RECONSTRUCTION OF QUASI-COMPACT QUASI-SEPARATED SCHEMES FROM CATEGORIES OF PERFECT COMPLEXES

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Abstract. We show how to recover the underlying topological space of a quasi-compact quasi-separated scheme from the tensor triangulated structure on its category of perfect complexes.

Introduction

It has been known for some time that the topology of a noetherian scheme is encoded in the tensor triangulated category of perfect complexes on the scheme. In [1], P. Balmer defines the spectrum of a tensor triangulated category, and shows that this construction recovers the underlying topological space of a noetherian scheme from its category of perfect complexes.

In this note, we show that the hypothesis of noetherianness can be relaxed at the cost of considering a slightly more complicated reconstruction procedure. Our main result is Theorem 3.5 which states that the underlying topological space of a quasi-compact and quasi-separated scheme $(X, \mathcal{O}_X)$ can be reconstructed up to homeomorphism as a certain space of subsets of thick subcategories of $D_{\text{perf}}(X)$, the tensor triangulated category of perfect complexes on $(X, \mathcal{O}_X)$.

The exposition is organized as follows. In §1, we establish the properties of perfect complexes that are needed in proving the reconstruction result. Inspired by M. Bökstedt (cf. [3]), we construct a map $\text{Loc}$, from tensor triangulated categories to lattices in §2. When applied to the category $D_{\text{perf}}(X)$, it turns out that this map recovers the lattice $\mathcal{U}(X)$ of open subsets of $X$ (cf. Theorem 2.1). In §3, we define a map $\text{pt}$, from lattices to topological spaces, and show that it gives a homeomorphism $\text{pt}(\mathcal{U}(X)) \simeq X$, for any sober topological space $X$ (cf. Proposition 3.4 (ii)). The map which recovers the quasi-compact and quasi-separated scheme $(X, \mathcal{O}_X)$ from $D_{\text{perf}}(X)$ is the composition $\text{pt} \circ \text{Loc}$ (cf. Theorem 3.5).

1. Preliminaries on perfect complexes

Let $(X, \mathcal{O}_X)$ be a quasi-compact and quasi-separated scheme. Recall that a complex $C$ of sheaves of $\mathcal{O}_X$-modules is perfect if $C$ is locally quasi-isomorphic to some bounded complex of locally free $\mathcal{O}_X$-modules of finite type. The derived category of the category of $\mathcal{O}_X$-modules carries a natural tensor triangulated structure, which descends to the full subcategory $D_{\text{perf}}(X)$ on the perfect complexes.

We define the support of a complex $C \in D_{\text{perf}}(X)$ to be the closed subset

$$\text{supph}C = \{ x \in X; C_x \neq 0 \},$$

of those points $x \in X$ at which the stalk complex $C_x$ of $\mathcal{O}_{X,x}$-modules is not acyclic.
Lemma 1.1. For any commutative unital ring \( R \), and any \( C \in D_{perf}(\text{Spec} R) \), the set 
\[ \text{Spec} R \setminus \text{supph} C \]
is quasi-compact.

Proof. Let \( S \subset R \) be a noetherian subring and \( C' \in D_{perf}(\text{Spec} S) \) a complex such that \( C = C' \otimes S R \). Let \( \varphi : \text{Spec} R \to \text{Spec} S \) be the natural map. Then \( \text{supph} C = \varphi^{-1}(\text{supph} C') \) corresponds to a finitely generated ideal \((f_1, ..., f_n) \subset R\). Thus
\[ \text{Spec} R \setminus \text{supph} C = \bigcup_{i=1}^n \text{Spec} R[f_i^{-1}] \]
is a finite union of quasi-compacts. \( \square \)

Lemma 1.2. Let \( a : A \to \mathcal{O}_X \) be a morphism in \( D_{perf}(X) \). If \( A \) is a thick triangulated subcategory of \( D_{perf}(X) \), and \( B \in D_{perf}(X) \) is such that \( B \otimes \text{cone}(a) \in A \), then
\[ B \otimes \text{cone}(\otimes^n a) \in A, \]
for all \( n \geq 1 \).

Proof. One has exact triangles
\[ A \xrightarrow{a} \mathcal{O}_X \]
\[ \otimes^n A \xrightarrow{\otimes^n a} \mathcal{O}_X \]
\[ A \otimes (\otimes^n A) \xrightarrow{1_A \otimes (\otimes^n a)} A \]
\[ A \otimes \text{cone}(\otimes^n a) \]
Since \( \otimes^{n+1} a \) can be identified with the composition \( a \circ (1_A \otimes (\otimes^n a)) \), one obtains the exact triangle
\[ A \otimes \text{cone}(\otimes^n a) \xrightarrow{} \text{cone}(\otimes^{n+1} a) \]
from the octahedral axiom. Using the exact triangle resulting from tensoring this one with \( B \), one obtains the desired result by induction on \( n \). \( \square \)

Lemma 1.3. Let \((X, \mathcal{O}_X)\) be a quasi-compact and quasi-separated scheme. Let \( A, B \in D_{perf}(X) \), and let \( a : A \to \mathcal{O}_X \) be a morphism in \( D_{perf}(X) \). If \( a \otimes k(x) = 0 \) in the derived category \( D(k(x)) \) of \( k(x) \)-modules, for all \( x \in \text{supph} B \), then there is an \( n \geq 1 \) such that \( 1_B \otimes (\otimes^n a) = 0 \) in \( D_{perf}(X) \).

Proof. This follows directly from Theorem 3.8 in [4]. \( \square \)
Proposition 1.4. Let \((X, \mathcal{O}_X)\) be a quasi-compact and quasi-separated scheme, and let \(B, D \in \text{D}^\text{perf}(X)\) be such that
\[
\text{supph} B \subset \text{supph} D.
\]
Then \(B\) is in the thick triangulated subcategory generated by \(D\).

Proof. Let \(\langle D \rangle\) denote the thick triangulated subcategory generated by \(D\). Since \(D\) is perfect, there is an isomorphism \(\text{RHom}(D, \mathcal{O}_X) \otimes D \cong \text{RHom}(D, D)\), so \(\text{RHom}(D, D) \in \langle D \rangle\). Let
\[
f : \mathcal{O}_X \to \text{RHom}(D, D)
\]
be the morphism corresponding to \(1_D : D \to D\), and let
\[
a : A \to \mathcal{O}_X
\]
be the edge opposite to the vertex \(\text{RHom}(D, D)\) in the exact triangle having \(f\) as an edge. Then \(\text{cone}(a) \cong \text{RHom}(D, D) \in \langle D \rangle\), so \(B \otimes \text{cone}(a) \in \langle D \rangle\). By Lemma 1.2, we have
\[
B \otimes \text{cone}(\otimes^n a) \in \langle D \rangle,
\]
for all \(n \geq 1\).

Next, we show that there is an \(n \geq 1\) such that \(1_B \otimes (\otimes^n a) = 0\) in \(\text{D}^\text{perf}(X)\). By Lemma 1.3, it suffices to show that for all \(x \in \text{supph} B\), we have \(a \otimes k(x) = 0\) in \(D(k(x))\). But if \(x \in \text{supph} B\), then \(D \otimes k(x) \neq 0\), since \(\text{supph} B \subset \text{supph} D\). Thus the map
\[
k(x) \to \text{RHom}_{k(x)}(D \otimes k(x), D \otimes k(x)),
\]
for all \(n \geq 1\), corresponding to \(1_{D \otimes k(x)} : D \otimes k(x) \to D \otimes k(x)\) is non-zero, and so is a split monomorphism in \(\text{D}^\text{perf}(\text{Spec}(k))\). But under the isomorphism \(\text{RHom}_{\mathcal{O}_X}(D, D) \otimes k(x) \cong \text{RHom}_{k(x)}(D \otimes k(x), D \otimes k(x))\) (cf. [2], I 7.1.2), this split monomorphism corresponds to \(f \otimes k(x)\). Since \(f \circ a = 0\), we get \(a \otimes k(x) = 0\).

Applying \(\text{Hom}_{\mathcal{D}(\mathcal{O}_X)}(-, B)\) to the exact triangle
\[
\begin{array}{ccc}
B \otimes (\otimes^n A) & \xrightarrow{1_B \otimes (\otimes^n a)} & B \otimes \mathcal{O}_X \\
\downarrow & & \downarrow \\
B \otimes \text{cone}(\otimes^n a) & & ,
\end{array}
\]
and considering the associated long exact Puppe sequence, one sees that
\[
B \simeq B \otimes \mathcal{O}_X \to B \otimes \text{cone}(\otimes^n a)
\]
is a split monomorphism. Hence \(B \in \langle D \rangle\), since \(\langle D \rangle\) is thick and \(B\) identifies with a direct summand of \(B \otimes \text{cone}(\otimes^n a) \in \langle D \rangle\). \(\square\)

2. Recovering open subsets from perfect complexes

Let \(\mathcal{K}\) be a tensor triangulated category. A thick subcategory \(\mathcal{A}\) of \(\mathcal{K}\) is said to be principal if it is the triangulated subcategory generated by an element \(C \in \mathcal{A}\). If this is the case, we use the notation \(\mathcal{A} = \langle C \rangle\). We denote the set of principal thick subcategories of \(\mathcal{K}\) by \(\text{PS}(\mathcal{K})\).

A subset \(S \subset \text{PS}(\mathcal{K})\) is said to be filtering if
\begin{itemize}
  \item [(FS1)] \(\langle C \rangle \in S\) and \(\langle C \rangle \subset \langle D \rangle\) implies \(\langle D \rangle \in S\) for all \(\langle D \rangle \in \text{PS}(\mathcal{K})\), and
  \item [(FS2)] for all \(\langle C \rangle, \langle D \rangle \in S\), there exists a \(\langle B \rangle \in S\) such that \(\langle B \rangle \subset \langle C \rangle\) and \(\langle B \rangle \subset \langle D \rangle\).
\end{itemize}
We denote the set of filtering subsets of \(\text{PS}(\mathcal{K})\) by \(\text{Loc}(\mathcal{K})\).
Theorem 2.1. If \((X, \mathcal{O}_X)\) is a quasi-compact and quasi-separated scheme, then there is an inclusion preserving bijection
\[ U(X) \leftrightarrow \text{Loc}(D_{\text{perf}}(X)) \]
between the set of open subsets of \(X\) and the set of filtering subsets of \(PS(D_{\text{perf}}(X))\).

Proof. Consider the maps
\[ \text{Loc}(D_{\text{perf}}(X)) \xrightarrow{f} U(X), \quad S \mapsto X \setminus \bigcap_{(C) \in S} \text{supph}C, \]
and
\[ U(X) \xrightarrow{g} \text{Loc}(D_{\text{perf}}(X)), \quad U \mapsto \{(C); X \setminus U \subset \text{supph}C\}. \]

Note first that \(f \circ g = \text{id}\), since given \(x \in U\) there is a \(C \in D_{\text{perf}}(X)\) with \(X \setminus U \subset \text{supph}C\) but \(x \notin \text{supph}C\). Indeed, let \(\text{Spec} R_x\) be an open affine neighborhood of \(x\) such that \(\text{Spec} R_x \subset U\), and choose \(C\) with \(\text{supph}C = X \setminus \text{Spec} R_x\) (cf. Lemma 3.4 in [4]).

In order to show that \(g \circ f = \text{id}\), let \(S \in \text{Loc}(D_{\text{perf}}(X))\) and \(D \in D_{\text{perf}}(X)\) be such that
\[ \bigcap_{(C) \in S} \text{supph}C \subset \text{supph}D. \]
We show that \(\langle D \rangle \in S\). By Lemma [1,1] and since \(X\) is quasi-compact, the subset \(X \setminus \text{supph}D\) is quasi-compact. By assumption
\[ \{X \setminus \text{supph}C\}_{(C) \in S} \]
form an open cover of \(X \setminus \text{supph}D\), so there are finitely many \(C_i\)’s such that
\[ \bigcap_{i=1}^n \text{supph}C_i \subset \text{supph}D. \]
By (FS2), there is a \(\langle B \rangle \in S\) such that \(\text{supph}B \subset \text{supph}D\). By Proposition [1,4] \(\langle B \rangle \subset \langle D \rangle\), so by (FS1), \(\langle D \rangle \in S\). \(\square\)

3. Recovering points from open subsets

Recall that a lattice is a partially ordered set \(\mathcal{L}\) which, considered as a category, has all binary products \(\land\), and all binary coproducts \(\lor\). In this section, we shall assume in addition, that lattices have initial and final objects, and that they satisfy the distributive law:
\[ U \land (V \lor W) = (U \land V) \lor (U \land W), \quad \text{for all } U, V, W \in \mathcal{L}. \]

Definition 3.1. Let \(\mathcal{L}\) be a lattice with all finite products and all coproducts. A point of \(\mathcal{L}\) is a map
\[ p : \mathcal{L} \rightarrow \{0, 1\}, \]
of partially ordered sets, which preserves finite products and infinite coproducts. The set of points of a lattice \(\mathcal{L}\) will be denoted by \(\text{pt}(\mathcal{L})\).

We define a topology on \(\text{pt}(\mathcal{L})\) by declaring all subsets which are of the form
\[ \{p; p(U) = 1\}, \quad \text{for some } U \in \mathcal{L}, \]
to be open.

The set \(\text{pt}(\mathcal{L})\) is in bijection with the set of proper primes of \(\mathcal{L}\), that is, the set of elements \(P \in \mathcal{L}\) which are not final in \(\mathcal{L}\), and which satisfy
\[ U \land V \leq P \text{ if and only if } U \leq P \text{ or } V \leq P, \quad \text{for all } U, V \in \mathcal{L}. \]
Under this bijection, a point \( p \in \text{pt}(L) \) corresponds to the coproduct
\[
\bigvee_{p(W)=0} W,
\]
of elements in the kernel of \( p \), and a proper prime \( P \) corresponds to the map
\[
p : L \to \{0,1\}, \quad p(U) = 0 \text{ if and only if } U \leq P.
\]

**Example 3.2.** Let \( X \) be a topological space, and consider the lattice \( \mathcal{U}(X) \), of open subsets of \( X \). A point \( x \in X \) determines the point
\[
p : \mathcal{U}(X) \to \{0,1\}, \quad p(U) = 0 \text{ if and only if } x \not\in U.
\]
Under the bijection with proper primes of \( \mathcal{U}(X) \), this point corresponds to the element \( X \setminus \{x\} \in \mathcal{U}(X) \).

**Definition 3.3.** A topological space \( X \) is **sober** if every non-empty irreducible closed set \( Z \) has a unique generic point, that is, there exists a \( z \in Z \) such that \( Z = \overline{\{z\}} \).

Examples of sober spaces include Hausdorff spaces and topological spaces underlying schemes.

**Proposition 3.4.** Let \( X \) be a topological space, and let \( F \) be the map \( x \mapsto X \setminus \{x\} \), from \( X \) to the set of proper primes of \( \mathcal{U}(X) \).

(i) The map \( F \) is a bijection if and only if \( X \) is sober.

(ii) If \( X \) is sober, then \( F \) induces a homeomorphism between \( X \) and the space \( \text{pt}(\mathcal{U}(X)) \) of points of \( \mathcal{U}(X) \).

**Proof.** (i) Note that an open set \( U \in \mathcal{U}(X) \) is a proper prime if and only if its complement \( X \setminus U \) is non-empty and irreducible.

(ii) Follows from (i), and the fact that the image of an open set \( U \subset X \) gets identified with the open set
\[
\{p; p(U) = 1\} \subset \text{pt}(\mathcal{U}(X)).
\]

Combining Theorem 2.1 and Proposition 3.4, we obtain the announced reconstruction result:

**Theorem 3.5.** Let \( (X, \mathcal{O}_X) \) be a quasi-compact and quasi-separated scheme. Then the underlying topological space \( X \) is homeomorphic to the space \( \text{pt} \circ \mathcal{L}(\mathcal{D}_{\text{perf}}(X)) \), of points of the lattice of filtering subsets of principal thick subcategories of \( \mathcal{D}_{\text{perf}}(X) \).

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