Uniform convergence of conditional distributions for one-dimensional diffusion processes

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Abstract

In this paper, we study the quasi-stationary behavior of the one-dimensional diffusion process with a regular or exit boundary at 0 and an entrance boundary at $\infty$. By using the Doob's $h$-transform, we show that the conditional distribution of the process converges to its unique quasi-stationary distribution exponentially fast in the total variation norm, uniformly with respect to the initial distribution. Moreover, we also use the same method to show that the conditional distribution of the process converges exponentially fast in the $\psi$-norm to the unique quasi-stationary distribution.

Keywords: One-dimensional diffusion processes; rate of convergence; quasi-stationary distributions; Doob's $h$-transform

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1 Introduction

Consider a one-dimensional diffusion process $X = (X_t)_{t \geq 0}$ on $[0, \infty)$ killed at 0, that is, for all $t \geq s$, if $X_s = 0$, then $X_t = 0$, given by the solution to the stochastic differential equation (SDE)

$$dX_t = dB_t - q(X_t)dt, \quad X_0 = x > 0,$$

where $(B_t)_{t \geq 0}$ is a standard one-dimensional Brownian motion and the drift $q$ is continuously differentiable on $(0, \infty)$, that is, $q \in C^1((0, \infty))$.

Let $T_0 := \inf\{t \geq 0 : X_t = 0\}$ be the first hitting time of 0. We write $\mathbb{P}_x$ for the probability measure of the process when $X_0 = x$. In this paper, we assume that absorption at 0 is certain, that is, for all $x > 0$,

$$\mathbb{P}_x(T_0 < \infty) = 1.$$
The concept of the *stationarity* will no longer be appropriate to depict the asymptotic behavior of such a process conditioned on long-term survival. A natural concept will be the *quasi-stationarity*. The main concept of this theory is the *quasi-stationary distribution*, that is, a probability measure $\alpha$ on $(0, \infty)$ such that, for any $t \geq 0$,

$$\mathbb{P}_\alpha(X_t \in \cdot | T_0 > t) = \alpha,$$

where $\mathbb{P}_\alpha(\cdot) = \int_0^\infty \mathbb{P}_x(\cdot) \alpha(dx)$. To be a quasi-stationary distribution, one possible approach is to look at the *quasi-limiting distribution* $\alpha$, that is, a probability measure $\alpha$ such that there exists an initial distribution $\mu$ on $(0, \infty)$,

$$\lim_{t \to \infty} \mathbb{P}_\mu(X_t \in \cdot | T_0 > t) = \alpha.$$  \hfill (1.4)

If (1.4) holds, we also say that $\mu$ is attracted to $\alpha$, or we say that $\mu$ is in the domain of attraction of $\alpha$. For a general overview on quasi-stationarity, the readers are urged to refer to [8, 21].

When studying quasi-stationary distributions of one-dimensional diffusion processes, the boundaries of an interval often need to be classified: *regular boundary*, *exit boundary*, *entrance boundary* and *natural boundary*. To do so, let us introduce some notations. For $x \in (0, \infty)$, we define the *scale function* $\Lambda$ and the *speed measure* $m$ of the process $X$ respectively by

$$\Lambda(x) = \int_1^x e^{Q(y)} dy, \quad m(dy) = e^{-Q(y)} dy,$$

where $Q(y) = \int_1^y 2q(x) dx$. Using the speed measure $m$ and the scale function $\Lambda$, the boundaries of $(0, \infty)$ can be classified. Let $a = 0$ or $\infty$ and fix a point $b \in (0, \infty)$. Set

$$I(a) = \int_b^a d\Lambda(y) \int_y^b dm(z), \quad J(a) = \int_b^a dm(y) \int_b^y d\Lambda(z).$$

According to [15], the boundary $a$ can be classified as follows:

The boundary $a$ is said to be

- regular if $I(a) < \infty$ and $J(a) < \infty$,
- exit if $I(a) < \infty$ and $J(a) = \infty$,
- entrance if $I(a) = \infty$ and $J(a) < \infty$,
- natural if $I(a) = \infty$ and $J(a) = \infty$.

When 0 is an exit boundary, the speed measure $m$ is an infinite measure. At this time, it is a great challenge to prove the existence of a quasi-stationary distribution, due to it is difficult to guarantee $\eta_1 \in L^1(m)$, which is the basic condition for constructing a quasi-stationary distribution. Here, $\eta_1$ is the eigenfunction corresponding to the first non-trivial eigenvalue of the operator of the process $X$. In this case, Cattiaux et al.[1] had proved the existence and uniqueness of quasi-stationary distributions for the process $X$ under a set of assumptions which are mainly to ensure that the spectrum of the infinitesimal operator is discrete and $\eta_1 \in L^1(m)$. 
When 0 is a regular boundary, the speed measure \( m \) is a finite measure, and then it is easy to get \( \eta_1 \in L^1(m) \). In this case, many scholars study quasi-stationary distributions of one-dimensional diffusion processes with a natural or entrance boundary at \( \infty \) (see, e.g., [16, 17, 20, 23, 29, 30]).

This paper is closely related to [1, 14, 16, 18, 30]. These papers only consider the existence, uniqueness and domain of attraction of quasi-stationary distributions. However, this does not imply that (1.4) is uniform convergence. The reverse is true. This paper, we are interested in the speed of convergence of (1.4). More formally, we are looking forward to the existence of a quasi-stationary distribution \( \alpha \) on \((0, \infty)\) such that, for all probability measure \( \mu \) on \((0, \infty)\) and all \( t \geq 0 \), there exist two positive constants \( C, \gamma \) such that

\[
\| P_\mu(X_t \in \cdot | T_0 > t) - \alpha \|_{TV} \leq Ce^{-\gamma t}.
\]  

(1.7)

Here, \( \| \cdot \|_{TV} \) is the total variation norm, defined by

\[
\| \nu - \mu \|_{TV} := \sup_{g \in B_1((0, \infty))} |\nu(g) - \mu(g)|,
\]

where \( \nu, \mu \) are any two probability measures on \((0, \infty)\), \( \mu(g) = \int_0^\infty g(x) \mu(dx) \) and \( B_1((0, \infty)) \) is the set of the measurable bounded functions defined on \((0, \infty)\) such that \( \|g\|_{\infty} \leq 1 \). Here, \( \|g\|_{\infty} = \sup_{x \in (0, \infty)} |g(x)| \). The total variation norm has been used by many scholars to quantify (1.4) (see, e.g., [2, 3, 5, 12, 24]). The exponential convergence (1.7) contains many interesting properties. One is that it implies the process admits a unique quasi-stationary distribution and all initial distributions are in the domain of attraction of this unique quasi-stationary distribution. See [2, 4, 13] for other more detailed results. For a one-dimensional diffusion in natural scale, the exponential convergence (1.7) has already been proved by Champagnat and Villemonais [3, 5] through probabilistic methods. But for the one-dimensional diffusion process \( X \) with a regular or exit boundary at 0 and an entrance boundary at \( \infty \), it is an open question that (1.4) is whether exponential convergence in the total variation norm [5]. In this case, we can use the Doob’s \( h \)-transform (or \( h \)-transform) to ensure the exponential convergence to a unique quasi-stationary distribution in the total variation norm, uniformly with respect to the initial distribution. Moreover, we also use this method to show that the conditional distribution of the process converges exponentially fast in the \( \psi \)-norm to the unique quasi-stationary distribution. This method fundamentally differ from [3, 5], which has been successfully applied to the birth and death process by the authors [12].

The following result is one of our main results.

**Theorem 1.1.** Let \( X \) be a one-dimensional diffusion process satisfying (1.2). Assume that 0 is a regular or exit boundary. Then, the following are equivalent:

(i) \( \infty \) is an entrance boundary.

(ii) There is precisely one quasi-stationary distribution.
(iii) There exist a probability measure $\alpha$ on $(0, \infty)$ and two positive constants $C, \gamma$ such that, for all initial distributions $\mu$ on $(0, \infty)$ and all $t \geq 0$,

\[ \| P_\mu(X_t \in \cdot | T_0 > t) - \alpha \|_{TV} \leq Ce^{-\gamma t}. \]

Moreover, the distribution $\alpha$ in (iii) is the unique quasi-stationary distribution for the process $X$.

For any measurable function $\psi : (0, \infty) \to [1, +\infty)$, we define the $\psi$-norm of a signed measure $\nu$ by $\| \nu \|_\psi = \sup_{|g| \leq \psi} | \nu(g) |$. We can see that when $\psi = 1$, it is nothing but the total variation norm.

We denote by $\| \cdot \|_2$ the $L^2(\beta)$-norm, defined as $\| g \|_2 = (\int_0^\infty g^2(x)\beta(dx))^{\frac{1}{2}}$, where $\beta$ is the unique invariant probability measure of the $Q$-process $Y$ defined in Section 2. For any positive measure $\mu$ on $(0, \infty)$ and any measurable function $g$ on $(0, \infty)$ satisfying $\mu(g) < \infty$, we denote by $g \circ \mu$ the probability measure defined as

\[ g \circ \mu(dx) := \frac{g(x)\mu(dx)}{\mu(g)}. \tag{1.8} \]

We denote by $E_\mu$ and $E_x$ the expectation with respect to $P_\mu$ and $P_x$ respectively. For any two probability measures $\mu, \nu$ on $(0, \infty)$, $\mu \ll \nu$ denotes $\mu$ is absolutely continuous with respect to $\nu$.

The second main result of this paper is the following stronger convergence result.

**Theorem 1.2.** Let $X$ be a one-dimensional diffusion process satisfying (1.2). Assume that 0 is a regular or exit boundary, $\infty$ is an entrance boundary and there exists a function $\psi : (0, \infty) \to [1, +\infty)$ such that $\alpha(\psi) < +\infty$ and $\alpha(\psi^2/\eta_1) < +\infty$, where $\alpha$ is the unique quasi-stationary distribution of the process $X$. Then, for any initial distribution $\mu$ on $(0, \infty)$ satisfying $\mu \ll \alpha$, there exist $t_\mu$ and $\gamma > 0$ such that, for any $t \geq t_\mu$,

\[ \sup_{|g| \leq \psi} | E_\mu[g(X_t) | T_0 > t] - \alpha(g) | \leq \max\{D_1, D_2\} \left[ \frac{\alpha(\psi^2/\eta_1)}{m(\eta_1)} \right]^{\frac{1}{2}} \| d(\eta_1 \circ \mu) / d(\eta_1 \circ \alpha) - 1 \|_2 e^{-\gamma t}, \]

where

\[ D_1 = \left( 1 + 1 + \frac{\alpha(\psi)}{1 - c} \right), \quad D_2 = 2 + \alpha(\psi) \quad \text{and} \quad c \in (0, 1). \]

As an application of the main results, we obtain the following result. This result implies that when $t$ goes to infinity, the polynomial convergence of conditional distributions $\frac{1}{t} \int_0^t P_\mu(X_s \in \cdot | T_0 > t) ds$ toward the quasi-ergodic distribution $\beta$, which improves the result obtained by He [10, Theorem 3.1] who only gave a weak convergence result. This corollary follows directly from [13, Corollary 3.2] or [4, Corollary 2.3]. A probability measure $\beta$ on $(0, \infty)$ is said to be a
quasi-ergodic distribution, if for all \( x \in (0, \infty) \), \( t > 0 \) and all bounded measurable function \( g \) on \((0, \infty)\),
\[
\lim_{t \to \infty} \mathbb{E}_x \left( \frac{1}{t} \int_0^t g(X_s) ds | T_0 > t \right) = \beta(g).
\]
There is an essential differences between a quasi-stationary distribution and a quasi-ergodic distribution. For work on this topic, we refer the reader to [4, 10, 11, 13] and the references therein.

**Corollary 1.3.** Let \( X \) be a one-dimensional diffusion process satisfying (1.2). Assume that 0 is a regular or exit boundary and \( \infty \) is an entrance boundary. Then, there exists a positive constant \( G \) such that, for all \( x \in (0, \infty) \), \( t > 0 \) and all bounded measurable function \( g \) on \((0, \infty)\),
\[
\left| \mathbb{E}_x \left( \frac{1}{t} \int_0^t g(X_s) ds | T_0 > t \right) - \beta(g) \right| \leq \frac{G\|g\|_{\infty}}{t},
\]
where \( \beta \) is as in (2.6).

The rest of this paper is arranged as follows. In Section 2, we present some preliminaries that will be needed in the proof of our main results. In Section 3, we are devoted to giving the proof of Theorem 1.1. We are devoted to giving the proof of Theorem 1.2 in Section 4. An example is given to study in Section 5.

## 2 Preliminaries

In this section, we mainly present some known results on the spectrum of the generator and quasi-stationary distributions of the process \( X \) and discuss some properties of the \( Q \)-process. Associated to the process \( X \), we consider the sub-Markovian semigroup \( (P_t)_{t \geq 0} \), that is, \( 0 \leq P_t g \leq 1 \) m.a.e. if \( 0 \leq g \leq 1 \), given by
\[
P_t g(x) = \mathbb{E}_x[g(X_t)1_{\{T_0 > t\}}].
\]
The generator of this semigroup is given by \( \mathcal{L} = \frac{1}{2} d^2 dx - q \frac{d}{dx} \).

To ensure the existence of a quasi-stationary distribution, the bottom of the spectrum of the generator is often necessary to be strictly positive and the lowest eigenfunction is integrable with respect to the reference measure. On the spectrum of \( \mathcal{L} \) and the quasi-stationary distribution for the process \( X \), we summarize the results proved in [14, 16, 18, 30] as follows. If 0 is a regular boundary and \( \infty \) is an entrance boundary, then it can be found in [16] and [30]. When 0 is an exit boundary and \( \infty \) is an entrance boundary, it mainly comes from [14] and [18].

**Proposition 2.1.** For the one-dimensional diffusion process \( X \) satisfying (1.2), if 0 is a regular or exit boundary and \( \infty \) is an entrance boundary, then
(i) \(-\mathcal{L}\) has a purely discrete spectrum. The eigenvalues \(0 < \lambda_1 < \lambda_2 < \cdots\) are simple, \(\lim_{n \to \infty} \lambda_n = +\infty\), and the eigenfunction \(\eta_n\) associated to \(\lambda_n\) has exactly \(n\) roots belonging to \((0, \infty)\). The eigenfunction sequence \((\eta_n)_{n \geq 1}\) is an orthonormal basis of \(L^2(m)\) and \(\eta_1\) can be taken to be strictly positive on \((0, \infty)\).

(ii) for any \(n \geq 1\), \(\eta_n \in L^1(m)\).

(iii) for all \(x > 0\), \(t > 0\) and all bounded measurable function \(g\) on \((0, \infty)\), there exists some density \(r(t, x, \cdot)\) satisfying

\[
\mathbb{E}_x[g(X_t)1_{\{T_0 > t\}}] = \int_0^\infty r(t, x, y)g(y)m(dy). \tag{2.1}
\]

Moreover, for all \(x > 0\) and \(t > 0\), \(r(t, x, \cdot) \in L^2(m)\).

(iv) there is precisely one quasi-stationary distribution \(\alpha\) for the process \(X\), given by

\[
\alpha(dx) = \frac{\eta_1(x)m(dx)}{m(\eta_1)}. \tag{2.2}
\]

Moreover, \(\alpha\) attracts all initial distributions \(\mu\) on \((0, \infty)\). Also, for any \(x \in (0, \infty)\) and any Borel subset \(B \subseteq (0, \infty)\),

\[
\lim_{t \to \infty} e^{\lambda_1 t}P_x(T_0 > t) = \eta_1(x)m(\eta_1),
\]

\[
\lim_{t \to \infty} e^{\lambda_1 t}P_x(X_t \in B, T_0 > t) = \alpha(B)m(\eta_1).
\]

Now, we introduce the \(Q\)-process, which is defined as a Doob’s \(h\)-transform of the semigroup \((P_t)_{t \geq 0}\). Here, we take \(h = \eta_1\). In other words, we describe the law of the process \(X\) conditioned to never be absorbed, usually called the \(Q\)-process. We denote by \(Y = (Y_t)_{t \geq 0}\) the \(Q\)-process. For the process \(X\) satisfying (1.2), if 0 is a regular or exit boundary and \(\infty\) is an entrance boundary, then we can use the same proof as in [1, Corollary 6.1] to show that the \(Q\)-process \(Y\) exists. In fact, the key elements of the proof of [1, Corollary 6.1] only need to know that \(-\mathcal{L}\) has a purely discrete spectrum, \(\eta_1 \in L^1(m)\) and the last part of Proposition 2.1 holds. More precisely, if 0 is a regular or exit boundary and \(\infty\) is an entrance boundary, then for all \(x > 0\), \(s \geq 0\) and all \(A\) Borel measurable subsets of \(C([0, s])\),

\[
Q_x(Y \in A) = \lim_{t \to \infty} P_x(X \in A|T_0 > t)
\]

is well-defined, and the process \(Y\) is a diffusion process on \((0, \infty)\) with transition probability densities (with respect to the Lebesgue measure) given by

\[
q(s, x, y) = \frac{e^{\lambda_1 s} \eta_1(y)}{\eta_1(x)} r(s, x, y)e^{-Q(y)},
\]

that is, \(Q_x\) is locally absolutely continuous with respect to \(P_x\) and

\[
Q_x(Y \in A) = \mathbb{E}_x \left( 1_A(X)e^{\lambda_1 s} \frac{\eta_1(X_s)}{\eta_1(x)} , T_0 > s \right). \tag{2.3}
\]
The equality (2.3) implies that
\[
\tilde{P}_t g(x) = e^{\lambda_1 t} \eta_1(x) P_t(\eta_1 g)(x),
\] (2.4)
where \((\tilde{P}_t)_{t \geq 0}\) denotes the semigroup of \(Y\). From (2.4), we obtain
\[
P_t g(x) = \eta_1(x) e^{-\lambda_1 t} \tilde{P}_t \left( \frac{g}{\eta_1} \right)(x).
\] (2.5)

So, it can be seen that the process \(Y\) is a Doob’s \(h\)-transform of the process \(X\). The Doob’s \(h\)-transform has a lot of nice properties. First, (2.5) naturally implies that the quasi-stationarity of the process \(X\) can be studied by using the Doob’s \(h\)-transform. In fact, this method has been used by many scholars successfully to discuss the quasi-stationarity for some classes of Markov processes (see, e.g., [9, 12, 23, 24, 27]). Besides, another useful piece of information is that the spectrum is invariant under Doob’s \(h\)-transform (see [25, Chapter 4, Sections 3 and 10]).

Under our assumptions, the same proof as in [1, Corollary 6.2] works, we know that the process \(Y\) is an ergodic diffusion process on \((0, \infty)\) and admits a unique invariant probability measure
\[
\beta(dx) = \eta_1^2(x) m(dx).
\] (2.6)

In fact, we can show that the process \(Y\) is strongly ergodic, that is, there exists a constant \(\gamma > 0\) such that
\[
\sup_{x \in (0, \infty)} \| q(t, x, \cdot) - \beta \|_{TV} = O(e^{-\gamma t}) \quad \text{as } t \to \infty.
\]

**Proposition 2.2.** For the one-dimensional diffusion process \(X\) satisfying (1.2), if 0 is a regular or exit boundary and \(\infty\) is an entrance boundary, then the process \(Y\) is strongly ergodic.

**Proof.** First, note that the process \(Y\) is the unique solution to the following SDE
\[
dY_t = dB_t - \left( q(Y_t) - \frac{\eta_1'(Y_t)}{\eta_1(Y_t)} \right) dt, \quad Y_0 = y > 0.
\] (2.7)

Thus, we define \(\tilde{Q}(y)\) analogous to \(Q(y)\):
\[
\tilde{Q}(y) = \int_1^y 2 \bar{q}(x) \, dx = \int_1^y 2 \left( q(x) - \frac{\eta_1'(x)}{\eta_1(x)} \right) \, dx = Q(y) - 2 \log \frac{\eta_1(y)}{\eta_1(1)}.
\]

Let \(\tilde{\mathcal{L}}\) denotes the generator of the semigroup \((\tilde{P}_t)_{t \geq 0}\) of the process \(Y\). Then, for any real-valued function \(g\) on \((0, \infty)\), we have
\[
\tilde{\mathcal{L}} g(x) = \frac{1}{\eta_1(x)} (\mathcal{L} + \lambda_1)(\eta_1 g)(x).
\]

We denote by \(\lambda_1\) the first non-trivial eigenvalue of the operator \(\mathcal{L}\). Note that the spectrum of \(\tilde{\mathcal{L}}\) and the spectrum of \(\mathcal{L}\) is invariant under Doob’s \(h\)-transform. Based on the formula above and
Proposition 2.1, we get that \( \tilde{\lambda}_1 = \lambda_2 - \lambda_1 > 0 \). In addition, according to [6, 26], \( \tilde{\lambda}_1 > 0 \) if and only if

\[
\tilde{\delta} := \sup_{b>0} \int_0^b e^{\tilde{Q}(y)} dy \int_b^\infty e^{-\tilde{Q}(z)} dz = \sup_{b>0} \int_0^b \frac{e^{Q(y)}}{\eta_1^2(y)} dy \int_b^\infty e^{-Q(z)} \eta_1^2(z) dz < \infty.
\]

So, for \( b \in (0, \infty) \), one has \( \int_0^b \frac{e^{Q(y)}}{\eta_1^2(y)} dy < \infty \). Thus

\[
J_1 := \int_0^b \frac{e^{Q(y)}}{\eta_1^2(y)} \left( \int_y^\infty e^{-Q(z)} \eta_1^2(z) dz \right) dy < \infty.
\]

Moreover, according to [18], \( \eta_1(x) \) is an increasing function on \((0, \infty)\). Also, we know from [28, Theorem 5.4] or [11, Proposition 2.3] that \( \eta_1(x) \) is bounded on \((0, \infty)\). Therefore, we have

\[
\bar{I}(\infty) := \int_0^\infty d\bar{\Lambda}(y) \int_0^y d\bar{m}(z)
\]

and

\[
\bar{J}(\infty) := \int_0^\infty d\bar{m}(y) \int_0^y d\bar{\Lambda}(z)
\]

Thus, according to [19, Theorem 2.1], the process \( Y \) is strongly ergodic. This completes the proof of Proposition 2.2.
In this paper, as mentioned above, it is natural for us to use the Doob’s $h$-transform to study the quasi-stationarity of one-dimensional diffusion processes. We point out that Proposition 2.2 plays an important key role in the proof of our main results.

3 Proof of Theorem 1.1

If 0 is a regular boundary, it’s a well-known fact that (i) and (ii) are equivalent (see [16, Theorem 4.14]), and if 0 is an exit boundary, (i) and (ii) are also equivalent (see [14, Corollary 1.14]). Now let’s turn our attention to prove that (i) and (iii) are equivalent. If (iii) holds, then we know from [2, Theorem 2.1] that there is precisely one quasi-stationary distribution for $X$, and that the distribution $\alpha$ defined in (2.2) is the unique quasi-stationary distribution. Therefore, (i) holds.

Conversely, if (i) holds, according to Proposition 2.2, the $Q$-process $Y$ is strongly ergodic, consequently there exist two positive constants $C', \gamma$ such that, for any $x \in (0, \infty)$ and $t \geq 0$,

$$||Q_x(Y_t \in \cdot) - \beta||_{TV} \leq C' e^{-\gamma t}. \quad (3.1)$$

Note that, if $g \in B_1(0, \infty)$, then for all $x \in (0, \infty)$ and $t \geq 0$, we have

$$\bar{P}_t[\frac{g}{\eta_1}](x) = \frac{e^{\lambda t}}{\eta_1(x)} P_t g(x) \leq \frac{e^{\lambda t}}{\eta_1(x)} < \infty.$$

Thus, for all probability measure $\mu$ on $(0, \infty)$, $g \in B_1(0, \infty)$ and all $t \geq 0$, we have

$$\mathbb{E}_\mu[g(X_t)|T_0 > t] = \frac{\int_0^\infty \mathbb{E}_x[g(X_t)1_{\{T_0 > t\}}] \mu(dx)}{\int_0^\infty \mathbb{E}_x[1_{\{T_0 > t\}}] \mu(dx)}$$

$$= \frac{\int_0^\infty \frac{e^{\lambda t}}{\eta_1(x)} \mathbb{E}_x[\eta_1(X_t) \frac{g(X_t)}{\eta_1(X_t)} 1_{\{T_0 > t\}}] \eta_1(x) \mu(dx)}{\int_0^\infty \frac{e^{\lambda t}}{\eta_1(x)} \mathbb{E}_x[\eta_1(X_t) 1_{\{T_0 > t\}}] \eta_1(x) \mu(dx)}$$

$$= \frac{\int_0^\infty \bar{P}_t[\frac{g}{\eta_1}](x) \eta_1(x) \mu(dx)}{\int_0^\infty \bar{P}_t[\frac{1}{\eta_1}](x) \eta_1(x) \mu(dx)}$$

$$= \frac{(\eta_1 \circ \mu) \bar{P}_t[\frac{g}{\eta_1}]}{(\eta_1 \circ \mu) \bar{P}_t[\frac{1}{\eta_1}]}.$$ 

Here, $1 = 1_{(0,\infty)}$. From Proposition 2.1, we know that $\eta_1 \in L^1(m)$. By (2.2) and (2.6), we have

$$\beta(\frac{g}{\eta_1}) = m(\eta_1)\alpha(g).$$

So, for any $g \in B_1(0, \infty)$, by (3.1), one has

$$|(\eta_1 \circ \mu) \bar{P}_t[\frac{g}{\eta_1}] - m(\eta_1)\alpha(g)| = |(\eta_1 \circ \mu) \bar{P}_t[\frac{g}{\eta_1}] - \beta(\frac{g}{\eta_1})| \leq C' e^{-\gamma t} \quad (3.2)$$
and
\[ |(\eta_1 \circ \mu \tilde{P}_t(\eta_1)) - m(\eta_1)| \leq C' e^{-\gamma t}. \] (3.3)

Set \( C_1 := \frac{C'}{m(\eta_1)} \). Then, by (3.2) and (3.3), for any \( t > \frac{\log C_1}{\gamma} \), one has
\[ \frac{\alpha(g) - C_1 e^{-\gamma t}}{1 + C_1 e^{-\gamma t}} \leq \mathbb{E}_\mu [g(X_t)|T_0 > t] \leq \frac{\alpha(g) + C_1 e^{-\gamma t}}{1 - C_1 e^{-\gamma t}}. \] (3.4)

In addition, for any \( t > \frac{\log C_1}{\gamma} \) and \( g \in B_1(0, \infty) \), we have
\[ \frac{\alpha(g) + C_1 e^{-\gamma t}}{1 - C_1 e^{-\gamma t}} = (\alpha(g) + C_1 e^{-\gamma t}) \left(1 + \frac{C_1 e^{-\gamma t}}{1 - C_1 e^{-\gamma t}}\right) \leq \alpha(g) + C_1 e^{-\gamma t} + (1 + 1) \frac{C_1 e^{-\gamma t}}{1 - C_1 e^{-\gamma t}} = \alpha(g) + C_1 e^{-\gamma t} \left(1 + \frac{2}{1 - C_1 e^{-\gamma t}}\right). \]

We can pick any value smaller than 1, denoted by \( d \), such that \( C_1 e^{-\gamma t} < d \). So
\[ \frac{\alpha(g) + C_1 e^{-\gamma t}}{1 - C_1 e^{-\gamma t}} \leq \alpha(g) + C_2 e^{-\gamma t}, \]
where
\[ C_2 := C_1 \left(1 + \frac{2}{1 - d}\right). \]

In a same way, for any \( t > \frac{\log C_1}{\gamma} \) and \( g \in B_1(0, \infty) \), one can obtain
\[ \frac{\alpha(g) - C_1 e^{-\gamma t}}{1 + C_1 e^{-\gamma t}} = (\alpha(g) - C_1 e^{-\gamma t}) \left(1 - \frac{C_1 e^{-\gamma t}}{1 + C_1 e^{-\gamma t}}\right) \geq \alpha(g) - C_1 e^{-\gamma t} - (\alpha(g) + C_1 e^{-\gamma t}) \frac{C_1 e^{-\gamma t}}{1 + C_1 e^{-\gamma t}} \geq \alpha(g) - 3C_1 e^{-\gamma t}. \]

Hence, for any \( t > \frac{\log C_1}{\gamma} \), we obtain
\[ \sup_{g \in B_1(0, \infty)} |\mathbb{E}_\mu [g(X_t)|T_0 > t] - \alpha(g)| \leq \max\{3C_1, C_2\} e^{-\gamma t}. \]

This ends the proof of Theorem 1.1.

4 Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. First, by (1.8), (2.2) and (2.6), one has
\[ \eta_1 \circ \alpha(dx) = \beta(dx). \]
It can be seen that if \( \|d\eta \mu/d\alpha - 1\|_2 = +\infty \), then the theorem is trivially true. So, we just need to consider initial distribution \( \mu \) on \((0, \infty)\) such that \( \|d\eta \mu/d\alpha - 1\|_2 < +\infty \).

Since \( \alpha(\psi^2/\eta_1) < +\infty \), for any measurable function \( g \) on \((0, \infty)\) such that \( |g| \leq \psi \), we have

\[
\|g/\eta_1\|_2 \leq \left[ m(\eta_1)\alpha(\psi^2/\eta_1) \right]^{1/2} < +\infty,
\]

that is, \( \frac{g}{\eta_1} \in L^2(\beta) \). Moreover, note that the measure \( \beta \) is a reversible measure for the semigroup \((\tilde{P}_t)_{t \geq 0}\). In fact, if \( f, g \in L^2(m) \), so that \( f, g \in L^2(\beta) \), then we have

\[
\int_0^\infty (\tilde{P}_tf)gd\beta = \int_0^\infty \eta_1^2 \left( \frac{e^{\lambda_1 t}}{\eta_1} \right) P_t\eta_1 f)d\eta = \int_0^\infty \eta_1^2 f \left( \frac{e^{\lambda_1 t}}{\eta_1} \right) P_t\eta_1 g)d\eta = \int_0^\infty f(\tilde{P}_tg)d\beta.
\]

Thus, for any initial distribution \( \mu \) on \((0, \infty)\) satisfying \( \mu \ll \alpha \), we have

\[
|\mu \tilde{P}_t[g/\eta_1] - m(\eta_1)\alpha(g)| = |\mu \tilde{P}_t[g/\eta_1] - \beta(g/\eta_1)| = \left| \beta \left( \frac{d\mu}{d\beta} P_t[g/\eta_1] - \frac{g}{\eta_1} \right) \right| = \left| \beta \left( \frac{g}{\eta_1} \tilde{P}_t \left( \frac{d\mu}{d\beta} \right) - \frac{g}{\eta_1} \right) \right| = \left| \beta \left[ \frac{g}{\eta_1} \left( \tilde{P}_t \left( \frac{d\mu}{d\beta} - 1 \right) \right) \right] \right|.
\]

Further, from Proposition 2.2, we know that the \( Q \)-process \( Y \) is strongly ergodic. Then, according to [7], \((\tilde{P}_t)_{t \geq 0}\) has \( L^2 \)-exponential convergence, that is, there exists a positive constant \( \gamma \) such that, for all \( g \in L^2(\beta) \) and \( t \geq 0 \),

\[
\|\tilde{P}_t g - \beta(g)\|_2 \leq e^{-\gamma t}\|g - \beta(g)\|_2.
\]

(4.1)

Hence, based on the Cauchy-Schwarz inequality and (4.1), we obtain

\[
\sup_{|g| \leq \psi} |\mu \tilde{P}_t[g/\eta_1] - m(\eta_1)\alpha(g)| \leq \left[ \beta(\psi^2/\eta_1^2) \right]^{1/2} \|d\mu/d\beta - 1\|_2 e^{-\gamma t} \left[ m(\eta_1)\alpha(\psi^2/\eta_1) \right]^{1/2} \|d\mu/d\beta - 1\|_2 e^{-\gamma t}.
\]

Moreover, according to the proof of Theorem 1.1, it holds that

\[
\mathbb{E}_\mu[g(X_t)\mid T_0 > t] = \frac{(\eta_1 \circ \mu) \tilde{P}_t[g/\eta_1]}{(\eta_1 \circ \mu) \tilde{P}_t[1/\eta_1]}.
\]

Therefore, for any \( t > \frac{\log[\left( \frac{\psi^2}{m(\eta_1)} \right)^{1/2} d(\eta_1 \mu)/d\alpha - 1\|_2]}{\gamma} \), we get

\[
\alpha(g) - \left[ \frac{(\psi^2}{m(\eta_1)} \right]^{1/2} \|d(\eta_1 \mu)/d\alpha - 1\|_2 e^{-\gamma t} e^{-\gamma t} \leq \mathbb{E}_\mu[g(X_t)\mid T_0 > t] \leq \alpha(g) + \left[ \frac{(\psi^2}{m(\eta_1)} \right]^{1/2} \|d(\eta_1 \mu)/d\alpha - 1\|_2 e^{-\gamma t}.
\]

(4.2)
We pick any value smaller than 1, denoted by \( c \), such that
\[
\frac{\alpha(\psi^2)}{m(\eta_1)} \frac{1}{2} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2e^{-\gamma t} < c.
\]
Thus, by an argument similar to that of Theorem 1.1, for any \( t > \frac{\log(\alpha(\psi^2))}{\gamma} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2 \) and \( |g| \leq \psi \), we have
\[
\alpha(g) + \frac{\alpha(\psi^2)}{m(\eta_1)} \frac{1}{2} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2e^{-\gamma t} \leq \alpha(g) + \frac{1 + \alpha(\psi)}{1 - c}.
\]
Similarly, for any \( t > \frac{\log(\alpha(\psi^2))}{\gamma} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2 \) and \( |g| \leq \psi \), one has
\[
\alpha(g) - \frac{\alpha(\psi^2)}{m(\eta_1)} \frac{1}{2} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2e^{-\gamma t} \geq \alpha(g) - \frac{2 + \alpha(\psi)}{1 - c}.
\]
where
\[
D_1 := \left( 1 + \frac{1 + \alpha(\psi)}{1 - c} \right).
\]
So by (4.2), for any \( t > \frac{\log(\alpha(\psi^2))}{\gamma} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2 \), we obtain
\[
\sup_{|g| \leq \psi} | \mathbb{E}_{\mu}[g(X_t)|T_0 > t] - \alpha(g) | \leq \max\{D_1, D_2\} \frac{\alpha(\psi^2)}{m(\eta_1)} \frac{1}{2} \| d(\eta_1 \circ \mu) \|_{d\beta} - 1 \| 2e^{-\gamma t}.
\]
Let
\[\varphi_t(\mu) := \mathbb{P}_{\mu}(X_t \in \cdot | T_0 > t).\]
As mentioned above, we get that there exists \( t_\mu \geq 0 \) such that, for any \( t \geq t_\mu \),
\[
\frac{\alpha(\psi^2)}{m(\eta_1)} \frac{1}{2} \| d(\eta_1 \circ \varphi_t(\mu)) \|_{d\beta} - 1 \| 2e^{-\gamma t} < c.
\]
Also, based on (1.8) and (2.4), for any probability measure \( \mu \) on \((0, \infty)\), any measurable function \( g \) on \((0, \infty)\) and \( t \geq 0 \), it is easy to show that
\[
\eta_1 \circ \varphi_t(\mu)(g) = \frac{\varphi_t(\mu)(\eta g)}{\varphi_t(\mu)(\eta_1)} = \frac{e^{\lambda t} \mu P_t(\eta_1 \circ g)}{\mu(\eta_1)} = (\eta_1 \circ \mu) \tilde{P}_t(g),
\]
that is,
\[
\eta_1 \circ \varphi_t(\mu) = (\eta_1 \circ \mu) \tilde{P}_t. \quad (4.3)
\]
Therefore, by (4.1) and (4.3), for any \( t \geq t_\mu \), we obtain

\[
\sup_{|g| \leq \psi} |\mathbb{E}_\mu[g(X_t)|T_0 > t] - \alpha(g)| \leq \max\{D_1, D_2\} \left[ \frac{\alpha(\psi^2/m_\eta)}{m_\eta} \right]^{1/2} \left\| \frac{d(\eta_1 \circ \varphi_{t_\mu}(\mu))}{d\beta} \right\|_2 e^{-\gamma(t-t_\mu)} \] 

\[
\leq \max\{D_1, D_2\} \left[ \frac{\alpha(\psi^2/m_\eta)}{m_\eta} \right]^{1/2} \left\| \frac{d(\eta_1 \circ \mu)}{d\beta} \right\|_2 e^{-\gamma t}.
\]

This ends the proof of Theorem 1.2.

5 An example

In this section, we apply our main results to the logistic Feller diffusion process on \((0, \infty)\) killed at 0, which is defined as the solution of the SDE

\[dZ_t = \sqrt{\sigma Z_t} dB_t + (rZ_t - kZ_t^2) dt, \quad Z_0 = z > 0,\]

where \(\sigma, r, k\) are positive constants and \((B_t)_{t \geq 0}\) is the standard one-dimensional Brownian motion. The logistic Feller diffusion process is a classic biological model and has strong biological background and applications. See [1] for more information.

Define \(X_t := 2\sqrt{Z_t/\sigma}\). By Itô’s formula, we get

\[dX_t = dB_t - \left( \frac{1}{2X_t} - \frac{rX_t}{2} + \frac{k\sigma X_t^2}{8} \right) dt, \quad X_0 = x = 2\sqrt{z/\sigma} > 0.\]

According to [22, Lemma 3.3], the point 0 is an exit boundary and the point \(\infty\) is an entrance boundary for the process \(X\). Therefore, Theorem 1.1 holds for the process \(X\), and then Theorem 1.1 holds for the process \(Z\). Further, if there exists a function \(\psi : (0, \infty) \to [1, +\infty)\) such that \(\alpha(\psi) < +\infty\) and \(\alpha(\psi^2/m_\eta) < +\infty\), where \(\alpha\) is the unique quasi-stationary distribution of the process \(X\), then for any probability measure \(\mu\) on \((0, \infty)\) satisfying \(\mu \ll \alpha\), Theorem 1.2 holds for the process \(X\), so that for the process \(Z\).

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