Generalized Local IV with Unordered Multiple Treatment Levels
Identification, Efficient Estimation, and Testable Implication*

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Abstract

This paper studies the econometric aspects of the generalized local IV framework defined using the unordered monotonicity condition, which accommodates multiple levels of treatment and instrument in program evaluations. The framework is explicitly developed to allow for conditioning covariates. Nonparametric identification results are obtained for a wide range of policy-relevant parameters. Semiparametric efficiency bounds are computed for these identified structural parameters, including the local average structural function and local average structural function on the treated. Two semiparametric estimators are introduced that achieve efficiency. One is the conditional expectation projection estimator defined through the nonparametric identification equation. The other is the double/debiased machine learning estimator defined through the efficient influence function, which is suitable for high-dimensional settings. More generally, for parameters implicitly defined by possibly non-smooth and overidentifying moment conditions, this study provides the calculation for the corresponding semiparametric efficiency bounds and proposes efficient semiparametric GMM estimators again using the efficient influence functions. Then an optimal set of testable implications of the model assumption is proposed. Previous results developed for the binary local IV model and the multivalued treatment model under unconfoundedness are encompassed as special cases in this more general framework. The theoretical results are illustrated by an empirical application investigating the return to schooling across different fields of study, and a Monte Carlo experiment.

Keywords: Generalized Local IV, Multi-valued Treatment, Unordered Monotonicity, Semiparametric Efficiency, Efficient Estimation, Optimal Testable Implication, Return to Schooling.

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1 Introduction

Since the seminal works of Imbens and Angrist (1994) and Angrist et al. (1996), local instrumental variable has become a popular method for causal inference in economics. Instead of imposing homogeneity of the treatment effects among the individuals, as in the classical IV regression model, the local IV framework allows for heterogeneous treatment effects. To achieve identification, however, in practice, the treatment has to be integrated into one binary indicator. But oftentimes, treatments in economic relevant programs are multi-leveled in nature. They can be ordered, as tax rates, years of schooling, and numbers of cigarettes smoked; or unordered, as different job training programs, fields of study in college, and vouchers to various housing opportunities. The unordered case is more general than the ordered one since ordered treatment levels can also be considered as unordered. The question now becomes how to finer evaluate programs in the local IV framework, incorporating the multiplicity in treatment levels. One possible solution is given in Heckman and Pinto (2018a) and Pinto (2019), which uses their unordered monotonicity condition to generalize the identification results in binary local IV to situations with multiple unordered levels of treatments. The extension from binary treatments to multi-valued ones further demonstrates the source of identification in the local IV model making use of the monotonicity conditions. In the current study, this broader framework is referred to as the generalized local IV model.

The marginal benefits for introducing multiple levels of treatment into the local IV model is twofold. First, as mentioned above, there are many empirical cases where treatments are explicitly multi-valued. Collapsing these levels together is not very useful for a detailed analysis of the program effects, covering estimation and inference for parameters of finer subpopulations defined by the way of the treatment choice varies with the instrument. Also, when the binary treatment is further divided into multiple values, efficiency gains are possible provided with overidentifying restrictions justified by the underlying economic theory. Conversely, the theories can be tested through these restrictions defined by parameters available only when the multiplicity of treatment is modeled.

This paper is concerned with the econometric aspects of the generalized local IV model, which turns the identification results into applicable methods in empirical research. The framework is first extended to allow for conditioning covariates, which is important because, often in observational
studies, the instrument is only valid conditioning on other factors. And the conditioning issue here suffers the same problem as in the binary local IV case since the subpopulations are not identified. Heckman and Pinto (2018a) and Pinto (2019) focused, among other things, on the identification of conditional local average structural function (LASF) and type probabilities. Additional identification results, in the unconditional sense, are obtained in this paper for more policy-relevant parameters, including the local average structural function on the treated (LASF-T). Semiparametric efficiency bounds are computed for a wide range of structural parameters, including the LASFs and LASF-Ts. For these parameters with explicit definition, conditional expectation projection (CEP) estimators, defined as semiparametric two-step estimators through the identification equations, are shown to achieve the efficiency bound, which is analogous to results in the literature (Frölich, 2007; Hahn, 1998; Hong and Nekipelov, 2010a). Efficiency is also proven for the double/debiased machine learning (DML) estimators (Chernozhukov et al., 2018) defined through the efficient influence functions. These estimators are well suited to modern high-dimensional cases since their moment conditions satisfy the Neyman orthogonality condition on the nonparametric nuisance parameters. More generally, for parameters implicitly defined by possibly non-smooth and overidentifying moment conditions, this study provides the calculation of the semiparametric efficiency bounds and proposes efficient semiparametric GMM estimators defined through the efficient influence functions. Two important cases can be incorporated, one is the quantile estimation (Melly and Wüthrich, 2017; Firpo, 2007), and the other is the aforementioned case where the underlying economic theory provides overidentification. Optimal joint inferences can be conducted across and between different treatment levels, based on the semiparametric efficient estimations.\footnote{The problem of joint inference is not salient in the binary local IV model since usually there is only a single parameter of interest.}

The assumption of the generalized local IV model is refutable but not verifiable. An optimal set of testable implications of the model assumptions is proposed, in the sense that the refutation of this particular set of implications necessarily leads to the rejection of all implications of the model assumptions.

The literature on semiparametric efficiency in program evaluation starts with the seminal work of Hahn (1998), which studies the benchmark case of estimating the average treatment effect (ATE) under unconfoundedness. When endogeneity is present, that is in the framework of local

\footnote{The problem of joint inference is not salient in the binary local IV model since usually there is only a single parameter of interest.}
IV, Frölich (2007) calculates the semiparametric efficiency bound for the local average treatment effect (LATE), and Hong and Nekipelov (2010a) extends to the estimation of general parameters defined by moment restrictions. For the case of multiple treatment levels, the extension of conditional LATE to situations with multiple choices is presented in Angrist and Imbens (1995), making use of the order monotonicity assumption to identify a weighted average of LATEs. More recently, Cattaneo (2010) studies the efficient estimation of multi-valued treatment effects, as implicitly defined through over-identified non-smooth moment conditions, under unconfoundedness. Nekipelov (2011) focused on the efficiency bound calculation in a case where ordered multi-valued treatment is allowed. Bajari et al. (2015) derived identification and efficient estimation results in a game-theoretic setting. In a more general framework encompassing missing data, Chen et al. (2004) and Chen et al. (2008) studies semiparametric efficiency bounds and efficient estimation of parameters defined through overidentifying moment restrictions. However, work is missing on semiparametric efficient estimation encompassing both unordered multiple treatment levels and local instruments. For testing, the assumptions underlying the local IV framework is in general refutable but not verifiable. Kitagawa (2015) proposes an set of conditions as the optimal testable implication for the binary local IV framework. Sun (2018) extends the idea to circumstances with multiple treatment levels under ordered monotonicity, and proposes more powerful tests.

The generalized local IV model is rarely applied in empirical research. An earlier work of Kline and Walters (2016) evaluates the cost-effectiveness of Head Start, classifying Head Start and other preschool programs as different treatment levels against the control group of no preschool. Pinto (2019) studies the neighborhood effects and voucher effects in housing allocation using data from the Moving to Opportunity experiment. One drawback of these works is that their typical methodology of simple two-stage least squares (2SLS) is not valid when the independence of the instrument is only satisfied conditioning on some covariates, which is often the case in observational studies. The empirical application in this paper studies the return to schooling across different fields of study in college, based on the analysis of Card (1993) which uses proximity to college as the instrument. Kitagawa (2015) shows that this instrument is only valid after conditioning on race and characteristics of the residence area. Different college majors are categorized into two treatment levels and the control is no college education. The joint estimation results of the LASFs and LASF-Ts show a significant gain in income after receiving a college education, which is similar for both
This paper proceeds as follows. Section 2 introduces the model and the main example. Section 3 discusses nonparametric identification results. Section 4 calculates the semiparametric efficiency bound and describes the efficient estimators proposed for parameters that can be derived from LASFs and LASF-Ts. Section 5 further extends the results to parameters defined by possibly overidentifying and non-smooth moment conditions. Section 6 proposes a set of optimal testable implications of the model assumptions. Section 7 presents the empirical illustration and Monte Carlo study. Section 8 concludes. In Appendix A, the results of binary LATE is presented in the general framework. Proofs are collected in Appendix B.

2 Setup and Main Example

The general framework is presented in this section. We have a treatment variable $T$ taking values in the unordered set $\mathcal{T} = \{t_1, \cdots, t_{N_T}\}$. Instrument $Z$ takes values in the unordered set $\mathcal{Z} = \{z_1, \cdots, z_{N_Z}\}$. The random variables $\left(Y_{t_1}, \cdots, Y_{t_{N_T}}\right)$, with $Y_t \in \mathcal{Y} \subset \mathbb{R}, t \in \mathcal{T}$, represent the potential outcomes under each treatment level. These are assumed to have finite second moments.\footnote{It is convenient to assume this in the beginning, since the main focus of the paper is on efficiency.}

And the random variables $(T_{z_1} \cdots, T_{z_{N_Z}})$, with $T_z \in \mathcal{T}, z \in \mathcal{Z}$, represent the potential treatment status under each instrument level. Random vector $X$ is a set of covariates, which takes value in $X \subset \mathbb{R}^{d_X}$. Also define a random vector $S = \left(T_{z_1}, \cdots, T_{z_{N_Z}}\right)$, which denotes the type of the individual. Let $S \subset \mathcal{T}^{N_Z}$ be the support of $S$. The observed treatment and observed outcome are defined as $T = \sum_{z \in \mathcal{Z}} 1\{Z = z\} T_z$ and $Y = \sum_{t \in \mathcal{T}} 1\{T = t\} Y_t$ respectively. An equivalent way to formulate is to use structural equations, as in Heckman and Pinto (2018a); Pinto (2019). By denoting the function and the determined random variable with the same notation, the treatment and outcome can be defined as $T = T(Z, X, V), Y = Y(T, X, V, \epsilon)$, where $Z, V, \epsilon$ are mutually independent conditional on $X$. Under this formulation, $T_z = T(z, X, V), Y_t = Y(t, X, V, \epsilon)$, and $S = (T(z_1, X, V), \cdots, T(z_{N_Z}, X, V))'$.

The following notations are used throughout the paper. Let $\pi(X) = (\pi_{z_1}(X), \cdots, \pi_{z_{N_Z}}(X))'$, where $\pi_z(X) = P(Z = z \mid X)$. For any $t \in \mathcal{T}$, let $P_{t,z}(X) = P(T = t \mid Z = z, X)$, and $P_{t,Z}(X) = \left(P_{t,z_1}, \cdots, P_{t,z_{N_Z}}\right)'$. For any measurable $g : \mathcal{Y} \to \mathbb{R}$, let $g_{t,z} = \mathbb{E}[g(Y) 1\{T = t\} \mid Z = z, X]$ and
$g_{t,Z} = \left( g_{t,z_1}, \ldots, g_{t,z_N} \right)$. Let $B_t$ be the $N_Z \times N_S$ binary matrix whose $i,j$th element is $1\{s_j[i] = t\}$, and its Moore-Penrose inverse is denoted by $B_t^+$.\(^3\) Let $\Sigma_{t,k} \subset S, k = 0, \ldots, N_Z$ be the set of types in which treatment $t$ appears exactly $k$ times, i.e. $\Sigma_{t,k} = \left\{ s \in S : \sum_{i=1}^{N_Z} 1\{s[i] = t\} = k \right\}$. Note that the $\Sigma_{t,k}$’s form a partition of the type configuration, i.e. $S = \bigsqcup_{k=0}^{N_Z} \Sigma_{t,k}$. Let $\tilde{b}_{t,k} = b_{t,k} B_t^+$, where $b_{t,k} = (1\{s_1 \in \Sigma_{t,k}\}, \ldots, 1\{s_{N_S} \in \Sigma_{t,k}\})$. The main assumption of the generalized local IV model is presented below.

**Assumption 1. Generalized Local IV:**

(i) **Conditional independence:** $(\{Y_t : t \in T\}, \{T_z : z \in Z\}) \perp Z \mid X$.

(ii) **Type constraint:** the type support $S$ is known and satisfies, for any $t \in T, z, z' \in Z$, either $1\{T_z = t\} \geq 1\{T_{z'} = t\}$ or $1\{T_z = t\} \leq 1\{T_{z'} = t\}$.

(iii) **First stage:** for all $z, z' \in Z$ and $t \in T$, it hold that $\pi_z(X) \geq \pi > 0$ and $P(T_z = t \mid X) \neq P(T_{z'} = t \mid X)$.

These assumptions are essentially the multi-valued analog of those used in Abadie (2003). Assumption 1(i) is on the validity of instrument conditioning on $X$. Assumption 1(ii) is the unordered monotonicity constraint on the configuration $S$ of type.\(^4\) It means that a shift in an instrument moves all agents uniformly toward or against each possible treatment choice (Heckman and Pinto, 2018a). As pointed out by Vytlacil (2002), the LATE type monotonicity condition is a restriction across individuals on the relationship between different hypothetical treatment choices defined in terms of an instrument. Assumption 1(iii) requires that the instrument has some effect on the selection of each treatment level\(^5\), and also implies that the support of $X$ does not change with the value of $Z$. The exclusion restrictions of the instrument from the outcomes are already imposed in the definition of potential outcomes. The observed data is assumed to be an IID sample $(Y_i, T_i, Z_i, X_i)_{i=1}^n$.

\(^3\)The concept of $B_t$ is defined using $S$, and hence are nonrandom and do not depend on $X$. The original definition of $B_t$ in Heckman and Pinto (2018a) involve random variables, and are difficult to state unambiguously with the presence of $X$. I thank Yixiao Sun for helpful suggestions on this.

\(^4\)An equivalent condition, shown by Heckman and Pinto (2018a), for unordered monotonicity is, for any $t \in T, B_t$ is lonesum.

\(^5\)The strong overlapping assumption is imposed here for estimation. For identification, it suffices to impose a weaker condition $\pi_z(X) \in (0,1)$. 
Consider a main example for demonstration purposes throughout the paper. There are three treatment levels and two instrument levels. Denote the treatment levels \( T = \{t_1, t_2, t_3\} \), and instrument levels \( Z = \{z_1, z_2\} \). Note that the indexing is only for convenience, they are intrinsically unordered. The type configuration \( S \) is specified below.\(^6\)

\[
\begin{array}{c|ccccc}
 T_{z_1} & s_1 & s_2 & s_3 & s_4 & s_5 \\
 t_1 & t_2 & t_3 & t_3 & t_3 \\
 T_{z_2} & t_1 & t_2 & t_3 & t_1 & t_2
\end{array}
\]

The unordered monotonicity is satisfied: \( 1\{T_{z_2} = t_1\} \geq 1\{T_{z_1} = t_1\} \), \( 1\{T_{z_2} = t_2\} \geq 1\{T_{z_1} = t_2\} \), and \( 1\{T_{z_2} = t_3\} \leq 1\{T_{z_1} = t_3\} \). The type partitions are, for \( t_1 \), \( \Sigma_{t_1,0} = \{s_2, s_3, s_5\} \), \( \Sigma_{t_1,1} = \{s_4\} \), \( \Sigma_{t_1,2} = \{s_1\} \); for \( t_2 \), \( \Sigma_{t_2,0} = \{s_1, s_3, s_4\} \), \( \Sigma_{t_2,1} = \{s_5\} \), \( \Sigma_{t_2,2} = \{s_2\} \); and for \( t_3 \), \( \Sigma_{t_3,0} = \{s_1, s_2\} \), \( \Sigma_{t_3,1} = \{s_4, s_5\} \), \( \Sigma_{t_3,2} = \{s_3\} \). \( B_t \)'s and their generalized inverses are

\[
B_{t_1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad B_{t_2} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1
\end{bmatrix}, \quad B_{t_3} = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
B_{t_1}^+ = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
-1 & 1 \\
0 & 0
\end{bmatrix}, \quad B_{t_2}^+ = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
-1 & 1
\end{bmatrix}, \quad \text{and } B_{t_3}^+ = \begin{bmatrix}
0 & 0 \\
0 & 0.5 \\
0 & -0.5
\end{bmatrix}.
\]

The \( b_{t,k} \)'s, and hence the \( \tilde{b}_{t,k} \)'s, are displayed in the following table.

|       | \( b_{t,1} \) | \( b_{t,2} \) | \( \tilde{b}_{t,1} \) | \( \tilde{b}_{t,2} \) |
|-------|----------------|----------------|-----------------------|-----------------------|
| \( t = t_1 \) | \((0, 0, 0, 1, 0)\) | \((1, 0, 0, 0, 0)\) | \((-1, 1)\) | \((1, 0)\) |
| \( t = t_2 \) | \((0, 0, 0, 1, 0)\) | \((0, 1, 0, 0, 0)\) | \((-1, 1)\) | \((1, 0)\) |
| \( t = t_3 \) | \((0, 0, 0, 1, 1)\) | \((0, 0, 1, 0, 0)\) | \((1, -1)\) | \((0, 1)\) |

\(^6\)These five groups are similarly defined in Kline and Walters (2016) for their analysis of the Head Start program.
Also, in this case, \( \pi_{z_1}(X) = P(Z = z_1 \mid X) \), \( \pi_{z_2}(X) = P(Z = z_2 \mid X) \), and

\[
\begin{align*}

P_{t_1,Z}(X) &= (P(T = t_1 \mid Z = z_1, X), P(T = t_1 \mid Z = z_2, X))^\prime, \\

P_{t_2,Z}(X) &= (P(T = t_2 \mid Z = z_1, X), P(T = t_2 \mid Z = z_2, X))^\prime, \\

P_{t_3,Z}(X) &= (P(T = t_3 \mid Z = z_1, X), P(T = t_3 \mid Z = z_2, X))^\prime. \\
\end{align*}
\]

For any measurable \( g \), we have

\[
\begin{align*}

g_{t_1,Z}(X) &= (\mathbb{E}[g(Y)1\{T = t_1\} \mid Z = z_1, X], \mathbb{E}[g(Y)1\{T = t_1\} \mid Z = z_2, X])^\prime, \\

g_{t_2,Z}(X) &= (\mathbb{E}[g(Y)1\{T = t_2\} \mid Z = z_1, X], \mathbb{E}[g(Y)1\{T = t_2\} \mid Z = z_2, X])^\prime, \\

g_{t_3,Z}(X) &= (\mathbb{E}[g(Y)1\{T = t_3\} \mid Z = z_1, X], \mathbb{E}[g(Y)1\{T = t_3\} \mid Z = z_2, X])^\prime. \\
\end{align*}
\]

3 Identification Results

Conditioning on \( X \), Heckman and Pinto (2018a) establishes the identification of the probabilities of \( S \) lying in any of the \( \Sigma_{t,k} \)’s and the conditional distribution of \( Y_t \) given \( S \in \Sigma_{t,k} \). This result is presented in Appendix B as Lemma 2. Bayes rule is applied to turn these conditional identification results into the unconditional ones. In particular, the difficulty here, that the conditional distribution of \( X \mid S \in \Sigma_{t,k} \) is unidentified, is similar to that of classical local IV. However, using the Bayes rule, this unidentified distribution can be represented as \( \frac{P(S \in \Sigma_{t,k} \mid X = x)}{P(S \in \Sigma_{t,k})} f_X(x) \) which is identified.

**Theorem 1.** For \( t \in T, k = 1, \cdots, N_Z \), and \( g \) measurable, the following quantities are identified.

(i) **Type probabilities:**

\[
\begin{align*}

p_{t,k} &\equiv P(S \in \Sigma_{t,k}) = \tilde{b}_{t,k} \mathbb{E}[P_{t,Z}(X)]. \\
\end{align*}
\]

(ii) **Mean potential outcome**\(^7\) **conditioning on type:**

\[
\begin{align*}

\mathbb{E}[g(Y_t) \mid S \in \Sigma_{t,k}] &= \frac{1}{p_{t,k}} \tilde{b}_{t,k} \mathbb{E}[g_{t,Z}(X)]. \\
\end{align*}
\]

The above theorem provides identification for quantities solely related to the type \( S \) of the

\(^7\)The term LASF is reserved for the more specific case of \( g \) being the identity map.
individual. Another set of policy-relevant parameters are the ones defined both by the type $S$ and the actual treatment received $T$. Especially, the conditional distribution of $Y_i$ given $T = t$ and $S \in \Sigma_{t,k}$ might be of more interest in the current setting. This is because the main source of identification lies in the structural functions, the $Y_i$’s, instead of the treatment effects. Hence it is more attractive to study the expectation of $Y_i$ inside the subpopulation whose treatment is actually $t$. The following theorem deals with the identification of distributions relevant to this idea.

**Theorem 2.** For $t \in T$, $k = 1, \cdots, N_Z$, and $g$ measurable, if for some $t' \in T$, there exists $W_{t',t,k} \subset Z$ such that $S \in \Sigma_{t,k}, T_z = t' \iff S \in \Sigma_{t,k}, z \in W_{t',t,k}$, and denote

$$
\pi_W(X) = \sum_{z \in W} \pi_z(X), \ W \subset Z,
$$

then the following quantities are identified.

(i) Treatment status and type probability:

$$
q_{t',t,k} \equiv P(T = t', S \in \Sigma_{t,k}) = \hat{b}_{t,k}E\left[ P_{t,Z}(X)\pi_{W_{t',t,k}}(X) \right]. \quad (3)
$$

(ii) Mean potential outcome conditioning on treatment status and type:

$$
E\left[ g(Y_t) \mid T = t', S \in \Sigma_{t,k} \right] = \frac{1}{q_{t',t,k}} \hat{b}_{t,k}E\left[ g_{t,Z}(X)\pi_{W_{t',t,k}}(X) \right]. \quad (4)
$$

In particular, the probability

$$
q_{t,k} \equiv P(T = t, S \in \Sigma_{t,k}), \quad (5)
$$

and the $t$-mean potential outcome

$$
E\left[ g(Y_t) \mid T = t, S \in \Sigma_{t,k} \right] = \frac{1}{q_{t,k}} \hat{b}_{t,k}E\left[ g_{t,Z}(X)\pi_{W_{t,k}}(X) \right], \quad (6)
$$

for the $t$-treated subpopulation with type $S \in \Sigma_{t,k}$, are always identified, where, $q_{t,k} = q_{t,t,k}$, and $W_{t,k} \equiv W_{t,t,k}$.

To the best of my knowledge, this result is new in the literature. The $W_{t',t,k}$ defined in the theorem contains the $z$’s such that, inside the subpopulation where type $S \in \Sigma_{t,k}$, individuals...
assigned with $z$ takes and only takes the treatment $t'$. The unordered monotonicity condition guarantees that such $W_{t,k}$ always exists for any pair of $t,k$. In the binary local IV case, it is shown previously in Hong and Nekipelov (2010a), as a special case, that inside the subpopulation of treated compliers, both of the structural functions are identified.

In most of the cases, the parameter of interest are the average structural functions identifiable in certain subpopulations. They are defined by taking $g = I$, the identity map on $\mathcal{Y}$, in equations (2) and (4), which are the LASFs and LASF-Ts displayed below.

\[
\beta_{t,k} = E[Y_t | S \in \Sigma_{t,k}] = \frac{1}{p_{t,k}} \hat{b}_{t,k} E[I_{t,Z}(X)].
\]

\[
\gamma_{t',t,k} = E[Y_t | T = t', S \in \Sigma_{t,k}] = \frac{1}{q_{t',t,k}} \tilde{b}_{t,k} E[I_{t,Z}(X)\pi_{W_{t',t,k}}(X)].
\]

As before, let $\gamma_{t,k} = \gamma_{t,t,k}$. To concretize ideas, the identification in the main example for the case of $\Sigma_{t_1,1} = s_4$ is computed, the other parameters can be computed in the same way. By Theorem 1,

\[
p_{t_1,1} = P(S = s_4) = E[P(T = t_1 | Z = z_2, X) - P(T = t_1 | Z = z_1, X)].
\]

\[
\beta_{t_1,1} = E[Y_{t_1} | S = s_4] = \frac{1}{p_{t_1,1}} E[E[Y_1(T = t_1) | Z = z_2, X] - E[Y_1(T = t_1) | Z = z_1, X]].
\]

For $\Sigma_{t_1,1} = s_4$, $W_{t_1,t_1,1} = \{z_1\}$, thus by Theorem 2,

\[
q_{t_1,1} = P(T = t_1, S = s_4) = E[(P(T = t_1 | Z = z_2, X) - P(T = t_1 | Z = z_1, X)) P(Z = z_2 | X)].
\]

\[
\gamma_{t_1,1} = E[Y_{t_1} | T = t_1, S = s_4] = \frac{1}{q_{t_1,1}} E[E[Y_1(T = t_1) | Z = z_2, X] - E[Y_1(T = t_1) | Z = z_1, X]] P(Z = z_2 | X)].
\]

A subtle thing here is that the parameters $p_{t,k}$’s and $q_{t',t,k}$’s can potentially be overidentified. For example, $\Sigma_{t_3,1} = \{s_4, s_5\} = \Sigma_{t_1,1} \cup \Sigma_{t_2,1}$, leading to two ways to calculate the probability $P(S \in \{s_4, s_5\})$. In this case, these ways give rise to identical expressions of $P(S \in \{s_4, s_5\})$. 


namely,

\[ P(S = s_4) + P(S = s_5) = \mathbb{E} [P(T = t_1 \mid Z = z_2, X) - P(T = t_1 \mid Z = z_1, X)] \]
\[ + \mathbb{E} [P(T = t_2 \mid Z = z_2, X) - P(T = t_2 \mid Z = z_1, X)] \]
\[ = \mathbb{E} [P(T \in \{t_1, t_2\} \mid Z = z_2, X) - P(T \in \{t_1, t_2\} \mid Z = z_1, X)] \]
\[ = \mathbb{E} [P(T = t_3 \mid Z = z_1, X) - P(T = t_3 \mid Z = z_2, X)] \]
\[ = P(S \in \{s_4, s_5\}). \]

As thus, it can be shown that there is no overidentifying restriction on the observable distribution in the main example. However, there are cases satisfying Assumption 1 that imposes overidentifying restrictions. Consider removing \( s_5 \) from the configurations \( S \), so that \( S \) can only take on the four values as shown in the following table.

|   | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) |
|---|---|---|---|---|
| \( T_{z_1} \) | \( t_1 \) | \( t_2 \) | \( t_3 \) | \( t_3 \) |
| \( T_{z_2} \) | \( t_1 \) | \( t_2 \) | \( t_3 \) | \( t_1 \) |

The unordered monotonicity is still satisfied. But it is automatically imposed that \( P(S = s_5) = 0 \) or equivalently

\[ \mathbb{E} [P(T = t_1 \mid Z = z_2, X) - P(T = t_1 \mid Z = z_1, X)] \]
\[ = \mathbb{E} [P(T = t_3 \mid Z = z_1, X) - P(T = t_3 \mid Z = z_2, X)]. \] (8)

which is an overidentifying restriction. These types of overidentifying restrictions are quite unnecessary, since benefit from the removal of impossible configurations is small compared to the cost of falsely eliminating a type that exists in the true DGP. In the next section on efficient estimation of local average structural functions, Assumption 1 is strengthened so that the parameters in Theorem 1 and Theorem 2 are exactly identified. This is not restrictive, since it is satisfied by the binary local IV, the main example in this paper, and examples in Heckman and Pinto (2018a). Moreover, when overidentification is present, the consistency and asymptotic normality of the proposed estimators, and it’s efficiency for the exactly-identified target parameter are valid regardless. Further
discussion on this issue is beyond the scope of this paper.\footnote{Some hints on more primitive assumptions for the non-existence of overidentifying conditions, within the structure of \( S \), can be found in A.14 of \textit{Heckman and Pinto (2018b)}, referring to the concept of “complete response matrix”.

4 Semiparametric Efficiency Bound and Efficient Estimation: Local Average Structural Function

This section calculates the semiparametric efficiency bound for \( \beta_{t,k}, \gamma_{t',t,k}, p_{t,k}, \) and \( q_{t',t,k} \), and proposes semiparametric efficient estimators. Other policy-relevant parameters that can be derived from the above quantities are also discussed afterwards. The more general case of non-smooth parameters with overidentifying constraints is considered in the next section. In this section and the next one, for simplicity, the notation for parameters, including the nuisance ones, are used to represent both the true value and a general value in the parameter space. When necessary, a superscript “o” is placed to signify the true value. The following theorem presents the efficiency bound using efficient influence functions.

\textbf{Theorem 3.} Consider any \( t \in T, k = 1, \cdots, N_Z \), and \( t' \) satisfying the condition in Theorem 2.

(i) The semiparametric efficiency bound for \( \beta_{t,k} \) is given by the variance of the efficient influence function

\[
\Psi_{\beta_{t,k}}(Y, T, Z, X, \beta_{t,k}, p_{t,k}, I_{t,Z}, P_{t,Z}, \pi) = \frac{1}{p_{t,k}} \tilde{b}_{t,k}(\zeta(Z, X, \pi) (\iota(Y1(T = t)) - I_{t,Z}(X)) + I_{t,Z}(X)) - \frac{\beta_{t,k}}{p_{t,k}} \tilde{b}_{t,k}(\zeta(Z, X, \pi) (\iota(1(T = t)) - P_{t,Z}(X)) + P_{t,Z}(X)).
\]

where \( \iota \) denotes a column vector of ones, and \( \zeta(Z, X, \pi) \) is a diagonal matrix with the diagonal elements being

\[
\left( \frac{1}{\pi_{z_1}(X)}, \ldots, \frac{1}{\pi_{z_N}(X)} \right)
\]

(ii) The semiparametric efficiency bound for \( \gamma_{t',t,k} \) is given by the variance of the efficient influence

}\]
(iii) The semiparametric efficiency bound for $p_{t,k}$ is given by the variance of the efficient influence function

$$\Psi_{p_{t,k}}(T, Z, X, p_{t,k}, P_{t,Z}, \pi) = \frac{1}{q_{t',t,k}} \tilde{b}_{t,k} \left( \zeta(Z, X, \pi) \left( \iota(Y \mathbb{1}\{T = t\}) - I_{t,Z}(X) \right) \pi_{W_{t',t,k}}(X) + I_{t,Z}(X) \mathbb{1}\{Z \in W_{t',t,k}\} \right) - q_{t',t,k}$$

(iv) The semiparametric efficiency bound for $q_{t',t,k}$ is given by the variance of the efficient influence function

$$\Psi_{q_{t',t,k}}(T, Z, X, q_{t',t,k}, P_{t,Z}, \pi) = \tilde{b}_{t,k} \left( \zeta(Z, X, \pi) \left( \iota(\mathbb{1}\{T = t\}) - P_{t,Z}(X) \right) \pi_{W_{t',t,k}}(X) + P_{t,Z}(X) \mathbb{1}\{Z \in W_{t',t,k}\} \right) - q_{t',t,k}$$

Notice that for the case of binary treatment and binary instrument, the first two parts of Theorem 3 reduces to Theorem 2 of Hong and Nekipelov (2010a). The structure of the efficient influence functions is interpretable in the view of Newey (1994). In $\Psi_{p_{t,k}}$, the terms $\tilde{b}_{t,k} \left( \zeta(Z, X, \pi) \left( \iota(Y \mathbb{1}\{T = t\}) - I_{t,Z}(X) \right) \right)$ and $\tilde{b}_{t,k} \left( \zeta(Z, X, \pi) \left( \iota(\mathbb{1}\{T = t\}) - P_{t,Z}(X) \right) \right)$ serve as the correction term due to the presence of unknown infinite dimensional nuisance parameters $I_{t,Z}$ and $P_{t,Z}$ respectively. In $\Psi_{q_{t',t,k}}$, the correction term also contains $\pi_{W_{t',t,k}}$ which accounts for the fact that $\pi$ is unknown. The derivation of this decomposition is more apparent in the proof of Theorem 4.

The program evaluation literature has become concerned about the role of propensity score in the efficient estimation. In the current context, $\pi$ represents the proper concept of propensity score. Observations in the proof of Theorem 3 indicate that the efficiency bound of $\beta_{t,k}$ is not affected by the knowledge of propensity score. However, the $\gamma_{t',t,k}$’s can be estimated more efficiently if the propensity score is known. To be more specific, the part of the score function that corresponds to

---

9See for example Hahn (1998); Frölich (2007); Hong and Nekipelov (2010a); Chen et al. (2008)
the conditional distribution of instrument only acts in the efficiency bound calculation for LASF-T, not LASF, because the identification of $\gamma_{t',t,k}$ explicitly involves the propensity score, hence does its pathwise derivative. Similarly, the efficiency in the estimation of the $q_{t',t,k}$’s is not the $p_{t,k}$ is affected by the knowledge of propensity score.

For efficient estimation, one possible way is to use the CEP estimators common in the literature (Hahn, 1998; Frölich, 2007; Chen et al., 2008; Hong and Nekipelov, 2010a). The methodology is to first estimate the conditional expectations $\pi$, $P_{t,Z}$, and $I_{t,Z}$, and then uses the identification results directly as moment conditions. The asymptotic linear representation of these estimators can be easily computed using the method developed in Newey (1994). More specifically, for $z \in \mathcal{Z}$, define two sets of conditional expectations by $h_{Y,t,z}(X) = \mathbb{E}[1\{Z = z\} Y_1|X]$, $h_{t,z}(X) = \mathbb{E}[1\{Z = z\} T |X]$. Let $\hat{h}_{Y,t,z}, \hat{h}_{t,z}, \hat{\pi}_z$ denote the nonparametric estimators, such as kernel estimators or series estimators. Notice that the conditional expectations are related by $I_{t,z} = h_{Y,t,z}/\bar{\pi}_z$ and $P_{t,z} = h_{t,z}/\bar{\pi}_z$, for $z \in \mathcal{Z}$, hence define $\hat{I}_{t,z} = \hat{h}_{Y,t,z}/\bar{\hat{\pi}}_z$ and $\hat{P}_{t,z} = \hat{h}_{t,z}/\bar{\hat{\pi}}_z$. Then the vector estimators $\hat{I}_{t,z}, \hat{P}_{t,z}, \hat{\pi}_Z$, and vector functions $h_{Y,t,Z}, h_{t,Z}$ are stacked in the obvious way. Also, let $\hat{\pi}_{W_{t',t,k}} = \sum_{z \in W_{t',t,k}} \hat{\pi}_z$. Define the estimators $\hat{p}_{t,k}, \hat{q}_{t',t,k}, \hat{\beta}_{t,k},$ and $\hat{\gamma}_{t',t,k}$ by

$$\hat{p}_{t,k} = \frac{1}{n} \sum_{i=1}^{n} \hat{b}_{t,k} \hat{P}_{t,Z}(X_i)$$  
$$\hat{q}_{t',t,k} = \frac{1}{n} \sum_{i=1}^{n} \hat{b}_{t,k} \hat{P}_{t,Z}(X_i) \hat{\pi}_{W_{t',t,k}}(X_i)$$  
$$\hat{\beta}_{t,k} = \frac{1}{\hat{p}_{t,k}} \frac{1}{n} \sum_{i=1}^{n} \hat{b}_{t,k} \hat{I}_{t,Z}(X_i)$$  
$$\hat{\gamma}_{t',t,k} = \frac{1}{\hat{q}_{t',t,k}} \frac{1}{n} \sum_{i=1}^{n} \hat{b}_{t,k} \hat{I}_{t,Z}(X_i) \hat{\pi}_{W_{t',t,k}}(X_i)$$

The efficient influence functions also provide a way for conducting optimal joint inferences. The efficient influence function for a vector of parameters is the collection of the efficient influence functions corresponding to the parameters. The variance-covariance matrix for efficient estimators can be calculated accordingly. More concretely, let $\kappa = (\kappa_1, \cdots, \kappa_K)$ be a vector of parameters selected from the identified ones $\{\beta_{t,k}, \gamma_{t',t,k}, p_{t,k}, q_{t',t,k}\}$. Then the efficient influence function of $\kappa$ is $\Psi_{\kappa} = (\Psi_{\kappa_1}, \cdots, \Psi_{\kappa_K})'$, and the efficiency bound is $\mathbb{E} [\Psi_{\kappa} \Psi_{\kappa}']$. A natural plug-in estimator $\hat{V}_{\kappa}$

\footnote{The reason for using $h_{Y,t,z}$ and $h_{t,z}$ as primitive estimators, instead of $I_{t,z}$ and $P_{t,z}$, is that they are simple conditional expectations with theoretical appeals.}
defined below can be employed to consistently estimate this bound under mild conditions,

\[ \hat{V}_\kappa = \frac{1}{n} \sum_{i=1}^{n} \hat{\Psi}_{\kappa,i} \hat{\Psi}'_{\kappa,i} \]  

(17)

where \( \hat{\Psi}_{\kappa,i} \) is defined by plugging-in the data observation of index \( i \), the CEP estimates \( \hat{\kappa} \), and the nonparametric estimates \( \hat{I}_{t,z}, \hat{P}_{t,z}, \text{ and } \hat{\pi}_z \). For example, when \( \kappa = \beta_{t,k} \), then

\[ \hat{\Psi}_{\beta_{t,k},i} = \Psi_{\beta_{t,k}}(Y_i, T_i, Z_i, X_i, \hat{\beta}_{t,k}, \hat{p}_{t,k}, \hat{I}_{t,Z}, \hat{P}_{t,Z}, \hat{\pi}). \]

Since the CEP estimators are efficient, their asymptotic covariance matrix can be estimated by the plug-in estimators for efficiency bounds. This leads to optimal joint inferences, where the optimality is by Section 25.6 of Vaart (1998), where it is stated that the semiparametric efficiency of the estimators leads to (locally) asymptotically uniformly most powerful tests.

The following theorem summarizes the properties of the CEP estimation procedure defined above. For each \( t \) and \( z \), let \( \mathcal{H}_{Y,t,z}, \mathcal{H}_{t,z}, \text{ and } \Pi_z \) be the space of functions containing the true nuisance parameters \( h^o_{Y,t,z}, h^o_{t,z}, \text{ and } \pi^o_z \) respectively. For any small enough \( \delta > 0 \), let \( \mathcal{H}_{Y,t,z}^\delta = \{ h_{Y,t,z} \in \mathcal{H}_{Y,t,z} : \| h_{Y,t,z} - h^o_{Y,t,z} \| \leq \delta \} \), and \( \mathcal{H}_{t,z}^\delta \text{ and } \Pi_z^\delta \) be defined analogously.

**Theorem 4.** Consider \( t \in T \). For any \( z \in Z \), assume the following conditions hold.

(i) The convergence rate of nonparametric estimators satisfy

\[ \sqrt{n} \| h_{Y,t,z} - h^o_{Y,t,z} \|^2 = o_p(1), \]

\[ \sqrt{n} \| \hat{h}_{t,z} - h^o_{t,z} \|^2 = o_p(1), \text{ and } \sqrt{n} \| \hat{\pi}_z - \pi^o_z \|^2 = o_p(1). \]

(ii) There exists some \( \delta > 0 \) such that the classes \( \mathcal{H}_{Y,t,z}^\delta, \mathcal{H}_{t,z}^\delta, \text{ and } \Pi_z^\delta \) are Donsker, with

\[ \mathbb{E} \left[ \sup_{h_{Y,t,z} \in \mathcal{H}_{Y,t,z}^\delta} |h_{Y,t,z}(X)| \right] < \infty. \]

Then, for \( k = 1, \cdots, N_Z \), estimators \( \hat{\beta}_{t,k}, \hat{\gamma}_{t',t,k}, \hat{p}_{t,k}, \text{ and } \hat{q}_{t',t,k} \) are semiparametric efficient for \( \beta_{t,k}, \gamma_{t',t,k}, p_{t,k}, \text{ and } q_{t',t,k} \), respectively. Moreover, the plug-in estimator \( \hat{V}_\kappa \) for efficiency bound, defined in equation(17), is consistent.

Condition 4(i) is a standard requirement on the convergence rate of nonparametric estimators in the semiparametric two-step estimation literature (Newey, 1994; Newey and McFadden, 1994). Condition 4(ii) is also standard that requires the functional spaces containing the infinite-dimensional nuisance parameters are not too complex, for the stochastic equicontinuity condition
to hold. The reason for this type of “limited information” estimators to work is well explained in Ackerberg et al. (2014). The estimation problem here falls into their general semiparametric model where the parameter of interest is defined by possibly overidentifying unconditional moment restrictions and the nuisance function are defined by exactly identifying conditional moment restrictions. They showed that the semiparametric two-step optimally weighted GMM estimators achieve the efficiency bound, which are the CEP estimators in this case since the parameters of interest are exactly identified. Discussions related to this phenomenon can also be found in Chen and Santos (2018).

Back to the main example, for $\beta_{t,1}$, the efficient influence function is

$$
\Psi_{\beta_{t,1}} = \frac{1}{p_{t,1}} \left( \frac{1}{p(Z = z_2 | X)} (Y_1 \{ T = t_1 \} - \mathbb{E} [ Y_1 \{ T = t_1 \} | Z = z_2, X]) + \mathbb{E} [ Y_1 \{ T = t_1 \} | Z = z_2, X] \right) \\
- \frac{1}{p_{t,1}} \left( \frac{1}{p(Z = z_1 | X)} (Y_1 \{ T = t_1 \} - \mathbb{E} [ Y_1 \{ T = t_1 \} | Z = z_1, X]) + \mathbb{E} [ Y_1 \{ T = t_1 \} | Z = z_1, X] \right) \\
- \frac{\beta_{t,1}}{p_{t,1}} \left( \frac{1}{p(Z = z_2 | X)} (1 \{ T = t_1 \} - \mathbb{E} [ 1 \{ T = t_1 \} | Z = z_2, X]) + \mathbb{E} [ 1 \{ T = t_1 \} | Z = z_2, X] \right) \\
+ \frac{\beta_{t,1}}{p_{t,1}} \left( \frac{1}{p(Z = z_1 | X)} (1 \{ T = t_1 \} - \mathbb{E} [ 1 \{ T = t_1 \} | Z = z_1, X]) + \mathbb{E} [ 1 \{ T = t_1 \} | Z = z_1, X] \right)
$$

and the CEP estimator is

$$
\beta_{t,1} = \frac{\sum_{i=1}^{n} (\hat{h}_{Y,T,z_2}(X_i) / \hat{\pi}_{z_2}(X_i) - (\hat{h}_{Y,T,z_1}(X_i) / \hat{\pi}_{z_1}(X_i)))}{\sum_{i=1}^{n} (\hat{h}_{T,z_2}(X_i) / \hat{\pi}_{z_2}(X_i) - (\hat{h}_{T,z_1}(X_i) / \hat{\pi}_{z_1}(X_i)))}.
$$

Besides the efficient estimation results shown above, it might also be of interest to efficiently estimate other policy-relevant parameters, whose identification can be derived from the aforementioned parameters $(\beta_{t,k}, \gamma_{t',t,k}, p_{t,k}, q_{t',t,k})$. Some examples are discussed here. The ratio $q_{t',t,k} / p_{t,k} = P(T = t' | S \in \Sigma_{t,k})$ can be understood as the conditional probability of taking treatment $t'$ given type $S$ belongs to $\Sigma_{t,k}$. The average structural function local in the subpopulation whose type $S$ belonging to any of the $\Sigma_{t,k}, k = 1, \ldots, N_Z - 1$ can be calculated by

$$
\beta_t \equiv \mathbb{E} [ Y_t | S \in \Sigma_t ] = \frac{\sum_{k=1}^{N_Z-1} \beta_{t,k} p_{t,k}}{\sum_{k=1}^{N_Z-1} p_{t,k}}
$$

(18)

where $\Sigma_t = \bigcup_{k=1}^{N_Z-1} \Sigma_{t,k}$ is referred to as $t$-switchers in Heckman and Pinto (2018a), which means
individuals in this subpopulation switches among $t$ and other treatments when given different levels of instruments. It is a generalization of the concept of compliers in the binary local IV framework. Similarly, one can also define

$$
\gamma_t = \mathbb{E}[Y_t \mid T = t, S \in \Sigma_t] = \frac{\sum_{k=1}^{N_s-1} \gamma_{t,k} p_{t,k}}{\sum_{k=1}^{N_s-1} p_{t,k}}
$$

which represents the average structural function local in the subpopulation of $t$-treated $t$-switchers. By Theorem 2, this parameter is always identified. Some treatment effects can also be identified and estimated through the parameters discussed in this section. This point is illustrated using the main example, which also appears in Heckman and Pinto (2018a). Consider the following quantity

$$
\beta_{t3,1} - \frac{\beta_{t1,1} p_{t1,1} + \beta_{t2,1} p_{t2,1}}{p_{t1,1} + p_{t2,1}}
= \frac{\mathbb{E}[Y_{t3} - Y_{t1} \mid S = s_4] P(S = s_4) + \mathbb{E}[Y_{t3} - Y_{t2} \mid S = s_5] P(S = s_5)}{P(S \in \{s_4, s_5\})}
$$

which represents the average treatment effect of $t_3$ against other treatments within the subpopulation of $t_3$-switchers. Analogously, the quantity

$$
\gamma_{t3,1} = \frac{\gamma_{t3,t1,1} p_{t3,t1,1} + \gamma_{t3,t2,1} p_{t3,t2,1}}{p_{t3,t1,1} + p_{t3,t2,1}}
= \frac{\mathbb{E}[Y_{t3} - Y_{t1} \mid T = t_3, S = s_4] P(T = t_3, S = s_4) + \mathbb{E}[Y_{t3} - Y_{t2} \mid T = t_3, S = s_5] P(T = t_3, S = s_5)}{P(T = t_3, S \in \{s_4, s_5\})}
$$

can be understood as the average treatment effect of $t_3$ against other treatments within the subpopulation of $t_3$-treated $t_3$-switchers.

More generally, let $\phi = \phi(p, q, \beta, \gamma)$ be a finite-dimensional parameter, where $\phi(\cdot)$ is a known continuously differentiable function, and $p$ is the vector containing all identifiable $p_{t,k}$'s, and $q, \beta, \gamma$ are defined analogously. A natural estimator can be defined through the CEP estimates, $\hat{\phi}(\hat{p}, \hat{q}, \hat{\beta}, \hat{\gamma})$. A delta method argument helps calculate the efficiency bound of $\phi$ and show the efficiency of $\phi(\hat{p}, \hat{q}, \hat{\beta}, \hat{\gamma})$. In fact, following Theorem 25.47 of Vaart (1998), and Theorem 3 and 4, the corollary below is immediate, which, in particular, solves the issue of efficient estimation for the several examples illustrated above.
Corollary 1. The semiparametric efficiency bound of $\phi$ is given by the variance of efficient influence function

$$
\Psi_\phi = \sum_{p} \frac{\partial \phi}{\partial p} \Psi_p + \sum_{q} \frac{\partial \phi}{\partial q} \Psi_q + \sum_{\beta} \frac{\partial \phi}{\partial \beta} \Psi_\beta + \sum_{\gamma} \frac{\partial \phi}{\partial \gamma} \Psi_\gamma
$$

(22)

where the partial derivatives are evaluated at the true parameter value. Moreover, the plug-in estimator $\phi(\hat{p}, \hat{q}, \hat{\beta}, \hat{\gamma})$, based on the CEP estimators $\hat{p}, \hat{q}, \hat{\beta}, \hat{\gamma}$, achieves the efficiency bound.

The role of the efficient influence functions discussed above is mainly on calculating the efficiency bound. They could also be used to generate a collection of moment conditions, to achieve efficient estimation directly. These moment conditions possess the feature that the first-step estimation of the nuisance function do not affect the asymptotic variance. This is straightforward to verify using Proposition 3 in Newey (1994). This feature can be further exploited in the DML methodology which is suitable in the high dimensional settings, where the Donsker properties as in condition 4(ii) can no longer be satisfied. More formally, the efficient influence function satisfies the Neyman orthogonality condition, which means reduced sensitivity with respect to the nuisance parameters $I_{t,Z}$'s, $P_{t,Z}$'s, and $\pi$. Together with appropriate data splitting methods, moment estimators constructed with Neyman orthogonal moment conditions are often employed in cases with data-rich environments where the nuisance parameters are “highly complex”, e.g. the dimension of covariates $X$ grows with sample size $n$. Here I explain how to implement in this specific setting the DML method introduced in Chernozhukov et al. (2018) to efficiently estimate $\beta_{t,k}$ when the dimension of $X$ is larger than the sample size. The cross-fitting method starts with taking a $L$-fold random partition of the data such that the size of each fold is $n/L$. Then, for $l = 1, \cdots, L$, let $I_l$ denote the observation indices in the $l$th fold and $I_l^c = \bigcup_{l \neq l} I_l$. Also, define $\hat{I}_l^{l^t}, \hat{P}_l^{l^t}, \hat{\pi}_l$ be the nonparametric machine learning estimates using data from $i \in I_l^c$. The associated moment condition is based on equation (9) that

$$
\mathbb{E} \left[ \hat{b}_{t,k} (\zeta(Z, X, \pi) (i (Y1(T = t)) - I_{t,Z}(X)) + I_{t,Z}(X)) \right. \\
\left. - \beta_{t,k} \tilde{b}_{t,k} (\zeta(Z, X, \pi) (i 1(T = t) - P_{t,Z}(X)) + P_{t,Z}(X)) \right] = 0.
$$

(23)

11I am grateful to Kaspar Wuthrich for suggesting this. Works on this topic include Belloni et al. (2014, 2017); Chernozhukov et al. (2018).

12The cases of estimating $\gamma_{t,1,k}$, $p_{t,1}$ and $q_{t,1,k}$ are essentially the same, thus omitted for brevity.
Using this moment condition, the estimator $\tilde{\beta}_{t,k}$ is defined by

$$
\tilde{\beta}_{t,k} = \frac{\sum_{l=1}^{L} \sum_{i \in I_l} b_{t,k} \left( \zeta(Z_i, X_i, \hat{\pi}) \left( \mathbb{I}(Y_i 1 \{ T_i = t \}) - \tilde{\mathcal{I}}_{t,Z}(X_i) \right) + \tilde{\mathcal{I}}_{t,Z}(X_i) \right)}{\sum_{l=1}^{L} \sum_{i \in I_l} b_{t,k} \left( \zeta(Z_i, X_i, \hat{\pi}) \left( \mathbb{I}(T_i = t) - \tilde{P}_{t,Z}(X_i) \right) + \tilde{P}_{t,Z}(X_i) \right)}.
$$

(24)

This is the DML2 estimator defined in Chernozhukov et al. (2018) with a $L$-fold cross-fitting. There is another estimation procedure called the DML1 estimator in Chernozhukov et al. (2018). It is not discussed here since DML1 and DML2 are asymptotically equivalent, and DML2 is generally recommended by the authors. The variance estimator is given by

$$
\hat{V}_{\beta_{t,k}} = \frac{1}{nL} \sum_{l=1}^{L} \sum_{i=1}^{n} \left( \Psi_{\beta_{t,k}}(Y, T, Z, X, \tilde{\beta}_{t,k}, \tilde{P}_{t,k}, \tilde{P}_{t,Z}, \tilde{P}_{t,Z}) \right)^2
$$

(25)

where

$$
\tilde{p}_{t,k} = \frac{1}{nL} \sum_{l=1}^{L} \sum_{i=1}^{n} b_{t,k} \left( \zeta(Z_i, X_i, \hat{\pi}) \mathbb{I}(T_i = t) - \tilde{P}_{t,Z}(X_i) \right) + \tilde{P}_{t,Z}(X_i)
$$

(26)

Theorem 5. Let $\delta_n \geq n^{-1/2}$ and $\Delta_n$ be some sequences of positive constants approaching zero. Also, let $C > 0$ and $q > 2$ be fixed constants, and $L \geq 2$ be fixed integer. Assume the following conditions hold for any joint distribution $P \in \mathcal{P}$ for the quadruple $(Y, T, Z, X)$.

(i) The variance bound for $\beta_{t,k}$ calculated in Theorem 3 is strictly positive.

(ii) $\max \left\{ \left\| T_{t,Z}^0 \right\|_q, \left\| \mathbb{I}(T = t) - T_{t,Z}^0 \right\|_q \right\} \leq C$.

(iii) With probability no less than $1 - \Delta_n$, $\max \left\{ \left\| \tilde{T}_{t,Z} - T_{t,Z}^0 \right\|_q, \left\| \tilde{P}_{t,Z} - P_{t,Z}^0 \right\|_q, \left\| \tilde{\pi} - \pi^0 \right\|_q \right\} \leq C$, $\max \left\{ \left\| \tilde{T}_{t,Z} - T_{t,Z}^0 \right\|_2, \left\| \tilde{P}_{t,Z} - P_{t,Z}^0 \right\|_2, \left\| \tilde{\pi} - \pi^0 \right\|_2 \right\} \leq \delta_n$, and for any $z \in Z$, $\tilde{\pi}_z$ strictly $\leq \pi_z$ and $\left\| \tilde{\pi}_z - \pi_z^0 \right\| \times \left( \left\| \tilde{T}_{t,z} - T_{t,z}^0 \right\|_2 + \left\| \tilde{P}_{t,z} - P_{t,z}^0 \right\|_2 \right) \leq n^{-1/2} \delta_n$.

Then the estimator $\tilde{\beta}_{t,k}$ obey

$$
V_{\beta_{t,k}}^{-1/2} \sqrt{n} (\tilde{\beta}_{t,k} - \beta_{t,k}) \Rightarrow N(0, 1),
$$

(27)

uniformly over $\mathcal{P}$, where $V_{\beta_{t,k}} = \mathbb{E} \left[ \Psi_{\beta_{t,k}}(Y, T, Z, X, \beta_{t,k}^0, P_{t,k}^0, T_{t,Z}^0, P_{t,Z}^0, \pi_{t,Z}^0) \right]$. Moreover, the results continue to hold when $V_{\beta_{t,k}}$ is replaced by $\hat{V}_{\beta_{t,k}}$. 

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The essential condition here is to restrict the convergence rate of estimating the nuisance parameters. These rates are often available for common machine learning methods. The Donsker properties are relaxed as discussed before. Since the convergence results hold uniformly in $\mathcal{P}$, it can be used for standard construction of uniformly valid confidence regions.

5 Semiparametric Efficiency Bound and Efficient Estimation: Non-smooth GMM

The previous section is on the efficient estimation of average structural functions inside some identifiable subpopulations. More generally, the parameter of interest could be defined through non-smooth and overidentifying moment conditions. One example is the case of quantile estimation. Another case of interest is when the underlying economic theory provides overidentifying constraints for the quantities of interest, which is very possible in the current framework with multiple levels of treatment and instrument.

Define a set of random variables $Y = \{Y^*_{t,k} : t \in T, k = 1, \cdots, N_z\}$ such that each $Y^*_{t,k}$ has the same marginal distribution as $Y_t | S \in \Sigma_{t,k}$. Their joint distribution is irrelevant. Let $Y^* = (Y^*_1, \cdots, Y^*_J)$, for $Y^*_j \in \mathcal{Y}$. Also, for notational convenience, use $j$ rather than $t, k$ to label $t_j$, $p_j$, and $\tilde{b}_j$ according to the ordering of the $Y^*_{t,k}$’s within the random vector $Y^*$. Let the parameter of interest be $\eta$, in the interior of $\Lambda \subset \mathbb{R}^{d_\eta}$, where $d_\eta \leq J$. The true value of the parameter $\eta_0$ satisfies the moment condition

$$E[m(Y^*, \eta_0)] = 0 \quad (28)$$

where $m : \mathcal{Y}^J \times \mathbb{R}^{d_\eta} \to \mathbb{R}^J$ is of the form

$$m(Y^*, \eta) = (m_1(Y^*_1, \eta), \cdots, m_J(Y^*_J, \eta))^t$$

In general, $m$ is allowed not to be differentiable with respect to $\eta$. Since the vector $\eta$ appears in each $m_j$, restrictions are allowed both within and across different subpopulation-conditional distributions. Another interesting feature of this specification is that the moment conditions are defined for the random variables whose distributions are not directly identified. The following theorem provides semiparametric efficiency bound for estimating $\eta$. Note that Assumption 1 is enough
for deriving the following results, since the distributions of $Y_{t,k}^*$’s are always exactly-identified.

Let $m_Z = (m_{1,t_1,z}, \ldots, m_{J,t_J,z})$, where $m_{j,t_j,z}(X, \eta) = (m_{j,t_j,z_1}(X, \eta), \ldots, m_{j,t_j,z}(X, \eta))'$, and $m_{j,t_j,z}(X, \eta) = \mathbb{E}[m_j(Y, \eta)1\{T = t_j\} \mid Z = z, X]$. 

**Theorem 6.** Assume the following conditions hold.

(i) For any $1 \leq j_1 \leq \cdots \leq j_{d_\eta} \leq J$, the subvector of moments $\mathbb{E}\left[\left(\begin{array}{c} m_{j_1}(Y_{j_1}^*, \eta), \ldots, m_{j_{d_\eta}}(Y_{j_{d_\eta}}^*, \eta) \end{array}\right)\right]'$ is zero if and only if $\eta = \eta^o$.

(ii) $\mathbb{E}[m(Y^*, \eta)^2] < \infty, \eta \in \Lambda$.

(iii) For each $j$ and $z$, $\mathbb{E}\left[m_{j,t_j,z}(Y, \eta)\right]$ is differentiable in some neighborhood of $\eta^o$, with the derivative continuous at $\eta^o$. Let $\Gamma$ be the $J \times d_\eta$ matrix whose $j$th row is $\tilde{b}_j \frac{\partial}{\partial \eta} \mathbb{E}\left[m_{j,t_j,z}(X, \eta)\right]' |_{\eta = \eta^o}$, and assume $\Gamma$ has full column rank.

Then for the estimation of $\eta$, the efficient influence function is

$$
\Psi_{\eta}(Y, T, Z, X, \eta^o, m_{Z}^o, \pi^o) = - (\Gamma'V^{-1}\Gamma)^{-1}\Gamma'V^{-1}\Psi_m(Y, T, Z, X, \eta^o, m_{Z}^o, \pi^o)
$$

(29)

$V = \text{Var}(\Psi_m(Y, T, Z, X, \eta^o, m_{Z}^o, \pi^o))$, and $\Psi_m(Y, T, Z, X, \eta^o, m_{Z}^o, \pi^o)$ is a $J \times 1$ random vector whose $j$th element is

$$
\tilde{b}_j \left(\zeta(Z, X, \pi^o) \left(\nu(m_j(Y, \eta^o)1\{T = t_j\}) - m_{j,t_j,z}(X, \eta^o)\right) + m_{j,t_j,z}(X, \eta^o)\right)
$$

(30)

Thus the semiparametric efficiency bound is $(\Gamma'V^{-1}\Gamma)^{-1}$.

Note that, for example, if $Y^* = Y_{t,k}^*$, and $m(Y_{t,k}^*, \eta) = Y_{t,k}^* - \eta$, then $\eta = \beta_{t,k}$, and the efficiency bound shown above reduces to the one computed in Theorem 3(i). If $T = Z$, that is under unconfoundedness, the above result reduces to Theorem 1 in Cattaneo (2010). The efficiency bound can be achieved by estimators in the same spirit of the EIFE proposed in Cattaneo (2010). Essentially, it is the optimally-weighted GMM estimator based on the moment conditions obtained from the efficient influence function $\Psi_m$. Let the criterion function be

$$
G_n(\eta, \pi, m_Z) = \frac{1}{n} \sum_{i=1}^{n} \Psi_m(Y_i, T_i, Z_i, X_i, \eta, \pi, m_Z)
$$

(31)
and its probability limit is

$$G(\eta, \pi, m_Z) = \mathbb{E} [\Psi_m(Y, T, Z, \eta, \pi, m_Z)]$$  \hspace{1cm} (32)$$

The main difficulty is that $G_n(\cdot, \pi, m_Z)$ could potentially be discontinuous, since we allow $m(Y^*, \cdot)$ to be discontinuous. To deal with this issue, the method developed in Chen et al. (2003) is employed, where the criterion function is allowed not to satisfy standard smoothness conditions and simultaneously depends on some nonparametric estimators.\textsuperscript{13}

The implementation procedure is as follows. Let $\hat{\pi}$ and $\hat{m}_Z$ be nonparametric estimators. One can first find a consistent GMM estimator $\tilde{\eta}$ using the identity matrix as the (non-optimal) weighting matrix, i.e.

$$\|G_n(\tilde{\eta}, \hat{\pi}, \hat{m}_Z)\| \leq \inf_{\eta \in \Lambda} \|G_n(\eta, \hat{\pi}, \hat{m}_Z)\| + o_p(1).$$  \hspace{1cm} (33)$$

Next use this estimate to form a consistent estimator $\hat{V}$ for the covariance matrix of $\Psi_m$ by

$$\hat{V} = \frac{1}{n} \sum_{i=1}^{n} \Psi_m(Y_i, T_i, Z_i, X_i, \tilde{\eta}, \hat{\pi}, \hat{m}_Z) \Psi_m(Y_i, T_i, Z_i, X_i, \tilde{\eta}, \hat{\pi}, \hat{m}_Z)'.$$  \hspace{1cm} (34)$$

Then define $\hat{\eta}$ as the optimally-weighted GMM estimator

$$\hat{\eta} = \arg \min_{\eta \in \Lambda} G_n(\eta, \hat{\pi}, \hat{m}_Z) \hat{V}^{-1} G_n(\eta, \hat{\pi}, \hat{m}_Z)'.$$  \hspace{1cm} (35)$$

Lastly, for estimating the asymptotic variance of $\hat{\eta}$, one can estimate $\Gamma$ using numerical derivatives as in Newey and McFadden (1994). Let $\varepsilon_n$ be a positive sequence such that $\varepsilon_n \to 0$ and $\varepsilon_n \sqrt{n} \to \infty$. Define the $J \times d_\eta$ matrix estimator by

$$\hat{\Gamma}_{jl} = \frac{1}{2\varepsilon_n} \hat{b}_j \left( \frac{1}{n} \sum_{i=1}^{n} \hat{m}_{j,t_i,Z}(X_i, \hat{\eta} + \varepsilon_n e_l) - \frac{1}{n} \sum_{i=1}^{n} \hat{m}_{j,t_i,Z}(X_i, \hat{\eta} - \varepsilon_n e_l) \right)$$  \hspace{1cm} (36)$$

where $e_l \in \mathbb{R}^{d_\eta}$ is the vector with the $l$th element being 1 and 0 on other entries. The following theorem summarizes the asymptotic properties of this estimation procedure. For each $j$ and $z$, let $\mathcal{M}_{j,z} \subset \mathbb{R}^{(X \times \Lambda)}$ be the vector space of functions, endowed with the sup-norm, containing the

\textsuperscript{13}Cattaneo (2010) instead uses the theory from Pakes and Pollard (1989). However, the general theory of Chen et al. (2003) is more straightforward to apply in this case, since they explicitly assumes the presence of infinite dimensional nuisance parameters, which can depend on the parameters to be estimated.
true unknown infinite dimensional nuisance parameter \( m_{j,t,j,z}^0 \). For any small enough \( \delta > 0 \), let
\[
\mathcal{M}_{j,z}^\delta = \left\{ m_{j,t,j,z} \in \mathcal{M}_{j,z} : \left\| m_{j,t,j,z} - m_{j,t,j,z}^0 \right\| \leq \delta \right\}.
\]

**Theorem 7.** Let the conditions in Theorem 6 hold. Further assume that, for each \( j \) and \( z \),

(i) \( \Lambda \) is compact and \( \eta^0 \in \text{int}(\Lambda) \).

(ii) Convergence rates of nonparametric estimators satisfy
\[
\left\| \hat{\pi}_z - \pi_z^0 \right\|, \left\| \hat{m}_{j,t,j,z} - m_{j,t,j,z}^0 \right\| = o_p(n^{-1/4}).
\]

(iii) The classes \( \left\{ m_{j,t,j,z}(\cdot, \eta) : \eta \in \Lambda, m_{j,t,j,z} \in \mathcal{M}_{j,z}^\delta \right\} \) and \( \Pi_z \) are Glivenko-Cantelli.

(iv) For some \( \delta > 0 \), the classes \( \left\{ m_{j,t,j,z}(\cdot, \eta) : \eta \in \Lambda, \| \eta - \eta^0 \| \leq \delta, m_{j,t,j,z} \in \mathcal{M}_{j,z}, \left\| m_{j,t,j,z} - m_{j,t,j,z}^0 \right\| \leq \delta \right\} \) and \( \Pi_z^\delta \) are Donsker.

(v) \( \mathbb{E}\left[ \sup_{\delta \in \Lambda, m_{j,t,j,z} \in \mathcal{M}_{j,z}} \left| m_{j,t,j,z}(\cdot, \eta) \right| \right] < \infty. \)

(vi) \( \mathbb{E}\left[ m_{j,t,j,z}(X, \cdot) \right] \) is continuous, for any \( m_{j,t,j,z} \in \mathcal{M}_{j,z}^\delta \).

Then \( \hat{V}, \hat{\eta}, \hat{\Gamma} \) are consistent and \( \sqrt{n}(\hat{\eta} - \eta^0) \Rightarrow N(0, (\Gamma' V^{-1} \Gamma)^{-1}). \)

As explained in the previous section, asymptotically optimal inferences can be conducted, based on this result, for joint hypothesis over \( \eta \). A possible case for the application of this non-smooth GMM methodology developed is illustrated below using the main example. The set \( \mathcal{Y} \) is defined by
\[
\mathcal{Y} = \left\{ Y^*_{t_1,1} \overset{d}{=} Y_1 \mid S = s_4, Y^*_{t_1,2} \overset{d}{=} Y_1 \mid S = s_1, Y^*_{t_2,1} \overset{d}{=} Y_2 \mid S = s_5, Y^*_{t_2,2} \overset{d}{=} Y_2 \mid S = s_2, Y^*_{t_3,1} = Y_3 \mid S \in \{s_4, s_5\}, Y^*_{t_3,2} \overset{d}{=} Y_3 \mid S = s_3 \right\}.
\]

The parameter of interest \( \eta \) could be defined by, say, the following moment conditions: \( \mathbb{E}\left[ Y^*_{t_1,1} - \eta \right] = 0, \mathbb{E}\left[ Y^*_{t_2,1} - \eta \right] = 0, \mathbb{E}\left[ \mathbf{1}\{Y^*_{t_1,1} \leq \eta\} - 0.5 \right] = 0, \) and \( \mathbb{E}\left[ \mathbf{1}\{Y^*_{t_1,1} \leq \eta\} - 0.5 \right] = 0. \) This means that \( Y^*_{t_1,1} \) and \( Y^*_{t_2,1} \) have the same expectations and medians, which all equal to \( \eta \). Note that both within and cross type restrictions are contained in this example.

Before ending this section, it is worth mentioning that the set \( \mathcal{Y} \) can be extended to include more random variables whose marginal distributions are identified. \( Y_t \mid T = t, S \in \Sigma_{t,k} \) and \( Y_t \mid S \in \Sigma_t \) are such examples. Similar arguments go through for efficient estimation, for which the details are not being repeated here.
6 Optimal Testable Implication of Model Assumption

In practice, one should check the validity of model assumptions before proceeding to estimation. As discussed previously, overidentifying restrictions may exist depending on the type configuration \( S \). This section discusses a systematic approach to generate testable implications for Assumption 1, even when no overidentifying type restriction exists. Based on works of Kitagawa (2015) and Sun (2018), an obvious generalization would be the set of conditions: for any \( t \in T, k = 1, \cdots, N_Z \), and measurable \( B \subset \mathcal{Y} \),

\[
\tilde{b}_{t,k} 1_{B_{t,Z}}(X) = P(Y_t \in B, S \in \Sigma_{t,k} \mid X) \in [0, 1], \text{ a.s. (37)}
\]

where \( 1B \) is the indicator function of \( B \). This means that the identifiable parts of the joint distribution of the potential outcomes and type should be a proper probability. However, more can be tested. For instance, both \( P(S = s_4) \) and \( P(S \in \{s_4, s_5\}) \) can be identified in the main example, then the former should be no greater than the latter. In fact, this intuition can be developed into a set of high-level conditions optimal for testing Assumption 1, including but not limited to both the implications (37) and the type configuration overidentifying conditions discussed in section 3.

Let \( \Sigma_t = \{\Sigma_{t,k} : k = 1, \cdots, N_Z\} \). For any \( t \in T \), define a function \( Q_t : \mathcal{X} \times (\mathcal{B}_Y \times \Sigma_t) \rightarrow \mathbb{R} \) by

\[
Q_t(X, (B, \Sigma_{t,k})) = \tilde{b}_{t,k} 1_{B_{t,Z}}(X)
\]  

**(Assumption 2).** There exist \( N_T \) functions \( \tilde{Q}_t : \mathcal{X} \times (\mathcal{B}_Y \times 2^S) \rightarrow \mathbb{R} \), such that, for all \( t, t' \in T \),

(i) \( \tilde{Q}_t \) is a probability kernel;

(ii) \( \tilde{Q}_t(\cdot, (\mathcal{Y}, \Sigma)) = \tilde{Q}_{t'}(\cdot, (\mathcal{Y}, \Sigma)) \) for all \( \Sigma \subset \mathcal{S} \);

(iii) \( \tilde{Q}_t(\cdot, (B, \Sigma)) = Q_t(\cdot, (B, \Sigma)) \), for all measurable \( B \subset \mathcal{Y} \), and \( \Sigma \in \Sigma_t \).

The probability kernel \( \tilde{Q}_t \) represents the unidentified joint distribution of \( (Y_t)_{t \in T} \) and \( S \) given \( X \), i.e. \( \tilde{Q}_t(X, (B, \Sigma)) = P(Y_t \in B, S \in \Sigma \mid X) \). The second condition in Assumption 2 ensures that \( P(S \in \Sigma \mid X) \) is well-defined, while the third condition assigns \( \tilde{Q}_t \) to its identified value whenever possible. The overidentification constraints on type configurations \( S \), e.g. equation (8), results from conditions (ii) and (iii). The constraints defined by equation (37) follows from conditions (i)
The optimality of Assumption 2 for testing is explained by the following theorem. Let $L_0$ be the underlying unidentified joint probability of $\left(\{Y_t : t \in T\}, \{T_z : z \in Z\}, Z, X\right)$ and $L$ be the observed joint probability of $\left(Y, T, Z, X\right)$. Each $L_0$ induces a $L$, but not the other way round. Denote such mapping from $L_0$ to $L$ by $O$.

**Theorem 8.** The following relationships hold between Assumption 1 on $L_0$ and Assumption 2 on $L$.

(i) If $L_0$ satisfies Assumption 1, then $O(L_0)$ satisfies Assumption 2. If $L$ satisfies Assumption 2, then there exists a $L_0$ that violates Assumption 1 and $O(L_0) = L$.

(ii) If $L$ satisfies Assumption 2, then there exists a $L_0$ that satisfies Assumption 1 and $O(L_0) = L$.

Therefore, if $C$ is another condition on $L$, such that $L_0$ satisfies Assumption 1 implies $O(L_0)$ satisfies Assumption $C$, then $L$ satisfies Assumption 2 implies $L$ satisfies condition $C$.

Part (i) of the theorem establishes that Assumption 2 is a necessary but not sufficient condition for Assumption 1. Part (ii) establishes that Assumption 2 is the optimal testable implication of Assumption 1, in the sense that any testable implication of Assumption 1 can be implied by the satisfaction of Assumption 2. Also note that Assumption 1 not only requires unordered monotonicity but the full specification of $S$. From this theorem and its proof, the idea becomes clear of the optimality for testable implications on nonverifiable hypothesis, which is another contribution of the paper. The set of conditions presented in Kitagawa (2015) is indeed a special case, the simplicity of which is specific to the binary case.

This section again ends with an illustration using the main example. Define

$$Q_{t_1}(X, (B, \Sigma)) = \begin{cases} P(Y \in B, T = t_1 | Z = z_2, X) - P(Y \in B, T = t_1 | Z = z_1, X) & , \text{if } \Sigma = \{s_4\} \\ P(Y \in B, T = t_1 | Z = z_1, X) & , \text{if } \Sigma = \{s_1\} \end{cases}$$

$$Q_{t_2}(X, (B, \Sigma)) = \begin{cases} P(Y \in B, T = t_2 | Z = z_2, X) - P(Y \in B, T = t_2 | Z = z_1, X) & , \text{if } \Sigma = \{s_5\} \\ P(Y \in B, T = t_2 | Z = z_1, X) & , \text{if } \Sigma = \{s_2\} \end{cases}$$
\[
Q_{t_3}(X,(B,\Sigma)) = \begin{cases} 
P(Y \in B, T = t_3 \mid Z = z_1, X) - P(Y \in B, T = t_3 \mid Z = z_2, X) & \text{, if } \Sigma = \{s_4, s_5\} \\
P(Y \in B, T = t_3 \mid Z = z_2, X) & \text{, if } \Sigma = \{s_3\}
\end{cases}
\]

By equation (8),
\[
Q_{t_1}(\cdot, \mathcal{Y} \times \{s_4\}) + Q_{t_2}(\cdot, \mathcal{Y} \times \{s_5\}) = Q_{t_3}(\cdot, \mathcal{Y} \times \{s_4, s_5\})
\]

Also note that \(Q_{t_1}\) on \(\{s_1\}\), \(Q_{t_2}\) on \(\{s_2\}\), and \(Q_{t_3}\) on \(\{s_3\}\) are already between 0 and 1. And \(Q_{t_1}\) on \(\{s_4\}\), \(Q_{t_2}\) on \(\{s_5\}\), and \(Q_{t_3}\) on \(\{s_4, s_5\}\) are already below 1. The true restrictions in this example are hence reduced to

\[
\begin{align*}
P(Y \in B, T = t_1 \mid Z = z_2, X) & \geq P(Y \in B, T = t_1 \mid Z = z_1, X) \\
P(Y \in B, T = t_2 \mid Z = z_2, X) & \geq P(Y \in B, T = t_2 \mid Z = z_1, X) \\
P(Y \in B, T = t_3 \mid Z = z_1, X) & \geq P(Y \in B, T = t_3 \mid Z = z_2, X)
\end{align*}
\]

These inequalities are very similar to the form of the conditions in Kitagawa (2015). This simplicity is due to the fact that \(2^S\) equals to the algebra generated by \(\Sigma\) and no overidentifying restriction exists. It is possible to generalize the variance-weighted Komolgorov-Smirnov test proposed in Kitagawa (2015) and the power-improved test proposed in Sun (2018), the implementation of which is beyond the scope of this study.

\section{Numerical Illustrations}

\subsection{Empirical Application}

In this section, the estimation methods discussed in the paper are applied to study the return to schooling using proximity to college as an instrument (Card, 1993). The data comes from the National Longitudinal Survey on the original cohort of young men at the time of 1966 and 1976. It is shown in Kitagawa (2015) that the instrument introduced is only valid after conditioning on covariates such as race and region of residence, making this example appropriate for illustrating the usefulness of including conditioning covariates in the framework, which is a step forward of the current paper from Heckman and Pinto (2018a) and Pinto (2019).

The model is the same as the main example and the variables are explained as follows. The
outcome $Y$ is the log of weekly earning in 1976. The instrument is the binary $Z$ indicating whether a four-year local college is present for the individual in 1966. It is relevant because the presence of a college nearby gets the individual to know about college in early life and reduces the cost of receiving some college education, and valid since the individual’s ability is presumed to be independent of the place of residence, given race and the extent to which this region is developed. The treatment $T$ describes the education received by the individual up to the time of 1976. Instead of being binary, $T$ takes on three unordered values $\{t_1, t_2, t_3\}$, where $t_1$ means the student receives college education in fields including engineering, mathematics, law, social sciences, and others, $t_2$ means the student receives college education in other fields including business, education, and public services among others, and $t_3$ means the student doesn’t receive college education.\footnote{This classification of the fields of study is for balancing the sample size in the dataset.} In the binary local IV case, $t_1$ and $t_2$ would be collapsed into one treatment indicating college-level education, for the study of return to college schooling. The unobserved type $S$ is defined by the way education decisions vary with the proximity to the college. The conditioning covariates $X$ includes race, whether residence in southern states and whether residence in the standard metropolitan area. The available sample size is 2930, 381 of which are treated by $t_1$, and 487 by $t_2$.

For estimation, no advanced nonparametric method is needed since $X$ is discrete. The $P_t, Z$’s are estimated using the linear probability model with five dummies as in Kitagawa (2015). The $\pi_z$’s are estimated with the sample means. Then in CEP estimators in Theorem 4 are used to evaluate the parameters of interest. The estimation results for the LASFs and LASF-Ts are displayed in Table 1. The asymptotic covariance matrix can be easily estimated using the efficient influence functions, as in Table 2, leading to joint inferences of the parameters, which is another benefit of the methodology developed in this paper. Table 3 shows the results of some selected statistical tests comparing both between and within LASFs and LASF-Ts. The differences between the pairs $\beta_{t_1,1}, \beta_{t_2,1}$ and $\gamma_{t_1,1}, \gamma_{t_2,1}$ are both insignificant, indicating that the income after receiving college education on the two categories of fields are similar for their own switchers respectively. As mentioned before, these facts can be turned into overidentifying restrictions ($\beta_{t_1,1} = \beta_{t_2,1}, \gamma_{t_1,1} = \gamma_{t_2,1}$) to improve efficiency, if supported by underlying economic theory. College education is generally perceived as a causal factor for increasing income. Thus it should be the case that the LASF-T is higher than the LASF when the treatment belongs to $\{t_1, t_2\}$, and lower when treatment is $t_3$, which is consistent with the
testing results. Also, both $\beta_{t1,1}$ and $\beta_{t1,1}$ are higher than $\beta_{t3,1}$, where the insignificance is due to the fact that these three parameters are averages over different subpopulations. Comparisons among $\beta_{t1,2}$, $\beta_{t2,2}$, and $\beta_{t3,2}$ shows similar intuition on the effect of schooling with a tendency that the outcomes for always-takers are less spread across treatments. The ratios $P(T = t_1 \mid S = s_4)$ and $P(T = t_2 \mid S = s_5)$ are estimated to be 0.68 and 0.92, revealing a significant amount of individuals receiving college education in the subpopulations of switchers. At this point, one might question whether the parameter estimates are of policy interest. The argument here is that clearly by the identification results, the parameters discussed here are at least as informative as the LATE (and LATT) parameters in the binary local IV case. And one of the themes of local IV and other causal inference models is to tradeoff informativeness with removing incredible assumptions.
Table 3: Differences between Parameter Values

|               | Two-sided  | One-sided  |
|---------------|------------|------------|
| $\beta_{t1,1} - \beta_{t2,1}$ | -0.467 (0.976) | $\beta_{t1,1} - \beta_{t3,1}$ | 0.353 (0.424) |
| $\gamma_{t1,1} - \gamma_{t2,1}$ | 0.617 (1.282) | $\beta_{t2,1} - \beta_{t3,1}$ | 0.821 (0.980) |
| $\gamma_{t1,1} - \beta_{t1,1}$ | 0.981 (0.573)* | $\gamma_{t2,1} - \beta_{t2,1}$ | -0.104 (0.291) |
| $\beta_{t3,1} - \gamma_{t3,1}$ | 2.056 (1.053)* |

Standard deviation of the estimate is presented in the parentheses, and “*” indicates significance at the 5% level.

7.2 Simulation Study

Monte Carlo simulations are conducted, for further understanding of the relationships between random variables in the model and finite-sample performances of the estimators. Two data generating processes (DGP), different only in the distribution of the covariate $X$, are specified as follows. In the first DGP, $X$ is drawn from the uniform distribution over $(0.5, 0.7)$; whereas in the second DGP, $X$ is a discrete random variable taking values in $(0.5, 0.55, 0.6, 0.65, 0.7)$ with equal probabilities. Then the two DGPs follow the same way to generate $S, Z, T$, and $Y$ from $X$. Type $S$ is drawn from $Binomial(4, X)$, the values $(0, 1, 2, 3, 4)$ matches with $(s_1, s_2, s_4, s_5, s_3)$ respectively. The instrument $Z$ is generated according to the distribution $Bernoulli(X)$. Then the treatment $T$ is determined by the realization of both $S$ and $Z$. The potential outcomes $(Y_1, Y_2, Y_3)$ are constructed as

$$
\begin{pmatrix}
  Y_1 \\
  Y_2 \\
  Y_3
\end{pmatrix}
= 1_{s_1} \begin{pmatrix}
  \xi_3 + \xi \\
  \xi_2 + \xi \\
  \xi_1 + \xi
\end{pmatrix}
+ 1_{s_2} \begin{pmatrix}
  \xi_3 + \xi \\
  \xi_2 + \xi \\
  \xi_1 + \xi
\end{pmatrix}
+ 1_{s_3} \begin{pmatrix}
  \xi_3 + \xi \\
  \xi_2 + \xi \\
  \xi_1 + \xi
\end{pmatrix}
+ 1_{s_4} \begin{pmatrix}
  \xi_3 + \xi \\
  \xi_2 + \xi \\
  \xi_1 + \xi
\end{pmatrix}
+ 1_{s_5} \begin{pmatrix}
  \xi_3 + \xi \\
  \xi_2 + \xi \\
  \xi_1 + \xi
\end{pmatrix}
$$

where $1_s$ is the indicator of type $s$, and the $\xi$’s are mutually independent random variables with distributions $\xi \sim N(0.1, 1)$, $\xi_1 \sim N(X, 1)$, $\xi_2 \sim N(X + 0.2, 1)$, and $\xi_3 \sim N(X + 0.4, 1)$. Normality is assumed for the bell-shaped empirical distribution of log-income. By construction, $S$ and $Z$ are independent conditional on $X$, and the $Y_i$’s depends on $S$ and $X$ not on $Z$. In the subpopulations with $S \in \{s_4, s_5\}$, the $Y_i$’s are mutually independent, while in other cases they are correlated through $\xi$. This feature resembles that of the data generating process in Hong and Nekipelov (2010a). The number of observations is $N = 3000$ with 10000 Monte Carlo replications. The simulation results
are shown in Table 4 for LASFs in subpopulations of switchers and their asymptotic standard deviations, indicating a good performance in the case of finite sample. It is also apparent that the standard deviations of \( \hat{\beta}_{t,k} \)'s are very close to the \( \hat{\sigma}_{\beta_{t,k}} \sqrt{N} \)'s, confirming the efficiency in estimation and optimality of tests with moderate sample size.

Table 4: Monte Carlo Results

| Parameter | \( P_X \) | Value | Mean Bias | Median Bias | Std Deviation | Root MSE |
|-----------|-----------|-------|-----------|-------------|---------------|----------|
| \( \beta_{t1,1} \) | Continuous | 1.00 | 0.0007 | 0.0010 | 0.0488 | 0.0488 |
| | Discrete | 1.00 | -0.0080 | -0.0078 | 0.0502 | 0.0508 |
| \( \beta_{t2,1} \) | Continuous | 1.00 | 0.0160 | 0.0149 | 0.0756 | 0.0773 |
| | Discrete | 1.00 | 0.0124 | 0.0126 | 0.0744 | 0.0754 |
| \( \beta_{t3,1} \) | Continuous | 0.60 | -0.0109 | -0.0105 | 0.0458 | 0.0471 |
| | Discrete | 0.60 | 0.0024 | 0.0025 | 0.0475 | 0.0476 |
| \( \sigma_{\beta_{t1,1}} \) | Discrete | 2.75 | -0.0183 | -0.0265 | 0.1316 | 0.1329 |
| \( \sigma_{\beta_{t2,1}} \) | Discrete | 4.12 | -0.0441 | -0.0555 | 0.2409 | 0.2449 |
| \( \sigma_{\beta_{t3,1}} \) | Discrete | 2.59 | -0.0095 | -0.0108 | 0.0843 | 0.0849 |

The \( \sigma_{\beta_{t,k}} \)'s are estimated using the plug-in estimators where the \( \beta_{t,k} \)'s are estimated by CEP.

8 Concluding Remarks

This paper has studied the semiparametric efficient estimation in the generalized local IV framework where treatment is allowed to take multiple values. A large class of parameters implicitly defined by a possibly over-identified non-smooth collection of moment conditions is considered, with a special focus on parameters derived through type probabilities and local average structural functions. The calculated efficient influence functions lead to the easy implementation of optimal joint inferences and the construction of estimators suitable under high-dimensional settings. The model assumptions of local IV, in general, is further understood through the optimal observable implications. The applicability of the methodology is demonstrated with examples for empirical research with a finite amount of sample data. For future studies, one could consider using the efficient estimation methods for, say, LASFs to extract information on the (non-local) average structural functions, as in Mogstad et al. (2018).
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A Binary Local IV with Conditioning Covariates

It may help enhance understanding of the notations and results in the main text by presenting them in the context of binary local IV. The identification of classical LATE, without conditioning covariates, is discussed in Section 4.1 of Heckman and Pinto (2018a) using binary matrices. The efficiency bound and efficient estimation are discussed in Frölich (2007); Hong and Nekipelov (2010a). Specification test is discussed in Kitagawa (2015). Here both the treatment and instrument are binary: $Z = \{z_0, z_1\}$, $T = \{t_0, t_1\}$. The type constraint is

| $T_{z_0}$ | $s_1$ | $s_2$ | $s_3$ |
|-----------|-------|-------|-------|
| $T_{z_1}$ | $t_1$ | $t_0$ | $t_0$ |

where $s_1$ is always-taker, $s_2$ is complier, and $s_3$ is never-taker. The unordered monotonicity condition is satisfied: $1\{T_{z_1} = t_1\} \geq 1\{T_{z_0} = t_1\}$ and $1\{T_{z_1} = t_0\} \leq 1\{T_{z_0} = t_0\}$. The type partitions are, for $t_0$, $\Sigma_{t_0,1} = \{s_2\}$, $\Sigma_{t_0,2} = \{s_3\}$; and for $t_1$, $\Sigma_{t_1,1} = \{s_2\}$, $\Sigma_{t_1,2} = \{s_1\}$. The $B_t$’s and their inverses are

$$B_{t_0} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \implies B_{t_0}^+ = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}; B_{t_1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \implies B_{t_1}^+ = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$$

and $b_{t,k}$’s are $b_{t_0,1} = b_{t_1,1} = (0,1,0)$, $b_{t_0,2} = (0,0,1)$, $b_{t_1,2} = (1,0,0)$. Thus $\tilde{b}_{t_0,1} = (1,-1)$, $\tilde{b}_{t_0,2} = (0,1)$, and $\tilde{b}_{t_1,2} = (1,0)$. The $\mathcal{I}_{t_i,Z}(X)$’s and $P_{t_i,Z}(X)$’s are

$$\mathcal{I}_{t_0,Z}(X) = (E[Y1{T = t_0} | Z = z_0, X], E[Y1{T = t_0} | Z = z_1, X])'$$
$$\mathcal{I}_{t_0,Z}(X) = (E[Y1{T = t_0} | Z = z_0, X], E[Y1{T = t_1} | Z = z_1, X])'$$
$$P_{t_1,Z}(X) = (P(T = t_1) | Z = z_0, X), P(T = t_1) | Z = z_1, X))'$$
$$P_{t_0,Z}(X) = (P(T = t_0) | Z = z_0, X), P(T = t_0) | Z = z_1, X))'$$
By Theorem 1,

\[ p_{t_1,2} = P(S = s_1) = \tilde{b}_{t_1,2} \mathbb{E}[P_{t_1,Z}(X)] = \mathbb{E}[P(T = t_1 \mid Z = z_0, X)] \]
\[ p_{t_1,1} = P(S = s_2) = p_{t_0,1} = \tilde{b}_{t_1,1} \mathbb{E}[P_{t_1,Z}(X)] = \mathbb{E}[-P(T = t_1 \mid Z = z_0, X) + P(T = t_1 \mid Z = z_1, X)] \]
\[ p_{t_0,2} = P(S = s_3) = \tilde{b}_{t_0,2} \mathbb{E}[P_{t_0,Z}(X)] = \mathbb{E}[P(T = t_0 \mid Z = z_1, X)] \]

and

\[
\begin{align*}
\beta_{t_1,1} &= \mathbb{E}[Y_t \mid S = s_2] = \frac{1}{p_{t_1,1}} \tilde{b}_{t_1,1} \mathbb{E}[I_{t_1,Z}(X)] = \frac{\mathbb{E} [\mathbb{E} [Y_1 \{T = t_1\} \mid Z = z_1, X] - \mathbb{E} [Y_1 \{T = t_1\} \mid Z = z_0, X]]}{\mathbb{E} [P(T = t_1 \mid Z = z_1, X) - P(T = t_1 \mid Z = z_0, X)]}
\beta_{t_1,2} &= \mathbb{E}[Y_t \mid S = s_1] = \frac{1}{p_{t_1,2}} \tilde{b}_{t_1,2} \mathbb{E}[I_{t_1,Z}(X)] = \frac{\mathbb{E} [\mathbb{E} [Y_1 \{T = t_1\} \mid Z = z_0, X]]}{\mathbb{E} [P(T = t_1 \mid Z = z_0, X)]}
\beta_{t_0,1} &= \mathbb{E}[Y_0 \mid S = s_2] = \frac{1}{p_{t_0,1}} \tilde{b}_{t_0,1} \mathbb{E}[I_{t_0,Z}(X)] = \frac{\mathbb{E} [\mathbb{E} [Y_1 \{T = t_0\} \mid Z = z_0, X] - \mathbb{E} [Y_1 \{T = t_0\} \mid Z = z_1, X]]}{\mathbb{E} [P(T = t_0 \mid Z = z_0, X) - P(T = t_0 \mid Z = z_1, X)]}
\beta_{t_0,2} &= \mathbb{E}[Y_0 \mid S = s_3] = \frac{1}{p_{t_0,2}} \tilde{b}_{t_0,1} \mathbb{E}[I_{t_0,Z}(X)] = \frac{\mathbb{E} [\mathbb{E} [Y_1 \{T = t_0\} \mid Z = z_1, X]]}{\mathbb{E} [P(T = t_0 \mid Z = z_1, X)]}
\end{align*}
\]

Thus we have the usual expression for LATE

\[
\mathbb{E}[Y_t - Y_0 \mid S = s_2] = \mathbb{E}[Y_t \mid S = s_2] - \mathbb{E}[Y_0 \mid S = s_2] = \beta_{t_1,1} - \beta_{t_0,1}
= \frac{\mathbb{E} [\mathbb{E} [Y \mid Z = z_1, X] - \mathbb{E} [Y \mid Z = z_0, X]]}{\mathbb{E} [P(T = t_1 \mid Z = z_1, X) - P(T = t_1 \mid Z = z_0, X)]}
\]

Focusing on \( \Sigma_{t_0,1} = \Sigma_{t_1,1} = \{s_2\} \), we can derive the LASF-Ts \( \mathbb{E}[Y_1 \mid T = t_1, S = s_2], \mathbb{E}[Y_0 \mid T = t_1, S = s_2] \).

In fact, \( W_{t_1,1} = \{z_1\} \). Thus, by Theorem 2,

\[
q_{t_1,t_1,1} = P(T = t_1, S = s_2) = \mathbb{E} [(-P(T = t_1 \mid Z = z_0, X) + P(T = t_1 \mid Z = z_1, X)) P(Z = z_1 \mid X)]
\]
\[
\gamma_{t_1,t_1,1} = \mathbb{E}[Y_1 \mid T = t_1, S = s_2] = \frac{\mathbb{E} [\tilde{b}_{t_1,1} I_{t_1,Z}(X) \pi_{W_{t_1,t_1,1}}(X)]}{\mathbb{E} [\tilde{b}_{t_1,1} P_{t_1,Z}(X) \pi_{W_{t_1,t_1,1}}(X)]}
= \frac{\mathbb{E} [(-P(Y_1 \mid T = t_1) \mid Z = z_0, X) + \mathbb{E} [Y_1 \{T = t_1\} \mid Z = z_1, X]) P(Z = z_1 \mid X)]}{\mathbb{E} [(-P(T = t_1 \mid Z = z_0, X) + P(T = t_1 \mid Z = z_1, X)) P(Z = z_1 \mid X)]}
\]
\[
\gamma_{t_0,t_1,1} = \mathbb{E}[Y_0 \mid T = t_1, S = s_2] = \frac{\mathbb{E} [\tilde{b}_{t_0,1} I_{t_0,Z}(X) \pi_{W_{t_1,t_0,1}}(X)]}{\mathbb{E} [\tilde{b}_{t_0,1} P_{t_0,Z}(X) \pi_{W_{t_1,t_0,1}}(X)]}
= \frac{\mathbb{E} [(-P(Y_1 \mid T = t_0) \mid Z = z_0, X) - \mathbb{E} [Y_1 \{T = t_0\} \mid Z = z_1, X]) P(Z = z_1 \mid X)]}{\mathbb{E} [(P(T = t_0 \mid Z = z_0, X) - P(T = t_0 \mid Z = z_1, X)) P(Z = z_1 \mid X)]}
\]

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Therefore

\[ \mathbb{E} [ Y_{t_1} - Y_{t_0} \mid T = t_1, S = s_2] = \mathbb{E} [ Y_{t_1} \mid T = t_1, S = s_2] - \mathbb{E} [ Y_{t_0} \mid T = t_1, S = s_2] = \gamma_{t_1,t_1} - \gamma_{t_0,t_1}, \]

\[
= \frac{\mathbb{E} \left[ \left( \mathbb{E} [ Y \mid Z = z_1, X] - \mathbb{E} [ Y \mid Z = z_0, X] \right) \mathbb{E} [ Z = z_1 \mid X] \right]}{\mathbb{E} \left[ \left( P(T = t_1 \mid Z = z_1, X) - P(T = t_1 \mid Z = z_0, X) \right) P(Z = z_1 \mid X) \right]}.
\]

The notation \( \zeta(Z, X) \) in Theorem 3 means \( \left( \frac{1_{Z=Z_0}}{P(Z=Z_0 \mid X)}, \frac{1_{Z=Z_1}}{P(Z=Z_1 \mid X)} \right) \). The semiparametric efficiency calculations give the efficient influence function for, say, \( \beta_{t_1,1} = \mathbb{E} [ Y_{t_1} \mid S = s_2], \) which is

\[
\Psi_{\beta_{t_1,1}} = \frac{1}{p_{t_1,1}} \left( \frac{1_{Z=Z_0}}{P(Z=Z_0 \mid X)} (Y_{1T(t_1)} - \mathbb{E} [ Y_{1T(t_1)} \mid Z = Z_0, X]) + \mathbb{E} [ Y_{1T(t_1)} \mid Z = Z_0, X] \right) \]

\[ + \frac{1}{p_{t_1,1}} \left( \frac{1_{Z=Z_1}}{P(Z=Z_1 \mid X)} (Y_{1T(t_1)} - \mathbb{E} [ Y_{1T(t_1)} \mid Z = Z_1, X]) + \mathbb{E} [ Y_{1T(t_1)} \mid Z = Z_1, X] \right) \]

\[ - \frac{\beta_{t_1,1}}{p_{t_1,1}} \frac{1_{Z=Z_0}}{P(Z=Z_0 \mid X)} (1_{T(t_1)} - \mathbb{E} [ 1_{T(t_1)} \mid Z = Z_0, X]) + \mathbb{E} [ 1_{T(t_1)} \mid Z = Z_0, X] \]

\[ + \frac{\beta_{t_1,1}}{p_{t_1,1}} \frac{1_{Z=Z_1}}{P(Z=Z_1 \mid X)} (1_{T(t_1)} - \mathbb{E} [ 1_{T(t_1)} \mid Z = Z_1, X]) + \mathbb{E} [ 1_{T(t_1)} \mid Z = Z_1, X] \]

The estimators are skipped for brevity. The optimal set of testable implications reduces to, for any measurable \( B \), almost surely

\[
\mathbb{E} [ B(Y)1_{T(t_1)} \mid Z = Z_1, X] - \mathbb{E} [ B(Y)1_{T(t_1)} \mid Z = Z_0, X] = \tilde{\beta}_{t_1,1} B_{t_1,Z}(X) \in [0, 1]
\]

\[
\mathbb{E} [ B(Y)1_{T(t_0)} \mid Z = Z_0, X] - \mathbb{E} [ B(Y)1_{T(t_0)} \mid Z = Z_1, X] = \tilde{\beta}_{t_0,1} B_{t_0,Z}(X) \in [0, 1]
\]

which is equation (3.3) of Kitagawa (2015).

B Proofs of Theorems and Regularity Conditions

Lemma 1. The following conditional independence relationships hold: \( S \perp Z \mid X \); and for any \( t \in T, Y_t \perp T \mid S, X. \)

Proof. The first statement follows from the definition of \( S \) and the fact that \( Z \) is independent of \( (T_{z_1}, \ldots, T_{z_{NZ}}) \) conditioning on \( X \). For the second statement, \( T \) is a function of \( (S, Z, X) \). Hence, given \( S \) and \( X \), \( T \) is independent of \( Y_t \), since \( Z \) is independent of \( (Y_{t_1}, \ldots, Y_{t_{NT}}) \) conditional on \( X \). \( \blacksquare \)
Lemma 2. For each \( t \in T, \ k = 1, \ldots, N_Z, \) and \( g \) measurable, the following identification results hold.

(i) \( P(S \in \Sigma_{t,k} \mid X) = \tilde{b}_{t,k}P_{t,Z}(X) \) a.s.

(ii) \( \mathbb{E}[g(Y_t) \mid S \in \Sigma_{t,k}, X] = \frac{\tilde{b}_{t,k}g_t(Z,X)}{\tilde{b}_{t,k}P_{t,Z}(X)} \) a.s.

Proof. Conditional on \( X, \) we have \( B_t[i,j] = 1\{T = t \mid Z = z_i, S = s_j\}, \) which is the definition of \( B_t \) in Heckman and Pinto (2018a). This means the quantity \( b_{t,k}B_t^+ \) defined in their paper is the constant (across different values of \( X \)) \( \tilde{b}_{t,k}. \) Hence the result is equivalent to Theorem 6 in Heckman and Pinto (2018a).

Proof of Theorem 1. (i) We can get the result by applying the law of iterated expectation on (i) of Lemma 2.

(ii) Using the Bayes rule, we have

\[
\mathbb{E}[g(Y_t) \mid S \in \Sigma_{t,k}] = \int \mathbb{E}[g(Y_t) \mid S \in \Sigma_{t,k}, X = x] f_X|S \in \Sigma_{t,k}(x)dx
\]

\[
= \int \mathbb{E}[g(Y_t) \mid S \in \Sigma_{t,k}, X = x] \frac{P(S \in \Sigma_{t,k} \mid X = x)}{P(S \in \Sigma_{t,k})} f_X(x)dx
\]

\[
= \frac{1}{pt,k} \mathbb{E}[\tilde{b}_{t,k}g_t(Z,X)].
\]

Proof of Theorem 2. (i) By the definition of \( W_{t',t,k}, \) we have

\[
P(T = t', S \in \Sigma_{t,k}) = P(Z \in W_{t',t,k}, S \in \Sigma_{t,k})
\]

\[
= \mathbb{E}\left[P(Z \in W_{t',t,k}, S \in \Sigma_{t,k} \mid X)\right]
\]

\[
= \mathbb{E}\left[P(Z \in W_{t',t,k} \mid X) P(S \in \Sigma_{t,k} \mid X)\right]
\]

\[
= \mathbb{E}\left[\tilde{b}_{t,k}P_{t,Z}(X)\pi_{W_{t',t,k}}(X)\right]
\]

where the third equality follows from \( Z \perp S \mid X.\)
Using the Bayes rule, we have

\[
\mathbb{E} \left[ g(Y_t) \mid T = t', S \in \Sigma_{t,k} \right] = \int \mathbb{E} \left[ g(Y_t) \mid T = t', S \in \Sigma_{t,k}, X = x \right] f_X(x) \, dx
\]

where the second equality follows from the Bayes rule and the third equality follows from

\[
Y_t \perp T \mid S, X.
\]

By Lemma L-16 of Heckman and Pinto (2018b), we know that under the unordered monotonicity assumption of \( S \), \( B_t[\cdot, i] = B_t[\cdot, i'] \) for all \( s, s' \in \Sigma_{t,k} \). Thus \( \mathbb{E} \left[ g(Y_t) \mid T = t, S \in \Sigma_{t,k} \right] \) is always identified.

The calculations for the semiparametric efficiency bound follow from Newey (1990). The likelihood of the statistical model can be specified as

\[
\mathcal{L}(Y, T, Z, X) = \left( \prod_{z \in Z} \left( f_z(Y, T \mid X) \pi_z(X) \right) 1\{Z = z\} \right) f_X(X)
\]

where \( f_z(\cdot, \cdot \mid X) \) denotes the conditional density of \( Y, T \) given \( Z = z \) and \( X \), and \( f_X(\cdot) \) denotes the marginal density of \( X \). In a regular parametric submodel, where the true underlying probability measure \( P \) is indexed by \( \theta^o \), using the following notations

\[
s_z(Y, Z \mid X) = \frac{\partial}{\partial \theta} \log (f_z(Y, T \mid X; \theta)) \bigg|_{\theta = \theta^o}, \quad z \in Z
\]

\[
s_\pi(Z \mid X; \theta^o) = \sum_{z \in Z} 1\{Z = z\} \frac{\partial}{\partial \theta} \log (\pi_z(X; \theta)) \bigg|_{\theta = \theta^o}
\]

\[
s_X(X) = \frac{\partial}{\partial \theta} \log (s_X(X; \theta)) \bigg|_{\theta = \theta^o}
\]

Then the following Lemma is immediate. It computes the score and tangent space, and is invoked many times below for the calculation of the semiparametric efficiency bound.
Lemma 3. The score in a regular parametric submodel is

\[ s_{\theta^o}(Y, T, Z, X) = \sum_{z \in Z} 1\{Z = z\} s_z(Y, T \mid X; \theta^o) + s_{\pi}(Z \mid X; \theta^o) + s_X(X; \theta^o) \]

Hence the tangent space of the original model is

\[ \mathcal{S}(P) = \left\{ s \in L_0^2(P) : s(Y, T, Z, X) = \sum_{z \in Z} 1\{Z = z\} s_z(Y, T \mid X) + s_{\pi}(Z \mid X) + s_X(X) \right\} \]

for some \( s_z, s_\pi, s_X \) such that \( \int s_z(y, t \mid X) f_z(y, t \mid X) dy dt = 0, \forall z; \)

\[ \sum_{z \in Z} s_{\pi}(z \mid X) \pi_z(X) = 0, \text{ and } \int s_X(x) f_X(x) dx = 0 \]

Proof of Theorem 3. We only prove (i) and (ii), (iii) and (iv) are easier cases that can be proved along the way.

(i) For the pathwise differentiability of \( \beta_{t,k} \), in any parametric submodel,

\[
\frac{\partial}{\partial \theta} \beta_{t,k}(\theta) \bigg|_{\theta = \theta^o} = \frac{\partial}{\partial \theta} \left( \frac{\hat{b}_{t,k} \mathbb{E}_\theta [I_{t,Z}(X)]}{p_{t,k}} \right) \bigg|_{\theta = \theta^o} = \frac{1}{p_{t,k}} \left( \frac{\partial}{\partial \theta} \mathbb{E}_\theta [I_{t,Z}(X)] \bigg|_{\theta = \theta^o} - \frac{\hat{b}_{t,k} \mathbb{E}_\theta [I_{t,Z}(X)]}{p_{t,k}} \frac{\partial}{\partial \theta} \frac{p_{t,k}}{p_{t,k}} \bigg|_{\theta = \theta^o} \right) = \frac{1}{p_{t,k}} \hat{b}_{t,k} \left( \frac{\partial}{\partial \theta} \mathbb{E}_\theta [I_{t,Z}(X)] \bigg|_{\theta = \theta^o} - \frac{\partial}{\partial \theta} \mathbb{E}_\theta [P_{t,Z}(X)] \bigg|_{\theta = \theta^o} \beta_{t,k} \right)
\]

where \( \frac{\partial}{\partial \theta} \mathbb{E}_\theta [I_{t,Z}(X)] \bigg|_{\theta = \theta^o} \) and \( \frac{\partial}{\partial \theta} \mathbb{E}_\theta [P_{t,Z}(X)] \bigg|_{\theta = \theta^o} \) are \( N_Z \times 1 \) random vectors whose typical elements are

\[
\int y 1\{\tau = t\} s_z(y, \tau \mid x; \theta^o) f_z(y, \tau \mid x; \theta^o) f_X(x; \theta^o) dy d\tau dx \\
+ \int y 1\{\tau = t\} s_X(x; \theta^o) f_z(y, \tau \mid x; \theta^o) f_X(x; \theta^o) dy d\tau dx
\]

and

\[
\int 1\{\tau = t\} s_z(y, \tau \mid x; \theta^o) f_z(y, \tau \mid x; \theta^o) f_X(x; \theta^o) dy d\tau dx \\
+ \int 1\{\tau = t\} s_X(x; \theta^o) f_z(y, \tau \mid x; \theta^o) f_X(x; \theta^o) dy d\tau dx
\]
respectively, for \( z \in \mathcal{Z} \). The efficient influence function has to satisfy

\[
\frac{\partial}{\partial \theta} \beta_{t,k}(\theta) \bigg|_{\theta = \theta_0} = \mathbb{E} \left[ \Psi_{\beta_{t,k}} s_{\theta_0} \right], \quad \text{and } \Psi_{\beta_{t,k}} \in \mathcal{S}(P)
\]

The expression of \( \Psi_{\beta_{t,k}} \) presented in the theorem meets the above requirements. In particular, correspondence between terms in the efficient influence function and pathwise derivative appears exactly as in Lemma 1 of Hong and Nekipelov (2010b).

(ii) The pathwise derivative of \( \gamma_{t',t,k} \) can be computed in a similar way.

\[
\frac{\partial}{\partial \theta} \gamma_{t',t,k}(\theta) \bigg|_{\theta = \theta_0} = \frac{1}{q_{t',t,k}} \beta_{t,k} \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} \left[ I_{t',Z}(X) \pi_{W_{t',t,k}}(X) \right] \bigg|_{\theta = \theta_0} - \frac{\gamma_{t',t,k}}{q_{t',t,k}} \beta_{t,k} \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} \left[ P_{t,Z}(X) \pi_{W_{t',t,k}}(X) \right] \bigg|_{\theta = \theta_0}
\]

where \( \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} \left[ I_{t',Z}(X) \pi_{W_{t',t,k}}(X) \right] \bigg|_{\theta = \theta_0} \) and \( \frac{\partial}{\partial \theta} \mathbb{E}_{\theta} \left[ P_{t,Z}(X) \pi_{W_{t',t,k}}(X) \right] \bigg|_{\theta = \theta_0} \) are \( N_{\mathcal{Z}} \times 1 \) random vectors whose typical elements are

\[
\int y 1 \{ \tau = t \} s_{z}(y, \tau \mid x; \theta_0) \pi_{W_{t',t,k}}(x; \theta_0) \psi_{x}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

\[
+ \int y 1 \{ \tau = t \} s_{X}(x; \theta_0) \pi_{W_{t',t,k}}(x; \theta_0) \psi_{y}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

\[
+ \int y 1 \{ \tau = t \} \left( \frac{\partial}{\partial \theta} \pi_{W_{t',t,k}}(X; \theta) \bigg|_{\theta = \theta_0} \right) \psi_{y}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

and

\[
\int 1 \{ \tau = t \} s_{z}(y, \tau \mid x; \theta_0) \pi_{W_{t',t,k}}(x; \theta_0) \psi_{x}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

\[
+ \int 1 \{ \tau = t \} s_{X}(x; \theta_0) \pi_{W_{t',t,k}}(x; \theta_0) \psi_{y}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

\[
+ \int 1 \{ \tau = t \} \left( \frac{\partial}{\partial \theta} \pi_{W_{t',t,k}}(X; \theta) \bigg|_{\theta = \theta_0} \right) \psi_{y}(y, \tau \mid x; \theta_0) f_{X}(x; \theta_0) dyd\tau dx
\]

respectively, for \( z \in \mathcal{Z} \). The main difference appears when dealing with the last terms in the above two expressions, which can be matched with terms in the efficient influence function of
the following two forms

\[
\begin{align*}
\mathbb{E} \left[ Y \mathbf{1} \{ T = t \} \mid Z = z, X \right] &= \left( \mathbf{1} \{ Z \in W_{t',t,k} \} - \pi_{W_{t',t,k}}(X) \right) \\
\mathbb{E} \left[ \mathbf{1} \{ T = t \} \mid Z = z, X \right] &= \left( \mathbf{1} \{ Z \in W_{t',t,k} \} - \pi_{W_{t',t,k}}(X) \right)
\end{align*}
\]

To further explain, take the latter one as an example. Notice that

\[
\mathbf{1} \{ Z \in W_{t',t,k} \} - \pi_{W_{t',t,k}}(X) = \sum_{z \in W_{t',t,k}} \left( \mathbf{1} \{ Z = z \} - \pi_z(X) \right)
\]

and

\[
\left( \mathbf{1} \{ Z = z \} - \pi_z(X) \right) \frac{\pi_z(Z \mid X; \theta^o)}{\pi_z(X)} = \mathbf{1} \{ Z = z \} \frac{\partial}{\partial \theta} \pi_z(X; \theta) \bigg|_{\theta = \theta^o} - \pi_z(X) \frac{\pi_z(Z \mid X; \theta^o)}{\pi_z(X)}
\]

Using the law of iterated expectation,

\[
\begin{align*}
\mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1} \{ T = t \} \mid Z = z, X \right] \left( \mathbf{1} \{ Z = z \} - \pi_z(X) \right) s_r(Z \mid X; \theta^o) \right] \\
= & \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1} \{ T = t \} \mid Z = z, X \right] \frac{\mathbf{1} \{ Z = z \} \frac{\partial}{\partial \theta} \pi_z(X; \theta) \bigg|_{\theta = \theta^o}}{\pi_z(X)} \right] \\
= & \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1} \{ T = t \} \mid Z = z, X \right] \frac{\partial}{\partial \theta} \pi_z(X; \theta) \bigg|_{\theta = \theta^o} \right] \\
= & \int \mathbf{1} \{ \tau = t \} \left( \frac{\partial}{\partial \theta} \pi_z(X; \theta) \bigg|_{\theta = \theta^o} \right) f_z(y, \tau \mid x; \theta^o) f_X(x; \theta^o) dy \, d\tau \, dx
\end{align*}
\]

Proof of Theorem 4. This proof is based on Section 5 in Newey (1994). We first focus on the case of \( \beta_{t,k} \). Some calculations are done for preparation. For the sake of brevity, let \( h_t = (h_{Y,t,Z}, h_{t,Z}, \pi) \). Notice that the estimator \( \hat{\beta}_{t,k} \) is defined by the following moment condition.

\[
M_{\hat{\beta}_{t,k}}(X, \beta_{t,k}, h_t) = \tilde{b}_{t,k} \left( \frac{h_{Y,t,z_1}(X)}{\pi_{z_1}(X)}, \ldots, \frac{h_{Y,t,z_N}(X)}{\pi_{z_N}(X)} \right) - \beta_{t,k} \hat{b}_{t,k} \left( \frac{h_{t,z_1}(X)}{\pi_{z_1}(X)}, \ldots, \frac{h_{t,z_N}(X)}{\pi_{z_N}(X)} \right)
\]
Then we can compute the following partial derivatives

\[
E \left[ \frac{\partial M_{\beta t,k}}{\partial \beta t,k} \right] = -\hat{b}_{t,k} E \left[ P_{t,Z}(X) \right] = -p^o_{t,k}
\]

\[
\frac{\partial M_{\beta t,k}}{\partial h_{Y,t,z_i}} \bigg|_{h_t^o} = \frac{\tilde{b}_{t,k}[i]}{\pi^o_{z_i}(X)} = \delta_{Y,t,z_i}(X)
\]

\[
\frac{\partial}{\partial h_{t,z_i}} M_{\beta t,k} \bigg|_{h_t^o} = -\frac{\tilde{b}_{t,k} \tilde{b}_{t,k}[i]}{\pi^o_{z_i}(X)} = \delta_{t,z_i}(X)
\]

\[
\frac{\partial}{\partial \pi_{z_i}} M_{\beta t,k} \bigg|_{h_t^o} = -\frac{\tilde{b}_{t,k}[i]P^o_{t,z_i}(X)}{\pi^o_{z_i}(X)} + \frac{\beta_{t,k} \hat{b}_{t,k}[i]P^o_{t,z_i}(X)}{\pi^o_{z_i}(X)} = \delta_{\pi,z_i}(X)
\]

where \(\tilde{b}_{t,k}[i]\) denotes the \(i\)th element of the vector \(\tilde{b}_{t,k}\). Let

\[
D_{\beta t,k}(X, h_t) = \sum_{z \in Z} \delta_{Y,t,z}(X) h_{Y,t,z}(X) + \sum_{z \in Z} \delta_{t,z}(X) h_{t,z}(X) + \sum_{z \in Z} \delta_{\pi,z}(X) \pi(z)
\]

\[
= \sum_{j=1}^{N_Z} \tilde{b}_{t,k}[j] \left[ h_{Y,t,z_j}(X) - \beta_{t,k}^o h_{t,z_j}(X) - \left( T_{t,z_j}(X) - P^o_{t,z_j}(X) \right) \pi(z) \right]
\]

and

\[
\alpha_{\beta t,k}(Y, T, Z, X) = \sum_{z \in Z} \delta_{Y,t,z}(X) \left( \mathbb{1}\{Z = z\} Y \mathbb{1}\{T = t\} - h_{Y,t,z}(X) \right)
\]

\[
+ \sum_{z \in Z} \delta_{t,z}(X) \left( \mathbb{1}\{Z = z\} Y \mathbb{1}\{T = t\} - h_{t,z}(X) \right) + \sum_{z \in Z} \delta_{\pi,z}(X) \left( \mathbb{1}\{Z = z\} - \pi(z) \right)
\]

\[
= \hat{b}_{t,k} \zeta(Z, X, \pi^o) \left( \mathbb{1}\{Y\{T = t\} - T^o_{t,z}(X) - \beta_{t,k}^o \hat{b}_{t,k} \zeta(Z, X, \pi^o) \{\mathbb{1}\{T = t\} - P^o_{t,z}(X) \right)
\]

Then we check in turns Assumption 5.1 to 5.3 in Newey (1994). For Assumption 5.1(i), the linearization \(D\) can be taken as \(D_{\beta t,k}\) by equation (4.2) in that paper, since \(M_{\beta t,k}\) depends on \(h_t\) only through its value \(h_t(X)\). Assumption 5.1(ii) is satisfied by our condition 4(i) on the convergence rate of \(\hat{h}_t\). Assumption 5.2 is the stochastic equicontinuity condition on \(D_{\beta t,k}\), which can be verified
by our condition 4(ii), since

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ D_{\beta_{t,k}}(X_i, \hat{h}_t - h_t^0) - \mathbb{E} \left[ D_{\beta_{t,k}}(X, \hat{h}_t - h_t^0) \right] \right]
\]

\[
= \sum_{j=1}^{N} \hat{b}_{t,k}[j] \left[ \left( \nu_n(h_{Y,t,z_j}) - \nu_n(h_{Y,t,z_j}^0) \right) - \beta_{t,k}^0 \left( \nu_n(\hat{h}_{t,z_j}) - \nu_n(h_{t,z_j}^0) \right) \right]
\]

\[
- \left( \bar{I}_{t,z_j}(X) - P_{t,z_j}(X) \right) \left( \nu_n(\hat{\pi}_{z_j}) - \nu_n(\pi_{z_j}^0) \right) \xrightarrow{p} 0
\]

where, for a generic function \( h : X \to \mathbb{R} \), \( \nu_n(h) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ h(X_i) - \mathbb{E}[h(X)] \right] \) is used to denote the empirical process. The \( \alpha(z) \) in Assumption 5.3 is constructed to be \( \alpha_{\beta_{t,k}}(Y, T, Z, X) \) using Proposition 4 in that paper.\(^{15}\) From Lemma 5.1 there, we can establish the asymptotically linear representation of \( \sqrt{n}M_{\beta_{t,k}}(X, \beta_{t,k}^0, \hat{h}_t) \) to be

\[
\sqrt{n}M_{\beta_{t,k}}(X, \beta_{t,k}^0, \hat{h}_t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ M_{\beta_{t,k}}(X_i, \beta_{t,k}^0, h_t^0) + \alpha(Y_i, T_i, Z_i, X_i) \right] + o_p(1).
\]

Also, the consistency of \( \hat{p}_{t,k} \) follows from \( \left\| \hat{h}_t - h_t \right\|_\infty \xrightarrow{p} 0 \) and the fact that the \( \pi_z \)'s are bounded away from zero and one. Then using Slutsky’s theorem, the above results can be combined to obtain asymptotic normality of \( \hat{\beta}_{t,k} \) since

\[
\sqrt{n}\left( \hat{\beta}_{t,k} - \beta_{t,k}^0 \right) = \sqrt{n}M_{\beta_{t,k}}(X, \beta_{t,k}^0, \hat{h}_t)/\hat{p}_{t,k}.\(^{16}\)
\]

Hence the influence function of \( \hat{\beta}_{t,k} \) should be \( \left( M_{\beta_{t,k}}(X, \beta_{t,k}^0, h_t^0) + \alpha_{\beta_{t,k}} \right)/\hat{p}_{t,k}^a \), which equals to \( \Psi_{\beta_{t,k}} \) evaluated at the true parameter values. The term \( \alpha_{\beta_{t,k}} \) corrects the bias in estimation due to the presence of the unknown infinite dimensional nuisance parameter \( (h_{Y,t,Z}, h_t, Z, \pi) \). The proofs for \( \hat{\gamma}_{t',t,k}, \hat{p}_{t,k}, \) and \( \hat{q}_{t',t,k} \) are essentially the same. For estimating the efficiency bound, consistency of the plug-in estimators follows directly from the consistency of both the nonparametric estimates and the CEP estimators, the continuity of the efficient influence functions in the parameters, and the fact that propensity scores are bounded away from zero and one.

Lastly, the consistency of \( \hat{V}_\kappa \) follows from Lemma 8.3 of Newey and McFadden (1994), where

\(^{15}\)More discussion on this “mean-square differentiability” condition can be found in Newey and McFadden (1994).

\(^{16}\)There is no remainder \( o_p(1) \) terms because \( M_{\beta_{t,k}} \) is linear in \( \beta_{t,k} \), and hence it’s unnecessary to check Assumptions 5.4 to 5.6 in Newey (1994).
the two extra conditions can be directly verified using the form of, say, $M_{\beta_{t,k}}$ and $\alpha_{\beta_{t,k}}$. ■

Proof of Theorem 5. The proof is similar to that of Theorem 5.1 and 5.2 in Chernozhukov et al. (2018), which verify the assumptions in their Theorem 3.1. First observe that the moment condition (23) is linear in $\beta_{t,k}$. Since $\tilde{b}_{t,k}$ is a finite vector, it suffice in our case to verify the conditions in their Theorem 3.1 for the following score function

$$
\psi(W, \beta_{t,k}, v) = \psi^o(W, P_t, \pi_z) \beta_{t,k} + \psi^b(W, T, \pi_z) \\
= \left( \frac{1\{Z = z\}}{\pi_z(X)} (1\{T = t\} - P_{t,z}(X)) + P_{t,z}(X) \right) \beta_{t,k} \\
- \frac{1\{Z = z\}}{\pi_z(X)} (Y 1\{T = t\} - I_{t,z}(X)) + I_{t,z}(X),
$$

where $W = (Y, T, Z, X)$ and $v = (I_{t,z}, P_t, \pi_z)$. To check the Neyman orthogonality condition, we can compute the Gateaux derivative

$$
\frac{\partial}{\partial T} \mathbb{E} \left[ \psi(W, \beta_{t,k}, v^o + r(v - v^o)) \right] \bigg|_{r=0} = \mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z^o(X)} (1\{T = t\} - P_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X)) \beta_{t,k} \\
+ \left( P_{t,z}(X) - P_{t,z}^o(X) - \frac{1\{Z = z\}}{\pi_z^o(X)} (P_{t,z}(X) - P_{t,z}^o(X)) \right) \beta_{t,k} \\
+ \frac{1\{Z = z\}}{\pi_z(X)} (Y 1\{T = t\} - I_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X)) \\
+ I_{t,z}(X) - I_{t,z}^o(X) - \frac{1\{Z = z\}}{\pi_z^o(X)} (I_{t,z}(X) - I_{t,z}^o(X)) \right].
$$

It is equal to zero since

$$
\mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z^o(X)} (1\{T = t\} - P_{t,z}^o(X)) \bigg| X \right] = \mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z^o(X)} (Y 1\{T = t\} - I_{t,z}^o(X)) \bigg| X \right] = 0, \quad (39)
$$

and $\mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z^o(X)} \bigg| X \right] = 1$. Inside the nuisance realization set such that the nuisance parameters take value in this set with probability $\Delta_n$, we verify their Assumption 3.2 as follows.

$$
\|\psi^o(W, P_t, \pi_z)\|_q = \left\| \frac{1\{Z = z\}}{\pi_z(X)} (1\{T = t\} - P_{t,z}(X)) + P_{t,z}(X) \right\|_q \\
\leq \left\| \frac{1\{Z = z\}}{\pi_z(X)} \right\|_q \left\| 1\{T = t\} - P_{t,z}(X) \right\|_q + \left\| P_{t,z}(X) \right\|_q \leq 1 + 1.
$$
\[
\|\psi(W, \beta_{t,k}, v)\|_q \leq \left\| \frac{1\{Z = z\}}{\pi_z(X)} (1\{T = t\} - P_{t,z}(X)) + P_{t,z}(X) \right\|_q \beta_{t,k} \\
+ \left\| \frac{1\{Z = z\}}{\pi_z(X)} (Y1\{T = t\} - I_{t,z}(X)) + I_{t,z}(X) \right\|_q \\
\leq (1/\pi + 1) \beta_{t,k} + \|I_{t,z}(X) - I^0_{t,z}(X)\|_q + \|I^0_{t,z}(X)\|_q \\
+ \left\| \frac{1\{Z = z\}}{\pi_z(X)} \left( \|Y1\{T = t\} - I^0_{t,z}(X)\|_q + \|I_{t,z}(X) - I^0_{t,z}(X)\|_q \right) \right\|_q \\
\leq (1/\pi + 1) + C + C + 1/\pi(C + C) = (2C + 1) (1/\pi + 1),
\]

\[
\left| \mathbb{E} [\psi^o(W, P_{t,z}, \pi_z)] - \mathbb{E} [\psi^o(W, P^o_{t,z}, \pi_z^o)] \right| \\
= \left| \mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z(X)} (1\{T = t\} - P_{t,z}(X)) + (P_{t,z}(X) - P^o_{t,z}(X)) \right] \right| \\
= \left| \mathbb{E} \left[ \frac{1\{Z = z\}}{\pi_z(X)} (P^o_{t,z}(X) - P_{t,z}(X)) + (P_{t,z}(X) - P^o_{t,z}(X)) \right] \right| \\
\leq \|P_{t,z}(X) - P^o_{t,z}(X)\|_2/\pi \leq \delta_n/\pi.
\]

To bound \(\|\psi(W, \beta_{t,k}, v) - \psi(W, \beta_{t,k}, v^o)\|_2\), note that we have

\[
\left\| \frac{1\{Z = z\}}{\pi_z(X)} (1\{T = t\} - P_{t,z}(X)) + P_{t,z}(X) - \frac{1\{Z = z\}}{\pi_z^o(X)} (1\{T = t\} - P^o_{t,z}(X) - P^o_{t,z}(X)) \right\|_2 \\
\leq \|P_{t,z}(X) - P^o_{t,z}(X)\|_2 + \left\| \frac{1}{\pi_z(X)} - \frac{1}{\pi_z^o(X)} \right\|_2 \left\| 1\{Z = z\} (1\{T = t\} - P_{t,z}(X)) \right\|_2 \\
+ \left\| \frac{1}{\pi_z^o(X)} (P_{t,z}(X) - P^o_{t,z}(X)) \right\|_2 \\
\leq (1 + 1/\pi) \|P_{t,z}(X) - P^o_{t,z}(X)\|_2 + \|\pi_z(X) - \pi_z^o(X)\|_2/\pi^2 \leq (1 + 1/\pi + 1/\pi^2) \delta_n,
\]
and similarly,

\[
\begin{align*}
\left\| \frac{1}{\pi_z(X)} (Y1\{ T = t \} - I_{t,z}(X)) + I_{t,z}(X) - \frac{1}{\pi^o_z(X)} (Y1\{ T = t \} - I_{t,z}^o(X) - I_{t,z}(X)) \right\|_2 \\
\leq \| I_{t,z}(X) - I_{t,z}^o(X) \|_2 + \left\| \left( \frac{1}{\pi_z(X)} - \frac{1}{\pi^o_z(X)} \right) 1\{ Z = z \} (Y1\{ T = t \} - I_{t,z}(X)) \right\|_2 \\
+ \left\| \frac{1}{\pi^o_z(X)} (I_{t,z}(X) - I_{t,z}^o(X)) \right\|_2 \\
\leq (1 + 1/\pi) \| I_{t,z}(X) - I_{t,z}^o(X) \|_2 + \| \pi_z(X) - \pi^o_z(X) \|_2 \left( \| Y1\{ T = t \} - I_{t,z}^o \|_q + \| I_{t,z} - I_{t,z}^o \|_q \right)/\pi^2 \\
\leq (1 + 1/\pi + 2C/\pi^2) \delta_n.
\end{align*}
\]

Thus we have

\[
\left\| \psi(W, \beta_{t,k}, \nu) - \psi(W, \beta_{t,k}, \nu^o) \right\|_2 \\
\leq \left\| \frac{1}{\pi_z(X)} (1\{ T = t \} - P_{t,z}(X)) + P_{t,z}(X) - \frac{1}{\pi^o_z(X)} (1\{ T = t \} - P_{t,z}^o(X) - P_{t,z}(X)) \right\|_2 \beta_{t,k} \\
+ \left\| \frac{1}{\pi^o_z(X)} (1\{ T = t \} - P_{t,z}(X)) + P_{t,z}(X) - \frac{1}{\pi^o_z(X)} (1\{ T = t \} - P_{t,z}^o(X) - P_{t,z}(X)) \right\|_2 \\
\leq \left( (1 + 1/\pi + 1/\pi^2) \beta_{t,k} + (1 + 1/\pi + 2C/\pi^2) \right) \delta_n.
\]

Lastly, for any \( r \in (0, 1) \), based on equation (39) we have

\[
\frac{\partial^2}{\partial r^2} E \left[ \psi(W, \beta_{t,k}, \nu^o + r(\nu - \nu^o)) \right] \\
\leq E \left[ \begin{aligned}
2 \times \frac{1}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^3 r (P_{t,z}(X) - P_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X))^2 \beta_{t,k} \\
- \frac{1}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^2 (P_{t,z}(X) - P_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X)) \beta_{t,k} \\
- \frac{1}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^2 r (P_{t,z}(X) - P_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X)) \beta_{t,k} \\
+ \frac{2}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^3 r (I_{t,z}(X) - I_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X))^2 \\
- \frac{1}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^2 (I_{t,z}(X) - I_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X)) \beta_{t,k} \\
- \frac{1}{\pi_z(X)} & (\pi_z(X) + r(\pi_z(X) - \pi_z^o(X)))^2 r (I_{t,z}(X) - I_{t,z}^o(X)) (\pi_z(X) - \pi_z^o(X))
\end{aligned} \right]
\]

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Hence we can bound the second-order derivative by
\[
\left| \frac{\partial^2}{\partial r^2} \mathbb{E} [\psi(W, \beta_{t,k}, v^\circ + r(v - v^\circ))] \right| \leq \text{Const.} \times \|z - z^\circ\|_2 \left( \|I_{t,z} - I_{t,z}^\circ\|_2 + \|P_{t,z} - P_{t,z}^\circ\|_2 \right) \\
\leq \text{Const.} \times n^{-1/2} \delta_n,
\]
which completes the proof.

**Proof of Theorem 6.** First note that the moment conditions can be equivalently represented by \( \tilde{b}_j \mathbb{E} [m_{j,t}, z(X, \eta)] = 0, 1 \leq j \leq J \). Then the rest of the proof is mainly based on the approach described in section 3.6 of Hong and Nekipelov (2010a) and the proof of Theorem 1 in Cattaneo (2010). We use a constant \( d_\eta \times d_m \) matrix \( A \) to transform the overidentified vector of moments into an exactly identified system of equations \( A \left( \tilde{b}_j \mathbb{E} [m_{j,t}, z(X, \eta)] \right)_{j=1}^J = 0 \), and find the \( A \)-dependent efficient influence function for the exactly-identified parameter. Then choose the optimal \( A \). In a parametric submodel, by the implicit function theorem, we have
\[
\frac{\partial}{\partial \eta} |_{\theta = \theta^0} = - (A \Gamma)^{-1} A \frac{\partial}{\partial \eta} \left( \tilde{b}_j \mathbb{E} [m_{j,t}, z(X, \eta^0)] \right)_{j=1}^J |_{\theta = \theta^0}
\]
where \( \frac{\partial}{\partial \eta} \mathbb{E} [m_{j,t}, z(X, \eta^0)] |_{\theta = \theta^0} \) is a \( N_Z \times 1 \) random vector with typical element
\[
\int m_j(y, \eta^0) 1\{r = t_j\} s_z(y, \tau | x; \theta^0) f_z(y, \tau | x; \theta^0) f_X(x; \theta^0) dy \, d\tau \, dx \]
\[
+ \int m_j(y, \eta^0) 1\{r = t_j\} s_X(x; \theta^0) f_z(y, \tau | x; \theta^0) f_X(x; \theta^0) dy \, d\tau \, dx
\]
for \( z \in Z \). So the efficient influence function for this exactly-identified parameter is
\[
\Psi_A(Y, T, Z, X, \eta^0, \pi^0, m_Z^\circ) = - (A \Gamma)^{-1} A \Psi_m(Y, T, Z, X, \eta^0, \pi^0, m_Z^\circ)
\]
where \( \Psi_m \) is defined by equation (30). It is straightforward to verify that \( \Psi_A \) satisfies \( \frac{\partial}{\partial \eta} |_{\theta = \theta^0} = \mathbb{E} [\Psi_A s_{\theta^0}], \) and \( \Psi_A \in \mathcal{S}(P) \). The optimal \( A \) is chosen by minimizing the sandwich matrix \( \mathbb{E} [\Psi_A \Psi_A'] = (A \Gamma)^{-1} A \mathbb{E} [\Psi_m \Psi_m'] A' (\Gamma' \Gamma)^{-1} \). Thus the efficient influence function for the generally over-identified parameter is obtained when \( A = \Gamma' V^{-1} p^\circ \). Plugging into \( \Psi_A \), we get equation (29).

**Proof of Theorem 7.** We follow the large sample theory in Chen et al. (2003) (hereafter CLK),
setting $\theta = \eta$, $h = (\pi, m_Z)$, $M(\theta, h) = G(\eta, \pi, m_Z)$, and $M_n(\theta, h) = G_n(\eta, \pi, m_Z)$.

Their Theorem 1 is applied first to show the consistency of $\tilde{\eta}$. Their (1.2) is satisfied since $\Lambda$ is compact, and $G(\eta, \pi^o, m^o_Z) = \left( \tilde{b}_j \mathbb{E} \left[ m^o_{j,t_j,z}(X, \eta) \right] \right)_{j=1}^J$, which has a unique zero $\eta^o$ by our condition 7(iii) and is continuous by our condition 7(iii). As for (1.3) of CLK, continuity of $G$ in $m_{j,t_j,z}$ and $\pi_z$ is verified by their way of entering (linear or by taking inverse, with $\pi_z$ bounded away from 0 and 1), and the uniformity in $\eta$ follows from the fact that $\mathbb{E} [m(Y^q, \eta)]$ is bounded as a function of $\eta$ (by its continuity and the compactness of $\Lambda$). Condition (1.4) of CLK is satisfied by our condition $\eta^o_1$, and the uniformity in $\eta$ follows from the law of iterated expectations and the fact that $\tilde{\eta}$ is Glivenko-Cantelli, which follows from our condition (2.2) in CLK is verified by our condition $\eta^o_1$, and the uniformity in $\eta$ follows from the fact that $\tilde{\eta}$ is Glivenko-Cantelli, which follows from our condition (2.2) in CLK is verified by our condition (1.5) of CLK is implied by the fact that, for any $j$ and $z$, the class

$$\left\{ \frac{1\{Z = z\}}{\pi_z(X)} \right\} \left\{ m_j(Y, \eta)1\{T = t_j\} - m_{j,t_j,z}(X, \eta) \right\} + m_{j,t_j,z}(X, \eta) : \eta \in \Lambda, m_{j,t_j,z} \in \mathcal{M}_j, \pi_z \in \Pi_\delta \right\}$$

is Glivenko-Cantelli, which follows from our condition 7(iii) and the results in Van Der Vaart and Wellner (2000), stating that Glivenko-Cantelli classes with integrable envelopes are preserved by a continuous function. Thus $\tilde{\eta} - \eta^o = o_p(1)$.

Then we use Corollary 1 in CLK to show the consistency of $\hat{V}$ and the asymptotic normality of $\hat{\eta}$. Condition (2.2) in CLK is verified by our condition 6(iii). As in the proof of Theorem 5, it is straightforward to show the moment condition $G$, based on the efficient influence functions, satisfies the Neyman orthogonality condition for the nuisance parameters $\pi$ and $m_Z$. For any $j$ and $z$, denote $\pi^r_z = \pi^o_z(X) + r(\pi_z(X) - \pi^o_z(X))$ and $m_{j,t_j,z}(X, \eta) = m^o_{j,t_j,z}(X, \eta) + r \left( m_{j,t_j,z}(X, \eta) - m^o_{j,t_j,z}(X, \eta) \right)$, we have

$$\left. \frac{\partial}{\partial r} \mathbb{E} \left[ \frac{1\{Z = z\}}{\pi^o_z(X)} \left( m_j(Y, \eta)1\{T = t_j\} - m^o_{j,t_j,z}(X, \eta) \right) + m^r_{j,t_j,z}(X, \eta) \right] \right|_{r=0}$$

$$= \mathbb{E} \left[ - \frac{1\{Z = z\}}{\pi^o_z(X)} \left( \pi_z(X) - \pi^o_z(X) \right) \left( m_j(Y, \eta)1\{T = t_j\} - m^o_{j,t_j,z}(X, \eta) \right) \right.$$

$$\left. + \left( m^o_{j,t_j,z}(X, \eta) - m_{j,t_j,z}(X, \eta) \right) \left( \frac{1\{Z = z\}}{\pi^o_z(X)} - 1 \right) \right] = 0$$

using the law of iterated expectations and the fact that

$$\mathbb{E} \left[ \frac{1\{Z = z\}}{\pi^o_z(X)} \left( m_j(Y, \eta)1\{T = t_j\} - m^o_{j,t_j,z}(X, \eta) \right) \right] = 0$$
Thus the pathwise derivative of $G$ with respect to $(\pi, m_Z)$ is zero in any direction, and hence condition (2.3) of CLK is verified. Their condition (2.4) is satisfied by our condition 7(ii). To show the stochastic equicontinuity condition (2.6), it suffice to show that the class

$$\left\{ \frac{1\{Z = z\}}{\pi_z(X)} (m_j(Y, \eta) 1\{T = t_j\} - m_j(t_j, z(X, \eta)) + m_j(t_j, z(X, \eta)) : \eta \in \Lambda, m_j(t_j, z(X, \eta) \in \mathcal{M}_{j, z}, \pi_z \in \Pi_z) \right\}$$

is Donsker. This follows from our condition 7(iv) and Theorem 2.10.6 (as well as examples 2.10.7-2.10.9) in Van Der Vaart and Wellner (1996). Condition (2.6) in CLK is trivially verified using the central limit theorem. For the condition in Corollary 1 of CLK, let

$$\Omega(\eta, \pi, m_Z) = \mathbb{E}[\Psi_m(Y, T, Z, X, \eta, \pi, m_Z)^\prime \Psi(Y, T, Z, X, \eta, \pi, m_Z)]$$

and

$$\Omega_n(\eta, \pi, m_Z) = \frac{1}{n} \sum_{i=1}^n \Psi_m(Y_i, T_i, Z_i, X_i, \eta, \pi, m_Z) \Psi_m(Y_i, T_i, Z_i, X_i, \eta, \pi, m_Z)^\prime.$$ 

Then $V = \Omega(\eta^0, \pi^0, m_Z^0)$ and $\hat{V} = \Omega_n(\hat{\eta}, \hat{\pi}, \hat{m}_Z)$. For any $\delta_n \downarrow 0$, in the shrinking neighborhoods $\Lambda^\delta_n, \Pi_{\pi_z}^\delta$, and $\mathcal{M}_{j, z}^\delta$, we have

$$\sup \| \Omega_n(\eta, \pi, m_Z) - V \| \leq \sup \| \Omega_n(\eta, \pi, m_Z) - \Omega(\eta, \pi, m_Z) - (\Omega_n(\eta^0, \pi^0, m_Z^0) - \Omega(\eta^0, \pi^0, m_Z^0)) \|$$

$$+ \sup \| \Omega(\eta, \pi, m_Z) - \Omega(\eta^0, \pi^0, m_Z^0) \| + \sup \| \Omega_n(\eta^0, \pi^0, m_Z^0) - \Omega(\eta^0, \pi^0, m_Z^0) \|$$

The first term on the RHS is $o_p(1)$ follows from the stochastic equicontinuity property on $\Omega_n - \Omega$, which results from the (element-wise) Donsker property of the matrix $\Psi_m \Psi_m^\prime$. The second term on the RHS is $o_p(1)$ since $\Omega$ is continuous in its arguments (equation (30) and condition 6(iii)), while the third term is $o_p(1)$ by the standard central limit theorem. Hence, we have shown that $\hat{V} - V = o_p(1)$ and $\hat{\eta} - \eta^0 = o_p(1)$.

Lastly, using the arguments in Theorem 7.4 in Newey and McFadden (1994), the numerical derivative $\hat{\Gamma}$ is consistent. ■

Proof of Theorem 8. (ii) The second part of the theorem is proved first. Suppose $\mathcal{L}$ satisfies Assumption 2, we want to find a $\mathcal{L}_0$ that induces $\mathcal{L}$ and satisfies Assumption 1. The strategy is to make the $Y_t$’s mutually independent. And set their conditional (on $S \in \Sigma_{t,k}$) distributions to the
identified value when identifiable, and to be arbitrary when unidentifiable.

Let \( \tilde{P}(\cdot \mid X) \) be an arbitrary conditional distribution on the support of \( Y \). Define joint distribution of \( (Z, X) \) as identified from \( L \). The goal is to construct the conditional distribution of \( (Y_t : t \in T, S) \mid Z, X \) not to depend on \( Z \). For any measurable sequence of sets \( \{B_1, \cdots, B_{N_T}\} \) on the support of \( Y, \Sigma \subset 2^S \), and \( z \in Z \),

\[
P \left( Y_{t_1} \in B_1, \cdots, Y_{t_{N_T}} \in B_{N_T}, S \in \Sigma \mid Z = z, X \right) = \left( \prod_{t \in T} \frac{\tilde{Q}_t(X, B_t \times \Sigma)}{Q_t(X, Y \times \Sigma)} \right) \tilde{Q}_{t_1}(X, Y \times \Sigma)
\]

For \( s \notin S \), let \( P \left( Y_{t_1} \in E_1, \cdots, Y_{t_{N_T}} \in E_{N_T}, S = s \mid Z = z, X \right) = 0 \). We have fully specified a joint distribution of \( \left( \{Y_t : t \in T, z \in Z\}, \{T_z : z \in Z\}, Z, X \right) \), \( L_0 \), that is consistent with \( L_0 \) and satisfies Assumption 1. Let \( \mathcal{C} \) be another condition on \( L \), such that \( L_0 \) satisfies Assumption 1 implies \( O(L_0) \) satisfies condition \( \mathcal{C} \). The contrapositive statement is that if \( L \) violates \( \mathcal{C} \), then any \( L_0 \) with \( O(L_0) = L \) has to violate \( \mathcal{C} \). Therefore, in the current case, where \( L \) satisfies Assumption 1, \( L \) has to satisfy \( \mathcal{C} \).

(i) The first statement is trivial. For the second statement, suppose \( L \) satisfies Assumption 2, we want to find a \( L_0 \) that induces \( L \) and violates Assumption 1. The strategy is to define the structural functions to be dependent on \( Z \). In particular, specify \( (Y_t : t \in T, S) \mid Z, X \) to be the same as before when conditioning on \( Z = z_1 \). When \( Z \neq z_1 \), let

\[
P \left( Y_{t_1} \in B_1, \cdots, Y_{t_{N_T}} \in B_{N_T}, S \in \Sigma \mid Z = z, X \right) = L_{Y \mid S}(B_1, \cdots, B_{N_T}) \tilde{Q}_{t_1}(X, Y \times \Sigma)
\]

where \( L_{Y \mid S} \) denotes a joint law of \( N_T \) not mutually independent random variables whose marginal distribution is equal to \( \frac{\tilde{Q}_t(X, B_t \times \Sigma)}{Q_t(X, Y \times \Sigma)} \). Clearly, the \( Y_t \)'s are \( Z \)-dependent. \( \blacksquare \)