EXACT VALUES OF COMPLEXITY FOR PAOLUZZI – ZIMMERMANN MANIFOLDS

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Abstract. There are found exact values of (Matveev) complexity for the 2-parameter family of hyperbolic 3-manifolds with boundary constructed by Paoluzzi and Zimmermann. Moreover, $\varepsilon$-invariants for these manifolds are calculated.

1. Preliminaries

Let $M$ be a compact 3-manifold with nonempty boundary. Recall [1] that a subpolyhedron $P \subset M$ is a spine of $M$ if the manifold $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$. A spine $P$ is said to be almost simple if the link of every of its points can be embedded into a complete graph $K_4$ with four vertices. A true vertex of an almost simple spine $P$ is a point with the link $K_4$. The complexity $c(M)$ of $M$ is defined as the minimum possible number of true vertices of an almost simple spine of $M$.

An almost simple spine $P$ is called simple if the link of each point $x \in P$ is homeomorphic to one of the following 1-dimensional polyhedra: (a) a circle (such a point $x$ is called nonsingular); (b) a circle with a diameter (such an $x$ is a triple point); (c) $K_4$. Components of the set of nonsingular points are said to be 2-components of $P$, while components of the set of triple points are said to be triple lines of $P$. A simple spine is special if each of its triple lines is an open 1-cell and each of its 2-components is an open 2-cell.

The problem of calculating the complexity for 3-manifolds is actual but it is very hard. Exact values of the complexity are presently known only for some computer-generated censuses [2], for two infinite series of hyperbolic manifolds with boundary [3, 4], for certain infinite series of lens spaces, and for generalized quaternion spaces [5, 6].

In the paper we find the exact values of the complexity for a family of manifolds $M_{n,k}$ with boundary constructed by Paoluzzi and Zimmermann [7]. Every $M_{n,k}$, $n \geq 4$, has one boundary component, its complexity $c(M_{n,k})$ is equal to $n$, and its Euler characteristic is equal to $2 - n$. Earlier known examples of manifolds with one boundary component and complexity $n$ have the Euler characteristic $1 - n$.

To prove the main result we use the $\varepsilon$-invariant of 3-manifolds (see [1] section 8.1.3]). Let $P$ be a special spine of a compact manifold $M$. Denote by $\mathcal{F}(P)$ the set of all simple subpolyhedra of $P$ including $P$ and the empty set. Let us associate to each simple polyhedron

2000 Mathematics Subject Classification. Primary: 57M25, Secondary: 20F36.

Key words and phrases. 3-manifold, complexity, hyperbolic manifold with boundary.

The authors were supported by the Russian Foundation for Basic Research, grants 10-01-91056 (A. Vesnin) and 11-01-00065 (E. Fominykh), and the Joint Program of the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences and the Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences.
For every \( n \geq 3 \) consider an \( n \)-gonal bipyramid which is the union of pyramids \( NL_0L_1 \ldots L_{n-1} \) and \( SL_0L_1 \ldots L_{n-1} \) along the common \( n \)-gonal base \( L_0L_1 \ldots L_{n-1} \). Let \( k \) be such integer that \( 0 \leq k < n \) and \( \gcd(n, 2 - k) = 1 \). We identify the faces of \( B_n \) in pairs: for each \( i = 0, \ldots, n - 1 \) the face \( L_iL_{i+1}N \) gets identified with the face \( SL_iL_{i+k}L_{i+k+1} \) by a transformation (homeomorphism of faces) which we shall denote by \( y_i \) (indices are taken \( \mod n \) and the vertices are glued together in the order in which they are written).

Identifications \( \{y_0, y_1, \ldots, y_{n-1} \} \) define the equivalence relations on the sets of faces, edges, and vertices of the bipyramid. It is easy to see that all the faces are split into pairs equivalent faces, all edges and all vertices become identified to a single edge resp. vertex (this is guaranteed by the above conditions on \( k \)). Denote the resulting identification spaces by \( M_{n,k}^* \). It is an orientable pseudomanifold with one singular point, since \( \chi(M_{n,k}^*) = 1 - 1 - n - 1 = n - 1 \neq 0 \). Cutting of a cone neighborhood of the singular point from \( M_{n,k}^* \) we get a compact manifold \( M_{n,k} \) with one boundary component. It arises from the truncated bipyramid \( A_n \) with vertices \( A_i, B_i, C_i, D_i, E_i, F_i, i = 0, \ldots, n - 1 \) (see Fig. 1 for the case \( n = 5 \), where the right and left sides are assumed identified).

Pairwise identifications \( \{y_0, y_1, \ldots, y_{n-1} \} \) of the faces of \( B_n \) induce pairwise identifications.
\{x_0, x_1, \ldots, x_{n-1}\} \text{ of the faces of } A_n, \text{ where } x_i \text{ identifies faces } X'_i = C_iD_iB_{i+1}A_{i+1}A_iB_i \text{ and } X'_{k+i} = F_{k+i}E_{k+i-1}C_{k+i}D_{k+i+1}E_{k+i}F_{k+i} \text{ (indices are taken mod } n \text{ and the vertices are glued together in the order in which they are written). In this case the faces } A_0A_1 \ldots A_{n-1}, F_0F_1 \ldots F_{n-1}, \text{ and } B_iC_iE_iD_{i-1}, i = 0, \ldots, n-1, \text{ form the boundary } \partial M_{n,k} \text{ of } M_{n,k}. \text{ It is easy to check that } \partial M_{n,k} \text{ is a closed orientable surface of genus } n-1. \text{ According to } [7], \text{ for every } n \geq 3 \text{ the manifold } M_{n,k} \text{ is hyperbolic and } M_{n,k} = \mathbb{H}^3/G_{n,k}, \text{ where } G_{n,k} \text{ is an } n\text{-generated group with one defining relation}

\begin{equation}
G_{n,k} = \langle x_0, \ldots, x_{n-1} \mid \prod_{i=0}^{n-1} x_{i(2-k)} x_{i(2-k)+1} x_{i+1}^{-1}(2-k)-1 = 1 \rangle.
\end{equation}

It is known [7] that } M_{n,k} \text{ and } M_{n',k'} \text{ are homeomorphic (or equivalently, isometric) if and only if } n = n' \text{ and } k \equiv k' \text{ mod } n. \text{ The manifold } M_{3,1} \text{ is constructed in [8] by identifying faces of 2 truncated hyperbolic tetrahedra. It was shown in [9] that } M_{3,1} \text{ is one of the compact manifolds which have minimal volume among all compact hyperbolic 3-manifolds with totally geodesic boundary, } \text{vol}(M_{3,1}) \approx 6, 451998. \text{ The volumes of } M_{n,k} \text{ for } n \leq 82 \text{ are presented in [10].}

3. Calculating the complexity

**Theorem.** For every integer } n \geq 4 \text{ we have } c(M_{n,k}) = n.

**Proof.** Let us prove that } c(M_{n,k}) \leq n \text{ for every integer } n \geq 3. \text{ To do that it suffices to construct a special spine of } M \text{ with } n \text{ true vertices. Cut } B_n \text{ into } n \text{ tetrahedra } T_i = NSL_iL_{i+1}, \text{ where } i = 0, 1, \ldots, n-1. \text{ For each } T_i \text{ consider the union of the links of all four vertices of } T_i \text{ in the first barycentric subdivision. } M_{n,k}^* \text{ can be obtained by gluing the tetrahedra } T_0, \ldots, T_{n-1} \text{ via affine homeomorphisms of the faces. This gluing determines a pseudotriangulation } T \text{ of } M_{n,k}^*, \text{ and induces a gluing of the corresponding polyhedra } R_i, i = 0, \ldots, n-1, \text{ together. We get a special spine } P_{n,k} = \cup_i R_i \text{ of } M_{n,k}. \text{ Since every } R_i \text{ is homeomorphic to a cone over } K_i \text{, the spine } P_{n,k} \text{ has exactly } n \text{ true vertices.}

\text{Demonstrate that the inequality } c(M_{n,k}) \geq n \text{ holds for every } n \geq 4. \text{ Since } M_{n,k} \text{ is hyperbolic, it is irreducible, has incompressible boundary, and contains no essential annuli. It follows from [11 Theorem 2.2.4] that there exists a special spine } P' \text{ of } M_{n,k} \text{ with } c(M_{n,k}) \text{ true vertices. Denote by } d \text{ the number of } 2\text{-components of } P'. \text{ Calculating the Euler characteristic of } M_{n,k}, \text{ we get } 2-n = \chi(P_{n,k}) = \chi(M_{n,k}) = \chi(P') = d-c(M_{n,k}). \text{ Hence } c(M_{n,k}) = n+d-2. \text{ It remains to show that } d \geq 2.

On the contrary, suppose that } d = 1. \text{ Let us calculate the } \varepsilon\text{-invariant } t(M_{n,k}) \text{ in two ways from the spines } P_{n,k} \text{ and } P'. \text{ By construction, } P_{n,k} \text{ has two 2-components, and each of them corresponds to an edge of } T. \text{ Denote by } \alpha \text{ and } \beta \text{ 2-components of } P_{n,k}, \text{ where } \alpha \text{ corresponds to the edge } NS. \text{ Note that for describing an arbitrary simple subpolyhedron } Q \text{ of } P_{n,k} \text{ it is sufficient to specify which 2-components of } P_{n,k} \text{ are contained in } Q \text{ (see [11 section 8.1.3])}. \text{ A simple analysis shows that there are three simple subpolyhedra of } P_{n,k}.

\begin{enumerate}
\item The empty subpolyhedron with the } \varepsilon\text{-weight 1.
\item The whole } P_{n,k} \text{ with the } \varepsilon\text{-weight equals to } (-1)^n \varepsilon^{2-2n}, \text{ since } V(P_{n,k}) = n \text{ and } \chi(P_{n,k}) = 2-n.
\end{enumerate}
(3) The subpolyhedron $P_{n,k} \setminus \alpha$. It contains no vertices and has the Euler characteristic $\chi(P_{n,k} \setminus \alpha) = \chi(P_{n,k}) - 1 = 1 - n$. Hence it has $\varepsilon$-weight $\varepsilon^{1-n}$.

Summing up, we get $t(M_{n,k}) = (-1)^n \varepsilon^{2-2n} + \varepsilon^{1-n} + 1$.

Now let us calculate the $\varepsilon$-invariant $t(M_{n,k})$ from $P'$. It is easy to see that $F(P') = \{\emptyset, P'\}$ and $\chi(P') = 2 - n$, since the spine $P'$ has only one 2-component and $n - 1$ true vertices. Summing up $w_\varepsilon(\emptyset) = 1$ and $w_\varepsilon(P') = (-1)^{n-1} \varepsilon^{3-2n}$, we get $t(M_{n,k}) = (-1)^{n-1} \varepsilon^{3-2n} + 1$.

Comparing two obtained formulas for $t(M_{n,k})$, we get $\varepsilon^{n-1} = (-1)^{n-1}(\varepsilon + 1)$ that is equivalent to $\varepsilon^{n-3} = (-1)^{n-1}$. It is obvious that this equality is true only for the case $n = 3$, which contradicts our assumption that $n \geq 4$. Hence, $d > 1$, and $c(M_{n,k}) \geq n$. □

**Proposition.** The following equality holds: $c(M_{3,1}) = 2$.

**Proof.** Since $B_3$ is the union of the tetrahedra $NL_0L_1L_2$ and $SL_0L_1L_2$, the manifold $M_{3,1}$ has a special spine with two true vertices and with only one 2-component. Hence $c(M_{3,1}) = 2$ (see [3]). □

**Corollary.** For every $n \geq 3$ we have $t(M_{n,k}) = (-1)^n \varepsilon^{2-2n} + \varepsilon^{1-n} + 1$.

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