Shock capturing staggered grid scheme for simulating dam-break flow and runup

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Abstract. This paper focuses on a numerical implementation for simulating complex flow phenomena such as shock wave and wave runup. We use the Nonlinear Shallow Water Equations (NSWE) for representing the flow. The wave model is implemented by using finite volume with momentum conservative staggered grid scheme. A relatively simple leap-frog scheme, upwind method, and special treatment in advection term are used in the numerical implementation. To test the numerical implementation, we reconstruct a physical experiment that was proposed by Aureli et al. 2000, i.e. a dam-break flow that creates shock wave and wave runup. Results of numerical simulation shows a good agreement with experimental data of.

1. Introduction
The Nonlinear Shallow Water Equations (NSWE) has been used in many areas for describing physical phenomena such as large scale ocean flow, tidal flow in estuary and coastal area, river hydraulics, open channel flow, dam-break, tsunami, even for simulating flooding in urban area. The NSWE is a non-dispersive model in which it is only valid for simulating rather long wave, yet it is a fully nonlinear model. It is considered as a simple model for describing nonlinear flow.

Shock wave is an important nonlinear phenomenon. It arises from a transcritical flow, in which a jump and discontinuity of the flow occurs. Scientifically, it is very crucial to track the generation and propagation of the shock wave, but numerically it can be very challenging. The problem is firstly noticed by von Neumann & Richtmyer in 1950 [5]. Inappropriate treatment in the numerical implementation, especially when handling the discontinuity, may lead to instability of the numerical simulation, i.e. solutions with complex numbers [7].

Many researchers addressing the problem for shock capturing numerical scheme is by adding an artificial dissipation by numerical viscosity in the numerical scheme. The idea is to spread the discontinuity during the occurrence of the shock over a certain length such that the numerical result is stabilized. Persson & Peraire [6] introduces an artificial dissipation by adding an artificial viscosity for capturing the shock wave, Huerta et al. 2012 [4] studies a discontinuous Galerkin (DG) method for solving 1D NSWE with data assimilation with minimax filter, especially for simulating the presence of shocks. Holdahl et al. 1999 [3] proposes a numerical scheme that balance between the internal force with source terms an adaptive grid refinement, shock tracking technique for constructing front tracking method, and use operator splitting for studying NSWE.
In this paper, we use a relatively simple numerical scheme that able to capture generation and propagation of shock wave, i.e. the finite volume method with momentum conservative staggered grid. NSWE is used for representing the nonlinear motion of the fluid. A rather simple leap-frog scheme in combination with upwind scheme is utilized in the numerical implementation. A special treatment is given for calculating the advection term.

To test the capability of the numerical implementation for simulating shock wave, we reconstruct the experiment of Aureli et al. 2000 [2]. In the experiment, a dam-break flow creates shock wave, wave runup, and reverses flow.

The organization of this paper is as follows. In Section 2, we discuss the nonlinear Shallow Water Equations and its proposed numerical implementation by using momentum conservative staggered grid scheme. The experimental setup and numerical simulation of the experiment of Aureli 2000 are discussed in Section 3. Results of numerical simulation are investigated qualitatively and quantitatively in Section 4. Finally, we finish the paper with some conclusions in the last section.

2. Wave model and numerical implementation

In this paper, we use the non-dispersive Nonlinear Shallow Water Equations (NSWE) for representing the dynamic of wave motion. We consider 1 dimensional (1D) problem. Let \( x, z \) and \( t \) be the horizontal and vertical spatial coordinate and the time, respectively. The surface elevation and the depth averaged horizontal velocity are denoted by \( \eta(x, t) \) and \( u(x, t) \), respectively. The initial depth \( d(x, t) \) is measured from the still water level at \( z = 0 \), such that the total water depth is denoted by \( h(x, t) = \eta(x, t) + d(x, t) \). The NSWE in conservative form consists of two equations, i.e. the continuity and the momentum equations, given as follows.

\[
\partial_t h + \partial_x (hu) = 0 \tag{2.1}
\]

\[
\partial_t (hu) + \partial_x \left( hu^2 + \frac{g}{2} h^2 \right) + gh \partial_x d - C_f u |u| = 0 \tag{2.2}
\]

The conservative form above can be rewritten in a familiar form as follows.

\[
\partial_t h = -\partial_x (hu) \tag{2.3}
\]

\[
\partial_t u = -g \partial_x \eta - u \partial_x u - C_f u |u| / h \tag{2.4}
\]

Where \( g = 9.81 \text{m/s}^2 \) denotes the gravity acceleration. The last right hand side term in (2.3) is representing the bottom roughness. Here \( C_f = n^2 g / h^{1/3} \) is the Chezy formula and \( n \) is the Manning’s coefficient. Note that for non-moving bottom, i.e. \( \partial_t d = 0 \), then the equation (2.3) can be simplified as,

\[
\partial_t \eta = -\partial_x (hu) \tag{2.5}
\]

Note that in the NSWE, there is no dispersion term; however, the NSWE is a fully nonlinear type of wave model. To describe a long wave, i.e. the wavelength \( \gg \) the water depth, and nonlinear propagation, the NSWE is suitable to be chosen as governing equations for simulation. To be able to simulate nonlinear phenomena such as shock wave and breaking wave such as a bore, it is necessary to choose an accurate and efficient numerical implementation that can simulate discontinuity such as in a shock wave and a bore. In this paper, we choose a numerical scheme that is firstly proposed by Stelling & Duinmeijer 2003 [9], i.e. the momentum conservative staggered grid scheme, this scheme is also used by [1, 10] for a Boussinesq type of model.

![Figure 1. Illustration of staggered grid scheme.](image)
In the staggered grid, instead of placing in the same discretized grid points for the surface elevation $\eta$, the horizontal velocity $u$ is placed in different grid points, i.e. in between the grid point of $\eta$. To illustrate the idea, let the domain of computational is defined as $x \in [0, L]$, then the domain is discretized by points $x_i$ such that $i = 1, 2, ..., N$, that is called as Full-grid. Here we defined grid points that are in between the Full-grid as Half-grid, i.e. $x_{i+1/2}$, $i = 1, 2, ..., N + 1$, as illustrated in figure 1. Let us denote subscribe $i$ and $n$ denote grid points in horizontal space and time, respectively, as $\eta_i^n \approx \eta(x_i, t^n)$. To implement the NSWE by using the staggered grid scheme, we use semi-implicit scheme in time discretization, in combination of center-space for spatial derivatives and upwind-scheme. We define the length of spatial discretization in horizontal space and temporal discretization in time as $\Delta x$ and $\Delta t$, respectively. The discretized form of the continuity equation (2.5) reads

$$\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} = -\left(\frac{-h_i^{n+1} + h_i^n}{\Delta x}u_i^{n+1/2} - h_i^n u_i^{n+1/2}\right)$$

(2.6)

Here, the horizontal velocity $u(x_{i+1/2}, t^n)$ is denoted by $u_i^{n+1/2}$. Note that the values of $h$ in the half-grid $h_i^{n+1/2}$ are not defined yet. To be able to simulate wave runup, its value is calculated by using upwind method as follows

$$\eta_i^{n+1} + \frac{h_i^n}{2}, \quad \text{if } u_i^{n+1/2} \geq 0$$

$$\eta_i^{n+1} + \frac{h_i^{n+1}}{2}, \quad \text{if } u_i^{n+1/2} < 0$$

(2.7)

The condition (2.7) implies that when the flow is propagating to the right ($u_i^{n+1/2} \geq 0$), the value of $h$ in the left is used for $h_i^{n+1/2}$, otherwise, when the flow is propagating to the left, the right value of $h$ in the full-grid is used. Note that the equation (2.6) are calculated in the full-grid. Meanwhile, the discretized form of the momentum equation (2.3) is defined in the half-grid as follows.

$$\frac{u_i^{n+1/2} - u_i^{n+1/2}}{\Delta t} = -g \left(\frac{q_i^{n+1} - q_i^n}{\Delta x}\right) - (u_i^{n+1/2}u_i^{n+1/2}) = \frac{C_f}{h} u_i^{n+1/2} \approx \frac{C_f}{h} u_i^{n+1/2}$$

(2.8)

Note that, here we use a leap-frog scheme, i.e. the newly calculated value $\eta_i^{n+1}$ from the continuity equation (2.6) is used for calculating the momentum equation (2.8). Special attention is given for calculating the so-called advection term $(u_i^{n+1/2}u_i^{n+1/2})$. Suggested by [9], the term is calculated by using horizontal momentum that is defined as $q = hu$, such that $\partial_x (hu) = \partial_x (hu) = u\partial_x (hu) = \partial_x (hu)$ or $u \partial_x u = \frac{1}{h} [\partial_x (hu) - \partial_x q]$. (2.9)

Here, the discretized form of the advection term should be defined in the half-grid, meanwhile the value of $q = hu$ is in the full-grid. To that end, we define the value of $q$ and $h$ in the half-grid by a simple averaging, as follows:

$$\bar{q}_i^{n+1} = \frac{1}{2} \left(q_{i+1/2}^n + q_{i-1/2}^n\right), \quad q_{i+1/2}^n = h_{i+1}^{n+1}u_{i+1/2}^n, \quad \bar{h}_{i+1/2}^n = \frac{1}{2} \left(h_i^n + h_i^{n+1}\right)$$

(2.10)

A consistent discretization of (2.9) is given by the following discretized form

$$\left(u \partial_x u\right)^{n+1}_{i+1} = \frac{1}{h_i^{n+1/2}} \frac{\bar{q}_{i+1/2}^{n+1} - \bar{q}_{i-1/2}^{n+1}}{\Delta x} - u_i^{n+1/2} \bar{q}_{i+1/2}^{n+1} - q_i^{n+1} - q_i^{n+1}$$

(2.11)

Analogous with $h_i^{n+1/2}$, the value of $u_i^{n+1/2}$ is calculated by using upwind method, as

$$u_i^{n+1/2} = \begin{cases} u_i^{n+1/2}, & \text{if } \bar{q}_i^n \geq 0 \\ u_i^{n-1/2}, & \text{otherwise} \end{cases}$$

(2.11)

Lastly, the bottom dissipation term in equation (2.8) is calculated by the following expression

$$C_f \frac{u_i^{n+1/2}u_i^{n+1/2}}{h_i^{n+1/2}}$$
In summary, to solve numerically the NSWE in equation (2.4, 2.5), we firstly solve the continuity equation by calculating the discretized form in (2.6). The resulting term $\eta_i^{n+1}$ is then used for calculating the discretized momentum equation in (2.8). New values of $\eta_i^{n+1}$ and $u_i^{n+1/2}$ are then used for calculating the next time step. As described in [9, 7], the Courant-Friedrichs-Lewy (CFL) stability condition for the derived system is $0 < C \leq 1$, with $C = \sqrt{gd_0 \Delta t / \Delta x}$. In the next section, we test the proposed numerical implementation for simulating a dam-break and runup test case that is proposed in [2].

3. Dam-break and runup test

To show the capability of the numerical implementation in representing a shock wave, we reconstruct the dam-break and runup experiment of Aureli et al. 2000 [2]. In the physical experiments that have been performed in [2], a complex hydrodynamic flow such as dam-break, reverse flow, and wetting and drying, has been simulated in a laboratory flume in Padma University. The physical experiments were done in a rectangular flume with 7.0 m long, 1.0 m wide, and 0.5 m high. The dam failure was created by instantaneous removal of a water gate at $x = 2.25$ m. There are 6 experiments that have been done, in this paper we only reconstruct the experiment N2 as in [2]. As illustrated in figure 2, an initial water height of 0.25 m is placed behind the water gate and the rest of the flume is set to be empty. In the right part of the flume, a sloping bottom with slope of 1:10 is placed stating at $x = 3.5$ m.

![Figure 2. Layout of dam-break experiment of Aureli et.al. 2000. Blue box illustrates the initial condition for water.](image)

As proposed in [2], the manning roughness $n$ for the bottom is set to be $n = 0.01$. The characteristics of the experiment are as follows: After the water gate is removed, the flow is propagating to the right, followed by a runup to the sloping bottom. At the same time the velocity of the wet front decreases and a shock emerges at the foot of the sloping bottom, propagating backward with respect to the flow. The height of the shock wave is then doubled as the wave hits the vertical wall in the left part of the flume and is moving to the sloping bottom. The back and forth of the flow is continued up to 40 s as the wave height is decreasing.

4. Numerical simulation

To reconstruct the physical experiment of [2], we use the following setting. To be able to represent accurately the shock wave, a sufficiently small grid sized is chosen, i.e. $\Delta x = 0.03$ m, with a temporal discretization that satisfies the CFL condition of the numerical implementation, i.e. $\Delta t = 0.0001$ s. The simulation is performed for 40 s. At $t = 0$ s, the initial condition of as shown in figure 2 is released without speed. As previously mentioned, the flow is firstly running up to the right side, while a shock wave is slowly created in the foot of the slope. As the wave running down, the shock wave
becomes larger and propagates to the hardwall boundary condition in the left. Snapshots of the numerical simulation at \( t = 0.05, 1, 2, \) and 4 s are shown in figure 3.

![Snapshots of numerical simulation](image)

Figure 3. Snapshot of dam-break simulation at \( t = 0.05, 1, 2, 4 \) s. Blue and gray indicate the water column and topography, respectively.

As in [2], signals at four different locations are measured as in the experiment, i.e. at \( x = 1.4, 2.25, 3.4, \) and 4.5 m. In figure 4, comparison between signal of results of the simulation and experiment are shown. Especially at location \( x = 1.4 \) m and 2.25 m, the shock wave is clearly captured by the numerical simulation during \( t = 5 \) s to 10 s. At all positions, the results of simulation can follow the experiment signal quite well, even at the shock wave. To show quantitative comparison between the signal of the experiment and the simulation, we calculate the Root Mean Square Error (\( RMSE \)) and Correlation (\( Corr \)) that are defined as follows

\[
RMSE = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N}} \quad \text{and} \quad Corr(y, \hat{y}) = \frac{\langle y, \hat{y} \rangle}{|y||\hat{y}|}
\]

where \( y \) and \( \hat{y} \) denote the measurement signal and the simulation signal, respectively. The notation \( \langle .., \rangle \) and \( |..| \) represent \( L_2 \) inner product and the \( L_2 \) norm, respectively. Comparison between the signals is summarized in table 1. From the table, the \( RMSE \) values show relatively small error between results of the simulation and the experiment, especially at the slope \( x = 3.40 \) m and 4.50 m. Similar trend is also seen from the correlation values.
Figure 4. Signal comparison between results of simulation (solid red line) and measurement (circles) at $x = 1.4, 2.25, 3.4$ and $4.5$ m.

Table 1. RMSE and correlation values at different positions.

| Position $x$ (m) | $RMSE$  | $Corr$ |
|------------------|---------|--------|
| 1.40             | 0.025954| 0.78   |
| 2.25             | 0.013838| 0.87   |
| 3.40             | 0.01013 | 0.97   |
| 4.50             | 0.010811| 0.96   |

5. Conclusion and discussion

Dam-break flow and runup are two important physical phenomena that should be able to be simulated by a good wave model and its numerical scheme. In addition, raises from a transcritical and nonlinear phenomena, a shock wave phenomenon is just as important as the other two phenomena. In this paper, we have presented a numerical scheme for implementing the Nonlinear Shallow Water Equations (NSWE) for simulating those three physical phenomena. The numerical scheme is based on finite volume method with momentum conservative staggered grid scheme. Here, a special treatment is
given when calculating the advection term, i.e. rather than calculating directly by using \( u \), the term is calculated via horizontal momentum \( q = hu \). Result of the numerical implementation is able to simulate a rather complex simulation, i.e. a dam-break flow with runup experiment as proposed by Aureli et al. 2000 [2]. The experiment has a complex hydrodynamic flow such as dam-break, reverse flow, wetting and drying or runup, and generation of a shock wave. The proposed numerical scheme does not require an ad-hoc setting for both a breaking wave phenomenon; bore, as well as wetting and drying during runup. It is able to simulate such phenomena with high accuracy, i.e. with RMSE values of 0.01013 to 0.025954 and correlation coefficient of 0.78 to 0.97.

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