Copulas for Streaming Data

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Abstract

Empirical copula functions can be used to model the dependence structure of multivariate data. This paper adapts the Greenwald and Khanna algorithm in order to provide a space-memory efficient approximation to the empirical copula function of a bivariate stream of data. A succinct space-memory efficient summary of values seen in the stream up to a certain time is maintained and can be queried at any point to return an approximation to the empirical copula function with guaranteed error bounds. This paper then gives an example of a class of higher dimensional copulas that can be computed from a product of these bivariate copula approximations. The computational benefits and the approximation error of this algorithm is theoretically and numerically assessed.

1 Introduction

Streaming data is found in many applications where data is acquired continuously. These characteristics, in addition to any space-memory constraints of the user, make such data a challenge for analyses. As data is acquired the analyser of the data must utilise it before the next piece of data is acquired and the entire stream cannot be stored. Therefore, given a particular statistical quantity of the data, a summary of the data with respect to this quantity must be maintained throughout time. This summary is typically much smaller in size than the entire stream. The idea of this summary is to allow an approximation of the desired statistical quantity to be made at any time with only a single pass of the data.

Estimating the quantiles of a data stream is a popular example of such a statistical quantity (Buragohain and Suri, 2009). A host of studies (Arandjelović et al., 2015; Greenwald and Khanna, 2001; Munro and Paterson, 1978; Manku et al., 1998) propose methods to construct succinct summaries of univariate data that can be queried at any time to obtain approximate quantiles within a guaranteed error bound $\epsilon$ (e.g. $\epsilon$-approximate quantile summaries). However data is rarely univariate. Copula functions (empirical) are a natural way to model the dependencies between multiple streams of data. This paper adapts the aforementioned Greenwald and Khanna algorithm (Greenwald and Khanna, 2001) to construct an alternative bivariate data summary, returning queries to the empirical copula function with guaranteed error bounds. Whilst the paper doesn’t directly extend the summary to higher dimensions, one can construct models of dependence for such multidimensional data using sets of pair-wise copulas (Aas et al., 2009). In particular, this paper gives an example of a class of multidimensional copulas that are formed from products of bivariate copulas (Mazo et al., 2015). Therefore, approximations to such a copula can be found by using the $\epsilon'$-accurate bivariate copula functions considered here.

This work is related to other studies that also consider the construction of summaries for multidimensional data. These summaries have been used to query multidimensional ranks and ranges (Hershberger et al., 2004; Suri et al., 2006; Yiu et al., 2006). Querying multidimensional ranges, such as a rectangle of points on the plane, is analogous to finding empirical copulas, only considering the actual data points on the plane rather than the marginal quantiles. This is where our motivation differs to that of Suri et al.
Empirical copulas

Copulas represent a joint probability distribution of a multidimensional random variable, and therefore can capture the dependence structure between components. The joint distribution is such that the marginal probability distributions of each component are uniform. Suppose we have two random variables $X \in \mathbb{R}$ and $Y \in \mathbb{R}$, with marginal cumulative distribution functions (CDF) $F_X$ and $F_Y$ respectively. Then the copula function $C(u_1, u_2)$ is defined by,

$$C(u_1, u_2) = F_{X,Y} \left(F_X^{-1}(u_1), F_Y^{-1}(u_2)\right),$$

where $(u_1, u_2) \in [0,1]^2$ and $F_{X,Y}$ is the joint CDF of $X$ and $Y$. There exist families of analytical copulas such as the Gaussian copula and Archimedean copulas, which can be fit to data. In many data sets however, one wishes to compute an empirical copula where the dependence structure is unknown in advance. This empirical copula is based on concordant and discordant ranks of data points and therefore is linked to Kendall Tau correlation.

Suppose $I(x \leq y)$ is the indicator function, taking the value of 1 if $x \leq y$, and 0 if $x > y$. Also let $\{\tilde{x}^i\}_{i=1}^n$ be the order statistics of the data stream $\{x^i\}_{i=1}^n \in \mathbb{R}^n$, such that $\tilde{x}^1 < \tilde{x}^2 < ... < \tilde{x}^n$. An empirical copula of the bivariate data stream $\{x^i_{(1)}, x^i_{(2)}\}_{i=1}^n \in \mathbb{R}^{2 \times n}$ is given by

$$\hat{C}(u_1, u_2) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^2 I(U_{(j)}^i \leq [u_n^j n]),$$

where $U_{(1)}$ and $U_{(2)}$ are the ranks (from 1 being the smallest element to $n$ being the largest element) of the samples $x^i_{(1)}$ and $x^i_{(2)}$ amongst all of the samples in entire stream. This copula weakly converges (with the number of samples $n$) to the true underlying dependence structure between the two components of the data stream [Deheuvels, 1980]. To simplify the analysis later on in the paper, if $\{x_{(2)}^{(i)}\}_{i=1}^{n_1}$, for $I \subset \{1,\ldots,n\}$, are the samples corresponding to the set of data points $\{x_{(1)}^{(i)}\}_{i=1}^{n_1}$ that have ranks less than or equal to $[u_1 n]$, then this can be expressed as

$$\hat{C}(u_1, u_2) = \frac{1}{n} \sum_{i=1}^{n_1} I(U_{(2)}^{(i)} \leq [u_2 n]) = \frac{1}{n} \sum_{i=1}^{n_1} I(x_{(2)}^{(i)} \leq \tilde{x}_{(2)}^{u_2 n}).$$

If we define the following empirical CDF,

$$\hat{F}_{n_{(1)},(2)}(y) = \frac{1}{n_1} \sum_{i=1}^{n_1} I(x_{(1)}^{(i)} \leq y),$$

then one can state

$$\hat{C}(u_1, u_2) = \frac{n_1}{n} \hat{F}_{n_{(1)},(2)}(\tilde{x}_{(2)}^{u_2 n}) = \frac{n_1}{n} \hat{F}_{n_{(1)},(2)}(\hat{F}_{n_{(2)},(1)}^{-1}(u_2)).$$

2 Empirical copulas
where $\hat{F}_{n,(2)}^{-1}(u) = \hat{x}_{(2)}^{[un]}$ is the empirical quantile function of $\{x_{(2)}^i\}_{i=1}^n$ ([Ma et al., 2011]). The problem that this paper considers is when elements are continuously added to the stream. In this case, one does not want to store the entire stream and cannot feasibly update the order statistics of $x_{(1)}^i$ and $x_{(2)}^i$ after every new data point is added. Therefore the following algorithm can be implemented to maintain an approximation of $\hat{C}(u_1, u_2)$.

3 The algorithm

3.1 Data structure

The algorithm uses a data structure very similar to both those in [Suri et al., 2006] and [Hershberger et al., 2004]. The latter uses a data structure for when the data stream elements are discrete. As done in the former, the data structure used here stores multiple versions of the data structure that was proposed in Greenwald and Khanna [2001]: summaries of certain values seen in a data stream $\{x^i\}_{i=1}^n$ that each ‘cover’ the quantities within a different range (e.g. the 0–0.1 quantiles). The size of these ranges are dependent on the approximation error that the user prescribes. On this note, define an $\epsilon$-approximate quantile summary as one that can be queried for the $\phi$-quantile and return a value $\hat{x}^j$, where $j \in [\phi n] - cn, [\phi n] + cn$. The algorithm proposed in this section starts by maintaining an $\epsilon$-approximate quantile summary, $S_{(1)}^\epsilon$, for $\{x_{(1)}^i\}_{i=1}^n$. Suppose this summary is $L$ elements long. The summary is composed from the following tuples: $(v_i, g_{(1)}^i, \Delta_{(1)}^i, S_{(2)}^i)$, for $i = 1, ..., L$. Here, $v_i \in \{x_{(1)}^k\}_{k=1}^L$ is the first component of a data point seen in the stream so far. The parameters $g_{(1)}^i$ and $\Delta_{(1)}^i$ enforce the range of quantiles that each element $v_i$ covers in the stream $\{x_{(1)}^k\}_{k=1}^L$. This will be explained in more depth in the next paragraph. Finally, $S_{(2)}^i$ is an $\epsilon$-approximate quantile subsummary for the second component of a selection of the data points seen in the stream so far. These points will not in general correspond to points with the first component $v_i$ (i.e. the coupling between the two components of each point is lost), however it is permissible for the motivation of this paper. Each subsummary is formed of tuples $(w_j, g_{(2)}^{i,j}, \Delta_{(2)}^{i,j})$, for $j = 1, ..., L_i$, where $w_j \in \{x_{(2)}^k\}_{k=1}^L$. The parameters $g_{(2)}^{i,j}$ and $\Delta_{(2)}^{i,j}$ work in the same way as $g_{(1)}$ and $\Delta_{(1)}$ in enforcing ranges of quantiles. Therefore, for each element in the summary $S_{(1)}$, there is a subsummary $S_{(2)}^i$, for $i = 1, ..., L$. This data structure resembles a grid of the joint ranks of the data, and is analogous to the grid of quantiles used in [Xiao, 2017] to this end. The summary $S_{(1)}$ will henceforth be referred to as the copula summary.

Given an $\epsilon$-approximate quantile summary $Q$ of the stream $\{x^i\}_{i=1}^n$, the parameters $g^i$ and $\Delta^i$ in all tuples within the summary are the only required information to infer the range of quantiles that each element in the summary, $v_i$, covers. On this note, let $r_{\text{min},Q}(v_i) = \sum_{j=1}^{v_i} g^j$ be the rank of the element that corresponds to the minimum quantile covered by $v_i$. Also, let $r_{\text{max},Q}(v_i) = r_{\text{min},Q}(v_i) + \Delta^i$ be the rank of the element that corresponds to the maximum quantile covered by $v_i$. Therefore these are minimum and maximum bounds on the rank that the element $v_i$ took in the original stream. This means that the upper bound on the number of elements in the original stream between $v_{i-1}$ and $v_i$ is $g^i + \Delta^i - 1$. The Greenwald and Khanna algorithm guarantees that

$$r_{\text{max},Q}(v_i) - r_{\text{min},Q}(v_{i-1}) = g^i + \Delta^i \leq 2cn,$$

at all times. Due to this guarantee, it follows that a query of the rank of an element $v$ in the original stream, where $v_{i-1} \leq v \leq v_i$, can be answered to within an $cn$ tolerance ([Greenwald and Khanna, 2001]).

Recall the data structure described in the previous paragraph. In light of this explanation, let

$$r_{\text{min},S_{(1)}}(v_i) = \sum_{j=1}^{i} g^j_{(1)} \quad \text{and} \quad r_{\text{max},S_{(1)}}(v_i) = r_{\text{min},S_{(1)}}(v_i) + \Delta_{(1)}^i,$$

for $i = 1, ..., L$. Also let

$$r_{\text{min},S_{(2)}}(w_j) = \sum_{k=1}^{j} g^k_{(2)} \quad \text{and} \quad r_{\text{max},S_{(2)}}(w_j) = r_{\text{min},S_{(2)}}(w_j) + \Delta_{(2)}^{i,j},$$

for $j = 1, ..., L_i$.
for \( j = 1, \ldots, L_i \). The following sections will describe how this data structure can be updated and used to answer copula function queries to a particular error tolerance.

### 3.2 The insert and combine operations

The insert and combine operations used to maintain the standard \( \epsilon \)-approximate quantile summaries in [Greenwald and Khanna (2001)](greenwald2001) are now explained and modified for use within this paper.

#### 3.2.1 Insert

When an element \((v_*, w_*)\) gets added to the stream the following occurs:

1. Define the following new subsummary \( S^*_2 = \{(w_*, 1, 0)\}\).
2. If \( v_* < v_1 \), then input \((v_*, 1, 0, S^*_2)\) at the start of \( S_1 \). Conversely, if \( v_* \geq v_L \), then input 
\((v_*, 1, 0, S^*_2)\) at the end of \( S_1 \).
3. Otherwise, find \( i \) where \( v_i \leq v_* < v_{i+1} \). Then compute \( \Delta^*_i = g^{i+1}_1 + \Delta_1^{i+1} - 1 \), and insert \((v_*, 1, \Delta^*_i, S^*_2)\) into \( S_1 \) inbetween \((v_i, g^i_1, \Delta_1^i, S^*_2)\) and \((v_{i+1}, g^{i+1}_1, \Delta_1^{i+1}, S^*_2)\) if \( S^*_2 \) is divisible by \( \epsilon_n \).

#### 3.2.2 Combine

The combine operation is implemented via the method below, whenever \( n \) is divisible by \( 1/(2\epsilon) \), and \( L \geq 3 \). Start with \( j = L \).

1. Set \( k = j \). If \( g^j_1 + \Delta^j \) \( \leq 2\epsilon n \), decrease \( k \) by one at a time assuring \( \sum_{i=0}^{k} g^j_i + \Delta^j \leq 2\epsilon n \). When this is no longer possible, or \( j - k = 2 \), stop decreasing \( k \).
2. Merge the \( \epsilon \)-approximate subsummaries \( S^j_2, \ldots, S^j_2 \) to form the new \( \epsilon \)-approximate subsummary \( Q = M(S^{j-k}_2, \ldots, S^j_2) \). The merge operation is explained in the next section.
3. Replace the tuples 
\((v_j-k, g^{j-k}_1, \Delta_1^{j-k}, S^j_2), \ldots, (v_j, g^j_1, \Delta_1^j, S^j_2)\)
by the tuple \((v_j, \sum_{i=0}^{k} g^j_{i}, \Delta^j_1, Q)\).
4. Finally, the tuples \((w, g^i_2, \Delta_2)\) within the subsummary \( Q \) are also combined in the same way as steps (1) and (3) to remove unnecessary tuples.
5. Set \( j = j - k \), then go back to step (1) if \( j > 2 \). This final constraint guarantees that the summary maintains the smallest and largest values of \( x_{(1)} \) seen in the stream.

The aim of this operation is to simultaneously refine all \( L + 1 \) \( \epsilon \)-approximate quantile summaries in the data structure, limiting the space-memory used. By requiring that the first tuple in each summary/subsummary is not combined with any other tuple during this operation, the algorithm preserves the smallest and largest elements seen in both \( \{x^1_{(1)}\}_{i=1}^n \) and \( \{x^1_{(2)}\}_{i=1}^n \).
3.3 The merge operation

The merge operation for merging quantile summaries, introduced in Greenwald and Khanna (2004), is now explained. This was utilised within the combine operation explained in the previous section. Suppose we can merge the $\epsilon$-approximate summaries $Q_1$, of length $L_{Q_1}$, and $Q_2$, of length $L_{Q_2}$, to obtain the summary $M(Q_1, Q_2)$. This summary is also $\epsilon$-approximate, and is of length $L_{Q_1} + L_{Q_2}$. In general, the merge operation works as follows. Suppose $M(Q_1, Q_2)$ has the elements $Q_1 \cup Q_2$. Suppose $w_k$ is an element in $M(Q_1, Q_2)$ from $Q_1$. Let $w_1$ be the largest element (if it exists) in $Q_2$ that is less than or equal to $w_k$. Let $w_2$ be the smallest element (if it exists) in $Q_2$ that is greater than $w_k$. Then the parameters $r_{\text{min},M(Q_1, Q_2)}(w_k)$ and $r_{\text{min},M(Q_1, Q_2)}(w_k)$ are given by

$$r_{\text{min},M(Q_1, Q_2)}(w_k) = \begin{cases} r_{\text{min},Q_2}(w_1) + r_{\text{min},Q_1}(w_k), & \text{if } w_1 \text{ exists} \\ r_{\text{min},Q_1}(w_k), & \text{otherwise,} \end{cases}$$

and

$$r_{\text{max},M(Q_1, Q_2)}(w_k) = \begin{cases} r_{\text{max},Q_2}(w_2) + r_{\text{max},Q_1}(w_k) - 1, & \text{if } w_2 \text{ exists} \\ r_{\text{max},Q_2}(w_1) + r_{\text{max},Q_1}(w_k), & \text{otherwise.} \end{cases}$$

In the implementations of the merge operation used in this paper, we treat a merge of a summary, $R$, with only one tuple $(v, g, \Delta) = (v, 1, 0)$ and a summary $Q$ slightly differently to this. The insert operation from Greenwald and Khanna (2001) is implemented on $(v, g, \Delta)$, into the summary $Q$. The merge operation can also be used recursively: the summary $M(Q_1, Q_2, Q_3)$ for example can be constructed by merging $M(Q_1, Q_2)$ and $Q_3$. For more information on merging summaries, including the accuracy when merging an $\epsilon$-approximate and $\epsilon'$-approximate summary, for any $\epsilon' \neq \epsilon$, turn to Greenwald and Khanna (2004).

3.4 Querying the summary

The previous sections described how to maintain the copula summary $S(1)$ over time. Now the following sections explain how this summary can be queried at any time to return an approximation to the empirical copula function. The data structure representing a grid of joint ranks, described earlier in this section, is inspired by the form of this query and the accompanying analysis, where one is required to evaluate the copula function at two uniform marginals $(u_1, u_2) \in [0, 1]^2$. First we recall how to query a single $\epsilon$-approximate quantile summary.

3.4.1 Querying a single quantile summary

A single $\epsilon$-approximate summary $Q$ can be queried for the $\phi$-quantile by the following method. Let $r = \lceil \epsilon n \rceil$. This is the rank of the element in the stream $\{x^i\}_{i=1}^n$ that one would like to approximate the value of.

1. Compute $\{r_{\text{max},Q}(v_i)\}_{i=1}^L$.
2. If $r \geq n - \lfloor \epsilon n \rfloor$, then return $v_L$.
3. If $r < n - \lfloor \epsilon n \rfloor$, then return $v_j$, where $r_{\text{max},Q}(v_j)$ is the smallest element in $\{r_{\text{max},Q}(v_i)\}_{i=1}^L$ that is greater than $r + \lfloor \epsilon n \rfloor$.

This query can be denoted by $\hat{F}_n^{-1}(u)$, as supposed to the standard empirical quantile function $\hat{F}_n^{-1}(u) = \hat{X}^r$. The work in Greenwald and Khanna (2001) showed that $\hat{F}_n^{-1}(u)$ returns a value $\hat{x}^j$, where $j \in [r - \epsilon n, r + \epsilon n]$.

3.4.2 Inverse querying

The idea of inversely querying a single $\epsilon$-approximate quantile summary $Q$ from Lall (2015) is now revisited. One wishes to approximate the rank $r$, where $\hat{x}^r$ is the largest element in a stream $\{x^i\}_{i=1}^n$ that is at most $y$. In other words, one wishes to approximate $\hat{F}_n(y)$, where $\hat{F}_n(y)$ is the empirical CDF of the stream. Let the approximation be obtained by querying the summary as below, and denote it by $\hat{F}_n(y)$. 

\begin{align*}
\hat{F}_n(y) &= \hat{x}^j, \\
\text{where } j &\in \left[\frac{r}{\epsilon n}, \frac{r + \epsilon n}{\epsilon n}\right].
\end{align*}
1. If \( y \geq v_L \), let \( r = n \). Conversely, if \( y < v_1 \), let \( r = 0 \).
2. Otherwise, find \( i \) where \( v_i \leq y < v_{i+1} \), and then set \( r = r_{\text{max},Q}(v_i) \).
3. Finally output \( \hat{F}_n(y) = r/n \).

The work in Lall (2015) showed that this obtains an approximation within the interval \( [\hat{F}_n(y) - 3\epsilon, \hat{F}_n(y) + 3\epsilon] \).

### 3.4.3 Marginal quantile queries

The marginal quantiles of the bivariate data can also be approximated by querying the copula summary. This returns two marginal quantile queries that are equivalent to the standard Greenwald and Khanna queries.

- To find an \( \epsilon \)-approximation to the \( \phi \)-quantile of \( \{x_{(1)}^i\}_{i=1}^n \), query the summary \( S_{(1)} \) using Sec. 3.4.1.
- To find an \( \epsilon \)-approximation to the \( \phi \)-quantile of \( \{x_{(2)}^i\}_{i=1}^n \), compute \( M(S_{(2)}^1, ..., S_{(2)}^L) \) and query this merged summary for the \( \phi \)-quantile using Sec. 3.4.1.

### 3.4.4 Copula queries

Finally using Sec. 3.4.1 and 3.4.2, the copula summary can be queried to return an approximation to \( \hat{C}(u_1, u_2) \) via the following method.

1. Query the \( \epsilon \)-approximate summary \( S_{(1)}^i \) for the \( r = \lceil u_1 n \rceil \) ranked element, where \( n \) is the number of elements so far in the stream, using Sec. 3.4.1. Let \( E \) take the value of the index \( i \) where \( r_{\text{max},S_{(1)}}(v_i) \) is the output from the query.
2. Merge the \( \epsilon \)-approximate subsummaries \( \{S_{(2)}^j\}_{j=1}^E \) to form \( P_1 = M(S_{(2)}^1, ..., S_{(2)}^E) \) and merge the \( \epsilon \)-approximate subsummaries \( \{S_{(2)}^j\}_{j=1}^L \) to form \( P_2 = M(S_{(2)}^1, ..., S_{(2)}^L) \). Let \( \hat{n}_1 = \sum_{j=1}^E \sum_{i=1}^{r_{\text{max},S_{(2)}}(v_i)} g_{(2)}^{ij} \) be the total number of elements that have been added to the subsummaries \( \{S_{(2)}^j\}_{j=1}^E \). Suppose these elements are \( \{x_{(2)}^i\}_{i=1}^{\hat{n}_1} \), where \( \hat{I} \subset \{1, ..., n\} \). On this note, let the empirical CDF \( \hat{F}_{\hat{n}_1,(2)}(y) \) be given by
   \[
   \hat{F}_{\hat{n}_1,(2)}(y) = \frac{1}{\hat{n}_1} \sum_{i=1}^{\hat{n}_1} I \left( x_{(2)}^i \leq y \right). 
   \]
3. Query the \( \epsilon \)-approximate summary \( P_2 \) for the \( \lceil u_2 n \rceil \) ranked element, and denote this query by \( \hat{F}_{\hat{n}_1,(2)}^{-1}(u_2) \).
4. Inversely query the \( \epsilon \)-approximate summary \( P_1 \) by \( \hat{F}_{\hat{n}_1,(2)}^{-1}(u_2) \), and denote this query by \( \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}_{\hat{n}_1,(2)}^{-1}(u_2) \right) \).
5. Then define the overall copula query as
   \[
   \hat{C}_S(u_1, u_2) = \frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}_{\hat{n}_1,(2)}^{-1}(u_2) \right). 
   \]

The next section provides a theoretical analysis of the error of this approximation.
4 Error and efficiency analysis

This section provides a theoretical analysis on the error and efficiency of the approximation $\hat{C}_S(u_1, u_2)$. The bound on the error of this approximation away from (1) is now stated and proved in the following theorems.

**Theorem 1** (Error bound). Let $\hat{C}(u_1, u_2)$ be the empirical copula function of the bivariate stream of data $\{x^{(1)}_i, x^{(2)}_i\}_{i=1}^n \in \mathbb{R}^{2 \times n}$ evaluated at $(u_1, u_2) \in [0, 1]^2$. Also suppose that $\hat{C}_S(u_1, u_2)$ is as it is defined in (6), then

$$|\hat{C}_S(u_1, u_2) - \hat{C}(u_1, u_2)| \leq 5\epsilon.$$

**Proof.** Note that

$$|\hat{C}_S(u_1, u_2) - \hat{C}(u_1, u_2)| = \left| \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right|,$$

due to (3) and therefore can be split up into three contributing parts by the triangle inequality,

$$|\hat{C}_S(u_1, u_2) - \hat{C}(u_1, u_2)| \leq \left( \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right) + \left( \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right) + \left( \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right).$$

The proof is now split up into sub-theorems (Theorems 2, 3, and 4) corresponding to the three parts above.

The error can therefore be framed as taking a sum of the errors from steps (3) and (4) in Sec. 3.2.2 (A and B) in addition to those from steps (1) in Sec. 3.2.2 (C). Each of these contributing errors are now bounded.

**Theorem 2** (Error bound on (A)). Suppose $u_2 \in [0, 1]$, then

$$\left| \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right| \leq 3\epsilon.$$

**Proof.** This is the guaranteed error bound for inversely querying an $\epsilon$-approximate summary, from [Lall (2015)].

**Theorem 3** (Error bound on (B)). Suppose $u_2 \in [0, 1]$, then

$$\left| \frac{\hat{n}_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) - \frac{n_1}{n} \hat{F}_{n_1, (2)} \left( \hat{F}^{-1}_{n_2, (2)}(u_2) \right) \right| \leq \epsilon.$$

**Proof.** Let $\xi = \left\lfloor u_2n \right\rfloor$. Then suppose the element returned by querying the $\epsilon$-approximate summary $M(S^{(2)}_1, ..., S^{(2)}_t)$ for the $u_2$-quantile is $\hat{x}^{(2)}_{\xi}$. Therefore $|\xi - \gamma| \leq \epsilon n$. Recall from Sec. 3.4.4 that $\hat{n}_1 \hat{F}_{n_1, (2)}(y)$ is simply the count of all samples in $\{\hat{x}^{(2)}_{\xi} \}_{i=1}^{\hat{n}_1}$ less than or equal to $y$. For $y = \hat{x}^{(2)}_{\xi}$, let this count be
denoted by $\xi_R$. For $y = \tilde{x}^{\gamma}_{(2)}$, let this count be denoted by $\gamma_R$. As $I \subset \{1, \ldots, n\}$, we have $|\xi_R - \gamma_R| \leq \epsilon n$ also. Therefore
\[
\left|\hat{n}_1 \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) - \hat{n}_1 \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) \right| \leq \epsilon n,
\]
and finally
\[
\left|\frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) - \frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) \right| \leq \epsilon.
\]

**Theorem 4** (Error bound on (C)). Suppose $u_2 \in [0, 1]$, then
\[
\left|\frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) - \frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) \right| \leq \epsilon.
\]

**Proof.** Let $A = \{x^{(i)}_{(2)}\}_{i=1}^{n}$. Recall from Sec. 4 that $B = \{x^{(i)}_{(2)}\}_{i=1}^{n_1}$ are the $n_1$ elements that have corresponding values $\{x^{(i)}_{(1)}\}_{i=1}^{n_1}$ with ranks less than or equal to $[u_1 n]$ in the original stream. We assume without loss of generality that if $n_1 < \hat{n}_1$ then $B \subset A$, and vice-versa if $\hat{n}_1 < n_1$. Define $\xi$ to be the count of all elements in $B$ that are less than or equal to $\hat{F}^{-1}_{n,(2)} (u_2)$, which is equivalent to $\hat{n}_1 \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right)$.

Then by the fact that $S_{(1)}$ is an $\epsilon$-approximate quantile summary of $\{x^{(i)}_{(1)}\}_{i=1}^{n}$, the count of all elements in $A$ that are less than or equal to $\hat{F}^{-1}_{n,(2)} (u_2)$, which is equivalent to $\hat{n}_1 \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right)$, is within the interval $[\xi - \epsilon n, \xi + \epsilon n]$. Therefore,
\[
\left|\frac{\hat{n}_1}{n} \hat{F}_{\hat{n}_1,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) - \frac{n_1}{n} \hat{F}_{n,(2)} \left( \hat{F}^{-1}_{n,(2)} (u_2) \right) \right| \leq \frac{\xi + \epsilon n}{n} - \frac{\xi}{n} = \frac{\epsilon n}{n} = \epsilon.
\]

The main benefit of this algorithm is that in streams of bivariate data acquired continuously one can compute the approximation to the empirical copula function, bounded in the theorems above, by maintaining a succinct summary of the data. It does this by storing a separate quantile summary $S_{(2)}$, for each $i = 1, \ldots, L$ elements in a single quantile summary $S_{(1)}$, all of which are $\epsilon$-approximate. From Greenwald and Khanna (2001), the length of an $\epsilon$-approximate summary constructed using the insert and combine operations discussed in Sec. 3.2.1 and 3.2.2 is at the worst-case $L = O \left( \frac{1}{\epsilon} \log(\epsilon n) \right)$. Therefore for the algorithm considered in this paper, the worst-case number of tuples stored in the copula summary at any one time is $O \left( \frac{1}{\epsilon} \log(\epsilon n) \right) = O \left( \frac{1}{\epsilon^2} \log(n \epsilon^2) \right)$. This is the same complexity as the queries of very similar data structures in Suri et al. (2006) and Hershberger et al. (2004) obtained for multidimensional range counting. It is worth noting, as seen in Greenwald and Khanna (2001), the space-memory of a single quantile summary is much better than this worst-case in practice. In many cases, such as when one implements the combine operation after an element is added to a single quantile summary rather than after every $1/\epsilon$ steps, the space-memory used is independent of $n$. Based on the analysis above, it is apparent that a direct extension of this algorithm to a higher dimension $d$, where such a data structure uses $O \left( \frac{1}{\epsilon^d} \log(n \epsilon^d) \right)$ space-memory, would be infeasible. This is noted in Hershberger et al. (2004) for a very similar data structure. However the next section gives an example of how one may model the dependence structure of high dimensional data streams by utilising these bivariate copula summaries.

## 5 Extension to a class of higher dimensional copulas

So far this paper has only discussed bivariate copulas for two streams of data. However, this section now gives a brief example of how the approximations from bivariate copula summaries can be used to construct approximations to a class of higher dimensional copulas. This class was introduced in Mazo et al. (2015), and are named Product of Bivariate Copulas (PBC). A benefit of these copulas is that they are associated...
with a graphical structure of dependence between components. Let $Z \in \mathbb{R}^d$ be a multidimensional random variable. Define the set $E$, with cardinality $|E|$ less than or equal to $d(d-1)/2$. This set is a subset of all of the index pairs, $(i,j)$, where $i,j \in [1,d]$, $j > i$ and $i \neq j$, representing dependence between two components. Let $n_i \in \{|e \in E, \text{ such that } i \in e\}|$ be the number of times $i$ appears within $E$. A graph can be associated to the set $E$ by allowing an index pair $(i,j) \in E$ to be an edge between two nodes $i$ and $j$.

A copula that models the dependence structure of the random variable $Z$ is then given by a product of the bivariate copulas between index pairs in $E$, only with transformed marginals $u_i \in [0,1]$, for $i = 1, ..., d$. On this note let, $v(u_i) = u_i^{1/n_i}$. Let the PBC of $Z$ be defined as

$$ C_{PBC}(u_1, ..., u_d) = \prod_{(i,j) \in E} C^{(i,j)}(v(u_i), v(u_j)),$$

where $C^{(i,j)}$ is a bivariate copula between the $i$'th and $j$'th component of $Z$. For example, consider $Z \in \mathbb{R}^3$ and $E = \{(1, 2), (2, 3), (1, 3)\}$. Then,

$$ C_{PBC}(u_1, u_2, u_3) = C^{(1,2)}(u_1^{1/2}, u_2^{1/2})C^{(2,3)}(u_2^{1/2}, u_3^{1/2})C^{(1,3)}(u_1^{1/2}, u_3^{1/2}). \quad (9) $$

For details on the theoretical background of this class of copulas, turn to Mazo et al. \textit{2015}.

Now suppose $\hat{C}_{PBC}(u_1, ..., u_d)$ is a copula constructed from a product of empirical bivariate copulas $\hat{C}^{(i,j)}(v(u_i), v(u_j))$, for $(i,j) \in E$. The algorithm presented in this paper can be used to approximate $\hat{C}_{PBC}(u_1, ..., u_d)$ by utilising products of queries from bivariate copula summaries. Define this approximation to $\hat{C}_{PBC}(u_1, ..., u_d)$ as

$$ \hat{C}_{PBC,S}(u_1, ..., u_d) = \prod_{(i,j) \in E} \hat{C}^{(i,j)}_{S}(v(u_i), v(u_j)),$$

where $\hat{C}^{(i,j)}_{S}$ is an approximation from a copula summary between the $i$'th and $j$'th components. Given that $|E| \leq d(d-1)/2$, it follows from the previous section that the worst-case length of this approximation will be $O\left(d(d-1)/2 \log(\epsilon n)^2\right)$. Unlike a direct extension of the algorithm to higher dimensions, this length is feasible even for large $d$. Recall that the error from each individual bivariate copula approximation in the product is bounded by $5\epsilon$. Now suppose that $\epsilon < 0.2$ and $(u_1, ..., u_d) \in [0,1]^d$ is fixed, then the error

$$ \left| \hat{C}_{PBC,S}(u_1, ..., u_d) - \hat{C}_{PBC}(u_1, ..., u_d) \right| \quad (10) $$

has the leading-order term of $O(|E|\epsilon)$. This is trivially shown by propagating multiplicative errors, and the proof is omitted. Therefore the approximation error grows linearly in the number of bivariate copulas within the product. If all $d$ components are dependent on one another, then $E$ is associated with a fully-connected graph and $|E| = d(d-1)/2$. Therefore the leading-order term of (10) is $O(d(d-1)/2)$. This class of copulas is a suitable example of how constructing approximations to bivariate copulas of streaming data has practical benefit to users estimating the dependence structure of high dimensional data. The approximations to products of empirical bivariate copulas will be considered in a numerical example in the following section.

6 Numerical demonstration

The copula summary proposed in this paper which produces guaranteed error estimates to an empirical bivariate copula function for streaming data is now explored. Consider the following random stream of data, $\{x_1^{(1)}, x_2^{(2)}\}_{i=1}^{n}$, where $x_1^{(1)} \sim N(0,1)$, $x_2^{(2)} \sim N(0,1)$ and $\rho(x_1^{(1)}, x_2^{(2)}) = -0.8$. Here, $\rho$ is the Pearson’s correlation coefficient. For the first experiment $\epsilon = 0.05$ is used alongside the algorithm in Sec. 3 and $n = 10^5$ is the length of the stream. The copula summary is constructed, with the size (in bytes) of the summary and the quantity $|\hat{C}_S(0.7,0.7) - \hat{C}(0.7,0.7)|$ being computed after every 100
elements are added to the stream. Figures 1(a) and 1(b) shows these two quantities over time respectively. The total space-memory used by the copula summary appears to be independent of $n$, beating the worst-case rate presented in Sec. 4. The theoretical error bound in Sec. 4 is also shown in Figure 1(b). Figure 3 shows the length of the subsummaries $S_i^{(2)}$, for $i = 1, ..., L$ within the copula summary at the end of the stream. There are a few subsummaries that only contain one element, and that correspond to the elements in $S^{(1)}$ that have been added to the stream since the last use of the combine operation (see Sec. 3.2.2). The copula summary is also shown queried at a grid of $(u_1, u_2)$ points in Figure 2(a) alongside the actual empirical copula function of the data stream in Figure 2(b). For the second experiment, the parameter $\epsilon$ is varied and copula summaries for five independent sets of $\{x_i^{(1)}, x_i^{(2)}\}_{i=1}^n$, with $n = 10000$, are constructed for each value of $\epsilon$. The average errors away from $\hat{C}(0.7, 0.7)$ are shown in Figure 4. The error decreases accordingly with $\epsilon$.

Finally the extension of the bivariate copula approximations to the Product of Bivariate Copulas, discussed in the previous section, is examined. Suppose the streams of data $\{x_i^{(1)}\}_{i=1}^n$ and $\{x_i^{(2)}\}_{i=1}^n$ are sampled from $N(0, 1)$ with correlation $\rho = 0.5$. Now suppose $\{x_i^{(2)}\}_{i=1}^n$ and another stream of data $\{x_i^{(3)}\}_{i=1}^n$, which is also sampled from $N(0, 1)$, have correlation $\rho = 0.5$. The correlation between $\{x_i^{(1)}\}_{i=1}^n$ and $\{x_i^{(3)}\}_{i=1}^n$ is $\rho = 0$. This inspires a graphical representation for the dependencies between the first, second and third components corresponding to the set $E = \{(1, 2), (2, 3)\}$ from the previous section.
Figure 2: The copula summary with $\epsilon = 0.05$ for a $10^5$-element data stream sampled from a bivariate Gaussian distribution with correlation $\rho = -0.8$, evaluated at a grid of $(u_1, u_2)$ values (a). The actual empirical copula function of the data stream is shown in (b).

Figure 3: The length of all subsummaries $S_{i(2)}$, for $i = 1, \ldots, L$, within the copula summary for a $10^5$-element data stream sampled from a bivariate Gaussian distribution with correlation $\rho = -0.8$.

Therefore define
\[ \hat{C}_{PBC}(u_1, u_2, u_3) = \hat{C}^{(1,2)}(u_1, u_2^{1/2})\hat{C}^{(2,3)}(u_2^{1/2}, u_3), \]
to model this dependence structure. This copula, evaluated at a grid of $(u_1, u_3)$ values, and for $u_2 = 0.1$ is shown in Figure 5b. The same, only with $u_2 = 0.9$, is also shown in Figure 6b. As described in the previous section, an approximation to this copula, $\hat{C}_{PBC,S}(u_1, u_2, u_3)$, can be constructed via a product of two bivariate copula approximations $\hat{C}^{S(1,2)}(u_1, u_2^{1/2})$ and $\hat{C}^{S(2,3)}(u_2^{1/2}, u_3)$. For the two cases of $u_2 = 0.1$ and $u_2 = 0.9$, this copula is shown in Figures 5a and 6a respectively. For fixed $u_2$, it is interesting to study this resulting marginal copula as it does not appear in the expression for $\hat{C}_{PBC}$ or $\hat{C}_{PBC,S}$, and the two data streams are independent of one another. The approximation from the copula summary matches the empirical copula well in both cases of $u_2$. 

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Figure 4: The average error $|\hat{C}_S(0.7, 0.7) - \hat{C}(0.7, 0.7)|$ of copula summaries, over a range of $\epsilon$ values, for five independent data streams sampled from a bivariate Gaussian distribution with correlation $\rho = -0.8$.

Figure 5: The approximation $\hat{C}_{PBC,S}(u_1, 0.1, u_3)$ (a), and the empirical PBC $\hat{C}_{PBC}(u_1, 0.1, u_3)$ (b) evaluated at a grid of $(u_1, u_3)$ values.

7 Conclusion

This paper has proposed an algorithm to approximate an empirical copula function of a bivariate data stream with a space-memory constraint. These approximations have a guaranteed error bound that has been presented here, and can be used to model the dependence structure between two streams of data. This paper has also shown how the bivariate approximations proposed here can be used to construct approximations to a class of higher dimensional copulas that are formed by the product of bivariate copulas with transformed marginals (Mazo et al., 2015). The extension of this algorithm to further classes of multidimensional copulas is an aim of future research by the author. Another natural extension of this work is to use these approximations to derive estimates of rank correlation coefficients with guaranteed error bounds, such as the Kendall Tau correlation coefficient (Xiao, 2017).

This algorithm is a generalisation to the one dimensional quantile summaries constructed via the Greenwald and Khanna algorithm (Greenwald and Khanna, 2001). The data structure is similar to the ones used to find multidimensional ranges in Suri et al. (2006) and Hershberger et al. (2004). It is formed via a particular combination of $L + 1$ different $\epsilon$-approximate quantile summaries and therefore the algorithm uses a worst-case $O\left(\frac{1}{\epsilon^2} \log(en)^3\right)$ space-memory after $n$ elements in the bivariate stream data. Numerical experiments in this paper (in two and three dimensional cases) have confirmed the space-memory
efficiency and the theoretical error bound of the approximation.

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