The Smoothness of kernel in Hardy spaces \(^*\dagger\)

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Abstract

This paper provides a study of problems related to Hardy spaces left by G. Weiss in [4]. First, We will prove that the Hardy spaces \(H^p(\mathbb{R}^n)\) can be characterized by a fixed Lipschitz function.

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1 Introduction

Fefferman and Stein in [1] showed that \(H^p(\mathbb{R}^n)\) can be defined as follows:

**Theorem 1.1.** For \(0 < p \leq \infty\), let \(f\) be a distribution, then the following conditions are equivalent:

1. There is a \(\phi \in S(\mathbb{R}^n)\) with \(\int \phi(x)dx \neq 0\) so that \(M_\phi f \in L^p(\mathbb{R}^n)\).
2. The distribution \(f\) is a bounded distribution and \(\sup_{|u-x| \leq t} (f * P_t)(u) \in L^p(\mathbb{R}^n)\).
3. \(M_f f(x) = \sup_{\phi \in S_0} \sup_{t \geq 0} (f * \phi_t)(x) \in L^p(\mathbb{R}^n)\), where \(F = \{\|\cdot\|_{a,b}\}\) is any finite collection of seminorms on \(S(\mathbb{R}^n)\), and \(S_F\) is the subset of \(S(\mathbb{R}^n)\) controlled by this collection of seminorms:

\[
S_F = \{\phi \in S(\mathbb{R}^n) : \|\phi\|_{a,b} \leq 1 \text{ for any } \|\cdot\|_{a,b} \in F\}.
\]

They also discussed the minimal conditions on \(\phi\) so that \(M_\phi f \in L^p(\mathbb{R}^n)\) with \(\|M_\phi f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}\) whenever \(f \in H^p\) (\(p \leq 1\)).

(a) For \(\phi\) that have compact support, it suffices to have \(\phi \in \Lambda^\gamma\) for some \(\gamma > n(p^{-1} - 1)\).

(b) For \(\phi\) not having compact support but vanishing at infinity and satisfying \(|\partial_\gamma^2 \phi(x)| \lesssim (1 + |x|)^{-N}\) for \(|\gamma| = n(p^{-1} - 1) + 1\), it suffices to have \(N_p > n\).

From [3] and [10], the Lipschitz spaces \(\Lambda^\gamma\) can be paired with \(H^p\) if \(\gamma = [n(p^{-1} - 1)]\) (\(0 < p < 1\)). That is for any \(f \in H^p\), the following holds:

\[
\|f\|_{H^p} = \sup_{\|g\|_{\Lambda^\gamma} \leq 1} \left| \int f(x)g(x)dx \right|.
\]

From [8], \(H^p(\mathbb{R}^n)\) \((\frac{1}{1+\gamma} < p \leq 1)\) spaces can also be defined as:

\[
\|f\|_{H^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f^*_p(x)|^p dx,
\]

where \(f \in (\Lambda^\gamma)'\), and \(f^*_p(x)\) is defined by (7).

From (1), (2) and Theorem 1.1, we could see that the smoothness of the kernel in Theorem 1.1 may be reduced. Thus the problem reducing smoothness of \(\phi\) in Hardy spaces was proposed by some mathematicians. The example that \(\phi = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)\) shows that the assumption of smoothness of \(\phi\) can not be removed in the definition of \(H^p\). This shows that the Hardy-Littlewood maximal
function $Mf$ can not characterize any $H^p$ for $0 < p \leq 1$. Thus we wish to replace the Schwartz function $\phi$ in Theorem 1.1 by a fixed Lipschitz function.

Our first result is Theorem 2.7. We will prove the following for $\frac{1}{1+\gamma} < p \leq 1$:

$$
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f \ast \phi)_+\|_{L^p(\mathbb{R}^n)},
$$

(3)

where $\phi$ is a fixed Lipschitz function with compact support satisfying $\int \phi(x)dx \sim 1$. $(f \ast \phi)_+ \nu(x)$ and $(f \ast \phi)_+(x)$ are non-tangential maximal function and radial maximal function defined as:

$$(f \ast \phi)_+(x) = \sup_{|x-u| < t} |f \ast \phi_t(u)|, \quad (f \ast \phi)_+(x) = \sup_{t > 0} |f \ast \phi_t(x)|,$$

where $\phi_t(x) = t^{-n}\phi(tx^{-1})$.

From the above (3), we could also see that the norm of $(f \ast \phi)_+$ and the norm of $(f \ast \phi)_+ \nu$ are equivalent when $\phi \in \Lambda^\gamma$ with a compact support. (3) is different to (2) that $\phi$ in (2) is not fixed.

In 1983, Han in [5] gave another characterization of $H^1_\gamma(\mathbb{R})$ with Carleson measure. Han also proved that there exists a $\phi$ which is a Lipschitz function satisfying the Formulas (4, 5) so that $(f \ast \phi)_\nu \in L^1(\mathbb{R}) \Rightarrow f(x) = 0 \text{ a.e.} x \in \mathbb{R}$.

$$
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{1+\gamma}},
$$

(4)

$$
|\phi(x+h) - \phi(x)| \lesssim \frac{|h|\gamma}{(1 + |x|)^{1+2\gamma}}, \quad \text{if } |h| \lesssim |x|/2.
$$

(5)

when $0 < \gamma \leq 1$. However, when we replace the Formulas (4, 5) with Formulas (42, 43), we could deduce the Proposition 3.1: $f \in H^p(\mathbb{R}^n) \Rightarrow (f \ast \phi)_\nu \in L^p(\mathbb{R}^n)$ for $\frac{1}{1+\gamma} < p \leq 1$.

Our second result is Theorem 3.7, that the following holds for $\frac{1}{1+\gamma} < p \leq 1$:

$$
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f \ast \phi)_+\|_{L^p(\mathbb{R}^n)},
$$

(6)

where $\phi$ is a fixed Lipschitz function without compact support satisfying Formulas (57, 58) and $\int \phi(x)dx \sim 1$.

Notation: Let $S(\mathbb{R}^n)$ be the space of $C^\infty$ functions on $\mathbb{R}^n$ with the Euclidean distance rapidly decreasing together with their derivatives (Schwartz Class), $S'(\mathbb{R}^n)$ the tempered distributions. In the following of this paper, we assume that $0 < \gamma \leq 1$, $\alpha \in \mathbb{N}^n$ satisfying:

$$
\alpha = (\alpha_i)_{i=1}^n, \quad \text{where } \alpha_i \in \mathbb{N}, \quad \text{and } |\alpha| = \sum_{i=1}^n \alpha_i.
$$

We use $S^n$ to denote the unit sphere in $\mathbb{R}^{n+1}$, $B(x, r)$ to denote the set: $B(x, r) = \{y : |x-y| < r\}$. $B(x, r_1) \setminus B(y, r_2)$ is denoted as the set: $B(x, r_1) \setminus B(y, r_2)^c$.

2 Lipschitz function with compact support in $\mathbb{R}^n$

Definition 2.1 (The Lipschitz function). For $\phi \in C(\mathbb{R}^n)$, $n \in \mathbb{N}$, $H^\gamma(\phi)$ is denoted as:

$$
H^\gamma(\phi) = \sup_{x, y \in \mathbb{R}^n, x \neq y} |\phi(x) - \phi(y)|/|x-y|^\gamma;
$$

The Lipschitz function $\Lambda^\gamma$ is defined as $\Lambda^\gamma = \{f : \sup_{x \in \mathbb{R}^n} |f(x) - f(y)| \leq C|y|^\gamma\}$, and $(\Lambda^\gamma)'$ is denoted as the dual space of $\Lambda^\gamma$.

For $f \in (\Lambda^\gamma)'$, the maximal function $f^*_\gamma(x)$ in $\mathbb{R}^n$ is defined as:

$$
f^*_\gamma(x) = \sup_{\phi, r} \left\{ \int_{\mathbb{R}^n} \frac{f(y)\phi(y)dy}{r^n} : r > 0, \supp \phi \subset B(x, r), H^\gamma(\phi) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\}.
$$

(7)

For $f \in S'(\mathbb{R}^n)$, $f^*_{\gamma_\nu}(x)$ is defined as:

$$
f^*_{\gamma_\nu}(x) = \sup_{\phi, r} \left\{ \int_{\mathbb{R}^n} \frac{f(y)\phi(y)dy}{r^n} : r > 0, \supp \phi \subset B(x, r), H^\gamma(\phi) \leq r^{-\gamma}, \phi \in S(\mathbb{R}^n), \|\phi\|_{L^\infty} \leq 1 \right\}.
$$

(8)
or
\[
 f_{S_{\gamma}}^*(x) = \sup_{\psi, r > 0} \left\{ \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y}{r} \right) dy : \psi(t) = \mathbb{S}(t) \in \mathbb{S}(\mathbb{R}^n), \sup \psi(t) \subseteq B(0,1), \|\psi\|_{L^\infty} \leq 1, H^\gamma \psi \leq 1 \right\}.
\]

**Definition 2.2 (M_\phi f(x)).** For \( f \in \mathbb{S}'(\mathbb{R}^n), M_\phi f(x) \) is defined as
\[
 M_\phi f(x) = \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} f(y) \phi \left( \frac{x-y}{r} \right) dy : \|\phi\|_r \leq 1, \psi \subseteq B(0,1), \psi \in \mathbb{S}(\mathbb{R}^n) \right\}.
\]

**Proposition 2.3.** [9] For fixed numbers \( 0 < b < a, F(x, r) \) is a function defined on \( \mathbb{R}^{n+1} \), its nontangential maximal function \( F_a^*(x) \) is defined as \( F_a^*(x) = \sup_{|x-y| < ar} |F(y, r)| \). If \( F_a^*(x) \) is \( L^1(\mathbb{R}^n) \) or \( F_a^*(x) = L^1(\mathbb{R}^n) \), then we could obtain the following inequality for \( p > 0 \):
\[
 \int_{\mathbb{R}^n} |F_a^*(x)|^p dx \leq c \left( \frac{a+b}{b} \right)^n \int_{\mathbb{R}^n} |F_b^*(x)|^p dx,
\]
where \( c \) is a constant independent on \( F, a, b \).

**Proposition 2.4.** For \( f \in L^1(\mathbb{R}^n), 0 < p < \infty \) we could obtain \( f_{S_{\gamma}}^*(x) = f_{S_{\gamma}}^*(x) \) a.e. \( x \in \mathbb{R}^n \). Further more, if \( \int_{\mathbb{R}^n} |f_a^*|^p dx \leq \infty \) or \( \int_{\mathbb{R}^n} |f_{S_{\gamma}}^*|^p dx \leq \infty \), then the following holds
\[
 \int_{\mathbb{R}^n} |f_a^*|^p dx \sim \int_{\mathbb{R}^n} |f_{S_{\gamma}}^*|^p dx.
\]

**Proof.** We will prove the following (10) first:
\[
 f_{S_{\gamma}}^*(x) = f_{S_{\gamma}}^*(x) \quad a.e. x \in \mathbb{R}^n.
\]
It is easy to see that \( f_{S_{\gamma}}^*(x) \leq f_{S_{\gamma}}^*(x) \). If \( \phi \) satisfies \( H^\gamma(\phi) \leq r^{-\gamma} \) and \( \sup \phi \subseteq B(x, r) \), then \( \phi \) is a continuous function with compact support. Thus there exists sequence \( \{\psi_k\}_{k \in \mathbb{N}} \subseteq \mathbb{S}(\mathbb{R}^n) \) with \( \lim_{k \to \infty} \|\psi_k - \phi\|_{\infty} = 0, \|\psi_k - \phi\|_{\infty} \neq 0 \). Denote \( \delta_k(x) = \delta_k(x) = \int_{B(x, r)} \mathcal{F}(y) (\phi(y) - \psi_k(y)) dy/r^n \), then \( \delta_k(x) \leq M f(x) \|\psi_k - \phi\|_{\infty} \). Let \( i_k \) be \( i_k = \|\psi_k - \phi\|_{\infty} \), then we could obtain:
\[
 \{ x : \delta_k(x) > \alpha \} \subseteq \left\{ x : M f(x) > \frac{\alpha}{i_k} \right\}.
\]
By the fact that \( M \) is weak-(1, 1) bounded, the following could be obtained from the above inequality for any \( \alpha > 0 \):
\[
 |\{ x : \delta_k(x) > \alpha \}| \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)} \|\psi_k - \phi\|_{\infty}.
\]
Thus by \( \lim_{k \to \infty} \|\psi_k - \phi\|_{\infty} = 0 \), we could have
\[
 \lim_{k \to \infty} |\{ x : \delta_k(x) > \alpha \}| = 0.
\]
Then there exists a sequence \( \{k_j\} \subseteq \{k\} \) such that
\[
 \int_{\mathbb{R}^n} f(y) \phi(y) dy/r^n = \lim_{k_j \to \infty} \int_{\mathbb{R}^n} f(y) \psi_{k_j}(y) dy/r^n, \quad a.e. x \in \mathbb{R}^n
\]
for \( f \in L^1(\mathbb{R}^n) \). Thus we could obtain:
\[
 \int_{\mathbb{R}^n} f(y) \phi(y) dy/r^n \leq f_{S_{\gamma}}^*(x) \quad a.e. x \in \mathbb{R}^n
\]
for any \( \phi \) satisfies \( H^\gamma(\phi) \leq r^{-\gamma} \) and \( \sup \phi \subseteq B(x, r) \). We could then deduce
\[
 \sup_{\phi, r > 0} \left| \int_{\mathbb{R}^n} f(y) \phi(y) dy/r^n \right| \leq f_{S_{\gamma}}^*(x) \quad a.e. x \in \mathbb{R}^n.
\]
Thus with the fact that $f^*_S(x) \leq f^*_N(x)$, we could have
\[ f^*_S(x) = f^*_N(x) \quad \text{a.e.} x \in \mathbb{R}^n. \]

Then we will prove the following (11):
\[ \int_{\mathbb{R}^n} |f^*_S(x)|^p \, dx \sim \int_{\mathbb{R}^n} |f^*_N(x)|^p \, dx \quad \text{(11)} \]

Let $E$ denote a set defined as $E = \{ x : f^*_S(x) = f^*_N(x) \}$. Next we will prove that for any $x_0 \in \mathbb{R}^n$, there is a point $\mathfrak{F}_0 \in \mathbb{R}$ such that
\[ f^*_S(x_0) \leq f^*_N(\mathfrak{F}_0). \quad \text{(12)} \]

Notice that for $x_0 \in \mathbb{R}^n$, there exist $\gamma > 0$ and $\phi_0$ satisfying: $\text{supp} \phi_0 \subset B(x_0, r_0)$, $\phi_0 \in S(\mathbb{R}^n)$, $H^\gamma(\phi_0) \leq r^{-\gamma}$, $\| \phi_0 \|_{L^\infty} \leq 1$, such that the following inequality holds:
\[ \left| \frac{1}{r^\gamma} \int f(y) \phi_0(y) \, dy \right| \geq \frac{1}{2} f^*_N(x_0). \]

Notice that $|\mathbb{R}^n \setminus E| = |E^c| = 0$ leads to the fact that $E$ is dense in $\mathbb{R}^n$, thus there exists a $\mathfrak{F}_0 \in E$ with $d(x_0, \mathfrak{F}_0) \leq \frac{\gamma}{4}$. Then $\text{supp} \phi_0 \subset B(\mathfrak{F}_0, 4r_0)$ holds, and we could obtain the following
\[ \left| \frac{1}{r^\gamma} \int f(y) \phi_0(y) \, dy \right| \leq C f^*_S(\mathfrak{F}_0), \]

where $C$ is a constant independent on $f$, $\gamma$ and $r_0$. Thus (12) could be obtained. By (12), we could deduce the following:
\[ \int_E |f^*_S(x)|^p \, dx < \infty \Rightarrow \int_{\mathbb{R}^n} |f^*_S(x)|^p \, dx \sim \int_E |f^*_N(x)|^p \, dx. \quad \text{(13)} \]

In the same way, we could conclude that
\[ \int_{\mathbb{R}^n} |f^*_N(x)|^p \, dx \sim \int_E |f^*_S(x)|^p \, dx. \quad \text{(14)} \]

From Formula (10) we could deduce:
\[ \int_E |f^*_S(x)|^p \, dx = \int_E |f^*_N(x)|^p \, dx. \quad \text{(15)} \]

The above Formula (15) together with Formulas (13, 14) lead to (11) if $\int_{\mathbb{R}^n} |f^*_N(x)|^p \, dx < \infty$ or $\int_{\mathbb{R}^n} |f^*_S(x)|^p \, dx < \infty$. This proves the proposition. \qed

**Proposition 2.5.** For $\phi(x) \in N^\gamma$, $\text{supp} \phi(x) \subset \{ x \in \mathbb{R}^n : |x| < 1 \}$, $\frac{1}{|x|^\gamma} < p \leq 1$, $|\phi(x)| \leq 1 \int \phi(x) \, dx = 1$, we could deduce the following for any $f \in H^p(\mathbb{R}^n)$:
\[ \| f \|_{H^p(\mathbb{R}^n)} \sim_{\gamma,p,\phi} \|(f * \phi)\|_{L^p(\mathbb{R}^n)}. \quad \text{(16)} \]

**Proof.** Let $f \in L^1(\mathbb{R}^n)$, $\psi \in S(\mathbb{R}^n)$ with $\int \psi(x) \, dx \sim 1$. There exists a sequence $\{ \phi^m(x) : \phi^m(x) \in S(\mathbb{R}^n) \}_{m \in \mathbb{N}}$ satisfying:
\[ \| \phi^m(x) - \phi(x) \|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{1}{m}, \int \phi^m(x) \, dx \sim 1. \]

$\varphi \in S(\mathbb{R}^n)$ is a fixed function so that
\[ \begin{cases} \varphi(\xi) = 0 & \text{for } |\xi| \geq 1 \\ \varphi(\xi) = 1 & \text{for } |\xi| \leq 1/2. \end{cases} \]

We use $\varphi^k \in S(\mathbb{R}^n)$ to denote as:
\[ \begin{cases} \varphi^k(\xi) = \varphi(\xi) & \text{for } k = 0, \\ \varphi^k(\xi) = \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi) & \text{for } k \geq 1. \end{cases} \]
Then we could have that

\[ 1 = \sum_{k=0}^{\infty} \phi^k(\xi). \]

Notice that \( \int \phi^m(x)dx \sim 1 \), thus \( (\mathcal{F}\phi^m)(2^{-k_0}\xi) \geq C \) for \( |\xi| \leq 1 \), where \( C \) and \( k_0 \) are independent on \( m \). Let \( \eta_m^k \) to be

\[ (\mathcal{F}\eta_m^k)(\xi) = \frac{\phi^k(\xi)(\mathcal{F}\psi)(\xi)}{(\mathcal{F}\phi^m)(2^{-k-k_0}\xi)}, \]

where \( \mathcal{F} \) denotes the Fourier transform. Then we could obtain that:

\[ (\mathcal{F}\psi)(\xi) = \sum_{k=0}^{\infty} \frac{\phi^k(\xi)(\mathcal{F}\psi)(\xi)}{(\mathcal{F}\phi^m)(2^{-k-k_0}\xi)}(\mathcal{F}\phi^m)(2^{-k-k_0}\xi) \]

\[ = \sum_{k=0}^{\infty} (\mathcal{F}\eta_m^k)(\xi)(\mathcal{F}\phi^m)(2^{-k-k_0}\xi). \]

Thus

\[ \psi(x) = \sum_{k=0}^{\infty} \eta_m^k * \phi^m_{2^{-k-k_0}}(x). \quad (17) \]

By the fact that \( \sup_{\xi \in \mathbb{R}^n} |\partial^{\alpha'}_x (\mathcal{F}\phi^m)(\xi)| \leq C_{\alpha'} \) and

\[ \sup_{\xi \in \mathbb{R}^n} \left| \xi^{\alpha} \partial^{\alpha'}_x (\phi^k(\xi)(\mathcal{F}\psi)(\xi)) \right| \leq C_{\alpha,\alpha',M} 2^{-kM} \text{ for any } M > 0, \quad (18) \]

where \( C_{\alpha} \) is a constant independent on \( m \), we could deduce that

\[ \sup_{\xi \in \mathbb{R}^n} \left| \xi^{\alpha} \partial^{\alpha'}_x (\mathcal{F}\eta_m^k)(\xi) \right| \leq C_{\alpha,\alpha',M,k_0} 2^{-kM} \text{ for any } M > 0, \quad (19) \]

where \( C_{\alpha,\alpha',M,k_0} \) is a constant independent on \( m \) and \( k \). Thus we could have:

\[ \left| \int_{\mathbb{R}^n} \eta_n^k(m) \left(1 + 2^{k+k_0}|u| \right)^N \right| \leq C_{k_0,N} 2^{-k}, \quad (20) \]

where \( C_{k_0,N} \) is a constant independent on \( m \). Then by Formulas (17) with the fact that \( f \in L^1(\mathbb{R}^n) \) we have

\[ M_{\psi} f(x) = \sup_{r>0} \int_{B(x,r)} f(y) \frac{1}{r^n} \psi \left( \frac{x-y}{r} \right) dy \]

\[ = C \sup_{r>0} \left| \int_{B(x,r)} f(y) \eta_n^k \left( \frac{s}{r} \right) \frac{1}{r^n} \psi \left( \frac{x-y-s}{2^{-k-k_0}r} \right) ds \right| \]

\[ \leq C \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}^n} \eta_n^k \left( \frac{s}{r} \right) \frac{1}{r^n} \psi \left( \frac{x-y-s}{2^{-k-k_0}r} \right) ds \right| \sup_{r>0} \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y-s}{r} \right) \left(1 + \frac{|s|}{r} \right)^{-N} ds, \]

where \( C \) is a constant independent on \( m \). From Formula (20) and Formula (21) we could obtain:

\[ M_{\psi} f(x) \leq \sup_{r>0} \int_{B(x,r)} f(y) \psi \left( \frac{x-y-s}{r} \right) \left(1 + \frac{|s|}{r} \right)^{-N} ds \]

\[ \leq \left( \sup_{0 \leq |s| \leq 2} \sup_{0 \leq |s| \leq r} \left| \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y-s}{r} \right) \left(1 + \frac{|s|}{r} \right)^{-N} dy \right| \right) \]

\[ \leq \sum_{k=0}^{\infty} 2^{-(k-1)N} \sup_{0 \leq |s| \leq 2} \left| \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y-s}{r} \right) dy \right|. \quad (22) \]

Formula (9) leads to

\[ \int_{\mathbb{R}^n} \sup_{0 \leq |s| \leq 2} \left| \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y-s}{r} \right) dy \right|^p dx \leq C (1 + 2^k)^n \int_{\mathbb{R}^n} \sup_{0 \leq |s| \leq 2} \left| \int_{\mathbb{R}^n} f(y) \psi \left( \frac{x-y-s}{r} \right) dy \right|^p dx. \quad (23) \]
Thus we could have:

\[ (F^m f) (x, r) = \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y}{r} \right) \frac{dy}{r^m}, \quad (F f) (x, r) = \int_{\mathbb{R}^n} f(y) \phi \left( \frac{x - y}{r} \right) \frac{dy}{r^m}. \]

Thus we could deduce the following:

\[ |(F^m f) (u, r) - (F f) (u, r)| \leq \int_{\mathbb{R}^n} |f(y)| \left| \phi^m \left( \frac{u - y}{r} \right) - \phi \left( \frac{u - y}{r} \right) \right| \frac{dy}{r^m} \]

Thus we could have:

\[ |(F^m f) (u, r) - (F f) (u, r)| \leq C \frac{m}{m} |M f(u)|, \]

where C is dependent on \( \gamma \) and \( M \) is the Hardy-Littlewood Maximal Operator. Let us set:

\[ \delta_m(u) = |(F^m f) (u, r) - (F f) (u, r)|. \]

Thus we could deduce the following:

\[ \{ x : \delta_m(x) > \alpha \} \subseteq \left\{ x : M f(x) > \frac{1}{c} \alpha \right\} \quad \text{for some constant } c. \]

Notice that \( M \) is weak-(1, 1) bounded, thus the following holds for any \( \alpha > 0 \):

\[ \left\| \{ x : \delta_m(x) > \alpha \} \right\|_{L^1(\mathbb{R}^n)} \leq \frac{c \| f \|_{L^1(\mathbb{R}^n)}}{m}. \]

Thus we could obtain:

\[ \lim_{m \to +\infty} \left\| \{ x : \delta_m(x) > \alpha \} \right\|_{L^1(\mathbb{R}^n)} = 0. \]

Thus there exists a sequence \( \{ m_j \} \subseteq \{ m \} \) such that the following holds:

\[ \lim_{m_j \to +\infty} (F^{m_j} f) (u, r) = (F f) (u, r), \quad \text{a.e.} \ u \in \mathbb{R}^n \]

for \( f \in L^1(\mathbb{R}^n) \). Let us set \( E \) as:

\[ E = \{ u \in \mathbb{R}^n : \lim_{m_j \to +\infty} (F^{m_j} f) (u, r) = (F f) (u, r) \}. \]

Thus it is clear that E is dense in \( \mathbb{R}^n \). For any \( x_0 \in \mathbb{R}^n \), there exists a \( (u_0, r_0) \) with \( r_0 > 0 \), \( u_0 \in \mathbb{R}^n \), \( |u_0 - x_0| < r_0 \) such that the following holds:

\[ |(F^{m_j} f) (u_0, r_0)| \geq \frac{1}{2} \sup_{|x - u_0| < r} |(F^{m_j} f) (u, r)|. \]

Notice that \( (F^{m_j} f) (u, r_0) \) is a continuous function in \( u \) variable and \( E \) is dense in \( \mathbb{R}^n \). There exists a \( \tilde{u}_0 \in E \) with \( |\tilde{u}_0 - x_0| < r_0 \) such that

\[ |(F^{m_j} f) (\tilde{u}_0, r_0)| \geq \frac{1}{2} \sup_{|x - \tilde{u}_0| < r} |(F^{m_j} f) (u, r)|. \]

Thus we could deduce that

\[ \sup_{u \in \mathbb{R}^n : |x - u| < r} \left| (F^{m_j} f) (u, r) \right| \sim \sup_{u \in \mathbb{R}^n : |x - u| < r} \left| (F^{m_j} f) (u, r) \right|. \]
Formula (26) together with the dominated convergence theorem, we could conclude:

\[
\lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|x-u|<r} |(F^{m_j}) (u, r)|^p \, dx \sim \lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|u-E|<|x-u|<r} |(F^{m_j}) (u, r)|^p \, dx \\
\leq C \int_{\mathbb{R}^n} \sup_{|u-E|<|x-u|<r} |(F^{m_j}) (u, r)|^p \, dx \\
\leq C \int_{\mathbb{R}^n} \sup_{(u, r) \in E} |(F f) (u, r)|^p \, dx \\
\leq C \int_{\mathbb{R}^n} \sup_{(u, r) \in E} |(F f) (u, r)|^p \, dx. \tag{27}
\]

Also it is clear that the following inequality holds for \( f \in L^1(\mathbb{R}^n) \):

\[
\| (f * \phi) \|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)} = \| f \|_{H^p(\mathbb{R}^n)}. \tag{28}
\]

Notice that \( H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) is dense in \( H^p(\mathbb{R}^n) \). Then by Formula (27) and Formula (28), we could deduce that the following inequality holds for \( f \in H^p(\mathbb{R}^n) \):

\[
\| f \|_{H^p(\mathbb{R}^n)} \sim_{p, \gamma, \phi} \| (f * \phi) \|_{L^p(\mathbb{R}^n)}. \tag{29}
\]

This proves our proposition. \( \square \)

**Proposition 2.6.** For \( \phi(x) \in \Lambda_\gamma \), \( \text{supp} \phi(x) \subseteq \{ x \in \mathbb{R}^n : |x| < 1 \} \), \( \frac{1}{1+\gamma} < p \leq 1 \), \( |\phi(x)| \leq 1 \), \( \int \phi(x) \, dx = 1 \), then we could obtain the following inequality for \( f \in H^p(\mathbb{R}^n) \):

\[
\| (f * \phi) \|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \phi, p} \| (f * \phi) \|_{L^p(\mathbb{R}^n)} \tag{30}
\]

**Proof.** Let us set \( 0 < \alpha < \gamma \leq 1 \), \( f \in L^1(\mathbb{R}^n) \) and \( 1 \geq p > \frac{1}{1+\gamma-\alpha} \) first. We use \( F(a, b, x, y, r) \) with \( |a-b| < r \) to denote as:

\[
F(a, b, y, z, r) = \left( \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right) - \left( \phi \left( \frac{a-z}{r} \right) - \phi \left( \frac{b-z}{r} \right) \right).
\]

We use \( T(a, b, y, r) \) with \( |a-b| < r \) to denote as:

\[
T(a, b, y, r) = \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right).
\]

Thus it is clear to see that the following inequalities hold for \( 0 < \alpha < \gamma \leq 1 \):

\[
|F(a, b, y, z, r)| \leq C \left( \frac{a-b}{r} \right)^\alpha \left( \frac{y-z}{r} \right)^{\gamma-\alpha}; \tag{31}
\]

\[
|T(a, b, y, r)| \leq C; \tag{32}
\]

\[
\text{supp} \, T(a, b, y, r) \subseteq B(x, 2r) \text{ when } a \in B(x, r), \, |a-b| < r. \tag{33}
\]

For \( p > \frac{1}{1+\gamma-\alpha} \), let \( F \) denote as:

\[
F = \left\{ x \in \mathbb{R}^n : f_{\gamma-\alpha}^* (x) \leq \sigma (f * \phi) \right\}
\]

It is clear that the following inequality holds for \( f \in L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n) \):

\[
\| f_{\gamma-\alpha}^* \|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \alpha} \| f_{\gamma-\alpha}^* \|_{L^p(\mathbb{R}^n)} \sim_{\gamma, \alpha} \| f_{\gamma-\alpha}^* \|_{L^p(\mathbb{R}^n)}. \tag{34}
\]

Then by Proposition 2.5, we could obtain

\[
\int_{\mathbb{R}^n} |(f * \phi) \, \varphi(x)|^p \, dx \leq \frac{C}{\sigma^p} \int_{\mathbb{R}^n} |f_{\gamma-\alpha}^* (x)|^p \, dx \leq \frac{C_{\gamma, \alpha} \sigma}{\sigma^p} \int_{\mathbb{R}^n} |f_{\gamma} (x)|^p \, dx \tag{34}
\]

\[
\leq \frac{C_{\gamma, \alpha, \phi}}{\sigma^p} \int_{\mathbb{R}^n} |(f * \phi) \, \varphi(x)|^p \, dx.
\]
Thus for any fixed \( s \) satisfying \( \frac{\alpha}{\beta} < \gamma \), we could deduce Formula (35)

\[
\int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx \lesssim_{\gamma, \alpha, \phi} \int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx.
\]

(35)

Denote \( Df(x) \) and \( F(x,r) \) as:

\[
Df(x) = \sup_{r > 0} |f \ast \phi_r(x)|, \quad F(x,t) = f \ast \phi_t(x).
\]

Next, we will show that for any \( q > 0 \),

\[
(f \ast \phi)\varphi(x) \lesssim M(q, (Df)^q(x))^{\frac{1}{q}} \quad \text{for } x \in F,
\]

where \( M \) is the Hardy-Littlewood maximal operator. Fix any \( x_0 \in F \), then there exists \((u_0, r_0)\) satisfying \(|u_0 - x_0| < r_0\) such that the following inequality holds:

\[
|F(u_0, r_0)| > \frac{1}{2} |(f \ast \phi)\varphi(x_0)|.
\]

(37)

Choosing \( \delta \) small enough and \( u \) with \(|u - u_0| < \delta r_0\), we could deduce that

\[
|F(u, r_0) - F(u_0, r_0)| = \left| \int_{\mathbb{R}^n} f(y) \phi\left(\frac{u - y}{r_0}\right) \frac{dy}{r_0^n} - \int_{\mathbb{R}^n} f(y) \phi\left(\frac{u_0 - y}{r_0}\right) \frac{dy}{r_0^n} \right| \leq \left| \int_{\mathbb{R}^n} f(y) T(u, u_0, y, r_0) \frac{dy}{r_0^n} \right|.
\]

We could consider \( T(u, u_0, y, r_0) \) as a new kernel. By Formulas (31, 32, 33) we could obtain:

\[
|F(u, r_0) - F(u_0, r_0)| \leq C \delta^\alpha f_{\gamma, \alpha}(x_0) \leq C \delta^\alpha \sigma(f \ast \phi)\varphi(x_0) \quad \text{for } x \in F.
\]

Taking \( \delta \) small enough such that \( C \delta^\alpha \sigma \leq 1/4 \), we could obtain

\[
|F(u, r_0)| \geq \frac{1}{4} |(f \ast \phi)\varphi(x_0)| \quad \text{for } u \in I(u_0, \delta r_0).
\]

Thus the following inequality holds for any \( x_0 \in F \),

\[
|(f \ast \phi)\varphi(x_0)|^q \leq \frac{1}{B(u_0, \delta r_0)} \left( \int_{B(u_0, \delta r_0)} 4^q |F(u, r_0)|^p \, du \right)^\frac{q}{p} \leq \frac{B(x_0, (1 + \delta) r_0)}{B(u_0, \delta r_0)} \left( \int_{I(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^p \, du \right)^\frac{q}{p} \leq \frac{(1 + \delta)^n}{\delta^n} \left( \int_{B(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^p \, du \right)^\frac{q}{p} \leq CM[(Df)^q](x_0)
\]

C is independent on \( x_0 \). Finally, using the maximal theorem for \( M \) when \( q < p \) leads to

\[
\int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx \leq C \int_{\mathbb{R}^n} M[(Df)^q](x)^{p/q} \, dx \leq C \int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx.
\]

(38)

Thus for any fixed \( \alpha \) satisfying \( 0 < \alpha < \gamma \) and \( 1 \geq p > \frac{1}{1 + \gamma - \alpha} \), the above Formula (38) combined with Formula (35) lead to

\[
\|(f \ast \phi)\varphi\|_{L^p(\mathbb{R}^n)} \leq C \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)} ^{\frac{\alpha}{\beta}}
\]

(39)

where \( C \) is dependent on \( p \) and \( \alpha \). Next we will remove the number \( \alpha \). For any \( 1 \geq p > \frac{1}{1 + \gamma} \), let \( p_0 = \frac{1}{2} \left( p + \frac{1}{1 + \gamma} \right) \) with \( p > p_0 > \frac{1}{1 + \gamma} \) and let \( \alpha = 1 + \gamma - \frac{1}{p_0} \). By Formula (39), we could obtain the following inequality holds for \( 1 \geq p > \frac{1}{1 + \gamma} \) and \( f \in L^1(\mathbb{R}^n) \)

\[
\|(f \ast \phi)\varphi\|_{L^p(\mathbb{R}^n)} \leq C \|(f \ast \phi)\|_{L^p(\mathbb{R}^n)}
\]

where \( C \) is dependent on \( p \) and \( \gamma \). Thus by the fact that \( L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n) \) is dense in \( H^p(\mathbb{R}^n) \), we could deduce Formula (22) holds for any \( f \in H^p(\mathbb{R}^n) \). This proves the Proposition. \( \square \)
Thus from Proposition 2.5 and Proposition 2.6, we could obtain the following theorem:

**Theorem 2.7.** For $\phi(x) \in \Lambda^\gamma$, $\text{supp} \phi(x) \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$, $1 \geq p > \frac{1}{1+\gamma}$, $|\phi(x)| \leq 1 \int \phi(x)dx = 1$, then we could obtain the following inequality for any $f \in H^p(\mathbb{R}^n)$:

$$\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|f * \phi\|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|f * \phi\|_{L^p(\mathbb{R}^n)}.$$  \hfill (40)

**Proposition 2.8.** For $\phi(x) \in S(\mathbb{R}^n)$, with $|\mathcal{F} \phi(\xi)| \geq C$ for $\xi \in B(\xi_0, r_0)$, we could obtain the following inequality for $f \in H^p(\mathbb{R}^n)$:

$$\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\xi_0,\phi,0} \|f * \phi\|_{L^p(\mathbb{R}^n)} \sim_{p,\xi_0,\phi,0} \|f * \phi\|_{L^p(\mathbb{R}^n)} \quad \text{when} \quad 0 < p \leq 1,$$

where $\tilde{\phi}(x) = \phi(r_0x)e^{-2\pi i(\xi_0,x)}$.

**Proof.** Notice that $\mathcal{F} \tilde{\phi}(\xi) = \mathcal{F} \phi \left( \frac{\xi + \xi_0}{r_0} \right)$, thus $|\mathcal{F} \tilde{\phi}(\xi)| \geq C$, when $\xi \in B(0,1)$. Then we can prove this proposition in a way similar to Proposition 2.5. \hfill $\square$

Thus in a way similar to Theorem 2.7, by Proposition 2.8, we could obtain the following corollary:

**Corollary 2.9.** For $\phi(x) \in \Lambda^\gamma$, $\text{supp} \phi(x) \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$, $1 \geq p > \frac{1}{1+\gamma}$, $|\phi(x)| \leq 1$, with $|\mathcal{F} \phi(\xi)| \geq C$, when $\xi \in B(\xi_0, r_0)$, we could obtain the following inequality for any $f \in H^p(\mathbb{R}^n)$:

$$\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\xi_0,\phi,0} \|f * \phi\|_{L^p(\mathbb{R}^n)} \sim_{p,\xi_0,\phi,0} \|f * \phi\|_{L^p(\mathbb{R}^n)},$$

where $\tilde{\phi}(x) = \phi(r_0x)e^{-2\pi i(\xi_0,x)}$.

## 3 Lipschitz function without compact support in $\mathbb{R}^n$

**Proposition 3.1.** $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$) without compact support in $\mathbb{R}^n$ satisfying the following:

$$|\phi(x)| \lesssim \frac{1}{(1+|x|)^{n+\gamma}}, \hfill (42)$$

$$|\phi(x+h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1+|x|)^{n+2\gamma}}, \quad \text{if} \quad |h| \lesssim 1 + |x|. \hfill (43)$$

Then we could deduce the following inequality for $\frac{1}{1+\gamma} < p \leq 1$:

$$\|f * \phi\|_{L^p(\mathbb{R}^n)} \lesssim_{p,n,\gamma} \|f\|_{H^p(\mathbb{R}^n)}.$$

**Proof.** Fix a positive $\varphi(t) \in S(\mathbb{R}^n)$ so that $\text{supp} \varphi(t) \subseteq B(0,1)$, and $\varphi(t) = 1$ for $t \in B(0,1/2)$. Let the functions $\psi_{k,x}(t)$ be defined as follows:

$$\psi_{0,x}(t) = \varphi \left( \frac{x - t}{r} \right),$$

$$\psi_{k,x}(t) = \varphi \left( \frac{x - t}{2^k r} \right) - \varphi \left( \frac{x - t}{2^{k+1} r} \right), \quad \text{for} \quad k \geq 1.$$  

Thus $\psi_{k,x}(t) \in S(\mathbb{R}^n)$ for $k \geq 0$, with $\text{supp} \psi_{0,x}(t) \subseteq B(x,r)$, $\text{supp} \psi_{k,x}(t) \subseteq B(x,2^{k+1}r) \setminus B(x,2^{k-1}r)$ for $k \geq 1$. It is clear that

$$\sum_{k=0}^\infty \psi_{k,x}(t) = 1.$$

Then we could write $(f * \phi)\nabla(x)$ as following:

$$(f * \phi)\nabla(x) = \sup_{|s-x| \leq r} \left| \int_{\mathbb{R}^n} \phi \left( \frac{s - y}{r} \right) \sum_{k=0}^\infty \psi_{k,x}(y)f(y)dy/r^n \right|$$

$$\leq \sum_{k=0}^\infty \sup_{|s-x| \leq r} \left| \int_{\mathbb{R}^n} \phi \left( \frac{s - y}{r} \right) \psi_{k,x}(y)f(y)dy/r^n \right|.$$
It is clear that the function \( y \to (1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) \) with \( |s-x| < r \) satisfies the following:

\[
\begin{align*}
&\| (1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) \| \lesssim 1 \\
&H^\gamma \left( (1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) \right) \lesssim (2^k r)^{-\gamma} \\
&\text{supp}(1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) \subseteq B(x, 2^{k+1} r) \setminus B(x, 2^{k-2} r) \text{ for } k \geq 1.
\end{align*}
\]

Then we could deduce that:

\[
(f \ast \phi)(x) = \sup_{|s-x| \leq r} \left| \int_{\mathbb{R}^n} \phi \left( \frac{s-y}{r} \right) f(y) dy \right| 
\leq \sum_{k=0}^{+\infty} \frac{(2^k)^n}{(1 + 2^k)^{n+\gamma}} \sup_{|s-x| \leq r} \left| \int_{\mathbb{R}^n} (1 + 2^k)^{n+\gamma} \phi \left( \frac{s-y}{r} \right) \psi_{k,x}(y) f(y) dy / (2^k r)^n \right|
\lesssim \sum_{k=0}^{+\infty} \frac{(2^k)^n}{(1 + 2^k)^{n+\gamma}} f^n(x)
\lesssim_{n, \gamma} f^n(x).
\]

Thus the following inequality holds for \( \frac{1}{1+\gamma} < p \leq 1 \) (0 < \( \gamma \leq 1 \)):

\[
\| (f \ast \phi)(x) \|_{L^p(\mathbb{R}^n)} \lesssim_{n, \gamma} \| f \|_{H^p(\mathbb{R}^n)}.
\]

This proves the proposition.

\[\square\]

**Proposition 3.2.** \( \phi(x) \) is a Lipschitz function (\( \phi(x) \in \Lambda^\gamma \)) without compact support in \( \mathbb{R}^n \) satisfying the following:

\[
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{n+\gamma}},
\]

\[
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{n+2\gamma}}, \text{ if } |h| \lesssim 1 + |x|.
\]

Then we could deduce the following inequalities for any fixed \( \alpha \) with \( 0 < \alpha < \gamma \leq 1 \), and \( r > 0 \):

\[
0 \leq \left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\alpha (1 + \frac{|x-y|}{r})^{-(\gamma-\alpha)-n},
\]

and

\[
\left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) - \left( \phi \left( \frac{a-z}{r} \right) - \phi \left( \frac{b-z}{r} \right) \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\alpha \left( \frac{|y-z|}{r} \right)^{\gamma-\alpha} \left( 1 + \frac{|x-y|}{r} \right)^{-(2(\gamma-\alpha)-n)},
\]

for \( |a-b| \lesssim r, \frac{|a-b|}{r} \leq C_3 \min \{1 + \frac{|a-y|}{r}, 1 + \frac{|a-z|}{r}\}, x \in B(a, 2r) \setminus B(b, 2r) \).

**Proof.** From the fact that \( |a-b| \lesssim r, \frac{|a-b|}{r} \leq C_3 \min \{1 + \frac{|a-y|}{r}, 1 + \frac{|a-z|}{r}\} \), the following relations could be obtained:

\[
1 + \frac{|a-y|}{r} \sim 1 + \frac{|b-y|}{r}, 1 + \frac{|a-z|}{r} \sim 1 + \frac{|b-z|}{r}, \text{ and } 1 + \frac{|a-z|}{r} \sim 1 + \frac{|a-b|}{r}.
\]

First, we will consider the case when \( |a-b| \leq |y-z| \). Then from Formula (45), we could get

\[
\left| \phi \left( \frac{a-y}{r} \right) - \phi \left( \frac{b-y}{r} \right) \right| \leq C \left( \frac{|a-b|}{r} \right)^\gamma \left( 1 + \frac{|a-y|}{r} \right)^{-2\gamma-n}
\leq C \left( \frac{|a-b|}{r} \right)^\gamma \left( 1 + \frac{|a-y|}{r} \right)^{-\gamma-\alpha} \left( 1 + \frac{|a-y|}{r} \right)^{-(\gamma-\alpha)-n}
\leq C \left( \frac{|a-b|}{r} \right)^\alpha \left( 1 + \frac{|a-y|}{r} \right)^{-(\gamma-\alpha)-n}.
\]
Also we could obtain
\[
|\phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{b - y}{r} \right)| \leq C \left( \frac{|a - b|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n},
\]
and
\[
|\phi \left( \frac{a - z}{r} \right) - \phi \left( \frac{b - z}{r} \right)| \leq C \left( \frac{|a - b|}{r} \right)^{\gamma} \left( 1 + \frac{|a - z|}{r} \right)^{-2\gamma - n}.
\]
Together with Formula (46), we could conclude
\[
\left| \phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{b - y}{r} \right) \right| - \left( \phi \left( \frac{a - z}{r} \right) - \phi \left( \frac{b - z}{r} \right) \right) \right|
\leq C \left( \frac{|a - b|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n}.
\]
By the fact \(|a - b| \leq |y - z|\) and \(1 \leq 1 + \frac{|a - y|}{r}\), we could obtain:
\[
\left( \frac{|a - b|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n} \leq \left( \frac{|a - b|}{r} \right)^{\alpha} \left( \frac{|y - z|}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{|a - y|}{r} \right)^{-2(\gamma - \alpha) - n}.
\]
Then for \(|a - b| \leq |y - z|\), the Formula
\[
\left| \phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{a - z}{r} \right) \right| \leq C \left( \frac{|y - z|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n}
\]
holds. In a similar way, we will obtain the Formula (48) for the case when \(|a - b| \geq |y - z|\). Notice that by Formula (46),
\[
|\phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{a - z}{r} \right)| \leq C \left( \frac{|y - z|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n},
\]
and
\[
|\phi \left( \frac{b - y}{r} \right) - \phi \left( \frac{b - z}{r} \right)| \leq C \left( \frac{|y - z|}{r} \right)^{\gamma} \left( 1 + \frac{|b - y|}{r} \right)^{-2\gamma - n}
\]
hold. Then we could obtain
\[
\left| \phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{b - y}{r} \right) \right| - \left( \phi \left( \frac{a - z}{r} \right) - \phi \left( \frac{b - z}{r} \right) \right) \right|
\leq C \left( \frac{|y - z|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n}.
\]
By the fact \(|a - b| \geq |y - z|\) and \(1 \leq 1 + \frac{|a - y|}{r}\), the following holds:
\[
\left( \frac{|y - z|}{r} \right)^{\gamma} \left( 1 + \frac{|a - y|}{r} \right)^{-2\gamma - n} \leq \left( \frac{|a - b|}{r} \right)^{\alpha} \left( \frac{|y - z|}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{|a - y|}{r} \right)^{-2(\gamma - \alpha) - n}.
\]
Then for \(|a - b| \geq |y - z|\), we could get
\[
\left| \phi \left( \frac{a - y}{r} \right) - \phi \left( \frac{b - y}{r} \right) \right| - \left( \phi \left( \frac{a - z}{r} \right) - \phi \left( \frac{b - z}{r} \right) \right) \right|
\leq C \left( \frac{|a - b|}{r} \right)^{\alpha} \left( \frac{|y - z|}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{|a - y|}{r} \right)^{-2(\gamma - \alpha) - n}.
\]
By the fact that \(x \in B(a, 2r) \cap B(b, 2r)\), we could deduce that:
\[
1 + \frac{|a - y|}{r} \sim 1 + \frac{|x - y|}{r}.
\]
Formulas (47, 48, 49, 50) yeald the Proposition.
Proposition 3.3. For $1 \geq p > \frac{1}{1 + \gamma}$, $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$) without compact support in $\mathbb{R}^n$ satisfying Formulas (44, 45). For $f \in L^1(\mathbb{R}^n)$, if the following inequality holds
\[\|(f \ast \phi)\varphi\|_{L^p(\mathbb{R}^n)} \sim \|f\gamma\|_{L^p(\mathbb{R}^n)}\]
then we could deduce that:
\[\|(f \ast \phi)\varphi\|_{L^p(\mathbb{R}^n)} \leq C\|(f \ast \phi)\|_{L^p(\mathbb{R}^n)},\]
where $C$ is dependent on $p$ and $\gamma$.

Proof. By Proposition 2.4, we could deduce that the following holds for $f \in L^1(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$:
\[\|f\gamma\ast\alpha\|_{L^p(\mathbb{R}^n)} \sim \gamma, \alpha \|f\gamma\|_{L^p(\mathbb{R}^n)} \sim \gamma, \alpha \|f\gamma\ast\alpha\|_{L^p(\mathbb{R}^n)} \sim \gamma, \alpha \|f\gamma\ast\alpha\|_{L^p(\mathbb{R}^n)}\]
For any fixed $\alpha$ satisfying $0 < \alpha < \gamma$ and $1 \geq p > \frac{1}{1 + \gamma - \alpha}$, we use $F$ to denote as:
\[F = \left\{ x \in \mathbb{R}^n : f\gamma\ast\alpha(x) \leq \sigma(f \ast \phi)\varphi(x) \right\}\]
Then it is clear that
\[\int_{F^c} |(f \ast \phi)\varphi(x)|^p \, dx \leq C \int_{F^c} |f\gamma\ast\alpha(x)|^p \, dx \leq C \int_{\mathbb{R}^n} |f\gamma(x)|^p \, dx \leq C \int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx.\] (51)
Choosing $\sigma^p \geq 2C \gamma$, then the following holds:
\[\int_{\mathbb{R}^n} |(f \ast \phi)\varphi(x)|^p \, dx \leq \int_{F} |(f \ast \phi)\varphi(x)|^p \, dx.\] (52)
We use $Df(x)$ and $F(x,r)$ to denote as:
\[Df(x) = \sup_{r>0} |f \ast \phi(x)|, \quad F(x,t) = f \ast \phi(x).\]
Next, we will show that for any $q > 0$,
\[(f \ast \phi)\varphi(x) \leq C [M (Df)^q(x)]^{1/q} \quad \text{for } x \in F,\] (53)
where $M$ is the Hardy-Littlewood maximal operator. For any fixed $x_0 \in F$, there exists $(u_0, r_0)$ satisfying $|u_0 - x_0| < r_0$ such that the following inequality holds:
\[|F(u_0, r_0)| > \frac{1}{2} (f \ast \phi)\varphi(x_0).\] (54)
Choosing $\delta < 1$ small enough and $u$ satisfying $|u - u_0| < \delta r_0$, we could deduce that
\[|F(u, r_0) - F(u_0, r_0)| = \left| \int_{\mathbb{R}^n} f(y) \phi \left( \frac{u - u_0}{r_0} \right) \frac{dy}{r_0^n} - \int_{\mathbb{R}^n} f(y) \phi \left( \frac{u_0 - y}{r_0} \right) \frac{dy}{r_0^n} \right| \leq \left| \int_{\mathbb{R}^n} f(y) \left[ \phi \left( \frac{u - u_0}{r_0} \right) - \phi \left( \frac{u_0 - y}{r_0} \right) \right] \frac{dy}{r_0^n} \right|.
\]
Notice that $\left( \frac{u - u_0}{r_0} - \phi \left( \frac{u_0 - y}{r_0} \right) \right)$ is a new kernel, thus by Proposition 3.2 and Proposition 3.1, we could obtain:
\[|F(u, r_0) - F(u_0, r_0)| \leq C \delta^\alpha f\gamma\ast\alpha(x_0) \leq C \delta^\alpha \sigma(f \ast \phi)\varphi(x_0) \quad \text{for } x_0 \in F.\]
Taking $\delta$ small enough such that $C \delta^\alpha \sigma \leq 1/4$, then
\[|F(u, r_0)| \geq \frac{1}{4} (f \ast \phi)\varphi(x_0) \quad \text{for } u \in B(u_0, \delta r_0).\]
Thus the following inequality holds for any \( x_0 \in F \),

\[
| (f * \phi) \varphi(x_0) |^q \leq \frac{1}{B(u_0, \delta r_0)} \int_{B(u_0, \delta r_0)} 4^q |F(u, r_0)|^q du \\
\leq \frac{B(x_0, (1 + \delta) r_0)}{B(u_0, \delta r)} \left| \frac{1}{B(x_0, (1 + \delta) r_0)} \right| \int_{B(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^q du \\
\leq \left( \frac{1 + \delta}{\delta} \right)^n \frac{1}{B(x_0, (1 + \delta) r_0)} \int_{B(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^q du \\
\leq CM[(Df)^q](x_0),
\]

where \( C \) is a constant independent on \( x_0 \). Finally, using the maximal theorem for \( M \) when \( q < p \) leads to

\[
\int_F |(f * \phi) \varphi(x)|^p dx \leq C \int_{\mathbb{R}^n} |M[(Df)^q](x)|^{p/q} dx \leq C \int_{\mathbb{R}^n} |(f * \phi) \varphi(x)|^p dx.
\] (55)

Thus for any fixed \( \alpha \) satisfying \( 0 < \alpha < \gamma \) and \( p > \frac{1}{1 + \gamma - \alpha} \), the above Formula (55) combined with Formula (52) leads to

\[
\| (f * \phi) \varphi \|_{L^p(\mathbb{R}^n)} \leq C \| (f * \phi) \|_{L^p(\mathbb{R}^n)},
\] (56)

where \( C \) is dependent on \( p \) and \( \alpha \). Next we will remove the number \( \alpha \). For any \( 1 \geq p > \frac{1}{1 + \gamma} \), let \( p_0 = \frac{1}{2} \left( p + \frac{1}{1 + \gamma} \right) \) with \( p > p_0 > \frac{1}{1 + \gamma} \) and let \( \alpha = 1 + \gamma - \frac{1}{p_0} \). Thus it is clear that

\[
p_0 = \frac{1}{1 + \gamma - \alpha}, \quad p > p_0.
\]

Thus by Formula (56), we could obtain the following inequality holds for \( 1 \geq p > \frac{1}{1 + \gamma} \)

\[
\| (f * \phi) \varphi \|_{L^p(\mathbb{R}^n)} \leq C \| (f * \phi) \|_{L^p(\mathbb{R}^n)},
\]

where \( C \) is dependent on \( p \) and \( \gamma \). This proves the Proposition. \( \square \)

**Proposition 3.4.** For \( N > \left[ \frac{5}{2} \right] + 1 \) (0 < \( p \leq 1 \), \( \phi(x) \) is a Lipschitz function \( \phi(x) \in \Lambda^r \) without compact support in \( \mathbb{R}^n \) satisfying:

\[
|\phi(x)| \lesssim \frac{1}{(1 + |x|)^{n + N + 1}},
\] (57)

\[
|\phi(x + h) - \phi(x)| \lesssim \frac{|h|^\gamma}{(1 + |x|)^{n + 2\gamma}}, \quad \text{if } |h| \lesssim 1 + |x|.
\] (58)

Then there exists sequence \( \{\phi^k(x) : \phi^k(x) \in C_{c}(\mathbb{R}^n)\}_{k=1}^{+\infty} \) such that:

(i) \( \text{supp} \phi^k(x) \subseteq B(0, k) \);

(ii) \( \lim_{k \to +\infty} \|\phi^k(x) - \phi(x)\| = 0 \);

(iii) \( |\phi^k(x)| \leq C_1 \frac{1}{(1 + |x|)^{n + N + 1}} \);

(iv) If \( |h| \lesssim 1 + |x| \), then

\[
|\phi^k(x + h) - \phi^k(x)| \leq C_2 \frac{|h|^\gamma}{(1 + |x|)^{n + 2\gamma}};
\]

(v) The following two inequalities hold:

\[
|\phi^k(x) - \phi(x)| \leq C_3 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{n + N + 1 - \frac{2}{r}}},
\]
\[
\int_{\mathbb{R}^n} |\phi^k(x) - \phi(x)| dx \lesssim \left( \frac{1}{k} \right)^{\gamma/2};
\]

where \( C_1, C_2, \) and \( C_3 \) are constants independent on \( k \).

**Proof.** Let \( \psi(t) \in S(\mathbb{R}^n) \) to be fixed satisfying \( 0 < \psi(t) \leq 1, \|H^\gamma \psi\|_{L^\infty} \leq C, \text{ supp } \psi(t) \subseteq B(0, 1) \), \( \psi(t) = 1 \) when \( t \in B(0, 1/2) \). The function \( \phi^k(x) \) is defined as:

\[
\phi^k(x) = \phi(x) \psi \left( \frac{x}{k} \right), \quad k = 1, 2, 3, \ldots .
\]

Then it is clear that the sequence \( \{\phi^k(x)\}_{k=1}^\infty \) satisfies (i), (ii), (iii), (iv). When \( |x| \leq \frac{2}{k} \), \( |\phi^k(x) - \phi(x)| = 0 \). When \( |x| \geq \frac{2}{k} \),

\[
|\phi^k(x) - \phi(x)| \leq C \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{\gamma+1}} \leq C \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{\gamma+1}}.
\]

Then

\[
|\phi^k(x) - \phi(x)| \leq C_3 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{\gamma+1}}.
\]

Thus

\[
\int_{\mathbb{R}^n} |\phi^k(x) - \phi(x)| dx \lesssim \left( \frac{1}{k} \right)^{\gamma/2}.
\]

By Proposition 3.4, we could obtain the following Proposition:

**Proposition 3.5.** \( \phi(x) \) is a Lipschitz function \( (\phi(x) \in \Lambda^\gamma)(0 < \gamma \leq 1) \) without compact support in \( \mathbb{R}^n \) satisfying Formulas (57, 58). Then there exists sequence \( \{\psi^k(x) : \psi^k(x) \in S(\mathbb{R}^n)\}_{k=1}^\infty \) satisfying the following:

(i) \( \text{ supp } \psi^k(x) \subseteq B(0, 2k) \);

(ii) \( \lim_{k \to \infty} \|\psi^k(x) - \psi(x)\|_\infty = 0 \);

(iii) For \( N \geq \left[ \frac{1}{\gamma} \right] + 1 (0 < p \leq 1), \|\psi^k(x)\|_p \leq C_1 \frac{1}{(1 + |x|)^{\gamma+1}}; \)

(iv) If \( |h| \lesssim 1 + |x| \), then \( |\psi^k(x+h) - \psi^k(x)| \leq C_2 \frac{|h|^\gamma}{(1 + |x|)^{\gamma+2}}; \)

(v) The following two inequalities hold:

\[
|\psi^k(x) - \phi(x)| \leq C_4 \left( \frac{1}{k} \right)^{\gamma/2} \frac{1}{(1 + |x|)^{\gamma+1}},
\]

\[
\int_{\mathbb{R}^n} |\psi^k(x) - \phi(x)| dx \lesssim \left( \frac{1}{k} \right)^{\gamma/2};
\]

where \( C_1, C_2, \) and \( C_4 \) are constants independent on \( k \).

**Proof.** Let \( \{\phi^k(x) : \phi^k(x) \in C_c(\mathbb{R}^n)\}_{k=1}^\infty \) to be the sequence in Proposition 3.4. Let

\[
\rho(x) = \begin{cases} 
\vartheta \exp \left( \frac{1}{|x|^2} \right), & \text{for } |x| < 1 \\
0, & \text{for } |x| \geq 1,
\end{cases}
\]

where \( \vartheta \) is a constant such that \( \int \rho(x) dx = 1 \). Let

\[
\phi^{k, \tau}(x) = \int_{\mathbb{R}^n} \phi^k(x-t) \rho \left( \frac{t}{\tau^\gamma} \right) dt.
\]
Thus (i) (ii) and (iii) hold. Next we prove (iv). Notice that $\text{supp}\rho(x) \subseteq \{x : |x| < 1\}$. Let us set $\tau = \frac{1}{k}$ for $k \in \mathbb{Z}, k \geq 1$, thus it is clear that $|h| \leq 1 + |x - t|$ holds if $|h| \leq 1 + |x|$. Thus the following holds for $|h| \leq 1 + |x|$: 

$$
\left| \phi^{k, \frac{1}{k}}(x + h) - \phi^{k, \frac{1}{k}}(x) \right| = \left| \int_{\mathbb{R}^n} \phi^k(x + h - t)\rho(kt) k^n dt - \int_{\mathbb{R}^n} \phi^k(x - t)\rho(kt) k^n dt \right| \quad (61)
$$

$$
\leq |t| \int_{\mathbb{R}^n} |\phi^k(x + h - t) - \phi^k(x - t)| \rho(kt) k^n dt \leq C_4 \frac{|h|}{(1 + |x|)^{n+2\tau}}.
$$

We could also deduce the following inequality:

$$
|\phi^{k, \frac{1}{k}}(x) - \phi(x)| \leq |\phi^{k, \frac{1}{k}}(x) - \phi^k(x)| + |\phi^k(x) - \phi(x)| \leq C_2 \frac{1}{(1 + \tau)^{n+2\gamma}} + C_3 \frac{1}{(1 + \tau)^{n+N+1-\frac{\gamma}{2}}} \leq C_4 \frac{1}{(1 + \tau)^{n+\frac{\gamma}{2}}}.
$$

Thus it is clear that

$$
\int_{\mathbb{R}^n} |\phi^{k, \frac{1}{k}}(x) - \phi(x)| dx \leq C \left( \frac{1}{k} \right)^{\gamma/2}.
$$

At last, we could set $\psi^k(x)$ as 

$$
\psi^k(x) = \phi^{k, \frac{1}{k}}(x).
$$

This proves our proposition. \hfill \Box

**Proposition 3.6.** $\phi(x)$ is a Lipschitz function ($\phi(x) \in \Lambda^\gamma$, $\int \phi(x) dx \sim 1$), $(0 < \gamma \leq 1), \phi(x) > 0$ without compact support in $\mathbb{R}^n$ satisfying Formulas (57, 58), then we could deduce the following inequality for $\frac{1}{1+\gamma} < p \leq 1$:

$$
\|f\|_{L^p(\mathbb{R}^n)} \leq \|f \ast \phi\|_{L^p(\mathbb{R}^n)}.
$$

**Proof.** Let $f \in L^1(\mathbb{R}^n)$ first. For $\phi(x) \in \Lambda^\gamma$, there exists sequence $\{\phi^m(x) : \phi^m(x) \in S(\mathbb{R}^n)\} m \in \mathbb{N}$ defined as Proposition 3.5. It is clear that $\int \phi^m(x) dx \sim C$, where $C$ is a constant independent on $k$, thus $(\mathcal{F}\phi^m)(2k_0 \xi) \geq C_0$, for $|\xi| \leq 1$ where $C_0$ is a constant independent on $m$, and $k_0$ is independent on $m$. Also by Proposition 3.5, we could deduce that the following inequality holds:

$$
|\partial_\xi^\alpha (\mathcal{F}\phi^m)(\xi)| \leq C_0
$$

for $0 \leq |\alpha| \leq N (N \geq \lceil \frac{\gamma}{2} \rceil + 1)$, where $C_0$ is a constant independent on $m$.

Let $\psi(x) \in S(\mathbb{R}^n)$ to be fixed satisfying $\int \psi(x) dx \sim 1$, $\varphi \in S(\mathbb{R}^n)$ and $\varphi^k \in S(\mathbb{R}^n)$ are defined as:

$$
\begin{cases} 
\varphi(\xi) = 0 \text{ for } |\xi| \geq 1, \\
\varphi(\xi) = 1 \text{ for } |\xi| \leq 1/2,
\end{cases}
$$

$$
\begin{cases} 
\varphi^k(\xi) = \varphi(\xi) \text{ for } k = 0, \\
\varphi^k(\xi) = \varphi(2^{-k} \xi) - \varphi(2^{1-k} \xi) \text{ for } k \geq 1.
\end{cases}
$$

Thus

$$
1 = \sum_{k=0}^{\infty} \varphi^k(\xi).
$$

Let $\eta^k_m$ to be defined as

$$
(\mathcal{F}\eta^k_m)(\xi) = \frac{\eta^k_m(\xi)}{(\mathcal{F}\phi^m)(2^{-k} \xi)}.
$$
where \( \mathcal{F} \) denotes the Fourier transform. Then we could obtain that:

\[
(\mathcal{F}\psi)(\xi) = \sum_{k=0}^{\infty} \frac{\psi^k(\xi)}{(\mathcal{F}\phi^m)(2^{-k-k_0}\xi)} (\mathcal{F}\phi^m)(2^{-k-k_0}\xi)
\]

\[
= \sum_{k=0}^{\infty} (\mathcal{F}\eta^k_m)(\xi) (\mathcal{F}\phi^m)(2^{-k-k_0}\xi).
\]

Thus

\[
\psi(x) = \sum_{k=0}^{\infty} \eta^k_m \ast \phi^m_{2^{-k-k_0}}(x).
\]

Notice that the following hold

\[
\sup_{\xi \in \mathbb{R}^n} |\xi|^{\alpha} \partial_\xi^{\alpha'} (\phi^k(\mathcal{F}\psi)(\xi)) \leq C_{\alpha,\alpha'} M 2^{-kM} \text{ for any } M > 0, 0 \leq |\alpha|,
\]

where \( C_{\alpha} \) is a constant independent on \( m \), together with Formula (62), we could deduce that

\[
\sup_{\xi \in \mathbb{R}^n} |\xi|^{\alpha} \partial_\xi^{\alpha'} ((\mathcal{F}\eta^k_m)(\xi)) \leq C_{\alpha,\alpha',k_0} N 2^{-kN} \text{ for } N \geq \left\lfloor \frac{m}{p} \right\rfloor + 1, 0 \leq |\alpha| \leq N
\]

where \( C_{\alpha,\alpha',k_0,N} \) is a constant independent on \( m \) and \( k \). Then

\[
\int_{\mathbb{R}^n} \eta^k_m(u) (1 + 2^{k+k_0}|u|)^N du \leq C_{k_0,N} 2^{-k},
\]

where \( C_{k_0,N} \) is a constant independent on \( m \). For \( f \in L^1(\mathbb{R}^n) \), from Formulas (63), we could have

\[
M_{\psi}f(x) = \sup_{r>0} \left| \int_{\mathbb{R}^n} f(y) \frac{1}{r^n} \psi \left( \frac{x-y}{r} \right) dy \right|
\]

\[
= C \sup_{r>0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) \eta^k_m \left( \frac{s}{r} \right) \frac{1}{r^n} \phi^m \left( \frac{x-y-s}{2^{-k-k_0}r} \right) \frac{ds}{(2^{-k-k_0}r)^n} dy
\]

\[
\leq C \sup_{r>0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \eta^k_m \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{k-k_0}r} \right)^N \frac{ds}{r^n} \sup_{\alpha,\alpha',k_0} \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} \frac{dy}{r^n},
\]

where \( C \) is a constant independent on \( m \). From Formulas (66, 67) we could obtain:

\[
M_{\psi}f(x) \lesssim \left( \sup_{0 \leq |s| < r} + \sum_{k=0}^{\infty} \sup_{0 \leq |s| < 2^{k-1}r \leq |s| < 2^{k}r} \right) \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} \frac{dy}{r^n} \right|
\]

\[
\lesssim \sum_{k=0}^{\infty} \sup_{0 \leq |s| < 2^{k}r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \frac{dy}{r^n} \right|.
\]

Formula (9) leads to

\[
\int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \frac{dy}{r^n} \right|^p dx \leq C (1 + 2^k)^n \int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \frac{dy}{r^n} \right|^p dx.
\]

Notice that \( N \geq \left\lfloor \frac{m}{p} \right\rfloor + 1 \), thus Formulas (68, 69) lead to

\[
\int_{\mathbb{R}^n} |M_{\psi}f(x)|^p dx \leq C \int_{\mathbb{R}^n} \sup_{0 \leq |s| < r} \left| \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x-y-s}{r} \right) \frac{dy}{r^n} \right|^p dx,
\]
where $C$ is a constant independent on $m$. We use $(F^m f) (x, r)$ and $(F f) (x, r)$ to denote as:

\[
(F^m f) (x, r) = \int_{\mathbb{R}^n} f(y) \phi^m \left( \frac{x - y}{r} \right) \frac{dy}{r^n}, \quad (F f) (x, r) = \int_{\mathbb{R}^n} f(y) \phi \left( \frac{x - y}{r} \right) \frac{dy}{r^n}.
\]

Thus by Proposition 3.5(v), we could deduce the following inequality:

\[
|(F^m f) (u, r) - (F f) (u, r)| \leq \int_{\mathbb{R}^n} |f(y)| \left| \phi^m \left( \frac{u - y}{r} \right) - \phi \left( \frac{u - y}{r} \right) \right| \frac{dy}{r^n}
\]

\[
\leq C \int_{\mathbb{R}^n} |f(y)| \left( \frac{1}{m} \right)^{\gamma/2} \left( 1 + \left| \frac{u - y}{r} \right| ^{-n-\frac{\gamma}{2}} \right) \frac{dy}{r^n}
\]

\[
\leq C \sum_{k=0}^{+\infty} \left( \frac{2^k}{m} \right)^{\gamma/2} \left( \sum_{j} |M f(u)| \left( \frac{1}{m} \right)^{\gamma/2}
\right.
\]

\[
\leq C |M f(u)| \left( \frac{1}{m} \right)^{\gamma/2}
\]

where $C$ is dependent on $\gamma$ and $M$ is the Hardy-Littlewood Maximal Operator. Let us set $\delta_m(u)$ as:

\[
\delta_m(u) = |(F^m f) (u, r) - (F f) (u, r)|.
\]

Thus we could deduce the following:

\[
\{ x : \delta_m(x) > \alpha \} \subseteq \left\{ x : M f(x) > \frac{1}{c} m^{\gamma/2} \alpha \right\}
\]

for some constant $c$. Notice that $M$ is weak-$\langle 1, 1 \rangle$ bounded, thus the following holds for any $\alpha > 0$:

\[
|\{ x : \delta_m(x) > \alpha \}| \leq \frac{C \| f \|_{L^1(\mathbb{R}^n)}}{\alpha} \left( \frac{1}{m} \right)^{\gamma/2}.
\]

Thus we could obtain:

\[
\lim_{m \to +\infty} |\{ x : \delta_m(x) > \alpha \}| = 0.
\]

Thus there exists a sequence $\{m_j\} \subseteq \{m\}$ such that the following holds:

\[
\lim_{m_j \to +\infty} (F^{m_j} f) (u, r) = (F f) (u, r), \quad a.e. u \in \mathbb{R}^n
\]

for $f \in L^1(\mathbb{R}^n)$. Let us set $E$ as:

\[
E = \{ u \in \mathbb{R}^n : \lim_{m_j \to +\infty} (F^{m_j} f) (u, r) = (F f) (u, r) \}.
\]

Thus it is clear that $E$ is dense in $\mathbb{R}^n$. For any $x_0 \in \mathbb{R}^n$, there exists a $(u_0, r_0)$ with $r_0 > 0$, $u_0 \in \mathbb{R}^n$, $|u_0 - x_0| < r_0$ such that the following holds:

\[
|(F^{m_j} f) (u_0, r_0)| \geq \frac{1}{2} \sup_{|u - u_0| \leq r} |(F^{m_j} f) (u, r)|.
\]

Notice that $(F^{m_j} f) (u_0, r_0)$ is a continuous function in $u$ variable and $E$ is dense in $\mathbb{R}^n$, thus there exists a $\tilde{u}_0 \in E$ with $|\tilde{u}_0 - x_0| < r_0$ such that

\[
|(F^{m_j} f) (\tilde{u}_0, r_0)| \geq \frac{1}{2} |(F^{m_j} f) (u_0, r_0)| \geq \frac{1}{4} \sup_{|u - u_0| < r} |(F^{m_j} f) (u, r)|.
\]

Thus we could deduce that for any $x_0 \in \mathbb{R}^n$

\[
\sup_{u \in E: |x_0 - u| < r} |(F^{m_j} f) (u, r)| \sim \sup_{u \in \mathbb{R}^n: |x_0 - u| < r} |(F^{m_j} f) (u, r)|.
\]
Let the \( \{Q_k\}_k \) to be the cubes with unit length satisfying \( Q_i \cap Q_j = \emptyset \) when \( i \neq j \), and \( \bigcup \{Q_k\} = \mathbb{R}^n \). From Formula (72) together with the dominated convergence theorem, we could conclude:

\[
\lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|x-u| < r} |(F^{m_j} f)(u, r)|^p \, dx \sim \lim_{m_j \to +\infty} \int_{\mathbb{R}^n} \sup_{|u-x| < r} |(F^{m_j} f)(u, r)|^p \, dx
\]

\[
\leq C \int_{\mathbb{R}^n} \sup_{|u-x| < r} |(F f)(u, r)|^p \, dx
\]

\[
\leq C \int_{\mathbb{R}^n} \sup_{|u-x| < r} |(F f)(u, r)|^p \, dx
\]

By Formulas (70, 73), we could deduce that the following inequality holds for \( f \in L^1(\mathbb{R}^n) \)

\[
\|f\|_{H^p(\mathbb{R}^n)} \lesssim_{p,n,\gamma,\phi} \|f * \phi\|_{L^p(\mathbb{R}^n)}. \tag{74}
\]

Notice that \( H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) is dense in \( H^p(\mathbb{R}^n) \). Then we could deduce that the Formula (74) holds for \( f \in H^p(\mathbb{R}^n) \). This proves our proposition. \( \square \)

Thus from Proposition 3.3, Proposition 3.6 and Proposition 3.1, we could obtain the following theorem:

**Theorem 3.7.** \( \phi(x) \) is a Lipschitz function \( \phi(x) \in \mathcal{A}^\gamma, \int \phi(x)dx \sim 1, (0 < \gamma \leq 1), \phi(x) > 0 \) without compact support in \( \mathbb{R}^n \) satisfying Formulas (57, 58), then we could deduce the following inequality for \( \frac{1}{1+\gamma} < p \leq 1, f \in H^p(\mathbb{R}^n) \):

\[
\|f\|_{H^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|(f * \phi)\|_{L^p(\mathbb{R}^n)} \sim_{p,\gamma,\phi} \|\|f * \phi\|\|_{L^p(\mathbb{R}^n)}. \tag{75}
\]

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