Stationary scalar configurations around extremal charged black holes

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We consider the minimally coupled Klein-Gordon equation for a charged, massive scalar field in the non-extremal Reissner-Nordström background. Performing a frequency domain analysis, using a continued fraction method, we compute the frequencies $\omega$ for quasi-bound states. We observe that, as the extremal limit for both the background and the field is approached, the real part of the quasi-bound states frequencies $\mathcal{R}(\omega)$ tends to the mass of the field and the imaginary part $\mathcal{I}(\omega)$ tends to zero, for any angular momentum quantum number $\ell$. The limiting frequencies in this double extremal limit are shown to correspond to a distribution of extremal scalar particles, at stationary positions, in no-force equilibrium configurations with the background. Thus, generically, these stationary scalar configurations are regular at the event horizon. If, on the other hand, the distribution contains scalar particles at the horizon, the configuration becomes irregular therein, in agreement with no hair theorems for the corresponding Einstein-Maxwell-scalar field system.

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I. INTRODUCTION

Scalar test fields in black hole (BH) geometries do not admit, generically, stationary configurations with an asymptotic decay and with real frequencies, i.e bound states. This follows from the physical requirement that only ingoing waves can exist at the horizon, therefore preventing a real equilibrium configuration between the field and the BH. Consequently, the configurations allowed in BH backgrounds are quasi-bound states, for which the frequencies are complex, with the imaginary part revealing a time dependence for the states, signaling either their absorption or, in the case of superradiant instabilities, their amplification by the BH $^{[1]}$.

The profile and some of the physical properties of quasi-bound states diverge at the horizon. This is intimately related to the inability that BH backgrounds have to accommodate, in a regular fashion, the scalar field, as an exact stationary solution, a property established by no-hair theorems $^{[2,3]}$. But even with this caveat such quasi-bound states are informative. For instance in $^{[4]}$, performing numerical simulations and starting with regular initial data for a scalar field around a Schwarzschild BH, there were found damped oscillating solutions with frequency and decay rate described by the real and imaginary parts of quasi-bound state frequencies. These decay rates can be very small $^{[2]}$ and thus long lived scalar field configurations could exist around BHs, even though eternal and regular configurations are, in general, precluded by no-hair theorems.

In this work we provide an example in which scalar field configurations around a BH can become stationary and an interpretation to justify why this is possible. We start by finding the quasi-bound states for a charged massive scalar field in a non-extremal Reissner-Nordström (RN) BH, and show that these states have a simple limiting behaviour as extremality for both the BH and the field is approached: the imaginary part of the frequency vanishes and the real part of the frequency tends to the test field mass (and charge). Then, the scalar field configurations obtained in this limit are understood as the electrostatic potential of a distribution of extremal scalar particles in equilibrium with the extremal BH, thus providing examples of scalar fields around extremal charged BHs which are regular on the event horizon. Such configurations do not preserve spherical symmetry around the BH and that is the way they circumvents no-hair theorems.

This paper is organized as follows. In Sec. II we discuss solutions of the minimally coupled Klein-Gordon equation on the RN background. Quasi-bound state solutions of this equation are considered in more detail in Sec. III where some explicit frequencies are computed and analyzed. In Sec. IV the extremal limit is discussed, by computing the states with frequencies equal to the limiting behaviour observed in Sec. III and an interpretation for these states is given. We draw some concluding remarks and comment on the non-linear solution including the backreaction of the scalar field in Sec. V.

II. BACKGROUND AND TEST FIELD

We consider a massive, charged scalar field, $\Phi$, with mass $\mu$ and charge $q$, obeying the wave equation

$$[\hat{D}_\nu \hat{D}^\nu - \mu^2] \Phi = 0 \; ,$$

where $\hat{D}_\nu \equiv D_\nu - iqA_\nu$. This field is propagating in the background of a Reissner-Nordström BH with charge $Q$ and mass $M$:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \; ,$$
where \( f(r) = (r - r_+)(r - r_-)/r^2 \), \( r_\pm \equiv M \pm \sqrt{M^2 - Q^2} \) and \( A_\mu dx^\mu = -Q/r \, dt \). Taking the standard ansatz for the scalar field which reflects the spherical symmetry and staticity of the background:

\[
\Phi = \sum_{\ell, m} \Phi^{\ell m} = \sum_{\ell, m} e^{-i\omega t} Y^{\ell m}(\theta, \phi) R_\ell(r),
\]

where \( Y^{\ell m} \) are the spherical harmonics and \( \omega \) the complex frequency of a wave, (1) yields the radial equation for each mode:

\[
r^2 f(r) \frac{d}{dr} \left( r^2 f(r) \frac{dR_\ell(r)}{dr} \right) + U R_\ell(r) = 0,
\]

where \( U = r^2 \left[(\omega r - qQ)^2 - f(\mu^2 r^2 + \ell(\ell + 1))\right] \). Observe that the azimuthal quantum number \( m \) is irrelevant due to spherical symmetry. The solution for the field \( \Phi \) is immediately obtained by solving the radial equation for each mode (4), due to the linearity of the wave equation (1). In terms of \( Z(r) = r R(r) \), and dropping the subscript \( \ell \) for notation simplicity, (4) becomes

\[
\frac{d^2}{dr^2} Z(r) + \frac{f'}{f} \frac{d}{dr} Z(r) + \frac{1}{f^2} \left[ \omega^2 - \nu_\ell(r) \right] Z(r) = 0,
\]

where we have defined the effective potential \( \nu_\ell(r) \), by \( dr^+ = dr/f(r) \), this wave equation is rewritten as

\[
\left[ \frac{d^2}{dr^2} + \nu_\ell(r) \right] Z(r) = \omega^2 Z(r),
\]

where \( r = r(r^+) \). The properties of the potential \( \nu_\ell(r) \) have been discussed in the past, see eg. [3]. In particular one can show that the height of the centrifugal barrier increases with the charge of the field and that the constant value of the potential near the outer horizon also increases with the charge of the field but only up to some maximum; then it starts decreasing. The main feature of this potential, however, is that for a given combination of the parameters it exhibits a well, that can be considered as one of the key ingredients to have quasi-bound states.

In order to solve the differential equation (5) we must provide a set of suitable boundary conditions at the horizon and at spacial infinity. To see the most relevant feature of the near horizon behaviour we note that in this region equation (2) becomes to leading order:

\[
\frac{d^2}{dr^2} Z(r) + (\omega - q\phi_+)^2 Z(r) \approx 0,
\]

where \( \phi_+ = Q/r_+ \) is the electrostatic potential of the external horizon. This equation is solved by a superposition of in and outgoing waves. Choosing the solution

\[
Z(r) \xrightarrow{r \to r_+} e^{-i(\omega - \omega_0)r^+},
\]

where \( \omega_0 \equiv q\phi_+ \), corresponding to an ingoing wave for \( q = 0 \), one observes the salient feature that it becomes an outgoing wave for \( \omega < \omega_0 \) (in this electromagnetic gauge). This is the condition for superradiance.

Asymptotically, keeping the terms of order 1/r in equation (1) one gets

\[
R(r) \xrightarrow{r \to \infty} e^{zr} \frac{e^{\chi r}}{r^{1/2}},
\]

where

\[
\sigma \equiv qQ\omega + M\mu^2 - 2M\omega^2 \chi, \quad \chi \equiv \pm \sqrt{\mu^2 - \omega_+^2}.
\]

From (10) one observes a qualitatively distinct behaviour depending on the sign of the real part of \( \chi \), \( \mathcal{R}(\chi) \). In particular, for \( \mathcal{R}(\chi) < 0 \) we have quasi-bound states. These are characterized by a decaying behaviour at spatial infinity. For \( \mathcal{R}(\chi) > 0 \) we have scattering states. Hereafter we will be interested in quasi-bound states.

### III. SEMI-ANALYTIC GLOBAL SOLUTION: QUASI-BOUND STATES FREQUENCIES

To find the solution of equation (5) in the region \( r > r_+ \) we will use a continued-fraction procedure developed by Leaver to find the quasinormal modes for the Schwarzschild and Kerr BHs [7]. This amounts to take a power series ansatz with a pre-factor adapted to the boundary conditions observed in the previous section

\[
Z(r) = e^{zr} u^0 (r - r_-)^{\sigma - 1} \sum_{n=0}^{\infty} a_n r^n,
\]

where

\[
u = \frac{r - r_+}{r - r_-}, \quad \rho = -i r_+^2 \left( \frac{\omega - \omega_+}{r_+ - r_-} \right).
\]

Substituting (12) into (5) we obtain a three term recurrence relationship for the \( a_n \) of the form:

\[
\begin{align*}
\alpha_0 a_1 + \beta_0 a_0 &= 0, \\
\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0, \quad n = 1, 2, 3, \ldots,
\end{align*}
\]

with

\[
\begin{align*}
\alpha_n &= (1 - Q^2)n^2 + c_n n - (1 - Q^2) + c_0, \\
\beta_n &= -2(1 - Q^2)n^2 + c_1 n + c_2, \\
\gamma_n &= (1 - Q^2)n^2 + c_3 n + c_4.
\end{align*}
\]

The constants \( c_i \) are lengthy expressions but otherwise straightforward to obtain. For a non charged massive

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1 In [3], Leaver considers a similar method for a Regge-Wheeler equation obtained from perturbing the Reissner-Nordström background.
scalar field on a Schwarzschild background, these expressions reduce to the ones (in the same limit) given in Ref. 9. The procedure to find the frequencies of the quasibound states consists in setting the three term relationship (13) in the form of a continued fraction algebraic equation and then solving it with a root finding procedure.

Furthermore, we have checked the value of the frequencies obtained with the continued fraction method by numerically integrating the radial equation. We took as initial condition the behaviour of the function close to the horizon and integrated up to some large \( r \) compared with the horizon radius.

The observed trend for the frequency of the quasibounded states is that, as the test field becomes extremal \( (\mu = |q|) \), the imaginary part of the frequency decreases. The imaginary part of the frequency is a measure of the rate at which the field falls into the horizon. For the charged massive scalar field, the closer the background black hole is to extremality the smaller this imaginary part is for the fundamental tone. The real part of the frequency, on the other hand, tends to increase and converge towards the field mass as the BH charge is increased towards extremality. This trend is shared by modes with different angular momentum quantum number. To make clear the behaviour of both the real and imaginary parts of the quasibound states frequencies as the double extremality is attained \( (|Q|, |q| \rightarrow M, \mu) \), we plot them against the BH charge for \( q/\mu = 1 \) in Fig. 1. Concerning the radial function as the double extremal limit is taken, the profile of the \( R_0(r) \) mode is qualitatively different from the profiles for the others modes. When \( \ell = 0 \) (s-wave), the radial function is localized in the region between the external horizon and the maximum of the potential barrier. As \( |Q|/M \rightarrow 1 \), \( R_0(r) \) becomes narrower in such a way that its maximum tends to the horizon. For \( \ell \neq 0 \), the functions \( R_\ell(r) \) tend to spread out as the double extremal limit is approached - Fig 2.

IV. EXTREMAL BLACK HOLE

The results of the previous section describe the limiting behaviour of the quasibound states as the double extremal limit is attained. We shall now consider exactly this limit by focusing on the extremal Reissner-Nordström BH, \( |Q| = M \) \( (\ell_\pm = M) \), with an extremal test field \( (\mu = |q|) \). Then, using a new radial coordinate \( \rho \equiv r - M \), for states with \( \omega = \mu = |q| \) and \( qQ > 0 \), the radial wave equation (14) reduces to

\[
\frac{d^2R_\ell}{d\rho^2} + \frac{2}{\rho} \frac{dR_\ell}{d\rho} - \frac{\ell(\ell + 1)}{\rho^2} R_\ell = 0 \, .
\]  

(15)

This equation is the radial part of the Laplace equation on Euclidean 3-space \( E^3 \) whose solution is \( R_\ell = A_\ell \rho^\ell + B_\ell / \rho^{\ell+1} \). The spacial part of the scalar field \( \Phi \) is therefore a linear combination of harmonic functions

\[
\Phi = e^{-i\mu t} \sum_{\ell,m} Y_{\ell m}^* (\theta, \phi) \left[ A_\ell \rho^\ell + B_\ell / \rho^{\ell+1} \right] \, .
\]  

(16)

Each of these partial waves, with appropriate \( A_\ell, B_\ell \), describes the double extremal limit of a quasibound state.

To interpret the meaning of the modes (15) and understand their appearance, it is useful to rewrite the extremal RN background using the coordinate \( \rho \); this corresponds to isotropic coordinates. Then the fields take the form

\[
ds^2 = -H^{-2} dt^2 + H^2 \delta_{ij} dx^i dx^j \, , \quad A = H^{-1} dt \, ,
\]  

(17)

where furthermore \( \rho = \sqrt{\delta_{ij} x^i x^j} \equiv |\mathbf{x}| \) and \( H \) is a harmonic function on Euclidean 3-space with a simple pole localised at the origin: \( H = 1 + M/|\mathbf{x}| \). In these coordinates \( \mathbf{x} = 0 \) is the location of the extremal RN BH horizon.

Taking the scalar field in the form \( \Phi(t, \mathbf{x}^i) = e^{-i\mu t} \tilde{H}(x^i) \), the wave equation (14) with \( \mu = |q| \) in the background (17), yields the harmonic equation \( \Delta_{E^3} \tilde{H} = 0 \). One solution is the harmonic function with a single pole at \( \mathbf{x}' : \tilde{H} = \mu/|\mathbf{x} - \mathbf{x}'| \). This describes the electric potential of one particle located at this pole. Expressed
For the latter, a pure s-wave corresponds to a source localised at the horizon, whereas in the former, the peak of the radial function tends to the origin, as displayed in the top panel of Fig. 2. A related observation is that the harmonic function $\tilde{H}$ is regular at the horizon except if the source is localised there.

A similar interpretation for the modes in (19) carries through if $\tilde{H}$ represents multiple scalar sources instead of a single one. We then have a superposition of harmonic functions with localised poles at fixed positions $x'_k$, corresponding to spherical coordinates $(\rho'_k, \theta'_k, \phi'_k)$,

$$\tilde{H} = \sum_k \tilde{H}_k = \mu \sum_k \frac{1}{|x - x'_k|},$$  \hspace{1cm} (20)

which may again be rewritten, in spherical coordinates, as the multipolar expansion in the right hand side of (18), with $B_\ell(\rho', \theta', \phi')$ replaced by $\sum_k B_\ell(\rho'_k, \theta'_k, \phi'_k)$, corresponding to replacing one particle by many particles.

The existence of stationary scalar states and their interpretation can, furthermore, be generalized to a background with multiple extremal BHs instead of a single one. This is achieved replacing $H$ by a superposition of harmonic functions with localised poles at different points, $x_i$, $H = 1 + \sum_i M_i / |x - x_i|$, since the scalar field equation still reduces to a harmonic equation on $\mathbb{E}^3$. Such solution of the Einstein-Maxwell system is the well known Majumdar-Papapetrou multi BH solution [10, 11], corresponding to a collection of BHs with mass and charge $M_i = Q_i$, placed at arbitrary positions $x_i$ [12], held in equilibrium by a balance between gravitational and electrostatic forces. A multiple scalar particle configuration will be regular on each horizon of the Majumdar-Papapetrou background as long as $x_i \neq x'_k$, for all $i, k$.

The scalar particles are in equilibrium with the BHs due to a ‘no-force’ condition, a balance between gravitational and electromagnetic forces, as can be easily checked by studying the orbits generated by the Lagrangian $L = \mu \sqrt{-g_{\alpha\beta} x'^\alpha x'^\beta} + \mu A_\alpha x'^\alpha$ in the background (17), where ‘dot’ denotes derivative with respect to proper time. This Lagrangian is adequate to describe the interaction of the scalar particles with the background (17) because, at linear level, there is no interaction mediated by the scalar field; only gravitational and electromagnetic interactions occur. The gravitational energy added to the system in equilibrium - the multi BH solution - by the massive scalar field is balanced by the electromagnetic energy carried by the field.

\section{V. Conclusions}

A scalar field on a BH geometry does not, generically, admit stationary configurations. In this note we have showed that an extremal scalar field in the background of a charged, extremal BH geometry does admit such configurations and we have provided a physical interpretation for them.
Our first observation was that the frequencies of quasi-bound states of a massive, charged, minimally coupled scalar field in the RN background have a well defined behaviour when a double extremal limit, for both the test field and the background is taken: the imaginary part vanishes and the real part becomes equal to the field mass. Then we showed that in such double extremal limit, configurations with a real frequency equal to the particle’s mass exist, corresponding to a distribution of extremal scalar particles, placed at arbitrary locations in the exterior of the extremal (multi-)BH solution. If none of these particles sits at the BH horizon, the configuration is regular therein. One may argue, however, that the field is irregular at the location of the sources. But this is the traditional problem in classical field theory associated to point-like sources.

The stationary scalar field states we have exhibited are due to no-force configurations between scalar sources and extremal BHs, at linear level: the gravitational attraction is being balanced by the electromagnetic repulsion. At non-linear level, however, the scalar field will back react on the geometry and, since it is charged, it will source the Maxwell field wherever the scalar field is non-trivial and not just at the location of the sources, in contrast to the typical multi-centre solutions found in Supergravity/String theory (see, e.g. [13]). It would will be interesting, but also challenging, to study the configurations we have analyzed herein at non-linear level, as solutions of the corresponding Einstein-Maxwell-scalar field theory.

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