Open String Fluctuations in AdS with and without Torsion

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Abstract

The equations of motion and boundary conditions for the fluctuations around a classical open string, in a curved space-time with torsion, are considered in compact and world-sheet covariant form. The rigidly rotating open strings in Anti de Sitter space with and without torsion are investigated in detail. By carefully analyzing the tangential fluctuations at the boundary, we show explicitly that the physical fluctuations (which at the boundary are combinations of normal and tangential fluctuations) are finite, even though the world-sheet is singular there. The divergent 2-curvature thus seems less dangerous than expected, in these cases.

The general formalism can be straightforwardly used also to study the (bosonic part of the) fluctuations around the closed strings, recently considered in connection with the AdS/CFT duality, on $\text{AdS}_5 \times S^5$ and $\text{AdS}_3 \times S^3 \times T^4$. 
1 Introduction

Semi-classical quantization of strings in Anti de Sitter space leads to the result that the energy $E$ scales with a quantum number $N$, $E \sim N$ (for large $N$). This result, which is independent of the dimensionality of Anti de Sitter space, was originally obtained almost a decade ago by considering fluctuations around the string center of mass [1] (with $N$ being a combination of spin angular momentum and oscillation number), by considering circular strings [2] (with $N$ being the oscillation number), and by considering rigidly rotating strings [3] (with $N$ being the spin angular momentum).

This result has recently received a lot of attention in connection with the conjectured duality [4, 5, 6] between super string theory on AdS$_5 \times$ S$^5$ and $\mathcal{N} = 4$ SU(N) super Yang-Mills theory in Minkowski space. In the case of rigidly rotating strings, it was noticed [7] that the subleading term is logarithmic in the spin $S$

$$E - S \sim \ln(S)$$

which is essentially the same behavior as found for certain operators on the gauge theory side [8, 9, 10, 11, 12]. The fluctuations around the rigidly rotating closed strings were considered in [13, 14] and confirmed the logarithmic behavior.

The $E$-$N$ relationship has been further investigated in a number of papers including [15, 16, 17, 18, 19, 20, 21, 22, 23, 24] for various string configurations. The subleading terms depend on the particular string. For instance, for a circular string it is the square root of $N$ [2, 18]. It is also known [25, 26] that torsion may change the subleading terms. So this needs further studies.

Another important point is that the $E$-$N$ relation in most cases is obtained in a purely classical way or from a simple WKB approximation [2, 4, 27, 18] of the path integral, using a method developed in [28]. Quantization of the fluctuations, which is notoriously a complicated problem for extended strings in curved spacetimes, has only been done in very few cases (for instance the already mentioned [13, 14, 17]).

In the present paper, we consider more generally the fluctuations around particular string configurations, which may be open or closed. Basically, one can proceed in two ways: One possibility is to take the Polyakov action in conformal gauge and to expand the action and the constraints to second order. However, in a generic curved spacetime, it is not possible to solve the constraints and thereby to eliminate the unphysical modes. The other
possibility is to start with the Nambu-Goto action and impose more physical
gauge conditions and thereby eliminate the unphysical modes from the very
beginning. More precisely, for a string in a generic $D$-dimensional spacetime,
one has, when using the Polyakov action in conformal gauge, $D+2$ equations
for a complicated mixture of physical and unphysical fluctuations. When
using the Nambu-Goto action with suitable gauge conditions, one has on the
other hand only $D-2$ equations for the physical fluctuations only. This
is why we prefer to use the Nambu-Goto action supplemented by a term
representing the torsion.

The fluctuations around rigidly rotating strings in spacetimes with to-

torsion were recently considered in [23, 29]. However, only closed string s

discussed and the unphysical modes were not eliminated.

The paper is organized as follows: In section 2, we derive the action, equa-
tions of motion and boundary conditions for quadratic fluctuations around
a classical string configuration in a curved spacetime with torsion. The fluc-
tuations are split into normal and tangential contributions, so as to clarify
the physical situation. In sections 3 and 4, we discuss in detail the particular
examples of rigidly rotating open strings in AdS with and without torsion. In
section 5, we comment on the quantization and we present our conclusions.

2 General Formalism

We are interested in the fluctuations around a classical open planar string,
i.e. a string that is extended only in a 2+1 dimensional section of a higher-
dimensional curved spacetime. It is straightforward to generalize to com-
pletely general configurations, but that seems to be an unnecessary compli-
cation at the present moment. As explained in the introduction, we prefer
to work with the Nambu-Goto action with an additional term representing
the torsion.

The action is thus given by

$$ S = \frac{-1}{2\pi \alpha'} \int_M d\tau d\sigma L \quad (2.1) $$

where

$$ L = \sqrt{-\det(G_{\mu\nu}X^\mu,_{\alpha}X^\nu,_{\beta}) - \frac{1}{2} \epsilon^{\alpha\beta} B_{\mu\nu} X^\mu,_{\alpha} X^\nu,_{\beta}} \quad (2.2) $$
The first variation of the Lagrangian leads to

\[
\delta L = -\sqrt{-g} \left( g^{\alpha\beta} \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} \hat{K}_{\alpha\beta} N_\mu \delta X^\mu \right) + \partial_\alpha \left[ \sqrt{-g} \left( g^{\alpha\beta} G_{\mu\nu} + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right) X^\mu_{\;\;\beta} \delta X^\nu \right]
\]

(2.3)

where \( g_{\alpha\beta} \) is the induced metric on the world-sheet

\[
g_{\alpha\beta} = G_{\mu\nu} X^\mu_{\;\;\alpha} X^\nu_{\;\;\beta}
\]

(2.4)

while \( \hat{K}_{\alpha\beta} \) is the generalized extrinsic curvature, as constructed from the generalized Christoffel-symbols

\[
\hat{K}_{\alpha\beta} = \left( X^\mu_{\;\;\alpha\beta} + \hat{\Gamma}^\mu_{\rho\sigma} X^\rho_{\;\;\alpha} X^\sigma_{\;\;\beta} \right) N_\mu = \hat{\nabla}_\rho \left( X^\mu_{\;\;\alpha} \right) X^\rho_{\;\;\beta} N_\mu
\]

(2.5)

with

\[
\hat{\Gamma}^\mu_{\rho\sigma} = \Gamma^\mu_{\rho\sigma} - \frac{1}{2} H^\mu_{\rho\sigma}
\]

(2.6)

The equation of motion corresponding to Eq.(2.2) is

\[
\left( g^{\alpha\beta} - \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} \right) \hat{K}_{\alpha\beta} = 0
\]

(2.7)

generalizing the usual condition on the extrinsic curvature for a minimal surface, to the case of a spacetime with torsion.

For an open string, we get the boundary conditions from the second term in Eq.(2.3) (obtained after projection on the normal and tangential vectors)

\[
0 = \left[ B_{\mu\nu} X^\mu_{\tau} N^\nu \right]_{\tau=0,\pi}
\]

(2.8)

\[
0 = \left[ \sqrt{-g} - B_{\mu\nu} X^\mu_{\tau} X^\nu_{\sigma} \right]_{\sigma=0,\pi}
\]

(2.9)

Now we turn to the second variation of the Lagrangian. The derivation of \( \delta^2 L \) is a straightforward exercise in differential geometry following \[30, 31, 32, 33\]. But now, keeping all the surface terms, the result is \((D_\alpha \) is the
Here we have expanded the variation of $X^\mu$ on the normal and tangential vectors

$$\delta X^\mu = \varphi N^\mu + \psi^\alpha X^\mu_{,\alpha} \quad (2.11)$$

such that the normal fluctuations are represented by a world-sheet scalar field $\varphi$, while the tangential fluctuations are represented by a world-sheet vector $\psi^\alpha$.

Several comments to Eq. (2.10) are now in order. First, we notice that only the symmetric part of the extrinsic curvature appears. Second, the curvature tensor $\hat{R}_{\sigma\mu\nu\rho}$ is the generalized one obtained from the generalized Christoffel symbols Eq. (2.6). Third, the tangential fluctuations obviously only contribute in the surface terms, i.e. at the boundary, because of the reparametrization invariance in the bulk. On the other hand, the surface terms depend explicitly on the torsion $B_{\mu\nu}$ since the action (2.1) is not invariant under gauge transformations $\delta B_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]}$, but picks up a surface term.

In obtaining the equation (2.10), we also used the first order equation of motion (2.7), but we have not yet used the first order boundary conditions (2.8), (2.9). The reason for not having used the first order boundary conditions at this stage, is that the basis $(N^\mu, X^\mu_{,\alpha})$ generally is not well-defined at the boundary. Thus, one has to be very careful when implementing the boundary conditions. This will be clarified in the following sections.

The equation of motion for the normal fluctuations is then given by

$$\left[ D_\alpha D^\alpha + \hat{K}_{(\alpha\beta)} \hat{K}^{(\alpha\beta)} - \left( g^{\alpha\beta} - \frac{\epsilon_{\alpha\beta}}{\sqrt{-g}} \right) \hat{R}_{\sigma\mu\rho\nu} X^\mu_{,\alpha} X^\nu_{,\beta} N^\rho N^\sigma \right] \varphi = 0 \quad (2.12)$$
while the boundary conditions are

\[
0 = \left[ \sqrt{-g} g^{\alpha \beta} (\varphi_{,\alpha} + \hat{K}_{(\alpha\beta)} \psi_{,\beta}) + B_{\mu\nu} N_{,\alpha}^\mu X_{,\nu}^\nu (-g^{\alpha \beta} \hat{K}_{(\beta\tau)} \varphi + D_{,\tau} \psi_{,\alpha}) \right. \\
+ \left. \nabla_\nu B_{\mu \rho} N_{,\alpha}^\mu X_{,\rho}^\rho (N_{,\nu} \varphi + X_{,\gamma}^\gamma \psi_{,\gamma}) \right]_{\sigma = (0, \pi)} (2.13)
\]

\[
0 = \left[ - (\sqrt{-g} - B_{\mu\nu} X_{,\tau}^\mu X_{,\sigma}^\nu) (-g^{\alpha \beta} \hat{K}_{(\alpha\tau)} \varphi + D_{,\tau} \psi_{,\sigma}) \right. \\
+ \left. B_{\mu\nu} X_{,\tau}^\mu N_{,\tau}^\nu (\varphi_{,\tau} + \hat{K}_{(\tau\alpha)} \psi_{,\alpha}) \right]_{\sigma = (0, \pi)} (2.14)
\]

\[
0 = \left[ (\sqrt{-g} - B_{\mu\nu} X_{,\tau}^\mu X_{,\sigma}^\nu) (-g^{\alpha \tau} \hat{K}_{(\alpha\tau)} \varphi + D_{,\tau} \psi_{,\tau}) \right. \\
+ \left. B_{\mu\nu} X_{,\sigma}^\mu N_{,\tau}^\nu (\varphi_{,\tau} + \hat{K}_{(\tau\alpha)} \psi_{,\alpha}) + \nabla_\nu B_{\mu \rho} X_{,\sigma}^\mu X_{,\rho}^\rho (N_{,\nu} \varphi + X_{,\gamma}^\gamma \psi_{,\gamma}) \right]_{\sigma = (0, \pi)} (2.15)
\]

In the following sections, we shall consider these equations in some particular cases.

**3 SL(2,R) Background**

As a first example of our general formalism, we consider the rigidly rotating open strings in the SL(2,R) ∼ AdS\(_3\) background. One can for instance imagine a string in the Anti de Sitter part of the spacetime AdS\(_3\) × SU(2) × T\(^4\).

The SL(2,R) background is given by

\[
ds^2 = -(1 + H^2 r^2) dt^2 + \frac{dr^2}{1 + H^2 r^2} + r^2 d\phi^2, \quad B = -2H r^2 dt \wedge d\phi \quad (3.1)
\]

There are no straight folded strings in this background \[25, 26\], since the torsion bends and unfolds them. The simplest open strings, which generalize the straight folded strings in Minkowski space, are given by \[25\]

\[
t = c_0 \tau \quad (3.2)
\]

\[
r = \frac{c_1}{n} \cos(n\sigma) \quad (3.3)
\]

\[
\phi = \frac{c_0 \sqrt{n^2 + H^2 c_1^2}}{c_1} \tau + \frac{n^2}{H c_1} \sigma \]

\[
- \frac{\sqrt{n^2 + H^2 c_1^2}}{H c_1} \cot^{-1} \left( \frac{\sqrt{n^2 + H^2 c_1^2}}{n} \cot(n\sigma) \right) \quad (3.4)
\]

where \(c_0\) is just introduced for dimensional reasons. It has no physical importance, so we have a continuous 1-parameter family of solutions parametrized
by $c_1$. The integer $n$ gives the number of string segments (for $H = 0$, the
number of foldings).

In the following we take $n = 1$, corresponding to the leading Regge tra-
jectory. Then the induced metric on the world-sheet is given by

$$g_{\tau\tau} = -c_0^2 \sin^2 \sigma ,$$  \hspace{1cm} (3.5)

$$g_{\tau\sigma} = \frac{-H c_0 c_1^2 \sqrt{1 + H^2 c_1^2 \cos^2 \sigma \sin^2 \sigma}}{1 + H^2 c_1^2 \cos^2 \sigma} ,$$  \hspace{1cm} (3.6)

$$g_{\sigma\sigma} = \frac{c_1^2 \sin^2 \sigma [1 + (2 - \cos^2 \sigma) H^2 c_1^2 \cos^2 \sigma]}{(1 + H^2 c_1^2 \cos^2 \sigma)^2}$$  \hspace{1cm} (3.7)

which is singular at the boundary $\sigma = (0, \pi)$. Indeed, the scalar curvature of
the world-sheet is

$$R_2 = \frac{2}{c_1^2 \sin^4 \sigma}$$  \hspace{1cm} (3.8)

which is the same as in Minkowski space ($H = 0$).

The extrinsic curvature is

$$\hat{K}_{(\tau\tau)} = H c_0^2 \cos^2 \sigma ,$$  \hspace{1cm} (3.9)

$$\hat{K}_{(\tau\sigma)} = \frac{c_0 \sqrt{1 + H^2 c_1^2} [H^2 c_1^2 \cos^2 \sigma \sin^2 \sigma + 1]}{1 + H^2 c_1^2 \cos^2 \sigma} ,$$  \hspace{1cm} (3.10)

$$\hat{K}_{(\sigma\sigma)} = \frac{H c_1^2 [2 - 3 \cos^2 \sigma + H^2 c_1^2 \cos^2 \sigma (2 \sin^4 \sigma - \cos^4 \sigma)]}{(1 + H^2 c_1^2 \cos^2 \sigma)^2}$$  \hspace{1cm} (3.11)

$$\hat{K}_{[\tau\sigma]} = H c_0 c_1 \sin \sigma$$  \hspace{1cm} (3.12)

and one can easily verify that Eqs.(2.7)–(2.9) are fulfilled.

The energy and spin angular momentum of these strings are given by

$$E = \frac{1}{2\pi \alpha'} \int_0^\pi \frac{\partial L}{\partial t} d\sigma = \frac{c_1}{2\alpha'}$$  \hspace{1cm} (3.13)

$$S = \frac{-1}{2\pi \alpha'} \int_0^\pi \frac{\partial L}{\partial \phi} d\sigma = \sqrt{1 + H^2 c_1^2} - 1$$  \hspace{1cm} (3.14)

It follows that

$$E = H S \sqrt{1 + \frac{1}{H^2 \alpha' S}}$$  \hspace{1cm} (3.15)
such that for long strings

\[ E/H - S = \frac{1}{2H^2\alpha'} + \ldots \]  

(3.16)

i.e. the dominant subleading term is just a constant \[25, 26\].

We now turn to the fluctuations. Using that \( \hat{R}_{\sigma\mu\nu} = 0 \) for the SL(2,R) background (it is a group manifold), we get the following equation of motion for the normal fluctuations

\[ \left[ D^\alpha D_\alpha + 2H^2 - \frac{2}{c_1^2 \sin^4 \sigma} \right] \varphi = 0 \]  

(3.17)

The boundary conditions become

\[ 0 = \left. \left[ \frac{2H^2c_0c_1(1 + H^2c_1^2)\cos^3 \sigma}{\sin \sigma(1 + H^2c_1^2 \cos^2 \sigma)^2} \varphi - \frac{c_0^2 \sqrt{1 + H^2c_1^2}}{c_1(1 + H^2c_1^2 \cos^2 \sigma)} \psi^\tau \right. \right. \]

\[ + \frac{Hc_0c_1[1 + 2H^2c_1^2 \cos^2 \sigma - H^2c_1^2 \cos^4 \sigma]}{(1 + H^2c_1^2 \cos^2 \sigma)^2} \psi^\sigma - \frac{H^2c_1^3 \sqrt{1 + H^2c_1^2} \cos^3 \sigma \sin \sigma}{(1 + H^2c_1^2 \cos^2 \sigma)^2} \psi^\tau \]

\[ \left. \left. + \frac{Hc_0c_1}{c_1} \sqrt{1 + H^2c_1^2} \cos \sigma \sin \sigma \left( \frac{1 + 2H^2c_1^2 \cos^2 \sigma - H^2c_1^2 \cos^4 \sigma}{c_1^2} \right) \right] \right]_{\sigma = (0,\pi)} \]

\[ 0 = \left. \left[ - \frac{c_0^2 c_1 \sin \sigma}{1 + H^2c_1^2 \cos^2 \sigma} \left( \frac{\sqrt{1 + H^2c_1^2}}{c_1^2 \sin \sigma} \varphi - \frac{H \cos \sigma}{c_0} \varphi + \frac{\sin \sigma}{c_0} \psi^\sigma \right) \right. \right. \]

\[ + \frac{c_0 \cos \sigma(1 + H^2c_1^2 \cos^2 \sigma)}{c_1^2} \psi^\tau - H \sqrt{1 + H^2c_1^2} \cos \sigma \sin^2 \sigma \psi^\sigma \right] \left. \right]_{\sigma = (0,\pi)} \]

\[ 0 = \left. \left[ \frac{Hc_0c_1[1 + 2H^2c_1^2 \cos^2 \sigma - H^2c_1^2 \cos^4 \sigma]}{(1 + H^2c_1^2 \cos^2 \sigma)^2} \varphi \right. \right. \]

\[ + \frac{c_0c_1 \cos \sigma \sin \sigma[1 + 2H^2c_1^2 \cos^2 \sigma - H^2c_1^2 \cos^4 \sigma]}{(1 + H^2c_1^2 \cos^2 \sigma)^2} \psi^\sigma \]

\[ + \frac{Hc_0^2 c_1 \sqrt{1 + H^2c_1^2} \cos \sigma \sin^3 \sigma}{1 + H^2c_1^2 \cos^2 \sigma} \psi^\tau + \frac{c_0c_1 \sin^2 \sigma}{1 + H^2c_1^2 \cos^2 \sigma} \psi^\tau \]

\[ - \frac{H^2c_1^3 \sqrt{1 + H^2c_1^2} \cos \sigma \sin \sigma}{(1 + H^2c_1^2 \cos^2 \sigma)^2} \varphi \right] \left. \right]_{\sigma = (0,\pi)} \]  

(3.20)

where dot and prime denote derivation with respect to \( \tau \) and \( \sigma \). Notice that we keep all the trigonometric functions in the boundary conditions. The
reason is that they have to be expanded, since the functions \((\varphi, \psi^\alpha)\) turn out to be singular at the boundary, as we will now show.

At this moment we are not interested in the general solution of Eq. (3.17), which will later be discussed in the Appendix. Here we only consider the solution near the boundary, where the boundary conditions come into play. It is convenient to separate the equations using

\[
\begin{align*}
\varphi(\tau, \sigma) &= e^{-i\frac{\omega}{c_1}(\tau + \xi(\sigma))} f_\omega(\sigma) \\
\psi^\alpha(\tau, \sigma) &= e^{-i\frac{\omega}{c_1}(\tau + \xi(\sigma))} g^\alpha_\omega(\sigma)
\end{align*}
\] (3.21)

where the function \(\xi\) is defined by

\[
\xi' = \frac{H c_1^2 \sqrt{1 + H^2 c_1^2 \cos^2 \sigma}}{c_0 (1 + H^2 c_1^2 \cos^2 \sigma)}
\] (3.23)

The equation for the normal fluctuations now reduces to

\[
f''_\omega + \left(\omega^2 + 2H^2 c_1^2 \sin^2 \sigma - \frac{2}{\sin^2 \sigma}\right) f_\omega = 0
\] (3.24)

The solution near the boundary \(\sigma = 0\) (the analysis near the boundary \(\sigma = \pi\) is completely similar) is

\[
f_\omega = \frac{k_1}{\sigma} + k_1 \left(\frac{\omega^2}{2} - \frac{1}{3}\right) \sigma + k_2 \sigma^2 + \ldots
\] (3.25)

where \(k_1\) and \(k_2\) are arbitrary constants. Now Eqs. (3.18) - (3.20) lead to

\[
g^\alpha_\omega = \frac{d_{11}^\alpha}{\sigma^2} + \frac{d_{12}^\alpha}{\sigma} + d_{32}^\alpha
\] (3.26)

where

\[
\begin{align*}
d_{11}^\alpha &= -\frac{k_1}{c_0 \sqrt{1 + H^2 c_1^2}} \\
d_{12}^\alpha &= -H k_1 \\
d_2 &= 0 \\
d_3^\alpha &= -i \frac{\omega k_1 \sqrt{1 + H^2 c_1^2}}{c_1}
\end{align*}
\] (3.27)
while the finite terms are related by

\[ 0 = \frac{k_1}{6} \left( -2 - 18H^2c_1^2 + 3\omega^2 - 10H^4c_1^4 + 6H^2c_1^2\omega^2 + 3H^4c_1^4\omega^2 \right) \\
- c_0(1 + H^2c_1^2)^{3/2}d_3^\tau + Hc_1^2(1 + H^2c_1^2)d_3^\sigma \right) \]  

(3.28)

It follows that both the normal and the tangential fluctuations are infinite at the boundary. However, if we consider the physical fluctuations, which at the boundary \( \sigma = 0 \), are given by

\[ \delta X^\mu(\sigma = 0) = [\varphi N^\mu + \psi^\alpha X^\mu_{,\alpha}]_{\sigma=0} \]  

(3.29)

we find that the singularities precisely cancel, and we get a finite result

\[ \delta t(\sigma = 0) = \left( \frac{k_1[3\omega^2(1 + H^2c_1^2) + 2H^2c_1^2 - 4]}{6(1 + H^2c_1^2)^{3/2}} + d_3^\tau c_0 \right) e^{-i\frac{c_0}{c_1}\omega(\tau + \xi(0))} \]  

(3.30)

\[ \delta r(\sigma = 0) = i\omega k_1 \sqrt{1 + H^2c_1^2} e^{-i\frac{c_0}{c_1}\omega(\tau + \xi(0))} \]  

(3.31)

\[ \delta \phi(\sigma = 0) = \left( \frac{k_1[3\omega^2(1 + H^2c_1^2) + 8H^2c_1^2 + 2]}{6c_1(1 + H^2c_1^2)} \right) \right) e^{-i\frac{c_0}{c_1}\omega(\tau + \xi(0))} \]  

(3.32)

It should be stressed that one can always obtain finite normal and tangential fluctuations at one of the boundaries by choosing appropriate values of the integration constants. For instance, taking \( k_1 = 0 \) in Eqs. (3.25)-(3.27) makes everything finite at \( \sigma = 0 \). However, one can easily show (by considering the exact solution in some simple case, say \( H = 0 \); see the Appendix) that both normal and tangential fluctuations then still blow up at the boundary \( \sigma = \pi \). What we have shown here is that the physical fluctuations in any case are finite everywhere.

### 4 Ordinary Anti de Sitter Space

As a second example, we now consider the fluctuations around the rigidly rotating open string in ordinary anti de Sitter space. Again, we take 2+1 dimensions, which might be envisioned as a slice of AdS\(_5\) in AdS\(_5\) \( \times S^5 \).
This solution was originally constructed in [3]. The radial coordinate is usually expressed in terms of an elliptic function, but for comparison with section 3, we now use the following gauge

\[ t = c_0 \tau, \quad r = \frac{c_1}{n} \cos(n\sigma) \] (4.1)

The remaining coordinate is obtained by the ansatz \( \phi = \omega \tau \), where \( \omega \) is determined by the boundary conditions. Actually, this is the solution for a rigidly rotating open string (rotating around its center of mass) in an arbitrary static cylindrically symmetric spacetime with line element

\[ ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\phi\phi}(r)d\phi^2 + 2g_{t\phi}(r)dtd\phi \] (4.2)

and without torsion, as one can easily verify since the induced metric is diagonal while the extrinsic curvature has only the \( \tau\sigma \)-component. In the case of Anti de Sitter space, where the line element is given in Eq.(3.1), one finds that

\[ \phi = \frac{c_0 \sqrt{n^2 + H^2 c_1^2}}{c_1} \tau \] (4.3)

which is exactly the same time dependency as in the case considered in section 3. It must be stressed, though, that the solution considered here is not a special case of the solution considered in section 3. The two solutions only overlap for \( H = 0 \), i.e. in Minkowski space.

In the present case, the induced metric on the world sheet becomes (again we take the leading Regge trajectory, \( n = 1 \))

\[ g_{\tau\tau} = -c_0^2 \sin^2 \sigma, \quad g_{\sigma\sigma} = \frac{c_1^2 \sin^2 \sigma}{1 + H^2 c_1^2 \cos^2 \sigma} \] (4.4)

which is again singular at the boundary \( \sigma = (0, \pi) \), as follows from the scalar curvature

\[ R_2 = \frac{-2}{c_1^2 \sin^4 \sigma} [H^2 c_1^2 \cos^4 \sigma - 2H^2 c_1^2 \cos^2 \sigma - 1] \] (4.5)

In the absence of torsion, the extrinsic curvature is just the usual one (symmetric in its indices), which we now call \( K_{\alpha\beta} \). In the present case it is given by

\[
\begin{align*}
K_{\tau\tau} & = K_{\sigma\sigma} = 0 \\
K_{\tau\sigma} & = -c_0 \frac{\sqrt{1 + H^2 c_1^2}}{1 + H^2 c_1^2 \cos^2 \sigma}
\end{align*}
\] (4.6)
and trivially $g^{\alpha\beta} K_{\alpha\beta} = 0$, as mentioned above.

In this case, the energy and spin are given by

$$E = c_1 \sqrt{1 + H^2 c_1^2} E \left( \frac{H^2 c_1^2}{1 + H^2 c_1^2} \right)$$  \hspace{1cm} (4.7)

$$S = \frac{1 + H^2 c_1^2}{\pi \alpha' H^2} \left[ E \left( \frac{H^2 c_1^2}{1 + H^2 c_1^2} \right) - \frac{1}{1 + H^2 c_1^2} K \left( \frac{H^2 c_1^2}{1 + H^2 c_1^2} \right) \right]$$  \hspace{1cm} (4.8)

It is impossible to express $E$ directly in terms of $S$, but for long strings where the modulus of the elliptic integrals goes towards 1, one finds

$$E/H - S = \frac{1}{2 \pi \alpha' H^2} \ln(2 \alpha' H^2 S) + \ldots$$  \hspace{1cm} (4.9)

which should be compared with Eq. (3.16). Notice that the factors of 2 are not present for the corresponding closed folded string ($n = 2$).

Eqs. (4.7)-(4.9) are well-known, but we now turn to the fluctuations. The fluctuations were actually already considered in some cases in [27], but without imposing any boundary conditions for the fluctuations. For the corresponding folded closed string, where there are of course no boundary conditions at all, the fluctuations were considered in [13, 14].

The equation of motion for the normal fluctuations is

$$\left[ D^\alpha D_\alpha - 2H^2 - 2 \frac{1 + H^2 c_1^2}{c_1^2 \sin^4 \sigma} \right] \varphi = 0$$  \hspace{1cm} (4.10)

while the boundary conditions are

$$0 = \left[ \varphi' - \frac{c_0 \sqrt{1 + H^2 c_1^2}}{\sqrt{1 + H^2 c_1^2 \cos^2 \sigma}} \psi^\tau \right]_{\sigma = (0, \pi)}$$  \hspace{1cm} (4.11)

$$0 = \left[ c_0 \sqrt{1 + H^2 c_1^2} \sqrt{1 + H^2 c_1^2 \cos^2 \sigma} \varphi + c_1^2 \sin^2 \sigma \psi^\tau + c_0 \cos \sigma \sin \sigma (1 + H^2 c_1^2 \cos^2 \sigma) \psi^\tau \right]_{\sigma = (0, \pi)}$$  \hspace{1cm} (4.12)

$$0 = \sin^2 \sigma \psi^\tau + \cos \sigma \sin \sigma \psi^\sigma$$  \hspace{1cm} (4.13)

First we write

$$\varphi(\tau, \sigma) = e^{-i \frac{c_0}{c_1} \omega \tau} f_\omega(\sigma), \quad \psi^\alpha(\tau, \sigma) = e^{-i \frac{c_0}{c_1} \omega \tau} g^\alpha_\omega(\sigma)$$  \hspace{1cm} (4.14)
Now Eq.(4.10) becomes

\[
0 = (1 + H^2 c_1^2 \cos^2 \sigma) \frac{d^2 f_\omega}{d\sigma^2} - H^2 c_1^2 \cos \sigma \sin \sigma \frac{df_\omega}{d\sigma} \\
+ \left( \omega^2 - 2 \frac{1 + H^2 c_1^2}{\sin^2 \sigma} - 2 H^2 c_1^2 \sin^2 \sigma \right) f_\omega \tag{4.15}
\]

This equation is actually well-known in the mathematical literature where it is known as “Heun’s equation”; see the Appendix. Again we only consider the solution near the boundary \( \sigma = 0 \). The solution is

\[
f_\omega = \frac{k_1}{\sigma} + \frac{-k_1}{6(1 + H^2 c_1^2)} \left( 5H^2 c_1^2 + 2 - 3\omega^2 \right) \sigma + k_2 \sigma^2 + \ldots \tag{4.16}
\]

while the boundary conditions (4.11)-(4.13) lead to

\[
g_\omega^\alpha = \frac{d_1^\alpha}{\sigma^2} + \frac{d_2^\alpha}{\sigma} + d_3^\alpha \tag{4.17}
\]

where

\[
d_1^\alpha = -\frac{k_1}{c_0} \\
d_2^\alpha = 0 \\
d_3^\alpha = -\frac{k_1}{6c_0(1 + H^2 c_1^2)} (2H^2 c_1^2 + 2 - 3\omega^2) \\
d_1^\sigma = 0 \\
d_2^\sigma = -\frac{i\omega k_1}{c_1} \tag{4.18}
\]

while \( d_3^\sigma \) is arbitrary. We notice that both the normal and tangential fluctuations are infinite at the boundary. However, the physical fluctuations given by Eq.(3.29) turn out to be finite also in this case

\[
\delta t(\sigma = 0) = -\frac{k_1}{1 + H^2 c_1^2} (1 + H^2 c_1^2 - \omega^2) e^{-i\frac{c_0}{c_1} \omega \tau} \tag{4.19}
\]

\[
\delta r(\sigma = 0) = i\omega k_1 e^{-i\frac{c_0}{c_1} \omega \tau} \tag{4.20}
\]

\[
\delta \phi(\sigma = 0) = \frac{k_1}{c_1 \sqrt{1 + H^2 c_1^2}} (-H^2 c_1^2 + \omega^2) e^{-i\frac{c_0}{c_1} \omega \tau} \tag{4.21}
\]

with similar conclusions as at the end of section 3.
If we take $n = 2$ in Eqs. (4.1), (4.3) the result can be interpreted as a folded closed string. This is of course the philosophy in [7, 13]. In that case, the tangential fluctuations completely decouple and we are left with the normal fluctuations which diverge at the boundaries. Our arguments therefore seem to hold only for open strings. This is a problem which deserves further study.

5 Concluding Remarks

We derived the equations of motion for normal fluctuations around a classical string in a curved spacetime with torsion. The boundary conditions, involving both normal and tangential fluctuations, for open strings were also obtained. In the two cases of rigidly rotating open strings in AdS with and without torsion, it was shown that the divergent part of the tangential fluctuations at the boundary is completely determined by the divergent part of the normal fluctuations in such a way that the physical fluctuations are everywhere finite. This is a non-trivial result since the classical world-sheet is singular at the boundary. It should be stressed that this does not work for the closed folded strings (which exist in the absence of torsion), since the tangential fluctuations decouple completely in that case.

The relation between normal and tangential fluctuations at the boundary also shows that one cannot in advance set the tangential fluctuations equal to zero everywhere, since this would kill also the normal fluctuations.

The quantization of the normal fluctuations is a difficult task. It is equivalent to the quantization of a scalar field in a curved spacetime. Thus, it is necessary to make some approximations. In the cases of rigidly rotating strings, the problem is that the mass-term is $\sigma$-dependent. It was argued in [13] that the mass-term can be approximated by a constant, though, at least for long strings. Moreover, earlier arguments [34, 35] suggest that at the quantum level one should actually skip completely the part of the mass-term corresponding to $R_2$. In any case, one then ends up with the quantization of an ordinary massive scalar field.

The general formalism developed in section 2 simplifies considerably for closed strings, where the tangential fluctuations decouple completely and there are no boundary conditions. The closed strings, recently considered in connection with the AdS/CFT duality, on $\text{AdS}_5 \times S^5$ and $\text{AdS}_3 \times S^3 \times T^4$ (for instance [15, 26] and references given therein), can then be straightforwardly analyzed. This is currently under investigation.
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A Appendix

In this appendix we consider the equations for the fluctuations (4.15) in more detail. Equation (4.15) is

\[
0 = (1 + H^2 c_1^2 \cos^2 \sigma) \frac{d^2 f_\omega}{d\sigma^2} - H^2 c_1^2 \cos \sigma \sin \sigma \frac{df_\omega}{d\sigma} + \left( \omega^2 - 2 \frac{1 + H^2 c_1^2}{\sin^2 \sigma} - 2 H^2 c_1^2 \sin^2 \sigma \right) f_\omega
\]  

(A.1)

First, we introduce the conformal coordinate \( \tilde{\sigma} \)

\[
\cos \sigma = \text{cn} \left[ \sqrt{1 + H^2 c_1^2} \tilde{\sigma}, m \right]
\]  

(A.2)

where \( \text{cn} \) is the Jacobi elliptic function with modulus \( m = H^2 c_1^2 / (1 + H^2 c_1^2) \). Then equation (A.1) becomes

\[
\frac{d^2 f_\omega}{d\tilde{\sigma}^2} + \left( \omega^2 - 2 \frac{1 + H^2 c_1^2}{\sin^2 \tilde{\sigma}} - 2 H^2 c_1^2 \sin^2 \tilde{\sigma} \right) f_\omega = 0
\]  

(A.3)

with \( \text{sn} = \text{sn} \left[ \sqrt{1 + H^2 c_1^2} \tilde{\sigma}, m \right] \). Notice that (A.3) generalizes the Lamé equation in the same way as eq.(3.24) generalizes the Pöschl-Teller equation \[36\]. However, we can go one step further by introducing the coordinate

\[
z = \sin^2 \sigma = \text{sn}^2 \left[ \sqrt{1 + H^2 c_1^2} \tilde{\sigma}, m \right]
\]  

(A.4)

Then eq.(A.3) reduces to

\[
0 &= \frac{d^2 g}{dz^2} + \left[ \frac{5/2}{z} + \frac{1/2}{z-1} + \frac{1/2}{z-(1/H^2 c_1^2 + 1)} \right] \frac{dg}{dz} + \frac{z + (\omega^2/4 H^2 c_1^2 - 1/H^2 c_1^2 - 2)}{[z - (1/H^2 c_1^2 + 1)]z(z-1)} g
\]  

(A.5)

where \( g = f_\omega / z \). Eq.(A.5) is a special case of Heun’s equation \[37\]

\[
0 = \frac{d^2 g}{dz^2} + \left[ \gamma \frac{\delta}{z} + \frac{\epsilon}{z-1} + \frac{\alpha \beta z - q}{z-a} \right] \frac{dg}{dz} + \frac{\alpha \beta z - q}{(z-a)z(z-1)} g
\]  

(A.6)
where $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. The parameters are $\alpha = 2$, $\beta = 1/2$, $\gamma = 5/2$, $\delta = 1/2$, $\epsilon = 1/2$. The accessory parameters are $q = -(\omega^2/4H^2c_1^2 - 1/H^2c_1^2 - 2)$ and $a = 1/H^2c_1^2 + 1$.

For $H = 0$ the solution to eq. (A.5) is just a hypergeometric function. For general $H$ the solution can be expanded on hypergeometric functions, but we shall not go into the details here.
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