Proving hamiltonian properties in connected 4-regular graphs: an ILP-based approach

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Abstract

In this paper we study some open questions related to the smallest order \( f(C, \neg H) \) of a 4-regular graph which has a connectivity property \( C \) but does not have a hamiltonian property \( H \). In particular, \( C \) is either connectivity, 2-connectivity or 1-toughness and \( H \) is hamiltonicity, homogeneous traceability or traceability. A standard theoretical approach to these questions had already been used in the literature, but in many cases did not succeed in determining the exact value of \( f() \). Here we have chosen to use Integer Linear Programming and to encode the graphs that we are looking for as the binary solutions to a suitable set of linear inequalities. This way, there would exist a graph of order \( n \) with certain properties if and only if the corresponding ILP had a feasible solution, which we have determined through a branch-and-cut procedure. By using our approach, we have been able to compute \( f(C, \neg H) \) for all the pairs of considered properties with the exception of \( C = 1 \)-toughness, \( H = \)traceability. Even in this last case, we have nonetheless significantly reduced the interval \([LB, UB]\) in which \( f(C, \neg H) \) was known to lie. Finally, we have shown that for each \( n \geq f(C, \neg H) \) (\( n \geq UB \) in the last case) there exists a 4-regular graph on \( n \) vertices which has property \( C \) but not property \( H \).

Keywords—4-regular graph; hamiltonian graph; traceable graph; homogeneously traceable graph; 1-tough graph; Integer Linear Programming; branch-and-cut.

1 Introduction

It is well-known that both the problems of deciding whether a graph is traceable, i.e., it admits a Hamilton path, and is hamiltonian, i.e., it admits a Hamilton cycle, are NP-complete \[10\] even when the graph is \( k \)-regular with \( k \geq 3 \) \[22\]. Similar problems concern the fact that a graph contains a Hamilton path starting at each vertex or Hamilton paths between each pair of vertices. In the former case the graph is called homogeneously traceable, in the second case Hamilton-connected.
All the above problems are difficult also for regular graphs and this fact gave rise to a wide search for conditions that are necessary and/or sufficient to guarantee a given hamiltonian property (see for instance [5, 6, 12, 26] for background and general surveys on the problems and the papers [2, 9, 13, 15] considering regular graphs). Clearly connectivity is a basic necessary condition for a graph satisfying each of the above properties; moreover, every hamiltonian graph must be 2-connected, i.e., at least two vertices have to be removed to disconnect the graph. A stronger necessary property for a graph to be hamiltonian was introduced by Chvátal [8] and is called 1-toughness. Given $t \in \mathbb{R}$, $t > 0$, a graph $G = (V, E)$ is $t$-tough if for each set of vertices $S$ whose removal disconnects the graph the number of connected components of the graph induced by $V \setminus S$ is at most $|S|/t$. As it is easy to verify, every hamiltonian graph is 1-tough, but the reverse statement does not hold in general. On the other hand, sufficient conditions are often based on the fact that the degree of the vertices of the graph is sufficiently high to guarantee a given hamiltonian property. In particular, when $k$-regular graphs are considered, no one of these properties may be guaranteed as the number of vertices increases. The issue of establishing the minimum order of a $k$-regular graph that has a given connectivity property $C$, i.e., is either 1-tough or 2-connected or simply connected but does not satisfy a given hamiltonian property $H$, i.e., is not Hamilton-connected or is not hamiltonian or is not homogeneously traceable or is not traceable has been considered in several papers. In particular, relevant theoretical results determine lower bounds for these minimum orders. Two relevant conditions that guarantee that a graph is either Hamilton-connected or hamiltonian were stated by Ore in the following result.

**Theorem 1.1.** (Ore [19]) Let $G$ be a graph with $n \geq 3$ vertices and let $d(v)$ denote the degree of vertex $v$. If for any pair of nonadjacent vertices $v$ and $w$ it holds that:

i) $d(v) + d(w) \geq n + 1$, then $G$ is Hamilton-connected;

ii) $d(v) + d(w) \geq n$ then $G$ is hamiltonian.

The above theorem implies, in particular, that every $k$-regular graph is Hamilton-connected if it has order $n \leq 2k - 1$ and is hamiltonian if $3 \leq n \leq 2k$.

The two stronger results concerning regular graphs are the following.

**Theorem 1.2.** (Cranston and Suil [9]) i) Every connected $k$-regular graph with at most $2k + 2$ vertices is hamiltonian. Furthermore, all connected $k$-regular graphs on $2k + 3$ vertices (when $k$ is even) and $2k + 4$ vertices (when $k$ is odd) that are nonhamiltonian can be characterized.

ii) Every connected $k$-regular graph with at most $3k + 3$ vertices has a Hamilton path. Furthermore, all connected $k$-regular graphs on $3k + 4$ vertices (when $k \geq 6$ is even) and $3k + 5$ vertices (when $k \geq 5$ is odd) that have no Hamilton path can be characterized.

**Theorem 1.3.** (Hilbig [13]) Let $G$ be a 2-connected, $k$-regular graph with at most $3k + 3$ vertices. Then $G$ is hamiltonian or $G$ is the Petersen graph $P$ or $G$ is the 3-regular graph obtained from $P$ by replacing one vertex with a triangle.

The above theorems do not cover some issues concerning 4-regular graphs. By Theorem 1.3 each 2-connected graph with $n \leq 15$ is hamiltonian and thus homogeneously traceable and traceable. On the other hand, as shown in [2], there exists a 4-regular 1-tough graph with 18 vertices which is not hamiltonian. These facts leave open the following question: Is it true that every 4-regular, 1-tough graph with at most 17 vertices is hamiltonian? Bauer, Broersma and Veldman conjectured in [2] that this should be the case. We also observe that Theorem 1.2 does not determine the minimum order of a connected $k$-regular graph which is not traceable when $k = 4$. These
two facts seem to suggest that to answer the above questions (and similar open questions involving other pairs of connectivity conditions and hamiltonian properties) for 4-regular graphs one might need a different approach than a “standard” mathematical proof. In this paper, extending our preliminary work [16], we address the issue by adopting an Integer Linear Programming (ILP) approach. We have proceeded as follows.

For each connectivity property \( C \in \{1\text{-toughness (1t), 2-connectivity (2c), connectivity (c)}\} \) and each hamiltonian property \( H \in \{\text{Hamilton-connectivity (HC), hamiltonicity (H), homogeneous traceability (HT), traceability (T)}\} \) we consider the problem \( P(n, C, \neg H) \) defined as

**Problem** \( P(n, C, \neg H) \): does there exist a 4-regular graph with \( n \) vertices satisfying property \( C \) and not satisfying property \( H \)?

For each pair of properties \( (C, H) \), we call any 4-regular graph having property \( C \) but not property \( H \) a \( (C, \neg H) \)-graph and denote by \( f(C, \neg H) \) the minimum \( n \) for which problem \( P(n, C, \neg H) \) has a positive answer, i.e., the minimum number of vertices in a \( (C, \neg H) \)-graph. For each unknown value \( f(C, \neg H) \) we have formulated problem \( P(n, C, \neg H) \) as an ILP problem whose feasible solutions correspond to the \( (C, \neg H) \)-graphs with \( n \) vertices. Then we have solved the problem for increasing values of \( n \) (chosen in a suitable range) so that \( f(C, \neg H) \) was determined as the minimum \( n \) for which the ILP model admits a feasible solution. Our computations allowed to almost complete Table 1 where the values \( f(C, \neg H) \) determined using our approach are written in bold and the bounds previously known are written in normal font. In particular, we have shown that the question posed by Bauer, Broersma and Veldman has a positive answer and that every connected 4-regular graph with less than 18 vertices is traceable. Furthermore, from our results it follows that for each considered pair of properties \( C, H \) and for every \( n \geq f(C, \neg H) \) there exists a \( (C, \neg H) \)-graph with \( n \) vertices. The only value that remains undetermined is \( f(1t, \neg T) \), i.e., the minimum order of a 4-regular 1-tough graph which is not traceable. However, even for this case, we were able to restrict the range to which this value belongs. We remark that the three values \( f(C, \neg HC) \), that we report in Table 1 for sake of completeness, were already known.

As it is well known, a feasibility ILP problem consists in finding an integer solution to a finite set of linear inequalities. Since the number of inequalities required to model every problem \( P(n, C, \neg H) \) happens to be exponential in \( n \), all the models were solved by using a branch and cut procedure.

| Hamilton-connectivity | hamiltonicity | homogeneous traceability | traceability |
|-----------------------|---------------|--------------------------|--------------|
| connectivity          | \( f(c, \neg HC) = 8 \) | \( f(c, \neg H) = 11 \) | \( f(c, \neg HT) = 11 \) | \( f(c, \neg T) \geq 16 \) |
| 2-connectivity        | \( f(2c, \neg HC) = 8 \) | \( f(2c, \neg H) \geq 16 \) | \( f(2c, \neg HT) \geq 16 \) | \( f(2c, \neg T) \geq 16 \) |
| 1-toughness           | \( f(1t, \neg HC) = 8 \) | \( 16 \leq f(1t, \neg H) \leq 18 \) | \( f(1t, \neg HT) \geq 16 \) | \( f(1t, \neg T) \geq 16 \) |

Table 1: Known and new bounds for the minimum order \( f(C, \neg H) \) of a 4-regular graph that satisfies property \( C \) but not property \( H \). The new bounds appear in bold.
Despite the values of $n$ used in the computations are relatively small (always less than 22), the dimension and the structure of the ILP models make their straightforward solution impossible in a reasonable time. For this reason we have adopted two fundamental strategies to reduce the computation times: a preliminary analysis that allowed us to conveniently split each model in few subproblems in which some variables may be fixed and the use of a symmetry-breaking technique called orbital branching [20] to reduce the symmetry of the subproblems.

The use of ILP as a technique to design a combinatorial object with given properties (such as, for instance, a counterexample to some hypothesis that one might have formulated) is not new, but is not as popular as it should probably be. For instance, Pulaj et al. applied ILP to study the size of counterexamples to the union-closed set conjecture ([23, 24]), while Caprara et al. [7] used ILP to find counterexamples to a property that fractional bin packing solutions should satisfy when rounded up to integer. Finally, in [27], Trevisan et al. used ILP to build “gadgets” that can turn a combinatorial problem into another. Through these gadgets, the authors were able to construct instances which they used to improve the approximability/inapproximability factors of some important combinatorial optimization problems.

The remainder of the paper is organized as follows. In Section 2 we introduce the notation and recall some known results. In Section 3 we describe a preliminary analysis about 4-regular, 2-connected graphs that are not 1-tough. Also this analysis is done using an ILP method. In Section 4 we present our ILP models, the branch-and-cut procedure to solve them and some strategies required to obtain an effective procedure. The obtained results are described in Section 5. Section 6 is devoted to discuss the main implementation issues and the computational experiments. Finally, we draw some conclusions in Section 7.

2 Notation and known results

Let $G = (V, E)$ be an undirected graph. The graph is called $k$-regular if every vertex has degree $k$. For each $S \subseteq V$ we denote by $\partial(S)$ the set of edges of $G$ having an endpoint in $S$ and the other in $V \setminus S$. Moreover, we denote by $G[S]$ the subgraph of $G$ induced by $S$, i.e., the graph with vertex set $S$ and edge set $E(S)$, the set containing all the edges of $E$ with both endpoints in $S$.

The graph $G$ is called connected if it contains a path between each pair of vertices, and is called 2-connected if the graph $G[V \setminus \{i\}]$ is connected for each vertex $i \in V$. Let $c(G)$ denote the number of connected components of $G$. The graph $G$ is called $t$-tough, $t \in \mathbb{R}_+$, if for every subset $S \subseteq V$ with $c(G[V \setminus S]) > 1$ it is $|S| \geq tc(G[V \setminus S])$. In particular, $G$ is 1-tough if one cannot create $c$ components by removing less than $c$ vertices. Clearly, every 1-tough graph is 2-connected and thus connected. We remark that the problem of deciding if a graph is $t$-tough is NP-hard even for $t = 1$ [3] and for regular graphs [4]. For an excellent survey on toughness in graphs the reader is referred to the paper by Bauer, Broersma, and Schmeichel [1].

A Hamilton cycle (or path) of $G$ is a cycle (respectively, a path) that visits each vertex of $V$ exactly once. A graph $G$ is called traceable if it contains a Hamilton path and is called hamiltonian if it contains a Hamilton cycle. Moreover, $G$ is called homogeneously traceable if for each vertex $i \in V$ it contains a Hamilton path beginning at $i$ and is called Hamilton-connected if for each pair of vertices $i, j \in V$ it contains a Hamilton path starting at $i$ and ending in $j$. The next claims collect some properties that immediately follow from the above definitions.
Fact 2.1. Let $G$ be a graph with at least three vertices. Then: if $G$ is Hamilton-connected then $G$ is hamiltonian, if $G$ is hamiltonian then $G$ is homogeneously traceable and if $G$ is homogeneously traceable then $G$ is traceable.

Fact 2.2. Every homogeneously traceable graph with at least three vertices is 1-tough.

Proof. Assume that the graph $G = (V, E)$ is homogeneously traceable and, given a nonempty $S \subset V$, let $P$ be any Hamilton path beginning at a vertex of $S$. Since by removing from $P$ the vertices of $S$ one obtains at most $|S|$ subpaths of $P$ and each node of $V \setminus S$ lies on exactly one of these subpaths, the graph $G[V \setminus S]$ has at most $|S|$ connected components. 

Let us now consider the hamiltonian properties of the 4-regular graphs and, in particular, what is already known about the minimum order $f(C, \neg H)$ of a 4-regular graph that satisfies property $C \in \{\text{connectivity (c), 2-connectivity (2c), 1-toughness (1t)}\}$ and does not satisfy the property $H \in \{\text{Hamilton-connectivity (HC), hamiltonicity (H), homogeneous traceability (HT), traceability (T)}\}$. As far as property $HC$ is concerned, it is easy to verify that the complete bipartite graph $K_{4,4}$ is 1-tough and does not contain any Hamilton path connecting two nonadjacent vertices, thus $f(1t, \neg HC) \leq 8$. On the other hand, by Theorem 1.1 i), $f(C, \neg HC) > 7$ for any considered connectivity property $C$. This implies $f(C, \neg HC) = 8$ for any property $C$. With regards to the other hamiltonian properties, we observe that any 2-connected 4-regular graph of order at most 15 is hamiltonian by Theorem 1.3 and, by Fact 2.1 this implies $f(C, \neg H) \geq 16$ for each $C \in \{2c, 1t\}$ and $H \in \{H, HT, T\}$. Moreover, $f(c, \neg H) \geq 11$ and $f(c, \neg T) \geq 16$ by Theorem 1.2. Since the nonhamiltonian graph with 11 vertices in Fig. 1 (reported in [9]) does not contain any Hamilton path starting at vertex $v$, we may conclude that $f(c, \neg HT) = f(c, \neg H) = 11$. Finally, the upper bound 18 for $f(1t, \neg H)$ is due to the 1-tough but not hamiltonian graph in Fig. 2 which has been proposed in [2].

Thanks to the above remarks we can fill Table 1 with the known lower and upper bounds on the values $f(C, \neg H)$ (reported not in bold).

![Figure 1: Connected non-homogeneously traceable graph with $n = 11$.](image1.png)

![Figure 2: 1-tough nonhamiltonian graph with $n = 18$.](image2.png)
3 Replacing 1-thougness by 2-connectivity: a preliminary analysis

Our strategy is based on the use of ILP to model each problem $P(n,C,\neg H)$, i.e., the problem to find if there exists a 4-regular graph satisfying property $C$ but not property $H$. The variables of the model represent the edges of the sought graph. While it is easy to state a set of constraints which imply that a graph is 2-connected (or just connected), dealing with the constraints which enforce a graph to be 1-tough is not a simple task. Indeed, to determine if a graph is 1-tough is NP-complete \[\text{\cite{1}}\]. Since every 1-tough graph is 2-connected, even when solving problems $P(n,1t,\neg H)$ it is then convenient to solve the relaxed problem $P(n,2c,\neg H)$. If the search fails, one can conclude that $f(1t,\neg H)$, as well as $f(2c,\neg H)$, is larger than $n$. Otherwise, if the model succeeds and finds a 2-connected graph which is not 1-tough, one should add suitable constraints to make this graph infeasible and continue the search for a 1-tough graph. As a final remark we observe that, since by Theorem \[\text{\cite{1}}\] every 2-connected 4-regular graph with $n \leq 15$ is 1-tough, one may expect that for slightly larger values of $n$ the 2-connected not 1-tough graphs are quite few and may be characterized. The knowledge of these graphs will be usefully exploited to reduce the computational effort required to solve problem $P(n,1t,\neg H)$, specially for $n=16,17$.

This preliminary analysis has the objective to study if it is possible to have 4-regular, 2-connected graphs with $n$ vertices which are not 1-tough and, in this case, to characterize their structure (in the following we call a graph with these properties a $\varphi$-graph). Also this analysis has been carried out using an ILP approach. Given $n$ and $k=2,\ldots,\lfloor \frac{n}{2} \rfloor$, let us denote by $v(n,k)$ the maximum number of connected components that can result by removing $k$ vertices from a 4-regular 2-connected graph with $n$ vertices. Clearly, if $v(n,k) \leq k$ for every $k$, then every 2-connected graph with $n$ vertices is 1-tough. Otherwise, at least for $n=16,17$, the structure of the $\varphi$-graphs can be easily determined.

Before presenting the ILP model, let us outline some simple properties. Given a 4-regular 2-connected graph $G=(V,E)$ and a subset $S \subseteq V$ with $|S|=k$, let $W_1,\ldots,W_t$ be the vertex-sets of the $t$ connected components of the graph $G[V \setminus S]$ and $n_r := |W_r|$, for $r=1,\ldots,t$.

**Proposition 3.1.** For each $r=1,\ldots,t$ it is $n_r \geq 5-k$.

**Proof.** Assume $n_r \leq 4-k$ for some $r$. Since each vertex $v$ of $W_r$ has degree 4 and can be adjacent to at most $n_r - 1 \leq 3-k$ vertices of $W_r$, $v$ must be adjacent to at least $k+1$ vertices in $S$, a contradiction. \[\square\]

**Proposition 3.2.** For each $r=1,\ldots,t$ it is $|\partial(W_r)| \geq m_r$ with $m_r := \max\{2, n_r(5-n_r)\}$. This in particular implies $\sum_{r=1}^{t} m_r \leq 4k$.

**Proof.** The 2-connectivity of $G$ implies $|\partial(W_r)| \geq 2$. If $n_r \leq 4$, each vertex of $W_r$ must be adjacent to at least $5-n_r$ vertices of $S$, so $|\partial(W_r)| \geq m_r$. Since $\sum_{r=1}^{t} |\partial(W_r)| = |\partial(S)| \leq 4k$ the second statement holds. \[\square\]

For each $n$ and $k$ we can compute an upper bound $v'(n,k)$ to the value $v(n,k)$ by solving the following ILP problem. Let $x_i$ be an integer variable representing the number of components of cardinality $i$ in the graph $G[V \setminus S]$ and $m_i := \max\{2,5i-i^2\}$. By Proposition 3.1 we can assume that $i$ goes from $s(k) := \max\{1,5-k\}$ to $n-k$. Let us consider the model $Q_{n,k}$:
The unique 4-regular 2-connected graph with $n = 16$ that is not 1−tough.

$$v'(n, k) := \max_{i=s(k)} \sum_{i=s(k)}^{n-k} x_i$$  \hspace{1cm} (1)

$$\sum_{i=s(k)}^{n-k} i x_i = n - k$$  \hspace{1cm} (2)

$$\sum_{i=s(k)}^{n-k} m_i x_i \leq 4k$$  \hspace{1cm} (3)

$$x_i \in \mathbb{N} \quad \forall i = s(k), \ldots, n - k.$$  \hspace{1cm} (4)

The objective function counts the number of components of the graph $G[V \setminus S]$, the constraints 2 state that the total number of vertices in these components must be $n - k$ and the constraints 3 require that the property stated in Proposition 3.2 is satisfied. If there exists a 2-connected graph with $n$ vertices which is not 1-tough, then it must be $v'(n, k) > k$ for some $k$.

By Theorem 1.3 every 2-connected, 4-regular graph with $n \leq 15$ is 1-tough. So, in order to close the conjecture by Bauer, Broersma and Veldman, we first focused on the cases $n = 16$ and $n = 17$.

By solving problem $Q_{n,k}$ for $n = 16$, it turns out that $v'(16, k) > k$ only for $k = 2$, in which case it is $v'(16, 2) = 3$. The optimal solution is $x^*_4 = 1, x^*_5 = 2, x^*_i = 0$ for $i \neq 4, 5$. It is easy to verify that there is just one $\varphi$-graph compatible with this solution, namely the graph in Figure 3. Note that if we remove the vertices in $S = \{v_1, v_2\}$ we obtain a graph with 3 components, one with 4 vertices and two with 5 vertices. By solving again the problem $Q_{16,2}$ with the additional constraint $x_3 \geq 1$ or the problem $Q_{16,2}$ with the additional constraint $\sum_{i \geq 6} x_i \geq 1$, we obtain optimal value 2. This means that $x^*$ is the unique solution of $Q_{16,2}$.

A similar analysis for $n = 17$ allows to identify seven $\varphi$-graphs with $n = 17$. Indeed, when solving problem $Q_{n,k}$ for $n = 17$, it turns out that $v'(17, k) > k$ for $k = 2, 4$ with optimal values, respectively, $v'(17, 2) = 3$ and $v'(17, 4) = 5$. In particular, one optimal solution of problem $Q_{17,2}$ is $x^*_3 = 3$ and $x^*_i = 0$ for $i \neq 5$. This solution determines the four graphs in Figure 4. By solving again $Q_{17,2}$ with the additional constraint $x_3 + x_4 \geq 1$ one obtains a different solution $\bar{x}$ of value 3 with $\bar{x}_4 = \bar{x}_5 = \bar{x}_6 = 1$ and $\bar{x}_i = 0$ for $i \neq 4, 5, 6$. This solution is compatible only with the two graphs in Figure 5. The solutions $x^*$ and $\bar{x}$ are the only solutions of $Q_{17,2}$ of value 3. Indeed, by adding to $Q_{17,2}$ either the constraint $x_3 \geq 1$ or the constraint $\sum_{i \geq 7} x_i \geq 1$ one obtains 2 as optimal value. Finally, the optimal solution found when solving $Q_{17,4}$ is $\tilde{x}$ with $\tilde{x}_1 = 3, \tilde{x}_5 = 2$ and $\tilde{x}_i = 0$, $i \neq 1, 5$. It has value 5 and corresponds to the graph in Figure 6. By solving again $Q_{17,4}$ with the additional constraint either $x_1 \leq 2$ or $x_1 \geq 4$ or $x_5 \leq 1$, one always obtains an optimal value at 7.
Figure 4: Four 4-regular 2-connected graphs with $n = 17$ that are not $1$–tough.

Figure 5: Two 4-regular 2-connected graphs with $n = 17$ that are not $1$–tough.

Figure 6: A 4-regular 2-connected graph with $n = 17$ that is not $1$–tough.

Figure 7: The graph $R_5$. 

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most 4. So \( \hat{x} \) is the unique optimal solution of problem \( \mathcal{Q}_{17,4} \).

We observe that all the eight graphs determined through the preliminary analysis contain one or more subgraphs isomorphic to the graph in Figure 7 which is obtained by removing one edge from the complete graph \( K_5 \). We call this graph \( R_5 \). This implies that any 2-connected 4-regular graph with \( n = 16 \) or \( n = 17 \) that does not contain any \( R_5 \) is 1-tough. This fact will play a role in reducing the computations needed to prove that \( f(1t, \neg H) = 18 \).

The preliminary analysis has been performed also for \( 18 \leq n \leq 20 \). For all these cases, we found that by removing \( k \) vertices the graph gets disconnected in at most \( k + 1 \) components. Defining \( K(G) := \{ k : \nu'(n,k) > k \} \) and denoting by \( r(G) \) the number of disjoint subgraphs of \( G \) isomorphic to \( R_5 \), the results obtained for \( 16 \leq n \leq 20 \) can be summarized as follows

- if \( n = 16 \) then \( r(G) = 2 \wedge K(G) = \{ 2 \} \);
- if \( n = 17 \) then \( (r(G) = 1 \wedge K(G) = \{ 2 \}) \lor (r(G) = 2 \wedge K(G) \subseteq \{ 2, 4 \}) \);
- if \( n = 18 \) then \( (r(G) = 1 \wedge K(G) \subseteq \{ 2, 4 \}) \lor (r(G) \in \{ 0, 2 \} \land K(G) = \{ 2 \}) \);
- if \( n = 19 \) then \( (r(G) \in \{ 0, 1 \} \land K(G) \subseteq \{ 2, 4 \}) \lor (r(G) = 2 \land K(G) \subseteq \{ 2, 5 \}) \);
- if \( n = 20 \) then

\[
(r(G) \in \{ 0, 2 \} \land K(G) \subseteq \{ 2, 4 \}) \lor (r(G) = 1 \land K(G) \subseteq \{ 2, 4, 5 \}) \lor (r(G) = 3 \land K(G) = \{ 3 \}).
\]

4 The ILP model for solving problem \( P(n, C, \neg H) \)

Let us consider the problems \( P(n, C, \neg H) \) defined in the introduction where

\[
\begin{align*}
C & \in \{ \text{connectivity, 2-connectivity, 1-toughness} \}, \\
\mathcal{H} & \in \{ \text{hamiltonicity, homogeneous traceability, traceability} \}.
\end{align*}
\]

We formulate any problem \( P(n, C, \neg H) \) as an ILP feasibility model \( \mathcal{M}(n, C, \neg H) \) whose feasible solutions correspond to those 4-regular graphs with \( n \) vertices that satisfy property \( C \) and do not satisfy property \( \mathcal{H} \). As a consequence, the value \( f(C, \neg H) \) corresponds to the minimum \( n \) for which model \( \mathcal{M}(n, C, \neg H) \) has a feasible solution. More in detail, let \( K^n = (V^n, E^n) \) denote the complete graph with \( V^n = \{ 1, \ldots, n \} \). We introduce a binary variable \( x_e \) for each edge \( e \in E^n \) and associate to any \( x \in \{ 0, 1 \}^{|E^n|} \) the graph \( G(x) = (V^n, E(x)) \) with edge-set \( E(x) = \{ e \in E^n : x_e = 1 \} \). The linear inequalities of the model are the following. The condition that \( G(x) \) is 4-regular is imposed by the family of \( n \) constraints, called degree constraints,

\[
\sum_{e \in \partial(i)} x_e = 4 \quad \forall \ i \in V^n
\] (5)

Let \( \mathcal{H}^n \) and \( \mathcal{P}_\ell(i) \) \( i \in V^n \), denote the set of the Hamilton cycles and, respectively, the set of the paths of length \( \ell \) starting at \( i \) in \( K^n \). Hence \( \mathcal{P}_{n-1}(i) \) denotes the set of the Hamilton paths starting at vertex \( i \). The condition that the graph \( G(x) \) is not hamiltonian can be imposed by the family of constraints

\[
\sum_{e \in H} x_e \leq |H| - 1 = n - 1 \quad \forall \ H \in \mathcal{H}^n.
\] (6)
Similarly, the condition that $G(x)$ is not not traceable can be imposed by the family of constraints

$$
\sum_{e \in P} x_e \leq |P| - 1 = n - 2 \quad \forall \ i \in V^n, \ P \in \mathcal{P}_{n-1}(i). \quad (7)
$$

The condition that $G(x)$ is not homogeneously traceable may be imposed by requiring that for a particular vertex, let us say vertex 1,

$$
\sum_{e \in P} x_e \leq |P| - 1 = n - 2 \quad \forall \ P \in \mathcal{P}_{n-1}(1). \quad (8)
$$

Finally, let us consider how to model each connectivity condition $C$. The family of constraints

$$
\sum_{e \in \partial(S)} x_e \geq 1 \quad \forall \ S \subset V^n, \ S \neq \emptyset \quad (9)
$$

guarantees that the graph $G(x)$ is connected while the family of constraints

$$
\sum_{e \in \partial(S) \setminus \partial(i)} x_e \geq 1 \quad \forall \ i \in V^n, \ S \subset V^n \setminus \{i\}, \ S \neq \emptyset \quad (10)
$$

guarantees that any graph obtained from $G(x)$ by removing any vertex is connected, i.e., that $G(x)$ is 2-connected. Finally, the 1-toughness of $G(x)$ may be imposed by introducing the family of constraints

$$
\sum_{1 \leq a < b \leq t} \sum_{e \in \partial(W_a) \cap \partial(W_b)} x_e \geq 1 \quad \forall \ \text{partition} \ S, W_1, \ldots, W_t \ \text{of} \ V^n \ \text{with} \ t > |S|. \quad (11)
$$

Each model $\mathcal{M}(n, C, \neg H)$ is defined by the degree constraints and the families of constraints that impose condition $C$ and forbid property $H$. We observe that all the previous families, except that of the degree constraints, contain a number of inequalities which is exponential with respect to $n$. Since this number is very high even for $n = 16$, we have adopted a cutting plane approach to generate and add these constraints to the model only when needed. This requires to be able to solve the separation problem corresponding to each family of constraints.

### 4.1 The separation problems

The separation problem with respect to a family $\mathcal{L}$ of inequalities is the following: given a solution $\bar{x}$ (not necessarily integer), find an inequality of $\mathcal{L}$ violated by $\bar{x}$ or determine that such inequality does not exist. An algorithm for this problem is called a separation algorithm for $\mathcal{L}$.

The separation problem for the $\neg H$-constraints

A solution $\bar{x}$ violates the non-hamiltonicity constraints [6] if and only if there exists a Hamilton cycle $H \in \mathcal{H}^n$ such that

$$
\sum_{e \in H} \bar{x}_e > n - 1 \Leftrightarrow \sum_{e \in H} (\bar{x}_e - 1) > -1 \Leftrightarrow \sum_{e \in H} (1 - \bar{x}_e) < 1.
$$
As a consequence, the separation problem for constraints (6) can be solved by finding the shortest Hamilton cycle in $K_n$ with respect to the lengths $c_e := 1 - \bar{x}_e$ for each $e \in E_n$. This is a Traveling Salesman Problem (TSP). If the optimal TSP solution $H^*$ has value smaller than 1 then the constraint
$$\sum_{e \in H^*} x_e \leq n - 1$$
is violated by $\bar{x}$ and the constraint is added to the model. Otherwise, $\bar{x}$ satisfies all constraints (6). Similarly, the constraints (7) (respectively, 8) are satisfied if and only if the shortest path with respect to the lengths $c_e$ in $P_{n-1}$ (respectively, in $P_{n-1}(i)$) has length strictly less than 1. It is well known that the TSP and the problem of finding the shortest path with a given number of edges are NP-hard. However, in our application $n$ is fixed and rather small. Solving these problems on such small graphs is quite simple and there are several effective algorithms to this end. In particular, we have used a simple branch-and-bound procedure.

The separation problem for the connectivity and 2-connectivity constraints

A solution $\bar{x}$ does not satisfy the 2-connectivity constraints (10) if and only for some vertex $i \in V^n$ and some subset $S \subset V^n \setminus \{i\}$, $S \neq \emptyset$, the sum $\sum_{e \in \partial(S)} \bar{x}_e$ over the cut $\partial(S)$ in $K^n[V^n \setminus \{i\}]$ is strictly smaller than 1. Thus the separation problem for the 2-connectivity constraints may be solved by finding, for each $i \in V^n$, a minimum-cut on the graph $K^n[V^n \setminus \{i\}]$ with respect to the weights $w_e := \bar{x}_e$. If for some $i$ the optimal value is smaller than 1, the 2-connectivity inequality defined by $i$ and the optimal solution $\bar{S}$ is violated by $\bar{x}$, otherwise $\bar{x}$ satisfies all inequalities (10). If $\bar{x}$ is integer, the separation problem can be alternatively solved in time $O(n)$ by searching for the articulation points of the graph $G(\bar{x})$, i.e., the vertices whose removal disconnects the graph [25]. Similar approaches may be used to separate the connectivity constraints (9) for fractional and integer solutions.

The separation problem for the 1-toughness constraints

The separation problem with respect to the 1-toughness constraints (11) is NP-hard. We separate these constraints only for binary solutions $\bar{x}$ that determine a 4-regular and 2-connected graph $G(\bar{x})$. For given $k$ and $t$, with $k < t$, we look for a partition $S, W_1, \ldots, W_t$ of $V^n$ such that $|S| = k$ and $W_1, \ldots, W_t$ are the vertex sets of the connected components of the graph $G(\bar{x})[V^n \setminus S]$ by solving the following ILP problem. Let $z_i, i \in V^n$, and $y_{ir}, i \in V^n$ and $r = 1, \ldots, t$, be binary variables such that

$$z_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad y_{ir} = \begin{cases} 1 & \text{if } i \in W_r \\ 0 & \text{otherwise} \end{cases}$$
Then the following constraints are satisfied only by partitions with the required properties

\[ T_{k,t} : \sum_{r=1}^{t} y_{ir} + z_i = 1 \quad \forall \, i \in V^n \]  
\[ y_{ir} \leq y_{jr} + z_j \quad \forall \, \{i,j\} \in E(\bar{x}), \quad r = 1, \ldots, t \]  
\[ y_{ir} \leq y_{jr} + z_i \quad \forall \, \{i,j\} \in E(\bar{x}), \quad r = 1, \ldots, t \]  
\[ \sum_{i \in V^n} y_{ir} \geq 1 \quad \forall \, r = 1, \ldots, t \]  
\[ \sum_{i \in V^n} z_i = k \]  
\[ z_i \in \{0, 1\} \quad \forall \, i \in V^n \]  
\[ y_{ir} \in \{0, 1\} \quad \forall \, i \in V^n, r = 1, \ldots, t. \]

Conditions (12) impose that each vertex belongs to exactly one set of the partition. Conditions (13) and (14) guarantee that each edge \( e \in E(\bar{x}) \) has either both endpoints in a same set \( W_r \) or at least one endpoint in \( S \). Finally, all the sets \( W_r \) are not empty by constraints (15) and \( |S| = k \) by constraint (16).

### 4.2 The branch-and-cut procedure

As already remarked, each model \( M(n, C, \neg H) \) has exponential size with respect to \( n \) and must therefore be solved with a constraint-generation approach. The standard way to do this is called \textit{branch-and-cut}. Branch-and-Cut is a version of branch and bound in which the constraint matrix at each node \( N \) of the search tree contains only a (small) subset of the constraints of the original model, while some of the missing constraints may be added at run time. Let us denote by \( M(N) \) the set of constraints of the subproblem corresponding to node \( N \). These are the constraints that were input at the root node, plus the branching constraints (fixing variables to 0 or 1) and all the constraints which were added in the nodes on the path from the root to \( N \).

Whenever the LP-relaxation of \( M(N) \) is solved, yielding a solution \( \bar{x} \), the feasibility of \( \bar{x} \) with respect to \( M(n, C, \neg H) \) must be checked. The solution \( \bar{x} \) could be infeasible either because it is fractional, or because it violates some of the constraints of \( M(n, C, \neg H) \) which are missing at \( N \). In order to find which constraints, if any, are not satisfied by \( \bar{x} \), we first run the separation algorithm described in Section 4.1 to possibly find one of the constraints enforcing \( \neg H \) which is violated. If this is not the case and \( \bar{x} \) is integer, we also run the separation algorithms which check if one of the constraints enforcing property \( C \) is violated. The properties of connectivity, 2-connectivity and 1-toughness are checked in this order. If we find any violated constraints, we add them to \( M(N) \) and solve the problem again. This phase is called constraint- (or cut-) generation.

The processing of the node terminates only when \( \bar{x} \) is integer and feasible for \( M(n, C, \neg H) \), or when \( \bar{x} \) is fractional but satisfies all constraints imposing property \( \neg H \). If \( \bar{x} \) is feasible, it induces a graph \( G(\bar{x}) \) with the sought properties and the search is terminated with a positive answer to problem \( P(n, C, \neg H) \). Otherwise, a branching is performed from \( N \), by picking a fractional component \( \bar{x}_j \) and creating two new subproblems, \( N' \) in which we fix \( x_j = 0 \), and \( N'' \) in which we fix \( x_j = 1 \).
4.3 Implementation decisions

To conclude the description of the ILP model, we briefly describe three implementation decisions that we have taken in order to speed-up the search.

4.3.1 Symmetries and orbital branching

Let us consider the generic ILP model $\mathcal{M}(n, \mathcal{C}, \neg \mathcal{H})$, a solution $x \in \{0, 1\}^{|E^n|}$ and the associated graph $G(x)$. For every permutation $\pi \in S_n$ we can define a new solution $\pi(x)$ by setting $\pi(x)_{ij} = x_{\pi(i)\pi(j)}$ for each $\{i, j\} \in E^n$. The graph $G(\pi(x))$ is clearly isomorphic to $G(x)$. Since all the connectivity properties and the hamiltonian properties that we are considering are preserved by graph isomorphisms, $x$ is feasible for $\mathcal{M}(n, \mathcal{C}, \neg \mathcal{H})$ if and only if $\pi(x)$ is. This implies that every permutation $\pi \in S_n$ induces a symmetry of the model, i.e., the model has many different, but in fact isomorphic, solutions. It is well known that even relatively small instances of ILP problems with large groups of symmetries can be extremely difficult to solve via branch and cut. For this reason several techniques have been proposed in the literature to reduce the impact of symmetries (see for instance the surveys of Margot [17] and Pfetsch and Rehn [21]). Among these techniques, a very effective one is Orbital Branching by Ostrovski and al. [20].

The orbital branching method requires to compute at each node $N$ of the branch and bound tree the group $G^N$ of the permutations of $S_n$ that stabilizes the sets $B_0(N)$ and $B_1(N)$ of the indices of the variables that have been fixed at 0 and 1 at $N$ (because of branching or some other reason). The orbit of an edge $\bar{e}$ under the action of $G^N$ is the set $O(\bar{e}) = \{\pi(\bar{e}) : \pi \in G^N\}$. The main idea of orbital branching is that, given a free variable $x_{\bar{e}}$, we can create two new nodes in the branch and bound tree based on the disjunction $(x_{\bar{e}} = 1) \lor (\sum_{e \in O(\bar{e})} x_e = 0)$. The orbital branching effectiveness can be strengthened by using a fixing technique introduced in [20]. In orbital branching we ensure that any two nodes are not equivalent with respect to the symmetries found at their first common ancestor. It is possible, however, that two child subproblems are equivalent with respect to a symmetry group found elsewhere in the tree. In order to overcome this situation, orbital fixing works as follows. Let $I_0$ and $I_1$ be the index sets of variables fixed to 0 and, respectively, to 1 at the root node. Note that $I_0 \subseteq B_0(N)$ and $I_1 \subseteq B_1(N)$ for each $N$. Given the group $G(B_1(N), I_0)$ of the permutations in $S_n$ that stabilize the sets $B_1(N)$ and $I_0$, let $O(e)$ denote the orbit of the edge $e$ under the action of $G(B_1(N), I_0)$. Consider the set $F_0 = \cup_{e \in B_0(N)} O(e)$ containing all the edges belonging to the orbits of edges in $B_0(N)$. The results concerning orbital branching and orbital fixing guarantee that, given a free variable $x_{\bar{e}} \notin F_0 \cup B_1(N)$, two new nodes may be created according to the disjunction

$$
(x_{\bar{e}} = 1 \land \sum_{e \in F_0} x_e = 0) \lor \left(\sum_{e \in O(\bar{e})} x_e = 0 \land \sum_{e \in F_0} x_e = 0\right).
$$

Clearly, the additional computational effort required by the method to compute the groups of symmetries $G^N$ and $G(B_1(N), I_0)$ and their orbits is worthwhile as long as it returns orbits of rather large size, in which case the orbital branching and the orbital fixing rule significantly limit the visit of isomorphic solutions. Since the branching constraints tend to reduce the symmetries of the problem, orbital branching is usually performed only at the first levels of the branch and bound tree.
4.3.2 Decomposition strategies for the solutions space

For every model $M(n,C,\neg H)$ we have adopted a decomposition of the set of feasible solutions based on the following idea. Given $k \leq n$, we define the problem $Q(n,C,\neg P_k)$ as the problem of finding a graph satisfying property $C$ which (1) contains a path of length $k-1$ and (2) does not contain a path of length $k$ (a Hamilton cycle if $k=n$). When $H=HT$ we require that these paths start at vertex 1. Clearly, problem $P(n,C,\neg H)$ is infeasible if and only if problem $Q(n,C,\neg P_k)$ is infeasible for every $k \leq r$, where $r = n$ if $H=H$ and $r = n-1$ if $H=T,HT$. Since each connected 4-regular graph with $n \geq 15$ vertices contains at least one path of length 8, we can start from the value $k = 8$. Condition (1) is imposed by fixing to 1 in the initial model the edges of a path of length $k-1$ in $K^n$ (for instance the edges $\{h,h+1\}$ for $h = 1, \ldots, k-1$). Condition (2) can be guaranteed by imposing that the constraints $[11]$ are satisfied for each path in $P_k(i)$ (in $P_k(1)$ if $H=HT$) instead than each path in $P_{n-1}(i)$. The separation routine for the TSP-constraints has been easily adjusted to separate the modified constraints. The use of this decomposition strategy allowed us to reduce by more than one order of magnitude the overall computational time required to solve model $M(n,1t,\neg H)$ with respect to the straightforward solution reported in [16].

A second type of decomposition was adopted for the model $M(n,1t,\neg H)$ with $16 \leq n \leq 18$. A main concern in solving this model is that the separation routine for the 1-toughness constraints $[11]$ takes a considerable time. In order to overcome this drawback, by exploiting the preliminary analysis of Section 3, we have identified a small number of cases in which we actually do have to impose these constraints and, for these cases, which problems $T_{2,3}$ have to be solved to separate the 1-toughness inequalities. In the remaining cases we can relax the constraints $[11]$ since they are implied by the 2-connectivity conditions. Let us now briefly describe this decomposition scheme. As it follows from the results of Section 3, every 4-regular 2-connected graph with 16 and 17 vertices which is not 1-tough contains at least one subgraph isomorphic to $R_5$. For $n = 18, 19, 20$ the graph may contain no $R_5$. Let $F_n$ be the set of solutions of $M(n,1t,\neg H)$. We partition $F_n$ into three sets, namely $F_n(2R5)$, $F_n(1R5)$ and $F_n(NOR5)$, which contain the solutions corresponding to graphs with, respectively, at least two (disjoint) copies, a single copy or no copy of $R_5$.

This partitioning allows us to fix many variables in the models in the first two cases. In particular, since the solutions in $F_n(2R5)$ contain two disjoint copies of $R_5$, we can fix to 1 the variables corresponding to the edges shown in Fig. 8(a). Similarly, for the solutions in $F_n(1R5)$ we can fix to 1 the variables corresponding to the edges in Fig. 8(b). Moreover, in the last two cases, we have to add to the model a set of inequalities which forbid the presence of any $R_5$ other than the one possibly fixed as above. These inequalities, called noR5-constraints (NOR5), are

$$
\sum_{e \in E(V')} x_e \leq 8 \quad \forall V' \subseteq W, \quad |V'| = 5,
$$

where $W = \{6, \ldots, n\}$ in the case of $F_n(1R5)$, and $W = V^n$ in the case of $F_n(NOR5)$. Since there are several hundred inequalities, we have decided not to add them all to the model, but to separate them only when needed. Based on the above decomposition, for $n = 16, 17, 18$ we have solved model $M(n,1t,\neg H)$ three times:

- with feasible set $F_n(NOR5)$ by removing the 1-toughness constraints $[11]$ for $n \leq 17$ and by solving problem $T_{2,3}$ for $n = 18$;
- with feasible set $F_n(1R5)$ by removing the 1-toughness constraints when $n = 16$, by solving
Figure 8: Variables fixed to 1 for the solutions in $F_n(2R5)$ (a) and in $F_n(1R5)$ (b).

- problem $T_{2,3}$ when $n = 17$ (to exclude the two graphs in Fig. 5) and by solving both the problems $T_{2,3}$ and $T_{4,5}$ when $n = 18$;
- with feasible set $F_n(2R5)$ by solving problem $T_{2,3}$ when $n = 16$ (to exclude the graph in Fig. 3) and when $n = 18$ and both the problems $T_{2,3}$ and $T_{4,5}$ when $n = 17$ (to exclude the graphs in Fig. 4 and in Fig. 6, respectively).

When $n \geq 19$ the above decomposition is not as useful. Indeed, based on the preliminary analysis, one has to solve both problems $T_{2,3}$ and $T_{4,5}$ also on the solution set $F_n(NOR5)$. This makes the decomposition no longer effective.

## 5 Overall results

Our computational study has determined the value $f(C, \neg H)$ for alla cases except when $C$ is the 1-toughness property and $H$ is the traceability property. Furthermore, based on the next fact, for each $n \geq f(C, \neg H)$ we are able to construct a 4-regular graph with $n$ vertices that satisfies property $C$ and does not satisfy property $H$.

### Fact 5.1

Let $G$ be a $(C, \neg H)$-graph with $n$ vertices. Then

(i) if $G$ contains a subgraph $H$ isomorphic to $K_4$ then for each $k \in \mathbb{N}$ the graph $G'$ obtained by replacing $H$ by any of the two graphs $T_{4+2k}$ and $T_{5+2k}$ in Fig. 9 is a $(C, \neg H)$-graph with either $n + 2k$ or $n + 1 + 2k$ vertices;

(ii) if $G$ contains a subgraph $H$ isomorphic to the graph $R_5$ then for each $k \in \mathbb{N}$ the graph $G'$ obtained by replacing $H$ by any of the two graphs $R_{5+2k}$ and $R_{6+2k}$ in Fig. 10 is a $(C, \neg H)$-graph with either $n + 2k$ or $n + 1 + 2k$ vertices.

**Proof.** The graphs $T_{4+2k}$ and $T_{5+2k}$ are 1-tough graphs having four vertices of degree 3 and the other vertices of degree 4. Thus each of them may be substituted for any complete subgraph of $G$ with 4 vertices leading to a graph $G'$ which still satisfies property $C$. The graph $G'$ cannot satisfy property $H$, otherwise, being $H$ a complete graph, also $G$ would satisfy this property. A similar argument can be used to prove statement (ii).

The results that we obtained for the different problems $P(n, C, \neg H)$ can be summarized as follows:

### Connectivity

As remarked in Section 2, the only open question about the hamiltonian properties of a connected 4-regular graph concerns the traceability. Moreover, by Theorem 1.2 ii) it is $f(c, \neg T) \geq 16$. 

15
Proposition 5.1. Every connected 4-regular graph with \( n \leq 17 \) is traceable. Moreover, for every \( n \geq 18 \) there exists a connected 4-regular graph with \( n \) vertices that is not traceable. Thus \( f(c, \neg T) = 18 \).

Indeed, the model \( M(n, c, \neg T) \) happened to be infeasible for \( n = 16 \) and \( n = 17 \). Moreover, for \( n = 18 \) our code produced the nontraceable graph in Fig. 11 which contains a subgraph isomorphic to \( R_5 \). Then the statement follows from Fact 5.1.
2-connectivity

By Theorem 1.3. every 2-connected 4-regular graph \( G \) with less than 16 vertices is hamiltonian, thus homogeneously traceable and traceable.

**Proposition 5.2.** For every \( n \geq 16 \) there exists a 2-connected 4-regular graph with \( n \) vertices which is not 1-tough, thus it is neither hamiltonian nor homogeneously traceable. This implies that \( f(2c, \neg H) = f(2c, \neg HT) = 16 \).

Indeed, the preliminary analysis found out the graph in Fig. 3 which is 2-connected but not 1-tough. Since this graph contains an \( R_5 \), the statement follows from Fact 5.1.

**Proposition 5.3.** Every 2-connected 4-regular graph with \( n \leq 21 \) is traceable. Moreover for every \( n \geq 22 \) there exists a 2-connected 4-regular graph with 22 vertices which is nontraceable. Thus \( f(2c, \neg T) = 22 \).

The result follows from Proposition 5.1 when \( n \leq 17 \). Moreover the model \( M(n, 2c, \neg T) \) happened to be infeasible for every \( 18 \leq n \leq 21 \). Finally, for \( n = 22 \) our code produced the 2-connected nontraceable graph in Fig. 12. Since this graph contains an \( R_5 \), the statement follows from Fact 5.1.

1-toughness

**Proposition 5.4.** Every 1-tough 4-regular graph with \( n \leq 17 \) is hamiltonian. Moreover for every \( n \geq 18 \) there exists a 1-tough 4-regular graph with \( n \) vertices which is not hamiltonian. Thus \( f(1t, \neg H) = 18 \).

Indeed, the model \( M(n, 1t, \neg H) \) happened to be infeasible for \( n = 16 \) and \( n = 17 \). Moreover, for \( n = 18 \) our code produced the same 1-tough nonhamiltonian graph in Fig. 2 proposed in [2]. Since this graph contains a \( K_4 \), the statement follows from Fact 5.1.

**Proposition 5.5.** Every 1-tough 4-regular graph with \( n \leq 19 \) is homogeneously traceable. Moreover for every \( n \geq 20 \) there exists a 1-tough 4-regular graph with \( n \) vertices which is not homogeneously traceable. Thus \( f(1t, \neg HT) = 20 \).

The result follows from Proposition 5.4 when \( n \leq 17 \). Moreover the model \( M(n, 1t, \neg HT) \) happened to be infeasible for \( n = 18 \) and \( n = 19 \). Finally, for \( n = 20 \) our code produced the 1-tough 4-regular graph in Fig. 13 which is not homogeneously traceable since it does not contain any Hamilton path from vertex \( v \). Since this graph contains both a \( K_4 \) and an \( R_5 \) the statement follows from Fact 5.1.

**Proposition 5.6.** Every 1-tough 4-regular graph with \( n \leq 21 \) is traceable. Moreover for every \( n \geq 40 \) there exists a 1-tough 4-regular nontraceable graph with \( n \) vertices. Thus \( 22 \leq f(1t, \neg T) \leq 40 \).

The bound \( f(1t, T) \geq 22 \) follows from Proposition 5.3. Moreover, the graph with 40 vertices in Fig. 14 that is obtained by suitably connecting two copies of the graph in Fig. 13 happens to be a 1-tough 4-regular graph which is not traceable. Since this graph contains both a \( K_4 \) and an \( R_5 \) the statement follows from Fact 5.1.

The above results allow us to fill the entries in boldface of Table 1 in Section 4.
Figure 13: A 4-regular 1-tough graph with $n = 20$ that is not homogeneously traceable. There is not a Hamilton path starting at $v$.

Figure 14: A 4-regular 1-tough graph with $n = 40$ that is not traceable.

6 Computational results

All the models $\mathcal{M}(n, C, \neg H)$ were solved within the SCIP 6.0.0 framework for branch and cut [11], using CPLEX 12.4 [14] as the LP solver and the software Nauty [18] to perform orbital branching. All the experiments were run on an Intel i7 CPU with 3.6GHz, 6 cores (4+2) and 16 GB RAM.

Next we list some details relative to the solution of model $\mathcal{M}(n, C, \neg H)$ for the different pairs of properties $(C, \neg H)$:

- $(c, \neg T)$: for $16 \leq n \leq 18$ we solved model $\mathcal{M}(n, c, \neg P_k)$, $k = 8, \ldots, n - 1$ (see Subsection 4.3.2). The separation of the connectivity constraints was performed on the integer solutions by checking the connectivity of the corresponding graph. The solution in Fig. 11 was returned when solving problem $Q(18, c, \neg P_{13})$.

- $(2c, \neg T)$: for $16 \leq n \leq 22$ we solved model $\mathcal{M}(n, 2c, \neg P_k)$, $k = 8, \ldots, n - 1$. The separation of the 2-connectivity constraints was performed on the integer solutions by checking the 2-connectivity of the corresponding graph. The solution in Fig. 12 was returned when solving problem $Q(22, 2c, \neg P_{17})$.

- $(1t, \neg H)$: for $16 \leq n \leq 18$ the feasible set of model $\mathcal{M}(n, 1t, \neg H)$ was partitioned in the three sets $\mathcal{F}(NOR5)$, $\mathcal{F}(1R5)$ and $\mathcal{F}(2R5)$ containing the feasible solutions with, respectively, no $R_5$, a single $R_5$ and at least two disjoint $R_5$. The separation of the 1-toughness constraints was performed as explained in Subsection 4.3.2 and the models with feasible set $\mathcal{F}(NOR5)$ were solved by solving the subproblems $Q(n, 1t, \neg P_k)$ for $k = 8, \ldots, n$. The solution in Fig. 13 was returned when solving the subproblem $Q(18, 1t, \neg P_{15})$. Since there exists a single 2-connected graph which disconnects in 4 components by removing three vertices and this graph contains a path of length at least 17 from any vertex, the problem $T_{3, 4}$ was not solved for $k \leq 15$.

- $(1t, \neg HT)$: for $18 \leq n \leq 20$ the model $\mathcal{M}(n, 1t, \neg HT)$ was solved by solving the subproblems $Q(n, 2c, \neg P_k(1))$, $k = 8, \ldots, n - 1$. The 1-toughness constraints have been separated by solving the problems $T_{\ell, \ell+1}$ with $\ell \in \{2, 4\}$ for $n = 18$ and $\ell \in \{2, 4, 5\}$ for $n = 19, 20$. The solution in Fig. 14 was returned when solving the subproblem $Q(20, 1t, \neg P_{15}(1))$. Since there exists a single 2-connected graph which disconnects in 4 components by removing three vertices and this graph contains a path of length at least 17 from any vertex, the problem $T_{3, 4}$ was not solved for $k \leq 15$. 18
Table 2: Computational results. The times are those reported by SCIP at the end of the computation.

| n    | result            | time (sec) | time (hours) |
|------|-------------------|------------|--------------|
| 16   | infeasible        | 572        | < 1          |
| 17   | infeasible        | 3344       | < 1          |
| 18   | Figure [11]       | 510        | < 1          |
| 18   | infeasible        | 12510      | < 4          |
| 19   | infeasible        | 66023      | < 19         |
| 20   | infeasible        | 221054     | < 62         |
| 21   | infeasible        | 1558501    | < 433        |
| 22   | Figure [12]       | 5128       | < 2          |
| 16   | infeasible        | 3978       | < 2          |
| 17   | infeasible        | 38856      | < 11         |
| 18   | Figure [2]        | 24620      | < 7          |
| 18   | infeasible        | 57356      | < 15         |
| 19   | infeasible        | 249100     | < 70         |
| 20   | Figure [13]       | 24700      | < 7          |

The computational times required to solve the different models $\mathcal{M}(n, C, \neg H)$ are reported in Table 2. For each pair of properties and each value $n$, the reported time is the sum of the times needed to solve all the subproblems used for that case. As expected, the computational times significantly increase with the number $n$ of vertices of the instances and with the type of constraints that define model $\mathcal{M}(n, C, \neg H)$.

7 Conclusions

It is known that the hamiltonian properties of a 4-regular graph $G$ depend on its connectivity properties and on its order, but for several pairs $\mathcal{H}, \mathcal{C}$ of such properties, determining the smallest order $n$ such that $G$ has $\mathcal{C}$ but does not have $\mathcal{H}$ is a challenging problem. In this paper we attacked this problem by using Integer Linear Programming to formulate the search of this type of graphs. We believe that using ILP to construct combinatorial objects (such as a graph) with given properties is a viable technique, which we want to support with our work. This technique can be used, for example, to try and settle a conjecture on the existence of an object of a particular type, provided its size is “small enough”. Indeed, a major limitation of the ILP approach is that the running times grow exponentially with the object’s size (as we experienced in our computations) and even for small instances, a certain amount of ingenuity and software engineering is required to make the approach work.

In our paper we were able to settle some open questions about the hamiltonian properties of
4-regular graphs of small order, but still large enough to make it quite hard to address them by a theoretical analysis. Indeed, even for these small graphs, we were able to complete the computations in a relatively small amount of time only thanks to the adoption of some strategic choices such as (i) the use of symmetry-breaking tools; (ii) a decomposition of the cases based on the existence/absence of paths of a given length; (iii) a preliminary analysis aimed at identifying subgraphs of the sought graph with a well defined structure, whose knowledge allowed us to considerably limit the search space. We remark that even the preliminary analysis was carried out by using an ILP approach, which demonstrates not only the the power, but also the flexibility of using Integer Programming when searching for combinatorial objects with given properties.

Although we were able to determine the smallest order for \((C, \neg H)\)-graphs for almost all pairs \((C, H)\), one remaining open problem concerns the nontraceability of 1-tough graphs. For this problem, the order of a smallest graph would still be too large for solving the corresponding ILP model within an acceptable time (it is difficult to estimate how much it would take, but, based on our experience, months or even years).

A future direction for our research would be to identify some other open problems or conjectures in graph theory concerning graphs of relatively small order, and try to tackle these problems with an ILP approach. We believe the use of this tool can be vary helpful in closing some of these questions.

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