LOG-PLURIGENERA IN STABLE FAMILIES

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ABSTRACT. We study the flatness of log-pluricanonical sheaves on stable families of varieties.

A key insight of [KSB88] is that, in dimension at least 2, the correct objects of moduli theory are flat morphisms $f : X \to S$ whose fibers are varieties with log canonical singularities and such that $mK_{X/S}$ is Cartier for some $m = m(X, S) > 0$. The latter assumption is not always easy to understand, but, if $S$ is reduced then it is equivalent to the condition

- $\omega_{X/S}^{[m]}$ is flat over $S$ and commutes with base change for every $m \in \mathbb{Z}$.

In studying the moduli of pairs, the right concept is less clear. One should consider morphisms $f : (X, \Delta) \to S$ such that $f : X \to S$ is flat, all fibers $(X_s, \Delta_s)$ are semi-log-canonical and $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier. Such morphisms are called locally stable; see [Kol13a] for a survey and [Kol17] for a detailed treatment.

However, these conditions are not sufficient; there are problems especially over non-reduced bases [AK16] and in positive characteristic [HK10, 14.7]. One difficulty is that the sheaf variant of the Cartier assumption is not well understood.

The best log-analog of $\omega_{X/S}^{[m]}$ is the twisted version $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$. The main theorem of this note shows that these sheaves also behave well, provided the coefficients of $\Delta$ are not too small.

For a divisor $\Delta = \sum_{i \in I} a_i D_i$, where the $D_i$ are distinct prime divisors, we write $\text{coeff } \Delta := \{ a_i : i \in I \}$. In the semi-log-canonical cases $\text{coeff } \Delta \subset [0, 1]$. We set $|\Delta| := \sum_{i \in I} |a_i| D_i$, $\lfloor \Delta \rfloor := \sum_{i \in I} \lfloor a_i \rfloor D_i$, and $\{\Delta\} := \sum_{i \in I} \{a_i\} D_i = \Delta - |\Delta|$.

**Theorem 1.** Let $S$ be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a locally stable morphism with normal generic fibers. Assume that $\text{coeff } \Delta \subset [\frac{1}{2}, 1]$. Then, for every $m \in \mathbb{Z}$, the sheaves

$$\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$$

are flat over $S$ and commute with base change (cf. Definition 14).

**Warning 1.** If some of the coefficients equal $\frac{1}{2}$, we have to be careful about what we mean by $\lfloor m\Delta \rfloor$ and commuting with base change; see Paragraph 12 for a general discussion and Paragraph 13 for the coefficient $\frac{1}{2}$ case.

The assumption on having normal generic fibers is probably superfluous, see Question 8. Aside from this, the theorem seems quite sharp. In (39.6) we give examples such that $\text{coeff } \Delta$ is arbitrarily close to $\frac{1}{2}$ yet $\omega_{X/S}^{[2]}(\lfloor 2\Delta \rfloor)$ does not commute with base change. Also, for $m \neq r$ the sheaves $\omega_{X/S}^{[m]}(\lfloor r\Delta \rfloor)$ usually do not have similar properties, see (39.5). In example (39.4) the pluricanonical sheaf $\omega_{X/S}^{[m]}$ commutes with base change only for $m = 0$ and $m = 1$. 

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In Theorem 1 we allow $\Delta$ to be an $\mathbb{R}$-divisor. In this case $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ need not be locally free for any $m \neq 0$; see (39.8).

In Section 1 we state some corollaries and variants while in Section 2 we recall some relevant facts of the moduli theory of pairs. In Section 3 we study small modifications of lc pairs in general. These show us how to transform divisors like $mK_X + \lfloor m\Delta \rfloor$ into a $\mathbb{Q}$-Cartier divisor in an economic way. Usually this can be done in several ways and we pin down some especially good choices. Most of these results do not hold for slc pairs by [Kol13b, 1.40]; see also Example 22.

In Section 4 we prove Proposition 28 which is a generalization of Proposition 5 and then in Section 5 we establish Corollary 3. We end by a collection of relevant examples in Section 6.

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1. Consequences and variants

The first corollary of Theorem 1 is the deformation invariance of the Hilbert function for certain locally stable morphisms. (Note that Warning 1.1 also applies to the results in this section; see Paragraph 13 if some of the coefficients of the boundary divisor equal $\frac{1}{2}$.)

**Corollary 2.** Let $S$ be a connected scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a proper, locally stable morphism with normal generic fibers such that $\text{coeff } \Delta \subset \left[\frac{1}{2}, 1\right]$. Then the Hilbert function of the fibers

$$\chi(X_s, \omega_{X_s}^{[m]}(\lfloor m\Delta_s \rfloor))$$

is independent of $s \in S$. □

In general the Hilbert function considered above is not a polynomial in $m$, but, if $r(K_{X/S} + \Delta)$ is Cartier, then it can be written as a polynomial whose coefficients are periodic functions in $m$ with period $r$. If we assume in addition that $K_{X/S} + \Delta$ is $f$-ample then, by Serre vanishing,

$$\chi(X_s, \omega_{X_s}^{[m]}(\lfloor m\Delta_s \rfloor)) = H^0(X_s, \omega_{X_s}^{[m]}(\lfloor m\Delta_s \rfloor))$$

for $m \gg 1$,

but it is not clear for which values of $m$ does this hold. However we get the optimal deformation invariance of plurigenera if we restrict the coefficients further.

**Corollary 3.** Let $S$ be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \to S$ a stable morphism with normal generic fibers such that $\text{coeff } \Delta \subset \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\right\}$. Then, for every $m \geq 2$,

1. $R^if_*\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor) = 0$ for $i > 0$ and

2. $f_*\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ is locally free and commutes with base change.

Examples 40–43 show that (3.1) can fail for any other coefficient. It is, however, possible that if we fix the relative dimension and coeff $\Delta$ is close enough to 1 then (3.1) holds. It would be good to know whether (3.2) holds more generally.

Rather general arguments involving hulls and husks reduce the proof of Theorem 1 to the cases when $S$ is smooth of dimension 1; see the proof of Theorem 6 and Proposition 16. In these cases $(X, \Delta)$ itself is an slc pair, and then Theorem 1 can be reformulated as a version of Serre’s $S_3$ property.
**Definition 4.** Recall that a sheaf $F$ on a scheme $X$ is called $S_m$ if
\[
\text{depth}_x F \geq \min\{m, \dim_x F\} \quad \forall \ x \in X.
\]
As a slight variant, we say that $F$ is $S_m$ along a subscheme $Z \subset X$ if depth$_x F \geq \min\{m, \dim_x F\}$ holds for every $x \in Z$.

We are mostly interested in the $S_3$ condition. Note that if $H \subset X$ is a Cartier divisor whose defining equation is not a zero-divisor on $F$ then $F$ is $S_3$ along $H$ iff $F|_H$ is $S_2$.

**Proposition 5.** Let $(X, H + \Delta)$ be an lc pair over a field of characteristic $0$ where $H$ is Cartier and $\text{coeff} \Delta \subset [\frac{1}{2}, 1]$. Then $\omega_X^{[m]}([m\Delta] - D)$ is $S_3$ along $H$ for every $m \in \mathbb{Z}$ and every divisor $D \subset [\Delta]$.

If, by happenstance, $mK_X + [m\Delta]$ is $\mathbb{Q}$-Cartier, then this follows from [Ale08, Kol11]; see also [Kol13b, 7.20] and [Fuj17, Sec.7.1]. Thus our main focus is on the cases when $mK_X + [m\Delta]$ is not $\mathbb{Q}$-Cartier. Having an extra divisor $D$ in Proposition 5 allows us to prove a stronger version of Theorem 1.

**Theorem 6.** Let $S$ be a reduced scheme over a field of characteristic $0$ and $f : (X, \Delta) \to S$ a locally stable morphism with normal generic fibers. Assume that $\text{coeff} \Delta \subset [\frac{1}{2}, 1]$ and let $D \subset [\Delta]$. Then, for every $m \in \mathbb{Z}$, the sheaves
\[
\omega_{X/S}^{[m]}([m\Delta] - D)
\]
are flat over $S$ and commute with base change.

Proof of Proposition 5 $\Rightarrow$ Theorem 6 $\Rightarrow$ Theorem 1. Setting $D = 0$ shows that Theorem 6 $\Rightarrow$ Theorem 1. In order to prove the first implication, we use the theory of hulls and husks; see [Ale08] or [Kol17, Chap.9].

Fix $m \in \mathbb{Z}$. As we note in [Kol17, 4.26], there is a closed subset $Z \subset X$ such that $K_{X/S}$ and $[m\Delta] - D$ are Cartier on $X \setminus Z$ and $Z \cap X_s$ has codimension $\geq 2$ in $X_s$ for every $s \in S$.

We aim to apply Proposition 16 to $U := X \setminus Z$ with injection $j : U \hookrightarrow X$ and $F := \omega_{U/S}^{[m]}([m\Delta|_U] - D|_U)$.

Lemma 15 says that Proposition 5 is equivalent to the assumption (16.2). Thus we get that (16.1) also holds, and the latter is just a reformulation of the claim of Theorem 6. \qed

There are several results [Ale08, Kol11] that guarantee that certain divisorial sheaves are Cohen–Macaulay (abbreviated as CM) or at least $S_3$. We recall these in Theorem 26; see also [Kol13b, 7.20] and [Fuj17, Sec.7.1] for detailed treatments. The following variant of Proposition 5 is closely related to them.

**Proposition 7.** Let $(X, H + \Delta)$ be an lc pair over a field of characteristic $0$ where $H$ is $\mathbb{Q}$-Cartier and $\text{coeff} \Delta \subset (\frac{1}{2}, 1]$. Let $B$ be a Weil $\mathbb{Z}$-divisor and $\Delta_3$ an effective $\mathbb{R}$-divisor such that $B \sim_{\mathbb{R}} -\Delta_3$, $\Delta_3 \leq [\Delta]$ and $[\Delta_3] \leq [\Delta]$.

Then $\mathcal{O}_X(B)$ is $S_3$ along $H$.

A stronger version, allowing some coefficients to be $\frac{1}{2}$, is proved in [Fuj17].

The main open question is the following.

**Question 8.** What happens for non-normal slc pairs?

Note that we derive Propositions 5 and 7—and hence also Theorems 1 and 6—using Proposition 19 which asserts that certain small modifications of $X$ exist.
The analogous small modifications need not exist for slc pairs; see [Kol13b, 1.40] or Example 22. However, Proposition 7 and Theorem 1 frequently hold even when Proposition 19 fails. Some non-normal cases are treated in [Kol18a].

The divisor $H$ does not play any role in the conclusion of Proposition 5, but it restricts the possible choices of $(X, \Delta)$. Example 38 shows that $H$ is necessary, but I have no counter example to the following variant.

**Question 9.** Let $(X, \Delta)$ be an slc pair over a field of characteristic 0 where $\text{coeff} \, \Delta \subset [\frac{2}{3}, 1]$. Let $x \in X$ be a point of codimension $\geq 3$ that is not an lc center of $(X, \Delta)$. Is it true that $\text{depth}_x \omega^m_X((m\Delta)) \geq 3$?

While the Serre dual of a CM sheaf is CM (cf. [KM98, 5.70]), the Serre dual of an $S_3$ sheaf need not be $S_3$; see [Pat13, 1.6]. Thus it is not clear whether the dual versions of our results also hold. The Serre dual of $\omega^m_X(\lfloor m\Delta \rfloor - D)$ is $\omega^{1-m}_X(-\lfloor m\Delta \rfloor + D)$. Changing $m$ to $-m$ we get the following.

**Question 10.** Using the notation and assumptions of Proposition 5, is the sheaf $\omega^{m+1}_X(\lceil m\Delta \rceil + D)$ $S_3$ along $H$?

**11 (Method of proof).** The idea is similar to the ones used in [Kol11, Kol14]. Let $g : Y \to X$ be a proper morphism of normal varieties, $F$ a coherent sheaf on $X$, $H \subset X$ a Cartier divisor and $H_Y := g^*H$. Assuming that $F$ is $S_m$ along $H_Y$, we would like to understand when $g_*F$ is $S_m$ along $H$. If (the local equation of) $H_Y$ is not a zero divisor on $F$ then the sequence

$$0 \to F(-H_Y) \to F \to F|_{H_Y} \to 0 \quad (11.1)$$

is exact. By push-forward we get the exact sequence

$$0 \to g_*F(-H) \to g_*F \to g_*(F|_{H_Y}) \to R^1g_*F(-H_Y) \cong \mathcal{O}_X(-H) \otimes R^1g_*F \quad (11.2)$$

Thus we see that $g_*F$ is $S_m$ along $H$ if

1. $R^1g_*F = 0$
2. $g_*(F|_{H_Y})$ is $S_{m-1}$ along $H$.

In many cases, for instance if $g$ is an isomorphism outside $H_Y$, these conditions are also necessary.

Our main interest is in the cases when $F$ is a divisorial sheaf. Using a Kodaira-type vanishing theorem, (3.a) needs some positivity condition on $F$. By contrast, we see in Lemma 29 that (3.b) needs some negativity condition on $F$.

In general one can not satisfy both of these restrictions, but choosing $Y$ carefully and varying the boundary divisor gives some wiggle room.

### 2. Comments on the moduli of pairs

The general theory of stable and locally stable maps is treated in [Kol17], see also [Kol13a]. Here we discuss one special aspect of it, the definition of the divisorial part of the fiber. This explains why the condition $\text{coeff} \, \Delta \subset [\frac{1}{2}, 1]$ is necessary and we also need later some of its easy but potentially confusing properties.
12 (Restriction and rounding down). Let \( f : (X, \Delta) \to S \) be a locally stable morphism. Here we consider the following problem.

**Question 12.1.** Given a point \( s \in S \), how can we compare the divisor-fibers \([m\Delta_s]\) and \([m\Delta]_s\)?

More generally, let \( \Theta \) be an \( R \)-divisor such that \( \text{Supp} \Theta \subset \text{Supp} \Delta \). What is the relationship between \([\Theta]_s\) and \([\Theta]_s\)?

The definition of local stability in [Kol17, Chap.3] is set up so that \( \Delta_s \) is a line with 2 points with weight \( 1 \). The solution is that we write the special fiber as \( \Delta_0 := 1/2(x = 0) + 1/2(x + u = 0) \).

**Claim 12.3.** Assume also that \( \text{Supp} \Theta \subset \text{Supp}(\Delta^{>1/2}) \). Then

(a) \( \text{coeff}(\Theta|_H) \subset \text{coeff} \Theta \) and

(b) \( |\Theta|_H = |\Theta||_H \). \( \square \)

If \( W \) is not an lc center of \((X, H + \Delta)\) then we get the stronger inequality \( \sum_i m_i a_i < 1 \), and the above conclusions also apply if we allow \( a_i = 1/2 \).

**Claim 12.4.** The conclusions (12.3.a–b) also hold if \( \text{Supp} \Theta \subset \text{Supp}(\Delta^{>1/2}) \) and none of the codimension 2 lc centers of \((X, H + \Delta)\) is contained in \( H \). \( \square \)

Next we discuss what happens if some of the coefficients equal \( 1/2 \).

13 (Problems with coefficient \( 1/2 \)). We have to be careful with bookkeeping if \( \Delta \) contains divisors with coefficient \( 1/2 \). This is best illustrated by some examples.

**13.1** Consider \( \pi : \mathbb{A}^2_{xu} \to \mathbb{A}^1_u \) with \( \Delta = 1/2(x - u = 0) + 1/2(x + u = 0) \). The generic fiber is a line with 2 points with weight \( 1/2 \), the special fiber over \((u = 0)\) is a line with 1 point with weight 1. The solution is that we remember that the boundary on the special fiber is \( \Delta_0 := 1/2(x = 0) + 1/2(x = 0) \).

**13.2** Consider \( \pi : \mathbb{A}^2_{xu} \to \mathbb{A}^1_u \) with \( \Delta = 1/2(x^2 - u = 0) \). Again the special fiber is a line with 1 point with weight 1. The solution is that we write the special fiber as \( \Delta_0 := 1/2(2(x = 0)) \).

These two examples describe what happens over a normal base scheme \( S \). In this case we write \( \Delta = \sum a_i D^i \) where the \( D^i \) are prime divisors. If \( T \to S \) is any base change then we write \( \Delta_T = \sum a_i D^i_T \). Even though the \( D^i_T \) need not be prime divisors, we set

\[
|m\Delta_T| := \sum |ma_i| D^i_T.
\]
If $a_i > \frac{1}{2}$ then $D^r_i$ has no multiple components and no irreducible components in common with any of the other $D^r_j$ by (12.3). Thus the only change is on how we count those divisors that have coefficient $\frac{1}{2}$.

Over reducible bases, other complication can arise.

(13.4) Consider $X := (uv = 0) \subset A^2_{uv}$ and $S := (uv = 0) \subset A^2_{uv}$. Set $\Delta := \frac{1}{2}(x - u = v = 0) + \frac{1}{2}(x + u = v = 0) + (x = u = 0)$. Here we can not view $|\Delta|$ as a family of divisors in any sensible way. Along the $u$-axis we should get $|\Delta| = 0$ but along the $v$-axis $|\Delta| = (x = u = 0)$. These can not be reconciled over the origin.

(13.5) The solution is to work with morphisms $f : (X, \Delta = \sum a_i D_i) \to S$ where each $D_i$ is a $\mathbb{Z}$-divisor on $X$ that is Cartier at the generic points of $X_\ast \cap \text{Supp } D_i$ for every $s \in S$. (This condition is automatic if $S$ is normal by [Kol17, 4.2].) This method is formalized by the concept of marked pairs defined in [Kol17, Sec.4.6].

Next we discuss some results that were used in the proof of Theorem 6.

**Definition 14.** Let $f : X \to S$ a morphism and $Z \subset X$ be a closed subset such that $Z \cap X_s$ has codimension $\geq 2$ in $X_s$ for every $s \in S$. Set $U := X \setminus Z$ with injection $j : U \inj X$ and let $F$ be a coherent sheaf on $U$. For every morphism $g : T \to S$ we have a base-change diagram

\[
\begin{array}{cccc}
U_T & \xrightarrow{j_T} & X_T & \xrightarrow{f_T} & T \\
g_U \downarrow & & g_X \downarrow & & \downarrow g \\
U & \xrightarrow{j} & X & \xrightarrow{f} & S
\end{array}
\]

and a natural base-change map

\[
g_X^\ast (j_T)_\ast (g_U^\ast F).
\]  

We say that $j_* F$ commutes with base change if (14.2) is an isomorphism for every $g : T \to S$.

Let $f : (X, \Delta) \to S$ be a locally stable morphism as in Theorem 6. By [Kol17, 4.26], there is a closed subset $Z \subset X$ such that $Z \cap X_s$ has codimension $\geq 2$ in $X_s$ for every $s \in S$ and both $K_{X/S}$ and $[m\Delta] - D$ are Cartier on $X \setminus Z$. In particular,

\[
\omega_{X/S}^\ast ([m\Delta] - D) = j_* (\omega_{U/S}^\ast ([m\Delta]_U - D|_U)).
\]

Thus we say that $\omega_{X/S}^\ast ([m\Delta] - D)$ commutes with base change if $j_* (\omega_{U/S}^\ast ([m\Delta]_U - D|_U))$ commutes with base change.

**Lemma 15.** (t, T) is the spectrum of a DVR, $f : X \to T$ a finite type morphism of pure relative dimension $n$ and $j : U \inj X$ as in (14). Let $F$ be a coherent sheaf on $U$ that is flat over $T$. The following are equivalent.

1. $j_* F$ commutes with arbitrary base change.
2. $j_* F$ commutes with base change to $i : \{t\} \to T$.
3. depth$_Z(j_* F) \geq 3$.

Proof. It is clear that (1) $\Rightarrow$ (2). The base-change map

\[r_t : (j_* F)|_{X_t} \to (j_t)_\ast (F|_{U_t}).\]

is an injection and an isomorphism over $U_t$. Furthermore depth$_Z(j_t)_\ast (F|_{U_t}) \geq 2$. Thus depth$_Z(j_* F) \geq 3$ iff $r_t$ is surjective.

Finally (3) $\Rightarrow$ (1) is straightforward, see [Kol17, 9.26].
Proposition 16. Let $S$ be a reduced scheme and $f : X \to S$ a finite type morphism of pure relative dimension $n$ and $j : U \hookrightarrow X$ as in (14). Let $F$ be a coherent sheaf on $U$ that is flat over $S$. The following are equivalent.

1. $j_* F$ is flat over $S$ and commutes with base change.
2. For every morphism $g : (t, T) \to S$, where $(t, T)$ is the spectrum of a DVR, $(j_T)_*(g_t^* F)$ commutes with base change to $\{t\} \hookrightarrow T$.

Proof. It is clear that (1) $\Rightarrow$ (2). To see the converse, we use the theory of hulls and husks; see [Kol08] or [Kol17, Chap.9]. In this language, our claim is equivalent to saying that $j_* F$ is its own hull; cf. [Kol17, 9.17].

If $X$ is projective over $S$, then [Kol08] shows that the hull of $j_* F$ is represented by a monomorphism $j : S^H \to S$, see also [Kol17, 9.59]. Assumption (2) and Lemma [15] imply that whenever $T$ is the spectrum of a DVR and $T \to S$ a morphism whose generic point maps to a generic point of $S$ then $T \to S$ factors through $S^H$. Thus $S^H \to S$ is an isomorphism and so $j_* F$ is its own hull; cf. [Kol17, 3.49].

A similar argument works in general. Assume to the contrary that $j_* F$ is not its own hull. Then there is a point $x \in X$ such that $j_* F$ is not its own hull at $x$. After completing $X$ at $x$ and $S$ at $f(x)$, we get $\hat{X} \to \hat{S}$ such that $j_* \hat{F}$ is not its own hull. Over a complete, local base scheme the hull functor is represented by a monomorphism $j : \hat{S}^H \to \hat{S}$ for local maps by [Kol17, 9.61]. We can now argue as before to get a contradiction. \qed

3. Small modifications of lc pairs

The local class group $\text{Cl}(x \in X)$ of a klt pair $(x \in X, \Delta)$ is finitely generated, and the vector space $\mathbb{Q} \otimes_\mathbb{Z} \text{Cl}(x \in X)$ has a finite chamber decomposition into cones such that the small modifications of $(x \in X)$ are in one-to-one correspondence with the chambers. The maximal dimensional chambers correspond to the $\mathbb{Q}$-factorial small modifications. These were first observed in [Rei83] for canonical 3-folds.

For an lc pair $(x \in X)$, the local class group $\text{Cl}(x \in X)$ need not be finitely generated. Even if it is and the analogous chamber decomposition into cones seems to exist, not every chamber corresponds to a small modification. Already cones over logCY surfaces exhibit many different patterns, as shown by Example [37].

Here we study the existence of small modifications of lc pairs. For our applications we need to consider potentially lc pairs as well.

Definition 17. Let $X$ be a normal variety and $\Delta$ an effective $\mathbb{R}$-divisor on $X$. Following [KK10] we say that $(X, \Delta)$ is potentially lc if there is an open cover $X = \bigcup_i X_i$ and effective $\mathbb{R}$-divisors $\Theta_i$ on $X_i$ such that $(X_i, \Delta|_{X_i} + \Theta_i)$ is lc for every $i$. Using that most lc surface singularities are rational, hence $\mathbb{Q}$-factorial (cf. [Kol13b, 10.4 and 10.9]) we get the following.

Claim [17]. Let $(X, \Delta)$ be potentially lc. Then there is a closed subset $Z \subset X$ of codimension $\geq 3$ such that $(X \setminus Z, \Delta|_{X \setminus Z})$ is lc. \qed

Potentially lc pairs appear quite frequently as auxiliary objects. For example, if $(Y, \Delta_Y)$ is lc and $\pi : Y \to X$ is a $(K_Y + \Delta_Y)$-negative birational contraction then $(X, \pi_* \Delta_Y)$ is potentially lc.

Canonical models of non-general type lc pairs have a natural potentially lc structure, cf. [Kaw98, Kol07].
(The definition can be naturally generalized to potentially slc pairs, but then (17.1) is not automatic. In order to get a good notion, one probably should impose (17.1) as an extra condition.)

**Definition 18.** (cf. [Kol13b, 1.32]) Let $X$ be a normal variety and $\Delta$ an effective $\mathbb{R}$-divisor on $X$. An lc modification of $(X, \Delta)$ is a proper, birational morphism $\pi : (X^c, \Delta^c + E^c) \to (X, \Delta)$ where $\Delta^c := \pi_*^{-1} \Delta$, $E^c$ is the reduced $\pi$-exceptional divisor, $(X^c, \Delta^c + E^c)$ is lc and $K_{X^c} + \Delta^c + E^c$ is $\pi$-ample.

An lc modification is unique. As for its existence, we clearly need to assume that $\text{coeff} \Delta \subset [0, 1]$. Conjecturally, this is the only necessary condition. [OX12] shows that lc modifications exist if $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Next we consider some cases when $K_X + \Delta$ is not $\mathbb{Q}$-Cartier.

**Proposition 19.** Let $(X, \Delta)$ be a potentially lc pair. Then

1. it has a projective, small, lc modification $\pi : (X^c, \Delta^c) \to (X, \Delta),$
2. $\pi$ is a local isomorphism at every lc center of $(X^c, \Delta^c)$ and
3. $\pi$ is a local isomorphism over $x \in X$ iff $K_X + \Delta$ is $\mathbb{R}$-Cartier at $x$.

Proof. Since an lc modification is unique, it is enough to construct it locally on $X$. We may thus assume that there is an effective $\mathbb{R}$-divisor $\Theta$ on $X$ such that $(X, \Delta + \Theta)$ is lc.

By [KK10, Fuj11], $(X, \Delta + \Theta)$ has a $\mathbb{Q}$-factorial dlt modification $(X', \Delta' + \Theta' + E') \to (X, \Delta + \Theta)$.

Next we have a canonical model for $(X', \Delta' + \Theta' + E') \to (X, \Delta)$ by [HX13, 1.6], call it $\pi : (X^c, \Delta^c + E^c) \to (X, \Delta)$. Thus $-\Theta^e \sim_{\mathbb{R}, \pi} K_{X^c} + \Delta^c + E^c$ is $\pi$-ample. In particular, $\text{Ex}(\pi) \subset \text{Supp} \Theta^c$ by Lemma 20. Thus $\pi$ is small, $E^e = 0$ and $\pi : (X^c, \Delta^c) \to (X, \Delta)$ is the required lc modification.

Let $W \subset X^c$ be an lc center of $(X^c, \Delta^c)$. Increasing $\Delta^c$ to $\Delta^c + \Theta^e$ decreases discrepancies, and the decrease is strict for divisors whose center is contained in $\text{Supp} \Theta^c$. Since $(X^c, \Delta^c + \Theta^e)$ is lc, this implies that $W \not\subset \text{Supp} \Theta^c$. Since $\text{Ex}(\pi) \subset \text{Supp} \Theta^c$, this shows that $W \not\subset \text{Ex}(\pi)$.

If $K_X + \Delta$ is $\mathbb{R}$-Cartier at $x$ then $K_{X^c} + \Delta^c \sim_{\mathbb{R}} \pi^*(K_X + \Delta)$ near $x$. Since $K_{X^c} + \Delta^c$ is $\pi$-ample, this implies that $\pi$ is a local isomorphism over $x$. □

**Lemma 20.** Let $\pi : Y \to X$ be a proper birational morphism, $X$ normal. Let $D$ be an effective divisor on $X$ such that $-\pi_*^{-1} D$ is $\pi$-nef. Then $\text{Supp} \pi_*^{-1} D = \text{Supp} \pi_*^{-1} D$. If $-\pi_*^{-1} D$ is $\pi$-ample then $\text{Ex}(\pi) \subset \text{Supp} \pi_*^{-1} D$ and $\pi$ is small. □

As we see in (17.1), an lc pair usually does not have a small, $\mathbb{Q}$-factorial modification. However, we can at least achieve that all irreducible components of $\Delta$ become $\mathbb{Q}$-Cartier.

**Corollary 21.** Let $(X, \Delta)$ be an lc pair. Then $(X, \Delta)$ has a projective, small modification $\pi : (X^w, \Delta^w := \pi_*^{-1}(\Delta)) \to (X, \Delta)$ such that

1. $(X^w, \Delta^w)$ is lc,
2. $K_{X^w}$ is $\pi$-nef,
3. every irreducible component of $\Delta^w$ is $\mathbb{Q}$-Cartier,
4. $\text{Ex}(\pi) \subset \text{Supp} \Delta^w$ and
5. $\pi$ is an isomorphism over $x \in X$ iff every irreducible component of $\Delta$ is $\mathbb{Q}$-Cartier at $x$. □
Proof. If \( \pi \) is small then \( K_{X^w} + \Delta^w \sim_\mathbb{R} \pi^*(K_X + \Delta) \) hence \( (X^w, \Delta^w) \) is lc.

First apply Proposition [19] to \((X, 0)\) to get \( \tau_1 : X_1 \rightarrow X \) such that \( K_{X_1} \) is \( \tau \)-ample and \( \text{Ex}(\tau) \) is contained in the support of \( \Delta_1 := (\tau_1)_*^{-1} \Delta \).

If \( \tau_2 : X_2 \rightarrow X_1 \) is any small modification then \( K_{X_2} \sim_\mathbb{R} \tau_2^* K_{X_1} \), thus \( K_{X_2} \) is \( \tau_1 \circ \tau_2 \)-nef. Similarly, since \( \Delta_1 \) is \( \mathbb{R} \)-Cartier, \( \Delta_2 := (\tau_2)_*^{-1} \Delta_1 \sim_\mathbb{R} \tau_2^* \Delta_1 \), hence (4) also holds for us.

In order to avoid multi-indices, we can thus assume that \( K_X \) is \( \mathbb{Q} \)-Cartier.

We need to achieve (3). Let \( a_1 D_1 \) be an irreducible component of \( \Delta \) and apply Proposition [19] to \((X, \Delta - a_1 D_1)\) to get \( \sigma : (X', \Delta' - a_1 D'_1) \rightarrow (X, \Delta - a_1 D_1) \).

Since \( K_{X'} + \Delta' - a_1 D'_1 \) and \( K_{X'} + \Delta' \sim_\mathbb{R} \sigma^*(K_X + \Delta) \) are \( \mathbb{R} \)-Cartier, so is \( D'_1 \) and \( \text{Ex}(\sigma) \subset \text{Supp} D'_1 \).

By induction on the number of irreducible components of \( \Delta \) we may assume that the claims (1–5) hold for \((X', \Delta' - a_1 D'_1)\) and we have

\[
\pi' : (X^w, \Delta^w - a_1 D^w_1) \rightarrow (X', \Delta' - a_1 D'_1).
\]

The composite

\[
\pi := \sigma \circ \pi' : (X^w, \Delta^w) \rightarrow (X, \Delta)
\]

is the required modification. \( \square \)

Note that \((X^w, \Delta^w)\) is not unique and the different ones are related to each other by flops. However, as shown by [Kol10, 96] and (37.3), flops do not always exist in the lc case, so we have less freedom than in the klt case.

Proposition [19] does not extend to potentially slc pairs. In codimension 2 this is related to the bookkeeping problem we already encountered in Paragraph [12]. For example, the pair \((S, D) := ((xy = 0), (x = z = 0))\) is potentially slc since \(((xy = 0), (z = 0))\) is slc, but \(K_S + D\) is not \( \mathbb{Q} \)-Cartier. Since a semi-normal surface does not have small modifications, there is nothing that can be done. However, even if our divisor is \( \mathbb{Q} \)-Cartier in codimension 2, there are higher codimension obstructions. The following is a slight modification of the second example in [Kol13b, 1.40].

**Example 22.** Let \( Z \subset \mathbb{P}^{n-1} \) be a smooth hypersurface of degree \( n \) and \( D_1 \subset X_1 := \mathbb{A}^n \) the cone over \( Z \). Next fix two points \( p, q \in \mathbb{P}^1 \), embed \( Z \times \mathbb{P}^1 \) into \( \mathbb{P}^{2n-1} \) by the global sections of \( \mathcal{O}_{\mathbb{P} \times \mathbb{P}^1}(1, 1) \) and let \( (X_2, D_2 + D'_2) \) be the cone over \(((Z \times \mathbb{P}^1), (Z \times \{p\}) + (Z \times \{q\}))\).

\((X_1, D_1)\) and \((X_2, D_2 + D'_2)\) are both lc by [Kol13b, 3.1], thus one can glue them using the natural isomorphism \( \sigma : D_1 \cong D_2 \) to obtain an slc pair \((X_1 \amalg_{\sigma} X_2, D'_2)\).

Thus \((X, 0) := (X_1 \amalg_{\sigma} X_2, 0)\) is potentially slc.

We check in [Kol13b, 1.40] that \((X, 0)\) has no slc modification. The reason is that \((X_1, D_1)\) is lc, hence its lc modification is itself, but the lc modification of \((X_2, D_2)\) is obtained by first taking the conical resolution and then contracting the \( \mathbb{P}^1 \)-factor of the exceptional divisor \( E \cong Z \times p^1 \). Thus the birational transform of \( D_1 \) in the lc modification is isomorphic to \( D_1 \) but the birational transform of \( D_2 \) in the lc modification is isomorphic to its blow-up. So the two lc modifications can not be glued together.

A similar example, where the canonical class is Cartier is the following.

Let \( v \in Q \subset \mathbb{A}^4 \) be a quadric cone with vertex \( v \) and \( v \in H \subset Q \) a hyperplane section. Assume that both \( Q \) and \( H \) have an isolated singularity at \( v \). We can glue \((Q, H)\) and \((H \times \mathbb{A}^1, H \times \{0\})\) to get an semi-dlt 3-fold \( X \) (cf. [Kol13b, 5.19]) and \( K_X \sim 0 \).
Next we add a boundary. Let \( A \subset Q \) be a plane. Then \( L := A \cap H \) is line through \( v \). The boundary \( \Delta \) is \( A \) on \( Q \) and \( L \times \mathbb{A}^1 \) on \( H \times \mathbb{A}^1 \). Note that \( L \times \mathbb{A}^1 \) is \( \mathbb{Q} \)-Cartier on \( H \times \mathbb{A}^1 \) but \( A \) is not \( \mathbb{Q} \)-Cartier on \( Q \). The lc modification of \( (Q, H + A) \) is one of the small modifications of \( Q \) and the birational transform of \( H \) is its minimal resolution. Thus again the 2 small modifications can not be glued together.

The next result shows that for locally stable morphisms, the existence of an slc modification depends only on the generic fiber.

**Proposition 23.** Let \( f : (X, \Delta + \Delta') \to B \) be a locally stable morphism to a smooth, irreducible curve \( B \). Assume that the generic fiber has an slc modification \( \pi_g : (X_g^c, \Delta_g^c) \to (X_g, \Delta_g) \). Then \((X, \Delta)\) also has an slc modification.

Proof. We follow the method of [Kol17, Sec.2.4]. That is, first we normalize \((X, \Delta)\), then use Proposition 19 to obtain the lc modification of the normalization and finally use the gluing theory of [Kol16] and [Kol13b, Chap.9] to get \( \pi : (X^c, \Delta^c) \to (X, \Delta) \).

Thus we start with the normalization \( \rho : (\tilde{X}, \tilde{\Delta} + \tilde{D} + \tilde{\Delta}') \to (X, \Delta + \Delta') \).

By Proposition 19 there is a projective, small, lc modification \( \sigma : (\tilde{X}^c, \tilde{\Delta}^c + \tilde{D}^c) \to (\tilde{X}, \tilde{\Delta} + \tilde{D}) \), whose generic fiber over \( B \) is the normalization of \((X_g^c, \Delta_g^c)\). Let \( \tilde{D}^{cn} \to \tilde{D}^c \) denote the normalization. On the generic fiber we have an involution \( \tau_g : (\tilde{D}_g^{cn}, \text{Diff} \tilde{\Delta}_g^c) \leftrightarrow (\tilde{D}_g^{cn}, \text{Diff} \tilde{\Delta}_g^c) \), where we take the different on \( \tilde{D}_g^{cn} \). By [Kol17, 2.12] \((\tilde{D}_g^{cn}, \text{Diff} \tilde{\Delta}^c)\) has no log centers supported over a closed point of \( B \), thus, by [Kol17, 2.45], \( \tau_g \) extends to an involution \( \tau : (\tilde{D}^{cn}, \text{Diff} \tilde{\Delta}^c) \leftrightarrow (\tilde{D}^{cn}, \text{Diff} \tilde{\Delta}^c) \), where we take the different on \( \tilde{D}^{cn} \).

By [Kol17, 2.54] we thus obtain \((X^c, \Delta^c)\) as the geometric quotient of \((\tilde{X}^c, \tilde{\Delta}^c + \tilde{D}^c)\) by the equivalence relation generated by \( \tau \). \( \square \)

4. Proof of Propositions 5 and 7

We prove Proposition 28 which is a generalization of Proposition 5. Its assumptions are somewhat convoluted, but probably sharp; see Examples 67–6. They were arrived at by looking at a proof of Proposition 3 and then trying to write down a minimal set of assumptions that make the arguments work. For now the only applications I know of are Propositions 5 and Theorem 1.

We start with the following application of the method of Section 3.

**Proposition 24.** Let \( (X, \Delta) \) be an lc pair and \( \Delta_1, \Delta_2 \) effective divisors such that \( \Delta_1 + \Delta_2 \leq \Delta \). Let \( B \) be a Weil \( \mathbb{Z} \)-divisor such that \( B \sim_{\mathbb{R}} K_X + L + \Delta_1 - c \Delta_2 \) where \( L \) is \( \mathbb{R} \)-Cartier and \( c \geq 0 \). Then there is a small, lc modification \( \pi : (X', \Delta') \to (X, \Delta) \) such that

1. \( B' \) is \( \mathbb{Q} \)-Cartier,
2. \( K_{X'} + \Delta'_1 \) is \( \mathbb{R} \)-Cartier,
3. \( \text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta'_1) \),
4. none of the lc centers of \((X', \Delta'_1)\) are contained in \( \text{Ex}(\pi) \),
Weil divisor is a Mumford divisor. Let \( \tau \) be a scheme. I call a Weil divisor \( \Delta ' = \sum a_i D_i \) a \( \tau \)-ample if for every \( \pi : X \to B \) is Cartier at all generic points of \( S \cap \text{Supp}(\Delta) \). Assume next that \( S \subset X \) is a Cartier divisor. Then (25.1) gives a natural injection

\[
\pi : \mathcal{O}_X(B)|_S \to \mathcal{O}_S(B|_S),
\]

which is an isomorphism on \( S \setminus Z \). Since \( \mathcal{O}_S(B|_S) \) is \( S_2 \) by definition, \( r \) is an isomorphism everywhere iff \( \mathcal{O}_X(B)|_S \) is \( S_2 \). As we noted in Theorem 4 this is equivalent to \( \mathcal{O}_X(B) \) being \( S_3 \) along \( S \). We thus obtain the following observation.

**Claim**. If \( S \subset X \) is a Cartier divisor then the sequence (25.1) is exact iff \( \mathcal{O}_X(B) \) is \( S_3 \) along \( S \).

We need to understand the \( S_3 \) condition for divisorial sheaves on slc pairs. The first part of the following is proved in [Ale08] and the second part in [Ko11]; see also [Kol13b] 7.20 and [Fuj17] Sec.7.1.

**Theorem 26.** Let \((X, \Delta)\) be an slc pair and \( x \in X \) a point that is not an lc center. Let \( B \) be a Mumford \( \mathbb{Z} \)-divisor on \( X \). Assume that

1. either \( B \) is \( \mathbb{Q} \)-Cartier,
2. or \( B \sim R - \Delta ' \) for some \( 0 \leq \Delta ' \leq \Delta \).

Proof. We construct \( \pi : (X', \Delta') \to (X, \Delta) \) in 2 steps. First we apply Proposition 19 to \((X, \Delta - \Delta_2)\). We get \( \tau_1 : (X, \Delta') \to (X, \Delta) \) such that

\[
-\Delta_2^* \sim R K_{X'} + \Delta^* - \Delta_2^* - \tau_1^*(K_X + \Delta) = \tau_1^*(K_X + \Delta) 
\]

is \( \tau_1 \)-ample and its support contains the exceptional set of \( \tau_1 \). Since \( \Delta_1^* \leq \Delta^* - \Delta_2^* \), we can next apply Proposition 19 to \((X^*, \Delta_1^*)\) to get \( \tau_2 : (X', \Delta') \to (X^*, \Delta^*) \). Set \( \pi := \tau_1 \circ \tau_2 \). By construction \( \Delta_2^* = \tau_2^* \Delta_1^* \) and \( K_{X'} + \Delta_1' \) are \( \mathbb{R} \)-Cartier, and so is \( B' \sim R K_{X'} + \Delta_1' + \pi^* L - c \Delta_2^* \).

Furthermore, Ex(\( \pi \)) is contained in \( \text{Supp}(\Delta_2^*) \cup \text{Supp}(\Delta' - \Delta_1') \). Since \( \Delta_2^* \leq \Delta_1' - \Delta_1^* \), this implies that \( \text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta_1') \). Thus none of the lc centers of \((X', \Delta_1')\) are contained in \( \text{Ex}(\pi) \). Since \( -\Delta_2^* = -\tau_2^* \Delta_1^* \) is \( \pi \)-nef, so is \( \pi^* L - \Delta_2^* \) and these in turn imply that \( R^i \pi_* \mathcal{O}_X(B') = 0 \) for \( i > 0 \) by Theorem 32. Finally the Leray spectral sequence shows (7). \( \square \)

It is convenient to state the following results using the notion of Mumford divisors; see [Kol18b] for a detailed treatment.

**Definition 25.** Let \( X \) be a scheme. I call a Weil divisor \( B \) on \( X \) a Mumford divisor if \( X \) is regular at all generic points of \( \text{Supp} B \). Thus on a normal variety, every Weil divisor is a Mumford divisor.

Let \( S \subset X \) be a closed subscheme. \( B \) is called a Mumford divisor along \( S \) if \( \text{Supp} B \) does not contain any irreducible component of \( S \), \( B \) is Cartier at all generic points of \( S \cap \text{Supp} B \) and \( S \) is regular at all generic points of \( S \cap \text{Supp} B \).

These imply that \( B|_S \) is a well-defined Mumford divisor on \( S \) and there is a subset \( Z \subset S \) of codimension \( \geq 2 \) such that \( B \) is Cartier at all points of \( S \setminus Z \). The restriction sequence

\[
0 \to \mathcal{O}_X(B)(-S) \to \mathcal{O}_X(B) \to \mathcal{O}_S(B|_S) \to 0, \tag{25.1}
\]

is left exact everywhere and right exact on \( X \setminus Z \). (We use this sequence only when \( X \) and \( S \) are both \( S_2 \); then there is no ambiguity in the definition of \( \mathcal{O}_X(B) \) and \( \mathcal{O}_S(B|_S) \).) Assume next that \( S \subset X \) is a Cartier divisor. Then (25.1) gives a natural injection

\[
r : \mathcal{O}_X(B)|_S \to \mathcal{O}_S(B|_S), \tag{25.2}
\]

which is an isomorphism on \( S \setminus Z \). Since \( \mathcal{O}_S(B|_S) \) is \( S_2 \) by definition, \( r \) is an isomorphism everywhere iff \( \mathcal{O}_X(B)|_S \) is \( S_2 \). As we noted in Definition 4 this is equivalent to \( \mathcal{O}_X(B) \) being \( S_3 \) along \( S \). We thus obtain the following observation.

**Claim.** If \( S \subset X \) is a Cartier divisor then the sequence (25.1) is exact iff \( \mathcal{O}_X(B) \) is \( S_3 \) along \( S \). \( \square \)
Proposition 28. Let $(X, \Delta)$ be an lc pair, $S$ a reduced $\mathbb{Q}$-Cartier, Mumford divisor and $B$ a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor that is Mumford along $S$. Assume that $B$ satisfies (28.1) or (28.2) and $\text{Supp} S \subset \text{Supp} \Delta$. Then the sequence
\[ 0 \to \mathcal{O}_X(B - S) \to \mathcal{O}_X(B) \to \mathcal{O}_S(B|_S) \to 0 \]
is exact.

Proof. Assume first that $S$ is Cartier. $(X, \Delta - \epsilon S)$ is also an lc pair for $0 < \epsilon \ll 1$ and none of its lc centers are contained in $S$ by [KM98, 2.27]. Thus $\mathcal{O}_X(B)$ is $S_3$ along $S$ by (26.1), hence the sequence is exact by (25.3).

If $S$ is not Cartier, let $m > 0$ be the smallest integer such that $mS \sim 0$. This linear equivalence defines a degree $m$ cyclic cover $\tau : \tilde{X} \to X$ such that $\tilde{S} := \tau^*(S)$ is Cartier; see [KM98, 2.49–53]. Set $B := \tau^*(B)$. We have already established that depth$_x \mathcal{O}_{\tilde{X}}(\tilde{B}) \geq \min\{3, \text{codim}_x \tilde{x}\}$ for every $\tilde{x} \in \tilde{S}$. Since $\mathcal{O}_X(B)$ is a direct summand of $\tau_*\mathcal{O}_{\tilde{X}}(\tilde{B})$, this implies that $\mathcal{O}_X(B)$ is $S_3$ along $S$. □

Now we come to the main technical result, which is a strengthening of Proposition 7. Examples 39.6–7 shows that the assumptions are likely optimal.

Proposition 29. Let $(X, S + \Delta)$ be an lc pair where $S$ is $\mathbb{Q}$-Cartier. Let $B$ be a Weil $\mathbb{Z}$-divisor that is Mumford along $S$ and $\Delta_3$ an effective $\mathbb{R}$-divisor such that
\begin{enumerate}
    \item $B \sim_R -\Delta_3$,
    \item $\Delta_3 \leq \lfloor \Delta \rfloor + \lfloor \Delta - 1/2 \rfloor$ and
    \item $\lfloor \Delta_3 \rfloor \leq \lfloor \Delta \rfloor$.
\end{enumerate}

Then $\mathcal{O}_X(B)$ is $S_3$ along $S$.

Proof. Assume first that $S$ is Cartier with equation $s = 0$. By (25.3) we need to show that the sequence (25.1) is exact.

By Theorem 28 we need to focus on the points where $B$ is not $\mathbb{Q}$-Cartier. This suggests that we should use Proposition 24. The question is local on $X$, so we may assume that $K_X + \Delta \sim_R 0$. By assumption (1)
\[ B \sim_R -\Delta_3 \sim_R K_X + \Delta - \Delta_3. \] (28.4)

Set $\Delta_1 := \lfloor (\Delta - \Delta_3)^{>0}_1 \rfloor$ and $\Delta_2 := \epsilon (\Delta - \Delta_3)^{<0}_1$ for some $0 < \epsilon \ll 1$. It is clear that $\text{Supp} \Delta_1$, $\text{Supp} \Delta_2$ have no common irreducible components and $\Delta_1 + \Delta_2 \leq \Delta$ for $0 < \epsilon \ll 1$. Furthermore, $B \sim_R K_X + \Delta_1 - \epsilon \Delta_2$ with $c := \epsilon^{-1}$. Using these $\Delta_1, \Delta_2$ in Proposition 24 we obtain $\pi : (X', \Delta') \to (X, \Delta)$.

Note that $B'$ is $\mathbb{Q}$-Cartier by (24.1), $(X', S' + \Delta')$ is lc and $\text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta'_1)$ by (24.3). Thus none of the lc centers of $(X', S' + \Delta'_1)$ are contained in $\text{Ex}(\pi)$ in particular, $S'$ is smooth at the generic points of all exceptional divisors of $\pi_{S'} := \pi|_{S'} : S' \to S$. Thus $B'$ is also Mumford along $S'$, hence the sequence
\[ 0 \to \mathcal{O}_{X'}(B') \xrightarrow{\pi_*} \mathcal{O}_{X'}(B') \to \mathcal{O}_{S'}(B'|_{S'}) \to 0 \] (28.5)
is exact by Theorem 27. Since $R^1\pi_*\mathcal{O}_{X'}(B') = 0$ by (24.6), pushing (28.5) forward gives an exact sequence
\[ 0 \to \pi_*\mathcal{O}_{X'}(B') \xrightarrow{\pi_*} \mathcal{O}_{X'}(B') \to (\pi_S)_*\mathcal{O}_{S'}(B'|_{S'}) \to 0. \] (28.6)

It remains to show that $((\pi_S)_*\mathcal{O}_{S'}(B'|_{S'})) = \mathcal{O}_S(B|_S)$. This holds if there is a $B'' \leq B'|_{S'}$ such that $(\pi_S)_*\mathcal{O}_{S'}(B'') = \mathcal{O}_S(B|_S)$. □
Assume first that $|\Delta_3| = 0$ and $\Delta_3 \leq [\Delta^{(>1/2)}]$. By assumption $B + \Delta_3 \sim_R 0$, hence $B' + \Delta'_3 \sim_R 0$. Thus $(B' + \Delta'_3)|_{S'} \sim_R 0$ and using Lemma 29 with $N := -(B' + \Delta'_3)|_{S'}$ and $H := 0$ gives that

\[ (\pi_S)_* O_{S'}((B'|_{S'} + \Delta'_3)|_{S'}) = O_S([B|_{S'} + \Delta_3|_{S}]). \tag{28.7} \]

Since $S'$ is Cartier, $B'|_{S'}$ is a $\mathbb{Z}$-divisor, so

\[ [B'|_{S'} + \Delta'_3|_{S'}] = B'|_{S'} + [\Delta'_3|_{S'}] \quad \text{and} \quad [B|_{S'} + \Delta_3|_{S}] = B|_{S'} + [\Delta_3|_{S}]. \]

Since we assume that $|\Delta_3| = 0$, (12.3) and our assumption $\Delta_3 \leq [\Delta^{(>1/2)}]$ imply that $[\Delta_3|_{S}] = [\Delta_3]|_{S} = 0$ and $[\Delta'_3|_{S'}] = [\Delta'_3]|_{S'} = 0$. Thus

\[ (\pi_S)_* O_{S'}(B'|_{S'}) = O_S(B|_{S}) \tag{28.8} \]

and we are done with this case.

In general, let $E \subset S'$ be the largest, reduced, $\pi_S$-exceptional divisor such that $\pi_S(Supp E) \subset Supp \Delta_3$. Set $H := [\Delta_3|_{S}]$ and let $H'$ denote the birational transform of $H$ on $X'$.

We apply Lemma 29 with $N := -(B' + \Delta'_3)|_{S'}$ and $H'$ to get that

\[ (\pi_S)_* O_{S'}((B'|_{S'} + \Delta'_3|_{S'}) - H' - \epsilon E)) = O_S([B|_{S'} + \Delta_3|_{S'} - H]). \tag{28.9} \]

This in turn gives (28.8) provided that we prove that

\[ [B'|_{S'} + \Delta'_3|_{S'} - H' - \epsilon E]) \leq B'|_{S'} \quad \text{and} \quad [B|_{S'} + \Delta_3|_{S'} - H] = B|_{S}. \tag{28.10} \]

The second of these holds by our choice of $H$ since

\[ B|_{S'} + \Delta_3|_{S'} - H = B|_{S'} + \Delta_3|_{S} - [\Delta_3|_{S}] = B|_{S'} + [\Delta_3|_{S}]. \tag{28.11} \]

The same argument shows that $[\Delta'_3|_{S'} - H']$ is $\pi_S$-exceptional. It remains to understand what happens along any $\pi_S$-exceptional prime divisor $F \subset S'$. If $F \subset \text{Supp} \Delta'_3$ then also $F \subset \text{Supp} \Delta'$ and, as we argue in the proof of (12.4), we are in one of the following cases.

(a) $\text{coeff}_{F} \Delta'_3|_{S'} < 1$,

(b) $F$ is contained in a unique irreducible component $D'_i$ of $\Delta'$, $S' \cap D'_i$ has multiplicity 1 along $F$ and $\text{coeff}_{D_i} \Delta_3 = \text{coeff}_{D_i} \Delta = 1$,

(c) $F$ is contained in a unique irreducible component $D'_j$ of $\Delta'$, $S' \cap D'_j$ has multiplicity 2 along $F$ and $\text{coeff}_{D_j} \Delta_3 = \text{coeff}_{D_j} \Delta = \frac{1}{2}$,

(d) $F$ is contained in two irreducible components $D'_i, D'_j$ of $\Delta'$, $S' \cap D'_i, S' \cap D'_j$ both have multiplicity 1 along $F$ and $\text{coeff}_{D_i} \Delta_3 = \text{coeff}_{D_j} \Delta_3 = \text{coeff}_{D_j} \Delta = \frac{1}{2}$.

In the first case $F \not\subset [\Delta'_3|_{S'}]$.

If (b) (resp. (c) or (d)) holds then $\text{coeff}_{F}(\Delta'_3|_{S'}) = 1$ by assumption (3) (resp. (2)). Moreover, $\pi_S(Supp F) \subset \text{Supp} (D_i|_{S})$ and $D_i|_{S}$ appears in $\Delta_3|_{S}$ with coefficient 1 or $\frac{1}{2}$. Thus $\pi_S(Supp F) \subset \text{Supp}(\pi_* (-(B' + \Delta'_3)|_{S'}) + \pi_* H')$, hence $F \subset E$. Thus

\[ \text{coeff}_{F}(\Delta'_3|_{S'} - H' - \epsilon E) = 1 - \epsilon. \]

Using this for every $F$ we get that

\[ [B'|_{S'} + \Delta'_3|_{S'} - H' - \epsilon E] = B'|_{S'}. \]

hence (28.8) holds. This completes the proof if $S$ is Cartier.

If $S$ is only $\mathbb{Q}$-Cartier, a suitable cyclic cover reduces everything to the Cartier case, as in the proof of Corollary 27. \hfill \Box
The following is a related to the negativity lemma \cite{KM98} 3.39.

**Lemma 29.** Let \( \pi : Y \to X \) be a proper, birational morphism of normal schemes. Let \( N, H \) be \( \mathbb{R} \)-divisors such that \( N \) is \( \pi \)-nef and \( H \) is effective and horizontal. Then
\[
\pi_* \mathcal{O}_Y([-N - H]) = \mathcal{O}_X([\pi_*(N - H)]).
\]
Furthermore, let \( E \) be the largest reduced, effective \( \pi \)-exceptional divisor such that \( \pi(Supp E) \subset Supp(\pi_*\{N\} + \pi_*H) \). Then
\[
\pi_* \mathcal{O}_Y([-N - H - \epsilon E]) = \mathcal{O}_X([\pi_*(N - H)])
\]
for \( 0 < \epsilon \ll 1 \).

Proof. The question is local on \( X \), so we may assume that \( X \) is affine. By the Chow lemma we may assume that \( \pi \) is projective. Let \( F \) denote the divisorial part of \( \text{Ex}(\pi) \).

Let \( s \) be a section of \( \mathcal{O}_X([\pi_*(N - H)]) \). Then \( \pi^*s \) is a section of \( \mathcal{O}_Y([-N - H] + mF) \) for some \( m \geq 0 \) and (29.1) is equivalent to saying that \( \pi^*s \) has no poles along \( F \). If this fails then we can cut \( Y \) with general hyperplanes, until (after Stein factorization) we get a counter example \( \tau : Y' \to X' \) of dimension 2. In this case \( N' := N|_{Y'} \) and \( H' := H|_{Y'} \) are both \( \tau \)-nef \( \mathbb{R} \)-divisors.

Another advantage of the 2-dimensional normal situation is that there is a well-defined, monotone pull-back for all Weil divisors. That is, if \( B_1 \leq B_2 \) then \( \tau^*B_1 \leq \tau^*B_2 \). Furthermore, there are effective \( \tau \)-exceptional divisors \( E_i \) such that
\[
\tau^*\tau_*N' = N' + E_1, \quad \tau^*\tau_*H' = H' + E_2 \quad \text{and}
\]
\[
\tau^*\tau_*([^N' + H'] - N' - H') = ([N' + H'] - N' - H')^h + E_3,
\]
where the first claim uses \cite{KM98} 3.39, the superscript \( h \) denotes the horizontal part and \( Supp E = Supp E_2 \cup Supp E_3 \). Note that
\[
[-N' - H'] = -N' - H' - ([N' + H'] - N' - H')
\]
Putting these together gives that
\[
\tau^*[\tau_*(-N' - H')] = -\tau^*\tau_*N' - \tau^*\tau_*H' - \tau^*\tau_*([^N' + H'] - N' - H')
\]
\[
= -N' - H' - (E_1 + E_2 + E_3 + ([N' + H'] - N' - H')^h)
\]
\[
\leq -N' - H' - \epsilon E
\]
for \( 0 < \epsilon \ll 1 \). Let now \( s' \) be any section of \( \mathcal{O}_{X'}([\tau_*(-N' - H')] \). Then we get that
\[
(\tau^*s') \leq \tau^*[\tau_*(-N' - H')] \leq -N' - H' - \epsilon E.
\]
Since \( (\tau^*s') \) is a \( \mathbb{Z} \)-divisor, we also have the stronger inequality
\[
(\tau^*s') \leq [-N' - H' - \epsilon E].
\]
This implies (29.2).

In some situations one may need the following variants of Lemma 29. The first one can be obtained by the same proof and the second one can be reduced to the normal case once we assume that the singularities do not interfere with the divisors much.

**Complement 29.7.** Using the above notation and assumptions, let \( E^* \) be the divisorial part of \( \text{Ex}(\pi) \) and fix a point \( x \in X \). Then, for \( 0 < \epsilon \ll 1 \), we have
(1) either \( \pi_* \mathcal{O}_Y([-N - H - \epsilon E^*]) = \mathcal{O}_X([\pi_*(N - H)]) \) near \( x \)
(2) or \( N \) is a principal divisor near \( \pi^{-1}(x) \) and \( Supp H \cap \pi^{-1}(x) = \emptyset \).
Complement 29.8. Let \( \pi : Y \to X \) be a proper, birational morphism of pure dimensional, reduced schemes. Assume that \( Y \) is \( S_2 \) and \( \pi \) is a local isomorphism at all codimension 1 singular points of \( X \) or \( Y \). Let \( N \) be a \( \pi \)-nef Mumford divisor on \( Y \) and \( H \) an effective, horizontal Mumford \( \mathbb{R} \)-divisor. Then (29.1–2) hold. \( \square \)

30 (Proof of Proposition 29). We may assume that \( X \) is affine. Let \( x \in H \) be a point of codimension 1. Then either \( H \) and \( X \) are both smooth at \( x \) or \( H \) has a node and \( X \) a Du Val singularity at \( x \). In the latter case \( x \notin \text{Supp} \Delta \). Thus \( mK_X + [m\Delta] \) is Cartier at \( x \), hence a general divisor \( B \sim mK_X + [m\Delta] - D \) is Mumford along \( H \).

In order to prove Proposition 4 we apply Proposition 28 to \( B \) with \( \Delta_3 := m\Delta - [m\Delta] + D \). Thus

\[
B \sim mK_X + |m\Delta| - D = m(K_X + \Delta) - \Delta_3 \sim_{\mathbb{R}} -\Delta_3,
\]

and the proof of Proposition 28 uses Proposition 24 with

\[
\Delta_1 := ([m\Delta] - (m-1)\Delta)^{\geq 0} \quad \text{and} \quad \Delta_2 := \epsilon([m\Delta] - (m-1)\Delta)^{< 0}.
\]

It is clear that \( \Delta_3 \leq \lfloor \Delta^{(1/2)} \rfloor \), but (28.2) needs a stronger statement for divisors whose coefficient is \( \frac{1}{2} \). If \( A \) is a prime divisor on \( X \) such that \( \text{coeff}_A \Delta = \frac{1}{2} \) then \( \text{coeff}_A \Delta_3 = 0 \) if \( m \) is even and \( \text{coeff}_A \Delta_3 = \frac{1}{2} \) if \( m \) is odd. Thus, in all cases,

\[
\text{coeff}_A \Delta_3 \leq \frac{1}{2} = \text{coeff}_A (\Delta^{(1/2)} + \lfloor \Delta^{(1/2)} \rfloor),
\]

as required for (28.2). So the assumptions of Proposition 28 are satisfied and we get Proposition 29. \( \square \)

5. Global applications

The previous constructions can also be used to obtain vanishing theorems for Weil divisors that are not assumed to be \( \mathbb{Q} \)-Cartier. It turns out that the Kawamata-Viehweg vanishing, even in its strong form given in [Fuj14, 1.10], holds without the \( \mathbb{Q} \)-Cartier assumptions, see Corollary 35. The proof below uses too much of MMP; it would be desirable to have an argument without such heavy tools.

Definition 31. Let \((X, \Delta)\) be a potentially lc pair. An irreducible subvariety \( W \subseteq X \) is an lc center of \((X, \Delta)\) if there is an open subset \( U \subseteq X \) containing the generic point of \( W \) such that \((K_U + \Delta|_U)\) is \( \mathbb{R} \)-Cartier (hence lc) and \( W \cap U \) is an lc center of \((U, \Delta|_U)\).

Let \( f : X \to S \) be a proper morphism and \( L \) an \( \mathbb{R} \)-Cartier, \( f \)-nef divisor on \( X \). Then \( L \) is called log \( f \)-big if \( L|_W \) is big on the generic fiber of \( f|_W : W \to f(W) \) for every lc center \( W \) of \((X, \Delta)\) and also for every irreducible component \( W \subseteq X \). Note that this notion depends on \( \Delta \).

If \( 0 \leq \Delta' \leq \Delta \) and \((X, \Delta)\) is potentially lc then so is \((X, \Delta')\) and every lc center of \((X, \Delta')\) is also an lc center of \((X, \Delta)\). Thus if \( L \) is log \( f \)-big on \((X, \Delta)\) then it is also log \( f \)-big on \((X, \Delta')\).

Note. There is another sensible way to define an “lc center” of a potentially lc pair \((X, \Delta)\) as an irreducible subvariety \( W \subseteq X \) such that \( W \cap U \) is an lc center of \((U, \Delta|_U + \Theta|_U)\) for every open subset \( U \subseteq X \) and for every effective divisor \( \Theta|_U \) such that \((U, \Delta|_U + \Theta|_U)\) is lc (and \( W \cap U \) is nonempty). With this definition, there are more “lc centers.”

As a typical example, set \( Q := (xy = uv) \subseteq \mathbb{A}^4 \), \( A_1 := (x = u = 0) \) and \( A_2 := (y = v = 0) \). Then \((Q, A_1 + A_2)\) is potentially lc. Its lc centers are the
divisors $A_1, A_2$. However, if $\Delta$ is any effective divisor such that $(Q, A_1 + A_2 + \Delta)$ is lc then the origin is also an lc center; cf. [Amb03] 4.8, [Kel13b] 4.41 or [Fuj17 6.3.11].

For our current purposes the first variant works better.

The following is proved in [Amb03] and [Fuj14, 1.10], see also [Fuj17, Sec.5.7], and [Fuj17 6.3.5], where it is called a Reid-Fukuda–type vanishing theorem.

**Theorem 32** (Ambro-Fujino vanishing theorem). Let $(X, \Delta)$ be an slc pair and $D$ a Mumford $\mathbb{Z}$-divisor on $X$. Let $f : X \rightarrow S$ be a proper morphism. Assume that $D \sim_\mathbb{R} K_X + L + \Delta$, where $L$ is $\mathbb{R}$-Cartier, $f$-nef and log $f$-big. Then

$$R^i f_* O_X(D) = 0 \quad \text{for } i > 0. \quad \square$$

Combining it with Proposition [19] gives the following variant, but only for normal varieties.

**Corollary 33.** Let $(X, \Delta)$ be a potentially lc pair, $f : X \rightarrow S$ a proper morphism and $D$ a Weil $\mathbb{Z}$-divisor on $X$. Assume that $D \sim_\mathbb{R} K_X + L + \Delta$, where $L$ is $\mathbb{R}$-Cartier, $f$-nef and log $f$-big. Then $R^i f_* O_X(D) = 0$ for $i > 0$.

Proof. Let $\pi : (X^c, \Delta^c) \rightarrow (X, \Delta)$ be as in Proposition [19]. Then $D^c \sim_\mathbb{R} K_{X^c} + \pi^* L + \Delta^c$. Since $\pi$ is a local isomorphism at every lc center of $(X^c, \Delta^c)$, we see that $\pi^* L$ is log $f \circ \pi$-big on $(X^c, \Delta^c)$. By Theorem [32] this implies that

$$R^i (f \circ \pi)_* O_{X^c}(D^c) = 0 \quad \text{for } i > 0 \quad \text{and} \quad R^i \pi_* O_{X^c}(D^c) = 0 \quad \text{for } i > 0.$$

The Leray spectral sequence now gives that $R^i f_* O_X(D) = 0$ for $i > 0. \quad \square$

**34 (Log plurigenera).** On an slc pair $(X, \Delta)$, the best analogs of the multiples of the canonical divisor are the divisors of the form $mK_X + \lfloor m\Delta \rfloor$. While it is probably not crucial, it is convenient to know when their higher cohomologies vanish. Even if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and ample, the divisors $mK_X + \lfloor m\Delta \rfloor$ need not be $\mathbb{Q}$-Cartier and, even if $\mathbb{Q}$-Cartier, need not be ample. In order to understand the situation, note that

$$mK_X + \lfloor m\Delta \rfloor \sim_\mathbb{R} K_X + (m - 1)(K_X + \Delta) + \lfloor m\Delta \rfloor - (m - 1)\Delta. \quad \text{[34.1]}$$

If $K_X + \Delta$ is nef and big then [34.1] has the expected form for vanishing theorems, except that $\lfloor m\Delta \rfloor - (m - 1)\Delta$ need not be effective and the whole divisor need not be $\mathbb{Q}$-Cartier.

If $\text{coeff } \Delta \subseteq \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$ then $0 \leq \lfloor m\Delta \rfloor - (m - 1)\Delta \leq \Delta$ for every $m$ and the second problem is remedied by an application of Corollary 33. This leads to the following.

**Proposition 35.** Let $(X, \Delta)$ be an lc pair with coeff $\Delta \subseteq \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$ and $f : X \rightarrow S$ a proper morphism. Assume that $K_X + \Delta$ is $f$-nef and log $f$-big. Then

$$R^i f_* O_X(mK_X + \lfloor m\Delta \rfloor) = 0 \quad \text{for all } m \geq 2, i > 0.$$

Proof. If $m \geq 2$ then $(m - 1)(K_X + \Delta)$ is $f$-nef and log $f$-big on $(X, \Delta)$. Since $0 \leq \lfloor m\Delta \rfloor - (m - 1)\Delta \leq \Delta$, the pair $(X, \lfloor m\Delta \rfloor - (m - 1)\Delta)$ is potentially lc and $(m - 1)(K_X + \Delta)$ is also log $f$-big on $(X, \lfloor m\Delta \rfloor - (m - 1)\Delta)$. Thus Corollary 33 shows the required vanishing. \hfill \square
36. (Proof of Corollary 36) By Theorem 11, the sheaves $\omega_X^{[m]}(\lfloor m\Delta \rfloor)$ are flat over $S$ and commute with base change. If all the fibers are normal then the higher cohomologies of the fibers vanish by Proposition 35. Thus (31–2) are implied by the Cohomology and Base Change theorem.

In general, pick $s \in S$, let $T$ be the spectrum of a DVR and $h : T \to S$ a morphism mapping the generic point of $T$ to a generic point of $S$ and the closed point of $T$ to $s$. After pull-back we get $f_T : (X_T, \Delta_T) \to T$. Here we can use Proposition 35 and conclude that

$$H^i(X_s, \mathcal{O}_{X_s}(mK_{X_s} + \lfloor m\Delta_s \rfloor)) = 0 \quad \text{for all} \quad m \geq 2, \ i > 0.$$ 

Thus the previous argument applies even if there are some non-normal fibers. □

6. Examples

First we give some examples of small modifications of lc singularities.

Example 37. (37.1) Let $(x \in X)$ be a cone over an Abelian variety $A$. We see next that small modifications of $(x \in X)$ correspond to Abelian subvarieties $A' \subset A$. So there are no small modifications if $A$ is simple but there are infinitely many small modifications if $A = E^m$ for some elliptic curve $E$ and $m \geq 2$.

Indeed, let $\pi : Y \to X$ be the cone-type resolution with exceptional divisor $F \cong A$. Note that $(Y, F)$ is canonical and $F$ is the unique divisor over $X$ with negative discrepancy. Let $\tau : Z \to X$ be a small modification and $Y' \to Z$ a $\mathbb{Q}$-factorial modification that extracts $F$; cf. [Kol13b, 1.38]. Then $Y' \to Y$ is an isomorphism in codimension 1, hence an isomorphism since $Y$ does not contain any proper rational curves, cf. [Kol96, VI.1.9]. Thus we get a morphism with connected fibers $\tau_A : A \to \operatorname{Ex}(\tau)$ and $Z$ is described by $\ker \tau_A$.

(37.2) Small modifications of cones over K3 surfaces can be obtained from elliptic structures (contract the cone-type resolution along the elliptic curves) and also from configurations of $(-2)$-curves (log-flop the curves and then contract the K3). Thus there can be infinitely many different small modifications. The existence of log flops between small modifications should follow from [Kov94].

(37.3) An example showing that log flops do not always exist is given in [Kol10, 96]. Here is another one.

Let $P \subset \mathbb{P}^2$ be a set of 9 points in very general position. Set $S := B_P\mathbb{P}^2$, let $E \subset S$ be the union of the 9 exceptional curves and $C \subset S$ the birational transform of the unique cubic through $P$. Since the points are in very general position, $H^0(S, \mathcal{O}_S(nC)) = 1$ for every $n \geq 0$.

Let $L \subset S$ be the birational transform of a general line; then $C \sim 3L - E$ and $H := C + L$ is ample.

Let $X := C_0(S, H)$ denote the affine cone over $S$ and $D := C_0(C, H|_C) \subset X$ the cone over $C$. Since $K_S + C \sim 0$, the pair $(X, D)$ is lc. (See [Kol13b, 3.1] for basic results on cones.) Let $g : Y \to X$ be the cone-type resolution with exceptional divisor $F \cong S$.

One can flop the 9 curves $E \subset F \subset Y$ and then contract the resulting $F' \cong \mathbb{P}^2$ to a point (of type $A^3/\mathbb{Z}(1, 1, 1)$). This gives a small, lc modification $\pi : X' \to X$ such that $D'$, the birational transform of $D$, is anti-ample. Thus $X' \cong \operatorname{Proj}_X \sum_{n=0}^\infty \mathcal{O}_X(-nD)$. We claim that $X' \to X$ does not have a log flop. The log flop would be given by $\operatorname{Proj}_X \sum_{n=0}^\infty \mathcal{O}_X(nD)$, but we check next that this ring is not finitely generated.
Since $X$ is a cone over $S$,

$$H^0(X, \mathcal{O}_X(nD)) = \sum_{m=0}^{\infty} H^0(S, \mathcal{O}_S(nC + mH)).$$

Thus we need to show that

$$\sum_{n,m=0}^{\infty} H^0(S, \mathcal{O}_S(nC + mH))$$

is not finitely generated. The problem is with the $m = 1$ summands. All of the

$$H^0(S, \mathcal{O}_S((n - r)C)) \times H^0(S, \mathcal{O}_S(rC + H)) \to H^0(S, \mathcal{O}_S(nC + H))$$

consists of sections that vanish along $C$.

Note also that if the 9 points $P$ are the base points of a cubic pencil, then $|C|$ is a base-point free elliptic pencil. The corresponding contraction $Y \to X'' \to X$ gives the log flop of $X' \to X$.

The following relates to Question 9.

**Example 38.** Set $Q := (xy = uv) \subset \mathbb{A}^4$ and let $|A|$ and $|B|$ denote the 2 families of planes on $Q$. Fix $n \geq 2$ and consider the klt pair.

$$(Q, \Delta := (\frac{2}{3} - \frac{1}{3n-1})(A_1 + A_2 + A_3) + (1 - \frac{1}{n})B_1 + (1 - \frac{1}{n(3n-1)})B_2).$$

By direct computation we see that

$$\omega^{|3n|}_Q([3n\Delta]) \cong \mathcal{O}_Q(-6A - 4B).$$

This has only depth 2 at the origin by [Kol13b, 3.15.2].

Next we describe all CM divisorial sheaves on certain singularities and see how this compares with the conclusions of Proposition 38 and Theorem 1. Note that for $Y$.

**Example 39.** Start with $Q := (xy = uv = 0) \subset \mathbb{A}^4$ and the divisors $B_0 := (y = v = 0), B := (x = u = 0)$ and $T := (v = u^n)$. Then $T \cong (xy - u^{n+1} = 0)$.

Next take quotient by a $\mu_n$-action $X := Q/\mathbb{Z}(1,0,1,0)$. The quotient map $\tau : Q \to X$ ramifies along $B$. In $X$ consider the (reduced) divisors $D_0 := \tau(B_0), D := \tau(B)$ and $S := \tau(T)$. Note that $(x/v)^n$ is a rational function on $X$ and it equals $(u/y)^n$. This shows that $D \sim nD_0$ and we compute that

$$\text{Cl}(X) = \mathbb{Z}[D_0] \quad \text{and} \quad K_X \sim -(n-1)D_0. \quad (39.1)$$

Furthermore, $\tau^*(K_X + S + aD_0 + bD) = K_Q + T - (n-1)B + aB_0 + nbB$ is $\mathbb{Q}$-Cartier if $a + nb - n + 1 = 0$. Using [Rei80] or [Kol13b, 2.43] and the convexity of lc boundaries gives the following.

**Claim 3.** For any $0 \leq \lambda \leq 1$ the pair

$$(X, S + \lambda(D_0 + \frac{a-2}{n}D) + (1 - \lambda)\frac{a-1}{n}D) \quad \text{is lc.} \quad \square$$

We can also get a complete description of all CM divisorial sheaves.

**Claim 4.** $\mathcal{O}_X(mD_0)$ is CM if $-n \leq m \leq 1$.

**Proof.** $Q$ is a cyclic cover of $X$ ramified along $D \sim nD_0$, thus

$$\pi_*\mathcal{O}_Q \cong \mathcal{O}_X + \mathcal{O}_X(-D_0) + \cdots + \mathcal{O}_X(-(n-1)D_0).$$
So \( \mathcal{O}_X(mD_0) \) is CM for \(- (n-1) \leq m \leq 0 \). For \( m = -n \) we use that \( \mathcal{O}_X(-nD_0) \cong \mathcal{O}_X(-D) \) which is the \( \mu_p \)-invariant part of \( \mathcal{O}_Q(-B) \) hence CM, cf. [Kol13b, 3.15.2]. By Serre duality we get that \( \mathcal{O}_X(D_0) \) is CM.

For \( m < -n \) set \( r := -m - n \). Note that \( D + rD_0 \) is not \( S_2 \), since the cokernel of \( \mathcal{O}_{D+rD_0} \to \mathcal{O}_D + r\mathcal{O}_{D_0} \) is supported at the origin. Then the exact sequence

\[ 0 \to \mathcal{O}_X(-(D - rD_0)) \to \mathcal{O}_X \to \mathcal{O}_{D+rD_0} \to 0 \]

shows that \( \mathcal{O}_X(mD_0) \) is not CM, cf. [Kol13b, 2.60]. By Serre duality this gives that \( \mathcal{O}_X(mD_0) \) is not CM for \( m \geq 2 \).

Corollary 39.4. If \( n \geq 3 \) then \( \omega_X^{[m]} \cong \mathcal{O}_X(-m(n-1)D_0) \) is CM only for \( m = 0, 1 \).

Next we see how this example compares with the conclusion of Theorem 1 for various choices of the boundary \( \Delta \).

Example 39.5. Assume that \( n \) is even and set \( \Delta := \frac{n-1}{n+1}(D_0 + D) \). Then \( K_X + \Delta \sim_{\mathbb{Q}} 0 \) and \( (X, S + \Delta) \) is lc by (39.2).

Observe that \( \frac{n+2}{n}K_X + \frac{n+2}{n}\Delta \sim_{\mathbb{Q}} -nD_0 \), thus Proposition 7 implies that \( \mathcal{O}_X(-nD_0) \) is CM. This coincides with the lower bound in (39.3). We also see that

\[
mK_X + [r\Delta] \sim_{\mathbb{Q}} (-m(n-1) + (n+1)\left\lfloor \frac{r(n-1)}{n+1} \right\rfloor)D_0
\]

Elementary estimates show that \( \omega_X^{[m]}([r\Delta]) \) is not CM whenever \( |m - r| \geq 2 \) and \( n \geq 4 \).

Example 39.6. Let \( \Delta := \frac{n-1}{2n+1}(D_0 + D') \). Then \( \mathcal{O}_X(B) \) is not CM. Thus the condition \( |\Delta_3| \leq |\Delta| \) is necessary in Proposition 28.

Next take \( \Delta := \frac{1}{2}(D'_0 + D''_0) + \frac{n-2}{n}D \) where \( D'_0, D''_0 \in |D_0| \) are two general divisors. Then \( K_X + \Delta \sim_{\mathbb{Q}} 0 \) and \( (X, S + \Delta) \) is lc by (39.2). Take \( B := -D_0 - D \) and \( \Delta_3 := \frac{1+\varepsilon}{n}(D'_0 + D'') + \frac{n-2}{n}D \).

Then \( |\Delta_3| = 0 \), \( \Delta_3 \leq |\Delta| \) but \( \mathcal{O}_X(B) \) is not CM. This shows that \( n=1/2 \) needs special handling in Proposition 28.

Example 39.8. Let \( 0 < \epsilon < 1 \) be irrational and set

\[ \Delta := \frac{n-1}{n+1}(1 + ne)D_0 + (1 - \epsilon)D. \]

Then \( K_X + \Delta \sim_{\mathbb{R}} 0 \) and \( (X, S + \Delta) \) is lc by (39.2). Also,

\[ mK_X + [m\Delta] \sim_{\mathbb{R}} \left( \left\lfloor m\frac{n-1}{n+1}(1 + ne) \right\rfloor + n\left\lfloor m\frac{n-1}{n+1}(1 - \epsilon) \right\rfloor \right)D_0, \]

and the right hand side is always a strictly negative multiple of \( D_0 \) for \( m \neq 0 \). Thus \( mK_X + [m\Delta] \) is not Cartier for every \( m \neq 0 \).
The next example shows that in Proposition 35 it is not enough to assume that the coefficients of $\Delta$ are close to 1. In fact, vanishing fails for every other value of the coefficients.

**Example 40.** Set $X = \mathbb{P}^n$ and choose $0 < \epsilon < \frac{1}{n+2}$. Let $H_i$ be hyperplanes in general position and set

$$\Delta = (1 - \frac{1+\epsilon}{n+3})(H_1 + \cdots + H_{n+2}).$$

Then $K_X + \Delta \sim_\mathbb{R} \frac{1-(n+2)}{n+3}H$ is ample and

$$|(n+3)\Delta| = (n+1)(H_1 + \cdots + H_{n+2}).$$

Therefore

$$(n+3)K_{\mathbb{P}^n} + |(n+3)\Delta| \sim_\mathbb{R} K_{\mathbb{P}^n},$$

which has nonzero higher cohomology.

Finally note that if $\epsilon' = \frac{1}{n+2}$ then $1 - \frac{1+\epsilon'}{n+3} = 1 - \frac{1}{n+2}$. Thus as $\epsilon$ varies in the interval $(0, \frac{1}{n+2})$, the values of $1 - \frac{1+\epsilon}{n+3}$ cover the open interval $(1 - \frac{1}{n+2}, 1 - \frac{1}{n+3})$.

Each of the above examples gives many more in one dimension higher.

**Example 41.** Let $Y'$ be a smooth projective variety of dimension $n+1$. Pick a point $y' \in Y'$ and let $H'_1, \cdots, H'_{n+2}$ be general smooth divisor passing through $y'$. Let $\pi : Y \to Y'$ be the blow-up of $y'$ with exceptional divisor $E$. Let $H_i$ denote the birational transform of $H'_i$. Our example is

$$(Y, E + \Delta) \quad \text{where} \quad \Delta = (1 - \frac{1+\epsilon}{n+3})(H_1 + \cdots + H_{n+2}).$$

Note that $(E, \text{Diff}_E \Delta)$ is isomorphic to the example in (40). In particular, our previous computations show that

$$R^n\pi_*\omega_Y^{[n+3]}((n+3)E + |(n+3)\Delta|) \cong k(y').$$

If the $H_i$ are chosen sufficiently ample then we get that

$$h^n(Y, \omega_Y^{[n+3]}((n+3)E + |(n+3)\Delta|)) = 1.$$ 

Note that such examples appear very naturally if we try to compactify the moduli space of $n+1$ dimensional varieties with $n+2$ divisors on them.

**Example 42.** Fix integers $n \geq m \geq 2$. Set $P = \mathbb{P}^{mn-n-1}$ and choose $0 < \epsilon < \frac{1}{mn}$. Let $H_0, \ldots, H_{mn}$ be hyperplanes in general position and set

$$\Delta = H_0 + (1 - \frac{1}{m} - \epsilon)(H_1 + \cdots + H_{mn}).$$

Then $K_P + \Delta \sim_\mathbb{R} (1 - mn\epsilon)H$ is ample and

$$|m\Delta| = mH_0 + (m-2)(H_1 + \cdots + H_{mn}).$$

Therefore

$$mK_P + |m\Delta| \sim_\mathbb{R} K_P - (n-m)H,$$

which has nonzero cohomology in degree $mn-n-1$ if $n \geq m$.

**Example 43.** Fix integers $n \geq m \geq 2$ and choose $0 < \epsilon < \frac{1}{mn}$. Let $Y'$ be a smooth projective variety of dimension $mn-n-1$. Pick a point $y' \in Y'$ and let $H'_1, \cdots, H'_{mn}$ be general smooth divisor passing through $(y', 0) \in Y' \times \mathbb{A}^1$. Let $\pi : X \to Y' \times \mathbb{A}^1$ be the blow-up of $(y', 0)$ with exceptional divisor $E$. Let $H_i$ denote the birational transform of $H'_i$. Set

$$\Delta = (1 - \frac{1}{m} - \epsilon)(H_1 + \cdots + H_{mn})$$
and let $f : (X, \Delta) \to \mathbb{A}^1$ be the composite of the coordinate projection with $\pi$.

If the $H_j$ are sufficiently ample then $f : (X, \Delta) \to \mathbb{A}^1$ is a stable morphism and $(X_t, \Delta_t)$ is a simple normal crossing pair for $t \neq 0$. The fiber over the origin has 2 irreducible components. One is $E$ and the other is $Y_0$, which is the blow-up of $y' \in Y_0'$.

Note that $(E, \text{Diff}_E(Y_0 + \Delta))$ is isomorphic to the example in (42). In particular, our previous computations show that

\[ R^{mn-n-1}\pi^*\omega^{|m\Delta|}_{Y} \neq 0 \]

and the other higher direct images are 0. Thus, if the $H_j$ are chosen sufficiently ample then we get that

\[ R^{mn-n-1}f_*\omega^{|m\Delta|}_{X_t} \]

is a torsion sheaf supported at the origin $0 \in \mathbb{A}^1$ and the other higher direct images are 0. In this case the functions

\[ t \mapsto h^i(X_t, \omega^{|m\Delta_t|}_{X_t}) \]

jump at $t = 0$ for $i = mn - n - 1$ and $i = mn - n - 2$ but are constant for other values of $i$.

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