Abstract

We prove that Weyl quantization preserves constant of motion of the Harmonic Oscillator. We also prove that if $f$ is a classical constant of motion and $\mathcal{O}p(f)$ is the corresponding operator, then $\mathcal{O}p(f)$ maps the Schwartz class into itself and it defines an essentially self-adjoint operator on $L^2(\mathbb{R}^n)$. As a consequence, we obtain detailed spectral information of $\mathcal{O}p(f)$. A complete characterization of the classical constants of motion of the Harmonic Oscillator is given and we also show that they form an algebra with the integral Moyal star product. We give some interesting examples of constants of motion and we analyze Weinstein-Guillemin average method within our framework.

Keywords  Constant of motion · Harmonic oscillator · Weyl calculus · Moyal product

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1 Introduction

Let $\Sigma$ be a symplectic manifold and $h_0 \in C^\infty(\Sigma)$ be a classical Hamiltonian. A classical constant of motion for $h_0$ is a classical observable $f \in C^\infty(\Sigma)$ such that $\{h_0, f\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(\Sigma)$. Similarly, if $H_0$ is a quantum Hamiltonian, a quantum constant of motion for $H_0$ is a quantum observable $F$ such that $[H_0, F] = 0$, which means that $H_0$ and $F$ are strongly commuting self-adjoint operators on certain Hilbert space $\mathcal{H}$. Equivalently, a constant of motion...
is an observable invariant by the evolution determined by the Hamiltonian. Moreover, classical and quantum constants of motion admit a decomposition through the reduction and diagonalization processes respectively (see Section 2.1).

Heuristically, a canonical quantization is a map $\mathcal{O}p$ that sends suitable classical observables into quantum observables in a physically meaningful manner. The analogy between classical and quantum constant of motion might suggest that a canonical quantization $\mathcal{O}p$ should map classical constants of motion of $h_0$ into quantum constants of motion of $H_0 = \mathcal{O}p(h_0)$. In such case we say that $\mathcal{O}p$ preserves constants of motion of $h_0$. However, at least when $\mathcal{O}p$ is the canonical Weyl quantization, since $\mathcal{O}p$ does not interchange the Poisson bracket of classical observables with the commutator of the corresponding operators (Groenewold-Van Hove’s no go Theorem [12]), we should expect that preservation of constants of motion occurs only for suitable Hamiltonians. We prove that Weyl quantization preserves constants of motion of the Harmonic Oscillator (Theorem 1). We combine the latter result with the $N-$representation Theorem to obtain important properties about the classical constants of motion of the Harmonic Oscillator and the corresponding operators, which we shall briefly summarize in the following paragraphs.

In Section 2, we begin fixing some notation. In Section 2.1, we summarize some known facts about general constants of motion. In Section 2.2, we restrict our attention to the Harmonic Oscillator and we introduce the natural class of symbols for our framework: the class of tempered constants of motion (see Definition 1).

We begin Section 3 interpreting a known fact about the metaplectic representation (Proposition 1) and we use it to prove that canonical Weyl quantization preserves constants of motion of the Harmonic Oscillator (for a precise statement, see Theorem 1). Using the $N-$representation Theorem, we prove that if $f$ is a tempered constant of motion then $\mathcal{O}p(f)$ maps the Schwartz class $S(\mathbb{R}^n)$ into itself (Theorem 2). Such property of the operator $\mathcal{O}p(f)$ is usually obtained requiring a different kind of conditions, which involve the growth of the observable $f$ and its derivatives. For instance, if $f$ belongs to a Hörmander class of symbols then $\mathcal{O}p(f)$ maps $S(\mathbb{R}^n)$ into itself [11, 14, 27]. Moreover, we also show that if $f$ is a real valued tempered constant of motion, then $\mathcal{O}p(f)$ is essentially self-adjoint on $L^2(\mathbb{R}^n)$ with domain $S(\mathbb{R}^n)$ (Theorem 3). Once again, such result is usually obtained imposing additional conditions on the growth of $f$; for instance $\mathcal{O}p(f)$ is essentially self-adjoint on the domain $C_c^\infty(\mathbb{R}^n)$ if $f$ is hypoelliptic (Theorem 26.2 [25]). Since constants of motion are decomposed through the spectral diagonalization of the Hamiltonian, we obtain detailed spectral information of $\mathcal{O}p(f)$ (Corollary 3). We also find an interesting criterion for boundedness of $\mathcal{O}p(f)$ (Proposition 2). Regarding the construction of examples, we show that the metaplectic representation maps the unitary group $U(n)$ into unitary quantum constants of motion (Corollary 4). In particular, we can represent the Lie algebra $u(n)$ as quantum constants of motion as well (Corollary 5) and we expect that the same holds for the enveloping algebra of $u(n)$. Furthermore, the operators corresponding to elements of $u(n)$ are the quantization of an explicit (and interesting) class of classical constants of motion (see (12)).
In Section 4, we use our techniques to study the class of tempered constants of motion. We begin giving a characterization of them: we show that every tempered constant of motion is of the form

\[ f = \sum_{|\alpha|=|\beta|} C_{\alpha, \beta} W(\phi_{\beta}, \phi_{\alpha}), \]

where \( \alpha, \beta \in \mathbb{N}^n \), \( \phi_{\alpha} \) and \( \phi_{\beta} \) are the corresponding Hermite functions, and \( W \) is the Wigner transform. Depending on the behavior of the constants \( C_{\alpha, \beta} \), the corresponding series might converge in the spaces \( S(\mathbb{R}^{2n}) \) or \( L^2(\mathbb{R}^{2n}) \) as well, and moreover \( C_{\alpha, \beta} = \langle \mathcal{D}(f) \phi_{\alpha}, \phi_{\beta} \rangle \) (Theorem 4 and Proposition 5). In Section 4.1, we consider the restriction of the Moyal star product \( \star \) to the set of tempered constants of motion. It is well known that \( f \star g \) is well-defined when \( f, g \in S(\mathbb{R}^n) \), but this is not always true when the two factors are tempered distributions. In order to avoid that issue, the so called multiplier algebra was introduced (for instance, see [23]). We prove that the class of tempered constants of motion endowed with the Moyal product forms a subalgebra of the multiplier algebra (Theorem 5 and Corollary 6).

In Section 5, we analyze the average method of Weinstein and Guillemin for constructing classical and quantum constants of motion [13, 28] within our framework. We recall how the classical and quantum averages are defined and we use our techniques to give explicit expressions of them (Propositions 7 and 8). Furthermore, we prove that Weyl quantization exchanges Weinstein-Guillemin average of observables (Theorem 6).

In Section 6, we give two explicit formulas to compute the Wigner transform \( W(\phi_{\alpha}, \phi_{\beta}) \) and the coefficients \( \langle \mathcal{D}(f) \phi_{\alpha}, \phi_{\beta} \rangle \) (Theorems 7 and 8).

Finally, for the sake of a self-contained presentation, we include an appendix containing some known facts about Weyl quantization.

## 2 Preliminaries

In this section, we summarize some general facts concerning constants of motion and the Harmonic Oscillator. Let us fix some notation:

- \( S(\mathbb{R}^m) \) is the Schwartz space of rapidly decreasing smooth functions on \( \mathbb{R}^m \) endowed with its canonical Fréchet topology.
- \( S'(\mathbb{R}^m) \) denotes the topological dual of \( S(\mathbb{R}^m) \) (the space of tempered distributions) endowed with the weak-* topology.
- We endow \( L^2(\mathbb{R}^m) \) with its canonical inner product, linear on the left, and for \( f \in S'(\mathbb{R}^m) \) and \( \phi \in S(\mathbb{R}^m) \) we use the notation \( (f, \phi) = f(\phi) \).
- If \( H \) is a linear operator, we denote its domain by \( D(H) \).
- \( \mathcal{B}(X, Y) \) is the space of continuous operators from \( X \) to \( Y \), where \( X \) and \( Y \) are locally convex spaces. When \( X = Y \) we just write \( \mathcal{B}(X) \).
- \( \mathcal{U}(\mathcal{H}) \) is the space of unitary operators on the Hilbert space \( \mathcal{H} \).
2.1 Constants of Motion

In this subsection we recall the concepts of classical and quantum constant of motion and some of their properties, emphasizing the analogies between them.

Let us fix a complete Hamiltonian $h_0 \in C^\infty(\mathbb{R}^{2n})$ i.e. the flow $\varphi_t$ along the corresponding Hamiltonian vector field is defined for all $t \in \mathbb{R}$. A classical constant of motion for $h_0$ is a function $f \in C^\infty(\mathbb{R}^{2n})$ (i.e. a classical observable) such that

$$\{h_0, f\} = 0,$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket corresponding to the canonical symplectic structure on $\mathbb{R}^{2n}$. Leibniz’s rule and Jacobi identity show that the set $\mathcal{A}$ of all constants of motion is a Poisson subalgebra of $C^\infty(\mathbb{R}^{2n})$. It is easy to show that $f$ belongs to $\mathcal{A}$ if and only if $f \circ \varphi_t = f$, for each $t \in \mathbb{R}$.

An important construction for the analysis of constants of motion is the following: Let $\lambda$ be a regular value of $h_0$. Thus, the constant energy level set $\hat{\Sigma}_\lambda := h_0^{-1}(\lambda) \subseteq \mathbb{R}^{2n}$ is a $(2n-1)$-submanifold invariant under $\varphi$. Moreover, considering $h_0$ as an equivariant moment map, the orbits space $\Sigma_\lambda := \hat{\Sigma}_\lambda/\varphi$ can be endowed with the symplectic form given by Marsden-Weinstein-Meyer reduction [1, 19, 20, 22] (our particular case is sometimes called Jacobi-Liouville Theorem).

Since $f \in \mathcal{A}$ is constant on each orbit, we can associate to $f$ the field of smooth functions $f_\lambda \in C^\infty(\Sigma_\lambda)$, defined by

$$f_\lambda([x, \xi]) = f(x, \xi),$$

where $(x, \xi) \in \hat{\Sigma}_\lambda$ is any element in the orbit $[x, \xi]$.

Clearly, if $a \in C^\infty(\mathbb{R})$ and $f$ is a constant of motion, then $a \circ f$ is also a constant of motion. In particular, $a \circ h$ is a constant of motion for any $a \in C^\infty(\mathbb{R})$.

Let us pass to the quantum description of constants of motion. Fix a self-adjoint operator $H_0$ on $L^2(\mathbb{R}^n)$ (a quantum Hamiltonian). A quantum constant of motion for $H_0$ is a self-adjoint operator $F$ on $L^2(\mathbb{R}^n)$ (i.e. a quantum observable) strongly commuting with $H_0$. If $F$ is a bounded self-adjoint operator, then $F$ is a constant of motion iff $e^{itH_0}F e^{-itH_0} = F$, for every $t \in \mathbb{R}$. Moreover, the set of bounded constants of motion is the self-adjoint part of a von Neumann algebra (the commutant of $H_0$).

In analogy with the classical framework, the following construction is useful to describe quantum constant of motion [4, 5]. Let $\sigma(H_0)$ the spectrum of $H_0$. There is a unique Borel measure $\eta$ (up to equivalence) on $\sigma(H_0)$, a unique measurable field of Hilbert spaces $\{\mathcal{H}(\lambda)\}_{\lambda \in \sigma(H_0)}$ (up to unitary equivalence on $\eta$-almost every fiber), and a unitary operator $T : L^2(\mathbb{R}^n) \to \int_{\sigma(H_0)}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ such that

$$[Ta(H_0)u](\lambda) = a(\lambda)(Tu)(\lambda) \quad \forall u \in D(a(H_0)),$$

where $a$ is any Borel function on $\sigma(H_0)$ and $a(H_0)$ denotes the corresponding operator given by the functional calculus. $T$ is called the spectral diagonalization of $H_0$. In particular, we have that

$$[TH_0u](\lambda) = \lambda(Tu)(\lambda) \quad \forall u \in D(H_0).$$

and

$$[Te^{itH_0}u](\lambda) = e^{it\lambda}(Tu)(\lambda) \quad \forall u \in \mathcal{H}, \ t \in \mathbb{R}.$$
Notice the similarities between the construction of $\mathcal{H}(\lambda)$ and $\Sigma_\lambda$: Both constructions are meant to make the Hamiltonian constant on each fiber and the corresponding dynamics trivial.

It is well-known that $F$ is a constant of motion if and only if it admits a decomposition through $T$ [4, 5], i.e. there is a measurable field of self-adjoint operators $\{F_\lambda\}_{\lambda \in \sigma(H_0)}$ such that $[TFu](\lambda) = F_\lambda[Tu(\lambda)]$. Heuristically, the quantum observable $F_\lambda$ is the counterpart of the classical observable $f_\lambda$ defined by (1).

It is easy to prove that, if $F$ is a quantum constant of motion, so is $a(F)$ and $a(F_\lambda) = a(F_\lambda)$, for any Borel function $a$ on $\mathbb{R}$.

If $h_0$ and $H_0$ are Hamiltonians describing the same physical system, we would like to have a canonical quantization that takes into account the described analogy between classical and quantum constants of motion of $h_0$ and $H_0$ respectively. The latter idea was used by one of the authors to construct canonical quantizations in [3]. However, one of the obstacles to follow that construction is the difference between the range of $h_0$ and the spectrum of $H_0$ (recall that $\Sigma_\lambda$ is defined when $\lambda$ is a regular value of $h_0$, while $H_\lambda$ is defined when $\lambda$ belongs to the spectrum of $H_0$). The Harmonic Oscillator is a simple but fundamental example where such difference is evident and important, so we expect that the results of this article might help us to improve our understanding of the problem of quantizing constants of motion.

### 2.2 The Harmonic Oscillator

In this subsection we restrict the discussion of the previous subsection to the case when $h_0$ is the Harmonic Oscillator, we summarize some well known facts about it, and we introduce notation that we use in the following sections. We also introduce the notion of tempered constant of motion for the Harmonic Oscillator.

The classical Harmonic Oscillator is the physical system with Hamiltonian given by $h_0(x, \xi) = \frac{1}{2}(\|x\|^2 + \|\xi\|^2)$. It is easy to check that the flow of $h_0$ is given by

$$
\phi_t(x, \xi) = \left(\begin{array}{cc}
\cos(t)I & \sin(t)I \\
-\sin(t)I & \cos(t)I
\end{array}\right) \left(\begin{array}{c} x \\ \xi \end{array}\right),
$$

where $I$ is the $n \times n$ identity matrix. Identifying $(x, \xi) \in \mathbb{R}^{2n}$ with $x + i\xi \in \mathbb{C}^n$, the flow is given by

$$
\phi_t(x + i\xi) = e^{-it}(x + i\xi).
$$

If $\lambda$ is a regular value for $h_0$, that is $\lambda \neq 0$, then the submanifolds $h_0^{-1}(\lambda)$ are precisely the spheres $S^{2n-1}_{\sqrt{2\lambda}}$. Hence, the orbit space $\Sigma_\lambda = S^{2n-1}_{\sqrt{2\lambda}}/\phi$ is diffeomorphic to the projective plane $\mathbb{CP}^{n-1}$.

There is a family of polynomial constants of motion of particular interest in the present study. Set $z = x + i\xi = (x_1 + i\xi_1, \ldots, x_n + i\xi_n) = (z_1, \ldots, z_n)$ and $\alpha, \beta \in \mathbb{N}^n$ n-tuples of non-negative integers with $|\alpha| = |\beta|$. The monomial

$$
m_{\alpha, \beta}(z) = z^\alpha \bar{z}^\beta
$$

is a constant of motion of the classical Harmonic Oscillator. For example, the coordinates of the angular momentum are constants of motion because they are a linear combination of some of the monomials $m_{\alpha, \beta}$. The set $\{m_{\alpha, \beta} : |\alpha| = |\beta| = k\}$ has
exactly $d_k^2 = \binom{n+k-1}{k}^2$ elements. Theorems 4 and 7 imply that suitable functions of this monomials generate the set of all tempered constants of motion (see Definition 1 below).

The flow $\varphi_t$ is a linear symplectomorphism. In particular, it preserves volume and its pullback maps $S(\mathbb{R}^n)$ into itself, so we can extend the pullback to $S'(\mathbb{R}^n)$.

**Definition 1** For each $f \in S'(\mathbb{R}^{2n})$, we define the tempered distribution $\varphi_t^* f$ by

$$(\varphi_t^* f)(\psi) = f(\psi \circ \varphi_t),$$

for any $\psi \in S(\mathbb{R}^{2n})$. We say that $f$ is tempered constant of motion if $\varphi_t^* f = f$.

The $n$-dimensional quantum Harmonic Oscillator is the self-adjoint operator $H_0 = \frac{1}{2}(-\Delta + ||x||^2)$ on $L^2(\mathbb{R}^n)$. The spectrum of $H_0$ is equal to $\mathbb{N} + \frac{n}{2}$ and it is discrete. Each eigenvalue $k + \frac{n}{2}$ has multiplicity $d_k = \binom{n+k-1}{k}$ and we denote by $\mathcal{H}_k$ the corresponding eigenspace. Its is well-known that the Hermite functions $\{\phi_\alpha\}_{\alpha \in \mathbb{N}^n}$ given by

$$\phi_\alpha(x_1, ..., x_n) = \frac{(-1)^{||\alpha||}}{\pi^{n/4} \sqrt{2^{||\alpha||!}}} e^{\frac{1}{4} ||x||^2} D_\alpha \left(e^{-||x||^2}\right),$$

form an orthonormal basis of $L^2(\mathbb{R}^n)$, where $||\alpha|| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1 \cdots \alpha_n$, and $D_\alpha = \frac{\partial^{||\alpha||}}{\partial x^{\alpha}}$. Moreover, the collection $\{\phi_\alpha : \alpha \in \mathbb{N}^n, ||\alpha|| = k\}$ is an orthonormal basis of $\mathcal{H}_k$. The spectral diagonalization (2) of $H_0$ is

$$T = \bigoplus_{k \in \mathbb{N}} P_k : L^2(\mathbb{R}^n) \longrightarrow \bigoplus_{k \in \mathbb{N}} \mathcal{H}_k,$$

where $P_k$ is the orthogonal projection over $\mathcal{H}_k$.

### 3 Preservation of Constants of Motion and its Consequences

In this section we prove our main results concerning the operators obtained after applying Weyl quantization to any tempered constants of motion.

An equivalent version of the next result can be found in equation 4.47 of [11]. However, it was not noticed there that $e^{i\mathcal{J}}$ is the flow for the classical Harmonic Oscillator.

**Proposition 1** The metaplectic representation maps the flow $\varphi_t$ to the one parameter group $e^{itH_0}$. In particular, $e^{itH_0}$ maps $S(\mathbb{R}^n)$ into itself and it can be extended to a strongly continuous one parameter group on $S'(\mathbb{R}^n)$.

Recall that $\mathcal{D}$ denotes Weyl canonical quantization (or Weyl calculus). In Appendix A we summarize some of the main properties of Weyl quantization.
**Theorem 1** Weyl quantization preserves constants of motion of the Harmonic Oscillator, more precisely, for each tempered constant of motion \( f \) and each \( t \in \mathbb{R} \), we have that,

\[
e^{itH_0} \mathcal{D} p(f) e^{-itH_0} = \mathcal{D} p(f).
\]

(4)

In particular, \([H_0, \mathcal{D} p(f)] = 0\) on \( S(\mathbb{R}^n)\).

**Proof** It is a direct consequence of Proposition 1 and (24) in the Appendix A. \(\square\)

It is well-known that \( \mathcal{D} p \) defines a unitary isomorphism between \( L^2(\mathbb{R}^{2n}) \) and the space of Hilbert-Schmidt operators on \( L^2(\mathbb{R}^n) \). In particular, for any \( f \in L^2(\mathbb{R}^{2n}) \), we have that

\[
\mathcal{D} p(f) = \sum_{\alpha, \beta} \langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle P_{\alpha, \beta},
\]

where \( \phi_\alpha \) is the Hermite function corresponding to the multiindex \( \alpha \in \mathbb{N}^n \), the convergence holds in the space of Hilbert-Schmidt operators and \( P_{\alpha, \beta} \) is the \( 1 \)-dimensional rank operator given by \( P_{\alpha, \beta} \phi = \langle \phi, \phi_\alpha \rangle \phi_\beta \). The particular properties of the Hermite orthonormal basis allow us to generalize such expression.

**Lemma 1** For any \( f \in S'(\mathbb{R}^{2n}) \), the kernel \( K_f \) of \( \mathcal{D} p(f) \) is given by

\[
K_f = \sum_{\alpha, \beta} \langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha,
\]

(5)

where \( (\phi \otimes \psi)(x, y) = \phi(x)\psi(y) \). There is a constant \( C > 0 \) and \( \gamma \in \mathbb{N}^{2n} \) such that, for any \( \alpha, \beta \in \mathbb{N}^n \), we have that

\[
|\langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle| \leq C((\alpha, \beta) + 1)^\gamma.
\]

(6)

Moreover, if \( f \in S(\mathbb{R}^{2n}) \), then for each \( m \in \mathbb{N} \), we have that

\[
\sup_{\alpha, \beta \in \mathbb{N}^n} |\langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle| (|\alpha| + |\beta|)^m < \infty.
\]

(7)

**Proof** Recall that \( \mathcal{D} p(f) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n) \) is continuous. The \( N \)-representation Theorem for \( S(\mathbb{R}^n) \) (Theorem V.13 [24]) implies that, for any \( \phi \in S(\mathbb{R}^n) \), the convergence of the series \( \sum \langle \phi, \phi_\alpha \rangle \phi_\alpha = \phi \) holds in \( S(\mathbb{R}^n) \) endowed with its canonical locally convex topology. Therefore, for any \( \phi, \psi \in S(\mathbb{R}^n) \) we have that

\[
\langle K_f, \psi \otimes \bar{\phi} \rangle = \langle \mathcal{D} p(f) \phi, \psi \rangle = \sum_{\alpha, \beta} \langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle \langle \phi_\beta, \psi \rangle \langle \phi, \phi_\alpha \rangle = \sum_{\alpha, \beta} \langle \mathcal{D} p(f) \phi_\alpha, \phi_\beta \rangle \langle \phi_\beta \otimes \phi_\alpha, \psi \otimes \bar{\phi} \rangle.
\]

The inequalities (6) and (7) follows from the \( N \)-representation Theorem for \( S'(\mathbb{R}^{2n}) \) and \( S(\mathbb{R}^{2n}) \) respectively (Theorem V.13 and Theorem V.14 in [24]). \(\square\)
Inequalities (6) and (7) characterize elements in $S'(\mathbb{R}^{2n})$ and $S(\mathbb{R}^{2n})$ respectively, but we shall leave the consequences of that fact for Section 4.

Let us restrict the previous results to the case when $f$ is a tempered constant of motion.

**Corollary 1** Let $f$ be a tempered constant of motion. If $|\alpha| \neq |\beta|$, then $\langle \mathcal{D}p(f)\phi_\beta, \phi_\alpha \rangle = 0$. In particular, the kernel of $\mathcal{D}p(f)$ is given by

$$K_f = \sum_{|\alpha| = |\beta|} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha.$$  \hfill (8)

There is a constant $C > 0$ and $M \in \mathbb{N}$ such that, for any $\alpha, \beta \in \mathbb{N}^n$, we have that

$$|\langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle| \leq (|\alpha| + 1)^M.$$  \hfill (9)

Moreover, if $f \in S(\mathbb{R}^{2n})$, then for each $m \in \mathbb{N}$, we have that

$$\sup_{\alpha, \beta \in \mathbb{N}^n} |\langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle| |\alpha|^m < \infty.$$  \hfill (10)

**Proof** Notice that

$$\langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle = \langle e^{itH_0} \mathcal{D}p(f) e^{-itH_0} \phi_\alpha, \phi_\beta \rangle = e^{it(|\alpha| - |\beta|)} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle.$$  

Thus, $\langle \mathcal{D}p(f)\phi_\beta, \phi_\alpha \rangle = 0$, unless $|\alpha| = |\beta|$. Inequality (10) follows directly from inequality (7), and inequality (9) follows from inequality (6) and noticing that $((\alpha, \beta) + 1)^\gamma \leq ((|\alpha| + 1)^\gamma$. \hfill $\square$

**Remark 1** Inequalities (6) and (7) characterize tempered and rapidly decreasing constants of motion respectively, because $((|\alpha| + 1)^M \leq ((\alpha, \beta) + 1)^\gamma_M$, where $\gamma_M = (M, \cdots, M)$, but we shall leave the consequences of that fact for Section 4 too.

The fact that the dimension of the eigenspaces $\mathcal{H}_k$ grows polynomially in $k$ and the $N$-representation Theorem implies the following remarkable result.

**Theorem 2** If $f$ is a tempered constant of motion, then $\mathcal{D}p(f)$ maps $S(\mathbb{R}^n)$ into itself.

**Proof** By Corollary 1, for any $\varphi \in S(\mathbb{R}^n)$ we have that

$$\mathcal{D}p(f)\varphi = \sum_{|\alpha| = |\beta|} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \langle \varphi, \phi_\beta \rangle \phi_\alpha$$

$$= \sum_{\alpha} \left( \sum_{|\beta| = |\alpha|} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \langle \varphi, \phi_\beta \rangle \right) \phi_\alpha.$$
Since \( \varphi \in S(\mathbb{R}^n) \) and \( d|_{\beta} \) is a polynomial of degree \( n - 1 \) on \( |\beta| \), inequality (9) implies that

\[
|\alpha|^m \left| \sum_{|\beta|=|\alpha|} \langle \mathcal{D} p(\cdot) \phi_\alpha, \phi_\beta \rangle \langle \varphi, \phi_\beta \rangle \right| \leq C \sum_{|\beta|=|\alpha|} |\alpha|^m (|\alpha| + 1)^M |\langle \varphi, \phi_\beta \rangle| \\
\leq C \sum_{|\beta|=|\alpha|} \frac{1}{d|_{\beta}} \sup_{\beta \in \mathbb{N}^n} \left( d|_{\beta} |\beta|^m (|\beta| + 1)^M |\langle \varphi, \phi_\beta \rangle| \right) \\
\leq C \sup_{\beta \in \mathbb{N}^n} \left( d|_{\beta} |\beta|^m (|\beta| + 1)^M |\langle \varphi, \phi_\beta \rangle| \right) < \infty,
\]

for every \( m \in \mathbb{N} \). Therefore, the \( N \)-representation Theorem implies that \( \mathcal{D} p(\cdot) \varphi \in S(\mathbb{R}^n) \). \( \square \)

The following straightforward consequence is an example of how the results of this article can be applied in the Fock-Bargmann model.

**Corollary 2** Let \( f \) be a tempered constant of motion and let \( B \) be the Bargmann transform. The operator \( B \mathcal{D} p(f) B^* \) maps homogeneous polynomials of degree \( k \) into homogeneous polynomials of degree \( k \), and it has Wick symbol \( f_W \) determined by

\[
f_W(z, \bar{z}) = 2^n \int f(x, \xi) e^{-2\pi [(s-\xi)^2 + (r-x)^2]} d\xi dx,
\]

where \( z = r - is \).

**Proof** It is well-known that \( BH_k \) is the space of homogeneous polynomials of degree \( k \). The last statement follows from Theorem 2 and Proposition 2.97 in [11]. \( \square \)

Recall that a tempered distribution \( f \) is called real if for every \( \varphi \in S(\mathbb{R}^m) \) we have \( f(\varphi) = \overline{f(\varphi)} \). It is well-known that if \( f \in S'(\mathbb{R}^{2n}) \) is real, then

\[
\langle \mathcal{D} p(f) \varphi, \psi \rangle = \langle \varphi, \mathcal{D} p(f) \psi \rangle.
\]

However, even if \( \mathcal{D} p(f) \) maps \( S(\mathbb{R}^n) \) into itself, the latter identity does not guarantee that \( \mathcal{D} p(f) \) defines a self-adjoint operator (or a essentially self-adjoint operator).

**Theorem 3** If \( f \) is a real tempered constant of motion, then \( \mathcal{D} p(f) \) is a essentially self-adjoint operator on \( L^2(\mathbb{R}^n) \) with domain \( S(\mathbb{R}^n) \).

**Proof** Let \( \mathcal{D} p^k(f) \) be the restriction of \( \mathcal{D} p(f) \) to \( H_k \). Since \( \mathcal{D} p^k(f) \) is a self-adjoint operator, \( U_k(t) := e^{it \mathcal{D} p^k(f)} \) is a unitary one parameter group on \( H_k \). Define

\[
U(t) := \bigoplus_{k \in \mathbb{N}} U_k(t).
\]
Clearly, $U(t)$ is a unitary one parameter group. Moreover, if $\varphi \in L^2(\mathbb{R}^n)$, then

$$\|U(t)\varphi - \varphi\|^2 = \sum_{k=1}^{\infty} \|U_k(t)P_k\varphi - P_k\varphi\|^2.$$ 

Since $\|U_k(t)P_k\varphi - P_k\varphi\|^2 \leq 4\|P_k\varphi\|^2$, the Lebesgue dominated convergence theorem implies that $\lim_{t \to 0} \|U(t)\varphi - \varphi\|^2 = 0$, i.e. $U$ is strongly continuous. By Stone's theorem, $U(t)$ has a self-adjoint infinitesimal generator $H$, we shall prove that $S(\mathbb{R}^n) \subset D(H)$ and $H\varphi = \mathcal{D}(f)\varphi$, for $\varphi \in S(\mathbb{R}^n)$. Observe that, by Fubini's theorem

$$\int_0^t iU(a)\mathcal{D}(f)\varphi da = \int_0^t \sum_{k=0}^{\infty} ie^{ia}\mathcal{D}(f)P_k\varphi da$$

$$= \sum_{k=0}^{\infty} \int_0^t ie^{ia}\mathcal{D}(f)P_k\varphi da$$

$$= \sum_{k=0}^{\infty} (U_k(t) - I)P_k\varphi$$

$$= (U(t) - I)\varphi.$$ 

The last equality implies that the function $t \to U(t)\varphi$ is differentiable and $\frac{dU(t)\varphi}{dt} = iU(t)\mathcal{D}(f)\varphi$, which proves our claim. The same argument in the proof of Theorem 2 shows that $U(t)$ maps $S(\mathbb{R}^n)$ into itself. Thus, Theorem VIII.10 of [24] implies that $\mathcal{D}(f)$ is essentially self-adjoint on $L^2(\mathbb{R}^n)$ with domain $S(\mathbb{R}^n)$. 

If $f$ is a tempered constant of motion, Corollary 1 implies that.

$$\mathcal{D}(f) = \bigoplus_k \mathcal{D}(f)^k = \bigoplus_k \sum_{|\alpha| = |\beta| = k} \langle \mathcal{D}(f)\phi_\alpha, \phi_\beta \rangle P_{\alpha,\beta}, \tag{11}$$

where $\mathcal{D}(f)^k$ is the restriction of $\mathcal{D}(f)$ to $H_k$. The following result is a straightforward consequence of the previous decomposition.

**Corollary 3** Let $f$ be a real tempered constant of motion, and denote by $\sigma(\mathcal{D}(f))$ and $\sigma_p(\mathcal{D}(f))$ the spectrum and the point spectrum of $\mathcal{D}(f)$ respectively. Then

$$\sigma_p(\mathcal{D}(f)) = \bigcup_k \sigma_p(\mathcal{D}(f)^k).$$

In particular, $\sigma_p(\mathcal{D}(f))$ is at most numerable and it is dense in $\sigma(\mathcal{D}(f))$.

**Proof** Clearly $\bigcup_k \sigma_p(\mathcal{D}(f)^k) \subseteq \sigma_p(\mathcal{D}(f))$. Conversely, if $\mathcal{D}(f)\varphi = \lambda\varphi$ with $\varphi \neq 0$, then $\mathcal{D}(f)^k P_k\varphi = \lambda P_k\varphi$ for each $k \in \mathbb{N}$. Since $\varphi \neq 0$, there is $k_0$ such
that \( P_k \varphi \neq 0 \), therefore \( \lambda \in \sigma_p(\mathcal{D}p_k(0)) \) and \( \sigma_p(\mathcal{D}p(f)) = \bigcup_k \sigma_p(\mathcal{D}p_k(f)) \). It is well-known that

\[
\sigma(\mathcal{D}p(f)) = \bigcup_k \sigma(\mathcal{D}p_k(f)) = \bigcup_k \sigma_p(\mathcal{D}p_k(f)).
\]

\[\square\]

**Remark 2** There exist tempered constants of motion \( u_t \) such that \( \mathcal{D}p(u_t) \) is the one parameter group \( U(t) = e^{it\mathcal{D}p(f)} \) (where \( \mathcal{D}p(f) \) is the closure of the operator \( \mathcal{D}p(f) \)). The latter claim follows from Theorem 4 in the next section.

The following result is a simple but interesting criterion for boundedness.

**Proposition 2** If \( f \) is a tempered constant of motion such that

\[
\sup_{\alpha, \beta \in \mathbb{N}^n} |\langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle| |\alpha|^{n-1} < \infty,
\]

then \( \mathcal{D}p(f) \) defines a bounded operator on \( L^2(\mathbb{R}^n) \).

**Proof** Recall that \( \mathcal{D}p(f) \) is bounded iff \( \sup_k ||\mathcal{D}p_k(f)|| < \infty \). Since each \( \mathcal{H}_k \) is finite dimensional, \( ||\mathcal{D}p_k(f)|| \leq d_k \max_{|\alpha|=|\beta|=k} |\langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle| \). Our criterion follows after recalling that \( d_k \) is a polynomial on \( k \) of degree \( n - 1 \). \[\square\]

Finally, using Proposition 1, we shall provide some interesting examples of constants of motion coming from the restriction of the metaplectic representation \( \mu \) to the complex unitary group \( U(n) \). The following result is a direct consequence of the comments after the proof of Proposition 4.75 in [11], but we shall approach it in a different way (without considering the Bargmann transform).

**Corollary 4** \( \mu(U(n)) \) is a unitary group of constant of motion of \( H_0 \).

**Proof** Recall that the flow \( \varphi_t \) in complex coordinates is given by \( \varphi_t = e^{itI} \). In particular, for each \( S \in U(n) \), we have that

\[
e^{itH_0} \mu(S)e^{-itH_0} = \mu(\varphi_t \circ S \circ \varphi_{-t}) = \mu(S).
\]

\[\square\]

The previous corollary allow us to define the representations \( \mu_k : U(n) \to \mathcal{U}(\mathcal{H}_k) \) given by \( \mu_k(S) = \mu(S)|_{\mathcal{H}_k} \), for each \( k \in \mathbb{N} \). Abusing of the notation, we also denote by \( \mu \) the restriction of the metaplectic representation to \( U(n) \). Hence, \( \mu \) admits the decomposition

\[
\mu = \bigoplus_k \mu_k.
\]
It is well-known that each $\mu_k$ is irreducible (for instance, see [15]). Moreover, since $d_k = \binom{n+k-1}{k}$, we expect that $\mu_1$ is equivalent to the canonical representation of $U(n)$ and $\mu_k$ is the corresponding symmetric $k$-th power representation.

Let us briefly consider a more general framework. Let $G$ be a Lie group with corresponding Lie algebra $\mathfrak{g}$ and $\pi$ any strongly continuous unitary representation of $G$ on a Hilbert space $\mathcal{H}$. For any $A \in \mathfrak{g}$, we can define the strongly continuous one parameter group $U(t) = \pi(e^{itA})$. Stone’s Theorem implies $U(t)$ has a self-adjoint infinitesimal generator which we denote by $d\pi(A)$. Moreover, the map $d\pi$ defines a Lie algebra homomorphism and it can be extended to the enveloping algebra of $\mathfrak{g}$, but self-adjointness of the emerging operators is not automatically guaranteed (a sort of ellipticity might be required, see chapter VI in [21]).

When $G = U(n)$, $\mathfrak{g} = u(n)$ and $\pi = \mu$, the operators $d\mu(A)$ also comes from Weyl quantization. Indeed, $d\mu(A) = \mathcal{O}(p_A)$, where $p_A$ is the polynomial of degree 2 given by
\[ p_A(w) = -\frac{1}{2} w \cdot A \mathcal{J} \cdot w, \tag{12} \]
and $\mathcal{J}$ is the canonical symplectic matrix. Moreover, the flow of $p_A$ is $e^{tA}$ (see Theorem 4.45 in [11]). The following result is a direct consequence of Theorem 3 and Corollary 4.

**Corollary 5** For each $A \in u(n)$, the self-adjoint operator $d\mu(A)$ is a constant of motion of the Harmonic Oscillator and it is essentially self-adjoint on the domain $S(\mathbb{R}^n)$. Moreover, the map $d\mu$ admits the decomposition
\[ d\mu = \bigoplus_k d\mu_k. \]

We would like to obtain refine spectral information concerning the operators $d\mu(A)$ using the latter decomposition, Corollary 3 and representation theory. Moreover, in view of Theorem 3, we expect that the operators obtained from the extension of $d\mu$ to the enveloping algebra of $u(n)$ are also essentially self-adjoint on the domain $S(\mathbb{R}^n)$. However, we shall leave the latter problems open for a future work.

## 4 Characterization of Classical Constants of Motion and Their Moyal Product

In this section we provide a characterization of tempered constants of motion. In Section 4.1, we use the latter result to study the restriction of the Moyal product to the class of tempered constants of motion.

Recall that $\mathcal{O}$ defines an isomorphism between the space of tempered distributions $S'(\mathbb{R}^{2n})$ and the space of continuous operators $\mathcal{B}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. Moreover, if $T \in \mathcal{B}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$ then $\mathcal{O}^{-1}(T) = \bar{W}(K_T)$, where $K_T$ is the kernel of $T$ given by the Schwartz’s kernel Theorem and $\bar{W}$ is the extended Wigner transform defined in (21) in Appendix A.
Proposition 3 Let $\Phi^{\alpha,\beta} = W(\phi_\beta, \phi_\alpha)$ where $W$ is the Wigner transform. For each $t \in \mathbb{R}$, we have that

$$\Phi^{\alpha,\beta} \circ \varphi_t = e^{it(|\alpha|-|\beta|)} \Phi^{\alpha,\beta}.$$  

(13)

In particular, $\Phi^{\alpha,\beta}$ is a constant of motion iff $|\alpha| = |\beta|$. Moreover, $\hat{\Omega}(\Phi^{\alpha,\beta}) = P_{\alpha,\beta}$.

Proof Proposition 4.28 in [11] (or (23) in Appendix A) implies that

$$\Phi^{\alpha,\beta} \circ \varphi_t = W(\mu(\varphi_{-t})\phi_\beta, \mu(\varphi_{-t})\phi_\alpha) = e^{it(|\alpha|-|\beta|)} \Phi^{\alpha,\beta}.$$  

The identity $\hat{\Omega}(\Phi^{\alpha,\beta}) = P_{\alpha,\beta}$ is a particular case of example (ii) in page 92 of [11].

The following result is interesting on its own right and it is a direct consequence of Lemma 1.

Proposition 4 For any $f \in S'(\mathbb{R}^{2n})$, we have that

$$f = \sum_{\alpha,\beta} \langle \hat{\Omega}(f) \phi_\alpha, \phi_\beta \rangle \Phi^{\alpha,\beta},$$

where the convergence of the series holds in $S'(\mathbb{R}^{2n})$ with its canonical topology. Conversely, if $\{C_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ is a family of constants such that there is $C > 0$ and $\gamma \in \mathbb{N}^{2n}$ satisfying the inequality $|C_{\alpha,\beta}| \leq C((|\alpha| + |\beta|)^\gamma$, then the series $\sum_{\alpha,\beta} C_{\alpha,\beta} \Phi^{\alpha,\beta}$ converges in $S'(\mathbb{R}^{2n})$. Similarly, if $\{C_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ is a family of constants such that $\sup_{\alpha,\beta} |C_{\alpha,\beta}|(|\alpha| + |\beta|)^m < \infty$ for each $m \in \mathbb{N}$, then the series $\sum_{\alpha,\beta} C_{\alpha,\beta} \Phi^{\alpha,\beta}$ converges in $S(\mathbb{R}^{2n})$.

The following results provide a complete characterization of tempered, square integrable and rapidly decreasing constants of motion.

Theorem 4 For any tempered constant of motion $f$, we have that

$$f = \sum_{|\alpha| = |\beta|} \langle \hat{\Omega}(f) \phi_\alpha, \phi_\beta \rangle \Phi^{\alpha,\beta},$$

where the convergence of the series holds in $S'(\mathbb{R}^{2n})$ with its canonical topology. Conversely, if $\{C_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ is a family of constants such that there is $C > 0$ and $M \in \mathbb{N}$ satisfying inequality $|C_{\alpha,\beta}| \leq C(|\alpha| + 1)^M$, then the series $\sum_{|\alpha| = |\beta|} C_{\alpha,\beta} \Phi^{\alpha,\beta}$ converges in $S'(\mathbb{R}^{2n})$ to a tempered constant of motion. Similarly, if $\{C_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$ is a family of constants such that $\sup_{\alpha,\beta} |C_{\alpha,\beta}| |\alpha|^m < \infty$ for each $m \in \mathbb{N}$, then the series $\sum_{|\alpha| = |\beta|} C_{\alpha,\beta} \Phi^{\alpha,\beta}$ converges in $S(\mathbb{R}^{2n})$ to a rapidly decreasing constant of motion.

The following result is the square integrable version of the latter Theorem.

Proposition 5 The set $\{\Phi^{\alpha,\beta}\}_{|\alpha| = |\beta|}$ forms an orthogonal basis of the Hilbert space of square integrable constants of motion.
Proof We only need to check orthogonality, but this is a direct consequence of Proposition 1.92 in [11].

4.1 The Moyal Product of Constants of Motion

In this subsection, we use our techniques to show that the Moyal product $\star$ is well-defined on the class of tempered constants of motion.

Let us briefly recall the definition of $\star$ and some of its properties. It is well known that if $f \in S(\mathbb{R}^{2n})$, then $\mathcal{D}p(f) : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$. In particular, for each $f, g \in S(\mathbb{R}^{2n})$, the composition $\mathcal{D}p(f)\mathcal{D}p(g)$ is well-defined. In fact, there is a unique $h \in S(\mathbb{R}^{2n})$ such that $\mathcal{D}p(f)\mathcal{D}p(g) = \mathcal{D}p(h)$. The Moyal product of $f$ and $g$ is by definition $f \star g = h$. For example, Proposition 3 implies that

$$\Phi^{\alpha,\beta} \star \Phi^{\alpha',\beta'} = \delta_{\alpha',\beta}\Phi^{\alpha,\beta'}.$$ (14)

It turns out that $\star$ defines a non-commutative and associative product.

Once Planck’s constant $\hbar$ is introduced (see Appendix A), $\star$ admits a power series expansion on $\hbar$. In fact, the Moyal product was originally defined through that expansion [8, 12]. The study of the latter expansion considered as a formal power series on $\hbar$ led to the theory of formal deformation quantization. Furthermore, under fairly general conditions, the series converges to the Moyal product defined above (for instance, see Theorems 1, 2 and 3 in [23]).

One of the main properties of the Moyal product is the identity

$$\int_{\mathbb{R}^{2n}} (f \star g)(x, \xi) d\xi dx = \int_{\mathbb{R}^{2n}} (fg)(x, \xi) d\xi dx.$$

The latter formula allow us to extend the definition of the Moyal product as follows: For each $f \in S'(\mathbb{R}^{2n})$ and $g \in S(\mathbb{R}^{2n})$, define $f \star g \in S'(\mathbb{R}^{2n})$ and $g \star f \in S'(\mathbb{R}^{2n})$ by

$$(f \star g)(h) = f(g \star h), \quad (g \star f)(h) = f(h \star g),$$

for any $h \in S(\mathbb{R}^{2n})$. However, we cannot define $f \star g$ for arbitrary $f, g \in S'(\mathbb{R}^{2n})$. In order to overcome the latter issue, the so called Moyal multiplier algebra $\mathcal{M}$ was introduced: we say that $f \in \mathcal{M}$ if $f \star h$ and $h \star f$ belong to $S(\mathbb{R}^n)$, for each $h \in S(\mathbb{R}^n)$. Therefore, for each $f, g \in \mathcal{M}$, we can define

$$(f \star g)(h) = f(g \star h).$$

With the latter extension of the Moyal product and complex conjugation, $\mathcal{M}$ becomes a $\ast$-algebra.

Theorem 2 implies that it is always possible to define the compositions $\mathcal{D}p(f)\mathcal{D}p(g)$ and $\mathcal{D}p(g)\mathcal{D}p(f)$ for any pair of tempered constants of motion $f, g$. The latter suggest that it should be always possible to define the Moyal products $f \star g$ and $g \star f$. Indeed, we will prove that any tempered constant of motion is an element of the multiplier algebra.
Lemma 2 If \( f \in S'(\mathbb{R}^{2n}) \) and \( g \in S(\mathbb{R}^{2n}) \) are such that \( f = \sum c_{\alpha, \beta} \Phi^{\alpha, \beta} \) and \( g = \sum d_{\alpha, \beta} \Phi^{\alpha, \beta} \) then

\[
f \ast g = \sum_{\alpha, \beta} \left( \sum_{\alpha'} c_{\alpha, \alpha'} d_{\alpha', \beta} \right) \Phi^{\alpha, \beta}\]

(15)

Proof This result is follows from formula (14) and using twice the continuity of the Moyal product when one of the factors is fixed.

Theorem 5 Every tempered constant of motion belongs to the multiplier algebra \( \mathcal{M} \).

Proof Let \( f \) be a tempered constant of motion and \( g \in S(\mathbb{R}^{2n}) \). Let \( a_{\alpha, \beta} \) and \( b_{\alpha, \beta} \) be the coefficients such that

\[
f = \sum_{\alpha, \beta} a_{\alpha, \beta} \Phi^{\alpha, \beta} \quad \text{and} \quad g = \sum_{\alpha, \beta} b_{\alpha, \beta} \Phi^{\alpha, \beta}.
\]

Since \( a_{\alpha, \alpha'} = 0 \) if \( |\alpha| \neq |\alpha'| \), (15) implies that

\[
f \ast g = \sum_{\alpha, \beta} \left( \sum_{|\alpha'| = |\alpha|} a_{\alpha, \alpha'} b_{\alpha', \beta} \right) \Phi^{\alpha, \beta}.
\]

Moreover, since \( g \in S(\mathbb{R}^{2n}) \) and \( d_{|\alpha|} \) is a polynomial on \( |\alpha| \) of degree \( n - 1 \), inequalities (9) and (7) imply that

\[
\sum_{|\alpha'| = |\alpha|} |a_{\alpha, \alpha'} b_{\alpha', \beta}| (|\alpha| + |\beta|)^m \\
\leq \sum_{|\alpha'| = |\alpha|} \frac{|a_{\alpha, \alpha'}|}{d_{|\alpha'|}(|\alpha'| + 1)^M} \sup_{\alpha', \beta} \left( d_{|\alpha'|}(|\alpha'| + 1)^M (|\alpha'| + |\beta|)^m |b_{\alpha', \beta}| \right) \\
\leq C \sup_{\alpha', \beta} \left( d_{|\alpha'|}(|\alpha'| + 1)^M (|\alpha'| + |\beta|)^m |b_{\alpha', \beta}| \right) < \infty,
\]

for all \( m \in \mathbb{N} \). Therefore, Lemma 1 implies that \( f \ast g \in S(\mathbb{R}^{2n}) \). A similar argument proves that \( g \ast f \in S(\mathbb{R}^{2n}) \).

Corollary 6 The space of tempered constants of motion of the Harmonic Oscillator endowed with the Moyal product forms a \( \ast \)-algebra.

Notice that the previous result does not imply that the Moyal product defines a formal \( \ast \) product on the Poisson algebra of smooth constants of motion \( \mathcal{A} \). The problem is that we do not know if the bidifferential operators defining the formal power series maps constants of motion into constants of motion. However, since \( \mathcal{A} \) can be identified with the space of smooth functions on the Poisson manifold \((\mathbb{R}^{2n} - \{0\})/\varphi\), the
Kontsevich’s formality Theorem implies that such star product exists \[16\]. We shall leave this problem open for the future.

5 The Weinstein-Guillemin Average Method

There is a canonical way to construct tempered constants of motion: For each \( f \in S'(\mathbb{R}^{2n}) \), the (classical) Weinstein-Guillemin average of \( f \) is the tempered distribution \( \tilde{f} \) defined by the Bochner integral

\[
\tilde{f} = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi_t^*(f) \, dt. \tag{16}
\]

Since \( \varphi_t \) is \( 2\pi \)-periodic, \( \tilde{f} \) is a well-defined tempered constant of motion.

Let us define the quantum version of the previous average. Notice that if \( n \) is odd \( e^{itH_0} \) is not \( 2\pi \) periodic, in fact \( e^{2\pi iH_0} = -I \). However, if \( F \) is a linear operator from \( S(\mathbb{R}^n) \) to \( S(\mathbb{R}^n) \), then the conjugation map \( t \rightarrow e^{itH_0}F e^{-itH_0} \) is \( 2\pi \) periodic, for every \( n \in \mathbb{N} \). The following result is a direct consequence of Proposition 1.

**Proposition 6** If \( F \in \mathcal{B}(S(\mathbb{R}^n)) \) then the average operator \( \tilde{F} \) given by

\[
\tilde{F} = \frac{1}{2\pi} \int_{\mathbb{T}} e^{itH_0}F e^{-itH_0} \, dt \tag{17}
\]

is well-defined in the strong sense. Similarly, if \( F \in \mathcal{B}(S'(\mathbb{R}^n)) \) then the average operator \( \tilde{F} \) above is well-defined in the weak sense.

The operator \( \tilde{F} \) is the quantum Weinstein-Guillemin average of \( F \), and by construction for every \( t \in \mathbb{R} \), \( e^{itH_0} \tilde{F} e^{-itH_0} = \tilde{F} \).

Let us use our techniques to compute \( \hat{f} \) and \( \hat{F} \). The following result is an interesting consequence of (13) and Proposition 4.

**Proposition 7** For any \( f \in S'(\mathbb{R}^{2n}) \) we have that

\[
\tilde{f} = \sum_{|\alpha|=|\beta|} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \Phi_{\alpha,\beta}. \]

**Proof** Notice that,

\[
\int_{\mathbb{T}} \Phi_{\alpha,\beta} \circ \varphi_t \, dt = \Phi_{\alpha,\beta} \int_{\mathbb{T}} e^{it(|\alpha|-|\beta|)} \, dt.
\]

Thus, \( \Phi_{\alpha,\beta}^* = \Phi_{\alpha,\beta} \) when \( |\alpha| = |\beta| \) and it vanishes otherwise. Therefore, the continuity of the average map implies that

\[
\tilde{f} = \sum_{\alpha,\beta} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \Phi_{\alpha,\beta}^* = \sum_{|\alpha|=|\beta|} \langle \mathcal{D}p(f)\phi_\alpha, \phi_\beta \rangle \Phi_{\alpha,\beta}.
\]
The following result is the quantum version of the previous one.

**Proposition 8** For any linear continuous operator $F : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$ we have that

$$\tilde{F}\phi_\alpha = P_{|\alpha|}F\phi_\alpha.$$  

Moreover, the kernel $K_{\tilde{F}}$ of $\tilde{F}$ is

$$K_{\tilde{F}} = \sum_{|\alpha| = |\beta|} \langle F\phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha.$$  

**Proof** Note that

$$\int_T e^{-it(H_0 - k - \frac{n}{2})} \phi_\alpha dt = \int_T e^{-it(|\alpha| - k)} \phi_\alpha dt = \left[ \int_T e^{-it(|\alpha| - k)} dt \right] \phi_\alpha = 2\pi \delta_{k,|\alpha|} \phi_\alpha.$$  

Therefore

$$\frac{1}{2\pi} \int_T e^{-it(H_0 - k - \frac{n}{2})} dt = P_k,$$

and

$$\tilde{F}\phi_\alpha = \frac{1}{2\pi} \int_T e^{itH_0} F e^{-itH_0} \phi_\alpha dt = \frac{1}{2\pi} \int_T e^{-it(H_0 - |\alpha| - \frac{n}{2})} F\phi_\alpha dt = P_{|\alpha|}F\phi_\alpha.$$  

Finally,

$$K_{\tilde{F}} = \sum_{\alpha,\beta} \langle \tilde{F}\phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha = \sum_{\alpha,\beta} \langle P_{|\alpha|}F\phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha = \sum_{|\alpha| = |\beta|} \langle F\phi_\alpha, \phi_\beta \rangle \phi_\beta \otimes \phi_\alpha.$$  

The main result of this section is a direct consequence of the previous results and Proposition 3.

**Theorem 6** Weyl quantization exchanges the Weinstein-Guillemin averaging of observables, that is $\mathcal{O}p(\tilde{f}) = \mathcal{O}p(f)$, for every $f \in \mathcal{S}'(\mathbb{R}^{2n})$.

The classical and quantum Weinstein-Guillemin average method have been successfully used to study the asymptotic behaviour of clusters of eigenvalues for Schrödinger operators on compact Riemannian manifolds [13, 28]. Using similar ideas, in [6, 18] it was shown that $\text{tr}(e^{-it(H_0 + P)})$ has the same singularities than $\text{tr}(e^{-itH_0})$, where $P$ is an isotropic pseudo-differential operator of order 1. We expect that the results of this article might become useful on the latter research topic.
6 Explicit Computations for $\Phi^{\alpha,\beta}$ and $\langle \mathcal{U}(f)\phi_\alpha, \phi_\beta \rangle$

Expressions for $\Phi^{\alpha,\beta}$ can be found in the literature for the case $n = 1$, for example in [11]. The following result is a closed formula for Wigner transform of $n$ dimensional Hermite functions.

**Theorem 7** If $|\alpha| = |\beta|$ we have the following formula

$$
\Phi^{\alpha,\beta}(x, \xi) = 2^n \pi^{\frac{n}{2}} e^{-h_0(x,\xi)} m_{\alpha',\beta'}(\xi + 2\pi i x) \prod_{k=1}^{n} P_{\alpha_k,\beta_k} \left( \frac{1}{\sqrt{2}} \|\xi_k + 2\pi i x_k\| \right)
$$

where $m_{\alpha',\beta'}$ are suitable monomials of the form (3) and $P_{\alpha_k,\beta_k}$ are suitable polynomials.

**Proof** We are interested in a closed expression for $W(\phi_\alpha, \phi_\beta)$ with $|\alpha| = |\beta|$. Due to the integral definition of the Wigner transform and its relation to the Fourier-Wigner Transform $V$ (Section 1.4. in [11]), we get

$$
W(\phi_\alpha, \phi_\beta)(x, \xi) = \prod_{k=1}^{n} W(\phi_{\alpha_k}, \phi_{\beta_k})(x_k, \xi_k) = 2^n (-1)^{|\alpha|} \prod_{k=1}^{n} V(\phi_{\alpha_k}, \phi_{\beta_k})(x_k, \xi_k),
$$

where $\phi_{\alpha_k}$ is the one-dimensional Hermite function with index $\alpha_k$. There are several expressions for $V(\phi_{\alpha_k}, \phi_{\beta_k})(x_k, \xi_k)$, one is given in [7] in terms of complex Hermite polynomials. After some straightforward computations we find that

$$
W(\phi_\alpha, \phi_\beta)(x, \xi) = 2^{n+|\alpha|} n \pi^{\frac{n}{2}} e^{-\|z\|^2} \prod_{k=1}^{n} H_{\alpha_k,\beta_k}(z_k)
$$

The complex Hermite polynomials admit a factorization of the form

$$
H_{\alpha_k,\beta_k}(z_k) = O_{\alpha_k,\beta_k}(z_k) P_{\alpha_k,\beta_k}(\|z_k\|)
$$

where $P_{\alpha_k,\beta_k}$ is the polynomial

$$
P_{\alpha_k,\beta_k}(\|z_k\|) = \sum_{s=0}^{\min(\alpha_k,\beta_k)} (-1)^s s! \binom{m}{s} \binom{n}{s} \|z_k\|^{|\alpha_k| - |\alpha_k| - s}
$$

and

$$
O_{\alpha_k,\beta_k}(z_k) = \begin{cases} z_k & \alpha_k > \beta_k \\ \frac{z_k}{z_k} & \alpha_k \leq \beta_k \end{cases}
$$

The latter implies that,

$$
W(\phi_\alpha, \phi_\beta)(x, \xi) = 2^{n+|\alpha|} n \pi^{\frac{n}{2}} e^{-\|z\|^2} m_{\alpha',\beta'}(z) \prod_{i=1}^{n} P_{\alpha_i,\beta_i}(\|z_i\|),
$$

for some indices $\alpha', \beta' \in \mathbb{N}^n$ with $|\alpha'| = |\beta'|$. \qed
We shall look for an integral expression of \( \langle \mathcal{O}(f) \phi_\alpha, \phi_\beta \rangle \). With that purpose in mind, in what follows we assume that \( f \) is a polynomially bounded measurable function (so it defines a tempered distribution).

In order to follow the general framework described in Section 2.1, we shall denote by \( \mathbb{CP}^{n-1}_\lambda \) the projective space obtained as the orbit space of the flow \( \varphi_t \) restricted to the sphere \( S^{2n-1}\sqrt{2\lambda} \). Recall that, since each \( \varphi_t \) is an isometry, \( \mathbb{CP}^{n-1}_\lambda \) admits a canonical Riemannian structure called the Fubini-Study metric, which by definition makes the projection \( \pi : S^{2n-1}\sqrt{2\lambda} \to \mathbb{CP}^{n-1}_\lambda \) a Riemannian submersion. The following identity is a direct consequence of Proposition A.III.5 in [17].

**Lemma 3** Let \( \nu^S_\lambda \) the canonical volume form of the sphere \( S^{2n-1}\sqrt{2\lambda} \) and \( \nu^F_\lambda \) the Fubini-Study volume form of the projective plane \( \mathbb{CP}^{n-1}_\lambda \). Also let \( \nu^\gamma_\lambda \) the volume form of the orbit \( \gamma \) (a great circle on \( S^{2n-1}\sqrt{2\lambda} \)) induced from the canonical Riemannian structure. For any \( g \in L^1(S^{2n-1}\sqrt{2\lambda}) \), we have that

\[
\int_{S^{2n-1}\sqrt{2\lambda}} g(z) \nu^S_\lambda(z) = \int_{\mathbb{CP}^{n-1}_\lambda} \left( \int_{[z]} g(z) \nu^S_\lambda(z) \right) \nu^F_\lambda([z])
\]

**Theorem 8** Let \( f \) be a polynomially bounded constant of motion. If \( |\alpha| = |\beta| \), then

\[
\langle \mathcal{O}(f) \phi_\alpha, \phi_\beta \rangle = 2^{n} \pi \int_0^\infty \left( \int_{\mathbb{CP}^{n-1}_\lambda} f_\lambda \Phi^\beta,\alpha_\lambda \nu^F_\lambda \right) d\lambda.
\]

**Proof** Using coarea formula [9, 10, 26] and the Wigner transform (for instance, see (22) in the Appendix A), we have that

\[
\langle \mathcal{O}(f) \phi_\alpha, \phi_\beta \rangle = \int_{\mathbb{R}^{2n}} f(x, \xi) \Phi^\beta,\alpha(x, \xi) dx d\xi
\]

\[
= \int_0^\infty \left( \int_{S^{2n-1}\sqrt{2\lambda}} f(z) \Phi^\beta,\alpha(z) \nu^S_\lambda(z) \right) \frac{1}{\sqrt{2\lambda}} d\lambda.
\]

Since \( f \Phi^\beta,\alpha \) is constant on each orbit, the previous lemma implies that

\[
\int_{S^{2n-1}\sqrt{2\lambda}} f(z) \Phi^\beta,\alpha(z) \nu^S_\lambda(z) = 2\pi \sqrt{2\lambda} \int_{\mathbb{CP}^{n-1}_\lambda} f_\lambda([z]) \Phi^\beta,\alpha_\lambda([z]) \nu^F_\lambda([z])
\]

and this finish the proof.

**Remark 3** Since all the projective planes are diffeomorphic, we can replace the integration over \( \mathbb{CP}^{n-1}_\lambda \) by integration over a single projective plane, but we would need to include a suitable Jacobian. Indeed, we have that

\[
\langle \mathcal{O}(f) \phi_\alpha, \phi_\beta \rangle = 2^{n} \pi \int_0^\infty \left( \int_{\mathbb{CP}^{n-1}_\lambda} f(\lambda z) \Phi^\beta,\alpha(\lambda z) \nu_F([z]) \right) \lambda^{n-1} d\lambda. \quad (18)
\]
Remark 4 Let $B$ be the Bargmann transform. Then $B(\mathcal{H}_k)$ is the space of homogeneous polynomials of degree $k$. Thus, $B(\mathcal{H}_k)$ can be identified with the Hilbert space $\Gamma^0(\mathbb{C}P^n - 1, \mathcal{O}(k))$ of global holomorphic sections of the vector bundle $\mathcal{O}(k) \to \mathbb{C}P^n - 1$. Since the operator $B\mathcal{O}_p(f)B^*$ maps $\Gamma^0(\mathbb{C}P^n - 1, \mathcal{O}(k))$ into itself, $\langle \mathcal{O}_p(f)\phi_\alpha, \phi_\beta \rangle = \langle B\mathcal{O}_p(f)B^*(B\phi_\alpha), B\phi_\beta \rangle$ can be expressed as an integral over $\mathbb{C}P^n - 1$, somehow exchanging the double integral (18) in the previous remark.

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Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

Appendix A: Weyl Quantization

In this appendix we recall the definition of Weyl quantization and some of its main features.

Weyl quantization [12, 29] (or Weyl calculus) is a map meant to transform functions in the canonical phase space $\mathbb{R}^{2n}$ (classical observables) into operators on $L^2(\mathbb{R}^n)$ (quantum observables) in a physically meaningful manner, that is, taking into account the analogies between the descriptions of classical and quantum mechanics. There are several approaches to introduce Weyl quantization, for instance [25, 27, 30], but for the purposes of this articles we will mainly follow [11]. We also recommend [2] for a quite complete review of quantization theory.

Formally, for certain function $f$ on phase space, we define the operator $\mathcal{O}_p\hbar(f)$ acting on $L^2(\mathbb{R}^n)$ given by

$$\mathcal{O}_p\hbar(f)u(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(\frac{x + y}{2}, \xi\right) e^{i(x - y) \cdot \xi} u(y) d\xi dy, \quad (19)$$

where $\hbar$ is a positive parameter interpreted as Planck’s constant. In this article, we will not need to consider the roll played by $\hbar$, so we will take $\hbar = 1$ and $\mathcal{O}_p := \mathcal{O}_1$.

The meaning of the expression (19) changes depending on the class of functions we are considering. More precisely, if $f \in S'(\mathbb{R}^{2n})$, then $\mathcal{O}_p(f) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$, so (19) defines a tempered distribution (in particular, the evaluation on $x$ is not well-defined). The standard way to rigorously define $\mathcal{O}_p(f)$ is the following: Notice that, the kernel of $\mathcal{O}_p(f)$ should be

$$K_f(x, y) = (I \otimes \mathcal{F})f\left(\frac{x + y}{2}, y - x\right) = \int_{\mathbb{R}^n} f\left(\frac{x + y}{2}, \xi\right) e^{i(x - y) \cdot \xi} d\xi, \quad (20)$$

where $(I \otimes \mathcal{F})$ is the Fourier transform in the momentum variable. The change of variables $(x, y) \to \left(\frac{x + y}{2}, y - x\right)$ is linear, injective and preserves measure. Thus composing with the latter change of variable maps $S(\mathbb{R}^{2n})$ into itself continuously, and it can be extended to $S'(\mathbb{R}^{2n})$. Since the $(I \otimes \mathcal{F})$ is well-defined on $S'(\mathbb{R}^{2n})$,
we can define $K_f$ for $f \in S'(\mathbb{R}^{2n})$ using the middle expression in (20). In fact, the map $f \mapsto K_f$ is an automorphism of the locally convex space $S'(\mathbb{R}^{2n})$. Finally, for $u, v \in S(\mathbb{R}^n)$, we define

$$[\mathcal{O}p(f) u](v) = K_f (v \otimes u)$$

The Wigner transform is a fundamental object in the description of Weyl quantization. It is defined by the map $W : S(\mathbb{R}^n) \times S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^{2n})$ given by

$$W(u, v) = \int_{\mathbb{R}^n} e^{-i \xi \cdot p} u \left( x + \frac{p}{2} \right) v \left( x - \frac{p}{2} \right) dp.$$ 

The Wigner transform can be regarded as the restriction to functions of two variables of the form $g(x, y) = u(x) v(y)$ of the map $\tilde{W}$ defined by

$$\tilde{W} g(x, \xi) = \int_{\mathbb{R}^n} e^{-i \xi \cdot p} g \left( x + \frac{p}{2}, x - \frac{p}{2} \right).$$  \hfill (21) 

Using the arguments in the definition of the kernel $K_f$, we can define $\tilde{W}$ on $S'(\mathbb{R}^{2n})$. In fact, $\tilde{W}$ is the inverse of the map $f \mapsto K_f$, and it is unitary on $L^2(\mathbb{R}^{2n})$. In particular, we have that

$$\langle \mathcal{O}p(f) u, v \rangle = \langle f, W(v, u) \rangle.$$ \hfill (22)

For instance, if $f$ is polynomially bounded, then

$$\langle \mathcal{O}p(f) u, v \rangle = \int_{\mathbb{R}^{2n}} f W(u, v).$$

One of the main properties of Weyl quantization is its relation to the so called metaplectic representation. Let $Sp(2n)$ be the real symplectic group, i.e. the group formed by all the linear and symplectic maps $S : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. There is a map $\mu : Sp(2n) \rightarrow U(L^2(\mathbb{R}^n))$, characterized up to a $\pm 1$ factor by the identity:

$$\rho(S(q, p)) = \mu(S) \rho(q, p) \mu(S)^{-1},$$

where $(q, p) \in \mathbb{H}_n$ is an element of Heisenberg group and $\rho$ is the Schrödinger representation. The map $\mu$ is called the metaplectic representation (see [11] for details). As consequences, we have the very important (and beautiful) identities

$$W(\mu(S) \phi, \mu(S) \psi) = W(\phi, \psi) \circ S^*$$ \hfill (23) 

$$\mathcal{O}p(f \circ S^*) = \mu(S) \mathcal{O}p(f) \mu(S)^{-1}$$ \hfill (24)

The metaplectic representation is not a true representation of $Sp(2n)$, because in general we only have that $\mu(ST) = \pm \mu(S) \mu(T)$, so it defines a representation of the double covering group $Sp_2(n)$ of $Sp(2n)$. However, restricted to the complex unitary group $U(n)$, $\mu$ does define a representation (notice that the Hamiltonian flow of the Harmonic Oscillator belongs to $U(n)$).

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