Boundary correlation functions of integrable vertex models

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Abstract

We review our recent work on the boundary correlation functions of integrable vertex models on an $N \times N$ lattice with domain wall boundary conditions. Particularly considered is the six vertex model. The general expression of the boundary correlation functions is obtained for the six vertex model by use of the quantum inverse scattering method. We also comment on the potential application of the boundary correlation functions, and the relation between the boundary correlation functions for the nineteen vertex model.

1 Introduction

In this note, we review our recent work [1] on the boundary correlation functions for the integrable vertex models with domain wall boundary conditions. We particularly focus on the six vertex model, which is one of the most fundamental exactly solved models in statistical physics [2, 3, 4, 5]. Not only the periodic boundary condition but also the domain wall boundary condition is an interesting boundary condition. For example, the partition function is deeply related to the norm [6] and the scalar product [7] of the XXZ chain, which plays a fundamental role in calculating correlation functions of the XXZ chain [10, 11, 12, 13]. The determinant formula also led to a deep advance in enumerative combinatorics [14, 15], and connections to Schur polynomial [16] and KP $\tau$ function [17] were also found. Domain wall boundary conditions are also interesting from the physical point of view since it exhibits phase separation phenomena [18, 19, 20, 21].

The calculation of correlation functions are also interesting in the domain wall boundary condition itself. By use of the quantum inverse scattering method, we obtained the general expression for the boundary correlation functions which includes some of the previously studied boundary one point functions, two point functions, boundary polarization, [22, 23, 24, 25] and the emptiness formation probability [26] as special cases. Moreover, our result has potential application to bulk magnetization and spin-spin correlation functions.

The obtained result has another application to higher spin vertex models such as the nineteen vertex model (Fateev-Zamolodchikov [27]), since by use of fusion, one can show that the boundary correlation functions for the nineteen vertex model can be reduced to those for the six vertex model.

In the next section, we review the boundary correlation for the six vertex model with domain wall boundary condition, and discuss (potential) applications of the result in section 3.

2 Six vertex model

The six vertex model is a model in statistical mechanics, whose local states are associated with edges of a square lattice, which can take two values. The Boltzmann weights are assigned to its vertices, and each weight is determined by the configuration around a vertex. The integrability is captured in

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weights over all possible configurations can be formally represented as
\[ R(\lambda, \nu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \text{sh}(\lambda - \nu) & \text{sh}(\lambda + \eta) & \text{sh}(\lambda - \nu + \eta) \\
0 & \frac{1}{\text{sh}(\lambda - \nu + \eta)} & \frac{1}{\text{sh}(\lambda + \eta)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \] (1)

which satisfies the Yang-Baxter equation
\[ R_{12}(\lambda, \nu)R_{13}(\lambda, \mu)R_{23}(\nu, \mu) = R_{23}(\nu, \mu)R_{13}(\lambda, \mu)R_{12}(\lambda, \nu). \] (2)

We consider the six vertex model on a \( N \times N \) lattice depicted in Figure 1. Here, the domain wall boundary condition is imposed, i.e., the spins are aligned all up at the bottom and right boundaries, and all down at the top and left boundaries. At the intersection of the \( \alpha \)-th row (from the bottom) and the \( k \)-th column (from the left), we associate the statistical weight
\[ \mathcal{L}_{\alpha k}(\lambda_\alpha, \nu_k) = \text{sh}(\lambda_\alpha - \nu_k + \eta/2) R_{\alpha k}(\lambda_\alpha - \eta/2, \nu_k) \]
where
\[ a(\lambda, \nu) = \text{sh}(\lambda - \nu + \eta/2), \quad b(\lambda, \nu) = \text{sh}(\lambda - \nu - \eta/2), \quad c = \text{sh} \eta. \] (4)

We refer to the \( \alpha \)-th row as the auxiliary space \( \mathcal{V}_\alpha \) and the \( k \)-th column as the quantum space \( \mathcal{H}_k \). Let us denote \( \{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \{\nu\} = \{\nu_1, \nu_2, \ldots, \nu_N\} \), and the basis (dual basis) of the spin-1/2 representation as \(|+\rangle, |−\rangle \) ((\(+\rangle, \langle−\)|).

The partition function of the six vertex model, which is the summation of products of statistical weights over all possible configurations can be formally represented as
\[ Z_N(\{\lambda\}, \{\nu\}) = \langle + || q^{-\eta} \prod_{\alpha, k=1}^N \mathcal{L}_{\alpha k}(\lambda_\alpha, \nu_k) || − \rangle_a < + \rangle_q. \] (5)

where \(|+\rangle = \otimes_{k=1}^N | + \rangle_k, \langle−\rangle = \otimes_{k=1}^N | − \rangle_k, \langle + || = \otimes_{k=1}^N \langle + |, \langle−|| = \otimes_{k=1}^N \langle − |\rangle_k\langle−\rangle_k\), and we distinguish the spins on the quantum and auxiliary spaces by the subscripts "\( q \)" and "\( a \)". The partition function has the following determinant form \([8, 9]\)
\[ Z_N(\{\lambda\}, \{\nu\}) = \prod_{\alpha=1}^N \prod_{k=1}^N a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k) \det M(\{\lambda\}, \{\nu\}), \] (6)
where
\[ d(\lambda, \nu) = \text{sh}(\lambda - \nu), \quad M_{\alpha k} = \varphi(\lambda_\alpha, \nu_k), \quad \varphi(\lambda, \nu) = \frac{c}{a(\lambda, \nu)b(\lambda, \nu)}. \] (7)

Introducing the monodromy matrix
\[ T_{\alpha}(\lambda_\alpha, \{\nu\}) = \mathcal{L}_{\alpha N}(\lambda_\alpha, \nu_N) \cdots \mathcal{L}_{\alpha 1}(\lambda_\alpha, \nu_1) \]
\[ = \begin{pmatrix}
A(\lambda_\alpha, \{\nu\}) & B(\lambda_\alpha, \{\nu\}) \\
C(\lambda_\alpha, \{\nu\}) & D(\lambda_\alpha, \{\nu\})
\end{pmatrix}, \] (8)

the partition function can be represented as
\[ Z_N(\{\lambda\}, \{\nu\}) = q^{-\eta} B(\lambda_N, \{\nu\}) \cdots B(\lambda_1, \{\nu\}) || + \rangle_q. \] (9)
From (10), one has

\[ R_{\alpha\beta}(\mu - \lambda)T_{\alpha}(\mu, \{\nu\})T_{\beta}(\lambda, \{\nu\}) = T_{\beta}(\lambda, \{\nu\})T_{\alpha}(\mu, \{\nu\})R_{\alpha\beta}(\mu - \lambda). \]  

(10)

From (10), one has

\[ A(\lambda, \{\nu\})B(\mu, \{\nu\}) = f(\lambda, \mu)B(\mu, \{\nu\})A(\lambda, \{\nu\}) + g(\mu, \lambda)B(\lambda, \{\nu\})A(\mu, \{\nu\}), \]  

(11)

\[ B(\lambda, \{\nu\})A(\mu, \{\nu\}) = f(\lambda, \mu)A(\mu, \{\nu\})B(\lambda, \{\nu\}) + g(\mu, \lambda)A(\lambda, \{\nu\})B(\mu, \{\nu\}), \]  

(12)

\[ B(\lambda, \{\nu\})B(\mu, \{\nu\}) = B(\mu, \{\nu\})B(\lambda, \{\nu\}), \]  

(13)

where

\[ f(\mu, \lambda) = \frac{\text{sh}(\lambda - \mu + \eta)}{\text{sh}(\lambda - \mu)}, \quad g(\mu, \lambda) = \frac{\text{sh} \eta}{\text{sh}(\lambda - \mu)}, \]  

(14)

for example.

We consider the following boundary correlation functions

\[ \mathcal{F}^{(r, \epsilon_1, \ldots, \epsilon_s)}(\lambda, \{\nu\}) = \frac{\mathcal{F}^{(r, \epsilon_1, \ldots, \epsilon_s)}_{N}(\lambda, \{\nu\})}{\mathcal{Z}_{N}(\{\lambda\}, \{\nu\})}, \]  

(15)

\[ \tilde{\mathcal{F}}^{(r, \epsilon_1, \ldots, \epsilon_s)}(\lambda, \{\nu\}) = a(+)q(-|| \prod_{a=r+1}^{N} \prod_{k=1}^{N} \mathcal{L}_{\alpha k}(\alpha, \nu_k) \prod_{k=1}^{s} \tau_{k}^{a} \prod_{a=1}^{r} \prod_{k=1}^{N} \mathcal{L}_{\alpha k}(\alpha, \nu_k)||-\rangle_{\alpha}||+\rangle_{q}, \]  

(16)

where \( \pi_{k}^{+} = |+\rangle_{kk}\langle+| \) and \( \pi_{k}^{-} = |-\rangle_{kk}\langle-| \) is a projection onto the up and down spin respectively. Some special cases of this general boundary correlation function reduces to the ones previously considered [22, 23, 24, 25, 26]. We calculated the boundary correlation functions [1] by solving recursive relations between different lattice sizes which can be derived from the quantum inverse scattering method [26].

To use the quantum inverse scattering method, we first translate the numerator (16) of the boundary correlation function (15) into the language of the quantum inverse scattering method as

\[ \tilde{\mathcal{F}}^{(r, \epsilon_1, \ldots, \epsilon_s)}(\lambda, \{\nu\}) = g(-||B(\lambda_N, \{\nu\}) \cdots B(\lambda_{r+1}, \{\nu\}) \prod_{k=1}^{s} \tau_{k}^{+} B(\lambda_{r}, \{\nu\}) \cdots B(\lambda_{1}, \{\nu\})||+\rangle_{q}. \]  

(17)

Figure 1: The six vertex model with domain wall boundary condition.
We also divide a monodromy matrix as

\[ T(\lambda, \{\nu\}) = T_2(\lambda, \{\nu\}) T_1(\lambda, \nu_1), \]

\[ T_2(\lambda, \{\nu\}) = L_{\alpha N}(\lambda, \nu_N) \cdots L_{\alpha 2}(\lambda, \nu_2) \]

\[ = \begin{pmatrix} A_2(\lambda, \{\nu\}) & B_2(\lambda, \{\nu\}) \\ C_2(\lambda, \{\nu\}) & D_2(\lambda, \{\nu\}) \end{pmatrix}, \]

\[ T_1(\lambda, \nu_1) = L_{\alpha 1}(\lambda, \nu_1). \]

One can show that the product of the \( B \) operators can be expressed in terms of the products of the \( B_2 \) operators as follows

\[ q\langle - | 1^r \cdots B(\lambda_r, \{\nu\}) \cdots B(\lambda_1, \{\nu\}) | + \rangle q \]

\[ = \sum_{\alpha=1}^r c \prod_{\beta=1, \beta \neq \alpha}^r b(\lambda_\beta, \nu_1) \prod_{\beta=1}^r f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^N a(\lambda_\alpha, \nu_k) \prod_{k=1}^r B_2(\lambda_k, \{\nu\}) \otimes \bigotimes_{k=2}^N | + \rangle, \]

\[ q\langle - | B(\lambda_N, \{\nu\}) \cdots B(\lambda_{r+1}, \{\nu\}) | + \rangle \]

\[ = \sum_{\alpha=r+1}^N c \prod_{\beta=r+1, \beta \neq \alpha}^N a(\lambda_\beta, \nu_1) \prod_{\beta=r+1}^N f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^N b(\lambda_\alpha, \nu_k) \otimes \bigotimes_{k=2}^N | - \rangle \prod_{k=r+1}^N B_2(\lambda_k, \{\nu\}) | \nu_1 \rangle. \]

From these relations, we can derive recursive relations for the numerators of the boundary correlation function.
functions between different lattice sizes \([1, 26]\]

\[
\tilde{F}_N^{(r, -, \epsilon_2, \ldots, \epsilon_s)}(\{\lambda\}, \{\nu\}) = \prod_{\beta = r+1}^{N} a(\lambda_{\beta}, \nu_1) \sum_{\alpha = 1}^{r} c \prod_{\beta = 1}^{\alpha} b(\lambda_{\beta}, \nu_1) \prod_{\beta = 1}^{r} f(\lambda_{\alpha}, \lambda_{\beta}) \prod_{k = 2}^{N} a(\lambda_{\alpha}, \nu_k) \\
\times \tilde{F}_{N-1}^{(r-1, -, \epsilon_2, \ldots, \epsilon_s)}(\{\lambda\} \setminus \lambda_\alpha, \{\nu\} \setminus \nu_1), \quad (23)
\]

\[
\tilde{F}_N^{(r, +, \epsilon_2, \ldots, \epsilon_s)}(\{\lambda\}, \{\nu\}) = \prod_{\beta = 1}^{r} b(\lambda_{\beta}, \nu_1) \sum_{\alpha = r+1}^{N} c \prod_{\beta = \alpha}^{N} a(\lambda_{\beta}, \nu_1) \prod_{\beta = r+1}^{N} f(\lambda_{\beta}, \lambda_{\alpha}) \prod_{k = 2}^{N} b(\lambda_{\alpha}, \nu_k) \\
\times \tilde{F}_{N-1}^{(r, \epsilon_2, \ldots, \epsilon_s)}(\{\lambda\} \setminus \lambda_\alpha, \{\nu\} \setminus \nu_1). \quad (24)
\]

We also have the following recursive relation \([8, 9, 26]\) for the denominator of the boundary correlation functions (partition function)

\[
Z_N(\{\lambda\}, \{\nu\}) = \sum_{\alpha = 1}^{N} c \prod_{\beta = 1}^{\alpha} b(\lambda_{\beta}, \nu_1) \prod_{\beta = 1}^{\alpha} f(\lambda_{\alpha}, \lambda_{\beta}) \prod_{k = 2}^{N} a(\lambda_{\alpha}, \nu_k) Z_{N-1}(\{\lambda\} \setminus \lambda_\alpha, \{\nu\} \setminus \nu_1). \quad (25)
\]

From these recursive relations (23), (24) and (25), one can show \([1]\) by induction the following general
expression for the boundary correlation functions

\[ F_N^{(r_1, \ldots, r_s)}(\{\lambda\}, \{\nu\}) \]

\[ = \frac{1}{\det M(\{\lambda\}, \{\nu\})} \prod_{j=1}^s \prod_{\beta=1}^{\lambda_j} a(\lambda_\beta, \nu_j) \prod_{j=1}^s \prod_{\beta=1}^{\nu_j} b(\lambda_\beta, \nu_j) \sum_{\alpha_1 \in S_{N, r}^N} \sum_{\alpha_2 \in S_{N, r}^{N-1}} \cdots \sum_{\alpha_s \neq \alpha_1, \ldots, \alpha_{s-1} \in S_{N, r}^{N-s}} \]

\[ \times (-1)^{\sum_{j<k} (\alpha_j, -\alpha_k)} + \sum_{k=1}^s (\alpha_k - 1 - r(\alpha_k + 1)/2) + \sum_{k=1}^s (\alpha_k + 1)(N-k)/2 \]

\[ \times \prod_{j=1}^s H_{r_j}^r (\lambda_{\alpha_j}) \prod_{1 \leq j < k \leq s} E_{r_j r_k}^r (\lambda_{\alpha_j}, \lambda_{\alpha_k}, \nu_j, \nu_k) \det M(\{\lambda\} \setminus \{\lambda_1, \ldots, \lambda_s\}, \{\nu\} \setminus \{\nu_1, \ldots, \nu_s\}) \]

where \( S_{N, r}^N = \{1, \ldots, r\}, S_{N, r}^{N-s} = \{r+1, \ldots, N\}, E(\lambda, \nu) = \text{sh}(\lambda - \nu + \eta). \)

\[ H_{r_j}^r (\lambda) = \prod_{\beta=1}^{\lambda_j} e(\lambda_\beta, \lambda) \prod_{\beta=r+1}^N d(\lambda_\beta, \lambda) \prod_{k=1}^N b(\lambda, \nu_k) \]

\[ H_{r_j}^r (\lambda) = \prod_{\beta=1}^{\lambda_j} a(\lambda_\beta, \nu_k) \prod_{\beta=r+1}^N e(\lambda, \lambda_\beta) \prod_{k=1}^N a(\lambda, \nu_k) \]

\[ E^{r+} (\lambda_{\alpha_k}, \lambda_{\alpha_k}, \nu_j, \nu_k) = \frac{a(\lambda_{\alpha_k}, \nu_k) a(\lambda_{\alpha_k}, \nu_j)}{d(\lambda_{\alpha_k}, \lambda_{\alpha_k})} \]

\[ E^{r-} (\lambda_{\alpha_k}, \lambda_{\alpha_k}, \nu_j, \nu_k) = \frac{a(\lambda_{\alpha_k}, \nu_k) b(\lambda_{\alpha_k}, \nu_j)}{e(\lambda_{\alpha_k}, \lambda_{\alpha_k})} \]

\[ E^{++} (\lambda_{\alpha_k}, \lambda_{\alpha_k}, \nu_j, \nu_k) = \frac{b(\lambda_{\alpha_k}, \nu_k) a(\lambda_{\alpha_k}, \nu_j)}{e(\lambda_{\alpha_k}, \lambda_{\alpha_k})} \]

\[ E^{+-} (\lambda_{\alpha_k}, \lambda_{\alpha_k}, \nu_j, \nu_k) = \frac{a(\lambda_{\alpha_k}, \nu_k) b(\lambda_{\alpha_k}, \nu_j)}{d(\lambda_{\alpha_k}, \lambda_{\alpha_k})} \]

Figure 4: Graphical description of the right hand side of (23).
and $\chi(\beta, \alpha) = 1$ for $\beta > \alpha$ and 0 otherwise. As a special case ($\epsilon_j = -, j = 1, \cdots, s$), the boundary correlation function reduces to the emptiness formation probability [26] (cf. [11]), which gives the probability of finding a sequence of all spins down of length $s$ from the left boundary.

3 Discussion

In the last section, we reviewed the derivation of boundary correlation functions for the six vertex model with domain wall boundary condition by use of the quantum inverse scattering method. Let us comment on the applications of the result.

First, note that the previous works [22, 23, 24, 25, 26] could only calculate some kind of boundary correlation functions, boundary magnetization for example. On the other hand, the general result for the boundary correlation functions enables us to express bulk magnetization and the spin-spin correlation functions in principle, since it can expressed as a linear sum of boundary correlation functions (15), (26). For example,

\[
G(r, \epsilon_j^1, \cdots, \epsilon_j^s)(N, j_1^1, \cdots, j_s^1)(\{\lambda\}, \{\nu\}) = \tilde{G}(r, \epsilon_j^1, \cdots, \epsilon_j^s)(N, j_1^1, \cdots, j_s^1)(\{\lambda\}, \{\nu\}),
\]

However, since this means we need to sum over intermediate spin states, the expression gets complicated. Simplifying the expression of the correlation functions is an important problem to be considered in the future.

Another application of the boundary correlation functions of the six vertex model is to higher spin vertex models such as the nineteen vertex (Fateev-Zamolodchikov) model. We can show by fusion...
that the boundary correlation functions of the nineteen vertex models reduce to those for the six vertex model. In particular, the emptiness formation probability of length $s$ for the nineteen vertex model on an $N \times N$ lattice reduces to that of length $2s$ for the six vertex model on a $2N \times 2N$ lattice (Fig. 6).

Figure 6: Relation between the boundary correlation function for the nineteen vertex model (left hand side) and the six vertex model (right hand side).

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