A Generalization of the Classical Kelly Betting Formula to the Case of Temporal Correlation

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Abstract—For sequential betting games, Kelly’s theory, aimed at maximization of the logarithmic growth of one’s account value, involves optimization of the so-called betting fraction $K$. In this paper, we extend the classical formulation to allow for temporal correlation among bets. For the example of successive coin flips with even-money payoff, used throughout the paper as an example demonstrating the potential for this new research direction, we formulate and solve a problem with memory depth $m$. By this, we mean that the outcomes of coin flips are no longer assumed to be i.i.d. random variables. Instead, the probability of heads on flip $k$ depends on previous flips $k-1, k-2, \ldots, k-m$. For the simplest case of $n$ flips, even-money payoffs and $m=1$, we obtain a closed form solution for the optimal betting fraction, denoted $K_n$, which generalizes the classical result for the memoryless case. That is, instead of fraction $K = 2p-1$ which pervades the literature for a coin with probability of heads $p \geq 1/2$, our new fraction $K_n$ depends on both $n$ and the parameters associated with the temporal correlation model. Subsequently, we obtain a generalization of these results for cases when $m > 1$ and illustrate the use of the theory in the context of numerical simulations.

I. INTRODUCTION

In Kelly’s 1956 seminal paper [1], the notion of Expected Logarithmic Growth (ELG) was introduced as the performance criterion for an optimal Markov-style repeated betting game. For a sequence of i.i.d. gambles, for example a coin flip with the probability of heads being $p > 1/2$, the theory leads to an optimal betting fraction $K^*$. That is with $V_k$ being the account value after $k$ plays of this game, the optimum for the $(k+1)$-th bet size should be $K^*V_k$. For the specific example of coin flipping with even-money payoff, this optimum turns out to be $K^* = 2p - 1$.

This innovative approach has resulted in a voluminous body of literature extending and applying the theory to other well-known gambling games such as the ones considered in [2], sports betting as in [3] and asset management and stock trading as in [4]–[8]. In addition, work along these lines also includes papers providing attention to many related issues such as asymptotics, problems related to aggressiveness of wagers and use of alternative risk metrics; e.g., see [4], [5], [9]–[13] and the extensive bibliography in the textbook [14].

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The main feature which differentiates this paper from existing work is our emphasis on the issue of temporal correlation among bets. While it is standard to assume correlation among components of a multi-dimensional bet as in modern portfolio theory, for example see [5], temporal correlation is an entirely different matter. Given that temporal effects have been shown to exist among financial time-series, for example see [15], [16], interestingly, this issue has received little attention in the context of bet sizing in the ELG framework; e.g., see [17] where a specific example incorporating correlation is considered but analysis of the general case is lacking. Motivated in part by the fact that a bettor may gain an “edge” by taking advantage of a model which includes temporal correlation, we consider a generalization of the ELG maximization in this setting. For our motivating example of repeated coin flipping upon which one is staking even-money bets, we use the descriptor history cognizant in the sense that each outcome is no longer i.i.d. but dependent on the previous $m$ results.

Given the considerations above, the remainder of the paper is organized as follows: After formalizing the notion of autcorrelated betting in Section [II] we consider the special case of a history-cognizant coin in Section [III]. Then, our main result and various extensions and generalizations are covered in Section [IV]. This includes, for $n$ bets and memory depth $m=1$, a closed-form solution for the optimal betting fraction $K_n$, generalizing the $K^* = 2p - 1$ result mentioned above. Unlike existing results in the literature for the memoryless case, this solution is seen to depend on both the number of bets $n$ and also the level of autocorrelation present in the series of results. We also include steady state analysis and discussion of issues which arise in statistical estimation of the model. Section [V] is devoted to proof of the main theorem. Then, in Section [VI] we provide some generalizations for the case of memory depth $m > 1$ followed by numerical simulations and conclusions in the remaining two sections.

II. AUTOCORRELATED KELLY BETTING

We start by considering a discrete-time betting game with arbitrary initial account value $V_0 > 0$. Letting $V_n$ denote the bettor’s account value after $n$ bets, we aim to develop a Kelly strategy which maximises the expected value of the logarithm of $V_n$, rather than simply its expected value. Initially we consider a simple coin-flipping game which involves repeated i.i.d. flips of a biased coin with even-money payoff. We let $X_k \in \{-1, 1\}$ be a random variable which
We proceed to consider a game of \( n \) terminal time. This being the case, the corresponding account value at \( K \) and an optimal betting fraction which we call \( m \) are henceforth referred to as the \( m \) generality, the probability distribution is totally general. In general case which is abstract and then specialize a scenario frequently encountered in practice. That is, the outcome of a given coin toss is only related to the recent history. In this case, it is straightforward to see that the probability \( P_X \) of a sequence \( X \) reduces to

\[
P_X = \prod_{k=0}^{n-1} \Pr(X_k | X_{k-1}, X_{k-2}, \ldots, X_{k-m})
\]

which is initialized by the \( m \) events \( X_{-m}, X_{-m+1}, \ldots, X_{-1} \) prior the first outcome \( X_0 \) at time \( k = 0 \).

III. THE HISTORY COGNIZANT COIN

Building on the above, we consider the simplest case of an autocorrelated bet: a coin whose next flip is affected by the previous flip. This is a coin with Markov memory, i.e., for \( k > 0 \), the probability of a head on the \( k \)th flip is

\[
\Pr(X_k = 1 | X_{k-1}, X_{k-2}, \ldots) = \Pr(X_k = 1 | X_{k-1}) = \omega_0 + \omega_1 X_{k-1},
\]

For the parameterised linear function above, it is readily verified that the conditions \( |\omega_1| < 0.5, |\omega_0| < 1 - |\omega_1| \) guarantee that \( \Pr(X_k = 1 | X_{k-1}) \in (0, 1) \). Now, via a straightforward calculation, these requirements reduce to

\[
|\omega_0 - \frac{1}{2}| + |\omega_1| < \frac{1}{2}
\]

which we recognize as the interior of the \( \ell^1 \) sphere, the so-called “diamond” with center \((1/2, 0)\) and radius \((1/2)\).

In this setting, we have memory depth \( m = 1 \), and we assume that we have observed one coin toss prior to betting, i.e., \( X_{-1} = x_{-1} \in \{1, -1\} \). Figure 2 shows some illustrative sample paths, consistent with the formula provided above.

IV. MAIN RESULT

In this section, we provide our main result whose proof is relegated to the next section. The first part of the theorem below provides an abstract characterization of the optimal betting fraction in terms of the expected number of heads \( E(H_n(X)) \) in the sample path \( X \) of length \( n \). It points the way to second part of the theorem which addresses the special case of a history cognizant coin and makes use of the following
notation: We initialize with \( X_{-1} = x_{-1} \) which is assumed to be known, and define
\[
p_0 = \omega_0 + \omega_1 x_{-1}
\]
corresponding to the unconditional probability that \( X_0 = 1 \),
\[
p_\infty = \frac{\omega_0 - \omega_1}{1 - 2\omega_1}
\]
which is later seen to be the steady state unconditional probability of heads, and
\[
\lambda_n \leq \frac{1}{n} \left[ \frac{1 - (2\omega_1)^n}{1 - 2\omega_1} \right]
\]
which satisfies the condition
\[
0 < \lambda_n < 1
\]
whenever \( |2\omega_1| < 1 \) and is called the relative weighting coefficient in that it tells us the relative roles of \( p_0 \) and \( p_\infty \) in the optimal solution.

**Theorem:** For \( n \) flips, the optimum betting fraction maximizing the expected logarithmic growth \( \text{ELG}(K) \) is given by
\[
K_n = 2 \left( \frac{E(H_n(X))}{n} \right) - 1,
\]
for the history-cognizant coin with memory depth \( m = 1 \) and associated conditional probability of heads
\[
\Pr(X_k = 1 | X_{k-1}) = \omega_0 + \omega_1 X_{k-1},
\]
the expected value above becomes the convex combination
\[
\frac{E(H_n(X))}{n} = \lambda_n p_0 + (1 - \lambda_n) p_\infty.
\]

**A. Special Cases, Generalizability and Remarks**

The remainder of this section focuses on finer details of our theory including its reduction to the classical Kelly formula in the presence of zero autocorrelation, results regarding the limiting values of the parameters used in our theory, and also the generalization of the concepts beyond the simple case of even-money two-outcome bets.

**When Coin Flips are Independent:** For the special case of classical Kelly betting with all payoffs \( X_k \) being independent and identically distributed, \( E(H_n) \) is simply a constant. Equivalently, for memory depth \( m \) with \( \omega_i = 0, 1 \leq i \leq m \), we have \( p_0 = p_\infty = \omega_0 \), and thus
\[
K_n = 2p_0 - 1,
\]
i.e., the original optimal betting fraction considered by Kelly, showing that our theory simplifies to the classical case in the absence of autocorrelation among bets.

**Long Run Steady State Considerations:** Of general interest are the dynamics in the limit as \( n \to \infty \), the first point to note is that \( \lambda_n(\omega_1) \to 0 \) which in turn implies that
\[
\lim_{n \to \infty} \frac{E(H_n(X))}{n} = p_\infty
\]
describes the limiting value underpinning our Kelly optimal fraction. The interpretation of this limit is quite simple: If we are playing forever, the long-run probability of a head, \( p_\infty \) which leads to our optimal betting fraction, denoted \( K_\infty \) is given by
\[
K_\infty = 2p_\infty - 1.
\]
This is the same betting fraction as that which one would obtain by ignoring correlation among the \( X_k \) and treating \( p_\infty \) as if it is the unconditional probability of a head along the lines of previous literature. On the other hand, if we are betting for a fixed time horizon \( n \), the difference between the memoryless equivalent probability \( E(H_n)/n \) and \( p_\infty \) is important. In other words, when playing for \( n \) bets, the bettor needs to be attentive as to how the optimal betting fraction \( K_n \) depends on \( n \) and also how the startup probability \( p_0 = \omega_0 + \omega_1 x_{-1} \) depends on \( x_{-1} \) whereas \( p_\infty \) does not. In practice, the importance of \( x_{-1} \) depends on the size of \( n \) and the strength of the autocorrelation as manifested by the \( \omega \) parameters.

**Generalization of the Theorem:** As indicated in the introduction, the assumption of even-money, two-outcome random variables \( X_k \in \{-1, 1\} \) was introduced to demonstrate the potential for further research on correlated Kelly Betting by considering the simplest possible scenario. To provide some flavor as to the type of generalizations which are possible, we consider the case when there are \( \ell \) possible outcomes \( x_1, x_2, ..., x_\ell \in (-1, \infty) \) for the \( X_k \). In this case, we take \( P_X \) to be an arbitrary probability mass function over sample paths. Then, for sample path \( X \), with \( H_{n,i}(X) \) being the random variable denoting the number of times \( X_k = x_i \) for \( i = 1, 2, ..., \ell \) in \( n \) plays, and using an argument which is quite similar to the one used in the proof of the theorem, we obtain
\[
\text{ELG}(K) = \sum_{i=1}^\ell \frac{E(\tilde{H}_{n,i}(X))}{n} \log(1 + K x_i).
\]
This leads us to define the notion of a memoryless equivalent bet: With probabilities
\[
p_i = \frac{E(\tilde{H}_{n,i}(X))}{n}
\]
for \( i = 1, 2, ..., \ell \), one can “pretend” as if the bet is memoryless and maximize
\[
\text{ELG}(K) = \sum_{i=1}^\ell p_i \log(1 + K x_i)
\]
in accordance with existing theory.

**V. Proof of the Theorem**

This section can be skipped by those readers solely interested in the application aspects of this work. Indeed, to determine the optimal betting fraction, we maximize the Expected Logarithm Growth. For simplicity of notation, we suppress
the dependence of \( H_n \) on the sample path \( X \) and calculate

\[
\text{ELG}(K) = \frac{1}{n} \mathbb{E} \{ \log [V_X(n)] \} \\
= \frac{1}{n} \mathbb{E} \left\{ \log \left[ \prod_{k=0}^{n-1} (1 + K X_k) \right] \right\} \\
= \frac{1}{n} \mathbb{E} \left\{ \sum_{k=0}^{n-1} \log(1 + K X_k) \right\} \\
= \frac{1}{n} \sum_{x \in X} P_X \left\{ \sum_{k=0}^{n-1} \log(1 + K X_k) \right\} \\
= \frac{1}{n} \sum_{x \in X} P_X \{ H_n \log(1 + K) + (n - H_n) \log(1 - K) \} \\
= \frac{\mathbb{E}(H_n)}{n} \log(1 + K) + \left(1 - \frac{\mathbb{E}(H_n)}{n}\right) \log(1 - K).
\]

Now, noting that
\[
K = K_n = 2 \left\{ \frac{\mathbb{E}(H_n)}{n} \right\} - 1.
\]
is the unique point of zero derivative and that ELG\((K)\) is concave with respect to \(K\), it follows that \(K_n\) is the unique maximizer.

It remains to show that for the special case of the memory-cognizant coin, \( \mathbb{E}(H_n)/n \) reduces to the formula given for \(K_n\). We define the unconditional probability that \( X_k = 1 \),

\[
p_k = \Pr(X_k = 1) = \mathbb{E} \left( \frac{X_k + 1}{2} \right),
\]

where \( (X_k + 1)/2 = 1(X_k = 1) \) and \( 1(\cdot) \) is an indicator function, and observe that

\[
\mathbb{E}(H_n) = \sum_{k=0}^{n-1} p_k.
\]

Now, to obtain a formula for the sum above, beginning with conditional probability \( \Pr(X_k = 1 \mid X_{k-1}) = \omega_0 + \omega_1 X_{k-1} \), using the law of total expectation, we first obtain a recursion

\[
p_k = \mathbb{E} \{ \Pr(X_k = 1 \mid X_{k-1}) \} = \omega_0 + \omega_1 \mathbb{E}(X_{k-1}) = 2 \omega_1 p_{k-1} + \omega_0 - \omega_1,
\]

where the last line follows from \( \mathbb{E}(X_{k-1}) = 2p_{k-1} - 1 \). Now initializing, with \( p_0 = \omega_0 + \omega_1 x_{-1} \) and using shorthand notation \( a = 2\omega_1 \) and \( b = \omega_0 - \omega_1 \), we view the equation above as a linear time-invariant system with state \( p_k \) which may be expressed in standard form as \( p_{k+1} = ap_k + bu_k \) with constant input \( u_k = 1 \). For this quantity, the well known solution for the simple scalar system is

\[
p_k = a^k p_0 + b \frac{1-a^k}{1-a},
\]

from which we obtain

\[
p_k = (2\omega_1)^k p_0 + \left[1 - (2\omega_1)^k\right] p_\infty
\]

where \( p_k \to p_\infty = (\omega_0 - \omega_1)/(1 - 2\omega_1) \) since \( |2\omega_1| < 1 \).

Now, we calculate our main quantity of interest

\[
\frac{\mathbb{E}(H_n)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} p_k
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} a^k p_0 + \frac{1}{n} (1 - a^k) \frac{b}{1-a}
\]

\[
= \frac{1}{n} \left( \frac{1-a^n}{1-a} \right) p_0 + \frac{1}{n} \left( n - 1 - a^n \right) \frac{b}{1-a}
\]

\[
= \lambda_n p_0 + (1 - \lambda_n)p_\infty. \square
\]

VI. Deeper Memory

In this section, we consider the general case of memory depth \( m > 1 \) and show how analytic expressions for \( \mathbb{E}(H_n) \) can be efficiently obtained. Indeed, we begin with parameterization of the conditional probability

\[
\Pr(X_k = 1 \mid X_{k-1}, X_{k-2}, \ldots, X_{k-m}) = \omega_0 + \sum_{i=1}^{m} \omega_i X_{k-i}
\]

with assumed initial conditions

\[
X_{-i} = x_{-i} \text{ for } i = 1, 2, \ldots, m.
\]

To ensure that \( \Pr(X_k = 1 \mid X_{k-1}, X_{k-2}, \ldots, X_{k-m}) \in (0, 1) \) we assume the \( \omega \) parameters to lie in the “hyperdiamond”

\[
\left| \omega_0 - \frac{1}{2} \right| + \sum_{i=1}^{m} |\omega_i| < \frac{1}{2}
\]

so as to assure that we have a probability measure above.

Then, noting the unconditional probability \( p_k \) that \( X_k = 1 \) can be expressed as

\[
p_k = \mathbb{E}\{\Pr(X_k = 1 \mid X_{k-1}, X_{k-2}, \ldots, X_{k-m})\},
\]

we take expectations on both sides in the first equation above to arrive at

\[
p_k = \omega_0 + \sum_{i=1}^{m} \omega_i \mathbb{E}(X_{k-i})
\]

for \( k = 0, 1, \ldots, n-1 \). Substituting \( \mathbb{E}(X_{k-i}) = 2p_{k-i} - 1 \), above and taking note of the “induced” initial conditions

\[
p_{-i} = \frac{(x_{-i} + 1)}{2} \text{ for } i = 1, 2, \ldots, m,
\]

we arrive at the recursion

\[
p_k = \omega_0 - \sum_{i=1}^{m} \omega_i + 2 \sum_{i=1}^{m} \omega_i p_{k-i}
\]

which holds for \( k = 0, 1, \ldots, n-1 \), and from which \( \mathbb{E}(H_n) \) and hence \( K_n \) may be calculated.

To solve the recursive equation above in closed form, for \( m \) and \( n \) not too large, it is straightforward to use symbolic computation to forward propagate \( p_k \). To illustrate, for the
memory depth \( m = 3 \) and \( n = 2 \) flips, a symbolic computation leads to expected number of heads given by
\[
E(H_2) = x_{-1}(2\omega_2^2 + \omega_1 + \omega_2) + x_{-2}(2\omega_1\omega_2 + \omega_2 + \omega_3) + x_{-3}(2\omega_1\omega_3 + \omega_3) + 2\omega_0\omega_1 + 2\omega_0 - \omega_1.
\]

An alternative to the above, which may perhaps prove useful in future research, involves using a standard state-space realization of the “delay system” to represent the scalar recursion. That is, by introducing the \( m \)-dimensional state vector \( v_k = [p_k-m+1, p_k-m+2, \ldots, p_k]^T \), we readily obtain a classical companion form realization \( v_{k+1} = Av_k + bu_k \) with triple \((A, b, c)\), input \( u(k) \equiv 1 \) and output being \( p_k \). To illustrate, for memory depth \( m = 3 \), we obtain
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
2\omega_3 & 2\omega_2 & 2\omega_1
\end{bmatrix}; \quad b = \begin{bmatrix}
\omega_0 - \omega_1 - \omega_2 - \omega_3
\end{bmatrix};
\]
and \( c = [0 \ 0 \ 1] \) with solution of the recursion given by
\[
p_k = c \left( A^k v_0 + \sum_{i=0}^{k-1} A^{k-1-i} b \right) = c \left( A^k v_0 + (I - A)^{-1}(I - A^k)b \right)
\]
and the matrix \( I - A \) is guaranteed to be invertible since its determinant is
\[
\det(I - A) = 1 - 2 \sum_{i=1}^{m} \omega_i.
\]

In addition, from Gerschgorin’s circle theorem [18] and the hyperdiamond constraint, each eigenvalue of \( A \) has magnitude less than 1, and so \( A^k \to 0 \) as \( k \to \infty \). Beginning with \( p_k \) above, this leads to the generalization of our steady-state unconditional probability to be
\[
p_{\infty} = c(zI - A)^{-1} b
\]
and recognizing that this corresponds to the transfer function \( H(z) \) for the triple \((A, b, c)\) evaluated at \( z = 1 \), we immediately arrive at
\[
p_{\infty} = \frac{\omega_0 - (\omega_1 + \cdots + \omega_m)}{1 - 2(\omega_1 + \cdots + \omega_m)}.
\]

**Remark:** In practice, prior to betting, it is necessary to obtain values for the \( \omega \) parameters. We define the “response” variable
\[
Y_k = \frac{1}{2}(X_k + 1)
\]
such that
\[
E(Y_k \mid X_{k-1}, X_{k-2}, \ldots, X_{k-m}) = \omega_0 + \sum_{i=1}^{m} \omega_1 X_{k-i}.
\]

Therefore, in a learning phase, prior to betting, we observe data \( x_{-\ell}, \ldots, x_{-1} \) from which we compute the \((\ell - m) \times 1\) response vector
\[
y = \frac{1}{2} \{ [x_{-\ell+m}, x_{-\ell+m+1}, \ldots, x_{-1}]^T + 1 \},
\]
and form the residual sum of squares
\[
\text{RSS}(\omega) = \sum_{k=-\ell+m}^{-1} \left( y_k - \omega_0 - \sum_{i=1}^{m} \omega_1 x_{k-i} \right)^2,
\]
to be minimized with respect to
\[
\omega = [\omega_0 \ \omega_1 \ \cdots \ \omega_m]^T.
\]

Whereas classical estimation theory with sufficiently exciting signals leads to *least squares* solution
\[
\hat{\omega} = \arg \min_{\omega} \text{RSS}(\omega) = (X^TX)^{-1}X^T y
\]
with \( X \) being the \((\ell - m) \times (m+1)\) matrix whose \( i \)th row is given by \([1 \ x_{-\ell+m-2} \ x_{-\ell+m-3} \ \cdots \ x_{-1}]^T \) and, in this case, enforcement of the hyperdiamond constraints leads to a positive-definite quadratic program to be solved.

**VII. Numerical Simulation**

In this section, we carry out simulations comparing the optimal ELG performance using the fraction \( K_n \) obtained from our new temporal correlation formula with that obtained using Kelly’s classical i.i.d. result \( K^* = 2p - 1 \). To this end, the performed simulations are driven by an underlying process of returns \( X_k = 1 \) given by conditional probability \( \omega_0 + \omega_1 X_{k-1} \). For the case of the classical bettor whom we call the *Kellier*, a model with independent and identically distributed returns is assumed. Accordingly, as seen in Section IV, this bettor mistakenly believes the probability of heads to be \( p = p_{\infty} = (\omega_0 - \omega_1)/(1 - 2\omega_1) \) and uses betting fraction \( K^* = 2p - 1 \). On the other hand, assuming that \( \omega_0 \) and \( \omega_1 \) have been perfectly estimated, the second better whom we call the *Autocorrelator* uses betting fraction \( K_n \) as per the aforementioned theorem.

With the setup above, a wide variety of interesting simulations are possible. To provide the flavor of our findings, we first consider the following scenario: We initialize with \( X_{-1} = 1 \), letting \( \omega_0 = 0.55 \) and \( \omega_1 = 0.20 \). In this case, a straightforward calculation leads to \( p_0 = 0.75, p_{\infty} = 0.5833, K^* \approx 0.1667, \lambda_n \approx 0.556(1 - 0.4^n)/n \) and finally,
\[
K_n \approx \frac{0.556}{n} (1 - 0.4^n) + 0.167.
\]

Then, for \( n = 1, 2, \ldots, 10 \), we calculate the two optimal performance levels \( ELG(K^*) \) and \( ELG(K_n) \), as a function of the number of flips and compare the two. As seen in Figure 3 for this positively correlated case, the Autocorrelator strongly outperforms the Kellier. However, when we re-run the simulation above with negative correlation \( \omega_1 = -0.2 \) instead (not shown), consistent with common sense considerations, results are rather different. The potential for more frequent fluctuations results in much smaller betting fractions and performance of the two bettors being more similar.

Another interesting scenario studied is obtained with parameters \( X_{-1} = 1, \omega_0 = 0.35 \) and \( \omega_1 = 0.33 \). In this case, rather dramatic performance differences between the two bettors result. As seen in Figure 4 in all the cases where
In this paper, we formulated a class of Kelly optimal ELG problems which account for temporal correlation from gamble to gamble. To demonstrate the potential for this new line of research, we highlighted the special case of repeated coin flips with even-money payoff and probability of heads line of research, we highlighted the special case of repeated coin flips with even-money payoff and probability of heads. To demonstrate the potential for this new line of research, we highlighted the special case of repeated coin flips with even-money payoff and probability of heads.

Regarding future research, many possibilities present themselves. Foremost among these, we envision full development of the preliminary non-even money, multiple-payoff case which was described following the theorem. A second research direction involves use of our binary lattice model with stock returns, for example see [5], to bring the ideas in this paper into the realm of algorithmic trading. A third direction involves extension of the theory in this paper to the so-called portfolio case. That is, instead of scalar returns, we have a vector \( X_k \) with correlation both temporally and across components. Given that \( K \) now becomes a vector, it is felt that the concave programming will play an important role in computation. Finally, a fourth promising direction involves use of a time-varying betting fraction instead of the constant \( K \) assumed here; e.g., see [19] for initial work along these lines.

\[ \text{ELG}(K^*) \text{ in Black; ELG}(K) \text{ in Red} \]

**Fig. 3.** Simulation with \( X_{-1} = 1, \omega_0 = 0.55 \) and \( \omega_1 = 0.20 \)

**Fig. 4.** Simulation with \( X_{-1} = 1, \omega_0 = 0.35 \) and \( \omega_1 = 0.33 \)

ten or less coins are bet on, the Kellier turns out to be a “loser” in the sense that the expected logarithmic growth, even when optimized, is negative. On the other hand, the Autocorrelator achieves positive growth throughout the entire range of interest.

**VIII. CONCLUSION**

In this paper, we formulated a class of Kelly optimal ELG problems which account for temporal correlation from gamble to gamble. To demonstrate the potential for this new line of research, we highlighted the special case of repeated coin flips with even-money payoff and probability of heads taken to be a function of the previous \( m \) outcomes. In the main theorem, for memory depth \( m = 1 \) and \( n \) flips, a closed form solution for the optimal betting fraction \( K_n \) was obtained. The paper also includes analysis for the case when \( n \to \infty \) and solutions for deeper memory \( m > 1 \) which can be obtained symbolically by either propagation of the recursive formula for \( p_k \) or use of the state space realization for the associated delay system. Finally, we provided numerical simulations which include comparison with classical Kelly betting results which do include temporal correlation. Regarding future research, many possibilities present themselves. Foremost among these, we envision full development of the preliminary non-even money, multiple-payoff case which was described following the theorem. A second research direction involves use of our binary lattice model with stock returns, for example see [5], to bring the ideas in this paper into the realm of algorithmic trading. A third direction involves extension of the theory in this paper to the so-called portfolio case. That is, instead of scalar returns, we have a vector \( X_k \) with correlation both temporally and across components. Given that \( K \) now becomes a vector, it is felt that the concave programming will play an important role in computation. Finally, a fourth promising direction involves use of a time-varying betting fraction instead of the constant \( K \) assumed here; e.g., see [19] for initial work along these lines.

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