ON LAMBDA-GRAF SYSTEMS FOR SUBSHIFTS OF SUBSHIFTS

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Abstract. Kengo Matsumoto has introduced \( \lambda \)-graph systems and strong shift equivalence of \( \lambda \)-graph systems [Doc.Math.4 (1999), 285-340]. We associate to a subshift of a subshift a \( \lambda \)-graph system. The strong shift equivalence class of the associated \( \lambda \)-graph system is an invariant of subsystem equivalence. Wolfgang Krieger and Kengo Matsumoto have introduced the \( \lambda \)-entropy of a \( \lambda \)-graph system and have shown its invariance under strong shift equivalence [Ergod.Th.&Dynam.Sys. 24 (2004) 1155 - 1172]. A separation entropy of a subshift of a subshift is introduced as the \( \lambda \)-entropy of the associated \( \lambda \)-graph system.

Keywords: subshift, \( \lambda \)-graph system, entropy

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1. Introduction

Let \( \Sigma \) be a finite alphabet, and let \( S \) denote the shift on \( \Sigma^\mathbb{Z} \),
\[
S((x_i)_{i \in \mathbb{Z}}) = ((x_{i+1})_{i \in \mathbb{Z}}), \quad (x_i)_{i \in \mathbb{Z}} \in \mathbb{Z}.
\]
A closed \( S \)-invariant set \( X \subset \Sigma^\mathbb{Z} \) with the restriction of \( S \) acting on it, is called a subshift. A finite word is said to be admissible for a subshift if it appears in a point of the subshift. A subshift is uniquely determined by its set of admissible words. A subshift is said to be of finite type if its admissible words are defined by excluding finitely many words from appearing as subwords in them. Subshifts are studied in symbolic dynamics. For an introduction to symbolic dynamics see [Ki] and [LM].

Given subshifts \( X \subset \Sigma^\mathbb{Z} \) and \( \tilde{X} \subset \tilde{\Sigma}^\mathbb{Z} \), we say that subshifts \( Y \subset X \) and \( \tilde{Y} \subset \tilde{X} \) are subsystem equivalent if there exists a topological conjugacy of \( X \) onto \( \tilde{X} \) that carries \( Y \) onto \( \tilde{Y} \). In this paper we associate to a subshift \( Y \) of a subshift \( X \) a \( \lambda \)-graph system [M1] whose strong shift equivalence class is an invariant of subsystem equivalence. Consequently the invariants of strong shift equivalence that are obtained from the \( \lambda \)-graph systems, e.g. their \( \lambda \)-entropy [KM2], are invariants of subsystem equivalence. In section 2 we recall relevant notions and results and we introduce terminology and notation. Section 3 contains a construction of \( \lambda \)-graph systems that will be instrumental in associating invariantly a \( \lambda \)-graph system to a subshift of a subshift, which is done in section 4. In section 5 we consider examples.
2. Labeled directed graphs and λ-graph systems

We recall from [M1] [KM1][N] some notions and results concerning labeled directed graphs, λ-graph systems and subshifts. We consider labeled directed graphs \((V, E, \lambda)\) with a set \(V\) of vertices, a set \(E\) of directed edges and a labeling function \(\lambda\) that assigns to every edge in \(E\) a symbol taken from a finite alphabet \(\Sigma\). (We will suppress the labeling function \(\lambda\) in the notation, and also the edge set will not always appear in the notation). A subset \(W\) of \(V\) determines a labeled directed sub-graph of \((V, E)\) with \(W\) as vertex set, in which the labeled edges in \(E\) are retained that have their initial and final vertex in \(W\). A labeled directed graph \((V, E)\) is called a Shannon graph if its labeling is 1-right resolving in the sense that for every vertex \(V \in V\) and every symbol \(\sigma \in \Sigma\) there is at most one edge leaving \(V\) that carries the label \(\sigma\). We denote the set of vertices that have an outgoing edge that carries the label \(\sigma\) by \(V(\sigma)\), and for \(V \in V(\sigma), \sigma \in \Sigma\) we denote by \(\tau_\sigma(V)\) the final vertex of the edge that leaves \(V\) and carries the label \(\sigma\). We call \((\tau_\sigma)_{\sigma \in \Sigma}\) the transition rules of the Shannon graph. A Shannon graph is determined by its transition rules. We say that a Shannon graph such that every vertex has an incoming and an outgoing edge presents a subshift \(X \subset \Sigma^\Z\) if the set of admissible words of the subshift coincides with the set of label sequences of finite paths in the graph. By a compact Shannon graph we mean a Shannon graph \((V, E)\) where \(V\) carries a compact topology such that the sets \(V(\sigma), \sigma \in \Sigma,\) are closed (or open) and such that the mappings \(V \to \tau_\sigma(V)(V \in V(\sigma)), \sigma \in \Sigma,\) are continuous. To a compact Shannon graph \((V, E)\) that presents a subshift there is associated the topological Markov chain \(M(V, E),\)

\[
M(V, E) = \bigcap_{i \in \Z} \{(V_i, x_i)_{i \in \Z} \in (V \times \Sigma)^\Z : V_{i+1} = \tau_{x_i}(V_i)\}.
\]

The subshift that is presented by a compact Shannon graph \((V, E)\) is identical to the set of label sequences of bi-infinite paths on the graph, in other words, \(M(V, E)\) projects onto the presented subshift. For a compact Shannon graph \((V, E)\) such that every vertex has an outgoing edge, let

\[
\tau(V) = \bigcup_{\sigma \in \Sigma} \tau_\sigma(V(\sigma)),
\]

and set inductively

\[
\tau^{(0)}(V) = \tau(V), \tau^{(n)}(V) = \tau(\tau^{(n-1)}(V)), \quad n \in \N.
\]

Here

\[
\tau^{(n)}(V) \subset \tau^{(n-1)}(V), \quad n \in \N,
\]

and the set \(\bigcap_{n \in \N} \tau^{(n)}(V)\) determines a labeled directed sub-Shannon graph of \((V, E)\) that presents a subshift, and that we denote by \((V, E)^\circ\).

A λ-graph system [M1] is a labeled directed graph in the shape of a Bratteli diagram with an additional structure. The vertex set of a λ-graph system is a disjoint union \(\bigcup_{n \in \Z_+} \mathcal{V}_n\) of finite sets, and its edge set is also a disjoint union \(\bigcup_{n \in \Z_+} \mathcal{E}_n\) of finite sets. Every edge in \(\mathcal{E}_n\) has its initial vertex in \(\mathcal{V}_n\) and its final vertex in \(\mathcal{V}_{n-1}, n \in \N\). (The edge set of a λ-graph system will not always appear in
It is assumed that all vertices have an incoming edge and that all vertices, except the vertices in \( V_0 \), have an outgoing edge. The additional structure that makes the directed labeled Bratteli diagram into a \( \lambda \)-graph system is a shift-like map \( \iota : \bigcup_{n \in \mathbb{N}} V_n \to V \) such that

\[
\iota(V_n) = V_{n-1}, \quad n \in \mathbb{N},
\]

that is compatible with the labeling. We say that a \( \lambda \)-graph system presents a subshift \( X \subset \Sigma^\mathbb{Z} \) if the set of label sequences of finite paths in the \( \lambda \)-graph system coincides with the set of admissible words of the subshift. To every Shannon \( \lambda \)-graph system \( \bigcup_{n \in \mathbb{N}^+} V_n \) there is associated the compact Shannon graph

\[
\{(V_n)_{n \in N} \in \prod_{n \in \mathbb{N}^+} V_n : \iota(V_n) = V_{n-1}, n \in \mathbb{N}\}
\]

of its \( \iota \)-orbits, where for \( \iota \)-orbits \((V_n)_{n \in \mathbb{N}^+}\) and \((W_n)_{n \in \mathbb{N}^+}\),

\[
\tau_\sigma((V_n)_{n \in \mathbb{N}^+}) = (W_n)_{n \in \mathbb{N}^+},
\]

if and only if for \( n \in \mathbb{N} \) \( \tau_\sigma(V_n) = W_{n-1} \). A \( \lambda \)-graph system is described by its symbolic matrix system \((M^{(n,n-1)}_{(n,n-1), c', c}), I^{(n,n-1)}_{(n,n-1), c', c})\), \( n \in \mathbb{N} \). Here

\[
M^{(n,n-1)}_{c,c'} = \sum_{\{\sigma \in \Sigma : \tau_\sigma(c) = c'\}} \sigma, \quad c \in V_n, c' \in V_{n-1},
\]

and

\[
I^{(n,n-1)}_{c,c'} = \begin{cases} 1 & \text{if } \iota(c) = c', \\ 0 & \text{otherwise}. \end{cases}
\]

The compatibility condition translates into the commutation relation

\[
M^{(n+1,n)} I^{(n,n-1)} = I^{(n+1,n)} M^{(n,n-1)}, \quad n \in \mathbb{N}.
\]

Given a finite alphabet \( \Sigma \) we denote by \( \iota^- \) the operation of removing the first symbol from a word in \( \Sigma^{[1,n]} \), \( n \in \mathbb{N} \), and by \( \iota^+ \) the operation of removing the last symbol. \( \iota^- \) also removes the first symbol of a sequence in \( \Sigma^\mathbb{N} \). We denote by \( \mathcal{V}(\Sigma) \) the set of closed subsets of \( \Sigma^\mathbb{N} \) with its Hausdorff subset topology, and for \( \sigma \in \Sigma \) we denote by \( \mathcal{V}(\sigma) \) the set of \( V \in \mathcal{V} \) that contain a sequence that begins with \( \sigma \). For \( \sigma \in \Sigma, V \in \mathcal{V}_n(\Sigma)(\sigma), n \in \mathbb{N} \), we set

\[
(1) \quad \tau_\sigma(V) = \{\iota^- (v) : v \in V, v_1 = \sigma\}.
\]
We have in this way described a Shannon graph with vertex set $\mathcal{V}(\Sigma)$ and transition rules $(\tau_{\sigma})_{\sigma \in \Sigma}$. We denote by $\mathcal{V}_n$ the set of subsets of $\Sigma^{[1,n]}$, $n \in \mathbb{Z}_+$, and for $\sigma \in \Sigma$ we denote by $\mathcal{V}_n(\sigma)$ the set of $V \in \mathcal{V}_n$ that contain a word that begins with $\sigma$, $n \in \mathbb{N}$. For $\sigma \in \Sigma$, $V \in \mathcal{V}_n(\Sigma)(\sigma)$, $n \in \mathbb{N}$, define $\tau_\sigma(V)$ again by (1). Using $\iota^+$ as the $\iota$-mapping, we have in this way described a forward separated $\lambda$-graph system with vertex set $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(\Sigma)$ and transition rules $(\tau_{\sigma})_{\sigma \in \Sigma}$. $\mathcal{V}(\Sigma)$ is the Shannon graph of $\iota$-orbits of the $\lambda$-graph system $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(\Sigma)$.

The forward context $\Gamma^+(V)$ of a vertex $V$ of a Shannon graph is defined as the set of sequences in $\Sigma^\mathbb{N}$ that are label sequences of a semi-infinite path that leaves $V$. A Shannon graph is called forward separated if distinct vertices have distinct forward contexts. We identify the vertices of a forward separated Shannon graph with their forward contexts, and use then on the vertex set the Hausdorff subset topology on $\mathcal{V}(\Sigma)$. A forward separated Shannon graph whose vertex set is compact is a compact Shannon graph. We say that a subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$ is transition complete if for $\sigma \in \Sigma$ and $V \in \mathcal{V}$ one has that $\tau_{\sigma}(V) \in \mathcal{V}$. There is a one-to-one correspondence between forward separated compact Shannon graphs and the Shannon graphs that are determined by transition complete closed subsets of $\mathcal{V}(\Sigma)$. A compact forward separated Shannon graph $\mathcal{V}$ is isomorphic to the Shannon graph of $\iota$-orbits of a Shannon $\lambda$-graph system that is a subsystem of $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(\Sigma)$: one maps a $V \in \mathcal{V}$ into the $\iota$-orbit $(V_{[1,n]})_{n \in \mathbb{Z}_+}$.

Let $X \subset \Sigma^\mathbb{Z}$ be a subshift. We denote

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, i, k \in \mathbb{Z}, i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}.$$

We use similar notation also if indices range in semi-infinite intervals. Blocks also stand for the words they carry. We set

$$\Gamma_n^+(x^-) = \{a \in X_{[1,n]} : (x^-, a) \in X_{(-\infty,n]}\}, \quad n \in \mathbb{N},$$

$$\Gamma^+(x^-) = \{x^+ \in X_{[1,\infty]} : (x^-, x^+) \in X\}, \quad x^- \in X_{(-\infty,0]},$$

$$\Gamma_n^+(a) = \{x^+ \in X_{[1,\infty]} : (a, x^+) \in X_{[-n,\infty)}\}, \quad a \in X_{[-n,0]}, n \in \mathbb{N}.$$

Every subshift $X \subset \Sigma^\mathbb{Z}$ is presented by its canonical $\lambda$-graph system with vertex set $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(X)$,

$$\mathcal{V}_n(X) = \{\Gamma_n^+(x^-) : x^- \in X_{(-\infty,0]}\}, \quad n \in \mathbb{Z}_+,$$

and by its word $\lambda$-graph system with vertex set $\bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}$ (or $\bigcup_{n \in \mathbb{Z}_+} \{x_{[1,n]} : x \in X\}$) if one insists that it be a sub-$\lambda$-graph system of $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(\Sigma)$). The $\lambda$-graph system $\bigcup_{n \in \mathbb{Z}_+} \mathcal{V}_n(X)$ accompanies the sub-Shannon graph of $\mathcal{V}(\Sigma)$ whose vertex set is the closure of the set $\{\Gamma^+(x^-) : x^- \in X_{(-\infty,0]}\}$, and the $\lambda$-graph system $\bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}$ accompanies the sub-Shannon graph of $\mathcal{V}(\Sigma)$ whose vertex set is $X_{[1,\infty]}$ (or $\{x^+ : x^+ \in X_{[1,\infty]}\}$).

We recall the notion of a bipartite subshift. Let $\Delta$ and $\tilde{\Delta}$ be finite disjoint alphabets, and let $Y \subset (\Delta \cup \tilde{\Delta})^\mathbb{Z}$ be a subshift. $Y$ is called bipartite if the admissible
words of length two of $Y$ are contained in $\Delta \tilde{\Delta} \cup \tilde{\Delta} \Delta$. If $Y$ is bipartite then $S^2_Y$ leaves the sets

$$X = \{(y_i)_{i \in \mathbb{Z}} \in Y : y_0 \in \Delta \},$$

and

$$\tilde{X} = \{(y_i)_{i \in \mathbb{Z}} \in Y : y_0 \in \tilde{\Delta} \},$$
i

invariant. Let $S$ resp. $\tilde{S}$ denote the restriction of $S^2_Y$ to $X$ resp. to $\tilde{X}$. $(X, S)$ and $(\tilde{X}, \tilde{S})$ are topologically conjugate: a topological conjugacy of $X$ onto $\tilde{X}$ is given by the restriction of

$S_Y$ to $X$. Denote the set of words in $\Delta \tilde{\Delta}$ resp. in $\tilde{\Delta} \Delta$ that are admissible for $Y$ by $\Sigma$ resp. by $\tilde{\Sigma}$. One has

$$X \subset \Sigma^\mathbb{Z}, \tilde{X} \subset \tilde{\Sigma}^\mathbb{Z},$$

and one has the injections

$$\varphi : \Sigma \hookrightarrow \Delta \tilde{\Delta}, \quad \tilde{\varphi} : \tilde{\Sigma} \hookrightarrow \tilde{\Delta} \Delta.$$

By applying $\varphi$ and $\tilde{\varphi}$ symbol by symbol one extends their domain of definition to finite words and right-infinite sequences. $\varphi$ and $\tilde{\varphi}$ satisfy the relation

$$(2) \quad \iota^-(\iota^+(\varphi(X_{[1,2]}))) = \tilde{\varphi}(\tilde{\Sigma}),$$

and are called specifications. Conversely, let $X \subset \Sigma^\mathbb{Z}$ and $\tilde{X} \subset \tilde{\Sigma}^\mathbb{Z}$ be subshifts, and let $\Delta$ and $\tilde{\Delta}$ be disjoint finite alphabets and let there be given injections

$$\varphi : \Sigma \hookrightarrow \Delta \tilde{\Delta}, \quad \tilde{\varphi} : \tilde{\Sigma} \hookrightarrow \tilde{\Delta} \Delta,$$

that are specifications, that is, they satisfy (2). Then

$$\varphi(X) \cup \tilde{\varphi}(\tilde{X}) \subset \{\Delta \tilde{\Delta} \cup \tilde{\Delta} \Delta\}^\mathbb{Z}$$

is a bipartite subshift and $X$ and $\tilde{X}$ are topologically conjugate. (1) implies that also

$$\iota^-(\iota^+(\varphi(\tilde{X}_{[1,2]}))) = \varphi(\Sigma)$$

and one has a 2-block code given by

$$a \rightarrow \tilde{\varphi}^{-1}(\iota^-(\iota^+(\varphi(a)))), \quad a \in X_{[1,2]},$$

that implements a topological conjugacy of $X$ onto $\tilde{X}$ with the inverse given by the 2-block map

$$\tilde{a} \rightarrow \varphi^{-1}(\iota^-(\iota^+(\tilde{\varphi}(\tilde{a})))), \quad \tilde{a} \in \tilde{X}_{[1,2]}.$$
set $\Sigma$ and a symbolic matrix system $(\tilde{M}^{(n,n-1)}, \tilde{I}^{(n,n-1)})_{n \in \mathbb{N}}$ with symbol set $\bar{\Sigma}$ is

given by specifications

$$\varphi : \Sigma \mapsto \Delta \bar{\Delta}, \quad \bar{\varphi} : \bar{\Sigma} \mapsto \bar{\Delta} \Delta,$$

together with symbolic matrices

$$K^{(n,n-1)} = (K_{C,\bar{C}})_{C \in \mathcal{V}_n, \bar{C} \in \mathcal{V}_{n-1}},$$

$$\tilde{K}^{(n,n-1)} = (\tilde{K}_{\bar{C},C})_{\bar{C} \in \mathcal{V}_n, C \in \mathcal{V}_{n-1}}, \quad n \in \mathbb{N},$$

such that

$$K^{(n+1,n)} K^{(n,n-1)} = \varphi(M^{(n+1,n)} I^{(n,n-1)}),$$

$$\tilde{K}^{(n+1,n)} \tilde{K}^{(n,n-1)} = \bar{\varphi}(\tilde{M}^{(n+1,n)} \tilde{I}^{(n,n-1)}),$$

$$K^{(n+1,n)} \bar{\varphi}(\tilde{M}^{(n,n-1)}) = \varphi(M^{(n+1,n)}) K^{(n,n-1)},$$

$$\tilde{K}^{(n+1,n)} \varphi(M^{(n,n-1)}) = \bar{\varphi}(\tilde{M}^{(n+1,n)}) \tilde{K}^{(n,n-1)},$$

$$K^{(n+1,n)} \tilde{I}^{(n,n-1)} = I^{(n+1,n)} K^{(n,n-1)},$$

$$\tilde{K}^{(n+1,n)} I^{(n,n-1)} = \tilde{I}^{(n+1,n)} \tilde{K}^{(n,n-1)}, \quad n \in \mathbb{N},$$

where the specifications act componentwise on the matrices. The specifications that come with a 1-step strong shift equivalence between Shannon $\lambda$-graph systems induce a bipartite coding between the subshifts that are presented by the Shannon $\lambda$-graph systems. If subshifts $X \subset \Sigma^Z$ and $\bar{X} \subset \bar{\Sigma}^Z$ that are bipartitely related by specifications

$$\varphi : \Sigma \mapsto \Delta \bar{\Delta}, \quad \bar{\varphi} : \bar{\Sigma} \mapsto \bar{\Delta} \Delta,$$

then their canonical $\lambda$-graph systems $\bigcup_{n \in \mathbb{Z}_+} V_n(X)$ and $\bigcup_{n \in \mathbb{Z}_+} V_n(\bar{X})$ are strong shift equivalent in one step: A 1-step strong shift equivalence between their symbolic matrix system is given by the symbolic matrices

$$K^{(n,n-1)} = (K_{C,\bar{C}})_{C \in \mathcal{V}_n(X), \bar{C} \in \mathcal{V}_{n-1}(\bar{X})},$$

$$\tilde{K}^{(n,n-1)} = (\tilde{K}_{\bar{C},C})_{\bar{C} \in \mathcal{V}_n(\bar{X}), C \in \mathcal{V}_{n-1}(X)}, \quad n \in \mathbb{N},$$

where

$$K_{C,\bar{C}}^{(n,n-1)} = \sum_{\{\delta \in \iota^+ (\varphi(\Sigma)): \bar{\varphi}^{-1}(\iota^+ (\tau_\delta (\varphi(C)))) = \bar{C}\}} \delta, \quad C \in \mathcal{V}_n(X), \bar{C} \in \mathcal{V}_{n-1}(\bar{X}), n \in \mathbb{N},$$

with the symmetric expression for the symbolic matrices $\tilde{K}^{(n,n-1)}, n \in \mathbb{N}$. The $\lambda$-graph systems $\bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}$ and $\bigcup_{n \in \mathbb{Z}_+} \bar{X}_{[1,n]}$ are also strong shift equivalent in one step: A 1-step strong shift equivalence between their symbolic matrix systems is given by the symbolic matrices

$$K^{(n,n-1)} = (K_{c,\bar{c}}^{(n,n-1)})_{c \in X_{[1,n]}, \bar{c} \in \bar{X}_{[1,n-1]}},$$

$$\tilde{K}^{(n,n-1)} = (\tilde{K}_{\bar{c},c}^{(n,n-1)})_{\bar{c} \in \bar{X}_{[1,n]}, c \in X_{[1,n-1]}}, \quad n \in \mathbb{N},$$

where for $c \in X_{[1,n]}, \bar{c} \in \bar{X}_{[1,n-1]}, n \in \mathbb{N},$

$$K_{c,\bar{c}}^{(n,n-1)} = \begin{cases} \delta, & \text{if } \iota^+ (\varphi(c_1)) = \delta, \bar{\varphi}^{-1}(\iota^+ (\tau_\delta (\varphi(c)))) = \bar{c}, \\ 0, & \text{otherwise}. \end{cases}$$

with the symmetric expression for $\tilde{K}^{(n,n-1)}, n \in \mathbb{N}$. 

3. Pair \( \lambda \)-graph systems

Given Shannon graphs \((V, E)\) and \((W, F)\), both with label set \(\Sigma\), we say that \((V, E)\) is subordinate to \((W, F)\) if for all \(V \in V\) there exists a \(W \in W\) such that \(\Gamma^+(V) \subset \Gamma^+(W)\). If \((V, E)\) is subordinate to \((W, F)\) then we form a pair Shannon graph \([\Gamma(V), \Gamma(W)]\) with vertex set the set of pairs of vertices \((V, W) \in V \times W\) such that \(\Gamma^+(V) \subset \Gamma^+(W)\), where the edges that leave the vertex \((V, W)\) are the pairs that consist of an edge in \(E\) that leaves \(V\) carrying a label \(\sigma\) and the edge in \(F\) that leaves \(W\) carrying the same label \(\sigma\). Given a \(\lambda\)-graph system \((\bigcup_{n \in \mathbb{Z}_+} V_n, \bigcup_{n \in \mathbb{Z}_+} E_n)\) that is subordinate to a \(\lambda\)-graph system \((\bigcup_{n \in \mathbb{Z}_+} W_n, \bigcup_{n \in \mathbb{Z}_+} F_n)\) as a Shannon graph one turns the pair Shannon graph \([\bigcup_{n \in \mathbb{Z}_+} V_n, \bigcup_{n \in \mathbb{Z}_+} E_n], \bigcup_{n \in \mathbb{Z}_+} F_n]\) into a pair \(\lambda\)-graph system by having the \(\iota\)-mappings of its constituent \(\lambda\)-graph systems acting on the components of its vertices.

Let \(X \subset \Sigma^\mathbb{Z}\) be a subshift and let \(V\) be a forward separated Shannon graph that presents \(X\). The Shannon word graph of the subshift \(X\) is subordinate to \(V\) and the word \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}\) is subordinate to the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}_+} V_{[1,n]}\). One has here

\[
([X_{[1,\infty)}, V])^o = [X_{[1,\infty)}, V]
\]

and

\[
([ \bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}, \bigcup_{n \in \mathbb{Z}_+} V_{[1,n]}])^o = [ \bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}, \bigcup_{n \in \mathbb{Z}_+} V_{[1,n]}]
\]

In these pair Shannon graphs resp. pair \(\lambda\)-graph systems every vertex has precisely one edge leaving it and this edge carries as label the first symbol of the word that is the word component of the vertex. We denote the vertex set of \([\bigcup_{n \in \mathbb{Z}_+} X_{[1,n]}, \bigcup_{n \in \mathbb{Z}_+} V_{[1,n]}]\) by \(\bigcup_{n \in \mathbb{Z}_+} \tilde{V}_n\).

**Theorem 3.1.** Let \(X \subset \Sigma^\mathbb{Z}\), and \(\tilde{X} \subset \tilde{X}\) be subshifts that are bipartitely related by specifications

\[
\varphi : \Sigma \leftrightarrow \Delta \tilde{\Delta}, \quad \tilde{\varphi} : \tilde{\Sigma} \leftrightarrow \tilde{\Delta} \Delta.
\]

Let \(V\) be a forward separated Shannon graph that presents \(X\), and let

\[
\tilde{V} = \{ \tilde{\varphi}^{-1}(\tau_\delta((\varphi(V)))) : V \in V, \delta \in \Delta\}.
\]

Then the \(\lambda\)-graph systems \(\bigcup_{n \in \mathbb{Z}_+} \tilde{V}_n\) and \(\bigcup_{n \in \mathbb{Z}_+} \tilde{V}_n\) are strong shift equivalent in one step.

**Proof.** Here

\[
\tilde{V}_{[1,n]} = \{ \tilde{\varphi}^{-1}(\tau_\delta((\varphi(V)))) : V \in (V_{[1,n+1]}, \delta \in \Delta)\}, \quad n \in \mathbb{N}.
\]

A 1-step strong shift equivalence between the \(\lambda\)-graph systems \(\bigcup_{n \in \mathbb{Z}_+} \tilde{V}_n\) and \(\bigcup_{n \in \mathbb{Z}_+} \tilde{V}_n\) is implemented by the symbolic matrices

\[
K^{(n,n-1)} = (K^{(n,n-1)}), \quad \tilde{K}^{(n,n-1)} = (\tilde{K}^{(n,n-1)}), \quad n \in \mathbb{N},
\]
where for \((c, C) \in \tilde{V}_n, (\tilde{c}, \tilde{C}) \in \tilde{V}_{n-1}, n \in \mathbb{N}\),

\[
K^{(n,n-1)}_{(c,C),(\tilde{c},\tilde{C})} = \begin{cases} 
\delta & \text{if } \iota(\varphi(c_1)) = \delta, \iota^+(\tau_\delta(\varphi(c))) = \tilde{c}, \tilde{\varphi}(\iota^+\tau_\delta(\varphi(C))) = \tilde{C}, \\
0, & \text{otherwise,}
\end{cases}
\]

with the symmetric expression for \(\tilde{K}^{(n,n-1)}, n \in \mathbb{N}\). \(\square\)

The \(\lambda\)-entropy \(h_\lambda\) of a \(\lambda\)-graph system was introduced in \([KM2]\) as the upper asymptotic growth rate of the number of vertices at the \(n\)-th level of the \(\lambda\)-graph system. The volume entropy of a \(\lambda\)-graph system is defined as the upper asymptotic growth rate of the number of paths in the \(\lambda\)-graph system from any vertex at its \(n\)-th level to its vertex at level 0 (see \([M2]\)). Inspection shows that there is a one-to-one correspondence between the set \(\{(a, C) \in X_{[1,n]} \times V_{[1,n]} : a \in C\}\) of vertices that one finds at the \(n\)-th level of the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} X_{[1,n]} \cup \bigcup_{n \in \mathbb{Z}^+} V_{[1,n]}\) and the set of paths in the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} V_{[1,n]}\) from any vertex at its \(n\)-th level to its vertex at level 0, \(n \in \mathbb{N}\). The \(\lambda\)-entropy of the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} X_{[1,n]} \cup \bigcup_{n \in \mathbb{Z}^+} V_{[1,n]}\) is therefore equal to the volume entropy of the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} V_{[1,n]}\).

4. Associating a \(\lambda\)-graph system to a subshift of a subshift

Consider a subshift \(X \subset \Sigma^\mathbb{Z}\) and a subshift \(Y \subset X\). The Shannon graph \(V(Y)\) and its \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} V_n(Y)\) are subordinate to the Shannon graph \(V(X)\) and its \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} V_n(X)\). \(V(Y, X)\) will denote the Shannon graph of \(\iota\)-orbits of the \(\lambda\)-graph system

\[
\left[ \bigcup_{n \in \mathbb{Z}^+} V_n(Y), \bigcup_{n \in \mathbb{Z}^+} V_n(X) \right]^\circ
\]

whose vertex set we denote by \(\bigcup_{n \in \mathbb{Z}^+} V_n(Y, X)\). Note that

\[
V_n(Y, X) = V(Y, X)_{[1,n]}, \quad n \in \mathbb{N}.
\]

We give an alternate description of the vertices of the \(\lambda\)-graph system \(\bigcup_{n \in \mathbb{Z}^+} V_n(Y, X)\).

**Theorem 4.1.** For all \(n \in \mathbb{N}\) and for

\[
(C_Y, C_X) \in V_n(Y) \times V_n(X)
\]

one has

\[
(C_Y, C_X) \in V_n(Y, X)
\]

precisely if

\[
C_Y \subset C_X,
\]

and if for all \(K \in \mathbb{N}\) there is a \(k \geq K\), together with

\[
b \in Y_{[-k,0]}, \quad a \in X_{[-k,0]},
\]

such that

\[
b_{[-k,0]} = a_{[-k,0]},
\]
and
\[ b^{(l)} \in Y_{[-k-l,0]}, \quad a^{(l)} \in X_{[-k-l,0]}, \quad l \in \mathbb{N}, \]
such that
\[ b^{(l)}_{[-k,0]} = b, \quad a^{(l)}_{[-k,0]} = a, \]
and
\[ \Gamma^+_Y(a^{(l)}) = \Gamma^+_X(a), \quad \Gamma^+_X(b^{(l)}) = \Gamma^+_Y(b), \quad l \in \mathbb{N}. \]

**Proof.** The fiber product of the closed subsystem
\[ \{(x_i, V_i)_{i \in \mathbb{Z}} \in M(V) : (x_i)_{i \in \mathbb{Z}} \in Y\} \]
of the topological Markov chain \( M(V) \) and of the topological Markov chain \( M(Y) \) with respect to the projection onto \( Y \) yields the topological Markov chain
\[ M(V(Y, X)) = \bigcap_{i \in \mathbb{Z}} \{ (V_i^Y, V_i^X, y_i)_{i \in \mathbb{Z}} \in (Y \times Y \times \Sigma)^{\mathbb{Z}} : V_i^Y \subset V_i^X, \tau_{y_i}(V_i^Y), \tau_{y_i}(V_i^X) \}. \]

An examination of the structure of a point in \( M(V(Y, X)) \) confirms the theorem. □

For subshifts \( Y \subset X \subset \Sigma^\mathbb{Z} \) we call the λ-entropy of the λ-graph system \( \bigcup_{n \in \mathbb{Z}_+} V_n(Y, X) \) the separation entropy of \( Y \) (as a subsystem of \( X \)). One has that
\[ h_\lambda ( \bigcup_{n \in \mathbb{Z}_+} V_n(Y)) \leq h_\lambda ( \bigcup_{n \in \mathbb{Z}_+} V_n(Y, X)). \]

**Theorem 4.2.** Let \( X \subset \Sigma^\mathbb{Z} \) and \( \tilde{X} \subset \Sigma^\mathbb{Z} \) be subshifts and let \( \varphi : X \to \tilde{X} \) be a topological conjugacy of \( X \) onto \( \tilde{X} \). Let \( Y \subset X \) be a subshift, and let \( \tilde{Y} = \varphi(Y) \).

Then the λ-graph systems \( \bigcup_{n \in \mathbb{Z}_+} V_n(Y, X) \) and \( \bigcup_{n \in \mathbb{Z}_+} V_n(\tilde{Y}, \tilde{X}) \) are strongly shift equivalent.

**Proof.** By Nasu’s theorem it is enough to consider the case that the subshifts \( X \subset \Sigma^\mathbb{Z} \) and \( \tilde{X} \subset \Sigma^\mathbb{Z} \) are bipartitely related by a specification
\[ \varphi : \Sigma \leftrightarrow \Delta \tilde{\Delta}, \quad \tilde{\varphi} : \tilde{\Sigma} \leftrightarrow \tilde{\Delta} \Delta. \]

Denote by \( \Sigma_Y(\tilde{\Sigma}_Y) \) the set of symbols in \( \Sigma(\tilde{\Sigma}) \) that are admissible for \( Y(\tilde{Y}) \), and denote by \( \varphi_Y(\tilde{\varphi}_Y) \) the restriction of \( \varphi_Y(\tilde{\varphi}_Y) \) to \( \Sigma_Y(\tilde{\Sigma}_Y) \). As is seen from Theorem (4.1) a 1-step strong shift equivalence between the λ-graph systems \( \bigcup_{n \in \mathbb{Z}_+} V_n(Y, X) \) and \( \bigcup_{n \in \mathbb{Z}_+} V_n(\tilde{Y}, \tilde{X}) \) is implemented by the symbolic matrices
\[ K^{(n,n-1)} = (K^{(n,n-1)}(C_Y, C_X))_{(\tilde{C}_Y, \tilde{C}_X) \in V_n(Y, X), (\tilde{C}_Y, \tilde{C}_X) \in V_{n-1}(\tilde{Y}, \tilde{X})}, \quad n \in \mathbb{N}, \]
where
\[ \tilde{K}^{(n,n-1)} = (\tilde{K}^{(n,n-1)}(\tilde{C}_Y, \tilde{C}_X))_{(\tilde{C}_Y, \tilde{C}_X) \in V_n(\tilde{Y}, \tilde{X}), (\tilde{C}_Y, \tilde{C}_X) \in V_{n-1}(Y, X)}, \quad n \in \mathbb{N}, \]
with \( \tilde{K}^{(n,n-1)} \) given by the symmetric expression, \( n \in \mathbb{N} \). □
5. Examples

Subsystem equivalence of subshifts of shifts of finite type was studied in [BK]. Here we consider some examples of subshifts of cartesian powers of a Dyck shift. We recall the construction of the Dyck shift $D_2$. Consider the inverse monoid with generators $\alpha^-, \alpha^+, \beta^-, \beta^+$ and relations
\begin{equation}
\alpha^- \alpha^+ = \beta^- \beta^+ = 1, \alpha^- \beta^+ = \beta^- \alpha^+ = 0.
\end{equation}

$D_2$ is a subshift of $\{\alpha^-, \alpha^+, \beta^-, \beta^+\}^\mathbb{Z}$. A point $(x_i)_{i \in \mathbb{Z}} \in \{\alpha^-, \alpha^+, \beta^-, \beta^+\}^\mathbb{Z}$, is in $D_2$ precisely if
\begin{equation}
\prod_{-I \leq i \leq I} x_i \neq 0, \quad I \in \mathbb{N}.
\end{equation}

Let $S_2$ denote the shift on $\{0, 1\}^\mathbb{Z}$. Setting
\[
\Phi^-(0) = \alpha^-, \Phi^-(1) = \beta^-, \Phi^+(0) = \alpha^+, \Phi^+(1) = \beta^+,
\]
and denoting the identity map on $\{\alpha^-, \beta^-, \alpha^+, \beta^+\}$ by $\Phi$, define $Y^-(Y^+)$ to be the subshift of $D_2 \times D_2$ that is the image of the embedding of $D_2 \times S_2$ into $D_2 \times D_2$ that is given by the 1-block map $\Phi \times \Phi^-(\Phi \times \Phi^+)$, and define $Y$ to be the subshift of $D_2 \times D_2 \times D_2$ that is the image of the embedding of $D_2 \times S_2 \times S_2$ into $D_2 \times D_2 \times D_2$ that is given by the 1-block map $\Phi \times \Phi^- \times \Phi^+$. It is
\begin{equation}
\lambda (V(D_2 \times D_2)) = \lambda (V(Y^+, D_2 \times S_2)) = \log 2,
\end{equation}
\begin{equation}
\lambda (V(Y^-, D_2 \times D_2)) = \lambda (V(D_2 \times D_2)) = \log 4,
\end{equation}
and
\[
\lambda (V(D_2 \times S_2 \times S_2)) = \log 2,
\]
\[
\lambda (V(Y, D_2 \times D_2 \times D_2)) = \log 4,
\]
\[
\lambda (V(D_2 \times D_2 \times D_2)) = \log 6.
\]

One notes that $D_2 \times D_2$ has a time reversal that carries $Y^-$ onto $Y^+$. However (3) and (4) imply that there does not exist an automorphism of $D_2 \times D_2$ that carries $Y^-$ onto $Y^+$.

Let $K \in \mathbb{N}$, and consider the semigroup with generators $\alpha^-, \alpha^+, \beta^-, \beta^+$, together with generators $\gamma^-(k), \gamma^+(k), 1 \leq k \leq K$, with the relations (1) and also the relations
\[
\gamma^-(k) \gamma^+(k) = 1 \quad 1 \leq k \leq K,
\]
\[
\gamma^-(l) \gamma^+(m) = 0, \quad 1 \leq l, m \leq K, l \neq m.
\]

and the relations
\[
\alpha^- \gamma^+(k) = \beta^- \gamma^+(k) = 0, \quad \gamma^-(k) \alpha^+ = \gamma^-(k) \beta^+ = 1, \quad 1 \leq k \leq K.
\]

(comp. [Kr]). Define a subshift $X$ of $(\{\alpha^-, \alpha^+, \beta^-, \beta^+\} \cup \{\gamma^-(k), \gamma^+(k), 1 \leq k \leq K\})^\mathbb{Z}$ that contains the points $(x_i)_{i \in \mathbb{Z}} \in (\{\alpha^-, \alpha^+, \beta^-, \beta^+\} \cup \{\gamma^-(k), \gamma^+(k), 1 \leq k \leq K\})^\mathbb{Z}$ such that (2) holds. Here $D_2 \subset X$ and one has
\[
\lambda (V(D_2, X)) = \log 2 + \log K.
\]

One has
\[
\lambda (X) = \log(2 + K).
\]

Note that here for $K > 2$ the separation entropy of the subsystem exceeds the $\lambda$-entropy of the receiving subshift.
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