Relaxation oscillations in model sandpiles

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Abstract

We introduce a simple one-dimensional sandpile model that undergoes relaxation oscillations. A single model can account for self-organized critical behavior and relaxation oscillations, depending on the manner in which it is driven, mirroring the experimental situation for real sandpiles. The relaxation oscillations are robust with respect to minor modifications of the avalanche rules, including the application of probabilistic rules.

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Many extended dissipative systems exhibit what may be called “avalanche dynamics”, meaning that they respond to slow external driving by undergoing rapid, discrete relaxation events, or avalanches. In general, such systems involve a quantity $\phi$ that is increased slowly by some external mechanism. When $\phi(x)$ or $\nabla \phi(x)$ becomes too large, an avalanche is initiated as $\phi$ is redistributed around a locally unstable region, possibly causing a chain reaction of local instabilities. The system then rapidly relaxes into a (temporarily) stable state in which $\Phi$, the volume integral of $\phi(x)$, has been reduced either by transport across the boundary of the system or by nonconservative dynamics in the bulk. This paper is concerned with the type of macroscopic behavior that might be observed in such systems, particularly the question of whether global relaxation oscillations should be expected.

One system that has received much attention recently is the proverbial sandpile. Here $\phi(x)$ is the height of the pile at position $x$ and the instability occurs when the local slope of the pile becomes too large. “Avalanches” consist of rearrangements on the surface of the pile corresponding to the ordinary meaning of the term, some of which transport mass off of an open boundary of the pile. Another example is the relaxation of the earth’s crust via earthquakes. Here $\phi(x)$ is the stress, which can be relieved by transport across an open boundary or by nonlinear processes that do not conserve $\Phi$.

Perhaps the most obvious feature of systems governed by avalanche dynamics that requires explanation is the distribution of avalanche sizes, $P(s)$, where $s$ is the total amount of $\Phi$ (mass, in the case of sandpiles) that leaves the system. By analogy with equilibrium statistical mechanics, one might expect that the qualitative features of $P(s)$ do not depend on microscopic details of the avalanche dynamics, but only on general features of the system such as its symmetries. If so, it should be possible to construct highly simplified numerical models that show behavior similar to real experiments on sand, just as the behavior of the Ising model accurately reflects the general features of a variety of microscopically complicated physical systems.

Following the seminal work of Bak et al. on self-organized criticality in a class of numerical models [1], Kadanoff et al. introduced several 1D models that could be taken as
candidates for the essential dynamics of sandpiles. The models introduced are extremely simple to write down, but show highly nontrivial behavior. By analogy with equilibrium systems, one would expect the generic behavior of $P(s)$ in the 1D system to be one of two possibilities, perhaps depending on the value of a system parameter: (1) $P(s)$ decays exponentially beyond some size $s$ associated with a correlation length in the steady state or, (2) $P(s)$ consists primarily of a large peak on the order of $L^2$, where $L$ is the length of the system, in which case the macroscopic slope of the pile would undergo oscillations whose amplitude does not decrease to zero in the infinite system-size limit; the macroscopic slope exhibits “relaxation oscillations”. The former possibility indicates a characteristic size for avalanches, independent of $L$, while the latter corresponds to a state dominated by huge avalanches that sweep away a finite fraction of the mass of the pile.

The results of simulations of a variety of sandpile models were remarkable in that although stationary slope profiles were attained and there were no relaxation oscillations, $P(s)$ was found to be a broad distribution in which no characteristic length scale other than some fractional power of the system size could be identified. In this sense, the models exhibit self-organized criticality (SOC), tuning themselves to a “critical point” between possibilities (1) and (2) by virtue of their own internal dynamics.

Stimulated in part by the apparent ubiquity of SOC in toy models, experiments have been performed on sandpiles in various configurations. The results to date remain inconclusive: while there is some evidence of SOC in heaps formed by grains dropped one at a time onto a flat plate, experiments employing a rotating drum half-filled with sand find relaxation oscillations.

Taking the empirical evidence for relaxation oscillations at face value, it appears that the numerical models previously studied all lacked some crucial ingredient that is present in real sandpiles. As a first step in discovering what that ingredient is, we consider here a class of models obtained by introducing modifications to the limited local sandpile (LLS) of Kadanoff et al. We find that relaxation oscillations do occur under appropriate conditions, as detailed below. Our conclusion is that the models originally studied are artificial in that
they do not incorporate generic effects that turn out to be relevant for the dynamics. Studies of our more realistic model should yield insights more generally applicable to real physical situations.

The LLS consists of a set of integer heights, \( h_i \), defined on a 1D lattice \( 1 \leq i \leq L \). The avalanche dynamics are extremely simple: whenever the slope \( z_i \equiv h_i - h_{i+1} \) exceeds 2, two units of height, called “grains”, are transferred from \( i \) to \( i + 1 \). Site \( L + 1 \) is an open boundary; grains transferred to it disappear and \( h_{L+1} \) is always 0. The LLS is driven by the slow addition of individual grains at randomly chosen sites, where “slow” means that after the addition of a grain, any avalanche that occurs is run to completion before the next grain is added. When we refer to a “state” below, we mean the stable configuration reached after a driving event. A variation of the LLS, called the “unlimited local sandpile” [2] (ULS) is the same except that when \( z_i > 2 \) the number of grains transferred is \( z_i - 1 \).

Our first observation is that the driving mechanisms used in the experiments differ dramatically, so the effects of different types of driving on the models must be explored. In particular, though the heap experiments do involve the addition of grains more or less as specified in the model, the rotating cylinder experiments involve uniform increases in the slope of the surface of the sand. A better representation of the latter is to drive the LLS or ULS by adding a unit of slope to a randomly chosen site rather than a unit of height. We will call models driven in these different ways “slope driven” and “height driven”.

The height driven LLS and ULS are known to evolve to self-organized critical states. [2] Though both are quite subtle in detail, exhibiting multifractal scaling properties, \( P(s) \) for finite size systems clearly decays exponentially for \( s > cL \), where \( c \) is a constant of order unity; the largest avalanche sizes observed in both models are of order \( L \).

The slope driven LLS turns out to be trivial. One can easily see that it will eventually reach a state in which \( z_i > 0 \) for all \( i \) and that all subsequent states will have the same property. If slope is then added to a site with \( z_i = 1 \) there is no avalanche. If slope is added where \( z_i = 2 \), the state reached will be identical to the original except that \( z_i \to 1 \) and the number of grains falling of the pile during the avalanche is just \( 2i \). Thus \( P(s) \) is independent
of $s$ for even $s < 2L$. As in the height driven LLS, avalanches bigger than $2L$ can never occur and relaxation oscillations are impossible.

The slope driven ULS is more interesting. As in the LLS, once a state with all $z_i > 0$ is reached, the system will always remain in such a state. For a state $z_1, z_2, z_3, \ldots, z_i = 2, \ldots, z_L$, making an addition of slope at $i$ results in the state $z_i+1, z_L, 1, 1, \ldots, 1$. The size of the avalanche in any particular case is easily calculated and for the case $z_j = 2$ for all $j$ one finds $s = i(2L - i + 1)/2$. Thus it is possible to have an avalanche size of order $L^2$, unlike in the LLS. Nevertheless, numerical evidence indicates that $P(s)$ decays exponentially for $s > cL^{1.45}$. (See Figure 1. [5]) This shows that the scaling properties of the system can be strongly affected by the form of the driving, but also that the slope driven ULS does not undergo relaxation oscillations.

We now introduce a new slope driven model for which $P(s)$ is dominated by a peak at $s \sim L^2$. The model is a variation of the LLS, which will be called the “dynamic, limited, local sandpile” (DLLS), designed to allow large avalanches to form by changing the local stability criteria for a site that has already participated in a given avalanche. In order to state the rules of the DLLS, it is useful to rephrase the LLS rules as follows: When a grain is added at site $i$ making $z_i = 3$, an avalanche is started. Let $j < i$ be the closest trap to the left of $i$ and $k > i$ be the closest trap to the right of $i$, where a trap is defined as a site with $z \leq 0$. The net result of the avalanche is that two grains per site are transferred from the sites $i$ through $j+1$ to the sites $k - (i - j) + 1$ through $k$. We say that a cluster of grains is destabilized, slides down the pile until its front reaches the first trap, and stops there.

The modification introduced in the DLLS is that sites with $z = 2$ are destabilized when the back of a cluster slides over them, and the 0 produced when two grains are removed from such a site is not counted as a trap for the duration of the avalanche. More precisely: an initial destabilized cluster is formed exactly as in the LLS; it then advances down the pile (to the right) one site at a time; if the last two grains in the cluster are moved from a site $a$ with $z_a = 2$, two grains are taken from $a$ and they, along with two grains per site between $a$ and the closest trap to the left, are added to the back of the sliding cluster; though $z_a$
is now 0, it does not act as a trap until the current avalanche is completed; the avalanche stops when the front reaches a trap. A site with \( z = 0 \) that is not acting as a trap will be called a “proto-trap” since it will become a trap upon the completion of the avalanche.

The avalanche size distribution for the slope driven DLLS is shown in Figure 2 for various system sizes. For large \( s \), the data are well-fit by the scaling form \( P(s) \sim L^{-\beta} f(s/L^\nu) \) with \( \nu = 2 \) and \( \beta = 3 \). It is clear that the position of the peak at \( s/L^2 \approx 0.25 \) does not shift, though its shape changes near the tip and the trailing edge is sharper for larger \( L \) (see inset). This implies that individual avalanches remove a finite fraction of the mass of the pile and that the average slope of the slope driven DLLS undergoes relaxation oscillations.

A plot of the total mass of the pile as a function of the number of drops is shown in Figure 3a. A feature of interest is the correlation between the size, \( s_n \), of the \( n^{th} \) avalanche and the time, \( t_n \), (measured in numbers of driving events, or “drops”) between it and the preceding avalanche. Figure 3(b) shows a contour plot of the probability distribution \( P(s_n, t_n) \), which clearly indicates a correlation. In contrast, a plot of \( P(s_n, t_{n+1}) \) shows a much weaker correlation; the width of the distribution for fixed \( s \) or \( t \) is significantly larger in (c) than in (b). This indicates that the large avalanches reset the average slope of the pile to some fixed reference value, but are triggered at a fairly broad distribution of average slopes. In the language used to describe fault dynamics, the avalanches in this model are “size-predictable” rather than “time-predictable”; i.e., the time elapsed since the last avalanche is useful information in predicting the size of the next one, but knowing the size of the last one does not help in predicting the time of the next one.

We note that the relaxation oscillations are robust with respect to the introduction of probabilistic rules. We have considered a modification in which each time a site with \( z = 2 \) would be destabilized in the DLLS, the destabilization occurs only with probability 1/2 and, furthermore, the probability that the resulting \( z = 0 \) site will be a proto-trap rather than a real trap is 1/2. The modified model also shows clear relaxation oscillations for system sizes up to 1000.

The height driven DLLS behaves quite differently, exhibiting self-organized criticality.
rather than relaxation oscillations. Figure 4 shows $P(s)$ for the height driven model for various system sizes. A reasonable data collapse is achieved by the same scaling form used above but with $\nu = 1.35$ and $\beta = 1.70$. In this case the average slope of the pile converges to a stationary value in the infinite system limit. Examination of the stationary slope profile reveals, however, that the mechanism of selection of the critical state is not the singular diffusion mechanism discussed by Carlson et al. The profile does not exhibit a power-law approach to a critical value at the open end of the system. (Preliminary results show a power-law convergence from above to some reference slope as the closed boundary is approached. Details will be discussed elsewhere.)

While it is too early to draw firm conclusions about the relevance of the DLLS to real sandpiles, there is a highly suggestive correspondence between the qualitative behavior observed in the model and the experiments of Held et al. and Jaeger et al. In the former, where individual grains were added at random positions spanning the surface of a heap, critical scaling was observed, as in the height driven DLLS. In the latter, a half-filled cylinder was slowly rotated so that the slope of the sand surface slowly increased everywhere. In the slope driven DLLS, this process is modeled by a discrete process in which the sequence in which the local slopes cross threshold values is determined randomly, but the basic fact that no local slope is decreased by the driving mechanism is faithfully represented. Both the experiment and the model show relaxation oscillations.

Furthermore, in experiments done by Held et al. on larger heaps in which the random positions where grains were added spanned only the top half of the pile, a power law distribution was not observed, but rather a distribution dominated by large avalanches. Remarkably, this also occurs in the height driven DLLS, as evidenced by the distribution shown in Figure 5. (The manner in which this distribution scales with increasing $L$ depends on what is held constant, the fraction of the pile covered by drops, the distance from the top that is covered, or the distance from the bottom that is uncovered.)

In an effort to determine which features of the DLLS are essential for generating relaxation oscillations, we have investigated one further model, the “dynamic, unlimited, local
sandpile” (DULS), which is a modification of the ULS in which sites that topple have a reduced toppling threshold for two lattice updates during an avalanche. (Details of the rules will be published elsewhere.) We have found that this dynamic effect does not produce relaxation oscillations in the slope driven model, though it does increase the exponent $\nu$ significantly. (Preliminary results give $\nu \simeq 1.75$.)

We conjecture that the ingredients crucial for generating relaxation oscillations are: (1) rules that allow large avalanches, which required the dynamic modification of the LLS, but would not require modification of the ULS; (2) slope driving; and (3) rules that generate effective traps in the wake of a large avalanche, an obvious feature of the DLLS that is not present in the ULS or DULS.

The coincidence between the qualitative behaviors observed in the DLLS and real sandpiles suggests that models of this type may yield useful insights into avalanche dynamics in generic physical systems. The question remains, however, as to whether the effects included in the DLLS are somehow artificial. In our view, the primary issue at this point is the temporal nonlocality implicit in the rules for proto-traps. As currently defined, the model requires that a proto-trap not turn into a real trap until the entire avalanche is completed. This may not be unreasonable, given the required separation in time scales between the avalanche dynamics and the driving rate. Nevertheless, it is important to find out whether a system governed by strictly local rules can produce the same behavior. In any case, investigation of the DLLS is significant in that it highlights certain features that can affect the qualitative macroscopic behavior of a broad class of slowly driven, dissipative systems.
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[5] We fit \( P(s) \) to the finite-size scaling form \( L^{-\beta} f(s/L^\nu) \). \( P(s) \) is the probability of an avalanche of size \( s \) occurring per driving event. With this normalization the sum rule given in Eq. (2.15) of Ref. [2] reads \( \int s P(s) ds = 1 \) (height driving) or \( L/2 \) (slope driving), implying \( \beta = 2\nu \) or \( 2\nu - 1 \), respectively.

[6] For the DLLS, the avalanche size \( s \) is defined as half the number of grains falling off the pile because the grains always fall off in pairs.

[7] In Figure 3(b), the nonzero density to the left of the main peak is due to the avalanches in the smallest bin in Figure 2. An analogous plot in which only avalanche sizes greater than \( L^2/500 \) and the elapsed times between such events are considered consists of a clean single peak along the diagonal.

[8] See, for example, C. Scholz, The Mechanics of Earthquakes and Faulting (Cambridge University Press, 1990), Chap. 5.

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FIGURES

FIG. 1. Avalanche size distributions for the slope driven ULS. Data are shown for $L = 250$, 500, 1000, and 2000. Each curve represents $\sim 5 \times 10^6$ avalanches and was obtained from $L \times 10^4$ driving events. Each point plotted represents an average over a fixed binwidth selected so that the curve contains $\sim 150$ points.

FIG. 2. Avalanche size distributions for the slope driven DLLS. Data are shown for $L = 250$, 500, 1000, and 2000. Each curve represents $\sim 3 \times 10^5$ avalanches and was obtained from $2L \times 10^5$ driving events. Each point plotted represents an average over a fixed binwidth selected so that the curve contains $\sim 150$ points.

FIG. 3. (a) Total mass of the pile for the slope driven DLLS. (b) $P(s_n, t_n)$ (see text). (c) $P(s_n, t_{n+1})$. Each contour line represents a change in $P$ by a factor of 3. Values of $s$ are binned as in Figure 2 and data was smoothed by averaging over $3 \times 3$ neighborhood.

FIG. 4. Avalanche size distributions for the height driven DLLS. Data are shown for $L = 250$, 500, 1000, and 2000. Each curve represents $\sim 1.2 \times 10^6$ avalanches and was obtained from $L \times 10^6$ driving events. Each point plotted represents an average over a fixed binwidth selected so that the curve contains $\sim 150$ points.

FIG. 5. Avalanche size distributions for the height driven DLLS with drops performed on top half of pile only. $L = 500$. Note the expanded horizontal scale, which includes all nonzero data points except $s = 0$. Each point plotted represents an average over a fixed binwidth selected so that the curve contains $\sim 150$ points.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5