LOGARITHMIC AND IMPROVED REGULARITY CRITERIA FOR THE 3D NEMATIC LIQUID CRYSTALS MODELS, BOUSSINESQ SYSTEM, AND MHD EQUATIONS IN A BOUNDED DOMAIN

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Abstract. In this paper, we prove some logarithmically improved regularity criteria for the 3D nematic liquid crystals models, Boussinesq system, and MHD equations in a bounded domain.

1. Introduction. In this paper we improve in different ways criteria concerning continuation of strong solutions of several models linked with the Navier-Stokes equations. Since the proofs have several points in common, we collect results for different models in a single paper, instead of scattering them through the literature. In particular, we prove with full details especially results for the nematic liquid crystals model below, and in the rest of the paper we show the changes needed to adapt the same techniques to different equations.

A particular feature of this paper is that we consider the problem in a bounded domain, with Dirichlet conditions on the velocity. Several results in literature concern the Cauchy problem (or the space periodic case) where proofs of regularity criteria are considerably less technical. In this paper we mainly focus in obtaining results with realistic boundary conditions, dealing carefully with the various boundary terms. To this end, let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial \Omega$, and $\nu$ is the unit outward normal vector to $\partial \Omega$. First, we consider the regularity criterion for the density-dependent, incompressible, nematic liquid crystals model [8, 10, 17, 19]:

\begin{align}
\partial_t \rho + \text{div}(\rho u) &= 0 & \text{in} & \Omega \times (0, \infty), \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla \pi - \Delta u &= -\nabla \cdot (\nabla d \otimes \nabla d) & \text{in} & \Omega \times (0, \infty), \\
\partial_t d + u \cdot \nabla d + |d|^2 d - d - \Delta d &= 0 & \text{in} & \Omega \times (0, \infty), \\
\text{div} u &= 0 & \text{in} & \Omega \times (0, \infty),
\end{align}

in $\Omega \times (0, \infty)$ with initial and boundary conditions

\begin{align}
(r, ru, d) (\cdot, 0) &= (\rho_0, \rho_0 u_0, d_0) & \text{in} & \Omega, \\
u &= 0, \partial_n d = 0 & \text{on} & \partial \Omega \times (0, \infty),
\end{align}

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where \( \rho \) denotes the density, \( u = (u_1, u_2, u_3) \) the velocity, \( \pi \) the pressure, and \( d = (d_1, d_2, d_3) \) represents the macroscopic molecular orientations. The for each component of \( d \), the symbol \( \nabla d \otimes \nabla d \) denotes a matrix whose \((i,j)\)-th entry is \( \partial_i d \partial_j d \). (We consider the viscosity and other physical parameters set equal to one, to avoid unessential complications)

When \( d = 0 \), then (1)-(3) represent the well-known density-dependent Navier-Stokes system, which has been object of many studies \([9, 16, 14]\). In particular, Kim \([16]\) proved in this case that the following assumption is enough to continue smooth solutions:

\[
\|u\|_{L^2(0,T;L^q_w(\Omega))} < \infty, \quad \text{with} \quad 3 < q \leq \infty,
\]

here \( L^q_w(\Omega) \) denotes the (Marcinkiewicz) weak-\( L^q(\Omega) \) space and \( L^\infty_w(\Omega) = L^\infty(\Omega) \). We recall that for the Navier-Stokes system this type of results in weak space date back to Sohr \([23]\), see also \([6]\), concerning Marcinkiewicz regularity with respect to the time variable.

When \( \rho = 1 \), (that is the nematic model with constant density) Guillén-González, Rojas-Medar, and Rodriguez-Bellido \([15]\) proved the following regularity criterion (which we are going to logarithmically improve): If

\[
\int_0^T \left( \|u(t)\|_{L^q_L}^{2q} + \|\nabla d(t)\|_{L^q_L}^{2q} \right) dt < \infty, \quad \text{with} \quad 3 < q \leq \infty,
\]

then strong solution can be continued beyond \( T > 0 \).

It is well-known that problem (1)-(6) has a unique local-in-time strong solution (see \([18]\)) and we omit the details here, we only recall that in particular a strong solution, we mean a quadruplet \((\rho, u, \pi, d)\) satisfying (1)-(4) almost everywhere with the initial-boundary conditions (5)-(6). (In particular the quantities \( \|u\|_{H^2}, \|d\|_{H^3}, \|\rho\|_{W^{1,p}} \) are bounded almost everywhere). The first aim of this paper is to prove the following regularity criterion.

**Theorem 1.1.** Let \( \rho_0 \in W^{1,p}(\Omega), \ u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \ d_0 \in H^3(\Omega) \) with \( 3 < p \leq 6 \), and \( \rho_0 \geq 0 \), \( \text{div} \, u_0 = 0 \) in \( \Omega \), with \( \partial_{\nu}d_0 = 0 \) on \( \partial\Omega \). Let also the following compatibility condition at time \( t = 0 \) holds true: \( \exists (\pi_0,g) \in H^1(\Omega) \times L^2(\Omega) \) such that

\[
\nabla \pi_0 - \Delta u_0 + \nabla \cdot (\nabla d_0 \otimes \nabla d_0) = \sqrt{\rho_0} g \quad \text{in} \quad \Omega.
\]

Let \((\rho,u,d)\) be a local strong solution to the problem (1)-(6). If \( u \) satisfies one of the following two conditions:

\[
(i) \quad \int_0^T \frac{\|u(t)\|_{L^q_L}^{2q}}{1 + \log(e + \|u(t)\|_{L^q_L})} dt < \infty \quad \text{with} \quad 3 < q \leq \infty, \\
(ii) \quad u \in L^2(0,T;BMO(\Omega)),
\]

with \( 0 < T < \infty \), then, the strong solution \((\rho,u,d)\) can be extended beyond \( T > 0 \).

**Remark 1.** The above compatibility condition means that the initial value has to satisfy \((\sqrt{\rho_0} u)(\cdot,0) \in L^2(\Omega))\.
In the case of constant density $\rho$, for simplicity set equal to 1, we consider also the following simplified liquid crystals model:

$$\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u &= -\nabla \cdot (\nabla d \otimes \nabla d) \quad \text{in } \Omega \times (0, \infty), \\
\partial_t d + u \cdot \nabla d + |d|^2 d - d &= \Delta d \quad \text{in } \Omega \times (0, \infty), \\
u &= 0, \quad \partial_n d = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
\nabla u &= 0, \quad \text{in } \Omega \times (0, \infty), \\
(u, d)(\cdot, 0) &= (u_0, d_0) \quad \text{in } \Omega.
\end{align*}$$

(9)

Clearly, for the initial boundary problem (9)-(13) the same regularity criterion (7) or (8) holds true. Furthermore, we will prove also the following result concerning a more classical scaling-invariant criterion in weak-spaces.

**Theorem 1.2.** Let $u_0 \in H^1_0(\Omega) \cap H^3(\Omega)$, $d_0 \in H^3(\Omega)$, with $\nabla u_0 = 0$ in $\Omega$, and $\partial_n d_0 = 0$ on $\partial \Omega$. Let $(u, d)$ be a local strong solution to the problem (9)-(13). If $u$ satisfies:

$$\int_0^T \left\| \nabla u(t) \right\|_{L^q}^{\frac{2q}{q-2}} dt < \infty \quad \text{for } \frac{3}{2} < q \leq \infty,$$

(14)

with $0 < T < \infty$. Then the solution $(u, d)$ can be extended beyond $T > 0$.

Observe that this condition is the same (since it does not involve the variable $d$) which holds true for the Navier-Stokes equations, see [23].

As a further application of related techniques, we next consider the 3D viscous/diffusive Boussinesq system:

$$\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u &= -\theta e_3 \quad \text{in } \Omega \times (0, \infty), \\
\partial_t \theta + u \cdot \nabla \theta &= \Delta \theta \quad \text{in } \Omega \times (0, \infty), \\
\nabla u &= 0 \quad \text{in } \Omega \times (0, \infty), \\
u &= 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
(u, \theta)(\cdot, 0) &= (u_0, \theta_0) \quad \text{in } \Omega.
\end{align*}$$

(15)

Here the scalar field $\theta$ is the temperature and $e_3 := (0, 0, 1)t$. As expected, the regularity criteria (7) and (8) are valid also for the Boussinesq initial boundary value problem (15)-(19). Extensions are proved in the following theorem.

**Theorem 1.3.** Let $u_0 \in H^1_0(\Omega) \cap H^3(\Omega)$ and $\theta_0 \in H^1_0(\Omega) \cap H^2(\Omega)$. Let $(u, \theta)$ be a local strong solution to the problem (15)-(19). If $u$ satisfies one of the following two conditions:

$$(i) \quad \int_0^T \frac{\|\nabla u(t)\|_{L^q}^{\frac{2q}{q-2}}}{1 + \log(e + \|\nabla u(t)\|_{L^q})} dt < \infty \quad \text{with } \frac{3}{2} < q \leq \infty,$$

(20)

$$(ii) \quad \nabla u \in L^1(0, T; BMO(\Omega)),$$

(21)

for some with $0 < T < \infty$. Then, the solution $(u, \theta)$ can be extended beyond $T > 0$.

**Remark 2.** The results of the previous theorem improve those collected in refs. [4, 5, 7]. Moreover, results extend to the bounded domain previous results from refs. [25, 13, 26, 11]
Next, we consider the regularity criterion for the 3D MHD system:

\begin{align}
\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u &= b \cdot \nabla b \quad \text{in } \Omega \times (0, \infty), \\
\partial_t b + u \cdot \nabla b - b \cdot \nabla u &= \Delta b \quad \text{in } \Omega \times (0, \infty), \\
\operatorname{div} u &= \operatorname{div} b = 0 \quad \text{in } \Omega \times (0, \infty), \\
u &= 0, \quad b \cdot \nu = 0, \quad \operatorname{curl} b \times \nu = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
(u, b)(\cdot, 0) &= (u_0, b_0) \quad \text{in } \Omega,
\end{align}

where the vector field \( b = (b_1, b_2, b_3) \) is the magnetic field. For the MHD initial boundary value problem (22)-(26) one can prove scaling invariant regularity criteria involving just the velocity field, as those with conditions (7) and (8). Furthermore, we will prove the following result.

**Theorem 1.4.** Let \( u_0 \in H^2_{01}(\Omega) \cap H^2(\Omega) \), \( b_0 \in H^2(\Omega) \), with \( \operatorname{div} u_0 = \operatorname{div} b_0 = 0 \) in \( \Omega \), and \( b_0 \cdot \nu = 0, \ \operatorname{curl} b_0 \times \nu = 0 \) on \( \partial \Omega \). Let \((u, b)\) be a local strong solution to the problem (22)-(26). If (14) holds true, then the solution \((u, b)\) can be extended beyond \( T > 0 \).

Next, we consider another liquid crystals model: (1), (2), (4), (5), (6) and

\[ \partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \]

with \(|d| \equiv 1\) in \( \Omega \times (0, \infty) \). Li and Wang [18] have proved that the problem has a unique local strong solution. When \( \Omega := \mathbb{R}^3 \), Fan-Gao-Guo [12] showed a blow-up criterion, which we are improving both logarithmically and both by studying the problem in a bounded domain. We will prove the following result.

**Theorem 1.5.** Let the initial data satisfy the same conditions as in Theorem 1.1 and assume in addition that \(|d_0| \equiv 1\) in \( \Omega \). Let \((\rho, u, d)\) be a local strong solution to the problem (1), (2), (4), (5), (6) and (27). If \( u \) and \( \nabla d \) satisfy

\[ \int_0^T \frac{\|u(t)\|_{L^q_{L^p}}^{\frac{2q}{s}} + \|\nabla d(t)\|_{L^q_{L^p}}^{\frac{2q}{s}}}{1 + \log(e + \|u(t)\|_{L^q_{L^p}} + \|\nabla d(t)\|_{L^q_{L^p}})} \, dt < \infty, \quad \text{with} \quad 3 < q \leq \infty, \]

with \( 0 < T < \infty \), then the solution \((\rho, u, d)\) can be extended beyond \( T > 0 \).

Finally, we consider the liquid crystals model with constant density (9), (10), (12), (13), and (27). We will prove the following result:

**Theorem 1.6.** Let the initial data satisfy the same conditions as in Theorem 1.2 and assume in addition that \(|d_0| \equiv 1\) in \( \Omega \). Let \((u, d)\) be a local strong solution to the problem (9), (10), (27), (12) and (13). If \( u \) and \( \nabla d \) satisfy

\[ \int_0^T (\|\nabla u(t)\|_{L^q_{L^p}}^{\frac{2q}{s}} + \|\nabla d(t)\|_{L^q_{L^p}}^{\frac{2q}{s}}) \, dt < \infty, \]

with \( \frac{3}{2} < s \leq \infty, \quad 3 < q \leq \infty, \) and \( 0 < T < \infty. \) Then, the solution \((u, d)\) can be extended beyond \( T > 0 \).

1.1. **Notation and some preliminary lemmas.** In the following we will use customary Lebesgue \( L^p(\Omega) \) and Sobolev spaces \( W^{k,p}(\Omega) \) and \( H^s(\Omega) = W^{s,2}(\Omega) \), without distinction between scalar, vector, and tensor fields. We will denote by \( C \) generic constants changing from line to line, but independent on relevant quantities and for simplicity we will also write \( \int \cdot \, dx := \int_\Omega \cdot \, dx. \) In our proofs, we will use the following basic lemmas, which are at the basis of the derivation of most of the
estimates. To handle boundary terms arising when looking for \(L^p\) estimates we will use the following integration by parts and trace inequality.

**Lemma 1.7** ([2, 3]). Let \(\Omega \subset \mathbb{R}^3\) be a smooth bounded domain, let \(b : \Omega \to \mathbb{R}^3\) be a smooth vector field, and let \(1 < p < \infty\). Then

\[
- \int_{\Omega} \Delta b \cdot b |b|^{p-2} \, dx = \frac{1}{2} \int_{\Omega} |b|^{p-2} |\nabla b|^2 \, dx + \frac{4p - 2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{2}{p}}|^2 \, dx
\]

and

\[
- \int_{\Omega} \Delta b \cdot b |b|^{p-2} \, dx = \frac{1}{2} \int_{\Omega} |b|^{p-2} |\nabla b|^2 \, dx + \frac{4p - 2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{2}{p}}|^2 \, dx
\]

\[
- \int_{\partial \Omega} |b|^{p-2} (\nu \cdot \nabla) b \cdot b \, d\sigma.
\]

**Lemma 1.8** ([1, 20]). Let \(\Omega\) be a smooth and bounded open set and let \(1 < p < \infty\). Then the following trace estimate

\[
\|b\|_{L^p(\partial \Omega)} \leq C\|b\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|b\|_{W^{1,p}(\Omega)}^{\frac{1}{p}}
\]

holds for any \(b \in W^{1,p}(\Omega)\).

When \(b\) satisfies \(b \cdot \nu = 0\) on \(\partial \Omega\), we will also use the identity

\[
(b \cdot \nabla) b - (b \cdot \nabla) \nu \cdot b = 0 \quad \text{on} \quad \partial \Omega,
\]

which is valid for any sufficiently vector field \(b\).

**Lemma 1.9** ([16]). Let \(f \in H^1(\Omega)\) and \(\Omega \subseteq \mathbb{R}^3\). Then there holds

\[
\|f\|_{L^{\frac{5}{3}}(\Omega)} \leq C\|f\|_{L^2(\Omega)}^{1-\frac{3}{5}} \|f\|_{L^3(\Omega)}^{\frac{3}{5}}, \quad \text{with} \quad 3 < s \leq \infty.
\]

We will also use the following generalization of Hölder and Sobolev inequality.

**Lemma 1.10** ([24]). There holds the generalized Hölder inequality:

\[
\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,r_1}} \|g\|_{L^{p_2,r_2}},
\]

with \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\).

**Lemma 1.11** ([22]). There holds the following logarithmic Sobolev inequality:

\[
\|\nabla f\|_{L^\infty} \leq C(1 + \|\nabla f\|_{BMO} \log(e + \|f\|_{W^{s,p}})), \quad \text{with} \quad s > 1 + \frac{3}{p}
\]

for any \(f \in W^{s,p}(\Omega)\) and \(\Omega \subseteq \mathbb{R}^3\).

**Lemma 1.12** ([21]). There holds

\[
\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \log^\frac{1}{2}(e + \|f\|_{W^{1,4}(\Omega)}))
\]

for any \(f \in W^{1,4}_0(\Omega)\) and \(\Omega \subseteq \mathbb{R}^3\).

**Proof.** When \(\Omega := \mathbb{R}^3\), (34) has been proved in Ogawa [21]. When \(\Omega\) is a bounded domain in \(\mathbb{R}^3\). We can define

\[
\hat{f} := \begin{cases} f & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \Omega^c := \mathbb{R}^3 \setminus \Omega. \end{cases}
\]
Then it is obvious that
\[ \|\tilde{f}\|_{L^\infty(\mathbb{R}^3)} = \|f\|_{L^\infty(\Omega)}, \quad \|\tilde{f}\|_{BMO(\mathbb{R}^3)} = \|f\|_{BMO(\Omega)}, \]
and we have, see [1, p. 71],
\[ \|\tilde{f}\|_{W^{1,q}(\mathbb{R}^3)} = \|f\|_{W^{1,q}(\Omega)}. \]
Thus (34) is proved.

2. On density-dependent, incompressible, nematic liquid crystals. Due to the aforementioned existence and uniqueness theorem for local strong-solutions, by using a standard continuation argument, we only need to establish a priori estimates on smooth enough local solutions. Hence, by using a standard argument we assume existence of strong solutions in the maximal interval \([0, T]\), and all the calculations we are going to perform are completely justified. By proving uniform bounds for all \(t \in [0, T]\), for appropriate norms of the unknowns, we will show that the solution can be continued beyond \(T\), contradicting its maximality. In particular, in this case it will be enough to show boundedness for the \(H^2(\Omega)\)-norm of the velocity and for the \(H^3(\Omega)\)-norm of \(d\).

Proof of Theorem 1.1. First, observe that, thanks to the maximum principle, it follows from (1) and (2) that
\[ 0 \leq \rho(x, t) \leq M < \infty \quad x \in \Omega, \ t \in [0, T]. \]

By testing (3) by \(u\), by suitable integration by parts (with boundary terms vanishing due to the boundary conditions (6)), and, by using (1) and (2), we see that
\[ \frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int |\nabla u|^2 dx = - \int (u \cdot \nabla) d \cdot \Delta d dx. \] (36)
Next, we test (4) by \(-\Delta d + |d|^2 d - d\) and by using (1), we easily obtain the equality
\[ \frac{d}{dt} \int \left( \frac{1}{2} |\nabla d|^2 + \frac{1}{2} (|d|^2 - 1)^2 \right) dx + \int (-\Delta d + |d|^2 d - d)^2 dx = \int (u \cdot \nabla) d \cdot \Delta d dx. \]

Summing up these two estimates we get the well-known energy inequality
\[ \frac{1}{2} \frac{d}{dt} \int \left( \rho|u|^2 + |\nabla d|^2 + \frac{1}{2} (|d|^2 - 1)^2 \right) dx + \int (|\nabla u|^2 + (-\Delta d + |d|^2 d - d)^2) dx \leq 0. \] (37)
Next, we prove the following estimate on \(d\):
\[ \|d\|_{L^\infty(0, T; L^\infty)} \leq \max(1, \|d_0\|_{L^\infty}). \] (38)
Without loss of generality, we can assume that \(1 \leq \|d_0\|_{L^\infty}\). Multiplying (4) by \(2d\), we get
\[ \partial_t \phi + u \cdot \nabla \phi - \Delta \phi + 2 |d|^2 \phi = -2 |d|^2 (\|d_0\|^2_{L^\infty} - 1) - 2 |\nabla d|^2 \leq 0 \] (39)
with \(\phi := |d|^2 - \|d_0\|^2_{L^\infty}\) and \(\phi(-\cdot) = |d|^2 - \|d_0\|^2_{L^\infty} \leq 0\) and \(\partial_t \phi = 0\) on \(\partial \Omega \times (0, \infty)\). Then, estimate (38) follows from (39) by the using the standard maximum principle.

We prove now the first part I of the theorem. Let (7) hold true. Taking \(\nabla\) to (4)_i, we deduce that
\[ \partial_t \nabla d_i + (u \cdot \nabla) \nabla d_i + \nabla (|d|^2 d_i) - \nabla d_i - \Delta \nabla d_i = \sum_j \nabla u_j \partial_j d_i. \] (40)
Testing (40) by $|\nabla d_i|^{p-2}\nabla d_i$ ($2 \leq p \leq 6$), using (1), (28), (30), (29), (31), (32), and (38) and summing over $i$, we derive

$$\frac{1}{p} \frac{d}{dt} \int_\Omega |\nabla d|^p dx + \frac{1}{2} \int_\Omega |\nabla d|^{p-2}|\nabla^2 d|^2 dx + 4 \frac{p-2}{p^2} \int_\Omega |\nabla |\nabla d|^\frac{p}{p-2}|^2 dx$$

$$= - \sum_i \int_{\partial \Omega} |\nabla d_i|^{p-2}(\nabla d_i \cdot \nabla) \nu \cdot \nabla d_i d\sigma - \sum_{i,j} \int_\Omega u_i \nabla (\partial_j d_i |\nabla d_i|^{p-2}\nabla d_i) dx$$

$$- \sum_i \int_\Omega \nabla (|d_i|^2 d_i - d_i) \cdot |\nabla d_i|^{p-2}\nabla d_i dx$$

$$\leq C \int_{\partial \Omega} |\nabla d|^p d\sigma + C \int_\Omega |u||\nabla d|^\frac{p}{p-2}|\nabla d|^\frac{p}{p-2} dx + C \int_\Omega |\nabla d|^p dx$$

$$+ C \int_\Omega |u||\nabla d|^\frac{p}{p-2} |\nabla d|^\frac{p}{p-2} |\nabla^2 d| dx$$

$$\leq C \|w\|_{L^2} \|w\|_{H^1} + C \|u\|_{L^\infty} \|w\|_{L^\frac{2n}{2n-2}} \|\nabla w\|_{L^2} + C \int_\Omega w^2 dx$$

$$+ C \|u\|_{L^\infty} \|w\|_{L^\frac{2n}{2n-2}} |||\nabla d|^\frac{p}{p-2} |\nabla^2 d||_{L^2}$$

$$\leq C \|w\|_{L^2} \|w\|_{H^1} + C \|u\|_{L^\infty} \|w\|_{L^\frac{2n}{2n-2}} \|w\|_{H^1}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}} + C \int_\Omega w^2 dx$$

$$+ C \|u\|_{L^\infty} \|w\|_{L^\frac{2n}{2n-2}} \|w\|_{H^1}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}} |\nabla d|^\frac{p}{p-2} |\nabla^2 d||_{L^2}$$

$$\leq \frac{2(p-2)}{p^2} \int_\Omega |\nabla |\nabla d|^\frac{p}{p-2}|^2 dx + \frac{1}{4} \int_\Omega |\nabla d|^{p-2} |\nabla^2 d|^2 dx + C |||w||_{L^2} + C \|u\|_{L^\infty} \|w\|_{L^2}^2.$$

With $y(t) := \sup_{[t_0,t]} \|u(\tau)\|_{W^{1,4}}$, this gives

$$\frac{d}{dt} \int_\Omega w^2 dx + \int_\Omega |\nabla w|^2 dx \leq C(1 + ||u||_{L^\infty}) \|w\|_{L^2}^2$$

$$\leq C \left( 1 + \frac{||u||_{L^\infty}^{\frac{2n}{2n-2}}}{1 + \log(e + ||u||_{L^\infty})} \right) \|w\|_{L^2}^2 (1 + \log(e + ||u||_{L^\infty}))$$

$$\leq C \left( 1 + \frac{||u||_{L^\infty}^{\frac{2n}{2n-2}}}{1 + \log(e + ||u||_{L^\infty})} \right) \|w\|_{L^2}^2 (1 + \log(e + y)).$$

From the latter, by using Gronwall lemma it follows that

$$||\nabla d||_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t \int_\Omega |\nabla d|^{p-2} |\nabla^2 d|^2 dx dx \leq C(e + y(t))^{C_0},$$

(41)

for any $0 < t_0 \leq t \leq T$ and $C_0$ is an absolute constant, provided that we choose $t_0 < T$ close enough to $T$ such that

$$\int_{t_0}^T \frac{||u||_{L^\infty}^{\frac{2n}{2n-2}}}{1 + \log(e + ||u||_{L^\infty})} dt \leq \epsilon << 1.$$  

(42)
We first estimate $I_2$ as follows.

\[
I_2 = \frac{d}{dt} \int \nabla d \otimes \nabla d : \nabla u \, dx - \int \partial_t (\nabla d \otimes \nabla d) : \nabla u \, dx
\]

\[
\leq \frac{d}{dt} \int \nabla d \otimes \nabla d : \nabla u \, dx + C \| \nabla d \|_{L^6} \| \nabla u \|_{L^2} \| \nabla u \|_{L^3}^\frac{1}{2}
\]

\[
\leq \frac{d}{dt} \int \nabla d \otimes \nabla d : \nabla u \, dx + C \| \nabla d \|_{L^6} \| \nabla u \|_{L^2} \| \nabla u \|_{L^3}^\frac{1}{2} \| u \|_{H^2}^\frac{5}{2}
\]

\[
\leq \frac{d}{dt} \int \nabla d \otimes \nabla d : \nabla u \, dx + \delta \| \nabla d \|_{L^2}^2 + \delta \| u \|_{H^2}^2 + C \| \nabla d \|_{L^6} \| \nabla u \|_{L^2}^2,
\]

for any $0 < \delta < 1$ and for some $C > 0$ depending on $\delta$.

We use (35), (31) and (32) to bound $I_1$ as follows.

\[
I_1 \leq \| \sqrt{\rho} u_t \|_{L^2} \| \sqrt{\rho} \|_{L^\infty} \| u \|_{L^6} \| \nabla u \|_{L^{\frac{26}{17}}}^2
\]

\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} \| u \|_{L^6}^\frac{1}{2} \| \nabla u \|_{L^2}^\frac{5}{2} \| u \|_{H^2}^2
\]

\[
\leq \delta \| \sqrt{\rho} u_t \|_{L^2}^2 + \delta \| u \|_{H^2}^2 + C \| u \|_{L^6}^\frac{26}{17} \| \nabla u \|_{L^2}^2,
\]

for any $0 < \delta < 1$. On the other hand, by using (2) and rewriting (3) as follows

\[
-\Delta u + \nabla \left( \pi - \frac{1}{2} | \nabla d |^2 \right) = -\rho \partial_t u - \rho (u \cdot \nabla) u - \nabla d^T \cdot \Delta d,
\]

by using the $H^2$-regularity theory of the Stokes system, we obtain

\[
\| u \|_{H^2} \leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| \nabla d^T \cdot \Delta d \|_{L^2}
\]

\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| u \|_{L^6} \| \nabla u \|_{L^{\frac{26}{17}}}^2 + C \| \nabla d^T \cdot \Delta d \|_{L^2}
\]

\[
\leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| u \|_{L^6} \| \nabla u \|_{L^2}^\frac{1}{2} \| u \|_{H^2}^\frac{3}{2} + C \| \nabla d^T \cdot \Delta d \|_{L^2},
\]

which gives

\[
\| u \|_{H^2} \leq C \| \sqrt{\rho} u_t \|_{L^2} + C \| \nabla d^T \cdot \Delta d \|_{L^2} + C \| u \|_{L^6}^\frac{26}{17} \| \nabla u \|_{L^2}.
\]

Testing (4) by $-\Delta d_t$, using (37), (38), (31) and (32), we have

\[
\frac{1}{2} \frac{d}{dt} \int |\Delta d |^2 \, dx + \int |\nabla d_t |^2 \, dx = \int (|d |^2 d - d + u \cdot \nabla d) \Delta d_t \, dx
\]

\[
\leq \int | \nabla (|d |^2 d - d) + \nabla (u \cdot \nabla d) | \nabla d_t | \, dx
\]

\[
\leq C (1 + \| u \|_{L^6} \| \Delta d \|_{L^{\frac{26}{17}}} + \| \nabla u \|_{L^6} \| \nabla d \|_{L^6}) \| \nabla d_t \|_{L^2}
\]

\[
\leq C (1 + \| u \|_{L^6} \| \Delta d \|_{L^{\frac{26}{17}}} \| d \|_{H^2}^\frac{3}{2} + \| u \|_{H^2}^\frac{3}{4} \| \nabla d \|_{L^6}) \| \nabla d_t \|_{L^2}
\]

\[
\leq \frac{1}{2} \| \nabla d_t \|_{L^2}^2 + \frac{\delta}{2} \| d \|_{H^2}^2 + \frac{\delta}{2} \| u \|_{H^2}^2 + C + C \| u \|_{L^6}^\frac{26}{17} \| \Delta d \|_{L^2}^2
\]

\[
+ C \| \nabla u \|_{L^6}^2 \| \nabla d \|_{L^6}^4,
\]
for any $0 < \delta < 1$. On the other hand, by the $H^3$-regularity theory of the elliptic equation, it follows from (4), (38), (37), (31) and (32) that

\[
\|d\|_{H^3}^2 \leq C(\|d\|_{L^2} + \|\nabla d\|_{L^3}) \\
\leq C(1 + \|\nabla (d_t + u \cdot \nabla d + |d|^2 d - d)\|_{L^2}) \\
\leq C + C\|\nabla d_t\|_{L^2} + C\|u\|_{L^6} \|\Delta d\|_{L^6}^{\frac{2}{k-2}} + C\|\nabla u\|_{L^3} \|\nabla d\|_{L^6} \\
\leq C + C\|\nabla d_t\|_{L^2} + C\|u\|_{L^6} \|\Delta d\|_{L^6}^{\frac{1}{k-2}} + \delta \|\nabla u\|_{L^3}^{\frac{3}{2}} \|d\|_{H^3}^3 + C\|\nabla u\|_{L^6}^2 \|u\|_{H^3}^2 \|\nabla d\|_{L^6}.
\]

This gives

\[
\|d\|_{H^3}^2 \leq C + C\|\nabla d_t\|_{L^2}^2 + C\|u\|_{L^6} \|\Delta d\|_{L^6}^2 + \delta \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^6}^2 \|\nabla d\|_{L^6}^2, \tag{48}
\]

for any $0 < \delta < 1$. It is also easy to compute that

\[
\frac{d}{dt} \int (|\nabla u|^2 + |\Delta d|^2 - \nabla d \otimes \nabla d : \nabla u + C_0|\nabla d|^2) \, dx \\
+ \int (\rho|u_t|^2 + |\nabla d_t|^2) \, dx + \|u_t\|_{H^2}^2 + \|d_t\|_{H^3}^2 \\
\leq C + C\|\nabla d_t\|_{L^6}^2 \|\nabla u\|_{L^2}^2 + C\|\nabla d^T \cdot \Delta d\|_{L^2}^2 + C\|u\|_{L^6} \|\nabla d\|_{L^6}^2 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \\
= C + C\|\nabla d_t\|_{L^6}^2 \|\nabla u\|_{L^2}^2 + C\|\nabla d^T \cdot \Delta d\|_{L^2}^2 \\
+ \frac{C \|u\|_{L^6}^2}{1 + \log(e + \|u\|_{L^6})} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)(1 + \log(e + \|u\|_{L^6})) \\
\leq C + C\|\nabla d_t\|_{L^6}^2 \|\nabla u\|_{L^2}^2 + C\|\nabla d^T \cdot \Delta d\|_{L^2}^2 \\
+ \frac{C \|u\|_{L^6}^2}{1 + \log(e + \|u\|_{L^6})} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)(1 + \log(e + y(t))).
\]

This gives, by using the Hölder inequality in the following way

\[
- \int \nabla d \otimes \nabla d : \nabla u \leq C\|u\|_{H^2}^2 + \delta \int |\nabla d|^4 \, dx,
\]

and by using the Gronwall lemma (by (41) and (42))

\[
\int (|\nabla u|^2 + |\Delta d|^2) \, dx + \int_t^0 \int (\rho|u|^2 + |\nabla d|^2) \, dx \, d\tau + \int_t^0 (\|u\|_{H^2}^2 + \|d\|_{H^3}^2) \, d\tau \\
\leq C(e + y(t))^{C_0} \tag{50}
\]

Next, we apply $\partial_t$ to (3) and, testing by $u_t$, using (1) and (2), we have

\[
\frac{1}{2} \frac{d}{dt} \int \rho|u|^2 \, dx + \int |\nabla u|^2 \, dx \\
= - \int \rho u \cdot \nabla |u|^2 \, dx - \int \rho u \cdot \nabla (u \cdot \nabla u) \, dx \\
- \int \rho u_t \cdot \nabla u \cdot u_t \, dx + \int \partial_t (\nabla d \otimes \nabla d) : \nabla u_t \, dx =: I_3 + I_4 + I_5 + I_6. \tag{51}
\]
We use (35) to bound $I_3, I_4, I_5$ and $I_6$ as follow, for any $0 < \delta < 1$ and with a constant $C$ depending on $\delta$:

$$I_3 \leq C \|u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^3} \|\nabla u_t\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$\leq \delta \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2},$$

$$I_4 \leq C \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2}$$

$$+ C \|u\|_{L^6} \|\Delta u\|_{L^2} \|u_t\|_{L^6} + C \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2} \|u_t\|_{L^2} \|\nabla u\|_{L^2}$$

$$\leq \delta \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2},$$

$$I_5 \leq C \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6}$$

$$\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|u_t\|_{L^2}$$

$$\leq \delta \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2},$$

$$I_6 \leq C \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^6} \|\nabla u_t\|_{L^2}$$

$$\leq C \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$\leq \delta \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2} + \delta \|\Delta d_t\|_{L^2} + C \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^2}$$

Applying $\partial_t$ to (4), testing by $-\Delta d_t$ and using (1) and (38), we get

$$\frac{1}{2} \int \frac{d}{dt} \int \nabla d_t|^2 dx + \int |\Delta d_t|^2 dx = \int \partial_t (u \cdot \nabla d + |d|^2 d - d) \cdot \Delta d_t dx$$

$$= -\sum_j \int \partial_j \partial_t (u \cdot \nabla d) \cdot \partial_j d_t + \int (|d|^2 d - d)_t \cdot \Delta d_t dx$$

$$= -\sum_j \int (\partial_t u \nabla \partial_j d + \partial_j u_t \cdot \nabla d + \partial_j u \cdot \nabla d_t) \partial_j d_t dx + \int (|d|^2 d - d)_t \Delta d_t dx$$

$$\leq \|u_t\|_{L^6} \|\Delta d\|_{L^2} \|\nabla d_t\|_{L^3} + \|\nabla u_t\|_{L^2} \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3}$$

$$+ \|\nabla u\|_{L^2} \|\nabla d_t\|_{L^2} + C \left( \int_{\Omega} \Delta d_t dx \right) \|\Delta d_t\|_{L^2}$$

$$\leq \|\nabla u_t\|_{L^2} \|\Delta d_t\|_{L^2} \|\nabla d_t\|_{L^2} + \|\nabla u_t\|_{L^2} \|\Delta d_t\|_{L^2} \|\nabla d_t\|_{L^2}$$

$$+ \delta \|\Delta d_t\|_{L^2} + C \|\nabla u_t\|_{L^2} \|\Delta d_t\|_{L^2} + C \|\Delta d_t\|_{L^2} \|\nabla d_t\|_{L^2}$$

$$\leq \delta \|\Delta d_t\|_{L^2} + C \|\nabla d_t\|_{L^2} + \delta \|\nabla u_t\|_{L^2} + C \|\Delta d_t\|_{L^2} \|\nabla d_t\|_{L^2}$$

$$+ C \|\nabla u\|_{L^2} \|\nabla d_t\|_{L^2}$$

for any $0 < \delta < 1$. Plugging the above estimates for $I_3, I_4, I_5$ and $I_6$ into (51) and using (52) and taking $\delta$ small enough, and integrating over $[t_0, t]$ and using (50), we have

$$\int (\rho |u_t|^2 + |\nabla d_t|^2) dx + \int_{t_0}^t \int (|\Delta u_t|^2 + |\Delta d_t|^2) dx d\tau \leq C(e + y(t))^{C_0 t}.$$
Similarly to (46) and (48), we have
\[ \|u\|_{H^2} \leq C\sqrt{\rho u} \|L^2 + C\|u\|_{L^6} \|\nabla u\|_{L^3} + C\|\nabla d\|_{L^6} \|\Delta d\|_{L^3}, \]
\[ \leq C\sqrt{\rho u} \|L^2 + C\|u\|_{L^6} \|\nabla u\|_{L^2}^{\frac{3}{2}} + C\|\nabla d\|_{L^6} \|\Delta d\|_{L^2}^{\frac{3}{2}} + C\|d\|_{H^3}. \]
This gives
\[ \|u\|_{H^2} \leq C\sqrt{\rho u} \|L^2 + C\|\nabla u\|_{L^2}^{3} + C\|\Delta d\|_{L^2} + C\|d\|_{H^3}, \tag{54} \]
and
\[ \|d\|_{H^3} \leq C + C\|\nabla d\|_{L^2} + C\|\nabla u\|_{L^2} \|\Delta d\|_{L^2} + C\|\nabla u\|_{L^2} \|\nabla d\|_{L^2}, \tag{55} \]
which gives
\[ \|d\|_{H^3} \leq C + C\|\nabla d\|_{L^2} + 2\|\nabla u\|_{L^2} \|\Delta d\|_{L^2}. \]
Combining (54) and (55) and using (50) and (53), we conclude that
\[ \|u(t)\|_{H^2} + \|d(t)\|_{H^3} \leq C(e + \sup_{\tau \in [t_0, t]} \|u(\tau)\|_{H^2})^{C_0}, \]
and thus, if \( C_0 \epsilon < 1 \) we can absorb terms from the right hand side to get
\[ \|u\|_{L^\infty(0, T; H^2)} + \|d\|_{L^\infty(0, T; H^3)} \leq C. \tag{56} \]
Having proved the above estimate, it is now standard proving that
\[ \|u\|_{L^2(0, T; H^2)} \leq C, \tag{57} \]
\[ \|\rho\|_{L^\infty(0, T; W^{1, p})} \leq C, \|\rho_t\|_{L^\infty(0, T; L^p)} \leq C, \tag{58} \]
ending the proof of statement i).

Concerning Part II, the proof is very close. In the second case, let (8) hold true. Similarly to (41), we take \( q = \infty \) and using (34), we still get (41) provided that
\[ \int_{t_0}^{T} \|u(t)\|_{HMO}^2 dt \leq \epsilon << 1. \tag{59} \]
The rest calculations are similar if we take \( q = \infty \) and using (59), and thus we arrive at (56), (57), and (58) ending the proof.

3. On a simplified liquid crystals model. This section is devoted to the proof of Theorem 1.2. As usual by a standard continuation argument, we only need to establish a priori estimates.

Proof of Theorem 1.2. First, similarly to (37) and (38), we get the energy and sup estimates
\[
\frac{1}{2} \frac{d}{dt} \int (|u|^2 + |\nabla d|^2 + \frac{1}{2} (|d|^2 - 1)^2) dx
\]
\[ + \int (|\nabla u|^2 + (-\Delta d + |d|^2d - d)^2) dx \leq 0, \tag{60} \]
\[ \|d\|_{L^\infty(0, T; L^\infty)} \leq \max(1, \|d_0\|_{L^\infty}). \tag{61} \]
Similarly to (41), testing (40) by \(|\nabla d_i|^{p-2} \nabla d_i| \) with \( 2 \leq p \leq 6 \), and summing over \( i \), we get
\[ \|\nabla d\|_{L^\infty(0, T; L^p)}^p + \int_0^T \int |\nabla d|^{p-2} |\nabla^2 d|^2 dx dt \leq C. \tag{62} \]
Next, by taking the time derivative of (9), testing by \( u_t \), and using (12), (31), and (32), we obtain for any \( 0 < \delta < 1 \)
\[
\frac{1}{2} \frac{d}{dt} \int |u_t|^2 \, dx + \int |\nabla u_t|^2 \, dx = \int \partial_t (\nabla d \otimes \nabla d) : \nabla u_t \, dx - \int u_t \cdot \nabla u \cdot u_t \, dx
\leq C \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^6}^2 \|u_t\|_{L^\frac{24}{7}}^2
\leq \delta \|\nabla u_t\|_{L^2}^2 + \delta \|\Delta d_t\|_{L^2} + C \|\nabla d_t\|_{L^2}^2 + C \|\nabla u\|_{L^6}^2 \|u_t\|_{L^2}^2.
\] (63)

Next, by taking the time derivative of (10), testing by \( -\Delta d_t \), we get for any \( 0 < \delta < 1 \)
\[
\frac{1}{2} \frac{d}{dt} \int |d_t|^2 \, dx + \int |\Delta d_t|^2 \, dx = \int \partial_t (u \cdot \nabla d) + \partial_t (|d|^2 d - d_t) \Delta d_t \, dx
\leq \|u_t\|_{L^3} \|\nabla d\|_{L^6} \|\Delta d_t\|_{L^2} + C \|\nabla u\|_{L^6} \|\nabla d_t\|_{L^\frac{24}{7}}^2 + C \|d_t\|_{L^2} \|\Delta d_t\|_{L^2}
\leq \delta (\|u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2})
\leq \delta \|\nabla u_t\|_{L^2}^2 + C \|\nabla d_t\|_{L^2}^2 + C \|\nabla u\|_{L^6} \|u_t\|_{L^2}^2.
\] (64)

Summing up (63) and (64), taking \( \delta > 0 \) small enough, and using the Gronwall lemma we get
\[
\|u_t\|^2 + \|\nabla u_t\|^2 + \int_0^T (\|u_t\|^2 + \|\Delta d_t\|^2) \, dt \leq C.
\] (65)

Next, testing (10) by \( u_t \), using (12), (62), (60), and (65), we have
\[
\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \|u_t\|^2 = \int \nabla d \cdot \nabla d : \nabla u_t \, dx + \int u \cdot \nabla u_t \, dx
\leq C + C \|\nabla u_t\|^2 + C \|\nabla u\|_{L^2}^2,
\]
which gives, by Gronwall lemma
\[
\|u\|_{L^\infty(0,T;H^1)} \leq C.
\] (66)

On the other hand, by the \( H^2 \)-theory of the Stokes system, it follows from (9), (65), (66), and (62) that
\[
\|u\|_{H^2} \leq C \|u_t + u \cdot \nabla u + \nabla (\nabla d \otimes \nabla d)\|_{L^2}
\leq C \|u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^3} + C \|\nabla d\|_{L^6} \|\Delta d\|_{L^3}
\leq C + C \|\nabla u\|_{L^3} + C \|\Delta d\|_{L^3},
\]
and by the previous inequalities
\[
\|u\|_{H^2} \leq C + C \|d\|_{H^3}.
\] (67)

By the theory of the elliptic equations, if follows from (11), (62), (65) and (66) that
\[
\|d\|_{H^3} \leq C \|d\|_{L^2} + \|\nabla \Delta d\|_{L^2}
\leq C (1 + \|\nabla (d_t + u \cdot \nabla d + |d|^2 d - d)\|_{L^2})
\leq C + C \|\Delta d\|_{L^3} + C \|\nabla d\|_{L^\infty}
\leq C + C \|\nabla d\|_{L^6}^\frac{3}{2} \|d\|_{H^3}^\frac{3}{2},
\]
implying with (67) that
\[ \|u\|_{L^\infty(0,T;H^2)} + \|d\|_{L^\infty(0,T;H^2)} \leq C, \]
ending the proof. \( \square \)

4. On the viscous/diffusive Boussinesq system. Also in this case we show just a-priori estimates for smooth solutions.

**Proof of Theorem 1.3.** By using the same techniques as before, from (16) and (17) it follows that
\[ \|\dot{\theta}\|_{L^\infty(0,T;L^\infty)} \leq \|\theta_0\|_{L^\infty}. \] (69)
Testing (15) by \( u \) and using (17) and (69), we see that
\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 \leq \int \theta e_3 u \, dx \leq C \|u\|_{L^2}, \]
which gives the energy estimate
\[ \|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C. \] (70)

Proof in the case I). In the case that (20) holds true, we test (16) by \( -\Delta \theta \) and, by using (17), (31), (32), and integrating by parts, we have
\[ \frac{1}{2} \frac{d}{dt} \|\nabla \theta\|^2 + \|\Delta \theta\|^2 dx = -\sum \int \partial_j u \cdot \nabla \theta \cdot \partial_j \theta \, dx \]
\[ \leq C \|\nabla u\|_2 \|\nabla \theta\| \frac{1}{L^2} \]
\[ \leq C \|\nabla u\|_2 \|\nabla \theta\| \frac{1}{L^2} \|\theta\|_{H^2} \]
\[ \leq \frac{1}{2} \|\Delta \theta\|_{L^2}^2 + C \left( 1 + \frac{\|\nabla u\|_2^{2n-3}}{1 + \log(e + \|\nabla u\|_2)} \right) \|\nabla \theta\|_{L^2}^2 (1 + \log(e + z(t))), \]
with \( z(t) := \sup_{[t_0,t]} \|u(\tau)\|_{H^3} \). This gives
\[ \|\theta\|_{L^\infty(t_0,t;H^1)}^2 + \int_{t_0}^t \|\Delta \theta\|^2 \, d\tau \leq C(e + z(t)) \]
(71)
with \( 0 < t_0 \leq t \leq T \) and \( C_0 \) is an absolute constant, provided that
\[ \int_{t_0}^T \frac{\|\nabla u\|_2^{2n-3}}{1 + \log(e + \|\nabla u\|_2)} \, dt \leq \epsilon << 1. \]

Taking the time-derivative of (15), testing by \( u_t \), using (16), (17), (69), (70), (31), and (32), we also obtain by suitable integration by parts
\[ \frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \|\nabla u_t\|^2 = \int \theta_t e_3 u_t \, dx - \int u_t \cdot \nabla u \cdot u_t \, dx \]
\[ = \int u \theta \nabla (e_3 u_t) \, dx - \int \nabla \theta \cdot \nabla (e_3 u_t) \, dx - \int u_t \cdot \nabla u \cdot u_t \, dx \]
\[ \leq \|\theta\|_{L^\infty} \|u\|_{L^2} \|\nabla u_t\|_{L^2} + \|\nabla \theta\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|u_t\|_{L^2}^{2n-3} \]
\[ \leq \frac{1}{2} \|\nabla u_t\|_{L^2}^2 + C + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_2^{2n-3} \|u_t\|_{L^2}^2, \]
which yields
\[
\frac{d}{dt} \|u_t\|^2 + \|\nabla u_t\|^2 \leq C + C\|\nabla \theta\|^2_{L^2} + C \frac{\|\nabla u\|^2_{L^2}}{1 + \log(e + \|\nabla u\|_{L^2}^2)} \|u_t\|^2_{L^2} (1 + \log(e + z(t))),
\]
and by Gronwall lemma
\[
\|u_t\|^2_{L^\infty([t_0, t]; L^2)} + \int_{t_0}^t \|\nabla u_t\|^2 \, dt \leq C(e + z(t))^{C_{\text{Gron}}}. \tag{72}
\]
Testing (15) by \(u_t\), using (17), (69), (70) and (71), we also have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u_t\|^2 + \|u_t\|^2 = \int \theta \epsilon_3 u_t \, dx - \int u \cdot \nabla u \cdot u_t \, dx \leq C \|u_t\|_{L^2} + C\|\nabla u_t\|_{L^2}^4 + C\|\nabla u_t\|^2_{L^2},
\]
which implies
\[
\|u_t\|^2_{L^\infty([t_0, t]; H^1)} \leq C(e + z(t))^{C_{\text{Gron}}}. \tag{73}
\]
Taking \(\partial_t\) to (15), testing by \(-P\Delta u_t\) and using (17), we infer that
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla u_t|^2 \, dx + \int |P\Delta u_t|^2 \, dx = \int \theta \epsilon_3 (-P\Delta u_t) \, dx + \int (u_t \cdot \nabla u + u \cdot \nabla u_t)(-\Delta u_t) \, dx \leq \|\theta_t\|_{L^2} \|\Delta u_t\|_{L^2} + C\|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^3} \|\Delta u_t\|_{L^2} \leq \|\theta_t\|_{L^2} \|\Delta u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^4 + C\|\nabla u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2. \tag{74}
\]
Taking \(\partial_t\) to (15), testing by \(\theta_t\), using (17) and (69), we find that
\[
\frac{1}{2} \frac{d}{dt} \|\theta_t\|^2 + \|\nabla \theta_t\|^2 = -\int u_t \cdot \nabla \theta \, dx = \int u_t \nabla \theta_t \, dx \leq \frac{1}{2} \|\nabla \theta_t\|^2_{L^2} + C\|\theta_t\|_{L^2}^2. \tag{75}
\]
Summing up (74) and (75) and using (73) and (72) and integrating it over \([t_0, t]\), we arrive at the estimate
\[
\|\nabla u_t\|^2 + \|\theta_t\|^2 + \int_{t_0}^t (\|\Delta u_t\|^2 + \|\nabla \theta_t\|^2) \, dt \leq C(e + z(t))^{C_{\text{Gron}}}. \tag{76}
\]
On the other hand, by the standard regularity-theory of the Stokes system, it follows from (15), (70), (71), and (76) that
\[
\|u\|_{H^3} \leq C(\|u\|_{L^2} + \|\nabla \Delta u\|_{L^2}) \leq C + C\|\nabla (u_t + u \cdot \nabla u - \theta \epsilon_3)\|_{L^2} \leq C + C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2} \cdot \|\nabla u\|_{L^3}^{\frac{3}{2}} + C\|\nabla u\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}.
\]
This gives
\[
\|u\|_{H^3} \leq C + C\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2} + C\|\nabla \theta\|_{L^2} \leq C(e + z(t))^{C_{\text{Gron}}},
\]
which implies
\[
\|u\|_{L^\infty(0, T; H^3)} \leq C, \tag{77}
\]
and it is then standard to prove that
\[ \|\theta\|_{L^\infty(0,T;H^2)} \leq C. \]  

(78)

Proof in the case II. Let (21) hold true. The same calculations as before, but taking \( q = \infty \) (and using (33), we still have (71), (72), (73), (74), (75), (76), (77), and (78)) lead to the same a-priori estimate, completing the proof. \( \square \)

5. On a magneto-hydrodynamics system. We follow the same path as before.

**Proof of Theorem 1.4.** First, testing (22) by \( u \) and (23) by \( b \) and using (24), we find that
\[ \frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|b\|^2) + (\|\nabla u\|^2 + \|\text{curl} \ b\|^2) \leq 0. \]  

(79)

By using \( |b|^{p-2}b \) (2 \( \leq p \leq 6 \) as test function in (22), and by using (24), (28), (29), (30), (31), and (32), we derive
\[ \frac{1}{p} \frac{d}{dt} \int \Omega |b|^p \, dx + \frac{1}{2} \int \Omega |b|^{p-2} |\nabla b|^2 \, dx + 4 \frac{p-2}{p^2} \int \Omega |\nabla |b|^2| \, dx = - \int_{\partial \Omega} |b|^{p-2} (b \cdot \nabla) u \cdot bd\sigma + \int \Omega (b \cdot \nabla) u \cdot |b|^{p-2} b \, dx \leq 2 \frac{p-2}{p^2} \int \Omega |\nabla |b|^2|^2 \, dx + C \|\nabla u\|^2_{L^p} + C \|\text{curl} \ b\|^2_{L^p}, \]

which gives
\[ \|b\|^p_{L^\infty(0,T;L^p)} + \int_0^T \int |b|^{p-2} |\nabla b|^2 \, dxdt \leq C, \quad \text{for any } 2 \leq p \leq 6. \]  

(80)

Taking \( \partial_t \) to (22), testing by \( u_t \), using (24), (31), (32), and (80), we have
\[ \frac{1}{2} \frac{d}{dt} \int \Omega |u_t|^2 \, dx + \int \Omega |\nabla u_t|^2 \, dx = - \int \Omega u_t \cdot \nabla u \cdot u_t \, dx - \int \partial_t (b \otimes b) : \nabla u_t \, dx \leq \delta \int \Omega |\nabla u_t|^2 \, dx + \delta \|\nabla b_t\|^2_{L^2} + C \|\nabla u\|^2_{L^p} \|b_t\|^2_{L^2} + C \|b_t\|^2_{L^2}, \]  

for any \( 0 < \delta < 1 \). Moreover, taking \( \partial_t \) to (23), testing by \( b_t \), using (24), (80), (31), and (32), we derive
\[ \frac{1}{2} \frac{d}{dt} \int \Omega |b_t|^2 \, dx + \int \Omega |\text{curl} \ b_t|^2 \, dx = - \int \Omega u_t \cdot \nabla b \cdot b_t + \int (b_t \cdot \nabla u + b \cdot \nabla u_t) b_t \, dx \leq \delta \|\text{curl} \ b_t\|^2_{L^2} + \delta \|\nabla b_t\|^2_{L^2} + C \|u_t\|^2_{L^2} + C \|b_t\|^2_{L^2} + C \|\nabla u\|^2_{L^p} \|b_t\|^2_{L^2}, \]  

(82)

for any \( 0 < \delta < 1 \). Combining (81) and (82) and taking \( \delta \) small enough and using the Gronwall inequality with the hypothesis (14), we have
\[ \|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \leq C, \]  

(83)

\[ \|b_t\|_{L^\infty(0,T;L^2)} + \|b_t\|_{L^2(0,T;H^1)} \leq C. \]  

(84)

Testing (22) by \( u_t \), using (24), (80), (79) and (83), we deduce that
\[ \frac{1}{2} \frac{d}{dt} \int \Omega |\nabla u|^2 \, dx + \int \Omega |u_t|^2 \, dx = - \int \Omega b \otimes b \cdot \nabla u_t \, dx + \int \Omega u \otimes u : \nabla u_t \, dx \leq \|b\|^2_{L^4} \|\nabla u_t\|_{L^2} + \|u\|^2_{L^4} \|\nabla u_t\|_{L^2} \leq \|\nabla u_t\|^2_{L^2} + C \|\nabla u\|^2_{L^2}, \]
which gives
\[ \|u\|_{L^\infty(0,T;H^1)} \leq C. \]
(85)

On the other hand, regularity theory and previous estimates imply that
\[ \|u\|_{H^2} \leq C \|u\|_{H^1} + u \cdot \nabla u - \operatorname{div}(b \otimes b) \|_{L^2} \leq C \|\nabla u\|_{L^3} + C \|\nabla b\|_{L^3}. \]
This gives \( \|u\|_{H^2} \leq C(1 + \|\nabla b\|_{L^3}) \).

Similarly, it follows from (23), (85) and (84) that
\[ \|b\|_{H^2} \leq C \|\Delta b\|_{L^2} \leq C \|b_t + u \cdot \nabla b - b \cdot \nabla u\|_{L^2} \leq C \|\nabla b\|_{L^3} + C \|b\|_{L^\infty}, \]
which implies \( \|b\|_{L^\infty(0,T;H^2)} \leq C \). By using the two latter estimates, we finally obtain that \( \|u\|_{L^\infty(0,T;H^2)} \leq C \), thus completing the proof.

6. On a simplified Ericksen-Leslie model. Also the results of this section are based on the same machinery, hence we need just a priori estimates to prove Theorem 1.5.

Proof of Theorem 1.5. First, observe that estimates (35) and (36) (proved in Theorem 1.1) continue to hold also in this case. Next, we easily infer that
\[ |d| \equiv 1 \text{ in } \Omega \times (0, \infty). \]
(86)

Testing (27) by \(-\Delta d - |\nabla d|^2 d\) and using (1) and (86), we find that
\[ \frac{1}{2} \frac{d}{dt} \int \Omega |\nabla d|^2 dx + \int \Omega |\Delta d + |\nabla d|^2 d|^2 dx = \int (u \cdot \nabla) d \cdot \Delta d dx. \]
(87)

Summing up (36) and (87), we have the well-known energy inequality
\[ \frac{1}{2} \frac{d}{dt} \int (\rho|u|^2 + |\nabla d|^2) dx + \int (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) dx \leq 0. \]
(88)

Taking the gradient of (27), testing by \( |\nabla d_i|^{p-2} \nabla d_i \) \((2 \leq p \leq 6)\), summing over \( i \), and using (1), (28), (29), (30), (31), (32) and (86) we have
\[ \frac{1}{p} \frac{d}{dt} \int_\Omega |\nabla d|^p dx + \frac{1}{2} \int_\Omega |\nabla d|^{p-2} |\nabla d|^2 dx + \frac{p-2}{p^2} \int_\Omega |\nabla |\nabla d|^{\frac{p}{2}}|^2 dx \]
\[ = - \sum_i \int_{\partial \Omega} |\nabla d_i|^{p-2} (\nabla d_i \cdot \nabla) \nu \cdot \nabla d_i ds - \sum_{i,j} \int_{\Omega} u_j \nabla (\partial_j d_i |\nabla d_i|^{p-2} \nabla d_i) dx \]
\[ + \int_\Omega \nabla (|\nabla d|^2 d) \cdot |\nabla d|^{p-2} \nabla d dx \]
\[ \leq C \int_{\partial \Omega} |\nabla d|^p ds + C \int_\Omega |u||\nabla d|^{\frac{p}{2}} |\nabla \nabla d|^{\frac{p}{2}} dx + C \int_\Omega |u||\nabla d|^{\frac{p}{2}} \cdot |\nabla d|^{\frac{p}{2}-1} |\nabla^2 d| dx \]
for any $0 < \delta < 1$, where $w = |\nabla d|^p/2$. This yields

$$\|\nabla d\|_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t \||\nabla d|^p - |\nabla d|^2|^2\| dx d\tau \leq C(e + h(t))C_0 \epsilon, \quad (89)$$

with $h(t) := \sup_{[t_0, t]}(\|u(\tau)\|_{W^{1,4}} + \|d(\tau)\|_{H^3})$, for any $0 < t_0 \leq t \leq T$ and $C_0$ is an absolute constant, provided that

$$\int_{t_0}^T \frac{\|u\|_{L^p}^{2q} + \|\nabla d\|_{L^p}^{2q}}{1 + \log(e + \|u\|_{L^p} + \|\nabla d\|_{L^p})} dt \leq \epsilon << 1. \quad (90)$$

Same calculations as before show (43), (44), (45), (46), and (49) hence we do not reproduce them here.

Similarly to (47), testing (27) by $-\Delta d_t$, using (86), (31) and (32), we have

$$\frac{1}{2} \frac{d}{dt}\|\Delta d\|^2 + \|\nabla d_t\|^2 = \int \langle u \cdot \nabla d - |\nabla d|^2 d \rangle \Delta d_t dx$$

$$= \int \nabla \|\nabla d\|^2 - u \cdot \nabla d \rangle \nabla d_t dx$$

$$\leq C(\|\nabla d\|_{L^p} \|\Delta d\|_{L^{2q} L^2} + \|\nabla d\|_{L^p}^3 \|\nabla d_t\|_{L^2})$$

$$+ C(\|u\|_{L^p} \|\Delta d\|_{L^{2q} L^2} + \|u\|_{L^p} \|\nabla d\|_{L^p} \|\nabla d_t\|_{L^2})$$

$$\leq \delta \|\nabla d\|_{L^2} + \|\Delta d\|_{L^2}^2 + \delta \|u\|_{L^2}^2 + C(\|\nabla d\|_{L^p}^3 \|\nabla d_t\|_{L^2}^2 + C(\|\nabla d\|_{L^p}^3 \|\nabla d_t\|_{L^2}^2$$

$$+ C(\|u\|_{L^p}^{2q} \|\Delta d\|_{L^2}^2 + C(\|\nabla d\|_{L^p}^{2q} \|\Delta d_t\|_{L^2}^2, \quad (91)$$

for any $0 < \delta < 1$. Combining (43), (44), (45), (46), (91), (49), by taking $\delta$ small enough, integrating over $[t_0, t]$, and using (89)-(90), we arrive at

$$\|\nabla u\|^2 + \|\Delta d\|^2 + \int_{t_0}^t \int \rho \|u\|^2 + |\nabla d_t|^2 dx d\tau$$

$$+ \int_{t_0}^t \int (\|u\|_{H^2}^2 + \|d\|_{H^3}^2) d\tau \leq C(e + h(t))C_0 \epsilon. \quad (92)$$
We still have (51) and the same estimates for $I_3, I_4, I_5$ and $I_6$. Moreover, similarly to (52), taking $\partial_x$ to (27), testing by $-\Delta d$, and using (4) and (86), we have

\[
\frac{1}{2} \frac{d}{dt} \| \nabla d_t \|^2 + \| \Delta d_t \|^2 = - \sum_j \int (\partial_t u \partial_j d + \partial_t u \nabla d + \partial_j u \nabla d_t) \partial_j d_t \, dx + \int \partial_t (|\nabla d|^2 d) \cdot \Delta d_t \, dx
\]

\[
\leq \delta \| \Delta d_t \|^2_{L^2} + \delta \| \nabla u_t \|^2_{L^2} + C \| \Delta d \|^2_{L^2} \| \nabla d_t \|^2_{L^2} + C \| \nabla u \|^2_{L^2} \| \nabla d_t \|^4_{L^2} + C \| \nabla d \|^4_{L^2} (\| \nabla d_t \|^2_{L^2} + 1),
\]

(93)

for any $0 < \delta < 1$. Next, by combining (51) and (93) and using the estimates for $I_3, I_4, I_5$ and $I_6$, taking $\delta$ small enough, integrating over $[t_0, t]$, and using (92), we have

\[
\int (\rho |u_t|^2 + |\nabla d_t|^2) \, dx + \int_{t_0}^t (\| \nabla u_t \|^2 + \| \Delta d_t \|^2) \, d\tau \leq C(e + h(t))e^\epsilon.
\]

(94)

Observe also that we still have (54) and, similarly to (55),

\[
\| d \|_{H^3} \leq C + C \| \nabla u \|_{L^2} + C(\| \nabla u \|_{L^2} + \| \nabla d \|_{L^2}) \| \Delta d \|_{L^2} \| \nabla d \|_{L^2}^{\frac{1}{2}} + C \| \nabla d \|_{L^2}^3,
\]

gives

\[
\| d \|_{H^3} \leq C + C \| \nabla d \|_{L^2} + C(\| \nabla u \|_{L^2} + \| \nabla d \|_{L^2}) \| \Delta d \|_{L^2} \| \nabla d \|_{L^2}^2 \| \Delta d \|_{L^2}.
\]

(95)

Combining (55) and (95) and using (94) and (92), we still have (56). Thus we have (57) and (58), ending the proof.

\[
\square
\]

7. On a simplified Ericksen-Leslie model with constant density. The proof of this last result is very similar to the previous one.

Proof of Theorem 1.6. Observe that also in this case we have (86). Similarly to (88), we have the well-known energy inequality:

\[
\frac{1}{2} \frac{d}{dt} \int (|u|^2 + |\nabla d|^2) \, dx + \int (|\nabla u|^2 + |\Delta d + |\nabla d|^2d|^2) \, dx \leq 0.
\]

Similarly to (89) we still have (62), (63), and (93). Thus we have (65), (66) and (67). Moreover, also (95) holds true. Thus, we arrive at (68), ending the proof.

\[
\square
\]

REFERENCES

[1] R. A. Adams and J. F. Fournier, Sobolev Spaces, 2nd ed., Elsevier/Academic Press, Amsterdam, 2003.
[2] H. Beirão da Veiga, Existence and asymptotic behavior for strong solutions of the Navier-Stokes equations in the whole space, Indiana Univ. Math. J., 36 (1987), 149–166.
[3] H. Beirão da Veiga and F. Crispo, Sharp inviscid limit results under Navier type boundary condition. An $L^p$ theory, J. Math. Fluid Mech., 12 (2010), 397–411.
[4] L. C. Berselli, On a regularity criterion for the solutions to the 3D Navier-Stokes equations, Differential Integral Equations, 15 (2002), 1129–1137.
[5] L. C. Berselli, Some criteria concerning the vorticity and the problem of global regularity for the 3D Navier-Stokes equations, Ann. Univ. Ferrara Sez. VII Sci. Mat., 55 (2009), 209–224.
[6] L. C. Berselli and R. Manfrin, On a theorem by Sohr for the Navier-Stokes equations, J. Evol. Equ., 4 (2004), 193–211.
[7] L. C. Berselli and S. Spirito, On the Boussinesq system: Regularity criteria and singular limits, Methods Appl. Anal., 18 (2011), 391–416.
[8] S. Chandrasekhar, Liquid Crystals, Cambridge University Press, 2nd edition, 1992.
[9] R. Danchin, Density-dependent incompressible fluids in bounded domains, J. Math. Fluid Mech., 8 (2006), 333–381.
[10] J. L. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Ration. Mech. Anal.*, 9 (1962), 371–378.

[11] J. Fan, Y. Fukumoto and Y. Zhou, Logarithmically improved regularity criteria for the generalized Navier-Stokes and related equations, *Kinet. Relat. Models*, 6 (2013), 545–556.

[12] J. Fan, H. Gao and B. Guo, Regularity criteria for the Navier-Stokes-Landau-Lifshitz system, *J. Math. Anal. Appl.*, 363 (2010), 29–37.

[13] J. Fan, S. Jiang, G. Nakamura and Y. Zhou, Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations, *J. Math. Fluid Mech.*, 13 (2011), 557–571.

[14] J. Fan and T. Ozawa, Regularity criteria for the 3D density-dependent Boussinesq equations, *Nonlinearity*, 22 (2009), 553–568.

[15] F. Guillén-González, M. A. Rojas-Medar and M. A. Rodríguez-Bellido, Sufficient conditions for regularity and uniqueness of a 3D nematic liquid crystal model, *Math. Nachr.*, 282 (2009), 846–867.

[16] H. Kim, A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations, *SIAM J. Math. Anal.*, 37 (2006), 1417–1434.

[17] F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.*, 28 (1968), 265–283.

[18] X. Li and D. Wang, Global strong solution to the density-dependent incompressible flow of liquid crystals, *Trans Amer. Math. Soc.*, to appear, arXiv:1202.1011 v1.

[19] F.-H. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Comm. Pure Appl. Math.*, 48 (1995), 501–537.

[20] A. Lunardi, *Interpolation Theory*, Lecture Notes. Scuola Normale Superiore di Pisa (New Series), 2nd ed., Edizioni della Normale, Pisa, 2009.

[21] T. Ogawa, Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow, *SIAM J. Math. Anal.*, 34 (2003), 1318–1330.

[22] T. Ogawa and Y. Taniuchi, A note on blow-up criterion to the 3D Euler equations in a bounded domain, *J. Diff. Equations*, 190 (2003), 39–63.

[23] H. Sohr, A regularity class for the Navier-Stokes equations in Lorentz spaces, *J. Evol. Equ.*, 1 (2001), 441–467.

[24] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[25] Y. Zhou and S. Gala, Logarithmically improved regularity criteria for the Navier-Stokes equations in multiplier spaces, *J. Math. Anal. Appl.*, 356 (2009), 498–501.

[26] Y. Zhou and J. Fan, Logarithmically improved regularity criteria for the 3D viscous MHD equations, *Forum Math.*, 24 (2012), 691–708.

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