A historical note on the 3/2-approximation algorithm for the metric traveling salesman problem

René van Bevern\textsuperscript{a,*}, Viktoriia A. Slugina\textsuperscript{b,c}

\textsuperscript{a}Department of Mechanics and Mathematics, Novosibirsk State University, Novosibirsk, Russian Federation
\textsuperscript{b}Institute of History of the Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russian Federation
\textsuperscript{c}Humanities Institute, Novosibirsk State University, Novosibirsk, Russian Federation

Abstract

One of the most fundamental results in combinatorial optimization is the polynomial-time 3/2-approximation algorithm for the metric traveling salesman problem. It was presented by Christofides in 1976 and is well known as “the Christofides algorithm”. Recently, some authors started calling it “Christofides-Serdyukov algorithm”, pointing out that it was published independently in the USSR in 1978. We provide some historic background on Serdyukov’s findings and a translation of his article from Russian into English.

Zusammenfassung

Eines der grundlegendsten Resultate auf dem Gebiet der kombinatorischen Optimierung ist der Polynomialzeit-3/2-Approximationsalgorithmus für das metrische Problem des Handlungsreisenden. Er wurde 1976 von Christofides vorgestellt und ist bekannt unter dem Namen „Christofides-Algorithmus“. In letzter Zeit bezeichnen ihn einige Autoren als „Christofides-Serdyukov-Algorithmus” mit dem Hinweis, er sei 1978 unabhängig in der UdSSR publiziert worden. In diesem Artikel beleuchten wir den historischen Hintergrund um Serdyukov’s Entdeckung und liefern eine Übersetzung seines Artikels aus dem Russischen ins Englische.

Аннотация

Одним из самых фундаментальных результатов в области комбинаторной оптимизации является полиномиальный 3/2-приближённый алгоритм для метрической задачи коммивояжёра. Он был представлен Никосом Кристофидесом в 1976 г. и хорошо известен под названием “алгоритм Кристофидеса”. В последнее время некоторые авторы стали назвать его “алгоритмом Кристофидеса-Сердюкова“, утверждая, что он был опубликован независимо в СССР в 1978 г. В этой статье рассматривается исторический контекст появления результата А. И. Сердюкова и приводится перевод его статьи с русского на английский язык.

Keywords: combinatorial optimization, Christofides algorithm, USSR, Novosibirsk Akademgorodok

2010 MSC: 90-03

1. Introduction

One of the most fundamental problems in combinatorial optimization is the traveling salesmen problem, formalized as early as 1832 (cf. Applegate et al., 2006, Chapter 1): given \( n \) cities and their pairwise distances, find a shortest tour to visit each city exactly once and return to the starting point.

Finding the shortest tour is a computationally intractable problem even in the special case where the distances between the cities satisfy the triangle inequality (Garey and Johnson, 1979). Christofides (1976) presented an \( O(n^3) \)-time 3/2-approximation algorithm for this special case: it yields a tour that is at most 3/2 times longer than the shortest one. It is a prime example for approximation algorithms that entered textbooks and encyclopedias as “the Christofides...
algorithm” or “the Christofides heuristic” (Garey and Johnson, 1979; Christofides, 1979; Gutin, 2009; Williamson and Shmoys, 2011; Bläser, 2016).

Quite some efforts have been made trying to improve it (cf. the surveys of Vygen, 2012; Svensson, 2013). One line of research aims for improving its running time: there are many faster heuristics, which cannot guarantee 3/2-approximate solutions (Johnson and McGeoch, 2007), yet (3/2 + \(\varepsilon\))-approximate solutions for any \(\varepsilon > 0\) are computable by a randomized algorithm in \(O(n^2 \log^4 n/\varepsilon^2)\) time (Chekuri and Quanrud, 2017). Another line of research aims for improving the approximation factor, which was successful only in special cases: in polynomial time, one can compute 8/7-approximate solutions if the distances are in \([1, 2]\) (Berman and Karpinski, 2006), 7/5-approximate solutions if the distances are lengths of shortest paths in an unweighted graph (Sebő and Vygen, 2014), and (1 + \(\varepsilon\))-approximate solutions for any fixed \(\varepsilon > 0\) if the cities are points in fixed-dimensional Euclidean (Arora, 1998; Mitchell, 1999) or doubling spaces (Bartal et al., 2016, using a randomized algorithm), or if the distances are lengths of shortest paths in a graph excluding some fixed minor (Demaine et al., 2011).

For general distances satisfying the triangle inequality, the 3/2 approximation factor of Christofides’ algorithm remains the state of the art.\(^1\) Recently, a small but growing group of authors started referring to it as “the Christofides-Serdyukov algorithm” (Burkard et al., 1998; Deineko and Tiskin, 2010; Bérczi et al., 2019; Gutekunst and Williamson, 2019; Sebő and van Zuylen, 2019; Tarnawski, 2019; Traub and Vygen, 2019),\(^2\) claiming that it was independently obtained in the USSR by Serdyukov (1978).

At the one hand, this claim is plausible: in the beginning of the 70s, a lot of research on computationally intractable problems was carried out parallelly in the USSR, leading to independent proofs of seminal results like the Cook-Levin theorem about the NP-completeness of the satisfiability problem for Boolean formulas (Trakhtenbrot, 1984). Moreover, the submission date given in the journal article of Serdyukov (1978), January 27th, 1976, predates the report of Christofides (1976), dated February, 1976. On the other hand, such claims should be treated with caution: for example, the wide-spread claim that Kuratovski’s theorem was earlier proved in the USSR has little support (Kennedy et al., 1985).

We give some historic background on Serdyukov’s findings, which indeed supports the claim of his independent discovery of the 3/2-approximation algorithm and sheds some light on the timely coincidence of the publications of Christofides and Serdyukov. We also provide a translation of Serdyukov’s article in the appendix.

2. Anatoliy I. Serdyukov (1951–2001)

The following information about Serdyukov can be found in Barvinok et al. (2007), Demidenko (2017),\(^3\) the archives of the Department of Mechanics and Mathematics of Novosibirsk State University, and the State Public Scientific Technological Library of the Russian Academy of Sciences.

Anatoliy Ivanovich Serdyukov was born on October 29th, 1951, in Prokopyevsk, a city in Kemerovo region (Western Siberia), USSR. He graduated from Novosibirsk State University in 1973, after which he was employed in the structures of the Siberian Branch of the Lenin Academy of Agricultural Sciences, then at the Institute of Cytology and Genetics of the Siberian Branch of the Academy of Sciences of the USSR (SB AS USSR), and finally at the Institute of Mathematics of the SB AS USSR (now named the Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences), where he was working until his death on February 7th, 2001.

In 1980, already working at the Institute of Mathematics, he was awarded the academic degree of candidate of physico-mathematical sciences. Serdyukov’s (1980) thesis is on the complexity of finding Hamiltonian and Eulerian cycles in graphs. His best known results are approximation algorithms for finding longest traveling salesmen tours (surveyed by Barvinok et al., 2007).

Taking into account his graduation year and the submission date of Serdyukov’s (1978) article, January 27th, 1976, Serdyukov must have obtained his 3/2-approximation algorithm as a young graduate student in about 1975.

\(^1\) Actually, Wolsey (1980) showed that the length of the computed tours is within a factor 3/2 not only of the optimum, but of a lower bound given by the optimal solution to a relaxation of an integer linear programming model.

\(^2\) We deliberately omit articles coauthored by Serdyukov’s former colleagues from this list.

\(^3\) The birth year 1952 given in the book edited by Demidenko (2017) is incorrect.
3. Circulation of Christofides’ result between 1976 and 1979

Authors usually refer to Christofides’ 1976 technical report at Carnegie-Mellon University (CMU) as the source of the 3/2-approximation algorithm for the metric traveling salesman problem, which some authors do not consider as published (cf. Bläser, 2016, who also claims that “Christofides never published his algorithm”).

Apparently, Christofides’ technical report was not known to a wide audience up to 1978. For example, Karp (1977) and Rosenkrantz et al. (1977) refer to Christofides’ abstract in the proceedings of a symposium held at CMU in April 1976. The proceedings were published only in December 1976 (Traub, 1976). Frederickson et al. (1976) refer to the same abstract, whereas later, in the journal version of their article, Frederickson et al. (1978) refer to the technical report. Christofides’ technical report could have been popularized in 1977, when its abstract stating the 3/2-approximation was indexed by the NASA abstract journal Scientific and Technical Aerospace Reports (Christofides, 1977).

Some authors of that time, for example Lenstra and Rinnooy Kan (1979), refer to a journal article of Christofides that is to appear in the journal Mathematical Programming. In a combinatorial optimization textbook, Christofides (1979) describes his algorithm without proving the approximation factor, referring to an article in press in Mathematical Programming for the proof, not mentioning his technical report. Interestingly, according to the archives of Mathematical Programming, his article was not published. The algorithm with complete proof details was published to a wide audience not later than in the seminal textbook of Garey and Johnson (1979).

Summarizing, Serdyukov (1978) submitted his journal article in January 1976, which predates all traces of Christofides’ publications on this topic. Thus, it is plausible that Serdyukov (1978) obtained the result independently.

4. Serdyukov’s work between 1974 and 1978

We give some historic background on the findings of Serdyukov to shed some light on the timely coincidence of the publications of Christofides (1976) and Serdyukov (1978). To this end, it is helpful to interpret the 3/2-approximation algorithm for the traveling salesman problem as follows: A first step computes a minimum-cost spanning tree that connects all the cities. A second step computes a shortest tour in the input graph that traverses the edges of the spanning tree.

The second step is solved using an approach earlier developed for the closely related Chinese postman problem of computing a shortest tour traversing all edges of a graph: Christofides (1973) and Serdyukov (1974), but also Edmonds and Johnson (1973), actively study the Chinese postman problem at that time. They all reduce it to the problem of finding a minimum-cost perfect matching on the complete edge-weighted graph on all odd-degree vertices of the input graph. Surprisingly, while Christofides, Edmonds and Johnson solve the matching problem using the polynomial-time algorithm of Edmonds (1965), Serdyukov reduces it to an exponential number of matching problems in bipartite graphs. Apparently, in 1974 neither Serdyukov nor his reviewers were aware of the work of Christofides (1973), Edmonds and Johnson (1973), or the polynomial-time algorithm for computing maximum-weight matchings in general graphs, published by Edmonds nine years earlier.

Since Serdyukov (1978) uses Edmonds’ algorithm to solve the matching problem in his 3/2-approximation algorithm for the traveling salesman problem but was unaware of it in 1974, he must have learned about Edmonds’ algorithm in 1974 or 1975. One scenario is that he learned about it via the article of Christofides (1973), which Serdyukov (1976) cites in an article studying reductions between matching, covering, the Chinese postman, and the traveling salesman problems. In this scenario, Serdyukov obtained his 3/2-approximation independently of Christofides but because of him. Another scenario is that Serdyukov (1978) learned about Edmonds’ algorithm from Karzanov (1976), whose \(O(n^3 \log n)\)-time implementation of Edmonds’ algorithm he uses in his 3/2-approximation. Karzanov’s article was probably not yet published in January 1976, when Serdyukov submitted his article, but he might have had access to a preliminary copy, which is supported by the fact that the titles given by Serdyukov and Karzanov differ slightly.

---

4 Notably, Serdyukov (1974) explicitly introduces the problem that forty years later is intensively studied as the Eulerian extension problem (Sorge et al., 2011; Höhn et al., 2012; Sorge et al., 2012; Dorn et al., 2013; Gutin et al., 2017; van Bevern et al., upcoming).

5 In contemporary terms of parameterized complexity theory (cf. Cygan et al., 2015), Serdyukov (1974) merely describes a fixed-parameter algorithm for the Chinese postman problem parameterized by the number of odd-degree vertices in the input graph.
5. Conclusion

Our findings support the claim that Serdyukov (1978) discovered the 3/2-approximation algorithm for the metric traveling salesman problem independently of Christofides (1976).

Concerning the timely coincidence of the publications of Christofides and Serdyukov, we conclude that, on the one hand, it was impossible for Serdyukov to find the algorithm much earlier than Christofides, being unaware of Edmonds’ polynomial-time matching algorithm up to 1974. On the other hand, actively working on the Chinese postman before, he found the 3/2-approximation for the traveling salesman problem as soon as he became aware of Edmonds’ algorithm.

An English abstract of Serdyukov’s (1978) article was indexed in zbMATH only in 1982 (Serdyukov, 1982). At this time, “the Christofides algorithm” had already entered fundamental textbooks like that of Garey and Johnson (1979). Moreover, the English abstract does not mention any approximation factors. Thus, it is not surprising that Serdyukov’s result remained largely unknown beyond the USSR.

Acknowledgments. We thank Edward Kh. Gimadi and Oxana Yu. Tsidulko for helpful input.

Funding. René van Bevern is supported by the Mathematical Center in Akademgorodok, agreement No. 075-15-2019-1675 with the Ministry of Science and Higher Education of the Russian Federation. Viktoriia A. Slugina is supported by grant No. 19-39-60006 of the Russian Foundation for Basic Research.

References

Applegate, D. L., Bixby, R. E., Chvátal, V., and Cook, W. J. (2006). The Traveling Salesman Problem — A Computational Study, volume 40 of Princeton Series in Applied Mathematics. Princeton University Press, Princeton, Oxford, USA. doi:10.1515/9781400841103.

Arora, S. (1998). Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. Journal of the ACM 45(5): pp. 753–782. doi:10.1145/290179.290180.

Bartal, Y., Gottlieb, L.-A., and Krauthgamer, R. (2016). The traveling salesman problem: Low-dimensionality implies a polynomial time approximation scheme. SIAM Journal on Computing 45(4): pp. 1563–1581. doi:10.1137/130913328.

Barvinok, A., Gimadi, E. Kh., and Serdyukov, A. I. (2007). The maximum TSP. In G. Gutin and A. P. Punnen, editors, The Traveling Salesman Problem and Its Variations, volume 12 of Combinatorial Optimization, pp. 585–607. Springer, Boston, USA. doi:10.1007/0-306-48213-4_12.

Berman, P. and Karpinski, M. (2006). 8/7-approximation algorithm for (1,2)-TSP. In Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm, p. 641–648. SIAM, USA.

van Bevern, R., Fluschnik, T., and Tsidulko, O. Yu. (upcoming). On approximate data reduction for the rural postman problem: Theory and experiments. Networks. Accepted for publication, https://arxiv.org/abs/1812.10131.

Bläser, M. (2016). Metric TSP. In M.-Y. Kao, editor, Encyclopedia of Algorithms, pp. 1276–1279. Springer, New York, USA. doi:10.1007/978-1-4939-2864-4_230.

Burkard, R. E., Deineko, V. G., and Woeginger, G. J. (1998). The travelling salesman and the PQ-tree. Mathematics of Operations Research 23(3): pp. 613–623. doi:10.1287/moor.23.3.613.

Bérczi, K., Berger, A., Mnich, M., and Vincze, R. (2019). Degree-bounded generalized polymatroids and approximating the metric many-visits TSP. ArXiv preprint, http://arxiv.org/abs/1911.08990.

Chekuri, C. and Quanrud, K. (2017). Approximating the Held-Karp bound for metric TSP in nearly-linear time. In 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2017), pp. 789–800. IEEE Computer Society, Los Alamitos, California, USA. doi:10.1109/FOCS.2017.78.

Christofides, N. (1973). The optimum traversal of a graph. Omega 1(6): pp. 719–732. doi:10.1016/0305-0483(73)90089-3.

Christofides, N. (1976). Worst-case analysis of a new heuristic for the traveling salesman problem. Management Science Research Report 388, Carnegie-Mellon University, Pittsburgh, Pennsylvania, USA.

Christofides, N. (1977). Worst-case analysis of a new heuristic for the traveling salesman problem. Scientific and Technical Aerospace Reports 15(1): p. 389. Accession number N77-12799.

Christofides, N. (1979). The travelling salesman problem. In N. Christofides, A. Mingozzi, P. Toth, and C. Sandi, editors, Combinatorial Optimization, pp. 131–149. John Wiley & Sons, Chichester, New York, Brisbane, Toronto.

Cygan, M., Fomin, F. V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., and Saurabh, S. (2015). Parameterized Algorithms. Springer, Cham, Switzerland. doi:10.1007/978-3-319-21275-3.

Deineko, V. and Tiskin, A. (2010). Fast minimum-weight double-tree shortcutting for metric TSP: Is the best one good enough? Journal of Experimental Algorithmics 14: art. 6. doi:10.1145/1498698.1594232.

Demaine, E. D., Hajiaghayi, M., and Kawarabayashi, K. (2011). Contraction decomposition in H-minor-free graphs and algorithmic applications. In STOC’11: Proceedings of the 43rd ACM Symposium on Theory of Computing, p. 441–450. ACM, New York, USA. doi:10.1145/1993636.1993696.

Demudenko, G. V., editor (2017). Oscheri ob Institute matematik im. S. L. Soboleva. Institute of Mathematics, Novosibirsk, Russian Federation.

Dorn, F., Moser, H., Niedermeier, R., and Weller, M. (2013). Efficient algorithms for Eulerian Extension and Rural Postman. SIAM Journal on Discrete Mathematics 27(1): pp. 75–94. doi:10.1137/110834810.

Edmonds, J. (1965). Maximum matching and a polyhedron with 0,1-vertices. Journal of Research of the National Bureau of Standards—B. Mathematics and Mathematical Physics 69: pp. 125–130. doi:10.6028/jres.069B.013.

Edmonds, J. and Johnson, E. L. (1973). Matching, Euler tours and the Chinese postman. Mathematical Programming 5: pp. 88–124. doi:10.1007/BF01580113.
Frederickson, G. N., Hecht, M. S., and Kim, C. E. (1976). Approximation algorithms for some routing problems. In *17th Annual Symposium on Foundations of Computer Science*, pp. 216–227. IEEE Computer Society, Long Beach, California, USA. doi:10.1109/SFCS.1976.6.

Frederickson, G. N., Hecht, M. S., and Kim, C. E. (1978). Approximation algorithms for some routing problems. *SIAM Journal on Computing* 7(2): pp. 178–193. doi:10.1137/0207017.

Garey, M. R. and Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, New York, USA.

Gutekunst, S. C. and Williamson, D. P. (2019). Semidefinite programming relaxations of the traveling salesman problem and their integrality gaps. ArXiv preprint, http://arxiv.org/abs/1907.09054.

Gutin, G. (2009). Traveling salesman problem. In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of Optimization*, pp. 3935–3944. Springer, Boston, USA. doi:10.1007/978-0-387-74759-0_687.

Gutin, G., Wahlström, M., and Yeo, A. (2017). Rural Postman parameterized by the number of components of required edges. *Journal of Computer and System Sciences* 83(1): pp. 121–131. doi:10.1016/j.jcss.2016.06.001.

Höhn, W., Jacobs, T., and Megow, N. (2012). On Eulerian extensions and their application to no-wait flowshop scheduling. *Journal of Scheduling* 15(3): pp. 295–309. doi:10.1007/s10951-011-0241-1.

Johnson, D. S. and McGeoch, L. A. (2007). Experimental analysis of heuristics for the STSP. In G. Gutin and A. P. Punnen, editors, *The Traveling Salesman Problem and Its Variations*, pp. 369–443. Springer, Boston, USA. doi:10.1007/978-0-306-48213-4_9.

Karp, R. M. (1977). Probabilistic analysis of partitioning algorithms for the traveling-salesman problem in the plane. *Mathematics of Operations Research* 2(3): pp. 209–224. doi:10.1287/moor.2.3.209.

Karzanov, A. V. (1976). Ekonomnye realizatsii algoritmov Edmondsa dlya nakhozhdeniya maksimal'noi mogushchosti i maksimal'nogo nogo vesa. In A. A. Fridman, editor, *Issledovaniya po diskretnomu optimal'nymu programmированию*, pp. 306–327. Nauka, Moscow, USSR.

Kennedy, J. W., Quintas, L. V., and Sysło, M. M. (1985). The theorem on planar graphs. *Historia Mathematica* 12(4): pp. 356–368. doi:10.1016/0315-0860(85)90045-X.

Lenstra, J. K. and Rinnooy Kan, A. H. G. (1979). Computational complexity of discrete optimization problems. In P. L. Hammer, E. L. Johnson, and B. Peleg, editors, *Combinatorial Optimization*, volume 1 of *Annals of Discrete Mathematics*, pp. 121–140. North-Holland Publishing Company, Amsterdam, New York, Oxford. doi:10.1016/S0167-5060(08)78021-5.

Mitchell, J. S. B. (1999). Guillotine subdivisions approximate polygonal subdivisions: A simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems. *SIAM Journal on Computing* 28(4): pp. 1298–1309. doi:10.1137/S0097539796307964.

Rosenkrantz, D. J., Stearns, R. E., and Lewis, P. M., II (1977). An analysis of several heuristics for the traveling salesman problem. *SIAM Journal on Computing* 6(3): pp. 563–581. doi:10.1137/0206041.

Serdyukov, A. I. (1976). O vzaimnoi svodimosti nekotorykh ekstremal'nykh zadach teorii grafov. *Issledovaniya po diskretnoi optimizatsii* 12: pp. 61–67. http://nas1.math.nsc.ru/aim/journals/us/us12/us12_008.pdf.

Serdyukov, A. I. (1977). O nekotorykh ekstremal'nykh obkhodakh v grafakh. *Upravlyaemye sistemy* 12: pp. 61–67. http://nas1.math.nsc.ru/aim/journals/us/us12/us12_008.pdf.

Serdyukov, A. I. (1974). O zadache nakhozhdeniya minimal'nogho Eulerova mu tigrafa dlya svyaznogo grafova so vzvshennymi rebrami. *Upravlyaemye sistemy* 15: pp. 69–73. http://nas1.math.nsc.ru/aim/journals/us/us15/us15_005.pdf.

Serdyukov, A. I. (1978). O nekotorykh ekstremal'nykh obkhodakh v grafakh. *Upravlyaemye sistemy* 17: pp. 76–79. http://nas1.math.nsc.ru/aim/journals/us/us17/us17_007.pdf.

Serdyukov, A. I. (1980). *Stochastichnost' otsykaniya gamiltonovykh i eilerovykh marshrutov v grafakh*. Abstract of thesis of a candiate of physico-mathematical sciences, Academy of Sciences of the USSR, Siberian Branch, Institute of Mathematics, Novosibirsk, USSR.

Serdyukov, A. I. (1982). On some extremal by-passes in graphs. Zentralblatt für Mathematik und ihre Grenzbereiche (zbMATH) 475: p. 520. Accession number 0473.90080.

Sorge, M., van Bevern, R., Niedermeier, R., and Weller, M. (2011). From few components to an Eulerian graph by adding arcs. In P. Kolman and J. Kratochvíl, editors, *Proceedings of WG 2011: Graph-Theoretic Concepts in Computer Science*, volume 6986 of *Lecture Notes in Computer Science*, pp. 307–318. Springer, Berlin, Heidelberg, Germany. doi:10.1007/978-3-642-25870-1_28.

Sorge, M., van Bevern, R., Niedermeier, R., and Weller, M. (2012). A new view on Rural Postman based on Eulerian Extension and Matching. *Journal of Discrete Algorithms* 16: pp. 12–33. doi:10.1016/j.jda.2012.04.007.

Svensson, O. (2013). Overview of new approaches for approximating TSP. In A. Brandstädt, K. Jansen, and R. Reischuk, editors, *WG 2013: Graph-Theoretic Concepts in Computer Science*, pp. 5–11. Springer, Berlin, Heidelberg, Germany. doi:10.1007/978-3-642-45043-3_2.

Tarnawska, J. (2019). *New Graph Algorithms via Polyhedral Techniques*. Ph.D. thesis, Ecole Polytechnique Fédérale de Lausanne, Switzerland.

Trakhtenbrot, B. A. (1984). A survey of Russian approaches to perebor (brute-force search) algorithms. *IEEE Annals of the History of Computing* 6(4): p. 384–400. doi:10.1109/MAHC.1984.10036.

Traub, J. F., editor (1976). *Algorithms and complexity: new directions and recent results*. Academic Press.

Traub, V. and Vyen, J. (2019). Approaching 3/2 for the s-t-path TSP. *Journal of them ACM* 66(2): art. 14. doi:10.1145/3309715.

Vyen, J. (2012). New approximation algorithms for the TSP. *OPTIMA Mathematical Optimization Society Newsletter* 90: pp. 1–13. http://www.mathopt.org/Optima-Issues/optima90.pdf.
Appendix

The following is a translation of the Russian article of A. I. Serdyukov, O nekotorykh ekstremal’nykh obkhodakh v grafakh, *Upravlyaemye sistemy* 17:76–79, 1978.

On some extremal walks in graphs

A. I. Serdyukov

1. Let \( G = (X, U) \) be an undirected \( n \)-vertex graph with edge weights \( \rho_{ij}, u_{ij} \in U, 1 \leq i, j \leq n \), where the \( \rho_{ij} \) are positive real numbers. Moreover, let there be a real number \( k \geq 1 \). By \( \mathcal{L} \) we denote the set of Hamiltonian cycles in the graph \( G \), and by \( \mathcal{M} \) the set of cycles that contain all vertices of \( G \). We consider two extremum problems.

**Problem A.** Find an element \( L^* \in \mathcal{L} \) with the property

\[
\rho(L^*) = \sum_{u_{ij} \in L^*} \rho_{ij} \leq k \cdot \min_{L \in \mathcal{L}} \rho(L).
\]

Note that, for \( k = 1 \), Problem A is nothing else but the traveling salesman problem.

**Problem B.** Find an element \( M^* \in \mathcal{M} \) with the property

\[
\rho(M^*) = \sum_{u_{ij} \in M^*} \rho_{ij} \leq k \cdot \min_{M \in \mathcal{M}} \rho(M).
\]

(herewith we take into account the multiplicity that each edge appears on the cycle).

For \( k = 1 \), Problems A and B are NP-complete [1]. Moreover, as shown in [2], Problem A is NP-complete for arbitrary \( k \geq 1 \). In [3], an algorithm with polynomial running time is presented for Problem A with \( k = 2 \) in complete undirected graphs whose edge weights respect the triangle inequality.

In this work, we present an algorithm for Problem A with \( k = \frac{3}{2} \) in the very same class of graphs, whose complexity is \( O(n^3 \ln n) \) operations.

Terminology related to graph theory used in this work can be found in [4].

2. Assume that the edge weights of a complete undirected \( n \)-vertex graph \( G \) satisfy the triangle inequality:

\[
\rho_{ij} \leq \rho_{ik} + \rho_{kj}, \quad 1 \leq i, j, k \leq n.
\]

We introduce some necessary notation:

- \( L_0 \) — a minimum Hamiltonian cycle in graph \( G \);
- \( M_0 \) — the shortest cycle containing all vertices of graph \( G \);
- \( \tilde{D}_0 \) — the shortest spanning tree in graph \( G \);
- \( X_1 \subseteq X \) — the set of odd-degree vertices in the tree \( \tilde{D}_0 \);
- \( \tilde{G} = (X_1, \tilde{U}) \) — the complete subgraph of \( G \) with vertex set \( X_1 \);
- \( L_0 \) — a minimum Hamiltonian cycle in \( \tilde{G} \);
- \( w_0 \) — a minimum perfect matching in graph \( \tilde{G} \).

We prove the following theorem.

**Theorem 1.** Independently of the edge weights of the input graph \( G \), which satisfy (1), the following relations hold:

\[
\rho(L_0) = \rho(M_0),
\]

\[
\rho(L_0) \geq \rho(L_0),
\]

\[
\rho(w_0) \leq \frac{1}{2}\rho(L_0).
\]
The proof of equality (2) can be found in [5] (cf. Lemma 1). Inequality (4) follows from the fact that graph $\tilde{G}$ has an even number of vertices (since the number of odd-degree vertices in a graph is even [4]) and that a Hamiltonian cycle can be partitioned into two edge-disjoint perfect matchings. We prove inequality (3). To this end, we write the Hamiltonian cycle $L_0$ as a sequence of vertices:

\[ \{x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_1\}. \] (5)

If $X_1 = X$, then inequality (3) is proven. Let $X_1 \subseteq X$. In this case, there must be vertices $x_i \in X_1, x_j \in X_1, i < j - 1$ such that $x_{i+1} \notin X_1, 1 \leq s < j - i$. Then, replacing the path between vertices $i$ and $j$ in the Hamiltonian cycle $L_0$ by the edge $u_{ij}$, we obtain a simple cycle visiting all vertices in $X_1$ with a weight not exceeding $\rho(L_0)$. This is possible since the graph $G$ is complete and (1) holds. By executing this process for all pairs of such vertices in the sequence (5), we obtain a Hamiltonian cycle in the graph $\tilde{G}$ of weight not exceeding $\rho(L_0)$. Theorem 1 is proved.

3. We now describe the algorithm for Problem A in the class of complete undirected graphs whose edge weights satisfy (1). The algorithm consists of five steps.

First step. Compute a minimum spanning tree $D_0$ in the graph $G$. To this end, one can use the algorithm of Prim [6], whose complexity is $O(n^2)$ operations.

Second step. Find all odd-degree vertices in the tree $D_0$, which form the set $X_1 \subseteq X$. Then build the subgraph $\tilde{G} = (X_1, \tilde{U})$ described above.

Third step. Find the perfect matching $w_0$ in graph $G$. To find such a perfect matching, one can use the algorithm described in [7], whose complexity is $O(n^3 \ln n)$ operations. (In fact, the algorithm in [7] solves the problem of finding a maximum-weight matching in a graph.) To reduce the problem of finding a minimum-weight perfect matching in the graph $\tilde{G}$ to this problem, it is enough to assign the edges of $G$ the weights $\rho_{ij} = 2a - \rho_{ij}, u_{ij} \in \tilde{U}$, where $a = \max_{u_{ij} \in \tilde{U}} \rho_{ij}$.) Note that the edge set $D_0 \cup w_0$ forms an Eulerian graph (taking into account multi-edges that may appear when $D_0 \cap w_0 \neq \emptyset$).

Fourth step. Find an Eulerian walk of all edges of the graph $D_0 \cup w_0$ using the algorithm described in [4].

Fifth step. Write the Eulerian walk for the edges in the set $D_0 \cup w_0$ as a sequence of vertices

\[ \{x_1, x_2, x_3, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_1\}. \] (6)

If there are no vertex repetitions in this sequence, then $D_0 \cup w_0$ is a Hamiltonian cycle. Then we set $L^* = D_0 \cup w_0$. In this case the algorithm stops. Assume that some vertex, without loss of generality $x_{k+1}$, is repeated in the sequence (6). Then we transform (6) into a cycle $\{x_1, x_2, x_3, \ldots, x_{k-1}, x_k, \ldots, x_1\}$ without increasing its weight. This is possible because $G$ is a complete graph whose edge weights satisfy (1). Executing this process for each vertex of graph $\tilde{G}$ as often as it is repeated in (6), we obtain a Hamiltonian cycle $L_1$. Set $L^* = L_1$. This concludes the description of the algorithm.

Theorem 2. The weight of the Hamiltonian cycle $L^*$ constructed by the described algorithm in the graph $G$ satisfies the inequality

\[ \rho(L^*) \leq \frac{3}{2} \rho(L_0). \] (7)

Proof. To prove this, it is enough to notice that $\rho(L^*) \leq \rho(D_0) + \rho(w_0) < \rho(L_0) + \rho(w_0)$ and to use Theorem 1. \(\square\)

4. We now consider Problem B in an arbitrary connected graph $\tilde{G} = (X, \tilde{U})$, the edges of which have weights $\tilde{\rho}_{ij}, u_{ij} \in \tilde{U}, 1 \leq i, j \leq n$, where $\tilde{\rho}_{ij}$ are positive real numbers. For $\tilde{G}$ we build the shortest-path graph $\tilde{G} = (X, \tilde{U})$, which is complete and whose edge weights satisfy (1) (cf. [5]). By $\tilde{M}_0$ denote the shortest cycle containing all vertices of $\tilde{G}$. Then we have the following equality [5]:

\[ \tilde{\rho}(\tilde{M}_0) = \rho(L_0). \] (8)

By replacing each edge $u_{ij} \in L^*$ by a shortest path between the vertices $i$ and $j$ in the graph $\tilde{G}$, we find some cycle $\tilde{M}^*$ containing all vertices of graph $\tilde{G}$, whose weight equals $\rho(L^*)$. Moreover, taking into account (7), (8), we obtain the following inequality:

\[ \tilde{\rho}(\tilde{M}^*) \leq \frac{3}{2} \tilde{\rho}(\tilde{M}_0). \]
This last inequality means that one can use $\tilde{M}^*$ as a solution to Problem B in the graph $\tilde{G}$.

For the complexity analysis of the algorithm, it is enough to note that each of its five steps requires no more than $Cn^3 \ln n$ operations. All subsequent computations related to finding a solution for Problem B can be executed using the algorithm of Dijkstra [8] in $O(n^3)$ operations. Thus, the number of operations required for realizing the described algorithms is proportional to $n^3 \ln n$.

Received by the editorial-publishing office
January 27, 1976

Literature

1. Karp R. M., Slozhnost’ resheniya ekstremal’nykh kombinatornykh problem. Kiberneticheskii sbornik, volume 12, Moscow, Mir, 1975.
2. Sahni S. and Gonzales T., P-complete approximation problems. Journal of Association for Computing Machinery, July 1976, No. 3, volume 23, pp. 555–565.
3. Rosenkrantz D. J., Stearns R. E. and Lewis P. M., Approximate algorithms for the travelling salesperson problem. 15th Annual IEEE Symp. on Switching and Automata Theory, 1974, pp. 33–42.
4. Berzh K., Teoriya grafov i ee prilozhenie. Moscow, IL, 1962.
5. Serdnyukov A. I., O vzaimnoi svodimosti nekotorykh ekstremal’nykh zadach teorii grafov. Upravlyaemye sistemy, No. 15, Novosibirsk, 1976, pp. 68–73.
6. Prim R. M., Kratchaishchtev vychyayuchchie seti i nekotorye oboshcheniya. Kiberneticheskii sbornik, volume 2, Moscow, Mir, 1961.
7. Karzanov A. V., Ekonomnye realizatsii algoritmov Edmondsa nakhozhdeniya parosochetanii maksimal’noi moshchnosti (maksimal’nogo vesa). In Issledovaniya po Diskretnoi Optimizatsii, Moscow, Mir, 1976, pp. 306–327.
8. Khu T., Tselochislennoe programmirovanie i potoki v setyakh. Moscow, Mir, 1974.