Strong renormalization scheme dependence in \( \tau \)-lepton decay: fact or fiction?

JIŘÍ CHÝLA

Institute of Physics, Academy of Sciences of Czech Republic
Prague, Czechoslovakia\(^1\)

Abstract

The question of the renormalization scheme dependence of the \( \tau \) semileptonic decay rate is revisited in response to a recent criticism. Particular attention is payed to a distinction between a consistent quantitative description of this dependence and the actual selection of a subset of "acceptable" renormalization schemes. It is argued that a reasonable universal measure of the renormalization scheme dependence can be formulated, which gives encouraging results when applied to various physical quantities, including the semileptonic \( \tau \) decay rate.

\(^1\)Postal address: Na Slovance 2, 180 40 Prague 8, Czech Republic
In a recent article [1] P. Rączka has questioned the optimism of several earlier papers [2, 3, 4] concerning the possibility to use the semileptonic decay rate of the $\tau$-lepton and namely the quantity $r_\tau$ appearing in the measured ratio

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau e^- \nu_e)} = 3 \left(1 + r_\tau\right) \left(1 + O(\alpha_{em})\right)$$

for an accurate determination of the basic QCD scale parameter $\Lambda_{\overline{\text{MS}}}$. Let me recall that this optimism is based on the following two favourable circumstances:

1. strong suppression of the potentially dangerous nonperturbative corrections [3, 4]
2. weak renormalization scheme (RS) dependence [2, 3]

The author of [1] claims, on the other hand, to have found for $r_\tau$ a strong RS dependence, at both the next-to-leading (NLO) and next-to-next-to-leading (NNLO) orders. He thus comes to the conclusion that it is impossible to use this quantity for an accurate determination of $\Lambda_{\overline{\text{MS}}}$. He motivates his own analysis of the RS-dependence by the observation that the previous analysis [2, 3], have considered only some of the possible RS’s, while “...for a full estimate of the RS dependence ambiguity one should compare predictions in all schemes which a priori seem to be admissible...”. The reason for writing this note is that although this observation is basically correct it, if taken literally, leads to complete arbitrariness of perturbative QCD predictions for any physical quantity. The $\tau$ decay rate plays certainly no special role in this respect. I shall, moreover, argue that the situation is not so hopeless and that one can formulate a plausible strategy how to construct reasonably “accurate” perturbative QCD predictions for many measurable quantities, including the $\tau$ decay rate.

The whole discussion, ongoing for more than a decade, of the RS dependence problem does in fact boil down to the question of finding some physically motivated algorithm restricting this arbitrariness. There can be no true solution of this problem, merely more or less reasonable assumptions can be made on how to select one, in one way or another “optimal”, RS [4, 5, 6], and how to measure the “strength” of the RS dependence, i.e. how to estimate the “theoretical” error associated with such a choice.

Before this can be attempted it is, however, necessary to have a consistent quantitative description of all the degrees of freedom associated with general renormalization group transformations. Choosing such a consistent labelling has nothing to do with the actual problem of “resolving” the RS dependence (in the above sense), but is an inevitable prerequisite for this latter step. There is no problem in this respect and there are various distinct, though in principle equivalent, ways how to do it. The lack of clear distinction between these two separate issues is, however, a frequent phenomenon and can be detected also in [1].

In the notation of [1] (with the exception of $Q$ which is used below to denote a generic external momentum variable instead $m_\tau$, appropriate to [1]), I shall discuss QCD predictions for any physical quantity depending on $Q$ and admitting perturbative expansion of the form (in massless QCD)

$$r(Q) = a(\mu) \left(1 + r_1(\mu/Q)a(\mu) + r_2(\mu/Q)a^2(\mu) + \cdots\right).$$

(2)
The renormalized couplant \( a(\mu) \equiv g^2(\mu)/4\pi^2 \) obeys the equation

\[
\frac{da(\mu)}{d \ln \mu} = -\beta_0 a^2 \left( 1 + c_1 a(\mu) + c_2 a(\mu)^2 + \cdots \right),
\]

(3)

where the first two coefficients \( c_1, c_2 \), are fixed by the number of quark flavours, but all the higher ones are essentially free, defining the so called renormalization convention (RC), RC=\(\{c_i, i \geq 2\} \). Although not written out explicitly, both the coefficients \( r_k \) and the couplant \( a(\mu) \) (when (3) is considered to the \( k \)-th order) do, however, depend on all \( c_i, i \leq k \) as well. Finally, they also depend on the specification which of the infinite number of solutions to (3) we have in mind. Combining this last information with that on \( c_i, i \leq k \) defines what is called the referential renormalization scheme (RRS) (for detailed discussion of all these points see, for instance, [10]). Only if this RRS is fixed does the specification of the scale \( \mu \) uniquely determine the RS. Instead of (2) we can write equivalently

\[
r(Q) = a(kQ) \left( 1 + r_1(k)a(kQ) + r_2(k)a^2(kQ) + \cdots \right),
\]

(4)

where the arbitrariness in \( \mu \) has been traded for that of the the dimensionless parameter \( k \in (0, \infty) \). As the essence of Rączka’s claim concerns both the NLO and NNLO, I shall in the rest of this note concentrate on the NLO case and furthermore set, purely for reasons of technical simplicity, \( c_1 = 0 \) in (3). All the following considerations hold for the realistic values of \( c_1 \) as well, only the formulae are more cumbersome.

Under these circumstances the solutions to (3) assume particularly simple form

\[
a(\mu, \text{RRS}) = \frac{1}{\beta_0 \ln(\mu/\Lambda_{\text{RRS}})},
\]

(5)

where the parameter \( \Lambda_{\text{RRS}} \) uniquely specifies the solution to (3). In this way of labelling the RS the selection of the RRS is just a matter of bookkeeping and so it is commonly accepted to use the \( \overline{\text{MS}} \) RRS for this purpose. Once this convention is made the scale \( \mu \), or the parameter \( k \), can be used to label uniquely all the available RS’s. In view of the arbitrariness in the selection of the RRS, no absolute meaning can, however, be given to the scale \( \mu \) or the parameter \( k \) and thus no arguments on the existence of a “natural” scale, given for instance by \( Q \), are sufficient to fix the RS.

I have repeated these simple facts as the author of [1] starts his criticism of the “conventional” approach with a correct but obvious and well-known observation just made, i.e. that the same \( \mu \), or equivalently \( k \), implies different \( r_i \)’s and \( a(\mu) \) in different RRS’s. Varying \( k \) around unity, or in any other finite interval, in any single RRS is certainly not sufficient to take properly into account all the RS available at the NLO, but \textbf{without} this restriction on \( k \) any such RRS is equally suitable for a consistent labelling of all RS’s.

It may, however, be preferable to use such a labelling of RS’s which avoids the concept of RRS and uses just one variable to uniquely fix a RS. One way of doing this is to use for this purpose the value of \( r_1 \) itself. The internal consistency of perturbation theory implies the following relation between \( \mu, a \) and \( r_1 \):

\[
r_1 = b \ln \left( \frac{\mu}{\Lambda_{\overline{\text{MS}}}} \right) - \rho = \frac{1}{a} - \rho \Rightarrow a = \frac{1}{r_1 + \rho}
\]

(6)
where $\rho$ is the renormalization group invariant [7]. It is solely through this invariant that the $Q$-dependence of $r(Q)$ enters [2]. Substituting (6) into (2) and truncating it to the NLO we get

$$r_{NLO}(\rho, r_1) = \frac{2r_1 + \rho}{(r_1 + \rho)^2}$$

(7)
as an explicit function of $r_1$ and $\rho$. In Fig.1 the dependence of $r_{NLO}$ on $r_1$ is displayed for several values of $\rho$ (only those points for which (7) stays positive are plotted). So far only the question of an exhaustive and consistent labelling of the RS’s has been discussed. Now the crucial moment arrives and some point (or some interval of points) on the curve corresponding to a given $\rho$ must be selected. Looking on the curves in Fig.1, we conclude that:

i/ there is one “exceptional” point on each of the curves, namely the stationary point, defining the PMS [7] choice of the RS (in our approximation this point coincides with the effective charges approach (ECH) of [8], defined by the condition $r_1 = 0$).

ii/The pattern of $r_1$ dependence is qualitatively the same for all values of $\rho$.

To give the word “strong” or “weak” RS dependence a quantitative content requires that we define the range of of “acceptable” $r_1$. In [3] we have taken as our preferred RS the PMS/ECH one and furthermore suggested to estimate the associated “theoretical error” by the difference

$$\Delta^{theory} = r_{NLO}(\text{PMS}) - r_{NLO}(\text{MS}).$$

(8)

This is of course somewhat arbitrary definition and we could certainly take some other measure of the RS-dependence. We, however, consider it meaningful to assume this, or some other, definition of the theoretical “error” and use it in analyses of all physical quantities for which the NLO and, if possible, also the NNLO, calculations are available. For $\Lambda_{\text{MS}}$ in the region of a few hundreds of MeV and taking into account that, for 3 quark flavours, $r_1(\mu = Q, \overline{\text{MS}}) = 5.2$ the range of $\rho$ appropriate to the $\tau$ decay rate (2) is roughly between 2 and 4.

After the $\tau$-lepton decay rate and $e^+e^-$ annihilation into hadrons [2] we have recently analyzed in the same way the Gross-Llewellyn-Smith sum rule [10] and intend to continue in this direction as further NNLO calculations become available. I stress the importance of comparing this kind of analysis at NLO and NNLO as only this comparison can tell us whether our choice of the RS leads to a reasonable convergence (in the pragmatical sense) as we proceed to higher orders and whether the associated “theoretical” error simultaneously decreases as it should if our procedure is sound. The results of [2, 10] are encouraging in both respects.

The dependence of $r_{NLO}$ on $r_1$ and $\rho$ as given in (7) is quite general and holds for any physical quantity admitting perturbative expansion of the form (2). The difference between various physical quantities enters entirely through the corresponding values of $\rho$ due to the fact that

$$\rho = b \ln \left( \frac{Q}{\Lambda_{\text{MS}}} \right) - r_1(\mu = Q, \overline{\text{MS}})$$

(9)
contains both possible differences in the scale $Q$ and in $r_1(\mu = Q, \overline{\text{MS}})$. 
In Fig. 2a the formula (7) is plotted as a function of $\rho$ for a number of values of $r_1$. It can be redrawn (as in [1]) for any particular physical quantity expressible in the form (2) as a function of $\ln (Q/\Lambda_{\overline{\text{MS}}})$ by simply shifting the origin of Fig. 2a by an appropriate value of $r_1$ and taking into account different value of external momentum $Q$. The content of Fig. 2a, with curves corresponding to $r_1 \in (-3.83, 8.32)$ has been interpreted in [1] as an evidence for the strong RS dependence of the $\tau$ decay rate (2). This interval of $r_1$ values correspond to the overlap between those obtained within the MS RRS varying $k$ in (4) between 1/2 and 2 and between 1/3 and 1 within the symmetric MOM RRS. No argument has, however, been put forward to justify the restriction to the above mentioned RRS and thus the selected interval of $r_1$ values must be considered as arbitrary as any other one. It must be born in mind that all values of $r_1$ are in principle equally acceptable and that even restricting the values of $k$ in (4) to the region around unity we can always find such a RRS in which $r_1$ is equal to any prescribed value. Consequently also the conclusions drawn in [1] and based on this restriction of “acceptable” RS is valid only within this particular definition. Taking into account sufficiently large upper and lower bounds on $r_1$ would, of course, make any physical quantity “strongly” RS-dependent.

To illustrate this point let me consider the same expression (7) in the region $\rho \in (10, 20)$, which is roughly the range covered by the PETRA experiments measuring the familiar $R$-ratio

$$R_{e^+e^-} = \frac{\Gamma(e^+e^- \rightarrow \text{hadrons})}{\Gamma(e^+e^- \rightarrow \mu^+\mu^-)} = 3(1 + e^{\pm}e^-)$$  \tag{10}$$

The quantity $r_{e^+e^-}$ differs from (2), beside the range of $\rho$ only by the value of $r_1(\mu = Q, \overline{\text{MS}}) = 1.41$. In Fig. 2b the same curves as in Fig. 2a are plotted in this high $\rho$ region (I continue to label as “MS” the curve corresponding to $r_1 = 5.2$ despite the fact that for (10) $r_1(\mu = Q, \overline{\text{MS}}) = 1.41$ in order to maintain direct relation to Fig. 2a). While the spread of the results corresponding to $r_1 \in (-3.83, 8.32)$ has decreased with respect to Fig. 2a, extending this interval of “acceptable” $r_1$ just a little bit down to moderately negative values around -5 or -6, would imply “strong” RS dependence even for the R-ratio (10) ! I don’t think there is any argument why $r_1 = -3.83$ should be acceptable while $r_1 = -5$ not. And even at LEP energies (corresponding to $\rho$ around 22) $r_1 = -11$ would be sufficient to yield $r_{e^+e^-} = 0$.

In the preceding paragraphs I have argued in favour of a particular measure of the strength of the RS-dependence. Compared to the measure based on an ad hoc choice of the interval of $r_1$, it has another advantage, which appears when going beyond the NLO. There, new parameters, one at each further order, are needed to describe the full RS dependence. We can choose $r_i; i \geq 2,$ (or $c_i; i \geq 2$) for that purpose but in any case if we attempt to define the “acceptable” RS by means of restrictions on these further parameters, new criteria have to be invented at each new higher order. For instance in [1] $c_2$ was considered in the interval $c_2 \in (-25, 25)$, with no particular reason given for these limits. In our approach, based on the selection of PMS/ECH and $\overline{\text{MS}}$ RS’s, we don’t face such problems as these criteria are reasonably (although not entirely) free of ambiguities.

Summarizing this note, it is fair to say that the considerations presented above are, or at least should be, nothing new. But neither is there any new idea in [1]. There is no real solution to the RS dependence ambiguity, short of calculating the expansions to all orders. I have tried to stress the distinction between the task of a full and consistent description of this
ambiguity, which presents no problem and the actual task of selecting one, or a subset, of the 
RS’s. In this latter step some arbitrariness and subjective choice is inevitable. Bearing this 
in mind the conclusions of [1] have their validity only within a particularly defined measure 
of the strength of the RS-dependence adopted by its author. They should certainly not 
discourage further attempts, both theoretical and experimental, to use the $\tau$ decay rate [1] 
as a potentially suitable place for quantitative tests of perturbative QCD.

References

[1] P. Rączka, Phys. Rev. D 23, R3699 (1992)
[2] J. Chýla, A. Kataev, and S. Larin, Phys. Lett. B 267, 269 (1991)
[3] F. Le Diberder and A. Pich, Phys. Lett. B 289, 165 (1992)
[4] A. Pich, in Proceedings of the XXVIIth Rancontre de Moriond, Les Arcs, March 1992, 
edited by Tran Than Van (Editions Frontieres, Gif-sur-Yvette, 1992)
[5] E. Braaten, Phys. Rev. Lett. 60, 1606 (1988); 63, 577 (1989)
[6] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. B 373, 581 (1992)
[7] P. M. Stevenson, Phys. Rev. D 23, 2916 (1981)
[8] G. Grunberg, Phys. Rev. D 29, 2315 (1984)
[9] S. Brodsky, G. P. Lepage, and P. Mackenzie, Phys. Rev. D 28, 228 (1983)
[10] J. Chýla and A. Kataev, Phys. Lett. B 297, 385 1992

Figure captions.

Fig.1: $r^{NLO}_\tau(r_1, \rho)$ as a function of $r_1$ for several values of $\rho$.

Fig.2: $r^{NLO}_\tau(r_1, \rho)$ as a function of $\rho$ for several values of $r_1$ in the region appropriate to 
(a) the $\tau$ decay rate and (b) the R-ration in $e^+e^-$ annihilations.