Leibniz complexity of Nash functions on differentiations

G. Ishikawa, and T. Yamashita

Abstract

The derivatives of Nash functions are Nash functions which are derived algebraically from their minimal polynomial equations. In this paper we show that, for any non-Nash analytic function, it is impossible to derive its derivatives algebraically, i.e., by using linearity and Leibniz rule finite times. In fact we prove algebraically the impossibility of algebraic computations, by using Kähler differentials. Then the notion of Leibniz complexity of a Nash function is introduced in this paper as the minimal number of usages of Leibniz rules to compute the total differential algebraically. We provide general observations and upper estimates on Leibniz complexity of Nash functions using the binary expansions and the complexity of Nash functions introduced by Ramanakoraisina.

1 Introduction

Let $f = f(x_1, \ldots, x_n)$ be a $C^\infty$ function on an open subset $U \subset \mathbb{R}^n$. Then $f$ is called a Nash function on $U$ if $f$ is analytic-algebraic on $U$, i.e. if $f$ is analytic on $U$ and there exists a non-zero polynomial $P(x, y) \in \mathbb{R}[x, y], \quad x = (x_1, \ldots, x_n)$, such that $P(x, f(x)) = 0$ for any $x \in U$ ([11][15][2]). If $U$ is semi-algebraic, then, $f$ is a Nash function if and only if $f$ is analytic and the graph of $f$ in $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is a semi-algebraic set ([2]). For a further significant progress on global study of Nash functions, see [7].

An analytic function $f$ on $U$ is called transcendental if it is not a Nash function. Then in this paper we show that, for any transcendental function, it is impossible to algebraically derive its derivatives by using linearity and Leibniz rule (product rule) finite times, even by using any $C^\infty$ function. In fact an analytic function $f$ is a Nash function if and only if its derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are computable algebraically (Theorem [2.1]). For example, for the transcendental function $f(x) = e^x$, the formula

$$ \frac{d}{dx} e^x = e^x $$

is never proved algebraically but is proved only by a “transcendental” method. The statement above is formulated in terms of Kähler differential exactly.

We begin by the simple example of Nash function $f(x) = \sqrt{x^2 + 1}$ of one variable. Then $f^2 - (x^2 + 1) = 0$. By differentiating the both sides of the relation, we have $2f'f - 2x = 0$
where \( f' = \frac{df}{dx} \). Here we have used Leibniz rule three times to get \((f^2)' = 2f'f \), \((x^2)' = 2x \) and \( 1' = 0 \) by setting \( dx/dx = 1 \). Then we have \( f'(x) = \frac{f(x)}{f(x)^2} = \frac{x}{\sqrt{x^2 + 1}} \). If we suppose \( c' = 0 \) for a constant function \( c \), then the usage of Leibniz rule is counted to be twice.

In general, let \( f \) be a Nash function on \( U \subset \mathbb{R}^n \). Then there is a non-zero polynomial \( P(x,y) \in \mathbb{R}[x,y], x = (x_1, \ldots, x_n) \) such that \( P(x,f(x)) = 0 \) for any \( x \in U \). We pose the condition that \( \frac{\partial P}{\partial y}(x,f(x)) \) is not identically zero on \( U \). The condition is achieved by choosing \( P \) which has the minimal total degree or the minimal degree on \( y \), among polynomials \( P \) satisfying \( P(x,f(x)) = 0 \) on \( U \). Then, by using Leibniz rule in several times, we have

\[
\frac{\partial P}{\partial x_i}(x,f(x)) + \frac{\partial P}{\partial y}(x,f(x)) \frac{\partial f}{\partial x_i}(x) = 0, \quad (1 \leq i \leq n).
\]

Therefore we have

\[
\frac{\partial f}{\partial x_i}(x) = -\frac{\partial P}{\partial x_i}(x,f(x)) / \frac{\partial P}{\partial y}(x,f(x)), \quad (1 \leq i \leq n).
\]

By our assumption that \( f \) is a Nash function on the assumption on \( P, \frac{\partial P}{\partial y}(x,f(x)) \) is a Nash function, which is not identically zero.

The problem on differentiations reminds us the problem on integrations. Note that the partial derivatives of Nash functions are Nash functions, while the integrals of Nash functions need not be Nash functions. This fact was one of reason to introduce the class of elementary functions in classical calculus. For related results, say, Liouville’s theorem on integrals of elementary functions, etc., refer [14] for instance. There the theory of differential fields play significant role likewise in the present paper (Proof of Theorem 2.1).

Then the **Leibniz complexity** of \( f \) is defined as the minimal number of usages of Leibniz rules to compute the total differential \( df \) algebraically. Nash functions are characterized by the finiteness of Leibniz complexity ([2] Theorem 2.1). In general it is a difficult problem to determine the exact value of the Leibniz complexity for a given Nash function. In §3 we provide general observations and estimates on Leibniz complexity of Nash functions using the binary expansions (Proposition 3.8) and discuss their relations with known notions on complexity of Nash functions ([12] [6]).

In §4 we generalize Theorem 2.1 to Nash functions on an affine Nash manifold (Theorem 4.1), by using the global results on Nash functions ([5] [8] [7]). In the last section §5 as an appendix, we recall known results which are necessary in this paper.

## 2 Algebraic computability of differentials

Let \( \mathcal{E}(U) \) (resp. \( \mathcal{O}(U), \mathcal{N}(U) \)) denote the set of all \( C^\infty \) functions (resp. analytic functions, Nash functions) on an open subset \( U \subset \mathbb{R}^n \). Regarding \( \mathcal{E}(U) \) as an \( \mathbb{R} \)-algebra, we take the space \( \Omega_{\mathcal{E}(U)} \) of Kähler differentials of \( \mathcal{E}(U) \) and the universal derivation \( d: \mathcal{E}(U) \to \Omega_{\mathcal{E}(U)} \). In fact, for any \( \mathbb{R} \)-algebra \( A \), \( \Omega_A \) can be constructed as follows: First consider the free \( A \)-module \( \mathcal{F}_A \) generated by elements \( df \), for any \( f \in A \), regarding as just symbols. Second consider the sub-\( A \)-module \( \mathcal{R}_A \subset \mathcal{F}_A \) generated by all relations of algebraic derivations:

\[
d(h+k) - dh - dk, \quad d(\lambda \ell) - \lambda d\ell, \quad d(pq) - pdq - qdp,
\]
h, k, ℓ, p, q ∈ A, λ ∈ R. Third we set Ω_A = ˜A/RA and define d : A → Ω_A by mapping each f ∈ A to the class of df in ˜A/RA.

If B is any A-module and D : A → B is any derivation, i.e. D is an R-linear map satisfying D(gh) = gD(h) + hD(g) for any g, h ∈ A, then there exists a unique A-homomorphism ρ : Ω_A → B such that D = ρ ∘ d.

Suppose U is connected. Consider the set S ⊂ N(U) of non-zero Nash functions i.e. Nash functions which are not identically zero on U. Then S is closed under the multiplication. Let ˜E(U) = E(U)_S denote the localization of E(U) by S. We consider the space Ω_˜E(U) of Kähler differentials of the R-algebra ˜E(U).

Then we have:

**Theorem 2.1** Let U be a semi-algebraic connected open subset of R^n. Then the following conditions on an analytic function f ∈ O(U) are equivalent to each other:
1. f is a Nash function on U.
2. There exists a non-zero Nash function g ∈ N(U) such that
   \[ g \left( df - \sum_{i=1}^{n} \frac{∂f}{∂x_i} dx_i \right) = 0, \]
   in the space Ω_˜E(U) of Kähler differentials of ˜E(U).
3. \[ df = \sum_{i=1}^{n} \frac{∂f}{∂x_i} dx_i, \]
   in the space Ω_˜E(U) of Kähler differentials of ˜E(U).
4. There exist f_1, . . . , f_n ∈ ˜E(U) such that
   \[ df = \sum_{i=1}^{n} f_i dx_i, \]
   in the space Ω_˜E(U) of Kähler differentials of ˜E(U).

**Proof:** (1) ⇒ (2) : Let f ∈ E(U) be a Nash function and P(x, y) be a non-zero polynomial satisfying P(x, f) = 0 and \( \frac{∂P}{∂y}(x, f) \neq 0 \). Then, by taking Kähler differential on both sides of the polynomial equality P(x, f) = 0, we have in Ω_˜E(U),

\[ 0 = df(P(x, f)) = \sum_{i=1}^{n} \frac{∂P}{∂x_i}(x, f)dx_i + \frac{∂P}{∂y}(x, f)df \]

\[ = \sum_{i=1}^{n} \left( -\frac{∂P}{∂y}(x, f) \frac{∂f}{∂x_i} \right) dx_i + \frac{∂P}{∂y}(x, f)df \]

\[ = \frac{∂P}{∂y}(x, f) \left( df - \sum_{i=1}^{n} \frac{∂f}{∂x_i} dx_i \right), \]
and that $\frac{\partial P}{\partial y}(x, f)$ is a non-zero Nash function on $U$.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1): Suppose $f$ is not a Nash function on $U$ and $df - \sum_{i=1}^{n} f_i dx_i = 0$ in $\Omega_{\xi(U)}$. Since $f$ is not a Nash function, there exists a point $a \in U$ such that $f \in \mathcal{F}_{\mathbb{R}^n, a} \subset Q(\mathcal{F}_{\mathbb{R}^n, a})$ is not algebraic. Here $\mathcal{F}_{\mathbb{R}^n, a} = \mathcal{E}_{\mathbb{R}^n, a}/m_{\mathbb{R}^n, a}$ is the $\mathbb{R}$-algebra of formal series, $M = Q(\mathcal{F}_{\mathbb{R}^n, a})$ is its quotient field and the Taylor series of $f$ at $a$ is written also by the same symbol $f$. Moreover, we have $df - \sum_{i=1}^{n} f_i dx_i = 0$ in the Kähler differentials $\Omega_M$ of $M$, via the homomorphism $\tilde{\mathcal{E}}(U) \to M$ defined by taking the Taylor series. Then, in the free $M$-module $\mathcal{F}_M$ generated by elements $\{dh \mid h \in M\}$, $df - \sum_{i=1}^{n} f_i dx_i$ is a finite sum of elements of type

$$a(d(h + k) - dh - dk), b(d(\lambda \ell) - \lambda d\ell), c(d(pq) - pdq - qdp).$$

Here $a, h, k, b, \ell, c, p, q \in M, \lambda \in \mathbb{R}$. Now we take the subfield $L \subset M$ generated over the rational function field $K = \mathbb{R}(x)$ by $f, f_i(1 \leq i \leq n)$ and those $a, h, k, b, \ell, c, p, q$ which appear in the above expression of $df - \sum_{i=1}^{n} f_i dx_i$: $L = K(f, h_1, \ldots, h_m)$, which is a finitely generated field over $K$ by $f$ and for some $h_1, \ldots, h_m \in M$. Then we have $df - \sum_{i=1}^{n} f_i dx_i = 0$ also in $\Omega_L$.

Take any non-zero element $u \in L$ and fix it. Since $f$ is transcendental over $K$ in the usual sense, we can define a derivation $D_0 : K(f) \to L$ on the extension field $K(f)$ over $K$ by $f$, by $D_0(x_j) = 0, 1 \leq j \leq n, D_0(f) = u$. In fact we can set $D_0(f) \in L$ freely. Then we define a derivation $D_1 : K(f, h_1) \to L$, $D_1|_{K(f)} = D_0$ as follows: If $h_1$ is transcendental over $K(f)$, then we set $D_1(h_1) = 0$. If $h_1$ is algebraic over $K(f)$, then we set $D_1(h_1)$ as the element in $K(f, h_1)$ which is determined by the algebraic relation of $h_1$ over $K(f)$ and $D_0$. In fact, if $\sum_{k=0}^{m} a_k h_1^{m-k} = 0$, $a_k \in K(f)$, is a minimal algebraic relation of $h_1$ over $K(f)$, then we would have

$$\sum_{k=0}^{m} D_0(a_k) h_1^{m-k} + (\sum_{k=0}^{m-1} (m-k) a_k h_1^{m-k-1}) D_1(h_1) = 0.$$ 

Since $\sum_{k=0}^{m-1} (m-k) a_k h_1^{m-k-1} \neq 0$ by the minimality assumption, $D_1(h_1)$ is uniquely determined by

$$D_1(h_1) = -\left( \sum_{k=0}^{m} D_0(a_k) h_1^{m-k} \right) / \left( \sum_{k=0}^{m-1} (m-k) a_k h_1^{m-k-1} \right).$$

Here it is essential that we discuss derivations over a field. Thus we extend $D_1$ into a derivation $D = D_m : L \to L$ by a finitely number of steps. Note that we need not to use Zorn’s lemma to show the existence of extension of derivation. Then by the universality of the Kähler differentials, there exists an $L$-linear map $\rho : \Omega_L \to L$ such that $\rho \circ d = D : L \to L$. Here $d : L \to \Omega_L$ is the universal derivation. Then we have

$$0 = \rho \left( df - \sum_{i=1}^{n} f_i dx_i \right) = D(f) = u.$$ 

This leads to contradiction with the assumption $u \neq 0$. Thus we have that $f$ is a Nash function.

$\square$
Remark 2.2 If $U$ is not connected, then Theorem 2.1 does not hold. In fact, let $U = \mathbb{R} \setminus \{0\}$ and set $f(x) = e^x$ if $x > 0$ and $f(x) = 1$ if $x < 0$. Then $f \in \mathcal{O}(U)$ and $f \not\in \mathcal{N}(U)$. However the condition (2) is satisfied if we take as $g$ the non-zero Nash function on $U$ defined by $g(x) = 0(x > 0), g(x) = 1(x < 0)$.

3 Estimates on Leibniz complexity

Let $U \subset \mathbb{R}^n$ be a connected open subset. For a Nash function $f \in \mathcal{N}(U)$, we define the Leibniz complexity of $f$ by the minimal number of terms corresponding to Leibniz rule for $g(df - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i)$ in the free $\mathcal{E}(U)$-module $\mathfrak{R}_\mathcal{E}(U)$ among all expressions for all non-zero $g \in \mathcal{N}(U)$. The definition is based on the statement (2) of Theorem 2.1. We do not care about the number of terms corresponding to linearity of the differential. Moreover we will do not count the term generated by the relation $d(1 \cdot 1) - 1d(1) - 1d(1)$. Therefore we use the relation $d(c) = 0$ for $c \in \mathbb{R}$ freely.

Let $\text{LC}(f)$ denote the Leibniz complexity of $f$.

First we show general basic inequalities:

Lemma 3.1 For $f, g \in \mathcal{N}(U)$, we have

1. $\text{LC}(f + g) \leq \text{LC}(f) + \text{LC}(g)$,
2. $\text{LC}(fg) \leq \text{LC}(f) + \text{LC}(g) + 1$.

Proof: Let $hdf \in \mathfrak{R}_\mathcal{E}(U)$ (resp. $kdg \in \mathfrak{R}_\mathcal{E}(U)$) be expressed using the terms of Leibniz rule minimally i.e. $\text{LC}(f)$-times (resp. $\text{LC}(g)$-times), for a non-zero $h \in \mathcal{N}(U)$ (resp. a non-zero $k \in \mathcal{N}(U)$), except for the term $d(1 \cdot 1) - 1d(1) - 1d(1)$. Then $hkd(f + g) = k(hdf) + h(dg) \in \mathfrak{R}_\mathcal{E}(U)$ is expressed using Leibniz rule at most $\text{LC}(f) + \text{LC}(g)$ times. Therefore we have (1). Moreover, by using Leibniz rule once, we have

$$hkdf = hk(df + gd) = kg(df) + hf(dg)$$

in $\Omega_\mathcal{E}(U)$. Then, using Leibniz rule $\text{LC}(f) + \text{LC}(g)$ times, we compute $df$ and $dg$, and thus $dfg$. Therefore we have (2). \qed

By the definition of Leibniz complexity, we have the affine invariance:

Lemma 3.2 Let $f \in \mathcal{N}(U)$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isomorphism. Then $f \circ \varphi \in \mathcal{N}(\varphi^{-1}(U))$ satisfies $\text{LC}(f \circ \varphi) = \text{LC}(f)$.

Proof: By the definition of Leibniz complexity $h(df - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i)$ is zero in $\Omega_\mathcal{E}(U)$ by using Leibniz rule $\text{LC}(f)$-times, for a non-zero $h \in \mathcal{N}(U)$. Let $x' = (x'_1, \ldots, x'_n)$ be new affine coordinate system on $\mathbb{R}^n$ defined by $x' = \varphi^{-1}(x)$. Then $(h \circ \varphi)(df \circ \varphi) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \circ \varphi dx_i$ is zero in $\Omega_\mathcal{E}(U)$ by using Leibniz rule $\text{LC}(f)$-times. Since we do not count the usage of Leibniz rule for $d(c) = 0, c \in \mathbb{R}$, we have that $(h \circ \varphi)(df \circ \varphi) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \circ \varphi dx_i$ is zero in $\Omega_\mathcal{E}(U)$ by using Leibniz rule the same $\text{LC}(f)$-times. Note that $h \circ \varphi \in \mathcal{N}(U)$ is non-zero. Therefore we have $\text{LC}(f \circ \varphi) \leq \text{LC}(f)$. Similarly, we have $\text{LC}(f) = \text{LC}((f \circ \varphi) \circ \varphi^{-1}) \leq \text{LC}(f \circ \varphi)$. Thus we have the required equality. \qed
In general it is a difficult problem to determine the exact value of the Leibniz complexity even for a polynomial function.

**Example 3.3** Let \( n = 1 \). \( \text{LC}(x + c) = 0 \). \( \text{LC}(x^2 + bx + c) = 1 \). \( \text{LC}(\sqrt{x^2 + 1}) = 2 \). Let \( n = 2 \). For \( \lambda \in \mathbb{R} \), we have

\[
\text{LC}(x_1^2 + x_2^2 + \lambda x_1 x_2) = \begin{cases} 
1 & \text{if } |\lambda| \geq 2 \\
2 & \text{if } |\lambda| < 2.
\end{cases}
\]

In fact, \( x_1^2 + x_2^2 + \lambda x_1 x_2 = (x_1 + \frac{\lambda}{2} x_2)^2 + (1 - \frac{\lambda^2}{4}) x_2^2 \). Moreover \( x_1^2 + x_2^2 + \lambda x_1 x_2 = (x_1 + \alpha x_2)(x_1 + \beta x_2) \) for some \( \alpha, \beta \in \mathbb{R} \) if and only if \( |\lambda| \geq 2 \).

Let \( n = 2 \) and write \( \lambda = x_1 \). Then we have \( \text{LC}(x_1) = \text{LC}(x_2) = 0 \), \( \text{LC}(x_3) = 2 \), \( \text{LC}(x_4) = 2 \). For example we calculate \( d(x_4) = 2x_2 d(x_2) = 4x_3 d(x_2) \) by using Leibniz rule twice, and we can check that it is impossible to calculate \( d(x_4) \) by using Leibniz rule just once.

To observe the essence of the problem to estimate the Leibniz complexity, let us digress to consider “the problem of strips”. Let \( k \) be a positive integer. Suppose we have a sheet of paper having width \( k \) and, using a pair of scissors, we make \( k \)-strips of width 1. We may cut several sheets of the same width at once by piling them. Then the problem is to minimize the total number of cuts. Clearly it is at most \( k - 1 \). Now we show one strategy for the problem. Consider the binary expansion of \( k \):

\[
k = 2^{\mu_r} + 2^{\mu_{r-1}} + \cdots + 2^{\mu_1},
\]

for some integers \( \mu_r > \mu_{r-1} > \cdots > \mu_1 \geq 0 \). We set \( \mu = \mu_r \). Then the number of digits (‘1’ or ‘0’) is given by \( \mu + 1 \), while \( r \) is the number of units, ‘1’, appearing in the binary expansion. Then first we cut the sheet into \( r \) sheets of width \( 2^{\mu}, 2^{\mu_{r-1}}, \ldots, 2^{\mu_1} \) by \( (r - 1) \)-cuts. Second divide the sheet of width \( 2^{\mu} \) into sheets of width \( 2^{\mu_{r-1}} \) by \( \mu - \mu_{r-1} \)-cuts. Third divide the piled sheets of width \( 2^{\mu_{r-1}} \) into sheets of width \( 2^{\mu_{r-2}} \) by \( \mu_{r-1} - \mu_{r-2} \)-cuts, and so on. Iterating the process, we have sheets of width \( 2^{\mu_1} \), which we divide into strips of width 1 by \( \mu_1 \)-cuts finally. The total number of cuts by this method is given by \( \mu + r - 1 \).

**Lemma 3.4** For a positive integer \( k \), we have

\[
\text{LC}(x^k) \leq \mu + r - 1.
\]

**Proof**: In general, the process of cutting of a pile of sheets of width \( \ell \) into those of width \( \ell' \) and \( \ell'' = \ell - \ell' \) respectively is translated into, for a constant \( \lambda \),

\[
d(\lambda x^{\ell'}) = \lambda d(x^{\ell'}) = \lambda x^{\ell''} d(x^{\ell'}) + \lambda x^{\ell'} d(x^{\ell''}).
\]

Here we use Leibniz rule just once and the linearity of the derivation. A piling is realized by just the distributive law of the module structure. Therefore the method described above in the strip problem implies the estimate of Leibniz complexity. \( \square \)
Lemma 3.5 For \( f \in \mathcal{N}(U) \) and a natural number \( k \geq 1 \), we have \( \text{LC}(f^k) \leq \text{LC}(f) + \text{LC}(x^k) \).

Proof: If \( f \) is a constant function, then \( \text{LC}(f^k) = 0 \), so the inequality holds trivially. We suppose \( f \) is not a constant function. By definition, for some non-zero \( g \in \mathcal{N}(\mathbb{R}) \), \( gd(x^k) \) is deformed into \( g kx^{k-1}dx \) in \( \Omega_{\mathcal{E}(\mathbb{R})} \) using Leibniz rules \( \text{LC}(x^k) \)-times. Using the same procedure, \( (g \circ f)d(f^k) \) is deformed into \( (g \circ f)kf^{k-1}df \) in \( \Omega_{\mathcal{E}(U)} \) using Leibniz rules \( \text{LC}(x^k) \)-times. Note that \( g \circ f \) is non-zero in \( \mathcal{N}(U) \). Moreover, using Leibniz rules \( \text{LC}(f) \) times, \( h(g \circ f)kf^{k-1}df \) is deformed into \( h(g \circ f) \sum_{i=1}^n k f^{k-1}(\partial f/\partial x_i)dx_i \) for some non-zero \( h \in \mathcal{N}(U) \). Since \( g \circ f \) is non-zero, \( h(g \circ f) \) is non-zero. \( \square \)

Remark 3.6 The estimate in Lemma 3.4 is, by no means, best possible. For example, let \( k = 31 \). Then \( 31 = 2^4 + 2^1 + 2^2 + 2^1 + 2^0 \). Therefore \( r = 5 \) and \( \mu = 4 \). So the estimate gives us that \( \text{LC}(x^{31}) \leq 8 \). However \( \text{LC}(x^{31}) \leq 6 \). In fact, since \( 32 = 2^5 \), we have by Lemma 3.4

\[
x d(x^{31}) = d(x^{32}) - x^{31}d(x) = 32x^{31}d(x) - x^{31}d(x) = 31x^{31}d(x),
\]

by using Leibniz rule 6 times. Then algebraically we have \( d(x^{31}) = 31x^{30}d(x) \).

As above, we consider “the problem of strips” starting from several number of sheets, say, \( s \), having width \( k_s \), \( k_{s-1} \), and \( k_1 \) respectively. Then we have

**Lemma 3.7** Let \( p = p(x) = a_s x^{k_s} + a_{s-1} x^{k_{s-1}} + \cdots + a_1 x^{k_1} \in \mathbb{R}[x] \) be a polynomial function of one variable, where \( a_j \neq 0 \) \((1 \leq j \leq s)\) and \( k_s > k_{s-1} > \cdots > k_1 \geq 0 \). Regarding the binary expansion, let \( \mu \) be (the number of digits of \( k_s \)) - 1, and \( r_j \) the number of units of \( k_j \), \( 1 \leq j \leq s \). Then, by using Leibniz rule \( \mu + \sum_{j=1}^s (r_j - 1) \)-times and linearity, and by supposing \( d(c) = 0 \), \( c \in \mathbb{R} \), we have \( d(p) = (dp(x)/dx)d(x) \) in \( \Omega_{\mathcal{E}(U)} \). In particular we have

\[
\text{LC}(p) \leq \mu + \sum_{j=1}^s (r_j - 1).
\]

Proof: Let \( \mu = \mu_1 > \mu_{t-1} > \cdots > \mu_1 \geq 0 \) be all of the exponents appearing in the binary expansions of \( k_s, k_{s-1}, \ldots, k_1 \). First, by using Leibniz rule \( \sum_{j=1}^s (r_j - 1) \)-times, we modify \( d(p) \) into a linear combination of \( d(x^\ell), \ell = 2^\mu = 2^{\mu_1}, 2^{\mu_{t-1}}, \ldots, 2^{\mu_1} \). Second, by using Leibniz rule \( \mu - \mu_{t-1} \)-times, we modify \( d(x^\ell), \ell = 2^\mu \) into \( d(x^{\ell'}), \ell' = 2^{\mu_{t-1}} \). Repeating the procedure, we modify \( d(p) \) into a multiple of \( d(x^{\ell'}), \ell = 2^{\mu_1} \). Finally, by using Leibniz rule \( \mu_1 \)-times, we modify \( d(p) \) into a multiple of \( d(x) \). \( \square \)

We estimate the Leibniz complexity for a polynomial of \( n \)-variables. Let \( p(x) = p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \). We set \( p(x) = \sum b_\alpha x^\alpha, b_\alpha \in \mathbb{R} \), by using multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of non-negative integers. It is trivial that \( \text{LC}(p) \) is at most the total number of multiplications of variables:

\[
\sum_{b_\alpha \neq 0} \max\{|\alpha| - 1, 0\}.
\]
Instead we consider the number
\[ \sigma(p) := \sum_{b_\alpha \neq 0} \max\{ \#\{ i \mid 1 \leq i \leq n, \alpha_i > 0 \} - 1, 0 \}, \]
which is needed just to separate the variables on differentiation, and we try to save the additional usage of Leibniz rule.

Suppose that, by arranging terms with respect to \(x_i\) for each \(1 \leq i \leq n\),
\[ p(x) = a_{i,s(i)}x_i^{k_{i,s(i)}} + a_{i,s(i)-1}x_i^{k_{i,s(i)-1}} + \cdots + a_{i,1}x_i^{k_{i,1}}, \]
where \(a_{i,j}\) is a non-zero polynomial of \(x_1, \ldots, x_n\) without \(x_i\), \(1 \leq j \leq s(i)\), and \(k_{i,s(i)} > k_{i,s(i)-1} > \cdots > k_{i,1} \geq 0\). The maximal exponent \(k_{i,s(i)}\) is written as \(\deg x_i\ p\), the degree of \(p\) in the variable \(x_i\). For the binary expansion of \(\deg x_i\ p\), let \(\mu_i\) denote (the number of digits of \(\deg x_i\ p\)) \(-1\). Moreover let \(r_{ij}\), \(1 \leq j \leq s(i)\) denote the number of units of the exponent \(k_{ij}\) for the binary expansion. Then we have

**Lemma 3.8** By using the linearly, \(d(c) = 0, c \in \mathbb{R}\), and Leibniz rule
\[ \sigma(p) + \sum_{i=1}^{n} \left( \mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right) \text{-times}, \]
we have \(d(p) = \sum_{i=1}^{n} (\partial p(x)/\partial x_i) d(x_i)\) in \(\Omega_{\mathcal{E}(U)}\).

In particular we have the estimate
\[ \text{LC}(p) \leq \sigma(p) + \sum_{i=1}^{n} \left( \mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right). \]

**Remark 3.9** We have, for any polynomial \(p(x) = \sum b_\alpha x^\alpha\),
\[ \sigma(p) + \sum_{i=1}^{n} \left( \mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right) \leq \sum_{b_\alpha \neq 0} \max\{ ||\alpha|| - 1, 0 \}. \]
and in almost cases the inequality is strict.

**Proof of Lemma 3.8**
By applying Leibniz rule to each term of \(p\), \(d(p)\) is deformed into a sum of forms \(a_{i,j}d(x_i^{k_{i,j}})\) with the differential of one variable \(x_i\) and a function \(a_{i,j}\) of other variables. For this process we need to use Leibniz rule \(\sigma(p)\)-times. Then \(d(p)\) is the sum of the form
\[ a_{i,s(i)}d(x_i^{k_{i,s(i)}}) + a_{i,s(i)-1}d(x_i^{k_{i,s(i)-1}}) + \cdots + a_{i,1}d(x_i^{k_{i,1}}), \]
\((i = 1, \ldots, n)\). By Lemma 3.7 for each \(i = 1, \ldots, n\), the form is deformed into \(\partial p/\partial x_i\) by using Leibniz rule \(\mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1)\). Thus we have the estimate. \(\square\)

Now we give an upper estimate of Leibniz complexities for Nash functions by those for polynomial functions in terms of its polynomial relation. Let \(f \in \mathcal{N}(U)\) be a Nash function on a connected open subset \(U\) of \(\mathbb{R}^n\). Let \(P(x, y) = P(x_1, \ldots, x_n, y)\) be a polynomial such
that $P(x, f(x)) = 0$ on $U$ and $\frac{\partial P}{\partial y}(x, f(x))$ is not identically zero. We set $x_0 = y$. Suppose that, by arranging with respect to $x_i$ for each $i, 0 \leq i \leq n$,

$$P(x, y) = a_{i, s(i)} x_i^{k_i, s(i)} + a_{i, s(i) - 1} x_i^{k_i, s(i) - 1} + \cdots + a_{i, 1} x_i^{k_i, 1},$$

where $a_{i,j}$ is a non-zero polynomial of $x_0, x_1, \ldots, x_n$ without $x_i$, $(1 \leq j \leq s(i))$, and $k_i, s(i) > k_i, s(i) - 1 > \cdots > k_i, 1 \geq 0$. For the binary expansion, let $\mu_i$ (resp. $r_{ij}, 1 \leq j \leq s(i)$) be the number of digits of $\deg x_i P$ (resp. the number of units of $k_{ij}$), $0 \leq i \leq n$, respectively. Write $\deg x_i P$ the degree of $P$ with respect to $x_i, 0 \leq i \leq n$ and use the same notation $\sigma(P)$ as in Lemma 3.8 for the polynomial $P$ of $n+1$ variables.

**Proposition 3.10** Under the above notations, we have the estimate

$$\text{LC}(f) \leq \sigma(P) + \sum_{i=0}^{n} \left( \mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right).$$

In particular we have

$$\text{LC}(f) \leq \sigma(P) + \sum_{i=0}^{n} \{(\deg x_i P + 2)(\log_2(\deg x_i P) - 1)\} + n + 1.$$

**Example 3.11** Let $n = 1, f = \frac{1}{\sqrt{x^2 + 1}}$ and $P(x, y) = y^2 - x^2 - 1$. Then $\sigma(P) = 0, \mu_0 = \mu_1 = 1$ and $r_{ij} = 1$. Therefore the first inequality gives us that $\text{LC}(f) \leq 2$ as is seen in Introduction.

**Proof of Proposition 3.10**

We write the right hand side by $\psi$ of the first inequality. By Lemma 3.8 we have, by using Leibniz rule $\psi$-times,

$$d(P(x, y)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x, y) dx_i + \frac{\partial P}{\partial y}(x, y) dy,$$

modulo several linearity relations and $dc, c \in \mathbb{R}$ in $\Omega_{E(U \times \mathbb{R})}$. Then, substituting $y$ by $f$, we have that

$$0 = d(P(x, f)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(x, f) dx_i + \frac{\partial P}{\partial y}(x, f) df,$$

in $\Omega_{E(U)}$, therefore that

$$\frac{\partial P}{\partial y}(x, f) \left( df - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right) = 0,$$

in $\Omega_{E(U)}$, by using Leibniz rule at most $\psi$-times. Thus we have the first inequality. The second equality is obtained from the first equality combined with the inequalities derived by the definitions:

$$2^{\mu_i} \leq \deg x_i P < 2^{\mu_i + 1}, s(i) \leq \deg x_i P + 1,$$

and $r_{ij} \leq \mu_i$. 

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(1 \leq j \leq s(i), 0 \leq i \leq n). \square

In [12], the complexity \( C(f) \) of a Nash function \( f \) is defined as the minimum the total degree \( \deg P \) of non-zero polynomials \( P(x, y) \) with \( P(x, f) = 0 \). Moreover we define

\[
S(f) := \min\{\sigma(P \circ \psi) \mid P(x, f) = 0, \deg P = C(f), \ \psi \text{ is an affine isomorphism on } \mathbb{R}^n+1\},
\]

i.e. the minimum of the number \( \sigma \) for any defining polynomial \( P \) of \( f \) with minimal total degree under any choice of affine coordinates. We can regard \( S(f) \) a complexity for the separation of variables in differentiation of \( f \). Then we have the following result:

**Corollary 3.12** Let \( f \in \mathcal{N}(U) \) be a Nash function on a connected open set \( U \subset \mathbb{R}^n \). Then we have an estimate on the Leibniz complexity \( \text{LC}(f) \) by the Ramanakoraisina’s complexity \( C(f) \) and another complexity \( S(f) \),

\[
\text{LC}(f) \leq S(f) + (n + 1)(C(f) + 2)(\log_2 C(f) - 1) + n + 1.
\]

**Proof:** Since \( \deg_x P \leq C(f) \) (0 \leq i \leq n) we have the above estimate by Proposition 3.10 and Lemma 3.2. \square

Naturally we would like to pose a problem to obtain any lower estimate of Leibniz complexity.

### 4 Algebraic differentiation on Nash manifolds

Let \( U \) be a connected semi-algebraic open subset of \( \mathbb{R}^n \) and \( M \subset U \) a Nash submanifold ([2][15]). Suppose \( M \) is a closed connected subset in \( U \). We consider the quotient \( \mathbb{R} \)-algebra \( \mathcal{N}(U)/I \) by the ideal \( I \) of \( \mathcal{N}(U) \) consisting of Nash functions on \( U \) which vanish on \( M \). Note that \( IO(U) \) is a prime ideal in \( O(U) \) and, locally formally prime, i.e. for each \( a \in U \), the ideal \( I_a \) in the formal algebra \( \mathcal{F}_{\mathbb{R}^n,a} \) generated by \( \{j^\infty h(a) \mid h \in I \} \) is prime.

Since \( \mathcal{N}(U) \) is Noetherian ([15][16]), \( I \) is generated by a finite number of Nash functions \( g_1, \ldots, g_\ell \in \mathcal{N}(U) \) over \( \mathcal{N}(U) \).

An element \([f] \in O(U)/IO(U)\) is called Nash if there exists a polynomial \( P(x, y) = a_m(x)y^m + a_{m-1}(x)y^{m-1} + \cdots + a_1(x)y + a_0(x) \in \mathbb{R}[x, y] \) satisfying that at least one of \( a_m([x]), a_{m-1}([x]), \ldots, a_1([x]), a_0([x]) \) is not zero in \( \mathcal{N}(U)/I \) and that \( P([x], [f]) = 0 \) in \( O(U)/IO(U) \). The condition is equivalent to that \([f] \) is algebraic over \( \mathbb{R}(x) \) via the composition \( \mathbb{R}(x) \hookrightarrow O(U) \rightarrow O(U)/IO(U) \) of natural homomorphisms. Also the condition is equivalent to that \([f] \) is algebraic over \( \mathcal{N}(U)/I \) via the natural homomorphism \( \mathcal{N}(U)/I \rightarrow O(U)/IO(U) \). Then there exist a non-zero polynomial \( P(x, y) \) and \( h_j \in O(U), 1 \leq j \leq \ell \) such that

\[
P(x, f(x)) = \sum_{j=1}^\ell h_j(x)g_j(x),
\]
for any $x \in U$ and that $\frac{\partial P}{\partial y}(x, f) \not\in IO(U)$. By differentiating both sides of the relation by $x_i$, we have that

$$\frac{\partial P}{\partial x_i}(x, f(x)) + \frac{\partial P}{\partial y}(x, f(x)) \frac{\partial f}{\partial x_i} = \sum_{j=1}^\ell g_j(x) \frac{\partial h_j}{\partial x_i}(x) + \sum_{j=1}^\ell h_j(x) \frac{\partial g_j}{\partial x_i}(x),$$

so that

$$\frac{\partial P}{\partial y}([x, [f]]) \left[ \frac{\partial f}{\partial x_i} \right] = -\frac{\partial P}{\partial x_i}([x, [f]]),$$

in $\mathcal{E}(U)/(I + \langle \partial g_1/\partial x_i, \ldots, \partial g_\ell/\partial x_i \rangle_{\mathcal{E}(U)})$, for $1 \leq i \leq n$. Note that $\frac{\partial P}{\partial y}([x, [f]])$ is non-null in $O(U)/IO(U)$ and algebraic over $N(U)/I$.

We consider the space $\Omega_{\mathcal{E}(U)/I\mathcal{E}(U)}$ of Kähler differentials of $\mathcal{E}(U)/I\mathcal{E}(U)$. Note that

$$\Omega_{\mathcal{E}(U)/I\mathcal{E}(U)} \cong \Omega_{\mathcal{E}(U)}/(\mathcal{E}(U) dI + I\Omega_{\mathcal{E}(U)}),$$

as an $\mathcal{E}(U)/I\mathcal{E}(U)$-module. For the set $S$ of non-zero Nash elements in $O(U)/IO(U)$, $(\mathcal{E}(U)/I\mathcal{E}(U))_S$ denote the localization of $\mathcal{E}(U)/I\mathcal{E}(U)$ by $S$. Then we have:

**Theorem 4.1** Let $U$ be a connected semi-algebraic open subset of $\mathbb{R}^n$ and $I$ a ideal in $N(U)$. Suppose that $IO(U)$ is prime in $O(U)$ and moreover locally formally prime, i.e. for each $a \in U$, the ideal $I_a$ is in the formal algebra $\mathcal{F}_{\mathbb{R}^n,a}$ generated by $\{j^\infty h(a) \mid h \in I\}$ is prime. (For example, $I$ is the ideal of Nash functions vanishing on a connected closed Nash submanifold $M \subset U$.) Then the following conditions on $[f] \in O(U)/IO(U)$ are equivalent to each other:

1. $[f]$ is Nash.
2. There exists a non-zero Nash element $[g] \in O(U)/IO(U)$ such that
   $$[g] \left( df - \sum_{i=1}^n \left[ \frac{\partial f}{\partial x_i} \right] dx_i \right) = 0,$$
   in the space $\Omega_{\mathcal{E}(U)/I\mathcal{E}(U)}$ of Kähler differentials of $\mathcal{E}(U)/I\mathcal{E}(U)$.
3. $$df = \sum_{i=1}^n \left[ \frac{\partial f}{\partial x_i} \right] dx_i,$$
   in the space $\Omega_{(\mathcal{E}(U)/I\mathcal{E}(U))_S}$ of Kähler differentials of $(\mathcal{E}(U)/I\mathcal{E}(U))_S$.
4. There exist $\alpha_1, \ldots, \alpha_n \in (\mathcal{E}(U)/I\mathcal{E}(U))_S$ such that
   $$df = \sum_{i=1}^n \alpha_i dx_i,$$
   in the space $\Omega_{(\mathcal{E}(U)/I\mathcal{E}(U))_S}$. 
Proof: (1) ⇒ (2): Suppose (1). Then we have

\[ 0 = \mathbf{d}(P([x], [f])) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}([x], [f]) \mathbf{d}[x_i] + \frac{\partial P}{\partial y}([x], [f]) \mathbf{d}[f] \]

\[ = \sum_{i=1}^{n} \left( -\frac{\partial P}{\partial y}([x], [f]) \left[ \frac{\partial f}{\partial x_i} \right] \right) \mathbf{d}[x_i] + \frac{\partial P}{\partial y}([x], [f]) \mathbf{d}[f] \]

\[ = \frac{\partial P}{\partial y}([x], [f]) \left( \mathbf{d}[f] - \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}([x], [f]) \mathbf{d}[x_i] \right), \]

in \( \Omega_{\mathcal{E}(U)/I\mathcal{E}(U)} \), and \( \frac{\partial P}{\partial y}([x], [f]) \in \mathcal{O}(U)/I\mathcal{O}(U) \) is non-zero and algebraic over \( \mathcal{N}(U)/I \).

The implications (2) ⇒ (3) ⇒ (4) are clear.

(4) ⇒ (1): Suppose (4) and \([f]\) is not Nash. Then, by Lemma 5.2 there exists a point \( a \in U \) such that \([f]\) is transcendental in \( \mathcal{F}_{R^{n}, a}/I_a \) via the \( R \)-algebra homomorphism \( \varphi: \mathcal{N}(U)/I \to \mathcal{F}_{R^{n}, a}/I_a \), where \( I_a \) is the ideal in the formal power series ring \( \mathcal{F}_{R^{n}, a} \) generated by \( g_1, \ldots, g_{\ell} \). Let \( K = Q(\varphi(\mathcal{N}(U)/I)) \) be the quotient field of the image of \( \mathcal{N}(U)/I \) by \( \varphi \).

Moreover, let \( L = K([f], [h_1], \ldots, [h_m]) \) be the extended field of \( K \) which is generated by all elements which appear in the relation \( \mathbf{d}[f] - \sum_{i=1}^{n} \alpha_i \mathbf{d}[x_i] = 0 \) in \( \Omega_{\mathcal{F}_{R^{n}, a}/I_a} \). Then the relation holds also in \( \Omega_L \).

Let \( u \) be any non-zero element of \( L \). We extend the zero derivation \( 0: K \to L \) to \( D_0: K([f]) \to L \) by setting \( D_0([f]) = u \), for the given non-zero element \( u \in L \). Moreover we extend \( D_0 \) to a derivation \( D: L \to L \). Then for an \( L \)-homomorphism \( \rho: \Omega_L \to L \) we have \( D = \rho \circ \mathbf{d}: L \to L \). Then we have

\[ 0 = \rho \left( \mathbf{d}[f] - \sum_{i=1}^{n} \alpha_i \mathbf{d}[x_i] \right) = D([f]) = u. \]

This leads a contradiction. Thus we have (1).

For a Nash element \([f] \in \mathcal{O}(U)/I\mathcal{O}(U)\), we define the Leibniz complexity of \([f]\) by the minimal number of terms corresponding to Leibniz rule for \([g]\) \( \left( \mathbf{d}[f] - \sum_{i=1}^{n} \left[ \frac{\partial g}{\partial x_i} \right] \mathbf{d}[x_i] \right) \)

in the free \( \mathcal{E}(U)/I\mathcal{E}(U) \)-module \( \mathfrak{F}_{\mathcal{E}(U)/I\mathcal{E}(U)} \) among all expressions for all non-zero Nash element \([g] \in \mathcal{O}(U)/I\mathcal{O}(U)\). The definition is based on the statement (2) of Theorem 4.1.

We do not care about the number of terms corresponding to linearity of the differential. Moreover we will not count the term generated by the relation \( \mathbf{d}([1 \cdot 1]) - [1]\mathbf{d}([1]) - [1]\mathbf{d}([1]) \). Therefore we use the relation \( \mathbf{d}([c]) = 0 \) for \( c \in R \) freely.

Let \( \text{LC}([f]) \) denote the Leibniz complexity of \([f]\). Similarly to Proposition 3.10 we have:

**Proposition 4.2** Under the situation of Theorem 4.1, let \( P(x, y) \) be a polynomial such that \( P(x, f) \in I\mathcal{O}(U) \) and \( \frac{\partial P}{\partial y}(x, f) \notin I\mathcal{O}(U) \). Then we have

\[ \text{LC}([f]) \leq \sigma(P) + \sum_{i=0}^{n} \left( \mu_i + \sum_{j=1}^{s(i)} (r_{ij} - 1) \right). \]
5 Appendix

We recall known basic results used in this paper.

Lemma 5.1 Let $U \subset \mathbb{R}^n$ be a semi-algebraic open subset and $f \in \mathcal{O}(U)$ be an analytic function on $U$. Then the following conditions are equivalent to each other:
(1) $f$ is a Nash function on $U$, i.e. there exists a non-zero polynomial $P(x, y)$ such that $P(x, f(x)) = 0$ for any $x \in U$.
(2) The graph of $f$ in $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is a semi-algebraic set.
(3) For any $a \in U$, the Taylor series $j^\infty f(a)$ of $f$ at $a$ is algebraic in the formal power series algebra $\mathcal{F}_{\mathbb{R}^n, a} = \mathbb{R}[[x-a]]$ over the polynomial algebra $\mathbb{R}[x-a] = \mathbb{R}[x]$, in other word, there exists a non-zero polynomial $P(x, y)$ such that $j^\infty P(x, f(a)) = 0$.
(4) For any connected component $U'$ of $U$, there exists a point $a \in U'$ such that the Taylor series $j^\infty f(a)$ of $f$ at $a$ is algebraic in the formal power series algebra $\mathcal{F}_{\mathbb{R}^n, a} = \mathbb{R}[[x-a]]$ over the polynomial algebra $\mathbb{R}[x-a]$.

Proof: The the equivalences (1) and (2) are well-known (see for instant [2]). The implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. To show the implication (4) $\Rightarrow$ (1), suppose (4). Note that the number of connected components of $U$ is finite. Let $U_1, \ldots, U_r$ are all connected components of $U$. Let $1 \leq i \leq r$. Then there exists $a_i \in U_i$ such that $f$ is expressed by the Taylor series at $a_i$ in a neighborhood $W \subset U_i$ of $a_i$ and there exists a non-zero polynomial $P_i(x, y)$ such that $P_i(x, f(x)) = 0$ for any $x \in W$. Since the function $P_i(x, f(x))$ is analytic on $U_i$ and $U_i$ is connected, $P_i(x, f(x)) = 0$ for any $x \in U_i$. Then it suffices to take $P = \prod_{i=1}^r P_i$ to get (1).}

The following characterization is proved using the extension theorem due to Efroymson or its generalization [3]:

Lemma 5.2 Let $U \subset \mathbb{R}^n$ be a connected semi-algebraic open subset and $I \subset \mathcal{N}(U)$ be an ideal. For any $f \in \mathcal{O}(U)$ the following conditions are equivalent to each other:
(1) $[f] \in \mathcal{O}(U)/I\mathcal{O}(U)$ is Nash (See §4).
(2) For any $a \in U$, the Taylor series $j^\infty f(a)$ of $f$ at $a$ is algebraic in $\mathcal{F}_{\mathbb{R}^n, a}/I_a$, in other word, there exists a polynomial $P(x, y) \in \mathbb{R}[x, y], \deg_y P > 0$, which possibly depends on $a$, such that $j^\infty P(x, f(a)) \in I_a$, where $I_a$ is the ideal in $\mathcal{F}_{\mathbb{R}^n, a}$ generated by $\{j^\infty h(a) | h \in I\}$.
(3) There exists a Nash function $g \in \mathcal{N}(U)$ such that $[g] = [f] \in \mathcal{O}(U)/I\mathcal{O}(U)$.

Proof: The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): Let $\mathcal{I}$ be the finite ideal sheaf generated by $I$ in the sheaf $\mathcal{N}_U$ of Nash functions. Then $f$ defines a section of the quotient sheaf $\mathcal{N}_U/\mathcal{I}$. By the extension theorem ([5],[8]) in non-compact case, there exists $g \in \mathcal{N}(U)$ which defines the same section of $\mathcal{N}_U/\mathcal{I}$ with that defined by $f$. Therefore $f - g \in \mathcal{O}(U)$ defines a section of $\mathcal{I}\mathcal{O}_U$, the ideal sheaf generated by $\mathcal{I}$ in the sheaf $\mathcal{O}_U$ of analytic functions. Then $f - g \in I\mathcal{O}(U)$, by Cartan’s theorem A for real analytic functions ([II]). Thus we have (3).

The implication (3) $\Rightarrow$ (1) is clear. \qed
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Goo ISHIKAWA,
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail : ishikawa@math.sci.hokudai.ac.jp

Tatsuya YAMASHITA,
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
e-mail : tatsuya-y@math.sci.hokudai.ac.jp