GENERALIZED VARIATIONAL CALCULUS IN TERMS OF MULTI-PARAMETERS INVOLVING ATANGANA-BALEANU’S DERIVATIVES AND APPLICATION

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Abstract. In this paper, the generalized variational calculus in terms of multi-parameters involving Atangana-Baleanu’s Derivatives are discussed. We consider the Hilfers generalized fractional derivative that in sense Atangana-Baleanu derivatives. We develop integration by parts formulas for the generalized fractional derivatives which are key to developing fractional variational calculus. It is shown that many derivatives used recently and their variational formulations can be obtained by setting different parameters to different values. The fractional Euler-Lagrange equations of fractional Lagrangians for constrained systems contains a fractional Hilfer-Atangana-Baleanu’s derivatives with multi parameters are investigated. We also define fractional generalized momenta and provide fractional Hamiltonian formulations in terms of the new generalized derivatives. An example is presented to show applications of the formulations presented here. Some possible extensions of this research are also discussed. We present a general formulation and a solution scheme for a class of Fractional Optimal Control Problems (FOCPs) for those systems. The performance index of a FOCP is considered as a function of both the state and the control variables, and the dynamic constraints are expressed by a set of FDEs. The calculus of variations, the Lagrange multiplier, and the formula for fractional integration by parts are used to obtain Euler-Lagrange equations for the FOCP.

1. Introduction. Fractional calculus represents a generalization of ordinary differentiation and integration to arbitrary order. During the last decades the fractional calculus started to be used in various fields, e.g. physics, engineering, biology and many important results were obtained [1]-[6],[25], [31].

There exists many definitions of a fractional derivative. A commonly known fractional derivatives are the classical Riemann-Liouville and Caputo derivatives. Fractional derivatives and integrals of these Riemann-Liouville and Caputo types have a vast number of applications across many fields of science and engineering. For example, they can be used to model controllability, viscoelastic flows, chaotic
systems, Stokes problems, thermo-elasticity, several vibration and diffusion processes, bioengineering problems, and many other complex phenomena (see e.g. [3]-[6] [12]-[20], [25], [27], [31]-[36] and references therein).

Fractional optimal control problems involving that classical derivatives have attracted several authors in the last two decades, and many techniques have been developed for solving such problems involving these classical fractional derivatives. Agrawal [3, 4] presented a general formulation and proposed a numerical method to solve such problems. In that paper, the fractional derivative was defined in the Riemann-Liouville sense and the formulation was obtained by means of fractional variation principle and the Lagrange multiplier technique. Using the same technique, Frederico et al. [27] obtained a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. In [39, 40], Mophu studied the fractional optimal control for the diffusion equation involving the classical Riemann-Liouville derivatives. In [41], Ozdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the classical Riemann-Liouville sense.

The FC brings new features in describing complex behaviors of the real-world phenomena with memory effects. However, the description of systems with memory effect is still a big challenge for researchers, since the classic type of FDs with singular kernel cannot characterize always properly the nonlocal dynamics. Hence, it seems there is a need of new FDs with nonsingular kernel to better describe the non-locality of complex systems. One of the best candidates among existing kernels is the one based on Mittag-Leffler function (see e.g. [1, 2, 9, 10, 11, 19, 21, 23, 24, 28, 29, 30, 37] and references therein).

Recently, Caputo and Fabrizio [22] have proposed a new definition of fractional derivatives:

$$CFD_{c+}^{\alpha}f(x) = \frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} \int_{c}^{x} \exp \left[ \frac{-\alpha}{1 - \alpha} (x - y) \right] f(y) dy,$$

valid for $0 < \alpha < 1$, with $B(\alpha)$ being a normalisation function satisfying $B(0) = B(1) = 1$. The basic challenge they were addressing was whether it is possible to construct another type of fractional operator which has nonsingular kernel and which can better describe in some cases the dynamics of non-local phenomena. The Caputo–Fabrizio definition has already found applications in areas such as diffusion modelling [34] and mass-spring-damper systems [8].

However, some issues were pointed out against both derivatives, including the one in Caputo sense and the one in Riemann-Liouville sense. As Sheikh [44] pointed out, the CF fractional derivative as the kernel in integral was non-singular but was still nonlocal. Some researchers also concluded that the operator was not a derivative with fractional order but a filter with fractional parameter. The fractional parameter can then be viewed as a filter regulator.

Atangana and Baleanu introduced a new operator with fractional order based upon the generalized Mittag-Leffler function [10]. Their operators have all the benefits of that of the CF derivative in addition to the kernel being nonlocal and non-singular. The non-locality of the kernel gives better description of the memory within the structure with different scale.

Here we shall mostly be considering a more recently developed definition for fractional differintegrals, due to [10]. This new type of calculus addresses the same underlying challenge as that of Caputo and Fabrizio, but it uses a kernel which is
non-local as well as non-singular, namely the Mittag-Leffler function:

\[
\begin{align*}
ABRD_{a+}^{\alpha} f(t) &= \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_{a}^{t} f(x) E_{\alpha} \left( \frac{-\alpha}{1-\alpha} (t-x)^{\alpha} \right) dx, \\
ABCD_{a+}^{\alpha} f(t) &= \frac{B(\alpha)}{1-\alpha} \int_{a}^{t} f'(x) E_{\alpha} \left( \frac{-\alpha}{1-\alpha} (t-x)^{\alpha} \right) dx.
\end{align*}
\]

In this way we are able to describe a different type of dynamics of non-local complex systems. In fact the classical fractional calculus and the one corresponding to the Mittag-Leffler nonsingular kernel complement each other in the attempt to better describe the hidden aspects of non-local dynamical systems. Fractional calculus based on the non-singular Mittag-Leffler kernel is more easily used from the numerical viewpoint, and this has been studied for example in [24].

We note that the Mittag-Leffler function is already known to be highly useful in fractional calculus [38]. Applications of the new AB formula have been explored in fields as diverse as chaos theory [7], heat transfer [10], and variational problems [2]. Furthermore, it is natural to address the same questions about the fractional integrator and applications of these new operators in the theory of control and related fractional variational Euler-Lagrange and Hamilton equations (see [2, 24, 28]). Besides, we expect to obtain some new terms in all generalized formulae from the classical fractional calculus, and this aspect will be important for the related applications.

Some basic properties of the new AB differintegrals have already been proven in several recent papers: for example, the original paper [10] established the formulae for Laplace transforms of AB differintegrals and some Lipschitz-type inequalities; the paper [2] considered integration by parts identities and Euler-Lagrange equations; and the paper [24] established, using Laplace transforms, analogues of the Newton–Leibniz formula for the integral of a derivative. However, much of the ground-level theory of this new model of fractional calculus has not yet been fully developed, and this paper aims to add to this basic theory by establishing new fundamental results in the field.

Our main results is that to compare the results of fractional Euler-Lagrange equations for the classical Hilfer Riemann–Liouville and Caputo fractional derivatives which stated in [32, 33] corresponding to Hilfer-Atangana-Baleanu’s fractional derivatives which defined in [10]. In this study, by using the Hilfer-Atangana-Baleanu’s derivative we propose to generalize the notion of equivalent Lagrangian for the fractional case. For a given classical Lagrangian there are several proposed methods to obtain the fractional Euler-Lagrange equations and the corresponding Hamiltonians. However, the fractional dynamics depends on the fractional derivatives used to construct the Lagrangian to start with, therefore the existence of several options can be used to treat a specific physical system. In this respect, application of the Hilfer-Atangana-Baleanu’s derivative to the fractional dynamics will bring new opportunities in studying the constrained systems mainly because the Atangana-Baleanu’s derivative contains both the left and the right derivatives. In addition, the fractional derivative of a function is given by a definite integral, thus depends on the values of the function over the entire interval. Therefore, the fractional derivatives are suitable to model systems with long range interactions in space and/or time (memory) and process with many scales of space and/or time involved.
The plan of this paper is as follows: In section 2 we collect notations, definitions involving the Atangana-Baleanu fractional time derivative. In section 3 we state some properties of AB derivatives and we give the integration by parts. Section 4 two parameter Hilfer fractional derivatives for Atangana-Baleanu fractional derivatives are defined and we state the integration by parts for these derivatives. In section 5, fractional variational principles within the new Hilfer-Atangana-Baleanu derivatives are defined and some examples are investigated in detail. In Section 6, we calculate the fractional canonical momenta and we generalize the fractional Lagrangian to n state equations in section 7. In section 8, the constrained system within Hilfer-Atangana-Baleanu’s derivatives are also discussed and some examples are investigated in detail. In Section 9, Fractional Optimal Control Problem FOCP involving Hilfer-Atangana-Baleanu’s derivatives are presented. Section 10 is dedicated to our conclusions.

2. Preliminaries. This section presents the basic definitions and properties of the new Atangana-Baleanu’s derivatives in the Caputo and Riemann–Liouville senses.

Definition 2.1. For a given function \( x(t) \in H^1(a,b), b > a \), \( \alpha \in (0,1) \), the left Atangana-Baleanu fractional derivative (AB derivative) of \( x(t) \) of order \( \alpha \) in Caputo sense \( ABCD \) with base point \( a \) is defined at a point \( t \in (a,b) \) by

\[
ABC_{a}D_{t}^{\alpha}x(t) = \frac{B(\alpha)}{1-\alpha} \int_{a}^{t} \frac{d}{ds}x(s)E_{\alpha}[\gamma(t-s)^{\alpha}]ds.
\]  

(1)

LHS (left ABCD) is a normalisation function satisfying

\[
B(0) = B(1) = 1,
\]

and \( B(\alpha) \) being a normalisation function satisfying

\[
B(\alpha) = (1 - \alpha) + \frac{\alpha}{\Gamma(\alpha)},
\]

(2)

where \( B(0) = B(1) = 1 \), \( \Gamma(\alpha) \) denotes thee Euler’s gamma function defined as

\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1}e^{-t}dt, \quad \Re(z) > 0.
\]

The Mittag-Leffler function \( E_{\alpha,\beta}(z) \) is a two-parameter family of entire functions of \( z \) of order \( \alpha^{-1} \). Furthermore we recall the following Lemma.

Lemma 2.2. Let \( \alpha, \beta \in \mathbb{C} \) such that \( \Re(\alpha) > 0 \) and \( \Re(\beta) > 0 \). Then we have that

\[
\left( \frac{d}{dz} \right) E_{\alpha,\beta}(z) = \frac{1}{\alpha} \left[ (1 + \alpha - \beta)E_{\alpha,\beta}(z) + E_{\alpha,\beta-1}(z) \right], \quad z \in \mathbb{C}.
\]

(3)

The left Atangana-Baleanu fractional derivative in Riemann-Liouville sense defined with:

\[
ABR_{a}D_{t}^{\alpha}x(t) = \frac{B(\alpha)}{1-\alpha} \int_{a}^{t} x(s)E_{\alpha}[\gamma(t-s)^{\alpha}]ds.
\]

(4)

For \( \alpha = 1 \) in (5) we consider the usual classical derivative \( \partial_t \).
The associated left AB fractional integral is also defined as
\[ A^B_I \alpha x(t) = \int_a^t x(s)(t-s)^{\alpha-1} ds, \]  
(6)
\[ = \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_a^t x(s)(t-s)^{\alpha-1} ds, \]  
(6)
where
\[ a^I \alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t x(s)(t-s)^{\alpha-1} ds, \]  
(7)
is the classical left Riemann-Liouville integral.

Notice that if \( \alpha = 0 \) in (6) we recover the initial function and if \( \alpha = 1 \) in (6) we consider the usual ordinary integral. So from the definition on [10] we recall the following definition

**Definition 2.3.** For a given function \( x(t) \in H^1(a,b), b > t > a \), the right Atangana-Baleanu fractional derivative of \( x(t) \) of order \( \alpha \) in Caputo sense with base point \( b \) is defined at a point \( t \in (a,b) \) by
\[ (ABC_t D^\alpha_b x)(t) = \frac{1}{1-\alpha} \int_t^b x(s)(s-t)^{\alpha-1} ds, \]  
(8)
and in Riemann-Liouville sense with:
\[ (ABR_t D^\alpha_b x)(t) = \frac{d}{dt} \frac{1}{1-\alpha} \int_t^b x(s)E_\alpha[-\gamma(s-t)^{\alpha}] ds, \]  
(9)
The associated right AB fractional integral is also defined as
\[ (AB^\alpha_i t D^\alpha_b x)(t) = \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_t^b x(s)(s-t)^{\alpha-1} ds, \]  
(10)
where
\[ i^I \alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b x(s)(s-t)^{\alpha-1} ds, \]  
(11)
is the classical right Riemann-Liouville integral.

3. Some properties of AB derivatives and integration by parts. In this section we state some important lemmas for properties of AB derivatives and integration by parts which can found in [19]. Some recent results and properties concerning this operator can be found [10] and papers therein.

**Lemma 3.1.** [19]. The left AB Caputo fractional derivatives and the left AB Riemann-Liouville derivative are related by the identity:
\[ (ABC_t D^\alpha_a x)(t) = (ABR_t D^\alpha_a x)(t) - \frac{B(\alpha)}{1-\alpha} x(a)E_\alpha[-\gamma(t)^{\alpha}]. \]  
(12)
The right AB Caputo fractional derivatives and the right AB Riemann-Liouville derivative are related by the identity:
\[ (ABC_t D^\alpha b x)(t) = (ABR_t D^\alpha b x)(t) - \frac{B(\alpha)}{1-\alpha} x(b)E_\alpha[-\gamma(b-t)^{\alpha}]. \]  
(13)
There are useful relations between the left and right AB FDs in the Riemann-Liouville and Caputo senses and the associated AB fractional integrals as the following formulas state.

\[
\begin{align*}
\mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) &= \mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) = x(t) \quad (14) \\
\mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (x) &= \mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) = x(t) - x(a) \quad (15) \\
\mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (b) &= \mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) = x(t) - x(b) \quad (16) \\
\mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) &= -\mathcal{A}_a^\beta D_0^\alpha x(t) \quad (17)
\end{align*}
\]

As a consequence, the backwards in time with the fractional-time derivative with nonsingular Mittag-Leffler kernel at the based point \( T \) is equivalently written as a forward in time operator with the fractional-time derivative with nonsingular Mittag-Leffler kernel – \( \mathcal{A}_a^\beta D_0^\alpha \).

Lemma 3.2. [19]. The AB integral operators and ABR differential operators form a commutative family of differintegral operators:

\[
\begin{align*}
\mathcal{A}_a^\beta D_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) &= \mathcal{A}_a^\beta D_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x) (t) = \mathcal{A}_a^\beta (\mathcal{A}_a^\beta D_0^\alpha x(t)) \\
\mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta I_0^\alpha x(t)) &= \mathcal{A}_a^\beta I_0^\alpha (\mathcal{A}_a^\beta I_0^\alpha x(t)) = \mathcal{A}_a^\beta (\mathcal{A}_a^\beta I_0^\alpha x(t)) \\
\mathcal{A}_a^\beta D_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x(t)) &= \mathcal{A}_a^\beta D_0^\alpha (\mathcal{A}_a^\beta D_0^\alpha x(t)) = \mathcal{A}_a^\beta (\mathcal{A}_a^\beta D_0^\alpha x(t))
\end{align*}
\]

for \( \alpha, \beta \in (0, 1) \) and \( a, x \) satisfying the conditions from Definition (2.1).

Lemma 3.3. The semigroup property [19]. The semigroup property for AB fractional differintegrals is not satisfied in general. For example, taking \( B(\alpha) = 1 \) we get

\[
\mathcal{A}_a^{1/3} I_{0+}^{2/3} (t) = \left( \frac{1}{3} + \frac{2}{3} \right) I_{0+}^{2/3} (t) = \frac{1}{3} t + \frac{2}{3} \frac{\Gamma(7/3)}{\Gamma(8/3)} t^{5/3}
\]

and yet

\[
\begin{align*}
\mathcal{A}_a^{1/3} I_{0+}^{1/3} (t) &= \mathcal{A}_a^{1/3} I_{0+}^{1/3} (t) = \frac{2}{3} \left( \frac{1}{3} t + \frac{1}{3} \frac{\Gamma(7/3)}{\Gamma(8/3)} t^{4/3} \right) + \frac{2}{3} \left( \frac{1}{3} t + \frac{1}{3} \frac{\Gamma(7/3)}{\Gamma(8/3)} t^{4/3} \right) \frac{\Gamma(7/3)}{\Gamma(8/3)} t^{5/3} \\
&= \frac{\Gamma(7/3)}{\Gamma(8/3)} t^{5/3}
\end{align*}
\]

– two entirely different expressions.

Can we find conditions for when the semigroup property does hold?

Firstly, note that it will be sufficient to consider fractional integrals only. Any function which satisfies the semigroup property for ABR fractional derivatives generates one which satisfies it for AB fractional integrals, and vice versa. This is because

\[
\mathcal{A}_a^{\alpha} I_{0+}^{\beta} f(t) = g(t) = \mathcal{A}_a^{\alpha+\beta} I_{0+}^{\beta} f(t)
\]

is exactly equivalent to

\[
\mathcal{A}_a^{\beta} D_{0+}^{\alpha} g(t) = f(t) = \mathcal{A}_a^{\alpha+\beta} D_{0+}^{\alpha} g(t).
\]

This is good to know, because the definition of AB fractional integrals is much simpler and easier to work with than that of ABR fractional derivatives.

Next we state the following proposition which gives the integration by parts.
Lemma 3.4. (Integration by parts) (see [1]). Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha(p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). Then for any $\phi(x) \in L^p(a, b)$, $\psi(x) \in L^q(a, b)$, we have

$$\int_a^b \phi(x) AB a I_b^\alpha \psi(x) dx = \int_a^b \psi(x) AB t I_b^\alpha \phi(x) dx$$

$$\int_a^b \phi(x) AB h^a \psi(x) dx = \int_a^b \psi(x) AB a t^\alpha \phi(x) dx$$

if $\phi(x) \in AB h^a (L^p)$ and $\psi(x) \in AB a t^\alpha (L^q)$, then

$$\int_a^b \phi(x) AB t D_t^\alpha \psi(x) dx = \int_a^b \psi(x) AB t D_t^\alpha \phi(x) dx$$

$$\int_a^b \phi(x) AB h^a D_t^\alpha \psi(x) dx = \int_a^b \psi(x) AB a D_t^\alpha \phi(x) dx$$

$$\int_a^b \phi(x) AB h^a D_t^\alpha \psi(x) dx = \int_a^b \psi(x) AB h^a D_t^\alpha \phi(x) dx$$

where the left generalized fractional integral operator

$$E_{\gamma, \mu, \omega, a}^\alpha x(t) = \int_a^t (t - \tau)^{\mu - 1} E_{\gamma, \mu}^\alpha [\omega(t - \tau)\gamma] x(\tau) d\tau, \ t > a,$$

and the right generalized fractional integral operator

$$E_{\gamma, \mu, \omega, b}^\alpha x(t) = \int_t^b (\tau - t)^{\mu - 1} E_{\gamma, \mu}^\alpha [\omega(\tau - t)\gamma] x(\tau) d\tau, \ t < b.$$

4. Two parameter Hilfer fractional derivatives for Atangana-Baleanu fractional derivatives. The left/forward and the right/backward two parameter fractional derivatives (P2FDs) are defined as, Left/forward two parameter fractional derivatives [32, 33].

$$(ABH a t^\alpha D_t^{\alpha, \beta} y)(t) = (AB a t,(1-\beta)(n-\alpha)) (D^n A B a t^\beta (n-\alpha) y)(t)$$

Right/backward two parameter fractional derivatives

$$(ABH h^a D_t^{\alpha, \beta} y)(t) = (AB h^a, (1-\beta)(n-\alpha)) (-D^n A B h^\beta (n-\alpha) y)(t)$$

Here $\alpha, n-1 < \alpha < n$, is the order of the fractional derivative, and $\beta, (0 \leq \beta \leq 1)$ is a parameter. These derivatives for the usual Riemann-Liouville derivatives were first introduced by Hilfer, here we defined them for AB derivatives, and therefore, we call them the left/forward and the right/backward AB Hilfer fractional derivatives.

Although, $\alpha$ could be any positive real number, we shall restrict ourselves to $0 < \alpha < 1$. The left/forward and the right/backward Hilfer fractional derivatives satisfy the following properties.

For $\beta = 0, n = 1$,

$$(ABH a t^\alpha D_t^{0, \alpha} y)(t) = (AB a t,(1-\alpha) D y)(t) = (ABC a t^\alpha y)(t)$$

$$(ABH h^a D_t^{0, \alpha} y)(t) = (AB h^a, (1-\alpha)(-D) y)(t) = (ABC h^a y)(t)$$
For $\beta = 1, n = 1$,

$$
(ABH_D^\alpha I_t^{1-\alpha}) (y)(t) = D (ABR_a I_t^{1-\alpha}) (y)(t) = (ABR_a D_t^\alpha y)(t)
$$

(29)

$$
(ABH_B^\alpha I_b^{1-\alpha}) (y)(t) = (-D) (ABH_B^\alpha I_b^{1-\alpha}) (y)(t) = (ABR_B^\alpha y)(t)
$$

(30)

Hence, the Hilfer fractional derivative $(ABH_D^\alpha y)(t)$ interpolates, in some sense, between the AB derivative in Riemann-Liouville sense $(ABR_D^\alpha y)(t)$ and the AB derivative in the Caputo sense $(ABC_D^\alpha y)(t)$, and $\beta$ is the interpolation parameter.

The following theorems can now be stated.

**Theorem 4.1.** If $y(a) = 0$, then the three left fractional derivatives $(ABH_D^\alpha y)(t)$, $(ABR_D^\alpha y)(t)$, $(ABC_D^\alpha y)(t)$ coincide, i.e.

$$
(ABH_D^\alpha y)(t) = (ABR_D^\alpha y)(t) = (ABC_D^\alpha y)(t)
$$

(31)

Similarly, if $y(b) = 0$, then the three right fractional derivatives $(ABH_D^\alpha y)(t)$, $(ABR_D^\alpha y)(t)$, $(ABC_D^\alpha y)(t)$ coincide, i.e.

$$
(ABH_D^\alpha y)(t) = (ABR_D^\alpha y)(t) = (ABC_D^\alpha y)(t)
$$

(32)

*Proof.* This follows by taking the relationship between the AB derivatives in Riemann-Liouville and Caputo senses given by (12), (13), and using the semi-group property of the fractional integrals Lemma (3.3).

Next we state the following proposition which gives the integration by parts [1].

**Theorem 4.2.** The Hilfer's two-parameter fractional derivative operators $(ABH_D^\alpha y)(t)$ and $(ABH_D^\alpha y)(t)$ satisfy the following integration by parts formula,

$$
\int_a^b f(t) (ABH_D^\alpha y)(t) dt = \int_a^b g(t) (ABH_D^\alpha y)(t) dt + \left( ARI_b^{(1-\beta)(1-\alpha)} (f) (t) \right)_a^b
$$

(33)

*Proof.* This follows by using Eq. (23), then ordinary integration by parts, and then again Eq. (23).

$$
\int_a^b f(t) (ABH_D^\alpha y)(t) dt = \int_a^b f(t) \left( ABH_D^\alpha y(t) \right) dt = \int_a^b g(t) \left( ABH_D^\alpha y(t) \right) dt + \left( ARI_b^{(1-\beta)(1-\alpha)} (f)(t) \right)_a^b
$$

(34)

In Eq. (33), we require that $f \in L^p(a,b)$ and $g \in L^q(a,b)$ where $1/p < (1 - \beta)(1 - \alpha)$ and $1/q < \beta(1 - \alpha)$, which simply state that the fraction integrals on the RHS of Eq. (33) are finite.

It could be verified that for $\beta = 0$ and 1, Eq. (33) reduces to Eqs. (20) and (21), respectively. Further, if $f$ and $g$ are bounded then Eq. (33) reduces to (22). Thus,
Eq. (33) is more general than Eq. (20) to (21). Eq. (33) will play a key role in the fractional variational calculus to follow.

5. Fractional variational principles within fractional Atangana-Baleanu’s derivatives. The fractional Euler-Lagrange and fractional Hamilton equations within Atangana-Baleanu’s derivatives are briefly presented in the following.

Theorem 5.1. Let $J[x]$ be a functional of the form

$$J[x] = \int_{a}^{b} L(t, x, ABH_{a}D_{t}^{\alpha,\beta}x(t))dt$$

(34)

defined by the set of functions which have continuous Atangana-Baleanu fractional derivative in the Caputo sense on the set of order $\alpha$ in $[a, b]$ and which satisfy the boundary conditions $x(a) = x_{a}$ and $x(b) = x_{b}$. Then a necessary condition for $J[x]$ to have a maximum for given function $x(t)$ is that

$$x(t)$$

must satisfy the following Euler-Lagrange equation

$$\frac{\partial L}{\partial x} + ABH_{t}D_{b}^{\alpha,\beta}(\frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))}) = 0$$

(35)

Proof. To obtain the necessary conditions for the extremum, assume that $x^{*}(t)$ is the desired function. Let $\varepsilon \in R$, and define a family of curves

$$x(t) = x^{*}(t) + \varepsilon \eta(t)$$

(36)

where $\eta(t)$ is an arbitrary curve except that it satisfies the boundary conditions, i.e. we require that

$$\eta(a) = \eta(b) = 0.$$  

(37)

To obtain the Euler-Lagrange equation, we substitute equation (36) into equation (34), differentiate the resulting equation with respect to $\varepsilon$ and set the result to 0. This leads to the following condition for extremum:

$$\int_{a}^{b} \left[ \frac{\partial L}{\partial x} \eta(t) + \frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))} ABH_{a}D_{t}^{\alpha,\beta} \eta(t) \right] dt = 0,$$

(38)

using equation (33), equation (38) can be written as

$$\int_{a}^{b} \left[ \frac{\partial L}{\partial x} + ABH_{t}D_{b}^{\alpha,\beta}(\frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))}) \right] \eta dt$$

(39)

$$+ AR_{t}I_{b}^{(1-\beta)(1-\alpha)}(\frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))}) \cdot AR_{a}I_{t}^{\beta(1-\alpha)} \eta(t)|_{a}^{b}$$

we call $AR_{t}I_{b}^{(1-\beta)(1-\alpha)}(\frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))})$, $AR_{a}I_{t}^{\beta(1-\alpha)} \eta(t)|_{a}^{b}$ 0, the natural boundary conditions, since $\eta(t)$ is arbitrary, it follows from a well established result in calculus of variations that

$$\frac{\partial L}{\partial x} + ABH_{t}D_{b}^{\alpha,\beta}(\frac{\partial L}{\partial (ABH_{a}D_{t}^{\alpha,\beta}x(t))}) = 0$$

(40)

Equation (40) is the Generalized Euler-Lagrange Equation GELE for the Fractional Calculus Variation (FCV) problem defined in terms of the Hilfer derivatives in the sense of Atangana-Baleanu Fractional Derivatives ABFD. Note that the left and right Hilfer derivatives in the Atangana-Baleanu derivatives sense appear in the resulting differential equations.
Example 5.1. Let us consider the following fractional Lagrangian is given by:

\[ L = \frac{1}{2}(x + A^B H_a^\alpha D_t^\beta x)^2, \]  

then independent fractional Euler-Lagrange equation (40) is given by

\[ x + A^B H_t^\alpha (1-\beta)(A^B H_a^\alpha D_t^\alpha x) = 0 \]  

Example 5.2. We consider now a fractional Lagrangian of the oscillatory system

\[ L = \frac{1}{2}m(A^B H_a^\alpha D_t^\beta x)^2 - \frac{1}{2}kx^2, \]

where \( m \) the mass and \( k \) is constant. Then the fractional Euler-Lagrange equation is

\[ m A^B H_t^\alpha (1-\beta)(A^B H_a^\alpha D_t^\alpha x) - kx = 0 \]

This equation reduces to the equation of motion of the harmonic oscillator when \( \alpha \to 1, \beta = 0 \).

6. The fractional canonical momenta. For a given fractional Lagrangian \( L = L(x, A^B H_a^\alpha D_t^\beta x(t)) \) the fractional canonical momenta are defined as

\[ P = \frac{\partial L}{\partial (A^B H_a^\alpha D_t^\beta x(t))} \]

Therefore, we construct the corresponding fractional Hamiltonian as follows,

\[ H(x, P) = P A^B H_a^\alpha D_t^\beta x(t) - L. \]

Then we have

\[ dH = P d(A^B H_a^\alpha D_t^\beta x(t)) + A^B H_a^\alpha D_t^\beta x(t) dP - \frac{\partial L}{\partial x} dx - \frac{\partial L}{\partial t} dt - \frac{\partial L}{\partial (A^B H_a^\alpha D_t^\beta x(t))} d(A^B H_a^\alpha D_t^\beta x(t)) \]

This suggests that \( H \) is a function of \( t, x, P \) only. Therefore, we can write

\[ dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial P} dP \]

By using (40),(45) and (46) we obtain the fractional Hamilton’s equations

\[ A^B H_a^\alpha D_t^\beta x(t) = \frac{\partial H}{\partial P}, \quad A^B H_a^\alpha D_t^\beta x(t) = -\frac{\partial H}{\partial x}, \]

and

\[ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \]

Equations (49) represent two fractional deferential equations of order \( \alpha \) for the system which is equivalent to the system (40). Because of their similarity with the canonical Euler equations for integer order systems, we call equations (49) the fractional canonical system of Euler equations or simply the fractional canonical Euler equations.

Example 6.1. Function \( L \) in equation (34) can be thought of as a function containing both the left and the right Hilfer Atangana-Baleanu Fractional Derivatives HABFDs

\[ L = L(x, A^B H_a^\alpha D_t^\beta x(t), A^B H_t^\beta x(t)) \]
for which the GELE is given as

$$\frac{\partial L}{\partial x} + \frac{\partial L}{\partial t} D^{\alpha,1-\beta}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha,\beta}_t x(t)})} + \frac{\partial L}{\partial a} D^{\alpha,1-\beta}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha,\beta}_t x(t)})} = 0$$

(51)

Also function $L$ in equation (34) can be thought of as a function containing both the left and the right Caputo Fractional Derivatives CFDs

$$L = L(x, \frac{C}{a} D^{\alpha}_t x(t), \frac{C}{b} D^{\alpha}_b x(t))$$

for which the GELE is given as

$$\frac{\partial L}{\partial x} + \frac{\partial L}{\partial t} D^{\alpha}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha}_t x(t)})} + \frac{\partial L}{\partial a} D^{\alpha}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha}_t x(t)})} = 0$$

(52)

For $\alpha = 1$, the Euler–Lagrange equation is given as

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial x} = 0$$

(53)

Equations (40), (51) (and its equivalent in terms of ABRFD) and (52) are similar and they all contain both forward and backward derivatives. Note that $-d/dt$ is essentially a backward derivative. Thus, backward derivatives in equations (40) and (51) appear explicitly, whereas they appear in equation (52) in a disguise form.

7. Extension to several dependent variables. We now give some generalizations of Theorem 5.1. We can generalize in a straightforward manner to problems containing several unknown functions. We denote by $\mathcal{F}_n$ the set of all functions $x_1(t), x_2(t), ..., x_n(t)$ which have continuous left ABC fractional derivative of order $\alpha$ and right ABC fractional derivative of order $\beta$ for $t \in [a, b]$ and satisfy the conditions

$$x_i(a) = x_{ia}, x_i(b) = x_{ib}, i = 1, 2, ..., n.$$

The problem can be defined as follows: find the functions $x_1, x_2, ..., x_n$ from $\mathcal{F}_n$, for which the functional

$$J[x_1, x_2, ..., x_n] =$$

$$\int_a^b L[t, x_1(t), x_2(t), ..., x_n(t), \frac{\partial x_1}{\partial t}, ..., \frac{\partial x_n}{\partial t}, \frac{\partial x_1}{\partial \alpha} D^{\alpha}_t x_1(t), ..., \frac{\partial x_n}{\partial \alpha} D^{\alpha}_t x_n(t), \frac{\partial x_1}{\partial \beta} D^{\beta}_b x_1(t), ..., \frac{\partial x_n}{\partial \beta} D^{\beta}_b x_n(t)] dt$$

has an extremum, where $L(t, x_1, ..., x_n, u_1, ..., u_n, v_1, ..., v_n)$ is a function with continuous first and second partial derivatives with respect to all its arguments. A necessary condition for $J[x_1, x_2, ..., x_n]$ to admit an extremum is that $x_1(t), x_2(t), ..., x_n(t)$ satisfy Euler-Lagrange equations: for $i = 1, 2, ..., n,$

$$\frac{\partial L}{\partial x_i} + \frac{\partial L}{\partial t} D^{\alpha,1-\beta}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha,\beta}_t x_i(t)})} + \frac{\partial L}{\partial a} D^{\alpha,1-\beta}_b \frac{\partial L}{\partial (\frac{\partial L}{\partial D^{\alpha,\beta}_t x_i(t)})} = 0$$

(54)

Example 7.1. If we consider the system of two planar pendule, both of length $l$ and mass $m$, suspended a same distance apart on a horizontal line so that they moving in the same plane. The fractional form of the Lagrangian is given by:

$$L(t, x_1, x_2, \frac{\partial x_1}{\partial t}, \frac{\partial x_1}{\partial \alpha} D^{\alpha}_t x_1, \frac{\partial x_2}{\partial t}, \frac{\partial x_2}{\partial \alpha} D^{\alpha}_t x_2) = \frac{1}{2} m \left[ (\frac{\partial x_1}{\partial \alpha} D^{\alpha}_t x_1)^2 + (\frac{\partial x_2}{\partial \alpha} D^{\alpha}_t x_2)^2 \right]$$

$$- \frac{1}{2} mg \left( x_1^2 + x_2^2 \right),$$

(55)
To obtain the fractional Euler-Lagrange equation, we use
\[ \frac{\partial L}{\partial x} + A^H_{i} D_{b}^{\alpha, (\beta-1)} \frac{\partial L}{\partial (A^H_{a} D_{t}^{\alpha, \beta} x(t))} + A^H_{i} D_{t}^{\alpha, (\beta-1)} \frac{\partial L}{\partial (A^H_{b} D_{t}^{\alpha, \beta} x(t))} = 0, \quad i = 1, 2. \] (56)

It follows
\[ A^H_{i} D_{b}^{\alpha, (\beta-1)} (A^H_{a} D_{t}^{\alpha, \beta} x_1) + \frac{g}{t} x_1 = 0, \quad A^H_{i} D_{b}^{\alpha, (\beta-1)} (A^H_{a} D_{t}^{\alpha, \beta} x_2) + \frac{g}{t} x_2 = 0. \] (57)

These equation reduces to the equation of motion of the harmonic oscillator when \( \alpha \rightarrow 1, \beta = 0, \)
\[ x_1 + \frac{g}{t} x_1 = 0, \quad x_2 + \frac{g}{t} x_2 = 0 \] (58)

8. Fractional variational principles and constrained systems within Hilfer-Atangana-Baleanu’s derivatives. The above fractional canonical equations are valid for the case when no primary constraints exist, namely all canonical momenta are linearly independent. Many dynamical systems possessing physical interest have constraints. The problem can be defined as follows. Find the extremum of the functional
\[ J[x] = \int_{a}^{b} L(t, x, A^H_{a} D_{t}^{\alpha, \beta} x(t)) dt, \]
subject to the dynamical constraint
\[ A^H_{a} D_{t}^{\alpha, \beta} x(t) = \phi(x), \]
with the boundary conditions
\[ x(a) = x_a, \quad x(b) = x_b. \]

In this case we define the functional
\[ S[x] = \int_{a}^{b} [L + \lambda \Phi] dt, \]
where
\[ \Phi(t, x, A^H_{a} D_{t}^{\alpha, \beta} x(t)) = \phi(x) - A^H_{a} D_{t}^{\alpha, \beta} x(t) = 0 \]
and \( \lambda \) is the Lagrange multiplier. Then equations (40) in this case takes the form:
\[ \frac{\partial S}{\partial x} + A^H_{i} D_{b}^{\alpha, (\beta-1)} \frac{\partial S}{\partial (A^H_{a} D_{t}^{\alpha, \beta} x(t))} = 0, \] (59)
which can be written as
\[ \frac{\partial L}{\partial x} + A^H_{i} D_{b}^{\alpha, (\beta-1)} \frac{\partial L}{\partial (A^H_{a} D_{t}^{\alpha, \beta} x(t))} + \lambda \left[ \frac{\partial \Phi}{\partial x} + A^H_{i} D_{b}^{\alpha, (\beta-1)} \frac{\partial \Phi}{\partial (A^H_{a} D_{t}^{\alpha, \beta} x(t))} \right] = 0. \] (60)

Example 8.1. Let take
\[ J[x] = \int_{0}^{1} (A^H_{0} D_{t}^{\alpha, \beta} x(t))^2 dt, \]
with the boundary conditions
\[ x(0) = 0, \quad x(1) = 0, \]
\[ K_1[x] = \int_{0}^{1} x dt = 0, \quad K_2[x] = \int_{0}^{1} t x dt = 1. \]
Then we have:
\[
S[x] = \int_0^1 \left[ \left( \frac{ABH_0 D_t^{\alpha,\beta} x(t)}{2} \right)^2 + \lambda_1 x + \lambda_2 t x \right] dt,
\]
where \( \lambda_1, \lambda_2 \) are the Lagrange multipliers. Then equations (60) takes the form:
\[
2 \frac{ABH_1 D_1^{\alpha,\beta-1} (\frac{ABH_0 D_t^{\alpha,\beta} x(t)}) - \lambda_1 - \lambda_2 t = 0,
\]
which can be written as
\[
\frac{ABH_1 D_1^{\alpha,\beta-1} (\frac{ABH_0 D_t^{\alpha,\beta} x(t)}) = \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_2 t.
\]

9. Fractional optimal control problem involving Hilfer derivatives in Atangana-Baleanu’s derivatives sense. Using the above definitions, the Fractional Optimal Control Problem FOCP under consideration can be defined as follows. Find the optimal control \( u(t) \) for a FDS that minimizes the performance index
\[
J[u] = \int_0^1 F(x, u, t) dt,
\]
subject to the dynamical constraint
\[
\frac{ABH_0 D_t^{\alpha,\beta} x(t)}{2} = G(x, u, t),
\]
with the boundary conditions
\[
x(0) = x_0.
\]
where \( x(t) \) is the state variable, \( t \) represents the time, and \( F \) and \( G \) are two arbitrary functions. Note that Equation (63) may also include some additional terms containing state variables at the end point. This term in not considered here for simplicity. When \( \alpha = 1 \), the above problem reduces to a standard optimal control problem. Here the limits of integration have been taken as 0 and 1. Furthermore, we consider \( 0 < \alpha < 1 \). These are not the limitations of the approach. Any limits can be considered and the derivative can be of any order. However, these conditions are considered for simplicity. To find the optimal control we follow the traditional approach and define a modified performance index as

In this case we define the functional
\[
J[x] = \int_0^1 [F(x, u, t) + \lambda (G(x, u, t) - \frac{ABH_0 D_t^{\alpha,\beta} x(t)))] dt,
\]
where \( \lambda \) is the Lagrange multiplier also known as a costate or an adjoint variable. Taking variation of Equation (66), we obtain
\[
\delta J[u] = \int_0^1 \left[ \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial u} \delta u + \delta \lambda (G(x, u, t) - \frac{ABH_0 D_t^{\alpha,\beta} x(t))} + \lambda \left( \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial u} \delta u - \delta (\frac{ABH_0 D_t^{\alpha,\beta} x(t))} \right) \right] dt,
\]
Using Equation (33), the last integral in Equation (67) can be written as
\[
\int_0^1 \lambda \delta (\frac{ABH_0 D_t^{\alpha,\beta} x(t))} dt = \int_0^1 \delta x(t) (\frac{ABH_1 D_t^{\alpha,\beta-1}}{2} \lambda) dt
\]
provided \( \delta x(0) = 0 \) or \( \lambda(0) = 0 \), and \( \lambda x(1) = 0 \) or \( \lambda(1) = 0 \). Because \( x(0) \) is specified, we have \( \delta x(0) = 0 \), and since \( x(1) \) is not specified, we require \( \lambda(1) \) to be zero. With these assumptions, the identity in Equation (68) is satisfied. Note that
we have assumed that the order of variation and the fractional derivative can be interchanged. Using Equations (67) and (68), we obtain

\[
\delta J[u] = \int_0^1 [\delta \lambda (G(x,u,t) - (ABH_0 D_t^{\alpha,\beta} x)(t)) + \delta x \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} - ABH_t D_t^{\alpha,\beta-1} \lambda] + \delta u \left[ \frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} \right] dt,
\]

(69)

Minimization of \( J[u] \) (and hence minimization of \( J(u) \)) requires that the coefficients of \( \delta x, \delta u, \) and \( \delta \lambda \) in Equation (69) be zero. This leads to

\[
(ABH_0 D_t^{\alpha,\beta} x)(t) = G(x,u,t)
\]

(70)

\[
ABH_t D_t^{\alpha,\beta-1} \lambda = \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x}
\]

(71)

\[
\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} = 0.
\]

(72)

and

\[
x(0) = x_0 \text{ and } \lambda(1) = 0.
\]

(73)

Equations (70)-(72) represent the Euler-Lagrange equations for the FOCP. These equations give the necessary conditions for the optimality of the FOCP considered here. They are very similar to the Euler-Lagrange equations for classical optimal control problems except that the resulting differential equations contain the left and the right fractional derivatives. Furthermore, the derivation of these equations is very similar to the derivation for an optimal control problem containing integral order derivatives. Determination of the optimal control for the fractional system requires solution of Equations (70)-(73).

Observe that Equation (70) contains Left Hilfer derivative in Atangana-Baleanu sense FD where as Equation (71) contains Right Hilfer derivatives in Atangana-Baleanu sense. This clearly indicates that the solution of optimal control problems requires knowledge of not only forward derivatives but also backward derivatives to account for end conditions. In classical optimal control theories, this issue is either not discussed or they are not clearly stated. This is largely because the backward derivative of order 1 turns out to be the negative of the forward derivative of order 1.

**Example 9.1.** As a special case, assume that the performance index is an integral of quadratic forms in the state and the control,

\[
J[u] = \frac{1}{2} \int_0^1 \left[ q(t)x^2(t) + r(t)u^2 \right] dt,
\]

(74)

where \( q(t) \geq 0 \) and \( r(t) > 0 \), and the dynamics of the system is described by the following linear fractional differential equation,

\[
(ABH_0 D_t^{\alpha,\beta} x)(t) = a(t)x + b(t)u,
\]

(75)

This linear system for \( \alpha = 1 \) and \( 0 < \alpha < 1 \) has been studied extensively, and formulations and solution schemes for this system within the classical Riemann-Liouville and Caputo derivatives are well documented in many textbooks and journal articles (see e.g. [3, 4]. Here we study it within the new Atangana-Baleanu’s derivatives. For \( 0 < \alpha < 1 \), the Euler-Lagrange Equations (70) to (72) lead to Equation (75) and

\[
ABH_t D_t^{\alpha,\beta-1} \lambda = q(t)x + a(t)\lambda,
\]

(76)
and
\[ r(t)u + b(t)\lambda = 0. \] (77)

From Equations (75) and (77), we get
\[ ({}^{c}A^{\alpha}B^{\beta} \frac{d^\alpha}{dt^\alpha} x)(t) = a(t)x - r^{-1}(t)b^2(t)\lambda. \] (78)

The state \( x(t) \) and the costate \( \lambda(t) \) are obtained by solving the fractional differential equations (76) and (78) subject to the terminal conditions given by Equation (72). Once \( \lambda(t) \) is known, the control variable \( u(t) \) can be obtained using Equation (77).

10. Conclusions. A general formulation has been presented for a class of fractional optimal control problems involving the Hilfer derivatives for the new Atangana-Baleanu's fractional derivatives. The formulation utilized the calculus of variations, the Lagrange multiplier technique, and the formula for fractional integration by parts to obtain the Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The formulation is specialized for a system with quadratic performance index subject to a fractional system dynamic constraint. From the above and other literature in the field of fractional calculus it is clear that many of the ideas of the ordinary calculus can be extended to fractional calculus with only minor changes. The advantage of the new fractional derivative has no singularity, which was not precisely illustrated in the previous definitions.

As a final remark, we note that very little progress has been made in the field of FOCP involving Hilfer derivatives for the new Atangana-Baleanu’s fractional derivatives. This is largely due to the fact that the underlying mathematics for fractional derivatives was not well developed. Recent development in the field of fractional derivatives has eliminated this barrier. From the formulation presented above, it is clear that many of the concepts of classical control theory can be directly extended to FOCPs. Although only one class of FOCPs involving the new Atangana-Baleanu’s fractional derivatives was considered here, the formulation can be extended to many other FOCPs involving the new Atangana-Baleanu’s fractional derivatives. It is hoped that this observation will initiate some interest in the areas of fractional variational calculus and fractional optimal control.

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