BRIDGELAND’S STABILITIES ON ABELIAN SURFACES

SHINTAROU YANAGIDA, KÔTA YOSHIOKA

ABSTRACT. In this paper, we shall study the structure of walls for Bridgeland’s stability conditions on abelian surfaces. In particular, we shall study the structure of walls for the moduli spaces of rank 1 complexes on an abelian surface with the Picard number 1.

0. Introduction

Let \( X \) be an abelian surface over a field \( k \). Denote by \( \text{Coh}(X) \) the category of coherent sheaves on \( X \), by \( \mathcal{D}(X) \) the bounded derived category of \( \text{Coh}(X) \) and by \( K(X) \) the Grothendieck group of \( \mathcal{D}(X) \).

For \( \beta \in \mathcal{NS}(X)_\mathbb{Q} \) and an ample divisor \( \omega \in \text{Amp}(X)_\mathbb{Q} \), Bridgeland [3] constructed a stability condition \( \sigma_{\beta, \omega} = (\mathfrak{A}_{\beta, \omega}, \mathcal{Z}_{\beta, \omega}) \) on \( \mathcal{D}(X) \). Here \( \mathfrak{A}_{\beta, \omega} \) is a tilting of \( \text{Coh}(X) \), and \( \mathcal{Z}_{\beta, \omega} : K(X) \to \mathbb{C} \) is a group homomorphism called the stability function. In terms of the Mukai lattice (\( \mathcal{D} \) is a projective scheme, if \( (\beta, \omega) \in \mathfrak{A} \)).

For abelian surfaces, the studies on the dependence of the parameters, i.e., those of wall and chamber structures, are started by two groups Maciocia and Meachan [5], and Minamide, Yanagida and Yoshioka [7], very natural to study the moduli of Bridgeland stable objects and its dependence on the parameter even for isomorphism of the moduli spaces of Bridgeland stable objects as a consequence of [3, Prop. 10.3]. So it is category of coherent sheaves is not preserved. On the other hand, the Fourier-Mukai transform induces an isomorphism of the moduli spaces. We shall show that the taking dual functor \( D_X \) also preserves the stability of objects (Theorem 2.5). These are done in sections 1 and 2.

In [14], we introduced a useful notion \textit{semi-homogeneous presentations}. It is a presentation of a coherent sheaf as the kernel or the cokernel of a homomorphism \( V_{-1} \to V_0 \) of semi-homogeneous sheaves \( V_{-1}, V_0 \) with some numerical conditions. Extracting the numerical conditions from \( V_{-1} \) and \( V_0 \), we also introduced the notion \textit{numerical solutions} and constructed moduli spaces of simple complexes \( V_{-1} \to V_0 \) associated to numerical solutions. In [7], we found a relation between these moduli spaces and the wall crossing behavior for Bridgeland stability conditions. In this paper, we give a supplement of this relation. Thus we shall relate a particular wall called a \textit{codimension 0 wall} for every numerical solution. If \( (\beta, \omega) \) belongs to a

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codimension 0 wall, then for any neighborhood $U$ of $(\beta, \omega)$, there is no $\sigma_{(\beta, \omega)}$-semi-stable object which is $\sigma_{(\beta', \omega')}$-semi-stable for all $(\beta', \omega') \in U$.

In section 4 we fix an ample divisor $H$ and study special stability conditions $\sigma_{(\beta + H, tH)}$ parametrized by the upper half plane $\{(s, t) \mid s, t \in \mathbb{R}, s > 0\}$. In the $(s, t)$-plane, the equations of walls are very simple. They define circles or lines forming a pencil of circles passing through imaginary points (Remark 4.3). Thus they do not intersect each other. For a principally polarized abelian surface with Picard number 1, these kind of results are obtained by Maciocia and Meachan [5]. Thus our results are generalization of theirs.

We also explain that there are one or two unbounded chambers which parametrize Gieseker semi-stable sheaves. In [17], we showed that Gieseker’s stability is preserved under the Fourier-Mukai transform, if the degree of the stable sheaf is sufficiently large. We shall explain the result as an application of this section (Proposition 1.24).

In section 5 we assume that $\text{NS}(X) = \mathbb{Z}H$. We are mainly interested in a Mukai vector $v$ which is a Fourier-Mukai transform of $1 - t_\mathbb{G}_X$. So we may assume that $v = 1 - t_\mathbb{G}_X$.

In [14], we described the algebraic part of the Mukai lattice $H^*(X, \mathbb{Z})_{\text{alg}}$ as a lattice in the vector space of quadratic forms of two variables. Then we can describe the action of Fourier-Mukai transforms as a natural $\text{GL}(2, \mathbb{R})$-action. By using these results, we shall study the structure of walls. In particular, we shall classify codimension 0 walls by using our description of Mukai lattice. We set $n := (H^2)/2$. If $\sqrt{\ell/n} \in \mathbb{Q}$, then there is a unique wall of codimension 0, since there is one numerical solution. Assume that $\sqrt{\ell/n} \not\in \mathbb{Q}$. In this case, these are infinitely many numerical solutions, thus we have infinitely many codimension 0 walls. Let $G_{n, \ell} \subset \text{GL}(2, \mathbb{R})$ be the subgroup generated by the cohomological action of (covariant or contravariant) Fourier-Mukai transforms preserving $\pm v$. $G_{n, \ell}$ acts on the set of walls. We show that there are infinitely many $G_{n, \ell}$-orbits of walls and the set of codimension 0 walls forms an orbit of $G_{n, \ell}$. Each orbit has two accumulation points $(\pm \sqrt{\ell/n}, 0)$ in the $(s, t)$-plane. An observant reader will see that the main part of section 5 essentially appeared in [13] without using Bridgeland stability condition.

In section 6 we shall study the structure of walls for $v = 1 - t_\mathbb{G}_X$, $\ell \leq 4$ on a principally polarized abelian surface $X$. As an application, we shall classify $M_H(v)$ for a primitive Mukai vector with $\langle v^2 \rangle / 2 \leq 4$.

In [9] and [10] Thm. 3, Mukai announced that $M_H(v) \cong X \times \text{Hilb}^{\langle v^2 \rangle / 2}(X)$ for a primitive Mukai vector with $\langle v^2 \rangle / 2 = 1, 2, 3$. Moreover he determined the Fourier-Mukai transform which induces the isomorphism. By using the structure of walls, we give an explanation of Mukai’s results for $\langle v^2 \rangle / 2 = 2, 3$. It is quite surprising that Mukai discovered his results 30 years ago without using Bridgeland’s stability conditions.

In appendix, we continue to assume that $X$ is an abelian surface with $\text{NS}(X) = \mathbb{Z}H$ and $(H^2)/2 = n$. We shall identify the period space with the upper half plane. Then we show that the action of auto-equivalences is the action of the modular group $\Gamma_0(n)$.

Finally we would like to mention related works which appeared during our preparation of this manuscript. We note that the examples of Bridgeland stability conditions in this paper are generalized to an arbitrary projective surfaces by Arcara and Bertram [1]. For these stability conditions, Maciocia [4] studied the structure of walls. In particular, he proved similar results to section 4 in a much more general context. For the stability conditions on principally polarized abelian surfaces, Meachan [6] studied the structure of walls in detail. In particular he independently found examples of walls with accumulation points.

1. Preliminaries on Bridgeland’s stability condition

As in the introduction, let $X$ be an abelian surface over a field $\mathfrak{k}$, and fix an ample divisor $H$ on $X$.

1.1. Notations for Mukai lattice. We set $A^*_\text{alg}(X) = \bigoplus_{i=0}^2 A^i_{\text{alg}}(X)$ to be the quotient of the cycle group of $X$ by the algebraic equivalence. Then we have $A^0_{\text{alg}}(X) \cong \mathbb{Z}$, $A^1_{\text{alg}}(X) \cong \text{NS}(X)$ and $A^2_{\text{alg}}(X) \cong \mathbb{Z}$. We denote the fundamental class of $A^2_{\text{alg}}(X)$ by $\var_{\mathfrak{k}}X$, and express an element $x \in A^2_{\text{alg}}(X)$ by $x = x_0 + x_1 + x_2\var_{\mathfrak{k}}X$ with $x_0 \in \mathbb{Z}$, $x_1 \in \text{NS}(X)$ and $x_2 \in \mathbb{Z}$. The lattice structure $\langle \cdot, \cdot \rangle$ of $A^2_{\text{alg}}(X)$ is given by

$$
\langle x, y \rangle := (x_1, y_1) - (x_0 y_2 + x_2 y_0),
$$

where $x = x_0 + x_1 + x_2\var_{\mathfrak{k}}X$ and $y = y_0 + y_1 + y_2\var_{\mathfrak{k}}X$. We shall call $(A^2_{\text{alg}}(X), \langle \cdot, \cdot \rangle)$ the Mukai lattice for $X$. In the case of $\mathfrak{k} = \mathbb{C}$, this lattice is sometimes denoted by $H^*(X, \mathbb{Z})_{\text{alg}}$ in literature. In this paper, we shall use the symbol $H^*(X, \mathbb{Z})_{\text{alg}}$ even when $\mathfrak{k}$ is arbitrary.

The Mukai vector $v(E) \in H^*(X, \mathbb{Z})_{\text{alg}}$ for $E \in \text{Coh}(X)$ is defined by

$$
v(E) := \text{ch}(E)\sqrt{td_X} = \text{ch}(E) = \text{rk}E + c_1(E) + \chi(E)\var_{\mathfrak{k}}X.
$$

We also use the vectorial notation

$$
v(E) = (\text{rk}E, c_1(E), \chi(E)).
$$

For an object $E$ of $\text{D}(X)$, $v(E)$ is defined by $\sum_k (-1)^k v(E^k)$, where $(E^k) = (\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots)$ is the bounded complex representing the object $E$. 

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We take an ample \( \mathbb{Q} \)-divisor \( H \). For \( v \in H^*(X, \mathbb{Z})_{\text{alg}} \) and \( \beta \in \text{NS}(X)_\mathbb{Q} \), we set

\[
(1.2) \quad r_\beta(v) := -(v, g_X), \quad a_\beta(v) := -(v, e^\beta), \quad d_\beta,H(v) := \frac{(v, H + (H, \beta)g_X)}{(H^2)}.
\]

If the choice of \( H \) is clear, then we write \( d_\beta(v) := d_\beta,H(v) \) for simplicity. By using (1.2), we have

\[
(1.3) \quad v = r_\beta(v)e^\beta + a_\beta(v)g_X + \left( d_\beta,H(v)H + D_\beta(v) \right) + \left( d_\beta,H(v)H + D_\beta(v), \beta \right)g_X, \quad D_\beta(v) \in H^+ \cap \text{NS}(X)_\mathbb{Q}.
\]

A Mukai vector \( v = (r, \xi, a) \neq 0 \) is positive, if (i) \( r > 0 \) or (ii) \( r = 0 \) and \( \xi \) is effective, or (iii) \( r = \xi = 0 \) and \( a > 0 \). We denote a positive Mukai vector \( v \) by \( v > 0 \). A Mukai vector \( v \) is called isotropic if \( v^2 = 0 \).

For \( \beta \in \text{NS}(X)_\mathbb{Q} \), we define the \( \beta \)-twisted semi-stability replacing the usual Hilbert polynomial \( \chi(E(nH)) \) by \( \chi(E(-\beta + nH)) \). Then \( v \) is positive if and only if \( \chi(E(-\beta + nH)) > 0 \) for \( E \in \text{D}(X) \) with \( v(E) = v \) and \( n > 0 \).

For a positive Mukai vector \( v \), \( M_\beta^v \) denotes the moduli stack of \( \beta \)-twisted semi-stable sheaves \( E \) on \( X \) with \( v(E) = v \). \( M_\beta^v \) is the moduli scheme of \( S \)-equivalence classes of \( \beta \)-twisted semi-stable sheaves \( E \) on \( X \) with \( v(E) = v \) and \( M_\beta^v \) denotes the open subscheme consisting of \( \beta \)-twisted stable sheaves. If \( \beta = 0 \), then we write \( M_\beta^v := M^v \).

For a proper morphism \( f : Z_1 \to Z_2 \), we denote the derived pull-back \( Lf^* \) and the derived direct image \( Rf_* \) by \( f^* \) and \( f_* \) respectively.

For \( E \in \text{D}(X \times Y), \Phi_{E,Y}^X : \text{D}(X) \to \text{D}(Y) \) denotes the integral functor whose kernel is \( E \):

\[
(1.4) \quad \Phi_{E,Y}^X(E) = \text{Hom}(p_X^*(E \otimes \Omega_X^\beta), E) = E \in \text{D}(X),
\]

where \( p_X \) and \( p_Y \) are projections from \( X \times Y \) to \( X \) and \( Y \) respectively. If \( \Phi_{E,Y}^X \) is an equivalence, it is called a Fourier-Mukai transform. If a Fourier-Mukai transform \( \Phi_{E,Y}^X \) exists and \( X \) is an abelian surface, then \( Y \) is also an abelian surface and \( \Phi_{E,Y}^X \) induces an isometry of Mukai lattices \( H^*(X, \mathbb{Z})_{\text{alg}} \to H^*(Y, \mathbb{Z})_{\text{alg}} \). We also denote this isometry by \( \Phi_{E,Y}^X \).

\( D_X(\ast) := \text{RHom}_{O_X}(\ast\ast, O_X) \) denotes the taking dual functor. It is a contravariant functor from \( \text{D}(X) \) to \( \text{D}(X) \). A contravariant Fourier-Mukai transform is a composite of a Fourier-Mukai functor and \( D_X \). If \( X \) is an abelian surface, then it is of the form \( \Phi_{E,Y}^X \circ D_X = D_Y \circ \Phi_{E,Y}^Y \) with \( E_{\ast} \ast := D_X \times Y(E) \).

### 1.2. Stability conditions and wall/ chamber structure

Let us recall the stability conditions given in [3] and [7] §1. Let

\[
\text{Amph}(X)_\mathbb{R} := \{ x \in \text{NS}(X)_\mathbb{R} \mid (x^2 > 0, (x, D) > 0) \}
\]

be the ample cone of \( X \), where \( D \) is an effective divisor. We take \( (\beta, \omega) \in \text{NS}(X)_\mathbb{R} \times \text{Amph}(X)_\mathbb{R} \) and \( H \in \mathbb{R}_{>0}\omega \).

For \( E \in K(X) \) with \( v = v(E) \) expressed as (1.3), we have

\[
Z_{(\beta, \omega)}(E) = \langle e^{\beta + \sqrt{-1}\omega}, v(E) \rangle = -a_\beta(v(E)) + \frac{(\omega^2)}{2}r_\beta(v(E)) + d_\beta,H(v(E))(H, \omega)\sqrt{-1}.
\]

Assume that \( (\beta, \omega) \in \text{NS}(X)_\mathbb{Q} \times \text{Amph}(X)_\mathbb{Q} \). Let \( \mathfrak{A}_{(\beta, \omega)} \) be the tilt of \( \text{Coh}(X) \) with respect to the torsion pair \( (\mathfrak{S}_{(\beta, \omega)}, \mathfrak{S}_{(\beta, \omega)}) \) defined by

- (i) \( \mathfrak{T}_{(\beta, \omega)} \) is generated by \( \beta \)-twisted stable sheaves with \( Z_{(\beta, \omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0} \).
- (ii) \( \mathfrak{S}_{(\beta, \omega)} \) is generated by \( \beta \)-twisted stable sheaves with \( -Z_{(\beta, \omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0} \), where \( \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} \) is the upper half plane. \( \mathfrak{A}_{(\beta, \omega)} \) is the abelian category in [3] and it depends only on \( \beta \) and the ray \( Q_{>0} \).

Then the pair \( \sigma_{(\beta, \omega)} = (\mathfrak{A}_{(\beta, \omega)}, Z_{(\beta, \omega)}) \) satisfies the requirement of stability conditions on \( \text{D}(X) \) [3]. In particular, the (semi-)stability of objects in \( \mathfrak{A}_{(\beta, \omega)} \) with respect to \( Z_{(\beta, \omega)} \) is well-defined.

**Definition 1.1.** For a non-zero Mukai vector \( v \in H^*(X, \mathbb{Z})_{\text{alg}} \), we define \( Z_{(\beta, \omega)}(v) \in \mathbb{C} \) and \( \phi_{(\beta, \omega)}(v) \in (-1, 1) \) by

\[
Z_{(\beta, \omega)}(v) = \langle e^{\beta + \sqrt{-1}\omega}, v \rangle = |Z_{(\beta, \omega)}(v)| e^{\pi \sqrt{-1}\phi_{(\beta, \omega)}(v)}.
\]

Then

\[
\phi_{(\beta, \omega)}(v(E)) = \phi_{(\beta, \omega)}(E)
\]

for \( 0 \neq E \in \mathfrak{A}_{(\beta, \omega)} \cup \mathfrak{A}_{(\beta, \omega)}[-1] \).

**Definition 1.2.** \( E \in \text{D}(X) \) is called semi-stable of phase \( \phi \), if there is an integer \( n \) such that \( E[-n] \) is a semi-stable object of \( \mathfrak{A}_{(\beta, \omega)} \) with \( \phi_{(\beta, \omega)}(E[-n]) = \phi - n \). If we want to emphasize the dependence on the stability condition, we say that \( E \) is \( \sigma_{(\beta, \omega)} \)-semi-stable.

**Definition 1.3.** For a Mukai vector \( v \), \( M_{(\beta, \omega)}(v) \) denotes the moduli stack of \( \sigma_{(\beta, \omega)} \)-semi-stable objects \( E \) of \( \mathfrak{A}_{(\beta, \omega)} \) with \( v(E) = v \). \( M_{(\beta, \omega)}(v) \) denotes the moduli scheme of the \( S \)-equivalence classes of \( \sigma_{(\beta, \omega)} \)-semi-stable objects \( E \) of \( \mathfrak{A}_{(\beta, \omega)} \) with \( v(E) = v \), if it exists.
Remark 1.4. If \( Z_{(\beta, \omega)}(-v) \in \mathbb{H} \cup \mathbb{R}_{<0} \), then we have
\[
M_{(\beta, \omega)}(v) = \{ E[-1] \mid E \in M_{(\beta, \omega)}(-v) \},
\]
since \( \phi_{(\beta, \omega)}(v) \in (-1,0) \).

Remark 1.5. If we take the phase of \( v \) as \( \phi_{(\beta, \omega)}(v) \in (1,2) \), then
\[
M_{(\beta, \omega)}(v) = \{ E[1] \mid E \in M_{(\beta, \omega)}(-v) \}.
\]

More generally, if we take the phase of \( v \) as \( \phi_{(\beta, \omega)}(v) \in (n, n+1) \), then we have
\[
M_{(\beta, \omega)}(v) = \{ E[n] \mid E \in M_{(\beta, \omega)}((-1)^n v) \}.
\]

Definition 1.6. Let \( v_1 \notin \mathbb{Q}v \) be a Mukai vector with \( \langle v_1^2 \rangle \geq 0 \), \( \langle (v - v_1)^2 \rangle \geq 0 \) and \( \langle v_1, v - v_1 \rangle > 0 \). We define a wall for \( v \) by
\[
W_{v_1} := \{ (\beta, \omega) \in \text{NS}(X) \times \text{Amp}(X) \mid \mathbb{R}Z_{(\beta, \omega)}(v_1) = \mathbb{R}Z_{(\beta, \omega)}(v) \}.
\]

A connected component of \( \text{NS}(X) \times \text{Amp}(X) \setminus \cup_{v_1} W_{v_1} \) is called a chamber for \( v \).

Replacing \( v \) by \(-v\) if necessary, we may assume that \( Z_{(\beta, \omega)}(v) \in \mathbb{H} \cup \mathbb{R}_{<0} \). Assume that \( (\beta, \omega) \in W_{v_1} \). By [8 Prop. 4.2.2], there are \( \sigma_{(\beta, \omega)} \)-semi-stable objects \( E_1 \) and \( E_2 \) with \( v(E_1) = v_1 \) and \( v(E_2) = v - v_1 \). Then \( \mathbb{R}_{\geq 0}Z_{(\beta, \omega)}(E_1) = \mathbb{R}_{>0}Z_{(\beta, \omega)}(E_2) \). Thus there is a properly \( \sigma_{(\beta, \omega)} \)-semi-stable object with the Mukai vector \( v \).

Proposition 1.7 ([8 Prop. 9.3]). Let \( v \) be a Mukai vector with \( \langle v^2 \rangle > 0 \). Let \( C \) be a chamber for \( v \). Then \( M_{(\beta, \omega)}(v) \) is independent of \( (\beta, \omega) \in C \).

Proposition 1.8. Let \( \xi : H^*(X, Z)_{\text{alg}} \to H^*(Y, Z)_{\text{alg}} \) be an isometry of Mukai lattices. Let \( W_u \) be a wall for \( v \) defined by \( u \). Then \( \xi(u) \) defines a wall for \( \xi(v) \).

Proof. We set \( w := v - u \). Then \( u \) defines a wall for \( v \) if and only if \( \langle u^2 \rangle, \langle w^2 \rangle \geq 0 \) and \( \langle u, w \rangle > 0 \). This condition is preserved under \( \xi \). So the claim holds.

Let \( v \) be a primitive Mukai vector with \( \text{rk} v > 0 \) and assume that \( M^\beta_H(v) \) consists of \( \beta \)-twisted stable sheaves (in the sense of Gieseker).

Lemma 1.9. Assume that \( \langle v^2 \rangle > 0 \) and \( r := -\langle v, \varphi_X \rangle \neq 0 \). Let \( v_1 \) be a Mukai vector such that \( rv_1(v_1) = r_1 c_1(v) = 0 \), where we set \( r_1 := -\langle v_1, \varphi_X \rangle \). Then \( \mathbb{R}Z_{(\beta, \omega)}(v) = \mathbb{R}Z_{(\beta, \omega)}(v_1) \) if and only if \( d_\beta(v) = 0 \). In particular, if \( v_1 \) defines a wall, then the defining equation of the wall \( W_{v_1} \) for \( v \) is \( d_\beta(v) = 0 \).

Proof. We have \( rv_1 - r_1 v = b_\varphi X, \) \( 0 \neq b \in \mathbb{Z} \). Hence \( rv_1Z_{(\beta, \omega)}(v_1) = r_1Z_{(\beta, \omega)}(v) = -b \in \mathbb{R} \). Then we have \( Z_{(\beta, \omega)}(v_1) = Z_{(\beta, \omega)}(v) = b \). Hence the condition is preserved under \( \xi \).

2. Fourier-Mukai transforms.

2.1. Stability conditions and Fourier-Mukai transforms. We shall recall the (twisted) Fourier-Mukai transform of \( \sigma_{(\beta, \omega)} \) and its relation to Bridgeland’s stability conditions explained in [8]. Let \( \Phi : D(X) \to D^{\alpha_1}(X) \) be a twisted Fourier-Mukai transform such that \( \Phi(r_1 e^{\gamma}) = -\varphi_X \) and \( \Phi(\varphi_X) = -r_1 e^{\gamma'} \), where \( \alpha_1 \) is a representative of a suitable Brauer class. Then we can describe the cohomological Fourier-Mukai transform as
\[
\Phi(re^{\gamma} + a\varphi_X + \xi + (\xi, \gamma)\varphi_X) = -\frac{r}{r_1}a\varphi_X - r_1e^{\gamma'} + \frac{r_1}{|r_1|}(\hat{\xi} + (\hat{\xi}, \gamma')\varphi_X),
\]
where \( \xi \in \text{NS}(X)_{\mathbb{Q}} \) and \( \hat{\xi} : = \frac{r_1}{|r_1|}c_1(\Phi(\xi + (\xi, \gamma)\varphi_X)) \in \text{NS}(X_1)_{\mathbb{Q}} \).

Remark 2.1. By taking a locally free \( \alpha_1 \)-twisted stable sheaf \( G \) with \( \chi(G, G) = 0 \), we have a notion of Mukai vector, thus, we have a map ([7 Rem. 1.2.10]):
\[
v_G : D^{\alpha_1}(X_1) \to H^*(X_1, \mathbb{Q})_{\text{alg}}.
\]

We set
\[
\bar{\omega} := -\frac{1}{|r_1|} \frac{((\beta - \gamma)^2 - (\omega_1^2)\omega)}{((\beta - \gamma)^2 - (\omega^2)^2) + (\beta - \gamma, \omega)^2},
\]
\[
\bar{\beta} := \gamma' - \frac{1}{|r_1|} \frac{((\beta - \gamma)^2 - (\omega_1^2)\omega)}{((\beta - \gamma)^2 - (\omega^2)^2) + (\beta - \gamma, \omega)^2}.
\]

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By [8 sect. 5.1], we get the following commutative diagram:

\[
\begin{array}{ccc}
\mathbf{D}(X) & \longrightarrow & \mathbf{D}^\alpha_1(X_1) \\
Z(\beta,\omega) & \downarrow & Z(\beta,\omega) \\
\mathbb{C} & \xrightarrow{\zeta^{-1}} & \mathbb{C}
\end{array}
\]

where

\[
\zeta = -r_1 \left( \frac{((\gamma - \beta)^2) - (\omega^2)}{2} + \sqrt{-1}(\beta - \gamma, \omega) \right).
\]

Let \( E \) be a complex such that \( \Phi = \Phi^{[1]}_{X \rightarrow X_1} \). Then

\[
\phi(\beta,\omega)(\Phi(E)) = \phi(\beta,\omega)(E) - \phi(\beta,\omega)(E_1 \times \{x_1\}).
\]

**Theorem 2.2.** (cf. [8 Thm. 2.2.1]) Assume that \( \mathbb{R}Z(\beta,\omega)(v) = \mathbb{R}Z(\beta,\omega)(e^\gamma) \). If \( r_1 e^\gamma \) does not define a wall, then \( \Phi \) induces an isomorphism

\[
\mathcal{M}(\beta,\omega)(v) \rightarrow \mathcal{M}_\mathbb{R}(u)^{ss},
\]

where \( u := \Phi(v) \).

As an application of Theorem 2.2 we give a different proof of Proposition 1.7.

**Proof.** For \( H \in \text{Amp}(X)_Q \), we take \( \delta \in \text{NS}(X)_Q \) such that \( (\delta, H) = 1 \). We set \( H(\delta) := \{ L \in \text{Amp}(X)_R \mid (\delta, L) = 1 \} \). Then \( \{ tL \in \text{Amp}(X)_R \mid L \in H(\delta), t \in \mathbb{R}_{>0} \} \) is an open neighborhood of \( H \). Assume that \( (\beta_0, \omega_0) \in \mathbb{C} \) and \( \omega_0 \in \mathbb{R}_{>0} H \). We shall show that \( \sigma(\beta,\omega) \)-semi-stability is independent of \( \beta \) in a neighborhood of \( (\beta_0,\omega_0) \).

We can take \( \gamma \in \mathbb{R}H + \beta_0 \subset \text{NS}(X)_R \) such that \( \mathbb{R}Z(\beta,\omega)(v) = \mathbb{R}Z(\beta_0,\omega_0)(e^\gamma) \), \( (\gamma - \beta_0, H) > 0 \) and \( (\gamma - c_1(v), H) \neq 0 \) ([8 sect. 4.1]). Replacing \( \omega_0 \) if necessary, we may assume that \( \gamma \in \text{NS}(X)_Q \). We take a neighborhood \( U \) of \( H(\delta) \) and \( J \subset \mathbb{R}_{>0} \) such that \( \omega_0 \in \{ tL \mid L \in U, t \in J \} \subset \mathbb{C} \). For a fixed \( L \in H(\delta)_Q \), the semi-stability is independent of \( (\beta, L) \in \mathbb{C} \).

Replacing \( U \) if necessary, we may assume that the following equation for \( t > 0 \) has a solution for each \( L \in U \) and a neighborhood \( V \) of \( \beta_0 \):

\[
t^2(L^2) = \frac{((e^\gamma, e^\beta)(c_1(v) - r_1) - \langle v, e^\beta(\gamma - \beta), L \rangle)}{\langle r_1 - c_1(v), L \rangle}.
\]

We set \( \omega := tL, \omega \) is a function on \( V \times U \) and we have \( \mathbb{R}Z(\beta,\omega)(v) = \mathbb{R}Z(\beta,\omega)(e^\gamma) \). For the proof of our claim, it is sufficient to show the independence of \( \sigma(\beta,\omega) \)-semi-stability, where \( (\beta, L) \in V \times U \).

We set \( X_1 := M_H(r_1 e^\gamma) \). For a universal family \( E \) on \( X \times X_1 \) as a twisted object, we consider the Fourier-Mukai transform \( \Phi^{[1]}_{X \rightarrow X_1} \). Then we have an isomorphism \( \mathcal{M}(\beta,\omega)(v) \rightarrow \mathcal{M}_\mathbb{R}(u)^{ss} \), where \( u := \Phi^{[1]}_{X \rightarrow X_1}(v) \).

Since \( (c_1(w^\beta), \tilde{L}) = 0 \) for all \( (\beta, L) \in V \times U \), we have \( E \in \mathcal{M}_\mathbb{C}(u)^{ss} \) contains a subsheaf \( F_1 \) with \( c_1(F_1(-\beta), \tilde{\omega}) = 0 \) and \( v(F_1) \notin \mathbb{R}Q \). Then \( \Phi^{[1]}_{X \rightarrow X_1}(F_1) \) defines a wall for \( v \). Therefore for any subsheaf \( F_1 \) \( E \in \mathcal{M}_\mathbb{R}(u)^{ss} \), \( (c_1(F_1(-\beta)), \tilde{\omega}) \neq 0 \) for any \( (\beta, L) \in V \times U \) or we have \( v(F_1) \in \mathbb{R}Q \). For \( E \in \mathcal{M}_\mathbb{R}(u)^{ss} \), there is a subsheaf \( F_1 \) such that \( (c_1(F_1(-\beta)), \tilde{\omega}) \geq 0 \) and \( (c_1(F_1(-\beta)), \tilde{\omega}) \leq 0 \). Since \( (c_1(F_1(-\beta)), \tilde{\omega}) \) is a continuous function on \( V \times U \), it is a contradiction. Therefore \( \mathcal{M}_\mathbb{R}(u)^{ss} \) is independent of \( (\beta, L) \). Then we see that \( \mathcal{M}(\beta,\omega)(v) \) is independent of \( \Phi^{[1]}_{X \rightarrow X_1} \).

**Definition 2.3.** Let \( \sigma(\beta,\omega) \) be a stability condition. For the contravariant Fourier-Mukai transform \( \Phi \circ D_X \), we set \( (\beta',\omega') := (-\beta,\omega) \) and attach the stability condition \( \sigma(\beta',\omega') \) associated to \( Z(\beta',\omega') \). We say \( \sigma(\beta',\omega') \) the stability condition induced by \( \Phi \circ D_X \).

**Lemma 2.4.** \( \phi(\beta,\omega)(E^{[1]}_\beta) = -\phi(\beta,\omega)(E) + 1 \).

**Proof.** For a non-zero object \( E \in \mathcal{D}(X) \), we have

\[
(2.4) \quad Z(\beta,\omega)(E^{[1]}_\beta) = -(e^\beta + \omega \sqrt{-1}, v(E^{[1]}_\beta)) = -\langle e^\beta + \omega \sqrt{-1}, v(E) \rangle = |Z(\beta,\omega)(E)| e^{\sqrt{-1}(\phi(\beta,\omega)(E))}.
\]

Hence \( \phi(\beta,\omega)(E^{[1]}_\beta) = -\phi(\beta,\omega)(E) + 1 \mod 2\mathbb{Z} \). Since \( \phi(\beta,\omega)((E[n])^{[1]}_\beta) = \phi(\beta,\omega)(E^{[1]}_\beta) - n \), we shall show that \( \phi(\beta,\omega)(E^{[1]}_\beta) \in \{0, 1\} \) for \( E \in \mathcal{A}(\beta,\omega) \).

We note that \( \mathcal{A}(\beta,\omega) \) is generated by (i) 0-dimensional object \( T \), (ii) \( F^1 \) where \( F \) is a locally free \( \mu \)-semi-stable sheaf with \( d_\beta(F) \leq 0 \), (iii) \( \mu \)-semi-stable sheaf \( E \) with \( d_\beta(E) > 0 \) and (iv) purely 1-dimensional sheaf \( E \).

(i) For a 0-dimensional sheaf \( T \), \( T^{[1]} = \mathcal{A}(-\beta, -1) \). Thus \( \phi(\beta,\omega)(T^{[1]}) = 0 \). (ii) For a locally free \( \mu \)-semi-stable sheaf \( F \) with \( d_\beta(F) \leq 0 \), \( (F[1])^{[1]} = F^1 \) is a \( \mu \)-semi-stable sheaf with \( d_\beta((F[1])^{[1]}) \geq 0 \).
Hence we also have an isomorphism $M_{\beta,\omega}(v) \to M_{\beta,\omega}(-v)$. Then we have an exact triangle
$$T^\vee[1] \to (E^{**})^\vee[1] \to T^\vee[1] \to T^\vee[2].$$
Since $(E^{**})^\vee$ is a locally free $\mu$-semi-stable sheaf with $d_\beta((E^{**})^\vee) = -d_\beta(E) < 0$ and $T^\vee[2]$ is a 0-dimensional sheaf, $E^\vee[1] \in \mathfrak{A}_{(-\beta,\omega)}$. (iv) If $E$ is a purely 1-dimensional sheaf, then $E^\vee[1]$ is a purely 1-dimensional sheaf, which implies $E^\vee[1] \in \mathfrak{A}_{(-\beta,\omega)}$. Therefore the claim holds.

2.2. Relations of moduli spaces under Fourier-Mukai transforms. We shall say that a pair $(\beta, \omega)$ is general with respect to $v$ if $(\beta, \omega)$ is not on any wall $W_n$, for $v$.

**Theorem 2.5.** Let $v$ be a Mukai vector with $(v^2) > 0$. Assume that $(\beta, \omega)$ is general with respect to $v$.

1. Any Fourier-Mukai transform $\Phi^E_{X \to X'}$ preserves Bridgeland’s stability condition.

2. Any contravariant Fourier-Mukai transform $\Phi^E_{X \to X', \sigma D_X}$ preserves Bridgeland’s stability condition.

The first claim is due to Bridgeland ([3 Prop. 10.3]). For the proof of (2), it is sufficient to prove the following claim.

**Proposition 2.6.** Let $v$ be a Mukai vector with $(v^2) > 0$. Assume that $(\beta, \omega)$ is general with respect to $v$. If $Z_{(\beta,\omega)}(v) \in \mathbb{H} \cup \mathbb{R}_{<0}$, then we have an isomorphism
$$M_{(\beta,\omega)}(v) \to M_{(-\beta,\omega)}(-v).$$

**Proof.** Let $C$ be a chamber containing $(\beta, \omega)$. We may assume that $d_{\beta,\omega}(v) > 0$. Replacing $(\beta, \omega) \in C$ if necessary, we can take $\gamma \in \text{NS}(X)_{\mathbb{Q}}$ such that $\mathbb{R}Z_{(\beta,\omega)}(v) = \mathbb{R}Z_{(\beta,\omega)}(v)$. We take a primitive vector $r_1 e^\gamma$ such that $d_{\beta,\omega}(r_1 e^\gamma) > 0$. We set $X_1 := M_{(\beta,\omega)}(r_1 e^\gamma)$. Let $E$ be the universal object on $X \times X_1$ as a complex of twisted sheaves. We set $w := \Phi^E_{X \to X_1}(v)$. By Theorem 2.2, $M_{\Sigma}^\alpha(w)$ consists of $\mu$-stable locally free sheaves and we have an isomorphism
$$\Phi^E_{X \to X_1} : M_{(\beta,\omega)}(v) \to M_{\Sigma}^\alpha(w),$$
where $w$ satisfies $(c_1(we^{-\beta}), \omega) = 0$. By taking the dual, we have an isomorphism
$$M_{\Sigma}^\alpha(w) \to M_{\Sigma}^\alpha(w^\vee).$$

We note that $F := E^\vee[1]$ is a family of stable objects with $v(F_{X \times X_1}) = -r_1 e^{-\gamma}$. For $\Phi^E_{X \to X_1}$ we define $(\tilde{\beta}, \tilde{\omega})$ by \((2.1)\). Thus we substitute $(\gamma, \gamma', \beta)$ in \((2.1)\) instead of $(\gamma, \gamma', \beta)$ for the definition of $(\tilde{\beta}, \tilde{\omega})$. Then we have $(\tilde{\beta}, \tilde{\omega}) = (-\tilde{\beta}, \tilde{\omega})$. Since $\mathbb{R}Z_{(-\beta,\omega)}(r_1 e^{-\gamma}) = \mathbb{R}Z_{(-\beta,\omega)}(-v^\vee)$, we have an isomorphism
\[(2.5) \quad \Phi^E_{X \to X_1} : M_{(-\beta,\omega)}(-v^\vee) \to M_{\Sigma}^{-\alpha}(w^\vee).\]

By the Grothendieck-Serre duality, we have
$$\Phi^E_{X \to X_1}(E^\vee[1]) = \Phi^E_{X \to X_1}(E^\vee[1]) = D_{X_1} \circ \Phi^E_{X \to X_1}(E).$$

Hence the claim holds. \(\square\)

**Remark 2.7.** By a similar argument, we can prove Theorem 2.6 (1): Let $C$ be a chamber containing $(\beta, \omega)$. We may assume that $d_{\beta,\omega}(v) > 0$. Replacing $(\beta, \omega) \in C$ if necessary, we can take $\gamma \in \text{NS}(X)_{\mathbb{Q}}$ such that $\mathbb{R}Z_{(\beta,\omega)}(v) = \mathbb{R}Z_{(\beta,\omega)}(v)$. We take a primitive vector $r_1 e^\gamma$ such that $d_{\beta,\omega}(r_1 e^\gamma) > 0$. For any Fourier-Mukai transform $\Phi : D(X) \to D(Y)$, let $v_0 \in \mathbb{Q} \Phi(e^\gamma)$ be a primitive and positive Mukai vector. If $r_0 := \text{rk} v_0 \neq 0$, then $v_0 = r_0 e^{\gamma}, \gamma \in \text{NS}(Y)$. We first assume that $r_0 \neq 0$. We set
\[(2.6) \quad \Phi(e^{\gamma} + \sqrt{-1}w) = -(e^{\gamma} + \sqrt{-1}w, \Phi^{-1}(\Phi_0))e^{\gamma} + \sqrt{-1}w'.\]

Then $X_1 \cong M_{r_0}(v_0)$ and the universal object $G$ on $X_1 \times Y$ satisfies $\Phi^G_{X_1 \to Y} = \Phi \circ \Phi^E_{X_1 \to X}$, $n \in \mathbb{Z}$. Since $\mathbb{R}Z_{(\beta,\omega)}(v) = \mathbb{R}Z_{(\beta,\omega)}(v_0) = \mathbb{R}Z_{(\beta',\omega')}(v)$. Then $\Phi^G_{X_1 \to Y}$ induces an isomorphism
$$M_{\Sigma}^\alpha((w) \to M_{(\beta',\omega')}(v)).$$
Hence we have an isomorphism
\[\Phi : M_{(\beta,\omega)}(v) \to M_{(\beta',\omega')}(v).\]
If $r_0 = 0$, then $\Phi$ is induced by isomorphisms on underlying abelian surfaces and the action of $\text{Pic}^0(X)$. Hence we also have an isomorphism
$$M_{(\beta,\omega)}(v) \to M_{(\beta,\omega)}(\Phi(v)).$$
If $(\beta, \omega)$ belongs to a wall, then $\Phi$ also preserves semi-stability. Indeed assume that $E$ is $S$-equivalent to $\otimes_i E_i$, where $E_i$ are $\sigma_{(\beta,\omega)}$-stable objects with $\sigma_{(\beta,\omega)}(E_i) = (\beta,\omega)(E)$. Then $\Phi(E_i)$ are $\sigma_{(\beta',\omega')}$-stable objects. By \(2.3\), $\sigma_{(\beta',\omega')}(E_i)$ are the same. Therefore the claim holds.
3. Numerical solutions and the walls.

3.1. Semi-homogeneous presentations and numerical solutions. Let us recall the notion of semi-homogeneous presentations introduced in [14].

Definition 3.1. A semi-homogeneous presentation of \( E \in \text{Coh}(X) \) is an exact sequence

\[
0 \to E \to E_1 \to E_2 \to 0 \quad \text{or} \quad 0 \to E_1 \to E_2 \to E \to 0,
\]

where \( E_i \) (\( i = 1, 2 \)) are semi-homogeneous sheaves satisfying the following condition: if we write \( v(E_i) = \ell_i v_i \) with \( \ell_i \) positive integers and \( v_i \) primitive Mukai vectors, then

\[
\langle \ell_1 - 1, \ell_2 - 1 \rangle = 0, \quad \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0, \quad \langle v_1, v_2 \rangle = -1.
\]

We call the first sequence the kernel presentation and the second one the cokernel presentation.

Since semi-homogeneous sheaves on abelian varieties are well-known objects, semi-homogeneous presentations give useful informations on \( E \). Moreover the property of having a semi-homogeneous presentation is an open condition, and describes the birational structure of the moduli spaces \( M_H(v) \). The numerical data appeared in Definition 3.1 are useful to find a semi-homogeneous presentation. So we introduce the following definition.

Definition 3.2. For a Mukai vector \( v \), the equation

\[
v = \pm(\ell_1 v_1 - \ell_2 v_2),
\]

\( \ell_1, \ell_2 \in \mathbb{Z}_{>0}, \quad v_1, v_2 \) positive primitive Mukai vectors,

is called the numerical equation of \( v \).

A solution \( (v_1, v_2, \ell_1, \ell_2) \) of this equation satisfying

\[
\langle \ell_1 - 1, \ell_2 - 1 \rangle = 0, \quad \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0, \quad \langle v_1, v_2 \rangle = -1,
\]

is called a numerical solution of \( v \).

Theorem 3.3 ([14] Thm. 3.9]). Suppose \( \text{NS}(X) = \mathbb{Z}H \) and let \( v \) be a Mukai vector with \( \langle v^2 \rangle > 0 \).

(i) If \( v \) has at least two numerical solutions, then a general member of \( M_H(v) \) has both kernel presentation and cokernel presentation. Each presentation is unique.

(ii) If \( v \) has only one numerical solution, then a general member of \( M_H(v) \) has either kernel presentation or cokernel presentation. Such a presentation is unique.

For each numerical solution \( (v_1, v_2, \ell_1, \ell_2) \) of \( v \), we constructed moduli spaces of simple two-term complexes. These moduli spaces plays an important role to prove [14] Thm. 3.9. We fix an ample divisor \( H \) on \( X \).

Theorem 3.4 ([14] Thm. 4.9]). Let \( v \) be a positive Mukai vector with \( \langle v^2 \rangle > 0 \) and \( (v_1, v_2, \ell_1, \ell_2) \) be a numerical solution of \( v \).

(i) We have the fine moduli space \( \mathfrak{M}^{-}(v_1, v_2, \ell_1, \ell_2) \) of simple complexes \( V^\bullet \) such that \( H^i(V^\bullet) = 0 \) \((i \neq -1, 0)\), \( H^{-1}(V^\bullet) \in \mathcal{M}_H(\ell v_1)_{\text{ss}} \) and \( H^0(V^\bullet) \in \mathcal{M}_H(\ell v_2)_{\text{ss}} \).

(ii) We have the fine moduli space \( \mathfrak{M}^{+}(v_1, v_2, \ell_1, \ell_2) \) of simple complexes \( V^\bullet \) such that \( V^\bullet \cong [W^{-1} \to W^0], W^{-1} \in \mathcal{M}_H(\ell v_1)_{\text{ss}} \) and \( W^0 \in \mathcal{M}_H(\ell v_2)_{\text{ss}} \).

Remark 3.5. Since \( \mathcal{M}_H(\ell v_i)_{\text{ss}} \) \((i = 1, 2)\) are independent of the choice of \( H \), \( \mathfrak{M}^{\pm}(v_1, v_2, \ell_1, \ell_2) \) are independent of the choice of \( H \).

The relation of \( \mathfrak{M}^{\pm}(v_1, v_2, \ell_1, \ell_2) \) is described as follows:

Proposition 3.6 ([14] Prop. 4.11]). For \( i = 1, 2 \), we denote \( Y_i := M_H(v_i) \), and let \( E_i \) be a universal family such that \( v(\Phi_{X \to Y_i}^E(v_j)) = (1, 0, 0) \) for \( j \neq i \). For \( V^\bullet \in \mathfrak{M}^{\pm}(v_1, v_2, \ell_1, \ell_2) \), we set

\[
\Psi(V^\bullet) := \begin{cases} 
\Phi^E_{Y_{i \to X} \to Y_i} \Phi^E_{X \to Y_i}(V^\bullet) & \ell_1 = \langle v_1^2 \rangle / 2, \\
\Phi^E_{Y_{i \to X} \to Y_i} \Phi^E_{X \to Y_i}(V^\bullet) & \ell_2 = \langle v_2^2 \rangle / 2.
\end{cases}
\]

Then \( \Psi \) induces an isomorphism

\[
\mathfrak{M}^{\pm}(v_1, v_2, \ell_1, \ell_2) \xrightarrow{\sim} \mathfrak{M}^{-}(v_1, v_2, \ell_1, \ell_2).
\]
3.2. Relation with the walls. The operation $\Psi$ is first introduced in [18] to construct birational map of moduli spaces, and it is reformulated as an isomorphism of $\mathfrak{M}^\pm(v_1, v_2, \ell_1, \ell_2)$ in [4]. $\Psi$ plays a fundamental role in these papers. In [7 sect. 4], we explained its relation with Bridgeland’s stability condition. In particular, we showed that $\mathfrak{M}^\pm(v_1, v_2, \ell_1, \ell_2)$ are moduli of stable objects if $v_1$ defines a wall. We shall slightly generalize this fact. Thus we follow the following.

**Lemma 3.7.** There is a stability condition $\sigma(\beta, \omega) = (\mathfrak{X}(\beta, \omega), Z(\beta, \omega))$ such that $\mathfrak{M}^\pm(v_1, v_2, \ell_1, \ell_2)$ are the moduli schemes of stable objects $M_{(\beta, \omega), (\pm v)}$, where $\omega^\pm = t^\pm \omega$, $t^- < 1 < t^+$, $t^+ - t^- < 1$.

**Proof.** Replacing $v$ by $-v$, we assume that $v = \ell_2 v_2 - v_1$. We set $v = re^{\gamma} + \xi$ and $v_i = r_i e^{\gamma} e^{\xi_i}$, $(i = 1, 2)$. Then $r = \pm (r_1 \ell_1 - r_2 \ell_2)$, $r_1 \ell_1 \xi_1 = r_2 \ell_2 \xi_2$ and $\xi = \pm (r_1 \ell_1 (\xi_1^2 - r_2 \ell_2 (\xi_2^2))$. Since $0 > (\ell_2 \xi_2 - \ell_1 \xi_1)< \ell_2 \xi_2 - \ell_1 \xi_1)$, there is an ample divisor $H$ with $(\xi_1 - \xi_2, H) \neq 0$. We may assume that $(\xi_1, H) < (\xi_2, H)$. We take $\beta(x) := \gamma + x (1 - x) \xi_2$ with $0 < x < 1$. Then

$$v_1 = r_1 e^{\beta(x)} (1 - x) (\xi_1 - \xi_2)$$

$$v_2 = r_2 e^{\beta(x)} (1 - x) (\xi_2 - \xi_1)$$

If $r_1 \ell_1 > r_2 \ell_2$, then $(\xi_2, H) > (\xi_1, H)$ implies that $(\xi_2, H) > (\xi_1, H) > 0$. Hence $d_{\beta(x)}(-v_1) > 0$ and $d_{\beta(x)}(v_2) > 0$. We have

$$\frac{d_{\beta(x)}(v_1) a_{\beta(x)}(v_2) - d_{\beta(x)}(v_2) a_{\beta(x)}(v_1)}{d_{\beta(x)}(v_1) r_{\beta(x)}(v_2) - d_{\beta(x)}(v_2) r_{\beta(x)}(v_1)} = (1 - x) \frac{x (\xi_1 - \xi_2)^2}{2} > 0.$$  

Hence there is a positive number $t$ such that $Z_{(\beta(x), tH)}(v_1) < 0$. Then $(\beta, \omega) := (\beta(x), tH)$ belongs to the wall defined by $-v_1$. By [14 Thm. 4.9] and [7 Prop. 4.1.5], $\mathfrak{M}^\pm(v_1, v_2, \ell_1, \ell_2) = M_{(\beta, \omega), (\pm v)}.$

If $r_1 \ell_1 > r_2 \ell_2$, then $\xi_1, H < (\xi_2, H) < 0$. Hence $d_{\beta(x)}(v_1) > 0$ and $d_{\beta(x)}(-v_2) = 0$. In this case, we also have

$$\frac{d_{\beta(x)}(v_1) a_{\beta(x)}(v_2) - d_{\beta(x)}(v_2) a_{\beta(x)}(v_1)}{d_{\beta(x)}(v_1) r_{\beta(x)}(v_2) - d_{\beta(x)}(v_2) r_{\beta(x)}(v_1)} = (1 - x) \frac{x (\xi_1 - \xi_2)^2}{2} > 0.$$  

Hence there is a positive number $t$ such that $Z_{(\beta(x), tH)}(v_1) < 0$. Then $(\beta, \omega) := (\beta(x), tH)$ belongs to the wall defined by $-v_1$. By [14 Thm. 4.9] and [7 Prop. 4.1.5], $\mathfrak{M}^\pm(v_1, v_2, \ell_1, \ell_2) = M_{(\beta, \omega), (\pm v)}.$

\[\Box\]

4. Stability conditions on a restricted parameter space.

4.1. The structure of walls. In this section, we shall partially generalize the structure of walls in [5]. We note that

$$\{\sigma(\xi, \omega) \mid \xi \in \text{NS}(X)_{\mathbb{R}}, \omega \in \text{Amp}(X)_{\mathbb{R}}\} = \bigcup_{(\beta, H) \in \text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}^0} \{\sigma(\beta, H, \omega) \mid s \in \mathbb{R}, t \in \mathbb{R}^+\}.$$

In this section, we fix $\beta \in \text{NS}(X)_{\mathbb{R}}$ and an ample divisor $H$ on $X$ and study the wall for the space of special stability conditions:

$$\{\sigma(\beta, H, \omega) \mid s \in \mathbb{R}, t \in \mathbb{R}^+\}.$$

**Definition 4.1.** Let $v_1$ be a Mukai vector with $\langle v_1^2 \rangle \geq 0$, $\langle (v - v_1)^2 \rangle \geq 0$ and $\langle v_1, v - v_1 \rangle > 0$. If $(\text{rk } v_1, d_{\beta}(v_1), a_{\beta}(v_1)) \notin \mathbb{Q}(\text{rk } v, d_{\beta}(v), a_{\beta}(v))$, we define a wall for $v$ by

$$W_{v_1, H} := \{(s, t) \in \mathbb{R}^2 \mid \mathbb{R} Z_{(\beta, H, tH)}(v_1) = \mathbb{R} Z_{(\beta, H, tH)}(v)\}.$$  

A connected component of $\mathbb{R} \times \mathbb{R}^+ \setminus \bigcup_{v_1} W_{v_1, H}$ is called a chamber for $v$.

For the study of walls, we collect elementary facts on a family of circles.

**Lemma 4.2.** We take $p \in \mathbb{R}$ and $q \in \mathbb{R}^+$. For $a \in \mathbb{R}$, let $C_a : (x + a)^2 + y^2 = (a + p)^2 - q$ be the circle in $(x, y)$-plane. We also set $C_\infty : x = p$. Thus $C_\infty$ is a line in $(x, y)$-plane.

(1) $C_a \cap C_{a'} = \emptyset$, if $a \neq a'$.

(2) $C_a \cap C_\infty = \emptyset$ for $a \in \mathbb{R}$.

**Proof.** (1) If $C_a \cap C_{a'} \neq \emptyset$, then the intersection satisfies $x = p$. Then $(p + a)^2 + y^2 = (a + p)^2 - q$, which implies that $y^2 = -q < 0$. Therefore the claim holds. The proof of (2) is similar.  

\[\Box\]
Lemma 4.4. Assume that $C_a : (x + a)^2 + y^2 = (a + p)^2 - q$ with $q > 0$ is non-empty. If $a + p > 0$, then $(p - \sqrt{q}, 0)$ is contained in $C_a$. If $a + p < 0$, then $(p + \sqrt{q}, 0)$ is contained in $C_a$.

Proof. Assume that $a + p > 0$. Then $(a + p)^2 - q - (p - \sqrt{q} + a)^2 = 2\sqrt{q}(a + p - \sqrt{q}) > 0$. Hence the claim holds. If $a + p < 0$, then $a + p < -\sqrt{q}$. Hence $(a + p)^2 - q - (p + \sqrt{q} + a)^2 = -2\sqrt{q}(a + p + \sqrt{q}) > 0$. Therefore the claim holds. □

Remark 4.3. By the proof, we see that $C_a$ $(a \in \mathbb{R})$ forms a pencil of conics passing through the imaginary points $\{(p, \pm \sqrt{-q})\}$.

We shall study the structure of walls. We first assume that $r \neq 0$ and set

$$v := r + D + b \tilde{g}_X \quad \text{and} \quad a_\beta := \frac{d_2^2(H^2) - ((v^2) - (D_\beta^2))}{2r}.$$ 

Then

$$a_\beta = 
\frac{d_2^2(H^2) - ((v^2) - (D_\beta^2))}{2r}, \quad \tilde{a}_\beta = \frac{(d_\beta - r s)(H^2) - ((v^2) - (D_\beta^2))}{2r}.$$ 

We also set

$$v_2 := r e_\beta + (d_2 H + D_\beta) + (d_2 H + D_\beta, \beta) \tilde{g}_X + a_2 \tilde{g}_X$$

Then

$$r = r e_\beta + (d_2 H + D_\beta) + (d_2 H + D_\beta, \beta) \tilde{g}_X + a_2 \tilde{g}_X$$

Proposition 4.5. Assume that $r \neq 0$ and $(v^2) > 0$.

1. Assume that $rd_2 - rd_\beta \neq 0$. Then $\mathbb{R}Z_{(\beta + s H, t H)}(v) = \mathbb{R}Z_{(\beta + s H, t H)}(v_2)$ holds for $(s, t) \in \mathbb{R}^2$ if and only if

$$i^2 + \left( s - \frac{a_2 r - a_\beta r_2}{(H^2)(rd_2 - rd_\beta)} \right)^2 = \left( \frac{d_2}{r} - \frac{a_2 r - a_\beta r_2}{(H^2)(rd_2 - rd_\beta)} \right)^2 - \frac{(v^2) - (D_\beta^2)}{(H^2)^2}.$$ 

2. Assume that $rd_2 - rd_\beta = 0$ and $a_2 r - a_\beta r_2 \neq 0$. Then $\mathbb{R}Z_{(\beta + s H, t H)}(v) = \mathbb{R}Z_{(\beta + s H, t H)}(v_2)$ holds for $(s, t) \in \mathbb{R}^2$ if and only if

$$rs - d_\beta = 0.$$

Proof. (1) We first note that

$$a_\beta d_2 - a_2 d_\beta = \frac{a_\beta}{r} (rd_2 - rd_\beta) - \frac{d_\beta}{r} (a_2 r - a_\beta r_2).$$

Then

$$(d_2 - rd_\beta) \tilde{a}_\beta - (d_\beta - rs) \tilde{a}_2 = -s \frac{(H^2)}{2} (rd_2 - rd_\beta) + s(a_2 r - a_\beta r_2) + (a_\beta d_2 - a_2 d_\beta)$$

$$= \frac{(H^2)}{2} \left\{ -s^2 (rd_2 - rd_\beta) + s \frac{2}{(H^2)} (a_2 r - a_\beta r_2) \right\} + \frac{2}{(H^2)^2} \frac{a_\beta}{r} (rd_2 - rd_\beta) - \frac{d_\beta}{r} (a_2 r - a_\beta r_2)$$

$$= \frac{(H^2)}{2} \left\{ -s^2 (rd_2 - rd_\beta) + s \frac{(a_2 r - a_\beta r_2)}{(H^2)(rd_2 - rd_\beta)} \right\} + \frac{2}{(H^2)^2} \frac{a_\beta}{r} (rd_2 - rd_\beta) - \frac{d_\beta}{r} (a_2 r - a_\beta r_2)$$

Hence the claim holds. (2) If $rd_2 - rd_\beta = 0$, then the claim follows from

$$(d_2 - rd_\beta) \tilde{a}_\beta - (d_\beta - rs) \tilde{a}_2 = -s \frac{(H^2)}{2} (rd_2 - rd_\beta) + s(a_2 r - a_\beta r_2) + (a_\beta d_2 - a_2 d_\beta).$$
By Lemma 4.4 we get the following corollary.

**Corollary 4.6.**  
(1) If \( \frac{d_2}{r} - \frac{a_2r - a_2 r_2}{(H^2/s_2 - r_2 d_2)} > 0 \), then \((s, t) = \left( \frac{d_2}{r} - \sqrt{\frac{(\nu^2) - (D_2^2)}{(H^2/r^2)}, 0} \right) \) is contained in the circle \( X \).

(2) If \( \frac{d_2}{r} - \frac{a_2r - a_2 r_2}{(H^2/s_2 - r_2 d_2)} < 0 \), then \((s, t) = \left( \frac{d_2}{r} + \sqrt{\frac{(\nu^2) - (D_2^2)}{(H^2/r^2)}, 0} \right) \) is contained in the circle \( Y \).

**Remark 4.7.**  
(1) If \( rd_2 - r_2 d_\beta < 0 \), then \((s, t)\) is surrounded by the circle in Proposition 4.5 if and only if \( \phi_{(\beta + H, H)}(v_2) \mod 2Z \) satisfies \( \phi_{(\beta + s, H, H)}(v_2) + 1 \) > \( \phi_{(\beta + s, H, H)}(v_2) \).

(2) If \( rd_2 - r_2 d_\beta > 0 \), then \((s, t)\) is surrounded by the circle in Proposition 4.5 if and only if \( \phi_{(\beta + s, H, H)}(v_2) \mod 2Z \) satisfies \( \phi_{(\beta + s, H, H)}(v_2) > \phi_{(\beta + s, H, H)}(v_2) + 1 \).

We next treat the case where \( r = 0 \).

**Proposition 4.8.** Assume that \( r = 0 \) and \( \langle \nu^2 \rangle > 0 \). If \( rd_2 - r_2 d_\beta \neq 0 \), then \( \Re Z_{(\beta + s, H, H)}(v) = \Re Z_{(\beta + s, H, H)}(v_2) \) holds for \((s, t) \in \mathbb{R}^2 \) if and only if

\[
i^2 + \left( s - \frac{a_\beta}{d_\beta(H^2)} \right)^2 = \left( \frac{a_\beta}{d_\beta(H^2)} - \frac{d_2}{r_2} \right)^2 - \frac{\langle \nu^2 \rangle - (D_2^2)}{r_2^2(H^2)}.
\]

**Proof.** We note that \( d_\beta \neq 0 \). Hence \( r_2 \neq 0 \). Then the claim follows from the following computation:

\[
(d_2 - r_2 s)\bar{a}_\beta - (d_\beta - rs)\bar{a}_2 = -s^2(H^2/2)(rd_2 - r_2 d_\beta) + (a_2 r - a_\beta r_2) + (a_\beta d_2 - a_2 d_\beta)
\]

\[
= \frac{r_2 d_\beta(H^2)}{2} \left( s^2 - s \cdot \frac{2a_\beta}{d_\beta(H^2)} + 2(a_\beta d_2 - a_2 d_\beta) / (H^2 r_2 d_\beta) \right)
\]

\[
= \frac{r_2 d_\beta(H^2)}{2} \left( s - \frac{a_\beta}{d_\beta(H^2)} \right)^2 - \frac{a_\beta}{d_\beta(H^2)} - \frac{d_2}{r_2} \right)^2 + \frac{\langle \nu^2 \rangle - (D_2^2)}{r_2^2(H^2)}.
\]

**Corollary 4.9.** If \( \sqrt{\frac{(\nu^2) - (D_2^2)}{(H^2)}} \in \mathbb{Q} \), then there are finitely many walls for \( v \).

**Proof.** For a fixed \( s \), there are finitely many walls. If \( r = 0 \), then \((s, t) = \left( \frac{d_2}{r}, 0 \right) \) is the center of \( X \). Hence every wall intersects with \( s = \frac{d_2}{r} - \sqrt{\frac{(\nu^2) - (D_2^2)}{(H^2/r^2)}} \) or \( s = \frac{d_2}{r} + \sqrt{\frac{(\nu^2) - (D_2^2)}{(H^2/r^2)}} \). Hence there are finitely many walls for \( v \).

**Lemma 4.10.** For \( v_2 = e^{\beta + \lambda H} \), the condition \( \Re Z_{(\beta + s, H, H)}(v) = \Re Z_{(\beta + s, H, H)}(v_2) \) for \((s, t) \in \mathbb{R}^2 \) is equivalent to the equation of the circle

\[
C_{v, \lambda} : t^2 + (s - \lambda) \left( \frac{1}{r(\lambda - d_\beta)} \left( \lambda d_\beta - \frac{2}{(H^2/2)a_\beta} \right) \right) = 0
\]

for \( r\lambda - d_\beta \neq 0 \) and

\[
rs - d_\beta = 0
\]

for \( r\lambda - d_\beta = 0 \). In particular, the circle \( C_{v, \lambda} \) passes the points \((\lambda, 0)\) and \((\lambda, 0) = \left( \frac{1}{(r\lambda - d_\beta)}(\lambda d_\beta - \frac{2}{(H^2/2)a_\beta}), 0 \right) \).

**Proof.** We note that

\[
v_2 = e^{\beta + \lambda H} = e^{\beta} + \lambda(H + (H, \beta)q_X) + \frac{(H^2)}{2} \lambda^2 q_X
\]

\[
= e^{\beta + sH} + \lambda sH + (H, \beta)q_X + \frac{(H^2)}{2} \lambda^2 q_X.
\]

Then we get

\[
(d_2 - r_2 s)\bar{a}_\beta - (d_\beta - rs)\bar{a}_2 = (\lambda - s)\bar{a}_\beta - (d_\beta - rs) \frac{(H^2)}{2}(\lambda - s)^2
\]

\[
= (\lambda - s) \left( \frac{(r\lambda - d_\beta)}{2}s \right)
\]

\[
= (\lambda - s) \left( \frac{(r\lambda - d_\beta)}{2}s - \frac{\lambda d_\beta}{2} - a_\beta \right).
\]
Assume that $r\lambda - d_\beta \neq 0$. Then the condition is given by the circle
\[ t^2 + (s - \lambda) \left( s - \frac{1}{r\lambda - d_\beta} \left( \lambda d_\beta - \frac{2}{(H^2)} a_\beta \right) \right) = 0. \]
In particular, the circle passes the points $(\lambda, 0)$ and $\left( \frac{1}{r\lambda - d_\beta} (\lambda d_\beta - \frac{2}{(H^2)} a_\beta), 0 \right)$. Assume that $r\lambda - d_\beta = 0$. Then by (4.3), we get
\[ 0 = (\lambda - s) \left( \lambda d_\beta (H^2) - a_\beta \right). \]
If $a_\beta = \lambda d_\beta (H^2) = r\lambda^2 (H^2)$, then we see that $v = r e^{\beta + \lambda H} + (D_\beta + (D_\beta + \lambda H)) g_X$. Hence $\langle v^2 \rangle = (D_\beta^2) \leq 0$, which is a contradiction. Therefore we get
\[ s = \lambda = \frac{d_\beta}{r}. \]

**Remark 4.11.** Assume that $r \neq 0$. Then
\[ \lambda \neq \frac{1}{r\lambda - d_\beta} \left( \lambda d_\beta - \frac{2}{(H^2)} a_\beta \right) \iff \lambda \neq \frac{d_\beta}{r} \pm \sqrt{\frac{\langle v^2 \rangle - (D_\beta^2)}{(H^2)r^2}}. \]
In particular, if $\sqrt{\frac{\langle v^2 \rangle - (D_\beta^2)}{(H^2)r^2}} \notin \mathbb{Q}$, then $C_{v, \lambda}$ is a circle.

**Corollary 4.12.** Assume that $v = r e^{\beta + a_\beta g_X} + d_\beta (H + (H, \beta) g_X)$. For a numerical solution
\[ (r_1 e^{\beta + \lambda_1 H}, r_2 e^{\beta + \lambda_2 H}, \ell_1, \ell_2), \]
we have $C_{v, \lambda_1} = C_{v, \lambda_2}$ and the equation is given by
\[ (4.4) \quad t^2 + (s - \lambda_1)(s - \lambda_2) = 0. \]

**Proof.** Since $v = \pm (\ell_1 r_1 e^{\beta + \lambda_1} - \ell_2 r_2 e^{\beta + \lambda_2})$, we have $C_{v, \lambda_1} = C_{v, \lambda_2}$. Since $(\lambda_i, 0) \in C_{v, \lambda_i}$ for $i = 1, 2$ and $\lambda_1 \neq \lambda_2$, we get the claim. \qed

### 4.2. Relation of stability conditions.
All walls except $rs = d_\beta$ are disjoint to the line $rs = d_\beta$. By Corollary 4.10 there are at most two unbounded chambers.

**Proposition 4.13.** Let $v$ be a positive and primitive Mukai vector such that $\langle v^2 \rangle > 0$. Assume that $(s, t) \in \mathbb{R}^2$ belongs to an unbounded chamber. Then
\[
M_{(\beta + sH, tH)}(v) \cong \begin{cases} 
M_H^\beta(v) & d_{\beta + sH}(v) > 0, \\
M_H^{-\beta}(v) & d_{\beta + sH}(v) \leq 0. 
\end{cases}
\]

**Proof.** If $d_{\beta + sH}(v) > 0$, then $\text{rk} v \geq 0$. By \cite[Cor. 2.2.9]{We}, we get $M_{(\beta + sH, tH)}(v) = M_H^\beta(v)$. If $d_{\beta + sH}(v) \leq 0$, then $d_{\beta + sH}(-v) \geq 0$ and $\text{rk}(-v) \leq 0$. If $\text{rk}(-v) = 0$, then $d_{\beta}(-v) = 0$ and $\langle v^2 \rangle \leq 0$, which is a contradiction. Thus we have $\text{rk}(-v) < 0$. Then \cite[Cor. 2.2.9]{We} implies that we have an isomorphism $M_H^{-\beta}(v') \cong M_{(\beta + sH, tH)}(v)$ via $F \to F'$. \qed

**Lemma 4.14.** Let $C_0$ and $C_1$ be two chambers such that $C_0$ is surrounded by $C_1$. We take $(s, t_0) \in C_0$ and $(s, t_1) \in C_1$. Let $x := (1 - x)t_0 + xt_1$ ($0 < x < 1$) be a segment connecting $t_0$ and $t_1$. If $E$ is $\sigma_{(\beta + sH, t_i H)}$-semi-stable for $i = 0, 1$, then $E$ is $\sigma_{(\beta + sH, t_i H)}$-semi-stable for all $x$.

**Proof.** Assume that $E$ is not $\sigma_{(\beta + sH, t_i H)}$-semi-stable for some $x \in (0, 1)$. Then there is a subobject $E_1$ of $E$ in $\mathfrak{M}_{(\beta + sH, t_i H)}$ such that $\phi_{(\beta + sH, t_i H)}(E_1) > \phi_{(\beta + sH, t_i H)}(E)$. Since $E$ is $\sigma_{(\beta + sH, t_i H)}$-semi-stable for $i = 0, 1$, $\phi_{(\beta + sH, t_i H)}(E_1) \leq \phi_{(\beta + sH, t_i H)}(E)$. Then there are two numbers $x_1, x_2$ such that $0 < x_1, x_2 < 1$ and $\phi_{(\beta + sH, t_i H)}(E_1) = \phi_{(\beta + sH, t_i H)}(E)$. Since $t_{x_i}$ is uniquely determined by $v(E_2)$, this does not occur. Therefore $E$ is $\sigma_{(\beta + sH, t_i H)}$-semi-stable for all $x$. \qed

**Definition 4.15.** Let $W$ be a wall for $v$ in $(s, t)$-plane. Let $(\beta, \omega)$ be a point of $W$ and $(\beta', \omega')$ be a point in an adjacent chamber. Then we define the codimension of the wall $W$ by
\[ \text{codim} W := \min_{v = \sum_i v_i} \left\{ \sum_{i < j} \langle v_i, v_j \rangle - \left( \sum_i (\dim M_H^\beta(v_i)) s_i - \langle v_i^2 \rangle \right) + 1 \right\}, \]
where $v = \sum_i v_i$ are decompositions of $v$ such that $\phi_{(\beta, \omega)}(v) = \phi_{(\beta, \omega)}(v_i)$ and $\phi_{(\beta', \omega')}(v_i) > \phi_{(\beta', \omega')}(v_j)$, $i < j$.

By using \cite[Lem. 4.2.4, Rem. 4.2.3]{We}, we get the following result.
Lemma 4.16. If $W$ is a codimension 0 wall, then $W$ is defined by $v_1$ such that

\[ v = n v_1 + v_2, \quad \langle v_1, v_2 \rangle = 1, \quad \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0. \]

If $v_1, v_2$ in $\text{Eq. } (4.6)$ satisfy $v_1 > 0, v_2 < 0$ or $v_1 < 0$ and $v_2 > 0$, then $(v_1, v_2, n, 1)$ or $(-v_1, v_2, n, 1)$ gives a numerical solution of $v$. Conversely for a numerical solution $(v_1, v_2, \ell_1, \ell_2)$, Lemma 3.7 implies that for a suitable $\beta$ and $\omega = tH$, $v_1$ defines a codimension 0 wall.

Proposition 4.17. Assume that $\text{NS}(X) = ZH$. We fix $\beta$. Then there is a bijective correspondence between a codimension 0 wall and a numerical solution.

Proof. Assume that $\text{NS}(X) = ZH$. For a numerical solution $(v_1, v_2, \ell_1, \ell_2)$, $\beta, c_1(v), c_1(v_1), c_1(v_2) \in \mathbb{Q}H$ implies that $C_{v, \lambda}$ in Corollary 4.12 gives a wall in the $(s, t)$-plane. Combining Lemma 4.16 we get the claim. \[\Box\]

Proposition 4.18 (cf. [7] Prop. 4.2.5]). If $(\beta_1, \omega_1)$ and $(\beta_2, \omega_2)$ are not separated by any codimension 0 wall, then $M(\beta_1, \omega_1)(v) \cap M(\beta_2, \omega_2)(v) \neq \emptyset$. In particular, $M(\beta_1, \omega_1)(v)$ and $M(\beta_2, \omega_2)(v)$ are birationally equivalent.

4.3. Semi-homogeneous presentation and the stability. For a semi-homogeneous presentation, Lemma 3.7 implies that we can relate a $\sigma_{(\beta, \omega)}$-semi-stability. We shall study Gieseker semi-stability of coherent sheaves with two semi-homogeneous presentations.

Lemma 4.19. Let $v$ be a primitive Mukai vector with $r := \text{rk} v > 0$. Assume that $M_H^\beta(v)^{ss}$ consists of $\beta$-stable sheaves. If a simple sheaf $E$ with $v(E) = v$ has two semi-homogeneous presentations and $d_\beta(v) - rs = 0$ is not a codimension 0 wall, then $E$ is a $\mu$-stable vector bundle.

Proof. Since $rs - d_\beta(v) = 0$ is not a codimension 0 wall, two semi-homogeneous presentations define two circles $C_1$ and $C_2$ which are separated by the line $rs - d_\beta(v) = 0$.

There is a Fourier-Mukai transform $\Phi := \Phi_{F \to X}$ and a complex $F$ such that $E = \Phi(F)$ and $F$ is semi-stable with respect to two chambers $C_0, C_1$ such that $C_0$ is surrounded by $C_1$ and $C_2$. Let $\mathcal{C}$ be an unbounded chamber between $\Phi(C_1)$ and $\Phi(C_2)$. Then $F$ is semi-stable with respect to $\Phi^{-1}(\mathcal{C})$ by Lemma 4.11. Hence $E$ is semi-stable with respect to all unbounded chambers.

We take an element $\omega \in \text{Amp}(X)_0$. Then $E$ is $\sigma_{(\beta + sH, \omega)}$-semi-stable if $1 \gg d_\beta(v) - rs > 0$. Hence $E$ is $\beta$-twisted semi-stable. Since $E$ is also $\sigma_{(\beta + sH, \omega)}$-semi-stable if $1 \gg -d_\beta(v) - rs > 0$, $E^\vee$ is $(-\beta)$-twisted semi-stable. Hence $E$ is locally free. Let $E_1$ be a locally free subsheaf of $E$. Then there is a generically surjective homomorphism $E^\vee \to E_1^\vee$. Since

\[ \frac{\chi(E_1(\beta))}{\text{rk } E_1} = \frac{\chi(E_1(\beta))}{\text{rk } E_1}, \]

\[ \frac{\text{d} \chi(E_1(\beta))}{\text{d} \text{rk } E_1} \]

imply $\frac{\chi(E_1(\beta))}{\text{rk } E_1} = \frac{\chi(E_1(\beta))}{\text{rk } E_1}$. Thus $E$ is properly $\beta$-twisted semi-stable, which is a contradiction. Therefore $E$ is $\mu$-stable. \[\Box\]

Lemma 4.20. Let $v$ be a primitive Mukai vector with $r := \text{rk} v > 0$. Assume that $d_\beta(v) - rs = 0$ defines a wall.

1. $d_\beta(v) - rs = 0$ is a codimension 0 wall if and only if
   (a) $v = re^\xi - a q_X$, $\xi \in \text{NS}(X), a \in \mathbb{Z}, (r-1)(a-1) = 0$ or
   (b) $v = v_1 + \ell v_2, \quad v_1 = r_1e^{\xi'}, r_1, r_2 > 0, \quad ((r_2\xi_1 - r_1\xi_2)^2) = r_1 r_2, \quad (r_2\xi_1 - r_1\xi_2, H) = 0$.

2. Assume that $v$ satisfies (a). We take $(s, t)$ such that $1 \gg d_\beta(v) - rs > 0$ and let $E$ be a $\sigma_{(\beta + sH, tH)}$-semi-stable object with $v(E) = v$. Then $r = 1$ and $E = I_Z(\xi)$ or $a = 1$ and $E = \ker(\oplus_{i=1}^n O_X(\xi_i) \to \mathcal{T})$.

3. If $\text{NS}(X) = ZH$, then (b) does not occur.

Proof. (1), (3) By Lemma 4.16 we have $v = v_1 + \ell v_2, \quad \langle v_1, v_2 \rangle = 1, \quad \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0$. Assume that $\text{rk } v_1 \text{ rk } v_2 = 0$. Then we can set $v_1 := r_1e^{\xi'} (r_2\xi_1 - r_1\xi_2)^2 = r_1 r_2$. Since the wall is $d_\beta - rs = 0$, we have $(r_2\xi_1 - r_1\xi_2, H) = 0$. By the Hodge index theorem, $r_1 r_2 > 0$. Thus $r_1, r_2 > 0$. If $r_1 = 0$ or $r_2 = 0$, then $r_2 d_\beta(v_1) - r_1 d_\beta(v_2) = 0$ and $\langle v_1, v_2 \rangle = 1$ implies that $\{v_1, v_2\} = \{0, 0, 1\}, (-1, 0, 0)$. Therefore (1) holds. If $\text{NS}(X) = ZH$ and $(r_2\xi_1 - r_1\xi_2, H) = 0$, then we have $r_2\xi_1 - r_1\xi_2 = 0$. Hence $r_1 r_2 = 0$. Thus (b) does not occur.

(2) By the choice of $(s, t)$, $\sigma_{(\beta + sH, tH)}$-semi-stability implies $\beta$-twisted semi-stability. Thus for $E \in M_{(\beta + sH, tH)}(v)$, we have a semi-homogeneous presentation

\[ 0 \to E \to E_0 \to E_1 \to 0, \]

where $\langle v(E_0), v(E_1) \rangle = \langle e^\xi, \ell q_X \rangle$ or $\langle v(E_0), v(E_1) \rangle = \langle \ell e^\xi, q_X \rangle$ with $\langle v^2 \rangle = 2\ell$. Hence the claim holds. \[\Box\]

Proposition 4.21. Let $X$ be an abelian surface with $\text{NS}(X) = ZH$. Let $E$ be a simple sheaf on $X$. If $E$ has two semi-homogeneous presentations, then $E$ is Gieseker semi-stable.
4.4. Relation with the Fourier-Mukai transforms. We shall study the Fourier-Mukai transform on our space of stability conditions. Let \( r_1 e \gamma \) be a primitive and isotropic Mukai vector. We set \( X_1 := M_{(\beta + s H, t H)}(r_1 e \gamma) \). Let \( E \) be the universal object on \( X \times X_1 \) as a complex of twisted sheaves. Assume that \( \gamma = \beta + \lambda H, \lambda \in \mathbb{Q} \). We consider the Fourier-Mukai transform \( \Phi := \Phi^E_{X \to X_1} : D(X) \to D^{\alpha_1}(X_1) \). We set 
\[
(\beta + s H, t H) = (\gamma' + s' \hat{H}, t' \hat{H})
\]
Then
\[
s' = \frac{1}{|r_1|} \frac{2(\lambda - s)}{((\lambda - s)^2 + t'^2)(H^2)},
\]
\[
t' = \frac{1}{|r_1|} \frac{2t}{((\lambda - s)^2 + t'^2)(H^2)}.
\]
Since \((s - \lambda)^2 + t^2) = \left( \frac{2}{|r_1|} \right)^2 \), the image of \((s - \lambda)^2 + t^2 = \frac{2}{|r_1|} \) is \( s'^2 + t'^2 = \frac{2}{|r_1|} \).

If \( \lambda r \neq d_\beta \), then Lemma 4.20 implies the condition \( \mathbb{R} Z_{(\beta + s H, t H)}(v) = \mathbb{R} Z_{(\beta + s H, t H)}(e^\beta + \lambda H) \) defines a circle
\[
C_{v, \lambda} : t^2 = (\lambda - s) \left( \frac{a_\beta - d_\beta \lambda}{2r} \frac{H^2}{H^2} + s \right).
\]
We have
\[
\frac{a_\beta - d_\beta \lambda}{2r} \frac{H^2}{H^2} + \lambda = \frac{(\lambda r - d_\beta)^2}{r(H^2)} - (D_\beta^2) = \frac{2a_\gamma}{d_\gamma(H^2)} - d_\gamma \frac{2(e^\beta + \lambda H, v)}{d_\gamma(H^2)}.
\]
Thus \( C_{v, \lambda} \) is
\[
(s - \left( \lambda + \frac{a_\gamma}{d_\gamma(H^2)} \right))^2 + t^2 = \left( \frac{a_\gamma}{d_\gamma(H^2)} \right)^2.
\]

Lemma 4.22. The image of
\[
t^2 \leq (\lambda - s) \left( \frac{a_\beta - d_\beta \lambda}{2r} \frac{H^2}{H^2} + s \right)
\]
by \( \Phi \) is
\[
\{(s', t') \mid |r_1| a_\gamma \frac{s'}{d_\gamma} \geq 1 \}.
\]

Proof. By \( 4.39 \) and \( 4.37 \), \((s, t)\) satisfies \( 4.10 \) if and only if
\[
0 \geq (s - \lambda)^2 + t^2 - (s - \lambda) \frac{2a_\gamma}{d_\gamma(H^2)}
\]
\[
= ((s - \lambda)^2 + t^2) \left( 1 + s' \left| r_1 \frac{a_\gamma}{d_\gamma} \right| \right).
\]
Hence the claim holds.

By Lemma 4.22, we have the following.

Proposition 4.23. Let \( C^\pm \) be the adjacent chamber of \( C_{v, \lambda} \) such that \( C^- \) is surrounded by \( C^+ \).

1. If \( \frac{a_\gamma}{d_\gamma} < 0 \), then \( \Phi(C^+) \) (resp. \( \Phi(C^-) \)) is the unbounded chamber satisfying \( s' < -\frac{d_\gamma}{|r_1| a_\gamma} \) (resp. \( s' > -\frac{d_\gamma}{|r_1| a_\gamma} \)).

2. If \( \frac{a_\gamma}{d_\gamma} > 0 \), then \( \Phi(C^-) \) (resp. \( \Phi(C^+) \)) is the unbounded chamber satisfying \( s' < -\frac{d_\gamma}{|r_1| a_\gamma} \) (resp. \( s' > -\frac{d_\gamma}{|r_1| a_\gamma} \)).

We set \( w := \Phi^E_{X \to X_1}(v) \). Proposition 4.13 implies \( M^{\alpha_1}_{(\gamma' + s' \hat{H}, t' \hat{H})} \) is isomorphic to the moduli space of semi-stable sheaves. Then Theorem 2.20 implies a generalization of [8, Thm. 3.3.3] for abelian surfaces.

For the preservation of Gieseker’s semi-stability, we also have the following, which is a generalization of [17].

Proposition 4.24. Let \( v \) be a positive Mukai vector and assume that there are walls for \( v \). Let \( W_{\text{max}} \) be the wall in the region \( rs < d_\beta \) such that \( W_{\text{max}} \) surround all walls in \( rs < d_\beta \), that is, the boundaries of the unbounded chamber is \( rs = d_\beta \) and \( W_{\text{max}} \). We set
\[
W_{\text{max}} \cap \{(s, 0) \mid s \in \mathbb{R} \} = \{(\lambda_1, 0), (\lambda_2, 0)\}, \quad \lambda_1 < \lambda_2.
\]
Let $\Phi : D(X) \to D(X_1)$ be the Fourier-Mukai transform as above. If $\lambda \leq \lambda_1$ or $\lambda_2 \leq \lambda < \frac{d_B}{r}$, then $\Phi$ or $\Phi \circ D_X$ preserves the Gieseker’s semi-stability.

In particular, if $\lambda$ is sufficiently small, we can apply this proposition, which is nothing but the main result of [17].

**Remark 4.25.** Assume that $r := \text{rk} v > 0$. We set
$$D := \min\{(\text{rk} v)c_1(v) - (\text{rk} v)c_1(w), H) > 0 \mid w \in H^*(X, Z)_{\text{alg}}\}$$
and assume that
$$D = \min\{(C, H) > 0 \mid C \in NS(X)\}.$$
We take $w_0 \in H^*(X, Z)_{\text{alg}}$ such that
$$(\text{rk} w_0)d_\beta(v) - (\text{rk} w_0)d_\beta(w_0))(H^2) = ((\text{rk} w_0)c_1(v) - (\text{rk} v)c_1(w_0), H) = D.$$Replacing $w_0$ by $w_0 + kv (k \in \mathbb{Z})$, we may assume that $\text{rk} v \geq \text{rk} w_0 > 0$. Then there is no wall for $d_\beta(w_0)/\text{rk} w_0 \leq s < d_\beta(v)/\text{rk} v$.

5. The chamber structure for an abelian surface $X$ with $NS(X) = ZH$.

From now on, we assume that $NS(X) = ZH$. Let $v$ be a primitive Mukai vector with a numerical solution. We shall study the walls and chambers for $v$. By our assumption, there is an isometry of Mukai lattice sending $v$ to $1 - \ell g_X$. So we may assume that $v = 1 - \ell g_X$. Since a generic classification of stable objects (it induces the birational classification) is most fundamental, we are mainly interested in codimension 0 walls.

5.1. Cohomological Fourier-Mukai transforms. Let $H_X$ be the ample generator of $NS(X)$. We shall describe the action of Fourier-Mukai transforms on the cohomology lattices in [14].

Two smooth projective varieties $Y_1$ and $Y_2$ are said to be Fourier-Mukai partners if there is an equivalence $D(Y_1) \simeq D(Y_2)$. We denote by $FM(X)$ the set of Fourier-Mukai partners of $X$. The set of equivalences between $D(X)$ and $D(Y)$ is denoted by $\text{Eq}(D(X), D(Y))$. For $Y, Z \in FM(X)$, we set
$$\text{Eq}_0(D(Y), D(Z)) := \{\Phi_{Y \to Z}^{E[2k]} \in \text{Eq}(D(Y), D(Z)) \mid E \in \text{Coh}(Y \times Z), k \in \mathbb{Z}\},$$

$$\mathcal{E}(Z) := \bigcup_{Y \in FM(Z)} \text{Eq}_0(D(Y), D(Z)), \quad \mathcal{E} := \bigcup_{Z \in FM(X)} \mathcal{E}(Z) = \bigcup_{Y, Z \in FM(X)} \text{Eq}_0(D(Y), D(Z)).$$

Note that $\mathcal{E}$ is a groupoid with respect to the composition of the equivalences. For $Y \in FM(X)$, we have $(H^2_Y) = (H^2_X)$. We set $n := (H^2_X)/2$.

In [14] sect. 6.4, we constructed an isomorphism of lattices
$$\iota_X : (H^*(X, Z)_{\text{alg}}, \langle \cdot, \cdot \rangle) \xrightarrow{\sim} (\text{Sym}_2(Z, n), B), \quad (r, dH_X, a) \mapsto \left(\frac{r}{d\sqrt{n}}, \frac{d\sqrt{n}}{a}\right),$$
where $\text{Sym}_2(Z, n)$ is given by
$$\text{Sym}_2(Z, n) := \left\{\left(\begin{array}{c} x \bigm/ \sqrt{n} \\ y \bigm/ \sqrt{n} \\ z \end{array}\right) \mid x, y, z \in \mathbb{Z}\right\},$$
and the bilinear form $B$ on $\text{Sym}_2(Z, n)$ is given by
$$B(X_1, X_2) := 2ny_1y_2 - (x_1z_2 + z_1x_2)$$
for $X_i = \left(\begin{array}{c} x_i \bigm/ \sqrt{n} \\ y_i \bigm/ \sqrt{n} \\ z_i \end{array}\right) \in \text{Sym}_2(Z, n)$ ($i = 1, 2$).

Each $\Phi_{X \to Y}$ gives an isometry
$$\iota_Y \circ \Phi_{X \to Y}^H \circ \iota_X^{-1} \in \text{O}(\text{Sym}_2(Z, n)),$$
where $\text{O}(\text{Sym}_2(Z, n))$ is the isometry group of the lattice $(\text{Sym}_2(Z, n), B)$. Thus we have a map
$$\eta : \mathcal{E} \to \text{O}(\text{Sym}_2(Z, n))$$
which preserves the structures of multiplications.

**Definition 5.1.** We set
$$\tilde{G} := \left\{\left(\begin{array}{cc} a\sqrt{r} & b\sqrt{s} \\ c\sqrt{s} & d\sqrt{r} \end{array}\right) \mid a, b, c, d, r, s \in \mathbb{Z}, r, s > 0, rs = n, adr - bcs = \pm 1\right\},$$
$$G := \tilde{G} \cap \text{SL}(2, \mathbb{R}).$$
We have a right action \( \cdot \) of \( \widehat{G} \) on the lattice \( \text{Sym}_2(\mathbb{Z}, n), B) \):

\[
(5.2) \quad \left( \frac{r}{d\sqrt{n}}, \frac{d\sqrt{n}}{a} \right) \cdot g := \left( \frac{r}{d\sqrt{n}}, \frac{d\sqrt{n}}{a} \right) g, \quad g \in \widehat{G}.
\]

Thus we have an anti-homomorphism:

\[
\alpha : \widehat{G}/\{\pm 1\} \to \text{O}(\text{Sym}_2(n, \mathbb{Z})).
\]

**Theorem 5.2** ([14] Thm. 6.16, Prop. 6.19). Let \( \Phi \in \text{Eq}(\text{D}(Y), \text{D}(X)) \) be an equivalence.

1. Let \( \nu := v(\Phi(O_Y)) \) and \( \nu_2 := \Phi(\nu) \) are positive isotropic Mukai vectors with \( \nu, \nu_2 = -1 \) and we can write

\[
\nu_1 = (p_1^2r_1, p_1q_1H_Y, q_1^2r_2), \quad \nu_2 = (p_2^2r_2, p_2q_2H_Y, q_2^2r_1),
\]

\[
(5.3) \quad p_1, q_1, p_2, q_2, r_1, r_2 \in \mathbb{Z}, \quad p_1, r_1, r_2 > 0,
\]

\[
\nu_1 \nu_2 = n, \quad p_1q_2r_1 - p_2q_1r_2 = 1.
\]

2. We set

\[
(5.4) \quad \theta(\Phi) := \pm \left( \frac{p_1\sqrt{r_1}}{p_2\sqrt{r_2}}, \frac{q_1\sqrt{r_2}}{q_2\sqrt{r_1}} \right) \in \widehat{G}/\{\pm 1\}.
\]

Then \( \theta(\Phi) \) is uniquely determined by \( \Phi \) and we have a map

\[
(5.5) \quad \theta : \mathcal{E} \to \widehat{G}/\{\pm 1\}.
\]

3. The action of \( \theta(\Phi) \) on \( \text{Sym}_2(n, \mathbb{Z}) \) is the action of \( \Phi \) on \( \text{Sym}_2(n, \mathbb{Z}) \):

\[
(5.6) \quad \iota_X \circ \Phi(v) = \iota_Y(v) \cdot \theta(\Phi).
\]

Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\theta} & \widehat{G}/\{\pm 1\} \\
\downarrow{\eta} & & \downarrow{\alpha} \\
\text{O}(\text{Sym}_2(n, \mathbb{Z})) & &
\end{array}
\]

From now on, we identify the Mukai lattice \( H^*(X, \mathbb{Z})_{\text{alg}} \) with \( \text{Sym}_2(n, \mathbb{Z}) \) via \( \iota_X \). Then for \( g \in \widehat{G} \) and \( v \in H^*(X, \mathbb{Z})_{\text{alg}} \), \( v \cdot g \) means \( \iota_X(v \cdot g) = \iota_X(v) \cdot g \).

We also need to treat the composition of a Fourier-Mukai transform and the dualizing functor \( D_X \). For a Fourier-Mukai transform \( \Phi^*_{X \to Y} \in \text{Eq}(\text{D}(X), \text{D}(Y)) \), we set

\[
(5.7) \quad \theta(\Phi^*_{X \to Y}D_X) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \theta(\Phi^*_{X \to Y}) \in \widehat{G}/\{\pm 1\}.
\]

Then the action of \( \theta(\Phi^*_{X \to Y}D_X) \) on \( \text{Sym}_2(\mathbb{Z}, n) \) is the same as the action of \( \Phi^*_{X \to Y}D_X \).

**Lemma 5.3** ([14] Lemma 6.18)]. If \( \theta(\Phi^*_{X \to Y}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then

\[
\theta(\Phi^*_{X \to Y}) = \pm \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad \theta(\Phi^*_{Y \to X}) = \pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \theta(\Phi^*_{X \to Y}) = \pm \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.
\]

5.2. The arithmetic group \( G \) and numerical solutions for the ideal sheaf. Let \( \ell \in \mathbb{Z}_{>0} \). We assume that \( \sqrt{n} \notin \mathbb{Z} \). Our next task is to describe the numerical solution of the ideal sheaf of \( 0 \)-dimensional subscheme. First we introduce an arithmetic group \( S_{n, \ell} \).

**Definition 5.4.** For \((x, y) \in \mathbb{R}^2\), set

\[
P(x, y) := \begin{pmatrix} y & \ell x \\ x & y \end{pmatrix}.
\]

We also set

\[
S_{n, \ell} := \left\{ \begin{pmatrix} y & \ell x \\ x & y \end{pmatrix} \mid x = a\sqrt{r}, y = b\sqrt{s}, a, b, r, s \in \mathbb{Z}, \quad r, s > 0, \quad rs = n, \quad y^2 - \ell x^2 = \pm 1 \right\}.
\]

**Lemma 5.5.**

1. \( S_{n, \ell} \) is a commutative subgroup of \( \text{GL}(2, \mathbb{R}) \).

2. We have a homomorphism

\[
\phi : \quad S_{n, \ell} \to \mathbb{R}^X, \quad P(x, y) \mapsto y + x\sqrt{\ell}.
\]
(3) For $\ell > 1$, $\phi$ is injective. For $\ell = 1$, we have

$$\text{Ker}\phi = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

(4) We set a subgroup $G_{n,\ell}$ of $\tilde{G}$ (Definition 5.1) to be

$$G_{n,\ell} := \left\{ g \in \tilde{G} \mid \ell g \begin{pmatrix} 1 \\ 0 \end{pmatrix} g = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$ 

Then

$$G_{n,\ell} = S_{n,\ell} \rtimes \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$ 

Proof. The proofs of (1) and (2) are straightforward.

For (3), assume that $x, y \in \mathbb{R}$ with $x^2, y^2, xy/\sqrt{n} \in \mathbb{Q}$ satisfy $y + x\sqrt{\ell} = 1$. Then $(y^2 + \ell x^2) + 2(xy/\sqrt{n})\sqrt{\ell n} = (y + x\sqrt{\ell})^2 = 1$. Our assumptions yields $y^2 + \ell x^2 = 1$ and $xy = 0$. If $x = 0$, then $y = \pm 1$. If $y = 0$, then $\ell = 1$ and $x = 1$. Hence the conclusion holds.

(4) follows from direct computations.

Then the Dirichlet unit theorem yields the following corollary.

**Corollary 5.6.** If $\ell > 1$, then $S_{n,\ell} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. Let $p_1, \ldots, p_m$ be the prime divisors of $\ell n$ and $\mathfrak{o}$ be the ring of algebraic integers in $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_m})$. By Dirichlet unit theorem $\mathfrak{o}^* \geq$ a finitely generated abelian group whose torsion subgroup is $\{\pm 1\}$. Hence $\phi(S_{n,\ell})$ is a finitely generated abelian group whose torsion subgroup is $\{\pm 1\}$. For $A \in S_{n,\ell}$, we have $\phi(A^2) \in \mathbb{Z}[\sqrt{n}]$. Since $\mathbb{Z}[\sqrt{n}]^* \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we get $S_{n,\ell} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

**Remark 5.7.** If $n = 1$ and $\ell > 1$, then $S_{1,\ell}$ is the group of units of $\mathbb{Z}[\sqrt{\ell}]$. Moreover if $\ell$ is square free and $\ell \equiv 2, 3 \pmod{4}$, then since $\mathbb{Z}[\sqrt{n}]$ is the ring of the integers of $\mathbb{Q}[\sqrt{n}]$, a generator of $S_{1,\ell}$ becomes a fundamental unit.

**Lemma 5.8.** For two positive isotropic Mukai vectors $w_0, w_1$ on the fixed abelian surface $X$, the condition

$$(1, 0, -\ell) = \pm (\ell w_0 - w_1), \quad \langle w_0, w_1 \rangle = -1$$

is equivalent to

$$w_0 = (p^2, -\frac{\ell}{\sqrt{n}} H, q^2), \quad w_1 = (q^2, -\frac{\ell}{\sqrt{n}} H, \ell^2 p^2), \quad P(p, q) \in S_{n,\ell}.$$ 

Proof. If there are isotropic Mukai vectors with the first condition, then we can write them as $w_0 = (r, dH, (r \ell \pm 1))$ and $w_1 = (r \ell \mp 1, dH, r \ell^2)$, where $d^2(H^2) = 2r(r \ell \mp 1)$. We set $p := \sqrt{r}$ and $q := \sqrt{\ell \mp 1}$. Then $w_0 = (p^2, -\frac{\ell}{\sqrt{n}} H, q^2)$, $w_1 = (q^2, -\frac{\ell}{\sqrt{n}} H, \ell^2 p^2)$ and $\ell p^2 - q^2 = \pm 1$. Thus $P(p, q) \in S_{n,\ell}$. The converse is obvious.

**Corollary 5.9.** Recall the action $\cdot$ of $\text{GL}(2, \mathbb{R})$ given in (5.2). By the correspondence

$$G_{n,\ell} \ni g \mapsto (w_0, w_1), \quad w_0 := (0, 0, 1) \cdot g, \quad w_1 := (1, 0, 0) \cdot g,$$

we have a bijective correspondence:

$$G_{n,\ell} / \{ \pm 1 \} \cong S_{n,\ell} / \{ \pm 1 \} \leftrightarrow \left\{ (w_0, w_1) \mid \langle w_0, w_1 \rangle = -1, \langle w_0^2 \rangle = \langle w_1^2 \rangle = 0, w_0, w_1 > 0, (1, 0, -\ell) = \pm (\ell w_0 - w_1) \right\}.$$ 

**Definition 5.10.** Assume that $\ell > 1$. Let

$$A_\ell := \begin{pmatrix} q & \ell p \\ p & q \end{pmatrix}, \quad p, q > 0$$

be the generator of $S_{n,\ell} / \{ \pm 1 \}$. We set $\epsilon := q^2 - \ell p^2 \in \{\pm 1\}$. For $m \in \mathbb{Z}$, we set

$$\begin{pmatrix} q & \ell p \\ p & q \end{pmatrix}^m = \begin{pmatrix} b_m & \ell a_m \\ a_m & b_m \end{pmatrix}.$$

By the definition we have

$$(a_0, b_0) = (0, 1), \quad (a_m, b_m) = \epsilon^m(-a_m, b_m), \quad m \in \mathbb{Z}_{>0}$$

and

$$S_{n,\ell} = \left\{ \pm \begin{pmatrix} b_m & \ell a_m \\ a_m & b_m \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$
Next we consider the right action of $\text{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$

\[(x, y) \mapsto (x, y)X, \quad X \in \text{GL}(2, \mathbb{R}).\]

Then the quadratic map

\[\mathbb{R}^2 \to \text{Sym}_2(\mathbb{R})\]

\[(x, y) \mapsto \begin{pmatrix} x & y \\ y & x \end{pmatrix}\]

is $\text{GL}(2, \mathbb{R})$-equivariant. Using this action, we have the next descriptions of the topological invariants of fine moduli spaces $\text{M}_H(v)$ of dimension 2.

\[
\begin{align*}
&\varphi_1: \begin{cases}
  v \in H^*(X, \mathbb{Z})_{\text{alg}}, \langle v^2 \rangle = 0, v > 0, \\
  \langle w, v \rangle = -1, \exists w \in H^*(X, \mathbb{Z})_{\text{alg}}
\end{cases} / \{\pm 1\} \to \{(0, 1)X \mid X \in G\} / \{\pm 1\} \\
&\varphi_2: \begin{cases}
  (a\sqrt{r}, b\sqrt{s}) \in \mathbb{R}^2, \quad a, b \in \mathbb{Z}, r, s \in \mathbb{Z}_{>0}, \\
  rs = n, \gcd(ar, bs) = 1
\end{cases} \to \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}
\end{align*}
\]

where we used the correspondences

\[v = (a^2r, abH, b^2s) \xrightarrow{\varphi_1} \pm(a\sqrt{r}, b\sqrt{s}) \xrightarrow{\varphi_2} \frac{\mu(v)}{2\sqrt{n}} = \frac{b\sqrt{n}}{ar}.
\]

Here we used the slope for the Mukai vector defined by $\mu(v) := (H, c_1(v))/\text{rk} \cdot v$. These correspondences are $\mathcal{G}$-equivariant under the action (5.9). Lemma 5.8 and (5.10) imply the following one to one correspondence:

\[
\begin{align*}
\{v_1, v_2\} &\quad \text{There is a numerical solution} \\
(v_1, v_2, \ell_1, \ell_2) &\quad \text{of } (1, 0, -\ell) \\
\{v_1, v_2\} &\quad \text{of } (1, 0, -\ell) \quad \Longleftrightarrow \quad \left\{ \begin{array}{c}
  b_m, a_m, \frac{\ell a_m}{b_m} \\
  \frac{\mu(v_1)}{2\sqrt{n}}, \frac{\mu(v_2)}{2\sqrt{n}}
\end{array} \right\} \in \mathbb{P}^1(\mathbb{R}) \quad m \in \mathbb{Z}
\end{align*}
\]

where $(\ell_1, \ell_2) = (\ell, 1)$ if and only if $\frac{\mu(v_1)}{2\sqrt{n}}, \frac{\mu(v_2)}{2\sqrt{n}} = \left( \frac{b_m}{a_m}, \frac{\ell a_m}{b_m} \right)$.

**Definition 5.11.** For $m \in \mathbb{Z}$, we set

\[
\begin{align*}
  u_m &:= a^2_m H = (a^2_m, h_m, b_m^2), \\
  u'_m &:= b^2_m H = (b^2_m, b_m, \ell^2 a_m).
\end{align*}
\]

**5.3. Codimension 0 walls and the action of $G_{n, \ell}$.**

**Definition 5.12.** $C_0$ is the wall associated to the numerical solution $(1, g_X, 1, \ell)$. For $m \neq 0$, let $C_m$ be the wall associated to the numerical solution $(u_m, u'_m, \ell, 1)$.

**Proposition 5.13.**

1. $C_0$ is the $t$-axis and $C_m$ ($m \neq 0$) is the circle defined by

\[
(s - \frac{1}{\sqrt{n}} \frac{b_m}{a_m}) (s - \frac{1}{\sqrt{n}} \frac{\ell a_m}{b_m}) + t^2 = 0.
\]

2. $\{C_m \mid m \in \mathbb{Z}\}$ is the set of codimension 0 walls.

**Definition 5.14.**

1. For $C_m$ ($m \in \mathbb{Z}$), we define adjacent chambers $C_m^\pm$ as follows:
   * $C_m^-$ is surrounded by $C_m^+$ for $m < 0$.
   * $C_0^+ \subseteq \{(s, t) \mid s < 0\}$ and $C_0^- \subseteq \{(s, t) \mid s > 0\}$.
   * $C_m^+$ is surrounded by $C_m^-$ for $m > 0$.

2. Let $M_{C_m^\pm}(v)$ be the moduli of stable objects $M_{(sH, tH)}(v)$ for $(s, t) \in C_m^\pm$.

Then we have

\[
M_{C_m^+}(v) = \begin{cases}
  \mathfrak{M}(u_m, u'_m, 1, \ell), & \frac{b_m}{a_m} < \frac{\ell a_m}{b_m} \\
  \mathfrak{M}(u'_m, u_m, 1, \ell), & \frac{b_m}{a_m} > \frac{\ell a_m}{b_m}
\end{cases}
\]

for $m < 0$ and

\[
M_{C_m^-}(v) = \begin{cases}
  \mathfrak{M}(u_m, u'_m, 1, -\ell), & \frac{b_m}{a_m} < \frac{\ell a_m}{b_m} \\
  \mathfrak{M}(u'_m, u_m, 1, -\ell), & \frac{b_m}{a_m} > \frac{\ell a_m}{b_m}
\end{cases}
\]

for $m \geq 0$. In particular, we have

\[
M_{C_0^-}(v) = \mathfrak{M}(1, g_X, 1, -1) = M_H(1, 0, -\ell).
\]
We note that $\Psi$ in Proposition 5.16 satisfies $\Psi^{-1} = \Psi$. By Proposition 5.16 we have the following isomorphisms.

$$\Psi_m : M_{C^\pm}(v) \to M_{C^\pm}(v).$$

Let $\Phi_{X_1 \to X}$ be a Fourier-Mukai transform such that $\Phi_{X_1 \to X}(\rho_{X_1}) = u_m$ and $\Phi_{X_1 \to X}(1) = u'_m$, where $X_1 = M_H(u_m)$. Then we have

$$\theta(\Phi_{X_1 \to X}) = \left( \begin{array}{cc} b_m & \ell a_m \\ c_m a_m & c_m b_m \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)^{(1 + m)} (q \ell_p)^m.$$  

Thus we get

$$\theta(\Psi_m) = \left( \begin{array}{cc} q \ell_p & 0 \\ p \ell_q & q \end{array} \right)^{-m} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)^{(1 + m)} (q \ell_p)^m.$$  

We also have

$$\left( \begin{array}{cc} q \ell_p & 0 \\ p \ell_q & q \end{array} \right)^{m+k} \theta(\Psi_m) = \left( \begin{array}{cc} q \ell_p & 0 \\ p \ell_q & q \end{array} \right)^{m} \left( \begin{array}{cc} q \ell_p & 0 \\ p \ell_q & q \end{array} \right)^{-m} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)^{(1 + m)} (q \ell_p)^m.$$  

Hence $\theta(\Phi_{X_1 \to X} \circ \Phi_{X \to X_1}) = \pm A^{2m}_t$. Therefore $M_H(u_{2m}) \cong X$.

**Proposition 5.17.**

(1) There are finitely many walls between $C_0$ and $C_{-1}$.

(2) By the action of $G_n$, every wall is transformed to a wall between $C_0$ and $C_{-1}$.

**Proof.** (1) We set $\lambda_0 := \frac{q}{\sqrt{p}} \lambda$. Then for $\beta = \lambda_0 H$, there is finitely many walls. Since every wall between $C_0$ and $C_{-1}$ are shifted with the line $s = \lambda_0$, the claim holds.

(2) Let $C$ be a wall between $C_m$ and $C_{m-1}$. Since $\Psi_0(C)$ is a wall between $C_{-m}$ and $C_{-m+1}$, we may assume that $m < 0$. By Proposition 5.15 we see that $\Psi_m(C)$ is a wall between $C_{m+1}$ and $C_{m}$. Therefore $\Psi_1 \circ \cdots \circ \Psi_{m-1} \circ \Psi_m(C)$ is a wall between $C_0$ and $C_{-1}$.

**Remark 5.18.**

$$\theta(\Psi_1 \circ \cdots \circ \Psi_{m-1} \circ \Psi_m) = \begin{cases} \theta(\Psi_{\ell}) A_{\ell}^{-m-1}, & 2 \not\mid m, \\ A_{\ell}^m, & 2 \mid m. \end{cases}$$

In Proposition 5.15 we did not specify the correspondence of complexes. Since $\phi_{(sH,tH)}(v) \mod 2\mathbb{Z}$ is well-defined, the correspondence is determined up to shift $[2k]$ ($k \in \mathbb{Z}$). We next fix the ambiguity of this shift. We note that $\theta(\Psi_m) = \theta([1] \circ D_X \circ \Phi_{X \to X}^{E_m})$, where $[1]$ is the shift functor. Since $Z_{(sH,tH)}(v) \in \mathbb{C} \setminus \mathbb{R}_{<0}$, we take $\phi_{(sH,tH)}(v) \in (-1,1)$ to consider the moduli space $M_{(sH,tH)}(v)$ as in Definition 1.1. Then the isomorphisms in Proposition 5.15 are given by the following proposition.

**Proposition 5.19.** Assume that $m < 0$. $[1] \circ D_X \circ \Phi_{X \to X}^{E_m}$ induces isomorphisms

$$M_{C^\pm_{m+k}}(v) \to M_{C^\pm_{m-k}}(v),$$

$$E \mapsto (\Phi_{X \to X}^{E_m}(E))^\vee [1].$$

**Proof.** Since $\ell_2 t_2 - \ell_2 t_2 = 1$, we have $\frac{b_{-m}}{\nu_{-m}} < \frac{a_{-m}}{\nu_{-m}} < 0$ for $m < 0$. For $(s,t) \in C_{2m}$, $\phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}) \in (-1,0]$. Hence $\phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}[1]) = \phi_{(sH,tH)}(E)$ for $(s,t) \in C_{2m}$ and $E \in M_{(sH,tH)}(v)$. For $E \in M_{(sH,tH)}(v)$, we also have

$$\phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}[1]) \phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}), \quad (s,t) \text{ is outside of } C_{2m},$$

$$\phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}[2]) \phi_{(sH,tH)}((E_{2m})_{X \times \{x\}}), \quad (s,t) \text{ is inside of } C_{2m}.$$
If \((s, t)\) is inside of \(C_{2m}\), then \(\phi_{(sH, tH)}(\Phi^{E_m}_{X \to X}(E)) \in (1, 2)\). Hence
\[
\phi_{(sH, tH)}((\Phi^{E_m}_{X \to X}(E))^\vee[1]) \in (-1, 0).
\]
Therefore the claim holds.

\[\square\]

**Remark 5.20.** Assume that \(m > 0\). Then \([-1] \circ D_X \circ \Phi^{E_m}_{X \to X}\) induces isomorphisms
\[
M_{C_{m+1}}(v) \to M_{C_{m-1}}(v)
\]
\[
E \to (\Phi^{E_m}_{X \to X}(E))^\vee[-1].
\]

**Proposition 5.21.** We set
\[
M_m := M_{C_{m+1}}(v) \cap M_{C_{m}}(v).
\]
(1) \(M_m \neq \emptyset\) and \(M_m\) is birationally equivalent to \(M_{C_{m-1}}(v)\) and \(M_{C_{m}}(v)\).

(2) We have a sequence of isomorphisms
\[
\cdots \to M_{-2} \to M_{-1} \xrightarrow{\Psi_m} M_0 \xrightarrow{\Psi_m} M_{1} \xrightarrow{\Psi_m} \cdots.
\]

**Proof.** (1) Since there is no codimension 0 wall between \(C_{m-1}\) and \(C_m\), \(M_m \neq \emptyset\). Since \(M_{C_{m-1}}(v)\) and \(M_{C_{m}}(v)\) are irreducible, \(M_m\) is birationally equivalent to \(M_{C_{m-1}}(v)\) and \(M_{C_{m}}(v)\).

(2) By Proposition 5.19 or Proposition 5.20 (and Remark 5.20), we have isomorphisms
\[
\Psi_m : M_{C_{m-1}}(v) \to M_{C_{m+1}}(v),
\]
\[
\Psi_m : M_{C_{m}}(v) \to M_{C_{m+1}}(v).
\]
Hence we have an isomorphism
\[
\Psi_m : M_{C_{m-1}}(v) \cap M_{C_{m}}(v) \to M_{C_{m+1}}(v) \cap M_{C_{m+1}}(v).
\]
Thus the claim holds.

\[\square\]

We note that \(M_0\) is an open subset of \(M_{C_0}(v) = \{I_Z \otimes L \mid I_Z \in \text{Hilb}^r(X), L \in \text{Pic}^0(X)\}\). Hence we have two semi-homogeneous presentations of \(I_Z \otimes L \in M_0:\)
\[
0 \to I_Z \otimes L \to L \to O_Z \to 0
\]
and
\[
0 \to E_{-1} \to E_0 \to I_Z \otimes L \to 0.
\]

Starting from these two semi-homogeneous presentations, we have a sequence of complexes \(F_m^* \in M_m\) such that \(F_0^* = I_Z \otimes L\) and \(\Psi_m(F_m^*) = F_{m+1}^*\). Then we have exact triangles
\[
V_{m-1}^+ \to F_m^* \to V_{m-1}^- \to V_{m-1}^+[1]
\]
\[
W_m^- \to F_m^* \to W_m^+ \to W_m^[1]
\]
such that
\[
\text{for } (s, t) \in C_{m-1}, \ V_{m-1}^{\pm} \text{ are } \sigma(sH+tH)\text{-semi-stable objects with the same phase and define the wall } C_{m-1},
\]
\[
\text{for } (s, t) \in C_m, \ V_{m-1}^{\pm} \text{ are } \sigma(sH+tH)\text{-semi-stable objects with the same phase and define wall } C_m.
\]

**5.4. Fourier-Mukai transforms of the families \(F_m^*\).** We first assume \(c = q^2 - \ell p^2 = -1\). In this case, the algebraic integers \(a_m, b_m\) in Definition 5.10 satisfy the following relations:
\[
\frac{b_{2k-1}}{a_{2k-1}} < \frac{\ell a_{2k}}{b_{2k}} < \frac{b_{2k+1}}{a_{2k+1}} < \sqrt{7} < \frac{\ell a_{2k+1}}{b_{2k+1}} < \frac{b_{2k}}{a_{2k}} < \frac{\ell a_{2k+1}}{b_{2k-1}} \quad (k \in \mathbb{Z}_{>0}),
\]
\[
\lim_{k \to \infty} b_k = \lim_{k \to \infty} \frac{\ell a_k}{b_k} = \sqrt{7}.
\]
Thus \(\pm \sqrt{7} m\) are the accumulation points of \(\cup_m C_m\).

We regard \((a_m : b_m)\) and \((b_m : \ell a_m)\) as elements of \(F^1(\mathbb{R})\). Then the inhomogeneous coordinates of these points give a sequence
\[
(5.20) \quad -\infty = a_0 < \frac{\ell p}{q} < \frac{\ell a_1}{b_1} < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_2}{a_2} < \frac{\ell a_2}{b_2} < \cdots < \sqrt{7} < \frac{\ell a_2}{b_2} < \frac{b_2}{a_2} < \frac{\ell a_1}{b_1} < \frac{b_1}{a_1} < \frac{\ell p}{q} < a_0 = \infty,
\]
where we write the inhomogeneous coordinate of \((0 : 1)\) as \(\infty\) or \(-\infty\).
For a Fourier-Mukai transform $\Phi_{X \to X'}: D(X) \to D(X')$, we write $c_1(G_{x'})/\text{rk}G_{x'} = (\lambda/\sqrt{n})H$. If $-\ell p/q < \lambda < -q/p$, then $\Phi_{X \to X'}^*(F^*)$ is not a sheaf for all $F^*$.

**Definition 5.22.** We set

$$I_1 := \left[0, \frac{b_0}{a_0}\right) \cup \left[\frac{b_1}{a_1}, \infty\right), \quad I_0 := \left[-\infty, -\frac{b_1}{a_1}\right) \cup \left[-\frac{b_0}{a_0}, 0\right),$$

$$I_{2k} := \left[\frac{b_{2k-1}}{a_{2k-1}}, \frac{b_{2k}}{a_{2k}}\right) \cup \left[\frac{b_{2k}}{a_{2k}}, \frac{b_{2k+1}}{a_{2k+1}}\right), \quad I_{-2k} := \left[\frac{b_{2k-1}}{a_{2k-1}}, \frac{b_{2k}}{a_{2k}}\right) \cup \left[\frac{b_{2k}}{a_{2k}}, \frac{b_{2k+1}}{a_{2k+1}}\right),$$

$$I_{2k+1} := \left[\frac{b_{2k}}{a_{2k+1}}, \frac{b_{2k+1}}{a_{2k+1}}\right) \cup \left[\frac{b_{2k+1}}{a_{2k+1}}, \frac{b_{2k+2}}{a_{2k+2}}\right).$$

For $I = \bigcup [s_i, t_i)$, we denote $I^* := \bigcup \{s_i, t_i\}$.

By (5.24), we have decompositions $\mathbb{P}^1(\mathbb{R}) \setminus \{\pm \sqrt{t}\} = \bigsqcup_{m \in \mathbb{Z}} \bar{I}_m = \bigsqcup_{m \in \mathbb{Z}} I^*_m$.

**Theorem 5.23.** (1) If $\lambda \in I_m (m \leq 0)$, then $\Phi_{X \to X'}^*(F^*_m)$ is a stable sheaf up to shift.

(2) If $\lambda \in I^*_m (m \leq 0)$, then $D_X, \Phi_{X \to X'}^*(F^*_m) = \Phi_{X \to X'}^*(G^*[2])$ is a stable sheaf up to shift.

**Proof.** (1) For a small number $t > 0$, $(\lambda, t)$ belongs to the interior of the annulus bounded by $C_{-m-1}$ and $C_{-m}$. By Lemma 4.12, $F^*_m$ is $\sigma(\lambda, t, H)$-semi-stable. By Proposition 4.23, $\Phi_{X \to X'}^*(F^*_m)$ is a stable sheaf, where $n = 1$ for $\lambda > -\sqrt{t}$ and $n = 2$ for $\lambda < -\sqrt{t}$. The proof of (2) is similar.

By this theorem, we have semi-homogeneous presentations for a general member of $M_H(w), w = \Phi_{X \to X'}^*(v)$.

**Remark 5.24.** The claim also follows from [13, Lem. 4.4].

We next assume that $\epsilon = q^2 - \ell p^2 = 1$. Then the algebraic integers $a_m, b_m$ in Definition 5.10 satisfy:

$$0 < \frac{b_m}{a_m} - \sqrt{t} < \frac{b_{m-1}}{a_{m-1}} - \sqrt{t}$$

for $m \in \mathbb{Z}_{>0}$. We also have the following sequence of inequalities:

$$-\infty = \frac{b_0}{a_0} < -\frac{q}{p} = \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < -\sqrt{t} < \cdots < \frac{\ell a_{-2}}{b_{-2}} < \frac{\ell a_{-1}}{b_{-1}} = \frac{\ell p}{q} = \frac{\ell a_0}{b_1} < \frac{\ell a_1}{b_1} = \frac{\ell p}{q} < \frac{\ell a_2}{b_2} < \cdots < \frac{\ell a_{-2}}{b_{-2}} < \frac{b_2}{a_2} < \frac{b_1}{a_1} < \frac{q}{p} < \frac{b_0}{a_0} = \infty.$$

**Definition 5.25.** We set

$$I_1 := \left[0, \frac{b_0}{a_0}\right) \cup \left[\frac{b_1}{a_1}, \infty\right), \quad I_0 := \left[-\infty, -\frac{b_1}{a_1}\right) \cup \left[-\frac{b_0}{a_0}, 0\right),$$

$$I_{m+1} := \left[\frac{b_m}{a_m}, \frac{b_{m+1}}{a_{m+1}}\right) \cup \left[\frac{b_{m+1}}{a_{m+1}}, b_m\right), \quad I_{-m} := \left[-\frac{b_m}{a_m}, \frac{b_{m+1}}{a_{m+1}}\right) \cup \left[-\frac{b_{m+1}}{a_{m+1}}, -\frac{b_m}{a_m}\right) \quad m \geq 1.$$

Then we have $\mathbb{P}^1(\mathbb{R}) \setminus \{\pm \sqrt{t}\} = \bigsqcup_{m \in \mathbb{Z}} \bar{I}_m = \bigsqcup_{m \in \mathbb{Z}} I^*_m$.

**Theorem 5.26.** For a Fourier-Mukai transform $\Phi_{X \to X'}: D(X) \to D(X')$, we write $c_1(G_{x'})/\text{rk}G_{x'} = (\lambda/\sqrt{n})H$.

(1) If $\lambda \in I_m (m \leq 0)$, then $\Phi_{X \to X'}^*(F^*_m)$ is a stable sheaf up to shift.

(2) If $\lambda \in I^*_m (m \leq 0)$, then $D_X, \Phi_{X \to X'}^*(F^*_m) = \Phi_{X \to X'}^*(G^*[2])$ is a stable sheaf.

The proof is based on the calculation in this subsection and is similar to that of Theorem 5.23. We omit the detail.

6. EXAMPLES

Let $X$ be a principally polarized abelian surface with $\text{NS}(X) = ZH$. We shall study walls for $v = 1 - \ell q_X$.

We note that $n := \frac{\ell q_X}{\sqrt{2}} = 1$. By using Corollary 4.5, it is easy to see that $s = 0$ is the unique wall for $\ell = 1$.

So we assume that $\ell \geq 2$. We use the notations in section 5.

(1) Assume that $\ell = 2$. In this case, $S_1, 2/\{\pm 1\}$ is generated by

$$A_2 := \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Hence we have a numerical solution $v = 2(1, -H, 1) - (1, -2H, 4)$. We have $u_{-1} = (1, -H, 1)$ and $C_{-1} = W_{u_{-1}}$ is the circle in the $(s, t)$-plane

$$\left(s + \frac{3}{2}\right)^2 + t^2 = \frac{1}{22}.$$

For $s = -1$, we get $c_1(v e^{-sH}) = H$. Thus there is no wall intersecting with $s = -1$. Then we see that there is no wall between $C_{-1}$ and $C_0 : s = 0$ (see Figure 1). Hence we get the following result.

**Lemma 6.1.** $C_n (n \in \mathbb{Z})$ are all the walls for $v = 1 - 2q_X$.

**Proposition 6.2.** Let $v$ be a positive and primitive Mukai vector with $\langle v^2 \rangle = 4$. Then $M_H(v)$ is isomorphic to $\text{Hilb}^2(X) \times X$.

**Proof.** There is a Fourier-Mukai transform $\Phi^{E_{X \to X}}_W : D(X) \to D(X)$ such that $\Phi^{E_{X \to X}}_W((1, 0, -2)) = v$. Then we have an isomorphism $M_{(\beta, \omega)}(1, 0, -2) \to M_H(v)$, where $\beta = c_1(E_{(x) \times X})/\text{rk} E_{(x) \times X}$ and $\langle \omega^2 \rangle \ll 1$. By Theorem 2.5 and Proposition 5.17, $M_{(\beta, \omega)}(1, 0, -2) \cong \text{Hilb}^2(X) \times X$, which implies the claim. \hfill $\square$

![Figure 1. Walls for $v = 1 - 2g_X$.](image)

(2) Assume that $\ell = 3$. In this case, $S_{1,3}/\{\pm 1\}$ is generated by

$$A_3 := \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$  

Hence we have a numerical solution $v = (4, -6H, 9) - 3(1, -2H, 4)$. We have $u_{-1} = (1, -2H, 4)$ and $W_{u_{-1}}$ is the circle in $(s, t)$-plane defined by

$$(s + \frac{7}{4})^2 + t^2 = \frac{1}{4}.$$  

We set $w_{-1} := (1, -H, 1)$. Then $v = w_{-1} + (0, H, -4)$ and $w_{-2}$ defines a wall $W_{w_{-1}}$ while the defining equation is

$(s + 2)^2 + t^2 = 1$.

**Lemma 6.3.** $W_{w_{-1}}$ is the unique wall between $C_0$ and $C_{-1} = W_{u_{-1}}$ (see Figure 2).

**Proof.** Since $W_{u_{-1}}$ passes the point $(s, t) = (-2, 0)$, it is sufficient to classify walls for $s = -2$. Assume that $we^{2H} = (r, dH, a)$ defines a wall for $s = -2$. Since $we^{2H} = (1, 2H, 1)$, we have $d = 1$. We set $w' := (v - w)$ and write $w'e^{2H} = (r', H, a')$. By the definition of walls, we have $\langle w^2 \rangle \geq 0, \langle w'^2 \rangle \geq 0$ and $\langle w, w' \rangle > 0$. Thus $r + r' = 1$, we may assume that $r > 0$ and $r' \leq 0$. Since $a + a' = 1$, we have $0 < 1 - ra' - r'a = (2a-1)r + (1-a)$. If $a \leq 0$, then $(2a-1)r + (1-a) \leq (2a-1) + (1-a) = a$. Hence $a \geq 1$. We set $u_1 = (1, -H, 1)$. We have $(s + 2)^2 + t^2 = 1$. Therefore $we^{2H} = (1, H, 1)$, which implies that $w = (1, -H, 1)$. \hfill $\square$

We have $u_0 = (0, 0, 1), u_{-1} = (1, -2H, 4), u_{-2} = (4^2, -28H, 7^2)$ and so on. We define $w_n \in H^*(X, \mathbb{Z})_{\text{alg}}$ by

$$w_n := (a_n, b_n) = (1, -1)A_{3}^{n-1}.$$  

Thus $w_{-1} = (1, -H, 1), w_{-2} = (9, -15H, 5^2)$ and so on. By Proposition 1.8 and Proposition 5.17, we get the following.

**Lemma 6.4.** $W_{u_n}$ and $W_{w_n}$ are all the walls for $v$.

Therefore all moduli spaces are isomorphic to $\text{Hilb}^3(X) \times X$.

**Proposition 6.5.** Let $v$ be a positive and primitive Mukai vector with $\langle v^2 \rangle = 6$. Then $M_H(v)$ is isomorphic to $\text{Hilb}^3(X) \times X$.

**Proof.** There is a Fourier-Mukai transform $\Phi^{E_{X \to X}}_W : D(X) \to D(X)$ such that $\Phi^{E_{X \to X}}_W((1, 0, -3)) = v$. Then we have an isomorphism $M_{(\beta, \omega)}(1, 0, -3) \to M_H(v)$, where $\beta = c_1(E_{(x) \times X})/\text{rk} E_{(x) \times X}$ and $\langle \omega^2 \rangle \ll 1$. By Theorem 2.5 and Proposition 5.17, $M_{(\beta, \omega)}(1, 0, -3) \cong \text{Hilb}^3(X) \times X$, which implies the claim. \hfill $\square$
(3) Assume that ℓ = 4. By Corollary \[4.16\] all walls in \( s < 0 \) intersect with the line \( s = -\sqrt{4} = -2 \). Assume that \( v_1 \) defines a wall for \( v \). Since \( ve^{2H} = (1, 2H, 0) \), \( u_1e^{2H} = (r, H, a) \). Hence \( v_1 = (r, (1 - 2r)H, a - 4 + 4r) \) and \( v - v_1 = (1 - r, (2r - 1)H, -a - 4r) \). Replacing \( v_1 \) by \( v - v_1 \) if necessary, we may assume that \( r > 0 \). We have \( 2n := \langle v_1^2 \rangle \geq 0 \), \( (\langle v - v_1 \rangle^2) \geq 0 \) and \( \langle v_1, v - v_1 \rangle > 0 \). Hence \( n = 1 - ra, 4 - 2n > a \geq -n \). Then \( n = 0, 1, 2, 3 \), which implies that \( ra = 1, 0, -1, -2 \). We also have \( |\frac{a - 4 + 4r}{2(1 - 2r)}| > 2 \), which implies that \( a(8(2r - 1) + a) > 0 \). In particular, \( a \neq 0 \). Then \( ra = 1, 0, -1, -2 \) and \( r > 0 \) implies that \( r = 1, 2 \). If \( r = 1 \), then \( a(a + 8) > 0 \) and \( ra = 1, 0, -1, -2 \) imply that \( a = 1 \). Thus \( v_1 = (1, -H, 1) \). If \( r = 2 \), then \( a = -1 \) and \( a(a + 24) > 0 \), which is impossible. Therefore \( v_1 = (1, -H, 1) \). Thus \( W_{v_1} \) is the unique wall which is defined by

\[
\left(s + \frac{5}{2}\right)^2 + t^2 = \frac{3^2}{2^2}.
\]

**Proposition 6.6.** Let \( v \) be a positive and primitive Mukai vector with \( \langle v^2 \rangle = 8 \). Then \( M_H(v) \) is isomorphic to \( \text{Hilb}^4(X) \times X \) or \( M_H(0, 2H, -1) \).

**Proof.** We first prove that \( M_H(1, 0, -4) \neq M_H(0, 2H, -1) \). We note that the Hilbert-Chow morphism of \( \text{Hilb}^4(X) \) induces a divisorial contraction of \( M_H(1, 0, -4) \). We note that \( M_H(3, H, -1) \) has a morphism to the Uhlenbeck compactification of the moduli of stable vector bundles, which contracts a \( P^2 \)-bundle over \( M_H(3, H, 0) \). Let \( P \) be the Poincaré line bundle on \( X \times X \), where we identify \( \text{Pic}^0(X) \) with \( X \). Then we have an isomorphism \( M_H(1, 0, -4) \cong M_H(1, H, -3) \cong M_H(3, H, -1) \) by sending \( E \) to \( \Phi_{X \rightarrow X}^H(E(H)) \) \[16\] Prop. 3.5]. Hence \( M_H(1, 0, -4) \) has another contraction. There is no other contraction by a similar argument in \[16\] Example 7.2]. On the other hand, \( M_H(0, 2H, -1) \) has a Lagrangian fibration. Therefore \( M_H(1, 0, -4) \neq M_H(0, 2H, -1) \).

For \( s = -2 \) and \( v = (1, 0, -4) \), we have two moduli spaces \( M(\text{Hilb}^4(H); 1, 0, -4) \) and \( M(-2H, t_2H); 1, 0, -4) \), where \( t_1 > \sqrt{2} \) and \( t_2 < \sqrt{2} \). For \( E \in M(-2H, t_2H); 1, 0, -4) \), \( \Phi_{X \rightarrow X}^H(E(2H)) \in M_H(0, 2H, -1) \) and we have an isomorphism \( M(-2H, t_2H); 1, 0, -4) \cong M_H(0, 2H, -1) \).

By \[13\] sect. 7.3], every quadratic form \( rx^2 + 2dxy + ay^2 \) is equivalent to \( x^2 - 4y^2 \). Since \( (1, 0, -4)^t = (1, 0, -4) \), there is an auto-equivalence \( \Phi \) such that \( \Phi(v) = (1, 0, -4) \). Then \( \Phi(M_H(v)) \cong M(s, H,tH); 1, 0, -4) \) for a suitable \( (s, t) \). Therefore the claim holds.

(4) Assume that \( \ell = 5 \). In this case, \( S_{1,5}/\{\pm 1\} \) is generated by

\[
A_5 := \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}.
\]

Hence we have a numerical solution \( v = 5(1, -2H, 4) - (4, -10H, 25) \). Then we have \( u_{-1} = (1, -2H, 4) \) and \( W_{u_{-1}} \) is the circle defined by

\[
\left(s + \frac{9}{4}\right)^2 + t^2 = \frac{1}{4^2}.
\]

(5) Assume that \( \ell = 6 \). In this case, \( S_{1,6}/\{\pm 1\} \) is generated by

\[
A_6 := \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}.
\]
Hence we have a numerical solution $v = (5^2, -60H, 12^2) - 6(4, -10H, 5^2)$. Then we have $u_{-1} = (5^2, -10H, 5^2)$ and $W_{u_{-1}}$ is the circle defined by

$$
\left( s + \frac{49}{20} \right)^2 + t^2 = \frac{1}{20^2}.
$$

**Remark 6.7.** Let $X$ be an arbitrary abelian surface and $H$ an ample divisor on $X$. Let $v$ be a primitive Mukai vector with $(v^2)/2 = 1$. Then we have an isomorphism $X \times \text{Pic}^0(X) \to M_H(v)$ by sending $(x, L) \in X \times \text{Pic}^0(X)$ to $T_x(E_0) \otimes L$, where $E_0$ is an element of $M_H(v)$ and $T_x$ is the translation by $x$ ([15 Cor. 4.3]).

### 7. Appendix

**7.1. The action of Fourier-Mukai transforms on $\mathbb{H}$.** Assume that $\text{NS}(X) = \mathbb{Z}H$. Then $\beta + \sqrt{-1}\omega = \frac{\sqrt{n}}{\sqrt{2}}H$ with $z \in \mathbb{H}$, where $z := x + \sqrt{-1}y$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$. Thus we have an identification of $\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}$ with $\mathbb{H}$. We set $Z_x := Z_{(\beta, \omega)}$.

We study the action of $\Phi$ with $\gamma \in \mathbb{Q}H$. We write

$$
\gamma' + \zeta + \sqrt{-1}\eta = \frac{z'}{\sqrt{n}}H, \ z' \in \mathbb{H}.
$$

Let $\Phi_{X \to X_1}^E : D(X) \to D(X_1)$ be a Fourier-Mukai transform such that $E$ is a coherent sheaf. Then there is

$$
A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G
$$

such that $\mu(\Phi_{X \to X_1}^E(t_x)) = \frac{a}{c} \sqrt{n}$ and $\mu(\Phi_{X \to X_1}^E(\mathcal{O}_X)) = \frac{a}{c} \sqrt{n}$. $X_1 = M_H(c^2H \zeta + \frac{d}{c})$ and $E$ is unique up to the action of $X \times \text{Pic}^0(X)$. Then

$$
\theta(\Phi_{X \to X_1}^E) = \left( \begin{array}{cc} d & b \\ c & a \end{array} \right) \in G.
$$

**Definition 7.1.** For $\Phi_{X \to X_1}^E$, we set

$$
\varphi(\Phi_{X \to X_1}^E) := \pm \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G/\{ \pm 1 \}.
$$

For $\Phi = \Phi_{X \to X_1}^E$, we have

$$
r_1e^\gamma = c^2 - \frac{cd}{\sqrt{n}}H + d^2q_X,
$$

$$
r_1e^{\gamma'} = c^2 + \frac{ac}{\sqrt{n}}H + a^2q_{X_1}.
$$

Hence

$$
\frac{z}{\sqrt{n}} + \frac{d}{c\sqrt{n}} = -\lambda + \sqrt{-1}t, \ \frac{z'}{\sqrt{n}} - \frac{a}{c\sqrt{n}} = \frac{\lambda + \sqrt{-1}t}{(\lambda^2 + t^2)nc^2}.
$$

Thus we get

$$
\frac{z'}{\sqrt{n}} = \frac{a}{c\sqrt{n}} - \frac{1}{c\sqrt{n}} - \frac{c^2n(z(1 - ad) - cz + b)}{c(cz + d)\sqrt{n}} = \frac{az + b}{\sqrt{n}(cz + d)}.
$$

Therefore the action of $A$ on $\mathbb{H}$ is the natural action of $\text{SL}(2, \mathbb{R})$.

$$
\zeta = -c^2(\lambda - t\sqrt{-1})^2H^2/2 = -c^2(\frac{z}{\sqrt{n}} + \frac{d}{c\sqrt{n}})^2H^2/2 = -(cz + d)^2.
$$

**Proposition 7.2.** For $\Phi_{X \to X_1}^E$ with $\varphi(\Phi_{X \to X_1}^E) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G$,

$$
-(cz + d)^2Z_{\frac{a\lambda + b}{c\sqrt{n}}} = Z_{\frac{z}{\sqrt{n}}} - (\Phi_{X \to X_1}^E)^{-1}.
$$

We also have

$$
\Phi_{X \to X_1}^E(\mathbb{Q}e^{\lambda\sqrt{n}}) = \mathbb{Q}e^{\frac{a\lambda + b}{c\sqrt{n}}}.\sqrt{n}.
$$
We now extend the action of $G$ to $\hat{G}$. We set
\[ \Delta := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

We note that
\[ \hat{G} = G \rtimes (\Delta) \]

with
\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \]

We define the action of $\Delta$ on $\mathbb{H}$ as $\Delta(z) := -\overline{z}$. Then we have
\[ \Delta(A(\Delta(z))) = \frac{az - b}{-cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

Thus we have an action of $\hat{G}$ on $\mathbb{H}$.

**Proposition 7.3.** We can extend the action of $G$ to the action of $\hat{G}$ by
\[ (g, \Delta^n) \cdot z := \begin{cases} g \cdot z, & 2|n, \\ -\overline{g} \cdot z, & 2 \nmid n, \end{cases} \]

where $g \in G$.

**Remark 7.4.** In [14], we showed that the cohomological action of $\text{Eq}_{0}(\mathcal{D}(X), \mathcal{D}(X))$ defines a normal subgroup of $\hat{G}$ which is a conjugate of $\Gamma_0(n)$ in $\text{GL}(2, \mathbb{R})$. More precisely, we set $G_0 := \theta(\text{Eq}_{0}(\mathcal{D}(X), \mathcal{D}(X)))$. Then
\[ \left( \sqrt{n} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)^{-1} G_0 \left( \sqrt{n} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right) = \Gamma_0(n). \]

We set $\beta + \sqrt{-1}\omega = wH$. Then $\text{Eq}_{0}(\mathcal{D}(X), \mathcal{D}(X))$ acts on $w$-plane as the action of $\Gamma_0(n)$ on $w$-plane.

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Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan
E-mail address: yanagida@math.kobe-u.ac.jp, yoshioka@math.kobe-u.ac.jp