Towards a theory of rapidly oscillating Ap stars

Douglas Gough,
Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA &
Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences, Wilberforce Road, Cambridge,
CB3 0WA.

Abstract

Peculiar A stars are so named because they exhibit abundance peculiarities in their atmospheres. It is believed that these arise as a result of differentiation of chemical species in large magnetic spots in which convective mixing is inhibited: there might be just two antipodal spots, whose axis is inclined to the axis of rotation. Many of the Ap stars that are rotating slowly also pulsate, with periods substantially shorter than the period of the fundamental radial mode. The pulsations appear to be non-radial, but axisymmetric, with their common axis usually aligned with the axis of the spots. In this lecture I shall first discuss the magnetic suppression of convection in the spots, and then I shall try to explain the pulsation phenomenon, reviewing some of the suggestions that have been made to explain the alignment and the excitation mechanism, and finally raising some issues that need to be addressed.

1 Introduction

It was my privilege to be the first research student to be registered by the University of Cambridge under Roger Tayler’s supervision. That did not make me Roger’s first student in practice, however, because Roger was a very kindly man and would supervise other astrophysics students in the department who needed more help than their registered supervisors provided; and so there were already several students under Roger’s wing when I arrived. Consequently Roger was at that time intellectually active in a variety of fields, embracing stellar structure and evolution, the physics of thermonuclear reactions, magnetohydrodynamics – particularly stability theory, with which Roger had a great deal of experience – and stellar pulsation. It is therefore appropriate that I devote this lecture to a subject that combines many of these fields. I have chosen to address some of the
issues that arise from trying to understand the phenomenon of rapid oscillations in peculiar A stars. Roger did not himself work in this area, but, recognizing his interests, one can easily imagine that he might well have done so had his attention not been drawn more strongly to other matters.

My own interest in rapidly oscillating Ap (roAp) stars was triggered by Donald Lynden-Bell, when in 1981, if I recall correctly, he told me on his return from a visit to the South African Astronomical Observatory of Don Kurtz’s recent, and yet unpublished, discovery. What intrigued me was that the stars were spotted and that the oscillations were apparently dipolar, with axes aligned with the spots. There were only a few examples, but nevertheless they must surely have at least hinted very strongly that the oscillations are always (almost) aligned with the spots, and therefore do not wander far off under the influence of Coriolis precession. Soon after that time, Nöel Dolez visited me from Paris, and we decided to carry out some simple calculations that might address the most obvious questions: why are only dipolar oscillations excited; why are their axes aligned with the spots; and why are only rapid oscillations observed, with frequencies much higher than that of the fundamental dipole mode? I address unashamedly in this lecture the picture that is emerging at least in my mind from that early work, and which has been substantially elaborated upon in recent years; I cannot here also review in detail the alternative suggestions that have been propounded. And I emphasize here that the line of argument that I follow is strongly influenced by what I learnt at Roger’s feet when I was a student.

2 The roAp-star phenomenon

Rapidly oscillating Ap stars are relatively slowly rotating A stars with chemical abundance peculiarities: they are rich particularly in Sr, Cr and Eu. The stars probably lie somewhat above the main sequence in the HR diagram, more-or-less in the pulsational instability strip. Their masses and radii are both roughly twice solar. They have convective cores, which will not concern me here, and shallow convection zones immediately beneath their surfaces. They appear to pulsate in high-overtone nonradial p modes of low degree, with periods generally in the range 4–15 minutes, the period of fundamental radial oscillation being about two hours. Typically only a very few modes are detected, sometimes only one, although HR 1217 is richly endowed with at least seven. The oscillation amplitudes, in radiative intensity, are typically of the order of 1mmag. This is some thousand times greater than the greatest amplitudes of stochastically excited intrinsically stable acoustic oscillations in stars like the Sun, so one might presume that the modes are intrinsically overstable. In most of the roAp stars that have been observed there are just one or two frequencies present in the spectrum of the oscillations (together, perhaps, with one or two higher harmonics); in a few others there are several. Fine structure, attributed to degeneracy splitting modulated by rotation, is usually present. The oscillations are often apparently dipole modes, although sometimes quadrupolar components appear
to be present too. Many Ap stars have large-scale, probably mainly dipolar, magnetic fields with field intensities ranging from several hundred Gauss to a few kiloGauss. The axis of the field is not, in general, aligned with the axis of rotation. For more details the reader is referred to the excellent reviews by Kurtz (1990, 1995) and Cunha (1998).

The picture of an Ap star that one might have in mind is perhaps somewhat like Figure 1, which illustrates the star and some of the lines of its external magnetic field. This field matches onto an essentially dipole field pervading the interior of the star (except, perhaps, the convective core). Near the magnetic axis, where the field is nearly vertical, convection is presumed to be suppressed by the field, probably throughout the entire radial extent of the unstably stratified zone. This permits chemical element segregation, by a combination of radiative levitation and gravitational settling against diffusion and by advection by a putative stellar wind (e.g. Vauclair, Dolez and Gough, 1991), leading to the appearance of two large antipodal spots of chemical peculiarity each of which might occupy some 10 or 20 per cent of its hemisphere. Severe line blanketing inhibits the radiant heat output, redistributing the optical spectrum in such a manner as to reduce the heat flux in the frequency ranges in which the stars are normally observed. However, the total heat flux is not altered substantially: because the spots are large, and because lateral heat transfer in the radiative region beneath is ineffectual, the radial stratification must adjust to let the heat out, unlike in sunspots. The large-scale magnetic field pervading the stellar interior presumably causes the star to rotate uniformly; Lorentz stresses
Figure 2: Distribution of the concentration $c$ of cobalt (upper row) and europium (lower row) over the surface of the roAp star HR 3831, inferred via Doppler imaging by Kochukhov et al. (2004). Shown are the star in five uniformly spaced rotation phases; a meridian is indicated by the dashed curve to indicate the relative orientation of the star. Contours are plotted at intervals of 1.0 dex; the darker areas are the regions of high concentration. The distribution is predominantly dipolar, with axis almost in the (rotational) equatorial plane (which is indicated in the diagrams). Matching a dipole to the magnetic-field data yield a polar surface field strength of 2.5kG inclined from the axis of rotation by 87°. The axis of rotation is inclined from the line of sight by 68°.

induced by putative differential rotation would act to oppose the shear on a timescale of only a few years.

Because the star is rotating, the spots can, in principle, be mapped by Doppler imaging; two recent examples of Doppler-imaged surface chemical-element abundance variations are presented in Figure 2.

As I mentioned in the introduction, a startling property of the oscillations is that they appear to be axisymmetric, with axes coinciding with the location of the spots. This kinematical description was first proffered by Kurtz (1982), who called it the oblique pulsator model. But there must be a dynamical explanation; surely, there must be an interaction with the magnetic field, either direct or indirect, which accounts for the alignment. A direct interaction might cause one of the three independent dipole modes of any given order to be aligned with the field; if that were so, there would remain the problem of explaining why that and only that member of the triplet is excited to an observable amplitude. The most likely explanation is that conditions in the spot, where the amplitude of the aligned mode is greatest, are conducive to mode excitation, whereas conditions elsewhere are not. Before addressing that central issue, I first offer some justification for the presumption that the magnetic field actually succeeds in suppressing convection in the spots.
3 The influence of a magnetic field on convection

This is the subject on which I worked with Roger when I was a student. Our goal was to find, by linearized perturbation analysis, conditions for the stabilization by a simple magnetic field of an otherwise convectively unstably stratified region of a star. In the event we approximated the potentially unstable region by a plane-parallel layer of ideal gas in hydrostatic equilibrium, and, for tractability, we chose the field to be either uniform but arbitrarily inclined, or vertical but varying with a single horizontal Cartesian coordinate $x$. Chandrasekhar (1961) had written extensively on a corresponding problem for a Boussinesq liquid, but there were few results for a compressible gas – results only for the case when the field is horizontal.

I had already learned from Roger’s lectures that energy arguments can be powerful in establishing sufficient conditions for stability, so I set about establishing an appropriate energy integral for the problem in hand. I think I must have worked for the better part of the eight weeks of a Cambridge Full Term\footnote{‘Full Term’ in Cambridge is the period within Term during which university lecturers, like Roger was at the time, were almost fully distracted from research by teaching and administration.} without discussing the matter seriously with Roger, but by the end I had succeeded in establishing an integral relation, which, together with an immediate deduction from it, I proudly presented to Roger when Full Term was over. I had had some very brief discussions with Roger over coffee during those weeks, and had learned that Roger had made some progress himself, so I was eager to discover whether Roger too had derived such an integral, and if so, how far he had progressed in extracting from it some useful criteria. It turned out, to my dismay, that he was well ahead of me. But when we compared notes, I was horrified to discover that Roger had not even attempted to construct an energy integral like the one that I had so painstakingly derived, but instead had simply written it down at the start of his investigation. He had done so simply by specializing a more general result that had been published in a well known (by plasma physicists) paper by Bernstein, Frieman, Kruskal and Kulsrud (1958), but of which I was quite unaware. Why had Roger not told me about it? Because, it subsequently transpired, Roger knew that I would appreciate the result more by deriving it myself than I would by merely reading it. Moreover, it taught me, as I realized only later, an important lesson about teaching, which I have since used to the benefit of my own students – although I’m not sure that at the time I implement it my students necessarily describe the service as a benefit!

3.1 A physical introduction

Before proceeding with the discussion, permit me first to anticipate the result. As is evident from the discussion in the paper that Roger and I wrote about this
investigation (Gough and Tayler, 1966) and a subsequent discussion concerned explicitly with roAp stars (Balmforth et al., 2001), the ability of a field to inhibit convection depends not only on its intensity but also on its orientation. A rough idea of the condition required of a magnetic field to stabilize an otherwise convectively unstable fluid layer can be obtained by comparing estimates of the (destabilizing) buoyancy force associated with a perturbation with the (stabilizing) restoring Lorentz force. Evidently, what one must demand immediately is that the magnetic field configuration be itself intrinsically stable; we are not interested here in the possible stabilization by a (presumably convectively stable) fluid layer of an otherwise unstable magnetic field: that is a subject on which Roger worked, with others, later. The buoyancy force per unit mass tending to drive convection is roughly minus the product of the square of the (imaginary) buoyancy frequency $N$ and the vertical displacement $\delta r$, namely $-N^2 \delta r = (\Gamma^{-1} - \gamma^{-1}) g^2 \rho / p$. Here, $g$ is the local acceleration due to gravity, $\rho$ is density, $\Gamma = \frac{d \ln p}{d \ln \rho}$ is a measure of the stratification of the background state, and $\gamma = \left( \frac{\partial \ln p}{\partial \ln \rho} \right)_s$, the partial derivative being taken at constant specific entropy $s$, is the first adiabatic exponent. This overestimates the destabilizing force somewhat, because it ignores the opposing pressure gradients associated with the horizontal component of the displacement of the fluid, which must necessarily be present. The magnetic restoring force per unit mass exerted on the displaced fluid, in SI units, is roughly $(B^2 k^2 / \mu \rho) \delta r$, where $B$ is the field strength, $k$ is a characteristic wavenumber of the magnetic field-line distortion, and $\mu$ is the magnetic permeability of the fluid. Thus one anticipates that for the field to prevent convection $v^2 \gtrsim -k^{-2} N^2$, where $v = B / \sqrt{\mu \rho}$ is the Alfvén speed. The criterion therefore depends critically on the greatest permissible lengthscale, $k_m^{-1}$, of the perturbation. That depends on the field geometry.

At the magnetic poles the field is almost vertical, and the characteristic wavenumber of the field-line perturbation is associated with the horizontal component of the putative motion. The scale of that motion is determined partly by an appropriate scale height of the background state, which for simplicity I take to be the pressure scale height $H_p = p / g \rho$, and partly by the depth $d$ of the unstably stratified zone: one might expect $k_m^{-1} = \min(H_p, d/2 \pi)$. In a typical roAp star, $H_p < d/2 \pi$, and consequently it is required that $v^2 \gtrsim -H_p^2 N^2$ for stability. This condition may be re-written

$$\frac{1}{\Gamma} - \frac{1}{\gamma} \lesssim \frac{B^2}{\mu \rho}. \quad (1)$$

It is satisfied in the model illustrated in Figure 3, for example, for $B$ of order $10^{-1}$T (1 kG), which is not atypical of roAp-star fields. Thus it seems not unreasonable that convection is suppressed near the magnetic poles. In the equatorial regions, on the other hand, if the magnetic field is predominantly dipolar, not only is the field somewhat weaker (by a factor of 2) than it is near the poles, but also it is nearly horizontal, and is therefore distorted by the vertical component of the convective motion, which is limited by neither the vertical scale heights nor the depth of the convection zone. Indeed, in principle
the length-scale could be very much larger than either, as had been found by Newcomb (1961) and Tayler (1961). In fact, Newcomb and Tayler found that a purely horizontal field in a horizontally infinite fluid layer did not influence the criterion for convective instability at all; and indeed sufficient conditions for stability derived subsequently by Roger and me (Gough and Tayler, 1966) depend only on the vertical component of the field. It is therefore much less likely that convection is suppressed near the magnetic equator; Lorentz stresses might succeed in imposing some degree of horizontal coherence to the motion, and in the nonlinear regime may even reduce the heat-flux-to-temperature-gradient ratio, thereby raising the temperature gradient above that in a corresponding nonmagnetic star. But it is likely that the unstably stratified layers are fully convective near the magnetic equator.

3.2 Locally applicable sufficient conditions for stability

Having painted the picture, let us now add a little rigour, and address a formal stability problem: a problem similar to those addressed by Roger and me, but tailored to suit the situation in hand. I adopt a simple spot model: namely, a plane layer of fluid, infinite in horizontal extent, and in hydrostatic equilibrium under gravity, pervaded by a vertical magnetic field that is axisymmetric about a vertical axis. For the purposes of discussing any putative instability, which I presume to occur on a dynamical timescale, the fluid is taken to have no viscosity and vanishing magnetic and thermal diffusion coefficients, although it must be acknowledged that diffusion was necessary for setting up the equilibrium state in the first place. The work $\delta W$ required to effect a displacement $\xi(r, \phi, z) = (\xi_r, \xi_\phi, \xi_z)$, with respect to cylindrical polar coordinates with $z$ vertical (and increasing downwards), is most easily derived from the work of Bernstein, Frieman, Kruskal and Kulsrud (1958), and may be written:

$$\delta W = \frac{1}{2} \int \left[ g \xi_z \nabla \cdot \xi + 2g \rho \xi_z \nabla \rho + (\gamma p + B^2) (\nabla \cdot \xi)^2 ight] dV$$

$$- 2B^2 \xi'_z \nabla \xi + B^2 \xi'_z \cdot \xi' \right] dV$$

$$\equiv \delta \tilde{W} + \frac{1}{2} \int B^2 \xi'_z dV ,$$

(2)

in which $B(r)$ is the equilibrium magnetic field (now in units in which the magnetic permeability is unity), $p$ and $\rho$ are the equilibrium pressure and density, and a prime denotes differentiation with respect to $z$. The integration is over the volume of the fluid, which I presume to be a convectively unstable (in the absence of a magnetic field) layer sandwiched between two stable layers. It is obvious on energetic grounds that the perturbation must decay with distance from the convectively unstable layer; therefore, I choose two boundaries, at $z = z_1$ and $z = z_2$ well into the stable layers, and set $\xi = 0$ on these boundaries. Also, I should point out that the convectively unstable layer need not be convectively unstable throughout; in the specific model considered below, it comprises two unstable zones separated by a stable zone.
If $\delta W > 0$ for all admissible functions $\xi$ (e.g., all suitably differentiable vector functions that vanish on the boundaries), then no displacement of the fluid is possible without the exertion of work by external forces, and the equilibrium state must be intrinsically stable. On the other hand, if there exist admissible functions $\xi$ for which $\delta W < 0$, one cannot infer instability, because those displacements may not be realizable: only displacements that satisfy the equations of magnetohydrodynamics (mhd) are actually permitted. Thus, from the energy integral one can derive only sufficient conditions for stability. But that will be adequate for the purpose here.

To obtain conditions that are both necessary and sufficient one must study the linearized mhd equations. A powerful way to proceed, in general, is via a variational principle derived from those equations, which can be used to bound below the growth rates $\eta$ of the linear modes. Much use of the method has been made by Chandrasekhar (1961). In the case considered here, the sign of $\eta$ rests on the sign of the integral (2), so the energy method and the variational method look very similar. It is still necessary to restrict attention to permissible functions $\xi$; and often this can be accomplished most readily by considering the eigenfunctions of the linearized equations, which satisfy the Euler-Lagrange equations for the stationarity of $\eta$: if a criterion can be found to render at least one eigenvalue $\eta$ positive, then the system is unstable. In most cases, to find a single condition that is both necessary and sufficient requires solving the Euler-Lagrange equations. This can be accomplished analytically in the absence of a magnetic field, and leads to the local criterion that Schwarzschild (1906) introduced into the astronomical literature (Lebowitz, 1965, 1966; Gough, 2001). However, no such criterion exists for the case being discussed here, because a magnetic field connects one region of the fluid to another, and the necessary and sufficient condition must necessarily be nonlocal, and possibly global. However, I shall not pursue that matter any further here, because the analysis I do present yields a sufficient condition for stability that is useful enough for discussing Ap-star spots.

Before proceeding with the analysis it is instructive (or perhaps merely only reassuring) to make a straightforward qualitative observation: the coefficient of $B^2$ in the integrand in equation (2) is positive definite. That implies that at least a state that is neutrally stable when $B = 0$ is rendered stable by the introduction of the field. Moreover, it is suggestive, although not yet demonstrated, that a convectively unstable state can be stabilized by a sufficiently intense field.

The objective of the analysis is thus to find a simple bound on $B$ above which the minimum value of $\delta W$ amongst all displacements $\xi$ is positive. Evidently that condition on $\delta W$ will be satisfied if $\delta W > 0$. The displacement $\xi$ can then be considered to be characterized by the functions $\xi_r$, $\xi_z$ and $\text{div}\xi$. The Euler-Lagrange equation for minimizing $\delta W$ with respect to $\text{div}\xi$ is

$$\left(\gamma p + B^2\right) \text{div}\xi = B^2 \xi_r' - g \rho \xi_z,$$

which can then be substituted into the formula for $\delta W$. It is evident that substantial simplification can be achieved by restricting attention to spot models.
in which \( \nabla \rho \) is vertical: then

\[
\delta \tilde{W} = \frac{1}{2} \int \left[ g \rho' \xi_z^2 - (\gamma p + B^2)^{-1} \left( B^2 \xi_z' - g \rho \xi_z \right)^2 + B^2 \left( \xi_z'^2 + \xi_z'^2 \right) \right] dV
\]

which is positive if \( \delta \tilde{W} > 0 \); the functional \( \delta \tilde{W} \) depends on \( \xi \) only through \( \xi_z \) and its \( z \) derivative (there are no horizontal derivatives), and therefore the condition \( \delta \tilde{W} > 0 \) can be applied separately on each field line, and, rather than being a partial differential equation, the minimizing Euler-Lagrange equation is then merely an ordinary differential equation. The condition for \( \delta \tilde{W} > 0 \) can therefore be written:

\[
\mathcal{E} (B^2; \xi_z) \equiv \int_{z_1}^{z_2} \left( F \xi_z'^2 + G \xi_z'^2 \right) dz > 0 \quad \text{on every field line,}
\]

where

\[
F = - \left( \frac{\gamma g p \rho}{\gamma p + B^2} \right)' + \frac{g^2 \rho^2}{\gamma p + B^2} + g' \rho \,,
\]

\[
G = \frac{\gamma p B^2}{\gamma p + B^2} \,,
\]

subject, of course, to the hydrostatic constraint \( dp/dz = g \rho \). An immediately obvious condition is \( F > 0 \) everywhere. However, it is more useful to introduce a function \( \Phi \), which at present is arbitrary, and rewrite inequality (6) as

\[
\mathcal{E} = \int_{z_1}^{z_2} \left[ (F - \Phi' - G^{-1} \Phi^2) \xi_z'^2 + G (\xi_z' - G^{-1} \Phi \xi_z')^2 \right] dz \,,
\]

for then a more general stability criterion can be written down, namely

\[
F - \Phi' - G^{-1} \Phi^2 > 0 \quad \text{everywhere.}
\]

In particular, because \( \gamma \) is a rapidly varying function of depth in the ionization zones of H and He, one might choose to set

\[
\Phi = \frac{-g \rho B^2}{\gamma p + B^2} + \Psi
\]

in which \( \Psi \) does not depend explicitly on derivatives of \( \gamma \) or \( g \). In that case, condition (10) does not depend explicitly on \( \gamma' \) or \( g' \). Then, for example, one could set \( \Psi = g \rho B^2 / (\alpha p + B^2) \), where \( \alpha \) is constant, as did Gough and Tayler (1966), to obtain, if \( \alpha = 1 \),

\[
\frac{1}{\gamma} - \frac{1}{\Gamma} < \frac{B^2}{\gamma (p + B^2)} \,,
\]

or
\[ \frac{1}{\gamma} - \frac{1}{\Gamma} < \left( 1 + \frac{1}{\gamma} - \frac{1}{\gamma_{\text{max}}} \right) \frac{B^2}{\gamma_{\text{max}} p + B^2}, \]

(13)

if \( \alpha = \gamma_{\text{max}} \), \( \gamma_{\text{max}} \) being the greatest value of \( \gamma \) in the region. Both criteria have a superficial resemblance to the approximate condition (11). The system is stable if either of these strict conditions is satisfied everywhere. It should be remarked that the left-hand sides of the conditions can be written in terms of the dimensionless superadiabatic temperature gradient \( \nabla - \nabla_{\text{ad}} \), which is more familiar to those working in the theory of stellar structure. For a gas composed principally of hydrogen and helium, the required relation is approximately

\[ \gamma - 1 - \Gamma - 1 = - (\nabla - \nabla_{\text{ad}} - \mu \ln \mu_0 / \ln p) \delta, \]

where \( \delta = -(\partial \ln \rho / \partial \ln T)_{\mu,\mu_0} > 0 \) is a dimensionless isobaric thermal expansion coefficient and \( \mu = (\partial \ln T / \partial \ln \mu_0)_{\rho,\mu}, \mu_0 \) being the mean molecular mass of the fluid when it is completely unionized. For a perfect gas, \( \delta = 1 \) and \( \mu = 1 \).

Although conditions (12) and (13) can be applied locally, they are actually global conditions, because stability is assured only if either of them is satisfied everywhere. However, some influence of the nonlocality has been thrown away by ignoring the contribution from \( \xi_z^2 \) to \( \mathcal{E} \), in addition to having already ignored the contribution from \( \xi_\phi \) and part of the contribution from \( \xi_r \) to \( \delta W \). In Figure 3 is plotted \( \gamma - 1 - \Gamma - 1 \), together with the right-hand sides of the inequalities (12) and (13) with \( B = 1 \text{kG} \); they are plotted through the outer layers of an Ap-star envelope model in radiative equilibrium similar to the model with the small accumulation parameter used by Balmforth et al., (2001) to represent a spot. Although the right-hand sides of both criteria (12) and (13) exceed the left-hand sides nearly everywhere, they fail to do so in the hydrogen ionization zone, in which there is a sharp density inversion. Moreover, augmenting the magnetic field does not change matters significantly, because both right-hand sides are bounded above by values that do not differ substantially from unity. Consequently, stability against convection is not assured by those locally applicable criteria. More account must be taken of the nonlocal interactions. For example, it is hardly likely that the term in the integrand in equation (9) that was jettisoned in deriving criteria (12) and (13) vanishes in the interval in which the criteria are not satisfied, and it is even likely to contribute substantially to stabilization, particularly when \( B \) is large. But to demonstrate that requires more detailed analysis.

### 3.3 A global sufficient condition

Progress can be made in a relatively simple manner by investigating more carefully the conditions under which the stability condition (6) is satisfied. To this end I seek the minimum of \( \mathcal{E} \) amongst all admissible functions \( \xi_z(z) \), and then find the smallest value of \( B \) for which that minimum is positive. This is equivalent to demanding that the minimum value of

\[ \lambda(B^2) = \frac{\mathcal{E}}{\int_{z_1}^{z_2} G \xi_z^2 dz} \]

(14)
Figure 3: Convective stability characteristics of the polar region of an Ap-star model similar to the model with low accumulation parameter considered by Balmforth et al. (2001). The dotted curve is $\gamma^{-1} - \Gamma^{-1}$, and the dot-dashed curve is the right-hand side of the approximate criterion (1) for a vertical field of strength $B = 1kG$. The solid and dashed curves are the right-hand sides of the sufficient conditions (12) and (13), respectively, also with $B = 1kG$. The hydrogen and first helium ionization zones (10%–90% ionization), which are merged, lie in the interval $\log p = (3.7, 4.9)$, the second helium ionization zone in $\log p = (5.3, 7.0)$.

be positive, for which the Euler-Lagrange equation is

$$\frac{d}{dz} \left( G \frac{d\xi_z}{dz} \right) - (F - \lambda G) \xi_z = 0 .$$  \hspace{1cm} (15)

Equation (15) is to be solved subject to the boundary conditions $\xi_z = 0$ at $z = z_1$, and $z = z_2$. Solving this eigenvalue problem is much simpler than solving the full stability problem, which, even for the idealized configuration considered here (an axisymmetric vertical magnetic field) is represented by a sixth-order partial differential equation in two dimensions ($z$ and $r$).

I have solved equation (15) for the spot model illustrated in Figure 3, taking $z_1 = 0$ and $z_2$ to be many scale heights beneath the lower region in which $\gamma^{-1} - \Gamma^{-1} > 0$ (i.e. well below the region where the stratification would be convectively unstable in the absence of a magnetic field). The result is that $\lambda(B^2) > 0$ for all $|B| > B_c = 1.11kG$; this critical magnetic field was found to be essentially independent of $z_1$ and $z_2$ provided that the boundaries are far enough away from the unstably stratified regions. Because it is only a sufficient
condition, this result implies that the region is linearly stable to convection for all vertical magnetic fields of strength greater than $B_c$. Thus, it is probably safe to conclude that convection is suppressed in regions of Ap stars that are pervaded by kiloGauss magnetic fields. Some Ap stars have been observed to have fields several times greater than that.

### 3.4 On the magnetic inhibition of convection

It is important to realize that this calculation does not prove that convection cannot occur in a spot. First, the analysis is linear, and it has been shown that under some circumstances instability can occur as a subcritical direct bifurcation; that is to say, convection at finite (i.e. not infinitesimal) amplitude can be sustained against more intense mean fields than the critical linearly stabilizing field. The mechanism is one of sweeping aside the field into columns between convective cells, leaving convection to run its course in the regions of relatively weak field between. It has been seen to occur in numerical simulations of convection in a Boussinesq fluid (Blanchflower and Weiss, 2002), and I see no reason why it should not occur in compressible convection too. Second, the field strength declines away from the centre of the spot, and there must be a radius beyond which convection can take place. Although formally the analysis requires the sufficient condition $\lambda(B^2) > 0$ to be satisfied everywhere, the fact that the criterion is applied separately on each field line suggests that lateral interactions have been adequately accounted for by the jettisoning of the term in $\delta W$ proportional to $\xi' \phi$ and by adopting the consequent Euler-Lagrange equation (4) relating the horizontal and vertical components of the displacement, and that convection is suppressed (at least linearly) where $\lambda > 0$, and not necessarily elsewhere. Indeed, our knowledge of the structure of sunspots adds some credence to this idea. Moreover, Ap-star spots are laterally very extensive compared to their depth, and it is hard to imagine that conditions at the lateral boundary have a material influence on convection dynamics in the middle. It is worth pointing out that the reduction of the sufficient conditions for stability to criteria on separate field lines suggests that the original assumption of axisymmetry is unimportant, and that the criteria are probably valid for any horizontal variation of $B$. This is supported by the observation that the same criteria apply to vertical fields that depend on a single horizontal Cartesian coordinate, as was shown by Gough and Tayler (1966).

In conclusion, it is very likely that convection is strongly inhibited, if not suppressed entirely, in the magnetic polar regions of Ap stars with kiloGauss fields, permitting the creation of large spots in which chemical segregation can lead to photospheric abundance anomalies (Vauclair, Dolez and Gough, 1991). That necessarily leads to a horizontal variation of the stratification, which has an influence on the structure of the eigenmodes of oscillation in addition to the direct influence by the field itself via the Lorentz stresses. It lifts the degeneracy amongst modes of like order and degree and so determines the orientations of the modes relative to the magnetic and the rotation axes, as will be discussed in §6. Is that at least part of the reason why it appears that only modes aligned
with the magnetic poles are observed? The other part of the reason would then be that it is the spots that lead to mode excitation. It is to these matters that I now turn my attention.

4 Mode excitation in starspots

I have already presumed that the modes of oscillation are likely to be overstable, driven principally in the convectively stable spots. The most likely source of excitation is the \( \kappa \) mechanism in ionization zones, as is considered to be the case in the classical variables; outside the spots convection dominates the heat transport and thereby quenches the \( \kappa \) mechanism. The key issue is to what extent convection drives or damps the oscillations outside the spots, via the pulsation-induced modulation of the convective heat-flux and the out-of-phase component of the Reynolds stresses. The issue was investigated by Dolez and Gough (1982) and Balmforth et al. (2001), using a composite Ap-star model comprising a (double) conical region of a spherically symmetrical stellar model with convection suppressed to represent the spot, or polar, regions, and a complementary, equatorial region of a corresponding normal stellar model in which convection was treated using a time-dependent mixing-length formalism. The direct influence of the perturbed Lorentz stresses on the oscillations was ignored. In both investigations it was found that monople modes are damped in the equatorial region, and it was argued that because the work integral characterizing the damping and driving varies significantly in only the outer layers of the star, the same conclusion must hold also for any other low-degree mode.

Studying the spot regions is somewhat complicated, because it is likely that a stellar wind emanates from the spots, outwardly advecting chemical species which are otherwise segregated by a combination of radiative levitation and gravitational settling against diffusion. It is not unlikely that the balance of these processes leads to a concentration of the helium in the vicinity of the zone of first ionization of helium (Vauclair, Dolez and Gough, 1991), which influences both the frequencies and the stability of the acoustic modes. In particular, it reduces, perhaps surprisingly, the excitation of high-order modes; but it adds to the driving of low-order modes. Despite the uncertainties in the model, it was concluded by Balmforth et al., that the spot regions are likely to contribute substantially to the driving of some of the modes, particularly some modes of relatively high frequency, as is observed. This was found to be the case even when the boundary conditions permitted energy leakage into the atmosphere, in contrast to the earlier findings by Gautschy and Saio (1998). Growth rates \( \eta \) for complete (i.e. spherical) polar and equatorial models are illustrated in Figure 4. Whether the mode is globally excited or damped depends, therefore, on the relative contributions to the growth rate from the polar and the equatorial regions. Before embarking on a discussion of that matter, I first warn that the growth rates plotted in Figure 4 are likely to be overestimated, because the perturbed Lorentz forces were omitted from the calculations; in the surface layers where Lorentz forces are significant the acoustic motion couples with
Figure 4: Contributions to the growth rates of radial modes from the polar and equatorial regions of the Ap-star model with $M/M_\odot = 1.87$, $\log T_{\text{eff}} = 3.910$ and $\log L/L_\odot = 1.164$ considered by Balmforth et al. (2001). The growth rate of a global low-degree mode of azimuthal order $m$ with respect to the spot axis is obtained as the weighted average $\Lambda_l^m \eta^p + (1 - \Lambda_l^m) \eta^{\text{eq}}$, where $\Lambda_l^m$ is depicted in Figure 5.
Alfvén waves which propagate downwards and subsequently dissipate in the deep interior of the star, thereby extracting energy from the modes (Roberts and Soward, 1983). However, the estimates of the energy loss by Bigot et al. (2000), Cunha and Gough (2000) and Saio and Gautschy (2004) are too low for this process to overcome the driving computed by Balmforth et al. (2001); therefore, with sufficiently large spots, one might expect some of the global modes to be overstable.

In order to estimate the growth rates of the global modes it is necessary to combine the separate calculations for the polar and the equatorial regions. The frequency $\nu_{nml}$ and the growth rate $\eta_{nml}$ of a mode of order $n$, degree $l$ and azimuthal order $m$ are given approximately by

$$\sigma_{nml} = \int_{0}^{2\pi} \left\{ \sigma_{nlm}^{p} \int_{\hat{\mu}}^{1} |S_{nlm}|^2 d\mu + \sigma_{nlm}^{eq} \int_{0}^{\hat{\mu}} |S_{nlm}|^2 d\mu \right\} d\phi ,$$

with respect to spherical angles ($\theta, \phi$) about the axis of symmetry of the spots. Here, $\mu = \cos \theta$, $\sigma_{nlm} = 2\pi \nu_{nlm} + i \eta_{nlm}$ is the complex frequency of the mode, the scalar components $\Psi_{nlm}$ of the eigenfunction of which are presumed to be factorized in the form $\Psi_{nlm}(r, \theta, \phi, t) = \psi_{nlm}(r)S_{nlm}(\theta, \phi)e^{-i\sigma_{nlm}t}$, in which $S_{nlm}$ is normalized such that $\int \int |S_{nlm}|^2 d\mu d\phi = 1$, the integration being over the entire sphere, and the superscripts $p$ and $eq$ denote corresponding values of $\sigma_{nlm}$ for spherically symmetrical stellar models that are entirely polar-like or entirely equator-like, respectively; in the composite Ap-star model, the spots occupy the regions $|\mu| > \hat{\mu}$. Were the modes to have been adiabatic, and the boundary condition perfectly reflecting, $\sigma_{nlm}$ would have been real and the pulsation equations self-adjoint, and equation (16) would follow immediately from the associated variational principle, provided that the acoustic differences between the polar and equatorial parts of the model were not too great (cf., Cunha and Gough, 2000). But the pulsations considered here are actually nonadiabatic and are only partially contained within the leaky surface of the star. Nevertheless, a variational principle can be constructed from a combination of the solutions of the actual problem and its adjoint, and once again equation (16) follows. Of course, in either case the function $S_{nlm}(\theta, \phi)$ should describe the actual eigenfunction, but, in view of the variational property of equation (16), it may be replaced by the unperturbed eigenfunctions $Y_{ml}(\theta'', \phi'') = P_{m}^{l}(\cos \theta'')e^{im\phi''}$, $P_{m}^{l}$ being the associated Legendre function of the first kind (with argument $\mu'' = \cos \theta''$) with respect to polar angles ($\theta'', \phi''$) about the principal axis of the pulsations. The orientation of that principal axis will be discussed in §6.

Suffice it to say now that if the frequency perturbation associated with the spots is much greater than that due to rotation, the mode axis is more-or-less aligned with the spots, in accord with Kurtz’s (1982) original assumption. In that case

$$\sigma_{nml} \simeq \Lambda_{l}^{m} \left( \sigma_{nl}^{p} + (1 - \Lambda_{l}^{m}) \sigma_{nl}^{eq} \right),$$

where

\[\Lambda_{l}^{m}\]
Figure 5: The ratio $\Lambda^m_l (1 - \Lambda^m_l)$ of the weighting coefficients for various low-degree modes, plotted against the angular spot radius $\tilde{\theta}$.

\[
\Lambda^m_l = (2l + 1) \frac{(l - m)!}{(l + m)!} \int_\mu (P^m_l)^2 d\mu ,
\]  

(18)

and $\sigma^p_{nl}$ and $\sigma^e_{nl}$ are respectively the $(m$-degenerate) complex eigenfrequencies of the corresponding modes of the entirely polar-like and the entirely equator-like models. Since for high-order modes $\nu_{nl} \simeq (n + \frac{1}{2} l) \nu_0$, where $\nu_0 \simeq \frac{1}{2} (\int c^{-1} dr)^{-1}$ in which $c$ is sound speed, and the excitation and damping take place predominantly in the outer layers of the star – and, one must note, the relative deviation of the inertia of the mode from that of the corresponding radial mode is only $O(l^2/n^2)$ – one can express the cyclic frequency $\nu_{nlm}$ and the growth rate $\eta_{nlm}$ in terms of the values for the corresponding radial modes:

\[
\nu_{nlm} \simeq (1 + l/2n) [\nu^e_{n0} + (\nu^p_{n0} - \nu^e_{n0}) \Lambda^m_l] ,
\]

(19)

\[
\eta_{nlm} \simeq \Lambda^m_l \eta^p_{n0} + (1 - \Lambda^m_l) \eta^e_{n0} .
\]

(20)

Figure 5 shows the ratio $\Lambda^m_l / (1 - \Lambda^m_l)$ determining the relative contributions to $\nu_{nlm}$ and $\eta_{nlm}$ from the polar and the equatorial regions for a variety of spot-aligned modes, plotted against the polar angle $\tilde{\theta}$ subtended by the spot radius.
Figure 6: Greyscale plot of the ratio $\Lambda_m^l/(1 - \Lambda_m^l)$ of the weighting coefficients for axisymmetric dipole (left) and quadrupole (right) modes inclined by an angle $\hat{\alpha}$ from the spot axis. Black denotes stability and white overstability; the shade for neutral stability depends on the ratio of the spot-driving and equatorial-damping coefficients, as described in the text.

The regions in which this ratio exceeds $-\eta_{int}^{eq}/\eta_{int}^{np}$, whose values can be obtained from Figure 4, define the critical spot sizes above which the corresponding modes are overstable. Because, according to Figure 4, the greatest growth rate of the polar component exceeds the corresponding decay rate of the equatorial component by a factor of about 4, this simple model predicts that there are overstable global modes if the spots have angular radii exceeding about 20°. For $20^\circ \lesssim \theta \lesssim 40^\circ$, the most unstable modes are axisymmetric quadrupole modes; if the spots are larger, the most unstable modes are the axisymmetric dipole modes. Moreover, provided $\theta \lesssim 55^\circ$, no nonaxisymmetric mode is unstable. In this regime cyclic frequency differences of up to about 20$\mu$ Hz were computed amongst dipole modes of like order and degree, and different azimuthal orders. These findings are promising, for they augur a more sophisticated theory that might explain Kurtz’s interpretation of the observations.

When the influence on the oscillation eigenfrequencies of the spots does not overwhelm that of the star’s rotation, the principal axis of pulsation is not well aligned with the spots. It has been pointed out by Shibahashi and Takata (1993) and by Bigot and Dziembowski (2002) that this can sometimes be the case: in particular, the oscillations observed by Kurtz et al., (1993) of HR 3831 may have this property. In that case excitation by the spots is less effectual, and one would expect possibly fewer modes to be excited. Some of the implications can be gleaned from Figure 6, which comprises grey-scale plots of $\Lambda_m^l/(1 - \Lambda_m^l)$ against $\theta$ and the angle $\hat{\alpha}$ subtended by the pulsation axis and the spots (see Figure 7) for axisymmetric dipole and quadrupole modes, where now $\Lambda_m^l$ is the first of the integrals in equation (16). Values on the $\theta$ axis correspond to the dotted and dashed curves in Figure 5. As $\hat{\alpha}$ increases, the range of $\theta$ for which
quadrupole modes are favoured over dipole modes shrinks, and disappears at $\hat{\alpha} = 27^\circ$.

5 The direct influence of the Lorentz stresses on the pulsations

There have been several studies of the direct influence of the magnetic fields on modes of stellar oscillation. Although Lorentz stresses are very much smaller than gas-pressure gradients throughout almost all the star, because the field penetrates the surface there must be a (shallow) region in which the field dominates. The influence of the field on the oscillation frequencies is small, but the problem cannot be treated by standard (nonsingular) perturbation theory. Instead, the full dynamical equations must be analysed in the subsurface region in which the magnitudes of the gas pressure and the Lorentz forces are comparable. The problem was considered in the stellar context by Roberts and Soward (1983), using matched asymptotic expansions for a plane parallel polytrope – curvature effects in the magnetically dominated superficial layers are small – and subsequently by Campbell and Papaloizou (1986). The essentially purely acoustic waves of the deep interior couple with Alfvén waves in the surface layers, and the Alfvén waves propagate downwards into the star, their wavenumbers increasing with depth as a result of the increase in density until, it is presumed, they dissipate, never to return to the surface. Thus they contribute to the damping of the modes. Moreover, the coupling modifies the
phase jump that the acoustic waves experience on (partial) reflection, thereby altering the frequencies of the resonant modes.

The problem was taken up by Dziembowski and Goode (1996), Bigot et al (2000), Bigot and Dziembowski (2002) and Bigot (2003) for more realistic stellar models. The boundary-layer equations in the magnetically important surface layers were solved numerically and patched onto acoustical solutions in the deep interior to obtain a representation of a complete mode. The procedure appeared to work well for kiloGauss magnetic fields, but for the stronger fields that appear to be present in some stars – Bagnulo et al. (1999) report 14 kG in HR 3831, although this has been challenged recently by Kochukhov et al. (2004) – the expansions may be invalid. Saio and Gautschy (2003, 2004) have extended the expansion to higher order, which, one hopes, extends the range of usefulness of the results to higher field strengths. Frequency perturbations of order \(10\mu\text{Hz}\) for a 1kG magnetic field have been reported by Bigot et al. (2000) and Bigot and Dziembowski (2002); Saio and Gautschy (2004) have reported values twice as great.

A difficulty with the formal expansions is that they rapidly become complicated as the expansion is taken to higher order. To obviate that difficulty, and at the same time, it was hoped, to increase the range of magnetic field strengths for which the results are useful, Cunha and Gough (2000) adopted a much simpler approach: they solved the full (planar) equations in the magnetic boundary layer numerically as before, but, instead of expanding the acoustic mode in the interior about the corresponding solution for a spherically symmetrical star, they simply adopted the usual asymptotic representation of that solution expressed as a function of a new radial coordinate uniformly stretched by a latitudinally (relative to the axis of the magnetic field) dependent factor chosen to assure phase agreement with the boundary-layer solution in a common domain of validity of the inner and outer solutions. The amplitude was left unperturbed. The justification for there being hope that the procedure might be reliable beyond the field strengths at which Dziembowski and Goode’s expansion is useful is that, for the high-order modes considered, an \(O(1)\) phase perturbation modifies the Eulerian perturbation to the eigenfunction by \(O(1)\), which may be beyond the radius of convergence of the formal expansion, whereas the vertical wavenumber, on which the frequency principally depends, changes by only \(O(n^{-1}) \ll 1\). However, no serious test of the procedure was presented. Moreover, Cunha and Gough carried out the analysis only for polytropically stratified boundary layers. Therefore the results should be treated with some caution.

An interesting property of the approximate solutions found by Cunha and Gough (2000) is that there are frequencies near which the acoustic-Alfvén coupling is extremely strong. Moreover, at these frequencies the magnetic frequency perturbation, plotted as a function of mode frequency, experiences a discontinuous drop. Therefore the mode frequencies do not adhere to an almost uniformly spaced spectrum, as one might otherwise have expected. This led Cunha (2001) to propose the nonuniformity of the coupling as an explanation of the nonuniformity in the spectrum of HR 1217, an explanation which implied that there
should in addition be another yet undiscovered mode nearer to the location of the discontinuity but which is too heavily damped to have been seen. I should point out that Dziembowski and Goode (1996) and Bigot et al., (2000) had not found this discontinuous behaviour in their expansions, which put the phenomenon in doubt (Bigot and Dziembowski, 2002). However, a new observational search by Kurtz and his collaborators (2002) has revealed that the mode is indeed present, at a lower amplitude than those of the modes discovered previously, which was encouraging. Moreover, a subsequent expansion somewhat similar to that adopted by Dziembowski and his colleagues but taken to higher order by Saio and Gautschy (2003, 2004) did produce such behaviour, suggesting that Cunha’s explanation might at least be qualitatively correct.

The variation of the surface amplitude, $S_{nml}(\theta, \phi)$, and the orientation of a mode relative to the axis of rotation of the star and the line of sight determine the relative amplitudes of the contribution of that mode to the power spectrum of the observed oscillations, analysis of which is carried out for the purpose of mode identification (e.g. Dziembowski and Goode, 1985, 1996; Shibahashi and Takata, 1993; Takata and Shibahashi, 1995). The perturbation expansions of Dziembowski and Goode (1996) and Saio and Gautschy (2004), within their range of validity, determine $S_{nml}$, which is caused to deviate from the corresponding spherical-harmonic variation that would have been the structure had the star been neither spotted nor rotating. That deviation comes about partly because the radial variation of the eigenfunction in the surface layers is modified by the spots in a latitudinally dependent fashion, and partly because the latitudinally dependent phase jump experienced by the mode on reflection modifies the interior eigenfunctions from a spherical-harmonic form. The approximation adopted by Cunha and Gough (2000) took account of only the first of these effects, an approximation which Montgomery and Gough (2003), using an argument based on an analysis of a rectangular isothermal toy stellar model, subsequently suggested might lead to substantial inaccuracy, because there might actually be serious horizontal mode trapping. Their more recent, unpublished, work suggests that in a real star wave refraction by the temperature gradient near the centre mitigates that trapping; consequently it seems likely that a generalization of the Cunha-Gough approximation can provide a workable, relatively simple procedure for computing Ap-star oscillation eigenfunctions.

6 The orientation of the axis of pulsation

Mode alignment is induced by the dynamical anisotropy caused by rotation and the spots, which split the degeneracy with respect to azimuthal order $m$ of modes of like order $n$ and degree $l$. The associated modifications to the pulsations are probably globally quite small, although in the surface regions they are certainly significant. If spherical symmetry were broken only by rotation, the principal axis of the modes would be the axis of rotation – that is to say, with respect to spherical polar coordinates $(r, \theta, \phi)$ about that axis the eigenfunctions $\psi_{nml}(r, \theta, \phi, t)$ could be written in the approximate form:
The main influence of the rotation is to assert that alignment and, in addition, to modify the eigenfrequencies and the eigenfunctions somewhat. It is normal to determine the modifications via a degenerate (singular) perturbation expansion about the nonrotating state. What follows is a description, not a derivation, of that theory.

In view of the variational principal that is satisfied by the eigenfrequencies, the first-order correction to the eigenfrequencies can be written in the form

$$\omega_{1m} = mC\Omega,$$

where $\Omega$ is the angular velocity of the star and $C$ is a factor which is independent of $m$ and which may be written as an integral depending on the zero-order eigenfunctions, and the unperturbed density $\rho$ (Cowling and Newing, 1948; Ledoux, 1951). It is given by

$$C = \frac{1}{I} \int \left( \xi^2 + \eta^2 \right) \rho r^2 dr,$$

where $L = l + \frac{1}{2}$ and $I = \int \left( \xi^2 + \eta^2 \right) \rho r^2 dr$ is a measure of the inertia of the mode, $\xi$ and $\eta$ being respectively the $r$-dependent factors in the vertical and horizontal displacement eigenfunctions. For the high-order acoustic modes of particular relevance to this discussion, $|\eta|/|\xi| = O(n^{-1})$, and both $\xi$ and $\eta$ are oscillatory functions (with approximately $n$ nodes) that are roughly $\pi/2$ out of phase; therefore $C = O(n^{-2})$.

The second-order corrections have been discussed by, for example, Gough and Thompson (1990) and Dziembowski and Goode (1992); they cannot distinguish between east and west, and therefore they do not depend on the sign of $m$. For high-order acoustic modes, at least, the centrifugal distortion of the star dominates the second-order correction; consequently the relative second-order correction $\omega_{2}/\omega_0$ (of, say, the $m = 0$ mode), where $\omega_0$ is the eigenfrequency of the corresponding unperturbed mode, is only weakly dependent on $n$, and tends to a finite constant as $n \to \infty$. In fact, $\omega_{2} \sim n\pi \left( \Omega^2 / \sigma^2 \right) D\sigma$, where $\sigma^2 = GM/R^3$ is the characteristic global dynamical frequency of the star and $D$ is a dimensionless $m$-dependent integral of order unity, which depends on (predominantly) the zero-order oscillation eigenfunctions.

It is convenient to separate the frequency perturbation into its mean (over $m$) value $\overline{\omega}_{2}$ and an $m$-dependent residual $\omega_{2m}$ satisfying $\sum_{m} \omega_{2m} = 0$. The mean value arises principally from the effect of the centrifugal force on the spherically averaged structure of the star, and, in the present state of the subject, cannot be distinguished from uncertainties in the structure of the corresponding nonrotating star, and so need not concern us here. If one specializes to dipole modes, which are the most pertinent to roAp stars, one can write $\omega_{20} = \omega_{2}$, and $\omega_{2m} = -\frac{1}{2}\omega_{2}$ when $m = \pm 1$.

Roughly speaking, the differential equation describing the oscillation modes $(\Psi_k, \omega_k)$ can be written in the form

$$\mathcal{L}\Psi_k + R\Psi_k - \lambda_k\Psi_k = 0,$$

where $\Psi_k$ represents eigenfunction $k$ of the $(2l + 1)$-fold multiplet of modes of given $n$ and $l$, and $\lambda_k = \omega_k^2$ is the square of its eigenfrequency; $\mathcal{L}$ is the differential operator describing the oscillations of the corresponding nonrotating star and $R$ is the operator describing the effect of rotation. For the purposes of this part of the discussion it is adequate to adopt the adiabatic approximation, in which case

\[21\]
is self-adjoint. (Generalization to nonadiabatic pulsations is straightforward, but because one must then deal with a non-self-adjoint operator the argument is a little more cumbersome.) To leading order in the rotational perturbation, eigenfunctions $\Psi_k$ can be expressed as a linear combination of the (orthogonal) zero-order eigenfunctions $\Psi_{nlm}^{(0)}$:

$$\Psi_k = \sum_m a_{km} \Psi_{nlm}^{(0)}, \quad \mathcal{L} \Psi_{nlm}^{(0)} - \lambda_k^{(0)} \Psi_{nlm}^{(0)} = 0,$$

which are normalized such that $\langle \Psi_{nlm}^{(0)} \Psi_{nlm}^{(0)*} \rangle = \delta_{m'm}$, where the angular brackets denote weighted integration over the star with the weight function with respect to which $\mathcal{L}$ is self-adjoint, and the asterisk denotes complex conjugate.

Substituting equations (22) into equation (21), writing $\lambda_k^{(1)} = \lambda_k - \lambda_k^{(0)}$, premultiplying by $\Psi_{nlm}^{(0)*}$ and integrating yields

$$\mathcal{R} a_k - \lambda_k^{(1)} a_k = 0,$$

where $\mathcal{R}$ now represents the matrix with components $\mathcal{R}_{m'm} = \langle \Psi_{nlm}^{(0)*} \mathcal{R} \Psi_{nlm}^{(0)} \rangle$ and $a_k$ is the $(2l + 1)$-dimensional vector with components $a_{km}$. In view of the discussion in the previous paragraph, $\mathcal{R}$ is diagonal, and in the simple case of dipole modes,

$$\mathcal{R} = -e\Omega A + \omega\Omega^2 B,$$

where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

The total effect of rotation on a group of modes of oscillation of like $n$ and $l$ can be summarized thus: The geometrical structure of each of the modes is perturbed only slightly, but the orientations of the modes are constrained relative to the axis of rotation. The frequencies are perturbed by centrifugal distortion by an amount that depends on the magnitude of $m$, but centrifugal Coriolis effects are proportional to $m$, and two modes of like $|m|$ (with the same amplitude), having structure proportional to $P_m^m(\cos \theta) \exp \left[-i \left(\omega_0 + \Omega t\right)t - im \left(\phi + C\Omega t\right)\right]$ and $P_m^m(\cos \theta) \exp \left[-i \left(\omega_0 + \Omega t\right)t + im \left(\phi + C\Omega t\right)\right]$ can be combined to yield two essentially identical standing modes having structure proportional to $P_m^m(\cos \theta) \cos \left[(\omega_0 + \Omega t) t \sin m \left(\phi + C\Omega t\right)\right]$, which, as a result of the Coriolis force, precess about the axis of rotation with angular velocity $-C\Omega$.

The effect of the asphericity associated with the spots is formally similar to that produced by the centrifugal force. If the star were not rotating, the modes would be aligned with the spots, and could not distinguish between east and west. Therefore, with respect to spherical polar co-ordinates $(r, \theta', \phi')$ about the spot axis, the eigenfunctions are given approximately by $\Psi_{nlm'}(r, \theta', \phi', t) =$
\[ \psi_{nl}(r) P^m_l(\cos \theta') \exp[i(m' \phi' - \omega t)], \]
where \( \omega = \omega_0 + \omega_s + \omega_m \) and \( \sum_m \omega_m = 0 \).
For dipole modes, \( \omega_0 = \omega_s \) and \( \omega_m = -\frac{1}{2} \omega_s \).
By analogy with the centrifugal effects, if \( M' \) is the matrix corresponding to \( R \) associated with the spot, but referred to the axis of the spots, then \( M' \) is diagonal, and is given by \( M' = \omega_B \).

When the effects of both the spots and the rotation are taken into account, the dipole eigenfunction can again be expressed to leading order as an axisymmetric dipole \( \cos \theta' \) with respect to polar coordinates \( (\theta', \phi') \) about an axis whose orientation is determined by the relative strengths of the perturbations.
If Coriolis effects were ignored, then, by symmetry, that axis would lie in the plane of the axis of rotation and the axis of the spots, inclined, say, by an angle \( \alpha \) from the axis of the spots. I call this axis \( \text{OP} \). Then if \( R \) represents the matrix that rotates \( M' \) by \( \alpha \) from the axis of the spots to the pulsation axis, and \( R' \) the matrix that rotates \( R \) from the rotation axis to the pulsation axis, the matrix \( R'^t M' R' + \lambda^{(1)}_\alpha I \), where \( I \) is the unit matrix and the superscript \( t \) represents transpose, is diagonal. The axis of pulsation is most easily found as the direction of the appropriate eigenvector of \( R + M \), referred to the rotation axis, or \( R' + M' \) referred to the spot axis. When the (small) Coriolis effects are included, the pulsation axis \( \text{OP} \) deviates slightly from \( \text{OP} \) (Figure 7). There is an axisymmetric dipole mode aligned with that axis, and two perpendicular linearly independent dipole oscillations precessing about it. Further details are given by Bigot and Dziembowski (2002).

It has usually been assumed that the spot perturbation is much greater in magnitude than the centrifugal perturbation: \( |\omega_s| \gg |\omega_{t2}| \). In that extreme it is evident that rotation has only a small influence on the dynamics, and there is always a mode that is almost aligned with the spots. That is essentially the oblique-rotator model discussed by Kurtz (1982), and is more-or-less what I had in mind when discussing the mode excitation in \$4 \) and \$5 \). Viewed in the frame of the rotating star, an axisymmetric mode of degree \( l \) and frequency \( \omega \) in the frame of the rotating star can be represented as a sum of oscillations of the form \( \sum_m A_m P^m_l(\cos \theta) e^{i(m \phi - \omega t)} \) about the axis of rotation, in which the (normalized) coefficients \( A_m \) depend solely on \( m \) (and \( l \)), and the inclination \( \alpha \) of the spot axis from the rotation axis. Viewed from the earth, the different contributions to the sum have different frequencies, approximately equal to \( \omega + m \Omega \), and can therefore be distinguished in a power spectrum of the observed oscillation signal. Their observed relative amplitudes \( a_m/a_0 \), which depend on \( A_m \), and the inclination \( \beta \) of the rotation axis relative to the line of sight, constrain \( \alpha \) and \( \beta \), and provide a consistency check of the oblique-rotator description (Kurtz and Shibahashi, 1986): if the description is correct, then \( a_{-1} = a_{+1} \).

It seems however, that not all roAp stars are that simple. In particular, it appears that in HR 3831, \( a_{-1} \neq a_{+1} \) (Kurtz et al., 1997). Shibahashi and Takata (1993) and Bigot and Dziembowski (2002) have attributed that inequality to the influence of the Coriolis effects. They pointed out that if the star is rotating rapidly enough the spot-induced perturbations do not overwhelm the rotational perturbations, and the axis of pulsation deviates substantially from the spot axis. Then Coriolis effects cause the amplitudes of the signal associated with the
$m = -1$ and the $m = +1$ components to differ. Moreover, there are additional components in the spectrum arising from the distortion of the eigenfunctions away from their zero-order spherically harmonic state (considered, for example, by Shibahashi and Takata, 1993, and Takata and Shibahashi, 1995, although they did not take centrifugal effects into account), the details of which I shall not discuss here.

When centrifugal effects are significant, the axis of the principal (nonprecessing) mode is no longer well aligned with the spots, and therefore the $\kappa$ mechanism is less effectual in exciting the oscillations (see Figure 6). Perhaps this is why roAp stars are generally observed to be slow rotators. In the extreme case when $|\omega_s| \ll |\omega_{\Omega_2}|$ and, as in Figure 2, $\alpha \approx \pi/2$, the principal mode is almost aligned with the rotation axis. But there are also two other linearly independent dipole modes, which precess about that axis, and whose axes of symmetry therefore pass through the spots. This is the case that was considered by Dolez and Gough (1982), who suggested that if the modes are indeed excited in the spots (Dolez and Gough did not actually find net excitation, principally because of a flaw in the opacity tables available at that time), when they are aligned, then, because the growth time when they are aligned and the decay time when they are not aligned are both likely to be much shorter than the time it takes for the mode to precess across the spot, the modes will always be found approximately aligned with the spots and might thereby be presumed not to be precessing. That possibility was dismissed by Dziembowski and Goode (1985) as being unnecessarily complicated, but maybe in the future it will be found that in some stars such complication is present.

7 Conclusion

It is my opinion that a theory of the roAp-star phenomenon is beginning to emerge. The stars are slow rotators. They are spotted as a result of there being a large-scale predominantly dipole magnetic field pervading the star, inclined from the axis of rotation, which suppresses subsurface convection to form two antipodal spots where the field is almost vertical. This permits the establishment of chemical abundance anomalies brought about a combination of radiative levitation and gravitational settling against diffusion. There might also be a stellar wind from the spot, where the magnetic field lines are effectively open. This would modify the chemical abundances, as also might transport by fingering in regions where the mean molecular mass increases upwards, although this process, unfortunately, is not well understood. The oscillations are low-degree p models of high order, which are probably axisymmetric with axes more-or-less aligned with the spots. They are excited by the $\kappa$ mechanism in the spots, and damped by convection elsewhere. Only a few high-order modes are overstable, in accord with observation. Some calculations have found overstability also in some relatively-low-order modes, with much lower growth rates; perhaps such oscillations are present too, but at amplitudes too low yet to have been observed.

An important matter that has never been seriously addressed is an explana-
tion of the amplitudes of the modes. If the modes are overstable, their growth must be limited by some nonlinear process. Is that process similar to that which limits the growth of the oscillations of Cepheids and RR Lyrae stars? If so, why does it operate effectively at so low and amplitude?

Only when the spots are sufficiently large, and the rotation of the star is sufficiently small, can axisymmetric oscillation modes be dynamically locked with their axes close enough to the spots for excitation to dominate over damping, causing overstability; nonaxisymmetric modes precess, due to Coriolis effects, and cannot be firmly locked. Because the oscillation amplitude is greatest on the axis of symmetry, overstability is most likely to result when the modes are aligned with the spots. Of course, the larger the spots, the less precise need be the alignment, and in the fullness of time a robust theoretical criterion for overstability might be found: a relationship between spot size and inclination of the oscillation axis from the spots. Nonaxisymmetric modes have lower growth rates, and are overstable only if the spots are very large; that they appear not to be observed would set an upper limit to the size of the spots, once a robust theory is available.

In some circumstances the stellar rotation might be so large that the principal axis of pulsation is more-or-less aligned with the rotation axis, and not with the spots; in that case, it might be that other modes of the same degree precess slowly across the spot and become overstable only when they are appropriately orientated, giving the impression of dynamical alignment.

The theoretical models are at present extremely primitive. The influence of the spots on the pulsations has been accommodated in a piecemeal way, and must in future be properly incorporated into the pulsation dynamics to produce a plausible quantitative model. There has been substantial progress in studying the influence of the Lorentz forces on the pulsations, but only in stellar models without spots. Lateral inhomogeneity of the background state, convection-pulsation interactions, Lorentz forces and nonadiabaticity must all be combined in a consistent way. And our understanding of the dynamics of the oscillations in the atmosphere, where nonlinearity in the dynamics can be important, and where the processes of amplitude limitation are probably the most effectual, must be improved. Only then can one reliably use future observations of mode frequencies and amplitudes, intensity-velocity phase relations and nonlinearities in the light-curves to draw reliable inferences about the structure of the stars. We have many (but not all) of the ingredients in hand. Therefore a preliminary theory might be almost in sight.

8 Acknowledgements

I am very grateful to J. Gilbert and D. Sword for preparing the typescript, and to R. Sword and G. Houdek for their help in preparing the diagrams.
9 references

Bagnulo, S., Landolfi, M. and Landi Degl’Innocenti, M. 1999, Astron.
Astrophys., 343, 865–871

Balmforth, N.J., Cunha, M.S., Dolez, N., Gough, D.O. and Vauclair, S. 2001,
Mon. Not. R. Astron. Soc., 323, 362–372

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud, R. M. 1958, Proc.
Roy. Soc., A 244, 17–40

Bigot, L. 2003, International Conference on magnetic fields in O, B and A stars,
(ed. L.A. Balona, H.F. Henrichs & T. Medupe), Astron. Soc. Pacific
Conf. Ser., 305, 73–82

Bigot, L. and Dziembowski, W.A. 2002, Astron. Astrophys., 391, 235–245

Bigot, L., Provost, J., Berthomieu, G., Dziembowski, W.A. and Goode, P.R.
2000, Astron. Astrophys., 356, 218–233

Blanchflower, S. and Weiss, N. 2002, Phys. Lett. A, 294, 297-303

Campbell, C. G. and Papaloizou, J. C. B. 1986, Mon. Not. R. Astron. Soc.,
220, 577-591

Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford
Univ. Press)

Cowling, T.G. and Newing, R.A. 1949, Astrophys. J., 109, 149–158

Cunha, M. S. 1998, Contrib. Astron. Obs. Skalnate Pleso, 27, no 3, 272-279

Cunha, M.S. 2001, Mon. Not. R. Astron. Soc., 325, 373–378

Cunha, M.S. and Gough, D.O. 2000, Mon. Not. R. Astron. Soc., 319, 1020–
1038

Dolez, N. and Gough, D.O. 1982, Pulsations in classical and cataclysmic
variable stars (ed. J.P. Cox &C.J. Hansen, JILA, Boulder), pp 248–256

Dziembowski, W.A. and Goode, P.R. 1985, Astrophys. J., 296, L27–L30

Dziembowski, W.A. and Goode, P.R. 1992 Astrophys. J., 394, 670–687

Dziembowski, W.A. and Goode, P.R. 1996, Astrophys. J., 458, 338–346

Gautschy, A. and Saio, H. 1998, Mon. Not. R. Astron. Soc., 301, 31–41

Gough, D.O. 2002 Radial and Nonradial Pulsations as Probes of Stellar Physics,
( Proc. IAU Colloq., 185, ed. C. Aerts, T.R. Bedding & J. Christensen-
Dalsgaard), Astron. Soc. Pacific Conf. Ser., 259, 37–57

Gough, D. O. and Tayler, R. J. 1966, Mon. Not. R. Astron. Soc. 133, 85

Gough, D.O. and Thompson, M.J. 1990, Mon. Not. R. Astron. Soc. 242, 25–55

Kochukhov, O., Drake, N. A., Piskunov, N. and de la Reza, R. 2004, Astron.
Astrophys., 424, 935-950

Kurtz, D. W. 1982, Mon. Not. R. Astron. Soc., 200, 807-859

Kurtz, D. W. 1990, Ann. Rev. Astron. Astrophys., 28 (A91-28201 10-90)

Kurtz, D. W. and Shibahashi, H. 1986, Mon. Not. R. Astron. Soc., 223, 557

Kurtz, D. W., Kanaan, A., Martinez, P. and Tripe, P. 1992, Mon. Not. R.
Astron. Soc., 255, 289

Kurtz, D. W. et al. 1997, Mon. Not. R. Astron. Soc., 287, 69–78

Kurtz, D. W. et al. 2002, Mon. Not. R. Astron. Soc., 330, L57-L61

Lebovitz, N. R. 1965, Astrophys. J., 142, 229-242
Lebovitz, N. R. 1966, Astrophys. J., 146, 946-949
Ledoux, P. 1951, Astrophys. J., 114, 373–384
Martinez, P. and Kurtz, D. W. 1995, Astrophysical applications of stellar pulsation (ed R. S. Stobie & P.A. Whitelock) Proc. IAU Colloq. 155, Astron. Soc. Pacific Conf. Ser., 83, 58
Montgomery, H.M. and Gough, D.O. 2003, Asteroseismology across the HR diagram, (ed. M.J. Thompson, M.S. Cunha & M.J.P.F.G. Monteiro, Kluwer, Dordrecht), pp 545–548
Newcomb, W.A. 1961, Phys. Fluids, 4, 391–396
Roberts, P. H. and Soward, A. M. 1983, Mon. Not. R. Astron. Soc., 205, 1171-1189
Saio, H. and Gautschy, A. 2003. International Conference on magnetic fields in O, B and A stars, (ed. L.A. Balona, H.F. Henrichs & T. Medupe ), Astron. Soc. Pacific Conf. Ser., 305, 140–145
Saio, H. and Gautschy, A. 2004, Mon. Not. R. Astron. Soc., 350, 485-505
Schwarzschild, K. 1906, Nach. Kön. Gesellsch. Wiss. Göttingen, 195, 41–53
Shibahashi, H. and Takata, M. 1993, Publ. Astron. Soc. Japan, 45, 617–641
Takata, M. and Shibahashi, H. 1995, Publ. Astron. Soc. Japan, 47, 219–231
Tayler, R.J. 1961, J. Nucl. Energy, 3, 266–272
Vauclair, S., Dolez, N. and Gough, D.O. 1991 Astron. Astrophys., 252, 618–624