ASYMPTOTICS AND ESTIMATES FOR EIGENELEMENTS OF LAPLACIAN WITH FREQUENT NONPERIODIC INTERCHANGE OF BOUNDARY CONDITIONS

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Abstract

We consider singular perturbed eigenvalue problem for Laplace operator in a two-dimensional domain. In the boundary we select a set depending on a character small parameter and consisting of a great number of small disjoint parts. On this set the Dirichlet boundary condition is imposed while on the rest part of the boundary we impose the Neumann condition. For the case of homogenized Neumann or Robin boundary value problem we obtain highly weak restrictions for distribution and lengths of boundary Dirichlet parts of the boundary under those we manage to get the leading terms of asymptotics expansions for perturbed eigenelements. We provide explicit formulae for these terms. Under weaker assumptions we estimate the degrees of convergence for perturbed eigenvalues.

Introduction

The object of this work is to study a two-dimensional boundary value problem with frequent nonperiodic interchange of type of boundary conditions. First we describe the formulation of such problems in general outline. The elliptic equation is considered in a domain with a boundary smooth enough. In the boundary the subset consisting of a great number of disjoint parts of small measure is selected. On this subset the boundary condition of one type (ex. Dirichlet condition) is imposed while on the rest part of the boundary the condition of another type (ex. Neumann condition) is set. The question is: What is the behaviour of the solution of such problem when a number of parts of selected boundary’s subset infinitely decreases while the measure of each part and distance between neighbouring ones tends to zero. It is also possible to formulate a problem, where such type of boundary condition described is imposed not on a whole boundary but only on its part while on the remaining part one of classic boundary condition is imposed.

The homogenization of problems with frequent interchange of boundary condition were widely investigated (see, for instance, [1]–[11]). The main object of
these works was to determine the limiting (homogenized) problems under minimal set of constraints for the structure of interchange of boundary conditions, i.e., for the behaviour of sets with different boundary conditions. Damlamian and Li Ta-Tsien in [1] considered Laplace equation in a bounded domain with frequent interchange of boundary conditions. They studied alternation of Dirichlet and Neumann condition and also the case when the former was replaced by integral boundary condition. The homogenized problems were obtained under severe constraints for the structure of alternation. In papers [2]-[9] for the problems with the alternation of Dirichlet and Neumann or Robin conditions the homogenized problems were obtained and simple conditions determining the dependence of homogenized problem’s type on the structure of alternation were adduced. The case when the Dirichlet part of boundary had the periodic structure was investigated in [3]-[6]. The convergence in nonperiodic case was studied in [2], [8], [9]. Barenbaltt, Bell and Crutchfield [10] and Dávila [11] considered nonlinear elliptic equations with frequent interchange of type of boundary condition. In [10] the problem was solved numerically; in [11] the homogenization was studied. The results obtained in investigating of problems with frequent interchange of boundary condition (both periodic and nonperiodic) can be briefly formulated as follows. Under general assumptions the elliptic boundary value problems with frequent interchange of boundary condition converge to classic problems. The type of boundary condition in homogenized problem depends on relationship between measures of parts of boundary with different type of boundary condition in the perturbed problem.

The homogenization of boundary value problem close to problem with interchange of boundary condition was studied in monograph [12]. Here they consider elliptic problems in whole space. Boundary condition (Dirichlet or Neumann one) was imposed on a boundary of a set consisting of a great number of small disjoint domains located closely each to other. Also it was considered the case when small domains were replaced by small curves on those Neumann boundary condition was imposed. Asymptotics expansions for solutions of problems having such geometry
of boundary condition were constructed by Gadyl’shin in the paper [13].

Besides the determining of homogenized problems for ones with frequent inter-
change also it is important and actual the question about estimates of degrees of
convergence. For periodic interchange of boundary conditions such estimates were
obtained by Chechkin and Gadyl’shin in [4], [15]. Nonperiodic interchange was
studied by Oleinik, Chechkin and Doronina; they considered the interchange of
Dirichlet condition with Robin condition (or Neumann one as a particular case).
The case of homogenized Dirichlet problem was treated in [14], the case of ho-
mogenized Robin problem (or Neumann one as a particular case) was studied in
[16].

In last years the papers appeared where the asymptotics of solutions of prob-
lems with periodic structure of interchange were constructed. First of all we stress
that this periodicity was essentially employed. Two-dimensional case is represented
by papers [17]-[22]. In these works they considered interchange of Dirichlet and
Neumann conditions. For the circle under some additional assumptions in [17] and
[19] the complete power asymptotics for eigenelements of Laplace operator in the
case of homogenized Dirichlet or Neumann problem were obtained. In the paper
[20] the results of [17] were generalized and having assumed only periodicity of in-
terchange Borisov obtained complete two-parametrical asymptotics of eigenvalues
of Laplace operator converging to simple limiting eigenvalues. The asymptotics
expansions for associated eigenfunctions were got, too. In papers [21] and [22]
for an arbitrary domain with periodic structure of interchange they constructed
the leading terms of asymptotics expansions for perturbed eigenelements, corre-
sponding eigenvalues were assumed to converge to simple limiting eigenvalues of
Neumann or Robin problem.

In papers [24]-[26] the authors studied problems for parabolic equations with
frequent interchange of Dirichlet and Robin condition assuming that measures of
parts of the boundary with different conditions have same smallness order. In ho-
mogenization it led to Dirichlet boundary condition. In [24], [25] for periodic inter-
change of boundary conditions and in [26] for almost periodic one they estimated
degrees of convergences and constructed first terms of asymptotic expansions for
solutions of the problems studied.

In the present paper we consider eigenvalue problem for Laplace in an arbi-
trary two-dimensional domain with frequent and, generally speaking, nonperiodic
interchange of boundary conditions. We study the interchange of Dirichlet and
Neumann boundary condition. In the problem we extract two character small
parameters governing lengths of Dirichlet and Neumann parts of boundary. We
give highly weak constraints for the set with Dirichlet condition under those it is
possible to construct leading terms of asymptotics expansions for eigenelements
converging to eigenelements of homogenized Neumann or Robin problem. These
expansions are simultaneously asymptotical with respect to both small parameters;
for leading terms the explicit formulae are obtained. For the case of homogenized
Neumann problem we carry out additional studying and show that leading terms
of asymptotics can be obtained under weaker constraints for the structure of interchange. These asymptotics include leading terms of asymptotics from [18], [21], [22] as a particular case.

Loosening the constraints for the structure of Dirichlet part, we obtain double-sided estimates for difference between perturbed and limiting eigenvalues. The cases of homogenized Dirichlet, Neumann and Robin problems are considered. These differences are estimated by infinitesimal having the order of smallness same with the order of smallness of first terms of asymptotics for eigenvalues of perturbed problems obtained, of course, under more severe constraints. Our constraints imposed to the interchange are more severe than ones in [14], [16]. At the same time, the estimates from these works are rougher than ones proved in this paper.

The results of this paper were announced in [23].

In conclusion of this section we mention that questions on homogenization and estimates of degree of convergences for three-dimensional problems with frequent interchange of boundary condition were studied in [1], [3]-[5], [7]-[11], [14]-[16]; asymptotics for eigenvalues of Laplace operators in cylinder with periodic frequent interchange of boundary conditions on narrow strips lying on lateral surface [27]-[29]. We note also that in the papers [30], [31] Chechkin studied boundary value problem for Poisson equation in n-dimensional layer with frequent periodic interchange of Dirichlet and Neumann conditions on parts of the boundary shrinking to a point. It was also assumed in addition that that measures of the parts of the boundary with different type of boundary condition have the same smallness order. For the solution of the problem considered the complete asymptotics expansions was obtained.

1. Description of the problem and the main results

Let $x = (x_1, x_2)$ be Cartesian coordinates, $\Omega$ be an arbitrary bounded simply-connected domain in $\mathbb{R}^2$ having smooth boundary, $s$ be a natural parameter of the curve $\partial \Omega$, and $S$ be a length of this curve, $s \in [0, S)$. We will describe the points of $\partial \Omega$ by natural parameter, fixing the direction of going around (counterclockwise) and choosing arbitrary a point in $\partial \Omega$ associated with a value $s = 0$. For convenience of presentation we additionally associate the points corresponding to values of $s$ close to $S$ or to zero with the values $(s - S)$ and $(S + s)$. We assume $N \gg 1$ to be a natural number, $\varepsilon = 2N^{-1}$ is a small positive parameter. For each value of $N$ we define a set $\gamma_\varepsilon$ in the boundary $\partial \Omega$ consisting of $N$ open disjoint connected (cf. fig.). Let us define the set $\gamma_\varepsilon$ more concretely. For each $N$ we define points $x_j^\varepsilon \in \partial \Omega$, $j = 0, \ldots, N - 1$, associated with values $s_j^\varepsilon \in [0, S)$ of natural parameter, where the distance between each two neighbouring points measured along the boundary of the domain $\Omega$ is of order $\varepsilon$. Next, we introduce two sets of $N$ functions: $a_j(\varepsilon)$ and $b_j(\varepsilon)$, $j = 0, \ldots, N - 1$, where the functions $a_j$ and $b_j$ are nonnegative and bounded. The set $\gamma_\varepsilon$ is defined as follows:

$$\gamma_\varepsilon = \bigcup_{j=0}^{N-1} \gamma_{\varepsilon,j}, \quad \gamma_{\varepsilon,j} = \{x : -\varepsilon a_j(\varepsilon) < s - s_j^\varepsilon < \varepsilon b_j(\varepsilon)\}, \quad j = 0, \ldots, N - 1.$$
Without loss of generality the sets $\gamma_{\epsilon,j}$ are assumed to be disjoint.

Remark 1.1. We stress that we does not exclude the situation when for some $\epsilon$ and $j$ function $a_j$ or $b_j$ vanishes. In this case the point $x_j^\epsilon$ does not belong to the set $\gamma_{\epsilon,j}$.

In the paper we consider singular perturbed eigenvalue problem:

$$-\Delta \psi_\epsilon = \lambda_\epsilon \psi_\epsilon, \quad x \in \Omega,$$

$$\psi_\epsilon = 0, \quad x \in \gamma_\epsilon,$$

$$\frac{\partial \psi_\epsilon}{\partial \nu} = 0, \quad x \in \Gamma_\epsilon,$$

(1.1) (1.2)

where $\nu$ is the outward unit normal for the boundary $\partial \Omega$, $\Gamma_\epsilon = \partial \Omega \setminus \gamma_\epsilon$. The object of the paper is to investigate the behaviour of solutions of the perturbed problem as $\epsilon \to 0$ (or, equivalently, $N \to \infty$).

We set $a_N(\epsilon) \equiv a_0(\epsilon)$, $b_N(\epsilon) \equiv b_0(\epsilon)$, $s_N^j \equiv s_0^\epsilon$. Everywhere in the paper the expressions of the form $f'$ denote the derivations on $s$.

Throughout the paper we suppose the following assumption to be held.

(C0). There exists a function $\theta_\epsilon(s)$, $\theta_\epsilon : [0,S] \to [0,2\pi]$, $\theta_\epsilon(0) = 0$, $\theta_\epsilon(S) = 2\pi$, such that

$$\theta_\epsilon(s_j^\epsilon) = \theta_\epsilon(s_0^\epsilon) + \epsilon \pi j, \quad j = 0, \ldots, N - 1,$$

$$\theta'_\epsilon \in C^\infty(\partial \Omega), \quad 0 < c_1 \leq \theta'_\epsilon(s) \leq c_2,$$

where $c_1$, $c_2$ are some constants independent on $\epsilon$. The function $\theta_\epsilon(s)$ converges to some function $\theta_0(s)$ in $C^1[0,S]$ as $\epsilon \to 0$, $\theta'_0 \in C^\infty(\partial \Omega)$. The norm $\|\theta'_\epsilon\|_{C^0(\partial \Omega)}$ is bounded on $\epsilon$.

Geometrically the assumption [C0] means that the boundary $\partial \Omega$ can be smoothly and in one-to-one manner mapped onto circumference of unit radius such that the set of points $\{x_j^\epsilon\}$ is mapped to a periodic set of points dividing unit circumference into $N$ arcs of length $\epsilon \pi$, and also this transformation may depend on $\epsilon$. The only constraints imposed to this dependence are convergence of $\theta_\epsilon$ and boundedness of $\|\theta'_\epsilon\|_{C^0(\partial \Omega)}$.

Remark 1.2. It should be stressed that the nonperiodic structure of interchange of boundary condition is not generated by transformation $\theta_\epsilon$, i.e., the function $\theta_\epsilon$, generally speaking, does not map $\gamma_\epsilon$ into periodic set. For instance, let $\Omega$ be a unit circle with center at the origin, $x_j^\epsilon = (\cos \epsilon \pi j, \sin \epsilon \pi j), \quad s_j^\epsilon = \epsilon \pi j, \quad a_j(\epsilon) = \epsilon j(1+\epsilon \sin j)/2, \quad b_j(\epsilon) = 1-\epsilon j/2$. Here the set $\gamma_\epsilon$ is a union of arcs having different lengths, laying in $\partial \Omega$, moreover, $j$-th arc contains the point $\{r = 1, \theta = \epsilon \pi j\}$, ($(r, \theta)$ are polar coordinates), but is not centered with respect to this point. The assumption [C0] for such set holds with $\theta_\epsilon(s) \equiv \theta_0(s) \equiv s \equiv \theta$, $a'(\epsilon) = a_j(\epsilon)$, $b'(\epsilon) = b_j(\epsilon)$.

We order the perturbed eigenvalues in ascending order counting simplicity: $\lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \ldots \leq \lambda_k^\epsilon \leq \ldots$. Associated eigenfunctions $\psi_\epsilon^k$ are supposed to be orthonormalized in $L_2(\Omega)$.

The first part of the main results of the article are the estimates for degree of convergence given in formulation of the following three theorems.
Theorem 1.1. Let the assumption \([C0]\) and the following ones hold:

1. There exists positive bounded function \(\eta = \eta(\varepsilon)\) satisfying an equality

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \eta(\varepsilon) = 0,
\]

such that estimates

\[
2c_1^{-1}\eta(\varepsilon) \leq \min_j a_j(\varepsilon) + \min_i b_i(\varepsilon),
\]

take place where \(c_1\) is from \([C0]\);

2. There exists \(d > 0\) such that a Hölder norm \(\|\theta'_\varepsilon\|_{C^{3+4}(\partial\Omega)}\) is bounded on \(\varepsilon\);

Then eigenvalues \(\lambda^k_{\varepsilon}\) of the perturbed problem converge to the eigenvalues \(\lambda^k_0\) (taken in ascending order counting multiplicity) of the limiting problem

\[
-\Delta \psi_0 = \lambda_0 \psi_0, \quad x \in \Omega, \quad \psi_0 = 0, \quad x \in \partial\Omega,
\]

and estimates

\[
C_{k,1} \varepsilon \ln \sin(\varepsilon) - C_{k,2} |\varepsilon \ln \eta(\varepsilon)|^{3/2} (\pi/2 - \eta(\varepsilon)) \leq \lambda^k_{\varepsilon} - \lambda^k_0 \leq 0,
\]

hold true, where \(C_{k,i}\) are some positive constants independent on \(\varepsilon\) and the function \(\eta(\varepsilon)\) is bounded above by a number \(\pi/2\).

Theorem 1.2. Let the assumption \([C0]\) and the following one hold:

1. There exist positive bounded functions \(\eta = \eta(\varepsilon)\) and \(\eta_0 = \eta_0(\varepsilon)\) satisfying equalities

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \eta_0(\varepsilon) = 0
\]

\[
\lim_{\varepsilon \to 0} (\varepsilon \ln \eta(\varepsilon))^{-1} = -A
\]

with \(A = \text{const} > 0\), such that estimates

\[
2c_1^{-1}\eta_0 \eta \leq a_j + b_j \leq 2c_2^{-1}\eta, \quad j = 0, \ldots, N - 1,
\]

take place with the constants \(c_1\) and \(c_2\) from \([C0]\).

Then eigenvalues \(\lambda^k_{\varepsilon}\) of the perturbed problem converge to the eigenvalues \(\lambda^k_0\) (taken in ascending order counting multiplicity) of the limiting problem

\[
-\Delta \psi_0 = \lambda_0 \psi_0, \quad x \in \Omega, \quad \left(\frac{\partial}{\partial \nu} + A\theta'_0(s)\right) \psi_0 = 0, \quad x \in \partial\Omega,
\]

and estimates

\[
C_{k,1} \mu(\varepsilon) + C_{k,2} \varepsilon \ln \eta_0(\varepsilon) - C_{k,3} \varepsilon - C_{k,4} \sigma(\varepsilon) \leq \lambda^k_{\varepsilon} - \lambda^k_0 \leq C_{k,5} \mu(\varepsilon) + C_{k,6} \varepsilon^{3/2} + C_{k,7} \sigma(\varepsilon),
\]

hold true, where \(\mu = \mu(\varepsilon) = - (\varepsilon \ln \eta(\varepsilon))^{-1} - A\), \(\sigma(\varepsilon) = \|\theta'_\varepsilon - \theta'_0\|_{C(\partial\Omega)}\), \(C_{k,i}\) are some positive constants independent on \(\varepsilon\) and the function \(\eta_0(\varepsilon)\) is bounded above by a unit.
Theorem 1.3. Let the assumption [(C0)] and the following one hold:

(1). There exists positive bounded function $\eta = \eta(\varepsilon)$ satisfying the equality (1.3) with $A = 0$ such that estimates

$$a_j + b_j \leq 2c_2^{-1}\eta, \quad j = 0, \ldots, N - 1,$$

take place with the constant $c_2$ from [(C0)].

Then eigenvalues $\lambda^k_\varepsilon$ of the perturbed problem converge to the eigenvalues $\lambda^k_0$ of the problem (1.4) with $A = 0$ and the estimates

$$0 \leq \lambda^k_\varepsilon - \lambda^k_0 \leq C_k\mu(\varepsilon),$$

hold true, where $\mu = \mu(\varepsilon) = - (\varepsilon \ln \eta(\varepsilon))^{-1}$, $C_k$ are some positive constants independent on $\varepsilon$.

Let us outline the geometrical meaning of the hypotheses of Theorems 1.1-1.3. The assumptions (1) of these theorems are posed to the lengths of individual components of the set $\gamma_\varepsilon$ and allow the sets $\gamma_{\varepsilon,j}$ to have lengths of different orders. Moreover, the estimate from assumption (1) of Theorem 1.3 admits the situation, when for some $\varepsilon$ and $j$ the equality $a_j(\varepsilon) + b_j(\varepsilon) = 0$ holds, i.e., corresponding set $\gamma_{\varepsilon,j}$ is empty and Neumann condition is imposed in a neighbourhood of the point $x^j_\varepsilon$. It should be noted that the constants in the assumptions (1) of Theorems 1.1-1.3 can be arbitrary, however, they can always be chosen in a shown way by multiplying the functions $\eta$ and $\eta_0$ by an appropriate numbers.

The second part of the article’s main results is asymptotics expansions for eigenelements of the perturbed problem. Clear, the restrictions for the set $\gamma_\varepsilon$ needed for constructing such expansions should be more severe in comparing with hypotheses of Theorems 1.1-1.3. One of such restriction for the set $\gamma_\varepsilon$ looks as follows:

(C1). There exists positive bounded function $\eta = \eta(\varepsilon)$ such that estimates

$$c_3\eta(\varepsilon) \leq a_j(\varepsilon) + b_j(\varepsilon) \leq 2c_2^{-1}\eta(\varepsilon), \quad j = 0, \ldots, N - 1,$$

hold true where positive constant $c_3$ is independent on $\varepsilon$, $\eta$ and $j$.

Geometrically the assumption [(C1)] means that all sets $\gamma_{\varepsilon,j}$ have length of order $\varepsilon \eta$ as $\varepsilon \to 0$, that, however, does not mean the coincidence of these lengths. Observe, in the right side of the inequality from this assumption we would have written just some constant $c_4$. However, multiplying $\eta(\varepsilon)$ by an appropriate number it is easy to make this constant equal to shown value.

In order to formulate main results of the work about asymptotics expansions we will need some auxiliary facts and additional notations.
We continue the function $\theta_\varepsilon$ to the values $s \in [-S, 2S]$ by a rule $\theta_\varepsilon(s) = \theta_\varepsilon(s - kS) + 2\pi k$, $s \in [kS, (k + 1)S)$, $k = -1, 0, 1$. Denote:

$$d_j(\varepsilon) = \frac{a_j(\varepsilon) + b_j(\varepsilon)}{2\eta(\varepsilon)}; \quad d^j(\varepsilon) = \frac{\theta_\varepsilon(s_j^+ + \varepsilon b_j(\varepsilon)) - \theta_\varepsilon(s_j^- - \varepsilon a_j(\varepsilon))}{2\varepsilon\eta(\varepsilon)};$$

$$\delta_j(\varepsilon) = d_{j+1}(\varepsilon) - d_j(\varepsilon), \quad \delta^j(\varepsilon) = d^{j+1}(\varepsilon) - d^j(\varepsilon).$$

Let $\chi(t)$ be an infinitely differentiable cut-off function equaling to one as $t < 1/4$ and vanishing as $t > 3/4$ whose values belong to a segment $[0, 1]$. We introduce one more function $f_\varepsilon(\theta)$:

$$f_\varepsilon(\theta) = d^{j+1}(\varepsilon) - \chi \left( (\theta - \theta_\varepsilon(s_j^+))/\varepsilon \right) \delta^j(\varepsilon),$$

as $\varepsilon\pi j \leq \theta - \theta_\varepsilon(s_j^0) \leq \varepsilon\pi (j + 1)$, $j = 0, \ldots, N - 1$. The eigenvalues of the problem \((\text{L}0)\), like above, are taken in ascending order counting multiplicity: $\lambda_0^1 \leq \lambda_0^2 \leq \ldots \leq \lambda_0^N \leq \ldots$, and we orthonormalize associated eigenfunctions $\psi_0^k$ in $L_2(\Omega)$.

The following proposition has an auxiliary character ant it will be proved in the second section.

**Lemma 1.1.** Let $\|\theta_\varepsilon' - \theta_0'\|_{C(\partial\Omega)} \to 0$. Then eigenvalues $\Lambda_0^k = \Lambda_0^k(\mu, \varepsilon)$ of a problem

$$-\Delta \Psi_0^k = \Lambda_0^k \Psi_0^k, \quad x \in \Omega, \quad (1.7)$$

$$\left( \frac{\partial}{\partial \nu} + (A + \mu)\theta_\varepsilon(s) \right) \Psi_0^k = 0, \quad x \in \partial\Omega, \quad (1.8)$$

where $A \geq 0$, taken in ascending order counting multiplicity converge to eigenvalues $\lambda_0^k$ of the problem \((\text{L}4)\) as $(\varepsilon, \mu) \to 0$. For each fixed value of $\varepsilon$ the eigenvalues $\Lambda_0^k$ and the associated orthonormalized in $L_2(\Omega)$ eigenfunctions $\Psi_0^k$ are holomorphic on $\mu$ (latter – in $H^1(\Omega)$ norm). If $\Lambda_0^k$ is a multiply eigenvalue, then the associated eigenfunctions can be additionally orthogonalized in $L_2(\partial\Omega)$ weighted by $\theta_\varepsilon'$.

Let us formulate the second part of the main results.

**Theorem 1.4.** Suppose the assumptions \((\text{C0}), (\text{C1}), (\text{C2}), (\text{C3})\), equality \((1.3)\) with $A \geq 0$ for the function $\eta$ from \((\text{C1})\) and

$$\max_j |\delta_j(\varepsilon)| \equiv \delta_*(\varepsilon) = o(\varepsilon^{1/2}(A + \mu)^{-1}), \quad (1.9)$$

where $\mu = \mu(\varepsilon) = - (\varepsilon \ln \eta(\varepsilon))^{-1} - A$, hold. Then the eigenvalue $\lambda_0^k$ of the perturbed problem converge to eigenvalue $\lambda_0^k$ of the limiting problem \((1.4)\) and has the asymptotics:

$$\lambda_0^k = \Lambda_0^k(\mu, \varepsilon) + \varepsilon \Lambda_1^k(\mu, \varepsilon) + o(\varepsilon(A + \mu)), \quad (1.10)$$

$$\Lambda_1^k(\mu, \varepsilon) = (A + \mu)^2 \int_{\partial\Omega} \left( \Psi_0^k(x, \mu, \varepsilon) \right)^2 \ln f_\varepsilon(\theta_\varepsilon(s)) \theta_\varepsilon'(s) \, ds, \quad (1.11)$$

where $\Lambda_0^k$ and $\Psi_0^k$ meet Lemma \((1.4)\). The function $\Lambda_1^k$ is non-positive and holomorphic on $\mu$ for each fixed $\varepsilon$.  

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Remark 1.3. For the case of simple limiting eigenvalue $\lambda_0^k$ in next section we will prove in addition that coefficients of Taylor series in powers of $\mu$ for the functions $\Lambda_0^k, \Psi_0^k$ are continuous as $\varepsilon \to 0$, and these functions are majorized by holomorphic on $\mu$ functions independent on $\varepsilon$. The function $\Psi_0^k$ is majorized in a sense of $H^1(\Omega)$ norm. Also it will be shown that $\Psi_0^k$ converges to $\psi_0^k$ in $H^1(\Omega)$ as $(\varepsilon, \mu) \to (0, 0)$.

Remark 1.4. Let us pay attention to the equality (1.9). The quantities $\delta_j$ characterize difference between lengths of two neighbouring sets $\gamma_{\varepsilon,j+1}$ and $\gamma_{\varepsilon,j}$, so, the equality (1.9) actually means that lengths of two neighbouring components of the set $\gamma_{\varepsilon}$ does not differ very much.

Along with asymptotics for $\lambda_0^k$ we will prove statements about asymptotics for associated eigenfunctions $\psi_0^k$ under the hypothesis of Theorem 1.4. In order to formulate these statements we have to introduce some additional notations and that’s why it is more convenient to formulate them in the end of the second section (see Theorems 2.1, 2.2).

In next theorem we give asymptotics of the perturbed eigenvalues in the case of breakdown of equality (1.9) and keeping other assumptions of Theorem 1.4.

**Theorem 1.5.** Suppose the assumptions [(C1), (C2)] and equality (1.3) with $A \geq 0$ for the function $\eta$ from (C1) hold. Then eigenvalue $\lambda_0^k$ of the perturbed problem converges to the eigenvalue $\lambda_0^k$ of the limiting problem (1.6) and has the asymptotics:

$$\lambda_0^k = \lambda_0^k + \mu \int_{\partial \Omega} (\psi_0^k(x))^2 \theta_0'(s)ds + O \left( \mu^2 + \mu (\sigma + \varepsilon^{3/2}) + A (\sigma + \varepsilon^{1/2}) \right), \quad (1.12)$$

where in the case of multiply eigenvalue $\lambda_0^k$ the associated eigenfunctions are additionally orthogonalized in $L^2(\partial \Omega)$ weighted by $\theta_0'$, $\sigma = \|\theta_0' - \theta_0''\|_{C(\partial \Omega)}$.

The asymptotics (1.12) is constructive as $A = 0$ and in the case $A > 0$ for $\sigma + \varepsilon^{1/2} = o(\mu)$.

The statement about asymptotics of eigenfunctions $\psi_0^k$ under hypothesis of last theorem will be proved in the third section (see Theorem 3.1).

The structure of the paper is as follows. In the second section we prove Theorem 1.4 and, under its hypothesis, Theorems 2.1, 2.2 about asymptotics of the perturbed eigenfunctions. The third section is devoted to the proof of Theorem 1.3 and Theorem 3.1 about asymptotics of perturbed eigenfunctions under hypothesis of Theorem 1.3. In the fourth section we will establish the correctness of auxiliary statement about asymptotics of perturbed eigenvalues in the case of limiting Dirichlet problem. This auxiliary statement will be employed in next section for the proof of Theorem 1.1. Furthermore, in the fifth section Theorems 1.2, 1.3 will be proved.

2. Asymptotics for the perturbed eigenvalues under hypothesis of Theorem 1.4
In this section we will obtain asymptotics for the eigenvalues of the perturbed problem. First we will establish the validity of some auxiliary statements. We start from Lemma 1.1.

**Proof of Lemma 1.1.** Boundary value problem (1.7), (1.8) is regular perturbed. Convergence of eigenvalues and maintained holomorphy on $\mu$ of eigenvalues is easily established by rewriting of (1.7), (1.8) to an operator equation and employing then the results of [35]. Keeping all stated properties, the eigenfunctions $\Psi_0^k$ can be orthonormalized in $L_2(\Omega)$. According to the theorem on diagonalization of two quadratic forms, the eigenfunctions associated with multiply eigenvalue can be additionally orthogonalized in $L_2(\partial\Omega)$ weighted by $\theta'_\varepsilon$. Since $\theta'_\varepsilon$ is independent on $\mu$, it is clear that such additional orthogonalization keeps holomorphy on $\mu$ of these eigenfunctions. The proof is complete.

If $\lambda_0^k$ is a simple eigenvalue of the problem (1.6), then exactly one eigenvalue $\Lambda_0^k$ of the problem (1.7), (1.8) converges to it, and associated eigenfunction $\Psi_0^k$ converges to $\psi_0^k$ in $H^1(\Omega)$. Represent $\Lambda_0$ and $\Psi_0^k$ as power on $\mu$ series, substitute them into (1.7), (1.8) and calculate the coefficients of the same powers of $\mu$. The recurrence system of boundary value problems derived in this way, as it is easy prove accounting simplicity $\lambda_0^k$, is uniquely solvable, its solutions are continuous as $\varepsilon \to 0$ and can be estimated uniformly on $\varepsilon$. Last estimates allows to construct independent on $\varepsilon$ and holomorphic on $\mu$ majorants for $\Lambda_0^k$ and $\Psi_0^k$. Thus, the statement of Remark 1.3 is proved.

Suppose the assumption [C0] holds. We denote:

$$a^i(\varepsilon) = (\theta_\varepsilon(s_j^\varepsilon - \varepsilon a_j(\varepsilon))) / \varepsilon, \quad b^i(\varepsilon) = (\theta_\varepsilon(s_j^\varepsilon + \varepsilon b_j(\varepsilon))) / \varepsilon.$$  

The functions $a^i$ and $b^i$ describe the image of the set $\gamma_{\varepsilon,j}$ under mapping $\theta_\varepsilon$: length of this image equals to $\varepsilon(a^i + b^i)$, and its end-points associated with angles $(\theta_\varepsilon(s_j^\varepsilon - \varepsilon a_j(\varepsilon)))$ and $(\theta_\varepsilon(s_j^\varepsilon + \varepsilon b_j(\varepsilon)))$. Suppose that the assumption [CT] holds, too. We set:

$$a^j(\varepsilon) = \frac{a^i(\varepsilon)}{2\eta(\varepsilon)}, \quad b^j(\varepsilon) = \frac{b^i(\varepsilon)}{2\eta(\varepsilon)}, \quad \delta^*(\varepsilon) = \max_j |\delta^j(\varepsilon)|.$$  

Note, that $a^j = \alpha^j + \beta^j$.

**Lemma 2.1.** Let the assumptions [C0] and [CT] hold. Then the estimates

$$c_1(a_j(\varepsilon) + b_j(\varepsilon)) \leq a^i(\varepsilon) + b^i(\varepsilon) \leq c_2(a_j(\varepsilon) + b_j(\varepsilon)), \quad \delta^*(\varepsilon) \leq C(\delta^*(\varepsilon) + \varepsilon).$$  

are true, where the constant $C$ are independent on $\varepsilon$ and $\eta$.

**Proof.** By Lagrange theorem and the definition of the functions $a^j$ and $b^j$ we have:

$$a^j(\varepsilon) + b^j(\varepsilon) = \theta'_\varepsilon(M_{j,\varepsilon}^{(1)})(a_j(\varepsilon) + b_j(\varepsilon)), \quad (2.1)$$  

where $M_{j,\varepsilon}^{(1)}$ is a midpoint belonging to an interval $(s_j^\varepsilon - \varepsilon a_j(\varepsilon), s_j^\varepsilon + \varepsilon b_j(\varepsilon))$. Employing now the estimate of the derivation $\theta'_\varepsilon$ from the assumption [C0], we arrive at the first inequality from the statement of the lemma.
Equality (2.1) and definition \( \delta^j(\varepsilon) \) imply:
\[
\delta^j = \theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon})(\alpha_j + \beta_{j+1}) - \theta'_\varepsilon(M^{(1)}_{j,\varepsilon})(\alpha_j + \beta_j) =
\]
\[
\theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon})\delta_j + (\alpha_j + \beta_j)(\theta'_\varepsilon(M^{(1)}_{j,\varepsilon}) - \theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon})).
\]
(2.2)

The quantity \((\theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon}) - \theta'_\varepsilon(M^{(1)}_{j,\varepsilon}))\) in accordance with Lagrange theorem can be represented in the form:
\[
\theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon}) - \theta'_\varepsilon(M^{(1)}_{j,\varepsilon}) = \theta''_{\varepsilon}(M^{(2)}_{j,\varepsilon})(M^{(1)}_{j+1,\varepsilon} - M^{(1)}_{j,\varepsilon}),
\]
where, recalling the definition of \( M^{(1)}_{j,\varepsilon} \), a midpoint \( M^{(2)}_{j,\varepsilon} \) lies in an interval \((s^\varepsilon_j - \varepsilon a_j(\varepsilon), s^\varepsilon_{j+1} + \varepsilon b_{j+1}(\varepsilon))\).

In view of last equality and the assumptions \([\text{C0}]\) and \([\text{C1}]\) the second term in right side of (2.2) is estimated as follows:
\[
| (\alpha_j + \beta_j)\theta''_{\varepsilon}(M^{(2)}_{j,\varepsilon})(M^{(1)}_{j+1,\varepsilon} - M^{(1)}_{j,\varepsilon}) | \leq C \left( s^\varepsilon_{j+1} - s^\varepsilon_j + \varepsilon(b_{j+1} + a_j) \right) \leq C\varepsilon,
\]
(2.3)
where constants \( C \) are independent on \( \varepsilon, \eta \) and \( j \). Here we also employed a relationship
\[
c_1 |s^\varepsilon_{j+1} - s^\varepsilon_j| \leq \theta_{\varepsilon}(s^\varepsilon_{j+1}) - \theta_{\varepsilon}(s^\varepsilon_j) = \varepsilon\pi,
\]
that is easy to prove. Substitution (2.3) into (2.2) and estimate of quantity \( \theta'_\varepsilon(M^{(1)}_{j+1,\varepsilon}) \) by the assumption \([\text{C0}]\) lead us to a second inequality from the statement of the lemma. The proof is complete.

The lemma proved in an obvious way yields

**Corollary 1.** Under hypothesis of theorem \([\text{I.4}]\) the equality \( \delta^*(\varepsilon) = o(\varepsilon^{1/2}(A+\mu)^{-1}) \) is true.

**Corollary 2.** The function \( \delta^*(\varepsilon) \) is bounded.

**Proof.** It arises from Lemma \([2.1]\) and \([\text{C1}]\) that
\[
|d^j(\varepsilon)| \leq c_2 \frac{a^j(\varepsilon) + b^j(\varepsilon)}{2\eta(\varepsilon)} \leq 1,
\]
what implies the boundedness of \( \delta^* \). The proof is complete.

**Proof of Theorem \([\text{I.4}]\).** Convergence of perturbed eigenvalues to ones of problem \((1.6)\) under assumptions \([\text{C0}], [\text{C1}]\) and equality \((1.9)\) can be easily established, using results and methods of papers \([2], [6], [8]\).

Our strategy in proving the asymptotics consists of two main steps. First we will formally construct the asymptotics for the eigenelements of the perturbed problem. Second step is to prove rigorously (to justify) that the asymptotics expansions formally constructed are really asymptotics of eigenelements. In formal construction we will use only the boundedness of the function \( \delta^*(\varepsilon) \) (see Corollary \([2]\) of Lemma \([2.1]\)), while equality \((1.3)\) will be employed only in justification of asymptotics for estimating the errors.
In formal construction we will show in detail only the case of simple limiting eigenvalue. Such a choice is explained by a desire to avoid an excessive cumber-someness of representation, on the one hand, and, on the other hand, the construction does not depend essentially on multiplicity of limiting eigenvalue. Below we will briefly outline the formal construction in the case of multiply limiting eigenvalue.

Now we proceed to the construction of the asymptotics. We suppose \( \lambda_0 \) to be a simple eigenvalue of the problem (1.6), \( \lambda_\varepsilon \) is an perturbed eigenvalue converging to \( \lambda_0 \), \( \psi_\varepsilon \) and \( \psi_0 \) are associated eigenfunctions. First we will demonstrate the scheme of constructing and formally obtain first terms of asymptotics. We seek for the asymptotics of eigenvalue as

\[
\lambda_\varepsilon = \Lambda_0(\mu, \varepsilon) + \varepsilon \Lambda_1(\mu, \varepsilon). \tag{2.4}
\]

The asymptotics for \( \psi_\varepsilon \) is constructed on the basis of combination of method of matching asymptotics expansions [32], method of composite expansions [33] and multiscaled method [34]. This asymptotics will be obtained as a sum of three expansions, namely, outer expansion, boundary layer and inner expansion. Exterior expansion is constructed as follows:

\[
\psi_{\varepsilon}^{ex}(x, \mu) = \Psi_0(x, \mu, \varepsilon) + \varepsilon \Psi_1(x, \mu, \varepsilon). \tag{2.5}
\]

Using method of composite expansions, we construct the boundary layer in the form:

\[
\psi_{\varepsilon}^{bl}(\xi, s, \mu) = \varepsilon v_1(\xi, s, \mu, \varepsilon) + \varepsilon^2 v_2(\xi, s, \mu, \varepsilon), \tag{2.6}
\]

where \( \xi = (\xi_1, \xi_2) = ((\theta_\varepsilon(s) - \theta_\varepsilon(s_0))/\varepsilon, \tau \theta_\varepsilon'(s)/\varepsilon) \) are ”scaled” variables. Here \( (s, \tau) \) are local variables defined in a neighbourhood of the boundary \( \partial \Omega \). \( \tau \) is a distance from the point to the boundary measured in the direction of inward normal. Such a definition of variables \( \xi \) will be explained below in Remark [2.7].

Interior expansion will be constructed by the method of matched asymptotics expansions in small neighbourhoods of points \( x_\varepsilon^j \) in the form:

\[
\psi_{\varepsilon}^{in,j}(\varsigma^j, s, \mu) = w_{0,0}(\varsigma^j, s, \mu, \varepsilon) + \varepsilon w_{1,0}(\varsigma^j, s, \mu, \varepsilon), \tag{2.7}
\]

where \( \varsigma^j = (\varsigma_1^j, \varsigma_2^j) = ((\xi_1 - \pi j)\eta^{-1}, \xi_2\eta^{-1}) \).

The aim of the formal construction is to determine the functions \( \Lambda_i, \Psi_i, v_i \) and \( w_{i,0}, \).

The equations for the functions \( \Psi_0 \) and \( \Psi_1 \) are derived by standard substitution of (2.4) and (2.5) into equation (1.1) with consequent writing out the coefficients of the same power of \( \varepsilon \). Such procedure leads to the equation (1.7) for the function \( \Psi_0 \) with \( \Lambda_0^k = \Lambda_0 \) and to the following equation for \( \Psi_1 \):

\[
(\Delta + \Lambda_0)\Psi_1 = -\Lambda_1 \Psi_0, \quad x \in \Omega. \tag{2.8}
\]

The boundary condition for the functions \( \Psi_0 \) and \( \Psi_1 \) will be deduced later in constructing of the boundary layer and inner expansion’s coefficients.
Let us determine the functions \(v_i\). First we should obtain the problems for them, in order to make it one needs to rewrite the Laplace operator in variables \((s, \tau)\):

\[
\Delta_x = \frac{1}{H} \left( \frac{\partial}{\partial \tau} \left( H \frac{\partial}{\partial \tau} \right) + \frac{\partial}{\partial s} \left( \frac{1}{H} \frac{\partial}{\partial s} \right) \right), \quad H = H(s, \tau) = 1 + \tau k(s),
\]

\(k = k(s) = (r''(s), \nu(s))_{\mathbb{R}^2}, \nu = \nu(s), r(s)\) is a two-dimensional vector-function describing the curve \(\partial \Omega\), \(k \in C^\infty(\partial \Omega)\). Now we substitute (2.4), (2.5) and the expression for Laplace operator in variables \((s, \tau)\) in (1.1), pass to the variables \(\xi\) and write out the coefficients of leading powers of \(\varepsilon\). This implies the equations for functions \(v_1\) and \(v_2\):

\[
\Delta_{\xi} v_1 = 0, \quad \xi_2 > 0,
\]

\[
\Delta_{\xi} v_2 = -\frac{\theta''_\varepsilon}{(\theta'_\varepsilon)^2} \left( \frac{\partial}{\partial \xi_1} + 2\xi_2 \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \right) v_1 - \frac{k}{\theta'_\varepsilon} \left( \frac{\partial}{\partial \xi_2} - 2\xi_2 \frac{\partial^2}{\partial \xi_2^2} \right) v_1 - \frac{2}{\theta'_\varepsilon} \frac{\partial^2}{\partial \xi_1 \partial s} v_1, \quad \xi_2 > 0.
\]

In accordance with method of composite expansions, the sum of functions \(\psi_\varepsilon^{ex}\) and \(\psi_\varepsilon^{bl}\) is to satisfy to homogeneous boundary condition everywhere in \(\partial \Omega\) except points \(x_{\varepsilon j}\):

\[
\frac{\partial}{\partial \nu} \psi_\varepsilon^{ex} - \frac{\partial}{\partial \tau} \psi_\varepsilon^{bl} = 0, \quad x \in \partial \Omega, \quad x \neq x_{\varepsilon j}.
\]

Now we rewrite second term in last equality to the variables \(\xi\) and replace the functions \(\psi_\varepsilon^{ex}\) and \(\psi_\varepsilon^{bl}\) by right sides of the equalities (2.5) and (2.6), after that we calculate the coefficient of the leading power of \(\varepsilon\) that is set equalling to zero. As a result, we have the boundary conditions for the functions \(v_i\):

\[
\frac{\partial v_1}{\partial \xi_2} = 1, \quad \xi \in \Gamma^0, \quad \xi_2 > 0,
\]

\[
\frac{\partial v_2}{\partial \xi_2} = 1, \quad \xi \in \Gamma^0,
\]

where \(\Gamma^0 = \{\xi : \xi_2 = 0, \xi_1 \neq \varepsilon \pi j, j \in \mathbb{Z}\}\),

\[
\Psi_\varepsilon^\nu = \Psi_i^\nu(s, \mu, \varepsilon) = \frac{\partial}{\partial \nu} \Psi_i(x, \mu, \varepsilon), \quad x \in \partial \Omega.
\]

**Remark 2.1.** It follows from the definition of the set \(\Gamma^0\) that the problems for the functions \(v_i\) are periodic on the variable \(\xi_1\) what will be essentially used in solving of the boundary value problems (2.9)–(2.12). One can easily check that the periodicity of \(\Gamma^0\) is a direct implication of the assumption \([C0]\) (namely, of the equation \(\theta_\varepsilon(s_{\varepsilon j}^0) = \theta_\varepsilon(s_{\varepsilon j}^0 + \varepsilon \pi j)\) and the definition of the variable \(\xi_1\) given above, what explains the indicated definition of the variable \(\xi_1\). The variable \(\xi_2\) was selected so that to obtain Poisson equations for the functions \(v_1\) and \(v_2\).
In accordance with method of composite expansions, we are to see \( k \) exponentially decaying as \( \xi_2 \to +\infty \) solutions to the problems (2.9), (2.11) and (2.10), (2.12). In constructing of boundary layer we additionally employ the multiscaled method: the variable \( s \) plays the slow time’s role. We also notice that the boundary layer ”controls” only Neumann boundary condition, exactly because of this we have passed to a formal limit as \( \eta \to 0 \) in deducing the boundary condition (2.11). Possibly, Dirichlet boundary condition is seemed to be satisfied simultaneously by a suitable choose of the functions \( v_1 \) and \( v_2 \). However, this way leads to unsolvable problems for the functions \( v_1 \) and \( v_2 \).

We indicate by \( \mathcal{V}_0 \) the space of \( \pi \)-periodic on the variable \( \xi_1 \) functions belonging \( C^\infty(\{\xi : \xi_2 > 0\} \cup \Gamma^0) \) and decaying exponentially with all their derivatives as \( \xi_2 \to +\infty \) uniformly on \( \xi_1 \).

Let us construct the solution of the problem (2.9), (2.11). We stress that this problem contains the variable \( s \) as a parameter. Consider a function
\[
X(\xi) = \text{Re} \ln \sin z + \ln 2 - \xi_2, \tag{2.13}
\]
where \( z = \xi_1 + i\xi_2 \) is a complex variable. By direct calculations one can check that \( X \in \mathcal{V}_0 \) is even on \( \xi_1 \) harmonic function meeting a boundary condition
\[
\frac{\partial X}{\partial \xi_2} = -1, \quad \xi \in \Gamma^0.
\] The representation
\[
X(\xi) = \ln \rho + \ln 2 - \xi_2 + \tilde{X}(\xi), \quad \tilde{X}(\xi) = O(\rho^2), \quad \xi \to \xi^{(j)}, \tag{2.14}
\]
holds, where \( \rho = |\xi - \xi^{(j)}|, \quad \xi^{(j)} = (\pi j, 0), \quad j \in \mathbb{Z}, \quad \tilde{X}(\xi) \in C^\infty(\{\xi : \xi_2 \geq 0\}) \). Bearing in mind all the facts counted, we conclude that the function \( v_1 \) has the form:
\[
v_1(\xi, s, \mu) = -\frac{1}{\theta'_{\varepsilon}(s)} \Psi_0^\nu(s, \mu, \varepsilon) X(\xi).
\]
The solutions for the problem (2.10), (2.12) can be constructed explicitly, too. By direct calculations we check that the function
\[
\tilde{v}_2 = \frac{\Psi_0^\nu}{2(\theta'_{\varepsilon})^2} \xi_2^2 \left( \frac{\theta''_{\varepsilon}}{\theta'_{\varepsilon}} \frac{\partial X}{\partial \xi_1} + k \frac{\partial X}{\partial \xi_2} \right) - \frac{1}{\theta'_{\varepsilon}} \left( \Psi_0^\nu \right)' \int_{\xi_2}^{+\infty} t \frac{\partial}{\partial \xi_1} X(\xi_1, t) \, dt
\]
is a solution of equation (2.10) satisfying homogeneous Neumann condition on \( \Gamma^0 \). The function \( \tilde{v}_2 \in \mathcal{V}_0 \) has the following (differentiable) asymptotics as \( \xi \to \xi^{(j)} \):
\[
\tilde{v}_2 = O(\rho \ln \rho). \tag{2.15}
\]
Taking into account all the described properties of the function \( \tilde{v}_2 \) and the properties of the function \( X \), we arrive at the formula for the function \( v_2 \):
\[
v_2(\xi, s, \mu, \varepsilon) = \tilde{v}_2(\xi, s, \mu, \varepsilon) - \frac{1}{\theta'_{\varepsilon}(s)} \Psi_1^\nu(s, \mu, \varepsilon) X(\xi). \tag{2.16}
\]
As it follows from (2.14)–(2.16), the functions $v_i$ have logarithmic singularities in neighbourhoods of points $\xi^{(j)}$, or, equivalently, in neighbourhoods of points $x_j^\varepsilon$. Moreover, the sum of the outer expansion and boundary layer does not satisfy (even asymptotically) to Dirichlet boundary condition on $\gamma_\varepsilon$. That’s why in neighbourhoods of the points $x_j^\varepsilon$ we introduce new "scaled" variables $\varsigma^j$, and the asymptotics of the eigenfunction is constructed as $\psi^i_\varepsilon$ by the method of matched asymptotics expansions. The using of term "scaled variables" for $\varsigma^j$ is correct, since owing to the equality (1.5) the function $\eta$ is of the form:

$$\eta(\varepsilon) = \exp\left( -\frac{1}{\varepsilon(A + \mu(\varepsilon))} \right), \quad (2.17)$$

where $\mu(\varepsilon)$ is defined in a statement of the theorem being proved. Thus, $\eta(\varepsilon)$ is exponentially small in comparing with $\varepsilon$.

First we carry out the matching procedure in a neighbourhood of the point $x_j^\varepsilon$. For the sake of brevity we denote $\varsigma = \varsigma^j$. Clear, the asymptotics

$$\psi^{ex}_\varepsilon(x) = \sum_{i=0}^1 \varepsilon^i \left( \Psi_i^D(s, \mu, \varepsilon) - \tau \Psi_i^\nu(s, \mu, \varepsilon) \right) + O(\tau^2), \quad (2.18)$$

holds true as $\tau \to 0$ where $\Psi_i^D$ indicates the values of the functions $\Psi_i$ as $x \in \partial \Omega$, and the variable $s$ ranges in a small neighbourhood of value $s_j^\varepsilon$. Bearing in mind the asymptotics (2.14), (2.15) and the formulae for $v_i$ we get that, as $\xi \to \xi^{(j)}$,

$$\psi^{bl}_\varepsilon(\xi, s, \mu) = -\varepsilon(\ln \rho + \ln 2 - \xi_2) \sum_{i=0}^1 \varepsilon^i \Psi_i^\nu(s, \mu, \varepsilon) \frac{\theta_\varepsilon'(s)}{\theta_\varepsilon(s)} + O(\rho \ln \rho). \quad (2.19)$$

Let us rewrite the asymptotics (2.18), (2.19) in the variables $\varsigma$ and take into account that due to (2.17) the equality $\varepsilon \ln \eta(\varepsilon) = -A + \mu)^{-1}$ is valid. Hence, we have that for $\frac{1}{4} \eta^{1/4} < \rho < \frac{3}{4} \eta^{-3/4}$ (or, equivalently, for $\frac{1}{4} \eta^{-3/4} < \varsigma < \frac{3}{4} \eta^{3/4}$)

$$\psi^{ex,\mu}_\varepsilon(x) + \psi^{bl}_\varepsilon(\xi, s, \mu) = W_{0,0}(s, \mu, \varepsilon) + \varepsilon W_{1,0}(\varsigma, s, \mu, \varepsilon) + O(\varepsilon^2 \ln |\varsigma|), \quad (2.20)$$

$$W_{0,0}(s, \mu, \varepsilon) = \Psi_0^D(s, \mu, \varepsilon) + \frac{\Psi_0^\nu(s, \mu, \varepsilon)}{(A + \mu) \theta_\varepsilon'(s)}, \quad (2.21)$$

$$W_{1,0}(\varsigma, s, \mu, \varepsilon) = -\frac{\Psi_0^\nu(s, \mu, \varepsilon)}{\theta_\varepsilon'(s)} (\ln |\varsigma| + \ln 2 + \Psi_1^D(s, \mu, \varepsilon) + \frac{\Psi_1^\nu(s, \mu, \varepsilon)}{(A + \mu) \theta_\varepsilon'(s)}. \quad (2.22)$$

In accordance with method of matched asymptotics expansions it arises from (2.20) that the functions $w_{i,0}^{(j)}$ must have the following asymptotics at infinity:

$$w_{i,0}^{(j)} = W_{i,0} + o(1), \quad \varsigma \to \infty. \quad (2.23)$$

The problems for the functions $w_{i,0}$ are deduced by standard substitution of (2.4) and (2.7) into boundary value problem (1.1), (1.2) and by writing out the coeffi-
cients of leading powers of \( \varepsilon \) and \( \eta \):

\[
\begin{align*}
\Delta_{\varepsilon} w_{i,0}^{(j)} &= 0, \quad \varsigma > 0, \\
w_{i,0}^{(j)} &= 0, \quad \varsigma \in \gamma_j^1, \\
\frac{\partial}{\partial \varsigma} w_{i,0}^{(j)} &= 0, \quad \varsigma \in \Gamma_j^1.
\end{align*}
\] (2.24)

Here \( \gamma_j^1 \) is an interval \((-2\alpha_j, 2\beta_j)\) lying in the axis \( O\varsigma_1 \), and \( \Gamma_j^1 \) is a complement of the closure of \( \gamma_j^1 \) with respect to a line \( \varsigma_2 = 0 \).

The problem (2.24) has no nontrivial solutions bounded at infinity, therefore, in view of (2.21), (2.23),

\[ w_{0,0}^{(j)} = 0. \]

This equality and the asymptotics (2.21), (2.23) yield the boundary condition (1.8) for the function \( \Psi_0 \). The eigenelements \( \Lambda_0 \) and \( \Psi_0 \) obey Lemma 1.1. The smoothness of domain’s boundary and of the function \( \theta_\varepsilon'(s) \) allows us to maintain that the function \( \Psi_0 \) is infinitely differentiable on the variables \( x \).

Let us determine the function \( w_{1,0} \). Let

\[ Y^{(j)}(\varsigma, \varepsilon) = \text{Re} \ln \left( y + \sqrt{y^2 - 1} \right), \] (2.25)

where \( y = (\varsigma_1 + i\varsigma_2 + \alpha^j - \beta^j)/(\alpha^j + \beta^j) \) is a complex variable. It is easy to establish that \( Y^{(j)} \in \mathcal{W} \), where

\[ \mathcal{W} \equiv C^\infty (\{ \varsigma : \varsigma_2 \geq 0, \varsigma \neq (-2\alpha_j, 0), \varsigma \neq (2\beta_j, 0) \}) \cap H^1 (\{ \varsigma : \varsigma_2 > 0, |\varsigma| < 5 \}). \]

The function \( Y^{(j)} \) is a solution of the problem (2.24) having the following asymptotics at infinity:

\[ Y^{(j)} = \ln |\varsigma| + \ln 2 - \ln(\alpha^j + \beta^j) + (\alpha^j - \beta^j)\varsigma_1|\varsigma|^{-2} + O(|\varsigma|^{-2}), \quad \varsigma \to \infty. \] (2.26)

Owing to the properties \( Y^{(j)} \) stated the function \( w_{1,0}^{(j)} \) is of the form:

\[ w_{1,0}^{(j)}(\varsigma, s, \mu, \varepsilon) = -\frac{\Psi_0^\nu(s, \mu, \varepsilon)}{\theta_\varepsilon'(s)} Y^{(j)}(\varsigma, \varepsilon). \]

It is obvious that \( w_{1,0}^{(j)} \in \mathcal{W} \). Now we write out the asymptotics of the function \( w_{1,0}^{(j)} \) at infinity (see (2.26)) and compare it with (2.22), (2.23). As a result we arrive at the equality

\[ \frac{\Psi_0^\nu(s, \mu, \varepsilon)}{\theta_\varepsilon'(s)} \ln(\alpha^j(\varepsilon) + \beta^j(\varepsilon)) = \Psi_1^D(s, \mu, \varepsilon) + \frac{\Psi_1^\nu(s, \mu, \varepsilon)}{(A + \mu)\theta_\varepsilon'(s)}. \] (2.27)

This equality actually is a boundary condition for the function \( \Psi_1 \). We just should correctly define the right side of this condition bearing in mind that, generally speaking, the quantities \( \ln(\alpha^j + \beta^j) \) depend on index \( j \) and the parameter \( \varepsilon \).
it has already been mentioned above, the variable $s$ in the equality (2.27) ranges in a small (of order $O(\varepsilon \eta^{1/4})$) neighbourhood of point $s^c_j$. Therefore, to satisfy the equality (2.27) it is sufficient to construct the function equalling to $(\alpha^j + \beta^j)$ in these neighbourhoods of the points $s^c_j$ and then replace $(\alpha^j + \beta^j)$ by this function in (2.27). The function $f_\varepsilon(\theta)$, as it is easy to prove, is infinitely differentiable on $\theta$ and equals to $d^j = \alpha^j + \beta^j$ for $|\theta - \theta_\varepsilon(s^c_j)| \leq \varepsilon \pi/4$. That's why as the function the sum $(\alpha^j + \beta^j)$ in (2.27) is replaced to we take $f_\varepsilon(\theta_\varepsilon(s))$ what immediately implies the boundary condition for $\Psi_1$:

\[
\left( \frac{\partial}{\partial \nu} + (A + \mu)\theta_\varepsilon'(s) \right) \Psi_1 = (A + \mu)\Psi_0^\nu(s, \mu, \varepsilon) \ln f_\varepsilon(\theta_\varepsilon(s)), \quad x \in \partial \Omega.
\]

Now we take into account that $\Psi_0^\nu = -(A + \mu)\theta_\varepsilon^D$, and finally we have:

\[
\left( \frac{\partial}{\partial \nu} + (A + \mu)\theta_\varepsilon'(s) \right) \Psi_1 = -(A + \mu)^2\Psi_0^D(s, \mu, \varepsilon)\theta_\varepsilon'(s) \ln f_\varepsilon(\theta_\varepsilon(s)), \quad x \in \partial \Omega.
\]

Problem (2.8), (2.28) is solvable under suitable choice of $\Lambda_1$. The solvability condition is deduced in a standard way, by multiplying equation (2.8) by $\Psi_0$ and integrating by parts with employing the boundary condition (2.28). Bearing in mind the normalization for $\Psi_0$, this condition implies formula (1.11) for the leading term of the asymptotics. It follows from Lemma 2.1, the assumption (C1) and the definition of $f_\varepsilon$ that:

\[
c_1 c_3 / 2 \leq f_\varepsilon(\theta) \leq 1,
\]

what due to formula (1.11) gives nonpositiveness of $\Lambda_1$. The maintained holomorphy on $\mu$ of $\Lambda_1$ is an implication of the corresponding properties of $\Psi_1$, boundedness of $\theta_\varepsilon'$ and $f_\varepsilon(\theta)$ and the estimate for the norm $\|\Psi\|_{L^2(\partial \Omega)}$ by $\|\Psi\|_{H^1(\Omega)}$.

The function $\Psi_1$ is defined up to an additive term $C\Psi_0$, $C = \text{const}$; we eliminate this arbitrariness by assuming $\Psi_1$ to be orthogonal to $\Psi_0$ in $L^2(\Omega)$. The function $\ln f_\varepsilon(\theta_\varepsilon(s))$ is smooth, that's why it is easy to show that $\Psi_1 \in C^\infty(\Omega)$. Moreover, the function $\Psi_1$ is holomorphic on $\mu$ in $H^1(\Omega)$ norm for each fixed value of $\varepsilon$. Using the simplicity of $\lambda_0$, one can establish that coefficients of Taylor series in powers of $\mu$ for $\Psi_1$ are continuous as $\varepsilon \to 0$, and $\Psi_1$ is majorized by holomorphic on $\mu$ function independent on $\varepsilon$.

The constructing done allowed to determine first terms of asymptotics expansions for $\lambda_\varepsilon$ and $\psi_\varepsilon$ (formally, of course). Now we should prove that the asymptotics constructed do provide the asymptotics for $\lambda_\varepsilon$ and $\psi_\varepsilon$. In order to make such a justification we need to prove first that the asymptotics constructed satisfy to the perturbed problem up to sufficiently small discrepancy. Exactly the proof of this statement will be our aim in this step. To guarantee the smallness of discrepancy needed we have to construct additional terms in asymptotics expansions for $\lambda_\varepsilon$ and $\psi_\varepsilon$.

We have to construct one more term in the outer expansion:

\[
\psi_\varepsilon^{ex}(x, \mu) = \Psi_0(x, \mu, \varepsilon) + \varepsilon \Psi_1(x, \mu, \varepsilon) + \varepsilon^2 \Psi_2(x, \mu, \varepsilon).
\]
In boundary layer it is should be constructed two additional terms; as a result the boundary layer reads as follows:

$$\psi_{bl}^\varepsilon (\xi, s, \mu) = \sum_{i=1}^{4} \varepsilon^i v_i(\xi, s, \mu, \varepsilon).$$  \hspace{1cm} (2.31)$$

With regard to the equality $w_{0,0}^{(j)} = 0$ and additional terms the inner expansion becomes:

$$\psi_{in,j}^\varepsilon (\zeta, s, \mu) = \sum_{i=1}^{3} \varepsilon^i w_{i,0}^{(j)}(\zeta, s, \mu, \varepsilon) + \eta \sum_{i=1}^{4} \varepsilon^i w_{i,1}^{(j)}(\zeta, s, \mu, \varepsilon).$$  \hspace{1cm} (2.32)$$

First terms of outer expansion (2.30) are known, we just need to determine the function $\Psi_2$. In what follows this function will be employed only for matching of additional terms of inner expansion. Like before, this matching will affect only the boundary condition of $\Psi_2$, hence, we have an arbitrariness in choosing the equation for $\Psi_2$, since its form does not influence very much on the estimate of discrepancy. We choose the equation for $\Psi_2$ so that to guarantee the solvability and to simplify the calculations. Both these aims are achieved by the following choice:

$$(\Delta - 1)\Psi_2 = -\Lambda_1 \Psi_1, \hspace{0.5cm} x \in \Omega.$$  \hspace{1cm} (2.33)$$

Additional terms of boundary layer are defined as follows:

$$v_3 = \frac{\Psi_2''}{2(\theta'_\varepsilon)^2} \xi_2 \left( \frac{\theta''_\varepsilon \partial X}{\theta'_\varepsilon \partial \xi_1} + k \frac{\partial X}{\partial \xi_2} \right) - \frac{1}{\theta'_\varepsilon} \left( \frac{\Psi_2'\prime}{\theta'_\varepsilon} \right)' \int_{\xi_2}^{+\infty} t \frac{\partial}{\partial \xi_1} X(\xi_1, t) \, dt +$$

$$+ a \int_{\xi_2}^{+\infty} t X(\xi_1, t) \, dt - \frac{1}{\theta'_\varepsilon} \Psi_2' X,$$

$$v_4 = \frac{\Psi_2''}{2(\theta'_\varepsilon)^2} \xi_2 \left( \frac{\theta''_\varepsilon \partial X}{\theta'_\varepsilon \partial \xi_1} + k \frac{\partial X}{\partial \xi_2} \right) - \frac{1}{\theta'_\varepsilon} \left( \frac{\Psi_2'\prime}{\theta'_\varepsilon} \right)' \int_{\xi_2}^{+\infty} t \frac{\partial}{\partial \xi_1} X(\xi_1, t) \, dt,$$

where $a = a(s, \mu, \varepsilon)$ is a some function that will be determined below, $\Psi_2'$ is a value of normal derivation for $\Psi_2$ on the boundary $\partial \Omega$. It is easy to check that $v_3, v_4 \in V_0$.

In order to match asymptotics expansions and to determine inner expansion,
one needs the following differentiable asymptotics held as $\rho \to 0$:

\begin{align*}
v_1(\xi, s, \mu, \varepsilon) &= -\frac{\Psi_0'}{\theta_\varepsilon'}(\ln \rho + \ln 2 - \xi_2) + O(\rho^2), \\
v_2(\xi, s, \mu, \varepsilon) &= -\frac{\Psi_1'}{\theta_\varepsilon'}(\ln \rho + \ln 2 - \xi_2) + V_\varepsilon(\xi - \xi^{(j)}, s, \Psi_0'(s, \mu, \varepsilon)) + O(\rho^2), \\
v_3(\xi, s, \mu, \varepsilon) &= -\frac{\Psi_2'}{\theta_\varepsilon'}(\ln \rho + \ln 2 - \xi_2) - \frac{\zeta(3)}{4}a(s, \mu, \varepsilon) + V_\varepsilon(\xi - \xi^{(j)}, s, \Psi_1'(s, \mu, \varepsilon)) + O(\rho^2 \ln \rho), \\
v_4(\xi, s, \mu, \varepsilon) &= V_\varepsilon(\xi - \xi^{(j)}, s, \Psi_2'(s, \mu, \varepsilon)) + O(\rho^2),
\end{align*}

where $\zeta(t)$ is Riemann zeta function, and it is indicated

\begin{align*}
V_\varepsilon(\xi, s, \Psi(s)) &= \frac{\Psi(s)}{2(\theta_\varepsilon'(s))^2 |\xi|^2} \left( \frac{\theta_\varepsilon''(s)}{\theta_\varepsilon'(s)} \xi_1 + k(s) \xi_2 \right) + \frac{1}{2\theta_\varepsilon'(s)} \left( \frac{\Psi(s)}{\theta_\varepsilon'(s)} \right)' \xi_1 (\ln |\xi| + \ln 2 - 1).
\end{align*}

The coefficients of the outer expansion satisfy the relationships

\begin{align*}
\Psi_i &= \Psi_i^D - \tau \Psi_i + O(\tau^2), \quad \tau \to 0, \quad i = 0, 1, 2
\end{align*}

in an neighbourhood of the boundary $\partial \Omega$. Rewriting now the asymptotics of the functions $v_i$ and $\Psi_j$ given above to the variables $\zeta$ in view of the equality (2.17), we obtain that for $\frac{1}{4} \eta^{1/4} < \rho < \frac{3}{4} \eta^{1/4}$ ($\frac{1}{4} \eta^{-3/4} < |\zeta| < \frac{3}{4} \eta^{-3/4}$) the equality

\begin{align*}
\psi^{ex}_\varepsilon(x, \mu) + \psi^{bl}_\varepsilon(\xi, s, \mu) &= \sum_{k=1}^{3} \varepsilon^k W_{k,0}(\zeta, s, \mu, \varepsilon) + \sum_{k=1}^{4} \varepsilon^k W_{k,1}(\zeta, s, \mu, \varepsilon) + O(\eta^2 |\zeta|^2 \ln |\zeta|),
\end{align*}

\begin{align*}
W_{2,0} &= -\frac{\Psi_1'}{\theta_\varepsilon'}(\ln |\zeta| + \ln 2) + \Psi_2^D + \frac{\Psi_2'}{(A + \mu)\theta_\varepsilon'}, \\
W_{3,0} &= -\frac{\Psi_2'}{\theta_\varepsilon'}(\ln |\zeta| + \ln 2) - \frac{\zeta(3)}{4}a, \\
W_{1,1} &= -\frac{1}{2(A + \mu)\theta_\varepsilon'} \left( \frac{\Psi_0'}{\theta_\varepsilon'} \right)' \xi_1, \\
W_{k,1} &= V_\varepsilon(\xi, s, \Psi_{k-2}) - \frac{1}{2(A + \mu)\theta_\varepsilon'} \left( \frac{\Psi_{k-1}'}{\theta_\varepsilon'} \right)' \xi_1, \quad k = 2, 3, \\
W_{4,1} &= V_\varepsilon(\xi, s, \Psi_2'),
\end{align*}

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holds. Here \( \Psi^D_2 = \Psi^D_2(s, \mu, \varepsilon) = \Psi_2(x, \mu, \varepsilon), x \in \partial \Omega \). Thus, the function \( w_{k,i}^{(j)} \) should meet the asymptotics

\[
w^{(j)}_{k,i} = W_{k,i} + o(|\varsigma|^i), \quad \varsigma \to \infty.
\]

Like before, the problems for the coefficients are deduced by the substitution of (2.4) and (2.32) into (1.1), (1.2) and writing out the coefficients of leading powers of \( \varepsilon \) and \( \eta \):

\[
\Delta_\varsigma w^{(j)}_{k,1} = -\frac{\theta''}{\theta'_\varepsilon} \left( \frac{\partial}{\partial \varsigma_1} + 2\varsigma_2 \frac{\partial^2}{\partial \varsigma_1 \partial \varsigma_2} \right) w^{(j)}_{k-1,0} - \frac{k}{\theta'_\varepsilon} \left( \frac{\partial}{\partial \varsigma_2} - 2\varsigma_2 \frac{\partial^2}{\partial \varsigma_2} \right) w^{(j)}_{k-1,0} - \frac{2}{\theta'_\varepsilon} \frac{\partial^2}{\partial \varsigma_1 \partial \varsigma_2} w^{(j)}_{k-1,0}, \quad \varsigma_2 > 0, \quad k = 2, 3, 4; \quad \text{and for } w^{(j)}_{1,1} \text{ and } w^{(j)}_{1,0}, \quad k = 2, 3 \text{ we obtain the same problem (2.24) as for } w^{(j)}_{1,0}. \text{ We define the functions } w^{(j)}_{k,0}, \quad k = 2, 3, \text{ as follows:}
\]

\[
w^{(j)}_{k,0} = -\frac{\Psi'^{\nu}}{\theta'_\varepsilon} \psi^{(j)}. \tag{2.38}
\]

The belongings \( w^{(j)}_{k,0} \in \mathcal{W} \) take place. Now we calculate the asymptotics for the functions \( w^{(j)}_{k,0} \) (see (2.26), (2.38)) and compare them with the asymptotics (2.36), (2.35). This procedure gives two equalities:

\[
\frac{\Psi'^{\nu}(s, \mu, \varepsilon)}{\theta'_\varepsilon(s)} \ln(\alpha^j(\varepsilon) + \beta^j(\varepsilon)) = \Psi^D_2(s, \mu, \varepsilon) + \frac{\Psi'^{\nu}(s, \mu, \varepsilon)}{(A + \mu)\theta'_\varepsilon(s)},
\]

\[
\frac{\Psi'^{\nu}(s, \mu, \varepsilon)}{\theta'_\varepsilon(s)} \ln(\alpha^j(\varepsilon) + \beta^j(\varepsilon)) = -\frac{\zeta(3)}{4} a(s, \mu, \varepsilon).
\]

The former leads us to a boundary condition for \( \Psi_2 \):

\[
\left( \frac{\partial}{\partial \nu} + (A + \mu)\theta'_\varepsilon(s) \right) \Psi_2 = (A + \mu)\Psi'^{\nu}(s, \mu, \varepsilon) \ln f_\varepsilon(\theta_\varepsilon(s)), \quad x \in \partial \Omega, \tag{2.39}
\]

while the latter determines the function \( a \):

\[
a(s, \mu, \varepsilon) = -\frac{4}{\zeta(3)\theta'_\varepsilon(s)} \Psi'^{\nu}(s, \mu, \varepsilon) \ln f_\varepsilon(\theta_\varepsilon(s)).
\]

Boundary value problem (2.33), (2.39) is uniquely solvable. The right sides of the equation in (2.33) and of the boundary condition (2.39) contain smooth on \( x \) and \( s \) functions, thus, \( \Psi_2 \in C^\infty(\Omega) \).

Now we return to the construction of the inner expansion. It is easy to check that the function

\[
Y^{(j)}_1(\varsigma, \varepsilon) = (\alpha^j(\varepsilon) + \beta^j(\varepsilon)) \Re \sqrt{y^2 - 1}
\]
In view of this asymptotics and other mentioned properties of the function $Y_1^{(j)}$ the function $w_{1,1}^{(j)}$ is given by

$$w_{1,1}^{(j)} = -\frac{1}{2(A + \mu)\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' Y_1^{(j)}.$$ 

By direct calculations one can establish that the function

$$\tilde{w}_{2,1} = \frac{\Psi_0^\nu}{2(\theta_\epsilon^j)^2} \frac{s_2^2}{|s|^2} \left( \frac{\theta_\epsilon^j}{\theta_\epsilon^j} \partial + k \frac{\partial}{\partial s_1} \right) Y^{(j)} + \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' \varsigma Y^{(j)},$$

belonging to $W$ is a solution of the problem (2.37) and satisfies to the following asymptotics at infinity ($\varsigma \to \infty$):

$$\tilde{w}_{2,1}^{(j)} = -\frac{\Psi_0^\nu}{2(\theta_\epsilon^j)^2} \frac{s_2^2}{|s|^2} \left( \frac{\theta_\epsilon^j}{\theta_\epsilon^j} \partial + k \frac{\partial}{\partial s_1} \right) Y^{(j)} + \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' \varsigma Y^{(j)} + O(1).$$

To get the function $w_{2,1}^{(j)}$ needed we should add harmonic function $Y^{(j)}$ with an suitable factor to $\tilde{w}_{2,1}^{(j)}$ so that the asymptotics of $w_{2,1}^{(j)}$ to contain the needed coefficient of $\varsigma_1$. Such a factor is a function:

$$\frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' (\ln(\alpha^j + \beta^j) - 1) - \frac{1}{2(A + \mu)\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' \varsigma Y^{(j)};$$

i.e.,

$$w_{2,1}^{(j)} = \tilde{w}_{2,1}^{(j)} + \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' (\ln(\alpha^j + \beta^j) - 1) - \frac{1}{2(A + \mu)\theta_\epsilon^j} \left( \frac{\Psi_0^\nu}{\theta_\epsilon^j} \right)' Y^{(j)}.$$

The functions $w_{3,1}^{(j)}$ and $w_{4,1}^{(j)}$ are determined similarly:

$$w_{3,1}^{(j)} = \frac{\Psi_1^\nu}{2(\theta_\epsilon^j)^2} \frac{s_2^2}{|s|^2} \left( \frac{\theta_\epsilon^j}{\theta_\epsilon^j} \partial + k \frac{\partial}{\partial s_1} \right) Y^{(j)} + \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_1^\nu}{\theta_\epsilon^j} \right)' \varsigma_1 Y^{(j)} +$$

$$+ \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_1^\nu}{\theta_\epsilon^j} \right)' (\ln(\alpha^j + \beta^j) - 1) - \frac{1}{2(A + \mu)\theta_\epsilon^j} \left( \frac{\Psi_1^\nu}{\theta_\epsilon^j} \right)' Y^{(j)};$$

$$w_{4,1}^{(j)} = \frac{\Psi_2^\nu}{2(\theta_\epsilon^j)^2} \frac{s_2^2}{|s|^2} \left( \frac{\theta_\epsilon^j}{\theta_\epsilon^j} \partial + k \frac{\partial}{\partial s_1} \right) Y^{(j)} +$$

$$+ \frac{1}{\theta_\epsilon^j} \left( \frac{\Psi_2^\nu}{\theta_\epsilon^j} \right)' (\varsigma_1 Y^{(j)} + (\ln(\alpha^j + \beta^j) - 1) Y^{(j)}.$$ 

Clear, $w_{k,1}^{(j)} \in W$. Employing asymptotics (2.20) and (2.40), we see that as $\varsigma \to \infty$

$$w_{k,i}^{(j)} = W_{k,i} + O(|\varsigma|^{i-1}).$$

(2.41)
The formal constructing of outer expansion (2.30), boundary layer (2.31) and inner expansion (2.32) is finished.

Next four auxiliary lemmas will be used in proving that the eigenelements’ asymptotics formally constructed is a formal asymptotics solution of the perturbed problem.

We denote \( \Omega^{bl} = \{ x : 0 < \tau < c_0 \} \) where \( c_0 \) is a some small fixed number so that in a domain \( \Omega^{bl} \) the coordinates \( (s, \tau) \) are defined correctly and the function \( H(s, \tau) \) has no zeroes. Throughout in what follows we will employ the symbol \( C \) for nonspecific constants independent on \( \varepsilon \) and \( \mu \).

**Lemma 2.2.** Suppose \( F = F(x, \mu, \varepsilon) \) and \( f = f(s, \mu, \varepsilon) \) are infinitely differentiable on \( x \) and \( s \) functions, \( a_0 = a_0(\mu, \varepsilon) \) is a some function uniformly bounded on \( \varepsilon \) and \( \mu \), and norms \( \| f \|_{C(\partial \Omega)}, \| F \|_{C(\Omega)} \) and \( \| F \|_{C_1(\Omega_1)} \), \( \Omega_1 \subset \Omega \) is an arbitrary subdomain, \( k \in \mathbb{N} \), are uniformly bounded on \( \varepsilon \) and \( \mu \). If the boundary value problem

\[
(\Delta + a_0)u = F, \quad x \in \Omega, \quad \left( \frac{\partial}{\partial \nu} + (A + \mu) \theta'_{\varepsilon} \right) u = f, \quad x \in \partial \Omega. \tag{2.42}
\]

has a solution whose \( H^1(\Omega) \) norm is uniformly bounded on \( \varepsilon \) and \( \mu \), then for this solution uniform on \( \varepsilon \) and \( \mu \) estimates hold:

- \( \| u \|_{C^1(\Omega)} \leq C(\| f \|_{C^1(\partial \Omega)} + 1), \)
- \( \| u^\nu \|_{C(\partial \Omega)} \leq C(A + \mu + \| f \|_{C(\partial \Omega)}), \)
- \( \| u^\nu \|_{C^i(\partial \Omega)} \leq C(\| f \|_{C^i(\partial \Omega)} + 1), \quad i = 1, 2, \)
- \( \| u^\nu \|_{C^2(\partial \Omega)} \leq C(\| f \|_{C^1(\partial \Omega)} + \| f \|_{C^3(\partial \Omega)} + 1), \)

where \( u^\nu = u^\nu(s, \mu, \varepsilon) = \frac{\partial u}{\partial \nu}(x, \mu, \varepsilon), \quad x \in \partial \Omega, \quad k \in \mathbb{Z}. \)

**Proof.** The smoothness \( f \) and \( F \) allows us to maintain that the solution \( u \) for the problem (2.42) is infinitely differentiable on \( x \). Moreover, by the boundedness of the norm \( \| u \|_{H^1(\Omega)} \) for each couple of strongly inner subdomains \( \Omega_1 \subset \Omega_2 \subset \Omega \) we have

\[
\| u \|_{H^{k+2}(\Omega_1)} \leq C(\| F \|_{H^k(\Omega_2)} + 1) \leq C, \quad k \in \mathbb{N}.
\]

Last inequalities and embedding theorems \( (C^k \subset H^{k+2}) \) imply that estimates

\[
\| u \|_{C^k(\Omega_1)} \leq C. \tag{2.43}
\]

take place. In a domain \( \Omega^{bl} \) we change the function \( u \):

\[
v(x, \mu, \varepsilon) = u(x, \mu, \varepsilon) e^{-(A + \mu) \theta'_{\varepsilon}(a_1 - a_2 \tau^2)},
\]

where \( a_1 \) and \( a_2 \) are some positive numbers. Owing to (2.42) the function \( v \) is a solution of the problem:

\[
\left( \Delta_x + a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + a_5 \right) v = L_1 v = \tilde{F}, \quad x \in \Omega^{bl},
\]
\[ v = a_6, \quad \tau = c_0, \quad \frac{\partial v}{\partial \tau} = -a_1 f, \quad \tau = 0, \]

where \( \bar{F} = e^{-(A+\mu)s}(a_1 - a_2\tau^2)F \), the functions \( a_i = a_i(x, \mu, \varepsilon), \quad i = 3, 4, 5, \quad a_6 = a_6(s, \mu, \varepsilon) \) are smooth on spatial variables and holomorphic on \( \mu \), and also \( \|a_6\|_{C(\tau = c_0)} \leq C \) (see (2.43)), \( \|a_i\|_{C^1(\Omega^d)} \leq C \). The functions \( a_i, \quad i = 3, 4, 5, \) can be easily got explicitly, we don’t adduce here these explicit formulae and just note that by a suitable choice of constant \( a_2, a_3 \) and constant \( c_0 \) from the definition of \( \Omega^d \) one can always achieve inequalities \( a_1 - a_2c_0^2 > 0, \quad a_5 \leq C < 0 \) for \( x \in \Omega^d \). Then for the operator \( L_1 \) and each function \( V \in C^2(\Omega^d) \) the statement holds: if

\[ L_1V < 0, \quad x \in \Omega^d, \quad V > 0, \quad \tau = c_0, \quad \frac{\partial V}{\partial \tau} < 0, \quad \tau = 0, \]

then \( V > 0 \). Indeed, assuming a contrary, at a point of minimum in \( \Omega^d \) the function \( V \) is negative, \( \Delta V \geq 0, \quad \nabla_x V = 0 \), i.e., at this point \( L_1V > 0 \). Clear, this point of minimum lies strongly inside the domain \( \Omega^d \); the contradiction obtained proves the statement. Now we take a "barrier" function \( (a_7 - a_8\tau - a_9\tau^2) \), \( a_7, a_8, a_9 \) are positive constant, and apply this statement to the functions \( V = (a_7 - a_8\tau - a_9\tau^2) + v \), each time choosing the constants \( a_i \) in a suitable way. As a result we have an estimate

\[
\|u\|_{C(\Omega^d)} \leq C\|v\|_{C(\Omega^d)} \leq C \left( \|\bar{F}\|_{C(\Omega^d)} + \|f\|_{C(\partial \Omega)} + \|a_6\|_{C(\tau = c_0)} \right) \leq C \left( \|F\|_{C(\Omega^d)} + \|f\|_{C(\partial \Omega)} + \|u\|_{C(\Omega^d)} \right) \leq C.
\]

Combining last inequality with (2.43), we finally get

\[ \|u\|_{C(\Omega)} \leq C. \quad (2.44) \]

In [34, Chapter 3, § 3, Theorem 3.1] the estimate is given, by that, dividing the equation and boundary condition in (2.42) to sufficiently great fixed number and taking into account the smoothness \( u \), we obtain:

\[ \|u\|_{C^2(\Omega)} \leq C \left( \|F\|_{C(\Omega)} + \|f\|_{C^1(\partial \Omega)} + \|u\|_{C(\Omega)} \right). \quad (2.45) \]

It follows from (2.44) and (2.45) that

\[ \|u\|_{C^2(\Omega)} \leq C \left( \|f\|_{C^1(\partial \Omega)} + 1 \right), \quad (2.46) \]

what, in particular, gives needed estimate for \( \|u\|_{C^1(\Omega)} \). Due to boundary condition \( u^\nu = -(A + \mu) \theta^\nu u + f \), hence, using (2.44) and (2.46) and bearing in mind the boundedness of \( \|\theta^\nu\|_{C^2(\partial \Omega)} \), we derive the estimate for the quantities \( \|u^\nu\|_{C^1(\partial \Omega)} \), \( i = 0, 1, 2 \), given in the statement of the lemma. Let us estimate \( \|u^\nu\|_{C^1(\partial \Omega)} \). For \( x \in \Omega^d \) we differentiate the problem (2.42) on \( s \). Then we have, that the function \( U = \frac{d}{dt} u \) is a solution of the boundary value problem:

\[
\left( \Delta_x + \frac{\partial}{\partial s} \left( H^{-2} \right) \frac{\partial}{\partial s} + a_0 \right) U = \frac{\partial F}{\partial s} - \frac{k' \partial u}{H^2} \frac{\partial}{\partial \tau} \equiv F_1, \quad x \in \Omega^d,
\]
\[
\left( \frac{\partial}{\partial \nu} + (A + \mu)\theta'_\mu \right) U = f' - (A + \mu)\theta''_\mu U \equiv f_1, \quad x \in \partial\Omega,
\]

\[
\frac{\partial U}{\partial \tau} = \frac{\partial^2 u}{\partial s \partial \tau}, \quad \tau = c.
\]

For such problem, leaning for \[\text{[30, Chapter 3, § 3, Theorem 3.1]}\], we can write the estimate of \((2.45)\) kind; here it is of the form:

\[
\|U\|_{C^2(\Omega')} \leq C \left( \|F_1\|_{C^1(\Omega')} + \|f_1\|_{C^1(\partial\Omega)} + \left\| \frac{\partial^2 u}{\partial s \partial \tau} \right\|_{C^1(\{\tau = \epsilon_0\})} + \|U\|_{C(\Omega')} \right).
\]

The quantity \(\left\| \frac{\partial^2 u}{\partial s \partial \tau} \right\|_{C^1(\{\tau = \epsilon_0\})}\) is estimated above by some constant \(C\) due to \((2.43)\); the sum of other three summands can be estimated by \((2.46)\):

\[
\|F_1\|_{C^1(\Omega')} + \|f_1\|_{C^1(\partial\Omega)} + \|U\|_{C(\Omega')} \leq C \left( \|F\|_{C^1(\Omega')} + \|f\|_{C^2(\partial\Omega)} + 1 \right).
\]

Substituting the estimate obtained into \((2.47)\), we arrive at the inequality

\[
\left\| \frac{\partial u}{\partial s} \right\|_{C^2(\Omega')} \leq C \left( \|F\|_{C^1(\Omega')} + \|f\|_{C^2(\partial\Omega)} + 1 \right),
\]

from what, the equality \(u'' = -(A + \mu)\theta'_\mu u + f\) and the boundedness of \(\|\theta'_\mu\|_{C^2(\partial\Omega)}\) the estimate for \(\|u''\|_{C^1(\partial\Omega)}\) follows. The proof is complete.

**Lemma 2.3.** The functions \(\Psi_1\) and \(\Lambda_1\) are represented in the form:

\[
\Psi_1(x, \mu, \varepsilon) = (A + \mu)^2\tilde{\Psi}_1(x, \mu, \varepsilon), \quad \Lambda_1(\mu, \varepsilon) = (A + \mu)^2\tilde{\Lambda}_1(\mu, \varepsilon),
\]

where \(\tilde{\Psi}_1\) is infinitely differentiable on \(x\), \(\tilde{\Psi}_1\) and \(\tilde{\Lambda}_1\) are holomorphic on \(\mu\) for each fixed value of \(\varepsilon\). The uniform on \(\varepsilon\) and \(\mu\) estimates \((i = 1, 2, 3)\)

\[
|\Lambda_0| \leq C, \quad \|\Psi_0\|_{H^1(\Omega)} \leq C, \quad \|\Psi'_0\|_{C^3(\partial\Omega)} \leq C(A + \mu),
\]

\[
|\Lambda_1| \leq C(A + \mu)^2, \quad \|\Psi_1\|_{H^1(\Omega)} \leq C(A + \mu)^2, \quad \|\Psi'_1\|_{H^1(\Omega)} \leq C(A + \mu)^3,
\]

\[
\|\Psi''_1\|_{C^1(\partial\Omega)} \leq C(A + \mu)^2, \quad \|\Psi''_1\|_{C^1(\partial\Omega)} \leq C(A + \mu)^2(\varepsilon^{-i}\delta^*(\varepsilon) + 1),
\]

\[
\|\Psi''_2\|_{C^1(\partial\Omega)} \leq C(A + \mu)^3, \quad \|\Psi''_2\|_{C^1(\partial\Omega)} \leq C(A + \mu)^3(\varepsilon^{-i}\delta^*(\varepsilon) + 1).
\]

hold true.

**Proof.** The proof of representations \((2.48)\) is very simple. Indeed, the representation for \(\Lambda_1\) is a direct implication of \((1.11)\). Employing this representation and presence of the factor \((A + \mu)^2\) in the boundary condition \((2.28)\), we arrive at the needed representation for \(\Psi_1\).

The proof of the estimates for \(\Lambda_0\) and \(\Psi_0\) from \((2.49)\) is elementary. The boundedness of \(\Lambda_0\) follows from the convergence \(\Lambda_0 \to \lambda_0\). Since \(\|\Psi_0\|_{L^2(\Omega)} = 1,\)
multiplying equation (1.7) by $Ψ_0$ and integrating once by parts, owing to boundedness of $Λ_0$ and $\|θ'_ε\|_{C^j(\partialΩ)}$ we get the needed estimate for the norm $\|Ψ_0\|_{H^1(Ω)}$. Now, applying Lemma 2.2 to the problem for the function $Ψ_0$, we obtain the estimate for $\|Ψ_0\|_{C^j(\partialΩ)}$, and also, $\|Ψ_0\|_{C^k(Ω)} ≤ C, \|Ψ_0\|_{C^k(Ω)} ≤ C$ for each subdomain $Ω₁ ∈ Ω$.

The estimate for $Λ_1$ arises from the proven estimates for $Ψ_0$, boundedness of the function $θ'_ε$ and $f_ε(θ_ε)$ and the formula (1.11).

We prove the inequalities for $Ψ_1$ and $Ψ_2$ from (2.49) on the basis of Lemma 2.2, too. Since $Ψ_1$ is orthogonal to $Ψ_0$ in $L_2(Ω)$, and the quantities $Λ_0$ and $Λ_1$ are bounded, an uniform estimate

$$\|Ψ_1\|_{H^1(Ω)} ≤ C(A + μ)^2 (\|Ψ_0\|_{L_2(Ω)} + \|Ψ_0^Dθ'_ε\ln f_ε(θ_ε)\|_{H^1(Ω)}) ≤ C(A + μ)^2$$

takes place. The right side of the equation (2.8) and the boundary condition (2.28) obey to hypothesis of Lemma 2.2. We also note that the estimating of the derivatives of boundary condition (2.28) actually reduces to the estimating of derivatives of (bounded) function $f_ε(θ_ε(s))$, since the derivatives of $θ'_ε$ are estimated by assumption (C0), while the estimates for the derivatives of $Ψ_0^D$ are deduced from the estimates for $Ψ_0^D$ proved already and the equality $Ψ_0^D = (A + μ)θ'_εΨ_0^D$. Obviously, the derivatives $f_ε(θ_ε)$ are estimated as follows

$$\|f_ε(θ_ε(s))\|_{C^j(\partialΩ)} ≤ C (ε^{-1}δ^s(δ) + 1), \quad i = 1, 2, 3.$$ 

Using this obvious fact and applying Lemma 2.2 to the problem for $Ψ_1$, we arrive at the estimates for $Ψ_1$ from (2.49). Besides, Lemma 2.2 implies inequalities

$$\|Ψ_1\|_{C^j(Ω)} ≤ C(A + μ)^2 (ε^{-1}δ^s(δ) + 1), \quad \|Ψ_1\|_{C^k(Ω)} ≤ C,$$

for all $k ∈ Z_+$ and all $Ω₁ ∈ Ω$. By obvious inequality

$$\|Ψ_2\|_{H^1(Ω)} ≤ C (|Λ_1|\|Ψ_1\|_{L_2(Ω)} + (A + μ)\|Ψ_1\|_{L_2(Ω)}),$$

and estimates for $Ψ_1$ and $Λ_1$ proved already we get the needed estimate for the norm $\|Ψ_2\|_{H^1(Ω)}$. Representing $Ψ_2$ as $Ψ_2 = (A + μ)^3Ψ_2$ and applying Lemma 2.2 to $Ψ_2$, we obtain other estimates for $Ψ_2$ from (2.49). The proof is complete.

We denote $\hat{Λ}_ε = Λ_0(μ, ε) + εΛ_1(μ, ε), Π^{(i)} = \{ξ : |ξ - πj| < π/2, ξ_2 > 0\}, Π_η^{(j)} = Π^{(j)} ∩ \{ξ : 4|ξ - ξ^{(j)}| > η^{1/4}\}, Ω^{bl}_η = Ω^{bl} ∩ \{x : ξ ∈ Π_η^{(j)}, j = 0, . . . , N - 1\}.

**Lemma 2.4.** For boundary layer (2.31) uniform estimates

$$\|ψ_ε^{bl}\|_{L_2(Ω^{bl})} ≤ Cε^{3/2}(A + μ), \quad \|θ_ε^{bl} - εv_1 - ε^2(v_2 - \tilde{v}_2)\|_{H^1(Ω^{bl})} ≤ Cε^{3/2}(A + μ)^2,$$

$$\|Δ_εθ_ε^{bl}\|_{L_2(Ω^{bl})} ≤ C(ε^{3/2}(A + μ) + ε^{1/2}δ^s(δ)(A + μ)^2)$$

hold as $ε → 0$. 

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**Proof.** Everywhere in the proof, not stressing it additionally, we will employ the fact that

\[
\xi_i^{i+k} \frac{\partial^i X}{\partial \xi_2^i}, \quad \xi_i^{i+k} \frac{\partial^i X}{\partial \xi_1 \partial \xi_2^{i-1}} \in L_2(\Pi^{(j)}) \cap \mathcal{V}_0, \quad i, k \in \mathbb{Z}, \quad i, k \geq 0.
\]

An estimate

\[
\|\psi_\varepsilon^{bl}\|_{L_2(\Omega^M)} \leq \varepsilon \left( \sum_{j=0}^{N-1} \|\psi_\varepsilon^{b\ell}\|_{L_2(\Omega^{(j)})}^2 \right)^{1/2}
\]

is true. We estimate the norms \(\|\psi_\varepsilon^{bl}\|_{L_2(\Omega^{(j)})}\) using explicit form of the functions \(v_i\) and estimates from Lemma 2.3:

\[
\|\psi_\varepsilon^{bl}\|_{L_2(\Omega^{(j)})} \leq C(\varepsilon + A + \mu).
\]

Last two estimates and the equality \(N = 2\varepsilon^{-1}\) yield first inequality from the statement of the lemma. Second inequality can be proved by analogy on the basis of explicit form of the functions \(v_i\) and Lemma 2.3. For the sake of brevity we denote: \(F_\varepsilon = (\Delta_x + \hat{\lambda}_\varepsilon)\psi_\varepsilon^{bl}\). Employing explicit form of the functions \(v_i\) we calculate:

\[
H^3F_\varepsilon = \varepsilon \sum_{i=1}^{3} \xi_i^2 \left( c_{i,0} \frac{\partial^i X}{\partial \xi_2^i} + c_{i,-1} \frac{\partial^i X}{\partial \xi_1 \partial \xi_2^{i-1}} \right) + \varepsilon c_{0,0} X + \varepsilon^2 c_{1,-1} \int_{\xi_2}^{+\infty} t \frac{\partial}{\partial \xi_1} X(\xi_1, t) \, dt + \varepsilon c_{0,-1} \int_{\xi_2}^{+\infty} t X(\xi_1, t) \, dt.
\]

Here \(c_{i,k} = c_{i,k}(\xi_2, s, \varepsilon, \mu)\) are polynomials on \(\xi_2\) whose coefficients depends on other variables and owing to Lemma 2.3 can be estimated above by a quantity \(C ((A + \mu) + \varepsilon^{-1} \delta^s(\varepsilon)(A + \mu)^2)\), where \(C\) are independent on \(\varepsilon, \mu, s\). Using these estimates for coefficients of polynomials \(c_{i,k}\) and the form of the function \(H^3F_\varepsilon\), we persuade to

\[
\|H^3F_\varepsilon\|_{L_2(\Omega^{(j)})} \leq C \left( \varepsilon (A + \mu) + \delta^s(\varepsilon)(A + \mu)^2 \right),
\]

where \(C\) is independent on \(\varepsilon, \mu, s\) and \(j\). Since for \(x \in \Omega^d\) the function \(H\) does not vanish it follows that

\[
\|F_\varepsilon\|_{L_2(\Omega^M)} \leq C \|H^3F_\varepsilon\|_{L_2(\Omega^M)} \leq C\varepsilon \left( \sum_{j=0}^{N-1} \|H^3F_\varepsilon\|_{L_2(\Omega^{(j)})}^2 \right)^{1/2},
\]

what with the estimates for the norms \(\|H^3F_\varepsilon\|_{L_2(\Omega^{(j)})}\) obtained already gives third inequality from the statement of Lemma. The proof is complete.

We denote \(\Omega_{j}^{in} = \{x : 4\eta^{3/4}|s^j| < 3\}\), \(\Omega_{j}^{mat} = \{x : 1 < 4|s^j|\eta^{3/4} < 3, j = 0, \ldots, N - 1\}\).

By analogy with Lemma 2.4 one can establish the validity of following statement.
Lemma 2.5. For the inner expansions (2.32) uniform on $\varepsilon$, $\mu$ and $\eta$ estimates
\[
\|\psi^{i,j}_\varepsilon\|_{L_2(\Omega^\mu)} \leq C\eta^{1/5}, \quad \|(\Delta_x + \hat{\lambda}_\varepsilon)\psi^{i,j}_\varepsilon\|_{L_2(\Omega^\mu)} \leq C\eta^{1/5},
\]
\[
\|\psi^{i,j}_\varepsilon - \varepsilon u_1^{(j)} - \varepsilon^2 w_2^{(j)}\|_{H^1(\Omega^\mu)} \leq C\varepsilon^2(A + \mu)^{5/2}
\]
take place as $\varepsilon \to 0$.

Let
\[
\hat{\psi}_\varepsilon(x) = (\psi^{ex}_\varepsilon(x, \mu) + \chi(\tau/c_0)\psi^{bl}_\varepsilon(\xi, s, \mu)) \chi_\varepsilon(x) + \sum_{j=0}^{N-1} \chi(|\xi^j|\eta^{3/4})\psi^{i,j}_\varepsilon(\xi^j, s, \mu),
\]
where $\psi^{ex}_\varepsilon$, $\psi^{bl}_\varepsilon$ and $\psi^{i,j}_\varepsilon$ are from (2.30)–(2.32),
\[
\chi_\varepsilon(x) = 1 - \sum_{j=0}^{N-1} \chi(|\xi^j|\eta^{3/4}).
\]

In next statement we will prove that formally constructed asymptotics $\hat{\lambda}_\varepsilon$ and $\hat{\psi}_\varepsilon$ are formal asymptotics solutions of the perturbed problem.

Lemma 2.6. The functions $\hat{\psi}_\varepsilon \in C^\infty(\Omega \cup \gamma_\varepsilon \cup \Gamma_\varepsilon) \cap H^1(\Omega)$ and $\hat{\lambda}_\varepsilon$ satisfy boundary value problem
\[
-\Delta u_\varepsilon = \lambda u_\varepsilon + f, \quad x \in \Omega, \quad u_\varepsilon = 0, \quad x \in \gamma_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad x \in \Gamma_\varepsilon, \quad (2.50)
\]
with $u_\varepsilon = \hat{\psi}_\varepsilon$, $\lambda = \hat{\lambda}_\varepsilon$ and $f = f_\varepsilon$, where for $f_\varepsilon$ the uniform estimate holds:
\[
\|f_\varepsilon\|_{L_2(\Omega)} \leq C (\varepsilon^{3/2}(A + \mu(\varepsilon)) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu(\varepsilon))^2). \quad (2.51)
\]
The function $\hat{\lambda}_\varepsilon$ converges to $\lambda_0$ as $\varepsilon \to 0$, and for $\hat{\psi}_\varepsilon$ the relationship $\|\hat{\psi}_\varepsilon - \Psi_0\|_{L_2(\Omega)} = o(1)$ holds true.

Proof. The maintained smoothness of the function $\hat{\psi}_\varepsilon$ is obvious. Convergence of $\hat{\lambda}_\varepsilon$ to $\lambda_0$ follows from Lemmas [1] and [2]. The relationship $\|\hat{\psi}_\varepsilon - \Psi_0\|_{L_2(\Omega)} = o(1)$ is a direct implication of Lemmas [2.3]-[2.5]. Let us check the boundary conditions from (2.50). Vanishing of the function $\hat{\psi}_\varepsilon$ on $\gamma_\varepsilon$ arises from vanishing of $\chi_\varepsilon$ on $\gamma_\varepsilon$ and of $\psi^{i,j}_\varepsilon$ on $\gamma^1_j$. It is easy to check that for $x \in \Gamma_\varepsilon$
\[
\frac{\partial \hat{\psi}_\varepsilon}{\partial \nu} = \left( \frac{\partial \psi^{ex}_\varepsilon}{\partial \nu} - \frac{\theta'}{\varepsilon} \frac{\partial \psi^{bl}_\varepsilon}{\partial \xi_2} \biggr|_{\xi \in \Gamma_0} \right) \chi_\varepsilon(x) - \frac{\theta'}{\varepsilon\eta} \sum_{j=0}^{N-1} \chi(|\xi^j|\eta^{3/4}) \frac{\partial \psi^{i,j}_\varepsilon}{\partial \xi_2} \biggr|_{\xi \in \Gamma^1_j} = \chi_\varepsilon(x) \sum_{i=0}^{2} \varepsilon^i \left( \psi^{i}_\varepsilon - \frac{\partial \psi^{i+1}_\varepsilon}{\partial \xi_2} \biggr|_{\xi \in \Gamma_0} \right) = 0.
\]
Let us estimate the function $f_\varepsilon$. Clear, it is of the form:

$$f_\varepsilon = -(\Delta_x + \widehat{\lambda}_\varepsilon)\widehat{\psi}_\varepsilon = - \sum_{i=1}^5 f_\varepsilon^{(i)},$$

$$f_\varepsilon^{(1)} = \chi_\varepsilon(\Delta_x + \widehat{\lambda}_\varepsilon)\psi_\varepsilon^{ex},$$
$$f_\varepsilon^{(2)} = \chi_\varepsilon\chi(\tau/c_0)(\Delta_x + \widehat{\lambda}_\varepsilon)\psi_\varepsilon^{bl},$$
$$f_\varepsilon^{(3)} = \chi_\varepsilon \left( 2(\nabla_x \psi_\varepsilon^{bl}, \nabla_x \chi(\tau/c_0)) + \psi_\varepsilon^{bl} \Delta_x \chi(\tau/c_0) \right),$$
$$f_\varepsilon^{(4)} = \sum_{j=0}^{N-1} \chi(|\varsigma^j|\eta^{3/4})(\Delta_x + \widehat{\lambda}_\varepsilon)\psi_\varepsilon^{in.j},$$
$$f_\varepsilon^{(5)} = \sum_{j=0}^{N-1} \left( 2(\nabla_x \psi_\varepsilon^{mat.j}, \nabla_x \chi(|\varsigma^j|\eta^{3/4})) + \psi_\varepsilon^{mat.j} \Delta_x \chi(|\varsigma^j|\eta^{3/4}) \right),$$

$$\psi_\varepsilon^{mat.j} = \psi_\varepsilon^{in.j} - \psi_\varepsilon^{ex} - \psi_\varepsilon^{bl}.$$

Employing equations (1.7), (2.8) and (2.33) we see that

$$(\Delta_x + \widehat{\lambda}_\varepsilon)\psi_\varepsilon^{ex} = \varepsilon^2(A_0 + \varepsilon A_1)\Psi_2,$$

thus, by Lemma 2.3,

$$\|f_\varepsilon^{(1)}\|_{L_2(\Omega)} \leq C\varepsilon^2(A + \mu)^3.$$

The function $f_\varepsilon^{(2)}$ is estimated by Lemma 2.4:

$$\|f_\varepsilon^{(2)}\|_{L_2(\Omega)} \leq \|f_\varepsilon^{(2)}\|_{L_2(\Omega^{\mu})} \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta(\varepsilon)(A + \mu)^2 \right).$$

The functions $v_i$ and, therefore, $\psi_\varepsilon^{bl}$ decay exponentially, and integrating of $f_\varepsilon^{(3)}$ over $\Omega$ due to definition $\chi$ actually reduces to integrating over a domain \{ $x : \frac{c_0\theta}{4\varepsilon} \leq \xi \leq \frac{3c_0\theta}{4\varepsilon}$ \}, thus

$$\|f_\varepsilon^{(3)}\|_{L_2(\Omega)} \leq C(A + \mu)e^{-1/eb},$$

where $b > 0$ is a some fixed number. Next, we estimate the function $f_\varepsilon^{(4)}$ on the basis of Lemma 2.5:

$$\|f_\varepsilon^{(4)}\|_{L_2(\Omega)} \leq C \sum_{j=0}^{N-1} \|\Delta_x + \widehat{\lambda}_\varepsilon\psi_\varepsilon^{in.j}\|_{L_2(\Omega_j^{\eta \mu})} \leq C\eta^{1/6}.$$

By the matching carried out (see (2.34), (2.41)) the function $\psi_\varepsilon^{mat.j}$ for $\eta^{-3/4} \leq 4|\varsigma^j| \leq 3\eta^{-3/4}$ has a differentiable asymptotics:

$$\psi_\varepsilon^{mat.j} = O(\eta^2|\varsigma^j|\ln|\varsigma| + \varepsilon|\varsigma|^{-1} + \varepsilon\eta),$$

(2.52)

using that we estimate $f_\varepsilon^{(5)}$:

$$\|f_\varepsilon^{(5)}\|_{L_2(\Omega)} \leq C \sum_{j=0}^{N-1} \|f_\varepsilon^{(5)}\|_{L_2(\Omega_j^{\text{mat}})} \leq C\eta^{1/5}.$$

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Collecting the estimates obtained for $f_\varepsilon^{(i)}$ we arrive at the statement of the lemma. The proof is complete.

Now we proceed to the justification of the asymptotics. By analogy with [17], [19], [28], [29], [37] it can be shown that for $\lambda$ close to $p$-multiply limiting eigenvalue $\lambda_0$, for the solution of the problem (2.50) with $f \in L_2(\Omega)$ the representation

$$u_\varepsilon = \sum_{k=q}^{q+p-1} \frac{\psi_k^\varepsilon}{\lambda_k^\varepsilon - \lambda} \int \Omega f \psi_k^\varepsilon \, dx + \tilde{u_\varepsilon},$$

(2.53)

takes place, where, recall, $\lambda_k^\varepsilon$, $k = q, \ldots, q + p - 1$, are perturbed eigenvalues converging to $\lambda_0$, $\psi_k^\varepsilon$ are associated orthonormalized in $L_2(\Omega)$ eigenfunctions, $\tilde{u_\varepsilon}$ is a holomorphic on $\lambda$ in $H^1(\Omega)$-norm function orthogonal to all $\psi_k^\varepsilon$, $k = q, \ldots, q + p - 1$, in $L_2(\Omega)$; for the function $\tilde{u_\varepsilon}$ a uniform on $\varepsilon$, $\lambda$, and $f$ estimate

$$\|\tilde{u_\varepsilon}\|_{H^1(\Omega)} \leq C\|f\|_{L_2(\Omega)}$$

(2.54)
is valid. In our case $\lambda_0$ is a simple eigenvalue. We set $u_\varepsilon = \hat{\psi_\varepsilon}$, $\lambda = \hat{\lambda_\varepsilon}$ and $f = f_\varepsilon$. Then from Lemma 2.6 and (2.53), (2.54) we obtain

$$\hat{\psi_\varepsilon} = \frac{\psi_\varepsilon}{\lambda_\varepsilon - \hat{\lambda_\varepsilon}} \int \Omega f \psi_\varepsilon \, dx + \tilde{u_\varepsilon},$$

(2.55)

$$\|\tilde{u_\varepsilon}\|_{H^1(\Omega)} \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu)^2 \right).$$

Since $\|\hat{\psi_\varepsilon} - \tilde{u_\varepsilon}\|_{L_2(\Omega)} = \|\Psi_0\|_{L_2(\Omega)}(1 + o(1)) = 1 + o(1)$ (see Lemma 2.6), by (2.55) we have:

$$C \leq \frac{\|f_\varepsilon\|_{L_2(\Omega)}}{|\lambda_\varepsilon - \hat{\lambda_\varepsilon}|} \Rightarrow |\lambda_\varepsilon - \hat{\lambda_\varepsilon}| \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu)^2 \right).$$

The Corollary 1 of Lemma 2.3 allows us to replace the function $\delta^*(\varepsilon)$ by $o(\varepsilon^{1/2}(A + \mu)^{-1})$ in last estimate, i.e., the asymptotics (1.10) is correct.

In general, the case of $p$-multiply eigenvalue $\lambda_0 = \lambda_0^0 = \ldots = \lambda_0^{q + p - 1}$ is proved similarly. In constructing the multiplicity of $\lambda_0$ becomes apparent in the fact that by same scheme we simultaneously construct several asymptotics corresponding to eigenvalues $\lambda_k^\varepsilon$ converging to $\lambda_0$. Besides, the multiplicity becomes apparent in solving the problem (1.7), (1.8), that has several eigenvalues $\Lambda_k$, converging to $\Lambda_0$, and, of course, several eigenfunctions $\Psi_k^\varepsilon$. These eigenfunctions are assumed to meet Lemma 2.3. In particular, the orthogonality of $\Psi_k^\varepsilon$ in $L_2(\partial\Omega)$ weighted by $\theta_\varepsilon$ is exactly a solvability condition of the problems for $\Psi_k^\varepsilon$, those again are chosen to be orthogonal to $\Psi_k^\varepsilon$. All other arguments of formal constructing hold true, including Lemmas 2.3-2.5. Thus, as a result of formal constructing we have functions $\hat{\lambda_k^\varepsilon}$ and $\hat{\psi_k^\varepsilon}$, $k = q, \ldots, p + q - 1$, those are defined as $\hat{\lambda_k^\varepsilon}$ and $\hat{\psi_k^\varepsilon}$ with replacement $\Lambda_0$ by $\Lambda_k^\varepsilon$ and $\Psi_0$ by $\Psi_k^\varepsilon$. For $\hat{\lambda_k^\varepsilon}$ and $\hat{\psi_k^\varepsilon}$ Lemma 2.6 is valid. By $f_\varepsilon^k$ we denote right sides of equations from (2.54) with $u_\varepsilon = \hat{\psi_k^\varepsilon}$, $\lambda = \hat{\lambda_k^\varepsilon}$. 

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Now we apply the representation (2.53) to the functions $\tilde{\psi}_k^\varepsilon$:

$$\tilde{\psi}_k^\varepsilon = \sum_{i=q}^{q+p-1} b_{ki}^\varepsilon \psi_i^\varepsilon + \tilde{\nu}_k^\varepsilon, \quad b_{ki}^\varepsilon = \frac{1}{\lambda_i^\varepsilon - \lambda_k^\varepsilon} \int_{\Omega} f_i^\varepsilon \psi_i^\varepsilon \, dx, \quad (2.56)$$

$$\|\tilde{\nu}_k^\varepsilon\|_{H^1(\Omega)} \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu)^2 \right).$$

Last estimate for $\tilde{\nu}_k^\varepsilon$ arises from (2.54) and Lemma 2.6. By (2.56) and the orthonormality of $\tilde{\nu}_k^\varepsilon$ to the functions $\psi_i^\varepsilon$ we get the assertions

$$b_{ki}^\varepsilon = \left(\tilde{\psi}_k^\varepsilon, \psi_i^\varepsilon\right)_{L_2(\Omega)}, \quad (2.57)$$

those imply boundedness of the quantities $b_{ki}^\varepsilon$. Let us prove the asymptotics (1.10) for the eigenvalues $\lambda_k^\varepsilon$, $k = q, \ldots, q + p - 1$. Assume a contrary, namely, suppose there exists a subsequence $\varepsilon_m$, on that for some of eigenvalues $\lambda_k^\varepsilon$, $k = q, \ldots, q + p - 1$, the asymptotics (1.10) are wrong, and for $k = q, \ldots, q + p - 1$, $i \in I \neq \emptyset$

$$|\lambda_i^\varepsilon - \lambda_k^\varepsilon| \geq m(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon_m)(A + \mu)^2), \quad (2.58)$$

where $I \subseteq \{q, \ldots, q + p - 1\}$ a subset of indices of eigenvalues not satisfying to asymptotics (1.10). By estimate for the functions $f_i^\varepsilon$, the formulae for $b_{ki}^\varepsilon$ from (2.56) and the inequalities (2.58) we deduce that

$$b_{ki}^{\varepsilon_m} \xrightarrow{m \to \infty} 0, \quad k = q, \ldots, q + p - 1, \quad i \in I. \quad (2.59)$$

Bearing in mind the boundedness $b_{ki}^{\varepsilon_m}$ and extracting a subsequence form $\varepsilon_m$ if it is needed, we assume that $b_{ki}^{\varepsilon_m} \to b_{ki}^0$, where due to (2.59) the equalities $b_{ki}^0 = 0$ are true for $k = q, \ldots, q + p - 1$, $k \in I$. By numbers $b_{ki}^{\varepsilon_m}$ we compose $p$ vectors $b_{ki}^\varepsilon$ by a rule: as components of vector $b_{ki}^{\varepsilon_m}$ we take consequently the numbers $b_{ki}^{\varepsilon_m}$, where index $i$ ranges in $q, \ldots, q + p - 1$ and does not takes values from the set $I$. In a similar way we compose $p$ vectors $b_{ki}^0$ from numbers $b_{ki}^0$. The dimension of the vectors composed are equal to $(p - |I|) < p$. Now multiply in $L_2(\Omega)$ the representations (2.56) for $\tilde{\psi}_k^\varepsilon$ each to other for all values of $k$ and take in account the equalities $\|\tilde{\psi}_k^\varepsilon - \Psi_0^k\|_{L_2(\Omega)} = o(1)$, the estimates for $\tilde{\nu}_k^\varepsilon$ and orthonormality of the functions $\Psi_0^k$ and $\psi_i^\varepsilon$. Then we get that

$$(b_{ki}^0, b_{kj}^0)_{L_2(\Omega)} = \lim_{m \to \infty} (b_{ki}^{\varepsilon_m}, b_{kj}^{\varepsilon_m})_{L_2(\Omega)} = \delta_{kj}, \quad k, j = q, \ldots, q + p - 1,$$

where $\delta_{kj}$ is a Kronecker delta, i.e., $b_{ki}^0$ make up a system of $p$ orthonormalized $(p - |I|)$-dimensional vectors. The contradiction obtained proves the estimates

$$|\lambda_k^\varepsilon - \lambda_k^\varepsilon| \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu)^2 \right),$$

what owing to the equality (1.9), Lemma 2.4, and Corollary 4 of this lemma leads us to the asymptotics (1.10) in the case of multiply eigenvalue $\lambda_0$. The proof of Theorem 1.4 is complete.
Let us clear up the asymptotics behaviour of perturbed eigenfunctions under assumption of Theorem 1.4. Under assumptions [C0], [C1] and equality (1.5) with 
\[ A \geq 0 \] one can establish following facts. If \( \lambda_0^k \) is a simple eigenvalue of problem 1.3, and \( \psi_0^k \) is an associated eigenfunction, then the eigenfunction \( \psi_\varepsilon^k \) converges to \( \psi_0^k \). If \( \lambda_0 = \lambda_0^0 = \cdots = \lambda_0^{q+p-1} \) is \( p \)-multiply eigenvalue and \( \lambda_\varepsilon \to \lambda_0^k \), \( k = q, \ldots, q+p-1 \), then for each associated eigenfunction \( \psi_\varepsilon^k \), \( k = q, \ldots, q+p-1 \), there exists a linear combination of eigenfunctions \( \psi_\varepsilon^l \), \( l = q, \ldots, q+p-1 \) converging to \( \psi_0^k \). This convergence is strong in \( L_2(\Omega) \) and weak in \( H^1(\Omega) \) if limiting problem is the Robin one \((A > 0)\) and it is strong in \( H^1(\Omega) \) if limiting problem is the Neumann one \((A = 0)\).

We will keep the notations of the proof of Theorem 1.4. Let \( \lambda_0 \) be a simple eigenvalue. It arises from Lemma 2.6 and Remark 1.3 that \( \psi_\varepsilon \) converges to \( \psi_0 \) in \( L_2(\Omega) \).

Multiplying (2.55) by \( \psi_\varepsilon \) in \( L_2(\Omega) \), owing to Lemmas 2.3-2.5 we see that

\[
\frac{1}{\lambda_\varepsilon - \lambda_\varepsilon^k} \int_{\Omega} f \psi_\varepsilon \, dx = (\hat{\psi}_\varepsilon, \psi_\varepsilon)_{L_2(\Omega)} = (\Psi_0 + \varepsilon \Psi_1, \psi_\varepsilon)_{L_2(\Omega)} + O(\varepsilon^{3/2}(A + \mu)).
\]

From last assertion, denoting

\[
\tilde{\psi}_\varepsilon = (\Psi_0 + \varepsilon \Psi_1, \psi_\varepsilon)_{L_2(\Omega)} \psi_\varepsilon,
\]
and from (2.55), (1.9) and Corollary 1.1 of Lemma 2.1 we derive that

\[
\|\tilde{\psi}_\varepsilon - \tilde{\psi}_\varepsilon\|_{L_2(\Omega)} \leq C \left( \varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2} \delta^*(\varepsilon)(A + \mu)^2 \right) = o(\varepsilon(A + \mu)),
\]

what, Remark 1.3 and Lemma 2.6 imply that the perturbed eigenfunction \( \tilde{\psi}_\varepsilon \), associated with \( \lambda_\varepsilon \), converges to \( \psi_0 \) in \( L_2(\Omega) \) and due to Lemmas 2.3-2.5 and the matching carried out has the following asymptotics in \( H^1(\Omega) \)-norm:

\[
\tilde{\psi}_\varepsilon(x) = \left( \begin{array}{c} \Psi_0(x, \mu, \varepsilon) + \varepsilon \Psi_1(x, \mu, \varepsilon) - \frac{\lambda(\tau/c_0)}{\theta^*(s)} \sum_{l=0}^{1} \varepsilon^l + 1 \Psi^\varepsilon_l(s, \mu, \varepsilon)X(\xi) \end{array} \right) \lambda_\varepsilon(x) - \end{array} \right.
\]

\[
- \sum_{j=0}^{N-1} \frac{\lambda(\xi)}{\theta^*(s)} \sum_{l=0}^{1} \varepsilon^l + 1 \Psi^\varepsilon_l(s, \mu, \varepsilon)Y^j(s, \xi, \varepsilon) + o(\varepsilon(A + \mu)).
\]

(2.61)

Let \( \lambda_0 = \lambda_0^0 = \cdots = \lambda_0^{q+p-1} \) be a \( p \)-multiply eigenvalue. Let us calculate the coefficients of linear combination of perturbed eigenfunctions converging to \( \psi_0^0, \psi_0^0, \psi_0^{q+p-1} \) and the asymptotics for them. First we will prove an auxiliary lemma.

**Lemma 2.7.** In \( H^1(\Omega) \) a convergence holds:

\[
\sum_{l=q}^{q+p-1} (\psi_0^l, \Psi_0^l)_{L_2(\Omega)} \Psi_0^l \to \psi_0^k.
\]
**Proof.** The eigenfunctions $\Psi_k^0$, $k = q, \ldots, q + p - 1$, converge to eigenfunction $\psi^k_0$ of the problem (1.6) in such sense that for each eigenfunction $\psi^k_0$, $k = q, \ldots, q + p - 1$, there exists a linear combination of eigenfunctions $\Psi^l_0$, $l = q, \ldots, q + p - 1$, converging to $\psi^k_0$ in $H^1(\Omega)$ (for $k$):

$$\sum_{l=q}^{q+p-1} b_{lk} \Psi^l_0 = \psi^k_0(1 + o(1)).$$

Multiplying this equality by $\Psi^l_0$ in $L_2(\Omega)$, we have: $b_{lk} = (\psi^k_0, \Psi^l_0)_{L_2(\Omega)}(1 + o(1))$, what proves the lemma. The proof is complete.

It follows from formulae (2.57) and Lemmas 2.3-2.5 that $b_{ki} = (\Psi_0 + \varepsilon \Psi_1, \psi^i_k)_{L_2(\Omega)} + O(\varepsilon^{3/2}(A + \mu))$, (2.62) moreover, last assertions hold under the assumption of boundedness of function $\delta^*(\varepsilon)$. Using these assertions and (2.50), we derive an estimate:

$$\left\| \sum_{i=q}^{q+p-1} (\Psi^l_0 + \varepsilon \Psi^l_1, \psi^i_k)_{L_2(\Omega)} \psi^i_\varepsilon - \psi^l_\varepsilon \right\|_{H^1(\Omega)} = O(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2} \delta^*(\varepsilon)(A + \mu)^2),$$

(2.63)

from that, Lemma 2.7 and the estimate $\| \psi^l_\varepsilon - \Psi^l_0 \|_{L_2(\Omega)} = o(1)$ (see Lemma 2.6) it follows the convergence in $L_2(\Omega)$:

$$\tilde{\psi}^k_\varepsilon \equiv \sum_{l=q}^{q+p-1} (\psi^k_0, \Psi^l_0)_{L_2(\Omega)} \sum_{i=q}^{q+p-1} (\Psi^l_0 + \varepsilon \Psi^l_1, \psi^i_k)_{L_2(\Omega)} \psi^i_\varepsilon \to \psi^k_0,$$

(2.64)

i.e., $\tilde{\psi}^k_\varepsilon$ is a linear combination of the perturbed eigenfunctions, converging to $\psi^k_0$ in $L_2(\Omega)$. On the other hand, by (2.63) for $\tilde{\psi}^k_\varepsilon$ the estimate

$$\left\| \psi^k_\varepsilon - \sum_{l=q}^{q+p-1} (\psi^k_0, \Psi^l_0)_{L_2(\Omega)} \tilde{\psi}^l_\varepsilon \right\|_{H^1(\Omega)} = O(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2} \delta^*(\varepsilon)(A + \mu)^2),$$

takes place, from that, equality (1.4) and Corollary 1 of Lemma 2.1 it arises that the asymptotics for $\tilde{\psi}^k_\varepsilon$ in $H^1(\Omega)$ has the following form:

$$\tilde{\psi}^k_\varepsilon(x) = \sum_{i=q}^{q+p-1} (\psi^k_0, \Psi^l_0)_{L_2(\Omega)} \left( \left( \Psi^l_0(x, \mu, \varepsilon) + \varepsilon \Psi^l_1(x, \mu, \varepsilon) - \frac{\chi((\tau/c_0)}{\theta^*_\varepsilon(s)} \sum_{l=0}^{1} \varepsilon^{l+1} \Psi^i_{l,\nu}(s, \mu, \varepsilon) X(\xi) \right) \chi(x) - \sum_{j=0}^{N-1} \frac{\chi((\gamma/\theta^*_\varepsilon(s)}{\theta^*_\varepsilon(s)} \sum_{l=0}^{1} \varepsilon^{l+1} \Psi^i_{l,\nu}(s, \mu, \varepsilon) Y(\gamma)(\varepsilon) \right) + o(\varepsilon(A + \mu)).$$

(2.65)

Thus, we have proved
Theorem 2.1. Suppose the hypothesis of Theorem 1.4 takes place. If $\lambda_0 = \lambda^k_0$ is a simple eigenvalue of the problem (1.6), then for each associated eigenfunction $\tilde{\psi}_k$, $\tilde{\psi}_k = 0$, with $\psi_k = \psi_0^k$ converges to $\psi_{0, 0}^k$ in $L_2(\Omega)$-norm and has the asymptotics (2.64) in $H^1(\Omega)$, where $\Psi_1$ is a solution of problem (2.8), (2.28) with $\Psi_0 = \psi_0^k$, $\Psi_1 = \psi_0^{k, \nu}$ are values of normal derivatives of functions $\Psi_1^k$ on $\partial\Omega$, $X$ and $Y^{(j)}$ are defined by equalities (2.13) and (2.23). If $\lambda_0 = \lambda_0^k = \ldots = \lambda_0^{k + p - 1}$ is a $p$-multiply eigenvalue of the problem (1.4), then for each associated eigenfunction $\psi_{0, 0}^k$, $k = q, \ldots, q + p - 1$, there exists a linear combination (2.64) of the perturbed eigenfunctions, converging to $\psi_{0, 0}^k$ in $L_2(\Omega)$ norm and having in $H^1(\Omega)$ the asymptotics (2.65).

From Theorem 1.4 and 2.1 it follows the validity of next statement.

Lemma 2.8. Suppose the assumptions [C0], [C1] and the equality (1.3) with $A > 0$ for the function $\eta$ from [C1] hold. Then the remainders in the asymptotics (1.10), (2.64) and (2.65) are of order $O(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}\delta^*(\varepsilon)(A + \mu)^2)$.

If $A = 0$, i.e., the limiting problem is the Neumann one, the statement of Theorem 2.1 can be strengthened as follows.

Theorem 2.2. Suppose the assumptions [C0], [C1] and the equality (1.3) with $A = 0$ for the function $\eta$ from [C1] hold. Then in the case of simple limiting eigenvalue – eigenfunction $\tilde{\psi}_k$ from (2.64) and in the case of multiply limiting eigenvalue – the linear combination of eigenfunctions $\tilde{\psi}_k$ from (2.64) converges to the limiting eigenfunction $\psi_{0, 0}^k$ in $H^1(\Omega)$.

Proof. Let us prove, that the equality

$$\| \tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k \|_{H^1(\Omega)} = o(1)$$

holds for all $k$ as $\varepsilon \to 0$. Since

$$\| \tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k \|_{H^1(\Omega)}^2 = \| \nabla_x (\tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k) \|_{L_2(\Omega)}^2 + \| \tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k \|_{L_2(\Omega)}^2,$$

and also, last term tends to zero as $\varepsilon \to 0$ by Lemma 2.8, it remains to estimate the gradient’s norm. Taking into account the form of $\psi_k^\varepsilon$, the gradient’s norm $(\tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k)$ is estimated as follows:

$$\| \nabla_x (\tilde{\psi}_k^\varepsilon - \Psi_{0, 0}^k) \|_{L_2(\Omega)}^2 \leq 2 \| \nabla_x (\psi_{0, 0}^k \psi_{0, 0}^\varepsilon) \|_{L_2(\Omega)}^2 + \sum_{j=0}^{N-1} \| \nabla_x \psi_{0, 0}^{\text{in}, j} \|_{L_2(\Omega_{0, 0}^{\eta, j})}^2 +$$

$$+ 2 \| \nabla_x \psi_{0, 0}^{bl} (\varepsilon/\mu_0) \|_{L_2(\Omega_{0, 0}^{\eta, j})}^2 + 2 \sum_{j=0}^{N-1} \| \nabla_x \psi_{0, 0}^{\text{mat}, j} (\varepsilon^3/4) \|_{L_2(\Omega_{0, 0}^{\eta, j})}^2,$$

(2.67)

where $\psi_{0, 0}^\varepsilon$, $\psi_{0, 0}^{bl}$, $\psi_{0, 0}^{\text{in}, j}$ and $\psi_{0, 0}^{\text{mat}, j}$ are the functions defined in formal constructing in the proof of Theorem 1.4 and associated with $\psi_k^0$. 

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In view of Lemma 2.3 and the definition $\psi_\varepsilon^{ex}$ we have:
\begin{equation}
\|\nabla_x (\psi_\varepsilon^{ex} - \Psi_0^k)\|_{L^2(\Omega)}^2 \leq C\varepsilon^2 \mu^4.
\end{equation}

It is easy to see
\begin{equation}
\|\nabla_x (\psi_\varepsilon^{bl} \chi(\tau/c_0))\|_{L^2(\Omega_{bl}^\varepsilon)}^2 \leq C \left( \|\nabla_x \psi_\varepsilon^{bl}\|_{L^2(\Omega_{bl}^\varepsilon \cap \Omega_{bl}^\varepsilon)}^2 + \|\psi_\varepsilon^{bl}\|_{L^2(\Omega_{bl}^\varepsilon)}^2 \right).
\end{equation}

Second term in the right side of the inequality obtained is estimated above by $C\varepsilon^{3/2} \mu$ (see Lemma 2.4). By direct calculations with employing explicit form of the functions $v_i$, the boundedness of function $\delta^*(\varepsilon)$, Lemma 2.3 and the equality $N = 2\varepsilon^{-1}$ one can check that
\begin{equation}
\sum_{j=0}^{N-1} \|\nabla_x \psi_\varepsilon^{in,j}\|_{L^2(\Omega_{in,j}^\varepsilon)}^2 \leq C \mu.
\end{equation}

Using the asymptotics (2.52), we prove that
\begin{equation}
\sum_{j=0}^{N-1} \|\nabla_x \psi_\varepsilon^{mat,j} \chi(|\varsigma_j|^3/4)\|_{L^2(\Omega_{j}^{\varepsilon,\mu})}^2 \leq C \eta^{1/5}.
\end{equation}

Collecting (2.67)–(2.71), we get (2.66). We stress that convergence (2.66) was proved without using the equality (1.9) and holds true for each bounded function $\delta^*(\varepsilon)$.

Let $\psi_0$ be associated with simple eigenvalue. Owing to convergence (2.66) and $\Psi_0 \xrightarrow{H^1(\Omega)} \psi_0$ (see Remark 1.3) we conclude that $\hat{\psi}_\varepsilon$ converges to $\psi_0$ strongly in $H^1(\Omega)$. Therefore, by Lemma 2.8 the eigenfunction $\hat{\psi}_\varepsilon$ from (2.66) satisfies an equality
\begin{equation}
\|\hat{\psi}_\varepsilon - \hat{\psi}_\varepsilon\|_{H^1(\Omega)} = o(1),
\end{equation}
from what it follows that eigenfunction $\psi_\varepsilon$ converges to $\psi_0$ in $H^1(\Omega)$ norm.

Let $\psi_0$ be associated with $p$-multiply eigenvalue $\lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1}$, that the eigenfunctions $\psi_0^k$, $k = q, \ldots, q+p-1$ are associated with. For the functions $\hat{\psi}_\varepsilon^k$ and $\hat{\psi}_0^k$ the relationships (2.66) hold. These relationships, Lemma 2.7 and (2.63) yield that the linear combination $\hat{\psi}_\varepsilon^k$ of perturbed eigenfunctions from (2.64) converges to $\psi_0^k$ in $H^1(\Omega)$. The proof is complete.
3. Asymptotics for the perturbed eigenelements under hypothesis of Theorem 1.5.

In this section we will obtain the asymptotics for the perturbed eigenelements in the case of breakdown of equality (1.9) of Theorem 1.4. First we will prove Theorem 1.5 about asymptotics for eigenvalues, and then we will establish Theorem 3.1 about asymptotics for associated eigenfunctions. Everywhere in the section, if it is not said specially, we keep the notations of the previous section.

Proof of Theorem 1.5. In proving we lean on the boundedness of the function $\delta^*(\varepsilon)$ established in Corollary 2 of Lemma 2.1. In Appendix we will show that eigenvalues of problem (1.7), (1.8) satisfy following asymptotics formulae

$$\Lambda_k^0(\mu, \varepsilon) = \lambda_k^0 + \mu \int_{\partial \Omega} (\psi_k^0)^2 \theta_0' \, ds + O(\mu^2 + (A + \mu)\sigma),$$  

(3.1)

where in the case of multiply eigenvalue $\lambda_k^0$ the associated eigenfunctions $\psi_k^0$ are additionally assumed to be orthogonal in $L^2(\partial \Omega)$ weighted by $\theta_0'\sigma = \sigma(\varepsilon) = o(1)$. From Lemmas 2.3, 2.8 and Corollary 2 of Lemma 2.1 it follows that $|\lambda - \Lambda_k^0| = O(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}(A + \mu)^2)$, what by asymptotics (3.1) implies the correctness of the theorem. The proof is complete.

Let us derive the asymptotics for the perturbed eigenfunctions under hypothesis of Theorem 1.5. We start from the case of simple eigenvalue $\lambda^0$. Assertion (2.60) and Lemma 2.3 imply that perturbed eigenfunction $\tilde{\psi}_\varepsilon = (\Psi_0, \psi_\varepsilon)_{L^2(\Omega)} \psi_\varepsilon$, satisfies an estimate

$$\| \tilde{\psi}_\varepsilon - \hat{\psi}_\varepsilon \|_{H^1(\Omega)} = O(\varepsilon^{3/2}(A + \mu) + \varepsilon^{1/2}(A + \mu)^2).$$

By direct calculations and employing Lemmas 2.3-2.5 and the results of matching procedure made in the previous section one can check see that $H^1(\Omega)$-norm of the function

$$\left( \varepsilon \Psi_1 - \varepsilon^2 \frac{\chi(\tau/c_0)}{\theta_\varepsilon'} \Psi_1^\nu X \right) \chi_\varepsilon - \varepsilon^2 \sum_{j=0}^{N-1} \frac{\chi(|s^j|\eta^{3/4})}{\theta_\varepsilon'} \Psi_0^\nu Y^{(j)}$$

is of order $O(\varepsilon^{1/2}(A + \mu))$. Hence, the function $\tilde{\psi}_\varepsilon$ from (3.2) converges to $\psi_0$ in $L^2(\Omega)$ and has the following asymptotics in $H^1(\Omega)$:

$$\tilde{\psi}_\varepsilon(x) = \left( \Psi_0(x, \mu, \varepsilon) + \varepsilon \frac{\chi(\tau/c_0)}{\theta_\varepsilon'} \Psi_0(\mu, \varepsilon) X(\xi) \right) \chi(x) -$$

$$+ \varepsilon \sum_{j=0}^{N-1} \frac{\chi(|s^j|\eta^{3/4})}{\theta_\varepsilon'} \Psi_0(\mu, \varepsilon) Y^{(j)}(\varepsilon^j, \varepsilon) + O(\varepsilon^{1/2}(A + \mu)).$$  

(3.3)
Now we proceed to the case of \( p \)-multiply eigenvalue \( \lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1} \). Due to (2.64) and Lemmas 2.3, 2.8 we see that a linear combination of perturbed eigenfunctions

\[
\hat{\psi}_\varepsilon^k(\Psi_0, \Psi_0^i)_{L_2(\Omega)}^{q+p-1} \sum_{i=q}^{q+p-1} (\Psi_0^i, \psi_\varepsilon^i)_{L_2(\Omega)} \psi_\varepsilon^i
\]

converges to \( L_2(\Omega) \) in \( \psi_0^k \), \( k = q, \ldots, q + p - 1 \) and its asymptotics in \( H^1(\Omega) \) reads as follows:

\[
\hat{\psi}_\varepsilon^k(x) = \sum_{i=q}^{q+p-1} (\psi_0^k, \Psi_0^i)_{L_2(\Omega)}^{q+p-1} \left( \Psi_0^i(x, \mu, \varepsilon) + \varepsilon \frac{\chi(\tau/c_0)}{\theta_\varepsilon(s)} \Psi_0^i(s, \mu, \varepsilon) X(\xi) \right) \chi(x) - \\
+ \varepsilon \sum_{j=0}^{N-1} \frac{\chi(\sqrt{\varepsilon})^{3/4}}{\theta_\varepsilon(s)} \Psi_0^i(s, \mu, \varepsilon) Y(\xi, \varepsilon) + O(\varepsilon^{1/2}(A + \mu))
\]

(3.5)

Similar to the case of simple limit eigenvalue, \( H^1(\Omega) \)-norm of neglected terms of \( \hat{\psi}_\varepsilon^k \) is of order \( O(\varepsilon^{1/2}(A + \mu)) \).

Lemmas 2.3 and Theorem 2.2 yield that Theorem 2.2 takes place for the functions (3.2), (3.4), too.

Thus, we have proved

**Theorem 3.1.** Suppose the hypothesis of Theorem 2.2 holds. If \( \lambda_0 = \lambda_0^k \) is a simple eigenvalue of problem (1.1), then the eigenfunction \( \psi_\varepsilon^k \) from (3.2) converges to \( \psi_0^k \) in \( L_2(\Omega) \) as \( A \geq 0 \) and in \( H^1(\Omega) \) as \( A = 0 \) and has in a sense of \( H^1(\Omega) \)-norm the asymptotics (3.3). If \( \lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1} \) is a \( p \)-multiply eigenvalue, then for each associated eigenfunction \( \psi_0^k \), \( k = q, \ldots, k + p - 1 \), there exists a linear combination (3.4), converging to \( \psi_0^k \) in \( L_2(\Omega) \) as \( A \geq 0 \) and in \( H^1(\Omega) \) as \( A = 0 \) having asymptotics (3.4) in \( H^1(\Omega) \)-norm. In asymptotics (3.3), (3.4) the notations of Theorem 2.1 are used.

4. Auxiliary statement

In this section we will prove an auxiliary statement that will be employed in next section in the proof of Theorem 1.1. Let us formulate this lemma.

**Lemma 4.1.** Suppose the assumptions (C0) and (C1) hold, the function \( \eta(\varepsilon) \) from (C1) is bounded above by a number \( \pi/2 \) and satisfies the equality (L.3), and, for each \( i, j \) and \( \varepsilon \) the equalities \( a^i+\varepsilon(b^i+\varepsilon) = 2\eta(\varepsilon), a^i(\varepsilon) = a^i(\varepsilon), b^i(\varepsilon) = b^i(\varepsilon) \) take place. Suppose also that there exists a fixed number \( d > 0 \) for that H"older norm \( \|\theta_\varepsilon\|_{C^{3+\delta}(\partial\Omega)} \) is bounded on \( \varepsilon \). Then the perturbed eigenvalue \( \lambda_\varepsilon^k \) converges to the eigenvalue \( \lambda_0^k \) of limiting problem (1.4) and has the asymptotics

\[
\lambda_\varepsilon^k = \lambda_0^k + \varepsilon \ln \sin(\varepsilon) \int_{\partial\Omega} \left( \frac{\partial\psi_0^k}{\partial\nu} \right)^2 ds + O\left( \varepsilon^{3/2} \left( \ln \eta(\varepsilon) \right)^{3/2} + 1 \right) \left( \frac{\pi}{2} - \eta(\varepsilon) \right).
\]
Proof. The convergence of eigenvalues is established by analogy with papers [2], [6], [8]. We will prove the asymptotics by the scheme employed in the second section. As before, first we will formally construct asymptotics and after we will justify them. It should be noted that formal construction of the asymptotics that will be used in general coincide with the scheme proposed in [17], [18]. The difference is a more general formulation of the problem considered here, the renunciation of additional assumptions made in [17], [18], and the estimate for the error with respect to both parameters \( \varepsilon \) and \( \eta \). We will consider in detail only the case of simple limiting eigenvalue; the case of multiply limiting eigenvalue is established by analogy.

Let \( \lambda_0 \) be a simple eigenvalue of limiting problem (1.4), \( \psi_0 \) be the associated eigenfunction (normalized in \( L_2(\Omega) \)), \( \lambda_\varepsilon \) be the perturbed eigenvalue converging to \( \lambda_0 \).

We seek for the asymptotics of \( \lambda_\varepsilon \) as follows:

\[
\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1(\varepsilon) \ln \sin \eta,
\]

and the asymptotics for associated eigenfunction is constructed as a sum of an outer expansion and a boundary layer:

\[
\begin{align*}
\psi_\varepsilon(x) &= \psi^{ex}_\varepsilon(x,\eta) + \chi(\tau/c_0)\psi^{bl}_\varepsilon(\xi, s, \eta), \\
\psi^{ex}_\varepsilon(x, \eta) &= \psi_0(x) + \varepsilon \psi_1(x, \varepsilon) \ln \sin \eta, \\
\psi^{bl}_\varepsilon(\xi, s, \eta) &= \varepsilon v_1(\xi, s, \varepsilon, \eta) + \varepsilon^2 v_2(\xi, s, \varepsilon, \eta),
\end{align*}
\]

where \( \xi = (\xi_1, \xi_2), \xi_1 = (\theta_\varepsilon(s) - \theta_\varepsilon(s_0))/\varepsilon - (b^i(\varepsilon) - a^j(\varepsilon))/2, \xi_2 = \tau \theta_\varepsilon'(s)/\varepsilon. \)

Observe, here it is possible to carry out the construction of asymptotics without employing method of matched asymptotics expansions.

We substitute (4.1) and (4.2) into equation (1.1) and write out the coefficient of \( \varepsilon \ln \sin \eta \):

\[
(\Delta + \lambda_0)\psi_1 = -\lambda_1 \psi_0, \quad x \in \Omega.
\]

Substitution (4.1) and (4.2) into equation (1.1) lead us to the equation (2.9) and (2.10) for the functions \( v_1 \) and \( v_2 \). Boundary conditions for these functions are derived from the claim the sum of (4.2) and (4.3) to satisfy both boundary conditions in (1.2):

\[
\begin{align*}
v_1 &= -\psi_1^D \ln \sin \eta, \quad \xi \in \gamma^\eta, \\
\frac{\partial v_1}{\partial \xi_2} &= \frac{1}{\theta_\varepsilon} \psi_0^\nu, \quad \xi \in \Gamma^\eta,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial v_2}{\partial \xi_2} &= \frac{1}{\theta_\varepsilon} \psi_1^\nu \ln \sin \eta, \quad \xi \in \Gamma^\eta
\end{align*}
\]

where \( \gamma^\eta \) is a union of intervals \( (\pi j - \eta, \pi j + \eta), j \in \mathbb{Z}, \) lying in the axis \( O\xi_1, \) and \( \Gamma^\eta \) is a complement of \( \overline{\gamma^\eta} \) on the axis \( O\xi_1, \psi_1^D \) and \( \psi_1^\nu \) are values of the functions \( \psi_1 \) and their normal derivatives on the boundary \( \partial \Omega. \) Problem (2.9), (4.3) is solved explicitly:

\[
v_1(\xi, s, \varepsilon, \eta) = -\frac{1}{\theta_\varepsilon(s)}\psi_0^\nu(s)X_\eta(\xi),
\]

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\[ X_\eta(\xi) = \text{Re} \ln \left( \sin z + \sqrt{\sin^2 z - \sin^2 \eta} \right) - \xi_2. \]

It is easy to check that \( X_\eta \in \mathcal{V}_\eta \cap H^1(\Pi^{(j)}) \) is even on \( \xi_1 \) harmonic function, where \( \mathcal{V}_\eta \) denotes the space of \( \pi \)-periodic on the variable \( \xi_1 \) functions decaying exponentially as \( \xi_2 \to +\infty \) uniformly on \( \xi_1 \) with all their derivatives and belonging to \( C^\infty(\{ \xi : \xi_2 > 0 \} \cup \gamma^\eta \cup \Gamma^\eta) \). The function \( X_\eta \) obeys boundary condition

\[ X_\eta(\xi) = \ln \sin \eta, \quad \xi \in \gamma^\eta, \quad \frac{\partial X}{\partial \xi_2} = -1, \quad \xi \in \Gamma^\eta. \quad (4.8) \]

The function \( v_1 \) defined by the equality (4.7) due to (4.8) meets the boundary condition

\[ v_1 = -\frac{1}{\mathbf{\theta}_\epsilon} \psi_0' \ln \sin \eta, \quad \xi \in \gamma^\eta, \]

comparing that with (4.5), we obtain the boundary condition for \( \psi_1 \):

\[ \psi_1 = \frac{1}{\mathbf{\theta}_\epsilon} \frac{\partial \psi_0}{\partial \nu}, \quad x \in \partial \Omega. \quad (4.9) \]

The solvability condition of boundary value problem (4.4), (4.9) gives the formula for \( \lambda_1 \):

\[ \lambda_1 = \int_{\partial \Omega} \left( \frac{\partial \psi_0}{\partial \nu} \right)^2 \frac{ds}{\mathbf{\theta}_\epsilon(s)}. \quad (4.10) \]

The function \( \psi_1 \) is chosen to be orthogonal to \( \psi_0 \) in \( L_2(\Omega) \). The function \( v_2 \) is defined as follows:

\[ v_2 = \frac{\psi_0''}{2(\mathbf{\theta}_\epsilon)^2} \xi_2^2 \left( \frac{\theta''_\epsilon}{\mathbf{\theta}_\epsilon} \frac{\partial X_\eta}{\partial \xi_1} + k \frac{\partial X_\eta}{\partial \xi_2} \right) - 2 \frac{\psi_0'}{\mathbf{\theta}_\epsilon} \left( \frac{\psi_0'}{\mathbf{\theta}_\epsilon} \right)' v_2^{\text{odd}} - \frac{1}{\mathbf{\theta}_\epsilon} \psi_0' \ln \sin \eta X_\eta, \quad (4.11) \]

where \( v_2^{\text{odd}} \) is an exponentially decaying solution for the boundary value problem

\[ \Delta_\epsilon v_2^{\text{odd}} = \frac{\partial X_\eta}{\partial \xi_1}, \quad \xi_2 > 0, \quad v_2^{\text{odd}} = 0, \quad \xi \in \gamma^\eta, \quad \frac{\partial v_2^{\text{odd}}}{\partial \xi_2} = 0, \quad \xi \in \Gamma^\eta. \quad (4.12) \]

The solution for problem (4.12) exists; this existence and also its evenness on \( \xi_1 \) and belonging to \( \mathcal{V}_\eta \cap H^1(\Pi^{(j)}) \) were proved in [17].

For justification of the asymptotics constructed formally we will use following lemmas.

**Lemma 4.2.** The properties takes place:

(1) for integer \( m \geq 0 \) the inequalities

\[ \| \xi_2^m X_\eta \|_{L_2(\Pi^{(j)})} \leq C \left( \frac{\pi}{2} - \eta \right)^2 \left( \left| \ln \left( \frac{\pi}{2} - \eta \right) \right|^{1/2} + 1 \right) \]

are true, where constants \( C \) are independent on \( \eta \).
(2). for integer \( m, p \geq 0 \) the estimates

\[
\left\| \xi_2^m \nabla_\xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L_2(\Pi^{(j)})} \leq C |\ln \sin \eta|^{1/2},
\]

\[
\left\| \xi_2^{m+p+1} \nabla_\xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L_2(\Pi^{(j)})} \leq C \left( \frac{\pi}{2} - \eta \right)^2 \left( \left| \ln \left( \frac{\pi}{2} - \eta \right) \right|^{1/2} + 1 \right),
\]

take place, where constants \( C \) are independent on \( \eta \).

**Proof.** First we prove the statement of item [1] for \( m = 0 \). It was shown in [20, §3] that \( \|X_\eta\|_{L_2(\Pi^{(j)})} \) is continuous on \( \eta \in [0, \pi/2] \) function. To prove the estimate needed it is sufficient to clear up the behaviour of this function as \( \eta \to \pi/2 \). It is easy to see that the function

\[
X^1_\eta(\xi) = -\frac{1}{2} \xi_2 \int_{\xi_2}^{+\infty} X_\eta(\xi_1, t) \, dt
\]

is even on \( \xi_1 \), belong to \( \mathcal{V}_\eta \) and is a solution for the equation \( \Delta_\xi X^1_\eta(\xi) = X_\eta \) in a domain \( \xi_2 > 0 \) satisfying boundary conditions:

\[
X^1_\eta = 0, \quad \frac{\partial X^1_\eta}{\partial \xi_2} = -\frac{1}{2} \int_{0}^{+\infty} X_\eta(\xi_1, t) \, dt, \quad \xi_2 = 0.
\]

Using these properties of the functions \( X^1_\eta \) and \( X_\eta \) and the equality

\[
\int_{\Pi^{(j)}} X_\eta^2 \, d\xi = \int_{\Pi^{(j)}} (X_\eta + \xi_2 - \ln \sin \eta) X_\eta \, d\xi
\]

proved in [20, §3] and integrating by parts we have:

\[
\int_{\Pi^{(j)}} X_\eta^2 \, d\xi = \int_{\Pi^{(j)}} (X_\eta + \xi_2 - \ln \sin \eta) \Delta_\xi X^1_\eta \, d\xi = \\
\quad \pi/2 \int_{\eta}^{\pi/2} (X_\eta(\xi_1, 0) - \ln \sin \eta) \int_{0}^{+\infty} X_\eta(\xi_1, t) \, dt \, d\xi_1. \tag{4.13}
\]

Since as \( \xi_1 \in (\eta, \pi/2] \)

\[
\frac{d^2}{d\xi_1^2} \int_{0}^{+\infty} X_\eta(\xi_1, t) \, dt = - \int_{0}^{+\infty} \frac{\partial^2}{\partial t^2} X_\eta(\xi_1, t) \, dt = -1,
\]

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due to evenness and \( \pi \)-periodicity of \( X_\eta \) on \( \xi_1 \) we get:

\[
\int_0^{+\infty} X_\eta(\xi_1, t) \, dt = -\frac{1}{2} \left( \xi_1 - \frac{\pi}{2} \right)^2 + \int_0^{+\infty} X_\eta \left( \frac{\pi}{2}, t \right) \, dt. \tag{4.14}
\]

Applying the estimate \( |\ln(1 + a)| \leq a, \, a \geq 0 \), to the integrand function

\[
X_\eta \left( \frac{\pi}{2}, t \right) = \ln \left( 1 + \frac{e^{-2t} - 1 + \sqrt{(1 - e^{-2t})^2 + 4e^{-2t} \cos^2 \eta}}{2} \right)
\]

in the right side of the equality (4.14) and integrating the integral obtained we deduce an assertion \((\eta \to \pi/2)\):

\[
\int_0^{+\infty} X_\eta \left( \frac{\pi}{2}, t \right) \, dt = O \left( \frac{\eta_1^2 \ln \eta_1}{\eta} \right), \tag{4.15}
\]

where \( \eta_1 = \pi/2 - \eta \). In [17] it was proved that:

\[
\int_{\gamma \eta \Pi} \frac{\partial X_\eta}{\partial \xi_2} \, d\xi_1 = \pi - 2\eta, \quad \int_{\Gamma \eta \Pi} X_\eta \, d\xi_1 = -2\eta \ln \sin \eta. \tag{4.16}
\]

Substituting (4.14)-(4.16) into (4.13), we arrive at equalities \((\eta \to \pi/2)\):

\[
\int_{\Pi^{(j)}} X^2 \, d\eta = -\frac{1}{2} \int_{\eta}^{\pi/2} \ln \left( \frac{\sin \xi_1}{\sin \eta} + \sqrt{\frac{\sin^2 \xi_1}{\sin^2 \eta} - 1} \right) \left( \xi_1 - \frac{\pi}{2} \right)^2 \, d\xi_1 +
\]

\[
+ \frac{1}{2} \int_{0}^{+\infty} X_\eta \left( \frac{\pi}{2}, \xi_2 \right) \, d\xi_2 \int_{\eta}^{\pi/2} \left( X_\eta(\xi_1, 0) - \ln \sin \eta \right) \, d\xi_1 =
\]

\[
= -\frac{1}{2} \int_{0}^{\eta_1} t^2 \left( \ln \left( \cos t + \sqrt{\cos^2 t - \sin^2 \eta} \right) - \ln \sin \eta \right) \, dt + O(\eta_1^4 \ln \eta_1) =
\]

\[
= O(\eta_1^4 \ln \eta_1).
\]

In calculations the change \( t = \pi/2 - \xi_1 \) has been done. The estimate for \( \|X_\eta\|_{L_2(\Pi^{(j)})}^2 \) obtained and the continuity of this function on \( \eta \in [0, \pi/2] \) imply the statement of item \((1)\) for \( m = 0 \).

It follows from explicit form of \( X \), its infinitely differentiability \((\xi, \eta)\) for \( \xi_2 \geq 1 \), continuity on \((\xi, \eta) \in \{\xi : \xi_2 > 0\} \times (0, \pi/2] \) and exponential decaying as \( \xi_2 \to +\infty \) that for \( m \geq 1 \) the quantity \( \|\xi_2^m X_\eta\|_{L_2(\Pi^{(j)})} \) is continuous on \( \eta \in (0, \pi/2] \) function, and the estimate:

\[
\|\xi_2^m X_\eta\|_{L_2(\Pi^{(j)}) \cap \{\xi_2 > 1\}} \leq C\eta_1^2
\]
holds, where constant $C$ is independent on $\eta$. Then by an inequality
\[
\|\xi_2^m X_\eta\|_{L^2(\Pi^{(j)} \cap \{\xi_2 < 1\})} < \|X_\eta\|_{L^2(\Pi^{(j)})} \leq C\eta_1^2 \left( |\ln \eta_1|^{1/2} + 1 \right)
\]
and the statement of item [1] for $m = 0$ we derive that this item takes place for $m > 0$, too.

Let us integrate by parts in the equality $\int X_\eta \Delta \xi X_\eta\,d\xi = 0$; as a result we have:
\[
\int_{\Pi^{(j)}} |\nabla \xi X_\eta|^2 \,d\xi = - \ln \sin \eta \int_{\Pi^{(j)}} \frac{\partial X_\eta}{\partial \xi_2} \,d\xi_1 + \int_{\Pi^{(j)}} X_\eta \,d\xi_1,
\]
from what and (4.16) it arises:
\[
\|\nabla \xi X_\eta\|_{L^2(\Pi^{(j)})} = \pi^{1/2} |\ln \sin \eta|^{1/2}, \tag{4.17}
\]
The chain of equalities $(m \geq 0, p \geq 0, m + p \geq 1, m, p \in \mathbb{Z})$:
\[
0 = \int_{\Pi^{(j)}} \xi_2^{2(m+p)} \frac{\partial^m X_\eta}{\partial \xi_2^m} \Delta \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \,d\xi = - \int_{\Pi^{(j)}} \xi_2^{2(m+p)} \left| \nabla \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right|^2 \,d\xi - 2(m+p) \int_{\Pi^{(j)}} \xi_2^{2(m+p)-1} \frac{\partial^m X_\eta}{\partial \xi_2^m} \frac{\partial^m X_\eta}{\partial \xi_2^m} \,d\xi = - \left\| \xi_2^{m+p} \nabla \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})}^2 + (m+p)(2(m+p)-1) \left\| \xi_2^{m+p-1} \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})}^2,
\]
gives the formulae:
\[
\left\| \xi_2^{m+p} \nabla \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})} = \sqrt{(m+p)(2(m+p)-1)} \left\| \xi_2^{m+p-1} \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})}, \tag{4.18}
\]
Employing these formulae for $p = 0$, $m \geq 1$ and with $p \geq 1$, $m \geq 0$, we get estimates
\[
\left\| \xi_2^m \nabla \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})} \leq C\|\nabla \xi X_\eta\|_{L^2(\Pi^{(j)})},
\]
\[
\left\| \xi_2^{m+p+1} \nabla \xi \frac{\partial^m X_\eta}{\partial \xi_2^m} \right\|_{L^2(\Pi^{(j)})} \leq C\|X_\eta\|_{L^2(\Pi^{(j)})},
\]
from those, the item [1] and the equality (4.17) it follows the statement of item [2]. The proof is complete.

**Lemma 4.3.** The function $v_2^{\text{odd}}$ satisfies estimates:
\[
\left\| \xi_2^p v_2^{\text{odd}} \right\|_{L^2(\Pi^{(j)})} \leq C|\ln \sin \eta|^{1/2},
\]
\[
\left\| \xi_2^p \nabla \xi v_2^{\text{odd}} \right\|_{L^2(\Pi^{(j)})} \leq C|\ln \sin \eta|^{1/2},
\]
\[
\left\| \xi_2^p \nabla \xi \frac{\partial}{\partial \xi_2} v_2^{\text{odd}} \right\|_{L^2(\Pi^{(j)})} \leq C|\ln \sin \eta|^{1/2},
\]
where $p \geq 0$, $p \in \mathbb{Z}$, and constants $C$ are independent on $\eta$. 41
Proof. Let \( v \in \mathcal{V}_\eta \cap H^1(\Pi^{(j)}) \) be an odd on \( \xi_1 \) function that is a solution of a boundary value problem

\[
\Delta \xi v = f, \quad \xi_2 > 0, \quad v = 0, \quad \xi \in \gamma^\eta, \quad \frac{\partial v}{\partial \xi_2} = 0, \quad \xi \in \Gamma^\eta, \tag{4.19}
\]

where \( f \in \mathcal{V}_\eta \cap L^2(\Pi^{(j)}) \) is odd on \( \xi_1 \). Since \( v \in \mathcal{V}_\eta \) is odd on \( \xi_1 \), it follows that \( v = 0 \) as \( \xi_1 = \pi k/2, k \in \mathbb{Z} \). Therefore,

\[
v(\xi) = \int_{-\pi/2+\pi j}^{\xi_1} \frac{\partial v}{\partial t}(t, \xi_2) \, dt,
\]

from what owing to Cauchy-Schwarz-Bunyakovskii inequality we derive an estimate:

\[
|v(\xi)|^2 \leq \pi \int_{-\pi/2+\pi j}^{\xi_1} \left| \frac{\partial v}{\partial \xi_1}(\xi) \right|^2 \, d\xi_1,
\]

employing that, we finally get:

\[
\|v\|_{L^2(\Pi^{(j)})} \leq \pi \|\nabla_\xi v\|_{L^2(\Pi^{(j)})}. \tag{4.20}
\]

We multiply equation in (4.19) by \( v \) and integrate by parts once:

\[
\|\nabla_\xi v\|_{L^2(\Pi^{(j)})} = - \int_{\Pi^{(j)}} vf \, d\xi,
\]

what by Cauchy-Schwarz-Bunyakovskii inequality and estimate (4.20) gives:

\[
\|v\|_{L^2(\Pi^{(j)})} \leq \pi \|f\|_{L^2(\Pi^{(j)})}, \quad \|\nabla_\xi v\|_{L^2(\Pi^{(j)})} \leq \pi \|f\|_{L^2(\Pi^{(j)})}. \tag{4.21}
\]

Applying estimates (4.21) to the solution of problem (4.12) and bearing in mind Lemma 4.2, we obtain uniform on \( \eta \) estimates:

\[
\|\nabla_\xi v_{\eta}^{odd}\|_{L^2(\Pi^{(j)})} \leq C \|\nabla \xi X_\eta\|_{L^2(\Pi^{(j)})} \leq C |\ln \sin \eta|^{1/2},
\]

\[
\|\nabla_\xi v_2^{odd}\|_{L^2(\Pi^{(j)})} \leq C |\ln \sin \eta|^{1/2}. \tag{4.22}
\]

Next, the functions \( \xi_2^p v_2^{odd} \) are solutions to problem (1.19), where \( \gamma^\eta \) coincides with axis \( O\xi_1 \); right sides are

\[
f = p(p-1)\xi_2^{p-2} v_2^{odd} + 2p \xi_2^{p-1} \frac{\partial v_2^{odd}}{\partial \xi_2} + \xi_2^{p} \frac{\partial X_\eta}{\partial \xi_1},
\]

thus, applying estimates (4.21) to \( \xi_2^p v_2^{odd} \) accounting (4.22) and Lemma 4.2, we have:

\[
\|\xi_2 v_2^{odd}\|_{L^2(\Pi^{(j)})} \leq C \left( \|\nabla v_2^{odd}\|_{L^2(\Pi^{(j)})} + \|\nabla X_\eta\|_{L^2(\Pi^{(j)})} \right) \leq C |\ln \sin \eta|^{1/2},
\]

\[
\|\xi_2^p v_2^{odd}\|_{L^2(\Pi^{(j)})} \leq C \left( \|\xi_2^{p-2} v_2^{odd}\|_{L^2(\Pi^{(j)})} + \|\xi_2^{p-1} \nabla v_2^{odd}\|_{L^2(\Pi^{(j)})} + \right)
\]

\[
+ \|\xi_2^{p} \nabla \xi X_\eta\|_{L^2(\Pi^{(j)})}, \quad p \geq 2. \tag{4.23}
\]

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Integrating by parts in equalities \((m \geq 1, m \in \mathbb{Z})\)

\[
\int_{\Pi^{(j)}} \xi_2 v_2^{\text{odd}} \frac{\partial X_\eta}{\partial \xi_1} \, d\xi = \int_{\Pi^{(j)}} \xi_2^{2m+1} v_2^{\text{odd}} \Delta \xi v_2^{\text{odd}} \, d\xi,
\]

\[
\int_{\Pi^{(j)}} \xi_2^{2(m+1)} \frac{\partial v_2^{\text{odd}}}{\partial \xi_2} \frac{\partial^2 X_\eta}{\partial \xi_1 \partial \xi_2} \, d\xi = \int_{\Pi^{(j)}} \xi_2^{2(m+1)} \frac{\partial v_2^{\text{odd}}}{\partial \xi_2} \Delta \xi v_2^{\text{odd}} \, d\xi,
\]

by analogy with how (4.18) was deduced, we derive inequalities:

\[
\|\xi_2^m \nabla_\xi v_2^{\text{odd}}\|_{L_2(\Pi^{(j)})} \leq C \left( \|\xi_2^{m-1} v_2^{\text{odd}}\|_{L_2(\Pi^{(j)})} + \left\|\xi_2^m \frac{\partial X_\eta}{\partial \xi_1}\right\|_{L_2(\Pi^{(j)})} \right),
\]

\[
\left\|\xi_2^{m+1} \nabla_\xi v_2^{\text{odd}}\right\|_{L_2(\Pi^{(j)})} \leq C \left( \|\xi_2^m \nabla_\xi v_2^{\text{odd}}\|_{L_2(\Pi^{(j)})} + \left\|\xi_2^{m+2} \frac{\partial^2 X_\eta}{\partial \xi_1 \partial \xi_2}\right\|_{L_2(\Pi^{(j)})} \right).
\]

The inequalities obtained, Lemma 4.2 and estimates (4.22), (4.23) by induction prove the lemma. The proof is complete.

Lemma 4.4. For each \(R > 0\) and integer \(m \geq 3\) the uniform on \(R\) and \(\eta\) estimates \((k = 0, 1, 2)\)

\[
\|X_\eta\|_{L_2(\Pi^{(j)} \cup \{\xi : \xi_2 > R\})} \leq CR^{-m} \left( \frac{\pi}{2} - \eta \right)^2 \left( \left| \ln \left( \frac{\pi}{2} - \eta \right) \right| + 1 \right),
\]

\[
\|\xi_2^k \nabla_\xi X_\eta\|_{L_2(\Pi^{(j)} \cup \{\xi : \xi_2 > R\})} \leq CR^{-m} \left( \frac{\pi}{2} - \eta \right)^2 \left( \left| \ln \left( \frac{\pi}{2} - \eta \right) \right| + 1 \right),
\]

\[
\left\|\xi_2^{k+1} \nabla_\xi \frac{\partial X_\eta}{\partial \xi_2}\right\|_{L_2(\Pi^{(j)} \cup \{\xi : \xi_2 > R\})} \leq CR^{-m} \left( \frac{\pi}{2} - \eta \right)^2 \left( \left| \ln \left( \frac{\pi}{2} - \eta \right) \right| + 1 \right),
\]

\[
\|v_2^{\text{odd}}\|_{L_2(\Pi^{(j)} \cup \{\xi : \xi_2 > R\})} \leq CR^{-m} |\ln \sin \eta|^{1/2},
\]

\[
\|\nabla_\xi v_2^{\text{odd}}\|_{L_2(\Pi^{(j)} \cup \{\xi : \xi_2 > R\})} \leq CR^{-m} |\ln \sin \eta|^{1/2}.
\]

take place.

Proof. By Cauchy-Schwarz-Bunyakovskii inequality each function \(v \in \mathcal{V}_\eta\) for \(\xi_2 \geq R\) obeys

\[
|v(\xi)| = \left| \int_{\xi_2}^{+\infty} \frac{\partial v}{\partial t}(\xi_1, t) \, dt \right| \leq \frac{1}{\sqrt{2m-3}} \xi_2^{-m+3/2} \left\|\xi_2^{m-1} \nabla_\xi v_2\right\|_{L_2(\mathbb{R}_+)}.
\]

Integrating this inequality over \(\Pi^{(j)} \cap \{\xi : \xi_2 > R\}\), we get:

\[
\|v\|_{L_2(\Pi^{(j)} \cap \{\xi : \xi_2 > R\})} \leq \frac{R^{-m}}{\sqrt{(2m-3)(2m-4)}} \left\|\xi_2^{m-1} \frac{\partial v}{\partial \xi_2}\right\|_{L_2(\Pi^{(j)} \cap \{\xi : \xi_2 > R\})}.
\]

Taking

\[
v = X_\eta, \quad v = \xi_2^k \frac{\partial X_\eta}{\partial \xi_i}, \quad v = \xi_2^{k+1} \frac{\partial^2 X_\eta}{\partial \xi_i \partial \xi_2}, \quad v = v_2^{\text{odd}}, \quad v = \frac{\partial v^{\text{odd}}}{\partial \xi_i}, \quad i = 1, 2,
\]

in this inequality we arrive at the statement of the lemma. The proof is complete.
Lemma 4.5. The functions $\lambda_1(\varepsilon)$ and $\psi_1(x, \varepsilon) \in C^\infty(\overline{\Omega})$ are uniformly bounded on $\varepsilon$:

$$|\lambda_1| \leq C, \quad \|\psi_1\|_{C^\infty(\overline{\Omega})} \leq C.$$  

Proof. The boundedness of $\lambda_1(\varepsilon)$ follows from assumption $[C0]$ and formula (4.10). The smoothness of the function $\psi_1$ is obvious. By well-known estimates for solutions of elliptic boundary value problems we have:

$$\|\psi_1\|_{H^2(\Omega)} \leq C \left(|\lambda_1| \|\psi_0\|_{L^2(\Omega)} + \|\psi'_0/\theta'_\varepsilon\|_{C^2(\partial\Omega)}\right) \leq C,$$

where $C$ is independent on $\varepsilon$, what by Theorem on embedding $H^2(\Omega)$ into $C(\Omega)$ implies:

$$\|\psi_1\|_{C(\overline{\Omega})} \leq C \left(|\lambda_1| \|\psi_0\|_{C^2(\overline{\Omega})} + \|\psi'_0/\theta'_\varepsilon\|_{C^3(\partial\Omega)}\right) \leq C,$$

where $C$ is independent on $\varepsilon$. The proof is complete.

We denote:

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \ln \sin \eta \lambda_1,$n
$$\hat{\psi}_\varepsilon(x) = \psi_0(x) + \varepsilon \ln \sin \eta \psi_1(x, \varepsilon) + \chi(\tau/c_0) \psi^{bl}_\varepsilon(\xi, s, \eta) + R_\varepsilon(x),$$n
$$R_\varepsilon(x) = \varepsilon^2 \ln^2 \sin \eta \chi(\tau/c_0) \psi^{bl}_\varepsilon/\theta'_\varepsilon,$$

where $\lambda_1$ is from (4.10), $\psi^{bl}_\varepsilon$ is from (4.3) with $v_1$ and $v_2$ from (4.7) and (4.11).

Next statement is an analogue of Lemma 2.6.

Lemma 4.6. The function $\hat{\psi}_\varepsilon \in C^\infty(\Omega \cup \gamma_\varepsilon \cup \Gamma_\varepsilon) \cap H^1(\Omega)$ converges to $\psi_0$ in $H^1(\Omega)$ and satisfies to the boundary value problem (2.50) with $u_\varepsilon = \hat{\psi}_\varepsilon$, $\lambda = \lambda_\varepsilon$, $f = f_\varepsilon$, where for $f_\varepsilon$ the uniform estimate

$$\|f_\varepsilon\|_{L^2(\Omega)} \leq C \varepsilon^{3/2} \left( |\ln \eta|^{3/2} + 1 \right) \left( \frac{\pi}{2} - \eta \right),$$

takes place, constant $C$ is independent on $\varepsilon$ and $\eta$. For the function $R_\varepsilon \in C^\infty(\overline{\Omega})$ a uniform on $\varepsilon$ and $\eta$ estimate

$$\|R_\varepsilon\|_{C^2(\overline{\Omega})} \leq C \varepsilon^2 \ln^2 \sin \eta$$

is valid.

Proof. The smoothness $\hat{\psi}_\varepsilon$ and $R_\varepsilon$ are direct implication of definitions of these functions. Maintained boundary condition for $\hat{\psi}_\varepsilon$ follows from (4.3), (4.6), (4.8), (4.9), (4.12). The proof of the estimate for $R_\varepsilon$ is based on Lemma 4.5 and the assumption $[C0]$:

$$\|R_\varepsilon\|_{C^2(\overline{\Omega})} \leq C \varepsilon^2 \ln^2 \sin \eta \|\psi_1\|_{C^0(\overline{\Omega})} \leq C \varepsilon^2 \ln^2 \sin \eta.$$
Let us prove the estimate for \( f_\varepsilon \). This function can be represented as
\[
f_\varepsilon = -(\Delta_x + \tilde{\lambda}_\varepsilon) \hat{\psi}_\varepsilon = -\sum_{i=1}^{3} f_\varepsilon^{(i)},
\]
where \( f_\varepsilon^{(1)} = \varepsilon^2 \ln^2 \sin \eta \left( \lambda_1 \psi_1 + \left( \Delta + \tilde{\lambda}_\varepsilon \right) \chi(\tau/c_0) \psi_0^{\varepsilon}/\theta_\varepsilon^{\prime} \right) \), \( f_\varepsilon^{(2)} = \chi(\tau/c_0)(\Delta_x + \tilde{\lambda}_\varepsilon) \psi_\varepsilon^d \), and \( f_\varepsilon^{(3)} = 2 \left( \nabla_x \psi_\varepsilon^d, \nabla_x \chi(\tau/c_0) \right) + \psi_\varepsilon^d \Delta_x \chi(\tau/c_0) \).

The function \( f_\varepsilon^{(1)} \) is easily estimated owing to Lemma 4.3:
\[
\| f_\varepsilon^{(1)} \|_{L^2(\Omega)} \leq C \varepsilon^2 \ln^2 \sin \eta,
\]
where \( C \) is independent on \( \varepsilon \) and \( \eta \). Since \( \nabla_x \chi(\tau/c_0) \) and \( \Delta_x \chi(\tau/c_0) \) are nonzero only for \( c_0/4 < \tau < 3c_0/4 \), taking into account the definition of the variables \( \xi \) and using Lemma 4.4 with \( m = 3 \) and \( R = c_0 c_1/(4\varepsilon) \) (here \( c_1 \) is from \([C0]\)), we arrive at an estimate:
\[
\| f_\varepsilon^{(2)} \|_{L^2(\Omega)} \leq C \varepsilon^{7/2} |\ln \eta|^{1/2},
\]
where \( C \) is independent on \( \eta \) and \( \varepsilon \). Employing the harmonicity \( X \) and the equation for \( \upsilon_2^{odd} \), we obtain a representation for the function \( f_\varepsilon^{(3)} \):
\[
f_\varepsilon^{(3)} = \varepsilon \sum_{k=0}^{2} \sum_{i=1}^{2} \left( \ln \sin \eta \right)^{\left[\frac{3+i}{2}\right]} p_{4k+2i-1} + \xi_2 p_{4k+2i} \right) \xi_2^{4\left[\frac{k+1}{2}\right]} \frac{\partial^{k+1} X}{\partial \xi_i \partial \xi_2^{k}} + 
\varepsilon (\varepsilon^2 \ln^2 \sin \eta \psi_{13} + \varepsilon \ln \sin \eta \psi_{14} + \psi_{15}) \chi(\tau/c_0) + \varepsilon \sum_{k=0}^{1} \sum_{i=1}^{2} \xi_2^k p_{2k+i+15} \frac{\partial^{k+1} \psi_{2}^{odd}}{\partial \xi_i \partial \xi_2^{k}} + 
+ \varepsilon^2 (\varepsilon \ln \sin \eta \psi_{20} + p_{21}) \upsilon_2^{odd},
\]
where \( p_i = \psi_i(\xi_2; s, \varepsilon) \) are polynomials on \( \xi_2 \) whose coefficients depending on \( s \) and \( \varepsilon \) are estimated uniformly on \( s \) and \( \varepsilon \) by Lemma 4.3. \( \[\] \) indicates the integral part of number. Bearing in mind these estimates and using Lemmas 4.2 and 4.3, we conclude that
\[
\| f_\varepsilon^{(2)} \|_{L^2(\Omega)} \leq C \varepsilon^{3/2} \left( \ln \eta \right)^{3/2} + 1 \left( \frac{\pi}{2} - \eta \right),
\]
where \( C \) is independent on \( \varepsilon \) and \( \eta \). Here we have also used obvious relationships:\n\( \ln \sin \eta = O(\eta), \eta \to 0; \ln \sin \eta = O((\pi/2 - \eta)^2), \eta \to \pi/2 \). The proof is complete.

The justification of the asymptotics constructed is carried out by analogy with one from the second section.

The formal construction of asymptotics in the case of multiply limiting eigenvalue does not differ in general from the case of simple limiting eigenvalue. The only difference is that we simultaneously construct asymptotics of all eigenvalues converging to multiply limiting eigenvalue; in whole the formal construction reproduces the arguments given above word for word. The justification of asymptotics in the case of multiply limiting eigenvalue is similar to the second section, too. The proof of Lemma 4.1 is complete.
5. Estimates for perturbed eigenvalues

In this section we will prove Theorems 1.1–1.3. Their proof will be based on the following auxiliary statement.

**Lemma 5.1.** Suppose sets $\gamma_1(\varepsilon), \gamma_2(\varepsilon) \subseteq \partial \Omega$ are such that $\gamma_1(\varepsilon) \subseteq \gamma_2(\varepsilon)$, $\lambda_{\varepsilon, 1}^k$, $\lambda_{\varepsilon, 2}^k$ are eigenvalues of the perturbed problems with $\gamma_\varepsilon = \gamma_1(\varepsilon)$ and $\gamma_\varepsilon = \gamma_2(\varepsilon)$, respectively, taken in ascending order counting multiplicity. Then for each $k$ the inequalities

$$\lambda_{\varepsilon, 1}^k \leq \lambda_{\varepsilon, 2}^k$$

hold true.

Lemma 5.1 is a standard statement about variational properties of eigenvalues for elliptic boundary value problems, the proof is based on the minimax property of eigenvalues and an obvious inclusions of functional spaces: $H^1(\Omega, \gamma_2(\varepsilon)) \subseteq H^1(\Omega, \gamma_1(\varepsilon))$, where $H^1(\Omega, \gamma_i(\varepsilon))$, $i = 1, 2$ is a set of functions belonging to $H^1(\Omega)$ and vanishing on $\gamma_i(\varepsilon)$.

**Proof of Theorem 1.1.** In accordance with Lagrange theorem, the functions $a_j^i$ and $b_j^i$ introduced in the second section, can be represented by $a_j$ and $b_j$ as follows:

$$a_j^i = \theta'_\varepsilon(M_{j, \varepsilon}^{(3)})a_j, \quad b_j^i = \theta'_\varepsilon(M_{j, \varepsilon}^{(4)})b_j,$$

where $M_{j, \varepsilon}^{(3)} \in (s_j^\varepsilon - \varepsilon a_j, s_j^\varepsilon)$, $M_{j, \varepsilon}^{(4)} \in (s_j^\varepsilon, s_j^\varepsilon + \varepsilon b_j)$ are midpoints. By representations obtained and the assumptions (C0) and (1) we deduce that

$$a_j^i \geq c_1 a_j, \quad b_j^i \geq c_1 b_j.$$

These estimates, the assumptions (C0) and (1) and the disjointness of sets $\gamma_{\varepsilon, j}$ yield:

$$2\eta(\varepsilon) = \min_j a_j^i(\varepsilon) + \min_i b_j^i(\varepsilon) \leq \pi,$$

i.e., the function $\eta$ is bounded above by the number $\pi/2$. Moreover, last inequalities imply the existence of functions $a_*(\varepsilon)$ and $b_*(\varepsilon)$ such that $a_* + b_* = 2\eta$, and for a set

$$\gamma_{\varepsilon,*} = \{ x : x \in \partial \Omega, -\varepsilon a_*(\varepsilon) < \theta_\varepsilon(s) - \theta_\varepsilon(s_j^\varepsilon) < \varepsilon b_*(\varepsilon), j = 0, \ldots, N - 1 \}$$

the inclusion $\gamma_{\varepsilon,*} \subseteq \gamma_\varepsilon$ holds. By $\lambda_{\varepsilon,*}^k$ we indicate the eigenvalues of the perturbed problems with $\gamma_\varepsilon = \gamma_{\varepsilon,*}$, taken in ascending order counting multiplicity. The set $\gamma_{\varepsilon,*}$ obeys the hypothesis of Lemma 4.1 with the function $\eta$ from assumption (1). According with Lemma 4.1, the eigenvalues $\lambda_{\varepsilon,*}^k$ converge to eigenvalues $\lambda_0^k$ of problem (1.4) and satisfy the asymptotics from this lemma. Applying Lemma 5.1 twice: with $\gamma_1(\varepsilon) = \gamma_{\varepsilon,*}$, $\gamma_2(\varepsilon) = \gamma_\varepsilon$ and $\gamma_1(\varepsilon) = \gamma_\varepsilon$, $\gamma_2(\varepsilon) = \partial \Omega$, we establish double-sided estimates:

$$\lambda_{\varepsilon,*}^k \leq \lambda_{\varepsilon}^k \leq \lambda_0^k.$$
Now we replace $\lambda_{\varepsilon,*}^k$ by their asymptotics from Lemma 5.1, which implies, first, convergence of $\lambda_{\varepsilon}^k$ to $\lambda_{0}^k$, and, second, needed double-sided of differences $(\lambda_{\varepsilon}^k - \lambda_{0}^k)$. The proof of Theorem 1.1 is complete.

**Proof of Theorem 1.2.** We deduce from the first estimate of Lemma 2.1 and the assumption (1) that

$$2\eta_0 \eta \leq a^j + b^j \leq 2\eta.$$

These inequalities imply that, first, the function $\eta_0$ is bounded above by one, and, second, there exist nonnegative bounded functions $a^j_\varepsilon(\varepsilon), b^j_\varepsilon(\varepsilon), a^{j,*}_\varepsilon(\varepsilon), b^{j,*}_\varepsilon(\varepsilon)$, such that $a^j_\varepsilon + b^j_\varepsilon = 2\eta_0 \eta$, $a^{j,*}_\varepsilon + b^{j,*}_\varepsilon = 2\eta$, and sets

$$\gamma_{\varepsilon,*} = \{x : x \in \partial \Omega, -\varepsilon a^j_\varepsilon(\varepsilon) < \theta_\varepsilon(s) - \theta_\varepsilon(s^j_\varepsilon) < \varepsilon b^j_\varepsilon(\varepsilon), j = 0, \ldots, N - 1\},$$

$$\gamma_{\varepsilon}^* = \{x : x \in \partial \Omega, -\varepsilon a^{j,*}_\varepsilon(\varepsilon) < \theta_\varepsilon(s) - \theta_\varepsilon(s^j_\varepsilon) < \varepsilon b^{j,*}_\varepsilon(\varepsilon), j = 0, \ldots, N - 1\}$$

meet inclusions

$$\gamma_{\varepsilon,*} \subseteq \gamma_{\varepsilon} \subseteq \gamma_{\varepsilon}^*.$$  \hspace{1cm} (5.1)

By $\lambda_{\varepsilon,*}^k$ and $\lambda_{\varepsilon}^{k,*}$ we denote the eigenvalues of the perturbed problem with $\gamma_{\varepsilon} = \gamma_{\varepsilon,*}$ and $\gamma_{\varepsilon} = \gamma_{\varepsilon}^*$, taken in ascending order counting multiplicity. The sets $\gamma_{\varepsilon,*}$ and $\gamma_{\varepsilon}^*$ obey the assumptions [C0] and [C1]: role of the function $\eta$ from [C1] for them is played by the functions $\eta_0\eta$ and $\eta$ from the assumption [1] respectively; the equality (1.3) for these functions holds with the same $A > 0$. The quantities $\delta^j(\varepsilon)$ for the sets $\gamma_{\varepsilon,*}$ and $\gamma_{\varepsilon}^*$ are zero, therefore, by Lemma 2.8 the eigenvalues $\lambda_{\varepsilon,*}^k$ and $\lambda_{\varepsilon}^{k,*}$ converge to the eigenvalues of the problem (1.6) and asymptotics

$$\lambda_{\varepsilon,*}^k = \Lambda_{0}^k(\bar{\mu}, \varepsilon) + \varepsilon \int_{\partial \Omega} (\Psi_0^k(x, \mu, \varepsilon))^2 \ln f_\varepsilon(\theta_\varepsilon(s)) \theta_\varepsilon'(s) \, ds + O(\varepsilon^{3/2}),$$

$$\lambda_{\varepsilon}^{k,*} = \Lambda_{0}^k(\mu, \varepsilon) + \varepsilon \int_{\partial \Omega} (\Psi_0^k(x, \bar{\mu}, \varepsilon))^2 \ln f_\varepsilon(\theta_\varepsilon(s)) \theta_\varepsilon'(s) \, ds + O(\varepsilon^{3/2}),$$ \hspace{1cm} (5.2)

hold, where $\mu = \mu(\varepsilon) = -(\varepsilon \ln \eta(\varepsilon))^{-1} - A,$

$$\bar{\mu} = \bar{\mu}(\varepsilon) = -(\varepsilon \ln \eta_0(\varepsilon) \eta(\varepsilon))^{-1} - A = \mu(\varepsilon) + \frac{(A^2 - \mu(\varepsilon)^2)\varepsilon \ln \eta_0(\varepsilon)}{1 + (A + \mu(\varepsilon))\varepsilon \ln \eta_0(\varepsilon)}.$$  \hspace{1cm}

Lemma 2.3 yields an estimate $\|\Psi_0^k\|_{L^2(\partial \Omega)} \leq C$ with constant $C$ independent on $\varepsilon$ and $\mu$. This estimate, (2.29) and the assumption [C0] allows to estimate the integrals in (5.2):

$$-C \leq \int_{\partial \Omega} (\Psi_0^k)^2 \ln f_\varepsilon(\theta_\varepsilon) \theta_\varepsilon' \, ds \leq 0,$$ \hspace{1cm} (5.3)

where $C > 0$ is independent on $\varepsilon$ and $\mu$. Lemma 5.1 due to inclusions (5.1) maintains the validity of estimates

$$\lambda_{\varepsilon,*}^k \leq \lambda_{\varepsilon}^k \leq \lambda_{\varepsilon}^{k,*},$$  \hspace{1cm}

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those, the asymptotics (5.2), (3.1) and the inequalities (5.3) imply the convergence \( \lambda_k^\varepsilon \to \lambda_0^k \) and needed double-sided estimates for the quantities \( (\lambda_k^\varepsilon - \lambda_0^k) \). The proof of Theorem 1.2 is complete.

**Proof of Theorem 1.3.** The main idea of proof is same with one in Theorems 1.1, 1.2. From the first estimate of Lemma 2.1 and the assumption (1) it follows the existence of nonnegative functions \( a_j^{\ast}(\varepsilon) \) and \( b_j^{\ast}(\varepsilon) \), such that

\[
 a_j^{\ast} + b_j^{\ast} = 2\eta,
\]

and needed double-sided estimates for the quantities \( \lambda_k^\varepsilon - \lambda_0^k \).

The proof of Theorem 1.3 is complete.

6. Appendix

In this section we will prove the formulae (3.1) for the eigenvalues of the problem (1.7), (1.8). Let

\[
 \tilde{\Lambda}_0^k = \lambda_0^k + \mu \lambda_1^k, \quad \tilde{\Psi}_0^k = \psi_0^k + \mu \psi_1^k + \psi^k, \quad \lambda_1^k = \int_{\partial \Omega} (\psi_0^k)^2 \theta_0' \, ds.
\]

The functions \( \psi_0^k \) associated with multiply eigenvalue are additionally chosen to be orthogonal in \( L_2(\partial \Omega) \) weighted by \( \theta_0' \). The functions \( \psi_1^k \) and \( \psi^k \) are defined as solutions of the problems:

\[
 (\Delta + \lambda_0^k) \psi_1^k = -\lambda_1^k \psi_1^k, \quad x \in \Omega, \\
 \left( \frac{\partial}{\partial \nu} + A \theta_0' \right) \psi_1^k = -\theta_0' \psi_0^k, \quad x \in \partial \Omega, \\
 (\Delta - 1) \psi^k = -\mu^2 \lambda_1^k \psi_1^k, \quad x \in \Omega, \\
 \left( \frac{\partial}{\partial \nu} + (A + \mu) \theta_0' \right) \psi^k = -(\theta_0' - \theta_0')((A + \mu) \psi_0^k + A \mu \psi_1^k) - \mu^2 \theta_0' \psi_1^k, \quad x \in \partial \Omega.
\]

The problem for \( \psi_1^k \) is solvable, the formula for \( \lambda_1^k \) and the assumption for \( \psi_0^k \) mentioned above are exactly the solvability condition. The functions \( \psi_1^k \) are selected
to be orthogonal to all eigenfunctions associated with $\lambda^k_0$. Clear, the problem for $\psi^k$ is uniquely solvable. General properties of solutions of elliptic boundary value problems yield that $\psi_1^k$ and $\psi^k$ are infinitely differentiable on $x$ functions, for those the estimates

$$\|\psi^k\|_{H^1(\Omega)} \leq C, \quad \|\psi^k\|_{H^1(\Omega)} \leq C(\mu^2 + (A + \mu)\sigma),$$

hold, where the constants $C$ are independent on $\varepsilon$ and $\mu$. Employing these estimates and the definition of $\lambda_1^k, \psi_1^k$ and $\psi^k$ one can check that the functions $\hat{\lambda}_0^k$ and $\hat{\psi}_0^k$ converge to $\lambda_0^k$ and $\psi_0^k$ and satisfy a problem

$$(\Delta + \hat{\lambda}_0^k)\hat{\psi}_0^k = \hat{F}_k, \quad x \in \Omega, \quad \left(\frac{\partial}{\partial \nu} + (A + \mu)\rho_{\varepsilon}^\prime\right)\hat{\psi}_0^k = 0, \quad x \in \partial\Omega,$$

$$\|\hat{F}_k\|_{L^2(\Omega)} \leq C(\mu^2 + (A + \mu)\sigma),$$

where the constant $C$ is independent on $\varepsilon$ and $\mu$. Let $\lambda_0 = \lambda_0^q = \ldots = \lambda_0^{q+p-1}$ be a $p$-multiply eigenvalue. By the problem for $\hat{\psi}_0^k$ and the estimate for the right side $\hat{F}_k$ employing results $[35]$, it is easy to show that for $k = q, \ldots, q + p - 1$ the representation and uniform on $\varepsilon$ and $\mu$ estimate

$$\hat{\psi}_0^k = \sum_{i=q}^{q+p-1} \frac{\Psi_i^0}{\Lambda_i^0 - \hat{\lambda}_0^k} \int \Psi_i^0 \hat{F}_k dx + \tilde{u}_k, \quad \|\tilde{u}_k\|_{H^1(\Omega)} \leq C(\mu^2 + (A + \mu)\sigma)$$

take place. By analogy with the justification from the second section on the base of last assertions we get the estimates

$$|\lambda_0^k - \hat{\lambda}_0^k| \leq C(\mu^2 + (A + \mu)\sigma),$$

those prove the equalities (3.1).

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