GROWING SMALL-WORLD NETWORKS GENERATED BY ATTACHING TO EDGES

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We introduce a minimal model of growing small-world network generated by attaching to edges. The produced network is a plane graph which exists in real-life world. We obtain the analytic results of degree distribution decaying exponentially with degree and average clustering coefficient

\[ C = \frac{1}{2} \ln 3 - 1 \approx 0.6479, \]

which are in good agreement with the numerical simulations. We also prove that the increasing tendency of average path length of the considered network is a little slower than the logarithm of the network order \( N \).

Keywords: Complex networks; Small-world Networks; Disordered systems.

1. Introduction

Many real-world systems take the form of networks—verices connected together by edges.\(^1,2,3,4,5\) Commonly cited examples include technological networks such as the Internet,\(^6\) information networks such as the World Wide Web,\(^7\) social networks such as co-author networks\(^8\) and sexual networks,\(^9\) biological networks such as metabolic networks\(^10\) and protein networks in the cell.\(^11\) Lots of empirical researches show that real-world networks have a small-world effect: they have both a small average path length (APL) like random graphs\(^12\) and a large clustering coefficient. Moreover, the clustering coefficient seems to be irrelevant to the network size.

How to model real-life networks with small-world properties? In the last few years there has been a substantial amount of interest in network structure and function within the physics community.\(^1,2,3,4,5\) The first successful attempt to generate networks with high clustering coefficients and small APL is that of Watts and Strogatz (WS model).\(^13\) The WS model starts with a ring lattice with \( N \) vertices in which every vertex is connected to its first \( 2m \) neighbors (\( m \) on either side). The small-world model is then created by randomly rewiring each edge of the lattice

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with probability \( p \) such that self-connections and duplicate edges are excluded. The rewiring edges are called long-range edges which connect vertices that otherwise would be part of different neighborhoods. The pioneering article of Watts and Strogatz started an avalanche of research on the properties of small-world networks and the WS model. A much-studied variant of the WS model was proposed by Newman and Watts,\(^{14,15}\) in which edges are added between randomly chosen pairs of sites, but no edges are removed from the regular lattice. In 1999, Kasturirangan proposed an alternative model to WS small-world network.\(^{16}\) The model starts also with one ring lattice, then we add a number of extra vertices in the middle of the lattice which are connected to a large number of sites chosen randomly on the main lattice. This model is similar to the WS model in that the addition of the extra vertices effectively introduces shortcuts between randomly chosen positions on the lattice. In fact, even in the case where only one extra vertex is added, the model shows the small-world effect if that vertex is sufficiently highly connected. This case has been solved exactly by Dorogovtsev and Mendes.\(^{17}\) To investigate the small-world effect further, Kleinberg has presented a generalization of the WS model which is based on a two-dimensional square lattice.\(^{18}\) Recently, in order to study other mechanisms for forming small-world networks, Blanchard and Krueger have studied a model where only local edge formation processes are involved and no long-range random edges at all are formed.\(^{19}\) Moreover, small-world networks can be also created by various deterministic techniques: modification of some regular graphs,\(^{20}\) addition and product of graphs,\(^{21}\) and other methods.\(^{22,23}\) Up to now, it is still significant how to generate networks with small-world effect.

In this paper, we present a small-world network model using a very simple method by attaching to edges, which was used by Dorogovtsev et al to generate scale-free networks\(^ {24,25}\) and by Zhang et al to create deterministic small-world networks\(^ {23}\). Our brief model displays small-world effect. We analyze the geometric characteristics of the model both analytically and by simulations. The network under consideration belongs to plane graphs existing in many real-life systems.

2. The Growing Network Model and Its Properties

The construction of the growing small-world network is shown in Fig. 1. Now we introduce its generation algorithm. Initially \(( t = 2 )\), the network is a triangle consist of three vertices, \( s = 0, 1, 2 \), each with degree 2. At each subsequent time step, a new vertex is added, which is attached via two links to both ends of one randomly chosen edge that has never been selected before. The growth process is repeated until the network grows to the desired size of \( N \) vertices. We can see easily at time \( t \), the network consists of \( t + 1 \) vertices and \( 2t - 1 \) edges. The total degree equals \( 4t - 2 \). Thus when \( t \) is large the average vertex degree at time \( t \) is equal approximately to a constant value 4, which shows our network is sparse like many real-life networks. It should be noted that our model is reduced to the scale-free network model proposed by Dorogovtsev et al,\(^ {24}\) if we have no restrictions on the edges previously selected.
2.1. Degree distribution

The degree distribution is one of the most important statistical characteristics of a network. In order to conveniently describe the computation of network characteristics, we label vertices by their birth times, $s = 0, 1, 2, \ldots, t$, and use $p(k, s, t)$ to denote the probability that at time $t$ a vertex created at time $s$ has a degree $k$. Additionally, we call the edges outer edges if they have never been chosen before. When a new vertex enter the network, it has two outer edges. At each time step, when one outer edge is selected, the number of outer edges of the two vertices connected by the selected edge will decrease by one, respectively; meanwhile the new vertex will link to them, so the number of their outer edges will remain unchanged. Thus at time $t$, there are $t + 1$ outer edges in the network and each vertex has two, respectively. The master equation governing the evolution of the degree distribution of individual vertices has the form

\[ p(k, s, t + 1) = \frac{2}{t+1} p(k - 1, s, t) + (1 - \frac{2}{t+1}) p(k, s, t) \]  \hspace{1cm} (1)

with the initial condition, $p(k, s = 0, 1, 2, t = 2) = \delta_{k,2}$ and the boundary one $p(k, t, t) = \delta_{k,2}$. This describes two possibilities for a vertex: first, with probability $\frac{2}{t+1}$, it may get an extra edge from the new vertex and increase its own degree by 1; and second, with the complimentary probability $1 - \frac{2}{t+1}$, the degree of the vertex may stays the same. Eq. (1) and all the following ones are exact for all $t \geq 2$.

The degree distribution of the network can be obtained as

\[ P(k, t) = \frac{1}{t+1} \sum_{s=0}^{t} p(k, s, t) \]  \hspace{1cm} (2)
Using this and applying $\sum_{s=0}^{t}$ to both sides of Eq. (1), we get the following master equation for the degree distribution,

$$(t + 2)P(k, t + 1) - (t + 1)P(k, t) = 2P(k - 1, t) - 2P(k, t) + \delta_{k,2}$$

(3)

The corresponding stationary equation, i.e., at $t \to \infty$, takes the form

$$3P(k) - 2P(k - 1) = \delta_{k,2}$$

(4)

Eq. (4) implies that $P(k)$ is the solution of the recursive equation

$$P(k) = \begin{cases} \frac{2}{3}P(k - 1) & \text{for } k > 2 \\ \frac{2}{3} & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

(5)

giving

$$P(k) = \frac{3}{4} \left( \frac{2}{3} \right)^k (k \geq 2)$$

(6)

Obviously, the degree distribution $P(k)$ is an exponential of a power of degree $k$ (see Fig. 2), so our network can be called an exponential one. Note that most small-world networks including WS network are exponential networks.

Fig. 2. Degree distribution of the network with order $N = 10^5$. The open black circles represent the simulation results according to our model algorithm and the solid line is the analytic calculation value given by Eq. (6).
2.2. Clustering coefficient

By definition, clustering coefficient $C_i$ of a vertex $i$ is the ratio of the total number $e_i$ of existing edges between all $k_i$ its nearest neighbors and the number $k_i(k_i - 1)/2$ of all possible edges between them, i.e. $C_i = 2e_i/[k_i(k_i - 1)]$. The clustering coefficient $C$ of the whole network is the average of all individual $C_i$’s. In our case, we can calculate the average clustering of the network exactly.

When a new vertex $i$ joins the network, its degree $k_i$ and $e_i$ is 2 and 1, respectively. Each subsequent addition of an edge to that vertex increases both $e_i$ and $k_i$ by one. Thus, $e_i$ equals to $k_i - 1$ for all vertices. So one can see that, in this network, there is a one-to-one correspondence between the clustering coefficient of a vertex and its degree: $C = 2/k$. This expression indicates that the local clustering scales as $C(k) \sim k^{-1}$, which is similar to that observed in some other models. Thus, the clustering coefficient $C$ of the whole network is given by

$$C = 2 \sum_{k=2}^{\infty} \frac{1}{k} P(k) = \frac{3}{2} \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{2}{3}\right)^k = \frac{3}{2} \ln 3 - 1 \approx 0.6479$$

(7)

So in the limit of large $t$ the clustering coefficient is large (Fig. 3). It is worth noting that this value is higher than for regular lattices with the same order and average vertex degree.5

Fig. 3. Clustering coefficient $C$ versus network order $N$. The open black circles denote the simulated results, while the dotted line is the predicted value given by Eq. (7).
2.3. Average path length

The average path length of a network is defined as the number of edges in the shortest path between two vertices averaged over all pairs of vertices. Below, we will discuss the APL of this growing network using the approach similar to that presented by Zhou et al.\textsuperscript{28}

First we note that it is not very difficult to prove that for this network and for any two arbitrary vertices \(i\) and \(j\) each shortest path from \(i\) to \(j\) does not pass through any vertices \(k\) satisfying that \(k > \max(i, j)\).

If \(d(i, j)\) denotes the distance between \(i\) and \(j\), we introduce the total distance of our network with order \(N\) as

\[
\sigma(N) = \sum_{0 \leq i < j \leq N-1} d(i, j)
\]  

and we denote the APL by \(L(N)\), defined as:

\[
L(N) = 2\sigma(N) / N(N-1)
\]  

According to the former remark, the addition of new vertices will not affect the distance between those already existing, so we have:

\[
\sigma(N+1) = \sigma(N) + \sum_{i=0}^{N-1} d(i, N)
\]  

Assume that the vertex \(N\) is added joining the edge \(E\) composed of vertices \(w_1, w_2\), then Eq. (10) can be rewritten as:

\[
\sigma(N+1) = \sigma(N) + \sum_{i=0}^{N-1} (D(i, w) + 1) = \sigma(N) + \sum_{i=0}^{N-1} D(i, w)
\]  

where \(D(i, w) = \min\{d(i, w_1), d(i, w_2)\}\). Constricting the edge \(E\) continuously into a single vertex \(w\) (here we assume that \(w \equiv w_1\)), we have \(D(i, w) = d(i, w)\). Since \(d(w_1, w) = d(w_2, w) = 0\), Eq. (11) can be rewritten as:

\[
\sigma(N+1) = \sigma(N) + \sum_{i \in \Gamma} d(i, w)
\]  

where \(\Gamma = \{0, 1, 2, \ldots, N-1\} - \{w_1, w_2\}\) is a vertex set with cardinality \(N-2\). The sum \(\sum_{i \in \Gamma} d(i, w)\) can be considered as the total distance from one vertex \(w\) to all the other vertices in our model with order \(N-1\), which can be roughly approximated in terms of \(L(N-1)\):

\[
\sum_{i \in \Gamma} d(i, w) \approx (N-2)L(N-1)
\]  

Note that, as \(L(N)\) increases monotonously with \(N\), it is clear that:

\[
(N-2)L(N-1) = \frac{2\sigma(N-1)}{N-1} < \frac{2\sigma(N)}{N}
\]  

\[
(14)
\]
Combining (11), (12) and (13), one can obtain the inequation:
\[ \sigma(N + 1) < \sigma(N) + N + \frac{2\sigma(N)}{N} \]  
(15)

From (15), the variation of \( \sigma(N) \) would be given by
\[ \frac{d\sigma(N)}{dN} = N + \frac{2\sigma(N)}{N} \]  
(16)

This equation leads to
\[ \sigma(N) = N^2 \ln N + \alpha \]  
(17)

where \( \alpha \) is a constant. As \( \sigma(N) \sim N^2 \ln N \), we have \( L(N) \sim \ln N \). Note that as we have deduced Eq. (17) from an inequality, then \( L(N) \) increases at most as \( \ln N \) with \( N \) (see Fig. 4). Therefore, there is a desired slow growth of APL with network size \( N \).

Based on the discussions above, the discussed network is a sparse one with high clustering and low APL, which is obvious a small-world network with the properties similar to other small-world networks.\(^{13,14,15,16,17,18,20,21,23}\)

3. Conclusion and Discussion

In summary, we have proposed a simple model of growing small-world networks. We obtain analytically some properties of the network, which are in good agreement with the simulation results. In spite of the simplicity of the considered model,
the results are close to those for usual small-world networks. It should be mentioned that the model under consideration is a plane graph which can be drawn on a plane without edges crossing and attracts little attention of physicists. Many real-life networks are plane graphs for technical or natural requirements, such as layout of printed circuits and vein networks clinging to cutis. We hope our model would help to understand some properties of real-world plane networks and be a catalyst for further studies.

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