SIEGEL AUTOMORPHIC FORM CORRECTIONS OF SOME LORENTZIAN KAC–MOODY LIE ALGEBRAS

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ABSTRACT. We find automorphic form corrections which are generalized Lorentzian Kac–Moody superalgebras without odd real simple roots (see R. Borcherds [Bo1] – [Bo7], V. Kac [Ka1] – [Ka3], R. Moody [Mo] and § 6 of this paper) for two elliptic Lorentzian Kac–Moody algebras of the rank 3 with a lattice Weyl vector, and calculate multiplicities of their simple and arbitrary imaginary roots (see an appropriate general setting in [N5]). These Kac–Moody algebras are defined by hyperbolic (i.e. with exactly one negative square) symmetrized generalized Cartan matrices

\[
G_1 = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}
\quad \text{and} \quad
G_2 = \begin{pmatrix}
4 & -4 & -12 & -4 \\
-4 & 4 & -4 & -12 \\
-12 & -4 & 4 & -4 \\
-4 & -12 & -4 & 4
\end{pmatrix}
\]

of the rank 3. Both these algebras have elliptic type (i.e. their Weyl groups have fundamental polyhedra of finite volume in corresponding hyperbolic spaces) and have a lattice Weyl vector. The correcting automorphic forms are Siegel modular forms. The form corresponding to \(G_1\) is the classical Siegel cusp form \(\Delta_5(Z)\) of weight 5 which is the product of ten even theta-constants. In particular we find an infinite product formula for \(\Delta_5(Z)\).

To find the correcting automorphic form for \(G_2\), we use the arithmetic lifting of Jacobi forms on domains of type IV which was constructed by first author [G1] and [G2]. It is proven by second author [N5] that the set of elliptic Lorentzian Kac–Moody algebras with a lattice Weyl vector is finite (but may be extremely big). Conjecturally all of them have automorphic form corrections.

\section{Introduction

First, we explain results of this paper formally. We find automorphic form corrections which are generalized Lorentzian Kac–Moody superalgebras without odd real simple roots (see works of R. Borcherds [Bo1] – [Bo7], V. Kac [Ka1] – [Ka3], R. Moody [Mo] and § 6 of this paper) for two elliptic Lorentzian Kac–Moody algebras, and calculate multiplicities of their simple and arbitrary imaginary roots. One can find an appropriate general setting on this subject in [N5]. The first Kac–Moody algebra has the symmetric generalized Cartan matrix

\[
G_1 = (\delta_i, \delta_j) = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}.
\]
For both these elliptic Lorentzian Kac–Moody algebras, the correcting automorphic forms are holomorphic modular forms on the Siegel upper half-plane. They are invariant (with a multiplier system) with respect to \( \text{Sp}_4(\mathbb{Z}) \) or the so-called paramodular group corresponding to the polarization \((1, 2)\). The first form is especially interesting because it is the classical cusp form \( \Delta_5(\mathbb{Z}) \) of weight 5 which is the product of all even theta-constants (see (1.2)). It is well known (see [F]) that \( \Delta_5(\mathbb{Z})^2 = F_{10}(\mathbb{Z}) \) where \( F_{10}(\mathbb{Z}) \) is one of the standard generators for the ring of Siegel modular forms with respect to \( \text{Sp}_4(\mathbb{Z}) \). As a result, we find the corresponding product formula for this classical Siegel cusp form \( \Delta_5(\mathbb{Z}) \).

To construct the product formula for \( \Delta_5(\mathbb{Z}) \), we use the Maass’s lifting for \( \text{Sp}_4(\mathbb{Z}) \) (see [M2], [EZ]). To find the product formula for correcting automorphic form of the algebra of \( G_2 \), we use the generalization of this lifting to the cases of the paramodular subgroups of \( \text{Sp}_4(\mathbb{Q}) \) and the orthogonal groups of signature \((2, n)\) (see [G1] – [G3]).

We should say that first examples of automorphic form corrections of Lorentzian Kac–Moody algebras were found by R. Borcherds (see [Bo1]—[Bo7]). Mainly, his examples are connected with automorphic forms on the complex domains of large dimension (for example the dimension 26) and the arithmetic of the Leech lattice.

Our examples above are first examples of automorphic form corrections of elliptic (or parabolic) Lorentzian Kac–Moody algebras (we use terminology from [N5]) when correcting automorphic forms are Siegel modular forms (necessarily of the genus 2). Moreover, we show that there are very classical automorphic forms like \( \Delta_5(\mathbb{Z}) \) above which give the automorphic form corrections of some Lorentzian Kac–Moody algebras. In forthcoming publication we hope to give other examples of automorphic forms which correct the corresponding Lorentzian Kac–Moody algebras.

Now we want to present our results in some details. We restrict considering the case (0.1).

Hyperbolic symmetric matrix \( G_1 \) defines an integral hyperbolic symmetric bilinear form (i.e. hyperbolic lattice) on the \( \mathbb{Z} \)-module \( M_{II} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3 \). Reflections \( s_{\delta_i} \) in \( \delta_i \in M_{II} \) generate a reflection subgroup of finite index \( W \subset O(M_{II}) \) which is discrete in corresponding hyperbolic space \( V^+(M_{II})/\mathbb{R}_{++} \) and has a fundamental polyhedron \( M_{II} \) of finite volume which is bounded by half-spaces orthogonal to \( \delta_i \). Here \( V^+(M_{II}) \) is a half-cone of the cone \( V(M_{II}) = \{ x \in M_{II} \otimes \mathbb{R} \mid (x, x) < 0 \} \). The half-cone \( V^+(M_{II}) \) is choosen uniquely by the following condition which is equivalent to finiteness of volume of \( M_{II}^1 \):

\[
V^+(M_{II}) \subset \mathbb{R}_+\delta_1 + \mathbb{R}_+\delta_2 + \mathbb{R}_+\delta_3.
\]

Moreover, \( \mathbb{R}_+M_{II} \) is the dual cone

\[
\mathbb{R}_+M_{II} = \{ x \in M_{II} \otimes \mathbb{R} \mid (x, \delta_i) \leq 0, \ i = 1, 2, 3 \} \subset V^+(M_{II^1}).
\]
to \( \mathbb{R}_+ \delta_1 + \mathbb{R}_+ \delta_2 + \mathbb{R}_+ \delta_3 \). We denote \( P(M_{II}) = \{ \delta_1, \delta_2, \delta_3 \} \subset M_{II} \). The group \( W \) and the set \( P(M_{II}) \) of vectors orthogonal to the fundamental polyhedron \( M_{II} \) of \( W \) have a lattice Weyl vector which is an element \( \rho \in M_{II} \otimes \mathbb{Q} \) with the property
\[
(\rho, \delta_i) = - (\delta_i, \delta_i) / 2.
\]
Evidently, \( \rho = (\delta_1 + \delta_2 + \delta_3) / 2 \). It is shown in [N5] (in fact, it easily follows from results of [N1] and [N2]) that the set of symmetrized generalized Cartan matrices and existence of a lattice Weyl vector for the set of vectors \( P(M) \) orthogonal to a fundamental polyhedron \( \mathcal{M} \) of \( W \) is finite (up to multiplication by a constant) for rank \( \geq 3 \). Thus, we are considering one of the finite set of cases.

The cone \( V^+(M_{II}) \) defines the corresponding complexified cone \( \Omega(V^+(M_{II})) = M_{II} \otimes \mathbb{R} + iV^+(M_{II}) \). We consider extended lattice \( L \) which is an orthogonal sum of \( M_{II} \) and unimodular even hyperbolic plane \( U \). One can consider \( \Omega(V^+(M_{II})) \) as a cusp of the domain \( \mathcal{H}_+ \) of type IV which is one of two connected components of
\[
\mathcal{H} = \{ C\omega \subset L \otimes \mathbb{C} \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) < 0 \}.
\]
Using natural identification of Siegel half-space \( \mathbb{H}_2 \) and \( \mathcal{H}_+ \) and natural isomorphism \( Sp_4(\mathbb{Z})/\{ \pm E_4 \} \subset SO^+_2(L) \), we can consider \( \Delta_5(Z) \) as an automorphic form with respect to \( SO^+_2(L) \). We show that \( \frac{1}{64} \Delta_5(Z) \) is anti-invariant with respect to \( W \), has integral Fourier coefficients and its Fourier coefficient corresponding to \( \rho \) is equal to 1. More exactly, we show (see Theorem 2.3) that
\[
\frac{1}{64} \Delta_5(z) = \sum_{w \in W} \det(w) \left( \exp(-\pi i (w(\rho), z)) - \sum_{a \in M_{II} \cap \mathbb{R}^+ M_{II}} m(a) \exp(-\pi i (w(\rho + a), z)) \right), \quad (0.3)
\]
where \( z = z_3 f_2 + z_2 f_3 + z_1 f_{-2} \in M_{II} \otimes \mathbb{R} + iV^+(M_{II}) \) and \( m(a) \in \mathbb{Z} \); for any primitive \( a_0 \in M_{II} \cap \mathbb{R}^+ M_{II} \) with \( (a_0, a_0) = 0 \), we have the identity
\[
1 - \sum_{t \in \mathbb{N}} m(ta_0) q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^{\tau(ka_0)}
\]
of power series of one variable \( q \), where all \( \tau(ka_0) = 9 \).

The matrix (0.1) is the Gram matrix of elements \( P(M_{II}) \). We use Fourier coefficients in (0.3) to extend the set \( P(M) \) to the set \( s\Delta = s\Delta^r \cup s\Delta^i \) where \( s\Delta^r = P(M) \) and
\[
s\Delta^i = \{ m(a)a \mid a \in M_{II} \cap \mathbb{R}^+ M_{II}, (a, a) < 0 \}
\]
\[
\cup \{ \tau(a)a \mid a \in M_{II} \cap \mathbb{R}^+ M_{II}, (a, a) = 0 \}.
\]
One can use Gram matrix of \( s\Delta \) to construct a generalized Kac–Moody superalgebra \( g \) with the set of simple roots \( s\Delta \). This construction is similar to [Bo1] (see also [Bo2], [Ka1]–[Ka3] and [Mo]). See § 3 and § 6 for details. The Weyl–Kac denominator function \( \Phi(z) \) of this Kac–Moody superalgebra is equal to \( \frac{1}{64} \Delta_5(2z) \) since our construction of \( g \). The Kac–Moody superalgebra \( g \) is called “corrected”
using the automorphic form $\frac{1}{64} \Delta_5(z)$. The Weyl–Kac denominator function $\Phi(z)$ for $\mathfrak{g}$ has the product formula

$$
\Phi(z) = \sum_{w \in W} \det(w) \left( \exp(-2\pi i (w(\rho), z)) - \sum_{a \in M_{II} \cap \mathbb{R}^+} m(a) \exp(-2\pi i (w(\rho + a), z)) \right) = \exp(-2\pi i (\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i (\alpha, z)))^{\text{mult} \alpha}.
$$

Thus, we should have similar product formula for $\Delta_5(z)$. Using automorphic properties of $\Delta_5(z)$ and Maass's lifting, we find this formula. In particular, we calculate multiplicities $\text{mult} \alpha$. See Theorem 4.1.

In [G1] – [G3], the arithmetic lifting of Jacobi forms on domains of IV type is constructed. This is a generalization of Maass's lifting [M2]. We use this lifting to get similar results for the Kac–Moody algebra corresponding to the matrix (0.2). See § 5.

Both our examples (0.1) and (0.2) belong to one series of elliptic Lorentzian Kac–Moody algebras related with reflection groups on hyperbolic plane which were classified in [N3]. They are groups $W^{(2)}(S)$ generated by reflections in all elements with square 2 of hyperbolic (i.e. of the signature $(n, 1)$) integral lattices $S$ of the rank 3 such that index $[O(S) : W^{(2)}(S)]$ is finite. We hope to consider other and may be all examples from this series in further publications.

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§ 1. **Siegel modular forms**

We remind that Siegel modular group $Sp_4(\mathbb{Z})$ is the group of all integral $4 \times 4$-matrices

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

such that $^t M J_4 M = J_4$ where $A, B, C, D$ are $2 \times 2$-integral matrices and

$$
J_4 = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.
$$

Siegel modular form $F$ of the weight $k$ is a holomorphic function on the Siegel domain

$$
\mathbb{H}_2 = \{ Z = ^t Z \in M_2(\mathbb{C}), \ Z = X + iY, \ Y > 0 \}
$$

with the invariance property

$$
F((AZ + B)(CZ + D)^{-1}) = \det (CZ + D)^k F(Z), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}).
$$

These forms define the ring

$$
\mathfrak{M}(Sp_4(\mathbb{Z})) = \bigoplus \mathfrak{M}_k(\mathfrak{M}(Sp_4(\mathbb{Z}))).
$$
where $\mathfrak{M}(Sp_4(\mathbb{Z}))$ denote the subspace of functions of the weight $k$. This ring has four generators

$$\mathfrak{M}(Sp_4(\mathbb{Z})) = \mathbb{C}[E_4, E_6, F_{10}, F_{12}],$$

(1.1)

where $E_4(\mathbb{Z})$, $E_6(\mathbb{Z})$ are the Eisenstein series and $F_{10}(\mathbb{Z})$ and $F_{12}(\mathbb{Z})$ are cusp forms (here below index denote weight). Moreover, $F_{10}(\mathbb{Z})$ is the square of a cusp form $\Delta_5(\mathbb{Z})$ of weight 5 with a multiplier system (see [F]). The function $\Delta_5(\mathbb{Z})$ is given by the product of all even theta-constants

$$\Delta_5(\mathbb{Z}) = \prod_{(a,b)} \vartheta_{a,b}(\mathbb{Z}),$$

(1.2)

where

$$\vartheta_{a,b}(\mathbb{Z}) = \sum_{l \in \mathbb{Z}^2} \exp(\pi i (Z[l + \frac{1}{2}a] + tbl)) \quad (Z[l] = tZl)$$

and the product is taken over all vectors $a, b \in (\mathbb{Z}/2\mathbb{Z})^2$ such that $t^ab \equiv 0 \mod 2$. (There are exactly ten different $(a, b)$.)

In this paper, it is more appropriate for our purpose to consider a Siegel modular form as a form with respect to the orthogonal group $SO(3, 2)$.

First, we construct some isomorphisms between symplectic and orthogonal groups.

We denote $L_4 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$. Any $\mathbb{Z}$-linear map $g : L_4 \rightarrow L_4$ induces a linear map $\wedge^2 g : L_4 \wedge L_4 \rightarrow L_4 \wedge L_4$.

The scalar product (the Pfaffian) on $L_4 \wedge L_4$ is defined by $u \wedge v = (u, v)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 L$, $x, y \in L_4 \wedge L_4$. This is an even unimodular integral symmetric bilinear form of the signature $(3, 3)$ on $L_4 \wedge L_4$. It is invariant with respect to the action of $SL(L_4)$ which was defined above.

We fix a skew-symmetric form $J_4$ on $L_4$ by the property:

$$J_4(x, y)e_1 \wedge e_2 \wedge e_3 \wedge e_4 = -x \wedge y \wedge (e_1 \wedge e_3 + e_2 \wedge e_4).$$

(Similarly, all elements of $L_4 \wedge L_4$ are identified with integral skew-symmetric bilinear forms on $L_4$.) Therefore

$$Sp_4(\mathbb{Z}) \cong \{ g : L_4 \rightarrow L_4 \mid (g \wedge g)(e_1 \wedge e_3 + e_2 \wedge e_4) = e_1 \wedge e_3 + e_2 \wedge e_4 \}.$$ 

Evidently, then $Sp_4(\mathbb{Z})$ keeps the lattice

$$L = (e_1 \wedge e_3 + e_2 \wedge e_4)^\perp \subset L_4 \wedge L_4, \quad L \cong U \oplus U \oplus < 2 >,$$

where $U$ is an unimodular integral hyperbolic plane, i.e. a lattice with the quadratic form

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$ 

The $< 2 >$ is a one dimensional $\mathbb{Z}$-lattice with the matrix (2). We fix a basis in $L$

$$\left(f_1 = e_1 \wedge e_4, f_2 = e_2 \wedge e_3, f_3 = e_2 \wedge e_4, f_4 = e_3 \wedge e_4, f_5 = e_1 \wedge e_3, f_6 = e_1 \wedge e_4 \right).$$
The real orthogonal group $O_R(L) = O(L \otimes \mathbb{R})$ acts on a domain

$$\mathcal{H}^{IV} = \{ Z \in \mathbb{P}(L \otimes \mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) < 0 \} = \mathcal{H}_+ \cup \overline{\mathcal{H}}_+,$$

where (using the basis above)

$$\mathcal{H}_+ = \{ Z =^t ((z_2^2 - z_1 z_3), z_3, z_2, z_1, 1) \cdot z_0 \in \mathcal{H}^{IV} \mid \text{Im}(z_1) > 0 \},$$

which is the classical homogeneous domain of type IV. The condition $(Z, \overline{Z}) < 0$ is equivalent to

$$y_1 y_3 - y_2^2 > 0, \quad \text{where } y_i = \text{Im}(z_i).$$

The domain $\mathcal{H}_+$ coincides with the Siegel upper half-plane $\mathbb{H}_2$ if we correspond to the point of $\mathcal{H}_+$ with the parameters $z_1, z_2, z_3$ above a symmetric matrix

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2.$$

The real orthogonal group $O_R(L)$ has four connected components. By $O_R^+(L)$ we denote the subgroup of $O_R(L)$ of index 2 consisting of elements, which leave $\mathcal{H}_+$ invariant. The kernel of the action of $O_R^+(L)$ on $\mathcal{H}_+$ is equal to $\pm E_5$. For our case (odd dimension of $L$), the group $O_R^+(L) = \pm E_5 SO_R^+(L)$, where $SO_R^+(L)$ is the subgroup of elements with real spin-norm equals to one.

We put $O^+(L) = O_R^+(L) \cap O(L)$ and $SO^+(L) = O(L) \cap SO_R^+(L)$.

One can easily find the images of generators of $Sp_4(\mathbb{Z})$:

$$\wedge^2 \left( \begin{array}{cc} 0 & E_2 \\ -E_2 & 0 \end{array} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\wedge^2 \left( \begin{array}{cc} E_2 & B \\ 0 & E_2 \end{array} \right) = \begin{pmatrix} 1 & -b_1 & 2b_2 & -b_3 & b_2^2 - b_1 b_2 \\ 0 & 1 & 0 & 0 & b_3 \\ 0 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$
Lemma 1.1. The correspondence $\wedge^2$ defines the isomorphism

$$\wedge^2 : Sp_4(\mathbb{Z})/\{\pm E_4\} \to SO^+(L) \cong O^+(L)/\{\pm E_5\}$$

which gives the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}_2 & \xrightarrow{g} & \mathbb{H}_2 \\
\downarrow & & \downarrow \\
\mathcal{H}_+ & \xrightarrow{g \wedge g} & \mathcal{H}_+
\end{array}$$

for the isomorphism $\mathbb{H}_2 \to \mathcal{H}_+$ above and any $g \in Sp_4(\mathbb{Z})$.

The modular form $\Delta_5(Z)$ has a non-trivial multiplier system. An exact formula for it was found by H. Maass in [M1]:

$$\Delta_5(M < Z >) = v(M) \det (CZ + D)^{5} \Delta_5(Z), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}),$$

(1.3)

where

$$v\left( \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \right) = 1, \quad v\left( \begin{pmatrix} E_2 & B \\ 0 & E_2 \end{pmatrix} \right) = (-1)^{b_1+b_2+b_3},$$

(1.4)

$$v\left( \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \right) = (-1)^{(1+a+d)(1+b+c)+ad}$$

(1.5)

with $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}), \ U^* = tU^{-1}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \in M_2(\mathbb{Z})$. According to (1.2) and (1.4), $\Delta_5(Z)$ has Fourier expansion

$$\Delta_5(Z) = \sum_{n, l, m \equiv 1 \mod 2, \ 4nm-l^2>0} f(n, l, m) \exp(\pi i (nz_1 + lz_2 + mz_3))$$

(1.6)

with integral $f(n, l, m)$. (We recall that $\Delta_5(Z)$ is a cusp form.) Moreover, it is not difficult to calculate that the Fourier coefficients

$$f(1, 1, 1) = 64 \quad \text{and} \quad 64 \mid f(n, l, m) \quad \text{for all} \quad n, l, m.$$  

(1.7)

To see the last property, one should remark that all Fourier coefficients of the theta-constants $\vartheta_{a,b}(Z)$ are multiple by 2 for $a \neq 0$. In §4 (see considerations after (4.8)) we prove the following identity for the formal power series

$$1 + \frac{1}{64} \sum_{t \in \mathbb{N}} f(1 + 2t, 1, 1) q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^9.$$  

(1.8)

In what follows we use the properties (1.3)–(1.8) of $\Delta_5(Z)$ for construction of a hyperbolic Kac-Moody Lie algebra.
§ 2. The hyperbolic lattice $M_{II}$, reflection group $W^{(2)}(M_{II})$ and Fourier expansion of $\Delta_5(Z)$

We fix a primitive hyperbolic sublattice $M_0 = \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_{-2} \cong U \oplus <2>$ of $L$. We extend an automorphism $\phi \in O(M_0)$ to be identical on $(M_0)^\perp$. This gives an embedding $O(M_0) \subset O(L)$. We are interested in automorphic properties of $\Delta_5(Z)$ with respect to $O(M_0)$. We remind that every primitive element $\delta \in M_0$ (or any other integral symmetric bilinear form) with $(\delta, \delta) > 0$ and $(\delta, \delta)|2(M_0, \delta)$ defines a reflection

$$s_\delta : x \rightarrow x - (2(x, \delta)/(\delta, \delta))\delta, \ x \in M_0.$$  

Here, $s_\delta(\delta) = -\delta$ and $s_\delta|\delta^\perp$ is identical. In particular, every $\delta \in M_0$ with $(\delta, \delta) = 2$ defines the reflection

$$s_\delta : x \rightarrow x - (x, \delta)\delta, \ x \in M_0.$$  

Proposition 2.1. We have:

$$\Delta_5(s_\delta(Z)) = -\Delta_5(Z)$$  

for reflections $s_\delta$ with respect to elements with square 2

$$\delta = \delta_1 = 2f_2 - f_3, \ \delta_2 = 2f_{-2} - f_3, \ \delta_3 = f_3 \in M_0$$  

and

$$\Delta_5(s_\delta(Z)) = \Delta_5(Z)$$  

for reflections $s_\delta$ with respect to elements

$$\delta = f_{-2} - f_2, \ f_2 - f_3, \ f_2 + f_3 \in M_0$$  

also with square 2.

Proof. To simplify notation, we denote

$$\tilde{U} = \wedge^2 \left( \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \right), \ U \in GL(2, \mathbb{Z}).$$

It is easy to see that $s_{f_{-2} - f_2} = -\left( \begin{small} 0 & 1 \\ 1 & 0 \end{small} \right)$, $s_{f_3} = -\left( \begin{small} 1 & 0 \\ 0 & -1 \end{small} \right)$, $s_{f_2 - f_3} = -\left( \begin{small} 1 & 1 \\ 0 & -1 \end{small} \right)$, therefore by Maass formula (1.3), we get the following relations

$$\Delta_5(s_{f_{-2} - f_2}(Z)) = \Delta_5(Z), \ \Delta_5(s_{f_3}(Z)) = -\Delta_5(Z), \ \Delta_5(s_{f_2 - f_3}(Z)) = -\Delta_5(Z).$$

We have $2f_2 - f_3 = s_{f_2 - f_3}(f_3); 2f_{-2} - f_3 = s_{f_{-2} - f_2}(2f_2 - f_3); f_2 + f_3 = s_{f_3}(f_2 - f_3)$. From above, it follows (by group’s arguments) that $\Delta_5(s_{2f_2 - f_3}(Z)) = -\Delta_5(Z), \ \Delta_5(s_{2f_{-2} - f_2}(Z)) = -\Delta_5(Z), \ \Delta_5(s_{f_2 + f_3}(Z)) = \Delta_5(Z)$.

It follows the statement.

Now we want to interpret Proposition 2.1 more invariantly, using the automorphism group $O(M_0)$. The lattice $M_0$ is hyperbolic, i.e. it has signature $(2, 1)$. Thus, $M_0$ defines a cone

$$V(M_0) = \{ \alpha \in M_0 \otimes \mathbb{R} \mid (\alpha, \alpha) < 0 \}.$$
which is the union of its two half-cones. (All general definitions and notations below hold for an arbitrary hyperbolic (i.e. with signature \((n, 1)\) lattice, and we shall use them later.) We choose one of this half-cones uniquely by the condition that

\[
\Omega(V^+(M_0)) := M_0 \otimes \mathbb{R} + iV^+(M_0) \subset \mathcal{H}_+.
\]

This means that if \(z = z_3 f_2 + z_2 f_3 + z_1 f_{-2} \in \Omega(V^+(M_0))\), then the point \(Z = t((z_2^2 - z_1 z_3), z_3, z_2, z_1, 1) \cdot z_0 \in \mathcal{H}_+\) in notations above. We denote by \(O^+(M_0)\) the subgroup of \(O(M_0)\) of index two which fixes the half-cone \(V^+(M_0)\). It is well-known that the group \(O^+(M_0)\) is discrete in the corresponding hyperbolic space \(\mathcal{L}^+(M_0) = V^+(M_0) / \mathbb{R}_{++}\) and has a fundamental domain of finite volume (below index ++ denote positive numbers, + denote non-negative numbers respectively).

Any reflection \(s_\delta \in O(M_0)\) with respect to an element \(\delta \in M_0\) with \((\delta, \delta) > 0\) is a reflection in the hyperplane

\[
\mathcal{H}_\delta = \{ \mathbb{R}_{++} x \in \mathcal{L}^+(M_0) \mid (x, \delta) = 0 \}
\]

of \(\mathcal{L}^+(M_0)\). This maps the half-space

\[
\mathcal{H}_\delta^+ = \{ \mathbb{R}_{++} x \in \mathcal{L}^+(M_0) \mid (x, \delta) \leq 0 \}
\]

to the opposite half-space \(\mathcal{H}_{-\delta}\) which are both bounded by the hyperplane \(\mathcal{H}_\delta\). Here \(\delta \in M_0\) is called orthogonal to \(\mathcal{H}_\delta\) and \(\mathcal{H}_\delta^+\). All reflections of \(M_0\) generate the reflection subgroup \(W(M_0) \subset O^+(M_0)\).

The hyperbolic lattice \(M_0\) of Proposition 2.1 is very special, and its automorphism group is well-known. Below we give results about this group which we need. These results one certainly can find in \([N3]\) where the reflection group of the lattice \(M_0\) and some its natural sublattices which are invariant with respect to its reflection subgroups of finite index were classified.

First, \(O^+(M_0) = W^{(2)}(M_0)\) where index 2 denote the subgroup generated by reflections in all elements of \(M_0\) with square 2. Similarly, we denote as \(\Delta^{(1)}(K)\) the set of all primitive elements \(\delta \in K\) of a lattice \(K\) with \((\delta, \delta) = t\) which define reflections \(s_\delta\) of \(K\), and as \(W^{(t)}(K)\) the group generated by reflections in all these elements. Thus, \(O^+(M_0)\) is generated by reflections in \(\Delta^{(2)}(M_0)\). An element \(\delta \in \Delta^{(2)}(M_0)\) and the corresponding reflection \(s_\delta\) has one of two types:

Type I: \((\delta, M_0) = \mathbb{Z}\).

Type II: \((\delta, M_0) = 2\mathbb{Z}\).

We introduce sublattices \(M_I\) and \(M_{II}\) which are generated by all elements \(\delta \in \Delta^{(2)}(M_0)\) of the type I and II respectively. We have

\[
M_I = \{ mf_2 + lf_3 + nf_{-2} \in M_0 \mid m + l + n \equiv 0 \mod 2 \}
\]

and

\[
M_{II} = \{ mf_2 + lf_3 + nf_{-2} \in M_0 \mid m \equiv n \equiv 0 \mod 2 \}.
\]

An element \(\delta \in \Delta^{(2)}(M_0)\) has the type I (respectively II) if and only if \(\delta \in M_I\) (respectively \(\delta \in M_{II}\)). It follows that the subgroup of \(O^+(M_0) = W^{(2)}(M_0)\) generated by all reflections \(s_\delta\) of the type I (respectively II) is equal to \(W^{(2)}(M_I)\) (respectively \(W^{(2)}(M_{II})\)). Obviously, sublattices \(M_I, M_{II}, M_{III}\) are \(W^{(2)}(M_0)\) invariant.
and both subgroups $W^{(2)}(M_I)$ and $W^{(2)}(M_{III})$ are normal in $W^{(2)}(M_0)$. The index $[W^{(2)}(M_0) : W^{(2)}(M_I)] = 2$ and $[W^{(2)}(M_0) : W^{(2)}(M_{III})] = 6$. Fundamental polyhedra $\mathcal{M} = \mathcal{M}_0, \mathcal{M}_I,$ and $\mathcal{M}_{III}$ for groups $W^{(2)}(M_0), W^{(2)}(M_I)$ and $W^{(2)}(M_{III})$ respectively are equal to $\mathcal{M} = \bigcap_{d \in P(\mathcal{M})_{pr}} \mathcal{H}_d^+$ where $P(\mathcal{M})_{pr}$ are minimal sets of primitive elements of $M_0$ with positive square (with square 2 for our case). They are called sets of (primitive) orthogonal vectors to polyhedra and are equal to

$$P(\mathcal{M})_{pr} = \{f_2 - f_3, f_2 - f_2, f_2 + f_3\};$$

$$P(\mathcal{M}_I)_{pr} = \{f_2 - f_3, f_2 - f_2, f_2 + f_3\};$$

and

$$P(\mathcal{M}_{III})_{pr} = \{\delta_1, \delta_2, \delta_3\}.$$

It follows that the groups $W^{(2)}(M_0), W^{(2)}(M_I), W^{(2)}(M_{III})$ are generated by reflections in $P(\mathcal{M})_{pr}, P(\mathcal{M}_I)_{pr}$ and $P(\mathcal{M}_{III})_{pr}$ respectively. We denote

$$A(P(\mathcal{M})) = \{g \in O^+(M_0) \mid g(P(\mathcal{M})) = P(\mathcal{M})\}$$

the “group of symmetries” of a fundamental polyhedron $\mathcal{M}$ and its set $P(\mathcal{M})$ of orthogonal vectors. The group $A(P(\mathcal{M})_{pr})$ is trivial, $A(P(\mathcal{M}_I)_{pr})$ has order two and is generated by $s_{f_3}$, the group $A(P(\mathcal{M}_{III})_{pr})$ is the group of symmetries of the right triangle (i.e. it is $S_3$). It is generated by $s_{f_2 - f_3}, s_{f_2 - f_2}$. We can write down the $O^+(M_0)$ as a semi-direct products:

$$O^+(M_0) = W^{(2)}(M_0) = W^{(2)}(M_I) \rtimes A(P(\mathcal{M}_I)_{pr}) = W^{(2)}(M_{III}) \rtimes A(P(\mathcal{M}_{III})_{pr}).$$

Using these considerations, we can reformulate the invariance properties of $\Delta_5(Z)$ of Proposition 2.1 with respect to the group $O^+(M_0)$ as follows:

**Proposition 2.2.** We have the following invariance properties of $\Delta_5(Z)$ with respect to the group $O^+(M_0) = W^{(2)}(M_0) = W^{(2)}(M_I) \rtimes A(P(\mathcal{M}_I)_{pr}) = W^{(2)}(M_{III}) \rtimes A(P(\mathcal{M}_{III})_{pr})$:

(a) For a reflection $s_\delta \in O^+(M_0)$ with $(\delta, \delta) = 2$ we have

$$\Delta_5(s_\delta(Z)) = \begin{cases} \Delta_5(s_\delta(Z)), & \text{if } \delta \text{ has the type I (i.e } \delta \in M_I), \\ -\Delta_5(s_\delta(Z)), & \text{if } \delta \text{ has the type II (i.e } \delta \in M_{III}). \end{cases}$$

(b) $\Delta_5(w.a(Z)) = \det(a)\Delta_5(Z)$ for $w \in W^{(2)}(M_I), a \in A(P(\mathcal{M}_I)_{pr})$;

in particular,

$$\Delta_5(g(Z)) = \Delta_5(Z) \text{ for } g \in O^+(M_0)$$

if and only if $g \in W^{(2)}(M_I)$.

(c) $\Delta_5(w.a(Z)) = \det(w)\Delta_5(Z)$ for $w \in W^{(2)}(M_{III}), a \in A(P(\mathcal{M}_{III})_{pr})$.

Because of the property (c), the reflection group $W^{(2)}(M_{III})$ will be important for us. We fix the fundamental polyhedron $\mathcal{M}_{III}$ above with the set of orthogonal vectors $P(\mathcal{M}_{III}) = P(\mathcal{M}_{III})_{pr} = \{\delta_1, \delta_2, \delta_3\}$. These elements have the Gram matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$  (2.1)
Let us consider the cone \( \mathbb{R}_+ Q_+ = \mathbb{R}_+ \delta_1 + \mathbb{R}_+ \delta_2 + \mathbb{R}_+ \delta_3 \) and the corresponding dual cone \( \mathbb{R}_+ Q_+^* = \{ x \in M_0 \otimes \mathbb{R} \mid (x, \delta_i) \leq 0 \} \). We know that \( \mathcal{M}_{II} \subset \mathcal{L}^+(M_0) \) has finite volume. Since the cone \( \overline{V^+(M_0)} = V^+(M_0)^* \) is self-dual, this is equivalent to the sequence of embeddings of cones
\[
\mathbb{R}_+ Q_+^* \subset \overline{V^+(M_0)} \subset \mathbb{R}_+ Q_+.
\]

This property (equivalent to finiteness of volume of \( \mathcal{M}_{II} \)) is very important. This gives a hope that all our further considerations may be successful (see [N5]). We will use this property many times further.

Another very important property of the group \( W^{(2)}(M_{II}) \) and the set \( P(\mathcal{M}_{II}) \) is that they have a lattice Weyl vector \( \rho \). (See general results on this subject and an explanation why this is important in [N5].) This is the element \( \rho \in M_{II}^* \) with the property
\[
(\rho, \delta_i) = -(\delta_i, \delta_i)/2 = -1 \text{ for any } \delta_i \in P(\mathcal{M}_{II}).
\]
By (2.1),
\[
\rho = (1/2)\delta_1 + (1/2)\delta_2 + (1/2)\delta_3 = f_2 - (1/2)f_3 + f_{-2}.
\]
Evidently, \( \rho \in \mathbb{R}_+ Q_+^* \), and by (2.2), \( \rho \in V^+(M_0) = V^+(M_{II}) \). Further we will work with the lattice \( M_{II} \) identifying \( M_0 \otimes \mathbb{Q} = M_{II} \otimes \mathbb{Q} \). Using the group \( W^{(2)}(M_{II}) \), we can rewrite \( \Delta_5(Z) \) as follows

\[\textbf{Theorem 2.3.} \ (a) \ \frac{1}{64}\Delta_5(Z) = \sum_{w \in W^{(2)}(M_{II})} \det(w) \left( \exp(-\pi i(w(\rho), z)) - \sum_{a \in M_{II} \cap \mathbb{R}_+ M_{II}} m(a) \exp(-\pi i(w(a), z)) \right),\]

where \( z = (f_2 - 2z_3) + (f_2 f_3 + z_1 f_{-2}) \in M_0 \otimes \mathbb{R} + iV^+(M_0) = M_{II} \otimes \mathbb{R} + iV^+(M_{II}) \) and \( m(a) \in \mathbb{Z} \).

(b) For any primitive \( a_0 \in M_{II} \cap \mathbb{R}_+ M_{II} \) with \( (a_0, a_0) = 0 \), we have the identity
\[
1 - \sum_{t \in \mathbb{N}} m(ta_0)q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^{\tau(ka_0)}
\]
of power series of one variable \( q \), where all \( \tau(ka_0) \in \mathbb{Z} \). More exactly,
\[
\tau(a) = 9 \text{ for any } a \in M_{II} \cap \mathbb{R}_+ M_{II} \text{ with } (a, a) = 0.
\]

\[\textbf{Proof.} \] Statement (a). The lattice \( M_{II}^* = \mathbb{Z}(f_2/2) + \mathbb{Z}(f_3/2) + \mathbb{Z}(f_{-2}/2) = \frac{1}{2} M_0 \). Thus, for \( n, l, m \in \mathbb{Z} \), we can rewrite
\[
(1/64)f(n, l, m) \exp(\pi i(nz_1 + lz_2 + mz_3)) = \sum_{a \in \mathbb{R}_+ M_{II}} m(a) \exp(\pi i(a, z)) = \sum_{a \in \mathbb{R}_+ M_{II}} m(a) \exp(\pi i(\rho + a, z)),
\]
where \( a = (n - 1)f_2 - (l - 1)f_3/2 + (m - 1)f_{-2} \in M_{II}^* = \mathbb{Z} M_{II} \) and \( m(a) = -(1/64)f(n, l, m) \). By (1.6) and (1.7), we have that \( \rho + a \in V^+(M_0) \), \( m(a) \in \mathbb{Z} \), and \( m(0) = -1 \). By Proposition 2.1, (c), we get
\[
\frac{1}{64}\Delta_5(Z) = \sum_{w \in W^{(2)}(M_{II})} \det(w) \left( - \sum_{a \in \mathbb{R}_+ M_{II}} m(a) \exp(-\pi i(w(\rho + a), z)) \right).
\]
Since for this sum $\rho + a \in M_{II}^* \cap \mathbb{R}_+ M_{II}$, we have that $(\rho + a, \delta_i) \leq 0$, $i = 1, 2, 3$. If $(\rho + a, \delta_i) = 0$, then the corresponding Fourier coefficient $m(a) = 0$, since $\Delta_5(Z)$ is anti-invariant with respect to $s_i$. Thus, we can suppose that $(\rho + a, \delta_i) < 0$, $i = 1, 2, 3$. By definition of $\rho$, we then get that $(a, \delta_i) \leq 0$. It follows that $a \in \mathbb{R}_+ Q^*_+$. By (2.2), $a \in \mathbb{R}_+ Q^*_+ \cap \overline{V(M_0)} = \mathbb{R}_+ M_{II}$. It follows that $a \in \mathbb{R}_+ M_{II}$ if $a \neq 0$. If $a = 0$, we have $m(a) = -1$. By the congruences $m, n, l \equiv 1 \mod 2$ of (1.6), we have $a \in 2M_0^* = M_{II}$.

Statement (b). Primitive elements $a_0 \in M_{II} \cap \mathbb{R}_+ M_{II}$ with $(a_0, a_0) = 0$ correspond to three infinite vertices of $\mathcal{M}_{II}$. By Statement (c) of Proposition 2.2, the group $A(P(M_{II}))$ is transitive on these three vertices and the corresponding primitive elements $a_0$ and preserves the Fourier expansion (a). Thus, it is sufficient to prove the identity (b) for $a_0 = 2f_2$ which is one of these three primitive elements of $M_{II} \cap \mathbb{R}_+ M_{II}$ (all these three primitive elements are $2f_2, 2f_2, 2f_2 - 2f_3 + 2f_2$). For $a_0 = 2f_2$, Statement (b) is equivalent to (1.8). It follows Statement (b).

§ 3. The Kac–Moody algebra $\mathfrak{g}(M_{II}, P(M_{II}))$ and its $\Delta_5(Z)$-correction $\mathfrak{g} = \mathfrak{g}(M_{II}, s\Delta)$ where $s\Delta = P(M_{II}) \cup s\Delta^{im}$.

The matrix (2.1) is the Gram matrix of elements $P(M_{II})$, and we denote this Gram matrix as $G(P(M_{II}))$. This is a symmetric generalized Cartan matrix (i.e. it is integral, has only 2 on the diagonal and only non-positive integers out of the diagonal). This matrix defines the corresponding Kac–Moody algebra $\mathfrak{g}(M_{II}, P(M_{II})) =: \mathfrak{g}(G(P(M_{II}))$ (see [Ka1] and [Mo]). We want to “correct” this algebra using Theorem 2.3. Some examples of this type constructions were first found by R. Borcherds (see [Bo3], [Bo5], [Bo6], [Bo7]). Its formulation in an appropriate general setting see in [N5]. This construction uses generalized Kac–Moody algebras which were introduced and studied by R. Borcherds in [Bo1] (see also [Bo2]). For our situation we need the corresponding superalgebra analog of this construction. It is a combination of Borcherd’s construction and of V. Kac ([Ka2], [Ka3]). We give this construction for our case below and refer to § 6, Appendix for proofs.

Using coefficients $m(a)$ and $\tau(a)$ of Theorem 2.3, we introduce (where the index $s$ means “simple”) the sets

$$s\Delta^{im}_0 = \{m(a)a \mid a \in M_{II} \cap \mathbb{R}_+ M_{II}, (a, a) < 0 \text{ and } m(a) > 0\} \cup \\
\cup \{\tau(a)a \mid a \in M_{II} \cap \mathbb{R}_+ M_{II}, (a, a) = 0 \text{ and } \tau(a) > 0\}$$

where $ka$ for $k \in \mathbb{N}$ means that we repeat $a$ exactly $k$ times, and

$$s\Delta^{im}_T = \{m(a)a \mid a \in M_{II} \cap \mathbb{R}_+ M_{II}, (a, a) < 0 \text{ and } m(a) < 0\} \cup \\
\cup \{\tau(a)a \mid a \in M_{II} \cap \mathbb{R}_+ M_{II}, (a, a) = 0 \text{ and } \tau(a) < 0\}$$

where $ka$ for $-k \in \mathbb{N}$ means that we repeat the element $a$ exactly $-k$ times. The minus sign of $k$ here denote that all this $-k$ elements $a$ are considered to be odd. The elements of $s\Delta^{im}_0$ and $s\Delta^{im}_T$ are considered to be even and odd respectively. They are called even and odd imaginary simple roots respectively. We denote

$$\Delta^{im} = \Delta^{im}_0 + \Delta^{im}_T.$$
Elements of $s\Delta^\text{im}$ are called \textit{imaginary simple roots}. To unify notations we denote $s\Delta^\text{re} = s\Delta^\text{re} = P(M_{II})$. Elements of $s\Delta^\text{re}$ are called \textit{real simple roots}. They are considered to be even (for our situation). We denote as $s\Delta = s\Delta^\text{re} \cup s\Delta^\text{im}$ the whole set of simple roots. Thus, simple roots are either real even or imaginary even or imaginary odd. By the construction, elements of $s\Delta$ give the corresponding elements of $M_{II} \subset M_{II} \otimes \mathbb{R}$. For the further construction of the corresponding Kac–Moody superalgebra we need that $(r, r) > 0$ if $r \in s\Delta^\text{re}$, $(r, r) \leq 0$ if $r \in s\Delta^\text{im}$, and $(r, r') \leq 0$ for all different $r, r' \in s\Delta$. This is true by the construction. Moreover, for $r, r' \in s\Delta$ we need

$$2(r, r')/(r, r) \in \mathbb{Z} \quad \text{if } (r, r) > 0.$$  

This is valid because here $(r, r) = 2$ and $r' \in M_{II}$ where $M_{II}$ is generated by elements of $s\Delta^\text{re}$ (which are $\delta_1, \delta_2, \delta_3$). The generalized Kac–Moody superalgebra without odd real simple roots $\mathfrak{g} = \mathfrak{g}''(M_{II}, s\Delta)$ (which is constructing) is a Lie superalgebra generated by $h_r, e_r, f_r$ where $r \in s\Delta$. Here all $h_r$ are even, $e_r, f_r$ are even (respectively odd) if $r$ is even (respectively odd). These elements have the following defining relations:

(1) The map $r \mapsto h_r$ for $r \in s\Delta$ gives an embedding of $M_{II} \otimes \mathbb{R}$ to $\mathfrak{g}''(M_{II}, s\Delta)$ as an abelian subalgebra (it is even since all $h_r$ are even). In particular, all elements $h_r$ commute.
(2) $[h_r, e_{r'}] = (r, r')e_{r'}$, and $[h_r, f_{r'}] = -(r, r')f_{r'}$.
(3) $[e_r, f_{r'}] = h_r$ if $r = r'$, and is 0 if $r \neq r'$.
(4) $(\text{ad } e_r)^{1-2(r, r')/2(r,r)}e_{r'} = (\text{ad } f_r)^{1-2(r, r')/2(r,r)}f_{r'} = 0$ if $r \neq r'$ and $(r, r) > 0$ (equivalently, $r \in s\Delta^\text{re}$).
(5) If $(r, r') = 0$, then $[e_r, e_{r'}] = [f_r, f_{r'}] = 0$.

The superalgebra $\mathfrak{g} = \mathfrak{g}''(M_{II}, s\Delta)$ is graded by $M_{II}$ as follows. Let

$$\tilde{Q}_+ = \sum_{\alpha \in s\Delta} \mathbb{Z}_+ \alpha \subset M_{II}$$

be the integral cone (semi-group) generated by all simple roots. (For our case $\tilde{Q}_+$ coincides with the integral cone $Q_+ = \mathbb{Z}_+ s\Delta^\text{re}$ generated by real simple roots $s\Delta^\text{re} = \{\delta_1, \delta_2, \delta_3\}$, compare with (2.2).) We have

$$\mathfrak{g} = \left( \bigoplus_{\alpha \in Q_+} \mathfrak{g}_\alpha \right) \bigoplus (M_{II} \otimes \mathbb{R}) \bigoplus \left( \bigoplus_{\alpha \in -\tilde{Q}_+} \mathfrak{g}_\alpha \right)$$

where $e_r$ and $f_r$ have degree $r \in \tilde{Q}_+$ and $-r \in -\tilde{Q}_+$ respectively, $r \in s\Delta$; and $\mathfrak{g}_0 = M_{II} \otimes \mathbb{R}$. The $0 \neq \alpha \in \pm Q_+$ is called a \textit{root} if $\mathfrak{g}_\alpha$ is non-zero. Let $\Delta$ be the set of all roots and $\Delta_\pm = \Delta \cap \pm Q_+$. For a root $\alpha \in \Delta$ we denote $\text{mult}_{\alpha} = \dim \mathfrak{g}_{\alpha, \bar{\alpha}}$; $\text{mult}_\alpha = -\dim \mathfrak{g}_{\alpha, \bar{\alpha}}$ and

$$\text{mult}_\alpha = \text{mult}_\alpha + \text{mult}_\alpha = \dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_\alpha,$$  

where $\sum_{\alpha \in \Delta_+} \text{mult}_\alpha = \dim M_{II}$.
By our construction and Statement 6.8 in §6, it follows the denominator identity for $g$: $\Phi(z) =$:

$$
\sum_{w \in W(2)(M_{II})} \det(w) \left( \exp(-2\pi i(w(\rho), z)) - \sum_{a \in M_{II} \cap \mathbb{R}^+_+ M_{II}} m(a) \exp(-2\pi i(w(\rho + a), z)) \right)
= \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i(\alpha, z)))^{\text{mult } \alpha},
$$

where $z \in \Omega(M_{II}) = M_{II} \otimes \mathbb{R} + iV^+(M_{II})$. The function $\Phi(z)$ is called the denominator function. Thus, using these considerations and Theorem 2.3, we get

**Proposition 3.1.** We have the following equality between the $\Delta_5(Z)$ and the denominator function $\Phi(z)$ of the correcting generalized Kac–Moody superalgebra $g = g''(M_{II}, s\Delta)$:

$$
\frac{1}{64} \Delta_5(2Z) = \Phi(z).
$$

Thus, the denominator function $\Phi(z)$ of the correcting generalized Kac–Moody superalgebra $g''(M_{II}, s\Delta)$ is a Siegel modular form. On the other hand, the classical Siegel modular form $\Delta_5(Z)$ is the denominator function of a generalized Kac–Moody superalgebra.

**Remark 3.2.** The denominator function $\Phi(z)$ is defined in the complexified cone $\Omega(V^+(M_{II}))$ (we remark that to have this property we again need (2.2) and also existence of a lattice Weyl vector $\rho \in M_{II} \otimes \mathbb{Q}$). This cone does not have a canonical embedding as a cusp to a IV type domain. This embedding is defined up to changing $z$ to $tz$ where $t \in \mathbb{N}$. This is an explanation of the coefficient 2 for the equality of Proposition 3.1. See [N5] for the appropriate general setting.

The equality (3.2) shows that there exists a product formula for $\Delta_5(Z)$

$$
\frac{1}{64} \Delta_5(Z) =
\sum_{w \in W(2)(M_{II})} \det(w) \left( \exp(-\pi i(w(\rho), z)) - \sum_{a \in M_{II} \cap \mathbb{R}^+_+ M_{II}} m(a) \exp(-\pi i(w(\rho + a), z)) \right)
= \exp(-\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-\pi i(\alpha, z)))^{\text{mult } \alpha}.
$$

On the other hand, using automorphic properties of $\Delta_5(Z)$, we may have a hope to calculate this product formula and calculate multiplicities $\text{mult } \alpha$ for products (3.2) and (3.3). We do it in §4. We will show that for $\alpha = 2nf_2 - lf_3 + 2mf_{-2} \in M_{II}$ (i.e. $n, m, l \in \mathbb{Z}$),

$$
\text{mult } \alpha = f(nm, l)
$$

where $f(k_1, k_2)$ are Fourier coefficients of the weak Jacobi function $\phi_{0,1}(z_1, z_2)$ of weight 0 and index 1. See (4.11) and Theorem 4.1.
§ 4. The product formula for $\Delta_5(Z)$ and root spaces multiplicities
mult $\alpha$ for by $\Delta_5(Z)$ corrected Kac–Moody algebra $\mathfrak{g}$

The modular form $\Delta_5(Z)$ has the following Fourier-Jacobi expansion

$$
\Delta_5\left(\begin{array}{ccc}
  z_1 \\
  z_2 \\
  z_3 \\
\end{array}\right) = \sum_{m>0 \atop m\equiv 1 \mod 2} \phi_{5,m}(z_1, z_2) \exp(\pi i m z_3).
$$

(4.1)

This is the Fourier expansion along the one dimensional cusp of domain $Sp_4(\mathbb{Z}) \setminus \mathbb{H}_2$ corresponding to the following maximal parabolic subgroup

$$
\Gamma_\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\
  * & * & * & * \\
  * & 0 & * & * \\
  0 & 0 & 0 & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}.
$$

The group $\Gamma^J = \Gamma_\infty / \{ \pm E_4 \} \cong SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ is called Jacobi group. We realize $SL_2(\mathbb{Z})$ as a subgroup of $Sp_4(\mathbb{Z})$ through the embedding

$$
* : \begin{pmatrix} a & b \\
  c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\
  c & d \end{pmatrix}^* = \begin{pmatrix} a & 0 & b & 0 \\
  0 & 1 & 0 & 0 \\
  c & 0 & d & 0 \\
  0 & 0 & 0 & 1 \end{pmatrix}.
$$

(4.2)

$H(\mathbb{Z})$ is the integral Heisenberg group

$$
H(\mathbb{Z}) \cong \left\{ h(p, q, r) = \begin{pmatrix} 1 & 0 & 0 & q \\
  p & 1 & q & r \\
  0 & 0 & 1 & -p \\
  0 & 0 & 0 & 1 \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}.
$$

(4.3)

The restriction of multiplier system $v$ on $\Gamma_\infty$ define a character

$$
v_\infty : SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z}) \to \{ \pm 1 \}.
$$

The functions $\phi_{5,m}(z_1, z_2)$ are Jacobi forms of half-integral index. They have the following invariant properties, which one can take as a definition of Jacobi forms of index $\frac{m}{2}$:

$$
\phi_{5,m}(z_1, z_2) \exp(\pi i m z_3)|_5 h(p, q, r) = (-1)^{p+q+pq+r} \phi_{5,m}(z_1, z_2) \exp(\pi i m z_3)
$$

(4.4)

$$
\phi_{5,m}(z_1, z_2) \exp(\pi i m z_3)|_5 g^* = v(g^*) \phi_{5,m}(z_1, z_2) \exp(\pi i m z_3)
$$

(4.5)

for $h(p, q, r) \in H(\mathbb{Z})$ and $g \in SL_2(\mathbb{Z})$. We set

$$
F|_k M(\mathbb{Z}) := \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})
$$

for any $M = \begin{pmatrix} A & B \\
  C & D \end{pmatrix} \in Sp_4(\mathbb{R})$ and any function $F : \mathbb{H}_2 \to \mathbb{C}$. One can rewrite the last two equations in the following form

$$
\phi_{5,m}(\frac{az_1 + b}{cz_1 + d}, \frac{z_2}{cz_1 + d}) = v(g^*)(cz_1 + d)^5 \exp(\pi i \frac{cmz_2^2}{cz_1 + d}) \phi_{5,m}(z_1, z_2),
$$

$$
\phi_{5,m}(z_1, z_2 + pq) = (-1)^{p+q+pq(m+1)} \exp(-\pi i m z_2)(z_1, z_2).
$$
where \( p, q \in \mathbb{Z} \) and \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \). One can easy check that
\[
v\left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right)^* = (-1)^b \quad \text{and} \quad v\left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^* = -1,
\]
since
\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^* = \left[ \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \begin{pmatrix} E_2 & B_1 \\ 0 & E_2 \end{pmatrix} \right]^3 \begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The first Fourier-Jacobi coefficient \( \phi_{5,1}(z_1, z_2) \) is connected with the denominator formula for a generalized affine Kac–Moody algebra. In order to get this connection let us take the following Jacobi theta-series
\[
\vartheta_{11}(z_1, z_2) = \sum_{n \in \mathbb{Z}} (-1)^n \exp \left( \frac{\pi i}{4} (2n + 1)^2 z_1 + \pi i (2n + 1)z_2 \right).
\]

We recall a well known variant of the Jacobi triple formula:
\[
\prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2} m(m-1)} r^m.
\]
From that the product formula for \( \vartheta_{11} \) follows
\[
\vartheta_{11}(z_1, z_2) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n). \quad (4.6)
\]
It is easy to see from definition that \( \vartheta_{11}(z_1, z_2) \) and \( \phi_{5,1}(z_1, z_2) \) have the same functional equation with respect to \( h(p, q; r) \):
\[
\vartheta_{11}(z_1, z_2 + p z_1 + q) = (-1)^{p+q} \exp (-\pi i (p^2 z_1 + 2pz_2)) \vartheta_{11}(z_1, z_2).
\]
Moreover \( \vartheta_{11} \) satisfies the following transformations formulae with respect to generators \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) of \( SL_2(\mathbb{Z}) \):
\[
\vartheta_{11}(\frac{1}{z_1}, z_2) = -i \sqrt{-iz_1} \exp (\pi i \frac{z_2^2}{z_1}) \vartheta_{11}(z_1, z_2),
\]
\[
\vartheta_{11}(z_1 + 1, z_2) = \exp \left( \frac{\pi i}{4} \right) \vartheta_{11}(z_1, z_2).
\]
The Dedekind eta-function
\[
\eta(\tau) = \exp \left( \frac{\pi i \tau}{12} \right) \prod_{n \geq 1} (1 - \exp (2\pi i n\tau))
\]
satisfies the similar functional equations
\[
\eta(-\frac{1}{\tau}) = -i \sqrt{-i\tau} \eta(\tau), \quad \eta(\tau + 1) = \exp \left( \frac{\pi i}{2\tau} \right) \eta(\tau).
\]
Thus \( \phi_{5,1}(z_1, z_2) \) and

\[
\psi_{5,\frac{1}{2}}(z_1, z_2) = \eta(z_1)^9 \sigma_{11}(z_1, z_2)
\]

are Jacobi cusp forms of index \( \frac{1}{2} \) with the same character \( v_\infty : \Gamma^J \to \{ \pm 1 \} \). The squares of these Jacobi forms are Jacobi cusp forms of weight 10 and index 1. Up to a constant there is only one such a form. This is the first Fourier-Jacobi coefficient \( \phi_{10,1}(z_1, z_2) \) of \( F_{10}(Z) \) from (1.1) (see [EZ]). Comparing a pair of their Fourier coefficients we obtain

\[
\frac{1}{12} \phi_{5,1}(z_1, z_2) = \psi_{5,\frac{1}{2}}(z_1, z_2) = -q^{1/2}r^{-1/2} \prod_{n \geq 1} \frac{(1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n)^{10},}
\]

where

\[
q = \exp (2\pi i z_1), \quad r = \exp (2\pi i z_2).
\]

If we write down the right hand side as a formal series in \( r \) then the \( q \)-sum in (1.8) is the coefficient of \( r^{1/2} \). Using the Jacobi triple formula we get (1.8).

Let \( \phi_{12,1}(z_1, z_2) \) be a Jacobi cusp form of weight 12 and index one (the first Fourier-Jacobi coefficient of \( F_{12}(Z) \) from (1.1)). It is known that up to a constant this is the only one such a form. Its Fourier coefficient can be calculated according to the following formula (see [EZ])

\[
\phi_{12,1}(z_1, z_2) = \frac{1}{144} \left( E_4^2(z_1) E_{4,1}(z_1, z_2) - E_6(z_1) E_{6,1}(z_1, z_2) \right),
\]

where

\[
E_4(z_1) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \quad E_6(z_1) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n
\]

are Eisenstein series for \( SL_2(\mathbb{Z}) \) and \( E_{k,1}(z_1, z_2) \) is Eisenstein series with integral coefficients of weight \( k \) and index one for the Jacobi group

\[
E_{k,1}(z_1, z_2) = \sum_{n, l \in \mathbb{Z}} \frac{H(k - 1, 4n - r^2)}{\zeta(3 - 2k)} \exp (2\pi i (n z_1 + l z_2)),
\]

where \( H(k, N) = L_N(1 - k) \) are H. Cohen’s numbers (see [EZ]). The form \( \phi_{12,1} \) has integral and coprime coefficients:

\[
\phi_{12,1}(z_1, z_2) = (r^{-1} + 10 + r)q + (10r^{-2} - 88r^{-1} - 132 - 88r + 10r^{-2})q^2 + \ldots .
\]

Let us introduce a function with integral Fourier coefficients

\[
\phi_{0,1}(z_1, z_2) := \phi_{12,1}(z_1, z_2)/\Delta_{12}(z_1) = \sum_{n \geq 0, l \in \mathbb{Z}} f(n, l) \exp (2\pi i (n z_1 + l z_2)),
\]

where

\[
\Delta_{12}(z_1) = q \prod_{n \geq 1} (1 - q^n)^{24}
\]

is the \( SL_2(\mathbb{Z}) \)-cusp form of weight 12. \( \phi_{0,1}(z_1, z_2) \) is a weak Jacobi function of weight 0 and index 1. This function satisfies the same functional equation as holomorphic Jacobi forms and has nonzero Fourier coefficients only with indexes \( (n, l) \in \mathbb{Z} \) such that \( n \geq 0 \) (since \( \phi_{12,1} \) is a cusp form) and \( 4n - l^2 \geq -1 \). The weight is even, thus \( f(n, l) = f(n, -l) \) and \( f(n, l) \) depends only on \( 4n - l^2 \). Moreover

\[
\phi_{0,1}(z_1, z_2) = (r^{-1} + 10 + r) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^{-2})q^2 + \ldots .
\]

All facts from the theory of Jacobi forms mentioned above one can find in [EZ].
Theorem 4.1. The following formula is valid:

\[
\frac{1}{64} \Delta_5(Z) = \exp(\pi i (z_1 + z_2 + z_3)) \prod_{\substack{n, l, m \in \mathbb{Z} \\
(n, l, m) > 0}} (1 - \exp(2\pi i (nz_1 + lz_2 + mz_3)))^{f(nm, l)},
\]

where \((n, l, m) > 0\) means that \(n \geq 0, m \geq 0, l\) is an arbitrary integral if \(n > 0\) or \(m > 0\) and \(l < 0\) if \(n = m = 0\).

Remark 4.2. The condition \((n, l, m) > 0\) means that the product is taken over the set of positive roots \(\Delta_+\) (see (3.2)).

Proof of Theorem. The holomorphic Jacobi form has the following Fourier expansion

\[
\phi_{12,1}(z_1, z_2) = \sum_{\mu \mod 2} \sum_{l \equiv \mu \mod 2} \sum_{4n-l^2 > 0} c_\mu(4n-l^2) \exp(2\pi i (nz_1 + lz_2))
\]

\[
= h_0(z_1) \vartheta_{1,0}(z_1, z_2) + h_1(z_1) \vartheta_{1,1}(z_1, z_2),
\]

where \(h_\mu(z_1) = \sum_{m > 0} c_\mu(m) \exp\left(\frac{2\pi i}{2} mz_1\right)\) are modular forms of weight \(\frac{23}{2}\) and \(\vartheta_{1,\mu}(z_1, z_2)\) are the standard Jacobi theta-functions. Thus the Fourier coefficients of \(\phi_{0,1}(z_1, z_2)\) have the following asymptotic

\[
f(n, l) = O(e^{\sqrt{4n-l^2}}).
\]

Using this estimate, (2.2) and arguments of [Ka1, § 10.6], we prove that the product of Theorem 4.1 absolutely converges on any neighborhood of the zero-dimensional cusp \(\infty\) of \(SP_4(Z)\).

Let us denote the product in the formula of Theorem 4.1 by \(P(z_1, z_2, z_3)\) and decompose it in two factors corresponding to the cases \(m = 0\) and \(m > 0\)

\[
P(z_1, z_2, z_3) = \exp(\pi i (z_1 + z_2 + z_3)) \prod_{\substack{n > 0, l \in \mathbb{Z} \\
n=0, l<0}} (1 - \exp(2\pi i (nz_1 + lz_2)))^{f(0, l)}
\]

\[
\times \prod_{\substack{n > 0, m > 0, l \in \mathbb{Z} \\
}} (1 - \exp(2\pi i (nz_1 + lz_2 + mz_3)))^{f(nm, l)},
\]

where \(f(0, 0) = 10, f(0, -1) = 1\) and \(f(0, l) = 0\) if \(l < -1\).

Let us introduce the following Hecke operators

\[
T_- (m) = \sum_{\alpha, \beta = m \mod 2} \Gamma_\alpha \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma_\beta = \sum_{ad = m, b \mod d} \Gamma_{ad} \begin{pmatrix} a & 0 & b & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

which are the “minus”-embedding of the usual Hecke operators \(T(m)\) for \(GL_2(\mathbb{Z})\) (see [G3]). By the definition

\[
\begin{pmatrix} a & 0 & b & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} < \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} > = \begin{pmatrix} a z_1 + b & a z_2 \\ a z_2 & m z_3 \end{pmatrix},
\]
therefore for each Jacobi form \( \tilde{\phi}(Z) = \phi(z_1, z_2) \exp(2\pi i tz_3) \) of weight \( k \) and index \( t \) \((t \in \mathbb{Q})\) the function

\[
(\tilde{\phi} |_k T_-(m))(Z) = m^{2k-3} \sum_{\substack{ad = m \\ b \mod d}} d^{-k} \phi\left( \frac{a z_1 + b}{d}, a z_2 \right) \exp(2\pi i mtz_3)
\]

is a Jacobi form of index \( mt \). (We put a normalizing factor \( m^{2k-3} \) in the definition of \(|_k T_-(m)\) like in \([G3]\)).

Let us calculate the Fourier expansion of the logarithm of the second factor from (4.12):

\[
\log \left( \prod_{n \geq 0, m > 0, l \in \mathbb{Z}} ... \right) = - \sum_{n \geq 0, m > 0} f(nm, l) \sum_{e \geq 1} \frac{1}{e} \exp \left( 2\pi i e (nz_1 + lz_2 + mz_3) \right) \]

\[
= - \sum_{a \geq 0, c > 0} \sum_{t \mid (a, b, c)} t^{-1} f \left( \frac{ac}{t^2}, \frac{b}{t} \right) \exp \left( 2\pi i (az_1 + bz_2 + cz_3) \right).
\]

The last sum can be written as a sum of Jacobi forms using the operators \( T_-(m) \).

By definition

\[
m^2 \left( \tilde{\phi}_{0,1} |_0 T_-(m) \right)(Z) = \sum_{ad = m} m^{-1} \sum_{n, l \bmod d} f(n, l) \exp \left( 2\pi i \left( n \frac{a z_1 + b}{d} + laz_2 + mz_3 \right) \right)
\]

\[
= \sum_{ad = m} a^{-1} \sum_{n, l} f(dn, l) \exp \left( 2\pi i (anz_1 + alz_2 + adz_3) \right).
\]

From that one easily gets

\[
\log \left( \prod_{n \geq 0, m > 0, l \in \mathbb{Z}} ... \right) = - \sum_{m \geq 1} m^2 \left( \tilde{\phi}_{0,1} |_0 T_-(m) \right)(Z).
\]

This expansion shows us that the second factor in (4.12) is invariant with respect to the action of the elements of \( \Gamma_\infty \).

The first factor in (4.12) is equal to \( \tilde{\psi}_{5,4}(Z) \) (see (4.8)). Therefore \( P(z_1, z_2, z_3) \) transforms like a \( \Gamma^J \)-modular form of weight 5 with a character \( \nu_\infty : \Gamma^J \to \{\pm 1\} \).

There is the only factor in \( P(z_1, z_2, z_3) \) with \( n = m = 0 \) and we may rewrite the product in the following form

\[
P(z_1, z_2, z_3) = (\exp(\pi i (z_1 + z_2 + z_3)) - \exp(\pi i (z_1 - z_2 + z_3))) \times \prod_{\substack{n, m \geq 0, l \in \mathbb{Z} \\ n > 0 \lor m > 0}} \left( 1 - \exp(2\pi i (nz_1 + lz_2 + mz_3)) \right) f(nm, l).
\]

Thus \( P(z_1, z_2, z_3) = P(z_3, z_2, z_1) \). The element

\[
I = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \quad \text{where } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

is the only element of \( \Gamma^J \) such that \( I \circ \psi_{5,4} = \psi_{5,4} \circ I \).
realizes this change of the variables. Hence

\[ P(z_1, z_2, z_3) |_I = -P(z_1, z_2, z_3) \quad \text{and} \quad v_P(I) = 1. \]

\( \Gamma_J \) and \( I \) generate the group \( PSp_4(\mathbb{Z}) \). Therefore \( P(z_1, z_2, z_3) \) is a Siegel modular form of weight 5 with the same multiplier system as \( \Delta_5(\mathbb{Z}) \). Comparing the first Fourier coefficients we finish the proof of the theorem.

There is a better way to define the Fourier expansion of \( \Delta_5(\mathbb{Z}) \), than the definition (1.2). This function belongs to the so-called Maass subspace (see [M3]). It means that \( \frac{1}{64} \Delta_5(\mathbb{Z}) \) can be written as a lifting of its “first” (it is better to say “one half”) Fourier-Jacobi coefficient

\[ \psi_{5, \frac{1}{2}}(z_1, z_2) = \eta(z_1)^9 \vartheta_{11}(z_1, z_2) = \sum_{n, l \equiv 1 \mod 2} g(n, l) \exp(\pi i (nz_1 + lz_2)). \]

More exactly

\[ \frac{1}{64} \Delta_5(\mathbb{Z}) = \sum_{n, l, m \equiv 1 \mod 2} \sum_{d \mid (n, l, m)} \frac{d^4 g(nm, l)}{d^2} \exp(\pi i (nz_1 + lz_2 + mz_3)) = \]

\[ \exp(\pi i (z_1 + z_2 + z_3)) \prod_{n, l, m \in \mathbb{Z}} (1 - \exp(2\pi i (nz_1 + lz_2 + mz_3))) f(nm, l) \quad (4.13) \]

where \( f(\cdot, \cdot) \) are the Fourier coefficients of the weak Jacobi form \( \phi_{0,1} \).

§ 5. The second example of Siegel automorphic form correction of a Lorentzian Kac–Moody algebra

Here we want to describe briefly another example of Siegel automorphic form correction of a Lorentzian Kac–Moody algebra. Actually, the case we have considered before and this example belong to one series which we hope to describe and study carefully in the corresponding publication later. The main difference of this example is that Siegel modular form we will use for this correction is not known. To construct this modular form we use the arithmetic lifting of Jacobi forms which was developed in [G1] – [G3]. This lifting is the advanced generalization of Maass’s lifting. This lifting is valid for an arbitrary domain of type 4, and we hope to apply this lifting for other cases later.

We start with the description of the Lorentzian Kac–Moody algebra we want to correct, its real roots and Weyl group (i.e. the corresponding reflection group in hyperbolic plane).

We consider an even unimodular hyperbolic plane \( \mathbb{Z}f_2 \oplus \mathbb{Z}f_{-2} \cong U \) with \( (f_2, f_2) = (f_{-2}, f_{-2}) = 0 \), \( (f_2, f_{-2}) = -1 \), and a one-dimensional lattice \( \mathbb{Z}f_3 \) with \( (f_3, f_3) = 4 \), and their orthogonal sum \( M_0 = \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_{-2} \). Thus, \( M_0 \cong U \oplus < 4 > \).

We consider its sublattice

\[ M_0 - A M_0^* = \mathbb{Z}(Af_2) \oplus \mathbb{Z}(Af_3) \oplus \mathbb{Z}(Af_{-2}) \cong U(16) \oplus < 4 >, \]
with the set of orthogonal vectors to $M_\delta$ where all elements $f$ have the square 4 and the Gram matrix

$$
(\delta_i, \delta_j) = \begin{pmatrix}
4 & -4 & -12 & -4 \\
-4 & 4 & -12 & -4 \\
-12 & -4 & 4 & -4 \\
-4 & -12 & -4 & 4
\end{pmatrix}.
$$

(5.1)

Here $M_{II}$ is a right quadrangle on the hyperbolic plane with all its vertices at infinity.

Then

$$A(P(M_{II})_{pr}) \cong D_4$$

is the group of symmetries of a right quadrangle and

$$O^+(M_0) = W^{(4)}(M_{II}) \rtimes A(P(M_{II})_{pr}).$$

We denote $P(M_{II}) = P(M_{II})_{pr} = \{\delta_1, \delta_2, \delta_3, \delta_4\}.$

This group $W^{(4)}(M_{II})$ is also elliptic which means (see [N5]) that $M_{II}$ has finite volume in the hyperbolic plane $V^+(M_0)/\mathbb{R}^+.$ Equivalently, for the cone $\mathbb{R}^+Q_+ = \mathbb{R}^+\delta_1 + \ldots + \mathbb{R}^+\delta_4$ and the corresponding dual cone $\mathbb{R}^+Q^*_+ = \{x \in M_0 \otimes \mathbb{R} \mid (x, \delta_i) \leq 0\}$ we have similar to (2.2) sequence of embeddings

$$\mathbb{R}^+Q^*_+ \subset V^+(M_0) \subset \mathbb{R}^+Q_+.$$

(5.2)

It follows that $\mathbb{R}^+Q^*_+ = \mathbb{R}^+M_{II}.$ Moreover, $P(M_{II})$ has a lattice Weyl vector $\rho$ which is defined by the property

$$(\rho, \delta_i) = -(\delta_i, \delta_i)/2 = -2.$$

One can easily calculate that

$$\rho = f_2 + (1/2)f_3 + f_{-2}.$$

(5.3)

We consider the corresponding “complexified cone”

$$\Omega(V^+(M_0)) = M_0 \otimes \mathbb{R} + iV^+(M_0)$$

with coordinates $t'(z'_1, z'_2, z'_3)$ where $z' = z'_3f_2 + z'_2f_3 + z'_1f_{-2} \in \Omega(V^+(M_0)).$ We consider another hyperbolic plane $\mathbb{Z}f_1 \oplus \mathbb{Z}f_{-1} \cong U$ with $(f_1, f_{-1}) = (f_{-1}, f_{-1}) = 0,$ $(f_1, f_{-1}) = -1,$ and its orthogonal sum $L$ with $M_0.$ Thus, we can consider $\Omega(V^+(M_0))$ as a cusp of the domain

$$\mathcal{H}_+ = \{Z' = t'((2z'_2)^2 - z'_1z'_3), z'_3, z'_2, z'_1, 1) \cdot z'_0 \in \mathcal{H}^{IV} \mid \text{Im}(z'_1) > 0\}.$$

Below we construct an automorphic form with a multiplier system $F_2(Z')$ on $\mathcal{H}_+$ with respect to the group $O^+(L)$ which has the following properties. By construction, this form will be anti-invariant relative to $W^{(4)}(M_{II}) \rtimes A(P(M_{II}))$ which means that

$$F_2(w.a(Z')) = \det(w)F_2(Z') \text{ for } w \in W^{(4)}(M_{II}), a \in A(P(M_{II})).$$

(5.4)

By (5.2)–(5.4) and (5.11), (5.12) below, the form $F_2(Z')$ will have the Fourier expansion with the properties:
\[ F_2(Z') = \sum_{w \in W^{(2)}(M_{II})} \det(w) \left( \exp\left( -\frac{\pi i}{2} (w(\rho), z') \right) - \sum_{a \in M_{II} \cap \mathbb{R}^+ M_{II}} m(a) \exp\left( -\frac{\pi i}{2} (w(\rho + a), z') \right) \right), \] (5.5)

where \( z' = z_2 f_2 + z_3 f_3 + z_1 f_{-2} \in M_0 \otimes \mathbb{R} + iV^+(M_0) = M_{II} \otimes \mathbb{R} + iV^+(M_{II}) \) and \( m(a) \in \mathbb{Z} \). For any primitive \( a_0 \in M_{II} \cap \mathbb{R}^+ M_{II} \) with \((a_0, a_0) = 0\), we will have the identity

\[ 1 - \sum_{t \in \mathbb{N}} m(ta_0)q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^\tau(ka_0), \] (5.6)

of power series of one variable \( q \), where all \( \tau(ka_0) \in \mathbb{Z} \). More exactly,

\[ \tau(a) = 3 \quad \text{for any} \quad a \in M_{II} \cap \mathbb{R}^+ M_{II} \quad \text{with} \quad (a, a) = 0. \] (5.7)

Like in § 3, we use Fourier coefficients (5.5) and (5.6) to construct the set of simple roots

\[ s\Delta = s\Delta^{re} \cup s\Delta^{im} \]

where \( s\Delta^{re} = P(M_{II}) \) and

\[ s\Delta^{im} = \{ m(a)a \mid a \in M_{II} \cap \mathbb{R}^+ M_{II}, \ (a, a) < 0 \} \]

\[ \cup \{ \tau(a)a \mid a \in M_{II} \cap \mathbb{R}^+ M_{II}, \ (a, a) = 0 \}, \] (5.8)

and the corresponding generalized Kac–Moody superalgebra without odd real simple roots \( g''(M_{II}, s\Delta) \). By construction, the denominator function \( \Phi(z') \) of this algebra and the automorphic form \( F_2(Z') \) are connected by the formula

\[ \Phi(z') = F_2(4Z'). \] (5.9)

Below (see (5.14)) we construct the automorphic form \( F_2(Z') \) and give its development in the infinite product. Like in § 3 (see (3.2) and (3.3)), this gives multiplicities for Weyl–Kac product formula of the denominator function \( \Phi(z') \).

Now we work with the Siegel upper-half plane:

\[ Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2. \]

Let

\[ \psi_{2, \frac{1}{2}}(z_1, z_2) = \eta(z_1)^3 \theta_{11}(z_1, z_2). \]

This is a Jacobi cusp form of weight 2 and index \( \frac{1}{2} \) with respect to the full Jacobi group and with a multiplier system \( v_2 : \Gamma^J \to \{ \pm 1, \pm i \} \). It means that the function \( \psi_{2, \frac{1}{2}}(Z) = \psi_{2, \frac{1}{2}}(z_1, z_2) \exp(\pi i z_3) \) satisfies the functional equations of types (4.4)–(4.5) with \( k = 2 \). We have the following exact formulae for the value of multiplier system (see (4.2), (4.3)):

\[ v_2(h(0, 1, 0)) = v_2(h(1, 0, 0)) = -1, \quad v_2\left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = v_2\left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = i. \] (5.10)
In accordance with (4.6) and the well known formula for $\eta(z)^3$, we have

$$
\psi_{2,1}(z_1, z_2) = q^{1/4}r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n)(1 - q^n)^4
$$

$$
= \left( \sum_{n \equiv 1 \text{mod} 2} (-1)^{n+1} \exp \left( \frac{\pi i}{4} n^2 z_1 + \pi i n z_2 \right) \right) \times \left( \sum_{m \equiv 1 \text{mod} 4} m \exp \left( \frac{\pi i}{4} m^2 z_1 \right) \right)
$$

$$
= \sum_{n \equiv 1 \text{mod} 4} c(n, l) \exp \left( \frac{\pi i}{2} (n z_1 + 2l z_2) \right).
$$

(5.11)

Analogues to [G3] – [G4], we can construct the arithmetical lifting of the Jacobi form $\psi_{2,1}$ to a modular form of weight 2 (with a multiplier system) with respect to the paramodular group

$$
\Gamma_2 := \left\{ \begin{pmatrix} * & 2* & * & * \\ * & * & * & 2^{-1} * \\ * & 2* & * & * \\ 2* & 2* & 2* & * \end{pmatrix} \in Sp_4(Q), \ all \ * \in \mathbb{Z} \right\}.
$$

This group is conjugate to the integral symplectic group keeping the skew-symmetric form

$$
\begin{pmatrix} 0 & T_2 \\ -T_2 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.
$$

Using the lifting constructed in [G3], we obtain a cusp form

$$
F_2(Z) = \text{Lift} \left( \tilde{\psi}_{2,1} \right)(Z)
$$

$$
= \sum_{n,m \equiv 1 \text{mod} 4} \sum_{d \mid (n,l,m)} d \frac{-4}{d} \exp \left( \frac{\pi i}{d} \left( n z_1 + 2l z_2 + 2mz_3 \right) \right),
$$

(5.12)

where

$$
\begin{pmatrix} -4 \\ d \end{pmatrix} = \begin{cases} 1 & \text{if } d \equiv 1 \text{ mod } 4 \\ -1 & \text{if } d \equiv -1 \text{ mod } 4 \end{cases}.
$$

The multiplier system of $F_2$ is defined by (5.10) and the equation

$$
v_2 \left( \begin{pmatrix} tU \\ 0 \end{pmatrix} \right) = 1 \quad \text{with} \quad U = \begin{pmatrix} 0 & \sqrt{2}^{-1} \\ \sqrt{2} & 0 \end{pmatrix}.
$$

To define an expansion in the infinite product we introduce a weak Jacobi form of weight zero and index 2

$$
\phi_{0,2}(z_1, z_2) = \frac{1}{288\Delta_{12}(z_1)} \left( E_4(z_1)E_4^2(z_1, z_2) - E_6^2(z_1, z_2) \right)
$$

$$
= \sum f_2(n, l) \exp(2\pi i nz_1 + lz_2).
$$

(5.13)
Combining the method of [G3] with the method used above, we get

\[ F_2(Z) = \exp \left( \frac{\pi i}{2} (z_1 - 2z_2 + 2z_3) \right) \prod_{n,l,m \in \mathbb{Z}} (1 - \exp (2\pi i (nz_1 + lz_2 + 2mz_3))) f_2(nm,l), \]

where \((n,l,m) > 0\) means that \(n \geq 0, m \geq 0, l\) is an arbitrary integral if \(n + m > 0\), and \(l > 0\) if \(n = m = 0\).

For any paramodular symplectic group the analogue of Lemma 1.1 is true (see [G2]). Using this isomorphism, one can easily rewrite the result about \(F_2\) above and obtain the analogue of the formula (4.13) in terms of the orthogonal group:

\[ F_2(Z') = \sum_{n,m \equiv 1 \mod 4} \sum_{l \equiv 1 \mod 2} \sum_{d | (n,l,m)} d \left( \frac{-4}{d} \right) c \left( \frac{nm}{d^2}, \frac{l}{d} \right) \exp \left( \frac{\pi i}{2} (nz'_1 + 2lz'_2 + mz'_3) \right) = \]

\[ \exp \left( \frac{\pi i}{2} (z'_1 - 2z'_2 + z'_3) \right) \prod_{n,l,m \in \mathbb{Z}} (1 - \exp (2\pi i (nz'_1 + lz'_2 + 2mz'_3))) f_2(nm,l), \quad (5.14) \]

where \(c(\cdot, \cdot)\) and \(f_2(\cdot, \cdot)\) are the Fourier coefficients of the corresponding Jacobi forms of weight 2 and 0 and \(z'_i\) are variables from the homogeneous domain of the orthogonal group. This gives the formula (5.9).

§ 6. Appendix: Generalized Kac–Moody superalgebras without odd real simple roots

We refer to the paper of V. Kac [Ka2] for basic definitions related with Lie superalgebras.

We fix a complex \(n \times n\) matrix \(A = (a_{ij})\) where \(i, j \in I = \{1, 2, \ldots, n\}\). Moreover, we fix a subset \(\tau \subset I\).

Like in [Ka1], one can define a graded Lie superalgebra \(\bar{g}(A, \tau)\), its maximal graded ideal \(\tau\) trivially intersecting the Cartan subalgebra \(h\) and Lie superalgebra \(g(A, \tau) = \bar{g}(A, \tau)/\tau\). The only difference is (see [Ka1, Ch. 1]) that the generators \(h\) and generators \(e_i, f_i, i \in I - \tau\), are even, and generators \(e_i, f_i, i \in \tau\), are odd. For \(\tau = \emptyset\), one obtains Chevalley Lie algebras \(g(A)\) which were considered in [Ka1].

The matrix \(A\) above is called a generalized generalized Cartan matrix if \(A\) is a real matrix which satisfies the following conditions:

\(\text{(C1') either } a_{ii} = 2 \text{ or } a_{ii} \leq 0;\)
\(\text{(C2') } a_{ij} \leq 0 \text{ if } i \neq j, \text{ and } a_{ij} \in \mathbb{Z} \text{ if } a_{ii} = 2;\)
\(\text{(C3') } a_{ij} = 0 \text{ implies } a_{ji} = 0.\)

If \(A\) is a generalized generalized Cartan matrix, then the Lie superalgebra \(g(A, \tau)\) is called a generalized Kac–Moody superalgebra. We assume the additional condition

\(\text{(C4') if } i \in \tau, \text{ then } a_{ii} \leq 0.\)

If \(A\) satisfies (C4'), then the generalized Kac–Moody superalgebra \(g(A, \tau)\) is called a generalized Kac–Moody superalgebra without odd real simple roots. In particular, for \(\tau = \emptyset\) one obtains a generalized Kac–Moody algebra \(g(A)\).
Results of the classical book by V. Kac [Ka1] are divided in 3 types:

**Type 1.** Results which are valid for Lie algebras $\mathfrak{g}(A)$ with an arbitrary or an arbitrary symmetrizable matrix $A$. These are results of Ch. 1 (Basic definitions); Ch. 2 (The invariant bilinear form and the generalized Casimir operator); Ch. 9 (Highest weight modules over Lie algebra $\mathfrak{g}(A)$).

**Type 2.** Results which are valid for an arbitrary or an arbitrary symmetrizable (i.e. $A$ is symmetrizable) generalized Kac–Moody algebra $\mathfrak{g}(A)$. These are results of Ch. 3 (Integrable representations and the Weyl group of a Kac–Moody algebra); Ch. 4 (Some properties of generalized Cartan matrices); Ch. 5 (Real and imaginary roots); Ch. 10 (Integrable highest weight modules: the character formula); Ch. 11 (Integrable highest weight modules: the weight system, the contravariant Hermitian form and the restriction problem). We mention that [Ka1, §11.13] contains changes one should make to adapt results of these Chapters (where Kac–Moody algebras (i.e. $a_{ii} = 2$ for any $i \in I$) are considered) to generalized Kac–Moody algebras. See also [Bo1] and [Bo2] where generalized Kac–Moody algebras were first introduced and studied, but we prefer to follow to [Ka1].

**Type 3.** Results on affine Kac–Moody algebras. Chs. 6, 7, 8, 12, 13, 14.

The general remark is that all type 1 results may be adapted to Lie superalgebras $\mathfrak{g}(A, \tau)$ with arbitrary or symmetrizable complex matrix $A$. All type 2 results may be adapted to Kac–Moody superalgebras without odd real simple roots $\mathfrak{g}(A, \tau)$. All type 3 results may be adapted to appropriate affine generalized Kac–Moody superalgebras without odd real simple roots $\mathfrak{g}(A, \tau)$. Here we consider changes in [Ka1] one should make for these adaptations of Type 1 and Type 2 results.

Type 1 results: One should introduce the following changes in Chs. 1, 2 and 9 of [Ka1] to adapt them to Lie superalgebras $\mathfrak{g}(A, \tau)$ with arbitrary complex $A$:

Instead of [Ka1, (1.3.3)], one has:

$$g_{\alpha_i} = C e_i, \quad g_{-\alpha_i} = C f_i, \quad g_{s\alpha_i} = 0, \quad \text{if } |s| > 1 \quad (6.1)$$

for $i \in I - \tau$ and for $i \in \tau$ and $a_{ii} = 0$;

$$g_{\alpha_i} = C e_i, \quad g_{-\alpha_i} = C f_i, \quad g_{s\alpha_i} = 0, \quad \text{if } |s| > 2, \quad g_{2\alpha_i} = \mathbb{C}[e_i, e_i] \neq \{0\}, \quad g_{-2\alpha_i} = \mathbb{C}[f_i, f_i] \neq \{0\} \quad (6.2)$$

for $i \in \tau$ and $a_{ii} \neq 0$.

Instead of [Ka1, Lemma 1.3], one has: If $\beta \in \Delta_+ - \{\alpha_i, 2\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+$.

Instead of Chevalley involution $\omega$ (and the corresponding involution $\bar{\omega}$) one should consider the Chevalley automorphism of the period 4 which is defined by the property: $e_i \mapsto -(-1)^{\deg e_i} f_i, \quad f_i \mapsto -e_i$ (i.e. $i \in I$), $h \mapsto -h$ ($h \in \mathfrak{h}$). It has $\omega^2$ identical on $\mathfrak{g}(A, \tau)_{\mathfrak{h}}$ and equal $-1$ on $\mathfrak{g}(A, \tau)_{\mathfrak{h}}$.

There are the following changes in [Ka1, Ch. 9]. Let $V$ be a $\mathfrak{g}(A, \tau)$-module from the category $\mathcal{O}$ and $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. There are two possibile definitions of the formal character of $V$:

$$\text{ch } V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e(\lambda); \quad (6.3)$$

and

$$\text{ch}_+ V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_{\lambda, 0} - \dim V_{\lambda, 1}) e(\lambda). \quad (6.4)$$
For a Verma module $M_{\overline{e}}(\Lambda)$ with the highest weight vector of the degree $\overline{e} \in \{\overline{\mathbb{U}}, \overline{\mathbb{T}}\}$ and $\Lambda \in h^*$, one has

$$
\text{ch } M_{\overline{e}}(\Lambda) = e(\Lambda) \prod_{\alpha \in \Delta_{+,\overline{e}}} (1 - e(-\alpha))^{\text{mult } \alpha} \prod_{\alpha \in \Delta_{+,\overline{e}}} (1 + e(\alpha))^{\text{mult } \alpha} = e(\Lambda) \text{ch } M_{\overline{0}}(0) \quad (6.5)
$$

and

$$
\text{ch}_{\pm} M_{\overline{e}}(\Lambda) = (-1)^{\overline{e}} e(\Lambda) \prod_{\alpha \in \Delta_{+}} (1 - e(-\alpha))^{\text{mult } \alpha} = (-1)^{\overline{e}} e(\Lambda) \text{ch}_{\pm} M_{\overline{0}}(0) \quad (6.6)
$$

where for both formulae (6.5) and (6.6)

$$
\text{mult } \alpha = \begin{cases} 
\dim g(A, \tau)_{\alpha} & \text{if } \alpha \in \Delta_{\overline{e}}, \\
-\dim g(A, \tau)_{\alpha} & \text{if } \alpha \in \Delta_{\overline{e}}.
\end{cases} \quad (6.7)
$$

One should consider only homomorphisms of $g(A, \tau)$-modules which preserve even and odd parts of modules.

Type 2 results: In [Ka1, § 11.13] one can find changes one should make to adapt Type 2 results of [Ka1] to generalized Kac–Moody algebras (i.e. algebras $g(A)$ where $A$ satisfies the conditions (C1'), (C2'), (C3') (and $\tau = \emptyset$). We give here additional changes one should make to adapt these results to generalized Kac–Moody superalgebras without odd real simple roots $g(A, \tau)$ (i.e. with condition (C4') instead of $\tau = \emptyset$).

Instead of [Ka1, Lemma 11.13.1], one has:

**Statement 6.1.** Let $V$ be an integrable $g(A)$-module, $\lambda \in h^*$ and $0 \neq v \in V_\lambda$ such that $f_i(v) = 0$. Then for $k \geq 1$,

$$
f_i e_i^k(v) = \begin{cases} 
-k(\langle \lambda, \alpha_i^\vee \rangle + \frac{1}{2} (k-1)a_{ii}) e_i^{k-1}(v) & \text{if } i \in I - \tau; \\
-[k/2]a_{ii} e_i^{k-1}(v), & \text{if } i \in \tau \text{ and } k \equiv 0 \mod 2; \\
-([k/2]a_{ii} + \langle \lambda, \alpha_i^\vee \rangle) e_i^{k-1}(v), & \text{if } i \in \tau \text{ and } k \equiv 1 \mod 2.
\end{cases}
$$

In particular, $e_i^k(v) \neq 0$ for all $k = 1, 2, \ldots$ if $a_{ii} \leq 0$, $\langle \lambda, \alpha_i^\vee \rangle < 0$ and either $i \in I - \tau$, or $i \in \tau$ and $a_{ii} < 0$.

Using this statement, one obtains the following statement instead of [Ka1, Corollary 11.13.1]:

**Statement 6.2.** Assume that $a_{ii} \leq 0$ and $i \in I - \tau$ for $a_{ii} = 0$. Assume that $\text{supp}(\alpha + \alpha_i)$ is connected. Then $\alpha + k\alpha_i \in \Delta_+$ for all $k \in \mathbb{Z}_+$.

All results of [Ka1, Ch. 4] are valid for generalized Kac–Moody superalgebras except that there exists one additional zero $1 \times 1$ matrix which gives a generalized Kac–Moody algebra for $\tau = \emptyset$ and a generalized Kac–Moody superalgebra for $\tau = \{1\}$.

To describe like in [Ka1, Ch. 5] the set of roots $\Delta$, we set $\tau_0 = \{i \in \tau \mid a_{ii} = 0\}$, $\tau_i = \{i \in \tau \mid a_{ii} < 0\}$, $\Pi_{\text{im}} = \{\alpha_i \in \Pi_{\text{im}} \mid i \in \tau_0\}$, $\Pi_{\text{im}} = \{\alpha_i \in \Pi_{\text{im}} \mid i \in \tau\}$.
\[
\Pi_{1, <0}^\im = \{ \alpha_i \in \Pi \mid i \in \tau_{<0} \}; \quad \Pi_{1, 0}^\im = \{ \alpha_i \in \Pi \mid i \in \tau_0 \}.
\]

Let

\[
K = \{ \alpha \in Q_+ \mid \alpha \in -C^\vee \text{ and supp } \alpha \text{ is connected} \}
- \left( \bigcup_{k \geq 2} k \Pi_{1, <0}^\im \right) \cup \left( \bigcup_{k \geq 3} k \Pi_{1, 0}^\im \right) \cup \left( \bigcup_{k \geq 2} k \Pi_{1, >0}^\im \right).
\]

Then we have

\[
\Delta_+^\im \subset \bigcup_{w \in W} w(K).
\]

Like in [Ka1, (11.13.3)] and using Statement 6.2, we get the opposite inclusion and thus

**Statement 6.3.**

\[
\Delta_+^\im = \bigcup_{w \in W} w(K).
\]

If \( \Pi_{1, 0}^\im = \emptyset \) (i.e. \( \tau_0 = \emptyset \)).

We don’t know what will be the analog of Statement 6.3 in general when \( \Pi_{1, 0}^\im \) is not empty.

Using [Ka1, Lemma 11.13.2] and superalgebras analog of [Ka1, Proposition 9.11], it follows that symmetrizable (i.e. with symmetrizable \( A \)) generalized Kac–Moody superalgebras without odd real simple roots \( g(A, \tau) \) have similar defining relations as generalized Kac–Moody algebras \( g(A) \).

**Statement 6.4.** A symmetrizable generalized Kac–Moody superalgebra without odd real simple roots \( g(A, \tau) \) is a Lie superalgebra with even generators \( h, e_i, f_i, i \in I-\tau \), and odd generators \( e_i, f_i, i \in \tau \), and defining relations

\[
[e_i, f_j] = \delta_{ij} \alpha_i^\vee \quad (i, j \in I),
\]
\[
[h, h'] = 0 \quad (h, h' \in \mathfrak{h}),
\]
\[
[h, e_i] = (\alpha_i, h)e_i,
\]
\[
[h, f_i] = - (\alpha_i, h)f_i, \quad (i \in I; \ h \in \mathfrak{h}),
\]
\[
(ad e_i)^{1-a_{ij}} e_j = 0, \quad (ad f_i)^{1-a_{ij}} f_j = 0, \quad \text{if } a_{ii} = 2 \text{ and } i \neq j,
\]
\[
[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad \text{if } a_{ij} = 0.
\]

Remark that the last relation is not trivial also for \( i = j \) if \( a_{ii} = 0 \) and \( i \in \tau \).

Let us consider a diagonal generalized generalized Cartan matrix \( B = (b_{ij}) \) with \( b_{ii} \leq 0 \) for any \( i \in I \) and some \( \tau \subset I \), and the generalized Kac–Moody superalgebra \( g(B, \tau) \). Using (6.1), (6.2), one obtains for \( g(B, \tau) \):

\[
\text{ch } L_\mathfrak{g}(0) = (\text{ch } M_\mathfrak{g}(0))^{-1}
= \prod_{i \in I-\tau} (1 - e(-\alpha_i)) \prod_{i \in \tau_{<0}} (1 - e(-2\alpha_i)) (1 + e(-\alpha_i))^{-1} \prod_{i \in \tau_0} (1 + e(-\alpha_i))^{-1}
= \prod_{i \in I-\tau} (1 - e(-\alpha_i)) \prod_{i \in \tau_{<0}} \left( \sum_{k \geq 0} (-1)^k e(-k\alpha_i) \right). \quad (6.8)
\]
It follows that \( \text{ch } L_{\overline{\sigma}}(0) \) is the sum
\[
\text{ch } L_{\overline{\sigma}}(0) = \sum \epsilon(s)e(-s),
\]
over all sums of simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^n \) if \( s \) is a sum of \( n \) simple pairwise perpendicular imaginary simple roots \( \alpha_i \) which are distinct if \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

Similarly, for the generalized generalized Cartan matrix \( B \) above and the character \( \text{ch}_\pm \), one has
\[
\text{ch}_\pm L_{\overline{\sigma}}(0) = (\text{ch}_\pm M_{\overline{\sigma}}(0))^{-1} = \prod_{i \in I - \tau} (1 - e(-\alpha_i)) \prod_{i \in \tau < 0} (1 - e(-2\alpha_i)) (1 - e(-\alpha_i))^{-1} \prod_{i \in \tau_0} (1 - e(-\alpha_i))^{-1}
\]
\[
= \prod_{i \in I - \tau} (1 - e(-\alpha_i)) \prod_{i \in \tau < 0} (1 + e(-\alpha_i)) \prod_{i \in \tau_0} \left( \sum_{k \geq 0} e(-k\alpha_i) \right).
\]
It follows that \( \text{ch}_\pm L_{\overline{\sigma}}(0) \) is the sum
\[
\text{ch}_\pm L_{\overline{\sigma}}(0) = \sum \epsilon(s)e(-s),
\]
over all sums of simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^{n_{\overline{\sigma}}} \) if \( s \) is a sum of pairwise perpendicular \( n_{\overline{\sigma}} \) imaginary simple roots \( \alpha_i, i \in I - \tau \), and \( n_{\overline{\tau}} \) imaginary simple roots \( \alpha_i, i \in \tau \), which are distinct if \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

Now we consider an arbitrary generalized generalized Cartan matrix \( A \). We consider an irreducible \( g(A, \tau) \)-module \( L_{\overline{\sigma}}(\Lambda) \) with a highest weight vector of the degree \( \overline{\tau} \in \{\overline{0}, \overline{T}\} \) and \( \Lambda \in \mathfrak{h}^* \) which satisfies the condition
\[
\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \text{ if } a_{ii} = 2, \quad \langle \Lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in I.
\]

We remind that \( \rho \in \mathfrak{h}^* \) is defined by relations
\[
\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii}, \text{ for all } i \in I.
\]
Equivalently, for a symmetrizable \( A \),
\[
\langle \rho | \alpha_i \rangle = \frac{1}{2} (\alpha_i | \alpha_i) \text{ for all } i \in I.
\]

From (6.5), (6.6), (6.8) — (6.11), like in [Ka1, § 11.13], we get Weyl–Kac–Borcherds character formulae for symmetrizable generalized Kac–Moody superalgebras without odd real simple roots \( g(A, \tau) \). Formulae (6.8) — (6.11) are particular cases of them.

**Statement 6.5.**
\[
\text{ch } L_{\overline{\tau}}(\Lambda) = \sum_{w \in W} e(w)w(S_{\Lambda, \overline{\tau}})/e(\rho) \prod_{\alpha \in \Delta_{+, \overline{\tau}}} (1 - e(-\alpha))^{\text{mult } \alpha} \prod_{\alpha \in \Delta_{+, \overline{\tau}}} (1 + e(-\alpha))^{\text{mult } \alpha}
\]
where \( S_{\Lambda, \overline{\tau}} \) is the sum
\[
S_{\Lambda, \overline{\tau}} = e(\Lambda + \rho) \sum \epsilon(s)e(-s)
\]
over all sums of imaginary simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^n \) if \( s \) is a sum of \( n \) pairwise perpendicular and perpendicular to \( \Lambda \) imaginary simple roots \( \alpha_i \) which are distinct for \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.
Statement 6.6.

\[ ch_\pm L_\tau(\Lambda) = \sum_{w \in W} \epsilon(w)w(S_{\Lambda, T}) / e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult} \alpha} \] (6.15)

where \( S_{\Lambda, T} \) is the sum

\[ S_{\Lambda, T} = (-1)^{t}e(\Lambda + \rho) \sum \epsilon(s)e(-s) \] (6.16)

over all sums of imaginary simple roots \( s \). Here \( \epsilon(s) = (-1)^{n_s} \) if \( s \) is a sum of \( n_{\Pi} + n_{T} \) pairwise perpendicular and perpendicular to \( \Lambda \) \( n_{\Pi} \) imaginary simple roots \( \alpha_i, i \in I - \tau \), and \( n_{T} \) imaginary simple roots \( \alpha_i, i \in \tau \), which are distinct for \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

In particular, for \( \Lambda = 0 \) and \( T = 0 \) we get the denominator identities for \( g(A, \tau) \).

Statement 6.7.

\[ e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult} \alpha} \prod_{\alpha \in \Delta_+} (1 + e(-\alpha))^{\text{mult} \alpha} = \sum_{w \in W} \epsilon(w)w(S) \] (6.17)

where \( S \) is the sum

\[ S = e(\rho) \sum \epsilon(s)e(-s) \] (6.18)

over all sums of imaginary simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^{n_s} \) if \( s \) is a sum of \( n \) pairwise perpendicular imaginary simple roots \( \alpha_i \) which are distinct for \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

Statement 6.8.

\[ e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult} \alpha} = \sum_{w \in W} \epsilon(w)w(S) \] (6.19)

where \( S \) is the sum

\[ S = e(\rho) \sum \epsilon(s)e(-s) \] (6.20)

over all sums of imaginary simple roots \( s \). Here \( \epsilon(s) = (-1)^{n_s} \) if \( s \) is a sum of pairwise perpendicular \( n_{\Pi} \) imaginary simple roots \( \alpha_i, i \in I - \tau \), and \( n_{T} \) imaginary simple roots \( \alpha_i, i \in \tau \), which are distinct for \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

Instead of a generalized Kac–Moody superalgebra without odd real simple roots \( g(A, \tau) \) one can consider the derived algebra \( g'(A, \tau) = [g(A, \tau), g(A, \tau)] \) which is also called generalized Kac–Moody superalgebra without odd real simple roots. By Statement 6.4, a symmetrizable \( g'(A, \tau) \) is generated by \( h_i = \alpha_i^\vee, e_i, f_i, i \in I \), where all \( h_i, i \in I \), are even, \( e_i, f_i \) are even if \( i \in I - \tau \), and \( e_i, f_i \) are odd if \( i \in \tau \), and has defining relations \((i, j \in I)\)

\[ [e_i, f_j] = \delta_{ij} h_i, \ [h_i, h_j] = 0 \ [h_i, e_j] = a_{ij} e_j, \ [h_i, f_j] = -a_{ij} f_j; \] (6.21)

\[ (\text{ad} e_i)^{1-a_{ij}} e_j = 0, \ (\text{ad} f_i)^{1-a_{ij}} f_j = 0 \quad \text{if} \ a_{ij} = 2 \text{ and} \ i \neq j; \]

\[ [e_i, \cdot] = 0, \ [f_i, \cdot] = 0 \quad \text{if} \ c_{ij} = 0. \]
We remind (see [Ka1]) that this algebra is graded by the root lattice

\[ Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad (6.22) \]

where

\[ \deg e_i = \alpha_i = -\deg f_i, \deg h_i = 0. \quad (6.23) \]

There is a pairing between \( Q \) and the coroot lattice

\[ Q^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i \]

which is defined by

\[ \langle h_i, \alpha_j \rangle = a_{ij}. \quad (6.25) \]

Remind that \( \mathfrak{h}' = Q^\vee \otimes \mathbb{C} \) is Cartan subalgebra of \( \mathfrak{g}'(A, \tau) \). Like in [Ka1], all results above (Statements 6.1—6.3, 6.5—6.8 and (6.21)) are valid for \( \mathfrak{g}'(A, \tau) \). Advantage of considering \( \mathfrak{g}'(A, \tau) \) is that one can consider also infinite matrices \( A \) with a countable set of indexes \( I \). One can consider \( \mathfrak{g}'(A, \tau) \) with a countable \( I \) as a union of \( \mathfrak{g}'(A', \tau') \) with finite \( I' \), \( I' \subset I \) and \( \tau' = \tau \cap I' \).

R. Borcherds [Bo1] (see also [Bo2]) uses another notation for generalized Kac–Moody algebras. One can use similar notation to define generalized Kac–Moody superalgebras without odd real simple roots as follows.

Suppose that we have (compare with [Bo1])

(i) A real vector space \( H \) with a symmetric bilinear inner product \( (, ) \).

(ii) A sequence of elements \( h_i \in H \) indexed by a countable set \( I \), such that \( (h_i, h_j) \leq 0 \) if \( i \neq j \) and \( 2(h_i, h_j)/(h_i, h_i) \) is an integer if \( (h_i, h_i) > 0 \).

(iii) A subset \( \tau \subset I \) such that \( (h_i, h_i) \leq 0 \) for any \( i \in \tau \).

The matrix \( (h_i, h_j), i, j \in I \), is called a symmetrized generalized generalized Cartan matrix. The generalized K–Moody superalgebra \( G \) associated to (i)—(iii) is defined to be the Lie superalgebra generated by \( H \) and elements \( e_i, f_i, i \in I \), where \( H \) and \( e_i, f_i, i \in I - \tau \), are even, and \( e_i, f_i, i \in \tau \), are odd. It has the following defining relations:

1. The image of \( H \) in \( G \) is commutative. (In fact the natural map from \( H \) to \( G \) is injective so we can consider \( H \) to be an abelian subalgebra of \( G \).)
2. If \( h \) is in \( H \), then \[ [h, e_i] = (h, h_i)e_i \text{ and } [h, f_i] = -(h, h_i)f_i. \]
3. \[ [e_i, f_i] = h_i \text{ if } i = j, 0 \text{ if } i \neq j. \]
4. If \( (h_i, h_i) > 0 \) and \( i \neq j \), then \((\text{ad } e_i)^{1-2(h_i, h_j)/(h_i, h_i)} e_j = 0 \) and \((\text{ad } f_i)^{1-2(h_i, h_j)/(h_i, h_i)} f_j = 0; \)
5. If \( (h_i, h_j) = 0 \), then \[ [e_i, e_j] = [f_i, f_j] = 0. \]

The algebra \( G \) is graded by

\[ \tilde{Q} = \bigoplus_{i \in I} \mathbb{Z}r_i, \quad (6.26) \]

where

\[ \deg e_i = r_i = -\deg f_i = \deg h_i = 0. \quad (6.27) \]
We set for $i, j \in I$,

$$a_{ij} = \begin{cases} (h_i, h_j) & \text{if } (h_i, h_i) \leq 0, \\ 2(h_i, h_j)/(h_i, h_i) & \text{if } (h_i, h_i) > 0. \end{cases}$$

The matrix $A = (a_{ij})$ is a symmetrizable generalized Cartan matrix. It defines the symmetrizable generalized Kac–Moody superalgebra without odd real simple roots $q'(A, \tau)$. Below we denote its standard generators $h_i, e_i, f_i$ in (6.21) as $\tilde{h}_i, \tilde{e}_i, \tilde{f}_i$ respectively.

We have the homomorphism

$$\pi : q'(A, \tau) \to G,$$ (6.28)

which is defined by

$$\pi(\tilde{h}_i) = \begin{cases} h_i, & \text{if } (h_i, h_i) \leq 0, \\ (2/(h_i, h_i))h_i & \text{if } (h_i, h_i) > 0; \end{cases}$$

$$\pi(\tilde{e}_i) = \begin{cases} e_i, & \text{if } (h_i, h_i) \leq 0, \\ +\sqrt{2/(h_i, h_i)}e_i & \text{if } (h_i, h_i) > 0; \end{cases}$$

$$\pi(\tilde{f}_i) = \begin{cases} f_i, & \text{if } (h_i, h_i) \leq 0, \\ +\sqrt{2/(h_i, h_i)}f_i & \text{if } (h_i, h_i) > 0; \end{cases}$$

which is evidently homogeneous for gradings (6.22), (6.23) and (6.28), (6.29). This homomorphism defines the isomorphism of $G$ with the quotient of $q'(A, \tau)$ by the ideal in the center $c = \{ h \in h' \mid (\pi(h), h_i) = 0, \text{ for any } i \in I \}$. Using this isomorphism, one can transfer all results above (Statements 6.1—6.3, 6.5—6.8 and (6.21)) to $G$.

Let us consider the canonical homomorphism of abelian groups

$$\pi : \bar{Q} \to H \ (r_i \mapsto h_i).$$ (6.29)

If $\pi$ has finite preimages on the semigroup $\bar{Q}_+ \subset \bar{Q}$, where

$$\bar{Q}_+ = \bigoplus_{i \in I} \mathbb{Z}_+ r_i,$$ 

and $\pi^{-1}(0) \cap \bar{Q}_+ = 0$, we can replace the $\bar{Q}$ grading above by the grading using

$$Q = \pi(\bar{Q}) \subset H, \quad Q_+ = \pi(\bar{Q}_+)$$ (6.30)

letting for $\alpha \in Q$

$$G_{\alpha} = G_{\alpha,0} \bigoplus G_{\alpha,1}.$$ 

For this grading, $G_0 = H$ and $[h, x_\alpha] = (h, \alpha)x_\alpha$ for any $h \in H, \ x_\alpha \in G_{\alpha}, \ \alpha \in Q$. Here $G_{\alpha} = G_{\alpha,0} \bigoplus G_{\alpha,1}$ where even and odd parts may be both non-zero. For $\alpha \in Q$ we set

$$\text{mult}_{\alpha} = (-1)^{i} \dim G_{\alpha,i}, \quad \text{mult } \alpha = \text{mult}_{0}\alpha + \text{mult}_{1}\alpha = \dim G_{\alpha,0} - \dim G_{\alpha,1}.$$ (6.21)
An \( \alpha \in Q \) is called a root if \( \dim G_\alpha = \dim G_{\alpha, \overline{\tau}} + \dim G_{\alpha, \tau} > 0 \). Let \( \Delta \subset Q \) be the set of all roots, \( \Delta_\pm = \Delta \cap \pm Q_+ \).

We formulate Statements 6.5—6.8 for \( G \) with this grading.

Let \( \Lambda \in H \) is such that \( (\Lambda, h_i) \geq 0 \) and \((\Lambda, h_i) \in \mathbb{Z}\) if \((h_i, h_i) > 0\) for any \( i \in I \). Let \( L(\Lambda) \) be an irreducible \( G \)-module with the highest weight \( \Lambda \in H \subset H^* \) and a highest weight vector of even degree. For \( \alpha \in Q \) we set \( L(\Lambda)_{\Lambda+\alpha} = U(G)_\alpha(L(\Lambda)_\Lambda) \) where \( \dim L(\Lambda)_\Lambda = 1 \). See [Ka1, § 9.10]. We remark that the \( Q \)-grading of \( G \) above induces the \( Q \)-grading of the universal enveloping algebra \( U(G) \). We set

\[
\text{ch } L(\Lambda) = \sum_{\alpha \in Q} \dim L(\Lambda)_{\Lambda+\alpha} e(\Lambda + \alpha).
\]

Similarly,

\[
\text{ch}_\pm L(\Lambda) = \sum_{\alpha \in Q} (\dim L(\Lambda)_{\Lambda+\alpha, 0} - \dim L(\Lambda)_{\Lambda+\alpha, \tau}) e(\Lambda + \alpha).
\]

**Statement 6.5’.**

\[
\text{ch } L(\Lambda) = \sum_{w \in W} e(w)w(S_\Lambda) / e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} \prod_{\alpha \in \Delta_+} (1 + e(-\alpha))^{\text{mult } \alpha}
\]

where \( S_\Lambda \) is the sum

\[
S_\Lambda = e(\Lambda + \rho) \sum s e(-\pi(s))
\]

over all sums of imaginary simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^n \) if \( s \) is a sum of \( n \) pairwise perpendicular and perpendicular to \( \Lambda \) imaginary simple roots \( r_i, i \in I \) (here imaginary means \((h_i, h_i) \leq 0\), which are distinct for \( i \in I_0 \); and \( \epsilon(s) = 0 \) otherwise.

**Statement 6.6’.**

\[
\text{ch}_\pm L(\Lambda) = \sum_{w \in W} e(w)w(S_\Lambda) / e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha}
\]

where \( S_\Lambda \) is the sum

\[
S_\Lambda = e(\Lambda + \rho) \sum s e(-\pi(s))
\]

over all sums of imaginary simple roots \( s \). Here \( \epsilon(s) = (-1)^n \) if \( s \) is a sum of \( n_\overline{\tau} + n_\tau \) pairwise perpendicular and perpendicular to \( \Lambda \) \( n_\overline{\tau} \) imaginary simple roots \( r_i, i \in I_\tau \), and \( n_\tau \) imaginary simple roots \( r_i, i \in \tau \), which are distinct for \( i \in I_\tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

The corresponding denominator identities are

**Statement 6.7’.**

\[
e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} \prod_{\alpha \in \Delta_+} (1 + e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} e(w)w(S)
\]

where \( S \) is the sum

\[
S = e(\rho) \sum s e(-\pi(s))
\]

over all sums of imaginary simple roots \( s \). Here the sign \( \epsilon(s) = (-1)^n \) if \( s \) is a sum of \( n \) pairwise perpendicular imaginary simple roots \( r_i, i \in I \), which are distinct for \( i \in I_\tau_0 \); and \( \epsilon(s) = 0 \) otherwise.
Statement 6.8'.

\[ e(\rho) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} \epsilon(w)w(S) \]

where \( S \) is the sum

\[ S = e(\rho) \sum_{s} \epsilon(s)e(-\pi(s)) \]

over all sums of imaginary simple roots \( s \). Here \( \epsilon(s) = (-1)^{n_{\tau}} \) if \( s \) is a sum of \( n_{\tau} + n_{\tau} \) pairwise perpendicular \( n_{\tau} \) imaginary simple roots \( r_i, i \in I - \tau \), and \( n_{\tau} \) imaginary simple roots \( r_i, i \in \tau \), which are distinct for \( i \in I - \tau_0 \); and \( \epsilon(s) = 0 \) otherwise.

We use the last statement in \( \S \) 3 and \( \S \) 5. Notation here and in \( \S \) 3, \( \S \) 5 are related as follows: \( H = M_{II} \otimes \mathbb{R}, I = s\Delta, \tau = s\Delta_{\tau}^\text{im}, G = g''(M_{II}, s\Delta) \).

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