Fault Identification via Non-parametric Belief Propagation

Danny Bickson, Member, IEEE, Dror Baron, Member, IEEE, Alexander Ihler, Member, IEEE, Harel Avissar, Student Member, IEEE, Danny Dolev, Member, IEEE

Abstract—We consider the problem of identifying a pattern of faults from a set of noisy linear measurements. Unfortunately, maximum a posteriori probability estimation of the fault pattern is computationally intractable. To solve the fault identification problem, we propose a non-parametric belief propagation approach. We show empirically that our belief propagation solver is more accurate than recent state-of-the-art algorithms. We formulate the fault identification problem as a combinatorial problem to identify the most likely pattern of faults given the system measurements.

In this work we shall consider a typical model of fault measurement which of a set of possible failures (faults) have occurred in a system given observations of its behavior. Such problems arise in a variety of computer-based engineering applications, including aerospace [1], [2], industrial process control [3], and automotive systems [4].

A key assumption of the current work is that the fault signature matrix, which is the linear transformation matrix applied to the fault vector, is sparse. This assumption is especially valid in large scale systems where different faults affect different parts of the system. As in fault identification, the system of linear equations is ill conditioned, and the assumption of sparsity (corresponding to the rarity of faults) is critical in the reconstruction. Because of this similarity between CS and fault identification, in the experimental results section we compare the performance of several CS algorithms with previously proposed methods for fault identification.

In this work, we develop a novel approach for fault identification based on a variant of the belief propagation (BP) algorithm called non-parametric belief propagation (NBP). Belief propagation approaches have been applied to solve many similar discrete, combinatorial problems in coding, and variants of NBP (which allows the algorithm to reason the observed system measurements in a linear additive way. Frequently there are far more potential faults than measurements, resulting in an ill conditioned set of linear equations. By assuming that faults are relatively rare, one can construct a combinatorial problem to identify the most likely pattern of faults given the system measurements.

Systems of binary faults have been extensively studied in the computer science community, for example posing the problem as constraint satisfaction and using heuristic search techniques [6], [7]. An alternative approach is to use a convex relaxation of the original combinatorial problem, for example using the interior point method of Zimnis et al. [8].

A key assumption of the current work is that the fault signature matrix, which is the linear transformation matrix applied to the fault vector, is sparse. This is the case in many applications, where each fault may affect only a small part (such as a single subsystem) of the whole [9], [10], [11], [12]. This assumption is especially valid in large scale systems where different faults affect different parts of the system.

The fault identification problem is closely related to several coding and signal reconstruction problems. For example, in multi-user detection for code division multiple access (CDMA) systems [13], [14], [15], the system measurements are given by the received noisy wireless signal and the goal is to estimate the transmitted bit pattern, which plays the role of the fault pattern. One important difference is that in multi-user detection, each bit typically has an equal probability of being 0 or 1, whereas in fault identification the prior probability that a bit is 1 (indicating that the fault is present) is typically much lower.

Compressed sensing (CS) problems are also closely related to fault identification. Informally, CS reconstructs a sparse signal from a set of noisy linear measurements of the signal. As in fault identification, the system of linear equations is ill conditioned, and the assumption of sparsity (corresponding to the rarity of faults) is critical in the reconstruction. Because of this similarity between CS and fault identification, in the experimental results section we compare the performance of several CS algorithms with previously proposed methods for fault identification.

In this work, we develop a novel approach for fault identification based on a variant of the belief propagation (BP) algorithm called non-parametric belief propagation (NBP). Belief propagation approaches have been applied to solve many similar discrete, combinatorial problems in coding, and variants of NBP (which allows the algorithm to reason
about real-valued variables) have been used in compressed sensing [16] and low-density lattice codes (codes defined over real-valued alphabets) [17], [18], [19]. Our method constructs a relaxation of the fault pattern prior using a mixture of Gaussians, which takes into account both the binary nature of the problem as well as the sparsity of the fault pattern.

Using an extensive experimental study, we show that our approach provides the best performance in identifying the correct fault patterns when compared to recent state-of-the-art algorithms, including interior point methods and semidefinite programming. To demonstrate the importance of each component of our model, we compare both to existing approaches for fault identification as well as to compressed sensing algorithms and a discrete BP formulation. We explain how to implement an efficient quantized version of NBP, and additionally provide the source code used in our experiments [20].

The structure of this paper is as follows. Section II introduces the fault identification problem in terms of maximum a posteriori (MAP) estimation, and describes graphical models and the BP algorithm. Section III presents our solution, based on NBP. Section IV explains implementation details and optimizations. Section V compares the accuracy of several state-of-the-art methods for fault identification, and shows that our proposed method has the highest accuracy. We shed light on the favorable performance of NBP from an information theoretic perspective in Section VI, and conclude with a discussion in Section VII.

II. Fault Identification Problem

In this section we describe the fault model in detail, and the basic MAP approach for estimating the fault pattern. Our goal is to infer the maximum a posteriori (MAP) fault value, the fault pattern most likely to have occurred given a set of observations. We then briefly review probabilistic graphical models and the belief propagation algorithm within this context.

A. Fault model and prior distribution

We consider a system in which there are \( n \) potential faults, any combination of which \((2^n - 1)\) in total can occur. A fault pattern, i.e., a set of faults, is represented by a vector \( x \in \{0, 1\}^n \), where \( x_s = 1 \) means that fault \( s \), \( s \in \{1, \ldots, n\} \) has occurred. We assume that faults are independent and identically distributed (i.i.d.), and that fault \( s \) occurs with known probability \( p \). Thus, the (prior) probability of fault pattern \( x \) occurring is

\[
p(x) = \prod_{s=1}^{n} p^{x_s} (1-p)^{1-x_s}.
\]

The fault pattern \( x = 0 \) corresponds to the null hypothesis, the situation in which no faults have occurred, with probability \( p(0) = (1-p)^n \), and the expected number of faults is \( \sum_{s=1}^{n} p = np \).

We assume that \( m \) scalar real measurements, denoted by \( y \), \( y \in \mathbb{R}^m \), are available. These measurements depend on the fault pattern \( x \in \{0,1\}^n \) linearly:

\[
y = Ax + v,
\]

where \( A \in \mathbb{R}^{m \times n} \) is the fault signature matrix, and the measurement noise \( v \in \mathbb{R}^m \) is random, with \( v_i \) independent of each other and \( x \), each with \( \mathcal{N}(0,\sigma^2) \) distribution. Typically the system of linear equation is under-determined \((n > m)\), which means the number of potential faults is greater than the number of measurements.

The fault signature matrix \( A \) is assumed to be known. Its \( s^{th} \) column \( a_s \in \mathbb{R}^m \) corresponds to the measurements, when only fault \( s \) has occurred and assuming no noise. For this reason \( a_s \) is called the \( s^{th} \) fault signature.

B. Posterior probability

Let \( p(x|y) \) denote the (posterior) probability of pattern \( x \) given the measurements \( y \). By Bayes rule we have

\[
p(x|y) \propto p(y|x)p(x) = p(x, y) = \mathcal{N}(y; Ax, \sigma^2 I) \prod_s p^{x_s} (1-p)^{1-x_s}.
\]

(1)

By letting \( C \) and \( C' \) indicate constants (values independent of \( x \)), we can define the log-loss function as the negative log probability \( l_y(x) = -\log p(x|y) \) and write

\[
l_y(x) = \lambda^T x + \frac{1}{2\sigma^2} \sum_{i=1}^{m} (y - Ax)^T (y - Ax) + C
\]

\[
= \frac{1}{2\sigma^2} x^T A^T Ax + (\lambda - \sigma^{-2} A^T y)^T x + C',
\]

(2)

where \( \lambda_s = \log((1-p)/p) \) denotes the log-odds ratio. Note that (2) is a convex quadratic function of \( x \) (a binary-valued vector), and is thus a convex integer quadratic program.

C. Graphical Models

Graphical models are used to represent and exploit the structure of the cost function (2), or its associated probability distribution (1), to develop efficient estimation algorithms. Specifically, we factor \( p(x) \) into a product of smaller functions, called factors, each of which is defined using only a few variables. This collection of smaller functions is then represented as a factor graph \( G \), in which each variable is associated with a variable node and each factor with a factor node. Factor nodes are connected to variable nodes that represent their arguments. Because (2) can be represented as the sum of terms involving at most two variables,

\[
l_y(x) = \sum_{s,t>s} J_{st} x_s x_t + \sum_s h_s x_s,
\]

(3)
functions $f$ correspond to factoring $f$ factors to variables and we can represent
and we can represent $p(x|y)$ or $l_y(x)$ as a pairwise graph with
an edge between $x_s$ and $x_t$ if and only if $J_{st}$ is non-zero. This corresponds to factoring $p(x|y)$ as
\[
p(x|y) \propto \prod_{i=(s,t)>s} f_i(x_s,x_t) \prod_s g_s(x_s) \\
= \prod_{(s,t)>s} \exp(-J_{st}x_sx_t) \prod_s \exp(-h_sx_s),
\]
where $f_i(\cdot)$ and $g_s(\cdot)$ are factors; we use the convention that functions $g$ and indices $s,t$ refer to local (single-variable)
factors while $f$ and indices $i,j$ refer to higher-order (in this case, pairwise) factors. For compactness, we generically write $f(x)$ to indicate a function $f$ over some subset of the $x_s$.

The graph structure is used to define efficient inference algoritms, including MAP estimation or marginalization. Both problems are potentially difficult, as they require optimizing or summing over a large space of possible configurations. However, structure in the graph may induce an efficient method of performing these operations. For example, in sequential problems the Viterbi algorithm \[21\] (and more generally, dynamic programming) provides efficient optimization, and can be represented as a message-passing algorithm on the graph $G$.

In graphs with more complicated structure such as cycles, exact inference is often inefficient; however, similar message-passing algorithms such as loopy belief propagation perform approximate inference \[22\]. The max-product algorithm is a form of BP that generalizes dynamic programming to an approximate algorithm. One computes messages $m_{is}^v$ from variables to factors and $m_{si}^e$ from factors to variables, factors,\n\[
m_{is}^v(x_s) \propto \max_{x_t \neq x_s} f_i(x) \prod_{t \in \Gamma_s \setminus s} m_{ti}^v(x_t), \\
m_{si}^e(x_s) \propto g_s(x_s) \prod_{j \in \Gamma_s \setminus i} m_{js}^f(x_s),
\]
where messages are normalized for numerical stability, and $\Gamma_s$ is the neighborhood of node $s$ in the graph (all nodes for which $\tilde{a}_i$ is non-zero, excluding $s$). One can also compute a “belief” $b_s(x_s)$ about variable $x_s$,\n\[
b_s(x_s) = g_s(x_s) \prod_{i \in \Gamma_s} m_{is}^f(x_s),
\]
which can be used to select the configuration of $x_s$ by choosing its maximizing value. If the graph $G$ is singly-connected (no cycles), then the max-product algorithm is equivalent to dynamic programming. However, the algorithm performs well even in graphs with cycles, and has been shown to be highly successful in many problems, most notably the decoding of low-density parity check (LDPC) codes \[23\]. Max-product and its so-called reweighted variants are closely related to linear programming relaxation techniques, but by exploiting the problem structure can be more efficient than generic linear programming packages \[24\].

The sum-product formulation of BP is intended for approximate marginalization, rather than optimization. Despite this, the sum-product algorithm has been frequently applied to MAP estimation problems, as it often exhibits better convergence behavior than max-product \[25\]. It has an almost identical message-passing formulation,
\[
m_{is}^v(x_s) \propto \sum_{x \neq x_s} f_i(x) \prod_{t \in \Gamma_s \setminus s} m_{ti}^v(x_t), \\
m_{si}^e(x_s) \propto g_s(x_s) \prod_{j \in \Gamma_s \setminus i} m_{js}^f(x_s).
\]
Again, one can estimate each $x_s$ by choosing the value that maximizes the belief $b_s(x_s)$, in this case corresponding to the maximum posterior marginal estimator. Although the sum-product beliefs are intended to approximate the marginal distributions of each $x_s$, if the most likely joint configurations all share a particular value for $x_s$, then this will be reflected in the marginal probability as well.

Variants of BP typically rely on graph sparsity (few edges) to ensure both efficiency and accuracy. When the graph has no cycles, these algorithms are exact; for graphs with cycles, they are typically only approximate but are often accurate in systems with long, weak, or irregular cycles \[26\]. Unfortunately, although the fault signature matrix $A$ may be sparse, the same is typically not true of the matrix $J = A^T A$, especially for large $m$ and $n$. In the dense case, a direct application of BP to \[3\] may fail. For example, in experiments, with $A$ sized $100 \times 200$ and approximately 10% non-zero values $\pm 1$, $A^T A$ is approximately 50% non-zero, and BP algorithms defined using \[3\] did less well than the current state of the art methods. As the dimension sizes $m$ and $n$ are increased, $A^T A$ becomes dominated by non-zeros. This motivates us to define an alternative graphical model that relies on the sparsity of $A$ itself.

III. BELIEF PROPAGATION FOR FAULT IDENTIFICATION

Since the structure of the matrix $A$ is sparse, let us define an alternative graphical model that uses $A$ explicitly. In particular, we can write the probability distribution
\[
p(x,y) = \prod_i f_i(y_i, x) \prod_s g_s(x_s),
\]
in terms of the factors,
\[
f_i(x) = \mathcal{N}(\tilde{a}_i x; y_i, \sigma^2), \\
g_s(x_s) = p^{x_s}(1 - p)^{1-x_s}.
\]
Notably, if $A$ is sparse (specifically, if each row $\tilde{a}_i$ is sparse), then the factors $f_i$ will depend on only the few $x_s$ for which $\tilde{a}_i$ is non-zero and the graph representing this factorization will be sparse as well. An example factor graph is shown in Fig.\[1\]a).

Using either max-product \[4\] or sum-product \[6\] on the resulting factor graph is computationally difficult, as it involves eliminating (maximizing or summing over) all exponentially many configurations of the neighboring variables.
of \( f_i \) \((2^d \) evaluations for factors over \( d \) variables). This can quickly become intractable for even moderate neighborhood sizes. Although the relationships among the \( x_s \) and \( y_i \) are simple and linear, our model defines a hybrid distribution over both continuous valued and discrete valued random variables. Observing \( y_i \) creates a combinatorial dependence among the neighboring discrete-valued \( x_s \).

Although it may seem counter-intuitive, we can avoid some of these difficulties by converting the graphical model to a fully continuous model. We abuse notation slightly to define continuous variables \( x_s \) in place of their discrete counterparts, with corresponding factors

\[
        g_s(x_s) = p_s \delta(x_s = 1) + (1 - p_s) \delta(x_s = 0).
\]

It will also prove convenient to relax the prior slightly into a Gaussian form, using the approximation

\[
        \tilde{g}_s(x_s) = p_s \mathcal{N}(x_s; 1, \nu) + (1 - p_s) \mathcal{N}(x_s; 0, \nu), \tag{8}
\]

where the variance \( \nu \) controls the quality of the approximation; as \( \nu \to 0 \), we recover the original prior on \( x_s \). The factors \( f_i(\cdot) \) and \( g_s(\cdot) \) are illustrated in Fig. 1(b).

The BP message-passing algorithm remains applicable to models defined over continuous random variables. However, message representation is more difficult as we must represent continuous functions over those variables. For non-Gaussian models, message representation typically requires some form of approximation.

In our continuous model, both sets of factors \( f_i \) and \( \tilde{g}_s \) consist of mixtures of Gaussians; thus their product, and the messages computed during BP, will also be representable as mixtures of Gaussians [22], [17]. Unfortunately, at each step of the algorithm, the number of mixture components required to represent the messages will increase at an exponential rate, and must be approximated by a smaller mixture. To handle the exponential growth of mixture components, approximations to BP over continuous variables were proposed in NBP [22] and variants in a broad array of problem domains [27], [17], [18], [16]. Those approximations use sampling to limit the number of components in each mixture. A number of sampling algorithms have been designed to ensure that this process is as efficient as possible [28], [29], [30]. Such sample based representations are particularly useful in high dimensional spaces, where discretization becomes computationally difficult. Various authors have also proposed message approximation methods based on dynamic quantization or discretization techniques [31], [32].

For the fault identification problem our variables \( x_s \) are one dimensional and can be reasonably restricted to a finite domain, for example \( x_s \in [-3\nu, 1 + 3\nu] \) where \( \nu \) is the assumed standard deviation of the noisy linear system, and 0.1 are the binary fault values. Thus, it is both computationally efficient and sufficiently accurate to use a simple uniform discretization over possible values, allowing our functions over \( x_s \) to be represented by fixed-length vectors [16], [17]. Although typically the term “non-parametric BP” refers to an algorithm using stochastic samples and Gaussian mixture approximations to the messages, rather than a fixed discrete quantization, here we use it more generically to distinguish BP in our fully continuous, Gaussian mixture model from a standard discrete BP performed directly on \( x_i \).

### IV. Implementation

In this section we discuss some details of our implementation of NBP and our overall fault identification algorithm. These include several techniques that make our algorithm more efficient, and two local optimization heuristics which may improve solution quality.

#### A. Computation of messages

We apply the BP algorithm presented in (6) using the factors (7) and the self potentials defined by the Gaussian approximation (8). The variable-to-factor messages are given by the usual formula,

\[
        m^v_{si}(x_s) \propto g_s(x_s) \prod_{j \in \Gamma_s \setminus i} m^f_{js}(x_s), \tag{9}
\]

and we compute the message product more efficiently by noting that

\[
        g_s \prod_{j \in \Gamma_s \setminus i} m^f_{js} = g_s \prod_{j \in \Gamma_s} m^f_{js} / m^f_{is}, \tag{10}
\]

where \( g_s \) is a shorthand to \( g_s(x_s) \) and the same for \( m^f_{is} \). The product on the r.h.s of (10) can be computed once and reused for each outgoing message [33], [27].

The factor-to-variable messages require a more detailed analysis. It is important to note that our factors \( f_i(x) \) are functions of several variables, say \( \Gamma_i = \{x_{i_1}, \ldots, x_{i_d}\} \). A message computation, for example from factor \( s_1 \) to variable \( s_1 \), can be explicitly written as

\[
        m^f_{s_1}(x_{s_1}) \propto \int_{x_{s_2}} \cdots \int_{x_{s_d}} f_i(y_i - \sum_{j=1}^d \hat{a}_{is_j} x_{s_j}) \prod_{j=2}^d m^v_{s_j i}(x_{s_j}) dx_{\Gamma_i}, \tag{11}
\]

Although an arbitrary function on \( d \) variables would require \( O(n^d) \) computation, where \( n \) is the number of discretization bins, the messages can be computed more efficiently, because \( f_i \) is a function of a linear combination of the \( x \), [34], [17]. In essence, one uses a change of variables to separate each integrand, defining variables for the cumulative sum, \( \bar{x}_{s_j} = \bar{a}_{s_j} x_{s_j} + \bar{x}_{s_{j+1}} \), and scaled messages \( \tilde{m}_{s_{j+1}}(x) = m_{s_{j+1}}(x/\bar{a}_{s_{j+1}}) \).

We re-express (11),

\[
        m^f_{s_1}(x_{s_1}) \propto \int_{\bar{x}_{s_2}} \cdots \int_{\bar{x}_{s_d}} \tilde{m}_{s_2}(\bar{x}_{s_2} - \bar{x}_{s_3}) \int_{\bar{x}_{s_3}} \cdots \tilde{m}_{s_d}(\bar{x}_{s_d}) d\bar{\Gamma}_i, \tag{12}
\]

in which each integral (approximated by a discrete sum) requires \( O(n^2) \) computation. Note also that, computing from the right-hand side, each step takes the form \( m_{s_{j+1}}(\bar{x}_{j+1}) = \int \tilde{m}(\bar{x}_{j} - \bar{x}_{j+1}) m_{j+1}(\bar{x}_{j+1}) d\bar{x}_{j+1}, \) and can thus be thought...
Fig. 1. Graphical model for the fault identification problem. (a) Factor graph representing dependency among variables, including faults \( \{ x_i \} \) and measurements \( \{ y_i \} \). Graph edges indicate non-zero values of \( \tilde{a}_{is} \). (b) Potential functions \( \hat{g}_s \), capturing the binary and sparse nature of the \( x_s \), and \( f_i \) representing the linear measurements with Gaussian noise.

of as a convolution operator, \( m(\tilde{x}_j) \otimes m(\tilde{x}_{j+1}) \) \[35\]. For convenience, we write this as

\[
m^f_{is}(x_s) \propto f_i(\tilde{a}_{is}x_s) \otimes \prod_{t \in \Gamma_i \setminus s} m^v_{ti}(x_t/\tilde{a}_{it}). \quad (12)
\]

Moreover, since convolution can be computed by an element-wise product in the Fourier domain, the factor-to-variable messages can be evaluated even more efficiently by first rescaling each incoming message, transforming into the Fourier domain, taking a product, transforming back, and unscaling, resulting in the update rule,

\[
m^f_{is}(x_s/\tilde{a}_{is}) \propto \mathcal{F}^{-1}\left(\mathcal{F}(f_i) \prod_{t \in \Gamma_i \setminus s} \mathcal{F}(m^v_{ti}(x_t/\tilde{a}_{it}))\right), \quad (13)
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are the discrete Fourier and inverse Fourier transforms, respectively. Again, the products are computed more efficiently using all terms and dividing as in \[10\].

This highlights the basic advantage over a formulation in which the \( x_s \) are explicitly discrete-valued. Although the exact calculations are exponential over the degree of the factor \( f_i \) in both cases, the continuous-valued formulation provides us the opportunity to approximate the intermediate quantities (in our implementation, using a discretization) and separates the computation into a simple and efficiently computed form \[13\]. We note that the rescaling step is equivalent to the stretch/unstretch operations proposed in LDLC decoding \[17\]. Additionally, if the scale factors \( \tilde{a}_{it} \) are bipolar (±1), then rescaling becomes trivial (e.g., for −1, the vector is reversed), and the algorithm simplifies further.

We proceed by computing all messages from variables to factors according to \[9\], then computing all messages from factors to variables by \[12\]. The algorithm is run for a predetermined number of iterations, or until convergence is detected locally. For detecting convergence, we use the \( \ell_2 \) norm of the product of all incoming messages in this round, relative to the product of all incoming messages in previous rounds. Finally, each variable node \( x_s \) computes its belief, and we estimate its value by rounding to the closest fault value (either zero or one),

\[
x_s = \text{round}\left(\arg\max_{x_s} \left\{ g_s(x_s) \prod_{i \in \Gamma_s} m^f_{is}(x_s) \right\} \right). \quad (14)
\]

B. Discretization of the Gaussian mixture

Recall that messages are Gaussian mixtures representing the posterior and are discretized for efficiency. To store each message, we allocate a vector of a fixed length \( q \). Entries in this vector are real positive values. Typically we used values of \( q \) in the range 512 − 1024. A higher value of \( q \) makes the algorithm more accurate but slows execution time. We evaluate the messages at fixed intervals \( \Delta \) within a range of interest, identical for each variable. This range should include, for example, both 0 and 1 when the fault pattern is binary and the actual real-valued measurements; because of noise, the range of discretization is further increased to include several standard deviations of uncertainty. We used the following formula to determine the scope of the discretization,

\[
R = \max\left\{ \max_i \{|y_i|, 1\} \right\} + 3\nu,
\]

and centered each message around zero, i.e., the range \([-R, R]\). This last point is useful for computing the scaling operation on negative-valued edges, by first computing the scale, then reversing the output around zero.

C. Local optimization procedures

We outline two useful heuristics proposed by Zymin et al. \[8\] that are used for improving the quality of our solutions: variable threshold rounding and a local search procedure. The purpose of these heuristics is to obtain an integer solution out of the fractional solution obtained using our BP solver.

a) Variable threshold rounding: Let \( x^* \) denote a soft decision, a vector of the same length as \( x \) whose entries are real-valued (rather than binary); \( x^* \) may correspond, for example, to the BP belief or to the optimal point of a convex
relaxation of the original problem. Our task is to round the
decision \( x^* \) to obtain a valid Boolean fault pattern (or
hard decision). Let \( \theta \in (0, 1) \) and set
\[
\hat{x} = \lfloor (x^* - \theta) \rfloor.
\]
To create \( \hat{x} \), we simply round all entries of \( x^* \) smaller than the
threshold \( \theta \) to zero. Thus \( \theta \) is a threshold for guessing that a
fault has occurred, based on the relaxed MAP solution \( x^* \). As
\( \theta \) varies from 0 to 1, this method generates up to \( n \) different
estimates \( \hat{x} \), as each entry in \( x^* \) falls below the threshold. We
can efficiently find them all by sorting the entries of \( x^* \), and
setting the values of each \( \hat{x}_s \) to one in the order of increasing
\( x^*_s \). We evaluate the loss for each of these patterns, and take
as our best fault estimate the one that has least loss, which we
denote by \( x^+ \).

\[ b) \text{Local search:} \] Further improvement of our estimate
can sometimes be obtained by performing a local search
around \( x^+ \). We use a simple search: initializing \( \hat{x} \) to \( x^+ \), we
cycle through indices \( s = 1, \ldots, n \), where at step \( s \) we replace
\( \hat{x}_s \) with \( 1 - \hat{x}_s \). If this leads to a reduction in the loss function,
then we accept the change and continue. Otherwise, we restore
\( \hat{x}_s \) to its original value and move on to the next index. We
continue in this manner until we have rejected changes in all
entries of \( \hat{x} \). At the end of this search, \( \hat{x} \) is at least 1-OPT,
which means that no change in any one entry will improve
the loss function. Numerical experiments show that this local
optimization method often has no effect, which means that \( x^+ \)
is often 1-OPT. In some cases, however, it can lead to modest
reduction of loss compared to \( x^+ \).

V. NUMERICAL EXAMPLES

A. Algorithms for comparison

We have implemented our NBP solver using Matlab; our
implementation is available online [20]. Table I lists the
different algorithms we evaluated. We compared NBP to
several groups of competing state-of-art algorithms. First, we
considered the interior point method (IP) for solving the fault
identification problem [8]. Second, we evaluated two other
variants of NBP: (i) CSBP [16]; and (ii) the low density lattice
decoder (LDLC) [17]. Third, we ran several non-Bayesian CS
algorithms: (i) CoSaMP [38]; (ii) GPSR [39]; and (iii) iterative
hard thresholding (HardIO) [40]. Fourth, we implemented a
semidefinite programming relaxation [36], [37]. Finally, for a
MAP problem over binary variables, it is natural to employ the
discrete BP max-product algorithm defined directly on the
model [3], and so we implemented this algorithm as well. In
practice, it performed significantly worse than the other
algorithms.

A technical subtlety that arises when handling the various
algorithms is that they use one of two equivalent formulations
of the problem: either a Boolean or bipolar representation.
Table II outlines the two models and the transformation needed
to shift between them; we use the notation 1 for the all-ones
vector of appropriate size. Without loss of generality we use
the binary form when describing the algorithms.

Within the BP group, max-product BP fails to exploit the
sparsity of the measurement matrix, and CSBP assumes a
sparse not necessarily binary signal. Our NBP uses the same
framework as CSBP, but with the correct fault prior. As we
show shortly, this results in a significant improvement in
identifying the correct fault pattern. LDLC does assume a
binary prior but assigns faults and non-faults equal probability,
which degrades performance. Within the linear programming
group, linear programming and semidefinite programming re-
lex the binary fault prior into the continuous domain, returning
fractional results, and sparsity of the fault pattern is not
assumed or exploited. Within the CS group, all algorithms
exploit the sparse nature of the signal, but not the sparse
measurement matrix or the binary nature of the signal.

B. Experimental settings

We consider an example with \( m = 100 \) sensors, \( n = 200 \)
possible faults, and linear measurements. The matrix A is
taken to be sparse with a fixed percentage of non-zero entries
\( q \) where the non-zero entries of A are drawn from a single
distribution which is either (i) Bernoulli, where elements of A

| Group | Algorithm | Abbreviation | Prior on x |
|-------|-----------|--------------|------------|
| BP    | NBP Solver (current work) | NBP | binary and sparse |
|       | Max-product BP (current work) | Max-prod | binary and sparse |
|       | Compressed sensing Belief Propagation [16] | CSBP | sparse |
|       | Low density lattice decoder [17] | LDLC | binary |
| LP    | Interior point (Newton method) [18] | IP | \( x \in [0, 1] \) |
|       | Semidefinite programming [35], [37] | SDP | \( x \in [0, 1] \) |
| CS    | Iterative signal recovery [38] | CoSaMP | sparse |
|       | Gradient Projection for Sparse Reconstruction [39] | GPSR | sparse |
|       | Iterative hard thresholding [40] | hardIO | sparse |
| Other | All zero hypothesis | Null | \( x \) is constant |

TABLE I
ALGORITHMS USED FOR COMPARISON, GROUPED INTO GENERAL CATEGORIES: BELIEF PROPAGATION METHODS (BP), LINEAR PROGRAMMING (LP), AND COMPRESSED SENSING (CS).

| Bipolar | Binary | Transformation |
|---------|--------|---------------|
| \( x \in \{-1, 1\}^n \) | \( y \in \{0, 1\}^n \) | \( \tilde{x} = \frac{x+1}{2} \) |
| \( y = Ax + v \) | \( v = (2A)^T + v \) | \( \tilde{y} = y + A1 \) |
| \( \min \|Ax - y\| \) | \( \min \|2A\| - \|\tilde{y}\| \) | \( s.t. x \in \{-1, 1\}^n \) |
| \( s.t. x \in \{-1, 1\}^n \) | \( s.t. \tilde{x} \in \{0, 1\}^n \) | |

TABLE II
TRANSFORMATION BETWEEN BIPOLAR AND BINARY REPRESENTATIONS
are chosen randomly and independently to be either $-1$ or $1$; and (ii) Gaussian, where non-zero elements of $A$ are chosen from a Gaussian distribution with zero mean and unit norm. The problem is under-determined in the sense that we have fewer measurements than potential faults, a similar setting to CS. These two matrix types correspond to models that commonly arise in practice. Bipolar matrices often occur, for example, in fault identification of linear analog circuits [41],[42], while Gaussian matrices were used in [8] and also appear in the CS literature [43]. In both models, we fix the measurement noise standard deviation to $\sigma = 1$. Where not otherwise stated, the fault probability is $p = 0.05$ (giving 10 faults on average), and the matrix sparsity is $q = 0.05$. Finally, we define a successful reconstruction as a run of an algorithm that resulted in a solution with likelihood greater than or equal to the true solution.

C. Discussion of results

We show two sets of results, varying the sparsity of the fault signatures (the matrix $A$) in Fig. 2 and varying the sparsity of the fault patterns (rarity of faults) in Fig. 3. In both cases, we compare all 10 algorithms listed in Table I under two conditions: with local optimization heuristics and without. At each sparsity level, the plots show average performance over 500 experiments.

We first note the effect of the local optimization heuristics on the quality of the solutions. In both figures, the optimization heuristics have a significant positive effect, and this effect is particularly powerful when the probability of a fault is low. For example, Fig. 2(a)–(b) corresponds to fault probability $0.05$ across a range of fault signature sparsities, and for many techniques the local optimization dramatically improves performance (note the change in scale between the two plots). Even the null hypothesis (which simply guesses no faults) attains 96% success in identifying the correct faults with local optimization, since most runs are well explained by only a few faults. As the fault patterns become denser, local optimization becomes less powerful; see, for example, the performance of the null hypothesis in Fig. 3(a), which decreases rapidly as faults become more common.

Since most algorithms did very well on such sparse fault
patterns with local optimization, we focus for the moment on the algorithms’ performance without local heuristics. In Fig. 2(b), we can distinguish between 3 general groups of algorithms. First, the LDLC algorithm has a relatively low success rate, because it uses a uniform prior distribution of faults vs. non-faults. Second, the non-Bayesian CS algorithms (CoSaMP, GPSR and hardIO) do not account for the Boolean nature of \( x \) and thus have inferior performance. Third, NBP, IP, and SDP are all targeted at our fault identification problem, and designed to perform well for problems that have both sparse and Boolean-valued fault patterns. CSBP also performed similarly to this third group. In all cases, NBP performs as well or better than the other tested methods.

We next evaluate the effect of fault pattern sparsity on solution quality. Fig. 3 shows the performance of each algorithm, this time when the sparsity level of \( A \) is fixed to be 0.05, and the fault prior varies between 3\% and 15\%. When the fault prior increases, the problem becomes harder, because there is an increasing number of \textit{a priori} likely fault combinations. For example, with a prior of \( 1/n \) we have on average only one fault, with \( n \) possible locations; when the prior is \( 2/n \) there are on average two faults and \( n(n-1)/2 \) possible locations, and so on. Increasing the prior fault probability shows a clear separation between the performance of the different algorithms, in which NBP outperforms IP in all cases (both with and without local optimization).

Finally, we illustrate the convergence of NBP, CSBP, LDLC, and IP in Fig. 4 where \( m = 10 \), \( n = 15 \), and two faults occurred. Each axis indicates the belief or soft decision about one of the two (correct) faults. Note that all algorithms converge to solutions greater than 0.5, and will thus round to the correct solution in post-processing. The NBP algorithm converges in two iterations to the correct solution while IP requires more iterations for converging to an approximate solution, although we should be cautious in giving this conclusion too much weight, since the computational cost per iteration of the NBP algorithm is higher. NBP converges accurately, because the prior distribution encourages it to converge to zero or one. CSBP converges to some positive but non-integral value, while LDLC also encourages binary values and is therefore the closest after NBP. Indeed, we have noticed in our experiments that NBP often converges faster than other algorithms. When many variables are involved, slow convergence or convergence to non-integral values may indicate that incorrect values are used to estimate other variables, making the overall process less accurate than when information on the binary nature is included (as in NBP).

Overall, our experiments show that NBP has better performance for fault identification than other tested algorithms in a variety of scenarios, including different sparsity levels of the matrix \( A \), different fault priors, and irrespective of the usage of local optimization procedures. As complementary material to this paper, we provide the Matlab source code of our experiments online [20].

VI. INFORMATION THEORETIC CHARACTERIZATION

Why does NBP perform so well relative to state-of-the-art algorithms? A partial answer can be obtained by considering recent information theoretic results in the related domains of multi-user detection [44], [45] and CS [46], [47]. Consider the signal and measurement models as before, where \( y = Ax + v \), \( x \) is i.i.d Bernoulli with \( \Pr(x_i = 1) = p \), and \( v \) is i.i.d. zero mean unit norm Gaussian. Suppose also that the measurement matrix \( A \) is assumed to be i.i.d. where the probability of non-zero entries is \( q \), and that non-zero matrix entries follow some unit norm distribution. We note in passing that the non-zero probability \( q \) must scale to zero as \( m \) and \( n \) grow; see details in Guo and Wang [45].

Following the conventions of Guo et al. [46], [45], [48], the signal to noise ratio (SNR) \( \gamma \) can be computed as,

\[
\gamma = \frac{m}{q}.
\]

We also define a distortion metric \( D \) that evaluates the approximation error between \( x \) and \( \hat{x} \) by averaging over per-element distortions,

\[
D(x, \hat{x}) = \frac{1}{n} \sum_{s=1}^{n} d(x_s, \hat{x}_s),
\]

where \( d(\cdot, \cdot) \) is a distortion metric such as square error or Hamming distortion. For this problem where a sparse measurement matrix \( A \) is used, Guo and Wang [45] provided the fundamental information theoretical characterization of the optimal performance that can be achieved, namely the minimal \( D \) that can be achieved as \( m \) and \( n \) scale to infinity with fixed aspect ratio \( \delta \), i.e., \( \lim_{m \to \infty} \frac{m}{n} = \delta \).

There are several noteworthy aspects in the analysis by Guo and Wang. First, Theorem 1 [45] proves that the problem behaves as if each individual input element \( x_s \) were estimated individually from a corrupted version \( z_s \), with \( z_s = x_s + w_s \) where \( w_s \) is Gaussian noise. That is, the vector estimation problem is related to an estimation problem defined over scalar
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