HARDY TYPE INEQUALITIES AND GAUSSIAN MEASURE

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Abstract. In this paper we prove some improved Hardy type inequalities with respect to the Gaussian measure. We show that they are strictly related to the well-known Gross Logarithmic Sobolev inequality. Some applications to elliptic P.D.E.’s are also given.

1. Introduction. The classical Hardy inequality states that, for every \( u \in W^{1,2}(\mathbb{R}^N) \), \( N > 2 \),

\[
\int_{\mathbb{R}^N} |Du|^2 \, dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx,
\]

where the constant \( (N-2)^2/4 \) is sharp and it is not attained in \( W^{1,2}(\mathbb{R}^N) \), even if one replaces \( W^{1,2}(\mathbb{R}^N) \) with \( W^{1,2}_0(\Omega) \), where \( \Omega \subseteq \mathbb{R}^N \) contains the origin. On the other hand, if \( u \in W^{1,2}_0(\Omega) \), inequality (1) can be improved by adding remainder terms. For example in [9] it has been proved that for every \( u \) in \( W^{1,2}_0(\Omega) \)

\[
\int_{\Omega} |Du|^2 \, dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx + \frac{z_0^2 \omega_N^2}{N |\Omega|^{2/N}} \int_{\Omega} u^2 \, dx,
\]

where \( \omega_N \) and \( |\Omega| \) denote, respectively, the Lebesgue measure of the unit ball and of \( \Omega \), and \( z_0 \) is the first zero of the Bessel function \( J_0(z) \). Since the error term in (2) is given in terms of a rearrangement-invariant norm of \( u \), Schwarz symmetrization allows to reduce to the radial case and, then, specify the constant. In [11] inequality (1) is further improved as follows

\[
\int_{\Omega} |Du|^2 \, dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx + C \left( \int_{\Omega} |Du|^q \, dx \right)^{2/q}, \quad q < 2,
\]

where \( C \) is a positive constant depending only on \( \Omega \) and \( q \).

Inequalities (1), (2) and (3) have been extended to the case \( p \neq 2 \), \( 1 < p < N \) (see, e.g., [13], [11]), obtaining for instance

\[
\int_{\Omega} |Du|^p \, dx \geq \left( \frac{N-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \, dx + C \left( \int_{\Omega} |Du|^q \, dx \right)^{p/q}, \quad q < p.
\]

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On the other hand, Hardy type inequalities, where the singularity is given in terms of the function
\[ \delta(x) = \text{dist}(x, \partial \Omega), \]
have been extensively studied. The related literature is wide and here we mention the following classical one-dimensional result (see [20], [21], [26], [22])
\[ \int_0^{\infty} |u'|^p dt \geq \left( \frac{p - 1}{p} \right)^p \int_0^{\infty} \frac{|u|^p}{t^p} dt, \quad \forall u \in W^{1,p}(0, \infty), \ u(0) = 0. \quad (5) \]
Inequality (5) has been extended to the case \( N \geq 2 \) when \( p = 2 \) in [14], while the case \( p \neq 2 \) is treated, for example, in [25], where the authors prove
\[ \int_{\Omega} |Du|^p dx \geq \left( \frac{p - 1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{\delta^p} dx, \quad \forall u \in W^{1,p}_0(\Omega), \quad (6) \]
when \( \Omega \) is an open, bounded, convex set in \( \mathbb{R}^N \) having \( C^1 \) boundary. When the domain is just Lipschitz (not necessarily convex) or when it is unbounded with smooth compact boundary, an inequality of the type (6) still holds true with a constant smaller than \( \left( \frac{p - 1}{p} \right)^p \) (see [24], [12]).

Hardy type inequalities have been applied in several contexts as, for instance, the study of the stability of solutions to semilinear elliptic and parabolic equations (see [9]) and the stability of eigenvalues in elliptic problems (see [10]). Finally, classical Hardy inequality (1) provides the embedding of \( W^{1,2}(\mathbb{R}^N) \) into the Lorentz space \( L^{2,2}(\mathbb{R}^N) \). It is rather natural to wonder if there is a relationship between Hardy inequality with respect to the Gaussian measure and the well-known Sobolev Logarithmic embedding theorem proved in [19]. Therefore, in this paper we investigate such a kind of inequalities with respect to the Gaussian measure
\[ d\gamma = \prod_{k=1}^N d\gamma_k, \]
where
\[ d\gamma_k = (2\pi)^{-1/2} \exp \left( -x_k^2/2 \right) dx_k. \]
In this setting it is natural to work with weighted Sobolev spaces, briefly recalled in Section 2. Our starting point is the following inequality
\[ \int_{\mathbb{R}} \left( (u')^2 + \frac{u^2}{2} \right) d\gamma_1 \geq \frac{1}{4} \int_{\mathbb{R}} x_1^2 u^2 d\gamma_1, \quad \forall u \in W^{1,2}(\mathbb{R}, d\gamma_1), \quad (7) \]
and its extension to \( \mathbb{R}^N \)
\[ \int_{\mathbb{R}^N} \left( |Du|^2 + \frac{N}{2} u^2 \right) d\gamma \geq \frac{1}{4} \int_{\mathbb{R}^N} |x|^2 u^2 d\gamma, \quad \forall u \in W^{1,2}(\mathbb{R}^N, d\gamma), \quad (8) \]
whose simple proofs are contained in Sections 3 and 4. The constants appearing in (7) and (8) are sharp and we will also verify that (7) and (8) cannot be improved by adding any \( L^p \) norm of \( u \).

As in the classical case one might ask whether, if we substitute \( \mathbb{R}^N \) with a proper subset, some improved Hardy inequalities hold. As we will see, this question has an affirmative answer. Our results rely on some one-dimensional inequalities, since the Gaussian measure is factorized and for it there exists a suitable notion of rearrangement, the so-called Gaussian rearrangement. It transforms the level sets of a positive function \( u \) into parallel half-spaces having the same Gaussian measure. In this way one obtains the Gaussian symmetrized of \( u, u^\# \), which is, therefore, a
one-dimensional, increasing function. This will be a useful tool in passing from $\mathbb{R}$ to $\mathbb{R}^N$, since a Pólya-Szegő principle with respect to the Gaussian measure holds.

Let $1 \leq q < p$, $a \in \mathbb{R}$ and let us denote by $I_a$ the interval $(a, \infty)$; then the following inequalities hold true for every $u \in W^{1,p}_0(I_a, d\gamma_1)$

$$ \int_{I_a} |u'|^p d\gamma_1 - \left( \frac{p-1}{p} \right)^p \int_{I_a} \varrho_p^p |u|^p d\gamma_1 \geq C \left( \int_{I_a} |u'|^q d\gamma_1 \right)^\frac{p}{q} $$

and

$$ \int_{I_a} |u'|^p d\gamma_1 - \left( \frac{p-1}{p} \right)^p \int_{I_a} \varrho_p^p |u|^p d\gamma_1 \geq C \left( \int_{I_a} |u|^q d\gamma_1 \right)^\frac{p}{q}. $$

The function $\varrho_p$ above is smooth in $I_a$ and it has the following asymptotic behavior

$$ \lim_{x \to -a^+} \varrho_p(x_1) (x_1 - a) = 1; \quad \lim_{x \to \infty} \frac{\varrho_p(x_1)}{x_1} = \frac{1}{p-1}. $$

So “near” $a$ our results are in agreement with the classical one-dimensional Hardy inequality $\mathbf{5}$, while at infinity we get a singularity of the type $x_1^p$; in agreement with $\mathbf{7}$. Moreover there exists $\tilde{x}_1 \in I_a$ such that $\varrho_p' < 0$ in $(a, \tilde{x}_1)$, $\varrho_p' > 0$ in $(\tilde{x}_1, \infty)$ with $\varrho_p(\tilde{x}_1) > 0$.

Thus, we generalize $\mathbf{10}$ to the $N$-dimensional case as follows

$$ \int_{\Omega} |Du|^p d\gamma - \left( \frac{p-1}{p} \right)^p \int_{\Omega} \varrho_T(x_1)^p |u|^p d\gamma $$

$$ \geq C \left( \int_{\Omega} |u|^q d\gamma \right)^\frac{p}{q} \quad \forall u \in W^{1,p}_0(\Omega, d\gamma), $$

where $\Omega$ is an open, connected subset of $\mathbb{R}^N$, with $\int_{\Omega} d\gamma < 1$ (i.e. $\Omega$ is essentially different from $\mathbb{R}^N$), and

$$ \varrho_T(x_1) = \begin{cases} \varrho_p(x_1) & \text{in } (a, \tilde{x}_1) \\ \varrho_p(\tilde{x}_1) & \text{in } [\tilde{x}_1, \infty). \end{cases} $$

We need to truncate the weight near $a$ so that the left hand side of $\mathbf{11}$ decreases under Gaussian rearrangement, while the right hand side does not change.

In order to extend $\mathbf{3}$ to $\mathbb{R}^N$ we assume that $\Omega$ is contained in an half-space and, thanks to factorization arguments, we prove that

$$ \int_{\Omega} |Du|^p d\gamma \geq C \left( \int_{\Omega} |Du|^q d\gamma \right)^\frac{p}{q} \quad \forall u \in W^{1,p}_0(\Omega, d\gamma). $$

As we will show, these Hardy type inequalities are, in some sense, equivalent to the Logarithmic Sobolev embedding (see Section 4).

This paper is organized as follows. In Section 2 some definitions and notation about weighted spaces, rearrangements and isoperimetric inequality are recalled. Then, Section 3 and Section 4 provide a detailed proof of the Hardy-type inequalities announced before. Finally Section 5 is devoted to investigating some applications of these Hardy inequalities to existence and uniqueness of solutions to Dirichlet problems for non-linear elliptic equations. We also give an example of pathological solution, in the same spirit of Serrin $\mathbf{27}$, of a one-dimensional problem.
2. Preliminaries. We begin this section by recalling some definitions about weighted spaces and Gaussian symmetrization that will be useful in the following. From now on $\Omega$ will be any connected, open subset of $\mathbb{R}^N$ such that $\int_{\Omega} d\gamma < 1$.

**Definition 2.1.** The weighted Sobolev space $W^{1,p}(\Omega, d\gamma)$ is the set of all functions $u \in W^{1,1}_{loc}(\Omega)$ such that $(u, |Du|) \in L^p(\Omega, d\gamma) \times L^p(\Omega, d\gamma)$, endowed with the norm
\[
\|u\|_{W^{1,p}(\Omega, d\gamma)} = \|u\|_{L^p(\Omega, d\gamma)} + \|Du\|_{L^p(\Omega, d\gamma)}.
\]
The weighted Sobolev space $W^{1,p}_0(\Omega, d\gamma)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega, d\gamma)$.

**Definition 2.2.** Let $1 < p < \infty$ and $\alpha \in \mathbb{R}$; the weighted Zygmund space $L^p(\log L)\alpha(\Omega, d\gamma)$ consists of all measurable functions on $\Omega$ for which
\[
\int_{\Omega} \left[ |f| \log^\alpha(|f|) \right]^{p} d\gamma < \infty
\]
(12)
(here $\log_+ x = \max\{\log x, 0\}$).

It is known (see, for example [6]) that a measurable function satisfies (12) if and only if
\[
\left( \int_0^{\|\Omega\|_\gamma} \left[ (1 - \log t)^\alpha f^*(t) \right]^{p} dt \right)^{1/p} < \infty,
\]
where $f^*(t)$ is the decreasing of $f$ with respect to the Gaussian measure (defined in the sequel) and
\[
\|\Omega\|_\gamma = \int_{\Omega} d\gamma
\]
is the Gaussian measure of $\Omega$. Obviously $\|\Omega\|_\gamma \in (0, 1)$ and $|\mathbb{R}^N|_\gamma = 1$. The last quantity contained in (13) defines a norm with respect to which the space $L^p(\log L)^\alpha(\Omega, d\gamma)$ is complete.

The Gaussian perimeter of a set $\Omega$ is defined as
\[
P_{\gamma}(\Omega) = \sup \left\{ \int_{\Omega} \text{div}_\gamma (v(x)) \gamma(x) dx : v \in C^1_0(\Omega, \mathbb{R}^N), \|v\|_{\infty} \leq 1 \right\},
\]
where $\text{div}_\gamma (v(x))$ denotes the Gaussian divergence given by
\[
\text{div}_\gamma (v(x)) = \text{div}(v(x)) - \langle x, v(x) \rangle.
\]
When $\Omega$ is $(N-1)$-rectifiable, then
\[
P_{\gamma}(\Omega) = \int_{\partial\Omega} \gamma(x) d\mathcal{H}^{N-1},
\]
where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

Now, let $\Omega^a$ be the half-space
\[
\Omega^a = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 > a\},
\]
(14)
where $a$ is taken such that $|\Omega^a|_\gamma = |\Omega|_\gamma$. A straightforward calculation gives
\[
a = k^{-1}(|\Omega|_\gamma),
\]
where $k(t)$ is the function
\[
k(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \exp(-\sigma^2/2) d\sigma.
\]
(15)

We recall the isoperimetric inequality for Gaussian measure (see [28], [7], [16] and [11]).
Theorem 2.3. Let $\Omega$ be any measurable subset of $\mathbb{R}^N$ having finite Gaussian perimeter and let $\Omega^\sharp$ be the half-space defined in (14) such that $|\Omega^\sharp|_\gamma = |\Omega|_\gamma$. Then
\[ P_\gamma(\Omega) \geq P_\gamma(\Omega^\sharp); \] (16)
moreover equality holds in (16) if and only if $\Omega = \Omega^\sharp$, modulo a rotation.

Now we can define the rearrangement, with respect to Gaussian measure, of any measurable function $u$. To this end let $\mu$ be the distribution function of $u$ defined by\[ \mu(t) = |\{ x \in \Omega : |u(x)| > \theta \}|_\gamma, \theta \geq 0, \]
and let $u^*$ be its decreasing rearrangement defined by\[ u^*(s) = \inf \{ \theta \geq 0 : \mu(\theta) \leq s \}, s \in \left( 0, |\Omega|_\gamma \right). \]
We will say that two functions $u$ and $v$ are equimeasurable, or equivalently that $v$ is a rearrangement of $u$, if they have the same distribution function.

The determination of the cases of equality in the isoperimetric inequality (16) leads to the determination of a particular rearrangement of a measurable function $u$ on $\Omega$: the Gaussian rearrangement $u^\sharp$. More precisely, $u^\sharp$ is that rearrangement of $u$ whose level sets are half-spaces. Since the measure of each level set of $u$ is given by the distribution function, then $u^\sharp(x) = u^*(k(x_1))$, $x \in \Omega^\sharp$,
where $k$ is defined in (15).

By definition $u^\sharp$ actually depends on one variable only (say $x_1$) and it is an increasing function with respect to it, therefore its level sets are half-spaces. Moreover, since, by definition, $u$ and $u^\sharp$ are equimeasurable, by Cavalieri’s principle (see [13], p. 30) we have\[ \|u\|_{L^p(\Omega, d\gamma)} = \|u^\sharp\|_{L^p(\Omega^\sharp, d\gamma)}. \] (17)
For our purposes we also need the following Hardy-Littlewood inequality

Lemma 2.4. Let $u$ be any function in $L^p(\Omega, d\gamma)$, with $1 \leq p \leq \infty$, and $v$ any function in $L^{p'}(\Omega, d\gamma)$, with $1/p + 1/p' = 1$; then
\[ \int_\Omega |u(x)v(x)| d\gamma \leq \int_0^{\gamma(\Omega)} u^*(s)v^*(s) ds = \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x) d\gamma. \] (18)

It has been shown in [16], [29] and [11] that the following Pólya-Szegő principle holds true.

Theorem 2.5. Let $u$ be a function in $W^{1,p}(\mathbb{R}^N, d\gamma)$, then
\[ \int_{\mathbb{R}^N} |Du|^p d\gamma \geq \int_{\mathbb{R}^N} |Du^\sharp|^p d\gamma. \] (19)
Moreover equality holds if and only if $u = u^\sharp$, modulo a rotation.

Now we recall the following generalization of the classical Hardy inequality due to Maz’ja (see [26], pages 40ff).

Lemma 2.6. Let $\mu$ and $\nu$ be nonnegative Borel measures on (a, $\infty$) and let $\nu^*$ be the absolutely continuous part of $\nu$. The inequality
\[ \left( \int_a^\infty \left( \int_a^x |f(t)|q d\mu(x) \right)^{1/q} dt \right)^{1/p} \leq C \left( \int_a^\infty |f(x)|p d\nu(x) \right)^{1/p} \] (20)
holds for all Borel functions and $1 \leq p \leq q \leq \infty$ if and only if
\[
B = \sup_{r > a} \mu((r, \infty))^{1/q} \left( \int_a^r \left( \frac{d\nu^s}{dx} \right)^{-1/(p-1)} dx \right)^{(p-1)/p} < \infty. \tag{21}
\]
Moreover, if $C$ is the best constant in (20), then
\[
B \leq C \leq B \left( \frac{q}{q-1} \right)^{(p-1)/p} q^{1/q}.
\]
If $p = 1$ or $q = \infty$, then $B = C$. In the case $q = \infty$ the condition (21) means that
\[
B = \sup\{r > a : \mu([r, \infty)) > 0\} < \infty
\]
and $\frac{d\nu^s}{dx} > 0$ for almost all $x \in [0, B]$.

We end this section by recalling some well-known convexity inequalities that will be used systematically in the paper (for the proofs see [23, 4]).

**Proposition 1.** For all $\xi_1, \xi_2 \in \mathbb{R}^N$ the following inequalities hold.

i) If $1 < p < 2$,
\[
|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2} < \xi_1, \xi_2 - \xi_1 \geq C(p) \frac{|\xi_2 - \xi_1|^2}{|\xi_2| + |\xi_1|}^{2-p}. \tag{22}
\]

ii) If $p \geq 2$,
\[
|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2} < \xi_1, \xi_2 - \xi_1 \geq C(p)|\xi_2 - \xi_1|^p. \tag{23}
\]

3. **Some Hardy type inequalities on the real line.** Let $I_a = (a, \infty)$, $a \in \mathbb{R}$ and set $d\gamma_1 = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} dx_1$. Let us define the function
\[
\varrho(a, p, x_1) = \frac{e^{x_1^2/2(p-1)}}{\int_a^{\infty} e^{\sigma^2/2(p-1)} d\sigma}.
\]
We will use the simpler notation $\varrho_p(x_1)$ when no possibility of ambiguity arises. It is elementary to verify that
\[
\lim_{x_1 \to a^+} \varrho_p(x_1)(x_1 - a) = 1, \quad \lim_{x_1 \to -\infty} \frac{\varrho_p(x_1)}{x_1} = \frac{1}{p-1}. \tag{25}
\]
Moreover, there exists $\bar{x}_1 \in I_a$ such that $\varrho_p' < 0$ in $(a, \bar{x}_1)$, $\varrho_p' > 0$ in $(\bar{x}_1, \infty)$ with $\varrho_p(\bar{x}_1) > 0$.

We begin by proving the following Hardy inequality.

**Theorem 3.1.** For every $u \in W^{1,p}_0(I_a, d\gamma_1)$ it holds
\[
\int_a^{\infty} |u'|^p d\gamma_1 \geq \left( \frac{p-1}{p} \right)^p \int_a^{\infty} |u|^p \varrho_p^p d\gamma_1. \tag{26}
\]

**Proof.** By density arguments we can argue with $C_0^\infty(I_a)$ functions. A trick, which we learned in [30], consists in integrating over the interval $I_a$ the following convexity inequality
\[
|u'|^p \geq \left( \frac{p-1}{p} \varrho_p u \right)^p + p \left( \frac{p-1}{p} \varrho_p u \right)^{p-2} \frac{p-1}{p} \varrho_p u \left( u' - \frac{p-1}{p} \varrho_p u \right),
\]
obtaining
\[
\int_a^\infty |u'|^p d\gamma_1 \geq \int_a^\infty \left| \frac{p-1}{p} \varphi_p u \right|^p + p \left| \frac{p-1}{p} \varphi_p u \right|^{p-2} \frac{p-1}{p} \varphi_p u (u' - \frac{p-1}{p} \varphi_p u) d\gamma_1
\]
\[= \left( \frac{p-1}{p} \right)^p (1-p) \int_a^\infty |u|^p \varphi_p^p d\gamma_1 - \left( \frac{p-1}{p} \right)^{p-1} \int_a^\infty |u|^p \varphi_p^{p-2} (x_1 \varphi_p + (p-1) \varphi_p') d\gamma_1.
\]
Observing that \( \varphi_p \) solves the differential equation
\[x_1 \varphi_p - (p-1) \varphi_p' = (p-1) \varphi_p^2
\]
we get the claim.

Let us show now that the constant \( \left( \frac{p-1}{p} \right)^p \) in (26) is sharp. Let us introduce the following sequence of functions (see also [3])
\[
u_n(x_1) = \begin{cases} (x_1 - a) \left( \int_a^{a+1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{\frac{p-1}{p}} & \text{if } a \leq x_1 \leq a + 1 \\
\left( \int_a^{x_1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{\frac{p-1}{p}} & \text{if } a + 1 < x_1 \leq n \\
\left( \int_{a+1}^n \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{\frac{p-1}{p}} & \text{if } x_1 > n.
\end{cases}
\]
Note that \( u_n \in W_0^{1,p}(I_n, d\gamma_1) \); we want to show that
\[
\lim_{n \to \infty} \int_a^\infty |u_n'|^p d\gamma_1 = \left( \frac{p-1}{p} \right)^p.
\]
A straightforward calculation gives
\[
\int_a^\infty |u_n'|^p d\gamma_1 = \left( \int_a^{a+1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{p-1} \left( \int_a^{a+1} d\gamma_1 \right)
\]
\[+ \left( \frac{p-1}{p} \right)^p \int_a^n \frac{x_1}{2(p-1)} \exp \left( \frac{x_1^2}{2(p-1)} \right) d\sigma d\gamma_1,
\]
and
\[
\int_a^\infty |u_n|^p \varphi_p^p d\gamma_1 = \left( \int_a^{a+1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{p-1} \left( \int_a^{a+1} |x_1 - a|^p \varphi_p^p d\gamma_1 \right)
\]
\[+ \int_{a+1}^n \left( \int_a^{x_1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{p-1} \varphi_p^p d\gamma_1
\]
\[+ \left( \int_n^\infty \varphi_p^p d\gamma_1 \right) \left( \int_{a+1}^n \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{p-1}.
\]
Now, since
\[
\lim_{n \to \infty} \left( \int_n^\infty \varphi_p^p e^{-x_1^2/2} dx_1 \right) \left( \int_{a+1}^n \exp \left( \frac{\sigma^2}{2(p-1)} \right) d\sigma \right)^{p-1} = \left( \frac{1}{p-1} \right)^{p-1},
\]
by using the definition of \( \rho_p \), we get the claim (28). \( \square \)

Now, let us show that we can improve the above result by adding a remainder term to the right hand side of (26) depending on the \( L^q \)-norm of \( u' \).

**Theorem 3.2.** For any \( q < p \) there exists a positive constant \( C \), depending on \( a \), \( p \) and \( q \), such that, for every \( u \in W^{1,p}_0(I_a, d\gamma_1) \), it holds

\[
\int_a^\infty |u'|^p \, d\gamma_1 - \left( \frac{p-1}{p} \right)^p \int_a^\infty |u|^p \rho_p^p \, d\gamma_1 \geq C \left( \int_a^\infty |u'|^q \, d\gamma_1 \right)^{p/q}.
\]  

(29)

In order to prove our theorem we need the following Lemma. From now on we will denote by \( C \) a positive constant whose value may change from line to line.

**Lemma 3.3.** Let \( 1 < q < p < \infty \), then

\[
\Phi(x_1) = \frac{e^{x_1^2/2(q-1)}}{\int_a^x e^{\sigma^2/2(q-1)} \, d\sigma} \geq C \frac{e^{x_1^2/2(p-1)}}{\int_a^{x_1} e^{\sigma^2/2(p-1)} \, d\sigma} = C \Phi_p(x_1).
\]

(30)

**Proof.** It suffices to observe that the function

\[
\Psi(x_1) = e^{x_1^2/2(p-1)} \int_a^{x_1} e^{\sigma^2/2(q-1)} \, d\sigma
\]

is bounded from above and this is an immediate consequence of the following two limits

\[
\lim_{x_1 \to a^+} \Psi(x_1) = 1, \quad \lim_{x_1 \to +\infty} \Psi(x_1) = \frac{q-1}{p-1}.
\]

\( \square \)

**Proof of Theorem 3.2 (The case \( p \geq 2 \)).** The function

\[
\chi(x_1) = \left( \int_a^{x_1} \exp \left( \frac{\sigma^2}{2(p-1)} \right) \, d\sigma \right)^{p-1}
\]

solves the differential equation

\[
\chi' = \frac{p-1}{p} \chi \rho_p \chi.
\]

(31)

Note that \( \chi \notin W^{1,p}_0(I_a, d\gamma_1) \) since \( \chi' \notin L^p(I_a, d\gamma_1) \) for any \( p \). Let \( u \in C_0^\infty(I_a) \); integrating by parts we can easily obtain

\[
\int_a^\infty \left[ |u'|^p - \left( \frac{|u|^p}{\chi^{p-1}} \right)' \chi^{p-1} \right] \, d\gamma_1
\]

\[
= \int_a^\infty |u'|^p \, d\gamma_1 + \frac{1}{2\pi} \int_a^\infty \frac{|u|^p}{\chi^{p-1}} \left( (\chi')^{p-1} e^{-x_1^2/2} \right)' \, dx_1
\]

\[
= \int_a^\infty |u'|^p \, d\gamma_1 + \int_a^\infty \frac{|u|^p}{\chi^{p-1}} (\chi')^{p-2} \left( (p-1)\chi'' - x_1 \chi' \right) \, dx_1
\]

\[
= \int_a^\infty |u'|^p \, d\gamma_1 - \left( \frac{p-1}{p} \right)^p \int_a^\infty |u|^p \rho_p^p \, d\gamma_1,
\]
We claim that

\[
\int_a^\infty \left| u' \right|^p - \left( \frac{|u|^p}{\chi} \right)' (\chi)'^p-1 d\gamma_1
\]

which is a Hardy type inequality with respect to the measure \(\chi\), where

\[
\phi = \frac{u}{\chi}
\]

with \(N = 1, \xi_1 = u\chi\) and \(\xi_2 = u'\), we have

\[
\int_a^\infty \left| u' \right|^p - p\left| u' \right|^{p-2} u u' \left( \frac{\chi'}{\chi} \right)^{p-1} + (p-1) \left| u \right|^p \left( \frac{\chi'}{\chi} \right) \right| d\gamma_1
\]

and then

\[
\int_a^\infty \left| u' \right|^p d\gamma_1 - \left( \frac{p-1}{p} \right)^p \int_a^\infty \left| u \right|^p \phi^p d\gamma_1 \geq C(p) \int_a^\infty \left| u' - \frac{\chi'}{\chi} \right|^p d\gamma_1.
\]

Moreover, setting \(u = v\chi\), by the convexity of the function \(|u|^q\) and \(\phi\), we get

\[
\int_a^\infty \left| u' \right|^q d\gamma_1 = \int_a^\infty |\chi'v + \chi v'|^q d\gamma_1
\]

\[
\leq C \left( \int_a^\infty (\chi')^q |v|^q d\gamma_1 + \int_a^\infty \chi^q |v'|^q d\gamma_1 \right)
\]

\[
\leq C \left( \int_a^\infty \chi^q g^q_p |v|^q d\gamma_1 + \int_a^\infty \chi^q |v'|^q d\gamma_1 \right)
\]

We claim that

\[
\int_a^\infty |v|^q g^q_p \chi^q d\gamma_1 \leq C \int_a^\infty |v'|^q \chi^q d\gamma_1,
\]

which is an Hardy type inequality with respect to the measure \(\chi^q e^{-x^2/2} dx_1\). In order to prove (33) we need the following auxiliary result.

**Lemma 3.4.** Let \(A > a, p > 1\), and let \(\phi : [a, \infty) \to [0, \infty)\) be a smooth function, with \(\phi(x_1) > 0\) for any \(x_1 \in (a, \infty)\); then for every \(u \in C^\infty_0(a, \infty)\)

\[
\int_a^\infty \left| u' \right|^p \phi \ dx_1 \geq \int_a^\infty |g|^p |u|^p \phi \ dx_1,
\]

where

\[
g(x_1) = \begin{cases} 
\phi^{\frac{1}{p-1}} (x_1) & \text{if } x_1 \in (a, A]
\left( \frac{\phi^{\frac{1}{p-1}} (A) - \frac{A}{p-1} \int_{x_1}^A \phi^{\frac{1}{p-1}} \ d\sigma} {\phi^{\frac{1}{p-1}} (x_1) + \frac{A}{p-1} \int_{x_1}^A \phi^{\frac{1}{p-1}} \ d\sigma} \right) & \text{if } x_1 \in (A, \infty).
\end{cases}
\]

**Proof.** The function \(g\) is a solution of the following Cauchy problem

\[
g(A) = 1
\]

\[
-(p-1) |g|^p - 2 g - |g|^{p-2} g \frac{d}{\phi} = p |g|^p
\]

in the interval \((a, A)\). For any function \(u \in C^\infty_0(I_a)\), it holds

\[
\left| u' \right|^p \geq |gu|^p + p |gu|^{p-2} gu(u' - gu).
\]
Equation in (35) and inequality (36), together with an integration by parts, yield
\[ \int_a^A |u'|^p \phi dx_1 \geq \int_a^A \left\{ |g|^p + p \left| |g|^{p-2} g \right| |u|^{p-2} u' - p |g|^p \right\} \phi dx_1 \tag{37} \]
\[ = \int_a^A |g|^p |u|^p \phi dx_1 + |u(A)|^p \phi(A). \]
On the other hand the function \( g \) solves problem (35) also in \((A, \infty)\). Arguing as before we obtain
\[ \int_a^A |u'|^p \phi dx_1 \geq \int_a^A \left\{ (1 - p) |g|^p |u|^p + \left( |u|^p \right)' |g|^{p-2} \right\} \phi dx_1 \tag{38} \]
\[ = \int_a^A |g|^p |u|^p \phi dx_1 - |u(A)|^p \phi(A) \]
Adding (37) and (38) we get the claim. \( \square \)

Now we come back to the proof of Theorem 3.2 and, in order to treat the term \( \int_a^\infty \chi^q g_p^q |v|^q \, d\gamma_1 \), we use Lemma 3.2 with \( \phi = \chi^q e^{-x^2/2} \) obtaining
\[ \int_a^\infty |v'|^q \chi^q d\gamma_1 \geq \int_a^\infty |g|^q |v|^q \chi^q d\gamma_1, \tag{39} \]
where
\[ g(x_1) = \begin{cases} \chi^{q-p}(x_1) e^{-x^2/(1-p)} & \text{if } x_1 \in (a, A) \\ \chi^{q-p}(A) e^{-A^2/(1-p)} - \frac{p}{p-q} \int_A^x \chi^{q-p}(\sigma) e^{-\sigma^2/(1-p)} d\sigma & \text{if } x_1 \in (A, \infty). \end{cases} \tag{40} \]
Claim (44) is proved once we show
\[ |g| \geq C g_p \text{ in } I_a. \tag{41} \]
By (34) and (29), we easily verify
\[ \lim_{x_1 \to a^+} \frac{|g(x_1)|}{g_p(x_1)} = \frac{p-q}{pq} > 0, \quad \lim_{x_1 \to +\infty} \frac{|g(x_1)|}{g_p(x_1)} = \frac{q/p - 1}{p q - 1} - 1 > 0, \]
which give (43). Combining (44) and (32), we conclude
\[ \int_a^\infty |u'|^q \cdot \chi^q d\gamma_1 \leq C \int_a^\infty |v'|^q |\chi|^q \cdot \chi^q d\gamma_1. \]
Then, by Hölder inequality
\[ \int_a^\infty |u'|^q \cdot \chi^q d\gamma_1 \leq C \left( \int_a^\infty |v'|^p |\chi|^p d\gamma_1 \right)^{q/p} = C \left( \int_a^\infty |u|^{p-1} \chi^{q-1} d\gamma_1 \right)^{q/p} \]
\[ \leq C \left( \int_a^\infty |u|^p \cdot \chi^q \cdot \chi^q d\gamma_1 \right)^{q/p}. \]
\( \square \)
Proof of Theorem 3.2 (The case $1 < p < 2$). Hölder inequality gives

\[
\int_a^\infty \left| u' - u \frac{\chi'}{\chi} \right|^q \, d\gamma_1 = \int_a^\infty \frac{\left| u' - u \frac{\chi'}{\chi} \right|^q}{\left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{(2-p)\frac{q}{2}}} \left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{(2-p)\frac{q}{2}} \, d\gamma_1 \tag{42}
\]

Arguing as in the case $p \geq 2$, we have

\[
\int_a^\infty \left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{(2-p)\frac{q}{2}} \, d\gamma_1 \leq C \left( \int_a^\infty \left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^q \, d\gamma_1 \right)^{\frac{2-p}{q}} \tag{43}
\]

At this point, recalling (30), we can use Hardy’s inequality (26) with respect to $q$ which gives

\[
\left( \int_a^\infty \left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^q \, d\gamma_1 \right)^{\frac{2-p}{q}} \leq C \left( \int_a^\infty \left| u' \right|^q \, d\gamma_1 \right)^{\frac{2-p}{q}} \tag{44}
\]

Collecting inequalities (42), (43) and (44) we have

\[
\int_a^\infty \left| u' - u \frac{\chi'}{\chi} \right|^q \, d\gamma_1 \leq C \left( \int_a^\infty \frac{\left| u' - u \frac{\chi'}{\chi} \right|^2}{\left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{2-p}} \, d\gamma_1 \right)^{\frac{q}{2}} \left( \int_a^\infty \left| u' \right|^q \, d\gamma_1 \right)^{1-\frac{q}{2}}.
\]

Arguing as in the case $p \geq 2$, setting $u = \chi v$, we obtain

\[
\int_a^\infty \left| u' \right|^q \, d\gamma_1 \leq C \int_a^\infty \left| u' \right|^q \chi^q \, d\gamma_1 = C \int_a^\infty \left| u' - u \frac{\chi'}{\chi} \right|^q \, d\gamma_1 \leq C \left( \int_a^\infty \frac{\left| u' - u \frac{\chi'}{\chi} \right|^2}{\left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{2-p}} \, d\gamma_1 \right)^{\frac{q}{2}} \left( \int_a^\infty \left| u' \right|^q \, d\gamma_1 \right)^{1-\frac{q}{2}},
\]

and therefore

\[
\left( \int_a^\infty \left| u' \right|^q \, d\gamma_1 \right)^{\frac{2}{q}} \leq C \left( \int_a^\infty \frac{\left| u' - u \frac{\chi'}{\chi} \right|^2}{\left( \left| u' \right| + \left| u \frac{\chi'}{\chi} \right| \right)^{2-p}} \, d\gamma_1 \right)^{\frac{q}{2}}.
\]
or equivalently
\[
\left( \int_a^\infty |u'|^q \, d\gamma \right)^{\frac{p}{q}} \leq C \int_a^\infty \frac{|u' - u \frac{\alpha'}{\alpha}|^2}{\left( |u'| + |u \frac{\alpha'}{\alpha}| \right)^{2-p}} \, d\gamma.
\]

On the other hand inequality (22), with \( N = 1, \xi_1 = u \frac{\alpha'}{\alpha} \) and \( \xi_2 = u' \), gives
\[
C \frac{|u' - u \frac{\alpha'}{\alpha}|^2}{\left( |u'| + |u \frac{\alpha'}{\alpha}| \right)^{2-p}} \leq |u'|^p - \left( \frac{|u|^p}{\chi^{p-1}} \right)' (\chi')^{p-1}.
\]

In the end we gather
\[
\left( \int_a^\infty |u'|^q \, d\gamma \right)^{\frac{p}{q}} \leq C \int_a^\infty \left[ |u'|^p - \left( \frac{|u|^p}{\chi^{p-1}} \right)' (\chi')^{p-1} \right] \, d\gamma.
\]

\[\square\]

**Remark 1.** From (24) and the embedding of \( W_0^{1,q}(I_a, d\gamma) \) in \( L^q(I_a, d\gamma) \) (see e.g. [15]) we can deduce that
\[
\int_a^\infty |u'|^p \, d\gamma - \left( \frac{p-1}{p} \right)^p \int_a^\infty |u|^p g_{\beta}^p \, d\gamma \geq C \left( \int_a^\infty |u|^q \, d\gamma \right)^{p/q}, \quad q < p. \quad (45)
\]

In general we cannot expect a remainder term of the kind \( ||u||_{L^p(I_a, d\gamma)} \) and, a fortiori, \( ||u||_{W_0^{1,p}(I_a, d\gamma)} \). Indeed, in the simplest case \( p = 2 \) and \( a = 0 \), setting as before
\[
u = v \chi = v \left( \int_0^{x_1} e^{\sigma^2/2} \, d\sigma \right)^{1/2},
\]
we get
\[
\int_0^\infty \frac{\left( u' \right)^2 - \frac{u^2}{4}}{u^2} \, d\gamma = \int_0^\infty \frac{\left( v' \right)^2 \chi^2}{v^2} \, d\gamma.
\]

Since
\[
\left( \int_r^\infty \chi^2 e^{-x^2/2} \, dx_1 \right) \left( \int_0^r \chi^{-2} e^{-x^2/2} \, dx_1 \right) = \infty, \quad \forall r > 0,
\]
by Lemma 2.6 the ratio above is not bounded from below by any positive constant.

**Remark 2.** Here we consider Sobolev functions defined on the whole real line. By using a standard substitution (see, for instance, [17], [3]) we get that for every \( u \in W^{1,2}(\mathbb{R}, d\gamma) \) the following inequality holds true
\[
\int_{\mathbb{R}} \left( u' \right)^2 + \frac{u^2}{2} \, d\gamma \geq \frac{1}{4} \int_{\mathbb{R}} x_1^2 u^2 \, d\gamma. \quad (46)
\]

Indeed, setting \( v = u e^{-x^2/4} \), integrating by parts we have
\[
0 \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| v' \right|^2 \, dx_1 = \int_\mathbb{R} \left( u' - \frac{x_1}{u} \right)^2 \, d\gamma = \int_\mathbb{R} \left( (u')^2 + \frac{u^2}{2} \right) \, d\gamma - \frac{1}{4} \int_{\mathbb{R}} x_1^2 u^2 \, d\gamma.
\]
The constant \( \frac{1}{q} \) appearing in the right hand side of (46) is sharp and inequality (46) cannot be improved. Indeed, it is easy to check that the function \( u_\varepsilon = e^{(\frac{1}{q} - \varepsilon)x_1^2} \) belongs to \( W^{1,2}(\mathbb{R}, d\gamma_1) \) and
\[
\int_{\mathbb{R}} \left( (u'_\varepsilon)^2 + \frac{u^2_\varepsilon}{2} \right) d\gamma_1 - \frac{1}{4} \int_{\mathbb{R}} x_1^2 u^2_\varepsilon d\gamma_1 \to 0 \quad \text{as} \quad \varepsilon \to 0;
\]
thus, any inequality of the type
\[
\int_{\mathbb{R}} \left( (u')^2 + \frac{u^2}{2} \right) d\gamma_1 - \frac{1}{4} \int_{\mathbb{R}} x_1^2 u^2 d\gamma_1 \geq ||u||_{L^q(\mathbb{R}, d\gamma_1)}
\]
cannot hold true for any \( q < 2 \).

**Remark 3.** All the arguments used in the case of the Gaussian measure can be adapted for a more general measure. Indeed, let \( a \in \mathbb{R}, p > 1 \), and let \( \phi : [a, \infty) \to (0, \infty) \) be a smooth function such that
\[
\int_a \phi^{\frac{1}{1-p}} dx_1 < \infty.
\]
Then for any \( u \in C^\infty_0 (a, \infty) \) it holds that
\[
\int_a^\infty |u|^p \phi dx_1 \geq \left( \frac{p-1}{p} \right)^p \int_a^\infty \frac{\phi^{\frac{1}{1-p}}}{(\int_a^x \phi^{\frac{1}{1-p}} d\sigma)^p} |u|^p \phi dx_1.
\]

4. **The \( N \)-dimensional case.** The results proved in Section 3 can be generalized to the \( N \)-dimensional case, depending on the domain. First of all, in the whole space \( \mathbb{R}^N \), by factorization arguments, it can be immediately deduced from (46)
\[
\int_{\mathbb{R}^N} \left( |Du|^2 + \frac{N}{2} u^2 \right) d\gamma \geq \frac{1}{4} \int_{\mathbb{R}^N} |x|^2 u^2 d\gamma,
\]
and, as in dimension one, (17) cannot be improved.

When \( \Omega \) is an half-space, or it is contained in an half-space, one can extend (20) using again factorization arguments.

**Theorem 4.1.** Let \( a \in \mathbb{R} \) and \( \Omega = \Omega^a = \{ x = (x_1, ..., x_N) \in \mathbb{R}^N : x_1 > a \} \); then there exists a positive constant \( C \) such that for every \( u \in W^{1,p}_0(\Omega, d\gamma) \)
\[
\int_\Omega \left( |Du|^p - \left( \frac{p-1}{p} \right)^p q_p(x_1)p|u|^p \right) d\gamma \geq C \left( \int_\Omega |Du|^q d\gamma \right)^{p/q}, \quad q < p.
\]

**Proof.** Recalling that smooth factorized functions with compact support are dense in \( W^{1,p}_0(\Omega, d\gamma) \), we just prove the claim for \( u = \prod_{j=1}^N u_j(x_j) \), with \( u_1 \in C^\infty_0 (I_a) \), \( u_j \in C^\infty_0 (\mathbb{R}), j \geq 2 \). Then, by (20), we have
\[
\int_a^\infty \left( \frac{du_1}{dx_1} \right)^p - \left( \frac{p-1}{p} \right)^p q_p(x_1)p|u_1|^p \right) d\gamma_1 \geq C \left( \int_a^\infty \left| \frac{du_1}{dx_1} \right|^q d\gamma_1 \right)^{p/q}, \quad q < p.
\]
Multiplying by \( \prod_{j=2}^N |u_j(x_j)|^p \) we get
\[
\int_a^\infty \left( \frac{\partial u}{\partial x_1} \right)^p - \left( \frac{p-1}{p} \right)^p q_p(x_1)p|u|^p \right) d\gamma_1 \geq C \left( \int_a^\infty \left| \frac{du_1}{dx_1} \right|^q d\gamma_1 \left( \prod_{j=2}^N |u_j(x_j)|^q \right) \right)^{p/q} = C \left( \int_a^\infty \left| \frac{\partial u}{\partial x_1} \right|^q d\gamma_1 \right)^{p/q}.
\]
Multiplying by $\prod_{j=2}^{N} e^{-x_j^2/2}$ and integrating with respect to $dx_2...dx_N$, by Jensen inequality we can deduce

$$\int_{\Omega} \left( \left| \frac{\partial u}{\partial x_1} \right|^p - \left( \frac{p-1}{p} \right)^p \varrho_p(x_1)|u|^p \right) d\gamma \geq C \left( \int_{\Omega} \left| \frac{\partial u}{\partial x_1} \right|^q d\gamma \right)^{p/q}.$$

Then, by Hölder inequality and some elementary convexity arguments, we obtain

$$\int_{\Omega} \left| Du \right|^p \geq C \left( \int_{\Omega} \left| \frac{\partial u}{\partial x_1} \right|^q d\gamma \right)^{p/q} + \int_{\Omega} \left[ \left( \frac{\partial u}{\partial x_2} \right)^2 + ... + \left( \frac{\partial u}{\partial x_N} \right)^2 \right]^{p/2} d\gamma$$

$$\geq C \sum_{j=1}^{N} \left( \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^q d\gamma \right)^{p/q}$$

$$\geq C \left( \int_{\Omega} \left| Du \right|^q d\gamma \right)^{p/q}.$$ 

More generally, when $\Omega \subset \mathbb{R}^N$ is an open, connected set such that $|\Omega|_{\gamma} < 1$, one can reduce to the one-dimensional case by means of Gaussian symmetrization. It transforms $\Omega$ in the half-space $\Omega^\sharp = \{ x = (x_1, ..., x_N) \in \mathbb{R}^N : x_1 > a \}$, with $a = k^{-1}(|\Omega|_{\gamma})$ and, recalling that $u^\sharp(x)$ depends only on $x_1$ and it is an increasing function, by (19) and (17) we get

$$\int_{\Omega} \left| Du \right|^p d\gamma \geq \int_{\Omega^\sharp} \left| Du \right|^p d\gamma = \int_{a}^{\infty} \left( \frac{du^\sharp}{dx_1} \right)^p d\gamma_1$$

and

$$\int_{\Omega} \left| u \right|^q d\gamma = \int_{\Omega^\sharp} \left| u \right|^q d\gamma.$$

On the other hand, since the function $\varrho_p(x_1) = \varrho(a, p; x_1)$ is not increasing with respect to $x_1$, the term

$$\int_{\Omega} \varrho_p(x_1)|u|^p d\gamma$$

in general, does not increase under Gaussian symmetrization. Therefore we consider the following new weight

$$\varrho^\sharp(x_1) = \varrho^\sharp(a, p; x_1) = \left\{ \begin{array}{ll} \min \varrho_p & a \leq x_1 \leq \bar{x}_1 \\ \varrho_p(x) & x_1 \geq \bar{x}_1, \end{array} \right.$$ 

where $\bar{x}_1$ denotes the point where $\varrho_p$ achieves its minimum. So we preserve the singularity at infinity, where the Gaussian function degenerates. Therefore, the following result appears meaningful when the domain is unbounded. These considerations, together with (19), are the main tools in the proof of the following result.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^N$ be an open, connected set such that $|\Omega|_{\gamma} < 1$; then there exists a positive constant $C$ such that for every $u \in W^{1,p}_0(\Omega, d\gamma)$

$$\int_{\Omega} \left( |Du|^p - \left( \frac{p-1}{p} \right)^p \varrho^\sharp(x_1)|u|^p \right) d\gamma \geq C \left( \int_{\Omega} |u|^q d\gamma \right)^{p/q}, \quad q < p. \quad (49)$$
Proof. By (19), (17) and (18) Gaussian symmetrization lowers the left hand side of (49) while leaves unchanged the right hand side, as said before. Hence (49) follows immediately from (45).

Remark 4. We do not know if the ratio
\[ \int_\Omega \left( |Du|^p - \left( \frac{p-1}{p} \right)^p \rho_T(x_1)^p |u|^p \right) d\gamma \quad \left( \int_\Omega |Du|^q d\gamma \right)^{p/q} \]
decreases under Gaussian symmetrization. Actually, the analogous question is still unsolved, as far as we know, for the classical Hardy inequality under Schwarz symmetrization.

Remark 5. The well-known Gross Logarithmic Sobolev inequality with respect to the Gaussian measure (see [19]) states that for every \( u \in W^{1,2}_0(\Omega, d\gamma) \)
\[ \int_\Omega u^2 \log |u| d\gamma \leq \int_\Omega |Du|^2 d\gamma + \frac{1}{2} \left( \int_\Omega u^2 d\gamma \right) \log \left( \int_\Omega u^2 d\gamma \right). \]
It ensures that the weighted Sobolev space \( W^{1,2}_0(\Omega, d\gamma) \) is contained in the Lorentz-Zygmund space \( L^2(\log L)^{1/2}(\Omega, d\gamma) \). We explicitly observe that, when \( \Omega \) is a connected open set, the same inclusion is provided by our inequality
\[ \int_\Omega |Du|^2 d\gamma \geq \frac{1}{4} \int_\Omega \rho_T(x_1)^2 u^2 d\gamma. \]
Indeed, by definition
\[ ||u||_{L^2(\log L)^{1/2}(\Omega, d\gamma)}^2 = \int_0^{\rho_T} (1 - \log t) u^*(t)^2 dt = \int_0^\infty (1 - \log k(x_1)) u^*(k(x_1))^2 d\gamma_1 \]
where \( a \) is chosen in such a way that the half-space \( \{ x_1 > a \} \) has the same Gaussian measure as \( \Omega \). Since
\[ \lim_{x_1 \to \infty} \frac{1 - \log k(x_1)}{\rho_T(x_1)^2} = \frac{1}{2}, \]
we get our assertion simply using (26) with \( p = 2 \).

5. Application to PDE’s. The inequalities contained in Section 4 can be used in order to prove the existence of a solution to the following Dirichlet problem
\[
\begin{align*}
-\text{div}(|Du|^{p-2}Du) & = \lambda \rho_T(x_1)^p u^{p-2} e^{-|x|^2/2} + f e^{-|x|^2/2} & \text{in } \Omega \\
u & = 0 & \text{on } \partial \Omega,
\end{align*}
\]
where the notation of the previous sections \( \rho_T \) and \( \Omega \) are in force. We explicitly observe that, when \( p = 2 \), the equation becomes
\[-\Delta u + <x, Du> = \lambda \rho_T(x_1)^2 u + f,\]
where on the left hand side one recognizes the well-known Hermite operator. In problem (50) we consider the subcritical case \( \lambda < \left( \frac{p-1}{p} \right)^p \), obtaining the existence of a solution in the energy space. While, when \( \Omega \) is the half-space \( \Omega^\# \), estimate (48) allows us to treat also the critical case \( \lambda = \left( \frac{p-1}{p} \right)^p \) obtaining the existence of a distributional solution belonging to the intersection of the Sobolev spaces \( W^{1,q}_0(\Omega, d\gamma) \), \( q < p \), without truncating the singularity.
Theorem 5.1. Let \( f \in L^{p'}(\Omega, d\gamma) \) and \( \lambda < \left( \frac{p-1}{p} \right)^p \); then problem \((50)\) admits a weak solution in \( W_0^{1,p}(\Omega, d\gamma) \).

Proof. The proof follows by means of standard techniques based on coerciveness arguments and Ekeland variational principle (see, for instance, [18]).

In the critical case \( \lambda = \left( \frac{p-1}{p} \right)^p \), in view of estimate \((48)\) we do not need to truncate the singularity \( \rho_p \), but we have to confine ourselves to the case of half-spaces or, clearly, domains contained in half-spaces.

Theorem 5.2. Let \( \Omega \) be the half-space \( \{ x_1 > a \} \) and \( f \in L^r(\Omega, d\gamma) \), with \( r > p' \); then the problem

\[
\begin{cases}
-\text{div}(|Du|^{p-2}Du) e^{-|x|^2/2} = \left( \frac{p-1}{p} \right)^p \rho_p(x_1)^p |u|^{p-2}ue^{-|x|^2/2} + fe^{-|x|^2/2} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

admits a distributional solution \( u \) belonging to \( W_0^{1,q}(\Omega, d\gamma) \) for any \( 1 \le q < p \).

Proof. Let us consider the following sequence of approximate problems

\[
\begin{cases}
-\text{div}(|Du_m|^{p-2}Du_m) e^{-|x|^2/2} = \left( \frac{p-1}{p} \right)^p P_m(x_1) |u_m|^{p-2}u_me^{-|x|^2/2} + fe^{-|x|^2/2} & \text{in } \Omega \\
u_m = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( P_m(x_1) = \min \{ m, (1-\frac{1}{m}) \rho_p(x_1)^p \} \), \( m \in \mathbb{N} \). Problem \((51)\) always admits at least a weak solution \( u_m \), i.e.

\[
\int_{\Omega} |Du_m|^{p-2}Du_m D\varphi d\gamma = \left( \frac{p-1}{p} \right)^p \int_{\Omega} P_m(x_1) |u_m|^{p-2}u_m \varphi d\gamma \quad \text{(52)}
\]

To this aim it suffices to show that the functional

\[
G_m : u \in W_0^{1,p}(\Omega, d\gamma) \rightarrow \frac{1}{p} \int_{\Omega} |Du|^p d\gamma - \frac{1}{p} \left( \frac{p-1}{p} \right)^p \int_{\Omega} P_m(x_1) |u|^p d\gamma - \int_{\Omega} f u d\gamma
\]

is coercive for every fixed \( m \). Since by definition

\[
\left( 1 - \frac{1}{m} \right) \rho_p(x_1)^p - P_m(x_1) \ge 0,
\]

applying [18], Holder inequality and Poincaré inequality we get

\[
G_m(u) = \frac{1}{mp} \int_{\Omega} |Du|^p d\gamma + \left( 1 - \frac{1}{m} \right) \frac{1}{p} \int_{\Omega} |Du|^p d\gamma
\]

\[
- \frac{1}{p} \left( \frac{p-1}{p} \right)^p \int_{\Omega} P_m(x_1) |u|^p d\gamma - \int_{\Omega} f u d\gamma
\]

\[
\ge \frac{1}{mp} \int_{\Omega} |Du|^p d\gamma + \frac{1}{p} \left( \frac{p-1}{p} \right)^p \int_{\Omega} |u|^p \left[ \left( 1 - \frac{1}{m} \right) \rho_p(x_1)^p - P_m(x_1) \right] d\gamma
\]

\[
\ge \frac{1}{mp} ||Du||_{L^p(\Omega,d\gamma)} ||f||_{L^p(\Omega,d\gamma)} - C ||Du||_{L^p(\Omega,d\gamma)} ||f||_{L^{p'}(\Omega,d\gamma)}.
\]
Choosing \( \varphi = u_m \) in (52), by (48) and Sobolev inequality we get
\[
C_1 ||u_m||_{W_0^{1,q}(\Omega, d\gamma)}^p \leq \int_{\Omega} |Du_m|^p d\gamma - \left( \frac{p-1}{p} \right)^p \int_{\Omega} \varrho p(x_1)^p |u_m|^p d\gamma
\]
\[
\leq \int_{\Omega} |Du_m|^p d\gamma - \left( \frac{p-1}{p} \right)^p \int_{\Omega} P_m(x_1) |u_m|^p d\gamma
\]
\[
= \int_{\Omega} f u_m d\gamma \leq C_2 ||f||_{L^{q'}(\Omega, d\gamma)} ||u_m||_{W_0^{1,q}(\Omega, d\gamma)}.
\]
where \( C_1, C_2 \) are positive constants. From the above inequality we deduce that \( u_m \) is uniformly bounded in \( W_0^{1,q}(\Omega, d\gamma) \) for any \( 1 \leq q < p \), so, up to a subsequence, we have
\[
\begin{cases}
  u_m \rightharpoonup u & \text{in} \ W_0^{1,q}(\Omega, d\gamma) \\
  u_m \to u & \text{in} \ L^q(\Omega, d\gamma) \text{ and a.e.}
\end{cases}
\]
When \( \varphi \in C_0^\infty(\Omega) \), we can pass to the limit into (52) proving that \( u \) is a distributional solution arguing as in [18]. \( \square \)

We end this section by showing an example of a one-dimensional equation which exhibits a trivial solution in the energy space and a pathological one. It is closely related to Serrin’s famous pathological solutions for elliptic equations in divergence form with bounded coefficients (see [27], see also [31]).

**Theorem 5.3.** Let \( 0 < \lambda < \frac{1}{4} \); then there exists a smooth function which solves the problem
\[
\begin{cases}
  - \left( w' e^{-x_1^2/2} \right)' - \lambda w(0, x_1) e^{-x_1^2/2} = 0 & \text{in} \ I_1 \\
  w(1) = \lim_{x_1 \to +\infty} w(x_1) = 0
\end{cases}
\]  
(53)
in the classical (i.e. everywhere) sense, but it does not belong to \( W_0^{1,2}(I_1, d\gamma_1) \).

**Proof.** As a straightforward calculation shows, the functions
\[
w_1(x_1) = \left( \int_0^{x_1} \exp \left( \frac{x_2^2}{2} \right) d\sigma \right)^{-\frac{1}{2}}, \quad w_2(x_1) = \left( \int_0^{x_1} \exp \left( \frac{x_2^2}{2} \right) d\sigma \right)^{-\frac{1}{2}},
\]
where \( c_1(\lambda) = -\frac{1}{2} + \sqrt{\frac{1}{4} - 4\lambda} \) and \( c_2(\lambda) = -\frac{1}{2} - \sqrt{\frac{1}{4} - 4\lambda} \), are independent solutions to the equation in problem (53). As it is easy to verify, \( w_1 \notin W_0^{1,2}(I_1, d\gamma_1) \) while \( w_2 \in W_0^{1,2}(I_1, d\gamma_1) \). Thus the function \( w = w_1 - w_2 \) satisfies the initial condition in (53) and clearly does not belong to the energy space. \( \square \)

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