ASYMPTOTIC ANALYSIS OF A SIZE-STRUCTURED CANNIBALISM POPULATION MODEL WITH DELAYED BIRTH PROCESS

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Abstract. In this paper, we study a size-structured cannibalism model with environment feedback and delayed birth process. Our focus is on the asymptotic behavior of the system, particularly on the effect of cannibalism and time lag on the long-term dynamics. To this end, we formally linearize the system around a steady state and study the linearized system by $C_0$-semigroup framework and spectral analysis methods. These analytical results allow us to achieve linearized stability, instability and asynchronous exponential growth results under some conditions. Finally, some examples are presented and simulated to illustrate the obtained stability conclusions.

1. Introduction. Population dynamics has been a central fixture in mathematical biology for more than two centuries, starting with Malthus' exponential model of population growth. The main focus of population dynamics has been a characterization of alterations in the numbers, sizes and age distribution of individuals and of potential internal or external causes provoking these changes. Traditionally, structured population models are typically formulated as partial differential equations for population densities. In the last three decades, linear or nonlinear age/size-structured population models have attracted a lot of interest among both theoretical biologists and applied mathematicians. Diekmann et al. have developed a general mathematical framework to study analytical questions for structured populations (see [6, 7]), including those pertaining to linear/nonlinear stability of population equilibria. In this context it was recently proven for large classes of structured population models, formulated as integral (or delay) equations, that the nonlinear stability/instability of a population equilibrium is completely determined by its linear stability/instability. In the recent years, Farkas and Hagen successfully applied linear semigroup methods to formulate biologically interpretable conditions for the linear stability/instability of equilibria of size-structured population models (see [11, 12, 13]). In these problems they assumed that any effect of intraspecific...
competition between individuals of different sizes on individual behavior is primarily due to a change in population size and that every individual in the population can influence the vital rates of other individuals.

It is well known that, cannibalism, or intraspecific predation, is a phenomenon which occurs in a wide variety of organisms, including many species of protozoa, rotifers, gastropods, copepods, insects, fish, and some species of amphibians, birds, and mammals (see [10]). Sophisticated population models are capable to elucidate a potentially stabilizing effect of cannibalism, underscoring that certain populations may benefit from cannibalism when resources are limited. Consequently, the effects of cannibalism on the long-term dynamics of population have attracted considerable interest and have been analyzed for various structured population models, see [4, 14, 19] for instance.

The above mentioned research was all based on the assumption that newborns in the population are fertile from birth on. This assumption is justified for a large population of primitive species, but is unrealistic in many other populations of multicellular organisms. Hence the population models with delay in birth process are considered naturally. Some equations, in which the birth process depends on the history of the population appear, for instance, in the models of host-parasite interactions (see [6, 7]). Here the delay is given by the time lag between laying and hatching of the parasite eggs. In general size-structured population equations with delay in the birth process occur in models where there is a time lag between conception and birth. Such problems have also been studied in the non-distributed delay case by Swick [30, 31], in the nonlinear case by Di Blasio [5], and in the linear case by Guo and Chan [22]. Recently, linear age-dependent model with delay in birth process was treated in [15, 28, 29], they applied Perron-Frobenius techniques (see [20]) and theory for positive semigroups to establish some stability criteria. In [16, 17] the authors have adopted the similar methods to study the linearized stability problems for nonlinear population models. We would like to mention the well-known books [25, 32] for references here.

In this paper, we study the asymptotic behavior for a cannibalism population model with delayed birth process to extend the work in [11, 12, 13, 14, 16]. That is, we will introduce a time lag into the birth process and explore how the time lag influences the asymptotic behaviors of the cannibalism population system. More precisely, we consider the problem

\[
\begin{cases}
\frac{\partial}{\partial t} n(s, t) + \frac{\partial}{\partial s} (\gamma(s, E(s, t))n(s, t)) = -(\mu(s, E(s, t)) + M(s, t))n(s, t), \\
n(0, t) = \int_0^m \int_{-\tau}^0 \beta(s, \sigma, E(s, t + \sigma))n(s, t + \sigma)d\sigma ds, \\
n(s, t) = n^0(s, t), \quad s \in [0, m], \quad t \in [-\tau, 0],
\end{cases}
\]

(1.1)

where the function \( n = n(s, t) \) denotes the density of individuals of size \( s \in [0, m] \) at time \( t \in [0, \infty) \). We shall employ the delay semigroup techniques and spectral methods to investigate the asymptotic behavior of solutions for the model (1.1).

The model equation involves the vital rates \( \mu = \mu(s, E(s, t)) \)-mortality, \( \beta = \beta(s, \sigma, E(s, t + \sigma)) \)-fertility and \( \gamma = \gamma(s, E(s, t)) \)-growth rate of individuals. All the vital rates are size dependent. Moreover, it is assumed that the mortality, growth and birth rates of individuals depend on the extra size-specific energy intake due to cannibalism at time \( t \) which is given by
\[ E(s, t) = \int_0^m c(y)\alpha(y, s)n(y, t)\,dy. \] (1.2)

Consequently, the model (1.1) is nonlinear. Here, as in [14], we assume that individuals eat their conspecifics and that this cannibalistic behaviour is modelled through the size-specific attack rate \( \alpha(y, s) \), which is the rate at which individuals of size \( s \) kill and eat individuals of size \( y \). Usually the victim of cannibalism is smaller than the attacker, so \( \alpha(y, s) \) should be zero for \( y > s \), but we will make no explicit use of this assumption in what follows. We assume that all individuals are born with the same size 0. While \( c(y) \) is the energetic value of an attacked individual of size \( y \). We assume that \( E \) is channelled into growth and affects ordinary mortality, that is, mortality not due to cannibalism but for example, due to starvation.

Cannibalism leads to an extra mortality in the population. The extra size-specific mortality rate due to cannibalism at time \( t \) is given by

\[ M(s, t) = \int_0^m \alpha(s, y)n(y, t)\,dy. \] (1.3)

(Note the switch of variables in \( \alpha \) in contrast to (1.2).)

In addition, for the remainder of this work we impose the following conditions on the model ingredients:

\[
\begin{align*}
\mu &= \mu(s, E) \in C([0, m]; C^1_b[0, \infty)), \\
\gamma &= \gamma(s, E) \in C^1([0, m]; C^1_b[0, \infty)), \quad \gamma \geq \gamma_0 > 0 \text{ for some constant } \gamma_0, \\
\beta &= \beta(s, \sigma, E) \in C([0, m] \times [-\tau, 0); C^1_b[-\tau, \infty)), \quad \beta \geq 0, \\
c &= c(s) \in C([0, m]), \quad c \geq 0, \\
\alpha &= \alpha(y, s) \in C([0, m]; C^1([0, m]), \quad \alpha \geq 0, \\
n^0(s, t) &\in C^1([0, m] \times [-\tau, 0)).
\end{align*}
\]

For clarity in later developments we will write \( D_2\alpha \) for the derivative of \( \alpha \) with respect to its second argument. The regularity assumptions above are tailored toward the linear analysis of this work. They might, however, not suffice to guarantee the existence and uniqueness of solutions of Eqs. (1.1), even in the steady-state case. Well-posedness of structured partial differential equation models with infinite dimensional environmental feedback variables is in general an open question. It has recently been shown in [1] that population models with infinite dimensional interaction variables may exhibit a more complicated dynamical behavior than the simple size-structured model of scramble competition.

Of interest in this work is to investigate the linearized stability and instability of stationary solutions of the system (1.1) by using semigroup techniques and spectral methods based on the characteristic equation, and meanwhile, the asynchronous exponential growth property (AEG, for short) of solutions is studied based on the spectral analysis as well. We mainly employ the Perron-Frobenius approaches to carry on our discussions and some results on linearized stability and AEG are obtained under proper conditions. It can be easily seen that these obtained results extend and develop the corresponding ones mentioned above. Moreover, the conditions for stability and AEG addressed in this paper are concrete and more applicable than those in paper [14] where the conditions are somewhat abstract.

This paper is organized as follows. In Section 2, we propose the nonlinear model (1.1) with delay in birth process and linearize the system. In Section 3, we set the linear system in the framework of semigroup theory, and prove the existence
and uniqueness of solutions for the simplified system by showing that the related abstract Cauchy problem gives rise to a strongly continuous semigroup. In Section 4, some regularity properties are derived for the linear system when the attack rate is separable, following that we deduce the characteristic equation in Section 5. Then we discuss in Section 6 the linearized stability and instability of the stationary solution under some conditions. And in Section 7 we devote to discussing the AEG property for the linearized system and present some conditions for AEG. Finally, in Section 8, we provide some simple examples to illustrate the obtained stability results through numerical simulations.

2. The linearized system. The system (1.1) admits obviously the trivial solution \( n_\ast \equiv 0 \). Realistically we also expect some additional positive (continuously differentiable) stationary solutions \( n_\ast \) for (1.1). In the following portion we formulate a necessary condition for the existence of a positive equilibrium solution of problem (1.1).

**Proposition 1.** For the given vital rate functions \( \beta, \mu, \gamma \), if \( n_\ast \) is a positive stationary solution of problem (1.1), then the functions \( E_\ast \) and \( M_\ast \), defined respectively by

\[
E_\ast(s) = \int_0^m c(y)\alpha(s,y)n_\ast(y)dy,
\]

\[
M_\ast(s) = \int_0^m \alpha(s,y)n_\ast(y)dy,
\]

satisfy

\[
1 = \int_0^m \gamma^*(0)\frac{\gamma^*(s)}{\gamma(s)}e^{-\int_0^s \mu(y,E_\ast(y)) + M_\ast(y)dy}\int_{-\tau}^0 \beta(s,\sigma,E_\ast(s))d\sigma ds,
\]

where we have set

\[
\gamma^*(s) = \gamma(s,E_\ast(s)), \ s \in [0,m].
\]

Or equivalently \( R(n_\ast) = 1 \) for any positive stationary solution \( n_\ast \), where the function

\[
R(t) = \int_0^m \frac{\gamma(0,E(0,\cdot))}{\gamma(s,E(s,\cdot))}e^{-\int_0^s \mu(y,E(0,\cdot)) + M_\ast(y)dy}\int_{-\tau}^0 \beta(s,\sigma,E(s,\cdot + \sigma))d\sigma ds
\]

can be regarded as the net reproduction rate of the standing population. In this case, the unique positive stationary solution \( n_\ast \) of problem (1.1) is given by

\[
n_\ast(s) = \frac{N_\ast e^{-\int_0^s \frac{\mu(y,E_\ast(y)) + M_\ast(y) + \gamma(0)}{\gamma(s)}dy}}{\int_0^m e^{-\int_0^s \frac{\mu(y,E_\ast(y)) + M_\ast(y) + \gamma(0)}{\gamma(s)}dy}ds},
\]

where \( N_\ast = \int_0^m n_\ast(s)ds \) represents the positive population quantity.

**Proof.** Suppose that \( n_\ast(s) \) is a positive stationary solution of the system (1.1). Then \( n_\ast \) satisfies the equations

\[
\frac{\partial}{\partial s} \gamma^*(s)n_\ast(s) = -(\mu(s,E_\ast(s)) + M_\ast(s))n_\ast(s),
\]

\[
n_\ast(0) = \int_0^m \int_{-\tau}^0 \beta(s,\sigma,E_\ast(s))n_\ast(s)d\sigma ds.
\]

The general solution of Eq. (2.7) is found as

\[
n_\ast(s) = e^{-\int_0^s \frac{\mu(y,E_\ast(y)) + M_\ast(y) + \gamma(0)}{\gamma(s)}dy}.
\]
Observing that
\[ e^{-\int_0^\tau \frac{\gamma^*(y)}{\gamma^*(y)} \, dy} = \frac{\gamma^*(0)}{\gamma^*(s)}, \]
and substituting (2.9) into (2.8), we obtain Eq. (2.3), that is \( R(n_\ast) = 1 \). Finally, integrating (2.9) from 0 to \( m \), we have
\[ n_\ast(0) = \frac{N_\ast}{\int_0^m e^{-\int_0^\tau \frac{\gamma^*(y)}{\gamma^*(y)} \, dy} \, ds}. \] (2.10)

Taking (2.10) into (2.9), we get the form of (2.6).

If, on the other hand, \( n_\ast \) is defined by (2.6), then \( n_\ast \) is readily seen to be a positive stationary solution. \( \square \)

**Remark 1.** Here and later on, starred quantities are stationary counterparts of the time dependent functions in Eqs. (1.1). For obvious reasons, we shall exclusively consider positive stationary solutions of the form (2.6) or the trivial solution \( n_\ast \equiv 0 \) in the following. Moreover, to be consistent with later developments, we shall assume throughout that stationary solutions \( n_\ast \) have the regularity \( W^{1,1}(0, m) \).

Given any stationary solution \( n_\ast \) of the system (1.1), we linearize the governing equations by introducing the infinitesimal perturbation \( u = u(s, t) \) and making the ansatz \( n = u + n_\ast \). Hence \( u \) has to satisfy the equations
\[
\begin{aligned}
\frac{\partial}{\partial t} u(s, t) + \frac{\partial}{\partial s} \left( \gamma(s, E(s, t))u(s, t) + (\mu(s, E(s, t)) + M(s, t))u(s, t) \\
+ \frac{\partial}{\partial s} (\gamma(s, E(s, t))n_\ast(s)) + (\mu(s, E(s, t)) + M(s, t))n_\ast(s) = 0, \\
u(0, t) = \int_0^m \int_0^\tau \beta(s, \sigma, E(s, t))u(s, t + \sigma) + n_\ast(s) d\sigma ds - n_\ast(0), \\
u(s, t) = u_0(s, t), \quad t \in [-\tau, 0], \quad s \in [0, m].
\end{aligned}
\] (2.11)

Now we linearize the vital rates. To this end, we note that the functional dependence of the vital rates on \( E \) rather than on \( n \) requires the linearization about \( E_\ast \). Thus by the approximations
\[
\begin{aligned}
\mu(s, E(s, t)) &= \mu(s, E_\ast(s)) + \mu_E(s, E_\ast(s))(E(s, t) - E_\ast(s)) + \text{h.o.t.}, \\
\beta(s, \sigma, E(s, t + \sigma)) &= \beta(s, \sigma, E_\ast(s)) + \beta_E(s, \sigma, E_\ast(s))(E(s, t + \sigma) - E_\ast(s)) + \text{h.o.t.}, \\
\gamma(s, E(s, t)) &= \gamma(s, E_\ast(s)) + \gamma_E(s, E_\ast(s))(E(s, t) - E_\ast(s)) + \text{h.o.t.}
\end{aligned}
\]
in system (2.11) and dropping all the nonlinear terms, we arrive at the linearized problem
\[
\begin{aligned}
\frac{\partial}{\partial t} u(s, t) + \gamma(s) \frac{\partial}{\partial s} u(s, t) \\
+ \rho^*(s) u(s, t) + \left[ (\gamma^*_E(s)n_\ast(s))_s + \mu_E(s, E_\ast(s))n_\ast(s) \right] F(s, t) + \gamma^*_E(s)n_\ast(s) F(s, t) \\
+ n_\ast(s) N(s, t) = 0, \quad t > 0, \quad s \in (0, m], \\
u(0, t) = \int_0^m \int_0^\tau \beta(s, \sigma, E_\ast(s))u(s, t + \sigma) d\sigma ds \\
+ \int_0^m n_\ast(s) \int_0^\tau \beta_E(s, \sigma, E_\ast(s)) F(s, t + \sigma) d\sigma ds, \quad t > 0, \\
u(s, t) = u^0(s, t) := n_\ast(s) - n^*(s), \quad t \in [-\tau, 0], \quad s \in [0, m],
\end{aligned}
\] (2.12)
where we have set
\[ F(s, t) = E(s, t) - E_*(s) = \int_0^m c(y)\alpha(y, s)u(y, t)dy, \quad (2.13) \]
\[ N(s, t) = M(s, t) - M_*(s) = \int_0^m \alpha(s, y)u(y, t)dy, \quad (2.14) \]
\[ \rho^*(s) = \gamma^*_E(s) + \mu(s, E_*(s)) + M_*(s). \quad (2.15) \]

3. **\(C_0\)-semigroup for linear system.** To analysis the asymptotic behavior for linearized system (2.12), we establish in this section the \(C_0\)-semigroup framework for this system and through which rewrite it into an abstract evolution equation.

Suppose \(n_*\) is any positive stationary solution of problem (1.1). We denote the Banach space \(X = L^1(0, m)\) with the usual norm \(\|\cdot\|\) and on this space we introduce the following operator
\[
(A_{m}f)(s) = -\gamma^*(s)f'(s) - \rho^*(s)f(s) \quad \text{for a.e. } s \in [0, m],
\]
with domain \(D(A_{m}) = W^{1,1}(0, m)\). Moreover, we call the map
\[
\mathcal{P} : D(A_{m}) \to \mathbb{C}, \quad \mathcal{P}f := f(0)
\]
the boundary operator, which is used to express the boundary condition (see [21, 28]). Define the bounded operator \(B_{m}\) as
\[
(B_{m}f)(s) = -\int_0^m f(y) \left[ c(y)\alpha(y, s) ((\gamma^*_E(s)n_*(s))s + \mu_{E}(s, E_*(s))n_*(s)) \right] dy \quad \text{for a.e. } s \in [0, m],
\]
with domain \(D(B_{m}) = L^1(0, m)\). The subscript ‘\(m\)’ reminds that the operators are defined on their maximal domain.

Look at the Banach space
\[ E := L^1([-\tau, 0], X) \cong L^1((0, m) \times [-\tau, 0]). \]

On this space we introduce the operator \(\Phi \in \mathcal{L}(E, \mathbb{C})\), by setting, for \(g \in E\),
\[
\Phi g = \int_0^m \int_{-\tau}^0 \beta(s, \sigma, E_*(s))g(s, \sigma)d\sigma ds
\]
\[
+ \int_0^m n_*(s) \int_{-\tau}^0 \beta_{E}(s, \sigma, E_*(s)) \int_0^m c(y)\alpha(y, s)g(y, \sigma)dyd\sigma ds.
\]

Then with these operators the linearized system (2.12) can be cast in the form of an abstract boundary delay problem:
\[
\begin{align*}
\frac{d}{dt}u(t) &= (A_{m} + B_{m})u(t), \quad t \geq 0, \\
P u(t) &= \Phi u_t, \quad t \geq 0, \\
u_0(t) &= u^0(t), \quad t \in [-\tau, 0],
\end{align*}
\quad (3.1)
\]

where \(u^0(t) := u^0(\cdot, t), \, u : [0, +\infty) \to X\) is defined as \(u(t) := u(\cdot, t)\) and \(u_t : [-\tau, 0] \to X\) is the history segment defined in the usual way as
\[ u_t(\sigma) := u(t + \sigma), \quad \sigma \in [-\tau, 0]. \]

In order to apply the \(C_0\)-semigroup theory, we rewrite (3.1) as an abstract Cauchy problem. For this, on the space \(E\), we consider the differential operator
\[ (G_{m}g)(\sigma) := \frac{d}{d\sigma}g(\sigma), \]
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with domain $D(G_m) = W^{1,1}([-\tau, 0], X)$. In addition, we introduce another boundary operator $Q : D(G_m) \to X$ defined as

$$Qg := g(0).$$

Finally, we consider the product space $X' := E \times X$, on which we define the operator matrix

$$A = A_1 + A_2,$$

where

$$A_1 := \begin{pmatrix} G_m & 0 \\ 0 & A_m \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 \\ 0 & B_m \end{pmatrix},$$

with domain

$$D(A) = D(A_1) = \{ \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in D(G_m) \times D(A_m) : \begin{pmatrix} \Phi & \Psi \end{pmatrix} = \begin{pmatrix} f \\ \Phi \end{pmatrix} \}.$$

With these notations, we obtain the following abstract Cauchy problem

$$\begin{cases} U'(t) = AU(t), & t \geq 0, \\ U(0) = U_0, \end{cases}$$

associated to the operator $(A, D(A))$ on the space $X$. Here the function $U : [0, +\infty) \to X$ is given by

$$U(t) = \begin{pmatrix} u_t \\ u(t) \end{pmatrix}.$$

To obtain the well-posedness of solutions for the abstract Cauchy problem (3.2), in the sequel we will verify that $(A_1, D(A_1))$ generates a $C_0$-semigroup on $X$ firstly, then we show that the operator $A = A_1 + A_2$ generates a $C_0$-semigroup by the perturbation theorem.

In the first step, we consider the Banach space $X' := E \times X \times X \times \mathbb{C}$ and the matrix operator

$$\mathcal{A} := \begin{pmatrix} G_m & 0 & 0 & 0 \\ -\Phi & 0 & Id & 0 \\ 0 & 0 & A_m & 0 \\ \Phi & 0 & -\Psi & 0 \end{pmatrix},$$

with domain $D(\mathcal{A}) = D(G_m) \times \{0\} \times D(A_m) \times \{0\}$.

**Lemma 3.1** (see [9, 27]). Let $(A, D(A))$ be a Hille-Yosida operator on a Banach space $X$ and $B$ be a bounded linear operator on $X$, then the sum $C = A + B$ is also a Hille-Yosida operator.

By this lemma we can prove that

**Proposition 2.** The operator $(\mathcal{A}, D(\mathcal{A}))$ is a Hille-Yosida operator on the Banach space $X'$.

**Proof.** The operator $\mathcal{A}$ can be written as the sum of two operators on $X'$ as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 := \begin{pmatrix} G_m & 0 & 0 & 0 \\ -\Phi & 0 & 0 & 0 \\ 0 & 0 & A_m & 0 \\ 0 & 0 & -\Psi & 0 \end{pmatrix}, \quad \mathcal{A}_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & 0 \end{pmatrix},$$

with $D(\mathcal{A}_1) = D(\mathcal{A})$ and $D(\mathcal{A}_2) = X'$. 
The restriction \((G_0, D(G_0))\) of \(G_m\) to the kernel of \(Q\) generates the nilpotent left shift semigroup \((S_0(t))_{t \geq 0}\) on \(E\) given by the formula
\[
((S_0(t))g)(s, \sigma) = \begin{cases} 
g(s, t + \sigma), & \text{if } t + \sigma \leq 0, \\
0, & \text{if } t + \sigma \geq 0.
\end{cases}
\]
Similarly, the restriction \((A_0, D(A_0))\) of \(A_m\) to the kernel of \(P\) generates a strongly continuous positive semigroup \((T_0(t))_{t \geq 0}\) on \(X\) given by
\[
(T_0(t)f)(s) = \begin{cases} 
e^{-\int_0^t (\Gamma(s) - t)} \frac{\gamma^*(\omega)}{\gamma^*(s)} d\gamma(f(\Gamma^{-1}(\Gamma(s) - t))), & \text{if } \Gamma(s) \geq t, \\
0, & \text{if } \Gamma(s) < t,
\end{cases}
\]
where
\[
\Gamma(s) = \int_0^s \frac{1}{\gamma^*(y)} dy.
\]
We claim that \(\mathscr{A}_1\) is a Hille-Yosida operator. In fact, for any \(\lambda \in \mathbb{C}\) and \(f_1 \neq 0\), let
\[
(\lambda I - A_0)f = f_1,
\]
we get
\[
f(s) = e^{-\int_0^s \frac{\lambda + \rho^*(s)}{\gamma^*(s)} dy} \int_0^s \frac{f_1(r)}{\gamma^*(r)} e^{\int_r^s \frac{\lambda + \rho^*(y)}{\gamma^*(y)} dy} dr,
\]
which shows \(\sigma(A_0) = \emptyset\) as \(\gamma^*(s) > 0\). And similarly, \(\sigma(G_0) = \emptyset\). So for every \(\lambda \in \mathbb{C}\), its resolvent is given by
\[
R(\lambda, \mathscr{A}_1) = \begin{pmatrix} R(\lambda, G_0) & \epsilon_\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R(\lambda, A_0) & \varphi_\lambda \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
where
\[
\epsilon_\lambda(\sigma) = e^{\lambda \sigma}, \quad \sigma \in [-\tau, 0], \quad \text{and} \quad \varphi_\lambda(s) = e^{-\int_0^s \frac{\lambda + \rho^*(y)}{\gamma^*(y)} dy}, \quad s \in [0, m].
\]
Moreover,
\[
\ker(\lambda - G_m) = \{ f : \epsilon_\lambda : f \in X \}, \quad \ker(\lambda - A_m) = \langle \varphi_\lambda \rangle.
\]
Let then \((g, f_1, f_2)^T \in \mathscr{X}^\prime\) and \(\lambda > 0\). It is true that
\[
\begin{align*}
\| R(\lambda, \mathscr{A}_1)(g f_1 f_2)^T \| &= \| R(\lambda, G_0)g + \epsilon_\lambda f_1 \|_E + \| R(\lambda, A_0)f_2 + x \varphi_\lambda \|_X \\
&\leq \| R(\lambda, G_0)f_1 \|_E + \| \epsilon_\lambda g \|_E + \| R(\lambda, A_0)f_2 \|_X + \| x \varphi_\lambda \|_X \\
&\leq \int_{-\tau}^0 \frac{1}{\lambda} \| g(\sigma) \|_X d\sigma + \frac{1}{\lambda} \| f_1 \|_X + \| f_2 \|_X \| x \|_X \\
&= \frac{1}{\lambda} \left( \| g \|_E + \| f_1 \|_X + \| f_2 \|_X + \| x \|_X \right).
\end{align*}
\]
Therefore, we have
\[
\| \lambda R(\lambda, \mathscr{A}_1) \| \leq 1,
\]
and \(\mathscr{A}_1\) is a Hille-Yosida operator.

Since the perturbation operator \(\mathscr{A}_2\) is clearly bounded, \(\mathscr{A}\) is a Hille-Yosida operator as well by Lemma 3.1.
Any Hille-Yosida operator gives rise to a strongly continuous semigroup on the closure of its domain, that is,

**Lemma 3.2** (see [27]). Let \((\mathcal{A}, D(\mathcal{A}))\) be a Hille-Yosida operator on the Banach space \(X\) and set \(X_0 := (\overline{D(\mathcal{A})}, \|\cdot\|)\), \(D(\mathcal{A}_0) = \{x \in D(\mathcal{A}) : Ax \in X_0\}\), \(A_0 x := Ax\) for \(x \in D(\mathcal{A}_0)\). Then the operator \((\mathcal{A}_0, D(\mathcal{A}_0))\), called the part of \(\mathcal{A}\) in \(X_0\), is the generator of a strongly continuous semigroup on \(X_0\) denoted by \((T_0(t))_{t \geq 0}\).

According to this lemma, we have actually obtained by Proposition 2 that the operator \((\mathcal{A}_0, D(\mathcal{A}_0))\), the part of the operator \((\mathcal{A}, D(\mathcal{A}))\) in the closure of its domain, generates a \(C_0\)-semigroup on the space \(E \times \{0\} \times X \times \{0\}\). Now we show that the operator \((\mathcal{A}_1, D(\mathcal{A}_1))\) generates a strongly continuous semigroup on \(\mathcal{X}\) by the following theorem.

**Theorem 3.3.** The operator \((\mathcal{A}_1, D(\mathcal{A}_1))\) is isomorphic to the part \((\mathcal{A}_0, D(\mathcal{A}_0))\) of the operator \((\mathcal{A}, D(\mathcal{A}))\) in the closure of its domain \(\overline{D(\mathcal{A})}\).

In particular, \((\mathcal{A}_1, D(\mathcal{A}_1))\) generates a \(C_0\)-semigroup on the space \(\mathcal{X}\).

**Proof.** From the arguments following Lemma 3.2, we know that the part \((\mathcal{A}_0, D(\mathcal{A}_0))\) of \((\mathcal{A}, D(\mathcal{A}))\) in the closure of its domain
\[
\mathcal{X}_0 := \overline{D(\mathcal{A})} = E \times \{0\} \times X \times \{0\}
\]
generates a strongly continuous semigroup. Observe that
\[
D(\mathcal{A}_0) = \left\{ \begin{pmatrix} g \\ 0 \\ f \\ 0 \end{pmatrix} : g \in D(G_m), \ f \in D(A_m), \ \mathcal{A} \begin{pmatrix} g \\ 0 \\ f \\ 0 \end{pmatrix} \in \mathcal{X}_0 \right\}
\]
\[
= \left\{ \begin{pmatrix} g \\ 0 \\ f \\ 0 \end{pmatrix} : g \in D(G_m), \ f \in D(A_m), \ Qg = f \right\}.
\]
Therefore, the operator \((\mathcal{A}_1, D(\mathcal{A}_1))\) is isomorphic to \((\mathcal{A}_0, D(\mathcal{A}_0))\) and thus generates a strongly continuous semigroup on the space \(\mathcal{X}\).

We can now formulate the main result of this section as follows.

**Theorem 3.4.** The operator \((\mathcal{A}, D(\mathcal{A}))\) of the abstract Cauchy problem (3.2) generates a strongly continuous semigroup \(\{\mathcal{T}(t)\}_{t \geq 0}\) of bounded linear operators on \(\mathcal{X}\).

**Proof.** Since the operator \(B_m\) is bounded on \(X\), we know by the definition of \(A_2\) that \(A_2\) is also bounded on \(\mathcal{X}\). Thus it is clearly seen that \(\mathcal{A} = A_1 + A_2\) generates a strongly continuous semigroup \((\mathcal{T}(t))_{t \geq 0}\) on \(\mathcal{X}\) due to Theorem 3.3 and the perturbation theory.

The following well-posedness result for (3.1) is then a direct consequence of Theorem 3.4.

**Corollary 1.** For initial data \(u^0 \in E\) the linear boundary delayed problem (3.1) has a unique solution \(u\) in \(C([-\tau, +\infty), \mathcal{X})\), given by \(u(s, t) = u^0(s, t)\) for \(t \in [-\tau, 0)\) and
\[
u(s, t) = \Pi_2 \left( T(t) \begin{pmatrix} u_0(s, 0) \\ u_0(s) \end{pmatrix} \right), \quad \text{for } t > 0,
\]
where $\Pi_2$ is the projection operator of $T(t)$ on the space $X$.

4. Spectral analysis and regularities. In this section, we will prove two regularity results about the $C_0$-semigroup generated by $(A, D(A))$ when the attack rate $\alpha$ is assumed to be a special form. The first result implies that the spectrally determined growth property holds true and that the linearized stability of the steady-state solution is governed by the location of the leading eigenvalue, while the second result establishes that under certain assumptions on the vital rates the leading eigenvalue will be real rather than complex, which enable us to achieve the concrete conditions for AEG.

As mentioned above, to carry on our further discussion in what follows, we make the assumption that the attack rate is separable, i.e.

$$\alpha(s_1, s_2) = \alpha_1(s_1)\alpha_2(s_2), \quad (s_1, s_2) \in [0, m] \times [0, m],$$  \hspace{1cm} (4.1)

where

$$\alpha_1, \alpha_2 \in C([0, m], \mathbb{R}^+).$$

We can interpret that $\alpha_1(s_1)$ represents the likelihood of being attacked at size $s_1$, while $\alpha_2(s_2)$ as a measure for the likelihood that individuals of size $s_2$ attack the others. The particular choice of the attack rate (4.1) allows us to cast the operator $B_m$ in the form

$$(B_m u)(s) = -\bar{u}_1 b_1(s) - \bar{u}_2 b_2(s),$$  \hspace{1cm} (4.2)

where we define

$$\bar{u}_1 = \int_0^m c(s)\alpha_1(s)u(s)ds, \quad \bar{u}_2 = \int_0^m \alpha_2(s)u(s)ds,$$

$$b_1(s) = \alpha_2(s)\left[(\gamma_E^r(s)n_\ast(s))_s + \mu_E(s, E_\ast(s))n_\ast(s)\right] + \alpha_2'(s)\gamma_E^r(s)n_\ast(s),$$

$$b_2(s) = \alpha_1(s)n_\ast(s).$$  \hspace{1cm} (4.3)

We now establish the first main result of this section.

**Theorem 4.1.** The semigroup $(T(t))_{t \geq 0}$ generated by the operator $(A, D(A))$ is eventually compact. Particularly, the semigroup operator $T(t)$ is compact for all $t > 2(\Gamma(m) + \tau)$.

**Proof.** Since, by the definition (4.2) of operator $B_m$, we know the operator $A_2$ is compact on $X$, it suffices to prove the claim for the operator $A_1$.

Next we show that the semigroup generated by $A_1$ is eventually compact. To this end, we observe the abstract Cauchy problem (3.2) with $A_1$ becomes

$$\begin{cases} 
\frac{d}{dt} \begin{pmatrix} u_t \\ u(t) \end{pmatrix} = A_1 \begin{pmatrix} u_t \\ u(t) \end{pmatrix}, & t \geq 0, \\
\mathcal{P} u(t) = \Phi u_t. 
\end{cases}$$

With the definition of $A_1$, we have

$$\begin{cases} 
\frac{d}{dt} u_t(s, \sigma) = G_m u_t(s, \sigma) = \frac{d}{d\sigma} u_t(s, \sigma), \\
\frac{d}{dt} u(t) = A_m u(t) = -\gamma^r(s)\frac{\partial}{\partial s} u(s, t) - \rho^r(s)u(s, t),
\end{cases}$$  \hspace{1cm} (4.4)

subject to the boundary condition in the system (1.1).

From the equation (4.4), we get the solution of $u_t$ that

$$u_t(s, \sigma) = u(s, t + \sigma), \quad \sigma \in [-\tau, 0].$$
To obtain the solution of \( u(s, t) \), let us introduce for \( t_0 > 0 \)
\[
v(s) = u(s, t_0 + s),
\]
where \( t(s) = t_0 + \Gamma(s) \) with \( \Gamma \) defined in (3.3). Then
\[
\frac{d}{ds} v(s) = \frac{\partial}{\partial s} u(s, t(s)) + \frac{\partial}{\partial t} u(s, t(s)) \Gamma'(s)
= \frac{\partial}{\partial s} u(s, t) + \frac{1}{\gamma'(s)} \frac{\partial}{\partial t} u(s, t(s)).
\]
With the equation (4.5), \( v \) satisfies
\[
\frac{d}{ds} v(s) + \frac{\rho^*(s)}{\gamma^*(s)} v(s) = 0,
\]
which yields that
\[
v(s) = v(0) e^{-\int_0^s \frac{\rho^*(\tau)}{\gamma^*(\tau)} d\tau},
\]
where \( v(0) = u(0, t_0) = \Phi(u_{t_0}) \). By the formula of \( \Phi \), for \( t_0 = t - \Gamma(s) > 0 \), one has that
\[
u(s, t) = e^{-\int_0^s \frac{\rho^*(\tau)}{\gamma^*(\tau)} d\tau} \left[ \int_0^s \int_{-\tau}^0 \beta(s, \sigma, E_s(\sigma)) u_{t-\Gamma(s)}(s, \sigma) d\sigma ds
+ \int_0^s \alpha_2(s) n_+(s) \int_0^s \int_0^m \beta_1(s, \sigma, E_s(\sigma)) \int_0^m c(y) \alpha_1(y) u(y, t - \Gamma(s) + \sigma) dy d\sigma ds \right].
\]
Therefore, if \( t > \Gamma(m) + \tau \), \( u \) is continuous in \( s \) and \( t \). Consequently, Eq. (4.5) implies that \( u \) is continuous differentiable if \( t > 2(\Gamma(m) + \tau) \). Hence the semigroup generated by \( A \) is differentiable for \( t > 2(\Gamma(m) + \tau) \). Since \( W^{1,1}(0, m) \) is compactly imbedded in \( X \), the claim follows.

\( \square \)

Theorem 4.1 has the following immediate and noteworthy consequence (see [9, 27]).

**Corollary 2.** The spectrum of the semigroup generator \((A, D(A))\) consists of isolated eigenvalues of finite multiplicity only and the Spectral Mapping Theorem holds true, i.e.,
\[
\sigma(T(t)) = \{0\} \cup e^{\sigma(A)}, \quad t > 0.
\]
Moreover, the semigroup is spectrally determined, i.e. the growth rate \( \omega(T(t)) \) of the \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) and the spectral bound \( s(A) \) of its generator coincide.

Because of Corollary 2 the linear stability of the steady-state solution is spectrally determined (see [9, 27]). Hence in the sequel it suffices to investigate the location of the leading eigenvalue of the generator of the \( C_0 \)-semigroup \((T(t))_{t \geq 0}\).

In order to state and prove the second main theorem of this section, we need state several existing lemmas and theorems. For this we introduce two operators as follows. For \( \lambda \in \rho(G_0) \cap \rho(A_0) \), we define \( K_\lambda : X \to E \) by \( K_\lambda := 1 \circ \varphi_\lambda \) and \( L_\lambda : E \to X \) by \( L_\lambda := (1 \circ \varphi_\lambda) \Phi \), or more precisely, \( K_\lambda(f) = f \circ \epsilon_\lambda \) for \( f \in X \), and \( L_\lambda(g) = \Phi(g) \varphi_\lambda \) for \( g \in E \). The next result was formulated in [26].

**Lemma 4.2** (see [26], Lemma 2.5). \( K_\lambda \in \mathcal{L}(X, E) \) and \( L_\lambda \in \mathcal{L}(E, X) \), moreover, \( \mathcal{Q}(K_\lambda(f)) = f \), \( \mathcal{P}(L_\lambda(g)) = \Phi(g) \) for all \( g \in D(G_m), f \in D(A_m) \), with \( \epsilon_\lambda \) and \( \varphi_\lambda \) given in (3.4).

The decomposition of the operator \( \lambda - A_1 \) below has also been proved in ([26], Lemma 2.6).
Lemma 4.3. For \( \lambda \in \rho(G_0) \cap \rho(A_0) \), we have
\[
(\lambda - A_1) = \begin{pmatrix} \lambda - G_0 & 0 \\ 0 & \lambda - A_0 \end{pmatrix} B_\lambda, \tag{4.6}
\]
where \( B_\lambda := \begin{pmatrix} I_d & -K_\lambda \\ -L_\lambda & I_d \end{pmatrix} \) is a bounded linear operator matrix on \( D(G_m) \times D(A_m) \) and the matrix \( \begin{pmatrix} \lambda - G_0 & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \) has domain \( D(G_0) \times D(A_0) \).

The next results are due to Nagel ([26], Theorem 2.7) and Engel ([8], Theorem 2.8).

Theorem 4.4. Let \( \lambda \in \rho(G_0) \cap \rho(A_0) \), for the following statements
(a) \( \lambda \in \rho(A_1) \),
(b) \( 1 \in \rho(K_\lambda L_\lambda) \) for the operator \( K_\lambda L_\lambda \in \mathcal{L}(E) \),
(c) \( 1 \in \rho(L_\lambda K_\lambda) \) for the operator \( L_\lambda K_\lambda \in \mathcal{L}(X) \).
Then one has the implications (a) \( \iff \) (b) \( \iff \) (c).
If, in particular, \( K_\lambda \) and \( L_\lambda \) are compact operators, then the statements (a), (b) and (c) are equivalent.

Since the operator \( L_\lambda \) here has one-dimensional range, it is compact, and hence \( K_\lambda L_\lambda \) and \( L_\lambda K_\lambda \) are compact too. Then, from Theorem 4.4 we obtain immediately that

Theorem 4.5. For the operator \((A_1, D(A_1))\), there hold that
1. \( \lambda \in \sigma(A_1) \iff 1 \in \sigma(L_\lambda K_\lambda) \iff 1 \in \sigma_p(L_\lambda K_\lambda) \iff \lambda \in \sigma_p(A_1) \); 
2. Furthermore, if \( \lambda \in \rho(A_1) \) (equivalently \( 1 \in \rho(L_\lambda K_\lambda) \)), then
\[
R(\lambda, A_1) = \begin{pmatrix} (1 - K_\lambda L_\lambda)^{-1} R(\lambda, G_0) & (1 - K_\lambda L_\lambda)^{-1} K_\lambda R(\lambda, A_0) \\ (1 - L_\lambda K_\lambda)^{-1} L_\lambda R(\lambda, G_0) & (1 - L_\lambda K_\lambda)^{-1} R(\lambda, A_0) \end{pmatrix}. \tag{4.7}
\]

Proof. We only need to verify (4.7). By the equation of (4.6) in Lemma 4.3, the inverse of \((\lambda - A_1)\) is
\[
R(\lambda, A_1) = B_\lambda^{-1} \begin{pmatrix} R(\lambda, G_0) & 0 \\ 0 & R(\lambda, A_0) \end{pmatrix}.
\]
By the definition of \( B_\lambda \), we have
\[
B_\lambda^{-1} = \begin{pmatrix} (1 - K_\lambda L_\lambda)^{-1} & (1 - K_\lambda L_\lambda)^{-1} K_\lambda \\ (1 - L_\lambda K_\lambda)^{-1} L_\lambda & (1 - L_\lambda K_\lambda)^{-1} \end{pmatrix}.
\]
Then the expression (4.7) follows.

Lemma 4.6 (see [9], Theorem VI.1.8), A strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach lattice \(X\) is positive if and only if the resolvent \(R(\lambda, A)\) of its generator \(A\) is positive for all sufficiently large \(\lambda\).

Now we conclude this section by formulating conditions for the positivity of the semigroup \((T(t))_{t \geq 0}\).

Theorem 4.7. Suppose that
\[
\overline{c} [\alpha_2(\gamma_E(\cdot)n_s)' + \mu_E(\cdot, E_\cdot(\cdot))n_s] + \alpha_2' \gamma_E(\cdot)n_s + \overline{c}_2 n_s \leq 0 \text{ for a.e. } s \in [0, m]. \tag{4.8}
\]
that the operator \((E, \mathbb{R})\) is positive as well for \(\Re \lambda \) invertible and its inverse \((1 - \lambda K)^{-1}\) is positive. Then the semigroup \((T(t))_{t \geq 0}\) generated by the operator \((A_1, D(A_1))\) is positive.

**Proof.** Condition (4.8) ensures that the operator \(A_2\) is positive. Hence we can restrict ourselves to the operator \(A_1\). As the proof of Theorem 4.1, we just verify the semigroup generated by the operator \(A_1\) is nonnegative.

To this end, we consider the operator \(K_\lambda L_\lambda\) first. By the definitions of \(K_\lambda\) and \(L_\lambda\) in Lemma 4.2, it is easy to see that

\[
\lim_{\Re \lambda \to +\infty} \|K_\lambda L_\lambda\| = 0.
\]

Therefore \(\|K_\lambda L_\lambda\| < 1\) for \(\Re \lambda\) sufficiently large. Thus the operator \((1 - K_\lambda L_\lambda)\) is invertible and its inverse \((1 - K_\lambda L_\lambda)^{-1}\) is given by the Neumann series. Obviously \(K_\lambda L_\lambda\) is a positive operator by the condition (4.9), and hence \((1 - K_\lambda L_\lambda)^{-1}\) is positive as well for \(\Re \lambda\) big enough. With the resolvent representation of \(A_1\) in (4.7), \(R(\lambda, A_1)\) is nonnegative for such \(\lambda\). Thus, using Lemma 4.6 above, we infer that the operator \((A_1, D(A_1))\) generates a positive semigroup on the Banach lattice \(E \times X\). Then we get the assertion. \(\square\)

The positivity and eventual compactness of the \(C_0\)-semigroup \((T(t))_{t \geq 0}\) enable us to draw the following important conclusion.

**Corollary 3.** Suppose that the conditions (4.8) and (4.9) hold true, then the spectral bound \(s(A) := \sup \{\Re \lambda \mid \lambda \in \sigma(A)\}\) belongs to the spectrum \(\sigma(A)\).

5. **The characteristic equation.** In the light of Corollary 2, the linearized stability of stationary solutions of the system (1.1) is entirely determined by the eigenvalues of the semigroup generator \((A, D(A))\) when the attack rate \(\alpha\) is assumed to be a special form in the section 4. Hence, in this section, we will derive a characterization of the eigenvalues in the form of zeros of a characteristic equation when the attack rate \(\alpha\) takes the form (4.1).

To determine the spectrum of the generator of the semigroup, we substitute \(u(s, t) = e^{\lambda t} U(s)\) into the linearized system (2.12). This ansatz gives the equations

\[
\begin{cases}
\gamma^*(s) U'(s) + (\lambda + \rho^*(s)) U(s) + b_1(s) U_1 + b_2(s) U_2 = 0, \\
U(0) = \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, E_*(s)) U(s) d\sigma ds + a(\lambda) U_1,
\end{cases}
\]

where

\[
\begin{align*}
U_1 &= \int_0^m c(s) \alpha_1(s) U(s) ds, \\
U_2 &= \int_0^m \alpha_2(s) U(s) ds,
\end{align*}
\]

and

\[
a(\lambda) = \int_0^m \alpha_2(s) n_*(s) \int_{-\tau}^0 e^{\lambda \sigma} \beta_E(s, \sigma, E_*(s)) d\sigma ds.
\]

The solution of equation (5.1) is found to be

\[
U(s) = U(0) \pi(s, \lambda) - U_1 \pi(s, \lambda) \int_0^s b_1(y) \gamma(y) \pi(y, \lambda) dy - U_2 \pi(s, \lambda) \int_0^s b_2(y) \gamma(y) \pi(y, \lambda) dy,
\]

asymptotic analysis of a structured population model
where we have made use of the notation
\[ \pi(s, \lambda) = e^{-\int_0^s \frac{\lambda s^*(y)}{\gamma^*(y)} dy} = \frac{\gamma^*(0)}{\gamma^*(s)} e^{-\int_0^s \frac{\lambda s^*(y)\pi(y, \lambda) + M_\lambda(y)}{\gamma^*(y)} dy}. \] (5.5)

We multiply equation (5.4) by \( c(s)\alpha_1(s) \) and \( \alpha_2(s) \), respectively, and integrate from 0 to \( m \) to arrive at
\[
\begin{align*}
U(0)\alpha_1(\lambda) + U_1(1 + a_2(\lambda)) + U_2 a_3(\lambda) &= 0, \quad (5.6) \\
U(0)\alpha_4(\lambda) + U_1 a_5(\lambda) + U_2(1 + a_6(\lambda)) &= 0, \quad (5.7)
\end{align*}
\]
Meanwhile, inserting the solution (5.4) into the boundary condition (5.2) we also have
\[
U(0)(1 + a_7(\lambda)) + U_1(a_8(\lambda) - a(\lambda)) + U_2 a_9(\lambda) = 0, \quad (5.8)
\]
where
\[
\begin{align*}
a_1(\lambda) &= -\int_0^m c(s)\alpha_1(s)\pi(s, \lambda) ds, \quad (5.9) \\
a_2(\lambda) &= \int_0^m c(s)\alpha_1(s)\pi(s, \lambda) \int_0^s \frac{b_1(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds, \quad (5.10) \\
a_3(\lambda) &= \int_0^m c(s)\alpha_1(s)\pi(s, \lambda) \int_0^s \frac{b_2(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds, \quad (5.11) \\
a_4(\lambda) &= -\int_0^m \alpha_2(s)\pi(s, \lambda) ds, \quad (5.12) \\
a_5(\lambda) &= \int_0^m \alpha_2(s)\pi(s, \lambda) \int_0^s \frac{b_1(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds, \quad (5.13) \\
a_6(\lambda) &= \int_0^m \alpha_2(s)\pi(s, \lambda) \int_0^s \frac{b_2(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds, \quad (5.14) \\
a_7(\lambda) &= -\int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, E_\sigma(s))\pi(s, \lambda) d\sigma ds, \quad (5.15) \\
a_8(\lambda) &= \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, E_\sigma(s))\pi(s, \lambda) d\sigma \int_0^s \frac{b_1(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds, \quad (5.16) \\
a_9(\lambda) &= \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, E_\sigma(s))\pi(s, \lambda) d\sigma \int_0^s \frac{b_2(y)}{\gamma^*(y)\pi(y, \lambda)} dy ds. \quad (5.17)
\end{align*}
\]
Now combining (5.6), (5.7) and (5.8) we can give a characterization of the point spectrum of the operator \( A \), that is

**Theorem 5.1.** For any \( \lambda \in \mathbb{C} \), \( \lambda \in \sigma_P(A) \) if and only if \( \lambda \) solves the equation
\[
K(\lambda) := \begin{vmatrix}
\alpha_1(\lambda) & 1 + \alpha_2(\lambda) & \alpha_3(\lambda) \\
1 + \alpha_7(\lambda) & a_8(\lambda) - a(\lambda) & a_9(\lambda)
\end{vmatrix} = 0, \quad (5.18)
\]

**Proof.** If \( \lambda \in \sigma_P(A) \), then Eqs. (5.1), (5.2) admit a nontrivial solution \( U \). Hence for this \( \lambda \) there exists a nonzero solution vector \( \langle U(0), U_1, U_2 \rangle \), such that Eqs. (5.6)-(5.8) hold true. Thus \( K(\lambda) = 0 \). Conversely, if \( K(\lambda) = 0 \) for some \( \lambda \) and \( \langle U(0), U_1, U_2 \rangle \) is a nonzero solution of Eqs. (5.6)-(5.8), then \( U \), given by Eq. (5.4), is a nonzero solution of Eqs. (5.1)-(5.2) with at least \( U(0) \neq 0 \). If, however,
$U(0) = 0$, the only possible situation for $U$ to vanish when defined by Eq. (5.4) would be the condition that
\[ \mathcal{U}_1 b_1(s) \equiv -\mathcal{U}_2 b_2(s) \]
holds true. Then Eqs. (5.3) would immediately give $\mathcal{U}_1 = 0 = \mathcal{U}_2$ in contradiction to our assumption on $(U(0), \mathcal{U}_1, \mathcal{U}_2)$.

6. Linear stability results. This section is devoted to proving asymptotic stability and instability of stationary solution in some intuitively interpretable and biologically relevant cases. Our first result addresses the stability/instability of the trivial stationary solution $n_\ast \equiv 0$.

**Theorem 6.1.** The trivial stationary solution $n_\ast \equiv 0$ is linearly asymptotically stable if $R(0) < 1$ and linearly unstable if $R(0) > 1$.

**Proof.** For $n_\ast \equiv 0$, in the characteristic equation (5.18), we observe that $b_1 = b_2 = 0$, $a_2(\lambda) = a_3(\lambda) = a_5(\lambda) = a_6(\lambda) = a_8(\lambda) = a_9(\lambda) = 0$, $a(\lambda) = 0$, then (5.18) becomes
\[
K(\lambda) = \begin{vmatrix}
    a_1(\lambda) & 1 & 0 \\
    a_4(\lambda) & 0 & 1 \\
    1 + a_7(\lambda) & 0 & 0
\end{vmatrix} = 1 + a_7(\lambda),
\]
following that we have
\[
K(0) = 1 + a_7(0) = 1 - \int_0^m \int_{-\tau}^0 \beta(s, \sigma, 0) \pi(s, 0) d\sigma ds
\]
\[
= 1 - \int_0^m \frac{\gamma(0, 0)}{\gamma(0,0)} e^{-\int_0^\lambda \sum_{i=1}^{\infty} \beta_i(s, \sigma, 0) d\sigma} ds
\]
\[
= 1 - R(0).
\]
Clearly the conditions (4.8) and (4.9) of Theorem 4.7 are fulfilled when the stationary solution $n_\ast(s) \equiv 0$. Then we can restrict ourselves to $\lambda \in \mathbb{R}$. Observe that
\[
\lim_{\lambda \to +\infty} K(\lambda) = 1, \quad (6.1)
\]
and
\[
K'(\lambda) = - \int_0^m \int_{-\tau}^0 \gamma(0, 0) \beta(s, \sigma, 0) \left( \sigma e^{\lambda \sigma} e^{-\int_0^\lambda \sum_{i=1}^{\infty} \beta_i(s, \sigma, 0) d\sigma} ds \right) d\sigma ds > 0. \quad (6.2)
\]
Therefore, if $R(0) < 1$ holds, $K(\lambda)$ increases monotonically by the inequality (6.2), then the characteristic function cannot have nonnegative roots as $K(0) > 0$. If, however, $R(0) > 1$ holds, then the Intermediate Value Theorem gives a positive root since $K(0) < 0$ and (6.1) holds.

**Theorem 6.2.** Let $n_\ast$ be any nontrivial stationary solution of Eqs. (1.1) with corresponding environmental quantity $E_\ast(s)$ and suppose the conditions (4.8) and (4.9) are fulfilled. Then the stationary solution is linearly unstable if $K(0) < 0$. \qed
Proof. It suffices to show that there exists a positive zero \( \lambda \) of the characteristic equation. In fact, it is easy to see that

\[
\lim_{\lambda \to +\infty} K(\lambda) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1,
\]

here the limit is taken in \( \mathbb{R} \), then we can formulate the above simple instability criterion, which follows immediately from the Intermediate Value Theorem.

Stability results of nonzero stationary solutions for this kind of model discussed here are much harder to obtain than instability results since a rigorous linear stability proof requires to show that all zeros of the characteristic equation are in the left half-plane of \( \mathbb{C} \). Hence it lies in the nature of the stability problem that any answer is generally hard to get by and is usually available for rather special or restricted cases only. Especially, when the birth process here contains time lags, it may bring about much more difficulty for us to deal with this situation. Therefore, only one special case is discussed in the following, in which stability conditions can be achieved relatively easily and simply.

Let us assume that the rate of an individual of size \( s \) to attack another individual is proportional to the product of the probability for an individual of size \( s \) to be attacked and its energetic value. Mathematically, this condition is modeled by the relation

\[
c(s)a_1(s) = pa_2(s), \quad s \in [0, m], \quad p \in \mathbb{R}_+.
\]

(6.3)

The constant \( p \) denotes here the proportionality factor. This condition is biologically relevant since environmental pressure conceivably makes individuals of higher energetic value, which are usually larger in size, not only easier to be attacked, but also more aggressive. In this case, the formulas (5.9)-(5.11) in the characteristic equation (5.18) turn out to be the expressions as follows,

\[
a_1(\lambda) = p\int_0^m a_2(s)\pi(s, \lambda)\,ds,
\]

(6.4)

\[
a_2(\lambda) = p\int_0^m a_2(s)\pi(s, \lambda)\int_0^s b_1(y)\gamma^*(y, \lambda)\pi(y, \lambda)\,dy\,ds,
\]

(6.5)

\[
a_3(\lambda) = p\int_0^m a_2(s)\pi(s, \lambda)\int_0^s b_2(y)\gamma^*(y, \lambda)\pi(y, \lambda)\,dy\,ds,
\]

(6.6)

and the representations of \( a_4(\lambda) - a_9(\lambda) \) remain unchanged. Thus one has that

**Theorem 6.3.** Let \( n_* \) be any nontrivial stationary solution of Eqs. (1.1). Suppose that the conditions (4.8) and (4.9) in Theorem 4.7 and the inequalities

\[
\beta_E(s, \sigma, E_*(s)) \leq 0,
\]

(6.7)

\[
pb_1(s) + b_2(s) \geq 0,
\]

(6.8)

hold true for all \( 0 \leq s \leq m \) and \(-\tau \leq \sigma \leq 0\). Then \( n_* \) is linearly asymptotically stable if and only if \( K(0) > 0 \).

Proof. First, from the condition (4.8) and (4.9), we see that Theorem 4.7 and Corollary 3 apply now. Thus we can restrict ourselves to \( \lambda \in \mathbb{R} \) as above. From the
representations (5.9)-(5.17) in this case, \( K(\lambda) \) in the characteristic equation (5.18) turns out to be

\[
K(\lambda) = \begin{vmatrix}
p a_4(\lambda) & 1 + p a_5(\lambda) & p a_6(\lambda) \\
a_4(\lambda) & a_5(\lambda) & 1 + a_6(\lambda) \\
1 + a_7(\lambda) & a_8(\lambda) - a(\lambda) & a_9(\lambda)
\end{vmatrix}
\]

\[
= \begin{vmatrix}
p a_4(\lambda) & 1 + p a_5(\lambda) & p a_6(\lambda) \\
0 & -\frac{1}{\mu} & 1 \\
1 + a_7(\lambda) & a_8(\lambda) - a(\lambda) & a_9(\lambda)
\end{vmatrix}
= (1 + a_7(\lambda))(1 + p a_5(\lambda) + a_6(\lambda)) - a_4(\lambda)(p a_8(\lambda) + a_9(\lambda) - p a(\lambda)).
\]

Then, it is easy to compute that \( \lim_{\lambda \to +\infty} K(\lambda) = 1 \). Thus if \( K(0) < 0 \), \( n_* \) is not linearly asymptotically stable as stated in Theorem 6.2. If now \( K(0) > 0 \), then \( n_* \) will be linearly asymptotically stable as long as we can show that \( K(\lambda) > 0 \) for every \( \lambda > 0 \). In fact, we observe that, for all \( \lambda > 0 \),

\[
(1 + a_7(\lambda))' > 0,
\]
and

\[
1 + a_7(0) = 1 - \int_0^m \int_{-\tau}^0 \gamma^*(s) \beta(s, \sigma, E_*(s)) e^{-\int_0^\sigma \frac{\gamma^*(y) \mu}{\gamma^*(s)} dy} ds d\sigma \\
= 1 - R(n_*) = 0,
\]

from which and making use of conditions (6.7) and (6.8), we deduce the following relations

\[
1 + a_7(\lambda) > 0, \quad (6.11)
\]

\[
1 + p a_5(\lambda) + a_6(\lambda) > 0, \quad p a_8(\lambda) + a_9(\lambda) - p a(\lambda) > 0. \quad (6.12)
\]

Therefore, from (6.11)-(6.12) and the fact \( a_4(\lambda) < 0 \), it yields readily that, for every \( \lambda > 0 \),

\[
K(\lambda) > 0.
\]

Because of \( K(0) > 0 \), we obtain that \( K(\lambda) > 0 \) for each \( \lambda \geq 0 \), then the result follows.

If the fertility rate \( \beta \) depends only on the environment feedback size at time \( t \) rather than at \( t + \sigma \), that is, \( \beta = \beta(s, \sigma, E(s, t)) \), then we can derive similarly that the characteristic equation is (5.18) with \( a(\lambda) \) replaced by \( (\text{while } a_1(\lambda) - a_9(\lambda) \text{ remain the same}) \)

\[
a(\lambda) = \int_0^m \alpha_2(s) n_*(s) \int_{-\tau}^0 \beta E(s, \sigma, E_*(s)) d\sigma ds.
\]

More precisely, we have that

**Theorem 6.4.** If \( \beta \) in the second equation of (1.1) is given by \( \beta(s, \sigma, E(s, t)) \). Let \( n_* \) be any nontrivial stationary solution of Eqs. (1.1). Suppose that the conditions (4.8), (4.9), (6.7) and (6.8) are fulfilled for all \( 0 \leq s \leq m \) and \( -\tau \leq \sigma \leq 0 \). Then \( n_* \) is linearly asymptotically stable if and only if \( K(0) > 0 \).
7. **Asynchronous exponential growth.** The purpose of this section is to gain a deeper insight into asymptotic properties of solutions of the linearized system (2.12). That is, we will use semigroup techniques and spectral analysis methods to obtain the property of asynchronous exponential growth (AEG for short) for (2.12) which is defined in the framework of semigroup theory as below.

**Definition 7.1.** The linear $C_0$-semigroup $(T(t))_{t \geq 0}$ on the Banach space $X$ with generator $A$ and spectral bound $s(A)$ is said to exhibit balanced exponential growth (BEG for short) if there exists a bounded linear projection $\Pi$ on $X$ such that

$$\lim_{t \to \infty} \| e^{-s(A)t}T(t) - \Pi \| = 0.$$  

(7.1)

The semigroup $(T(t))_{t \geq 0}$ is said to exhibit asynchronous exponential growth if it exhibits BEG with a rank one projection $\Pi$.

The phenomenon of AEG appears frequently in age/size-structured populations models (see [2, 18, 32]). It describes the situation when the population grows exponentially in time but the proportion of individuals within any range of age compared to the total population tends, as time tends to infinity, to a limit which just depends on the chosen range. This is an important characteristic of solutions of population equations both from the theoretical and application point of view.

For positive semigroups there exists the well-known characterization of AEG (see [3]). Our analytical approach will be guided toward the result as follows.

**Lemma 7.2** (see [3], Theorems 9.10 and 9.11). Let $T = (T(t))_{t \geq 0}$ be a positive and irreducible strongly continuous semigroup on a Banach lattice space $X$, satisfying the inequality $\omega_{\text{ess}}(T) < \omega_0(T)$. Then, there exist a rank one projector $\Pi$ into $X$ and an $\epsilon > 0$, such that, for all $\eta \in (0, \epsilon)$, there exists $M(\eta) \geq 1$ satisfying

$$\| e^{-\omega_0(T)t}T(t) - \Pi \|_{L(X)} \leq M(\eta)e^{-\eta t}, \ t \geq 0.$$  

So a strongly continuous semigroup $(T(t))_{t \geq 0}$ satisfying Lemma 7.2 possesses an asynchronous exponential growth with essential growth constant $\omega_0(T)$.

Let us recall here that, a positive $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ is called irreducible if $\{0\}$ and $X$ are the only $\{T(t)\}$–invariant closed ideals. A useful characterization of irreducibility on $L_1(\Omega, \mu)$ lies in

**Lemma 7.3** (see [9], Theorem VI.1.12). Let $T = (T(t))_{t \geq 0}$ be a positive strongly continuous semigroup with generator $A$ on a Banach lattice space $X = L_1(\Omega, \mu)$. If for any $f \in X$ with $f \geq 0$, there has $(\lambda I - A)^{-1} f(s) > 0$, for almost all $s \in \Omega$ and some $\lambda > s(A)$ sufficiently large, then the semigroup $T = (T(t))_{t \geq 0}$ is irreducible.

Using this lemma we obtain immediately that

**Theorem 7.4.** Suppose that the positivity conditions (4.8) and (4.9) in Theorem 4.7 are fulfilled. Then the semigroup $(T(t))_{t \geq 0}$ generated by $A$ is irreducible.

**Proof.** From the proof of Theorem 4.7, $A_2$ and $R(\lambda, A_1)$ (for $\lambda$ large enough) are all positive due to the conditions (4.8) and (4.9). Since

$$(\lambda I - A)^{-1} = (\lambda I - (A_1 + A_2))^{-1} = \sum_{n=0}^{\infty} ((\lambda I - A_1)^{-1} A_2)^n (\lambda I - A_1)^{-1},$$

it suffices to prove the irreducibility of the semigroup generated by $(A_1, D(A_1))$. This fact, however, follows immediately from the expression (4.7) as it indicates that the operator $R(\lambda, A_1)$ verifies the condition of Lemma 7.3 on $X$ exactly.  \(\square\)
Before we formulate the main result of this section, let us review the notions of essential norm, growth bound, and essential growth bound, and some of their properties (see [9] for more details). Suppose that \( A \) is the infinitesimal generator of the strongly continuous semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \). Then the growth bound of the semigroup is defined by

\[
\omega_0(A) := \lim_{t \to \infty} \frac{\ln \| T(t) \|}{t}.
\]

For a linear operator \( L \) on \( X \), the essential growth bound \( \omega_{\text{ess}}(L) \) is defined by

\[
\omega_{\text{ess}}(L) := \lim_{t \to +\infty} \frac{\ln \left( \alpha[e^t L] \right)}{t},
\]

where \( \alpha \) is the measure of noncompactness defined as

\[
\alpha[T] := \inf \left\{ \| T - K \| : K \in \mathcal{K}(X) \right\}
\]

for \( T \in L(X) \). It is readily seen that, for \( S \in \mathcal{K}(X) \),

\[
\omega_{\text{ess}}(A) = \omega_{\text{ess}}(A + S).
\]

The significance of the essential growth bound lies in the central fact that

\[
\omega_0(A) = \max \{ \omega_{\text{ess}}(A), s(A) \}.
\]

Based on the \( C_0 \)-semigroup and the spectral analysis, we are on the position now to show the system (2.12) has AEG under some conditions, namely,

**Theorem 7.5.** Given a positive stationary solution \( n_* \), suppose that conditions (4.8) and (4.9) hold true. If

\[
K(0) < 0,
\]

and

\[
\int_0^m \left[ \int_0^m a_2(s)n_*(s) \int_{-\tau}^{\alpha_0} \beta(E(s, \sigma, E_*(s)))d\sigma ds + \int_{-\tau}^{0} \beta(s, \sigma, E_*(s)) e^{-\int_0^\sigma \int_{-\tau}^{0} \rho(y) \gamma(y) dy d\sigma} d\sigma \right] ds < 1,
\]

then the semigroup \( (T(t))_{t \geq 0} \) generated by \( A \) exhibits AEG.

**Proof.** First we note that \( \mathcal{A}_1 \) has nonempty spectrum and generates a positive semigroup by the proof of Theorem 4.1 and Theorem 4.7. By similar computation it is easy to obtain the characteristic equation of operator \( \mathcal{A}_1 \) as follows:

\[
\Delta_1(\lambda) := \int_0^m \left[ a(\lambda) + \int_{-\tau}^{\alpha_0} e^{\lambda \sigma} \beta(s, \sigma, E_*(s)) e^{-\int_0^\sigma \int_{-\tau}^{0} \rho(y) \gamma(y) dy d\sigma} d\sigma \right] ds - 1 = 0.
\]

Clearly, if restricted to \( \mathbb{R} \), \( \Delta_1(\lambda) \) is a continuous, strictly decreasing, real function having that

\[
\lim_{\lambda \to -\infty} \Delta_1(\lambda) = +\infty \text{ and } \lim_{\lambda \to +\infty} \Delta_1(\lambda) = -1.
\]

Therefore, \( \Delta_1 \) has a unique real zero \( \lambda_0 \) which is the spectral bound \( s(\mathcal{A}) \), from which, together with (7.4) and Derndinger’s Theorem (see [9]), it follows that

\[
\omega_0(\mathcal{A}_1) = s(\mathcal{A}_1) \leq 0,
\]

On the other hand, we infer from (7.3), Theorem 4.7 and Derndinger’s Theorem that (also see the proof of Theorem 6.2

\[
\omega_0(\mathcal{A}) = s(\mathcal{A}) > 0.
\]
Consequently, in the light of (7.2), (7.5) and (7.6), we have
\[ \omega_{\text{ess}}(A) = \omega_{\text{ess}}(A_1 + A_2) = \omega_{\text{ess}}(A_1) \leq \omega_0(A_1) < \omega_0(A). \]
Hence by Theorem 4.7 and Theorem 7.4, the semigroup \( (T(t))_{t \geq 0} \) is positive and irreducible with essential growth bound strictly smaller than its growth bound. The assertion follows now immediately from Lemma 7.2.

**Remark 2.** We point out that, all the results on stability and AEG of solutions for the system (1.1) established in Section 6 and Section 7, as well as the characteristic equation, involve the time delay \( \tau' \), which shows clearly the dependence of these long time behavior of solutions on the time lag.

8. Examples and simulations. In this section, we present some examples and the corresponding simulations to demonstrate the stability/instability results of the trivial and nontrivial solutions given in Theorems 6.1, 6.2 and 6.3 respectively.

**Example 1.** Let the vital rates and the parameters of (1.1) be as follows:
\[
\begin{align*}
\beta(s, \sigma, E(s, t + \sigma)) &= 1 + E(s, t + \sigma), \quad s \in [0, m], \quad \sigma \in [-\tau, 0], \\
\mu(s, E(s, t)) &= 0.5 - 0.01E(s, t), \quad M(s, t) = 0.01E(s, t), \quad s \in [0, m], \\
\gamma(s, E(s, t)) &\equiv 1, \quad \alpha_1(s) \equiv 0.1, \quad \alpha_2(s) \equiv 1, \quad c(y) \equiv 100, \quad s \in [0, m].
\end{align*}
\]
Then \( R(0) = 2(1 - e^{-40}) \tau/2(1 - e^{-40}) \approx 0.5 \). Thus, due to Theorem 6.1, we infer that, if \( \tau < \frac{1}{2(1 - e^{-40})} \approx 0.5 \), then the trivial stationary solution \( n_0 \equiv 0 \) is (locally) asymptotically stable, see Figure 1 below (in all the figures here \( n(t) \) and \( N(t) \) always denote respectively the density of individuals and the total number of population at time \( t \)). If, however, \( \tau > \frac{1}{2(1 - e^{-40})} \approx 0.5 \), then the null solution \( n_0 \equiv 0 \) is unstable, see Figure 2. This example illustrates very well the effects of the delay \( \tau' \) on the stability of the trivial solution.

**Example 2.** We now put in the system (1.1) that
\[
\begin{align*}
\beta(s, \sigma, E(s, t + \sigma)) &= \frac{1}{2}e^{\sigma - s}E(s, t + \sigma), \quad s \in [0, m], \quad \sigma \in [-\tau, 0], \\
\mu(s, E(s, t)) &= 0.05 - 0.01E(s, t), \quad M(s, t) = 0.01E(s, t), \quad s \in [0, m], \\
\gamma(s, E(s, t)) &\equiv 1, \quad \alpha_1(s) \equiv 0.1, \quad \alpha_2(s) \equiv 1, \quad c(y) \equiv 100, \quad s \in [0, m].
\end{align*}
\]
Then the unique positive steady-state solution of the problem is

\[ n_*(s) = \frac{0.05 N_* e^{-0.05 s}}{1 - e^{-0.05 m}}, \]

with \( N_* = \frac{0.21}{(1-e^{-\tau})(1-e^{-1.00m})} \), \( E_* = 10 N_* \). It is readily observed that

\[
\beta_E(s, \sigma, E_*(s)) = \frac{1}{2} e^{\sigma - s} > 0, \\
\tau [\tau_2(\gamma_E(s) n_*)' + \mu_E(s, E_*(s)) n_* + \alpha_2' \gamma_E(s) n_*] + \tau_2 n_* = 0, \\
\int_{-\tau}^{0} \beta(s, \sigma, E_*(s)) d\sigma + \frac{\tau_2}{\tau} \int_{0}^{m} \alpha_2(s) n_* d\sigma \int_{-\tau}^{0} \beta_E(\alpha, \sigma, E_*(\alpha)) d\sigma d\alpha > 0.
\]

Then we have \( K(0) < 0 \). Thus, in the light of Theorem 6.2, the stationary solution \( n_* \) is linearly unstable. See Figure 3.

**Example 3.** For the choice of vital rates and the parameters

\[
\beta(s, \sigma, E(s, t + \sigma)) = \frac{1}{E(s, t + \sigma)} e^\sigma, \quad s \in [0, m], \quad \sigma \in [-\tau, 0], \\
\mu(s, E(s, t)) = 0.05 - 0.01 E(s, t), \quad M(s, t) = 0.01 E(s, t), \quad s \in [0, m], \\
\gamma(s, E) \equiv 1, \quad \alpha_2(s) \equiv 1, \quad \bar{c} = 100, \quad p = 100 \tau_1, \quad s \in [0, m].
\]
Substituting these parameters into Eqs. (2.1) and (2.3), we obtain that
\[ E^* = pN^*, \quad N^* = \frac{20}{p}(1 - e^{-\tau})(1 - e^{-0.05m}). \]

Then the population model has a unique positive stationary solution given by
\[ n^*_s(s) = \frac{0.05N^*e^{-0.05s}}{1 - e^{-0.05m}} = \frac{1}{1.05}(1 - e^{-\tau}). \]

And it is not hard to get the following relations:
\[
\tau[\bar{\sigma}_2((\gamma^*_E(\cdot)n^*_s) + \mu_E(\cdot,E^*_s)n^*_s) + \alpha^*_E(\cdot)n^*_s] + \sigma_2n^*_s = 0,
\]
\[
\beta_E(s,\sigma,E^*_s(s)) = \frac{e^\sigma}{E^*_s} < 0, \quad pb_1(s) + b_2(s) \geq 0,
\]
\[
\beta(s,\sigma,E^*_s(s)) + \bar{\sigma}_1 \frac{1}{\rho} \int_0^m \alpha_2(\alpha)n^*_s(\alpha)\beta(\alpha,\sigma,E^*_s(\alpha))d\alpha = 0.
\]

Then \( K(0) > 0 \). According to Theorem 6.3, the nontrivial stationary solution \( n^*_s \) is linearly asymptotically stable as simulated in Figure 4.

**FIGURE 4.** Choosing the parameters with \( p = 1, \tau = 1, m = 80, N^* \approx 12.41 \), “a” represents \( n^*_s \) and \( N^* \), the initial conditions corresponding to curves b to d are: (b) \( n_0(s) = \frac{1}{1+s} \); (c) \( n_0(s) = \frac{1}{1+s} + 0.1 \); (d) \( n_0(s) = \frac{1}{1+s} + 0.15 \), and \( N(t) \) denotes the total number of the population at time \( t \).

9. **Conclusion.** In this work we have given a careful analysis of an important linearized size-structured cannibalism population model with delayed birth process. The vital rates in this model depend on a structuring variable (size), which takes values in a bounded set, and on the interaction variable (environment), describing the environmental feedback on individuals. Population models of this type are notoriously difficult to analyze. We would like to point out that the emphasis in the present work was to demonstrate how analytical techniques can be developed and used to treat qualitative questions of physiologically structured population models. In the analysis, we used \( C_0 \)-semigroup theory and spectral methods that allowed us to give a rigorous characterization of the linearized dynamical behavior of initially small perturbations of steady state via roots of the associated characteristic equation when the attack rate is separable. The positivity result for the semigroup was based on the decomposition of the operator matrix and the discussion of the resolvent operator under certain conditions for the vital rates. Here, we have formulated linear stability and instability criteria for equilibrium solutions of the model. Besides Theorem 6.1, we have obtained for non-zero stationary solutions an instability...
result (Theorem 6.2) and two stability criteria (Theorem 6.3 and 6.4) in the case of a separable attack rate. The spectral analysis of the linearized operator allowed us to gain successfully deeper insights into the asymptotic behavior of solutions of the linearized system. In particular, we investigated for the system whether the solutions of the linearized problem exhibit AEG property and gave concrete sufficient conditions as an affirmative answer.

The size-structured population model with delayed birth process studied in this paper improves and extends the earlier problems given in [7], [14] and [28] and elsewhere for simpler population models. And obviously, the effects of delay on asymptotic behaviors of the systems can be explored readily in these obtained results.

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