Noncommutative quantum mechanics and Bohm’s ontological interpretation

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We carry out an investigation into the possibility of developing a Bohmian interpretation based on the continuous motion of point particles for noncommutative quantum mechanics. The conditions for such an interpretation to be consistent are determined, and the implications of its adoption for noncommutativity are discussed. A Bohmian analysis of the noncommutative harmonic oscillator is carried out in detail. By studying the particle motion in the oscillator orbits, we show that small-scale physics can have influence at large scales, something similar to the IR-UV mixing.

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I. INTRODUCTION

The natural appearance of noncommutativity of the canonical type in string theory has been motivating an intensive investigation of its implications for quantum field theory and quantum mechanics. The theoretical relevance of this new and growing branch of physics was soon recognized, since it gives us the opportunity to understand very interesting phenomena. Among them are nonlocality and IR-UV mixing, new physics at very short distances, and the possible implications of Lorentz violation. From the experimental point of view, a great deal of effort has been devoted to the search for evidence of possible manifestations of noncommutative effects in cosmology and high-energy and low-energy experiments. Noncommutative quantum mechanics (NCQM) has also been an issue of great interest. In addition to its possible phenomenological relevance, the study of NCQM is motivated by the opportunity it gives us to understand problems that are present in noncommutative quantum field theory (NCQFT), and perhaps in string theory, in a framework easy to handle.

In previous work, a new interpretation for the canonical commutation relation consideration,

$$[\hat{X}_\mu, \hat{X}_\nu] = i\theta^{\mu\nu},$$

was proposed. According to the point of view exposed there, it is possible to interpret the commutation relation as a property of the particle coordinate observables, rather than of the spacetime coordinates. This fact was shown to have implications for the way of performing the calculations of NCQFT and enforced a reinterpretation of the meaning of the wave function in NCQM.

The aim of this work is to investigate the possibility of developing a Bohmian interpretation for NCQM. We shall benefit from the ideas presented in to develop a deterministic theory of hidden variables that exhibit canonical noncommutativity between the particle position observables. Presently, there are several motivations for the reconsideration of hidden-variable theories. We are now sure that the Copenhagen interpretation is not the unique framework where quantum phenomena can be described. Many theoreticians consider it more as a provisory set of rules than as the fundamental theory of quantum physics. Alternative points of view have been proposed that claim to solve some alleged difficulties of the Copenhagen interpretation.

It is a result of Bell that any hidden-variables model that leads to the same results predicted by quantum theory must itself be nonlocal. Historically, the possibility that a measurement process at one
point can have an immediate effect at a point separated in space was uncomfortable for physicists, and totally abhorrent to Einstein in particular [18]. For long time, local models were by far the preferred ones, but nowadays there is no fundamental reason for that. Noncommutative theories, as well as string theory, have been shown not to be local, at least in the usual sense [4, 19]. Thus, as long as one considers a noncommutative theory, the (historical) objection against hidden-variables models for their nonlocal character is clearly senseless. In reality, the study of noncommutative theories using hidden variables is strongly motivated by the detailed information such a description can provide us if compared to the one of the Copenhagen interpretation. Currently, however, there is a lack of investigation in this direction, and we are aware of just one work combining noncommutative geometry with a hidden-variables model [20], where stochastic quantization is employed.

Among the hidden-variables theories, the Bohmian one occupies a distinguished position. It has been an object of intensive investigation and application in a wide range of branches of physics, like quantum field theory [21], the phenomenology of high-energy physics [22], condensed matter and atomic-molecular physics [23], among others [24]. The enormous resurgence of interest in the Bohmian interpretation comes from multiple directions. From the experimental point of view, the possibility is under consideration of testing the limits and discussing the foundations of quantum theory in the realm of condensed matter and atomic-molecular physics [18, 25]. Techniques using ion traps may allow information on the behavior of individual particles to be obtained [18]. This makes the Bohmian formulation using particle trajectories especially attractive for investigation, since it allows the description of individual systems. Moreover, the applicability of this interpretation transcends the discussion of the foundations of quantum theory, since the Bohmian formalism can be adopted for “non-Bohmian” physicists as a tool to get intuition about the nature of quantum phenomena by the detailed description of the underlying dynamics it provides (see, e.g., [23]). In the theoretical framework, there is a large number of phenomena that do not fit comfortably within the standard operator quantum formalism of the orthodox Copenhagen interpretation. Among them, we quote dwell and tunneling times [26], escape times and escape positions [27], and scattering theory [28]. They are easily handled by Bohm’s ontological approach [29].

The main motivation to develop a Bohmian interpretation for NCQM in this work is the variety of evidence indicating that noncommutativity must, in some way, be related to a quantum theory of gravitation (see, e.g., [2, 3, 30, 31]); consequently, it may have implications for quantum cosmology [32]. The inadequacy of the application of the Copenhagen interpretation for quantum cosmology has been stressed for a long time by many prominent physicists, like Feynman [33] (a review of the subject may be found in [34]). As an alternative to the Copenhagen interpretation, the Bohmian one is employed in several works of quantum cosmology (see [35] and references therein). Thus, it is important to investigate, having in mind future applications for this area, if canonical noncommutativity is compatible with the Bohmian interpretation of quantum theory.

The organization of this work is the following. In Sec. II, we summarize the essential concepts of NCQM and develop Bohmian noncommutative quantum mechanics (BNCQM). After an informal presentation of the construction of the theory of motion, we formalize it in a simple and compact form. An application of the theory for the noncommutative harmonic oscillator is presented in Sec. III. Finally, in Sec.IV, we end up with a general discussion and a summary of the main results.

II. BOHMIAN INTERPRETATION FOR NCQM

A. Background on NCQM

The essential NCQM necessary for this work (for details see [14]) is summarized in what follows. According to the Weyl quantization procedure [2, 3], the realization of the commutation relation [11] between position observables is given by the Moyal star product defined as below:

$$ (f \star g)(x) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_x p_x + k_\mu p_\mu)} x^{\mu} \frac{1}{2} k_\mu \theta^{\mu\nu} p_\nu f(k) g(p) $$

$$ = e^{\frac{\theta^{\mu\nu}}{\theta^{\mu\nu}} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p_\nu}} f(x + \xi) g(x + \eta) \mid_{\xi = \eta = 0}. $$

(2)
The commutative coordinates $x^i$ are called the Weyl symbols of position operators $\hat{X}^i$, and, if the interpretation for canonical noncommutativity of $[14]$ is adopted, they can be considered as spacetime coordinates. In this work we shall assume that $\theta^{ii} = 0$. The Hilbert space of states of NCQM can consistently be taken as the same as in the commutative quantum mechanics, and the noncommutative Schrödinger equation is given by

$$i\hbar \frac{\partial \Psi(x^i, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x^i, t) + V(x^i) \star \Psi(x^i, t)$$

$$= -\frac{\hbar^2}{2m} \nabla^2 \Psi(x^i, t) + V \left( x^i + i\frac{\theta^{ij}}{2} \partial_j \right) \Psi(x^i, t).$$

(3)

The operators

$$\hat{X}^i = x^i + i\frac{\theta^{ij} \partial_j}{2}$$

(4)

are the observables that correspond to the physical positions of the particles, and $x^i$ are the associated canonical coordinates. Methodologically, the NCQM formulated with Eqs. 3 and 4 can be considered as the “usual” quantum mechanics with a Hamiltonian not quadratic in momenta and “unusual” position operators defined in Eq. 4. From this point of view, the BNCQM developed below can be considered as an extension of the usual Bohmian quantum mechanics along the same lines.

Since the $\hat{X}^i$ do not commute and satisfy the relation $[14]$, the particles cannot be localized in a measurement process. Any attempt to localize the particles must obey the uncertainty relation

$$\Delta X^i \Delta X^j \geq \frac{|\theta^{ij}|}{2}.$$  

(5)

The expression for the definition of probability density $\rho(x^i, t) = |\Psi(x^i, t)|^2$ has a meaning that differs from that of ordinary quantum mechanics. The quantity $\rho(x^i, t)d^3x$ must be interpreted as the probability that the system is found in a configuration such that the canonical coordinate of the particle is contained in a volume $d^3x$ around the point $\vec{x}$ at time $t$. Computation of the expected values can be done in a similar way as in the usual formalism. Given an arbitrary physical observable, characterized by a Hermitian operator $\hat{A}(\hat{x}^i, \hat{p}^i)$ [this naturally includes $\hat{A}(\hat{X}^i(\vec{x}, \vec{p}), \vec{p})$], its expected value is defined as

$$\langle \hat{A} \rangle_t = \int d^3x \Psi^*(x^i, t) \hat{A}(x^i, -i\hbar \partial_i) \Psi(x^i, t).$$

(6)

A Hamilton-Jacobi formalism for NCQM is found by writing the wave function in its polar form $\Psi = Re^{iS/\hbar}$, replacing it in Eq. 3, and splitting its real and imaginary parts. For the real part, we obtain

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_K + Q_I = 0.$$  

(7)

The three new potential terms are defined as

$$V_{nc} = V \left( x^i - \frac{i\theta^{ij}}{2\hbar} \partial_j S \right) - V \left( x^i \right).$$

(8)

\(^2\) An intuition about the meaning of these coordinates may be found in the dipole picture $[14]$. For the case of NCQM, it would consist in considering that, instead of a particle, the elementary object of the theory is a “half dipole” whose extent is proportional to its canonical momentum, $\Delta x^i = \theta^{ij} p_j/2\hbar$. One of its end points carries its mass and is responsible for its interactions. The other extreme is empty. According to this intuitive view, the change of variables $X^i = x^i - \theta^{ij} p_j/2\hbar$ corresponds to a change of coordinates of the interacting extreme of the dipole $X^i$, where the corresponding physical particle is located, to its empty one $x^i$. However, since $x^i$ is not an interacting extremum, we are adopting the point particle interpretation as preferential.
\[
Q_K = \text{Re}\left(-\frac{\hbar^2}{2m} \nabla^2 \frac{\Psi}{\Psi}\right) - \left(\frac{\hbar^2}{2m} (\nabla S)^2\right) = -\frac{\hbar^2}{2m} \nabla^2 \frac{\Psi}{\Psi},
\]
and
\[
Q_I = \text{Re}\left(\frac{V [x^i + (i\theta^{ij}/2) \partial_j] \Psi}{\Psi}\right) - V \left(x^i - \frac{\theta^{ij}}{2\hbar} \partial_j S\right).
\]

\(V_{nc}\) is the potential that accounts for the noncommutative classical interactions, while \(Q_K\) and \(Q_I\) account for the quantum effects. The noncommutative contributions contained in the latter two can be split out by defining

\[
Q_{nc} = Q_K + Q_I - Q_c,
\]
where

\[
Q_c = -\frac{\hbar^2}{2m} \nabla^2 R_c, \quad R_c = \sqrt{\Psi_c^* \Psi_c}.
\]

\(\Psi_c\) is the wave function obtained from the commutative Schrödinger equation containing the usual potential \(V(x^i)\), that is, the equation obtained by setting \(\theta^{ij} = 0\) in Eq. (3) before solving it. The imaginary part of the Schrödinger equation, which yields the differential probability conservation law,

\[
\frac{\partial R^2}{\partial t} + \nabla \cdot \left(\frac{R^2 \nabla S}{m}\right) + \Sigma_\theta = 0,
\]
where

\[
\Sigma_\theta = -\frac{2R}{\hbar} \text{Im} \left[e^{-iS/\hbar} V \star \left(Re^{iS/\hbar}\right)\right].
\]

By integrating Eq. (13) over the space we find

\[
\frac{d}{dt} \int R^2 d^3x = 0,
\]
and \(R^2\) vanishes at infinity.

### B. Constructing the ontological theory of motion

The formalism to be presented from now on is along the same line as the one adopted by Bohm and followers (see for example [37, 38]). Before developing the BNCQM, we briefly summarize the essential ideas that lie behind the Bohmian interpretation.

The Bohmian approach to quantum theory is founded on the assumption that the complete characterization of a quantum system cannot be provided by a wave function alone. For the description of individual processes, which are not statistical in character, an objective view of matter is adopted. In order to reconcile the notion
of objective reality with the known results from quantum theory, an individual physical system is assumed to be composed of a wave propagating with a particle. The particle moves under the guidance of the wave, which satisfies the ordinary Schrödinger equation and contains the information on how the energy of the particle must be directed.

As in the commutative counterpart, in the formulation of BNCQM we shall assume the system as composed of a wave function and a point particle. Contrary to the commutative case, however, in BNCQM the position observables satisfy the relation \( i \), and the wave function satisfies the noncommutative Schrödinger equation \( \). Having the equation for the evolution of the guiding wave \( \Psi \), one still has to determine the particle motion. In principle, there is an arbitrariness in this procedure. However, necessary conditions for the theory to be capable of reproducing the same statistical results as the standard interpretation of NCQM constrain the admissible form for the functions \( X(t) \) that describe the particle motion. Notice that the wave function is valued on canonical coordinates. Therefore, the use of these coordinates in intermediary calculations to determine the \( X(t) \)'s is unavoidable. Before determining the procedure to find these functions, we must define the rules for the computation of expectation values in BNCQM.

With an arbitrary physical observable, characterized by a Hermitian operator \( \hat{A}(\hat{x}, \hat{p}) \), it is possible to associate a function \( A(x^i, t) \), the “local expectation value” of \( \hat{A} \) \( , \) which when averaged over the ensemble of density \( \rho(x^i, t) = |\Psi(x^i, t)|^2 \) gives the same expectation value obtained by the standard operatorial formalism. It is natural to define the ensemble average by

\[
\langle \hat{A} \rangle_t = \int \rho(x^i, t) A(x^i, t) d^3x.
\]

For Eq. \( \) to agree with Eq. \( \), \( A(x^i, t) \) must be defined as

\[
A(x^i, t) = \frac{\text{Re} \left[ \Psi^*(x^i, t) \hat{A}(x^i, -i\hbar \partial_x) \Psi(x^i, t) \right]}{\Psi^*(x^i, t) \Psi(x^i, t)} = A(x^i, t) + Q_A(x^i, t),
\]

where the real value was taken to account for the hermiticity of \( \hat{A}(\hat{x}, \hat{p}) \) and \( A(x^i, t) = A(x^i, p^i = \partial_i S(x^i, t)) \), that is, a function obtained from \( \hat{A}(\hat{x}, \hat{p}) \) by replacing \( \hat{x} \to x^i, \hat{p} \to \partial^i S(x^i, t) \). \( Q_A \) is defined by

\[
Q_A = \text{Re} \left[ \frac{\hat{A}(x^i, -i\hbar \partial_x) \Psi(x^i, t)}{\Psi(x^i, t)} \right] - A(x^i, t)
\]

and is the quantum potential that accompanies \( A(x^i, t) \) (for details of the procedure to identify quantum effects, see, e.g., \( \)). From Eq. \( \) we find that the local expectation value of Eq. \( \) is

\[
X^i = x^i - \frac{\theta^{ij}}{2\hbar} \partial_j S(x^i, t).
\]

The strategy to find the \( X^i(t) \)'s now becomes clear. The relevant information for particle motion can be extracted from the guiding wave \( \Psi(x^i, t) \) by first computing the associated canonical position tracks \( x^i(t) \), and then evaluating Eq. \( \) at \( x^i = x^i(t) \). In order to find a good equation for the \( x^i(t) \)'s, it is interesting to consider the Heisenberg formulation and the equations of motion for the observables (see, for example, \( \)). For the variables \( \hat{x}^i \) they are given by

\[
\frac{d\hat{x}^i_H}{dt} = \frac{1}{i\hbar} [\hat{x}^i_H, \hat{H}] = \frac{\hat{p}^i_H}{m} + \frac{\theta^{ij}}{2\hbar} \frac{\partial \hat{V}(\hat{X}^j_H)}{\partial \hat{X}^i_H}.
\]

By passing the right-hand side (RHS) of Eq. \( \) to the Schrödinger picture it is possible to define the velocity operators

\[
\hat{v}^i = \frac{1}{i\hbar} [\hat{x}^i, \hat{H}] = \frac{\hat{p}^i}{m} + \frac{\theta^{ij}}{2\hbar} \frac{\partial \hat{V}(\hat{X})}{\partial \hat{X}^j}.
\]

\[4\] Our notation differs from that of Holland \( \) who denotes local expectation value of \( \hat{A} \) by \( A(x^i, t) \).
The differential equation for the canonical positions $x^i(t)$ is found by identifying $dx^i(t)/dt$ with the local expectation value of $\hat{v}^i$:

$$\frac{dx^i(t)}{dt} = \left[ \frac{\partial^i S(x^i, t)}{m} + \frac{\theta^{ij} \frac{\partial V(X^j)}{\partial X^j}}{2\hbar} + \frac{Q^i}{2} \right] \bigg|_{x^i=x^i(t)}, \quad (23)$$

where $X^i$ is given in Eq. (20), $S(x^i, t)$ is the phase of $\Psi$, and

$$Q^i = \text{Re} \left( \frac{(\theta^{ij}/\hbar) \left[ \frac{\partial \hat{V}(\hat{X}^i)}{\partial \hat{X}^j} \right] \Psi(x^i, t)}{\Psi(x^i, t)} \right) - \frac{\theta^{ij}}{\hbar} \frac{\partial V(X^j)}{\partial X^j}. \quad (24)$$

The potentials $Q^i$ account for quantum effects coming from derivatives of order 2 and higher contained in $\partial \hat{V}(\hat{X}^i)/\partial \hat{X}^j$.

Once the $x^i(t)$ are known, the particle trajectories are given by

$$X^i(t) = x^i(t) - \frac{\theta^{ij}}{2\hbar} \partial_j S(x^i(t), t). \quad (25)$$

One important property of Eq. (25) is that the particles' positions are not defined on nodal regions of $\Psi$, where $S$ is undefined. Thus, the particles cannot run through these regions. An interesting consequence of this property is that, although the wave function is valued on the canonical position variables, its vanishing can be adopted as a boundary condition, implying that the particles do not run through a region. This is a nontrivial conclusion, since, as stressed before, $|\Psi(x^i, t)|^2 d^3x$ refers to the canonical variables, and thus does not represent the probability that the particles are in the volume $d^3x$ around the point $\vec{x}$ at time $t$. Indeed, it must exclusively be attributed to the fact that the particles, in the theory under consideration, are objective and their trajectories are given by Eq. (25). Had one considered, for example, the problem of how to apply boundary conditions in NCQM to calculate the energy levels of a particle in an infinite square well potential from the point of view of the orthodox Copenhagen interpretation, there would be no preferred answer.

The difficulty of introducing well-defined lines with boundary conditions in noncommutative theories was previously stressed in [40]. In that context, noncommutativity was considered as an intrinsic property of the spacetime. Part of the difficulty in conceiving boundary conditions on well-defined lines is automatically removed if the interpretation for the noncommutativity proposed in [14] is adopted, since the spacetime in that work is assumed to be pointwise. For the determination of the appropriate boundary condition for the particles not to run through a region in NCQM, the Bohmian approach is hereby providing the unambiguous prescription one would request.

We close this subsection by commenting how the uncertainty (5) can be understood in the Bohmian interpretation. In the ordinary de Broglie-Bohm theory, the impossibility of simultaneously determining the position and momentum of a particle is attributed to the perturbation introduced on $p^i = \partial^i S$ by the evolution of the wave function during the measurement process [34]. The uncertainty (3) is generated by a similar mechanism, since the $X^i$'s contain $\partial^i S$ in their definition. Notice that, contrary to the ordinary de Broglie-Bohm theory, where the initial particle positions can be perfectly known in measurement (by paying the price of disturbing the system and modifying the wave function), the initial positions of the particles in BNCQM are experimentally undeterminable.

C. The basic postulates

In the previous subsection, we proposed an objective quantum theory of motion for NCQM. Let us now summarize the complete theory in a formal structure. This is done with the help of the following postulates.

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5 The relevance of this procedure for the determination of the equation of motion will be clearer in the next subsection.
The spacetime is commutative and has a pointwise manifold structure with canonical coordinates \( x^i \). The observables corresponding to operators of position coordinates \( \hat{X}^i \) of particles satisfy the commutation relation
\[
[\hat{X}^i, \hat{X}^j] = i\theta^{ij}.
\] (26)
The position observables can be represented in the coordinate space as \( \hat{X}^i = x^i + i\theta^{ij}\partial_j/2 \), and the \( x^i \) are canonical coordinates associated with the particle.

(2) A quantum system is composed of a point particle and a wave \( \Psi \). The particle moves in spacetime under the guidance of the wave, which satisfies the Schrödinger equation
\[
i\hbar \frac{\partial \Psi(x^i, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x^i, t) + V(\hat{X}^i)\Psi(x^i, t).
\] (27)

The position observables can be represented in the coordinate space as
\[
\rho(x^i, t) = \int d^3x' \Psi(\vec{x}', t)^* \Psi(\vec{x}', t) = \int d^3x' \hat{\rho}(\vec{x}', t)
\]
where \( \hat{\rho}(\vec{x}', t) \) is the canonical probability current
\[
\hat{J}^i = i\hbar \frac{\partial \Psi(x^i, t)}{\partial x^i} = \int d^3x' \hat{\rho}(\vec{x}', t) \partial_{x^i} \Psi(\vec{x}', t)
\] (28)
independent of observation, where \( S \) is the phase of \( \Psi \) and the \( x^i(t) \) describe the canonical position trajectories, which are found by solving
\[
\frac{dx^i(t)}{dt} = \frac{\partial S(\vec{x}, t)}{m} + \frac{\theta^{ij}}{2\hbar} \frac{\partial V(X^j)}{\partial X^j} + \frac{Q^i}{2} \bigg|_{x' = x^i(t)}.
\] (29)
To find the path followed by a particle, one must specify its initial canonical position \( x^i(0) \), solve Eq. (29), and then obtain the physical path via Eq. (28).

The three postulates presented above constitute on their own a consistent theory of motion. However, the theory presented is intended to be a finer view of quantum mechanics, able to give a detailed description of the individual physical processes and provide the same statistical predictions. In ordinary commutative Bohmian mechanics, in order to reproduce the statistical predictions of the Copenhagen interpretation, the additional requirement that, at a certain instant of time \( t_0 \), \( \rho(x^i, t_0) = |\Psi(\vec{x}, t_0)|^2 \) is imposed. This assumption and the equivariance of the probability distribution assure that \( \rho(x^i, t) = |\Psi(x^i, t)|^2 \) for all \( t \). A distribution \( \rho(x^i, t) = |\Psi(x^i, t)|^2 \) is said to be equivariant if it retains its form as a functional of \( \Psi(x^i, t) \) under evolution of the ensemble particles satisfying \( \dot{x}^i(t) = f^i(x^j, t) \). In other words, equivariance is achieved if, departing from an ensemble of physical systems, each one containing a single particle, whose associated canonical probability density at initial time \( t_0 \) is given by \( \rho(x^i, t_0) = |\Psi(x^i, t_0)|^2 \), and evolving according to \( \dot{x}^i(t) = f^i(x^j, t) \), then \( \rho(x^i, t) = |\Psi(x^i, t)|^2 \) for all \( t \). In such a case the probability distribution \( \rho(x^i, t) \) satisfies the transport equation
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \dot{x}^i)}{\partial x^i} = 0.
\] (30)

In ordinary commutative quantum mechanics the equivariance property is satisfied thanks to the equality \( \dot{x}^i(t) = J^i/\rho \), which is a consequence of the identification between \( \dot{x}^i(t) \) and the local expectation value of the \( \hat{v}^i \). In the BNCQM proposed in this work, the same identification is valid. However, this is not sufficient to guarantee equivariance in all cases. This is rendered evident by computing the canonical probability current \( J^i(x^i, t) \), which is defined by
\[
J^i(x^i, t) = \text{Re} \left[ \Psi^*(x^i, t) \dot{\Psi}(x^i, t) \right] = |\Psi(x^i, t)|^2 \left[ \frac{\partial S(\vec{x}, t)}{m} + \frac{\theta^{ij}}{2\hbar} \frac{\partial V(X^j)}{\partial X^j} + \frac{Q^i}{2} \right] = \rho \dot{x}^i,
\] (31)
and regrouping the terms in Eq. (30) in such a way that the canonical probability flux \( J^i \) appears explicitly,\(^6\)

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\(^6\) The notion of conserved current in noncommutative theories is little different from the one in commutative theories, as was pointed out in [12], in the context of field theory. From the global U(1) symmetry of the noncommutative Schrödinger Lagrangian, the maximum that can be said is that \( \partial \rho/\partial t + \partial J^i(x^i, t)/\partial x^i + F_0 = 0 \), where \( F_0 \) is some function containing the Moyal product and that satisfies \( \int d^3xF_0 = 0 \).
obtaining

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \dot{x}^i)}{\partial x^i} - \frac{\partial}{\partial x^i} \left[ \rho \left( \frac{\theta^{ij}}{2\hbar} \frac{\partial V(X^j)}{\partial X^j} + \frac{Q^i}{2} \right) \right] + \Sigma_\theta = 0.
\] (32)

For equivariance to occur, an additional condition that the sum of the last two terms in the RHS of Eq. (32) vanishes is required. When \(V(X^i)\) is a linear or quadratic function, as in the application problem of the next section, such a condition is trivially satisfied,\(^7\) and thus \(\rho(x^i, t) = |\Psi(x^i, t)|^2\) is certainly equivariant. The same may also occur for special states when other potentials are considered in Eq. (27), but it is not a general property of Eq. (32). We shall return to this point in the final discussion.

III. BOHMIAN NONCOMMUTATIVE HARMONIC OSCILLATOR

Here, we show a simple application of the BNCQM for the analysis of a two-dimensional harmonic oscillator. We shall follow the approach previously discussed in Ref. \[14\]. Other relevant work on the noncommutative harmonic oscillator may be found in Ref. \[12\]. In two dimensions, Eq. (1) is reduced to

\[
[\hat{X}^\mu, \hat{X}^\nu] = i\theta e^{\nu\mu}.
\] (33)

The position observables of the particles can therefore be represented by

\[
\hat{X}^i = x^i - \theta \epsilon^{ij} \hat{p}_j / 2\hbar,
\]

where \(m\) and \(w\) are the mass and frequency of the associated commutative oscillator, respectively. The corresponding Schrödinger equation in polar coordinates is

\[
i\hbar \frac{\partial \Psi_\theta (r, \varphi, t)}{\partial t} = H_\theta \Psi_\theta (r, \varphi, t)
\]

\[
= -\frac{\hbar^2}{2m} \left[ 1 + \left( \frac{mw\theta}{2\hbar} \right)^2 \right] \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\varphi^2 \right) \Psi_\theta (r, \varphi, t)
\]

\[
+ \left( \frac{i}{2} \partial_\varphi \partial_r + \frac{m}{2} w^2 r^2 \right) \Psi_\theta (r, \varphi, t),
\]

whose solution is Ref. \[14\]

\[
\Psi_\theta (r, \varphi, t) = (-1)^n \sqrt{\frac{n!\zeta}{\pi (n + |\alpha|)!}} \exp \left( -\frac{\zeta r^2}{2} \right) \left( \sqrt{\zeta} r \right)^{|\alpha|} L_{n,\theta}^{(|\alpha|)} \left( \zeta r^2 \right) e^{i\alpha \varphi - iE t/\hbar},
\] (36)

where \(L_{n,\theta}^{(|\alpha|)} (\zeta r^2)\) are the associated Laguerre polynomials

\[
L_{n,\theta}^{(|\alpha|)} (\zeta r^2) = \sum_{l=0}^{n} (-1)^l \binom{n + |\alpha|}{n - l} \frac{(\zeta r^2)^l}{l!}, \quad \zeta^2 = \frac{(mw/\hbar)^2}{1 + (mw\theta/2\hbar)^2},
\] (37)

\(n = 0, 1, 2, \ldots\) is the principal quantum number, and \(\alpha = 0, \pm 1, \pm 2, \ldots\) is the canonical angular momentum quantum number.

\(\)\(^7\) This is easily seen by substituting Eq. (4) in Eq. (27), regrouping the terms, and noticing that in these cases \(\hat{H}\) is reduced to a “familiar” Hamiltonian quadratic in the canonical momenta, as occurs in commutative quantum mechanics. The noncommutative effects, however, are still present, as we shall show in the next section.
The energy levels are given by

\[ E_{n,\alpha,\theta} = 2\hbar w \left[ 1 + \left( \frac{mw\theta}{2\hbar} \right)^2 \right]^{1/2} \left( n + |\alpha| + \frac{1}{2} \right) - \frac{m\theta w^2 \alpha}{2}. \]  

(38)

Notice that, due to the noncommutative effects, the degeneracy of the energy levels corresponding to the right- and left-handed polarizations for the same \( n \) is removed. When the noncommutativity is assumed as originating from the action of a strong background field, like the Neveu-Schwartz field in the stringy context, or a magnetic field when a condensed matter system is projected onto its lowest Landau level, the lifting of the degeneracy can be intuitively understood as the consequence of a chirality introduced by the background field.

For simplicity, let us consider the state where \( n = 1 \). In this state, Eq. (36) is simplified to

\[ \Psi_\theta (r, \varphi, t) = \sqrt{\frac{\zeta}{\pi |\alpha|!}} \exp \left( -\frac{\zeta r^2}{2} \right) \left( \sqrt{\zeta r} \right)^{|\alpha|} \left( 1 + |\alpha| - \zeta r^2 \right) e^{i\alpha \varphi - iEt/\hbar}, \]  

(39)

and the corresponding \( V, V_{nc}, Q_c, Q_{nc}, \) and \( \Sigma_\theta \) are

\[ V = \frac{1}{2} mw^2 r^2, \]
\[ V_{nc} = \left( \frac{mw\theta}{2\hbar} \right)^2 \alpha^2 \hbar^2 \frac{2}{2mr^2} - \frac{m\theta w^2 \alpha}{2}, \]
\[ Q_c = -\frac{1}{2} mw^2 r^2 + \hbar w (|\alpha| + 3) - \alpha^2 \hbar^2 \frac{2}{2mr^2}, \]
\[ Q_{nc} = \left[ \sqrt{1 + \left( \frac{mw\theta}{2\hbar} \right)^2} - 1 \right] \hbar w (|\alpha| + 3) - \left( \frac{mw\theta}{2\hbar} \right)^2 \alpha^2 \hbar^2 \frac{2}{2mr^2}, \]
\[ Q^i = \Sigma_\theta = 0. \]

The canonical trajectories are found by solving the equations

\[ \frac{dx}{dt} = \frac{1}{m} \frac{\partial S}{\partial x} + \frac{\theta}{2\hbar} mw^2 \left( x - \frac{\theta}{2\hbar} \frac{\partial S}{\partial y} \right), \quad \frac{dy}{dt} = \frac{1}{m} \frac{\partial S}{\partial y} - \frac{\theta}{2\hbar} mw^2 \left( y + \frac{\theta}{2\hbar} \frac{\partial S}{\partial x} \right). \]  

(41)

Changing into polar coordinates and substituting \( S = \alpha h \varphi - Et \) in Eq. (41), we find

\[ \frac{dr}{dt} = 0, \quad \frac{d\varphi}{dt} = \frac{h\alpha}{mr^2} + \frac{mw^2 \theta^2 \alpha}{4hr^2} - \frac{mw^2 \theta}{2h}, \]  

(42)

whose solutions are

\[ r = r_0, \quad \varphi = \varphi_0 + w_\theta t, \quad w_\theta = \left( \frac{h\alpha}{mr^2} + \frac{mw^2 \theta^2 \alpha}{4hr^2} - \frac{mw^2 \theta}{2h} \right). \]

(43)

The physical radius and angle are

\[ R(t) = \sqrt{X^2(t) + Y^2(t)} = \left[ \left( x - \frac{\theta}{2\hbar} \frac{\partial S}{\partial y} \right)^2 + \left( y + \frac{\theta}{2\hbar} \frac{\partial S}{\partial x} \right)^2 \right]^{1/2} = r_0 \left| 1 - \frac{\alpha \theta}{2r_0^2} \right| = R_0 \]  

(44)

and

\[ \Phi(t) = \arctan \left[ \frac{Y(t)}{X(t)} \right] = \arctan \left[ \frac{y(t)}{x(t)} \right] = \varphi(t). \]  

(45)
The velocity of the particles is tangential to their circular orbits, being given by

\[
|\mathbf{V}(t)| = \sqrt{\dot{X}^2(t) + \dot{Y}^2(t)} = R_0 |\phi| = R_0 |w_\theta|.
\]

(46)

As expected, the particle trajectories are circles. Contrary to the commutative case, however, the absolute value of the tangential velocity \(|\mathbf{V}(t)|\) is not the same for both the right- and left-polarized states. This is due to a difference in the angular velocity \(w_\theta\) and in the radius \(R_0\) of their corresponding orbits. While the orbits associated with the right-handed excitations have the energy levels shifted downward, and their velocities and radii reduced with respect to the commutative ones, the left-handed excitations have their energy levels shifted upward, and move with larger velocities and radii. Notice from Eq. (46) that, when the system is in the lowest-energy state, characterized by \(\alpha = 0\), the particle is still moving,\(^8\) unless \(R_0 = 0\). Such a motion is absent in ordinary Bohmian theory and originates from a noncommutative term contained in \(w_\theta\).

From Eq. (40), it is possible to see that the condition \(V_{nc} + Q_{nc} \rightarrow 0\) is satisfied if \(\theta \ll \frac{2\hbar}{mw}\), as shown in [14]. In that work, there was also an assumption that \(\theta\) should be sufficiently small in order that Eq. (5) could not be directly verified by experimentation. The length scales considered until now, therefore, were assumed to be many times larger than that of \(\sqrt{\theta}\), which is the characteristic length of noncommutativity. On scales of \(\sqrt{\theta}\) order or smaller, noncommutativity effects associated with Eq. (5) are expected to drastically modify the behavior of the system \[\square R.\] Let us ignore, for the time being, the previous assumption on the dimensions of our system, and allow the noncommutative harmonic oscillator to live at arbitrarily small length scales, or, equivalently, allow \(\theta\) to assume a large value. In this case, it is interesting to consider the individual behavior of \(V_{nc}\) and \(Q_{nc}\). To study the oscillator orbits, we compare the behavior of the variable \(R\), which describes its physical radius, with that of the canonical variable \(r\). A plot of \(R(r)\) is found in Fig. 1 for the cases where \(\alpha \theta = 1\) and \(\alpha \theta = -1\).

![FIG. 1. The typical behavior of the radius of the oscillator orbit, \(R(r)\), for \(\alpha > 0\) (thick line) and \(\alpha < 0\) (thin line) illustrated for the cases where \(\alpha \theta = 1\) and \(\alpha \theta = -1\).](image)

From Fig. 1 we can see that, when the scale of the system is sufficiently large as compared to \(\sqrt{|\alpha \theta|}\), the distinction between \(r\) and \(R\) is not relevant. However, for length scales around \(\sqrt{|\alpha \theta|}\), the distinction between \(R\)

\[^8\] This is not the source of any inconsistency. Since \(\alpha\) is a quantum number related to the canonical position variables, it is not directly connected to the physical angular momentum.
and $r$ begins to become important, since these variables can differ by a large amount. The asymmetry between the states for which $\alpha > 0$ and $\alpha < 0$ is also important at these scales. If $\alpha > 0$, there is a specific orbit radius in which canonical and physical coordinates are completely identified, corresponding to the point where the straight line $R = r$ crosses the $\alpha > 0$ thick line. From Eq. (41), it is easy to see that this point is $r_0 = \sqrt{\alpha \theta} / 2$. By substituting this value for $r$ in Eq. (40), one finds $V_{nc} = 0$. Since $Q_{nc} \neq 0$, one still has noncommutative effects in the corresponding orbit, but they have a genuinely quantum nature. The minimum value allowed for $R_0$ occurs when $r_0 = \sqrt{\alpha \theta} / 2$, for both $\alpha > 0$ and $\alpha < 0$ states. Its value is $R_{OM} = 0$ for $\alpha > 0$, and $R_{OM} = \sqrt{2 |\alpha \theta|} / 2$ for $\alpha < 0$.

There is an interesting property of the noncommutative harmonic oscillator that is rendered evident by its Bohmian description. Observe in Fig. 1 that, for each value of $R$, there correspond two values of $r$, one smaller than $\sqrt{|\alpha \theta|} / 2$ and the other larger. The smallest values of $r$ contained in the interval $(0, \sqrt{|\alpha \theta|} / 2)$ have as their partners exactly the largest ones in the interval $(\sqrt{|\alpha \theta|} / 2, \infty)$. In the ordinary commutative Bohmian interpretation, there is a contextual character in the information provided by the wave function to guide the particle motion (for details see, for example, [39]). The information for the particle to move, however, is extracted by the particle from the part of the wave function that is spread over the spatial points which cover the particle position and a small neighborhood around it. What we find in the noncommutative Bohmian mechanics is a new and interesting property, over and above the contextuality of ordinary Bohmian mechanics. For the case where the system has a large physical radius, it may be receiving information from part of the wave function that is concentrated in a small region far beyond the position where the particle is located. This property is similar to the IR-UV mixing that occurs in field theory [2] and is a manifestation of the specific kind of nonlocality found in NCQM due to the “shift” in the interaction point in Eq. (3).

IV. DISCUSSION AND OUTLOOK

In this work we carried out an investigation of the possibility of developing a Bohmian interpretation for NCQM. The theory was proposed to be based on a commutative spacetime containing point particles, whose position observables should satisfy the canonical relation [11]. Such a realization of noncommutativity only between particle position observables means that, although the particles are point like, their complete localization in a process of measurement is forbidden by the disturbances caused by the measurement apparatus interacting with the quantum system. The intrinsic uncertainty of the particle localization during a measurement process must be faced on the same footing as the one forbidding the simultaneous determination of the momentum and the position of the particle in ordinary Bohmian theory.

In the manner it was constructed, BNCQM was conceived to reproduce the same statistical predictions of NCQM in the Copenhagen interpretation with a continuous evolution law for particle motion. As a result of our investigation of the possibility for this to occur, we found that, when linear and quadratic potentials are considered in the Hamiltonian, the theory in its present form is certainly statistically equivalent to NCQM in the Copenhagen interpretation. The same may also occur for other specific potential terms or physical states where Eqs. (30) and (32) do match. When this is not the case, for the theories’ predictions to agree, it is necessary to modify the particle evolution law. One interesting way this may be done is by considering a “hybrid” evolution law constituted of a continuous part governed by a differential equation added to a stochastic piece, which allows the particles to jump, along the lines adopted in [44]. Equation (32) in this case should be understood as a probability transport equation of a piecewise-deterministic jump process.

Among the physical systems of interest where BNCQM with a continuous particle motion is able to reproduce the Copenhagen interpretation results is the harmonic oscillator. Although very specific, the harmonic potential is of great relevance for physics, which justifies the enormous variety of work on NCQM devoted to it (see [12] and references therein). Indeed, in this work, the harmonic oscillator proved to be useful to illustrate essential properties of NCQM in the spirit of the Bohmian interpretation. In its fine description of the harmonic oscillator, BNCQM revealed the interesting possibility that the small-scale physics can influence the large-scale phenomena in the quantum-mechanical context, in close similarity to the IR-UV mixing that appears in field theory. Another important contribution of the Bohmian interpretation comes from its capability to give well-defined predictions in situations where the Copenhagen interpretation is vague, as discussed in Sec. III.
An interesting environment where the predictions of the Bohmian interpretation may be confronted in future with experimentation is that of quantum cosmology. Recently, noncommutativity at early times of the universe was introduced by deforming the commutation relation among the minisuperspace variables in a cosmological model based on the Kantowski-Sachs metric \cite{32}, originating a noncommutative Wheeler-DeWitt equation. Since in the formalism of minisuperspace the Wheeler-DeWitt equation is essentially quantum mechanical, the application of the Bohmian interpretation developed in this work for models like the one of \cite{32} is almost immediate \cite{45}. It may reveal unknown features or unexpected results, since it will allow a description of the primordial quantum universe following a well-defined “trajectory” in minisuperspace. In the case of conceiving a quantum cosmology based on the canonical noncommutativity of the spatial coordinates \cite{11}, for example, the ideas presented in this work may also be a good starting point.

Since the ontological interpretations have variants and are still under construction, this work should not be considered a closed structure. Many of the rules stated here are open and may be subject to reformulation after further discussion. In addition to the interesting possibility for a Bohmian description with stochastic particle jumps to shed light on the interpretation of all terms in Eq. \cite{52}, which compels us to carry on an extension of the theory presented here, there are many open questions to be exploited in the formulation. Among them, we quote the extension of the theory to incorporate a many-body approach, where some care must be taken when considering charged particles \cite{10}, for example.

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