Boundary conformal invariants and the conformal anomaly in five dimensions

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Abstract

In odd dimensions the integrated conformal anomaly is entirely due to the boundary terms \cite{1}. In this paper we present a detailed analysis of the anomaly in five dimensions. We give the complete list of the boundary conformal invariants that exist in five dimensions. Additionally to 8 invariants known before we find a new conformal invariant that contains the derivatives of the extrinsic curvature along the boundary. Then, for a conformal scalar field satisfying either the Dirichlet or the conformal invariant Robin boundary conditions we use the available general results for the heat kernel coefficient $a_5$, compute the conformal anomaly and identify the corresponding values of all boundary conformal charges.

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1 Introduction

The presence of the boundaries brings the new features to the quantum field theory. One of the features is the modification of local quantities such as the heat kernel coefficients due to the boundary terms that results in the boundary terms in the quantum effective action. Those terms were studied for already some time in the literature and the findings, available to the date, were summarized in a review [3]. The related new feature is a modification of the conformal transformations of the quantum effective action [2] and of the conformal anomaly. In the absence of the boundaries the local conformal anomaly is non-trivial only if the dimension \(d\) of the space-time is even [4]. It is because it is presented by a combination of the Euler density and the local conformal anomalies constructed form the Riemann curvature of dimension \(d\). No such invariants exist if \(d\) is odd. A related fact is that the Euler number of an odd-dimensional compact manifold is zero.

In the presence of the boundaries two new things happen. Note that in this case it is more appropriate to consider the integrated conformal anomaly. Then, in even dimension \(d = 2n\) there appear certain boundary terms in the integrated anomaly. Those boundary terms are constructed from the Riemann curvature of the bulk metric and the extrinsic curvature \(K_{ij}\) of the boundary. Since \(K_{ij}\) has dimension 1 one can now construct terms of dimension \(2n - 1\) that can be further integrated over the boundary. These boundary terms in \(d = 4\) dimensions were identified in [5], [6]. The other new feature is that the anomaly in odd dimension \(d = 2n + 1\) is non-trivial. It is entirely due to the boundary terms [1] that can be of two types: the topological Euler number of the boundary \(E_{2n}\) and the local conformal invariants constructed from the Riemann tensor and the extrinsic curvature \(K_{ij}\) and their derivatives. In dimension \(d = 3\) there is only one such invariant so that the anomaly is determined by two conformal charges: one for the topological term \(E_2\) and the other for the conformal invariant. In [1] these charges were determined for the conformal scalar both for the Dirichlet and conformal Robin boundary conditions and in [7] for the Dirac fermions.

The primary goal of the present note is to advance the analysis to the next odd dimension \(d = 5\). Some preliminary list of the conformal invariants in this dimension was given in [1]. This list, as it was clear already in the time of writing the paper [1], was rather incomplete since it was anticipated that the other, unknown at that time, invariants may exist. The essential missing element was one or more invariants that could be constructed from the derivatives of the extrinsic curvature along the boundary and that would not vanish if the bulk 5d spacetime were flat. The differential invariant presented in [1] does not have this last property since, as we show this in the present paper, it happens to be not an independent invariant as it reduces to a combination of the other invariants constructed from the various contractions of the Weyl tensor and the extrinsic curvature tensor. One of the main results of the present paper is that we have now found this new invariant, see eq. (29), that contains derivatives of the extrinsic
curvature tensor and that was absent in [1]. With the new invariant the list of the conformal boundary invariants in five dimensions is now complete.

In a wider context the present paper is a part of the on-going study of the various manifestations of the boundaries, defects and interfaces in the conformal field theories, [8] - [21]. It should be noted that contrary to the situation with the conformal invariants in the bulk of space-time, where all such invariants are classified, no such a classification theorem exists in general for the boundary conformal invariants. Discussing the preliminary work we should note the earlier results in mathematical literature [22] where the conformal invariant boundary structures were studied in various dimensions. The other preliminary work relevant to our present study is the calculation of the heat kernel coefficient $a_5$ for the various boundary conditions and the different elliptic operators that was performed in a series of papers [23], [24], [25] that was summarized in [26]. It is our other principle goal in the paper to use these earlier results, compute the conformal anomaly in the case of a conformal scalar both for the Dirichlet and conformal Robin boundary conditions and express the anomaly in terms of the complete set of the invariants thus determining the exact values of the boundary conformal charges.

It should be noted that our findings can be also relevant to the study of the logarithmic terms in the entanglement entropy of a conformal field theory in $d = 6$. Indeed, the entangling surface then has dimension 4 and the logarithmic terms are due to the possible conformal invariants constructed from the curvature of spacetime and the extrinsic curvature of the surface. The previous study of the logarithmic terms within the holographic paradigm includes [27]. In the paper we make contact with this previous study.

The holographic aspect of the conformal anomaly in five dimensions in the presence of the boundaries is interesting. We, however, leave it for future work.

2 Boundary conformal invariants in $d = 5$

2.1 Notations and conformal transformations

We denote by $n^\mu$ the components of the out-ward normal vector to the boundary $\partial \mathcal{M}_5$, $n^\mu n_\mu = 1$. The metric induced on the boundary is $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. In this and the subsequent sections we use the notations that the latin indices $i, j, k \ldots$ should be understood as projections along the boundary, they take values 1, 2, 3, 4. The index projected on the normal direction is denoted by $n$. With these notations for any tensor $X_{\mu\alpha\beta}$ one has that $X_{nij} = n^\mu \gamma_i^\alpha \gamma_j^\beta X_{\mu\alpha\beta}$. The extrinsic curvature is defined as $K_{ij} = \gamma_i^\mu \gamma_j^\nu \nabla_{(\mu} n_{\nu)}$, where $\nabla_\mu$ is the covariant derivative defined with respect to the 5d metric $g_{\mu\nu}$. The covariant derivative defined with respect to the intrinsic metric $\gamma_{ij}$ is denoted by $\bar{\nabla}_i$ and the respective curvature by $\bar{R}$, $\bar{R}_{ij}$ and $\bar{R}_{ijkl}$. The relations between the intrinsic curvature of the boundary and the curvature in the 5d space-time are given by the Gauss-Codazzi identities presented in Appendix A.
Under the infinitesimal conformal transformations \( \delta g_{\mu\nu} = 2\sigma g_{\mu\nu}, \delta n_\mu = \sigma n_\mu \) the Weyl tensor transforms as \( \delta W_{\alpha\mu\beta\nu} = 2\sigma W_{\alpha\mu\beta\nu} \). The extrinsic curvature transforms as follows

\[
\delta K_{ij} = \sigma K_{ij} + \gamma_{ij} \nabla_n \sigma, \quad \delta \hat{K}_{ij} = \sigma \hat{K}_{ij},
\]

where \( \nabla_n = n^\mu \nabla_\mu \) and \( \hat{K}_{ij} = K_{ij} - \frac{1}{4} \gamma_{ij} K \) is the trace free part of the extrinsic curvature. The basic conformal tensors are, thus, the bulk Weyl tensor \( W_{\alpha\mu\beta\nu} \) and the trace free extrinsic curvature of the boundary \( \hat{K}_{ij} \). The intrinsic Weyl tensor of the boundary metric is expressed in terms of the 5-dimensional Weyl tensor and the extrinsic curvature by means of the Gauss-Codazzi relations.

### 2.2 Conformal anomaly

The integrated conformal anomaly in five dimensions can be presented in the form,

\[
\int_{\mathcal{M}_5} (T_{\mu\nu}) g^{\mu\nu} = \frac{1}{5760(4\pi)^2} (a E_4 + \sum_{k=1}^{8} c_k I_k),
\]

where \( E_4 \) is the 4d Euler density integrated over the boundary \( \partial \mathcal{M}_5 \), so that \( \chi[\partial \mathcal{M}_5] = \frac{1}{32\pi} E_4 \) is the Euler number of the boundary, and \( I_k, k = 1, \ldots, 8 \) are the conformal invariants constructed from the 5-dimensional Weyl tensor and the trace free part of the extrinsic curvature of the boundary \( \hat{K}_{ij} = K_{ij} - \frac{1}{4} \gamma_{ij} K \). We note that the Euler number of an odd-dimensional open manifold is \( 1/2 \) of the Euler number of the boundary. This explains why the Euler number \( E_5 \) is not an independent quantity and, thus, is absent in the anomaly (2). The numerical pre-factor in (2) is chosen for the further convenience. \( (a, c_1, \ldots, c_8) \) are the boundary conformal charges to be determined below in the paper.

### 2.3 Euler number of the boundary

The Euler density of the 4-dimensional boundary has the standard expression in terms of the intrinsic curvature of the boundary,

\[
E_4 = \int_{\partial \mathcal{M}_5} \left( \bar{R}^2_{ijk\ell} - 4\bar{R}_{ij}^2 + \bar{R}^2 \right).
\]

Applying the Gauss-Codazzi equations, it can be expanded in terms of the curvature tensors of the bulk manifold as well as the extrinsic curvature of the four-dimensional boundary,

\[
E_4 = \int_{\partial \mathcal{M}_5} \left( R^2_{ijkl} - 4R_{ij}^2 + R^2 - 4R_{injn}^2 + 8R^{ij} R_{injn} + 4R_{nn}^2 - 4RR_{nn} + 4K^{ij} K^{kl} R_{ikj\ell} 
+ 8(K^2)^{ij} R_{ij} - 8K K^{ij} R_{ij} - 2 \text{Tr} K^2 R + 2K^2 R - 8(K^2)^{ij} R_{injn} + 8K K^{ij} R_{injn} 
+ 4 \text{Tr} K^2 R_{nn} - 4K^2 R_{nn} - 6 \text{Tr} K^4 + 8K \text{Tr} K^3 + 3(\text{Tr} K^2)^2 - 6K^2 \text{Tr} K^2 + K^4 \right),
\]

(4)
where we defined \((K^2)^{ij} = K^i_k K^{kj}\).

Now we list the conformal invariants which can be called algebraic. These invariants do not contain the derivatives of the extrinsic curvature.

### 2.4 Invariants constructed from the extrinsic curvature tensor

The first group of the invariants of this type is constructed from the trace free extrinsic curvature \(\hat{K}_{ij}\),

\[
I_1 = \int_{\partial \mathcal{M}_5} (\text{Tr} \hat{K}^2)^2 = \int_{\partial \mathcal{M}_5} \left[ (\text{Tr} K^2)^2 - \frac{1}{2} K^2 \text{Tr} K^2 + \frac{1}{16} K^4 \right],
\]

and

\[
I_2 = \int_{\partial \mathcal{M}_5} \text{Tr} \hat{K}^4 = \int_{\partial \mathcal{M}_5} \left( \text{Tr} K^4 - K \text{Tr} K^3 + \frac{3}{8} K^2 \text{Tr} K^2 - \frac{3}{64} K^4 \right).
\]

### 2.5 Invariants constructed from the Weyl tensor

The next set of the invariants is constructed from the 5d Weyl tensor,

\[
I_3 = \int_{\partial \mathcal{M}_5} W^2_{ikj\ell} = \int_{\partial \mathcal{M}_5} \left( R^2_{ikj\ell} - \frac{16}{9} R^2_{ij} + \frac{5}{18} R^2 + \frac{8}{3} R^{ij} R_{injn} + \frac{4}{9} R^2_{nn} - \frac{8}{9} R R_{nn} \right),
\]

and

\[
I_4 = \int_{\partial \mathcal{M}_5} W^2_{injn} = \int_{\partial \mathcal{M}_5} \left( \frac{1}{9} R^2_{ij} - \frac{1}{36} R^2 + R^2_{injn} - \frac{2}{3} R^{ij} R_{injn} - \frac{4}{9} R^2_{nn} + \frac{2}{9} R R_{nn} \right).
\]

### 2.6 Invariants constructed from the Weyl tensor contracted with the extrinsic curvature tensor

Contractions of the Weyl tensor and the trace free extrinsic curvature give us a new set of the conformal invariants,

\[
I_5 = \int_{\partial \mathcal{M}_5} \hat{K}^{ij} \hat{K}^{k\ell} W_{ikj\ell} = \int_{\partial \mathcal{M}_5} \left( K^{ij} K^{k\ell} R_{ikj\ell} + \frac{2}{3} (K^2)^{ij} R_{ij} - \frac{5}{6} K K^{ij} R_{ij} - \frac{1}{12} \text{Tr} K^2 R + \frac{1}{8} K^2 R \right.
\]

\[
+ \frac{1}{2} K K^{ij} R_{injn} - \frac{1}{6} K^2 R_{nn} \right),
\]
and

\[ I_6 = \int_{\partial M_5} \tilde{K}^i_k \tilde{K}^k_j W_{injn} \]
\[ = \int_{\partial M_5} \left( -\frac{1}{3} K^i_k R_{ij} + \frac{1}{6} K K^{ij} R_{ij} + \frac{1}{12} \text{Tr} K^2 R - \frac{1}{24} K^2 R \right) \]
\[ K^i_k K^{kj} R_{injn} - \frac{1}{2} K K^{ij} R_{injn} - \frac{1}{3} \text{Tr} K^2 R_{nn} + \frac{1}{6} K^2 R_{nn} \bigg). \]  

(10)

### 2.7 Invariants with derivatives and their independence

The invariants of this group will be expressed in terms of the derivatives of the extrinsic curvature along the boundary. The first invariant of this kind is

\[ I_7 = \int_{\partial M_5} W^2_{nijk}. \]  

(11)

Using the Gauss-Codazzi equation, \( W_{nijk} \) can be expanded in terms of the derivatives of the extrinsic curvature tensor

\[ W_{nijk} = 2 \bar{\nabla}_k K^i_j + \frac{2}{3} h_{i[j} \left( \bar{\nabla}_i K^\ell_k \right) - \bar{\nabla}_k |K| \]. \]  

(12)

This identity helps us to re-write the invariant \( I_7 \) in the form that contains only the extrinsic curvature tensor and its derivatives along the boundary,

\[ I_7 = \int_{\partial M_5} \left( 2 \bar{\nabla}_k K^i_j \bar{\nabla}^k \bar{K}^{ij} - 2 \bar{\nabla}_k K^{ij} \bar{\nabla}^j K^k_i + \frac{4}{3} \bar{\nabla}_i K \bar{\nabla}^j K^{ij} - \frac{2}{3} \bar{\nabla}_i K^{ij} \bar{\nabla}^k K^j_k - \frac{2}{3} (\bar{\nabla} K)^2 \right) \], \]  

(13)

using (A.10), one re-writes this expression as follows

\[ I_7 = \int_{\partial M_5} \left[ 2 \bar{\nabla}_k K^i_j \bar{\nabla}^k \bar{K}^{ij} - \frac{8}{3} \bar{\nabla}_i K^i_j \bar{\nabla}^k K^k_j + \frac{4}{3} \bar{\nabla}_i K \bar{\nabla}^j K^{ij} - \frac{2}{3} (\bar{\nabla} K)^2 \right.
\[ \left. - 2 K^{ij} K^{kl} R_{ik\ell} - 2 (K^2)^{ij} R_{injn} + 2 (K^2)^{ij} R_{ij} + 2 K \text{Tr} K^3 - 2 (\text{Tr} K^2)^2 \right]. \]  

(14)

It is instructive to analyse the relation of this invariant to another conformal invariant that is written in terms of the conformal invariant operator acting on a tensor of rank two. The following term is introduced in [11] and is known to be a conformal invariant in 5 dimensions,

\[ I^{(D)} = \int_{\partial M_5} \text{Tr} \tilde{K} \mathcal{F} \tilde{K}, \]  

(15)

where the differential operator

\[ \mathcal{F}_{k\ell}^{ij} = \delta_{(k} \delta_{\ell)} \Box - \frac{4}{3} \bar{\nabla}^{(i} \bar{\nabla}_{(k} \delta_{\ell)}^{j)} - \frac{8}{6} \bar{R}_i^{i j} \bar{R}_{k \ell}^{j} - \frac{2}{3} \bar{R}_{(k \ell)}^{ij} + \frac{1}{6} \bar{R} \delta_{(k} \delta_{\ell)}^{i j} \]  

(16)
acts on a tensor with conformal weight +1. It should be noted that there is only one such
differential operator that respects the conformal symmetry \[28, 29\]. The explicit form of this
invariant reads

\[
I^{(D)} = \int_{\partial M_5} \text{Tr} \, \hat{K} F \hat{K} = \int_{\partial M_4} \hat{K}^{kl} F_{kl} \hat{K}_{ij}
\]

\[
= \int_{\partial M_5} [K^{ij} \Box K_{ij} - \frac{1}{3} K \Box K + \frac{1}{3} K^{ij} \nabla_i \nabla_j K + \frac{1}{3} K \nabla_i \nabla_j K_{ij} - \frac{4}{3} K^{ij} \nabla_j \nabla_k K^k
\]

\[
+ 2 (K^2)^{ij} R_{injn} - KK^{ij} R_{injn} - 2 (K^2)^{ij} R_{ij} + KK^{ij} R_{ij} - \frac{1}{3} \text{Tr} K^2 R_{nn} + \frac{1}{3} K^2 R_{nn}
\]

\[
+ \frac{1}{6} \text{Tr} K^2 R - \frac{1}{6} K^2 R + 2 \text{Tr} K^4 - 3K \text{Tr} K^3 - \frac{1}{6} (\text{Tr} K^2)^2 + \frac{4}{3} K^2 \text{Tr} K^2 - \frac{1}{6} K^4].
\]

(17)

It can be shown using the Gauss-Codazzi relations that in flat bulk spacetime \( R_{\mu \nu \alpha \beta} = 0 \) all
derivatives of the extrinsic curvature in (17) are mutually cancelled.

In what follows it will be shown that this term is not an independent invariant. To see this,
let us start with the observation that

\[
I^{(D)} = -\frac{1}{2} I_7 - \int_{\partial M_5} \hat{K}^{ij} \hat{K}^{kl} \hat{W}_{ikj}.
\]

(18)

Note also that using the Gauss-Codazzi relations one has that

\[
\int_{\partial M_5} \hat{K}^{ij} \hat{K}^{kl} \hat{W}_{ikj} = I_5 + \int_{\partial M_5} \left[ - (R_{injn} - \frac{1}{3} R_{ij})( (K^2)^{ij} - \frac{1}{2} KK^{ij})
\right.

\[
+ \frac{1}{3} (R_{nn} - \frac{1}{4} R)( \text{Tr} K^2 - \frac{1}{2} K^2) - 2 \text{Tr} K^4 + 2K \text{Tr} K^3 + \frac{7}{6} (\text{Tr} K^2)^2 - \frac{4}{3} K^2 \text{Tr} K^2 + \frac{1}{6} K^4].
\]

(19)

The integral in the right hand side of the above equation can be written in terms of \( I_3, I_4 \)
and \( I_6 \). So finally we conclude that

\[
I^{(D)} = -\frac{1}{2} I_7 - I_5 + I_6 - \frac{7}{6} I_3 + 2 I_4.
\]

(20)

This indicates that it is not an independent invariant and, thus, it has to be excluded from the
list. Note that all derivative terms in \( I^{(D)} \) come just from \( I_7 \), so it is natural that all derivative
terms disappear in the flat spacetime case, where \( W_{nijk} = 0 \).

### 2.8 Invariants with derivatives: new invariant

To construct the last invariant, we start with the normal derivative of the Weyl tensor, con-
tracted with the normal vector and the traceless extrinsic curvature tensor

\[
I_8^{(1)} = \int_{\partial M_5} \hat{K}^{ij} \nabla_n W_{injn},
\]

(21)
where by $\nabla_n W_{injn}$ we mean $n^\rho n^\mu n^\nu \nabla_\rho W_{\mu j\nu}$. We can show that

$$\delta I_8^{(1)} = -2 \int_{\partial M_5} \left( \hat{K}^{ij} W_{injn} \partial_n \sigma + \hat{K}^{ij} W_{nijk} \nabla^k \sigma \right).$$  \hspace{1cm} (22)

The first term in the conformal variation can be removed if we simply add

$$I_8^{(2)} = \frac{1}{2} \int_{\partial M_5} K \hat{K}^{ij} W_{injn}. \hspace{1cm} (23)$$

Let us focus now on the second term in (22). We can construct an integral with the same transformation

$$I_8^{(3)} = \int_{M_5} \left( \frac{1}{9} \nabla_i \hat{K}_{ij} \nabla_k \hat{K}^{kj} - (\hat{K}^2)^{ij} \bar{S}_{ij} + \frac{1}{2} \text{Tr} \hat{K}^2 \bar{S}_i^i \right), \hspace{1cm} (24)$$

where

$$\bar{S}_{ij} = \frac{1}{2} (\bar{R}_{ij} - \frac{1}{6} \bar{R} \gamma_{ij}) \hspace{1cm} (25)$$

is the 4d Schouten tensor computed with respect to the intrinsic boundary metric $\gamma_{ij}$. Under the conformal transformations one has

$$\delta(\nabla_i \hat{K}_{ij} \nabla_k \hat{K}^{kj}) = -4 \sigma \nabla_i \hat{K}_{ij} \nabla_k \hat{K}^{kj} + 6 \hat{K}^{ij} \nabla_k \hat{K}^{kj} \nabla_i \sigma,$$

$$\delta \bar{S}_{ij} = -\nabla_i \nabla_j \sigma, \hspace{0.5cm} \delta \bar{S}_i^i = -2 \sigma \bar{S}_i^i - \Box \sigma. \hspace{1cm} (26)$$

Therefore one finds that

$$\delta I_8^{(3)} = \int_{\partial M_5} \left[ \frac{2}{3} \hat{K}^{ij} \nabla_k \hat{K}^{kj} \nabla_i \sigma + (\hat{K}^2)^{ij} (\nabla_i \nabla_j \sigma - \frac{1}{2} \gamma_{ij} \Box \sigma) \right]. \hspace{1cm} (27)$$

Interestingly, we may recast the above expression as follows,

$$\delta I_8^{(3)} = \int_{\partial M_5} \hat{K}^{ij} W_{nijk} \nabla^k \sigma. \hspace{1cm} (28)$$

This is precisely what we need to cancel the second term in the conformal transformation of $I_8^{(1)}$. So we can now construct the following conformal invariant, $I_8 = I_8^{(1)} + I_8^{(2)} + 2I_8^{(3)}$.

$$I_8 = \int_{M_5} \left( \hat{K}^{ij} \nabla_n W_{injn} + \frac{1}{2} K \hat{K}^{ij} W_{injn} + \frac{2}{9} \nabla_i \hat{K}_{ij} \nabla_k \hat{K}^{kj} - 2(\hat{K}^2)^{ij} \bar{S}_{ij} + \text{Tr} \hat{K}^2 \bar{S}_i^i \right). \hspace{1cm} (29)$$

Using the Gauss-Codazzi equations and the differential relations (A.9), (A.8) and (A.10) the new invariant $I_8$ can be rewritten as follows,

$$I_8 = \int_{\partial M_5} \left[ \frac{2}{3} K^{ij} \nabla_n R_{injn} - \frac{1}{12} K \nabla_n R + \frac{2}{3} K^{ij} K^{kl} R_{ikjl} - K^i K^{ijk} R_{ij} + \frac{1}{3} \text{Tr} K^2 R - \frac{5}{48} K^2 R^2 \right. \left. + \frac{5}{3} K^i K^{ijk} R_{injn} - \frac{1}{3} K K^{ij} R_{injn} - \text{Tr} K^2 R_{nn} + \frac{11}{24} K^2 R_{nn} \right. \left. + \text{Tr} K^4 - \frac{11}{6} K \text{Tr} K^3 + \frac{47}{48} K^2 \text{Tr} K^2 \right. \left. - \frac{7}{48} K^4 \right. \left. - \frac{1}{3} \nabla_k K_{ij} \nabla^k K^{ij} + \frac{8}{9} \nabla_i \hat{K}_{ij} \nabla_k \hat{K}^{kj} - \frac{7}{9} \nabla_i K^{ij} \nabla_j K - \frac{25}{72} (\nabla K)^2 \right]. \hspace{1cm} (30)$$
To the best of our knowledge the invariant $I_8$ is a new invariant that was not available in the literature before. It has been for a long time an important missing element in the discussion of the conformal anomalies in five dimensions. Having said that we should notice that in some particular case this invariant reduces to the one that already appeared in the literature on the holographic computation of the entanglement entropy in $d = 6$. Indeed, provided the 5d manifold is flat and using the Gauss-Codazzi equations, one has that

$$R_{nijk} = 0 \rightarrow \bar{\nabla}_k K_{ij} = \bar{\nabla}_j K_{ik}. \quad (31)$$

In this case $I_8$ takes a simpler form,

$$I_{8\text{ flat}} = \int_{\partial \mathcal{M}_5} \left[ \frac{1}{8} (\bar{\nabla} K)^2 + \text{Tr} K^4 - \frac{3}{2} K \text{Tr} K^3 - \frac{1}{3} (\text{Tr} K^2)^2 + \frac{47}{48} K^2 \text{Tr} K^2 - \frac{7}{48} K^4 \right]. \quad (32)$$

Interestingly, in this form, it is related to invariant $T_3$ found by Safdi [27] in a holographic calculation of the entanglement entropy on a 6 dimensional flat manifold,

$$I_{8\text{ flat}} = \frac{1}{8} (T_3 + \frac{10}{3} I_3 - I_4). \quad (33)$$

It should be noted that no conformal invariant form of $T_3$ was given in [27]. Thus, our result (29), (30), among other things, offers a conformal invariant form, valid in a curved spacetime, for this holographic calculation.

3 Conformal anomaly for a scalar field

The free conformal scalar field in five dimensions is described by a conformal Laplace operator,

$$D = -(\nabla^2 + E), \quad E = -\frac{3}{16} R. \quad (34)$$

In the space-time with boundaries it should be supplemented by a boundary condition. There are two possible boundary conditions consistent with the conformal symmetry: the Dirichlet condition and the conformal Robin condition,

Dirichlet b.c. : $\phi|_{\partial \mathcal{M}_5} = 0$, 

Robin b.c. : $(\partial_n - S)\phi|_{\partial \mathcal{M}_5} = 0, \quad S = -\frac{3}{8} K. \quad (35)$

The important object that encodes the main information about the quantum field theory is the heat kernel $K(D, s) = e^{-sD}$ and its small $s$ expansion,

$$\text{Tr} K(D, s) = \sum_{p=0} a_p(D) s^{(p-5)/2}, \quad s \rightarrow 0 \quad (36)$$
where $a_p(D)$ are the heat kernel coefficients that are represented by integrals over the manifold and its boundary of the local invariants constructed from the curvature of the bulk metric, the extrinsic curvature of the boundary and the quantities $E$ and $S$.

The integrated conformal anomaly in dimension $d = 5$ is determined by the coefficient $a_5$,

$$
\int_{\mathcal{M}_5} \langle T_{\mu\nu} \rangle g^{\mu\nu} = a_5 .
$$

(37)

As was explained above, there is no a bulk term in $a_5$ and the entire contribution comes only from the boundary $\partial \mathcal{M}_5$. The exact form of the coefficient $a_5$ is available in the literature for a rather general elliptic operator with the boundary condition being a mixture of the Dirichlet and the Robin conditions, see [23], [24], [25], [26]. In what follows we use the general form of $a_5$ presented in [23].

For the conformal operator $D$ and the Dirichlet boundary condition we find that

$$
a_5^{(D)} = - \frac{1}{5760(4\pi)^2} \int_{\partial \mathcal{M}_5} \left( R^2_{\mu\nu\alpha\beta} - 8R^2_{\mu\nu} + \frac{5}{16} R^2 - 10R^2_{inj} + 16R_{ij} R_{inj} - 17R_{nn}^2 + \frac{5}{2} R R_{nn} 
+ 48\Box R - 45\Box R - \frac{111}{2} \nabla_i^2 R + 24\Box R_{nn} + 15\nabla_n^2 R_{nn} - \frac{339}{8} K \nabla_i R 
+ 32K^{ij}K^{k\ell} R_{ikj\ell} + 16K^{ij}K^{k\ell} R_{ij} + 14KK^{ij}R_{ij} + \frac{25}{8} \text{Tr} K^2 R - \frac{35}{16} K^2 R 
- \frac{47}{2} K^{ij}K^{k\ell} R_{inj} + \frac{215}{8} \text{Tr} K^2 R_{nn} - \frac{215}{16} K^2 R_{nn} 
- \frac{327}{8} \text{Tr} K^4 + \frac{17}{2} K \text{Tr} K^3 + \frac{777}{32} (\text{Tr} K^2)^2 + \frac{141}{32} K^2 \text{Tr} K^2 - \frac{65}{128} K^4 
+ \frac{355}{8} \nabla_i K_{ij} \nabla^k K^{ij} - \frac{11}{4} \nabla_i K_j \nabla^k K^{jk} - \frac{29}{4} \nabla_i K_j \nabla_k \nabla^i K^{jk} + 58 \nabla_i K^{ij} \nabla_j K - 30 K \nabla_i \nabla_j K^{ij} 
- \frac{75}{2} K^{ij} K^{k\ell} K_j + \frac{285}{4} \nabla_i K_j \nabla_j K + 54 K^{ij} \Box K_{ij} - 6 \Box K - \frac{413}{16} \nabla_i K \nabla^i K \right), 
$$

(38)

while for the conformal Robin boundary condition we have that

$$
a_5^{(R)} = \frac{1}{5760(4\pi)^2} \int_{\partial \mathcal{M}_5} \left( R^2_{\mu\nu\alpha\beta} - 8R^2_{\mu\nu} + \frac{5}{16} R^2 - 10R^2_{inj} + 16R_{ij} R_{inj} - 17R_{nn}^2 + \frac{5}{2} R R_{nn} 
+ 48\Box R - 45\Box R - \frac{111}{2} \nabla_i^2 R + 24\Box R_{nn} + 15\nabla_n^2 R_{nn} - \frac{309}{8} K \nabla_i R 
+ 32K^{ij}K^{k\ell} R_{ikj\ell} + 16K^{ij}K^{k\ell} R_{ij} + \frac{43}{2} K K^{ij} R_{ij} - \frac{5}{8} \text{Tr} K^2 R - \frac{5}{4} K^2 R 
+ \frac{133}{2} K^{ij}K^{k\ell} R_{inj} + \frac{275}{8} \text{Tr} K^2 R_{nn} - \frac{185}{16} K^2 R_{nn} 
+ \frac{231}{8} \text{Tr} K^4 - \frac{125}{4} K \text{Tr} K^3 + \frac{375}{32} (\text{Tr} K^2)^2 + \frac{147}{32} K^2 \text{Tr} K^2 - \frac{37}{64} K^4 
+ \frac{535}{8} \nabla_i K_{ij} \nabla^k K^{ij} + \frac{49}{4} \nabla_i K_j \nabla^k K^{jk} + \frac{151}{4} \nabla_k K_j \nabla^i K^{jk} + 58 \nabla_i K^{ij} \nabla_j K - \frac{75}{2} K \nabla_i \nabla_j K^{ij} 
- \frac{15}{2} K^{ij} K^{k\ell} K_j + \frac{315}{4} \nabla_i \nabla_j K + 114 K^{ij} \Box K_{ij} - \frac{3}{2} K \Box K - \frac{503}{16} \nabla_i K \nabla^i K \right), 
$$

(39)
We shall insert here

\[ R_{\mu
u\rho\beta} = R_{skjt}^2 + 4R_{nijk}^2 + 4R_{injn}^2, \quad R_{i\mu}^2 = R_{ij}^2 + 2R_{in}^2 + R_{nn}^2. \]  

\[ (40) \]

The other important point is that the general expression for the coefficient \( a_5 \) given in [23] contains some sort of redundancy since the differential relation \( (A.11) \) generalises the Gauss-Codazzi identities, was not taken into account. In fact, a similar redundancy is present in the coefficient \( a_4 \) given in [3], where the relation \( (A.8) \) was not taken into account. Since these relations contain the higher order normal derivatives one might worry whether one would have to specify an extension for the normal vector our-side the boundary to make these relations valid. We, however, have checked that the relations \( (A.7) \) - \( (A.11) \) do not depend on the way the normal vector is extended. Thus, using the Gauss-Codazzi equations and the identities \( (A.7) \), \( (A.11) \) we finally arrive at the expression

\[ a_5^{(D)} = \frac{1}{5760(4\pi)^2} \int_{\partial M_5} \left( -8R_{skjt}^2 + 8R_{ij}^2 - \frac{5}{16} R_{i\mu}^2 - 22R_{injn}^2 - R_{ij}^2 R_{injn} + 10R_{in}^2 - \frac{5}{2} RR_{nn} 
- 15K^i j \nabla_n R_{injn} + \frac{15}{8} K \nabla_n R \right. 
\]

\[ + \frac{277}{4} K^i j K^k l R_{skjt} - \frac{289}{4} K^i_l K^k_j R_{ij} + K K^i j R_{ij} - \frac{25}{8} Tr K^2 R + \frac{35}{16} K^2 R
\]

\[ + \frac{499}{4} K^i_k K^k_j R_{injn} - \frac{109}{4} K K^i j R_{injn} - \frac{25}{8} Tr K^2 R_{nn} - \frac{25}{16} K^2 R_{nn}
\]

\[ + \frac{327}{8} Tr K^4 - \frac{379}{4} K Tr K^3 + \frac{1983}{32} (Tr K^2)^2 + \frac{141}{32} K^2 Tr K^2 + \frac{65}{128} K^4
\]

\[ - \frac{555}{8} \nabla K_{kij} \nabla K_{kij} + \frac{165}{2} \nabla_i K_{kij} \nabla_k K_{ijk} - \frac{135}{4} \nabla_i K_{kij} \nabla_j K + \frac{285}{16} \nabla_i K \nabla_i K \right) \]

for the conformal scalar operator with the Dirichlet boundary condition and

\[ a_5^{(R)} = \frac{1}{5760(4\pi)^2} \int_{\partial M_5} \left( 8R_{skjt}^2 - 8R_{ij}^2 + \frac{5}{16} R_{i\mu}^2 + 22R_{injn}^2 + R_{ij}^2 R_{injn} - 10R_{in}^2 + \frac{5}{2} RR_{nn} 
- 15K^i j \nabla_n R_{injn} + \frac{15}{8} K \nabla_n R \right. 
\]

\[ - \frac{97}{4} K^i j K^k l R_{skjt} + \frac{109}{4} K^i_l K^k_j R_{ij} + \frac{13}{2} K K^i j R_{ij} - \frac{5}{8} Tr K^2 R - \frac{5}{4} K^2 R
\]

\[ + \frac{41}{4} K^i_k K^k_j R_{injn} - \frac{101}{4} K K^i j R_{injn} - \frac{35}{8} Tr K^2 R_{nn} + \frac{55}{16} K^2 R_{nn}
\]

\[ + \frac{231}{8} Tr K^4 + 10K Tr K^3 - \frac{945}{32} (Tr K^2)^2 + \frac{147}{32} K^2 Tr K^2 - \frac{37}{64} K^4
\]

\[ + \frac{255}{8} \nabla K_{kij} \nabla K_{kij} - \frac{105}{2} \nabla_i K_{kij} \nabla K_{ijk} + \frac{135}{4} \nabla_i K_{kij} \nabla_j K - \frac{255}{16} \nabla_i K \nabla_i K \right) \]

for the conformal scalar operator with the conformal Robin boundary condition \( (35) \).

\[ ^1 \text{The relation} \ (A.8) \text{ has appeared earlier in} \ [30] \text{ and} \ [5]. \]
Table 1: Conformal charges for Dirichlet and Robin boundary conditions

| Conformal charges | Dirichlet b.c. | Robin b.c. |
|-------------------|---------------|------------|
| $a$               | $\frac{17}{8}$ | $-\frac{17}{8}$ |
| $c_1$             | $-\frac{681}{32}$ | $\frac{39}{32}$ |
| $c_2$             | $\frac{609}{8}$ | $\frac{309}{8}$ |
| $c_3$             | $-\frac{81}{8}$ | $\frac{81}{8}$ |
| $c_4$             | $-\frac{27}{2}$ | $\frac{27}{2}$ |
| $c_5$             | $-\frac{9}{8}$ | $\frac{189}{8}$ |
| $c_6$             | $\frac{819}{8}$ | $\frac{441}{8}$ |
| $c_7$             | $-\frac{615}{16}$ | $\frac{195}{16}$ |
| $c_8$             | $-\frac{45}{2}$ | $-\frac{45}{2}$ |

Following the earlier works [23] - [26], we adopted in this paper the convention that the components of the normal vector $n^\mu$ are always outside the covariant derivatives, for instance $\nabla_n R_{mn} \equiv n^\alpha n^\mu n^\nu \nabla_\alpha R_{\mu\nu}$, $\nabla_n R_{mnij} \equiv n^\alpha n^\mu n^\nu \nabla_\alpha R_{\mu\nu ij}$ etc. This guarantees that these terms do not depend on how the components of the normal vector are extended outside the boundary. It should be noted that the elliptic operators in question do not require any such extension so that the respective heat kernel coefficients contain only terms that are independent of the way the vectors normal to the boundary are extended to the nearest vicinity of the boundary. This, in particular, explains why in the heat kernel coefficients there do not appear any terms that contain the normal derivative of the extrinsic curvature, $\nabla_n K_{ij}$, since for these terms one would need to specify the extension of the normal vector outside the boundary.

The expressions for $a_5$, obtained above, should be now compared with the general form (2) of the anomaly decomposed over the possible conformal invariants. We have checked that the decomposition is indeed possible and unique both for $a_5^{(D)}$ and $a_5^{(R)}$ so that the list of conformal invariants $E_4, I_1, \ldots, I_8$ is indeed complete. The details of the analysis are given in Appendix B. The decomposition allows us to determine the values of the boundary conformal charges in the case of a conformal scalar field. For each boundary condition, one arrives at 27 equations on 9 charges. That solution of this highly overdetermined system of algebraic equations exists is a nice check on our formulas. The result is presented in Table 1. The values found for the Euler charge $a$ agree with the earlier computations [14], [31].

\footnote{We thank D. Vassilevich for correspondence on this point.}
4 Conclusions

In this note we have presented an exhaustive discussion of the boundary conformal invariants in five dimensions. In particular, and this is one of our main results, we have found a new conformal invariant $I_8$ that contains the derivatives of the extrinsic curvature tensor along the boundary. In a flat space limit this invariant is related to the one found holographically by Safdi. We note that the invariants discussed in this paper do not depend on the way the normal vector is extended outside the boundary. Only invariants of this type may appear in the heat kernel of an elliptic operator that knows nothing about such an extension. We, however, note here for the record that there exists a family of the conformal tensors and the respective conformal invariants that can be defined provided an extension of the normal vector is given. These invariants have not been discussed in this paper.

We use the available in the literature general results for the heat kernel coefficient $a_5$ and compute the integrated conformal anomaly for a conformal scalar satisfying either the Dirichlet or the conformal Robin boundary conditions. The anomaly is then uniquely decomposed over the set of conformal invariants we have found. We then compute the respective conformal charges. This is our second main result. It would be interesting to extend our results and compute the conformal anomaly and the respective boundary charges for the conformal fields of higher spin. Although we do not anticipate any principle difficulties we do not present this analysis here leaving it to the future.

Among other possible applications, our results will be useful in classifying the possible conformal invariants that appear in the logarithmic terms of entanglement entropy of a $d = 6$ conformal field theory. The holographic derivation of the conformal anomaly in five dimensions is yet another interesting subject to explore. We leave these directions for future work.

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A Gauss-Codazzi relations

The Gauss-Codazzi identities give us relations between the bulk curvature $R$, the intrinsic curvature on the boundary $\bar{R}$ and the extrinsic curvature $K_{ij}$,

$$ R_{ikjl} = \gamma_i^\mu \gamma_k^\alpha \gamma_j^\nu \gamma_l^\beta R_{\mu\nu\alpha\beta} = \bar{R}_{ikjl} - (K_{ij} K_{kl} - K_{ik} K_{jl}), \quad (A.1) $$

$$ R_{nijk} = \gamma_i^\mu \gamma_j^\nu \gamma_k^\rho \gamma_l^n R_{\mu\nu\rho\gamma} = (\bar{\nabla}_k K_{ij} - \bar{\nabla}_j K_{ik}), \quad (A.2) $$

where $\gamma_i^\mu$ represents the projection operation.

**Contracted Equations**

$$ R_{in} = R_{ni} = (\bar{\nabla}_j K_i^j) - \bar{\nabla}_i K. \quad (A.3) $$

$$ R_{ij} = \bar{R}_{ij} + R_{injn} + (K_{ij}^2 - KK_{ij}). \quad (A.4) $$

**Doubly Contracted Equations**

$$ R = \bar{R} + 2R_{nn} + (\text{Tr} K^2 - K^2). \quad (A.5) $$

Thus, one finds for $G_{nn} = G_{\mu\nu} n^\mu n^\nu$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$,

$$ G_{nn} = -\frac{1}{2} \bar{R} - \frac{1}{2} (\text{Tr} K^2 - K^2). \quad (A.6) $$

**Differential Equations**

$$ \Box R = \bar{\Box} R + \nabla^2_n R + K \nabla_n R, \quad (A.7) $$

$$ \nabla_n G_{nn} = \nabla_n (R_{nn} - \frac{1}{2} R) = K^{ij} R_{ij} - K R_{nn} - \bar{\nabla}_i {\bar{\nabla}}_j K^{ij} + \Box K, \quad (A.8) $$

$$ \nabla_n R_{ij} = \nabla_n R_{injn} - 2K^{k\ell} R_{ik\ell} - K^{k} R_{jk} + K_{j}^k R_{lk} + K_{j}^k R_{lk} + K_{ij} R_{nn} $$

$$ - \text{Tr} K^2 K_{ij} + K K_{ik} K_{jk} + \bar{\nabla}_i \bar{\nabla}_j K^k + \bar{\nabla}_i \bar{\nabla}_j K^k + \Box K_{ij} - \bar{\nabla}_i \bar{\nabla}_j K, \quad (A.9) $$

$$ \nabla_k K_{ij} \bar{\nabla}^i K^{jk} = \bar{\nabla}_i K^{ij} \bar{\nabla}_k K^{jk} + K^{ij} K^{kl} R_{ik\ell} - K^{ik} K^{j} \bar{R}_{ij} $$

$$ + K^{ik} K^{j} R_{injn} - K \text{Tr} K^2 + (\text{Tr} K^2)^2 + T.D., \quad (A.10) $$

$$ \nabla^2_n G_{nn} = -R_{ij} R_{injn} + R_{nn}^2 - K_{k}^i K^{kj} R_{ij} + \text{Tr} K^2 R_{nn} $$

$$ + K^{ij} \nabla_n R_{ij} - K \nabla_n R_{nn} - \nabla_i K^{ij} \nabla_j K + (\nabla K)^2 + T.D., \quad (A.11) $$

where we defined $\nabla_n G_{nn} = n^\alpha n^\mu n^\nu \nabla_\alpha G_{\mu\nu}$ and $\nabla^2_n G_{nn} = n^\alpha n^\beta n^\mu n^\nu \nabla_\alpha \nabla_\beta G_{\mu\nu}$. 

14
B  Algebraic equations for the conformal charges

Matching the coefficients of the various terms in $a_5^{(D,R)}$ with the corresponding factors in the decomposition over the basis of the conformal invariants one arrives at the following algebraic equations for the conformal charges:

\[
\begin{align*}
\text{Tr } K^4 & : -6a + c_2 + c_8 = \left( \frac{327}{8}, \frac{231}{8} \right), \\
K \text{Tr } K^3 & : 8a - c_2 + 2c_7 - \frac{11}{6} c_8 = \left( -\frac{379}{4}, 10 \right), \\
(\text{Tr } K^2)^2 & : 3a + c_1 - 2c_7 = \left( \frac{1983}{32}, -\frac{945}{32} \right), \\
K^2 \text{Tr } K^2 & : -6a - \frac{1}{2} c_1 + \frac{3}{8} c_2 + \frac{47}{48} c_8 = \left( \frac{141}{32}, \frac{147}{32} \right), \\
K^4 & : a + \frac{1}{16} c_1 - \frac{3}{64} c_2 - \frac{7}{48} c_8 = \left( \frac{65}{128}, \frac{37}{64} \right), \\
R_{ikj}^2 & : a + c_3 = (-8, 8), \\
R_{ij}^3 & : -4a - \frac{16}{9} c_3 + \frac{1}{9} c_4 = (8, -8), \\
R^2 & : a + \frac{5}{18} c_3 - \frac{1}{36} c_4 = \left( \frac{5}{16}, \frac{5}{16} \right), \\
R_{injn}^4 & : -4a + c_4 = (-22, 22), \\
R^{ij} R_{injn} & : 8a + \frac{8}{3} c_3 - \frac{2}{3} c_4 = (-1, 1), \\
R_{injn}^2 & : 4a + \frac{4}{9} c_3 - \frac{4}{9} c_4 = (10, -10), \\
RR_{inn} & : -4a - \frac{8}{9} c_3 + \frac{2}{9} c_4 = \left( \frac{5}{2}, \frac{5}{2} \right), \\
K^{ij} K^{k\ell} R_{ikj} & : 4a + c_5 - 2c_7 + \frac{2}{3} c_8 = \left( \frac{277}{4}, \frac{-97}{4} \right), \\
K_K K^{ij} R_{ij} & : 8a + \frac{2}{3} c_5 - \frac{1}{3} c_6 + 2c_7 - c_8 = \left( -\frac{289}{4}, \frac{109}{4} \right), \\
K K^{ij} R_{ij} & : -8a - \frac{5}{6} c_5 + \frac{1}{6} c_6 = (1, \frac{13}{2}), \\
\text{Tr } K^2 R & : -2a - \frac{12}{12} c_5 + \frac{12}{12} c_6 + \frac{3}{8} c_8 = \left( \frac{-25}{8}, -\frac{5}{8} \right), \\
K^2 R & : 2a + \frac{1}{8} c_5 - \frac{1}{24} c_6 - \frac{5}{48} c_8 = \left( \frac{35}{16}, \frac{-5}{4} \right), \\
K_K K^{ij} R_{injn} & : -8a + c_6 - 2c_7 + \frac{5}{3} c_8 = \left( \frac{499}{4}, \frac{-41}{4} \right), \\
K K^{ij} R_{injn} & : 8a + \frac{1}{2} c_5 - \frac{1}{2} c_6 - \frac{1}{3} c_8 = \left( \frac{-109}{4}, \frac{-101}{4} \right), \\
\text{Tr } K^2 R_{inn} & : 4a - \frac{1}{3} c_6 - c_8 = \left( \frac{25}{8}, \frac{-35}{8} \right), \\
K^2 R_{inn} & : -4a - \frac{1}{6} c_5 + \frac{1}{6} c_6 + \frac{11}{24} c_8 = \left( \frac{-25}{16}, \frac{55}{16} \right), \\
\nabla K_K \nabla^{ij} K^{ij} & : 2c_7 - \frac{1}{3} c_8 = \left( -\frac{555}{8}, \frac{255}{8} \right), \\
\nabla_i K^{ij}_K \nabla^{ij}_K K^{ij} & : -\frac{8}{3} c_7 + \frac{8}{9} c_8 = \left( \frac{165}{2}, \frac{-105}{2} \right), \\
\nabla_i K^{ij}_K \nabla^{ij}_K K_K & : \frac{4}{3} c_7 - \frac{7}{9} c_8 = \left( -\frac{135}{4}, \frac{135}{4} \right), \\
(\nabla K)^2 & : \frac{2}{3} c_7 + \frac{25}{72} c_8 = \left( \frac{285}{16}, \frac{-255}{16} \right), \\
K^{ij} \nabla_{injn} & : \frac{2}{3} c_8 = (-15, -15), \\
K \nabla_n R & : \frac{1}{12} c_8 = \left( \frac{15}{8}, \frac{15}{8} \right).
\end{align*}
\]
The numbers in the parentheses correspond to the (Dirichlet, Robin) boundary conditions, respectively. These 27 equations can be solved consistently to determine uniquely 9 conformal charges. The results have been listed in Table 1.
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