ELEMENTARY PROOFS OF THE GAUSS-BONNET THEOREM
AND OTHER INTEGRAL FORMULAS IN $\mathbb{R}^3$

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ABSTRACT. For a compact differentiable surface with boundary embedded in
$\mathbb{R}^3$, we give simple proofs of the Gauss-Bonnet theorem, Poincaré-Hopf theo-
rem, and several other integral formulas. We complete all of the proofs without
using fundamental or differential forms.

1. INTRODUCTION

In this note we prove several integral formulas involving the principal curvatures
of a surface, including the Gauss-Bonnet theorem. All of the proofs are elementary
in the sense that we do not place any co-ordinates on the surface, and hence we
do not need to invoke the theory of fundamental or differential forms. The results
will follow from applying Stokes’ theorem directly to the surface, and studying
objects associated with the surface that are co-ordinate invariant. In particular,
the principal curvatures of the surface will arise as the directional derivatives of the
normal vector to the surface.

2. PRELIMINARIES

All of the identities we will prove are for a compact, oriented $C^\infty$ surface $M$
with boundary $\partial M$ embedded in $\mathbb{R}^3$ with volume element $dA$. At each point on $M$
we denote the position vector by $X$ and the normal vector by $N$. Our results will
follow from this variation of Stokes’ Theorem:

**Claim.** Let $f$ and $g$ be differentiable functions that map an $\mathbb{R}^3$-neighborhood of
$M$ to $\mathbb{R}$. If $P$ and $Q$ are orthonormal vector fields on $M$ that are not necessarily
continuous, and are oriented so that $P \times Q = N$ everywhere on $M$, then:

$$\int_{\partial M} f \, dg = \int_{M} \left[ (\nabla_P f)(\nabla_Q g) - (\nabla_Q f)(\nabla_P g) \right] dA$$

**Proof.** We start with:

$$\int_{\partial M} f \, dg = \int_{\partial M} f \, \nabla g \cdot dX.$$ 

Using Stokes’ Theorem and the identity:

$$\text{curl} (f \nabla g) = \nabla f \times \nabla g$$

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gives:
\[
\int_{\partial M} f \, dg = \int_M \nabla f \times \nabla g \cdot N \, dA
= \int_M \nabla f \cdot \nabla g \times N \, dA.
\]

Since \( P \) and \( Q \) form an orthonormal basis for the tangent plane at any point on \( M \), any tangent vector can be written as the sum of its projection onto \( P \) and its projection onto \( Q \). Therefore:
\[
\nabla g \times N = (\nabla g \cdot N \times P)P + (\nabla g \cdot N \times Q)Q
= (\nabla g \cdot Q)P - (\nabla g \cdot P)Q
= (\nabla g Q)P - (\nabla g P)Q.
\]

We finally get:
\[
\int_{\partial M} f \, dg = \int_M \left[ (\nabla g Q) (\nabla f \cdot P) - (\nabla g P) (\nabla f \cdot Q) \right] \, dA
= \int_M \left[ (\nabla g Q f) - (\nabla g f Q) \right] \, dA.
\]

□

In what follows we will assume that \( P \) and \( Q \) are orthonormal principal directions on \( M \) with corresponding principal curvatures \( \kappa_1 \) and \( \kappa_2 \), and that they are oriented so that \( P \times Q = N \) everywhere on \( M \). We will use the subscripts \( p \) and \( q \) to denote the directional derivatives of a function in the directions \( P \) and \( Q \), respectively. An immediate corollary of the above claim is that if \( V \) and \( W \) are vector fields on \( \mathbb{R}^3 \) then:

\[
\int_{\partial M} V \cdot dW = \int_M (V_p \cdot W_q - V_q \cdot W_p) \, dA.
\]

Since \( P \) and \( Q \) are not required to be continuous in the above equation, we will not run into difficulties if \( M \) contains umbilical points. With these definitions we have:

\[
N_p = -\kappa_1 P, \quad N_q = -\kappa_2 Q.
\]

Finally, we define the mean curvature \( H \) and Gaussian curvature \( K \) at points on \( M \) by:

\[
H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1 \kappa_2.
\]

3. Two Curvature Identities

We begin with two simple consequences of identity (1):

Claim. If \( V \) is a vector field on \( \mathbb{R}^3 \) then:

\[
(3) \quad \int_{\partial M} (V \times N) \cdot dX = - \int_M (V_p \cdot P + V_q \cdot Q + 2H V \cdot N) \, dA
\]

\[
(4) \quad \int_{\partial M} (V \times N) \cdot dN = \int_M (\kappa_2 V_p \cdot P + \kappa_1 V_q \cdot Q + 2K V \cdot N) \, dA.
\]
Proof. Using equations (1) and (2) we get:

\[ \int_{\partial M} (V \times N) \cdot dX = \int_{M} (V_p \times N - \kappa_1 V \times P - V_q \times N \cdot P + \kappa_2 V \times Q) \cdot dA \]

\[ = \int_{M} (V_p \cdot N + \kappa_1 V \cdot P + \kappa_1 V \times N \cdot P - \kappa_2 V \times Q) \cdot dA \]

\[ = \int_{M} (V_p \cdot P - V_q \cdot Q - 2H V \cdot N) \cdot dA \]

\[ = \int_{\partial M} (V \times N) \cdot dN = \int_{M} (\kappa_2 V_p \cdot N + \kappa_1 V_q \cdot Q + 2K V \cdot N) \cdot dA. \]

Equation (3) is a generalization of the divergence theorem for surfaces to vector fields on \( \mathbb{R}^3 \) that appears for example in §7 of [2]. With these two identities we can quickly prove several well-known results.

Let \( \{e_1, e_2, e_3\} \) denote the standard basis for \( \mathbb{R}^3 \). Successively setting \( V \) equal to these basis elements in (3) and (4) gives the vector identities:

\[ \int_{\partial M} N \times dX = -2 \int_{M} H N \cdot dA \]

\[ \int_{\partial M} N \times dN = 2 \int_{M} K N \cdot dA. \]

Setting \( V \) equal to the vectors \( e_1 \times X, e_2 \times X \) and \( e_3 \times X \) in (3) and (4) gives the vector identities:

\[ \int_{\partial M} X \times (N \times dX) = -2 \int_{M} H X \times N \cdot dA \]

\[ \int_{\partial M} X \times (N \times dN) = 2 \int_{M} K X \times N \cdot dA. \]

Setting \( V = X \) in (3) and (4) gives Minkowski’s formulas (cf. [3], pp. 181-185):

\[ \int_{\partial M} X \times N \cdot dX = -2 \int_{M} (1 + H X \cdot N) \cdot dA \]

\[ \int_{\partial M} X \times N \cdot dN = 2 \int_{M} (H + K X \cdot N) \cdot dA. \]

4. The Gauss-Bonnet And Poincaré-Hopf Theorems

We next prove the Gauss-Bonnet theorem and the Poincaré-Hopf theorem using (1) and (4). We start with the following simplified version of Liouville’s formula (cf. [1], problem 11.19):
Claim. Suppose that there exists a constant vector $C$ with $\|C\| = 1$ such that $C \cdot N \neq \pm 1$ on $\partial M$. Let $s$ denote arc length, with the subscript $s$ denoting the derivative with respect to arc length. Let $\theta$ denote the angle between the unit tangent vector $X_s$ to $\partial M$ and the vector $C \times N$ tangent to $M$. There on $\partial M$ there holds:

$$\theta_s = \kappa_g - \frac{C \cdot N}{1 - (C \cdot N)^2} C \times N \cdot N_s$$

where $\kappa_g$ is the geodesic curvature of $\partial M$.

Proof. We use $[ ]$ to denote the triple product of vectors: $[v_1 v_2 v_3] = v_1 \cdot v_2 \times v_3 = v_1 \times v_2 \cdot v_3 = \det(v_1, v_2, v_3)$.

We have the relations:

$$X_s \cdot X_s = 1, \quad X_s \cdot N = 0, \quad N \cdot N = 0.$$  

Differentiating these with respect to $s$ gives:

$$X_s \cdot X_{ss} = 0, \quad X_{ss} \cdot N = -X_s \cdot N_s, \quad N_s \cdot N = 0.$$ 

The geodesic curvature $\kappa_g$ is defined by:

$$\kappa_g = [X_s X_{ss} N]$$

and the angle $\theta$ satisfies the relation:

$$\tan \theta = \frac{X_s \cdot C}{[X_s \cdot CN]}.$$ 

Differentiating $\tan^{-1} \theta$ gives:

$$\theta_s = \frac{[X_s \cdot CN][X_{ss} \cdot C] - (X_s \cdot C)([X_{ss} \cdot CN] + [X_s \cdot CN_s])}{[X_s \cdot CN]^2 + (X_s \cdot C)^2}.$$ 

By projecting onto the orthonormal basis $\{X_s, N, X_s \times N\}$ for $\mathbb{R}^3$ we get:

$$X_{ss} = -(X_s \cdot N_s)N - \kappa_g X_s \times N$$

$$N_s = (X_s \cdot N_s)X_s + [N_s, X_s N] X_s \times N, \quad N_s \times X_s = [N_s, X_s N] N$$

$$\|C\|^2 = (C \cdot X_s)^2 + (C \cdot N)^2 + (C \cdot X_s \times N)^2 = 1.$$ 

Therefore:

$$\theta_s = \frac{[X_s \cdot CN][\kappa_g X_s \cdot C] - (X_s \cdot N)(X_s \cdot N_s))}{[X_s \cdot CN]^2 + (X_s \cdot C)^2} + \frac{\kappa_g (X_s \cdot C)^2 - (X_s \cdot C)(C \cdot N_s \times X_s)}{[X_s \cdot CN]^2 + (X_s \cdot C)^2}$$

$$= \kappa_g - \frac{C \cdot N}{1 - (C \cdot N)^2} ([X_s \cdot CN] (X_s \cdot N_s) + (X_s \cdot C) [N_s, X_s N])$$

$$= \kappa_g - \frac{C \cdot N}{1 - (C \cdot N)^2} ((X_s \times N) \cdot (X_s \cdot C) N_s - (X_s \cdot N_s) C)$$

$$= \kappa_g - \frac{C \cdot N}{1 - (C \cdot N)^2} ((X_s \times N) \cdot (X_s \times (N_s \times C)))$$

$$= \kappa_g - \frac{C \cdot N}{1 - (C \cdot N)^2} [CNN_s]$$

□

We can now prove:
**Theorem** (Gauss-Bonnet).

(6) \( \int_{\partial M} \kappa_g \, ds + \int_M K \, dA = 2\pi \chi(M) \)

*Proof.* We triangulate \( M \) as is done in classical proofs of the theorem (see for example [1] pp. 242-246). For our proof, we choose the triangles small enough so that the range of the Gauss map on each triangle lies properly within an open hemisphere. This will ensure that, for each triangle, we can find a constant vector \( C \) with \( \|C\| = 1 \) and \( C \cdot N \neq \pm 1 \) everywhere in the triangle.

We integrate (5) over the boundary of each triangle. As in the classical proof, the \( \theta \) term in (5) gives rise to the Euler characteristic term \( \chi(M) \) when the exterior angles at the vertices of each triangle are enumerated. To derive the surface integral we substitute:

\[ V = \frac{C \cdot N}{1 - (C \cdot N)^2} C \]

in (5). The resulting integrand over each triangle is then:

\[
\frac{1 + (C \cdot N)^2}{(1 - (C \cdot N)^2)^2} [(C \cdot P)^2 + (C \cdot Q)^2] K - \frac{2(C \cdot N)^2}{1 - (C \cdot N)^2} K,
\]

which equates to \( K \) because \( \{P, Q, N\} \) is an orthonormal basis for \( \mathbb{R}^3 \) and:

\[ \|C\|^2 = (C \cdot P)^2 + (C \cdot Q)^2 + (C \cdot N)^2 = 1. \]

The Poincare´-Hopf theorem will follow from the Gauss-Bonnet theorem and the following integral formula:

**Claim.** If \( V \) is a vector field on \( M \) with \( \|V\| = 1 \) then:

\[ \int_{\partial M} V \times N \cdot dV = \int_M K \, dA \]

*Proof.* Using (1) gives:

\[
\int_{\partial M} V \times N \cdot dV = \int_M (-\kappa_1 V \times P \cdot V_q + \kappa_2 V \times Q \cdot V_p - 2 V_p \times V_q \cdot N) \, dA.
\]

Differentiating the relation \( V \cdot V = 1 \) gives:

\[ V_p \cdot V = V_q \cdot V = N \cdot V = 0, \]

which implies that all of \( V_p, V_q \) and \( N \) lie in a plane perpendicular to \( V \) and the last term in the integral over \( M \) is zero. From the relation \( V \cdot N = 0 \) we get:

\[ V \times P = -(V \cdot Q)N, \quad V \times Q = (V \cdot P)N \]

and differentiating \( V \cdot N = 0 \) gives:

\[ V_p \cdot N = \kappa_1 V \cdot P, \quad V_q \cdot N = \kappa_2 V \cdot Q. \]

The surface integral therefore reduces to:

\[ \kappa_1 \kappa_2 [(V \cdot P)^2 + (V \cdot Q)^2] = K. \]

\( \square \)
Corollary. If $\mathbf{V}$ is a vector field on $M$ that does not vanish on $\partial M$ and has only isolated singularities on $M$ at points in the set $S$ then:

$$\int_{\partial M} \frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} = \int_M K dA - 2\pi \sum_{s \in S} \text{Index}_s(\mathbf{V})$$

Proof. This follows from:

$$\frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} = \frac{\mathbf{V}}{\|\mathbf{V}\|} \times \mathbf{N} \cdot d\left(\frac{\mathbf{V}}{\|\mathbf{V}\|}\right)$$

and the fact that near singularities $s$ of $\mathbf{V}$, the boundary integral approaches $-\pi d\theta$ for the vector field projected onto the tangent plane to $M$ at $s$. □

Combining this with the Gauss-Bonnet theorem to eliminate the total curvature term gives:

**Theorem (Poincaré-Hopf).** If $\mathbf{V}$ is a vector field on $M$ satisfying the same conditions as in the above corollary then:

$$\chi(M) - \sum_{s \in S} \text{Index}_s(\mathbf{V}) = \frac{1}{2\pi} \int_{\partial M} \left(\frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} + \kappa_g ds\right).$$

The above result extends the Poincaré-Hopf theorem to vector fields that are not perpendicular to $\partial M$. When $\mathbf{V}$ is perpendicular to $\partial M$, the boundary integral reduces to zero and gives the traditional result.

5. **Identities Containing The Difference Of Principal Curvatures**

We conclude by proving two identities that contain a term equal to the difference of the principal curvatures, instead of their sum or product:

**Claim.** For any twice differentiable function $F : \mathbb{R}^3 \to \mathbb{R}$ there holds:

$$\int_{\partial M} \nabla F(\mathbf{X}) \cdot d\mathbf{N} = \int_M (\kappa_2 - \kappa_1)(\nabla^2 F(\mathbf{X})) \cdot \mathbf{P} \cdot \mathbf{Q}$$

$$\int_{\partial M} \nabla F(\mathbf{N}) \cdot d\mathbf{X} = \int_M (\kappa_2 - \kappa_1)(\nabla^2 F(\mathbf{N})) \cdot \mathbf{P} \cdot \mathbf{Q}.$$

Proof. These follow from (1) and:

$$\nabla F(\mathbf{X})_p = (\nabla^2 F(\mathbf{X})) \mathbf{P} \quad \nabla F(\mathbf{N})_p = -\kappa_1(\nabla^2 F(\mathbf{N})) \mathbf{P}$$

$$\nabla F(\mathbf{X})_q = (\nabla^2 F(\mathbf{X})) \mathbf{Q} \quad \nabla F(\mathbf{N})_q = -\kappa_2(\nabla^2 F(\mathbf{N})) \mathbf{Q}.$$

□

**References**

[1] Martin Lipschutz, *Schaum’s Outline of Differential Geometry*, McGraw-Hill, 1969.

[2] Leon Simon, *Lectures on Geometric Measure Theory*, Australian National University, 1984.

[3] Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 5, Publish or Perish, Houston, 1999.

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