GLOBAL STRONG SOLUTION TO A THERMODYNAMIC COMPRESSIBLE DIFFUSE INTERFACE MODEL IN 1-D

YAZHOU CHEN, QIAOLIN HE, BIN HUANG, AND XIAODING SHI

Abstract. In this paper, we investigate the wellposedness of the heat-conducting Navier-Stokes/Allen-Cahn system, which is formed by nonlinear strong coupling of Navier-Stokes equations and Allen-Cahn equation. This system describes the flow of a two-phase inmiscible heat-conducting viscous compressible mixture, the interface of inmiscible two-phase flow is regarded as a thin diffusion layer between different fluids represented by the phase field function. The phases are allowed to shrink or grow due to changes of density in the fluid and incorporates their transport with the current. We established the global existence and uniqueness of strong solutions for this system in 1-D, which means no vacuum, phase separation, shock wave, mass or heat concentration will be developed in finite time, although the motion of the two-phase inmiscible flow has large oscillations and the interaction between the hydrodynamic and phase-field effects is complex.

1. Introduction

The inmiscible two-phase flows appear widely in many fields, such as thermal power engineering, nuclear energy engineering, cryogenic engineering and aerospace, etc. An important feature of inmiscible two-phase flow is the coexistence of two fluids with different phase states or components and the existence of the interface. The theoretical analysis of the inmiscible two-phase flow is much more difficult than that of single-phase flow. A common treatment method is the so-called separated flow model, which holds that the concept and method of single-phase flow can be applied to each phase of the two-phase flow system, and the interface properties and motion between the two phases are considered.

Understanding the geometry and distribution of the interface is very important for determining the flow of inmiscible two-phase flow. The treatment of such two-phase flow interface is derived from the idea of physicist J.D. Van der Waals \cite{29}, who regarded the interface of inmiscible two-phase flow as a region with a certain thickness. Mathematical models based on this idea are often called diffusion interface models, such as the famous Navier-Stokes/Allen-Cahn system, which can be used to study the inmiscible two-phase flow, such as phase transformation, chemical reactions, etc. see \cite{1}–\cite{15}, \cite{23}, \cite{25}, and the references therein. In these literatures, by introducing diffusion interface instead of sharp interface, the authors can avoid the difficulty of processing interface boundary conditions.

Here is a brief review of the model. For compressible inmiscible two-phase flow, taking any one of the volume particles in the flow, we assume $M_i$ the mass of the components in the representative material volume $V$, $\phi_i = \frac{M_i}{\rho}$ the mass concentration, $\rho_i = \frac{M_i}{V}$ the

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\end{itemize}
apparent mass density of the fluid \( i \) \((i = 1, 2)\). The total density is given by \( \rho = \rho_1 + \rho_2 \), and \( \phi = \phi_1 - \phi_2 \). We call \( \phi \) the difference of the two components for the fluid mixture. Obviously, \( \phi \) describes the distribution of the interface. Therefore, on this basis, Blesgen [6], Heida-Málek-Rajagopal [15] coupled the Navier-Stokes system describing the flow of the single fluid and the Allen-Cahn equation describing the change of the phase field (Allen-Cahn [1]), the Navier-Stokes/Allen-Cahn system is proposed (hereinafter called as NSAC equations):

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) &= \text{div}\mathbb{T}, \\
(\rho \phi)_t + \text{div}(\rho \phi \mathbf{u}) &= -\mu, \\
\rho \mu = \rho \frac{\partial f}{\partial \phi} - \text{div}(\rho \frac{\partial f}{\partial \nabla \phi}), \\
(\rho E)_t + \text{div}(\rho E \mathbf{u}) &= \text{div}(\mathbb{T} \mathbf{u} + \kappa(\theta) \nabla \theta - \mu \frac{\partial f}{\partial \nabla \phi}),
\end{align*}
\]

(1.1)

where \( \mathbf{x} \in \Omega \subset \mathbb{R}^N \), \( N \) is the spatial dimension, \( t > 0 \) is the time. The unknown functions \( \rho(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t), \phi(\mathbf{x}, t), \theta(\mathbf{x}, t) \) denote the total density, the velocity, the difference of the two components for the fluid mixture, and the absolute temperature respectively. \( \mu(\mathbf{x}, t) \) is called the chemical potential of the fluid. \( \epsilon > 0 \) is the thickness of the diffuse interface. \( \text{div} \) and \( \nabla \) are the divergence operator and gradient operator respectively. The Cauchy stress-tensor is represented by

\[
\mathbb{T} = 2\nu \mathbb{D}(\mathbf{u}) + \lambda (\text{div}\mathbf{u}) \mathbb{I} - p \mathbb{I} - \rho \nabla \phi \otimes \frac{\partial f}{\partial \nabla \phi},
\]

(1.2)

where \( f \) is the fluid-fluid interfacial free energy density, and it has the following form (Lowengrub-Truskinovsky [25], Heida-Málek-Rajagopal [15]):

\[
f(\rho, \phi, \nabla \phi) \triangleq \frac{1}{4\epsilon}(1 - \phi^2)^2 + \frac{\epsilon}{2\rho} |\nabla \phi|^2,
\]

(1.3)

\( \mathbb{I} \) is the unit matrix, \( \mathbb{D} \mathbf{u} \) is the so-called deformation tensor

\[
\mathbb{D} \mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla^\top \mathbf{u}),
\]

(1.4)

here and hereafter, superscript \( \top \) denotes the transpose and all vectors are column ones. \( \nu > 0, \lambda > 0 \) are viscosity coefficients, satisfying

\[
\nu > 0, \quad \lambda + \frac{2}{N} \nu \geq 0.
\]

(1.5)

The total energy density \( \rho E \) is given by

\[
\rho E = \rho e + \rho f + \frac{1}{2} \rho \mathbf{u}^2,
\]

(1.6)

where \( \rho e \) is the internal energy, \( \frac{\rho \mathbf{u}^2}{2} \) is the kinetic energy, \( p = p(\rho, \theta) \), \( e = e(\rho, \theta) \) and \( f = f(\rho, \phi, \nabla \phi) \) obey the second law of thermodynamics (Lions [24]):

\[
ds = \frac{1}{\theta} (d(e + f) + p d(\frac{1}{\rho}))
\]

(1.7)
where \( s \) is the entropy. Then we deduce from (1.7):
\[
\frac{\partial s}{\partial \theta} = \frac{1}{\theta} \frac{\partial (e + f)}{\partial \theta}, \quad \frac{\partial s}{\partial \rho} = \frac{1}{\theta} \left( \frac{\partial (e + f)}{\partial \rho} - \frac{p}{\rho^2} \right),
\] (1.8)
which implies the following compatibility equation
\[
p = \rho^2 \frac{\partial (e + f)}{\partial \rho} + \theta \frac{\partial p}{\partial \theta} = \rho^2 \frac{\partial e}{\partial \rho} - \frac{\epsilon}{2} |\nabla \phi|^2 + \theta \frac{\partial p}{\partial \theta}.
\] (1.9)
Throughout the paper, we concentrate on ideal polytropic gas, that is, \( p \) satisfies
\[
p(\rho, \theta) = R\rho \theta - \frac{\epsilon}{2} |\nabla \phi|^2,
\] (1.10)
and \( e \) satisfies
\[
e = c_v \theta + \text{constant},
\] (1.11)
where \( c_v \) is the specific heat capacity. Substituting (1.2), (1.6), (1.3), (1.9) and (1.11) into (1.1), we get
\[
\begin{align*}
\rho \rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho (u \cdot \nabla) u - 2\nu \div \mathbb{D} u - \lambda \nabla \div u &= -\div \left( \epsilon \nabla \phi \otimes \nabla \phi - \frac{\epsilon}{2} |\nabla \phi|^2 + \theta \frac{\partial p}{\partial \theta} \right), \\
\rho \phi_t + \rho u \cdot \nabla \phi &= -\mu, \\
\rho \mu &= \frac{\rho}{\epsilon} (\phi^3 - \phi) - \epsilon \Delta \phi, \\
c_v (\rho \theta_t + \rho u \cdot \nabla \theta) + \theta p_0 \div u - \div (\kappa (\theta) \nabla \theta) &= 2\nu |\mathbb{D} u|^2 + \tilde{\lambda} (\div u)^2 + \mu^2.
\end{align*}
\] (1.12)
Moreover, on this basis, the compressible heat-conducting NSAC system (1.12) in one dimensional space becomes
\[
\begin{align*}
\rho_t + (\rho u)_\tilde{x} &= 0, \\
\rho u_t + \rho uu_\tilde{x} + (R\rho \theta)_\tilde{x} &= \nu u_{\tilde{x}\tilde{x}} - \frac{\epsilon}{2} (\phi^2)_\tilde{x}, \\
\rho \phi_t + \rho \phi_\tilde{x} &= -\mu, \\
\rho \mu &= \frac{\rho}{\epsilon} (\phi^3 - \phi) - \epsilon \phi_{\tilde{x}\tilde{x}}, \\
c_v (\rho \theta_t + \rho u \theta_\tilde{x}) + R\rho \theta u_\tilde{x} - \kappa \theta_{\tilde{x}\tilde{x}} &= \nu u^2_\tilde{x} + \mu^2,
\end{align*}
\] (1.13)
for \((\tilde{x}, t) \in [0, 1] \times [0, +\infty)\), where \( \nu = 2\tilde{\nu} + \tilde{\lambda} > 0 \). We consider (1.13) with the natural boundary condition for the concentration, no-slip boundary condition for viscous fluids, the adiabatic boundary condition for temperature
\[
(u, \phi_\tilde{x}, \theta_\tilde{x})(0, t) = (u, \phi_\tilde{x}, \theta_\tilde{x})(1, t) = 0,
\] (1.14)
and the initial value condition
\[
(\rho, u, \theta, \phi)|_{t=0} = (\rho_0, u_0, \theta_0, \phi_0), \quad \tilde{x} \in (0, 1).
\] (1.15)
Without loss of generality, we assume that
\[
\int_0^1 \rho_0(\tilde{x}) d\tilde{x} = 1.
\]
In Lagrange coordinates
\[
x = \int_0^{\tilde{x}} \rho(\xi, t) d\xi,
\] (1.16)
the system (1.13)–(1.15) can be rewritten as

\[
\begin{align*}
\frac{v_t}{v_x} - u_x &= 0, \\
u_t + (R\theta \frac{\theta}{v})_x &= \left(\nu \frac{u_x}{v} - \frac{\varepsilon}{2} \frac{\phi_x^2}{v_x^2}\right)_x, \\
\phi_t &= -v\mu, \\
\mu &= \frac{1}{\varepsilon} (\phi^3 - \phi) - \varepsilon \left(\frac{\phi_x}{v}\right)_x, \\
\varepsilon_{\nu} \theta_t + \frac{R\theta \varepsilon_{\nu}}{v} u_x - \left(\kappa \frac{\theta_x}{v}\right)_x &= \nu \frac{u_x^2}{v} + v\mu^2,
\end{align*}
\] (1.17)

with the boundary condition

\[(u, \phi_x, \theta_x)(0, t) = (u, \phi_x, \theta_x)(1, t) = 0, \quad t \geq 0.\] (1.18)

and the initial value condition

\[(v, u, \theta, \phi)(x, 0) = (v_0, u_0, \theta_0, \phi_0)(x), \quad x \in (0, 1),\] (1.19)

where \(v = \frac{1}{\rho}\).

There are a lot of works on the research of the global existence and large time behavior of solutions to the compressible heat-conducting Navier-Stokes system. Kazhikhov-Shelukhin [21] first proposed the global existence of solutions with large initial data. In the year since, significant progress has been made, see [2–5], [8], [18], [19], [22] and the references therein.

For the isentropic case of the problem (1.1), Feireisl-Petzeltová-Rocca-Schimperna [13] proved the global existence of a weak solution based on the renormalization weak solution framework for the compressible Navier-Stokes system introduced by Lions [24]. Kotschote [23] showed the existence and uniqueness of local strong solutions for arbitrary initial data. Ding-Li-Lou [11] established the existence and uniqueness of global strong solution in 1D for initial density without vacuum states. Chen-Guo [10] generalized the result of Ding-Li-Lou [11] to the case that the initial vacuum is allowed.

So far as we know, there are only a few studies on compressible non-isentropic two-phase immiscible mixture, and all of them confined to incompressible fluids, see Sun-Liu-Xu [28], Wu [30], Eleuteria-Roccabc-Schimperna [12] and the reference therein. In this paper, we focus on the compressible non-isentropic flow for two-phase immiscible mixture. Our purpose is to study the existence and uniqueness of global strong solution for the non-isentropic NSAC systems (1.17) even with large initial data. More specifically, for general initial conditions without vacuum state, we study the global existence of the solution for the system (1.17)–(1.19). Moreover we show that the phase variable \(\phi\) belongs to the physical interval \([-1, 1]\). Some new techniques are developed to establish the up and low bounds of the density \(v\) and \(\theta\). Now we give our main result as following

**Theorem 1.1.** Assume that

\[(v_0, \theta_0) \in H^1(0, 1), \quad \phi_0 \in H^2(0, 1), \quad u_0 \in H^1_0(0, 1),\] (1.20)

and

\[
\inf_{x \in (0, 1)} v_0(x) > 0, \quad \inf_{x \in (0, 1)} \theta_0(x) > 0, \quad \phi_0(x) \in [-1, 1].\] (1.21)
Then the initial boundary value problem (1.17)-(1.19) has a unique strong solution \((v, u, \theta, \phi)\) such that for fixed \(T > 0\), satisfying

\[
\begin{align*}
v, \theta &\in L^\infty(0, T; H^1(0, 1)), \quad u, \phi_x \in L^\infty(0, T; H^1_0(0, 1)), \\
v_t &\in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \\
\phi_t &\in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \\
u_t, \theta_t, \phi_{xt}, u_{xx}, \theta_{xx}, \phi_{xxx} &\in L^2((0, 1) \times (0, T)).
\end{align*}
\]  

(1.22)

Moreover, there exist a positive constant \(C\) depending on the initial data and \(T\), satisfying

\[
C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C, \quad \phi(x, t) \in [-1, 1]. 
\]

(1.23)

Remark 1.1. Our result can be regarded as a natural generalization of Ding-Li-Luo’s result \([7]\) for the isentropic case to non-isentropic case.

Remark 1.2. The global existence and uniqueness of strong solutions for NSAC system means no vacuum, phase separation, shock wave, mass or heat concentration will be developed in finite time, although the motion of the two-phase immiscible flow has large oscillations and the interaction between the hydrodynamic and phase-field effects is complex.

Now we give some notes on the proof of the main theorem. The key to the proof is to get the upper and lower bounds of \(v, \theta\) and \(\phi\), see (2.16), (2.23), (2.72). Inspired by the idea of Kazhikhov [20], we first obtain a key expression of \(v\), see (2.8), combining with the energy inequality (2.2), Jensen’s inequality, we obtain the lower bound of \(v\) and \(\theta\). Then after getting the upper and lower bounds on \(\phi\), observing that \(\max_{x \in [0, 1]} (\frac{\phi}{\epsilon})^2(x, t)\) can be bounded by \(C(\max_{x \in [0, 1]} \theta(x, t) + 1 + V(t))\) (see (2.30)), we obtain the upper bound of \(v\) by the expression of \(v\). Further, after observing that the inequalities (2.33) and (2.50), we get the estimates on the \(L^\infty(0, T; L^2)\)-norm of \(\theta_x\), and the upper bound of temperature \(\theta\) is achieved.

2. The Proof of Theorem

The following Lemma is the existence and uniqueness of local strong solutions which can be obtained by the fixed point method.

Lemma 2.1. Let (1.20) and (1.21) hold. Then there exists some \(T_s > 0\) such that the initial boundary value problem (1.17)-(1.19) has a unique strong solution \((v, u, \theta, \phi)\) satisfying

\[
\begin{align*}
v, \theta &\in L^\infty(0, T_s; H^1(0, 1)), \quad u, \phi_x \in L^\infty(0, T_s; H^1_0(0, 1)), \\
v_t &\in L^\infty(0, T_s; L^2(0, 1)) \cap L^2(0, T_s; H^1(0, 1)), \\
\phi_t &\in L^\infty(0, T_s; L^2(0, 1)) \cap L^2(0, T_s; H^1(0, 1)), \\
u_t, \theta_t, \phi_{xt}, u_{xx}, \theta_{xx}, \phi_{xxx} &\in L^2((0, 1) \times (0, T_s)).
\end{align*}
\]  

(2.1)

Theorem 1.1 can be achieved by extending the local solutions globally in time based on the following series of prior estimates. Without loss of generality, we assume that \(\nu = R = c_v = \kappa = 1\), and

\[
\int_0^1 v_0 dx = 1, \quad \int_0^1 \left( \frac{u_0^2}{2} + \theta_0 + \frac{1}{4\epsilon}(\phi_0^2 - 1)^2 + \frac{\epsilon}{2} \frac{\phi_0^2}{v_0} \right) dx = 1.
\]

(2.2)
From here to the end of this paper, $C > 0$ denotes the generic positive constant depending only on $\|(v_0, u_0, \theta_0)\|_{H^1(0, 1)}$, $\|\phi_0\|_{H^2(0, 1)}$, $\text{inf}_{x \in [0, 1]} v_0(x)$, and $\text{inf}_{x \in [0, 1]} \theta_0(x)$.

**Lemma 2.2.** Let $(v, u, \theta, \phi)$ be a smooth solution of (1.17)-(1.19) on $[0, 1] \times [0, T]$. Then it holds

$$\sup_{0 \leq t \leq T} \int_0^1 \left( \frac{u^2}{2} + \frac{1}{4\epsilon} (\phi^2 - 1)^2 + \frac{\epsilon \phi_x^2}{2v} + (v - \ln v) + (\theta - \ln \theta) \right) dx + \int_0^T V(s) ds \leq E_0, \quad (2.3)$$

where

$$V(t) \overset{\text{def}}{=} \int_0^1 \left( \frac{\theta_x^2}{v \theta^2} + \frac{u_x^2}{v \theta} + \frac{v \mu_x^2}{\theta} \right) dx, \quad (2.4)$$

and

$$E_0 \overset{\text{def}}{=} \int_0^1 \left( \frac{u^2}{2} + \frac{1}{4\epsilon} (\phi_0^2 - 1)^2 + \frac{\epsilon \phi_0^2}{2v} + (v_0 - \ln v_0) + (\theta_0 - \ln \theta_0) \right) dx. \quad (2.5)$$

**Proof.** From (1.17) and (1.18), combining with (2.2), we have

$$\int_0^1 v dx = 1, \quad \int_0^1 \left( \frac{u^2}{2} + \theta + \frac{1}{4\epsilon} (\phi^2 - 1)^2 + \frac{\epsilon \phi_x^2}{2v} \right) dx = 1. \quad (2.6)$$

Multiplying (1.17) by $1 - \frac{1}{\nu}$, (1.17) by $u$, (1.17) by $\mu$, (1.17) by $1 - \frac{1}{\nu}$, and adding them together, we get

$$\left( \frac{u^2}{2} + \frac{1}{4\epsilon} (\phi^2 - 1)^2 + \frac{\epsilon \phi_x^2}{2v} + (v - \ln v) + (\theta - \ln \theta) \right) t + \left( \frac{\theta_x^2}{v \theta^2} + \frac{u_x^2}{v \theta} + \frac{v \mu_x^2}{\theta} \right) = u_x + (uu_x - u\theta_x)_x + ((1 - \theta^{-1}) \theta_x)_x + \epsilon (\frac{\phi_x \phi_t}{v})_x - \frac{\epsilon \phi_x^2 u_x}{v^2}, \quad (2.7)$$

integrating (2.7) over $[0, 1] \times [0, T]$ by parts, together with the boundary condition (1.18), we obtain (2.3), and the proof of Lemma 2.2 is finished. \hfill \Box

**Lemma 2.3.** Let $(v, u, \theta, \phi)$ be a smooth solution of (1.17)-(1.19) on $[0, 1] \times [0, T]$. Then it has the following expression of $v$

$$v(x, t) = D(x, t) Y(t) + \int_0^t D(x, t) Y(t) \left( \theta(x, \tau) + \frac{\epsilon \phi_x^2}{v \theta^2} \right) d\tau, \quad (2.8)$$

where

$$D(x, t) = v_0(x) e^{\int_0^1 \left( u(y, t) - u_0(y) \right) dy} e^{\left( -\int_0^1 v \int_0^t u \omega dy dx + \int_0^1 u_0 \int_0^t u \omega dy dx \right)}, \quad (2.9)$$

and

$$Y(t) = e^{-\int_0^1 \int_0^t \left( u^2 + \theta + \frac{\epsilon \phi_x^2}{2v^2} \right) dx ds}. \quad (2.10)$$

**Proof.** We introduce the following $\sigma$

$$\sigma \overset{\text{def}}{=} \frac{u_x}{v} - \frac{\theta}{v} - \frac{\epsilon \phi_x^2}{2v^2}. \quad (2.11)$$

Firstly, integrating (1.17) over $[0, x]$, we have

$$\left( \int_0^x u dy \right)_t = \sigma - \sigma(0, t), \quad (2.12)$$
multiply both sides of (2.12) by \( v \), we get

\[
v\sigma(0, t) = v\sigma - v \left( \int_0^x udy \right)_t,
\]

Integrating the above equation over \([0, 1]\), combining with (2.2) and (1.18), we obtain

\[
\sigma(0, t) = \int_0^1 v\sigma dx - \int_0^1 v \left( \int_0^x udy \right)_t dx
= \int_0^1 (u_x - \theta - \frac{\varepsilon \phi_r^2}{2v}) dx - (\int_0^1 v \int_0^x udy dx)_t + \int_0^1 u_x \int_0^x udy dx
= - \left( \int_0^1 v \int_0^x udy dx \right)_t - \int_0^1 \left( u^2 + \theta + \frac{\varepsilon \phi_r^2}{2v} \right) dx.
\]

(2.13)

Secondly, from (1.17), we have

\[
\sigma = (\ln v)_t - \frac{\theta}{v} - \frac{\varepsilon \phi_r^2}{2v^2},
\]

which combining with (2.12) and (2.13), we get

\[
\left( \int_0^x udy \right)_t = (\ln v)_t - \frac{\theta}{v} - \frac{\varepsilon \phi_r^2}{2v^2} + \left( \int_0^1 v \int_0^x udy dx \right)_t + \int_0^1 \left( \theta + u^2 + \frac{\varepsilon \phi_r^2}{2v} \right) dx,
\]

integrating both sides of the above equation with respect to \( t \), we have

\[
v(x, t) = D(x, t)Y(t)e^{\int_0^t \left( \frac{\theta}{v} + \frac{\varepsilon \phi_r^2}{2v^2} \right) dt},
\]

(2.14)

with \( D(x, t) \) and \( Y(t) \) as defined in (2.9) and (2.10) respectively.

Finally, introducing the following \( g \)

\[
g = \int_0^t \left( \frac{\theta}{v} + \frac{\varepsilon \phi_r^2}{2v^2} \right) ds,
\]

(2.15)

by using (2.14), we get the following ordinary differential equation for \( g \)

\[
g_t = \frac{\theta(x, t) + \frac{\varepsilon \phi_r^2(x, t)}{2v(x, t)} v(x, t)}{D(x, t)Y(t)e^g},
\]

and this gives us an expression for \( e^g \)

\[
e^g = 1 + \int_0^t \frac{\theta(x, \tau) + \frac{\varepsilon \phi_r^2(x, \tau)}{2v(x, \tau)}}{D(x, \tau)Y(\tau)} d\tau,
\]

substituting it into the expression (2.14), and thus the proof of Lemma 2.3 is finished. \( \square \)

**Lemma 2.4.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.17)-(1.19) on \([0, 1] \times [0, T]\). Then it holds that for \( \forall (x, t) \in [0, 1] \times [0, T] \)

\[
v(x, t) \geq C^{-1}, \quad \theta(x, t) \geq C^{-1}.
\]

(2.16)

**Proof.** Firstly, since the function \( x - \ln x \) is convex, by using Jensen’s inequality, we have

\[
\int_0^1 \theta dx - \ln \int_0^1 \theta dx \leq \int_0^1 (\theta - \ln \theta) dx,
\]

(2.17)
combining with (2.3) and (2.6), we get
\[ \bar{\theta}(t) \overset{\text{def}}{=} \int_0^1 \theta(x,t)dx \in [\alpha_1, 1], \] (2.18)
where \( 0 < \alpha_1 < \alpha_2 \) are the two roots of the following algebraic equation
\[ x - \ln x = E_0. \] (2.19)

Secondly, from (2.6), by using Cauchy’s inequality, we have
\[ \left| \int_0^1 v \int_0^x udydx \right| \leq \int_0^1 v \left( \int_0^1 u^2dy \right)^{1/2} \leq C, \]
and then by the definition (2.9) of \( D \), we get
\[ C^{-1} \leq D(x,t) \leq C. \] (2.20)
Moreover, from (2.6), combining with (2.18), we also have
\[ \left| \int_0^1 \ln v dx \right| + \int_0^1 (u^2 + \theta + \frac{\epsilon^2}{2v}) dx \leq C, \]
which follows that, for \( \forall \tau \in [0, t) \),
\[ e^{-2t} \leq Y(t) \leq 1, \text{ and } e^{-2(t-\tau)} \leq \frac{Y(t)}{Y(\tau)} \leq e^{-\alpha_1(t-\tau)}. \] (2.21)

By using (2.14), (2.20) and (2.21), we obtain, there exist some positive constant \( C \), such that
\[ v(x,t) \geq C^{-1}, \quad \forall (x,t) \in [0,1] \times [0,T]. \] (2.22)

Finally, \( \forall p > 2 \), multiplying (1.17) by \( \theta^p \), integrate over \([0,1]\) with respect to \( x \), by using (2.22), we have
\[
\frac{1}{p-1} \frac{d}{dt} \int_0^1 (\theta^{-1})^{p-1} dx + \int_0^1 \frac{u_x^2}{v^p} dx \leq \int_0^1 \frac{u_x}{v^{p-1}} dx \\
\leq \frac{1}{2} \int_0^1 \frac{u_x^2}{v^p} dx + \frac{1}{2} \int_0^1 \frac{1}{v^{p-2}} dx \\
\leq \frac{1}{2} \int_0^1 \frac{u_x^2}{v^p} dx + C \left\| \theta^{-1} \right\|_{L^{p-1}}^{p-2}.
\]
Applying Gronwall’s inequality to the above result, we obtain
\[ \sup_{0 \leq t \leq T} \left\| \theta^{-1}(\cdot,t) \right\|_{L^{p-1}} \leq C, \quad \forall p > 2, \]
moreover, letting \( p \) tends to infinity, we do eventually get the lower bound of \( \theta \). The proof of Lemma 2.4 is completed. \( \square \)

**Lemma 2.5.** Let \((v,u,\theta,\phi)\) be a smooth solution of (1.17)-(1.19) on \([0,1] \times [0,T]\). Then it holds that for \( \forall (x,t) \in [0,1] \times [0,T] \)
\[ |\phi(x,t)| \leq C, \quad v(x,t) \leq C. \] (2.23)
Proof. Firstly, from (2.3), we have
\[
\frac{\epsilon}{4} \int_0^1 (\phi^2 - 1)^2 \leq E_0,
\]
which implies that
\[
\int_0^1 \phi^4 \leq C. \tag{2.24}
\]
Moreover, for \(\forall (x, t) \in [0, 1] \times [0, T]\),
\[
|\phi(x, t)| \leq \left| \int_0^1 (\phi(x, t) - \phi(y, t)) dy \right| + \left| \int_0^1 \phi(y, t) dy \right|
\leq \left| \int_0^1 (\int_y^x \phi_\xi(\xi, t) d\xi) dy \right| + CE_0
\leq (\int_0^1 \phi_x^2 dx)^{\frac{1}{2}} + CE_0
\leq C. \tag{2.25}
\]
Secondly, from (2.6) once again, by using Hölder inequality, we get
\[
\left| \theta^x(x, t) - \bar{\theta}^x(t) \right| = \left| \int_0^x \frac{d}{dy} (\theta^x(y, t) - \bar{\theta}^x(t)) dy \right|
\leq \frac{1}{2} \left( \int_0^1 \theta_y^2 dx \right)^{1/2} \left( \int_0^1 \theta v dx \right)^{1/2}
\leq CV^{1/2}(t) \max_{x \in [0, 1]} v^{1/2}(x, t),
\]
thus we obtain, for all \((x, t) \in [0, 1] \times [0, \infty)\), the following inequality holds
\[
\alpha_1 - CV(t) \max_{x \in [0, 1]} v(x, t) \leq \theta(x, t) \leq C + CV(t) \max_{x \in [0, 1]} v(x, t). \tag{2.26}
\]
Finally, we can give the upper bounds of \(v\). In fact, combining the expression of \(v\) (2.8) with the upper and lower bound estimates (2.20)–(2.26), we have the following inequality
\[
v(x, t) = D(x, t)Y(t) + \int_0^t D(x, t)Y(t) \left( \theta(x, \tau) + \frac{\phi_x^2(x, \tau)}{v(x, \tau)} \right) d\tau
\leq C + C \int_0^t e^{-t+\tau} \left( \max_{x \in [0, 1]} \theta(x, \tau) + \max_{x \in [0, 1]} \left( \frac{\phi_x(x, \tau)}{v(x, \tau)} \right)^2 \max_{x \in [0, 1]} v(x, \tau) \right) d\tau.
\tag{2.27}
\]
In view of (1.17), we derive that
\[
\epsilon \left( \frac{\phi_x}{v} \right)_x = -\mu + \frac{1}{\epsilon} (\phi^3 - \phi), \tag{2.28}
\]
and then combining with (2.3), (2.6), (2.16), (2.25), we obtain
\[
\int_0^1 \left( \frac{\phi_x}{v} \right)_x^2 v dx \leq C(1 + V(t)). \tag{2.29}
\]
Using (2.3), (2.6), (2.22), (2.25), (2.29), we get
\[
\max_{x \in [0,1]} \left( \frac{\phi_x}{v} \right)^2 (x,t) \leq C \int_0^1 \frac{\phi_x}{v} \left( \frac{\phi_x}{v} \right)_x dx \\
\leq C \int_0^1 \frac{\theta}{v^2} \phi_x^2 dx + \int_0^1 \left( \frac{\phi_x}{v} \right)_x \theta dx \\
\leq C \left( \max_{x \in [0,1]} \theta(x,t) + 1 + V(t) \right).
\] (2.30)
Substituting (2.30) into (2.27), we obtain
\[
v(x,t) \leq C + C \int_0^t V(\tau) \left( \max_{x \in [0,1]} v(x,\tau) + (\max_{x \in [0,1]} v(x,\tau))^2 \right) d\tau,
\] (2.31)
by using the Gronwall inequality, we get the upper bound of \( v \)
\[
v(x,t) \leq C, \quad \forall (x,t) \in [0,1] \times [0,T].
\] (2.32)
The proof of Lemma 2.5 is completed. \( \square \)

**Lemma 2.6.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.17)-(1.19) on \([0,1] \times [0,T]\). Then it holds that for \(\forall (x,t) \in [0,1] \times [0,T]\)
\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx \leq C, \quad \int_0^T \int_0^1 \phi_x^2 dx \leq C, \quad \int_0^T \int_0^1 \phi_t^2 dx \leq C.
\] (2.33)
**Proof.** Firstly, we rewrite the momentum equation in Lagrange coordinate system (1.17) as follows
\[
\left( u - \frac{v_x}{v} \right)_t = -\frac{\theta}{v} + \frac{\epsilon}{2} \left( \frac{\phi_x}{v} \right)_x^2
\] (2.34)
Multiplying (2.34) by \( u - \frac{v_x}{v} \), integrating by parts over \([0,1]\) with respect to \(x\), we have
\[
\frac{1}{2} \int_0^1 \left( u - \frac{v_x}{v} \right)^2 (x,t) dx - \frac{1}{2} \int_0^1 \left( u - \frac{v_x}{v} (x,0) \right)^2 dx \\
= \int_0^t \int_0^1 \left( \frac{\theta v_x}{v^2} - \frac{\theta_x}{v} - \epsilon \frac{\phi_x}{v} \left( \frac{\phi_x}{v} \right)_x \right) \left( u - \frac{v_x}{v} \right) dx dt \\
= - \int_0^t \int_0^1 \frac{\theta v_x^2}{v^2} dx dt + \int_0^t \int_0^1 \frac{\theta u v_x}{v^2} dx dt \\
- \int_0^t \int_0^1 \frac{\theta}{v^2} (u - v_x) dx dt - \int_0^t \int_0^1 \epsilon \frac{\phi_x}{v} \left( \frac{\phi_x}{v} \right)_x (u - v_x) dx dt.
\] (2.35)
Now we give the last three terms on the right side of (2.35). By using (2.3), (2.22), (2.32), (2.26), we have
\[
\left| \int_0^t \int_0^1 \frac{\theta u v_x}{v^2} dx dt \right| \leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta v_x^2}{v^2} dx dt + \frac{1}{2} \int_0^t \int_0^1 \frac{u^2 \theta}{v} dx dt \\
\leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta v_x^2}{v^2} dx dt + C \int_0^t \max_{x \in [0,1]} \theta dt \\
\leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta v_x^2}{v^2} dx dt + C \int_0^t \left( C + CV(t) \max_{x \in [0,1]} v(x,t) \right) dt
\]
Moreover, by using (2.3), (2.22), we obtain
\[
\left| \int_0^t \int_0^1 \frac{\theta_x}{v} (u - \frac{v_x}{v}) \, dx \, dt \right| \leq \int_0^t \int_0^1 \frac{\theta_x^2}{v^2} \, dx \, dt + \frac{1}{2} \int_0^t \int_0^1 \left( u - \frac{v_x}{v} \right)^2 \, dx \, dt \\
\leq C + C \int_0^t \max_{x \in [0,1]} \theta^2 \int_0^1 \left( u - \frac{v_x}{v} \right)^2 \, dx \, dt. \tag{2.37}
\]

Since
\[
\int_0^T \max_{x \in [0,1]} \theta^2 \, dt \leq C \int_0^T \max_{x \in [0,1]} \left| \frac{\theta^2}{v} - \int_0^1 \theta^2 \, dx \right| \, dt + C \int_0^T \max_{x \in [0,1]} \theta \, dt \\
\leq \int_0^T \int_0^1 \theta |\theta_x| \, dx \, dt + C \int_0^T \left( C + CV(t) \max_{x \in [0,1]} v(x, t) \right) \, dt \\
\leq C(\delta) \int_0^T \int_0^1 \frac{\theta_x^2}{v^2} \, dx \, dt + \delta \int_0^T \int_0^1 v \theta^3 \, dx \, dt + C \\
\leq C + C\delta \int_0^1 \max_{x \in [0,1]} \theta^2 \, dx \, dt, \tag{2.38}
\]

which follows that
\[
\int_0^T \max_{x \in [0,1]} \theta^2 \, dt \leq C. \tag{2.39}
\]

Integrating (1.17)5 over \([0,1] \times [0, T]\), combining with (2.22), (2.32), (2.39), we have
\[
\int_0^T \int_0^1 \frac{u_x^2 + (v \mu)^2}{v} \, dx \, dt = \int_0^1 \theta \, dx - \int_0^1 \theta_0 \, dx + \int_0^T \int_0^1 \frac{\theta}{v} u_x \, dx \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{u_x^2}{v} \, dx \, dt, \tag{2.40}
\]

which implies that
\[
\int_0^T \int_0^1 \left( u_x^2 + (v \mu)^2 \right) \, dx \, dt \leq C, \tag{2.41}
\]

and moreover, by using (2.16), (2.29), (2.32), we get
\[
\left| \int_0^t \int_0^1 \frac{\phi_x}{v} \left( \frac{\phi_x}{v} \right)_x \left( u - \frac{v_x}{v} \right) \, dx \, dt \right| \\
\leq C \int_0^t \int_0^1 \left( | \frac{\phi_x}{v} |^2 + \left| \frac{\phi_x}{v} \right|^2 \left( u - \frac{v_x}{v} \right)^2 \right) \, dx \, dt \\
\leq C + C \int_0^t \max_{x \in [0,1]} \left| \frac{\phi_x}{v} \right|^2 \int_0^1 \left( u - \frac{v_x}{v} \right)^2 \, dx \, dt \tag{2.42}
\]

Substituting (2.36), (2.37), (2.42) into (2.35), combining with (2.30) (2.38) and Gronwall’s inequality, we have
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( u - \frac{v_x}{v} \right)^2 \, dx + \int_0^T \int_0^1 \frac{\theta v_x^2}{v^3} \, dx \, dt \leq C, \tag{2.43}
\]
together with (2.3), (2.16) and (2.23), we derive that
\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx \leq C.
\]

Secondly, we rewrite (1.17) as follows
\[
\phi_t - \epsilon \phi_{xx} = -\epsilon \frac{\phi_xv_x}{v} - \epsilon (\phi^3 - \phi), \quad (2.44)
\]

integrating (2.47) over [0, 1], we arrive
\[
\int_0^1 \phi_t^2 dx \leq C \left( \int_0^1 \phi_{xx}^2 dx + \int_0^1 \phi_x^2 dx + \int_0^1 (\phi^3 - \phi)^2 dx \right)
\leq C \left( \int_0^1 \phi_{xx}^2 dx + \int_0^1 \phi_x^2 dx + \int_0^1 \phi_{xx}^2 dx + 1 \right)
\leq C \left( \int_0^1 \phi_{xx}^2 dx + 1 \right), \quad (2.48)
\]

then, by using (2.46), we arrive
\[
\int_0^T \int_0^1 \phi_t^2 dx \leq C. \quad (2.49)
\]

The proof of Lemma 2.6 is finished. \(\Box\)

In order to get the a priori estimates on \(u\), we present some prior estimates for higher derivatives of \(\phi\).
Lemma 2.7. Let \((v, u, \theta, \phi)\) be a smooth solution of (1.17)-(1.19) on \([0, 1] \times [0, T]\). Then it holds that for \(\forall (x, t) \in [0, 1] \times [0, T]\)

\[
\sup_{0 \leq t \leq T} \int_0^1 \phi_{xx}^2 \, dx + \int_0^T \int_0^1 \left( \phi_{xt}^2 + \left( \frac{\phi_x}{v} \right)_{xx}^2 \right) \, dx \leq C. \tag{2.50}
\]

**Proof.** We rewrite (2.44) as

\[
\frac{\phi_t}{v} - \epsilon \left( \frac{\phi_x}{v} \right)_x = -\frac{1}{\epsilon} (\phi^3 - \phi), \tag{2.51}
\]

then, differentiating (2.51) with respect to \(x\), we have

\[
\left( \frac{\phi_x}{v} \right)_t - \epsilon \left( \frac{\phi_x}{v} \right)_{xx} = -\frac{1}{\epsilon} (\phi^3 - \phi)_x + \frac{\phi_t v_x}{v^2} - \frac{\phi_x u_x}{v^2}, \tag{2.52}
\]

multiplying (2.52) by \(\left( \frac{\phi_x}{v} \right)_t\) and integrating the resultant over \([0, 1]\), by using (2.3), (2.16), (2.23), (2.33), (2.41), (2.48), we obtain

\[
\int_0^1 \left( \frac{\phi_x}{v} \right)^2 \, dx + \epsilon \frac{d}{dt} \int_0^1 \left( \frac{\phi_x}{v} \right)^2 \, dx \leq C \left( \int_0^1 (3\phi^2 - 1)^2 \phi_x^2 \, dx + \int_0^1 \phi_t^2 \phi_x^2 \, dx + \int_0^1 \phi_x^2 u_x^2 \, dx \right) + \frac{1}{3} \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx
\]

\[
\leq C \left( 1 + \|\phi_t\|_{L^\infty}^2 \int_0^1 u_x^2 \, dx + \|\phi_x\|_{L^\infty}^2 \int_0^1 u_x^2 \, dx \right) + \frac{1}{3} \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx
\]

\[
\leq C \left( 1 + \int_0^1 (\phi_t^2 + 2|\phi_t \phi_x|) \, dx + \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx \right) + \frac{1}{3} \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx
\]

\[
\leq C \left( 1 + \int_0^1 \phi_{xx}^2 \, dx + \int_0^1 \phi_{xt}^2 \, dx + \int_0^1 \left( \frac{\phi_x}{v} \right)^2 \, dx \right) + \frac{1}{3} \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx
\]

\[
\leq C \left( 1 + \int_0^1 \phi_{xx}^2 \, dx + \int_0^1 u_x^2 \, dx \right) + \frac{1}{2} \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx, \tag{2.53}
\]

where in the last inequality we have used \(\phi_{xt} = (\left( \frac{\phi_x}{v} \right)_t + \frac{\phi_x u_x}{v^2}) v\). Thus, by Gronwall’s inequality, we have

\[
\sup_{t \in [0, T]} \int_0^1 \left( \frac{\phi_x}{v} \right)_x^2 \, dx + \int_0^T \int_0^1 \left( \frac{\phi_x}{v} \right)_t^2 \, dx \leq C, \tag{2.54}
\]

combining with (2.3), (2.33), we get

\[
\sup_{0 \leq t \leq T} \int_0^1 \phi_{xx}^2 \, dx + \int_0^T \int_0^1 \phi_{xt}^2 \, dx \leq C. \tag{2.55}
\]

Moreover, by using the Sobolev embedding theorem for one dimension, (2.54) also implies

\[
\max_{(x, t) \in [0, 1] \times [0, T]} \left| \frac{\phi_x}{v} \right|^2 \leq C + C \sup_{t \in [0, T]} \int_0^1 \left( \frac{\phi_x}{v} \right)_x^2 \, dx \leq C. \tag{2.56}
\]
Further, by using (2.52) and the estimates obtained above, we achieve
\[\int_0^T \int_0^1 (\phi_x v)_x^2 dx \leq C,\]  
(2.57)
the proof of Lemma 2.7 is completed. \(\square\)

After the above preparation work, now we can give a prior estimate of the velocity \(u\).

**Lemma 2.8.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.17)-(1.19) on \([0,1] \times [0,T]\). Then it holds that for \(\forall (x,t) \in [0,1] \times [0,T]\)
\[
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 dx + \int_0^T \int_0^1 (u_t^2 + u_{xx}^2) dx \leq C. \tag{2.58}
\]

**Proof.** Multiplying (1.17)\(_2\) by \(u_{xx}\) and integrating the resultant over \((0,1) \times (0,T)\), by using (2.16), (2.23), (2.33), (2.39), (2.54), (2.56), we obtain
\[
\frac{1}{2} \int_0^1 u_t^2 dx + \int_0^T \int_0^1 \frac{u_{xx}^2}{v} dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 u_{xx}^2 dx dt + C \int_0^T \int_0^1 \left( \theta_x^2 + \theta^2 v_x^2 + \left| \frac{\phi_x}{v} \right| (\phi_x)_x^2 + u_x^2 v_x^2 \right) dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 u_{xx}^2 dx dt + C \int_0^T \int_0^1 \theta_x^2 dx + C \int_0^T \max_{x \in [0,1]} \theta^2 \int_0^1 v_x^2 dx dt \\
+ C \max_{(x,t) \in [0,1] \times [0,T]} \left| \frac{\phi_x}{v} \right|^2 \int_0^T \int_0^1 \left| (\phi_x)_x \right|^2 dx dt + C \int_0^T \max_{x \in [0,1]} u_x^2 \int_0^1 v_x^2 dx dt \\
\leq C + \frac{3}{4} \int_0^T \int_0^1 u_{xx}^2 dx dt + C_1 \int_0^T \int_0^1 \theta_x^2 dx dt, \tag{2.59}
\]
where we have used the following and the Sobolev embedding inequality in one dimension
\[
\int_0^T \max_{x \in [0,1]} u_x^2 dt \leq C(\delta) \int_0^T \int_0^1 u_{xx}^2 dx dt + \delta \int_0^T \int_0^1 \frac{u_x^2}{v} dx dt \\
\leq C(\delta) + \delta \int_0^T \int_0^1 \frac{u_x^2}{v} dx dt. \tag{2.60}
\]
Multiplying (1.17)\(_5\) by \(\theta\), integrating the resultant over \((0,1) \times (0,T)\), by using (2.39), (2.41) and (2.60), we have
\[
\frac{1}{2} \int_0^1 \theta^2 dx + \int_0^T \int_0^1 \frac{\theta^2}{v} dx dt \\
\leq C + C \int_0^T \int_0^1 \theta |u_x| dx + C \int_0^T \int_0^1 (u_x^2 + \mu^2) \theta dx dt \\
\leq C + C \int_0^T \int_0^1 \theta u_x^2 dx dt + + C \int_0^T \int_0^1 \theta \theta^2 dx dt + \int_0^T \max_{x \in [0,1]} \theta^2 dt \\
\leq C + C \int_0^T \max_{x \in [0,1]} u_x^2 dt + C \int_0^T \max_{x \in [0,1]} \theta^2 dt.
\]
\[\leq C(\delta) + C\delta \int_0^T \int_0^1 \frac{u_{xx}^2}{v} dx dt. \tag{2.61}\]

From (2.59) and (2.61), choosing \(\delta\) small enough, we obtain

\[\sup_{t \in [0, T]} \int_0^1 (\theta^2 + u_x^2) dx + \int_0^T \int_0^1 \theta_x dx dt + \int_0^T \int_0^1 u_{xx}^2 dx dt \leq C. \tag{2.62}\]

Then, rewriting (1.17) as

\[u_t = -\left(\frac{\theta}{v}\right)_x + \frac{u_{xx}}{v} - \frac{u_x v_x}{v^2} - \epsilon \frac{\phi_x}{v} \left(\frac{\phi_x}{v}\right)_x, \tag{2.63}\]

combining with (2.33), (2.39), (2.56), (2.60), (2.50) and (2.62), we achieve

\[\int_0^T \int_0^1 u_t^2 dx dt \leq C \int_0^T \int_0^1 \left(u_{xx}^2 + u_x^2 v_x^2 + \theta_x^2 + \theta_x^2 v_x^2 + \epsilon \frac{\phi_x}{v} \left(\frac{\phi_x}{v}\right)_x^2 \right) dx dt \]

\[\leq C, \tag{2.64}\]

together with (2.62), the energy inequality (2.58) is obtained. The proof of Lemma 2.8 is completed. \(\square\)

**Lemma 2.9.** Let \((v, u, \theta, \phi)\) be a smooth solution of (1.17)-(1.19) on \([0, 1] \times [0, T]\). Then it holds that for \(\forall (x, t) \in [0, 1] \times [0, T]\)

\[\sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 dx + \int_0^T \int_0^1 (\theta_t^2 + \theta_{xx}^2) dx dt \leq C. \tag{2.65}\]

**Proof.** Multiplying (1.17) by \(\theta_t\) and integrating the resultant over \((0, 1)\), by using (2.16), (2.23), (2.62), we have

\[\int_0^1 \theta_t^2 dx + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\theta_x^2}{v} dx\right) = -\frac{1}{2} \int_0^1 \frac{\theta_x^2 u_x}{v^2} dx + \int_0^1 \theta_t \left(- \theta u_x + u_x^2 + v^2 \mu^2\right) dx \]

\[\leq C \max_{x \in [0, 1]} |u_x| \int_0^1 \theta_x^2 dx + \frac{1}{2} \int_0^1 \theta_t dx + C \int_0^1 \theta^2 u_x^2 dx + C \int_0^1 (\theta_x^4 + \mu^4) dx \]

\[\leq C \int_0^1 \theta_x^2 dx + \frac{1}{2} \int_0^1 \theta_t^2 dx + C \max_{x \in [0, 1]} (\theta^2 + u_x^4 + \mu^4) + C. \tag{2.66}\]

Now we deal with the term \(\max_{x \in [0, 1]} (\theta^2 + u_x^4 + \mu^4)\) in the last inequality of (2.66). Combining Lemma 2.8, direct computation shows that

\[\int_0^T \max_{x \in [0, 1]} u_x^4 dt \]

\[\leq C \int_0^1 \int_0^1 u_x^4 dx dt + C \int_0^T \int_0^1 |u_{xx}^3 u_{xx}| dx dt \]

\[\leq C \int_0^T \max_{x \in [0, 1]} u_x^2 \int_0^1 u_x^2 dx dt + C \int_0^T \max_{x \in [0, 1]} u_x^2 \left(\int_0^1 u_x^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}} dt \]
\[ \leq \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_x^4 \, dt + C \int_0^T \int_0^1 (u_x^2 + u_{xx}^2) \, dx \, dt \]
\[ \leq \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_x^4 \, dt + C, \quad (2.67) \]
so we get immediately that
\[ \int_0^T \max_{x \in [0,1]} u_x^4 \, dt \leq C. \quad (2.68) \]
By the same way above, combining with (2.50), we have
\[ \int_0^T \max_{x \in [0,1]} \mu^4 \, dt \leq C. \quad (2.69) \]
Moreover, by using Sobolev embedding theorem, we get
\[ \max_{x \in [0,1]} \theta^2 \leq C + C \int_0^1 \theta_x^2 \, dx, \quad (2.70) \]
Substituting (2.68), (2.69), (2.70) into (2.66), by using Gronwall’s inequality and (2.5), we obtain
\[ \sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 \, dx + \int_0^T \int_0^1 \theta_t^2 \, dx \, dt \leq C. \quad (2.71) \]
Therefore, in view of (2.70), we achieve
\[ \max_{(x,t) \in [0,1] \times [0,T]} \theta \leq C. \quad (2.72) \]
Let’s go back to (1.17) again, we have
\[ \frac{\theta_{xx}}{v} = \theta_t + \frac{\theta_x v_x}{v^2} + \frac{\theta u_x}{v} - \frac{u_x^2}{v} - v \mu^2, \quad (2.73) \]
which yields that
\[ \int_0^T \int_0^1 \theta_{xx}^2 \, dx \, dt \leq C \int_0^T \int_0^1 (\theta_x^4 + \theta_x^2 v_x^2 + u_x^4 + \mu^4 + u_x^2 + \theta_t^2) \, dx \, dt \]
\[ \leq C + C \int_0^T \max_{x \in [0,1]} \theta_x^2 \, dt \]
\[ \leq C + C \int_0^T \int_0^1 \theta_{xx}^2 \, dx \, dt. \quad (2.74) \]
Meanwhile, by using maximum principle, \(-1 \leq \phi \leq 1\). The proof of Lemma 2.9 is completed. \(\square\)

**Proof of Theorem 1.1.** By Lemma 2.1, there exists a \(T_0 > 0\), such that the system (1.17)-(1.19) has a unique strong solution \((v, u, \theta, \phi)\) on \((0, T_0]\) satisfying (2.1). Suppose that \(T_0\) is the maximum existence time of the unique strong solution \((v, u, \theta, \phi)\) to (1.17)-(1.19). Therefore, \(T_0 \geq T_0 > 0\). We claim that
\[ T_0 = +\infty. \quad (2.75) \]
If not, $T_0 < +\infty$, then by using Lemma 2.2–2.9, the global a priori estimates of the solutions ensure $(v, u, \theta, \phi) \in C([0, T_0]; H^1)$ and that
\[ \|(v, u, \theta, \phi)(t)\|_{H^1} \leq C(T_0) < +\infty, \quad \forall t \in [0, T_0], \] (2.76)
where $C(T_0)$ is a positive constant depending only on $T_0$, $\inf_{x \in [0,1]} v_0(x)$, $\inf_{x \in [0,1]} \theta_0(x)$, and $\|(v_0, u_0, \theta_0, \phi_0)\|_{H^1([0,1])}$. Thus, $(v, u, \theta, \phi)(x, T_0)$ is finite and well-defined. Thus, follows from Lemma 2.1, there exists a positive constant $T_1 > 0$ such that (1.17)-(1.19) has a unique strong solution on $[T_0, T_0 + T_1]$, which contradicts the definition of $T_0$. Thereby, the claim (2.75) is true. The proof of Theorem 1.1 is completed.

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