right)$$
$$\text{of preferably oriented edges toward } v$$
$$= \sum_{e \in E(G)} \mu^2(e).$$

The last equality holds since any preferably oriented edge of $G$ points to some trivalent vertex of $G$. \qed

**Theorem 5.2.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then the mod 2 reduction $\xi_G^2$ is non-trivial. In particular, we have $\xi_G \neq 0$.

To prove this theorem, we need the following lemma.

**Lemma 5.3.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then $G$ contains an edge cycle of odd length.

**Proof.** We introduce a terminology; a pair of consecutive oriented edges of $G$ is called a corner of $G$. There are $3\#V^{\text{int}}(G) = 3(4g - 1)$ corners. We number them as $c_1, \ldots, c_{3(4g-1)}$, so that $c_1$ contains the preferably oriented tail of $G$, and for each $i$, $c_i$ and $c_{i+1}$ share an oriented edge in common. There are $n_o := 6g - 1$ odd numbered corners, and $n_e := 6g - 2$ even numbered corners.

Since $n_o$ and $n_e$ are not divisible by 3, there exist distinct indices $i$ and $j$ with $1 \leq i < j \leq 3(4g - 1)$ such that the corners $c_i$ and $c_j$ are around the same vertex and $i - j \equiv 1 \mod 2$. We can write $c_i$ and $c_j$ as $c_i = (c_i, c_i')$ and $c_j = (c_j, c_j')$ with $c_i < c_i'$ and $c_j < c_j'$. Consider the edge cycle following consecutive oriented edges of $G$ from $c_i'$ to $c_j$. Since $i$ and $j$ have different parity, the length of this edge cycle must be odd. This completes the proof. \qed
Proof of Theorem 5.2. By Lemma 5.3, $G$ contains an edge cycle $\gamma$ of odd length. By Proposition 5.1, the mod 2 intersection pairing of $\xi_2^G$ and $\gamma$ is computed as
\[
(\xi_2^G, \gamma) = \left( \sum_{e \in E(G)} \mu^2(e) \cdot \gamma \right) = (\text{the length of } \gamma) = 1.
\]
Therefore, $\xi_2^G \neq 0$.

An element $x \in H$ is called primitive if there do not exist $m \in \mathbb{Z}$ and $y \in H$ such that $|m| \geq 2$ and $x = my$. Note that $x$ is primitive if and only if there exists an element $x' \in H$ such that $(x \cdot x') = \pm 1$. With Theorem 5.2 in mind, we would like to pose the following question.

**Question 5.4.** For any trivalent fatgraph spine $G \subset \Sigma_{g,1}$, is $\xi_G$ a primitive element of $H$?

The answer to this question is affirmative if $g \leq 2$. In fact, there is only one combinatorial isomorphism class of trivalent fatgraph spines for $g = 1$, and there are 105 combinatorial isomorphism classes for $g = 2$. By a direct computation, we can show the primitivity of $\xi_G$ for these cases. The question remains open for $g \geq 3$.

In the below, we show that $\xi_G$ is primitive if $G$ is of a few special types. We recall that for a fatgraph spine $G$ of $\Sigma_{g,1}$, the greedy algorithm [1] gives a maximal tree $T_G$ of $G$. By definition, the set of vertices of $T_G$ is $V(G)$. For $e \in E(G)$, give the preferred orientation to $e$ and let $v$ be the end point of $e$. Then $e$ is an edge of $T_G$ if and only if $e \prec e'$ for any $e' \in E^\text{ori}(G)$ with $e' \neq e$. One can see that $T_G$ is a maximal tree of $G$ and contains the tail of $G$.

A trivalent fatgraph spine $G \subset \Sigma_{g,1}$ is a chord diagram of genus $g$ if the first $4g$ elements in $E^\text{ori}(G)$ have the preferred orientation. To work with chord diagrams, we set up some notation. Name and order the first $4g$ elements in $E^\text{ori}(G)$ as $e_0, e_1, \ldots, e_{4g-2},$ and $f_0$. Their underlying unoriented edges are distinct. Let $\{f_k\}_{k=0}^{2g-1} \subset E(G)$ be the other unoriented edges of $G$. We give the preferred orientation to $f_k$, and let $v_k$ and $v_k'$ be the start and end points of $f_k$. Also, let $v_0'$ be the end point of $f_0$. We have $V^\text{int}(G) = \{v_k, v_k'\}_{k=0}^{2g-1} \sqcup \{v_0'\}$. Note that the maximal tree $T_G$ is straight, and the set of edges is $\{e_i\}_{i=0}^{4g-2}$. We can regard $G$ as a linear chord diagram constructed by attaching $2g$ chords $\{f_k\}_{k=0}^{2g-1}$ suitably to the interval $[0, 4g]$ at the integer points $\{1, \ldots, 4g\}$, where we identify $e_i$ with the interval $[i, i+1]$ (when we see it as a fatgraph spine, we remove the bivalent vertex at $4g \in [0, 4g]$). This is an explanation for our terminology. See also [1] [2] [3].

**Theorem 5.5.** If $G$ is a chord diagram of genus $g$, then $\xi_G$ is a primitive element of $H$.

**Proof.** We use the notation in the paragraph before the statement of the theorem.

We fix an index $k$ with $1 \leq k \leq 2g - 1$ and compute the contribution from $v_k$ and $v_k'$ to $\xi_G$. Let $i_k$ and $i_k'$ be indices such that $f_k$ is next to $\bar{e}_{i_k}$ in
$E^{\text{ori}}(G)$ and $e_{i_k}'$ is next to $f_k$ in $E^{\text{ori}}(G)$. We have either $i_k > i_k'$ or $i_k < i_k'$. See Figure 9.

If $i_k > i_k'$, the contribution from $v_k$ is $\mu(\bar{e}_{i_k}) - \mu(\bar{f}_k) = -\mu(e_{i_k}) + \mu(f_k)$, and that from $v_k'$ is $\mu(e_{i_k'}) - \mu(f_k)$. As a total, the contribution from $v_k$ and $v_k'$ is $\mu(e_{i_k'}) - \mu(e_{i_k})$. Moreover, the edge cycle

$$\gamma_k := f_k e_{i_{k-1}} e_{i_k} \cdots e_{i_k}$$

represents the homology class $\mu(e_{i_k'}) - \mu(e_{i_k})$. If $i_k < i_k'$, we can argue similarly and the contribution from $v_k$ and $v_k'$ is represented by the edge cycle

$$\gamma_k = f_k e_{i_{k+1}} \cdots e_{i_k-1}.$$

We observe that for any $1 \leq k \leq 2g - 1$ and $0 \leq k' \leq 2g - 1$, one has

$$(\gamma_k \cdot f_k') = -\delta_{kk'},$$

where $\delta_{kk'}$ is Kronecker’s delta, and for simplicity we write $f_{k'}$ instead of $\mu(f_{k'})$. This is because $\gamma_k \cap (G \setminus \text{Int}(f_{k'})) = \emptyset$ for $k \neq k'$, and $\gamma_k$ intersects $f_k$ once.

Let $i_0$ be an index such that $e_{i_0}$ is next to $f_0$ in $E^{\text{ori}}(G)$. The vertex $v_0'$ is of type 2 and its contribution is given by $\mu(e_{i_0}) - \mu(f_0)$.

We conclude that

$$(\gamma_k \cdot f_0) = -\delta_{kk},$$

By $(e_{i_0} \cdot f_0) = -1$ and (5.1), we have $(\xi_G \cdot f_0) = -1$. This shows that $\xi_G$ is primitive. 

We can also prove the primitivity of $\xi_G$ for a trivalent fatgraph spine $G \subset \Sigma_{g,1}$ which is obtained from a chord diagram of genus $g$ by a single flip.

**Theorem 5.6.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$ and assume that the first $4g - 1$ elements in $E^{\text{ori}}(G)$ have the preferred orientation, and that the $4g$-th element does not. Then $\xi_G$ is a primitive element of $H$.

**Proof.** Name and order the first $4g - 1$ elements in $E^{\text{ori}}(G)$ as $e_0, e_1, \ldots, e_{4g - 2}$. There exist an integer $n$ with $0 \leq n \leq 4g - 3$ and $h \in E(G)$ such that if $v$ is the end point of $e_n$, then $e_n, \bar{e}_{n+1}$, and $\bar{h}$ are in this order with respect to the cyclic ordering given to $E^{\text{ori}}(G)$, and the set of edges of $T_G$ is $\{e_1\}_{i=0}^{4g-3} \cup \{h\}$. We denote by $v'$ the end point of $h$ with the preferred orientation. Let $f_1, f_2 \in E(G)$ be edges which are different from $h$ and have
Figure 10. The proof of Theorem 5.6

Let $G'$ be the result of flip along $h$, and let $h' \in E(G')$ be the edge corresponding to $h \in E(G)$. Then $G'$ is a chord diagram in the sense of Theorem 5.5, and \{\e_{i}\}_{i=0}^{4g-3} \cup \{h'\}$ becomes the set of edges of $T_{G'}$. Extending $f_1$ and $f_2$ to \{\f_{j}\}_{j=1}^{2g-1}$ as in the proof of Theorem 5.5, by (5.2) we have

\begin{equation}
(5.3)
\xi_{G'} = \sum_{k=1}^{2g-1} \gamma_{k} + \mu(\e_{i_0}) - \mu(\e_{4g-2}),
\end{equation}

where $i_0$ is an index such that $\e_{i_0}$ is next to $\e_{4g-2}$ in $E_{ori}(G')$. By the last part of the proof of Theorem 5.5 we have

\begin{equation}
(5.4)
(\xi_{G'} \cdot \e_{4g-2}) = -1.
\end{equation}

Note that for $j = 1, 2$, we have

\begin{equation}
(5.5)
(e_{i_0} \cdot f_{j}) = 0 \text{ and } \left(\sum_{k=1}^{2g-1} \gamma_{k} \cdot f_{j}\right) = -1.
\end{equation}

In fact, the first equation of (5.5) follows from $\e_{i_0} \prec f_{j}$, and the second one follows from (5.1).

By Proposition 3.1 (2), we have

\begin{equation}
(5.6)
\xi_{G} = \xi_{G'} - 2j'(W_{h}) + m(W_{h}).
\end{equation}

Furthermore we have $m(W_{h}) = \mu(\e_{n}) + \mu(\f_{1}) = \mu(\e_{n}) - \mu(\f_{1})$, and

\begin{equation}
\begin{aligned}
j'(W_{h}) &= (e_{n} \cdot \e_{n+1})\mu(\f_{1}) + (\e_{n+1} \cdot \f_{1})\mu(\e_{n}) + (f_{1} \cdot e_{n})\mu(\e_{n+1}) \\
&= (e_{n} \cdot \e_{n+1})\mu(\f_{1}) + (\e_{n+1} \cdot \f_{1})\mu(\e_{n}) + (f_{1} \cdot e_{n})\mu(\e_{n+1}).
\end{aligned}
\end{equation}

Now we have either $\f_{1} < \e_{n}$ or $\e_{n} < \f_{1}$ in $E_{ori}(G)$.

Case 1: assume that $\f_{1} < \e_{n}$. See the left part of Figure 10. Then any two elements of \{\mu(\e_{n}), \mu(\e_{n+1}), \mu(\f_{1}), \mu(\f_{2})\} have the trivial intersection

$v'$ as an end point. We arrange that $f_1$ with the preferred orientation is next to $h$ in $E_{ori}(G)$.
pairing. By a direct computation using equations (5.3) to (5.6), we obtain

\[
\begin{aligned}
(\xi_G \cdot e_{4g-2}) &= -1 + (e_n \cdot e_{4g-2}) - (f_1 \cdot e_{4g-2}), \\
(\xi_G \cdot f_1) &= (f_1 \cdot e_{4g-2}) - 1, \\
(\xi_G \cdot f_2) &= (f_2 \cdot e_{4g-2}) - 1.
\end{aligned}
\]  

If \((f_2 \cdot e_{4g-2}) = 0\), we have \((\xi_G \cdot f_2) = -1\). Otherwise, we see that \((e_n \cdot e_{4g-2}) = (f_1 \cdot e_{4g-2}) = 0\) and hence \((\xi_G \cdot e_{4g-2}) = -1\).

Case 2: assume that \(\bar{e}_n < \bar{f}_1\). See the right part of Figure 10. We have \((e_n \cdot e_{n+1}) = (f_1 \cdot e_{n+1}) = (f_2 \cdot e_{n+2}) = 0\) and \((e_n \cdot f_1) = (e_n \cdot f_2) = (f_1 \cdot f_2) = -1\).

Again by using equations (5.3) to (5.6), we obtain

\[
\begin{aligned}
(\xi_G \cdot e_{4g-2}) &= -1 + (e_n \cdot e_{4g-2}) - (f_1 \cdot e_{4g-2}) - 2(e_{n+1} \cdot e_{4g-2}), \\
(\xi_G \cdot f_1) &= (f_1 \cdot e_{4g-2}) - 2, \\
(\xi_G \cdot f_2) &= (f_2 \cdot e_{4g-2}) - 1.
\end{aligned}
\]  

If \((f_2 \cdot e_{4g-2}) = 0\), we have \((\xi_G \cdot f_2) = -1\). Otherwise, we have \((f_2 \cdot e_{4g-2}) = 1\) and we obtain \((\xi_G \cdot f_1) = -1\).

In any case, we can find an element \(x' \in H\) such that \((\xi_G \cdot x') = -1\). Therefore, \(\xi_G\) is primitive. \(\square\)

In the case of trivalent fatgraph spines of a once punctured surface \(\Sigma_g^1\), it can happen that \(\xi_G = 0\). Two examples of \(G \subset \Sigma_g^1\) with \(\xi_G = 0\) are given in Figure 11.

Let \(G\) be a trivalent fatgraph spine of \(\Sigma_g^1\). Recall from the proof of Corollary 14.3 that if \(G \subset \Sigma_g^1\), then \(E^{ori}(G)\) is cyclically ordered. Hence it is possible to talk about corners of \(G\) in a way similar to the case of trivalent fatgraph spines of \(\Sigma_g^1\). Now we give labels \(\alpha\) or \(\beta\) to each corner of \(G\) so that any pair of consecutive corners of \(G\) have distinct labels. Since the number of corners of \(G\) is even, this labeling is always possible and is determined once we choose the label of a fixed corner.

We say that \(G\) is \textit{balanced} if for any vertex of \(G\), the three corners around the vertex have the same label. For example, trivalent fatgraph spines in Figure 10 and Figure 11 are balanced.
Theorem 5.7. Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then the mod 2 reduction $\xi_G^2 = \xi_G \otimes (1 \mod 2)$ is trivial if and only if $G$ is balanced.

Proof. Pick a corner $c$ of $G$ and write it as $c = (e, e')$, where $e'$ is next to $e$ in the cyclic ordering given to $E^{ori}(G)$. We give the label $\alpha$ to $c$ and extend this labeling to all other corners as above. Take an admissible arc $\ell$ for $e$ and set $\tilde{G} = \tilde{G}(e, \ell)$. The oriented edge $e$ is split at the middle point $v_e$ into two oriented edges. We name them as $e_1, e_2 \in E^{ori}(\tilde{G})$ so that $v_e$ is the end point of $e_1$. We extend the labeling of corners of $G$ to that of corners of $\tilde{G}$ by giving $\alpha$ to $(e_1, \ell)$ and $(e_2, e_1)$, and $\beta$ to $(\ell, e_2)$. 

In view of Corollary 4.3, the condition $\xi_G^2 = 0$ is equivalent to $\xi_G^2 = \mu^2(e_2)$. Furthermore, since the mod 2 homology markings $\{\mu^2(f)\}_{f \in E(\tilde{G})}$ generate the mod 2 homology $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$, this condition is equivalent to the condition that $(\xi_G^2 \cdot \mu^2(f)) = (\mu^2(e_2) \cdot \mu^2(f))$ for any $f \in E(\tilde{G})$.

Assume that $G$ is balanced. For any vertex of $\tilde{G}$ other than $v_e$, the three corners about it is labeled by the same symbol. Let $f \in E(\tilde{G})$. Let $\gamma(f)$ be the edge cycle following consecutive oriented edges of $\tilde{G}$ from $f$ to $\tilde{f}$, where we give the preferred orientation to $f$. The mod 2 homology class $\mu^2(f)$ is represented by $\gamma(f)$. By the property of the labeling, the length of this edge cycle is odd if $f \prec e_2 \prec \tilde{f}$ (this also implies $f \neq e_2$), and is even otherwise. Note that the condition $f \prec e_2 \prec \tilde{f}$ is equivalent to $(\mu^2(e_2) \cdot \mu^2(f)) = 1$. Hence $(\xi_G^2 \cdot \mu^2(f)) = (\text{length of } \gamma(f)) = 1$ if and only if $(\mu^2(e_2) \cdot \mu^2(f)) = 1$. Therefore, $\xi_G^2 = 0$.

On the other hand, assume that $\xi_G^2 = 0$. Then for $f \in E(\tilde{G})$, the length of $\gamma(f)$ is odd if and only if $f \prec e_2 \prec \tilde{f}$. Now we remove the tail from $\tilde{G}$ and go back to $G$. Then $\gamma(f)$ is reduced to an edge cycle of $G$. Its length is 1 less than the length of $\gamma(f)$ if $f \prec e_2 \prec \tilde{f}$, and is the same as the length of $\gamma(f)$ otherwise. This implies that the reduced edge cycle of $G$ has even length. Since $f$ can be arbitrary, this shows that $G$ is balanced. \hfill \Box

6. Mod 2 reduction and spin structures

In this section, we give a topological interpretation of the mod 2 reduction $\xi_G^2$. Namely, we show that to any trivalent fatgraph spine $G \subset \Sigma_{g,1}$ one can associate two distinct spin structures on $\Sigma_{g,1}$, and that $\xi_G^2$ is the difference of them.

We use the following description of the mod 2 homology of $\Sigma_{g,1}$.

Lemma 6.1. Let $G$ be a fatgraph spine of $\Sigma_{g,1}$. For $v \in V^{int}(G)$, let $\{e_i^v\}_{i \in I}$ be the unoriented edges of $G$ having $v$ as an end point. If there is an edge loop based at $v$, we count it twice. Then the mod 2 homology marking induces an isomorphism

$$H_1(\Sigma_{g,1}, \mathbb{Z}_2) \cong \bigoplus_{e \in E(G)} \mathbb{Z}_{2e} / \sum_{v \in V^{int}(G)} \mathbb{Z}_2 \left( \sum_{i} e_i^v \right).$$

Proof. Recall from 2.2 that we associate an oriented simple loop $\hat{e}$ to each (oriented) edge $e$. In the proof of this lemma we forget the orientation of $e$ and $\hat{e}$. We can arrange that the simple loops $\{\hat{e}\}_{e \in E(G)}$ share only one
point \( q \in \partial \Sigma_{g,1} \), and that if \( t \) is the tail of \( G \) then \( \hat{t} = \partial \Sigma_{g,1} \) with basepoint \( q \). Then we obtain a cell decomposition of \( \Sigma_{g,1} \) whose 1-cells coincide with \( \{ e \}_{e \in E(G)} \). Now the right hand side of the assertion can be identified with the first mod 2 cellular homology group of this cell decomposition. \( \square \)

Recall that a spin structure on \( \Sigma_{g,1} \) is an element \( w \in H^1(UT \Sigma_{g,1}; \mathbb{Z}_2) \), where \( UT \Sigma_{g,1} \) is the unit tangent bundle of \( \Sigma_{g,1} \) (with respect to some Riemannian metric), such that the restriction of \( w \) to a fiber of the projection \( UT \Sigma_{g,1} \to \Sigma_{g,1} \) is non-trivial. As Johnson [10] showed, the set of spin structures on \( \Sigma_{g,1} \) is naturally identified with the set of quadratic forms on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \). Here, a map \( q: H_1(\Sigma_{g,1}; \mathbb{Z}_2) \to \mathbb{Z}_2 \) is called a quadratic form on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \) if it satisfies

\[
q(x + y) = q(x) + q(y) + (x \cdot y)
\]

for any \( x, y \in H_1(\Sigma_{g,1}; \mathbb{Z}_2) \). The set of spin structures on \( \Sigma_{g,1} \) is a torsor under the action of \( H^1(\Sigma_{g,1}; \mathbb{Z}_2) \). In other words, the difference of two quadratic forms on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \) can be written as a uniquely determined element of \( \text{Hom}(H_1(\Sigma_{g,1}; \mathbb{Z}_2), \mathbb{Z}_2) \cong H^1(\Sigma_{g,1}; \mathbb{Z}_2) \). Note that using the mod 2 intersection pairing, we have a natural isomorphism

\[
H_1(\Sigma_{g,1}; \mathbb{Z}_2) \cong \text{Hom}(H_1(\Sigma_{g,1}; \mathbb{Z}_2), \mathbb{Z}_2), \quad x \mapsto \{ y \mapsto (x \cdot y) \}.
\]

In what follows, \( G \) is a trivalent fatgraph spine of \( \Sigma_{g,1} \). The following result gives an identification of certain \( \mathbb{Z}_2 \)-valued functions on \( E(G) \) with the set of quadratic forms on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \), thus with the set of spin structures on \( \Sigma_{g,1} \) via Johnson’s result stated above.

**Theorem 6.2.** Let \( G \) be a trivalent fatgraph spine of \( \Sigma_{g,1} \). Let \( Q(G) \) be the set of maps \( q: E(G) \to \mathbb{Z}_2 \) such that for any \( v \in V^\text{int}(G) \), the sum of values of \( q \) at the three edges having \( v \) as an end point is 0 if \( v \) is of type 1, and is 1 if \( v \) is of type 2. Then there is a natural bijection from \( Q(G) \) to the set of quadratic forms on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \).

**Proof.** Given a map \( q: E(G) \to \mathbb{Z}_2 \), we extend \( q \) to a map from the free \( \mathbb{Z}_2 \)-module generated by \( E(G) \) by

\[
q \left( \sum_{e \in E(G)} m_e e \right) := \sum_{e \in E(G)} m_e q(e) + \sum_{e \prec e'} m_e m_{e'} (\mu^2(e) \cdot \mu^2(e')), \tag{6.2}
\]

for \( m_e \in \mathbb{Z}_2 \), \( e \in E(G) \). Here \( ( \cdot, \cdot ) \) is the mod 2 intersection pairing and we give the preferred orientation to each element of \( E(G) \). By a direct computation, we can check that for any \( x, y \in \bigoplus_{e \in E(G)} \mathbb{Z}_2 e \),

\[
q(x + y) = q(x) + q(y) + (x \cdot y). \tag{6.3}
\]

Here \( (x \cdot y) \) is the mod 2 intersection pairing of the homology class determined by \( x \) and \( y \) through the isomorphism in Lemma 6.1.

We claim that if \( q \in Q(G) \), then for any \( v \in V^\text{int}(G) \),

\[
q(e_v^1 + e_v^2 + e_v^3) = 0. \tag{6.4}
\]

By (6.2), this condition is equivalent to the following.

\[
\sum_{i=1}^3 q(e_v^i) + (\mu^2(e_v^1) \cdot \mu^2(e_v^3)) + (\mu^2(e_v^2) \cdot \mu^2(e_v^3)) + (\mu^2(e_v^2) \cdot \mu^2(e_v^1)) = 0. \tag{6.4}
\]
If \( v \) is of type 1, then \((\mu^2(e_i^v) \cdot \mu^2(e_j^v)) = 0\) for any \( 1 \leq i, j \leq 3 \). If \( v \) is of type 2, then \((\mu^2(e_i^v) \cdot \mu^2(e_j^v)) = 1\) for any \( 1 \leq i, j \leq 3 \) with \( i \neq j \). See Figure 3. Therefore, the condition (6.3) is exactly equivalent to the condition for \( q \) being an element of \( Q(G) \). This proves the claim.

By the claim, Lemma 6.1, and (6.3), it follows that the map \( q \) induces a quadratic form on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \). The above construction gives a map from \( Q(G) \) to the set of quadratic forms on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \), and the inverse of this map is given by composing any quadratic form on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \) with the mod 2 homology marking \( \mu^2: E(G) \to H_1(\Sigma_{g,1}; \mathbb{Z}_2) \).

We record how the set \( Q(G) \) changes under a flip.

**Proposition 6.3.** Let \( W = W_e \) be a flip from \( G \) to \( G' \). Then the bijection in Theorem 6.2 induces a bijection from \( Q(G) \) to \( Q(G') \), which maps a given \( q \in Q(G) \) to the element \( q' \in Q(G') \) defined as follows.

- For any edge \( f \) in \( E(G') \setminus \{e'\} \cong E(G) \setminus \{e\} \), we have \( q'(f) = q(f) \).
- We adopt the notation in Figure 3 and assume that in each case \( G \) and \( G' \) correspond to the left and right pictures, respectively. Then the value \( q'(e') \) is given by the following formula.

\[
\begin{align*}
I : q'(e') &= q(b) + q(c) = q(a) + q(d), \\
II : q'(e') &= q(b) + q(c) = q(a) + q(d) + 1, \\
III : q'(e') &= q(b) + q(c) + 1 = q(a) + q(d), \\
IV : q'(e') &= q(b) + q(c) + 1 = q(a) + q(d), \\
V : q'(e') &= q(b) + q(c) = q(a) + q(d) + 1, \\
VI : q'(e') &= q(b) + q(c) + 1 = q(a) + q(d) + 1.
\end{align*}
\]

By a suitable replacement of labels of edges, one can similarly obtain a formula for \( q' \) in terms of \( q \) for the case where \( G \) and \( G' \) correspond to the right and left pictures, respectively, in each case in Figure 4.

**Proof.** To prove the first condition, note that the mod 2 homology marking of \( f \) as an edge of \( E(G) \) is the same as that of \( f \) as an edge of \( E(G') \). The second condition follows from the first condition and the defining relation for elements of \( Q(G') \). For example, in case VI, two end points of \( e' \) are of type 2, and hence we have \( q'(b) + q'(c) + q'(e') = q'(a) + q'(d) + q'(e') = 1 \). □

**Remark 6.4.** The description of spin structures on \( \Sigma_{g,1} \) given in Theorem 6.2 and how it changes under a flip as in Proposition 6.3 was pointed out by Robert Penner [20]. Recently, Penner and Zeitlin [27] give another natural description of spin structures on a punctured surface in terms of orientations on a trivalent fatgraph spine of the surface, and they also show how it changes under a flip. In other words, Penner and Zeitlin give a lift of the action of the mapping class group on the set of quadratic forms to the action of the Ptolemy groupoid, and the present construction gives another lift. It should be remarked that while their description works for any surfaces with multiple punctures, our description here is for a once (punctured/bordered) surface. It is an interesting question whether ours generalizes to any (punctured/bordered) surface.
Now we give the preferred orientation (Definition 3.2) to each unoriented edge of $G$ and use the same letter for the resulting oriented edge of $G$. For example, if we write $e \prec f \prec \bar{e}$ or $e \prec f \prec \bar{e}$ for $e, f \in E(G)$, we understand that $e$ and $f$ have the preferred orientation. Take $e \in E(G)$. We define an element $q_G(e), \bar{q}_G(e) \in \mathbb{Z}_2$ by

$$q_G(e) = \# \{ f \in E(G) | e \prec f \prec \bar{e} \} \mod 2,$$

and

$$\bar{q}_G(e) = \# \{ f \in E(G) | e \prec \bar{f} \prec \bar{e} \} \mod 2.$$

Here $\#$ means the number of elements of a set.

**Proposition 6.6.** The maps $q_G$ and $\bar{q}_G$ are elements of $Q(G)$. In particular, they induce quadratic forms on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$.

**Proof.** We consider the case of $q_G$ only.

We work with Figure 3. Suppose that $v$ is of type 1. Then $e_1, \bar{e}_2,$ and $\bar{e}_3$ have the preferred orientation, and we have a disjoint sum decomposition

$$\{ f \in E(G) | e_1 \prec f \prec \bar{e}_1 \} = \{ e_2, e_3 \} \cup \{ f \in E(G) | \bar{e}_2 \prec f \prec e_2 \} \cup \{ f \in E(G) | e_3 \prec f \prec e_3 \}.$$

This implies that $q_G(e_1) = q_G(e_2) + q_G(e_3)$.

Suppose that $v$ is of type 2. Then $e_1, \bar{e}_2,$ and $\bar{e}_3$ have the preferred orientation, and we have a disjoint sum decomposition

$$\{ f \in E(G) | \bar{e}_2 \prec f \prec e_2 \} = \{ \{ f \in E(G) | e_1 \prec f \prec \bar{e}_1 \} \setminus \{ e_2 \} \} \cup \{ f \in E(G) | e_3 \prec f \prec \bar{e}_3 \}.$$

This implies that $q_G(e_2) = q_G(e_1) + q_G(e_3) + 1$.

Therefore, $q_G \in Q(G)$. By Theorem 6.2, $q_G$ induce a quadratic form on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$. \hfill $\Box$

For simplicity, we use the same letter $q_G$ and $\bar{q}_G$ for the induced quadratic forms. This construction of quadratic forms is $\mathcal{M}_{g,1}$-equivariant in the following sense.

**Proposition 6.6.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$, and let $\varphi \in \mathcal{M}_{g,1}$. Then we have

$$q_{\varphi(G)} \circ \varphi_* = q_G,$$

$$\bar{q}_{\varphi(G)} \circ \varphi_* = \bar{q}_G,$$

where $\varphi_*$ is the automorphism of $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ induced from $\varphi$.

**Proof.** We consider the case of $q_G$ only. Consider a homomorphism

$$\Phi: \bigoplus_{e \in E(G)} \mathbb{Z}_2 e \to \bigoplus_{e' \in E(\varphi(G))} \mathbb{Z}_2 e', \quad \Phi(e) = \varphi(e).$$

Since $\varphi$ gives a combinatorial isomorphism from $G$ to $\varphi(G)$, we have $q_{\varphi(G)} \circ \Phi = q_G$. Now $\Phi$ induces the map $\varphi_*$ on the level of homology, and we conclude $q_{\varphi(G)} \circ \varphi_* = q_G$. \hfill $\Box$

Finally, we compute the difference of $q_G$ and $\bar{q}_G$. 
Theorem 6.7. Under the isomorphism (6.1), we have

\[ q_G - \bar{q}_G = \xi_G^2. \]

Moreover, we have \( q_G \neq \bar{q}_G \).

Proof. For \( e \in E(G) \), we have

\[ q_G(e) - \bar{q}_G(e) = q_G(e) + \bar{q}_G(e) \]
\[ = \# \{ f \in E^{\text{ori}}(G) \mid e \prec f \prec e \} \mod 2 \]
\[ = \left( \sum_{f \in E(G)} \mu^2(f) \cdot \mu^2(e) \right) = (\xi_G^2 \cdot \mu^2(e)), \]

where the last equality follows from Proposition 5.1. Since \( \{\mu^2(e)\}_{e \in E(G)} \) generates \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \), we obtain \( q_G - \bar{q}_G = \xi_G^2. \) The second statement follows from Theorem 5.2. \( \square \)

Appendix A. A non-singular vector field associated to a once bordered trivalent fatgraph spine

Let \( G \) be a trivalent fatgraph spine of \( \Sigma_{g,1} \). In this appendix, we define a non-singular vector field \( \mathcal{X}_G \) on \( \Sigma_{g,1} \), and then consider the induced quadratic form on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \). In particular, we discuss a relationship between this quadratic form, \( q_G \), and \( \bar{q}_G \).

The following construction of \( \mathcal{X}_G \) was communicated to the author by Gwénaëll Massuyeau.

Let \( \text{Vect}(\Sigma_{g,1}) \) be the homotopy set of non-singular vector fields on \( \Sigma_{g,1} \). In other words, \( \text{Vect}(\Sigma_{g,1}) \) is the homotopy set of sections of the projection \( \pi : UT\Sigma_{g,1} \rightarrow \Sigma_{g,1} \). For \( \mathcal{X} \in \text{Vect}(\Sigma_{g,1}) \), the winding number

\[ \text{wind}_{\mathcal{X}} : \pi_1(UT\Sigma_{g,1}) \rightarrow \mathbb{Z} \]

is defined as follows. Let \( \tilde{\gamma} : S^1 \rightarrow UT\Sigma_{g,1} \) be a (based) loop. For any \( t \in S^1 \), there uniquely exists an element \( \Phi_t = \Phi(\mathcal{X}, \tilde{\gamma}, t) \in S^1 = U(1) \) such that \( \mathcal{X}(\pi \circ \tilde{\gamma}(t))\Phi_t = \tilde{\gamma}(t) \). Then \( \text{wind}_{\mathcal{X}}(\tilde{\gamma}) \) is defined to be the mapping degree of the map \( S^1 \rightarrow S^1, \ t \mapsto \Phi_t \). The map \( \text{wind}_{\mathcal{X}} \) is a group homomorphism, and its mod 2 reduction

\[ w_{\mathcal{X}} \in \text{Hom}(\pi_1(UT\Sigma_{g,1}), \mathbb{Z}_2) \cong H^1(UT\Sigma_{g,1}; \mathbb{Z}_2) \]

is a spin structure on \( \Sigma_{g,1} \).

Now we give the preferred orientation to any unoriented edge of \( G \). Let \( v \in V^{\text{int}}(G) \). According to the type of \( v \), we realize a small neighborhood \( N_v \) of \( v \) in the \( xy \)-plane as in Figure 14 and then restrict the horizontal vector field \( \partial / \partial x \) to \( N_v \). We extend the vector field on \( \bigsqcup_v N_v \) thus obtained to a globally defined non-singular vector field \( \mathcal{X}_G \), so that outside \( \bigsqcup_v N_v \), each trajectory of \( \mathcal{X}_G \) is perpendicular to \( G \).

Let \( q_{\mathcal{X}_G} \) be the quadratic form on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \) corresponding to \( w_{\mathcal{X}} \). Following Johnson [10], one can compute it as follows. Let \( \gamma \) be an oriented simple closed curve. Consider the lift \( \tilde{\gamma} = (\gamma, \dot{\gamma}) \) of \( \gamma \) to a loop in \( UT\Sigma_{g,1} \).
Figure 13. $X_G$ on $N_v$

(a vertex of type 1)  (a vertex of type 2)

\[ \gamma \] is the velocity vector of $\gamma$ normalized to have the unit length. Then one has

\[ q_{X_G}(\gamma) = \text{wind}_{X_G}(\tilde{\gamma}) + 1 \mod 2. \]

We apply this formula to $\gamma = \hat{e}$, where $e \in E^\text{ori}(G)$. Assume that $e$ has the preferred orientation. Let $L(e)$ be the set of corners $(f, f')$ of $G$ (see the proof of Lemma 5.3) such that

1. $e \preceq f \prec f' \preceq \bar{e}$, and
2. exactly one of $f$ and $f'$ have the preferred orientation.

Here, $e \preceq f$ means $e \prec f$ or $e = f$. For example, if $v$ is a vertex of type 1 as in the left part of Figure 13, only $(\bar{e}_2, e_3)$ is an element of $L(e)$ among the three corners around $v$. Set $\lambda(e) = \# L(e)$.

**Lemma A.1.** We have $\text{wind}_{X_G}(\hat{e}) = (1 - \lambda(e))/2$.

**Proof.** Take a small regular neighborhood $N(G)$ of $G$, and we arrange that $\hat{e}$ stays inside $N(G)$ throughout. Every time when $\hat{e}$ goes through a common vertex of a member of $L(e)$, the velocity vector of $\hat{e}$ rotates by an angle $-\pi$ with respect to $X_G$. Also, when $\hat{e}$ goes through the middle point of $e$, the velocity vector of $\hat{e}$ rotates by an angle $\pi$ with respect to $X_G$. This proves the lemma.

In particular, using the fact that $\lambda(e)$ is odd (since $e$ has the preferred orientation and $\bar{e}$ does not), we have from (A.1) that

\[ q_{X_G}(e) = q_{X_G}([\hat{e}]) = \frac{1 - \lambda(e)}{2} + 1 \mod 2 = \frac{1 + \lambda(e)}{2} \mod 2. \]

**Proposition A.2.** Let $G$ be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then the quadratic forms $q_{X_G}$, $q_G$, and $\bar{q}_G$ are distinct to each other.

**Proof.** By Theorem 6.7, it is sufficient to prove $q_{X_G} \neq q_G$ and $q_{X_G} \neq \bar{q}_G$.

Let $e_1 \in E^\text{ori}(G)$ be the “last” preferably oriented edge. Namely, $e_1$ is the unique element such that $e_1$ has the preferred orientation and if $e_1 \prec f$, $f$ does not have the preferred orientation. We have $\lambda(e_1) = 1$ and $q_{X_G}(e_1) = \lambda(e_1)$.
(1 + 1)/2 = 1. On the other hand, since there are no preferably oriented edge \( f \) with \( e_1 \prec f \prec \bar{e}_1 \), we have \( q_G(e_1) = 0 \). Hence \( q_{X_G} \neq q_G \).

Let \( e_2 \in E^{ori}(G) \) be the unique element such that \( e_2 \) has the preferred orientation and if \( f \prec \bar{e}_2 \), \( f \) has the preferred orientation. We have \( \lambda(e_2) = 1 \) and \( q_{X_G}(e_2) = 1 \). On the other hand, since any edge \( f \in E^{ori}(G) \) with \( e_2 \prec f \prec \bar{e}_2 \) has the preferred orientation, we have \( \bar{q}_G(e_2) = 0 \). Hence \( \bar{q}_{X_G} = \bar{q}_G \).

\[ \square \]

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