Evolving statistical systems: application to academic courses

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Abstract

Statistical systems are conceived from the standpoint of statistical mechanics, as made of a (generally large) number of identical units and exhibiting a (generally large) number of different configurations (microstates), among which only equivalence classes (macrostates) are accessible to observations. Further attention is devoted to evolving statistical systems, and a simple case including only a possible event, E, and related opposite event, ¬E, is examined in detail. In particular, the expected evolution is determined and compared to the random evolution inferred from a sequence of random numbers, for different sample populations. The special case of radioactive decay is considered and results are expressed in terms of the fractional time, \( t/\Delta t \), where the time step, \( \Delta t \), is related to the decay probability, \( p = p(\Delta t) \). An application is made to data collections from selected academic courses, focusing on the extent to which expected evolutions and model random evolutions fit to empirical random evolutions inferred from data collections. Results could be biased by the assumed number of students who abandoned their course, defined as suitable impostors (SI). Extreme cases related to a lower and an upper limit of the SI number are considered for a time step, \( \Delta t = (1/12)y \), where

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fitting expected evolutions relate to $0.003 \leq p \leq 0.200$. In conclusion, evolving statistical systems made of academic courses are similar to poorly populated samples of radioactive nuclides exhibiting equal probabilities, $p$, and time steps, $\Delta t$, where inferred mean lifetimes, $\tau$, and half-life times, $t_{1/2}$, range within $0.37 < \tau/y < 27.73$ and $0.25 < t_{1/2}/y < 19.22$, respectively, and upper limits are related to incomplete data collections.

**Keywords:** Systems: statistical; events: microstates, macrostates; evolutions: expected, random.

## 1 Introduction

From the standpoint of statistical mechanics, statistical systems can be conceived as made of a (in general, extremely large) number of identical “atomic” units and exhibiting a (in general, extremely large) number of distinct configurations (microstates), among which only equivalence classes (macrostates) are accessible to observations. More specifically, a single microstate arises after a selected physical process, or attempt, is applied to each unit, which can attain two distinct configurations related to a possible event, $E$, and the opposite event, $\neg E$, respectively. Accordingly, microstates can be defined as sequences of $E$ and $\neg E$, or $n$-tuples if the system under consideration is made of $n$ units, and macrostates can be defined as subsets of $n$-tuples exhibiting $k$ possible events, $E$, and $(n - k)$ opposite events, $\neg E$, $0 \leq k \leq n$.

For instance, a unit may be a box containing $N$ identical (leaving aside the colour) spheres, numbered on the inside in arithmetic progression, among which $N_W$ are white and $(N - N_W)$ are black. Accordingly, the attempt is the extraction of a sphere; the possible event, $E$, is a white sphere extracted, and the opposite event, $\neg E$, is a black sphere extracted.

If the system is made of $n$ boxes, numbered on the inside in arithmetic progression, a microstate can be denoted by a $n$-tuple, $\{i_1, i_2, \ldots, i_n\}$, where $i_k$ denotes the $i$th sphere within the $k$th box.

If numbered spheres and numbered boxes are not accessible to observations, macrostates are denoted by $n$-tuples of events (or colours), $\{\{E^k, \neg E^{n-k}\}\}$, where inner brackets represent a single microstate made of $k$ specified white spheres and $(n - k)$ specified black spheres, each extracted from $k$ specified boxes and $(n-k)$ specified boxes, respectively; middle brackets represent the whole set of microstates made of $k$ unspecified white spheres and $(n-k)$ unspecified black spheres, each extracted from $k$ specified boxes and $(n-k)$ specified boxes, respectively; outer brackets represent the whole set of microstates made of $k$ unspecified white spheres and $(n-k)$ un-
specified black spheres, each extracted from \( k \) unspecified boxes and \((n - k)\) unspecified boxes, respectively.

The probability of macrostates is expressed by the binomial distribution, as \( P_n(k) = \binom{n}{k} p^k q^{n-k} \), where \( p \) is the probability of the possible event, \( E \), and \( q = 1 - p \) is the probability of the opposite event, \( \neg E \), made of a white and a black sphere extracted, respectively, in the above mentioned example. Related expectation value and variance are \( k^* = np \) and \( \sigma_B = npq \), respectively.

Statistical systems where attempts are repeated at a fixed frequency, \((\Delta t)^{-1}\), can either remain unchanged or evolve. For instance, boxes where extracted spheres are reintroduced before the next extraction remain unchanged, while boxes are evolving if the contrary holds. A special class of evolving statistical systems relates to units which are maintained or removed according if the possible event, \( E \), or the opposite event, \( \neg E \), respectively, occurs after an attempt is performed. For instance, the extraction of white or black spheres could imply box preservation or removal, respectively. Typical evolving statistical systems are (i) samples of radioactive nuclides, and (ii) selected academic courses.

The current paper is aimed to highlight some aspects of evolving statistical systems with regard to the fractional number of surviving units, \( n(t)/n_0 \), where \( n_0 \) is the initial number. In particular, attention is devoted to radioactive decay and an application is made to data collection related to selected academic courses.

Basic considerations on statistical systems, in the light of statistical mechanics, are outlined in Section 2. Evolving statistical systems are considered in Section 3 where further attention is devoted to radioactive decay. An application to data collections related to selected academic courses is performed in Section 4. The discussion is presented in Section 5. The conclusion is drawn in Section 6. Further details on data collection and input parameters are shown in the Appendix.

## 2 Statistical systems

### 2.1 General remarks

Statistical systems can be defined as able to attain a number of different configurations after experiencing specified physical or conceptual processes in ordinary or abstract space, respectively. Configurations attained by statistical systems can be defined as possible events, and the whole amount of selected physical or conceptual processes can be defined as attempt.
Let $S_1$ be a simple statistical system i.e. made of a single unit. Let the effect of an attempt, $A_1$, performed on $S_1$, be the possible event, $E_1$, or the opposite event, $\neg E_1$, i.e. the union of all possible events other than $E_1$. Let $S_{u,1}$ be an underlying statistical system which exhibits the following features.

(i) $E_1$ occurs in $S_1$ if and only if $(E_{u,1})_i$, $1 \leq i \leq N_u(E_{u,1})$, occurs in $S_{u,1}$, where $N_u(E_{u,1})$ is the number of possible occurrences of $E_{u,1}$ in $S_{u,1}$;

(ii) $\neg E_1$ occurs in $S_1$ if and only if $(\neg E_{u,1})_j$, $1 \leq j \leq N_u(\neg E_{u,1})$, occurs in $S_{u,1}$, where $N_u(\neg E_{u,1})$ is the number of possible occurrences of $\neg E_{u,1}$ in $S_{u,1}$;

(iii) There is no physical or conceptual reason for which, after performing an attempt, $A_{u,1}$, on $S_{u,1}$, $(E_{u,1})_i$ has to be preferred with respect to $(\neg E_{u,1})_j$ or $(E_{u,1})_j$, and $(\neg E_{u,1})_j$ has to be preferred with respect to $(E_{u,1})_i$ or $(E_{u,1})_i$.

Accordingly, the probability of the possible event, $E_1$, can be defined as $p = N_u(E_{u,1})/N_u$, and the probability of the opposite event, $\neg E_1$, can be defined as $q = N_u(\neg E_{u,1})/N_u$, where $N_u = N_u(E_{u,1}) + N_u(\neg E_{u,1})$, which implies the normalization condition, $p + q = 1$.

In the light of quantum mechanics, physical processes involve discrete quantities, which implies $p$ is a rational number. On the other hand e.g., in the light of geometry, conceptual processes could involve continuous quantities, which implies $p$ could be an irrational number. If it is the case, both $N_u(E_{u,1})$ and $N_u(\neg E_{u,1})$ must be infinite to ensure a ratio, $N_u(E_{u,1})/N_u$ and $N_u(\neg E_{u,1})/N_u$, infinitely close to $p$ and $q$, respectively, via Dedekind’s axiom.

Let $S_n$ be a complex statistical system made of $n$ units $S_i$. Let $A_n$ be an attempt performed on $S_n$, made of $n$ attempts $A_1$ each performed on a different $S_i$. Let $s_{u,k} = E_{u,k} = \{E_{u,1}^k, \neg E_{u,1}^{n-k}\}$, $0 \leq k \leq n$, denote a configuration of $S_n$ after $A_n$ has been performed. More specifically, $E_{u,k}$ is a complex event resulting from the union of $k$ independent events, $(E_{u,1})_i$, $1 \leq i \leq N_u$, each related to a specified $S_n$, and $(n - k)$ independent events, $(\neg(E_{u,1}))_j$, $1 \leq j \leq N_u$, each related to a specified $S_n$.

Similarly, $s'_{u,k} = \{E_{u,k}\} = \{E_{u,1}^k, \neg E_{u,1}^{n-k}\}$, $0 \leq k \leq n$, denotes the whole set of $E_{u,k}$, in number of $[N_u(E_{u,1})]^k[N_u - N_u(E_{u,1})]^{n-k}$, where $k$ independent events, $E_{u,1}$, and $(n - k)$ independent events, $\neg E_{u,1}$, are no longer specified.

Finally, $s_{n,k} = \{\{E_{u,1}^k, \neg E_{u,1}^{n-k}\}\} = \{E_{i}^k, \neg E_{i}^{n-k}\} = \{E_{i,1}\}$, $0 \leq k \leq n$, denotes the whole set of $E_{i,k}$, in number of $\binom{n}{k}$, where $k$ units, $S_i$, related to $E_i$, and $(n - k)$ units, $S_j$, related to $\neg E_i$, are no longer specified.

In short, $s_{u,k}$ can be conceived as a microstate and $s_{n,k}$ as a macrostate, with regard to an attempt, $A_n$, performed on a complex statistical system,
\( S_n \). The whole set of microstates yields the sure event, \( E_{S,n} \). The empty set of microstates yields the unpossible event, \( E_{U,n} \). In the case under discussion, the number of microstates related to \( s'_{u,k}, s_{n,k}, E_{S,n}, E_{U,n} \), reads:

\[
N(s'_{u,k}) = [N_u(E_{u,1})]^{n-k} ; \quad (1a)
\]

\[
N(s_{n,k}) = \binom{n}{k}[N_u(E_{u,1})]^{n-k} ; \quad (1b)
\]

\[
N(E_{S,n}) = N_u^n ; \quad N(E_{U,n}) = 0 ; \quad (1c)
\]

and the corresponding probability is:

\[
P(s'_{u,k}) = \frac{N(s'_{u,k})}{N(E_{S,n})} = p^k q^{n-k} ; \quad (2a)
\]

\[
P(s_{n,k}) = \binom{n}{k} p^k q^{n-k} ; \quad (2b)
\]

\[
P(E_{S,n}) = \frac{N(E_{S,n})}{N(E_{S,n})} = 1 ; \quad P(E_{U,n}) = \frac{N(E_{U,n})}{N(E_{S,n})} = 0 ; \quad (2c)
\]

according to the above considerations.

Let the macrostate, \( s_{n,k} \), be designed after performing an attempt, \( A_n \), on a complex statistical system, \( S_n \). The variable, \( k \), \( 0 \leq k \leq n \), is a random variable, and the probability of a macrostate, \( P_n(k) \), is the related distribution. In particular, the distribution expressed by Eq. (2b):

\[
P_n(k) = \binom{n}{k} p^k q^{n-k} ; \quad 0 \leq k \leq n ; \quad (3)
\]

is known as binomial distribution or Bernoulli distribution. By definition, related expectation value and variance read:

\[
k^* = \sum_{k=0}^{n} k P_n(k) = np ; \quad (4)
\]

\[
\sigma_B^2 = \sum_{k=0}^{n} (k - k^*)^2 P_n(k) = npq ; \quad (5)
\]

where the index, B, denotes binomial distribution.

In the limit, \( n \to +\infty \), \( p = \text{const} \), the binomial distribution takes the expression:

\[
\lim_{n \to +\infty} P_n(k) = \lim_{n \to +\infty} \frac{h}{\sqrt{n}} \exp\left[-h^2(k - k^*)^2\right] ; \quad h = \frac{1}{2npq} ; \quad p = \text{const} ; \quad (6)
\]
which is known as Gauss distribution (with regard to a single source of accidental errors). Related expectation value and variance are divergent via Eqs. (4) and (5).

In the limit, \( n \to +\infty \), \( np = \text{const} \), the binomial distribution takes the expression:

\[
\lim_{n \to +\infty} P_n(k) = \frac{(k^*)^k}{k!} \exp(-k^*) \quad ; \quad k^* = np = \text{const} \quad ; \quad (7)
\]

which is known as Poisson distribution. Related expectation value and variance are expressed via Eqs. (4) and (5), the last reduced to:

\[
\sigma_P^2 = np = k^* \quad ; \quad (8)
\]

where the index, \( P \), denotes Poisson distribution. For further details, an interested reader is addressed to specific textbooks e.g., [3] Chap. 2.

2.2 A guidance example

The following guidance example is aimed to better understanding considerations outlined in Subsection 2.1. In this view, \( S_{u,1} \) is conceived as a box, \( B_1 \), containing \( N \) identical spheres numbered on the inside in arithmetic progression \((i = 1, 2, ..., N)\), among which \( N_W \) are white and \( N - N_W \) are black.

The possible event, \( E_{u,1} \), is a white sphere extracted from \( B_1 \); the opposite event, \( \neg E_{u,1} \), is a black sphere extracted from \( B_1 \); a single extraction is the related attempt.

Requirements mentioned in Subsection 2.1 are satisfied, as (i) \( E_{1} \) is the union of \( N_W \) white spheres which can be extracted from \( B_1 \), \((E_{u,1})_i, 1 \leq i \leq N_B\); (ii) \( \neg E_{1} \) is the union of \((N - N_W)\) black spheres which can be extracted from \( B_1 \), \((\neg E_{u,1})_i, 1 \leq i \leq N - N_B\); (iii) there is no physical or conceptual reason for which, after performing an extraction from \( B_1 \), a specified sphere has to be preferred with respect to another one. Accordingly, the probability of extracting a specified (white or black) sphere is \( p_i = q_i = 1/N \), and the probability of extracting an unspecified white or black sphere is \( p = N_W/N \) or \( q = (N - N_W)/N \), respectively.

The possible event, \( E_{1} \), is statistically equivalent to a white sphere extracted from \( B_1 \) and the opposite event, \( \neg E_{1} \), is statistically equivalent to a black sphere extracted from \( B_1 \), in the sense that \( p(E_{1}) = p = N_W/N \) and \( p(\neg E_{1}) = q = (N - N_W)/N \), respectively.

With regard to the complex statistical system, \( B_n \), made of \( n \) units, \( B_1 \), and to the attempt, \( A_n \), made of \( n \) attempts, \( A_1 \), performed each on a different \( B_1 \), the microstate, \( s_{u,k} \), is made of \( k \) specified white spheres and \((n - k)\) specified black spheres, each extracted from a different specified box.
Similarly, $s'_{u,k}$, is made of the whole set of $s_{u,k}$, where there are $N_W$ different white spheres and $(N - N_W)$ different black spheres to be extracted from each specified box, for a total of $N_W^k (N - N_W)^{n-k}$.

Finally the macrostate, $s_{n,k}$, is made of the whole set of $s'_{u,k}$, where there are $\binom{n}{k}$ different ways of extracting $k$ white spheres and $(n-k)$ black spheres from $n$ identical boxes.

For $n \gg 1$, microstates are virtually indistinguishable (leaving aside Laplace’s daemon) and only macrostates can be detected. In the case under discussion, microstates are denoted by numbers specifying spheres and boxes, and macrostates by colours. Natural limits intrinsic to observers makes numbers undetected and colours distinguishable.

With regard to a generic statistical system, $S_1$, where an attempt, $A_1$, is performed yielding either the possible event, $E_1$, or the opposite event, $\neg E_1$, of probability, $P(E_1) = p$, $P(\neg E_1) = q = 1 - p$, respectively, the underlying statistical system, $S_{u1}$, may be conceived as a box, $B_1$, containing $N$ identical spheres among which $N_W$ are white and $N - N_W$ are black, where $p = N_W/N$ and $q = (N - N_W)/N$, respectively. For further details, an interested reader is addressed to specific textbooks e.g., [3] Chap. 2.

3 Evolving statistical systems

3.1 Basic considerations

With regard to a statistical system, $S_n$, made of $n$ units, $S_1$, and to an attempt, $A_n$, made of the union of $n$ attempts, $A_1$, each in connection with a different $S_1$, performing $A_n$ on $S_n$ yields a macroscopical state, $s_{n,k}$, $0 \leq k \leq n$. Related probability obeys binomial distribution, as expressed by Eq. (3).

If $S_n$ maintains unchanged after undergoing a succession of $A_n$, no evolution occurs. If the contrary holds, some kind of evolution takes place and an arrow of the time can be defined.

For instance, let $S_1$ be a box containing identical white and black spheres, $A_1$ a single extraction, $E_1$ a white sphere extracted and $\neg E_1$ a black sphere extracted. Let $A_n$ be performed on $S_n$ and $s_{n,k_1}$ be denoted. Let the additional condition hold that all boxes, from which white spheres are extracted, be removed before the next attempt. Accordingly, $S_n$, $A_n$, are changed into $S_{n-k_1}$, $A_{n-k_1}$, respectively, and so on until $n - k_1 - \cdots - k_L = 0$ after $L$ successive attempts. Statistical systems of the kind considered can be conceived as evolving.

Let $S_n$ be an evolving statistical system, and let a trial, $A_n$, be successively performed on $S_n$ after a time step, $\Delta t$, has been elapsed. Let $s_{n,t,k_t}$ be a
macrostate at the $\ell$th step. Related binomial distribution via Eq. (3) reads:

$$P_{n\ell}(k\ell) = \binom{n\ell}{k\ell} p^{k\ell} q^{n\ell-k\ell} \quad 0 \leq k\ell \leq n\ell \quad ;$$

which exhibits expectation value and variance via Eqs. (4) and (5), respectively, as:

$$k^*_\ell = n\ell - 1 p \quad ;$$
$$\sigma^2_{B,\ell} = n\ell - 1 pq \quad ;$$

where $p = p(\Delta t)$ and $q = q(\Delta t) = 1 - p(\Delta t)$ is the probability of $E_1$ and $\neg E_1$, respectively, after a time step, $\Delta t$, has been elapsed.

### 3.2 Expected evolution

With regard to an evolving statistical system, $S_n$, and an attempt, $A_n$, let $p = p(\Delta t)$ be the probability of the event, $E_1$, in connection with a generic unit, $S_1$, within a time step, $\Delta t$, where the occurrence of $E_1$ implies related $S_1$ is removed from $S_n$.

The expected number of surviving units, at the end of the $\ell$th step, via Eq. (10) reads:

$$n^*_\ell = n^*_{\ell-1} - k^*_\ell = n^*_{\ell-1}(1 - p) \quad ; \quad n^*_0 = n_0 \quad ;$$

which, after $\ell$ iterations, yields the fractional number of surviving units as:

$$\frac{n^*_\ell}{n_0} = (1 - p)^\ell \quad ;$$
$$\ell = \frac{t_\ell - t_0}{\Delta t} \quad ;$$

where $t_\ell$ is the time at the end of the $\ell$th step and $t_0$ is the initial time. It is worth noticing $p$ depends on the time step only, while $\ell$ depends on both the time elapsed and the time step.

Using the logarithmic Taylor series e.g., [8] Chap. 20 §20.17:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad ; \quad -1 < x \leq 1 \quad ;$$

the following identity holds:

$$(1-p)^\ell = \exp \ln(1-p)^\ell = \exp[\ell \ln(1-p)] = \exp \left[ -\ell \left( p + \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4} + \ldots \right) \right] ;$$

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which, in the limit of negligible $p$, reduces to:

$$(1 - p)^{\ell} = \exp(-p\ell) \ ; \ p \ll 1 \ ;$$

(17)

and the substitution of Eq. (17) into (13) yields:

$$\frac{n^*_\ell}{n_0} = \exp \left( -p \frac{t_\ell - t_0}{\Delta t} \right) ;$$

(18)

where the product, $p\ell = p(\Delta t)(t_\ell - t_0)/\Delta t$, for fixed $t_\ell$ remains unchanged provided the ratio, $p(\Delta t)/\Delta t$, remains unchanged i.e. $p$ is directly proportional to $\Delta t$. The ratio, $(1 - p)^{\ell}/\exp(-p\ell)$, by use of Eq. (16) reads:

$$\frac{(1 - p)^{\ell}}{\exp(-p\ell)} = \exp \left[ -\ell \left( p + \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4} + \ldots \right) \right] \exp(p\ell)$$

$$= \exp \left[ -\ell \left( \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4} + \ldots \right) \right] ;$$

(19)

which tends to zero as $\ell \to +\infty$ i.e. an infinite time. Accordingly, the power, $(1 - p)^{\ell}$, is overstimated by the exponential, $\exp(-p\ell)$, and the former is infinitesimal of higher order with respect to the latter, as $\ell \to +\infty$.

The above results hold for the expected evolution, where the number of surviving $S_1$ at the end of any step equals related expected number. According to Bernoulli’s theorem, discrepancies between random evolution and expected evolution may safely thought of as negligible for large $S_1$ populations, $n \gg 1$.

The expected evolution of fractional number of surviving units, $n^*_\ell$, is shown in Fig.1 for both the exact power law, expressed by Eq. (13), and the exponential approximation, expressed by Eq. (18). Different curves relate to different probabilities, $p = P(E_1)$, within the range, $0.1 \leq p \leq 0.99$. The limit, $p = 0$, corresponds to $n^*(t)/n_0 = 1$. The limit, $p = 1$, corresponds to $(t - t_0)/\Delta t = 0$.

A generic point on the $\{O[(t - t_0)/\Delta t][n^*(t)/n_0]\}$ plane depends on three parameters, namely the time elapsed, $t - t_0$, the time step, $\Delta t$, and the probability, $p = p(\Delta t)$. The changes, $(t - t_0) \to k(t - t_0)$, $\Delta t \to k\Delta t$, leave a generic point unchanged provided $p(\Delta t) \to p(k\Delta t)$. It is worth of note the expected evolution needs an infinite time via Eq. (13) or (18), while the random evolution ends within a finite time unless $n_0 \to +\infty$.

### 3.3 Random evolution

With regard to an evolving statistical system, $S_n$, and an attempt, $A_n$, let $p = p(\Delta t)$ be the probability of the event, $E_1$, within a time step, $\Delta t$, where the
occurrence of $E_i$ on a generic unit, $S_1$, implies removal from $S_n$. Accordingly, the random evolution can be determined along the following steps e.g., [10][2].

(i) Take $\ell = 0$ at the beginning of the first step.

(ii) Generate a succession of $n_\ell$ random numbers, $\xi_1, \xi_2, \ldots, \xi_{n_\ell}$, within the range, $0 \leq \xi_i \leq 1$, $1 \leq i \leq n_\ell$.

(iii) Perform a one-to-one correspondence, $(S_1)_i \leftrightarrow \xi_i$, $1 \leq i \leq n_\ell$.

(iv) Remove $(S_1)_i$ unit from $S_n$ if $\xi_i < p$ and preserve if otherwise, yielding $S_n(n_\ell) \rightarrow S_{n_\ell-\Delta n}(n_\ell - \Delta n)$, where $\Delta n$ is the number of removed units at the end of the $(\ell + 1)$th step.

(v) End if $n_{\ell+1} = 0$ or replace $n_\ell$ with $n_{\ell+1}$ and return to (ii) if $n_{\ell+1} > 0$.

Owing to finite $n_0$, the random evolution ends when, after $L$ steps, $n_L = 0$ and $S_{n_L} = S_0$ is the empty set of units.

An example of model random evolution is shown in Fig. 2 for $\log n_0 = 1, 2, 3, 4$ (listed on each panel) and $p = 0.01$ (diamonds), $0.10$ (crosses), $0.90$ (saltires). Related expected evolution is represented as full curves. An inspection of Fig. 2 discloses statistical fluctuations are large for $n_0 < \sim 10$, small for $10 \ll n_0 < \sim 100$, negligible for $100 \ll n_0 < \sim 1000$ and $n_0 > 1000$.

### 3.4 Mean lifetime and half-life time

With regard to an evolving statistical system, $S_n$, where the exponential approximation holds to an acceptable extent, let the mean lifetime, $\tau$, and the mean half-life time, $t_{1/2}$, be defined in connection with the following values of the fractional number of surviving units:

\[
\frac{n^*(\tau)}{n_0} = \exp(-1) ;
\]

\[
\frac{n^*(t_{1/2})}{n_0} = \frac{1}{2} ;
\]

where Eq. (20) via (18) implies an explicit expression of the mean lifetime as:

\[
\tau = \frac{\Delta t}{p} ; \quad p \ll 1 ;
\]

accordingly, Eq. (18) reads:

\[
\frac{n^*_0}{n_0} = \exp\left(-\frac{t_0 - t_0}{\tau}\right) ;
\]
where the special case, \( t - t_0 = t_{1/2} \), yields \( 1/2 = \exp(-t_{1/2}/\tau) \), or:

\[
t_{1/2} = \tau \ln 2 \; ;
\]

which implies \( t_{1/2} < \tau \).

In the case under discussion, \( p(\Delta t) = \Delta t/\tau \) via Eq. (22), where \( \Delta t \leq \tau \) to preserve the statistical meaning of \( p \). Accordingly, \( \Delta t \ll \tau \) implies \( p = p(\Delta t) \ll 1 \) i.e. the validity of the exponential approximation.

The mean lifetime, \( \tau \), can be inferred from the intersection of related curve with the horizontal line, \( n^*(t)/n_0 = 1/e \), as shown in Fig. 1. The same holds for the mean half-life time, \( t_{1/2} \), with regard to the horizontal line, \( n^*(t)/n_0 = 1/2 \), as shown in Fig. 1.

The extension of the above definitions to the general case, via Eqs. (13)-(14) yields:

\[
\tau = -\frac{\Delta t}{\ln(1 - p)} \; ;
\]

and the substitution of Eq. (25) into (13) yields (23). In the limit, \( p \ll 1 \), \(-\ln(1 - p) \approx p \) via Eq. (15) and Eq. (25) reduces to (22).

In conclusion, Eq. (23) holds in general for evolving statistical systems, provided the mean lifetime is defined by Eq. (25) instead of (22). The same holds for the mean half-life time via Eq. (24), which follows from (23).

Values of the mean lifetime, \( \tau \), and mean half-life time, \( t_{1/2} \), are listed in Table 1 for the general case, Eq. (25), and the exponential approximation, Eq. (22). An inspection of Table 1 shows the exponential approximation holds to a good extent in the limit, \( p \ll 1 \), while the contrary holds for \( p \approx 1 \), as expected.

3.5 Radioactive decay

A sample of radioactive nuclides is a special case of evolving statistical system, where \( S_1 \) is a single nuclide, \( A_1 \) is waiting for a time step, \( \Delta t \), \( E_1 \) is a nuclide after radioactive decay, \( \neg E_1 \) is a surviving radioactive nuclide, and \( p = p(\Delta t) \) is the probability a radioactive nuclide decays within a time step, \( \Delta t \).

With regard to a selected nuclide, the probability of radioactive decay within a time step, \( \Delta t \), can be inferred from the mean lifetime, \( \tau \), or the mean half-life time, \( t_{1/2} \), via Eqs. (24)-(25) as:

\[
p = 1 - \exp\left(-\frac{\Delta t}{\tau}\right) = 1 - \exp\left(-\frac{\Delta t}{t_{1/2}}\right) \; ;
\]

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Table 1: Fractional mean lifetime, $\tau/\Delta t$, and fractional half-life time, $t_{1/2}/\Delta t$, as a function of the probability, $p = p(\Delta t)$, for the general case and the exponential approximation, indexed as “gen” and “app”, respectively. See text for further details.

| $p$   | $\tau_{\text{gen}}/\Delta t$ | $\tau_{\text{app}}/\Delta t$ | $(t_{1/2})_{\text{gen}}/\Delta t$ | $(t_{1/2})_{\text{app}}/\Delta t$ |
|-------|------------------|------------------|------------------|------------------|
| 0.00  | $\infty$        | $\infty$        | $\infty$        | $\infty$        |
| 0.01  | 9.9499D+01      | 1.0000D+02      | 6.8968D+01      | 6.9315D+01      |
| 0.02  | 4.9498D+01      | 5.0000D+01      | 3.4310D+01      | 3.4657D+01      |
| 0.03  | 3.2831D+01      | 3.3333D+01      | 2.2757D+01      | 2.3105D+01      |
| 0.04  | 2.4497D+01      | 2.5000D+01      | 1.6980D+01      | 1.7329D+01      |
| 0.05  | 1.9496D+01      | 2.0000D+01      | 1.3513D+01      | 1.3863D+01      |
| 0.06  | 1.6162D+01      | 1.6667D+01      | 1.1202D+01      | 1.1552D+01      |
| 0.07  | 1.3780D+01      | 1.4286D+01      | 9.5513D+00      | 9.9021D+00      |
| 0.08  | 1.1993D+01      | 1.2500D+01      | 8.3130D+00      | 8.6643D+00      |
| 0.09  | 1.0603D+01      | 1.1111D+01      | 7.3496D+00      | 7.7016D+00      |
| 0.10  | 9.4912D+00      | 1.0000D+01      | 6.5788D+00      | 6.9315D+00      |
| 0.15  | 6.1531D+00      | 6.6667D+00      | 4.2650D+00      | 4.6210D+00      |
| 0.20  | 4.814D+00       | 5.0000D+00      | 3.1063D+00      | 3.4657D+00      |
| 0.30  | 2.8037D+00      | 3.3333D+00      | 1.9434D+00      | 2.3105D+00      |
| 0.40  | 1.9576D+00      | 2.5000D+00      | 1.3569D+00      | 1.7329D+00      |
| 0.50  | 1.4427D+00      | 2.0000D+00      | 1.0000D+00      | 1.3863D+00      |
| 0.60  | 1.0914D+00      | 1.6667D+00      | 7.5647D−01      | 1.1552D+00      |
| 0.70  | 8.3058D−01      | 1.4286D+00      | 5.7572D−01      | 9.9021D−01      |
| 0.80  | 6.2133D−01      | 1.2500D+00      | 4.3068D−01      | 8.6643D−01      |
| 0.90  | 4.3429D−01      | 1.1111D+00      | 3.0103D−01      | 7.7016D−01      |
| 0.99  | 2.1715D−01      | 1.0101D+00      | 1.5052D−01      | 7.0015D−01      |
| 1.00  | 0.0000D−01      | 1.0000D+00      | 0.0000D−01      | 6.9315D−01      |
where \( \exp_a(x) = a^x \) and \( \exp_e(x) = \exp(x) = e^x \) according to the standard notation.

Then \( p \) depends on the ratio, \( \Delta t/\tau \) or \( \Delta t/t_{1/2} \): for instance, \( p = 9.995 \cdot 10^{-4} \) relates to \( \Delta t/t_{1/2} = 10^{-3} \), regardless the sample of radioactive nuclides is made of, say, neutrons \(^{0}\text{n}, t_{1/2} = 10.183\text{m} = 1.9374 \cdot 10^{-5}\text{y} \) [5], or cobalt \(^{60}\text{Co}, t_{1/2} = 5.274\text{y} \) [5], or uranium \(^{238}\text{U}, t_{1/2} = 4.468 \cdot 10^9\text{y} \) [5], implying different time steps, \( \Delta t \). In other words, radioactive nuclides with equal \( \Delta t/\tau \) exhibit same \( p = p(\Delta t) \).

For extensive results on nuclide mean half-life times and related theory, an interested reader is addressed to current data collections e.g., [7] [9] [6] and specific textbooks e.g., [4] [1], respectively.

3.6 Passed exam

An academic course is a special case of evolving statistical system, where \( S_1 \) is a single student, \( A_1 \) is trying an exam, \( E_1 \) is a student who has passed an exam, \( \neg E_1 \) is a student who unsucceeded in passing an exam, and \( p = p(\Delta t) \) is the probability of passing an exam within a time step, \( \Delta t \). Clearly an academic course is largely less populated than a sample of radioactive nuclides, which implies considerable fluctuations with respect to the expected evolution, as shown in Fig. 2 for \( n_0 = 10 \). Nevertheless, the mean lifetime and the mean half-life time of an academic course can be defined as in the case of radioactive decay.

4 Application to academic courses

Available data concern experimentation-of-physic (EOP) courses performed every academic year (AY) within the range, 1979/80-1998/99, with two additional occurrences related to 2005/06 and 20013/14, respectively. Different periods, namely 1979/80-1992/93, 1993/94-1998/99, and 2005/06 + 2013/14, obey different guidelines.

Students are admitted to exam after a selection, where a negative response implied attendance at course on the next AY. Students selected as suitable for exam are considered for random evolution of their course. Related numbers, \( n_0 \), for each AY, are listed in Table 2 together with additional data: for complete explanation, an interested reader is addressed to Appendix A.

Empirical random evolutions related to data collections are plotted in Figs. 3 and 4 for AY within the range, 1979/80-1992/93 and 1993/94-1998/99 + 2005/06 + 2013/14, respectively. Also shown therein are expected evolutions related to \( p = 10^{-k_i}, 1 \leq i \leq 9, 1 \leq k \leq 3 \), where curves from up
Table 2: With regard to a selected experimentation-of-physics (EOP) course, the following number of graduate students are listed per academic year: first registration, $N_I$; additional registration: unsuitable for exam, $N_R$, suitable for exam, $N_R^*$, total, $N_I + N_R + N_R^* = N_T$; suitable for exam at the end of course: first registration, $N_IP$, additional registration, $N_RP$, total, $N_IP + N_RP = N_TP$; suitable for exam at the end of course but transferred elsewhere: first registration, $N_IF$, additional registration, $N_RF$, total, $N_IF + N_RF = N_TP$; unsuitable for exam at the end of course: first registration, $N_IN$, additional registration, $N_RN$, total, $N_IN + N_RN = N_TN$; inferred suitable impostors, $N_SI$. By definition, $N_T = N_TP + N_TF + N_TN$. Blank boxes correspond to lack of data. To save space, academic years are labelled by the last two digits and number columns by related subscripts in small letters. Courses obeying different guidelines are subgrouped by horizontal lines. See text for further details.

| a.year | i  | r  | r' | t  | ip | rp | tp | if | rf | tf | in | rn | tn | si |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 79/80  | 38 | 38 | 30 | 30 | 0  | 0  | 8  | 8  | 0  | 22 |     |     |    |    |
| 80/81  | 23 | 24 | 12 | 13 | 2  | 2  | 9  | 9  | 0  | 4  |     |     |    |    |
| 81/82  | 31 | 33 | 23 | 23 | 1  | 1  | 7  | 2  | 9  | 7  |     |     |    |    |
| 82/83  | 26 | 28 | 20 | 22 | 0  | 0  | 6  | 0  | 6  | 9  |     |     |    |    |
| 83/84  | 35 | 38 | 34 | 37 | 0  | 0  | 1  | 0  | 1  | 23 |     |     |    |    |
| 84/85  | 38 | 41 | 30 | 32 | 0  | 0  | 8  | 1  | 9  | 13 |     |     |    |    |
| 85/86  | 46 | 47 | 45 | 46 | 1  | 0  | 1  | 0  | 0  | 27 |     |     |    |    |
| 86/87  | 41 | 46 | 35 | 40 | 0  | 1  | 5  | 0  | 5  | 16 |     |     |    |    |
| 87/88  | 40 | 45 | 33 | 38 | 0  | 0  | 7  | 0  | 7  | 23 |     |     |    |    |
| 88/89  | 48 | 55 | 44 | 51 | 0  | 0  | 4  | 0  | 4  | 32 |     |     |    |    |
| 89/90  | 56 | 57 | 45 | 46 | 0  | 0  | 11 | 0  | 11 | 28 |     |     |    |    |
| 90/91  | 50 | 57 | 42 | 49 | 0  | 0  | 8  | 0  | 8  | 29 |     |     |    |    |
| 91/92  | 38 | 41 | 32 | 35 | 0  | 0  | 6  | 0  | 6  | 16 |     |     |    |    |
| 92/93  | 44 | 45 | 30 | 31 | 0  | 0  | 14 | 0  | 14 | 18 |     |     |    |    |
| 93/94  | 93 | 102| 79 | 88 | 0  | 0  | 14 | 0  | 14 | 51 |     |     |    |    |
| 94/95  | 64 | 88 | 39 | 41 | 0  | 0  | 25 | 0  | 25 | 25 |     |     |    |    |
| 95/96  | 71 | 112| 30 | 12 | 42 | 0  | 0  | 41 | 0  | 41 | 29 |     |     |    |    |
| 96/97  | 20 | 48 | 15 | 15 | 30 | 0  | 0  | 5  | 0  | 5  | 20 |     |     |    |    |
| 97/98  | 64 | 71 | 53 | 55 | 0  | 0  | 11 | 0  | 11 | 49 |     |     |    |    |
| 98/99  | 56 | 66 | 48 | 50 | 0  | 0  | 8  | 1  | 9  | 47 |     |     |    |    |
| 05/06  | 37 | 37 | 36 | 36 | 0  | 0  | 1  | 1  | 9  |    |     |     |    |    |
| 13/14  | 46 | 46 | 39 | 39 | 0  | 0  | 7  | 7  | 17 |    |     |     |    |    |
to down correspond to increasing $p$. An inspection of Figs. 3 and 4 shows empirical random evolutions are affected by considerable statistical fluctuations, as expected from the low population of related samples. In addition, empirical random evolutions never exceed a threshold above zero, which is at odds with an assumed absence of students who transferred elsewhere or abandoned university, or “suitable impostors” (SI), $N_{SI} = 0$. For further details, an interested reader is addressed to Appendix A.

Taking an upper value, $N_{SI} = N_{TP}(t_L)$, where $t_L$ is the ending time related to data collection, yields empirical random evolutions plotted in Figs. 5 and 6, respectively, while expected evolutions remain unchanged. An inspection of Figs. 5 and 6 shows empirical random evolutions decline to a comparable extent with respect to expected evolutions. Statistical fluctuations look similar to their counterparts exhibited by model random evolutions inferred from a sequence of random numbers in connection with low-population samples, as depicted in Fig. 2.

Empirical random evolutions, shown collectively in Figs. 5 and 6, are plotted separately in Figs. 7 and 8, respectively, where related AY is labelled on each panel. An inspection of Figs. 7 and 8 shows empirical random evolutions fit to expected evolutions to a different extent, ranging between close agreement e.g., AY 1984/85 and marked disagreement e.g., AY 1990/91. The bottom right panel in Figs. 7 and 8 relates to Figs. 5 and 6 respectively, where SI number has been assumed equal to zero, and is placed for comparison.

The effect of statistical fluctuations is outlined in Figs. 9 and 10, where empirical random evolutions are as in Figs. 3 (squares) - 5 (diamonds) and in Figs. 4 (squares) - 6 (diamonds), respectively, with the addition of a single expected evolution (curve) related to an assigned probability, $p$, and a model random evolution (triangles) inferred from a sequence of random numbers, as in Fig. 2. An inspection of Figs. 9 and 10 discloses empirical random evolutions are consistent with related expected evolutions, in that statistical fluctuations appear to be of comparable order with respect to their counterparts exhibited by model random evolutions.

For data collections spanning over sufficiently large time intervals e.g., AY 1979/80-1988/89, expected evolutions are closer to empirical random evolutions where an upper limit to SI number is assumed; for sufficiently short time intervals e.g., AY 1993/94-1998/99, expected evolutions are closer to empirical random evolutions where a lower limit (equal to zero) to SI number is assumed; for intermediate time intervals e.g., AY 1989/90-1992/93, expected evolutions are close to empirical random evolutions regardless of assumed SI number; for the shortest time intervals e.g., AY 2005/06 and 2013/14, expected evolutions are closer again to empirical random evolutions where an upper limit to SI number is assumed.
In conclusion, empirical random distributions, related to EOP courses under consideration, can safely be described via the binomial distribution, as outlined in Section 3, where probabilities lie within the range, $0.003 \leq p \leq 0.200$, with regard to a time step, $\Delta t = (1/12)y$.

5 Discussion

Evolving statistical systems, where only a possible event and the opposite event are involved, have a wide range of applications in spite of their intrinsic simplicity, in particular radioactive decay for high-population samples ($N \gg 10$) and academic courses for low-population samples ($N \approx 10$).

The expected evolution of the fractional number of surviving units, $n_t^*/n_0$, is expressed by Eq. (23) in the limit, $p \ll 1$, if the mean lifetime is expressed as $\tau = \Delta t/p$, where $p$ is the probability of the possible event during a time step, $\Delta t = t_k - t_{k-1}$. On the other hand, Eq. (23) is exact provided the mean lifetime is defined as $\tau = -\Delta t/\ln(1-p)$. It is worth emphasizing $p$ depends on the ratio, $\tau/\Delta t$, as shown in Table 1 or in other words expected statistical evolution can be scaled to the ratio, $\tau/\Delta t$.

Concerning the application to passed exams, the key factors for interpreting academic courses as evolving statistical systems, where $p$ is the probability of passing an exam within a time step, $\Delta t$, are essentially two, namely estimation of SI number, $N_{SI}$, and role of statistical fluctuations.

A lower limit to $N_{SI}$ is clearly zero, while an upper limit can be inferred from a flat tail shown by empirical random evolution, $n_t/n_0$, for sufficiently long times. The true evolution lies between the above mentioned extreme cases. SI take origin from several independent occurrences, such as incomplete knowledge on students who decided to transfer or dismiss or suspend academic studies, lack of data on passed exams, and so on.

An upper limit to $N_{SI}$ can safely be assumed for data collections spanning over a wide time range, while a lower limit could be preferred in connection with narrower intervals, as depicted in Figs. 9 and 10. Displacements of empirical random evolutions from related expected evolutions are comparable to their counterparts related to model random evolutions inferred from sequences of random numbers, as shown in Figs. 9 and 10.

In addition, empirical random evolutions are weakly dependent on course guidelines, in the sense that discrepancies between empirical random evolutions related to EOP courses following equal guidelines (AY 1979/80-1992/93; 1993/94-1998/99; 2005/2006 + 2013/14) are comparable to their counterparts related to EOP courses following different guidelines, as depicted in Figs. 9 and 10. Expected evolutions plotted therein are arbitrarily selected.
regardless of fitting procedures, aiming to show they mimic related empirical random evolutions to an acceptable extent.

Accordingly, the probability of passing a EOP exam within a time step, $\Delta t = (1/12)y$, may safely be assumed as time independent. Inferred mean lifetimes, $\tau$, and half-life times, $t_{1/2}$, are listed in Table S where larger values relate to data collections spanning over a short time interval, with the exception of AY 2005/06 and 2013/14 which were subjected to different guidelines. Then long mean lifetimes and half-life times are probably overestimated, due to incomplete data collections.

6 Conclusion

Statistical systems have been conceived from the standpoint of statistical mechanics, as made of a (generally large) number of identical units and exhibiting a (generally large) number of different configurations (microstates), among which only equivalence classes (macrostates) are accessible to observations.

Further attention has been devoted to evolving statistical systems and a simple attempt, involving only a possible event and the opposite event, is examined in detail. In particular, the expected evolution has been determined and compared to the random evolution inferred from a sequence of random numbers, for different sample populations.

The special case of radioactive decay has been considered and results have been expressed in terms of fractional time, $t/\Delta t$, where the time step, $\Delta t$, is related to the decay probability, $p = p(\Delta t)$.

An application to data collection related to experimentation-of-physics (EOP) courses per academic year (AY) has shown related empirical random evolution of the fractional number of surviving units, $n(t)/n_0$, could be biased by lack of data on students who, for some reason, dismissed the course, defined as suitable impostors (SI). The extreme cases, related to a null and an inferred upper limit to SI number, $N_{SI}$, have been considered. A comparison has been performed with expected evolutions and model random evolutions inferred from sequences of random numbers, for different values of the probability, $p$, of passing a EOP exam. The main results are listed below.

(1) At least one among the empirical random evolutions, related to a null and an inferred upper limit to $N_{SI}$, is fitted by an appropriate expected evolution to an acceptable extent.

(2) For data collections spanning over a sufficiently wide time range, a closer
Table 3: Mean lifetime, $\tau$, and mean half-life time, $t_{1/2}$, of experimentation-of-physics (EOP) courses per academic year (AY), inferred from the probability, $p = p(\Delta t)$, $\Delta t = (1/12)y$, related to expected evolutions fitting to empirical random evolutions plotted in Figs. 9-10. To save space, AYs are labelled by the last two digits. Courses performed following different guidelines are subgrouped by horizontal lines.

| a.year | $p$  | $\tau/y$      | $t_{1/2}/y$     |
|--------|------|----------------|-----------------|
| 79/80  | 0.09 | 8.83604D−1     | 6.12468D−1      |
| 80/81  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 81/82  | 0.02 | 4.12486D+0     | 2.85913D+0      |
| 82/83  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 83/84  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 84/85  | 0.09 | 8.83604D−1     | 6.12468D−1      |
| 85/86  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 86/87  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 87/88  | 0.05 | 1.62464D+0     | 1.12612D+0      |
| 88/89  | 0.02 | 4.12486D+0     | 2.85913D+0      |
| 89/90  | 0.004| 2.07916D+1     | 1.44117D+1      |
| 90/91  | 0.005| 1.66250D+1     | 1.15235D+1      |
| 91/92  | 0.03 | 2.73590D+0     | 1.89638D+0      |
| 92/93  | 0.007| 1.18630D+1     | 8.22284D+0      |
| 93/94  | 0.009| 9.21753D+0     | 6.38910D+0      |
| 94/95  | 0.01 | 8.29160D+0     | 5.74730D+0      |
| 95/96  | 0.007| 1.18630D+1     | 8.22284D+0      |
| 96/97  | 0.02 | 4.12486D+0     | 2.85913D+0      |
| 97/98  | 0.003| 2.77361D+1     | 1.92252D+1      |
| 98/99  | 0.003| 2.77361D+1     | 1.92252D+1      |
| 99/00  |     |                |                 |
| 00/01  |     |                |                 |
| 01/02  |     |                |                 |
| 02/03  |     |                |                 |
| 03/04  |     |                |                 |
| 04/05  |     |                |                 |
| 05/06  | 0.1 | 7.90935D−1     | 5.48234D−1      |
| 06/07  |     |                |                 |
| 07/08  |     |                |                 |
| 08/09  |     |                |                 |
| 09/10  |     |                |                 |
| 10/11  |     |                |                 |
| 11/12  |     |                |                 |
| 12/13  |     |                |                 |
| 13/14  | 0.2 | 3.73452D−1     | 2.58857D−1      |
agreement between empirical random evolution and expected evolution takes place assuming the upper limit to SI number, \( N_{SI} = N_{TP}(t_L) \).

3. For data collections spanning over a sufficiently narrow time range, a closer agreement between empirical random evolution and expected evolution takes place assuming the lower limit to SI number, \( N_{SI} = 0 \). An exception arises from the shortest intervals (AY 2005/06 and 2013/14), possibly due to large lack of data.

4. For data collections spanning over an intermediate time range, the agreement between empirical random evolution and expected evolution is weakly dependent on the assumed \( N_{SI} \).

5. Empirical random evolutions exhibit weak dependence on course guidelines, in the sense that discrepancies from AY obeying equal guidelines are comparable to discrepancies from AY obeying different guidelines.

6. Statistical fluctuations exhibited by empirical and model random evolution, with respect to related expected evolution, are of comparable order.

7. Inferred values of the probability, \( p \), related to the time step, \( \Delta t = (1/12)y \), lie within the range, \( 0.003 \leq p \leq 0.200 \).

In conclusion, the evolving statistical system made of an academic course is similar to a poorly populated sample of radioactive nuclides exhibiting equal values of probability, \( p \), and time step, \( \Delta t \).

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Appendix

A Data collections and input parameters

Data collections relate to 14 experimentation-of-physics (EOP) academic courses per academic year (AY). Owing to different guidelines implying different materials and methods, the sample can be divided into three subsamples, namely (a) 1979/80-1992/93, (b) 1993/94-1998/99, and (c) 2005/06 + 2013/14. More specifically, EOP courses are structured in the following way. Lessons and experimentations range along (a) two consecutive AY; (b) one AY; (c) one half AY. In any case, selection is made on attending experimentations: students exhibiting two absences or less are suitable for exam, while more than two absences implies students are unsuitable for exam.

EOP courses can be attended by new students (first registration), in number of \(N_I\), and students who already attended (additional registration), in the last case either unsuitable or suitable for exam, in number of \(N_R\) and \(N_R^*\), respectively, for a total of \(N_T = N_I + N_R + N_R^*\).

At the end of the course, students suitable for exam are in number of \(N_{IP}\) among initial \(N_I\) and \(N_{RP}\) among initial \(N_R\), for a total of \(N_{TP} = N_{IP} + N_{RP}\). Students among initial \(N_R^*\) are already suitable for exam and then are not considered to this respect. Suitable students among initial \(N_I\) and \(N_R\), who decided to transfer elsewhere or to abandon university, are counted apart as \(N_{IF}\) and \(N_{RF}\), respectively, for a total of \(N_{TF} = N_{IF} + N_{RF}\). Finally,
students unsuitable for exam are in number of \(N_{IN}\) among initial \(N_I\) and \(N_{RN}\) among initial \(N_R\), for a total of \(N_{TN} = N_{IN} + N_{RN}\).

Accordingly, the evolving statistical system made of a selected EOP course has an initial number of units, \(n_0 = N_{TP}\), which has to be considered as an upper limit due to lack of information about students who decided to transfer elsewhere or to abandon university. Then the above mentioned “suitable impostors” (SI), in number of \(N_{SI}\), should be subtracted from suitable students, yielding \(n_0 = N_{TP} - N_{SI}\), \(0 \leq N_{SI} \leq N_{TR}(t_L)\), where \(t_L\) is the time at the end of data collection. For \((t_L - t_0)/\Delta t \gg 1\) and \(N_{TP}(t_L) \gg 1\), \(N_{SI}\) may safely be assumed equal to \(N_{TP}(t_L)\).

In conclusion, with regard to an evolving statistical system made of an academic course, the fractional number of surviving units lies between the extreme cases:

\[
\frac{n(t)}{n_0} = \frac{N_{TP}(t)}{N_{TP}(0)} \ ; \ n_0 = N_{TP}(0) \ ; \\
\frac{n(t)}{n_0} = \frac{N_{TP}(t) - N_{TP}(t_L)}{N_{TP}(0) - N_{TP}(t_L)} \ ; \ n_0 = N_{TP}(0) - N_{TP}(t_L) \ ;
\]

(27)

(28)

where the total number of suitable (for exam) students at the end of the course, \(N_{TP} = N_{TP}(0)\), and the upper limit of SI, \(N_{SI} = N_{TP}(t_L)\), are listed in Table 2. The time step is fixed as \(\Delta t = (1/12)y\), in the sense that students were allowed to try exam every month.

In addition to the parameters shown in Table 2, data collections include evolution of surviving units, \(n(t) = N_{TP}(t)\), which can be visualized in Figs. 3 and 4 regardless of the vertical scale. Related tables exhibit about one hundred lines per AY, and for this reason are not presented here.
Figure 1: Expected evolution of fractional number of surviving units, $n^*(t)/n_0$, related to both exact power law (full curves) and exponential approximation (dotted curves) for different values of probability, $p = p(\Delta t) = (i - \delta_{i,10}/10)/10$, $1 \leq i \leq 10$. The limit, $p = 0$, corresponds to the top side of the box. The limit, $p = 1$, corresponds to the left side of the box. The fractional time, $(t - t_0)/\Delta t$, corresponds to the number of steps, $\ell$. The changes, $(t - t_0) \to k(t - t_0)$, $\Delta t \to k\Delta t$, $k > 0$, imply $p(\Delta t) \to p(k\Delta t)$ for any point. The mean lifetime, $\tau$, and the mean half-life time, $t_{1/2}$, can be inferred from the intersection of the selected curve with the horizontal line, $n^*(t)/n_0 = 1/e \approx 0.3679$ and $n^*(t)/n_0 = 1/2$, respectively. See text for further details.
Figure 2: Example of model random evolution of fractional number of surviving units, $n(t)/n_0$, for $\log n_0 = 1, 2, 3, 4$, listed on each panel, and $p = 0.01$ (diamonds), 0.1 (crosses), 0.9 (saltires). Related expected evolution is shown by full lines. The fractional time, $(t_\ell - t_0)/\Delta t$, corresponds to the number of steps, $\ell$. See text for further details.
Figure 3: Empirical random evolution of the fractional number of surviving units, \( n(t)/n_0 = N_{TP}(t)/N_{TP}(0) \), inferred from data collections related to experimentation-of-physics (EOP) courses per selected academic year (AY), as listed in Table 2. The number of suitable impostors is underestimated as \( N_{SI} = 0 \). The time step is \( \Delta t = (1/12) y \). Symbol caption per AY: 1979/80 - crosses; 1980/81 - diamonds; 1981/82 - triangles; 1982/83 - squares; 1983/84 - saltires; 1984/85 - asterisks; 1985/86 - crosses & squares; 1986/87 - crosses & triangles; 1987/88 - crosses & diamonds; 1988/89 - saltires & squares; 1989/90 - saltires & triangles; 1990/91 - saltires & diamonds; 1991/92 - asterisks & squares; 1992/93 - asterisks & diamonds; where “&” has to be read as “superimposed to”. Curves represent expected evolution related to different probabilities, \( p = \exp_{10}(-k)i \), \( 1 \leq i \leq 9 \), \( 1 \leq k \leq 3 \), where larger values denote lower curves and vice versa. See text for further details.
Figure 4: Empirical random evolution of the fractional number of surviving units, $n(t)/n_0 = N_{TP}(t)/N_{TP}(0)$, inferred from data collections related to experimentation-of-physics (EOP) courses per selected academic year (AY), as listed in Table 2. The number of suitable impostors is underestimated as $N_{SI} = 0$. The time step is $\Delta t = (1/12)y$. Symbol caption per AY: 1993/94 - crosses; 1994/95 - diamonds; 1995/96 - triangles; 1996/97 - squares; 1997/98 - saltires; 1998/99 - asterisks; 2005/06 - crosses & squares; 2013/14 - crosses & diamonds; where “&” has to be read as “superimposed to”. Curves represent expected evolution related to different probabilities, $p = \exp_{10}(-k)i$, $1 \leq i \leq 9$, $1 \leq k \leq 3$, where larger values denote lower curves and vice versa. See text for further details.
Figure 5: Random evolution of the fractional number of surviving units, $n(t)/n_0 = \left[ N_{TP}(t) - N_{TP}(t_L) \right] / \left[ N_{TP}(0) - N_{TP}(t_L) \right]$, inferred from data collections related to experimentation-of-physics (EOP) courses per selected academic year (AY), as listed in Table 2. The number of suitable impostors is overestimated as $N_{SI} = N_{TP}(t_L)$. The time step is $\Delta t = (1/12)y$. Symbol caption and curves as in Fig. 3. See text for further details.
Figure 6: Empirical random evolution of the fractional number of surviving units, \( n(t)/n_0 = \frac{[N_{TP}(t) - N_{TP}(t_L)]/[N_{TP}(0) - N_{TP}(t_L)]}{N_{TP}(0) - N_{TP}(t_L)} \), inferred from data collections related to experimentation-of-physics (EOP) courses per selected academic year (AY), as listed in Table 2. The number of suitable impostors is overestimated as \( N_{SI} = N_{TP}(t_L) \). The time step is \( \Delta t = (1/12) y \). Symbol caption and curves as in Fig. 4. See text for further details.
Figure 7: Empirical random evolution shown in Fig. 5 plotted on different panels for different academic years (top labels). Curves as in Fig. 5. The bottom right panel, related to Fig. 3, is placed for comparison. See text for further details.
Figure 8: Empirical random evolution shown in Fig. 6 plotted on different panels for different academic years (top labels). Curves as in Fig. 6. The bottom right panel, related to Fig. 4, is placed for comparison. See text for further details.
Figure 9: Empirical random evolution shown in Fig.3 (squares) and in Fig.5 (diamonds), plotted on different panels for different academic years (top labels); expected evolution (curve) for a selected value of the probability, $p$ (bottom labels); related model random evolution (triangles) inferred from a sequence of random numbers. The bottom right panel is a repetition of related empirical random evolution, but with different expected and model random evolution plotted therein. See text for further details.
Figure 10: Empirical random evolution shown in Fig. 4 (squares) and in Fig. 6 (diamonds), plotted on different panels for different academic years (top labels); expected evolution (curve) for a selected value of the probability, $p$ (bottom labels); related model random evolution (triangles) inferred from a sequence of random numbers. The bottom right panel is a repetition of related empirical random evolution, but with different expected and model random evolution plotted therein. See text for further details.