COMPLEXITY GROWTH OF A TYPICAL TRIANGULAR BILLIARD IS WEAKLY EXPONENTIAL

By

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Abstract. We prove that for any $\epsilon > 0$ the growth rate $P_n$ of generalized
diagonals of a typical (in the Lebesgue measure sense) triangular billiard satisfies
$P_n < C e^{n\epsilon}$. This makes further progress towards a solution of problem 3 in Katok’s
list “Five most resistant problems in dynamics”.

1 Overview

A generalized diagonal of a polygonal billiard is an orbit which connects two
vertices. One of the characteristics of billiard dynamics is a complexity function $P_n$,
which is a number of generalized diagonals of length no greater than $n$. Here,
by length we mean the number of reflections. There are also other versions of
billiard complexity, such as position complexity, directional complexity and word
complexity which are closely related to our definition (see [1], [2], [3], [4]).

Katok [6] proved the subexponential estimate.

Theorem 1 (Katok, 1987). For any polygon, $\lim \ln(P_n)/n = 0$.

Masur [9, 10] using Teichmüller theory, proved quadratic estimates for any
rational-angled polygon:

Theorem 2 (Masur, 1990). For any polygon with angles in $\pi \mathbb{Q}$ there are
constants $C_1, C_2 > 0$ such that: $C_1 \cdot n^2 < P_n < C_2 \cdot n^2$.

Since the local geometry of polygonal billiards is similar and the result of
Masur gives quadratic growth in the rational case, Katok formulated the following
conjecture, included in his list “Five most resistant problems in dynamics” [7]:

Conjecture 1 (Katok). For any polygon and any $\epsilon > 0$, $P_n < C n^{2+\epsilon}$.
The explicit subexponential upper bound was an open problem for a long time, and it is still not fully resolved. The key problem here is a lack of structure of an irrational polygonal billiard as, unlike the rational case, one cannot use tools from Teichmüller theory. Recently, we provided an explicit subexponential estimate for a full measure set of triangular billiards [11]:

**Theorem 3** (Scheglov, 2012). *For a typical triangle and any $\epsilon > 0$ there is a constant $C > 0$ such that $P_n < Ce^n\sqrt{3-1+\epsilon}$.*

In the proof of Theorem 3 we introduced a natural technique of indexed partitions for dealing with generalized diagonals. The key idea was to show that if there are too many generalized diagonals, then by some kind of a pigeonhole principle there is a sequence of relatively short billiard trajectories which stay too close to one of the vertices at three different time moments. The last fact implies that some sequence of trigonometric polynomials must take very small values, which is not possible on the set of full measure by the result of Kaloshin and Rodnianski [5].

In the current paper we again use indexed partitions as a convenient framework, however we will rely on quite different geometric ideas which require more delicate analysis of billiard dynamics. We will still need an abstract result by Kaloshin and Rodnianski about trigonometric polynomials, however it will not play one of the main roles as it did in our previous paper. From this point of view we now use more dynamical arguments rather than abstract properties of trigonometric polynomials. The aim of the paper is to prove the following theorem:

**Theorem 4** (Weakly exponential estimate). *For a typical triangle and any $\epsilon > 0$ there is a constant $C > 0$ such that $P_n < Ce^{n\epsilon}$.*

In the proof we use a bootstrap technique. We first assume that for large $n$, $P_n < e^{n\nu}$, where $\nu > 0$ is any constant. We then prove by contradiction that for any $\epsilon > 0$ there is a subsequence $n_k$ such that $Q_{n_k} < e^{n_k\mu(\nu)\epsilon}$, where $0 < \mu(\nu) < \nu$ is some explicit function and $Q_n$ is a complexity function for generalized diagonals, emanating from a fixed vertex. The key geometric idea here is that for two suitably chosen generalized diagonals, the segment connecting the endpoints of their unfoldings is also a generalized diagonal of a relatively short length. So the large number of long generalized diagonals produces too many short generalized diagonals, which gives a principal contradiction. We then show that the subsequence gaps $n_{k+1} - n_k$ grow relatively slowly, which allows us to obtain a full bootstrap and show that for any $\epsilon > 0$ and large $n$, $P_n < e^{n\mu(\nu)\epsilon}$. Then iterating the bootstrap theorem we get the desired conclusion.
2 Interval partitions

We consider a triangle, a fixed vertex and the corresponding angular segment located at the vertex, which we naturally associate with an interval \( I \subseteq [0, \pi] \) using the angular distance on it. Points on the interval then correspond to directions of rays emanating from the vertex. We introduce a useful reduced quantity \( Q_n \) as a number of generalized diagonals emanating from the vertex of algebraic length no greater than \( n \). By algebraic length here we mean the number of reflections from sides. Now let us create a decreasing sequence of finite indexed partitions \( \xi_n \) of \( I \) on subintervals as follows. The intervals of \( \xi_n \) are formed by the points of \( I \) corresponding to the generalized diagonals of algebraic length no greater than \( n \) and the endpoints of \( I \). We define the index of a cutting point of \( \xi_n \) as the algebraic length of the corresponding generalized diagonal. Two observations immediately follow from this construction:

1. Inside each interval of the partition \( \xi_n \) there is at most one point of the partition \( \xi_{n+1} \).
2. \( Q_n \) equals the number of cutting points of \( \xi_n \).

Lemma 2.1. Let \( \xi_n, \xi_{n+1}, \ldots, \xi_{n+c} \) be a finite sequence of indexed partitions such that \( Q_{n+c} > 2Q_n + 1 \). Then there are at least \( Q_{n+c} - 2Q_n - 1 \) intervals of the partition \( \xi_{n+c} \) such that the indices of their endpoints belong to the interval \([n+1, n+c]\).

Proof. Let us consider intervals of the partition \( \xi_n \) and the cutting points of the partition \( \xi_{n+c} \) inside these intervals. Let \( X \) be the set of intervals of \( \xi_n \) which have no points of \( \xi_{n+c} \) inside, \( Y \) be the set of intervals of \( \xi_n \) which have 1 point of \( \xi_{n+c} \) inside, and \( Z \) be the set of intervals which have 2 or more points of \( \xi_{n+c} \) inside. Let us denote the cardinalities of these sets as \( |X| = x, |Y| = y, |Z| = z \). We immediately have \( x + y + z = Q_n + 1 \).

Let us now consider a fixed interval \( I_i \in Z \), where \( i \) belongs to the index set, parameterizing \( Z \). If \( S_i \) is a number of points of \( \xi_{n+c} \) inside \( I_i \), then we have at least \( N_i = S_i - 1 \) intervals of \( \xi_{n+c} \) with indices in the interval \([n+1, n+c]\) located inside \( I_i \). Then all the intervals from \( Z \) contain at least

\[
N = \sum N_i = \sum S_i - z
\]

intervals of \( \xi_{n+c} \) with the required property.

Counting all the points of \( \xi_{n+c} \) we have \( y + \sum S_i + Q_n = Q_{n+c} \). As we noted that \( x + y + z = Q_n + 1 \) then by subtracting we obtain \( \sum S_i - z = Q_{n+c} - 2Q_n - 1 + x \) so \( N \geq Q_{n+c} - 2Q_n - 1 \).
3 Combinatorial geometry of orbits

3.1 Unfolding of a billiard trajectory. We recall a useful unfolding construction associated to any polygonal billiard [8]. We fix a polygon on the plane and consider a time moment when a particular billiard orbit hits a polygon side. Then instead of reflecting the orbit, we continue it as a straight line and reflect the polygon along the line. As we continue this process indefinitely, the sequence of polygons obtained is called unfolding of the polygon along the orbit. Figure 1 below illustrates the unfolding of a triangle along an orbit.

![Figure 1. Triangle unfolding.](image1)

For a given triangle, the shape obtained from a triangle by reflection about one side is called a kite. It is clear that for any triangle unfolding there is an associated kite unfolding. We will use both unfoldings having in mind the natural correspondence between them. Figure 2 illustrates the corresponding kite unfolding.

![Figure 2. Kite unfolding.](image2)

From Figure 2, we see that any kite unfolding along the orbit consists of consecutive rotations of the kite along one of the two kite vertices, corresponding to the angles $\alpha$ and $\beta$ of the original triangle with corresponding angles $2\alpha$ and $2\beta$.

We now assume that a kite is located in the standard Euclidean $xy$ coordinate plane and introduce several notations.
The \( \alpha \)-vertex and \( \beta \)-vertex are kite vertices corresponding to the angles \( 2\alpha \) and \( 2\beta \). The two other vertices are called side vertices.

The kite diagonal is a vector going from the \( \alpha \)-vertex to the \( \beta \)-vertex.

The kite angle is a counterclockwise angle between the \( x \)-axis and the kite diagonal.

In Figure 3, \( A \) and \( B \) are \( \alpha \) and \( \beta \) the corresponding vertices, vector \( \overrightarrow{AB} \) is a kite diagonal, \( C \) and \( D \) are side vertices. Figure 4 shows a kite in standard position on the \( xy \) plane.

![Figure 3. A kite on the xy coordinate plane.](image)

![Figure 4. A kite in the standard position on the xy plane.](image)

Note that any unfolding of \( K \) is uniquely characterized by the sequence of rotations on angles \( \pm 2\alpha \) or \( \pm 2\beta \) depending on the kite vertex we rotate about and the direction of rotation.

The combinatorics of a kite unfolding is a corresponding sequence of \( \alpha \) or \( \beta \) vertices and rotation directions.
The combinatorics of a generalized diagonal is a combinatorics of the corresponding kite unfolding.

**Lemma 3.1.** Assume a kite $K$ with a diagonal length 1 is in standard position and a kite $K'$ is obtained from $K$ by means of a fixed unfolding of length $n$. Let $x^a_n, y^a_n$ and $x^b_n, y^b_n$ be the coordinates of $\alpha$ and $\beta$ vertices of $K'$, and let $x_n, y_n$ be the coordinates of either of the two side vertices of $K'$. Then:

1. The coordinates $x^a_n, y^a_n, x^b_n$ and $y^b_n$ are represented as $P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))$, where $P(x, y, u, v)$ is a polynomial with integer coefficients, depending on the coordinate, such that:
   - $\deg(P) \leq 2n - 2$.
   - The absolute values of the coefficients of $P$ are bounded from above by $n^4$.

2. $x_n = P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)) + \frac{\sin(\beta)}{\sin(\alpha + \beta)} \cdot \cos(m \alpha + l \beta)$,
   $y_n = Q(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)) + \frac{\sin(\beta)}{\sin(\alpha + \beta)} \cdot \sin(m \alpha + l \beta)$,
   where $|m| + |l| \leq 2n - 1$ and $P(x,y,u,v)$, $Q(x,y,u,v)$ are polynomials with integer coefficients, such that:
   - $\deg(P) \leq 2n - 2$, $\deg(Q) \leq 2n - 2$.
   - The absolute values of the coefficients of $P, Q$ are bounded from above by $n^4$.

**Proof.** The proof of the first statement goes by an easy induction on $n$. For $n = 1$ the statement is trivial. If $\phi_n$ is the kite angle on the $n$-th step and on the $n+1$-th step, we rotate, say, about the $\alpha$-vertex, then $\phi_{n+1} = \phi_n \pm 2\alpha$ and $x^a_{n+1} = x^a_n$, $y^a_{n+1} = y^a_n$, $x^b_{n+1} = x^a_n + \cos(\phi_{n+1})$, $y^b_{n+1} = y^a_n + \sin(\phi_{n+1})$. The case when we rotate about the $\beta$-vertex is entirely analogous. The estimate on the absolute values of the coefficients is an immediate consequence of the fact that the corresponding polynomials for $\sin(k\alpha + l\beta)$ and $\cos(k\alpha + l\beta)$ with $|k| + |l| \leq 2n$ have integer coefficients with absolute values bounded from above by $2^{2n}$.

This completes the induction step.

The second statement easily follows from the first one by noticing that the length of the side, adjacent to the $\alpha$-vertex, is $\frac{\sin(\beta)}{\sin(\alpha + \beta)}$, and so if $\phi_n$ is the kite angle of $K'$, then

$$x_n = x^a_n + \frac{\sin(\beta)}{\sin(\alpha + \beta)} \cdot \cos(\phi_n \pm \alpha)$$

and

$$y_n = y^a_n + \frac{\sin(\beta)}{\sin(\alpha + \beta)} \cdot \sin(\phi_n \pm \alpha).$$
3.2 Local geometry near singular points. In this section we study local dynamics of generalized diagonals near singular points. We consider a triangle with a side 1 and adjacent angles $\alpha, \beta$ in the standard position on the plane as shown in Figure 5 and a generalized diagonal.

The diagonal angle is an oriented angle between the $x$-axis and a diagonal. We want to emphasize that the generalized diagonal should not necessarily start from the ‘zero’ vertex of a triangle in the standard position. In Figure 5, $\xi$ is a diagonal angle.

![Figure 5. The diagonal angle.](image)

Now let us look more carefully at the behavior of billiard trajectories near the end of the generalized diagonal. In Figure 6 we see an unfolding corresponding to all trajectories emanating from the vertex in the clockwise small neighborhood of the generalized diagonal. Of course an analogous picture and all the arguments below hold for a counterclockwise neighborhood.

The unfolding about the end vertex of the diagonal is called a rose. It represents the unfolding of all billiard trajectories emanating from the vertex $A$ and clockwise close to the diagonal.

In Figure 6, the dashed line is a formal geometric continuation of the generalized diagonal. The only triangle $BDE$ in the rose which it intersects is called the exit triangle. Notice that the side $BE$ of the exit triangle may lie on the formal continuation of the diagonal. This possibility causes no difficulties in our arguments.

The relative position of the triangle $BDE$ on the $xy$-plane up to parallel translations is called the exit position of the exit triangle corresponding to the diagonal $AB$. Notice that many different diagonals emanating from $A$ may have exit triangles in the same exit position.

The oriented angle $\theta$ between the $x$-axis and the side $BD$ of the exit triangle is called the exit angle.
Figure 6. The rose.

Now we use the concept of a rose to analyze the local behavior of the diagonals. For any $\delta > 0$ small enough let $\Delta_\delta$ be the set of triangles with a fixed side of length 1 such that all angles are greater than $\delta$. From now on we assume that we consider triangles from the set $\Delta_\delta$ for some $\delta > 0$. As $\delta$ can be chosen arbitrarily small our arguments would ultimately work for the whole set of triangles. Let us now prove a technical statement that for two close enough diagonals, the greater algebraic length implies the strictly greater geometric length.

**Lemma 3.2.** Let $\zeta_n$ be an indexed partition and $I \in \zeta_n$ be an interval with endpoints $x, y$ with indices $p < q$. There are universal constants $b, r > 0$ such that if $L_p$ and $L_q$ are geometric lengths of the diagonals corresponding to points $x, y$ and the length $|I| < b/p$, then $L_q > L_p + r$.

**Remark.** The assumption on $|I|$ in Lemma 3.2 is mild. In particular, if $P_n$ grows faster than linearly, then it will be satisfied for most partition elements. This follows from the simple observation that the number of partition elements of length greater than $p/n$ is bounded from above by $\pi n/p$. Informally speaking, the lemma follows from the fact that adjacent diagonals travel in the same unfolding triangles up until the smaller index.

**Proof.** There is a constant $D_\delta$ such that for any triangle from $\Delta_\delta$ and any diagonal of algebraic length $p$, its geometric length is bounded from above by $D_\delta p$. This immediately follows from the fact that the diameters of the triangles from $\Delta_\delta$ are uniformly bounded by some constant, depending only on $\delta$. In figure 7, $AP$ is a diagonal corresponding to $x$ and $\phi$ is a circular segment of the angular beam, corresponding to $I$ on the distance $L_p + r$ from $A$ where $r$ is a small constant to be chosen later.
Figure 7. The $\phi$-segment.

The point $B$ lies outside the $\phi$-segment because $p < q$, and so all the trajectories from the $\phi$-segment intersect $BC$ as by the definition of the partition $\xi_n$ they all have the same unfolding combinatorics of length at least $q$. The side lengths of all triangles from $\Delta_\delta$ are bounded from below by some constant $R_\delta$.

Assuming that $|I| < b/p$ we have

$$|\phi| = |I|(L_p + r) \leq \frac{b}{p}(D_\delta p + r) = bD_\delta + br/p,$$

and it is now clear that there are small enough constants $b, r > 0$ depending only on $\delta$ such that $|\phi| + r < R_\delta$. It implies that the points of the rose cannot lie inside the $\phi$-segment because, on the one hand, they are separated from the $\phi$-segment by the point $B$ and, on the other hand, their distances to $P$ are bounded from below by $R_\delta$.

**Remark.** From the proof of Lemma 3.2 it follows that the whole open angular $\phi$-segment in Figure 8 consists of orbits with the same unfolding of algebraic length at least $q$. This happens because any combinatorics change inside the $\phi$-segment must happen at the algebraic time greater than $q$, which by Lemma 3.2 happens geometrically outside the $\phi$-segment of radius $L_q + r$.

Our next lemma proves a simple but important observation, that for two close enough diagonals the segment connecting their endpoints is also a diagonal.

**Lemma 3.3.** Let $\xi_n$ be an indexed partition and $I \in \xi_n$ be an interval with endpoints with indices $p < q$. There is a universal constant $b$ such that if $|I| < b/q$ then:
(a) the segment connecting the endpoints of the diagonals on Figure 8 is either a side of the triangle or an unfolding of some generalized diagonal of algebraic length bounded by $q - p$.

(b) The combinatorics of the diagonal $PQ$ coincides with the tail of the combinatorics of the diagonal $AQ$ on Fig 8.

**Proof.** (a) We take $b$ from Lemma 3.2. By the Remark after Lemma 3.2, as the combinatorics does not change, then any straight line segment located strictly inside the angular $\phi$-segment does not hit any vertex and so either represents the unfolding of some billiard trajectory or belongs to the side of one of the unfolding triangles. In particular, the segment $PQ$ represents the unfolding of some generalized diagonal or coincides with a side of the triangle. In the first case its algebraic length is obviously bounded above by $q - p$.

(b) By Lemma 3.2 the $\phi$-segment is covered by the finite union of triangles, representing the unfolding along any trajectory emanating from $A$ and staying inside the $\phi$-segment. Consider the triangle $APQ$ and the family of segments, connecting $Q$ and the side of the exit triangle for the diagonal $AP$ at the point $P$. These segments correspond to the billiard trajectories emanating from $Q$ and have the same combinatorics. By considering the boundary segments of this family the proof is complete. □

![Figure 8. Connecting diagonal.](image)

The following technical Lemma will be used in our later partition estimates.

**Lemma 3.4.** Assume that there are fixed constants $r$, $D > 0$ and a triangle $ABC$ (Figure 9) such that $|AB| < |AC|$ and $|AC| < Dn$ for some positive integer $n$ and $|BC| > r$. Then there are constants $K$, $b > 0$ depending only on $r$, $D$ such that if $\phi < b/n$, then $\psi < K\phi n$. 
Proof. By the sinus theorem $\sin(\psi) = |AC| \sin(\phi)/|BC|$ and since $\sin(x) \approx x$ as $x \to 0$, then for small enough $b$ the conclusion follows. \hfill \Box

4 Complexity estimate

In this section we use the tools described above to get a global complexity estimate. At first we need an important lemma, which says that for typical triangles from $\Delta_\delta$ two diagonals cannot be too close to each other.

Lemma 4.1. There is a full measure set of triangles $X \subseteq \Delta_\delta$ such that for any triangle from $X$ there is a constant $a > 0$ with the following property:

If $I \in \xi_n$ is an interval of the partition $\xi_n$ corresponding to any vertex of the triangle, then $|I| > e^{-an^2}$.

Proof. We consider some triangle $\Delta \in \Delta_\delta$ in the standard position on the $xy$-plane with angles $\alpha, \beta$ adjacent to the fixed side of length 1 and two diagonals from $\xi_n$ corresponding to some vertex. Let us assume that $A$ is the vertex of the original triangle from which two given diagonals start and $B, C$ are the end points of the diagonal unfoldings on the $xy$-plane. By Lemma 3.1 the $x$ and $y$ coordinates of points $A, B, C$ can be represented as

$$x = \frac{P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))}{\sin(\alpha + \beta)}, \quad y = \frac{Q(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))}{\sin(\alpha + \beta)},$$

where $P, Q$ are polynomials with integer coefficients of degree at most $2n$, and with the absolute values of coefficients bounded from above by $n^{4n+1}$ so the area $S$ of the triangle $ABC$ can be represented as

$$S = \frac{M(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))}{2 \sin^2(\alpha + \beta)},$$

where $M(x, y, u, v)$ is a polynomial with integer coefficients of degree at most $4n$ and with the absolute values of coefficients bounded from above by $n^{10}2^{2n+6}$. When
it does not lead to ambiguity, we will use the short notation

\[ M(\alpha, \beta) = M(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)). \]

As the diagonals are different \( M(\alpha, \beta) \neq 0 \). Since \( \text{Deg}(M) \leq 4n \), then easy calculation shows that

\[ \max \left| p_k(x_\alpha, y_\alpha, x_\beta, y_\beta) \right| \leq n^{14} 4^{2n+7} < m^2 4^m, \quad \text{for} \ (x_\alpha, y_\alpha, x_\beta, y_\beta) \in [-1, 1]^4, \]

and where \( m = F_n, F > 23 \).

Let \( \phi \) be an angle between the diagonals \( AB \) and \( AC \); then \( S = |AB||AC| \sin(\phi)/2 \). This implies that

\[ \phi > \frac{2S}{|AB||AC|} > l M(\alpha, \beta)/n^2 \]

for some constant \( l > 0 \), because \( |AB|, |AC| < D_\beta n \). We now refer to the theorem by Kaloshin and Rodnianski [5] (see Proposition 1, page 962 and precise conditions on the polynomial on page 965) which can be formulated as follows:

**Theorem 4.1** (Kaloshin and Rodnianski). **There are universal constants** \( R, h > 0 \) **such that for any non-zero polynomial with integer coefficients** \( P(x_\alpha, y_\alpha, x_\beta, y_\beta) \) **which satisfies:**

- \( \text{Deg}(P) \leq 2m \),
- \( P = \sum_{k=0}^m p_k(x_\alpha, y_\alpha, x_\beta, y_\beta)y_\gamma^k \),
- \( \max \left| p_k(x_\alpha, y_\alpha, x_\beta, y_\beta, y_\gamma) \right| \leq (m2^m)^2, \text{for} \ (x_\alpha, y_\alpha, x_\beta, y_\beta, y_\gamma) \in [-1, 1]^5 \).

**The following estimate holds:**

\[ \text{Leb}(\alpha, \beta, \gamma) : |P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta), \cos(\gamma), \sin(\gamma))| < e^{-Rm^2} < e^{-hm}. \]

Any trigonometric polynomial \( P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)) \) in 2 variables can be considered as a polynomial \( P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta), \cos(\gamma), \sin(\gamma)) \) in three variables of the same degree, where the variable \( \gamma \) is not present. Moreover, any level set for \( P \) in variables \( \alpha, \beta, \gamma \) is a product of the level set for \( P \) in variables \( \alpha, \beta \) and \([0, 2\pi]\) in variable \( \gamma \). Then an easy use of Fubini’s theorem implies the following corollary:

**Corollary 4.1.** **There are universal constants** \( R, h > 0 \) **such that for any non-zero polynomial with integer coefficients** \( P(x_\alpha, y_\alpha, x_\beta, y_\beta) \) **which satisfies:**

- \( \text{Deg}(P) \leq 2m \),
- \( \max \left| p(x_\alpha, y_\alpha, x_\beta, y_\beta) \right| \leq (m2^m)^2, \text{for} \ (x_\alpha, y_\alpha, x_\beta, y_\beta) \in [-1, 1]^4 \),

**the following estimate holds:**

\[ \text{Leb}(\alpha, \beta) : |P(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))| < e^{-Rm^2} < e^{-hm}. \]
Let $\mathcal{F}_n$ be a set of all trigonometric polynomials $M$ as in the proof of Lemma 4.1. Any such polynomial is determined by the choice of two generalized diagonals of algebraic length no greater than $n$. So the cardinality $|\mathcal{F}_n| < e^m$, for some $t > 0$.

We now take $m = F_n$, where $F > 23$ is a constant to be chosen later, and pick some polynomial $M \in \mathcal{F}_n$. By Corollary 4.1,

$$\text{Leb}\{(\alpha, \beta) : |M(\alpha, \beta)| < e^{-RF^2n^2}\} < e^{-hF_n}.$$  

Now take the set of angles $B_n = \{(\alpha, \beta) | \exists M \in \mathcal{F}_n : |M(\alpha, \beta)| < e^{-RF^2n^2}\}$. Since $|\mathcal{F}_n| < e^m$, then $\text{Leb}(B_n) < e^{(t-h)F_n}$. Picking $F > t/h$ we have that $\sum \text{Leb}(B_n) < \infty$, so by a Borel–Cantelli argument for almost any pair $(\alpha, \beta)$ and all large enough $n$ for any polynomial $M \in \mathcal{F}_n$ we have $|M(\alpha, \beta)| \geq e^{-RF^2n^2}$. Picking large enough $a > RF^2$ (depending on $(\alpha, \beta)$) we complete the proof.  

For the rest of the section we assume that we have fixed a particular triangle $\Delta \in X$ satisfying the conclusion of Lemma 4.1 with some constant $a > 0$.

### 4.1 Subsequence complexity and a bootstrap.

In this subsection we fix a triangle vertex and recall that a reduced quantity $Q_n$ counts only generalized diagonals emanating from it. As usual $P_n$ denotes the global (non-reduced) complexity for $\Delta$.

**Theorem 4.2** (Bootstrap on subsequence complexity). Assume that for some constant $\nu$, $0 < \nu \leq 1$ and all $n$ large enough, $P_n < e^{n\nu}$. Then for any $\mu, \gamma < \mu < \nu$, \liminf $Q_n e^{-n\gamma} = 0$, where $\gamma = \frac{-\nu^2 + \sqrt{\nu^4 + 4\nu^2}}{2}$.

**Remark.** One can easily check that $\gamma = \frac{-\nu^2 + \sqrt{\nu^4 + 4\nu^2}}{2} < \nu$ for any $\nu > 0$ so the assumption on $\mu$ makes sense.

**Proof.** We are going to prove the theorem by contradiction. In order to do so we assume for the rest of the proof that for some $\mu, \gamma < \mu < \nu$, and all $n$ large enough, $Q_n > e^{n\nu}$. We first prove a technical lemma.

**Lemma 4.2.** For any numbers $\nu, \mu, \gamma$ satisfying the assumptions of Theorem 4.2 there exists $\epsilon > 0$ satisfying $\mu > \epsilon \nu$ and $(\epsilon + \nu)\mu > \nu$.

**Proof.** Consider the system:

$$\begin{cases} 
\mu > \epsilon \nu, \\
(\epsilon + \nu)\mu > \nu.
\end{cases}$$
Considering the extreme case and substituting $\epsilon$, we obtain $(\frac{n}{k} + v)\mu = v$ or $(\mu + v^2)\mu = v^2$. Solving this quadratic equation with $v$ as a parameter we get that $\gamma = -\frac{v^2 + \sqrt{v^4 + 4\epsilon^2}}{2}$ is a critical value for $\mu$, so for any $\mu > \gamma$ we may find $\epsilon$ satisfying the above inequalities.

Let us introduce the notations $k(n) = n^\nu$, $c(n) = n^\xi$ with $\epsilon$ from Lemma 4.2. When it does not lead to ambiguity we will use the notations $k(n) = k$, $c(n) = c$ for readability.

**Lemma 4.3.** For any $n$ large enough there exists $s$, $n \leq s \leq n + c(n)(k(n) - 1)$, such that $Q_{s+c(n)} \geq 3Q_s$.

**Proof.** Divide the index set $[n, n+k\epsilon]$ into $k$ subsets of length $c$: $I_1 = [n, n+c]$, $I_2 = [n+c, n+2c]$, $\ldots$, $I_k = [n + (k-1)c, n+k\epsilon]$, and prove that for one of the sets $I_i$, $1 \leq i \leq k$, $Q_{n+i}/Q_{n+(i-1)c} > 3$. Here, by $[n, m]$ we mean the finite set $[n, m] = \{n, n+1, \ldots, m\}$. Assuming the opposite we have $Q_{n+k\epsilon}/Q_n \leq 3^k$. On the other hand, $Q_{n+k\epsilon}/Q_n > \epsilon^{(n+k\epsilon)^\mu - n^\nu}$. Taking the logarithm of both inequalities we obtain $(n+k\epsilon)^\mu - n^\nu < k\ln(3)$. As $(\epsilon + \nu)\mu > \nu$, we get a contradiction for large enough $n$. More precisely, $(\epsilon + \nu)\mu = \nu + \delta$ for some $\delta > 0$, so

$$(n+k\epsilon)^\mu > (k\epsilon)^\mu = n^{\nu + \delta} > n^\nu + n^\nu \ln(3) = n^\nu + k\ln(3)$$

for $n$ large enough. □

We now need the following abstract lemma.

**Lemma 4.4.** Assume there is a set $I$ of $n$ non-intersecting closed subintervals $I_1, I_2, \ldots, I_n$ of interval $[0, 1]$ satisfying $L^{-1} \leq \frac{|I_i|}{|I|} \leq L$ for some constant $L > 0$ and a set $J$ of closed intervals $J_1, J_2, \ldots, J_n$ such that for each $i$, $1 \leq i \leq n$, intervals $J_i$ and $I_i$ have the same left endpoints and $|J_i| \leq m|I_i|$ for some $m > 0$ such that $n \geq Lm$. Then $J$ contains a subset of $\frac{n}{Lm}$ non-intersecting intervals.

**Proof.** We assume that the set $I$ is naturally ordered by the left points of intervals $I_i$. Then $J_1 = [x_1, x_2]$ contains not more than $mL$ intervals from $I$. We pick $J_1$ and then consider the reduced interval $[x_2, 1]$ with at least $n - mL$ intervals from $I$ left. We can repeat this procedure at least $\frac{\nu}{mL}$ times and so the proof is complete. □

**Remark.** Of course analogous statement holds if in Lemma 4.4 assumptions $J_i$ and $I_i$ have the same right points.

We now fix $n$ large enough and, using Lemma 4.3, find $s$ such that the conclusion of Lemma 4.3 holds. Consider the sequence of interval partitions $\xi_s$, $\xi_{s+1}$, $\ldots$, $\xi_{s+c}$. 


Lemma 4.5. There is a set $X_1$ of intervals belonging to the partition $\tilde{\xi}_{s+c}$ such that:

1. $|X_1| = Q_s - 1$.
2. For any interval $I_i = [x_i, y_i]$ from $X_1$ the indices of the endpoints $x_i, y_i$ belong to the set $[s + 1, s + c]$.

Proof. By Lemma 4.3, $Q_{s+c} > 3Q_s$ and so by Lemma 2.1 applied to the sequence of partitions $\tilde{\xi}_s, \tilde{\xi}_{s+1}, \ldots, \tilde{\xi}_{s+c}$ we can find the set $X_1$ of at least $Q_s - 1$ intervals $[x_i, y_i]$ of $\tilde{\xi}_{s+c}$ with indices of endpoints in $[s + 1, s + c]$. \hfill \Box

Lemma 4.6. There is a set $X_2 \subset X_1$ such that:

1. $|X_2| = \frac{1}{2}(Q_s - 1 - \frac{n+kc(n)}{b})$.
2. For all the intervals of $X_2$ the indices of the left endpoints are smaller than the indices of the right endpoints, or vice versa.
3. For any interval $I \in X_2$ the length $|I| < \frac{b}{n+kc(n)}$, where $b$ is a universal constant from Lemma 3.4.

Proof. As $|X_1| = Q_s - 1$, then there exists at least $Q_s - 1 - \frac{n+kc}{b}$ intervals from $X_1$ with lengths smaller than $\frac{b}{n+kc}$. By the pigeonhole principle at least half of these intervals have left indices larger than the right ones, or vice versa. \hfill \Box

Lemma 4.7. There is a set $X_3 \subset X_2$ such that:

1. $|X_3| = \frac{Q_s-1}{2a(n+kc(n))},$ where $a$ is a constant from Lemma 4.1.
2. For any pair of intervals $I_i, I_j \in X_3$, $|I_i| < e|I_j|$.

Proof. We divide the set $X_2$ into $a(n+kc)^2$ subsets by interval length, namely,

\[ Y_1 = \{I \in X_2 : e^{-1} < |I| \leq 1\}, \]
\[ Y_2 = \{I \in X_2 : e^{-2} < |I| \leq e^{-1}\}, \]
\[ \vdots \]
\[ Y_{a(n+kc)^2} = \{I \in X_2 : e^{-a(n+kc)^2} < |I| \leq e^{-a(n+kc)^2+1}\}. \]

By the pigeonhole principle at least one of these sets $Y_i$ has the required cardinality. \hfill \Box

Any interval $I \in X_3$ corresponds to a pair of generalized diagonals with indices $p < q$ such that $s \leq p, q \leq s + c$. Let the points $P, Q$ represent the endpoints of unfoldings of these diagonals on the $xy$-plane.

Lemma 4.8. There is a set $X_4 \subset X_3$ such that:

1. $|X_4| = \frac{Q_s-1}{4a(n+kc(n))},$.
2. All the exit triangles for the $P$-points are in the same exit position.
Proof. By Lemma 3.3 the segment $PQ$ is either an unfolding of a generalized diagonal of algebraic length bounded by $c$ from above, or a triangle side. For each point $P$ corresponding to each interval $I \in X_3$ we consider the exit triangle in the $P$-rose. Each such triangle corresponds to particular combinatorics of length at most $n + kc$. The kite angle of each exit triangle equals $i2\alpha + j2\beta$, where $|i| + |j| \leq n + kc$. So there are less than $(n + kc)^2$ angular kite positions for such exit triangles. As there are also two positions of an exit triangle inside a corresponding kite we get that there are at most $2(n + kc)^2$ exit positions for exit triangles of $P$-points for intervals $I \in X_3$. Applying the pigeonhole principle again we get that there is a set of intervals $X_4 \subseteq X_3$ such that all the exit triangles for the $P$-points are in the same exit position and, moreover, the cardinality $|X_4| = \frac{Q_i - 1 - \frac{m\kappa}{4a(n+k)c}}{2}$. □

Now we take a closer look at the set of intervals $X_4$. For a moment we fix a particular interval $I_i \in X_4$, where $1 \leq i \leq |X_4|$, and look at the unfolding sequences, corresponding to the endpoints of $I_i$. As usual we denote the corresponding points on the $xy$-plane as $P$ and $Q$. From Figure 10 we see that $\angle CPQ = \xi - \psi - \theta$. We recall that for all intervals $I \in X_4$ the exit triangles $PBC$ are in the same exit position and so the oriented exit angle $\theta$ is the same for all intervals under consideration. The segment $PQ$ is an unfolding of a generalized diagonal which is uniquely characterized by $\angle CPQ$. In particular, any two such segments for different intervals $I$ correspond to different generalized diagonals if the corresponding angles $\angle CPQ$ are different.

Now it is time to make a very useful ‘local’ definition. For an element $I_i \in X_4$ the connecting diagonal $\gamma_i$ is the angle $\gamma_i = \angle CPQ$ from Figure 10. Here we intentionally identify the diagonal $PQ$ from the vertex $P$ and the corresponding angle as they are in one-to-one correspondence. By Lemma 4.6 and Lemma 3.3, $PQ$ is indeed a diagonal of length not greater than $c$. In other words, we have a map $\Gamma$.
from \( X_4 \) to the set of cutting points of \( \xi_c \) (with vertex at \( P \)) defined by \( \Gamma(I_i) = \gamma_i \).

Our aim now is to estimate from below the cardinality \(|\Gamma(X_4)|\) in terms of \(|X_4|\). We will do it by finding a large enough set \( X_5 \subseteq X_4 \) such that \( \Gamma \) is injective on \( X_5 \). By the definition of \( \Gamma \) this will in turn imply that \(|X_5| \leq P_c \).

**Lemma 4.9.** There is a set \( X_5 \subseteq X_4 \) such that:

1. \(|X_5| = \frac{|X_4|}{eK(n+kc)}\).
2. \( \Gamma \) is injective on \( X_5 \).

**Proof.** Consider two elements \( I_i, I_j \in X_4 \). It is suggested to have in mind Figure 10 for each of the elements. By the definition of \( \Gamma \) we have \( \Gamma(I_i) = \zeta_i - \psi_i - \theta \) and 
\[
\Gamma(I_j) = \zeta_j - \psi_j - \theta.
\]
From this we have that \( \Gamma(I_i) \neq \Gamma(I_j) \) whenever \( \zeta_i - \psi_i \neq \zeta_j - \psi_j \).

We now consider two finite sets of angular segments with the vertex at \( A \). The set of angular segments \( I \) consists of the segments \( I_i = [\zeta_i - \phi_i, \zeta_i] \) for each \( I_i \in X_4 \). In other words, \( I = X_4 \) as sets. By the definition of \( X_4 \), \( I \) then consists of \(|X_4|\) non-intersecting segments of lengths \( \phi_i \).

The second set \( J \) consists of segments \( J_i = [\zeta_i - \psi_i, \zeta_i] \) for each \( I_i \in X_4 \). In other words, \( J \) consists of the segments which have the same right endpoints as the segments from \( I \) and with the corresponding lengths \( \psi_i \). In these terms it is obvious that the desired inequality \( \zeta_i - \psi_i \neq \zeta_j - \psi_j \) takes place as soon as the intervals \( J_i \) and \( J_j \) do not intersect. We are now exactly in the position to apply Lemma 4.4 to the sets of angular segments \( I \) and \( J \).

Fix some element \( I_i \in X_4 \) and consider the triangle \( APQ \) (Figure 10). By Lemma 3.4 there is a constant \( K > 0 \) such that \( \psi_i < K\phi_i(n+kc) \).

At this time we apply Lemma 4.4 to the systems of segments \( I, J \) with the same right endpoints and with constants \( L = e \) and \( m = K(n+kc) \). By Lemma 4.4 we can find at least \( \frac{|X_4|}{eK(n+kc)} \) non-intersecting intervals \( J_i \). \( \square \)

Combining Lemmas 4.5–4.9 we see that there is a set \( X_5 \) of segments \( I_i \), such that \( X_5 \subseteq X_4 \), and \(|X_5| = \frac{\rho - 1 - \frac{a+b}{4aeK(n+kc)}}{e^4} \) such that the corresponding connecting segments \( PQ = \Gamma(I_i) \) correspond to different generalized diagonals of algebraic length bounded by \( c \). But this in turn implies that \(|X_5| \leq P_c \). By the assumptions of Theorem 4.2, \( Q_n \geq Q_n > e^{\nu^r} \) and \( P_c \leq e^{\nu^r} = e^{\nu^r} \) which gives 
\[
\frac{e^{\nu} - 1 - \frac{a+b}{4aeK(n+kc)}}{e^4} \leq e^{\nu^r}.
\]
However as \( \mu > \epsilon \nu \) we get a contradiction by taking \( n \) large enough. \( \square \)

### 4.2 Gap estimates and global complexity bootstrap.

In this subsection we still assume that for some \( \nu > 0 \) and large \( n \) we have \( P_n < e^{n\nu} \). As a result of Theorem 4.2, for any \( \mu > \gamma \) there exists an infinite sequence of all the times \( n_i \) characterized by the property \( Q_{n_i} < e^{\nu^r} \). We are now going to estimate the gaps \( n_{i+1} - n_i \).
Theorem 4.3. Let \( v > 0, \mu > \gamma = \frac{-v^2 + \sqrt{v^4 + 4v^2}}{2} \), and \( n_i \) is a sequence characterized by the property \( Q_{n_i} < e^{\mu} \). Then for any \( \epsilon > 0 \) and for any \( i \) large enough \( n_{i+1} - n_i < n_i^{1+\epsilon} \).

**Proof.** We fix small enough \( \epsilon > 0 \) and introduce the notations \( k(n) = n^\mu \), \( c(n) = n^{1-\mu+\epsilon} \). When it does not lead to ambiguity we will for brevity use the notations \( k = k(n_i) \) and \( c = c(n_i) \). We are going to prove the theorem by contradiction, so let us now assume that there exists a subsequence of \( n_i \) satisfying \( n_{i+1} - n_i > n_i^{1+\epsilon} \). We still denote this subsequence by \( n_i \) in order not to overabuse the notations.

**Lemma 4.10.** For any \( i \) large enough there exists \( s, n_i \leq s \leq n_i + kc \) such that \( Q_{s+c} > 3Q_s \).

**Proof.** We divide index interval \([n_i, n_i + kc]\) into \( k \) subintervals of length \( c \):
\[
I_1 = [n_i, n_i + c], I_2 = [n_i + c, n_i + 2c], \ldots, [n_i + (k-1)c, n_i + kc].
\]
We would like to prove that for one of the intervals \( I_j, 1 \leq j \leq k, \frac{Q_{n_i+jc}}{Q_{n_i+(j-1)c}} > 3 \). Assuming the opposite we have \( Q_{n_i+kc}/Q_{n_i} \leq 3^k \). On the other hand, from the assumptions of Theorem 4.3 we have \( Q_{n_i+kc}/Q_{n_i} > e^{(n_i+kc)^\mu - n_i^\mu} \). Taking the logarithm of both inequalities we obtain \((n_i+kc)^\mu - n_i^\mu < k \ln(3)\). As \((\mu + (1-\mu+\epsilon))\mu > \mu\) we get a contradiction for large enough \( n_i \).

At this moment we entirely repeat all the arguments in the proof of Theorem 4.2 only changing \( n \) to \( n_i \), and \( k(n), c(n) \) in the Subsection 4.1 to the newly introduced \( k(n_i), c(n_i) \). We can do it because in the proof of Theorem 4.2 the only requirements needed to prove Lemmas 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9 were a fixed \( n \) large enough and a fixed \( s \), depending on \( n \), which satisfies the conclusion of Lemma 4.3. We now change \( n \) large enough to \( n_i \) large enough and fix it. Instead of Lemma 4.3 we use Lemma 4.10 which produces \( s \), satisfying the same conclusion as in Lemma 4.3, namely \( Q_{s+c} > 3Q_s \). One may note that the newly introduced power-like functions \( k(n), c(n) \) differ from those in Theorem 4.2, however in the proof of Theorem 4.2 up to Lemma 4.8 we did not use the precise expressions for \( k(n) \) and \( c(n) \) at all. These arguments allow us to repeat the proofs of Lemmas 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9 word for word until we get the set of generalized diagonals \( \Gamma(X_5) \) of cardinality
\[
|X_5| = \frac{Q_{s+1+k}}{4^c k(n_i+kc)}
\]
which must satisfy \( |X_5| < P_c \).

At this moment we stop the formal repeating of the arguments in the proof of Theorem 4.2, because now it does become important how the functions \( k \) and \( c \) depend on \( n_i \). As \( Q_s \geq Q_{n_i} \geq \frac{Q_{n_i+1}}{2} \geq \frac{e^{n_i+1}}{2} \) and \( P_c \leq e^{c^{\epsilon}} = e^{e^{(1-\mu+\epsilon)v}} \) we have a contradiction for large enough \( n_i \) if \( \mu > (1-\mu+\epsilon)\nu \). It is clear that the last condition
holds for any small enough $\epsilon > 0$ if $\mu > (1 - \mu)\nu$ or $\mu > \frac{\nu}{1 + \nu}$ which immediately follows from the easily checked inequality $\mu > \gamma = \frac{-\nu^2 + \sqrt{\nu^4 + 4\nu^2}}{2} > \frac{\nu}{1 + \nu}$. □

Now we combine the two previous theorems and get the global complexity bootstrap.

**Theorem 4.4.** Assume that for some constant $\nu$, $0 < \nu \leq 1$, and all $n$ large enough, $P_n < e^{n\nu}$. Then for any $\mu > \gamma = \frac{-\nu^2 + \sqrt{\nu^4 + 4\nu^2}}{2}$ and all $n$ large enough $P_n < e^{n\mu}$.

**Proof.** It is enough to prove that $Q_n < e^{n\mu}$ for all $n$ large enough. Pick any $\epsilon > 0$ small enough and then pick positive $\delta \ll \epsilon$ small enough. Consider a sequence $n_i$ corresponding to $\mu = \gamma + \delta/2$. By Theorem 4.3, for all $i$ large enough, $n_{i+1} < n_i + n_i^{1+\delta/2} = n_i(1 + n_i^{\delta/2}) < n_i^{1+\delta}$.

Now for large $n$ we take $i$ such that $n_i \leq n < n_{i+1}$. By monotonicity $Q_n \leq Q_{n_i} \leq Q_{n_{i+1}}$, so we get: $Q_n \leq e^{n\gamma} \leq e^{n_{i+1}^\delta} \leq e^{n_{i+1}^{(1+\delta)(1+\delta)}} \leq e^{n^\gamma + \epsilon}$. Here we assumed that $\delta$ is small enough, so that $(1 + \delta)(\gamma + \delta) < \gamma + \epsilon$. As $\epsilon$ can be chosen arbitrarily small the proof is complete. □

One can easily see that the bootstrap function $\gamma = f(\nu) = \frac{-\nu^2 + \sqrt{\nu^4 + 4\nu^2}}{2}$ is a monotone function satisfying $0 < f(\nu) < \nu$ for all $\nu > 0$ and so the iterations $f^k(\nu)$ converge to 0. In particular, for any $\epsilon > 0$ there exists $k$ such that $f^k(1) < \epsilon$. As for any triangle and any $n$ large enough $P_n < e^{n\gamma}$, then the $k$ times application of Theorem 4.4 implies our main result:

**Theorem 4.5 (Weakly exponential estimate).** For a typical triangle and any $\epsilon > 0$ there is a constant $C > 0$ such that $P_n < Ce^{n\epsilon}$.

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