Notes on thermodynamics in special relativity

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Abstract
The foundations of thermodynamics in the special theory of relativity are considered. We argue that, from the phenomenological point of view, the most natural relativistic transformations of heat and absolute temperature are given by the formulae proposed by Ott, Arzeliès and Møller. It is shown that the same transformation rules can also be found from the relativistic Gibbs distribution for an ideal gas. This distribution has recently been verified by computer simulations. Phenomenological and statistical thermometers in relativistic thermodynamics are analyzed.

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1. Introduction
The present paper is an extended and improved version of the previous work of one of us (MP) [1] and is devoted to the foundations of relativistic thermodynamics in the special theory of relativity; see the next section for details, we note here that there is a long-standing problem concerning the relativistic transformation rules for heat and temperature. First, Einstein [2], Planck [3], von Mosengeil [4], Pauli [5], von Laue [6] and many others found transformation formulae (2.18) and (2.28). In 1952, Einstein changed his primary opinion in favor of new formulae (2.17) and (2.27) [7], which were also found later on by Ott [8], Arzeliès [9] and Møller [10–12]. (It is worthwhile to point out that in the first edition of Møller’s book of 1952 the ‘classical’ transformation rules (2.18) and (2.28) were considered to be correct, and in the second edition from 1972 [12] things have changed and formulae (2.17) and (2.27) have been given as the correct ones.) Note that perhaps Landau and Lifshitz [13] were the first authors who mentioned the transformation formula for $T$ equivalent to (2.27) in the Russian edition of their famous monograph on statistical physics in 1951 (see equation (27.4) in [13]).

But it is not the complete story. Landsberg and co workers in [14–17] argue that heat and temperature are Lorentz invariants, i.e. (2.19) and (2.29) hold true, and in other works [18–20], they claim that there does not exist any universal relativistic transformation of temperature. Recently, this point of view was strongly supported in [21, 22]. The idea that heat and temperature are Lorentz invariants has also been analyzed by van Kampen [23].

As shown by Dunkel et al [24], these three different classes of transformation rules follow from the choice of isochronous spacetime hypersurfaces as well as definitions of heat and work.

The above very brief review of the development of opinions on the transformation rules for heat and temperature in relativistic thermodynamics does not sound optimistic. It seems that the eventual relativistic thermodynamics depends very much on accepted conventions and that there does not exist any natural system of conventions which leads to a correct construction of relativistic thermodynamics. However, in our opinion there exist preferred definitions of internal energy, heat and temperature. A strong confirmation of this opinion we found in distinguished papers by Møller [10, 11] and, recently, in a nice work by Requardt [25]. In the mentioned works it is argued that there exists a natural definition of thermodynamic work and then, by the first law of thermodynamics, the natural definition of heat. Moreover, from the second law of thermodynamics one arrives at the natural definition of absolute temperature. This temperature can be experimentally determined with the use of an appropriate cyclic thermodynamic engine. Having defined heat and temperature it is shown that the respective transformation rules are given by (2.17) and (2.27). In section 2, we analyze all these questions not only for ideal liquid but also for an arbitrary continuous medium and finally we arrive at the conclusion that the most natural
transformation formulae for heat and absolute temperature in relativistic thermodynamics indeed are given by (2.17) and (2.27).

It is expected that, analogously to nonrelativistic thermodynamics, deeper insight into the problems of phenomenological relativistic thermodynamics can be acquired when the respective statistical thermodynamics is applied. But here we are in trouble since, as yet, there does not exist a general relativistic statistical thermodynamics.

In 1911, Jüttner [26] found the relativistic generalization of the Maxwell distribution to ideal gas. The Jüttner distribution was generalized to the case of a moving ideal gas and the Bose–Einstein or Fermi–Dirac ideal gases [27, 28]. The Jüttner distribution is commonly accepted [5, 27–31] although results not compatible with this distribution have also been published [32–36]. At this moment there is no experimental evidence to settle which point of view is correct.

Nevertheless we meet a light at the end of this tunnel provided by recent computer simulations. Numerical simulation results are in perfect agreement with the Jüttner distribution and its generalization to the moving gas [24, 37–41]. Therefore assuming that the Jüttner distribution and its generalization to a moving system are the correct relativistic statistical distributions we define the relativistic Gibbs equilibrium distributions (2.27). Then in section 3 we use the well known nonrelativistic statistical physics machinery [13] leading from the Gibbs distribution to phenomenological thermodynamics and we prove that from this point of view the natural transformations of heat and absolute temperature are given again by (2.17) and (2.27), respectively.

In section 4, some useful identities in phenomenological relativistic thermodynamics are found and methods of measuring the absolute temperature are proposed. Section 5 is devoted to searching for a natural definition of the statistical thermometer. In fact, one can find many ‘natural’ statistical thermometers. One of them can be constructed with the use of the generalized principle of equipartition of energy (5.12).

However, if one searches for some parameter which independently of the reference frame characterizes systems that are in thermal equilibrium, then this parameter is certainly the empirical temperature \( T_0 \) and this is what has been pointed out in [37]. From this point of view the natural statistical thermometer is the one defined by (5.16) (see also [37]).

We can also define a statistical thermometer that measures the absolute temperature \( T_0 \) of the ideal gas at rest (see (5.16), which is equivalent to the empirical temperature, or another one that measures the absolute temperature \( T \) (see (5.15)). Many other possibilities are also acceptable (e.g. (5.17)).

In conclusion, we find that there are two notions of relativistic temperature. One of them follows from the zeroth law of relativistic thermodynamics and this is the empirical temperature, which is a relativistic scalar and which can be identified with the absolute temperature \( T_0 \) of a thermodynamic system at rest. It seems that the analogous point of view concerning of a thermodynamic system. The second one is a consequence of the second law of relativistic thermodynamics and this is the absolute temperature \( T \) of the transformation law (2.27). Although in nonrelativistic thermodynamics one also meets these two notions of temperature, it is only relativistic thermodynamics which shows explicitly the deep difference between them.

We use the Einstein summation convention. The symbol \( p \) denotes either a momentum or a pressure but from the context its meaning is always clear. The subindex ‘0’ corresponds to the proper frame \( K_0 \) of the thermodynamic system.

2. Phenomenological relativistic thermodynamics

Consider a continuous medium in thermodynamic equilibrium. We choose a bounded domain of the medium of the volume \( V_0 \) with respect to the proper inertial frame \( K_0 \). We also assume that the medium moves with a constant velocity \( \vec{u} \) with respect to the laboratory inertial frame \( K \). The space axes of \( K_0 \) and \( K \) are assumed to be mutually parallel. The spacetime metric is \( g_{ij} = η_{ij} = diag(−, −, −, +) \).

Then the coefficients \( Λ^k_{\ j \ k} \) are defined in the Lorentz transformation \( x^j = Λ^j_{\ k} x^k \) by the formulae

\[
Λ^0_{\ j} = δ^0_{\ j}, \quad Λ^k_{\ j} = Λ^k_{\ j0} = \frac{v^k}{c^2} (1 − γ(v)),
\]

\[
Λ^k_{\ j0} = \frac{v^k}{c^2} γ(v), \quad A^k_{\ j} = \frac{v^k}{c^2} γ(v), \tag{2.1}
\]

\[
Λ^k_{\ j} = γ(v), \quad μ, \ ν = 1, 2, 3,
\]

where

\[
y(v) := \frac{1}{\sqrt{1 − \frac{v^2}{c^2}}},
\]

The energy–momentum tensor \( Θ^j_{\ k} \) for a continuous medium can be written in the form [12, 43, 44]

\[
Θ^j_{\ k} = η_{0j} u_k + τ^j_{\ k} = Θ_{kj}, \tag{2.2}
\]

where \( η_{0j} \) is the energy density in the proper inertial frame \( K_0 \), \( u^j = (\nu, γ(v), γ(v)) \) is the four-vector of velocity of \( K_0 \) relative to \( K \), and \( τ^j_{\ k} \) is determined by the stress tensor \( τ_{\ \mu \nu} \) as follows:

\[
τ^j_{\ k} = Λ^j_{\ l} Λ^l_{\ m} τ_{\ \nu \ \mu} \quad \tau_{\ \nu \ \mu} = \frac{η_{\ \mu \nu}}{c^2} \tag{2.3}
\]

We put \( τ^j_{\ k} = τ_{\ \nu \ \mu} = 0 \) because for a system in rest the stream of strength \( τ_{\ \nu \ \mu} = 0 \). Moreover, \( Θ_{k4} = η_{00} \) so that \( τ_{\ \nu \ 4} = 0 \).

Inserting (2.1) into (2.3) and then (2.2), one quickly finds the total energy \( E \) and momentum \( P^\mu \)

\[
E = \int_V Θ^{μν} \, dx^1 \, dx^2 \, dx^3 = γ(v) \left( E_0 + \frac{v^j v^j}{c^2} Υ_{0\ ν \ 0} \right), \tag{2.4a}
\]

\[
P^\mu = \frac{1}{c} \int_V Θ^{μν} \, dx^1 \, dx^2 \, dx^3 = \frac{v^μ}{c^2} γ(v) E_0 + \frac{v^ν}{c^2} \left[ γ(v) − 1 \right] \frac{v^μ v^ν}{c^2} Υ_{0\ ν \ 0}, \tag{2.4b}
\]

where \( Υ_{0\ ν \ 0} := \int_V τ_{0\ ν \ 0} \, dx^1 \, dx^2 \, dx^3 \).
As yet we have not used the conservation law \( \frac{\partial \tau_{\mu \nu}}{\partial x^\lambda} = 0 \). Since our thermodynamic system is in equilibrium, this law in the proper frame \( K_0 \) takes the form

\[
\frac{\partial \tau_{\mu \nu}}{\partial x^\nu} = 0.
\]

(2.5)

With the use of (2.5), one obtains [43]

\[
\gamma_{0 \mu \nu} = \int_{V_0} \frac{\partial}{\partial x^0} \frac{1}{x_0^0} \partial x_0^0 \partial x^0 \partial x_0^0 \partial x_0^0 = - \int_{\partial V_0} \tau_{0 \mu \nu} n_0^\mu d \Sigma_0.
\]

(2.6)

where \( n_0^\mu \) is the \( \rho \)’s component of the unit outward normal vector to the boundary \( \partial V_0 \) of \( V_0 \) and \( d \Sigma_0 \) is the surface element of \( \partial V_0 \).

By symmetrization of (2.6), we finally obtain

\[
\gamma_{0 \mu \nu} = - \frac{1}{2} \int_{V_0} \left( \tau_{0 \rho \nu} x_0^0 + \tau_{0 \nu \rho} x_0^0 \right) n_0^\rho d \Sigma_0.
\]

(2.7)

In particular, in the case of ideal liquid one has

\[
\Theta_{jk} = (p_0 + \epsilon_0) u_j u_k - p_0 \eta_{jk},
\]

\[
\tau_{0 \mu \nu} = - p_0 \eta_{\mu \nu}, \quad \gamma_{0 \mu \nu} = - p_0 V_0 \eta_{\mu \nu},
\]

(2.8)

where \( p_0 \) stands for pressure. The tensor \( \Theta_{jk} \) in (2.8) can be rewritten in the form of (2.2) as follows:

\[
\Theta_{jk} = \epsilon_0 u_j u_k + p_0 (u_j u_k - \eta_{jk})
\]

(compare [43, 44]).

Substituting (2.8) into (2.4a) and (2.4b), we obtain

\[
E = \gamma (v) \left( E_0 + \frac{v^2}{c^2} p_0 V_0 \right),
\]

(2.9a)

\[
P^\mu = \gamma (v) \left( E_0 + p_0 V_0 \right) \frac{v^\mu}{c^2}.
\]

(2.9b)

Remembering that pressure is a Lorentz invariant, i.e. \( p = p_0 \), and the volume \( V_0 = V \gamma (v) \), we can rewrite (2.9a) as follows:

\[
E + p V = \gamma (v) \left( E_0 + p_0 V_0 \right).
\]

(2.10)

From (2.9b) and (2.10) with (2.1) one concludes that \( (p^1, p^2, p^3, \frac{E}{c}) \) constitutes a four-vector. This crucial point has been considered by many authors [5, 10, 11, 25, 27, 28, 45, 46].

We assume that the first law of thermodynamics for our thermodynamic system in its proper frame \( K_0 \) has the usual form

\[
dE_0 = \delta Q_0 + \delta L_0,
\]

(2.11)

where \( \delta Q_0 \) is the amount of heat entering the system and \( \delta L_0 \) denotes the thermodynamic work done over the system. For reversible processes one has

\[
\delta Q_0 = T_0 dS_0, \quad \delta L_0 = - \int_{\partial V_0} \tau_{0 \mu \nu} n_0^\mu d \Sigma_0
\]

(2.12)

with \( T_0 \) being the absolute temperature and \( S_0 \) the entropy of the system. For example, in the case of an ideal liquid, from (2.8) and (2.12) we obtain

\[
\delta L_0 = p_0 \int_{\partial V_0} \eta_{\mu \nu} n_0^\mu d \Sigma_0 = - p_0 d V_0.
\]

(2.13)

Differentiating (2.4a), (2.4b) and applying (2.11), one obtains the first law of thermodynamics in the inertial frame \( K \)

\[
dE = \gamma (v) dQ_0 + \gamma (v) \left( \delta L_0 + \frac{v^\mu v^\rho}{c^2} d\gamma_{0 \mu \rho} \right).
\]

(2.14a)

\[
dP^\mu = \gamma (v) \frac{v^\mu}{c^2} \delta Q_0 + \gamma (v) \frac{v^\mu}{c^2} \delta L_0
\]

\[+ \frac{v^\mu}{c^2} \left[ \gamma (v) - 1 \right] \frac{v^\mu v^\rho}{c^2} - \eta_{\mu \rho} \right] \gamma_{0 \mu \nu}.
\]

(2.14b)

A ‘reversible adiabatic process’ should be independent of the choice of a system of frames. Consequently we claim that \( \delta Q_0 = 0 \) if and only if for any inertial frame \( K \) the heat \( \delta Q \) supplied to the thermodynamic system vanishes, i.e. \( \delta Q = 0 \). Therefore (2.14a) can be rewritten in the form

\[
dE = \delta Q + \delta L,
\]

\[
\delta Q = \gamma (v) \delta Q_0, \quad \delta L = \gamma (v) \left( \delta L_0 + \frac{v^\mu v^\rho}{c^2} \gamma_{0 \mu \rho} \right)
\]

(2.15)

with \( \delta Q \) standing for the heat supplied to the system and \( \delta L \) denoting the thermodynamic work performed on the system with respect to the frame \( K \). Quantities \( \delta Q \) and \( \delta L \) are well defined although it may be complicated to measure them directly in the frame \( K \).

Employing also (2.14b) one can express the first law of relativistic thermodynamics in 4D form

\[
dP^j = \delta Q^j + \delta L^j, \quad P^j = \left( P^1, P^2, P^3, \frac{E}{c} \right) = \left( \bar{P}, \frac{E}{c} \right),
\]

\[
\delta Q^j = \frac{\delta Q_0}{c} \epsilon^j, \quad \delta L^j = \gamma (v) \frac{v^\mu}{c^2} \delta L_0
\]

\[+ \frac{v^\mu}{c^2} \left[ \gamma (v) - 1 \right] v^\rho - \eta_{\mu \rho} \right] \gamma_{0 \mu \nu}, \quad \delta L^4 = \frac{\delta L}{c}
\]

(2.16)

with \( \delta Q \) standing for the heat supplied to the system and \( \delta L \) denoting the thermodynamic work performed on the system with respect to the frame \( K \). Thus we arrive at the four-vector of heat \( \delta Q^j \) and the four-object of relativistic thermodynamic work \( \delta L^j \). The work \( \delta L^j \) is not a four-vector. The spatial part \( \frac{\partial}{\partial x^0} \delta L^j \) of the four-vector of heat represents the change of momentum \( d \bar{P} \) caused by the transfer of heat. The fourth component of \( \delta Q^j \) is proportional (with the coefficient \( \frac{1}{c} \)) to the change of energy following from the heat transfer.
Formula (2.15) or (2.16) leads to the relativistic transformation of heat found by Ott [8] and independently by Arzeliès [9] and Møller [10, 11]

\[ \delta Q = \gamma(v) \delta Q_0 \]  
(2.17)

(see also [12, 25]).

This transformation rule differs dramatically from the one given by Einstein [2], Planck [3], von Mosengeil [4], Pauli [5] and von Laue [6]

\[ \delta Q^{(\text{Planck})} = \frac{1}{\gamma(v)} \delta Q_0. \]  
(2.18)

(We must note that, as pointed out by [7] after his studies of the letters between Einstein and Laue, Einstein in 1952 changed his opinion on the validity of (2.18) in favor of (2.17).)

If we mention also that the heat supplied to the system is considered by some authors to be a Lorentz invariant (see Landsberg and coworkers [14–17] and van Kampen [23])

\[ \delta Q^{(\text{L})} = \delta Q_0, \]  
(2.19)

then one obtains a variety of approaches to the problem of relativistic transformation of heat. However, in our opinion according to (2.14a) the most natural transformation rule is given by equation (2.17). Our conviction comes from the fact that the infinitesimal heat \( \delta Q \) and the infinitesimal work \( \delta L \) must be of the form (2.15).

The next question is: what about the relativistic transformation of temperature? Before we consider this question we write down the first law of relativistic thermodynamics (2.14a), (2.14b) in the case of an ideal liquid. Inserting (2.8) and (2.13) into (2.14a) and (2.14b), one obtains

\[ dE = \gamma(v) \delta Q_0 - \rho \, dV + \left( \gamma(v) \right)^2 \frac{\nu^2}{c^2} V \, dp, \]  
(2.20a)

\[ d\bar{P} = \gamma(v) \bar{v} \delta Q_0 + \left( \gamma(v) \right)^2 \frac{\bar{v}^2}{c^2} V \, dp. \]  
(2.20b)

Hence

\[ \delta L = - \rho \, dV + \left( \gamma(v) \right)^2 \frac{\nu^2}{c^2} V \, dp, \quad \delta \bar{L} = \left( \gamma(v) \right)^2 \frac{\bar{v}^2}{c^2} V \, dp \]  
(2.21)

(see [10–12, 25]).

Consider now the relativistic transformation of temperature. In nonrelativistic phenomenological thermodynamics we meet two main notions of temperature. The first one is the empirical temperature introduced by the zeroth law of thermodynamics as a number determining the equivalence class of all thermodynamic systems being in thermal equilibrium.

Let us analyze a system consisting of a vessel and an unbounded reservoir. The vessel and the reservoir can exchange heat. At the beginning these two elements were in thermodynamical equilibrium and they did not move. The vessel and the reservoir contained water in the triple point. The vessel has been quasi-statically accelerated to a velocity \( \bar{v} \). In both the vessel and the reservoir water remains in the triple point. Moreover, the vessel and the reservoir are still in thermal equilibrium with each other. The empirical temperature determined by the triple point of water of the two components of our system has not changed. Hence the empirical temperature is well defined and can be identified with the absolute temperature \( T_0 \) in the proper frame. Thus the empirical temperature in relativistic thermodynamics is a Lorentz invariant. An analogous conclusion can be found in [25].

The second notion, the absolute temperature, is introduced by the second law of thermodynamics as the unique (up to a constant factor) integrating factor of the Pfaffian form \( \delta Q_0 \) depending only on the empirical temperature. This leads to the Clausius equality

\[ \oint \frac{\delta Q_0}{T_0} = 0 \]  
(2.22)

for every cyclic reversible process. We assume that the second law of thermodynamics is in force in relativistic thermodynamics in any inertial frame. Consequently, the absolute temperature of a thermodynamic system moving with a constant velocity \( \bar{v} \) with respect to the frame \( K \) should be some smooth function of the form

\[ T = T(T_0, v), \lim_{v \to 0} T(T_0, v) = T_0 \]  
(2.23)

and the Clausius equality in \( K \) reads

\[ \oint \frac{\delta Q_0}{T} = 0. \]  
(2.24)

Substituting (2.17) and (2.23) into (2.24) we obtain

\[ \oint \frac{\delta Q_0}{T_0 (\gamma(v))^{-1}} = 0. \]  
(2.25)

Comparing (2.25) with (2.22), one arrives at the conclusion that

\[ (\gamma(v))^{-1} T(T_0, v) = b T_0, \]  
(2.26)

where \( b \) is some constant. Taking the limit of equation (2.26) for \( v \to 0 \) and applying (2.23), we find that \( b = 1 \), so finally

\[ T = \gamma(v) T_0. \]  
(2.27)

This is the relativistic transformation of absolute temperature consistent with (2.17) and the second law of relativistic thermodynamics for reversible processes. This transformation rule is dramatically different from the transformation law [2–6]

\[ T^{(\text{Planck})} = \left( \gamma(v) \right)^{-1} T_0 \]  
(2.28)

following from (2.18) or the transformation rule [14–17]

\[ T^{(\text{L})} = T_0 \]  
(2.29)

consistent with (2.19).

In the next section, we are going to give some justification for (2.17) and (2.27) which follows from the Jüttner distribution.

Note also that by (2.12), (2.17) and (2.27), one obtains

\[ \delta Q = T \, dS_0, \]  
(2.30)

which means that according to Planck [3], entropy is a Lorentz invariant

\[ S = S_0 \]  
(2.31)

(see also [10, 11, 28]).
3. From the Jüttner distribution to phenomenological relativistic thermodynamics

In 1911, Jüttner [26] proposed a relativistic counterpart of the famous Maxwell distribution for an ideal gas in its proper inertial frame $K_0$. The Jüttner distribution had been accepted for many years [5, 27–31], but from the 1980s onwards several alternatives have been suggested [32–36].

Doubts also appear when the Jüttner distribution is generalised to the case when a vessel containing a gas moves with a constant velocity $\vec{v}$ with respect to the laboratory frame $K$. Hence, the natural question arises if the Jüttner formula and its generalization to the moving thermodynamic system (ideal gas) are correct. Nowadays we are not able to find any experimental solution of this question. Nevertheless a partial answer comes from recent outstanding works [24, 37–41] where some computer simulations have been presented. The numerical results of these simulations show perfect agreement with both the Jüttner distribution in the proper frame $K_0$ and its generalization to the moving vessel.

Finally, the computer simulations in 3D given in [24] and [41] also confirm the 3D Jüttner distribution and its generalization to the moving gas.

If so, it is an easy matter to get from these one-particle distributions the corresponding relativistic Gibbs distributions. Finally, the relativistic Gibbs distribution for an ideal gas contained in the vessel moving with a constant velocity $\vec{v}$ and four-velocity $\vec{u} = (v, \frac{\vec{v}}{c})$ reads

$$dw = \frac{1}{(2\pi\hbar)^3N!Z} \exp \left\{-\beta cu_j \Pi^j \right\} d^3N p \, d^3N q, \quad (3.1)$$

where $N$ stands for the number of particles, $k = \frac{1}{\sqrt{\gamma}}$, with $\gamma$ being the Boltzmann constant, $\Pi^j = (\vec{\Pi}, \vec{\Pi})$ is the total momentum of the gas, $d^3N pd^3Nq$ is the phase space volume element and $Z$ denotes the partition function

$$Z = \frac{V^n}{(2\pi\hbar)^3N!} \int_{\mathbb{R}^3} \exp \left\{-\beta cu_j \Pi^j \right\} d^3N p. \quad (3.2)$$

Noting that the measure $d^3p$ transforms as follows [28]:

$$d^3p = \gamma(v) \left(1 + \frac{\vec{v} \cdot \vec{p}_0}{c\sqrt{\vec{p}_0^2 + m^2c^2}}\right) d^3p_0, \quad (3.3)$$

one obtains

$$Z = \frac{V^n}{(2\pi\hbar)^3N!} \left(\int_{\mathbb{R}^3} \exp \left\{-\beta cu_j \Pi^j \right\} d^3p \right)^N = \frac{V^n}{(2\pi\hbar)^3N!} \left(\int_{\mathbb{R}^3} \exp \left\{-\beta c\sqrt{\vec{p}_0^2 + m^2c^2} \right\} \gamma(v) \left(1 + \frac{\vec{v} \cdot \vec{p}_0}{c\sqrt{\vec{p}_0^2 + m^2c^2}}\right) d^3p_0 \right)^N = \frac{V^n}{(2\pi\hbar)^3N!} \left(\int_{\mathbb{R}^3} \exp \left\{-\beta c\sqrt{\vec{p}_0^2 + m^2c^2} \right\} d^3p_0 \right)^N = Z_0, \quad (3.4)$$

where, as in section 2, the subindex ‘0’ corresponds to the proper frame $K_0$. Therefore, the partition function $Z$ is a Lorentz invariant [28]. Then the entropy of the gas $S$ reads

$$S = -k:\ln \left(\frac{1}{Z} \exp \left\{-\beta cu_j \Pi^j \right\} \right) = k\ln (Z + \beta c\Pi^j), \quad (3.5)$$

with $\langle \cdot \rangle$ denoting the expected value (average) with respect to the Gibbs distribution (3.1).

Analogous calculations to those performed by Pathria [27, 28] show that the gas pressure is a Lorentz invariant, i.e. $p = p_0$ exactly as in the case of a continuous medium from section 2 and also that the following formulae, which are closely related to (2.9a), (2.9b), hold true:

$$\langle \mathcal{E} \rangle = \gamma(v) \left(\mathcal{E}_{00} + \frac{v^2}{c^2} p_0 V_0 \right), \quad (3.6a)$$

$$\langle \Pi \rangle = \gamma(v) \left(\mathcal{E}_{00} + p_0 V_0 \right) \frac{\vec{v}^2}{c^2}, \quad (3.6b)$$

(see also [46]). Hence, as before, $\langle (\Pi, \frac{v^2}{c^2} p_0 V_0) \rangle$, i.e. the average total momentum $\langle \Pi \rangle$ and the enthalpy of the gas divided by $c$, namely $\langle \mathcal{E} \rangle/c$, constitute a four-vector.

Employing (3.6a) and (3.6b) one quickly finds

$$u_j/\Pi^j = \frac{\mathcal{E}_{00}}{c}. \quad (3.7)$$

Thus, although $\langle \Pi^j \rangle$ is not a four-vector, the ‘scalar product’ $u_j/\Pi^j$ is a Lorentz invariant.

Inserting (3.4) and (3.7) into (3.5), we obtain the result

$$S = k\langle \ln Z_0 + \beta \mathcal{E}_{00} \rangle = S_0, \quad (3.8)$$

which confirms the Planck formula (2.31).

Differentiating (3.6a) and (3.6b) and performing the same calculations as in section 2, one arrives at the first law of relativistic thermodynamics for an ideal gas in the form (2.20a) and (2.20b) with

$$E := \langle \mathcal{E} \rangle, \quad \vec{P} := \langle \Pi \rangle. \quad (3.9)$$

Hence, again we are led to the relativistic transformations of heat (2.17) and absolute temperature (2.27).

However, we are going to give a slightly deeper insight into the problem by applying some statistical considerations which are well known in nonrelativistic statistical physics and which lead from the Gibbs distributions to the first law of thermodynamics [13].

To this end, we rewrite the relativistic Gibbs distribution (3.1) in quantum form

$$w_n = \frac{1}{Z} \exp \left\{-\beta cu_j \Pi^j \right\}, \quad n = 1, 2, \ldots, \quad (3.10)$$

where the subindex ‘$n$’ denotes a quantum state. One assumes that the total four-momentum eigenvalues $\Pi^j$ depend on some external thermodynamic parameters $\lambda_1, \ldots, \lambda_s$. Differentiating the formula

$$\sum_{n=1}^{\infty} w_n = 1 \quad (3.11)$$
with respect to $T_0, \lambda_1, \ldots, \lambda_s$, and performing some simple
manipulations we obtain

$$u_i d (\Pi^i) = \frac{1}{k\beta c} dS + u_j (d \Pi^j).$$  (3.12)

Keeping in mind that $u_j = (- \gamma(v) \frac{\vec{v}}{c}, \gamma(v))$ and $\Pi^i_m = (\Pi_m, \frac{\vec{v}_m}{c})$, one can rewrite (3.12) in the following form:

$$d[\mathcal{E}] = \frac{T_0}{\gamma(v)} dS + \vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]) + d[\mathcal{E}].$$  (3.13)

Assuming that, analogously to the proper frame $K_0$, the statistical definition of thermodynamic work $\delta L$ in any inertial frame reads

$$\delta L = \langle d[\mathcal{E}] \rangle,$$  (3.14)

(see [47]) and comparing (2.15) with (3.13) we obtain

$$\delta Q = \frac{T_0}{\gamma(v)} dS + \vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]).$$  (3.15)

From (3.15) with (2.12) and (3.8) it follows that statistical considerations give the transformation rule (2.18) for heat if either $\vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]) = 0$ or one decides to change definition (3.14) of $\delta L$ into

$$\delta L \rightarrow \delta L' = \langle d[\mathcal{E}] \rangle + \vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]).$$  (3.16)

The latter possibility explains from the statistical physics point of view the reason why Einstein in 1907, Planck in 1908 and others found transformation formula (2.18). Employing the detailed phenomenological analysis of Møller [10], one can expect that

$$\langle d[\bar{\Pi}] \rangle = \gamma(v) \frac{\vec{v}}{c} V d\rho.$$  (3.17)

(As yet we have not been able to derive (3.17) using laws of statistical physics only.)

Then from (2.20) with (3.9) and (3.17), we have

$$\vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]) = \gamma(v) \frac{v^2}{c^2} \delta Q_0.$$  (3.18)

Finally, inserting (3.18) into (3.15) one arrives at the transformation rule (2.17) of Ott, Arzelés and Møller.

Moreover, (2.20a) and (2.20b) with (3.9), (2.12), (2.16) and (3.13)–(3.18) lead to the Lorentz invariant formulation of the first law of relativistic thermodynamics

$$d[\Pi^i] - d[\Pi^i] = T^i dS, \quad T^i := \frac{T_0}{c} h_i.$$  (3.19)

$T^i$ is the four-vector of temperature [10, 11]. One quickly finds that by (2.27)

$$T^4 = T_4 = \gamma(v) \frac{T_0}{c} = \frac{T}{c}.$$  (3.20)

(An analogous approach to the relation between relativistic statistical physics and relativistic phenomenological thermodynamics was presented by Bergmann in 1951 [48] (see also [49]) many years before computer simulations were done by Cubero et al [24, 37] and Rasinariu [38].) It seems that the statistical considerations of this section, which follow directly from the relativistic Gibbs distribution (3.1) and consequently from the Jüttner distribution, give evidence that the natural transformation rules for heat and temperature are given by (2.17) and (2.27), respectively.

(Remark: observe that in order to obtain the transformation rule (2.17), one should define

$$\delta L = \langle d[\mathcal{E}] \rangle - \vec{v} \cdot (d[\bar{\Pi}] - d[\bar{\Pi}]) + \gamma(v) \frac{v^2}{c^2} V d\rho.$$  (3.21)

Then with (3.17) one obtains (3.14).)

4. On some identities in relativistic thermodynamics: operational definition of the absolute temperature $T$

First, using (2.12), (2.27) and (2.31) we rewrite (2.20a) and (2.20b) in the form

$$dE = T dS - p dV + \gamma(v) \frac{v^2}{c^2} V d\rho.$$  (4.1a)

$$dP = \left( \frac{v}{c^2} V dS + \gamma(v) \frac{v^2}{c^2} V d\rho \right).$$  (4.1b)

From (4.1a) one quickly finds

$$d \left( E \left[ \gamma(v) \frac{v^2}{c^2} V P \right] = T dS - \gamma(v) \frac{v^2}{c^2} V d\rho. \right.$$  (4.2)

Hence

$$T = \langle \frac{\partial E}{\partial S} \rangle_{V, \rho} - \gamma(v) \frac{v^2}{c^2} V \left( \frac{\partial p}{\partial S} \right)_V.$$  (4.3a)

$$p = \gamma(v) \frac{v^2}{c^2} V \left( \frac{\partial E}{\partial V} \right)_S = - \left( \frac{\partial E}{\partial V} \right)_S.$$  (4.3b)

and

$$\frac{\partial T}{\partial V} = - \gamma(v) \frac{v^2}{c^2} V \left( \frac{\partial p}{\partial V} \right)_V.$$  (4.4)

Moreover, from (4.2) and (4.1b) we obtain

$$T = \gamma(v) \left( \frac{\partial E}{\partial S} \right)_{V, \rho}. \quad (4.5)$$

(compare with formula (9) from the paper [28].)

As we showed in section 2, the sum $E + p V$ transforms according to the rule (2.10). We identify it with enthalpy $H$. Hence, we see that

$$H = \gamma(v) \frac{v^2}{c^2} V.$$  (4.6)

Applying (4.2) and the definition of enthalpy, we obtain

$$T = \left( \frac{\partial H}{\partial S} \right)_p.$$  (4.7)

We define the free energy $F$

$$F := E - TS - \gamma(v) \frac{v^2}{c^2} PV = \gamma(v) F_0.$$  (4.8)
(the last equality follows from (2.27), (2.31) and (2.9a)). \(F_0 = E_0 - T_0 S_0\) is the free energy of the system in its proper frame \(K_0\). Inserting (4.8) into (4.2) we obtain
\[
dF = -S \, dT - (\gamma(v))^2 \, p \, dV.
\] (4.9)

From (4.9) one finds
\[
S = -\left(\frac{\partial F}{\partial T}\right)_v, \quad (\gamma(v))^2 \, p = -\left(\frac{\partial F}{\partial V}\right)_T,
\] (4.10)
\[
E = F - T \left(\frac{\partial F}{\partial T}\right)_T - \frac{\gamma(v)}{c^2} \, V \left(\frac{\partial F}{\partial V}\right)_T
\]
and one of the relativistic Maxwell identities
\[
\left(\frac{\partial S}{\partial V}\right)_T = (\gamma(v))^2 \left(\frac{\partial p}{\partial T}\right)_v.
\] (4.11)

We define the Gibbs function \(G\) in a standard way
\[
G := E - TS + pv = \gamma(v)G_0
\] (4.12)
(use (2.27), (2.31) and (2.9a)), with \(G_0 = E_0 - T_0 S_0 + p_0 v_0\).
Differentiating (4.12) and substituting (4.1a), we obtain the formula
\[
dG = -S \, dT + (\gamma(v))^2 \, V \, dp
\] (4.13)
from which we obtain the relations
\[
S = -\left(\frac{\partial G}{\partial T}\right)_p, \quad (\gamma(v))^2 \, V = \left(\frac{\partial G}{\partial p}\right)_T,
\] (4.14)
\[
E = G - T \left(\frac{\partial G}{\partial T}\right)_p - (\gamma(v))^2 \, p \left(\frac{\partial G}{\partial p}\right)_T
\]
and the Maxwell identity
\[
\left(\frac{\partial S}{\partial p}\right)_T = -\gamma(v))^2 \left(\frac{\partial V}{\partial T}\right)_p.
\] (4.15)

We conclude that the thermodynamic potentials \(H, F, G\) are defined like in a rest frame and satisfy the same transformation rule.

In the celebrated monograph of Landau and Lifshitz [13], formula (4.15) with \(\gamma(v) = 1\) is employed to find the absolute temperature as a function of empirical temperature in nonrelativistic thermodynamics. Modifying slightly those considerations we can obtain an analogous result in relativistic thermodynamics. Namely, writing
\[
\left(\frac{\partial V}{\partial T}\right)_p = \left(\frac{\partial V}{\partial T_0}\right)_p \frac{dT_0}{dT}
\]
and noting that \(p = p_0, S = S_0\), from (4.15), one obtains
\[
dT_0 = \frac{1}{T_0} \left(\frac{\partial V}{\partial T_0}\right)_p \, dT
\] (4.16)
This formula gives the derivative \(\frac{dT_0}{dT}\) in terms of measurable quantities. Indeed, \(\left(\frac{\partial V}{\partial T_0}\right)_p\) is determined by the change of the volume \(V\) under the change of \(T_0\) for the constant pressure \(p\). Moreover, \(T_0\left(\frac{\partial V}{\partial T_0}\right)_p\) is determined by the heat \(\delta Q_0\) supplied to the thermodynamic system so that under the change of pressure \(p_0\) the temperature \(T_0\) remains unchanged:
\[
T_0 \left(\frac{\partial S_0}{\partial p_0}\right)_T = \lim_{p_0 \to 0} \left(\frac{Q_0}{\Delta p_0}\right)_T.
\]
Assuming also that
\[
\lim_{T_0 \to 0} T = 0,
\] (4.17)
which is consistent with the third law of thermodynamics (note (2.31)), one finds that formula (4.16) determines \(T = T(T_0, v)\).

Therefore, formula (4.16) indicates an operational definition of the absolute temperature \(T\).

Another operational definition of the absolute temperature \(T\) is provided by the Clausius equation (2.24) when applied to an appropriate thermodynamic engine. This idea was presented by Møller [11] and then considered in [1, 25]. We recall briefly the main points of Møller’s construction. Let \(R\) and \(R_0\) be two reservoirs. \(R_0\) is at rest with respect to the inertial frame \(K_0\) and \(R\) is at rest with respect to the system \(K\). The absolute temperature of \(R_0\) with respect to \(K_0\) is \(T_0\) and it is equal to the absolute temperature of the reservoir \(R\) with respect to the system \(K\).

Let us consider a thermodynamic engine operating between the reservoirs \(R_0\) and \(R\) according to some Carnot cycle. The whole process is analyzed from the frame \(K_0\). First, the engine absorbs isothermally the amount of heat \(Q_0\) from the reservoir \(R_0\) at the absolute temperature \(T_0\), being at rest with respect to the reference system \(K_0\). Then the engine is accelerated adiabatically to the velocity of the frame \(K\) equal to \(\vec{v}\). An amount of heat \(Q\) (with respect to the system \(K_0\), i.e. \(Q_0\) with respect to the frame \(K\) is released isothermally from the engine to the reservoir \(R\) at the temperature \(T\) (with respect to the reference frame \(K_0\), i.e. the temperature \(T_0\) with respect to the system \(K\)). The final step is adiabatic deceleration of the engine so that it returns to its initial state.

The efficiency \(\eta\) of this cycle reads
\[
\eta = 1 - \frac{Q}{Q_0} = 1 - \frac{T}{T_0}.
\] (4.18)

Consequently, by measuring \(\eta\) one finds transformation rules of heat and of temperature. Thus both (2.17) and (2.27) can be verified experimentally. (For details of the relation between the absolute temperature and the efficiency of the Carnot cycle, see also [50].)

Taking the scalar product of both the sides of equation (4.1b) with \(\vec{v}\) and subtracting the result from (4.1a), we obtain
\[
dE = (\gamma(v))^{-1} T_0 \, dS - \rho \, dV + \vec{v} \cdot d\vec{P}.
\] (4.19)
In the case of an ideal gas, one can employ (3.18) and then (4.19) is brought to the form
\[
dE = \gamma(v) T_0 \, dS - \rho \, dV + \vec{v} \cdot [d\vec{H}].
\] (4.20)
The quantity \([d\vec{H}]\) is interpreted as the mechanical momentum supplied to the system [10, 11]. With this interpretation, formula (4.20) is also true for an ideal liquid.
\[ dE_0 = T_0 \, dS_0 - p_0 \, dV_0 + \mu_0 \, dN_0, \]

where \( \mu_0 \) is the chemical potential with respect to the rest frame and \( N_0 = N \), after some simple manipulations one obtains

\[ dE = T \, dS - p \, dV + (\gamma(v))^2 \frac{v^2}{c^2} V \, dp \]
\[ + (\gamma(v))^3 \frac{E_0 + p_0 V_0}{c^2} \, \vec{v} \cdot d\vec{v} + \mu \, dN, \]

where \( T \) is given by (2.27) and

\[ \mu = \gamma(v) \mu_0. \]

From (4.22a) and (4.22b) one infers that analogously to the four-vector of temperature \( T^j \) (see (3.19) and (3.20)), we can introduce the four-vector of chemical potential

\[ \mu = \frac{\mu_0}{c} \Rightarrow u^i = \gamma(v) \mu_0 \frac{c}{\gamma(v)} = \mu \]

Then, from (4.22a) and (4.22b) we quickly find that the thermodynamic work \( \delta L \) reads

\[ \delta L = -p \, dV + (\gamma(v))^2 \frac{v^2}{c^2} V \, dp + (\gamma(v))^3 \frac{E_0 + p_0 V_0}{c^2} \, \vec{v} \cdot d\vec{v} \]

and \( \delta \tilde{L} \) (compare with (2.21)) is

\[ \delta \tilde{L} = (\gamma(v))^2 \frac{v^2}{c^2} V \, dp \]
\[ + (\gamma(v))^3 \frac{E_0 + p_0 V_0}{c^2} \left[ d\vec{v} + \frac{1}{c^2} \vec{v} \times (\vec{v} \times d\vec{v}) \right] . \]

It is an easy matter to generalize all results obtained before. For example, in the present case the identity (4.13) reads

\[ dG = -S \, dT + (\gamma(v))^2 \frac{v^2}{c^2} V \, dp \]
\[ + (\gamma(v))^3 \frac{E_0 + p_0 V_0}{c^2} \, \vec{v} \cdot d\vec{v} + \mu \, dN. \]

Hence

\[ \mu = \left( \frac{\partial G}{\partial N} \right)_{T,p,\vec{v}} . \]

Writing \( G \) in the form

\[ G = Ng(T, p, \vec{v}) \Rightarrow g(T, p, \vec{v}) = \left( \frac{\partial G}{\partial N} \right)_{T,p,\vec{v}} \]

and comparing (4.28) with (4.29), we get the formula well known in nonrelativistic thermodynamics

\[ G = N \mu. \]

Then the potential \( \Omega := F - \mu N \) takes the form (use (4.8) and (4.12))

\[ \Omega = -\gamma(v)^2 \frac{v^2}{c^2} E V - (E - TS + pV) = -\gamma(v)^2 pV. \]

Of course

\[ \Omega = \gamma(v)^2 V \mu_0 \]

and

\[ N = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{T,\vec{v}} = -\left( \frac{\partial \mu_0}{\partial \mu} \right)_{T,\vec{v}} = N_0. \]

Now we can easily find an identity that has been used by many authors as a starting point for relativistic thermodynamics \([10, 28, 51]\). Multiplying (4.22b) by \( \vec{v} \) and subtracting from (4.22a) one obtains

\[ dE = \gamma(v) \frac{1}{c^2} T \, dS - p \, dV + \vec{v} \cdot d\vec{P} + (\gamma(v))^2 \mu \, dN. \]

In this approach, the energy \( E \) is considered to be a function of \( (S, V, \vec{P}, N) \). A motion of the system with velocity \( \vec{v} \) increases its number of degrees of freedom. New conjugated parameters \( \vec{p} \) and \( \vec{P} \) appear. Variables \( S, V, \vec{P} \) and \( N \) are natural variables for energy \( E \). Unfortunately, neither \( \frac{dE}{d\vec{v}} \) nor \( \frac{dE}{d\vec{P}} \) are known in nonrelativistic thermodynamics. Writing

\[ \left( \frac{dE}{d\vec{P}} \right)_{\vec{v}} = E + \gamma(v)^2 \frac{v^2}{c^2} E V \]

and

\[ \vec{v} = \frac{c^2 \vec{P}}{(E_0 + p_0 V_0)^2 + c^2 P^2} , \]

keeping in mind that \( S = \text{const}, \quad p = p_0 = \text{const}, \quad V = \text{const} \), one finds \( \vec{v} \) as a function of \( (S, V, \vec{P}, N) \). This enables us to pass from (4.22a) where \( E = E(S, V, \vec{P}, N) \) to (4.34) where \( E = E(S, V, \vec{P}, N) \).
that it is more convenient and natural to use the independent variables \((S, p, N)\) rather than \((S, V, N)\) and the enthalpy \(H = E + pV\) instead of energy \(E\). This is because \(S, p\) and \(N\) are Lorentz invariants (but \(V\) is not) and \((\dot{P}, \frac{\mu}{c^2}, \frac{\nu}{c})\) constitute a four-vector (see section 2).

Adding \(d(pV)\) to both sides of (4.22a), one obtains
\[
dH = T\, dS + (\gamma(v))\, V\, dp + (\gamma(v))\, \frac{3}{c^2} H_0 c^2 \cdot d\nu + \mu \, dN.
\] 
(4.36)

Here \(H = H(S, p, \nu, \bar{N})\).

Analogously, adding \(d(pV)\) to both sides of (4.34) we have
\[
dH = (\gamma(v))^{-2} T\, dS + V\, dp + \bar{\nu} \cdot d\bar{P} + (\gamma(v))^{-2} \mu \, dN,
\]
where \(H = H(S, p, \bar{P}, \bar{N})\) is given by (4.35b)
\[
H = \sqrt{(E_0 + p_0 V_0)^2 + c^2} \bar{P}^2 = \sqrt{H_0^2 + c^2} \bar{P}^2.
\] 
(4.37)

In particular, from (4.37) with (4.38) one finds
\[
V = \frac{\partial H}{\partial p} \bigg|_{S, \rho, N} = V_0 \frac{H_0}{H} = V_0 \frac{E_0 + p_0 V_0}{\sqrt{(E_0 + p_0 V_0)^2 + c^2} \bar{P}^2}.
\]
\[
\bar{\nu} = \frac{\partial H}{\partial \bar{P}} \bigg|_{S, \rho, N} = \frac{c^2 \bar{P}}{H} = \frac{c^2 \bar{P}}{(E_0 + p_0 V_0)^2 + c^2} \bar{P}^2.
\] 
(4.39)

As can be seen, (4.39b) is exactly (4.35c).

We end the present section with an important remark. Thermodinamic identity (4.22a) with (4.25) lead to the Ott–Arzelie–Moller transformation of absolute temperature (2.27) and to the transformation of chemical potential as given by (4.20). However, the identity (4.34) suggests the Planck transformation of temperature (2.28) \(T(\text{Planck}) = (\gamma(v))^{-2} T = (\gamma(v))^{-1} T_0\) and the transformation of the chemical potential given by \(\mu(\text{Planck}) = (\gamma(v))^{-2} \mu = (\gamma(v))^{-1} \mu_0\) under the assumption that the thermodinamic work reads
\[
\delta L' = -p\, dV + \bar{\nu} \cdot d\bar{P}.
\] 
(4.40)

Substituting \(\bar{\nu} \cdot d\bar{P}\) calculated from (4.22b), we obtain
\[
\delta L' = -p\, dV + (\gamma(v))\, V\, dp + (\gamma(v))\, \frac{3}{c^2} H_0 c^2 \cdot d\nu + \mu \, dN.
\] 
(4.41)

Comparing (4.41) with (4.25) one concludes that the difference between the transformation rules of the absolute temperature and the chemical potential derived from identity (4.22a) and the rules derived from (4.34) is caused by the fact that \(\delta L'\) given by (4.40) cannot be interpreted as the relativistic thermodinamic work. In fact, \(\delta L'\) consists of three groups of terms. The term \(-p\, dV + (\gamma(v))\, V\, dp + (\gamma(v))\, \frac{3}{c^2} H_0 c^2 \cdot d\nu\) defines the thermodinamic work \(\delta L\), the component equal to \(\frac{\gamma}{2} T\, dS\) corresponds to a part of relativistic heat supplied to the system and, finally, the part \(\frac{\gamma}{2} \mu \, dN\) is an element of \(dE\) which corresponds to the increment \(dN\) of the number of particles.

If \(dN = 0\), then a statistical interpretation of \(\delta L'\) is given by (3.16) with (3.14).

5. Statistical thermometers

In the preceding sections, we have argued that in relativistic phenomenological thermodynamics one meets two operationally well-defined notions of temperature: the empirical temperature which can be identified with the proper absolute temperature \(T_0\) and which is a Lorentz invariant, and the absolute temperature \(T\) which follows from the second law of thermodynamics for reversible processes. According to the Clausius equation (2.24) this absolute temperature can be experimentally determined as indicated by measurement of the efficiency \(\eta\) (4.18) of an appropriate thermodinamic engine. We have also argued that the transformation rule for \(T\) is given by formula (2.27) and, consequently, \(T\) be experimentally determined as is indicated by equation (4.16).

Now we are going to consider the question: how one can define temperature in relativistic statistical thermodynamics of an ideal gas? To this end we employ the results of sections 2–4 to find some thermodinamic functions for the ideal gas.

First, performing integration in (3.4) one obtains
\[
Z = Z_0 = \left(\frac{m^2 c V_0}{2\pi^2 \hbar^2 \beta} K_2(\beta mc^2)\right)^N,
\]
(5.1)

where \(K_2(x)\) denotes the modified Bessel function of the second kind of order 2. Its integral representation is of the form
\[
K_v(x) = \int_0^\infty e^{-x \cosh t} \cosh vt \, dt, \quad x > 0.
\]
(5.2)

Inserting (5.1) into the thermodinamic equation
\[
\langle \mathcal{E}_0 \rangle_0 = -\left(\frac{\partial \ln Z_0}{\partial \beta}\right)_{V_0}
\]
(5.3)

and employing the recurrence relation
\[
\frac{dK_v(x)}{dx} + \frac{v}{x} = -K_{v-1}(x),
\]
(5.4)

we easily obtain
\[
E_0 \equiv \langle \mathcal{E}_0 \rangle_0 = \frac{3N}{\beta} + Nmc^2 K_1(\beta mc^2) K_2(\beta mc^2).
\]
(5.5)

Substituting (5.1) and (5.5) into (3.8) and for large \(N\) applying the well-known Stirling formula \(\ln N! \approx N \ln \frac{N}{e}\), one obtains
\[
S = S_0 = kN \ln \left(\frac{m^2 c^3}{2\pi^2 \hbar^2 \beta} K_2(\beta mc^2)\right) + \beta mc^2 K_1(\beta mc^2) K_2(\beta mc^2) + \frac{3N}{\beta}.
\]
(5.6)

From (3.6a), (3.8), (3.9) and (4.8) we have
\[
F = -kT \ln Z = \gamma(v)\left(-kT_0 \ln Z_0\right) = \gamma(v) F_0.
\]
(5.7)

Then from (4.10) with (5.1) one obtains
\[
p = p_0 = kT_0 \left(\frac{\partial \ln Z_0}{\partial V_0}\right)_{T_0} = \frac{NkT_0}{V_0} \implies p_0 V_0 = NkT_0.
\]
(5.8)
Note that formula (5.7) justifies the name 'free energy' for $F$ (see section 4, equation (4.8)).

We are ready to propose a model of a ‘natural’ statistical thermometer. We start from the observation that the average energy of the ideal gas in its proper frame $K_0$ in nonrelativistic thermodynamics is given by

$$ E_{\text{nonrel}} = \lim_{c \to \infty} \left( \langle E_0 \rangle_0 - Nmc^2 \right), \quad \text{(5.9)} $$

where $\langle E_0 \rangle_0$ is defined by (5.5). Applying the principle of equipartition of energy

$$ \left( p^\mu \frac{\partial H_{\text{nonrel}}}{\partial p^\mu} \right) = kT_0 \quad \text{(do not sum over } \mu!), \quad \text{(5.10)} $$

where $H_{\text{nonrel}}$ denotes the Hamilton function of a system, we immediately obtain

$$ T_0 = \frac{E_{\text{nonrel}}}{2Nk} \quad \text{(5.11)} $$

By analogy to (5.11) one can introduce the statistical temperature in relativistic thermodynamics.

The generalized principle of equipartition of energy in a moving frame for a fixed index $j$ is of the form (see also [15, 39])

$$ \left( p^\mu \frac{\partial H}{\partial p^\mu} \right)_j = kT_0 + v^\mu \langle p^\mu \rangle \quad \text{(do not sum over } \mu!). \quad \text{(5.12)} $$

Then in a similar way to (5.11), we propose

$$ T^{(\text{stat})} := \frac{1}{3Nk} \left( \langle \mathcal{E} \rangle - Nmc^2 \left( \frac{m^2 c^4}{H_{\text{particle}}} \right) - \vec{v} \cdot \vec{p} \right). \quad \text{(5.13)} $$

Thus $T^{(\text{stat})} = T^{(\text{Planck})} = \frac{T_0}{\gamma(v)}$. The statistical temperature $T^{(v)}$ measures the average kinetic energy modulo the contribution of translatory motion of the vessel. The term $\left( \frac{m^2 c^4}{H_{\text{nonrel}}} \right) = \frac{1}{\gamma^2(v)}$ with $H_{\text{particle}}$ being the Hamilton function of an individual particle says how relativistic is the motion of the system. In the rest frame $K_0$ from (5.13), we can reconstruct both the nonrelativistic $T_0 = \frac{E_{\text{nonrel}}}{2Nk}$ and the ultrarelativistic $T_0 = \frac{\sum_{i=1}^{K} \sqrt{p_i^2 c^2}}{3Nk}$ limits. Note that there is some freedom of choice of the factor in definition (5.13). Instead of $\frac{1}{\gamma^2}$ we can use $\frac{1}{\gamma^2(v)^2}$. Thus for $r = 1$ the statistical temperature is an invariant and for $r = 2$ we obtain the transformation rule (2.27).

Another possibility is suggested by the general formula (5.8). First, we recall that the pressure $p$ is defined as the average momentum transported per second per unit area through the surface element in the direction given by the unit positive normal $\hat{n}$ to this element. It can be easily shown that using the above definition one obtains [28]

$$ p = p_0 = \frac{N}{V} \langle \vec{p} \cdot \hat{n} \rangle (\vec{w} - \vec{v}) \cdot \hat{n} \rangle = \frac{N}{V} \langle m\gamma(w)(\vec{w} \cdot \hat{n})((\vec{w} - \vec{v}) \cdot \hat{n}) \rangle, \quad \text{(5.14)} $$

where $\vec{w}$ is the velocity of the particle with respect to the frame $K$, $\vec{p}$ as before is the particle momentum in the system $K$ and $\vec{v}$ denotes the velocity of the vessel containing a gas relative to $K$.

From (2.27), (5.8) and (5.14), noting that $V = (\gamma(v)^{-1})V_0$ we find

$$ T = (\gamma(v)^{-1})^2 \frac{1}{k} \frac{1}{m\gamma(w)(\vec{w} \cdot \hat{n})((\vec{w} - \vec{v}) \cdot \hat{n})} \quad \text{(5.15)} $$

This formula shows that the absolute temperature $T$ can be measured by the 'statistical thermometer' defined by the right-hand side of (5.15). However, the term $(\gamma(v)^{-1})^2$ appearing in (5.15) is somewhat artificial and is taken into account only in order to reproduce the absolute temperature $T$. Other terms $(\gamma(v)^{-1})$ can also be used.

Thus one can find the temperature as a Lorentz invariant [15, 37]

$$ T^{(\text{stat})} = T_0 = \frac{1}{k} m\gamma(w)(\vec{w} \cdot \hat{n})((\vec{w} - \vec{v}) \cdot \hat{n}) \quad \text{(5.16)} $$

or the temperature transforming according to (2.28) [38] (a moving system appears to be cooler)

$$ T^{(\text{Planck})} = \frac{1}{k} m\gamma(w)(\vec{w} \cdot \hat{n})((\vec{w} - \vec{v}) \cdot \hat{n}) \quad \text{(5.17)} $$

Analogous arbitrariness appears when one looks for the temperature of a moving black body by employing the Planck distribution [18–20].

### 6. Concluding remarks

Concluding this paper we should elucidate some points. We are very far from the statement that only transformations (2.17) for heat and (2.27) for absolute temperature are correct, and all the remaining ones such as, for example, (2.18) and (2.28) are not correct. What we would rather say is that the transformations (2.17) and (2.27) are perhaps the most natural ones in the sense that they follow from both phenomenological and statistical relativistic thermodynamics as the result of considerations which are very close to the ones known in nonrelativistic thermodynamics. To be more concrete, let us take the formula (2.20a) under the assumption that the pressure $p = \text{const}$. Then (2.20a) gives

$$ dE = \gamma(v)\delta(Q_0 - p dV). $$

In close analogy to the nonrelativistic case, one assumes that thermodynamic work $\delta L$ is expressed by the differentials of exterior thermodynamic parameters as a suitable Pfaff differential form. In the present case $\delta L = - p dV$. Therefore, the remaining term $\gamma(v)\delta Q_0$ defines the heat and this is just the transformation rule (2.17).

It is worthwhile to point out another important fact. In analogy to the nonrelativistic thermodynamics, we deal with the equal-time statistical ensembles. However, such an approach cannot be applied in general relativity for a thermodynamic system in an arbitrary gravitational field [13, 24, 44] since then the notion of an equal-time hypersurface is not well defined. Consequently, one should also consider some more ‘invariant’ approaches. One of them given in a distinguished work [24] consists in statistical averaging the thermodynamic observables.
over backward-lightcones. This leads to the relativistic thermodynamics called by the authors ‘photographic thermodynamics’. As shown in [24] this procedure ends with the Ott–Arzelès–Møller transformations (2.17), (2.27) or with the Landsberg–van Kampen transformations (2.19), (2.29). Note that the authors of [24] consider this latter possibility as ‘more appealing’.

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