SQUARED WEYL AND DIRAC FIELDS WITH SOMMERS – SEN CONNECTION, ASSOCIATED WITH $V^3_4$ DISTRIBUTION

Volodymyr Pelykh

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Abstract

The generalization of Sommers–Sen spinor connection for spinor fields, associated with a distribution $V^3_4$ and on this basis the equations for Weyl and Dirac null vector fields on complexificated $V^3_4$ are obtained. We interpret the obtained results by examining the interaction of spinor fields with the inertial forces.

1 INTRODUCTION

As known, the four–dimensional description of relativistic fields for the row of problems, especially under the comparison of theoretical previsions and experimental results, must by exchanged by 3+1 description. For the spinor fields the original method of their description in 3+1 form was proposed in [1] and developed in [2]. The obtained in [2] Sen–Witten equation has a wide usage for the positive signification of gravitational field energy problem investigations [3, 4, 5]. But this method does not allow to study all the variety of physical effects in the interactions of spinor fields with inertial forces, because in the base of the method lies the foliation of curved space-time by space-like hypersurfaces and thus the nonrotatory frames of reference. In our work [6] we introduced a covariant derivative of spinor fields, associated with the space-like distribution $V^3_4$, which generalizes the Sommers–Sen covariant
derivative, and obtained on this base the 3+1 equation for Weyl and Rarita–Schwinger fields in arbitrary frames of reference, not only in nonrotatory. In this work we obtain for arbitrary frame of reference the equations for complex 3-vector, which are correspondent to Weyl spinors and bispinors. This squared equations of Weyl and Dirac field are in fact the 3+1 splitting of Penrose–Rindler tensor form of spinor differential equations.

In section 2 we briefly review the technique for obtaining the 3+1 spinor equations with spinor connection on nonintegrable manifolds, in section 3 we give the spinor representation of tensor fields on the $V_4^3$ distribution.

2 THE SOMMERS–SEN GENERALIZED DERIVATIVE ON DISTRIBUTIONS

Let us consider the oriented manifold $V_4$ of class $C^\infty$, which is noncompact or of zero Euler characteristic; then it accepts Lorentz metric. Let us denote, as usually, the tangent bundle over $V_4$ by $TV_4$.

Definition 1. A vector subbundle $V_4^m$, $1 \leq m \leq 3$, of $TV_4$ is called a distribution over $V_4$. Let us denote by $C^\infty(V_4)$ a ring of functions of $C^\infty$ class and by $\Gamma(V_4^m)$ — a $C^\infty$ module of sections of distribution $V_4^m$ over $V_4$.

Definition 2. The mapping

$$\Gamma(V_4^m) \times \Gamma(V_4^m) \longrightarrow \Gamma(V_4^m),$$

$$(X, X_1) \longrightarrow D_X X_1, \quad X, X_1 \in \Gamma(V_4^m),$$

which for arbitrary vectors $X, X_1$ and functions $g, f \in C^\infty(V_4)$ satisfies the conditions

$$D_X(X_1 + X_2) = D_X X_1 + D_X X_2,$$

$$D_X(fX_1) = X(f)X_1 + fD_X X_1,$$

$$D_{fX + gX_1}X_2 = fD_X X_2 + gD_{X_1} X_2.$$

is called a covariant derivative or anholonomic connection on distribution $V_4^m$.

The homomorphism

$$\Gamma(B_0) : \Gamma(TV_4) \longrightarrow \Gamma(V_4^m)$$

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of cross-sections is corresponding with the projections $B_0$ of bundle $TV_4$ on subbundle $V_4^m$.

**Definition 3.** $\text{im } C_0$, where $C_0 = 1_{TV_4} - B_0$, is called a rigging $V_4^{1-m}$ of distribution $V_4^m \subset TV_4$.

Let $V_4^1$ be a one-dimensional time-like distribution over $V_4$; it’s unitary cross-section $u$ is identified with the field of 4-velocity of some frame of reference, and the integral curve of cross-section $u$ — with is time lines. The normal rigging $V_4^3$ of the distribution $V_4^1$ is the geometrical image of the physical space for appropriate frame of reference and is nonintegrable in general. The vectors, which belong to $V_4^1$, are called the time vectors, and those, which belong to $V_4^3$ — the spatial vectors.

Further we require that the second Stiefel–Whitney class $w_2$ of manifold $V_4$ equals zero. Then $V_4$ permits the $SL(2, C)$ spinor structure. Let us denote by $S_{r,s}(V_4)$ the $C^\infty$ module of spinor fields of $(r, s)$ valence on $V_4$.

Let us introduce $C^\infty$ module of $S_{r,s}(V_3^4)$ spinor fields of $(r, s)$ valence on $V_4$, associated with the distribution $V_3^4$, in the next way: its elements are the spinor fields of the form $\varphi T^{A...L}_{M...Q}$ and

$$\psi T^{A...K}_{M...RP...Q} = \left(-\sqrt{2}u^A_M\right) \cdots \left(-\sqrt{2}u^K_R\right) \left(\sqrt{2}u^A_M\right) \cdots \left(\sqrt{2}u^K_R\right) \times$$

where $\varphi$ and $\psi$ are the arbitrary $C^\infty(V_4)$ functions. In such way defined the $C^\infty$ module $S_{r,s}(V_3^4)$ is the module of $SU(2)$ spinor fields. A module $S_{r,s}(V_3^4)$ with the basis limited to the hypersurface $\Sigma$ let call a module of $SU(2)$ spinor fields on the anholonomic hypersurface $\Sigma$. In the partial case, when this hypersurface is ordinary and space-like, the module of $SU(2)$ spinor fields on it coincides with the module of Sommers–Sen spinor fields.

We introduce the antisymmetric tensor $A$ anholonomicity of $V_3^4$, $A \in V_3^4$.

Let $T : \Gamma(V_3^4) \times \Gamma(V_3^4) \to \Gamma(V_1^4)$,

$$T = \Gamma(C_0) [X_1, X_2]; \quad X_1, X_2 \in \Gamma(V_3^4).$$

Then $T = 4A \otimes u$. In coordinate basis on some open domain in $V_4$ the tensor $A$ has the components \[1\]

\[1\]For expressions written in coordinates, Greek indices are global and Latin indices local. Our Lorentz metric will have signature (-2)
\[ A_{\mu\lambda} = \frac{1}{2} h^\nu_{\mu} h^\delta_{\lambda} \nabla_{[\nu} u_{\delta]} . \]

Let us introduce the spatial covariant derivative for spinor fields, associated with the distribution \( V_4^3 \), as the mapping

\[ \Gamma(V_4^3) \times S_{1,0}(V_4^3) \rightarrow S_{1,0}(V_4^3), \]

\[ (X, \lambda) \rightarrow D_X \lambda, \quad X \in V_4^3, \quad \lambda \in S_{1,0}(V_4^3) \quad (4) \]
determined by the condition

\[ D_{AB} \lambda_C = \sqrt{2} u_{(A} \nabla_{B)} \lambda_C - \frac{1}{\sqrt{2}} (\pi_{ABC} + A_{ABC}) \lambda_D, \quad (5) \]

where \( \nabla_{B\dot{A}} \) is the spinor representation of the operator of covariant derivative on \( V_4 \), in agreement with metrical connection. The action \( D_{AB} \) on spinors of higher valence extends in accordance with the Leibnitz rule, and the action on vector fields satisfies the condition \( (1) \rightarrow (3) \).

Reducing the \( SL(2, C) \) operator of covariant derivative to \( SU(2) \) operator, we obtain:

\[ \nabla_{AB} = \sqrt{2} \left( u_{(A} \dot{A} \nabla_{B)} + u_{(B} \nabla_{A)} \dot{A} \right) = \frac{\sqrt{2}}{2} \varepsilon_{AB} u^A_{A} \nabla_{A} + \frac{1}{2} u_{(B} \nabla_{A) \dot{A} . \]

The first term, denoted by \( \frac{\sqrt{2}}{2} \varepsilon_{AB}(u \cdot \nabla) \), is the time derivative, the second is represented in terms space derivative \( D_{AB} \) in the rigging. Finally, we obtain the action of \( \nabla_{AB} \) on spinor \( \lambda_C \) in form

\[ \nabla_{AB} \lambda_C = \frac{\sqrt{2}}{2} \varepsilon_{AB} u^A \nabla_A \lambda_C + D_{AB} \lambda_C - \frac{\sqrt{2}}{2} \left( \pi_{AB} \dot{C} D + A_{AB} \dot{C} D \right) \lambda^D. \quad (6) \]

The generalized 3+1 form of Weyl equation

\[ \nabla_{A\dot{A}} \lambda^A = 0 \]

we obtain, fulfilling the \( SL(2, C) \rightarrow SU(2) \) reduction and using \( (3) \). Then we have:

\[ (u \cdot \nabla) \lambda_A + \sqrt{2} D_{AB} \lambda^B + \frac{1}{2} \pi \lambda_A - A_{BA} B_{D} \lambda^D = 0. \quad (7) \]
Therefore the Weyl spinor \( \lambda_A \in S_{1,0}(V^3_4) \) is determined by both geometric and equally physical properties of \( V^3_4 \) and \( V^1_4 \). These properties are determined by the acceleration spinor \( F_{AB} \in S_{2,0}(V^3_4) \), the angular velocity spinor \( A_{ABCD} \in S_{4,0}(V^3_4) \) and the rate–of–strain spinor \( \pi_{ABCD} \in S_{4,0}(V^3_4) \). These spinors are uniquely expressed by the Schouten first order curvature tensors of \( V^3_4 \) and vector \( u \in V^1_4 \).

3 Spinor Representation of Tensors Fields on the \( V^3_4 \) Distributions

Let spinor field \( T \in S_{2r,2s}(V^3_4) \). If \( T \) is symmetric in all pairs of indices, then \( T \in V^3_4 \). The projector from \( TV_4 \) into \( V^1_4 \) is \( u \otimes u \), the projector from \( TV_4 \) in \( V^3_4 \) is \( h = g - u \otimes u \).

It is easy to characterize the \( SU(2) \) representation of a spatial projected tensor:

\[
\mathbf{V}_{AB} = \mathbf{V}_{\dot{A}\dot{A}} \sqrt{2} u_{B} \dot{A} = v_{\nu}(g_{\mu}^{\nu} - u_{\nu} u_{\mu}) h_{m}^{\nu} \sigma_{\dot{A}\dot{A}}^{m} = v_{\nu}(g_{\mu}^{\nu} - u_{\nu} u_{\mu}) \sigma_{\dot{A}\dot{A}}^{\mu} =
\]

\[
\mathbf{\overline{V}}_{AB} = \mathbf{\overline{V}}_{\dot{A}\dot{A}} \sqrt{2} u_{B} \dot{A} = \mathbf{\overline{V}}_{\mu} \sigma_{\dot{A}\dot{A}}^{\mu} \]

Overline denotes the components of spatial projected tensors, the symbol \(^{\dot{}}\) denotes the components of time projected tensors, \( \sqrt{2} \sigma_{\dot{A}\dot{A}}^{\mu} \) are the Pauli spin matrices and the unit matrix which are referred to a space–time tetrad. The \( \sigma_{AB}^{\lambda} \) and \( \sigma_{AB}^{\nu} \) matrices are given by the formulas

\[
\sigma_{AB}^{\lambda} = \sigma_{\dot{A}\dot{A}}^{\lambda} \sqrt{2} u_{B} \dot{A} = \sqrt{2} u_{m} \sigma_{\dot{A}\dot{A}}^{m} \sigma_{mB} \dot{A}
\]

and respectively

\[
\sigma_{AB}^{\nu} = \sigma_{\dot{A}\dot{A}}^{\nu} \sqrt{2} u_{B} \dot{A} = \sqrt{2} u_{\mu} \sigma_{\dot{A}\dot{A}}^{\mu} \sigma_{\mu B} \dot{A}.
\]

For the matrices \( \sigma_{AB}^{\lambda} \) we obtain necessary in the following consideration identities, which exchange the normalization and orthogonalization identities of Pauli matrices:

\[
\sigma_{AB}^{\lambda} \sigma_{\lambda}^{DC} = \sigma_{AB}^{\nu} \sigma_{\nu}^{DC} = \sqrt{2} u_{m} \sigma_{\mu B} \dot{A} \sigma_{\dot{A}\dot{A}}^{\nu} \sqrt{2} u_{\nu} \sigma_{\nu}^{C} \sigma_{\lambda}^{D} \sigma_{\lambda}^{C} = \sqrt{2} u_{\nu} \sigma_{\nu}^{C} B \delta_{\dot{A}}^{D} = \]

\[
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\]
\[ \sqrt{2} \sigma_C^D \delta_A^B, \quad \sigma^\lambda A B \sigma_\mu A B = \delta_\mu^\lambda. \]

Let us also obtain the necessary for the further investigations tensor representation of the spinor form

\[ y_C^{(A \bar{z} B)} C = \sum_{m=0}^{3} y_m z_m \sigma_m^{(A \bar{z} m B)} C + \sum_{m \neq l}^{3} y_m \bar{z}_l \sigma_m^{(A \bar{l} B)} C. \quad (8) \]

The matrices \( \sigma_{CA}^m \) we obtain in the form

\[ \sigma_{CA}^m = \sqrt{2} (n^0 \omega_{m,CA}^n + n^1 \tau_{m,CA}^n + n^2 \chi_{m,CA}^n + n^3 \psi_{m,CA}^n), \quad \sigma_{AB}^C = \sigma_{CA}^B \varepsilon^{BC}. \]

where

\[ \omega_{m,CA}^n = \sigma_{CC}^m \sigma_{0A}^n \chi^C, \quad \tau_{m,CA}^n = \sigma_{CC}^m \sigma_{1A}^n \chi^C, \quad \chi_{m,CA}^n = \sigma_{CC}^m \sigma_{2A}^n \chi^C, \]

\[ \psi_{m,CA}^n = \sigma_{CC}^m \sigma_{3A}^n \chi^C. \]

The first sum in (8) equals zero; therefore further we consider only the second sum, in which we distinguish the terms with the products \( y_1 z_2 \). In this case the products of necessary matrices give:

\[
\begin{align*}
\omega_{1,CA}^1 \omega_{2,CA}^2 C &= -i \omega_{AB}^3 \\
\omega_{1,CA}^1 \tau_{2,CA}^2 C &= \chi_{AB}^1 \\
\omega_{1,CA}^1 \chi_{2,CA}^2 C &= -i \chi_{AB}^3 \\
\omega_{1,CA}^1 \psi_{2,CA}^2 C &= -i \psi_{AB}^3 \\
\chi_{1,CA}^1 \omega_{2,CA}^2 C &= -i \chi_{AB}^3 \\
\chi_{1,CA}^1 \tau_{2,CA}^2 C &= -\tau_{AB}^1 \\
\chi_{1,CA}^1 \chi_{2,CA}^2 C &= i \psi_{AB}^0 \\
\chi_{1,CA}^1 \psi_{2,CA}^2 C &= -\psi_{AB}^2 \\
\psi_{1,CA}^1 \omega_{2,CA}^2 C &= \omega_{AB}^3 \\
\psi_{1,CA}^1 \tau_{2,CA}^2 C &= i \tau_{AB}^3 \\
\psi_{1,CA}^1 \chi_{2,CA}^2 C &= -i \psi_{AB}^1 \\
\psi_{1,CA}^1 \psi_{2,CA}^2 C &= i \psi_{AB}^0.
\end{align*}
\]

After this the term with \( y_1 z_2 \) we obtain in the form:

\[ \sqrt{2} y_1 z_2 [ -i n^0 \sigma_{(AB)}^3 + i n^3 \sigma_{(AB)}^0 ]. \]

By analogy for terms with \( y_2 z_1 \) we have

\[ \sqrt{2} y_2 z_1 [ i n^0 \sigma_{(AB)}^3 - i n^3 \sigma_{(AB)}^0 ]. \]
After summarizing of all terms of the form $\bar{y}_m \bar{z}_l \quad (m \neq l)$ we find, that

$$y_C(\bar{z}_B)^C = -i\sqrt{2}\varepsilon^{asm} k n_a \bar{y}_s \bar{z}_n \sigma^k_{(AB)}.$$  \hfill (9)

Let us consider the evolution of complex null spatial vector field $L = -\lambda_A \lambda_B$, corresponding to the Weyl spinor field. Then as first step we obtain

$$(u \cdot \nabla)L_a = 2\sqrt{2}\lambda_{(B} D_{C)} \lambda^C - \pi L_a + 2\lambda_{(B} A_{A)} C^C D \lambda^D.$$

After the using of the Sommers identity

$$\lambda_C D_{AB} \lambda_C = \lambda^C D_C(\lambda_B) - \lambda(A D_B) C \lambda^C$$

we have

$$(u \cdot \nabla)L_a = -\sqrt{2} \lambda^C D_{AB} \lambda_C - \pi L_a - \sqrt{2} D_{C}(A L_B)^C + 2\lambda_{(B} A_{A)} C D \lambda^D.$$

For the vector representation of $\sqrt{2} D_{C}(A L_B)^C$ we obtain:

$$D_{C}(A L_B)^C = -i\sqrt{2}\varepsilon^{asm} k n_a \sigma^k_{(AB)} D_s L_n - i\sqrt{2}\varepsilon^{as n} k A_{as} D_{n} \sigma^k_{(AB)} =$$

$$= -\sqrt{2} \cdot 3! i *(A \wedge L + u \wedge D \wedge L).$$

The first term on the left can be re-expressed \[\boxed{\text{ as } \bar{L}^k D_k L_c, \text{ where unit real spatial vector } \bar{L} \text{ is in the direction of propagation of the neutrino field and is}}\]

$$\bar{L}^c = -i(D_4 \bar{L}_d)^{-1} \varepsilon^{a} \cdot n_a L^a \bar{D}^a.$$

Finally, we find that squared neutrino equation on $V^1_4$ and $V^3_4$ distributions, i.e. in a arbitrary frame of reference is

$$<dL, u> = <\bar{L}, D L> = -\pi L + 2 \cdot 3! i *(u \wedge D \wedge L) +$$

$$4 \cdot 3! *(u \wedge w \wedge L) - 2 \cdot 3! i *(u \wedge F \wedge L) - \sqrt{2} \cdot 3! i *(A \wedge L).$$

With the help of squared Weyl equations we can simply obtain the squared Dirac equation in 3+1 form. The Dirac equation is equivalent to the pair of equations

$$\nabla_{AA} \xi^A = \frac{m}{\sqrt{2}} \eta_{\dot{A}} \quad \text{and} \quad \nabla^{\dot{A}} \eta_{\dot{A}} \xi^A = -\frac{m}{\sqrt{2}} \xi^A$$
or in terms of $\xi^A \in S_{0,1}(V_4^3)$, $\eta_B \in S_{1,0}(V_4^3)$:
\[
\nabla_{AB}\xi^A = \frac{m}{\sqrt{2}}\eta_B \quad \text{and} \quad \nabla^{AB}\eta_B = -\frac{m}{\sqrt{2}}\xi^A.
\]

Let $X = -\xi_A\xi_B$ and $Y = -\eta_A\eta_B$ are two complex null spatial vector fields. Then the squared Dirac equation, which describes the evolution of $X$ and $Y$ fields and which is written in terms of the tensors determined on the distribution and in the rigging, is the system of two equations:
\[
< dX, u > = < \tilde{X}, DX > - \pi X - 2 \cdot 3! i \ast [u \wedge (F - D + 2iw) \wedge X - A \wedge X] - 3! \sqrt{2}im < -X, Y >^{-1/2} (u \wedge X \wedge Y)
\]
and
\[
< dY, u > = - < \tilde{Y}, DY > - \pi Y + 2 \cdot 3! i \ast [u \wedge (F + D - 2iw) \wedge Y + A \wedge Y] - 3! \sqrt{2}im < -X, Y >^{-1/2} (u \wedge X \wedge Y).
\]

4 DISCUSSION

The proposed in [6] and in this paper method for investigation of the interaction between spinor fields and inertial forces requires the usage of nonintegrable subbundle. In difference to [1, 2] the spinor derivatives are determined here by intrinsic geometry of distribution. This is defined by the physical sense of the problem and not by application of tetrad formalism, which was not necessary. In partial case of integrable $V_4^3$ distribution both tetrad and monad methods determine the spinors in terms of the intrinsic geometry of foliation.

We ascertain the appearance of additional difference between the evolution of Maxwell field, Weyl field and Dirac field, since the interaction of latter with the inertial field is described by the term, which includes not only the angular velocity vector of the frame of reference $w$, but also its angular velocity tensor $A$. 

8
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