Some Important Properties of Multiple G-Itô Integral in the G-Expectation Space

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Abstract

In the G-expectation space, we propose the multiple Itô integral, which is driven by multi-dimensional G-Brownian motion. We prove the recursive relationship of multiple G-Itô integrals by G-Itô formula and mathematical induction, and we obtain some computational formulas for a kind of multiple G-Itô integrals.

Keywords

G-Brownian Motion, Multiple G-Itô Integrals, G-Itô Formula, Recursive Relationship

1. Introduction

With the rapid development of financial markets, traditional linear expectations cannot explain its uncertainty sometimes. In 2007, Peng [1] introduced a new sublinear expectation—G-expectation, and he introduced G-normal distribution and G-Brownian motion under the G-expectation framework. In 2008, Peng [2] proved the law of large numbers and the central limit theorem under the sublinear expectation, and he defined the Itô integral about G-Brownian motion. Later, Peng [3] obtained the G-Itô formula and proved the existence and uniqueness of solution for the stochastic differential equations driven by G-Brownian motion (G-SDEs for short) and the backward stochastic differential equations driven by G-Brownian motion (G-BSDEs for short).

Since then, G-expectation space and the applications of G-Itô integral have been extensively studied by many researchers. In 2014, Hu, Ji, Peng and Song [4] studied the comparison theorem, nonlinear Feynman-Kac formula and Girsanov transformation of G-BSDE. In 2016, Hu, Wang and Zheng [5] obtained the Itô-Krylov formula under the G-expectation framework. Then they proved the reflection principle of G-Brownian motion, and they got the reflection principle of G-Brownian motion by Krylov’s estimate in [6]. [7] studied rough path properties of stochastic integrals of Itô’s type and Stratonovich’s type.
with respect to G-Brownian motion. Then, Hu, Ji and Liu [8] studied the strong Markov property for G-SDEs in 2017. Wu [9] introduced the multiple Itô integrals driven by one-dimensional Brownian motion in G-expectation space. He also obtained the relationship between Hermite polynomials and multiple G-Itô integral. In 2012, Yin [10] introduced the Stratonovich integral with respect to G-Brownian motion, and she also researched the properties of G-Stratonovich integrals. In 2014, Sun [11] studied multiple stochastic integrals under one-dimensional G-Brownian motion and developed the $L^p$ estimation of maximal inequalities for iterated integrals by the property of Hermite polynomials. The more contents about multiple random integrals can be found in the literature [12].

A nature question is how to define and calculate the multiple G-Itô integral of multi-dimensional G-Brownian motion. This problem will be solved in this paper. We define multiple Itô integrals driven by multi-dimensional G-Brownian motion under G-expectation space. And we prove the recursive relationship between multiple G-Itô integrals strictly by using G-Itô formula and mathematical induction method. Then we obtain some important formulas for calculating multiple G-Itô integrals and make some preparations for further study on scientific calculation of G-SDEs.

The remainder of this paper is organized as follows: In Section 2, we introduce some concepts and lemmas such as G-Brownian motion, G-Itô formula and so on. In Section 3, we define multiple Itô integrals driven by multi-dimensional G-Brownian motion, and prove the recursive relationship between multiple G-Itô integrals. Then we give some important formulas for calculating multiple G-Itô integrals. Finally, several concluding remarks are given in Section 4.

2. Preliminaries and Notation

In this section, we will give some basic theories about G-Brownian motion and multi-indices. Some more details can be found in literatures [1–3] and [12]. Let $\Omega$ be a given set, and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$. For each $c$ we suppose that $c \in \mathcal{H}$, and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables.

2.1. G-Brownian Motion and G-Itô Formula

Firstly, we introduce some notations about G-Brownian motion.

**Definition 1.** [9] A $d$-dimensional process $(B_t)_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied:

(i) $B_0(\omega) = 0$;

(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t \sim N(0 \times s \sum)$ is independent from $\{B_{t_1}, B_{t_2}, \ldots, B_{t_n}\}$, for each $n \in \mathbb{N}, 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$.

Let $G(\cdot) : \mathbb{S}(d) \to \mathcal{R}$ be a given monotonic and sublinear function. We denote by $\mathbb{S}(d)$ the collection of all $d \times d$ symmetric matrices. There exists a bounded, convex and closed subset $\sum \subset \mathbb{S}_+(d) = \{\theta \in \mathbb{S}(d), \theta \geq 0\}$, such that $G(A) = \frac{1}{2} \sup_{B \in \sum} (A, B)$. $A \in \mathbb{S}(d)$.

In the following sections, we denote by $\Omega = C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with
the distance $\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t\in[0,i]} |\omega^1_t - \omega^2_t|] \wedge 1$. For each fixed $T \in [0, \infty)$, we set $\Omega_T := \{\omega_{\wedge T} : \omega \in \Omega\}$,

$L_{ip}(\Omega_T) := \{\phi(B_{t_1} \wedge \ldots, B_{t_n} \wedge T) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, \infty), \phi \in C_{lip}(\mathbb{R}^n)\}$,

$L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n),$

where $B_t$ is a canonical process, that is $B_t(\omega) = \omega_t$. For a given $p \geq 1$, we also denote $L^p_G(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $||X||_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

We recall some important notions about G-Itô formula, product rule and so on (see [3]).

**Definition 2.** [3] We denote the set of simple process

$$M^p_G([0,T]) := \{\eta(\omega) := \sum_{j=0}^{N-1} \xi_{t_j}(\omega)I_{[t_j, t_{j+1})}(t); \xi_{t_j}(\omega) \in L^p_G(\Omega_{t_j})\},$$

\[ \forall N \geq 1, 0 = t_0 < \ldots < t_N = T, j = 0, 1, \ldots, N - 1 \}.$$

And for each $p \geq 1$, we denote by $M^p_G([0,T])$ the completion of $M^p_G([0,T])$ under the norm

$$||\eta||_{M^p_G([0,T])} = \{\mathbb{E}\int_0^T |\eta|^p dt\}^{\frac{1}{p}}.$$

**Definition 3.** [3] For each $\eta \in M^2_G([0,T])$, we define the Itô integral of G-Brownian motion is as follows:

$$I(\eta) = \int_0^T \eta dB_t := \sum_{j=0}^{N-1} \xi_{t_j}(\omega)(B_{t_{j+1}} - B_{t_j}).$$

**Definition 4.** [3] We first consider the quadratic variation process of one-dimensional G-Brownian motion $(B_t)_{t \geq 0}$ with $B_1 = N(\{0\} \times [\sigma^2, \sigma^2])$. Let $\pi^N_t, N = 1, 2, \ldots$, be a sequence of partitions of $[0,t]$. We consider

$$B^2_t = \sum_{j=0}^{N-1} (B^2_{t+1,j} - B^2_{t,j}) = \sum_{j=0}^{N-1} 2B_{t,j}(B_{t+1,j} - B_{t,j}) + \sum_{j=0}^{N-1} (B_{t+1,j} - B_{t,j})^2.$$

As $\mu(\pi^N_t) \to 0$, the first term of the right side converges to $2\int_0^t B_s dB_s$ in $L^2_G(\Omega)$. The second term must be convergent. We denote its limit by $\langle B \rangle_t$, i.e.,

$$\langle B \rangle_t := \lim_{\mu(\pi^N_t) \to 0} \sum_{j=0}^{N-1} (B_{t+1,j} - B_{t,j})^2 = B^2_t - 2\int_0^t B_s dB_s.$$

By the above construction, $(\langle B \rangle_t)_{t \geq 0}$ is an increasing process with $\langle B \rangle_0 = 0$. We call it the quadratic variation process of the G-Brownian motion $B$.

Now let us introduce the following two important lemmas.
Lemma 1. [9] We denote $B_t$ be a $m$-dimensional G-Brownian motion. Let $\Phi \in C^2(\mathbb{R}^n)$ be bounded with bounded derivatives and $\partial^2_{x \sigma^j} \Phi$ are uniformly Lipschitz. Let $s \in [0, T]$ be fixed and let $X^i_t$ be the $i$ ($i = 1, \ldots, d$)-th component of $X_t = (X^1_t, \ldots, X^d_t)^\top$ satisfying

$$X^i_t = X^i_0 + \int_0^t a^i_s ds + \sum_{j=1}^m \int_0^t \eta^{i,j}_s dB^j_s + \sum_{j=1}^m \int_0^t \sigma^{i,j}_s dB^j_s,$$

where $a^i$ be the $i$-th of $a = (a^1, \ldots, a^d)^\top$, $\eta^{i,j}$ and $\sigma^{i,j}$ is the lines $i$-th and $j$-th of $\eta = (\eta^{i,j})_{d \times m}$ and $\sigma = (\sigma^{i,j})_{d \times m}$, and they are bounded process on $M^2_0(0, T)$. For $t, s \geq 0$, then we have

$$\Phi(X_t) - \Phi(X_s) = \sum_{i=1}^d \left[ \int_s^t \partial_{x^i} \Phi(X_u) a^i_u du + \sum_{j=1}^m \int_s^t \partial_{x^i} \Phi(X_u) \eta^{i,j}_u dB^j_u \right]$$

$$+ \int_s^t \sum_{i=1}^m \sum_{j=1}^m \partial_{x^i} \Phi(X_u) \eta^{i,j}_u dB^j_u$$

$$+ \frac{1}{2} \sum_{i,j=1}^m \int_s^t \partial^2_{x^i \sigma^j} \Phi(X_u) \sigma^{i,j}_u dB^j_u.$$

Lemma 2. [1, 3] In G-expectation space, the following product rule is established:

$$dB^i_t d\sigma^j_t = \delta_{ij} \begin{cases} d(B^i_t), & i = j \\ 0, & i \neq j \end{cases},$$

$$dt dt = 0, \quad dt d(B^i_t) = 0, \quad d(B^i_t) dt = 0, \quad d(B^i_t) d(B^j_t) = 0,$$

$$d(B^i_t) d\sigma^j_t = 0, \quad d\sigma^i_t dt = 0, \quad d\sigma^i_t d\sigma^j_t = 0,$$

$$d(B^i_t) d\sigma^j_t = 0, \quad d\sigma^i_t d\sigma^j_t = 0, \quad d\sigma^i_t d\sigma^j_t = d(B^i_t).$$

2.2. Multi-Indices

Let us introduce some notations about multi-indices for simplify statements and proof. We shall call a row vector $\alpha = (j_1, j_2, \ldots, j_l)$, where $j_i \in \{ -m, -(m - 1), \ldots, -1, 0, 1, 2, \ldots, m \}$, $i \in \{ 1, 2, \ldots, l \}$ and $m, l = 1, 2, 3, \ldots$ a multi-index of length $l := l(\alpha) \in \{ 1, 2, \ldots \}$.

Definition 5. [12] We denote the set of all multi-indices by $\mathcal{M}$, so

$$\mathcal{M} = \{ (j_1, j_2, \ldots, j_l) \colon j_i \in \{ -m, -(m - 1), \ldots, -1, 0, 1, 2, \ldots, m \}, \quad i \in \{ 1, 2, \ldots, l \}, l \in \{ 1, 2, 3, \ldots \} \cup \{ v \} \},$$

where $v$ is the multi-index of length zero.

We write $n(\alpha)$ for the number of components of a multi-index $\alpha$ that are equal to 0 and $s(\alpha)$ for the number of components of a multi-index $\alpha$ that are equal to $-1$. Moreover, we write $\alpha^-$ for the multi-index obtained by deleting the first component of $\alpha$ and $-\alpha$ for the multi-index obtained by deleting the first component of $\alpha$. $\alpha - (j)$ for the multi-index obtained by deleting the last component of $\alpha = (j_1, j_2, \ldots, j_k)$ so we can get the multi-index $(j_1, j_2, \ldots, j_k)$. Additionally, given two multi-indices $\alpha_1 = (j_1, j_2, \ldots, j_k)$ and $\alpha_2 = (i_1, i_2, \ldots, i_l)$, we introduce the concatenation operator $*$ on $\mathcal{M}$ defined by

$$\alpha_1 * \alpha_2 = (j_1, \ldots, j_k, i_1, \ldots, i_l),$$

where $\alpha_1, \alpha_2 \in \mathcal{M}$. The operator allows us to combine two multi-indices. For instance, assuming $m = 2$ one obtains

$$l((0, -1, 1)) = 3, \quad n((0, 1, -1, 2, 0)) = 2, \quad s((0, 1, -1, 2, 0)) = 1,$$

$$(0, -1, 1)^- = (0, -1), \quad (0, 1, -1) * (0, 2) = (0, 1, -1, 0, 2).$$
3. Main Results

In this section, by a component \( j \in \{1, 2, \ldots, m\} \) of a multi-index we will denote in a multiple stochastic integral the integration with respect to the \( j \)-th Wiener process. A component \( j = 0 \) will denote integration with respect to time. Lastly, a component \( j \in \{-m, -(m-1), \ldots, -1\} \) refer to an integration with respect to quadratic variation process. We shall define three sets of adapted right continuous stochastic processes \( g = \{g(t, \omega), t \in [0, T]\} \) with left hand limits.

\[
\mathcal{H}_0 = \{g: \sup_{t \in [0, T]} \mathbb{E}(|g(t, \omega)|) < \infty\};
\]

\[
\mathcal{H}(0) = \{g: \mathbb{E}\left(\int_0^T |g(t, \omega)| ds \right) < \infty\};
\]

\[
\mathcal{H}(j) = \{g: \mathbb{E}\left(\int_0^T |g(s, \omega)|^2 ds \right) < \infty\},
\]

where \( j \in \{1, 2, \ldots, m\} \).

**Definition 6.** Let \( g \) and \( \tau \) be two stopping times with \( 0 \leq g \leq \tau \leq T \) a.s.. Then for a multi-index \( \alpha \in \mathcal{M} \) and a process \( g(\cdot) \in \mathcal{H}_\alpha \), we define the multiple \( G-\)Itô integral \( I_\alpha[g(\cdot)]_{\rho, \tau} \) recursively by

\[
I_\alpha[g(\cdot)]_{\rho, \tau} = \begin{cases} 
\frac{g(\tau)}{\alpha}, & l = 0, \\
\int_\rho^\tau I_\alpha[g(\cdot)]_{\rho, \tau} d\zeta, & l \geq 1; j_l = 0, \\
\int_\rho^\tau I_\alpha[g(\cdot)]_{\rho, \tau} dB_{l\zeta}, & l \geq 1; j_l \in \{1, 2, \ldots, m\}, \\
\int_\rho^\tau I_\alpha[g(\cdot)]_{\rho, \tau} d\langle B^{-j_l}\rangle_{\zeta}, & l \geq 1; j_l \in \{-m, -(m-1), \ldots, -1\},
\end{cases}
\]

where \( g(\cdot) = g(\cdot, v_1, \ldots, v_{n(\alpha)}) \).

We use the following example to illustrate Definition 6: \( I_{(1)}[g(\cdot)]_{0, t} = g(t), I_{(0)}[g(\cdot)]_{0, t} = \int_0^t g(z)dz \),

\[
I_{(1)}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau g(z)dB_z,
\]

\[
I_{(2, 0)}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau \int_\rho^{z_2} g(z_1)dB_{z_1}dz_2,
\]

\[
I_{(0, -1)}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau \int_\rho^{z_2} g(z_1)d\langle B^{-j_l}\rangle_{z_2}.
\]

For a multi-index \( \alpha = (j_1, j_2, \ldots, j_l) \in \mathcal{M} \) and \( l(\alpha) > 1 \), we define the set \( \mathcal{H}_\alpha \) to be the totality of adapted right continuous processes \( g = \{g(t), t \geq 0\} \) with left hand limits such that the integral process \( \{I_\alpha[g(\cdot)]_{\rho, t}, t \in [0, T]\} \) considered as a function of \( t \) satisfies \( I_\alpha[g(\cdot)]_{\rho, t} \in \mathcal{H}_{(j_l)} \). For convenience we write \( I_{\alpha, t} = I_\alpha[1]_{0, t} \) and \( B_0^t = t \) for \( \alpha \in \mathcal{M}, t \geq 0 \).

Now, we will give our main theorems.

**Theorem 1.** For multi-index \( \alpha^n = (j_1, j_2, \ldots, j_n), j_i \in \{1, 2, \ldots, m\} \), where \( j_1, j_2, \ldots, j_n \) are not equal with each other. The set \( C(\alpha^n) \) be the all of the \( n \) level arrangement of \( \alpha^n \). We define

\[
C(\alpha^n) = \{(a_1, a_2, \ldots, a_n) | a_i \in \{j_1, j_2, \ldots, j_n\}, i = 1, \ldots, n, 2 \leq n \leq m\},
\]

such that

\[
H_{C(\alpha^n)} = \sum_{\alpha \in C(\alpha^n)} I_{\alpha, t} = \prod_{i=1}^n B_i^{n_t}.
\]
For $n = 2$, we have $I_{(i,j),t} + I_{(j,i),t} = \int_0^t \int_0^t dB_i^t dB_j^t = B_i^t B_j^t$;

For $n = k$ we have $H_{C^{(α^k)}} = \sum_{α ∈ C^{(α^k)}} I_{α,t} = \prod_{i=1}^k B_i^j$. We need to prove that

$$H_{C^{(α^{k+1})}} = \sum_{α ∈ C^{(α^{k+1})}} I_{α,t} = \prod_{i=1}^{k+1} B_i^j.$$ 

Actually, we only need to prove that

$$\sum_{l=1}^{k+1} \int_0^t H_{C^{(α^{k+1} - (j_l))},t} dB_i^j = \sum_{l=1}^{k+1} \int_0^t \prod_{i=1, i \neq l}^k B_i^j dB_i^j = \prod_{i=1}^{k+1} B_i^j. \quad (2)$$

where $α = (j_1, j_2, \ldots, j_k, j)$ and $α^{k+1} - (j_l)$ for the $l$-index obtained by deleting the last component $j_l$ of $α^{k+1}$. Applying $G$-Itô formula and independence of Brown motion, one has

$$d \prod_{i=1}^{k+1} B_i^j = \sum_{l=1, i \neq l}^{k+1} B_i^j dB_i^j. \quad (3)$$

Taking integral on Equation (3) and combined with Equation (2), the proof is completed.

**Example 1.** For $i, j, k ∈ \{1, 2, 3, \ldots, m\}$, and $i, j, k$ are different with each other. Using $G$-Itô formula and Theorem 1, we can get

$$I_{(i,j,k),t} + I_{(i,k,j),t} + I_{(k,i,j),t} + I_{(i,j,i),t} + I_{(j,k,i),t} + I_{(j,i,j),t}$$

$$= \int_0^t B_i^t B_j^t dB^k + \int_0^t B_j^t B_i^t dB^k + \int_0^t B_i^t B_j^t dB^k$$

$$= B_i^t B_j^t B_k^t.$$ 

Now we shall prove the recursive relationship between multiple $G$-Itô integrals.

**Theorem 2.** Let $j_1, \ldots, j_l ∈ \{0, 1, \ldots, m\}$ and $α = (j_1, \ldots, j_l) ∈ M$, where $l = 1, 2, 3, \ldots$ Then for $t ≥ 0$,

$$B_i^t I_{α,t} = \sum_{i=0}^{l} I_{(j_1, j_2, \ldots, j_{l-1}, j_l),t} + \sum_{i=1}^{l} I_{(j_1, j_2, \ldots, j_l),t}$$

$$= \prod_{i=1}^{k+1} B_i^j.$$ 

**Proof.** We consider multi linear $G$-Itô process $X = \{X_t, t ≥ 0\}$, which defined as follows

$$X_t = (X_t^{(0)}, X_t^{(m)}, X_t^{(j_1, j_2)}, \ldots, X_t^{(j_1, j_2, j_3)}, \ldots, X_t^{(j_1, \ldots, j_l)})^T$$

$$= (I_{(0),t}, I_{(m),t}, I_{(j_1, j_2),t}, \ldots, I_{(j_1, j_2, j_3),t}, \ldots, I_{(j_1, \ldots, j_l),t})^T, \quad (5)$$

where each component of $X_t$ is a multi $G$-Itô integral. For $β$-th component $β(j_1', \ldots, j_r')$ the coefficients are

$$a^β = \left\{ \begin{array}{ll} x^β, & j_r' = 0, \\ 0, & \text{otherwise} \end{array} \right., \quad b^β = \left\{ \begin{array}{ll} x^β, & j = j_r' ∈ \{1, \ldots, m\}, \\ 0, & \text{otherwise} \end{array} \right.,$$

$$c^β = \left\{ \begin{array}{ll} x^β, & j = j_r' ∈ \{-m, -(m-1), \ldots, -1\}, \\ 0, & \text{otherwise} \end{array} \right..$$
In this work, we define $G$-ple integrals. As discussed in Section 1, this effort focuses on multi-dimensional $G$-Itô integral driven by multi-dimensional $G$-Brownian motion in $G$-expectation space. And we use $G$-Itô formula and mathematical induction to obtain a kind of multiple $G$-Itô integrals. For $j = 1, \alpha = (0, 1)$, from Theorem 2 it follows that

$$B_1^1 I_{(0,1),t} = I_{(j_1,j_1,j_2),t} + I_{(j_1,j_2,j_2),t} + I_{(j_1,j_2,j_3),t} + I_{(j_1,j_2,j_3),t} + I_{(j_2,j_1),t} \cdot I_{(j_1,-j_2),t}$$

For $j = 2, \alpha = (0, 1, 3)$, applying the Theorem 2 we can get

$$B_2^1 I_{(0,1,3),t} = I_{(j_1,j_2,j_3,j_1),t} + I_{(j_1,j_2,j_3,j_2),t} + I_{(j_1,j_2,j_3,j_3),t} + I_{(j_1,j_2,j_3,j_4),t} + I_{(j_2,j_1),t} \cdot I_{(j_1,j_2,j_3),t}$$

The proof is completed.

Example 2. Particularly, for $j = 1, \alpha = (0, 1, 3)$, from Theorem 2 it follows that

$$B_1^1 I_{(0,1,3),t} = I_{(j_1,j_1,j_2),t} + I_{(j_1,j_2,j_2),t} + I_{(j_1,j_2,j_3),t} + I_{(j_1,j_2,j_3),t} + I_{(j_2,j_1),t} \cdot I_{(j_1,-j_2),t}$$

For $j = 2, \alpha = (0, 1, 3)$, applying the Theorem 2 we can get

$$B_2^1 I_{(0,1,3),t} = I_{(j_1,j_2,j_3,j_1),t} + I_{(j_1,j_2,j_3,j_2),t} + I_{(j_1,j_2,j_3,j_3),t} + I_{(j_1,j_2,j_3,j_4),t} + I_{(j_2,j_1),t} \cdot I_{(j_1,j_2,j_3),t}$$

4. Concluding Remarks and Future Work

In this work, we define $G$-Itô integral driven by multi-dimensional $G$-Brownian motion in $G$-expectation space. And we use $G$-Itô formula and mathematical induction to obtain a kind of multiple $G$-Itô integrals. As discussed in Section 1, this effort focuses on multiple $G$-Itô integrals driven by multi-dimensional $G$-Brownian motion rather than one-dimensional $G$-Brownian motion. Our future efforts will focus on introducing the properties of Stratonovich integral driven by multi-dimensional $G$-Brownian motion, and exploring the relationship between Stratonovich integral and $G$-Itô integral under the $G$-expectation framework.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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