QUASIPROJECTIVE THREE-MANIFOLD GROUPS AND COMPLEXIFICATION
OF THREE-MANIFOLDS

INDRANIL BISWAS AND MAHAN MJ

Abstract. We characterize the quasiprojective groups that appear as fundamental groups of compact 3-manifolds (with or without boundary). We also characterize all closed 3-manifolds that admit good complexifications. These answer questions of Friedl–Suciu, [FrSu], and Totaro [To].

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1. Introduction

A group is called quasiprojective (respectively, Kähler) if it is the fundamental group of a smooth complex quasiprojective variety (respectively, compact Kähler manifold). Kähler and quasiprojective 3-manifold groups have attracted much attention of late [DiSu, Ko1, BMS, DPS, FrSu, Ko2]. In this paper we characterize quasiprojective 3-manifold groups.

We shall follow the convention that our 3-manifolds have no spherical boundary components. Capping such boundary components off by 3-balls does not change the fundamental group, which is really what interests us here.

Theorem 1.1 (See Theorem 3.4). Let $N$ be a compact 3-manifold (with or without boundary). If $\pi_1(N)$ is a quasiprojective group, then $N$ is either Seifert-fibered or $\pi_1(N)$ is one of the following

- virtually free, or

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• virtually a surface group.

Finer results leading to a complete characterization are given in Section 3.1 and Section 5 (see Theorem 5.6). We omit stating these here as they are slightly more complicated to do so.

This characterization of quasiprojective 3-manifold groups answers Questions 8.3 and Conjecture 8.4 of [FrSu]; see Corollary 5.7 and Corollary 5.9.

The following theorem provides an answer to Question 8.1 of [FrSu] under mild hypotheses.

**Theorem 1.2 (See Theorem 5.13).** Suppose $A$ and $B$ are groups, such that the free product $G = A * B$ is a quasiprojective group. In addition suppose that both $A$ and $B$ admit nontrivial finite index subgroups, and at least one of $A, B$ has a subgroup of index greater than 2. Then each of $A, B$ are free products of cyclic groups. In particular both $A$ and $B$ are quasiprojective groups.

A good complexification of a closed smooth manifold $M$ is defined to be a smooth affine algebraic variety $U$ over the real numbers such that $M$ is diffeomorphic to $U(\mathbb{R})$ and the inclusion $U(\mathbb{R}) \rightarrow U(\mathbb{C})$ is a homotopy equivalence [To]. Totaro asks whether a closed smooth manifold $M$ admits a good complexification if and only if $M$ admits a metric of non-negative curvature [To, p. 69, 2nd para]. As an application of Theorem 1.1, we prove this in the following strong form for 3-manifolds.

**Theorem 1.3 (See Theorem 4.5).** A closed 3-manifold $M$ admits a good complexification if and only if one of the following hold:

1. $M$ admits a flat metric,
2. $M$ admits a metric of constant positive curvature, and
3. $M$ is covered by the (metric) product of a round $S^2$ and $\mathbb{R}$.

Curiously, the proof of Theorem 1.3 is direct and there is virtually no use of the method or results of [Ka, To, DPS, FrSu]. Our main tools from recent developments in 3-manifolds are:

1. The Geometrization Theorem and its consequences (cf. [AFW]).
2. Largeness of 3-manifold groups [Ag, Wi, LoNi, CLR, La].

The basic complex geometric tool is a theorem of Bauer, [Bau], regarding existence of irrational pencils for quasiprojective varieties (the theorem of Bauer is recalled in Theorem 2.7). It is a useful existence result in the same genre as the classical Castelnuovo-de Franchis Theorem and a theorem of Gromov [Gr, ABCKT].

As a consequence of our results we deduce, in Section 3.1.1, the restrictions on quasiprojective 3-manifold groups obtained by the authors of [DPS, FrSu, Ko2] and the restrictions on good complexifications of 3-manifolds deduced in [To]. We also indicate, in Remark 3.12 how to deduce the classification of (closed) 3-manifold Kähler groups [DiSu, Ko1, BMS] using the techniques of Theorem 1.1 thus providing a unified treatment of known results.

2. Preliminaries

2.1. **Three-manifold groups.** We collect together facts about 3-manifold groups that will be used here.

**Definition 2.1.**

1. A group $G$ is quasiprojective (respectively, quasi-Kähler) if it can be realized as the fundamental group of a smooth quasiprojective complex variety (respectively, quasi-Kähler manifold).
2. A group $G$ is a 3-manifold group if it can be realized as the fundamental group of a compact real 3-manifold (possibly with boundary).
3. A group $G$ is large if it has a finite index subgroup that admits a surjective homomorphism onto a non-abelian free group (and hence in particular onto $F_3$).

A prime 3-manifold (possibly with boundary) is a 3-manifold that cannot be decomposed as a non-trivial connected sum. **Graph manifolds** are prime 3-manifolds obtained by gluing finitely many Seifert-fibered JSJ components along boundary tori. In particular, torus bundles over a circle are graph
manifolds. Among the graph manifolds, Sol and Seifert manifolds are geometric; the rest are non-geometric. It follows that the gluing maps between the Seifert components in non-geometric manifolds do not identify circle fibers. (See [AFW] p. 59 and [He1] Ch. 3.)

The following omnibus theorem is the consequence of the Geometrization theorem of Thurston-Perelman and work of a large number of people culminating in the resolution of the virtual Haken problem by Agol and Wise. See [AFW] (especially Diagram 1, p. 36) for an excellent account.

**Theorem 2.2.** If a 3-manifold $M$ has a prime component $N$ satisfying one of the following three conditions, then the fundamental group of $M$ is large.

1. $N$ is a compact orientable irreducible 3-manifold with non-empty boundary such that $M$ is not an $I$-bundle (“$I$” is a closed interval) over a surface with non-negative Euler characteristic [CLR, La].
2. $N$ is closed hyperbolic [Ag, Wi].
3. $N$ is a closed, non-geometric graph manifold [LoNi].

If $\pi_1(M)$ is a nontrivial free product $G_1 \ast G_2$ (e.g., if $M$ is not prime), where at least one $G_i$ has order greater than 2, then the fundamental group of $M$ is large. The exceptional case $(\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z})$ is realized only by the connected sum of two real projective spaces.

As an immediate corollary we have the following:

**Corollary 2.3.** If the fundamental group of $M$ is not large, then $M$ is Seifert-fibered or a Sol manifold.

A finitely presented group is coherent if any finitely generated subgroup is finitely presented.

**Theorem 2.4 ([Sc]).** Fundamental groups of compact 3-manifolds are coherent.

A consequence is the following [He1, Ch. 11].

**Proposition 2.5.** Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite finitely generated groups with $G$ the fundamental group of a compact orientable 3-manifold (possibly with boundary). Then

1. either $H$ is infinite cyclic and $Q$ is the fundamental group of a 2-orbifold,
2. or $H$ is the fundamental group of a closed surface and $Q$ is virtually cyclic.

Another theorem that will be used is:

**Theorem 2.6 ([Bas]).** A finitely generated group $G$ is virtually free if and only if $G$ can be represented as the fundamental group of a finite graph of groups where all vertex and edge groups are finite.

### 2.2. Fibrations.

We shall require a generalization, due to Bauer, of the classical Castelnuovo-de Franchis theorem on the existence of an irrational pencil on a projective variety to the case of a quasiprojective variety.

**Theorem 2.7 ([Bau, p. 442]).** Let $X$ be a smooth complex quasiprojective variety such that $\pi_1(X)$ admits a surjection onto a group $G$ that admits a finite presentation with $n$ generators and $m$ relations, where $n - m \geq 3$. Then there exists an integer $\beta \geq m - n$ and a quasiprojective curve $C$ with first Betti number $\beta$ and a logarithmic irrational pencil (in particular a holomorphic fibration) $f : X \to C$ with connected fibers.

The proof of Theorem 2.7 in [Bau] combined with Remark 2.3(1) in [Bau] furnishes the following:

**Proposition 2.8 ([Bau]).** Let $X$ be a smooth quasiprojective variety, and let $\overline{X}$ denote a smooth compactification such that $\overline{X} \setminus X$ is a divisor with normal crossings. Further suppose that $\pi_1(X)$ admits a surjection onto a group $G$ that admits a finite presentation with $n$ generators and $m$ relations, where $n - m \geq 3$. Let $C, f$ be the quasiprojective curve and holomorphic fibration obtained in Theorem 2.7. Let $\overline{C}$ denote the projective completion of $C$. Then there exists $f_1 : \overline{X} \to \overline{C}$ such that $f_1|_X = f$. In particular, the fibers of $f_1$ are quasiprojective.

We shall also require the following:
Theorem 2.9 ([Bau, Theorem 2.1]). Let $X$ be a smooth quasiprojective variety, and let $\overline{X}$ denote a smooth compactification such that $\overline{X} \setminus X$ is a divisor with normal crossings. Every maximal real isotropic subspace $V \subset H^1(X, \mathbb{C})$ determines a logarithmic irrational pencil $f : X \to C$. The curve $C$ is complete if and only if $V$ is contained in $H^1(\overline{X}, \mathbb{C})$.

Proof. Only the last statement is not explicitly mentioned in [Bau].

To prove the last statement, note that the fibers of $f$ are intersections of fibers of $f_1$ with $X$. Fibers of $f_1$ are projective varieties as $f_1$ is algebraic. Hence fibers of $f$ are quasiprojective.

Corollary 2.10. Let $X, C, f$ be as in Theorem 2.9. Then there is an exact sequence
\[ 1 \to H \to \pi_1(X) \to \pi_1(C) \to 1 \]
with $H$ finitely generated.

Proof. The subgroup $H$ is the kernel of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(C)$. The only statement that requires a proof therefore is that $H$ is finitely generated.

The subgroup $H$ is the image of the fundamental group of a regular fiber of $f$ in $\pi_1(X)$. Since a regular fiber of $f$ is quasiprojective by Proposition 2.8, its fundamental group is finitely generated. So $H$ is a quotient of a finitely generated group. This implies that $H$ is finitely generated.

3. Quasiprojective three-manifold groups

In this section, we combine Theorem 2.2 with Theorem 2.7 to completely characterize quasiprojective 3-manifold groups.

We shall use the following restriction on quasiprojective groups due to Arapura and Nori which says that solvable quasiprojective groups are virtually nilpotent.

Theorem 3.1 ([ArNo]). Let $N$ be a closed 3-manifold such that $\pi_1(N)$ is a quasiprojective group. Then $N$ is not a Sol manifold.

Theorem 3.2. Let $N$ be a closed 3-manifold, such that $\pi_1(N)$ is a quasiprojective group. Then $N$ is either Seifert-fibered or $N$ is finitely covered by $\#_m S^2 \times S^1$.

Proof. By Theorem 3.1 we can exclude the case where $N$ is a Sol manifold. Hence it follows that if $\pi_1(N)$ is not large, then, by Corollary 2.8 the manifold $N$ is Seifert-fibered.

Next suppose $\pi_1(N)$ is large. Then there exists a finite index subgroup $G$ of $\pi_1(N)$ such that $G$ admits a surjection onto the free group $F_3$.

Since $\pi_1(N)$ is quasiprojective, so is $G$. Let $X$ be a smooth quasiprojective variety with fundamental group $G$. By Theorem 2.7, there exists a holomorphic fibration $f$ (with connected fibers) of $X$ over a quasiprojective curve $C$ with first Betti number greater than two. By passing to a finite sheeted (orbifold) cover of the base if necessary, we can assume without loss of generality that $f$ has no multiple fibers.

By Proposition 2.8 the generic fiber $F$ is quasiprojective and hence has finitely generated fundamental group. Let $H$ denote the image of $\pi_1(F)$ in $\pi_1(X)$. Now we have an exact sequence
\[ 1 \to H \to \pi_1(X) \to \pi_1(C) \to 1. \]

If $C$ is closed, it follows from Proposition 2.10 that $X$ is Seifert fibered.

Else $C$ is quasiprojective non-compact. Hence by Proposition 2.8 again, the subgroup $H$ is finite and $G$ is virtually free. By Grushko’s theorem, [He1, p. 25, Theorem 3.4], the manifold $N$ is finitely covered by a connected sum $\#_m S^2 \times S^1$.

Proposition 3.3. Let $N$ be a 3-manifold with at least one boundary component of positive genus. Assume that $\pi_1(N)$ is an infinite quasiprojective group. Then $\pi_1(N)$ is either virtually free or virtually of the form $\mathbb{Z} \times F_n$ ($n \geq 1$) or virtually a surface group.
Proof. By Theorem 2.2(1), either $N$ is an $I$–bundle over a surface of non-negative Euler characteristic or it is large. If $N$ is an $I$–bundle over a surface of non-negative Euler characteristic, then $\pi_1(N)$ is either $\mathbb{Z}$ or virtually $\mathbb{Z} \oplus \mathbb{Z}$.

Else, by the same argument as in the proof of Theorem 3.2 we have an exact sequence

$$1 \to H \to \pi_1(X) \to \pi_1(C) \to 1$$

with $H$ either $\mathbb{Z}$ or finite, and $C$ a (possibly noncompact) surface. If $H$ is finite, then $\pi_1(N)$ is either virtually free or virtually a surface group.

If $H$ is $\mathbb{Z}$, then $N$ is Seifert-fibered with base a compact orbifold surface with boundary. Consequently, $\pi_1(N)$ is either virtually cyclic or virtually of the form $\mathbb{Z} \times F_n$ with $n \geq 1$.

Combining Theorem 3.2 and Proposition 3.3 we have the following:

**Theorem 3.4.** Let $N$ be a compact 3-manifold (with or without boundary) such that $\pi_1(N)$ is a quasiprojective group. One of the following is true:

1. $N$ is closed Seifert-fibered,
2. $\pi_1(N)$ is virtually free,
3. $\pi_1(N)$ is virtually of the form $\mathbb{Z} \times F_n$ with $n \geq 1$,
4. $\pi_1(N)$ is virtually a surface group.

### 3.1. Refinements and consequences.

**Remark 3.5.** The proof of Theorem 3.4 gives us a bit more. A standing assumption in this section is that $N$ is a compact 3-manifold (with or without boundary) and $\pi_1(N)$ is quasiprojective.

**Case 1:** $N$ is closed prime. Then Theorem 3.2 forces $M$ to be Seifert-fibered.

**Case 2:** $N$ is closed but not prime. Then from Theorem 3.2 the fundamental group $\pi_1(M)$ is virtually free and hence by Theorem 2.6, $\pi_1(M)$ is the fundamental group of a graph of groups with edge and vertex groups finite. Hence each prime component of $M$ has fundamental group that is either finite or virtually cyclic. By the classification of such 3-manifold groups (see [AFW, Theorems 1.1, 1.12], [He1, Theorem 9.13]), $\pi_1(M)$ is of the form $G_1 \ast G_2 \ast \cdots \ast G_k$, where each $G_i$ is either the fundamental group of a spherical 3-manifold or $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.

**Case 3:** $N$ is an $I$–bundle over a surface of non-negative Euler characteristic. Then $\pi_1(N)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ or the fundamental group of a Klein bottle. It turns out (see below) that all these three groups are quasiprojective.

**Case 4:** $N$ has a boundary component of positive genus and $\pi_1(N)$ contains an infinite cyclic normal subgroup. Then by Proposition 3.3, the manifold $N$ is Seifert-fibered with base a compact orbifold surface with boundary. In this case a subgroup $G$ of index at most 2 in $\pi_1(N)$ (if $N$ is non-orientable) or one, i.e., $\pi_1(N)$ itself (if $N$ is orientable) contains an infinite cyclic central subgroup $\langle t \rangle$ such that the quotient $G/\langle t \rangle$ is a free product of cyclic groups (finite or infinite) [He1, p. 118].

**Case 5:** $N$ has a boundary component of positive genus and $\pi_1(N)$ does not contain an infinite cyclic normal subgroup. Then by Proposition 3.3

1. either $\pi_1(N)$ is virtually a surface group in which case $N$ is an $I$–bundle over a surface,
2. or after compressing the boundary as far as possible, $N = M \# H$, where $H$ is a (possibly non-orientable) handlebody and hence $\pi_1(H)$ is free, and $M$ is a closed manifold covered by Case 2.

We now demonstrate the converse to Theorem 3.4 by describing examples of smooth quasiprojective varieties that realize the groups occurring in Remark 3.5 as their fundamental groups. To do this we shall restrict ourselves to orientable compact 3-manifolds with or without boundary.
We start with a lemma that is probably well-known to experts. We were unable to find an exact reference and hence provide a proof (see [BiMj, Section 5.3] for a closely related construction).

**Lemma 3.6.** Let $X$ be a smooth complex quasiprojective variety, and let $G$ be a finite group acting by automorphisms on $X$. Then the orbifold fundamental group of $X/G$ is quasiprojective.

**Proof.** Let $W$ be a smooth projective variety admitting a free $G$–action by automorphisms. Such varieties exist by a theorem of Serre, [ABCKT, Example 1.11], which says that any finite group is realizable as the fundamental group of a smooth projective variety.

Let $Y = X \times W$. Then the diagonal action of $G$ on $Y$ is free and the (usual) fundamental group of the quotient $Y/G$ coincides with the orbifold fundamental group of $X/G$. □

The next proposition addresses Cases (1) and (4) in Remark 3.5.

**Proposition 3.7.** Let $N$ be Seifert-fibered with fiber subgroup in the center of $\pi_1(N)$ such that the base surface is orientable (with or without boundary). Then $\pi_1(N)$ is quasiprojective.

**Proof.** Let $Q$ be the orientable base orbifold of $N$. Then $Q$ admits the structure of an algebraic curve (projective or quasiprojective according as $Q$ is without boundary or with boundary). Consider the quasiprojective orbifold given by $Q$ (after we put a quasiprojective structure on it). Let $\mathcal{L}$ be an orbifold algebraic line bundle on $Q$ such that

- for each point $x \in Q$, the action of the isotropy group for $x$ on the fiber $\mathcal{L}_x$ is faithful, and
- the degree of $\mathcal{L}$ is the degree of the Seifert-fibration.

Let $L$ denote the underlying variety for the orbifold $\mathcal{L}$. Let $\Sigma \subset L$ be the image of the zero-section of $\mathcal{L}$. Then the complement $L \setminus \Sigma$ is a smooth quasiprojective variety with the same fundamental group as $N$. □

To address Case (3), we observe first that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are both quasiprojective. So only the fundamental group of a Klein bottle remains. Let

$$\phi : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$$

be defined by $(z_1, z_2) \mapsto (\frac{1}{z_1}, -z_2)$. Let $Q \subset \text{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ be the order 2 subgroup generated by $\phi$. Then $Q$ acts freely on $C$, and the quotient $C/Q$ has the same homotopy type as a Klein bottle.

In order to completely answer the question “Which 3-manifold groups are quasi-projective?” it remains to deal with virtually free groups or virtually surface groups. These will be addressed in Section 5 after developing some further tools in Section 4.

### 3.1.1. Consequences.

We deduce some of the results that preceded this paper from Theorem 3.4.

**Theorem 3.8** ([DPS, Theorem 1.1]). Let $G$ be the fundamental group of a closed orientable 3-manifold $M$. Assume $M$ is formal. Then the following are equivalent.

1. The Malcev completion of $G$ is isomorphic to the Malcev completion of a quasi-Kähler group.
2. The Malcev completion of $G$ is isomorphic to the Malcev completion of the fundamental group of $S^3 \# n(S^1 \times S^2)$, or $S^1 \times \Sigma_g$, where $\Sigma_g$ denotes a closed orientable surface of genus $g$ with $g \geq 1$.

**Proof.** This follows from Theorem 3.2 by observing that a Seifert-fibered space is formal if and only if it is finitely covered by a trivial circle bundle. □

**Theorem 3.9** ([FrSu, Theorem 1.2]). Let $N$ be a 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all the closed prime components of $N$ are graph manifolds.

**Proof.** All the closed prime components of $N$ are in fact Seifert-fibered by Theorem 3.2 and Remark 3.6 Case (5). □
Theorem 3.10 ([Ko2]). Let \( N \) be a 3-manifold with non-empty boundary. If \( \pi_1(N) \) is a projective group, then \( N \) is an \( I \)-bundle over a closed orientable surface.

Proof. Case 3 and Case 5(1) of Remark 3.5 give that \( N \) is an \( I \)-bundle over a closed surface \( S \). If \( S \) is non-orientable, then \( \pi_1(S) \) is not projective, hence \( \pi_1(N) \) is not projective.

Case 4 of Remark 3.5 forces a finite index subgroup \( H \) of \( \pi_1(N) \) to be isomorphic to \( F_n \times \mathbb{Z} \), with \( n > 1 \). The group \( H \) is not projective and hence \( \pi_1(N) \) is not projective.

Case 5 (2) of Remark 3.5 along with Theorem 2.6 forces a finite index subgroup \( H \) of \( \pi_1(N) \) to be isomorphic to \( F_n \), with \( n > 1 \). The group \( H \) is not projective and hence \( \pi_1(N) \) is not projective. \( \square \)

Remark 3.11. Kotschick proves Theorem 3.10 in the context of Kähler groups. The proof we have given above works equally well in the Kähler case. The only point to be noted is that we have to replace the use of Theorem 2.7 by the analogous theorem in the Kähler context ensuring existence of irrational pencils as in [Gr] or [DeGr].

Remark 3.12. In order to recover the main Theorems of [DiSu] or [Ko1] from Theorem 3.4 with the modifications mentioned in Remark 3.11 it remains to show that fundamental groups of circle bundles \( N \) over closed surfaces of positive genus are not Kähler. If the bundle is trivial, then \( b_1(N) \) is odd. If the bundle is non-trivial, then the cup product vanishes identically on \( H^1 \). Hence the maximal isotropic subspace of \( H^1 \) has dimension \( 2g \), which would imply that \( \pi_1(N) \) would admit a surjection onto the fundamental group of a surface of genus \( 2g \), a contradiction.

Following [To] p. 69, define a good complexification of a closed manifold \( M \) without boundary to be a smooth affine algebraic variety \( U \) over \( \mathbb{R} \) such that \( M \) is diffeomorphic to the space \( U(\mathbb{R}) \) of real points and the inclusion \( U(\mathbb{R}) \hookrightarrow U(\mathbb{C}) \) is a homotopy equivalence.

Using Theorem 3.2 we have an alternative proof of the following theorem of Totaro.

Theorem 3.13 ([To, Section 2]). Let \( M \) be a closed orientable 3-manifold with a good complexification. Then either the cup product \( H^1(M, \mathbb{Q}) \otimes H^1(M, \mathbb{Q}) \to H^2(M, \mathbb{Q}) \) is 0 or \( M \) is formal.

Proof. By Theorem 3.2 \( M \) is

1. either finitely covered by \( \#_n(S^1 \times S^2) \) in which case the above cup product is 0,
2. or \( M \) is Seifert-fibered and finitely covered by either \( S^3 \) or a trivial circle bundle over a closed orientable surface; in this case \( M \) is formal,
3. or \( M \) is finitely covered by a non-trivial circle bundle over a closed surface of positive genus; in this case, the above cup product is zero.

This completes the proof. \( \square \)

Remark 3.14. In the definition of a good complexification, if the affine variety over \( \mathbb{R} \) is weakened to a Stein manifold equipped with an antiholomorphic involution, then all manifolds admit such a complexification. Indeed, given a manifold \( M \), the total space of the cotangent bundle \( T^*M \) admits a Stein manifold structure such that the multiplication by \(-1\) on \( T^*M \) is an antiholomorphic involution.

4. Classification of three-manifolds with good complexification

The definition of a good complexification was recalled prior of Theorem 3.13. In this Section we shall describe all 3-manifolds admitting a good complexification.

Lemma 4.1. If a closed smooth manifold \( M \) admits a good complexification, and \( M_1 \) is a finite-sheeted étale cover of \( M \), then \( M_1 \) also admits a good complexification.

Proof. Let \( U \) be a good complexification of \( M \). Fix a diffeomorphism of \( M \) with \( U(\mathbb{R}) \). Since the inclusion \( U(\mathbb{R}) \hookrightarrow U(\mathbb{C}) \) induces an isomorphism of fundamental groups, the covering \( M_1 \) of \( M = U(\mathbb{R}) \) has a unique extension to a covering \( U_1 \) of \( U(\mathbb{C}) \). For any point \( x \in U(\mathbb{R}) \), the Galois (antiholomorphic) involution \( \sigma \) of \( U(\mathbb{C}) \) for the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) induces the identity map of \( \pi_1(U(\mathbb{C}), x) \).
because \( \sigma|_{U(\mathbb{R})} = \text{Id}_{U(\mathbb{R})} \) and the inclusion \( U(\mathbb{R}) \hookrightarrow U(\mathbb{C}) \) induces an isomorphism of \( \pi_1(U(\mathbb{C}), x) \) with \( \pi_1(U(\mathbb{R}), x) \). Therefore, \( \sigma \) has a unique lift \( \sigma' \) to \( U'_1 \) that fixes \( M_1 \) pointwise.

The pair \((U'_1, \sigma')\) defines a smooth affine variety over \( \mathbb{R} \) (cf. [FuSiu] p. 157, Lemma 4.1). Now the variety \((U'_1, \sigma')\) defined over \( \mathbb{R} \) is a good complexification of \( M_1 \).

Let \( M \) be a closed 3-manifold admitting a good complexification. From Theorem 3.2 it follows that \( M \) is either closed Seifert-fibered or is finitely covered by \( \#_m S^2 \times S^1 \). We shall therefore consider separately the following problems:

1. Which Seifert-fibered manifolds admit good complexifications?
2. Does \( \#_m S^2 \times S^1 \) (\( m > 1 \)), admit a good complexification?

Seifert-fibered 3-manifolds split into three further sub-cases according to the genus of the orbifold base \( S \) of the fibration:

- (1a) \( \text{genus}(S) = 0 \),
- (1b) \( \text{genus}(S) = 1 \), and
- (1c) \( \text{genus}(S) > 1 \).

First we consider case (1a). If \( \text{genus}(S) = 0 \), then \( M \) is covered by \( S^3 \) or \( S^2 \times S^1 \) (this follows from the Poincaré conjecture and classical 3-manifold topology [AFMW] Theorem 1.12). It is known that manifolds covered by \( S^3 \) or \( S^2 \times S^1 \) admit good complexification [To], [Ku].

4.1. Seifert-fibered manifolds with base hyperbolic. Now we consider case (1c).

**Proposition 4.2.** Let \( N \) be Seifert-fibered with hyperbolic base orbifold. Then \( N \) does not admit a good complexification.

**Proof.** Seifert-fibered manifolds are finitely covered by circle bundles over surfaces. Since a finite cover of a good complexification is a good complexification (see Lemma 1.1), it suffices to rule out principal \( S^1 \)-bundles \( N \) over surfaces \( S \) with \( \text{genus}(S) = g > 1 \) and trivial orbifold structure.

So \( N \) is now a principal \( S^1 \)-bundle over a compact oriented surface \( S \) with \( \text{genus}(S) = g > 1 \).

Let, if possible, \( X \) be a good complexification of \( N \). Let \( X_C = X(\mathbb{C}) \) be the base change of \( X \) to \( \mathbb{C} \).

If the principal \( S^1 \)-bundle \( N \rightarrow S \) is nontrivial, then the fundamental group \( \pi_1(N) \) admits a presentation

\[
\langle a_1, \ldots, a_g, b_1, \ldots, b_g, t \mid \prod_{i=1}^{g} [a_i, b_i]^{t^n} \rangle.
\]

In that case, by Theorem 2.7 there exists an irrational logarithmic pencil

\[
\tilde{f} : X_C \rightarrow C
\]
onto a quasiprojective curve \( C \) with \( b_1(C) \geq (2g + 1 - 1) = 2g \). If \( C \) is non-compact, then \( \pi_1(N) \) must admit a surjection onto the free group \( F_{2g} \), which is impossible. Hence \( C \) is compact.

Alternatively, if \( N \) is the trivial principal \( S^1 \)-bundle over \( S \), then \( \pi_1(N) \) admits a surjection onto \( \pi_1(S) \). Hence by Theorem 2.7 there exists a holomorphic fibration as in \( 4.1 \) onto a quasiprojective curve \( C \) with \( b_1(C) \geq (2g - 1) \). If \( C \) is non-compact, then \( \pi_1(N) \) must admit a surjection onto \( F_{2g-1} \) which is impossible. Hence \( C \) is compact also in this case.

In either case the genus of \( C \) is \( g \).

Let

\[
\sigma : X_C \rightarrow X_C
\]
denote the antiholomorphic involution corresponding to the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \). Fix an identification of \( N \) with \( X_C^\sigma = X(\mathbb{R}) \). The action of \( \sigma \) on \( H^1(X_C, \mathbb{C}) \) is trivial because the inclusion \( X_C^\sigma \hookrightarrow X_C \) is a homotopy equivalence. There is a natural bijection between the irrational logarithmic pencils as in \( 4.1 \) and the maximal real isotropic subspaces of \( H^1(X_C, \mathbb{C}) \) satisfying certain conditions (see the first paragraph in [Ban] p. 442)). In view of this bijective correspondence, from the fact that the
action of \( \sigma \) on \( H^1(X_C, \mathbb{C}) \) is trivial we conclude that the fibration in (4.1) commutes with \( \sigma \). In other words, \( \sigma \) descends to an antiholomorphic involution
\[
\sigma_1 : C \rightarrow C
\]
of \( C \). Note that inclusion
\[
C^{\sigma_1} \supset f(X_C^\sigma)
\]
holds, where \( C^{\sigma_1} \) is the fixed point set for \( \sigma_1 \).

The restriction of \( f \) to \( N = X_C^\sigma \) is surjective because at the level of fundamental groups it simply annihilates the cyclic normal fiber subgroup. Therefore, from (4.3) it follows that \( C^{\sigma_1} = C \). This is a contradiction because the identity map of \( C \) is not antiholomorphic. Hence \( N \) cannot admit a good complexification. \( \square \)

4.2. Nil manifolds. We now consider the second case where the orbifold base of the Seifert fibration is flat (the genus of the orbifold is 1).

Non-trivial circle bundles over Euclidean orbifolds are also called *nil manifolds*.

**Proposition 4.3.** Let \( N \) be a nil manifold. Then \( N \) does not admit a good complexification.

**Proof.** As before, in view of Lemma 4.1 it suffices to rule out non-trivial principal \( S^1 \)-bundles \( N \) over the torus with trivial orbifold structure.

So \( N \) is a nontrivial principal \( S^1 \)-bundle over a surface of genus one.

Suppose \( X \) is a good complexification of \( N \). As before, let
\[
\sigma : X_C \rightarrow X_C
\]
denote the antiholomorphic involution corresponding to the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \).

Let
\[
\text{Alb} : X_C \rightarrow C
\]
be the (quasi) Albanese map. Then \( C \) has fundamental group \( \mathbb{Z} \oplus \mathbb{Z} \) and hence \( C \) is either an elliptic curve or the semiabelian variety \( \mathbb{C}^* \times \mathbb{C}^* \) (cf. \([\text{NWY}]\)).

**Case 1:** If \( C \) is an elliptic curve, then the same arguments as in Section 4.1 now go through as before. It leads to the conclusion that the real dimension of the fixed set of the involution \( \sigma \) is 4, which is a contradiction.

**Case 2:** Assume therefore that \( C \) is the semiabelian variety \( \mathbb{C}^* \times \mathbb{C}^* \). If \( \dim_{\mathbb{C}}(\text{Alb}(X)) = 1 \), then \( \text{Alb}(X) \) is a curve with fundamental group \( \mathbb{Z} \oplus \mathbb{Z} \) and the same argument as in the proof of Proposition 4.2 goes through.

**Case 3:** Hence suppose that \( \dim_{\mathbb{C}}(\text{Alb}(X)) = 2 \), in which case all the fibers of \( \text{Alb} \) are quasiprojective curves.

**Case 3A:** If some fiber of \( \text{Alb} \) is a singular curve, the same (complex Morse theoretic) arguments as in \([\text{Ka}, \text{Lemmas 4, 7}]\) (see also \([\text{BMP}, \text{Theorem 7.9}]\)) show that the kernel of \( \text{Alb} : \pi_1(X) \rightarrow \pi_1(C) \) is infinitely presented.

**Case 3B:** Hence the fibers of \( \text{Alb} \) must all be regular. This forces \( \pi_1(F) = \mathbb{Z} \) and hence \( F = \mathbb{C}^* \) (since \( F \) is a curve). Thus \( X \) is a holomorphic \( \mathbb{C}^* \)-bundle over \( \mathbb{C}^* \times \mathbb{C}^* \).

We note that the involution \( \sigma \) commutes with \( \text{Alb} \). This is because \( \text{Alb} \) is the base change to \( \mathbb{C} \) of a morphism between varieties defined over \( \mathbb{R} \). Therefore, \( \sigma \) descends to an antiholomorphic involution
\[
\sigma_1 : C \rightarrow C
\]
Since the fixed point set \( C^{\sigma_1} \subset C \) for the involution \( \sigma_1 \) contains \( \text{Alb}(X_C^\sigma) \), and \( X_C^\sigma \) is nonempty, we know that \( C^{\sigma_1} \) is nonempty. Consequently,
\[
C^{\sigma_1} = S^1 \times S^1.
\]

Therefore, \( X_C^\sigma = N \) is a principal \( S^1 \)-bundle over \( C^{\sigma_1} = S^1 \times S^1 \). We will show that the first Chern class of this principal \( S^1 \)-bundle on \( C^{\sigma_1} \) vanishes.
The first Chern class of the above principal $S^1$–bundle over $C^{\sigma_1}$ coincides with the first Chern class of the principal $\mathbb{C}^*$–bundle $X_C$ in (1.3) after we identify $H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})$ with $H^2(C^{\sigma_1}, \mathbb{Z})$ using the inclusion of $C^{\sigma_1}$ in $C$. Therefore, it suffices to show that the first Chern class of an algebraic line bundle over $\mathbb{C}^* \times \mathbb{C}^*$ vanishes.

Take any algebraic line bundle $L$ over $\mathbb{C}^* \times \mathbb{C}^*$. The line bundle $L$ extends to an algebraic line bundle over the projective surface $\mathbb{P}^1 \times \mathbb{P}^1$. To see this, take the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of any divisor in $\mathbb{C}^* \times \mathbb{C}^*$ representing $L$. Let $L' \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be an extension of $L$. Therefore, $c_1(L) = \iota^*c_1(L')$, where $\iota: \mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the inclusion map. But

$$\iota^*(H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})) = 0.$$ 

Therefore, $c_1(L) = 0$.

Since $X_C^\sigma = N$ is the trivial $S^1$–bundle over $S^1 \times S^1$, we conclude that $N = S^1 \times S^1 \times S^1$. This contradicts the given condition that $N$ is a nil manifold. \hfill $\Box$

### 4.3. Connected sum of copies of $S^2 \times S^1$

Now we consider case (2).

**Proposition 4.4.** Let $N$ be any closed 3-manifold with virtually free fundamental group and suppose that $\pi_1(N)$ is not virtually cyclic. Then $N$ does not admit a good complexification.

**Proof.** Any closed 3-manifold with virtually free fundamental group is covered by a connected sum of copies of $S^2 \times S^1$. Therefore, in view of Lemma 4.1 it is enough to rule out $N = \#_mS^2 \times S^1$, where $m > 1$.

The argument here follows that in Section 4.1. We continue with the same notation. By passing to a finite-sheeted cover, we can assume that $m \geq 3$. So Theorem 2.7 applies to give

$$f: X_C \rightarrow C,$$

where $C$ is a quasiprojective curve with $b_1(C) \geq m \geq 3$. This forces $C$ to be noncompact and $f_* : \pi_1(X_C) \rightarrow \pi_1(C)$ to be an isomorphism.

As shown in the proof of Proposition 4.2, the morphism $f$ commutes with the antiholomorphic involution $\sigma$ of $X_C$. Therefore, $\sigma$ descends to an involution $\sigma_1$ of $C$ (as in (1.2)). The fixed point locus $C^{\sigma_1}$ is a disjoint union of (real) one dimensional proper (embedded) submanifolds of $C$. The image $f(X_C^\sigma) \subset C^{\sigma_1}$ is a connected component of $C^{\sigma_1}$, in particular, $f(X_C^\sigma)$ is a connected proper (embedded) submanifold of $C$ of dimension one.

The inclusion $f(X_C^\sigma) \hookrightarrow C$ induces an isomorphism of fundamental groups. On the other hand, we have $b_1(C) \geq m \geq 3$. Therefore, there is no connected proper (embedded) submanifold of $C$ of dimension one such that the inclusion induces an isomorphism of fundamental groups. In view of this contradiction, the proof of the proposition is complete. \hfill $\Box$

Combining Theorem 3.2 with Propositions 4.2, 4.3 and 4.4 (along with the Geometrization Theorem) we obtain:

**Theorem 4.5.** If a closed 3-manifold $M$ admits a good complexification, then one of the following is true:

1. The manifold $M$ admits the structure of a Seifert-fibered space over a spherical orbifold and is therefore covered by $S^3$ or $S^2 \times S^1$. Hence $M$ either admits a metric of constant positive curvature or is covered by the (metric) product of a round $S^2$ and $\mathbb{R}$.
2. The manifold $M$ is finitely covered by $S^1 \times S^1 \times S^1$. Hence $M$ admits a flat metric.

### 5. Virtually free groups and virtually surface groups

The genus of a complex quasiprojective curve $C$ is defined to be the genus of its smooth compactification $\overline{C}$. 
Lemma 5.1. Let $X$ be a smooth complex quasiprojective variety and

$$f : X \rightarrow C$$

a nonconstant algebraic map to a quasiprojective complex curve of positive genus. Let $i : S \hookrightarrow X$ be a smooth curve in $X$ such that $f \circ i$ is a nonconstant map. Then the dimension of the image of the pullback homomorphism

$$i^* : H^1(X, \mathbb{R}) \rightarrow H^1(S, \mathbb{R})$$

is at least two.

Proof. Let $\overline{X}$ be a smooth compactification of $X$ such that $f$ extends to a morphism

$$\overline{f} : \overline{X} \rightarrow \overline{C}$$

with the image of the extension

$$\overline{i} : \overline{S} \rightarrow \overline{X}$$

being smooth.

We have $(\overline{f} \circ \overline{i})^* (H^0(\overline{C}, \Omega_{\overline{C}})) \subset \tau^*(H^0(\overline{X}, \Omega_{\overline{X}}))$, and $(\overline{f} \circ \overline{i})^* : H^0(\overline{C}, \Omega_{\overline{C}}) \rightarrow H^0(\overline{S}, \Omega_{\overline{S}})$ is injective. Therefore,

$$\dim \tau^*(H^0(\overline{X}, \Omega_{\overline{X}})) \geq 1.$$  

This implies that

$$(5.1) \quad \dim_{\mathbb{R}} i^*(H^1(X, \mathbb{R})) = 2 \dim_{\mathbb{C}} \tau^*(H^0(\overline{X}, \Omega_{\overline{X}})) \geq 2.$$  

The restriction homomorphism $H^1(\overline{X}, \mathbb{R}) \rightarrow H^1(X, \mathbb{R})$ is injective, and $\overline{i}|_S = i$. Therefore, from Lemma 5.1, it follows that $\dim_{\mathbb{R}} i^*(H^1(X, \mathbb{R})) \geq 2$.  

A slight modification of the techniques developed in the proofs of Proposition 4.2, 4.3 and 4.4 yield the following general result. (This might be regarded as a (weak) “maps” version of a theorem of Catanese [C], Theorem A’ which provides the analogue for spaces.)

Proposition 5.2. Let $X$ be a smooth complex quasiprojective variety and $f : X \rightarrow C$ a fibration over a curve $C$ with $b_1(C) \geq 3$. Let $F$ be any regular fiber of $f$ and $i : F \hookrightarrow X$ the inclusion map. Suppose that the image $i_\ast (\pi_1(F))$ is either infinite cyclic or finite. Let $A$ be an algebraic automorphism of $X$. Then $A(F)$ is a fiber of $f$. Hence $A$ induces an algebraic automorphism $A_0 : C \rightarrow C$.

Proof. By lifting to a further Galois cover of the base $C$ if necessary, we can assume that the smooth projective curve $\overline{C}$ has genus greater than one.

Let $i$ denote the inclusion of $A(F)$ in $X$. Assume that $f \circ i$ is not a constant map. Applying Lemma 5.1 to any smooth curve $S \subset A(F)$ such that $f|_S$ is not constant, we conclude that the dimension of the image of the homomorphism

$$(5.2) \quad i^* : H^1(X, \mathbb{R}) \rightarrow H^1(A(F), \mathbb{R})$$

is at least two.

Since $A$ is a homeomorphism, from the given condition on $F$ it follows that $i_\ast (\pi_1(A(F))) \subset \pi_1(X)$ is either infinite cyclic or finite. Therefore, the dimension of the image of the homomorphism

$$i_\ast : H_1(S, \mathbb{R}) \rightarrow H_1(X, \mathbb{R})$$

is at most one. But this contradicts the observation that the image of the homomorphism in (5.2) is at least two. Therefore, $f \circ i$ is a constant map.  

The next proposition imposes restrictions on quasiprojective groups that are virtually free groups or virtually surface groups.

Proposition 5.3. Let $G$ be a quasi-projective group that is virtually a non-abelian free group or virtually the fundamental group of a closed orientable surface of genus greater than one. Then there is a short exact sequence of the form

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1,$$

where $K$ is finite and $H$ is the fundamental group of one of the following two
(1) an orientable orbifold surface,
(2) a finite quotient of $\mathbb{C}^* \times \mathbb{C}^*$.

Proof. Let $X$ be a smooth quasiprojective variety with fundamental group $G$. Let $X_1$ be a finite Galois étale cover of $X$ with fundamental group $H_1$ such that

- either $H_1$ is non-abelian free, or
- $H_1$ is isomorphic to the fundamental group of a closed orientable surface of genus greater than one.

Let $f : X_1 \to C$ be a fibration given by Theorem 2.7 and let $i : F \to X_1$ be a regular fiber of $f$. Then $i_*\pi_1(F)$ is finite. The quotient group $Q = G/H_1$ acts by algebraic automorphisms on $X_1$ and hence, by Proposition 5.2, on $C$ via algebraic automorphisms. Let $K$ be the kernel of the action of $Q$ on $C$. Let $H$ be the orbifold fundamental group of the quotient $C/Q$. Then we have an exact sequence $1 \to K \to G \to H \to 1$. Also since $Q$ acts on $C$ by holomorphic automorphisms, the quotient $C/Q$ is orientable.

Proposition 5.4. Let $G$ be a quasi-projective 3-manifold group that is virtually free. Then $G$ is one of the following:

(1) $G = \mathbb{Z}$ or $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$
(2) $G = \ast_i G_i$ where each $G_i$ is cyclic.

Proof. If $G$ is virtually cyclic, then by the classification of such 3-manifold groups, $G$ is one of $\mathbb{Z}$ or $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ or $(\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z})$.

Else $G$ is virtually a non-abelian free group. Let $N$ be a 3-manifold with $G = \pi_1(N)$. Then we are in Case (2) or Case 5(2) of Remark 3.5. In either case, $G = \ast_i G_i$ where each $G_i$ is either finite or $\mathbb{Z}$ or $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. By ScWa, Theorem 3.11, the group $G$ contains no finite normal subgroup. Hence by Proposition 5.3 the group $G$ is isomorphic to the fundamental group of an orientable orbifold surface $S$. Since $G$ is virtually a non-abelian free group, the orbifold surface $S$ must have boundary. The orbifold fundamental group $G$ of such an $S$ is of the form $G = \ast_i G_i$, where each $G_i$ is cyclic. This is because $S$ deformation retracts onto a wedge $(\vee_i S^1) \vee (\vee_j D_j)$, where each $D_j$ is a quotient of the unit disk by a finite cyclic group acting with a single fixed point at the origin.

Proposition 5.5. Let $G$ be a quasi-projective 3-manifold group that is virtually the fundamental group of a closed orientable surface of genus greater than one. Then $G$ is isomorphic to the fundamental group of a closed orientable surface of genus greater than one.

Proof. If $G$ is not isomorphic to the fundamental group of a closed orientable surface of genus greater than one, then by Case 5(1) of Remark 3.5 the group $G$ contains an index 2 subgroup $H$ that is isomorphic to the fundamental group of a closed orientable surface of genus greater than one. Also $G$ is isomorphic to the fundamental group of a closed non-orientable surface of genus greater than one.

Since such a $G$ contains no finite normal subgroup, by Proposition 5.3 the group $G$ is isomorphic to the fundamental group of an orientable orbifold surface $S$. No orientable orbifold surface $S$ has the same fundamental group as a closed non-orientable surface. Therefore, the proposition follows.

Combining the observations in Section 5.1 with those of this section, we have the following classification result for quasiprojective 3-manifold groups.

Theorem 5.6. Let $G$ be a quasiprojective group that can be realized as the fundamental of a compact 3-manifold $N$ with or without boundary. Then either $N$ is Seifert-fibered, or $G$ satisfies one of the following:

(a) $G$ is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ or the fundamental group of a Klein bottle or the fundamental group of a closed orientable surface of genus greater than one.
(b) $G = \ast_i G_i$ where each $G_i$ is cyclic.
Each of the groups appearing in above alternatives (a) and (b) are quasiprojective. If \( N \) is closed Seifert-fibered, and \( N \) is spherical, flat or covered by \( S^2 \times \mathbb{R} \), then \( \pi_1(N) \) is quasiprojective. If \( N \) is an orientable closed Seifert-fibered with hyperbolic base orbifold \( B \), then \( \pi_1(N) \) is quasiprojective if and only if \( B \) is an orientable orbifold.

**Proof.** All the statements except for the last two are contained in Remark 3.5, the examples constructed in Section 3.7, or in Proposition 3.4 and Proposition 3.5. The penultimate statement is a consequence of the fact that such manifolds admit good complexifications [10].

It remains to deal with \( N \) an orientable, Seifert-fibered with hyperbolic base orbifold \( B \). That an orientable, Seifert-fibered space \( N \) with orientable hyperbolic base orbifold \( B \) has quasiprojective fundamental group follows from Proposition 3.7 and the last statement in the first paragraph of [He1, p. 118]. We will prove the converse statement.

Let \( X \) be a smooth quasiprojective variety with \( \pi_1(X) = \pi_1(N) \). Let \( B' \) be an orientable hyperbolic surface (without orbifold points) that (Galois) covers \( B \) and with \( b_1(B') > 2 \). There is a corresponding finite (Galois) cover \( N' \) of \( N \) which is a circle bundle over \( B' \). Let \( X' \) be the Galois étale cover of \( X \) corresponding to the subgroup \( \pi_1(N') \). By Theorem 2.7 (or more precisely by Theorem A of Ca which is its generalization to the quasi-Kähler context), there is a fibration \( f : X' \rightarrow C \) with \( C \) a closed curve (as \( N \) is closed). We are now in the situation of Proposition 5.2 the deck transformation group \( Q \) induces an algebraic action on \( C \) forcing the quotient orientable orbifold \( C/Q \) to be orientable.

The following immediate Corollary of Theorem 5.6 answers Question 8.3 of [PS, p. 166].

**Corollary 5.7.** Let \( G \) be a quasiprojective group that can be realized as the fundamental of a closed graph manifold \( M \). Then \( M \) is Seifert-fibered.

Friedl and Suciu conjecture the following in [PS]:

**Conjecture 5.8 ([PS, p. 166, Conjecture 8.4]).** Let \( N \) be a compact 3-manifold with empty or toroidal boundary. If \( \pi_1(N) \) is a quasiprojective group and \( N \) is not prime, then \( N \) is the connected sum of spherical 3-manifolds and manifolds which are either diffeomorphic to \( S^1 \times D^2 \), \( S^1 \times S^1 \times [0,1] \), or the 3-torus.

Following is a strong positive answer to it.

**Corollary 5.9.** Let \( N \) be a compact 3-manifold with empty or toroidal boundary such that \( \pi_1(N) \) is a quasiprojective group and \( N \) is not prime. Then \( N \) is the connected sum of lens spaces, \( S^1 \times S^2 \) and manifolds which are diffeomorphic to disk bundles over the circle.

**Proof.** We are in Case (b) of Theorem 5.6. Then by the prime decomposition theorem for 3-manifolds [He1, Ch. 3], the manifold \( M \) is a connected sum of manifolds with cyclic fundamental group. A complete list of such manifolds is: lens spaces, \( S^1 \times S^2 \) and manifolds which are diffeomorphic to disk bundles over the circle.

From Theorem 5.6 it follows that a closed non-orientable Seifert-fibered manifold \( N \) with hyperbolic base orbifold such that its orientable double cover \( N' \) is a Seifert-fibered manifold with non-orientable hyperbolic base orbifold cannot have quasiprojective fundamental group, because otherwise \( \pi_1(N') \) is quasiprojective contradicting Theorem 5.6. The only case that thus remains unanswered by Theorem 5.6 is the following:

**Question 5.10.** Let \( N \) be a closed non-orientable Seifert-fibered space with hyperbolic base orbifold such that its orientable double cover is a Seifert-fibered space with orientable hyperbolic base orbifold. Is \( \pi_1(N) \) quasiprojective?

5.1. Quasiprojective free products. In [PS], Friedl and Suciu ask the following:

**Question 5.11 ([PS, p. 165, Question 8.1]).** Suppose \( A \) and \( B \) are groups, such that the free product \( A \ast B \) is a quasiprojective group. Does it follow that \( A \) and \( B \) are already quasiprojective groups?
Lemma 5.12. Suppose $A$ and $B$ are groups, such that the free product $A \ast B$ is a quasiprojective group. In addition suppose that both $A, B$ admit nontrivial finite index subgroups and at least one of $A, B$ has a subgroup of index greater than 2. Then $A \ast B$ is virtually free.

Proof. Since $A, B$ admit nontrivial finite index subgroups, they also admit finite index normal subgroups. By the hypothesis, there exist finite quotients $A_1$ and $B_1$ (of $A$ and $B$ respectively) of which at least one has order more than 2. So $A \ast B$ admits a surjection onto $A_1 \ast B_1$, and hence a finite index subgroup $G$ of $A \ast B$ admits a surjection onto a non-abelian free group with greater than 2 generators.

Let $X$ be a smooth quasiprojective variety with fundamental group $G$. By Corollary 2.10, there exists an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow F_n \rightarrow 1$$

with $n \geq 3$ and $H$ finitely generated. Hence $H$ is trivial [ScWa, Theorem 3.11]. It follows that $A \ast B$ is virtually free. □

Following is a positive answer to Question 5.11 under mild hypotheses.

Theorem 5.13. Suppose $A$ and $B$ are groups, such that the free product $G = A \ast B$ is a quasiprojective group. In addition suppose that both $A, B$ admit nontrivial finite index subgroups and at least one of $A, B$ has a subgroup of index greater than 2. Then each of $A, B$ are free products of cyclic groups. In particular both $A$ and $B$ are quasiprojective.

Proof. By Lemma 5.12 and Proposition 5.3, there is a short exact sequence of the form

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1,$$

where $K$ is finite and $H$ is the fundamental group of an orientable orbifold surface. The subgroup $K$ is trivial by [ScWa, Theorem 3.11], and $H$ is virtually free. Hence as in the proof of Proposition 5.4, we have $G = \ast_i G_i$, where each $G_i$ is cyclic. Therefore, since both $A$ and $B$ are free factors of $G$, they are free product of cyclic groups. Hence $A$ and $B$ are fundamental groups of orientable orbifold surface. In particular, both $A$ and $B$ are quasiprojective. □

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

RKM Vivekananda University, Belur Math, WB 711202, India

E-mail address: mahan.mj@gmail.com, mahan@rkmvu.ac.in