Singularity of classical and quantum correlations at critical points of the Lipkin-Meshkov-Glick model in bipartition and tripartition of spins

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Abstract. We study the classical correlation (CC) and quantum discord (QD) between two spin subgroups of the Lipkin-Meshkov-Glick (LMG) model in both binary and trinary decompositions of spins. In the case of bipartition, we find that the classical correlations and all the quantum correlations including the QD, the entanglement of formation (EoF) and the logarithmic negativity (LN) are divergent in the same singular behavior at the critical point of the LMG model. In the case of tripartition, however, the classical correlation is still divergent but all the quantum correlation measures remain finite at the critical point. The present result shows that the classical correlation is very robust but the quantum correlation is much frangible to the environment disturbance. The present result may also lead to the conjecture that the classical correlation is responsible for the singularity behavior of physics quantities at critical points of a many-body quantum system.

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1. Introduction

In a many-body quantum system, the interplay of various energies leads to different phases. When one of the energies becomes dominant over all the others by varying either adjustable interaction constants or applied external fields in the Hamiltonian, the system undergoes a phase transition and some observable display the singular behavior at a critical point. Since this phase transition occurs at zero temperature and is induced purely by quantum fluctuations in the system, in contrast to usual phase transitions induced by the thermal fluctuation, it is called quantum phase transition (QPT) \[1\]. During the last decade, the QPT has attracted a lot of attention and become an important research domain \[1\]. On the other hand, quantum many-body systems have genuinely "quantum" correlations or entanglement in contrast to classical correlations \[2, 3\]. Therefore, it becomes natural to connect QPTs to quantum entanglement. At present, many measures for entanglement have been proposed such as the relative entropy \[4\], the concurrence \[5\], the entanglement of formation \[6\], the logarithmic negativity \[7\] and so on. As observables for identifying QPTs, those quantities indeed display the singular behaviour at zero temperatures \[8, 9, 10\]. When calculating the measures of entanglement, one need to divide a system into several subsystems and then investigate quantum correlations between the subsystems. For a bipartition, two subsystems are of complementary parts one another and the entire system is always in a pure state. For a more multi partition, however, any two subsystems no longer forms a whole system and are in general in a mixed state. In this case, the other parts play a role of environments to the two subsystems under consideration and may strongly affect the critical behavior of quantum correlations between the chosen subsystems. In fact, Osborne et al. \[11\] investigated the two-spin entanglement in the XY spin model and found that the entanglement remains finite and displays a peak at the critical point. Vidal et al. \[12\] studied the entanglement of a \(L\) spins block and found that the entanglement entropy displays a logarithmic divergence for large \(L\). Recently, Werlang et al. \[13\] showed that the quantum discord (QD) \[14, 15\] and the entanglement of formation (EoF) between nearest-neighbors spins in an XXZ infinite spin chain at finite temperatures are no longer divergent but become finite at the critical point.

The Lipkin-Meshkov-Glick model (LMG) is one of few solvable many-body systems. In recent years, a lot of efforts have been devoted to the study of quantum correlations such as the entanglement entropy, the concurrence and the logarithmic negativity in the LMG model \[16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\]. Latorre et al. \[18\] investigated the entanglement entropy in the LMG model and found that the entropy displays a singularity at the critical point. Morrison et al. \[21\] studied the dynamical QPTs with the spin-spin entanglement in a dissipative LMG. Orús et al. \[23\] investigated the many-body entanglement in the LMG model and showed that the critical scaling laws for the single-copy entanglement and the global geometric entanglement are equivalent. Wichterich et al. \[27\] found that the logarithmic negativity of two macroscopic sets of spins becomes finite at the critical point in any tripartition of mutually interacting...
spins described by the LMG model whereas it displays a logarithmic divergence in a complementary bipartition.

From previous studies, a question is raised which correlations are responsible for the divergent behavior of many-body systems at critical points. To clarify this question, one need to distinguish classical correlation from quantum one since the later one is much sensitive to the disturbance of environments. By recognizing the discrepancy between quantum extensions of two equivalent expressions for the classical mutual information [28], Olliver and Zurek [14, 15] introduced quantum discord (QD) that is sufficiently to qualify the total amount of quantum correlation including entanglement in a composite system and classical correlation (CC). In this paper, in order to analytically and clearly answer the raised question, taking the LMG model as an example, we compute the QD and the CC of two macroscopic sets of the mutually interacting spins in the cases of both bipartition and tripartition. For completeness and comparison, the EoF and the logarithmic negativity (LN) are also computed. We find that at the critical point both the CC and quantum correlations, including QD, EoF and LN, are always divergent in the bipartition. However, in the tripartition the quantum correlations remain finite and the CC is still divergent. The result may lead to a conjecture that the singular behaviour of observables of a many-body system with finite temperatures at critical points comes from the CC divergency.

This paper is organized as follows. In section 2, the model is introduced. In section 3, correlations in a bipartition setting are studied and detailed discussions are given. In section 4, correlations in a tripartition setting are investigated. Finally, a brief summary is given in section 5.

2. The Model

The LMG model describes a collection of mutually interacting N spins-1/2 on the x-y plane with an external field applied along the z direction. The Hamiltonian of LMG model reads

\[
H = -\frac{1}{N} \sum_{i<j=1}^{N} \left( \sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y \right) - h \sum_{i=1}^{N} \sigma_i^z,
\]

where \( \sigma_i^{\beta} \) (\( \beta = x, y, z \)) are the Pauli matrices for a spin at position \( i \), \( N \) is the total number of spins, \( 0 \leq \gamma < 1 \) is an anisotropy parameter and \( h \) is an external magnetic field applied along the z direction.

In terms of the total spin operators \( S_\beta = \sum_i \sigma_i^\beta / 2 \), the Hamiltonian (1) can be rewritten as

\[
H = -\frac{1}{N} \left( S_x^2 + \gamma S_y^2 \right) - hS_z.
\]

The ground-state properties of the LMG model have been found by use of a mean-field approach [19, 29]. The LMG model undergoes a second-order phase transition at \( h = 1 \). For \( h > 1 \), the ground state is a symmetrical and fully polarized state where all the
spins are along the external field direction. For \( h < 1 \), the corresponding ground state is two-fold degenerate \[19, 29\].

3. The quantum and classical correlations in a bipartition

In this section, we divide the \( N \) spins into two groups and investigate the ground-state quantum and classical correlations between the two spin groups. To do so, we first need to determine the lowest energy state of the LMG model for a given external field. Thus, a rotation transformation to the total spin operator \( s \) around the first need to determine the lowest energy state of the LMG model for a given external field. Thus, a rotation transformation to the total spin operators around the \( y \) axis is introduced as follows

\[
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_0 & 0 & \sin \theta_0 \\
0 & 1 & 0 \\
-\sin \theta_0 & 0 & \cos \theta_0
\end{pmatrix}
\begin{pmatrix}
\tilde{S}_x \\
\tilde{S}_y \\
\tilde{S}_z
\end{pmatrix}.
\]

(3)

In (3), \( \theta_0 \) stands for the value of the rotation angle which is chosen to make the expectation of the Hamiltonian (2) in the ground state \( \langle H \rangle \) be minimum. \( \theta_0 = 0 \) for the symmetrical phase with \( h > 1 \), and \( \theta_0 = \arccos h \) for the broken phase with \( 0 \leq h < 1 \) \[19, 29\]. Substituting Eq. (3) into (2), one obtains

\[
H = -\frac{1}{2N} \cos \theta_0 \sin \theta_0 \left( \tilde{S}_+ \tilde{S}_z + \tilde{S}_- \tilde{S}_z + \tilde{S}_z \tilde{S}_+ + \tilde{S}_z \tilde{S}_- \right)
-\frac{1}{4N} \left( \left( \cos^2 \theta_0 - \gamma \right) \left( \tilde{S}_+^2 + \tilde{S}_-^2 \right) + \left( \cos^2 \theta_0 + \gamma \right) \left( \tilde{S}_+ \tilde{S}_- + \tilde{S}_- \tilde{S}_+ \right) \right)
-\frac{1}{N} \sin^2 \theta_0 \tilde{S}_z^2 + \frac{h \sin \theta_0}{2} \left( \tilde{S}_+ + \tilde{S}_- \right) - h \cos \theta_0 \tilde{S}_z.
\]

(4)

When working out Eq. (4), we have used the relations \( \tilde{S}_x = \left( \tilde{S}_+ + \tilde{S}_- \right) / 2 \) and \( \tilde{S}_y = \left( \tilde{S}_+ - \tilde{S}_- \right) / (2i) \).

We now split the \( N \) spins into two groups and consequently write the total spin operators as \( \tilde{S}_{\beta} = \tilde{S}_{\beta}^{(1)} + \tilde{S}_{\beta}^{(2)} \). In the Holstein-Primakoff representation \[19\], the spin operators \( \tilde{S}_{\beta}^{(k)} \) \((k = 1, 2)\) for each of the spin groups can be written as

\[
\tilde{S}_{\beta}^{(k)} = N_k/2 - a_k^\dagger a_k,
\]

(5)

\[
\tilde{S}_{+}^{(k)} = \left( N_k - a_k^\dagger a_k \right)^{1/2} a_k,
\]

(6)

\[
\tilde{S}_{-}^{(k)} = a_k^\dagger \left( N_k - a_k^\dagger a_k \right)^{1/2},
\]

(7)

where \( a_k \) and \( a_k^\dagger \) are bosonic annihilation and create operators and \( N_k \) denotes the spin number in the \( k \)th group under the condition \( N = N_1 + N_2 \).

Upon substituting Eqs. (5)-(7) into (4) and expanding \( H \) as a series of powers \( 1/N_k \), and keeping the lowest order, one obtains

\[
H = -\frac{m^2 - \gamma}{4N} \left( N_1 a_1^2 + \sqrt{N_1 N_2} a_1 a_2 + \sqrt{N_1 N_2} a_2^2 \right)
-\frac{m^2 + \gamma}{4N} \left( N_1 a_1^2 + \sqrt{N_1 N_2} a_1 a_2 + \sqrt{N_1 N_2} a_2^2 \right)
+ \left( 1 - m^2 + hm \right) \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) + \text{Const},
\]

(8)
with $m = \cos \theta_0$. The Hamiltonian $H$ is of a quadratic form of the bosonic annihilation and creation operators. It can be diagonalized by introducing the Bogoliubov transformation

$$a_1 = \left( \cosh \frac{\Theta}{2} b_1 + \sinh \frac{\Theta}{2} b_1^\dagger \right) \sqrt{\tau_1} + b_2 \sqrt{\tau_2},$$

$$a_2 = \left( \cosh \frac{\Theta}{2} b_1 + \sinh \frac{\Theta}{2} b_1^\dagger \right) \sqrt{\tau_2} - b_2 \sqrt{\tau_1},$$

where $b_i (i = 1, 2)$ are new bosonic operators and $\tau_k = N_k / N$ with $\sum_k \tau_k = 1$. If choosing $\tanh \Theta = -s / r$,

$$\tau_1 = \frac{N_1}{N}, \quad \tau_2 = \frac{N_2}{N},$$

with $s = \gamma - m^2$, and $r = 2hm - 3m^2 + 2 - \gamma$, the Hamiltonian $H$ can be written in the diagonal form except an irrelevant constant to the present investigation

$$H = \Delta_1 b_1^\dagger b_1 + \Delta_2 b_2^\dagger b_2,$$

where

$$\Delta_1 = \frac{1}{2} \left[ \left( 2hm - 3m^2 + 2 - \gamma \right) \cosh \Theta - \left( m^2 - \gamma \right) \sinh \Theta \right],$$

$$\Delta_2 = \frac{1}{2} \left( 2hm - 3m^2 + 2 - \gamma \right).$$

The Hamiltonian $H$ represents two independent harmonic oscillators which ground state $|\psi_0\rangle$ is a Gaussian state, defined as $b_i |\psi_0\rangle = 0$. Therefore, the ground state of the LMG model can be fully characterized by the covariance matrix with the elements $\Gamma_{ij} = \langle \psi_0 | \{ \hat{R}_i, \hat{R}_j \} | \psi_0 \rangle$, where $\hat{R} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$ with canonical coordinates $\hat{x}_i = \left( a_i^\dagger + a_i \right) / \sqrt{2}$ and momenta $\hat{p}_k = i \left( a_k^\dagger - a_k \right) / \sqrt{2}$. By use of the Bogoliubov transformation (9) and (10), one can obtain the explicit expression for the covariance matrix

$$\Gamma = \begin{pmatrix}
A_1 \tau_1 + 1 & 0 & A_1 \sqrt{\tau_1 \tau_2} & 0 \\
0 & B_1 \tau_1 + 1 & 0 & B_1 \sqrt{\tau_1 \tau_2} \\
A_1 \sqrt{\tau_1 \tau_2} & 0 & A_1 \tau_2 + 1 & 0 \\
0 & B_1 \sqrt{\tau_1 \tau_2} & 0 & B_1 \tau_2 + 1
\end{pmatrix} = \begin{pmatrix} G_1 & C_1 \\ C_1 & G_2 \end{pmatrix},$$

where

$$A_1 = \sqrt{(r - s) / (r + s) - 1},$$

$$B_1 = \sqrt{(r + s) / (r - s) - 1},$$

and $G_i, C_i$ are $2 \times 2$ matrices. For the simplicity of expressions in the following, we set $\sqrt{(r + s) / (r - s)} = \alpha$. In the symmetrical ($h \geq 1$) and broken ($0 \leq h < 1$) phases the parameter $\alpha$ reads

$$\alpha = \begin{cases} 
\sqrt{(h - 1) / (h - \gamma)}, & h \geq 1 \\
\sqrt{(1 - h^2) / (1 - \gamma)}, & 0 \leq h < 1
\end{cases}$$
By performing a like-Bogoliubov transformation [30], the covariance matrix (15) can be written in the standard form

$$\Gamma_{sf} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix},$$

(19)

where $a, b, c_1$ and $c_2$ are determined by

$$a^2 = \det G_1 = A,$$

(20)

$$b^2 = \det G_2 = B,$$

(21)

$$c_1 c_2 = \det C_1 = C,$$

(22)

$$\left(ab - c_1^2\right) \left(ab - c_2^2\right) = \det \Gamma = D,$$

(23)

with

$$A = B = \alpha^{-1} \left[\alpha \tau_1 + \left(1 - \tau_1\right)\right] \left(\tau_1 + \alpha \left(1 - \tau_1\right)\right),$$

(24)

$$C = \left[2 - \alpha - \alpha^{-1}\right] \left[1 - \tau_1\right] \tau_1,$$

(25)

and $D = 1$.

Correspondingly, the simplectic eigenvalues of the covariance matrix (19) are given by

$$\nu_{\pm}^2 = \left(M \pm \sqrt{M^2 - 4D}\right)/2,$$

(26)

with $M = A + B + 2C$.

According to Ref. [31], the classical and quantum correlations of the Gaussian state characterized by the covariance matrix (19) can be respectively calculated by the formula

$$CC = f\left(\sqrt{A}\right) - f\left(\sqrt{E_{\text{min}}}\right),$$

(27)

$$QD = f\left(\sqrt{B}\right) - f\left(\nu_-\right) - f\left(\nu_+\right) + f\left(\sqrt{E_{\text{min}}}\right),$$

(28)

where $E_{\text{min}} = \left(2C^2 + (B - 1)\left(D - A\right) + 2|C|\sqrt{C^2 + (B - 1)\left(D - A\right)}\right)/(B - 1)^2$ for $(D - AB)^2 \leq (1 + B)C^2\left(A + D\right)$, and when it comes to other cases $E_{\text{min}}$ is determined by $\left(AB - C^2 + D - \sqrt{C^4 + (D - AB)^2 - 2C^2\left(AB + D\right)}\right)/(2B)$, and

$$f\left(x\right) = \left(\frac{1 + x}{2}\right) \ln \left(\frac{1 + x}{2}\right) - \left(\frac{x - 1}{2}\right) \ln \left(\frac{x - 1}{2}\right).$$

Upon substituting Eqs. (24)-(26) into Eqs. (27) and (28), the explicit expressions for the CC and the QD can be obtained as

$$CC = QD = \frac{\sqrt{A} + 1}{2} \ln \frac{\sqrt{A} + 1}{2} - \frac{\sqrt{A} - 1}{2} \ln \frac{\sqrt{A} - 1}{2}.$$  

(29)

It is noted that the expression of the CC and the QD are the same as that of the entanglement entropy obtained in Ref. [10],
From (19), we can also obtain the entanglement of formation (EoF) between the two divided spin groups [6]

\[ EoF = f (\Delta), \]  \hspace{1cm} (30)

where

\[ f (\Delta) = c_+ (\Delta) \log_2 (c_+ (\Delta)) - c_- (\Delta) \log_2 (c_- (\Delta)), \]  \hspace{1cm} (31)

\[ c_\pm (\Delta) = \left(\Delta^{-1/2} \pm \Delta^{1/2}\right)^2 / 4, \]  \hspace{1cm} (32)

\[ \Delta = a - c, \]  \hspace{1cm} (33)

\[ c = \alpha^{-1/2} \sqrt{(-1 + \alpha)^2 (1 - \tau_1)\tau_1}. \]  \hspace{1cm} (34)

From Eq. (19), one can also calculate the logarithmic negativity (LN) [7] which has been obtained by Wichterich et al in Ref. [27].

In Fig. 1, the various correlations such as CC, QD, EoF and LN are plotted against the external field \( h \). It is clearly shown that all the correlations between the two spin groups diverge at the critical point \( h = 1 \). By comparing the two figures, one may find that the anisotropic parameter \( \gamma \) has little impact on the singular behaviour of these correlations and the CC, QD, EoF and LN display the same divergency at the critical point although they describe the different correlations. In Fig. 2, the correlations versus the external field are shown for different divisions of bipartition. It is observed that the divergent behavior of the correlations at the critical point is hardly affected by the bipartition way.

In order to analytically investigate the critical behavior of the CC, the QD and the EoF, we expand Eqs. (29) and (30) at the critical point \( (h = 1) \) and obtain

\[ CC = QD = -\frac{1}{4} \ln (h - 1) + \frac{1}{4} \ln (1 - \gamma) + \frac{1}{2} \ln \tau_1 (1 - \tau_1) - \ln 2. \]  \hspace{1cm} (35)

and

\[ EoF = -\frac{1}{4} \log_2 (h - 1) + \frac{1}{4} \log_2 (1 - \gamma) + \frac{1}{2} \log_2 \tau_1 (1 - \tau_1) - 1. \]  \hspace{1cm} (36)

Eq. (35) shows that when reaching the critical point the CC and QD diverge as \(-\frac{1}{4} \ln (h - 1)\) which is consistent with that appears in Figs. 1 and 2. Interestingly, the singular behavior of the CC and QD is really the same as that of the logarithmic negativity [27], the entanglement entropy [10] and the single-copy entanglement [23]. From Eq. (36) we know that the EoF diverges as \(-\frac{1}{4} \log_2 (h - 1)\) at the critical point and behaves slightly different from the CC and the QD.

Based on the scaling hypothesis proposed in Refs. [10] [16], the finite-size scaling behavior of the CC and QD can be straightforwardly extracted from Eq. (35)

\[ CC = QD \sim \frac{1}{6} \ln N + \frac{1}{6} \ln (1 - \gamma) + \frac{1}{2} \ln \tau_1 (1 - \tau_1). \]  \hspace{1cm} (37)

This finite-size scaling behavior is identical to that of the logarithmic negativity [27], the entanglement entropy [10], the geometric entanglement and the single-copy entanglement [23]. Therefore, all the correlations between artificial divided two parts of the mutually interacting spins in the LMG model obey the same critical scaling law.
4. The classical and quantum correlations in a tripartition

In this section, we divide the mutually interacting \( N \) spins in the LMG model into three groups, each of which has \( N_i \) spins under the condition \( N = N_1 + N_2 + N_3 \), and investigate correlations between any two groups. In this case, if we consider the first and third groups, we need to trace the variable of the second group. Thus, the spins in the second group plays a role of the environment to the spins in the first and third groups, and the spins in the groups under consideration is generally in a mixed state.

Following the same procedure as shown in the preceding section, we can diagonalize the Hamiltonian \((1)\) and obtain the ground state of the LMG model, from which the density matrix of the ground state can be built. By tracing the density matrix over the variable of spins in the second group, one can obtain the reduced density matrix for spins in the first and third groups. It is obvious that the reduced density matrix is also of a Gaussian state. The covariance matrix of the reduced density matrix is found to be

\[
\Gamma = \begin{pmatrix}
A_1 \tau_1 + 1 & 0 & A_1 \sqrt{\tau_1 \tau_3} & 0 \\
0 & B_1 \tau_1 + 1 & 0 & B_1 \sqrt{\tau_1 \tau_3} \\
A_1 \sqrt{\tau_1 \tau_3} & 0 & A_1 \tau_3 + 1 & 0 \\
0 & B_1 \sqrt{\tau_1 \tau_3} & 0 & B_1 \tau_3 + 1
\end{pmatrix}.
\] (38)

If one sets \( \tau_1 = \tau_3 = \tau < 1/2 \), the standard form of \((38)\) is the same as \((19)\) which elements are determined by

\[
A = B = \alpha^{-1} (\alpha \tau + (1 - \tau)) (\tau + \alpha (1 - \tau)),
\] (39)

\[
C = -\alpha^{-1} (\alpha - 1)^2 \tau^2,
\] (40)

\[
D = \alpha^{-1} (\alpha + 2 (\alpha - 1)^2 (\tau - 2\tau^2))
\] (41)

according to Eqs. \((20)-(23)\). In this case, the reduced density matrix is of a symmetrical Gaussian state \([30]\). The symplectic eigenvalues of Eq. \((38)\) are found to be

\[
\nu_- = 1,
\]

\[
\nu_+ = \alpha^{-1/2} \sqrt{\alpha + 2 (\alpha - 1)^2 (\tau - 2\tau^2)}
\] (42)

Upon substituting Eqs. \((39)-(42)\) into Eqs. \((27)-(28)\), one can work out the the CC and the QD between spins in the first and third groups with \( E_{\text{min}} = 1 \) for \( h = \sqrt{\gamma} \), and \( E_{\text{min}} = \left( -2\alpha^2 - 4d\alpha \tau - d (1 + (\alpha - 8) \alpha) \tau^2 + 2d^2 \tau^3 + |d^{3/2}| (\alpha + 1) \tau^2 (1 - 2\tau) \right) / \mu \) for other circumstance with \( d = (\alpha - 1)^2 \) and \( \mu = 2\alpha (\alpha (\tau - 1) - \tau) (1 + (\alpha - 1) \tau) \). Since the analytical expressions for the CC and the QD are much lengthy, we here have to give up to explicitly write them out.

The entanglement of formation (EoF) can be obtained from Eq. \((30)\) with

\[
\Delta = \sqrt{(\sqrt{A} - k_1)(\sqrt{A} - k_2)}
\] (43)
where
\[ k_1 = \sqrt{\frac{(\alpha - 1)^2 \tau^2 (1 + (\alpha - 1) \tau)}{\alpha (\alpha + \tau - \alpha \tau)}}. \quad (44) \]
\[ k_2 = \frac{(\alpha - 1)^2 \tau^2}{\alpha \times k_1}. \quad (45) \]

In Fig. 3, the various correlations for an equal tripartition \( \tau_1 = \tau_3 = 1/3 \) are plotted as a function of the magnetic field \( h \). It is clearly observed that the CC diverges whereas all the quantum correlation measures such as QD, EoF and LN remain finite at the critical point.

To clearly look into the behavior of the CC and QD at the critical point, we expand the analytical expression of the QD at \( h = 1 \) and obtain
\[ QD = \ln \frac{1 - \tau}{2\sqrt{2}} + \frac{1}{2} \ln \left( \frac{\sqrt{2 (1 - \tau)} + 1}{\sqrt{2 (1 - \tau) - 1}} \right)^{\sqrt{2(1-\tau)}}. \quad (46) \]
Eq. (46) shows that the QD indeed remains finite at the critical point. Moreover, the value of the QD is irrelative to the anisotropy parameter \( \gamma \). This universal character is much similar to that of the logarithmic negativity as found in Ref. [27]. Eq. (46) also shows that when \( \tau \) approaches to 1/2, QD diverges as that in the bipartition setting.

In the similar way, we can find the analytical expression for the classical correlation of the LMG model around the critical point
\[ CC = -\frac{1}{4} \ln (h - 1) + \frac{1}{4} \ln (1 - \gamma) \\
+ \frac{1}{2} \ln \left\{ \frac{\tau (1 - \tau)}{(1 - 2\tau)} \left( \frac{\sqrt{2 (1 - \tau)} - 1}{\sqrt{2 (1 - \tau)} + 1} \right)^{\sqrt{2(1-\tau)}} \right\}. \quad (47) \]
In contrast to the QD, the classical correlation between the two spin groups diverges as \(-\frac{1}{4} \ln (h - 1)\) at the critical point. This divergent behavior is the same as that obtained in the bipartition setting.

In quantum information theory, the total correlation of a bipartite quantum system is measured by the mutual information [32, 33]. Qualitatively, the total correlation equals to the QD plus the CC. From the present result, it is very clear that in a tripartite setting the classical correlation is responsible for the divergency of the total correlations at the critical point [27].

The critical behaviour of EoF can also be investigated from Eqs.(30), (43)-(45). However, the analytical expression of it is too lengthy to be explicitly written here. We just give the numerical results. In Fig. 4, the EoF and QD are plotted as a function of the partition parameter \( \tau \) at the critical point \( h = 1 \). It clearly shows that when \( \tau \) is less than 1/2 the EoF and QD remain finite. When \( \tau \) reaches 1/2 and the tripartition reduces to the bipartition, the EoF and QD go from finite to infinity.
5. Summary

The Lipkin-Meshkov-Glick (LMG) model describes a collection of mutually interacting spins-1/2 in an external magnetic field. By dividing spins of the LMG model into two or three parts, we study the classical correlation (CC) and quantum correlation measures such as the quantum discord (QD), the entanglement of formation (EoF) and the logarithmic negativity (LN) between the two spin groups. In the case of bipartition, where the two spin groups are complementary and their ground state must be of a pure state, we find that the classical correlations and all the quantum correlations are divergent in the same singular behaviour at the critical point of the LMG model. In the case of tripartition, however, the classical correlation is still divergent but all the quantum correlation measures remain finite at the critical point. In a tripartition, the spin group traced out plays a role of environments and the other two spin groups are general in a mixed state. The present result shows that the classical correlation is very robust but the quantum correlation is much frangible to the environment disturbance. In the real situation, a many-body quantum system is unavoidably to be coupled to its surroundings and is in a mixed state. Therefore, the present result may lead to the conjecture that the classical correlation is responsible for the singularity behaviour of physics quantities at critical points of a many-body quantum system.

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References

[1] Sachdev S 1999 Quantum Phase Transition (England: Cambridge)
[2] Schrödinger E 1935 Naturwiss. 23 807; Schrödinger E 1935 Naturwiss. 23 823; E. Schrödinger 1935 Naturwiss. 23 844
[3] Einstein A, Podolski L, and Rosen N 1935 Phys. Rev. 47 777
[4] Vedral V, Plenio M B, Rippin M A, and Knight P L 1998 Phys. Rev. Lett. 78 2275
[5] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[6] Giedke G, Wolf M M, Krüger O, Werner R F, and Cirac J I 2003 Phys. Rev. Lett. 91 107901
[7] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
[8] Lambert N, Emary C, and Brandes T 2004 Phys. Rev. Lett. 92 073602
[9] Amico L, Fazio R, Osterloh A, and Vedral V 2008 Rev. Mod. Phys. 80 517
[10] Barthel T, Dusuel S, and Vidal J 2006 Phys. Rev. Lett. 97 220402
[11] Osborne T J and Nielsen M A 2002 Phys. Rev. A 66 032110
[12] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Phys. Rev. Lett. 90 227902
[13] Werlang T, Trippe C, Ribeiro G A P, and Rigolin G 2010 Phys. Rev. Lett. 105 095702
[14] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[15] Zurek W H 2000 Ann. Phys (Berlin) 9 855
[16] Dusuel S, Vidal J 2004 Phys. Rev. Lett 93 237204
Unanyan R G, Ionescu C, and Fleischhauer M 2005 Phys. Rev. A 72 022326
Latorre J I, Orús R, Rico E, and Vidal J 2005 Phys. Rev. A 71 064101
Dusuel S, Vidal J 2005 Phys. Rev. B 71 224420
Ribeiro P, Vidal J, and Mosseri R 2007 Phys. Rev. Lett. 99 050402
Morrison S and Parkins A S 2008 Phys. Rev. Lett. 100 040403
Kwok H M, Ning W Q, Gu S J and Lin H Q 2008 Phys. Rev. E 78 032103
Orús R, Dusuel S and Vidal J 2008 Phys. Rev. Lett. 101 025701
Morrison S and Parkins A S 2008 Phys. Rev. A 77 043810
Ribeiro P, Vidal J and Mosseri R 2008 Phys. Rev. E 78 021106
Filippone M, Dusuel S, and Vidal J, 2011 Phys. Rev. A 83 022327
Wichterich H, Vidal J, and Bose S 2010 Phys. Rev. A 81 032311
Henderson L and Vedral V 2001 J. Phys. A 34 68899
Botet R and Jullien R 1983 Phys. Rev. B 28 3955
Duan L M, Giedke G, Cirac J I, and Zoller P 2000 Phys. Rev. Lett. 84 2722
Adesso G, Datta A 2010 Phys. Rev. Lett. 105 030501
Groisman B, Popescu S, and Winter A 2005 Phys. Rev. A 72 032317
Schumacher B, Westmoreland M D 2006 Phys. Rev. A 74 042305
Figure Captions

Fig. 1 Various correlations as a function of the magnetic field $h$ for the bipartition with $\tau_1 = 1/3$. The symbols shown in the inset of Fig. 1(a) are applicable to the curves of Fig. 1(b).

Fig. 2 Various correlations as function of the magnetic field $h$ for the different divisions of bipartition with $\tau_1 = 1/2, 1/6, 1/100$ and $\gamma = 0.5$. The symbols shown in the inset of Fig. 2(a) are applicable to the curves of Fig. 2(b) and Fig. 2(c).

Fig. 3 The various correlations as function of the magnetic field $h$ for an equal tripartition $\tau_1 = \tau_2 = \tau_3 = 1/3$. The symbols shown in the inset of Fig. 3(a) are applicable to the curves of Fig. 3(b).

Fig. 4 The EoF and QD versus the partition parameter $\tau$ at the critical point $h = 1$. 
Fig 1
Fig2
Fig3
