An isoperimetric inequality for the harmonic mean of the Steklov eigenvalues in hyperbolic space

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Abstract. In this article, we prove an isoperimetric inequality for the harmonic mean of the first \((n - 1)\) nonzero Steklov eigenvalues on bounded domains in \(n\)-dimensional hyperbolic space. Our approach to prove this result also gives a similar inequality for the first \(n\) nonzero Steklov eigenvalues on bounded domains in \(n\)-dimensional Euclidean space.

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1. Introduction. Let \(\Omega\) be a bounded domain in a complete Riemannian manifold \((M, ds^2)\) with smooth boundary \(\partial \Omega\). Consider the Steklov eigenvalue problem on \(\Omega\)

\[
\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \mu u \quad \text{on } \partial \Omega.
\]

Here \(\Delta := -\text{div(\text{grad } u)}\), \(\nu\) is the outward unit normal to \(\partial \Omega\), and \(\frac{\partial u}{\partial \nu}\) denotes the directional derivative of \(u\) in the direction \(\nu\).

This problem was introduced by Steklov [11] for bounded domains in the plane in 1902. Its importance lies in the fact that the set of eigenvalues of the Steklov problem is the same as the set of eigenvalues of the well known Dirichlet-Neumann map. This map associates to each function \(u\) defined on \(\partial \Omega\) the normal derivative of its harmonic extension on \(\Omega\). It is known that the Steklov eigenvalues are discrete and form an increasing sequence \(0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \nearrow \infty\).

The interplay between the geometry of the manifold and the Steklov eigenvalues has recently attracted substantial attention. See [3–5, 7, 9, 10] and the references therein for recent developments. The problem of finding a domain...
under some geometric constraints, which optimizes the eigenvalues (or some combination of eigenvalues), is a classical question in spectral geometry. In this direction, the first result for the Steklov eigenvalues was given by Weinstock [12] in 1954. Using conformal map technique, he proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes $\mu_1$. Hersch and Payne [6] noticed that Weinstock’s proof gives a sharper isoperimetric inequality

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \geq \frac{P(\Omega)}{\pi},$$

where $P(\Omega)$ represents the perimeter of $\Omega \subset \mathbb{R}^2$.

Later F. Brock [2] generalized result (1) to $\mathbb{R}^n$ by fixing the volume of the domain and proved the following inequality: for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$,

$$\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \geq \frac{n}{\mu_1(B(R))},$$

where $B(R) \subset \mathbb{R}^n$ is a ball of radius $R$ such that $\text{vol}(\Omega) = \text{vol}(B(R))$.

In this paper, we extend Brock’s result and prove an isoperimetric inequality for the sum of reciprocals of the first $(n-1)$ nonzero Steklov eigenvalues on bounded domains in $n$-dimensional hyperbolic space. Further, our technique also provides a geometric proof of inequality (2). The main results of this article are as follows.

**Theorem 1.1.** Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space with constant curvature $-1$ and $\Omega \subset \mathbb{H}^n$ be a bounded domain with smooth boundary $\partial \Omega$. Then

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \sum_{i=1}^{n-1} \frac{1}{\mu_i(B(R))},$$

where $B(R) \subset \mathbb{H}^n$ is a geodesic ball of radius $R > 0$ such that $\text{vol}(\Omega) = \text{vol}(B(R))$. Further, equality holds if and only if $\Omega$ is a geodesic ball.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$ and $B(R) \subset \mathbb{R}^n$ be a geodesic ball of radius $R > 0$ such that $\text{vol}(\Omega) = \text{vol}(B(R))$. Then

$$\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \geq nR = \sum_{i=1}^{n} \frac{1}{\mu_i(B(R))}.$$  

Further, equality holds if and only if $\Omega$ is a geodesic ball.

In [1], Binoy and Santhanam considered the Steklov eigenvalue problem on bounded domains of a noncompact rank-1 symmetric space and proved that among all domains of fixed volume, the ball maximizes the first nonzero Steklov eigenvalue. In order to prove their result, authors constructed $n$ test functions for the first nonzero Steklov eigenvalue and derived an estimate for the Rayleigh quotient. We follow a similar approach to prove our results. However, in our proof, we construct test functions for the first $n$ nonzero Steklov eigenvalues and then simplify the Rayleigh quotient which requires different
arguments than [1] (for more details, see Remark 2.4 and 3.2). To the best of our knowledge, this is the first attempt to study the harmonic mean of the Steklov eigenvalues on bounded domains in hyperbolic space.

The rest of the paper is organized as follows. In Sect. 2, we state several known facts related to the first nonzero Steklov eigenvalue of a geodesic ball and also prove some results which are used to prove the main theorems. Proofs of Theorems 1.1 and 1.2 are given in Sect. 3.

2. Preliminaries. In this section, we begin with providing some facts related to the first $n$ nonzero Steklov eigenvalues on a geodesic ball in $\mathbb{H}^n$ and $\mathbb{R}^n$ and then prove some properties of its eigenfunction.

Let $M = \mathbb{H}^n$ or $\mathbb{R}^n$ with the Riemannian metric $ds^2 = dr^2 + \sin^2_k r g_{S^{n-1}}$, where $g_{S^{n-1}}$ represents the canonical metric on the $(n-1)$-dimensional unit sphere $S^{n-1}$ and

$$\sin_k r = \begin{cases} r & \text{if } M = \mathbb{R}^n, \\ \sinh r & \text{if } M = \mathbb{H}^n. \end{cases}$$

Define

$$\cos_k r = \begin{cases} 1 & \text{if } M = \mathbb{R}^n, \\ \cosh r & \text{if } M = \mathbb{H}^n. \end{cases}$$

2.1. Properties of the first $n$ nonzero Steklov eigenvalues on a ball. Let $B(R)$ be a geodesic ball in $(M, ds^2)$, then $\mu_1(B(R)) = \mu_2(B(R)) = \cdots = \mu_n(B(R))$ and $\mu_i(B(R)), 1 \leq i \leq n$, satisfies

$$\mu_i(B(R)) = \frac{\int_{B(R)} g^2(r) \frac{n-1}{\sin_k r} + (g'(r))^2 dV}{g^2(R) \text{vol}(S(R))},$$

where $g(r)$ is the radial function satisfying

$$g''(r) + \frac{(n-1)}{\sin_k r} g'(r) - \frac{(n-1)}{\sin_k r} g(r) = 0, \quad r \in (0, R),$$

$$g(0) = 0, \quad g'(R) = \mu_1(B(R)) g(R). \tag{3}$$

The equation $g''(r) + \frac{(n-1)}{\sin_k r} g'(r) - \frac{(n-1)}{\sin_k r} g(r) = 0$ can be rewritten as

$$g''(r) + \left( \frac{(n-1) \cos_k r}{\sin_k r} g(r) \right)' = 0.$$

Simplifying the above equation, we get

$$g(r) = \frac{1}{\sin_k^{n-1} r} \int_0^r \sin_k^{n-1} t dt. \tag{4}$$

See [1] for further details.

Remark 2.1. In case of $\mathbb{R}^n$, $g(r) = \frac{r}{n}$ and $\mu_1(B(R)) = \frac{1}{R}$.

The function $g$ defined in (4) satisfies the following properties:

Lemma 2.2. For $r > 0$,

$(i)$ $0 < g'(r) \leq \frac{g(r)}{\sin_k r}$,
(ii) \((g')^2(r) + \frac{n-1}{\sinh r}g^2(r)\) is a decreasing function of \(r\).

**Proof.** (i) It can be seen easily that in case of \(M = \mathbb{R}^n\), it holds true. Let \(M = \mathbb{H}^n\). Since \(g'(r) = 1 - (n-1)g(r)\coth r\), it follows that \(g'(r) \leq \frac{g(r)}{\sinh r}\) if and only if

\[
(1 + (n-1)\cosh r) \left( \int_0^r \sinh^{n-1} t \, dt \right) \geq \sinh^n r.
\]

Let \(h_1(r) = (1 + (n-1)\cosh r)\left( \int_0^r \sinh^{n-1} t \, dt \right)\) and \(h_2(r) = \sinh^n r\). Since \(h_1(0) = h_2(0) = 0\) and \(h_1(r), h_2(r)\) are both increasing functions of \(r\), to show that \(h_1(r) \geq h_2(r)\), it is enough to prove that \(h_1'(r) \geq h_2'(r)\) for \(r > 0\). Next \(h_1'(r) \geq h_2'(r)\) if and only if

\[
(n-1) \int_0^r \sinh^{n-1} t \, dt \geq (\cosh r - 1)\sinh^{n-2} r.
\]

Using a similar argument, we have that the above inequality is true if

\[
\sinh^{n-1} r \geq (\cosh r - 1)\sinh^{n-3} r \cosh r.
\]

This can be rewritten as \(\cosh r \geq 1\), which is true. Hence \(g'(r) \leq \frac{g(r)}{\sinh r}\).

In similar way, it can be proved that \(g'(r) > 0\) for \(r > 0\).

(ii) This can be proved by using a similar argument as in (i). We refer to [1] for a complete proof. \(\square\)

Now using the Borsuk-Ulam theorem, we construct test functions for the first \(n\) nonzero Steklov eigenvalues on a bounded domain \(\Omega \subset M\).

### 2.2. Construction of test functions.

Let \(A\) be a domain in \(M\) and let \(CA\) denote the convex hull of \(A\). Let \((X_1, X_2, \ldots, X_n)\) be a geodesic normal coordinate system at \(p\). We identify \(CA\) with \(\exp_{p}^{-1}(CA)\) and \(d_{p}^2(X, X)\) as \(\|X\|^2_{p}\) for \(X \in T_{p}(M)\). The following lemma gives the existence of a center of mass of \(A\) in \(M\) (see [8]).

**Lemma 2.3.** Let \(G : [0, \infty) \rightarrow \mathbb{R}\) be a continuous function which is positive on \((0, \infty)\). Then there exists a point \(q \in CA\) such that

\[
\int_{A} G(\|X\|_q)X \, ds = 0,
\]

where \((X_1, X_2, \ldots, X_n)\) is a geodesic normal coordinate system at \(q\).

The above lemma will be used to construct a test function for the first nonzero Steklov eigenvalue on bounded domains in \(M\). Let \(\Omega\) be a bounded domain in \(M\) and \(B(R)\) be a ball in \(M\) such that \(\text{vol}(\Omega) = \text{vol}(B(R))\). We denote by \(\{u_i\}_{i=0}^{\infty}\) a sequence of orthonormal eigenfunctions corresponding to eigenvalues \(\{\mu_i(\Omega)\}_{i=0}^{\infty}\). The variational characterization of \(\mu_i(\Omega), 1 \leq i < \infty\), is given by

\[
\mu_i(\Omega) = \inf_{0 \neq u \in H^1(\Omega)} \left\{ \frac{\int_{\Omega} \|\nabla u\|^2 \, dV}{\int_{\partial \Omega} u^2 \, dA} : \int_{\partial \Omega} uu_j \, dA = 0, 0 \leq j \leq i - 1 \right\}. \quad (5)
\]
Then by Lemma 2.3, there exists a point $p \in C(\partial \Omega)$ such that
\[
\int_{\partial \Omega} g(r) \frac{x_i}{r} dA = 0, \quad 1 \leq i \leq n.
\]
Here $g(r)$ is the function defined in (4), $(x_1, x_2, \ldots, x_n)$ is a geodesic normal coordinate system at $p$, and $r = \|\exp_p^{-1}(q)\|_p$, $q \in \partial \Omega$.

Next define mapping $f_n : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ as
\[
f_n(\sigma) = \left( \int_{\partial \Omega} g(r) \frac{\langle \exp_p^{-1}(q), \sigma \rangle}{r} u_1 dA, \int_{\partial \Omega} g(r) \frac{\langle \exp_p^{-1}(q), \sigma \rangle}{r} u_2 dA, \ldots, \int_{\partial \Omega} g(r) \frac{\langle \exp_p^{-1}(q), \sigma \rangle}{r} u_{n-1} dA \right).
\]
Since the function $f_n$ is antipode preserving, by the Borsuk-Ulam theorem, there exists $\sigma_n \in \mathbb{S}^{n-1}$ such that $f_n(\sigma_n) = 0$. Then we repeat a similar process with functions $f_{n-1}, f_{n-2}, \ldots, f_2$, where the function $f_m : \mathbb{S}^{m-1} \to \mathbb{R}^{m-1}$, $2 \leq m \leq (n-1)$, is defined componentwise as
\[
f_{m,k}(\sigma) = \int_{\partial \Omega} g(r) \frac{\langle \exp_p^{-1}(q), \sigma \rangle}{r} u_k dA, 1 \leq k \leq (m-1).
\]
Here $\sigma \in \mathbb{S}^{m-1}$ ranges over all unit vectors perpendicular to the ones already fixed. In particular, $f_{n-1}$ acts on $\sigma \in \mathbb{S}^{n-2}$, which is perpendicular to the already fixed unit vector $\sigma_n$ in the previous step. By repeating this inductive process, once $\sigma_2$ is chosen, $\sigma_1$ is effectively fixed.

Next we define a new normal coordinate system at $p$ by considering $\sigma_1, \sigma_2, \ldots, \sigma_n$ as $x_1$-axis, $x_2$-axis, $\ldots$, $x_n$-axis respectively. Since the new coordinate system is obtained by performing rotations, we have $\int_{\partial \Omega} g(r) \frac{x_i}{r} dA = 0, 1 \leq i \leq n$, in this coordinate system. Also in this new coordinate system,
\[
\int_{\partial \Omega} g(r) \frac{x_i}{r} u_j dA = 0, \quad 0 \leq j \leq i - 1. \quad (6)
\]

Remark 2.4. In [1], the authors use Lemma 2.3 to find test functions for the first nonzero Steklov eigenvalue but to find test functions for the first $n$ nonzero Steklov eigenvalues, we also need the Borsuk-Ulam theorem.

3. Proof of the main results. Let $\Omega$ be a bounded domain in $M$ with smooth boundary and $p$ be the center of mass defined as in the above section. Let $B(R)$ be the geodesic ball of radius $R > 0$ in $M$ centered at $p$ such that $\text{vol}(\Omega) = \text{vol}(B(R))$. We denote the geodesic sphere of radius $R > 0$ in $M$ centered at $p$ by $S(R)$. The following lemma is needed to prove the main result (see [1]).

Lemma 3.1. Let $g$ be the function defined as in (4). Then
\[
\int_{\partial \Omega} g^2(r) dA \geq \text{vol}(S(R)) g^2(R),
\]
and equality holds if and only if $\partial \Omega$ is a geodesic sphere of radius $R$ centered at $p$. 
From the variational characterization of $\mu_i(\Omega), 1 \leq i \leq n$, and (6), we have
\[
\mu_i(\Omega) \int_{\partial \Omega} \left( g(r) \frac{x_i}{r} \right)^2 dA \leq \int_{\Omega} \left\| \nabla \left( g(r) \frac{x_i}{r} \right) \right\|^2 dV.
\]
Substituting $\left\| \nabla \left( g(r) \frac{x_i}{r} \right) \right\|^2 = \left( g'(r) \right)^2 \frac{x_i^2}{r^2} + \frac{g^2(r)}{\sinh^2 r} \left( 1 - \frac{x_i^2}{r^2} \right)$ in the above equation, we get
\[
\int_{\partial \Omega} \left( g(r) \frac{x_i}{r} \right)^2 dA \leq \frac{1}{\mu_i(\Omega)} \int_{\Omega} \left( (g'(r))^2 \frac{x_i^2}{r^2} + \frac{g^2(r)}{\sinh^2 r} \left( 1 - \frac{x_i^2}{r^2} \right) \right) dV.
\]
Summing over $i$ from 1 to $n$ in the above equation, we get
\[
\int_{\partial \Omega} g^2(r) dA \leq \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{g^2(r)}{\sinh^2 r} dV + \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \left( (g'(r))^2 - \frac{g^2(r)}{\sinh^2 r} \right) \frac{x_i^2}{r^2} dV.
\]
(7)

3.1. Proof of Theorem 1.1. Using the fact that $\sum_{i=1}^{n} x_i^2 = r^2$, we have
\[
\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \frac{x_i^2}{r^2} = \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \frac{x_i^2}{r^2} + \frac{1}{\mu_n(\Omega)} \frac{x_n^2}{r^2} = \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \frac{x_i^2}{r^2} + \frac{1}{\mu_n(\Omega)} \left( 1 - \sum_{i=1}^{n-1} \frac{x_i^2}{r^2} \right) = \sum_{i=1}^{n-1} \left( \frac{1}{\mu_i(\Omega)} - \frac{1}{\mu_n(\Omega)} \right) \frac{1}{r^2} \frac{x_i^2}{r^2} + \frac{1}{\mu_n(\Omega)}.
\]
(8)
Since $0 < g'(r) \leq \frac{g(r)}{\sinh r}$ by Lemma 2.2, we have $(g'(r))^2 - \frac{g^2(r)}{\sinh^2 r} \leq 0$. Thus
\[
\sum_{i=1}^{n-1} \left( \frac{1}{\mu_i(\Omega)} - \frac{1}{\mu_n(\Omega)} \right) \int_{\Omega} \left( (g'(r))^2 - \frac{g^2(r)}{\sinh^2 r} \right) \frac{x_i^2}{r^2} dV \leq 0.
\]
(9)
Substituting (8) and (9) in (7), we have
\[
\int_{\partial \Omega} g^2(r) dA \leq \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{g^2(r)}{\sinh^2 r} dV + \frac{1}{\mu_n(\Omega)} \int_{\Omega} \left( (g'(r))^2 - \frac{g^2(r)}{\sinh^2 r} \right) dV
\]
\[
= \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{g^2(r)}{\sinh^2 r} dV + \frac{1}{\mu_n(\Omega)} \int_{\Omega} (g'(r))^2 dV
\]
\[
\leq \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{g^2(r)}{\sinh^2 r} dV + \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{(g'(r))^2}{(n-1)} dV
\]
\[
= \frac{1}{(n-1)} \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \left( \frac{n-1}{\sinh^2 r} g^2(r) + (g'(r))^2 \right) dV.
\]
(10)
Define $F(r) = \frac{n-1}{\sinh^2 r} g^2(r) + (g'(r))^2$. Since $F(r)$ is a decreasing function of $r$,

$$\int_{\Omega} F(r) dV = \int_{\Omega \cap B(R)} F(r) dV + \int_{\Omega \setminus (\Omega \cap B(R))} F(r) dV$$

$$\leq \int_{B(R)} F(r) dV - \int_{B(R) \setminus (\Omega \cap B(R))} F(r) dV + \int_{\Omega \setminus (\Omega \cap B(R))} F(r) dV$$

$$\leq \int_{B(R)} F(r) dV - \int_{B(R) \setminus (\Omega \cap B(R))} F(R) dV + \int_{\Omega \setminus (\Omega \cap B(R))} F(R) dV$$

$$= \int_{B(R)} F(r) dV.$$

Substituting this and $\int_{\partial \Omega} g^2(r) dA$ from Lemma 3.1 in (10), we get

$$g^2(R) \text{vol}(S(R)) \leq \frac{1}{(n-1)} \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{B(R)} \left( \frac{n-1}{\sinh^2 r} g^2(r) + (g'(r))^2 \right) dV.$$

This gives

$$\frac{1}{(n-1)} \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \frac{\text{vol}(S(R)) g^2(R)}{\int_{B(R)} \left( \frac{n-1}{\sinh^2 r} g^2(r) + (g'(r))^2 \right) dV} = \frac{1}{\mu_1(B(R))}.$$

Since $\mu_1(B(R)) = \mu_2(B(R)) = \cdots = \mu_n(B(R))$, we have the desired inequality. Moreover, equality holds if and only if $\Omega$ is a geodesic ball.

**Remark 3.2.** One major difference between our proof and the proof in [1] is as follows. In [1], the authors find an upperbound for the expression

$$\sum_{i=1}^{n} \left\| \nabla \left( g(r) \frac{x_i}{r} \right) \right\|^2$$

by using the fact that $\frac{x_i}{r}, i = 1, 2, \ldots, n,$ are eigenfunctions of the Laplace operator on a geodesic sphere and the property that $(g')^2(r) + \frac{n-1}{\sinh^2 r} g^2(r)$ is a decreasing function. Whereas in our proof, we need to find an upperbound for the expression $\sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)} \left\| \nabla \left( g(r) \frac{x_i}{r} \right) \right\|^2$. To get this, we first write an expression for $\left\| \nabla \left( g(r) \frac{x_i}{r} \right) \right\|^2$ explicitly for each $i = 1, 2, \ldots, n$, and then use various properties of the function $g$ (different from [1]) and the eigenvalues $\mu_i(\Omega), i = 1, 2, \ldots, n$.

**3.2. Proof of Theorem 1.2.** Since $g(r) = \frac{r}{n}$ and $\sinh r = r$ for $M = \mathbb{R}^n$, Equation (7) can be written as

$$\int_{\partial \Omega} r^2 dA \leq \text{vol}(\Omega) \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)}.$$

By Lemma 3.1, we get

$$R^2 \text{vol}(S(R)) \leq \text{vol}(B(R)) \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)}.$$
Since \( \text{vol}(B(R)) = \frac{R}{n} \text{vol}(S(R)) \), the above inequality thus becomes

\[
n R \leq \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)}.
\]

Recall that the first \( n \) nonzero Steklov eigenvalues of \( B(R) \) are \( \mu_1(B(R)) = \mu_2(B(R)) = \cdots = \mu_n(B(R)) = \frac{1}{R} \). Hence we have the desired result,

\[
\sum_{i=1}^{n} \frac{1}{\mu_i(B(R))} \leq \sum_{i=1}^{n} \frac{1}{\mu_i(\Omega)}
\]

and equality holds if and only if \( \Omega = B(R) \).

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