Finite size effects in metallic superlattice systems

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Abstract

Clean metallic superlattice systems composed of alternating layers of superconducting and normal materials are considered, particularly aspects of the proximity effect as it affects the critical temperature. A simple model is used to address the question of when a finite–sized system theoretically approximates well a true infinite superlattice. The methods used in the analysis afford some tests of the approximation used that the pair amplitude of the Cooper pairs is constant over a superconducting region. We also use these methods to construct a model of a single superconducting layer which intends to incorporate a more realistic form of the pair amplitude than a simple constant.
I. INTRODUCTION

Metallic superlattice systems consisting of alternating layers of superconducting and normal films have attracted wide interest, both experimentally and theoretically [1]. Much of the theoretical work has gone into the understanding of the proximity effect in such systems. This effect between adjacent layers of materials with different superconducting properties has been studied extensively for the case of two such neighbouring films. Close to the transition point Ginzburg–Landau theory can be employed [2–7], but away from this point other approaches such as the constant pair amplitude approximation [8–15], a “Cooper” proximity effect argument [16], the tunneling model of McMillan [17], a WKB approximation [18], or a semi–classical approximation [19] can be used.

Implicit in these approaches when applied to superlattices is the argument that, from symmetry considerations, the form of the pair amplitude of the Cooper pairs for a bilayer is the same as that for a superlattice consisting of an infinite number of adjoining bilayers. While true for an infinite superlattice, experimentally these systems are of a finite extent, and one could then question the reasonableness of the bilayer assumption for finite systems. This issue could also arise in the general context of the applicability of Bloch’s theorem for periodic potentials in cases of large but finite–sized systems, as for instance as applied to the Kronig–Penney model [20]. In this paper we will address some aspects of this question of how many bilayers are effectively needed to have an infinite superlattice system, particularly as related to how the critical temperature of such systems is affected.

The paper is organized as follows. To introduce the approach and to establish notation, in Sec. II we review a method of analyzing the proximity effect based on the constant pair amplitude approximation as applied to the simple case of a single superconducting film immersed in a normal metal [13]. In Sec. III we extend this method to an infinite superlattice composed of alternating superconducting and normal layers. To test some of the ideas we shall later employ, we consider the superlattice as the limit of a finite sized system as the number of layers approaches infinity; the result we find for the critical temperature agrees
with that obtained by arguments which explicitly incorporate Bloch’s theorem using the same general approach [21]. In Sec. IV we consider finite–sized “superlattices”. We argue from the calculated form of the pair amplitude that the approximation of a constant pair potential is reasonable in two distinct cases: for thick superconducting films over a wide range of thicknesses of the normal film, and for thin superconducting and normal films. As related to finding the critical temperature in this approach, in the first case a relatively small number of layers are needed before the finite–sized system approximates well an infinite superlattice, whereas in the second case quantitative differences arise between the finite and infinite lattices. In Sec. V we reconsider the single superconducting layer of Sec. II regarding the issue raised in the previous section of the reasonableness of the constant pair potential assumption. We study a model based on these methods which aims to incorporate in a simple manner a more realistic form of the pair potential, and find that the resulting zero temperature critical thickness gets enhanced compared to that of Sec. II found using the constant pair potential approximation. Sec. VI contains some concluding remarks.

II. A SINGLE FILM

We first study the case of a single superconducting film embedded in a normal metal [13]. The film, of width $2a$, is centered at the origin $x = 0$, as in Fig. 1. The Hamiltonian for this system in terms of the electron wave function $\psi$ is given in the mean field approximation by

$$H = \int d^3r \, g(x) \, |F(x)|^2 + \int d^3r \, \phi^\dagger(r) \left[ -\frac{\nabla^2}{2m} - \mu \right] \tau_3 \phi(r)$$

$$- \int d^3r \, g(x) F(x) \phi^\dagger(r) \tau_1 \phi(r),$$

where $\tau_i$ are the Pauli matrices, $F(x) = F^*(x) = -\langle \psi^\uparrow(x) \psi^\downarrow(x) \rangle$ is the pair amplitude,

$$\phi(r) = \begin{cases} \psi^\uparrow(r) \\ \psi^\dagger(r) \end{cases},$$

and $g(x)$ denotes the spatially inhomogeneous BCS coupling constant. We assume the superconducting and normal metals are clean and have the same effective electron masses,
Fermi velocities, Debye temperatures, etc.

The Nambu doublet is then decomposed as

$$\phi(\mathbf{r},t) = \sum_i \frac{1}{(2\pi)^3} \int dE \int d^2l \left[ u^{(i)}(E,l,x)a^{(i)}(E,l)e^{-i(Et-l\cdot\rho)} + v^{(i)}(E,l,x)b^{(i)\dagger}(E,l)e^{i(Et-l\cdot\rho)} \right],$$

(3)

where $\mathbf{l} = (k_y,k_z)$, $\rho = (y,z)$, and “$i$” labels the particular linearly independent solution present. If we now introduce

$$u^{(i)}(E,l,x) = \chi^{(i)}(E,l,x)e^{iqx},$$

(4)

where $q = \sqrt{k_F^2 - l^2}$, then assuming $\chi(E,l,x)$ to be smooth on the atomic scale results in the Bogoliubov equation following from Eq.(1):

$$\left\{ E + i\tau_3 q \frac{d}{m} \frac{dx}{x} + g(x)F(x)\tau_1 \right\} \chi^{(i)}(E,l,x) = 0.$$  

(5)

The corresponding wave function $v(E,l,x) = u(-E,l,x)$ can then be readily obtained.

The non–linearity introduced by $F(x) = -\langle \psi_\uparrow(x)\psi_\downarrow(x) \rangle$ makes Eq.(5) difficult to solve for inhomogeneous geometries. To obtain a simple solution we therefore make the approximation

$$g(x)F(x) = \begin{cases} \Delta & \text{for } -a < x < a \\ 0 & \text{for } x < -a \text{ or } x > a \end{cases},$$

(6)

where $\Delta$ is a constant. The solutions to Eq.(5) under this approximation are readily found to be

$$\chi(E,l,x) = \begin{cases} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ikx}, & B \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ikx} & \text{for } x < -a \text{ or } x > a, \\ C \begin{pmatrix} 1 + \gamma \\ \delta \end{pmatrix} e^{ipx}, & D \begin{pmatrix} \delta \\ 1 + \gamma \end{pmatrix} e^{-ipx} & \text{for } -a < x < a \end{cases},$$

(7)

where

$$p = \gamma k = mE\gamma/q = mE\gamma/\sqrt{k_F^2 - l^2},$$

$$\delta = -\Delta/E,$$

$$\gamma = \sqrt{1 - \delta^2}.$$  

(8)
The coefficients of the wave function entering into the solutions of Eq.(7) can be determined by imposing continuity at the interface \( x = \pm a \) together with the normalization conditions

\[
\int_{-\infty}^{\infty} dx \, \pi(E, l, x) u(E', l, x) = 2\pi \delta(E - E') = \int_{-\infty}^{\infty} dx \, \pi(E, l, x) v(E', l, x),
\]

\[
\int_{-\infty}^{\infty} dx \, \pi(E, l, x) v(E', l, x) = 0.
\]

(9)

Thus far these solutions contain the free parameter \( \Delta \). We fix this parameter by requiring that it minimize the free energy of the system:

\[
\left\langle \frac{\partial H}{\partial \Delta} \right\rangle = 0,
\]

(10)

where the angular brackets denote the thermal average. Using Eq.(1) we find that this condition translates into the self–consistent gap equation

\[
\Delta = \frac{g}{2a} \int_{-a}^{a} F(x, \Delta) \, dx.
\]

(11)

In other words, we demand that the spatial average of the calculated pair amplitude be equal to the constant \( \Delta \) assumed in finding the solution.

In the limit \( \Delta \to 0 \) the gap equation (11) will determine the critical temperature \( T_c(a) \) as a function of the thickness \( a \) of the superconducting film. Using the solutions and boundary conditions discussed previously, we find this equation becomes, assuming \( T_c(a) \ll T_d \),

\[
\frac{1}{g N(0)} = H(\omega_d) - \frac{\beta}{2} \int_{0}^{\infty} d\omega \, H(\omega) \cosh^{-2}(\beta \omega/2),
\]

(12)

where \( \beta = 1/k_B T \), \( N(0) = mk_F/2\pi^2 \) is the density of states of one spin projection at the Fermi surface and, with \( \Lambda = 2k_F a \) and \( \Omega = \omega/E_F \), the function \( H(\omega) = H(\Lambda \Omega) \) is given by

\[
H(x) = \gamma_e + \ln(x) - \frac{1}{2} G(x),
\]

\[
G(x) = \cos(x) - \frac{\sin(x)}{x} + 2 \text{Ci}(x) + x \text{si}(x).
\]

(13)

Here \( \text{Ci}(x) \) and \( \text{si}(x) \) are the Cosine and Sine integrals defined as

\[
\text{Si}(x) = \text{si}(x) + \frac{\pi}{2} = \int_{0}^{x} \frac{\sin t}{t} \, dt,
\]

\[
\text{Ci}(x) = \gamma_e + \ln x + \int_{0}^{x} \frac{\cos t - 1}{t} \, dt,
\]

(14)
and $\gamma_e$ is Euler’s constant. The function $H(x)$ of Eq.(13) has the following limiting forms:

$$
\lim_{x \to \infty} H(x) = \gamma_e + \ln(x) + O(1/x),
$$
$$
\lim_{x \to 0} H(x) = \frac{\pi x}{4} + O(x^2). 
$$

(15)

For comparison with other geometries the numerical solution to Eq.(12) giving the critical temperature $T_c(a)$ will be discussed in Sec. IV. For now we only note that Eq.(12) can be solved analytically in the zero temperature limit, which gives the critical thickness $a_c$ below which superconductivity cannot be maintained at any temperature: $T_c(a_c) = 0$. One finds, assuming $\Lambda_c \Omega_d \gg 1$,

$$
a_c = \frac{1}{\pi k_F} \left( \frac{T_F}{T_c} \right) = \frac{1}{2} \pi e^{-\gamma_e} \xi_0 \approx 0.882 \xi_0, 
$$

(16)

where $T_c$ is the bulk critical temperature and $\xi_0 = v_F / \pi \Delta_0$ is the bulk coherence length. This value of the critical thickness compares favourably with that found using other approaches [17], and will be used in subsequent sections as a convenient length scale when discussing other geometries. It is also useful for later comparisons with other geometries to introduce $a_c$ directly into the gap equation of Eq.(12); doing so leads to the following equivalent forms:

$$
\ln \left[ \frac{a}{a_c} \right] = I(\tau),
$$
$$
\ln \left[ \frac{T_c}{T_c(a)} \right] = \frac{\pi}{8\tau} + \frac{1}{2} K(\tau), 
$$

(17)

where

$$
\tau = \frac{2 a T_c(a)}{\pi a_c T_c},
$$
$$
I(x) = \frac{1}{2} \pi x \ln 2 - x \int_0^\infty \frac{dt \sin(2t)}{t^2} \ln[1 + \exp(-2t/x)],
$$
$$
K(x) = \int_1^\infty dt \left[ 1 - \frac{1}{t^2} \right] \ln \tanh \left( \frac{\pi x t}{2} \right). 
$$

(18)

The feature of these and the following gap equations that the thicknesses of the superconducting and normal layers scale with $a_c$ makes it possible that dirty materials might also be described within the present framework by a simple reinterpretation of $a_c$, as was found in the single layer case by a comparison with the tunneling model [15,17].
III. INFINITE SUPERLATTICE

We next study an infinite superlattice consisting of alternating superconducting films of width $2a$ separated by normal films of width $2b$. This is illustrated in Fig. 2. Using the approach of the previous section, we first consider $N$ such films and eventually take $N \to \infty$. We assume the pair potential $g(x)F(x)$ in each superconducting layer is a constant $\Delta$ and vanishes in each normal layer. This allows us to solve the corresponding Bogoliubov equations (5) in each region subject to the appropriate boundary conditions. The parameter $\Delta$ is then determined by the minimization of the free energy of the system. This condition, in the limit $\Delta \to 0$, gives the critical temperature $T_c(a, b)$ as a function of the thicknesses $a$ and $b$ of the films. Taking the limit $N \to \infty$ then results in the gap equation

$$\ln(a/a_c) = I(\tau) - J(\tau, y),$$

where $y \equiv 1 + b/a$, $\tau$ and $I(x)$ appear in Eq.(18), and

$$J(x, y) = x \int_0^\infty \frac{dt}{t} \frac{\sin^2(t)}{t^2} \cot[ty] \ln \cosh(t/x).$$

For infinite $b$ the function $J(\tau, y)$ vanishes, resulting in the gap equation of Eq.(17) for the single layer. $J(\tau, y)$ thus represents coherence effects the other layers introduce to the single film geometry. Eq.(19) agrees with the results of an analysis directly incorporating Bloch’s theorem in this context for periodic systems [21]. We present in the next section the numerical solution to Eq.(19) in various cases and compare these results with those corresponding to a finite number of layers using this same general approach.

IV. FINITE SUPERLATTICE

In this section we examine by these methods a “superlattice” with a finite number of layers. As with the true infinite superlattice this consists of alternating superconducting films of width $2a$ separated by normal films of width $2b$, as in Fig. 2, but now after the last superconducting layer on each end we assume a normal metal fills the remainder of space.
We again assume the pair potential $g(x)F(x)$ in each superconducting layer is a constant, but unlike the infinite superlattice there is no reason a priori to assume the same constant in each layer. We therefore assume $g(x)F(x)$ is some constant $\Delta_j$ for the $j^{th}$ superconducting layer, and as before vanishes in each normal layer. This is illustrated in Fig. 3. With appropriate boundary conditions the Bogoliubov equations (3) are readily solved, and the parameters $\Delta_j$ are again determined by the minimization of the free energy of the system. This again translates into equating the spatial average of the calculated pair potential in the $j^{th}$ layer to the assumed constant $\Delta_j$. These conditions now are a coupled set of $N + 1$ equations, which in the limit $\Delta_j \to 0$ and $\Lambda \Omega_d \gg 1$ are given by

$$\ln \left[ \frac{T_c(a, b, N)}{T_c} \right] = -\frac{\pi}{16\tau}$$

$$-\frac{1}{4} \left\{ K(\tau) - 2 j y K(2 j y \tau) + (1 + 2 j y) K[(1 + 2 j y) \tau] \right\}$$

$$+ \frac{1}{4} \sum_{l=j+1}^{N} \frac{\Delta_l}{\Delta_j} \left\{ 2(l + j) y K[(l + j) y \tau] + 2(l - j) y K[(l - j) y \tau] \right\}$$

$$- [1 + (l + j) y] K[1 + (l + j) y \tau] - [1 + (l - j) y] K[1 + (l - j) y \tau]$$

$$- [-1 + (l + j) y] K[-1 + (l + j) y \tau] - [-1 + (l - j) y] K[-1 + (l - j) y \tau] \right\},$$

where $j = 0, 1, 2, \ldots, N$, $y = 1 + b/a$, and $\tau$ and $K(x)$ appear in Eq.(18). As a check, if we consider the single layer limit with $N = 0$, then the system of equations (21) reduces to a single equation, which agrees with the gap equation (17) derived before.

We first consider the $j = N^{th}$ equation of the set (21), for which the term involving the summation over $\Delta_l/\Delta_j$ does not contribute. This equation will determine the critical temperature $T_c(a, b, N)$ as a function of the thicknesses $a$ and $b$ of the films and of the number of superconducting layers present, $2N + 1$. In Figs. 4 and 5 we present the numerical solution to this equation for different values of $b$ and $N$, along with the results for the infinite superlattice of Eq.(19) and for the single layer of Eq.(17). Although for all values of $b$ and $N$ the correct bulk limit as $a \to \infty$ is reached, quantitative differences with the infinite superlattice are obtained for smaller values of $a/a_c$ for the finite lattice. In particular, as $b/a_c$ increases in the finite lattice the curves do not approach that of the single layer case,
as might be expected and is found for the infinite superlattice.

Before concluding anything from these results, however, let us examine the other equations of the set (21). For \( j = 0, 1, 2, \ldots, N - 1 \) these equations involve the term containing the summation over \( \Delta_l/\Delta_j \), and can be used to express, for example, \( \Delta_1, \Delta_2, \ldots, \Delta_N \) in terms of the value of the center layer, \( \Delta_0 \). In Figs. 6–9 we present the numerical solution to these equations at zero temperature for some representative values of \( b/a_c \) and \( N \). We would expect that as one moved out from the central region the superconductivity would become weaker in some sense, which would translate into decreasing values of the ratio \( \Delta_j/\Delta_0 \) as \( j \) increases. This expectation is borne out for smaller values of \( b/a_c \) for a large range of \( N \), but not for larger values of \( b/a_c \). At finite temperature, illustrated in Figs. 10 and 11, the expected behaviour is again found for smaller values of \( b/a_c \) but not completely for the larger values, although the situation improves as the temperature increases. What one might conclude from this analysis is that the method used in this context seems reliable for thin normal films over a wide range of thicknesses of the superconducting layers, but for thicker normal films the method seems reasonable only for thicker superconducting layers (i.e., larger critical temperatures). This could indicate that the approximation of a constant pair potential in the superconducting layer appears reasonable in this range of validity, for which little spatial variation in the true potential might be expected.

With the preceding discussion in mind we now return to a comparison of the finite superlattice results of the critical temperature in Figs. 4 and 5 with that of the infinite superlattice. We restrict this comparison to a potentially large but still finite number of layers in the finite system, as in the limit \( N \to \infty \) our initial assumption of the independence of the gap parameters \( \Delta_j \) in each layer will no longer be valid. Given this, for thicker superconducting layers \( (a/a_c > 10) \), where the approximation of a constant pair potential seems reasonable over a wide range of thicknesses \( b/a_c \) of the normal layer, we could conclude for these methods that a finite superlattice approximates well an infinite superlattice for a reasonably small number of layers \( (N \sim 10) \). On the other hand, for thinner superconducting layers \( (a/a_c < 10) \), the approximation of a constant pair potential seems reasonable only for
thin normal layers \((b/a_c < 0.1)\). In this range we could conclude that there are noticeable differences between the finite lattice and the infinite superlattice in this approach, as seen from Fig. 4, even with a relatively large number of layers for the finite system. Qualitatively what is found is that, for a given thickness \(a/a_c\), the corresponding critical temperature in the finite lattice is smaller compared to the case of the infinite superlattice. This feature could be understood in terms of the form of the pair potential suggested by Fig. 3 and found in Figs. 6, 7, 10, and 11, where the gap parameters decrease relative to the central value as one moves towards the edges. If one assumes this gap parameter ratio measures in a way the relative strength of the binding of the Cooper pairs in each region, then from this one might infer that the superconductivity is in a sense “weaker” towards the edges for the finite potential of Fig. 3 compared to the case of the same constant gap parameter assumed over each superconducting region in an infinite superlattice, as in Fig. 2. Thermal excitations would then more readily break superconductivity in these outer regions, leading to a lower critical temperature for a given thickness \(a/a_c\) in the finite lattice compared to the infinite superlattice. Such a tendency of weaker superconductivity in the finite lattice continues to hold except as the thickness of the superconducting layer decreases further \((a/a_c < 0.1)\); in the extreme limit this is a reflection of the fact that the approximation \(\Lambda \Omega_d = 2k_F a \Omega_d \gg 1\) used in deriving Eq.\((21)\) is breaking down.

V. THE SINGLE LAYER REVISITED

The results of the last section raised questions concerning the appropriateness of the assumption of a constant pair amplitude in a superconducting layer. In this section we will attempt to improve on this assumption for the case of the single layer described in Sec. II. Specifically, we imagine for the situation of \(2N + 1\) layers in Fig. 2 letting the width \(b\) of the normal layer approach zero. In effect we have then artificially divided a superconducting single layer of width \(2A\), as in Fig. 12, into \(2N + 1\) layers, each of width \(2a\). As in the last section we next assume the gap parameter in each layer is some constant \(\Delta_j\), and then set
up and solve the system of self-consistent gap equations found by a minimization of the free energy. This again leads to a coupled set of equations, and as in the previous section the \( j = N^{th} \) equation for the layer on the extreme edge will determine the transition temperature as a function of the film thickness \( a \) and the number of layers \( N \). One obtains

\[
\ln \left( \frac{a}{a_c} \right) = \frac{1}{2} \left[ 2N \ln(2N) - (1 + 2N) \ln(1 + 2N) \right] + \frac{1}{4} \{ -\alpha \pi (1 + 2N) \\
+ \frac{\alpha \cos(\alpha) - \sin(\alpha)}{\alpha} - \frac{2N \cos(2N\alpha) - \sin(2N\alpha)}{\alpha} \\
+ \frac{(1 + 2N) \alpha \cos[(1 + 2N)\alpha] - \sin[(1 + 2N)\alpha]}{\alpha}
\}
\]

\[ + \{2 \text{Ci}(\alpha) + \alpha \text{Si}(\alpha)\} - (2N) \{2 \text{Ci}(2N\alpha) + 2N\alpha \text{Si}(2N\alpha)\} \]

\[ + (1 + 2N) \{2 \text{Ci}[(1 + 2N)\alpha] + (1 + 2N)\alpha \text{Si}[(1 + 2N)\alpha]\} \]

\[ + \frac{1}{2\tau} \int_0^{\infty} dx L(x) \cosh^{-2}(x/2\tau), \] (22)

where \( \tau \) appears in Eq.(18),

\[
\alpha = 2\Lambda \Omega_d = \frac{2}{\pi} \frac{a}{a_c} \frac{T_d}{T_c}; \]

\[
L(x) = \gamma_e + \ln(x) - \frac{1}{2} \left[ 2N \ln(2N) - (1 + 2N) \ln(1 + 2N) \right] \\
- \frac{1}{4} \left\{ G(x) - 2NG(2Nx) + (1 + 2N)G[(1 + 2N)x] \right\}, \] (23)

\( G(x) \) is given in Eq.(13), and the Cosine and Sine integrals \( \text{Ci}(x) \) and \( \text{Si}(x) \) appear in Eq.(14). In Eq.(22) we then take \( a \to 0 \) and \( N \to \infty \) in such a way as to keep the product \( 2A = (2N + 1)2a \) fixed. We first consider the zero temperature case, by which this limiting procedure will yield the critical thickness \( A_c \) in this model below which superconductivity cannot be maintained at any temperature. We find

\[
\frac{A_c}{a_c} = \frac{2}{\pi} \frac{T_d}{T_c} e^{\gamma_e - 1} = \exp \left[ \frac{1}{gN(0)} - 1 \right]. \] (24)

As for BCS superconductors \( gN(0) \) ranges from about 0.2 to 0.4, this critical thickness \( A_c \) is larger by about a factor of 4 to 50 compared to the critical thickness \( a_c \) obtained by the method described in Sec. II, where the pair potential was assumed to be constant over the entire superconducting layer. As well, Eq.(24) reinforces the trend that the critical thickness
would be larger for weaker BCS superconductors with smaller values of $gN(0)$, as one might expect.

As with the case of thin normal and superconducting layers of the finite lattice in the last section, these features of the critical thickness can be interpreted in terms of the form of the pair potential suggested by Fig. 12, where the gap parameters decrease relative to the central value as one moves towards the edges. From this one might infer that the superconductivity is in a sense “weaker” towards the edges for such a potential compared to the case of a constant gap parameter assumed over the entire superconducting region, as in Fig. 1. If one pictures the breaking of superconductivity as the “leaking” of Cooper pairs into the normal region, then one would expect that the form of the more “realistic” pair potential in Fig. 14 would lead to an earlier destruction of superconductivity compared to the constant pair potential of Fig. 1.

One can also analyze the gap equation \((22)\) at finite temperature. For example, in the vicinity of zero temperature one finds

$$
\frac{T_c(A)}{T_c} = \frac{\pi}{4} \frac{T_c}{T_d} A^{1-\gamma_e} \left[ \frac{A}{A_c} - 1 \right].
$$

However, at higher temperatures this method of analyzing the single layer case fails, and in particular gives the wrong bulk limit as $A \to \infty$. This would indicate the assumption in this model of a constant pair amplitude for thick initial superconducting film layers is invalid.

### VI. CONCLUSIONS

We have considered the question of when a finite sized system composed of alternating layers of superconducting and normal metals can effectively be thought of as a true infinite superlattice. Concentrating on the calculation of the critical temperature, we extended some methods used previously for a single superconducting layer and an infinite superlattice to such finite-sized systems. By examining the form of the pair potential found we argued that the approximation of a constant Cooper pair potential in each superconducting region seems
reasonable for thick superconducting films \((a/a_c > 10)\) over a wide range of thicknesses of the normal films or else for thin superconducting \((a/a_c < 10)\) and thin normal \((b/a_c < 0.1)\) films. The results suggest that for thick superconducting films it takes a relatively small number of layers \((N \sim 10)\) using these methods to effectively make an infinite superlattice. However, for thin superconducting and normal films noticeable differences were found in this approach between the finite and infinite superlattices, even for a relatively large number of layers for the finite lattice. The corresponding decrease in the critical temperature of the finite lattice at a fixed thickness of the superconducting layer was attributed to the “weaker” superconductivity in the outer regions compared to the case of the constant pair potential assumption of the infinite superlattice. One might argue that such “weaker” zones in finite but large systems would be true for more realistic forms of the pair amplitude incorporating some spatial variation. We also applied these methods in the case of a single superconducting layer to construct a model which attempts to incorporate a more realistic form of the pair potential than the assumed constant. The model suggested that the zero temperature critical thickness will be enhanced with such a pair potential compared to that found by assuming a constant pair potential, and also exhibited the trend that weaker BCS superconductors will have a larger value of this thickness. These features were again attributed to the “weaker” superconductivity near the edges of a more realistic pair potential.

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Figure Captions

Fig. 1: The assumed form of the pair potential for a single superconducting layer.

Fig. 2: The assumed form of the pair potential for an infinite superlattice.

Fig. 3: The assumed form of the pair potential for a finite “superlattice” of $2N + 1$ layers.

Fig. 4: The critical temperature as a function of $a/a_c$ for a single layer, an infinite superlattice of $b/a_c = 0.05$, and a finite “superlattice” with $b/a_c = 0.05$ and $N = 10, 40$ and $100$.

Fig. 5: The critical temperature as a function of $a/a_c$ for a single layer, an infinite superlattice of $b/a_c = 5.0$, and a finite “superlattice” with $b/a_c = 5.0$ and $N = 10, 40$ and $100$.

Fig. 6: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at zero temperature, with $N = 40$ and $b/a_c = 0.01$.

Fig. 7: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at zero temperature, with $N = 10$ and $b/a_c = 0.01$.

Fig. 8: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at zero temperature, with $N = 40$ and $b/a_c = 0.5$.

Fig. 9: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at zero temperature, with $N = 10$ and $b/a_c = 0.5$.

Fig. 10: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at finite temperature $T_c(a, b, N)/T_c = 0.283$, with $N = 40$ and $b/a_c = 0.5$.

Fig. 11: The gap parameter ratio $\Delta_j/\Delta_0$ for the finite “superlattice” as a function of the $j^{th}$ layer at finite temperature $T_c(a, b, N)/T_c = 0.282$, with $N = 10$ and $b/a_c = 0.5$.

Fig. 12: The assumed form of the pair potential for the single layer superconducting film of width $2A$ artificially divided into $2N + 1$ layers.
Figure 5: $b/\alpha c = 5.0$

- Finite lattice ($N=100$)
- Finite lattice ($N=40$)
- Finite lattice ($N=10$)
- Infinite lattice
- Single layer
Figure 6: N=40, b/ac=0.01
Figure 7: N=10, b/ac=0.01
Figure 8: N=40, b/ac=0.5
Figure 9: $N=10$, $b/ac=0.5$
Figure 10: N=40, b/ac=0.5

T/Tc=0.283
Figure 11: $N=10$, $b/ac=0.5$
Figure 1
Figure 2
Figure 3
Figure 12