ON THE RIEMANN-ROCH FORMULA WITHOUT PROJECTIVE HYPOTHESES

A. NAVARRO AND J. NAVARRO

To our father

Abstract. Let $S$ be a finite dimensional noetherian scheme. For any proper morphism between smooth $S$-schemes, we prove a Riemann-Roch formula relating higher algebraic $K$-theory and motivic cohomology, thus with no projective hypotheses either on the schemes or on the morphism. We also prove, without projective assumptions, an arithmetic Riemann-Roch theorem involving Arakelov’s higher $K$-theory and motivic cohomology as well as an analogous result for the relative cohomology of a morphism.

These results are obtained as corollaries of a motivic statement that is valid for morphisms between oriented absolute spectra in the stable homotopy category of $S$.

INTRODUCTION

Let $f: Y \to X$ be a proper morphism between non-singular quasi-projective varieties over a field $k$. The original Grothendieck’s Riemann-Roch theorem (cf. [BS58]) states that the following square commutes:

$$
\begin{array}{ccc}
K_0(Y) & \xrightarrow{f_!} & K_0(X) \\
Td(T_Y) \cdot \text{ch} & \downarrow & Td(T_X) \cdot \text{ch} \\
CH^\bullet(Y) \otimes \mathbb{Q} & \xrightarrow{f_*} & CH^\bullet(X) \otimes \mathbb{Q}
\end{array}
$$

or, in other words, that for any element $a \in K_0(Y)$, the following formula holds:

$$
Td(T_X) \cdot \text{ch}(f_!(a)) = f_! \left( Td(T_Y) \cdot \text{ch}(a) \right).
$$

This formula relates the exceptional direct image $f_!$ on the Grothendieck group of vector bundles $K_0$ to the direct image $f_*$ on the Chow ring with rational coefficients. This relation involves the Chern character $\text{ch}$, adequately twisted with the Todd series $Td$ of the corresponding tangent bundles.

Later on, Grothendieck aimed to generalise this Riemann-Roch formula in three directions: considering schemes defined over a general base, replacing the smoothness condition on the schemes by a regularity condition on the morphism, and removing any projective assumption both from the morphism and from the schemes. The first two problems were among the main questions addressed in [SGA6] (cf. [SGA6, O.1]), whereas the third one—namely, proving a Riemann-Roch formula...
for proper complete intersection morphisms between (non-projective) schemes—remained at that time as an open problem ([SGA6 XIV.2]).

In subsequent years there were various results regarding this question. If \( X \) is a smooth and proper variety over a field \( k \) of characteristic zero, the Chow lemma and the resolution of singularities allow us to construct a projective variety \( \overline{X} \) and a birational map \( \overline{X} \to X \) that is a composition of blowing-ups along smooth centers. In ([Ful77]), Fulton used this fact to prove the Riemann-Roch formula for the projection \( X \to \text{Spec}(k) \). Also, the general case of a proper morphism between smooth varieties over a field of characteristic zero follows with similar reasoning (cf. [Nav81], [FG83]).

The case of positive characteristic was proved in [FG83] using a different approach. Fulton-Gillet’s reasoning avoids the resolution of singularities due to the formulation of the Riemann-Roch theorem developed in [BFM75], the technique of envelopes ([Gil81]), and a little use of higher \( K \)-theory. This result of [FG83 Corollary 2] was the furthest generalization we knew of the Riemann-Roch formula without projective hypotheses.

On the other hand, despite a Riemann-Roch formula for higher \( K \)-theory being already known at that moment (cf. [Gil81]), Fulton-Gillet’s approach did not apply in this setting. In fact, they explicitly posed the problem of extending these results to higher \( K \)-theory (cf. [FG83 3.1.4]).

As Grothendieck already pointed out,

Il semble clair que la démonstration de la formule Riemann-Roch dans ce cas général [i.e., without projective hypotheses] demandera l’introduction d’idées essentiellement nouvelles ([SGA6 XIV.2])

and each proof restricts to the situation where its new ideas may be applied.

In this paper we prove a higher Riemann-Roch formula without projective hypotheses, either on the schemes or on the morphism. Our statement is valid for absolute oriented ring spectra and smooth schemes defined over a finite dimensional, noetherian scheme (cf. Theorem 3.4) and therefore relates homotopy invariant \( K \)-theory with rational motivic cohomology. Hence, we deduce a Riemann-Roch theorem for classic higher \( K \)-theory and (Levine’s extension of) higher Chow groups for smooth schemes over a Dedekind domain. To be more precise, let \( S \) be the spectrum of a Dedekind domain. For any smooth \( S \)-scheme \( X \), let us write \( \text{ch}: K(X) \to CH(X)_\mathbb{Q} \) for the higher Chern character, relating higher \( K \)-groups to higher Chow groups with rational coefficients (see definitions in Example 1.2):

**Theorem 3.6** Let \( S \) be the spectrum of a Dedekind domain and let \( f: Y \to X \) be a proper morphism between smooth \( S \)-schemes. The following diagram commutes:

\[
\begin{array}{ccc}
K(Y) & \xrightarrow{f^*} & K(X) \\
\downarrow \text{Td}(T_Y)\text{ch} & & \downarrow \text{Td}(T_X)\text{ch} \\
CH(Y)_\mathbb{Q} & \xrightarrow{f^*} & CH(X)_\mathbb{Q}.
\end{array}
\]

In other words, for any element \( a \in K(Y) \), the following formula holds:

\[
\text{Td}(T_X)\text{ch}(f_*(a)) = f_*(\text{Td}(T_Y)\text{ch}(a)).
\]
Note that the restriction of the formula above to the $K_0$ group produces a Grothendieck-Riemann-Roch statement that is valid for proper morphisms between general (non-projective) smooth $S$-schemes.

In addition, we also prove other Riemann-Roch statements for the relative cohomology of a morphism (cf. Corollary 3.3) and for the arithmetic counterparts of higher $K$-theory and motivic cohomology (cf. Corollary 3.9). These results also improve previous versions of [Nav16b] and [HSt15], respectively.

Our proof relies on a duality theorem of Ayoub (Ayo07), as refined by Cisinski-Déglise in [CD19], and on the Riemann-Roch formula for closed embeddings. Roughly speaking, since duality holds without projective assumptions, the Riemann-Roch formula can also be proved in this generality. The main idea—namely, that the Riemann-Roch theorem is a comparison of dualities—comes from classic algebraic topology (cf. [Dye62]).

The paper is written in Morel-Voevodsky’s language of $\mathbb{A}^1$-homotopy theory ([MV99], [Voe98]). In addition, our development of the Riemann-Roch is directly inspired by Panin’s work on orientation theory ([Pan03], [Pan04], [Pan09], [Nav16]).

Finally, let us briefly comment on the plan of this article. The first, preliminary section recalls definitions and statements from stable homotopy, in particular regarding cohomology, orientations, Borel-Moore homology, and Thom spaces. In Section 2 we construct a functorial direct image in cohomology as the dual of the direct image in Borel-Moore homology and prove its main properties. Finally, in the last section we establish the aforementioned Riemann-Roch theorems.

1. Preliminaries

All the schemes we will consider throughout this paper are smooth over a finite dimensional noetherian base $S$. We use the same notation as [Nav16b] and recall the indispensable results.

Let $X$ be a smooth $S$-scheme and denote by $\text{SH}(X)$ the stable homotopy category of Voevodsky, whose objects are called spectra (cf. [Voe98]).

These categories are symmetric monoidal, meaning that there is a symmetric product $\wedge$ and a unit object $\mathbb{1}_X$. There is also a shift functor $[1]: \text{SH}(X) \to \text{SH}(X)$ and a Tate object $\mathbb{1}_X(1)$ that defines a twist $\wedge \mathbb{1}_X(1): \text{SH}(X) \to \text{SH}(X)$. If $E$ is a spectrum of $\text{SH}(X)$ and $p, q \in \mathbb{Z}$, we write $E(q)[p]$ for the result of shifting $p$ times and twisting $q$ times.

Moreover, the family of categories obtained if one varies $X$ satisfies Grothen- dieck’s six functors formalism (cf. [Ayo07] and [CD19]). In particular, for any morphism $f: Y \to X$ there exists a pair of adjoint functors

$$f^*: \text{SH}(X) \rightleftarrows \text{SH}(Y): f_*.$$

If $f$ is separated of finite type, then there also exist mutually adjoint exceptional functors

$$f_!: \text{SH}(Y) \rightleftarrows \text{SH}(X): f^!.$$

Finally, if $\pi: X \to T$ is a smooth morphism, then there exists an “extension by zero” functor $\pi_*: \text{SH}(X) \to \text{SH}(T)$ that is left adjoint to $\pi^*$. These functors satisfy adequate functorial properties (cf. [Ayo07] 1.4.2 and the refinement in [CD19] 2.4.50]).

Definition 1.1. A ring spectrum is an associative commutative unitary monoid in the stable homotopy category $\text{SH}(X)$ of a scheme $X$.
An absolute ring spectrum $E$ is a ring spectrum on the stable homotopy category of the base scheme $SH(S)$. Equivalently, an absolute ring spectrum is a family of ring spectra $E_X$ on $SH(X)$ for every scheme $X$ that is stable under pullback, i.e., such that for every morphism of schemes $f: Y \to X$ we have fixed an isomorphism $\epsilon_f: f^*E_X \to E_Y$ satisfying the usual compatibility conditions (cf. [Deg18 1.2.1]).

A morphism of absolute ring spectra $\varphi: E \to F$ is a morphism of spectra in $SH(S)$ or, equivalently, a collection of morphisms of ring spectra $\varphi_X: E_X \to F_X$, one for each scheme $X$, that is stable under pullback.

If $E$ is an absolute ring spectrum, the $E$-cohomology of a scheme $X$ is, for $p, q \in \mathbb{Z}$:

$$E^{p,q}(X) := \text{Hom}_{SH(X)}(\mathbb{1}_X, E_X(q)[p]), \quad E(X) := \bigoplus_{p,q} E^{p,q}(X).$$

An oriented absolute ring spectrum is a pair $(E, c_1)$, where $E$ is an absolute ring spectrum and $c_1 \in \pi^{2,1}(\mathbb{P}^{\infty})$ is a class satisfying that $i_1^*(c_1) = \eta$, for $i_1: \mathbb{P}_1 \to \mathbb{P}^{\infty}$ and $\eta \in \pi^{2,1}(\mathbb{P}_1)$, the canonical class defined by the Tate object (cf. [Nav16b §1.3]).

**Example 1.2.** There is a pleiad of cohomology theories represented by oriented absolute ring spectra. Let us recall here those that we will mention later on, although the reader may consult [Nav16b 1.4] for more examples to which the results of this paper apply.

- The $K$-theory spectrum $KGL$ is an absolute ring spectrum defined in [Voe98 and Rio10]. It represents Weibel’s homotopy invariant $K$-theory (cf. [Cis13 2.15]), which will be denoted $KH(\_)$, and therefore also represents Quillen’s algebraic $K$-theory for regular schemes, which will be denoted $K(\_)$.

- The $\mathbb{Q}$-localization of the $K$-theory spectrum admits a decomposition induced by the Adams operations, i.e., $KGL_{\mathbb{Q}} = \bigoplus_{i \in \mathbb{Z}} KGL_{\mathbb{Q}}^{(i)} \in SH(S)_{\mathbb{Q}}$, where $KGL_{\mathbb{Q}}^{(i)}$ denotes the eigenspaces for the Adams operations ([Rio10 5.3]).

Beilinson’s motivic cohomology spectrum is defined as $H_B = KGL_{\mathbb{Q}}^{(0)} \in SH(S)_{\mathbb{Q}}$ and it is also an absolute ring spectrum. It represents motivic cohomology with rational coefficients, which we denote $H^p_M(\_ , \mathbb{Q}(q))$. On smooth schemes over a Dedekind domain, Beilinson’s motivic cohomology spectrum coincides with Spitzweck’s motivic cohomology spectrum ([Spi12 7.14]); therefore it coincides with rational Levine’s motivic cohomology ([Spi12 7.20]) and in particular with higher Chow groups ([Blo86, Lev, Lev01]). More concretely, if $S$ is the spectrum of a Dedekind domain, for $X$ a smooth $S$-scheme we have

$$\text{Hom}_{SH(X)}(\mathbb{1}_X, H_B(q)[p]) = H^p_M(\_ , \mathbb{Q}(q)) = CH_{2q-p}(X, q)_{\mathbb{Q}}.$$

- Let $g: T \to X$ be a morphism of schemes and let $E$ be an absolute ring spectrum.

The spectrum $\text{hofib}_E(g) := \text{hofib}(E_X \to g_*g^*E_X)$ defines cohomology groups that are called the relative cohomology of $g$:

$$E^{*,*}(g) := \text{Hom}_{SH(X)}(\mathbb{1}_X, \text{hofib}(g)[\ast](\ast)).$$

They fit into long exact sequences

$$\cdots \to E^{p,q}(g) \to E^{p,q}(X) \to E^{p,q}(T) \to E^{p+1,q}(g) \to \cdots$$
and generalize many constructions: the cohomology with proper support, the cohomology with support on a closed subscheme, and the reduced cohomology are, respectively, the relative cohomology of a closed immersion, an open immersion, and the projection over a base point (cf. [Nav16b 1.19]).

Let \( f : Y \to X \) be another morphism and denote \( g_Y : T \times_X Y \to Y \). If either \( g \) is proper or \( f \) is smooth, then the absolute spectrum over \( X \) defined by \( \text{hofib}_E(g) \) represents \( E(g_Y) \) (cf. [Nav16b §1.17]).

**Definition 1.3.** Let \( E \) in \( \text{SH}(X) \) be a ring spectrum. An \( E \)-module is a spectrum \( M \) in \( \text{SH}(X) \) together with a morphism of spectra \( \nu : E \land M \to M \) in \( \text{SH}(X) \) satisfying the usual module condition. An absolute \( E \)-module is a module over an absolute ring spectrum \( E \).

Let \( \phi : E \to F \) be a morphism of ring spectra and let \((M', \nu')\) be an \( F \)-module. A \( \phi \)-morphism of modules \( \Phi : M \to M' \) is a morphism of spectra in \( \text{SH}(X) \) such that \( \nu' \circ (\phi \land \Phi) = \Phi \circ \nu \).

**Example 1.4.** Holmstrom-Scholbach defined in [HS15] the arithmetic \( K \)-theory and the arithmetic motivic cohomology ring spectra, \( \text{KGL} \) and \( \text{H}_{\text{B}} \). They are absolute modules over \( \text{KGL} \) and \( \text{H}_{\text{B}} \), respectively. In addition, there is an arithmetic Chern character \( \hat{\text{ch}} : \text{KGL} \to \text{H}_{\text{B}} \) that is a ch-morphism of modules (cf. [HS15 4.2]).

**Remark 1.5.** The machinery of the six functors formalism provides the \( \mathbb{E} \)-cohomology defined by oriented absolute spectra with the classic properties of cohomologies (such as inverse image, cup product, Chern classes, etc.). For a complete account of these properties the reader may consult [Dég18].

Let us recall here the construction of the morphism of forgetting support, since it will be used later on. Let \( Z \xrightarrow{j} Y \xrightarrow{i} X \) be closed immersions and let \( \mathbb{M} \) be an absolute \( \mathbb{E} \)-module. We define a morphism

\[
j_* : \mathbb{M}_Z(X) \to \mathbb{M}_Y(X)
\]
as follows: the adjunction of \((j^*, j_*)\) induces a morphism

\[
i_*(\text{ad}) : i_*(\mathbb{I}_Y) \to i_*(j_*j^* \mathbb{I}_Y) = (ij)_* \mathbb{I}_Z.
\]

Let \( a : (ij)_* \mathbb{I}_Z \to \mathbb{M}_X \) be in \( \mathbb{M}_Z(X) \). The element \( j_*(a) \in \mathbb{M}_Y(X) \) is defined as

\[
i_* (\mathbb{I}_Y) \xrightarrow{i_*(\text{ad})} (ij)_* \mathbb{I}_Z \xrightarrow{a} \mathbb{M}_X.
\]

**1.6 (Borel-Moore homology).** Let \( \mathbb{E} \) be an absolute ring spectrum, and let \( \mathbb{M} \) be an \( \mathbb{E} \)-module. The Borel-Moore homology of a scheme \( \pi_X : X \to S \) is defined as

\[
\mathbb{M}^{\text{BM}}_{p,q}(X) := \text{Hom}_{\text{SH}(X)}(\mathbb{I}_X, \pi_X^! \mathbb{M}_{\text{S}}(-q)[-p]) \quad \mathbb{M}^{\text{BM}}(X) := \bigoplus_{p,q} \mathbb{M}^{\text{BM}}_{p,q}(X).
\]

Observe the equivalent descriptions, using adjunction:

\[
\mathbb{M}^{\text{BM}}_{p,q}(X) = \text{Hom}_{\text{SH}(S)}(\pi_{X!} \mathbb{I}_X, \mathbb{M}_{\text{S}}(-q)[-p])
\]

\[
= \text{Hom}_{\text{SH}(S)}(\mathbb{I}_S, \pi_{X*} \pi_X^! \mathbb{M}_{\text{S}}(-q)[-p]).
\]

If \( f : Y \to X \) is a proper morphism, then the direct image with compact support, \( f_* \), is canonically isomorphic to \( f_* \) (cf. [CD19 2.2.7]), so that the adjunction of \((f_!, f^!\)) induces a natural transformation

\[
\pi_{Y*} \pi_Y^! = \pi_{X*} f_* f^! \pi_X^! \to \pi_{X*} \pi_X^! \to \pi_{X*} \pi_X^!.
\]
which in turn produces a direct image in BM-homology

\[ f_* : BM_{p,q}(Y) \rightarrow BM_{p,q}(X). \]

As expected Borel-Moore homology is a module, in the classic sense, over the \( E \)-cohomology. More precisely, there is a cap product:

\[ E^{p,q}(X) \times BM_{r,s}(X) \rightarrow BM_{p+r-s,q}(X), \]

where a pair \((a, m)\) is mapped into the element \(a \cdot m \in BM_{p+r-s,q}(X)\) defined as

\[ \llbracket X \rrbracket \mapsto \llbracket E_X(q)[p] \rrbracket \]
\[ \mapsto \pi^*_X(E_S) \land \pi^*_X(E_S)(q-s)[p-r] \]
\[ \cong \pi^*_X(E_S \land E_S)(q-s)[p-r] \]
\[ \mapsto \pi^*_X(E_S)(q-s)[p-r]. \]

Here, \( \mu \) stands for the structural map of \( E \), and the first isomorphism is due to the fact that the structural map of the bilateral module \( \llbracket \pi^*, \pi^! \rrbracket \) is an isomorphism (cf. [Ayo07, 2.3.27]).

1.7 (The Thom space). Let \( V \rightarrow X \) be a vector bundle of rank \( d \) and let us write \( \bar{V} := \mathbb{P}(V \oplus \llbracket \rrbracket) \) for the projective completion.

The Thom space of \( V \) is defined as

\[ Th(V) := V/V - \{0\} \simeq \bar{V}/\mathbb{P}(V) \in SH(X). \]

Let \((E, c_1)\) be an oriented absolute ring spectrum. Let us write

\[ E^{*,*}(Th(V)) := \text{Hom}_{SH(X)}(Th(V), E_X[\ast][\ast]). \]

There exists a long exact sequence

\[ \cdots \rightarrow E^{*,*}(Th(V)) \xrightarrow{i^*} E^{*,*}(\bar{V}) \xrightarrow{i^*} E^{*,*}(\mathbb{P}(V)) \rightarrow \cdots. \]

The Thom class is the following element in cohomology:

\[ t(V) := \sum_{i=0}^d (-1)^i c_i(V)x^i \in E^{2d,d}(\bar{V}) \quad x := c_1(O_{\bar{V}}(-1)). \]

As a consequence of the projective bundle theorem ([Dég18, 2.1.13]), \( i^*(t(V)) = 0 \), so that there exists a unique class \( t^{\text{ref}}(V) \in E^{2d,d}(Th(V)) \), called the refined Thom class, such that \( \pi(t^{\text{ref}}(V)) = t(V) \). Moreover, the cohomology of the Thom space \( E(Th(V)) \) is a free \( E(X) \)-module of rank one generated by \( t^{\text{ref}}(V) \in E^{2d,d}(Th(V)) \).

Also, let us recall that the deformation to the normal bundle induces an isomorphism

\[ E(Th(V)) \xrightarrow{\cong} E_X(\bar{V}) \]
\[ t^{\text{ref}}(V) \mapsto \tilde{\eta}_X^V, \]

where \( \tilde{\eta}_X^V \) denotes the refined fundamental class of \( X \) on \( V \) (cf. [Dég18, 2.3.1]).

Similar statements are true for \( E \)-modules. Due to the lack of a reference, let us sketch the proof of the projective bundle theorem in this case:

Lemma 1.8. Let \((E, c_1)\) be an oriented absolute ring spectrum, let \( M \) be an absolute \( E \)-module, and let \( V \rightarrow X \) be a vector bundle.

It holds that

\[ M(\mathbb{P}(V)) = M(X) \otimes_{E(X)} E(\mathbb{P}(V)). \]

\(^1\)For the definition of left and right module in this context see [Ayo07, 2.1.94].
Proof. As in [Dég08, 3.2], due to Mayer-Vietoris we can reduce to the case of a trivial vector bundle. To prove $M(P^n_\mathbb{P}^n) = M(X) \otimes_{E(X)} E(P^n_\mathbb{P}^n)$ we can assume that $X = S$ and proceed by induction on $n$. The case $n = 0$ is trivial; for the induction step consider the homotopy cofiber sequence

$$\mathbb{P}^{n-1} \overset{i}{\to} \mathbb{P}^n \overset{\pi}{\to} \mathbb{P}^n/\mathbb{P}^{n-1}$$

that produces a long exact sequence

$$\cdots \to M(\mathbb{P}^n/\mathbb{P}^{n-1}) \to M(\mathbb{P}^n) \overset{i^*}{\to} M(\mathbb{P}^{n-1}) \to \cdots.$$  

Since $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{I}(n)(2n)$ (cf. [MV99, 3.2.18]), $M^{p,q}(\mathbb{P}^n/\mathbb{P}^{n-1}) = M^{p-2n,q-n}(S)$. By the induction hypothesis we deduce that $i^*$ is surjective (since we have $i^*(m \cdot c_1(O_{\mathbb{P}^n}(-1))^j) = m \cdot c_1(O_{\mathbb{P}^n}(-1))^j$ for $m \in M(X)$). The map $\pi : M^{p-2n,q-n}(S) \to M^{p,q}(\mathbb{P}^n)$ maps $m \mapsto m \cdot c_1(O_{\mathbb{P}^n}(-1))^n$ (cf. [Dég18, Proof 2.1.3]). Hence, the thesis follows.

As a consequence of this lemma, for any vector bundle $V \to X$, the $M$-cohomology of its Thom space is isomorphic to the $M$-cohomology of $X$ through the morphism

$$M(X) \xrightarrow{\sim} M^{p,q}(\text{Th}(V))$$

$$m \quad \mapsto \quad m \cdot t^\text{ref}(V).$$

Remark 1.9. More generally, all the statements above remain true if $V$ is a virtual vector bundle. Indeed, as shown by Riou ([Riou10]), the Thom space construction in $\text{SH}$ extends to a canonical functor

$$\text{Th} : K(X) \to \text{SH}(X)$$

defined on the category $K(X)$ of virtual vector bundles ([Del87, 4.12]).

In particular, this means that for every short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ in $X$ there is a canonical isomorphism in $\text{SH}(X)$:

$$\text{Th}(E') \wedge \text{Th}(E'') \xrightarrow{\sim} \text{Th}(E).$$

It induces an isomorphism (cf. [Dég18, 2.4.8])

$$E(\text{Th}(E')) \otimes E(\text{Th}(E'')) \xrightarrow{\sim} E(\text{Th}(E)), \quad t^\text{ref}(E') \otimes t^\text{ref}(E'') \mapsto t^\text{ref}(E') \cdot t^\text{ref}(E'').$$

Thus, the Thom class of a virtual vector bundle $\xi = [E] - [E']$ is defined to be the class $t^\text{ref}(\xi) = t^\text{ref}(E) \cdot t^\text{ref}(E')^{-1}$.

2. Direct image for proper morphisms

In what follows, we will make a strong use of the following statement, which is the duality theorem in the motivic setting proved by Ayoub (cf. [Ayo07, 1.4.2], [SGA4, XVIII.3.2.5]). More concretely, we will use its extension to the non-quasi-projective setting of [CD19, 2.4.50].

Theorem 2.1. If $\pi : X \to T$ is a smooth morphism, there exists a functorial isomorphism

$$\pi^*(\_\_\_) \wedge \text{Th}(T_\pi) \xrightarrow{\sim} \pi^!(\_\_),$$

where $T_\pi$ stands for the virtual tangent bundle of $\pi$ (cf. [Ful98, B.2.7]).
In the particular case of a smooth $S$-scheme, $\pi_X: X \to S$, the tangent bundle of $\pi_X$ is the tangent bundle $T_X$ of $X$. Hence, the theorem above implies the following isomorphism:

$$\text{Hom}_{\text{SH}(X)}(\text{Th}(-T_X), M_X(q)[p]) \simeq \text{Hom}_{\text{SH}(X)}(\iota_X, \pi_X^* M_S(q)[p])$$

which relates the cohomology of the Thom space of the tangent bundle to its Borel-Moore homology; that is to say,

$$M^{p,q}(\text{Th}(-T_X)) \simeq M^{BM}_{-p,-q}(X).$$

Combining these facts with the computation of the cohomology of the Thom space $\text{Th}(\mathcal{E})$, we obtain duality isomorphisms between the $M$-cohomology and the Borel-Moore homology of a smooth scheme:

$$M^{p,q}(X) \simeq M^{BM}_{p+2n,q+n}(\text{Th}(-T_X)) \simeq M^{BM}_{-p-2n,-q-n}(X).$$

With these isomorphisms, it is possible to define a direct image in cohomology without projective assumptions:

**Definition 2.2.** Let $(\mathcal{E}, c_1)$ be an oriented absolute ring spectrum and let $M$ be an absolute $\mathcal{E}$-module. The direct image—in $M$-cohomology—of a proper morphism $f: Y \to X$ between smooth $S$-schemes is defined as the composition

$$M^{p,q}(Y) \simeq M^{BM}_{-p-2n,-q-n}(Y) \xrightarrow{f_*} M^{BM}_{-p-2n,-q-n}(X) \simeq M^{p+2d,q+d}(X),$$

where $n = \dim_S Y$ and $d = n - \dim_S X$.

**Remark 2.3.** Due to Ayoub’s duality, it is not surprising that we can express the direct image in terms of Thom spaces. Let us describe in detail the case of a closed immersion $i: Z \to X$.

Let $N_{Z/X}$ be the normal bundle and recall that $T_Z = i^* T_X - N_{Z/X} \in K_0(Z)$. Then

$$\text{Th}(T_Z) = \text{Th}(-N_{Z/X}) \wedge \text{Th}(i^* T_X) = \iota^!(\iota_X) \wedge \iota^*(\text{Th}(T_X)),$$

because of isomorphism (11) and the fact that $\iota^!(\iota_X) = \text{Th}(-N_{Z/X})$ (cf. [Dég18, 1.3.5.3]) and $\text{Th}(i^* T_X) = i^*(\text{Th}(T_X))$.

Therefore, there is a map

$$\text{Hom}_{\text{SH}(Z)}(\iota_Z, \text{Th}(T_Z) \wedge \mathcal{E}_Z) \xrightarrow{\text{ad}_{i^!, i^*}} \text{Hom}_{\text{SH}(X)}(\iota_X, \iota^! \iota^!(\iota_X) \wedge \text{Th}(T_X) \wedge \mathcal{E}_X) \xrightarrow{\text{ad}_{i^!, i^*}} \text{Hom}_{\text{SH}(X)}(\text{Th}(T_Z), \mathcal{E}_X),$$

where $\text{ad}_{i^!, i^*}$ is obtained by composing with the natural adjunction $i^! i^* \to \text{Id}$.

Denote $\pi = \pi_X$ and $\pi' = \pi_Z$. Observe that $T_{\pi i} = T_{\pi'} = T_Z$; then $\text{Th}(T_{\pi i}) = \text{Th}(T_Z) = \iota^!(\iota_X) \wedge \iota^*(\text{Th}(T_{\pi'}))$. The compatibility of the above map $\mathcal{E}(\text{Th}(-T_Z)) \to \mathcal{E}(\text{Th}(-T_X))$ and the direct image in Borel-Moore homology derive
from the commutative square

\[ \begin{array}{c}
\pi_1 i_! \pi_! (\_ ) \\
\downarrow \\
\pi_1 i_! (\text{Th}(T_{\pi}) \wedge i^* \pi^*(\_ ) ) \\
\downarrow \\
\pi_1 (i_! (\_ I_X) \wedge i^* (\text{Th}(T_{\pi}) \wedge \pi^*(\_ ) ) ) \\
\downarrow \\
\pi_1 (i_! (\_ I_X)) \wedge \pi^*(\_ ) \xrightarrow{\text{ad}_{i_! i^*}} \pi_1 \pi^*(\_ ) ,
\end{array} \]

where the vertical isomorphisms arise from duality and the projection formula (cf. [Ayo07, 2.3.10]).

Remark 2.4. If \( i : Z \to X \) is a closed immersion of codimension \( d \), then the direct image can be refined to a direct image with support. More precisely, let us write \( \pi_! : E^*(Z) \to E^*+2d,*,+d(Z) \) for the morphism defined as follows:

\[ \begin{array}{c}
\text{Hom}_{SH}(Z)(\_ I_Z, \text{Th}(T_Z) \wedge E_Z) \\
\downarrow \\
\text{Hom}_{SH}(Z)(\_ I_Z, \text{Th}(-N_{Z/X}) \wedge \text{Th}(i^* T_X) \wedge E_Z) \\
\downarrow \\
\text{Hom}_{SH}(Z)(\_ I_Z, \text{Th}(-N_{Z/X}) \wedge E_Z) \sim \to E(\text{Th}(N_{Z/X})),
\end{array} \]

where the right equality is a consequence of the deformation to the normal bundle (cf. [Dég18, §1.3]).

Theorem 2.5. Let \( (E, c_1) \) be an oriented absolute ring spectrum and let \( M \) be an absolute \( E \)-module.

The direct image in \( M \)-cohomology defined in Definition 2.2 between smooth \( S \)-schemes satisfies:

1. **Functoriality:** for any proper morphisms \( Z \xrightarrow{u} Y \xrightarrow{f} X \),
   \[ (fg)_* = f_* g_* . \]

2. **Normalization:** for any closed immersion \( i : H \to X \) of codimension one,
   \[ i_*(a) = i_b(a \cdot c_1^H(L_H)) , \]
   where \( i_b \) is the morphism of forgetting support (cf. Remark 1.3).

3. **Key formula:** for any closed immersion \( i : Z \to X \) of codimension \( d \),
   \[ \pi^* i_*(a) = j_*(c_{d-1}(K) \cdot \pi^*(a)) , \]
   where \( \pi^* : B_Z X \to X \) is the blowing-up of \( Z \) in \( X \) with exceptional divisor
   \( j : P(N_{Z/X}) \to B_Z X \).

4. **Projection formula:** for any proper morphism \( f : Y \to X \) and any elements \( b \in E(X) \) and \( m \in M(Y) \), it holds that
   \[ f_*(f^*(b) \cdot m) = b \cdot f_*(m) . \]
   (An analogous formula holds for general projective morphisms; cf. Corollary 2.6.)

---

2The key formula also holds for general projective morphisms; cf. Corollary 2.6.
Proof. Functoriality is evident from the definition.

Both normalization and the projection formula follow from the comparison made in Remark 2.3. Indeed, for a closed immersion \(i: Z \to X\), the morphism \(E(\text{Th}(-T_Z)) \to E(\text{Th}(-T_X))\) maps \(a \cdot t^{\text{ref}}(-T_Z)\) into \(i_*(a \cdot t^{\text{ref}}(N_{Z/X})) \cdot t^{\text{ref}}(-T_X)\) (cf. [Dég18, 2.4.8]). Hence

\[
i_*(1) = i_*(t^{\text{ref}}(N_{Z/X})).
\]

In the case of codimension one, one can explicitly compute that \(t^{\text{ref}}(N_{H/X}) = c^H(I_H)\), where \(I_H\) is the sheaf of ideals defining \(H\) (cf. [Nav16b, 2.19]). In addition, formula (7) also shows that our direct image coincides with that of [Dég18, 2.3.1]. Therefore it also satisfies the key formula (cf. [Dég18, 2.4.2]).

Finally, the projection formula is equivalent to its analogue in Borel-Moore homology. To be precise, we have to check that for \(b \in E^{p,q}(X)\) and \(m \in M_{r,s}^{BM}(Y)\),

\[
f_*(f^*(b) \cdot m) = b \cdot f_*(m).
\]

Let \(\pi_X: X \to S\) and \(\pi_Y: Y \to S\) be the structural morphisms. Without loss of generality, assume that \(p = q = r = s = 0\). By definition, the left hand side of the equation is the adjoint to the morphism

\[
\pi_X|\pi_X^{-1}S \xrightarrow{ad} \pi_Y!\pi_Y^*|\pi_X^{-1}S \xrightarrow{f^*(b)} \pi_Y!E_Y \simeq E_S \land \pi_Y!(\mathbb{1}_Y) \xrightarrow{(1^\land m)} E_S \land M_S \xrightarrow{\mu} M_S.
\]

The right hand side is the adjoint of

\[
\pi_X|\pi_X^{-1}S \xrightarrow{\pi_X^{-1}(b)} \pi_X|E_X \simeq E_S \land \pi_X|\pi_X^{-1}S \xrightarrow{1^\land ad} E_S \land \pi_Y!(\mathbb{1}_Y) \xrightarrow{1^\land m} E_S \land M_S \xrightarrow{\mu} M_S.
\]

These two morphisms agree due to the same argument of [Dég18, 1.2.10.E7], replacing inverse images by exceptional images and using the analogous compatibility.

\[\square\]

Corollary 2.6. Let \((E, c_1)\) be an oriented absolute ring spectrum and let \(f: Y \to X\) be a projective morphism between smooth \(S\)-schemes.

The direct image \(f_*: E(Y) \to E(X)\) defined in Definition 2.2 coincides with that of [Dég18, 3.2.6] and [Nav16b, 2.32].

Proof. These three definitions satisfy the conditions of the uniqueness result from [Nav16b, 2.34], restricted to smooth \(S\)-schemes. The result follows from the same arguments of the cited result.

\[\square\]

A direct image for the cohomology defined by absolute modules over oriented absolute ring spectra was defined in [Nav16b, §2]. For the sake of completeness, let us prove here a uniqueness result.

Theorem 2.7. Let \((E, c_1)\) be an oriented absolute ring spectrum and let \(M\) be an absolute \(E\)-module.

There exists a unique way of assigning, for any projective morphism \(f: Y \to X\) between smooth \(S\)-schemes, a group morphism \(f_*: M(Y) \to M(X)\) satisfying the following properties:

1. **Functoriality**: for any projective morphisms \(Z \xrightarrow{g} Y \xrightarrow{f} X\), \((fg)_* = f_*g_*\).
2. **Normalization**: for any closed immersion \(i: H \to X\) of codimension one, \(i_*(a) = i_*(a \cdot c^H_1(L_H))\).
(3) **Key formula**: for any closed immersion \( i: Z \to X \) of codimension \( n \),
\[
\pi^* i_*(a) = j_*(c_{n-1}(K) \cdot \pi'^*(a)),
\]
where \( \pi^*: B_Z X \to X \) is the blowing-up of \( Z \) in \( X \) with exceptional divisor
\( j: \mathbb{P}(N_{Z/X}) \to B_Z X \).

(4) **Projection formula**: for any projective morphism \( f: Y \to X \) and any
elements \( b \in \mathbb{E}(X) \) and \( m \in \mathbb{M}(Y) \), it holds that
\[
f_*(f^*(b) \cdot m) = b \cdot f_*(m).
\]

**Proof.** Any projective morphism \( f: Y \to X \) factors as the composition of a closed
embedding \( i: Y \to \mathbb{P}_X^n \) and a projection \( p: \mathbb{P}_X^n \to X \).

Properties (1), (2), and (3) characterize the direct image for closed immersions. The projection formula characterizes the direct image for a projection \( p \) due to Lemma 1.8

**Remark 2.8.** A uniqueness result for direct images of proper morphisms is also true, due to work in progress of F. Dégilde. However, such a statement requires us to take into account Borel-Moore homology, and therefore the framework of bivariant theories is more convenient (cf. [FM81]).

2.1. **Direct image on \( K \)-theory.** Let us prove that the direct image for proper
morphisms of Definition 2.2 coincides, when applied to the spectrum \( \mathrm{KGL} \) and
regular schemes, with the usual definition of direct image for higher \( K \)-theory (cf. [TT90, 3.16.4]).

To this end, let us firstly introduce a general definition.

**Definition 2.9.** Let \((E, c_1)\) be an oriented absolute ring spectrum and let \( \mathbb{M} \) be an
absolute \( E \)-module. We define the inverse image, in the Borel-Moore \( \mathbb{M} \)-homology,
of a smooth \( S \)-morphism \( f: Y \to X \) to be
\[
\mathbb{M}_{p, q}^\mathrm{BM}(X) \simeq \mathbb{M}^{-p+2n, -q+n}(X) \xrightarrow{f^*} \mathbb{M}^{-p+2n, -q+n}(Y) \simeq \mathbb{M}_{p+2d, q+n}^\mathrm{BM}(Y),
\]
where \( n = \dim_S X \) and \( d = n - \dim_S Y \).

Let us recall a recent construction by Jin of a spectrum which represents the \( G \)-theory of a scheme and its comparison result with the \( K \)-theory spectrum (cf. [Jin16b], [Jin17]).

**Theorem 2.10** (Jin). Let \( S \) be a finite dimensional noetherian scheme.

1. There exists a spectrum \( \mathrm{GGL}_S \in \mathrm{SH}(S) \) representing Thomason’s \( G \)-theory. In other words, for any smooth \( S \)-scheme \( X \) and any \( n \in \mathbb{Z} \) there exist canonical isomorphisms
\[
G_n(X) = \mathrm{Hom}_{\mathrm{SH}(S)}(X[n], \mathrm{GGL}_S).
\]

2. If \( S \) is regular and \( \pi: X \to S \) is a separated morphism of finite type, there are canonical isomorphisms
\[
\mathrm{GGL}_S \simeq \mathrm{KGL}_S, \quad \mathrm{GGL}_X \simeq \pi^! \mathrm{KGL}_S
\]
compatible with the proper covariance of \( G \)-theory and smooth contravariance of \( G \)-theory (cf. [TT90, 3.14.1]). In other words, for \( f: T \to S \) a
smooth morphism of relative dimension \( d \), \( p : Y \to X \) a proper morphism, and \( n, m \in \mathbb{Z} \), the following squares commute:

\[
\begin{array}{ccc}
KGL_{n,m}^BM(Y) & \xrightarrow{\sim} & G_{n-2m}(Y) \\
p_* & & \downarrow \ \\
KGL_{n,m}^BM(X) & \xrightarrow{\sim} & G_{n-2m}(X),
\end{array}
\]

\[
\begin{array}{ccc}
KGL_{n+2d,m+d}^BM(T) & \xrightarrow{\sim} & G_{n-2m}(T) \\
f^* & & \downarrow f^* \\
KGL_{n,m}^BM(S) & \xrightarrow{\sim} & G_{n-2m}(S).
\end{array}
\]

If \( X \) is a smooth \( S \)-scheme, the isomorphism of (6) together with Jin’s result \((\pi^!KGL_S \simeq GGL_X)\) allows us to define an isomorphism

\[
\begin{array}{ccc}
KGL^BM(X) & \xrightarrow{\sim} & G(X) \\
\phi & & \downarrow \ \\
\end{array}
\]

On the other hand, let us recall that for every scheme \( X \) there is a natural map \( \Phi : K(X) \to G(X) \), which is an isomorphism whenever \( X \) is regular.

**Proposition 2.11.** If \( X \) is a smooth \( S \)-scheme, then the two isomorphisms \( \varphi, \Phi : K(X) \to G(X) \) considered above coincide.

**Proof.** Denote by \( \eta_f \in KGL^BM(X) \) the image of \([\mathcal{O}_X] \otimes t_{\text{ref}}(-T_X) \in K(X) \otimes K(\text{Th}(-T_X))\) through the isomorphism \( (6) \). Since both \( \Phi \) and \( \varphi \) are morphisms of \( K(X) \)-modules (in the classical sense), we only need to check that the image of \( \eta_f \) through the second isomorphism of the construction of \( \varphi \) is \([\mathcal{O}_X] \in G(X)\).

Let \( \pi : X \to S \) be the structural map. The compatibility with smooth contravariance of Theorem 2.10 amounts to the commutativity of both squares of the diagram

\[
\begin{array}{ccc}
KGL^BM(X) & \xleftarrow{\sim} & K(X) \otimes K(\text{Th}(X)) \\
\pi^* & & \downarrow \pi^* \\
KGL^BM(S) & \xrightarrow{\sim} & K(S) \xrightarrow{\sim} G(S).
\end{array}
\]

Chasing \([\mathcal{O}_S] \in K(S)\) along that diagram allows us to conclude. \( \square \)

Let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes. Using the notation introduced above, the direct image \( f_* : K(Y) \to K(X) \) defined by Thomason in \([TT90\text{ 3.16.4}]\) can be described as the composition

\[
K(Y) \xrightarrow{\Phi} G(Y) \xrightarrow{\Phi^{-1}} K(X).
\]

**Corollary 2.12.** Let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes. The direct image \( f_* : K(Y) \to K(X) \) of Definition 2.2 coincide with the direct image \( f_* : K(Y) \to K(X) \) of \([TT90\text{ 3.16.4}]\).³

³We warn the reader that the direct image \( f_* \) is denoted by \( f_* \) in \([TT90\).
Proof. The statement follows from the commutativity of the following diagram:

\[
\begin{array}{ccc}
K(Y) & \xrightarrow{f_*} & K(X) \\
\sim & & \sim \\
\Phi & \downarrow & \Phi^{-1} \\
KGL^{BM}(Y) & \xrightarrow{\sim} & KGL^{BM}(X) \\
\sim & & \sim \\
G(Y) & \xrightarrow{\sim} & G(X)
\end{array}
\]

3. Riemann-Roch theorems

Let \((E, c_1)\) and \((F, c_1)\) be oriented absolute ring spectra. To avoid confusion, we overline notation to refer to elements and morphisms in the \(F\)-cohomology.

3.1. If \(\varphi: (E, c_1) \to (F, c_1)\) is a morphism of oriented absolute ring spectra, then

\[\varphi(c_1) = G(\bar{c}_1) \cdot \bar{c}_1\]

for a unique invertible series \(G(t) \in (\mathbb{F}(S)[[t]])^* = \mathbb{F}(\mathbb{P}^\infty_F)^*\) (Nav16b, 1.34).

Let us write \(G_X\) for the multiplicative extension of \(G(t)\). To be more precise, if \(V \to X\) is a vector bundle, the splitting principle assures the existence of a base change \(\pi: X' \to X\),injective in cohomology, such that \(\pi^*V = L_1 + \cdots + L_r\) is a sum of line bundles in \(K_0(X')\) (cf. Nav16b, 1.35). We use the notation

\[G_X(V) := G(L_1) \cdot \ldots \cdot G(L_r) \in \mathbb{F}(X),\]

where \(G(L)\) stands for \(G(c_1(L))\).

Remark 3.2. Let \(\varphi: (E, c_1) \to (F, c_1)\) be a morphism of oriented absolute ring spectra and let \(G \in \mathbb{F}[S[[t]]\) be the unique series such that \(\varphi(c_1) = G(\bar{c}_1) \cdot \bar{c}_1\).

If \(i: Z \to X\) is a closed immersion and \(N_{Z/X}\) stands for its normal bundle, then Déglise (Dég18, 4.2.3) proved that, for any \(a \in E(Z)\), the following Riemann-Roch formula holds:

\[\varphi(p_i(a)) = p_i(G_X^{-1}(-N_{Z/X}) \cdot \varphi(a))\] (8)

To be more precise, the following lemma proves the formula we will require later.

Lemma 3.3. Let \(\varphi: (E, c_1) \to (F, c_1)\) be a morphism of oriented absolute ring spectra such that \(\varphi(c_1) = G(\bar{c}_1) \cdot \bar{c}_1\).

If \(V \to X\) is a vector bundle, the following equality holds on \(\mathbb{F}(\text{Th}(V))\):

\[\varphi(t^{\text{ref}}(V)) = G_X^{-1}(-V) \cdot t^{\text{ref}}(V)\]

Proof. If \(s: X \to V\) denotes the zero section, then \(p_s(1) = t^{\text{ref}}(V) \in \mathbb{E}(\text{Th}(V))\) (cf. equation (4) and Remark 2.4). Moreover, \(s\) is a closed immersion whose normal bundle equals \(V\); thus, the Riemann-Roch formula (8) unfolds in this case into

\[\varphi(t^{\text{ref}}(V)) = \varphi(p_s(1)) \equiv p_s(G_X^{-1}(-V) \varphi(1)) = p_s(G_X^{-1}(-V)) = G_X^{-1}(-V) t^{\text{ref}}(V)\]

\(\square\)
Theorem 3.4. Let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes. Let \( \varphi : (E, c_1) \to (F, c_1) \) be a morphism of oriented absolute ring spectra such that \( \varphi(c_1) = G(c_1) \cdot \tilde{c}_1 \) and let \( G^{-1}_x \) stand for the multiplicative extension of \( G^{-1} \in \mathbb{F}(S)[[t]] \).

The following square commutes:

\[
\begin{array}{c}
\mathbb{E}(Y) \xrightarrow{f_*} \mathbb{E}(X) \\
\downarrow \mathbb{F}(T_Y) \varphi \downarrow \mathbb{F}(T_X) \varphi \\
\mathbb{F}(Y) \xrightarrow{f_*} \mathbb{F}(X).
\end{array}
\]

Proof. The statement follows from the commutativity of the following diagram:

\[
\begin{array}{c}
\mathbb{E}(Y) \xrightarrow{\sim} \mathbb{E}(\text{Th}(-T_Y)) \simeq \mathbb{E}^\text{BM}(-n,-q-n) \xrightarrow{f_*} \mathbb{E}^\text{BM}(X) = \mathbb{E}(\text{Th}(-T_X)) \xrightarrow{\sim} \mathbb{E}(X) \\
\downarrow \mathbb{F}^{-1}(T_Y) \varphi \downarrow \mathbb{F}(T_X) \varphi \\
\mathbb{F}(Y) \xrightarrow{\sim} \mathbb{F}(\text{Th}(-T_Y)) \simeq \mathbb{F}^\text{BM}(Y) \xrightarrow{f_*} \mathbb{F}^\text{BM}(X) \simeq \mathbb{F}(\text{Th}(-T_X)) \xrightarrow{\sim} \mathbb{F}(X).
\end{array}
\]

Let us check each of the three squares separately.

For any element \( a \in \mathbb{E}^{p,q}(Y) \), we have

\[
a \cdot \mathbf{t}^\text{ref}(-T_Y) \in \mathbb{E}(\text{Th}(-T_Y)) \simeq \mathbb{E}^{BM}_{-p-2n,-q-n}(Y).
\]

Since \( \varphi \) is a morphism of rings and due to the previous lemma,

\[
\varphi(a \cdot \mathbf{t}^\text{ref}(-T_Y)) = \varphi(a) \cdot \mathbf{t}^\text{ref}(-T_Y) = \varphi(a) \cdot G^{-1}_x(T_Y) \cdot \tilde{t}^\text{ref}(-T_Y).
\]

Hence, if \( n = \dim_S Y \), the following diagram commutes:

\[
\begin{array}{c}
\mathbb{E}^{p,q}(Y) \xrightarrow{\sim} \mathbb{E}^{p+2n,q+n}(\text{Th}(-T_Y)) \simeq \mathbb{E}^{BM}_{-p-2n,-q-n}(Y) \\
\downarrow \mathbb{F}^{-1}(T_Y) \varphi \downarrow \mathbb{F}(T_X) \varphi \\
\mathbb{F}^{p,q}(Y) \xrightarrow{\sim} \mathbb{F}^{p+2n,q+n}(\text{Th}(-T_Y)) \simeq \mathbb{F}^{BM}_{-p-2n,-q-n}(Y).
\end{array}
\]

Analogously, if \( d = n - \dim_S X \) the diagram

\[
\begin{array}{c}
\mathbb{E}^{BM}_{-p-2n,-q-n}(X) \simeq \mathbb{E}^{p+2n,q+n}(\text{Th}(-T_X)) \xrightarrow{\sim} \mathbb{E}^{p+2d,q+d}(X) \\
\downarrow \mathbb{F}(T_X) \varphi \\
\mathbb{F}^{BM}_{-p-2n,-q-n}(X) \simeq \mathbb{F}^{p+2n,q+n}(\text{Th}(-T_X)) \xrightarrow{\sim} \mathbb{F}^{p+2d,q}(X)
\end{array}
\]

commutes.

Finally, the following square also commutes:

\[
\begin{array}{c}
\mathbb{E}^{BM}_{-p-2n,-q-n}(Y) \xrightarrow{f_*} \mathbb{E}^{BM}_{-p-2n,-q-n}(X) \\
\downarrow \varphi \\
\mathbb{F}^{BM}_{-p-2n,-q-n}(Y) \xrightarrow{f_*} \mathbb{F}^{BM}_{-p-2n,-q-n}(X)
\end{array}
\]

because direct image in Borel-Moore homology is defined out of the adjunction \( f_! f^* \to \text{Id} \), which is a natural transformation and hence compatible with morphisms of spectra. \( \square \)
The analogous statement for absolute modules is a consequence of the theorem above.

**Corollary 3.5.** Let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes.

Let \( \varphi : (E, c_1) \to (F, \bar{c}_1) \) be a morphism of oriented absolute ring spectra such that \( \varphi(c_1) = G(\bar{c}_1) \cdot \bar{c}_1 \in F(\mathbb{P}^\infty_S) \). Also, let \( \Phi : M \to \bar{M} \) be a \( \varphi \)-morphism of absolute modules.

The following diagram commutes:

\[
\begin{array}{ccc}
M(Y) & \xrightarrow{f^*} & M(X) \\
G^{-1}_\infty(T_Y) \Phi & \downarrow & G^{-1}_\infty(T_X) \Phi \\
M(Y) & \xrightarrow{f^*} & M(X)
\end{array}
\]

**Riemann-Roch for \( K \)-theory and motivic cohomology.** Let \( \text{KGL}_Q \) be the \( Q \)-localization of the \( K \)-theory ring spectrum and let \( H_B \) be the Beilinson’s motivic cohomology spectrum (cf. Example 1.2).

The higher Chern character \( \text{ch} : \text{KGL}_Q \to \bigoplus_i H_B[2i](i) \) is an isomorphism of absolute ring spectra (cf. [Ri10, 6.2.3.9]). It does not preserve orientations but satisfies

\[
\text{ch}(c_1) = \frac{1 - e^{\bar{c}_1}}{\bar{c}_1}.
\]

Consequently, let us consider the inverse of this series, called the **Todd series**:

\[
\text{Td}(t) := \left( \frac{1 - e^t}{t} \right) = \frac{t}{1 - e^t}.
\]

**Corollary 3.6.** Let \( S \) be the spectrum of a Dedekind domain and let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes.

The following diagram, relating higher \( K \)-theory and higher Chow groups with rational coefficients, commutes:

\[
\begin{array}{ccc}
K(Y)_Q & \xrightarrow{f^*} & K(X)_Q \\
\text{Td}(T_Y) \text{ch} & \downarrow & \text{Td}(T_X) \text{ch} \\
CH(Y)_Q & \xrightarrow{f^*} & CH(X)_Q
\end{array}
\]

**Remark 3.7.** To our knowledge, this is the first Riemann-Roch statement involving higher \( K \)-theory that does not assume any projective hypotheses (cf. [Gil15], [Dég18], [HS15], [Nav16b]).

Regarding Grothendieck-Riemann-Roch statements involving the \( K_0 \) group, the most general formula without projective assumptions we know, due to Fulton and Gillet ([FG83]), applies to schemes defined over a field.

Theorem 3.5 also applies to the modules we described in Example 1.2. In particular, we obtain the following result:

**Corollary 3.8.** Let \( S \) be a finite dimensional noetherian scheme and let \( f : Y \to X \) be a proper morphism between smooth \( S \)-schemes. Let \( g : T \to X \) be a morphism of schemes and let us write \( g_Y : T \times_X Y \to Y \) for the induced map.
If either $g$ is proper or $f$ is smooth, then the following diagram commutes:

$$
\begin{array}{c}
KH(g_Y)_\mathbb{Q} \xrightarrow{f_*} KH(g)_\mathbb{Q} \\
\text{Td}(T_Y)\text{ch} \downarrow \quad \downarrow \text{Td}(T_X)\text{ch} \\
H_M(g_Y, \mathbb{Q}) \xrightarrow{f_*} H_M(g, \mathbb{Q}).
\end{array}
$$

Arithmetic Riemann-Roch. Let $\widehat{\text{KGL}}$ and $\widehat{H}_B$ be the arithmetic $K$-theory and motivic cohomology ring spectra defined by Holmstrom-Scholbach in [HS15] (see Example 1.4). Let us also consider the arithmetic Chern character (cf. [HS15, 4.2], [Nav16b, §1.2]), which is a ch-morphism of modules:

$$
\widehat{\text{ch}} : \widehat{\text{KGL}} \to \widehat{H}_B.
$$

As a consequence of Corollary 3.5, there also follows an arithmetic formula:

**Corollary 3.9.** Let $f : Y \to X$ be a proper morphism between smooth schemes over an arithmetic ring.

The following diagram commutes:

$$
\begin{array}{c}
\widehat{KH}(Y)_\mathbb{Q} \xrightarrow{f_*} \widehat{KH}(X)_\mathbb{Q} \\
\text{Td}(T_Y)\widehat{\text{ch}} \downarrow \quad \downarrow \text{Td}(T_X)\widehat{\text{ch}} \\
\widehat{H}_M(Y, \mathbb{Q}) \xrightarrow{f_*} \widehat{H}_M(f, \mathbb{Q}).
\end{array}
$$

**Acknowledgments**

The authors would like to thank J. Ayoub, J. I. Burgos Gil, F. Déglise, H. Gillet, and F. Jin for many helpful discussions and comments.

**References**

[Ayo07] Joseph Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I* (French, with English and French summaries), Astérisque 314 (2007), x+466 pp. (2008). MR2423375; II 315, (2007), vi+364 pp. (2008). MR2438151

[BFM75] Paul Baum, William Fulton, and Robert MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. 45 (1975), 101–145. MR412190

[Blo86] Spencer Bloch, *Algebraic cycles and higher $K$-theory*, Adv. in Math. 61 (1986), no. 3, 267–304, DOI 10.1016/0001-8708(86)90081-2. MR852815

[BS58] Armand Borel and Jean-Pierre Serre, *Le théorème de Riemann-Roch* (French), Bull. Soc. Math. France 86 (1958), 97–136. MR116022

[Cis13] Denis-Charles Cisinski, *Descente par éclatements en $K$-théorie invariante par homotopie* (French, with English and French summaries), Ann. of Math. (2) 177 (2013), no. 2, 425–448, DOI 10.4007/annals.2013.177.2.2. MR3010804

[CD19] Denis-Charles Cisinski and Frédéric Déglise, *Triangulated categories of mixed motives*, Springer Monographs in Mathematics, Springer, Cham, [2019] ©2019. MR3971240

[CD12] Denis-Charles Cisinski and Frédéric Déglise, *Mixed Weil cohomologies*, Adv. Math. 230 (2012), no. 1, 55–130, DOI 10.1016/j.aim.2011.10.021. MR2900540

[Dég08] Frédéric Déglise, *Around the Gysin triangle. II*, Doc. Math. 13 (2008), 613–675. MR2466188

[Dég18] Frédéric Déglise, *Orientation theory in arithmetic geometry. K-Theory—Proceedings of the International Colloquium, Mumbai, 2016*, Hindustan Book Agency, New Delhi, 2018, pp. 239–347. MR3930052
[Dég18b] Frédéric Déglise, Bivariant theories in motivic stable homotopy, Doc. Math. 23 (2018), 997–1076. MR3874952

[Del87] P. Deligne, Le déterminant de la cohomologie (French), Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 93–177, DOI 10.1090/comm/067/902592. MR902592

[Dye62] E. Dyer, Relations between cohomology theories, Colloquium on Algebraic Topology, Aarhus University, 1962, pp. 89–93.

[FG83] William Fulton and Henri Gillet, Riemann-Roch for general algebraic varieties (English, with French summary), Bull. Soc. Math. France 111 (1983), no. 3, 287–300. MR735307

[Ful77] William Fulton, A Hirzebruch-Riemann-Roch formula for analytic spaces and non-projective algebraic varieties, Compositio Math. 34 (1977), no. 3, 279–283. MR460323

[Ful98] William Fulton, Intersection theory, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323

[FM81] William Fulton and Robert MacPherson, Categorical framework for the study of singular spaces, Mem. Amer. Math. Soc. 31 (1981), no. 243, vi+165, DOI 10.1090/memo/0243. MR609831

[ Gil81] Henri Gillet, Comparison of K-theory spectral sequences, with applications, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), Lecture Notes in Math., vol. 854, Springer, Berlin-New York, 1981, pp. 141–167, DOI 10.1007/BFb0089520. MR618303

[ Gil81b] Henri Gillet, Riemann-Roch theorems for higher algebraic K-theory, Adv. in Math. 40 (1981), no. 3, 203–289, DOI 10.1016/S0001-8708(81)80006-0. MR624666

[HS15] Andreas Holmstrom and Jakob Scholbach, Arakelov motivic cohomology I, J. Algebraic Geom. 24 (2015), no. 4, 719–754, DOI 10.1090/jag/648. MR3383602

[Jin16] Jin Fangzhou, Bord-Moore motivic homology and weight structure on mixed motives, Math. Z. 283 (2016), no. 3-4, 1149–1183, DOI 10.1007/s00209-016-1363-7. MR3519998

[Jin16b] F. Jin, Quelques aspects sur l’homologie de Bord-Moore dans le cadre de l’homotopie motivique : poids et G-théorie de Quillen, PhD Dissertation, E.N.S. de Lyon, 2016.

[Jin17] F. Jin, Algebraic G-theory in motivic homotopy categories, arXiv:1806.03927, 2018.

[Lev] M. Levine, K-theory and motivic cohomology of schemes, I, https://www.uni-duis.de/~bm0032/publ/RthoMotI12.01.pdf.

[Lev01] Marc Levine, Techniques of localization in the theory of algebraic cycles, J. Algebraic Geom. 10 (2001), no. 2, 299–363. MR1811558

[LM07] M. Levine and F. Morel, Algebraic cobordism, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR2286826

[MV99] Fabien Morel and Vladimir Voevodsky, A1-homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45–143 (2001). MR1813224

[Nav81] J. A. Navarro González, Cálculo de las clases de Chern de los esquemas lisos y singulares, PhD Dissertation, Universidad de Salamanca, 1981.

[Nav16] Alberto Navarro, On Grothendieck’s Riemann-Roch theorem, Expo. Math. 35 (2017), no. 3, 326–342, DOI 10.1016/j.exmath.2016.09.005. MR3689905

[Nav16b] A. Navarro, Riemann-Roch for homotopy invariant K-theory and Gysin morphisms, Adv. Math. 328 (2018), 501–554, DOI 10.1016/j.aim.2018.01.001. MR3771136

[Pan03] I. Panin, Oriented cohomology theories of algebraic varieties, K-Theory 30 (2003), no. 3, 265–314, DOI 10.1023/B:KTHE.0000019788.33790.cb. MR2064242

[Pan04] I. Panin, Riemann-Roch theorems for oriented cohomology, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 261–333, DOI 10.1007/978-94-007-0948-5_8. MR2061857

[Pan09] Ivan Panin, Oriented cohomology theories of algebraic varieties. II (After I. Panin and A. Smirnov), Homology Homotopy Appl. 11 (2009), no. 1, 349–405. MR2529164

[PPR08] Ivan Panin, Konstantin Pimenov, and Oliver Röndigs, A universality theorem for Voevodsky’s algebraic cobordism spectrum, Homology Homotopy Appl. 10 (2008), no. 2, 211–226. MR2475610

[PPR09] Ivan Panin, Konstantin Pimenov, and Oliver Röndigs, On the relation of Voevodsky’s algebraic cobordism to Quillen’s K-theory, Invent. Math. 175 (2009), no. 2, 435–451, DOI 10.1007/s00222-008-0155-5. MR2470112
[Rio10] Joël Riou, *Algebraic K-theory, $\mathbb{A}^1$-homotopy and Riemann-Roch theorems*, J. Topol. 3 (2010), no. 2, 229–264, DOI 10.1112/jtopol/jtq005. MR2651359

[SGA4] M. Artin, A. Grothendieck, J-L Verdier, eds: *Séminaire de Géométrie Algébrique du Bois Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas* - (SGA 4) - vol. 3 (PDF). Lecture Notes in Mathematics (in French). 305. Berlin; New York: Springer-Verlag. pp. vi+640 (1972).

[SGA6] P. Berthelot, A. Grothendieck, L. Illusie: *Séminaire de Géométrie Algébrique du Bois Marie — 1966–67 — Théorie des intersections et théorème de Riemann-Roch — (SGA 6)*. Lecture notes in mathematics, vol. 225 (1971)

[Spi12] Markus Spitzweck, *A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. (N.S.) 157 (2018), 110, DOI 10.24033/msmf.465. MR3865569

[TT90] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435, DOI 10.1007/978-0-8176-4576-2_10. MR1106918

[Voe98] Vladimir Voevodsky, *$\mathbb{A}^1$-homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), Doc. Math. Extra Vol. I (1998), 579–604. MR1648048

Departamento de Matemáticas Aplicada, E. T. S. de Arquitectura, Universidad Politécnica de Madrid, 28040 Madrid, Spain

Departamento de Matemáticas, Universidad de Extremadura, 06006 Badajoz, Spain