Abstract

In this talk, it is discussed the derivation of low-frequencies part of quark determinant and partition function. As a first application, quark condensate is calculated beyond chiral limit with the account of $O(m)$, $O(\frac{1}{N_c})$, $O(\frac{1}{N_c} m)$ and $O(\frac{1}{N_c} m \ln m)$ corrections. It was demonstrated complete correspondence of the results to chiral perturbation theory.

Introduction

Instanton vacuum model assume that QCD vacuum is filled not only by perturbative but also very strong non-perturbative fluctuations – instantons. This model provides a natural mechanism for the spontaneous breaking of chiral symmetry (SBCS) due to the delocalization of single-instanton quark zero modes in the instanton medium. The model is described by two main parameters – the average instanton size $\rho \sim 0.3 \text{ fm}$ and average inter-instanton distance $R \sim 1 \text{ fm}$. These values was found phenomenologically \cite{1} and theoretically \cite{2} and was confirmed by lattice measurements. \cite{3, 4, 5, 6, 7}. On the base of this model was developed effective action approach \cite{8, 9, 10}, providing reliable method of the calculations of the observables in hadron physics at least in chiral limit.

On the other hand, chiral perturbation theory makes a theoretical framework incorporating the constraints on low-energy behavior of various observables based on the general principles of chiral symmetry and quantum field theory \cite{11}.

It is natural expect, that instanton vacuum model leads to the results compatible with chiral perturbation theory.

One of the most important quantities related with SBCS is the vacuum quark condensate $\langle \bar{q}q \rangle$, playing also important phenomenological role in various applications of QCD sum rule approach. Previous investigations \cite{12} shows that beyond chiral limit and at small current quark mass $m \sim \text{few MeV}$ these quantity receive large so called chiral log contribution $\sim \frac{1}{N_c} m \ln m$ with fixed model independent coefficient. On the typical scale $1 \text{GeV}$ it become leading correction since $|\frac{1}{N_c} \ln m| \geq 1$. It was shown, that this correction is due to pion loop contribution \cite{12, 11}.

So, to be consistent we have to calculate simultaneously all of the corrections of order $m$, $\frac{1}{N_c}$, $\frac{1}{N_c} \ln m$ in order to find quark condensate beyond chiral limit.

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In our previous papers \cite{13, 14} on the base of low–frequencies part of light quark determinant $\text{Det}_{\text{low}}$, obtained in \cite{15, 16}, was derived effective action. In this framework was investigated current quark mass $m$ dependence of the quark condensate, but without meson loop contribution \cite{14}.

In the present work we refine the derivation of the low–frequencies part of light quark determinant $\text{Det}_{\text{low}}$. The following averaging of $\text{Det}_{\text{low}}$ over instanton collective coordinates is done independently over each instanton thanks to small packing parameter $\pi(p)^4 \sim 0.1$ and also by introducing constituent quarks degree of freedoms $\psi$. This procedure leads to the light quarks partition function $Z[m]$. We apply bosonisation procedure to $Z[m]$, which is exact one for our case $N_f = 2$ and calculate partition function $Z[m]$ with account of meson loops. This one provide us the quark condensate with desired $O(m)$, $O(\frac{1}{N_c})$, $O(\frac{1}{N_c} m \ln m)$ corrections.

**Low–frequencies part of light quark determinant**

The main assumption of previous works \cite{8, 9, 10} (see also review \cite{16}) was that at very small $m$ the quark propagator in the single instanton field $A_i$ can be approximated as:

$$S_I(m \sim 0) \approx \frac{1}{i\partial} + \frac{\langle \Phi_{0I} \rangle}{i m}$$  \hspace{1cm} (1)

It gives proper value for the $\langle \Phi_{0I} | S_I(m \sim 0) | \Phi_{0I} \rangle = \frac{1}{im}$, but in $S_I(m \sim 0) | \Phi_{0I} \rangle = \frac{\langle \Phi_{0I} \rangle}{im} + \frac{1}{i\partial} | \Phi_{0I} \rangle$ second extra term has a wrong chiral properties. We may neglect by this one only for the $m \sim 0$.

At the present case of non-small $m$ we assume:

$$S_I \approx S_0 + S_0 i\partial \frac{\langle \Phi_{0I} \rangle}{i\partial S_0}, \quad S_0 = \frac{1}{i\partial + im}$$  \hspace{1cm} (2)

where

$$c_I = - \langle \Phi_{0I} | i\partial S_0 | \partial S_0 | \Phi_{0I} \rangle = im \langle \Phi_{0I} | S_0 i\partial | \Phi_{0I} \rangle$$  \hspace{1cm} (3)

The matrix element $\langle \Phi_{0I} | S_I | \Phi_{0I} \rangle = \frac{1}{im}$, more over

$$S_I | \Phi_{0I} \rangle = \frac{1}{im} | \Phi_{0I} \rangle, \quad < \Phi_{0I} | S_I = < \Phi_{0I} | \frac{1}{im}$$  \hspace{1cm} (4)

as it must be.

In the field of instanton ensemble, represented by $A = \sum_I A_I$, full quark propagator, expanded with respect to a single instanton, and with account Eq. (2) is:

$$S = S_0 + \sum_I (S_I - S_0) + \sum_{I \neq J} (S_I - S_0) S_0^{-1} (S_J - S_0)$$

$$+ \sum_{I \neq J, J \neq K} (S_I - S_0) S_0^{-1} (S_J - S_0) S_0^{-1} (S_K - S_0) + ...$$

$$= S_0 + \sum_{I, J} S_0 i\partial | \Phi_{0I} \rangle \left( \frac{1}{C} + \frac{1}{C} T \frac{1}{C} + ... \right)_{I,J} | \Phi_{0J} | i\partial S_0$$

$$= S_0 + \sum_{I, J} S_0 i\partial | \Phi_{0I} \rangle \left( \frac{1}{C - T} \right)_{I,J} | \Phi_{0J} | i\partial S_0$$  \hspace{1cm} (5)
where

\[ C_{IJ} = \delta_{IJ} c_I = -\delta_{IJ} c_I = -\delta_{IJ} \hat{c}_I = -\delta_{IJ} \hat{c}_I < \Phi_0 | i\hat{\partial} S_0 i\hat{\partial} | \Phi_0 > \]

\[ (C - T)_{IJ} = -< \Phi_0 | i\hat{\partial} S_0 i\hat{\partial} | \Phi_0 > \]  \hfill (6)

We are calculating \( \text{Det}_{\text{low}} \) using the formula:

\[ \ln \text{Det}_{\text{low}} = \text{Tr} \int_{M_1} dm' (\tilde{S}(m') - \tilde{S}_0(m')) \]  \hfill (7)

Within zero-mode assumption (Eq. (2)) the trace is restricted to the subspace of instantons:

\[ \text{Tr}(S - S_0) = -\sum_{I,J} < \Phi_0 | i\hat{\partial} S_0 i\hat{\partial} | \Phi_0 > < \Phi_0 | (\frac{1}{i\hat{\partial} S_0 i\hat{\partial}}) | \Phi_0 > \]  \hfill (8)

Introducing now the matrix

\[ B(m)_{IJ} = < \Phi_0 | i\hat{\partial} S_0 i\hat{\partial} | \Phi_0 > \]  \hfill (9)

it is easy to show that

\[ \ln \text{Det}_{\text{low}} = \text{Tr} \int_{M_1} dm' B(m')_{IJ} = \sum_I \int_{B(M_1)} (dB(m'))_{II} \]

\[ = \text{Tr} \ln \frac{B(m)}{B(M_1)} = \ln \det B(m) - \ln \det B(M_1) \]  \hfill (10)

which is desired answer. The determinant \( \det B(m) \) from Eq. (10) is the extension of the Lee-Bardeen result \[15\] for the non-small values of current quark mass \( m \).

Light quark effective action beyond chiral limit

Averaged \( \text{Det}_{\text{low}} \) leads to the partition function \( Z[m] \), which for \( N_f = 2 \) has the form:

\[ Z[m] = \int d\lambda_+ d\lambda_- D\psi D\psi^\dagger \exp[\int d^4 x \sum_{f=1}^{2} \psi_f^\dagger (i\hat{\partial} + i m_f) \psi_f + \lambda_+ Y_2^+ + \lambda_- Y_2^- + N_+ \ln \frac{K}{\lambda_+} + N_- \ln \frac{K}{\lambda_-}] \]  \hfill (11)

here \( \lambda_\pm \) are dynamical couplings (\( K \) is unessential constant, which provide under-logarithm expression dimensionless) \[9\] \[13\] \[14\]. Values of them are defined by saddle-point calculations. \( Y_2^\pm \) are t’Hooft type interaction terms \[10\]:

\[ Y_2^\pm = \frac{1}{N_c^2 - 1} \int d^4x [(1 - \frac{1}{2N_c}) \det iJ^\pm (\rho, x) + \frac{1}{8N_c} \det iJ^\pm_{\mu\nu} (x)] \]  \hfill (12)

\[ J_{f\rho}^\pm (x) = \int \frac{d^4k_f d^4l_g}{(2\pi)^8} \exp i(k_f - l_g)xq_f^\dagger (k_f)\frac{1 \pm \gamma_5}{2} q_g(l_g) \]

\[ J_{\mu\nu,f\rho}^\pm (x) = \int \frac{d^4k_f d^4l_g}{(2\pi)^8} \exp i(k_f - l_g)xq_f^\dagger (k_f)\frac{1 \pm \gamma_5}{2} \sigma_{\mu\nu} q_g(l_g) \]
where \( q(k) = 2\pi \rho F(k) \psi(k) \). The form-factor \( F(k) \) is due to zero-modes and has explicit form \( F(k) = -\frac{d}{dt}[I_0(t)K_0(t) - I_1(t)K_1(t)]_{t=|k|/2} \). In the following we will neglect by \( J^\pm_{\mu \nu, f_0}(x) \) interaction term, since it gives a \( O(\frac{1}{N_c^2}) \) contribution to the quark condensate. Since

\[
q(x) = \int \frac{d^4k}{(2\pi)^4} \exp(ikx) \, q(k), \quad J^\pm_{f_0}(x) = q^\pm(x) \frac{1 \pm \gamma_0}{2} \hat{q}_0(x),
\]

(13)

\[
\det \frac{iJ^+(x)}{g} + \det \frac{iJ^-(x)}{g} = \frac{1}{8g^2} (-q^+(x)q(x))^2 - (q^+(x)i\gamma_5\tau q(x))^2 + (q^+(x)\tau q(x))^2 + (q^+(x)i\gamma_5\bar{q}(x))^2).
\]

Here color factor \( g^2 = \frac{(N_c^2-1)2N_c}{(2N_c-1)} \).

In the following we will take equal number of instantons and antiinstantons \( N_+ = N_- = N/2 \) and corresponding couplings \( \lambda_\pm = \lambda \).

Now it is natural to bosonize quark-quark interaction terms (13) by introducing meson fields. For \( N_f = 2 \) case it is exact procedure. We have to take into account the changes of \( q \) and \( q^\dagger \) under the \( SU(2) \) chiral transformations:

\[
\delta q = i\gamma_5\bar{\tau}\bar{\alpha}q, \quad \delta q^+ = q^+ i\gamma_5\bar{\tau}\bar{\alpha}
\]

to introduce appropriate meson fields, changing under \( SU(2) \) chiral transformations as:

\[
\delta \sigma = 2\bar{\alpha}\bar{\bar{\sigma}}, \quad \delta \bar{\sigma} = -2\bar{\alpha}\sigma, \quad \delta \eta = -2\bar{\alpha}\bar{\sigma}, \quad \delta \bar{\eta} = 2\eta\bar{\bar{\eta}}.
\]

Then \( \delta q^+(\sigma + i\gamma_5\bar{\tau}\phi)q = 0, \quad \delta q^+(\bar{\tau}\bar{\sigma} + i\gamma_5\eta)q = 0 \) means that these combinations of fields are chiral invariant. So, the interaction term has an exact bosonized representation:

\[
\int d^4x \exp[\lambda(\det \frac{iJ^+(x)}{g} + \det \frac{iJ^-(x)}{g})]
\]

(14)

\[
= \int D\sigma D\bar{\sigma} D\eta D\bar{\eta} \exp \int d^4x \frac{\lambda^0.5}{2g} q^+(\sigma + i\gamma_5\bar{\tau}\phi + i\bar{\tau}\bar{\sigma} + \gamma_5\eta q - \frac{1}{2}(\sigma^2 + \phi^2 + \bar{\sigma}^2 + \eta^2))
\]

Then the partition function is

\[
Z[m] = \int d\lambda D\sigma D\bar{\sigma} D\eta D\bar{\eta} \exp[\lambda \ln \frac{K}{\lambda} - N - \frac{1}{2} \int d^4x (\sigma^2 + \phi^2 + \bar{\sigma}^2 + \eta^2) + \text{Tr} \ln \frac{\hat{p} + im + i\frac{\lambda^{0.5}}{2g}(2\pi\rho)^2 F(\sigma + i\gamma_5\bar{\tau}\phi + i\bar{\tau}\bar{\sigma} + \gamma_5\eta)F}{\hat{p} + im}]
\]

(15)

\[
(\text{Tr}(\cdots) \text{ means here } \text{tr}_{\gamma,c,f} \int d^4x <x|(\cdots)|x> \text{, where } \text{tr}_{\gamma,c,f} \text{ is the trace over Dirac, color, and flavor indexes.}) \text{ In the following we assume } m_u = m_d = m. \text{ Then common saddle point on } \lambda, \sigma (= \text{ const}) \text{ (others = 0)} \text{ is defined by Eqs. } \frac{\partial V[m,\lambda,\sigma]}{\partial \lambda} = \frac{\partial V[m,\lambda,\sigma]}{\partial \sigma} = 0, \text{ where the potential}
\]

\[
V[m,\lambda,\sigma] = -N \ln \frac{K}{\lambda} + N + \frac{1}{2} V\sigma^2 - \text{Tr} \ln \frac{\hat{p} + im + M(\lambda,\sigma)F^2(p)}{\hat{p} + im}
\]

(16)

\[
\text{Certainly, quark-quark interaction term Eq. (13) is non-invariant over } U(1) \text{ axial transformations, as it must be.}
and we defined \( M(\lambda, \sigma) = \frac{\lambda^5}{2g} (2\pi \rho)^2 \sigma \). Then the common saddle-point on \( \lambda \) and \( \sigma \) is given by Eqs.:

\[
N = \frac{1}{2} \text{Tr} \frac{iM(\lambda, \sigma)F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma)F^2(p))} = \frac{1}{2} V\sigma^2. \tag{17}
\]

The solutions of this Eqs. are \( \lambda_0 \) and \( \sigma_0 = (2^{N/2})^{1/2} = 2^{1/2} R^{-2} \). It is clear that \( M_0 = M(\lambda_0, \sigma_0) \) has a meaning of dynamical quark mass, which is defined by this Eqs.. At typical values \( R^{-1} = 200 \text{ MeV}, \rho^{-1} = 600 \text{ MeV} \) we have \( \sigma_0^2 = 2(200 \text{ MeV})^4 \), and in chiral limit \( m = 0 \) \( M_0 \rightarrow M_{00} = 358 \text{ MeV} \), \( \lambda_{00} \approx M_{00}^2 \).

**Vacuum with account of quantum corrections**

The account of the quantum fluctuations around saddle-points \( \sigma_0, \lambda_0 \) will change the potential \( V[m, \lambda, \sigma] \) to \( V_{\text{eff}}[m, \lambda, \sigma] \) (it is clear that the difference between these two potentials is order of \( 1/N_c \)). Then, the partition function is given by Eq.

\[
Z[m] = \int d\lambda \exp(-V_{\text{eff}}[m, \lambda, \sigma]) \tag{18}
\]

There is important difference between this instanton generated partition function \( Z[m] \) and traditional \( NJL \)-type models – we have to integrate over the coupling \( \lambda \) here. As was mentioned before, this integration on \( \lambda \) by saddle-point method leads to exact answer. This saddle-point is defined by Eq.:

\[
\frac{dV_{\text{eff}}[m, \lambda, \sigma]}{d\lambda} = 0 \tag{19}
\]

which leads to the \( \lambda \) as a function of \( \sigma \), i.e. \( \lambda = \lambda(\sigma) \).

Then, the vacuum is the minimum of the effective potential \( V_{\text{eff}}[m, \sigma] \), which is given by a solution of the equation

\[
\frac{dV_{\text{eff}}[m, \sigma, \lambda(\sigma)]}{d\sigma} = \frac{\partial V_{\text{eff}}[m, \sigma, \lambda(\sigma)]}{\partial \sigma} = 0. \tag{20}
\]

where it was used Eq. \([17]\).

We denote a fluctuations as a primed fields \( \Phi' \). The action and corresponding \( V_{\text{eff}} \) now has a form:

\[
S[m, \lambda, \sigma, \Phi'] = S_0[m, \lambda, \sigma] + S_V[m, \lambda, \sigma, \Phi'], \tag{21}
\]

\[
S_0[m, \lambda, \sigma] = V[m, \lambda, \sigma] = \frac{1}{2} V\sigma^2 - \text{Tr} \ln \frac{\hat{p} + i(m + M(\lambda, \sigma)F^2)}{\hat{p} + im} - N \ln \frac{K}{\lambda} + N \]

\[
S_V[m, \lambda, \sigma, \Phi'] = \int d^4x \frac{1}{2} (\sigma'^2 + \bar{\phi}^2 + \bar{\sigma}^2 + \eta'^2)
- \frac{1}{2\sigma^2} \text{Tr} \left[ \frac{iM(\lambda, \sigma)F^2}{\hat{p} + i(m + M(\lambda, \sigma)F^2)} (\sigma' + i\gamma_5 \bar{\phi} + i\bar{\sigma}' + \gamma_5 \eta') \right]^2,
\]

and

\[
V_{\text{eff}}[m, \lambda, \sigma] = S_0[m, \lambda, \sigma] + V_{\text{eff}}^{\text{mes}}[m, \lambda, \sigma] \tag{22}
\]
Here second term in Eq. (22) is explicitly represented by

\[
V_{\text{eff}}^{\text{mes}}[m, \lambda, \sigma] = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S_V[m, \lambda, \sigma, \Phi']}{\delta \Phi_i(x) \delta \Phi'_j(y)} = \frac{V}{2} \sum_i \int \frac{d^4q}{(2\pi)^4} \ln[1 - \frac{1}{\sigma^2}] \int \frac{d^4p}{(2\pi)^4}
\]

\[
\times \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma], \tag{23}
\]

where the factors \( \Gamma_i = (1, i\gamma_5\vec{\tau}, i\vec{\tau}, \gamma_5) \) and the sum on \( i \) is counted all corresponding meson fluctuations contribution \( \sigma', \vec{d}', \vec{d}', \eta' \) tr here means the trace over flavor, color and Dirac indexes. Integrals in Eq. (23) are completely convergent one due to the presence of the form-factors \( F \).

Certainly the quantum fluctuations contribution will move the the coupling \( \lambda \) from \( \lambda_0 \) to \( \lambda_0 + \lambda_1 \) and \( \sigma \) as \( \sigma_0 \to \sigma_0 + \sigma_1 \), where \( \frac{\lambda_1}{\lambda_0} \) and \( \frac{\sigma_1}{\sigma_0} \) are of order \( 1/N_c \).

First, consider Eq. (19):

\[
\frac{dV_{\text{eff}}[m, \lambda, \sigma]}{d\lambda} = N - \frac{1}{2} \text{Tr} \frac{iM(\lambda, \sigma) F^2}{\hat{p} + i(m + M(\lambda, \sigma) F^2)} + \frac{V}{2} \sum_i \int \frac{d^4q}{(2\pi)^4} \tag{24}
\]

\[
\times [\sigma^2 - \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i^{-1}]
\]

\[
+ \text{itr} \int \frac{d^4p}{(2\pi)^4} \left( \frac{M(\lambda, \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda, \sigma) F^2(p))} \right)^2 \Gamma_i \frac{M(\lambda, \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda, \sigma) F^2(p + q))} \Gamma_i = 0
\]

From this saddle-point Eq. we get \( \lambda = \lambda(\sigma) \).

From vacuum Eq. (20) we in similar manner arrive to:

\[
\frac{\partial V_{\text{eff}}[m, \sigma, \lambda(\sigma)]}{\partial \sigma} = V \sigma^2 - \text{Tr} \frac{iM(\lambda(\sigma), \sigma) F^2}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2)} + \frac{V}{2} \sum_i \int \frac{d^4q}{(2\pi)^4} \tag{25}
\]

\[
\times [\sigma^2 - \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{M(\lambda(\sigma), \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2(p))} \Gamma_i \frac{M(\lambda(\sigma), \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda(\sigma), \sigma) F^2(p + q))} \Gamma_i^{-1}]
\]

\[
+ 2\text{itr} \int \frac{d^4p}{(2\pi)^4} \left( \frac{M(\lambda(\sigma), \sigma) F^2(p)}{\hat{p} + i(m + M(\lambda(\sigma), \sigma) F^2(p))} \right)^2 \Gamma_i \frac{M(\lambda(\sigma), \sigma) F^2(p + q)}{\hat{p} + \hat{q} + i(m + M(\lambda(\sigma), \sigma) F^2(p + q))} \Gamma_i = 0
\]

Since we are believing to \( \frac{1}{N_c} \) expansion, it is natural inside quantum fluctuations contribution (under the integrals over \( q \)) to take \( \sigma = \sigma_0 \), \( M(\lambda(\sigma), \sigma) = M_0 \).

To simplify the expressions introduce vertices \( V_{2i}(q), V_{3i}(q) \) and meson propagators \( \Pi_i(q) \), which are defined as:

\[
V_{2i}(q) = \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \Gamma_i \frac{M_0(p + q)}{\hat{p} + \hat{q} + i\mu_0(p + q)} \tag{26}
\]

\[
V_{3i}(q) = \text{tr} \int \frac{d^4p}{(2\pi)^4} \left( \frac{M_0(p)}{\hat{p} + i\mu_0(p)} \right)^2 \Gamma_i \frac{M_0(p + q)}{\hat{p} + \hat{q} + i\mu_0(p + q)} \tag{27}
\]

\[
\Pi_i^{-1}(q) = \frac{2}{R^4} - V_{2i}(q). \tag{28}
\]
Here $M_0(p) = M_0 F^2(p)$, $\mu_0(p) = m + M_0(p)$ and was taken into account that $\sigma_0^2 = 2 R^{-1}$.

From Eqs. (24) and (25) we have

$$\frac{M_1}{M_0} \left[ \frac{2}{R^4} + \frac{1}{V} \text{Tr} \left( \frac{M_0(p)}{p + i\mu_0(p)} \right)^2 \right] = \sum_i \int \frac{d^4 q}{(2\pi)^4} (i V^3_i(q) - V_{2i}(q)) \Pi_i(q) \tag{29}$$

$$\frac{\sigma_1}{\sigma_0} = - \frac{R^4}{4} \sum_i \int \frac{d^4 q}{(2\pi)^4} V_{2i}(q) \Pi_i(q) \tag{30}$$

The vertices $V_{2i}(q)$, $V_{3i}(q)$ and the meson propagators $\Pi_i(q)$ are well defined functions, providing well convergence of the integrals in Eqs. (29), (30).

It is of special attention to the contribution of pion fluctuations $\bar{\phi}'$ at small pion momentum $q$. We shall demonstrate that this contribution leads to the famous chiral log term with model independent coefficient in the correspondence with previous calculations in NJL-model [18].

Pion inverse propagator of $\Pi^{-1}_\phi(q)$ at small $q \sim m_\pi$ is: $\Pi^{-1}_\phi(q) = f_{\text{kin}}^2 (m_\pi^2 + q^2)$. At lowest order on $m$, $f_{\text{kin},m=0} \approx f_{\pi} = 93 \text{ MeV}$, $m_\pi^2 \sim m$.

The vertices in the right side of Eq. (29) at $q = 0$ and in chiral limit are:

$$V_{2\bar{\phi}',m=0}(0) = \frac{2}{R^4}, \quad i V_{3\bar{\phi}',m=0}(0) - V_{2\bar{\phi}',m=0}(0) = 8 N_c \int \frac{d^4 p}{(2\pi)^4} \frac{p^2 M_0^2(p)}{(p^2 + M_0^2(p))^2} \tag{31}$$

We see that the factor in the left side of Eq. (29) in the chiral limit is equal to:

$$\text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{i \hat{p} M_0(p)}{(\hat{p} + i M_0(p))^2} = -2 (i V_{3\bar{\phi}',m=0}(0) - V_{2\bar{\phi}',m=0}(0)) \tag{32}$$

Collecting all the factors we get small $q \leq \kappa$ contribution of pion fluctuations $\bar{\phi}'$:

$$\frac{\sigma_1}{\sigma_0} |_{\bar{\phi}', \text{small } q} = \frac{M_1}{M_0} |_{\bar{\phi}', \text{small } q} = - \frac{3}{2 f_\pi^2} \int_0^\kappa \frac{d^4 q}{(2\pi)^4} \frac{1}{m_\pi^2 + q^2}$$

$$= - \frac{3}{32 \pi^2 f_\pi^2} \int_0^\kappa q^2 dq^2 \frac{1}{f_\pi^2(m_\pi^2 + q^2)} = - \frac{3}{32 \pi^2 f_\pi^2} \left( \kappa^2 + m_\pi^2 \ln \frac{m_\pi^2}{\kappa^2 + m_\pi^2} \right)$$

Here we put $m = 0$ everywhere except $m_\pi$. We see that the coefficient in the front of of $m_\pi^2 \ln m_\pi^2$ is a model independent as it must be.

Quick estimate, assuming $\kappa = \rho^{-1}$, gives

$$\frac{\sigma_1}{\sigma_0} |_{\bar{\phi}} \approx \frac{M_1}{M_0} |_{\bar{\phi}} \approx - \frac{3}{32 \pi^2 f_\pi^2 \rho^2} (1 + m_\pi^2 \rho^2 \ln m_\pi^2 \rho^2) \approx -0.4 (1 + 0.054 \ln 0.054) \tag{34}$$

So, we expect that pion loops is provided not only non-analytical $\frac{1}{N_c} m \ln m$ term but also very large contribution to $\frac{1}{N_c} m$ term.

This one dictate the strategy of the following calculations of $\sigma_1$ and $M_1$:
1. we have to extract analytically $\frac{1}{N_c} m \ln m$ term from pion loops;
2. rest part of $\sigma_1$ and $M_1$ can be calculated numerically and expanded over $m$, paying special attention to the pion loops and keeping $\frac{1}{N_c}$ and $\frac{1}{N_c} m$ terms.
For actual numerical calculations we are using simplified version of the form-factor \( F(p) \) from [17] (with corrected high momentum dependence):

\[
F(p < 2\text{GeV}) = \frac{L^2}{L^2 + p^2}, \quad F(p > 2\text{GeV}) = \frac{1.414}{p^3} \tag{35}
\]

where \( L \approx \sqrt{\frac{2}{p}} = 848\text{MeV} \).

At \( N_c = 3 \) semi-numerical calculations of \( M_1 \) and \( \sigma_1 \) lead to:

\[
\frac{M_1}{M_0} = -0.662 - 4.64m - 4.01m \ln m \tag{36}
\]

\[
\frac{\sigma_1}{\sigma_0} = -0.523 - 4.26m - 4.00m \ln m \tag{37}
\]

Here \( m \) is given in \( \text{GeV} \). Certainly, in (38) the \( m \ln m \) term is completely correspond to Eq. (33). \( \frac{M_1}{M_0} \) is \(-66\%\) in chiral limit and reach its maximum \( \sim -20\% \) at \( m \sim 0.115 \text{GeV} \).

The relative shift of the vacuum \( \frac{\sigma_1}{\sigma_0} \) is \(-52\%\) at the chiral limit and reach its maximum \( \sim -2\% \) at \( m \sim 0.125 \text{GeV} \).

The main contribution to both quantities \( \frac{M_1}{M_0} \) and \( \frac{\sigma_1}{\sigma_0} \) come from pion loops. Other mesons give the contribution \( \sim 10\% \) to \( O\left(\frac{1}{N_c}\right) \) and \( O\left(\frac{1}{N_c^2m}\right) \) terms.

**Quark condensate**

We have to calculate quark condensate beyond chiral limit taking into account \( O(m), O\left(\frac{1}{N_c}\right), O\left(\frac{1}{N_c^2m}\right) \) and \( O\left(\frac{1}{N_c^3m \ln m}\right) \) terms. Quark condensate is extracted from the partition function:

\[
<\bar{q}q> = \frac{1}{2V} \frac{dV_{\text{eff}}[m, \lambda, \sigma]}{dm} = \frac{1}{2V} \partial_m \left[ V[m, \lambda, \sigma] + V_{\text{eff}}^\text{mes}[m, \lambda_0, \sigma_0] \right]
\]

\[
= -\frac{1}{2V} \text{Tr} \left( \frac{i}{\hat{p} + i\mu(p)} - \frac{i}{\hat{p} + im} \right) + \frac{1}{2V} \frac{\partial V_{\text{eff}}^\text{mes}[m, \lambda_0, \sigma_0]}{\partial m} \tag{38}
\]

here \( \lambda = \lambda_0 + \lambda_1, \quad \sigma = \sigma_0 + \sigma_1, \quad M = M_0 + M_1, \quad \mu(p) = m + MF^2(p) \). First term of Eq. (38) is

\[
-\frac{1}{2V} \text{Tr} \left( \frac{i}{\hat{p} + i\mu(p)} - \frac{i}{\hat{p} + im} \right) \tag{39}
\]

\[
= -4N_c \int \frac{d^4p}{(2\pi)^4} \left( \frac{\mu_0(p)}{p^2 + \mu_0^2(p)} - \frac{m}{p^2 + m^2} + \frac{M_1}{M_0} \frac{M_0(p)(p^2 - \mu_0^2(p))}{(p^2 + \mu_0^2(p))^2} \right)
\]

Second term of Eq. (38) — meson loops contribution to the condensate is

\[
\frac{1}{2V} \frac{\partial V_{\text{eff}}^\text{mes}[m, \lambda_0, \sigma_0]}{\partial m} = \frac{i}{2} \sum_i \int \frac{d^4q}{(2\pi)^4} \left[ \text{Tr} \int \frac{d^4p}{(2\pi)^4} \frac{M_0(p)}{\hat{p} + q + i\mu_0(p + q)} \Gamma_i \right] \frac{M_0(p + q)}{\hat{p} + q + i\mu_0(p + q)} \Gamma_i \tag{40}
\]

\[
\times \left( \frac{2N}{V} - \text{Tr} \int \frac{d^4p}{(2\pi)^4} \frac{M_0(p)}{\hat{p} + q + i\mu_0(p + q)} \Gamma_i \right)^{-1}
\]
At $m = 0$ and without meson loops the condensate is

$$<\bar{q}q>_{00} = -4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}(p)}{p^2 + M_{00}^2(p)}$$

(41)

Here $M_{00} \equiv M_{0,m=0}$.

Let us consider now the contribution of pion fluctuations $\bar{\phi}$ to the quark condensate at small $q$. First we consider:

$$\frac{1}{2V} \frac{\partial V_{eff, small \ q}[m, \lambda_0, \sigma_0]}{\partial m} = 12N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}^3(p)\mu_0(p)}{(p^2 + M_{00}^2(p))^2} \int_0^\infty \frac{d^4q}{(2\pi)^4} \frac{d^4q}{f_{kin}^2(m_q^2 + q^2)}$$

(42)

We keep $m$ only in $m_q^2$. Then at $m = 0\mu_0(p) \Rightarrow M_0(p) \Rightarrow M_{00}(p)$, $f_{kin} \Rightarrow f_\pi$ and we have

$$<\bar{q}q> = <\bar{q}q>_{00} - \frac{M_1}{M_0} 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M_{00}(p)(p^2 - M_{00}^2(p))}{(p^2 + M_{00}^2(p))^2} \int_0^\infty \frac{d^4q}{(2\pi)^4} \frac{1}{f_\pi^2(m_q^2 + q^2)}$$

(43)

$$= <\bar{q}q>_{00} \left(1 - \frac{3}{2} \int_0^\infty \frac{d^4q}{(2\pi)^4} \frac{1}{f_\pi^2(m_q^2 + q^2)}\right)$$

(44)

Eq. (44) for $\frac{M}{M_0}$ was applied here. We see that Eq. (41) is in the full correspondence with [11, 12].

Detailed numerical calculations lead to the semi-analytical formula for the quark condensate including all $O(m)$, $O(\frac{1}{N_c})$ and $O(\frac{1}{N_c}m \ln m)$-corrections:

$$<\bar{q}q> = <\bar{q}q>_{m=0} (1 - 18.53 m - 7.72 m \ln m)$$

(45)

Here $<\bar{q}q>_{m=0} = 0.52 <\bar{q}q>_{00}$. Certainly, the $m \ln m$ term in Eq. (45) is in full correspondence with Eq. (44), as it must be. $<\bar{q}q> / <\bar{q}q>_{m=0}$ is a rising function of $m$ until $m \sim 0.04 GeV$ and it falls again in the region $m > 0.04 GeV$.

The main contribution to $O(\frac{1}{N_c})$, $O(\frac{1}{N_c}m)$ and $O(\frac{1}{N_c}m \ln m)$ terms in $<\bar{q}q>_{m=0}$ is due to pion loops. Other mesons give the contribution $\sim$ few % to $O(\frac{1}{N_c})$ and $O(\frac{1}{N_c} m)$ terms.

$m_d - m_u$ effects in quark condensate

Current quark mass become diagonal $2 \times 2$ matrix with $m_1 = m_u$, $m_2 = m_d$, $m = m_1 + m_2 = m + \delta m$. Here $m = \frac{1}{2} [m_1 + m_2]$, $\delta m = m_1 - m_2$. Let us introduce external field $s_i$. In our particular case it is $s_3 = \frac{1}{2} m_i$, $s_1 = s_2 = 0$. Our aim is to find the asymmetry of the quark condensate $<\bar{q}u> - <\bar{q}d>$, taking into account only $O(\delta m)$ terms and neglecting $O(\frac{1}{N_c}\delta m)$, $O(\frac{1}{N_c}\delta m \ln m)$. It means that we neglect at all by meson loops contribution.

In the presence of the external field $\vec{s}$ we expect also vacuum field $\vec{\sigma}$. Effective potential within requested accuracy is

$$V_{eff}[\sigma, \vec{\sigma}, m] \approx S_0[m, \lambda, \sigma, \vec{\sigma}]$$

$$= \frac{V}{2} (\sigma^2 + \vec{\sigma}^2) - \text{Tr} \ln \frac{\hat{p} + i\vec{s} + i(m + M(\lambda, \sigma, \vec{\sigma})F^2)}{\hat{p} + im + i\vec{s}} - N \ln \frac{K}{\lambda} + N.$$

(46)
\(\lambda, \sigma, \vec{\sigma}\) are defined by the vacuum equations:

\[
\frac{\partial V_{\text{eff}}}{\partial \lambda} = 0, \quad \frac{\partial V_{\text{eff}}}{\partial \sigma} = 0, \quad \frac{\partial V_{\text{eff}}}{\partial \sigma_i} = 0.
\]  

(47)

They can be reduced to the following form:

\[
\frac{1}{2} \Tr \frac{F^2(p) M_i(m_i + M_i F^2(p))}{p^2 + (m_i + M_i F^2(p))^2} = N
\]

(48)

where \(M_i = \frac{x_0}{2g} (2\pi \rho)^2 (\sigma \pm \sigma_3)\). Solution of these equations leads to \(\lambda = \lambda[m, \vec{s}], \sigma = \sigma[m, \vec{s}] \sigma_i = \sigma_i[m, \vec{s}]\). We have to put them into \(V_{\text{eff}}\) and find \(V_{\text{eff}} = V_{\text{eff}}[m, \vec{s}]\). Desired correlator is

\[
\left. \frac{\partial V_{\text{eff}}[m, \vec{s}]}{\partial s_3} \right|_{s_3=\frac{4m}{2}, s_{1,2}=0}
\]

(49)

We calculate this correlator within requested accuracy, taking into account only \(O(\delta m)\) terms. So, the difference of the vacuum quark condensates of \(u\) and \(d\) quarks is

\[
< \bar{u}u > - < \bar{d}d > = \frac{1}{V} \left[ \Tr \left( \frac{-i}{\hat{p} + i(m_u + M_u F^2)} - \frac{-i}{\hat{p} + i m_u} \right) - \Tr \left( \frac{-i}{\hat{p} + i(m_d + M_d F^2)} - \frac{-i}{\hat{p} + i m_d} \right) \right]
\]

We expect that \(< \bar{d}d > < < \bar{u}u > \) if \(m_d > m_u\).

Typical values of light current quark masses \([19]\) are \(m_u = 5.1MeV, m_d = 9.3MeV\) on the scale \(1GeV\) (which is in fact close to our scale \(\rho^{-1} = 0.6 GeV\)) leads to the asymmetry

\[
\frac{< \bar{u}u > - < \bar{d}d >}{< \bar{u}u >} = 0.026
\]

(51)

From this asymmetry and using sum-rules \([20]\) we estimate strange quark condensate at \(m_s = 120 MeV\) as:

\[
\frac{< \bar{s}s >}{< \bar{u}u >} = 0.43
\]

(52)

which is rather small. The reason that the asymmetry \([51]\) is rather large.

**Conclusion**

In the framework of instanton vacuum model it was calculated simplest possible correlator – quark condensate with complete account of \(O(m), O(\frac{1}{N_c}), O(\frac{1}{N_c} m)\) and \(O(\frac{1}{N_c} m \ln m)\) terms, demanding the calculation of meson loops contribution. Since initial instanton generated quark-quark interactions are nonlocal and contain corresponding form-factor induced by quark zero-mode, these loops correspond completely convergent integrals. The main loop corrections come from the pions, as it was expected. We found that \(O(\frac{1}{N_c})\) corrections are very large \(\sim 50\%\), which request the \(\sim 10\%\) changing of the basic parameters – average inter-instanton distance \(R\) and average instanton size \(\rho\) to restore chiral limit value of the quark condensate \(< \bar{q}q >_{m=0}\) and other important quantities as \(f_\pi\) and \(m_\pi\) to their phenomenological values. This work in the progress.

In general, it was demonstrated, that instanton vacuum model is well working tool also beyond chiral limit and satisfy chiral perturbation theory.
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