ON A CLASS OF SOLUTIONS TO THE GENERALIZED KDV TYPE EQUATION

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Abstract. We consider the IVP associated to the generalized KdV equation with low degree of non-linearity
\[ \partial_t u + \partial_x^3 u \pm |u|^\alpha \partial_x u = 0, \quad x, t \in \mathbb{R}, \quad \alpha \in (0, 1). \]
By using an argument similar to that introduced by Cazenave and Naumkin we establish the local well-posedness for a class of data in an appropriate weighted Sobolev space. Also, we show that the solutions obtained satisfy the propagation of regularity principle proven in solutions of the \( k \)-generalized KdV equation.

1. Introduction

In this work we study the initial value problem (IVP) for the generalized Korteweg-de Vries (KdV) type equation
\[
\begin{aligned}
\partial_t u + \partial_x^3 u &\pm |u|^\alpha \partial_x u = 0, \quad x, t \in \mathbb{R}, \quad \alpha \in (0, 1), \\
u(x, 0) &= u_0(x),
\end{aligned}
\]
where \( u = u(x, t) \) is a real-valued or complex-valued unknown function.

The equation in (GK) is a lower nonlinearity version of the celebrate Korteweg-de Vries equation (KdV) \[2\]
\[
\partial_t u + \partial_x^3 u + u \partial_x u = 0, \quad x, t \in \mathbb{R},
\]
and its \( k \)-generalized form
\[
\partial_t u + \partial_x^k u + u \partial_x u = 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+.
\]
The IVP and the periodic boundary value problem (pbvp) associated to the equation in \[1,2\] have been extensively studied. In fact, sharp local and global well-posedness, stability of special solutions and blow-up results have been established in several publications (for a more detail account of them we refer to \[7\] Chapters 7-8).

Formally, real valued solutions of (GK) satisfy three conservation laws:
\[
\begin{aligned}
I_1(u) &= \int_{-\infty}^{\infty} u(x, t) \, dx, \\
I_2(u) &= \int_{-\infty}^{\infty} u^2(x, t) \, dx, \\
I_3(u) &= \int_{-\infty}^{\infty} ((\partial_x u)^2 \pm \frac{2}{(\alpha + 1)(\alpha + 2)} |u|^{\alpha+2})(x, t) \, dx.
\end{aligned}
\]
Roughly speaking, the nonlinearity in (GK) is non-Lipchitz in any Sobolev space \( H^s(\mathbb{R}) = (1 - \partial_x^2)^{s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R}, \) or in the weighted versions \( H^s(\mathbb{R}) \cap L^2(\mathbb{R} : \langle x \rangle^r dx), \quad k, r \in \mathbb{R} \) as a consequence local well-posedness can not be established in these spaces.
Our first goal is to establish the local well-posedness for the IVP (GK) in a class of initial data. To present our result we first describe our motivation and the ideas behind the proofs.

In [2] Cazenave and Naumkin studied the IVP associated to semi-linear Schrödinger equation,

\[
\begin{cases}
    \partial_t u = i(\Delta u \pm |u|^\alpha u), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \ \alpha > 0, \\
    u(x,0) = u_0(x),
\end{cases}
\]

with initial data \( u_0 \in H^s(\mathbb{R}^n) \). For every \( \alpha > 0 \) they constructed a class of initial data for which there exist unique local solutions for the IVP (1.3). Also, they obtained a class of initial data for \( \alpha > \frac{2}{n} \) for which there exist global solutions that scatter.

One of the ingredients in the proofs of their results, is the fact that solutions of the linear problem satisfy

\[
\inf_{x \in \mathbb{R}^n} \langle x \rangle^m |e^{it\Delta}u_0(x)| > 0,
\]

for \( t \in [0,T] \) with \( T \) sufficiently small whenever the initial data satisfy

\[
\inf_{x \in \mathbb{R}^n} \langle x \rangle^m |u_0(x)| \geq \lambda > 0.
\]

This is reached for \( m = m(\alpha) \) and \( u_0 \in H^s(\mathbb{R}^n) \) with \( s \) sufficiently large with appropriate decay. To prove the inequality (1.4) the authors in [2] rely on Taylor’s power expansion to avoid applying the Sobolev embedding since the nonlinear \( |u|^\alpha u \) is not regular enough and it would restrict the argument to dimensions \( n \geq 4 \).

In [8], the arguments introduced in [2] were modified to study the IVP associated to the generalized derivative Schrödinger equation

\[
\partial_t u = i\partial_{xx}^2 u + \mu |u|^\alpha \partial_x u, \quad x, t \in \mathbb{R}, \ 0 < \alpha \leq 1 \text{ and } |\mu| = 1,
\]

establishing local well-posedness for a class of small data in an appropriate weighted Sobolev space.

In the case considered here, the non-linearity is non-Lipschitz. Motivated by the results in [2] and using the smoothing effects of Kato type [4] we shall obtain the desired local well-posedness result for the IVP (GK) for a class of data satisfying (1.5).

The first aim of this paper is to show the following:

**Theorem 1.1.** Fix \( m = \left\lfloor \frac{1}{\alpha} \right\rfloor + 1 \). Let \( s \in \mathbb{Z}^+ \) satisfy \( s \geq 2m + 4 \). Assume \( u_0 \) is a complex-valued function such that

\[
\begin{align*}
    & u_0 \in H^s(\mathbb{R}), \ \langle x \rangle^m u_0 \in L^\infty(\mathbb{R}), \ \langle x \rangle^m \partial_x^{j+1} u_0 \in L^2(\mathbb{R}), \ j = 0, 1, 2, 3, \\
    & \|u_0\|_{H^s} + \|\langle x \rangle^m u_0\|_{L^\infty} + \sum_{j=0}^3 \|\langle x \rangle^m \partial_x^{j+1} u_0\|_{L^2} < \delta
\end{align*}
\]

for some \( \delta > 0 \) and

\[
\inf_{x \in \mathbb{R}} \langle x \rangle^m |u_0(x)| =: \lambda > 0.
\]

Then there exists \( T = T(\alpha; \delta; s; \lambda) > 0 \) such that (GK) has a unique local solution

\[
\begin{align*}
    & u \in C([0,T]; H^s(\mathbb{R})), \ \langle x \rangle^m \partial_x^{j+1} u \in C([0,T]; L^2(\mathbb{R})), \ j = 0, 1, 2, 3
\end{align*}
\]
THE GENERALIZED KDV TYPE EQUATION

with

\[ (x)^m u \in C([0,T]; L^\infty(\mathbb{R})), \quad \partial_x^{j+1} u \in L^\infty(\mathbb{R}; L^2([0,T])) , \quad j = 0,1 \]

and

\[ \sup_{0 \leq t \leq T} \| (x)^m (u(t) - u_0) \|_{L^\infty} \leq \frac{\lambda}{2} \]

Moreover, the map \( u_0 \mapsto u(t) \) is continuous in the following sense: For any compact \( I \subset [0,T] \), there exists a neighborhood \( V \) of \( u_0 \) satisfying (1.6) and (1.8) such that the map is Lipschitz continuous from \( V \) into the class defined by (1.9) and (1.10).

Above we have used the following notation: \( \langle x \rangle = (1 + |x|^2)^{1/2} \), \( x \in \mathbb{R} \), and \([x] \) denotes the greatest integer less than or equal to \( x \).

Remark 1.2. The solution in Theorem 1.1 is in fact unique in the class \( C([0,T]; H^{2m+2}(\mathbb{R})), \quad (x)^m \partial_x^{j+1} u \in C([0,T]; L^2(\mathbb{R})), \quad j = 0,1 \)

with

\[ \langle x \rangle^m u \in L^\infty([0,T]; L^\infty(\mathbb{R})), \quad \partial_x^{2m+3} u \in L^\infty(\mathbb{R}; L^2([0,T])) , \]

and

\[ \sup_{0 \leq t \leq T} \| (x)^m (u(t) - u_0) \|_{L^\infty} \leq \frac{\lambda}{2} \]

Remark 1.3. Inequality (1.11) gives us

\[ \frac{\lambda}{2} \leq -\frac{\lambda}{2} + \langle x \rangle^m |u_0(x)| \leq \langle x \rangle^m |u(x,t)| \leq \langle x \rangle^m |u_0(x)| + \frac{\lambda}{2} \]

for any \((x,t) \in \mathbb{R} \times [0,T] \).

Remark 1.4. As in \[1.7\] from \[1.8\] and \[1.12\] it follows that

\[ \langle x \rangle^{m-1/2} u_0 \notin L^2(\mathbb{R}) \quad \text{and} \quad \langle x \rangle^{m-1/2} u(t) \notin L^2(\mathbb{R}), \quad t \in (0,T] \]

A typical data \( u_0 \) satisfying the hypotheses in Theorem 1.1 is:

\[ u_0(x) = \frac{2\lambda e^{i\theta}}{\langle x \rangle^m} + \varphi(x), \quad \theta \in \mathbb{R}, \]

with \( \varphi \in \mathcal{S}(\mathbb{R}) \) and

\[ \| (x)^m \varphi \|_\infty \leq \lambda. \]

Remark 1.5. We observe that the IVP (GK) has (real) traveling wave solutions (positive, even and radially decreasing) with speed \( c > 0 \)

\[ \phi_{c,\alpha}(x,t) = c^{1/\alpha} \phi(\sqrt{c}(x-ct)), \]

\[ \phi(x) = \left( \frac{(\alpha + 1)(\alpha + 2)}{2} \right) \text{sech} \left( \frac{\alpha x}{2} \right)^{2/\alpha}. \]

We observe that the data \( \phi \) does not satisfy (1.8) so the traveling wave is not in the class of solutions provided by Theorem 1.1. This is similar to the situation for the log-KdV equation

\[ \partial_t v + \partial^2_x v + \partial_x (v \log |v|) = 0, \quad x,t \in \mathbb{R} \]
described in [1], where the well-posedness obtained there does not include the uniqueness and continuous dependence for the gaussian traveling wave solution of (1.13).

Our second result is concerned with the propagation of regularity in the right hand side of the data for positive times of real solutions in the class provided by Theorem 1.1. It affirms that this regularity moves with infinite speed to its left as time evolves.

**Theorem 1.6.** In addition to hypotheses (1.6) - (1.8), suppose $u_0$ is real valued and there exist $l \in \mathbb{Z}^+$ and $x_0 \in \mathbb{R}$ such that

$$\left| u_0 \right|_{(x_0, \infty)} \in H^{s+l}((x_0, \infty)).$$

Then for any $\epsilon' > 0$, $v > 0$, $R > 0$ and $j = 1, \ldots, l$

(1.14) \[ \sup_{0 \leq t \leq T} \int_{x_0 + \epsilon' - vt}^{\infty} (\partial_x^{s+l} u(x, t))^2 \, dx < C^s(\epsilon'; v; x_0; j; l; s) \]

and

(1.15) \[ \int_0^T \int_{x_0 + \epsilon' - vt}^{x_0 + R - vt} (\partial_x^{s+l+1} u(x, t))^2 \, dx \, dt < C^{ss}(\epsilon'; R; v; x_0; j; l; s). \]

**Remark 1.7.** In the proof of Theorem 1.1 we shall only consider the integral equation version of the IVP (GK) and estimates which hold for complex and real valued solutions. In the case of Theorem 1.6 the proof is based on weighted energy estimates performed in the differential equations for which we need to have real-valued solutions.

Below we shall use the notation:

$$\|F\|_{L^\infty_t L^2_x} = \sup_{t \in \mathbb{R}} \|F(t)\|_2, \quad \|F\|_{L^\infty_t L^2_x} = \sup_{t \in \mathbb{R}} (\int_{-\infty}^{\infty} |f(x, t)|^2 \, dx)^{1/2},$$

with $\|F\|_{L^\infty_t L^2_x}$ and $\|F\|_{L^\infty_t L^2_x}$ denoting the corresponding norms restricted to the time interval $[0, T]$.

In section 2 we shall state the necessary estimates for the proofs of Theorems 1.1 and 1.6, which will be given in section 3.

2. Preliminaries

We start this section presenting some linear estimates. The first one is concerning the sharp (homogeneous) version of Kato smoothing effect found in [3].

**Lemma 2.1.** Let $\{U(t) : t \in \mathbb{R}\} = \{e^{-i\partial_x^3} : t \in \mathbb{R}\}$ denote the unitary group describing the solution of the associated linear problem to (GK). Then, for any $f \in L^2(\mathbb{R})$ complex or real valued

(2.1) \[ \|U(t)f\|_{L^\infty_t L^2_x} + \|\partial_x U(t)f\|_{L^\infty_t L^2_x} = \left(1 + \frac{1}{\sqrt{3}}\right) \|f\|_{L^2}. \]

Next, we collect some estimates necessary to prove the main results.
Lemma 2.2. Let $\mu > 0$, $r \in \mathbb{Z}^+$. Then for any $\theta \in [0, 1]$ with $(1 - \theta)r \in \mathbb{Z}^+$
\begin{equation}
\label{2.2}
\left\| \left( x \right)^{\beta \mu} \partial_x^{\beta(1 - \theta)r} f \right\|_{L^2_x} \leq C \left\| \left( x \right)^{\theta \mu} f \right\|_{L^2_x} \left\| \partial_x f \right\|_{L^2_x}^{1 - \theta} + L.O.T.
\end{equation}
where the lower order terms $L.O.T.$ in (2.2) are bounded by
\[
\sum_{0 \leq \beta \leq 1, (1 - \beta)(r - 1) \in \mathbb{Z}^+} \left\| \left( x \right)^{\beta(\mu - 1)} \partial_x^{\beta(1 - \beta)(r - 1)} f \right\|_{L^2_x}.
\]
\begin{proof}
The proof of this estimate follows by successive integration by parts. \hfill \Box
\end{proof}

The inequality (2.2) is related with the following estimates found in [9].

Lemma 2.3 (9 Lemma 4]). For any $a, b > 0$ and $\gamma \in (0, 1)$, there exists $C > 0$ such that
\[
\left\| J^a \left( \left( x \right)^{1 - \gamma} b f \right) \right\|_{L^2_x} \leq C \left\| \left( x \right)^{\beta \mu} f \right\|_{L^2_x} \left\| J^a f \right\|_{L^2_x},
\]
\[
\left\| \left( x \right)^{\gamma a} \left( J^{1 - \gamma} b f \right) \right\|_{L^2_x} \leq C \left\| J^b f \right\|_{L^2_x} \left\| \left( x \right)^{\gamma a} f \right\|_{L^2_x}.
\]

3. Proof of the main results.

Proof of Theorem 1.1. To simplify the exposition and without lost of generality we shall consider real valued functions. Let us introduce the complete metric space
\[
X_T = \{ u \in C([0, T]; H^{s}(\mathbb{R})); \| u \|_{X_T} := \| u \|_{L^\infty_x H^s_x} + \| \left( x \right)^m u \|_{L^\infty_x L^\infty_x} + \sum_{l=1}^{4} \| \left( x \right)^m \partial_x^l u \|_{L^\infty_x L^\infty_x} + \| \partial_x^{r+1} u \|_{L^\infty_x L^\infty_x} \leq 5C_1 \delta,
\]
\[
\sup_{0 \leq t \leq T} \left\| \left( x \right)^m \left( u(t) - u_0 \right) \right\|_{L^\infty_x} \leq \frac{\lambda}{2}
\}
\]
equipped with the distance function
\[
d_{X_T}(u, v) = \| u - v \|_{X_T}
\]
for any $s \geq 2m + 4$ with $s \in \mathbb{Z}^+$, $m = \left( \frac{1}{\alpha} \right) + 1$. Notice that we have
\begin{equation}
\label{3.1}
\frac{\lambda}{2} \leq \left\langle x \right\rangle^m |u(x, t)| \leq \left\langle x \right\rangle^m |u_0(x)| + \frac{\lambda}{2}
\end{equation}
for any $(x, t) \in \mathbb{R} \times [0, T]$ as long as $u \in X_T$. Here the constant $C_1$ will be chosen later. Set
\begin{equation}
\label{3.2}
\Phi(u(t)) = U(t)u_0 \mp \int_0^t U(t-s)\left( |u|^a \partial_x u \right)(s) ds.
\end{equation}
We will prove that $\Phi$ is a contraction map in $X_T$. Let us first show that $\Phi$ maps from $X_T$ to itself. Recall that
\[
\| f \|_{H^s_x} \approx \| \partial_x^a f \|_{L^2_x} + \| f \|_{L^2_x}.
\]
By using (2.1) and the Leibniz rule, one has
\[
\|\partial_x^s \Phi(u)\|_{L_T^\infty L^2_x} + \|\partial_x^{s+1} \Phi(u)\|_{L_T^\infty L^2_x} \\
\leq C_0 \|\partial_x^s u_0\|_{L^2_T L^2_x} + c \|\partial_x^s (|u|^\alpha \partial_x u)\|_{L^1_T L^2_x}
\]
(3.3)
\[
\leq C_0 \|\partial_x^s u_0\|_{L^2} + c \sum_{j=0}^{s} \|\partial_x^j (|u|^\alpha)\partial_x^{s+1-j} u\|_{L^1_T L^2_x}
\]
\[
= 2C_0 \|\partial_x^s u_0\|_{L^2} + c \sum_{j=0}^{s} A_j.
\]

We shall consider \(A_j\). A use of the Hölder inequality gives us
\[
A_0 = \|u|^{\alpha} \partial_x^{s+1} u\|_{L^1_T L^2_x}
\]
\[
\leq CT^{1/2} \|\partial_x^{s+1} u\|_{L_T^\infty L^2_x} \|\langle x \rangle |u|^\alpha\|_{L_T^\infty L^\infty_x} \|\langle x \rangle^{-1}\|_{L^2}
\]
\[
\leq CT^{1/2} \|\partial_x^{s+1} u\|_{L_T^\infty L^2_x} \|\langle x \rangle \|^m u\|_{L_T^\infty L^\infty_x},
\]
which yields
\[
(3.4)
A_0 \leq CT^{1/2} \delta^{s+1}.
\]
The estimates for the intermediate terms \(A_j\) (\(2 \leq j \leq s - 1\)) can be obtained by the interpolation between the terms in \(A_0\) and \(A_s\). Hence, we shall consider \(A_s\). One sees that
\[
A_s = \|\partial_x^s (|u|^\alpha) \partial_x u\|_{L^1_T L^2_x}
\]
\[
\leq CT \left( \| |u|^{\alpha-s} \partial_x u \partial_x^s u \|_{L_T^\infty L^2_x} + \cdots + \| |u|^\alpha \partial_x^s u \|_{L_T^\infty L^1_x} \right)
\]
\[
= : CT (A_{s,1} + \cdots + A_{s,s}).
\]
Since the middle term \(A_{s,j}\) (\(2 \leq j \leq s - 1\)) can be estimated by the interpolation between \(A_{s,1}\) and \(A_{s,s}\), it suffices to estimate \(A_{s,1}\) and \(A_{s,s}\).

Using that
\[
\langle x \rangle^m \geq c \lambda |u(x,t)|^{-1}
\]
and Sobolev embedding we deduce that
\[
A_{s,1} = \| |u|^{\alpha-1} \partial_x^s u \partial_x u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \| |u|^{\alpha-1} \partial_x u \|_{L_T^\infty L^\infty_x} \| \partial_x^s u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \| \langle x \rangle^m (1 - \alpha) \partial_x u \|_{L_T^\infty L^\infty_x} \| \partial_x^s u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L^2_x} \| \partial_x^s u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \left( \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L^2_x} + \| \langle x \rangle^m \partial_x^s u \|_{L_T^\infty L^2_x} \right) \| \partial_x u \|_{L_T^\infty L^2_x}.
\]

By using (3.5) and Sobolev embedding again, one has
\[
A_{s,s} = \| |u|^{\alpha-s} \partial_x u \partial_x^s u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \| \langle x \rangle^m (s - \alpha) \partial_x u \partial_x^s u \|_{L_T^\infty L^2_x}
\]
\[
\leq C \left( \| \langle x \rangle^m \partial_x u \partial_x^s u \|_{L_T^\infty L^2_x} + \| \langle x \rangle^m \partial_x^s u \|_{L_T^\infty L^2_x} \right) \| \partial_x u \|_{L_T^\infty L^2_x}.
\]
\[ \leq C \| \langle x \rangle^m \partial_x u \|^2_{L^2} + \| \langle x \rangle^m \partial_x^2 u \|^2_{L^2} \] 
\[ \leq C \left( \| \langle x \rangle^m \partial_x u \|^2_{L^2} + \| \langle x \rangle^m \partial_x^2 u \|^2_{L^2} \right) \| \partial_x u \|^2_{L^2}. \]

Therefore it holds that
\[ A_h \leq C T \left( \| \langle x \rangle^m \partial_x u \|^2_{L^2} + \| \langle x \rangle^m \partial_x^2 u \|^2_{L^2} \right) \| \partial_x u \|^2_{L^2} \]
\[ + C T \left( \| \langle x \rangle^m \partial_x u \|^2_{L^2} + \| \langle x \rangle^m \partial_x^2 u \|^2_{L^2} \right) \| \partial_x u \|^2_{L^2} \]
\[ \leq C T (\delta^2 + \delta^{s+1}). \]

Combining (3.3) with (3.4) and (3.6), we obtain
\[ \| \partial_x \Phi(u) \|_{L^2} \leq C T \| u \|^2_{L^2} \]
\[ \leq 2 C_0 \delta + C T^{1/2} \delta^{s+1} + C T \delta (\delta + \delta^s). \]

One also sees from Sobolev embedding that
\[ \| \Phi(u) \|_{L^2} \leq \| u_0 \|_{L^2} + C T \| u \|^2_{L^2} \]
\[ \leq \| u_0 \|_{L^2} + C T \| u \|^2_{L^2} \]
\[ \leq \delta + C T \delta^{s+1}. \]

Let us next consider
\[ \| \langle x \rangle^m \partial_x^l \Phi(u) \|_{L^2} \]
for any \( l \in [1, 4] \). Note that
\[ U(-t)x U(t) = x f - 3 t \partial_x^2 f. \]

This implies
\[ x^j U(t) f = U(t) (x - 3 t \partial_x^2)^j f \]
for any \( j \in \mathbb{Z}^+ \). Therefore, by interpolation, we have
\[ \| \langle x \rangle^j U(t) f \|_{L^2} \leq C \left( \| U(t) f \|_{L^2} + \| x^j U(t) f \|_{L^2} \right) \]
\[ \leq C \| f \|_{L^2} + C \| x^j f \|_{L^2} \]
\[ \leq C \| f \|_{L^2} + C \| x^j f \|_{L^2} + C t^j \| \partial_x^j f \|_{L^2}, \]
which yields
\[ \| \langle x \rangle^j U(t) f \|_{L^2} \leq C \| \langle x \rangle^j f \|_{L^2} + C t^j \| \partial_x^j f \|_{L^2} \]
for any \( j \in \mathbb{Z}^+ \). Thus, from (3.9), we deduce that
\[ \| \langle x \rangle^m \partial_x^l \Phi(u) \|_{L^2} \]
\[ \leq \| \langle x \rangle^m U(t) \partial_x^l u_0 \|_{L^2} + \int_0^t \| \langle x \rangle^m U(t-s) \partial_x^l (|u|^\alpha \partial_x u) (s) \|_{L^2} \]
\[ \leq C \| \langle x \rangle^m \partial_x^l u_0 \|_{L^2} + C T^m \| \partial_x^{2m+l} u_0 \|_{L^2} \]
\[ + C T \| \langle x \rangle^m \partial_x^l (|u|^\alpha \partial_x u) \|_{L^2} + C T^{m+1} \| \partial_x^{2m+l} (|u|^\alpha \partial_x u) \|_{L^2}. \]
Since \( s \geq 2m + 4 \), as in the proof of (3.4) and (3.6), the interpolation argument gives us
\[
\| \partial_x^{2m+1} \langle |u|^\alpha \partial_x u \rangle \|_{L_T^\infty L_x^2} \leq C \| \langle |u|^{\alpha+1} \partial_x u \rangle \|_{L_T^\infty L_x^2} + C \| \partial_x^\alpha \langle |u|^{\alpha} \partial_x u \rangle \|_{L_T^\infty L_x^2}
\]
(3.10)
\[
\leq C \| \langle |u|^{\alpha+1} \rangle \|_{L_T^\infty L_x^2} + C\partial + C\delta^{\alpha+1}.
\]

By means of interpolation once more, we deduce that
\[
\| \langle x \rangle^m \partial_x^l \langle |u|^\alpha \partial_x u \rangle \|_{L_T^\infty L_x^2}
\leq C \| \langle x \rangle^m \langle |u|^\alpha \partial_x^l u \rangle \|_{L_T^\infty L_x^2} + C \| \langle x \rangle^m \partial_x^\alpha \langle |u|^\alpha \partial_x u \rangle \|_{L_T^\infty L_x^2}
\leq C \| \langle x \rangle^m \langle |u|^\alpha \partial_x^l u \rangle \|_{L_T^\infty L_x^2}
\]
\[
+ C \| \langle x \rangle^m \langle |u|^\alpha \partial_x^l u \rangle \|_{L_T^\infty L_x^2}
\]
\[
=: I_1 + I_2 + I_3.
\]

Firstly, \( I_1 \) is estimated as
\[
I_1 \leq C \| \langle x \rangle^m u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^{\alpha-1} \partial_x^l \partial_x u \|_{L_T^\infty L_x^2}
\leq C \| \langle x \rangle^m u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^{m-1} \partial_x^l \partial_x u \|_{L_T^\infty L_x^2}
\leq C\delta^2.
\]

Note that when \( l = 4 \), we work with (2.2). Further, one sees from Sobolev embedding and \( m(l - \alpha) < ml \) that
\[
I_2 \leq C \| \langle x \rangle^{m(l-\alpha)} \partial_x u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^2}
\leq C \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^2}
\leq C \left( \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^2} + \| \langle x \rangle^m \partial_x^2 u \|_{L_T^\infty L_x^2} \right)^{l} \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^2}
\leq C\delta^{l+1}.
\]

On the other hand, by using Sobolev embedding again, we obtain
\[
I_3 \leq C \| \langle x \rangle^{m-1} \partial_x u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^m \partial_x^l u \|_{L_T^\infty L_x^2}
\leq \left( \| \langle x \rangle^m \partial_x u \|_{L_T^\infty L_x^2} + \| \langle x \rangle^m \partial_x^2 u \|_{L_T^\infty L_x^2} \right) \| \langle x \rangle^m \partial_x^l u \|_{L_T^\infty L_x^2}
\leq C\delta^2.
\]

Combining these estimates, it holds that
\[
\| \langle x \rangle^m \partial_x^l \Phi(u) \|_{L_T^\infty L_x^2} \leq C\delta^2 + C\delta^{l+1}.
\]

Hence, we obtain
\[
\| \langle x \rangle^m \partial_x^l \Phi(u) \|_{L_T^\infty L_x^2} \leq C\delta + C\delta(1 + \delta^\alpha)
\]
as long as \( T \leq 1 \).
Next we estimate
\[ ||\langle x \rangle^m \Phi(u)||_{L_T^\infty L_x^\infty}. \]
By using (3.9) and the fact
\[ \frac{d}{dt} U(t)u_0 = -\partial_x^3 U(t)u_0, \]
together with Sobolev embedding, we obtain
\[ \leq ||\langle x \rangle^m (U(t)u_0 - u_0)||_{L_x^\infty} \]
\[ \leq \left\| \int_0^t \frac{d}{ds} (\langle x \rangle^m U(s)u_0) \, ds \right\|_{L_x^\infty} \]
\[ \leq \left\| \int_0^t \langle x \rangle^m U(s)\partial_x^2 u_0 \, ds \right\|_{L_x^\infty} \]
\[ \leq CT \left( ||\langle x \rangle^m \partial_x^3 u_0||_{L_x^2} + ||\langle x \rangle^m \partial_x^4 u_0||_{L_x^2} \right) + CT^{m+1} ||u_0||_{H^{2m+4}}, \]
which implies
\[ (3.12) \quad ||\langle x \rangle^m (U(t)u_0 - u_0)||_{L_T^\infty L_x^\infty} \leq CT\delta. \]
if \( T \leq 1 \). From (3.9) it follows that
\[ \left\| \langle x \rangle^m \int_0^t U(t-s) (|u|^{\alpha} \partial_x u) (s) \, ds \right\|_{L_x^\infty} \]
\[ \leq C \left\| \langle x \rangle^m \int_0^t U(t-s) (|u|^{\alpha} \partial_x u) (s) \, ds \right\|_{L_x^2} \]
\[ + C \left\| \langle x \rangle^m \int_0^t U(t-s) \partial_x (|u|^{\alpha} \partial_x u) (s) \, ds \right\|_{L_x^2} \]
\[ \leq C \int_0^t ||\langle x \rangle^m (|u|^{\alpha} \partial_x u) (s)||_{L_x^2} \, ds \]
\[ + C \int_0^t |t-s|^m \left\| \partial_x^{2m} (|u|^{\alpha} \partial_x u) (s) \right\|_{L_x^2} \, ds \]
\[ + C \int_0^t \left\| \langle x \rangle^m \partial_x (|u|^{\alpha} \partial_x u) (s) \right\|_{L_x^2} \, ds \]
\[ + C \int_0^t |t-s|^m \left\| \partial_x^{2m+1} (|u|^{\alpha} \partial_x u) (s) \right\|_{L_x^2} \, ds \]
\[ \leq CT \left( ||\langle x \rangle^m |u|^{\alpha} \partial_x u||_{L_T^\infty L_x^2} + ||\langle x \rangle^m \partial_x (|u|^{\alpha} \partial_x u)||_{L_T^\infty L_x^2} \right) \]
\[ + CT^{m} \left( \left\| \partial_x^{2m} (|u|^{\alpha} \partial_x u) \right\|_{L_T^\infty L_x^2} + T \left\| \partial_x^{2m+1} (|u|^{\alpha} \partial_x u) \right\|_{L_T^\infty L_x^2} \right). \]
Also, Sobolev embedding provides that
\[ \left\| \langle x \rangle^m \partial_x (|u|^{\alpha} \partial_x u) \right\|_{L_T^\infty L_x^2} \]
\[ \leq C \left\| \langle x \rangle^m |u|^{\alpha-1} (\partial_x u)^2 \right\|_{L_T^\infty L_x^2} + \left\| \langle x \rangle^m |u|^{\alpha} \partial_x^2 u \right\|_{L_T^\infty L_x^2} \]
\[ \leq C \left\| \langle x \rangle^{2m+m-\alpha} (\partial_x u)^2 \right\|_{L_T^\infty L_x^2} + \left\| \langle x \rangle^m |u|^{\alpha} \partial_x^2 u \right\|_{L_T^\infty L_x^2} \]
\[ \leq C \left\| \langle x \rangle^m \partial_x u \right\|_{L_T^\infty L_x^2} \left\| \langle x \rangle^m \partial_x u \right\|_{L_T^\infty L_x^2} + C ||u||_{L_T^\infty L_x^2} \left\| \langle x \rangle^m \partial_x^2 u \right\|_{L_T^\infty L_x^2}. \]
Similarly to the above, by the interpolation argument, it suffices to deal with (3.10) and (3.14), one establishes that whenever $T \leq 1$. Therefore, by using (3.7) and (3.8) together with (3.11) and (3.12), one establishes that

\[
\|u\|_{X_T} \leq 2C_0\delta + CT^{1/2}\delta^{\alpha+1} + CT\delta(\delta + \delta^s) + \delta + CT\delta^{\alpha+1} + C\delta + CT\delta(1 + \delta^s) + \delta + CT(\delta + \delta^{s+1}) \\
\leq 5C_1\delta,
\]

where $C_1 = \max(2C_0, C, 1)$ as long as $T = T(\alpha, \delta, s)$ is small enough.

Moreover, as in the proof of (3.14), it holds that

\[
\sup_{0 \leq t \leq T} \|\langle x \rangle^m (\Phi(u(t)) - u_0)\|_{L^\infty} \\
\leq \|\langle x \rangle^m (U(t)u_0 - u_0)\|_{L^\infty} + \|\langle x \rangle^m \int_0^t U(t-s) (|u|^\alpha \partial_s u) (s) ds\|_{L^\infty} \\
\leq CT(\delta + \delta^{1+s}) \leq \frac{\lambda}{2}
\]

if $T = T(\delta, s, \lambda)$ is sufficiently small. Thus, $\Phi(u) \in X_T$ holds.

Let us show $\Phi$ is a contraction map in $X_T$. By using (2.4), we first estimate

\[
\|\partial_x^\alpha (\Phi(u) - \Phi(v))\|_{L^\infty L^2} + \|\partial_x^{\alpha+1} (\Phi(u) - \Phi(v))\|_{L^\infty L^2} \\
\leq \sum_{j=0}^s \|\partial_x^\alpha (|u|^\alpha)\partial_x^{\alpha+1-j} u - \partial_x^\alpha (|v|^\alpha)\partial_x^{\alpha+1-j} v\|_{L^1 L^2} \\
=: \sum_{j=0}^s B_j.
\]

Similarly to the above, by the interpolation argument, it suffices to deal with $B_0$ and $B_s$. Here we observe that

\[
\|u|^\alpha - |v|^\alpha| = \alpha(\theta|u| + (1-\theta)|v|)^{\alpha-1})(|u| - |v|) \leq c \langle x \rangle^{m(1-\alpha)}|u - v|
\]
for \( \theta \in (0, 1) \) and any \( \alpha \in \mathbb{R} \). Together with \((3.15)\), a similar computation as in \((3.14)\) shows that

\[
B_0 = \left\| u^{\alpha} \partial_x^{s+1} u - |v|^{\alpha} \partial_x^{s+1} v \right\|_{L^1_T L^2_x}^2 \\
\leq C \left\| u^{\alpha} \partial_x^{s+1} u - |v|^{\alpha} \partial_x^{s+1} v \right\|_{L^1_T L^2_x} + C \left( \left\| u^{\alpha-1} + |u|^{\alpha-1} \right\| u - v \partial_x^{s+1} v \right\|_{L^1_T L^2_x} \\
\leq C T^{1/2} \left\| \langle x \rangle^m u \right\|_{L^\infty_T L^\infty_x} \left\| \langle x \rangle^{-1} \right\|_{L^2_x} \left\| \partial_x^{s+1} u - v \right\|_{L^\infty_T L^2_x} \\
+ C T^{1/2} \left\| \langle x \rangle^m (u - v) \right\|_{L^\infty_T L^\infty_x} \left\| \langle x \rangle^{-1} \right\|_{L^2_x} \left\| \partial_x^{s+1} v \right\|_{L^\infty_T L^2_x}.
\]

This implies

\[
(3.17) \quad B_0 \leq C T^{1/2} \delta d_{X^r}(u, v).
\]

On the other hand, we see from \((3.15)\) that

\[
B_s \leq C T \left\| u^{\alpha-\delta} (\partial_x u)^{s+1} - |v|^{\alpha-\delta} (\partial_x v)^{s+1} \right\|_{L^\infty_T L^2_x} \\
+ \ldots \\
+ C T \left\| u^{\alpha-1} \partial_x u - |v|^{\alpha-1} \partial_x v \right\|_{L^\infty_T L^2_x} \\
=: C T(B_{s,1} + \ldots + B_{s,s}).
\]

Similarly to \((3.6)\), the middle terms \(B_{s,j} (2 \leq j \leq s-1)\) can be estimated by the interpolation between \(B_{s,1}\) and \(B_{s,s}\), so it suffices to estimate \(B_{s,1}\) and \(B_{s,s}\). By using Sobolev embedding and \((3.10)\), one has that

\[
B_{s,1} \leq \left\| u^{\alpha-\delta} ((\partial_x u)^{s+1} - (\partial_x v)^{s+1}) \right\|_{L^\infty_T L^2_x} \\
+ \left\| (u^{\alpha-\delta} - |u|^{\alpha-\delta}) (\partial_x v)^{s+1} \right\|_{L^\infty_T L^2_x} \\
\leq \left( \left\| \langle x \rangle^m \partial_x u \right\|_{L^\infty_T L^2_x} + \left\| \langle x \rangle^m \partial_x v \right\|_{L^\infty_T L^2_x} \right) \left\| \partial_x u - v \right\|_{L^\infty_T L^2_x} \\
+ \left( \left\| \langle x \rangle^m \partial_x v \right\|_{L^\infty_T L^2_x} + \left\| \langle x \rangle^m \partial_x^2 v \right\|_{L^\infty_T L^2_x} \right) \\
\times \left\| \partial_x v \right\|_{L^\infty_T L^2_x} \left\| \langle x \rangle^m (u - v) \right\|_{L^\infty_T L^\infty_x} \\
\leq \delta(1 + \delta) d_{X^r}(u, v).
\]

We also obtain that

\[
B_{s,s} = \left\| u^{\alpha-1} \partial_x u - |v|^{\alpha-1} \partial_x v \right\|_{L^\infty_T L^2_x} \\
\leq \left\| (u^{\alpha-1} \partial_x u - u) \right\|_{L^\infty_T L^2_x} + \left\| u^{\alpha-1} \partial_x(u - v) \right\|_{L^\infty_T L^2_x} \\
+ \left\| (|u|^{\alpha-1} - u^{\alpha-1}) \partial_x v \right\|_{L^\infty_T L^2_x} \\
=: E_1 + E_2 + E_3.
\]

To estimate \(E_1\) and \(E_2\) we follow an argument similar to the one used in \((3.6)\), so that one obtains

\[
E_1 + E_2 \leq C \left( \left\| \langle x \rangle^m \partial_x (u - v) \right\|_{L^\infty_T L^2_x} + \left\| \langle x \rangle^m \partial_x^2 (u - v) \right\|_{L^\infty_T L^2_x} \right) \left\| \partial_x^s u \right\|_{L^\infty_T L^2_x} \\
+ C \left( \left\| \langle x \rangle^m \partial_x u \right\|_{L^\infty_T L^2_x} + \left\| \langle x \rangle^m \partial_x^2 u \right\|_{L^\infty_T L^2_x} \right) \left\| \partial_x^s (u - v) \right\|_{L^\infty_T L^2_x} \\
\leq C \delta d_{X^r}(u, v).
\]
From Sobolev embedding and (3.10), $E_3$ is estimated as

$$E_3 = \|(u^{|\alpha - 1|} - v^{|\alpha - 1|})\partial_x v \partial_x u\|_{L^\infty_T L^2_x} \leq C \|(u^{|\alpha - 2|} + |v^{|\alpha - 2|} - u - v|\partial_x v \partial_x v\|_{L^\infty_T L^2_x} \leq C \|(x)^{2m-1} |u - v|\partial_x v \partial_x v\|_{L^\infty_T L^2_x} \leq C \|(x)^{m} (u - v)\|_{L^\infty_T L^\infty_x} \|\partial_x v\|_{L^\infty_T L^2_x} \times \left( \|(x)^{m} \partial_x v\|_{L^\infty_T L^2_x} + \|(x)^{m} \partial_x^2 v\|_{L^\infty_T L^2_x} \right) \leq C \delta^2 d_{X_T}(u, v).$$

Thus, since

$$B_{s,s} \leq E_1 + E_2 + E_3 \leq C \delta (1 + \delta) d_{X_T}(u, v),$$

we have

(3.18) \hspace{1cm} B_s \leq C T \delta (1 + \delta^s) d_{X_T}(u, v).

Combining (3.15) with (3.17) and (3.18), it holds that

(3.19) \hspace{1cm} \|\partial_x^s (\Phi(u) - \Phi(v))\|_{L^\infty_T L^2_x} + \|\partial_x^{s+1} (\Phi(u) - \Phi(v))\|_{L^\infty_T L^2_x} \leq C T^{1/2} \delta d_{X_T}(u, v) + C T \delta (1 + \delta^s) d_{X_T}(u, v).

Moreover, by using (3.16) and Sobolev embedding, together with (3.15), we deduce that

$$\|\Phi(u) - \Phi(v)\|_{L^\infty_T L^2_x} \leq C T \|u^{|\alpha|} \partial_x (u - v)\|_{L^\infty_T L^2_x} + C T \|(u^{|\alpha|} - |v^{|\alpha|})\partial_x v\|_{L^\infty_T L^2_x} \leq C T \|u^{|\alpha|} \partial_x H^1_T \| \partial_x (u - v)\|_{L^\infty_T L^2_x} + C T \|(u^{|\alpha - 1|} + |v^{|\alpha - 1|})|u - v|\partial_x v\|_{L^\infty_T L^2_x} \leq C T \|u^{|\alpha|} \partial_x H^1_T \| \partial_x (u - v)\|_{L^\infty_T L^2_x} + C T \|(x)^{m} |u - v|\|_{L^\infty_T L^2_x} \|\partial_x v\|_{L^\infty_T L^2_x},$$

which yields

(3.20) \hspace{1cm} \|\Phi(u) - \Phi(v)\|_{L^\infty_T L^2_x} \leq C T (\delta + \delta^s) d_{X_T}(u, v).

Let us estimate

$$\|\langle x \rangle^{m} (\Phi(u) - \Phi(v))\|_{L^\infty_T L^\infty_x}.$$
Arguing as in (3.13), by means of the Sobolev embedding and (3.9), we first compute
\[
\| \langle x \rangle^m (\Phi(u) - \Phi(v)) \|_{L^\infty_t L^\infty_x} \\
\leq C \left\| \int_0^t \langle x \rangle^m U(t-s)(|u|^\alpha \partial_x u - |v|^\alpha \partial_x v)(s)ds \right\|_{L^\infty_t L^2_x} \\
+ C \left\| \int_0^t \langle x \rangle^m U(t-s)\partial_x(|u|^\alpha \partial_x u - |v|^\alpha \partial_x v)(s)ds \right\|_{L^\infty_t L^2_x} \\
(3.21) \\
\leq C \| \langle x \rangle^m (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L^1_t L^2_x} \\
+ C \| \langle x \rangle^m \partial_x(|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L^1_t L^2_x} \\
+ CT \| \partial_x^m (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L^1_t L^2_x} \\
+ CT \| \partial_x^{m+1}(|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L^1_t L^2_x} \\
= : J_1 + J_2 + J_3 + J_4.
\]

Using interpolation the terms $J_3$ and $J_4$ can be handled applying the same argument as in (3.19) and (3.20). Thus
\[
J_3 + J_4 \leq CT^{1/2} \delta d_{X_T}(u, v) + CT \delta(1 + \delta^s) d_{X_T}(u, v) \\
\text{whenever } T \leq 1. \text{ Further, it holds that}
\]
\[
J_1 \leq CT \| \langle x \rangle^m u \|_{L^\infty_t L^\infty_x} \left\| \langle x \rangle^{m-1} \partial_x (u - v) \right\|_{L^\infty_t L^2_x} \\
+ CT \| \langle x \rangle^m (u - v) \|_{L^\infty_t L^\infty_x} \left\| \langle x \rangle^{m-1} \partial_x v \right\|_{L^\infty_t L^2_x},
\]
which implies that
\[
J_1 \leq CT \delta d_{X_T}(u, v).
\]

Next, we estimate $J_2$, indeed,
\[
J_2 \leq C \| \langle x \rangle^m (|u|^\alpha - (\partial_x u)^2 - |v|^{\alpha-1}(\partial_x v)^2) \|_{L^1_t L^2_x} \\
+ C \| \langle x \rangle^m (|u|^\alpha \partial_x^2 u - |v|^\alpha \partial_x^2 v) \|_{L^1_t L^2_x} \\
= : J_{2,1} + J_{2,2}.
\]

Hence, it comes from (3.16) and Sobolev embedding that
\[
J_{2,1} \leq C \| \langle x \rangle^m |u|^{\alpha-1}((\partial_x u)^2 - (\partial_x v)^2) \|_{L^1_t L^2_x} \\
+ C \| \langle x \rangle^m (|u|^{\alpha-1} - |v|^{\alpha-1})(\partial_x v)^2 \|_{L^1_t L^2_x} \\
\leq CT \delta(1 + \delta) d_{X_T}(u, v).
\]

On the other hand, we obtain employing (3.16) once again and the argument leading to (3.17) that
\[
J_{2,2} \leq C \| \langle x \rangle^m (|u|^{\alpha} - |v|^{\alpha}) \partial_x^2 v \|_{L^1_t L^2_x} + C \| \langle x \rangle^m |u|^{\alpha} \partial_x^2 (u - v) \|_{L^1_t L^2_x} \\
\leq CT \delta d_{X_T}(u, v).
\]

Hence one establishes that
\[
J_2 \leq CT \delta(1 + \delta) d_{X_T}(u, v).
\]
Thus, collecting (3.21), (3.22), (3.23) and (3.24), it follows that
\[ \| \langle x \rangle^m (\Phi(u) - \Phi(v)) \|_{L_T^\infty L_x^\infty} \]
(3.25)
\[ \leq CT^{1/2} \delta d_{X_T}(u,v) + CT \delta (1 + \delta^s) d_{X_T}(u,v) \]
as long as \( T \leq 1 \). Finally, we turn to consider
\[ \| \langle x \rangle^m \partial_x^l (\Phi(u) - \Phi(v)) \|_{L_T^\infty L_x^2} \]
for any \( l \in [1, 4] \). By using (3.9) one has
\[ \leq C \| \langle x \rangle^m \partial_x^l (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L_T^1 L_x^2} \]
(3.26)
\[ + CT^m \| \partial_x^{2m+1} (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L_T^1 L_x^2} \]
\[ =: F_1 + F_2. \]

\( F_2 \) can be estimated following the argument in (3.22). Hence we have
\[ F_2 \leq CT^{1/2+m} \delta d_{X_T}(u,v) + CT^{m+1} \delta (1 + \delta^s) d_{X_T}(u,v). \]
(3.27)

Let us estimate \( F_1 \). A use of the triangle inequality tells us that
\[ F_1 \leq C \| \langle x \rangle^m (|u|^\alpha \partial_x u)^{l+1} - (|v|^\alpha \partial_x v)^{l+1} \|_{L_T^1 L_x^2} \]
\[ + C \| \langle x \rangle^m (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L_T^1 L_x^2} \]
\[ + C \| \langle x \rangle^m (|u|^\alpha \partial_x^{l+1} u - |v|^\alpha \partial_x^{l+1} v) \|_{L_T^1 L_x^2} \]
\[ =: F_{1,1} + F_{1,2} + F_{1,3}. \]

It comes from (3.1), (3.16) and Sobolev embedding that
\[ F_{1,1} \leq C \| \langle x \rangle^m (|u|^\alpha \partial_x u)^{l+1} - (|v|^\alpha \partial_x v)^{l+1} \|_{L_T^1 L_x^2} \]
\[ + C \| \langle x \rangle^m (|u|^\alpha \partial_x u - |v|^\alpha \partial_x v) \|_{L_T^1 L_x^2} \]
\[ \leq CT \delta^2 (1 + \delta) d_{X_T}(u,v). \]

On the other hand, combining (3.1) and (3.16), we obtain
\[ F_{1,3} \leq C \| \langle x \rangle^m (|u|^\alpha - |v|^\alpha) \partial_x^{l+1} v \|_{L_T^1 L_x^2} + C \| \langle x \rangle^m (|u|^\alpha \partial_x^{l+1} u - v) \|_{L_T^1 L_x^2} \]
\[ \leq CT \delta d_{X_T}(u,v). \]

Observe that if \( l = 4 \) we use (3.22) to estimate \( F_{1,3} \). As for \( F_{1,2} \), one sees from (3.1) and (3.16) that
\[ F_{1,2} \leq CT \| \langle x \rangle^m |u|^{\alpha-1} \partial_x^4 u \partial_x (u-v) \|_{L_T^\infty L_x^2} \]
\[ + CT \| \langle x \rangle^m |u|^{\alpha-1} \partial_x^4 (u-v) \partial_x v \|_{L_T^\infty L_x^2} \]
\[ + CT \| \langle x \rangle^m (|u|^{\alpha-2} - |v|^{\alpha-2}) |u-v| \partial_x^2 v \partial_x v \|_{L_T^\infty L_x^2} \]
Combining these estimates, it holds that

\begin{equation}
F_1 \leq CT \delta d_{X_T}(u,v) + CT \delta^1(1 + \delta)d_{X_T}(u,v).
\end{equation}

Therefore, by unifying (3.26) and (3.28), we establish

\begin{equation}
\langle x \rangle^m \partial_s^j \Phi(u) \leq CT \delta^{1/2} d_{X_T}(u,v) + CT \delta^1(1 + \delta^s)d_{X_T}(u,v).
\end{equation}

if \( T \leq 1 \). In conclusion, combining (3.19) with (3.20), (3.25) and (3.29), we see that

\[
d_{X_T}(\Phi(u), \Phi(v)) \leq \frac{1}{2} d_{X_T}(u,v)
\]

as long as \( T = T(\delta, s) \) is sufficiently small. This tells us \( \Phi \) is a contraction map in \( X_T \), that is, (GK) has a unique local solution in \( X_T \). The reminder of the proof is standard, so we omit the detail. This completes the proof. \( \square \)

**Proof of Theorem 1.6** Without loss of generality we shall assume \( x_0 = 0 \). First, we introduce a two parameter family of cut-off functions \( \chi_{\epsilon, b} \) : for any \( \epsilon > 0, b \geq 5\epsilon \)

\[
\chi_{\epsilon, b}(x) = \begin{cases} 
0, & x < \epsilon, \\
1, & x > b - \epsilon,
\end{cases}
\]

with

\[
\left\{
\begin{array}{l}
\chi'_{\epsilon, b}(x) \geq 0, \quad \text{supp}(\chi_{\epsilon, b}) \subseteq [\epsilon, \infty), \\
\text{supp}(\chi'_{\epsilon, b}) \subseteq [\epsilon, b - \epsilon], \\
\chi'_{\epsilon, b}(x) \geq \frac{1}{b - \epsilon}, \quad x \in [2\epsilon, b - 2\epsilon], \\
\chi_{\epsilon/2, b}(x) \geq c \chi_{\epsilon, b}(x) + \chi'_{\epsilon, b}(x), \quad x \in \mathbb{R}.
\end{array}
\right.
\]

By formally taking the \( s + j, (j = 1, \ldots, l) \) derivative of the equation in (GK), multiplying the result by \( \partial_s^{s+j} u(x, t) \chi_{\epsilon, b}(x + vt) \) for arbitrary \( \epsilon > 0, v > 0 \) and \( b \geq 5\epsilon \) and integrating the result in the space variable, after
some integration by parts, it follows that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int (\partial_x^{s+j} u)^2(x,t) \chi_{\epsilon,b}(x+vt) \, dx \\
- \frac{v}{2} \int (\partial_x^{s+j} u)^2(x,t) \chi_{\epsilon,b}',(x+vt) \, dx \\
+ \frac{3}{2} \int (\partial_x^{s+j+1} u)^2(x,t) \chi_{\epsilon,b}',(x+vt) \, dx \\
(3.30) \\
- \frac{1}{2} \int (\partial_x^{s+j} u)^2(x,t) \chi_{\epsilon,b}''(x+vt) \, dx \\
\pm \int \partial_x^{s+j}(|u|^{\alpha} \partial_x u) \partial_x^{s+j} u(x,t) \chi_{\epsilon,b}(x+vt) \, dx = 0.
\end{equation}

Note that the above formal computation is justified by arguing as in [3, Section 3]. The idea is to use the formula (3.30) and induction argument in \( l \in \mathbb{Z}^+ \) to establish (1.14) and (1.15).

**Case: \( l = 1 \):** We observe that the term \( A_2 \) in (3.30) is positive. Also, after integration in the time interval \([0, T]\), the terms \( A_1 \) and \( A_3 \) are bounded by using the second statement in (1.10). Hence, one just needs to consider the contributions of the term \( A_4 \) in (3.30).

Thus, we write

\begin{equation}
\partial_x^{s+1}(|u|^{\alpha} \partial_x u) = |u|^{\alpha} \partial_x^{s+2} u + 2\alpha |u|^{\alpha-1} \partial_x u \partial_x^{s+1} u + \ldots \\
+ c_{\alpha,s} |u|^{\alpha-(s+1)} (\partial_x u)^{s+1} \partial_x u.
\end{equation}

By integration by parts, one sees that

\begin{equation}
\int |u|^{\alpha} \partial_x^{s+2} u \partial_x^{s+1} u(x,t) \chi_{\epsilon,b}(x+vt) \, dx \\
= - \frac{\alpha}{2} \int |u|^{\alpha-2} \partial_x u (\partial_x^{s+1} u(x,t))^2 \chi_{\epsilon,b}(x+vt) \, dx \\
- \frac{1}{2} \int |u|^{\alpha} (\partial_x^{s+1} u(x,t))^2 \chi_{\epsilon,b}'(x+vt) \, dx = : B_1 + B_2.
\end{equation}

Since for \( x \in \mathbb{R} \)

\begin{equation}
|u|^{\alpha-1} |\partial_x u|(x,t) \leq c \| (x)^{m(1-\alpha)} \partial_x u(t) \|_{\infty},
\end{equation}

combining the facts that after integration in the time interval \([0, T]\),

\begin{equation}
\int (\partial_x^{s+1} u(x,t))^2 \chi_{\epsilon,b}'(x+vt) \, dx
\end{equation}

is bounded with

\begin{equation}
\sup_{0 \leq t \leq T} \| u(t) \|_{\infty} < \infty,
\end{equation}

one can control the contributions of the terms \( B_1 \) and \( B_2 \) in (3.32) for \( l = 1 \). The argument to estimate the second term in the right hand side (r.h.s.)
of (3.31) is similar to that already done for $B_1$ in (3.32). So it remains to consider the third term in the r.h.s. of (3.31). For this we write
\[
| \int |u|^{-\alpha-(s+1)}(\partial_x u)^{s+1}\partial_x u\partial_x^{s+1}u(x,t)\chi_{\epsilon,b}(x+vt)\,dx |
\]
(3.36)
\[
\leq c \langle x \rangle^{m(s+1-\alpha)}|\partial_x u|^s|\partial_x^{s+1}u(x,t)||\chi_{\epsilon,b}(x+vt)||dx|
\]
\[
\leq c \langle x \rangle^m \partial_x u \|^{s+1}_{L^2} \| \partial_x u \|_2 (\int (\partial_x^{s+1}u(x,t))^2 \chi_{\epsilon,b}(x+vt) \,dx)^{1/2},
\]
whose contribution can be bounded after using Gronwall’s inequality in (3.30).

This basically completes the proof of the case $l = 1$ in the induction argument, i.e. (1.14) and (1.15) with $j = l = 1$. We remark that the terms omitted in (3.31) and in the proof of Theorem 1.1 can be handled as they will be done in the next step, see (3.42)-(3.52).

Now assuming the result (1.14) and (1.15) for $l = r + 1$, we write
\[
\langle x \rangle^m \partial_x^j u \in L^2(\mathbb{R}), \quad k = 1, \ldots, 4,
\]
and by hypothesis of induction for any $\epsilon > 0$ and $b \geq 5\epsilon$
\[
(3.37) \quad \partial_x^{j+1}u(x,t)\varphi_{\epsilon,b}(x+vt) \in L^2(\mathbb{R}), \quad j = 1, \ldots, r, \quad \varphi_{\epsilon,b}(x) = \sqrt{\chi_{\epsilon,b}(x)}.
\]
Using that
\[
(3.38) \quad \langle x \rangle^m \chi'_{\epsilon,b}(x+vt) \leq c\chi_{\epsilon/2,b}(x+vt), \quad c = c(m;v;t;\epsilon; b),
\]
successive integration by parts show that for any $\theta \in [0,1]$ with $\theta k + (1-\theta)(s+r) \in \mathbb{Z}$, $k = 1, 2, 3, 4$
\[
(3.39) \quad \langle x \rangle^{\theta m} \partial_x^{\theta k+1-(1-\theta)(s+r)}u(x,t)\varphi_{\epsilon,b}(x+vt) \in L^2(\mathbb{R}),
\]
and by Sobolev embedding
\[
(3.40) \quad \langle x \rangle^{\theta m} \partial_x^{\theta k+1-(1-\theta)(s+r)-1}u(x,t)\chi_{\epsilon,b}(x+vt) \in L^\infty(\mathbb{R}),
\]
since the one already has the estimates for the lower order terms.

Thus, we need to estimate the term (3.31) in (3.30)
\[
(3.41) \quad \int \partial_x^{s+r+1}|u|^{\alpha}(\partial_x u)\partial_x^{s+r+1}u(x,t)\chi_{\epsilon,b}(x+vt)\,dx,
\]
so, we write
\[
\partial_x^{s+r+1}(|u|^{\alpha}\partial_x u) = |u|^\alpha\partial_x^{s+r+2}u + 2\alpha u|^{\alpha^{-1}}\partial_x u\partial_x^{s+r+1}u + D_{s+r+1},
\]
(3.42)
The argument to handle the contribution of the first two terms in the r.h.s. of (3.42) is similar to that described in (3.32)-(3.36) so it will be omitted. Then it remains to consider the contribution of $D_{s+r+1} = :D$ in (3.42) when
it is inserted in the term $A_4$ in (3.32) with $j = r + 1$. We observe that $D$ is the sum of terms which are product of factors involving at derivatives of order at most $s + r$. In fact, one has that
\begin{equation}
D = \sum_{n + \beta_0 = s + r + 1} \sum_{\beta_1 + \cdots + \beta_n = n + 1} \sum_{0 \leq \beta_0 \leq s + r - 1} \frac{c_{\beta}}{|u|^{\alpha - n} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0 + 1} u}.
\end{equation}

When $D$ is inserted in the term $A_4$ in (3.30), this yields an expression of the form
\begin{equation}
\int |u|^{\alpha - n} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0 + 1} u \partial_x^{\epsilon + r + 1} u(x, t) \chi(x, t) dx
\end{equation}
which are bounded in absolute value by
\begin{equation}
\int \langle x \rangle^{m(n - \alpha)} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0 + 1} u \partial_x^{\epsilon + r + 1} u(x, t) \chi(x, t) dx.
\end{equation}
Consequently, it suffices to see that
\begin{equation}
E_1 = \| \langle x \rangle^{m(n - \alpha)} \partial_x^{\beta_1} u \cdots \partial_x^{\beta_n} u \partial_x^{\beta_0 + 1} u \varphi(x, t) \|_2
\end{equation}
is controlled by a product of the terms in (3.39)-(3.40) which have been already bounded in the previous case $l = r$. By an appropriate modification of the parameters in $\chi$, see (3.37)-(3.38), one gets
\begin{equation}
E_1 \leq \prod_{j=1}^{n} \| \langle x \rangle^{m\theta_j} \partial_x^{(1 - \theta_j)(s + r) + \theta_j k_j - 1} u \chi_{\epsilon,b} \|_{L^\infty} \times \| \langle x \rangle^{m\theta_0} \partial_x^{(1 - \theta_0)(s + r) + \theta_0 k_0} u \varphi \|_2,
\end{equation}
where
\begin{equation}
k_0, k_j \in \{1, \ldots, 4\}, \quad j = 1, \ldots, n,
\end{equation}
\begin{equation}
\beta_j = (1 - \theta_j)(s + r) + \theta_j k_j - 1, \quad j = 1, \ldots, n,
\end{equation}
\begin{equation}
\beta_0 = (1 - \theta_0)(s + r) + \theta_0 k_0 - 1.
\end{equation}

We shall complete the proof by establishing that
\begin{equation}
\theta_0 + \theta_1 + \cdots + \theta_n \geq n - \alpha.
\end{equation}
After some computations from (3.48) one finds that
\begin{equation}
n + 2 = (n - (\theta_0 + \theta_1 + \cdots + \theta_n))(s + r) + \theta_0 k_0 + \theta_1 k_1 + \cdots + \theta_n k_n,
\end{equation}
which implies that
\begin{equation}
\theta_0 + \theta_1 + \cdots + \theta_n = n + \frac{\theta_0 k_0 + \theta_1 k_1 + \cdots + \theta_n k_n - (n + 2)}{r + s}
\end{equation}
\begin{equation}
\geq n + \frac{\theta_0 + \theta_1 + \cdots + \theta_n - (n + 2)}{r + s},
\end{equation}
since $k_0, k_j \in \{1, \ldots, 4\}$ and $j = 1, \ldots, n$. Recalling that $n > 1$ ($n \geq 2$) and $s \geq 2/\alpha + 4$, one gets the desired result
\begin{equation}
\theta_0 + \theta_1 + \cdots + \theta_n \geq \frac{n(s + r) - (n + 2)}{s + r - 1} \geq n - \frac{2}{r + 3 + \frac{2}{\alpha}} > n - \alpha.
\end{equation}
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References

[1] Rémi Carles and Dmitry Pelinovsky, On the orbital stability of Gaussian solitary waves in the log-KdV equation, Nonlinearity 27 (2014), no. 12, 3185–3202. MR3291147
[2] Thierry Cazenave and Ivan Naumkin, Local existence, global existence, and scattering for the nonlinear Schrödinger equation, Commun. Contemp. Math. 19 (2017), no. 2, 1650038, 20. MR3611666
[3] Pedro Isaza, Felipe Linares, and Gustavo Ponce, On the propagation of regularity and decay of solutions to the k-generalized Korteweg-de Vries equation, Comm. Partial Differential Equations 40 (2015), no. 7, 1336–1364. MR3341207
[4] Tosio Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR759907
[5] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620. MR1211741
[6] Diederik Johannes Korteweg and Gustav de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. (5) 39 (1895), no. 240, 422–443. MR3363408
[7] Felipe Linares and Gustavo Ponce, Introduction to nonlinear dispersive equations, 2nd ed., Universitext, Springer, New York, 2015. MR3308874
[8] Felipe Linares, Gleison N Santos, and Gustavo Ponce, On a class of solutions to the generalized derivative Schrödinger equations, preprint (2017), available at arXiv:1712.00863
[9] Joules Nahas and Gustavo Ponce, On the persistent properties of solutions to semilinear Schrödinger equation, Comm. Partial Differential Equations 34 (2009), no. 10–12, 1208–1227. MR2581970

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