Characters and Brauer trees of the covering group of $2E_6(2)$

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Abstract

Let $G$ be the finite simple Chevalley group of type $2E_6(2)$. It has a Schur multiplier of type $C_2^2 \times C_3$. We determine the ordinary character tables of the central extensions $3G, 6G, (2^2 \times 3).G$ of $G$ and their extensions by an automorphism of order 2, that is $3G.2, 6G.2$ and $(2^2 \times 3).G.2$.

Furthermore we determine all Brauer trees of all groups of type $Z.G.A$ (where $Z$ is central in $Z.G \trianglelefteq Z.G.A$ and $A \cong Z.G.A/Z.G$) for which the ordinary character table is known.

1 Introduction

Let $p$ be a prime and $G$ a simple reductive algebraic group over an algebraic closure of the finite field $\mathbb{F}_p$ with $p$ elements. Let $q$ be a power of $p$ and $G$ be defined over $\mathbb{F}_q$. We denote $G(q)$ the group of $\mathbb{F}_q$-rational points—a (twisted or untwisted) finite group of Lie type. Let $Z$ be the center of $G(q)$.

If $G$ is of simply connected type then except for finitely many cases $G(q)/Z$ is a finite simple group. If $G(q)/Z$ is simple then in most cases $Z$ is isomorphic to the Schur multiplier of the simple quotient and $G(q)$ is a universal covering. There are 18 cases with an exceptional Schur multiplier, see [CCN+85, 3. and Table 5.]. For only one of these cases the (ordinary) character table of the full covering group was not determined by the ATLAS project, this is the case where $G(q)/Z$ is of type $2E_6(2)$. In the first part of this paper we describe the computation of this missing character table. This also yields the character table of this group extended by an outer automorphism of order 2 and of some other related groups.

Now let $G$ be the finite simple group of type $2E_6(2)$.

Previously known character tables. The character tables of the central extension $2^2.G$ of $G$ with the exceptional part of the Schur multiplier (and so of its factor groups $2.G$ and $G$), as well as of the larger groups of form $G.2, G.3, G.S_3, 2.G.2, 2^2.G.2, 2^2.G.3$ and $2^2.G.S_3$ were available in the GAP character table library [Bre04], the table of $2.G.2$ is also printed in the ATLAS [CCN+85, p.191].

The essential step to get the character table of the covering group $(2^2 \times 3).G$ of $G$ is to find the table of its quotient of type $3.G$. The group $3.G$
arises as a twisted finite group of Lie type $G(2)$ where $G$ is the simple simply connected reductive group of type $E_6$ over an algebraic closure of the field with 2 elements. For this group we can use Deligne-Lusztig theory (see [DL76], [Car85]) to construct some faithful characters, which together with the known table of the simple quotient $G$ enables us to compute the whole table of $3.G$. We explain the details of the construction in section 2.

As an application we describe in section 3 the determination of all Brauer trees (that is the decomposition numbers for all prime blocks with a cyclic defect group) of $(2^2 \times 3).G$ and the available extensions of this and its factor groups by outer automorphisms.

Since the newly computed character tables are huge (e.g., $(2^2 \times 3).G$ has 934 conjugacy classes and irreducible characters) we will not include the new character tables in this article. The tables are available in newer versions of the GAP-package CTbLib, see [GAP06] and [Bre04, since version 1.2.0], and were already used for various applications. Information on conjugacy classes and character degrees of $3.G$ is also given on the webpage [Lüb07].

We note that the ATLAS-tables which are relevant here could be recomputed and checked recently [BMO16] using some explicit representations and a generic function in Magma [C+C+15]. But the case of the table for $3.G$ seems out of reach of this generic function currently. Therefore, we document the computation of the tables of groups with simple composition factor $G$ which are not contained in the ATLAS in this note.

The character tables of the related groups $3.G.3$, $3.G.S_3$, $(2^2 \times 3).G.3$ and $(2^2 \times 3).G.S_3$ are still not known. Their computation will be addressed elsewhere.

2 The character table of $3.G$

As mentioned in the Introduction the group $3.G$ can be constructed as twisted group $G(2)$ of $F_2$-rational points of a simply connected algebraic group $G$ of type $E_6$ over an algebraic closure of $F_2$.

To compute its character table we use the following data which were previously known:

(a) The character table of the simple quotient $G$ from the ATLAS [CCN+85]. This contains the centralizer orders (or class lengths), the power maps (that is for each conjugacy class $C$ with representative $g$ and integer $n$ one can read off the conjugacy class of $g^n$, in particular the orders of representatives of the conjugacy classes are known) and the character values.

(b) A parameterization of the conjugacy classes of $G(q)$ for all prime powers $q$ with $q \equiv 2 \mod 3$ and part of the values of unipotent characters were determined and used in [HL98]. This can easily be specialized to the 346 conjugacy classes in case $q = 2$. Those computations also gave explicit representatives for the semisimple classes (those with elements of odd order) in some fixed maximal torus of the algebraic group $G$, in particular we get the element orders for the semisimple classes.
(c) A list of all irreducible character degrees of $G(q)$ for all prime powers $q$ was computed for the results in [Lüb01], again this can easily be specialized to $q = 2$.

The specialized data are available on [Lüb07].

To find some values of faithful characters for the group $3.G$ we extend (b) and compute the Deligne-Lusztig characters of $G(2)$. For this we use a practical variant of the character formula [Car85, 7.2.8], as described in [Lüb93, Satz 2.1(b)]. In principle this allows one to compute the generic values of Deligne-Lusztig characters, that is a parameterized form of the values of $G(q)$ for all prime powers $q \equiv 2 \mod 6$. But this would need too much memory and computation time, even with current computers. Fortunately, only a small part of this generic character table is relevant for $q = 2$ (for example, only a few conjugacy classes of maximal tori of $G(2)$ contain regular elements; more precisely 342 of the 494 columns of the generic character table are not needed for $q = 2$). Computing just the part of the generic table of Deligne-Lusztig characters relevant for $q = 2$ is feasible. The values of the Green functions occurring in the character formula are needed for groups of type $2E_6$, $D_4$ and $A_l$, $l \leq 5$. They are available from [Mal93] and CHEVIE [GHL+96].

Some irreducible characters of $G(2)$ are known linear combinations of Deligne-Lusztig characters, this uses Lusztig’s parameterization of irreducible characters [Lus85, 4.22]. More details are given in [Lüb93, 7.]. There are also some further uniform almost characters which are known as explicit linear combinations of irreducible characters and of Deligne-Lusztig characters; the multiplicities of irreducibles are rational numbers, and we can multiply by some integers to get generalized characters. This way we get:

(d) The values of about 200 irreducible characters of $G(2)$ in terms of the parameterization in (b), some of them are trivial on the center (that is they lead to known characters of the simple quotient $G$), and some are not (that is they are faithful characters of $3.G$).

(e) The values of about 150 generalized characters of $G(2)$.

It remains to explain how to merge the information from (a) to (e) to find the whole character table of $G(2)$. This involves further computations with GAP [GAP06] which provides powerful functions for computing with character tables.

First we need the fusion of the conjugacy classes as described in (b) to the classes of the ATLAS-table in (a). There are two possible behaviours of a class, either an element multiplied by the center elements yields representatives of one or of three different classes; that is under the canonical map $3.G \rightarrow G$ the preimage of each class of $G$ contains either one or three classes. Using the element orders and centralizer orders in both cases and identifying some irreducible characters from (d) which are trivial on the center with characters in (a) only very few possibilities remain. Namely, on a few tuples of classes all
characters in (d) and (e) have the same value, in these cases we just choose some fusion.

Now we can lift all irreducible characters of $G$ as given in (a) to the table of $3.G$.

It turns out that the class functions we have found so far are sufficient to determine all power maps for $3.G$ using the GAP-function for computing all power maps compatible with a set of given characters (there are several possibilities but these are all equivalent modulo simultaneous renaming of some conjugacy classes and irreducible characters).

The next step is to compute many tensor products of known virtual characters and to apply GAPs implementation of the LLL-algorithm to find class functions of small norm in the lattice of all available class functions. With this technique we find 322 of the 346 irreducible characters as well as 20 virtual characters of norm 2 or 3 and 4 virtual characters of norm 24.

There are some further standard tricks to produce more class functions from known ones, like computing symmetrizations or inducing class functions from subgroups (e.g., in our case there is a subgroup of type $3.F_{i22}$). But all of these do not improve the state as described above.

To find the remaining irreducible characters we can now take advantage of (c), the list of known character degrees. This tells us that we are missing irreducible characters of degrees 7194825, 1929727800, 4583103525 and 11972188800, each occurring 6 times. It turns out that for all of our virtual characters of norm 2 or 3 there is only one possibility to write their degree as sum or difference of the missing degrees. So, we know the degrees and the multiplicities ($\pm 1$) of their constituents. Now we consider the scalar products of our non-irreducible virtual characters with themselves and their complex conjugates. This shows that all their constituents are different from their complex conjugates, and that any two of the virtual characters of norm 2 and 3 and their complex conjugates have at most one common constituent. This yields a labelling of the missing characters and a decomposition of the 20 virtual characters of norm 2 and 3 in terms of this labelling. For the remaining four virtual characters of norm 24 there are several possible decompositions which are compatible with the computed scalar products. But the number of possibilities is small enough that we can try all of them and compute the potential set of all irreducible characters. Some of these possibilities could be ruled out easily, because some random tensor product has scalar products with irreducibles which are not non-negative integers.

In the end there are 12 possibilities left, and it turns out that they are all equivalent modulo table automorphism (that is, the resulting tables are the same modulo simultaneous permutations of conjugacy classes and irreducible characters).

### 2.1 Character tables of 6.$G$, (2$^2 \times 3$).$G$, 3.$G$.2, 6.$G$.2 and (2$^2 \times 3$).$G$.2

Having constructed the character table of 3.$G$ as described above and using the previously known character tables for 2.$G$, 2$^2$.3.$G$, 2.$G$.2, 2$^2$.2.$G$.2 it turns
out to be straightforward to compute the new tables mentioned in the header. This can be done with utility functions from the character table GAP-package CTblLib [Bre04].

Using the known tables of $G$, $2.G$ and $3.G$ as input we can now compute the table of $6.G$ with the function `CharacterTableOfCommonCentralExtension` (this function computes as many irreducibles of the common extension as it can find, in this case all irreducibles of $6.G$ are found).

In the next step we can construct the table of the group $(2^2 \times 3).G$ with the function `PossibleCharacterTablesOfTypeV4G`. The input for this function are the tables of $3.G$ and $6.G$ as well as the permutation induced by an automorphism of order 3 on the conjugacy classes of $3.G$. Using the function `AutomorphismsOfTable` on the table of $3.G$ it turns out that the result contains a unique subgroup of order 3 and this way we find the needed permutation. It takes several hours to find the possible tables for the larger extension, it turns out that there is a unique possibility which then must be the table we are looking for.

Finally, the tables for the groups of form $M.G.2$ can be computed with the function `PossibleCharacterTablesOfTypeMGA`. This function gets as input the tables of $M.G$, $G$ and $G.2$ and a list of orbits of the group $M.G.2$ acting on the classes of $M.G$. In all of our cases these orbits are easily found with the function `PossibleActionsForTypeMGA` which only returns a unique possibility. Applying `PossibleCharacterTablesOfTypeMGA` on this input yields a unique possible result and so this must be the correct table.

3 Determination of Brauer trees

As an application of the new character tables constructed in the previous section we want to determine all Brauer trees of the corresponding groups which encode for primes $l$ the $l$-modular decomposition numbers of blocks with non-trivial cyclic defect group.

For the following facts about Brauer trees and their computation from character tables we refer to the first three chapters of the book [HL89]. By results of Feit [Fei84] and Külshammer (unpublished, see [HL89, 1.3]) the shape of any Brauer tree of any finite group occurs as Brauer tree of a central extension of an automorphism group of a finite simple group. The latter are now known for most finite simple groups, the cases considered here being among the few missing ones.

We briefly recall some basic facts about Brauer trees. Let $H$ be a finite group and $l$ be a prime, and let $(K, R, k)$ be a splitting $l$-modular system for $H$. The set of irreducible characters $\text{Irr}(H)$ is partitioned into $l$-blocks, let $B = \{\chi_1, \ldots, \chi_k\}$ be such a block. We assume that the defect group of $B$ (a subgroup of some Sylow $l$-subgroup of $H$) is a non-trivial cyclic group of order $l^d$. Let $e$ be the number of irreducible $l$-modular Brauer characters of $B$. Then we have:

(a) $e \mid (l - 1)$. 

(b) There may be several characters in $B$ with the same restriction to $l$-regular classes. If this happens, the corresponding characters are called the exceptional characters of $B$ and there are $m = (l^d - 1)/e$ exceptional and $e$ non-exceptional characters in $B$ (so $k = e + m$). Otherwise, there are no exceptional characters and $k = e + 1$ non-exceptional characters.

(c) All projective indecomposable characters corresponding to $B$ are of the form $\chi + \chi'$ where $0 \neq \chi \neq \chi' \neq 0$ and $\chi$ and $\chi'$ are either the sum of all exceptional characters in $B$ or a non-exceptional character in $B$.

(d) The decomposition matrix of $B$ can be encoded in a graph. Its $e + 1$ vertices are labelled by the non-exceptional characters in $B$ and, if there are exceptional characters, the sum of all exceptional characters. There is an edge joining two vertices $\chi$ and $\chi'$ if and only if $\chi + \chi'$ is projective.

(e) The graph defined in (d) is a tree, that is it is connected and has $e$ edges. This is called the Brauer tree of $B$.

(f) Let $\alpha$ be an automorphism of $H$. We also denote by $\alpha$ the induced map on class functions of $H$, and for a character $\chi$ we write $\bar{\chi}$ for its complex conjugate. Assume that $\chi \mapsto \bar{\chi}^\alpha$ maps $B$ to $B$. Then this map induces a graph automorphism on the Brauer tree of $B$ and the subgraph of the invariant vertices ($\chi = \bar{\chi}^\alpha$) forms a line (in case $\alpha = 1$ this is called the real stem of the tree).

(g) Let $\alpha$ be an automorphism as in (f) which maps $B$ to another block $B'$. Then $\alpha$ induces a graph isomorphism from the Brauer tree of $B$ to the Brauer tree of $B'$.

Now we describe the strategy which allowed us to find the Brauer trees we are considering here. Let $H$ be a group such that we know its ordinary character table (the character values, the centralizer orders and the power maps) and let $l$ be a prime divisor of $H$ (otherwise all $l$-blocks contain a single irreducible character). We assume that we have the character table available in GAP [GAP06] such that we can use the GAP-functions for computing with character tables.

1. The $l$-blocks of the character table of $H$ and the orders of the corresponding defect groups can be computed with the GAP-function PrimeBlocks. (This uses that two irreducibles are in the same block if and only if their central characters modulo $l$ are equal, the order of the defect group can be found from the character degrees.) If the defect group is of order $l$ then it is clearly cyclic and non-trivial. In the cases we consider here it is easy to see that all other blocks do not have a non-trivial cyclic defect group (because it is trivial or the numerical conditions in facts (a), (b) are not fulfilled).

2. If $G$ is the finite simple group of type $^2E_6(2)$ and $H$ is one of the groups $G$, $2^2,G$ or $(2^2 \times 3).G$ then $H$ has an outer automorphism group of type $S_3$ [CCN+85]. In GAP we can compute the automorphisms of the character table of $H$ (a permutation of the classes, compatible with power maps, that leaves the table invariant). It turns out that this has a unique subgroup of
type $S_3$. Since the outer automorphism group acts faithfully on the conjugacy classes we find the explicit action of the outer automorphisms of $H$ on $\text{Irr}(H)$, as well as the induced action on the $l$-blocks of $H$.

The following steps are applied to one block $B$ of cyclic defect in each orbit under the outer automorphisms of $H$ (this is sufficient because of fact (g)).

(3) We start with listing for each vertex in the tree all possible vertices they may be connected to. For the initial list we take into account that projective characters are zero on $l$-singular classes. So, if one character has a non-zero value on some $l$-singular class it can only be connected to other characters which have the negative of that value on the same class. (If the tree has the shape of a star, that is, there is one vertex connected to all the others, then we have already found the tree in this step.)

(4) Whenever we find a new edge of the tree during the following steps we may be able to reduce the possibilities for further edges using the facts (e) (there cannot be a further edge between vertices in the same connected component with respect to the known edges) and (f) (an invariant vertex can only be connected to at most two other invariant vertices). If the number of known edges and possible further edges is $e$, we are done.

In the situation of fact (f) each edge between $\chi$ and $\chi'$ involving a non-invariant character implies a further edge between $\bar{\chi}^\alpha$ and $(\bar{\chi}')^\alpha$.

(5) We use that the defect-zero characters (the single characters in blocks with trivial defect group) are projective and that tensor products of projective characters with arbitrary characters are again projective. This way we can easily compute a huge number of projective characters and compute their scalar products with the irreducibles (and the sum of the exceptional characters) in our block $B$. For each projective this yields a list of multiplicities $m_i$ for each vertex $v_i$ in our tree ($m_i$ is the sum of the multiplicities of all projective indecomposable characters which correspond to an edge of the tree involving $v_i$). Let $\{v_j \mid j \in J\}$ the subset of vertices which are possibly connected to $v_i$, and let $j' \in J$ such that $m_{j'}$ is maximal among $\{m_j \mid j \in J\}$. If now $\sum_{j \in I, j \neq j'} m_j < m_i$ then we can conclude that $v_i$ must be connected to $v_{j'}$; we have found an edge of the tree. In particular we have found a new projective character corresponding to this edge, we use this for further iterations of this step. (Heuristically, we find a few (and sometimes all) edges very quickly in this step but nothing new later. So, we stop this procedure when we have not found new edges for a while.)

(6) In all our cases we find enough edges in (5) such that it is now feasible to enumerate all trees which are consistent with the edges already found. Many of the possible trees can be ruled out easily by computing the degrees of the irreducible Brauer characters which are implied by the tree; these must be positive integers but incorrect trees often yield other numbers.

(7) For each possible tree left in step (6) we now compute again some random projective characters (by tensoring known projectives with irreducibles) and check if the multiplicities with the characters in $B$ are consistent with the tree (the multiplicities of the projective indecomposables are recursively determined by the tree, starting from the leaves of the tree). This quickly rules out more trees.
(8) In very few cases we need to induce projective indecomposable characters from a subgroup of index 2 or 3 to find a tree. In one case this was also not enough, but the induced projective characters with only four irreducible constituents allowed us to reduce the possibilities which were initialized in step (5).

(9) It happens that the previous steps do not rule out all but one tree. In these cases we are always left with two or four possible trees and it turns out that the remaining trees are the same modulo permutation of algebraically conjugate characters in the block. In such a case any of the possible trees is correct for some choice of the modular system (or equivalently, for some choice of identification of certain conjugacy classes).

The cases to consider.

The simple group $^2E_6(2)$ has order $2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, the orders of the related groups we consider have additional factors 2 or 3. We handle the following cases where $G = ^2E_6(2)$:

- $(2^2 \times 3).G$ and $l = 5, 7, 11, 13, 17, 19$.
- $2^2.G$ and $l = 3$.
- $(2^2 \times 3).G.2$ and $l = 5, 7, 11, 13, 17, 19$.
- $2^2.G.2$ and $l = 3$.
- $2^2.G.3$ and $l = 5, 7, 11, 13, 17, 19$.
- $2^2.G.S_3$ and $l = 5, 7, 11, 13, 17, 19$.

Note that in a larger extension we also see the blocks of the smaller extensions which are quotient groups. The special cases for $l = 3$ are needed because the blocks with non-trivial cyclic defect have no longer cyclic defect in the larger extension. For $l = 2$ there are no blocks with non-trivial cyclic defect in any of these groups. In $2^2.G.3$, $2^2.G.S_3$ there are no blocks with non-trivial cyclic defect for $l = 3$.

3.1 The Brauer trees

Here is the list of all Brauer trees we obtain. We give the name of the character tables as they can be accessed in GAP via the CTblLib-package. The nodes of the trees are labeled by the position of the corresponding character in that table. A notation $(i + j)$ denotes a sum of the exceptional characters in a block.

All characters that do not appear here are either not in a block with cyclic defect (this happens for $p \in \{3, 5, 7\}$) or of defect zero.

Some trees are only determined up to algebraic conjugacy, that is any of the possible trees is correct with respect to some choice of $p$-modular system or to some choice of labeling for certain conjugacy classes. In such cases we mention the permutation group on the characters in the block induced by Galois automorphisms.
If the characters of a block are not faithful we indicate the order $|K|$ of their kernel.

### 3.1.1 GAP table $(2^2 \times 3).2E6(2), l = 5$

|   |   |   |   |   | $|K| = 12$   |
|---|---|---|---|---|----------------|
| 3 | 52 | 87 | 82 | 15 | $|K| = 12$   |
| 6 | 88 | 94 | 76 | 45 | $|K| = 12$   |
| 16| 112| 115| 67 | 33 | $|K| = 12$   |
| 39| 105| (103+104)| | | $|K| = 12$   |
| 130| 168| 198| 192| 149| $|K| = 6$   |
| 131| 147| 167| 169| 159| $|K| = 6$   |
| 134| 190| (188+189)| | | $|K| = 6$   |
| 139| 158| 163| 162| 150| $|K| = 6$   |
| 213| 251| 347| 349| 283| $|K| = 4$   |
| 214| 252| 348| 350| 284| $|K| = 4$   |
| 215| 257| 363| 365| 303| $|K| = 4$   |
| 216| 258| 364| 366| 304| $|K| = 4$   |
| 339| 391| (375+377)| | | $|K| = 4$   |
| 340| 392| (376+378)| | | $|K| = 4$   |
| 425| 449| 507| 511| 453| $|K| = 2$   |
| 426| 450| 508| 512| 454| $|K| = 2$   |
| 431| 505| 533| 503| 471| $|K| = 2$   |
| 432| 506| 534| 504| 472| $|K| = 2$   |
| 441| 527| 539| 493| 443| $|K| = 2$   |
| 442| 528| 540| 494| 444| $|K| = 2$   |
| 483| 509| (485+487)| | | $|K| = 2$   |
| 484| 510| (486+488)| | | $|K| = 2$   |
| 546| 584| 614| 608| 565| $|K| = 6$   |
| 547| 563| 583| 585| 575| $|K| = 6$   |
| 550| 606| (604+605)| | | $|K| = 6$   |
| 555| 574| 579| 578| 566| $|K| = 6$   |
| 621| 645| 703| 707| 649| $|K| = 2$   |
| 622| 646| 704| 708| 650| $|K| = 2$   |
| 627| 701| 729| 699| 667| $|K| = 2$   |
| 628| 702| 730| 700| 668| $|K| = 2$   |
| 637| 723| 735| 689| 639| $|K| = 2$   |
| 638| 724| 736| 690| 640| $|K| = 2$   |
| 679| 705| (681+683)| | | $|K| = 2$   |
| 680| 706| (682+684)| | | $|K| = 2$   |
| 742| 780| 810| 804| 761| $|K| = 6$   |
3.1.2 GAP table \((2^2\times3).2\mathbb{E}6(2), l = 7\)

| Number ||
|--------|--------|
| 6      | 40     | 88     | 116    | 112    | 64     | 16     |
| 9      | 46     | 79     | 89     | 71     | 60     | 27     |
| 10     | 47     | 80     | 90     | 72     | 61     | 28     |
| 11     | 48     | 81     | 91     | 73     | 62     | 29     |
| 42     | 87     | 113    | (103+104) | (|K| = 12) |
| 128    | 160    | 193    | 197    | 173    | 152    | 133    |
| 147    | 191    | 201    | (188+189) | (|K| = 6) |
| 203    | 223    | 265    | 339    | 349    | 319    | 251    |
| 204    | 224    | 266    | 340    | 350    | 320    | 252    |
| 213    | 247    | 283    | 391    | 397    | 323    | 227    |
| 214    | 248    | 284    | 392    | 398    | 324    | 228    |
| 217    | 401    | 353    | (|K| = 4) |
| 221    | 355    | (|K| = 4) |
| 220    | 354    | |
| 218    | 402    | 352    | (|K| = 4) |
| 222    | 356    | (|K| = 4) |
| 253    | 347    | 393    | (375+377) | (|K| = 4) |
| 254    | 348    | 394    | (376+378) | (|K| = 4) |
| 425    | 473    | (485+487) | 459    | (|K| = 2) |
| 426    | 474    | (486+488) | 460    | (|K| = 2) |
\begin{align*}
443 &- 455 - 461 - 489 - 527 - 531 - 491 \quad (|K| = 2) \\
444 &- 456 - 462 - 490 - 528 - 532 - 492 \quad (|K| = 2) \\
544 &- 576 - 609 - 613 - 589 - 568 - 549 \quad (|K| = 6) \\
563 &- 607 - 617 - (604+605) \quad (|K| = 6) \\
621 &- 669 - (681+683) - 655 \quad (|K| = 2) \\
622 &- 670 - (682+684) - 656 \quad (|K| = 2) \\
639 &- 651 - 657 - 685 - 723 - 727 - 687 \quad (|K| = 2) \\
640 &- 652 - 658 - 686 - 724 - 728 - 688 \quad (|K| = 2) \\
740 &- 772 - 805 - 809 - 785 - 764 - 745 \quad (|K| = 6) \\
759 &- 803 - 813 - (800+801) \quad (|K| = 6) \\
817 &- 865 - (877+879) - 851 \quad (|K| = 2) \\
818 &- 866 - (878+880) - 852 \quad (|K| = 2) \\
835 &- 847 - 853 - 881 - 919 - 923 - 883 \quad (|K| = 2) \\
836 &- 848 - 854 - 882 - 920 - 924 - 884 \quad (|K| = 2) \\
175 &\\
186 &\\
140 &- 179 - 141 \\
187 &\\
176 & \quad \text{modulo } \langle (175, 176), (186, 187) \rangle \quad (|K| = 6) \\
591 &\\
602 &\\
556 & - 595 - 557 \\
603 &\\
592 & \quad \text{modulo } \langle (591, 592), (602, 603) \rangle \quad (|K| = 6) \\
787 &\\
798 &\\
752 & - 791 - 753 \\
799 &\\
788 & \quad \text{modulo } \langle (787, 788), (798, 799) \rangle \quad (|K| = 6) \\
\end{align*}

3.1.3 \textbf{GAP table } $(2^2 \times 3).2E6(2), l = 11$

\begin{align*}
1 &- 14 - 49 - 106 - (107+108) - 56 \quad (|K| = 12)
\end{align*}
2 — 20 — 50 — 125 — (23+124) — 13 \quad (|K| = 12)
6 — 33 — 76 — 112 — (109+110) — 44 \quad (|K| = 12)
127 — 131 — 139 — 150 — 159 — (154+155) \quad (|K| = 6)
128 — 144 — 166 — 200 — 197 — (156+157) \quad (|K| = 6)
129 — 135 — 148 — 181 — 195 — (177+178) \quad (|K| = 6)
203 — 213 — 255 — 349 — 397 — (379+381) \quad (|K| = 4)
204 — 214 — 256 — 350 — 398 — (380+382) \quad (|K| = 4)
215 — 285 — 365 — (383+385) — 321 — 237 \quad (|K| = 4)
216 — 286 — 366 — (384+386) — 322 — 238 \quad (|K| = 4)
225 — 309 — 403 — (415+417) — 357 — 249 \quad (|K| = 4)
226 — 310 — 404 — (416+418) — 358 — 250 \quad (|K| = 4)
427 — 433 — 479 — 529 — (519+521) — 467 \quad (|K| = 2)
428 — 434 — 480 — 530 — (520+522) — 468 \quad (|K| = 2)
435 — 461 — 535 — 531 — (445+447) — 439 \quad (|K| = 2)
436 — 462 — 536 — 532 — (446+448) — 440 \quad (|K| = 2)
441 — 511 — (523+525) — 497 — 493 — 453 \quad (|K| = 2)
442 — 512 — (524+526) — 498 — 494 — 454 \quad (|K| = 2)
543 — 547 — 555 — 566 — 575 — (570+571) \quad (|K| = 6)
544 — 560 — 582 — 616 — 613 — (572+573) \quad (|K| = 6)
545 — 551 — 564 — 597 — 611 — (593+594) \quad (|K| = 6)
623 — 629 — 675 — 725 — (715+717) — 663 \quad (|K| = 2)
624 — 630 — 676 — 726 — (716+718) — 664 \quad (|K| = 2)
631 — 657 — 731 — 727 — (641+643) — 635 \quad (|K| = 2)
632 — 658 — 732 — 728 — (642+644) — 636 \quad (|K| = 2)
637 — 707 — (719+721) — 693 — 689 — 649 \quad (|K| = 2)
638 — 708 — (720+722) — 694 — 690 — 650 \quad (|K| = 2)
739 — 743 — 751 — 762 — 771 — (766+767) \quad (|K| = 6)
740 — 756 — 778 — 812 — 809 — (768+769) \quad (|K| = 6)
741 — 747 — 760 — 793 — 807 — (789+790) \quad (|K| = 6)
819 — 825 — 871 — 921 — (911+913) — 859 \quad (|K| = 2)
820 — 826 — 872 — 922 — (912+914) — 860 \quad (|K| = 2)
827 — 853 — 927 — 923 — (837+839) — 831 \quad (|K| = 2)
828 — 854 — 928 — 924 — (838+840) — 832 \quad (|K| = 2)
833 — 903 — (915+917) — 889 — 885 — 845 \quad (|K| = 2)
3.1.4 **GAP table** $(2^2 \times 3) \cdot 2E6(2), \ l = 13$

\[
\begin{array}{c}
| K | = 12
\end{array}
\]

\[
\begin{array}{cccccccccc}
1 & 8 & 38 & 70 & 125 & 126 & 106 & 68 & 31 & 7 & 2 \\
& & & & & & & & & & \\
& & & & & & & 25 & & & \\
| K | = 13
\end{array}
\]

\[
\begin{array}{cccccccccc}
207 & 277 & 369 & 421 & 261 & & & & & \\
& & & & & & & 259 & 373 & & \\
& & & & & & & & & & \\
| K | = 4
\end{array}
\]

\[
\begin{array}{cccccccccc}
208 & 278 & 370 & 422 & 262 & & & & & \\
& & & & & & & 260 & 374 & & \\
& & & & & & & & & & \\
| K | = 4
\end{array}
\]

\[
\begin{array}{cccccccccc}
425 & 427 & 441 & 485 & 495 & 509 & 527 & 501 & & & \\
& & & & & & & & & & \\
| K | = 6
\end{array}
\]

\[
\begin{array}{cccccccccc}
426 & 428 & 442 & 486 & 496 & 510 & 528 & 502 & & & \\
& & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
543 & 551 & 574 & 599 & 616 & 606 & 576 & 567 & 597 & 598 & 566 & 550 & 544 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
621 & 623 & 637 & 681 & 691 & 705 & 723 & 697 & 675 & 695 & 707 & 683 & 651 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
622 & 624 & 638 & 682 & 692 & 706 & 724 & 698 & 676 & 696 & 708 & 684 & 652 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
543 & 551 & 574 & 599 & 616 & 606 & 576 & 567 & 597 & 598 & 566 & 550 & 544 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 6
\end{array}
\]

\[
\begin{array}{cccccccccc}
425 & 427 & 441 & 485 & 495 & 509 & 527 & 501 & & & \\
& & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
426 & 428 & 442 & 486 & 496 & 510 & 528 & 502 & & & \\
& & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
543 & 551 & 574 & 599 & 616 & 606 & 576 & 567 & 597 & 598 & 566 & 550 & 544 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
621 & 623 & 637 & 681 & 691 & 705 & 723 & 697 & 675 & 695 & 707 & 683 & 651 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]

\[
\begin{array}{cccccccccc}
622 & 624 & 638 & 682 & 692 & 706 & 724 & 698 & 676 & 696 & 708 & 684 & 652 & & & \\
& & & & & & & & & & & & & & & & \\
| K | = 2
\end{array}
\]
\begin{align*}
739 &= 747 - 770 - 795 - 812 - 802 - 772 - 763 - 793 - 794 - 762 - 746 - 740 \\
740 &= \text{modulo } \langle (794, 795) \rangle \quad (|K| = 6) \\
817 &= 819 - 833 - 877 - 891 - 893 - 871 - 891 - 903 - 879 - 847 \\
818 &= \text{modulo } \langle (877, 879), (891, 893) \rangle \quad (|K| = 2) \\
817 &= 820 - 834 - 878 - 892 - 920 - 894 - 872 - 892 - 904 - 880 - 848 \\
818 &= \text{modulo } \langle (878, 880), (892, 894) \rangle \quad (|K| = 2)
\end{align*}

\begin{table}
\begin{center}
\begin{tabular}{cccc}
3.1.5 & GAP&table& $(2^2\times 3).2E6(2)$, $l = 17$ \\
1 & 7 & 20 & 63 \quad (121+122) - 125 - 68 - 49 - 5 \quad (|K| = 12) \\
127 & 133 & 150 & 200 - 202 - 197 \quad (182+183) - 163 - 146 \quad (|K| = 6) \\
203 & 239 & 319 & 367 \quad (411+413) - 397 - 307 - 247 - 205 \quad (|K| = 4) \\
204 & 240 & 320 & 368 \quad (412+414) - 398 - 308 - 248 - 206 \quad (|K| = 4) \\
543 & 549 & 566 & 616 - 618 - 613 \quad (598+599) - 579 - 562 \quad (|K| = 4) \\
739 & 745 & 762 & 812 - 814 - 809 \quad (794+795) - 775 - 758 \quad (|K| = 4) \\
423 & 445 & (499+501) - 531 - 497 - 443 - 441 - 447 - 437 \\
\text{modulo} \quad ((445,447)) \quad (|K| = 2) \\
424 & 446 & (500+502) - 532 - 498 - 444 - 442 - 448 - 438 \\
\text{modulo} \quad ((446,448)) \quad (|K| = 2) \\
619 & 641 & (695+697) - 727 - 693 - 639 - 637 - 643 - 633 \\
\text{modulo} \quad ((641,643)) \quad (|K| = 2) \\
620 & 642 & (696+698) - 728 - 694 - 640 - 638 - 644 - 634 \\
\text{modulo} \quad ((642,644)) \quad (|K| = 2) \\
815 & 837 & (891+893) - 923 - 889 - 835 - 833 - 839 - 829 \\
\text{modulo} \quad ((837,839)) \quad (|K| = 2) \\
816 & 838 & (892+894) - 924 - 890 - 836 - 834 - 840 - 830 \\
\text{modulo} \quad ((838,840)) \quad (|K| = 2) \\
\end{tabular}
\end{center}
\end{table}

\begin{table}
\begin{center}
\begin{tabular}{cccc}
3.1.6 & GAP&table& $(2^2\times 3).2E6(2)$, $l = 19$ \\
1 & 3 & 24 & 82 \quad (118+119) - 125 - 56 - 13 \quad (|K| = 12) \\
\text{modulo} \quad ((25,26)) \\
\end{tabular}
\end{center}
\end{table}
\[
217 \rightarrow 351 \rightarrow (405+407) \rightarrow 267
\]
\[
271 \rightarrow 355 \rightarrow 219 \quad (|K| = 4)
\]
\[
270 \rightarrow 354 \rightarrow 218 \rightarrow 352 \rightarrow (406+408) \rightarrow 268
\]
\[
220 \quad (|K| = 4)
\]
\[
128 \rightarrow 130 \rightarrow 145 \rightarrow 179 \rightarrow 182 \rightarrow 192 \rightarrow 200 \rightarrow (186+187)
\text{modulo } \langle (182,183) \rangle \quad (|K| = 6)
\]
\[
423 \rightarrow 425 \rightarrow 465 \rightarrow 499 \rightarrow (515+517) \rightarrow 495 \rightarrow 463 \rightarrow 473 \rightarrow 507 \rightarrow 501
\text{modulo } \langle (499,501) \rangle \quad (|K| = 2)
\]
\[
424 \rightarrow 426 \rightarrow 466 \rightarrow 500 \rightarrow (516+518) \rightarrow 496 \rightarrow 464 \rightarrow 474 \rightarrow 508 \rightarrow 502
\text{modulo } \langle (500,502) \rangle \quad (|K| = 2)
\]
\[
544 \rightarrow 546 \rightarrow 561 \rightarrow 595 \rightarrow 598 \rightarrow 589 \rightarrow 599 \rightarrow 608 \rightarrow 616 \rightarrow (602+603)
\text{modulo } \langle (598,599) \rangle \quad (|K| = 6)
\]
\[
619 \rightarrow 621 \rightarrow 661 \rightarrow 695 \rightarrow (711+713) \rightarrow 691 \rightarrow 659 \rightarrow 669 \rightarrow 703 \rightarrow 697
\text{modulo } \langle (695,697) \rangle \quad (|K| = 2)
\]
\[
620 \rightarrow 622 \rightarrow 662 \rightarrow 696 \rightarrow (712+714) \rightarrow 692 \rightarrow 660 \rightarrow 670 \rightarrow 704 \rightarrow 698
\text{modulo } \langle (696,698) \rangle \quad (|K| = 2)
\]
\[
740 \rightarrow 742 \rightarrow 757 \rightarrow 791 \rightarrow 794 \rightarrow 785 \rightarrow 795 \rightarrow 804 \rightarrow 812 \rightarrow (798+799)
\text{modulo } \langle (794,795) \rangle \quad (|K| = 6)
\]
\[
815 \rightarrow 817 \rightarrow 857 \rightarrow 891 \rightarrow (907+909) \rightarrow 887 \rightarrow 855 \rightarrow 865 \rightarrow 899 \rightarrow 893
\text{modulo } \langle (891,893) \rangle \quad (|K| = 2)
\]
\[
816 \rightarrow 818 \rightarrow 858 \rightarrow 892 \rightarrow (908+910) \rightarrow 888 \rightarrow 856 \rightarrow 866 \rightarrow 900 \rightarrow 894
\text{modulo } \langle (892,894) \rangle \quad (|K| = 2)
\]

3.1.7 \textbf{GAP table } 2^*2.2E6(2), \ l = 3

\[
69 \rightarrow 120 \rightarrow 116 \quad (|K| = 4)
107 \rightarrow 123 \rightarrow 109 \quad (|K| = 4)
108 \rightarrow 124 \rightarrow 110 \quad (|K| = 4)
\]

3.1.8 \textbf{GAP table } (2^*2x3).2E6(2).2, \ l = 5
3.1.9  GAP table $\left(2^2\times3\right).2E6(2).2$, $l = 7$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
|11 | 64 | 139 | 174 | 166 | 100 |
|12 | 65 | 140 | 175 | 167 | 101 |
|17 | 76 | 127 | 141 | 115 |  96 |
|18 | 77 | 128 | 142 | 114 |  95 |

$|K| = 2$
\[\begin{align*}
19 &- 78 - 129 - 143 - 116 - 97 - 46 \ (|K| = 2) \\
68 &- 137 - 168 - 157 - 169 - 138 - 69 \ (|K| = 2) \\
192 &- 248 - 302 - 310 - 273 - 237 - 202 \ (|K| = 2) \\
193 &- 247 - 301 - 309 - 274 - 238 - 203 \ (|K| = 2) \\
216 &- (279+280) - 293 - 277 \ (|K| = 2) \\
227 &- 297 - 317 - 294 - 318 - 298 - 228 \ (|K| = 2) \\
321 &- 331 - 352 - 389 - 394 - 379 - 345 \ (|K| = 2) \\
326 &- 343 - 361 - 415 - 418 - 381 - 333 \ (|K| = 2) \\
329 &- 396 \\
328 &- 420 - 395 \\
330 &- 397 \ (|K| = 2) \\
346 &- 393 - 416 - (407+408) \ (|K| = 2) \\
432 &- 456 - (462+463) - 449 \ (|K| = 2) \\
441 &- 447 - 450 - 464 - 483 - 485 - 465 \ (|K| = 2) \\
492 &- 524 - 557 - 561 - 537 - 516 - 497 \\
511 &- 555 - 565 - (552+553) \\
569 &- 617 - (629+631) - 603 \\
570 &- 618 - (630+632) - 604 \\
587 &- 599 - 605 - 633 - 671 - 675 - 635 \\
588 &- 600 - 606 - 634 - 672 - 676 - 636 \\
539 &\quad \\
550 &\quad \\
504 &- 543 - 505 \\
551 &\quad \\
540 &\quad \text{modulo } \langle(539, 540), (550, 551)\rangle
\end{align*}\]

**3.1.10 GAP table** 
\((2^{2\times3}) \cdot 2E6(2) \cdot 2, l = 11\)

\[\begin{align*}
90 &\quad \\
1 &- 24 - 79 - 160 - 162 - 161 - 80 - 25 - 2 \ (|K| = 2) \\
91 &\quad \\
22 &\quad \\
3 &- 33 - 81 - 186 - 185 - 187 - 82 - 34 - 4 \ (|K| = 2) \\
23 &\quad \\
\end{align*}\]
\[
\begin{array}{cccccccccccc}
11 & 53 & 121 & 166 & 163 & 167 & 122 & 54 & 12 & 72 & | & (|K| = 2) \\
190 & 198 & 215 & 234 & 246 & 241 & 245 & 233 & 214 & 199 & 191 & | (|K| = 2) \\
192 & 221 & 259 & 315 & 309 & 242 & 310 & 316 & 260 & 222 & 193 & | (|K| = 2) \\
194 & 206 & 229 & 284 & 306 & 278 & 305 & 283 & 230 & 207 & 195 & | (|K| = 2) \\
321 & 326 & 347 & 394 & 418 & | (|K| = 2) \\
327 & 362 & 402 & | (411+412) & 380 & 338 & | (|K| = 2) \\
332 & 374 & 421 & | (427+428) & 398 & 344 & | (|K| = 2) \\
433 & 436 & 459 & 484 & | (479+480) & 453 & | (|K| = 2) \\
437 & 450 & 487 & 485 & | (442+443) & 439 & | (|K| = 2) \\
440 & 475 & | (481+482) & 468 & 466 & 446 & | (|K| = 2) \\
491 & 495 & 503 & 514 & 523 & | (518+519) \\
492 & 508 & 530 & 564 & 561 & | (520+521) \\
493 & 499 & 512 & 545 & 559 & | (541+542) \\
571 & 577 & 623 & 673 & | (663+665) & 611 \\
572 & 578 & 624 & 674 & | (664+666) & 612 \\
579 & 605 & 679 & 675 & | (589+591) & 583 \\
580 & 606 & 680 & 676 & | (590+592) & 584 \\
585 & 655 & | (667+669) & 641 & 637 & 597 \\
586 & 656 & | (668+670) & 642 & 638 & 598 \\
\end{array}
\]

3.1.11 **GAP table** \((2^2 \times 3).2E6(2).2, l = 13\)

\[
\begin{array}{ccccccccc}
1 & 15 & 60 & 112 & 186 & 188 & 161 & 109 & 50 & 14 & 4 & | \ (|K| = 2) \\
2 & 16 & 61 & 113 & 187 & 189 & 160 & 108 & 49 & 13 & 3 & | \ (|K| = 2) \\
\end{array}
\]

18
3.1.12 GAP table \((2^2 \times 3).2E6(2).2, l = 17\)

\[
\begin{align*}
1 &- 13 - 33 - 98 - (182+184) - 187 - 109 - 80 - 9 \\
2 &- 14 - 34 - 99 - (181+183) - 186 - 108 - 79 - 10 \\
190 &- 202 - 233 - 315 - 319 - 310 - (286+288) - 253 - 225 \\
191 &- 203 - 234 - 316 - 320 - 309 - (285+287) - 254 - 226 \\
321 &- 339 - 379 - 403 - (425+426) - 418 - 373 - 343 - 322 \\
491 &- 497 - 514 - 564 - 566 - 561 - (546+547) - 527 - 510 \\
431 &- 442 - (469+470) - 485 - 468 - 441 - 440 - 443 - 438 \\
\end{align*}
\]
modulo \(\langle \langle 442,443 \rangle \rangle\) \(\langle |K| = 2 \rangle\)

\[
\begin{align*}
567 &- 589 - (643+645) - 675 - 641 - 587 - 585 - 591 - 581 \\
\end{align*}
\]
modulo \(\langle 589,591 \rangle\)

\[
\begin{align*}
568 &- 590 - (644+646) - 676 - 642 - 588 - 586 - 592 - 582 \\
\end{align*}
\]
modulo \(\langle 590,592 \rangle\)

3.1.13 GAP table \((2^2 \times 3).2E6(2).2, l = 19\)

19
\begin{align*}
&\begin{array}{c}
22 \\
40 \\
90 \\
42 \\
1 & 5 & 38 & 130 & 186 & 178 & 187 & 131 & 39 & 6 & 2 \\
91 \\
41 & 43 \\
23 \\
\end{array} \\
&(|K| = 2) \\
&\begin{array}{c}
328 & 395 & (422 + 423) & 353 \\
354 & 396 \\
328 & 395 & (422 + 423) & 353 \\
355 & 397 \\
329 & (|K| = 2) \\
192 & 196 & 223 & 279 & 286 & 274 & 287 & 299 & 315 & 293 & 316 & 300 \\
193 & 197 & 224 & 280 & 285 & 273 & 288 \\
\end{array} \\
&\text{modulo } \langle (279, 280), (285, 287) (286, 288) \rangle \quad (|K| = 2) \\
&\begin{array}{c}
431 & 432 & 452 & 469 & (477 + 478) & 467 & 451 & 456 & 473 & 470 \\
492 & 494 & 509 & 543 & 546 & 537 & 547 & 556 & 564 & (550 + 551) \\
567 & 569 & 609 & 643 & (659 + 661) & 639 & 607 & 617 & 651 & 645 \\
568 & 570 & 610 & 644 & (660 + 662) & 640 & 608 & 618 & 652 & 646 \\
\end{array} \\
&\text{modulo } \langle (469, 470) \rangle \quad (|K| = 2) \\
&\begin{array}{c}
567 & 569 & 609 & 643 & (659 + 661) & 639 & 607 & 617 & 651 & 645 \\
568 & 570 & 610 & 644 & (660 + 662) & 640 & 608 & 618 & 652 & 646 \\
\end{array} \\
&\text{modulo } \langle (643, 645) \rangle \\
&\text{modulo } \langle (644, 646) \rangle \\
\end{align*}

\textbf{3.1.14 GAP table } 2^2 \cdot 2^\text{E6}(2) \cdot 2, \ l = 3

\begin{align*}
&\begin{array}{c}
110 & 179 & 174 \quad (|K| = 2) \\
111 & 180 & 175 \quad (|K| = 2) \\
162 & 185 & 163 \quad (|K| = 2) \\
\end{array} \\
\end{align*}

\textbf{3.1.15 GAP table } 2^2 \cdot 2^\text{E6}(2) \cdot 3, \ l = 5

\begin{align*}
&\begin{array}{c}
7 & 105 & 168 & 162 & 35 \\
8 & 106 & 169 & 163 & 36 \\
9 & 107 & 170 & 164 & 37 \\
16 & 171 & 180 & 153 & 92 \\
17 & 172 & 181 & 154 & 93 \\
18 & 173 & 182 & 155 & 94 \\
\end{array} \\
\end{align*}
3.1.16 **GAP table** $2^*2.2\text{E}6(2).3$, $l = 7$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 16 | 77 | 171 | 232 | 219 | 124 | 38 |
| 17 | 78 | 172 | 233 | 220 | 125 | 39 |
| 18 | 79 | 173 | 234 | 221 | 126 | 40 |
| 31 | 114 | 214 | 231 | 189 | 143 | 70 |
| 83 | 168 | 222 | (190+193) |
| 84 | 169 | 223 | (191+194) |
| 85 | 170 | 224 | (192+195) |
| 268 | 300 | 333 | 337 | 313 | 292 | 273 |
| 287 | 331 | 341 | (328+329) |
| 315 |
| 326 |
| 280 | 319 | 281 |
| 327 |
| 316 modulo $\langle(315, 316), (326, 327)\rangle$ |

3.1.17 **GAP table** $2^*2.2\text{E}6(2).3$, $l = 11$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 1 | 32 | 95 | 199 | (202+205) | 108 |
| 2 | 33 | 96 | 200 | (203+206) | 109 |
| 3 | 34 | 97 | 201 | (204+207) | 110 |
| 4 | 41 | 98 | 259 | (253+256) | 28 |
| 5 | 42 | 99 | 260 | (254+257) | 29 |
| 6 | 43 | 100 | 261 | (255+258) | 30 |
3.1.18 GAP table $2^2.2E6(2).3$, $l = 13$

\[
\begin{array}{cccccccccc}
47 \\
1 & 22 & 71 & 144 & 259 & 263 & 199 & 137 & 58 & 19 & 4 \\
50 \\
48 \\
2 & 23 & 72 & 145 & 260 & 264 & 200 & 138 & 59 & 20 & 5 \\
51 \\
49 \\
3 & 24 & 73 & 146 & 261 & 265 & 201 & 139 & 60 & 21 & 6 \\
52 \\
267 & 275 & 298 & 323 & 340 & 330 & 300 & 291 & 321 & 322 & 290 & 274 & 268 \\
\end{array}
\]

modulo $\langle (322,323) \rangle$

3.1.19 GAP table $2^2.2E6(2).3$, $l = 17$

\[
\begin{array}{cccccccccc}
1 & 19 & 41 & 121 & (247+250) & 259 & 137 & 95 & 13 \\
2 & 20 & 42 & 122 & (248+251) & 260 & 138 & 96 & 14 \\
3 & 21 & 43 & 123 & (249+252) & 261 & 139 & 97 & 15 \\
267 & 273 & 290 & 340 & 342 & 337 & (322+323) & 303 & 286 \\
\end{array}
\]

3.1.20 GAP table $2^2.2E6(2).3$, $l = 19$

\[
\begin{array}{cccccccccc}
47 \\
1 & 7 & 44 & 162 & 259 & (238+241) & 108 & 28 \\
50 \\
\end{array}
\]
\begin{align*}
\begin{array}{cccccccc}
& & & & & 48 & & & \\
2 & 8 & 45 & 163 & 260 & (239+242) & 109 & 29 \\
\hline
& & & & & 51 & & & \\
& & & & & 49 & & & \\
3 & 9 & 46 & 164 & 261 & (240+243) & 110 & 30 \\
\hline
& & & & & 52 & & & \\
& & & & & 268 & -270 & -285 & -319 & -322 & -313 & -323 & -332 & -340 & - (326+327) \\
& & & & & & & & & & & & & \text{modulo } ((322,323)) \\
\end{array}
\end{align*}

3.1.21 GAP table $2^2.2E6(2).3.2$, $l = 5$

\begin{align*}
7 & 110 - 174 - 168 - 37 \\
8 & 111 - 173 - 167 - 36 \\
38 & 169 - 9 - 112 - 175 \\
16 & 159 - 97 - 176 - 186 \\
17 & 158 - 96 - 177 - 185 \\
18 & 160 - 98 - 178 - 187 \\
68 & 139 - 39 - 215 - 224 \\
40 & 138 - 67 - 216 - 225 \\
69 & 217 - 41 - 140 - 226 \\
78 & 196 - 79 - 197 - 193 \\
80 & 198 - (194+195) \\
266 & 333 - 302 - 370 - 382 \\
267 & 334 - 301 - 369 - 381 \\
268 & 298 - 316 - 336 - 332 \\
269 & 297 - 315 - 335 - 331 \\
274 & 365 - 275 - 366 - 364 \\
284 & 313 - 304 - 322 - 324 \\
285 & 314 - 303 - 321 - 323 \\
\end{align*}

3.1.22 GAP table $2^2.2E6(2).3.2$, $l = 7$

\begin{align*}
16 & 81 - 176 - 229 - 215 - 127 - 39 \\
17 & 82 - 177 - 230 - 216 - 128 - 40 \\
18 & 83 - 178 - 231 - 217 - 129 - 41 \\
31 & 119 - 208 - 227 - 192 - 148 - 74 \\
32 & 120 - 209 - 228 - 191 - 147 - 73 \\
87 & 173 - 218 - 193 - 219 - 174 - 88 \\
89 & 175 - 220 - (194+195) \\
\end{align*}
262 — 318 — 372 — 380 — 343 — 307 — 272
263 — 317 — 371 — 379 — 344 — 308 — 273
286 — (349+350) — 363 — 347
297 — 367 — 387 — 364 — 388 — 368 — 298

3.1.23  GAP table $2^2 \cdot 2E6(2) \cdot 3.2$, $l = 11$

\begin{verbatim}
113
1 — 33 — 99 — 199 — 202 — 200 — 100 — 34 — 2
114
3 — 35 — 101 — 201 — (203+204) — 115
28
4 — 42 — 102 — 250 — 247 — 251 — 103 — 43 — 5
29
6 — 44 — 104 — 252 — (248+249) — 30
93
16 — 67 — 158 — 215 — 205 — 216 — 159 — 68 — 17
94
18 — 69 — 160 — 217 — (206+207) — 95
260 — 268 — 285 — 304 — 316 — 311 — 315 — 303 — 284 — 269 — 261
262 — 291 — 329 — 385 — 379 — 312 — 380 — 386 — 330 — 292 — 263
264 — 276 — 290 — 354 — 376 — 348 — 375 — 353 — 300 — 277 — 265
\end{verbatim}

3.1.24  GAP table $2^2 \cdot 2E6(2) \cdot 3.2$, $l = 13$

\begin{verbatim}
49
1 — 22 — 75 — 149 — 250 — 255 — 200 — 142 — 62 — 20 — 5
52
48
2 — 23 — 76 — 150 — 251 — 256 — 199 — 141 — 61 — 19 — 4
51
50
3 — 24 — 77 — 151 — 252 — 257 — 201 — 143 — 63 — 21 — 6
53
\end{verbatim}

24
\[260 - 276 - 313 - 358 - 386 - 366 - 317 - 305 - 353 - 356 - 304 - 275 - 263\]
\[\text{modulo } ((356,358))\]
\[261 - 277 - 314 - 357 - 385 - 365 - 318 - 306 - 354 - 355 - 303 - 274 - 262\]
\[\text{modulo } ((355,357))\]

3.1.25 GAP table 2\(^2\).2E6(2).3.2, \(l = 17\)

\[
\begin{align*}
1 & - 19 - 42 - 124 - (242 + 245) - 251 - 142 - 100 - 13 \\
2 & - 20 - 43 - 125 - (241 + 244) - 250 - 141 - 99 - 14 \\
3 & - 21 - 44 - 126 - (243 + 246) - 252 - 143 - 101 - 15 \\
260 & - 272 - 303 - 385 - 389 - 380 - (356 + 358) - 323 - 295 \\
261 & - 273 - 304 - 386 - 390 - 379 - (355 + 357) - 324 - 296
\end{align*}
\]

3.1.26 GAP table 2\(^2\).2E6(2).3.2, \(l = 19\)

\[
\begin{align*}
&28 \\
&| \\
&48 - 113 - 51 \\
&| \\
1 & - 7 - 45 - 167 - 250 - 235 - 251 - 168 - 46 - 8 - 2 \\
&| \\
&49 - 114 - 52 \\
&| \\
&29 \\
&| \\
&50 \\
&| \\
3 & - 9 - 47 - 169 - 252 - (236 + 237) - 115 - 30 \\
&| \\
&53 \\
&| \\
262 & - 266 - 293 - 349 - 356 - 344 - 357 - 369 - 385 - 363 - 386 - 370 \\
&| \\
&263 - 267 - 294 - 350 - 355 - 343 - 358
\end{align*}
\]

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