Quantum algorithms for formula evaluation

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Abstract. We survey the recent sequence of algorithms for evaluating Boolean formulas consisting of NAND gates.

Keywords. Quantum computing, quantum algorithms, quantum walks, formula evaluation.

1. Introduction

One of the most famous quantum algorithms is Grover’s search [20] which can search among \( N \) possibilities in \( O(\sqrt{N}) \) steps. This provides a quadratic speedup over the naive classical algorithm for a variety of search problems [4].

Grover’s algorithm can be recast as computing OR of \( N \) bits \( x_1, \ldots, x_N \), with \( O(\sqrt{N}) \) queries to a black box storing \( x_1, \ldots, x_N \). A natural generalization of this problem is computing the value of an AND-OR formula of \( x_1, \ldots, x_N \).

Grover’s algorithm easily generalizes to computing AND-OR formulas of small depth \( d \). Then, \( O(\sqrt{N} \log^{d-1} N) \) queries are sufficient to evaluate the formula [11]. For balanced formulas (with each AND and OR in the formula having the same fan-in), this can be improved to \( O(\sqrt{N}) \) [21] which is optimal [2].

A different case is when, instead of a constant depth, we have a constant fan-in. This case has been much harder and, until 2007, there has been no progress on it at all. If we restrict to binary AND-OR trees, the classical complexity of evaluating a full binary AND-OR tree is \( \Theta(N^{7.54}) \) [25,24,31] and there was no better quantum algorithm known.

In a breakthrough result, Farhi et al. [19] showed that the full binary AND-OR tree can be evaluated in \( O(\sqrt{N}) \) quantum time in an unconventional continuous-time Hamiltonian query model of [18,23].

Several improvements followed soon. Ambainis et al. [15,6,16,7] translated the algorithm of [19] to the conventional discrete time quantum query model and extending it to evaluating arbitrary Boolean formulas in \( O(N^{1/2+o(1)}) \) steps. Reichardt and Špalek [30,29,27] then further extended the algorithm to evaluating span programs, a generalization of Boolean logic formulae. This resulted in a surprising result: Reichardt [26] showed that the span-program based approach...
gave nearly-optimal query algorithms for any Boolean function. Also using the span program approach, Reichardt [28] gave a better formula evaluation algorithm, which can evaluate any Boolean formula in $O(\sqrt{N \log N})$ steps (instead of $O(N^{1/2+o(1)})$ in [7]).

In this paper, we give a simple description of the basic technical ideas behind this sequence of quantum algorithms, by describing how the algorithms of [19, 15, 6, 16, 7] work for the simplest particular case - the full binary tree. Besides the two published algorithms [19, 7], we also describe two intermediate versions which appeared in the technical reports [15, 6].

2. Technical preliminaries

2.1. The problem and motivation

We consider evaluating a Boolean formula of variables $x_1, \ldots, x_N$ consisting of ANDs and ORs, with each variable occurring exactly once in the formula. Such a formula can be described by a tree, with variables $x_i$ at the leaves and AND/OR gates at the internal nodes. This problem has many applications because Boolean formulas can be used to describe a number of different situations. The most obvious one is determining if the input data $x_1, \ldots, x_N$ satisfy certain constraints that can be expressed by AND/OR gates.

For a less obvious application, we can view formula evaluation as a black-box model for a 2-player game (such as chess) if both players play their optimal strategies. In this case, the game can be represented by a game tree consisting of possible positions. The leaves of a tree correspond to the possible end positions of the game. Each of them contains a variable $x_i$, with $x_i = 1$ if the 1st player wins and $x_i = 0$ otherwise. Internal nodes corresponding to positions which the 1st player makes the next move contain a value that is OR of the values of their children. (The 1st player wins if he has a move that leads to a position from which he can win.) Internal nodes for which the 2nd player makes the next move contain a value that is AND of the values of their children. (The 1st player wins if he wins for any possible move of the 2nd player.)

The question is: assuming we have no further information about the game beyond the position tree, how many of the variables $x_i$ do we have to examine to determine whether the 1st player has a winning strategy?

2.2. The model

By standard rules from Boolean logic (de Morgan’s laws), we can replace both AND and OR gates by NAND gates. A NAND gate $NAND(y_1, \ldots, y_k)$ outputs 1 if $AND(y_1, \ldots, y_k) = 0$ (i.e., $y_i = 0$ for at least one $i \in \{1, \ldots, k\}$) and 0 otherwise. Then, we have a tree with $x_1, \ldots, x_N$ at the leaves and NAND gates at the internal vertices. The advantage of this transformation is that we now have to deal with just one type of logic gates (instead of two - AND and OR).

We work in the quantum query model. In the discrete-time version of this model [4, 12], the input bits $x_1, \ldots, x_N$ can be accessed by queries $O$ to a black box.
To define $O$, we represent basis states as $|i, z\rangle$ where $i \in \{0, 1, \ldots, N\}$. The query transformation $O_x$ (where $x = (x_1, \ldots, x_N)$) maps $|0, z\rangle$ to $|0, z\rangle$ and $|i, z\rangle$ to $(-1)^{x_i}|i, z\rangle$ for $i \in \{1, \ldots, N\}$ (i.e., we change phase depending on $x_i$, unless $i = 0$ in which case we do nothing).

Our algorithm can involve queries $O_x$ and arbitrary non-query transformations that do not depend on $x_1, \ldots, x_N$. The task is to solve a computational problem (e.g., to compute a value of a NAND formula) with as few queries as possible.

In the continuous time Hamiltonian model (first considered by Farhi and Gutman [18]), instead of unitary oracle $O_x$, we have a Hamiltonian oracle $H_x$. We can define $H_x$ as [17]

$$H|i\rangle = x_i|i\rangle$$

where $i$ is a register that can hold values 0, 1, \ldots, $N$. (Similarly to the discrete case, $H_x|0\rangle = |0\rangle$, i.e. we have the option of not querying any $x_i$.) We are allowed to combine $H_x$ with an arbitrary time-dependent Hamiltonian $H_0(t)$ that does not depend on $x_1, \ldots, x_N$. The task is to solve the computational problem by running $H_x$ for as little time as possible.

The continuous time and the discrete query models are roughly equivalent [13, 14, 17]. A discrete query can be simulated using a Hamiltonian oracle. An Hamiltonian query algorithm that uses $T$ queries can be transformed into a discrete query algorithm with $O(T \log T / \log \log T)$ queries [17].

3. Continuous time quantum algorithms

3.1. Farhi et al.: quantum walk on an infinite line

Assume that we have a formula described by a full binary NAND tree of depth $d$. (That is, all variables $x_i$ are at depth $d$. At levels 0, 1, \ldots, $d - 1$, we have NAND gates, each of which evaluates NAND of two gates from the next level.) We augment this tree:

1. For each leaf $v$ that contains a variable $x_i = 1$, we create a new vertex $v'$, with an edge $(v, v')$.
2. We take an infinite line of vertices indexed by integers $x$, with a vertex $x$ connected to vertices $x - 1$ and $x + 1$. We connect the root $r$ of our NAND tree to the vertex 0 on this line.

An example of an augmented tree (for depth $d = 2$) is shown in Figure 1.

We now interpret the adjacency matrix of this augmented tree as a Hamiltonian $H$. As shown by Farhi et al. [19], if we apply this Hamiltonian $H$ for time $O(\sqrt{N})$ to an appropriately chosen starting state

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*A finite segment of the line that is sufficiently long in both directions can be used as a good approximation to the infinite line. But, for simplicity of the presentation, we will assume that an infinite line is being used.*
Figure 1. An tree augmented by an infinite line and extra edges. The numbers at leaves show the values of the variables.

\[ |\psi\rangle = \sum_{i \leq 0} \alpha_i |i\rangle \]

that has non-zero amplitude only at the locations to the left of 0 \((-1, -2, \ldots)\), we get a state \(|\psi'\rangle\) with the following properties:

1. If the tree evaluates to 1, almost of the state \(|\psi'\rangle\) consists of locations to the right of 0;
2. If the tree evaluates to 0, almost of the state \(|\psi'\rangle\) consists of locations to the left of 0;

Thus, by measuring the state \(|\psi'\rangle\), we can determine whether the tree evaluates to 0 or 1. The behaviour of the algorithm can be understood by expressing \(|\psi'\rangle\) as a superposition of the energy eigenstates of the Hamiltonian \(H\). Let \(|\psi_E\rangle\) be an energy eigenstate of \(H\) with an energy \(E = 2\theta\). We express

\[ |\psi_E\rangle = \sum_{n \in \mathbb{Z}} \alpha_n |n\rangle + \sum_{v \in T} \alpha_v |v\rangle, \]

with \(|n\rangle\) being the vertices on the infinite line ( "runway") and \(|v\rangle\) being the vertices in the tree. One can show that the amplitudes of the vertices on the runway are

\[ \alpha_n = \begin{cases} e^{i\theta n} + R(E)e^{-i\theta n} & \text{if } n < 0 \\ T(E)e^{i\theta n} & \text{if } n \geq 0 \end{cases}, \]

(1)

where coefficients \(R(E)\) and \(T(E)\) depend on the energy \(E\) and the structure of the tree (i.e., which leaves of the tree contain \(x_i = 1\) and, therefore, have an extra edge attached to them).

\(R(E)\) and \(T(E)\) are called reflection and transmission coefficients of the tree, by a following physical analogy. We can view tree as an obstacle attached to the runway at \(n = 0\). If we have a particle propagating rightwards from the \(n < 0\) side, the particle may either get reflected back to \(n < 0\) (in which can it starts moving to the left) or it may pass to \(n > 0\) and keep moving to the right. The reflection and the transmission coefficients describe the amplitudes of these two possibilities. We have

**Theorem 1** [19]
1. If $F = 0$, then $T(0) = 0$ and $R(0) = -1$.
2. If $F = 1$, then $T(0) = 1$ and $R(0) = 0$.

Thus, if $F = 0$, we have an eigenstate $|\psi_0\rangle$ which has zero amplitudes for $n \geq 0$. This eigenstate is

$$|\psi_0\rangle = \sum_{k \geq 0} (| -4k \rangle - | -4k - 2 \rangle) + \sum_{v \in T} \alpha_v |v\rangle$$

for some amplitudes $\alpha_v$. If we start in $|\psi_0\rangle$ and apply $H$, the state $|\psi_0\rangle$ stays unchanged. In contrast, if $F = 1$, the same state $|\psi_0\rangle$ is not an eigenstate and applying $H$ for sufficiently long time leads to nonzero amplitudes for $n \geq 0$. Thus, we can distinguish $F = 0$ and $F = 1$ by preparing $|\psi_0\rangle$, applying $H$ and measuring the state. If we find $|n\rangle$, $n \geq 0$, we know that $F = 1$.

A slight complication is that the states $|\psi_0\rangle$ and $|\psi_{\text{start}}\rangle$ has equal amplitudes on infinitely many states $| -4k \rangle$ and $| -4k - 2 \rangle$ and, thus has an infinite norm. This can be avoided by using

$$|\psi'_{\text{start}}\rangle = \frac{1}{\sqrt{2L}} \sum_{k=0}^{L-1} (-1)^k |2k\rangle.$$

This state turns out to be a sufficiently good approximation of $|\psi_{\text{start}}\rangle$.

3.2. Childs et al.: finite segment

A modification of FGG algorithm was proposed by Childs et al. [15].

We can, instead of an infinite line, attach a finite segment of length $2L$ to the root of the tree (see figure 2). The starting state is similar to the infinite line algorithm:

$$|\psi'_{\text{start}}\rangle = \frac{1}{\sqrt{L+1}} \sum_{k=0}^{L} (-1)^k |2k\rangle$$

More precisely, we start in a slightly different state $|\psi_{\text{start}}\rangle = \sum_{k \geq 0} (| -4k \rangle - | -4k - 2 \rangle)$ instead of $|\psi_0\rangle$, since we do not know the amplitudes $\alpha_v$. It can be shown that $|\psi_{\text{start}}\rangle$ is a sufficiently good approximation of $|\psi_0\rangle$. 

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**Figure 2.** A tree augmented by a finite tail and extra edges.
where \( L = O(\sqrt{N}) \). (Here, \(|0\rangle\) denotes the root of the tree and \(|1\rangle, \ldots, |2L\rangle\) denote the vertices in the tail.)

In this case, we have the following behaviour. If \( F = 0 \), the state \(|\psi'_\text{start}\rangle\) remains almost unchanged by \( H \). If \( F = 1 \), then, after \( O(\sqrt{N}) \) steps, the state becomes sufficiently different from \(|\psi'_\text{start}\rangle\). More formally:

**Lemma 1**

1. If \( F = 0 \), then there exists \(|\psi\rangle\) such that \( H|\psi\rangle = 0 \) and 
\[
\langle\psi|\psi'_\text{start}\rangle \geq 1 - \epsilon.
\]
2. If \( F = 1 \), then, for any eigenstate \(|\psi\rangle\) of Hamiltonian \( H \), either the corresponding eigenvalue \( \lambda \) satisfies 
\[
|\lambda| = \Omega(\frac{1}{\sqrt{N}})
\]
or \(|\psi\rangle \perp |\psi_{\text{start}}\rangle\).

Because of that, we can distinguish between the two cases by running the eigenvalue estimation [22][9] on state \(|\psi_{\text{start}}\rangle\), with precision \( \frac{1}{\sqrt{N}} \). If \( F = 0 \), we get the answer \( \lambda = 1 \) with a high probability 1. If \( F = 1 \), we get a value \( \lambda \) with \( |\lambda| \geq \frac{1}{\sqrt{N}} \). To estimate the eigenvalue with precision \( \frac{1}{\sqrt{N}} \), it is sufficient and necessary to run the Hamiltonian \( H \) for time \( O(\sqrt{N}) \).

The computer experiments suggest that we can also distinguish between the two cases simply by running the Hamiltonian for time \( O(\sqrt{N}) \) and measuring the final state. If \( F = 0 \), we find one of basis states \(|2k\rangle\) with a high probability. If \( F = 1 \), we find one of basis states \(|v\rangle\) in the tree.

4. Discrete time algorithms

There are two ways to transform the above algorithm into a discrete time quantum algorithm, discovered independently by Ambainis [6] and Childs et al. [16].

4.1. Discrete time algorithm by eigenspace decomposition

The basic idea behind the algorithm of [6] is as follows. We can decompose the continuous time Hamiltonian \( H \) as \( H = H_{\text{tree}} + H_{\text{input}} \) where \( H_{\text{tree}} \) is the part of \( H \) that is independent of \( x_1, \ldots, x_N \) and \( H_{\text{input}} \) consists of extra edges that are added to the tree if \( x_i = 1 \). To obtain similar behaviour in discrete time, we define two unitary transformations \( U_{\text{tree}} \) and \( U_{\text{input}} \) that correspond to \( H_{\text{tree}} \) and \( H_{\text{input}} \) and then consider \( U = U_{\text{input}}U_{\text{tree}} \).

In the continuous case, we had \( F = 0 \) iff there exists \(|\psi\rangle \approx |\psi_{\text{start}}\rangle\) with 
\[
H|\psi\rangle = H_{\text{tree}}|\psi\rangle + H_{\text{input}}|\psi\rangle = 0.
\]

In the discrete time, this corresponds to \( U|\psi\rangle = U_{\text{tree}}U_{\text{input}}|\psi\rangle = |\psi\rangle \).

To define \( U_{\text{tree}} \) and \( U_{\text{input}} \), we consider a tree that is augmented by a finite tail of length \( 2L \) (as in the previous algorithm) but with no extra edges at the leaves. \( U_{\text{tree}} \) is defined by

\[
U_{\text{tree}}|\psi\rangle = \begin{cases} 
|\psi\rangle & \text{if } H_{\text{tree}}|\psi\rangle = 0 \\
-|\psi\rangle & \text{if } H_{\text{tree}}|\psi\rangle = \lambda |\psi\rangle, \lambda \neq 0
\end{cases}
\]

\( U_{\text{input}} \) is defined as follows. \( U_{\text{input}}|v\rangle = -|v\rangle \) if \( v \) is a leaf containing \( i : x_i = 1 \) and \( U_{\text{input}}|v\rangle = |v\rangle \) if \( v \) is a leaf containing \( i : x_i = 0 \) or if \( v \) is an internal vertex.
Figure 3. A 0-eigenstate of $H$. $c$ is a number that is chosen so that the norm of the state is 1.

If $F = 0$, there is a state $|\psi_0\rangle$ satisfying $U_{\text{tree}}|\psi_0\rangle = U_{\text{input}}|\psi_0\rangle = |\psi_0\rangle$ and $|\psi_0\rangle \approx |\psi_{\text{start}}\rangle$. In figure 3 we show this state for a NAND tree of depth 2, with particular values of variables ($x_1 = x_3 = 1$, $x_2 = x_4 = 0$).

If $F = 1$, there is no such state. To see that, we first notice that $U_{\text{tree}}$ and $U_{\text{input}}$ are both reflections. Therefore, the spectrum of $U_{\text{input}}U_{\text{tree}}$ can be analyzed within the "two reflections" framework of Aharonov [1] and Szegedy [32]. We show

**Lemma 2** [6] Let $H_1$ be the subspace spanned by $|v\rangle$ for leaves $v$ containing variables $x_i = 1$ and let $P_1$ be the projection to $H_1$. Assume that, for any 1-eigenvector $|\psi\rangle$: $U_{\text{tree}}|\psi\rangle = |\psi\rangle$ which is not orthogonal to $|\psi_{\text{start}}\rangle$, we have

$$\|P_1|\psi\rangle\| \geq \epsilon\|\psi\|.$$  

Then, any eigenvector $|\psi\rangle$: $U|\psi\rangle = \lambda|\psi\rangle$ which is not orthogonal to $|\psi_{\text{start}}\rangle$ must have $|\lambda - 1| \geq \epsilon$.

Therefore, to show the desired bound on the eigenvalues of $U$ ($\lambda = e^{i\theta}$, $\theta = \Omega(1/\epsilon)$), it suffices to lower-bound the projections of eigenvectors of $U_{\text{tree}}$ to $H_1$. This is done by analyzing the amplitudes of $|\psi\rangle$ at vertices of the tree that evaluate to 1 (either internal vertices which contain a NAND gate whose output is 1 or leaves that contain $x_i = 1$). We start by analyzing the amplitude of the root (which evaluates to 1 because $F = 1$) and then move up the tree. See [6] for details.

4.2. **Discrete time algorithm via Szegedy quantization**

An alternative approach is to construct a quantum algorithm based on discrete time quantum walk. As in the previous section, we consider a NAND tree augmented by a finite tail of even length $L = O(\sqrt{N})$. We consider a coined quantum walk on this tree defined in the natural way [3]. The coined walk has the state space consisting of states $|v,d\rangle$ where $v$ is a vertex of the augmented tree, $d \in \{\text{down, left, right}\}$ and:

- If $v = L$ (i.e., $v$ is the end of the tail), then $d = \text{left}$;
- If $v \in \{1, \ldots, L - 1\}$ (i.e., $v$ is in the tail but is not the end of the tail), then $d \in \{\text{left, right}\}$;
- If $v$ is a non-leaf vertex in the tree, then $d \in \{\text{down, left, right}\}$;
• If \( v \) is a leaf, then \( d = \text{down} \).

Each such state \( |v,d\rangle \) can be associated with one of edges incident to \( v \), in a natural way. If \( v \) is a vertex in the tail, then states \( |v,\text{left}\rangle \) and \( |v,\text{right}\rangle \) correspond to edges \((v,v+1)\) and \((v,v-1)\), respectively. If \( v \) is a vertex in the tree, \( |v,\text{down}\rangle \) corresponds to the edge \((v,p)\) where \( p \) is \( v \)'s parent and \( |v,\text{left}\rangle \) and \( |v,\text{right}\rangle \) correspond to edges \((v,l)\) and \((v,r)\) where \( l \) and \( r \) are \( v \)'s left and right child.

A discrete-time quantum walk (also called coined quantum walk) consists of two steps: a coin flip \( C \) and a shift operator \( S \). The coin flip \( C \) is defined as follows:

- If \( v = L \), \( C_v = I \).
- If \( v \) is a leaf, \( C_v = (-1)^{x_i}I \) where \( x_i \) is the variable at the leaf \( v \).
- Otherwise, \( C_v = 2|\psi\rangle\langle\psi| - I \) where \( |\psi\rangle \) is the uniform superposition over all possible \( |d\rangle \).

A shift \( S \) is just the transformation that, for every edge \((v,v')\) in the augmented tree, swaps the two basis states \( |v,d\rangle \) and \( |v',d'\rangle \) associated with the edge \((v,v')\).

There is a close correspondence between the eigenvalues of Hamiltonian \( H \) (described in section 3.2) and the eigenvalues of the unitary transformation \( SC \). Namely, let the starting state be

\[
|\psi_{\text{start}}\rangle = \frac{1}{\sqrt{4L}} \sum_{k=0}^{L-1} (-1)^k (|2k,\text{right}\rangle - i|2k+1,\text{left}\rangle + i|2k+1,\text{right}\rangle - |2k+2,\text{left}\rangle). \tag{3}
\]

Then, we have:

1. If \( F = 0 \), there exists \( |\psi\rangle \) such that \( \| |\psi\rangle - |\psi_{\text{start}}\rangle \| \leq \epsilon \) and \( SC|\psi\rangle = i|\psi\rangle \).
2. If \( F = 1 \), then, for any eigenstate \( |\psi\rangle \) (with \( CS|\psi\rangle = \lambda|\psi\rangle \)), either \( |\psi\rangle \perp |\psi_{\text{start}}\rangle \) or \( \text{Re} \lambda = c/\sqrt{N} \) for some constant \( c > 0 \).

Running the eigenvalue estimation with \( |\psi_{\text{start}}\rangle \) as the starting state and accuracy \( c/2\sqrt{N} \) allows us to distinguish between the two cases.

The finite tail can be shortened from \( L \) vertices to just 2. Namely, we can attach a tail of length 2 consisting of two vertices 1 and 2. For the tree vertices, we define quantum walk in the same way as before. For the vertex 1, we define \( C_1 = 2|\psi\rangle\langle\psi| - I \) where

\[
|\psi\rangle = \frac{1}{\sqrt{N}} |\text{right}\rangle + \sqrt{1 - \frac{1}{\sqrt{N}}} |\text{left}\rangle.
\]

For the vertex 2, we define \( C_2 = I \).

Then, taking the starting state \( |\psi_{\text{start}}\rangle = |2\rangle \) gives us the same behaviour as we had for the starting state given by (3) for the walk with tail of length \( L \).
5. Further developments

All of those algorithms can be generalized to evaluating NAND trees of arbitrary structure. The key idea is to use a *weighted adjacency matrix* as the Hamiltonian. That is, for every edge \((u, v)\) in the tree, we define a weight \(w_{uv}\) that depends on the number of leaves in the subtree above \(u\) and the number of leaves in the subtree above \(v\). We then take the matrix \(H\) defined by \(H_{uv} = 0\) if \((u, v)\) is not an edge and \(H_{uv} = w_{uv}\) if \((u, v)\) is an edge.

If a formula \(F\) is of depth \(d\), we can choose weights \(w_{uv}\) so that a similar quantum algorithm (with the weighted \(H\) as the Hamiltonian) evaluates \(F\) with \(O(\sqrt{Nd})\) queries \[10,6,7\]. For formulas \(F\) of large depth \(d\), one should first reduce their depth using a following result of Bshouty, Cleve and Eberly \[10\]:

**Theorem 2** \[10\] For any NAND formula \(F\) of size \(N\), there exists a NAND formula \(F'\) of size \(N' = O(N^{1+\Theta(\sqrt{\log N})})\) and depth \(d' = O(N^{O(\sqrt{\log N})})\) such that \(F' = F\).

By first applying this theorem to reduce the depth and then using the \(O(\sqrt{Nd'})\) query algorithm to evaluate the resulting \(T'\), we can evaluate any Boolean formula \(T\) with \(O(N^{1/2+O(1/\sqrt{\log N})})\) queries \[10,6,7\].

The NAND tree evaluation algorithms have been generalized by Reichardt and Špalek \[30\] and Reichardt \[26,27,28\]. They have discovered that similar ideas can be used to evaluate span programs (an algebraic computation model which generalizes Boolean logic formulas) with the number of queries being square root of the *witness size* of the span program \[30\].

The span program framework is very powerful. It has been used to design better quantum algorithms for many specific Boolean formulas (by designing span programs for them) \[30\]. Moreover, Reichardt \[26,20\] has shown that span programs are nearly optimal for any Boolean function. That is, if a Boolean function can be evaluated with \(t\) queries by some quantum algorithm, then:

- It can be evaluated by a span-program based quantum algorithm, using a generalization of the algorithm from section 4.2 with \(O(t \log^2 t)\) queries \[20\];
- It can be evaluated by a span-program based quantum algorithm, using a generalization of the algorithm from section 4.2 with \(O(t)\) queries \[29\].

Span programs also can be used to evaluate any NAND formula (of any depth) with \(O(\sqrt{N \log N})\) queries \[28\].

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