A geometrical action for dilaton gravity

Alberto Saa

Departamento de Física de Partículas,
Universidade de Santiago de Compostela,
E-15706, Santiago de Compostela, Spain

Abstract

We study the gravitational interaction involving the dilaton and the anti-symmetrical \( B_{\mu\nu} \) fields that arises in the low-energy limit of string theory. It is shown that such interaction can be derived from a geometrical action principle, with the scalar of curvature of a non-Riemannian, but metric-compatible, connection as the lagrangian, and with a non-parallel volume-element. This action is contrasted with the recently proposed geometrical action for the 4-dimensional axi-dilaton gravity.

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Recently, Derely and Tucker [1] have shown that the 4-dimensional axi-dilaton gravity equations can be obtained from a geometrical action principle, where the lagrangian is given by the scalar of curvature of a non-Riemannian connection. Besides of be an elegant way of understanding the axi-dilaton gravity, such result provides us with a geometrical interpretation for the massless states of the closed bosonic string. Since the equations for all the massless states of the closed string (namely the dilaton, the graviton, and the anti-symmetrical $B_{\mu\nu}$ field) arise in the same footing from the requirement of conformal invariance up to one-loop order of the closed string in background fields [2,3], such common interpretation in geometrical terms to these states is natural and pertinent. The seeds of this geometrical interpretation can be encountered in the middle 70’s works of Scherk and Schwarz [4]. In the model presented in [1], the torsion tensor is proportional to the anti-symmetrical field $H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}$, in agreement with previous works on sigma models [5], and the non-metricity to the derivative of the dilaton field, as in the interpretation proposed in [1].

The purpose of this Letter is to present another geometrical formulation based in a metric-compatible connection, valid for any space-time dimension. The lagrangian will be the scalar of curvature of a metric-compatible connection, and the interpretation of the $H_{\alpha\mu\nu}$ field is compatible with the previous results. We remember that the usual interpretation of the torsion tensor $S_{\alpha\mu\nu}$ as being proportional to the totally anti-symmetrical tensor $H_{\alpha\mu\nu}$ determines only the traceless part of $S_{\alpha\mu\nu}$. In our proposal, the dilaton field will determine the trace of the torsion tensor and the volume-element. This is the key point, the action principle will be formulated by using a non-parallel volume-element. Non-standard volume-elements have appeared sometimes in the literature, as for example in the characterization of half-flats solutions of Einstein equations [6] and in the description of field theory on Riemann-Cartan spacetimes [7]. The proposed formulation has two advantages in comparison with the proposed in [1]. First, the space-time geometry is more simpler because it involves a metric-compatible connection, and second, and more important, our geometrical formulation applies also in the presence of matter fields, elucidating the dilaton peculiar way of coupling.
We recall that the gravitational interaction involving the dilaton and the $B_{\mu \nu}$ fields is given by the equations

$$
\beta \Phi = 4 D_\mu \Phi D^\mu \Phi - 4 D^2 \Phi - R + \frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} = 0, \\
\beta^\nu_{\mu \nu} = R_{\mu \nu} - \frac{1}{4} H^\lambda_{\mu \nu} H_{\nu \lambda \rho} + 2 D_\mu D_\nu \Phi = 0, \\
\beta^B_{\mu \nu} = D_\lambda H^\lambda_{\mu \nu} - 2 (D_\lambda \Phi) H^\lambda_{\mu \nu} = 0,
$$

(1)

where $R_{\mu \nu}$, $R$, and $D_\mu$ are respectively the Ricci tensor, the scalar of curvature, and the covariant derivative in the 26-dimensional background manifold of metric $g_{\mu \nu}$, all these quantities being calculated by using the Levi-Civita connection. The following conventions are adopted: $\text{sign}(g_{\mu \nu}) = (+, -, -,...)$, $R^\alpha_{\alpha \nu \mu} = \partial_\alpha \Gamma^\beta_{\nu \mu} + \Gamma^\beta_{\alpha \rho} \Gamma^\rho_{\nu \mu} - (\alpha \leftrightarrow \nu)$, $R_{\nu \mu} = R^\alpha_{\alpha \nu \mu}$, and $g = |\text{det} g_{\mu \nu}|$. One can check that the equations (1) follow from the minimization of the action:

$$
S = - \int d^N x \sqrt{|g|} e^{-2\Phi} \left( R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} \right). 
$$

(2)

In spite of dilaton gravity is restricted by its origin in string theory to space-times of dimension $N = 26$, one can define stringy inspired models by assuming that (2) is valid also for other dimensions $N > 2$. As to the two dimensional case, the totally anti-symmetrical third-rank tensor $H_{\alpha \beta \gamma}$ vanishes identically, and we have the following action for the two dimensional dilaton gravity

$$
S = - \int d^2 x \sqrt{|g|} e^{-2\Phi} (R + 4 \partial_\mu \Phi \partial^\mu \Phi). 
$$

(3)

We will discuss latter the incorporation of matter fields in (2) and (3).

We wish to expound how the actions (2) and (3) can be understood as Hilbert-Einstein actions in a manifold endowed with a metric-compatible connection and with a special volume-element. To this end, let us introduce briefly some results on Riemann Cartan (RC) manifolds. A RC manifold is a $N$-dimensional differentiable manifold endowed with a metric tensor $g_{\alpha \beta}(x)$ and with a metric-compatible connection $\Gamma^\mu_{\alpha \beta}$, which is non-symmetrical in its lower indices. From the anti-symmetric part of the connection one can define the torsion...
tensor \( S_{\alpha\beta} = \frac{1}{2} \left( \Gamma_{\alpha\beta}^{\gamma} - \Gamma_{\beta\alpha}^{\gamma} \right) \). The metric-compatible connection, that is used to define the covariant derivative \( D_{\mu} \), can be written as \( \Gamma_{\alpha\beta}^{\gamma} = \left\{ \gamma \right\}_{\alpha\beta} - K_{\alpha\beta}^{\gamma} \), where \( \left\{ \gamma \right\}_{\alpha\beta} \) are the usual Christoffel symbols, and \( K_{\alpha\beta}^{\gamma} \) is the contorsion tensor, which is given in terms of the torsion tensor by

\[
K_{\alpha\beta\gamma} = -S_{\alpha\beta\gamma} + S_{\beta\gamma\alpha} - S_{\gamma\alpha\beta}.
\]

The contorsion tensor (4) can be covariantly split in a traceless part and in a trace

\[
K_{\alpha\beta\gamma} = \tilde{K}_{\alpha\beta\gamma} - \frac{2}{N-1} (g_{\alpha\gamma} S_{\beta} - g_{\alpha\beta} S_{\gamma}),
\]

where \( \tilde{K}_{\alpha\beta}^\alpha = 0 \) and \( S_{\beta} \) is the trace of the torsion tensor, \( S_{\beta} = S_{\alpha\beta}^\alpha \). The RC curvature tensor, calculated by using the connection \( \Gamma_{\alpha\beta}^{\mu} \), according to our conventions is given by

\[
\mathcal{R}_{\alpha\nu\mu}^{\beta} = \partial_{\alpha} \Gamma_{\nu\mu}^{\beta} - \partial_{\nu} \Gamma_{\alpha\mu}^{\beta} + \Gamma_{\alpha\rho}^{\beta} \Gamma_{\nu\mu}^{\rho} - \Gamma_{\nu\rho}^{\beta} \Gamma_{\alpha\mu}^{\rho},
\]

and after some manipulations we get the following expression for the scalar of curvature, valid for \( N > 2 \),

\[
\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\alpha\nu\mu}^{\alpha} = R - 4D_{\mu} S^{\mu} + \frac{4N}{N-1} S_{\mu} S^{\mu} - \tilde{K}_{\nu\rho\alpha} \tilde{K}^{\nu\rho\alpha},
\]

where \( R \) is the Riemannian scalar of curvature, calculated from the Christoffel symbols. For \( N = 2 \) the RC scalar of curvature is given simply by

\[
\mathcal{R} = R - 4D_{\mu} S^{\mu}.
\]

A volume-element in a differentiable orientable \( N \)-dimensional manifold \( \mathcal{M} \) is an \( N \)-form that does not vanish anywhere. If an \( N \)-form \( \Omega \) defines a volume-element, \( f\Omega \) also does it if \( f > 0 \). We see that the volume-forms compatible with the orientation of \( \mathcal{M} \) form a equivalence class, \( \{\Omega\} \). Assuming that \( \mathcal{M} \) is endowed with a metric, the local expression for an element of \( \{\Omega\} \) is given by

\[
d^{N} \mu = f \sqrt{g} d^{N} x,
\]
with \( f > 0 \). In the case that \( \mathcal{M} \) has a metric, we have a privileged volume-element, that one given by \( \Omega = 1^* \), which corresponds to the choice \( f = 1 \) in (10). If the manifold is endowed with a linear connection, another \( N \)-form \( \Omega \) can be singled out with the criterion of parallelism of the volume-form. We know that in general cases they do not coincide [4]. We will consider for convenience the one-parameter family of volume-elements

\[
d^N \mu(\alpha) = e^{\alpha \Theta} \sqrt{g} d^N x, \tag{10}
\]

with \( \alpha \) a real parameter and \( \Theta \) a scalar field.

As it was already said, we will try to identify the trace of the torsion tensor as a derivative of the dilaton field. To this end, let us consider the anzats

\[
S_\beta(x) = \partial_\beta \Theta(x). \tag{11}
\]

We can now construct a Hilbert-Einstein action in a RC manifold using the volume (10) and the condition (11). For the case \( N > 2 \) we obtain

\[
S = -\int d^N \mu(\alpha) \mathcal{R} = -\int d^N x \sqrt{g} e^{\alpha \Theta} \left( R + 4 \left( \frac{N}{N-1} + \alpha - 2 \right) \partial_\mu \Theta \partial^\mu \Theta - \tilde{K}_{\nu \rho \alpha} \tilde{K}^{\alpha \nu \rho} \right) + \text{surf. terms.} \tag{12}
\]

We have that (12) is identical to (2) if one assumes that

\[
\Theta = -\frac{2}{\alpha} \Phi, \tag{13}
\]

\[
\tilde{K}_{\alpha \beta \gamma} = \frac{\sqrt{3}}{6} H_{\alpha \beta \gamma},
\]

\[
\alpha = \begin{cases} 
2 + 2/\sqrt{N - 1} \text{ or} \\
2 - 2/\sqrt{N - 1}
\end{cases}
\]

As to the two-dimensional case, the action will be

\[
S = -\int d^2 x \sqrt{g} e^{\alpha \Theta} \left( R + 4 \alpha \partial_\mu \Theta \partial^\mu \Theta \right), \tag{14}
\]

and to get (3) one needs \( \Theta = -\frac{2}{\alpha} \Phi \), and \( \alpha = 4 \). The two dimensional case can be contrasted also with the ref. [8], where the two-dimensional dilaton gravity is interpreted by means of a symmetrical non-Riemannian connection.
The simplest way to introduce matter fields into the discussion is to add to the Hilbert-Einstein lagrangian a minimally coupled term describing their dynamics. The total lagrangian would be $\mathcal{L} = \mathcal{R} + \mathcal{L}_{\text{matt}}$. One can check that the action $S = -\int d^N \mu(\alpha) \mathcal{L}$ with the assumptions (13) describes precisely the interaction of the dilaton field with matter. Similar arguments can be applied to the introduction of a cosmological constant.

As the conclusion, we summarize that the action (2) is a Hilbert-Einstein action for the metric-compatible connection and volume-element given respectively in terms of the dilaton and the $B_{\mu\nu}$ fields by

$$
\Gamma^\gamma_{\alpha\beta} = \{\gamma_{\alpha\beta}\} - \sqrt{3/6}H_{\alpha\beta}^\gamma - \frac{2}{N - 1 + \sqrt{N} - 1}(\delta^\gamma_\alpha \partial_\beta \Phi - g_{\alpha\beta}g^{\gamma\nu} \partial_\nu \Phi), \text{ or }
\Gamma^\gamma_{\alpha\beta} = \{\gamma_{\alpha\beta}\} - \sqrt{3/6}H_{\alpha\beta}^\gamma - \frac{2}{N - 1 - \sqrt{N} - 1}(\delta^\gamma_\alpha \partial_\beta \Phi - g_{\alpha\beta}g^{\gamma\nu} \partial_\nu \Phi),
$$

$$
d^N \mu = d^N \sqrt{g} e^{-2\Phi},
$$

for space-time dimensions greater than two. The two possible connections correspond to the two values of $\alpha$ in (13). The two-dimensional dilaton gravity can be obtained from a Hilbert-Einstein action constructed from the following connection and volume-element,

$$
\Gamma^\gamma_{\alpha\beta} = \{\gamma_{\alpha\beta}\} - (\delta^\gamma_\alpha \partial_\beta \Phi - g_{\alpha\beta}g^{\gamma\nu} \partial_\nu \Phi),
\Gamma^\gamma_{\alpha\beta} = \{\gamma_{\alpha\beta}\} - (\delta^\gamma_\alpha \partial_\beta \Phi - g_{\alpha\beta}g^{\gamma\nu} \partial_\nu \Phi),
\quad d^2 \mu = d^2 \sqrt{g} e^{-2\Phi}.
$$

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