ZERO–BRANES IN 2+1 DIMENSIONS\textsuperscript{1,2}

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Abstract

We discuss zero-brane solutions in three-dimensional Minkowski space-time annihilated by half of the supersymmetries. The other half of the supersymmetries which should generate the supersymmetric multiplet are ill-defined and lead to a non bose-fermi degenerate solitonic spectrum.

1 Introduction

There exists a boson-fermion degeneracy of the supermultiplet in supersymmetric theories. However, there are cases where such degeneracies do not appear. A U(1) gauge theory for example in 2+1 dimensions has Nielsen-Olesen vortex solutions\textsuperscript{1} annihilated by half of the supersymmetries\textsuperscript{2,3}. The other half generates zero modes which form a supermultiplet of the unbroken N=1 supersymmetry. This happens as long as one restricts himself to global supersymmetry. In the supergravity context, Nielsen-Olesen vortex solutions which break half of the supersymmetries...
also exist. However, the other half of the supersymmetries now generates non-normalizable zero modes and thus the latter do not appear in the physical Hilbert space \( \mathcal{H} \). As a result, the U(1) vortex solutions do not have fermionic partners which are necessary to form a supermultiplet of the unbroken supersymmetry. Thus, one finds a solitonic spectrum without bose-fermi degeneracies.

The reason of this peculiarity is the conical structure of space-time in 2+1 dimensions \([1, 3]\). It is known that away from the sources the space-time is flat and there is a deficit angle at spatial infinity. The deficit angle does not allow the existence of Killing spinors \([3, 4]\) and, consequently, there are no states which carry fermionic supercharge. The latter cannot be consistently defined. In the presence of gauge fields, Killing spinors may exist as in the U(1) case mentioned above and states with fermionic supercharge do exist. However, they are not normalizable.

Here we will examine a similar case, namely, we will show that the solitonic spectrum of zero-branes in 2+1 dimensions has no bose-fermi degeneracies. These considerations are also related to Witten’s observation \([8]\) that in 2+1 dimensions the vacuum can have exactly zero cosmological constant because of supersymmetry while excited states may not come in boson-fermion pairs \([3, 9]\).

## 2 Global supercharges

Charges are defined as spatial integrals of the time component of a conserved current. However, in many cases there are currents which satisfy covariant conservation laws due to some local symmetry. One may recall for example the energy–momentum tensor in general relativity or the Yang-Mills current in gauge theories. In the first case an ordinarily conserved energy–momentum tensor may be constructed if there exist Killing vectors \([10]\). For an asymptotically D+1-dimensional Minkowski space, there exist \((D+1)(D+2)/2\) Killing vectors which give rise to \((D+1)(D+2)/2\) conserved charges \([11]\).

\[
P_0 = \oint_{r \to \infty} dS_i (g_{ij,i} - g_{jj,i}),
\]

\[
P^j = \oint_{r \to \infty} dS_i \pi^{ij},
\]

\[
J^{ik} = \oint_{r \to \infty} dS_k (x_i \pi^{jk} - x^k \pi^{ij}),
\]

\[
J_{0k} = \oint_{r \to \infty} dS_i \left( g^{ik} (g_{jj,j} - g_{jj,i}) - g_{ik} + g_{jj} \delta_{ik} \right).
\]

One may prove that they satisfy the Poincaré algebra while enlarged with the spinor charges they all form the supersymmetric algebra.

In order to define the spinor charges \([12]\) let us consider the vector-spinor density in \( D + 1 \) dimensions

\[
j^\mu = \Gamma^{\mu \nu \lambda} D_\nu \psi_\lambda,
\]

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where as usual, $\Gamma^{\mu\nu\lambda} = \Gamma^{[\mu\nu}\Gamma^{\lambda]}$, $(\mu, \nu = 0, \ldots, D, m, n = 1, \ldots, D)$. The current $j^\mu$ is not conserved but rather, as the Yang-Mills current and the energy–momentum tensor, it is covariantly constant $D_\mu j^\mu = 0$. However, one may construct a conserved quantity out of $j^\mu$ if the space-time admits a covariantly constant (Killing) spinor $\epsilon$, i.e.,

$$D_\mu \epsilon = 0.$$  \hfill (6)

Then, one may form the vector $J^\mu = \bar{\epsilon} j^\mu$ which may be written as

$$J^\mu = \partial_\nu (\bar{\epsilon} \Gamma^{\mu\nu\kappa} \psi^\kappa),$$  \hfill (7)

and thus it satisfies $\partial_\mu J^\mu = 0$. As a result, the spinor charge

$$Q(\epsilon) = \int d^D x \sqrt{|g|^{(D)}} J^0 = \oint_\infty dS_i \bar{\epsilon} \Gamma^{0ij} \psi_j,$$  \hfill (8)

is a conserved quantity. One may observe that in order to have a non-zero spinor charge, the covariantly constant spinor $\epsilon$ must approach a constant spinor $\epsilon_0$ at spatial infinity. In this case, there exist at most $2^{[D+1/2]}$ spinor charges, one for each non-zero component of $\epsilon_0$. However, global charges may not always be possible to be defined in gravitational as well as in gauge theories. One may recall for example that global colour does not exist in a monopole background \cite{12} and momentum cannot be defined in the gravitational field of a massive object in 2+1 dimensions \cite{4}.

In our case, the lack of global charges may be traced to the absence of the associated Killing vectors and spinors which are essential for the definition of such charges as we will see below.

### 3 Global supercharges in 2+1 dimensions

Let us now consider the rather special 2+1-dimensional case. Here, the space is flat out of the sources since the Riemann tensor is completely determined by the Einstein tensor. For localized sources, there exists a peculiar conical structure at spatial infinity due to a deficit angle. This leads to breakdown of the Poincaré symmetry at spatial infinity although the space-time is flat there \cite{13}. Asymptotically one should expect, in view of the local flatness, the Killing vector

$$\xi^\mu = a^\mu + \omega_\mu^\nu x^\nu,$$  \hfill (9)

to generate space-time rotations and translations. If a deficit angle $\beta$ is present, however, one has to identify $x^i = \Omega_i^j(\beta) x^j$ where $\Omega(\beta)$ is an SO(2) rotation matrix with parameter $\beta$ \cite{5}. Then, it may easily be proved that this condition breaks some space-time symmetries and the only surviving ones are the time-translations and space rotations. The other symmetries fail to satisfy the continuity conditions along the seams. As a result, the only conserved charges at spatial infinity are the energy
and the angular momentum while the ordinary momenta do not exist. One of course expects energy and momenta to form a 3-vector under Lorentz transformations and by boosting energy states to obtain states with non-zero momentum. However, this is not possible since the momentum of the boosted states diverges which simply reflects the fact that the asymptotic Poincaré invariance has been broken [13]. On the other hand, the spinor charges are given by

\[ Q(\epsilon) = \oint_{r \to \infty} dS j^i \bar{\epsilon} \varepsilon^{ij} \psi_j, \]  

(10)

and since there are no covariantly constant spinors \( \epsilon \) in conical space-times they do not exist as well. The reason of the non-existence of Killing spinors is the non half-integral phase acquired by \( \epsilon \) as it goes around a circle at infinity [6].

Covariant constant spinors and, consequently spinor charges, may be defined if one couples the spinors to a U(1) gauge field. Then Killing spinors may exist if the non half-integral phase acquired by the spinors as we parallel transport them around a circle at infinity is canceled by an Aharonov-Bohm phase due to the U(1) gauge field. This kind of situation has been considered in Ref. 3. It is similar to the generalized spin structures on manifolds with no ordinary spin structure. It is also possible that the U(1) gauge field cancels exactly the spin connection [14, 15]. In this case, covariantly constant spinors exist as well and these are the only cases we know where global supercharges may consistently be defined on a conical space-time.

The bosonic part of the action in 2+1 dimensions where only the dilaton \( \phi \) and an axion \( \alpha \) are not vanishing is

\[ I = \int d^3x \sqrt{-g} \left( R - \frac{1}{2} \frac{\partial \mu S \partial^\mu \bar{S}}{(S - S)^2} \right). \]  

(11)

It describes a SL(2,R)/U(1) \( \sigma \)-model coupled to gravity and can be obtained by wrapping the seven-brane [13] of type IIB around a six-cycle. It can also be obtained by compactifying a five-dimensional theory on a two torus of constant volume [16]. The complex scalar \( S \) is given in terms of the dilaton and the axion as \( S = \alpha + i e^{-\phi} \) and belongs to the upper half plane (\( \text{Im} S > 0 \)). The action (11) is invariant under the following N=2 supersymmetry transformations [17, 15] on a pure bosonic background

\[ \delta \lambda = -\frac{1}{S^2} \left( \frac{S - i}{S + i} \right) \gamma^\mu \partial_\mu S \epsilon^*, \]

\[ \delta \psi_\mu = D_\mu \epsilon, \]

(12)

where \( \epsilon = \epsilon_1 + i \epsilon_2 \) is a complex spinor and

\[ D_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^a \gamma^b - \frac{i}{2} Q_\mu, \]

(13)
is a covariant derivative containing the spin connection as well as the composite U(1) gauge field

\[ Q_\mu = -\frac{1}{S^2} \left[ \frac{(S - i)}{(S - i)} \partial_\mu \bar{S} + \frac{(S + i)}{(S + i)} \partial_\mu S \right]. \quad (14) \]

The field equations as follows from the action (11) are

\[ R_{\mu \nu} = \frac{\partial_\mu S \partial_\nu \bar{S}}{(S - S)^2} + \frac{\partial_\nu S \partial_\mu \bar{S}}{(S - S)^2} \quad (15) \]
\[ 0 = \frac{1}{\sqrt{-g}} \partial_\mu \left( g^{\mu \nu} \sqrt{-g} \frac{1}{(S - S)^2} \partial_\nu S \right) + \frac{\partial_\mu S \partial_\nu \bar{S}}{(S - S)^2}. \quad (16) \]

We are looking for supersymmetric solutions of the form

\[ ds^2 = -dt^2 + e^{\rho(z, \bar{z})} dz d\bar{z}, \quad (17) \]

and, as usual, the conditions for unbroken supersymmetry are

\[ \delta \lambda = 0 \quad , \quad \delta \psi_\mu = 0. \quad (18) \]

The field equations for the S-field in the background of eq.(17) is

\[ \partial \bar{\partial} S + 2 \frac{\partial S \partial \bar{S}}{(S - S)} = 0, \quad (19) \]

and it is obvious that it is solved by holomorphic (or anti-holomorphic) functions \( S = S(z) \) \( (S = S(\bar{z})) \). In this case, the \( \delta \lambda = 0 \) condition is satisfied if

\[ \sigma^3 \epsilon = \epsilon. \quad (20) \]

Moreover, if \( \epsilon \) satisfies eq.(20), the integrability condition for holomorphic S-field of \( D_\mu \epsilon = 0 \) turns out to be

\[ \partial \bar{\partial} \rho = \frac{\partial S \partial \bar{S}}{(S - S)^2}. \quad (21) \]

This equation coincides with the Einstein equations in eq.(16) and thus, it is the only one which remains to be solved. However, before doing that, let us consider the energy of these configurations which is given by

\[ E = -\frac{i}{2} \int d^2z \frac{\partial S \partial \bar{S}}{(S - S)^2}. \quad (22) \]

Since the S-field belongs to the upper half plane, the energy in eq.(22) is infinite. In order to find finite energy solutions one has to restrict S to the fundamental domain \( \mathcal{F} \) of PSL(2,Z) \[16\]. Thus, S has discontinuous jumps done by PSL(2,Z) transformations \( S \rightarrow S + 1 \) as we go around the source. These jumps as well as the holomorphicity require that near the sources

\[ S \simeq \frac{1}{2\pi i} \ln z. \quad (23) \]
The energy in this case is indeed finite and in particular

\[ E = \frac{\pi}{6} N, \tag{24} \]

where \( N \) is the number of times the \( z \)-plane covers the fundamental domain \( \mathcal{F} \).

Turning now to eq. (21), one may easily verify that it is solved by

\[ \rho = S_2 |h(S)|^2, \tag{25} \]

where \( S_2 = \text{Im} S \) and \( h(S) \) is an arbitrary integration function. It can be specified by demanding modular invariance and nowhere vanishing metric. These two conditions give the supersymmetric solution \[ \tag{26} \]

\[ e^\rho = S_2 \eta(S)^2 \bar{\eta}(S)^2 \prod_{i=1}^{N} (z - z_i)^{-1/12}, \]

where \( \eta(S) \) is the Dedekind’s \( \eta \)-function. The asymptotic form of the space-time is then

\[ ds^2 \sim -dt^2 + |z\bar{z}|^{-N/12} d\bar{z}d\bar{z}, \tag{27} \]

so that the space-time develops a deficit angle \( \delta = \pi N/6 \). The constraint eq. (21) breaks half of the supersymmetries and although the space-time is conical, we managed to find a Killing spinor because the spin connection is exactly canceled by the \( \text{U}(1) \) gauge field of eq. (14). This possibility has to be anticipated to the one in [3] where an Aharonov-Bohm phase due to the gauge field cancels the phase due to the spin connection.

There exist now Golstone fermions for the broken supersymmetry which are the zero modes

\[ \begin{align*}
\delta \lambda &= -\frac{1}{S_2} \left( \frac{\bar{S} - i}{S + i} \right) \gamma^\mu \partial_\mu S \epsilon^*, \\
\delta \psi_z &= D_z \epsilon, \\
\delta \bar{\psi}_z &= D_{\bar{z}} \epsilon,
\end{align*} \tag{28} \]

where now the spinor \( \epsilon \) satisfies

\[ \sigma^3 \epsilon = -\epsilon. \tag{29} \]

In order the supersymmetry parameter \( \epsilon \) to generate the zero modes, it must have the asymptotic behaviour

\[ \epsilon \rightarrow e^{-\rho/2} \epsilon_0, \tag{30} \]

at spatial infinity. In this case, however, the norm of the zero mode gets a divergent contribution

\[ \int d^2 z \delta \psi_z^* \delta \bar{\psi}_z e^{-\rho} \sim \int d^2 z \frac{1}{z \bar{z}}, \tag{31} \]

and thus, it does not appear in the physical Hilbert space.
4 Conclusions

We discussed above solitonic solutions describing zero-branes in 2+1 dimensions which preserve half of the supersymmetries. Massive particles in planar gravity produce a deficit angle which also breaks the other half of the supersymmetries. Here we have seen that in the particular supersymmetric theory in 2+1 dimensions we have considered, it was possible to define Killing spinors and consequently, the unbroken supersymmetry survives despite the conical structure. The Golstone fermions of the broken supersymmetry however, are not normalizable and thus do not fill any representation of the unbroken one. On the other hand, in the vacuum all supersymmetries are unbroken and the cosmological constant vanishes. We see here a concrete example of Witten’s observation for the connection between the vanishing of the cosmological constant and the bose-fermi degeneracy of the excited states.

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