IRRATIONALLITY OF GENERIC QUOTIENT VARIETIES
VIA BOGOMOLOV MULTIPLIERS

UBER JEZERNIK AND JONATAN SÁNCHEZ

Abstract. The Bogomolov multiplier of a group is the unramified Brauer group associated to the quotient variety of a faithful representation of the group. This object is an obstruction for the quotient variety to be stably rational. The purpose of this paper is to study these multipliers associated to nilpotent pro-$p$ groups by transporting them to their associated Lie algebras. Special focus is set on the case of $p$-adic Lie groups of nilpotency class 2, where we analyse the moduli space. This is then applied to give information on asymptotic behaviour of multipliers of finite images of such groups of exponent $p$. We show that with fixed $n$ and increasing $p$, a positive proportion of these groups of order $p^n$ have trivial multipliers. On the other hand, we show that by fixing $p$ and increasing $n$, log-generic groups of order $p^n$ have non-trivial multipliers. Whence quotient varieties of faithful representations of log-generic $p$-groups are not stably rational. Applications in non-commutative Iwasawa theory are developed.

1. Introduction

1.1. The rationality problem. Let $X$ be a smooth connected projective complex variety. The famous rationality problem asks whether or not $X$ is birational to a projective space. In terms of function fields, this means that its field of rational functions $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}$. This problem is especially interesting if one assumes that $X$ is a priori unirational, meaning that there is an inclusion of $\mathbb{C}(X)$ into some $\mathbb{C}(t_1, \ldots, t_n)$. Under this additional assumption, the rationality problem is known as the Lüroth problem. In its stable form, it asks whether unirational varieties are stably rational, meaning that some purely transcendental extension of $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}$.

1.2. Counterexamples to the Lüroth problem. The answer to the Lüroth problem turns out to be positive for curves and surfaces, and negative in general. The first counterexamples were constructed independently by Clemens and Griffiths [11], Iskovskikh and Manin [27], and by Artin and Mumford [2]. The latter even constructed counterexamples to the stable version of the problem. This is done by showing that the Brauer group, which in this situation coincides with $\text{Br}(X) = \text{Tors} \, H^3_{\text{sing}}(X, \mathbb{Z})$, of the specific 3-folds they consider is non-trivial, and

Date: January 18, 2023.

Key words and phrases. Bogomolov multiplier, nilpotent pro-$p$ group, $p$-adic group, Lazard correspondence, Grassmannian variety, exterior algebra, decomposable form.

The first author has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 748129. He has also been supported by the Spanish Government grant MTM2017-86802-P and by the Basque Government grant IT974-16. The second author was supported by the Spanish Government grant MTM2014-54804-P and the Basque Government grant IT1094-16.
this is an obstruction for such a variety to be stably rational. Many more counterexamples have recently been produced relying on the much finer intermediate property of possessing a decomposition of the diagonal of the Chow group, providing means to show that very general varieties in many interesting families are not stably rational (see [49]). The latter ultimately leads to a far-reaching generalization of the counterexamples of Artin and Mumford.

1.3. Unramified Brauer group. In this paper, we address questions of a similar flavour as the very general celebrated result mentioned above, but we stay with the Brauer group as an obstruction to stable rationality. This obstruction can be made to work even for varieties that are not smooth by passing to a smooth birational model and computing the Brauer group there. The reason for this is that the Brauer group of a smooth variety can be expressed as the subgroup of the classical $\text{Br}(\mathbb{C}(X)) = H^2(\text{Gal}(\mathbb{C}(X)), \mathbb{G}_m)$ consisting of the cocycles

$$\text{Br}(X) = \bigcap_{A \text{ DVR of } \mathbb{C}(X)} \text{ker}(\text{Br}(\mathbb{C}(X)) \xrightarrow{\partial_A} H^1(\kappa_A, \mathbb{Q}/\mathbb{Z})),$$

where $\partial_A$ is the residue map (see [21]). This description of the Brauer group depends only on the function field $\mathbb{C}(X)$ and so it can be computed in any birational model of $X$. The group on the right hand side is the unramified Brauer group of $X$, denoted by $\text{Br}_{nr}(X)$.

1.4. Quotients by group actions. Throughout the paper, we will deal with varieties $X$ that are GIT quotients of affine spaces $V$ by faithful linear actions of finite groups $G$ (see [12]). Thus $\mathbb{C}(X) = \mathbb{C}(V)^G$. The Lüroth problem for such varieties is known as Noether’s problem, especially when considered over $\mathbb{Q}$ rather than $\mathbb{C}$, and it is tightly related to a constructive approach to the inverse Galois problem (see [28]). In the setting of quotients by group actions, Bogomolov was able to transfer the above formula for the unramified Brauer group into purely group-theoretical cohomology (see [5]),

$$\text{Br}_{nr}(\mathbb{C}(V)^G) \cong \bigcap_{A \leq G, [A,A]=1} \ker \left( H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{res}_A^G} H^2(A, \mathbb{Q}/\mathbb{Z}) \right).$$

The group on the right hand side is hence known in the literature as the Bogomolov multiplier, denoted by $B_0(G)$. This description of the unramified Brauer group as the Bogomolov multiplier has made it much more tractable to compute these invariants and thus produce new counterexamples to rationality as well as understand these objects better (see [34, 26, 29]). It also explains that, in a way, the simplest obstruction for a quotient variety $V/G$ to be stably rational is that the group $G$ possesses some special 2-cocycles, a property that might be seen as rather special.

1.5. Profinite groups. The notion of the Bogomolov multiplier can be extended in a natural way to the category of all profinite groups using the same cohomological definition. Much work has been done in understanding Bogomolov multipliers of
absolute Galois groups of function fields, particularly in relation to recent developments in anabelian geometry (see [7]). One of the spectacular achievements of the latter is that (see [6], Theorem 13.2) abelian subgroups of absolute Galois groups \( G = \text{Gal}(K^s/K) \) are determined already by considering the Galois group only up to the second term of its lower central series, \( G^c = G/[[G,G],G] \). Passing to a maximal pro-\( p \) quotient \( G_p \) of \( G \), we find the \( p \)-part of the Bogomolov multiplier as

\[
B_0(G)_p = B_0(G_p) = B_0(G^c_p) = \lim_{N \geq 2, G^c_p} B_0(G^c_p/N).
\]

Understanding these Bogomolov multipliers is therefore reduced to considering either pro-\( p \) groups of nilpotency class 2, or equivalently their finite quotients.

1.6. Contributions and reader’s guide. Motivated by the above, the aim of this paper is to investigate generic behaviour of Bogomolov multipliers of nilpotent pro-\( p \) groups of sufficiently small nilpotency class, especially those of nilpotency class 2, as well as their finite quotients. In doing so, we are able to extract the following rather surprising result (Theorem 5.8.2).

**Theorem** \((n \to \infty)\). Fix a prime \( p > 2 \) and a positive integer \( M \). Let \( \#_{\text{all}}(n) \) be the number of all \( p \)-groups of order \( p^n \), and let \( \#_{B_0 \geq M}(n) \) be the number of those \( p \)-groups of order \( p^n \) whose Bogomolov multiplier is of order at least \( M \). Then

\[
\lim_{n \to \infty} \frac{\log_p \#_{B_0 \geq M}(n)}{\log_p \#_{\text{all}}(n)} = 1.
\]

**Corollary.** Quotient varieties of faithful representations of log-generic \( p \)-groups are not stably rational.

This is quite in contrast with thinking that groups possessing the Bogomolov 2-cocycles are rather special. We now explain our technique together with side applications.

In Section 2 we first dualize the Bogomolov multiplier \( B_0(G) \) into its homological version (Subsection 2.3). This dualized version appears naturally in non-commutative Iwasawa theory as \( SK_1(\mathbb{Z}_p[G]) \) (Subsection 2.1). We also give some remarks on general structural properties of this object (Subsection 2.5).

In Section 3 we restrict to nilpotent pro-\( p \) groups with a special regard to Lie groups over the \( p \)-adic field \( \mathbb{Q}_p \). We transport the dualized object from the previous section to the category of Lie algebras (Subsections 3.3 and 3.7) by exploiting Hopf formulae for Schur multipliers (Subsection 3.4). We then restrict to objects of nilpotency class at most 2 and no torsion (Subsection 3.9). We show how to parametrize such objects by using the Grassmannian variety \( \text{Gr}(r, (d^2)) \) as the moduli space (Subsection 3.10). To every point \( L \in \text{Gr}(r, (d^2)) \) we associate a Lie algebra \( \mathcal{L} \) and then a group \( G_L \) via the Lazard correspondence. Here, \( d \) and \( r \) are the natural parameters encoding the number of generators and essential relators of the objects. Using the above, we establish a Lie criterion over \( \mathbb{Q}_p \) for finiteness of the Bogomolov multiplier (Subsection 3.11).
In Section 4 we express the general Bogomolov multiplier $B_0(G_L)$ in terms of the image of some rational map that we call the decomposability map (Subsection 4.1). It is defined on some quasiprojective variety over $\mathbb{Q}_p$ and maps into the moduli space. This map is the heart of this paper. We find a subvariety of the moduli space of small codimension (more precisely, the ratio of its codimension and the dimension of the moduli space is asymptotically zero) that misses the image of this map and thus consists entirely of groups with large Bogomolov multipliers (Subsection 4.3). We study the generic behaviour by analysing the local behaviour of this map using methods of differential geometry (Subsection 4.4), and we give a precise criterion on when the tangent map is generically injective or surjective in terms of $r$ and $d$ (Subsections 4.5 and 4.6). The crucial step here is analysing the case $r = \left(\frac{d-2}{2}\right) + 1$, when the decomposability map turns out to be a local isomorphism.

In Section 5 we finally go back to the original motivation and transfer results from the analytic world to their finite quotients of exponent $p$ (Subsections 5.1 and 5.3). Groups with few generators are dealt with separately, where we can exploit the presence of an action of a general linear group due to the dimensions being sufficiently small (Subsection 5.5). Some of the general phenomena are visible in these calculations, particularly when the decomposability map is a local isomorphism. In order to deal with general groups, we expand the map describing the general Bogomolov multiplier into a scheme map (Subsection 5.7). The value of the parameters $r$ and $d$ that is most relevant for our understanding of generic $p$-groups corresponds to the situation when the $p$-adic base change of the decomposability map is a local surjection. We use some arithmetic geometry to then conclude that in this case, the proportion of elements in the image of the relevant map over the finite field $\mathbb{F}_p$ is bounded away from 0 as $p$ tends to infinity (Theorem 5.7.2).

**Theorem** ($p \to \infty$). Fix $r$ and $d$ with $r \geq \left(\frac{d-2}{2}\right) + 1$. Then

$$\liminf_{p \to \infty} \frac{\left|\{L \in \text{Gr}(r, \binom{d}{2}) \mid B_0(G_L) = 0\}\right|}{\left|\text{Gr}(r, \binom{d}{2})\right|} \geq \left(\frac{1}{Cd-2}\right)^r,$$

where $C_{d-2}$ is the Catalan number.

Thus there are many groups with trivial Bogomolov multipliers. The first issue with the above result is that it only applies to large enough primes $p$, while we are interested in finite $p$-groups for a fixed $p$. The second issue is that even if the above inequality would hold at every fixed $p$, it would still not enable us to provide numerically many finite $p$-groups $G_L$ with vanishing Bogomolov multipliers. The reason for this is that the constant on the RHS of the above theorem converges to zero too quickly. In fact, it comes as a surprise that just the opposite can be achieved (Subsection 5.8). By fixing $p$ and letting the parameters $r$ and $d$ suitably tend to infinity, the produced finite groups $G_L$ will be of order $p^n$ with $n$ depending only on $r$ and $d$. The parameters can be set in such a way that the number of $p$-groups produced is log-comparable to the number of all groups of order $p^n$, and we show that with increasing $n$, log-generic groups $G_L$ have non-trivial Bogomolov
multipliers (Theorem \((n \to \infty) / 5.8.2\)). This relies on transferring the small codimension subvariety mentioned above from the analytic world into finite fields. Whence log-generic \(p\)-groups produce quotient varieties that are very much not stably rational.

The paper concludes with an application of our technique to another problem involving commutators (Subsection 5.9). We show that in log-generic \(p\)-groups, not every element of the derived subgroup is a simple commutator.

Acknowledgements. The authors thank Oihana Garaialde Ocaña for conversations on the Lazard correspondence and many other nice remarks, Jon González Sánchez for providing the preprint [22], Immanuel Halupczok for sharing his knowledge of general algebraic geometry, Geoffrey Janssens for helpful thoughts on the exposition, and Daniel Loughran for pointing out the relevant version of the fibre dimension theorem and patiently responding to related questions.

2. Overture

This section is devoted to recognizing the dual of the Bogomolov multiplier within Iwasawa theory and collecting some known and unknown structural properties of this object. We will henceforth switch depending on the context between this homological interpretation and the original Bogomolov multiplier.

2.1. Iwasawa theory. Let \(G\) be a compact analytic pro-\(p\) group over \(\mathbb{Q}_p\), and assume \(p > 2\). Associated to \(G\) is its completed group ring

\[
\mathbb{Z}_p[G] = \lim_{\leftarrow N \leq G} \mathbb{Z}_p[G/N]
\]

over the \(p\)-adic integers \(\mathbb{Z}_p\), where the inverse limit runs over all open normal subgroups \(N\) of \(G\). Non-commutative Iwasawa theory deals with investigating this ring from the point of view of \(K\)-theory. More precisely, one is interested in understanding the abelian group \(K_1(\mathbb{Z}_p[G])\) in relation with constructing \(L\)-functions for \(p\)-adic representations (see [51]). The continuous representation theory of \(G\) is captured by the completed group algebra over the \(p\)-adic field

\[
\mathbb{Q}_p[G] = \lim_{\leftarrow N \leq G} \mathbb{Q}_p[G/N].
\]

This algebra decomposes into a product of matrix algebras over finite extensions of \(\mathbb{Q}_p\) (see [45, Section 3]), making it easy to understand its \(K_1\)-group. There is a natural scalar extension map

\[
K_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{Q}_p[G]).
\]

The deficit of this map being injective is measured by its kernel \(\text{SK}_1(\mathbb{Z}_p[G])\). This object can be represented as a limit

\[
\text{SK}_1(\mathbb{Z}_p[G]) = \lim_{\leftarrow N \leq G} \text{SK}_1(\mathbb{Z}_p[G/N]).
\]

It follows from [36, Theorem 3] that the groups \(\text{SK}_1(\mathbb{Z}_p[G/N])\) are finite \(p\)-groups, and so \(\text{SK}_1(\mathbb{Z}_p[G])\) is an abelian pro-\(p\) group.
2.2. Dualizing into the Bogomolov multiplier. There is a homological way of interpreting the finite \( SK_1 \)'s, studied in detail by Oliver via the \( p \)-adic logarithm (see [36]). This description can be dualized (see [34]) and extended to continuous Galois cohomology (see [44, Corollary 2.2]), yielding the following isomorphism:

\[
(2.1) \quad SK_1(\mathbb{Z}_p[G])^* \cong \bigcap_{A \leq G, [A,A]=1} \ker \left( H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\text{res}_G} H^2(A, \mathbb{Q}_p/\mathbb{Z}_p) \right).
\]

Here the dual is taken with respect to \( \mathbb{Q}_p/\mathbb{Z}_p \). Thus \( SK_1(\mathbb{Z}_p[G])^* \) is nothing but \( B_0(G) \).

2.3. Homological interpretation. We will work with the homological version of \( B_0(G) \). This can be obtained directly by dualizing (2.1). The abelian pro-\( p \) group \( SK_1(\mathbb{Z}_p[G]) \) coincides with its double dual (see [38, Theorem 2.9.6]). Thus we obtain an exact sequence

\[
\left( \prod_{A \leq G, [A,A]=1} H^2(A, \mathbb{Q}_p/\mathbb{Z}_p) \right)^* \xrightarrow{(\prod \text{res}_G^*)} H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^* \rightarrow SK_1(\mathbb{Z}_p[G]) \rightarrow 0.
\]

The Universal Coefficient Theorem (see [37, Section 7.6]) gives a natural isomorphism \( H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)^* \cong H_2(G, \mathbb{Z}_p) \), and therefore we have an exact sequence

\[
(2.2) \quad \bigoplus_{A \leq G, [A,A]=1} H_2(A, \mathbb{Z}_p) \xrightarrow{\Theta, \text{res}_G^*} H_2(G, \mathbb{Z}_p) \rightarrow SK_1(\mathbb{Z}_p[G]) \rightarrow 0.
\]

2.4. Exterior squares and commutator relations. As in the case of finite groups (see [34, Theorem 3.2]), the sequence (2.2) can be interpreted via the notion of non-abelian exterior squares (see [37, Section 8.6]). The description goes as follows. Let \( G \hat{\times} G \) be the profinite group topologically generated by symbols \( g \triangleleft h \) for \( g, h \in G \) subject to the universal relations

\[
xy \triangleleft z = (x^y \triangleleft z^y)(y \triangleleft z), \quad x \triangleleft yz = (x \triangleleft z)(x^z \triangleleft y^z), \quad x \triangleleft x = 1
\]

for all \( x, y, z \in G \), where \( x^y \) denotes the conjugate \( y^{-1}xy \). This group may be thought of as the universal model for the derived subgroup \([G,G]\). Correspondingly, there is a commutator map

\[
\kappa: G \hat{\times} G \rightarrow [G,G], \quad g \triangleleft h \mapsto [g,h].
\]

The deficit that the above relations between wedges are in fact all relations between commutators is measured by the kernel \( M(G) := \ker \kappa \). It follows from [37, Section 8.6.5] that there is a natural isomorphism \( M(G) \cong H_2(G, \mathbb{Z}_p) \). Moreover, corestrictions from abelian subgroups \( A \leq G \) correspond to natural maps \( A \hat{\times} A \rightarrow G \hat{\times} G \). Set \( M_0(G) := \langle g \triangleleft h \mid g, h \in G, [g,h] = 1 \rangle \). Equation (2.2) can therefore be interpreted as

\[
(2.3) \quad SK_1(\mathbb{Z}_p[G]) \cong \frac{M(G)}{M_0(G)},
\]

the group of relations between commutators in \( G \) modulo the universal relations and commuting pairs (see [29]).
2.5. **Bounding the rank and exponent.** Both for the purposes of Iwasawa theory and the rationality problem, it is of particular interest to know whether or not the object $SK_1(\mathbb{Z}_p[[G]])$ is trivial. Exploiting the fact that in our context, the group $G$ is of finite rank (this means that there is an upper bound on the minimal number of generators of its closed subgroups, see [13, Corollary 8.34]), we provide some general bounds on the structure of this object. Recall that such a group $G$ is equivalently described as a finitely generated pro-$p$ group that possesses a finite index subgroup $H$ that is uniform. The latter means that $H$ is finitely generated, torsion-free and powerful, i.e., $[H,H] \leq H^p$ (see [13]).

We first show that the rank of the Bogomolov multiplier can be bounded in terms of the rank of $G$.

**Proposition 2.5.1.** $SK_1(\mathbb{Z}_p[[G]])$ is a finitely generated abelian pro-$p$ group.

**Proof.** Let $r$ be the rank of the group $G$. All finite quotients $G/N$ as well as their subgroups can be generated by $r$ elements. Furthermore, every such finite quotient contains a powerful subgroup of index bounded by a function of $r$ independently of $N$ (see [13, Theorem 2.13]). The rank of the second integral homology of this powerful subgroup can be bounded in terms of $r$ (see [33, Corollary 2.2]), and therefore the same is true for its Bogomolov multiplier. Passing from this subgroup to $G$, the rank of the multiplier gets at worst multiplied by a constant depending only on its index and therefore only on $r$ (see [30, Proposition 6.1]). Thus the rank of the Bogomolov multiplier of the finite quotients $G/N$ is bounded by a function of $r$. It follows that the limit $SK_1(\mathbb{Z}_p[[G]])$ is topologically finitely generated (see [50, Proposition 4.2.1]). □

A similar argument proves that the exponents of Bogomolov multipliers of finite quotients of $p$-adic analytic groups can asymptotically be controlled.

**Proposition 2.5.2.** There exists a function $f(r)$ such that for every pro-$p$ $p$-adic analytic group $G$ of rank $r$, we have $\exp B_0(G/N) \leq f(r) \cdot \exp G/N$ for every $N \triangleleft G$.

**Proof.** As in the proof of Proposition 2.5.1, there is a powerful subgroup of $G/N$ of rank and index bounded in terms of $r$. The exponent of the Bogomolov multiplier of such a subgroup is bounded by the exponent of $G/N$ (see [33, Theorem 2.4]). It follows that the exponent of $B_0(G/N)$ can be bounded by a function of $r$ multiplied by $\exp G/N$ (see [30, Proposition 6.2]). □

It follows from the above proofs that when the uniform subgroup of $G$ is abelian, the group $SK_1(\mathbb{Z}_p[[G]])$ is of finite exponent bounded in terms of the rank of $G$, and so it is a finite $p$-group. This is the case, for example, when $G$ is a pro-$p$ group of finite coclass (see [13, Theorem 10.1] and [18, Theorem 4.4]).

3. **Via Lie Theory**

In this section we linearise the description of the Bogomolov multiplier to Lie algebras of uniform pro-$p$ groups. This description, however, is not entirely equivalent
in the categorical sense to the Bogomolov multiplier. In order to properly transfer homology, we will need to work with nilpotent pro-$p$ groups of bounded class. Relying on Hopf formulae for both groups and Lie algebras, it is possible to transfer the whole Schur multiplier and thus also the Bogomolov multiplier. For nilpotent uniform pro-$p$ groups, we obtain a simple Lie criterion for triviality of the Bogomolov multiplier, that does coincide with the naive notion introduced at the beginning of this section.

3.1. The Lie algebra and the commutator map. Let $G$ be a uniform pro-$p$ group. One can associate to it a Lie algebra $\mathcal{L} = \log G$ over $\mathbb{Z}_p$ (see [13, Chapter 9]). The Lie bracket $[,]_L$ determines a mapping

$$\kappa_L : \mathcal{L} \wedge \mathcal{L} \to \mathcal{L}, \quad x \wedge y \mapsto [x, y]_L.$$ 

This map may be thought of as a linearised version of (2.3).

3.2. Decomposable wedges and $\text{SK}_1$. One can linearise the description (2.4) of $\text{SK}_1(\mathbb{Z}_p[G])$. Let

$$D\mathcal{L} = \{x \wedge y | x, y \in \mathcal{L}\} \subseteq \mathcal{L} \wedge \mathcal{L}$$

denote the set of decomposable wedges in $\mathcal{L}$. It is clear that whenever $x, y \in \mathcal{L}$ commute, we have $x \wedge y \in D\mathcal{L} \cap \ker \kappa_L$.

It is shown in [44, Theorem 5.5] that $\text{SK}_1(\mathbb{Z}_p[G])$ is trivial if and only if $\ker \kappa_L$ can be generated by elements of $D\mathcal{L} \cap \ker \kappa_L$. This is done by passing to $\mathbb{F}_p$-coefficients in cohomology and using quite heavy calculations relating the structure of $G$ to that of $\mathcal{L}$. Thus the intuitive way of transferring the Bogomolov multiplier to Lie algebras makes sense. However, this only works with the triviality criterion. We will now transfer the whole object and thus provide a more conceptual clarification of this triviality criterion.

3.3. The Lazard correspondence. We can avoid the above mentioned calculations by restricting to the class of nilpotent pro-$p$ groups of small nilpotency class by using the Lazard correspondence (see [32]). This is an isomorphism of categories between the category of finitely generated nilpotent pro-$p$ groups of nilpotency class less than $p$ and finitely generated nilpotent Lie algebras over $\mathbb{Z}_p$ of nilpotency class less than $p$. To a Lie algebra $\mathcal{L}$, one associates the pro-$p$ group $\exp \mathcal{L}$, which coincides with $\mathcal{L}$ as a set and whose multiplication arises from the Baker-Campbell-Hausdorff formula

$$x \cdot y = \log(\exp(x) \cdot \exp(y))$$

for $x, y \in \mathcal{L}$. The logarithm and exponential here arise from formal power series, truncated from $p$ on due to the restriction on the nilpotency class. Conversely, to a pro-$p$ group $G$ one associates the Lie algebra $\log G$ which coincides with $G$ as a set and with

$$x + y = \exp(\log(x) + \log(y)), \quad [x, y] = \exp(\log(x), \log(y)).$$
for $x, y \in G$. The functors exp and log are inverse to each other and induce an isomorphism of the relevant categories. They transform subgroups to subalgebras, normal subgroups to ideals, commuting pairs to commuting pairs, centre to centre, generating sets to generating sets, etc. We remark that these functors are the truncated versions of functors that transform uniform groups to their Lie algebras and vice versa.

3.4. **Hopf formulae.** Our objective is to transport the functor $\text{SK}_{1}$ from the category of groups to that of Lie algebras. In light of (2.2), we will do this by first transporting the second homology. This relies on the following universal expression of the second homology in both group and Lie algebra categories.

**Lemma 3.4.1 (Hopf formulae).** Let $F/R$ be a free presentation of either a pro-$p$ group $G$ or a $\mathbb{Z}_p$-Lie algebra $L$. Then

$$H_2(F/R, \mathbb{Z}_p) \cong \frac{[F,F] \cap R}{[F,R]}.$$ 

**Proof.** This is classical for groups. See [37, Section 8.2.3] for pro-$p$ groups, the heart of the argument being the 5-term exact sequence from the LHS spectral sequence. The same works for Lie algebras, see also [17]. A universal explanation of this phenomenon is given in [3]. \[\square\]

3.5. **Universal central extensions.** The objects described by Hopf formulae can be interpreted as universal objects associated to the group or the Lie algebra in the following sense. Let $F/R$ be a free presentation of a finitely generated pro-$p$ group $G$ with $d(F) = d(G)$. Here $d(G)$ is the minimal number of topological generators of $G$. There exists a largest $d(G)$-generated central extension of $G$, and this is precisely the extension

$$1 \to \frac{R}{[F,R]} \to \frac{F}{[F,R]} \to G \to 1.$$ 

3.6. **Transporting homology.** Denote $E = F/[F,R]$ and let $p : E \to G$ be the projection. The group $E$ comes equipped with the additional projection $a : E \to E/[E,E]$. Combine the two projections $p, a$ into a product map

$$\frac{F}{[F,R]} = E \to G \times (E/[E,E]) = \frac{F}{R} \times \frac{F}{[F,F]}$$

and let $K$ be the kernel of this map. Note that $K = ([F,F] \cap R)/[F,R]$. Therefore the object $K \cong H_2(G, \mathbb{Z}_p)$ can be expressed in purely categorical terms. Note that since we are assuming $G$ to be finitely generated, all the relevant objects of the construction stay in the category of finitely generated nilpotent pro-$p$ groups. Moreover, if $G$ is of nilpotency class $c$, then $E$ is of nilpotency class at most $c + 1$. Using the Lazard isomorphism of categories, we can therefore fully transport the construction, starting from the universal central extension. In order for the correspondence to apply to all the relevant objects, we need to assume that $c+1 < p$. 
In particular, starting with a group of nilpotency class 2, this applies to all primes
\( p > 3 \). Finally we obtain the natural isomorphism
\[
\log H_2(G, \mathbb{Z}_p) \cong H_2(\mathcal{L}, \mathbb{Z}_p).
\]

Remark 3.6.1. Let \( G \) be a uniform pro-\( p \) group, not necessarily nilpotent. In this situation, one can still define a Lazard correspondence. The problem of making our construction work in this more general setting is that the additional object \( E \) that appears in the construction might not be uniform and it is not clear what its associated Lie algebra should be. For a unified explanation of why the above homological transport works, one would therefore need a more general Lazard correspondence. See [22, Section 9.4].

Remark 3.6.2. The above argument of transporting homology does not work for finite \( p \)-groups, since the object \( E \) falls out of the relevant category. It is nevertheless possible to obtain the same result for the case of finite \( p \)-groups, either using Schur’s theory of covers (see [15]) or more explicitly via the usual interpretations of low-dimensional cohomology (see [20]).

3.7. Transporting \( SK_1 \). Given an abelian subgroup \( A \leq G \), there is a corestriction map \( \text{cores}_A^G : H_2(A, \mathbb{Z}_p) \to H_2(G, \mathbb{Z}_p) \). It follows from the above that there is an induced corestriction map \( \log(\text{cores}_A^G) : H_2(\log A, \mathbb{Z}_p) \to H_2(\log G, \mathbb{Z}_p) \). It follows from (2.2) that

\[
SK_1(\mathbb{Z}_p[G]) \cong \frac{H_2(G, \mathbb{Z}_p)}{\langle \text{im cores}_A^G \ | \ A \leq G, [A, A] = 1 \rangle},
\]

and the latter can be transferred with \( \log \) to
\[
\log(SK_1(\mathbb{Z}_p[G])) \cong \frac{H_2(\mathcal{L}, \mathbb{Z}_p)}{\langle \text{im log(cores}_A^G) \ | \ A \leq G, [A, A] = 1 \rangle}.
\]

Abelian subgroups of \( G \) correspond precisely to abelian Lie subalgebras of \( \mathcal{L} \), and so we obtain
\[
\log(SK_1(\mathbb{Z}_p[G])) \cong \frac{H_2(\mathcal{L}, \mathbb{Z}_p)}{\langle \text{im cores}_A^\mathcal{L} \ | \ A \leq \mathcal{L}, [A, A] = 1 \rangle}.
\]

Following [22, Remark 9.19], there is a direct way of computing \( H_2(\mathcal{L}, \mathbb{Z}_p) \) via the standard complex (see [25, Theorem 4.2]). To do this, consider the Jacobi map
\[
\theta_\mathcal{L} : \mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}, \quad x \wedge y \wedge z \mapsto [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y.
\]

We have a natural isomorphism
\[
H_2(\mathcal{L}, \mathbb{Z}_p) \cong \frac{\ker \kappa_\mathcal{L}}{\text{im} \theta_\mathcal{L}}.
\]

For an abelian Lie subalgebra \( A \leq \mathcal{L} \), we therefore have \( H_2(A, \mathbb{Z}_p) \cong \ker \kappa_A = A \wedge A \). We thus obtain a natural isomorphism
\[
\log(SK_1(\mathbb{Z}_p[G])) \cong \frac{\ker \kappa_\mathcal{L}}{\text{im} \theta_\mathcal{L} + \langle D\mathcal{L} \cap \ker \kappa_\mathcal{L} \rangle}
\]

and we correspondingly denote the left hand side by \( SK_1(\mathcal{L}) \). We have therefore transferred \( SK_1 \) into the category of nilpotent Lie algebras over \( \mathbb{Z}_p \) of nilpotency class less than \( p - 2 \).
Remark 3.7.1. The object $SK_1(L)$ as we have defined it makes sense for any Lie algebra $L$. In fact, it coincides with the quotient (in the sense of (2.4)) of the Lie bracket induced map $\ker(L \wedge L \to L)$, where $\wedge$ is the non-abelian exterior square of the Lie algebra $L$ (see [17]). This object has been explored independently in [41] (see also [40] and [9]).

3.8. Triviality of $SK_1$. We show how the above can be utilized in order to give a quick argument that triviality of $SK_1(Z_p[[G]]$ is precisely detected by decomposability of wedges (see [44, Theorem 5.5]). We are assuming that $G$ is both a uniform pro-$p$ group and that $G$ is nilpotent of nilpotency class at most $p-2$.

Note that since $L$ is a powerful Lie algebra, we have $[L,L] \subseteq pL$, and therefore $\text{im } \theta_L \subseteq pL \wedge L \cap \ker \kappa_L$. The algebra $[L,L] \cong L \wedge L / \ker \kappa_L$ is (additively) torsion-free, and so $\text{im } \theta_L \subseteq p \ker \kappa_L$. Note that $SK_1(L)$ is trivial if and only $\ker \kappa_L = \text{im } \theta_L + \langle DL \cap \ker \kappa_L \rangle$. Since $\text{im } \theta_L$ is contained in the Frattini subgroup of $\ker \kappa_L$, we have the following.

Corollary 3.8.1. Let $G$ be a uniform pro-$p$ group of nilpotency class at most $p-2$. Then $SK_1(Z_p[[G]] = 0$ if and only if $\ker \kappa_L = \langle DL \cap \ker \kappa_L \rangle$.

3.9. Nilpotency class 2. We now specialize all of the above to groups and algebras of nilpotency class 2. Thus we assume throughout that $p > 3$ and we restrict to the category of pro-$p$ groups that are finitely generated and nilpotent of class at most 2. We will additionally assume that our groups and algebras are torsion-free in order to pass from $Z_p$ to $Q_p$, and we will also assume that their abelianizations are torsion-free in order to have a uniform description. On the other hand, our construction will not need the more powerful restriction that the groups and algebras are powerful. Our final objective will be to analyse the behaviour of $SK_1$ in the generic case. More precisely, we want to consider the Bogomolov multiplier of a random pro-$p$ group from the above category. The main reason why we restrict to objects of nilpotency class 2 is that these can be parametrized in a natural way as we explain below, and this makes it possible to talk about random objects.

3.10. Parametrizing Lie algebras of nilpotency class 2. Let $V$ be a finitely-generated free $Z_p$-module. Consider $W = V \wedge V$ and let $L$ be a $Z_p$-submodule of $W$. The natural projection map $\pi: W \to W/L$ can be thought of as determining the Lie product on the $Z_p$-module $L = V \oplus W/L$,

$$[v_1 + w_1, v_2 + w_2]_L = \pi(v_1 \wedge v_2).$$

Thus we obtain a Lie algebra of nilpotency class at most 2. Conversely, every Lie algebra $L$ of nilpotency class 2 with $L/[L,L]$ torsion-free is obtained in this way, where we take $L = \ker(L \wedge L \to [L,L])$ to be the kernel of the Lie bracket map.

We will require the algebra $L$ to be torsion-free. This occurs precisely when $W/L$ has no torsion. We will call such $Z_p$-submodules $L \leq W$ co-torsion-free. The following lemma shows that such submodules naturally correspond to $Q_p$-subspaces of $W \otimes Q_p$. We will prefer to work in this setting.
Lemma 3.10.1. There is a bijection between \( r \)-dimensional vector subspaces of \( \mathbb{Q}_p^k \) and co-torsion-free \( \mathbb{Z}_p \)-submodules of \( \mathbb{Z}_p^k \) of rank \( r \). The bijection is given as

\[
\mathbb{Q}_p^k \ni L \mapsto L \cap \mathbb{Z}_p^k, \quad \mathbb{Z}_p^k \ni L \mapsto L \otimes \mathbb{Q}_p.
\]

Proof. Let us first verify that the map is well-defined. Given \( L \subseteq \mathbb{Q}_p^k \), the \( \mathbb{Z}_p \)-module \( L \cap \mathbb{Z}_p^k \) is a torsion-free abelian pro-\( p \) group. Its rank equals the rank of \( (L \cap \mathbb{Z}_p^k) \otimes \mathbb{Q}_p \). Since there is an \( n \geq 0 \) such that \( p^n L \subset \mathbb{Z}_p^k \), the latter rank is equal to the dimension of \( L \), that is \( r \). Let us now verify that \( \mathbb{Z}_p^k/(L \cap \mathbb{Z}_p^k) \) is torsion-free. Were there an element \( \ell \in \mathbb{Z}_p^k \) with \( p\ell \in L \cap \mathbb{Z}_p^k \), then \( p\ell \in L \) and so \( \ell \in L \), and so \( \ell \in L \cap \mathbb{Z}_p^k \).

We now check that the above map is a bijection. To this end, we only need to verify that for a co-torsion-free \( \mathbb{Z}_p \)-module \( L \leq \mathbb{Z}_p^k \), we have \( (L \otimes \mathbb{Q}_p) \cap \mathbb{Z}_p^k = L \). Let \( \ell = \sum_i \alpha_i \ell_i \in (L \otimes \mathbb{Q}_p) \cap \mathbb{Z}_p^k \) with \( \alpha_i \in \mathbb{Q}_p \) and \( \ell_i \in L \). Therefore there is an \( n \geq 0 \) such that \( p^n \alpha_i \in \mathbb{Z}_p \) for all \( i \), and so \( p^n \ell \in \sum_i \mathbb{Z}_p \ell_i \subseteq L \). It follows from the assumption on \( L \) that \( \ell \in L \).

Once such a subspace \( L \) is chosen, we can construct the corresponding Lie algebra \( \mathcal{L} \). In this setting, it is also easy to produce uniform algebras or groups. This is achieved by simply replacing an algebra \( \mathcal{L} \) as above with \( p\mathcal{L} \).

All in all, we have designed a map that takes as input a subspace \( \mathcal{L} \) of \( \mathbf{Gr}(r, W) \) and constructs a Lie algebra of nilpotency class 2 that are torsion-free and have torsion-free abelianizations:

\[
\Lambda : \mathbf{Gr}(r, W) \to \text{N}_2\text{Lie}, \quad L \mapsto \mathcal{L}.
\]

To every such a Lie algebra \( \mathcal{L} \) we associate via the Lazard correspondence a group that we denote \( \exp \mathcal{L} = G_L \). The map \( \Lambda \) is therefore also parametrizing the relevant groups.

3.11. \textbf{SK}_1 \textbf{ of the Lie algebra.} Given a subspace \( L \subseteq W \), we determine the \( \text{SK}_1 \) of the Lie algebra \( \mathcal{L} = \Lambda(L) \) using (3.1). Note that in this case, \( \text{im} \theta_{\mathcal{L}} \subseteq \langle D\mathcal{L} \cap \ker \kappa_{\mathcal{L}} \rangle \), and we therefore have

\[
\text{SK}_1(\mathcal{L}) = \frac{L}{\langle D\mathcal{L} \cap L \rangle}.
\]

Nothing changes if we replace the Lie algebra by the powerful one \( p\mathcal{L} \),

\[
\text{SK}_1(p\mathcal{L}) = \frac{\ker \kappa_{p\mathcal{L}}}{\langle (D(p\mathcal{L}) \cap \ker \kappa_{p\mathcal{L}}) \rangle} = \frac{p^2 \ker \kappa_{\mathcal{L}}}{\langle p^2 (D\mathcal{L} \cap \ker \kappa_{\mathcal{L}}) \rangle} \cong \frac{L}{\langle D\mathcal{L} \cap L \rangle}.
\]

Using the correspondence from Lemma 3.10.1, we can return from submodules to subspaces and consider the Lie algebra \( \mathcal{L} \otimes \mathbb{Q}_p \) with its own Bogomolov multiplier

\[
\text{SK}_1(\mathcal{L} \otimes \mathbb{Q}_p) = \frac{L \otimes \mathbb{Q}_p}{\langle (\mathcal{L} \otimes \mathbb{Q}_p) \cap (L \otimes \mathbb{Q}_p) \rangle}.
\]

The following lemma establishes the connection between the two objects.
Lemma 3.11.1. $SK_1(\mathcal{L})$ is finite if and only if $SK_1(\mathcal{L} \otimes \mathbb{Q}_p) = 0$.

Proof. Suppose first that $SK_1(\mathcal{L})$ is finite. Thus there exists an $n \geq 0$ such that for every element $\ell$ of the $\mathbb{Z}_p$-basis of $L$, we have $p^n \ell \in (D(\mathcal{L}) \cap L)$. It follows that $\ell \in (D(\mathcal{L} \otimes \mathbb{Q}_p) \cap (L \otimes \mathbb{Q}_p))$. This implies that $L \subseteq (D(\mathcal{L} \otimes \mathbb{Q}_p) \cap (L \otimes \mathbb{Q}_p))$, and so $SK_1(\mathcal{L} \otimes \mathbb{Q}_p) = 0$.

Conversely, assume that $SK_1(\mathcal{L} \otimes \mathbb{Q}_p) = 0$. Then every element $\ell$ of the $\mathbb{Z}_p$-basis of $L$ belongs to $(D(\mathcal{L}) \cap L)$ for all basis elements of $L$, and therefore $p^n L \subseteq (D(\mathcal{L}) \cap L)$. This means that $SK_1(\mathcal{L})$ is a finitely generated torsion abelian pro-$p$ group, and so finite. □

Here is a neat example showing how torsion can indeed occur.

Example 3.11.2. Consider the case when $V$ is the free $\mathbb{Z}_p$-module of dimension 4 generated by $\{e_1, e_2, e_3, e_4\}$. Let $L$ be the $\mathbb{Z}_p$ submodule

$$L = \langle e_1 \wedge e_2, e_3 \wedge e_2 + e_1 \wedge e_4 + p e_3 \wedge e_4 \rangle \leq W.$$  

Note that $L \otimes \mathbb{Q}_p$ is generated by the vectors $e_1 \wedge e_2$ and $(e_1 + pe_3) \wedge (e_2 + pe_4)$, and so $SK_1(\mathcal{L} \otimes \mathbb{Q}_p) = 0$. The same two vectors do not generate $L$. In fact, the only decomposable wedges in $L$ are $\mathbb{Z}_p$-multiples of $e_1 \wedge e_2$ and $(e_1 + pe_3) \wedge (e_2 + pe_4)$.

Therefore $SK_1(\mathcal{L}) \cong C_p$.

Relying on Lemma 3.11.1, it is therefore possible to completely linearise the question of whether or not $SK_1(\mathbb{Z}_p[[G]])$ is finite, i.e., we are reduced to analysing whether or not $SK_1(\mathcal{A}(L) \otimes \mathbb{Q}_p)$ is trivial for a generic subspace $L \subseteq V \wedge V$, and this is in turn a question about decomposability in the exterior algebra $V \wedge V$ over $\mathbb{Q}_p$.

We emphasize that we are dealing with nilpotent algebras. This complements the existing literature where it is shown that kernels of commutator maps for semisimple Lie algebras are always generated by decomposable elements, both over $\mathbb{Q}_p$ (see [44, Theorem 3.1]) and $\mathbb{C}$ (see [31, Corollary 5.1]).

4. The decomposability map over $\mathbb{Q}_p$

Our aim in this section is to analyse the behaviour of $SK_1$ in the generic case. Based on the previous chapter, this boils down to understanding the points in the Grassmannian $\text{Gr}(r, W)$ that can be generated by decomposable elements $D\mathcal{L} \cap \text{ker}_L$. We therefore establish a decomposability map whose image are precisely the points for which the Bogomolov multiplier of the objects associated to it under our parametrization vanishes. We study the local properties of this map and give a precise description of its differential. The bulk of the work is devoted to finding points in which the differential is a local isomorphism under a certain restriction on the parameters of the construction.
4.1. The decomposability map. Let $V$ be a vector space over $\mathbb{Q}_p$ of dimension $d$. Fix some $1 \leq r \leq d$. The variety $\text{Gr}(r, W)$ consists of vector subspaces of $W = V \wedge V$ of dimension $r$. The subspaces that can be generated by decomposable wedges have a basis consisting of $r$ decomposable wedges. The set of decomposable wedges in $W$ itself forms a variety $\text{Gr}(2, V)$. We therefore have a rational map of projective varieties

$$\Psi : \text{Gr}(2, V)^r \rightarrow \text{Gr}(r, W), \quad (L_1, \ldots, L_r) \mapsto \bigwedge^2 L_1 \oplus \cdots \oplus \bigwedge^2 L_r.$$  

Note that $\Psi$ is not regular everywhere, since the spaces $L_i$ might overlap. It is, however, defined on a Zariski dense open subset. The subspaces of $W$ that can be generated by decomposable wedges, i.e., those whose corresponding Lie algebra has a trivial SK$_1$, are precisely the elements of $\text{im} \, \Psi$.

The map (4.1) can be expressed in terms of Plücker coordinates on the Grassmannians (see [43]), giving a more explicit map

$$\Psi : \text{Gr}(2, V)^r \rightarrow \text{Gr}(r, W), \quad (x_1 \wedge y_1, \ldots, x_r \wedge y_r) \mapsto \langle x_1 \wedge y_1, \ldots, x_r \wedge y_r \rangle$$

with

$$\text{Gr}(2, V) \subseteq \mathbb{P}^{\binom{d}{2} - 1}, \quad \text{Gr}(r, W) \subseteq \mathbb{P}^{\binom{r}{2} - 1}.$$  

We call $\Psi$ the decomposability map. Observe that the dimension of the domain of $\Psi$ is equal to $r \cdot 2(d - 2)$, while the dimension of the codomain of $\Psi$ is equal to $r\binom{d}{2} - r$. The difference between the two is

$$\dim(\text{codom} \, \Psi) - \dim(\text{dom} \, \Psi) = r \left( \binom{d - 2}{2} + 1 - r \right).$$  

4.2. Few relators. In order to understand the map $\Psi$ better, we first deal with the case when the dimension $r$ of the selected subspace is not too close to the full dimension $\binom{d}{2}$.

**Lemma 4.2.1.** Suppose $r \leq \binom{d - 2}{2}$. Then $\text{im} \, \Psi$ is contained in a proper subvariety of $\text{Gr}(r, W)$.

**Proof.** The inequality in the statement is equivalent to saying that the dimension of the domain of $\Psi$ is strictly smaller than the dimension of its codomain.

Therefore a generic subspace of a small dimension $r$ produces a Lie algebra whose SK$_1$ is not trivial, and therefore a group whose SK$_1$ is not finite. In other words, fixing the dimension $r$ and letting the number of generators $d$ tend to infinity, a generic group will have an infinite SK$_1$.

4.3. Plenty of non-decomposable subspaces. In general, the image $\text{im} \, \Psi$ will not be contained in a proper subvariety of $\text{Gr}(r, W)$. However, we can (almost) always find an asymptotically large subset of $\text{Gr}(r, W)$ that does not intersect $\text{im} \, \Psi$. 

Proposition 4.3.1. Suppose that $\rho = \binom{d}{2} - r \geq 5$. Then there is a subvariety $Q_r \subseteq \text{Gr}(r, W)$ such that

$$Q_r \cap \text{im} \Psi = \emptyset \quad \text{and} \quad \lim_{\rho \to \infty} \frac{\dim Q_r}{\dim \text{Gr}(r, W)} = 1.$$  

Proof. Let $\{v_i \mid 1 \leq i \leq d\}$ be a basis of $V$ and let $\tilde{V} = \langle v_1, v_2, v_3, v_4 \rangle$. Set $X = \langle x \rangle \leq \bar{V} \wedge \tilde{V}$ to be a 1-dimensional subspace that is not generated by decomposable wedges, for example $x = v_1 \wedge v_2 + v_3 \wedge v_4$. Now let $U \leq W$ be the standard complement of $\tilde{V} \wedge \tilde{V}$, so that $v_k \wedge v_l \in U$ as long as not both $v_k, v_l$ belong to $\tilde{V}$. Thus $W = (\tilde{V} \wedge \tilde{V}) \oplus U$.

Fix a subspace $L = \langle \ell_1, \ldots, \ell_{r-1} \rangle \leq U$ of dimension $r - 1$ and its complement $K$, so that $U = L \oplus K$. Now select a set of $r$ vectors $k_0, \ldots, k_{r-1}$ in $K$ and associate to it the vector subspace

$$S = \langle x + k_0, \ell_1 + k_1, \ldots, \ell_{r-1} + k_{r-1} \rangle \in \text{Gr}(r, W).$$

Note that $S + U = X \oplus U$, which is not generated by decomposable wedges. Whence $S$ has the same property.

It follows that every choice of vectors as above produces a subspace of $W$ of codimension $\rho$ that is not generated by decomposable wedges. Moreover, these subspaces form a quasiprojective subvariety of the Grassmannian, call it $Q_r$. This subvariety is contained in the open subset given by the non-vanishing of the $r \times r$ minor associated to $(x, \ell_1, \ldots, \ell_r)$. Inside this open subset, $Q_r$ is determined by the vanishing of the minors associated to selecting a basis vector of a complement of $X$ in $\tilde{V} \wedge \tilde{V}$ and $r - 1$ vectors in $(x, \ell_1, \ldots, \ell_r)$.

Counting dimensions, we have $\dim Q_r = r(\binom{d}{2} - 6 - (r - 1))$ and so

$$\frac{\dim Q_r}{\dim \text{Gr}(r, W)} = 1 - \frac{5}{\rho},$$

whence the proposition. \qed

4.4. The differential of decomposability. We now proceed with our analysis of $\Psi$. Let $L = (L_1, \ldots, L_r)$ be a point in the domain of $\Psi$. We are interested in the behaviour of $\Psi$ around $L$, so we will determine its differential

$$d\Psi_L : T_L \text{Gr}(2, V)^\prime \to T_{\Psi(L)} \text{Gr}(r, W).$$

Tangent spaces of Grassmannians. Tangent vectors $T_L \text{Gr}(2, V)$ can be identified with linear morphisms $\text{hom}(L_i, V/L_i)$ in the following way (see [48]). Let $A \in \text{hom}(L_i, V)$ be a linear map. Consider its associated curve

$$\gamma : Q_p \to \text{Gr}(2, V), \quad t \mapsto \text{im}(id_{L_i} + tA)$$

In coordinates, this can be expressed as follows. Suppose $L_i = \langle x_i, y_i \rangle$. Then

$$\text{im}(id_{L_i} + tA) = \langle x_i + tAx_i, y_i + tAy_i \rangle.$$  

Correspondingly, we obtain the tangent vector $[\gamma] \in T_L \text{Gr}(2, V)$. This is the vector associated to the linear map $A$. Replacing $A$ by a map $B \in \text{hom}(L_i, V)$ gives the
same tangent vector if and only if \( \text{im}(A - B) \leq L_i \). This explains the identification
\[
T_{L_i} \text{Gr}(2, V) \equiv \text{hom}(L_i, V/L_i),
\]
and the same reasoning gives the identification on the right hand side,
\[
T_{\Psi(L)} \text{Gr}(r, W) \equiv \text{hom}(\Psi(L), W/\Psi(L)).
\]
Both of these will henceforth be used.

The matrix of the differential. The differential \( d\Psi_L \) is completely determined by its values on each of the components \( T_{L_i} \text{Gr}(2, V) \) of its domain. Let us therefore fix a linear map \( A \in \text{hom}(L_i, V) \). Its associated curve \( \gamma \) induces a curve in \( \text{Gr}(2, V)^r \) by means of
\[
\tilde{\gamma}: t \mapsto (L_1, \ldots, L_{i-1}, \gamma(t), L_{i+1}, \ldots, L_r) \in \text{Gr}(2, V)^r.
\]
In order to determine \( d\Psi_L[\tilde{\gamma}] \), we first map the curve \( \tilde{\gamma} \) with \( \Psi \) into
\[
\Psi \circ \tilde{\gamma}: \mathbb{Q}_p \to \text{Gr}(r, W), \quad t \mapsto \wedge^2 \gamma(t) \oplus \bigoplus_{j \neq i} \wedge^2 L_j.
\]
The volume form \( \gamma(t) \wedge \gamma(t) \) can be expressed in terms of coordinates as
\[
(x_i + tAx_i) \wedge (y_i + tAy_i) = x_i \wedge y_i + t(Ax_i \wedge y_i + x_i \wedge Ay_i) + t^2 Ax_i \wedge Ay_i.
\]
The tangent vector \( [\Psi \circ \tilde{\gamma}] \) can therefore be represented by the linear map \( A \in \text{hom}(\Psi(L), W/\Psi(L)) \) given as
\[
A|_{L_i \wedge L_i} = (x_i \wedge y_i \mapsto Ax_i \wedge y_i + x_i \wedge Ay_i),
\]
\[
A|_{L_j \wedge L_i} = 0 \text{ for } j \neq i.
\]
In this notation, we therefore have \( d\Psi_L(A) = A \).

4.5. Submersiveness of decomposability. Recall from Lemma 4.2.1 that as long as \( r \leq \left( \frac{d-2}{2} \right) \), the image \( \text{im} \Psi \) is contained in a proper subvariety. We will now show that for all other values of \( r \), \( \Psi \) is a local surjection.

Theorem 4.5.1. The map \( \Psi \) is a submersion on a dense open subset of its domain if and only if \( r \geq \left( \frac{d-2}{2} \right) + 1 \).

The bound value is particularly interesting. It will follow from our proof that when \( r = \left( \frac{d-2}{2} \right) + 1 \), the decomposability map \( \Psi \) is a local isomorphism.

First of all, it is clear that the points in which \( \Psi \) is a submersion form a Zariski open subset of the domain of \( \Psi \), since they correspond to points in which the differential is of full rank (see [46, Section III.10]). In order to show that the principal minors of the differential are not all identically zero, we will provide explicit points in which the differential is indeed of full rank, and so the set of points in which \( \Psi \) is a submersion is indeed non-empty and dense. This will be done by a process of understanding more precisely the condition for \( \Psi \) to be a submersion.

Remark 4.5.2. It will follow from the proof that the same result holds in any complete field of characteristic zero, in particular \( \mathbb{C} \).
Affine charting the Grassmannian. We will use the following chart on the Grassmannian $\text{Gr}(2, V)$. Suppose $K \leq V$ is a subspace of codimension 2. Choose a complement $L \leq V$ so that $V = K \oplus L$. Any other complement can be obtained as the image of the map

$$\varphi_K: \text{hom}(L, K) \to \text{Gr}(2, V), \quad f \mapsto \text{im}(\text{id}_L + f).$$

This map is an immersion of the affine space $\text{hom}(L, K) \cong \mathbb{Q}_p^{2(d-2)}$ into $\text{Gr}(2, V)$ with a Zariski open image. We will denote the image of this open chart by $U_K$. Note that

$$U_K = \{L' \in \text{Gr}(2, V) \mid L' \cap K = 0\},$$

which is independent of the choice of the initial complement $L$.

Criterion for submersiveness. We now develop a criterion on when $\Psi$ is a submersion at a point. This relies on knowing the differential of $\Psi$. Based on Subsection 4.4, we have

$$\text{im } d\Psi = \{A \in \text{hom}(\Psi(L), W/\Psi(L)) \mid \forall i. \text{ im } A|_{L_i \wedge L_i} \subseteq \Psi(L) + (L_i \wedge V)\}.$$

Whence the condition for $\Psi$ to be a submersion is equivalent to

$$\forall i. \quad \Psi(L) + (L_i \wedge V) = W.$$  

Consider the natural projections $\pi_i: V \to V/L_i$. These induce maps

$$\bigwedge^2 \pi_i: W \to \bigwedge^2(V/L_i),$$

using which conditions (4.2) can be restated as

$$\forall i. \quad \left(\bigwedge^2 \pi_i\right)(\Psi(L)) = \bigwedge^2(V/L_i).$$

This is the condition that we will use in what follows. Note that since

$$\dim \Psi(L) \leq r \quad \text{and} \quad L_i \wedge L_i \subseteq \Psi(L) \cap \ker \left(\bigwedge^2 \pi_i\right),$$

a necessary condition for (4.3) is that

$$r - 1 \geq \binom{\dim(V/L_i)}{2} = \binom{d-2}{2}.$$  

Condition for submersiveness in a chart. We will now develop the condition (4.3) further by restricting the subspaces $L_i$ to belong to an open subset $U_K \subseteq \text{Gr}(2, V)$ for some fixed subspace $K \leq V$ of codimension 2. Select a complement $L$ of $K$.

Each of the spaces $L_i$ therefore arises uniquely from a linear map $f_i: L \to K$ via the affine chart $\varphi_K: \text{hom}(L, K) \to \text{Gr}(2, V)$. We will take $f_1$ to be the zero map, so that $L_1$ is just the distinguished complement $L$.

The advantage of restricting to $U_K$ is that we can replace the quotient $V/L_i$ by the concrete model $K$ in each of the natural projections $\pi_i: V \to V/L_i$. This corresponds to considering the alternative projections

$$\tilde{\pi}_i = f_i - \text{id}_K: L \oplus K = V \to K$$

with kernels $\text{im}(\text{id}_L + f_i) = L_i$. In these terms, we can rephrase the condition (4.3) as

$$\forall i. \quad \left(\bigwedge^2 \tilde{\pi}_i\right)(\Psi(L)) = K \wedge K.$$
In order to determine the left hand side, we capitalize on the fact that the subspaces \( L_i \) are determined by the maps \( f_i \). For a subspace \( L_j \in U_K \), we have
\[
\tilde{\pi}_l (L_j) = \text{im}(f_j - id_K) \circ (id_L + f_j) = \text{im}(f_j - f_i) \subseteq K.
\]
Condition (4.4) is therefore saying that the volume forms of \( \text{im}(f_j - f_i) \) should generate \( K \wedge K \) for any fixed \( i \),
\[
\forall i. \quad K \wedge K = \sum_{j=1}^r \wedge^2 \text{im}(f_j - f_i).
\]

Coordinates and the canonical choice. We will now focus on the particular case when \( r \) is equal to the bound value \( \binom{d-2}{2} + 1 \), and resolve the submersiveness condition in this case. Our objective is to find examples of tuples \( \mathbf{L} \) that satisfy the condition and this can be achieved, as shown below, by a fairly standard choice of subspaces \( L_i \).

We begin by introducing coordinates in the above construction. Let \( e_1, \ldots, e_d \) be a basis of \( V \), and take
\[
L = \langle e_1, e_2 \rangle, \quad K = \langle e_3, \ldots, e_d \rangle.
\]
Note that \( L_1 = L \). The remaining \( r-1 = \binom{d-2}{2} \) subspaces \( L_i \) will be parametrized by the set
\[
\mathcal{I} = \{ I \subseteq \{3, \ldots, n\} \mid |I| = 2 \}
\]
in the following way. For each \( I = \{a, b\} \in \mathcal{I} \) with \( a < b \), we set \( \omega_I = e_a \wedge e_b \in \wedge^2 K \).

The set \( \{ \omega_I \mid I \in \mathcal{I} \} \) forms a standard basis of the space \( \wedge^2 K \). The subspace of \( K \) whose volume form is represented by the wedge \( \omega_I \) will be denoted by \( \Lambda_I = \langle e_a, e_b \rangle \subseteq K \). We will look for the subspaces \( L_I \), now parametrized by the set \( \mathcal{I} \), via their corresponding \( f_I \) maps. In view of our understanding of condition (4.4) via volume forms of images of the \( f_I \) maps, we will restrict ourselves to the case when these maps are embeddings with canonical images, i.e.,
\[
\forall I \in \mathcal{I}. \quad f_I : L = \langle e_1, e_2 \rangle \to \Lambda_I = \langle e_a, e_b \rangle
\]

Note that the set \( \mathcal{I} \) can be seen as the vertex set of the Johnson graph \( J(n, 2) \). Two vertices \( I, J \) in this graph are connected, written as \( I - J \), if and only if \( |I \cap J| = 1 \). We will use this terminology in what follows.

Condition for submersiveness in the canonical choice. Let us now determine the conditions on the maps \( f_I \) so that the projections \( \tilde{\pi}_1 \) and \( \tilde{\pi}_l \) satisfy (4.4).

First of all, when projecting according to \( \tilde{\pi}_1 : V \to K \) with kernel \( L_1 \), the volume forms \( \wedge^2 \text{im}(f_J - f_l) = \omega_J \) for \( J \in \mathcal{I} \) clearly generate \( \wedge^2 K \). This takes care of the exceptional case.

Now consider projecting according to \( \tilde{\pi}_I \) for some \( I \in \mathcal{I} \). The volume forms of \( \text{im}(f_1 - f_I) = \text{im}(f_I) \) and of \( \text{im}(f_J - f_I) \) for all \( J \in \mathcal{I} \) generate \( \wedge^2 K \) if and only if
\[
\forall H \in \mathcal{I}. \quad \omega_H \in \langle \omega_I \rangle + \sum_{J \in \mathcal{I}} \wedge^2 \text{im}(f_J - f_I)
\]
For every \( J \in \mathcal{I} \), we can write the volume form of \( \text{im}(f_J - f_I) \) explicitly as
\[
(f_1(e_1 \wedge e_2) + f_J(e_1 \wedge e_2)) - (f_1(e_1) \wedge f_J(e_2) + f_J(e_1) \wedge f_1(e_2))
\]
Observe that
\[ f_I(e_1 \wedge e_2) - (f_I(e_1) \wedge f_I(e_2) + f_I(e_1) \wedge f_I(e_2)) \in \langle \omega_I \rangle + \{\{\omega_H \mid H \in \mathcal{I}, H - I\}\} \]
and that \( f_I(e_1 \wedge e_2) \) is a non-trivial multiple of \( \omega_J \). This means that as long as (4.6)
is shown to be satisfied for all subsets \( H \in \mathcal{I} \) with that are connected to \( I \), it willalsobe satisfied for all the other subsets \( H \in \mathcal{I} \). We can therefore restrict ourselves to inspecting thesituation when \( I = \{a, b\} \) and \( H \) is either \( \{a, c\} \) or \( \{b, c\} \). In suchacase, consider the corresponding part of the space in (4.6),
\[ \langle \omega_{(a,b)} \rangle + \bigwedge^2 \text{im}(f_{(a,c)} - f_{(a,b)}) + \bigwedge^2 \text{im}(f_{(b,c)} - f_{(a,b)}). \]
Observe that the space (4.7) is a subspace of \( \langle \omega_{(a,b)}, \omega_{(a,c)}, \omega_{(b,c)} \rangle \). These two spaces are equal if and only if both \( \{a, c\} \) and \( \{b, c\} \) satisfy (4.6). Whence (4.4) issatisfied if and only if the vectors
\[ \{\omega_I, \bigwedge^2 \text{im}(f_J - f_I), \bigwedge^2 \text{im}(f_H - f_I)\} \]
are linearly independent for all triangles \( I - J - H - I \) in the graph.

**Transition maps.** In order to express this last condition more clearly, set \( f_I^{[-1]}: \Lambda_I \rightarrow L \) tobe the inverse of \( f_I \) defined on its image \( \Lambda_I \), and let
\[ \delta_I^J: \Lambda_J \rightarrow \Lambda_I, \quad \delta_I^J = f_I \circ f_{I^{[-1]}}. \]
These maps should be thought of as transition maps as will be clear from whatfollows. For all \( a, b \), we can pull back the basis of \( \Lambda_{(a,b)} \) into
\[ u_{(a,b),a} = f_{(a,b)}^{-1}(e_a), \quad u_{(a,b),b} = f_{(a,b)}^{-1}(e_b). \]
The advantage of this new basis is that we can replace the volume forms in (4.8)with their scalar multiples
\[ (f_{(a,c)} - f_{(a,b)})(u_{(a,c),a} \wedge u_{(a,c),c}), (f_{(b,c)} - f_{(a,b)})(u_{(b,c),b} \wedge u_{(b,c),c}). \]
The first of these vectors can be rewritten as
\[ \left( e_a - \delta_{(a,c)}^{(a,b)}(e_a) \right) \wedge e_c - \delta_{(a,c)}^{(a,b)}(e_c), \]
which is simply
\[ \left( e_a - \delta_{(a,c)}^{(a,b)}(e_a) \right) \wedge e_c \mod \langle \omega_{(a,b)} \rangle. \]
We have a similar expression for the second vector. The vectors (4.8) are thereforeindependent if and only if the vectors
\[ \{e_a - \delta_{(a,c)}^{(a,b)}(e_a), e_b - \delta_{(b,c)}^{(a,b)}(e_b)\} \]
are independent.

Let \( X = \{3, 4\} \in \mathcal{I} \). In order to resolve the last condition, Note that we canexpress \( \delta_I^J = (\delta_X^Y)^{-1} \circ \delta_J^Y \). The subfamily \( \Delta_X = \{\delta_X^I \mid I \in \mathcal{I}\} \) of maps into \( \Lambda_X \)therefore uniquely determines all the \( \delta_I^J \) maps. Moreover, we can apply \( \delta_X^{(a,b)} \) toconvert the vectors (4.9) into
\[ \{\delta_X^{(a,b)} - \delta_X^{(a,c)}(e_a), (\delta_X^{(a,b)} - \delta_X^{(b,c)}(e_b))\} \]
obtaining a condition expressed solely in terms of the maps belonging to \( \Delta_X \).
It is possible to invert the described procedure and recover the maps $f_I$ from the transition maps $\delta^X_I$ and the map $f_X$. To achieve this, we can simply define $f_I = (\delta^X_I)^{-1} \circ f_X$ for any $I \in \mathcal{I}$. We can even assume $f_X$ is given as the standard map $e_1 \mapsto e_3$, $e_2 \mapsto e_4$. Our problem is thus reduced to finding a family of maps $\Delta^X$ such that the vectors (4.10) are independent.

Finding the family $\Delta^X$. We can take $\delta^X_X = \text{id}_{\Lambda^X}$. We search for the remaining $\delta^X_I$ maps in terms of their matrices written in standard basis of $\Lambda^I$ and $\Lambda^X$. Set $\delta^X_I \equiv \begin{pmatrix} \alpha^I & \gamma^I \\ \beta^I & \delta^I \end{pmatrix} \in M_2(\mathbb{Q}_p)$.

The maps $\delta^X_I$ have to be invertible, giving the conditions $\alpha^I \delta^I - \beta^I \gamma^I \neq 0$ for all $I \in \mathcal{I}$. The condition that the vectors (4.10) be independent can be written as $\forall (I, J, H) \in \mathbb{I}^3$. $I - J - H - I \implies \det \begin{pmatrix} \alpha^I - \alpha^J & \gamma^I - \gamma^H \\ \beta^I - \beta^J & \delta^I - \delta^H \end{pmatrix} \neq 0$.

The set of solutions to these conditions forms a complement of a finite union of quadrics in $M_2(\mathbb{Q}_p)$, and so the set of solutions to our problem is Zariski dense.

To write down an explicit solution, we can simply take scalar matrices with the conditions that $\alpha^I \neq 0$ and $\alpha^I \neq \alpha^J$ for $I - J$. Taking this back to the subspaces $L_i$, we obtain the following quite elementary example.

Example 4.5.3. Let $V = \langle e_1, \ldots, e_n \rangle$. Set $L_1 = \langle e_1, e_2 \rangle$ and for any $3 \leq i < j \leq n$, set $L_{i,j} = \langle e_1 + \lambda_{i,j}e_i, e_2 + \lambda_{i,j}e_j \rangle$ for non-zero distinct scalars $\lambda_{i,j} \in \mathbb{Q}_p$. Then $\Psi$ is a submersion in a neighbourhood of $L$.

Larger than bound. In order to complete the proof of Theorem 4.5.1, it remains to inspect the case when $r > \binom{d-2}{2} + 1$. To this end, let $Z$ be any subset of $\{1, \ldots, r\}$ with $|Z| = \binom{d-2}{2} + 1$. Consider the coordinate projection

$\text{pr}_Z : \text{Gr}(2, V)^r \to \text{Gr}(2, V)^{\binom{d-2}{2}+1}$

onto the $Z$-axes. We have shown above that there is a dense neighbourhood $U_Z \subseteq \text{Gr}(2, V)^{\binom{d-2}{2}+1}$ consisting of those tuples $L$ in which the associated map $\Psi$ is a submersion, i.e., satisfies the condition (4.5). Let

$U = \bigcap_{Z \subseteq \{1, \ldots, r\} \atop |Z| = \binom{d-2}{2}+1} \text{pr}_Z^{-1}(U_Z) \subseteq \text{Gr}(2, V)^r$.

Note that $U$ is a Zariski dense open subset of $\text{Gr}(2, V)^r$. Clearly every point $L \in U$ satisfies (4.5). The proof of Theorem 4.5.1 is thus complete.

4.6. Immersiveness of decomposability. We now deal with complementing the previous section by proving the following.

Theorem 4.6.1. The map $\Psi$ is an immersion on a dense open subset of its domain if and only if $r \leq \binom{d-2}{2} + 1$. 
Our proof will rely on exploiting that in the bound case when $r = \binom{d-2}{2} + 1$, we already know that $\Psi$ is a local isomorphism on a dense open subset. Recall that as before, it suffices to find one good point of the domain of $\Psi$. To this end, we connect the general case with the bound one. First of all, consider the coordinate projection

$$\text{pr}: \text{Gr}(2, V)^{(d-2)/2 + 1} \to \text{Gr}(2, V)^r$$

onto the first $r$-axes. Both the source and the target of $\text{pr}$ have an associated rational map $\Psi$,

\[
\text{Gr}(2, V)^{(d-2)/2 + 1} \xrightarrow{\text{pr}} \text{Gr}(2, V)^r \xrightarrow{\Psi} \text{Gr}(r, W).
\]

These induce differential maps,

\[
T_L \text{Gr}(2, V)^{(d-2)/2 + 1} \xrightarrow{d \text{pr}_L} T_{\text{pr}(L)} \text{Gr}(2, V)^r \xrightarrow{d \Psi_L} T_{\Psi(L)} \text{Gr}((d-2)/2 + 1, W) \xrightarrow{\sigma} T_{\Psi(pr(L))} \text{Gr}(r, W).
\]

Note that $d \text{pr}_L$ is surjective and has a natural splitting $\sigma$. As long as $L$ belongs to some dense open subset $U \subseteq \text{Gr}(2, V)^{(d-2)/2 + 1}$, the map $d \Psi_L$ is an isomorphism. Select complements $W = \Psi(L) \oplus X$ and $\Psi(L) = \Psi(\text{pr}(L)) \oplus Y$. After identifying the tangent spaces with hom sets, the map $\iota$ is the natural composition

\[
\iota: \text{hom}(\Psi(L), X) \xrightarrow{res} \text{hom}(\text{pr}(L), X) \to \text{hom}(\text{pr}(L), X \oplus Y)
\]

Now, as $\iota$ is an embedding on the image of $d \Psi_L \circ \sigma$, it follows that $d \Psi_{\text{pr}(L)}$ is injective. This completes the proof of Theorem 4.6.1. Example 4.5.3 gives an explicit point $L \in U$, from which points $\text{pr}(L)$ can be produced.

**Remark 4.6.2.** One can think of searching for a point $L$ in which $\Psi$ is a local immersion by searching for points over which the map $\Psi$ has a 0-dimensional fibre. This means that we can pass to the target of $\Psi$ and search there. Consider $\Psi(L)$ as a subspace of $W$ of dimension $r$. This subspace is spanned by its intersections with the image of the Plücker embedding of $\text{Gr}(2, V)$ into $W$,

$$\Psi(L) = (\Psi(L) \cap \text{Gr}(2, V)).$$

The latter condition in fact characterizes $\text{im} \Psi$, since for $H \in \text{Gr}(r, W)$ we have $H \in \text{im} \Psi$ if and only if $H = (H \cap \text{Gr}(2, V))$. Our search for the $r$ points therefore corresponds to finding a linear section of the Grassmannian $\text{Gr}(2, V)$ that is 0-dimensional and is generated by its finitely many points. Over the algebraic closure $\overline{\mathbb{Q}}$, one can use Bertini’s theorem (see [1, Theorem 12.1 (i) and (ii)]) together with Bézout to deduce that generic subspaces have this property. More precisely, there is a dense open subset $V \subseteq \text{Gr}((d-2)/2 + 1, d)$ such that every element $\Lambda \in V$ has the property that $\dim(\Lambda \cap \text{Gr}(2, V)) = 0$. In this language, our Example 4.5.3 shows
that such properties can be obtained from rational points both in $\text{Gr}(\binom{d-2}{2} + 1, d)$ and $\text{Gr}(2, V)$.

5. The decomposability map over $\mathbb{F}_p$

We now descend from groups and algebras over $\mathbb{Z}_p$ to their finite quotients. The role of torsion-freeness will be played here by the assumption that the finite groups are of exponent $p$. All the structure of these groups is therefore captured in the relations between commutators. We apply the same approach as above to set up a parametrization of these groups, express their Bogomolov multipliers and define the decomposability map. We first illustrate what happens with a small number of generators, and inspect when this map is surjective. We analyse the generic behaviour for large primes $p$ by using the results in characteristic 0. On the other hand, when fixing $p$, we show how the subvariety of large dimension representing groups with non-trivial Bogomolov multipliers from the previous section can be made to exist in the finite case as well. We conclude with an application of our technique regarding a more elementary problem in commutators in finite groups, which can also be expressed in the language of Grassmannians. Throughout this section, we will prefer to use the more standard term Bogomolov multiplier rather than $\text{SK}_1$.

5.1. Groups and the Grassmannian variety. Let $G$ be a finite $p$-group of exponent $p$ (assume $p > 2$) and nilpotency class 2. Suppose $G$ is of rank $d$, so that $V = G/[G, G] \cong \mathbb{F}_p^d$. The structure of $G$ is then completely determined by the set of relations between its commutators. These form a certain linear subspace in $V \wedge V$. Fix the dimension $r$ of this space of relations. Thus we are looking at $r$-dimensional subspaces of the $\binom{d}{2}$ dimensional $\mathbb{F}_p$-space $W = V \wedge V$. These form the Grassmannian variety $\text{Gr}(r, W)$, and each of its points $L$ determines a group $G_L$ obtained by imposing precisely the relations of the corresponding subspace $L \leq W$. This is a mod $p$ version of the parametrization (3.2), so we stay with the same notation as in the previous parts.

5.2. Bogomolov multiplier. Let $L \in \text{Gr}(r, W)$ be a subspace. The Bogomolov multiplier of the corresponding group $G_L$ can be recognized as follows (see [5]). Let $DW$ denote the set of decomposable wedges of $W$. These form a variety $\text{Gr}(2, V)$. Then we have

$$B_0(G_L) = \frac{L}{\langle DW \cap L \rangle}.$$ 

Therefore deciding whether or not the Bogomolov multiplier is trivial reduces to deciding whether or not the subspace $L$ is generated by decomposable wedges. This is just the same as in the previous section, the only difference being that we are now considering vector spaces over $\mathbb{F}_p$. 

5.3. Decomposable subspaces. In order to study the subspaces generated by decomposable wedges, we consider, as in (4.1), the rational map
\[ \psi : \text{Gr}(2,V) \rightarrow \text{Gr}(r,W) \]
mapping an \( r \)-tuple of decomposable vectors of \( W \) into their span in \( W \). The locus of indeterminacy consists of the tuples which do not span an \( r \)-dimensional subspace of \( W \). The image of \( \psi \) consists precisely of the subspaces of \( W \) whose corresponding groups have trivial Bogomolov multipliers.

5.4. Bounds for sizes of Grassmannians. The size of \( \text{Gr}(k,n) \) is the number of subspaces of dimension \( k \) in an \( n \)-dimensional vector spaces over a finite field with \( p \) elements. This is equal to the \( p \)-binomial coefficient
\[ \binom{n}{k}_p = \frac{(p^n - 1)(p^{n-1}) \cdots (p^n - p^{k-1})}{(p^k - 1)(p^{k-1}) \cdots (p^2 - p^1)} \]

We will require the following straightforward bounds for these coefficients.

Lemma 5.4.1. Let \( k \leq n \). Then
\[ p^{k(k-k)} - p^{(n-1)k-k^2} \leq \binom{n}{k}_p \leq p^{n(k-k-1)k} = p^{k(n-k)+k}. \]

Proof. In the definition of the binomial coefficient, use the bounds \( p^n - 1 \leq p^n - p^i \leq p^n \) and collect. \( \square \)

We will also need the following bound. It is stronger than the one above, but it includes implied constants and will only be used for asymptotics when \( p \) tends to infinity.

Lemma 5.4.2. Let \( k \leq n \). Then
\[ |\text{Gr}(k,n)| = p^{k(k-k)} + O(p^{k(k-k)-1}), \]
where the implied constant is independent of \( p \).

Proof. Immediate from [4, Section 13.5, Theorem 6]. \( \square \)

5.5. Few generators. Here we inspect the behaviour of the map \( \psi \) when the dimension of \( V \) is as small as possible. First we show that \( \psi \) is surjective for \( d \leq 3 \).

Proposition 5.5.1. Suppose \( d \leq 3 \). Then \( \psi \) is surjective.

Proof. We are claiming that \( B_0(G_L) = 0 \) for all \( L \leq W \). This is clear for \( d \leq 2 \), since \( V \wedge V \) is at most 1-dimensional. When \( d = 3 \), we either have that \( L = 0 \), and so \( B_0(G_L) = 0 \), or \( L \neq 0 \), in which case \( |G_L| \leq p^5 \). As \( G_L \) is also of nilpotency class 2, it follows from [35] that \( B_0(G_L) = 0 \). \( \square \)

The smallest interesting case is when \( d = 4 \). We analyse it in detail by exploiting the presence of the action of \( GL(V) \).

Proposition 5.5.2. Suppose \( d = 4 \).
We can directly compute the size of the codomain of \( \psi \)
with \( L \) is equal to \( \text{GL}((\text{invertible block matrices}) \). It follows that
\[
X = \text{Gr} \langle v \rangle
\]
the scalar multiples of \( V \). Note that
\[
|X| \approx 1 - \frac{1}{p}.
\]
On the other hand, we can directly compute the size of the domain of \( \psi \); it is equal to
\[
|\text{Gr}(2, V)|^2 = \left( \frac{d^2}{2} \right)_p = (p^2 + 1)^2 (p^2 + p + 1)^2.
\]
Note that \(|X|\) is asymptotically comparable to \(|\text{Gr}(2, V)|^2|\), so that \( X \) is in fact the orbit of a generic point.

We now consider the image of \( \psi \). Since the restriction of \( \psi \) to \( X \) has fibres of size 2, we have
\[
(5.1) \quad \frac{1}{2} |X| \leq |\im \psi| \leq (|\text{Gr}(2, V)|^2 - |X|) + \frac{1}{2} |X|.
\]
We can directly compute the size of the codomain of \( \psi \),
\[
|\text{Gr}(2, W)| = \left( \frac{6}{2} \right)_p = (p^2 - p + 1) (p^2 + p + 1) (p^4 + p^3 + p^2 + p + 1).
\]
Dividing inequality (5.1) by $|\text{Gr}(2, W)|$, we get:

$$\frac{p^6 + p^4}{2(p^6 + p^4 + p^3 + p^2 + 1)} \leq \frac{|\text{im} \psi|}{|\text{Gr}(2, W)|} \leq \frac{p^6 + 2p^5 + 5p^4 + 4p^3 + 6p^2 + 2p + 2}{2(p^6 + p^4 + p^3 + p^2 + 1)}.$$ 

Both sides converge to $\frac{1}{2}$ as $p \to \infty$ and the claim is proved.

(3) In this case, $|G_L| = p^7$. Suppose that $B_0(G_L) \neq 0$. It follows from [29, Corollary 2.14, Theorem 2.13] that there are only two possibilities up to isoclinism for such a group. When restricted to groups of exponent $p$, these are in fact the only two possibilities up to isomorphism. Therefore each one of these corresponds to an orbit of $\text{GL}(V)$ acting on subspaces of dimension 3 in $W$. We analyse both cases in more detail.

The first group $G_1$ is determined by the subspace

$$A = \langle v_1 \wedge v_2 - v_3 \wedge v_4, v_2 \wedge v_4 - \omega v_1 \wedge v_3, v_1 \wedge v_4 \rangle \leq W,$$

where $\omega$ is a generator of $\mathbb{F}_p^*$. The stabilizer of $B$ under the action of $\text{GL}(V)$ contains the set of matrices of the form

$$\begin{pmatrix} a & * & * & b \omega \\ 0 & c & d \omega & 0 \\ 0 & d & c & 0 \\ b & * & * & a \end{pmatrix},$$

where $0 \neq (a^2 - \omega b^2)(c^2 - \omega d^2)$ and $*$ is any element of $\mathbb{F}_p$. For any $b$, there are at most two roots of the equation $a^2 - \omega b^2 = 0$, so there are at least $p - 2$ options for selecting $a$. The same is true for the pair $c, d$. The stabilizer therefore contains at least

$$(p(p - 2))^2 \cdot p^4$$

elements, and so the orbit of $A \in \text{Gr}(3, W)$ is of size at most

$$\frac{|\text{GL}(V)|}{(p - 2)^4 p^6} = \frac{(p - 1)^4(p + 1)^2(p^2 + 1)(p^2 + p + 1)}{(p - 2)^2}.$$ 

The second group $G_2$ is determined by the subspace

$$B = \langle v_1 \wedge v_2 - v_3 \wedge v_4, v_1 \wedge v_3, v_1 \wedge v_4 \rangle \leq W.$$ 

The stabilizer of $B$ under the action of $\text{GL}(V)$ contains the set of matrices of the form

$$\begin{pmatrix} a & * & * & * \\ 0 & b & 0 & 0 \\ 0 & x & y & 0 \\ 0 & z & w & 0 \end{pmatrix},$$

where $0 \neq ab = xw - yz$ and $*$ is any element of $\mathbb{F}_p$. There are $(p - 1)^2$ options for selecting $a, b$, and after these there are $(p^2 - 1)p$ options for selecting $x, y, z, w$. The stabilizer therefore contains at least

$$(p - 1)^2 \cdot (p^2 - 1)p \cdot p^5$$

elements, and so the orbit of $B \in \text{Gr}(3, W)$ is of size at most

$$\frac{|\text{GL}(V)|}{(p - 1)^2(p^2 - 1)p^6} = (p - 1)(p + 1)(p^2 + 1)(p^2 + p + 1).$$
Taking both orbits into account, we conclude that the number of subspaces of $W$ that are not in the image $\text{im} \psi$ is at most the sum of the two bounds derived above, which is proportional to $p^8$ as $p \to \infty$.

On the other hand, the number of all subspaces $\text{Gr}(3, W)$ is equal to
\[
|\text{Gr}(3, W)| = \binom{6}{3}_p = (p + 1)(p^2 + 1)(p^2 - p + 1)(p^4 + p^3 + p^2 + p + 1),
\]
which is proportional to $p^9$ as $p \to \infty$.

We therefore obtain
\[
\frac{|\text{im} \psi|}{|\text{Gr}(3, W)|} \geq 1 - a_p,
\]
where $a_p$ is a sequence proportional to $1/p$. The claim follows.

(4) In this case, $|G_L| \leq p^6$. It follows from [29, Corollary 2.14] that there are no such $p$-groups of nilpotency class 2 with non-trivial Bogomolov multipliers.

Remark 5.5.3. It is clear from the proof that the action of $\text{GL}(V)$ could be exploited precisely because of the restriction on the number of generators. As $d$ grows, the orbits become proportionately too small and the action does not make a difference. We will see this explicitly in the last proofs of this section.

5.6. Surjectivity of decomposability. We record a consequence of the proof of Proposition 5.5.2 revealing how the map $\psi$ is almost never surjective.

Proposition 5.6.1. The map $\psi$ is surjective if and only if $(\frac{d}{2}) - r \leq 2$.

Proof. Let $G_0$ be one of the $p$-groups from [29, Theorem 2.13]. The group $G_0$ is of order $p^7$, generated by 4 elements and $B_0(G_0) \neq 0$. Put $G = G_0 \times C_p^{d-4}$ for any $d \geq 4$. Then $G$ is a $d$-generated group of exponent $p$ and nilpotency class 2 with $B_0(G) \neq 0$. Hence not every subspace of $W$ of codimension 3 is generated by decomposable wedges, and so $\psi$ is not a surjection for $r \leq (\frac{d}{2}) - 3$.

On the other hand, assume that $r \geq (\frac{d}{2}) - 2$ and let $L \leq W$ be of codimension at most 2. Suppose that $L$ is not generated by decomposable elements. Then the set of its decomposable elements is contained in a subspace $\bar{L} \leq L$ of codimension 1. Suppose that $w \in L - \bar{L}$. Now, the group $G_{\bar{L}}$ is of nilpotency class 2 with $|G_{\bar{L}}^L| \leq p^3$. It follows from [39, Theorem A] that every element of $G_{\bar{L}}^L$ is a commutator, and so in particular $w$ is decomposable modulo $\bar{L}$. Therefore there is a decomposable vector in $L$ not belonging to $\bar{L}$. This is a contradiction. We conclude that $L$ is generated by decomposable elements, and so $\psi$ is indeed surjective.

5.7. Asymptotics of decomposability. We now turn to analysing the asymptotic behaviour of $\psi$. This is strongly correlated with submersiveness and immersiveness of $\Psi$ over $\mathbb{Q}_p$, although not entirely equivalent (see Example 3.11.2).

Few relators. As long as the subspace $L$ is not of a very small codimension in $W$, the generic Bogomolov multiplier is trivial. This is an easy application of counting rational points, whose asymptotic behaviour coincides with that over closed fields.
Proposition 5.7.1. Suppose \( r \leq \binom{d-2}{2} \). Then proportion of elements in the image of \( \psi \) tends to 0 as \( p \to \infty \). Setting \( \delta = \binom{d-2}{2} - r - 3 \), we moreover have

\[
\frac{|\text{im} \psi|}{|\text{Gr}(r,W)|} \leq p^{-\delta r}.
\]

Proof. The asymptotic part follows immediately from Lemma 5.4.2 applied to the source and target of \( \psi \). As for the second part, we can use the exact upper and lower bounds from Lemma 5.4.1 to obtain

\[
|\text{Gr}(2,V)| \leq p^{2r(d-1)} \quad \text{and} \quad |\text{Gr}(r,W)| \geq p^{\binom{d}{2} - r - 2}.
\]

It follows that

\[
\frac{|\text{im} \psi|}{|\text{Gr}(r,W)|} \leq p^{2 - r - \frac{d^2 - 5d + 2}{2}} = p^{-\delta r}. \tag*{□}
\]

Many decomposable subspaces. We now focus on the case when \( r \geq \binom{d-2}{2} + 1 \). In such a situation, the map \( \Psi \) over \( \mathbb{Q}_p \) is a submersion almost everywhere, meaning that its image is not contained in a proper subvariety. These differential techniques can not help us directly with understanding the situation over \( \mathbb{F}_p \), since small neighbourhoods of points over \( \mathbb{Q}_p \) project entirely to a single point in \( \mathbb{F}_p \). In order to transfer results over \( \mathbb{Q}_p \) to the case of finite fields, we will need to consider both maps \( \psi \) and \( \Psi \) as particular instances of a scheme rational map \( \Psi \). We will succeed in proving that the image of \( \psi \) is very large indeed.

Theorem 5.7.2. Suppose \( r \geq \binom{d-2}{2} + 1 \). Then

\[
\liminf_{p \to \infty} \frac{|\text{im} \psi|}{|\text{Gr}(r,W)|} \geq \left( \frac{1}{C_{d-2}} \right)^{r},
\]

where \( C_{d-2} \) is the Catalan number. In particular, the proportion of elements in the image of \( \psi \) is bounded away from 0 as \( p \to \infty \).

Grassmannian schemes and the decomposability map. To begin with, consider the schemes \( \text{Gr}_Z(2,d)^r \) and \( \text{Gr}_Z(r,\binom{d}{2}) \) over \( \text{Spec} \mathbb{Z} \) (see [16]). The rational map (4.1) is defined over \( \mathbb{Z} \) and extends to a rational map of schemes

\[
\Psi : \text{Gr}_Z(2,d)^r \dashrightarrow \text{Gr}_Z(r,\binom{d}{2}). \tag{5.2}
\]

We will tacitly assume throughout that \( \Psi \) is restricted to the open subset on which it is regular. Each Grassmannian scheme \( \text{Gr}_Z(k,n) \) can be viewed as a projective scheme via Plücker coordinates. To be more precise, consider the polynomial ring \( P_k^n = \mathbb{Z}[...,X_I,...] \) in \( \binom{n}{k} \) variables indexed by the subsets \( I \subseteq \{1,2,\ldots,n\} \) with \( k \) elements. Consider the ring generated by matrix coefficients \( A^n_k = \mathbb{Z}[...,x_{i,j},...] \) of a generic matrix \( M = [x_{i,j}]_{1 \leq i \leq k, 1 \leq j \leq n} \). To each \( X_I \) we can associate the minor corresponding to columns indexed by \( I \). This gives a ring homomorphism \( P_k^n \to A^n_k \) with kernel \( J^n_k \). Then we have

\[
\text{Gr}_Z(k,n) = \text{Proj} \frac{P_k^n}{J^n_k} \subseteq \text{Proj} P_k^n = \mathbb{P}(z)^1.
\]

Now, for the source scheme \( \text{Gr}_Z(2,d)^r \), we will also need to use the Segre embedding of the product of schemes into a projective space. This corresponds to taking tensor
products of rings, and we have

$$\text{Gr}_Z(2, d)^r = \text{Proj} \prod_{i=1}^r P_2^d / J_2^d \subset \text{Proj} \prod_{i=1}^r P_2^d = \mathbb{P}^{(d^2)-1}.$$  

Coordinates on the ring $\bigotimes_{i=1}^r P_2^d$ are indexed as tuples $(I_1, \ldots, I_r)$ with each $I_i$ a subset of $\{1, 2, \ldots, n\}$ of size 2. Fix a natural bijection between $\{1, 2, \ldots, (d^2)\}$ and $\mathcal{I} = \{I \subseteq \{1, 2, \ldots, d\} \mid |I| = 2\}$. Coordinates on the ring corresponding to $\text{Gr}_Z(r, W)$ can therefore be indexed as $X_{(I_1, \ldots, I_r)}$ with $I_i \in \mathcal{I}$. In this language, our map $\Psi$ is given as a restriction of the scheme version of the ring map

$$P_2^{(d^2)} \to \bigotimes_{i=1}^r P_2^d, \quad X_{(I_1, I_2, \ldots, I_r)} \mapsto \sum_{\sigma \in \text{Sym}_r} \text{sgn}(\sigma) \cdot X_{(I_{\sigma(1)}, I_{\sigma(2)}, \ldots, I_{\sigma(r)})}$$

Over fields, this corresponds to the projectivization of the natural projection of $\otimes^r W$ onto $\wedge^r W$ symmetrizing a tensor.

**Generic dimension of fibres.** Our main objective is to inspect the image of $\Psi$ over finite fields. This will be done in two steps. The first step is to say something about the generic behaviour of this map. We will do this by using the following version of the fibre dimension theorem (see [47, Lemma 05F7]).

**Lemma 5.7.3.** Let $f : X \to Y$ be a morphism of schemes. Assume $Y$ irreducible with generic point $\eta$ and $f$ of finite type. If $X_\eta$ has dimension $n$, then there exists a non-empty open $V \subseteq Y$ such that for all $y \in V$ the fibre $X_y$ has dimension $n$.

The dimension of the generic fibre $X_\eta$ will be obtained using submersiveness based on the following.

**Lemma 5.7.4.** Suppose $r \geq (d^2 - 2^2) + 1$. Then the rational map $\Psi$ is dominant.

**Proof.** Since dominance is witnessed on open subsets (see [47, Lemma 0CC1]), we can replace $X$ and $Y$ by their affine open subsets, and hence assume that $X = \text{Spec} A$ and $Y = \text{Spec} B$ for finitely generated rings $A, B$ over $\mathbb{Z}$. Changing the base from $\mathbb{Z}$ to $\mathbb{C}$, we have a diagram

$$
\begin{array}{ccc}
X_\mathbb{C} & \xrightarrow{\Psi_\mathbb{C}} & Y_\mathbb{C} \\
\downarrow{\beta_X} & & \downarrow{\beta_Y} \\
X_\mathbb{Z} & \xrightarrow{\Psi_\mathbb{Z}} & Y_\mathbb{Z},
\end{array}
$$

where $\beta_\ast$ are base change projections induced by the scalar extensions $A \to A \otimes_\mathbb{Z} \mathbb{C}$ and similarly for $B$. Note that since $Y$ is irreducible and $Y_\mathbb{Q} \neq \emptyset$, the ring $B$ has no additive torsion. Therefore $B \to B \otimes_\mathbb{Z} \mathbb{C}$ is an injection, meaning that $\beta_Y$ is dominant. It follows that the generic point of $Y_\mathbb{Z}$ belongs to the image of $\beta_Y$ (see [47, Lemma 0CC1]). Now, by Theorem 4.5.1 the image of $\Psi_\mathbb{C}$ is not contained in any proper subvariety, whence the rational map $\Psi_\mathbb{C}$ is also dominant. The generic point of $Y_\mathbb{C}$ is therefore in the image of $\Psi_\mathbb{C}$. By commutativity of the diagram, we now have that the generic point of $Y_\mathbb{Z}$ is in the image of $\Psi_\mathbb{Z}$. This completes the proof. \(\square\)
In order to express $\dim X_\eta$, we can pass to affine open subsets of both $X$ and $Y$ (see [23, Exercise 3.20 (e)]) and hence assume that $X = \text{Spec} A$ and $Y = \text{Spec} B$ with a ring map $B \to A$. This map is injective by Lemma 5.7.4, and so we can assume that $B \leq A$. Therefore we have

$$X_\eta = \{ p \in \text{Spec} A \mid p \cap B = 0 \} = \text{Spec}(B - \{0\})^{-1}A.$$ 

The latter corresponds to the spectrum of $(B \otimes \mathbb{Q} - \{0\})^{-1}(A \otimes \mathbb{Q})$, which is the same as the dimension of the generic fibre of $\Psi : X_\mathbb{Q} \to Y_\mathbb{Q}$. As this morphism is dominant and its source and target are irreducible, the latter dimension is equal to $\dim X_\mathbb{Q} - \dim Y_\mathbb{Q}$ (see [23, Exercise 3.22 (b)]). Whence we have

$$\dim X_\eta = \dim X_\mathbb{Q} - \dim Y_\mathbb{Q}.$$ 

Bounding rational points on fibres. The second step in our reasoning will be to bound the sizes of the fibres over finite fields. In order to achieve this, we will utilize the following bound, known in the literature as the Schwarz-Zippel type bound.

**Lemma 5.7.5.** Let $Z$ be a variety in $\mathbb{F}_p^k$ defined over $\mathbb{F}_p$ of degree $d$ and dimension $k$. Then $|Z(\mathbb{F}_p)| \leq d \cdot p^k$.

We are now ready for the proof.

**Proof of Theorem 5.7.2.** Set $X$ and $Y$ to denote the source and target of the rational map $\Psi$. We can replace $X$ and $Y$ by their affine open subsets, and hence assume that both are affine varieties. Let $\eta$ be the generic point of $Y$. It follows from Lemma 5.7.4 that $\dim X_\eta = \dim X_\mathbb{Q} - \dim Y_\mathbb{Q}$. Thanks to Lemma 5.7.3 there is a non-empty dense subset $V \subseteq Y$ containing $\eta$ such that $\dim \Psi^{-1}(y) = \dim X_\mathbb{Q} - \dim Y_\mathbb{Q}$ for all $y \in V$. Set $U = \Psi^{-1}(V)$. It follows from Theorem 5.7.3 applied to $U \to \text{Spec}(\mathbb{Z})$ that for all but finitely many primes $p$, we have $\dim U_{\mathbb{F}_p} = \dim U_{\mathbb{Q}} = \dim X_\mathbb{Q}$. Express

$$|U(\mathbb{F}_p)| = \sum_{y \in \Psi(U(\mathbb{F}_p))} |\Psi^{-1}(y)|.$$ 

It follows from Lemma 5.7.5 that we can bound

$$|\Psi^{-1}(y) \cap X(\mathbb{F}_p)| \leq \deg \Psi^{-1}(y)_{\mathbb{F}_p} \cdot p^{\dim X_\mathbb{Q} - \dim Y_\mathbb{Q}}.$$ 

Note that $\deg \Psi^{-1}(y)_{\mathbb{F}_p} \leq \deg X_{\mathbb{F}_p}$ by Bézout (see [19, Chapter 8]) since $\Psi$ is linear. The latter degree is equal to $\deg X_{\mathbb{Q}} = (C_{d-2})^r$ (see [10, Section 6]). We thus obtain

$$|U(\mathbb{F}_p)| \leq (C_{d-2})^r \cdot p^{\dim X_\mathbb{Q} - \dim Y_\mathbb{Q} \cdot |\Psi(U(\mathbb{F}_p))|}.$$ 

The number of rational points of $X - U$ can be bounded by Lemma 5.7.5. Together with Lemma 5.4.2, we obtain

$$|U(\mathbb{F}_p)| \geq p^{\dim X_\mathbb{Q}} + O(p^{\dim X_\mathbb{Q} - 1}),$$

where the implied constant is independent of $p$. It now follows that

$$|\text{im} \psi| \geq |\Psi(U(\mathbb{F}_p))| \geq \left(\frac{1}{C_{d-2}}\right)^r \cdot p^{\dim Y_\mathbb{Q}} + O(p^{\dim Y_\mathbb{Q} - 1})$$

given $d > 2$. This completes our proof.
and the proof is complete. □

Remark 5.7.6. In general, it is not possible to replace the lower bound $1/(C_{d-2})^r$ with 1 for all values of $r$. This is clear from Proposition 5.5.2, where the bound appears in the following way. A general point in $\text{Gr}(2, W)$ is of the form $\langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle$ for a basis $\{e_1, e_2, e_3, e_4\}$ of $V$. The fibre over such a point consists of the pair $\langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle$ and its permutation $\langle e_3 \wedge e_4, e_1 \wedge e_2 \rangle$. The generic fibre in this case consists of $(C_2)! = 2$ points, while our method bounds these from above as $(C_2)^r = 4$. Proposition 5.5.2 indicates that the situation even improves with increasing $r$, and not worsens as our bound does.

Remark 5.7.7. Our method provides bounds that are interesting for increasing $p$ with $d, r$ fixed. Doing the opposite, i.e., fixing $p$ and increasing $d, r$, might result in the large value of $(C_{d-2})^r$ as well as the implied constants interacting with the numerical dimensions, and so the obtained bound will be useless, possibly even negative for large enough $d, r$. We will show in Theorem 5.8.2 that indeed in such a setting it is possible to produce many elements outside $\im \psi$.

5.8. Log-generic $p$-groups. Once the parameters $d$ and $r$ establishing our context are selected, each of the constructed groups $G_L$ will be of order

$$|G_L| = |V| \cdot |W/L| = p^{d + (d^2 - 2r)}. $$

Now, rather than independently selecting $d$ and $r$, suppose that we fix $n := d + (d^2 - 2r)$ and with it the sizes of the groups we are considering. Fix a parameter $0 < \alpha < 1$ and set $d := \lceil \alpha n \rceil$. Expressing $r = (d^2 - 2r) + d - n$, we can therefore produce groups of orders $p^n$ as $n$ varies and $\alpha$ is fixed. The number of groups we obtain in this way is of the form (see [24])

$$p^{d((d^2 - 2r) + O(n^2))} = p^{\delta n^3 + O(n^2)},$$

and since the size of $\text{GL}_n(\mathbb{F}_p)$ is at most $p^{O(n^3)}$, the same expression gives an estimate on the number of non-isomorphic groups we obtain. In order to maximize the number of $p$-groups we obtain in this way, one takes $\alpha = \frac{3}{5}$, with which the above simplifies into

$$p^{3n^3 + O(n^2)}.$$

It is well known (see [42]) that the latter also gives an asymptotical upper bound for the number of all $p$-groups of order $p^n$.

We will now relate this to the previous section. Using the above notation, we have $r - (d^2 - 2r) = 3d - n - 4$, which is asymptotically positive for values of $\alpha > \frac{1}{4}$. Thus for the optimal value $\alpha = \frac{3}{5}$, we are in the situation of Theorem 5.7.2. However, here we are fixing the prime $p$ and varying the parameters $d$ and $r$, whence the theorem does not apply directly. We can instead use quite the opposite Theorem 5.7.1. In order to maximize its use, we should select $\alpha$ as large as possible so that $\delta = (d^2 - 2) - r - 3$ is positive. The latter is equivalent to $\alpha \leq \frac{1}{4}$. Thus we can parametrize groups with $\alpha = \frac{1}{4}$, and it follows from Theorem 5.7.1 that in this case,
the number of elements in the image of $\psi$ is at most $\frac{1}{2}|\text{Gr}(r, W)|$. Thus there are at least $\frac{1}{2}|\text{Gr}(r, W)|/|\text{GL}(V)|$ non-isomorphic $p$-groups with non-trivial Bogomolov multipliers. The size of the last Grassmannian can be bounded by

$$|\text{Gr}(r, W)| \geq p^{\binom{d}{2} - 1} = p^{-\frac{d^2}{2} + \frac{d}{2} + O(n^2)}.$$

As a function of $n$, the leading term of the exponential is $d^2(n - d)/2$, which is of order $(\frac{1}{2}n^2 \cdot \frac{3}{2}n)/2 = \frac{3}{2}n^3$. Factoring by the action of $\text{GL}(V)$ does not change this order, since the log-size of $\text{GL}(V)$ is only quadratic in $n$. Therefore it follows that as $n$ grows, we obtain many groups with non-trivial Bogomolov multipliers, their number is of an order of magnitude log-comparable to that of the number of all $p$-groups of order $p^n$.

The above is slightly unexpected, particularly so since we are in the situation when $r \geq \binom{d-2}{2} + 1$ and so the corresponding $\mathbb{Q}_p$-map is a local surjection by Theorem 4.5.1, meaning that many Bogomolov multipliers vanish. The explanation is morally given by the fact that the constant in Theorem 5.7.2 converges to zero too quickly, so we can not obtain good numerical bounds for the number of groups of order $p^n$ with vanishing multipliers. At the same time, recall that by Proposition 4.3.1, we can find a large subvariety avoiding the image of $\Psi$. We now show how the same subvariety can be constructed over finite fields at the value $\alpha = \frac{2}{3}$. This ultimately produces log-generic $p$-groups with non-vanishing Bogomolov multipliers.

Our construction is based on the following lemma. For a subspace $L \leq W$, set

$L^\wedge := \langle DW \cap L \rangle = \langle v \wedge w \mid v, w \in V, \ v \wedge w \in L \rangle$.

**Lemma 5.8.1.** Suppose $\rho := \binom{d}{2} - r \geq 3$. For every positive integer $N$, we have

$$|\{L \mid L \leq W, \dim L = r, \dim(L/L^\wedge) \geq N\}| \geq p^{(r-3N)(\rho-3N-1)}$$

as long as $d > 4N, \ r > 3N$ and $\rho > 3N + 1$. In particular, taking $N = 1$ gives

$$|\text{Gr}(r, W) - \text{im} \psi| \geq p^{(r-3)(\rho-4)}.$$

**Proof.** Let $\{v_i \mid 1 \leq i \leq d\}$ be a basis of $V$ and let $\bar{V}_i \leq V$ for $1 \leq i \leq N$ be the 4-dimensional subspaces $\langle v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i} \rangle$. Set $X_i \leq \bar{V}_i \wedge \bar{V}_i$ to be one of the subspaces of codimension 3 that are not generated by decomposable wedges (cf. the proof of Proposition 5.5.2 (3)). Now let $U \leq W$ be the standard complement of $\oplus_i (\bar{V}_i \wedge \bar{V}_i)$, so that $v_k \wedge v_l \in U$ as long as not both $v_k, v_l$ belong to the same $\bar{V}_i$. Thus $W = \oplus_i (\bar{V}_i \wedge \bar{V}_i) \oplus U$.

Select any subspace $L \leq U$ of dimension $r - 3N$. Now

$$(\oplus_i X_i \oplus L)^\wedge \leq (\oplus_i X_i \oplus U)^\wedge$$

and, as $U$ is generated by decomposable wedges, it follows that we have a surjection

$$\oplus_i X_i \oplus L \to (\oplus_i X_i \oplus L)^\wedge \to (\oplus_i X_i \oplus U)^\wedge$$

The latter space is of dimension at least $N$, and therefore

$$\dim(\oplus_i X_i \oplus L)/(\oplus_i X_i \oplus L)^\wedge \geq N.$$
It follows that every subspace of $U$ of dimension $r - 3N$ produces a subspace of $W$ of dimension $r$ whose decomposable wedges belong to a subspace of codimension at least $N$. Thus the number of all such subspaces is at least
\[
\left| \text{Gr}(r - 3N, U) \right| \geq p^{(r - 3N)(p - 3N - 1)}.
\]
\[\square\]

**Theorem 5.8.2.** Fix a prime $p > 2$ and a positive integer $M$. Let $\#_{all}(n)$ be the number of all $p$-groups of order $p^n$, and let $\#_{B_{\geq M}}(n)$ be the number of $p$-groups of order $p^n$ whose Bogomolov multiplier is of order at least $M$. Then
\[
\lim_{n \to \infty} \frac{\log_p \#_{B_{\geq M}}(n)}{\log_p \#_{all}(n)} = 1.
\]

**Proof.** Set $\alpha = \frac{2}{3}$ and consider the $p$-groups of order $p^n$ obtained with the above parametrization. Thus $d = \frac{2}{3}n + O(1)$, $r = \frac{2}{3}n^2 + O(n)$. Set $N = \log_p M$. Now, note that we have
\[
\#_{B_{\geq M}}(n) \geq \frac{|\{L \mid L \leq W, \dim L = r, \dim(L/L^\alpha) \geq N\}|}{|\text{GL}(V)|}.
\]
Using Lemma 5.8.1 with $\rho = n - d = \frac{1}{3}n + O(1)$, it follows that
\[
\log_p \#_{B_{\geq M}}(n) \geq r\rho + O(n^2) = \frac{2}{27}n^3 + O(n^2).
\]
The latter is also a log-upper bound for $\#_{all}(n)$ and the proof is complete. \[\square\]

5.9. **Commutators in the end.** The methods in the previous section can be used in other types of problems involving commutators. A sample application is the following elementary statement about commutators in $p$-groups. Its proof captures the heart of the argument of Theorem 5.8.2.

**Proposition 5.9.1.** Fix a prime $p > 2$. Let $\#_{all}(n)$ be the number of all $p$-groups of order $p^n$, and let $\#_{[G,G] \neq K(G)}(n)$ be the number of $p$-groups of order $p^n$ in which not every element of the derived subgroup is a simple commutator. Then
\[
\lim_{n \to \infty} \frac{\log_p \#_{[G,G] \neq K(G)}(n)}{\log_p \#_{all}(n)} = 1.
\]

**Proof.** As in the proof of Theorem 5.8.2, consider only $p$-groups of order $p^n$ obtained by setting $\alpha = \frac{2}{3}$. Let $\tilde{V} \leq V$ be a proper fixed subspace of dimension 4 and let $U$ be the standard complement of $V \wedge \tilde{V}$ in $W$ just like in the proof of Lemma 5.8.1. Every $r$-dimensional subspace $L \leq U$ gives a group $G_L$ of order $p^n$. Factoring by the normal subgroup represented by $U$, we obtain a surjection $G_L \to G_U$. The latter group is the free group of nilpotency class 2 and exponent $p$ on 4 generators, and not every element of its derived subgroup is a simple commutator. The latter property therefore also holds for the group $G_L$. Thus we have
\[
\#_{[G,G] \neq K(G)}(n) \geq \frac{|\text{Gr}(r, U)|}{|\text{GL}(V)|}.
\]
It follows that
\[
\log_p \#_{[G,G] \neq K(G)}(n) \geq r \left( \left( \frac{d}{2} \right) - r - 1 \right) + O(d^2) = \frac{2}{27}n^3 + O(n^2).
\]
\[\square\]
REFERENCES

[1] E. Arrondo, *Introduction to projective varieties*, lecture notes, version November 26, 2017, http://www.mat.ucm.es/~arrondo/projvar.html.

[2] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) 25 (1972), 75–95.

[3] A. Bak, G. Donadze, N. Inassaridze, M. Ladra, *Homology of multiplicative Lie rings*, Journal of Pure and Applied Algebra 208.2 (2007), 761–777.

[4] G. Berman and K. Fryer, *Introduction to Combinatorics*, Academic Press, 1972.

[5] F. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk SSSR Ser. Mat 51 (1987), no. 3, 485–516.

[6] F. Bogomolov, *Abelian subgroups of Galois groups*, Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), no. 1, p. 32-67.

[7] F. Bogomolov and Y. Tschinkel, *Introduction to birational anabelian geometry*, Current developments in algebraic geometry, 17–63, Math. Sci. Res. Inst. Publ., 59, Cambridge Univ. Press, Cambridge, 2012.

[8] F. Bogomolov and Y. Tschinkel, *Universal spaces for unramified Galois cohomology*, Brauer groups and obstruction problems, volume 320 of Progress in Mathematics, 57–86. Birkhäuser/Springer, Cham, 2017.

[9] Y. Chen and R. Ma, *Bogomolov multipliers of some groups of order $p^6$*, Comm. Algebra 49 (2021), no. 1, 242–255.

[10] J. V. Chipalkatti, *Notes on Grassmannians and Schubert varieties*, Queen’s Papers in Pure and Applied Mathematics 13.119 (2000).

[11] C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Mathematics 95 (1972), 281–356.

[12] J.-L. Colliot-Thélène and J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 113–186.

[13] J.D. Dixon, M.P.F. du Sautoy, A. Mann, and D. Segal, *Analytic Pro-p Groups*, Cambridge University Press, 1999.

[14] Z. Dvir, J. Kollár, and S. Lovett, *Variety evasive sets*, Comput. Complexity 23 (2014), no. 4, 509–529.

[15] B. Eick, M. Horn and S. Zandi, *Schur multipliers and the Lazard correspondence*, Archiv der Mathematik 99 (2012), 217–226.

[16] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Graduate Texts in Mathematics, Springer Verlag, 2000.

[17] G. J. Ellis, *Non-abelian exterior products of Lie algebras and an exact sequence in the homology of Lie algebras*, Journal of Pure and Applied Algebra 46.2-3 (1987), 111–115.

[18] G. A. Fernandez-Alcober and U. Jezernik, *Bogomolov multipliers of p-groups of maximal class*, Quart. J. Math. 71 (2020), 121–138.

[19] W. Fulton, *Intersection theory*, Springer Science & Business Media, 2013.

[20] O. Garaialde Ocaña and J. González-Sánchez, *Transporting cohomology in Lazard correspondence*, J. Algebra Appl. 16, 1750119 (2017).

[21] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*. Vol. 165. Cambridge University Press, 2017.

[22] J. González-Sánchez and Th. Weigel, *The Malcev correspondence and Schur multipliers*, preprint, private communication, 2018.

[23] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.

[24] G. Higman, *Enumerating p-groups: I*, Proc. London Math. Soc. (3) 10 (1960), 24–30.

[25] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, GTM 4, Springer, New York, 1971.

[26] A. Hoshi, M. Kang, and B. Kunyavskii, *Noether’s problem and unramified Brauer groups*, Asian J. Math. 17 (2013), 689–713.

[27] V. A. Iskovskikh and J. I. Manin, *Three-dimensional quartics and countereamples to the Lüroth problem*, Math. USSR Sb. 15 (1971), 141–166.

[28] C. U. Jensen, A. Ledet, and N. Yui, *Generic polynomials: constructive aspects of the inverse Galois problem*. Vol. 45. Cambridge University Press, 2002.

[29] U. Jezernik and P. Moravec, *Universal commutator relations, Bogomolov multipliers, and commuting probability*, J. Algebra 428 (2015), 1–25.

[30] U. Jezernik and P. Moravec, *Commutativity preserving extensions of groups*, Proc. Roy. Soc. Edinburgh Sect. A 148 (3) (2018), 575–592.
[31] B. Kostant, *Eigenvalues of a Laplacian and commutative subalgebras*, Topology **3**, 147–159 (1965).

[32] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. École Norm. Sup. (3) **71** (1954), 101–190.

[33] A. Lubotzky and A. Mann, *Powerful $p$-groups I: Finite groups*, J. Algebra **105** (1987), 484–505.

[34] P. Moravec, *Unramified Brauer groups of finite and infinite groups*, Amer. J. Math. **134** (2012), no. 6, 1679–1704.

[35] P. Moravec, *Groups of order $p^5$ and their unramified Brauer groups*, J. Algebra **372** (2012), 420–427.

[36] R. Oliver, *SK$_1$ for finite group rings: II*, Math. Scand. **47** (1980), 195–231.

[37] T. Porter, *Profinite Algebraic Homotopy*, preprint, ncatlab.org/timporter/, 2009.

[38] L. Ribes and P. A. Zaleskii, *Profinite groups*, Ergebnisse der Math. **40**, Springer, Berlin – Heidelberg, 2000.

[39] D. M. Rodney, *Commutators and abelian groups*, J. Austral. Math. Soc. **24** (Series A) (1977), 79–91.

[40] Z. A. Rostami, M. Parvizi, and P. Niroomand, *Bogomolov multiplier and the Lazard correspondence*, Comm. Algebra **48** (2020), no. 3, 1201–1211.

[41] Z. A. Rostami, M. Parvizi, and P. Niroomand, *The Bogomolov multiplier of Lie algebras*, Hacet. J. Math. Stat. **49** (2020), no. 3, 1190–1205.

[42] C. C. Sims, *Enumerating $p$-groups*, Proceedings of the London Mathematical Society **3.1** (1965), 151–166.

[43] I. R. Shafarevich, *Basic algebraic geometry 1: Varieties in projective space*, translated from the 1988 Russian edition and with notes by Miles Reid, (1994).

[44] P. Schneider and O. Venjakob, *SK$_1$ and Lie algebras*, Math. Ann. **357** (2013), no. 4, 1455–1483.

[45] P. Schneider and O. Venjakob, *A splitting for $K_1$ of completed group rings*, Comment. Math. Helv. **88** (2013), no. 3, 613–642.

[46] J.-P. Serre, *Lie algebras and Lie groups: 1964 lectures given at Harvard University*, Springer, 2009.

[47] The Stacks project authors, *The Stacks project*, https://stacks.math.columbia.edu, 2018.

[48] C. Voisin, *Hodge theory and complex algebraic geometry I*, Cambridge Studies in Advanced Mathematics **76**, Cambridge University Press, Cambridge, 2002.

[49] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Invent. Math. **201** (1) (2015), 207–237.

[50] J. S. Wilson, *Profinite groups*, Clarendon Press, Oxford, 1998.

[51] M. Witte, *Non-commutative $L$-functions for $p$-adic representations over totally real fields*, preprint, arXiv:1710.09133, 2017.