Lectures on Designing Screening Experiments

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Preface

Designing Screening Experiments (DSE) is a class of information-theoretical models for multiple-access channels (MAC). In Sect. 2-4, we discuss the combinatorial model of DSE called a disjunct channel model. This model is the most important for applications and closely connected with the superimposed code concept. In Sect. 2, we give a detailed survey of lower and upper bounds on the rate of superimposed codes. The best known constructions of superimposed codes are considered in Sect. 3-4, where we also discuss the development of these codes (non-adaptive pooling designs) intended for the clone-library screening problem. In Sect. 5, we obtain lower and upper bounds on the rate of binary codes for the combinatorial model of DSE called an adder channel model. In Sect. 6, we consider the concept of universal decoding for the probabilistic DSE model called a symmetric model of DSE.

Contents

1 Introduction (p. 1).
   1.1 Statement of problem (p. 1).
   1.2 List of notations (p. 3).
2 Disjunct channel model and bounds on the rate of superimposed codes (p. 4).
   2.1 Notations, definitions of superimposed codes and their properties (p. 4).
   2.2 Upper and lower bounds on $R(s, L)$ and $R_D(s)$ (p. 6).
      2.2.1 Lower bounds on $R(s, 1)$ and $R_D(s)$ (p. 6).
      2.2.2 Upper bounds on $R(s, L)$ and $R_D(s)$ (p. 7).
      2.2.3 Lower bound on $R(s, L)$ (p. 8).
   2.3 Kautz-Singleton codes (p. 13).
      2.3.1 Upper bound on $R_{KS}(s)$ (p. 14).
      2.3.2 Lower bound on $R_{KS}(s)$ (p. 14).
   2.4 Symmetrical superimposed codes (p. 16).
3 Constructions of superimposed codes (p. 17).

3.1 Notations and definitions (p. 17).
3.2 Application to DNA library screening (p. 17).
3.3 Generalized Kautz-Singleton codes (p. 18).
3.4 Superimposed concatenated codes (p. 21).

4 Optimal superimposed codes and designs for Renyi search model (p. 26).

4.1 Notations and definitions (p. 26).
4.2 Lower bound (p. 26).
4.3 Optimal parameters (p. 27).
4.4 Homogeneous q-ary codes (p. 29).
4.5 Proof of Theorem 1 (p. 31).
4.6 Proof of Theorem 2 (p. 32).
4.7 Proof of Theorem 3 (p. 34).
4.8 On \((q,k,3)\)-homogeneous 3-separable and 3-hash codes. Proof of Theorem 4 (p. 35).

4.8.1 Characteristic matrices (p. 35).
4.8.2 Examples of hash, separable and hash&separable codes (p. 36).
4.8.3 Existence of hash and hash&separable codes (p. 37).
4.8.4 Product of characteristic matrices (p. 38).
4.9 Proof of Theorem 5 (p. 40).

5 Adder channel model and \(B_s\)-codes (p. 42).

5.1 Statement of the problem and results (p. 42).

5.1.1 Upper bounds (p. 42).
5.1.2 Lower bounds (p. 43).

5.2 Proof of upper bounds on \(R_A(s)\) and \(R(s)\) (p. 44).

5.2.1 Proof of Theorem 1 (p. 44).
5.2.2 Proof of inequality \(R(2) \leq 3/5\) (p. 46).

5.3 Proof of Theorem 3 (p. 48).

6 Universal decoding for random design of screening experiments (p. 51).

6.1 Statement of the problem, formulation and discussion of results (p. 51).
6.2 Proof of Theorem 1 (p. 57).

References (p. 62).
1 Introduction

1.1 Statement of problem

Let $1 \leq s < t$ be fixed integers, $[t]$ be the set of integers from 1 to $t$. Let $e \triangleq (e_1, e_2, \ldots, e_s)$, where $e_i \in [t]$, $1 \leq e_1 < e_2 < \ldots < e_s \leq t$, be an arbitrary $s$-subset of $[t]$ and, here and below, the symbol $\triangleq$ denote the equation by definition. Introduce $\mathcal{E}(s, t)$ as the collection of all such subsets. Note that the cardinality (number of elements)

$$|\mathcal{E}(s, t)| = \binom{t}{s} = \frac{t!}{s!(t-s)!}.$$ 

Suppose that among $t$ factors, numbered by integers from 1 to $t$, there are some $s < t$ unknown factors called significant factors. Each collection of significant factors is identified as an $s$-subset $e \in \mathcal{E}(s, t)$. The problem of screening experiment design (DSE) is to find all significant factors, i.e. to detect an unknown subset $e$.

To search $e$ one can carry out $N$ experiments. Each experiment is a test of a subset of $[t]$. These tests could be described by a binary $N \times t$-matrix $X = \|x_i(u)\|$, $x_i(u) \in \{0; 1\}$, $i = 1, 2, \ldots, N$, $u = 1, 2, \ldots, t$, where

$$x_i(u) \triangleq \begin{cases} 1, & \text{if the $u$-th factor is included into the } i \text{-th test,} \\ 0, & \text{otherwise.} \end{cases}$$

Matrix $X$ is called a code (design of experiments) of length $N$ and size $t$.

Fix an arbitrary $i$, $i = 1, 2, \ldots, N$. Let $x_i \triangleq (x_i(1), x_i(2), \ldots, x_i(t))$ be the $i$-th row of $X$. The $i$-th row $x_i$ is identified with a subset of $[t]$, which consists of positions where this row contains 1’s. We say that the $i$-th experiment is a group test of this subset.

Let code $X$ be fixed and the symbols

$$x(u) \triangleq (x_1(u), x_2(u), \ldots, x_N(u)) \in \{0; 1\}^N, \quad u = 1, 2, \ldots, t,$$

denote the columns (codewords) of code $X$. For the given $s$-subset $e = (e_1, e_2, \ldots, e_s)$ called a message, consider a non-ordered $s$-collection of codewords

$$x(e) \triangleq (x(e_1), x(e_2), \ldots, x(e_s)).$$

We say that $x(e)$ encodes $e$. Let

$$x_i(e) \triangleq (x_i(e_1), x_i(e_2), \ldots, x_i(e_s)) \in \{0; 1\}^s, \quad i = 1, 2, \ldots, N,$$

be the $i$-th row of $s$-collection $x(e)$.

Let $z_i$ be an output (or result) of the $i$-th test and $z = z(e, X) \triangleq (z_1, z_2, \ldots, z_N)$. To describe the model of such a test output, we use the terminology of a memoryless multiple-access channel (MAC), which has $s$ inputs and one output $[20]$. Let all $s$ input alphabets of MAC will be the same and coincide with $\{0, 1\}$. Denote by $Z$ a finite output alphabet. In Sections 2-5 we will consider the deterministic model of MAC. This MAC is defined by the function

$$z = f(x_1, x_2, \ldots, x_s), \quad z \in Z, \quad x_k \in \{0, 1\}, \quad k = 1, 2, \ldots, s,$$

and, by definition, the result $z_i$ of the $i$-th test is

$$z_i \triangleq f(x_i(e)) = f(x_i(e_1), x_i(e_2), \ldots, x_i(e_s)), \quad i = 1, 2, \ldots, N.$$

\(^1\)This terminology was suggested by I. Csiszar in 1978.
The corresponding deterministic model of DSE is called an \( f \)-model. The problem of DSE for the probabilistic model of MAC will be discussed in Section 6.

On the basis of \( z(e, X) \) an observer makes a decision about the unknown \( s \)-subset \( e \). To identify \( e \) on the basis of \( z(e, X) \), a code \( X \) is assigned the following definition.

**Definition.** We say that a code \( X \) is an \((s, N)\)-design of size \( t \) (or an \((s, t)\)-design of length \( N \)) for the \( f \)-model, if all \( z(e, X) \), \( e \in \mathcal{E}(s, t) \) are distinct.

Let \( t_f(s, N) \) \((N_f(s, t))\) be the maximal possible size of \((s, N)\)-design (minimal possible length of \((s, t)\)-design) for \( f \)-model. For fixed \( s \geq 2 \) define the number

\[
R_f(s) \triangleq \lim_{N \to \infty} \frac{\log_2 t_f(s, N)}{N},
\]

which is called a design rate of the \( f \)-model. Using the terminology of the Shannon coding theory, the number \( R_f(s) \) is called a zero error capacity for the \( f \)-model of DSE.

Let \( \sum_{i=1}^{s} x_i \) denote the arithmetic sum, i.e. the number of 1’s in the sequence \( x_1, x_2, \ldots, x_s \). We introduce the following three combinatorial models of DSE:

- A–model (adder channel model), where the output alphabet \( Z = \{0, 1, 2, \ldots, s\} \) and
  \[
f(x_1, x_2, \ldots, x_s) = f_A(x_1, x_2, \ldots, x_s) \triangleq \sum_{i=1}^{s} x_i; \tag{1}
\]

- D–model (disjunct channel model), where the output alphabet \( Z = \{0, 1\} \) and
  \[
f(x_1, x_2, \ldots, x_s) = f_D(x_1, x_2, \ldots, x_s) \triangleq \begin{cases} 1, & \sum_{i=1}^{s} x_i \neq 0, \\ 0, & \sum_{i=1}^{s} x_i = 0, \end{cases}; \tag{2}
\]

- SD–model (symmetrical disjunct channel model), where the output alphabet \( Z = \{0, 1, *\} \) (the symbol * denotes erasure) and
  \[
f(x_1, x_2, \ldots, x_s) = f_{SD}(x_1, x_2, \ldots, x_s) \triangleq \begin{cases} 1, & \sum_{i=1}^{s} x_i = n, \\ 0, & \sum_{i=1}^{s} x_i = 0, \\ *, & 1 \leq \sum_{i=1}^{s} x_i \leq n - 1. \end{cases} \tag{3}
\]

Let \( t_A(s, N), N_A(s, t), R_A(s) \) be the parameters of A–model, \( t_D(s, N), N_D(s, t), R_D(s) \) be the parameters of D–model and \( t_{SD}(s, N), N_{SD}(s, t), R_{SD}(s) \) be the parameters of SD–model. Obviously, if \( s = 2 \), then the A–model and SD–model are the same. Hence

\[
t_A(2, N) = t_{SD}(2, N), \quad N_A(2, t) = N_{SD}(2, t), \quad R_A(2) = R_{SD}(2).
\]

In Sections 2 and 5, we will discuss the properties of these optimal characteristics. Sections 2–4 focus on the disjunct channel model. This model is the most important for applications and closely connected with the superimposed code concept introduced by Kautz and Singleton [6]. In Section 2, we give a detailed survey of the best lower and upper bounds [25, 31, 41] on the rate of superimposed codes. The best known constructions of superimposed codes [6, 52, 54] are considered in Sections 3–4, where we also discuss the development of these codes (non-adaptive pooling designs) intended for the clone-library screening problem [51, 53]. In Section 5, we obtain lower and upper bounds on the rate of binary codes for the adder channel model. These bounds are based on a refinement of the results published in papers [13, 21] and [23, 30]. In Section 6, we consider the concept of universal decoding [20, 42] for the probabilistic DSE model called a symmetric model of DSE.
1.2 List of notations

- $[t] = \{1, 2, \ldots, t\}$—the set of integers from 1 to $t$;
- $[b]$—the least integer $\geq b$, $[b]$—the largest integer $\leq b$;
- $\triangleq$—equation by definition;
- $\lor$—the symbol of Boolean summation, $|A|$—the number of elements in set $A$;
- $\log$—logarithm to the base 2, $\exp\{a\} = 2^a$, $\ln$—logarithm to the base $e$;
- $X = \|x_i(u)\|, x_i(u) \in \{0; 1\} \quad i = 1, 2, \ldots, N, u = 1, 2, \ldots, t,$—code (design);
- $x(u) = (x_1(u), x_2(u), \ldots, x_N(u)), u = 1, 2, \ldots, t,$—codewords;
- $x_i = (x_i(1), x_i(2), \ldots, x_i(t), i = 1, 2, \ldots, N$—the $i$-th row of code $X$—is identified with the $i$-th group test.
- $N$—the code (design) length, $t$—the code (design) size, $s$—the code (design) strength;
- $(N, R)$-code—the code of length $N$, rate $R$ and size $t = \lfloor \exp\{RN\} \rfloor$;
- for arbitrary integers $s$ and $t$, we introduce the generalized binomial coefficients,
\[
\binom{t}{s} \triangleq \left\{ \begin{array}{ll}
\frac{t!}{s!(t-s)!}, & \text{if } 0 \leq s \leq t, \\
0, & \text{otherwise};
\end{array} \right.
\]
- $\Pr\{E\}$—the probability of event $E$, $p(x) = \Pr\{X = x\}$—the probability distribution of a discrete random variable $X$;
- $H(X) \triangleq -\sum_x p(x) \log p(x)$—the Shannon entropy of $X$;
- $h(u) \triangleq -u \log u - (1 - u) \log(1 - u)$—binary entropy;
2 Disjunct Channel Model and Bounds on the Rate of Superimposed Codes

In this section, we consider the connection between the D–model of DSE and the theory of superimposed codes introduced by Kautz and Singleton [6].

2.1 Notations, definitions of superimposed codes and their properties

Let $1 < s < t$, $1 \leq L \leq t - s$, $N > 1$ be integers, and let $y(j) = (y_1(j), y_2(j), \ldots, y_N(j))$, $j = 1, 2, \ldots, s$ denote the binary columns of length $N$. The Boolean sum

$$y = \bigvee_{j=1}^{s} y(j) = y(1) \lor y(2) \lor \cdots \lor y(s)$$

of columns $y(1), y(2), \ldots, y(s)$ is the binary column $y = (y_1, y_2, \ldots, y_N)$ with components

$$y_i = \begin{cases} 0, &\text{if } y_i(1) = y_i(2) = \cdots = y_i(s) = 0, \\ 1, &\text{otherwise.} \end{cases}$$

Let us say that column $y$ cover column $z$ if $y \lor z = y$.

Let $X = \|x_i(u)\|$, $i = 1, 2, \ldots, N$, $u = 1, 2, \ldots, t$ be a binary $N \times t$–matrix (code). Later on the matrix (code) $X$ is interpreted as a set of $t$ binary columns (codewords) $x(1), x(2), \ldots, x(t)$.

**Definition** [31]. An $N \times t$–matrix $X$ is called a list-decoding superimposed code (LDSC) of length $N$, size $t$, strength $s$, and list-size $\leq L - 1$ if the Boolean sum of any $s$-subset of codewords $X$ can cover not more than $L - 1$ codewords that are not components of the $s$-subset. This code also will be called an $(s, L, N)$-code of size $t$, or an $(s, L, t)$-code of length $N$.

Note, that in the most important particular case $L = 1$, the definition is equivalent to the following condition. The Boolean sum of any $s$-subset of columns $X$ covers those and only those columns that are the components of given Boolean sum. Some generalizations of LDSC were studied in [41, 46].

Superimposed $(s, 1, N)$-codes were introduced in [6]. Applied problems leading to the definition of $(s, 1, N)$-codes and some methods of construction of such codes are described in [6]. New constructions and applications developed in papers [52, 53] will be given in Section 4.

**Remark.** Each column of the matrix $X$ is identified with the subset of $[N]$, which consists of positions where this column contains 1’s. Then using the terminology of sets, the construction of an $(s, L, N)$-code of size $t$ is equivalent to the following combinatorial problem. A family of $t$ subsets of the set $[N]$ should be constructed in which no union of $L$ members of the family is covered by the union of $s$ others.

Let $t(s, L, N)$ be the maximal possible size of LDSC and $N(s, L, t)$ be the minimal possible length of LDSC. For fixed $s$ and $L$, define the rate of LDSC

$$R(s, L) \triangleq \lim_{N \to \infty} \frac{\log t(s, L, N)}{N}.$$  

For the optimal parameters of D–model, we also use the notations of Sect. 1, i.e., $t_D(s, N)$, $N_D(s, t)$ and $R_D(s)$. Propositions 1–3 follow easily from definitions of $(s, N)$-design and LDSC.
Proposition 1.  
\[ N_D(s, t) \geq \left\lceil \log \left( \frac{t}{s} \right) \right\rceil, \quad R_D(s) \leq \frac{1}{s} \]

Proposition 2. Any \( s+1, L, N \)-code is an \( (s, L, N) \)-code, and any \( (s, L, N) \)-code is an \( (s, L+1, N) \)-code. Hence,  
\[ t(s+1, L, N) \leq t(s, L, N) \leq t(s, L+1, N). \]

Proposition 3. For any \( s \)-subset of columns of an \( (s, L, t) \)-code, there are not more than \( \binom{s+L-1}{s} \) \( s \)-subsets of columns, such that the Boolean sums of columns of these \( s \)-subsets coincide with the Boolean sum of columns of a given \( s \)-subset. Hence,  
\[ N(s, L, t) \geq \left\lceil \log \left( \frac{t}{s} \right) - \log \left( \frac{s+L-1}{s} \right) \right\rceil, \quad R(s, L) \leq \frac{1}{s}. \]

Let \( t = \binom{N}{[N/2]} \) and \( X \) be an \( N \times t \) matrix whose columns are all distinct and contain the same number \([N/2]\) of 1’s. Then \( X \) is a \( (1, N) \)-design and a \( (1, L, N) \)-code simultaneously. Hence, \( R_D(1) = R(1, L) = 1, \ L = 1, 2, \ldots \).

The following Propositions 4–5 are proved easily by contradiction.

Proposition 4. Any \( (s, 1, N) \)-code is a \( (s, N) \)-design, and any \( (s, N) \)-design is a \( (s-1, 2, N) \)-code, i.e.,  
\[ t(s, 1, N) \leq t_D(s, N) \leq t(s-1, 2, N), \quad R(s, 1) \leq R(s) \leq R(s-1, 2). \]

Proposition 5. The matrix \( X \) simultaneously satisfies the definitions of the \( (s-1, 1, N) \)-code and the \( (s, N) \)-design iff all the Boolean sums composed of not more than \( s \) columns of \( X \) are distinct.

Proposition 5 is very important for the DSE. It allows us to define the rate of the code satisfying the condition of Proposition 5 as \( \min\{R_D(s); R(s-1, 1)\} \).

Proposition 6. [25] (L.A. Bassalygo, 1975)  
\[ N(s, 1, t) \geq \min \left\{ \frac{(s+1)(s+2)}{2}; \ t \right\} \]

and, therefore, \( N(s, 1, t) = t \) if \( s \geq \sqrt{2t} \). In other words, for \( s \geq \sqrt{2t} \), no \( (s, 1, t) \)-code is better than the trivial one of length \( N = t \), whose matrix \( X \) is diagonal.

Proof. Consider an arbitrary \( (s, 1, t) \)-code \( X \). Let \( w_j, j = 1, 2, \ldots, t \), called the weight, denote the number of 1-entries in codeword \( x(j) \) and let \( t(w), w = 1, 2, \ldots, N \), denote the number of codewords of the weight \( w \). Evidently,  
\[ \sum_{w=1}^{N} t(w) = t, \quad 0 \leq t(w) \leq t. \]

Lemma 1 [6]. If \( X \) is an \( (s, 1, t) \)-code of length \( N \), then the number of its codewords of the weight \( \leq s \) does not exceed \( N \), i.e.,  
\[ \sum_{w=1}^{s} t(w) \leq N. \]

Proof of Lemma 1. We fix an arbitrary codeword \( x(j) \) containing \( w \leq s \) 1’s, and consider rows of \( X \) that contain 1’s of \( x(j) \). The definition of \( (s, 1, t) \)-code implies that among these rows there exists at least one row in which all the remaining elements except for the element of codeword \( x(j) \), are zero. To prove this, it is sufficient to note that otherwise code \( X \) would have \( s \) codewords, whose Boolean sum covers \( x(j) \). Using the similar arguments with other codewords of \( X \), whose weight \( w \leq s \), we obtain Lemma 1.
Lemma 2. Let $X$ be a $(s, 1, t)$-code of length $N$. If there exists at least one codeword of weight $w$, i.e., $t(w) > 0$, then

$$w \leq N - N(s - 1, 1, t - 1).$$

Proof of Lemma 2. Consider codeword $x(j)$ of weight $w$. We fix $w$ rows of $X$ containing 1’s positions of $x(j)$, and we delete them from $X$ together with $x(j)$. To obtain Lemma 2, it is sufficient to check that the remaining matrix $\tilde{X}$ will be $(s - 1, 1, t - 1)$-code of length $N - w$. This property of matrix $\tilde{X}$ is easily checked by contradiction.

Lemma 2 is proved.

Let $X$ be a $(s, 1, t)$-code of length $N \leq t - 1$. From Lemma 1 it follows that code $X$ contains at least one codeword of weight $\geq s + 1$, Therefore, Lemma 2 yields the inequality

$$N(s, 1, t) \geq s + 1 + N(s - 1, 1, t - 1).$$

Using the induction on $s$ it follows Proposition 6.

Below, we formulate (without proofs) the generalizations of of Lemma 1 and Proposition 6 for the case of $(s, L, t)$-code. The generalization of Lemma 2 is evident.

Lemma 1'. Let $X$ be a $(s, L, N)$-code of size $t$. If $L \leq s$, then

$$\sum_{w=1}^{\lfloor s/L \rfloor} t(w) \leq N + L - 1.$$

Proposition 6'. For $1 \leq L \leq s < t$,

$$N(s, L, t) \geq \min \left\{ \frac{s(s + 1) - L(L - 1)}{2L}; t - L + 1 \right\}.$$

In the next section, we give a survey of the best known upper and lower bounds on $R(s, L)$ and $R_D(s)$, $s \geq 2$.

2.2 Upper and lower bounds on $R(s, L)$ and $R_D(s)$

2.2.1 Lower bounds on $R(s, 1)$ and $R_D(s)$

Using the random ensemble of constant-weight codes of length $N$, size $t$ and weight $w = \lceil QN \rceil$, $0 < Q < 1$, one can prove the following theorem.

Theorem 1.

$$R(s, 1) \geq R(s, 1) = \frac{A(s)}{s},$$

where

$$A(s) = \max_{0 < u < 1, 0 < Q < 1} \left\{ -(1 - Q) \log(1 - u^s) + s \left( Q \log \frac{u}{Q} + (1 - Q) \log \frac{1 - u}{1 - Q} \right) \right\}.$$

Theorem 1 is the particular case of the result which was proved in [41]. If $s \to \infty$, then

$$\frac{R(s, 1)}{s} = \frac{1}{s^2 \log e} (1 + o(1)) = \frac{0.693}{s^2} (1 + o(1)).$$
The same random ensemble of constant-weight codes also yields (unpublished) a lower bound on 
\( R_D(s) \). We denote this lower bound by \( \tilde{R}_D(s) \). The definition of \( \tilde{R}_D(s) \) (it is omitted here) is similar to the definition of \( \tilde{R}(s, 1) \). If \( s \to \infty \), then the corresponding asymptotic behavior has the following form
\[
\tilde{R}_D(s) = \frac{2}{s^2 \log e}(1 + o(1)) = \frac{1.386}{s^2}(1 + o(1)).
\]

In Table 1, we give some numerical values of \( R(s, 1) \) and \( \tilde{R}_D(s) \).

| \( s \) | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|-----|-----|-----|-----|-----|-----|-----|
| \( R(s, 1) \) | .182 | .079 | .044 | .028 | .019 | .014 | .011 |
| \( \tilde{R}_D(s) \) | .302 | .142 | .082 | .053 | .037 | .027 | .021 |
| \( R(s, 2) \) | .236 | .115 | .068 | .046 | .032 | .024 | .019 |
| \( R(s, 2) \) | 1/2 | 1/3 | 1/4 | 1/5 | .163 | .141 | .117 |

Table 1.

2.2.2 Upper bounds on \( R(s, L) \) and \( R_D(s) \)

We remind that \( h(u) \overset{\Delta}{=} -u \log u - (1 - u) \log(1 - u) \) denotes the binary entropy. For integers \( m = 1, 2, \ldots \), define
\[
f(m, v) \overset{\Delta}{=} h(v/m) - v h(1/m), 0 < v < 1.
\]

Theorem 2 is a generalization of the upper bound on \( R(s, 1) \) from [25].

**Theorem 2.** (Unpublished). For any fixed \( L \geq 2 \), the rate
\[
R(s, L) \leq \mathcal{R}(s, L),
\]
where the sequence \( \mathcal{R}(s, L), s = 2, 3, \ldots \), is defined by recurrence relations:

- if \( s \leq L \), then \( \mathcal{R}(s, L) = 1/s \),
- if \( s \geq L + 1 \), then \( \mathcal{R}(s, L) = \min\{1/s; r(s, L)\} \), and \( r(s, L) \) is the unique solution of the equation
\[
r(s, L) = \max_{(*)} f(\lfloor s/L \rfloor, v),
\]

where the maximum is taken over all \( v \) satisfying
\[
0 < v < 1 - \frac{r(s, L)}{\mathcal{R}(s - 1, L)}
\]

The following properties (Theorems 2.1 and 2.2) of \( \mathcal{R}(s, L) \) take place.

**Theorem 2.1.** For any \( L \geq 2 \), there exists the integer \( s(L) \geq L + 1 \) such that
\[
\mathcal{R}(s, L) = \begin{cases} 
1/s, & \text{if } s < s(L), \\
< 1/s, & \text{if } s \geq s(L),
\end{cases}
\]
and \( s(L) = L \log L(1 + o(1)) \) for \( L \to \infty \).

Therefore, Theorem 2 improves the upper bound of Proposition 3 provided that \( s \geq s(L) \). The computations give \( s(1) = 2, s(2) = 6, s(3) = 12, s(4) = 20, s(5) = 25, s(6) = 36, \ldots \). For \( L = 2 \) and \( s = 6, 7, \ldots, 13 \), the values of \( \mathcal{R}(s, 2) \) are given in Table 2.
Table 2.

| s | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|----|----|----|----|
| $\bar{R}(s,2)$ | .163 | .141 | .117 | .102 | .086 | .076 | .066 | .059 |

For $s = 2, 3, \ldots, 8$, the values of $\bar{R}(s,1)$ and $\bar{R}(s,2)$ are given in Table 1.

**Theorem 2.2.** If $L \geq 2$ is fixed and $s \to \infty$, then

$$\bar{R}(s, L) = \frac{2L \log s}{s^2} (1 + o(1)).$$

With the help of Proposition 4, it follows an upper bound on the design rate $R_D(s)$:

**Corollary.** For $s \geq 2$, the design rate $R_D(s) \leq \bar{R}_D(s) \triangleq \bar{R}(s-1, 2)$. If $s \to \infty$, then

$$\bar{R}_D(s) = \frac{4 \log s}{s^2} (1 + o(1)).$$

From the numerical values of Table 2, we can conclude $\bar{R}(s-1, 2) < 1/s$, if $s \geq 11$. Hence, Theorem 4 implies that for $s \geq 11$, the design rate $R_D(s) < 1/s$. This improves the upper bound of Proposition 1.

### 2.2.3 Lower bound on $R(s, L)$

To obtain the random coding (lower) bound on $R(s, L)$, for $L \geq 2$, we use the ensemble of random codes, which was suggested in [38] for the particular case $L = 1$. To formulate our results, we need some notations. Let $K \geq s$ be an integer, $\xi_1, \xi_2, \ldots, \xi_s$ and $\eta_1, \eta_2, \ldots, \eta_L$ be independent identically distributed random variables with uniform distribution on the set $[K]$, i.e.,

$$\Pr\{\xi_i = k\} = \Pr\{\eta_j = k\} \triangleq \frac{1}{K}, \quad i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, L, \quad k = 1, 2, \ldots, K.$$

Define the probability

$$Q_K(s, L) \triangleq \Pr\left\{ \bigcap_{j=1}^L \bigcup_{i=1}^s (\eta_j = \xi_i) \right\}$$

and put

$$E(s, L) \triangleq \max_{K \geq s} \left\{ -\frac{\log Q_K(s, L)}{K} \right\}.$$ 

**Theorem 3.** [43]. For any $s \geq 2$ and $L \geq 2$, the rate

$$R(s, L) \geq R(s, L) \triangleq \frac{E(s, L)}{s + L - 1},$$

Theorems 3.1 and 3.2 give the properties of the lower bound $\underline{R}(s, L)$.

**Theorem 3.1.** For fixed $s \geq 2$, define

$$r_s \triangleq \max_{K \geq s} \left\{ \frac{\log(K/s)}{K} \right\} = \frac{\log([es]/s)}{|es|},$$

where we took into account that the maximum is achieved at $K = [es]$. The following statements are true:
1. For any \( s \) and \( L \),
\[
\frac{Lr_s}{s + L - 1} \leq R(s, L) \leq \frac{Lr_s}{s + L - 1} + \frac{\log e}{s + L - 1}.
\]

2. For fixed \( s \geq 2 \), there exists
\[
R(s, \infty) \equiv \lim_{L \to \infty} R(s, L) = r_s;
\]

3. For any \( s \geq 2 \), \( R(s, \infty) \geq \log e / \lceil e s \rceil \) and at \( s \to \infty \)
\[
R(s, \infty) = \frac{\log e}{e \cdot s} (1 + o(1)) = \frac{0.5307}{s} (1 + o(1)).
\]

**Theorem 3.2.** If \( L \geq 1 \) is fixed and \( s \to \infty \), then
\[
R(s, L) = \frac{L}{s^2 \log e} (1 + o(1)).
\]

For the fixed values of \( s \geq 2 \) and \( L \geq 2 \), the bound of Theorem 3 could be improved \([40]\) (see, also, \([48]\)) by the random coding method of \([41]\). This method uses the random ensemble of constant-weight codes. In Table 1, for \( s = 2, 3, \ldots, 8 \), we give the corresponding values of \( R(s, 1) \) and \( R(s, 2) \). The asymptotic lower bounds, obtained in Theorems 3.1-3.2, are the best known.

**Proof of Theorem 3.** Let \( n \geq 1 \) be integer and \([K]\) be the set of integers from 1 to \( K \). Denote by \( Y = ||y_m(j)||, m = 1, 2, \ldots, n; j = 1, 2, \ldots, t \) an arbitrary \((n \times t)\) matrix with elements \( y_m(j) \in [K] \). Let \( N = n \cdot K \). For matrix \( Y \), we denote by
\[
X_K = (x(1), x(2), \ldots, x(t))
\]
a binary \( N \times t \) matrix, whose columns have the following form
\[
x(j) = (x^1(j), x^2(j), \ldots, x^n(j)), \quad j = 1, 2, \ldots, t,
\]
\[
x^m(j) = (x^m_1(j), x^m_2(j), \ldots, x^m_n(j)), \quad m = 1, 2, \ldots, n,
\]
\[
x^m_k(j) = \begin{cases} 1, & \text{if } k = y_m(j), \\ 0, & \text{if } k \neq y_m(j). \end{cases}
\]

Obviously, each column (codeword) \( x(j) \) of matrix (code) \( X_K \) contains \( n = N/K \) 1’s and \((N - n)\) 0’s.

We say that a column \( x(j) \) is “bad” or code \( X \) if the \( x(j) \) does not satisfy the definition of LDSC. It follows that among the rest \( t - 1 \) columns there exist \( L - 1 \) columns \( x(j_1), x(j_2), \ldots, x(j_{L-1}) \) and \( s \) columns \( x(l_1), x(l_2), \ldots, x(l_s) \) for which
\[
\bigvee_{i=1}^s x(l_i) \text{ covers } x(j) \lor \bigvee_{i=1}^{L-1} x(j_i).
\]

Let \( Y \) be a random matrix whose components be independent with distribution
\[
\Pr\{y_m(j) = k\} \equiv \frac{1}{K}, \quad k \in [K].
\]

This ensemble was suggested in \([38]\). It is not hard to see that for any column \( x(j) \), the ensemble probability to be “bad” does not exceed
\[
\binom{t-1}{s+L-1} \binom{s+L-1}{s} [Q_K(s, L)]^{N/K}.
\]

Hence, the arguments of the random coding gives the statement of Theorem 3.
Proof of Theorem 3.1. Let $K \geq s$. We need to prove only Statement 1). Using the definition of $Q_K(s, L)$ and the theorem of total probability, we have

$$Q_K(s, L) \triangleq \Pr \left\{ \bigcap_{j=1}^{L} \bigcup_{i=1}^{s} (\eta_j = \xi_i) \right\} =$$

$$= K^{-s} \sum_{k_1, k_2, \ldots, k_s} \left[ \Pr \left\{ \bigcup_{i=1}^{s} (\eta_1 = k_i) \right\} \right]^L,$$

where the summation is taken over all $K^s$ ordered $s$-collections of integers $(k_1, k_2, \ldots, k_s)$, $k_i \in [K]$.

From this formula it follows the evident bound $Q_K(s, L) \leq \left( \frac{s}{K} \right)^L$. Hence,

$$E(s, L) \triangleq \max_{K \geq s} \left\{ -\frac{\log Q_K(s, L)}{K} \right\} \geq L \max_{K \geq s} \left\{ -\frac{\log(s/K)}{K} \right\} = r_s$$

and the lower bound of statement 1) is proved.

To prove the upper bound, we introduce the concept of composition for a word $(k_1, k_2, \ldots, k_s)$, i.e., the collection of nonnegative integers $\|n_k\|$, $k = 1, 2, \ldots, K$, where $n_k$ is the number of positions $i$, $i = 1, 2, \ldots, s$, for which $k_i = k$. We have

$$0 \leq n_k \leq s, \quad \sum_{k=1}^{K} n_k = s.$$ 

Let the summation in 1) over $(k_1, k_2, \ldots, k_s)$ be replaced by the summation over $\|n_k\|$, i.e.,

$$Q_K(s, L) = K^{-s} \sum_{\|n_k\|} \frac{s!}{\prod_{k=1}^{s} n_k!} \left( \frac{\sum_{k=1}^{K} u_k}{K} \right)^L,$$

where

$$u_k \triangleq \begin{cases} 1, & n_k > 0, \\ 0, & n_k = 0. \end{cases}$$

We can rewrite formula 2) in the form

$$Q_K(s, L) = K^{-s} \sum_{m} B_K(s, m) \left( \frac{m}{K} \right)^L,$$

where

$$B_K(s, m) \triangleq \binom{K}{m} \frac{s!}{\prod_{i=1}^{m} n_i!}.$$

The summation is taken over all ordered $m$-collections of positive integers $(n_1, \ldots, n_m)$, for which

$$n_1 + n_2 + \cdots + n_m = s, \quad n_i > 0, \quad i = 1, 2, \ldots, m.$$ 

Obviously,

$$B_K(s, s) = \binom{K}{s} s! = \prod_{i=0}^{s-1} (K - i).$$

If we restrict the summation in 3) by the member $m = s$, then we obtain the lower bound

$$Q_K(s, L) \geq \left( \frac{s}{K} \right)^L \frac{\prod_{i=0}^{s-1} (K - i)}{K^s} > \left( \frac{s}{K} \right)^L \frac{K!}{K^s} > \left( \frac{s}{K} \right)^L e^{-K}.$$
The last inequality is the consequence of the Stirling inequality $K! > K^K e^{-K}$. It follows

$$-\log Q_K(s, L) \leq L \log(K/s) + K \log e.$$ 

This inequality yields the upper bound of Statement 1).

Theorem 3.1 is proved.

Remark. It is known (see, for example [24], problem 3.9) that

$$\sum_{(n_1, n_2, \ldots, n_m)} \frac{s!}{\prod_{i=1}^{m} n_i!} = \left\{ \begin{array}{ll} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^s, & \text{if } s \geq m, \\
0, & \text{if } s < m. \end{array} \right.$$ 

Hence, from 3) it follows that

$$Q_K(s, L) = \operatorname{Pr}\left\{ \bigcap_{j=1}^{L} \bigcup_{i=1}^{s} (\eta_j = \xi_i) \right\} =$$

$$= K^{-s} \sum_{m=1}^{\min\{K; s\}} \binom{K}{m} \left( \frac{L}{K} \right)^m \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^s.$$ 

Proof of Theorem 3.2. Let $L \leq s \leq K$. For independent random variables $\xi_1, \xi_2, \ldots, \xi_s$, we introduce binary random variables $\alpha_1, \alpha_2, \ldots, \alpha_K$, where

$$\alpha_k \triangleq \begin{cases} 1, & \text{if there exists } i = 1, 2, \ldots, s \text{ and } \xi_i = k, \\
0, & \text{otherwise}. \end{cases}$$

Let the overline denote the averaging over random variables $\xi_1, \xi_2, \ldots, \xi_s$. It is easy to check that formula 2) can be rewritten in the form

$$Q_K(s, L) = K^{-L} \left( \sum_{k=1}^{K} \alpha_k \right)^L = K^{-L} \sum_{m=1}^{K} B_K(L, m) q_K(s, m),$$

where coefficients $B_K(L, m)$, $m = 1, 2, \ldots, L$, are defined by 4) and

$$q_K(s, m) \triangleq \left( \prod_{i=1}^{m} \alpha_i \right), \quad m = 1, 2, \ldots, L.$$

Lemma. Let $L \leq s \leq K$. For $m = 1, 2, \ldots, L$, the following formula is true

$$q_K(s, m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left( \frac{K-i}{K} \right)^s.$$ 

Proof of Lemma. Let $m = 2, 3, \ldots, L$. We have

$$q_K(s, m) = \operatorname{Pr}\{\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1\} = \operatorname{Pr}\left\{ \bigcap_{k=1}^{m} \bigcup_{i=1}^{s} (\xi_i = k) \right\} =$$

$$= \sum_{j=1}^{s-(m-1)} \left( 1 - \frac{m}{K} \right)^{j-1} \frac{m}{K} \operatorname{Pr}\left\{ \bigcap_{k=1}^{m-1} \bigcup_{i=j+1}^{s} (\xi_i = k) \right\}.$$
Hence, the following recurrent formula takes place

\[ q_K(s, m) = \sum_{j=1}^{s-(m-1)} \left( 1 - \frac{m}{K} \right)^{j-1} \frac{m}{K} q_K(s-j, m-1). \]

With the help of 7), we can check 6) by induction on \( m = 1, 2, \ldots, L \). If \( m = 1 \), then

\[ q_K(s, 1) = \mathcal{Q}_s = \Pr \left\{ \bigcup_{i=1}^{s} (\xi_i = 1) \right\} = 1 - \left( 1 - \frac{1}{K} \right)^s, \]
i.e., for \( m = 1 \), formula 6) is true. Let 6) be true for \( q_K(s, m-1) \). Consider

\[ q_K(s, m) = \sum_{j=1}^{s-(m-1)} \left( 1 - \frac{m}{K} \right)^{j-1} \frac{m}{K} \left[ \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( \frac{K-l}{K} \right) \right]^{s-j} = \]

\[ = 1 - \left( \frac{K-m}{K} \right)^{s-(m-1)} + \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( \frac{K-l}{K} \right)^s \left( 1 - \left( \frac{K-m}{K-l} \right)^{s-(m-1)} \right) \frac{K-l}{m-l} = \]

\[ = 1 - \left( \frac{K-m}{K} \right)^{s-(m-1)} + \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( \frac{K-l}{K} \right)^s \left[ 1 - \left( \frac{K-m}{K-l} \right)^{s-(m-1)} \right] = \]

\[ = \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( \frac{K-l}{K} \right)^s - \left( \frac{K-m}{K} \right)^{s-(m-1)} \cdot \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( \frac{K-l}{K} \right)^{m-1}. \]

In order to complete the proof of Lemma, we need to check that

\[ \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \left( 1 - \frac{l}{K} \right)^{m-1} = 0, \]

provided that \( K \geq m \).

It is known (see, [24], problem 3.10) that for any \( k = 1, 2, \ldots, m-1 \)

\[ \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} l^k = \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} (m-l)^k = 0. \]

It follows 8). Lemma is proved.

To complete the proof of Theorem 3.2, we fix an arbitrary \( L \geq 1 \). Consider the following asymptotic conditions

\[ s \to \infty, \quad K \to \infty, \quad s/K = \lambda \leq 1 - \text{const}. \]

Let \( m = 1, 2, \ldots, L \) be fixed. From 4) and 6) it follows that

\[ \lim_{m \to 0} \frac{B_K(L, m)}{K^L} = \begin{cases} 0, & \text{if } m \leq L-1, \\ 1, & \text{if } m = L. \end{cases} \]

\[ \lim_{m \to 0} q_K(s, m) = (1 - e^{-\lambda})^m. \]

Therefore, formula 5) yields

\[ \lim_{m \to 0} Q_K(s, L) = (1 - e^{-\lambda})^L. \]

From the definition of \( \mathcal{R}(s, L) \) and 10) it follows that

\[ \lim_{s \to \infty} s^2 \mathcal{R}(s, L) = L \max_{0 \leq \lambda \leq 1} \left\{ -\lambda \log (1 - e^{-\lambda}) \right\} = \frac{L}{\log e}, \]

and the extreme value of \( \lambda = (\log e)^{-1} = \ln 2 = 0.6931 \).

Theorem 3.2 is proved.
2.3 Kautz-Singleton codes

Theorems 1 and 3 of Sect. 2.2 are only theorems of existence. They do not give any method for the construction of the “good” codes. The first question, arising when one tries to apply Theorem 1, is the following. How many steps $S$ of computation one must make to check, that a given matrix $X$ with parameters corresponding to the bound of Theorem 1, satisfies the definition of an $(s, 1, t)$-code? If one step is the computation of a Boolean sum and checking of covering of the two binary codewords of length $N$, then the number $S$ evidently has the order of $ts + 1$. For $t = 10^3, \ldots, 10^4$, and $s = 5, \ldots, 15$, which occur in applications [6, 23, 37], $S$ becomes astronomically great

$$S = 10^{18}, \ldots, 10^{64}.$$ 

Is it possible to find any simple sufficient condition for matrix $X$ to be an $(s, 1, t)$-code and the checking of this condition takes essentially less computation steps? The important sufficient condition is given in [6] and formulated below as Theorem 4.

**Theorem 4** [6]. Let $X$ be a constant-weight code, i.e., $X$ be a binary $N \times t$ matrix, whose columns (codewords) $x(j)$ have the same number of 1’s

$$w = \sum_{i=1}^{N} x_i(j), \quad j = 1, 2, \ldots, t,$$

and

$$\lambda \triangleq \max_{k \neq j} \sum_{i=1}^{N} x_i(k) x_i(j)$$

be the maximal correlation of codewords. Then the matrix $X$ is $(s, 1, t)$-code for any $s$, satisfying the inequality

$$s \leq \left\lceil \frac{w - 1}{\lambda} \right\rceil.$$ 

The computation of the number $\left\lceil \frac{w - 1}{\lambda} \right\rceil$ for the matrix $X$, whose columns have the same weight $w$ takes $S = \left( t^2 / 2 \right)$ steps, where one step is the computation of

$$\lambda_{kj} \triangleq \sum_{i=1}^{N} x_i(k) x_i(j).$$

For the above-mentioned values of parameters $s$ and $t$, this number has the order

$$S = 10^6, \ldots, 10^8,$$

which is acceptable from the practical point of view. The most of known constructions [6, 47] of $(s, 1, N)$-codes were obtained with the help of Theorem 4. Below, in this section, we consider the upper and lower bounds on the optimal parameters of such codes.

**Definition.** [31]. Let $1 \leq \lambda \leq w \leq N$ be given integers, and let $X$ be a code of size $t$, length $N$ with parameters $w$ and $\lambda$. A code $X$ will be called an *KS-superimposed code (KSSC)* of length $N$, size $t$ and strength $s$, if inequality $s \leq \left\lceil \frac{w - 1}{\lambda} \right\rceil$ holds. This code also will be called an $(s, N)$-KS-code of size $t$, or an $(s, t)$-KS-code of length $N$.

Let $t_{KS}(s, N)$ be the maximal possible size of KSSC and $N_{KS}(s, t)$ be the minimal possible length of KSSC. For fixed $s \geq 1$, define the *rate* of KSSC.

$$R_{KS}(s) \triangleq \lim_{N \to \infty} \frac{\log t_{KS}(s, N)}{N}.$$
From Theorem 4 it follows that any \((s, N)\)-KS-code is also \((s, 1, N)\)-code. Therefore,

\[ N_{KS}(s, t) \geq N(s, 1, t), \quad t_{KS}(s, N) \leq t(s, 1, N), \quad R_{KS}(s) \leq R(s, 1). \]

Thus, the lower bound on \(N(s, 1, t)\), given by Proposition 6 can be considered as the lower bound on \(N_{KS}(s, t)\). The following Proposition 7 gives an improved (roughly twice) lower bound on \(N_{KS}(s, t)\).

**Proposition 7.** \([31]\]

\[ N_{KS}(s, t) \geq \min \left\{ t; \frac{s(s+1)}{1+s/t} \right\} \geq \min \{ t; s^2 \}. \]

**Proof.** Let \(X\) be a constant-weight code of length \(N\) and size \(t\) with parameters \(w\) and \(\lambda\). The well-known Johnson bound \([4]\) yields the inequality

\[ N \geq \frac{tw^2}{\lambda(t-1)+w}. \]

By virtue of Lemma 1, if \(X\) is a \((s, 1, t)\)-code of length \(N \leq (t-1)\), then weight \(w \geq (s+1)\). Hence, the above-mentioned inequality yields the statement of Proposition 7.

### 2.3.1 Upper bound on \(R_{KS}(s)\)

We give here (without proof) the best known upper bound on \(R_{KS}(s)\). This bound was obtained with the help of the best known \([15]\) upper bound on the rate for the constant-weight codes.

**Theorem 5.** For any \(s \geq 1\)

\[ R_{KS}(s) \leq \overline{R}_{KS}(s) \equiv h \left( \frac{1}{2} - \frac{\sqrt{s(s-1)}}{2s-1} \right). \]

We have \(\overline{R}_{KS}(2) = 0.187\), \(\overline{R}_{KS}(3) = 0.081\) and as \(s \to \infty\)

\[ \overline{R}_{KS}(s) = \frac{\log s}{8s^2}(1 + o(1)). \]

### 2.3.2 Lower bound on \(R_{KS}(s)\)

The following Theorem 6 is called a random coding bound on \(R_{KS}(s)\).

**Theorem 6** \([31]\). For any \(s \geq 1\)

1) \[ R_{KS}(s) \geq \overline{R}_{KS}(s) \equiv \max_{0 < p < s^{-1}} E(s, p), \]

where

2) \[ E(s, p) = h(p) - ph(s^{-1}) - (1-p)h \left( \frac{p(s-1)}{(1-p)s} \right). \]

Let the maximum in 1) be achieved at \(p = p_s\). The equation for computation \(p_s\) could be written in the form

3) \[ p = \frac{(1-p)^2(s-1)^2(s^{-1})}{s - (2s-1)p}. \]
If $s = 1$, then the root of 3) is $p_1 = 1$. If $s \geq 2$, then 3) can be solved numerically by the method of consecutive approximation. For $s = 2$ and $s = 3$, the following values are obtained:

$$p_2 = 0.13846, \ R_{KS}(2) = 0.09415; \quad p_3 = 0.08222, \ R_{KS}(3) = 0.03495.$$ 

In addition, if $s \to \infty$, then

$$p_s = \frac{a}{s}(1 + o(1)),$$

where $a = .203188$ is the unique solution of equation $a = e^{2(a-1)}$. It follows that if $s \to \infty$, then the lower bound

$$R_{KS}(s) = \frac{-a \log[ae^{1-a}]}{s^2}(1 + o(1)) = \frac{0.23358}{s^2}(1 + o(1)).$$

**Proof of Theorem 6.** Let $X = \|x_i(j)\|$ be a fixed $N \times 2t$ matrix, whose columns have weight $w$, $w = 1, 2, \ldots, N$. A column $x(j)$ is called “bad”, if there exists a column $x(k)$, $k \neq j$ that the correlation

$$\lambda_{kj} \geq \left\lfloor \frac{w-1}{s} \right\rfloor \triangleq \lambda_0.$$ 

Otherwise, the column $x(j)$ is called “good”. Denote by $t_1 = t_1(X)$ ($t_2 = t_2(X)$) the number of “good” (“bad”) columns of $X$. Obviously, $t_1 + t_2 = 2t$.

Let $X$ be the random $N \times 2t$ matrix, whose $2t$ columns are selected with replacement from the set of $\binom{N}{w}$ constant-weight columns. Introduce the events

$$B_{kj} = \{\lambda_{kj} \geq \lambda_0\}, \quad B^j = \bigcup_{k=1, k \neq j}^{2t} B_{kj}.$$ 

The probability of event $B_{kj}$ does not depend on $k$ and $j$ and

$$\Pr(B_{kj}) = \binom{w}{m} \binom{N-w}{w-m} \binom{N}{w} \triangleq q_s(N, w).$$ 

Hence, $\Pr(B^j) \leq 2tq_s(N, w)$ and the expectation

$$M(t_2(X)) = \sum_{j=1}^{2t} \Pr(B^j) \leq 4t^2q_s(N, w).$$ 

With the help of the arguments of the random coding method it follows

**Lemma.** If $q_s(N, w) < \left(4t\right)^{-1}$, then there exists $(s, N)$-KS-code of size $t$.

Let $p$, $0 < p < s^{-1}$ be fixed. From Lemma it follows that the rate of KSSC

$$R_{KS}(s) \geq \lim_{N \to \infty} -\frac{\log q_s(N, [Np])}{N} = h(p) - ph(s^{-1}) - (1-p)h\left(\frac{p(s-1)}{(1-p)s}\right).$$

To obtain the last equality we applied the well-known asymptotic behavior of the binomial coefficients [9]. Theorem 6 is proved.
2.4 Symmetrical superimposed codes

Let \( y(j) = (y_1(j), y_2(j), \ldots, y_N(j)), \) \( j = 1, 2, \ldots, s \) be binary columns of length \( N \). By definition, the symmetrical Boolean sum

\[
y = y(1) \odot y(2) \odot \cdots \odot y(s)
\]

of columns \( y(1), y(2), \ldots, y(s) \) is the binary column \( y = (y_1, y_2, \ldots, y_N) \) with components

\[
y_i = \begin{cases} 
0, & \text{if } y_i(1) = y_i(2) = \cdots = y_i(s) = 0, \\
1, & \text{if } y_i(1) = y_i(2) = \cdots = y_i(s) = 1, \\
\ast, & \text{otherwise.}
\end{cases}
\]

Let us say that the symmetrical Boolean sum \( y(1) \odot y(2) \odot \cdots \odot y(s) \) cover binary column \( z \), if column \( z \) do not change this sum, i.e.,

\[
y(1) \odot y(2) \odot \cdots \odot y(s) = y(1) \odot y(2) \odot \cdots \odot y(s) \odot z.
\]

**Definition.** An \( N \times t \)-matrix \( X \) is called a list-decoding symmetrical superimposed code (LDSSC) of length \( N \), size \( t \), strength \( s \), and list-size \( \leq L - 1 \) if the symmetrical Boolean sum of any \( s \)-subset of codewords \( X \) can cover not more than \( L - 1 \) codewords that are not components of the \( s \)-subset. This code also will be called an \((s, L, N)\)-code of size \( t \), or \((s, L, t)\)-code of length \( N \).

Let \( t_\circ(s, L, N) \) be the maximal possible size of LDSSC and \( N_\circ(s, L, t) \) be the minimal possible length of LDSSC. For fixed \( s \) and \( L \), define the rate of LDSSC

\[
R_\circ(s, L) \triangleq \lim_{N \to \infty} \frac{\log t_\circ(s, L, N)}{N}.
\]

We apply the notations of Sect. 1 for the optimal parameters of SD-model, i.e., \( t_{SD}(s, N), N_{SD}(s, t) \) and \( R_{SD}(s) \).

Below we give two evident propositions, showing the connection between codes for symmetrical Boolean sum and superimposed codes of Sect 2.1.

**Proposition 8.** Any superimposed code is a code for the symmetrical Boolean sum.

**Proposition 9.** Let an \( N \times t \) matrix \( X = \|x_i(j)\|, i = 1, 2, \ldots, N, j = 1, 2, \ldots, t \) be a symmetrical superimposed code of length \( N \). Consider \( N \times t \) matrix \( X' = \|x'_i(j)\| \) with elements

\[
x'_i(j) \triangleq \begin{cases} 
1, & \text{if } x_i(j) = 0, \\
0, & \text{if } x_i(j) = 1.
\end{cases}
\]

Then the \( 2N \times t \) matrix composed of the rows of the two matrices \( X \) and \( X' \) is a superimposed code.

From propositions 9 and 10 it follows that

\[
\frac{1}{2} N(s, L, t) \leq N_\circ(s, L, t) \leq N(s, L, t), \quad R(s, L) \leq R_\circ(s, L) \leq 2R(s, L).
\]

These inequalities permit to obtain the bounds on the length and rate of symmetrical superimposed codes using the corresponding bounds which are presented in Sect. 2.2-2.3.
3 Constructions of Superimposed Codes

Kautz-Singleton (1964) [6] suggested a class of binary superimposed codes which are based on the \( q \)-ary Reed-Solomon codes (RS-codes) [28]. Applying a concatenation of the binary constant-weight error-correcting codes [28] and the shortened RS-codes, we obtain new constructions of superimposed codes. Tables of their parameters are given. From the tables it follows that the rate of obtained codes exceeds the corresponding random coding bound [41].

3.1 Notations and definitions

Let \( 1 \leq s < t, N > 1 \) be integers and \( X = \{x_i(u)\}, i = 1,2,\ldots,N, u = 1,2,\ldots,t, \) be a binary \((N \times t)\)-matrix (code) of size \( t \) and length \( N \) with columns (codewords) \( x(1), x(2),\ldots,x(t) \), where \( x(u) = (x_1(u), x_2(u),\ldots,x_N(u)) \). Let \( \triangleq \) denote the equation by definition. For code \( X \), let \( w \) and \( \lambda \) be defined by

\[
    w \triangleq \min_u \sum_{i=1}^N x_i(u), \quad \lambda \triangleq \max_{u,v} \sum_{i=1}^N x_i(u)x_i(v).
\]

\( w \) is the minimal weight of codewords and \( \lambda \) is the maximal correlation of codewords.

We say that the binary column \( x \) covers the binary column \( y \) if the Boolean sum \( x \lor y = x \). The code \( X \) is called a superimposed \((s, N, t)\)-code, or \( s \)-disjunct code if the Boolean sum of any \( s \)-subset of columns of \( X \) covers only those columns of \( X \) which are the terms of the given Boolean sum.

One can consider an arbitrary fixed code \( X \) as the incidence matrix of a \( t \)-family of subsets of the \( N \)-set. In this interpretation, the \( s \)-disjunct code \( X \) one-to-one corresponds to the family in which no set is covered by the union of \( s \) others.

Superimposed codes were introduced by Kautz - Singleton [6] who worked out the constructive methods and some applications. Dyachkov - Macula - Rykov [53] investigated the development of constructions for superimposed codes (non-adaptive pooling designs) intended for the clone-library screening problem. (See Knill, Bruno, Torney [51]). In Section 3.2, we give a brief introduction to the problem.

In Sect. 3.3 and 3.4, we study the most important class of superimposed codes which are based on the \( q \)-ary Reed-Solomon codes (RS-codes) [28]. The given class was invented by Kautz-Singleton [6]. We introduce some generalizations of the Kautz-Singleton codes and identify the parameters of the best known superimposed codes.

3.2 Application to DNA library screening

To understand what a DNA library is, think of several copies of an identical but incredible long word (of length \( \sim 10^8 \), e.g., a chromosome) from letters of the quaternary alphabet \{A, C, G, T\}. Each copy of the word has been cut in thousands of contiguous pieces (of length \( \sim 10^4 \), e.g., chromosome fragments). Take those pieces and copy their letter strings onto their own separate small piece of paper. The thousands of little pieces of paper (i.e., clones) that result essentially constitute a DNA library. In other words, each clone represents some contiguous subpiece of a contiguous superpiece of DNA. The DNA library, or the clone-library consists of thousands separate clones.

An unique and contiguous sub-subpiece of DNA (of length \( \sim 10^2 \)) is called a sequenced tagged site (STS). For a fixed STS, a clone is called positive (negative) for that STS if it contains (does not contain) that given STS.
Example. Let the following $s = 4$ copies of the DNA superpiece be given and \{$C_1, C_2, C_3, C_4, C_5\}$ be the library of 5 clones.

\[
\begin{align*}
C_1 & : \text{AAA} \quad \text{GGTCT} \quad \text{TAA} \quad \text{CCGATAGGCAACCTTG}, \\
C_2 & : \text{AAA} \quad \text{GGTCT} \quad \text{TAA} \quad \text{CCGATAGGCAACTTG}, \\
C_3 & : \text{AAA} \quad \text{GGTCT} \quad \text{TAA} \quad \text{CCGATAGGCAACTTG}, \\
C_4 & : \text{AAA} \quad \text{GGTCT} \quad \text{TAA} \quad \text{CCGATAGGCAACTTG}, \\
C_5 & : \text{AAA} \quad \text{GGTCT} \quad \text{TAA} \quad \text{CCGATAGGCAACTTG}.
\end{align*}
\]

Clones \{C_1, C_3\} could be taken from the same copy of the DNA superpiece. Clones \{C_2, C_4\} are taken from the different copies. Let \(\text{STS}_1 = \text{AAA}\) and \(\text{STS}_2 = \text{TAA}\). Then \(C_1\) is positive for \(\text{AAA}\) and \(C_1, C_2\) and \(C_4\) are positive for \(\text{TAA}\). Note that \(C_1\) is positive for both \(\text{AAA}\) and \(\text{TAA}\). Clones \(C_3, C_5\) are negative for both \(\text{AAA}\) and \(\text{TAA}\).

A pool is a subset of clones. Each pool is tested as a group by exposing that entire group to a chemical probe (e.g. polymerase chain reaction) which can detect a given STS. A pool is called positive for the STS if the probe indicates that some member of that group contains the given STS. In other words, if the tests are error-free, then a pool is positive for an STS if that pool contains at least one clone that contains the given STS.

Let \(1 \leq s < t, \ N > 1\) be integers. Mathematically, clone-library screening for positive clones is modeled by searching a \(t\)-set of objects (clone-library) for a particular \(p\)-subset, \(p \leq s\), called a subset of positive clones. A non-adaptive pooling design is a series of \(N\) a priori group tests that can often be carried out simultaneously. Every parallel pooling design is non-adaptive. The pooling design can be described by a binary \(N \times t\)-matrix \(X = \langle x_i(u) \rangle\), \(i = 1, 2, \ldots, N, \ u = 1, 2, \ldots, t\), where an element \(x_i(u) = 1\) if the \(u\)-th clone is in the \(i\)-th pool and \(x_i(u) = 0\), otherwise. A pool outcome (result of the group test) is said to be positive if one of the pool’s clones is positive, negative otherwise. Using this binary \(N\)-sequence of outcomes, an investigator has to identify the \(p\)-subset, \(p \leq s\), of positive clones.

Let \(p \leq s\) be the number of positive clones in a clone-library of size \(t\). To identify an unknown \(p\)-subset of positive clones, we apply the pooling design \(X\) which is the superimposed \((s, N, t)\)-code. Obviously, the binary \(N\)-sequence \(y\) of the pool outcomes is the Boolean sum of the unknown \(p\)-subset of columns of \(X\). The definition of \((s, N, t)\)-code means that the unknown \(p\)-subset is represented by all columns which are below \(y\). Thus, we need to carry out \(\leq t\) successive comparisons of the Boolean sum \(y\) with codewords of \(X\). Hence, the identification complexity of \((s, N, t)\)-code does not exceed \(t\).

### 3.3 Generalized Kautz-Singleton codes

Let \(\mathcal{P}\) be the set of all primes or prime powers \(\geq 2\), i.e.,

\[
\mathcal{P} \triangleq \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, \ldots\}.
\]

Let \(q_0 \in \mathcal{P}\) and \(2 \leq k \leq q_0 + 1\) be fixed integers for which there exists the \(q_0\)-ary Reed-Solomon code (RS-code) \(B\) of size \(q_0^k\), length \((q_0 + 1)\) and distance \(d = q_0 - k + 2 = q_0 + 1 - (k - 1)\) [28]. We will identify the code \(B\) with an \(((q_0 + 1) \times q_0^k)\)-matrix whose columns, (i.e., \((q_0 + 1)\)-sequences from the alphabet \(\{0, 1, 2, \ldots, q_0 - 1\}\)) are the codewords of \(B\). Therefore, the maximal possible number of positions (rows) where its two codewords (columns) can coincide, called a coincidence of code \(B\), is equal to \(k - 1\).

Fix an arbitrary integer \(r = 0, 1, 2, \ldots, k - 1\) and introduce the shortened RS-code \(\bar{B}\) of size \(t = q_0^{k-r}\), length \(q_0 + 1 - r\) that has the same distance \(d = q_0 - k + 2\). Code \(\bar{B}\) is obtained by the shortening of the
subcode of $B$ which contains 0’s in the first $r$ positions (rows) of $B$. Obviously, the coincidence of code $\tilde{B}$ is equal to
\[
\lambda_0 \triangleq (q_0 + 1 - r) - d = q_0 + 1 - r - (q_0 - k + 2) = k - r - 1. \tag{1}
\]

Consider the following standard transformation of the $q_0$-ary code $\tilde{B}$ into the binary constant-weight code $X$ of size $q_0^{k-r}$, length $(q_0 + 1 - r)q_0$ and weight $w = q_0 + 1 - r$. Each symbol of the $q_0$-ary alphabet $\{0, 1, 2, \ldots, q_0 - 1\}$ is substituted for the corresponding binary column of the length $q_0$ and the weight 1, namely:
\[
0 \leftrightarrow \underbrace{(1, 0, 0, \ldots, 0)}_{q_0}, \quad 1 \leftrightarrow \underbrace{(0, 1, 0, \ldots, 0)}_{q_0}, \quad \ldots, \quad q_0 - 1 \leftrightarrow \underbrace{(0, 0, 0, \ldots, 1)}_{q_0}.
\]

From (1) it follows that for binary code $X$, the maximal correlation of codewords is $\lambda = \lambda_0 = k - r - 1$.

Let $X$ be a binary code with parameters $w$ and $\lambda$. Kautz-Singleton [6] suggested the following evident sufficient condition of the $s$-disjunct property:
\[
s\lambda \leq w - 1. \tag{2}
\]

Hence, by virtue of (1) and (2), code $X$ is the $s$-disjunct code if
\[
s(k - r - 1) = s\lambda_0 \leq w - 1 = q_0 - r. \tag{3}
\]

For the particular case $r = 0$, this construction of $s$-disjunct codes was suggested in [6].

Denote by $\lceil b \rceil$ the least integer $\geq b$. Let $m \geq 1$ and $2 \leq s < 2^m$ be arbitrary fixed integers. Define
\[
\mathcal{P}(m, s) \triangleq \left\{ q : q \in \mathcal{P}, \ s \left( \left\lfloor \frac{m}{\log_2 q} \right\rfloor - 1 \right) \leq q \right\}. \tag{4}
\]

Consider a binary code $X$ identified by parameters $0 \leq r \leq k - 1 \leq q_0$ , $q_0 \in \mathcal{P}(m, s)$. The sufficient condition (3) for $s$-disjunct property of $X$ could be written in the form $k - r \leq \frac{q_0 + 1 - k}{s - 1} + 1$. Hence, if
\[
\left\lfloor \frac{m}{\log_2 q_0} \right\rfloor \leq k - r \leq \frac{q_0 + 1 - k}{s - 1} + 1, \tag{5}
\]

then code $X$ has $s$-disjunct property and its size $t = q_0^{k-r} \geq 2^m$.

For fixed value $q_0 \in \mathcal{P}(m, s)$, denote by
\[
N(q_0, s, m) = \min \left\{ q_0(q_0 + 1 - r) \right\}, \tag{6}
\]

the minimal possible length of $s$-disjunct codes of size $\geq 2^m$ which are based on the $q_0$-ary shortened RS-codes. The minimum in (6) is taken over all parameters $0 \leq r \leq k - 1 \leq q_0$ for which (5) is true.

**Lemma.** The minimum in (6) is achieved at
\[
k = q_0 + s - (s - 1) \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor, \tag{7}
\]
\[
r = k - \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor = q_0 - s \left( \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor - 1 \right) \geq 0 \tag{8}
\]
and the optimal length is given by the formula
\[
N(q_0, s, m) = q_0 \left[ s \left( \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor - 1 \right) + 1 \right]. \tag{9}
\]

**Proof.** Fix an arbitrary $q_0 \in \mathcal{P}(m, s)$. Let the integer $k$ be defined by (7). One can easily check that $k$ is the root of equation $\left\lfloor \frac{m}{\log_2 q_0} \right\rfloor = \frac{q_0 + 1 - k}{s - 1} + 1$. Hence, the given value of $k$ is the maximal possible
integer satisfying (5). From the left-hand side of (5) it follows $r \leq k - \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor$. It means that the maximal possible value of $r$ is given by (8). Therefore, the minimum in (6) is achieved at $r$ defined by (8) and the corresponding minimal code length is equal to the right-hand side (9).

Lemma is proved.

We can summarize as follows.

**Proposition 1.** Fix an arbitrary $q_0 \in \mathcal{P}(m, s)$, where the subset $\mathcal{P}(m, s) \subset \mathcal{P}$ is defined by (4). For the binary code $X$ based on the $q_0$-ary shortened RS-code with parameters (7)–(8), we have:

1. $X$ is the $s$-disjunct constant-weight code of size $t$, length $N$, weight $w$ and the maximal correlation of codewords $\lambda$, where

$$
\lambda = \lambda_0 = k - r - 1 = \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor - 1, \quad w = q_0 + 1 - r = s\lambda_0 + 1,
$$

$$
t = q_0^{k-r} = q_0^\lambda + 1 = q_0^\lambda \left\lceil \frac{m}{\log_2 q} \right\rceil \geq 2^m, \quad N = q_0(s\lambda_0 + 1);
$$

2. the length $N$ of code $X$ coincides with the minimal possible length $N(q_0, s, m)$ defined by (9).

Denote by $N(s, m)$ the minimal possible length of $s$-disjunct codes of size $t \geq 2^m$ which are based on the shortened RS-codes, i.e.,

$$
N(s, m) = \min_{\mathcal{P}(s, m)} N(q, s, m) = \min_{\mathcal{P}(s, m)} \left\{ q \left( \left\lceil \frac{m}{\log_2 q} \right\rceil - 1 \right) + 1 \right\},
$$

$$
\mathcal{P}(m, s) = \left\{ q : q \in \mathcal{P}, s \left( \left\lceil \frac{m}{\log_2 q} \right\rceil - 1 \right) \leq q \right\}.
$$

In Table 1, we give numerical values of $N = N(s, m)$, $q_0$ and $\lambda_0$, when $s = 2, 3, \ldots, 8$, $m = 5, 6, \ldots, 30$ and the code size $t = q_0^\lambda + 1$ satisfies the inequalities $2^m \leq t < 2^{m+1}$. The optimal parameters are identified as follows:

$$
q_0 \geq s\lambda_0, \quad r = q_0 - s\lambda_0, \quad k = r + \lambda_0 + 1, \quad \lambda_0 = \left\lfloor \frac{m}{\log_2 q_0} \right\rfloor - 1,
$$

$$
w = n_0 = s\lambda_0 + 1, \quad N = q_0(s\lambda_0 + 1), \quad t = q_0^\lambda + 1, \quad 2^m \leq t < 2^{m+1}. \tag{10}
$$

Table 1 shows the solutions of (10), i.e., $\lambda_0$ and $q_0$, yielding the minimal length $N = N(s, m)$ if for the given integers $s$ and $m$, these solutions exist. The optimal solutions were calculated with the help of a computer program.

**Example.** For the case $s = 3$, $m = 10$, Table 1 gives $q_0 = 11$, $\lambda_0 = 2$, $N = 77$. It means that there exists 3-disjunct constant-weight code with parameters

$$
\lambda = \lambda_0 = 2, \quad w = s\lambda_0 + 1 = 7, \quad t = q_0^\lambda + 1 = 11^3 = 1331, \quad N = 11 \cdot 7 = 77.
$$

This code is obtained from shortened RS-code with parameters $q_0 = 11$, $k = 7$ and $r = 4$.

**Remark 1.** By the boldface type, we marked two examples of the superimposed code parameters which were known from [6, 31].

**Remark 2.** In Sect. 4, based on paper [54], we give the detailed investigation of superimposed codes with parameter $\lambda = 1$.

**Discussion.**

Table 1 contains the values $R(s)$ (lower bound on the rate $R(s)$ of the optimal code), the values $\bar{R}(s)$ (upper bound on the rate $R(s)$ of the optimal code) and the values of the rate for several obtained codes, namely: the values of fraction $\frac{w}{n}$, $m = 12, 20, 25, 29$. The comparison yields the following conclusions:

- If $s = 2$ and $m \leq 15$, then the values $\frac{w}{n}$ exceed the random coding rate $R(2) = .182$.
- If $s \geq 3$ and $m \leq 30$, then the values $\frac{w}{n}$ exceed the random coding rate $R(s)$. 

20
Table 1. Parameters of constant-weight \((s,N,t)\)-codes of strength \(s\), \(2 \leq s \leq 8\), weight \(w\), length \(N\), size \(t = q_0^{m+1}, 2^m \leq t < 2^{m+1}, 5 \leq m \leq 30\), based on the \(q_0\)-ary shortened Reed-Solomon codes.

| \(s\) | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|------|-----|-----|-----|-----|-----|-----|-----|
| \(R(s)\) | .182 | .079 | .044 | .028 | .019 | .014 | .011 |
| \(\overline{R(s)}\) | .322 | .199 | .140 | .106 | .083 | .067 | .056 |
| \(m\) | \(q_0, \lambda_0, N\) | \(q_0, \lambda_0, N\) | \(q_0, \lambda_0, N\) | \(q_0, \lambda_0, N\) | \(q_0, \lambda_0, N\) | \(q_0, \lambda_0, N\) |
| 5   | 7   | .1, 28 | 7   | .1, 35 | 7   | .1, 42 | 7   | .1, 49 | –   | –   |
| 6   | 4   | .2, 20 | 8   | .1, 32 | 8   | .1, 40 | 8   | .1, 48 | 8   | .1, 56 | 9   | .1, 72 | 11  | .1, 99 |
| 7   | –   | –   | –   | 13   | .1, 65 | 13   | .1, 78 | 13   | .1, 91 | 13   | .1, 104 | 13   | .1, 117 |
| 8   | 7   | .2, 35 | 7   | .2, 49 | –   | 16   | .1, 96 | 16   | .1, 112 | 16   | .1, 128 | 16   | .1, 144 |
| 9   | 8   | .2, 40 | 8   | .2, 56 | 8   | .2, 72 | –   | 23   | .1, 161 | 23   | .1, 184 | 23   | .1, 207 |
| 10  | –   | –   | –   | 11   | .2, 77 | 11   | .2, 99 | 11   | .2, 121 | –   | –   | –   | –   |
| 11  | 7   | .3, 49 | –   | 13   | .2, 117 | 13   | .2, 143 | 13   | .2, 169 | –   | –   | –   | –   |
| 12  | 8   | .3, 56 | 9   | .3, 90 | 16   | .2, 144 | 16   | .2, 176 | 16   | .2, 208 | 16   | .2, 240 | 16   | .2, 272 |
| 13  | –   | –   | –   | 11   | .3, 110 | –   | 23   | .2, 253 | 23   | .2, 299 | 23   | .2, 345 | 23   | .2, 391 |
| 14  | –   | –   | –   | 13   | .3, 130 | 13   | .3, 169 | –   | 27   | .2, 351 | 27   | .2, 405 | 27   | .2, 459 |
| 15  | 8   | .4, 72 | –   | –   | –   | –   | –   | 32   | .2, 480 | 32   | .2, 544 | –   | –   |
| 16  | –   | 16   | .3, 160 | 16   | .3, 208 | 16   | .3, 256 | 19   | .3, 361 | –   | –   | –   | –   |
| 17  | 11  | .4, 99 | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   |
| 18  | 13   | .4, 117 | 13   | .4, 169 | –   | 23   | .3, 368 | 23   | .3, 437 | 23   | .3, 506 | 25   | .3, 625 |
| 19  | –   | –   | –   | 27   | .3, 432 | 27   | .3, 513 | 27   | .3, 594 | 27   | .3, 675 | –   | –   |
| 20  | 11   | .5, 121 | 16   | .4, 208 | 16   | .4, 272 | –   | 32   | .3, 608 | 32   | .3, 704 | 32   | .3, 800 |
| 21  | –   | –   | –   | 19   | .4, 323 | –   | –   | –   | 41   | .3, 1025 | –   | –   | –   |
| 22  | 13   | .5, 143 | –   | 23   | .4, 391 | 23   | .4, 483 | –   | –   | –   | –   | –   | –   |
| 23  | –   | –   | –   | 25   | .4, 425 | 25   | .4, 525 | 25   | .4, 625 | –   | –   | –   | –   |
| 24  | –   | 16   | .5, 256 | –   | 27   | .4, 609 | 29   | .4, 725 | 29   | .4, 841 | –   | –   | –   |
| 25  | 13   | .6, 169 | 19   | .5, 304 | –   | –   | 32   | .4, 800 | 32   | .4, 928 | 32   | .4, 1056 | –   | –   |
| 26  | –   | –   | –   | 37   | .4, 925 | 37   | .4, 1073 | 37   | .4, 1221 | –   | –   | –   | –   |
| 27  | –   | –   | –   | 23   | .5, 483 | –   | –   | 43   | .4, 1247 | 43   | .4, 1419 | –   | –   |
| 28  | 16   | .6, 208 | –   | 27   | .5, 702 | 25   | .5, 650 | –   | –   | 49   | .4, 1617 | –   | –   |
| 29  | –   | 19   | .6, 361 | 29   | .5, 609 | 29   | .5, 754 | 31   | .5, 961 | –   | –   | –   | –   |
| 30  | –   | –   | –   | 32   | .5, 832 | 32   | .5, 992 | –   | –   | –   | –   | –   | –   |

3.4 Superimposed concatenated codes

A further extension of the Kautz-Singleton superimposed \((s,N,t)\)-codes is based on the following concatenated codes which were suggested in [6] and also were discussed in [32] and [45].

Consider the \(q_0\)-ary shortened Reed-Solomon code with parameters (10) where \(q_0\) is a prime power. Let \(q_0\)-ary symbols of this code be substituted, i.e., be coded, for the binary codewords of a known constant-weight \(s\)-disjunct code of size \(q' \geq q_0\), length \(q \leq q_0\) and weight \(w' < q\). Denote this binary superimposed code as \((n,q,q')\)-code. We will apply \((s,q,q')\)-codes which are standard constant-weight \((n,D,w')\)-codes of size \(A(n,D,w') = q'\), length \(n = q\), weight \(w' = s\lambda' + 1\), distance \(D = 2(w' - \lambda')\) and the maximal correlation \(\lambda'\). It is easy to check that the result of Sect. 2 can be generalized as follows.

Proposition 2. The given substitution yields the concatenated code which is the binary constant-weight superimposed \((s,qn_0,q_0^{m+1})\)-code of weight \(w = w'n_0\).
In Sect. 3.3, we used only the trivial substitution, where \( q_0 = q = q' \) and \( w' = 1 \).

**Remark 3.** If we apply the trivial substitution \( q_0 = q = q' \), i.e., \( w' = 1 \), then we obtain the concatenated code which is a standard constant-weight \((N, D, w)\)-code of size \( t \), where

\[
N = q_0n_0, \quad w = n_0, \quad t = q^{k-r}, \quad D = 2(n - \lambda_0) = 2(q_0 - k + 2) = 2d.
\]

**Remark 4.** If \( q < q_0 \) and \( w' > 1 \), then one knows only the weight \( w = w'n_0 \) of the constant-weight concatenated code. We cannot identify its distance \( D \) and the maximal correlation \( \lambda \).

### Table 2. Parameters for 2- and 3-disjunct codes based on \((N, D, w)\)-codes

| \( N \) | \( D \) | \( w \) | \( \lambda \) | \( t \) | \( N \) | \( D \) | \( w \) | \( \lambda \) | \( t \) |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 4 | 3 | 1 | 12 | 16 | 6 | 4 | 1 | 20 |
| 10 | 4 | 3 | 1 | 13 | 17 | 6 | 5 | 2 | 42 |
| 11 | 4 | 3 | 1 | 17 | 18 | - | - | - | - |
| 12 | 4 | 3 | 1 | 20 | 19 | 6 | 5 | 2 | 76 |
| 13 | 4 | 3 | 1 | 26 | 20 | 6 | 5 | 2 | 84 |
| 14 | 6 | 5 | 2 | 28 | 21 | 8 | 7 | 3 | 120 |
| 15 | 6 | 5 | 2 | 48 | 22 | 8 | 7 | 3 | 176 |
| 16 | 6 | 5 | 2 | 48 | 23 | 8 | 7 | 3 | 253 |
| 17 | 6 | 5 | 2 | 68 | 24 | - | - | - | - |
| 18 | 6 | 5 | 2 | 68 | 25 | 8 | 7 | 3 | 254 |
| 19 | 6 | 5 | 2 | 76 | 26 | 6 | 5 | 2 | 260 |
| 20 | 6 | 5 | 2 | 84 | 27 | 8 | 7 | 3 | 278 |
| 21 | 8 | 7 | 3 | 120 | 28 | 8 | 7 | 3 | 296 |
| 22 | 8 | 7 | 3 | 176 | 29 | 8 | 7 | 3 | 300 |
| 23 | 8 | 7 | 3 | 253 | 30 | 8 | 7 | 3 | 306 |
| 24 | - | - | - | - | 31 | 8 | 7 | 3 | 306 |
| 25 | - | - | - | - | 32 | 8 | 7 | 3 | 403 |
| 26 | - | - | - | - | 33 | 8 | 7 | 3 | 442 |
| 27 | - | - | - | - | 34 | 8 | 7 | 3 | 494 |
| 28 | - | - | - | - | 35 | 8 | 7 | 3 | 555 |
| 29 | - | - | - | - | 36 | 8 | 7 | 3 | 622 |
| 30 | - | - | - | - | 37 | 8 | 7 | 3 | 696 |
| 31 | - | - | - | - | 38 | 8 | 7 | 3 | 785 |
| 32 | - | - | - | - | 39 | 8 | 7 | 3 | 869 |
| 33 | - | - | - | - | 40 | 8 | 7 | 3 | 965 |
| 34 | - | - | - | - | 41 | 8 | 7 | 3 | 1095 |
| 35 | - | - | - | - | 42 | 8 | 7 | 3 | 1206 |
| 36 | - | - | - | - | 43 | 8 | 7 | 3 | 1344 |
| 37 | - | - | - | - | 44 | 8 | 7 | 3 | 1471 |
| 38 | - | - | - | - | 45 | 8 | 7 | 3 | 1795 |
| 39 | - | - | - | - | 46 | 8 | 7 | 3 | 1976 |
| 40 | - | - | - | - | 47 | 8 | 7 | 3 | 1976 |
| 41 | - | - | - | - | 48 | 8 | 7 | 3 | 2401 |
| 42 | - | - | - | - | 49 | 8 | 7 | 3 | 2401 |
| 43 | - | - | - | - | 50 | 8 | 7 | 3 | 2401 |
| 44 | - | - | - | - | 51 | 8 | 7 | 3 | 2401 |
| 45 | - | - | - | - | 52 | 8 | 7 | 3 | 2401 |
| 46 | - | - | - | - | 53 | 8 | 7 | 3 | 2401 |
| 47 | - | - | - | - | 54 | 8 | 7 | 3 | 2401 |
| 48 | - | - | - | - | 55 | 8 | 7 | 3 | 2401 |
| 49 | - | - | - | - | 56 | 8 | 7 | 3 | 2401 |

22
Let \( D = 2, 4, 6, \ldots, D \leq n \) and \( w \leq n \) be arbitrary integers. Denote by \( A(n, D, w) \) the maximal size of the corresponding constant-weight code which is known up to now. The tables of \( A(n, D, w) \) called Standard Tables (ST) are available [44] and:

http://www.research.att.com/~njas/codes/Andw/index.html

Tables 2–4 give the numerical values of the best known parameters for superimposed concatenated \((s, N, t)\)-codes, when \( s = 2 \) and \( s = 3 \).

Table 3. Parameters of constant-weight concatenated \((2, N, t)\)-codes
of weight \( w \), length \( N \) and size \( t \), \( 2^m \leq t < 2^{m+1} \), \( 10 \leq m \leq 32 \)

| \( m \) | \( q \) | \( q' \) | \( w' \) | \( q_0 \) | \( k \) | \( r \) | \( t = q_0^{s-r} \) | \( n_0 \) | \( \lambda_0 \) | \( N = qn_0 \) | \( w \) |
|------|------|------|------|------|------|------|----------|------|------|--------|------|
| 10   | 15   | 42   | 5    | 41   | 41   | 39   | \( 41^2 \) | 3    | 1     | 45     | 15   |
| 10   | 15   | 42   | 5    | Latin square \( 42 \times 42 = 1764 \) | 45  | 15   |
| 11   | 16   | 48   | 5    | 47   | 47   | 45   | \( 47^2 \) | 3    | 1     | 48     | 15   |
| 11   | 16   | 48   | 5    | Latin square \( 48 \times 48 = 2304 \) | 48  | 15   |
| 11   | 7    | 7    | 1    | 7    | 5    | 1    | \( 7^4 \) | 7    | 3     | 49     | 7    |
| 12   | 17   | 68   | 5    | 67   | 67   | 65   | \( 67^2 \) | 3    | 1     | 51     | 15   |
| 12   | 17   | 68   | 5    | Latin square \( 68 \times 68 = 4624 \) | 51  | 15   |
| 12   | 19   | 76   | 5    | 73   | 73   | 71   | \( 73^2 \) | 3    | 1     | 57     | 15   |
| 12   | 19   | 76   | 5    | Latin square \( 76 \times 76 = 5776 \) | 57  | 15   |
| 12   | 20   | 84   | 5    | 83   | 83   | 81   | \( 83^2 \) | 3    | 1     | 60     | 15   |
| 12   | 20   | 84   | 5    | Latin square \( 84 \times 84 = 7056 \) | 60  | 15   |
| 13   | 21   | 120  | 7    | Latin square \( 120 \times 120 = 14400 \) | 63  | 21   |
| 13   | 9    | 12   | 3    | 11   | 9    | 5    | \( 11^4 \) | 7    | 3     | 63     | 21   |
| 13   | 13   | 26   | 3    | 25   | 24   | 21   | \( 25^3 \) | 5    | 2     | 65     | 15   |
| 15   | 23   | 253  | 7    | 251  | 251  | 249  | \( 251^2 \) | 3    | 1     | 69     | 21   |
| 15   | 23   | 253  | 7    | Latin square \( 253 \times 253 = 64009 \) | 69  | 21   |
| 16   | 11   | 17   | 3    | 17   | 15   | 11   | \( 17^4 \) | 7    | 3     | 77     | 21   |
| 17   | 9    | 12   | 3    | 11   | 8    | 3    | \( 11^9 \) | 9    | 4     | 81     | 27   |
| 17   | 17   | 68   | 5    | 67   | 66   | 63   | \( 67^3 \) | 5    | 2     | 85     | 25   |
| 18   | 10   | 13   | 3    | 13   | 10   | 5    | \( 13^9 \) | 9    | 4     | 90     | 27   |
| 18   | 13   | 26   | 3    | 25   | 23   | 19   | \( 25^4 \) | 7    | 3     | 91     | 21   |
| 19   | 9    | 12   | 3    | 11   | 7    | 1    | \( 11^{16} \) | 11   | 5     | 99     | 33   |
| 21   | 12   | 20   | 3    | 19   | 16   | 11   | \( 19^9 \) | 9    | 4     | 108    | 27   |
| 22   | 10   | 13   | 3    | 13   | 9    | 3    | \( 13^{16} \) | 11   | 5     | 110    | 33   |
| 23   | 23   | 253  | 7    | 251  | 250  | 247  | \( 251^4 \) | 5    | 2     | 115    | 35   |
| 24   | 11   | 17   | 3    | 17   | 13   | 7    | \( 17^9 \) | 11   | 5     | 121    | 33   |
| 25   | 10   | 13   | 3    | 13   | 8    | 1    | \( 13^9 \) | 11   | 6     | 130    | 39   |
| 28   | 11   | 17   | 3    | 17   | 12   | 5    | \( 17^9 \) | 13   | 6     | 143    | 39   |
| 29   | 12   | 19   | 3    | 19   | 12   | 5    | \( 19^9 \) | 13   | 6     | 156    | 39   |
| 31   | 23   | 253  | 7    | 251  | 249  | 245  | \( 251^4 \) | 7    | 3     | 161    | 49   |
| 32   | 11   | 17   | 3    | 17   | 11   | 3    | \( 17^9 \) | 15   | 7     | 165    | 45   |

Description of Table 2.
We tabulate the values of size \( t > N \) for \((s, N, t)\)-superimposed codes of strength \( s = 2, 3 \) and length \( N = 9, 10, \ldots, 47 \). The symbol \( t = A(N, D, w) \) denotes the code size, i.e., the number of codewords (columns), \( N \) is the code length, \( w = s \lambda + 1 \) is the code weight and \( D = 2^w - \lambda \) is the code distance. Table 2 is calculated as follows. Let \( N = 9, 10, \ldots, 47 \) be fixed. In ST, we are looking for the pair \((D, w)\) such that

\[
w = s \lambda + 1, \quad D = 2(w - \lambda) \implies w(s - 1) = \frac{D}{2} s - 1
\]
and the value of size $A(N, D, w)$ is maximal. Table 2 gives these maximal values if $s = 2, 3$. For $s = 2$, $N = 45, 48$ and $s = 3$, $N = 49$, we use the boldface type because the corresponding codes are concatenated. We will repeat them in Table 3 and Table 4. For $s = 2, 3$ and $N = 49$, the concatenated codes are the standard constant-weight codes because they are obtained by the trivial substitution. These codes are new for the Standard Tables and should be included in ST.

**Description of Tables 3 and 4.**

We tabulate the parameters of the best known concatenated ($s, N, t$)-codes of strength $s = 2, 3$ and length $N \geq 45$. For values $N$ of the form $N = qn_0$, we give the parameters of the $(s, q, q')$-code and the shortened RS-code yielding the maximal possible size $t$ of the form $t = q_0^{k-r}$. To calculate the parameters of the shortened RS-code, we apply the particular cases of formulas (10) corresponding to $s = 2$ or $s = 3$. The groups of $t$ values for which $2^m \leq t < 2^{m+1}$, $m = 8, 9, \ldots$, are separated. The Reed-Solomon codes are a known class of maximum-distance separable codes (MDS-codes) [5, 28] which could be applied for our concatenated construction. In Table 3, we give the parameters of several concatenated codes, which could be obtained with the help of the Latin squares $q_0 \times q_0$, when $q_0 \geq 2$ is an arbitrary integer. We remind [5, 28] that any Latin square $q_0 \times q_0$ yields the $q_0$-ary MDS-code of size $q_0^n$, length $n_0 = 3$ and distance $d = 2$, i.e., $\lambda_0 = 1$.

**Table 4. Parameters of constant-weight concatenated $(3, N, t)$-codes of weight $w$, length $N$ and size $t$, $2^m \leq t < 2^{m+1}$, $m = 8, 9, \ldots, 62$**

| $m$ | $q$ | $q'$ | $w$ | $q_0$ | $k$ | $r$ | $t = q_0^{k-r}$ | $n_0$ | $\lambda_0$ | $N = qn_0$ | $w$ |
|-----|-----|-----|-----|------|-----|-----|----------------|------|---------|------------|-----|
| 8   | 7   | 7   | 7   | 4    | 1   | 4   | $7^3$         | 7    | 2       | 49         | 7   |
| 9   | 8   | 8   | 8   | 5    | 2   | 8   | $8^2$         | 7    | 2       | 56         | 7   |
| 10  | 11  | 11  | 11  | 8    | 5   | 11  | $11^2$        | 7    | 2       | 77         | 7   |
| 12  | 9   | 9   | 9   | 6    | 3   | 9   | $9^2$         | 10   | 3       | 90         | 10  |
| 13  | 11  | 11  | 11  | 6    | 2   | 11  | $11^2$        | 10   | 3       | 110        | 10  |
| 14  | 13  | 13  | 13  | 8    | 4   | 13  | $13^2$        | 10   | 3       | 130        | 10  |
| 15  | 22  | 37  | 37  | 34   | 3   | 34  | $34^2$        | 7    | 2       | 154        | 28  |
| 16  | 16  | 16  | 16  | 11   | 7   | 16  | $16^2$        | 10   | 3       | 160        | 10  |
| 17  | 13  | 13  | 13  | 6    | 1   | 13  | $13^2$        | 13   | 4       | 169        | 13  |
| 19  | 20  | 30  | 29  | 24   | 20  | 29  | $29^2$        | 10   | 3       | 200        | 40  |
| 21  | 16  | 20  | 19  | 12   | 7   | 19  | $19^2$        | 13   | 4       | 208        | 52  |
| 23  | 19  | 25  | 25  | 18   | 13  | 25  | $25^2$        | 13   | 4       | 247        | 52  |
| 25  | 16  | 20  | 19  | 10   | 4   | 19  | $19^2$        | 16   | 5       | 256        | 64  |
| 26  | 22  | 37  | 37  | 30   | 25  | 37  | $37^2$        | 13   | 4       | 286        | 52  |
| 29  | 16  | 20  | 19  | 8    | 1   | 19  | $19^2$        | 19   | 6       | 304        | 76  |
| 31  | 22  | 37  | 37  | 28   | 22  | 37  | $37^2$        | 16   | 5       | 352        | 64  |
| 37  | 19  | 25  | 25  | 12   | 4   | 25  | $25^2$        | 22   | 7       | 418        | 88  |
| 41  | 19  | 25  | 25  | 10   | 1   | 25  | $25^2$        | 25   | 8       | 475        | 100 |
| 46  | 22  | 37  | 37  | 22   | 14  | 37  | $37^2$        | 28   | 9       | 616        | 112 |
| 52  | 22  | 37  | 37  | 20   | 10  | 37  | $37^2$        | 28   | 9       | 616        | 112 |
| 62  | 22  | 37  | 37  | 16   | 4   | 37  | $37^2$        | 34   | 11      | 748        | 136 |

**Discussion.**

The preliminary conclusions could be formulated as follows.

- For fixed integers $s = 2, 3, \ldots$ and $m \geq 1$, we constructed the family $F = F(s, m)$ of binary concatenated $s$-disjunct constant-weight codes of size $t$, $2^m \leq t < 2^{m+1}$, based on the $q_0$-ary shortened RS-codes ($q_0$ satisfies the inequality in (10)) and the standard binary constant-weight codes.

- For $s = 2$, Table 3 shows the stability of the code rate, i.e., $\frac{w}{n} \approx 0.2 > 0.182$, $m = 11, 12, \ldots, 20$. It means that for $s = 2$, the code rate of the family $F$ exceeds the random coding rate $R(2) = 0.182$ [41].
• For $s = 3$ and $m = 15, 16, \ldots, 21$, the stable code rate $\frac{m}{N} \approx 0.1$ also exceeds the corresponding random coding rate $R(3) = 0.079$ [41].

• It seems to us that the calculations of the best parameters of $F = F(3, m)$, $m \geq 8$, are not completed and some values from Table 4 could be improved.

Example. For the length $N = 45 = 3 \cdot 15$, the $(s = 2, N = 45, t = 42^2 = 1764)$-code is the concatenated code based on the binary constant-weight $(n = 15, D = 6, w = 5)$-code of size $A(15, 6, 5) = 42$ and the Latin square $42 \times 42 = 1764$. 

4 Optimal Superimposed Codes and Designs for Renyi’s Search Model

In 1965, Renyi [7] suggested a combinatorial group testing model, in which the size of a testing group was restricted. In this model, Renyi considered the search of one defective element (significant factor) from the finite set of elements (factors). The corresponding optimal search designs were obtained by Katona [8]. In this section, we study Renyi’s search model of several significant factors. This problem is closely related to the concept of binary superimposed codes, which were introduced by Kautz-Singleton [6] and were investigated by Dyachkov-Rykov[31]. Our goal is to prove a lower bound on the search length and to construct the optimal superimposed codes and search designs. The results of Sect. 4 will be published in paper [54].

4.1 Notations and definitions

Let \( 1 \leq s < t, 1 \leq k < t, N > 1 \) be integers and \( X = \{x_i(u)\}, i = 1, 2, \ldots, N, u = 1, 2, \ldots, t, \) be a binary \((N \times t)\)-matrix (code) with columns (codewords) \( x(1), x(2), \ldots, x(t) \) and rows \( x_1, x_2, \ldots, x_N, \) where \( x(u) = (x_1(u), x_2(u), \ldots, x_N(u)) \) and \( x_i = (x_i(1), x_i(2), \ldots, x_i(t)) \) Let

\[
\begin{align*}
    w &= \min_u \sum_{i=1}^{N} x_i(u), \quad k = \max_i \sum_{u=1}^{t} x_i(u), \quad \lambda = \max_{u,v} \sum_{i=1}^{N} x_i(u)x_i(v)
\end{align*}
\]

be the minimal weight of codewords, the maximal weight of rows and the maximal dot product of codewords.

We say that the binary column \( x \) covers the binary column \( y \) if the Boolean sum \( x \vee y = x \). The code \( X \) is called [6, 31] a superimposed \((s, t)\)-code if the Boolean sum of any \( s \)-subset of columns of \( X \) covers those and only those columns of \( X \) which are the terms of the given Boolean sum. The code \( X \) is called [6, 31] a superimposed \((s, t)\)-design if all Boolean sums composed of not more than \( s \) columns of \( X \) are distinct.

Definition 1. An \((N \times t)\)-matrix \( X \) is called a superimposed \((s, t, k)\)-code (design) of length \( N \), size \( t \), strength \( s \) and constraint \( k \) if code \( X \) is a superimposed \((s, t)\)-code (design) whose the maximal row weight is equal to \( k \).

The above-mentioned constraint \( k \) was introduced by Renyi [7] and was studied by Katona [8] for the search designs.

4.2 Lower bound

Proposition 1. Let \( t > k \geq s \geq 2 \) and \( N > 1 \) be integers.

1. For any superimposed \((s - 1, t, k)\)-code \(((s, t, k) - \text{design}) X \) of length \( N \), the following inequality takes place

\[
N \geq \left\lceil \frac{st}{k} \right\rceil.
\]

2. If \( k \geq s + 1, st = kN \) and there exists the optimal superimposed \((s - 1, t, k)\)-code \( X \) of length \( N = st/k \), then
(a) code $X$ is a constant weight code of weight $w = s$, for any $i = 1, 2, \ldots, N$, the weight of row $x_i$ is equal to $k$ and the maximal dot product $\lambda = 1$;
(b) the following inequality is true
\[
k^2 - \frac{k(k-1)}{s} \leq t. \tag{2}
\]

**Proof.** 1. It is known [31] that code $X$ is a superimposed $(s, t, k)$-design if and only if $X$ is superimposed $(s-1, t, k)$-code and all $\binom{s}{i}$ Boolean sums composed of $s$ columns of $X$ are distinct. Hence, we need to prove inequality (1) for superimposed $(s-1, t, k)$-codes only. Let $s \geq 2$, $1 \leq k < t$ be fixed integers. Consider an arbitrary superimposed $(s-1, t, k)$-code $X$ of length $N$. Let $n, 0 \leq n \leq t$, be the number of codewords of $X$ having a weight $\leq s-1$. From definition of superimposed $(s-1, t)$-code it follows (see, [6]) that $n \leq N$ and, for each codeword of weight $\leq s-1$, there exists a row in which all the remaining elements, except for the element of this codeword, are 0’s. We delete these $n$ rows from $X$ together with $n$ codewords of weight $\leq s-1$. Consider the remaining $(N-n) \times (t-n)$ matrix $X'$.

Obviously, each column of $X'$ has a weight $\geq s$ and each its row contains $\leq k$ 1’s. Since $k \geq s$, we have
\[
s(t-n) \leq k(N-n), \quad ts \leq kN - n(k-s) \leq kN. \tag{3}
\]

Statement 1 is proved.

2. Let $k \geq s + 1$, $st = kN$ and $X$ be the optimal superimposed $(s-1, t, k)$-code of length $N = st/k$.

- Since $k \geq s + 1$, inequality (3) has signs of equalities if and only if $X$ is the constant weight code of weight $w = s$ and for any $i = 1, 2, \ldots, N$, the weight of row $x_i$ is equal to $k$. By contradiction, using the constant weight property $w = s$ one can easily check that the maximal dot product $\lambda = 1$.

Statements (2a) is proved.

- To prove Statement (2b), we apply the the well-known Johnson inequality
\[
t \binom{w}{\lambda + 1} \leq \binom{N}{\lambda + 1}
\]
which is true for any constant weight code $X$ of length $N$, size $t$, weight $w$ and the maximal dot product $\lambda$. In our case, $\lambda = 1$, $w = s$, $tw = kN$ and $N = st/k$. This gives
\[
tw(w-1) \leq N(N-1), \quad k(s-1) \leq N-1 = \frac{st}{k} - 1, \quad k^2(s-1) + k \leq st, \quad k^2 - \frac{k(k-1)}{s} \leq t.
\]

Proposition 1 is proved.

Denote by $N(s, t, k)$, $(\hat{N}(s, t, k))$ the minimal possible length of superimposed $(s, t, k)$-code ($(s, t, k)$-design). From Proposition 1 it follows:

- if $k \geq s + 1$, then
  \[
  \hat{N}(s, t, k) \geq N(s-1, t, k) \geq \left\lceil \frac{st}{k} \right\rceil.
  \]

- if $k \leq s$, then $N(s-1, t, k) = \hat{N}(s, t, k) = t$.

### 4.3 Optimal parameters

Let $s \geq 2$ and $k \geq s + 1$ be fixed integers. Denote by $q \geq 2$ an arbitrary integer. We shall consider the *optimal* superimposed $(s-1, kq, k)$-codes and *optimal* superimposed $(s, kq, k)$-designs of length $N = sq$ whose parameters satisfy (1) with the sign of equality. By virtue of (2)

- if $q \geq k - \frac{k-1}{s}$, then there exists a possibility to find the optimal superimposed $(s-1, kq, k)$-code of length $N = sq$;
• if \( q < k - \frac{k-1}{s} \), then lower bound (1) is not achieved and the interesting open problem is how to obtain a new nontrivial lower bound on \( N(s-1, t, k) \) provided that

\[
k^2 - \frac{k(k-1)}{s} > t.
\]

Some constructions of superimposed \((2, kq, k)\)-designs of length \( N = 2q \) and superimposed \((2, kq, k)\)-codes of length \( N = 3q \) were obtained in [6]. By virtue of Proposition 1, they are optimal. We give here the parameters of these designs and codes. The following statements are true:

• if \( k-1 \geq 2 \) is a prime power and \( q = k^2 - k + 1 \), then there exists an superimposed \((2, kq, k)\)-design of length \( N = 2q \);

• for pair \((k = 3, \ q = 5)\) and pair \((k = 7, \ q = 25)\), there exists an superimposed \((2, kq, k)\)-design of length \( N = 2q \);

• if \( k \geq 4 \) and \( q = k-1 \), or \( q = k \), then there exists an superimposed \((2, kq, k)\)-code of length \( N = 3q \).

Our aim – to prove Theorems 1–4.

**Theorem 1.** Let \( s = 2 \) and \( k \geq 3 \) be integers. Then

1) for any integers \( q \geq k \geq 3 \) there exists an optimal superimposed \((1, kq, k)\)-code of length \( N = 2q \), i.e., \( N(1, kq, k) = 2q, \ q \geq k \);

2) for any integer \( q \geq 2^k - 1 \) there exists an optimal superimposed \((2, kq, k)\)-design of length \( N = 2q \), i.e., \( N(2, kq, k) = 2q, \ q \geq 2^k - 1 \).

**Theorem 2.** Let \( s \geq 3 \), \( k \geq s + 1 \) be fixed integers and \( q = k^{s-1} \). Then there exists an optimal superimposed \((s, kq, k)\)-design \( X \) of length \( N = sq \), i.e., \( N(s, kq, k) = sk^{s-1} \).

**Theorem 3.** Let \( k = 4, 5, \ldots \) be a fixed integer. For any integer \( q \geq k + 1 \), there exists an optimal superimposed \((2, kq, k)\)-code of length \( N = 3q \), i.e., \( N(2, kq, k) = 3q, \ q \geq k + 1 \).

**Remark.** Let \( s \geq 3 \). For the case of superimposed \((s, kq, k)\)-codes, Theorem 3 is generalized (the proof is omitted) as follows. Let \( p_i, \ i = 1, 2 \ldots I \), be arbitrary prime numbers and \( r_i, \ i = 1, 2 \ldots I \), be arbitrary integers. If

\[
q = p_1^{r_1} p_2^{r_2} \cdots p_I^{r_I}, \quad 3 \leq s \leq \min_i \{p_i^{r_i}\} - 1,
\]

then for any \( k, \ s + 1 \leq k \leq q + 1 \), the optimal length \( N(s, kq, k) = (s+1)q \).

The following theorem supplements Theorem 2 if \( s = 3 \) and \( k = 4 \).

**Theorem 4.** If \( k = 4 \) and \( q \geq 12 \), then there exists an optimal superimposed \((3, kq, k)\)-design of length \( N = 3q \), i.e.,

\[
\tilde{N}(3, 4q, 4) = 3q, \quad q \geq 12.
\]

To prove Theorems 1–4, we apply concatenated codes using a class of homogeneous \( q \)-nary codes of size \( t = kq \). The description of cascade construction, definitions and properties of homogeneous \( q \)-nary codes will be given Sect. 4.4. The proofs of Theorems 1-4 will be given in Sect. 4.5–4.8.

The following theorem yields a different family of optimal superimposed \((s, t, k)\)-codes. It will be proved in Sect. 4.9.

**Theorem 5.** Let \( s \geq 1, \ k \geq s+2 \) be fixed integers. Then there exists an \((s, t, k)\)-code of size \( t = \binom{k+s}{s+1} \) and length

\[
N = \frac{(s+1)t}{k} = \frac{(s+1)\binom{k+s}{s+1}}{k} = \binom{k+s}{s},
\]

i.e., the optimal length

\[
N \binom{s}{k,s+1}, k = \binom{k+s}{s}.
\]

For Theorem 5, the optimal code constructions were invented in [49].
4.4 Homogeneous \( q \)-nary codes

Let \( q \geq s \geq 1 \), \( k \geq 2 \), \( k \leq t \leq kq \), \( J \geq 2 \) be integers, \( A_q = \{a_1, a_2, \ldots, a_q\} \) be an arbitrary \( q \)-ary alphabet and \( B = \{b_j(u)\}, \ j = 1, 2, \ldots, J \), \( u = 1, 2, \ldots, t \), be an \( q \)-nary \((b_j(u) \in A_q) (J \times t)\)-matrix (code) with \( t \) columns (codewords) and \( J \) rows

\[
b(u) = (b_1(u), b_2(u), \ldots, b_J(u)), \ u = 1, 2, \ldots, t, \quad b_j = (b_j(1), b_j(2), \ldots, b_j(t)), \ j = 1, 2, \ldots, J.
\]

Denote the number of \( a \)-entries in the \( j \)-th row \( b_j \) by \( n_j(a) \), where \( a \in A_q \), \( j = 1, 2, \ldots, J \). We suppose that for any \( j = 1, 2, \ldots, J \) and any \( a \in A_q \), the value \( n_j(a) \leq k \).

**Definition 2.** Let \( t = kq \). Code \( B \) is called an \((q, k, J)\)-homogeneous code if for any \( j = 1, 2, \ldots, J \) and any \( a \in A_q \), the number \( n_j(a) = k \).

**Definition 3.** Code \( B \) will be called an \( s \)-disjunct if for any codeword \( b(u) \) and any \( s \)-subset of codewords \( \{b(u_1), b(u_2), \ldots, b(u_s)\} \), there exists a coordinate \( j = 1, 2, \ldots, J \) for which \( b_j(u) \neq b_j(u_i) \), \( i = 1, 2, \ldots, s \).

For two codewords \( b(u), b(v) \), \( u \neq v \), define the \( q \)-ary Hamming distance

\[
D(b(u); b(v)) = \sum_{j=1}^{J} \chi(b_j(u); b_j(v)),
\]

\[
\chi(a; b) = \begin{cases} 
1, & \text{if } a \neq b, \\
0, & \text{if } a = b.
\end{cases}
\]

Let \( D = D(B) = \min_{u \neq v} D(b(u); b(v)) \) be the Hamming distance of code \( B \). By contradiction, one can easily prove the following statement which gives the analog of the well-known Kautz-Singleton [6] condition.

**Proposition 2.** If \( s(J - D(B)) \leq J - 1 \), then code \( B \) is \( s \)-disjunct. In addition, \((q, k, s)\)-homogeneous code \( B \) is \((s - 1)\)-disjunct code if and only if \( D(B) = s - 1 \).

Let \( n \leq t \) be a fixed integer and \( e = \{e_1, e_2, \ldots, e_n\}, \ 1 \leq e_1 < e_2 < \cdots < e_n \leq t \) be an arbitrary \( n \)-subset of the set \([t] = \{1, 2, \ldots, t\}\). For a given code \( B \) and any \( j = 1, 2, \ldots, J \), denote by \( A_j(e, B) \subseteq A_q \) the set of all pairwise distinct elements of the sequence \( b_j(e_1), b_j(e_2), \ldots, b_j(e_n) \). The set \( A_j(e, B) \) is called the \( j \)-th, \( 1 \leq j \leq J \), coordinate set of subset \( e \subseteq [t] \) over code \( B \). For its cardinality \( |A_j(e, B)| \), we have

\[
1 \leq |A_j(e, B)| \leq \min\{n, q\}.
\]

**Definition 4.** Let \( s \geq 1 \), \( n \leq s \), \( m \leq s \) be arbitrary integers. Code \( B \) is called an \( s \)-separable code if for any two distinct subsets

\[
e = \{e_1, e_2, \ldots, e_n\}, \ 1 \leq e_1 < e_2 < \cdots < e_n \leq t, \quad e' = \{e'_1, e'_2, \ldots, e'_m\}, \ 1 \leq e'_1 < e'_2 < \cdots < e'_m \leq t,
\]

of the set \([t]\), there exists \( j = 1, 2, \ldots, J \), for which the corresponding coordinate sets are distinct, i.e., \( A_j(e, B) \neq A_j(e', B) \). In other words, for an arbitrary \( n \)-subset \( e = \{e_1, e_2, \ldots, e_n\} \) of the set \([t]\), there exists the possibility to identify this \( n \)-subset \( e = \{e_1, e_2, \ldots, e_n\} \) (or the corresponding \( n \)-subset of codewords \( \{b(e_1), b(e_2), \ldots, b(e_n)\} \) of code \( B \)) on the basis of sets:

\[
A_1(e, B), A_2(e, B), \ldots, A_J(e, B), \quad A_j(e, B) \subseteq A_q.
\]

**Remark.** In Definitions 3 and 4, we used the terminology of [47]. One can easily prove (by contradiction) the following ordering among these properties:

\[
s\text{-disjunct} \implies s\text{-separable} \implies (s - 1)\text{-disjunct.}
\]
Definition 5. Code $B$ is called an $s$-hash [33] if for an arbitrary $s$-subset
\[ e = \{e_1, e_2, \ldots, e_s\}, \quad 1 \leq e_1 < e_2 < \cdots < e_s \leq t, \]
of the set $[t]$, there exists a coordinate $j = 1, 2, \ldots, J$, where the cardinality $|A_j(e, B)| = s$, i.e., the elements $b_j(e_1), b_j(e_2), \ldots, b_j(e_s)$ are all different.

Obviously, the following ordering takes place: $s$-hash $\implies$ $(s-1)$-disjunct.

Definition 6. Code $B$ is called an $s$-hash&separable if it has both of these properties.

Let $q$-nary alphabet $A_q = [q] = \{1, 2, \ldots, q\}$. To illustrate Definitions 2–6 and the proof of Theorem 1, we give two examples of disjunct and separable codes.

Example 1. Let $k = q = 2, 3, \ldots$ be fixed integers. The evident $(k, k, 2)$-homogeneous 1-disjunct code $B$ of distance $D = 1$ has the following $t = k^2$ columns (codewords):
\[
B = \begin{pmatrix}
111 & \cdots & 1 & 2 & \cdots & k & k & k
\end{pmatrix}.
\]

Example 2. For $k = 3$, $q = 7$, the $(7, 3, 2)$-homogeneous 2-hash&separable code $B$ of distance $D = 1$ has $kq = 21$ codewords:
\[
B = \begin{pmatrix}
111 & 222 & 333 & 444 & 555 & 666 & 777
123 & k & 123 & k & 123 & k & 123 & k
123 & 456 & 789 & 123 & 456 & 789 & 123 & 456 & 789
\end{pmatrix}.
\]

The idea of the following two examples of $(q, k, 3)$-homogeneous 3-hash&separable codes will be used to prove Theorem 2.

Example 3. For $k = 3$, $q = 9$, the $(9, 3, 3)$-homogeneous 3-hash&separable code $B$ of distance $D = 2$ has $kq = 27$ columns (codewords):
\[
B = \begin{pmatrix}
111 & 222 & 333 & 444 & 555 & 666 & 777
123 & 123 & 123 & 456 & 456 & 456 & 789 & 789 & 789
123 & 456 & 789 & 123 & 456 & 789 & 123 & 456 & 789
\end{pmatrix}.
\]

Code $B$ contains $k = 3$ groups of codewords. In the first and second rows, we use the construction idea which could be called an alphabet separating between groups.

Remark. Obviously, 3-separable code $B$ from example 3 is not 3-disjunct code. Hence, in general, the ordering $s$-separable$\implies$s-disjunct is not true.

Example 4. For $k = 4$, $q = 16$, the $(16, 4, 3)$-homogeneous 3-hash&separable code $B$ of distance $D = 2$ has $kq = 64$ columns (codewords) which are divided into $k = 4$ groups:

- the first 16 codewords take the form
  \[
  \begin{pmatrix}
  1111 & 2222 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4
  1234 & 1234 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4
  1234 & 5678 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
  \end{pmatrix}
  \]

- the construction of the last 48 codewords of $B$ applies the same method of alphabet separating:
  - for $j = 1, 2$ and $u = 16m + l$, $m = 1, 2, 3$, $l = 1, 2, \ldots, 16$, the element
    \[ b_j(u) = b_j(16m + l) = b_j(l) + 4m, \]
  - for $j = 3$ and $u = 16m + l$, $m = 1, 2, 3$, $l = 1, 2, \ldots, 16$, the element
    \[ b_3(u) = b_3(16m + l) = b_3(l) + l. \]
Let \( q \)-nary alphabet \( A_q = [q] = \{1, 2, \ldots, q\} \). For code \( B \), we denote by
\[
X_B = (x(1), x(2), \ldots, x(t)), \quad k \leq t \leq kq,
\]
a binary \( Jq \times t \) matrix (code), whose columns (codewords) have the form
\[
x(u) = (x^1(u), x^2(u), \ldots, x^s(u)), \quad u = 1, 2, \ldots, t,
\]
\[
x^j(u) = (x^j_1(u), x^j_2(u), \ldots, x^j_q(u)), \quad j = 1, 2, \ldots, J,
\]
\[
x^j_t(u) = \begin{cases} 1, & \text{if } l = b_j(u), \\ 0, & \text{if } l \neq b_j(u), l = 1, 2, \ldots, q. \end{cases}
\]

In other words, a symbol \( b \in [q] \) of \( q \)-ary matrix \( B \) is replaced by the binary \( q \)-sequence in which all elements are 0’s, except for the element with number \( b \). Obviously, each codeword \( x(u) \) of (code) \( X_B \) contains \( J \) 1’s and \( (Jq - J) \) 0’s and each row \( x_t \) of code \( X_B \) contains \( k \) 1’s. For \((q, k, J)\)-homogeneous code \( B \), each row \( x_i \) of code \( X_B \) contains \( k \) 1’s and \((kq - k) \) 0’s. In addition, the stated below Proposition 3 follows easily by Definitions 2–4 and Propositions 1–2.

**Proposition 3.** Let \( q > k \geq s + 1 \) and \( B \) be a \((q, k, s)\)-homogeneous code. The following two statements are true.

- If \( B \) is a \((s - 1)\)-disjunct code \( X_B \), then \( X_B \) will be the optimal superimposed \((s - 1, kq, k)\)-code of length \( N = sq \).
- If \( B \) is a \( s \)-separable code, then \( X_B \) will be the optimal superimposed \((s, kq, k)\)-design of length \( N = sq \).

Hence, to prove Theorems 1–4, it is sufficient to construct the corresponding \((q, k, s)\)-homogeneous codes. In particular, the constructive method of examples 3 and 4 yields Theorem 2 for the case \( s = 3 \), i.e., \( N(3, k^3, k) = 3k^2 \), \( k = 4, 5, \ldots \).

### 4.5 Proof of Theorem 1

Let \( s = 2, q \geq k \), \( q \)-nary alphabet \( A_q = [q] = \{1, 2, \ldots, q\} \) and \( B = (b(1), b(2), \ldots, b(kq)) \) be an arbitrary \((q, k, 2)\)-homogeneous code 1-disjunct code, i.e., \( B \) has pairwise distinct codewords \( b(u) = (b_1(u), b_2(u)), \quad u = 1, 2, \ldots, kq \). Following [6], we introduce the binary characteristic \((q \times q)\)-matrix \( C = \|c_i(j)\|, \quad i = 1, 2, \ldots, q, \quad j = 1, 2, \ldots, q, \) where
\[
c_i(j) = \begin{cases} 1, & \text{if there exists codeword } b(u) = (i, j), \\ 0, & \text{otherwise}. \end{cases}
\]

**Example 5.** For \((7, 3, 2)\)-homogeneous code \( B \) of Example 2, the characteristic \((7 \times 7)\)-matrix is
\[
C = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Obviously, the 1-disjunct code \( B \) is a \((q, k, 2)\)-homogeneous code if and only if the weight of each row and the weight of each column of \( C \) are equal to \( k \). It is not difficult to understand that this condition is true for any circulant matrix. The circulant matrix \( C \) is defined as follows:
• the first row \( c_1 = (c_1(1), c_1(2), \ldots, c_1(q)) \) of circulant matrix \( C \) is an arbitrary binary sequence of length \( q \) and weight \( k \leq q \),

• the \( m \)-th row \( c_m = ((c_{m}(1), c_{m}(2), \ldots, c_{m}(q)) \) of \( C \) is the cyclic shift of the \((m-1)\)-th row, i.e.,
\[
c_m(j) = \begin{cases} c_{m-1}(q), & \text{if } j = 1, \\ c_{m-1}(j-1), & \text{if } j = 2, 3, \ldots, q. \end{cases}
\]

The first statement of Theorem 1 is proved.

To prove the second statement of Theorem 1, we apply the evident necessary and sufficient condition of 2-separable property which is given in [6]: no two 1’s in \( C \) must occupy the same pair of rows and columns as two other 1’s; that is, no row of \( C \) can contain a pair of 1’s in the same two positions as another row.

It is easy to check that the circulant matrix \( C \) of Example 5 satisfies this condition. Let \( q \geq 2^k \). As the simple generalization, we consider the circulant matrix \( C \) whose first row \( c_1 = (c_1(1), c_1(2), \ldots, c_1(q)) \) is defined as follows
\[
c_1(j) = \begin{cases} 1, & \text{if } j = 2^{n-1}, n = 1, 2, \ldots, k, \\ 0, & \text{otherwise}. \end{cases}
\]

Theorem 1 is proved.

4.6 Proof of Theorem 2

The following Proposition 4 gives the recurrent construction method of \((s+1)\)- separable codes with the help of \( s \)-separable codes.

**Proposition 4.** If there exists an \((q, k, s)\)-homogeneous \( s \)-separable code \( B(q, k, s) \) with elements from \( A_q = [q] \), then there exists the \((kq, k, s+1)\)-homogeneous \((s+1)\)-separable code \( B(kq, k, s+1) \) with elements from \( A_{kq} = [kq] \).

**Proof.** Let
\[
B(q, k, s) = \|b_j^s(u)\|, \quad b_j^s(u) \in [q], \quad j = 1, 2, \ldots, s, \quad u = 1, 2, \ldots, kq
\]
be an arbitrary \((q, k, s)\)-homogeneous \( s \)-separable code. Consider the following two-step recurrent construction (cf. examples 3 and 4) for \((kq, k, s+1)\)-homogeneous code
\[
B(kq, k, s+1) = \|b_j^{s+1}(u)\|, \quad b_j^{s+1}(u) \in [kq], \quad j = 1, 2, \ldots, s + 1, \quad u = 1, 2, \ldots, k^2q.
\]

• The first \( kq \) codewords of \( B(kq, k, s+1) \) have the form
\[
\begin{array}{cccc}
b_1^1(1) & b_1^1(2) & \cdots & b_1^1(qk) \\
b_2^1(1) & b_2^1(2) & \cdots & b_2^1(kq) \\
\vdots & \vdots & \ddots & \vdots \\
& & & \\
b_s^1(1) & b_s^1(2) & \cdots & b_s^1(kq) \\
1 & 2 & \cdots & kq,
\end{array}
\]
i.e., for \( u = 1, 2, \ldots, kq \), the element \( b_j^{s+1}(u) = b_j^s(u) \), if \( j = 1, 2, \ldots, s \), and \( b_{s+1}^{s+1}(u) = u \).

• If the number \( u = kqm + l, \quad m = 1, 2, \ldots, k-1, \quad l = 1, 2, \ldots, kq \), then
  - for \( j = 1, 2, \ldots, s \), the element \( b_j^{s+1}(u) = b_j^s(kqm + l) = b_j^{s+1}(l) + qm \),
  - for \( j = s + 1 \), the element \( b_{s+1}^{s+1}(u) = b_{s+1}^{s+1}(kqm + l) = b_{s+1}^{s+1}(l) = l \).
Note that \( t = k^2q \) codewords of \( B(kq, k, s + 1) \) (or the set \([k^2q]\)) could be divided into \( k \) groups of the equal cardinality \( kq \) where the \( m \)-th group \( G_m(q, k, s) \), \( m = 1, 2, \ldots, k \), has the form

\[
G_m(q, k, s) = \begin{pmatrix}
b_1^1(1) + (m - 1)q & b_1^2(1) + (m - 1)q & \ldots & b_1^s(kq) + (m - 1)q \\
b_2^1(1) + (m - 1)q & b_2^2(1) + (m - 1)q & \ldots & b_2^s(kq) + (m - 1)q \\
\vdots & \vdots & \ddots & \vdots \\
b_s^1(1) + (m - 1)q & b_s^2(1) + (m - 1)q & \ldots & b_s^s(kq) + (m - 1)q \\
\end{pmatrix}.
\]

Let

\[
A_{kq}^{(m)} = \{(m - 1)kq + 1, (m - 1)kq + 2, \ldots, mkq\}, \quad |A_{kq}^{(m)}| = kq, \quad \bigcup_{m=1}^{k} A_{kq}^{(m)} = [k^2q],
\]

be the set of numbers of codewords which belong to \( G_m(q, k, s) \). Consider the \( s \times kq \) matrix \( B_m(q, k, s) \) composed of the first \( s \) rows of \( G_m(q, k, s), m = 1, 2, \ldots, k \). Obviously, \( B_m(q, k, s) \) is \((q, k, s)\)-homogeneous code. In addition, all elements of \( B_m(q, k, s) \) belong to the alphabet

\[
A_q^{(m)} = \{(m - 1)q + 1, (m - 1)q + 2, \ldots, mq\}, \quad |A_q^{(m)}| = q, \quad \bigcup_{m=1}^{k} A_q^{(m)} = [kq],
\]

and, hence, they do not may occur in \( B_n(q, k, s) \), if \( n \neq m, n = 1, 2, \ldots, k \). On account of the \( s \)-separable property of \( B(q, k, s) \), it follows the \( s \)-separable property of \( B_m(q, k, s), m = 1, 2, \ldots, k \).

To prove the \((s + 1)\)-separable property of \( B(kq, k, s + 1) \), we consider an arbitrary \((s + 1)\)-subset of the set \([k^2q] \): \( \mathbf{e} = \{e_1, e_2, \ldots, e_{s+1}\}, \quad 1 \leq e_1 < e_2 < \cdots < e_{s+1} \leq k^2q \). Let

\[
A_1(\mathbf{e}, B), A_2(\mathbf{e}, B), \ldots, A_s(\mathbf{e}, B), A_{s+1}(\mathbf{e}, B)
\]

be the corresponding subsets of the set \([kq] \) and

\[
\mathbf{e} = \sum_{m=1}^{k} \mathbf{e}_m, \quad \mathbf{e}_m = \mathbf{e} \cap A_{kq}^{(m)}.
\]

The above-mentioned property of groups \( G_m(q, k, s), m = 1, 2, \ldots, k \) implies that for any \( j = 1, 2, \ldots, s \), the set \( A_j(\mathbf{e}, B) \) could be written in the form

\[
A_j(\mathbf{e}, B) = \sum_{m=1}^{k} A_j(\mathbf{e}_m, B),
\]

where \( A_j(\mathbf{e}_m, B) \subseteq A_q^{(m)} \). Hence, for any fixed \( j = 1, 2, \ldots, s \), all nonempty sets \( A_j(\mathbf{e}_m, B), m = 1, 2, \ldots, k \), could be identified on the basis of the set \( A_j(\mathbf{e}, B) \).

We have two possibilities.

- There exists the unique value \( m = 1, 2, \ldots, k \) such that \( \mathbf{e}_m = \mathbf{e}, \ |\mathbf{e}_m| = s + 1 \). It follows that for any \( j = 1, 2, \ldots, s \), the set \( A_j(\mathbf{e}_m, B) \neq \emptyset \) and, for any \( n \neq m \), the set \( A_j(\mathbf{e}_n, B) = \emptyset \). Hence, one can identify the set \( \mathbf{e} \) on the basis of the set \( A_{s+1}(\mathbf{e}, B) \).

- For any \( m = 1, 2, \ldots, k \), the cardinality \( |\mathbf{e}_m| \leq s \). Accounting the \( s \)-separating property of \( B_m(q, k, s) \), the set \( \mathbf{e}_m \) could be identified on the basis of \( s \) subsets \( A_j(\mathbf{e}_m, B), j = 1, 2, \ldots, s \).

It follows the possibility to identify \( \mathbf{e} = \sum_{m=1}^{k} \mathbf{e}_m \).

Proposition 4 is proved.

Let an \((k, k, 2)\)-homogeneous 1-separable code \( B(k, k, 2) \) be the code from example 1. Consider the corresponding \((k^2, k, 3)\)-homogeneous code \( B(k^2, k, 3) \) obtained from \( B(k, k, 2) \) on the basis of Proposition 4. For \( k = 3, 4 \), constructions of \( B(k^2, k, 3) \) are given in examples 3 and 4. To prove Theorem 2, it is sufficient to establish the 3-separable property of code \( B = B(k^2, k, 3) \) for \( k = 4, 5, \ldots \).
We shall use symbols which were introduced to prove Proposition 4. Note that \( t = k^3 \) codewords of \( B \) could be divided (in increasing order) into \( k \) groups \( G_m(k, k; 2), \ m = 1, 2, \ldots, k \) of the equal cardinality \( k^2 \). Consider the \( 2 \times k^2 \) matrix \( B_m(k, k; 2) \) composed of the first 2 rows of \( G_m(k, k; 2), \ m = 1, 2, \ldots, k \). Obviously, \( B_m(k, k; 2) \) is the \((k, k, 2)\)-homogeneous 1-separable code. In addition, all elements of \( B_m(k, k, s) \) belong to the alphabet

\[
A_k^{[m]} = \{(m-1)k+1, (m-1)k+2, \ldots, mk\}, \quad |A_k^{[m]}| = k, \quad \bigcup_{m=1}^{k} A_k^{[m]} = \{k^2\},
\]

and, hence, they do not may occur in \( B_n(k, k; 2) \), if \( n \neq m, n = 1, 2, \ldots, k \).

Let \( e = \{e_1, e_2, e_3\}, \ 1 \leq e_1 < e_2 < e_3 \leq k^3 \) be an arbitrary fixed 3-subset of the set \( [k^3] \) and \( \{b(e_1), b(e_2), b(e_3)\} \) be the corresponding triple of codewords of code \( B \). To identify the codewords \( b(e_i), i = 1, 2, 3 \), using the properties of \( B_m(k, k; 2), \ m = 1, 2, \ldots, k \), mentioned above, it suffices to analyze the following three cases.

- There are known three numbers \( 1 \leq m_1 < m_2 < m_3 \leq k \) such that the codeword \( b(e_i), i = 1, 2, 3 \) belongs to the group \( G_{m_i}(k, k; 2) \). In this case, \( b(e_i) \) could be identified on the basis of 1-separable property of \( B_{m_i}(k, k, 2) \).

- There is known the number \( m = 1, 2, \ldots, k \) such that all three codewords \( b(e_1), b(e_2), b(e_3) \) belong to the group \( G_m(k, k; 2) \). In this case, the triple \( \{b(e_1), b(e_2), b(e_3)\} \), can be identified on the basis of the set \( A_3(e, B) \) whose cardinality \( |A_3(e, B)| = 3 \).

- There are known two numbers \( 1 \leq m < n \leq k \) such that (without loss of generality) codeword \( b(e_1) \) belongs to the group \( G_m(k, k; 2) \) and two other codewords \( b(e_2) \) and \( b(e_3) \) belong to the group \( G_n(k, k; 2) \). In this case, we have the following three-step identification:
  
  - the codeword \( b(e_1) = (b_1(e_1), b_2(e_1), b_3(e_1)) \) is identified on the basis of 1-separable property of \( B_m(k, k, 2) \),
  
  - the set \( \{b_3(e_2), b_3(e_3)\} \) evidently identified on the basis of symbol \( b_3(e_1) \) and the set \( A_3(e, B) \),

  - codewords \( b(e_2) \) and \( b(e_3) \) are identified on the basis of the set \( \{b_3(e_2), b_3(e_3)\} \).

Theorem 2 is proved.

4.7 Proof of Theorem 3

Let \( q \geq k + 1, \ k \geq 4, \) and \( q \)-ary alphabet \( A_q = [q] = \{1, 2, \ldots, q\} \). We need to construct \((q,k,3)\)-homogeneous code \( B \) of distance \( D(B) = 2 \). Consider the construction of \((q,k,3)\)-homogeneous code \( B = \|b_j(u)\| \) whose rows \( b_j = (b_j(1), b_j(2), \ldots, b_j(kq)) \), \( j = 1, 2, 3 \), are defined as follows:

1. for \( j = 1 \), the first row \( b_1 = (b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(q)}), \ b_1^{(m)} = (m, m, \ldots, m), \ m = 1, 2, \ldots, q; \)

2. for \( j = 2 \), the second row \( b_2 = (b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(k)}), \ b_2^{(m)} = (1, 2, \ldots, q), \ m = 1, 2, \ldots, k; \)

3. for \( j = 3 \), the third row \( b_3 = (b_3^{(1)}, b_3^{(2)}, \ldots, b_3^{(k)}), \) where the subsequence \( b_3^{(m)} \) of length \( q \) is the \((m-1)\)-step cyclic shift of the sequence \( (1, 2, \ldots, q) \):

\[
b_3^{(m)} = \begin{cases} 
(1, 2, \ldots, q), & \text{if } m = 1, \\
(m, m+1, \ldots, q-1, q, 1, 2, \ldots, m-1), & \text{if } m = 2, 3, \ldots, k.
\end{cases}
\]
Obviously, this construction guarantees the distance \( D(B) = 2 \). From Proposition 2 it follows 2-disjunct property of \( B \).

Theorem 3 is proved.

Example 6. As an illustration, we yield the \((6,4,3)\)-homogeneous 2-disjunct code \( B \) with \( kq = 24 \) codewords

\[
B = \begin{pmatrix}
111122 & 223333 & 444455 & 556666 \\
123456 & 123456 & 123456 & 123456 \\
123456 & 234561 & 345612 & 456123
\end{pmatrix}.
\]

Remark. For 2-disjunct code \( B \) of Example 6, it is easy to check the following properties.

- The 3-subsets \( e = \{2, 8, 13\} \) and \( e' = \{2, 7, 13\} \) of the set \([24]\) have the same coordinate sets, namely: \( A_1 = \{1, 2, 4\} \), \( A_2 = \{1, 2\} \) and \( A_3 = \{2, 3\} \). From this it follows that the code \( B \) is not 3-separable code, i.e., in general, the ordering \((s-1)\)-disjunct \(\Rightarrow s\)-separable is not true.

- The 3-subset \( e = \{1, 2, 7\} \) of the set \([24]\) has the equal coordinate sets \( A_1 = A_2 = A_3 = \{1, 2\} \) of cardinality 2. Hence, the code \( B \) is not 3-hash code, i.e., in general, the ordering \((s-1)\)-disjunct \(\Rightarrow s\)-hash is not true.

4.8 On \((q,k,3)\)-homogeneous 3-separable and 3-hash codes.

Proof of Theorem 4

4.8.1 Characteristic matrices

Consider an arbitrary \((q,k,3)\)-homogeneous 2-disjunct code \( B \). From Proposition 2 it follows that we can introduce characteristic \((q \times q)\)-matrix \( C = \|c_{i,j}\| \), \( i = 1, 2, \ldots, q, \) \( j = 1, 2, \ldots, q \), with elements from alphabet \( A_{q+1} = \{*, [q]\} = \{*, 1, 2, \ldots, q\} \), where

\[
c_{i,j} = \begin{cases} 
a, & \text{if there exists codeword } b(u) = (a, i, j), 
\ast, & \text{otherwise}. 
\end{cases}
\]

We shall say that code \( B \) is identified by the (characteristic) matrix \( C \) which will be called \( C(q,k)\)-matrix.

Example 7. For \( k = 4 \), \( q = 6 \), the \((6,4,3)\)-homogeneous 2-disjunct code \( B \) of Example 6 is identified by \( C(q,k)\)-matrix

\[
C = \begin{pmatrix}
1 & 2 & 4 & 5 & * & * \\
* & 1 & 2 & 4 & 5 & * \\
* & * & 1 & 3 & 4 & 6 \\
6 & * & * & 1 & 3 & 4 \\
5 & 6 & * & * & 2 & 3 \\
3 & 5 & 6 & * & * & 2
\end{pmatrix}.
\]

The evident characterization of \( C(q,k)\)-matrix is given by Proposition 5.

Proposition 5. The matrix \( C \) is \( C(q,k)\)-matrix if and only if \( C \) has the following properties:

- for any \( a \in [q] \), the number of \( a \)-entries in \( C \) is equal to \( k \),
- for any row (column) of \( C \), the number of \(*\)-entries in the row (column) is equal to \( q - k \),
- for any \( a \in [q] \) and any row (column) of \( C \), the number of \( a \)-entries in the row (column) does not exceed 1.
Remark. If \( q = k \), then \( C(q, q) \)-matrix is called the Latin square.

Characteristic matrix \( C \) of hash, separable and hash&separable code will be called \( C_H(q, k) \)-matrix, \( C_S(q, k) \)-matrix and \( C_{HS}(q, k) \)-matrix.

One can easily check the following characterization of \( C_H(q, k) \)-matrix.

**Proposition 6.** Matrix \( C \) is \( C_H(q, k) \)-matrix if and only if \( C \) has the properties of Proposition 5 and the following two equivalent conditions take place:

- if for \( i \neq m \) and \( j \neq n \), the element \( c_i(j) = c_m(n) = a \neq * \), then \( c_i(n) = c_m(j) = * \).
- if for \( i \neq m \), \( j \neq n \) and \( a \neq b \), code \( B \) contains codewords \((a, i, j)\) and \((a, m, n)\), then \( B \) does not contain the word \((b, m, j)\).

The evident characterization of \( C_{HS}(q, k) \)-matrix is given by Proposition 7.

**Proposition 7.** Let \( a, b \) and \( c \) be arbitrary pairwise distinct elements of \([q]\). Matrix \( C \) is \( C_{HS}(q, k) \)-matrix if and only if \( C \) has properties of Propositions 5 and 6 and the following property is true. Matrix \( C \) does not contain any \((3 \times 3)\)-submatrix of the form:

\[
\begin{pmatrix}
* & a & c \\
 a & * & * \\
 c & * & *
\end{pmatrix},
\begin{pmatrix}
 c & a & * \\
 b & * & * \\
 * & b & c
\end{pmatrix},
\begin{pmatrix}
* & c & a \\
 a & * & * \\
b & * & c
\end{pmatrix},
\begin{pmatrix}
 c & * & a \\
 a & c & * \\
 b & c & *
\end{pmatrix},
\begin{pmatrix}
 a & * & c \\
 * & a & b \\
 b & c & *
\end{pmatrix},
\begin{pmatrix}
 a & * & * \\
 * & a & b \\
 b & c & *
\end{pmatrix},
\begin{pmatrix}
 c & * & * \\
 * & c & b \\
 b & * & *
\end{pmatrix}.
\]

These prohibited matrices are the permutations of the same three columns.

**Remark.** The characterization of \( C_S(q, k) \)-matrix has a tedious form and it is omitted here. Below, we give the examples of \( C_S(q, k) \)-matrices which are not \( C_{HS}(q, k) \)-matrices.

### 4.8.2 Examples of hash, separable and hash&separable codes

Let an integer \( k \geq 2 \) be fixed. How to find the minimal possible integer \( q_k \geq k \) such that there exists \( C_S(q_k, k) \)-matrix, \( C_H(q_k, k) \)-matrix or \( C_{HS}(q_k, k) \)-matrix? From Examples 3 and 4 it follows that one can put \( q_k = k^2 \). For \( k = 2, 3, 4 \), the following Examples 8-10 improve this result and yield \( C_S(q_k, k), C_H(q_k, k) \) and \( C_{HS}(q_k, k) \)-matrices for which \( q_k < k^2 \). If \( k = 2, 3, 4 \) and \( q_k < q < k^2 \), then the corresponding characteristic matrices could be given also.

**Example 8.** For \( k = 2 \), \( C_S(q_2, 2) \)-matrix, \( C_H(q_2, 2) \)-matrix and \( C_{HS}(q_2, 2) \)-matrix are

\[
\begin{pmatrix}
 1 & 2 & * \\
 * & 1 & 3 \\
 3 & * & 2
\end{pmatrix},
\begin{pmatrix}
 3 & 1 & * \\
 1 & 2 & * \\
 3 & 2 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 3 \\
 * & 1 & 3 \\
 4 & 2 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 2 \\
 * & 1 & 5 \\
 1 & 7 & *
\end{pmatrix},
\begin{pmatrix}
 1 & 5 & * \\
 2 & 5 & * \\
 2 & 4 & 6
\end{pmatrix},
\begin{pmatrix}
 3 & 4 & 6 \\
 2 & 4 & 6 \\
 5 & 4 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 2 \\
 * & 1 & 5 \\
 2 & 7 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 3 \\
 * & 3 & 4 \\
 5 & 4 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 2 \\
 * & 1 & 5 \\
 2 & 7 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 3 \\
 * & 3 & 4 \\
 5 & 4 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 2 \\
 * & 1 & 5 \\
 2 & 7 & *
\end{pmatrix},
\begin{pmatrix}
 1 & * & 3 \\
 * & 3 & 4 \\
 5 & 4 & *
\end{pmatrix}.
\]

The first matrix \((q_2 = 3)\) identifies the separable (not hash) code. The second matrix \((q_2 = 3)\) identifies the hash (not separable) code. The third matrix \((q_2 = 4)\) is the particular case of Proposition 4.

**Example 9.** Let \( k = 3 \). For hash code \( q_3 = 6 \) and for hash&separable code \( q_3 = 7 \). The corresponding characteristic matrices are

\[
\begin{pmatrix}
 * & 1 & 2 & 3 & * \\
 * & 1 & 5 & 3 & * \\
 1 & * & * & 5 & 4 \\
 * & 2 & 5 & * & 6 \\
 2 & 4 & 6 & * & *
\end{pmatrix},
\begin{pmatrix}
 * & 1 & 2 & 3 & * \\
 * & 1 & 5 & 7 & * \\
 1 & * & * & 7 & 3 \\
 2 & 7 & * & * & 6 \\
 5 & 4 & 6 & * & *
\end{pmatrix}.
\]

\[1\text{Here and below, for 3-hash codes, we mark the pairs of “bad” triples which break the 3-separable property.}\]
Example 10. Let \( k = 4 \). For hash codes, \( q_4 = 8 \) and for hash-separable codes, \( q_4 = 13 \). The corresponding characteristic \( C_H(8,4) \) and \( C_{HS}(13,4) \)-matrices are
\[
\begin{pmatrix}
* & * & * & 1 & * & \frac{2}{3} & \frac{5}{6} \\
* & 1 & * & \frac{2}{4} & * & 6 & * \\
* & 1 & * & 3 & 4 & * & 7 \\
1 & * & * & 5 & 6 & 7 & * \\
* & 2 & 3 & 4 & * & * & 8 \\
2 & * & 5 & 6 & * & 8 & * \\
3 & 5 & * & 7 & 8 & * & * \\
4 & 6 & 7 & * & 8 & * & *
\end{pmatrix},
\]

\[
\begin{pmatrix}
* & * & * & * & 4 & 2 & 3 & 8 & * \\
* & * & * & * & 7 & 13 & 5 & 1 & * \\
* & * & * & * & 9 & 10 & * & 1 & 8 \\
* & * & * & * & 6 & 12 & 11 & * & 5 & * \\
* & * & * & * & * & 10 & 11 & 13 & 3 \\
* & * & * & * & * & * & 9 & 12 & 7 & 2 \\
1 & * & 13 & * & * & * & 6 & * & 4 \\
2 & 7 & 9 & 12 & * & * & * & * & * & * \\
3 & 6 & 11 & 10 & * & * & * & * & * & * \\
5 & * & * & 11 & 9 & 13 & * & * & * & * \\
8 & 4 & * & * & 10 & 12 & * & * & * & * \\
* & * & 4 & 5 & 6 & 7 & * & * & * & * \\
* & * & 8 & 3 & 2 & 1 & * & * & * & *
\end{pmatrix}.
\]

Open problems. 1. Is it possible to construct a \( C_{HS}(q,4) \)-matrix if \( q < 13 \)? 2. Is it possible to construct \( C_{HS}(q,k) \)-matrices, if \( k \geq 5 \) and \( q < k^2 \)?

4.8.3 Existence of hash and hash-separable codes

The following obvious Proposition 8 can be used to construct the new characteristic matrices using the known ones.

Proposition 8. (S.M. Yekhanin, 1998). Let \( v = 1,2 \) and there exist \( C_H(q_v,k) \)-matrix
\[
C^v = \|c^v_i(j)\|, \ i,j \in [q_v], \ c^v_i(j) \in \{*, [q_v]\}.
\]

Let \( \hat{C}^2 = \|\hat{c}^2_i(j)\| \) be the matrix whose element
\[
\hat{c}^2_i(j) = \begin{cases} q_1 + c^2_i(j), & \text{if } c^2_i(j) \neq *, \\ * , & \text{otherwise.} \end{cases}
\]

Then matrix
\[
C = \begin{pmatrix} C^1 & * \\ * & \hat{C}^2 \end{pmatrix}
\]
is a \( C_H(q_1+q_2,k) \)-matrix. The similar statement is also true for characteristic matrices of hash-separable codes.

With the help of the computer checking, we constructed the finite collection of ”non-regular” \( C_H(q,4) \)-matrices, \( q \geq 8 \), and \( C_{HS}(q,4) \)-matrices, \( q \geq 13 \). Taking into account Proposition 8, we obtain

Proposition 9. 1. If \( q \geq 8 \), then there exists \( C_H(q,4) \)-matrix. 2. If \( q \geq 13 \), then there exists \( C_{HS}(q,4) \)-matrix.
The following statement is a generalization of the hash&separable construction of Examples 3 and 4.

**Proposition 10.** If $q \geq k^2$, then there exists $(q, k, 3)$-homogeneous 3-hash code.

**Proof.** Let $k = 2, 3, \ldots$ and $q \geq k^2$. Consider the following construction of $(q, k, 3)$-homogeneous code $B = \|b_j(u)\|$ whose rows

$$b_j = (b_j(1), b_j(2), \ldots, b_j(kq)), \ j = 1, 2, 3,$$

are defined as follows:

1. for $j = 1$, the first row

$$b_1 = (b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(q)}), \quad b_1^{(m)} = (m, m, \ldots, m), \quad m = 1, 2, \ldots, q;$$

2. for $j = 2$, the second row

$$b_2 = (b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(k)}), \quad b_2^{(m)} = (1, 2, \ldots, q), \quad m = 1, 2, \ldots, k;$$

3. for $j = 3$, the third row $b_3 = (b_3^{(1)}, b_3^{(2)}, \ldots, b_3^{(k)})$, where the subsequence $b_3^{(m)}$, $m = 1, 2, \ldots, k$ of length $q$ is the $k(m-1)$-step cyclic shift of the sequence $(1, 2, \ldots, q)$:

$$b_3^{(m)} = \begin{cases} (1, 2, \ldots, q), & \text{if } m = 1, \\ (k(m-1) + 1, k(m-1) + 2, \ldots, q-1, q, 1, 2, \ldots, k(m-1)), & \text{if } m = 2, 3, \ldots, k. \end{cases}$$

As an illustration, we yield the $(11, 3, 3)$-homogeneous code

$$\begin{pmatrix} 111 & 222 & 333 & 444 & 555 & 666 & 777 & 888 & 999 & \text{aaa} & \text{bbb} \\ 123 & 456 & 789 & \text{ab1} & 234 & 567 & 89a & b12 & 345 & 678 & \text{9ab} \\ 123 & 456 & 789 & \text{ab4} & 567 & 89a & b12 & 378 & 9ab & 123 & 456 \end{pmatrix},$$

where, for convenience of notations, we put $a = 10$, $b = 11$.

If $q \geq k^2$, then this construction of $(q, k, 3)$-homogeneous code $B$ has an evident property of alphabet separation, which can be formulated as follows. Let the symbol $\oplus$ denote modulo $kq$ addition and $u = 1, 2, \ldots, kq$ be an arbitrary fixed integer. Then $q$-ary elements of the $k$-subsequence

$$b_3(u), b_3(u \oplus 1), b_3(u \oplus 2), \ldots, b_3(u \oplus (k-1))$$

do not may occur in the $k$-subsequence

$$b_3(u \oplus q), b_3(u \oplus (q+1)), b_3(u \oplus (q+2)), \ldots, b_3(u \oplus (q+k-1)).$$

By virtue of the second condition of Proposition 6, it implies 3-hash property of code $B$. Proposition 10 is proved.

**Conjecture.** The construction of Proposition 10 yields hash&separable codes.

### 4.8.4 Product of characteristic matrices

In this section, we consider a construction of homogeneous codes, which makes possible to obtain the new (more complicated) codes using the known ones.

Let $v = 1, 2$ and $C^v = \|c_i^v(j)\|$, $i, j \in [q_v]$, $c_i^v(j) \in \{*, [q_v]\}$, be $C(q_v, k_v)$-matrix of code $B_v$. Denote by

$$C = C^1 \circ C^2 = \|c_r(u)\|, \ r, u \in [q_1q_2], \ c_r(u) \in \{*, [q_1q_2]\}$$
the product of characteristic matrices of code $B_1$ and code $B_2$. Matrix $C$ is defined as follows: for arbitrary $i, j \in [q_1]$ and $l, m \in [q_2]$, put

$$r = q_2(i - 1) + l, \quad u = q_2(j - 1) + m,$$

$$c_r(u) = \begin{cases} q_2(c^1_i(j) - 1) + c^2_l(m), & \text{if } c^1_i(j) \neq * \text{ and } c^2_l(m) \neq *, \\ * & \text{otherwise.} \end{cases}$$

Example 11. Let $k_1 = k_2 = 2$, $q_1 = q_2 = 3$, and

$$C_H(q_1, k_1) = C_H(q_2, k_2) = \begin{pmatrix} * & 1 & 2 \\ 1 & * & 3 \\ 2 & 3 & * \end{pmatrix}.$$ 

$$C_H(q_1q_2, k_1k_2) = C_H(q_1, k_1) \odot C_H(q_2, k_2) =$$

$$= \begin{pmatrix} * & * & * & 1 & 2 & * & 4 & 5 \\ * & * & 1 & * & 3 & 4 & * & 6 \\ * & * & 2 & 3 & 5 & 6 & * & * \\ 1 & 2 & * & * & * & 7 & 8 & * \\ 2 & 3 & * & * & * & 8 & 9 & * \\ * & 4 & 5 & * & 7 & 8 & * & * \\ 4 & 6 & 7 & 9 & * & * & * & * \\ 5 & 6 & 8 & 9 & * & * & * & * \end{pmatrix}.$$

Such product of matrices remains the hash property.

Example 12. Let $k_1 = k_2 = 2$, $q_1 = q_2 = 3$, and

$$C_S(q_1, k_1) = C_S(q_2, k_2) = \begin{pmatrix} 1 & 2 & * \\ * & 1 & 3 \\ 3 & * & 2 \end{pmatrix}.$$ 

The product of matrices

$$C(q_1q_2, k_1k_2) = C_S(q_1, k_1) \odot C_S(q_2, k_2) =$$

$$= \begin{pmatrix} 1 & \hat{2} & 4 & 5 & * & * & * \\ * & 1 & 3 & \hat{4} & 6 & * & * \\ 3 & 2 & 6 & 5 & * & * & * \\ * & * & 1 & \hat{2} & * & 7 & 8 \\ * & * & 3 & 2 & \hat{9} & * & 8 \\ 7 & \hat{8} & * & * & 4 & 5 & * \\ * & \hat{7} & \hat{9} & * & * & 4 & 6 \\ 9 & \hat{8} & * & * & \hat{6} & * & 5 \end{pmatrix}$$

does not remain the separable properties of factors. In the figure, we have marked three pairs of "bad" triples, namely:

$$\{(1, \hat{2}, 4) (1, \hat{2}, \hat{4})\}, \quad \{(\hat{4}, \hat{7}, \hat{8}) (\hat{1}, \hat{7}, \hat{8})\}, \quad \{(\hat{6}, \hat{8}, \hat{9}) (\hat{6}, \hat{8}, \hat{9})\}.$$ 

This example shows the reason why the separable property of the product of two separable matrices is not true. To guarantee the separable property of the product of two separable matrices, at least one of two factors should have hash&separable property. The following Proposition takes place.
Proposition 11. 1. The product of $C_H(q_1, k_1)$-matrix and $C_H(q_2, k_2)$-matrix is $C_H(q_1q_2, k_1k_2)$-matrix. 2. The product of $C_S(q_1, k_1)$-matrix and $C_{HS}(q_2, k_2)$-matrix is $C_S(q_1q_2, k_1k_2)$-matrix. In addition, if the product of two separable matrices has the separable property, then at least one of these factors should have the hash & separable property. (S.M. Yekhanin, 1998).

To explain the second statement of Proposition 11, we give the following example.

Example 13. Let $k_1 = k_2 = 2$, $q_1 = 3$, $q_2 = 4$, and

$$C_S(q_1, k_1) = \begin{pmatrix} 1 & 2 & * \\ * & 1 & 3 \\ 3 & * & 2 \end{pmatrix}, \quad C_{HS}(q_2, k_2) = \begin{pmatrix} 1 & * & 3 \\ * & 1 & * \\ 4 & * & 2 \\ * & 4 & 2 \end{pmatrix}. $$

The product $C_S(q_1, k_1) \circ C_{HS}(q_2, k_2)$ has the form

$$\begin{pmatrix} 1 & * & 3 & * & 5 & * & 7 & * & * & * & * \\ * & 1 & * & 3 & 5 & * & 7 & * & * & * \\ 4 & * & 2 & * & 8 & 6 & * & * & * & * \\ * & 4 & * & 2 & 8 & 6 & * & * & * & * \\ * & * & * & 1 & 3 & 9 & 11 & * & * & * \\ * & * & * & 1 & 3 & 9 & 11 & * & * & * \\ * & * & * & 4 & 2 & 12 & 10 & * & * & * \\ 9 & 11 & * & * & * & 5 & 7 & * & * & * \\ 9 & 11 & * & * & * & 5 & 7 & * & * & * \\ 12 & 10 & * & * & * & 8 & 6 & * & * & * \\ 12 & 10 & * & * & * & 8 & 6 & * & * & * \end{pmatrix}$$

which illustrates its separable property.

The changed order product $C_{HS}(q_2, k_2) \circ C_S(q_1, k_1)$ also remains the separable property and has the form

$$\begin{pmatrix} 1 & 2 & * & * & * & 7 & 8 & * & * & * \\ * & 1 & 3 & * & * & 7 & 9 & * & * & * \\ 3 & * & 2 & * & * & 9 & 8 & * & * & * \\ * & * & 1 & 2 & * & * & 7 & 8 & * \\ * & * & 1 & 3 & * & * & 7 & 9 & * \\ * & * & 3 & 2 & * & * & 9 & 8 & * \\ 10 & 11 & * & * & * & 4 & 5 & * & * & * \\ * & 10 & 12 & * & * & 4 & 5 & * & * & * \\ * & * & 10 & 11 & * & * & 4 & 5 & * \\ * & * & 10 & 12 & * & * & 4 & 6 & * \\ * & * & 12 & 11 & * & * & 6 & 5 & * \end{pmatrix}.$$ 

From Propositions 9, 11 and Example 13 it follows the statement of Theorem 4.

4.9 Proof of Theorem 5

Let $s \geq 2$, $l \geq 1$ be fixed integers and $n > 2s + l$ be an arbitrary integer. Let $[n]$ be the set of integers from 1 to $n$ and $E(s, n)$ be the collection of all $\binom{n}{s}$ $s$-subsets of $[n]$. Following [49], we define the binary code $X = \|x_B(A)\|$, $B \in E(s, n)$, $A \in E(s + l, n)$, of size $t = \binom{n}{s+l}$ and length $N = \binom{n}{s}$, whose element
$x_B(A) = 1$ if and only if $B \subset A$. One can easily understand that $X$ is the constant weight code with parameters:

\[ t = \binom{n}{s+l}, \quad N = \binom{n}{s}, \quad k = \binom{n-s}{l}, \quad w = \binom{s+l}{s}, \quad \lambda = \binom{s+l-1}{s}, \]

where $t$–code size, $N$–code length, $w$–weight of columns (codewords), $k$–weight of rows and $\lambda$–the maximal dot product of codewords. In addition, let $A_0, A_1, \ldots, A_s, A_i \in \mathcal{E}(s+l,n)$ be an arbitrary $(s+1)$-collection of pairwise different $(s+1)$-subsets of $[n]$. Since $A_0 \neq A_i$, for any $i = 1, 2, \ldots, s$, there exists an element $a_i \in A_0$ and $a_i \notin A_i$. Hence, there exists a $s$-subset $B \subset A_0$ and for any $i = 1, 2, \ldots, s$, $B \not\subset A_i$. It follows that $X$ is a superimposed $(s, t, k)$-code. For the particular case $l = 1$, these properties yield Theorem 5.
5 Adder channel model and $B_s$-codes

5.1 Statement of the problem and results

For the optimal parameters of A-model, we use the notations of Sect. 1, i.e., $t_A(s, N)$, $N_A(s, t)$ and $R_A(s)$. Let $1 \leq u_1 \leq u_2 \leq \ldots \leq u_s \leq t$. In what follows, the sum of columns $\sum_{k=1}^{s} x(u_k)$ is defined as a column of length $N$ whose $i$-th component, $i = 1, 2, \ldots, N$, is equal to the arithmetic sum $\sum_{k=1}^{s} x_i(u_k)$.

**Definition.** Matrix $X$ is called a $B_s$-code of length $N$ and size $t$, if all $\binom{t+s-1}{s}$ sums of its columns $\sum_{k=1}^{s} x(u_k)$, where $1 \leq u_1 \leq u_2 \leq \ldots \leq u_s \leq t$, are distinct.

Denote by $N(s, t)$, $(t(s, N))$ the minimal (maximal) possible number of rows (columns) of $B_s$-code. For fixed $s \geq 2$ define the number

$$R(s) \triangleq \lim_{N \to \infty} \frac{\log t(s, N)}{N},$$

which is called a rate of $B_s$-code.

Obviously, if $s = 2$, then the definition of $B_2$-code and the definition of $(2, N)$-design are equivalent. Hence, $N(2, t) = N_A(2, t)$, $t(2, N) = t_A(2, N)$, $R(2) = R_A(2)$, while for $s \geq 3$

$$t_A(s, N) \geq t(s, N), \quad N_A(s, t) \leq N(s, t), \quad R_A(s) \geq R(s).$$

The concept of $B_s$-code (called a $B_s$-sequence in [11]) was motivated by the concept of $B_s$-sequence, introduced by Erdos in [1]. In this section we give a survey of the known upper and lower bounds on $R_A(s)$ and $R(s)$.

5.1.1 Upper bounds

The upper bounds are given as Theorems 1 and 2. To formulate Theorem 1, we introduce some notations. Let

$$b_s(k, p) \triangleq \binom{s}{k} p^k (1 - p)^{s-k}, \quad 0 \leq p \leq 1, \quad 0 \leq k \leq s \quad (1)$$

be binomial probabilities and

$$H_s(p) \triangleq - \sum_{k=0}^{s} b_s(k, p) \log b_s(k, p) \quad (2)$$

be the Shannon entropy of the binomial distribution with parameters $(s, p)$.

**Theorem 1.** For any $s \geq 2$ the rate

$$R(s) \leq R_A(s) \leq H_s/s, \quad (3)$$

where

$$H_s \triangleq H_s(1/2) = - \sum_{k=0}^{s} \binom{s}{k} 2^{-s} \log \left( \binom{s}{k} 2^{-s} \right). \quad (4)$$
Theorem 1 is called an entropy bound and its proof will be given in Sect. 5.2.

Remark. Let \( p, 0 \leq p \leq 1 \) be fixed and \( s \to \infty \). Using the Moivre-Laplace local limit theorem, one can prove [50] that

\[
H_s(p) = \frac{1}{2} \log s + \frac{1}{2} \log 2\pi e(1 - p) + O(1/s),
\]

and for the particular case \( p = 1/2 \)

\[
H_s(1/2) = \frac{1}{2} \log s + \frac{1}{2} \log \frac{\pi e}{2} + O(1/s^2).
\]

It follows that for \( s \gg 1 \) upper bound (3) can be written as

\[
R(s) \leq R_A(s) \leq \log \frac{s}{2s} + \frac{1}{2s} \log \frac{\pi e}{2} + O(1/s^3).
\]

Theorem 2. [21, 30] For any \( s = 1, 2, \ldots \) the rate

\[
R(2s) \leq \left[ sH_s^{-1} + sh_s^{-1} \right]^{-1},
\]

and for any \( s = 2, 3, \ldots \) the rate

\[
R(2s - 1) \leq \left[ sH_s^{-1} + (s - 1)h_s^{-1} \right]^{-1},
\]

where \( H_s \) is defined by (4) and

\[
h_s \equiv \log(s + 1) + s(s + 1)^{-1}.
\]

For \( B_2 \)-codes, Theorem 2 gives \( R(2) \leq 3/5 \). For the first time this result was obtained in [11]. Direct calculations show that for \( B_s \)-codes, where \( s = 2, 3, \ldots, 10 \), bounds (6) and (7) are better than entropy bound (3). Bounds (6) and (7) demonstrate the possibility of improving the entropy bound only for \( B_s \)-codes. Analogous improvement of the entropy bound for \( (s, N) \)-designs, when \( s \geq 3 \), remains an open question. In Sect. 5.2, we give the proof of Theorem 2 only for the simplest case \( s = 2 \).

Remark. For the particular case of \( B_4 \)-codes, bound (6) was slightly improved in [23].

5.1.2 Lower bounds

It was shown in [12] that the Bose theorem from additive number theory [3] yields \( B_s \)-codes with parameters \( N = ks, t = 2^k \), where \( k = 1, 2, \ldots \). This result gives the lower bound

\[
R_A(s) \geq R(s) \geq 1/s.
\]

The following theorem will be proved in Sect. 5.3 by the random coding method.

Theorem 3. [21, 36]. For any \( s = 2, 3, \ldots \)

\[
R_A(s) \geq R(s) \geq \frac{\tilde{H}_s}{2s - 1},
\]

where

\[
\tilde{H}_s \equiv \log \frac{2^{2s}}{\binom{2s}{s}} = \log \frac{(2s)!!}{(2s - 1)!!}.
\]
It is easy to see that bound (9)-(10) improves the Bose bound (8) for \( s \geq 3 \). If \( s = 2 \), then the Bose bound is better than (9)-(10).

**Remark.** Let \( s \to \infty \). With the help of the Stirling formula \( s! \sim s^s e^{-s} \sqrt{2\pi s} \), one can prove that

\[
\tilde{H}_s = \frac{1}{2} \log \pi s + \frac{1}{8s \ln 2} + O(s^{-2}).
\]

It follows that for \( s \gg 1 \) the rate lower bound (9) could be written as

\[
R_A(s) \geq \frac{\log s}{4s} + \frac{1}{4s} \log \pi + o(s^{-1}).
\] (11)

Hence, as \( s \to \infty \), the ratio of upper bound (5) to lower bound (11) tends to 2.

### 5.2 Proof of upper bounds on \( R_A(s) \) and \( R(s) \)

#### 5.2.1 Proof of Theorem 1

**Lemma.** [16]. Entropy (2) takes its maximal possible value at \( p = 1/2 \), i.e.,

\[
\max_{0 \leq p \leq 1} H_s(p) = H_s(1/2) = H_s.
\]

**Proof of Lemma.** We will use logarithms to the base \( e \). Denote by the symbols \( f'(p) \) and \( f''(p) \) the first and the second derivatives of \( f(p) \) with respect to \( p \). Taking into account (1) and the definition of generalized binomial coefficients (see Sect. 1.3), one can easily check that

1) \( b'_s(k, p) = sb_{s-1}(k-1, p) - sb_s(k, p) \).

In addition, it is evident that

2) \( \sum_{k=0}^{s} b'_s(k, p) = 0, \quad 0 \leq p \leq 1. \)

It follows from 1) and 2) that

3) \( H'_s(p) = -\sum_{k=1}^{s} sb_{s-1}(k-1, p) \ln b_s(k, p) + \sum_{k=0}^{s-1} sb_{s-1}(k, p) \ln b_s(k, p). \)

The formula 3) could be written as

4) \( H'_s(p) = s \sum_{k=0}^{s-1} b_{s-1}(k, p) \ln \frac{(k+1)(1-p)}{(s-k)p}. \)

Put

5) \( A_s(p) \triangleq \sum_{k=0}^{s} b_s(k, p) \ln \frac{k+1}{p}. \)

With the help of 5) we can rewrite 4) in the form

6) \( H'_s(p) = s[A_{s-1}(p) - A_{s-1}(1-p)]. \)
Therefore, \( H'(1/2) = 0 \). Further, we want to prove that the function \( H_s(p) \) is convex \( \cap \). It is sufficient to prove Lemma. The equality \( 6 \) means that the second derivative \( H'_s'(p) = s[A'_s-1(p) + A'_s-1(1-p)] \). Hence, we need to prove that \( A'_s-1(p) < 0 \) for any \( p \), \( 0 < p < 1 \). The equalities \( 1 \), \( 2 \) and \( 5 \) yield

\[
A'_s(p) = \frac{1}{p} + \sum_{k=0}^{s} \ln(k+1)b'_s(k, p) =
\]

\[
= -\frac{1}{p} + s \left[ \sum_{k=1}^{s} b_{s-1}(k-1, p) \ln(k+1) - \sum_{k=0}^{s-1} b_{s-1}(k, p) \ln(k+1) \right] = -\frac{1}{p} + s \sum_{k=0}^{s-1} b_{s-1}(k, p) \ln \frac{k+2}{k+1} =
\]

\[
= -\frac{1}{p} + \frac{1}{p} \sum_{k=0}^{s-1} \frac{s!(1-p)^{s-k-1}p^{k+1}}{(k+1)!(s-k-1)!} (k+1) \ln \frac{k+2}{k+1} = \frac{1}{p} \left[ -1 + \sum_{k=0}^{s-1} b_s(k+1, p) \ln \left( \frac{k+2}{k+1} \right)^{k+1} \right].
\]

Taking into account the well-known inequality \( (\frac{k+2}{k+1})^{k+1} = \left( 1 + \frac{1}{k+1} \right)^{k+1} < e \), we have

\[
A'_s(p) < \frac{1}{p} \left[ -1 + \sum_{k=1}^{s} b_s(k, p) \right] < 0.
\]

Lemma is proved.

**Proof of Theorem 1.** Let \( X \) be an arbitrary \((s,t)\)-design of length \( N \). Introduce the random variable (message)

\[
e = (e_1, e_2, \ldots, e_s), \quad e_i \in \{t\}, \quad 1 \leq e_1 < e_2 < \cdots < e_s \leq t,
\]

which takes equiprobable values in the set \( \mathcal{E}(s,t) \). The Shannon entropy of \( e \) is \( H(e) = \log \left( \begin{bmatrix} t \\ s \end{bmatrix} \right) \). For \( X \) and \( e \), consider the random variable \( z = z(e, X) = (z_1, z_2, \ldots, z_N) \). The definition of an \((s,t)\)-design \( X \) implies that the Shannon entropy of \( z \) is

\[
H(z) = H(z(e, X)) = H(e) = \log \left( \begin{bmatrix} t \\ s \end{bmatrix} \right).
\]

Therefore, the well-known subadditivity property \([9, 20]\) of the Shannon entropy gives

\[
1) \quad \log \left( \begin{bmatrix} t \\ s \end{bmatrix} \right) \leq \sum_{i=1}^{N} H(z_i).
\]

Let \( r_i, \ i = 1, 2, \ldots, N \) denote the number of 1's in the \( i \)-th row of the matrix \( X \). It is easy to see that the random variable \( z_i \) has hypergeometric probability distribution

\[
\Pr\{z_i = k\} = \binom{k}{s} \binom{t-r_i}{s-k} / \binom{t}{s}, \quad \max\{0, s + r_i - t\} \leq k \leq \min\{s, r_i\}.
\]

Hence, inequality 1) implies that the design length

\[
2) \quad N \geq \log \left( \begin{bmatrix} t \\ s \end{bmatrix} / \max_{r_i} H(z_i) \right).
\]

Let \( s \geq 2 \) and \( p, \ 0 < p < 1 \) be fixed, \( t \to \infty \) and \( r_i \sim pt \). One can easily check that the entropy of hypergeometric distribution \( H(z_i) \sim H_s(p) \). Lemma and inequality 2) yield inequality (3).

Theorem 1 is proved.
5.2.2 Proof of inequality $R(2) \leq 3/5$

Consider an arbitrary $(2,N)$-design $X = (x(1), x(2), \ldots, x(t))$ of size $t$. Let $n, 1 \leq n \leq N-1$ be fixed. Denote by

$$z = (z_1, z_2, \ldots, z_n) \in \{0,1\}^n \quad (y = (y_1, y_2, \ldots, y_{N-n}) \in \{0,1\}^{N-n})$$

an arbitrary column of length $n \cdot (N-n)$ with components from the alphabet $\{0,1\}$. Denote by $\mu(z, y)$ the number of columns of $X$, which have the form $(z, y)$. Obviously $\mu(z, y) \in \{0,1\}$. Let

$$\mu(z, \cdot) \triangleq \sum_y \mu(z, y)$$

be the number of columns, the first $n$ components of which coincide with $z$. We have

1) $$\sum_z \sum_y \mu(z, y) = \sum_z \mu(z, \cdot) = t.$$ 

For an $(N-n)$-ary column $w = (w_1, w_2, \ldots, w_{N-n}) \in \{-1,0,1\}^{N-n}$ and $(2,N)$-design $X$, define the number

2) $$m(w) \triangleq \sum_z \sum_y \mu(z, w+y) \mu(z, y).$$

**Lemma.** Let $X$ be $(2,N)$-design. Then the following two statements are true. 1) If $w = 0$, then $m(0) = t$. 2) If $w \neq 0$, then $m(w) \in \{0,1\}$. 

**Proof of Lemma.** The first statement follows from 1) and 2). The second statement is proved by contradiction. Let there exists $w \neq 0$ for which $m(w) \geq 2$. By symbols $u, u', v, v'$ we denote arbitrary elements of the set $[t]$. The definition 2) implies that there exist two pairs $(u,v), (u',v')$, $u \neq v$, $u' \neq v'$ for which the corresponding columns of $X$ have the following form

3) $$x(u) = (z, w+y), \quad x(v) = (z, y), \quad x(u') = (z', w+y'), \quad x(v') = (z', y'),$$

where either $z \neq z'$ or $y \neq y'$. It follows from 3) that

$$x(u) + x(v') = x(v) + x(u').$$

By virtue of the $(2,N)$-design definition, the last equality yields $u = u'$, $v = v'$, i.e., the both of inequalities $z = z'$, $y = y'$ are true.

Lemma is proved.

Define the number

4) $$l_n \triangleq \sum_{x \in \{0,1\}^n} \mu^2(x, \cdot) \geq t^2 2^{-n}.$$ 

Inequality 4) follows from 1) and the convex $\cup$ property of the function $y = x^2$. 1) and 2) implies

$$\sum_w m(w) = \sum_z \mu^2(z, \cdot) = l_n.$$ 

Therefore, we can define an $(N-n)$-ary random variable $\xi = (\xi_1, \xi_2, \ldots, \xi_{N-n}) \in \{-1,0,1\}^{N-n}$ with the distribution

5) $$\Pr\{\xi = w\} \triangleq \frac{m(w)}{l_n}.$$
Let the set \( W_n \triangleq \{ w : m(w) = 1 \} \). For the Shannon entropy of the random variable \( \xi \), we have
\[
H(\xi) \triangleq - \sum_w \Pr\{ \xi = w \} \log \Pr\{ \xi = w \} \geq - \sum_{{w \in W_n}} \Pr\{ \xi = w \} \log \Pr\{ \xi = w \} = (1 - t/l_n) \log l_n.
\]
Here we used 5) and Lemma. By virtue of 4),
\[
6) \quad H(\xi) \geq (1 - 2^n/t)(2 \log t - n).
\]
The subadditivity of the Shannon entropy gives
\[
7) \quad H(\xi) \leq \sum_{i=1}^{N-n} H(\xi_i).
\]
For any \( y = 0, 1 \) and any \( i = 1, 2, \ldots, N - n \) define the set
\[
Y_i(y) \triangleq \{ y = (y_1, y_2, \ldots, y_{N-n}) : y_i = y \}
\]
and introduce
\[
\mu_i(z, y) = \sum_{y \in Y_i(y)} \mu(z, y).
\]
We have \( \mu_i(z, 0) + \mu_i(z, 1) = \mu(z, \cdot) \) and, hence,
\[
8) \quad \mu_i(z, 0) + \mu_i(z, 1) \geq \mu^2(z, \cdot)/2.
\]
By virtue of 2) and 5), the probability
\[
\Pr\{ \xi_i = w \} = \sum_{z} \sum_{y=0}^{1} \mu_i(z, w + y) \mu_i(z, y)/l_n, \quad w \in \{ -1, 0, 1 \}.
\]
It follows, that \( \Pr\{ \xi_i = 1 \} = \Pr\{ \xi_i = -1 \} = (1 - p)/2 \), and
\[
9) \quad p \triangleq \Pr\{ \xi_i = 0 \} = \frac{\sum_z \sum_{y=0}^{1} \mu_i^2(z, y)}{l_n} \geq \frac{1}{2},
\]
where we used inequality 8) and definition 4).

Inequality 9) implies that
\[
10) \quad H(\xi_i) \leq \max_{1/2 \leq p \leq 1} \{ -(1 - p) \log \frac{1 - p}{2} - p \log p \}.
\]
It is easy to check, that maximum in 10) is equal 3/2 and this maximum is achieved at \( p = 1/2 \). On account of 6) and 7), it follows
\[
11) \quad (1 - 2^n/t)(2 \log t - n) \leq \frac{3}{2} (N - n), \quad n = 1, 2, \ldots, N - 1.
\]
Put \( n \triangleq \lfloor \log(t/ \log t) \rfloor \). Inequality 11) yields
\[
\log t \leq \frac{3}{5} \frac{N}{1 - 1/ \log t} + \frac{1}{2} \log \log t.
\]
Taking into account the definition of \( R(2) \), it follows the inequality \( R(2) \leq 3/5 \).

For \( B_2 \)-codes, Theorem 2 is proved.
5.3 Proof of Theorem 3

We say that the codeword \( x(u) \) is “bad” for code \( X \), if it is not satisfied the definition of \( B_s \)-code. This means, that there exist integers

\[
1 \leq i_1 \leq i_2 \leq \ldots \leq i_s \leq t, \quad 1 \leq j_1 \leq j_2 \leq \ldots \leq j_s \leq t,
\]

such that

\[
x(u) + x(i_1) + x(i_2) + \cdots + x(i_s) - x(j_1) - x(j_2) - \cdots - x(j_s).
\]

Consider the random matrix (code) \( X = \| x(u) \|, \quad i = 1, 2, \ldots, N, \quad u = 1, 2, \ldots, t \), whose components are independent identically distributed random variables with distribution

\[
\text{e}
\]

The following upper bound on the probability

\[
\Pr\{x_i(u) = 0\} = \Pr\{x_i(u) = 1\} = 1/2.
\]

For the codeword \( x(u) \), denote by \( P_s(N, t) \) the probability to be “bad”. Obviously, this probability does not depend on \( u = 1, 2, \ldots, N \) and the average number of “bad” words does not exceed \( t P_s(N, t) \). It follows

**Lemma 1.** If \( P_s(N, t) < 1/2 \), then there exists \( B_s \)-code of length \( N \) and size \( t/2 \).

The definition of the rate \( R(s) \) and Lemma 1 yield

**Lemma 2.** For any \( s \geq 2 \)

\[
R(s) \geq \sup \{ R : \lim_{N \to \infty} -\log P_s(N, \exp\{RN\}) \neq 0 \}
\]

Let \( m \leq k \leq s \) be integers. Denote by \( E(m, k) \) the collection of \( \binom{k-1}{m-1} \) ordered sequences of integers \( e = (e_1, e_2, \ldots, e_m) \) that satisfy the conditions

\[
1 \leq e_i \leq k, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} e_i = k.
\]

Let \( \beta_1, \beta_2, \ldots, \beta_s; \beta_1', \beta_2', \ldots, \beta_s' \) be the collection of \( 2 \cdot s \) independent random variables with the same distribution 1). The following upper bound on the probability \( P_s(N, t) \) is true:

\[
P_s(N, t) \leq K_s \sum_{1 \leq n \leq m \leq s} t^{n+m-1} Q_s(n, m),
\]

where the value \( K_s \) depends only on \( s \). To prove 2), we used the theorem that the probability of a union of events does not exceed the sum of their probabilities, and we also took into account that the number of all possible pairs of the form \( (A_m, A_m) \) (or \( A_n, A_m \)), \( 1 \leq n \leq m \leq s \), where \( A_k, \ k = 1, 2, \ldots \) is an \( k \)-subset of the set \( \{t\} \) and \( A_n \cap A_m = \emptyset \) (or \( A_n \cap A_m = \emptyset \)), is not greater than \( t^{n+m-1} \).

Inequality 2) implies that

\[
P_s(N, \exp\{RN\}) \leq K_s \sum_{1 \leq n \leq m \leq s} \exp\{-N[-\log Q_s(n, m) - (m + n - 1)R]\}.
\]

Lemma 2 and inequality 3) yield

**Lemma 3.** The \( B_s \)-code rate

\[
R(s) \geq R_s \triangleq \min_{1 \leq n \leq m \leq s} \frac{\log Q_s(n, m)}{(m + n - 1)}.
\]

48
Our subsequent aim will be to prove that minimum in 4) is achieved at \( n = m = s \) and the minimal value \( R_s = (2s - 1)^{-1} \tilde{H}_s \). It is sufficient to establish Theorem 3.

We introduce the independent random variables \( \mu_i = \beta_i - \beta'_i, \ i = 1, 2, \ldots, s \) with distribution

5) \[ \Pr\{\mu_i = 0\} = 1/2, \ \Pr\{\mu_i = -1\} = \Pr\{\mu_i = 1\} = 1/4. \]

**Lemma 4.** For any \( 1 \leq n \leq m \leq s \)

6) \[ Q_s(n, m) \leq \sqrt{Q_s(n, n)} \cdot \sqrt{Q_s(m, m)}, \]

where the sign of equality iff \( n = m \) and

7) \[ Q_s(m, m) \triangleq \max_{m \leq k \leq s} \max_{e \in E(m, k)} \Pr\{\sum_{i=1}^{m} e_i \mu_i = 0\}. \]

**Proof of Lemma 4.** Inequality 6) follows from the Cauchy inequality [2]:

\[ \sum p_i q_i \leq \sqrt{\sum p_i^2} \cdot \sqrt{\sum q_i^2}, \]

if the Cauchy inequality is used to obtain an upper bound on the probability of coincidence of two independent discrete random variables.

**Corollary.** Number \( R_s \), defined by 4), could be written in the form

8) \[ R_s = \min_{1 \leq n \leq m \leq s} - \log(Q_s(n, n) Q_s(m, m))^{1/2(m+n-1)}. \]

**Lemma 5.** For any collection of integers \( e_1, e_2, \ldots, e_m, e_i \geq 1 \), the probability

\[ \Pr\{\sum_{i=1}^{m} e_i \mu_i = 0\} \leq \Pr\{\sum_{i=1}^{m} \mu_i = 0\} = \frac{(2m - 1)!!}{(2m)!!}, \]

where the sign of equality iff \( e_1 = e_2 = \cdots = e_m \).

**Proof of Lemma 5.** Consider the characteristic functions of random variables \( e_j \mu_j \) and \( \mu_j \):

\[ \chi_j(u) = Me^{iue_j \mu_j} = \frac{1}{2}(1 + \cos e_j u), \ \chi(u) = Me^{i\mu_j} = \frac{1}{2}(1 + \cos u) = \cos^2(u/2), \]

where we used 5). The inversion formula [10] and the inequality between the geometric mean and the arithmetic mean [2] yield

\[ \Pr\{\sum_{i=1}^{m} e_i \mu_i = 0\} = (2\pi)^{-1} \int_{-\pi}^{\pi} (\prod_{j=1}^{m} \chi_j^m(u))^{1/m} \, du \leq \]

\[ \leq \frac{1}{m} \sum_{j=1}^{m} (2\pi)^{-1} \int_{-\pi}^{\pi} \chi_j^m(u) \, du = (2\pi)^{-1} \int_{-\pi}^{\pi} \chi^m(u) \, du = \]

\[ = (\pi)^{-1} \int_{-\pi/2}^{\pi/2} \cos^{2m} u \, du = \Pr\{\sum_{i=1}^{m} \mu_i = 0\} = \frac{(2m - 1)!!}{(2m)!!}. \]

The last equality follows from the well-known formula of the calculus:

\[ (\pi)^{-1} \int_{-\pi/2}^{\pi/2} \cos^{2m} u \, du = \frac{(2m - 1)!!}{(2m)!!}. \]
Lemma 5 is proved.

Lemma 5 and definition 7) imply that

\[ Q_s(m, m) = (2\pi)^{-1} \int_{-\pi}^{\pi} \chi^m(u) \, du = \frac{(2m - 1)!!}{(2m)!!}. \]  

Formula 9) and the property of monotonicity of the generalized mean [2] give

**Lemma 6.** Sequence \( Q_s(m, m)^{1/m}, m = 1, 2, \ldots, s, \) increases monotonically with \( m. \)

Put, for brevity, \( a_m \triangleq Q_s(m, m) < 1. \) For \( n \leq m, \) Lemma 6 gives \( a_n^{1/n} < a_m^{1/m}, \) where the sign of equality iff \( n = m. \) In addition, for \( n \leq m, \) the following inequalities are true

\[
    a_n \leq a_m^{n/m} \leq 1, \quad \frac{n + m}{n + m - 1} \geq \frac{2m}{2m - 1},
\]

where the sign of equality iff \( n = m. \) It follows that for any \( n \leq m, \)

\[
    (a_n a_m)^{1/2(n+m-1)} \leq (a_m^{1/2(m+n-1)})^{m-n+1} \leq (a_m^{1/2m})^{\frac{m}{m+n-1}} = a_m^{1/2m-1},
\]

where the sign of equality iff \( n = m. \) By virtue of 8) and 10), we have

\[
    R_s = \min_{m \leq s} -\frac{\log a_m}{2m - 1}.
\]

For \( m \leq s, \) by virtue of Lemma 6,

\[
    a_m^{1/2m-1} = (a_m^{1/2m})^{\frac{m}{m+n-1}} \leq (a_m^{1/2m})^{\frac{m}{m+n-1}} \leq a_m^{1/2m-1}.
\]

Hence

\[
    R_s = -\frac{\log a_s}{2s - 1} = \frac{\tilde{H}_s}{2s - 1}.
\]

Theorem 3 is proved.
6 Universal Decoding for Random Design of Screening Experiments

In this section, we consider the problem of screening experiment design (DSE) for the probabilistic model of multiple access channel (MAC). We will discuss the random design of screening experiments (random DSE) and present a method of universal decoding (U-decoding) which does not depend on transition probabilities of MAC. The logarithmic asymptotic behavior of error probability for the symmetric model of random DSE is obtained. Sect. 6 is based on the results of paper [42] which completed the series of preceding works [17, 18, 22].

6.1 Statement of the problem, formulation and discussion of results

We need to remind some notations from Section 1. Let \( 2 \leq s < t \) be fixed integers and
\[
eq (e_1, e_2, \ldots, e_s), \quad e_i \in [t], \quad 1 \leq e_1 < e_2 < \ldots < e_s \leq t,
\]
be an arbitrary \( s \)-subset of \([t]\). Introduce \( \mathcal{E}(s, t) \), the collection of all such subsets. Note that the cardinality \( |\mathcal{E}(s, t)| = \binom{t}{s} \).

Suppose that among \( t \) factors, numbered by integers from 1 to \( t \), there are some \( s < t \) unknown factors called significant factors. Each \( s \)-collection of significant factors is prescribed as an \( s \)-subset \( e \in \mathcal{E}(s, t) \). The problem of designing screening experiments (DSE) is to find all significant factors, i.e. to identify an unknown subset \( e \). To look for \( e \), one can carry out \( N \) experiments. Each experiment is a group test of a prescribed subset of \([t]\). These \( N \) subsets are interpreted as \( N \) rows of a binary \( N \times t \) matrix
\[
X = ||x_i(u)||, \quad x_i \in \{0; 1\}, \quad i = 1, 2, \ldots, N, \quad u = 1, 2, \ldots, t,
\]
where the elements of the \( i \)-th row \( x_i \triangleq (x_i(1), x_i(2), \ldots, x_i(t)) \) are defined as follows
\[
x_i(u) \triangleq \begin{cases}
1, & \text{if the } u \text{-th factor is included into the } i \text{-th test,} \\
0, & \text{otherwise.}
\end{cases}
\]
The matrix \( X \) is called a code or a design of experiments. The detailed discussion of the DSE problem is presented in the survey [29]. English translation of [29] is included in the book [35].

Let the symbol \( x(u) \triangleq (x_1(u), x_2(u), \ldots, x_N(u)) \in (0, 1)^N \), \( u = 1, 2, \ldots, t \), denote a column called a codeword of \( X \). The number of 1’s \( w_u \triangleq \sum_{i=1}^{N} x_i(u) \) is called a weight of the codeword \( x(u) \). We say that the given \( s \)-subset \( e = (e_1, e_2, \ldots, e_s) \) called a message is encoded into the non-ordered \( s \)-collection of codewords
\[
x(e) \triangleq (x(e_1), x(e_2), \ldots, x(e_s)).
\]

Let \( z = (z_1, z_2, \ldots, z_N) \) be a test outcome. To describe the model of such a test outcome, we will use the terminology of a memoryless multiple-access channel (MAC) [20], which has \( s \) inputs and one output. Let all \( s \) input alphabets of MAC are identical and coincide with \([0, 1]\). Denote by \( Z \) a finite output alphabet. This MAC is prescribed by a matrix of transition probabilities
\[
||P(z | x_1, x_2, \ldots, x_s)||, \quad z \in Z, \quad x_k \in \{0, 1\}, \quad k = 1, 2, \ldots, s.
\]
If the collection $x(e)$ was transmitted over MAC, then

$$P_N(z \mid x(e)) \triangleq \prod_{i=1}^N P(z_i \mid x_i(e_1), x_i(e_2), \ldots, x_i(e_s))$$

is the conditional probability to receive the word (outcome of test) $z \in Z^N$ at the output of MAC.

We will focus on the symmetric model of DSE, which is identified as a symmetric MAC. This means that for each fixed $z$, the conditional probability $P(z \mid x_1^s)$, where $x_i^s = (x_1, x_2, \ldots, x_s) \in \{0, 1\}^s$, depends only on the cardinal number of 1’s in the sequence $x_i^s$, i.e. on the sum $\sum_{k=1}^s x_k$. Thus, the conditional probability $P_N(z \mid x(e))$ does not depend on the order of MAC inputs.

Let $i = 1, 2, \ldots, N$ and $x_i(e) \triangleq (x_i(e_1), x_i(e_2), \ldots, x_i(e_s)) \in \{0, 1\}^s$ be the $i$-th row of the $s$ collection $x(e)$. Introduce the concept of composition $C(x(e), z)$ of a pair $(x(e), z)$, i.e. the collection of integers $\|n(x_1^s, z)\|, x_i^s \in \{0, 1\}^s, z \in Z$, where the element $n(x_1^s, z)$ is the cardinal number of positions $i = 1, 2, \ldots, N$, in which $x_i(e) = x_1^s, z_i = z$. Using this concept, the definition of MAC can be written in the form

$$P_N(z \mid x(e)) = \prod_{x_1^s} \prod_{z} P(z \mid x_1^s)^{n(x_1^s, z)}, \quad \text{where} \quad \|n(x_1^s, z)\| \triangleq C(x(e), z).$$

(1)

Notice that

$$\sum_{x_1^s} \sum_{z} n(x_1^s, z) = N.$$

Let $1 \leq v \leq k \leq s$ and the symbol $x_v^k \triangleq (x_v, x_{v+1}, \ldots, x_k)$ be a part of the sequence $x_1^s$. For the codeword $x(e_k)$, we introduce a marginal composition

$$C(x(e_k)) \triangleq \|n(x_k)\|, \quad x_k \in \{0, 1\}, \quad k = 1, 2, \ldots, s,$

where

$$n(x_k) \triangleq \sum_{x_1^k} \sum_{x_{k+1}^s} \sum_{z} n(x_1^s, z)$$

(2)

is the cardinal number of positions $i = 1, 2, \ldots, N$, in which $x_i(e_k) = x_k$. We will also use the similar symbols to denote another marginal composition. For instance, the marginal composition $\|n(x_{k+1}^s, z)\|$, $k = 1, 2, \ldots, s - 1$, is the composition with elements

$$n(x_{k+1}^s, z) \triangleq \sum_{x_1^s} n(x_1^s, z).$$

Let $Q \triangleq (Q(0), Q(1))$, $0 < Q(0) < 1$, $Q(1) = 1 - Q(0)$, be a fixed probability distribution on $\{0, 1\}$. For any pair $(x(e), z)$ we introduce an universal decoding (U-decoding) $D_Q$, which does not depend on transition probability of MAC:

$$D_Q(x(e), z) = D_Q(\|n(x_1^s)\|) \triangleq \prod_{x_1^s} \prod_{z} n(x_1^s, z)! \prod_{k=1}^s \prod_{x_k} Q(x_k)^{n(x_k)}, \quad \text{if} \quad C(x(e), z) = \|n(x_1^s, z)\|.$$ 

An average error probability of code $X$ and decoding $D_Q$ is defined as follows:

$$P_Q(X) \triangleq \left(\frac{t}{s}\right)^{-1} \sum_{e} \sum_{z} P_N(z \mid x(e)) \left[1 - \phi_Q(e, z)\right],$$

(3)

where $\phi_Q(e, z)$ is the characteristic function of the set

$$A_Q^e \triangleq \{z : \text{for any} \ e' \neq e \ \text{the value} \ D_Q(x(e), z) > D_Q(x(e'), z)\}.$$

Let the distribution $Q$ be fixed. We will consider two ensembles of codes with independent and identically distributed codewords $x(u), u = 1, 2, \ldots, t$. 

52
1. For a completely randomized ensemble (CRE), each codeword \( x(u) \) is taken with probability
\[
Q_N(x(u)) \triangleq \prod_{i=1}^{N} Q(x_i(u)).
\] (4)

2. For a constant-weight ensemble (CWE), the probability is
\[
Q_N(x(u)) \triangleq \begin{cases} 
\left( \prod_{i=1}^{N} Q(x_i(u)) \right)^{-1}, & \text{if the weight } w_u = \sum_{i=1}^{N} x_i(u) = \lfloor NQ(1) \rfloor, \\
0, & \text{otherwise}.
\end{cases}
\] (5)

Denote by \( \overline{P_Q(X)} \) the average error probability over ensembles (4) and (5). This probability is called a random coding bound for DSE [22].

Fix an arbitrary number \( R > 0 \) called a rate and consider the following asymptotic conditions
\[
N \to \infty, \quad t \to \infty, \quad \frac{\ln t}{N} \sim R, \quad s = \text{const}.
\] (6)

Our aim is to investigate the logarithmic asymptotic behavior of the random coding bound for CRE and CWE under conditions (6).

To formulate the results, we need the following notations. Let
\[
\tau \triangleq \{ \tau(x_i^1, z), \ x_1^s \in \{0,1\}^s, \ z \in Z, \ P(z|x_1^s) = 0 \Rightarrow \tau(z|x_1^s) = 0 \}
\]
be an arbitrary probability distribution on the product \( \{0,1\}^s \cdot Z \), such that the conditional probability \( \tau(z|x_1^s) = 0 \) if \( P(z|x_1) = 0 \). Consider the functions
\[
\mathcal{H}(Q, \tau) \triangleq \sum_{x_i^s \cdot z} \tau(x_i^s, z) \ln \frac{\tau(x_i^s, z)}{P(z|x_i^s) \cdot \prod_k Q(x_k)} \quad \text{and} \quad I^{(k)}(\tau) \triangleq \sum_{x_i^s \cdot z} \tau(x_i^s, z) \ln \frac{\tau(x_k^k|x_{k+1}^s, z)}{\prod_{i=1}^{k} Q(x_i)}
\]
where \( k = 1, 2, \ldots, s \) and the standard symbols for conditional probabilities are used. Let \( [a]^+ \triangleq \max\{0; a\} \). The following main result will be proved in Section 6.2.

**Theorem 1.** If conditions (6) are fulfilled, then
\[
\overline{P_Q(X)} = \exp\{-N[E(R, Q) + o(1)]\},
\]
where
\[
E(R, Q) \triangleq \min_{k=1} \ E_k(R, Q),
\] (8)
\[
E_k(R, Q) \triangleq \min\{\mathcal{H}(Q, \tau) + [I^{(k)}(\tau) - kR]^+]\}.
\] (9)

For CRE (4) the minimum in (9) is taken over all \( \tau \), and for CWE (5) the minimum in (9) is taken over distributions \( \tau \), for which the marginal probabilities on \( x_k \) are fixed and coincide with \( Q \), i.e., \( \tau(x_k) = Q(x_k), k = 1, 2, \ldots, s \).

**Remark 1.** One can easily understand that Theorem 1 remains also true for the optimal maximum likelihood decoding (ML-decoding), when the function (1) is used as the decoding rule. Hence, this theorem completes our preceding investigation [17, 18, 22] of the error probability asymptotic behavior for the symmetric model of random DSE.

**Remark 2.** Similar to the discrete memoryless channel [26] one can show that Theorem 1 for CRE remains true also, if (instead of CRE) we consider the ensemble, when all \( N \) rows of length \( t \) are chosen independently from the set of all binary \( t \)-sequences containing \( \lfloor Q(0) t \rfloor \) zeros and \( \lfloor Q(1) t \rfloor \) ones.
Note that the functions (7) are non-negative and convex as functions of \( \tau \). Hence, using the standard arguments [20], we can easily obtain the following properties of (9) which are given in the form of Propositions 1-3.

**Proposition 1.** Introduce the distribution
\[
\tau_Q \triangleq \left\{ P(z|x_i^s) \cdot \prod_{i=1}^{s} Q(x_i), \quad x_i^s \in \{0,1\}^s, \quad z \in Z \right\}.
\]
Then for \( R \geq I^{(k)}(\tau_Q)/k \), the function \( E_k(R, Q) = 0 \), and for \( 0 < R < I^{(k)}(\tau_Q)/k \), the function \( E_k(R, Q) \) is positive, convex and monotonically decreasing with increasing \( R \).

**Proposition 2.** For each \( k = 1, 2, \ldots, s \) there exists the unique distribution \( \tau_{cr}^{(k)} \), for which
\[
\min \{ \mathcal{H}(Q, \tau) + I^{(k)}(\tau) \} = \mathcal{H}(Q, \tau_{cr}^{(k)}) + I^{(k)}(\tau_{cr}^{(k)}).
\]
Conditions of minimization in (10) for CRE and CWE are pointed out in the formulation of Theorem 1. In addition,
\[
0 < I^{(k)}(\tau_{cr}^{(k)}) \leq I^{(k)}(\tau_Q).
\]

It is evident that the extreme distribution \( \tau_{cr}^{(k)} \) for CRE may not coincide with the extreme distribution \( \tau_{cr} \) for CWE.

**Proposition 3.** If \( 0 < R \leq I^{(k)}(\tau_{cr}^{(k)})/k \), then
\[
E_k(R, Q) = \mathcal{H}(Q, \tau_{cr}) + I^{(k)}(\tau_{cr}) - kR.
\]
If \( I^{(k)}(\tau_{cr}^{(k)})/k \leq R < I^{(k)}(\tau_Q)/k \), then the minimum in (9) is achieved at the unique distribution \( \tau_k \), for which
\[
R = \frac{I^{(k)}(\tau_k)}{k}; \quad E_k(R, Q) = \mathcal{H}(Q, \tau_k).
\]

Some useful relations between the functions \( I^{(k)}(\tau), k = 1, 2, \ldots, s \) are given by

**Proposition 4.** Fix arbitrary \( k = 2, 3, \ldots, s \). Let \( \tau_k \) be the extreme distribution satisfying (12). Then
\[
\frac{I^{(k)}(\tau_k)}{k} \leq \frac{I^{(k-1)}(\tau_k)}{k-1} \leq \ldots \leq I^{(1)}(\tau_k).
\]

**Proof.** By virtue of the MAC symmetry, the extreme distribution
\[
\tau_k = \{ \hat{x}(x_1^k, x_{k+1}^s, z), \quad x_1^k \in \{0,1\}^k, \quad x_{k+1}^s \in \{0,1\}^{s-k}, \quad z \in Z \}
\]
has the symmetry property as relative to components of \( x_1^k \) so to components of \( x_{k+1}^s \). For instance, for any \( v = 1, 2, \ldots, k \), probabilities \( \hat{\tau}(x_v, x_{v+1}^s, z) \) coincide with the corresponding probabilities \( \hat{\tau}(x_1, x_{v+1}^s, z) \) and the marginal probabilities \( \hat{\tau}(x_v) \) do not depend on \( v \). So
\[
\frac{I^{(v)}(\tau_k)}{v} = -\sum_{x_1} \hat{\tau}(x_1) \ln Q(x_1) - \frac{H_{\tau_k}(x_1|x_{v+1}^s, z)}{v},
\]
where we use the standard notation of the Shannon entropy of distribution \( \tau_k \). We have [20]
\[
H_{\tau_k}(x_1|x_{v+1}^s, z) = \sum_{i=2}^{v+1} H_{\tau_k}(x_i|x_{i-1}^s, z).
\]
Since
\[ H_{r_k}(x_i|x^*_i, z) \geq H_{r_k}(x_i|x^*_{i-1}, z), \quad i = 3, 4, \ldots, v + 1, \]
then by virtue of (14) and the monotone property of the arithmetic mean, we obtain the inequality (13). Proposition 4 is proved.

The behaviour of the random coding exponent (8) is described by

**Proposition 5.** The following properties are true.

- If \( R \geq I^s(\tau_Q)/s \), then \( E(R, Q) = 0 \), and if \( 0 < R < I^s(\tau_Q)/s \), then \( E(R, Q) \) is positive and monotonically decreasing with increasing \( R \).
- There exists the interval \( 0 < R \leq R_0 \), where \( E(R, Q) = E_1(R, Q) \).
- There exists the interval \( R_1 \leq R \leq I^s(\tau_Q)/s \), in which \( E(R, Q) = E_s(R, Q) \).

**Proof.** Using the definition (7), it is easy to verify that for any fixed distribution \( \tau \), the sequence \( I^k(\tau) \) is monotonically increasing with increasing \( k = 1, 2, \ldots, s \). Therefore, the sequence
\[ \mathcal{H}(Q, \tau^{(k)}_c) + I^k(\tau^{(k)}_c), \quad k = 1, 2, \ldots, s, \]
defined by (10) is also increasing. The monotonic decreasing of the sequence \( I^k(\tau_Q)/k \), \( k = 1, 2, \ldots, s \), follows from (13).\(^2\) Hence, taking into account the definition (8) and Propositions 1 and 3, we obtain all three assertions of Proposition 5.

Proposition 5 is proved.

Now we will consider the following question. How to solve the extreme problem (9), i.e., how to evaluate the exponent \( E_k(R, Q) \), \( k = 1, 2, \ldots, s \)? Introduce the function of parameter \( \rho \geq 0 \)
\[ B_k(\rho, Q) \triangleq \min \{ \mathcal{H}(Q, \tau) + \rho I^k(\tau) \} \]
\[ = \mathcal{H}(Q, \tau^{(k)}_c) + \rho I^k(\tau^{(k)}_c), \quad k = 1, 2, \ldots, s, \tag{15} \]
where for CRE and CWE, the minimum in (15) is taken over the distribution \( \tau \), pointed out in the formulation of Theorem 1, and \( \tau^{(k)}_c(\rho) \) is the extreme distribution in (15).

**Theorem 2.** Let \( k = 1, 2, \ldots, s \). The function \( B_k(\rho, Q) \), \( \rho \geq 0 \), and the exponent \( E_k(R, Q) \), \( R > 0 \), satisfy the following assertions.

1. \( B_k(0, Q) = 0 \) and at \( \rho > 0 \) the function \( B_k(\rho, Q) \) is positive monotonically increasing and concave.
2. The derivative \( \partial B_k(\rho, Q)/\partial \rho = I^k(\tau^{(k)}_c(\rho)) \), \( \rho \geq 0 \).
3. \( E_k(R, Q) = \max_{0 \leq \rho \leq 1} \{ B_k(\rho, Q) - k\rho R \} \).
4. \( I^k(\tau_Q) = \partial B_k(\rho, Q)/\partial \rho|_{\rho = 0} \), \( I^k(\tau^{(k)}_c) = \partial B_k(\rho, Q)/\partial \rho|_{\rho = 1} \).
5. Formulas (11) and (12) from Proposition 3 may be written as follows:
   a. if \( 0 < R < I^k(\tau^{(k)}_c)/k \), then
   \[ E_k(R, Q) = B_k(1, Q) - kR; \tag{11'} \]
   b. if \( I^k(\tau^{(k)}_c)/k \leq R \leq I^k(\tau^{(k)}_c)/k \), then the function \( E_k(R, Q) \) has the following parametric form
   \[ R = k^{-1}\partial B_k(\rho, Q)/\partial \rho, \quad 0 \leq \rho \leq 1, \]
   \[ E_k(R, Q) = B_k(\rho, Q) - \rho \partial B_k(\rho, Q)/\partial \rho. \tag{12'} \]

\(^2\)This particular case of (13) was established in [19].
6. For CWE, the function
\[ B_k(\rho, Q) = \max_{Q_1, Q_2} \left\{ G_k(\rho, Q_1, Q_2) - k(1 + \rho)K(Q, Q_1) - (s - k)K(Q, Q_2) \right\} \] (16)

where \( Q_j = (Q_j(0), Q_j(1)), j = 1, 2 \), are probability distributions at \( \{0, 1\} \), and

\[ G_k(\rho, Q_1, Q_2) \triangleq -\ln \left\{ \sum_{x_{i+1} = k+1} x_k \prod_{i=k+1}^N Q_2(x_i) \right\} \]
\[ \cdot \sum_z \left[ \sum_{x_k^1} Q_1(x_i)P(z|x_i^1)^{1/(1+\rho)} \right] \] \[ K(Q, Q_j) \triangleq Q(0) \ln \frac{Q(0)}{Q_j(0)} + Q(1) \ln \frac{Q(1)}{Q_j(1)}. \] (17)

7. For CRE, the function \( B_k(\rho, Q) = G_k(\rho, Q, Q) \).

**Proof.** 1. The positivity and monotone increase of the function \( B_k(\rho, Q) \) for \( \rho > 0 \) are evident from definition (15). If \( \rho = 0 \), then the minimum in (15) is achieved at \( \tau = \tau_Q \) and this minimum equals zero, so \( B_k(0, Q) = 0 \). To prove the concavity of \( B_k(\rho, Q) \) we consider \( \rho = \lambda \rho_1 + (1 - \lambda)\rho_2 \), where \( \rho \geq 0 \), and \( 0 < \lambda < 1 \). We have
\[ \mathcal{H}(Q, \tau) + \rho I^{(k)}(\tau) = \lambda[\mathcal{H}(Q, \tau) + \rho_1 I^{(k)}(\tau)] + (1 - \lambda)[\mathcal{H}(Q, \tau) + \rho_2 I^{(k)}(\tau)]. \] (18)

Since the minimum of the sum of two functions is more than or equal to the sum of their minimums, then from (18) and definition (15) we obtain

\[ B_k(\rho, Q) \geq \lambda B_k(\rho_1, Q) + (1 - \lambda)B_k(\rho_2, Q). \]

Statement 1 is proved.

To establish the other statements of Theorem 2, we can apply the standard Lagrange method. In our case, functions (7) are convex, and the Khun-Tucker theorem is applied. We omit the detailed proofs, because the similar proofs were described in [27] for the more simple situation of discrete memoryless channels.

Let
\[ B_s \triangleq \max_{Q} B_1(1, Q) = \max_{Q} E(0, Q), \] (19)

where \( E(0, Q) \triangleq \lim_{R \to 0} E(R, Q) \) and the second equality in (19) follows from Proposition 5. Now we consider the evaluation problem of \( B_s \) for the important particular case of DSE.

**Example.** (The disjunct channel model of DSE, see Sect. 1). This model is the most interesting for applications [29, 18]. It is specified by the deterministic MAC, the output of which \( z \) is the Boolean sum of MAC inputs \( x_1, x_2, \ldots, x_s \), i.e.
\[ z = \begin{cases} 0, & \text{if } x_1 = x_2 = \cdots = x_s = 0, \\ 1, & \text{otherwise.} \end{cases} \]

We conclude from (16) and (17) that for CWE
\[ B_s = \max_{(\beta_1, \beta_2, Q)} \left\{ -\ln \left[ 1 - 2\beta_1 (1 - \beta_1)\beta_2^{-1} \right] + 2 \left[ Q \ln \frac{\beta_1}{Q} + (1 - Q) \ln \frac{1 - \beta_2}{1 - Q} \right] + \\ + (s - 1) \left[ Q \ln \frac{\beta_2}{Q} + (1 - Q) \ln \frac{1 - \beta_2}{1 - Q} \right] \right\}, \] (20)

where the maximum is taken over \((\beta_1, \beta_2, Q)\), provided that
\[ 0 < \beta_j = Q_j(0) < 1, \quad j = 1, 2, \quad \text{and} \quad 0 < Q = Q(0) < 1. \]

The following table gives the extreme parameters \((\beta_1, \beta_2, Q)\) at which the maximum in (20) is achieved:
If $s \to \infty$, then
\[
\beta_1 = 1 - \frac{2\ln 2}{s} + O(s^{-2}), \quad \beta_2 = 1 - \frac{\ln 2}{s} + \frac{2(\ln 2)^2}{s^2} + O(s^{-3}), \tag{21a}
\]
\[
Q = 1 - \frac{\ln 2}{s} + O(s^{-2}), \quad B_s = \frac{2(\ln 2)^2}{s} (1 + o(1)) = \frac{0.9609}{s} (1 + o(1)). \tag{21b}
\]
With the help of Theorem 2 (statement 7) one can check that for CRE the maximum in (19) is achieved at $Q = (s/(1+s); 1/(1+s))$, and
\[
B_s = -\ln \left[ 1 - \frac{2s^s}{(1+s)^{1+s}} \right] = \frac{2}{e} (1 + o(1)) = \frac{0.7358}{s} (1 + o(1)).
\]
Let $N(s, t)$ be the minimal possible length of the code $X$ with zero error probability for the Boolean model of DSE, and
\[
R_0^0 = \limsup_{t \to \infty} \frac{\ln t}{N(s, t)}
\]
be the zero error capacity [29]. It is easy to understand that applying (11'), we obtain the lower bound
\[
R_s^0 \geq \max_Q \min_{k=1, 2, \ldots, s} \frac{B_k(1, Q)}{s + k}. \tag{22}
\]
Using Theorem 2, it is not difficult to prove that for the Boolean model of DSE the minimax in (22) is asymptotically achieved, when $k = 1$ and $Q$ satisfies (21). So the above mentioned asymptotic formula for $B_s$ means that
\[
R_s^0 \geq \frac{0.9609}{s^2} (1 + o(1)) \quad \text{for} \quad \text{CWE}; \quad R_s^0 \geq \frac{0.7358}{s^2} (1 + o(1)) \quad \text{for} \quad \text{CRE}. \tag{23}
\]
The bound (23) for CRE was obtained in [14], and the bound (23) for CWE was obtained in [42]. Note that the best known upper bound [31] has the form
\[
R_s^0 \leq \frac{2\ln s}{s^2} (1 + o(1)), \quad s \to \infty.
\]

### 6.2 Proof of Theorem 1

Fix an arbitrary message $e \in \mathcal{E}(s, t)$ and an integer $k = 1, 2, \ldots, s$. Let $e^{(s-k)} \subset e$, $|e^{(s-k)}| = s - k$, be an arbitrary fixed $(s - k)$–subset of $e$. Consider the collection of messages
\[
\mathcal{E}_k \triangleq \mathcal{E}_k(e^{(s-k)}) \triangleq \{ e' \in \mathcal{E}(s, t) : e' \cap e = e^{(s-k)} \}.
\]
It is clear that $|\mathcal{E}_k| = \binom{s}{k}$. For each $e' \in \mathcal{E}_k$ and $z \in Z^N$, we introduce the ensemble events
\[
X(z, e') \triangleq \{ X : D_Q(x(e), z) \leq D_Q(x(e'), z) \}, \quad X^{(k)}(z) \triangleq \bigcup_{\mathcal{E}_k} X(z, e'). \tag{24}
\]
Let $x^s_i \triangleq (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_s) \triangleq (x^s_k, x^s_{k+1})$ be a fixed collection of $s$ binary columns $x_i \in \{0, 1\}^N$. We define the probabilities
\[
q_k \triangleq \sum_{e} \sum_{x^s_i} Q_N(x^s_i) P_N(z|x^s_i) q_k(x^s_i, z),
q_k(x^s_i, z) \triangleq \Pr \{ X^{(k)}(z) \leq X^{(k)}(e) = x^s_i \}, \tag{25}
Q_N(x^s_i) \triangleq \prod_{i=1}^s Q_N(x_i).
Here and below, we will use the symbol $\Pr\{\cdot\}$ to denote the probability of an event in the ensemble. Probabilities $Q_N(x_i)$ for CRE and CWE are calculated accordingly to (4) and (5), respectively. Note that $q_k$, $k = 1, 2, \ldots, s$, is the average ensemble probability of the following event: components $e^{(s-k)}$ of the transmitted message were decoded correctly, and the components $e \setminus e^{(s-k)}$ were decoded incorrectly. Hence, by virtue of the symmetry of MAC and definition (3), we obtain

$$\max_{k=1, 2, \ldots, s} q_k \leq \bar{P}_Q(X) \leq \sum_{k=1}^s \binom{s}{k} q_k. \quad (26)$$

Under conditions (6), the following statements take place.

**Lemma 1.** For each $k = 1, 2, \ldots, s$

$$q_k \leq \exp\{-N[E_k(R, Q) + o(1)]\}. \quad (27)$$

**Lemma 2.** For fixed $k$, $k = 1, 2, \ldots, s$, and $0 < R < I^k(\tau^{(k)}_{cr})/k$, the probability

$$q_k \geq \exp\{-N[H(Q, \tau^{(k)}_{cr}) + I^k(\tau^{(k)}_{cr}) - kR + o(1)]\}. \quad (28)$$

**Lemma 3.** For each $k$, $k = 1, 2, \ldots, s$

$$\bar{P}_Q(X) \geq \exp\{-N[\hat{E}_k(r, Q) + o(1)]\}. \quad (29)$$

$$\hat{E}_k(R, Q) \doteq \min H(Q, \tau), \quad (30)$$

where for CRE and CWE the minimum in (30) is taken over distributions $\tau$, which were mentioned in the formulation of Theorem 1, and such that $I^k(\tau) \leq kR$.

With the help of arguments used in [20] for investigating the sphere-packing exponent of a discrete memoryless channel it is easy to understand that

$$\hat{E}_k(R, Q) = E_k(R, Q) \quad \text{when} \quad \frac{I^k(\tau^{(k)}_{cr})}{k} \leq R \leq \frac{I^k(\tau_Q)}{k}.$$ 

Therefore, the statement of Theorem 1 arises from inequality (26), Lemmas 1-3, and Proposition 3 (see [34]). To complete the proof of Theorem 1 we need to establish the proofs of Lemmas 1-3.

**Proof of Lemma 1.** Denote by $\mathcal{N}(Q)$ the set of all compositions $\|n(x_i^*, z)\|$ for which marginal compositions $\|n(x_k)\|$, $k = 1, 2, \ldots, s$ are the same and

$$n(x_k) \doteq \begin{cases} [NQ(0)], & \text{if } x_k = 0, \\ [NQ(1)], & \text{if } x_k = 1. \end{cases}$$

Let $1 \leq v \leq k \leq s$ be integers. In the case of CRE we define

$$\delta^k_v(\|n(x_i^*, z)\|) \doteq \prod_{i=v}^k \prod_{x_i} Q(x_i)^{n(x_i)}, \quad (31a)$$

and in the case of CWE we define

$$\delta^k_v(\|n(x_i^*, z)\|) \doteq \begin{cases} \left[NQ(1)\right]^{-(k-v+1)}, & \text{if } \|n(x_i^*, z)\| \in \mathcal{N}(Q), \\ 0, & \text{otherwise}. \end{cases} \quad (31b)$$

Replace in (25) the sum over the pairs $(z, x_i^*)$ by the sum over the compositions $\|n(x_i^*, z)\|$. Taking into account (1), we have

$$q_k = \sum_{\|n(x_i^*, z)\|} b(\|n(x_i^*, z)\|) \cdot q^{(k)}(\|n(x_i^*, z)\|), \quad (32)$$

58
By virtue of (33), (35), and (37), the Stirling formula yields

\[ b(\|n(x^*_1, z)\|) \triangleq N! \cdot \delta^*_t(\|n(x^*_1, z)\|) \prod_{x^*_1} \frac{P(z|x^*_1)^{n(x^*_1, z)}}{n(x^*_1, z)!}, \tag{33} \]

\[ q^{(k)}(\|n(x^*_1, z)\|) \triangleq \Pr \left\{ X^{(k)}(z) | C(x(e), z) = \|n(x^*_1, z)\| \right\}. \tag{34} \]

Fix any composition \( \|n(x^*_1, z)\| \) and introduce for this composition the set \( M_k = M_k(\|n(x^*_1, z)\|) \), \( k = 1, 2, \ldots, s \), of compositions

\[ M_k \triangleq \left\{ \|m(x^*_1, z)\| : \sum_{x^*_1} m(x^*_1, z) = n(x^*_1, z), D_Q(\|m(x^*_1, z)\|) \geq D_Q(\|n(x^*_1, z)\|) \right\}, \]

where \( D_Q \) is the considered U-decoding. Then for \( e' \in E_k = E_k(e^{(s-k)}) \) we have

\[ \Pr \{ X(z, e') | C(x(e), z) = \|n(x^*_1, z)\| \} = \sum_{M_k} d_k(\|m(x^*_1, z)\|), \]

where

\[ d_k(\|m(x^*_1, z)\|) \triangleq \prod_{x^*_1} \prod_{z} \frac{n(x^*_1, z)!}{m(x^*_1, z)!} \cdot \delta^*_t(\|m(x^*_1, z)\|). \tag{35} \]

For the composition \( \|m(x^*_1, z)\| \in M_k \) the inequality

\[ d_k(\|m(x^*_1, z)\|) \leq d_k(\|n(x^*_1, z)\|) \]

arises from the definition of U-decoding \( D_Q \). Therefore, applying the additive upper bound on the probability of a union of events (34) we find that

\[ q^{(k)}(\|n(x^*_1, z)\|) \leq \min \left\{ 1; (N + 1)^A \cdot \binom{t - s}{k} \cdot d_k(\|n(x^*_1, z)\|) \right\}, \]

where \( A \triangleq 2^s \cdot |Z| \), and we use the fact that the number of all compositions \( \|m(x^*_1, z)\| \) does not exceed \( (N + 1)^A \). Hence, in virtue of (32),

\[ q_k \leq (N + 1)^A \cdot \max \left\{ b(\|n(x^*_1, z)\|) \cdot \min \left\{ 1; (N + 1)^A \binom{t - s}{k} d_k(\|n(x^*_1, z)\|) \right\} \right\}. \tag{36} \]

Note that for CRE the maximum in (36) is taken over all compositions \( \|n(x^*_1, z)\| \) and for CWE, by virtue of (31) and (32), this maximum is taken only over those compositions \( \|n(x^*_1, z)\| \) which belong to \( \mathcal{N}^t(Q) \).

Fix an arbitrary distribution \( \tau = \{ \tau(x^*_1, z), x^*_1 \in \{0, 1\}^s, z \in Z \} \), which belongs to the set of distributions pointed out in the formulation of Theorem 1. Suppose that the elements of the composition \( \|n(x^*_1, z)\| \) satisfy the asymptotic equalities

\[ \|n(x^*_1, z)\| = N[\tau(x^*_1, z) + o(1)], \quad N \to \infty. \tag{37} \]

By virtue of (33), (35), and (37), the Stirling formula yields

\[ b(\|n(x^*_1, z)\|) = \exp \left\{ -N[\mathcal{H}(Q, \tau) + o(1)] \right\}, \tag{38} \]

\[ d_k(\|n(x^*_1, z)\|) = \exp \left\{ -N[q^{(k)}(\tau) + o(1)] \right\}, \quad k = 1, 2, \ldots, s, \tag{39} \]

where the exponents in the right-hand sides are defined by (7). From (36)-(39) it follows the asymptotic bound (27).

Lemma 1 is proved.
Proof of Lemma 2. Fix an arbitrary composition \( \|n(x_1^s, z)\| \) and an integer \( k, k = 1, 2, \ldots, s \). For each \( e' \in \mathcal{E}_k = \mathcal{E}_k(e^{(k)}) \) and \( z \in Z^N \) we introduce the ensemble event

\[
\bar{X}(z, e') \triangleq X(z, e') \cap Y(z, e'), \quad Y(z, e') \triangleq \{ X : C(x(e'), z) = \|n(x_1^s, z)\| \},
\]

where \( X(z, e') \) was defined by (24). Note, that for any \( e' \in \mathcal{E}_k \) the conditional probability

\[
\Pr \left\{ \bar{X}(z, e') \mid C(x(e'), z) = \|n(x_1^s, z)\| \right\} = d_k(\|n(x_1^s, z)\|), \quad (40)
\]

where the notation of (35) is used.

Let \( e' \neq e'' \in \mathcal{E}_k \). Introduce \( e'_k \triangleq e' \setminus e \) and \( e''_k \triangleq e'' \setminus e \). Consider the collection \( S = S(s, t, k, v) \) of all pairs \((e', e'')\) for which \( |e'_k \cap e''_k| = k - v \), where \( v = 1, 2, \ldots, k \). Note that the cardinality of this collection is

\[
|S| = \left( \frac{t - s}{k - v} \right) \left( \frac{t - s - (k - v)}{2v} \right) \left( \frac{2v}{2v} \right) < t^{k + v}, \quad (41)
\]

Further, for any pair \((e', e'')\) from \( S \) the conditional probability

\[
\Pr \left\{ \bar{X}(z, e') \cap \bar{X}(z, e'') \mid C(x(e), z) = \|n(x_1^s, z)\| \right\} =
\]

\[
= \prod_{z} \prod_{x_{k+1}^s \in z} n(x_{k+1}^s, z)! \prod_{n(x_{k+1}^s, z)!} \delta_{k+1}(\|n(x_1^s, z)\|) [d_v(\|n(x_1^s, z)\|)]^2 =
\]

\[
= d_v(\|n(x_1^s, z)\|) \cdot d_k(\|n(x_1^s, z)\|)
\]

where the notations of (31) and (35) are used. It is evident that the conditional probability (34) is greater than or equal to the conditional probability of the union of events \( \bar{X}(z, e'), e' \in \mathcal{E}_k \). Hence, applying the standard lower bound

\[
\Pr \left\{ \bigcup_{\theta_i} \right\} \geq \sum_{\theta_i} \Pr\{\theta_i\} - \sum_{i < j} \Pr\{\theta_i \cap \theta_j\}
\]

and taking into account (40)-(42), we obtain

\[
q^{(k)}(\|n(x_1^s, z)\|) \geq \left( \frac{t - s}{k} \right) \cdot d_k(\|n(x_1^s, z)\|) - \sum_{v=1}^{k} k^{k+v} d_k(\|n(x_1^s, z)\|) \cdot d_v(\|n(x_1^s, z)\|). \quad (43)
\]

Consider the distribution \( \tau^{(k)}_{cr} \) introduced in Proposition 2. Restrict the summation in (32) by one and only one composition \( \|n(x_1^s, z)\| \), which satisfies the asymptotic equality (37) when \( \tau = \tau^{(k)}_{cr} \). By substituting (38) and (39) in (43) and (32) we have

\[
q_k \geq \exp \{-N[H(Q, \tau^{(k)}_{cr}) + I^{(k)}(\tau^{(k)}_{cr}) - R + o(1)]\},
\]

if for any \( v, v = 1, 2, \ldots, k \), the rate \( R < I^{(v)}(\tau^{(k)}_{cr})/v \). Applying Proposition 4 for distribution \( \tau^{(k)}_{cr} \), we obtain (28).

Lemma 2 is proved.

Proof of Lemma 3. Fix arbitrary composition \( \|n(x_1^s, z)\| \) and define on the product \( \{0, 1\}^s \cdot Z \) the probability distribution

\[
\tau = \left\{ \tau(x_1^s, z) = \frac{n(x_1^s, z)}{N}, \quad x_1^s \in \{0, 1\}^s, \ z \in Z \right\}.
\]

Let

\[
I_r(x_k^s, z|x_{k+1}^s) \triangleq \sum_{x_1} \sum_{x_1^s} \tau(x_1^s, z) \ln \frac{\tau(z|x_1^s)}{\tau(z|x_{k+1}^s)}, \quad K_p(\tau) \triangleq \sum_{x_1} \sum_{x_1^s} \tau(x_1^s, z) \ln \frac{\tau(z|x_1^s)}{P(z|x_1^s)}
\]

60
be the information functions of Shannon and Kullback [20]. For the correctness of the $K_p(\tau)$ definition we consider only such compositions for which $\tau(z|x_1^s) = 0$ if $P(z|x_1^s) = 0$.

For a given composition $\|n(x_1^s, z)\|$, introduce the corresponding marginal compositions $\|n(x_1^s)\|$ and $\|n(x_{k+1}^s)\|$. Fix an arbitrary $(N \times t)$ code $X$. Denote by

$$\mathcal{E}_X(\|n(x_1^s)\|) = \{ e : C(x(e)) = \|n(x_1^s)\| \}$$

the set of messages $e \in \mathcal{E}(s, t)$ which are encoded by $X$ into $s$–collection $x(e)$ with composition $\|n(x_1^s)\|$. Also introduce

$$\mathcal{E}_X(\|n(x_{k+1}^s)\|) = \{ e^{(s-k)} : C(x(e^{(s-k)}) = \|n(x_{k+1}^s)\| \}$$

the set of elements $e^{(s-k)} \in \mathcal{E}(s-k, t)$ which are encoded by $X$ into $(s-k)$–collection $x(e^{(s-k)})$ with composition $\|n(x_{k+1}^s)\|$. For each $k = 1, 2, \ldots, s$ and any composition $\|n(x_1^s, z)\|$, the error probability (3) satisfies the inequality

$$P_{Q}(X) \geq \exp\{ -N K_p(\tau) \cdot \left[ \frac{\mathcal{E}_X(\|n(x_1^s)\|)}{(N+1)^t} - \frac{t^k}{\binom{t}{t-k}} \frac{\mathcal{E}_X(\|n(x_{k+1}^s)\|)}{\binom{t}{t-k}} \cdot \exp\{ -N[kR - I(1, x_1^s, z|x_{k+1}^s)] \} \right],$$

where $R = \ln t/N$, $A = 2^s|Z|$. We omit the proof of (44) and observe that it may be proved with the help of arguments which were used in [26]. For comparison we point also to the similar lower bounds which presented in [34, 39] for the error probability of an individual code pair in MAC. Averaging over ensembles (4) and (5), we have

$$\frac{\mathcal{E}_X(\|n(x_1^s)\|)}{\binom{t}{t}} = \frac{N!}{\prod_{x_1^s} n(x_1^s)!} \cdot \delta_1^s(\|n(x_1^s, z)\|),$$

$$\frac{\mathcal{E}_X(\|n(x_{k+1}^s)\|)}{\binom{t}{t-k}} = \frac{N!}{\prod_{x_{k+1}^s} n(x_{k+1}^s)!} \cdot \delta_{k+1}^s(\|n(x_1^s, z)\|),$$

where notations of (31) are used.

Fix an arbitrary distribution $\tau$ which satisfies the conditions (depending on CRE or CWE) pointed out in the formulation of Theorem 1. In virtue of (6), (31), and (45), the averaging of inequality (44) yields

$$\overline{P_{Q}(X)} \geq \{ -N[\mathcal{H}(Q, \tau) + o(1)] \cdot [1 - \exp\{ -N[kR - I^{(k)}(\tau) + o(1)] \} \},$$

$k = 1, 2, \ldots, s$, where the exponents of the right-hand side are defined by (7). From (46) it follows the inequality (29).

Lemma 3 is proved.
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