Refinement of the Rotational Deformation of the Viscoelastic Earth

Skorobogatyy I.V., Myo Zo Aung, Perepelkin V.V.

Moscow Aviation Institute, Volokolamskoe shosse, 4, Moscow, Russia

E-mail: igorsko4@yandex.ru

Abstract. Using a modal approach, this work determined the deformations caused by the pole tide for the axisymmetric model of a deformable Earth with a viscoelastic layer. We compared centrifugal moments of inertia variations with generally accepted expressions. The main properties of the theoretical pole tide are considered in comparison with those observed on the basis of processing high-precision measurements of the gravity acceleration.

The influence of variations in the centrifugal moments of inertia due to the displacement of the pole on the motion of the satellite is quite small. So, according to [4], it leads to a disturbing acceleration of the order of 10^{-11} m/s^2 for high-orbit satellites with an orbital altitude of about 20,000 km (GLONASS, GPS) and of the order of m/s^2 for low-orbit satellites with an orbital altitude of about 350-400 km (ISS).

In contrast to the small influence of the terms of the pole tide on the orbit of satellites, in the problem of the rotation of the Earth and the movement of its pole, such deformations are already more significant - decisive. The consequence of the inelasticity of the Earth's mantle is a small displacement of the pole tide and a phase shift of the oscillations of the centrifugal moments of inertia relative to the oscillations of the earth's pole. Such a small bias does not play any role in practical problems. However, it determines the amplitude of the necessary external disturbance to excite the main motion of the earth's pole.

Let us consider a method for determining deformations during the motion of an axisymmetric Earth around the center of mass by inertia. To calculate the variations of its inertia tensor, we use the equations describing the Earth's deformations in the form [3]:

\[
D(Q + \dot{b}Q) = P, \quad Q = (p_{0m}, q_{1m}, p_{1m}, q_{2m}, p_{2m})^T,
\]

\[
D = \text{diag}(\nu_{0m}^2, \nu_{1m}^2, \nu_{2m}^2, \nu_{3m}^2),
\]

\[
P = (\omega_1^2 + \omega_2^2 + 2\omega_3^2)c_{0m1} + (\omega_1^2 + \omega_3^2)c_{0m31} - 2\omega_1\omega_3c_{1m2},
\]

\[
-2\omega_1\omega_2c_{1m3}, -2\omega_2\omega_3c_{2m1}, (\omega_1^2 + \omega_2^2)c_{2m1})^T.
\]

In these equations \(\omega_1, \omega_2, \omega_3\) are components of the Earth's angular velocity; \(\nu_{lm}^2\) is the frequency of natural vibrations corresponding to the modes \(V_{lm}, W_{lm}, C_{0m1}, C_{0m3}, C_{1m3}, C_{2m1}, B_{1m2}, B_{2m3}\) are constant coefficients which depend on the geometry of the domain \(\Omega\) - in other words, on the shape of
the Earth; \( \chi \) - dimensionless dissipative coefficient, \( \chi \ll 1 \); \( b \) is a positive constant and \( \chi b \) is the relaxation time. The elastic displacement vector \( u \) is represented as an infinite series of modes of elastic oscillations of the Earth:

\[
\mathbf{u} = \sum_{k,l=0}^{\infty} (q_{kl} \mathbf{V}_{kl} + p_{kl} \mathbf{W}_{kl}),
\]

(2)

where vectors \( \mathbf{V}_{kl}, \mathbf{W}_{kl} \) are eigenforms and quantities \( q_{kl}, p_{kl} \) are normal coordinates. The proper forms are an orthonormalized basis, that is, they obey the conditions

\[
(V_{kl}, V_{lm}) = \int_{\Omega} V_{kl} V_{lm} dx = \delta_{(kl)(lm)},
\]

\[
(W_{kl}, W_{lm}) = \int_{\Omega} W_{kl} W_{lm} dx = \delta_{(kl)(lm)},
\]

\[
(V_{kl}, W_{lm}) = 0,
\]

(3)

Here the Kronecker symbol \( \delta_{(kl)(lm)} \) has indices \((ki) \ (lm)\). The configuration space of the system is a Hilbert space embedded in the Sobolev space and being the linear hull of vectors. In cylindrical coordinates, the shapes are written in the form [3]:

\[
\mathbf{V}_{km}(\rho, \varphi, z) = (U_{km}(\rho, z) \sin k\rho, V_{km}(\rho, z) \cos k\rho, W_{km}(\rho, z) \sin k\rho),
\]

\[
\mathbf{W}_{km}(\rho, \varphi, z) = (U_{km}(\rho, z) \cos k\rho, -V_{km}(\rho, z) \sin k\rho, W_{km}(\rho, z) \cos k\rho),
\]

For variables, the equations have the same form (1), where there is zero on the right-hand side; therefore, in the quasi-static approximation, we have

\[
q_{km} = p_{km} = 0, \quad k > 0.
\]

We find an approximate solution to system (1) in the form

\[
\rho_{om} = \nu_{2}^{-1}[(\omega_{1}^{2} + \omega_{2}^{2} + 2\omega_{3}^{2})c_{0m11} + (\omega_{1}^{2} + \omega_{3}^{2})c_{0m13} - \chi b_{v_{2}}^{-1}2(\omega_{1}\omega_{2} \dot{\omega}_{1} + \omega_{2}\omega_{3} \dot{\omega}_{2} + 2\omega_{3}\omega_{3} \dot{\omega}_{3})c_{0m13} + 2(\omega_{1}\omega_{1} \dot{\omega}_{1} + \omega_{2}\omega_{2} \dot{\omega}_{2})c_{0m13},
\]

\[
q_{im} = -\nu_{2}^{-1}2\omega_{2} \omega_{3} \dot{\omega}_{2} b_{m2} + \chi b_{v_{2}}^{-1}2(\dot{\omega}_{2}\omega_{2} + \omega_{2}\dot{\omega}_{2}),
\]

\[
p_{im} = -\nu_{2}^{-1}2\omega_{1} \omega_{3} \dot{\omega}_{1} c_{m3} + \chi b_{v_{2}}^{-1}2c_{m3} (\dot{\omega}_{2}\omega_{2} - \omega_{2}\dot{\omega}_{2}),
\]

\[
q_{om} = -\nu_{2}^{-1}2\omega_{3} \dot{\omega}_{3} b_{m2} + \chi b_{v_{2}}^{-1}2c_{0m13} (\dot{\omega}_{2}\omega_{2} + \omega_{2}\dot{\omega}_{2}),
\]

\[
p_{om} = -\nu_{2}^{-1}(\omega_{1}^{2} - \omega_{2}^{2})c_{2m11} + \chi b_{v_{2}}^{-1}2c_{2m11} (\dot{\omega}_{2}\omega_{2} + \omega_{2}\dot{\omega}_{2}),
\]

(4)

Then for the displacement vector \( \mathbf{u} \) we obtain the expression

\[
\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = p_{0m} \begin{pmatrix} \frac{V_{om} \cos \varphi}{U_{om} \sin \varphi} + \frac{U_{1m} - V_{1m} \sin \varphi}{W_{om} \sin \varphi} \sin \varphi \sin \phi \\ \frac{U_{1m} \sin \varphi + V_{1m} \cos \varphi}{W_{om} \sin \varphi} \cos \varphi \sin \phi \sin \phi + \frac{U_{1m} \cos \phi + V_{1m} \sin \phi}{W_{om} \sin \phi} \cos \phi \sin \phi \sin \phi \\ \frac{U_{2m} \sin 2\phi \cos \phi - V_{2m} \cos 2\phi \sin \phi}{W_{om} \sin \phi} + \frac{U_{2m} \cos 2\phi \cos \phi + V_{2m} \sin 2\phi \sin \phi}{W_{om} \sin \phi} \sin \phi \sin \phi \sin \phi + \frac{U_{2m} \cos 2\phi \sin \phi - V_{2m} \sin 2\phi \cos \phi}{W_{om} \sin \phi} \cos \phi \sin \phi \sin \phi \end{pmatrix} + \begin{pmatrix} u_{1m} \\ u_{2m} \\ u_{3m} \end{pmatrix}.
\]

The Earth's inertia tensor will depend on the displacement vector \( \mathbf{u} \), that is, \( J = J[\mathbf{u}] \) and, without taking into account the quadratic terms in \( \mathbf{u} \), is represented as
\[ J[u] = J_0^{-1} - J_0^{-1} J_1[u] J_0^{-1}, J_0 = \text{diag}\{A, A, C\}, \]

where \( J_1[u] \) is the component of the inertia tensor of the deformed Earth, linear in \( u \):

\[ J_1[u] = \|J_\theta[u]\|, \]

\[ J_{11} = 2\int_\Omega (x_2 u_2 + x_3 u_3) \rho_0 dx, \quad J_{22} = 2\int_\Omega (x_1 u_1 + x_3 u_3) \rho_0 dx \]

\[ J_{33} = 2\int_\Omega (x_1 u_1 + x_2 u_2) \rho_0 dx, \quad J_{22} = 2\int_\Omega (x_1 u_1 + x_3 u_3) \rho_0 dx \]

\[ J_{12} = J_{21} = -2\int_\Omega x_2 u_2 \rho_0 dx, \quad J_{13} = J_{31} = -2\int_\Omega x_1 u_1 \rho_0 dx, \]

\[ J_{13} = J_{23} = -2\int_\Omega x_1 u_1 \rho_0 dx, \quad J_{23} = J_{32} = -2\int_\Omega x_3 u_3 \rho_0 dx \]

From formulas (4) and (5) it follows

\[ J_{12} = J_{21} = -\pi \int_{\Omega'} \rho_0 r^2 (U_{2m} + V_{2m}) dx^* q_{2m}, \]

\[ J_{13} = J_{31} = -2\pi \int_{\Omega'} \rho_0 r^2 W_{1m}^* dx^* p_{1m}, \]

\[ J_{23} = J_{32} = -2\pi \int_{\Omega'} \rho_0 r^2 W_{1m}^* dx^* q_{1m}. \]

We denote (similarly to [3]):

\[ \pi \int_{\Omega'} \rho_0 r^2 W_{1m} dx^* = b_{h_{m2}} = c_{1m1}, \]

\[ \frac{\pi}{2} \int_{\Omega'} \rho_0 r^2 (U_{2m} + V_{2m}) dx^* = b_{h_{m2}} = b_{2m2} = c_{2m1} = c_{2m2}. \]

Then

\[ J_{12} = J_{21} = -\rho_0 b_{2m2} q_{2m}, \]

\[ J_{13} = J_{31} = -\rho_0 b_{1m2} p_{1m}, \]

\[ J_{23} = J_{32} = -\rho_0 b_{1m2} q_{1m}. \]

The largest variations will be \( \delta J_{13}, \delta J_{23} \), which using formulas (4) can be represented as:

\[ \delta J_{13} = a \phi_1 + b \phi_1, \quad \delta J_{23} = a \phi_2 + b \phi_2, \quad a < 0, \quad b > 0. \]
The coefficients in expression (8) are determined by the viscoelastic properties of the Earth. Thus, the differential equations of motion of the Earth's pole contain small dissipative terms determined by the pole tide.

In accordance with the recommendations of the International Earth Rotation Service [1], the variations in centrifugal moments of inertia due to the influence of the pole tide, according to, can be represented as:

$$
\delta J_{13} = a\omega_1 - c\omega_1, \quad \delta J_{23} = a\omega_2 + c\omega_2, \quad a < 0, \quad b > 0.
$$ (9)

These expressions differ from (8) in small terms with coefficients b and c due to the dissipative properties of the Earth's mantle. Dissipative terms in (8) lead to both a "retardation" of the pole tide and the presence of a small component of the incoherent oscillation of the pole, while the corresponding terms from (10) lead only to a "retardation" of the pole tide. Let's consider these expressions in more detail.

The approximate time dependence of the components $\omega_1$, $\omega_2$ of the angular velocity vector is obtained from the equations of the main two-frequency motion of the earth's pole and has the form:

$$
\omega_1 = a_{ch}\cos\alpha_{ch} + a_h\cos\alpha_h,
\omega_2 = a_{ch}\sin\alpha_{ch} + a_h\sin\alpha_h.
$$ (10)

Here $a_{ch}$, $a_h$ are amplitudes of Chandler and yearly oscillations accordingly, and $\alpha_{ch}$, $\alpha_h$ are their phases which correspond to Chandler ($N = 0.843$ cycle/year) and yearly (1 cycle/year) frequencies.

In (11), we make the change of variables $\omega_1 = A\cos\psi$, $\omega_2 = A\sin\psi$. The phase of the amplitude modulation of the pole tide coincides with the phase of the amplitude modulation of the pole oscillations.

Fig.1. Variations in the acceleration of gravity $\delta g$ (black line) associated with the movement of the earth's pole in comparison with smoothed variations $\delta g_{pt}$ (red line) and oscillations of the earth's pole along the coordinate $\xi_p$ passing through the observation point (blue discrete points).

according to (10) since $\sqrt{\left(\delta J_{13}\right)^2 + \left(\delta J_{23}\right)^2} = \sqrt{a^2 + c^2} A$. The amplitude modulation of the pole tide described by expressions (8) does not coincide with the phase of the amplitude modulation of the pole oscillations:
The proposed model shows that if the real pole tide differs from its model (8) there will be a shift in the amplitude modulation of its oscillations. This may be an important characteristic for refining the pole tide model and the Earth rheology model. So, for example, in fig. 1 compares the variations in the acceleration of gravity $\delta g$ arising due to the pole tide with the oscillations of the pole in the projection onto the coordinate $\xi_p = x_p \cos \lambda - y_p \sin \lambda$, passing through the observation point at longitude $\lambda$.

Variations $g$ were isolated from the data of high-precision measurements on an SG gravimeter (Superconducting Gravimeter of the GGP project - Global Geodynamic Project [8, 9]) located in Membach, Belgium. ($\theta = 50.6^\circ$, $\lambda = 6^\circ$). Smoothed dependance

$\delta g_{st} = a_\xi + b_\eta$, where $\eta_p = -x_p \sin \lambda - y_p \cos \lambda$, determined using frequency filtering of observation data $\delta g$.

Fig. 2. Variations in the amplitude of gravity acceleration (red line) and amplitude of oscillations of the earth’s pole $a$ (black line)

Using the smoothed curve $\delta g_{st}$ and its Hilbert transform $\delta g_{st}^\perp$ we construct the change in amplitude (Fig. 2). A phase shift of $\delta g_{st}$ can be observed from a comparison of the graphs. It can be both a consequence of the nonlinearity of the pole tide due to the inhomogeneous structure of the Earth or a consequence of unfiltered close harmonics of geophysical origin.

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