MODULI SPACE OF INSTANTON SHEAVES ON THE FANO 3-FOLD $V_5$

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Abstract. We study semi-stable sheaves of rank 2 with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on the Fano 3-fold $V_5$ of Picard number 1, degree 5 and index 2. We show that the moduli space of such sheaves is isomorphic to $\mathbb{P}^5$ by identifying it with the moduli space of semi-stable quiver representations. This provides a natural smooth compactification of the moduli space of Ulrich bundles of rank 2.

1. Introduction

Instanton bundles first appeared in [AHDM] as a way to describe Yang-Mills instantons on a 4-sphere $S^4$. They provide extremely useful links between mathematical physics and algebraic geometry. The notion of mathematical instanton bundles was first introduced on $\mathbb{P}^3$. Since then the irreducibility [T], rationality [MT] and smoothness [JV] of their moduli spaces were heavily investigated. Faenzi [F] and Kuznetsov [Ku12] generalized this notion to Fano threefolds, we recall

Definitions 1.1. [Ku03] Let $Y$ be a Fano threefold of index 2. An instanton bundle on $Y$ is a stable vector bundle $E$ of rank 2 with $c_1(E) = 0$ such that $H^1(Y, E(-1)) = 0$ and $c_2(E) \geq 2$.

We mention that if $c_2(E) = 0$, in which case $E$ is called a minimal instanton, the condition $H^1(Y, E(-1)) = 0$ is automatically satisfied. The moduli spaces of instanton bundles were discussed in the aforementioned papers using quadric nets. They are in general not projective.

An instanton sheaf is by definition a semi-stable sheaf of rank 2 sharing the same Chern classes with the (minimal) instanton bundle: $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$. The moduli space of such sheaves provides a natural compactification of the moduli space of (minimal) instantons.

On the other hand, Ulrich bundles are defined as vector bundles on a smooth projective variety $X$ of dimension $d$ so that

$$H^t(X, E(-t)) = 0$$

for all $t = 1, \ldots, d$. They first appeared in commutative algebra and entered the world of algebraic geometry via [ES]. The existence and moduli space of Ulrich bundles provide great amount of information about the original variety. For example, in the case when $X$ is a smooth hypersurface, the existence of Ulrich bundle is equivalent to the fact that $X$ can be defined set-theoretically by a linear determinant [B]. Inspired by [Ku12], Lahoz, Macri and Stellari [LMS1], [LMS2] studied moduli spaces of Ulrich bundles on cubic threefolds and fourfolds using derived categories. In their recent paper [LP], Lee and Park described the

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moduli space of Ulrich bundles on $V_5$. As an easy consequence of their work, we see on $V_5$, (minimal) instanton bundles and Ulrich bundles of rank 2 differ only by a twist of the very ample divisor. Thus they share the same moduli space. As a result, the moduli space of instanton sheaves provides also a natural compactification of the moduli space of rank 2 Ulrich bundles on $V_5$.

Our first result is the classification of instanton sheaves on $V_5$. [D] classified instanton sheaves on cubic threefold and proved that their moduli space is isomorphic to the blow-up of the intermediate Jacobian in the Fano surface of lines. We follow his method and prove that the parallel classification happens on $V_5$.

**Theorem 1.2.** Let $E$ be an instanton sheaf on $V_5$. If $E$ is stable, then either $E$ is locally free or $E$ is associated to a smooth conic $C \subset V_5$ such that we have the exact sequence:

$$0 \to E \to H^0(\theta(1)) \otimes O_{V_5} \to \theta(1) \to 0$$

where $\theta$ is the theta-characteristic of $C$.

If $E$ is properly semi-stable, then $E$ is the extension of two ideal sheaves of lines.

Unfortunately, the method to study the moduli space in [D] does not transfer well to $V_5$. However, we note that $\mathcal{D}(V_5)$ has a semi-orthogonal decomposition:

$$\mathcal{D}(V_5) = \langle U, Q^\vee, O_{V_5}, O_{V_5}(1) \rangle$$

where $U$ is the restriction of the universal subbundle and $Q$ the universal quotient bundle. It is well-known that the subcategory $\mathcal{B}_{V_5} := \langle U, Q^\vee \rangle$ is equivalent to the derived category of finite dimensional representations of the Kronecker quiver with three arrows. Our next result establish the relation between instantons and such representations:

**Theorem 1.3.** Let $E$ be an instanton sheaf on $V_5$, then $E \in \mathcal{B}_{V_5}$ and there exists a short exact sequence:

$$0 \to U^\oplus 2 \to Q^\vee^\oplus 2 \to E \to 0$$

Moreover, the induced representation inherits the stability of $E$.

Using this relation, we construct a morphism between the moduli space of instanton sheaves $M^{\text{inst}}$ and the moduli space of semistable representations $M$ and prove it is an isomorphism.

**Theorem 1.4.** There exists a morphism $\phi: M^{\text{inst}} \to M$ which is an isomorphism. As a result, the moduli space of instanton sheaves on $V_5$ is $\mathbb{P}^5$.

We believe our results can be generalized to find compactification of moduli spaces of Ulrich bundles of higher ranks on $V_5$. Also similar idea should work in finding the moduli space of instanton sheaves on Fano threefolds other than $V_5$ and cubics.

This paper is organized as follows. In the second section the reader can find some preliminary results and definitions that are used throughout the paper. In the third section we classify the instanton sheaves, showing the parallel result as on cubic threefolds holds. In the fourth section we connect instanton sheaves to representations of the Kronecker quiver using derived category. In the last section we describe the moduli space of instantons.

**Notations and conventions.**

- We work over the complex numbers $\mathbb{C}$.
- Let $E$ be a sheaf on $V_5$. We use $H^i(E)$ to denote $H^i(V_5, E)$ for simplicity. Also we use $h^i(E)$ to denote the dimension of $H^i(V_5, E)$ as a complex vector space.
• Let $F$ be a sheaf or a representation with certain characterization, we will use $[F]$ to denote the point it corresponds to in the moduli space.

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2. Preliminaries

2.1. Fano 3-fold $V_5$. Let $V$ be a complex vector space of dimension 5, let $A \subset \Lambda^2 V^*$ be a 3-dimensional subspace of 2-forms on $V$. It is well known that if $A$ is generic, then the intersection of $\text{Gr}(2,V)$ with $\mathbb{P}(A^\perp)$ in $\mathbb{P}(\Lambda^2 V)$ is a smooth Fano threefold of Picard number 1, degree 5 and index 2. Varying $A$ generically will provide projectively equivalent varieties. We use $V_5$ to denote this unique smooth threefold. Note $V_5$ comes with a natural choice of a very ample line bundle $\mathcal{O}_{V_5}(1)$. We will always use this polarization for the rest of this paper. Let $S \in |\mathcal{O}_{V_5}(1)|$ be a generic hyperplane section. Then $S$ is a smooth del Pezzo surface of degree 5.

The cohomology group of $V_5$ is isomorphic to $\mathbb{Z}^4$:

$$H^*(V_5, \mathbb{Z}) = H^0(V_5, \mathbb{Z}) \oplus H^2(V_5, \mathbb{Z}) \oplus H^4(V_5, \mathbb{Z}) \oplus H^6(V_5, \mathbb{Z})$$

$$= \mathbb{Z}[V_5] \oplus \mathbb{Z}[h] \oplus \mathbb{Z}[l] \oplus \mathbb{Z}[p]$$

with $h \cdot l = p, h^2 = 5l, h^3 = 5p$.

Let $U$ be the restriction of the universal subbundle on $\text{Gr}(2,5)$ to $V_5$ and $Q$ be the restriction of the universal quotient bundle, we have an exact sequence:

$$0 \to U \to \mathbb{C}^5 \otimes \mathcal{O}_{V_5} \to Q \to 0.$$  

$U$ and $Q^\vee$ will play important roles in this paper. We first note $\text{Hom}(U, Q^\vee) = A$ (see [Or]). Their cohomology groups are computed in [LP, Lemma 2.9]:

**Lemma 2.1.** The cohomology groups $H^i(V_5, U(j))$ and $H^i(V_5, Q^\vee(j))$ for $j = -2, -1, 0$ are as follows:

$$H^*(V_5, U) = H^*(V_5, U(-1)) = 0$$

$$H^*(V_5, Q^\vee) = H^*(V_5, Q^\vee(-1)) = 0$$

$$H^i(V_5, U(-2)) = \begin{cases} \mathbb{C}^5 & \text{if } i=3 \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(V_5, Q^\vee(-2)) = \begin{cases} \mathbb{C}^5 & \text{if } i=3 \\ 0 & \text{otherwise} \end{cases}$$

The Chern classes of $U$ and $Q^\vee$ are as follows:

$$\text{rk}(U) = 2, c_1(U) = -h, c_2(U) = 2l$$

$$\text{rk}(Q^\vee) = 3, c_1(Q^\vee) = -h, c_2(Q^\vee) = 3l, c_3(Q^\vee) = -p$$

Moreover, we have

$$\text{td}(\mathcal{T}_{V_5}) = 1 + h + \frac{8}{3} l + p.$$
The Fano variety of lines on $V_5$ is $\mathbb{P}(A) \simeq \mathbb{P}^2$. It parametrizes the ideal sheaves of lines in $V_5$. Moreover each ideal sheaf corresponds to a representation of $\mathbb{Q}_3$ with dimension vector $(1,1)$ (see section 2.4), thanks to the following result:

**Lemma 2.2.** [Ku12 Lemma 4.2] For each point $a \in \mathbb{P}(A)$, we have an exact sequence:

$$0 \to U \to Q^\vee \to I_L \to 0$$

where $L$ is the line corresponding to $a$.

The restriction of any vector bundles on $V_5$ to a line $L$ splits into direct sum of line bundles, we have:

**Lemma 2.4.** [Sa Lemma 2.17] For any line $L$ in $V_5$, we have

$$U|_L = O_L(-1) \oplus O_L$$

$$Q^\vee|_L = O_L(-1) \oplus O_L \oplus O_L.$$

Since a smooth conic $C$ is isomorphic to $\mathbb{P}^1$, any restriction of vector bundle also splits:

**Lemma 2.5.** [Sa Lemma 2.40] Let $C$ be any smooth conic in $V_5$, we have

$$U|_C = O_C(-1) \oplus O_C(-1)$$

$$Q^\vee|_C = O_C(-1) \oplus O_C(-1) \oplus O_C.$$

### 2.2. Stability of sheaves.

Let $X$ be a smooth projective variety of dimension $n$ and $O_X(1)$ be a fixed ample line bundle. Let $E$ be a coherent sheaf of rank $r$, then the slope of $E$ is defined as:

$$\mu(E) = \frac{c_1(E)c_1(O_X(1))^{n-1}}{rc_1(O_X(1))^n}.$$ 

The sheaf $E$ is called (semi)-stable if it is torsion free and for any torsion free subsheaf $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$, we have

$$\frac{\chi(F(n))}{\text{rk}(F)}(\leq) < \frac{\chi(E(n))}{\text{rk}(E)}$$

for $n >> 0$.

The sheaf $E$ is called $\mu$-(semi)-stable if it is torsion free and for any torsion free subsheaf $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$, we have

$$\mu(F)(\leq) < \mu(E)$$

We have the following implications:

$$\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semi-stable} \Rightarrow \mu - \text{semi-stable}$$

We recall Hoppe’s criterion.

**Lemma 2.6.** Assume the Picard group of $X$ is $\mathbb{Z}$ and its ample divisor $O_X(1)$ has global sections. Let $F$ be a vector bundle of rank $r$ on $X$ so that for each $1 \leq k \leq r-1$, $(\Lambda^k(F))_{\text{norm}}$ has no global sections. Then $F$ is stable.

Here, for a sheaf $F$, $F_{\text{norm}}$ is the unique twist $F(n)$ such that $-1 < \mu(F(n)) \leq 0$. Using this result, we can easily check

**Lemma 2.7.** The vector bundles $U, U^\vee, Q, Q^\vee$ are all stable.
2.3. Derived Categories. Let $X$ be an algebraic variety, we use $\mathcal{D}^b(X)$ to denote the bounded derived categories of coherent sheaves on $X$. We denote $\text{Ext}^n(F, G) = \text{Hom}(F, G[\cdot])$ and $\text{Ext}^\bullet(F, G) = \oplus_{p \in \mathbb{Z}} \text{Ext}^p(F, G)[-p]$. Recall that objects $E_1, \ldots, E_n$ in the bounded derived category of coherent sheaves $\mathcal{D}^b(X)$ forms a full exceptional collection if

1. $\text{Hom}(E_i, E_j[m]) = k$ if $m = 0$ and is $0$ otherwise;
2. $\text{Hom}(E_i, E_j[m]) = 0$ for all $m \in \mathbb{Z}$ if $j < i$;
3. The smallest triangulated subcategory of $\mathcal{D}^b(X)$ containing $E_1, \ldots, E_n$ is itself.

An exceptional collection is strong if in addition: $\text{Hom}(E_i, E_j[m]) = 0$ for all $i, j$ if $m \neq 0$. On a Fano variety $X$, Kodaira vanishing theorem implies:

$$H^i(X, O_X) = 0$$

for all $i > 0$. Thus all line bundles on $V_5$ are exceptional objects. Moreover [Or] showed that $\mathcal{D}^b(V_5)$ has a full exceptional collection:

$$\mathcal{D}^b(V_5) = \langle U, Q', O_{V_5}, O_{V_5}(1) \rangle$$

We use $\mathcal{B}_{V_5}$ to denote the triangulated subcategory $\langle U, Q' \rangle = \langle O_{V_5}, O_{V_5}(1) \rangle$. The following result can be found in [Ku12].

**Lemma 2.8.** We have canonical isomorphism \[\text{Ext}^\bullet(U, Q') = A\]

As a result, we have an equivalence of category \[\Psi : \mathcal{B}_{V_5} \simeq \mathcal{D}^b(Q_5)\].

where $\mathcal{D}^b(Q_5)$ is the derived category of finite dimensional representations of the Kronecker quiver with 2 vertices and 3 arrows from the first vertex to the second.

For $F \in \mathcal{B}_{V_5}$, $\Psi(F)$ is the representation $(M_1^*, M_2^*)$ with

$$M_1^* = \text{Ext}^\bullet(F, U[1])^*$$

$$M_2^* = \text{Ext}^\bullet(Q', F)$$

2.4. Quivers representations and their moduli. A quiver $Q$ is given by two sets $Q_{vx}$ and $Q_{ar}$, where the first set is the set of vertices and the second is the set of arrows, along with two functions $s, t : Q_{ar} \rightarrow Q_{vx}$ specifying the source and target of an arrow. The path algebra $kQ$ is the associative $k$-algebra whose underlying vector space has a basis consisting of elements of $Q_{ar}$. The product of two basis elements is defined by concatenation of paths if possible, otherwise 0. The product of two general elements is defined by extending the above linearly.

Let $Q$ be a quiver. A quiver representation $R = (R_v, r_a)$ consists of a vector space $R_v$ for each $v \in Q_{vx}$ and a morphism of vector spaces $r_a : R_{s(a)} \rightarrow R_{t(a)}$ for each $a \in Q_{ar}$. A subrepresentation of $R$ is a pair $R' = (R'_v, r'_a)$ where $R'_v$ is a subspace of $R_v$ for each $v \in Q_{vx}$ and $r'_a : R'_{s(a)} \rightarrow R'_{t(a)}$ is a morphism of vector spaces for each $a \in Q_{ar}$ such that

$$r'_a = r_a|_{R'_{s(a)}}$$

and

$$r_a(R'_{s(a)}) \subseteq R'_{t(a)}.$$
Thus we have the commutative diagram

\[
\begin{array}{ccc}
R'_i & \xrightarrow{r'_u} & R'_j \\
\downarrow{i} & & \downarrow{j} \\
R_i & \xrightarrow{r_u} & R_j
\end{array}
\]

for any arrow \( a \) from \( i \) to \( j \). We use \( R' \subset R \) to denote that \( R' \) is a subrepresentation of \( R \).

Given a quiver \( Q \), a weight is an element \( \theta \in \mathbb{Z}^N \) where \( N = |Q_{vx}| \). For a weight \( \theta \), the weight function is defined by:

\[
\theta(S) = \sum_{i=1}^{N} d_i \theta_i,
\]

where \( S \) is a representation of \( Q \) and \( d_i \) and \( \theta_i \) are the \( i \)-th entries of \( \vec{d} \) and \( \theta \) respectively. We recall the definition of semi-stability:

**Definition 2.10.** A representation \( R \) is \( \theta \)-semi-stable if for any subrepresentation \( R' \subset R \)

\[
\theta(R') \geq \theta(R)
\]

\( R \) is \( \theta \)-stable if all the above inequalities are strict.

Let \( \vec{d} \) be a dimension vector, the set of representations of \( Q \) with dimension vector \( \vec{d} \) forms an affine space \( \text{Rep}(Q)_{\vec{d}} \). For a weight \( \theta \), the set of \( \theta \)-semi-stable representations forms an open subscheme \( \text{Rep}(Q)_{\vec{d}}^{\theta-SS} \) of \( \text{Rep}(Q)_{\vec{d}} \); the set of \( \theta \)-stable representations forms an open subscheme \( \text{Rep}(Q)_{\vec{d}}^{\theta-S} \) of \( \text{Rep}(Q)_{\vec{d}}^{\theta-SS} \).

The group \( G_0 = \Pi_i GL(d_i) \) acts by incidence on \( \text{Rep}(Q) \), in other words, it acts by \( (g \cdot a) = g_{i(a)} r_u g_{s(a)}^{-1} \). Apparently, the diagonal subgroup \( k^*_{\text{diag}} \) of \( (k^*)^{Q_{vx}} \) consisting of elements of the form \((k,k,\ldots,k)\) for \( k \in k^* \) acts trivially on \( \text{Rep}(Q) \). So it is natural to only consider the action of \( G_0 = \Pi_i GL(d_i)/k^*_{\text{diag}} \).

**Definition 2.11.** Two representations with dimension vector \( \vec{d} \) are isomorphic if they are in the same orbit under the action of \( G \).

Let \( \vec{d} \) be a dimension vector, the set of representations of \( Q \) with dimension vector \( \vec{d} \) forms an affine space \( \text{Rep}(Q)_{\vec{d}} \). For a weight \( \theta \), the set of \( \theta \)-semi-stable representations forms an open subscheme \( \text{Rep}(Q)_{\vec{d}}^{\theta-SS} \) of \( \text{Rep}(Q)_{\vec{d}} \); the set of \( \theta \)-stable representations forms an open subscheme \( \text{Rep}(Q)_{\vec{d}}^{\theta-S} \) of \( \text{Rep}(Q)_{\vec{d}}^{\theta-SS} \).

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**Definition 2.11.** Two representations with dimension vector \( \vec{d} \) are isomorphic if they are in the same orbit under the action of \( G \).

Give a weight \( \theta \), we may interpret \( \theta \) as a multiplicative character of the group \( G \) by the formula \( g \mapsto \det(g_1)^{\theta_1} \cdots \det(g_n)^{\theta_n} \). The moduli space of \( \theta \)-semi-stable representations with dimension vector \( \vec{d} \) is the GIT quotient

\[
M_{\vec{d}}^{\theta-ss} : = \text{Rep}(Q)_{\vec{d}}^{\theta-S}/G.
\]

We mention a few facts about \( M_{\theta} \). An equivalent definition of \( M_{\vec{d}}^{\theta-ss} \) is to consider the graded ring

\[
B_{\theta} = \bigoplus_{r \geq 0} B(r\theta)
\]

where \( B(r\theta) \) is the space of \( r\theta \)-semi-invariant functions in the coordinate ring of \( \text{Rep}(Q) \). Then the GIT quotient is defined as

\[
M_{\theta} = \text{Proj}(B_{\theta})
\]

From this definition, it is easy to see that \( M_{\theta} \) is an irreducible normal projective scheme.
2.5. Representations of Kronecker quiver $Q_3$. Let $Q_3$ be the Kronecker quiver with two vertices $1, 2$ and three arrows $a_1, a_2, a_3$ from $1$ to $2$. We let $\theta = (-1, 1)$ be a weight for $Q_3$. We know that the moduli space of $\theta$-semi-stable representations of $Q_3$ with dimension vector $(1, 1)$ is isomorphic to $\mathbb{P}^2$. Also, it is well-known that the moduli space of $\theta$-semi-stable representation of $Q_3$ with dimension vector $(2, 2)$ is isomorphic to $\mathbb{P}^5$. We provide a proof that will be useful later on.

**Proposition 2.12.** We have

$$M_{(2, 2)}^{\theta-ss} \simeq \mathbb{P}^5$$

**Proof.** A representation of $Q_3$ with dimension vector $(2, 2)$ is given by three $2 \times 2$ matrices

$$Y_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

for $i = 1, 2, 3$, each corresponding to the arrow $a_i$. Then the affine representation scheme $\text{Rep}(Q_3(2, 2)) = \text{Spec} \mathbb{C}[a_i, b_i, c_i, d_i, i = 1, 2, 3] \simeq \mathbb{A}^{12}$. It is not hard to check $B(\theta)$ is spanned by

$$\{\det(Y_1), \det(Y_2), \det(Y_3), a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1, a_1d_3 + a_2d_1 - b_1c_2 - b_2c_1, a_3d_2 + a_2d_3 - b_3c_2 - b_2c_3\}$$

Note the last three terms come from $\det(Y_1 + Y_j) - \det(Y_i) - \det(Y_j)$. And for $r \geq 2$, $B(r\theta)$ is generated by $B(\theta)$. This shows $B_\theta = \mathbb{C}[x_0, \ldots, x_5]$, and hence $M_{(2, 2)}^{\theta-ss} \simeq \mathbb{P}^5$. \qed

**Remark 2.13.** From now on, we use $M$ to denote $M_{(2, 2)}^{\theta-ss}$. We use

$$\left( Y_1, Y_2, Y_3 \right)$$

to denote a representation where $Y_i$ corresponds to the arrow $a_i$.

2.6. Instanton Sheaves. Let $Y$ be a Fano threefold of index 2. By definition, an (minimal) instanton bundle is a stable vector bundle $E$ of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$. We generalize this definition as follows:

**Definitions 2.14.** An instanton sheaf is a semi-stable sheaf of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$.

We use $M_{\text{inst}}$ to denote the moduli space of such sheaves. It is clear that the moduli space of minimal instanton bundles $M_{I_2}(Y)$ is an open subscheme of $M_{\text{inst}}$. [Ku12, Theorem 4.7] gave a concrete description of $M_{I_2}(V_5)$ using quadric nets. But it is not clear what $M_{I_2}(V_5)$ looks like as a variety.

We also recall the definition of an Ulrich bundle.

**Definitions 2.15.** Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $d$. An Ulrich bundles $E$ is a vector bundle on $X$ satisfying

$$H^*(X, E(-t)) = 0$$

for all $t = 1, \ldots, d$.

We list a few well-known facts about Ulrich bundles that will be useful to us:

- There are no Ulrich line bundles on a variety $X$ with $\text{Pic}(X) = \mathbb{Z}O_X(1)$.
- An Ulrich bundle is semistable. If it is not stable, it is an extension of Ulrich bundles of smaller ranks.
To see the relation between (minimal) instanton bundles and Ulrich bundles of rank 2 on $V_5$, we first recall the computation:

**Lemma 2.16.** [Ku12] Let $E$ be an (minimal) instanton bundle on a Fano threefold of index 2. Then

$$H^r(E(t)) = 0$$

for $t = 0, -1, -2$.

On the other hand, Lee and Park obtained the following result in their recent paper:

**Proposition 2.17.** [LP] Proposition 3.4 | For any $r \geq 2$, an Ulrich bundle $E$ of rank $r$ on $V_5$ corresponds to the following quiver representation.

$$E(-1) = \text{Coker}(U^r \to Q^{r^2})$$

Combine these two results, we immediately obtain:

**Corollary 2.18.** A vector bundle $E$ on $V_5$ is an (minimal) instanton bundle if and only if $E(1)$ is an Ulrich bundle of rank 2.

In addition, [LP] described the moduli space of stable Ulrich bundles of rank $r$.

**Theorem 2.19.** [LP] Theorem 3.11 | The moduli space of stable Ulrich bundles of rank $r$ $M^U_r$ is a smooth $(r^2 + 1)$-dimensional open subscheme of $M_{(r,r)}^{(−1,1)−}$.

By the above corollary, we obtain the following relation when $r = 2$,

$$MI_2(V_5) \simeq M^U_2 \subset M \simeq \mathbb{P}^5$$

3. Classification of Instanton Sheaves on $V_5$

[D] classified instanton sheaves on cubic threefolds. In this section we classify instanton sheaves on $V_5$, closely following the argument of [D]. When the proof transfers almost verbatim, we will only point out the changes in our situation and refer the readers to [D].

**Lemma 3.1.** [D] Lemma 2.1 | Let $X \subset \mathbb{P}^n$ be a variety of dimension $\geq 2$ and $E$ a $\mu$-semi-stable vector bundle of rank 2 with Chern classes $c_1(E) = 0$. If $h^0(E) \neq 0$, then either the zero locus of a nonzero global section is pure of codimension 2 or the section does not vanish and $c_2(E) = 0$.

**Lemma 3.2.** Let $S \subset \mathbb{P}^5$ be a del Pezzo surface of degree 5 and $E$ a semi-stable vector bundle of rank 2 with Chern classes $c_1(E) = 0$ and $c_2(E) = 2$. If $h^0(E) = 0$, then $h^1(E(n)) = 0$ for $n \in \mathbb{Z}$ and $h^2(E(n)) = 0$ for $n \geq 1$. If $h^0(E) \neq 0$, then $h^0(E) = 1$, $h^1(E(n)) = 0$ for $n \leq -2$ and $n \geq 1$, $h^1(E(-1)) = h^1(E) = 1$ and $h^2(E(n)) = 0$ for $n \geq 0$.

**Proof.** See [D] Lemma 2.2]

**Lemma 3.3.** Let $S \subset \mathbb{P}^5$ be a del Pezzo surface of degree 5 and $E$ a semi-stable vector bundle of rank 2 with Chern classes $c_1(E) = 0$ and $c_2(E) = 1$. If $h^0(E) \neq 0$, then $h^0(E) = 1$, $h^1(E(n)) = 0$ for all $n \in \mathbb{Z}$ and $h^2(E(n)) = 0$ for all $n \geq 0$.

**Proof.** See [D] Lemma 2.3]

**Proposition 3.4.** Let $E$ be an instanton sheaf on $V_5$. Let $F$ be the double dual of $E$. Then either $E$ is locally free or $F$ is locally free with second Chern classes $c_2(F) = 1$ and $h^0(F) = 1$ or $F = H^0(F) \otimes \mathcal{O}_{V_5}$. 
Proof. See [D, Proposition 3.1]. We have

\[
\begin{align*}
\chi(E(n)) &= \frac{5}{3} n^3 + 5n^2 + \frac{10}{3} n \\
\chi(\mathcal{O}_{V_5}(n)) &= \frac{5}{6} n^3 + \frac{5}{2} n^2 + \frac{8}{3} n + 1 \\
\chi(I_p(n)) &= \frac{5}{6} n^3 + \frac{5}{2} n^2 + \frac{8}{3} n
\end{align*}
\]

where \( p \) is a point on \( V_5 \). \(\square\)

Lemma 3.5. [D, Lemma 3.2] Let \( R \) be a coherent sheaf on \( \mathbb{P}^n \), so that \( h^0(R(-1)) = 0 \) and \( \chi(R(n)) = n + 1 \). There exists a straight line \( l \subset \mathbb{P}^n \) so that \( R = \mathcal{O}_l \).

Lemma 3.6. [D, Lemma 3.3] Let \( R \) be a coherent sheaf on \( \mathbb{P}^n \) so that \( h^0(R(-1)) = 0 \) and \( \chi(R(n)) = 2n + 2 \). Then there exist two lines \( l_1, l_2 \) so that \( R \) is the extension of \( \mathcal{O}_{l_2} \) by \( \mathcal{O}_{l_1} \) or \( R(-1) \) is a theta-character on a smooth conic \( C \subset \mathbb{P}^n \).

Lemma 3.7. Suppose \( \theta \) is the theta-characteristic of a smooth conic \( C \subset V_5 \). We consider the sheaf \( E \) which is the kernel of the surjection \( H^0(\theta(1)) \otimes \mathcal{O}_{V_5} \to \theta(1) \). Then \( E \) is stable with Chern classes \( c_1(E) = 0 \), \( c_2(E) = 2 \) and \( c_3(E) = 0 \).

Proof. See [D, Lemma 3.4]. We have

\[
\chi(I_C(n)) = \frac{5}{6} n^3 + \frac{5}{2} n^2 + \frac{2}{3} n
\]

\(\square\)

Theorem 3.8. Let \( E \) be an instanton sheaf on \( V_5 \). If \( E \) is stable, then either \( E \) is locally free or \( E \) is associated to a smooth conic \( C \subset V_5 \) such that we have the exact sequence:

\[
0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_5} \to \theta(1) \to 0
\]

where \( \theta \) is the theta-characteristic of \( C \).

If \( E \) is properly semi-stable, then \( E \) is the extension of two ideal sheaves of lines in \( V_5 \).

Proof. See [D, Theorem 3.5] \(\square\)

4. Relation to representation of \( Q_3 \)

[LP] proved that for any Ulrich bundle \( E \) of rank \( r \geq 2 \) on \( V_5 \), \( E(-1) \) is a representation of \( Q_3 \) with dimension vector \((r, r)\). By Corollary 2.18 this implies all instanton bundles are of representations of \( Q_3 \) with dimension vector \((2, 2)\). We generalize this result in the following theorem:

Theorem 4.1. Let \( E \) be an instanton sheaf on \( V_5 \), then there exists a short exact sequence:

\[
0 \to \mathcal{U}^{\oplus 2} \to Q^{\oplus 2} \to E \to 0.
\]

In particular, \( E \in \mathcal{B}_{V_5} \) and \( \Psi(E) \) is isomorphic to a representation of \( Q_3 \) with dimension vector \((2, 2)\).

We need two lemmas for the proof.

Lemma 4.2. Let \( E \) be an instanton sheaf on \( V_5 \). Then \( E \in \mathcal{B}_{V_5} \).
Proof. It suffices to show that $H^*(E(-1)) = H^*(E) = 0$. If $E$ is a vector bundle, the result follows from [Ku03, Lemma B.3].

If $E$ is associated to a smooth conic, we have short exact sequence:

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_5} \to \theta(1) \to 0$$

Since $H^*(\mathcal{O}_{V_5}(-1)) = H^*(\theta) = 0$, we immediately obtain $H^*(E(-1)) = 0$. On the other hand, we have $h^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_5}) = 2$ and $h^0(\theta(1)) = 2$. It is clear that the map $H^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_5}) \to H^0(\theta(1))$ is surjective. Moreover, $H^i(\mathcal{O}_{V_5}) = H^i(\theta(1)) = 0$ for all $i > 0$. Thus $H^*(E) = 0$.

If $E$ is the extension of the ideal sheaves of lines in $V_5$, we note by [Ku12, Lemma 4.2], $I_l \in B_{V_5}$ for all lines $l$. The lemma follows immediately. \qed

Lemma 4.3. Let $C$ be a smooth conic on $V_5$. Consider the short exact sequence:

$$0 \to Q \otimes I_C \to Q \to Q|_C \to 0$$

The induced map

$$f : H^0(Q) \to H^0(Q|_C)$$

is an isomorphism.

Proof. We have exact sequence:

$$0 \to H^0(Q \otimes I_c) \to H^0(Q) \xrightarrow{f} H^0(Q|_C) \to H^1(Q \otimes I_c)$$

Note $h^0(Q) = h^3(Q'(-2)) = 5$ and $h^0(Q|_C) = h^0(\mathcal{O}_C(1) \oplus \mathcal{O}_C(1) \oplus \mathcal{O}_C) = 5$. To show $f$ is an isomorphism, it suffices to show $H^1(Q \otimes I_C) = 0$. Consider the short exact sequence:

$$0 \to U \to V \otimes \mathcal{O}_{V_5} \to Q \to 0$$

we have

$$H^1(I_C) \oplus 5 \to H^1(Q \otimes I_C) \to H^2(U \otimes I_C)$$

It is clear $H^1(I_C) = 0$, so it remains to show $H^2(U \otimes I_C) = 0$. Consider the short exact sequence:

$$0 \to U \otimes I_C \to U \to U|_C \to 0$$

we have

$$H^1(U|_C) \to H^2(U \otimes I_C) \to H^2(U)$$

But $H^2(U) = 0$ and $H^1(U|_C) = H^1(C, \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)) = 0$, thus $H^2(U \otimes I_C) = 0$ and the proof is complete. \qed

Proof of Theorem 4.1. By Lemma 2.8, it suffices to show $\text{Ext}^*(E, U[1]) = \mathbb{C}^2$ and $\text{Ext}^*(Q', E) = \mathbb{C}^2$.

If $E$ is a vector bundle, the result follows from [LP, Proposition 3.4]. If $E$ is associated to a smooth conic, we have the short exact sequence:

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_5} \to \theta(1) \to 0$$

Taking the long exact sequence and noting $H^*(U) = 0$, we have

$$\text{Ext}^i(E, U) = \text{Ext}^{i+1}(\theta(1), U)$$
It is easy to see the only nonzero part is
\[ \text{Ext}^1(E, \mathcal{U}) = \text{Ext}^2(\theta(1), \mathcal{U}) \]
\[ = \text{Ext}^1(\mathcal{U}^\vee \otimes \theta(-1)) \]
\[ = \text{Ext}^2_C(\mathcal{O}_C(-2) \oplus \mathcal{O}_C(-2)) \]
\[ = \mathbb{C}^2 \]

This proves \( \text{Ext}^*(E, \mathcal{U}[1]) = \mathbb{C}^2 \). We now compute \( \text{Ext}^*(Q^\vee, E) \). We have \( Q \otimes \theta(1) = \mathcal{O}_C(1) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \) and \( H^k(Q \otimes \theta(1)) = 0 \) unless \( k = 0 \). Moreover, \( H^k(Q) = 0 \) unless \( k = 0 \). Thus \( \text{Ext}^k(Q^\vee, E) = 0 \) for \( k \geq 2 \) and we have exact sequence:
\[ 0 \rightarrow \text{Hom}(Q^\vee, E) \rightarrow H^0(\theta(1)) \otimes H^0(Q) \rightarrow H^0(Q \otimes \theta(1)) \rightarrow \text{Ext}^1(Q^\vee, E) \rightarrow 0 \]

Since \( C \simeq \mathbb{F}^1 \), \( g \) factors as
\[ H^0(\theta(1)) \otimes H^0(Q) \xrightarrow{id \otimes f} H^0(\theta(1)) \otimes H^0(Q|_C) \rightarrow H^0(Q \otimes \theta(1)) \]
where \( f \) is as in Lemma 4.3, so \( g \) is surjective. Thus \( \text{Ext}^1(Q^\vee, E) = 0 \) and \( \text{Hom}(Q^\vee, E) = \mathbb{C}^2 \) by counting dimension.

If \( E \) is the extension of ideal sheaves of lines in \( V_5 \), then we have short exact sequence:
\[ 0 \rightarrow I_{l_1} \rightarrow E \rightarrow I_{l_2} \rightarrow 0 \]

By [Ku12, Lemma 4.2], we have
\[ \text{Ext}^*(I_{l_k}, \mathcal{U}[1]) = \mathbb{C} \]
\[ \text{Ext}^*(Q^\vee, I_{l_k}) = \mathbb{C} \]

hence the result follows by looking at the long exact sequence induced by (4.4). \( \square \)

We now try to show all representations in Theorem 4.1 are semi-stable.

**Lemma 4.5.** Let \( E \) be an instanton sheaf on \( V_5 \), then \( H^0(E(-2)) = 0 \) and \( \chi(E(-2)) = 0 \).

**Proof.** If \( E \) is a vector bundle, the result follows from [Ku03, Lemma B.3]. If \( E \) is associated to a smooth cubic, we have exact sequence:
\[ 0 \rightarrow E(-2) \rightarrow H^0(\theta(1)) \otimes \mathcal{O}_{V_5}(-2) \rightarrow \theta(-1) \rightarrow 0 \]

Then \( H^0(E(-2)) = H^0(\mathcal{O}_{V_5}(-2)) = 0 \) and
\[ \chi(E(-2)) = 2\chi(\mathcal{O}_{V_5}(-2)) - \chi(\theta(-1)) \]
\[ = -2\chi(\mathcal{O}_{V_5}) - \chi(\mathcal{O}_C(-3)) \]
\[ = -2 - (-2) \]
\[ = 0 \]

If \( E \) is the extension of ideal sheaves of lines in \( V_5 \), the lemma follows from the fact that \( H^0(I_{l_1}(-2)) \subset H^0(\mathcal{O}_{V_5}(-2)) = 0 \) and \( \chi(I_{l_1}(-2)) = \chi(\mathcal{O}_{V_5}(-2)) - \chi(\mathcal{O}_{l_1}(-2)) = \chi(\mathcal{O}_{l_1}) - \chi(\mathcal{O}_{V_5}) = 1 - 1 = 0 \). \( \square \)

**Proposition 4.6.** Let \( E \) be a stable(strictly semi-stable) instanton sheaf on \( V_5 \), then the corresponding representation in Theorem 4.1 is stable(strictly semi-stable).
Proof. Using the first part of the proof of [LP, Proposition 3.10] along with Lemma 4.5, we see any instanton sheaf corresponds to a semi-stable representation.

If $E$ is strictly semi-stable, we have short exact sequence:

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0,$$

where $l_1, l_2$ are two lines in $V_5$. By [Ku12, Lemma 4.2], we have exact sequence:

$$0 \to U \to Q^\vee \to I_{l_1} \to 0,$$

Since $\{U, Q^\vee\}$ is an exceptional pair, $\text{Hom}(Q^\vee, Q^\vee \oplus 2) = \text{Hom}(Q^\vee, E)$, we obtain the commutative diagram:

$$
\begin{array}{cccccc}
0 & 0 & 0 & & & \\
\downarrow & \downarrow & \downarrow & & & \\
0 & U & Q^\vee & I_{l_1} & 0 & \\
\downarrow & \downarrow & \downarrow & & & \\
0 & U^{\oplus 2} & Q^{\vee \oplus 2} & E & 0 & \\
\end{array}
$$

This shows any representation $R_E$ associated to $E$ has a subrepresentation with dimension vector $(1, 1)$, thus it is not stable.

Suppose $E$ is stable and the corresponding representation $R_E$ is not stable, then $R_E$ must have a subrepresentation of dimension $(1, 1)$. Note all nonzero morphisms from $U$ to $Q^\vee$ are injective and has $I_l$ as cokernel for some line $l$ in $V_5$. Thus we have the commutative diagram:

$$
\begin{array}{cccccc}
0 & 0 & 0 & & & \\
\downarrow & \downarrow & \downarrow & & & \\
0 & U & Q^\vee & I_{l_1} & 0 & \\
\downarrow & \downarrow & \downarrow & & & \\
0 & U^{\oplus 2} & Q^{\vee \oplus 2} & E & 0 & \\
\downarrow & \downarrow & \downarrow & & & \\
\text{Ker}(f) & U & Q^\vee & \text{Coker}(f) & \\
\end{array}
$$

Again by the injectivity of nonzero morphisms from $U$ to $Q^\vee$, we have either $\text{Ker}(f) = 0$ or $\text{Ker}(f) = U$. But the second case would imply there is an injection $U^{\oplus 2} \to Q^\vee$, which is absurd. Thus $\text{Ker}(f) = 0$ and $\text{Coker}(f) = I_{l'}$ for some line $l'$. And by the snake lemma, we see $I_{l'} \hookrightarrow E$, contradicting the stability of $E$.

□

By now we have established a well-behaved correspondence between (semi)-stable instanton sheaves on $V_5$ and $\theta$-(semi)-stable representations of $Q_3$ with dimension vector $(2, 2)$. Next is to use this correspondence to analyze the two moduli spaces.

5. Moduli Space of Instantons

We start by showing the smoothness of $M^{\text{inst}}$. To do this we first compute some related invariants.

Lemma 5.1. Let $\theta$ be the theta characteristic of a smooth conic $C$ in $V_5$. Let $E$ be the kernel of the natural surjection $H^0(\theta(1)) \otimes \mathcal{O}_{V_5} \to \theta(1)$. Then $\text{Ext}^2(E, E) = 0$ and $\text{Ext}^1(E, E)$ has dimension 5.
Proof. Consider the exact sequence:
\[
\text{Ext}^2(H^0(\theta(1)) \otimes O_{V_5}, E) \rightarrow \text{Ext}^2(E, E) \rightarrow \text{Ext}^3(\theta(1), E)
\]
We have \(\text{Ext}^2(H^0(\theta(1)) \otimes O_{V_5}, E) \simeq H^0(\theta(1)) \otimes H^2(E) = 0\) and \(\text{Ext}^3(\theta(1), E) \simeq \text{Hom}(E, \theta(-1))^*\).
Note we have injection \(\text{Hom}(E, \theta(-1)) \rightarrow \text{Hom}(Q^* \oplus Q^*, \theta(-1)) = 0\) by looking at the splitting type of \(Q^*\). Thus \(\text{Ext}^2(E, E) = 0\). Moreover, \(\text{Ext}^3(E, E) \simeq \text{Hom}(E, (E(-2))^*) = 0\) and \(\text{Hom}(E, E) = \mathbb{C}\). By Riemann-Roch, \(\chi(E, E) = -4\). Thus \(\text{Ext}^1(E, E)\) is five dimensional.

\[\square\]

Lemma 5.2. Let \(l_1, l_2 \subset V_5\) be two lines. Then \(\text{Ext}^2(I_{l_1}, I_{l_2}) = 0\) and \(\dim \text{Ext}^1(I_{l_1}, I_{l_2}) = 1\) if \(l_1 \neq l_2\) and \(2\) if \(l_1 = l_2\).

Proof. Consider the exact sequence:
\[
\text{Ext}^2(O_{V_5}, I_{l_2}) 
\rightarrow \text{Ext}^2(I_{l_1}, I_{l_2}) 
\rightarrow \text{Ext}^3(O_{l_1}, I_{l_2})
\]
We have \(\text{Ext}^2(O_{V_5}, I_{l_2}) = 0\) and \(\text{Ext}^3(O_{l_1}, I_{l_2}) \simeq \text{Hom}(I_{l_2}, O_{l_1}(-2))^*\). Note we have injection
\[
\text{Hom}(I_{l_2}, O_{l_1}(-2)) \rightarrow \text{Hom}(Q^*, O_{l_1}(-2)) = 0
\]
by looking at the splitting type of \(Q^*\). Thus \(\text{Ext}^2(I_{l_1}, I_{l_2}) = 0\). Moreover, \(\text{Ext}^3(I_{l_1}, I_{l_2}) \simeq \text{Hom}(I_{l_2}, O_{l_1}(-2))^* = 0\). By Riemann-Roch, \(\chi(I_{l_1}, I_{l_2}) = -1\). Thus the lemma follows.

\[\square\]

Let \(N \geq 1\) be an integer and \(V\) be a complex vector space. Let \(Q\) be the Hilbert scheme of the quotient \(V \otimes O_X(-N) \rightarrow E\) of \(X\) with rank \(2\) and Chern classes \(c_1(E) = 0, c_2(E) = 2\) and \(c_3(E) = 0\) and \(L\) the natural polarization [Sl]. The group \(G = \text{PGL}(V)\) acts on \(Q\) and a suitable power of \(L\) is \(G\)-linearized. Let \(Q^{ss}_e\) be the \(\text{PGL}(V)\)-semi-stable points corresponding to quotients without torsion and \(Q_e\) the closure of \(Q^{ss}_e\) in \(Q\). When the integer \(N\) and the vector space \(V\) are suitably chosen, the following properties are satisfied. The map \(V \otimes O_X \rightarrow E(N)\) induce an isomorphism \(V \simeq H^0(E(N))\) and \(h^i(E(k)) = 0\) for \(k \geq N\) and \(i \geq 1\) and for all \(E\) in \(Q_e\). The point \([E] \in Q_e\) is semi-stable if and only if the sheaf \(E\) is semi-stable if and only if \(E \in Q^{ss}_e\). The stabilizer of \([E]\) in \(GL(V)\) is identified with the group of automorphisms of the sheaf \(E\) and moduli space is then the GIT quotient \(Q^{ss}_e//G\).

Lemma 5.3. With the above hypothesis, the scheme \(Q^{ss}_e\) is smooth.

Proof. The tangent space of \(Q^{ss}_e\) at a point \([E]\) is isomorphic to \(\text{Hom}(F, E)\) where \(F\) is the kernel of the map \(V \otimes O_X(-N) \rightarrow E\). The scheme \(Q^{ss}_e\) is smooth at the point if \(\text{Ext}^1(F, E) = 0\). Consider the exact sequence:
\[
\text{Ext}^1(V \otimes O_X(-N), E) \rightarrow \text{Ext}^1(F, E) \rightarrow \text{Ext}^2(E, E)
\]
We then obtain an inclusion \(\text{Ext}^1(F, E) \rightarrow \text{Ext}^2(E, E)\) since \(h^1(E(N)) = 0\). It suffices then to prove \(\text{Ext}^2(E, E) = 0\). If \(E\) is stable and locally free, then \(\text{Ext}^2(E, E) = \text{Ext}^2_{\text{D}^b(Q_e)}(R_E, R_E) = 0\) since the path algebra of a quiver is hereditary. If \(E\) is stable but not locally free, we apply Lemma 6.1. If \(E\) is strictly semi-stable, then \(E\) is the extension of \(I_1\) and \(I_2\), the vanishing follows from Lemma 5.2.

\[\square\]

Theorem 5.4. The moduli space of semi-stable sheaves of rank \(2\) with Chern classes \(c_1(E) = 0, c_2(E) = 2, c_3(E) = 0\) on \(V_5\) is smooth of dimension \(5\).

Proof. See [D, Theorem 4.6]
We now construct a morphism from $M^{\text{inst}}$ to $M$. $M^{\text{inst}}$ is the GIT quotient $Q_5^g//G$. Let $\mathcal{E}$ be a universal family on $Q_5^{ss} \times V_5$. For any $t \in Q_5^{ss}$, by Lemma 2.8 and Lemma 4.11, $\Psi(\mathcal{E}_t)$ is isomorphic to a representation of $Q_5$ with dimension vector $(2, 2)$. We use $[\Psi(\mathcal{E}_t)]$ to denote the corresponding point in $M$ of such a representation. By Lemma 4.6 the map
\[
\Phi : Q_5^{ss} \to M
\]
\[
t \mapsto [\Psi(\mathcal{E}_t)]
\]
is well defined and algebraic. (To see the map is algebraic, we can use $\mathcal{E}$ to construct family of representations on $Q_5^{ss}$, see Remark 4.11 in [S]). To see this morphism is invariant under $G$, it suffices to check for any $t$ that corresponds to strictly semi-stable instanton sheaf, i.e. $\mathcal{E}_t$ is the extension of $I_{l_1}$ and $I_{l_2}$, we have $\Phi(t) = \Phi(t_0)$ where $t_0$ corresponds to $I_{l_1} \oplus I_{l_2}$. This is the content of the next lemma.

**Lemma 5.5.** Let $E$ be a sheaf defined by
\[
0 \to I_{l_1} \to E \to I_{l_2} \to 0
\]
where $l_1, l_2$ are two lines in $V_5$. Then $\Phi([E]) = \Phi([I_{l_1} \oplus I_{l_2}])$.

**Proof.** Recall $F(V_5) = \mathbb{P}(A) \simeq \mathbb{P}^2$. Suppose $l_1$ corresponds to $[a_1 : b_1 : c_1]$ and $l_2$ corresponds to $[a_2 : b_2 : c_2]$. Then $\Psi(E)$ is given by
\[
\left( \begin{array}{ccc}
 a_1 & \alpha & b_1 \\
 0 & a_2 & 0 \\
 b_2 & \beta & c_2
\end{array} \right)
\]
where $\alpha, \beta, \gamma \in \mathbb{C}$. Let
\[
g_n = \left( \begin{array}{cc}
 \frac{1}{n} & 0 \\
 0 & 1
\end{array} \right) \in G
\]
then
\[
g_n \cdot \Psi(E) = \left( \begin{array}{ccc}
 a_1 & \frac{\alpha}{n} & b_1 \\
 0 & a_2 & 0 \\
 b_2 & \frac{\beta}{n} & c_2
\end{array} \right)
\]
and
\[
\lim_{n \to \infty} g_n \cdot \Psi(E) = \left( \begin{array}{ccc}
 a_1 & 0 & b_1 \\
 0 & a_2 & 0 \\
 0 & b_2 & c_2
\end{array} \right) = \Psi(I_{l_1} \oplus I_{l_2})
\]

This lemma shows $\Phi$ is invariant under $G$. Thus $\Phi$ descends to a morphism $\phi : M^{\text{inst}} \to M$.

**Theorem 5.6.** $\phi : M^{\text{inst}} \to M$ is an isomorphism. As a result, the moduli space of instanton sheaves on $V_5$ is $\mathbb{P}^5$.

**Proof.** Since both $M^{\text{inst}}$ and $M$ are projective, $\phi$ is proper. We claim $\phi$ is injective. Let $\phi([E_1]) = \phi([E_2])$. By Proposition 4.10 either both $E_i$ are stable or both $E_i$ are strictly semi-stable. If $E_i$ are stable, then $\phi([E_1]) = \phi([E_2])$ implies $E_1 \simeq E_2$, i.e. $[E_1] = [E_2]$. If $E_i$ are semi-stable, we can assume $E_1 = I_{l_1} \oplus I_{l_2}$ where $l_1$ corresponds to the point $[a_1 : a_2 : a_3]$, $l_2$ corresponds to the point $[a_1 : a_2 : a_3]$ in $F(V_5)$ and $l_2$ corresponds to the point $[a_1 : a_2 : a_3]$. Then $\Psi(E)$ is given by
\[
\left( \begin{array}{ccc}
 a_1 & 0 & a_2 \\
 0 & d_1 & 0 \\
 a_3 & 0 & d_3
\end{array} \right)
\]
Note $M \simeq \mathbb{P}^5$. Using the coordinates in the proof of Proposition 2.12, $\Psi(E)$ corresponds to the point with coordinate:

$[a_1 d_1 : a_2 d_2 : a_3 d_3 : a_1 d_2 + a_2 d_1 : a_3 d_1 : a_2 d_3 + a_3 d_2]$ \hfill (5.7)

The point now is to consider the symmetric power $(\mathbb{P}^2 \times \mathbb{P}^2)/S_2$. Suppose the first component has coordinates $[a_1 : a_2 : a_3]$ and the second has $[d_1 : d_2 : d_3]$, then we have a closed embedding

$$(\mathbb{P}^2 \times \mathbb{P}^2)/S_2 \hookrightarrow \mathbb{P}^5$$

by using the coordinates

$\{a_1 d_1, a_2 d_2, a_3 d_3, a_1 d_2 + a_2 d_1, a_3 d_1, a_2 d_3 + a_3 d_2\}$

This shows we can recover the set $\{l_1, l_2\}$ from (5.7). Thus $\phi$ is also injective on strictly semi-stable points.

Since $\phi$ is injective and $M$ is integral, we see $M^{\text{inst}}$ is connected. Along with Theorem 5.3 we know $M^{\text{inst}}$ is a smooth variety. By [LP] Theorem 3.11, $\phi$ is birational and the image of $\phi$ contains an open subscheme of $M$, thus $f$ is surjective.

Now $\phi$ is a bijective birational proper morphism between smooth varieties, by Zariski Main Theorem, it is an isomorphism.

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