Krajewski diagrams and spin lifts

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Abstract

A classification of irreducible, dynamically non-degenerate, almost commutative spectral triples is refined. It is extended to include centrally extended spin lifts. Simultaneously it is reduced by imposing three constraints: (i) the condition of vanishing Yang-Mills and mixed gravitational anomalies, (ii) the condition that the fermion representation be complex under the little group, while (iii) massless fermions are to remain neutral under the little group. These constraints single out the standard model with one generation of leptons and quarks and with an arbitrary number of colours.

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1 Introduction

A small class of Yang-Mills theories with fermions admit an interpretation in terms of almost commutative 4-dimensional spectral triples \([1, 2, 3, 4]\). In these theories the Yang-Mills forces are pseudo-forces induced by gravity via transformations belonging to the automorphism group of the algebra defining the noncommutative geometry. In the almost commutative case this group consists of timespace diffeomorphisms and gauge transformations. In the same way that in general relativity the diffeomorphisms produce the gravitational field, the gauge transformations produce a scalar field, the metric of internal space. In the same vein, the Einstein-Hilbert action can be generalized to noncommutative geometry, the so-called spectral action. In the almost commutative case it contains the Yang-Mills action. On the internal space the spectral action reduces to the Higgs-potential and breaks the gauge group spontaneously. The minimum of the Higgs-potential induces the fermion masses and they satisfy the Einstein equation in internal space. A natural question to ask is whether these solutions are stable under renormalization flow. This question motivates our definition of dynamical non-degeneracy \([5]\). At this point we must note that the standard model of electromagnetic, weak and strong forces is a Yang-Mills theory that can be interpreted as a spectral triple, that this interpretation produces the phenomenologically correct Higgs field, a colorless isospin doublet, and that its spectral triple is dynamically non-degenerate. Furthermore, after restriction to one generation of leptons and quarks, this spectral triple is irreducible. In \([5]\) we started the classification of irreducible, dynamically non-degenerate triples for finite algebras with one, two and three simple summands. We simplified the task by lifting unitaries instead of automorphisms. Lifting automorphism to the Hilbert space is delicate for algebras \(M_n(\mathbb{C})\), \(n \geq 2\). Indeed all its automorphisms connected to the identity are inner and form the group \(U(n)/U(1)\). When we want to lift such automorphisms to the Hilbert space we encounter a continuous infinity of multi-values parameterized by \(U(1)\). We avoid this obstruction by a central extension \([6]\). Redoing the classification \([5]\) with centrally extended automorphisms rather than unitaries implies two complications: (i) the extension is not unique, (ii) there are more unitaries than extended automorphisms, therefore a triple, which is dynamically degenerate with unitaries might be non-degenerate with extended automorphisms. To keep the classification manageable three additional, physically motivated constraints are introduced. We only keep those triples and central extensions,

- that are free of Yang-Mills anomalies and free of mixed gravitational-Yang-Mills anomalies,
- whose fermionic representation is complex under the little group in each irreducible component,
- whose massless fermions are neutral under the little group, i.e. transform trivially.

Of course the standard model with an arbitrary number of colours satisfies these criteria and we will show that within noncommutative geometry it is essentially unique as such.
2 Statement of the result

Consider a finite, real, $S^0$-real, irreducible spectral triple whose algebra has one, two or three simple summands and the extended lift as described below. Consider the list of all Yang-Mills-Higgs models induced by these triples and lifts. Discard all models that have (i) a dynamically degenerate fermionic mass spectrum, (ii) Yang-Mills or gravitational anomalies, (iii) a fermion multiplet whose representation under the little group is real or pseudo-real, or (iv) a neutrino transforming non-trivially under the little group. The remaining models are the following, $p$ is the number of colours, $p \geq 2$, the gauge group is on the left-hand side of the arrow, the little group on the right-hand side:

\[
\begin{align*}
SU(2) \times U(1) \times SU(p) \quad &\rightarrow \quad U(1) \times SU(p) \\
\mathbb{Z}_2 \times \mathbb{Z}_p &\rightarrow \mathbb{Z}_p
\end{align*}
\]

The left-handed fermions transform according to a multiplet $2 \otimes p$ with hypercharge $q/(2p)$ and a multiplet $p$ with hypercharge $-q/2$. The right-handed fermions sit in two multiplets $p$ with hypercharges $q(1+p)/(2p)$ and $q(1-p)/(2p)$ and one singlet with hypercharge $-q$, $q \in \mathbb{Q}$. The elements in $\mathbb{Z}_2 \times \mathbb{Z}_p$ are embedded in the center of $SU(2) \times U(1) \times SU(p)$ as

\[
\left( \exp \frac{2\pi ik}{2} 1_2, \exp[2\pi i(pk - 2\ell)/q], \exp \frac{2\pi i\ell}{p} 1_p \right), \quad k = 0, 1, \quad \ell = 0, 1, \ldots, p - 1. \quad (1)
\]

The Higgs scalar transforms as an $SU(2)$ doublet, $SU(p)$ singlet and has hypercharge $-q/2$.

With the number of colours $p = 3$, this is the standard model with one generation of quarks and leptons.

We also have in our list two submodels of the above model defined by the subgroups

\[
\begin{align*}
SO(2) \times U(1) \times SU(p) \quad &\rightarrow \quad U(1) \times SU(p) \\
\mathbb{Z}_2 \times \mathbb{Z}_p &\rightarrow \mathbb{Z}_p \\
SU(2) \times U(1) \times SO(p) \quad &\rightarrow \quad U(1) \times SO(p) \\
\mathbb{Z}_2 &\rightarrow \mathbb{Z}_2
\end{align*}
\]

They have the same particle content as the standard model, in the first case only the $W^\pm$ bosons are missing, in the second case roughly half the gluons are lost.

\[
p = 2, 4, \ldots
\]

\[
SU(2) \times U(1) \times SU(p) \quad &\rightarrow \quad U(1) \times SU(p) \\
\mathbb{Z}_p &\rightarrow \mathbb{Z}_p
\]

with the same particle content as for odd $p$. But now we have three possible submodels:

\[
\begin{align*}
SO(2) \times U(1) \times SU(p) \quad &\rightarrow \quad U(1) \times SU(p) \\
\mathbb{Z}_p &\rightarrow \mathbb{Z}_p
\end{align*}
\]
\[ SU(2) \times U(1) \times SO(p) \rightarrow U(1) \times SO(p), \]
\[ SU(2) \times U(1) \times USp(p/2) \rightarrow U(1) \times USp(p/2). \]

3 The set up

Let \((A, \mathcal{H}, D, J, \varepsilon, \chi)\) be a real, \(S^0\)-real, finite spectral triple. \(A\) is a finite dimensional real algebra represented faithfully on a finite dimensional complex Hilbert space \(\mathcal{H}\) via \(\rho\). Four additional operators are defined on \(\mathcal{H}\): the Dirac operator \(D\) is self-adjoint, the real structure (or charge conjugation) \(J\) is antiunitary, and the \(S^0\)-real structure \(\varepsilon\) and the chirality \(\chi\) are both unitary involutions. These operators satisfy:

- \(J^2 = 1, \quad [J, D] = [J, \chi] = [\varepsilon, \chi] = [\varepsilon, D] = 0, \quad \varepsilon J = -J \varepsilon, \quad D \chi = -\chi D, \)
- The chirality can be written as a finite sum \(\chi = \sum_i \rho(a_i) J \rho(b_i) J^{-1}\). This condition is called orientability.
- The intersection form \(\cap_{ij} := \text{tr}(\chi \rho(p_i) J \rho(p_j) J^{-1})\) is non-degenerate, \(\det \cap \neq 0\). The \(p_i\) are minimal rank projections in \(A\). This condition is called Poincaré duality.
- The kernel of \(D\) has no nontrivial \(A\)-invariant subspace.

We choose a basis of \(\mathcal{H}\) such that the five operators take the form

\[ \rho = \begin{pmatrix} \rho_L & 0 & 0 & 0 \\ 0 & \rho_R & 0 & 0 \\ 0 & 0 & \rho'_L & 0 \\ 0 & 0 & 0 & \rho'_R \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M} \\ 0 & 0 & \mathcal{M}^* & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \chi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The algebra is a finite sum of simple algebras, \(A = \bigoplus_i M_{n_i}(\mathbb{K}_i)\) and \(\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}\) where \(\mathbb{H}\) denotes the quaternions. Except for complex conjugation in \(M_n(\mathbb{C})\) and permutations of identical summands in the algebra \(A\), every algebra automorphism \(\sigma\) is inner, \(\sigma(a) = u a u^{-1} =: i_u(a)\) for a unitary \(u \in U(A)\). This unitary is ambiguous by any central unitary \(u_c \in U(A) \cap \text{Center}(A)\), indeed \(i_{u_c u} = i_u\). \(M_n(\mathbb{R})\) and \(M_n(\mathbb{H})\) do not have central unitaries close to the identity. We therefore start with the complex case \(A = \bigoplus_i M_{n_i}(\mathbb{C})\). Since \(M_1(\mathbb{C}) = \mathbb{C}\) has no automorphisms close to the identity, we have to distinguish the cases \(n_i = 1\) and \(n_i \geq 2\), and write

\[ A = \mathbb{C}^M \oplus \bigoplus_{k=1}^N M_{n_k}(\mathbb{C}) \ni a = (b_1, \ldots b_M, c_1, \ldots, c_N), \quad n_k \geq 2. \tag{2} \]
Its group of unitaries is
\[ U(\mathcal{A}) = U(1)^M \times \prod_{k=1}^N U(n_k) =: U(1)^M \times G_A \ni u = (v_1, \ldots, v_M, w_1, \ldots, w_N), \quad (3) \]

\( G_A \) is the group of ‘noncommutative unitaries’. The subgroup of automorphisms of \( \mathcal{A} \) connected to the identity is the group of inner automorphisms, \( \text{Int}(\mathcal{A}) = G_A/ZG_A \) where \( ZG_A = U(1)^N \ni (w_{c1}1_{n_1}, \ldots, w_{cN}1_{n_N}) \) is the subgroup of noncommutative, central unitaries, \( w_{ck} \) is a root of the determinant of \( w_k \). The spin lift is a group homomorphism \( L \) from the automorphism group of the algebra \( \mathcal{A} \) (or its connected component \( \text{Int}(\mathcal{A}) \) for simplicity) into the group of unitary operators on \( \mathcal{H} \) satisfying \( [L(\sigma), J] = [L(\sigma), \chi] = 0 \) for all \( \sigma \in \text{Int}(\mathcal{A}) \) and satisfying the covariance condition
\[ i_{L(\sigma)}\rho(a) = \rho(\sigma(a)) \quad \text{for all } a \in \mathcal{A}. \quad (4) \]

The ambiguity of the lift by the noncommutative, central unitaries forces us to centrally extend \( \text{Int}(\mathcal{A}) \) to \( G_A \). Then the group homomorphism \( L : G_A \to U(\mathcal{H}) \) must satisfy \( [L(w), J] = [L(w), \chi] = 0 \) for all \( w \in G_A \) and
\[ i_{L(w)}\rho(a) = \rho(i_w(a)) \quad \text{for all } a \in \mathcal{A}. \quad (5) \]

The following map qualifies as extended lift \[ L(w) = \rho(\hat{u})J\rho(\hat{u})J^{-1}, \]
\[ \hat{v}_j := \prod_{k=1}^N (\det w_k)^q_{1,k}, \quad j = 1, \ldots, M, \]
\[ \hat{w}_\ell := w_\ell \prod_{k=1}^N (\det w_k)^q_{M+\ell,k}, \quad \ell = 1, \ldots, N, \quad (6) \]

where the \( q \)'s are arbitrary rational numbers, ‘charges’. They form a \((M+N)\times N\) matrix. This lift is multi-valued, but thanks to the central extension it only has a finite number of values. The maps \( L \) are not the only possible extensions, if the representation decomposes into several blocks different charges may be chosen in each block.

The induced Yang-Mills model has \( G_A \) as gauge group and the (extended) lift \( L \) defines the fermionic representation. We will use the conditions of vanishing Yang-Mills and mixed gravitational-Yang-Mills anomalies to reduce the possible charges in the definition of the lift. To spell out these conditions for the general \( L(w) \), equation \[ (6) \], we need its infinitesimal version, \( \ell(X) \) defined by
\[ L(\exp X) = \exp \ell(X), \quad X = (X_1, \ldots, X_N) \in \bigoplus_{k=1}^N u(n_k) =: \mathfrak{g}_A. \quad (7) \]

The vanishing of the Yang-Mills anomalies and of the gravitational-Yang-Mills anomalies is equivalent to
\[ \text{tr}[\ell(X)^3\chi(1+\varepsilon)/2] = 0 \quad \text{and} \quad \text{tr}[\ell(X)\chi(1+\varepsilon)/2] = 0, \quad \text{for all } X \in \mathfrak{g}_A. \quad (8) \]
Let us write the lift in components:

\[
L(w) := \begin{pmatrix}
L_L(w) & 0 & 0 & 0 \\
0 & L_R(w) & 0 & 0 \\
0 & 0 & \bar{L}_L(w) & 0 \\
0 & 0 & 0 & \bar{L}_R(w)
\end{pmatrix}
\] and

\[
\ell(X) := \begin{pmatrix}
\ell_L(X) & 0 & 0 & 0 \\
0 & \ell_R(X) & 0 & 0 \\
0 & 0 & \bar{\ell}_L(X) & 0 \\
0 & 0 & 0 & \bar{\ell}_R(X)
\end{pmatrix}.
\] (9)

Then the anomaly conditions read:

\[
\text{tr}[-\ell_L(X)^3 + \ell_R(X)^3] = 0,
\text{tr}[-\ell_L(X) + \ell_R(X)] = 0,
\text{for all } X \in \mathfrak{g}_A.
\] (11)

The scalar field is obtained by fluctuating the internal Dirac operator \(D\), that is a finite linear combination:

\[
\Phi := \sum_j r_j L(jw) D L(jw)^{-1}, \quad r_j \in \mathbb{R}, \ jw \in G_A.
\] (12)

After the decomposition

\[
\Phi := \begin{pmatrix}
0 & \phi & 0 & 0 \\
\phi^* & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{\phi} \\
0 & 0 & \bar{\phi}^* & 0
\end{pmatrix},
\] (13)

we have

\[
\phi := \sum_j r_j L_L(jw) M L_R(jw)^{-1}.
\] (14)

The action of the scalar field is the Higgs potential

\[
V(\Phi) = \lambda \text{tr} [\Phi^4] - \frac{\mu^2}{2} \text{tr} [\Phi^2] = 4\lambda \text{tr} [(\phi^* \phi)^2] - 2\mu^2 \text{tr} [\phi^* \phi],
\] (15)

where \(\lambda\) and \(\mu\) are positive constants [4, 7]. Our task is to find the minima \(\Phi\) of this action, their spectra and their \textit{little groups}

\[
G_\ell := \left\{ w \in G_A, L(w) \Phi L(w)^{-1} = \Phi \right\}.
\] (16)

If we replace one of the \(M_{n_k}(\mathbb{C}), n_k \geq 2\), by \(M_{n_k}(\mathbb{R})\) or \(M_{n_k}(\mathbb{H})\) in the algebra \(\mathfrak{g}\) then the lift (13) simplifies: \(q_{M+k,j} = 0\) for all \(1 \leq j \leq N\) and \(q_{j,k} = 0\) for all \(1 \leq j \leq M + N\). If we replace say the \(\mathbb{K}_1 = \mathbb{C}\) by \(\mathbb{K}_1 = \mathbb{R}\) then \(q_{1,j} = 0\) for all \(1 \leq j \leq N\).

In \[\mathbb{H}\] we had lifted the entire unitary group \(U(A) \ni u\) by means of the lift \(L(u) = \rho(u)J \rho(u)J^{-1}\) which up to phases coincides with the present lift (6). In all cases where
\( \mathcal{A} \) has no commutative summand, \( \mathbb{C} \) or \( \mathbb{R} \), which means \( M = 0 \), the phases can be re-absorbed into the noncommutative unitaries and there is no modification to the configuration space, the affine space of all scalars. In the following we go through all irreducible Krajewski diagrams with \( N + M \leq 3 \) as given in [5] using the most general anomaly free lift and work out the modifications for \( M \geq 1 \). All induced Yang-Mills-Higgs models have one massless particle, a ‘neutrino’. Our aim is to list all those models whose fermion masses are dynamically non-degenerate, whose fermions transform as complex representations under the little group in each irreducible component, and whose neutrino is neutral under the little group.

Let us briefly recall the definition of dynamical degeneracy [5]. The mass matrix \( \mathcal{M} \) decomposes into blocks, which are represented by the arrows in the Krajewski diagram. Each arrow comes with an orientation and three algebras, a left-handed algebra, a right-handed algebra and a colour algebra. If the sizes of the matrices, elements of the three algebras, are \( k, \ell \) and \( p \), then the block corresponding to this arrow is \( \mathcal{M} \otimes 1_p \) with \( M \) being a complex \( k \times \ell \) matrix. The spectrum of the (internal) Dirac operator \( \mathcal{D} \) is always degenerate: all nonvanishing eigenvalues come in pairs of opposite sign due to the chirality that anticommutes with \( \mathcal{D} \), ‘left-right degeneracy’. All eigenvalues appear twice due to the real structure that commutes with \( \mathcal{D} \), ‘particle anti-particle degeneracy’. There is a third degeneracy, \( p \)-fold for the block above, that comes from the first order axiom. Let us call it colour degeneracy. It is absent if and only if the colour algebras of all arrows are commutative. We call these degeneracies kinematical because these come from the axioms. Because of the axioms, these three degeneracies survive the fluctuations of the Dirac operator and the minimization of the Higgs potential. By dynamical non-degenerate we mean that no minimum of the Higgs potential has degeneracies other than the above three. The first two degeneracies survive quantum fluctuations as well. We also want the colour degeneracies to be protected from quantum fluctuations. A natural protection is unbroken gauge invariance, a requirement that we include in the definition of dynamical non-degeneracy. More precisely, the irreducible spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \epsilon, \chi)\) is dynamically non-degenerate if all minima \( \Phi \) of the Higgs potential define again a spectral triple \((\mathcal{A}, \mathcal{H}, \Phi, J, \epsilon, \chi)\) and if the spectra of all minima have no degeneracies other than the three kinematical ones. We also suppose that the colour degeneracies are protected by the little group. By this we mean that all eigenvectors of \( \Phi \) corresponding to the same eigenvalue are in a common orbit of the little group (and scalar multiplication and charge conjugation).

Recall also that a unitary representation is real if it is equal to its complex conjugate and pseudo-real if it is unitarily equivalent to its complex conjugate. Otherwise the representation is complex. For example the representations of \( SO(n) \) are real, those of \( SU(2) \) are pseudo-real. An irreducible, unitary representation of \( U(1) \) is complex if and only if its charge is non-zero. The physical motivation for complex representations is that they allow to distinguish between particles and anti-particles.
4 One and two summands

Of all simple algebras only $\mathcal{A} = M_n(\mathbb{C})$ admit real spectral triples \cite{[5]}. The irreducible triples with $n \geq 2$ are dynamically degenerate. If $n = 1$ then $\mathcal{A} = \mathbb{C}$. Being commutative this algebra has no automorphisms close to the identity and there is nothing to extend nor to lift.

For the sum of two algebras only $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C} \ni (a, b)$ and its real or quaternionic subalgebras admit dynamically non-degenerate triples. These are of the form

$$\rho(a, b) = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & \beta b & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & b
\end{pmatrix}, \quad \beta = \pm 1, \quad 1b := b, \quad -1b := b, \quad \mathcal{M} = \begin{pmatrix} 0 \\ m \end{pmatrix}, \quad m \in \mathbb{R}. \quad (17)$$

We have one noncommutative unitary $w \in U(2)$ and the lift (10) reduces to

$$L(w) = \rho(w(det w)^{q_{21}}, (det w)^{q_{11}})Jp(w(det w)^{q_{21}}, (det w)^{q_{11}})J^{-1} \quad (18)$$
or equivalently

$$L_L(w) = w(det w)^{q_{21} - q_{11}}, \quad L_R(w) = w(det w)^{-\beta q_{11} + 1}. \quad (19)$$

On infinitesimal level with $X \in u(3)$ and $X_0 := X - \frac{1}{2} trX \mathbf{1}_2$ we have

$$\ell_L = X_0 + (q_{21} - q_{11} + \frac{1}{2}) trX \mathbf{1}_2, \quad \ell_R = - (\beta + 1)p trX \mathbf{1}_2, \quad (20)$$

and the lift is anomaly free if and only if $q_{21} = -1/2$, $q_{11} = 0$ for the representation $\beta = 1$ and $q_{21} = q_{11} - 1/2$ for $\beta = -1$. Therefore all fermionic hypercharges vanish and the gauge group is $SU(2)$. The scalar field is a doublet, $\phi = (x, y)^T$, $x, y \in \mathbb{C}$ and $\tilde{\phi} = (\mu/(4\lambda)^{1/2}, 0)^T$ minimizes the Higgs potential with little group $G_\ell = \{1\}$. Replacing $M_2(\mathbb{C})$ by the quaternions $\mathbb{H}$ leads to the same Yang-Mill-Higgs model: $SU(2) \rightarrow \{1\}$ with a left-handed doublet of fermions, a right-handed singlet and a doublet of scalars. If we take $M_2(\mathbb{R})$ then only the gauge group changes: $SO(2) \rightarrow \{1\}$. In all cases the little group is trivial and has no complex representation.

5 Three summands

We use the list of the 41 irreducible Krajewski diagrams with three simple summands from \cite{[5]}, figure 1. This list becomes exhaustive upon permutations of the three algebras $\mathcal{A}_1 = M_n(\mathbb{K}_1) \ni a$, $\mathcal{A}_2 = M_m(\mathbb{K}_2) \ni b$, $\mathcal{A}_3 = M_q(\mathbb{K}_3) \ni c$, upon permuting left and right, i.e. changing the directions of all arrows simultaneously, and upon permutations of particles and antiparticles independently in every connected component of the diagram.

Let $k, \ell, p$ be the sizes of the matrices $a, b, c$, for example $k = n$ if $\mathbb{K}_1 = \mathbb{C}$ or $\mathbb{R}$, $k = 2n$ if $\mathbb{K}_1 = \mathbb{H}$. To write the lift we will use the following letters:

$$\hat{u} := u (det u)^{q_{11}}(det v)^{q_{12}}(det w)^{q_{13}} \in U(\mathcal{A}_1), \quad (21)$$
$$\hat{v} := v (det u)^{q_{21}}(det v)^{q_{22}}(det w)^{q_{23}} \in U(\mathcal{A}_2), \quad (22)$$
$$\hat{w} := w (det u)^{q_{31}}(det v)^{q_{32}}(det w)^{q_{33}} \in U(\mathcal{A}_3). \quad (23)$$
It is understood that for instance if \( k = 1 \) we set \( u = 1 \) and \( q_{j1} = 0, j = 1, 2, 3 \). If \( \mathbb{K}_1 = \mathbb{H} \) or \( \mathbb{R} \) we set \( q_{j1} = 0 \) and \( q_{1j} = 0 \).

**Diagram 1** yields:
\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 \\ 0 & b \otimes 1_\ell \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 \\ 0 & c \otimes 1_\ell \end{pmatrix},
\]
\[
\rho^c_L(a, b, c) = \begin{pmatrix} 1_k \otimes \alpha' a & 0 \\ 0 & 1_\ell \otimes \beta' b \end{pmatrix}, \quad \rho^c_R(a, b, c) = \begin{pmatrix} 1_\ell \otimes \alpha' a & 0 \\ 0 & 1_p \otimes \beta' b \end{pmatrix},
\]
(24)

where \( \beta, \alpha', \beta' \) take values \( \pm 1 \) to indicate whether the fundamental representation, 1, or its complex conjugate, \(-1\), is meant. The fermionic mass matrix has the form
\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ 1_\ell \otimes M_3 & M_2 \otimes 1_\ell \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \ M_2 \in M_{p \times \ell}(\mathbb{C}), \ M_3 \in M_{\ell \times k}(\mathbb{C}).
\]
(25)

If \( M_3 \) is nonzero the first order axiom implies \( \beta = 1 \). We consider only this case, the other case, \( M_3 = 0 \) is treated as diagram 2. The first two summands, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are colour algebras, they are both broken and must therefore be 1-dimensional, \( k = \ell = 1 \). Then we must take \( p = 2 \), for \( p \geq 3 \) we would have two or more neutrinos.

Vanishing anomalies imply \( q_{33} + 1/2 + \beta' q_{23} = 0, q_{13} = -\beta' q_{23} \). The little group then is \( U(1) \) or trivial, but in the former case the neutrino is charged.

**Diagram 2** yields:
\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 \\ 0 & c \otimes 1_\ell \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 \\ 0 & b \otimes 1_\ell \end{pmatrix},
\]
\[
\rho^c_L(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 \\ 0 & 1_p \otimes b \end{pmatrix}, \quad \rho^c_R(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 \\ 0 & 1_\ell \otimes b \end{pmatrix},
\]
(26)

and
\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ 0 & M_2 \otimes 1_\ell \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \ M_2 \in M_{p \times \ell}(\mathbb{C}).
\]
(27)

The possible complex conjugations in the representations are irrelevant in this diagram.

As in diagram 1 be must take \( k = \ell = 1, p = 2 \).

Let us write the fluctuations in the form:
\[
\phi = \begin{pmatrix} \varphi_1 \otimes 1_k & 0 \\ 0 & \varphi_2 \otimes 1_\ell \end{pmatrix}, \quad \varphi_1 \in M_{k \times \ell}(\mathbb{C}), \ \varphi_2 \in M_{p \times \ell}(\mathbb{C})
\]
(28)

with
\[
\varphi_1 = \sum_j r_j \hat{u}_j M_1 \hat{v}_j^{-1}, \quad \varphi_2 = \sum_j r_j \hat{w}_j M_2 \hat{v}_j^{-1},
\]
(29)
We can decouple the two scalars \( \varphi_1 \) and \( \varphi_2 \) by means of the fluctuation: \( r_1 = \frac{1}{2}, \)
\( \hat{u}_1 = 1_k, \hat{v}_1 = 1_{\ell}, \hat{w}_1 = 1_p, r_2 = \frac{1}{2}, \hat{u}_2 = 1_k, \hat{v}_2 = 1_{\ell}, \hat{w}_2 = -1_p. \) Note that \( \hat{w}_2 = -1_p \) is possible because \( \det(-1_p) = 1. \) Since the arrows \( M_1 \) and \( M_2 \) are disconnected, the Higgs potential is a sum of a potential in \( \varphi_1 \) and of a potential in \( \varphi_2. \) Proceeding as in the preceding section we find the minimum \( \hat{\varphi}_1 = \mu (4\lambda)^{-\frac{1}{2}} \) and \( \hat{\varphi}_2 \) has one eigenvalue \( \mu (4\lambda)^{-\frac{1}{2}} \) and one vanishing eigenvalue. All triples associated to the first diagram are therefore dynamically degenerate.

Similarly, we can discard diagrams 3, 4, 6.

**Diagram 5** yields:

\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 \\ 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = b \otimes 1_k,
\]

\[
\rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 \\ 0 & 1_\ell \otimes c \end{pmatrix}, \quad \rho_R^c(a, b, c) = 1_\ell \otimes a,
\]

\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k \\ 1_\ell \otimes M_2 \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \quad M_2 \in M_{p \times k}(\mathbb{C}).
\] (30)

The colour algebra is indexed by \( k \) and \( \ell. \) Both summands are broken, therefore \( k = \ell = 1 \) forcing \( p = 1 \) to avoid two or more neutrinos.

**Diagrams 7, 9, 11, 12** fall in the same way.

**Diagram 8** yields the representations

\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & c \otimes 1_k & 0 \\ 0 & 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} \beta b \otimes 1_k & 0 \\ 0 & 0 & \gamma_c \otimes 1_p \end{pmatrix},
\] (31)

\[
\rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes \alpha' a & 0 & 0 \\ 0 & 1_p \otimes \alpha' a & 0 \\ 0 & 0 & 1_\ell \otimes \gamma'_c \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes \alpha' a & 0 \\ 0 & 0 & 1_p \otimes \gamma'_c \end{pmatrix},
\] (32)

where \( \beta, \alpha', \gamma' \) take values \( \pm 1 \) to indicate whether the fundamental representation, 1, or its complex conjugate, \( -1, \) is meant. The primes refer to colour representations. These leave the Higgs scalars invariant. The mass matrix is

\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ M_2 \otimes 1_k & 0 \\ 0 & M_3^* \otimes 1_p \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \quad M_2, M_3 \in M_{p \times k}(\mathbb{C}).
\] (33)

Requiring at most one zero eigenvalue (up to a possible colour degeneracy) implies \( k = 1, \ ell = p + 1 \) or \( k = 1, \ ell = p. \) The colour group consists of the \( us \) and \( ws. \) As they are spontaneously broken we must have \( k = p = 1, \) leaving \( \ell = 2. \) Then the fluctuations are

\[
\varphi_1 = \sum_j r_j \hat{u}_j M_1 \beta \hat{v}_j^{-1}, \quad \varphi_2 = \sum_j r_j \hat{w}_j M_2 \beta \hat{v}_j^{-1}, \quad \varphi_3 = \sum_j r_j \hat{w}_j M_3 \hat{v}_j^{-1},
\] (34)
\[ \hat{u} = (\det v)^{q_{12}}, \quad \hat{v} = v (\det v)^{q_{22}}, \quad \hat{w} = (\det v)^{q_{32}}. \] (35)

With \( C_i := \varphi_i^* \varphi_i \) the Higgs potential reads
\[ V(C_1, C_2, C_3) = 4[\lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2)] + 4[\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)]. \] (36)

If we impose an anomaly free lift then in all but four cases the little group is trivial, \( G_\ell = \{1_2\} \). The exceptions have \( K_1 = K_2 = K_3 = \mathbb{C} \), i.e. \( A = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \), \( q_{12} = -q_{32}, \ q_{22} = -1/2 \) and \( \beta, \gamma, \gamma', \alpha' = - - - +, - - + - , - + - + , - + - +, + - - +, + - - +, + - - + \) or \( + + + - \). These four triples induce the electro-weak model \( (SU(2) \times U(1))/\mathbb{Z}_2 \to U(1) \) of protons, neutrons, neutrinos and electrons. One chiral part of the neutron is an \( SU(2) \) singlet and therefore a real representation under the little group. It is nevertheless amusing to note a mass relation in these models: in the first and the third triple the neutron is slightly heavier than the proton,
\[ m_p = m_n \sqrt{\left(1 + \frac{m_e^4}{m_n^4}\right) / \left(1 + \frac{m_e^4}{m_p^4}\right)}, \] (37)
in the second and fourth triple the neutron is slightly lighter than the proton,
\[ m_n = m_p \sqrt{\left(1 + \frac{m_e^4}{m_p^4}\right) / \left(1 + \frac{m_e^4}{m_n^4}\right)}. \] (38)

**Diagram 10** is similar to diagram 8. It needs \( k = p = 1 \) and \( \ell = 2 \) and has representations
\[ \rho_L(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} \beta b & 0 \\ 0 & c \end{pmatrix}, \] (39)
\[ \rho_L^c(a, b, c) = \begin{pmatrix} \alpha' a & 0 & 0 \\ 0 & \alpha' a & 0 \\ 0 & 0 & \gamma' c_1 2 \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} \alpha' a_{12} & 0 \\ 0 & \gamma' c \end{pmatrix}. \] (40)

The mass matrix and Higgs potential are as for diagram 8, while the fluctuations read
\[ \varphi_1 = \sum_j r_j \hat{u}_j M_1^{\beta \hat{v}_j^{-1}}, \quad \varphi_2 = \sum_j r_j^{\alpha} \hat{u}_j M_2^{\beta \hat{v}_j^{-1}}, \quad \varphi_3 = \sum_j r_j \hat{w}_j M_3^{\hat{v}_j^{-1}}. \] (41)

Only four triples are anomaly free and have non-trivial little group: they all have \( K_1 = K_2 = K_3 = \mathbb{C} \) and \( q_{22} = -1/2 \) and are given by (i) \( \alpha, \beta, \alpha', \gamma' = - - - +, q_{12} = -q_{32}, \) (ii) \( - - + +, q_{12} = q_{32}, \) (iii) \( - + - +, q_{12} = -q_{32}, \) (iv) \( - + + +, q_{12} = q_{32}. \) As before they induce the electro-weak model with one mass relation.
**Diagram 13** has *ladder form*, i.e. it consists of horizontal arrows, vertically aligned. Its representations are

\[ \rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & \alpha_1 a \otimes 1_k & 0 \\ 0 & 0 & \alpha_2 a \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 \\ 0 & \beta b \otimes 1_p \end{pmatrix} \quad (42) \]

\[ \rho^\ell_L(a, b, c) = \begin{pmatrix} 1_k \otimes \alpha' a & 0 \\ 0 & 1_k \otimes \alpha' a \\ 0 & 0 & 1_k \otimes c \end{pmatrix}, \quad \rho^\ell_R(a, b, c) = \begin{pmatrix} 1_\ell \otimes \alpha' a & 0 \\ 0 & 1_\ell \otimes c \end{pmatrix} \quad (43) \]

The mass matrix is

\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ M_2 \otimes 1_k & 0 \\ 0 & M_3 \otimes 1_p \end{pmatrix}, \quad M_1, M_2, M_3 \in M_{k \times \ell}(\mathbb{C}). \quad (44) \]

The fluctuations read

\[ \varphi_1 = \sum_j r_j \hat{u}_j M_1 \beta_j^{-1}, \quad \varphi_2 = \sum_j r_j \alpha_1 \hat{u}_j M_2 \hat{v}_j^{-1}, \quad \varphi_3 = \sum_j r_j \alpha_2 \hat{u}_j M_3 \beta_j^{-1}, \quad (45) \]

and the action is

\[ V(C_1, C_2, C_3) = 4k \left[ \lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2) \right] + 4p \left[ \lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3) \right]. \quad (46) \]

The neutrino count implies \( k = 1, \ell = 1 \) or \( 2 \).

1: If \( \ell = 1 \) we must take \( \mathbb{K}_1 = \mathbb{C} \) and \( \alpha_1 = -1 \), otherwise the kernel of \( D \) would have a nontrivial \( \mathcal{A} \)-invariant subspace. We also must take \( \mathbb{K}_3 = \mathbb{C} \) and \( p \geq 2 \), otherwise there would be no automorphism to lift nor to extend. Then the extended lift reduces to

\[ \hat{u} = (\det w)^{q_{13}}, \quad \hat{v} = (\det w)^{q_{23}}, \quad \hat{w} = w (\det w)^{q_{33}}. \quad (47) \]

All remaining triples that admit an anomaly free extension have little group \( SU(p) \). Its representation on the first fermion is trivial.

2: If \( \ell = 2 \) we must take \( \mathbb{K}_1 = \mathbb{R} \) or \( \mathbb{C} \) and \( p = 1 \), otherwise the neutrino would have unbroken colour. The lift has

\[ \hat{u} = (\det v)^{q_{12}}, \quad \hat{v} = v (\det v)^{q_{22}}, \quad \hat{w} = (\det v)^{q_{32}}. \quad (48) \]

All anomaly free lifts have a trivial little group.

**Diagram 18** goes down the same drain. **Diagrams 14, 16, 19, 21** must have \( \ell = 1 \) and \( k = 1 \) or \( 2 \). The first case falls as diagram 13 with \( \ell = 1 \), the second has broken colour. **Diagrams 15, 20, 23 and 24** must have \( k = \ell = 1 \) and are rejected as diagram 13 with \( \ell = 1 \).
Diagram 17 provides models satisfying all criteria, in particular the standard model. In order to obtain the latter in conventional physics notations, ‘left things left’, we interchange left with right and $A_1$ with $A_2$. Then with $(a, b, c) \in M_k(\mathbb{C}) \oplus M_\ell(\mathbb{C}) \oplus M_p(\mathbb{C})$ the representation reads:

$$\rho_L = \begin{pmatrix} a \otimes 1_p & 0 \\ 0 & a \otimes 1_\ell \end{pmatrix}, \quad \rho_R = \begin{pmatrix} b \otimes 1_p & 0 & 0 \\ 0 & \beta_1 b \otimes 1_p & 0 \\ 0 & 0 & \beta_2 b \otimes 1_\ell \end{pmatrix},$$

$$\rho^c_L = \begin{pmatrix} 1_k \otimes c & 0 \\ 0 & 1_k \otimes \beta' b \end{pmatrix}, \quad \rho^c_R = \begin{pmatrix} 1_\ell \otimes c & 0 & 0 \\ 0 & 1_\ell \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes \beta' b \end{pmatrix}.$$  \hspace{1cm} (49)

The mass matrix is:

$$M = \begin{pmatrix} M_1 \otimes 1_p & M_2 \otimes 1_p & 0 \\ 0 & M_3 \otimes 1_\ell \end{pmatrix}.$$ \hspace{1cm} (51)

To avoid more than one neutrino we must take $k = 1$ or $2$ and $\ell = 1$. The first case forces $p \geq 2$ implying that the neutrino has unbroken colour. Therefore we take $k = 2$ and $\ell = 1$. If $p \geq 2$ we have

$$\hat{u} = u \det u^{q_{11}} \det w^{q_{13}}, \quad \hat{v} = \det u^{q_{21}} \det w^{q_{23}}, \quad \hat{w} = w \det u^{q_{31}} \det w^{q_{33}}.$$ \hspace{1cm} (52)

Only four sign choices, $\alpha, \beta_1, \beta_2, \beta' = +---, --++, ---+,$ and $----$, admit an anomaly free lift with complex fermion representations under the little group. All four choices lead to the same model,

$$\frac{SU(2) \times U(1) \times SU(p)}{\mathbb{Z}_2 \times \mathbb{Z}_p} \rightarrow \frac{U(1) \times SU(p)}{\mathbb{Z}_p}, \quad p = 3, 5, ...,$$

$$\frac{SU(2) \times U(1) \times SU(p)}{\mathbb{Z}_p} \rightarrow \frac{U(1) \times SU(p)}{\mathbb{Z}_p}, \quad p = 2, 4, ...$$

For example with the first choice we get:

$$q_{11} = -\frac{1}{2}, \quad q_{21} = \frac{x}{2}, \quad q_{31} = \frac{x}{2p},$$

$$q_{13} = 0, \quad q_{23} = \frac{y}{2}, \quad q_{33} = \frac{y}{2p} - \frac{1}{p},$$ \hspace{1cm} (53) \hspace{1cm} (54)

where $x$ and $y$ are rational numbers, $x + y \neq 0$. The hypercharges of the five irreducible fermion representations under the gauge group are:

$$\frac{x+y}{2p}, \quad -\frac{x+y}{2}; \quad (x+y)\frac{1+p}{2p}, \quad (x+y)\frac{1-p}{2p}, \quad -(x+y).$$ \hspace{1cm} (55)

A few other choices of the fields $K_1, K_2, K_3$ induce submodels of the above one that still satisfy all of our criteria:
\[ H, C, C \] entails \( \alpha = 1, q_{i1} = 0, i = 1, 2, 3, x = 1 \) and reproduces the standard model, \( SU(2) \times U(1) \times SU(p) \rightarrow U(1) \times SU(p) \), where to alleviate notations we suppress the quotient by discrete groups.

\[ R, C, C \] entails \( \alpha = 1, q_{i1} = 0, i = 1, 2, 3, x = 1 \) and induces the standard model with the Ws missing, \( U(1) \times U(1) \times SU(p) \rightarrow U(1) \times SU(p) \).

\[ C, C, H \] entails \( q_{i3} = 0, i = 1, 2, 3, y = 1 \) and induces the standard model with a few gluons, roughly half of them, missing, \( SU(2) \times U(1) \times USp(p/2) \rightarrow U(1) \times USp(p/2), p \) even.

\[ C, C, R \] entails \( q_{i3} = 0, i = 1, 2, 3, y = 1 \) and induces the standard model with some, roughly half, the gluons missing, \( SU(2) \times U(1) \times SO(p) \rightarrow U(1) \times SO(p) \).

All other possibilities have all fermion hypercharges vanish.

Finally for \( p = 1 \) we have \( K_i = C, i = 1, 2, 3 \) and \( q_{i3} = 0, q_{23} = 0, q_{33} = -1 \). The only models satisfying our criteria have again \( \alpha, \beta_1, \beta_2, \beta' = +--\, -, ++-, -++ \) or \(----\). These four triples induce the electro-weak model \( SU(2) \times U(1) \rightarrow U(1) \) of protons, neutrons, neutrinos and electrons, this time without a mass relation. E.g. for the first choice, we have \( q_{11} = -1/2, q_{21} = x/2, q_{31} = x/2 \).

**Diagram 22** induces the same models as diagram 17.

The following diagrams can be excluded simply by imposing broken colour to be 1-dimensional and by requiring the model to have at most one neutrino: **diagrams 25, 26, 27, 28, 31, 32, 33**.

**Diagram 29** is similar to diagram 8. It takes \( k = p = 1 \) and \( \ell = 2 \). The representations are

\[
\rho_L = \begin{pmatrix} b & 0 \\ 0 & \alpha a \end{pmatrix}, \quad \rho_R = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \beta b \end{pmatrix},
\]

and

\[
\rho_L' = \begin{pmatrix} \alpha a \ell_2 & 0 \\ 0 & \bar{c} \end{pmatrix}, \quad \rho_R' = \begin{pmatrix} \alpha a & 0 & 0 \\ 0 & \alpha a & 0 \\ 0 & 0 & \bar{c} \end{pmatrix}
\]

with a mass matrix

\[
\mathcal{M} = \begin{pmatrix} M_1 & M_2 & 0 \\ 0 & 0 & M_3' \end{pmatrix}.
\]

Only three triples are anomaly free and have non-trivial little group: they all have \( K_1 = K_2 = K_3 = C \) and are given by (i) \( \alpha, \beta = --, q_{12} = q_{32}, q_{22} = -1/2, \) (ii) \( ++, q_{12} = q_{32}, q_{22} = -1/2, \) (iii) \( +-, q_{22} = -1/2 - q_{12}/2, q_{32} = 0 \). As before they induce the electro-weak model with one mass relation.

**Diagram 30** takes \( k = \ell = 1 \) and \( p = 2 \). The representations are

\[
\rho_L = \begin{pmatrix} c & 0 \\ 0 & \alpha a \end{pmatrix}, \quad \rho_R = \begin{pmatrix} \bar{b} & 0 & 0 \\ 0 & \bar{b} & 0 \\ 0 & 0 & \gamma c \end{pmatrix},
\]

14
\[
\rho^c_L = \begin{pmatrix} a_{12} & 0 \\ 0 & a_b \end{pmatrix}, \quad \rho^c_R = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a b_{12} \end{pmatrix}
\]

(60)

with a mass matrix

\[
\mathcal{M} = \begin{pmatrix} M_1 & M_2 & 0 \\ 0 & 0 & M^*_3 \end{pmatrix}. \quad (61)
\]

Note that in this model \( M_1 \) and \( M_2 \) fluctuate in the same way. This entails that the little group is trivial.

For diagram 34, broken colour and neutrino count imply \( k = \ell = 1 \). All remaining triples with extended, but anomaly free lift have a non-trivial invariant subspace in the kernel of the Dirac operator or a real representation under the little group.

Diagrams 35, 38 and 39 share this fate.

Diagram 36, 37, 40 and 41 must have \( k = \ell = 1 \) for broken colour and neutrino count and \( p = 1 \) to avoid a coloured neutrino.

At this point we have exhausted all irreducible triples for algebras with one, two and three simple summands.

## 6 Outlook

We conjectured \[\text{[5]}\] that for five and more summands there is no irreducible, dynamically non-degenerate spectral triple without mass relations and that for four summands we only have the standard model with two simple colour algebras. Jureit and Stephan have written a computer algorithm that allows to compute the irreducible Krajewski diagrams with four summands and letter changing arrows. So far the conjecture for four summands resists their findings. The extension of the above list of Yang-Mills-Higgs models to four algebras is also in work.

The aim is of course to realize the old dream of Grand Unification \[\text{[9]}\] within noncommutative geometry. The dream was to describe particle physics by means of a Yang-Mills-Higgs model with (i) a simple group or at least a group as simple as possible, (ii) an irreducible fermion representation, or at least irreducible in one generation of quarks and leptons; that the fermion representation be (iii) anomaly free and (iv) complex. Let us mention two main differences between the two approaches. First, being an extension of Riemannian geometry, noncommutative geometry unifies the standard model of electromagnetic, weak and strong forces with gravity already at non-quantum level. Second, while Grand Unification predicts new forces at the unification scale of \( 10^{17} \) GeV, noncommutative geometry predicts no new forces but a new uncertainty relation in timespace at this energy scale. This uncertainty relation should shed new light on quantum field theory.

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Figure 1: The 41 irreducible Krajewski diagrams for three summands