ADDITIVITY OF SUPPORT NORM OF TIGHT CONTACT STRUCTURES

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ABSTRACT. The support norm $sn(\xi)$ of a contact structure $\xi$ is the minimum of the negative Euler characteristics of the pages of the open books supporting $\xi$. In this paper we prove additivity of the support norm for tight contact structures.

1. INTRODUCTION

Since Giroux’s discovery [4] of the relation between open books and contact structures, one of the main focus of research in 3-dimensional contact geometry has been to deduce contact invariants from the combinatorics of their supporting open books. The most obvious such invariant is the support genus or $sg(\xi)$ of a contact structure, which is the minimal possible genus of open books supporting $\xi$. Etnyre [3] showed that the support genus of overtwisted contact structures is zero, and he gave obstructions of having support genus zero in terms of the symplectic fillings of the contact structure. Ozsváth, Stipsicz and Szabó [10] gave another obstruction for $sg(\xi) = 0$. Namely, that the contact invariant in Heegard Floer homology has to be reducible. These criteria give rise to examples of contact 3-manifolds whose support genus is at least one. However there is no confirmed examples of contact structures with support genus greater than one.

Another combinatorial invariant of similar type for a contact structure $\xi$ is the support norm or $sn(\xi)$, which is the minimum of the negative Euler characteristic of pages of open books supporting $\xi$. Since an open book gives rise to a Heegaard decomposition whose genus is the Euler characteristic of the open book minus one, we can immediately give contact 3-manifolds with arbitrarily large support norm. Heegaard decompositions, that are constructed from these open books have been also studied by Özbağcı [9], and their minimal genus was denoted by $Hg(\xi)$, where $Hg(\xi) = sn(\xi) - 1$.

As always, it is important to understand how these two invariants behave under connected sum. Since the connected sum can always be done in a small ball that intersects the binding of the open book the invariants are sub-additive. More precisely $sg(\xi_1 \# \xi_2) \leq sg(\xi_1) + sg(\xi_2)$ and $sn(\xi_1 \# \xi_2) \leq sn(\xi_1) + sn(\xi_2) + 1$ (or $Hg(\xi_1 \# \xi_2) \leq Hg(\xi_1) + Hg(\xi_2)$). The
additivity of the support genus would immediately provide examples for contact structures with arbitrarily high support genus. In general neither the support genus nor the support norm is additive. The connected sum of a contact structure with support genus one and an overtwisted contact structure gives an overtwisted contact structure which has support genus zero. Özbağcı [9] gave a similar example for the non-additivity of support norm using the homological classification of Eliashberg [2] for overtwisted contact structures. He observes that any overtwisted contact structure on an integer homology sphere is isotopic to its connected sum with the overtwisted contact sphere $(S^3, \xi_{-\frac{1}{2}})$ with $d_3 = -\frac{1}{2}$, which has $Hg(\xi_{-\frac{1}{2}}) = 2$. In this paper we prove that for tight contact structures the support norm is additive:

**Theorem 1.1.** (additivity of Euler characteristic for tight contact structures) Let $(Y_1, \xi_1)$ and $(Y_2, \xi_2)$ be tight contact 3-manifolds and let $(Y, \xi)$ be their contact connected sum. Then

$$sn(\xi_1) + sn(\xi_2) + 1 = sn(\xi)$$

This statement is another fine appearance of the tight–overtwisted dichotomy of 3-dimensional contact structures. The proof uses cut and paste arguments for open books, that is more generally phrased and understood by the machinery, built up in a preprint of the author with Licata [7] of foliated open books.

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**2. PRELIMINARIES**

In this section we review basic notions about contact structures, open books and the foliations they induce on embedded surfaces.

An open book $(B, \pi)$ of a closed oriented 3-manifold $Y$ is a pair of an embedded 1-manifold $B$ in $Y$ and a map $\pi: Y \setminus B \to S^1$ that restricts to $\vartheta$ in some tubular neighborhood $N(B) \cong B \times D^2(v, \vartheta)$ of the binding $B$. The closures of the preimages $\pi^{-1}(t)$ are called the pages of the open book. By the previous condition $B$ is the oriented boundary of each page. The Euler characteristic of a page of $(B, \pi)$ is denoted by $\chi(B, \pi)$, and the norm

An abstract open book is a pair $(S, h)$ of a genus $g$ surface $S$ with boundary and a diffeomorphism $h: S \to S$, called the monodromy, that is the identity
near $\partial S$. We can recover a 3-manifold $Y_{(S,h)}$ by factoring the mapping cone of $h$ by the extra relation $(x, t) \sim (x', t)$ for $x \in \partial S$ and $t, t' \in S^1$. An abstract open book $(S, h)$ corresponds to an (embedded) open book $(B, \pi)$ if there is a diffeomorphism between $Y$ and $Y_{(S,h)}$ that maps $B$ onto the equivalence class of $\partial S$ and restricts to a bundle map on $Y \setminus B$.

As it is described by Ito and Kawamuro [5, 6] an open book decomposition $(B, \pi)$ induces a foliation on (generic) embedded surfaces $\iota: F \hookrightarrow Y$. An open book foliation $\mathcal{F}_{ob} = \mathcal{F}_{ob}(\iota)$ is an oriented singular foliation on $F$ whose leaves are defined by the level-sets $(\pi \circ \iota)^{-1}(t)$. By definition\(^1\) open book foliations satisfy the following properties:

- The transverse intersection points of $B$ with $\iota(F)$ are exactly the elliptic singularities of $\mathcal{F}_{ob}$ (See the first two pictures of Figure 1). An elliptic point is positive if $B$ coorients $F$, and negative otherwise. Positive elliptic points are sources, while negative elliptic points are sinks of $\mathcal{F}_{ob}$. The set of elliptic points on $F$ is $E = E_+ \cup E_-;$
- The pull back $\tilde{\pi}$ of the map $\pi$ onto $F \setminus E$ is a circle-valued Morse function such that each critical point have different values;
- The maxima and minima of $\tilde{\pi}$ are center singularities of $\mathcal{F}_{ob}$ (See the third and fourth pictures of Figure 1);
- The saddles of $\tilde{\pi}$ are hyperbolic points of $\mathcal{F}_{ob}$ (See Figure 2). A hyperbolic point is positive if $\nabla \pi$ coorients $F$, and it is negative otherwise. The set of hyperbolic points on $F$ is $H = H_+ \cup H_-.$

By using Thom transversality theorem one can prove that any surface can be isotoped so that it admits an open book foliation. Moreover, by a further isotopy one can assume that $\mathcal{F}_{ob}$ is circle-free:

**Proposition 2.1** ([5] removing circles). *Suppose that the embedding $\iota: F \hookrightarrow Y$ has an open book foliation with respect to the open book $(B, \pi)$. Then there is an isotopy of the embedding $\iota$ to $\iota'$ such that the foliation $\mathcal{F}_{ob}(\iota')$ has no circles.\(^\square\)*

As a corollary, in such an embedding $\mathcal{F}_{ob}$ has no center singularities. In the sequel we will always assume that $\mathcal{F}_{ob}$ has no circles. This property will turn out to be essential when one relates open book foliations to characteristic foliations (See Proposition 2.3) which is one of the main tools in this paper.

Here we describe how one can alter the foliation $\mathcal{F}_{ob}$ by isotoping the embedding of $F$. Note that even though there are more possibilities in changing the open book foliation even in a general open book, here we will only concentrate on the change we need in $S^3$ with the standard open book. The standard open book $(B_0, \pi_0)$ of $S^3$ is defined on $S^3 \subset C^2(z_1, z_2)$ with

\(^1\)The original definition in [5] is weaker, and does not require $\pi \circ \iota$ to be a Morse function. Later in the proof of Theorem 2.21 of [5], however the authors use this property.
$B_0 = \{z_1 = 0\} \cap S^3$ and $\pi_0 : S^3 \setminus B_0 \to S^1$ is $\frac{z_1}{|z_1|}$. This open book has disc pages.

**Proposition 2.2 (Change in foliation).** Let $\mathcal{F}_{\text{ob}}(\iota)$ be the open book foliation induced by the open book $(B_0, \pi_0)$ on the embedding $\iota : F \hookrightarrow S^3$. Suppose that there is a disc $D \subset F$ so that the foliation restricted to $D$ is as on the first picture in Figure 3. Then there is an isotopy of the embedding $\iota$ to $\iota'$ (or $\iota''$) that is constant away from $D$ such that the open book foliation $\mathcal{F}_{\text{ob}}(\iota')$ is the second (or the third) picture on $D$. □

These changes are achieved by exchanging the $\pi$-values of the hyperbolic singularities.

A contact structure $\xi$ on an oriented 3-manifold $Y$ is a plane field given as the kernel of a 1-form $\alpha$ for which $\alpha \wedge d\alpha$ is a volume form. The characteristic foliation $\mathcal{F}_\xi = \mathcal{F}_\xi(\iota)$ on $F$ corresponding to the embedding $\iota : F \hookrightarrow Y$ is an oriented singular foliation given by the dual of the 1-form $\iota^* \alpha$. The leaves of this foliation are given by the pull-back of the integral of the oriented line field $T_p \lambda(F) \cap \xi_p$. The singular points of $\mathcal{F}_\xi$ are the points where $T_p \lambda(F) = \xi_p$, and one can assign signs to them depending on whether the orientation of $T_p \lambda(F)$ and $\xi_p$ agree (positive) or disagree (negative). Positive elliptic points are sources, while negative ones are sinks.

![Figure 1. Singularities of open book foliations. The first row depicts the embedding $\iota : F \hookrightarrow Y$ with respect to the open book. While the second row depicts the foliation $\mathcal{F}_{\text{ob}}$ on $F$. The first and second pictures are intersections of the binding with $\iota(F)$. The third and fourth pictures depict a minimum and a maximum of $\pi \circ \iota$. The surface $F$ is depicted yellow when its orientation is counterclockwise, and it is orange when its orientation is clockwise.](image-url)
Later we will use a graph associated to an oriented singular foliation $\mathcal{F}$ with compact leaves, only elliptic and hyperbolic singular points and with the extra information of “sign” for hyperbolic singularities. The vertices of the graph $G_{++} = G_{++}(\mathcal{F})$ are the positive elliptic points (sources) of $\mathcal{F}$, and the edges connect those positive elliptic points that lie on the ends of stable separatrices of the same positive hyperbolic point. See Figure 4.

An open book $(B, \pi)$ supports a contact structure $\xi = \ker \alpha$ if $\alpha$ is positive on $B$ and $d\alpha$ pulls back as a volume form to the pages. The relation between open book foliations and characteristic foliations is described in the following. We say that two oriented singular foliations on a surface $\mathcal{F}$ are topologically conjugate if there is a homomorphism of the surface that takes one foliation to the other. For foliations with compact leaves this means that
the combinatorial information of which elliptic points do the separatrices of the hyperbolic points go is the same.

**Proposition 2.3** ([5] connection of open book foliations and characteristic foliations). Let \((B, \pi)\) be an open book of the 3-manifold \(Y\). Suppose that for an embedding \(\iota: F \hookrightarrow Y\) the open book foliation has no circles. Then there is a contact structure \(\xi\) supported by the open book \((B, \pi)\) such that \(\mathcal{F}_\xi\) and \(\mathcal{F}_{ob}\) are topologically conjugate.

**Remark 2.4.** In most (maybe all) papers about braid- or open book foliations, the authors treated these singular foliations merely as a collection of leaves with some singularities with no additional structure. On the other hand Giroux [4] defined characteristic foliations as the dual of the pullback of the contact form to the surface. Thus characteristic foliations have an extra structure of a singular vector field up to multiplicity with a smooth positive function. The sign of the divergence of such structures is well-defined in singular points, and as proved by Giroux these divergences are always nonzero. A more precise definition of open book foliations would define them as a pair \((E, \tilde{\pi})\) and the corresponding vector field away from the elliptic points is the dual of \(d\tilde{\pi}\). But in this case the divergence would be zero everywhere. Thus one cannot even hope that the two singular foliations agree. That is why the above statement only states topological conjugacy. As Giroux proved [8], nonzero divergence on the singularities implies that the set of singular points and the leaves determine the equivalence class of the singular foliations, thus for characteristic foliations we do not lose information by using this vague definition.

The proof of Proposition 2.3 uses local models for \((B, \pi)\) near the singular and regular points of \(\mathcal{F}_{ob}\), and alters the Thurston and Winkelnkemper [11] construction just slightly near \(B \cap F\).

Let \((B, \pi)\) and \((B', \pi')\) be open books for the 3-manifolds \(Y\) and \(Y'\) respectively. Assume that for the embeddings \(\iota: F \hookrightarrow Y\) and \(\iota': F \hookrightarrow Y'\) the open book foliations \(\mathcal{F}_{ob}(\iota)\) and \(\mathcal{F}_{ob}(\iota')\) agree and have no circle leaves. If the surfaces \(\iota(F)\) and \(\iota'(F)\) are separating then, as we prove in [7], one can...
obtain new contact 3-manifolds with open books by open book surgery\(^2\) as follows. Cut open the 3-manifolds \(Y\) and \(Y'\) along \(\iota(F)\) and \(\iota'(F)\) to obtain 3-manifolds with boundaries. Denote the parts of \(Y\) by \(Y_1\) with \(\partial Y_1 \cong F\) and \(Y_2\) with \(\partial Y_2 \cong -F\). Similarly denote the parts of \(Y'\) by \(Y'_1\) with \(\partial Y'_1 \cong F\) and \(Y'_2\) with \(\partial Y'_2 \cong -F\). Then using the maps \(\iota^{-1} \circ \iota'\) between the boundaries one can form the new objects:

\[
\begin{align*}
Z &= Y_1 \cup Y'_2 \quad \text{and} \quad Z' = Y'_1 \cup Y_2 \\
\zeta &= \xi|_{Y_1} \cup \xi'|_{Y'_2} \quad \text{and} \quad \zeta' = \xi'|_{Y'_1} \cup \xi|_{Y_2} \\
C &= (B \cap Y_1) \cup (B' \cap Y'_2) \quad \text{and} \quad C' = (B' \cap Y'_1) \cup (B \cap Y_2)
\end{align*}
\]

\(\varrho = \pi|_{Y_1 \setminus B} \cup \pi'|_{Y'_2 \setminus B'} : Z \setminus C \to S^1\) and \(\varrho' = \pi'|_{Y'_1 \setminus B'} \cup \pi|_{Y_2 \setminus B} : Z' \setminus C' \to S^1\)

Then it is proved in [7]:

**Theorem 2.5** ([7] open book surgery). With the above notations and conditions the contact structures \((Z, \zeta)\) and \((Z', \zeta')\) are supported by the open books \((C, \varrho)\) and \((C', \varrho')\).

3. **PROOF OF THE MAIN THEOREM**

The inequality \(\text{sn}(\xi_1) + \text{sn}(\xi_2) + 1 \geq \text{sn}(\xi)\) is obvious, as one can form connected sum of open books for \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\) in the neighbourhood of a point on their bindings, which induces boundary connected sum on the pages.

Let \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\) be tight contact 3-manifolds and let \((B, \pi)\) be a supporting open book with minimal norm for their connected sum \((Y, \xi) = (Y_1, \xi_1) \# (Y_2, \xi_2)\). In the following two lemmas and using open book surgery we will construct open books \((B_1, \pi_1)\) and \((B_2, \pi_2)\) for \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\) with norms that add up to \(\chi(B, \pi) + 1\).

Since \(Y = Y_1 \# Y_2\) there is a separating embedded sphere \(\iota : S^2 \hookrightarrow Y\) along which the connected sum is formed. By a possible isotopy of the embedding we can make sure that the sphere admits an open book foliation \(\mathcal{F} = \mathcal{F}_{ob}\) with no circles.

**Lemma 3.1** \((G_{++})\) is a tree. Let \(\mathcal{F}\) be an open book foliation with no circles on a sphere, induced by an open book \((B, \pi)\) that supports a tight contact structure. Then the graph \(G_{++}(\mathcal{F})\) is a tree.

**Proof.** By Proposition 2.3 and the uniqueness of contact structures supported by a given open book, \(\xi\) can be isotoped so that \(\mathcal{F}_{\xi}\) and \(\mathcal{F}\) are topologically conjugates of each other. In particular the graphs \(G_{++}(\mathcal{F})\) and \(G_{++}(\mathcal{F}_{\xi})\) are homomorphic. The dividing curve \(\Gamma\) on \(S^2\) is given by \(\partial N(G_{++}(\mathcal{F}_{\xi}))\). As \(\xi\)

\(^2\)For being able to do open book surgery on the nose one needs that the actual functions \(\pi \circ \iota\) and \(\pi' \circ \iota'\) agree on \(F\), which is very restrictive. Instead we relax this condition, and only require that the singular foliations agree, but then for the gluing we need to build up a general theory, that is worked out in [7].
is tight $\Gamma$ has only one component. Thus $G_{++}(F_\ell)$ is a tree, and then $G_{++}(F)$ is a tree as well.

**Proposition 3.2** (realisation of a foliation on $S^2$ as open book foliation in $(B_0, \pi_0)$). Let $F$ be an open book foliation on a sphere with no circles. If $G_{++}(F)$ is a tree, then $F$ can be realised as an open book foliation induced by the standard open book $(B_0, \pi_0)$ for some embedding $\iota'$: $S^2 \hookrightarrow S^3$.

**Proof.** Let $e_\pm$ denote the number of positive/negative elliptic points and $h_\pm$ be the number of positive/negative hyperbolic points in $F$. Since the negative elliptic points are paired to the positive elliptic points in $Y_1$ by the arcs $B_1 \cap Y_1$ we have $e_+ = e_-$. As $G_{++}$ is a tree we get $e_+ - 1 = h_+$, and by the Poincaré–Hopf Theorem we have $(e_+ + e_-) - (h_+ + h_-) = 2$. Thus $h_- = e_+ - 1$. From now on let $k = e_+$ and we will use induction on $k$.

If $k = 1$ then there are no hyperbolic points, and the standard embedding of $S^2$ with image in the pure imaginary part of $S^3 \subset \mathbb{C}^2$ works. Otherwise assume that $k > 1$ and take a degree one vertex $p$ of the graph $G_{++}$. Consider the star $D_p$ of $p$. As shown on the first picture of Figure 5, $D_p$ is a disc whose boundary consists of $n$ hyperbolic points with their unstable separatrices connecting them to $n$ negative elliptic points. Note that since each of the hyperbolic points are connected to $p$ they all happen in different times (i.e. they have different $\pi$-value), thus the boundary of $D_p$ is embedded, and even a sufficiently small neighbourhood of $D_p$ is embedded in $S^2$. Also, since $p$ has degree one in the graph $G_{++}$ all but one of the hyperbolic points of $D_p$ are negative.

If $n = 2$, then $F$ can be produced from a foliation $F'$ where one exchanges the neighbourhood of $D_p$ with the neighbourhood of a single negative elliptic point. The new $G_{++}(F')$ is still a tree, thus by induction $F'$ is induced by an embedding of $S^2$ into $(B_0, \pi_0)$. Now performing a finger move, as in Figure 6, pushing the neighbourhood of the negative elliptic point.

![Figure 5](image-url)
point, that we replaced $D_p$ with, through the $z$-axis gives a new embedding of $S^2$ into $(B_0, \pi_0)$ with open book foliation $\mathcal{F}$ on it. While doing this finger move we can achieve any identification of the boundary of $D_p$ and $D - D_p$, thus we can ensure that we get back the same leaves as we started with.

![Figure 6. Changing the open book foliation with a finger move.](image)

If $n > 2$, then since all but one of the hyperbolic points of $D_p$ are negative we can perform a local change of the foliation on two consecutive hyperbolic points as on the last two pictures of Figure 5 and get a new open book foliation with smaller $n$. Thus by a sequence of these local change moves we can construct a foliation $\mathcal{F}'$ with $n = 2$, and then as before $\mathcal{F}'$ can be realised by an embedding of $S^2$, and by Proposition 3.2 we can do the reverse of the local change moves, now by changing the $\pi$-values of the hyperbolic points in question. This change gives us the required embedding.

End of the proof for Theorem 1.1. We continue the argument from the beginning of the section of constructing open books for $(Y_1, \xi_1)$ and $(Y_2, \xi_2)$ from an open book $(B, \pi)$ for $(Y, \xi)$. Let $\mathcal{F}$ be an open book foliation with no circles induced on the separating sphere $\iota: S^2 \hookrightarrow Y = Y_1 \# Y_2$, using Lemma 3.1 and Proposition 3.2 we can construct an embedding $\iota': S^2 \hookrightarrow S^3$ such that the standard open book $(B_0, \pi_0)$ induces the same open book foliation $\mathcal{F}$ on $S^2$. Then by Theorem 2.5 we can do open book surgery on the two open books $(B, \pi)$ and $(B_0, \pi_0)$ along $\mathcal{F}$ to obtain open books $(B_1, \pi_1)$ and $(B_2, \pi_2)$ for the contact structures $(Y_1, \xi_1)$ and $(Y_2, \xi_2)$. Since the gluing happened along the same foliation the sum of the Euler characteristics for the old and new open books are the same:

$$sn(\xi_1) + sn(\xi_2) \leq \chi(B_1, \pi_1) + \chi(B_2, \pi_2) = \chi(B_0, \pi_0) + \chi(B, \pi) = sn(\xi) - 1.$$  

This inequality together with the obvious inequality in the beginning of the section proves the statement. □
4. FUTURE DIRECTIONS AND UPCOMING PAPERS

However simple this cut and paste technique for open books is, it has not yet been fully explored. By understanding the embeddings of $S^2$ into $(B_0, \pi_0)$ that induce a given open book foliation the results of the paper could possibly be extended to prove the additivity of genus for open books of tight contact structures. Another direction is to generalise this embedding result to some foliations on tori, and get an understanding of open books in terms of JSJ decompositions of contact $3$-manifolds. In a paper in preparation with Licata [7] we describe the full machinery that is needed for cutting and pasting open books by defining the notion of open books that have been cut. These open books are called foliated open books, and similarly to ordinary or partial open books they satisfy all desirable properties of existence and uniqueness.

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