On integration of the system of MHD equations modeling wave processes in a rotating liquid with arbitrary magnetic Reynolds number

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Abstract. The paper is concerned with the dynamics of large-scale wave processes in a rotating layer of inviscid conducting incompressible liquid of variable depth. The problem is modelled as a system of partial differential equations with necessary boundary conditions. With the help of auxiliary functions, the above system of partial differential magnetohydrodynamic equations is reduced to a single scalar partial differential equation. An exact analytic solution of the small perturbation problem is obtained. It is shown that if the external magnetic field is parallel to the axis of rotation of the layer, then the magnetic field decays for finite values of the magnetic Reynolds number.

1. Introduction

The paper is concerned with the dynamics of large-scale wave processes in a rotating layer of inviscid conducting incompressible liquid of variable depth.

The problem of magnetic field generation is a classical problem. The corresponding theory has been developed since 1950s mostly in the UK. J. J. Larmor was the first to mention the possibility of excitation of the magnetic field of celestial bodies. He thought that the solar magnetic field is maintained by axially symmetric motion of a substance in meridional planes. However, Cowling put forward the conjecture that bounded motions are incapable of generating axially symmetric or two-dimensional fields. It is worth pointing out that Cowling did not object the potency of two-dimensional velocity fields to generate the magnetic field. Later, the dynamo effect has been extensively studied in connection with astrophysical and geophysical problems in magnetic fields; but this phenomenon is known to have more general value in magnetic hydrodynamics.

The theory has been considerably advanced by H. Alfven [1], T. G. Cowling, I. M. Kirko, A. G. Kulikovskii, G. A. Lyubimov, J. A. Shercliff, L. Hartmann, V. M. Kontorovich, and by others. First achievements in this field were concerned with the study of propagation of small perturbations in ideal media and with representation of solutions of classical hydrodynamic problems of flow of conducting fluids in channels and pipes. The effect of waves on the liquid sodium surface in the BH-600 reactor pressure chamber was noted by I. M. Kirko [2, 3]. For Alfven waves and generation of oscillations of the magnetic field in fast-neutron reactors with liquid-metal heat-transfer fluid [4, 5].
A. Herzenberg and G. Backus gave an experimental evidence of the possibility of self-
excitation of the magnetic field with conducting medium motion [2,3]. I. M. Kirko experimentally
described the modification of the Herzenberg–Backus model; calculation showed that the internal
magnetic field is much stronger than the external one.

Advances in computers paved the way for direct numerical solution of dynamo problems in
various geometric topologies. Most seminal results in this direction are due to G. Glatzmaier
with coauthors [6], who succeeded in solving problems with a magnetic field whose properties
have similar properties with that of the real observed Earth magnetic field. The principal issue
with this approach is that modern supercomputers can be used for numerical solution of three-
dimensional problems with magnetic Reynolds numbers much smaller than those required in
practical applications of astrophysics, for which large values of the magnetic Reynolds number
are characteristic. Besides, from separate numerical calculations it is as a rule impossible to
qualitatively assess the wave process in the entire space–time scale; however, in the context of
the analytic solution (if available) they are capable of providing a quantitative estimate of the
wave process properties. Hence analytic approaches also have great value.

Due to the complexity of the equations describing the magnetohydrodynamic processes, the
research is mainly aimed at solving the Maxwell equation for given distributions of the velocity
field. Among such kinematic models, of special interest is S. I. Braginskii’s dynamo, because it
was constructed for extremely large Reynolds numbers. Since the generation of the magnetic field
can be caused by many various motions, the kinematic theory proves insufficient for identifying
the real motion pattern, thereby requiring the development of complete hydromagnetic theory.
At present, two principal types of magnetohydrodynamic dynamo have been studied: the
two-scale one and the near-symmetric highly conducting one; these two approaches depend
substantially on the ideas of S. I. Braginskii.

A possibility of the growth of a magnetic field maintained by the motion of liquid metal was
supported by experimental studies performed at Institute of Continuous Media Mechanics of the
Ural Branch of Russian Academy of Science (Perm’, Russia) and at Commissariat à l’Énergie
Atomique et aux Énergies Alternatives (CEA). The possibility of such generation has not only
astrophysically theoretical value, but it also of practical importance in the study of flows of liquid-
metal melts in cooling systems of reactors of nuclear power plants and in naval hydrodynamics,
thereby enabling one to determine magnetohydrodynamic characteristics of a liquid medium.
This is a modern important integrated hydrophysical problem.

In general, magnetic force lines are partially transformed by the liquid flow and partially
diffuse through it. The present study is concerned with this general case; that is, the problem
is solved with arbitrary values of the magnetic Reynolds number. The mathematical model is
constructed from the complete system of magnetic hydrodynamics equations with due account of
the inertia forces in the motion equations (such forces are neglected in available studies involving
applications of rapid revolution theory); in the limit case of infinitely large values of the magnetic
Reynolds number the results obtained are reduced to the available earlier results.

2. Basic equations of the model under study and their analytic realization

For the above problem, the basic equations and boundary value conditions of magnetic
hydrodynamics read as

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} - 2\omega \times \mathbf{v} - g \mathbf{z} + \frac{1}{\mu \rho} \left[ \text{rot} \mathbf{b} \times \mathbf{b} \right],
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \text{rot} \left[ \mathbf{b} \times \mathbf{v} \right] + \frac{1}{\mu \sigma} \Delta \mathbf{b},
\]
\[
\text{div} \; \mathbf{b} = 0, \\
p(x, y, -h_B) = p_0(x, y, t), \quad \mathbf{b}(x, y, -h_B) = \mathbf{b}_0(x, y, t), \\
v_z(x, y, -Z, t) = -v_x \frac{\partial Z}{\partial x} - v_y \frac{\partial Z}{\partial y}, \\
v_z(x, y, -h_B, t) = -\frac{\partial h_B}{\partial t} - v_x \frac{\partial h_B}{\partial x} - v_y \frac{\partial h_B}{\partial y}, \\
b_n(x, y, -h_B, t) = B_{n0}(x, y, t), \\
b_x \frac{\partial Z}{\partial x} + b_y \frac{\partial Z}{\partial y} + b_z = b_{n0}'(x, y, t),
\]

where \( \mathbf{v} \) is the liquid velocity in the frame rotating with rate \( \omega \), \( p \) is the pressure, \( \rho \) is the density, \( \mathbf{b} \) is the field magnetic induction vector, \( \mu \) is the magnetic permeability, \( \sigma \) is the electrical conductivity, \( p_0, \mathbf{b}_0, B_{n0}, b_{n0}'(x, y, t), Z \) are given functions, \( h_B \) is the unknown free layer surface.

For long–wave perturbations, the system of partial differential equations is reduced to system

\[
\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 2 \omega v_y - \frac{\partial}{\partial x} \left( \frac{p_0}{\rho} + \frac{\mu^2_0}{2\mu\rho} - gh_B \right) + \frac{1}{\mu\rho} \left( b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_y}{\partial y} \right), \\
\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -2 \omega v_x - \frac{\partial}{\partial y} \left( \frac{p_0}{\rho} + \frac{\mu^2_0}{2\mu\rho} - gh_B \right) + \frac{1}{\mu\rho} \left( b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_y}{\partial y} \right), \\
\frac{\partial h_B}{\partial t} + \frac{\partial}{\partial x} [\Omega h_B - Z v_x] + \frac{\partial}{\partial y} [\Omega h_B - Z v_y] = 0,
\]

Let

\[ H = h_B - Z = H_0(x, y) + \eta(x, y, t), \quad \eta \ll H_0, \quad \mathbf{v} = \mathbf{v}'(x, y, t), \quad \mathbf{b} = \mathbf{b}_0 + \mathbf{b}'(x, y, t). \]

For small amplitude the system of partial differential equations is reduced to system

\[
\frac{\partial v_x}{\partial t} - \alpha v_y = g \frac{\partial \eta}{\partial x} - \frac{1}{\rho} \frac{\partial}{\partial x} \left( \rho_0 + \rho_0 \frac{\mu_0^2}{2\rho} \right) \left( \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} \right) + \frac{1}{\mu\rho} \left( b_{0x} \frac{\partial b_x}{\partial x} + b_{0y} \frac{\partial b_y}{\partial y} \right), \\
\frac{\partial v_y}{\partial t} + \alpha v_x = g \frac{\partial \eta}{\partial y} - \frac{1}{\rho} \frac{\partial}{\partial y} \left( \rho_0 + \rho_0 \frac{\mu_0^2}{2\rho} \right) \left( \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} \right) + \frac{1}{\mu\rho} \left( b_{0x} \frac{\partial b_x}{\partial x} + b_{0y} \frac{\partial b_y}{\partial y} \right), \\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (H_0 v_x) + \frac{\partial}{\partial y} (H_0 v_y) = 0,
\]

\[
H_0 \left( \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} \right) + b_x \frac{\partial H_0}{\partial x} + b_y \frac{\partial H_0}{\partial y} + \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y} + b_{0x} \frac{\partial H_0}{\partial x} + b_{0y} \frac{\partial H_0}{\partial y} = \\
= B_{n0} \sqrt{1 + [\nabla H_0]^2 + [\nabla Z]^2 + 2 [\nabla H_0, \nabla (Z + \eta)] + [\nabla Z, \nabla \eta]} - b_{n0}'(x, y, t),
\]

\[
\frac{\partial b_x}{\partial t} - b_{0x} \frac{\partial v_x}{\partial x} - b_{0y} \frac{\partial v_y}{\partial y} = \frac{1}{R_m} \Delta b_x, \\
\frac{\partial b_y}{\partial t} - b_{0x} \frac{\partial v_x}{\partial x} - b_{0y} \frac{\partial v_y}{\partial y} = \frac{1}{R_m} \Delta b_y.
\]
Consider the auxiliary functions
\[ \tilde{\eta}(x, y, t), \quad \tilde{b}_x(x, y, t), \quad \tilde{b}_y(x, y, t), \quad \tilde{p}(x, y, t) \]
defined by
\[ \eta(x, y, t) = \frac{1}{g} D_t (D_t^2 + \alpha^2) \tilde{\eta}(x, y, t), \]
\[ x(x, y, t) = \mu \rho D_t (D_t^2 + \alpha^2) \tilde{b}_x(x, y, t), \]
\[ y(x, y, t) = \mu \rho D_t (D_t^2 + \alpha^2) \tilde{b}_y(x, y, t), \]
\[ p(x, y, t) = D_t (D_t^2 + \alpha^2) \tilde{p}(x, y, t), \]
where
\[ D_t = \frac{\partial}{\partial t}, \quad H = h_B - Z = H_0(x, y) + \eta(x, y, t). \]

Using these functions, for small amplitude long-wave perturbations, the system of partial differential magnetohydrodynamic equations under consideration is reduced to a single scalar partial differential equation
\[
D \left( D_t^2 + \alpha^2 \right)^2 \left\{ \left[ \left( D_t - \frac{\Delta}{R_m} \right) D_t - \frac{D_t^2}{\mu \rho} \right] \left[ \Delta_2 \xi + \langle \nabla \ln H_0, \nabla \xi \rangle \right] - \frac{D}{D(x, y)} \left( \ln H_0, \alpha \left( D_t - \frac{\Delta}{R_m} \right) \xi \right) \right\} + \frac{1}{g(\mu \rho)^2 H_0} \left( D_t^2 + \alpha^2 \right) \left[ F^2 + (\alpha D_t)^2 \right] \left[ D \xi + \frac{B_{n0}}{2 H_0 \sqrt{f}} \langle \nabla \xi, \nabla (H_0 + Z) \rangle \right] = 0
\]
\[
= - \frac{D H_0 + h_{n0}^{(c)}}{(\mu \rho)^2 H_0} - D \left( D_t^2 + \alpha^2 \right)^2 \left[ \left( D_t - \frac{\Delta}{R_m} \right) D_t - \frac{D_t^2}{\mu \rho} \right] \left[ \Delta_2 P - \langle \nabla \ln H_0, \nabla P \rangle \right] + \alpha D \left( D_t^2 + \alpha^2 \right)^2 \left( D_t - \frac{\Delta}{R_m} \right) \frac{D (\ln H_0, P)}{D(x, y)},
\]
where \( \alpha = 2\omega, \ D \) is the differential operator,
\[ P = \frac{1}{\rho} \left( p_0 + \frac{b_0^2}{2\mu} \right), \]
\[ \eta(x, y, t) = \left( F^2 + (\alpha D_t)^2 \right) \xi(x, y, t), \]
\[ F = \mu \rho \left( D_t^2 + \alpha^2 \right) \left( D_t - \frac{\Delta}{R_m} \right) - D_t^2 D_t, \]
\[ R_m \] is the magnetic Reynolds number,
\[ f = 1 + |\nabla Y_0|^2 + |\nabla Z|^2 + 2 \langle \nabla H_0, \nabla Z \rangle. \]

An exact analytic solution of the small perturbation problem is obtained in terms of the function
\[ \xi = \text{Re} A \exp (i (kx + ly - \sigma t)). \]
the frequency oscillations satisfying the dispersion relations

\[
\sigma^4 + i \frac{2(k^2 + l^2)}{R_m} \sigma^3 + \left[ gH_0(k^2 + l^2) - \alpha^2 - \frac{2(b_0xk + b_0yl)^2}{\mu \rho} - \frac{2(k^2 + l^2)^2}{R_m^2} \right] \sigma^2 + \\
+ \left[ \frac{i gH_0(k^2 + l^2)^2 - 2\alpha^2(k^2 + l^2)}{\mu \rho R_m} + i \frac{2(k^2 + l^2)(b_0xk + b_0yl)^2}{\mu \rho R_m} \right] \sigma + \\
+ \left[ \frac{\alpha^2(k^2 + l^2)^2}{R_m} - \frac{gH_0(k^2 + l^2)(b_0xk + b_0yl)^2}{\mu \rho} + \frac{(b_0xk + b_0yl)^4}{(\mu \rho)^2} \right].
\]

The above analysis shows that if the external magnetic field is parallel to the axis of rotation of the layer, then the magnetic field decays for finite values of the magnetic Reynolds number.

3. Conclusions

Based on the above study it can be concluded that a periodic process can be observed for sufficiently large values of the magnetic Reynolds number. But if the only nonzero component of the external magnetic field vector is the normal component to the liquid boundary, then there exists a wave regime caused not only by the magnetic forces, but also by the forces of gravity, Coriolis, and the corresponding boundary effects. Besides, there is a possibility of existence of the induced magnetic field on a relatively long wave interval, as well as the existence thereof without the background field.

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