PHYSICAL VACUUM PROPERTIES AND INTERNAL SPACE DIMENSION

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Abstract

The paper addresses matrix spaces, whose properties and dynamics are determined by Dirac matrices in Riemannian spaces of different dimension and signature. Among all Dirac matrix systems there are such ones, which nontrivial scalar, vector or other tensors cannot be made up from. These Dirac matrix systems are associated with the vacuum state of the matrix space. The simplest vacuum system realization can be ensured using the orthonormal basis in the internal matrix space. This vacuum system realization is not however unique. The case of 7-dimensional Riemannian space of signature 7(-) is considered in detail. In this case two basically different vacuum system realizations are possible: 1) with using the orthonormal basis; 2) with using the oblique-angled basis, whose base vectors coincide with the simple roots of algebra $E_8$.

Considerations are presented, from which it follows that the least-dimension space bearing on physics is the Riemannian 11-dimensional space of signature 1 (-) &10 (+). The considerations consist in the condition of maximum vacuum energy density and vacuum fluctuation energy density.
1. Introduction

The problem under discussion in this paper consists in construction of matrix spaces (MS) in the Riemannian spaces of different dimensions and signatures and in the study of those MS properties, which can have a physical meaning as applied to the physical MS vacuum state. Before we formulate the problem more specifically, elucidate what is meant by the MS theory (see [1]-[3]). In so doing we restrict our consideration to minimum explanations needed for the consistent treatment.

The MS theory is a theory of internal degrees of freedom of Riemannian spaces. The properties of the internal degrees are introduced through Dirac matrices (DM) $\gamma_A$ satisfying relations

$$\gamma_A \gamma_B + \gamma_B \gamma_A = 2 g_{AB} \cdot E. \quad (1)$$

Here: $g_{AB}$ is the metric tensor of the Riemannian space of dimension $n$ $(A, B = 1, 2, \ldots, n)$; $\gamma_A$, $\gamma_B$ are square matrices $N \times N$; $E$ is a unit matrix in the space of internal degrees of freedom.

The internal degrees of freedom are related, first, to transformations

$$\gamma_A \rightarrow \gamma'_A = S(x) \gamma_A S^{-1}(x) \quad (2)$$

and, second, to the transition to the Riemannian spaces of higher dimensions and different signatures.

Illustrate the MS properties by the example of the Riemannian space used in the general relativity. The space has dimension $n = 4$ and signature $1 (-) \& 3 (+)$. In this space, DM can be realized above real, complex, quaternion and octonion fields.

One of the best-known DM systems is that in the Majorana (real) representation:

$$\gamma_0 = -i \rho_2 \sigma_1; \quad \gamma_1 = \rho_1; \quad \gamma_2 = \rho_2 \sigma_2; \quad \gamma_3 = \rho_3. \quad (3)$$

The DM realized above the complex field can be exemplified with:

DM in the representation typically used in quantum electrodynamics:

$$\gamma_0 = -i \rho_3; \quad \gamma_1 = \rho_2 \sigma_1; \quad \gamma_2 = \rho_2 \sigma_2; \quad \gamma_3 = \rho_2 \sigma_3. \quad (4)$$

DM in the helicity representation:

$$\gamma_0 = -i \rho_1; \quad \gamma_1 = \rho_2 \sigma_1; \quad \gamma_2 = \rho_2 \sigma_2; \quad \gamma_3 = \rho_2 \sigma_3. \quad (5)$$
The well-known rho-sigma matrices $4 \times 4$ are used hereinafter in the notation of the expressions for DM. For the space used in the general relativity, the vector subscript $A$ will be denoted in a most habitual manner, viz. by lower case Greek letter “$\alpha$” $(\alpha = 0, 1, 2, 3)$. If an arbitrary field $\gamma_\alpha (x)$ is taken, then scalar and tensor fields can be constructed from it, for example, the real scalar, such as

$$Sp (\gamma^\alpha \gamma^\alpha_\alpha).$$

(6)

Similar quantities will be, generally speaking, functions of coordinates. However, among the DM systems there are such systems, from which it proves impossible to construct coordinate dependent fields. Only trivial fields can be constructed from them, i.e. the fields, whose components reduce to integers constant in space. As it is easy to verify, such DM systems include systems [4], [1], [5]. We will term these DM systems as vacuum DM (VDM). We will therewith use this term not only in the case of the Riemannian space used in the general relativity, but also in any other case.

The vacuum DM systems in spaces of different dimensions and signatures are just the subject of consideration in this paper. It will be shown that in most cases the VDM are realized through using the orthonormal basis in the internal space. However, for some dimensions and signatures this vacuum system realization is not unique. The case of 7-dimensional Riemannian space of signature 7(-) is considered in detail. In this case two basically different vacuum system realizations are possible: [1] with using the orthonormal basis; [2] with using the oblique-angled basis, whose base vectors coincide with the simple roots of algebra $E_8$. In the second case the vacuum DM system offers a unique feature. The feature is that it has maximum vacuum energy density and vacuum fluctuation energy density in comparison with other vacuum DM systems. The compactification of the internal 7-dimensional space leads to the 11-dimensional Riemannian space of signature 1 (-) &10 (+).

The question arises: Can any other Riemannian space of a dimension higher than 11 exist, which would admit a denser “zero” energy packing than the 11-dimensional space of signature 1 (-) &10 (+) with compactified 7-dimensional subspace? From the general theory of lattices it is known that densest packings are admitted by 8-dimensional lattice $E_8$ and 24-dimensional Leech lattice. Hence, the obtained result regarding the dimensional regularization admits, in principle, only one correction. The correction can appear only in consideration of the lattice VDM configurations described
by the Leech lattice. However, even on the transition to a space, in which
the VDM are described by the simple roots of the Leech lattice, the 11-
dimensional space of signature 1 (−) & 10 (+) will preserve the sense of the
space, in which the local minimum of action on vacuum is realized.

The obtained results have a direct bearing, for example, on a so-called
problem of dimensional regularization in multidimensional schemes (super-
gravity; superstring theories). In essence, this is a forcible argument in favor
of the fact that the dimension of the World is 11 and its signature is of form
1 (−) & 10 (+).

2. MS theory

In what follows the following facts pertaining to the MS theory are considered
known1:

The realization of DM above the complex and quaternion field reduces to
the special cases of the realizations above the real field, but in Riemannian
spaces of a higher dimension. So in what follows it is sufficient to restrict
our consideration to the realization of DM only above the real field.

In the realization of DM above the real field, without loss of generality
we can consider only the Riemannian spaces of odd dimension \( n \),

\[
\begin{align*}
   n &= 2k + 1, \\
   &\quad (7)
\end{align*}
\]

where \( k \) is a positive integer. The signature of the Riemannian spaces under
consideration should therewith be of form \( k(-) & (k + 1)(+) \) or differ from it
by a number of “minuses” multiple of four.

The DM which can be introduced in the Riemannian space possessing the
above mentioned properties are square matrices \( N \times N \), with the \( N \) being
related with \( k \) as

\[
N = 2^k. \\
&\quad (8)
\]

The hermitizing (or anti-hermitizing) matrix \( D \) determined as

\[
D\gamma_\alpha D^{-1} = -\gamma_\alpha^+, \\
&\quad (9)
\]

can be introduced in any MS. The matrix \( D \) is a Hermitean matrix and,
hence, it can be used as a metric in the internal space, that is can be used
to introduce the Hermitean quadratic form.

1The facts mentioned are substantiated in refs. [1]-[3].
The best-known vacuum DM type is the systems of DM possessing the following properties (we will refer to them as standard VDM):
- They can be written in the form of monomials in terms of the rho-sigma matrices.
- They possess certain symmetric, real properties.
- They have either zeroes or numbers ±1 as their elements.

3. Physical criteria for comparison of VDM properties

Ask ourselves the question: Are there any criteria to select those multidimensional Riemannian spaces with different values of \( n \) and all possible signatures among all the ones, in which the VDM could be comparable to the physical vacuum?

The answer to the question is of the fundamental nature, as it bears on the problem of dimensional regularization in the multidimensional theories. Without an answer to this question the multidimensional theories become a subject of investigation, whose relation to the reality is unknown.

It is clear that the question posed cannot be answered without going beyond the standard VDM class. In fact, when solving the question of physical advantages or disadvantages of the VDM, we can appeal either to any scalar characteristics of the internal space like

\[
\frac{1}{N} \text{Sp}(D)
\] (10)

or to any integrals over the elementary cell, which the VDM is constructed on. The quantity of the second type can be exemplified with

\[
\frac{1}{NV_{\text{Cell}}} \int_{\text{Cell}} Z \sqrt{\text{det}(D)} \, dZ.
\] (11)

The integration in (11) is performed over all the elementary cell points in the internal space.

The quantities of type (10) will be interpreted as the vacuum energy density and those of type (11) as the vacuum fluctuation energy density.
For all standard VDM these characteristics are the same,

\[ \frac{1}{N} \text{Sp}(D) = 1, \]

\[ \frac{1}{NV_{\text{Cell}}} \int_{Cell} Z \sqrt{\det(D)} \, dZ = \text{Const.} \]

4. Explicit form of nonstandard VDM system

We have found that the standard VDM do not constitute a unique VDM class. It turns out that for some dimensions and signatures basically another type of the VDM systems can exist, which we will term as an alternative VDM type. As this paper reveals, the comparison between the standard and alternative VDM properties allows us to find those criteria, which lead to essentially unique solution to the dimensional regularization problem.

The analysis of the DM structure in spaces of different dimensions and signatures, which is performed by us in ref. [3], suggests that the nonstandard VDM type can exist only with a definite signature of the matrix degree of freedom subspace part, which is described by the nonstandard DM type. From the general theory of lattices suitable for physical use (see, e.g., [4]) it is known that the lattice generated in the matrix degree of freedom space should be autodual. It is also known (see, e.g., [5]) that the least dimension of the Riemannian space admitting introduction not only of a dual lattice based on the orthonormal basis, but also a basically different lattice, is 7. The “different” lattice structure is therewith based on the simple root vectors of the algebra $E_8$.

The satisfaction of all the above conditions needed for the nonstandard VDM appearance cannot be achieved with arbitrary Riemannian space dimension and signature. We have found that the 7-dimensional definite space admitting the description with DM $8 \times 8$ can be included as a compactified internal space in the ordinary 4-dimensional space of signature $1 (-) & 3 (+)$. The signature of the 7-dimensional subspace should be $7 (-)$ and that of the complete space $1 (-) & 10 (+)$.

The simplest method to prove the existence of the alternative VDM along with the standard VDM system is to present the explicit form of both the systems. We will do this now.
Below are the explicit forms of two DM systems in the 7-dimensional space of signature 7 (−). Either system is a set of seven matrices $8 \times 8$. The first DM system refers to the standard VDM category; we will denote it by $\{\gamma_A\}$ ($A = 1, 2, \cdots, 7$). The second system refers to the alternative VDM category; we will denote it by $\{g_A\}$ ($A = 1, 2, \cdots, 7$).

The system $\{\gamma_A\}$ can be written in terms of sigma matrix $2 \times 2$ and rho-sigma matrix $4 \times 4$, i.e. as

$$
\begin{align*}
\gamma_1 &= \sigma_3 \otimes i\rho_2\sigma_1 \\
\gamma_2 &= \sigma_3 \otimes i\sigma_2 \\
\gamma_3 &= \sigma_3 \otimes i\rho_2\sigma_3 \\
\gamma_4 &= \sigma_1 \otimes i\rho_1\sigma_2 \\
\gamma_5 &= \sigma_1 \otimes i\rho_2 \\
\gamma_6 &= \sigma_1 \otimes i\rho_3\sigma_2 \\
\gamma_7 &= i\sigma_2 \otimes E \\
D &= E \otimes E
\end{align*}
$$

Recall that the anti-hermitizing matrix $D$ is determined as

$$
\gamma_A^+ = -D\gamma_A D^{-1}.
$$

Now this is the explicit form of DM $\{g_A\}$.

\[g_1 = \begin{array}{cccccccc}
-1 & -1 & -1 & -2 & -3 & -2 & -1 & -2 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 2 & 3 & 4 & 5 & 3 & 1 & 2 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{array}\]

\[g_2 = \begin{array}{cccccccc}
1 & -1 & -2 & -2 & -3 & -2 & -1 & -2 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 2 & 3 & 4 & 4 & 3 & 2 & 2 \\
1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 \\
\end{array}\]
\[ g_3 = \begin{bmatrix}
-1 & -2 & -2 & -2 & -3 & -2 & -1 & -2 \\
1 & 1 & 1 \\
-1 & -1 & -2 & -3 & -4 & -3 & -1 & -2 \\
1 & 2 & 3 & 4 & 5 & 4 & 2 & 2 \\
-1 & -1 & -1 \\
-1 & -2 & -3 & -4 & -4 & -2 & -1 & -2 \\
-1 & -2 & -2 & -2 & -2 & -1 & -1
\end{bmatrix} \]

\[ g_4 = \begin{bmatrix}
-1 & -2 & -3 & -4 & -5 & -4 & -2 & -2 \\
1 & 2 & 3 & 4 & 5 & 3 & 1 & 2 \\
-1 & -2 & -3 & -4 & -4 & -2 & -1 & -2 \\
1 & 2 & 3 & 3 & 3 & 2 & 1 & 2 \\
-1 & -2 & -2 & -3 & -2 & -1 & -2 \\
1 & 1 & 1 & 2 & 3 & 2 & 1 & 2 \\
-1 & -2 & -3 & -2 & -1 & -2 \\
1 & 1 & 1 & 2 & 1 & 1 & 1
\end{bmatrix} \]

\[ g_5 = \begin{bmatrix}
-1 & -2 & -3 & -4 & -4 & -3 & -2 & -2 \\
1 & 2 & 2 & 3 & 4 & 2 & 1 & 2 \\
-1 & -2 & -3 & -2 & -1 & -2 \\
-1 & 1 & 2 & 2 & 1 & 1
\end{bmatrix} \]

\[ g_6 = \begin{bmatrix}
-1 & -2 & -3 & -4 & -4 & -2 & -1 & -2 \\
1 & 2 & 3 & 4 & 4 & 3 & 2 & 2 \\
-1 & -2 & -3 & -4 & -5 & -4 & -2 & -2 \\
1 & 1 & 2 & 3 & 4 & 3 & 1 & 2 \\
-1 & -2 & -3 & -2 & -1 & -2 \\
1 & 2 & 2 & 3 & 2 & 1 & 2 \\
-1 & -2 & -2 & -3 & -2 & -1 & -2 \\
1 & 2 & 1 & 1 & 1
\end{bmatrix} \]
Systems \( \{ \gamma_A \}, \{ g_A \} \) satisfy relations (1), which in this case have form

\[
\gamma_A \gamma_B + \gamma_B \gamma_A = -2\delta_{AB}; \quad g_A g_B + g_B g_A = -2\delta_{AB}
\]  

and, hence, according to the Pauli theorem, are related as

\[
\gamma_A \rightarrow g_A = R \gamma_A R^{-1}. \tag{18}
\]

The matrix \( R \) is found uniquely (with accuracy to common multiplier) and takes the form

\[
R = \begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2
\end{bmatrix}
\]  

\[
R^{-1} = \frac{1}{2} \begin{bmatrix}
1 & 0 & -1 & -2 & -3 & -2 & -1 & -2 \\
-1 & 0 & -1 & -2 & -3 & -2 & -1 & -2 \\
-1 & -2 & -1 & -2 & -3 & -2 & -1 & -2 \\
-1 & -2 & -3 & -2 & -3 & -2 & -1 & -2 \\
-1 & -2 & -3 & -4 & -5 & -2 & -1 & -2 \\
-1 & -2 & -3 & -4 & -5 & -4 & -1 & -2 \\
-1 & -2 & -3 & -4 & -5 & -4 & -3 & -2
\end{bmatrix}
\]  

For DM \( \{ g_A \} \) the anti-hermitizing matrix \( D \) is found from relation

\[
D = R^{-1} + DR^{-1}. \tag{21}
\]
and proves equal to

\[
D = \begin{bmatrix}
2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\
3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\
4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\
5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\
6 & 12 & 18 & 24 & 30 & 20 & 14 & 7 & 10 \\
4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\
2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\
3 & 6 & 9 & 12 & 15 & 10 & 5 & 8
\end{bmatrix}
\]

(22)

The inverse matrix is

\[
D^{-1} = \begin{bmatrix}
2 & -1 & & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & & \\
\end{bmatrix}
\]

(23)

The question arises: In what way can the 7-dimensional subspace, whose vacuum state is described by latticed VDM $g_A$, be integrated with the ordinary 4-dimensional space-time used in the general relativity? In principle, there are two possibilities. The first possibility is to make the 3-dimensional general relativity subspace a subspace of the 7-dimensional space. This possibility leads to the new question: Why are only 3 dimensions of 7 observable? We do not know any answer to the question, so we will not consider this possibility. The second possibility is to assume that the complete space is 11-dimensional and factorized to the direct product of two subspaces: 4-dimensional general relativity subspace and 7-dimensional subspace. It is this possibility that is considered by us in what follows.

From refs. [1]-[3] it follows that in the MS of dimension $n = 2k + 1$ the real DM are of dimension $N \times N$, where $N = 2^k$. Among the set of all MS there is a subset of such MS, in which the DM structure is of the following form:

\[
Matrix\ (N \times N) \Rightarrow Matrix\ \left(2^{k-2} \times 2^{k-2}\right) \otimes Matrix\ \left(2^2 \times 2^2\right),
\]

(24)
As applied to the 11-dimensional MS of signature \(1(-)\&10(+)\) the DM systems have dimension \(32 \times 32\) and subset \([24]\) is factorized as follows:

\[
\text{Matrix}_{32\times32} = \text{Matrix}_{8\times8} \otimes \text{Matrix}_{4\times4}.
\] (25)

Here is the explicit form of the DM LS:

\[
\Gamma_\alpha = e \otimes \gamma_\alpha \quad \alpha = 0, 1, 2, 3 \\
\Gamma_4 = g_1 \otimes \gamma \quad \Gamma_5 = g_2 \otimes \gamma \\
\Gamma_7 = g_1 \otimes \gamma \\
\Gamma_{10} = g_7 \otimes \gamma
\] (26)

Here \(\gamma\) denotes matrix \(4 \times 4\) \(\gamma \equiv \gamma_0\gamma_1\gamma_2\gamma_3\) (it is typically denoted as \(\gamma_5\)), \(e\) is the unit matrix \(8 \times 8\), and \(g_1, ..., g_7\) are matrices \(8 \times 8\) determined by relations \([10]\). The \(g_1, ..., g_7\) are the DM for the 7-dimensional space of signature \(7(-)\). When the 7-dimensional space as a subspace is included in the 11-dimensional space, the signature of the 7-dimensional space changes from \(7(-)\) to \(7(+)\).

The table below lists the characteristics of the matrix space being discussed in this paper.

Table 1. List of characteristics of the MS under discussion in this paper

| Riemannian space dimension | 11 |
|---------------------------|----|
| Riemannian space signature | \(1(-)\&10(+)\) |
| Matrix degree of freedom space dimension | \(32 \times 32\) |
| Dimension of the compactified part of the Riemannian subspace | 7 |
| Signature of the compactified part of the Riemannian subspace | \(7(+)\) |
| Matrix structure in the matrix degree of freedom space | \(M_{32\times32} = M_{8\times8} \otimes M_{4\times4}\) |

5. Comparison of different VDM properties

Compare properties of systems \(\{\gamma_A\}, \{g_A\}\).
First and foremost note that in either system the DM elements are integer numbers. Matrices \( D, D^{-1}, D, D^{-1} \) are integer matrices as well.

Matrix \( R \) is not a unitary matrix, therefore the system \( \{\mathbf{g}_A\} \) definitely does not reduce to \( \{\gamma_A\} \) through the orthogonal transformation either of base vectors of the 7-dimensional space or those of the internal 8-dimensional space.

The structure of DM \( \{\gamma_A\} \) and matrices \( D, D^{-1} \) possesses the following properties:

- Each of the above matrices is written in the form of a monomial, which is a direct product of the rho-sigma matrices.
- As few as one element is nonzero in each row and each column, with the nonzero element being equal to \( \pm 1 \).
- Each of the above matrices is either symmetrical or antisymmetrical.

The necessary condition for conversion of DM to VDM:

For some DM system to be a vacuum system, it is necessary that both matrix \( D \) and matrix \( D^{-1} \) be integer matrices.

If matrices \( D, D^{-1} \) are integer, then their determinants are the same and equal either to \(+1\) or \(-1\).

The statement is proved as follows. First the evident fact is noted: in an integer matrix its determinant is also an integer. Let

\[
\det (D^{-1}) \equiv p, \quad \det (D) \equiv q,
\]

where \( p, q \) are integers. As \( D, D^{-1} \) are matrices reciprocal to each other, then the following relation should hold: \( \det (D^{-1}) = \frac{1}{\det (D)} \), that is \( pq = 1 \).

The only combinations of integers \( p, q \) satisfying the relation are

\[ p = q = \pm 1. \]

Thus, for VDM

\[
\det (D^{-1}) = \det (D) = \pm 1. \tag{28}
\]

The following statements constitute the consequence of the above VDM definition:

1. If some DM system has been found to be a VDM, then the system derived from the original through arbitrary \( T \) - and \( L \)-transformations will also be a VDM.

2. There is a \( T \)-representation, in which the VDM is constant over the entire space.
3. From VDM it is impossible to construct not only a nontrivial tensor, but also a nontrivial tensor function.

The straightforward check of properties of the matrices \( \{ \gamma_A \} \), \( \{ g_A \} \) indicates that the systems \( \{ \gamma_A \} \), \( \{ g_A \} \) satisfy the above definition of VDM. At the same time, these two VDM systems are not interrelated with the \( L \)- and \( O \)-transformations, with accuracy to which the VDM are determined.

6. Weyl group

From the general theory of lattices it is known (see, e.g., [5]) that the densest packings are admitted by the 8-dimensional lattice \( E_8 \) and 24-dimensional Leech lattice \( \Lambda_{24} \). Hence, the case of the 8-dimensional lattice \( E_8 \) definitely has an independent meaning. The results can be used later on in consideration of a more complex case of the 24-dimensional Leech lattice \( \Lambda_{24} \).

The basis of the lattice \( E_8 \) consists of eight simple roots in this algebra. It is common practice to depict the simple root sets in the form of Dynkin’s diagrams (see, e.g., [6]-[8]). In Fig. 1 the diagrams are shown for two root systems: \( e^k \) for the orthonormal algebra; \( r^a \) for the algebra \( E_8 \).

Fig. 1. Dynkin’s diagrams for the basic systems \( e^k \), \( r^a \)

Each basic system consists of 8 vectors denoted with the open circle in the diagrams. The dark circle is correspondent with the vector norm equal to 1 and the open circle with that equal to \( \sqrt{2} \). If there is no line between the circles, then they are orthogonal. If the circles are connected with a solid line, then the angle between the respective vectors is \( 120^0 \).
The complete set of the root vectors of the algebra $E_8$ consists of 240 vectors and any system of the simple roots in the algebra of 8 vectors. From the set of all roots the simple root system is constructed nonuniquely. By way of example consider matrix $\tilde{R}$ describing a simple root system distinct from that described by matrix $R$ of form (19).

$$\tilde{R} = \begin{pmatrix}
1 & -1 & & & & \\
1 & -1 & 1 & -1 & & \\
1 & -1 & 1 & -1 & & \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1 & 1 & & & & \\
\end{pmatrix} \quad (29)$$

Matrix $W$ relating matrices $R$ of form (19) and $\tilde{R}$ of form (29) according to relation

$$\tilde{R} = W \cdot R,$$  \hspace{1cm} (30)

is of the form:

$$W == \begin{pmatrix}
-1 & & & & & \\
-1 & 1 & 2 & 3 & 4 & 4 \\
-1 & -1 & 1 & & & \\
-1 & -1 & & 4 & 2 & 1 \\
-1 & -2 & -3 & -4 & 5 & -4 \\
-1 & -2 & -3 & -4 & 5 & -3 \\
\end{pmatrix} \quad (31)$$

The above matrix $W$ is one of the elements of a so-called Weyl group for the root system in algebra $E_8$. Any matrix $W$ is distinguished with two properties: the determinant equal to one and integer elements. None of the matrices $W$ is orthogonal. The set of the matrices $W$ determining the transition from one simple root system to another comprises a finite group of unimodular matrices above the integer field. The order of the Weyl group coincides with the quantity of different simple root systems and is equal to (see, e.g., [6])
\[ \rho = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600. \] (32)

7. Comparison of VDM by magnitudes of physical quantities for vacuum

Thus, we have obtained two solution types for VDM. One type corresponds to the choice of Euclidean basis in the local internal spaces \( M_{\text{local}} \), the second type corresponds to the choice of Euclidean basis from the simple roots in algebra \( E_8 \). It is natural to pose the question as to what of these VDM describes the physical vacuum.

One of the criteria for the selection could be the magnitude of the simplest scalar

\[ Sp\left( D^{-1} \right) = 8; \quad Sp\left( D^{-1} \right) = 16 \] (33)

calculated for both the VDM. For VDM the magnitudes of scalars (33) are the same, which is easy to find, as the values of \( Sp\left( M \right) \), where \( M \) is the polarization density matrix. Its average value \( Sp\left( M \right) \) in the MS vacuum state has the meaning of background ("zero") physical vacuum energy density \( E_{\text{vac}} \),

\[ Sp\left( M \right) = E_{\text{vac}}. \] (34)

From this it follows that the "zero" vacuum energy packing density \( E_{\text{vac}} \) is two times higher for the VDM constructed on the roots of the algebra \( E_8 \) than that for the VDM constructed on the orthonormal basis. A denser packing is more beneficial energetically, hence, the vacuum solution is that VDM, in which the matrix \( D^{-1} \) is of form (23).

Another possible criterion for the selection of the true vacuum solution for VDM can be elementary cell averaged magnitudes of scalars

\[ \tilde{E}_{\text{vac fluc}} \equiv \left\langle Sp\left( Z^+ D Z \right) \right\rangle_{\text{cell}}; \quad \tilde{E}_{\text{vac fluc}} \equiv \left\langle Sp\left( Z^+ D Z \right) \right\rangle_{\text{cell}}. \] (35)

Averaging \( \left\langle \right\rangle_{\text{cell}} \) in the evaluation of (35) implies that not only the values of the matrix \( Z \) at the lattice nodes, but also those at all points within the
elementary cell are considered. Therefore, whereas scalars (33) characterize the background vacuum energy density, scalars (35), apparently, have the meaning of vacuum fluctuation energy density.

In space $M_{\text{local}}^8$ single out the elementary cells corresponding to the orthonormal lattice and lattice $E_8$. For the orthonormal lattice the cell volume is

$$V = \int_0^1 dx^1 \cdots \int_0^1 dx^8 = 1.$$  \hfill(36)

For the lattice $E_8$ the cell volume is

$$V = \sqrt{2} \int_0^\sqrt{2} dy^1 \cdots \int_0^\sqrt{2} dy^8 \sqrt{\text{det}(D)} = 2^4.$$ \hfill(37)

In (37) the integration is performed in the limits from 0 to $\sqrt{2}$, since the length of each of the basic spinor matrices is $\sqrt{2}$ (see Fig. 1); besides, $\sqrt{\text{det}(D)} = 1$ is taken into account.

Now calculate values (37) for the orthonormal cell and for cell $E_8$.

The value of the quadratic form $\tilde{E}_{\text{vac fluc}}$ over the elementary orthonormal cell is

$$\tilde{E}_{\text{vac fluc}} = \left( \frac{1}{V} \right) \int_0^1 dx^1 \cdots \int_0^1 dx^8 \left\{ x_1^2 + x_2^2 + \ldots + x_8^2 \right\} = \frac{8}{3}. \hfill(38)$$

The value of the quadratic form $\tilde{E}_{\text{vac fluc}}$ over the elementary cell constructed on the simple roots in the algebra $E_8$ is

$$\tilde{E}_{\text{vac fluc}} = \left( \frac{1}{V} \right) \sqrt{2} \int_0^\sqrt{2} dy^1 \cdots \int_0^\sqrt{2} dy^8 \left\{ 2y_1^2 + 2y_2^2 + 2y_3^2 + 2y_4^2 + 2y_5^2 + 2y_6^2 + 2y_7^2 + +2y_8^2 - 2y_1^2y_2^2 - 2y_2^2y_3^2 - 2y_3^2y_4^2 - 2y_4^2y_5^2 + 2y_5^2y_6^2 + +2y_6^2y_7^2 - 2y_7^2y_8^2 - 2y_8^2y_1^2 \right\} = \frac{11}{3}. \hfill(39)$$

The comparison of (39) with (38) shows that the density of elementary cell space filling with vacuum fluctuation energy is $11/8$ times higher for the lattice $E_8$ than that for the orthonormal lattice, that is

$$\tilde{E}_{\text{vac fluc}} / \tilde{E}_{\text{vac fluc}} = 11/8 \hfill(40)$$

Note that this result agrees with the results of the consideration of the classic problem of the search for a method of densest sphere packing in 8-dimensional space (see, e.g., [5]).
8. Conclusion

The paper has demonstrated that a Riemann space, in which it becomes possible to introduce a nonstandard type of autodual vacuum DM system, is the space of dimension 7 and signature 7 (−). For the Riemannian space to be integrated with the conventional Riemann space of the general relativity, it should be assumed that the 7-dimensional space is not only an internal subspace of a complete space, but also a compactified space. The resultant complete Riemannian space is of dimension 11 and signature 1 (−) &10 (+).

In the 7-dimensional space of the constructed 11-dimensional space, the DM vacuum system is the system \{g_A\} (A = 1, 2, · · ·, 7), whose explicit form is given in (16). The DM \{g_A\} are matrices 8 × 8 and have integer elements. In the 8-dimensional space of the matrix degrees of freedom of DM \{g_A\} the simple root vectors of the algebra $E_8$ are used for the basis. The transition of the orthonormal basis to the basis of the simple root vectors of the algebra $E_8$ is performed by matrix $R$ of form (19).

The vacuum state corresponding to the DM based on the simple root vectors of the algebra $E_8$ is in fact degenerate, with the degeneracy degree being the same as the order of Weyl group, a group of symmetry of the root system of the algebra. For $E_8$ the Weyl group order is determined as (32).

The introduction of the lattice VDM based on the simple root vectors of the algebra $E_8$ changes the MS vacuum state structure in a qualitative sense. It turns out that “zero” energy density for these VDM is higher than that for the VDM in their conventional representation. From the general principles of physics it follows that the vacuum state of the highest “zero” energy density is most stable and, hence, it is this state that must be used for the description of physical effects. This, in its turn, singles out the 11-dimensional space of signature 1 (−) &10 (+) among other spaces of lower dimensions and other signatures.

Thus, this paper has arrived at a direct answer to the question of what space dimensions and signatures should be considered in the physical theories, where the need arises to refer to multidimensional Riemannian spaces (supergravity; superstring theories). A candidate for the highest-priority consideration in the multidimensional version of the interaction theories should be the 11-dimensional Riemannian space of signature 1 (−) &10 (+) as a space of a most stable vacuum state. The stability of the vacuum state excludes appearance of tachyons causing a reconstruction of the state. The group of
symmetry of the internal 7-dimensional subspace is the group $O_8$ well-known in the superstring theory (see, e.g., [4]).

The results of this paper on the Dirac matrices constructed on the basis from lattice $E_8$ can be also used in the construction of Dirac matrices on the basis from Leech lattice $\Lambda_{24}$. This follows from the following two facts. First, according to [9], [10], the Leech lattice admits splitting $16+8$, i.e. the splitting into a sublattice of two $E_8$ and a sublattice of one $E_8$. Second, the internal Riemannian space dimension can be in principle not necessarily equal to 7, it can be multiple of 7. With dimension 21, the internal subspace description will require the Dirac matrices constructed from matrices $g_A$ in the form of their direct sum.

Pay attention to the fact that to obtain the above nontrivial results it was not necessary to resort to exotic schemes, to complication of the class of numbers, groups of symmetry, geometries, etc. It was sufficient to use the adequately studied lattice properties, the most fundamental class of numbers, i.e. integer real numbers, and intuitively clear matrix space vacuum state properties, i.e. impossibility to produce any coordinate dependent tensor quantities from the Dirac matrix vacuum systems. After that it turned out that the requirements to the DM vacuum systems can be met not only on the way of using the standard orthonormal basis, but also the oblique-angled basis corresponding to the system of simple roots of the algebra $E_8$. The properties of lattices, integers, and vacuum state that have been used in this paper are of “super-quantum” nature in the sense that they will hardly be measured in any quantization procedure.

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