On the number of transversals in latin squares

Vladimir N. Potapov

Sobolev Institute of Mathematics, 4 Acad. Koptyug Avenue, Novosibirsk, Russia

Abstract

The logarithm of the maximum number of transversals over all latin squares of order $n$ is greater than $\frac{2}{3}(\ln n + O(1))$.

Keywords: transversal, latin square, Steiner triple system.

2010 MSC: 05B15

1. Introduction

A latin square of order $n$ is an $n \times n$ array of $n$ symbols in which each symbol occurs exactly once in each row and in each column. This property ensures that a latin square is the Cayley table of a finite quasigroup. It is often convenient to represent a latin square as a graph of the corresponding quasigroup, i.e. as a set of ordered triples. Without loss of generality suppose that the set of symbols of the latin square is $\{0, 1, \ldots, n-1\}$. A latin square $A$ of order $n$ is said to contain a proper latin subsquare of order $m$, $1 < m < n$, if there exists an intersection of $m$ rows and $m$ columns within $A$ that is also a latin square.

A diagonal of a square is a set of entries that contains exactly one representative of each row and column. A transversal is a diagonal in which no symbol is repeated. A pair of latin squares $A = (a_{ij})$ and $B = (b_{ij})$ of order $n$ are said to be orthogonal mates if the $n^2$ ordered pairs $(a_{ij}, b_{ij})$ are distinct. Thus, if we look at all $n$ occurrences of a given symbol in $B$, the corresponding positions in $A$ must form a transversal.

Email address: vpotapov@math.nsc.ru (Vladimir N. Potapov)

The work was funded by the Russian Science Foundation (grant No 14-11-00555).
Denote by \( t(A) \) the number of transversals of a latin square \( A \). Define \( T(n) = \max t(A) \) to be the maximum number of transversals over all the latin squares of order \( n \). The best upper bound of \( T(n) \) is due to Taranenko \[4\]
\[
T(n) \leq n^n e^{-2n+o(n)}.
\] (1)

Let \( B_n \) be the Cayley table of the cyclic group of order \( n \). Vardi supposed that there exist two real constants \( c_1 \) and \( c_2 \) such that \( c_1n! \leq t(B_n) \leq c_2n! \), where \( 0 < c_1 < c_2 < 1 \) and \( n > 3 \) is odd, but the known lower bound of \( T(n) \) is only exponential (see \[1\], \[2\]). Cavenagh and Wanless \[3\] proved that if \( n \) is a sufficiently large odd integer then \( t(B_n) > (3.246)^n \). By MacNeish’s theorem \[5\], the following is true.

**Proposition 1.** If \( n = p_1^{i_1} \cdots p_k^{i_k} \), where numbers \( p_i \) are prime, and \( m = \min p_j^{i_j} \), then there exists a set of \( m-1 \) mutually orthogonal latin squares which include \( B_n \).

2. Transversals in Steiner latin squares

A set of 3-element subsets (triples) of \( n \)-element set is called Steiner triple system (STS) if each pair of elements is contained in exactly one triple. A STS consists of \( n(n-1)/6 \) triples. A well-known necessary and sufficient condition for the existence of STS is that \( n \equiv 1 \) or 3 mod 6.

As mentioned above, a latin square can be represented as the graph of a quasigroup, i.e. as a set of ordered triples of the \( n \)-element set such that each pair of elements occurs in each pair of positions and a pair of elements of any triple defines the third element of the triple. The first and the second elements of triples define row and column, and the third element defines the symbol in the corresponding entry of a latin square. Thus, given a STS, we can obtain a latin square by replacing each unordered triple with the six ordered triples and by adding \( n \) triples of the form \((a, a, a)\). This latin square is called Steiner (it is a table of a Steiner quasigroup). By the inclusion-exclusion principle and the definition of STS, it is easy to prove the following proposition.
Proposition 2. For any STS of order $n$ the union of $p$ disjoint triples of the STS intersects with at most $s(p) = 3p\left(\frac{n-1}{2} - \frac{3p-1}{3}\right)$ triples of the STS.

Proof. Let $V$ be the union of $p$ disjoint triples. For each point $v \in V$ there are $(n-1)/2$ triples of the STS that include $v$. Each pair $v_1, v_2 \in V$ is contained in some triple of the STS. Consequently, $\frac{3p(3p-1)}{2}$ triples of the STS occur again. The number of triples $v_1, v_2, v_3 \in V$ included in the STS isn’t more than $3p(\frac{n-1}{2} + \frac{3p-1}{6})$. So, by the inclusion-exclusion principle $V$ intersects with at most $3p(\frac{n-1}{2} - \frac{3p-1}{3} + \frac{3p-1}{6})$ triples of the STS. ▲

Theorem 1. If $S_n$ is a Steiner latin square then $t(S_n) \geq 6 \left(\left\lceil \frac{(n-1)/6 - 1}{\frac{3p-1}{6}}\right\rceil !\left\lceil \frac{n-1}{3}\right\rceil !\left\lfloor \frac{n-1}{3}\right\rfloor !\left\lceil \frac{n-1}{6}\right\rceil !\!\right.$.

Proof. Consider the STS corresponding to $S_n$. By definition, for each triple $(a, b, c)$ of the STS there is the latin subsquare
\[
\begin{pmatrix}
  a & c & b \\
  c & b & a \\
  b & a & c 
\end{pmatrix}
\]
in the intersection of rows and columns labeled by $a, b, c$. This subsquare has three transversals. We will construct transversal $T$ of $S_n$ recursively. In the each step we will take a triple of STS that is disjoint with triples taken before and we will add three elements corresponding to this triple to $T$. Let $K(p)$ be the set of elements of triples taken after $p$ steps, $K(0) = \emptyset$ and $|K(p)| = 3p$. By Proposition 2, elements of $K(p)$ belong to $s(p) = 3p\left(\frac{n-1}{2} - \frac{3p-1}{3}\right)$ triples at most. Thus at the $(p + 1)$th step, it is possible to take one of $n(n-1)/6 - s(p) = (n-3p)(n-6p-1)/6$ triples that don’t intersect with $K(p)$. If we take the triple $(a, b, c)$ then we get $K(p+1) = K(p) \cup \{a, b, c\}$ and add 3 entries (ordered triples) $\{(a, c, b), (b, a, c), (c, b, a)\}$ or $\{(a, b, c), (b, c, a), (c, a, b)\}$ to $T$. If $n(n-1)/6 \leq s(p)$ i.e. $p \geq p_0 = \lceil (n-1)/6 \rceil$ then we add to $T$ entries $(e, e, e)$ for all $e \not\in K(p)$. Consequently, there exist more than $\frac{p_0!}{p_0!} \prod_{p=0}^{p_0-1} 2[(n-3p)(n-6p-1)/6] \geq 6^{p_0-1}\left(\frac{n}{3}\right)!/(\left\lfloor \frac{n}{3}\right\rfloor !p_0!)$ variants to choose a transversal. ▲

In the last part of the note the proposed construction of transversals is adapted to latin squares of any large order.
Bose (see [5]) proposed the following construction of STS. Let $A$ be a latin square of order $n$ corresponding to idempotent, commutative quasigroup. Put

$$S^1 = \{(x, i), (y, i), (z, i + 1 \mod 3) \mid (x, y, z) \in A, i \in \{0, 1, 2\}\},$$

$$S^2 = \{(x, 0), (x, 1), (x, 2) \mid x \in \{0, 1, \ldots, n - 1\}\}.$$

Then $S^1 \cup S^2$ is a STS of order $3n$. Consider a latin square $S_{3n}$ of order $3n$ corresponding to a STS obtained by Bose’s construction from a latin square $A$ of order $n$. Each transversal $T$ of $A$ generates the transversal $T'$ of $S_{3n}$ by the following rule. If $(a, b, c) \in T$ then $((a, 0), (b, 0), (c, 1)), ((a, 1), (b, 1), (c, 2)), ((a, 2), (b, 2), (c, 0)) \in T'$, if $(a, a, a) \in T$ then $((a, 0), (a, 1), (a, 2)), ((a, 1), (a, 2), (a, 0)), ((a, 2), (a, 0), (a, 1)) \in T'$.

Let $A$ be a latin square of order $n$ with $k$ disjoint transversals $T_0, \ldots, T_{k-1}$. Construct the latin square $\hat{A}_k$ of order $n + k$ in the following way. If $(a, b, c) \in A \setminus \cup_{i=0}^{k-1} T_i$ then $(a, b, c) \in \hat{A}_k$. If $(a, b, c) \in T_i$ then $(a, b, n + i), (a, n + i, c), (n + i, b, c) \in \hat{A}_k$. Moreover in the intersection of rows and columns labeled by additional symbols $n, \ldots, n + k - 1$ we substitute a latin square $C$ of order $k$ on the alphabet $\{n, \ldots, n + k - 1\}$. For example, if $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and $T_0 = \{(0, 1, 1), (1, 2, 0), (2, 0, 2)\}, T_1 = \{(0, 2, 2), (1, 0, 1), (2, 1, 0)\}$ then $\hat{A}_2 = \begin{pmatrix} 0 & 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 2 & 0 \\ 2 & 1 & 0 & 3 & 4 \\ 1 & 0 & 2 & 4 & 3 \end{pmatrix}$. Let $t(A; T_0, \ldots, T_{k-1})$ be the number of transversals of $A$ which don’t intersect with transversals $T_0, \ldots, T_{k-1}$. It is easy to see that $t(\hat{A}_k) \geq t(C)t(A; T_0, \ldots, T_{k-1})$. If $k \neq 2$ then there exists $C$ with $t(C) \geq 1$.

It is easy to see that the quasigroup $q$ defined by the rule $q(x, y) + q(x, y) \equiv x + y \mod n$ is idempotent, commutative and isotopic to cyclic group of order $n$ as $n$ is odd. By Proposition 1, the latin square $B'_n$ that corresponds to quasigroup $q$ has $n$ disjoint transversals. By means of Bose’s construction, we can obtain
the Steiner latin square $S_{3n}$ from $B'_n$. $S_{3n}$ has at least $n$ disjoint transversals. Thus we can construct a latin square $D = (S_{3n})_k$ of order $3n + k$, $k \leq n$, as described above. For $D$ we can repeat the reasonings of the proof of Theorem 1. The only difference is the number of variants to choose a triple at each step. The number of variants for $D$ is $kn$ less than for $S_{3n}$.

**Corollary 1.** \( \frac{2}{9}(\ln n + O(1)) \leq \ln T(n) \leq n(\ln n - 2 + o(1)) \) as $n \to \infty$.

The upper bound is provided by (1). The lower bound follows from Theorem 1 and Stirling’s formula if $n \equiv 1$ or $3 \mod 6$, and it is proved analogously in other cases.

After submitting this paper to the journal, another paper [6] has appeared that contains a better lower bound for the number of transversals in latin squares. This lower bound is asymptotically equal to the upper bound (1). However, the result in [6] is obtained using a non-constructive approach as oppose to the method in our paper.

3. Acknowledgments

The author is grateful to Denis Krotov and Anna Taranenko for useful discussions.

References

[1] I. M. Wanless, Transversals in Latin Squares, Quasigroups and Related Systems 15 (2007) 169–190.

[2] I. M. Wanless, Transversals in Latin squares: a survey, Surveys in Combinatorics 2011, London Math. Soc. Lecture Note Ser. 392 (2011) Cambridge Univ. Press, Cambridge, 403–437.

[3] N. J. Cavenagh, I. M. Wanless, On the number of transversals in Cayley tables of cyclic groups, Discrete Applied Mathematics 158 (2010) 136–146. DOI: 10.1016/j.dam.2009.09.006
[4] A. A. Taranenko, Multidimensional permanents and an upper bound on the number of transversals in latin squares, Journal of Combinatorial Designs 23 (7) (2015) 305–320. DOI: 10.1002/jcd.21413

[5] Handbook of combinatorial designs. Edited by Charles J. Colbourn and Jeffrey H. Dinitz. Second edition. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007. xxii+984 pp.

[6] R. Glebov, Z. Luria, On the maximum number of Latin transversals, arXiv:1506.00983 [math.CO]