Gauge dependence in the nonlinearily realized massive $SU(2)$ gauge theory

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Abstract

The implementation of the ’t Hooft $\alpha$-gauge in the symmetrically subtracted massive gauge theory based on the nonlinearily realized $SU(2)$ gauge group is discussed. The gauge independence of the self-mass of the gauge bosons is proven by cohomological techniques.

1 Introduction

A consistent subtraction scheme for massive non-Abelian gauge theories based on a nonlinearily realized gauge group has been recently proposed in [5]. The symmetric subtraction algorithm was already successfully applied to the four-dimensional nonlinear sigma model in the flat connection formalism in [4]–[7].

The Feynman rules of the nonlinearily realized massive gauge theory entail that already at one loop level there is an infinite number of divergent amplitudes involving the pseudo-Goldstone fields $\phi_a$ [5]. The latter amplitudes are uniquely fixed by implementing a defining local functional equation [5, 7] which encodes the invariance of the path-integral Haar measure under local $SU(2)_L$ transformations

$$\Omega' = U_L \Omega, \quad A'_\mu = U_L A_\mu U_L^\dagger + iU_L \partial_\mu U_L^\dagger$$

$\Omega$ is the element of the nonlinearily represented $SU(2)_L$ gauge group

$$\Omega = \frac{1}{v_D}(\phi_0 + i\phi_a \tau_a), \quad \phi_0 = \sqrt{v_D^2 - \phi_a^2}$$

$v_D$ is a D-dimensional mass scale and $\tau_a$ are the Pauli matrices. $A_\mu = A_{a\mu} \tau_a^\mu$ is the $SU(2)_L$ gauge connection. It is also convenient to introduce the $SU(2)_L$ flat connection

$$F_\mu = F_{a\mu} \tau_a^\mu = i\Omega \partial_\mu \Omega^\dagger$$

with the following $SU(2)_L$ transformation induced by the transformation of $\Omega$

$$F'_\mu = U_L F_\mu U_L^\dagger + iU_L \partial_\mu U_L^\dagger$$

The amplitudes not involving the pseudo-Goldstone fields are named ancestor amplitudes since they are at the top of the hierarchy induced by the local functional equation. At every loop order there is only a finite number of divergent ancestor amplitudes (weak power-counting theorem [5, 10]). The requirement of physical unitarity is satisfied since the Slavnov-Taylor (ST) identity holds [5, 9]. The ghost equation and the Landau gauge equation are also preserved by the

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2 the subscript $L$ stands for the left action on the group element.
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These symmetries, supplemented by global $SU(2)_R$ invariance and the weak power-counting, uniquely fix the tree-level vertex functional of the nonlinearly realized theory [5].

In [5] the Landau gauge was used for the sake of simplicity and conciseness. The aim of this note is to implement the ’t Hooft $\alpha$-gauge in a way compatible with all the symmetries required for the definition of the model (local functional equation, ST identity, ghost equation, $B$-equation for a general $\alpha$-gauge [15]) and the weak power-counting.

The validity of the local functional equation requires that the gauge-fixing functional transforms in the adjoint representation of $SU(2)_L$. In the Landau gauge this was achieved by introducing an external vector source $V_{a\mu}$ and by making use of the gauge-fixing functional

$$ \int d^Dx \, B_a(D^\mu[V](A - V)_\mu)_a $$

where $(D_\mu[V])_{ac} = \partial_\mu \delta_{ac} + \epsilon_{abc} V_{b\mu}$ is the covariant derivative w.r.t. the vector source $V_{a\mu}$. $B_a$ is the Nakanishi-Lautrup field [15]. It transforms in the adjoint representation of $SU(2)_L$. The local functional equation is preserved by the gauge-fixing (1.5).

It should be stressed that the local functional equation associated with the $SU(2)_L$ local invariance is not the standard background Ward identity [1]-[8]. The essential difference is that the local functional equation is bilinear in the vertex functional $\Gamma$, due to the presence of the nonlinear constraint $\phi_0$ in Eq (1.2), which needs to be coupled to the scalar source $K_0$ in the tree-level vertex functional.

The ’t Hooft $\alpha$-gauge is defined by the condition of the cancellation (once the Nakanishi-Lautrup field is eliminated via its equation of motion) of the mixed $A_\mu \phi$ terms arising in the nonlinear theory from the mass invariant

$$ \int d^Dx \, \frac{M^2}{2} (A_{a\mu} - F_{a\mu})^2 $$

The ’t Hooft ($\phi$-dependent) gauge-fixing functional must transform in the adjoint representation of $SU(2)_L$ in order to preserve the local functional equation. For that purpose one needs to introduce an auxiliary matrix $\hat{\Omega}$

$$ \hat{\Omega} = \frac{1}{v_D} (\hat{\phi}_0 + i \hat{\phi}_a \tau_a) $$

with the same $SU(2)_L$ transformation as $\Omega$:

$$ \hat{\Omega}' = U_L \hat{\Omega} $$

The combination

$$ \mathcal{F} = \mathcal{F}_a \frac{\tau_a}{2}, \quad \mathcal{F}_a = D^\mu[V](A - V)_{a\mu} + M^2 \frac{2}{2\alpha} Tr[i \hat{\Omega}^\dagger \tau_a \Omega + h.c] $$

has the correct transformation properties. Therefore one can consider the following gauge-fixing functional

$$ \int d^Dx \left[ - \frac{1}{2\alpha} B_a^2 + B_a \mathcal{F}_a \right] $$

where $\alpha$ is the gauge parameter. With the choice in Eq (1.10) the local functional equation is respected and by integrating the Nakanishi-Lautrup field $B_a$ the mixed $A_\mu \phi$-terms arising from Eq (1.6) are canceled. The propagators obtained by using the gauge-fixing functional in Eq (1.10) have a UV behaviour compatible with the weak power-counting.

It is important to realize that $\hat{\Omega}$ is not an element of $SU(2)$. The reason is that the amplitudes involving $\hat{\phi}_0$ and $\hat{\phi}_a$ must be ancestors. Already at one loop level one cannot have a finite number
of divergent amplitudes involving \( \hat{\phi}_a \) if \( \hat{\phi}_0 \) is given by the \( SU(2) \) constraint \( \hat{\phi}_0^2 + \hat{\phi}_a^2 = v_D^2 \). One must split in a linear way the constant component of \( \hat{\phi}_0 \) by setting \( \hat{\phi}_0 \equiv v_D + \hat{\sigma} \). Since \( \hat{\sigma} \) and \( \hat{\phi}_a \) are independent, by inspecting the Feynman rules one can then check that the weak power-counting is preserved. Moreover, since \( \hat{\phi}_0, \hat{\phi}_a \) are independent variables, the BRST transformation can be extended to these sources by pairing them to external source ghosts \( \theta_0, \theta_a \) as follows

\[
\tag{1.11}
\]

Then \((\hat{\phi}_0, \theta_0), (\hat{\phi}_a, \theta_a)\) form BRST doublets \([15, 16]\) and therefore they do not contribute to the cohomology \( H(s) \) of the BRST differential \( s \). Hence they are not physical, as expected. The same technique can be used to prove that the vector source \( V_{a\mu} \) does not modify the physical observables too \([5]\).

The ghost-antighost part of the tree-level vertex functional is generated as usual by

\[
\tag{1.12}
\]

We define as usual the connected generating functional \( W \) by the Legendre transformation of \( \Gamma \) w.r.t. the quantized fields (collectively denoted by \( \Phi \))

\[
\tag{2.3}
W = \Gamma + \int d^Dx \ K \ \Phi
\]

where \( K \) is a short-hand notation for the sources of the quantized fields. Eq (2.2) yields \( (K(\varphi) \) stands for the source coupled to the field \( \varphi)\)

\[
\tag{2.4}
\]

By differentiating Eq (2.4) w.r.t. \( \zeta \) and a set of sources \( \beta_1, \ldots, \beta_n \) coupled to physical BRST-invariant local operators \( O_1, \ldots, O_n \) one finds by going on-shell (all external sources set to zero)

\[
\tag{2.5}
\frac{\partial}{\partial \alpha} W_{\beta_1 \ldots \beta_n} \bigg|_{\text{on-shell}} = 0
\]

i.e. the physical Green function \( W_{\beta_1 \ldots \beta_n} \) is on-shell gauge-independent.
The Nielsen identities [13, 12] can also be obtained from the extended ST identity (2.2). We discuss here in detail the Nielsen identity for the two point 1-PI function of the gauge bosons. By differentiating Eq (2.2) w.r.t. $A_{a_1 \mu_1}$, $A_{a_2 \mu_2}$ and $\zeta$ and by setting all the fields and external sources to zero one gets

$$
\Gamma_{\zeta A_{a_1 \mu_1}} A_{a_{1}}^{\mu_1} A_{a_{2}}^{\mu_2} + \Gamma_{\zeta A_{a_2 \mu_2}} A_{a_{1}}^{\mu_1} A_{a_{2}}^{\mu_2} + \partial_\alpha \Gamma_{A_{a_1 \mu_1} A_{a_2 \mu_2}} = 0 \quad (2.6)
$$

We decompose $\Gamma_{AA}$ and $\Gamma_{\zeta A}$ into their transverse and longitudinal components as follows

$$
\Gamma_{A_{a_1 \mu_1} A_{a_2 \mu_2}} = \delta_{ab} \left( \Sigma_{AA}^{T}(p^2) T_{\mu \nu} + \Sigma_{AA}^{L}(p^2) L_{\mu \nu} \right)
$$

$$
\Gamma_{\zeta A_{a_1 \mu_1} A_{a_2 \mu_2}} = \delta_{ab} \left( \Sigma_{\zeta A}^{T}(p^2) T_{\mu \nu} + \Sigma_{\zeta A}^{L}(p^2) L_{\mu \nu} \right)
$$

$$
T_{\mu \nu} = g_{\mu \nu} - \frac{p_\mu p_\nu}{p^2}, \quad L_{\mu \nu} = \frac{p_\mu p_\nu}{p^2} \quad (2.7)
$$

By taking the transverse part of Eq (2.6) one finds (notice that the terms proportional to $\Gamma_{A_\mu \phi}$ only contribute to the longitudinal part and thus drop out)

$$
\partial_\alpha \Sigma_{AA}^{T}(p^2) = -2\Sigma_{\zeta A}^{T}(p^2) \Sigma_{AA}^{T}(p^2) \quad (2.8)
$$

The self-mass $M^2$ is defined as the zero of $\Sigma_{AA}(p^2)$:

$$
\Sigma_{AA}^{T}(M^2) = 0 \quad (2.9)
$$

if no tadpoles are present as in the case under consideration. In passing it is worth noticing that the presence of tadpoles requires that the self-mass is defined as the pole of the transverse part of the connected two-point function (as it happens in the linearly realized theory).

By Eq (2.8) one finds

$$
\partial_\alpha \Sigma_{AA}^{T}(M^2) = 0 \quad (2.10)
$$

Moreover invertibility of $\frac{\partial \Sigma_{AA}^{T}}{\partial p^2}$ is guaranteed in the loop expansion since

$$
\frac{\partial \Sigma_{AA}^{T}}{\partial p^2} = -1 + O(\hbar)
$$

The above equation together with Eq (2.10) implies

$$
\frac{\partial M^2}{\partial \alpha} = 0 \quad (2.11)
$$

i.e. the self-mass is gauge-independent. This behaviour is a typical property of the nonlinear theory. In the linear case the zero of the two-point 1-PI function is in general gauge-dependent. Gauge independence can only be recovered by taking into account the Higgs tadpole contributions. For a detailed comparison of the two-point 1-PI function in the linear and the nonlinear case see [6].

3 Conclusions

The formulation of the nonlinearly realized $SU(2)$ massive gauge theory in the `t Hooft gauge has been achieved in a way consistent with all the symmetries of the model and the weak power-counting. This requires the introduction of auxiliary external sources $\hat{\sigma}, \hat{\phi}$. We have shown that this procedure does not alter the physical content of the model. Gauge independence of physical observables has been established by using cohomological methods. The self-mass, which can be computed in the nonlinearly realized theory as the zero of the transverse part of the 1-PI two-point function, has been proven to be gauge-independent.
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