Equivalences of operads over symmetric monoidal categories

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Abstract. In this paper, we study conditions for extending Quillen model category properties, between two symmetric monoidal categories, to their associated category of symmetric sequences and of operads. Given a Quillen equivalence \( \lambda : \mathcal{C} \rightleftarrows \mathcal{D} : R \), so that \( \mathcal{D} \) is any symmetric monoidal category and the adjoint pair \((\lambda, R)\) is weak monoidal, we prove that the categories of connected operads \( \text{Op}_C \) and \( \text{Op}_D \) are Quillen equivalent. This expands an analogous result of Schwede-Shipley ([SS03]) when we replace these categories of operads with the sub-categories of \( C\text{-Monoid} \) and \( D\text{-monoid} \).

Key words. Model category, coalgebra, cosimplicial frame, operad.

Introduction

This paper was inspired by the work of Schwede-Shipley who studied conditions for extending Quillen equivalences of two symmetric monoidal model categories to Quillen equivalences on the associated sub-categories of monoids. With good assumption, the circle product endows the category of symmetric sequences with a monoidal structure, and in that case an operad over the underlying category is simply a monoid in the category of symmetric sequences. The study of Schwede-Shipley still do not apply in this case since the circle product is not symmetric.

However, putting the same assumption as in their paper, we are able to come out with an analogue result when the left underlying category is \( \mathcal{C} = Ch_{k,t} \), \((t \in \mathbb{Z} \cup \{-\infty\})\). Our method consists of constructing first of all an explicit model of the realization functor for simplicial operads over \( Ch_{k,t} \).

Theorem A. Let \( P_* \) be a simplicial operad on chain complexes \( \mathcal{C} = Ch_{k,t} \), \((t \in \mathbb{Z} \cup \{-\infty\})\). Then for any integer \( r \geq 0 \), there is a quasi-isomorphism

\[
\Gamma : |B^c(B(P_*))(\underline{\tau})|_C \xrightarrow{\simeq} |P_*|_{\text{Op}_C}(\underline{\tau})
\]

Now given any cofibrant operad, we consider its simplicial resolution to prove, using theorem A, the next proposition (see prop 21).

Proposition A-1. Let \( \lambda : \mathcal{C} = Ch_{k,t} \rightleftarrows \mathcal{D} : R \) be a weak monoidal Quillen pair between the category \( (Ch_{k,t} \otimes k) \), \((t \in \mathbb{Z} \cup \{-\infty\})\), and any other symmetric monoidal category \( (\mathcal{D}, \wedge, I_D) \). If \( P \) is a cofibrant operad in \( Ch_{k,t} \), then the morphism \( \Lambda(U(P)) \to UL(P) \), which is adjoint to the unit \( \eta : P \to \overline{RL}(P) \), is a weak equivalence.

This later proposition is the key ingredient to prove the main result of this paper (see theorem 20).

Theorem B. Let \( \lambda : \mathcal{C} = Ch_{k,t} \rightleftarrows \mathcal{D} : R \) be a weak monoidal Quillen pair between the category \( (Ch_{k,t} \otimes k) \), \((t \in \mathbb{Z} \cup \{-\infty\})\), and any other symmetric monoidal category \( (\mathcal{D}, \wedge, I_D) \). If the pair \((\lambda, R)\) is a Quillen equivalence, then so is the pair \( L : \text{Op}_C \rightleftarrows \text{Op}_D : \overline{R} \).
Outline of the paper

In section 1, we give the background on chain complexes, coalgebras, lax symmetric monoidal functor, and symmetric sequence. In section 2, we study the extension of weak monoidal Quillen pair and Quillen equivalence, between two symmetric monoidal model categories, to the associated category of symmetric sequences. In section 3, we define the cosimplicial frame for the realization functor of simplicial operads, and we prove theorem A. We end by giving a simplicial resolution of any arbitrary cofibrant operad. Section 4 is dedicated to the proof of theorem B.

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1 preliminaries

1.1 Chain complexes

Let \( k \) be a field of characteristic 0. In this note, we denote by \( \text{Ch}_k \) the category of differential \( \mathbb{Z} \)-graded chain complexes over \( k \).

This category has a symmetric monoidal structure given by:

\[
(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q
\]

and differential, \( \forall x \otimes y \in V_p \otimes W_q, d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y) \);

The switch morphism \( T : V \otimes W \to W \otimes V \) must respect the Koszul sign: \( T(x \otimes y) = (-1)^{pq} y \otimes x \).

The unit of the monoid \( - \otimes - \), that we denote abusively \( k \), is the chain complex having \( k \) in degree 0 and is trivial in the above degrees.

We denote by \( \text{Ch}_k, t \) the sub-category of \( \text{Ch}_k \) which consist of \( t \)-below truncated chain complexes with \( t \in \mathbb{Z} \cup \{-\infty\} \), where \( \text{Ch}_k, -\infty = \text{Ch}_k \).

1.2 Cofree coalgebra construction on chain complexes

**Definition 1** (Coalgebra). A coalgebra on \( \text{Ch}_k, 0 \) is a cotriple \( (C, \Delta, \varepsilon) \) with \( C \) an object in \( \text{Ch}_k, 0 \), \( \Delta : C \to C \otimes C \) a map called diagonalization or comultiplication and \( \varepsilon : C \to k \) a map called the augmentation or counit, which satisfies the coassociativity and counitary conditions (see [Swe69, Section 1]). The category of coalgebra is denoted \( \text{coAlg} \).

There is a fundamental theorem of coalgebras due to Getzler-Goerss:

**Theorem 2.** [GG99, Corollary 1.6] Every coalgebra on \( \text{Ch}_k, 0 \) is a filtered colimit of its finite dimensional sub-coalgebras.

One of the consequences of this theorem is the following proposition which justifies the cofree construction of coalgebras.

**Proposition 3.** [GG99, Prop 1.10] The forgetful functor \( U : \text{coAlg} \to \text{Ch}_k, 0 \) has a right adjoint \( S : \text{Ch}_k, 0 \to \text{coAlg} \).

A good property of the cofree functor \( S(-) \) is that if \( V \) is an acyclic non negatively graded chain complex (i.e: \( H_*(V) = 0 \)), then there is a quasi-isomorphism \( S(V) \to k \) (see [GG99, Prop 2.2]).
1.3 Weak monoidal adjoint pair

Definition 4 (Quillen pair). (1) A pair of adjoint functors
\[ \lambda : C \rightleftarrows D : R \]

between two model categories is a Quillen pair if the right adjoint R preserves fibrations and trivial fibrations.

A Quillen pair \((\lambda, R)\) induces a derived adjoint pair \((L\lambda, L\lambda)\) of functors between the homotopy categories of \(C\) and \(D\) (see [Qui67, I.4.5]).

(2) A Quillen pair is a Quillen equivalence if the associated derived functors are equivalences between the homotopy categories.

Definition 5 (Weak monoidal). A weak monoidal Quillen pair between two monoidal model categories \((C, \wedge, I_C)\) and \((D, \wedge, I_D)\) consists of a Quillen pair \(\lambda : C \rightleftarrows D : R\) satisfying the following conditions:

a) For any two objects \(X\) and \(Y\) in \(D\), there is a morphism
\[ \varphi_{X,Y} : R(X) \wedge R(Y) \to R(X \wedge Y), \] and a morphism \(\nu : I_C \to R(I_D)\),

natural in \(X, Y\) and coherently associative, commutative and unital (see diagrams 6.27 and 6.28 of [Bor94]).

b) If \(X\) and \(Y\) are cofibrant, the adjoint of the composition
\[ X \wedge Y \xrightarrow{\eta \wedge \eta} R\lambda(X) \wedge R\lambda(Y) \xrightarrow{\varphi_{X,Y}} R(\lambda(X) \wedge \lambda(Y)) \]

and denoted
\[ \tilde{\varphi}_{X,Y} : \lambda(X \wedge Y) \to \lambda(X) \wedge \lambda(Y) \]

is a weak equivalence.

c) for some (hence any) cofibrant replacement \(\tilde{I}_C \to I_C\) of the unit object in \(C\), the composite map \(\lambda(\tilde{I}_C) \to \lambda(I_C) \xrightarrow{\tilde{\nu}} I_D\) is a weak equivalence in \(D\), where
\[ \tilde{\nu} : \lambda(I_C) \to I_D \]

is the adjoint of \(\nu\).

In the literature (see [SS03]), one says that the functor \(R\) is lax symmetric monoidal when condition a) is satisfied. In this note we sometimes refer to maps \(\varphi_{X,Y}\) as the lax monoidal structure morphisms associated to \(R\).

1.4 Symmetric sequence

We give here the definition of a symmetric sequence along with the monoidal structure on the category of symmetric sequences. We refer to [Chi12] for more on this topic.

Let \((\mathcal{C}, \wedge, I_\mathcal{C})\) be a pointed symmetric monoidal category.

Definition 6 (Symmetric sequence). A symmetric sequence in the category \(\mathcal{C}\) is a functor \(M : \text{FinSet} \to \mathcal{C}\) from the category FinSet, whose objects are finite sets and whose morphisms are bijections, to \(\mathcal{C}\). Denote the category of all symmetric sequences in \(\mathcal{C}\) by \([\text{FinSet}, \mathcal{C}]\) (in which morphisms are natural transformations).
Let $\text{FinSet}_0$ be the category whose objects are the finite sets $r := \{1, \ldots, r\}$ for $r \geq 0$ (with $0$ the empty set), and whose morphisms are bijections. $\text{FinSet}_0$ is clearly a subcategory of $\text{FinSet}$, and any symmetric sequence $M : \text{FinSet} \to \mathcal{C}$ is determined, up to canonical isomorphism, by its restriction $M : \text{FinSet}_0 \to \mathcal{C}$. This restriction consists of the sequence $M(0), M(1), M(2), \ldots$ of objects in $\mathcal{C}$, together with an action of the symmetric group $\Sigma_r$ on $M(r)$, hence the name "symmetric sequence."

**Definition 7.** For a finite set $J$, we define the category $J/\text{FinSet}_0$ as follows. The class of objects of $J/\text{FinSet}_0$ consists of all functions (not necessary bijection) $f : J \to I$ for some $I \in \text{FinSet}_0$, and the set of morphisms from $(f : J \to I)$ to $(f' : J \to I')$ is the set of bijections $\sigma : I \to I'$ such that $f' = \sigma \circ f$.

**Definition 8.** Let $M$ and $N$ be two symmetric sequences in $\mathcal{C}$. For each finite set $J$, we define a functor

$$(M, N) : J/\text{FinSet}_0 \to \mathcal{C}$$

on objects by

$$(M, N)(f : J \to I) := M(I) \wedge \bigwedge_{i \in I} N(f^{-1}(i)).$$

For morphism $\sigma : I \to I'$ in $J/\text{FinSet}_0$ we define

$$(M, N)(\sigma) := M(I) \wedge \bigwedge_{i \in I} N(f^{-1}(i)) \to M(I') \wedge \bigwedge_{i' \in I'} N(f^{-1}(i')).$$

by combining map $M(\sigma)$ with the permutation of the smash product identifying the term corresponding to $i \in I$ with the term corresponding to $\sigma(i) \in I'$.

**Definition 9** (Composition product). For symmetric sequences $M, N$, we define a symmetric sequence $M \circ N$ by

$$(M \circ N)(J) := \operatorname{colim}_{f \in J/\text{FinSet}_0} (M, N)(f).$$

A bijection $\theta : J \to J'$ determines a map $(M, N)(f) \to (M, N)(f \circ \theta^{-1})$ that is the map

$$M(I) \wedge \bigwedge_{i \in I} N(f^{-1}(i)) \to M(I) \wedge \bigwedge_{i \in J} N(\theta(f^{-1}(I)))$$

via the identity on $M(I)$ and the action of the symmetric sequence $N$ and the bijections $f^{-1}(i) \cong \theta(f^{-1}(i))$ given by restricting $\theta$. We thus obtain induced maps

$$(M \circ N)(\theta) := (M \circ N)(f) \to (M \circ N)(f')$$

that make $M \circ N$ into a symmetric sequence in $\mathcal{C}$.

**Definition 10** (Unit symmetric sequence). The unit symmetric sequence $\mathbb{I}$ in the symmetric monoidal category $(\mathcal{C}, \wedge, \mathbb{1}_\mathcal{C})$ is given by

$$\mathbb{I}(J) = \mathbb{1}_\mathcal{C}, \text{ if } |J| = 1, \text{ and } \mathbb{I}(J) = 0 \text{ otherwise;}$$

where $0$ is the initial object in $\mathcal{C}$. The map $\mathbb{I}(J) \to \mathbb{I}(J')$ induced by a bijection $J \to J'$ is the identity morphism on $\mathbb{I}_\mathcal{C}$ or $0$ as appropriate.
If the category \((\mathcal{C}, \land, I_{\mathcal{C}})\) is such that \(\land\) commutes with finite colimits, then the composition product \(\circ\) is a monoidal product and \(([\text{FinSet}, \mathcal{C}], \circ, I)\) is a monoidal category, but not symmetric (see [Chi05, Prop 2.9]). For instance the category \((\text{Ch}_{\mathbb{k}, t}, \otimes, k)\) is closed symmetric monoidal, thus the tensor product \(\otimes\) has a right adjoint, so it preserves all colimits. Therefore \(([\text{FinSet}, \text{Ch}_{\mathbb{k}, t}], \circ, I)\) is a monoidal category.

In the sequel of this paper, we equip the category of symmetric sequences \([\text{Finset}, \mathcal{C}]\) (viewed as functor categories) with the projective model structure which consists of:

- fibrations and weak equivalences are natural transformations that are objectwise such morphisms in \(\mathcal{C}\).

### 2 Equivalence of symmetric sequences on Quillen equivalent categories

The composition product of symmetric sequences has a monoidal compatibility with the model structure of the underlying category as we state in the following lemma.

**Lemma 11.** Let \((\mathcal{C}, \land, I_{\mathcal{C}})\) be a symmetric monoidal model category, and cofibrantly generated. If \(i : A \hookrightarrow A'\) and \(j : B \hookrightarrow B'\) are cofibrations in \([\text{Finset}, \mathcal{C}]\), so that \(B\) and \(B'\) are cofibrant, then the natural morphism

\[
(i_{\ast}, j_{\ast}) : A' \circ B \coprod_{A \circ B} A \circ B' \longrightarrow A' \circ B'
\]

is a cofibration. In addition if \(i\) or \(j\) is acyclic, then so is \((i_{\ast}, j_{\ast})\).

**Proof.** We will make a progressive proof when \(i\) is respectively a generating cofibration, a cellular morphism, and finally a cofibration.

1. Suppose that \(K \otimes I(\Sigma_{\alpha}) \hookrightarrow K' \otimes I(\Sigma_{\alpha})\) is constructed from a generating cofibration \(K \hookrightarrow K'\) of \(\mathcal{C}\), and let \(J\) be a finite set. We have

\[
\begin{align*}
(K \otimes I(\Sigma_{\alpha}) \circ B)(J) & \cong \bigvee_{\substack{J_1 \cup \ldots \cup J_r, \quad J_1 \cup \ldots \cup J_r \leq J}} K \land B(J_1) \land \ldots \land B(J_r) \\
& \cong K \land T_r^{B}(J)
\end{align*}
\]

where, \(T_r^{B}(J) = \bigvee_{\substack{J_1 \cup \ldots \cup J_r, \quad J_1 \cup \ldots \cup J_r \leq J}} B(J_1) \land \ldots \land B(J_r)\).

Since the objects \(B(J_i)\) and \(B'(J_i)\) are cofibrants and that \(\mathcal{C}\) is a symmetric monoidal model category, one deduce that \(T_r^{B}(J) \longrightarrow T_r^{B'}(J)\) is a cofibration in \(\mathcal{C}\). It then follows that the the induced morphism

\[
K' \land T_r^{B}(J) \prod_{K \land T_r^{B}(J)} K \land T_r^{B'}(J) \longrightarrow K' \land T_r^{B'}(J)
\]

is a cofibration in \(\mathcal{C}\). This induces the result in this specific case of \(i\). In addition, Since the functor \(- \circ -\) preserves left colimits, this result generalizes to any cofibration of the form \(\bigvee_{\alpha} K_{\alpha} \otimes I(\Sigma_{\alpha}) \hookrightarrow \bigvee_{\alpha} K'_{\alpha} \otimes I(\Sigma_{\alpha})\).

2. We assume now that \(i : A \longrightarrow A'\) is a cellular morphism presented by the \(\beta\)-sequence \((\beta\) is an ordinal) :
where $A' = \text{colim}_\rho A < \rho >$, with pushout diagrams (cell attachment):

$$K = \bigsqcup_{\alpha} K_{\alpha} \otimes \mathbb{I}(\Sigma_{\alpha}) \longrightarrow A < \rho >$$

$$K' = \bigsqcup_{\alpha} K'_{\alpha} \otimes \mathbb{I}(\Sigma_{\alpha}) \longrightarrow A < \rho + 1 >$$

One deduce from this diagram the following pushout:

$$K' \circ B \coprod_{K \circ B'} A < \rho + 1 > \circ B \coprod_{A < \rho > \circ B'} A < \rho > \circ B'$$

where the left vertical map is a cofibration using 1. One then deduce that the right vertical map is also a cofibration.

We deduce from this, the cofibration

$$A < \rho + 1 > \circ B \coprod_{A < \rho - 1 > \circ B'} A < \rho - 1 > \circ B'$$

$$\cong$$

$$A < \rho + 1 > \circ B \coprod_{A < \rho > \circ B'} A < \rho > \circ B'$$

$$\cong$$

$$A < \rho + 1 > \circ B \coprod_{A < \rho - 1 > \circ B'} A < \rho - 1 > \circ B'$$

We then deduce the result by induction on $\rho$.

3. Finally let us assume that $i$ is an arbitrary cofibration. $i$ is by definition the retract of a cellular map $X \xrightarrow{i'} Y$. Using the universal property of colimits, one deduce that $(i_*, j_*): A' \circ B \coprod_{A \circ B'} A' \circ B' \longrightarrow A' \circ B'$ is a retract of $Y \circ B \coprod_{X \circ B'} Y \circ B'$ which is a cofibration according to 2. Therefore one deduce that $(i_*, j_*)$ is also a cofibration, and is acyclic if one of $i$ or $j$ is.

\[\square\]

**Lemma 12.** Let $(\mathcal{C}, \wedge, \mathbb{I}_\mathcal{C})$ be a symmetric monoidal model category, and cofibrantly generated. If $i : A \rightarrow A'$ is a cofibration in $[\text{Finset}, \mathcal{C}]$, and $B$ is a cofibrant symmetric sequence in $\mathcal{C}$, then the natural morphism

$$A \circ B' \longrightarrow A' \circ B'$$
is a cofibration.

Proof. We apply the result of lemma 11 in specific case $j : I_C \hookrightarrow B'$.

Let $\lambda : C \leftrightarrow D : R$ be a Quillen pair between two cofibrantly generated model categories. One can always lift the pair $(\lambda, R)$ to a Quillen pair

$$\overline{\lambda} : [\text{Finset}, C] \leftrightarrow [\text{Finset}, D] : \overline{R}$$

preserving in this process some properties of the adjunction $(\lambda, R)$.

**Proposition 13.** Under the above notation we have the following results.

1. If $\lambda : C \leftrightarrow D : R$ is a Quillen equivalence, then so is $(\overline{\lambda}, \overline{R})$.

2. If $(C, \land, I_C)$ and $(D, \land, I_D)$ are two symmetric monoidal categories so that the categories $([\text{Finset}, C], \circ, I)$ and $([\text{Finset}, D], \circ, I)$ are monoidal categories, and if the pair $(\lambda, R)$ is a weak monoidal Quillen pair, then so is the pair $(\overline{\lambda}, \overline{R})$.

Proof. 1. Let $C$ be a cofibrant object in $[\text{Finset}, C], D$ a fibrant object in $[\text{Finset}, D]$ and a weak equivalence $\overline{\lambda}(C) \xrightarrow{\simeq} D$.

Then for any finite set $J$, one have the levelwise weak equivalence $\lambda(C(J)) \xrightarrow{\simeq} D(J)$.

Since $C(J)$ is cofibrant and $D(J)$ fibrant, its adjoint $C(J) \xrightarrow{\simeq} R(D(J))$ is also a weak equivalence in $D$. Therefore one have the weak equivalence $C \xrightarrow{\simeq} \overline{R}(D)$.

2. Let $A$ and $B$ be two symmetric sequences in $D$.

$$(\overline{R}(A), \overline{R}(B))(f : J \rightarrow I) = R(A(I)) \land \bigwedge_{i \in I} R(B(f^{-1}(i)))$$

$$\varphi_{A,B}^R : R(A(I)) \land \bigwedge_{i \in I} B(f^{-1}(i))) = \overline{R}(A, B)(f : J \rightarrow I)$$

where $\varphi_{A,B}^R$ is the natural iteration of the lax monoidal structure morphism associated to $\overline{R}$.

The adjunction of this morphism gives

$$\overline{\lambda}(\overline{R}(A), \overline{R}(B))(f : J \rightarrow I) \rightarrow (A, B)(f : J \rightarrow I)$$

By applying $\text{colim}$ one get the map

$$\overline{\lambda} \text{ colim}_{f \in J/\text{FinSet}_0} (\overline{R}(A), \overline{R}(B))(f) \rightarrow \text{colim}_{f \in J/\text{FinSet}_0} (A, B)(f)$$

and it adjoint gives what we claim to be the lax monoidal structure morphism

$$\varphi_{A,B}^\overline{R} : \overline{R}(A) \circ \overline{R}(B) \rightarrow \overline{R}(A \circ B)$$

Consider now that $(\lambda, R)$ is a weak monoidal Quillen pair. Let $A, B$ be two cofibrant objects in $[\text{Finset}, C]$.

The adjoint of the composition $A \circ B \rightarrow \overline{R} \overline{\lambda} A \circ \overline{R} \overline{\lambda} B \xrightarrow{\varphi_{A,B}^\overline{R}} \overline{R}(\overline{\lambda}(A) \circ \overline{\lambda}(B))$ is obtained, according to our above construction, from the collection of maps: for $f : J \rightarrow I$,

$$\overline{\lambda}(A, B)(f) \rightarrow \overline{\lambda}(\overline{R} \overline{\lambda} A, \overline{R} \overline{\lambda} B)(f) \rightarrow (\overline{\lambda}(A), \overline{\lambda}(B))(f)$$

and these later are weak equivalences by assumption. We would like to conclude that when we apply the colimit functor we obtain the weak equivalence $\overline{\lambda}(A \circ B) \rightarrow \overline{\lambda}(A) \circ \overline{\lambda}(B)$.
- We assume that $A = \operatorname{colim}_{\rho} A < \rho >$, with single cell attachment

$$K = \bigvee_{\alpha} K_{\alpha} \otimes \mathbb{I}(\Sigma_{\alpha}) \longrightarrow A < \rho >$$

$$K' = \bigvee_{\alpha} K'_{\alpha} \otimes \mathbb{I}(\Sigma_{\alpha}) \longrightarrow A < \rho + 1 >$$

When we apply the functor $- \circ B$ and the functor $\lambda$, we obtain the following cube where the top and back faces are the pushout diagrams

$$\begin{array}{ccc}
\bar{\lambda}(K \circ B) & \xrightarrow{\sim} & \bar{\lambda}(A < \rho > \circ B) \\
\downarrow & & \downarrow \\
\bar{\lambda}(K) \circ \bar{\lambda}(B) & \xrightarrow{\sim} & \bar{\lambda}(A < \rho >) \circ \bar{\lambda}(B) \\
\downarrow & & \downarrow \\
\bar{\lambda}(K') \circ \bar{\lambda}(B) & \xrightarrow{\sim} & \bar{\lambda}(A < \rho + 1 > \circ B) \\
\downarrow & & \downarrow \\
\bar{\lambda}(K') \circ \bar{\lambda}(B) & \xrightarrow{\sim} & \bar{\lambda}(A < \rho + 1 >) \circ \bar{\lambda}(B)
\end{array}$$

where the horizontal weak equivalences are induced by equation (1). The left vertical cofibrations are induced by lemma 12 since $\bar{\lambda}(B)$ is cofibrant and $\bar{\lambda}$ preserves cofibrations. The front and back faces are then homotopy pushout squares, and one deduce that $h$ is a weak equivalence.

- If $A$ is any arbitrary cofibrant object, it is a retract of a cell object $X$, and one deduce easily that $\bar{\lambda}(A \circ B) \longrightarrow \bar{\lambda}(A) \circ \bar{\lambda}(B)$ is as weak equivalence as the retract of the weak equivalence $\bar{\lambda}(X \circ B) \longrightarrow \bar{\lambda}(X) \circ \bar{\lambda}(B)$.

\[\square\]

3 Simplicial operads

3.1 Cofibrant resolution of chain operads

Let $\mathcal{C}$ be a cofibrantly generated and pointed model category, and let $(\mathcal{C}, \wedge, \mathbb{I}_{\mathcal{C}})$ be a symmetric monoidal model category. We denote by $\mathcal{O}_{\mathcal{PC}}$ the category of connected operads. These are operads $P$ so that $P(0) = *$ (the terminal object) and $P(1) = \mathbb{I}_{\mathcal{C}}$. The category $\mathcal{O}_{\mathcal{PC}}$ has a model structure given by

- Fibrations (resp. weak equivalences) are levelwise fibrations (resp. weak equivalences) in the underlying category $\mathcal{C}$.

- Cofibrations are morphisms which have the left lifting property with respect to trivial fibrations.

When $\mathcal{C} = Ch_{k,t}, t \in \mathbb{Z} \cup \{-\infty\}$, then it is showed in [BM06, Thm 5.1.and Thm 8.5.4.] that the reduced $W$-construction $W^{red}(N^k_*(\Delta^1), P) \cong B^c B(P)$ of a connected operad $P$ is a cofibrant model of $P$; In particular, there is a quasi-isomorphism

$$B^c B(P) \xrightarrow{\cong} P$$
3.2 Realization of simplicial operads

We will be working here in the specific context when $\mathcal{C} = Ch_{k,t}, t \in \mathbb{Z} \cup \{-\infty\}$. To define the realization functor on simplicial operads, it is fundamental to define first a cosimplicial frame associated to operads.

Lemma 14. There exists a cosimplicial cocommutative non negatively differential graded coalgebra $C(\Delta^*)$ so that, given any integer $n \geq 0$, one have a diagram

$$\mathbb{k} \otimes sk_0\Delta^n \xrightarrow{\beta_n} C(\Delta^n) \xrightarrow{\sim} \mathbb{k}$$

where $\mathbb{k} \otimes sk_0\Delta^n := \bigoplus_{i=0}^n \mathbb{k} g_i$ is the cocommutative coalgebra with the coproduct $\Delta g_i := g_i \otimes g_i$.

Proof. We give an inductive construction of the cosimplicial coalgebra $C(\Delta^*)$. We start with $C(\Delta_0) := \mathbb{k}$. The morphism of chain complexes $C(sk_0\Delta^1) := \mathbb{k}g_0 \oplus \mathbb{k}g_1 \rightarrow 0$ factors as:

$$C(sk_0\Delta^1) \xrightarrow{\alpha_1} 0 \xrightarrow{\cong} V^1$$

Since the domain of $\alpha_1$ is a coalgebra, using the universal property of cofree coalgebras (see §1.2), we co-extends $\alpha_1$ to:

$$\mathbb{k}g_0 \oplus \mathbb{k}g_1 \xrightarrow{\cong} S(V^1)$$

and since the domain of $\alpha_1$ is cocommutative, its co-restriction produces:

$$\mathbb{k}g_0 \oplus \mathbb{k}g_1 \xrightarrow{\beta_1} C(V^1) \xrightarrow{\gamma_1} \mathbb{k}$$

where $C(V^1)$ denotes the greatest cocommutative sub-coalgebra of $S(V^1)$. We then set:

- $C(cosk_0\Delta^1) := C(\Delta^0) = \mathbb{k}$
- $C(\Delta^1) := C(V^1) \otimes C(cosk_0\Delta^1) \cong C(V^1)$; The cofaces $d^0, d^1 : C(\Delta^0) \rightarrow C(\Delta^1)$ are given the respective restrictions of $\beta_1$ on $\mathbb{k}g_0$ and on $\mathbb{k}g_1$. The codegeneracy $s^0 : C(\Delta^1) \rightarrow C(\Delta^0)$ is given by $\gamma_1$.

If $k = 2$, we first use the notation $C(sk_1\Delta^2)$ to be the colimit as a coalgebra of the diagram

where for any $i, j \in \{0, 1, 2\}$, $C(\Delta^0_{ij}) = \mathbb{k}g_i$ and refers to the vertices of the standard simplex $\Delta^2$; $C(\Delta^1_{(ij)})$ are copies of $C(\Delta^1)$ whose indices refer to 1-simplices of the standard simplex $\Delta^2$. As in the previous case, we consider the following factorization of the trivial map:
and using the same previous trick, one recovers from $\alpha_2$ a coalgebra morphism

$$C(sk_1\Delta^2) \xrightarrow{\alpha_2} V^2 \xrightarrow{\simeq} k$$

where $C(V^2)$ denotes the greatest cocommutative sub-coalgebra of $S(V^2)$. In addition one have an inclusion $k \otimes sk_0\Delta^2 \to C(sk_1\Delta^2)$, thus one form the morphism

$$k \otimes sk_0\Delta^2 \xrightarrow{\beta_2} C(V^2) \xrightarrow{\simeq} k$$

We then set

- $C(cosk_1\Delta^2) := \lim_{n \to \infty} C(\Delta^n) \xrightarrow{\delta^0} C(\Delta^0) \xrightarrow{s^0} C(\Delta^1) \xrightarrow{\simeq} k$
- $C(\Delta^2) := C(V^2) \otimes C(cosk_1\Delta^2)$: The cofaces $d^0, d^1, d^2 : C(\Delta^1) \to C(\Delta^2)$ are given, with our construction, by the respective restriction of $\alpha_2$ on the cofaces $C(\Delta^1_{(ij)})$. The codegeneracies $s^0, s^1 : C(\Delta^2) \to C(\Delta^1)$ are obtained from the projection $C(V^2) \otimes C(cosk_1\Delta^2) \to C(cosk_1\Delta^2)$ followed respectively by the projections $C(cosk_1\Delta^2) \to C(\Delta^1)$.

This construction generalizes inductively to higher values according to the shape of the standard simplexes $\Delta^n(n \geq 0)$, as follows:

- $C(sk_{n-1}\Delta^n) := \colim_{m \to \infty} C(\Delta^m)$; One form the factorisation

  $$C(sk_{n-1}\Delta^n) \xrightarrow{} C(V^n) \xrightarrow{\simeq} k$$

  and then deduce $\beta_n : k \otimes sk_0\Delta^n \to C(\Delta^n)$ by precomposing with the sequence $k \otimes sk_0\Delta^n \to C(sk_1\Delta^n) \to ... \to C(sk_{n-1}\Delta^n)$.
- $C(cosk_{n-1}\Delta^n) := \lim_{n \to \infty} C(\Delta^m) \xrightarrow{\simeq} k$
- $C(\Delta^n) := C(V^n) \otimes C(cosk_{n-1}\Delta^n)$; The cofaces $d^i : C(\Delta^{n-1}) \to C(\Delta^n)$ and the codegeneracies $s^i : C(\Delta^n) \to C(\Delta^{n-1})$ are obtained, as in the lower cases, by construction.

Let $P$ be an operad. Since the tensor product of a cocommutative coalgebra with a cooperad is again a cooperad, we deduce from lemma 14 the following diagram in the category $coOprC$ of cooperads on $\mathcal{C} = Ch_{k,F}$:

$$B(P)\boxtimes sk_0\Delta^k \xrightarrow{\beta_k} B(P)\boxtimes C(\Delta^k) \xrightarrow{\simeq} B(P)$$

where,

- $B(P)$ denotes the bar construction of $P$;
- $\forall r \geq 0, B(P)\boxtimes C(\Delta^k)(r) := B(P)(r) \otimes C(\Delta^k)$;
By applying the cobar construction functor $B^c(-)$ to this sequence one get the operad diagram:

$$B^c(B(P) \otimes \Delta) \xrightarrow{\beta} B^c(B(P) \otimes C(\Delta)) \xrightarrow{\approx} B^c(B(P)) \xrightarrow{\approx} P$$

One deduce without too much effort from this later sequence that

$$P \otimes \Delta := B^c(B(P) \otimes C(\Delta))$$

satisfies all the hypothesis of cosimplicial frames for $B^cB(P)$.

**Definition 15** (Realization). If $P_\ast$ is a simplicial operad on chain complexes $\mathcal{C} = \text{Ch}_{k,t}$, $(t \in \mathbb{Z} \cup \{-\infty\})$, then the realization of $P_\ast$ is given by the coend in $\text{Op}_\mathcal{C}$

$$|P_\ast|_{\text{Op}_\mathcal{C}} = \int_{k \in \Delta} P_k \otimes \Delta^k$$

**Notation 1.** Let $T$ be a tree. If $M$ is any symmetric sequence in chain complexes, it will be practical to adopt the notation $T(M) := \otimes_{v \in V(T)} M(I_v)$, where $V(T)$ denotes the set of the vertices of $T$ and given a vertex $v$, $I_v$ is the set of all incoming edges to $v$.

**Definition 16** (Filtering the cobar construction). Let $C$ be a cooperad. We define the filtration $\mathcal{F}_u B^c(C)$ of the cobar construction of the form:

$$B^c(C) = \mathcal{F}_0 B^c(C) \supseteq \mathcal{F}_1 B^c(C) \supseteq ... \supseteq \mathcal{F}_u B^c(C) \supseteq ...$$

as follows:

$$\mathcal{F}_u B^c(C) := \bigoplus_{T, |V(T)| \geq u} T(\Sigma^{-1} \mathcal{C}),$$

where $|V(T)|$ denotes the number of internal vertices of the tree $T$.

The cobar differential $\partial_{\text{cobar}}$, which is defined from the cooperad coproduct $C \longrightarrow C \otimes C$, increases the number of vertices of the tree. Therefore $\partial_{\text{cobar}} \mathcal{F}_u B^c(C) \subseteq \mathcal{F}_{u+1} B^c(C)$.

The good characteristic of this filtration is that it isolates the cobar differential $\partial_{\text{cobar}}$ from the first page of the associate spectral sequence. Namely, the associated bigraded complex $\mathcal{F}_{*,*} B^c(C)$ is given by

$$\mathcal{F}_{*,*} B^c(C) \cong F(\Sigma^{-1} \mathcal{C}).$$

**Theorem 17.** Let $P_\ast$ be a simplicial operad on chain complexes $\mathcal{C} = \text{Ch}_{k,t}$, $(t \in \mathbb{Z} \cup \{-\infty\})$. Then for any integer $r \geq 0$, there is a quasi-isomorphism

$$\Gamma : |B^c(B(P_\ast))(r)|_{\mathcal{C}} \xrightarrow{\approx} |P_\ast|_{\text{Op}_\mathcal{C}}(r)$$

**Proof.** One can observe that:

$$|P_\ast|_{\text{Op}_\mathcal{C}}(r) \cong B^c(\int_{co\text{Op}_\mathcal{C}} B(P_k) \otimes C(\Delta^k))(r) \ (\text{This is because } B^c(-) \text{ is a left adjoint functor})$$

$$\cong B^c(\int_{\text{Finset}, \mathcal{C}} B(P_k) \otimes C(\Delta^k))(r),$$

where:

- $co\text{Op}_\mathcal{C}$ denotes the category of cooperads on $\mathcal{C}$;
- The isomorphism \((1)\) is justified by the fact that the forgetful functor \(U : \text{coOp}_C \longrightarrow \[\text{Finset}, C\]\) preserves colimits as a left adjoint.

On the other hand the model of the realization functor in chain complexes \(|B^c(B(P_*))(r)|_C\) that we use in this proof is the coend

\[
|B^c(B(P_*))(r)|_C \simeq \int^k B^c(B(P_k))(r) \otimes C(\Delta^k)
\]

After this setting we want to construct a chain map

\[
\Gamma : \int^k B^c(B(P_k))(r) \otimes C(\Delta^k) \longrightarrow B^c(\int^k B(P_k) \otimes C(\Delta^k))(r)
\]

The construction is made on the set of trees. Namely let \(T\) be a tree with \(r\) leaves. We fix an order on the set of the vertices of the tree \(T\) and we define the chain map \(\Gamma_T\) as follows:

\[
\Gamma_T : \int^k T(s^{-1}B(P_k)) \otimes C(\Delta^k) \longrightarrow T(s^{-1} \int^k B(P_k) \otimes C(\Delta^k))
\]

where

\[
\Delta : C(\Delta^k) \longrightarrow C(\Delta^k)^{\otimes V(T)}
\]

\[
v \longmapsto \bigotimes_{v \in V(T)} c_v \quad \text{is the coproduct of the coalgebra } C(\Delta^k).
\]

The collection of the maps \(\Gamma_T\) produces naturally the chain map \(\Gamma\) that we want.

We now have to prove that \(\Gamma\) is a quasi-isomorphism. To prove it we use a spectral sequence argument from the filtration of the cobar construction of definition 16. Namely, let us build the filtrations

\[
\mathcal{F}_u|B^c(B(P_*))(r)|_C := \int^k \mathcal{F}_u B^c(B(P_k)) \otimes C(\Delta^k)
\]

and

\[
\mathcal{G}_u|P_*|_{\text{Op}_C} := \mathcal{F}_u \int^k B(P_k) \otimes C(\Delta^k)
\]

One can see that the associated bigraded complexes \(\mathcal{F}_{s,*}|B^c(B(P_*))(r)|_C\) and \(\mathcal{G}_{s,*}|P_*|_{\text{Op}_C}\) are equivalent to:

\[
\mathcal{F}_{s,*}|B^c(B(P_*))(r)|_C \cong \int^k F(\Sigma^{-1} \mathcal{B}(P_k)) \otimes C(\Delta^k)
\]

and

\[
\mathcal{G}_{u,t}|P_*|_{\text{Op}_C} = F(\Sigma^{-1} \int^k B(P_k) \otimes C(\Delta^k))
\]

From a classical theorem of spectral sequences, to prove that \(\Gamma\) is a quasi-isomorphism, it will be enough to show that its restriction to the bigraded complexes \(\mathcal{F}_{s,*}|B^c(B(P_*)|(r)|_C\) and \(\mathcal{G}_{s,*}|P_*|_{\text{Op}_C}\) is a quasi-isomorphism with the internal differential of these complexes. This later condition is true if for any fixed tree \(T\), the associated chain map \(\Gamma_T\) is a quasi-isomorphism with the internal differential.

Let us consider now the following commutative diagram
\[
\begin{align*}
\int C \left( s^{-1} B(P_k) \otimes C(\triangle^k) \right) & \xrightarrow{\Gamma_T} T(\int C \left( s^{-1} B(P_k) \otimes C(\triangle^k) \right)) \\
\int C \left( s^{-1} B(P_k) \otimes C(\triangle^k) \otimes N(\triangle^k) \right) & \xrightarrow{\mathcal{AW}_*} T(\int C \left( s^{-1} B(P_k) \otimes C(\triangle^k) \otimes N(\triangle^k) \right))
\end{align*}
\]

where,
\[
\begin{align*}
\int C \left( s^{-1} B(P_k) \otimes C(\triangle^k) \otimes N(\triangle^k) \right) & \xrightarrow{\mathcal{AW}_*} T(\int C \left( s^{-1} N_* B(P_\bullet) \otimes C(\triangle^k) \otimes N(\triangle^k) \right))
\end{align*}
\]

1. Given any simplicial chain complex \( K_\bullet : \int C \left( s^{-1} K_n \otimes N_*(\triangle^n) \right) \rightarrow N_* K_\bullet \) is the chain complex defined by: \( \alpha([x \otimes \sigma]) := [x \sigma^*] \), with \( \sigma^* : \triangle^* \rightarrow \triangle^n \) induced by \( \sigma : \triangle \rightarrow \triangle^n \). The inverse of \( \alpha \) is the morphism \( \alpha' : [x] \mapsto [x \otimes \iota] \), where \( \iota \) denotes the top cell of \( \triangle_n \).

2. \( \mathcal{AW}_* : N_* T(\int C \left( s^{-1} N_* B(P_\bullet) \otimes C(\triangle^k) \otimes N(\triangle^k) \right)) \rightarrow T(\int C \left( s^{-1} N_* B(P_\bullet) \otimes C(\triangle^k) \otimes N(\triangle^k) \right)) \) is the generalization (at the level of trees) of the Alexander Whitney map defined as follows: given any two simplicial chain complexes \( K_L \) and \( L_\bullet \), one have

\[
\mathcal{AW}_* : \int C \left( s^{-1} \sum_k \otimes \bigoplus \int C \right) K_n \otimes N_* \Delta^n \rightarrow \bigoplus \int C \left( s^{-1} \sum_k \otimes \bigoplus \int C \right) K_k \otimes N_* \Delta^k \otimes \int C \left( s^{-1} \sum_l \otimes \bigoplus \int C \right) L_l \otimes N_* \Delta^l
\]

where \( \iota_j \) is the top cell of \( \Delta^j \); \( \iota_n(0...k) : k \rightarrow n \) and \( \iota_n(0...k) : l \rightarrow n \) are the canonical monotone injections defined by: \( \iota_n(0...k)(i) = i, \iota_n(0...k)(i) = k + i \).

This is the same map though defined with a different notation in [ML63, Corollary 8.6.], and is known to be a quasi-isomorphism (see [SS03, § 2.3.]).

3. The vertical weak equivalences in each column are due to the fact that the terms in the integral are all good model for simplicial framing.

From this diagram, since the morphism \( \mathcal{AW}_* \) at the bottom is a quasi-isomorphism, it follows inductively that \( \Gamma_T \) is a quasi-isomorphism.

\[\Box\]

### 3.3 Simplicial resolution of operads

Let \((C, \wedge, \mathbb{I}_C)\) be a symmetric monoidal category equipped with colimits such that \( \wedge \) distributes over colimits. In this context we are able to define free operads. There are the following adjoint functors:

\[
F : [\text{Finset}, C] \rightleftarrows Opc : U
\]

where \( F \) denotes the free operad functor, and \( U \) is the forgetful functor. This adjunction gives the comonad \( T := FU : Opc \rightarrow Opc \).

Let \( P \) be an operad in \( C \). We define the simplicial operad \( \text{Res}_\bullet(P) \) associated to the comonad \( T \) and the operad \( P \) as follows:
For any integer \( k \), \( \text{Res}_k(P) := T^{k+1}(P) \)

The counit \( \varepsilon : T \rightarrow 1 \) of the comonad \( T \) is used in the classical way to construct the faces \( d_i : \text{Res}_k(P) \rightarrow \text{Res}_{k-1}(P) \), \( 0 \leq i \leq k \). The degeneracies \( s_j : \text{Res}_k(P) \rightarrow \text{Res}_{k+1}(P) \), \( 0 \leq j \leq k \), are induced by the comonadic coproduct \( \mu : T \rightarrow T^2 \).

**Remark 18.** The simplicial operad \( \text{Res}_\bullet(P) \) has a natural augmentation \( \text{Res}_0(P) = T(P) \varepsilon \rightarrow P \).

The associated augmented simplicial chain complex sequence \( \varepsilon : U\text{Res}_\bullet(P) \rightarrow UP \) has extra degeneracies \( s_{-1} : U\text{Res}_k(P) \rightarrow U\text{Res}_{k+1}(P) \), \( (\forall k \geq -1) \) given by:

\[
\begin{array}{ccc}
UP & \xrightarrow{s_{-1} = \eta_U\cdot P} & UFU(P) \\
& \| & \| \\
U\text{Res}_0(P) & \rightarrow & U\text{Res}_1(P)
\end{array}
\]

A straight consequence of these extra degeneracies when \( C = Ch_{k,t} \) is that one have the quasi-isomorphisms : \( \forall r \geq 0 \),

\[
\varepsilon : \left| \text{Res}_\bullet(P)(r) \right|_C \xrightarrow{\simeq} P(r)
\]

**Proposition 19.** Let us consider the category \( (C = Ch_{k,t}, \otimes, k) \). If \( P \) be a cofibrant operad on \( C \), then the augmentation \( \varepsilon : \left| \text{Res}_\bullet(P) \right|_{\text{Op}_C} \rightarrow P \) is a weak equivalence.

**Proof.** Let us consider the following commutative diagram: \( \forall r \geq 0 \),

\[
\begin{array}{ccc}
|B^cB(Res_\bullet(P))(r)|_C & \xrightarrow{\simeq} & |Res_\bullet(P)|_{\text{Op}_C}(r) \\
\downarrow^{(2)} & & \downarrow^{(4)} \\
|Res_\bullet(P)(r)|_C & \xrightarrow{\simeq} & P(r)
\end{array}
\]

where

- The quasi-isomorphism (1) is induced by theorem 17;
- The quasi-isomorphism (2) is induced by the following fact:
  \( P \) is cofibrant, therefore \( Res_\bullet(P) \) and \( B^cB(Res_\bullet(P)) \) are Reedy cofibrant. We conclude using [Fre17b, Thm 3.3.7.] that (2) is a quasi-isomorphism.
- (3) is induced by remark 18.

We then deduce by the 2-out of 3 property of weak equivalence that (4) is a quasi-isomorphism.

\( \square \)
4 Equivalence of operads on Quillen equivalent categories

Let \( \lambda : \mathcal{C} = Ch_{k,t} \rightleftarrows \mathcal{D} : R \) be a Quillen pair between the category of chain complexes \( Ch_{k,t} \) \((t \in \mathbb{Z} \cup \{-\infty\})\) with its classical projective model structure, and any model category \( \mathcal{D} \). If in addition the category \( \mathcal{D} \) is monoidal and the pair \((\lambda, R)\) is a weak symmetric monoidal Quillen pair, then the functor \( R \) extends naturally to a functor \( \overline{R} : Op_{\mathcal{D}} \rightarrow Op_{\mathcal{C}} \) given by:

\[
\forall k \text{ and } P \in Op_{\mathcal{D}}, (\overline{R}(P))(k) := R(P(k)).
\]

It is proved in [Fre17a, Prop 3.1.5.- (a)] that the functor \( R \) has a left adjoint \( L : Op_{\mathcal{C}} \rightarrow Op_{\mathcal{D}} \) given by:

(a) If \( P = F(M) \) is a free operad in chain complexes generated by a symmetric sequence \( M \), then

\[
L(F(M)) := F(\overline{\lambda}(M)),
\]

where \( \overline{\lambda} : [\text{Finset}, \mathcal{C}] \rightarrow [\text{Finset}, \mathcal{D}] \) is the aritywise left composition with \( \lambda \);

(b) If \( P \) is any operad on chain complexes, then one make the following identification:

\[
P \cong coeq(FUFU(P) \xrightarrow{id_0} FU(P))
\]

where

- \( U : Op_{\mathcal{C}} \rightarrow [\text{Finset}, \mathcal{C}] \) is the forgetful functor;
- \( d_0 : FUFU(P) \rightarrow FU(P) \) is the morphism of operads adjoint of the identity morphism of symmetric sequences \( Id : UFU(P) \rightarrow UFU(P) \);
- \( d_1 = FU(\varepsilon) : FUFU(P) \rightarrow FU(P) \), with \( \varepsilon : FU(P) \rightarrow P \) being the morphism of operads adjoint to the identity of symmetric sequences \( Id : U(P) \rightarrow U(P) \).

We now set

\[
L(P) := coeq(F(\overline{\lambda}(UFU(P)))) \xrightarrow{d_0} F(\overline{\lambda}(U(P))))
\]

It is good to remark that the pair \((L, \overline{R})\) remains a Quillen pair. Using this notation and construction, we state the following theorem:

**Theorem 20.** Let \( \lambda : \mathcal{C} = Ch_{k,t} \rightleftarrows \mathcal{D} : R \) be a weak monoidal Quillen pair between the category \( (Ch_{k,t}, \otimes, k) \), \((t \in \mathbb{Z} \cup \{-\infty\})\), and any other symmetric monoidal category \((\mathcal{D}, \wedge, I_{\mathcal{D}})\). If the pair \((\lambda, R)\) is a Quillen equivalence, then so is the pair \( L : Op_{\mathcal{C}} \rightleftarrows Op_{\mathcal{D}} : \overline{R} \).

To prove this theorem, we will need the result of the following lemma.
Proposition 21. Let \( \lambda : C = \text{Ch}_{k,t} \leftrightarrow D : R \) be a weak monoidal Quillen pair between the category \( (\text{Ch}_{k,t}, \otimes, k), \) \( t \in \mathbb{Z} \cup \{ -\infty \} \), and any other symmetric monoidal category \( (D, \land, \mathbb{I}_D) \). If \( P \) is a cofibrant operad in \( \text{Ch}_{k,t} \), then the morphism \( \overline{\lambda}(U(P)) \rightarrow UL(P) \), which is adjoint to the unit \( \eta : P \rightarrow \overline{R}L(P) \), is a weak equivalence.

Proof. We form the following diagram which is commutative from the natural transformations \( |\text{Res}_\bullet(-)|_{\text{Op}_C} \rightarrow 1_{\text{Op}_C} \) and \( -|c \rightarrow U| -|_{\text{Op}_C} : \forall r \geq 0, \)

\[
\begin{array}{ccc}
\lambda(P(r)) & \longrightarrow & L(P)(r) \\
\left(1\right) \downarrow & \searrow & \left(2\right) \\
\lambda(|\text{Res}_\bullet(P)|_{\text{Op}_C}(r)) & \longrightarrow & L(|\text{Res}_\bullet(P)|_{\text{Op}_C}(r)) \xrightarrow{(3)} |L(\text{Res}_\bullet(P))|_{\text{Op}_C}(r) \\
\left(4\right) \downarrow & \searrow & \left(5\right) \\
|\lambda(\text{Res}_\bullet(P)(r))|_c & \longrightarrow & L(\text{Res}_\bullet(P)(r))|_c
\end{array}
\]

where,

- The weak equivalence of \( 1 \) is justified by the weak equivalence of corollary 19, and the fact that the functor \( \lambda \) is a left Quillen adjoint so preserves weak equivalences (all chain complexes are cofibrant);

- The weak equivalence of \( 2 \) is justified by the following facts: the functor \( L \) is a left Quillen adjoint, thus preserves weak equivalence between cofibrant operads, and the operads \( P \) and \( |\text{Res}_\bullet(P)|_{\text{Op}_C} \) are cofibrant, thus by applying \( L \) to the weak equivalence of proposition 19, one obtain \( 2 \);

- The weak equivalences of \( 4 \) and \( 6 \) are a straight use of Theorem 17;

- The isomorphisms \( 3 \) and \( 5 \) come from then fact that the functors \( L \) and \( \lambda \) commute with colimits as left adjoint.

The orizontal map at the bottom is obtained literally by applying the functor \( -|c \) to the simplicial map

\[ \lambda((FU)^{r+1}(P))(r) \rightarrow L((FU)^{r+1}(P))(r) = F\overline{\lambda}(FU)^{r+1}(P)(r) \]

and this later map is a weak equivalence since it is built out of the lax monoidal morphisms \( V, W \in \text{Ch}^+_k \), \( \lambda(V \otimes W) \xrightarrow{\simeq} \lambda(V) \land \lambda(W) \).

One therefore deduce from the bottom weak equivalence of the diagram that the middle and then the top map are weak equivalences. \( \square \)

Proof of theorem 20. Let \( P \) be a cofibrant operad on \( C = \text{Ch}_{k,t}, Q \) be a fibrant operad in \( D \), and consider a weak equivalence \( \overline{f} : L(P) \xrightarrow{\simeq} Q. \)

The adjoint of this map fits into the commutative diagram

\[
\begin{array}{ccc}
P & \longrightarrow & \overline{R}(Q) \\
\downarrow & & \downarrow \overline{R}(\overline{f}) \\
\overline{R}L(P) & \longrightarrow & \overline{R}(Q)
\end{array}
\]

We aim to prove that the map \( P \rightarrow \overline{R}(Q) \) is a quasi-isomorphism. It will be enough in showing that the morphism \( UP \rightarrow U\overline{R}(Q) \) (where \( U : \text{Op}_C \rightarrow [\text{FinSet}, C] \) is the forgetful functor) is a quasi-isomorphism.

The adjoint in \([\text{FinSet}, D]\) of the diagram
We proved in lemma 21 that (1) is a weak equivalence, therefore (2) is a weak equivalence between the fibrant $UQ$ (in $[\text{FinSet}, \mathcal{D}]$) and the cofibrant $UP$ (in $[\text{FinSet}, \mathcal{C}]$) object. We now use the fact that the adjoint pair $(\lambda, R)$ is a Quillen equivalence (proposition 13) to deduce that $UP \to U\overline{R}(Q)$ is a weak equivalence.

Conversely, consider a quasi-isomorphism $g : P \xrightarrow{\sim} R(Q).$ We want to prove that $\tilde{g} : L(P) \xrightarrow{\sim} Q,$ and it will be sufficient to prove that $\tilde{g} : UL(P) \xrightarrow{\sim} UQ.$ Let us consider the commutative diagram

\[
\begin{array}{ccc}
\overline{\lambda}(UP) & \xrightarrow{\alpha_0} & UL(P) \\
\downarrow^{\overline{\lambda}(g)} & & \downarrow^{\gamma_0} \\
\overline{\lambda}RQ & \xrightarrow{\alpha_1} & ULRQ \\
\end{array}
\]

- $\alpha_0$ is a quasi-isomorphism according to lemma 21;

- $\gamma_1 \circ \alpha_1 : \overline{\lambda}R \to UQ$ is the unit of the adjunction $$(\overline{\lambda}, \overline{R}),$$ therefore $\gamma_1 \alpha_1 \overline{\lambda}(g) : \overline{\lambda}(UP) \to UQ$ is the adjoint of $g.$ One then deduce that $\gamma_1 \alpha_1 \overline{\lambda}(g) = \tilde{g} \alpha_0$ is a weak equivalence, since $(\overline{\lambda}, \overline{R})$ is a Quillen equivalence.

Therefore from the 2 out of 3 property, one deduce that $\tilde{g}$ is a weak equivalence.

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