Quantum Groups and Twisted Spectral Triples

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Abstract

Through the example of the quantum symplectic 4-sphere, we discuss how the notion of twisted spectral triple fits into the framework of quantum homogeneous spaces.

Keywords: Noncommutative geometry, quantum groups, Dirac operator, twisted spectral triples.
1 Introduction

In a recent paper [4] it is explained how a simple twist in the original definition of spectral triple [2, 3] makes it possible to deal with algebras with no (or few) traces. It was also suggested that this notion has potential applications to quantum groups and quantum homogeneous spaces, a domain where to construct a spectral triple may sometimes be problematic.

We investigate the connection between twisted spectral triples and quantum homogeneous spaces using as guiding example the algebra $\mathcal{A}(S^4_q)$ of the quantum symplectic 4-sphere constructed in [12], on which up to now no spectral triples are known. A possible application of a ‘twisted’ Dirac operator is in the construction of a differential calculus, which in the particular case of $S^4_q$ is fundamental for the study of noncommutative instantons. Another point is the study of the spectral action [1] on $S^4_q$: a problem shared by most quantum homogeneous spaces is that the axioms for a ‘real structure’ are fulfilled only modulo an ideal of ‘infinitesimals’, and we are not able to give meaning to the ‘adjoint representation’ of 1-forms (see e.g. [6, 7, 8, 9], and [5] for a example which doesn’t suffer from this problem). We’ll explain how the notion of real structure can be (trivially) extended to the case of twisted spectral triples, and then construct a real structure in the example of $S^4_q$.

We start with a reformulation of the notion of twisted spectral triple which seems to be appropriate when studying quantum homogeneous spaces.

We call the data $(\mathcal{A}, \mathcal{H}, D, K)$ a twisted spectral triple if (i) $\mathcal{A}$ is a complex associative involutive algebra with unity (for short $\ast$-algebra) represented by bounded operators on a separable Hilbert space $\mathcal{H}$, (ii) $D$ is a (unbounded) selfadjoint operator on $\mathcal{H}$ with dense domain and compact resolvent, (iii) $K$ is an invertible linear operator on $\text{dom} \, D$, (iv) the ‘1-form’

$$da := K^{-1}(Da - (K^{-1} a K) D)$$

extends to a bounded operator on $\mathcal{H}$ for any $a \in \mathcal{A}$. If $K$ is the identity, one gets the original definition of spectral triple [2]. If $K$ is bounded the data $(\mathcal{A}, \mathcal{H}, D', \sigma)$, with $D' := K^{-1} D$ and $\sigma(a) := K^{-2} a K^2$, is a $\sigma$-spectral triple in the sense of [4], i.e. $D'$ has compact resolvent and $D' a - \sigma(a) D'$ is bounded (being equal to $da$); notice that not all automorphisms $\sigma$ are implementable, so the notion of $\sigma$-spectral triple of [4] is more general. If $K$ and $K^{-1}$ are both bounded, we can ‘untwist’ the Dirac operator by defining $D'' = KD$; since $D''$ has compact resolvent and $[D'', a] = K^2 da$ is bounded, the data $(\mathcal{A}, \mathcal{H}, D'')$ is an ordinary spectral triple. As usual, we will refer to $D$ as the ‘Dirac operator’, in analogy with the commutative situation where spectral triples are canonically associated to spin structures. Moreover, we’ll identify $\mathcal{A}$ with its representation and omit the representation symbol.

In this paper we construct a triple over the quantum symplectic 4-sphere $S^4_q$ which satisfies all the axioms of a twisted spectral triple, but for the compact resolvent condition. To compute the spectrum of the Dirac operator is not an easy task, mainly because $\mathcal{A}(S^4_q)$ has no known symmetries and we cannot use the powerful tools of representation theory. We stress that the compact resolvent condition has no role in the construction of the differential calculus associated to $D$ (not even in the construction of the spectral action), which we describe in the following.
Given a twisted spectral triple \((\mathcal{A}, \mathcal{H}, D, K)\), a differential calculus \((\Omega^\bullet, d)\) can be constructed as explained in [4]. We define \(\Omega^\bullet\) as the \(\mathbb{N}\)-graded algebra generated by degree 0 elements \(a \in \mathcal{A}\) and degree 1 elements \(db\) given by (1.1), \(b \in \mathcal{A}\), with \(\mathcal{A}\)-bimodule structure

\[
a \cdot \omega = \sigma(a)\omega, \quad \omega \cdot a = \omega a
\]

where \(\sigma(a) = K^{-2}aK^2\) and for all \(a \in \mathcal{A}\), \(\omega \in \Omega^\bullet\) (on the right hand sides the multiplication in \(\mathcal{B}(\mathcal{H})\) is understood). Thanks to the modified bimodule structure, the Leibniz rule is satisfied

\[
d(ab) = da \cdot b + a \cdot db, \quad \forall a, b \in \mathcal{A}.
\]

The notion of even and real spectral triple can be extended in a straightforward way to the twisted case. A (twisted) spectral triple is called even if there exists a \(\mathbb{Z}_2\)-grading \(\gamma\) on \(\mathcal{H}\) (i.e. a bounded selfadjoint operator satisfying \(\gamma^2 = 1\)) such that the Dirac operator is odd and the algebra \(\mathcal{A}\) is even:

\[
\gamma D + D\gamma = 0, \quad a\gamma = \gamma a \quad \forall a \in \mathcal{A}.
\]

A real structure on a (twisted) spectral triple is a bounded antilinear operator \(J\) on \(\mathcal{H}\) satisfying

\[
J^2 = \pm 1, \quad JD = \pm DJ,
\]

and such that for all \(a, b \in \mathcal{A}\)

\[
[a, JbJ^{-1}] = 0, \quad [da, JbJ^{-1}] = 0.
\]

We refer to last equation as the ‘first order condition’. If the spectral triple is even, we impose the further condition \(J\gamma = \pm \gamma J\). The signs ‘\(\pm\)’ in previous equations are determined by the dimension of the geometry [3]; a real spectral triple of dimension 4, for example, corresponds to the choices \(J^2 = -1, JD = DJ\) and \(J\gamma = \gamma J\).

For the reader’s ease, we recall also the notions of module algebra and of crossed product algebra (see e.g. [11]), which will be used throughout the paper. Let \(\mathcal{A}\) be a \(*\)-algebra and \((\mathcal{U}, \Delta, \epsilon, S)\) a \(*\)-Hopf algebra. We say that \(\mathcal{A}\) is a (left) \(\mathcal{U}\)-module \(*\)-algebra if there is a (left) action ‘\(\triangleright\)’ of \(\mathcal{U}\) on \(\mathcal{A}\) satisfying

\[
h \triangleright ab = (h(1) \triangleright a)(h(2) \triangleright b), \quad h \triangleright 1 = \epsilon(h)1, \quad h \triangleright a^* = \{S(h)^* \triangleright a\}^*,
\]

for all \(h \in \mathcal{U}\) and \(a, b \in \mathcal{A}\). If \(\mathcal{A}\) is a (left) \(\mathcal{U}\)-module \(*\)-algebra, the (left) crossed product \(\mathcal{A} \rtimes \mathcal{U}\) is defined as the \(*\)-algebra generated by \(\mathcal{A}\) and \(\mathcal{U}\) with crossed commutation relations

\[
h a = (h(1) \triangleright a)h(2), \quad \forall h \in \mathcal{U}, a \in \mathcal{A}.
\]

Right module algebras and right crossed product algebras are defined similarly. As usual we use Sweedler notation for the coproduct, \(\Delta(h) = h(1) \otimes h(2)\).

The plan of the paper is the following. In Section 2, we present the algebra of the quantum symplectic 7-sphere \(S_q^7\) and its symmetry Hopf algebra \(U_q(so(5))\). In Section 3, we construct
the algebra of the symplectic 4-sphere $S_q^4$ of [12] as the subalgebra of $\mathcal{A}(S_q^7)$ which is invariant for the left action of a sub $*$-Hopf algebra of $U_q(so(5))$ isomorphic to $U_q(su(2))$. In Section 4, we construct a bounded $*$-representation of $\mathcal{A}(S_q^4)$ on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ which is a deformation of the space of $L^2$-spinors on the round 4-sphere. In Section 5, we construct a Dirac operator $D$ and prove that, leaving aside the compact resolvent condition, the data $(\mathcal{A}(S_q^4), \mathcal{H}, D, K)$ satisfies all other axioms for a twisted spectral triple. In Section 6, we complete the picture by constructing a real structure $J$. The study of the differential calculus on $\mathcal{A}(S_q^4)$ associated to $D$ is postponed to forthcoming papers. If finite dimensional, this differential calculus would provide a framework for the study of $q$-deformations of instantons.

2 The symplectic 7-sphere and its symmetries

We now introduce the main characters of this paper, the quantum universal enveloping algebra $U_q(so(5))$ and the algebra $\mathcal{A}(S_q^7)$ of the symplectic 7-sphere [12].

For $0 < q < 1$, we call $U_q(so(5))$ the ‘compact’ real form of the Hopf algebra denoted $\tilde{U}_q(so(5))$ in [11]. As a $*$-algebra, it is generated by $\{K_i = K_i^+, K_i^{-1}, E_i, F_i := E_i^\ast\}_{i=1,2}$ with relations

\[
[K_1, K_2] = 0 , \quad K_iK_i^{-1} = K_i^{-1}K_i = 1 , \quad [E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}},
\]

\[
K_iE_iK_i^{-1} = q^i E_i , \quad K_iE_jK_i^{-1} = q^{-1} E_j \text{ if } i \neq j ,
\]

plus Serre relations, explicitly, given by

\[
E_1E_2^2 - (q^2 + q^{-2})E_2E_1E_2 + E_2^2E_1 = 0 , \quad (2.1a)
\]

\[
E_3^3E_2 - (q^2 + 1 + q^{-2})(E_3^2E_2E_1 - E_1E_2E_3^2) - E_2E_3^3 = 0 . \quad (2.1b)
\]

Serre relations can be written in a more compact form by defining $[a, b]_q := q^a b - ba$. Then, (2.1) are equivalent to

\[
[E_2, [E_1, E_2]_q] = 0 , \quad [E_1, [E_1, [E_2, E_1]_q]] = 0 .
\]

The Hopf algebra structure $(\Delta, \epsilon, S)$ of $U_q(so(5))$ is given by

\[
\Delta K_i = K_i \otimes K_i , \quad \Delta E_i = E_i \otimes K_i + K_i^{-1} \otimes E_i , \quad \epsilon(K_i) = 1 , \quad \epsilon(E_i) = 0 , \quad S(K_i) = K_i^{-1} , \quad S(E_i) = -q^i E_i .
\]

Notice that the elements $(K_1, K_1^{-1}, E_1, F_1)$ generate a sub-Hopf-$*$-algebras of $U_q(so(5))$, which for obvious reasons we'll denote $U_q(su(2))$ in the following.

For each non negative $n_1, n_2$ such that $n_2 \in \frac{1}{2}\mathbb{Z}$ and $n_2 - n_1 \in \mathbb{N}$, there is an irreducible representation of $U_q(so(5))$, whose representation space we denote $V_{(n_1, n_2)}$ (here $\mathbb{N}$ denotes non-negative integers, i.e. it includes 0) and whose highest weight vector is an eigenvector of $K_1$ and $K_2$ with eigenvalues $q^{n_1}$ and $q^{n_2-n_1}$ respectively.
As an example, let us draw the weight diagrams of \(V_{(\frac{1}{2}, \frac{1}{2})}, V_{(0,1)}\) and \(V_{(1,1)}\).

\[
\begin{array}{c}
V_{(\frac{1}{2}, \frac{1}{2})}, \text{ dim } 4.
\end{array}
\]

\[
\begin{array}{c}
V_{(0,1)}, \text{ dim } 5.
\end{array}
\]

\[
\begin{array}{c}
V_{(1,1)}, \text{ dim } 10.
\end{array}
\]

A solid arrow indicates points that can be joined by applying \(E_1\) (the reverse arrow corresponds to \(F_1\)), a dashed arrow indicates points that can be joined by applying \(E_2\) (the reverse arrow corresponds to \(F_2\)). In each diagram, the highest weight vector is denoted by its weight \((n_1, n_2)\). A bullet indicates a weight with multiplicity 1, an empty circle a weight with multiplicity 2.

The Hopf algebra \(U_q(so(5))\) describes the symmetries of an algebra which is a deformation of the algebra of polynomial functions on the 7-sphere \(S^7\).

**Definition 2.1.** We call symplectic 7-sphere the ‘virtual space’ underlying the \(*\)-algebra \(A(S^7_q)\) with generators \(z_i, z_i^*(i = 1, \ldots, 4)\), commutation relations

\[
\begin{align*}
z_1z_2 &= q^{-1}z_2z_1, & z_2z_4 &= q^{-1}z_4z_2, & z_2z_4^* &= q^{-1}z_4^*z_2, \\
z_1z_3 &= q^{-1}z_3z_1, & z_3z_4 &= q^{-1}z_4z_3, & z_3z_4^* &= q^{-1}z_4^*z_3, \\
z_1z_4 &= q^{-2}z_4z_1, & z_1z_4^* &= q^{-2}z_4^*z_1, & z_2z_3^* &= q^{-2}z_3z_2, \\
z_2z_3 - q^2z_3z_2 &= (q - q^2)z_1z_4, & z_1z_2^* - q^{-1}z_2^*z_1 &= (q - q^{-1})z_1^*z_4, & z_1z_3^* - q^{-1}z_3^*z_1 &= q(1 - q^2)z_4^*z_2, \\
[z_4^*, z_4] &= 0, & [z_2^*, z_2] &= (1 - q^2)z_4z_4^*, & [z_3^*, z_3] &= z_2^*z_2 - q^4z_2z_2^*.
\end{align*}
\]

and

\[
z_1z_1^* + z_2z_2^* + z_3z_3^* + z_4z_4^* = z_1^*z_1 + q^6z_2^*z_2 + q^2z_3^*z_3 + q^8z_4^*z_4 = 1.
\]

The generators used in [12], which we denote \(x_i'\), are related to ours by the equations \(x_1' = q^4z_4, x_2' = q^3z_2, x_3' = -qz_3\) and \(x_4' = z_1\).

**Proposition 2.2.** The algebra \(A(S^7_q)\) is an \(U_q(so(5))\)-module \(*\) algebra for the action defined on generators by

\[
\begin{align*}
K_1 \triangleright z_1 &= q^{1/2}z_1, & K_1 \triangleright z_2 &= q^{1/2}z_2, & K_1 \triangleright z_3 &= q^{-1/2}z_3, & K_1 \triangleright z_4 &= q^{-1/2}z_4, \\
K_2 \triangleright z_1 &= z_1, & K_2 \triangleright z_2 &= q^{-1}z_2, & K_2 \triangleright z_3 &= qz_3, & K_2 \triangleright z_4 &= z_4, \\
E_1 \triangleright z_1 &= 0, & E_1 \triangleright z_2 &= 0, & E_1 \triangleright z_3 &= z_1, & E_1 \triangleright z_4 &= z_2, \\
E_2 \triangleright z_1 &= 0, & E_2 \triangleright z_2 &= z_3, & E_2 \triangleright z_3 &= 0, & E_2 \triangleright z_4 &= 0, \\
F_1 \triangleright z_1 &= z_3, & F_1 \triangleright z_2 &= z_4, & F_1 \triangleright z_3 &= 0, & F_1 \triangleright z_4 &= 0, \\
F_2 \triangleright z_1 &= 0, & F_2 \triangleright z_2 &= 0, & F_2 \triangleright z_3 &= z_2, & F_2 \triangleright z_4 &= 0.
\end{align*}
\]

Let \(U_q(u(1))\) be the Hopf \(*\) algebra generated by \(K_1, K_1^{-1}\). Then \(A(S^7_q)\) is also a right \(U_q(u(1))\)-module \(*\) algebra for the action defined by \(z_i \triangleleft K_1 = q^{1/2}z_i\). Left and right actions commute.
Proof. We consider first the free $*$-algebra generated by $\{z_i, z^*_i\}$. The elements $z_i$ carry the fundamental representation $V_{(\frac{1}{2}, \frac{1}{2})}$ of $U_q(so(5))$, while the action is extended to $z^*_i$ by compatibility with the involution using the rule $h \triangleright a^* = \{S(h)^* \triangleright a\}^*$. Thus the free $*$-algebra is trivially an $U_q(so(5))$-module $*$-algebra.

Let $V := V_{(\frac{1}{2}, \frac{1}{2})} \oplus V_{(\frac{1}{2}, \frac{1}{2})}$. Inside the decomposition of Hopf tensor product $V \otimes V$ we consider the following vectors (tensor product symbol implied)

$$
\begin{align*}
\ v_1 & := z_1z^*_1 - q^{-2}z^*_1z_1, & v_5 & := q^2z_4z_4 - q^{-2}z_4z_1 + q^{-1}z_2z_3 - qz_3z_2, \\
\ v_2 & := z_1z_3 - q^{-1}z_3z_1, & v_6 & := v^*_5, \\
\ v_3 & := (z_2z_4 - q^{-1}z_4z_2)^*, & v_7 & := z_1^*z_1 + q^6z^*_2z_2 + q^2z^*_3z_3 + q^8z^*_4z_4 - 1, \\
\ v_4 & := z_1^*z_2 + z_3z_4^* - q^{-1}(z_2^*z_1 + q^2z_4^*z_3), & v_8 & := z_1^*z_1 + z_2^*z_2 + z_3^*z_3 + z_4^*z_4 - 1.
\end{align*}
$$

As one check by direct computation, $v_i$’s are eigenvectors of $K_i$’s and annihilated by both $E_1$ and $E_2$, hence they are highest weight vectors of irreducible representations of $U_q(so(5))$ inside $V \otimes V$. In particular, $v_1$ has weight $(1, 1)$, $\{v_2, v_3, v_4\}$ have weight $(0, 1)$, $\{v_5, v_6, v_7, v_8\}$ have weight $(0, 0)$. By applying $F_1$ and $F_2$ to these highest weight vectors one proves that a linear basis for the real 29-dimensional representation

$$
4V_{(0,0)} \oplus 3V_{(0,1)} \oplus V_{(1,1)} \subset V \otimes V
$$

is given just by the degree $\leq 2$ polynomials appearing in Proposition 2.1 and by their conjugated. Thus the ideal they generate is $U_q(so(5))$-invariant and $A(S^7_q)$, being the quotient of an $U_q(so(5))$-module $*$-algebra by a two-sided invariant $*$-ideal, is itself an $U_q(so(5))$-module $*$-algebra.

The remaining part of the Proposition is trivial.

From the general theory of compact matrix quantum groups (cf. Section 11 of [11]) we know that there exists a (unique) positive faithful $U_q(so(5))$-invariant linear functional $\varphi : A(S^7_q) \to \mathbb{C}$. This functional comes from the Haar state of the Hopf algebra $A(Sp_q(2))$ dual to $U_q(so(5))$, and satisfies

$$
\varphi(ab) = \varphi(b \kappa(a)), \tag{2.2}
$$

for all $a, b \in A(S^7_q)$, where $\kappa : A(S^7_q) \to A(S^7_q)$ is called the ‘modular automorphism’ and is given by (cf. Section 11.3.4 of [11])

$$
\kappa(a) = K_1^8K_2^6 \triangleright a < K_1^8. \tag{2.3}
$$

3 The quantum symplectic 4-sphere

Consider the algebra

$$
A(S^4_q) := \{ a \in A(S^7_q) \mid h \triangleright a = \epsilon(h)a \ \forall \ h \in U_q(su(2)) \} \tag{3.1}
$$

where $U_q(su(2))$ is the sub-Hopf-$*$-algebra of $U_q(so(5))$ generated by $(K_1, K_1^{-1}, E_1, F_1)$. We now show that this is just the algebra of ‘functions’ of the 4-sphere constructed in [12].
Dually to the left action of $U_q(su(2))$ we can define a right coaction of $A(SU_q(2))$ such that $A(S^4_q)$ is a right comodule $*$-algebra. Recall that $A(SU_q(2))$ is the $*$-algebra generated by $\alpha, \beta$ and the adjoints with relations

$$\beta\alpha = q\alpha\beta, \quad \beta^*\alpha = q\alpha^*\beta, \quad [\beta, \beta^*] = 0, \quad \alpha^*\alpha + q^2\beta^*\beta = 1, \quad \alpha\alpha^* + \beta\beta^* = 1,$$

with Hopf algebra structure

$$\Delta \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right) \hat{\otimes} \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right),$$

$$\epsilon \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right) = 1, \quad S \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right) = \left( \begin{array}{cc} \alpha^* & -q\beta \\ \beta^* & \alpha \end{array} \right),$$

and with obvious $*$-structure. We use the same notation of [7], but for greek letters instead of latin ones. The dotted tensor product is defined as $(A \hat{\otimes} B)_{ij} = \sum_k A_{ik} \otimes B_{kj}$.

The coaction $\Delta_R(a) = a_{(0)} \otimes a_{(1)}$ dual to the right action of $U_q(su(2))$ is determined by:

$$h \triangleright a =: a_{(0)} \langle h, a_{(1)} \rangle,$$

with $\langle \ , \ \rangle$ the dual pairing between $U_q(su(2))$ and $A(SU_q(2))$, given by (cf. Section 4.4.1 of [11])

$$\langle K_1, \alpha \rangle = q^{1/2}, \quad \langle K_1, \alpha^* \rangle = q^{-1/2}, \quad \langle E_1, \beta \rangle = \langle F_1, -q\beta^* \rangle = 1,$$

while $\langle h, a \rangle = 0$ for any other pair of generators. Using these we compute the coaction dual to the left action of $U_q(su(2))$, and get

$$\Delta_R(z_1) = z_1 \otimes \alpha - qz_3 \otimes \beta^*,$$
$$\Delta_R(z_2) = z_2 \otimes \alpha - qz_4 \otimes \beta^*,$$
$$\Delta_R(z_3) = z_3 \otimes \alpha^* + z_1 \otimes \beta,$$
$$\Delta_R(z_4) = z_4 \otimes \alpha^* + z_2 \otimes \beta.$$

Let $\Psi$ be the matrix

$$\Psi = \left( \begin{array}{ccc} -qz_3^* & z_1^* \\ z_1 & z_3 \\ qz_2 & qz_4 \\ -q^2z_4^* & q^2z_2^* \end{array} \right).$$

(3.2)

The coaction can be encoded in the compact formula

$$\Delta_R(\Psi) = \Psi \hat{\otimes} \left( \begin{array}{cc} \alpha & \beta \\ -q\beta^* & \alpha^* \end{array} \right).$$

(3.3)

Since $\Psi^\dagger\Psi = 1$, the matrix $P := \Psi\Psi^\dagger$ is a projection and by construction its elements are $A(SU_q(2))$-coinvariant. By duality between the action of $U_q(su(2))$ and the coaction of $A(SU_q(2))$, the algebra of coinvariant is just $A(S^4_q)$, which then coincides with the algebra
called symplectic 4-sphere in [12]. In particular, in [12] it was proved that $\mathcal{A}(S^4_q)$ is generated by the matrix elements $P_{ij}$ of $P$, with relations and *-structure $P = P^* = P^2$.

We choose the generators

$$x_0 := z_2 z_2^* + z_4 z_4^*, \quad x_1 := q(z_1 z_2^* + z_3 z_4^*), \quad x_2 := z_2 z_3 - q z_1 z_4,$$

and notice that the projection $P$ becomes

$$P = \begin{pmatrix}
1 - q^6 x_0 & 0 & -q^2 x_2^* & q^2 x_1^* \\
0 & 1 - x_0 & x_1 & x_2 \\
-q^2 x_2 & x_1^* & q^2 x_0 & 0 \\
q^2 x_1 & x_2^* & 0 & q^4 x_0
\end{pmatrix}.$$  \hspace{1cm} (3.4)

With this expression, we compute the relations among the $x_i$'s, which we summarize in the next proposition.

**Proposition 3.1.** The *-algebra $\mathcal{A}(S^4_q)$ is generated by $x_0 = x_0^*$, $x_i$ and $x_i^*$ ($i = 1, 2$) with relations

$$x_0 x_i = q^{2i} x_i x_0, \quad x_1 x_2 = x_2 x_1, \quad x_1^* x_2 = q^4 x_2 x_1^*, \quad x_i^* x_i - q^4 x_i x_i^* = (1 - q^{2i})(q^2 x_0)^i,$$

and

$$x_0^2 + x_1 x_1^* + x_2 x_2^* = x_0.$$  

The generators used in [12] are given by $t := q^4 x_0$, $a := x_1^*$ and $b := q^2 x_2$.

## 4 The modules of chiral spinors

The left regular representation of a compact matrix quantum group is bounded with respect to the inner product induced by the Haar state $\varphi$. Being $\mathcal{A}(S^7_q) \subset \mathcal{A}(Sp_q(2))$, if we define the inner product of two elements $v = (v_1, v_2)$ and $w = (w_1, w_2)$ of $\mathcal{A}(S^7_q)^2$ as

$$\langle v, w \rangle := \varphi(v_1^* w_1) + \varphi(v_2^* w_2),$$  \hspace{1cm} (4.1)

then the representation of $\mathcal{A}(S^7_q)$ by left multiplication on this inner product space is a bounded *-representation. We now define two subspaces of $\mathcal{A}(S^7_q)^2$ which are invariant when multiplied by elements in the subalgebra $\mathcal{A}(S^4_q) \subset \mathcal{A}(S^7_q)$.

Let $\sigma_+ : U_q(so(2)) \to \text{Mat}_4(\mathbb{C})$ be following representation

$$\sigma_+(K_1) = \begin{pmatrix} q^{1/2} & 0 \\
0 & q^{-1/2}
\end{pmatrix}, \quad \sigma_+(E_1) = \begin{pmatrix} 0 & 1 \\
0 & 0
\end{pmatrix}, \quad \sigma_+(F_1) = \begin{pmatrix} 0 & 0 \\
0 & 1
\end{pmatrix},$$

where as before $U_q(su(2))$ denotes the *-Hopf algebra generated by $(K_1, K_1^{-1}, E_1, F_1)$. A second (unitary equivalent) representation is given by

$$\sigma_-(h) = \begin{pmatrix} 0 & -1 \\
1 & 0
\end{pmatrix} \sigma_+(h) \begin{pmatrix} 0 & 1 \\
-1 & 0
\end{pmatrix}.$$
for all \( h \in U_q(su(2)) \).

If we write \( v = (v_1, v_2) \in \mathcal{A}(S_q^7)^2 \) as a two vector, two actions of \( h \in U_q(su(2)) \) on \( \mathcal{A}(S_q^7)^2 \) can be defined through the formula

\[
h^{\updownarrow \pm} v := (h_{(1)} \triangleright v)\sigma_{\pm}(S(h_{(2)}) ) ,
\]

(4.2)

where on the right row by column multiplication is understood. With these actions, we define two subspaces \( \mathcal{M}_{\pm} \) of \( \mathcal{A}(S_q^7)^2 \) as follows

\[
\mathcal{M}_{\pm} := \{ v \in \mathcal{A}(S_q^7)^2 \mid h^{\updownarrow \pm} v = \epsilon(h)v \forall h \in U_q(su(2)) \} ,
\]

(4.3)

which in particular means that

\[
K_1 \triangleright (v_1, v_2) = (q^{\frac{3}{2}}v_1, q^{-\frac{1}{2}}v_2) , \quad E_1 \triangleright (v_1, v_2) = (0, qv_1) , \quad F_1 \triangleright (v_1, v_2) = (q^{-1}v_2, 0) ,
\]

(4.4)

for all \((v_1, v_2) \in \mathcal{M}_+ \) and

\[
K_1 \triangleright (w_1, w_2) = (q^{-\frac{1}{2}}w_1, q^{\frac{1}{2}}w_2) , \quad E_1 \triangleright (w_1, w_2) = (-qw_2, 0) , \quad F_1 \triangleright (w_1, w_2) = (0, -q^{-1}w_1) ,
\]

(4.5)

for all \((w_1, w_2) \in \mathcal{M}_- \). Conditions (4.4) are necessary and sufficient for a vector \((v_1, v_2) \) to be an element of \( \mathcal{M}_+ \), and conditions (4.5) are necessary and sufficient for a vector \((w_1, w_2) \) to be an element of \( \mathcal{M}_- \).

**Lemma 4.1.** The linear spaces \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) are orthogonal.

**Proof.** From (4.4) and (4.5) we get

\[
K_1 \triangleright v_1^\ast w_1 = q^{-1}v_1^\ast w_1 , \quad K_1 \triangleright v_2^\ast w_2 = qv_2^\ast w_2 .
\]

for all \( v \in \mathcal{M}_+ \) and \( w \in \mathcal{M}_- \). Applying the Haar functional to both sides of the equations and using its invariance we get

\[
\phi(v_1^\ast w_1) = q^{-1}\phi(v_1^\ast w_1) , \quad \phi(v_2^\ast w_2) = q\phi(v_2^\ast w_2) ,
\]

which imply \( \phi(v_1^\ast w_1) = \phi(v_2^\ast w_2) = 0 \) and then \( \langle v, w \rangle = 0 \) for all \( v \in \mathcal{M}_+ \) and \( w \in \mathcal{M}_- \). \( \square \)

**Lemma 4.2.** The linear spaces \( \mathcal{M}_{\pm} \) are \( \mathcal{A}(S_q^4) \)-bimodules.

**Proof.** By (3.1) we have

\[
h^{\updownarrow \pm} (av) = (h_{(1)} \triangleright a)(h_{(2)}^{\updownarrow \pm} v) = a\{h_{(1)}h_{(2)}^{\updownarrow \pm} v\} = a(h^{\updownarrow \pm} v)
\]

\[
h^{\updownarrow \pm} (va) = h_{(1)}^{\updownarrow \pm} (va)\sigma_{\pm}(S(h_{(2)})) = \{h_{(1)}^{\updownarrow \pm} v\sigma_{\pm}(S(h_{(2)}))\}(h_{(2)} \triangleright a) = (h^{\updownarrow \pm} v)a
\]

for all \( h \in U_q(su(2)) \), \( a \in \mathcal{A}(S_q^4) \) and \( v \in \mathcal{A}(S_q^7)^2 \). In particular, if \( v \) is an invariant element for the action \( h^{\updownarrow \pm} \), \( av \) and \( va \) are invariant elements too. Hence, \( \mathcal{M}_{\pm} \) are invariant subspaces of \( \mathcal{A}(S_q^7)^2 \) with respect to left/right multiplication by \( \mathcal{A}(S_q^4) \). \( \square \)
Proposition 4.3. \( M \) is even with respect to the natural grading \( \gamma \), to which in the next sections we will add a Dirac operator \( D \) and a real structure \( J \).

Note that since \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) are orthogonal, their direct sum is just their linear span.

Usually, an additional requirement for a spectral triple is that the domain of the Dirac operator contains a dense subspace of \( \mathcal{H} \) which is a finitely generated projective (left) \( \mathcal{A} \)-module (this is called ‘finiteness axiom’ in Section 10.5 of [10]). With this in mind, we now prove that \( \mathcal{M}_+ \) is finitely generated and projective both as left and right \( \mathcal{A}(S_q^4) \)-modules.

Proposition 4.3. \( \mathcal{M}_+ \) is isomorphic to \( \mathcal{A}(S_q^4)^4P \) as a left \( \mathcal{A}(S_q^4) \)-module, with \( P = \Psi \Psi^\dagger \) and \( \Psi \) the matrix defined in Equation (3.2).

Proof. A linear map \( \rho : \mathcal{M}_+ \to \mathcal{A}(S_q^4)^4P \) is defined by
\[
\rho(v_1, v_2) = (v_1, v_2) \cdot \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \Psi^\dagger,
\]
where matrix multiplication on the right hand side is understood, and writing elements of \( \mathcal{A}(S_q^4)^4P \) as row vectors. Since \( \Psi^\dagger P = \Psi^\dagger \) it is clear that the image of \( \rho \) is in \( \mathcal{A}(S_q^4)^4P \). By construction \( \rho \) is a left \( \mathcal{A}(S_q^4) \)-module map.

A second left \( \mathcal{A}(S_q^4) \)-module map \( \rho^{-1} : \mathcal{A}(S_q^4)^4P \to \mathcal{M}_+ \) is given by
\[
\rho^{-1}(a_1, a_2, a_3, a_4) := (a_1, a_2, a_3, a_4) \cdot \Psi \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix}.
\]
From (3.2), the invariance of \( a_i \)'s and the explicit expression of the action of \( (K_1, E_1, F_1) \) on \( \mathcal{A}(S_q^7) \), one proves that \( \rho^{-1}(a_1, a_2, a_3, a_4) \) satisfies (4.4). Hence the image of \( \rho^{-1} \) is in \( \mathcal{M}_+ \).

Since \( \Psi^\dagger \Psi = 1 \) and right multiplication for \( \Psi \Psi^\dagger \) is the identity operator on \( \mathcal{A}(S_q^4)^4P \), the maps \( \rho \) and \( \rho^{-1} \) are one the inverse of the other (as the notation suggests) and so \( \rho \) is a bijective left \( \mathcal{A}(S_q^4) \)-module map, i.e. an isomorphism of left \( \mathcal{A}(S_q^4) \)-modules.

Proposition 4.4. \( \mathcal{M}_+ \) is isomorphic to \( PA(S_q^4)^4 \) as a right \( \mathcal{A}(S_q^4) \)-module.

Proof. We write elements of the right projective module \( PA(S_q^4)^4 \) as column vectors. The proof is similar to the proof of Proposition 4.3, with the only difference that now the maps realizing the isomorphism \( \mathcal{M}_+ \simeq PA(S_q^4)^4 \), which we denote again \( \rho : \mathcal{M}_+ \to PA(S_q^4)^4 \) and \( \rho^{-1} : PA(S_q^4)^4 \to \mathcal{M}_+ \), are given by
\[
\rho(v_1, v_2) = \Psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_1, v_2)^t,
\]
\[
\rho^{-1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Psi^\dagger \begin{pmatrix} a_1 \\ 0 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}^t.
\]
Clearly \( \rho \) and \( \rho^{-1} \) are right \( \mathcal{A}(S^4_q) \)-linear, are one the inverse of the other, the image of \( \rho \) is in \( P \mathcal{A}(S^4_q) \) simply because \( P \Psi = \Psi \), and the image of \( \rho^{-1} \) is \( \mathcal{M}_+ \) since \( \rho^{-1}(a) \) satisfies (4.4) for all \( a \in \mathcal{A}(S^4_q) \).

### 5 The Dirac operator

Let \( v = (v_1, v_2) \in \mathcal{A}(S^7_q)^2 \) and consider (for \( i = 1, 2, 3 \)) the linear maps

\[
v \mapsto X^j v := \sum_{j=1,2} (X^i_{1,j} \triangleright v_j, X^i_{2,j} \triangleright v_j)
\]

with \( (X^i_{jk}) \) the matrices

\[
(X^1_{jk}) = \begin{pmatrix}
q[2]E_2 & q[E_1, E_2]_q \\
q^{-1}[E_2, E_1]_q & -[E_1, [E_2, E_1]_q]
\end{pmatrix},
\]

\[
(X^2_{jk}) = \begin{pmatrix}
-[F_1, [F_1, F_2]_q] & q[F_1, F_2]_q \\
q^{-1}[F_1, F_2]_q & -q[2]F_2
\end{pmatrix},
\]

\[
(X^3_{jk}) = K_2^{-1} \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.
\]

**Lemma 5.1.** The operator \( X^i \) maps \( \mathcal{M}_+ \) into \( \mathcal{M}_- \), for all \( i = 1, 2, 3 \).

**Proof.** Let \( v \) be a vector in \( \mathcal{M}_+ \), thus satisfying (4.4). We want to prove that the vector \( (w_1, w_2) := X^i(v_1, v_2) \) satisfies the conditions (4.5) defining \( \mathcal{M}_- \). If \( i = 3 \) the check is trivial.

Let us focus on the cases \( i = 1, 2 \).

From the defining relations of \( U_q(\text{so}(5)) \) we derive the following commutation rules

\[
K_1 A = q^{-1}AK_1 \quad \text{for } A = E_2, [E_1, [E_2, E_1]_q],
\]

\[
K_1 A = qAK_1 \quad \text{for } A = F_2, [F_1, [F_1, F_2]_q],
\]

\[
K_1 A = AK_1 \quad \text{for } A = [E_2, E_1]_q, [F_1, F_2]_q,
\]

\[
[F_1, [E_2, E_1]_q] = -[2]E_2 K_1^2,
\]

\[
[E_1, [F_1, F_2]_q] = q^2[2]F_2 K_1^2,
\]

\[
E_1[E_1, [E_2, E_1]_q] = q^{-2}[E_1, [E_2, E_1]_q]E_1 \quad \text{(Serre relation)},
\]

\[
F_1[F_1, [F_1, F_2]_q] = q^2[F_1, [F_1, F_2]_q]F_1 \quad \text{(Serre relation)}.
\]

The first three equations imply \( F_1 \triangleright (w_1, w_2) = (q^{-1/2}w_1, q^{1/2}w_2) \), which is the first condition of (4.5), while the remaining equations gives us for \( i = 1 \)

\[
F_1 \triangleright w_1 = q[2]E_2 F_1 \triangleright v_1 + q[F_1, [E_2, E_1]_q] \triangleright v_2 \quad \text{(since } F_1 \triangleright v_2 = 0)
\]

\[
= q[2]E_2 F_1 \triangleright v_1 - q[2]E_2 K_1^2 \triangleright v_2
\]

\[
= q[2]E_2 \triangleright (F_1 \triangleright v_1 - q^{-1}v_2) = 0,
\]

\[
E_1 \triangleright w_1 = q[2]E_1 E_2 \triangleright v_1 + qE_1[E_2, E_1]_q \triangleright v_2
\]

\[
= -[E_2, E_1]_q \triangleright v_1 + q[E_1, [E_2, E_1]_q] \triangleright v_2 \quad \text{(since } E_1 \triangleright v_j = q\delta_{j2} v_2)\]
= -qw_2,
E_1 \triangleright w_2 = q^{-1}E_1[E_2, E_1]_q \triangleright v_1 - E_1[E_1, [E_2, E_1]_q] \triangleright v_2
= q^{-1}E_1[E_2, E_1]_q \triangleright v_1 - q^{-2}[E_1, [E_2, E_1]_q]E_1 \triangleright v_2
= q^{-1}[E_2, E_1]_q E_1 \triangleright v_1 = 0 ,

and for \( i = 2 \)

\[ F_1 \triangleright w_1 = -F_1[F_1, [F_1, F_2]_q] \triangleright v_1 + q F_1[F_1, F_2]_q \triangleright v_2 \]
\[ = q^{-2}[F_1, [F_1, F_2]_q]F_1 \triangleright v_1 + q F_1[F_1, F_2]_q \triangleright v_2 \]
\[ = q[F_1, F_2]_q F_1 \triangleright v_2 = 0 , \]
\[ F_1 \triangleright w_2 = q^{-1}F_1[F_1, F_2]_q \triangleright v_1 - q[2]F_1 F_2 \triangleright v_2 \]
\[ = q^{-1}[F_1, [F_1, F_2]_q] \triangleright v_1 - [F_1, F_2]_q \triangleright v_2 \quad \text{(since } F_1 \triangleright v_j = q^{-1}\delta_{j1}v_2) \]
\[ = -q^{-1}w_1 , \]
\[ E_1 \triangleright w_2 = q^{-1}[E_1, [F_1, F_2]_q] \triangleright v_1 - q[2]F_2 E_1 \triangleright v_2 \quad \text{(since } E_1 \triangleright v_1 = 0) \]
\[ = q[2]F_2 \triangleright (qv_1 - E_1 \triangleright v_2) = 0 . \]

These equations implies
\[ q F_1 \triangleright w_2 = -F_1 E_1 \triangleright w_1 = [E_1, F_1] \triangleright w_1 = \frac{k^3 - k^{-2}}{q-q^{-1}} \triangleright w_1 = -w_1 \quad \text{if } i = 1 , \]
\[ q^{-1} E_1 \triangleright w_1 = -E_1 F_1 \triangleright w_2 = -[E_1, F_1] \triangleright w_2 = -\frac{k^3 - k^{-2}}{q-q^{-1}} \triangleright w_2 = -w_2 \quad \text{if } i = 2 . \]

Thus, \((w_1, w_2)\) satisfies all the conditions in Equation (4.5), and this concludes the proof. \( \square \)

**Lemma 5.2.** The operator \((X^i)^*\) maps \( \mathcal{M}_- \) into \( \mathcal{M}_+ \), for all \( i = 1, 2, 3 \).

*Proof.* Similar to the proof of Lemma 5.1. \( \square \)

**Lemma 5.3.** The operators
\[ K_2^{-1}(X^i a - (K_2^{-1} \triangleright a)X^i) \quad \text{and} \quad (X^i a - (K_2^{-1} \triangleright a)X^i)K_2^{-1} \quad (5.1) \]
are bounded on \( \mathcal{M}_+ \), for all \( a \in \mathcal{A}(S_4^q) \) and for all \( i = 1, 2, 3 \).

*Proof.* The case \( i = 3 \) is trivial, being \( X^3 a = (K_2^{-1} \triangleright a)X^3 \) by covariance of the action. For the remaining cases, we proceed as follows. Firstly, by direct computation one proves
\[ \Delta(E_2) = K_2^{-1} \otimes E_2 + E_2 \otimes K_2 \quad (5.2a) \]
\[ \Delta(F_2) = K_2^{-1} \otimes F_2 + F_2 \otimes K_2 \quad (5.2b) \]
\[ \Delta([E_2, E_1]_q) \sim K_2^{-1} \otimes [E_2, E_1]_q + [E_2, E_1]_q \otimes K_1 K_2 \quad (5.2c) \]
\[ \Delta([F_1, F_2]_q) \sim K_2^{-1} \otimes [F_1, F_2]_q + [F_1, F_2]_q \otimes K_1 K_2 \quad (5.2d) \]
\[ \Delta([E_1, [E_2, E_1]_q]) \sim K_2^{-1} \otimes [E_1, [E_2, E_1]_q] + [E_1, [E_2, E_1]_q] \otimes K_1 K_2 \quad (5.2e) \]
\[ \Delta([F_1, [F_1, F_2]_q]) \sim K_2^{-1} \otimes [F_1, [F_1, F_2]_q] + [F_1, [F_1, F_2]_q] \otimes K_1 K_2 \quad (5.2f) \]
where the notation $A \sim B$ means that the difference $A - B$ acts trivially on $\mathcal{A}(S_q^4) \otimes \mathcal{A}(S_q^7)$. For example

$$\Delta([E_2, E_1]_q) = (K_1K_2)^{-1} \otimes [E_2, E_1]_q + [E_2, E_1]_q \otimes K_1K_2 + (q^2 - q^{-2})K_2^{-1}E_1 \otimes E_2K_1,$$

hence using $E_1 \otimes 1 \sim 0$ and $K_1 \otimes 1 \sim 1$ we get the expression in Equation (5.2c).

Then, we observe that each matrix element $X^i_{jk}$ of $X^i$ $(i, j, k = 1, 2)$ is proportional to one of the elements whose coproduct has been computed in (5.2). Thus,

$$\Delta(X^i_{jk}) \sim K_2^{-1} \otimes X^i_{jk} + X^i_{jk} \otimes K_1^{n_{ijk}}K_2$$

for all $i, j, k$ and for a suitable $n_{ijk} \geq 0$. This equation together with (1.5) implies

$$K_2^{-1}(X^i_{jk}a - (K_2^{-1} \triangleright a)X^i_{jk}) = (K_2^{-1}X^i_{jk} \triangleright a)K_1^{n_{ijk}}, \quad (5.3a)$$

$$(X^i_{jk}a - (K_2^{-1} \triangleright a)X^i_{jk})K_2^{-1} = (X^i_{jk} \triangleright a)K_1^{n_{ijk}}. \quad (5.3b)$$

Now, $X^i_{jk} \triangleright a$ and $K_2^{-1}X^i_{jk} \triangleright a$ are elements of $\mathcal{A}(S_q^7)$ for all $a \in \mathcal{A}(S_q^4)$, hence are bounded since the left regular representation of $\mathcal{A}(S_q^7)$ is bounded. The restriction of $K_1$ to $\mathcal{M}_+$ is bounded too. So, the left hand sides of (5.3), which are the matrix elements of the operators in (5.1), are bounded and this concludes the proof. \hfill $\square$

For arbitrary (but fixed) $\lambda, \mu, \delta \in \mathbb{C}$ we define

$$D_+ := \lambda X^1 + \mu X^2 + \delta X^3, \quad D_- := (D_+)^*. \quad (5.4)$$

The operator

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

symmetric on $\mathcal{M}_+ \oplus \mathcal{M}_-$ and odd with respect to the grading, is our candidate for a Dirac operator. By Lemma 5.3, the operators

$$K_2^{-1}(D_+ a - (K_2^{-1} \triangleright a)D_+), \quad (D_+ a - (K_2^{-1} \triangleright a)D_+)K_2^{-1}$$

are bounded on $\mathcal{M}_+$. From this we deduce that the operator

$$K_2^{-1}(D_- b - (K_2^{-1} \triangleright b)D_-) = K_2^{-1}(a^*D_- - D_-(K_2 \triangleright a^*))$$

$$= K_2^{-1}(a^*D_- - D_-(K_2^{-1} \triangleright a)^*) = \{(D_+ a - (K_2^{-1} \triangleright a)D_+)K_2^{-1}\}^*$$

is bounded on $\mathcal{M}_-$ for all $b \in \mathcal{A}$ (we called $a = -K_2 \triangleright b^*$). Now $K_2$ does not map $\mathcal{M}_\pm$ into itself, so we cannot take $K_2$ as twist operator. But if we define

$$K := K_2 \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad (5.5)$$

then $K$ maps $\mathcal{M}_\pm$ into itself, $K^{-1}aK = K_2^{-1} \triangleright a$ and the operator

$$da = K^{-1}(Da - (K^{-1}aK)D) = \begin{pmatrix} 0 & q^{-1}K_2^{-1}(D_+ a - (K_2^{-1} \triangleright a)D_+) \\ K_2^{-1}(D_- b - (K_2^{-1} \triangleright b)D_-) & 0 \end{pmatrix}$$
is bounded for all $a \in \mathcal{A}$.

Although $D$ is only a symmetric operator, any odd symmetric densely defined operator has a canonical selfadjoint extension associated with the grading. Indeed, let $W$ be the restriction of $\gamma$ to the range of $D + i$; since $W(D + i) = -(D - i)W$ the map $W$ is the Cayley transform of $D$. Selfadjoint extensions of $D$ are in bijection with unitary extensions of $W$, but the grading $\gamma$ is a unitary extension of $W$, and this provides the canonical extension that we need.

For any choice of the parameters $\lambda, \mu, \delta$ in Equation (5.4), the data $(\mathcal{A}(S_q^4), \mathcal{H}, D, K)$ satisfies all the axioms of an even twisted spectral triple, except for the compact resolvent condition. Our hope is that it is possible to tune these three parameters so as to obtain an operator $D$ with compact resolvent. To compute the spectrum of $D$ seems quite problematic, and is postponed to future works.

### 6 The real structure

The next step is to define the real structure. Let $\kappa^{\frac{1}{2}}(a) = K_1^4 K_2^3 \triangleright a < K_1^4$ be the square root of the modular automorphism (2.3), and call $T$ the antilinear operator on $\mathcal{A}(S_q^7)^2$ given by

$$T(v_1, v_2) := (\kappa^{\frac{1}{2}}(v_2^*), -\kappa^{\frac{1}{2}}(v_1^*)) \quad (6.1)$$

Since $\mathcal{A}(S_q^7)$ is an $U_q(su(2))$-module $*$-algebra, we have $h \triangleright v_i^* = \{S(h)^* \triangleright v_i\}^*$. Moreover

$$K_1 \triangleright \kappa^{\frac{1}{2}}(a) = \kappa^{\frac{1}{2}}(K_1 \triangleright a) \quad E_1 \triangleright \kappa^{\frac{1}{2}}(a) = q^{-1/2}(E_1 \triangleright a) \quad F_1 \triangleright \kappa^{\frac{1}{2}}(a) = q \kappa^{\frac{1}{2}}(F_1 \triangleright a)$$

Using these properties one checks that the operator $J_+$ (resp. $J_-$), defined by

$$J_+(v_1, v_2) = T(q v_1, q^{-1} v_2) \quad J_-(w_1, w_2) = T(q^{-1} w_1, q w_2)$$

maps $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) into itself.

Since $\kappa^{\frac{1}{2}} \circ * \circ \kappa^{\frac{1}{2}} \circ * = id$, trivially $J_\pm^2 = -1$. Furthermore, using the property (2.2) and the invariance of the Haar functional one proves that

$$\langle Tv, Tw \rangle = \sum_i \varphi(\kappa^{\frac{1}{2}}(v_i^*) \kappa^{\frac{1}{2}}(w_i^*)) = \sum_i \varphi(\kappa^{-\frac{1}{2}}(v_i) \kappa^{\frac{1}{2}}(w_i^*))$$

so that $T$ is an isometry and $J_\pm$ are bounded antilinear operators and extend to $\mathcal{H}_\pm$.

**Lemma 6.1.** If the parameters in Equation (5.4) satisfy $\lambda = \bar{\mu}$ and $\delta = 0$, the operator $J = J_+ \oplus J_-$ is a real structure for the data $(\mathcal{A}(S_q^4), \mathcal{H}, D, K)$.

**Proof.** The operator $J$ satisfies two of the three conditions in (1.3), namely $J^2 = -1$ and $J \gamma = \gamma J$. Since both $a$ and $da$ are operators of left multiplication for matrices with entries in $\mathcal{A}(S_q^7)$ (cf. the proof of Lemma 5.3), since

$$J b J^{-1} v = v \kappa^{\frac{1}{2}}(b^*)$$
and since left and right multiplication commute, also the conditions in Equation (1.4) are satisfied. To prove that $J$ is a real structure we still have to prove that $DJ = JD$. Since $J \cdot D_- = D_- \cdot J = (D_+ J_+ - J_+ D_+)^*$, it is sufficient to prove that $D_+ J_+ = J_+ D_+$. The first step is to observe that
\[
\text{Th}
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{pmatrix}
T^{-1}
= K_1^4 K_2^3 S(h)^* K_1^{-4} K_2^{-3}
\begin{pmatrix}
  -\bar{c}_{22} & -\bar{c}_{21} \\
  \bar{c}_{12} & \bar{c}_{11}
\end{pmatrix}
\]
for all $h \in U_q(so(5))$ and $c_{ij} \in \mathbb{C}$. With this formula we compute
\[
JX^1 J^{-1} = X^2, \quad JX^2 J^{-1} = X^1, \quad JX^3 J^{-1} = K_2^2 \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} X^3.
\]
Thus if $\lambda = \bar{\mu}$ and $\delta = 0$, $D_+ = \lambda X^1 + \bar{\lambda} X^2$ and $J D_+ = D_+ J$. This concludes the proof.

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