DEGREES OF FREEDOM
OF
ARBITRARILY HIGHER-DERIVATIVE FIELD THEORIES

by

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ABSTRACT

As an example of what happens with physically relevant theories like effective
gravity, we consider the covariant relativistic theory of a scalar field of arbitrarily
higher differential order. A procedure based on the Legendre transformation and
suitable field redefinitions allows to recast it as a theory of second order with one
explicit independent field for each degree of freedom. The physical and ghost fields are
then apparent. The full (classical) equivalence of both Higher and Lower Derivative
versions is shown. An artifact of the method is the appearance of irrelevant spurious
fields which are devoid of any dynamical content.
1. Introduction

Theories of gravity with terms of any order in curvatures arise as part of the low energy effective theories of the strings [1] and from the dynamics of quantum fields in a curved spacetime background [2].

Theories of second order (4-derivative theories in the following) have been studied more closely in the literature because they are renormalizable [3] in four dimensions and have nice renormalization group properties [4]. In particular a procedure based on the Legendre transformation was devised [5] to recast them as an equivalent theory of second differential order. A suitable diagonalization of the resulting theory was found later [6] that yields the explicit independent fields for the dynamical degrees of freedom involved. In [7] the simplest example of this procedure was given using a model of one scalar field with a massless and a massive degree of freedom. In an appendix, Barth and Christensen [8] gave the splitting of the higher derivative (HD) propagator into quadratic ones for the 4th, 6th and 8th differential-order scalar theories though not devising the systematic procedure for the general case.

The problem remained open of how to tackle arbitrary derivative-order theories, as it is the general case, since the Legendre transformation procedure then becomes far from trivial. Classical treatises [9] face the Lagrangian and Hamiltonian theories of systems including higher time derivatives of the generalized coordinates and the definition of canonical momenta. Later work has considered the variational problem of those theories with the tools of the Cartan form, k-jets, symplectic geometry and Legendre mappings [10]. The difficulties of the seemingly unavoidable trading of unitarity against non locality have been also studied [11]. Recently, S.Hamamoto [12] has proven the equivalence of the path integral for the theory formulated in terms of constrained systems and Dirac’s method, and for the classical Ostrogradski’s treatment. However the particular case of relativistic covariant field theories, though involving only even differential orders, has complications of its own and, of course, is not trivially covered by those general treatments.

We address this issue by using a simplified model with scalar fields as in [7][8]. Our presentation highlights the Lorentz covariance and the particle aspect of the theory, with emphasis in the structure of the propagators and the coupling to other matter sources. In Section 2 we study the case of the 4-derivative theory for arbitrary masses, which exemplifies the use of the Helmholtz Lagrangian and the crucial diagonalization of the fields. In Section 3 we work out the 6-derivative case where the complications characterising the general case appear for the first time, including the occurrence of a new family of spurious fields. The notation needed to deal with the previous case is fully generalized in Section 4 where the general 2N-derivative theory is considered. Four Appendices are devoted to the technical details of some proofs and the cohomological interpretation of a result.
2. Four-derivative theory and notations.

In this section we will introduce a convenient notation, and try to get the reader acquainted with our treatment.

We adopt the Minkowski signature \((1, -1, -1, -1)\).
Masses are ordered such that \(m_i > m_j\) when \(i<j\).

\([i]\) \(\equiv (\Box + m_i^2)\), the Klein–Gordon (KG) operator for mass \(m_i\).

\(\langle ij \rangle \equiv (m_i^2 - m_j^2)\) is positive when \(i<j\). It will always be written with this ordering unless we wish to highlight some symmetry property.
Notice that \([j] = [i] - \langle ij \rangle\).

Here we generalize the example in [7] to arbitrary masses. Consider the 4–derivative scalar theory

\[
L^{(4)} = -\frac{1}{2} \frac{1}{(m_1^2 - m_2^2)} \phi^0 (\Box + m_1^2) (\Box + m_2^2) \phi^0 - j \phi^0 .
\] (2.1)

It yields the propagator

\[
- \frac{(m_1^2 - m_2^2)}{(\Box + m_1^2)(\Box + m_2^2)} = \frac{1}{\Box + m_1^2} - \frac{1}{\Box + m_2^2} ,
\] (2.2)
where the pole at \(m_2\) is physical and the one at \(m_1\) is a poltergeist.

Dropping total derivatives Eq.(2.1) may be written as

\[
L^{(4)}[\phi^0, [1]\phi^0] = -\frac{1}{2} \frac{1}{\langle 12 \rangle} \left( [1]\phi^0 \right)^2 - \langle 12 \rangle \phi^0 [1]\phi^0 \right] - j \phi^0 .
\] (2.3)

Define the canonical conjugate variable

\[
\pi \equiv \frac{\partial L^{(4)}}{\partial ([1]\phi^0)} = -\frac{1}{\langle 12 \rangle} \left[ [1]\phi^0 - \frac{1}{2} \langle 12 \rangle \phi^0 \right] ,
\] (2.4)
from which

\[
[1]\phi^0 = \langle 12 \rangle (-\pi + \frac{1}{2} \phi^0) \equiv Q \] .
(2.5)

The "Hamiltonian" is

\[
\mathcal{H}[\phi^0, \pi] \equiv \pi Q - L^{(4)}[\phi^0, Q] ,
\] (2.6)
that is
\[ \mathcal{H} = -\frac{1}{2} \langle 12 \rangle (-\pi + \frac{1}{2} \phi^o)^2 + j \phi^o \quad . \] (2.7)

The Helmholtz Lagrangian (the Euler equations of which are the canonical ones from \( \mathcal{H} \)) is
\[ \mathcal{L}^H \equiv \pi[1] \phi^o - \mathcal{H} \quad . \] (2.8)

Upon diagonalization by changing to new variables
\[ \phi^o = \phi_1^1 + \phi_2^1 \]
\[ \pi = \frac{1}{2} (\phi_1^1 - \phi_2^1) \quad , \] (2.9)
or conversely
\[ \phi_2^1 = -\pi + \frac{1}{2} \phi^o \quad , \]
\[ \phi_1^1 = \pi + \frac{1}{2} \phi^o \quad , \] (2.10)

one obtains the equivalent 2–derivative theory
\[ \mathcal{L}^{(2)} = \frac{1}{2} \phi_1^1[1] \phi_1^1 - \frac{1}{2} \phi_2^1[2] \phi_2^1 - j (\phi_1^1 + \phi_2^1) \quad . \] (2.11)

The propagators stemming from (2.11) are just the ones in the r.h.s. of (2.2). Notice that our notation is such that, here and in the following, the physical (negative) sign is beared by the lightest field, namely \( \phi_2^1 \) or generally the field with the highest subindex. This seemingly unnecessary wealth of upper and lower index labels has been introduced for further generalization.

The splitting of the quartic propagator displayed in (2.2) tells us that the emission of a "particle" endowed with such a propagator is actually equivalent to the emission of two particles (one physical and one with nonphysical norm) with quadratic propagators. They are made explicit in (2.11) by two independent fields the sum of which couples to the source.

The equivalence between the equations of motion can be tested as well. For (2.1) the HD Euler equation is
\[ -\frac{1}{\langle 12 \rangle} [1][2] \phi^o = j \quad . \] (2.12)
Hamilton’s canonical equations from (2.6) are, by definition, the same as Euler’s from (2.8) for $\phi^o$ and $\pi$, which respectively are

\[
[1]\pi = -\frac{1}{2} \langle 12 \rangle (-\pi + \frac{1}{2} \phi^o) + j \\
[1]\phi^o = \langle 12 \rangle (-\pi + \frac{1}{2} \phi^o).
\] (2.13)

Finally for the lower derivative (LD) theory (2.11) one has

\[
[1]\phi^1 = j \\
-[2]\phi^2 = j.
\] (2.14)

By changing variables according to (2.10) one may recast (2.14) into (2.13). Then working $\pi$ out of the 2nd equation (2.13) and substituting in the 1st one, after a little algebra one recovers (2.12). This proves the full (classical) equivalence of the theories (2.1) and (2.11).

3. 6-derivative theories

The conveniently normalized 6-derivative Lagrangian is

\[
\mathcal{L}_6^{\phi^o} = -\frac{1}{2} \frac{1}{\langle 12 \rangle \langle 13 \rangle \langle 23 \rangle} \phi^o[1][2][3]\phi^o - j\phi^o,
\] (3.1)

which may be rewritten as

\[
\mathcal{L}_\psi = -\frac{1}{2} \frac{1}{\langle 12 \rangle \langle 13 \rangle \langle 13 \rangle} \psi [1][2]\psi - j[3]^{-\frac{1}{2}}\psi,
\] (3.2)

where $\psi \equiv [3]^\frac{1}{2} \phi^o$. The Lagrangian (3.1) has mass dimension 2, so a further dimensional constant in front of it must be understood.

Applying the procedure described in the previous section to (3.2), with the diagonalization

\[
\psi^1_1 = \pi + \frac{1}{2} \psi \\
\psi^1_2 = -\pi + \frac{1}{2} \psi,
\] (3.3)

one obtains
\[ \mathcal{L}_{\psi_1^1 \psi_2^1} = \frac{1}{2} \frac{1}{(13)(23)} \psi_1^1 [1] \psi_1^1 - \frac{1}{2} \frac{1}{(13)(23)} \psi_2^1 [2] \psi_2^1 - j [3]^{-\frac{1}{2}} (\psi_1^1 + \psi_2^1) , \quad (3.4) \]

that with

\[ \phi_1^1 = \frac{[3]}{3}^{-\frac{1}{2}} \psi_1^1 \]
\[ \phi_2^1 = \frac{[3]}{3}^{-\frac{1}{2}} \psi_2^1 \],

(3.5)

gives

\[ \mathcal{L}_{\phi_1^1 \phi_2^1} = \frac{1}{2} \frac{1}{(13)(23)} \phi_1^1 [1] [3] \phi_1^1 - \frac{1}{2} \frac{1}{(13)(23)} \phi_2^1 [2] [3] \phi_2^1 - j (\phi_1^1 + \phi_2^1) . \quad (3.6) \]

We now repeat the procedure of Section 2 for both fields \( \phi_1^1 \) and \( \phi_2^1 \). Notice that the factors \( \frac{1}{(23)} \) and \( \frac{1}{(13)} \) are spectators when performing this process for the first and second fields respectively. The diagonalizations to be made now are given by

\[ \phi_1^2 = \pi_1^1 + \frac{1}{2} \phi_1^1 \]
\[ \Phi_3^2 = -\pi_1^1 + \frac{1}{2} \phi_1^1 \],

(3.7)

and

\[ \phi_2^2 = \pi_2^1 + \frac{1}{2} \phi_2^1 \]
\[ \Theta_3^2 = -\pi_2^1 + \frac{1}{2} \phi_2^1 \],

(3.8)

that lead to the following 2-derivative Lagrangian

\[ \mathcal{L}^2 = -\frac{1}{2} \frac{1}{(23)} \phi_1^2 [1] \phi_1^2 + \frac{1}{2} \frac{1}{(23)} \Phi_3^2 [3] \Phi_3^2 + \frac{1}{2} \frac{1}{(13)} \phi_2^2 [2] \phi_2^2 - \frac{1}{2} \frac{1}{(13)} \Theta_3^2 [3] \Theta_3^2 - j (\phi_1^2 + \phi_2^2 + \Phi_3^2 + \Theta_3^2) . \quad (3.9) \]

The surprise has come up of the duplication of particles with the same mass \( m_3 \). This can be dealt with by observing that only the linear combination \( \phi_3^2 = \Phi_3^2 + \Theta_3^2 \) couples to the source, whereas the linearly independent one \( \zeta_3^2 = C_{2} \frac{1}{(23)} \Phi_3^2 + C_{2} \Theta_3^2 \) is
decoupled, where $C_2$ is real and $\neq 0$. In terms of these new fields the theory gets its final transparent form

$$
\mathcal{L}^2 = -\frac{1}{2}\langle 23 \rangle \phi_1^2 + \frac{1}{2}\langle 13 \rangle \phi_2^2 + \frac{1}{2}\langle 12 \rangle \phi_3^2 - \frac{1}{2}\langle 23 \rangle \phi_1^2 - j(\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{1}{2}\langle 23 \rangle (C_2)^2 \langle 12 \rangle \langle 13 \rangle \zeta_3^2
$$

(3.10)

Contrarily to the expectations we have ended up with four degrees of freedom instead of three. However the field $\zeta_3^2$ is devoid of any dynamical content since it does not couple either to the source or to the other fields, it may then be arbitrarily normalized and does not propagate between sources. Therefore it must be regarded as an spurious field. A trivial way to dispose of it is realizing that (3.10) is invariant under the local Abelian transformations $\delta \zeta_3^2 = \lambda$, with $\lambda$ obeying $\langle 3 \rangle \lambda(x) = 0$, and using them to gauge away the field.

The equations of motion of the 2-derivative theory (3.10) are

$$
-\frac{1}{\langle 23 \rangle} \phi_1^2 = j
$$

$$
\frac{1}{\langle 13 \rangle} \phi_2^2 = j
$$

$$
-\frac{1}{\langle 12 \rangle} \phi_3^2 = j
$$

(3.11)

plus the trivial decoupled one

$$
\langle 3 \rangle \zeta_3^2 = 0
$$

(3.12)

Recovering the HD equation of motion corresponding to (3.1), namely

$$
-\frac{1}{\langle 12 \rangle \langle 13 \rangle \langle 23 \rangle} \phi^o = j
$$

(3.13)

can be achieved by adding and substracting the LD ones (3.11) and (3.12) between themselves and undoing the various diagonalizations and field redefinitions done at several stages.

Here also the dynamically irrelevant role of $\zeta_3^2$ is shown by the fact that its equation of motion adds or substracts 0 to the others in the first step. Moreover, even a coupling $-\lambda j \zeta_3^2$ added to (3.10) would be immaterial as long as the ensuing modification of the equation of motion (3.12) cancels out when going back to (3.13). This is shown in Appendix A.
Finally the propagators stemming from (3.10) are the pieces found in the algebraic splitting of the HD one, namely

\[-\frac{\langle 12 \rangle \langle 13 \rangle \langle 23 \rangle}{[1][2][3]} = -\frac{\langle 23 \rangle}{[1]} + \frac{\langle 13 \rangle}{[2]} - \frac{\langle 12 \rangle}{[3]} . \tag{3.14}\]

This completes the proof of the full (classical) equivalence of both theories (3.1) and (3.10).

4. 2N-derivative general theory.

Once the 6-derivative theory has been worked out, we face the general 2N-derivative case along the same lines. The Lagrangian is

\[\mathcal{L}_{\phi^o}^{2N} = -\frac{1}{2} \left( \frac{1}{(N-1)N} \right) \phi^o \prod_{(ij)=(12)}^{N} \langle ij \rangle \phi^o - j \phi^o . \tag{4.1}\]

The product in the denominator must be calculated for all the ordered pairs \((ij)\) ranging from \((12)\) to \((N-1)N\) with \(i < j\), so that always \(\langle ij \rangle > 0\). A dimensional constant has been omitted.

The HD propagator stemming from (4.1) can be expanded as follows

\[-\frac{\prod_{(ij)=(12)}^{(N-1)N} \langle ij \rangle}{\prod_{i=1}^{N} [i]} = \sum_{i=1}^{N} (-1)^{N+i+1} \frac{\prod_{k,l=(12)}^{(N-1)N} \langle kl \rangle}{\prod_{k,l \neq i}^{i} [i]} . \tag{4.2}\]

On the r.h.s. of (4.2), alternating plus and minus signs occur, the one for the smallest mass term being negative. So (4.2) gives us the splitting of the propagator with \(N\) poles in terms of simple Klein-Gordon quadratic propagators. The signs give unphysical character to many poles, but it is always possible to choose the one with the smallest mass as a physical pole, as done here.

In order to prove (4.2), we follow the induction method. Assume it holds for \(N\), then for \(N+1\) we have
where we have used (4.2) for the second step.

Taking into account that \( i \leq N \), and that as done in (2.2) for the poles 1 and 2, for \( i < j \) one has that

\[
\frac{1}{[i][j]} = -\frac{1}{\langle ij \rangle [i]} + \frac{1}{\langle ij \rangle [j]},
\]

the last expression in (4.3), may be written as

\[
\sum_{i=1}^{N} (-1)^{N+1+i+1} \prod_{\substack{k,l=(12) \\ k,l \neq \langle ij \rangle}}^{(N-1)N} \langle kl \rangle \prod_{m=1}^{N} \langle m N + 1 \rangle \frac{1}{[i]} \\
+ \left( \sum_{i=1}^{N} (-1)^{N+1+i} \prod_{\substack{k,l=(12) \\ k,l \neq \langle ij \rangle}}^{(N-1)N} \langle kl \rangle \prod_{m=1}^{N} \langle m N + 1 \rangle \right) \frac{1}{[N+1]},
\]

Now, in both terms the product symbols can be merged into one symbol so that (4.5) becomes

\[
\sum_{i=1}^{N} (-1)^{N+1+i+1} \prod_{\substack{k,l=(12) \\ k,l \neq \langle ij \rangle}}^{(N+1)N} \langle kl \rangle \prod_{m=1}^{N} \langle m N + 1 \rangle \frac{1}{[i]} [i][N+1] \\
+ \left( \sum_{i=1}^{N} (-1)^{N+1+i} \prod_{\substack{k,l=(12) \\ k,l \neq \langle ij \rangle}}^{(N+1)N} \langle kl \rangle \prod_{m=1}^{N} \langle m N + 1 \rangle \right) \frac{1}{[N+1]}. \]

Next the 2nd term in (4.6) could be embodied in the 1st one by extending there the summation to the value \( N + 1 \), (4.6) then becoming
\[\sum_{i=1}^{N+1} (-1)^{N+1+i+1} \frac{\prod_{(kl)=(12)}^{(N-1,N)}}{[i]} \langle kl \rangle, \quad (4.7)\]

provided we could show that
\[\sum_{i=1}^{N} (-1)^{N+1+i} \prod_{(kl)=(12)}^{(N-1,N)} \langle kl \rangle = (-1)^{N+N+3} \prod_{(kl)=(12)} \langle kl \rangle . \quad (4.8)\]

But (4.7) equals the l.h.s. of (4.3), so (4.2) is proved for \(N+1\) once (4.8) has been shown to hold. This is done in Appendix B. From (4.8), an interesting cohomological result is obtained in Appendix C.

Once the validity of the splitting formula for the 2N-order propagator has been shown, we steer to the problem of deriving the LD Lagrangian that yields the quadratic propagators.

The starting HD lagrangian (4.1), namely
\[\mathcal{L}^{2N} = -\frac{1}{2} \sum_{(ij)=(12)}^{(N-1,N)} \frac{1}{[ij]} \cdot \sum_{N} \phi^0 [1][2] \cdots [N] \phi^0 - j \phi^0 , \quad (4.9)\]
can be handled, as in the case of the 6-derivative theory, by successive Legendre transformations and one ends up with the following 2-derivative theory:
\[\mathcal{L}^2 = \frac{1}{2} \sum_{i=1}^{N} (-1)^{N-i+1} \left( \prod_{(mn)=(12)}^{(N-1,N)} \frac{1}{[mn]} \right) \phi_i^{N-1} [i] \phi_i^{N-1} - j \left( \sum_{i=1}^{N} \phi_i^{N-1} \right) \]
\[+ \frac{1}{2} \sum_{M=3}^{N} \left( \sum_{l=1}^{(2M-2)} (-1)^{N-M+l-1} \zeta_{Ml}^{N-1} [M] \zeta_{Ml}^{N-1} \right). \quad (4.10)\]

Here, the upper and lower indices in the fields \(\phi_i^{N-1}\) stand to indicate that they are obtained from \(\phi^0\) after \(N-1\) Legendre transformations and have mass \(m_i\). These fields couple to the source, and their free Lagrangians exactly fit what is needed to get the particle poles occurring in the r.h.s. of (4.2). Therefore the degrees of freedom are conserved and the physical or ghostly character of the fields in (4.10) are the same as in (4.2). A crowd of spurious fields \(\zeta_{Ml}^{N-1}\) arise, but again they are irrelevant. They are degenerate in mass, being their number \((2^M-2)\) for the mass \(m_M\), where \(M\)
ranges from 3 to $N$, and their Lagrangians bear the sign given in (4.10). The proof of the dynamical equivalence of (4.10) and (4.9) is carried out in Appendix D.

The LD equations of motion, namely

\begin{equation}
(-1)^{N-i+1} \left( \prod_{(mn)=\{12\}}^{(N-1,N)} \frac{1}{\langle mn \rangle} \right) [i] \phi_i^{N-1} = j \quad (i = 1, \ldots, N) \\
= \left( \prod_{(ij)=\{12\}}^{(N-1,N)} \frac{1}{\langle ij \rangle} \right) [1][2] \cdots [N] \phi^a = j 
\end{equation}

(4.11)

can be traced back to the HD one

\begin{equation}
(-1)^{N-M+l-1} [M] S_M^{N-1} = 0 \quad \left( \begin{array}{c} M = 3, \ldots, N \\ l = 1, \ldots, 2^{M-1} - 1 \end{array} \right) ,
\end{equation}

(4.12)

as in the 4-derivative and 6-derivative cases. This establishes the full classical equivalence between the HD and the LD theories also for the general 2N-derivative case.

5. Conclusions

Starting from a general HD relativistic covariant theory for a scalar field, we have devised the procedure for translating it into an equivalent 2-derivative theory with as many independent scalar fields as degrees of freedom the HD theory had. By studying the equations of motion we have assessed the full classical equivalence of both versions of the theory. The physical picture stemming from this result is that the emission of one "particle" by a source in a 2N-derivative theory, is equivalent to the emission of $N$ particles described by the usual Klein-Gordon 2-derivative theory.

The procedure followed here, based on the Legendre transformation, works only when all the masses involved are different, many expressions becoming singular otherwise as a consequence of the system not being regular. The case of the (conformally invariant in four dimensions) HD theories of gravity based on the squared Weyl tensor, where only the highest derivative terms occur since all the masses are zero, has this kind of difficulty. On the other hand, besides the alternating sign of the norm of the states in the LD theory, the scheme may also accommodate tachionic and/or massless states. In fact both the HD and the LD formulations depend only on the differences of the squared masses involved. So they are invariant under the shifting of all the squared masses by an arbitrary real quantity. Therefore any (but only one) of them can be brought to zero, the greater ones remaining positive and the lesser ones becoming negative (i.e. tachionic).
A key technique for the 2N-derivative theories, with $N \geq 3$, is the use of Legendre transformations involving analytical functions of the space-time derivatives. The typical example is the definition of the conjugate variable appearing in equation (3.3), namely $\pi \equiv \frac{\partial L_\phi}{\partial (\partial_1 \phi)} = \frac{\partial L_\phi}{\partial (\partial_1 [3] \phi)}$. The mathematics of this kind of transformations deserves further study in relation with the formalism developed in refs.[10]. The model presented here provides a working example.

An unavoidable feature of HD field theories is the occurrence of negative norm (poltergeist) states, which is synonymous of instability. The ensuing loss of unitarity seems hopeless unless the full quantum corrections are taken into account. Renormalization group calculations for 4-derivative gravity have failed to solve this difficulty. The problem is intrinsically associated to the finite differential order of the theory, but may be absent if infinitely higher order terms are considered [11] ($N \to \infty$), as it is the actual case of the effective theory stemming from the string and quantum field theory in curved background. Our simple scalar HD field model could provide a suitable test bed to implement these ideas.

The occurrence of spurious fields is an unexpected byproduct and is likely an artifact of our method. They are physically irrelevant once they turn out to be decoupled from the source and from the other true dynamical degrees of freedom. Several arguments stressing their irrelevance have been presented above, stressing the idea they are indeed an artifact. A refined version of the procedure we have followed might cope with them from the very beginning at the price of losing some clearness of presentation. It might also happen that they are naturally absent in the framework of the alternative formulation of Dirac’s method for constrained systems as adopted in [12].

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Appendix A

For use in the following we repeat the derivation of the eqn’s of motion from (2.14) back to (2.12), starting now from the suitably modified 2-derivative field eqn’s

\[ [1] \phi_1^1 = lj \]
\[ -[2] \phi_2^1 = kj \]

(A.1)

where \( l \) and \( k \) are arbitrary constants. In terms of \( \pi \) and \( \phi^o \) they translate into

\[ [1] \pi = -\frac{1}{2} \langle 12 \rangle (\pi + \frac{1}{2} \phi^o) + \frac{l + k}{2} j \]
\[ [1] \phi^o = \langle 12 \rangle (\pi + \frac{1}{2} \phi^o) + (l - k) j \]

(A.2)

Again, as in (2.13), working \( \pi \) out of the 2nd equation (A.2) and substituting it into the 1st, one obtains the HD eqn.

\[-\frac{1}{\langle 12 \rangle [1][2]} \phi^o = (l - \frac{l - k}{\langle 12 \rangle [1]}) j \]

(A.3)

to which corresponds the Lagrangian

\[ \mathcal{L}^{(4)}_{lk} = \frac{1}{2} \langle 12 \rangle \phi^o [1][2] \phi^o - (l - \frac{l - k}{\langle 12 \rangle [1]}) j \]

(A.4)

Let us come back now to the equations of motion of the 6-derivative theory with, eventually, a non-zero coupling constant \( \lambda \) of the spurious field to the source.

Equation (3.10) gets a term \(-j \lambda \zeta^2 \) so that (3.9) has the new contribution

\(-j (\lambda C_2 \langle 13 \rangle \Phi_3^2 + \lambda C_2 \Theta_3^2) \). Thus the equations of motion for \( \Phi_3^2 \) and \( \Theta_3^2 \) become

\[ \frac{1}{\langle 23 \rangle [3]} \Phi_3^2 = \left( 1 + \lambda C_2 \langle 13 \rangle \right) j \]
\[ -\frac{1}{\langle 13 \rangle [3]} \Theta_3^2 = \left( 1 + \lambda C_2 \right) j \]

(A.5)
which together with the field equations for $\phi_1^2$ and $\phi_2^2$ derived from (3.9), namely

$$-\frac{1}{\langle 23 \rangle}[1]\phi_1^2 = j$$

$$\frac{1}{\langle 13 \rangle}[2]\phi_2^2 = j$$

(A.6)

and equations (3.7),(3.8), combine to yield

$$\frac{1}{\langle 13 \rangle\langle 23 \rangle}[1][3]\phi_1^2 = \left(1 - \lambda \frac{C_2}{\langle 23 \rangle}[1]\right)j$$

$$-\frac{1}{\langle 13 \rangle\langle 23 \rangle}[2][3]\phi_2^2 = \left(1 - \lambda \frac{C_2}{\langle 23 \rangle}[2]\right)j$$

(A.7)

Using (3.5) they may be written as

$$\frac{1}{\langle 13 \rangle\langle 23 \rangle}[1]\psi_1^1 = l\left[3\right]^{-\frac{1}{2}}j$$

$$-\frac{1}{\langle 13 \rangle\langle 23 \rangle}[2]\psi_2^1 = k\left[3\right]^{-\frac{1}{2}}j$$

(A.8)

where $l = (1 - \lambda \frac{C_2}{\langle 23 \rangle}[1])$ and $k = (1 - \lambda \frac{C_2}{\langle 23 \rangle}[2])$. Now the same derivation that leads from (A.1) to (A.3) brings (A.8) to the equivalent equation

$$-\frac{1}{\langle 12 \rangle\langle 13 \rangle\langle 23 \rangle}[1][2]\psi = \left(l - \frac{l - k}{\langle 12 \rangle}\right)\left[3\right]^{-\frac{1}{2}}j$$

(A.9)

or, with $\psi = \left[3\right]^\frac{1}{2}\phi^\alpha$, to the final form

$$-\frac{1}{\langle 12 \rangle\langle 13 \rangle\langle 23 \rangle}[1][2][3]\phi^\alpha = \left(l - \frac{l - k}{\langle 12 \rangle}\right)j$$

(A.10)
But substituting the actual values of $k$ and $l$ one has that

$$
\left( l - \frac{l - k}{\langle 12 \rangle \langle 1 \rangle} \right) = 1 ,
$$

(A.11)

so that (A.10) is exactly the same unaltered HD field equation (3.13).

Appendix B

To prove (4.8), some arrangements are in order. First, the upper limit of the r.h.s. of (4.8) can be extended to the pair $(N N + 1)$ with the restriction $k, l \neq N + 1$; this property has already been used when getting (4.7) from (4.6). Secondly, a factor $(-1)^{N+2}$ can be deleted and, after bringing all the terms to the l.h.s., (4.8) adopts the simpler form

$$
\sum_{i=1}^{N+1} (-1)^{i-1} \prod_{\langle kl \rangle = \langle 12 \rangle \, \, k, l \neq i} \langle kl \rangle = 0 .
$$

(B.1)

This compact version will be given a meaning in Appendix C.

Proving (B.1) can be better done by recasting it in an even more convenient form. A common factor can be extracted from the sum, obtaining

$$
\prod_{\langle kl \rangle = \langle 12 \rangle} \langle kl \rangle \left[ \sum_{i=1}^{N+1} \left( \prod_{j=1}^{N+1} \frac{1}{\langle ij \rangle} \right) \right] = 0 .
$$

(B.2)

Notice that in writing (B.2), the couples $\langle ij \rangle$ have been let not to respect the ordering convention $i < j$. Thus, $i - 1$ of them are negative, which explains why the sign factor $(-1)^{i-1}$ in (B.1) does not occur in (B.2).

A sufficient condition for (B.2) to hold is that

$$
\sum_{i=1}^{N+1} \left( \prod_{j=1}^{N+1} \frac{1}{\langle ij \rangle} \right) = 0 ,
$$

(B.3)

and this will be proven again by induction. Let us assume (B.3) to be valid for $N$ terms, namely
\[
\sum_{i=1}^{N} \left( \prod_{j=0}^{N-1} \frac{1}{\langle i \, j+1 \rangle} \right) = 0 ,
\]
(B.4)

in which we have renamed \( j \) by \( j + 1 \). From eq.(B.4), assigning to the indices \( i = 1, 2, 3, ..., N \) new values \( i = 1, 3, 4, ..., N + 1 \) we also have

\[
\sum_{i=1}^{N+1} \left( \prod_{j=0}^{N} \frac{1}{\langle i \, j+1 \rangle} \right) = 0 ,
\]
(B.5)

and with the values \( i=2,3,4,...,N+1 \) one obtains

\[
\sum_{i=2}^{N+1} \left( \prod_{j=1}^{N} \frac{1}{\langle i \, j+1 \rangle} \right) = 0 .
\]
(B.6)

The equation (B.3) can be written as

\[
\frac{1}{\langle 12 \rangle} \left[ \prod_{j=3}^{N+1} \frac{1}{\langle 1 \, j \rangle} - \prod_{j=3}^{N+1} \frac{1}{\langle 2 \, j \rangle} \right] + \sum_{i=3}^{N+1} \left( \prod_{j=1}^{N} \frac{1}{\langle ij \rangle} \right) = 0 ,
\]
(B.7)

or as

\[
\frac{1}{\langle 12 \rangle} \left[ \prod_{j=2}^{N} \frac{1}{\langle 1 \, j+1 \rangle} - \prod_{j=2}^{N} \frac{1}{\langle 2 \, j+1 \rangle} \right] + \sum_{i=3}^{N+1} \left( \prod_{j=1}^{N} \frac{1}{\langle ij \rangle} \right) = 0 .
\]
(B.8)

The two terms inside the squared brackets in (B.8) have been arranged to coincide with the first ones in the sums of (B.5) and (B.6) respectively, so (B.8) can be written as

\[
\frac{1}{\langle 12 \rangle} \left[ - \sum_{i=3}^{N+1} \left( \prod_{j=0}^{N} \frac{1}{\langle i \, j+1 \rangle} \right) + \sum_{i=3}^{N+1} \left( \prod_{j=1}^{N} \frac{1}{\langle i \, j+1 \rangle} \right) \right] + \sum_{i=3}^{N+1} \left( \prod_{j=1}^{N} \frac{1}{\langle ij \rangle} \right) = 0 ,
\]
(B.9)
and renaming $j$ by $j + 1$ in the last sum we get

$$\frac{1}{\langle 12 \rangle} \left[ - \sum_{i=3}^{N+1} \left( \prod_{j=0}^{N} \langle i \ j + 1 \rangle \right) + \sum_{i=3}^{N+1} \left( \prod_{j=1}^{N} \langle i \ j + 1 \rangle \right) \right] + \sum_{i=3}^{N+1} \left( \prod_{j=0}^{N} \langle i \ j + 1 \rangle \right) = 0 .$$

(B.10)

Now we can check that (B.10) is true because it is exactly verified for each fixed index value $i=3,...,N+1$.

For the case $N = 2$, we have

$$\frac{1}{\langle 12 \rangle \langle 13 \rangle} + \frac{1}{\langle 21 \rangle \langle 23 \rangle} + \frac{1}{\langle 31 \rangle \langle 32 \rangle} = 0 .$$

(B.11)

This is immediately seen because (B.11) reduces to the trivial identity $\langle 23 \rangle - \langle 13 \rangle + \langle 12 \rangle = 0$, after multiplying by $\langle 12 \rangle \langle 13 \rangle \langle 23 \rangle$.

For any $m \geq 3$, we have in the same way that

$$\frac{1}{\langle 12 \rangle \langle 1m \rangle} + \frac{1}{\langle 21 \rangle \langle 2m \rangle} + \frac{1}{\langle m1 \rangle \langle m2 \rangle} = 0 .$$

(B.12)

Next we consider the terms with fixed $i = m \geq 3$ in (B.10), which we also claim to add to zero, namely

$$- \frac{1}{\langle 12 \rangle} \prod_{j=0}^{N} \langle m \ j + 1 \rangle + \frac{1}{\langle 12 \rangle} \prod_{j=1}^{N} \langle m \ j + 1 \rangle + \prod_{j=1}^{N+1} \frac{1}{\langle mj \rangle} = 0 ,$$

(B.13)

because, with $n = j + 1$ in the second and first terms, (B.13) can be written as

$$- \frac{1}{\langle 12 \rangle} \prod_{n=3}^{N+1} \frac{1}{\langle mn \rangle} + \frac{1}{\langle 12 \rangle} \prod_{n=3}^{N+1} \frac{1}{\langle mn \rangle} + \prod_{n=3}^{N+1} \frac{1}{\langle mn \rangle} = 0 ,$$

(B.14)

or else as
\[ \left[ \frac{1}{\langle 12 \rangle \langle 1m \rangle} + \frac{1}{\langle 21 \rangle \langle 2m \rangle} + \frac{1}{\langle m1 \rangle \langle m2 \rangle} \right] \prod_{n=3}^{N+1} \frac{1}{\langle mn \rangle} = 0 , \]

which trivially holds because of (B.12). Then (B.10) is true, and we have proven (B.3) for \( N + 1 \) terms, provided it holds for \( N \) terms; but (B.3) is true for \( N = 2 \), which is nothing but equation (B.11), so (B.3) is satisfied for any \( N \).

**Appendix C**

Consider the cohomological space spanned by the simplices

- 0 - simplices \( P_i \)
- 1 - simplices \( (P_i P_j) \)
- 2 - simplices \( (P_i P_j P_k) \)
- 3 - simplices \( (P_i P_j P_k P_l) \)
- ...... ......

corresponding to points \( i = 1, 2, \ldots \); ordered couples of points \( i < j \); ordered triads \( i < j < k \); ordered tetrads \( i < j < k < l \); etc., and endowed with the boundary operator \( \partial : \)

\[
\partial P_i = 0 \\
\partial (P_i P_j) = P_i - P_j \\
\partial (P_i P_j P_k) = (P_j P_k) - (P_i P_k) + (P_i P_j) \\
\partial (P_i P_j P_k P_l) = (P_j P_k P_l) - (P_i P_k P_l) + (P_i P_j P_l) - (P_i P_j P_k) \\
...... ...... 
\]

It can be trivially checked that \( \partial^2 = 0 \).

Any n-chain may be given a weight by assigning the following weights to the simplices:

\[
P_i \rightarrow m_i^2 \\
(P_i P_j) \rightarrow \langle ij \rangle \equiv m_i^2 - m_j^2 \\
(P_i P_j P_k) \rightarrow \langle ij \rangle \langle jk \rangle \langle ik \rangle \\
(P_i P_j P_k P_l) \rightarrow \langle ij \rangle \langle jk \rangle \langle ik \rangle \langle il \rangle \langle jl \rangle \langle kl \rangle \\
...... ...... 
\]
Then equation (B.1) can be read as the following statement: For \( n \geq 1 \), the weight of any closed \( n \)-chain is zero.

The lower (trivial) case of this statement is \( \langle 23 \rangle - \langle 13 \rangle + \langle 12 \rangle = \langle 12 \rangle + \langle 23 \rangle + \langle 31 \rangle = 0 \).

**Appendix D**

We will prove the equivalence of (4.9) and (4.10) again by the induction method. First note that (4.10) for \( N = 3 \) is just (3.10) where the coefficient in the spurious Lagrangian has been brought down to just \( \frac{1}{2} \) by a rescaling of the spurious field. Next take

\[
\psi = \left[ [3][4] \cdots [N + 1] \right]^{\frac{1}{2}} \phi^o ,
\]

and, following a similar procedure to the one used in the 6-derivative theory, define

\[
\pi \equiv \frac{\partial \mathcal{L}^{2N+1}}{\partial ([1] \psi)} ,
\]

go to the Helmholtz Lagrangian, and choose

\[
\psi_1^1 = \pi + \frac{1}{2} \psi ,
\]
\[
\psi_2^1 = -\pi + \frac{1}{2} \psi .
\]

The lagrangian for the \( 2(N + 1) \) case then reads:

\[
\mathcal{L}^{2(N+1)} = -\frac{1}{2} \prod_{(ij) = (13)}^{(N,N+1)} \frac{1}{ij} \left[ (-1) \psi_1^1 [1] \psi_1^1 + \psi_2^1 [2] \phi^1 \right] - j \left[ [3][4] \cdots [N+1] \right]^{-\frac{1}{2}} (\psi_1^1 + \psi_2^1) ,
\]

(D.4)
where the factor \( \frac{1}{\langle 12 \rangle} \) has been used to split the KG operators \([1]\) and \([2]\). With

\[
\phi_1 = \left[ [3][4] \cdots [N+1] \right]^{-\frac{1}{2}} \psi_1
\]

\[
\phi_2 = \left[ [3][4] \cdots [N+1] \right]^{-\frac{1}{2}} \psi_2 ,
\]

we have

\[
\mathcal{L}^{2(N+1)} = -\frac{1}{2} \prod_{(ij) = (13)}^{(N \, N+1)} \frac{1}{\langle ij \rangle} \left[ (-1) \phi_1 \left( \prod_{k=1}^{N+1} [k] \right) \phi_1 + \phi_2 \left( \prod_{k=2}^{N+1} [k] \right) \phi_2 \right] - j \phi_1^1 - j \phi_2^1 ,
\]

(D.5)

that can be written as

\[
\mathcal{L}^{2(N+1)} = \left[ (-1) \prod_{k=3}^{N+1} \frac{1}{(2k)} \right] \left[ -\frac{1}{2} \prod_{(ij) = (13)}^{(N \, N+1)} \frac{1}{\langle ij \rangle} \phi_1 \left( \prod_{k=1}^{N+1} [k] \right) \phi_1 \right] - j \phi_1^1 \\
+ \left[ \prod_{k=3}^{N+1} \frac{1}{(1k)} \right] \left[ -\frac{1}{2} \prod_{(ij) = (23)}^{(N \, N+1)} \frac{1}{\langle ij \rangle} \phi_2 \left( \prod_{k=2}^{N+1} [k] \right) \phi_2 \right] - j \phi_2^1 .
\]

(D.6)

Now, observe that inside the first bracket we have the expression for a 2N derivative theory, with \( \phi_1^1 \) in the place of \( \phi^o \), and with the KG operators \([1][3][4] \cdots [N]\) \([N+1]\]. The factor that multiplies the kinetic term, does not play any role as in the N=3 case. Inside the second bracket we have also a 2N derivative theory with \( \phi_2^1 \) in place of \( \phi^o \), and with the operators\([2][3] \cdots [N][N+1]\). Then, with the assumption that (4.10) is true for the 2N-derivative theory, we have
\( \mathcal{L}^{2(N+1)} = \left[ (-1)^{N+1} \prod_{k=3}^{N+1} \frac{1}{(2k)} \right] \left[ \frac{1}{2} (-1)^N \prod_{(mn) = (34), m, n \neq 1, 2} \frac{1}{\langle mn \rangle} \phi_i^N [1] \phi_i^N \right. \\
+ \frac{1}{2} \sum_{i=3}^{N+1} (-1)^{N-i} \prod_{(mn) = (13), m, n \neq i, m, n \neq 2} \frac{1}{\langle mn \rangle} \phi_i^N [i] \phi_i^N \\
+ \frac{1}{2} \sum_{M=4}^{N+1} \sum_{l_1=1}^{2M-3-1} (-1)^a \zeta_{M_{l_1}}^N [M] \zeta_{M_{l_1}}^N \\
+ \left. \left[ \prod_{k=3}^{N+1} \frac{1}{(1k)} \right] \left[ \frac{1}{2} (-1)^N \prod_{(mn) = (34), m, n \neq 1, 2} \frac{1}{\langle mn \rangle} \phi_2^N [2] \phi_2^N \right. \\
+ \frac{1}{2} \sum_{i=3}^{N+1} (-1)^{N-i} \prod_{(mn) = (23), m, n \neq i} \frac{1}{\langle mn \rangle} \phi_i^N [i] \phi_i^N \\
+ \frac{1}{2} \sum_{M=4}^{N+1} \sum_{l_2=1}^{2M-3-1} (-1)^b \zeta_{M_{l_2}}^N [M] \zeta_{M_{l_2}}^N \right] - j \left( \phi_1^N + \sum_{i_1=3}^{N+1} \phi_i^N \right) + \left. \sum_{i_1=3}^{N+1} \phi_i^N \right), \]

where \( a = N + 1 - M + l_1 - 1 \) and \( b = N + 1 - M + l_2 - 1 \).

To get to (D.8), one needs to notice that the number of the spurious fields associated to the operator \([M]\) depends on the place it occupies in the set \([1][3] \cdots [N+1]\) or in \([2][3] \cdots [N+1]\), that for \( i \geq 3 \) is \( M - 1 \); the same is true for the signs of the kinetic terms.

In (D.8), we see that there are two contributions to the \( i \)-th KG Lagrangian for \( i \geq 3 \). To disentangle this point we define

\[
\begin{align*}
\phi_i^N &\equiv \phi_{i_1}^N + \phi_{i_2}^N \\
\phi_{i_2}^N &\equiv C_{i_1} \phi_{i_1}^N + C_{i_2} \phi_{i_2}^N,
\end{align*}
\]

(D.9)
i.e.
\[ \phi_{i_1}^N = \frac{1}{C_{i_2} - C_{i_1}} (C_{i_2} \phi_{i_1}^N - \phi_{i_2}^N) \]
\[ \phi_{i_2}^N = \frac{1}{C_{i_2} - C_{i_1}} (\phi_{i_1}^N - C_{i_1} \phi_{i_2}^N) \]

for all \( i \geq 3 \). The KG Lagrangians for any \( i, i_1, i_2 \geq 3 \) can be then written as

\[
(-1)^{N-i+1} \frac{1}{2} \prod_{k=3}^{N+1} \frac{1}{\langle 2k \rangle} \prod_{(m,n) = (13)}^{(N,N+1)} \frac{1}{\langle mn \rangle} \phi_{i_1}^N [i] \phi_{i_1}^N
\]
\[
+ (-1)^{N-i} \frac{1}{2} \prod_{k=3}^{N+1} \frac{1}{\langle 1k \rangle} \prod_{(m,n) = (23)}^{(N,N+1)} \frac{1}{\langle mn \rangle} \phi_{i_2}^N [i] \phi_{i_2}^N
\]

\[
= (-1)^{N-i+1} \frac{1}{2} \prod_{(m,n) = (12)}^{(N,N+1)} \frac{1}{\langle mn \rangle} \left[ \frac{\langle 12 \rangle}{\langle 2i \rangle} \phi_{i_1}^N [i] \phi_{i_1}^N - \frac{\langle 12 \rangle}{\langle 1i \rangle} \phi_{i_2}^N [i] \phi_{i_2}^N \right],
\]

and with (D.10) we get

\[
(-1)^{N-i+1} \frac{1}{2} \prod_{(m,n) = (12)}^{(N,N+1)} \frac{1}{\langle mn \rangle} \left[ \left( \frac{\langle 12 \rangle}{\langle 2i \rangle} \frac{(C_{i_2})^2}{(C_{i_2} - C_{i_1})^2} - \frac{\langle 12 \rangle}{\langle 1i \rangle} \frac{(C_{i_1})^2}{(C_{i_2} - C_{i_1})^2} \right) \phi_{i_1}^N [i] \phi_{i_1}^N
\]
\[
+ \left( \frac{\langle 12 \rangle}{\langle 2i \rangle} \frac{1}{(C_{i_2} - C_{i_1})^2} - \frac{\langle 12 \rangle}{\langle 1i \rangle} \frac{1}{(C_{i_2} - C_{i_1})^2} \right) \phi_{i_2}^N [i] \phi_{i_2}^N
\]
\[
- \frac{\langle 12 \rangle}{\langle 2i \rangle} \frac{2C_{i_2}}{(C_{i_2} - C_{i_1})^2} \phi_{i_1}^N [i] \phi_{i_1}^N + \frac{\langle 12 \rangle}{\langle 1i \rangle} \frac{2C_{i_1}}{(C_{i_2} - C_{i_1})^2} \phi_{i_2}^N [i] \phi_{i_2}^N \right].
\]

Choosing

\[ C_{i_1} = \frac{\langle 1i \rangle}{\langle 2i \rangle} C_{i_2} \]

the crossed terms in (D.12) desappear, and after inserting the expression for \( C_{i_1} \) we get
\[ (-1)^{N+1-i+1} \frac{1}{2} \prod_{(mn) = (12), m,n \neq i}^{(N,N+1)} \frac{1}{\langle mn\rangle} \phi_i^N \phi_i^N + (-1)^{N-i+1} \frac{1}{2} \prod_{(mn) = (12), m,n \neq i}^{(N,N+1)} \frac{1}{\langle mn\rangle} \frac{1}{\langle 1i\rangle} \frac{1}{(C_{i2})^2} \phi_i^N \phi_i^N \]

(D.14)

So we observe, that the kinetic term for \( i \geq 3 \), is exactly the one we need to reach (4.10) for \( N + 1 \). On the other side, it is trivial to check that the kinetic terms in (D.8), for \( \phi_1^N \) and \( \phi_2^N \), are the appropriated ones to fulfill (4.10) for the desired \( N \).

A brief statistics of the spurious fields is in order. In (D.14) we get a rather complicated coefficient that is immaterial because we can arbitrarily normalize these fields since they do not couple to the other fields and sources. Just notice that it is positive for \( i = N + 1 \), and alternating in sign as \( i \) gets lesser. The mass degeneracy is the following: With mass \( m_M \), i.e. associated to the KG operator \([M]\) in (D.8), we have \( 2^{M-3} - 1 \) spurious terms with positive norm and \( 2^{M-3} - 1 \) with negative norm. Equation (D.14) yields a further one, rising the total number to \( 2^{M-2} - 1 \). For \( i = N + 1 \) the positive terms outnumber the negative ones by one unit, with this balance alternating for dwindling \( i \).

This proves (4.10), because it holds for \( N + 1 \) if it does for \( N \), and the procedure to prove the case \( N = 3 \) is legitimate.
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