COUNTING SHEAVES ON CALABI-YAU 4-FOLDS, I

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ABSTRACT. Borisov-Joyce constructed a real virtual cycle on compact moduli spaces of stable sheaves on Calabi-Yau 4-folds, using derived differential geometry.

We construct an algebraic virtual cycle. A key step is a localisation of Edidin-Graham's square root Euler class for $SO(2n, \mathbb{C})$ bundles to the zero locus of an isotropic section, or to the support of an isotropic cone.

We prove a torus localisation formula, making the invariants computable and extending them to the noncompact case when the fixed locus is compact.

We give a $K$-theoretic refinement by defining $K$-theoretic square root Euler classes and their localised versions.

In a sequel we prove our invariants reproduce those of Borisov-Joyce.

CONTENTS

1. Introduction
2. (Special) orthogonal bundles
3. Localised Edidin-Graham class
4. Moduli of sheaves on CY$^4$ via orthogonal bundles
5. $K$-theoretic virtual class
6. Virtual Riemann-Roch
7. Torus localisation
8. Local Calabi-Yau 4-folds
Appendix A. Gauge theoretic motivation
References

1. INTRODUCTION

Let $X$ be a Calabi-Yau 4-fold: a smooth complex projective variety with trivial canonical bundle $K_X \cong \mathcal{O}_X$. Let $M$ denote a moduli space of Gieseker stable sheaves on $X$ of fixed topological type. We assume throughout that there are no strictly semistable sheaves, so that $M$ is projective.

First order deformations of a sheaf $F \in M$ are given by $\text{Ext}^1(F, F)$ and there is a natural obstruction space $\text{Ext}^2(F, F)$. These fit together over $M$ to give it an “obstruction theory” in the sense of [BF, LT]; see Section 4.4 for more details. But this obstruction theory is not “perfect” because the higher obstruction space $\text{Ext}^3(F, F)$ need not vanish. Therefore the theories of Behrend-Fantechi and Li-Tian do not give $M$ a virtual cycle, and we must find another approach.
Formal picture. The crucial feature of the set-up is that morally we should think of $M$ as the zero locus of an isotropic section $s \in \Gamma(A, E)$ of an $SO(r, \mathbb{C})$ bundle $E$ over a smooth ambient variety $A$,

\begin{equation}
(E, q) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oco
Therefore we can consider $M$ as being cut out (set-theoretically) by $s^+ = 0$ instead of $s = 0$. So we might give $M$ a (real!) virtual cycle by intersecting the graph of $s^+$ with the zero section inside $E_{\mathbb{R}}$. Equivalently by the projection formula for $E \to E_{\mathbb{R}}$ we can intersect $\Gamma_s$ or our isotropic cone $C(2)$ with $iE_{\mathbb{R}}$ inside $E$; the result lies in the zero section $M$ since $q$ is definite on $iE_{\mathbb{R}}$ but vanishes identically on $\Gamma_s$ or $C$.

This is effectively what Borisov-Joyce do in their seminal work \cite{BJ}. They work with real derived differential geometry to glue (in a weak categorical sense) the finite dimensional local models $(\mathcal{A}, E_{\mathbb{R}}, s^+)$ to produce a Kuranishi structure on $M$ and hence a virtual cycle $\nu M_{\mathcal{A}}E_{\mathbb{R}}s^+$ or our isotropic cone $C(2)$ with $\Gamma_s$ inside $E_{\mathbb{R}}$; the result lies in the zero section $M$ since $q$ is definite on $iE_{\mathbb{R}}$ but vanishes identically on $\Gamma_s$ or $C$.

Square root Euler classes. Morally then — pretending for now the model $(\Pi)$ is global — the Borisov-Joyce virtual cycle is the Euler class of $E_{\mathbb{R}}$, localised to the zeros of $s^+$. When $r$ is odd — equivalently when $\nu d$ is odd — this Euler class is 2-torsion, and in \cite{OT2} we show the Borisov-Joyce class vanishes when we use $\mathbb{Z}_{\frac{1}{2}}$ coefficients.

However, real derived differential geometry is both unfamiliar and complicated, leading to invariants which seem almost impossible to calculate.

It is natural to ask if it lifts to the Chow cohomology group $A^n(Y, \mathbb{Z})$. While Field and Totaro \cite{Fi} proved it does not in general, Edidin-Graham \cite{EG1, Theorem 3} proved that it does lift if we use $\mathbb{Z}_{\frac{1}{2}}$ coefficients.

In place of the real subbundle $E_{\mathbb{R}} \subset E$, Edidin-Graham use a maximal isotropic holomorphic subbundle $\Lambda \subset E$. Notice that projection to $E_{\mathbb{R}}$ gives an isomorphism $\Lambda \cong E_{\mathbb{R}}$ by the computation \cite{EG1}. Thus $e(E) = (-1)^n e(\pi)_{\mathbb{R}}^2$ so we can think of $e(E_{\mathbb{R}})$ as defining a “square root Euler class” for $E$,

$$\sqrt{e}(E) := e(E_{\mathbb{R}}).$$

We denote the resulting Edidin-Graham class by

$$\sqrt{e}(E) := (-1)^n c_n(\Lambda) \in A^n(Y).$$

This is independent of $\Lambda$. The sign $(-1)^{|\Lambda|}$ is specified in Definition \ref{def:sign}.
Cosection localisation. This suggests defining a version of the Edidin-Graham class localised to the zero locus of an isotropic section, or to the support of an isotropic cone, to produce an algebraic version of Borisov-Joyce’s class in the Chow group of $M$.

So given an isotropic section $s$, we try to intersect its graph $\Gamma_s$ with a maximal isotropic subbundle $\Lambda \subset E$. This is entirely analogous to the real setting where we intersected $\Gamma_s$ with $iE_{\mathbb{R}} \subset E$. But while the computation (4) showed that the intersection $\Gamma_s \cap iE_{\mathbb{R}}$ lies in the zero section $0_E$, this does not hold for $\Gamma_s \cap \Lambda$ in general.

Instead we note that the normal bundle to $\Lambda \subset E$ is (the pullback of) $\Lambda^*$ so the tautological section of $\Lambda$ defines a natural cosection $N\Lambda\{E\}$. The isotropic condition implies that the cone $C_{\Gamma_s \cap \Lambda} \subset N\Lambda\{E\}$ lies in the kernel of this cosection. Therefore by Kiem-Li [KL1] the Fulton-MacPherson intersection of $\Gamma_s$ and $\Lambda$ inside $E$ can be localised to the zeros of this cosection on $\Gamma_s \cap \Lambda$, i.e. to the zeros of $s$.

Results. We summarise some of our results, leaving full explanations and definitions to the main text.

Theorem. Fix an $SO(2n, \mathbb{C})$ bundle $E$ over a scheme $Y$ and an isotropic section $s$ with zeros $\iota: Z(s) \hookrightarrow Y$. There is an operator 
$$\sqrt{e}(E, s) : A_*(Y, \mathbb{Z}[\frac{1}{2}]) \rightarrow A_{* - n}(Z(s), \mathbb{Z}[\frac{1}{2}])$$

such that $\iota_* \circ \sqrt{e}(E, s)$ is cap product with the Edidin-Graham class $\sqrt{e}(E)$.

The Fulton-MacPherson construction of the localisation of the classical Euler class to the zeros of a section uses the Gysin operator $0^E_!$ associated to the zero section $0_E$ of the bundle. In our setting there is also a square-rooted analogue.

Theorem. Fix an $SO(2n, \mathbb{C})$ bundle $E$ over a scheme $Y$ and an isotropic cone $C \subset E$ supported over $Z \subseteq Y$. There is an operator 
$$\sqrt{0^E_!} : A_*(C, \mathbb{Z}[\frac{1}{2}]) \rightarrow A_{* - n}(Z, \mathbb{Z}[\frac{1}{2}])$$

with good properties. When $C$ is contained in a maximal isotropic subbundle $\Lambda \subset E$ then $\sqrt{0^E_!} = (-1)^{|\Lambda|} 0^E_\Lambda$.

Precise details are in Section [3.1] There we prove various compatibilities and desirable properties of these operators such as their behaviour under passing to the reduction $K^2/K$ by an isotropic subbundle $K \subset E$, and a Whitney sum formula.

We also prove $K$-theoretic analogues. For explanations and full details see Section 5 for now let us simply recall that the $K$-theoretic Euler class of a bundle $E$ on a scheme $Y$ is
$$e(E) := \sum_{i=0}^{\text{rank } E} (-1)^i \Lambda^i E^* \in K^0(Y).$$
When $E$ has a section $s$ transverse to its zero section $0_E$, this is the class $[\mathcal{O}_{Z(s)}]$ of the structure sheaf of its zero locus. When $E$ is an $SO(2n, \mathbb{C})$ bundle we define a square-rooted analogue $\sqrt{e}(E)$.

**Theorem.** Fix an $SO(2n, \mathbb{C})$ bundle $E$ over a scheme $Y$. There is a natural class

$$\sqrt{e}(E) \in K^0(Y, \mathbb{Z}[\frac{1}{2}])$$

such that $\sqrt{e}(E)^2 = (-1)^n e(E)$.

When $E$ admits a maximal isotropic subbundle $\Lambda \subset E$ then

$$\sqrt{e}(E) = (-1)^{|\Lambda|} e(\Lambda) \otimes \sqrt{\det \Lambda}.$$

The use of $\mathbb{Z}[\frac{1}{2}]$ coefficients ensures the existence and uniqueness of $\sqrt{\det \Lambda}$; see Lemma 5.4. In various special cases — such as when the $SO(2n, \mathbb{C})$ bundle is Zariski locally trivial, admits a spin bundle or admits a maximal isotropic subbundle — there are many related classes in the literature [An, CLL, Ch, EG1, KO, OS, PV]. Some use different twistings, some use rational coefficients, some are localised to the zeros of an isotropic section, and most are defined only up to an overall sign. Our class unifies them, fixing the overall sign and twisting to ensure its square is $(-1)^n e(E)$, while removing any assumptions of Zariski local triviality, the existence of maximal isotropics or of spin bundles.

**Theorem.** Fix an $SO(2n, \mathbb{C})$ bundle $E$ over a scheme $Y$ and an isotropic section $s$ with zeros $\iota: Z(s) \to Y$. There is an operator

$$\sqrt{e}(E, s) : K_0(Y, \mathbb{Z}[\frac{1}{2}]) \to K_0(Z(s), \mathbb{Z}[\frac{1}{2}])$$

such that $\iota_* \circ \sqrt{e}(E, s)$ is tensor product with $\sqrt{e}(E) \in K^0(Y, \mathbb{Z}[\frac{1}{2}])$.

The $K$-theoretic analogue of the Gysin operator $0_E^!$ is the (derived) pullback operator $0_E^*$. Again, there is a square-rooted version.

**Theorem.** Fix an $SO(2n, \mathbb{C})$ bundle $E$ over a scheme $Y$ and an isotropic cone $C \subset E$ supported over $Z \subset Y$. There is an operator

$$\sqrt{0_E^*} : K_0(C, \mathbb{Z}[\frac{1}{2}]) \to K_0(Z, \mathbb{Z}[\frac{1}{2}])$$

with good properties. When $C$ is contained in a maximal isotropic subbundle $\Lambda \subset E$ then $\sqrt{0_E^*} = (-1)^{|\Lambda|} \sqrt{\det \Lambda} \cdot 0_\Lambda^*$.

**Virtual cycles.** Proposition 4.3 proves the cone $C \subset E$ of (2) is isotropic. Therefore we may apply $\sqrt{0_E^*}$ to $[C]$, and $\sqrt{0_\Lambda^*}$ to $[\mathcal{O}_C]$, to define our virtual cycle and twisted virtual structure sheaf.

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4The twisting by $\sqrt{\det T_\Lambda}$ is necessary to get a well-defined class; see the discussion after Definition 5.9.
**Theorem.** The following are well-defined, independent of choices, and deformation invariant,

\[ [M]_{\text{vir}} := \sqrt{0}_E [C] \in A^{1/2}_{\text{vd}}(M, \mathbb{Z}[1/2]), \]

\[ \hat{O}_M^{\text{vir}} := \sqrt{0}^* [\mathcal{O}_C] \cdot \sqrt{\det T^*_A} \in K_0(M, \mathbb{Z}[1/2]). \]

They are related over \( \mathbb{Q} \) by a virtual Riemann-Roch formula,

\[ \tau_M(\hat{O}_M^{\text{vir}}) = \sqrt{\text{td}}(T^*_M) \cap [M]_{\text{vir}}, \]

where \( T^*_M |_F = \text{Ext}^1(F, F) - \text{Ext}^2(F, F) + \text{Ext}^3(F, F) \). In particular,

\[ \chi(\hat{O}_M^{\text{vir}}) = \int_{[M]_{\text{vir}}} \sqrt{\text{td}}(T^*_M). \]

In fact we can define \([M]_{\text{vir}}, \hat{O}_M^{\text{vir}}\) when \( X \) is quasi-projective and \( M \) parameterises compactly supported sheaves thereon. (Projectivity is only required to prove deformation invariance.) To show they are nontrivial we calculate them on local Calabi-Yau 4-folds \( X = K_Y \) in Section 8. There they reduce to the usual 3-fold virtual cycle — and Nekrasov-Okounkov-twisted virtual structure sheaf — on the moduli space of sheaves on \( Y \) when all stable sheaves on \( X \) are pushed forward from the zero section \( Y \hookrightarrow X \) (for instance when \( Y \) is Fano).

In the sequel \([OT2]\) we show that on projective Calabi-Yau 4-folds \( X \) our virtual cycle agrees with Borisov-Joyce’s \([BJ]\) on inverting 2.

**Theorem (OT2).** For \( X \) projective and \( \text{vd} \) even our class \([M]_{\text{vir}}^{\text{vir}}\) and Borisov-Joyce’s \([M]_{BJ}^{\text{vir}} \in H_{\text{vd}}(M, \mathbb{Z}) \) have the same image under the maps

\[ [M]_{\text{vir}} \in A^{1/2}_{\text{vd}}(M, \mathbb{Z}[1/2]) \twoheadrightarrow H_{\text{vd}}(M, \mathbb{Z}[1/2]). \]

\[ [M]_{\text{vir}}^{\text{vir}} \in H_{\text{vd}}(M, \mathbb{Z}) \twoheadrightarrow H_{\text{vd}}(M, \mathbb{Z}). \]

If \( \text{vd} \) is odd then Borisov-Joyce’s class is zero after inverting 2.

In particular the resulting invariants — given by integrating insertions in integral cohomology over the virtual cycle — lie in \( \mathbb{Z} \subset \mathbb{Z}[1/2] \), which is not immediately obvious from our construction.

As a further indication that our classes should be more computable than Borisov-Joyce’s, we prove torus localisation formulae for them.

**Theorem.** Suppose \( \mathbb{T} := \mathbb{C}^* \) acts on a quasi-projective Calabi-Yau 4-fold \( X \) preserving the holomorphic 4-form. Let \( \iota : M^T \hookrightarrow M \) denote the fixed locus of the induced \( \mathbb{T} \) action on \( M \). Then

\[ [M]_{\text{vir}}^{\text{vir}} = \iota_* \left[ \frac{[M^T]_{\text{vir}}}{\sqrt{\text{td}}_{\mathbb{T}}(N_{\text{vir}})} \right] \in A^{1/2}_{\text{vd}}(M, \mathbb{Q})[t^{-1}], \]

\[ \hat{O}_M^{\text{vir}} = \iota_* \left[ \frac{\hat{O}_M^{\text{vir}}}{\sqrt{\text{td}}_{\mathbb{T}}(N_{\text{vir}})} \right] \in K_0(M) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Q}(t^{1/2}). \]
Here $t$ is the 1-dimensional weight 1 representation of $T$ and $t = c_1(t)$. For full details and explanations see Section 7. In particular this allows us to make sense of CY$^4$ sheaf counting (for insertions which lift to equivariant cohomology) and $K$-theoretic sheaf counting in noncompact equivariant settings, so long as the fixed locus $M^T$ is compact. Finally we show in (137) that the isomorphism $\tau_{M^T}: K_0(M^T)_{\mathbb{Q}} \to A_* (M^T)_{\mathbb{Q}}$ maps the second localisation formula to the first $\times \sqrt{td(T^\vir_M)}$. Thus the equivariant holomorphic Euler characteristic $\chi_t$ (the $T$-character of $R\Gamma$) can be described by

$$\chi_{e^t} \left( \frac{\hat{\Omega}^\vir_M}{\sqrt{\epsilon_T (N^\vir)}} \right) = \int_{[M^T]}^{\vir} \frac{\sqrt{td(T^\vir_M)}}{\sqrt{\epsilon_T (N^\vir)}}.$$

Furthermore Kiem and Park [KP] have now proved a version of cosection localisation for $[M]^\vir$, $\hat{\Omega}^\vir_M$, writing them as the pushforward of classes supported on the zero locus of an isotropic section of the obstruction sheaf.

There are already a number of papers making calculations of DT$^4$ invariants by assuming the Borisov-Joyce virtual cycle takes a nice algebro-geometric form in special cases [CK1, CK2, CKM1, CKM2, CMT1, CMT2, CT1, CT2, DSY, Ne, NP]. This paper gives those references a theoretical foundation, justifying many of their results.

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**Notation.** We use $E^*$ for the dual of a vector bundle $E$, reserving $E^\vee$ for the derived dual of a complex or object of the derived category.

Given a codimension $d$ regular embedding $\iota: Z \hookrightarrow Y$ of schemes, we use $\iota^!$ to denote both the Gysin map $A_*(Y) \to A_{*-d}(Z)$ and the refined Gysin map $A_*(Y') \to A_{*-d}(Z')$ for any scheme $Y' \to Y$ and $Z' := Y' \times_Y Z$. These are denoted $\iota^*$ and $\iota^!$ respectively in [Fu, Section 6.2].
For determinants, duals and signs we adopt the conventions of [KM]. For vector bundles $A, B$ of ranks $n, m$ we use the standard isomorphism

$$\det(A \oplus B) \cong \det A \otimes \det B,$$

which we write as $\mathbf{a} \wedge \mathbf{b} \mapsto \mathbf{a} \otimes \mathbf{b}$. Combined with the isomorphism $A \oplus B \cong B \otimes A$, $(a, b) \mapsto (b, a)$ we find that the identification

$$\det B \otimes \det A \cong \det(B \otimes A) \cong \det(A \oplus B) \cong \det A \otimes \det B$$

is given by the Koszul sign rule $\mathbf{b} \otimes \mathbf{a} \mapsto (-1)^{mn} \mathbf{a} \otimes \mathbf{b}$. For line bundles $L, M$ we identify $(L \otimes M)^* \cong M^* \otimes L^*$ in the obvious way [KM, page 31]. Combining these two conventions leads to the pairing

$$\det A \otimes \det A^* \rightarrow O, \quad (a_1 \wedge \cdots \wedge a_n) \otimes (a_1^* \wedge \cdots \wedge a_n^*) \rightarrow 1,$$

where $\{a_i\}_{i=1}^n$ is a local basis of sections of $A$ and $\{a_i^*\}_{i=1}^n$ is the dual basis. Beware this pairing privileges $\det A$ over $\det A^*$; if we swap their roles and apply (7) we get another map $\det A \otimes \det A^* \rightarrow O$ which differs from (8) by the sign $(-1)^n$.

These conventions ensure the commutativity of the diagram

$$\begin{array}{c}
\det A \otimes \det B \otimes \det B^* \otimes \det A^* \\
\downarrow S_B \\
\det A \otimes \det A^* \\
\downarrow S_A \\
O.
\end{array}$$

We write (8) as $\mathbf{a} \otimes \mathbf{a}^* \mapsto 1$. That is, we have set

$$\mathbf{a}^* := a_n^* \wedge \cdots \wedge a_1^*.$$  

The alternative convention of setting $\mathbf{a}^*$ to be $a_1^* \wedge \cdots \wedge a_n^*$ has the advantage of eliminating the unpleasant sign in Definition 2.1 below, but at the (worse) expense of making (9) commute only up to a sign.

Finally we note that where we refer to specific numbering in published papers, we use the arXiv version where possible.

2. **(Special) Orthogonal Bundles**

In this paper a central role will be played by orthogonal bundles, their maximal isotropic subbundles, their maximal positive definite real subbundles, and the relationship between the two. Throughout we work over a complex quasi-projective scheme $Y$.  

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5In [KM] this is made explicit by decorating $\det A$ by the integer $n = \text{rank}(A)$; we silently remember $\text{rank}(A) \mod 2$ without recording it.
2.1. $O(r)$ and $SO(r)$ bundles. By an orthogonal bundle we mean a pair $(E, q)$, where $E$ is a rank $r$ vector bundle over $Y$ and

$$q : E \otimes E \to \mathcal{O}_Y$$

is a nondegenerate quadratic form. By the Gram-Schmidt process (which uses square roots) we can pick an étale-local orthonormal basis for $(E, q)$,

$$e_1, \ldots, e_r \text{ such that } q(e_i, e_j) = \delta_{ij}.$$  

(11)

Transition functions from one such normal form to another lie in $O(r, \mathbb{C})$, giving a 1-1 correspondence between isomorphism classes of pairs $(E, q)$ and $O(r, \mathbb{C})$ principal bundles in the étale topology.

The quadratic form induces an isomorphism $q : E \cong E^*$ and thus an isomorphism $\det q : \det E \to \det E^*$, giving

$$\det E \otimes \det E^* \cong \mathcal{O}_Y,$$

where the final arrow is $\sim$. Choosing $\{f_i\}$ to be the dual basis to the étale local orthonormal trivialisation (11) we have $q(e_i) = f_i$, so (12) takes

$$(e_1 \wedge \cdots \wedge e_r) \otimes (f_1 \wedge \cdots \wedge f_r) \mapsto (-1)^{\binom{r-1}{2}}.$$  

(12)

We define an orientation to be a global choice of $\pm e_1 \wedge \cdots \wedge e_r$, when one exists. Notice this is a $\mathbb{Z}/2$ choice, not a $\mathbb{C}^*$ choice.

Definition 2.1. An orientation on $(E, q)$ is a trivialisation

$$o : \mathcal{O}_Y \to \Lambda^r E$$

(13)

whose square maps to $(-1)^{\binom{r-1}{2}}$ under (12).

Transition functions from one orthonormal basis to another, preserving $o$, lie in $SO(r, \mathbb{C})$. Therefore an orientation is equivalent to a reduction of structure group of our étale principal frame bundle from $O(r, \mathbb{C})$ to $SO(r, \mathbb{C})$, giving a 1-1 correspondence between isomorphism classes of $SO(r, \mathbb{C})$ principal bundles and oriented orthogonal bundles $(E, q, o)$.

Note that if $(E, q)$ is a Zariski locally trivial orthogonal bundle then its $\mathbb{Z}/2$-bundle of local orientations is Zariski locally trivial and hence trivial. Thus $(E, q)$ is automatically orientable.

2.2. Maximal positive definite real subbundle. Let $E_R \subset E$ denote a maximal real subbundle on which $q$ is real and positive definite. For instance we can use a partition of unity argument to patch together the real spans of the local orthonormal frames (11). We get a real orthogonal splitting

$$E \cong E_R \oplus iE_R = E_R \otimes_{\mathbb{R}} \mathbb{C}.$$  

\footnote{By this we mean $E$ admits $q$-orthonormal frames Zariski locally. Such bundles $(E, q)$ are in 1-1 correspondence with Zariski locally trivial $O(r, \mathbb{C})$ bundles, and admit two reductions to $SO(r, \mathbb{C})$ bundles. Beware that [EG1] use a different definition of Zariski local triviality, probably for characteristic 2 reasons that we can ignore.}
Conversely, given a positive definite real quadratic form $q_\mathbb{R}$ on $E_\mathbb{R}$, we get a negative definite complex quadratic form on $iE_\mathbb{C}$, and their direct sum is a non-degenerate complex quadratic form $q = q_\mathbb{R} \otimes \mathbb{C}$ on $E$. Thus, topologically, $O(r, \mathbb{C})$ and $O(r, \mathbb{R})$ bundles are equivalent, reflecting the homotopy equivalence $O(r, \mathbb{R}) \subset O(r, \mathbb{C})$.

Since $SO(r, \mathbb{R}) \subset SO(r, \mathbb{C})$ is also a homotopy equivalence, an orientation on $E$ in the sense of Definition 2.1 must be equivalent to an orientation on $E_\mathbb{R}$ in the usual sense. Obviously a trivialisation $o_\mathbb{R}$ of $\Lambda^r_\mathbb{R} E_\mathbb{R}$ induces a trivialisation $o_\mathbb{C} \otimes o_\mathbb{R}$ of $\Lambda^r E \cong (\Lambda^r_\mathbb{R} E_\mathbb{R}) \otimes \mathbb{C}$. In our local orthonormal basis this takes $e_1 \wedge \cdots \wedge e_r$ to $e_1 \wedge \cdots \wedge e_r$, which satisfies Definition 2.1. Conversely, given an orientation $o$ in the sense of (13), its real part $o_\mathbb{R}$ under $\Lambda^r E \cong \Lambda^r_\mathbb{R} E_\mathbb{R} \otimes i\Lambda^r_\mathbb{R} E_\mathbb{R}$ gives a classical orientation on $E_\mathbb{R}$ — an element of $\Lambda^r_\mathbb{R} E_\mathbb{R}$ satisfying $o_\mathbb{R} \otimes \gamma \mapsto (-1)^{r(r-1)/2}$ under

$$\Lambda^r_\mathbb{R} E_\mathbb{R} \otimes \Lambda^r_\mathbb{R} E_\mathbb{R} \xrightarrow{1 \otimes \Lambda^r_\mathbb{R} q_{\mathbb{R}}} \Lambda^r_\mathbb{R} E_\mathbb{R} \otimes \Lambda^r_\mathbb{R} E_\mathbb{R} \xrightarrow{\otimes o_\mathbb{R}} O_Y.$$

2.3. **Maximal isotropic subbundles.** Write $r$ as either $2n$ or $2n + 1$. We call an algebraic subbundle $\Lambda \subset (E, q)$ *isotropic* if $q|_\Lambda \equiv 0$ and *maximal isotropic* if it also has maximal rank $n$. The quadratic form gives a surjection $q: E/\Lambda \to \Lambda^*$. When $r = 2n$ and $\Lambda$ is maximal isotropic, we thus get a short exact sequence

$$0 \to \Lambda \to E \to \Lambda^* \to 0.$$  

Zariski locally we may pick a basis of sections $e_1, \ldots, e_n$ for $\Lambda$ with dual basis $f_1, \ldots, f_n$ for $\Lambda^*$. By a version of the Gram-Schmidt process (without square roots!\footnote{Replace any lift of $f_1$ to $E$ by $f_1 - \frac{1}{2} q(f_1, f_1) e_1$ to make it isotropic. Do the same for $f_2$, then replace it by $f_2 - q(f_1, f_2) e_1$ to ensure it is orthogonal to $f_1$. Etc.}) we can lift the $f_i$ to $E$ in such a way that their span $\Lambda^* \subset E$ is also maximal isotropic. Thus, Zariski locally, (14) splits and $E$ is trivial with basis of sections $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying

$$q(e_i, e_j) = 0 = q(f_i, f_j), \quad q(e_i, f_j) = \delta_{ij}.$$  

When $r = 2n + 1$, instead $E/\Lambda$ sits inside an exact sequence

$$0 \to \Lambda^\perp/\Lambda \to E/\Lambda \to \Lambda^* \to 0,$$

where $\Lambda^\perp/\Lambda$ is a line bundle with a nondegenerate quadratic form inherited from $q$. Working étale locally this time, we may choose an orthonormal section $e$ of $\Lambda^\perp/\Lambda$ and split as before to get an étale local basis of sections $e_1, \ldots, e_n, f_1, \ldots, f_n, e$ satisfying (15) and

$$q(e, e_i) = 0 = q(e, f_i), \quad q(e, e) = 1.$$  

Thus we get normal forms (15) and (16) for $(E, q)$ which differ from the orthonormal form (11). (They correspond to writing the first $2n$ terms of the quadratic form $\sum_{i=1}^r x_i^2$ as $\sum_{i=1}^n x_i y_i$ by a change of basis.)
Signs of maximal isotropics. Fix \( r = 2n \). Continuing with the same local basis \( \{ e_i, f_i \} \) from \([15]\), let \( \{ E_i, F_i \} \) denote the dual basis of \( E^* \). Then \( q: E \to E^* \) maps \( e_i \mapsto F_i, f_i \mapsto E_i \) so \([12]\) takes \((e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n)^\otimes 2\) to

\[
(e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n) \otimes (F_1 \wedge E_1 \wedge \cdots \wedge F_n \wedge E_n)
\]

\[
= (e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n) \otimes (F_n \wedge E_n \wedge \cdots \wedge F_1 \wedge E_1) \overset{[8]}{\to} 1.
\]

So in this normal form, an orientation \( o \) is one of \( \pm i^n e_1 \wedge f_1 \wedge \cdots e_n \wedge f_n \) whose square maps to \((-1)^n = (-1)^{\frac{r(r-1)}{2}}\) under \([12]\).

**Definition 2.2.** Given a maximal isotropic subbundle \( \Lambda \subset (E, q, o) \), define its sign \((-1)^{|\Lambda|}\) by 

\[
o = (-1)^{|\Lambda|}(-i)^n e_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n.
\]

We call \( \Lambda \) positive if \((-1)^{|\Lambda|} = +1\) and negative if \((-1)^{|\Lambda|} = -1\).

Notice \( \Lambda \) is isomorphic (as a real bundle) to \( E_\mathbb{R} \) via the composition

\[
\Lambda \hookrightarrow E \cong E_\mathbb{R} \oplus iE_\mathbb{R} \xrightarrow{} E_\mathbb{R},
\]

since \( q \) is negative definite on \( iE_\mathbb{R} \) and zero on \( \Lambda \) so their intersection is \( \{0\} \); see also \([CK1\text{, Proposition 2.9}]\). The choice of sign in Definition 2.2 ensures that \( \Lambda \) is positive if and only if its real orientation (induced by its complex structure) agrees with the real orientation \( o_\mathbb{R} \) on \( E_\mathbb{R} \) under \([17]\); see Proposition 2.3 below.

It is easy to check that picking a different local basis \( e_i \) for \( \Lambda \) we get the same notion of sign. Replacing \( o \) by \(-o\) swaps signs. The sets of positive and negative maximal isotropic subspaces of \( \mathbb{C}^{2n} \) (with its standard quadratic form) form the two connected components of the orthogonal Grassmannian \( OGr(n, 2n) \).

Put another way, given a maximal isotropic \( \Lambda \subset (E, q) \), the exact sequence \([14]\) gives a canonical isomorphism

\[
\Lambda^{2n}E \cong \Lambda^n(\Lambda) \otimes \Lambda^n(\Lambda^*) \cong \mathcal{O}_Y,
\]

taking \((e_1 \wedge \cdots \wedge e_n) \otimes (f_n \wedge \cdots \wedge f_1)\) in the central term to \(1\) in \( \mathcal{O}_Y \) according to our convention \([8]\). The \(2n\)-forms corresponding to \( \pm i^n \in \Gamma(\mathcal{O}_Y) \) are orientations in the sense of Definition 2.1. Choosing

\[
(-i)^n \in \Gamma(\mathcal{O}_Y) \cong \Gamma(\Lambda^{2n}E)
\]

defines an orientation \((E, q, o)\) with respect to which \( \Lambda \) is positive. In particular the existence of a maximal isotropic reduces the structure group of \((E, q)\) from \( O(2n, \mathbb{C}) \) to \( SO(2n, \mathbb{C}) \). (In fact more is true: \((E, q)\) is Zariski locally trivial, as we showed in \([15]\).)

So on \( \mathbb{C}^2 \) with its standard quadratic form \( z_1^2 + z_2^2 \) and orientation \((\frac{1}{0}) \wedge (\frac{0}{1})\) our conventions are that \( \Lambda_+ = \langle \langle \frac{1}{0} \rangle \rangle \) is positive and \( \Lambda_- = \langle \langle 1 \rangle \rangle \) is negative. Let \( \mathbb{R}^2 \subset \mathbb{C}^2 \) be the standard maximal positive definite real subspace \( \text{Im} z_i = 0 \). From the orientation on \( \mathbb{C}^2 \) it inherits the real orientation \((\frac{1}{0}) \wedge \mathbb{R} (\frac{0}{1})\).
Under (17) we see \( \binom{1}{-1} \) and \( i \binom{1}{-1} \) map to \( \binom{1}{0} \) and \( \binom{0}{1} \) respectively. So (17) preserves orientations. Taking products of this standard model proves the following.

**Proposition 2.3.** Fix an \( \text{SO}(2n, \mathbb{C}) \) bundle \((E, q, o)\). A maximal isotropic \( \Lambda \subset E \) is positive if and only if the orientation it induces on \( E_\mathbb{R} \) is \( o_\mathbb{R} \). \( \square \)

### 3. Localised Edidin-Graham class

Fix a quasi-projective scheme \( Y \). Since \( \text{SO}(r, \mathbb{C}) \cong \text{SO}(r, \mathbb{R}) \) is a homotopy equivalence, the Euler class \( e(E_\mathbb{R}) \in H^r(Y, \mathbb{Z}) \) provides a topological characteristic class for \( \text{SO}(r, \mathbb{C}) \) bundles \((E, q, o)\) over \( Y \). We call it a "square root Euler class" \( \sqrt{e}(E) \) since

\[
e(E) = (-1)^{\frac{r(r-1)}{2}} e(E_\mathbb{R})^2.
\]

For the rest of this Section we fix \( r = 2n \). It is then natural to ask if \( e(E_\mathbb{R}) \) lifts to the Chow cohomology group \( A^n(Y, \mathbb{Z}) \). Field and Totaro \([\text{Fi}]\) proved it does not in general, but Edidin-Graham \([\text{EG1}]\) Theorem 3\] proved that it *does* lift if we use \( \mathbb{Z}[\frac{1}{2}] \) coefficients.

In place of the maximal real positive definite subbundle \( E_\mathbb{R} \subset E \), Edidin-Graham use maximal isotropic subbundles \( \Lambda \subset E \). Since these need not exist on \( Y \) in general, they pull back to a bundle \( \widetilde{Y} \rightarrow Y \) on which they do (cf. the splitting principle). Pulling back to \( \widetilde{Y} \) is what loses the 2-primary information, forcing us to invert 2.

#### 3.1. Edidin-Graham square root Euler class

Given an \( \text{SO}(2n, \mathbb{C}) \) bundle \((E, q, o)\) we let \( \rho : \widetilde{Y} \rightarrow Y \) be the bundle of length \( n-1 \) isotropic flags in \( E \) \([\text{EG1}]\) Section 6\]. So as a set,

\[(19) \quad \widetilde{Y} := \{ (y, E_1 \subset \cdots \subset E_{n-1} \subset E_y) : y \in Y, \dim E_i = i, \ q|_{E_i} \equiv 0 \}. \]

Let \( \mathcal{E}_i \subset \rho^* E \) denote the tautological isotropic subbundle of rank \( i \). Then \( \mathcal{E}_{n-1}^\perp / \mathcal{E}_{n-1} \) inherits an orthogonal structure from \( q \). Via the composition

\[
\rho^* \Lambda^{2n} E \cong \Lambda^2 (\mathcal{E}_{n-1}^\perp / \mathcal{E}_{n-1}) \otimes \Lambda^{n-1} \mathcal{E}_{n-1} \otimes \Lambda^{n-1} (\mathcal{E}^*_{n-1}) \cong \Lambda^2 (\mathcal{E}_{n-1}^\perp / \mathcal{E}_{n-1})
\]

the trivialisation \( i^{n-1} \rho^* (o) \) of the left hand side induces an orientation on the right hand side. Thus \( \mathcal{E}_{n-1}^\perp / \mathcal{E}_{n-1} \) is an \( \text{SO}(2, \mathbb{C}) \) bundle, and so splits as a direct sum of isotropic line bundles \( L \oplus L^{-1} \), with \( L \) positive and \( L^{-1} \) negative. (Locally this is just the observation that the quadratic form \( xy \) on \( \mathbb{C}^2 \) admits precisely two isotropic lines \( x = 0 \) and \( y = 0 \); globally we use the orientation to choose the positive line and call it \( L \)).

---

8An oriented basis \( e_1, \ldots, e_r \) for \( E_\mathbb{R} \) gives a real basis \( e_1, \ldots, e_r, ie_1, \ldots, ie_r \) for \( E = E_\mathbb{R} \oplus i E_\mathbb{R} \). The induced orientation differs by \( (-1)^{\frac{r(r-1)}{2}} \) from the natural real orientation induced by the complex structure on \( E \), which has oriented real basis \( e_1, ie_1, \ldots, e_r, ie_r \).
Pulling \( L \subset \mathcal{E}_{n-1}^\perp / \mathcal{E}_{n-1} \) back to \( \mathcal{E}_{n-1}^\perp \) defines a positive maximal isotropic\(^9\)

\[ \Lambda_\rho := L + \mathcal{E}_{n-1}^\perp \subset \mathcal{E}_{n-1}^\perp \subset \rho^* E \]

and Edidin-Graham prove that \( c_n(\Lambda_\rho) \) descends to \( Y \) if we invert 2.

More precisely, there exists a distinguished class with degree \( 2^{n-1} \) over \( Y \),

\[ h \in A^{n(n-1)}(\bar{Y}, \mathbb{Z}) \quad \text{with} \quad \rho_* h = 2^{n-1} \]

by [EG1 Proposition 5]. Using this Edidin-Graham define

\[ \sqrt{e}(E) := \frac{1}{2^{n-1}} \rho_*(h \cup c_n(\Lambda_\rho)) \in A^n(Y, \mathbb{Z}[\frac{1}{2}]) \]

and show it satisfies

\[ \rho^* \sqrt{e}(E) = c_n(\Lambda_\rho). \]

Moreover it is the unique class with this property since \( \rho^* \) is injective with left inverse \( 2^{1-n} \rho_* (h \cup \cdot) \) on \( A^*(\cdot, \mathbb{Z}[\frac{1}{2}]) \). The exact sequence \( 0 \to \Lambda_\rho \to \rho^* E \to \Lambda_\rho^* \to 0 \) gives \( \rho^* c_{2n}(E) = c_n(\Lambda_\rho) c_n(\Lambda_\rho^*) = (-1)^n c_n(\Lambda_\rho)^2 \), so

\[ e(E) = (-1)^n (\sqrt{e}(E))^2. \]

Moreover, since the composition \( \Lambda_\rho \subset \rho^* E \to \rho^* E_{\mathbb{R}} \) preserves orientations, \( c_n(\Lambda_\rho) = e(E_{\mathbb{R}}) \) in cohomology, so

\[ \sqrt{e}(E) = e(E_{\mathbb{R}}) \quad \text{in} \quad H^{2n}(Y, \mathbb{Z}[\frac{1}{2}]). \]

Finally if \( E \) (rather than its pullback \( \rho^* E \)) admits a maximal isotropic \( \Lambda \subset E \) then by [EG1 Theorem 1(c)] we have

\[ c_n(\Lambda) = (-1)^{|\Lambda|} \sqrt{e}(E). \]

As a consequence, given two oriented orthogonal bundles \((E_i, g_i, o_i)\) we can compute \( \sqrt{e}(E_1 \oplus E_2) \) using the positive maximal isotropic \( \Lambda_{\rho_1} \oplus \Lambda_{\rho_2} \) on the bundle \( \tilde{Y} := \tilde{Y}_1 \times Y \tilde{Y}_2 \) over \( Y \). Here \( \rho_i : \tilde{Y}_i \to Y \) is the cover associated to \( E_i \).

Since \( e(\Lambda_{\rho_1} \oplus \Lambda_{\rho_2}) = e(\Lambda_{\rho_1}) e(\Lambda_{\rho_2}) \) on \( \tilde{Y} \) we find a Whitney sum formula

\[ \sqrt{e}(E_1 \oplus E_2) = \sqrt{e}(E_1) \sqrt{e}(E_2). \]

Given an isotropic subbundle \( K \subset E \) of an orthogonal bundle \((E, q)\), a standard operation is to take its reduction

\[ K^\perp / K. \]

Locally \( E = K^\perp / K \oplus (K \oplus K^*) \), and we can give the orthogonal bundle \( K \oplus K^* \) the standard orientation of [8] so that \( K \to K \oplus K^* \) is a positive maximal isotropic. Thus an orientation on \( E \) then induces an orientation on \( K^\perp / K \). Let \( \rho : \tilde{Y} \to Y \) be the bundle of isotropic flags for \( K^\perp / K \), with

\(^9\)Picking local bases \( e_1, \ldots, e_{n-1} \) for \( \mathcal{E}_{n-1} \) and \( e_n \) for \( L \) with the dual bases \( f_1, \ldots, f_{n-1} \) for \( \mathcal{E}_{n-1}^\perp \) and \( f_n \) for \( L^* \), [8] takes \(-ie_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n \) to \(-ie_1 \wedge f_n + f_n \). The positivity of \( L \) implies then \( \iota^{n-1} \rho^*(o) = -ie_1 \wedge f_1 \wedge \cdots \wedge e_n \wedge f_n \), which tells us \( \Lambda_\rho \) is positive.
positive maximal isotropic $\Lambda_\rho \subset K^\perp/K$. Then we get an induced positive maximal isotropic

\[ \Lambda := \Lambda_\rho \times_{K^\perp/K} K^\perp \subset \rho^*E \]

sitting in an exact sequence $0 \to K \to \Lambda \to \Lambda_\rho \to 0$. This gives $c_n(\Lambda) = c_k(K)c_{n-k}(\Lambda_\rho)$ where $k := \text{rank } K$, and so

\[ \sqrt{c(E)} = \sqrt{c(K^\perp/K)} c(K). \]

3.2. Localisation by an isotropic section. Let $(E, q, o)$ be an oriented $SO(2n, \mathbb{C})$-bundle over $Y$ and suppose $s \in \Gamma(E)$ is an isotropic section: $q(s, s) = 0$. Let $i: Z(s) \to Y$ denote its zero scheme. In this section, we will construct a localised square root Euler class — an operator $\gamma^E_{q,s}$ such that $i_* \gamma^E_{q,s} = \sqrt{c(E)} \cap (\cdot)$.

**Special case.** To begin with we suppose that $E$ admits a maximal isotropic $\Lambda \subset E$. Here we can construct a localised class with integer coefficients. By [KO] it coincides, over $\mathbb{Q}$ at least, with the one constructed by Polishchuk-Vaintrob [PV] by different methods.

We have the exact sequence

\[ 0 \to \Lambda \to E \xrightarrow{\pi} \Lambda^* \to 0. \]

By (22) we know $(-1)^{n+|\Lambda|}\sqrt{c(E)} = (-1)^n c_n(\Lambda) = c_n(\Lambda^*)$, which is represented by the Fulton-MacPherson intersection of the graph of $s^* := \pi(s) \in \Gamma(\Lambda^*)$ with the 0-section of $\Lambda^*$.

In turn this is described by first linearising the graph $\Gamma_{s^*} \subset \Lambda^*$ about the zero locus $Z^*$ of $s^*$, replacing it with the cone

\[ C_{Z^*/Y} \subset \Lambda^*|_{Z^*}. \]

This is the limit of the graphs $\Gamma_{ts^*} \subset \Lambda^*$ as $t \to \infty$; see [Fu, Remark 5.1.1] for instance.

Intersecting $C_{Z^*/Y}$ with the zero section of $\Lambda^*|_{Z^*}$ defines the Fulton-MacPherson localisation of $c_n(\Lambda^*)$ to $Z^*$. But we would like to localise it further to $Z(s) \subset Z^*$ by using the “other half” of the section $s$.

Note that on restriction to $Z^*$ the section $s \in \Gamma(E)$ factors through $\Lambda|_{Z^*}$, thus defining a map of bundles over $Z^*$

\[ \tilde{s} : \Lambda^*|_{Z^*} \to \mathcal{O}_{Z^*} \]

\[ 10 \text{ In fact it can be constructed as a bivariant class [Fu, Chapter 17] } \sqrt{c(E, s)} \in A^n(i: Z(s) \to Y, Z[\frac{1}{2}]) \text{ whose composition with } i_* \text{ gives the bivariant class } \sqrt{c(E)} \in A^n(id: Y \to Y, Z[\frac{1}{2}]), \text{ but we will not strictly need this language.} \]

\[ 11 \text{ As a subscheme of the total space } \Lambda^*. \text{ That is, we are taking the unique limit in the Hilbert scheme of } \Lambda^*. \]
with zero scheme $Z(s) \subset \Lambda^*$. In the language of [KL1], (28) is a cosection of $\Lambda^*$. By Lemma 3.1 below, the fact that $s$ is isotropic forces $\tilde{s}$ to be identically zero on $C_{Z^*/Y} \subset \Lambda^*|_{Z^*}$. Therefore, by [KL1] Proposition 1.3], the intersection of $C_{Z^*/Y}$ with the zero section $0_{\Lambda^*|_{Z^*}}$ can be cosection localised to the zero locus $Z(s)$ of $\tilde{s}$ by an operator

$$0_{\Lambda^*, \tilde{s}}^{1, \text{loc}} : A_* (C_{Z^*/Y}) \longrightarrow A_{*-n} (Z(s))$$

sitting in the following commutative diagram with the Fulton-MacPherson Gysin operator $0^!_{\Lambda^*}$,

$$\begin{array}{ccc}
A_* (Y) & \longrightarrow & A_* (C_{Z^*/Y}) \\
\downarrow & & \downarrow \\
A_* (\Lambda^*|_{Z^*}) & \longrightarrow & A_{*-n} (Z(s)) \\
\downarrow & & \downarrow \\
A_* (Y) & \longrightarrow & A_{*-n} (Y).
\end{array}$$

Here the vertical arrows, and the arrows in the triangle, are the obvious push-forward maps. Letting the first horizontal arrow be the specialisation map $W \mapsto C_{W \cap Z^*/W}$ of [Fu] Proposition 5.2, the composition right-down-right-right $A_* (Y) \rightarrow A_{*-n} (Y)$ is cap product with $c_n (\Lambda^*) = (-1)^{n+|\Lambda|} \sqrt{\epsilon} (E)$ by [Fu] Example 6.3.4. Therefore denoting the horizontal composition across the top of the diagram by $(-1)^{n+|\Lambda|} \sqrt{\epsilon} (E, s, \Lambda)$ defines an operator

$$(29) \quad \sqrt{\epsilon} (E, s, \Lambda) : A_* (Y) \longrightarrow A_{*-n} (Z(s))$$

such that

$$(30) \quad i_* \circ \sqrt{\epsilon} (E, s, \Lambda) = \sqrt{\epsilon} (E) \cap (\cdot \).
$$

**Lemma 3.1.** The cosection $\tilde{s}$ of (28) is zero on $C_{Z^*/Y} \subset \Lambda^*|_{Z^*}$. 

*Proof.* We may work locally, where (27) splits as in (15) so that $E \cong \Lambda \oplus \Lambda^*$ with the canonical quadratic form. Therefore $s = s_\Lambda \oplus s^*$ also splits, and the isotropic condition $q (s, s) = 0$ becomes

$$\langle s^*, s_\Lambda \rangle = 0.
$$

Multiplying by $t$ and considering $s_\Lambda$ as a (fibrewise linear) function $\tilde{s}_\Lambda$ on the total space of $\Lambda^*$, this says that

$$(31) \quad \tilde{s}_\Lambda |_{\Gamma_{s^*}} = 0.
$$

Since $\tilde{s}_\Lambda$ restricts on $\Lambda^*|_{Z^*}$ to the function $\tilde{s}$ of (28), taking $\lim_{t \to Z} (31)$ gives $\tilde{s}|_{C_{Z^*/Y}} \equiv 0$. 

**General case.** We can now define the localised square root Euler class in general by working on the cover $\rho : \tilde{Y} \rightarrow Y$ (19). As before, to descend back to $Y$ via (21) we have to invert 2.
Definition 3.2. Given an isotropic section \( s \in \Gamma(E) \) of an \( SO(2n, \mathbb{C}) \) bundle \((E, q, o)\) we define the localised square root Euler class
\[
\sqrt{e}(E, s) : A_{*}(Y, Z[\frac{1}{2}]) \to A_{*}(Z(s), Z[\frac{1}{2}])
\]
by
\[
\sqrt{e}(E, s) := \frac{1}{2^{n-1}} \rho_{*}(h \cup \sqrt{e}(\rho^{*}E, \rho^{*}s, \Lambda_{\rho})).
\]

By (30) and (21) this satisfies
\[
i_{*} \circ \sqrt{e}(E, s) = \sqrt{e}(E) \cap (\cdot).
\]

When \( E \) admits a positive maximal isotropic \( \Lambda \subset E \), the operators (29) and (32) become equal on inverting 2,
\[
\sqrt{e}(E, s) = \sqrt{e}(E, s, \Lambda).
\]
This can be shown by combining Lemmas 3.6 and 3.9 below with the deformation of \( \Gamma_{s} \subset E \) to \( C_{Z(s)/Y} \subset E \) through \( (\Gamma_{ts})_{t \in C} \). Since we do not need it we omit the details, but see [KP, Theorem 5.2] for a complete proof.

3.3. Localisation by an isotropic cone. We continue with an \( SO(2n, \mathbb{C}) \) bundle \((E, q, o)\) over \( Y \). Given an isotropic section we have described how to localise \( \sqrt{e}(E) \) to \( Z(s) \). In our application, \( Z(s) \) will provide a local model for the moduli space of sheaves on a Calabi-Yau 4-fold. We will not be able to see \( E, Y \) or \( s \), but the obstruction theory on the moduli space will enable us to see the limiting data\(^{12}\) \( C_{Z(s)/Y} \subset E|_{Z(s)} \). So we would like to make sense of the idea that the cone \( C_{Z(s)/Y} \) should be isotropic in \( E|_{Z(s)} \), and recover the localised operator \( \sqrt{e}(E, s) \) from this data alone.

So suppose given the data of a subscheme \( Z \subset Y \) and a cone \( C \subset E|_{Z} \) supported on \( Z \). We call \( C \) isotropic if \( q \), thought of as a function on the total space of \( E \) (quadratic on the fibres), vanishes on the subscheme \( C \).

Letting \( p : E \to Y \) be the projection map from the total space, and denoting the tautological section of \( p^{*}E \) by
\[
\tau_{E} \in \Gamma(E, p^{*}E),
\]
we see that \( C \) is isotropic if and only if \( \tau_{E}|_{C} \) is an isotropic section of \( p^{*}E|_{C} \).

Definition 3.3. For an isotropic cone \( C \subset E|_{Z} \) we define the square root Gysin map by
\[
\sqrt{0}_{E} := \sqrt{e}(p^{*}E|_{C}, \tau_{E}|_{C}) : A_{*}(C, Z[\frac{1}{2}]) \to A_{*}(Z, Z[\frac{1}{2}]),
\]
noting that the support \( Z \) of the cone is the zero locus of \( \tau_{E}|_{C} \).

\(^{12}\)Note this is a different cone from the one \( C_{Z(s)/Y} \) considered in the previous Section.
In the special case that the isotropic cone $C \subset E$ factors through a maximal isotropic subbundle $\Lambda \subset E$, this operator is familiar. It will follow from Lemma \ref{lem:isotropic} below that it is
\[
\sqrt{0^1_E} = (-1)^{|\Lambda|} 0^1_{\Lambda} : A_*(C, Z[\frac{1}{2}]) \to A_{*-n}(Z, Z[\frac{1}{2}]).
\]
First we need a preliminary result giving an expression for $\sqrt{0^1_E}$, that does not use cosection localisation. It is a square-rooted version of \cite[Proposition 3.3]{Fu}. Using the zero section and projection
\[
Z \xrightarrow{\pi} C,
\]
let’s pretend for a minute that $\pi$ is proper so that $\pi_* 0_{C_*} = \text{id}$ and $0_{C_*}$ does not lose any information. This is useful because after pushing forward, our cosection localised operator becomes the usual Edidin-Graham class, so
\[
0^1_E a = \pi_* 0_{C_*} (\sqrt{e(\pi^* E, \tau_{E|C})} a) \overset{\text{33\footnote{To define $\overline{C} = P(C \oplus \mathcal{O}_Z)$ write $C = \text{Spec} A^*$ for some positively graded algebra $A^*$.
Take its graded tensor product with $B^* := \bigoplus_{i\geq 0} \mathcal{O}_Z$ and set $\overline{C} := \text{Proj}(A^* \otimes B^*)$.}}} = \pi_* (\sqrt{e(\pi^* E) \wedge a}).
\]
We can turn this fantasy into reality by replacing $\pi: C \to Z$ by its projective completion $\overline{\pi}: \overline{C} \to Z$ and extending the pair $(\pi^* E, \tau_{E|C})$ of an orthogonal bundle and an isotropic section from $C$ to $\overline{C}$.
To do this we give $\overline{\pi}^*(E \oplus \mathcal{O}_Z \oplus \mathcal{O}_Z)$ the quadratic form $\overline{\pi}^* q \oplus (xy)$, in the obvious notation. This makes $\mathcal{O}_{\overline{C}}(-1)$ an isotropic subbundle orthogonal to $\{0\} \oplus \mathcal{O}_Z \oplus \{0\}$. Thus on $\overline{C}$ we get the data of
\begin{itemize}
  \item[(i)] an orthogonal bundle $\overline{E} := \mathcal{O}(-1) / \mathcal{O}(-1)$, \\
  \item[(ii)] an isotropic section $\overline{\tau}$ of $\overline{E}$ given by the image of the section $(0, -1, 0)$ of $\overline{\pi}^*(E \oplus \mathcal{O}_Z \oplus \mathcal{O}_Z)$, such that \\
  \item[(iii)] the zero locus of $\overline{\tau}$ is the 0-section $0_{\overline{C}}$, and \\
  \item[(iv)] on the cone $j: C \subset \overline{C}$ the pair $(\overline{E}, \overline{\tau})$ restricts to $(\pi^* E, \tau_{E|C})$.
\end{itemize}
This is enough to revive our argument to get a formula for $\sqrt{0^1_E}$ like \eqref{eq:isotropic} in terms of a global (rather than cosection localised) Edidin-Graham class.

\textbf{Lemma 3.4.} \textit{Using the above notation, given $a \in A_*(C, Z[\frac{1}{2}])$ choose any $\overline{a} \in A_* (\overline{C}, Z[\frac{1}{2}])$ such that $j^* \overline{a} = a$. Then}
\[
\sqrt{0^1_E} a = \pi_* [\sqrt{e(\overline{E})} \wedge \overline{\tau}].
\]

\textit{Proof.} Since the zero locus $0_{\overline{C}}$ of $\overline{\tau}$ lies in the image of $j$ it is immediate from the construction of the localised class that
\[
\sqrt{e(\overline{E}, \overline{\tau})(\overline{a})} = \sqrt{e(j^* \overline{E}, j^* \overline{\tau})(j^* \overline{a})}.
\]
Since this is $\sqrt{e(\pi^* E, \tau_{E|C})} (a) = \sqrt{0^1_E} a$ we obtain
\[
\sqrt{0^1_E} a = \pi_* 0_{C_*} \sqrt{e(\overline{E}, \overline{\tau}) (\overline{\tau})} \overset{\text{33\footnote{To define $\overline{C} = P(C \oplus \mathcal{O}_Z)$ write $C = \text{Spec} A^*$ for some positively graded algebra $A^*$.
Take its graded tensor product with $B^* := \bigoplus_{i\geq 0} \mathcal{O}_Z$ and set $\overline{C} := \text{Proj}(A^* \otimes B^*)$.}}} = \pi_* [\sqrt{e(\overline{E})} \wedge \overline{\tau}].
\]
\hfill $\Box$
Our first application is the following.

**Lemma 3.5.** Suppose \( \Lambda \subset E \) is a maximal isotropic subbundle. Thinking of it as an isotropic cone supported on \( Y \), we have

\( \sqrt{0_E} = (-1)^{|\Lambda|} 0_A : A_*(\Lambda, \mathbb{Z}[\frac{1}{2}]) \to A_{*-n}(Y, \mathbb{Z}[\frac{1}{2}]). \)

**Proof.** We suppress the \( \mathbb{Z}[\frac{1}{2}] \) coefficients throughout. Applying the construction above to the isotropic cone \( C = \Lambda \subset E \) gives us the data (i)-(iv) and a diagram of maps

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{j} & \overline{\Lambda} \\
\pi \downarrow & & \downarrow \pi \\
0 & \underset{0} \rightarrow & Y \end{array}
\]

Moreover in this situation we can construct a maximal isotropic subbundle of \( E \) over \( \Lambda \) that restricts on \( \Lambda \subset \pi^* \Lambda \subset \pi^* E \). We start with the maximal isotropic subbundle \( \pi^* (\Lambda \oplus \mathcal{O}_Y) \to \pi^* E \oplus \mathcal{O}_\overline{\Lambda} \oplus \mathcal{O}_\overline{\Lambda} \).

The tautological line bundle \( \mathcal{O}(1) := \mathcal{O}_{\pi^*(\Lambda \oplus \mathcal{O}_Y)}(-1) \) is a line subbundle of both, and \( \pi^* (\Lambda \oplus \mathcal{O}_Y) \) lies in its orthogonal \( \mathcal{O}(1)^\perp \) in \( \pi^* E \oplus \mathcal{O}_\overline{\Lambda} \oplus \mathcal{O}_\overline{\Lambda} \). So dividing both by \( \mathcal{O}(1) \) gives, by the relative Euler sequence, the following maximal isotropic subbundle of the orthogonal bundle \( E \),

\[ T_{\overline{\pi}}(1) \hookrightarrow \pi^* E = \mathcal{O}(1)^\perp/\mathcal{O}(1) \] on \( \overline{\Lambda} \).

The section \( (0, -1) \) of \( \pi^* (\Lambda \oplus \mathcal{O}_Y) \) projects, under quotienting by \( \mathcal{O}(1) \), to a section \( \pi \) of \( T_{\overline{\pi}}(-1) \). Its image in \( \overline{E} \) is the isotropic section \( \pi \) of (ii). It restricts over \( \Lambda \subset \overline{\Lambda} \) to the Euler vector field on \( \Lambda \to Y \), i.e. the tautological section \( \pi \Lambda \) of \( \pi^* \Lambda \cong T_{\overline{\pi}} \).

Furthermore \( \pi \) cuts out the zero section \( 0_{\overline{\Lambda}} \subset \overline{\Lambda} \), and transversally when thought of as a section of \( T_{\overline{\pi}}(-1) \). Thus

\[ 0_{\overline{\Lambda}} 0_{\overline{\Lambda}}^1 = e(T_{\overline{\pi}}(-1)) \cap (\cdot). \]

Therefore, choosing any class \( \overline{a} \in A_*(\overline{\Lambda}) \) such that \( J\overline{a} = a \),

\[ \sqrt{0_E} a \overset{39}{=} \pi_* [\sqrt{e(\overline{E}) \cap \overline{a}}] \overset{22}{=} \pm \pi_* e(T(1) \cap \overline{a}) \]

\[ \overset{30}{=} \pm \pi_* 0_{\Lambda} 0_{\Lambda}^1 \overline{a} = \pm 0_{\Lambda}^1 \overline{a} = \pm 0_{\overline{\Lambda}}^1 \pi_j^*(J\overline{a}) = \pm 0_{\overline{\Lambda}}^1 a, \]

where \( \pm \) is the sign \( (-1)^{|\Lambda|} \).

Recall our definition \( \sqrt{e} : p^* E_{/C} \to \tau_{E_{/C}} \). If \( E \) admits a maximal isotropic subbundle \( \Lambda \subset E \) there is an obvious alternative definition using the operator \( \sqrt{e(p^* E_{/C}, \tau_{E_{/C}})} \) of (29). Our second application of Lemma 3.4 is to show that they’re the same.
Lemma 3.6. Suppose $E$ admits a maximal isotropic subbundle $\Lambda \subset E$ and an isotropic cone $C \subset E$. Let $\pi : C \to Z \subseteq Y$ denote the projection. Then
\[
\sqrt{0^1_E} = \sqrt{e}(\pi^*E, \tau_E|_C, \pi^*\Lambda) : A_*(C, Z[\frac{1}{2}]) \to A_{*-n}(Z, Z[\frac{1}{2}]).
\]

Proof. Lemma 3.4 rests on extending $(\pi^*E, \tau_E|_C)$ from $C$ to its projective completion $\widetilde{C}$, giving the orthogonal bundle and isotropic section $(\widetilde{E}, \widetilde{\tau})$. There may be no extension $\widetilde{\Lambda} \subset \widetilde{E}$ of the maximal isotropic $\pi^*\Lambda \subset \pi^*E$, but there is one if we replace $\widetilde{C}$ by a certain blow up (away from $C$).

Namely, $\pi^*\Lambda \subset \pi^*E$ defines a section of the orthogonal Grassmannian bundle $OGr(C) \to \overline{C}$ over $C \subset \overline{C}$. Taking its closure defines the blow up $b : \tilde{C} \to \overline{C}$ on which the universal subbundle on $OGr(\widetilde{E})$ restricts to give the maximal isotropic $\tilde{\Lambda} \subset b^*E$.

So now replacing $C \stackrel{j}{\to} \overline{C} \stackrel{\pi}{\to} Z$ by $C \stackrel{j}{\to} \tilde{C} \stackrel{\tilde{\pi} = b \circ \pi}{\to} Z$, the zero section $0_{\overline{C}}$ by $0_{\tilde{C}}$, the orthogonal bundle and isotropic section $(\overline{E}, \overline{\tau})$ by $(\tilde{E}, \tilde{\tau}) := (b^*E, b^*\tilde{\tau})$, and $\pi$ by any $\tilde{\alpha}$ such that $j^*\tilde{\alpha} = \alpha$, the same proof gives
\[
\sqrt{e}(\pi^*E, \tau_E|_C)(\alpha) = \tilde{\pi}_*0_{\tilde{C}}[\sqrt{e}(\tilde{E}, \tilde{\tau})(\tilde{\alpha})] \overset{(33)}{=} \tilde{\pi}_*[\sqrt{e}(\tilde{E}) \cap \tilde{\alpha}].
\]
But replacing (33) by (30) we similarly get
\[
\sqrt{e}(\pi^*E, \tau_E|_C, \pi^*\Lambda)(\alpha) = \tilde{\pi}_*0_{\tilde{C}}[\sqrt{e}(\tilde{E}, \tilde{\tau}, \tilde{\Lambda})(\tilde{\alpha})] \overset{(30)}{=} \tilde{\pi}_*[\sqrt{e}(\tilde{E}) \cap \tilde{\alpha}].
\]
Thus the left hand sides of these two expressions are equal. \hfill \Box

Our third application of Lemma 3.4 is to derive a Whitney sum formula for the localised operators $\sqrt{0^1_E}$ by using the global Whitney sum formula (23) for Edidin-Graham classes. From now on we work over all of $Y$ instead of specialising to the support $Z \subseteq Y$ of a cone, i.e. we set $Z = Y$.

Suppose $C_1$, $C_2$ are isotropic cones in $SO(2n, \mathbb{C})$ bundles $E_1$, $E_2$ and set
\[
C := C_1 \oplus C_2 \subset E_1 \oplus E_2 =: E.
\]
Let $p_i : E_i \to Y$ denote the projections from the total spaces. The total space of $E \to Y$ is the total space of $p_1^*E_2 \to E_1$. So we may think of the isotropic cone $C \subset E$ as lying in $p_1^*E_2|_{C_1}$, with 0-section $C_1$, and form the composition
\[
A_*(C, Z[\frac{1}{2}]) \overset{\sqrt{0^1_{p_1^*E_2}}}{\longrightarrow} A_{*-n_2}(C_1, Z[\frac{1}{2}]) \overset{\sqrt{0^1_{p_1^*E_1}}}{\longrightarrow} A_{*-n_1-n_2}(Y, Z[\frac{1}{2}]).
\]

Proposition 3.7. The composition \((11)\) is $\sqrt{0^1_E}$,
\[
\sqrt{0^1_{E_1}} \circ \sqrt{0^1_{p_1^*E_2}} = \sqrt{0^1_E} : A_*(C, Z[\frac{1}{2}]) \to A_{*-n_1-n_2}(Y, Z[\frac{1}{2}]).
\]

Proof. Associated to the cones $C_i \subset E_i$ we have the data $(\overline{C_i}, \overline{E_i}, \overline{\tau_i}, j_i)$ from (i)-(iv) above. Thus we can complete $C = C_1 \oplus C_2$ with $C_1 \times_Y C_2$ instead of $\overline{C}$. On this we have the orthogonal bundle $\widetilde{E}_1 \oplus \widetilde{E}_2$ and isotropic section $\tilde{\Lambda}$.

\[14\] We suppress some obvious pullback maps for the sake of clarity. The functoriality $\sqrt{e}(p^*F) = p^*\sqrt{e}(F)$ of the Edidin-Graham class means this adds no hidden dangers.
\[ \tau := (\tau_1, \tau_2) \] with zero locus the 0-section \[ \tau_{1 \times Y} \tau_2 \]. Applied to these, the argument of Lemma 3.4 then gives

\[ \sqrt{0^1_E a} = (\tau_1 \times_Y \tau_2)_* [\sqrt{e(E_1 + E_2)} \cap \tilde{\alpha}] \]

for any \( \tau \in A_*(\overline{C_1 \times Y} \overline{C_2}) \) whose pullback \( (j_1 \times_Y j_2)^* \tau \) to \( C_1 \oplus C_2 \) is \( a \). By \( (23) \) and the projection formula this is

\[ \tau_1* \left[ \sqrt{e(E_1)} \cap \tau_2* \left( \sqrt{\overline{E_1(E_1)} \cap \overline{E_2}} \right) \right] \]

where \( \tilde{\alpha} \) is the restriction of \( \alpha \) to \( \overline{C_1 \times Y} \overline{C_2} \). Finally applying \( (37) \) to \( \tau_1* \) gives

\[ \sqrt{0^1_E a} = \sqrt{0^1_{E_1} j_1^* \sqrt{0^1_{E_2} \tilde{\alpha}}} = \sqrt{0^1_{\overline{E}_1} \sqrt{0^1_{\overline{E}_2} \tilde{\alpha}}}. \]

\[ b_0^! E \overset{\alpha}{\longrightarrow} E \]

for any \( a \in A_* \overline{C_1 \times Y} \overline{C_2} \). By \( (23) \) and the projection formula this is

\[ \tau_1* \left[ \sqrt{e(E_1)} \cap \tau_2* \left( \sqrt{\overline{E_1(E_1)} \cap \overline{E_2}} \right) \right] \]

where \( \tilde{\alpha} \) is the restriction of \( \alpha \) to \( \overline{C_1 \times Y} \overline{C_2} \). Finally applying \( (37) \) to \( \tau_1* \) gives

\[ \sqrt{0^1_E a} = \sqrt{0^1_{E_1} j_1^* \sqrt{0^1_{E_2} \tilde{\alpha}}} = \sqrt{0^1_{\overline{E}_1} \sqrt{0^1_{\overline{E}_2} \tilde{\alpha}}}. \]

\[ \square \]

A closely variant of this result is the following. Suppose we have \( K \subset C \subset K^\perp \subset (E, q, o) \) with \( K \) a rank \( k \) isotropic subbundle of a rank 2 oriented orthogonal bundle \((E, q, o)\) and \( C \) an isotropic cone. Then \( C \) descends to an isotropic cone \( C/K \subset K^\perp/K \)

in the reduction \( (24) \) of \( E \) by \( K \). Using the orientation described after \( (24) \), the formula \( (26) \) for the Edidin-Graham class of \( K \) then gives the following. Note the projection \( p: C \rightarrow C/K \) is flat, so \( p^* \) is defined on cycles.

**Proposition 3.8.** The square root Gysin classes of \( E \) and its reduction \( K^\perp/K \) are related by

\[ \sqrt{0^1_{K^\perp/K}} = \sqrt{0^1_E \circ p^*: A_*(C/K, \mathbb{Z}[\frac{1}{2}]) 
\rightarrow A_{*+k-n}(Y, \mathbb{Z}[\frac{1}{2}]).} \]

**Proof.** This result can be proved by deforming the pair \((C \subset E)\) through (isotropic cones in orthogonal bundles) to \( C/K \subset K \subset (K^\perp/K) \subset (K \oplus K^*) \). Then we can apply Proposition 3.7 which amounts to working in the compactification \( \overline{C/K \times Y} \overline{K} \) of \( C/K \oplus K \).

Alternatively we can generalise the proof of Proposition 3.7 by working in the right compactification of the undeformed \( C \) from the beginning. It is locally isomorphic to \( \overline{C/K \times Y} \overline{K} \) but is globally twisted.

We start with the projective completion \( \overline{\pi}: \overline{C/K} \rightarrow Y \) of Footnote 13 with the data (i)-(iv) over it. In particular we have the isotropic line subbundle \( \mathcal{O}_{\overline{C/K}}(-1) \hookrightarrow \overline{\pi}^*(K^\perp/K \oplus \mathcal{O} \oplus \mathcal{O}) \) and the orthogonal bundle

\[ 15 \text{If } C \text{ is an isotropic subbundle then } K \subset C \text{ implies } C \subset K^\perp, \text{ but for cones this need not be true, e.g. if } K := \{x = 0\} \text{ in } C := \{xy = 0\} \text{ inside } \mathbb{C}^2 \text{ with quadratic form } xy. \]
\( \overline{K/K} := \mathcal{O}_{C/K}(-1)^{\perp}/\mathcal{O}_{C/K}(-1) \) extending \( K/K \) over \( C/K \subset \overline{C/K} \). Via the surjection \( K^{\perp} \to K^{\perp}/K \) we get the Cartesian diagram

\[
\begin{array}{c}
q^*\mathcal{O}_{C/K}(-1) \hookrightarrow \overline{\mathcal{P}}^*(K^{\perp} \oplus \mathcal{O}) \hookrightarrow \overline{\mathcal{P}}^*(E \oplus \mathcal{O} \oplus \mathcal{O}) \\
\mathcal{O}_{C/K}(-1) \hookrightarrow \overline{\mathcal{P}}^*(K^{\perp}/K \oplus \mathcal{O}).
\end{array}
\]

Our compactification of \( C \) is the projective bundle \( \overline{C} := \mathbb{P}(q^*\mathcal{O}_{C/K}(-1)) \) over \( \overline{C/K} \). The inclusion \( C \subset \overline{C} \) takes \( c \in C \) to the point \([c : 1] \) in the projectivisation of \( K^{\perp} \oplus \mathcal{O} \) sat over the point \([c : 1] \in \mathbb{P}(C/K \oplus \mathcal{O}_Y) = \overline{C/K} \).

We use the projections

\[
\begin{array}{c}
\overline{C} \xrightarrow{\overline{\mathcal{P}}} \overline{C/K} \xrightarrow{\overline{\pi}} Y.
\end{array}
\]

As the projectivisation of a vector bundle, \( \overline{C} := \mathbb{P}(q^*\mathcal{O}_{C/K}(-1)) \) carries a tautological line bundle

\[
\mathcal{O}_{\overline{C}}(-1) \hookrightarrow \overline{\mathcal{P}}^*(q^*\mathcal{O}_{C/K}(-1)) \hookrightarrow \overline{\mathcal{P}}^*(K^{\perp} \oplus \mathcal{O}) \hookrightarrow \overline{\mathcal{P}}^*(E \oplus \mathcal{O} \oplus \mathcal{O}),
\]

where we have applied \( \overline{\mathcal{P}}^* \) to (42). Since \( \mathcal{O}_{\overline{C}}(-1) \) is isotropic in \( \overline{\mathcal{P}}^*(E \oplus \mathcal{O} \oplus \mathcal{O}) \) we may define the orthogonal bundle

\[
\overline{E} := \overline{\mathcal{P}}^*(q^*\mathcal{O}_{C/K}(-1))^{\perp}/\mathcal{O}_{\overline{C}}(-1) \quad \text{over} \quad \overline{C}.
\]

The image of the section \((0, -1, 0)\) of \( \overline{\mathcal{O}_{\overline{C}}(-1)^{\perp}} \subset \overline{\mathcal{P}}^*(E \oplus \mathcal{O} \oplus \mathcal{O}) \) is an isotropic section \( \overline{\pi} \) of \( \overline{E} \) which cuts out precisely the zero section \( \overline{Y} \hookrightarrow \overline{C} \hookrightarrow \overline{\mathcal{C}} \). Therefore (37) applies again, giving, for \( a \in A_*(C) \),

\[
(43) \quad \sqrt{\partial_{\overline{E}}a} = \overline{\pi}^*\left[\sqrt{c(E)} \cap \overline{\pi}\right],
\]

for any \( \overline{\pi} \in A_*(\overline{C}) \) which restricts to \( a \) on \( C \subset \overline{C} \). We can also define

\[
\overline{K} := \overline{\mathcal{P}}^*(q^*\mathcal{O}_{C/K}(-1))^{\perp}/\mathcal{O}_{\overline{C}}(-1) \subset \overline{\mathcal{O}_{\overline{C}}(-1)^{\perp}}/\mathcal{O}_{\overline{C}}(-1) = \overline{E}
\]

and check that over \( C \subset \overline{C} \) this gives \( K \subset E \). By elementary linear algebra

\[
(44) \quad \overline{K}^{\perp}/\overline{K} \cong \overline{\mathcal{P}}^*(K^{\perp}/K),
\]

since being orthogonal to \( q^*\mathcal{O}_{C/K}(-1) \) is equivalent to being orthogonal to \( K \) and \( \mathcal{O}_{C/K}(-1) \). Moreover, \( \overline{C/K} = \overline{C/K} \).

So now suppose \( a = p^*b \) for \( b \in A_*(C/K) \). Choose \( \overline{b} \in A_*(\overline{C/K}) \) restricting to \( b \) and take \( \overline{\pi} = \overline{\mathcal{P}}^*\overline{b} \). Then by (26) and (44), equation (43) becomes

\[
\sqrt{\partial_{\overline{E}}p^*b} = \overline{\pi}^*\overline{\pi}^*[\overline{\mathcal{P}}^*\sqrt{c(K^{\perp}/K)} e(\overline{K}) \cap \overline{\mathcal{P}}^*\overline{b}].
\]
Now $\pi_* e(K)$ is a constant which can be computed on any fibre of $\pi$. Since each fibre is a projective space $\mathbb{P}^k$ on which $K$ is $T_{\mathbb{P}^k}(-1)$ (cf. (39)) we find the constant is 1. Therefore, by the projection formula,

$$\sqrt{0^1_E p^* b} = \pi_* \left[ \sqrt{e(K^1/K)} \cap b \right]$$

Our final application of Lemma 3.4 is to check that the square root Gysin operator $\sqrt{0^1_E}$ commutes with refined Gysin operators $f^!$. Suppose we have a Cartesian diagram with $f$ a regular embedding of codimension $d$,

$$\begin{array}{ccc}
\quad C & \xrightarrow{f^!} & X' \\
\pi' \downarrow & & \downarrow f' \\
X & \xrightarrow{f} & Y,
\end{array}$$

where $C$ is an isotropic cone in an $SO(2n, C)$ bundle $(E, q, o)$ over $Y'$. Given $a \in A_*(C)$ we get the cycle $f^! a$ in the isotropic cone $f^* C \subset f^* E$.

Lemma 3.9. In the above notation we have

$$f^! \sqrt{0^1_E} = \sqrt{0^1_{f^* E}} f^! : A_*(C, \mathbb{Z}[\frac{1}{2}]) \longrightarrow A_{*-d-n}(X', \mathbb{Z}[\frac{1}{2}]).$$

Proof. Using the notation of Lemma 3.4 we have projective completions $\pi: C \rightarrow Y'$ and $\pi': f^* C \rightarrow X'$ of the cones $C, f^* C$, with a map $\pi'$ between them covering $f^!$. By (37),

$$\sqrt{0^1_E a} = \pi_* \left[ \sqrt{e(E)} \cap \alpha \right],$$

where as usual $\alpha$ is any extension of $a$ to $A_*(\mathcal{C})$.

Since we are using $\mathbb{Z}[\frac{1}{2}]$ coefficients we may replace $\mathcal{C}$ by the bundle $\rho$ (19) over it (and replace $f^* C$ by the corresponding flat basechange) to assume that $E$ admits a positive maximal isotropic $\Lambda$. Thus by (22),

$$\sqrt{0^1_E a} = \pi_* [e(\Lambda) \cap \pi].$$

By the usual commutativity of Chern classes and refined Gysin maps [Ful, Proposition 6.3],

$$f^! [e(\Lambda) \cap \pi] = e(f^* \pi^* \Lambda) \cap f^! \pi.$$ 

Now pushing down by $\pi'_* f^!$ and using $\pi'_* f^! = f^! \pi_*$ [Ful, Theorem 6.2(a)] gives

$$f^! \sqrt{0^1_E a} = \pi'_* \left[ \sqrt{e(f^* \pi^* \Lambda)} \cap f^! \pi \right],$$

which by another application of (22) and (37) is precisely $\sqrt{0^1_{f^* E}} f^! a$. □
4. Moduli of sheaves on CY$^4$ via orthogonal bundles

Let $(X, \mathcal{O}_X(1))$ be a smooth projective 4-fold with a fixed trivialisation of $K_X$. Fix a class $c \in H^*(X, \mathbb{Q})$ such that $\mathcal{O}_X(1)$-Gieseker semistable sheaves with Chern character $c$ are all stable. Then there is a projective moduli space $M = M(X, c)$ of stable sheaves $F$ of charge $c$. We let $\mathbb{L}_M \in D(M)$ denote the truncated cotangent complex in the bounded derived category of coherent sheaves on $M$. We use $\pi$ to denote any projection $X \times N \to N$ down $X$.

4.1. Obstruction theory. Let $\mathcal{E}$ be any universal twisted sheaf on $X \times M$, whose existence is proved in [Ca, Propositions 3.3.2, 3.3.4]. The twistings cancel in $R\mathbb{H}om(\mathcal{E}, \mathcal{E})$, giving a complex of sheaves in $D(M \times X)$. By [HT, Theorem 4.1] the truncated Atiyah class of [HT, Equation 4.2] defines an obstruction theory for $M$,

$$E := \tau^{[-2,0]}(R\pi_* R\mathbb{H}om(\mathcal{E}, \mathcal{E})[3]) \xrightarrow{\text{At}} \mathbb{L}_M,$$

in the sense of [BF, Definition 4.4]. That is, $h^0(\text{At}) : h^0(\mathcal{E}) \to \Omega_M$ is an isomorphism, and $h^{-1}(\text{At}) : h^{-1}(\mathcal{E}) \to h^{-1}(\mathbb{L}_M)$ is a surjection. The obstruction theory (46) has virtual dimension

$$vd := \text{rank} E = 2 - \chi(F, F),$$

for any sheaf $F$ on $X$ of the same Chern character $c$. Relative Serre duality down the map $\pi$ gives an isomorphism

$$\theta : E \cong \mathcal{E} \overset{\cdot}[2] \in \text{Hom}_{D(M)}(\mathcal{E}, \mathcal{E} \overset{\cdot}[2]).$$

Since $E$ is perfect of amplitude $[-2,0]$, rather than $[-1,0]$, the obstruction theory $\text{At}$ is not perfect. (In general $\text{Ext}^3(F, F) = \text{Ext}^1(F, F)^*$ is nonzero for $F \in M$, so $h^{-1}(\mathcal{E})$ can be nonzero.) So we cannot apply [BF, LT] to get a virtual cycle. Instead we will follow the Behrend-Fantechi recipe as far as producing a cone in a vector bundle — in fact an isotropic cone in an oriented orthogonal bundle — then we will replace their intersection with the zero section $0^1_E$ by the square-rooted analogue $\sqrt{0^1_E}$ of Definition 3.3.

To show the cone (2) is isotropic, we do not know if it is sufficient to have the duality (47). Instead our proof will use a lifting of this symmetry to a $(-2)$-shifted symplectic structure [PTVV], whereupon we can employ the results of [BBBJ, BBJ, BC]. (Contraction with the shifted symplectic form then induces a shifted duality $\mathbb{L}_M^\text{vir} \cong (\mathbb{L}_M^\text{vir})^\vee [2]$ on the virtual cotangent bundle $\mathbb{L}_M^\text{vir} := E$, recovering [17].) For this we will need an alternative description of the obstruction theory (46) using the wonders of derived stacks.
Derived description. Let $\mathcal{M}^{\text{der}}$ denote the derived stack $\mathbb{I} \mathcal{V} \mathcal{A}$ of stable sheaves of charge $c$, with underlying Artin stack $\mathcal{M}$ and coarse moduli space $M$. Since stable sheaves are simple the projection $\mathcal{M} \to M$ is a $BC^*$-bundle. Let $\mathcal{E}'$ denote the universal sheaf on $X \times \mathcal{M}^{\text{der}}$. The (derived) cotangent bundle of $\mathcal{M}^{\text{der}}$ is

$\mathbb{L}_{\mathcal{M}^{\text{der}}} = (R\pi_* R\mathcal{H}\text{om}(\mathcal{E}', \mathcal{E})[1])^\vee \cong R\pi_* R\mathcal{H}\text{om}(\mathcal{E}', \mathcal{E})[3],$

by relative Serre duality and the fixed trivialisation of the dualising sheaf $K_X$ of $\pi$. Restricting to $\mathcal{M} \subset \mathcal{M}^{\text{der}}$ and truncating gives the composition

$\tau[-2,0](R\pi_* R\mathcal{H}\text{om}(\mathcal{E}', \mathcal{E})[3]) \to \tau[-2,0]\mathbb{L}_M \to \tau[-1,0]\mathbb{L}_M.$

The latter is the pullback of $\mathbb{L}_M$ from $M$\cite{17} and by [STV, Appendix A] the result is the pullback of the map $A: \mathbb{E} \to \mathbb{L}_M$ of $\mathbb{E}$.\footnote{Since $\mathcal{E}'$ and $\mathcal{E}'$ differ locally by a line bundle, their derived endomorphisms coincide.}

Normal form for $\mathbb{E}$. We begin by getting the virtual cotangent bundle $\mathbb{E}$ into a normal form. Call a 3-term complex of locally free sheaves $E^*$ self-dual if it has the form

$$E^* := \{ T \xrightarrow{\alpha} E \xrightarrow{a^*} T^* \},$$

where $(E, q)$ is an orthogonal bundle, inducing the isomorphism $E \cong E^*$ used in forming the map $a^*$ above. Such a complex has an obvious duality

$$E^*[2] := (E^*)^\vee [2] = \begin{array}{cccc}
T & a & E & a^* & T^* \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(T^*)^* & (a^*)^* & E^* & a^* & T^* \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
E & \xrightarrow{i} & \xrightarrow{q} & \xrightarrow{\alpha} & \xrightarrow{\alpha^*} \\
& & & & \\
\end{array}$$

We use the standard notation for complexes $E^*$ that $E^i$ appears in degree $i$, with $E_{-i} := (E^i)^\vee$ in degree $-i$ in the dual complex $E_*$. Given a map of complexes $\psi*: E^* \to F^*$ we denote its dual by $\psi_* := (\psi^*)^\vee: F_* \to E_*$. We will show there is a quasi-isomorphism from $\mathbb{E}$ to a self-dual complex $E^*$ intertwining the Serre duality map $\theta: \mathbb{E}^\vee [2] \to \mathbb{E}$ of $\mathbb{E}$ with the duality $\mathbb{E}$. To state this we are careful to distinguish between morphisms in $\mathbb{D}(M)$ and genuine maps of complexes. We denote the former by single letters such as $\alpha$, and the latter by $\alpha^*$.\footnote{Since $\mathcal{E}$ and $\mathcal{E}'$ differ locally by a line bundle, their derived endomorphisms coincide.}

**Proposition 4.1.** There is a self-dual 3-term complex of locally free sheaves $E^*$\cite{18} and an isomorphism $\alpha: E^* \to \mathbb{E}$ in $\mathbb{D}(M)$, such that the following diagram commutes in $\mathbb{D}(M)$,

$$\begin{array}{ccc}
E^* & \xrightarrow{19} & E_*[2] \\
\alpha & \downarrow & \alpha^*[2] \\
\mathbb{E} & \xrightarrow{\theta} & \mathbb{E}^\vee [2].
\end{array}$$
Furthermore, given an embedding $M \subset A$ in a smooth scheme with ideal $I$, we may assume that $A^\bullet \xrightarrow{\alpha} E^\bullet \to \mathbb{L}_M$ is represented by a genuine map of complexes $E^\bullet \to \{I/I^2 \to \Omega_A|_M\}$ which is surjective in each degree.

**Proof.** Choose a locally free resolution $\phi: A^\bullet \xrightarrow{\sim} \mathbb{E}$ in $D(M)$. Then $\theta: \mathbb{E} \to \mathbb{E}\vee[2]$ becomes the morphism $\phi^\vee[2] \circ \theta \circ \phi: A^\bullet \to A^\bullet[2]$ in $D(M)$.

This need not be a genuine map of complexes, however. So we now further resolve $A^\bullet$ by a map of complexes $\psi^\bullet: B^\bullet \to A^\bullet$, where $B^\bullet$ is a *sufficiently negative* locally free locally free resolution that there exist maps of complexes $f^\bullet, g^\bullet$ filling in commutative diagrams in $D(M)$,

\[
\begin{array}{ccc}
B^\bullet & \xrightarrow{\psi^\bullet} & A^\bullet[2] \\
\downarrow{\phi} & & \downarrow{\phi^\vee[2]} \\
E & \xrightarrow{\theta} & \mathbb{E}\vee[2]
\end{array}
\]

where $L^\bullet_M$ is defined to be the complex $\{I/I^2 \xrightarrow{d} \Omega_A|_M\} \cong \mathbb{L}_M$ made from the fixed embedding $M \subset A$ with ideal $I$. Composing then gives another commutative diagram in $D(M)$,

\[
\begin{array}{ccc}
B^\bullet & \xrightarrow{\psi^\bullet[2] \circ f^\bullet} & B^\bullet[2] \\
\downarrow{\phi \circ \psi^\bullet} & & \downarrow{\phi^\vee \circ \phi^\vee[2]} \\
\mathbb{E} & \xrightarrow{\theta} & \mathbb{E}\vee[2]
\end{array}
\]

representing $\theta$ by a genuine map of complexes $\theta^\bullet : \psi^\bullet[2] \circ f^\bullet: B^\bullet \to B^\bullet[2]$,

\[
\begin{array}{ccc}
\cdots & \xrightarrow{d_3} & B^{-2} & \xrightarrow{d_2} & B^{-1} & \xrightarrow{d_1} & B^0 & \xrightarrow{d_0} & \cdots \\
\cdots & \xrightarrow{\theta_0} & B_0 & \xrightarrow{d_3} & B_1 & \xrightarrow{\theta_1} & B_2 & \xrightarrow{\theta_2} & \cdots
\end{array}
\]

Since $\theta^\bullet = \theta^\bullet[2]$ as morphisms in $D(M)$, we may replace $\theta^\bullet$ by $\frac{1}{2}(\theta^\bullet + \theta^\bullet[2])$ to ensure $\theta_i = \theta_{2-i}^\bullet$ while (50) still commutes.

The restriction of $E$ to any closed point has cohomology only in degrees $[-2, 0]$, so the truncation $\tau^{[-2,0]}$ of (51) gives canonically quasi-isomorphic complexes of locally free sheaves. That is we replace $B^{-2}$ and $B^0$ by the locally free sheaves $\text{coker } d_3$ and $\text{ker } d_0$ respectively. Dually we replace $B_2$ by $\text{ker } d_3$ and $B_0$ by $\text{coker } d_0$. The maps $\theta_i$ induce maps of complexes on the truncations. Thus, by applying $\tau^{[-2,0]}$ to (50), we may assume $B^{-i} = 0 = B_i$ for $i \notin [0, 2]$ in (51). Similarly replacing $g^\bullet$ by $\tau^{[-2,0]} g^\bullet$ we also keep the map to $\tau^{[-2,0]} L^\bullet_M = L^\bullet_M$. 

---

COUNTING SHEAVES ON CALABI-YAU 4-FOLDS, I 25
So we now have a commutative diagram in $D(M)$ like (50) intertwining $\theta$ with the map of complexes

\[
\begin{array}{ccc}
B^{-2} & \overset{d_2}{\longrightarrow} & B^{-1} \overset{d_1}{\longrightarrow} B^0 \\
\downarrow{\theta_2^*} & & \downarrow{\theta_1^*} = \theta_1^* \\
B_0 & \overset{d_2^*}{\longrightarrow} B_1 & \longrightarrow B_2.
\end{array}
\]

Since $\theta$ is a quasi-isomorphism the total complex of (52) is acyclic. In particular $d_2^* \oplus \theta_2$ is onto and

\[
E := \ker (B_1 \oplus B^0 \overset{d_2^* \oplus \theta_2}{\longrightarrow} B_2) \cong \coker (B^{-2} \overset{d_2^* \oplus \theta_2^*}{\longrightarrow} B^{-1} \oplus B_0) \cong E^*
\]
defines a bundle with orthogonal structure $E \cong E^*$, i.e. an $O(r, \mathbb{C})$ bundle.

Now the acyclicity of (52) is equivalent to the vertical map of complexes

\[
\begin{array}{ccc}
B^{-2} & \overset{d_2 \oplus \theta_2}{\longrightarrow} B^{-1} \oplus B_0 & \overset{d_1 \oplus 0}{\longrightarrow} B^0 \\
\downarrow{\theta_1 \oplus -d_1^*} & & \downarrow{\theta_2} \\
B_1 & \overset{d_2^*}{\longrightarrow} B_2
\end{array}
\]

being a quasi-isomorphism. Dividing by the injection $d_2 \oplus \theta_2^*$ shows the top row is just the complex $E \to B^0$, quasi-isomorphic to the bottom row $B_1 \to B_2$. Therefore the vertical quasi-isomorphism of horizontal complexes (52) factors through the vertical quasi-isomorphisms of complexes

\[
\begin{array}{ccc}
B^{-2} & \longrightarrow & B^{-1} \longrightarrow B^0 \\
\downarrow{g^*} & & \downarrow{g^*} =: E^* \cong \mathbb{E} \\
B_0 & \longrightarrow E & \longrightarrow B^0 \\
\downarrow{\eta^*} & \downarrow{\eta^*} & \downarrow{\eta^*} \\
B_0 & \longrightarrow B_1 & \longrightarrow B_2.
\end{array}
\]

By their construction the two arrows in $E^* := \{B_0 \to E \to B^0\}$ are dual to each other, so it is a self-dual complex. So setting $\alpha$ to be composition of the inverse of $\eta^* : B^* \overset{\sim}{\longrightarrow} E^*$ with the morphism $\phi \circ \psi^* : B^* \to \mathbb{E}$ of (50) gives the result claimed.

Finally, the map of complexes $g^*$,

\[
\begin{array}{ccc}
B^{-2} & \overset{g_1}{\longrightarrow} B^{-1} & \overset{g^0}{\longrightarrow} B^0 \\
\downarrow{d} & & \downarrow{\Omega_A|_M}
\end{array}
\]

\[
\begin{array}{ccc}
I/T^2 & \longrightarrow & \Omega_A|_M
\end{array}
\]
factors through $E^\bullet$ by

$$
\begin{array}{cccccc}
B^{-2} & \rightarrow & B^{-1} & \rightarrow & B^0 & \rightarrow & E^* \\
\theta_2 & \downarrow & (1,0) & \downarrow & B^0 & \rightarrow & \eta^* \\
B_0 & \rightarrow & B_0^1 & \rightarrow & B^0 & \rightarrow & E^* \\
g' \downarrow & \downarrow & g^0 & \downarrow & \downarrow & \downarrow & \downarrow \\
I/I^2 & \rightarrow & \Omega_A|_M & \rightarrow & L_M^\bullet.
\end{array}
$$

By adding copies of the acyclic self-dual complex

$O_M(N) \rightarrow O_M(N) \oplus O_M(-N) \rightarrow O_M(-N)$, $N \geq 0$,

to $E^\bullet$, mapping to $L_M^\bullet$, by sections of $I/I^2(N)$, we may assume that $E \rightarrow I/I^2$ is a surjection. Since $h^0(E^\bullet) \rightarrow h^0(L_M^\bullet)$ is also a surjection (in fact an isomorphism) this shows that $E^\bullet \rightarrow L_M^\bullet$ is a surjection in each degree. □

**Determinants.** We defined orientations for orthogonal bundles in Definition 2.1. To get a similar definition for $E^\bullet$ and $E^\bullet$ we first need to review determinants of complexes. Fix a bounded complex of bundles $A^\bullet$ over a quasi-projective scheme $Y$, with $A^j$ in degree $j$,

$$
A^\bullet = \ldots \rightarrow A^{2i-1} \rightarrow A^{2i} \rightarrow A^{2i+1} \rightarrow \ldots.
$$

Following the conventions of [KM] we define its determinant by

$$
\det A^\bullet := \Lambda^{\text{top}} \left[ \bigoplus_i A^{2i} \oplus \bigoplus_j (A^{2j+1})^* \right].
$$

In [KM] this is shown to extend to give a determinant functor, unique up to canonical isomorphism, from the category of perfect complexes (with morphisms quasi-isomorphisms) to the category of line bundles (with morphisms the isomorphisms) satisfying natural compatibilities. The key is to describe the right isomorphism $\det f: \det A^\bullet \xrightarrow{\sim} \det b^\bullet$ induced by a quasi-isomorphism $f: A^\bullet \rightarrow B^\bullet$ between complexes of bundles. We may write $f$ as a composition $b^\bullet \circ (a^\bullet)^{-1}$ or roof

(54)

where $a^\bullet$ and $b^\bullet$ are genuine chain maps of complexes which are also surjective and quasi-isomorphisms. By symmetry it is then enough to describe $\det a^\bullet: \det C^\bullet \xrightarrow{\sim} \det A^\bullet$. As a map of sheaves (in fact line bundles) it is sufficient to describe it locally. Here we may assume $a^\bullet$ is split by an injective map of complexes $A^\bullet \rightarrow C^\bullet$ so it becomes

(55)

$$
C^\bullet = K^\bullet \oplus A^\bullet \xrightarrow{(0,1)} A^\bullet,
$$

where $a^\bullet$ and $b^\bullet$ are genuine chain maps of complexes which are also surjective and quasi-isomorphisms.
where \( K^* := \ker a^* \) is acyclic. Since we are working locally we can write \( K^* \) as a sum of 2-term complexes of the form
\[
K[i] \xrightarrow{\sim} K[i - 1].
\]
Using (8) it is now sufficient to trivialise the determinant of this by
\[
\det(K[i] \xrightarrow{\sim} K[i - 1]) = \det(K \oplus K^* \xrightarrow{\sim} \mathcal{O}_Y), \quad k \wedge k^* \longrightarrow 1.
\]
Here, as in (8) and (10), our convention is that if \( \{k_i\}_{i=1}^n \) is a local basis of sections of \( K \) with dual basis \( \{k_i^*\}_{i=1}^n \) of \( K^* \) and \( k := k_1 \wedge \cdots \wedge k_n \in \det K \), then its dual is \( k^* = k_1^* \wedge \cdots \wedge k_n^* \in \det K^* \). The resulting trivialisation (56) appears in [KM, p32] for \( i \) odd and [KM] p33 for \( i \) even.

**Pairing.** Fix a complex of bundles \( A^* \) and set \( A := \oplus_i A^i \). We define
\[
(57) \quad p_A^* : \det A^* \otimes \det(A^*)^\vee \cong \det(A \oplus A^*) \longrightarrow \mathcal{O}_Y, \quad a \otimes a^* \longrightarrow 1.
\]
This definition reduces to (8) when \( A^* = A[0] \) is a bundle. Like (8) it privileges \( A^* \) over its dual \( (A^*)^\vee \): if we use (7) and \( p_{(A^*)^\vee} \) to produce another map \( \det A^* \otimes \det(A^*)^\vee \rightarrow \mathcal{O}_Y \) then it differs from (57) by the sign \((-1)^{\operatorname{rank} A^*}\).

The pairing (57) induces a pairing on the derived category of perfect complexes, so we can write \( \det A \otimes \det a^\vee \rightarrow \mathcal{O}_Y \) for any \( A \in D^b(\text{Perf } Y) \). Again the key point is to handle the roofs (54) locally, where we check that (54, 56) induce an isomorphism \( \det(A \oplus K \oplus (A \oplus K)^*) \rightarrow \det(A \oplus A^*) \) intertwining \( p_{A^* \oplus K^*} \) and \( p_A^* \). By repeated use of (6) we get
\[
\begin{array}{ccc}
\det A \otimes \det A^* \otimes \det K \otimes \det K^* & \xrightarrow{\text{(7)}} & \det A \otimes \det K \otimes \det K^* \otimes \det A^*\\
\text{det A \otimes \det A^*} & \xrightarrow{\text{p_A^*}} & \mathcal{O}_Y,
\end{array}
\]
which by inspection acts commutatively as follows,
\[
\begin{array}{ccc}
a \otimes a^* \otimes k \otimes k^* & \longrightarrow a \otimes k \otimes k^* \otimes a^* \\
a \otimes a^* & \longrightarrow 1.
\end{array}
\]

**Orientations.** The duality \( \theta : E \xrightarrow{\sim} E^\vee [2] \) of (17) induces an isomorphism
\[
(58) \quad (\det E)^{\otimes 2} \xrightarrow{1 \otimes \det \theta} \det E \otimes \det E^\vee \xrightarrow{p_E} \mathcal{O}_M
\]
on the moduli space \( M = M(X, c) \) of stable sheaves of Chern character \( c \) and virtual dimension \( \operatorname{vd} := \operatorname{rank} E \). By analogy with Definition 2.11 we define an orientation on \( E \) to be an isomorphism
\[
(59) \quad o : \mathcal{O}_M \xrightarrow{\sim} \det E \text{ such that } (58) \circ o^{\otimes 2} = (-1)^{\operatorname{vd} \operatorname{vd} - 1}. \tag{58}
\]
Using different sign conventions, Team Joyce [CGJ, Corollary 1.17] has constructed an orientation on \( E \), canonical up to sign. Throughout this paper
we arbitrarily fix one of these two orientations; choosing the other would simply multiply our virtual cycle by $-1$.

By Proposition 4.1 there exists an orthogonal bundle $(E, q)$ and a quasi-isomorphism

$$
E \cong E^* := \{T \to E \to T^*\}
$$

to a self-dual complex $E^*$ in degrees $-2, -1, 0$, intertwining $\theta$ with

$$
\theta_{E^*} := (\text{id}_T, q, \text{id}_{T^*}) : E^* \simto E^*[2].
$$

**Proposition 4.2.** The choice of orientation (59) defines a canonical orientation (13) on $pE, q$. Thus $E$ is an $SO_{r, C}$ bundle.

**Proof.** Let $Q$ be the obvious quadratic form on $T \oplus T^* \oplus E^*$ — the direct sum of the pairing on $T \oplus T^*$ with the inverse $q^{-1} : E^* \to E$ of the quadratic form $q : E \to E$. Comparing with (61) we find

$$
\det \theta_{E^*} = \det Q : \det(T \oplus T^* \oplus E^*) \longrightarrow \det(E \oplus T \oplus T^*).
$$

Therefore, under this identification, an orientation (59) on $E^*$ is the same as an orientation on $pT, q$. Now $T \oplus T^*$ has a canonical orientation — the one with respect to which $T$ is a positive maximal isotropic subbundle. By Definition 2.2 it is

$$
o_T := (-i)^m t_1 \wedge t_1^* \wedge \cdots \wedge t_m \wedge t_m^* = (-i)^m t \wedge t^*,
$$

where $\{t_i\}_{i=1}^m$ is a local basis of sections of $T$ with dual basis $\{t_i^*\}$ for $T^*$. With respect to

$$
\det E^* \cong \det(T \oplus T^*) \otimes \det E^*
$$

we can write any orientation on $E^*$ as $o_T \otimes o^*$, where $o^*$ is an orientation on $E^*$. Thus (59) endows $E^*$ with a canonical orientation $o^*$. Via $(\det q)^{-1} : \det E^* \to \det E$ this induces a canonical orientation $o'$ on $E$. □

Suppose we change the representative $E^*$ of $E$ by adding an acyclic complex $K^* = \{K \to K \oplus K^* \to K^*\}$,

$$
E^* \oplus K^* = \{T \oplus K \to E \oplus K \oplus K^* \to T^* \oplus K^*\},
$$

where the quadratic form on $E \oplus K \oplus K^*$ is the direct sum of $q$ and the pairing. Then the orientation on $E \oplus K \oplus K^*$ prescribed by Proposition 4.2 works out to be

$$
o' \otimes o_K \in \det E \otimes \det(K \oplus K^*),
$$

with $o_K$ the canonical orientation (63). That is, under the isomorphism (8) $\det(K \oplus K^*) \simto \mathcal{O}_M$ we use the trivialisation $(-i)^\text{rank} K$ rather than 1, just as in (13).
4.2. Virtual cycle. In an act of violence, we denote by $\tau E^\bullet$ the stupid truncation $\{E \to T^*\}$ of the complex $E^\bullet$. Composing with the obstruction theory (46),

\[(66) \quad \tau E^\bullet \longrightarrow E^\bullet \xrightarrow{\alpha} E \xrightarrow{\Lambda_t} L_M\]

defines a perfect obstruction theory $\tau E^\bullet \to L_M$ for $M$ since each map induces an isomorphism on $h^0$ and a surjection on $h^{-1}$. (Note that $\tau E^\bullet \to L_M$ is only a morphism in $D(M)$; we do not require it to be described via a map of complexes.)

Therefore by [BF, Section 4] we get a cone $C_{E^\bullet} \subset E^* \cong E$ by pulling back the intrinsic normal cone $C_M \subset h^1/h^0((\tau E^\bullet)^\vee) = [E^*/T]$,

\[(67) \quad C_{E^\bullet} := C_M \times_{[E^*/T]} E^* \subset E^* \cong E.\]

The Behrend-Fantechi virtual cycle would then be the Fulton-MacPherson intersection $0_E[C_{E^\bullet}] \in A_{vd}(M)$ of $C_{E^\bullet}$ with the zero section $0_E$, but our use of the truncation $\tau$ makes this the wrong thing to do — the result (in fact even its virtual dimension) is not a quasi-isomorphism invariant of $E^\bullet \to E$.

Instead the following result will allow us to replace $0_E[C_{E^\bullet}]$ by $0_E[C_{E^\bullet}]$, giving a cycle in $M$ of $\frac{1}{2}$ the virtual dimension.

**Proposition 4.3.** The cone $C_{E^\bullet} \subset E$ is isotropic.

*Proof.* Since this is a local statement it is sufficient to prove it on any étale neighbourhood $U$ of a point $[F] \in M$. The proof is quite lengthy; we first use [BBBJ] to get the nice local form (69) for the obstruction theory (46). Then we compare the resulting self-dual complex $Q^\bullet$ to our representative $E^\bullet$ from Proposition 4.1.

We work on a “minimal standard form neighbourhood” of $F \in \mathcal{M}^\text{der}$ provided by [BBBJ, Theorem 2.8]. This is a result about the structure of derived Artin stacks that gives a derived affine scheme $\mathcal{U}$ and a smooth morphism $\mathcal{U} \to \mathcal{M}^\text{der}$ of relative dimension $1 = \dim \text{Aut}(F)$ such that $L_{\mathcal{U}/\mathcal{M}^\text{der}} = \mathcal{L}$ is a line bundle on $\mathcal{U}$. So at the level of cdgas, $\mathcal{U} \to \mathcal{M}^\text{der}$ modifies only the degree 1 piece governing stabilisers.

If we ignore the derived structure, $\mathcal{U} \to \mathcal{M}^\text{der}$ becomes an atlas $\mathcal{U} \to \mathcal{M}$ for an open neighbourhood of $F \in \mathcal{M}$. Here $\mathcal{U}$ is the scheme underlying $\mathcal{U}$, and the composition $\mathcal{U} \to \mathcal{M} \to M$ is an étale neighbourhood of $[F] \in M$.

By [PTVV], $\mathcal{M}^\text{der}$ carries a $(-2)$-shifted symplectic structure. Its pull-back to $\mathcal{U}$ is no longer $(-2)$-shifted symplectic because we have removed $\mathcal{L}[-1]$ from $L_{\mathcal{M}^\text{der}}$ but not yet its shifted dual $\mathcal{L}^*[3]$. This is done by [BBBJ, Theorem 2.10(c)], replacing $\mathcal{U}$ by another affine derived scheme $\mathcal{U}'$ with the same underlying affine scheme $\mathcal{U}$ but with its cdga replaced by a sub-cdga such that

$$L_{\mathcal{U}/\mathcal{U}'} \cong L_{\mathcal{U}/\mathcal{M}^\text{der}}^\vee \cong \mathcal{L}^*[3],$$

\[\text{18} \text{In particular this does not use the shifted symplectic structure.}\]
by [BBBJ] Equation 2.14. After possibly shrinking the étale neighbourhood $U$ of $[F] \in M$ if necessary, [BBBJ] prove that the pullback to $\mathcal{U}$ of the $(-2)$-shifted symplectic structure on $\mathcal{M}^{der}$ descends to a $(-2)$-shifted symplectic structure on $\mathcal{U}'$ which can be written in the standard Darboux form of [BBJ] Theorem 5.18(ii) and Example 5.16; we describe this next.

Since $U$ is also the underlying scheme of $U_1$ it inherits an obstruction theory from the derived structure of $U_1$. This is the pullback of the obstruction theory $\mathbb{E} \to \mathbb{L}_M$ [40] to our étale neighbourhood $U \to M$ of $[F] \in M$ on which we will show the cone $C_E \subset E$ is isotropic. The Darboux form is described in terms of the tangent and obstruction spaces of $U_1$ at a point lying over $[F] \in M$, which are

$$h^0(\mathcal{L}_{\mathcal{M}^{der}}|_F) = \text{Ext}^1(F, F) \quad \text{and} \quad h^1(\mathcal{L}_{\mathcal{M}^{der}}|_F) = \text{Ext}^2(F, F).$$

Let $q$ denote the quadratic form on the latter given by Serre duality. Then by [BBBJ] Example 5.16 there is an open neighbourhood $V$ of $0 \in \text{Ext}^1(F, F)$ and an isotropic map

$$s : V \longrightarrow \text{Ext}^2(F, F) \quad \text{such that} \quad ds|_0 = 0, \quad q(s, s) = 0 \quad \text{and} \quad s^{-1}(0) \cong U$$

compatible with the obstruction theory $\mathcal{L}_{\mathcal{M}^{der}}|_U \to \mathcal{L}_U$ in the following sense. Let $Q \cong Q^*$ denote the trivial orthogonal bundle with fibre $\text{Ext}^2(F, F)$ over $V$, and think of $s$ as an isotropic section of $Q$. Then $\mathcal{L}_{\mathcal{M}^{der}}|_U \to \mathcal{L}_U$ is isomorphic to

$$Q^* := \{T_V|_U \xrightarrow{ds} Q \xrightarrow{(ds)^*} \Omega_V|_U\} \cong E|_U$$

Here $I$ is the ideal of $Z(s) \subset V$ generated by $s$. Finally, this isomorphism intertwines the Serre duality [47] with the obvious self-duality $Q^* \cong Q_*[2]$ [49] of the above complex.

Let $E^* \cong E$ be the self-dual complex given to us by Proposition 4.1. Since we are working locally, where morphisms in $D(M)$ can be lifted to homotopy classes of maps of complexes of locally free sheaves, the isomorphism [69] and dualities it intertwines can be represented by a map of complexes $f^*$ such that

$$Q^* \xrightarrow{69} Q_*[2] \quad \text{and} \quad E^*|_U \xrightarrow{69} E_*|_U[2]$$

commutes only up to homotopy.

On restriction to the point $[F] \in U$ the differentials $ds, (ds)^*$ in $Q^*$ vanish, so the three terms of $Q^*|_{[F]}$ equal its cohomology groups $\text{Ext}^i(F, F)$ and the
composition \( f_\ast \circ f^\ast \) is precisely \( \text{id}_{\mathcal{Q}^\ast} \). Thus the \( f^{-i} \) are injections and the \( f_{2-i} \) are surjections at \([F]\) — and therefore also over \( U \), after shrinking if necessary. In particular \( K_\ast := \ker f_\ast[2] \) is a 3-term acyclic complex of vector bundles \([K_{-2} \rightarrow K_{-1} \rightarrow K_0]\). By the definition of \( f_\ast \) it is orthogonal to \( f^\ast(Q^\ast) \) under the identification \( E^\ast \cong E_\ast[2] \), giving an orthogonal splitting
\[
E^\ast = f^\ast(Q^\ast) \oplus \ker (f_\ast[2]) \\
\cong Q^\ast \oplus K_\ast
\]
(70)
over \( U \). The pairing on \( E^\ast \) makes \( K_\ast \) isomorphic to its shifted dual via the composition
\[
K_\ast = \ker f_\ast[2] \hookrightarrow E_\ast[2] \cong E^\ast \twoheadrightarrow \text{coker} f^\ast = (K_\ast)^\vee[2],
\]
so \( K_\ast \) is a self-dual complex.

The quasi-isomorphism \( Q^\ast \cong \mathbb{E} \) over \( U \) gives a cone \( C_{Q^\ast} \subset Q \) by Behrend-Fantechi’s recipe (67). As in [Fu, Remark 5.1.1] it is the flat limit (in the Hilbert scheme of subschemes of the total space of \( E \)) of the graphs of \( ts \),
\[
(71) \quad C_{Q^\ast} = \lim_{t \to \mathcal{E}} \Gamma_{ts} \subset Q.
\]
Considering the quadratic form \( q \) on \( Q \) to be a function on its total space (quadratic on the fibres), it vanishes on \( \Gamma_{ts} \subset Q \) since \( ts \) is an isotropic section. Taking the limit as \( t \to \infty \) shows that \( C_{Q^\ast} \) is an isotropic cone.
(Notice the similarity to the proof of Lemma 3.1)

By the recipe (67) and the splitting (70) we deduce the Behrend-Fantechi cone \( C_{E^\ast} \subset E \cong E^\ast \) induced from the quasi-isomorphism \( E^\ast \cong \mathbb{E} \) is
\[
(72) \quad C_{E^\ast} \cong C_{Q^\ast} \oplus K^0 \subset Q \oplus K^1.
\]
Since \( K_\ast \) is self-dual and exact over \( U \),
\[
\text{im} (K^0 \hookrightarrow K^1) = \ker (K^1 \twoheadrightarrow K^2) \cong \ker (K_{-1} \twoheadrightarrow K_0)
\]
shows \( \text{im}(K^0) \) is a maximal isotropic in \( K^1 \). Therefore the cone (72) is isotropic over \( U \).

By Propositions 4.1 and 4.2, \( E \) is an \( SO(r, \mathbb{C}) \) bundle on \( M \), where \( r \) has the same parity as the virtual dimension \( \text{vd} \) (5). So Proposition 4.3 and Definition 3.3 finally allow us to define a virtual cycle for \( M \).

**Definition 4.4.** If \( \text{vd} \) is odd define \( [M]^{\text{vir}} = 0 \). If \( \text{vd} \) is even, define
\[
(73) \quad [M]^{\text{vir}} = \sqrt{0_E^1} [C_{E^\ast}] \in A_4^{0,\text{vd}}(M, \mathbb{Z}[\frac{1}{2}]).
\]

We need to check that this definition is independent of choices: given two self-dual complexes \( E^\ast \), \( F^\ast \) with quasi-isomorphisms to \( \mathbb{E} \) intertwining \( \theta \) as in Proposition 4.1, we would like to show that
\[
(74) \quad \sqrt{0_E^1} [C_{E^\ast}] = \sqrt{0_F^1} [C_{F^\ast}].
\]

By Propositions 4.1 and 4.2, \( E \) is an \( SO(r, \mathbb{C}) \) bundle on \( M \), where \( r \) has the same parity as the virtual dimension \( \text{vd} \) (5). So Proposition 4.3 and Definition 3.3 finally allow us to define a virtual cycle for \( M \).*
We may represent the quasi-isomorphism $E^\bullet \cong E \cong F^\bullet$ by a roof of maps of complexes

\[(75) \quad \xymatrix{ e^* & A^\bullet & f^* \ar[ll] \ar[rr] & & F^\bullet }\]

Replacing $A^\bullet$ by its truncation $\tau^{[-2,0]} A^\bullet$, we may assume it is also a 3-term complex of locally free sheaves. It will not, in general, be self-dual, however. It inherits two representatives $\theta_e^\bullet := e_*[2] \circ e^\bullet$ and $\theta_f^\bullet := f_*[2] \circ f^\bullet : A^\bullet \to A_*[2]$, both of which are self-dual:

$$\theta_e^\bullet \quad \text{and} \quad \theta_f^\bullet = (\theta_e^\bullet)^\vee [2].$$

From such data — a 3-term complex with a self-dual map to its dual — we defined in (53) a canonical self-dual complex. Applied to $(A_e^\bullet, \theta_e^\bullet)$ and $(A_f^\bullet, \theta_f^\bullet)$ we get self-dual complexes $A_e^\bullet, A_f^\bullet$ respectively. By (53),

\[\begin{align*}
A_e^\bullet &= \{ A_0 \to A_e \to A^0 \}, \quad \text{where} \quad A_e := \frac{A^{-1} \oplus A_0}{A^{-2}} \cong A_e^\bullet
\end{align*}\]

is naturally an orthogonal bundle. Note $A_f^\bullet$ takes the same form, but the arrows (and so the orthogonal bundle $A_f^\bullet$) induced by $\theta_f^\bullet$ will be different.

Though this construction is canonical — and applied to the self-dual complexes $E^\bullet, F^\bullet$ it returns them unmodified — it is not functorial due to the appearance of both covariant terms $A^{-t}$ and the contravariant term $A_0$. Thus the roof (75) does not induce maps amongst $A_e^\bullet, A_f^\bullet, E^\bullet, F^\bullet$. Nonetheless Proposition 4.5 below proves the virtual cycles made from $A_e^\bullet$ are the same. By symmetry this also equates the virtual cycles associated to $A_f^\bullet$ and $F^\bullet$. So finally to prove the desired equality (74) we must check the virtual cycles associated to $A_e^\bullet$ and $A_f^\bullet$ are the same.

This follows from a simple deformation argument. Since $\theta_e^\bullet$ and $\theta_f^\bullet$ describe the same morphism in $D^b(M)$, for each $t \in \mathbb{C}$ so does

$$\theta_t^\bullet := t \theta_e^\bullet + (1 - t) \theta_f^\bullet,$$

which is also self-dual. Applying (53) gives orthogonal bundles $A_t$ and self-dual complexes

\[\begin{align*}
A_t^\bullet &= \{ A_0 \to A_t \to A^0 \},
\end{align*}\]

all of which vary in a flat family over $\mathbb{C} \ni t$. Let $A$ denote the bundle on $M \times \mathbb{C}$ whose restriction to $M \times \{t\}$ is $A_t$. Truncating (77) gives a perfect relative obstruction theory [BP Section 7] $\{ A \to A^0 \}$ for $M \times \mathbb{C} \to \mathbb{C}$. The resulting absolute perfect obstruction theory has a local model just as in Proposition 4.3, given by allowing (68) to vary with $t \in \mathbb{C}$. That is, we replace the ambient space $V$ by $V \times \mathbb{C}$, and the section $s$ by $s_t$. This is isotropic in $A$ because $s_t$ is isotropic in $A_t$ for each fixed $t \in \mathbb{C}$.
Therefore we get an isotropic cone $C \subset A$ over $M \times C$ just as in (71).

We apply the square root Gysin map of Definition 3.3 defined via cosection localisation. By [KL1, Theorem 5.2] it satisfies

$$(78) \quad \sqrt{0_{A_t}^1[C_A^*]} = \iota_t^1 \left( \sqrt{0_A^1[C]} \right), \quad t \in \mathbb{C},$$

where $\iota_t$ is the inclusion of any point $\{t\} \hookrightarrow \mathbb{C}$. Hence it is independent of $t$ and setting $t = 0, 1$ shows

$$\sqrt{0_{A_t}^1[C_A^*]} = \sqrt{0_{A_e}^1[C_A^*]}.$$

So we can finally conclude the well-definedness (74) of our virtual cycle by proving the following.

**Proposition 4.5.** In the above notation, $\sqrt{0_E^1[C_{E^*}]} = \sqrt{0_{A_e}^1[C_A^*]}$.

**Proof.** Start with the map of complexes $A^* \to E^*$ (75). By choosing a sufficiently negative vector bundle $B$ and a surjection $B \to A_0$ and adding the acyclic self-dual complex

$$(79) \quad B \xrightarrow{(1,0)} B \oplus B^* \xrightarrow{(0,1)} B^*$$

to $E^*$ as in (64), we may assume that $A^0 \to E^0 = T^*$ is an injection. By (67) the addition of (79) replaces $C_{E^*} \subset E$ by

$$C_{E^*} \oplus B \subset E \oplus B \oplus B^*$$

to which we can apply Proposition 3.7. Since $\sqrt{0_{B \oplus B^*}^1[B]} = 0_B^1[B]$ is just the zero section by Lemma 3.5 the virtual cycle (73) is unaffected:

$$(80) \quad \sqrt{0_{E \oplus B \oplus B^*}^1[C_{E^*} \oplus B]} = \sqrt{0_E^1[C_{E^*}]}.$$

So without loss of generality $A^* \to E^*$ and its dual $E_\bullet[2] \to A_\bullet[2]$ take the form

$$A^{-2} \longrightarrow A^{-1} \longrightarrow A^0 \longrightarrow \cdots \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2.$$

Set $K := \ker(T \to A^0)$. Since $E^\bullet \to A_\bullet[2]$ is a quasi-isomorphism, the composition $K \hookrightarrow T \to E$ is also an injection, with isotropic image. We get

---

We endow $B \oplus B^*$ with its obvious quadratic form and canonical orientation $o_B$ (63) with respect to which $B \hookrightarrow B \oplus B^*$ is a positive maximal isotropic.
the commutative diagrams with exact columns

\[
\begin{array}{ccc}
K & \longrightarrow & K \\
\downarrow & & \downarrow \\
T & \longrightarrow & T^* \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & A^0 \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A_0 & \longrightarrow & A_e \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & A^0 \\
\end{array}
\]


Letting \( K^\perp \) denote the orthogonal to \( K \subset E \), by \((76)\) the diagrams become

\[
\begin{array}{ccc}
K & \longrightarrow & K \\
\downarrow & & \downarrow \\
T & \longrightarrow & E \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & (K^\perp)^* \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A_0 & \longrightarrow & A^0 \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & A_e \\
\end{array}
\]


Let \( T^* \) denote the complex \( A_0 \to (K^\perp)^* \to T^* \). The diagrams on the left and right describe quasi-isomorphisms \( E^\bullet \to T^\bullet \leftarrow A^*_e \) respectively, corresponding to the reduction \( K^\perp/K \leftarrow K^\perp \to E \) of \( E \) by \( K \).

Letting \( \tau E^\bullet \) denote the stupid truncation \( \{E \to T^*\} \) we get the perfect obstruction theory

\[
\tau E^\bullet \longrightarrow E^\bullet \longrightarrow E \xrightarrow{\text{At}} \mathbb{L}_M
\]

of \((66)\) and similarly for \( \tau A^*_e \) and \( \tau T^\bullet \). This removes the first columns of the diagrams \((81)\). We then dualise to find, by \((67)\), the exact sequences

\[
\begin{array}{ccc}
K^\perp & \longrightarrow & C_{E^\bullet} \\
\downarrow & & \downarrow \\
K & \longrightarrow & C_{A^*_e} \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
K^\perp & \longrightarrow & C_{T^\bullet} \\
\downarrow & & \downarrow \\
K & \longrightarrow & C_{T^*} \\
\end{array}
\]

The upshot is that \( C_{E^\bullet} \subset E \) lies in \( K^\perp \subset E \) and contains \( K \); quotienting gives \( C_{A^*_e} = C_{E^\bullet}/K \subset K^\perp/K \). Thus applying Proposition \((3.8)\) to \( [C_{E^\bullet}] \) gives

\[
(82) \quad \sqrt{0}_{E} \left[ C_{E^\bullet} \right] = \sqrt{0}_{A_e} \left[ C_{A^*_e} \right]. \quad \square
\]

**Deformation invariance.** Suppose \( X \) moves in a smooth projective family \( X \to C \) over a smooth connected curve \( C \). Then we can repeat our working relative to \( C \). We get a coarse moduli space \( \mathcal{M} \to C \) with a relative obstruction theory \( \text{At: } E \to \mathbb{L}_{\mathcal{M}/C} \) as in \((16)\). We can resolve it by a self-dual 3-term complex \( E^\bullet = \{ T \to E \to T^* \} \) over all of \( \mathcal{M} \). Thus by \([BF, \text{Section 7}]\) we get a cone \( C_{E^\bullet} \subset E^* \approx E \).

Choices of orientation on the fibres of \( \mathcal{M} \to C \) form a \( \mathbb{Z}/2 \)-local system on \( C \), essentially because the proof of their existence in \([CGJ]\) depends only
on the (locally trivial) topology of the fibres of $\mathcal{X} \to C$. Replacing $C$ by a $\mathbb{Z}/2$-cover we may assume this has trivial monodromy. Therefore a choice of orientation on one fibre induces one on all of them, thus orienting $E$ as in Proposition 4.2.

Now the key point is that the Darboux theorems \[BBBJ, BG, BBJ\] also work with parameters (in fact \[BG\] explicitly works over a base) allowing us to write $\mathcal{M}$ locally as the zeros of an isotropic section of an orthogonal bundle such that the derivative of this description recovers the relative obstruction theory. That is, Proposition 4.3 goes through in this family setting to show that $C_{E^*} \subset E$ is isotropic.

The argument \(78\) then produces a cycle $[\mathcal{M}]^{\text{vir}} \in A_*(\mathcal{M}, \mathbb{Z}[\frac{1}{2}])$ with the deformation invariance property that

\[
[M_i]^{\text{vir}} = i_*^![\mathcal{M}]^{\text{vir}}
\]

for any point $i_*: \{t\} \hookrightarrow C$.

### 4.3. Generalisations.

We sketch some extensions of the above theory.

**Fixed determinant.** In Definition 4.4 we allowed our Calabi-Yau 4-fold to have nonzero $H^i(O_X)$ for $i = 1, 2, 3$. However when $H^2(O_X) \neq 0$ it is easy to see that $[M]^{\text{vir}} = 0$. In this case we can restrict attention to sheaves of strictly positive rank $r > 0$ and define a kind of reduced virtual class on the subscheme $M_L \subset M$ of stable sheaves with determinant a fixed line bundle $L$ on $X$. By \[STV, Definition 3.1\] there is a derived determinant map to the derived Picard stack,

\[
\mathcal{M}_L^{\text{der}} \xrightarrow{R^{\text{det}}} R\text{Pic}(X).
\]

Letting $\mathcal{M}_L^{\text{der}}$ denote the fibre over $\{L\}$, it is a derived Deligne-Mumford stack with stabilisers $\mathbb{Z}/r\mathbb{Z}$ at every point. Étale locally, tensoring with $r$th roots of degree 0 line bundles splits $R^{\text{det}}$, giving a local product structure $\mathcal{M}_L^{\text{der}} \cong \mathcal{M}_L^{\text{der}} \times R\text{Pic}(X)$. By \[STV, Proposition 3.2\] this induces the splitting of derived cotangent complexes

\[
R\pi_\# R\mathcal{H}om(\mathcal{E}, \mathcal{E})[3] \cong R\pi_\# R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0[3] \oplus R\pi_\# O_{X \times \mathcal{M}_L^{\text{der}}},
\]

splitting the trace map

\[
\text{tr} : R\pi_\# R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \longrightarrow R\pi_\# O_{X \times \mathcal{M}_L^{\text{der}}},
\]

by $\frac{1}{r} \text{id}$. We expect that it is obvious to experts\footnote{Related results are explained in \[PTVV, Remarks 3.1, 3.2\] for $X$ a K3 surface, as Mauro Porta pointed out to us.} that the $(-2)$-shifted symplectic structure splits accordingly, thus endowing $\mathcal{M}_L^{\text{der}}$ — and thus the underlying Deligne-Mumford stack $\mathcal{M}_L$ and its coarse moduli scheme $M_L$ — with a $(-2)$-shifted symplectic structure. Consider the restriction map

\[
R\pi_\# R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0[3] = \mathbb{L}_{\mathcal{M}_L^{\text{der}}} \longrightarrow \mathbb{L}_{\mathcal{M}_L}.
\]
Since the latter is the pullback of $\mathbb{L}_M$ this gives an amplitude $[-2, 0]$ obstruction theory

\[(84) \quad \mathbb{E} := R\pi_* R\text{Hom}(\mathcal{E}, \mathcal{E})_0[3] \longrightarrow \mathbb{L}_M,\]

replacing (46). Here $\mathcal{E}$ is again any choice of twisted universal sheaf on $X \times M_L$. The shifted symplectic isomorphism $\mathbb{L}^{\text{der}}_{M_L} \cong \mathbb{L}^{\text{der}}_{M_L}[2]$ becomes the relative Serre duality $\theta: \mathbb{E} \rightarrow \mathbb{E}^\vee[2]$ on (84). Thus we have all the same ingredients as before. Resolving $\mathbb{E}$ by a self-dual 3-term locally free complex $E^\bullet$ we get an isotropic cone $C_{E^\bullet} \subset E$, and a virtual cycle

\[(85) \quad [M_L]^\text{vir} := \sqrt{0}_E^1[C_{E^\bullet}] \in A_{\frac{2}{vd}} h^1(\mathcal{O}_X) + \frac{1}{2} h^2(\mathcal{O}_X)(M_L, \mathbb{Z}[\frac{1}{2}]).\]

Deformation invariance of this class holds only for families $X \rightarrow C$ over which the line bundle $L$ (on the central fibre $X_0 = X$) extends to a line bundle $L$ over all of $X$. This allows us to form a family moduli space of fixed determinant sheaves and repeat the working relative to $C$ to give the deformation invariance (83).

**Quasi-projective Calabi-Yau 4-folds.** Though we have worked with projective $X$ out of habit, all of our results — except deformation invariance — work for moduli spaces of *compactly supported* stable sheaves on a quasi-projective Calabi-Yau 4-fold $X$. The paper [BD] extends the $(-2)$-shifted symplectic structure of [PTVV] to this setting, and the obstruction theory $\mathbb{E} \rightarrow \mathbb{L}_M$ is identical to the projective case. Therefore [BBBJ] applies to the stack of stable compactly supported sheaves and we can deduce Proposition 4.3 as before: the cone $C_{E^\bullet} \subset E$ is isotropic. The orientation result [CGJ] on $\mathbb{E}$ is extended to the quasi-projective case in [Bo]. Thus we have all the same ingredients to produce the virtual cycle in this setting.

\((-2)\)-shifted symplectic schemes.\) Though we have not worked in such generality, our results apply to any quasi-projective scheme $M$ with a $(-2)$-shifted symplectic structure (in the sense of [PTVV]) and an orientation. Quasi-projectivity ensures $M$ has enough locally free sheaves to allow us to produce a self-dual 3-term locally free resolution of the virtual cotangent bundle. Then the key point is that the papers [BG, BBBJ] only use the $(-2)$-shifted symplectic structure, so Proposition 4.3 works in this generality, producing an isotropic cone $C_{E^\bullet} \subset E$ to which we can apply $\sqrt{0}_E^1$ to define the virtual cycle.

**Theorem 4.6.** Let $M$ be a quasi-projective $(-2)$-shifted symplectic derived scheme with an orientation in the sense of (59). If $vd$ is odd, set $[M]^\text{vir} = 0$. If $vd$ is even then

\[[M]^\text{vir} := \sqrt{0}_E^1[C_{E^\bullet}] \in A_{\frac{2}{vd}} h^1(\mathcal{O}_X) + \frac{1}{2} h^2(\mathcal{O}_X)(M_L, \mathbb{Z}[\frac{1}{2}])\]

is independent of the choice of self-dual resolution $E^\bullet$.

Moreover if $M$ is projective then $[M]^\text{vir}$ is deformation invariant in the sense that (83) holds.
In particular we can get virtual cycles on projective moduli spaces of simple complexes in the derived category $D(X)$ of sheaves on any projective Calabi-Yau 4-fold. This includes the moduli space $P_n(X, \beta)$ of stable pairs $(F, s)$ of [PT], where

- $F$ is a pure 1-dimensional sheaf on $X$,
- $F$ has curve class $[F] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(F) = n$,
- $s \in H^0(F)$ has 0-dimensional cokernel.

The same applies to the moduli space $J_p(X, c)$ of Joyce-Song pairs $(F, s)$ of [JS], where

- $F$ is a semistable sheaf on $X$ of Chern character $c$,
- $n \gg 0$ is sufficiently large that $H^{\geq 1}(F(n)) = 0$ for all semistable $F$,
- $s \in H^0(F(n))$ factors through no semi-destabilising subsheaf of $F$.

Both moduli spaces $P_n(X, \beta), J_p(X, c)$ are naturally projective schemes with obstruction theory governed by the self-dual 3-term complex

$$R \text{Hom}_X(I^\bullet, I^\bullet)_0,$$

where $I^\bullet \in D(X)$ denotes the complexes $\{\mathcal{O}_X \to F\}$ and $\{\mathcal{O}_X(-n) \to F\}$ respectively. By [CGJ] $P_n(X, \beta), J(X, c)$ admit orientations. Choosing one, our theory endows them with virtual cycles.

**Theorem 4.7.** If $\chi(c, c) \in 2\mathbb{Z}$ we have algebraic virtual cycles

$$[P_n(X, \beta)]^{\text{vir}} \in A_n(P_n(X, \beta), \mathbb{Z}[\frac{1}{2}]),$$

$$[J(X, c)]^{\text{vir}} \in A_{\chi(c(n)) - \frac{1}{2} \chi(c, c)}(J(X, c), \mathbb{Z}[\frac{1}{2}])$$

which, by the sequel [OT2], map to the integral classes of [BJ] in homology. They are deformation invariant in the sense of (83).

The first class puts the Calabi-Yau 4-fold conjectures of [CMT2, CT1, CT2, CK2, CKM1, CKM2] on a firmer footing. When stability = semistability for sheaves $F$ of Chern character $c$, the Joyce-Song moduli space $J(X, c)$ is a projective bundle over $M = M(X, c)$ with fibre $\mathbb{P}^{\chi(F(n)) - 1}$, so we expect a simple relationship between $[J(X, c)]^{\text{vir}}$ and $[M]^{\text{vir}}$, just as in the Calabi-Yau 3-fold case. More generally — again by analogy with the Calabi-Yau 3-fold case — one could hope for a universal wall-crossing formula like [JS Theorem 5.27]. This would allow the definition of invariants valued in $\mathbb{Q}$ counting strictly semistable sheaves $F$ by using virtual counts (valued in $\mathbb{Z}$) of the Joyce-Song pairs $I^\bullet$. This will be addressed in forthcoming work of Dominic Joyce.

5. $K$-theoretic virtual class

Again let $M$ be a quasi-projective $(-2)$-shifted symplectic derived scheme with an orientation. As usual we have in mind the moduli space of compactly supported stable sheaves — or stable pairs or Joyce-Song pairs — of fixed

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21 The section rigidifies the pairs $(F, s)$; there are no semistables and no automorphisms.
total Chern character $c$ on a smooth quasi-projective 4-fold $X$ with $K_X \cong \mathcal{O}_X$. In this Section we will endow $M$ with a virtual structure sheaf with good properties.

For this we will need a $K$-theoretic analogue of the Edidin-Graham class of an $SO(2n, \mathbb{C})$ bundle $(E, q, o)$. There are candidate classes in the literature in various special cases \cite{An, CLL, Ch, EG1, KO, OS, PV}. We find a class which works in maximal generality — with no assumption of Zariski local triviality, the existence of a maximal isotropic subbundle, or the existence of a spin bundle — and which uses Definition 2.2 to fix the sign ambiguity present in many of these classes. To do so we find only that we need to invert 2 in the coefficients of $K$-theory.

We make crucial use of Anderson’s class \cite[Appendix B]{An}. We define a variant $?_E$ of it with the desirable property that $p_1 ?_E =$ is the $K$-theoretic Euler class; see \eqref{eq:K-theoretic-Euler-class}. Our class and his are supported in codimension $n$, agree in codimension $n - 1$, but differ in codimension $n - 2$; see \eqref{eq:Anderson-Edidin-Graham-class}

5.1. $K$-theoretic Edidin-Graham class. On a quasi-projective scheme $Y$, let $K^0(Y)$ and $K_0(Y)$ denote the Grothendieck groups of vector bundles and coherent sheaves respectively. The former is a ring under (derived) tensor product, and the latter is a module over it. On inverting 2 we will find that any line bundle has a distinguished square root in $K^0_0(Y)$, set $a_i := 2^{1 - 2i}C_i - 1 \in \mathbb{Z}[\frac{1}{2}]$.

This is minus the $i$th Taylor coefficient of

\[
(1 - x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \ldots = 1 - \sum_{i \geq 1} a_i x^i \in \mathbb{Z}[\frac{1}{2}][[x]].
\]

Squaring gives the identity

\[
\left(1 - \sum_{i = 1}^{n - 1} a_i x^i\right)^2 = (1 - x) + 2a_n x^n + O(x^{n + 1}),
\]

where $O(x^{n + 1})$ denotes the product of $x^{n + 1}$ and a polynomial in $x$.

For $i > 0$ let $K^0(Y)^i \subset K^0(Y)$ denote the subring of elements which may be written as a formal difference of sheaves supported in codimension $i$. Since $K^0(Y)^i \cdot K^0(Y)^j \subset K^0(Y)^{i+j}$ this subring is nilpotent.

Lemma 5.1. For $E \in 1 + K^0(Y)^1$, the finite sum

\[
\sqrt{E} := 1 - \sum_{i \geq 1} a_i (1 - E)^{\otimes i} \in K^0(Y, \mathbb{Z}[\frac{1}{2}])
\]

is the unique class in $1 + K^0(Y)^1$ which squares to $E$. In particular, $\sqrt{EF} = \sqrt{E} \sqrt{F}$.

\footnote{Anderson works in connective $K$-theory \cite[Appendix A]{An}; we specialise his Bott class $\beta$ to $-1$ to work in $K^0$. He also restricts to Zariski locally trivial $SO(2n, \mathbb{C})$ bundles.}
Proof. That \((\sqrt{E})^2 = E\) is trivial from \((88)\) for \(n > \text{dim} \, Y\). For the uniqueness statement, suppose \(M \in 1 + K^0(Y)^1\) is another class with \(M^2 = E\). Working in \(K^0(Y, \mathbb{Z}[\frac{1}{2}])\) at all times, define \(M_n\) by
\[
(88) \quad M = 1 - \sum_{i=1}^{n-1} a_i(1 - E)^{\otimes i} + M_n.
\]
Assume inductively that \(\text{codim} \, M_n \geq n\). (The base case \(n = 1\) is trivial.) Squaring gives
\[
M^{\otimes 2} = E = \left(1 - \sum_{i=1}^{n-1} a_i(1 - E)^{\otimes i}\right)^{\otimes 2} + 2M_n + O(n+1)
\]
\[
\overset{(88)}{=} E + 2a_n(1 - E)^{\otimes n} + 2M_n + O(n+1),
\]
where \(O(n+1)\) means a class supported in codimension at least \(n+1\). Thus
\[
M_n = -a_n(1 - E)^{\otimes n} + O(n+1),
\]
which by \((88)\) shows \(\text{codim} \, M_{n+1} \geq n + 1\). For \(n \geq \text{dim} \, Y\) this shows that \(M_{n+1} = 0\) and \(M = \sqrt{E}\). \(\square\)

Remark 5.2. In particular any line bundle \(L\) has a canonical square root \(\sqrt{L} \in K^0(Y, \mathbb{Z}[\frac{1}{2}])\),

since \(L = 1 + (L - 1) \in 1 + K^0(Y)^1\). It is unique in \(1 + K^0(Y)^1\), so if \(M\) is any line bundle on \(Y\) such that \(M^{\otimes 2} = L\) then \(M = \sqrt{L}\) in \(K^0(Y, \mathbb{Z}[\frac{1}{2}])\).

The \(K\)-theory of complex vector bundles is an “oriented cohomology theory” — it admits a notion of Chern classes. The \(K\)-theoretic first Chern class of a line bundle \(L\) is
\[
(89) \quad c_1(L) := 1 - L^* \in K^0(Y)^1,
\]
which is the class \([\mathcal{O}_D]\) of the structure sheaf of any divisor \(D \in |L|\). The splitting principle and Whitney sum formula for bundles
\[
(90) \quad c(E_1 \oplus E_2, t) = c(E_1, t)c(E_2, t), \quad \text{where } c(E, t) = 1 + \sum_{i=1}^{\text{rank} \, E} t^i c_i(E),
\]
then determine the Chern classes of any object of \(K^0(Y)\). In particular the \(r\)th Chern class \(c_r(E)\) (or Euler class \(c(E)\)) of a rank \(r\) bundle \(E\) is
\[
K^0(Y)^r \ni c_r(E) := \Lambda^r E^* := \sum_{i=0}^{r} (-1)^i \Lambda^i E^*
\]
\[
(91) \quad = 0_r^* 0_{E^*}[\mathcal{O}_Y] = 0_r^* [\mathcal{O}_{\Gamma_s}],
\]
where \(0_E : Y \hookrightarrow E\) is the 0-section, \(0_r^*\) is the derived pullback, and \(\Gamma_s \subset E\) is the graph of any section \(s \in \Gamma(E)\). If \(s\) is transverse to \(0_E\) with zero locus \(Z\) then this gives \(c_r(E) = [\mathcal{O}_Z]\). We also have
\[
(92) \quad c_r(E^*) := \Lambda^r E = (-1)^{\text{rank} \, E} c_r(E) \cdot \det E.
\]
Special case. We first define our $K$-theoretic Edidin-Graham class in the presence of a maximal isotropic subbundle.

**Definition 5.3.** Let $(E, q, o)$ be an $SO(2n, \mathbb{C})$ bundle admitting a maximal isotropic subbundle $\Lambda \subset E$. We define

\begin{equation}
\sqrt{\varepsilon}(E) := (-1)^{|\Lambda|} c_n(\Lambda) \cdot \sqrt{\det \Lambda} \in K^0(Y, \mathbb{Z}[\frac{1}{2}]).
\end{equation}

Note that by (92),

\begin{equation}
(-1)^{n+|\Lambda|} c_n(\Lambda^*) \cdot \sqrt{\det \Lambda^*} = (-1)^{|\Lambda|} c_n(\Lambda) \cdot \sqrt{\det \Lambda} = \sqrt{\varepsilon}(E).
\end{equation}

So if $E = \Lambda \oplus \Lambda^*$ and we choose $\Lambda^*$ as our maximal isotropic, of sign $(-1)^{|\Lambda|+n}$, we get the same class. More generally we have the following.

**Proposition 5.4.** Definition 5.3 is well-defined: $\sqrt{\varepsilon}(E)$ (93) is independent of the choice of $\Lambda \subset E$ and is really a $K$-theoretic square root Euler class:

\begin{equation}
\sqrt{\varepsilon}(E)^2 = (-1)^n c_{2n}(E).
\end{equation}

**Proof.** The exact sequence $0 \to \Lambda \to E \to \Lambda^* \to 0$ and (94) give

\[ \sqrt{\varepsilon}(E)^2 = (-1)^n c_n(\Lambda) c_n(\Lambda^*) = (-1)^n \varepsilon(E). \]

As noted in Footnote 7 the existence of $\Lambda \subset E$ means $(E, q, o)$ is Zariski-locally trivial, so Anderson’s $K$-theoretic Edidin-Graham class $\varepsilon(E)$ is defined [An, Appendix B]. To show $\sqrt{\varepsilon}(E)$ is well-defined we relate it to $\varepsilon(E)$, using some identities we learned from [An] Appendix B.

From (89) it is immediate that for any line bundle $L$,

\[ c(L, -\frac{1}{2}) \otimes L = c(L^*, -\frac{1}{2}). \]

By the splitting principle and Whitney sum formula (90) this gives

\[ c(\Lambda, -\frac{1}{2}) \otimes \det \Lambda = c(\Lambda^*, -\frac{1}{2}). \]

Combined with the exact sequence $0 \to \Lambda \to E \to \Lambda^* \to 0$ this gives

\[ c(E, -\frac{1}{2}) \otimes \det \Lambda = c(\Lambda, -\frac{1}{2}) c(\Lambda^*, -\frac{1}{2}) \otimes \det \Lambda = c(\Lambda^*, -\frac{1}{2})^2. \]

Since both lie in $1 + K^0(Y, \mathbb{Z}[\frac{1}{2}])^1$ we deduce that

\[ \sqrt{c(E, -\frac{1}{2}) \sqrt{\det \Lambda} = c(\Lambda^*, -\frac{1}{2}). \]

Now Anderson’s class is $\varepsilon(E) := (-1)^{|\Lambda|} c(\Lambda^*, -\frac{1}{2}) c_n(\Lambda)$ and is independent of $\Lambda$ by [An] Appendix B, Theorem 3. Therefore

\begin{equation}
\sqrt{\varepsilon}(E) = \sqrt{c(E, -\frac{1}{2})^{-1}} \varepsilon(E)
\end{equation}

is also independent of $\Lambda$. \qed
General case. Fix any étale locally trivial $SO(2n, \mathbb{C})$ bundle $(E, q, o)$. We work, as usual, on the bundle $\rho: \tilde{Y} \to Y$ of \cite[139]{19}, where $\rho^*E$ admits a canonical positive maximal isotropic $\Lambda_\rho \subset \rho^*E$. Thus $\sqrt{\epsilon}(\rho^*E)$ is defined by Proposition \ref{5.4}. We show it is the pullback by $\rho^*$ of a class we will define to be $\sqrt{\epsilon}(E)$.

**Proposition 5.5.** $\sqrt{\epsilon}(\rho^*E) = \rho^*\rho_*(\sqrt{\epsilon}(\rho^*E))$. Furthermore, if $E$ admits a maximal isotropic then $\sqrt{\epsilon}(E)$ defined by \eqref{93} equals $\rho_*\sqrt{\epsilon}(\rho^*E)$.

**Proof.** We use the Cartesian diagram of flat maps

$$
\begin{array}{ccc}
\rho_1 \times_Y \tilde{Y} & \overset{\rho_2}{\longrightarrow} & \tilde{Y} \\
\rho_1 \downarrow \quad & & \downarrow \\
\tilde{Y} & \overset{r_2}{\longrightarrow} & Y,
\end{array}
$$

where $r_1 = \rho = r_2$. Let $P: \tilde{Y} \times_Y \tilde{Y} \to Y$ be $r_2 \circ \rho_1 = r_1 \circ \rho_2$. Then $P^*E$ has two positive maximal isotropic subbundles $\rho_1^*\Lambda_\rho$ and $\rho_2^*\Lambda_\rho$, so by Proposition \ref{5.3}

$$c_n(\rho_1^*\Lambda_\rho) \cdot \sqrt{\det(\rho_1^*\Lambda_\rho)} = c_n(\rho_2^*\Lambda_\rho) \cdot \sqrt{\det(\rho_2^*\Lambda_\rho)}$$

in $K^0(\tilde{Y} \times_Y \tilde{Y}, \mathbb{Z}[\frac{1}{2}])$. Equivalently,

$$\rho_1^*(\sqrt{\epsilon}(\rho^*E)) = \rho_2^*(\sqrt{\epsilon}(\rho^*E)). \tag{97}$$

Since $\rho_1$, like $\rho$, is an iterated bundle of smooth quadrics, we have $\rho_{1*}\rho_1^*O_{\tilde{Y}} = O_{\tilde{Y}}$ in the derived category and so also in $K$-theory. (As usual we use derived pushforward.) Therefore by the projection formula, \eqref{97} and flat basechange,

$$\sqrt{\epsilon}(\rho^*E) = \rho_{1*}\rho_1^*(\sqrt{\epsilon}(\rho^*E)) = \rho_{1*}\rho_2^*(\sqrt{\epsilon}(\rho^*E)) = r_2^*r_1^*(\sqrt{\epsilon}(\rho^*E)).$$

That is, $\sqrt{\epsilon}(\rho^*E) = \rho^*\rho_*(\sqrt{\epsilon}(\rho^*E))$, as required.

If $\Lambda \subset E$ is a maximal isotropic subbundle then by Proposition \ref{5.4} the class $\sqrt{\epsilon}(\rho^*E)$ — defined by $\Lambda_\rho$ — is the same as the class \eqref{93} defined by $\rho^*\Lambda$. That is, $\sqrt{\epsilon}(\rho^*E) = \rho^*\sqrt{\epsilon}(E)$. Applying the projection formula gives $\rho_*\sqrt{\epsilon}(\rho^*E) = \sqrt{\epsilon}(E)$.

It follows that the following is well-defined, and gives the same as Definition \ref{5.3} when $E$ admits a maximal isotropic.

**Definition 5.6.** For an $SO(2n, \mathbb{C})$ bundle $(E, q, o)$ define

$$\sqrt{\epsilon}(E) := \rho_*(\sqrt{\epsilon}(\rho^*E)) \in K^0(Y, \mathbb{Z}[\frac{1}{2}]).$$

By \eqref{95} this satisfies $\sqrt{\epsilon}(E)^2 = (-1)^n\epsilon_{2n}(E)$. 

We note the curious contrast with the cohomological Edidin-Graham class, where we had to invert 2 in order to descend from \( \tilde{Y} \) to \( Y \). In \( K \)-theory the descent works over the integers, but we had to invert 2 earlier to define the class \( \langle 93 \rangle \) even in the presence of a maximal isotropic subbundle.

5.2. **Localisation by an isotropic section.** Fix an isotropic section \( s \) of our \( SO(2n, \mathbb{C}) \)-bundle \( (E, q, o) \), with zero scheme \( i: Z(s) \hookrightarrow Y \). We will construct a *localised* \( K \)-theoretic square root Euler class

\[
\sqrt{\mathcal{E}}(E, s) : K_0(Y, \mathbb{Z}[\frac{1}{2}]) \longrightarrow K_0(Z(s), \mathbb{Z}[\frac{1}{2}])
\]

whose pushforward \( i_* \circ \sqrt{\mathcal{E}}(E, s) \) is tensor product with \( \sqrt{\mathcal{E}}(E) \in K^0(Y, \mathbb{Z}[\frac{1}{2}]) \).

**Special case.** To begin with we suppose that \( E \) admits a maximal isotropic \( \Lambda \subset E \). We follow closely what we did in Chow in Section 3.2 adapting to \( K_0 \). Via the exact sequence

\[
0 \longrightarrow \Lambda \longrightarrow E \overset{\pi}{\longrightarrow} \Lambda^* \longrightarrow 0
\]

we set \( s^* := \pi(s) \in \Gamma(\Lambda^*) \). Then by \( \langle 91 \rangle \) we have

\[
c_n(\Lambda^*) = 0^\Lambda \cdot \mathcal{O}_{\Gamma_{s^*}} \in K^0(Y).
\]

By deformation to the normal cone we can deform the graph \( \Gamma_{s^*} \subset \Lambda^* \) to its linearisation \( C_{Z^*/Y} \subset \Lambda^*|_{Z^*} \) about the zero locus \( j: Z^* \hookrightarrow Y \) of \( s^* \). Using the \( K \)-theoretic specialisation map \( K_0(Y) \rightarrow K_0(C_{Z^*/Y}) \) [Fu, p 352] we get the localisation

\[ K_0(Y) \longrightarrow K_0(C_{Z^*/Y}) \overset{0^\Lambda|_{Z^*}}{\longrightarrow} K_0(Z^*) , \]

whose pushforward to \( K_0(Y) \) is \( c_n(\Lambda^*) \otimes \).

To localise further to \( Z(s) \subset Z^* \) by using the "other half" of the section \( s \), we use Kiem-Li’s \( K \)-theoretic cosection localisation [KL2]. By Lemma 5.1 the cosection \( \tilde{s} \) (28) defined by \( s \) is zero on \( C_{Z^*/Y} \subset \Lambda^*|_{Z^*} \), so [KL2, Theorem 4.1] defines the arrow across the top of the commutative diagram

\[
\begin{array}{ccc}
K_0(Y) & \longrightarrow & K_0(C_{Z^*/Y}) \overset{0^\Lambda|_{Z^*}}{\longrightarrow} K_0(Z(s)) \\
\downarrow & & \downarrow \\
K_0(\Lambda^*|_{Z^*}) & \overset{0^\Lambda}{\longrightarrow} & K_0(Z^*)
\end{array}
\]

in which the vertical maps are the obvious pushforwards. Composing the first two arrows on the top row defines an operator

\[
c_n(\Lambda^*, s) : K_0(Y) \longrightarrow K_0(Z(s))
\]

such that \( i_* \circ c_n(\Lambda^*, s) \) is tensor product with \( c_n(\Lambda^*) \in K^0(Y) \). Thus we get a localised \( K \)-theoretic Edidin-Graham operator

\[
\sqrt{\mathcal{E}}(E, s, \Lambda) := (-1)^{n+|\Lambda|} c_n(\Lambda^*, s) \cdot \sqrt{\det \Lambda^*}
\]
from $K_0(Y, Z[\frac{1}{2}]) \to K_0(Z(s), Z[\frac{1}{2}])$ such that $i_* \circ \sqrt{\epsilon}(E, s, \Lambda)$ is tensor product with $\sqrt{\epsilon}(E) \in K^0(Y, Z[\frac{1}{2}])$.

**General case.** We can now define the localised $K$-theoretic Edidin-Graham class in general by using the cover $\rho: \tilde{Y} \to Y$ (19).

**Definition 5.7.** Given an isotropic section $s \in \Gamma(E)$ of an $SO(2n, \mathbb{C})$ bundle $(E, q, o)$ we define the localised operator

$$\sqrt{\epsilon}(E, s) := \rho_*(\sqrt{\epsilon}(\rho^*E, \rho^*s, \Lambda_\rho)) : K_0(Y, Z[\frac{1}{2}]) \to K_0(Z(s), Z[\frac{1}{2}]).$$

By construction its pushforward to $Y$ is tensor product with $\rho_*\sqrt{\epsilon}(\rho^*E) \in K^0(Y)$, so by Proposition 5.3 and Definition 5.6 it follows that

$$i_* \circ \sqrt{\epsilon}(E, s) = \sqrt{\epsilon}(E) \otimes : K_0(Y, Z[\frac{1}{2}]) \to K_0(Y, Z[\frac{1}{2}]).$$

This allows us to define a $K$-theoretic square root version of the intersection between an isotropic cone $C \subset E$ (supported over $Z \subset Y$) and the zero section $0: Y \to E$. As usual we let $\tau_E$ denote the tautological (isotropic) section of $\pi^*E$ on $\pi: C \to Y$. Its zero locus is $Z$.

**Definition 5.8.** Given an isotropic cone $C \subset E$ we define

$$\sqrt{\epsilon}_E := \sqrt{\epsilon}(\pi^*E, \tau_E) : K_0(C, Z[\frac{1}{2}]) \to K_0(Z, Z[\frac{1}{2}]).$$

There are obvious $K$-theoretic versions of Lemma 3.4

$$\sqrt{0^*_E} = \pi_*(\sqrt{\epsilon}(E) \otimes (\cdot)),$$

and — when $C$ lies in a maximal isotropic subbundle $\Lambda \subset E$ — Lemma 3.5

$$\sqrt{0^*_E} = (-1)^{\frac{|\Lambda|}{2}} \sqrt{\det \Lambda \cdot 0^*_\Lambda}.$$

The proofs are almost identical on replacing each $e \otimes (\cdot)$ by $e \otimes (\cdot)$, equation (10) by $0 \cdot \frac{0^*E}{\pi_*}$, and using (100) in place of (33).

Similarly the proof of Lemma 3.9 goes through without change to give

$$f^!\sqrt{0^*_E} = \sqrt{0^*f^!E} f^! : K_0(C, Z[\frac{1}{2}]) \to K_0(X', Z[\frac{1}{2}]).$$

Here $f^!$ is the $K$-theoretic refined Gysin map of [AP Section 3] and [Lee Section 2.1]. By construction this commutes with tensor product and product with $K$-theoretic Chern classes. It also commutes with proper pushforward by [AP Lemma 3.1], so the result $\pi_*f^! = f^!\pi_*$ used in the proof of Lemma 3.9 also holds in $K$-theory.

5.3. **Virtual structure sheaf.** Let $M$ be a quasi-projective $(-2)$-shifted symplectic derived scheme with an orientation. By Proposition 4.4 its obstruction theory admits a self-dual resolution $E^* = \{ T \to E \to T^* \} \to \mathbb{L}_M$. By Proposition 4.2 $E$ is an $SO(r, \mathbb{C})$ bundle, where $r$ has the same parity as the virtual dimension $vd(M)$. If this is odd we define the virtual structure sheaf to be zero. Suppose now it is even. By (67) and Proposition 4.3 we get an isotropic cone $C_{E^*} \subset E$. 
Definition 5.9. We define the twisted virtual structure sheaf of $M$ to be
\begin{equation}
\hat{\mathcal{O}}_M^{\text{vir}} := \sqrt{\mathcal{O}_{C_{E^*}}} \cdot \sqrt{\det T^*} \in K_0(M, \mathbb{Z}[\frac{1}{2}]).
\end{equation}

Proposition 5.10 below shows the twist by $\sqrt{\det T^*}$ ensures independence from the choice of self-dual complex $E^*$. (Since the other terms in (103) involve only the stupid truncation $\{E \to T^*\}$ of $E^*$, we should expect some contribution from the omitted term $T$, and this turns out to be it.)

In special situations — one being the local Calabi-Yau 4-fold case of Section 8 — $E$ admits a maximal isotropic subbundle $\Lambda \subset E$ such that $A_t : E^* \to \mathbb{L}_M$ factors through the following quotient of $E^*$,
\begin{equation}
\frac{1}{2}E^* := \{\Lambda^* \to T^*\}.
\end{equation}

That is, in this situation $\{T \to \Lambda\}$ can be though of as a virtual tangent bundle for $M$; taking determinants gives
\begin{equation}
K_M^{\text{vir}} := \det T^* \otimes \det \Lambda.
\end{equation}

Then $C_{E^*}$ lies in $\Lambda \subset E$ and, by (101), the class (103) can be written in the perhaps more suggestive form
\begin{equation}
(-1)^{|\Lambda|} \mathcal{O}_{C_{E^*}} \cdot \sqrt{K_M^{\text{vir}}}
\end{equation}
in this situation. The last term is the “Nekrasov-Okounkov twist” [NO] that has proved profitable to use in $K$-theoretic DT theory. By contrast it is necessary in DT theory — only the twisted virtual structure sheaf is well-defined, as was already anticipated in the papers [Ne, NP, CKM].

Proposition 5.10. Definition 5.9 is independent of the choice of self-dual resolution $E^*$ of the obstruction theory $E$. If $M$ is projective then $\hat{\mathcal{O}}_M^{\text{vir}}$ is deformation invariant: in the setting of (83) we have $\hat{\mathcal{O}}_M^{\text{vir}} = \iota_! \hat{\mathcal{O}}_{M_t}^{\text{vir}}$.

Proof. The proof follows that of the cycle version (74) very closely. That required invariance under three moves.

1. The deformation invariance (78) was proved by [KL1, Theorem 5.2]. The precise $K$-theoretic analogue is given by [KL2, Proposition 5.5].
2. The invariance (80) under the addition of the acyclic complex $B^*$ (79) to $E^*$ was proved by Proposition 3.7 and Lemma 3.4. Replacing $e$ by $\mathcal{E}$ and $\sqrt{0_E^*}$ by $\sqrt{0_E^*}$ in the proof of Proposition 3.7 gives the precise $K$-theoretic analogue. Lemma 3.3 has $K$-theoretic analogue (101), which introduces the twist $\sqrt{\det B}$. But this is cancelled by the $\sqrt{\det T^*}$ twist in (103), since we have replaced $T^*$ by $T^* \otimes B^*$.
3. Finally, we used Proposition 3.8 to pass from the deformation complex $E^*$ made from the orthogonal bundle $E$ to the complex $A^*_{e}$ made from $A_e = K^{-}/K$ as in (82).

To get the $K$-theoretic analogue of the last step we first replace $e$ by $\mathcal{E}$ in (26). That is, fix an isotropic subbundle $K \subset E$ and orient $K^{-}/K$.
as described in [24]. Then, in the notation of [25], the exact sequence 
\(0 \rightarrow K \rightarrow \Lambda \rightarrow \Lambda_\rho \rightarrow 0\) gives 
\[\Lambda^* (\Lambda^*) \otimes \sqrt{\det \Lambda} = \Lambda^* (\Lambda_\rho^*) \otimes \sqrt{\det \Lambda_\rho} \otimes \Lambda^* (K^*) \otimes \sqrt{\det K},\]
which is
\[(106) \quad \sqrt{\epsilon}(E) = \sqrt{\epsilon}(K_{\perp}/K) \epsilon(K) \cdot \sqrt{\det K}.
\]
Then the proof of Proposition [3.8] goes through as before on replacing \(e\) by \(\epsilon\) and \(\sqrt{0^*_E}\) by \(\sqrt{0^*_E}\) to give
\[(107) \quad \sqrt{\det K} \cdot \sqrt{0^*_K_{K^*}} = \sqrt{0^*_E} \circ p^*.
\]
Again the \(\sqrt{\det T^*}\) twist in (103) cancels the \(\sqrt{\det K}\) since in passing from 
\(E^*\) to \(A^*_E\) we replaced \(E\) by \(A^0 = (T/K)^*\).

**Remark 5.11.** It is natural to ask if there are analogues of Siebert’s formula
[Sie, Theorem 4.6] for the virtual cycle — or the \(K\)-theoretic analogue
[Th3, Theorem 4.2] for the virtual structure sheaf — in this square-rooted setting.
When the cone \(C_E\) is contained in a maximal isotropic subbundle \(\Lambda \subset E\)
we are in a Behrend-Fantechi setting (cf. (104)), so we can deduce
\[[M]^{\text{vir}} = (-1)^{|A|} [c(\Lambda - T) \cap c_F(M)]_{k_{\text{vd}}},\]
\[\hat{O}_M^{\text{vir}} = (-1)^{|A|} \sqrt{\det A} \cdot \sqrt{\det T^*} \cdot [\Lambda^*(\Lambda^* - T^*) \otimes \Lambda_\rho]_{k_{\text{vd}}}.
\]
Here \(c_F(M) \in A_*(M)\) is Fulton’s class [Ful, Example 4.2.6] while \(\Lambda_M \in K_0(M)[t]\) is the \(K\)-theoretic analogue of [Th3]. In general the situation is
more complicated [Oh].

6. **Virtual Riemann-Roch**

Using rational coefficients, in this Section we will relate \(\hat{O}_M^{\text{vir}}\) in
\(K\)-theory to \([M]^{\text{vir}}\) in Chow homology via the isomorphism
\[(108) \quad \tau_M : K_0(M)_{\mathbb{Q}} \xrightarrow{\sim} A_*(M)_{\mathbb{Q}}\]
that holds for any quasi-projective scheme \(M\) [Ful, Corollary 18.3.2]. Here \(\tau_M = \text{ch}(\cdot) \cap (\text{td}(T_M) \cap [M])\) when \(M\) is smooth. More generally choose an embedding \(i: M \hookrightarrow P\) in a smooth variety and set
\[\tau_M(F) := \text{ch}_M^P(F^*) \cap (\text{td}(T_P) \cap [P]) \in A_*(M)_{\mathbb{Q}},\]
where \(F^* \rightarrow i_* F\) is a locally free resolution and \(\text{ch}_M^P(F^*) \in A_*(M \rightarrow P)_{\mathbb{Q}}\) is its localised Chern character of [Ful Theorem 18.1].

First we recall the Riemann-Roch theorems of Fantechi-Göttsche [FG, Lemma 3.5] and Ciocan-Fontanine-Kapranov [CFK, Theorem 4.4.1]. Suppose \(M\) has a perfect obstruction theory \(E^* \rightarrow L_M\) with dual the virtual tangent bundle \(E_\ast\). We get a Behrend-Fantechi virtual cycle \([M]^{\text{vir}} := 0^!_{E_1} [C_{E^*}]\)
and virtual structure sheaf \(O_M^{\text{vir}} := 0^!_{E_1} [O_{C_{E^*}}]\). They prove
\[(109) \quad \tau_M(O_M^{\text{vir}}) = \text{td}(E_\ast) \cap [M]^{\text{vir}}.
\]
In our setting we fix instead a 3-term self-dual obstruction theory
\[ E^* = \{ T \to E \to T^* \} \to \mathbb{L}_M \]
of even rank and with a fixed orientation. From this we get the virtual cycle
\[ [M]^{vir} := \sqrt{0} E [C_{E^*}] \] of (73) and the twisted virtual structure sheaf \( \hat{\mathcal{O}}^{vir}_M := \sqrt{0} E [\mathcal{O}_{C_{E^*}}] \cdot \sqrt{\det T^*} \) of (103). Since any Todd class \( \text{td} \in 1 + K^0(M)^1 \) its square root is uniquely defined and multiplicative as in Lemma 5.1. The analogue of (109) is the following.

**Theorem 6.1.** In \( A_*(M)_\mathbb{Q} \) we have \( \tau_M(\hat{\mathcal{O}}^{vir}_M) = \sqrt{\text{td}(E^*)} \cap [M]^{vir} \).

**Proof.** We first recall the definition (103) of \( \hat{\mathcal{O}}^{vir}_M \). Using the stupid truncation \( \{ E \to T^* \} \) of \( E^* \) as a perfect obstruction theory for \( M \) gives an isotropic cone \( \pi: C_{E^*} \to M \) in the \( SO(2n, \mathbb{C}) \) bundle \( E \) by Proposition 4.3. Unravelling Definitions 5.8 and 5.7 and equations (99, 98) we see \( \hat{\mathcal{O}}^{vir}_M \) is the image of \( (-1)^n [\mathcal{O}_{C_{E^*}}] \cdot \pi^* \sqrt{\det T^*} \) along the top row of the diagram

\[
\begin{align*}
K_0(C_{E^*}) \xrightarrow{\rho^*} K_0(\widetilde{\mathcal{C}}) \xrightarrow{\tau} K_0(C_{Z^*/\mathcal{C}}) \xrightarrow{\sqrt{\det \Lambda^*_{\rho}} \cdot \theta_{\text{loc}}^*} K_0(\widetilde{M}) \xrightarrow{\rho^*} K_0(M) \\
A_*(C_{E^*}) \xrightarrow{\text{td}(T^*) \cdot \rho^*} A_*(\widetilde{\mathcal{C}}) \xrightarrow{\tau} A_*(C_{Z^*/\mathcal{C}}) \xrightarrow{\sigma \cap \theta_{\text{loc}}^*} A_*(\widetilde{M}) \xrightarrow{\rho^*} A_*(M).
\end{align*}
\]

We explain the notation and maps. We use \( \rho \) to denote any basechange of the flat map \( \rho: \widetilde{M} \to M \) of (19) such as \( \tilde{C} := \rho^* C_{E^*} \to C_{E^*} \). Recall \( \Lambda^*_{\rho} \) denotes the canonical maximal isotropic in the pullback of \( E \). We suppress some pullback maps for clarity, so \( \Lambda^*_{\rho} \subset E \) makes sense on \( \tilde{M} \) or \( \tilde{C} \) (which one should be clear from the context). Since \( \tilde{C} \) is a subscheme of (the pullback of) \( E \), it inherits a tautological section \( \tilde{\tau} \) of \( E \). Projecting this to \( E/\Lambda^*_{\rho} \cong \Lambda^*_{\rho} \) gives the section \( \tilde{\tau}^* \) whose zero locus we denote \( Z^* \subset \tilde{C} \).

The second horizontal arrows are the specialisation maps defined by the deformation of \( \tilde{C} \) to the normal cone of \( Z^* \subset \tilde{C} \),

\[
\tilde{C} \xrightarrow{\tau} C_{Z^*/\mathcal{C}}.
\]

This normal cone embeds in (the pullback of) \( \Lambda^*_{\rho}|_{Z^*} \) on which \( \tau \) factors through \( \Lambda^*_{\rho} \subset E \) and so defines a cosection \( \tilde{\tau}|_{Z^*}: \Lambda^*_{\rho}|_{Z^*} \to \mathcal{O}_{Z^*} \). The third horizontal arrows use this to define the pullback \( 0^*_{\Lambda^*_{\rho}} \) and intersection \( 0^*_{\Lambda^*_{\rho}} \) cosection-localised to the zero locus \( \tilde{M} \) of \( \tilde{\tau}|_{Z^*} \) by [KL2] and [KL1] respectively. Finally, we use \( \mathbb{Q} \) coefficients throughout, and the vertical maps are the \( \tau \) maps of (108).

The first square of the diagram commutes by [Fu] Theorem 18.2(3)] (since \( \rho \) is flat), the second by [Fu] Example 18.3.8] and the fourth by the Grothendieck-Riemann-Roch theorem of [Fu] Theorem 18.2(1)] (since \( \rho \) is
By [KL2, Equation 5.21] the third commutes on setting
\[ \sigma := \text{ch} \sqrt{\det \Lambda_p} \text{td}(-\Lambda_p). \]
Since \( \text{td}(F^*) = \text{td}(F) \text{ch}(\det F^*) \) for any bundle \( F \) we have the identity
\[ (\sqrt{\text{td}(F \oplus F^*)} = \text{td}(F) \text{ch} \sqrt{\det F^*}. \]
It follows that
\[ \sigma = (\sqrt{\text{td} E})^{-1}, \]
so it is pulled back from \( M \).

By a similar unravelling of Definitions 5.4 and 3.3 and equations (32, 29), we get
\[ r_{M^s \text{vir}} \text{ by starting with } \frac{1}{2^m - 1} h \text{ of (20) before applying } \rho_* \text{ to give } [M]^{\text{vir}} \in A_* (M). \]
This is the same as starting with
\[ (111) \quad (-1)^n \sqrt{\text{td}(E)} \cap [C_{E^*}] \]
in the first group and moving along to the last one, because \( \text{td}(T^\rho) \) plays the same role as \( \frac{1}{2^m - 1} h \) on classes pulled back by \( \rho^* \):
\[ \rho_*(\text{td}(T^\rho) \cap \rho^* a) = a = \frac{1}{2^n - 1} \rho_* (h \cap \rho^* a). \]

Here we have used the projection formula and \( \rho_* \text{td}(T^\rho) = 1 \) by Grothendieck-Riemann-Roch, and we are applying the formula to \( (-1)^n 0^1,_{\text{loc}} [C_{E^*}] \), which is \( \rho^*[M]^{\text{vir}} \) by Lemma 3.6.

So we now chase \( (-1)^n [O_{C_{E^*}}] \sqrt{\det T^*} \) through the diagram. Applying the key result [FG, Proposition 3.1] to the stupidly truncated perfect obstruction theory \( \tau_{E^*} \) gives
\[ \tau_{C_{E^*}} [O_{C_{E^*}}] = \pi^* \text{td}(T) \cap [C_{E^*}], \]
Combined with the module property [Elm, Theorem 18.2(2)] of \( \tau \) this shows the first vertical map gives
\[ (-1)^n \text{ch}(\sqrt{\det T^*}) \text{td}(T) [C_{E^*}] \in A_* (C_{E^*}). \]
This is \( \text{ch}(\sqrt{\det T^*}) \text{td}(T) \sqrt{\text{td}(E)}^{-1} \cap (111) \), so across the bottom of the diagram it maps to
\[ \text{ch}(\sqrt{\det T^*}) \text{td}(T) \sqrt{\text{td}(E)}^{-1} \cap [M]^{\text{vir}} \in A_* (M). \]
Applying (110) to \( F = T \) shows this is
\[ \sqrt{\text{td}(T \oplus T^*)} \sqrt{\text{td}(E)}^{-1} \cap [M]^{\text{vir}} = \sqrt{\text{td}(E_\rho)} \cap [M]^{\text{vir}}. \]
7. Torus localisation

Let $T := \mathbb{C}^*$ be the one dimensional algebraic torus. If $T$ acts on a scheme $Y$ then it is elementary to do everything in this paper in $T$-equivariant Chow (co)homology, for instance by replacing $Y$ by the $Y$-bundle $Y \times_T (\mathbb{C}^{N+1} \setminus \{0\})$ over the finite dimensional approximation $\mathbb{P}^N$ to the classifying space $BT$ (and taking the limit as $N \to \infty$). In particular we have $T$-equivariant versions of the (localised) Eddidin-Graham classes and square root Gysin operators, satisfying the identities proved in Section 3.

So now suppose $T$ acts on a quasi-projective Calabi-Yau 4-fold $(X, \mathcal{O}_X(1))$ preserving the holomorphic 4-form. Thus it acts on any moduli space $M$ of compactly supported sheaves — or stable pairs, or Joyce-Song pairs — on $X$ by pull back of sheaves. There is a lifting of the $T$ action to the (twisted) universal sheaf $\mathcal{E}$ by [Ri, Proposition 4.2]. Thus the complex of (untwisted) sheaves $E$ (46) is also $T$-equivariant, as is the obstruction theory $At: \mathcal{E} \to \mathbb{L}_M$ given by the Atiyah class of $\mathcal{E}$ [Ri, Theorem 4.3]. The orientation on $E$ is a $\mathbb{Z}/2$ choice; since $T$ is connected it preserves it.

Since each step of Proposition 4.1 (resolution, truncation, Serre duality, etc) can be done $T$-equivariantly, we get a $T$-equivariant self-dual 3-term complex of locally free sheaves $E^\bullet$ and a map in $D(M)$,

$$E^\bullet = \{ T \to E \to T^* \} \to \mathbb{L}_M$$

resolving $At: \mathcal{E} \to \mathbb{L}_M$, with $E = (E, q, o)$ a $T$-equivariant $SO(2n, \mathbb{C})$ bundle. The cone $C_{E^\bullet} \subset E$ is also $T$-equivariant, so we get a lift of the virtual cycle (73) to equivariant homology,

$$[M]^{\text{vir}} := \sqrt{0!_E^T [C_{E^\bullet}] \in A^T_{\text{vir}}(M, \mathbb{Z}[\frac{1}{2}])}.$$

The restriction of $E^\bullet$ to $\iota: M^T \hookrightarrow M$ splits into fixed and moving parts,

$$\iota^* E^\bullet = E^\bullet_f \oplus (N^{\text{vir}})^\vee,$$

which are also self-dual 3-term complexes of locally free sheaves,

$$E^\bullet_f = \{ T^f \to E^f \to (T^f)^* \},$$

$$N^{\text{vir}} = \{ T^m \to E^m \to (T^m)^* \}.$$

The duality on $N^{\text{vir}}$ weights $w > 0$ with weights $-w < 0$, so $r := \text{rank } N^{\text{vir}}$ is even \footnote{We assume that $\text{vd (5)}$ is even since otherwise $[M]^{\text{vir}} = 0$.} and $\det N^{\text{vir}}$ is trivial. (We thank Davesh Maulik for this observation.) Thus it is orientable; choosing one of its two orientations means the orientation on $E^\bullet$ induces one on $E^\bullet_f$. Though it does not matter which we choose (ultimately the signs will cancel in the localisation formula), a canonical choice is to apply the convention (63) to the positive weight

\footnote{It is important to note — especially when interpreting the formulae (114, 115) — that $r$ may not be constant but can vary from one connected component of $M^T$ to another.}
subbundles of \((E^m)^*, T^m\) and \((T^m)^*\). That is, we recall that by \([02]\) an orientation on \(N^\text{vir}\) is the same as an orientation on

\[ T^m \oplus (T^m)^* \oplus (E^m)^* = N^{>0} \oplus (N^{>0})^*, \]

where \(N^{>0} := T^{>0} \oplus (T^{<0})^* \oplus (E^{<0})^*\), and we use \(o_{N^{>0}}\) \([63]\). By Proposition \([4, 2]\) this induces the orientation \(o_{E^{>0}}\) on

\[ E^m = E^{>0} \oplus (E^{>0})^*, \]

with respect to which \(E^{>0}\) is a positive maximal isotropic.

By \([GP\text{, Proposition 1]}\) the induced map \(E^\bullet_f \to \mathbb{L}_{M^T}\) is an obstruction theory. In Footnote \([28]\) below we observe that \(C_{E_f^\bullet}\) is contained in the fixed part of \(C_{E^\bullet}|_{M^T}\), which is isotropic in \(E|_{M^T}\). Thus \(C_{E_f^\bullet}\) is isotropic in \(E^\bullet_f\).

(Alternatively one could prove that the fixed part of the \((-2)\)-shifted symplectic form on \(M\) induces one on \(M^T\) and then invoke Proposition \([4, 3]\)) Together with the orientation on \(E^\bullet_f\) this therefore defines a virtual cycle on the fixed locus by \((73)\).

\[ [M^T]^\text{vir} = \sqrt{0_{E^\bullet_f}}(C_{E_f^\bullet}) \in A^r_{T/2(\text{vd} - r)}(M^T, \mathbb{Z}[\{\frac{1}{T}\}]). \]

The complex dimension \(\frac{1}{2}(\text{vd} - r)\) of this cycle can vary from one connected component of \(M^T\) to another along with that of

\[ \sqrt{e_T}(N^\text{vir}) := \frac{e_T(T^m)}{\sqrt{e_T(E^m)}} \in A^{r/2}_{T}(M^T, \mathbb{Q})[t^{-1}]. \]

Here we have used the fact that \(T^m\) and \(E^m\) split into weight spaces all of whose weights are nonzero, so the equivariant Chow cohomology classes \(e_T(T^m), \sqrt{e_T}(E^m)\) and \(\sqrt{e_T}(N^\text{vir})\) are all invertible once we localise by inverting \(t = e_1(1)\), where \(t\) is the standard one dimensional representation of \(T\). Using \([20]\) it is elementary to show that \((115)\) is independent of choices; since we do not strictly need this we leave the details to the reader.

**Theorem 7.1.** We can localise \([M]^\text{vir}\) to \(\iota: M^T \hookrightarrow M\) by

\[ [M]^\text{vir} = \iota_* \left[ \frac{[M^T]^\text{vir}}{\sqrt{e_T}(N^\text{vir})} \right] \in A^r_{T/2\text{vd}}(M, \mathbb{Q})[t^{-1}]. \]

**Proof.** We follow \([GP\text{, Section 3]}\). The usual construction of the moduli space \(M\) as a GIT quotient of an open set in a Quot scheme (of an equivariant compactification of \(X\)) can be done \(T\)-equivariantly since \(O_X(1)\) is a \(T\)-linearised line bundle. Thus GIT endows \(M\) with a \(T\)-linearised ample line bundle \(O_M(1)\). For large \(N \gg 0\) there is a \(T\)-equivariant finite dimensional linear system \(V \subset H^0(O_M(N))\) inducing a \(T\)-equivariant embedding \(M \hookrightarrow P := \mathbb{P}(V^*)\) into a smooth \(T\)-variety \(P\). The \(T\)-fixed locus \(i : P^T \hookrightarrow P\) is also smooth and so regularly embedded (but with codimension \(c\) which can
vary from one connected component to another). It fits into a Cartesian diagram

\[(116)\]

\[
\begin{array}{c}
M^T \xleftarrow{i} \xrightarrow{\iota} M \\
P^T \xleftarrow{i} \xrightarrow{\iota} P.
\end{array}
\]

Furthermore, by Proposition 4.1 we may assume that the map \(E^* \to \mathbb{L}_M\) of (112) is represented by a \(T\)-equivariant surjection of complexes

\[
\begin{array}{c}
T \xrightarrow{\iota} E \xrightarrow{\iota} T^* \\
\quad \quad \xrightarrow{d} \quad \quad \quad I/I^2 \xrightarrow{\iota \circ i} \Omega_{P|M}.
\end{array}
\]

In particular, \(T_P|_M \to T\) is injective over \(M\).

By [Kr, Theorem 6.3.5], for instance,

\[(117)\]

\[
\iota_* : A^*_P(M^T, \mathbb{Q})[t^{-1}] \xrightarrow{\sim} A^*_P(M, \mathbb{Q})[t^{-1}]
\]

is an isomorphism, so we may write \([M]^{\text{vir}} = \iota_* a\) for some \(a \in A^*_P(M^T)[t^{-1}]\). Then by [Fu, Corollary 6.3],

\[(118)\]

\[
i^! [M]^{\text{vir}} = i^! \iota_* a = e_T(N_{P^T/P}) \cap a,
\]

where \(i^!\) is the refined Gysin map associated to the Cartesian diagram (116). Splitting \(N_{P^T/P}\) into weight spaces, all its weights are nonzero, so \(e_T(N_{P^T/P})\) is invertible. Thus we get the standard localisation formula

\[(119)\]

\[
\iota_* \left( \frac{i^! [M]^{\text{vir}}}{e_T(N_{P^T/P})} \right) = \iota_* a = [M]^{\text{vir}}.
\]

We split \(T_P|_{P^T} = T_P^f \oplus T_P^m\) into fixed and moving parts, where \(T_P^m = N_{P^T/P}\). Substituting in the definition (73) of \([M]^{\text{vir}}\) now gives

\[(120)\]

\[
[M]^{\text{vir}} = \iota^! \sqrt{0_{E^*}^! [C_{E^*}]} \quad \text{(15)} \quad \iota_* \sqrt{0_{E^*}^! \iota^! [C_{E^*}]} = \iota_* \frac{\sqrt{0_{E^*}^! \iota^! [C_{E^*}]}^|_{T_P^m}}{e_T(T_P^m)},
\]

where the isotropic cone \(C_{E^*} \subset E\) is defined from the intrinsic normal cone \(\mathfrak{C}_M\) by (61). Since \(\mathfrak{C}_M\) is by definition the stack quotient \(C_{M/P}/T_P|_M\) we see from (67) that

\[(121)\]

\[
C_{E^*} = \frac{C_{M/P} \oplus T}{T_P|_M},
\]
where we take the diagonal action of $T_P|_M$. Expanding our Cartesian diagram to

$$
t^*C_{M/P} \longrightarrow C_{M/P}$$

$$M^T \longleftarrow \mathcal{L} \quad \mathcal{L} \longleftarrow P^T \longleftarrow P,$$

Vistoli’s rational equivalence [Vi] implies

$$i^!\left[C_{M/P}\right] = \left[C_{M^T/\mathcal{L}}\right] \quad \text{in} \quad A_*(t^*C_{M/P}),$$

as in [GP] Equation 15, or the equation before Proposition 3.3 in [BF]. Thus

$$i^!\left[C_{M/P} \oplus T\right] = \left[C_{M^T/\mathcal{L}} \oplus t^*T\right],$$

by [Fu] Theorem 6.2(b). We now quotient by $T_P|_M$ using the diagram

$$A^t_{*+t}(C_{M/P} \oplus T) \longleftarrow A^t_{*+t-c}(t^*C_{M/P} \oplus t^*T)$$

$$q^* \longleftarrow i^! \quad \mathcal{L} \longleftarrow A^t_{*+c}(C_{\mathcal{L}^*}) \longleftarrow A^t_{*+c}(t^*C_{\mathcal{L}^*}).$$

Here $q: C_{M/P} \oplus T \to C_{\mathcal{L}^*}$ is an affine bundle of dimension $t := \text{rank } T_P|_M$, giving the vertical isomorphisms by [Kr] Corollary 3.6.4. Since $q$ is flat, the diagram commutes by [Fu] Theorem 6.2(b)]. By (121) and (123) we find

$$q^*i^!\left[C_{\mathcal{L}^*}\right] = \left[C_{M^T/\mathcal{L}} \oplus t^*T\right].$$

We would like to deform $C_{M^T/\mathcal{L}} \oplus t^*T$ to something invariant under the action of $T_P|_{M^T} = T_P^l \oplus T_P^m$ in order to apply $(q^*)^{-1}$ to both sides. To do this we may assume, without loss of generality, that the inclusion $T_P^l \hookrightarrow T_P^m$ is split. This is an application of the Jouanolou trick [J], as used in [GT], Section 1.1] for instance, replacing $M^T$ by an affine variety on which all extensions split automatically. That is, there is an affine variety

$$\widetilde{M^T} \longrightarrow M^T$$

which is an affine bundle over $M^T$. It is therefore a homotopy equivalence inducing an isomorphism on Chow groups [Kr] Corollary 3.6.4], so any relation we prove in the Chow group upstairs also holds downstairs on $M^T$.

The splitting $T^m \hookrightarrow T_P^m$ induces a map $f: T^m \hookrightarrow t^*C_{M/P}$. Deforming the inclusion induced by the Cartesian diagram (116)

$$C_{M^T/\mathcal{L}} \oplus T^l \oplus T^m \hookrightarrow t^*C_{M/P} \oplus T^l \oplus T^m$$

Alternatively one can work entirely “upstairs” on $C_{M^T/\mathcal{L}} \oplus t^*T$ via an analogue of [GP] Lemma 1].
by \( t.f \), for \( t \in \mathbb{C} \), gives a rational equivalence from \((125)\) at \( t = 0 \) to a \( TP|_{MT} \)-invariant cycle at \( t = 1 \). Thus \((125)\) becomes

\[
q^*i'[C_E] = q^*\left[ \frac{C_{MT/P_T} + t^*T}{TP|_{MT}} \right],
\]

where on the right the \( T^m \) in \( t^*T \) embeds in \( t^*C_{M/P} + t^*T \) via \((f,1)\) instead of \((0,1)\). Thus the same relation is true with \( q^* \) removed. So by Proposition 3.7 \((120)\) has now become

\[
\sqrt[1]{e_T(T^m_{P_T})} = \sqrt[1]{e_T(T^m_{P_T})} = \frac{1}{e_T(T^m_{P_T})}.
\]

On the right we have used Proposition 3.8 to write

\[
\sqrt[1]{e_T(T^m_{P_T})} = \sqrt[1]{e_T(T^m_{P_T})} = \frac{1}{e_T(T^m_{P_T})} = \frac{1}{e_T(T^m_{P_T})}.
\]

where \( K = T^m/T^m_{P_T} \) injects into \( E|_{MT} \) by taking moving parts of the inclusion

\[
\frac{C_{MT/P_T} + t^*T|_{MT}}{TP|_{MT}} = \frac{C_{MT/P_T} + t^*T|_{MT}}{TP|_{MT}} = C_E|_{MT} \subset E|_{MT}.
\]

Thus by \((114)\),

\[
[M]_{\text{vir}} = \frac{[M]_{\text{vir}}}{e_T(T^m_{P_T})} = \frac{[M]_{\text{vir}}}{e_T(T^m_{P_T})}.
\]

We would like to do something similar with \( \hat{O}^\text{vir}_M \) in the equivariant \( K \)-theory groups \( K^0_T, K^0_T \). But our definition \((87)\) of \( \sqrt[L]{L} \) does not immediately work equivariantly because \( 1 - L \) is usually not nilpotent in \( K^0_T(Y) \), so \((87)\) becomes an infinite series. (For instance \( 1 - t \) is not nilpotent, where \( t \) is the weight one irreducible representation of \( T \) ). We thank Andrei Okounkov for pointing this out, and Noah Arbesfeld for a suggestion on how best to fix it.

Note first that given a \( T \)-equivariant line bundle on a \( T \)-fixed scheme such as \( M^T \), it takes the form \( L' \otimes t^w \) for some \( T \)-fixed line bundle \( L' \) and locally constant integer-valued weight \( w \). Thus we may use \((87)\) to define its square root as

\[
\sqrt[1]{L' \otimes t^w} = K^0_T(M^T) \otimes_{\mathbb{Z}[t^{1\pm1}]} \mathbb{Q}(t^{1/2}).
\]

Combining this with the localisation formula will allow us to construct an operator \( \sqrt[L]{L} \) on a localised version of \( K^0_T \) (rather than an element of a localisation of \( K^0_T \)). So for any quasi-projective \( T \)-scheme \( Y \), set

\[
K^T_0(Y)_{\text{loc}} := K^T_0(Y) \otimes_{\mathbb{Z}[t^{1\pm1}]} \mathbb{Q}(t^{1/2}).
\]

\( \footnote{Notice the fixed part of the left hand side is \( C_{E^f} \) by the same argument as in \((121)\). So \( C_{E^f} \) is contained in the fixed part of \( C_{E^f}|_{MT} \) which is isotropic in \( E|_{MT} \). This shows that \( C_{E^f} \subset E^f \) is isotropic, as used earlier.} \)
Denoting the fixed locus by $\iota : Y^T \hookrightarrow Y$, we have the $K$-theoretic localisation theorem \cite[Theorem 3.3(a)]{EG2},
\[ \iota_* : K^T_0(Y^T)_{\text{loc}} \longrightarrow K^T_0(Y)_{\text{loc}}, \]
and its inverse $\iota^{-1}_*$. Given a $T$-equivariant line bundle $L$ this allows us to define an operator
\[ \sqrt{L} \ast : K^T_0(Y)_{\text{loc}} \longrightarrow K^T_0(Y)_{\text{loc}} \]
by specifying it to be \cite{S7} on $Y^T$, where $\iota^* L \cong L' \cdot t^w$. That is, we define
\begin{equation}
\label{eq:128}
\sqrt{L} \ast F := \iota_* \left( \sqrt{L} \cdot t^{w/2} \otimes \iota^{-1}_*(F) \right).
\end{equation}
We check this reduces to our previous definition \cite{S7} in the non-equivariant limit $t^{1/2} \to 1$. Let $L_1 \in K^0(Y)$ denote the line bundle $L|_{t=1}$ given by forgetting about the $T$ action.

**Proposition 7.2.** If $F \in K^0_0(Y)_{\text{loc}}$ has no pole at $t^{1/2} = 1$ then setting
\[ F_1 := \lim_{t^{1/2} \to 1} (F) \in K_0(Y) \otimes \mathbb{Q}, \]
\[ \lim_{t^{1/2} \to 1} \sqrt{L} \ast F \text{ exists and equals } \sqrt{L_1} \otimes F_1 \in K_0(Y) \otimes \mathbb{Q}. \]

**Proof.** Let $S_{q_k}(x)$ denote the approximation to the square root given by truncating the power series \cite{S7},
\[ S_{q_k}(x) := 1 - \sum_{i=1}^{k} a_i(1-x)^i \in \mathbb{Q}[x]. \]
Applied to $\iota^* L$ this approximates $\sqrt{L'} \cdot t^{w/2}$ with error
\[ E_k := S_{q_k}(L' \cdot t^w) - \sqrt{L} \cdot t^{w/2} \in K^0(Y^T) \otimes \mathbb{Q}[t^{1/2}]. \]

Working in $K^T_0(Y)_{\text{loc}}$, by \eqref{eq:128} we have
\begin{equation}
\label{eq:129}
\sqrt{L} \ast F = \iota_* \left( \sqrt{L} \cdot t^{w/2} \otimes \iota^{-1}_*(F) \right) = \iota_* \left( S_{q_k}(\iota^* L) \otimes \iota^{-1}_*(F) \right) - \iota_* (E_k \otimes \iota^{-1}_*(F)) = S_{q_k}(L) \otimes F - \iota_* (E_k \otimes \iota^{-1}_*(F)),
\end{equation}
by the projection formula. We will show that, for $d \in \mathbb{N}$ specified below and then any fixed $k \geq \dim Y^T + d$,
\begin{itemize}
\item[(a)] $E_k \in (1 - t^{1/2})^{d+1} K^0(Y^T) \otimes \mathbb{Q}[t^{1/2}]$ and
\item[(b)] $(1 - t)^d \iota^{-1}_* F \in K^0_0(Y^T)_{\text{loc}}$ has no pole at $t^{1/2} = 1$.
\end{itemize}
Thus $(1 - t^{1/2})$ divides $E_k \otimes \iota^{-1}_*(F)$, so $\lim_{t^{1/2} \to 1}$ of either side of \eqref{eq:129} exists and equals $S_{q_k}(L_1) \otimes F_1$. Since $k \geq \dim Y^T$ this is $\sqrt{L_1} \otimes F_1$, as required.

We first prove (a). By \cite{S6} we see that $(1 - x)^{k+1}$ divides
\[ S_{q_k}(x)^2 - x = (S_{q_k}(x) - x^{1/2})(S_{q_k}(x) + x^{1/2}) \in \mathbb{Q}[x^{1/2}]. \]
Since the second factor is nonzero at \( x^{1/2} = 1 \) and \( \mathbb{Q}[x^{1/2}] \) is a principal ideal domain, we deduce that \( (1 - x^{1/2})^{k+1} \) divides the first, so

\[
S_{q_k}(x) - x^{1/2} \in (1 - x^{1/2})^{k+1} \mathbb{Q}[x^{1/2}].
\]

Taking the branch of \( \sqrt{x} \) which is 1 at \( x = 1 \) and Taylor expanding the identity \( \sqrt{xy} = \sqrt{x} \sqrt{y} \) about \( (x, y) = (1, 1) \) shows that the formal power series \( S_q(x) := 1 - \sum_{i=1}^\infty a_i(1 - x)^i \in \mathbb{Q}[1 - x] \) satisfies

\[
S_q(x) = S_q(x)S_q(y) \in \mathbb{Q}[1 - x, 1 - y].
\]

Truncating, it follows that

\[
S_{q_k}(xy) - S_{q_k}(x)S_{q_k}(y) \in m^{k+1} \mathbb{Q}[x, y],
\]

where \( m := (1 - x, 1 - y) \subset \mathbb{Q}[x, y] \) is the maximal ideal at \( (1, 1) \). Hence there is a polynomial \( f(x, y) \) such that

\[
S_{q_k}(L' \cdot t^w) - S_{q_k}(L')S_{q_k}(t^w) = f(1 - L', 1 - t),
\]

where \( f \) is a sum of monomials all of which have degree \( \geq k + 1 \). All of these monomials are divisible by \( (1 - t)^{k+1 - \dim Y^T} \) because \( (1 - L')^{\dim Y^T + 1} = 0 \). So combining (131) and (130) gives, for \( k \geq \dim Y^T + d \),

\[
S_{q_k}(L' \cdot t^w) - S_{q_k}(L')t^{w/2} \in (1 - t^{1/2})^dK^0(Y^T) \otimes \mathbb{Q}[t^{\pm 1/2}].
\]

Since the left hand side is \( E_k \) this proves (a).

To prove (b) we use a \( T \)-equivariant embedding of \( Y \) in a smooth projective \( T \)-scheme \( P \) with fixed locus \( i: P^T \hookrightarrow P \) as in (128). Let \( N := N_{PT/P|Y^T} \). The refined Gysin map \( i^1: K_0(Y) \rightarrow K_0(Y^T) \) of [Lee] Section 2.1 satisfies

\[
\iota_*^{-1} = \frac{i^1}{\Lambda^* N^*} : K_0^T(Y)_{\text{loc}} \rightarrow K_0^T(Y^T)_{\text{loc}}.
\]

Writing \( N = \bigoplus_{i=1}^c N_i t^{u_i} \) as a sum of Chern roots, where \( c \) is the locally constant function \( \dim P - \dim P^T \leq \dim P \), we have

\[
\Lambda^* N^* = \bigotimes_{i=1}^c \left( 1 - N_i^*(t^{-u_i}) \right) = \bigotimes_{i=1}^c \left( 1 + \frac{t^{-u_i}}{1-t^{-u_i}} (1 - N_i^{-1}) \right).
\]

The first bracket contributes a pole of order \( c \leq \dim P \) to \( 1/\Lambda^* N^* \). Expanding the product of the reciprocals of the second bracket gives a pole of order \( \leq \dim Y^T \) because any product of \( > \dim Y^T \) terms of the form \( (1 - N_i^{-1}) \) is zero. Thus \( 1/\Lambda^* N^* \) has a pole of order \( \leq d := \dim Y^T + \dim P \) at \( t = 1 \). In particular, for \( F \) whose limit \( F_1 \) exists as \( t^{1/2} \rightarrow 1 \), we see that \( (1 - t)^d \iota_*^{-1} F \) has no pole at \( t^{1/2} = 1 \), as required.

Thus we get a \( T \)-equivariant analogues of everything in Section 5 replacing each occurrence of \( \sqrt{L} \in K^0 \) by the operator \( \sqrt{L \bullet}: K_0^T \rightarrow K_0^T \) [128]. For instance, given a \( T \)-equivariant \( SO(2n, \mathbb{C}) \) bundle \( (E, q, o) \) over a \( T \)-scheme.
Y, the bundle \( \rho: \tilde{Y} \rightarrow Y \) (19) is naturally \( T \)-equivariant, as is \( \Lambda_{\rho} \subset \rho^* E \).

We then define
\[
\sqrt{e}_T(E) := \rho_* \left( \sqrt{\det \Lambda_{\rho} \star \epsilon_T(\Lambda_{\rho}) \cap (\cdot)} \right) : K^T_0(Y)_{\text{loc}} \rightarrow K^T_0(Y)_{\text{loc}}.
\]

Since the \( K \)-theoretic cosection localisation of [KL2] operates on \( K_0 \) anyway, we can work equivariantly in Definition 5.7 to get a \( T \)-equivariant localised operator
\[
\sqrt{e}(E, s) : K^T_0(Y)_{\text{loc}} \rightarrow K^T_0(Z(s))_{\text{loc}}
\]
when \( s \) is a \( T \)-equivariant isotropic section. Applying this to a \( T \)-equivariant cone \( C \subset E \), Definition 5.8 gives an operator
\[
\sqrt{0_E^T} : K^T_0(C)_{\text{loc}} \rightarrow K^T_0(Y)_{\text{loc}},
\]
Combining this with the \( T \)-equivariant resolution (112) defines, by Definition 5.9, an equivariant virtual structure sheaf on the moduli space \( M \),
\[
\hat{O}_{\text{vir}}^M := \sqrt{\det T^* \star \sqrt{0^T_E[O_{C, \nu^*}]} \in K^T_0(M)_{\text{loc}}}
\]
whose \( t^{1/2} \rightarrow 1 \) limit is (103). So we have the ingredients to give a virtual localisation result for \( \hat{O}_{\text{vir}}^M \) in \( T \)-equivariant \( K \)-theory. This is a square-rooted analogue of the usual virtual localisation formula in \( K \)-theory [Qu].

In the notation of (113) we set
\[
\sqrt{e}_T(N^\text{vir}) := \frac{\epsilon_T(T^m)}{\sqrt{e}_T(E^m)} \in K^T_0(M^T, \mathbb{Q}),
\]
where \( \sqrt{\det T^m} \) is defined as in (57).

**Theorem 7.3.** We can localise \( \hat{O}_{\text{vir}}^M \) to \( \iota: M^T \rightarrow M \) by
\[
\hat{O}_{\text{vir}}^M = \iota_* \frac{\hat{O}_{\text{vir}}^M}{\sqrt{e}_T(N^\text{vir})} \in K^T_0(M)_{\text{loc}}.
\]

**Proof.** We follow the proof of Theorem 7.1 closely, highlighting small differences. The localisation formula (117) is replaced by its \( K \)-theoretic analogue
\[
\iota_* : K^T_0(M^T)_{\text{loc}} \xrightarrow{\sim} K^T_0(M)_{\text{loc}}
\]
of [EG2] Theorem 3.3(a)]. By (132),
\[
\iota^1 \iota_* = \Lambda^* N^* P_{P^T/P}|_{M^T} \otimes,
\]
giving the \( K \)-theoretic analogue of (118), so the same argument there gives
the following replacement for (119),
\[
\iota_* \left( \frac{\iota^1 \hat{O}_{\text{vir}}^M}{\Lambda^* N^* P_{P^T/P}|_{M^T}} \right) = \hat{O}_{\text{vir}}^M.
\]

---

29 Thus the right hand side of Theorem 7.3 has no poles at \( t^{1/2} = 1 \) and specialises there to (103). Beware, however, that before taking \( \iota_* \) we typically have poles at \( t^{1/2} = 1 \).
Splitting $T_P|_{P^T} = T_P^f \oplus T_P^m$ into fixed and moving parts, where $T_P^m = N_{P^T/P}$, and substituting in the definition (133) of $\hat{\mathcal{O}}_M^\vir$ now gives

\begin{equation}
\iota_* \left( \frac{i^!(\sqrt{0_E^\vir} \mathcal{O}_{E_1^\vir}) \cdot \sqrt{\det T^m}}{\varepsilon_T(T^m) \cdot \sqrt{\det t^* T}} \right) = \iota_* \frac{\sqrt{0_E^\vir}^\vir i^! \mathcal{O}_{E_1^\vir}}{\varepsilon_T(T^m) \cdot \sqrt{\det t^* T}},
\end{equation}

where we have used the following consequence of (132),

\begin{equation}
i^!(\sqrt{\det T^m} \ast (\cdot)) = \sqrt{\det T^m} \otimes i^!(\cdot),
\end{equation}

and the $K$-theoretic analogue $i^!(\sqrt{0_E^\vir}) = \sqrt{0_E^\vir}^K i^!(102)$ of (195).

From [Lee, Lemma 2] we get the following $K$-theoretic analogue of Vistoli’s rational equivalence (122),

\begin{equation}i^! \left[ \mathcal{O}_{C_0/M} \right] = \left[ \mathcal{O}_{C_0^{\vir}/M^{\vir}} \right] \in K_0^\vir(t^*C_0/M)_{\text{loc}}.
\end{equation}

The $K$-theoretic analogue of (124) is the isomorphism $K_0^\vir(A)_{\text{loc}} \cong K_0^\vir(B)_{\text{loc}}$ of [CG] Theorem 5.4.17 when $A \to B$ is a $T$-equivariant affine bundle. Thus the same arguments as before show the analogue of (126), namely $i^! \left[ \mathcal{O}_{E_1^\vir} \right]$ is the structure sheaf of

\begin{equation}\frac{C_{M^{\vir}/P^{\vir}} \oplus t^* T}{t^* T_P} = \frac{C_{M^{\vir}/P^{\vir}} \oplus T^f_T}{T^f_T} \oplus \frac{T^m_T}{T^m_T},\end{equation}

inside $E^f \oplus E^m$. Therefore (134) becomes

\begin{equation}
\hat{\mathcal{O}}_M^\vir = \iota_* \frac{\sqrt{0_E^\vir}^\vir \mathcal{O}_{E_1^\vir} \sqrt{0_E^\vir}^\vir \mathcal{O}_{T^m/T^m}}{\varepsilon_T(T^m) \cdot \sqrt{\det t^* T}} = \iota_* \frac{\hat{\mathcal{O}}_M^\vir \sqrt{\det T^f_T} \sqrt{\varepsilon_T(E^m)} \varepsilon_T(T^m)}{\varepsilon_T(T^m) \cdot \sqrt{\det t^* T}} = \iota_* \frac{\hat{\mathcal{O}}_M^\vir \sqrt{\varepsilon_T(E^m)}}{\varepsilon_T(T^m) \cdot \sqrt{\det T^m}} = \iota_* \frac{\hat{\mathcal{O}}_M^\vir}{\sqrt{\varepsilon_T(N^\vir)}},
\end{equation}

where the second equality follows from applying (107) and (106) to the isotropic $K := T^m/T^m_P \subset E^m$. \hfill \Box

**Remark 7.4.** This allows us to define invariants for $X$ — even though $X$ and $M$ were only assumed quasi-projectile — so long as $M$ parameterises compactly supported sheaves and $M^T$ is projective. Over $[M]^{\vir}$ we integrate lifts of insertions to the equivariant cohomology of $M$, yielding invariants in $\mathbb{Q}[t, t^{-1}]$. Or in $K$-theory we lift the natural invariant $\chi(\hat{\mathcal{O}}_M^\vir)$ to its equivariant $T$-character in $\mathbb{Q}(t^{1/2}) = K_0^\vir(\text{point})_{\text{loc}}$. Unless $M$ is projective the natural numerical specialisation of these invariants (to $t = 0$ in the cohomological case and to $t^{1/2} = 1$ in $K$-theory) may not be defined. However
the localisation theorems show the results are invariant under deformations through other T-varieties satisfying the same conditions.

In [KR] Kool and Rennemo give a simple formula (in particular explicitly determining the sign) for the contribution \(\hat{\mathcal{O}}^\text{vir}_M\) of isolated reduced points of \(M^T\) with virtual dimension zero, i.e. when \(\text{vd} - r = 0\) in \(\text{(114)}\).

Finally we note that over \(\mathbb{Q}\) the two localisation formulae are equivalent under the Riemann-Roch map of Section 6. Firstly one can show that for any \(SO(2n, \mathbb{C})\) bundle \(F\),
\[
\text{(136)} \quad \text{ch}(\sqrt{\varepsilon}(F)) = \sqrt{\text{td}(F)}^{-1}\sqrt{\varepsilon}(F)
\]
by working on the cover \((19)\) with the maximal isotropic \(\Lambda_\rho\). Applying \((110)\) to \(\Lambda_\rho\) quickly reduces \((136)\) to the identity \(c_n(\Lambda_\rho) = \text{ch}(\Lambda^\ast \Lambda^\ast_\rho) \text{td}(\Lambda_\rho)\). An easy calculation using \((136)\) then gives
\[
\text{ch}(\sqrt{\varepsilon_T}(N^\text{vir})) = \frac{\sqrt{\varepsilon_T}(N^\text{vir})}{\sqrt{\text{td}(N^\text{vir})}}.
\]
By the module property [Fu, Theorem 18.2(2)] of \(\tau_{M^T}\) this shows
\[
\tau_{M^T}\left(\frac{\hat{\mathcal{O}}^\text{vir}_{M^T}}{\sqrt{\varepsilon_T}(N^\text{vir})}\right) = \sqrt{\text{td}(N^\text{vir})} \cap \frac{\tau_{M^T}\left(\hat{\mathcal{O}}^\text{vir}_{M^T}\right)}{\sqrt{\varepsilon_T}(N^\text{vir})}.
\]
Applying Theorem 6.1 to \(M^T\), with virtual tangent bundle \(E^f\), this gives
\[
\tau_{M^T}\left(\frac{\hat{\mathcal{O}}^\text{vir}_{M^T}}{\sqrt{\varepsilon_T}(N^\text{vir})}\right) = \sqrt{\text{td}(E^m)} \sqrt{\text{td}(E^f)} \cap \frac{[M^T]^\text{vir}}{\sqrt{\varepsilon_T}(N^\text{vir})}
\]
\[
\text{(137)} \quad = \sqrt{\text{td}(E^f)} \cap \frac{[M^T]^\text{vir}}{\sqrt{\varepsilon_T}(N^\text{vir})}.
\]

8. LOCAL CALABI-YAU 4-FOLDS

We thank Yukinobu Toda for suggesting we study the example of local Calabi-Yau 4-folds \(X = K_Y\), i.e. canonical bundles of 3-folds \(Y\). These have already been treated in [CL, DSY] by conjecturing the form of the Borisov-Joyce virtual cycle in this case. We show briefly how to prove these conjectures for our virtual cycle. When combined with the sequel [OT2] this makes many of the results in those papers rigorous.

The correct abstract setting is to start with a projective scheme \(M\) with a perfect obstruction theory \(A_t : F^\ast \to L_M\). Writing \(F^\ast\) as a 2-term complex of vector bundles \(F^{-1} \to F^0\) we get the Behrend-Fantechi cone \(j : C_{F^\ast} \to F_1\) [67], whose intersection with the zero section \(0_{F_1}\),
\[
\text{(138)} \quad 0^1_{F_1}[C_{F^\ast}] = [M]^\text{vir},
\]
defines the usual Behrend-Fantechi virtual cycle of $M$. Intersecting in $K$-theory instead, then twisting by a square root of $K_{vir}$ gives the twisted virtual structure sheaf

$$\hat{O}_M^{vir} = 0_{F_1}^* [O_{C_F^{•}}] \otimes K_{vir}^{\frac{1}{2}}.$$  

The $(−2)$-shifted cotangent bundle of $(M, F^{•})$ is the same scheme $M$ with the different, non-perfect, obstruction theory

$$E^{•} := F^{•} \oplus F_{•} [2] \xrightarrow{(At, 0)} L_M.$$  

Here $E^{•} = \{ F_0 \to F_1 \oplus F^{-1} \to F^0 \}$ is a self-dual 3-term complex of vector bundles. By (62) an orientation on $E^{•}$ is the same as one on $F_0 \oplus F^0 \oplus (F_1 \oplus F^{-1})$ which in turn admits the canonical orientation of $F_0 \oplus F_1$ of (63). By the construction of Proposition 4.2 this induces the orientation on $E^{•}$ with respect to which $F_1 \subset F_1 \oplus F^{-1}$ is a positive maximal isotropic.

Therefore $M$ inherits a new virtual cycle from Definition 4.4. We recall its construction. The procedure of Section 4.2 uses the truncation $\tau E^{•} = \{ F^{-1} \oplus F_1 \to F^0 \}$. Since the arrow is zero on the second summand $F_1$, the prescription (67) gives the cone

$$C_{F^{•}} \xrightarrow{(j, 0)} F_1 \oplus F^{-1} = E_1$$

lying in the maximal isotropic $F_1 \subset E_1$. Therefore the virtual cycle of Definition 4.4 is

$$\sqrt{\det F^0} \cdot \sqrt{\det F^{-1}} \cdot \sqrt{\det F^{•}} = \sqrt{\det F_1^{•}} \cdot \sqrt{\det F_1^{•}} = \sqrt{\det F_1^{•}} \cdot \sqrt{\det F_1^{•}}.$$  

Similarly we find the virtual structure sheaf (103) is

$$\sqrt{\det F_1^{•}} \cdot \sqrt{\det F_1^{•}} = \sqrt{\det F_1^{•}} \cdot \sqrt{K_{vir}}.$$  

That is, the virtual cycle and virtual structure sheaf of the $(−2)$-shifted cotangent bundle of $(M, F^{•})$ recover the original virtual cycle (138) and the usual Nekrasov-Okounkov-twisted virtual structure sheaf (139).

This abstract situation arises in nature by letting $M = M_Y$ be a projective moduli space of stable sheaves of fixed Chern character on a smooth

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30Nekrasov-Okounkov show such a square root exists [NO, Section 6.2] in some situations. By Lemma 5.1 any choice equals $\sqrt{K_{vir}}$ once we invert 2 in our coefficients.

31It would be nice to know the relationship between this orientation and those constructed for noncompact Calabi-Yau 4-folds in [32].
projective 3-fold $Y$ with $\deg K_Y < 0$. This condition ensures the standard obstruction theory

$$F^\bullet = \tau^{\leq 0}(R\text{Hom}_{\pi_Y}(E, E)^\vee[-1]) \xrightarrow{\text{At}_Y} \mathbb{L}_{M_Y}$$

of $M_Y$, based on $\text{Ext}_Y^2(E, E)$ at $E \in M_Y$, is perfect because $\text{Ext}_Y^2(E, E) = \text{Hom}_Y(E, E \otimes K_Y)^*$ = 0. (It is also possible to handle the case that $Y$ is Calabi-Yau by using the trace-free obstruction theory $\text{Ext}_{Y}^*(E, E)_0$.) Here $\text{At}_Y$ is the Atiyah class of the (twisted) universal sheaf $E$ over $\pi_Y: Y \times M_Y \to M_Y$ as in [HT, Equation 4.2].

Now let $X = K_Y$ with zero section $i: Y \to X$. Let $M_X$ denote the moduli space of compactly supported sheaves on $X = K_Y$ with the same Chern character as $i_*E$ for $E \in M_Y$. Since $Li^*i_*= id \oplus K_Y^{-1}[1]$, adjunction shows that

$$\text{Ext}_Y^i(i_*E, i_*E) = \text{Ext}_Y^i(E, E) \oplus \text{Ext}_Y^{i-1}(E, E \otimes K_Y).$$

Done in a family over moduli space this shows that $i_*: M_Y \to M_X$ is an isomorphism of schemes and that the virtual cotangent bundle of $M_X$ is

$$E^\bullet := \tau^{[-2,0]}(R\text{Hom}_{\pi_X}(i_*E, i_*E)^\vee[-1]) = F^\bullet \oplus F[2].$$

Moreover the standard obstruction theory given by the Atiyah class map [HT, Equation 4.2],

$$E^\bullet = F^\bullet \oplus F[2] \xrightarrow{\text{At}_X} \mathbb{L}_{M_X},$$

is shown in [DSY, Proposition 3.2] to be precisely $(\text{At}_Y, 0)$. Thus we are in the situation of [130] with $M_X$ being the $(-2)$-shifted cotangent bundle of $M_Y$. We conclude from [111, 112] that our virtual cycle [73] for $M_X$ is the same as the 3-fold virtual cycle [Th2] for $M_Y$, and our virtual structure sheaf [103] is just the Nekrasov-Okounkov-twisted 3-fold virtual structure sheaf:

$$[M_X]^\text{vir} = [M_Y]^\text{vir}, \quad \hat{O}_{M_X}^\text{vir} = \hat{O}_{M_Y}^\text{vir}.$$

So for local Calabi-Yau 4-folds $X = K_Y$ the 4-fold theory reduces to the now classical 3-fold theory of $Y$.

**APPENDIX A. GAUGE THEORETIC MOTIVATION**

On a complex manifold $X$, consider holomorphic bundles which are topologically isomorphic to a fixed $C^\infty$ bundle $F$. From a gauge theory point of view we study them as $\overline{\partial}$ operators on $F$ satisfying the integrability condition

$$\overline{\partial}^2 = 0 \in \Omega^{0,2}(\text{End } F),$$

all modulo the action of the gauge group of $C^\infty$ automorphisms of the bundle. In dimensions greater than three this set-up has “Fredholm index” $-\infty$: the problem is over-determined. We could try to throw away infinitely many

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32If instead we consider stable pairs on $X$, these are not pushed forward from $Y$, so the analysis is different. The results are much the same for irreducible curve classes, however [CMT2, Proposition 3.3].
of the equations (143) by noting, by the Bianchi identity, that they take values in the kernel of \( \tilde{\omega} : \Omega^{0,2}(\text{End } F) \to \Omega^{0,3}(\text{End } F) \), but this need not be a vector bundle — its rank can jump as we vary the \( \tilde{\omega} \)-operator.

Equivalently, from an algebro-geometric point of view, the deformation and obstruction spaces \( H^i(\text{End } F) = \text{Ext}^i(F,F) \), \( i = 1, 2 \), do not fit together to form a “perfect obstruction theory” over the moduli space of holomorphic bundles since the higher obstruction groups \( H^{\geq 3}(\text{End } F) = \text{Ext}^{\geq 3}(F,F) \) need not vanish.

Let \((X, \Omega)\) now be a Calabi-Yau 4-fold with a holomorphic \((4,0)\)-form \( \Omega \). Then \( \Omega^{0,2}(\text{End } F) \) carries a gauge-invariant complex quadratic form

\[
q(\cdot, \cdot) = \int_X \text{tr}(\cdot \wedge \cdot) \wedge \Omega.
\]

There is an obvious topological obstruction to the existence of a holomorphic structure on \( F \),

\[
\int_X p_1(F) \wedge [\Omega] = 0
\]

since the left hand side is proportional to the integral of \( \text{tr} \tilde{\omega}^2 \wedge \Omega \). So we are only interested in \( C^\infty \) bundles \( F \) satisfying (145). By construction this condition is equivalent to \( \tilde{\omega}^2 \in \Omega^{0,2}(\text{End } F) \) being isotropic with respect to \( q \). So we are in the model situation described in (1), with an ambient space (of \( \tilde{\omega} \)-operators modulo gauge), a bundle with quadratic form over it, and an isotropic section

\[
s = \tilde{\omega}^2, \quad q(s, s) = 0,
\]

cutting out the moduli space of integrable \( \tilde{\omega} \)-operators or, equivalently, holomorphic bundles\(^{33}\) on \( X \).

Now give \( X \) its Ricci-flat metric, and endow \( F \) with a hermitian metric. Then there is a complex anti-linear Hodge star operator

\[
\tilde{*} := * (\cdot \wedge \Omega)
\]

on \( \Omega^{0,2}(\text{End } F) \), where \( * \) is the usual Hodge star and \( \tilde{*} \) denotes complex conjugation on forms tensored with hermitian transpose in \( \text{End } F \). Since \( \tilde{*}^2 = \text{id} \), it splits \( \Omega^{0,2} \) into real (not complex) \( \pm 1 \) eigenspaces

\[
\Omega^{0,2}(\text{End } F) = \Omega^{0,+}(\text{End } F) \oplus \Omega^{0,-}(\text{End } F).
\]

These are maximal positive (respectively negative) definite real subspaces for the quadratic form \( q \), which restricts to the \( L^2 \) hermitian metric on \( \Omega^{0,+} \) and its negative on \( \Omega^{0,-} \). Splitting the section (146) accordingly into \( s = s^+ \oplus s^- \), the isotropic condition (146) becomes the identity

\[
\|s^+\|^2 - \|s^-\|^2 = 0
\]

\(^{33}\)One can also handle sheaves by spherically twisting them about \( O_X(-N), \ N > 0 \), as in [JS, Section 8] to make them into bundles.
on their $L^2$-norms. Therefore $\|s\|^2 = \|s^+\|^2 + \|s^-\|^2$ equals $2\|s^+\|^2$ so
$$s = 0 \iff s^+ = 0,$$
just as in (11). That is, we are led to consider “half” of the equations (143) by projecting them to $\Omega^{0,+}(\text{End } F)$,
\begin{equation}
(\bar{\gamma}^2)^+ = 0 \in \Omega^{0,+}(\text{End } F).
\end{equation}
Then we have recovered the standard result (see [Th1, page 28] for instance) that solutions are the same as solutions of the original equations (143),
\begin{equation}
\bar{\gamma}^2 = 0 \iff (\bar{\gamma}^2)^+ = 0
\end{equation}
when the topological condition (145) holds.

Unlike the overdetermined equations (143), the equations (148) are elliptic modulo gauge, forming part of the $SU(4)$ instanton equations [DT]. Therefore they have the advantage of endowing $M$ with a Kuranishi structure and so a virtual cycle, as shown by Borisov-Joyce. Since the equations (148) are real rather than holomorphic they have to use real derived geometry [BJ]. (In reality they use finite dimensional local models (1) projected to a real subbundle $E_R$ of $E$ in place of the infinite dimensional gauge theory approach.)

**Deformation complex.** Linearising the the action of the gauge group and the Newlander-Nirenberg equations (143) at a fixed integrable $\bar{\gamma}$-operator gives the first line of the elliptic complex
\begin{equation}
0 \longrightarrow \Omega^0(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,1}(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,2}(\text{End } E)
\end{equation}
\begin{equation}
\xrightarrow{\bar{\gamma}} \Omega^{0,3}(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,4}(\text{End } E) \longrightarrow 0.
\end{equation}
The existence of the second line is what makes the equations over-determined. But the whole complex is self-dual, with sections of $\Omega^{0,3}$ being dual to sections of $\Omega^{* 0,1}$ by integrating against the holomorphic $(4,0)$-form $\Omega^{0,2}$. Via (147) we can split the complex into two halves, both of which are also elliptic. The first is
\begin{align*}
0 \longrightarrow & \Omega^0(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,1}(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,2}(\text{End } E) \\
& \xrightarrow{\bar{\gamma}} \Omega^{0,3}(\text{End } E) \xrightarrow{\bar{\gamma}} \Omega^{0,4}(\text{End } E) \longrightarrow 0.
\end{align*}
This is the deformation complex of the equations (148). Its exactness is what makes Borisov-Joyce theory work. It is the complex analogue of Donaldson theory on a real 4-manifold, where the deformation complex of the asd equations is half of the elliptic complex governing flat connections.

In this paper we halve the equations in a different way, effectively by intersecting the graph of the isotropic section $\bar{\gamma}^2$ of $\Omega^{0,2}(\text{End } E)$ with a choice of maximal isotropic subbundle of $\Omega^{0,2}(\text{End } E)$ instead of either of the maximal real definite subbundles $\Omega^{0,\pm}(\text{End } E)$. Since this choice breaks the $SU(4)$

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This gives both the quadratic form (144) and the Serre duality $\text{Ext}^i(E, E)^* \cong \text{Ext}^{4-i}(E, E)$ so crucial to this paper.
symmetry of the problem it makes it less likely to have gauge-theoretic motivation in general, although Nikita Nekrasov pointed out that one can think of working on the cover $\tilde{M}$ of $\mathcal{M}$ as averaging over all maximal isotropics — which is $SU(4)$ invariant — and is reminiscent of the twistor approach to solving gauge theory equations (replacing the twistor space of complex structures by the orthogonal Grassmannian of maximal isotropics).

References

[An] D. Anderson, K-theoretic Chern class formulas for vexillary degeneracy loci, Adv. Math. 350 (2019), 440–485. arXiv:1701.00126

[AP] D. Anderson and S. Payne, Operational K-theory, Documenta Math. 20 (2015), 357–399. arXiv:1301.0425

[BF] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. 128 (1997), 45–88. alg-geom/9601010

[BBBJ] O. Ben-Bassat, C. Brav, V. Bussi and D. Joyce, A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications, Geom. Topol. 19 (2015), 1287–1359. arXiv:1312.0090

[Bo] A. Bojkov, Orientations for DT invariants on quasi-projective Calabi–Yau 4-folds, Adv. Math. 388 (2021), 107859. arXiv:2008.08441

[BJ] D. Borisov and D. Joyce, Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds, Geom. Topol. 21 (2017), 3231–3311. arXiv:1504.00690

[BG] E. Bouaziz and I. Grojnowski, A d-shifted Darboux theorem, arXiv:1309.2197

[BBJ] C. Brav, V. Bussi and D. Joyce, A Darboux theorem for derived schemes with shifted symplectic structure, JAMS 32 (2019), 399–443. arXiv:1305.6302

[BD] C. Brav and T. Dyckerhoff, Relative Calabi-Yau structures II: Shifted Lagrangians in the moduli of objects, Selecta Mathematica 27 (2021), 63. arXiv:1812.11913

[Ca] A. Căldăraru, Derived Categories of Twisted Sheaves on Calabi-Yau Manifolds, Ph.D. thesis, Cornell University (2000). https://www.math.wisc.edu/~andreic/publications/ThesisSingleSpaced.pdf

[Cao] Y. Cao, private communication, Oxford, 2015.

[CGJ] Y. Cao, J. Gross and D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds, Adv. Math. 368 (2020), 107134. arXiv:1811.09658

[CK1] Y. Cao and M. Kool, Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds, Adv. Math. 338 (2018), 601–648. arXiv:1712.07347

[CK2] Y. Cao and M. Kool, Curve counting and DT/PT correspondence for Calabi-Yau 4-folds, Adv. Math. 375 (2020), 107371. arXiv:1903.12171

[CKM1] Y. Cao, M. Kool and S. Monavari, K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds, arXiv:1906.07856

[CKM2] Y. Cao, M. Kool and S. Monavari, Stable pair invariants of local Calabi-Yau 4-folds, IMRN 2022 (2022), 4753–4798. arXiv:2004.09355

[CL] Y. Cao and N. C. Leung, Donaldson-Thomas theory for Calabi-Yau 4-folds, arXiv:1407.7659

[CMT1] Y. Cao, D. Maulik and Y. Toda, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, Adv. Math. 338 (2018), 41–92. arXiv:1801.02513

[CMT2] Y. Cao, D. Maulik and Y. Toda, Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, JEMS 24 (2022), 527–581. arXiv:1902.00003

[CT1] Y. Cao and Y. Toda, Curve counting via stable objects in derived categories of Calabi-Yau 4-folds, arXiv:1909.04897

[CT2] Y. Cao and Y. Toda, Tautological stable pair invariants of Calabi-Yau 4-folds, Adv. Math. 396 (2022), 108176. arXiv:2009.03553
[CLL] H.-L. Chang, J. Li and W.-P. Li, Wittens top Chern class via cosection localization, Invent. Math. 200 (2015), 1015–1063. arXiv:1303.7126.

[Ch] A. Chiodo, The Witten top Chern class via K-theory, Jour. Alg. Geom. 15 (2006), 681–707. math.AG/0210398.

[CG] N. Chriss and V. Ginzburg, Representation theory and complex geometry, Springer 2009.

[CFK] I. Ciocan-Fontanine and M. Kapranov, Virtual fundamental classes via dg-manifolds, Geom. Topol. 13 (2009), 1779–1804. math.AG/0703214.

[DSY] E. Diaconescu, A. Sheshmani and S.-T. Yau, Atiyah class and sheaf counting on local Calabi Yau fourfolds, Adv. Math. 368 (2020), 107–132. arXiv:1810.09382.

[DT] S. Donaldson and R. P. Thomas, Gauge theory in higher dimensions, in “The Geometric Universe; Science, Geometry, And The Work Of Roger Penrose”, OUP (1998), 31–47.

[EG1] D. Edidin and W. Graham, Characteristic classes and quadratic bundles, Duke Math. Jour. 78 (1995), 277–299. alg-geom/9412007.

[EG2] D. Edidin and W. Graham, Nonabelian localization in equivariant K-theory and Riemann-Roch for quotients, Adv. Math. 198 (2005), 547–582. math.AG/0411213.

[FG] B. Fantechi and L. Göttsche, Riemann-Roch theorems and elliptic genus for virtually smooth schemes, Geom. Topol. 14 (2010), 83–115. arXiv:0706.0988.

[Fi] R. Field, The Chow ring of the classifying space BSOn, C, Jour. Algebra 350 (2012), 330–339. math.AG/0411424.

[Fu] W. Fulton, Intersection theory, 2nd ed, Springer, Berlin (1998).

[GP] T. Graber and R. Pandharipande, Localization of virtual classes, Invent. Math. 135 (1999), 487–518. alg-geom/9708001.

[HT] D. Huybrechts and R. P. Thomas, Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes, Math. Ann. 346 (2010), 545–569. arXiv:0805.3527.

[JS] D. Joyce and Y. Song, A theory of generalized Donaldson-Thomas invariants, Mem. AMS 217 (2012). arXiv:0810.5645.

[KL1] Y.-H. Kiem and J. Li, Localizing virtual cycles by cosections, JAMS 26 (2013), 1025–1050. arXiv:1007.3085.

[KL2] Y.-H. Kiem and J. Li, Localizing virtual structure sheaves by cosections, IMRN 2020 (2020), 8387–8417. arXiv:1705.09458.

[KP] Y.-H. Kiem and H. Park, Localizing virtual cycles for Donaldson-Thomas invariants of Calabi-Yau 4-folds, arXiv:2012.13167.

[KO] B. Kim and J. Oh, Localized Chern Characters for 2-periodic complexes, Selecta Mathematica 28 (2022), 23. arXiv:1804.03774.

[KM] F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves I: preliminaries on “det” and “div”, Math. Scand. 39 (1976), 19–55.

[KR] M. Kool and J. Rennemo, Proof of a magnificent conjecture, preprint.

[Kr] A. Kresch, Cycle groups for Artin stacks, Invent. Math. 138 (1999), 495–536. math.AG/9810166.

[Lee] Y.-P. Lee, Quantum K-theory, I: Foundations, Duke Math. Jour. 121 (2004), 389–424. math.AG/0105014.

[LT] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, JAMS 11 (1998), 119–174. alg-geom/9602007.

[Ne] N. Nekrasov, Magnificent Four, Ann. Inst. Henri Poincaré D 7 (2020), 505–534. arXiv:1712.08128.
COUNTING SHEAVES ON CALABI-YAU 4-FOLDS, I

N. Nekrasov and A. Okounkov, Membranes and Sheaves, Algebraic Geometry 3 (2016), 320–369. arXiv:1404.2323

N. Nekrasov and N. Piazzalunga, Magnificent four with colors, Commun. Math. Phys. 372 (2019), 573–597. arXiv:1808.05206

J. Oh, A Siebert formula for the DT4 virtual cycle, in preparation.

J. Oh and B. Sreedhar, Localization by 2-periodic complexes and virtual structure sheaves, to appear in Jour. Inst. Math. Jussieu (2020). arXiv:1909.12164

J. Oh and R. P. Thomas, Counting sheaves on Calabi-Yau 4-folds, II, preprint

https://www.ma.imperial.ac.uk/~rpwt/papers.html

R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), 407–447. arXiv:0707.2348

T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, Shifted symplectic structures, Publ. Math. IHES 117 (2013), 271–328. arXiv:1111.3209.

A. Polishchuk and A. Vaintrob, Algebraic construction of Witten’s top Chern class, in Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 229–249, AMS, Providence, RI, 2001. math.AG/0011032

F. Qu, Virtual Pullbacks In K-Theory, Ann. Inst. Fourier, Grenoble 68 (2018), 1609–1641. arXiv:1608.02524

A. Ricolfi, The equivariant Atiyah class, Comptes Rendus. Math. 359 (2021), 257–282. arXiv:2003.05440

T. Schürg, B. Toën and G. Vezzosi, Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes, J. Reine Angew. 702 (2015), 1–40. arXiv:1102.1150

B. Siebert, Virtual fundamental classes, global normal cones and Fulton’s canonical classes, in “Frobenius manifolds”, eds. K. Hertling and M. Marcolli, Aspects Math. 36 (2004), 341–358, Vieweg. math.AG/0509076

R. P. Thomas, Gauge theory on Calabi-Yau manifolds, D. Phil thesis, University of Oxford (1997).

R. P. Thomas, A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations, Jour. Diff. Geom. 54 (2000), 367–438. math.AG/9806111

R. P. Thomas, A K-theoretic Fulton class, to appear in “Facets of Algebraic Geometry: A Volume in Honour of William Fulton’s 80th Birthday”, Cambridge University Press, 2022. arXiv:1810.00079.

B. Toën and M. Vaquie, Moduli of objects in dg-categories, Annales de l’ÉNS 40 (2007), 387–444. math.AG/0503269.

A. Vistoli, Intersection theory on algebraic stacks and on their moduli spaces, Invent. Math. 97 (1989), 613–670.

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