Fault-Tolerant Center Problems with Robustness and Fairness

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Abstract

We study a family of clustering problems that require fault-tolerant solutions that are also robust with the presence of outliers. We consider robust fault-tolerant $k$-center, matroid center and knapsack center, and develop pure or multi-criteria approximation algorithms for them. In order to address the fairness concern, we also consider variants of the aforementioned problems, namely fair robust fault-tolerant center problems. In these problems, each client $j$ has a value $e_j$, and we need to stochastically open a set of facilities such that the expected number of facilities that are assigned to $j$ is at least $e_j$. We develop a pure approximation for fair robust fault-tolerant $k$-center and multi-criteria approximation algorithms for the knapsack and matroid variations.

1 Introduction

The $k$-center problem is one of the most studied fundamental clustering problems, and still gaining attention for its various generalizations. We are given a finite metric space $(X, d)$, and required to find a subset $S \subset X$ with $|S| \leq k$, in order to minimize the maximum distance between any point $j \in X$ and its closest neighbor in $S$. It is NP-hard to approximate $k$-center to any factor smaller than 2 [14], and tight approximation algorithms have been found as well [12]. For the fault-tolerant $k$-center problem, every $j \in X$ needs to be assigned to at least $r_j$ distinct facilities in $S$, and the cost of $j$ is some function of the distances from $j$ to its connected facilities. We consider the function to be the $\ell_p$-norm, where $p \in \mathbb{Z}_+$ or the infinity $\ell_\infty$-norm, and several constant-factor approximation algorithms [6, 17] have been developed for $\ell_\infty$-norm fault-tolerant $k$-center.

In this paper, we study the robust fault-tolerant center and the fair robust fault-tolerant center problems. In these problems, every client $j$ has a lower bound $l_j$ and an upper bound $r_j$, and we need to connect a number $f_j \in [l_j, r_j]$ of distinct open facilities to client $j$.

Consider 5G networks as an example. 5G provides far better latency and connection speeds, but the signal is much easier to be blocked by some obstacles. Hence if we want to maintain good connection quality, we need to hold multiple connections with nearby signal stations. However, it is often infeasible or too expensive to satisfy the maximum connection capacity $r_j$, so we may drop some connections, but no lower than the threshold $l_j$. In order to sustain a good overall service quality, the total number of connections for all clients is also required to be at least $m$, where $m \in \mathbb{Z}_+$ is also given in the input.

We propose the following formal definition for robust fault-tolerant $k$-center (abbreviated RFT-$k$Center). For completeness, our definition contains objective functions of $\ell_p$-norms for any $p \in \mathbb{Z}_+$ and $\ell_\infty$-norm. The knapsack and matroid variants are defined similarly except that we replace the cardinality constraint $|S| \leq k$ with a knapsack constraint and matroid constraints. We use $C$ and $F$ to denote the set of clients and candidate facility locations, respectively, and $d$ the metric on $C \cup F$. For facilities $S$ and a radius $R \geq 0$, let the set of facilities within distance $R$ from $j$ be $\text{Ball}_S(j, R) = \{i \in S : d(i, j) \leq R\}$, and the number of valid connections be $\text{range}_S(j, R, r_j) = \min\{|\text{Ball}_S(j, R)|, r_j\}$.

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Definition 1. (RFT-$k$Center$_\infty$, RFT-$k$Center$_p$) An instance of robust fault-tolerant $k$-center is specified by metric space $(\mathcal{C} \cup \mathcal{F}, d)$, non-negative integers $l_j \leq r_j$ for $j \in \mathcal{C}$ and $k, m \in \mathbb{Z}_+$. We are required to identify a subset $S \subset \mathcal{F}, |S| \leq k$ and determine $f_j \in [l_j, r_j] \cap \mathbb{Z}$ such that $\sum_{j \in \mathcal{C}} f_j \geq m$. In the $\ell_\infty$-norm variant RFT-$k$Center$_\infty$ and $\ell_p$-norm variant RFT-$k$Center$_p \ (p \in \mathbb{Z}_+)$, the objectives are, respectively,

$$\max_{j \in \mathcal{C}} \min_{F_j \in S, |F_j| = f_j} \|\{d(j, i) : i \in F_j\}\|_\infty, \quad \max_{j \in \mathcal{C}} \min_{F_j \in S, |F_j| = f_j} \|\{d(j, i) : i \in F_j\}\|_p.$$ 

In fair robust fault-tolerant $k$-center (abbreviated FairRFT-$k$Center), we let each $j$ submit a real value $e_j \in [l_j, r_j]$, and we need to stochastically decide which facilities to open and how to establish connections for all clients, such that the expected number of connections for $j$ is at least $e_j$. Harris et al. [11] study an easier version of this problem, known as fair robust $k$-center and provide a multi-criteria approximation. Anegg et al. [1] use the primal-dual schema and show a true approximation for fair colorful $k$-center, improving the previous result.

Definition 2. (FairRFT-$k$Center) An instance of fair robust fault-tolerant $k$-center is specified by metric space $(\mathcal{C} \cup \mathcal{F}, d)$, non-negative integers $l_j \leq r_j$, real $e_j \in [l_j, r_j]$ for $j \in \mathcal{C}$ and $k, m \in \mathbb{Z}_+$. We are required to find the minimum radius $R \geq 0$, such that there exists a distribution $D$ on subsets of $\mathcal{F}$, and when $S$ is sampled from $D$, it satisfies the following constraints:

Coverage constraints: With probability 1, $\forall j \in \mathcal{C}$, $|\text{Balls}(j, R)| \geq l_j$, and the total number of valid service connections is at least

$$\sum_{j \in \mathcal{C}} \text{range}_S(j, R, r_j) \geq m,$$

Fairness constraint: $\forall j \in \mathcal{C}, E[\text{range}_S(j, R, r_j)] \geq e_j$,

Cardinality constraint: $|S| \leq k$ with probability 1.

1.1 Our Results

Our first main results are constant-factor approximation algorithms for various versions of robust fault-tolerant center problems, where the cardinality constraint $|S| \leq k$ may be replaced either by a knapsack constraint $w(S) \leq W$ or a matroid constraint $S \in \mathcal{M}$ for $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ being a given matroid. Our algorithms are built on the natural LP relaxations of the problems. Rounding is processed using a packing partition of the LP solution and an auxiliary LP. For the $\ell_\infty$-norm version of the problem, our subroutine ALG-Filter is a generalized version of the filtering algorithm in [11]. We select a subset of clients $\mathcal{C}' \subset \mathcal{C}$ and for every $j \in \mathcal{C}'$, consider the facilities $j$ connects to, which is a subset of $\text{Balls}(j, R)$, where $R$ is the optimal radius. We make sure that clients in $\mathcal{C}'$ are far from each other, so that $\text{Balls}(j, R) \cap \text{Balls}(j_1, R) = \emptyset$ for any $j \neq j_1$ in $\mathcal{C}'$. We provide the details of ALG-Filter in Appendix B.

For the $\ell_p$-norm variant, we use another subroutine ALG-Bundle for partitioning the LP solution into so-called “bundles”, which is used in [10][19]. We strengthen the algorithm by allowing partial bundles and provide a stronger rounding technique for the coverage constraint of making at least $m$ service connections. By defining a “profitable” factor $n_U \in \mathbb{Z}_+$ for a partial bundle $U$, if we do not have any open facility in $U$, we cannot have extra connections via $U$; if we have one inside $U$, we gain $n_U$ additional connections. The auxiliary LP for rounding is then carefully designed to open exactly one facility in some of the partial bundles and provide at least $m$ connections. We present the algorithm ALG-Bundle in Appendix B.

As a second result, we show a pure constant factor approximation algorithm for FairRFT-$k$Center. We use the primal-dual schema in [1], and design more involved algorithms for the rounding and dynamic programming subroutines. More specifically, the algorithm uses the ellipsoid algorithm, and in the description of the separating-hyperplane oracle, if the packing of fractional facilities has total value at most $k - 1$, we use a stronger auxiliary LP for rounding, similar to the one used in our robust algorithms; if the packing has total value more than $k - 1$, we use a more general dynamic programming method than the binary program.
Lemma 1. Let \( E \) be a finite ground set, \( \mathcal{L} \) be a laminar family on \( E \) and \( \mathcal{M} \) be a matroid on \( E \) with rank function \( r \). If the polytope \( \mathcal{P} = \{ x \in \mathbb{R}^{|E|} : x \geq 0, A_1x \leq b_1, A_2x \geq b_2, x(S) \leq r(S) \forall S \subseteq E \} \) satisfies that each row of \( A_1, A_2 \) corresponds to the characteristic vector \( \chi_L \) of some \( L \in \mathcal{L} \) and all entries of \( b_1, b_2 \) are integers, then \( \mathcal{P} \) either is empty or has integral extreme points.

3 A Unified Approach for Robust Fault-Tolerant Center

In this section, we present constant-factor approximations for the \( \ell_\infty \)-norm versions of robust fault-tolerant center problems. We also show constant-factor approximation algorithms for their \( \ell_p \)-norm variants in Appendix C.

Let \( R \) be the largest connection distance in the optimal solution and obviously \( R = \text{OPT} \), where \( \text{OPT} \) is the optimum. Since \( R \) only takes a polynomial number of possible values, we assume that \( R \) is exactly guessed, and consequently the associated LP relaxations are all feasible.
3.1 The $\ell_\infty$-norm Base Case: RFT-$k\text{Center}_\infty$

To start with, we consider RFT-$k\text{Center}_\infty$. The LP relaxation of this problem is the following, 

$$\mathcal{P}_{0,k} = \left\{ (u, x, y) \in \mathcal{P}_0 \left| \sum_{i \in \mathcal{F}} y_i \leq k \right. \right\}.$$ 

Since $R$ is optimal, the corresponding optimal solution induces an integral solution $(u^*, x^*, y^*)$ to $\mathcal{P}_{0,k}$, thus the polytope is non-empty. We find an arbitrary solution $(u, x, y)$ and apply ALG-Filter($x, y, 2R$). We obtain the output $C'$, the partition $\{D_j : j \in C'\}$ and $\{c_j : j \in C'\}$.

Now, for any $j'$ in $D_j$, because $x_j \geq x_{j'}$ and $d(j, j') \leq 2R$ (see ALG-Filter), using triangle inequality, the current extent of $j'$ can be fully satisfied in $\text{Ball}_F(j, R)$ at a distance at most $3R$. For every $j \in C'$, we arbitrarily pre-select $|x_j|$ facilities within $\text{Ball}_F(j, R)$ and define $m_j = \sum_{j' \in D_j} \min\{r_{j'}, |x_j|\}$, the number of total connections that is satisfied in $D_j$ by these facilities. Recall that $c_j$ is defined as $|\{j' \in D_j : r_{j'} \geq |x_j|\} |$ and for a vector $v$ indexed by $S$, $v(S) = \sum_{i \in S} v_i$.

From the solution $(u, x, y)$, we create another solution $y'$ by assigning $y'_i = u_{ij}$ for any $i \in \text{Ball}_F(j, R), j \in C'$ and 0 otherwise. It is easy to see that $y'((\text{Ball}_F(j, R)) = x_j$ and we have the following lemma.

Lemma 2. 

$$\sum_{j \in C'} [c_j \cdot (y'((\text{Ball}_F(j, R) - |x_j|) + m_j]) \geq m.$$

Proof. We only need to prove the LHS is at least $x(C) = \sum_{j \in C} x_j$, since $x$ satisfies the LP constraints in $\mathcal{P}_0$ and $x(C)$ is at least $m$. Consider any $j' \in D_j$, its contribution in $x(C)$ is exactly $x_{j'}$.

If $r_{j'} \leq |x_j|, j'$ is not counted into $c_j$, so its contribution in the LHS is $\min\{r_{j'}, |x_j|\} = r_{j'} \geq x_{j'}$; if we have $r_{j'} \geq |x_j| + 1$, it is counted in $c_j$, so its contribution in the LHS is $y'(\text{Ball}_F(j, R)) - |x_j| + \min\{r_{j'}, |x_j|\} = y'(\text{Ball}_F(j, R)) = x_j \geq x_{j'}$,

where the last inequality is due to the order of selection in ALG-Filter.

Therefore, the contribution of any $j' \subseteq C$ to the LHS is greater than or equal to that in $x(C)$, and the LHS is at least $x(C) \geq m$. 

We proceed and seek to find a way that opens $|x_j|$ or $|x_j|$ facilities in $\text{Ball}_F(j, R)$. Our new auxiliary LP is defined as follows,

$$\text{max: } \sum_{j \in C'} [c_j \cdot (z(\text{Ball}_F(j, R)) - |x_j|) + m_j]$$

s.t. 

$$z(\text{Ball}_F(j, R)) \in [\lfloor x_j \rfloor, \lceil x_j \rceil] \quad \forall j \in C'$$

$$z_i \in [0, 1] \quad \forall i \in \mathcal{F}$$

$$z(\mathcal{F}) \leq k,$$

and it is easy to see that $y'$ satisfies this LP and the constraints form a laminar family, so the optimal solution $z^*$ is integral with objective value at least that of $y'$, and from Lemma 2 at least $m$. We open $F = \{i \in \mathcal{F} : z_i^* = 1\}$, and according to the triangle inequality, $F$ is a solution that maintains at least $m$ connections within radius $3R$ with $|F| \leq k$, therefore we have the following theorem.

Theorem 3. There exists a polynomial-time $\beta$-approximation algorithm for RFT-$k\text{Center}_\infty$.

3.2 The $\ell_\infty$-norm Case: Extensions

As a simple corollary, we notice that the algorithm above can be directly applied to RFT-MatCenter$_\infty$ by only changing the cardinality constraints in $\mathcal{P}_{0,k}$ and $\text{Aux}_k$ to $z(S) \leq r(S) \forall S \subseteq \mathcal{F}$ as the matroid constraints. Using a similar argument, the new LP $\text{Aux}_{\text{max}}$ is non-empty and its set of constraints consists of the characteristic vectors of a laminar family and a matroid, hence using Lemma 4 and Lemma 5 the optimal solution $z^*$ is integral with objective value at least $m$. 

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Corollary 1. There exists a polynomial-time 3-approximation algorithm for \( \text{RFT-MatCenter}_\infty \).

Then we turn to \( \text{RFT-KnapCenter}_\infty \). Notice that if we change \( \text{Aux}_n \) such that the last constraint becomes the knapsack constraint \( w(F) \leq W \), the constraints of \( \text{Aux}_{\text{knap}} \) now describe the intersection of a laminar family with a knapsack polytope, therefore the optimal solution \( z^* \) has at most \( 2 \) fractional entries. We define \( F = \{ i \in \mathcal{F} : z^*_i > 0 \} \). Compared to \( z^* \), the total weight of \( F \) is increased by the amount of at most \( 2 \) facilities, therefore we have the following standard lemma (for rounded-up facilities, see, e.g., [8, 11]).

Lemma 3. \( F \) satisfies the coverage constraints with an objective value at most \( 3 \cdot \text{OPT} \), and \( w(F) \leq W + 2 \max_{i \in \mathcal{F}} w_i \).

Next, to make the total weight of the solution arbitrarily close to \( W \), for any fixed \( \epsilon > 0 \) we can guess the set of open facilities in the optimal solution that have weights at least \( \epsilon \cdot W \). Denote this set by \( F_0 \). Obviously \( |F_0| = O(1/\epsilon) \), and the number of possible guesses is \( n^{O(1/\epsilon)} \). We then simply decrease the demands of \( j \) within distance \( R \) from any \( i \in F_0 \). More specifically, for any \( j \in C \), define

\[
I'_j = \max\{0, l_j - |\text{Ball}_{F_0}(j, R)|\}, \quad R'_j = \max\{0, r_j - |\text{Ball}_{F_0}(j, R)|\},
\]

and we also reduce the coverage constraint \( m \) accordingly,

\[
m' = m - \sum_{j \in C} \text{range}_{\text{Ball}}(j, R, r_j).
\]

After this pre-processing step, we consider the instance on the new facility set \( \mathcal{F}_{<\epsilon} = \{ i \in \mathcal{F} : w_i < \epsilon \cdot W \} \). One easily sees that the modified instance is also feasible with optimal solution \( \leq \text{OPT} \) and knapsack constraint \( W - w(F_0) \). Therefore, by applying the algorithm above to the new instance, and combining the overall output as \( F \cup F_0 \), the weight is at most

\[
w(F \cup F_0) = w(F) + w(F_0) \leq W - w(F_0) + 2 \max_{i \in \mathcal{F}_{<\epsilon}} w_i + w(F_0) \leq (1 + 2\epsilon)W.
\]

We summarize the procedures above in the following corollary.

Corollary 2. For any fixed \( \epsilon > 0 \), there exists a 3-approximation algorithm for \( \text{RFT-KnapCenter}_\infty \) that violates the knapsack constraint by a multiplicative factor \( \epsilon \) and runs in time \( n^{O(1/\epsilon)} \), where \( n \) is the input size.

## 4 Fair Robust Fault-Tolerant \( k \)-Center

### 4.1 The Primal-Dual Schema

Our primal-dual schema uses the framework from [11], and is technically more involved in the design of LP rounding and dynamic programming subroutines. We restate our feasible polytope \( \mathcal{P}_{0,k}(R) \) here for a fixed radius \( R \geq 0 \), which we do NOT assume is optimal this time. We also drop the fairness constraint and proceed to add it later.

\[
\mathcal{P}_{0,k}(R) = \left\{ (u, x, y) \in \mathcal{P}_0 : \sum_{i \in \mathcal{F}} y_i \leq k \right\}.
\]

Recall that \( \text{range}_{\mathcal{S}}(j, R, r_j) = \min\{|\text{Ball}_{\mathcal{S}}(j, R)|, r_j\} \) is the number of valid connections for \( j \) within radius \( R \). Define \( \mathcal{F}(R) \) as the collection of subsets of \( \mathcal{F} \) that corresponds to the integral solutions of \( \mathcal{P}_{0,k}(R) \). By definition, the distribution \( \mathcal{D} \) exists if and only if \( \mathcal{F}(R) \) is non-empty, and there exists a distribution over
$\mathcal{F}(R)$ that further satisfies the fairness constraint. Specifically, define the following LP and its dual program.

\[ \begin{align*}
\text{min:} & \quad 0 & \quad (\text{Primal}(R)) \\
\text{s.t.} & \quad \sum_{S \in \mathcal{F}(R)} \lambda_S \cdot \text{range}_S(j, R, r_j) \geq e_j & \quad \forall j \in \mathcal{C} \\
& \quad \sum_{S \in \mathcal{F}(R)} \lambda_S = 1 \\
& \quad \lambda_S \geq 0 & \quad \forall S \in \mathcal{F}(R); \\
\text{max:} & \quad \sum_{j \in \mathcal{C}} \alpha_j \cdot e_j - \mu & \quad (\text{Dual}(R)) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{C}} \alpha_j \cdot \text{range}_S(j, R, r_j) \leq \mu & \quad \forall S \in \mathcal{F}(R) \\
& \quad \alpha_j \geq 0 & \quad \forall j \in \mathcal{C} \\
& \quad \mu \in \mathbb{R}.
\end{align*} \]

Clearly if $\text{Primal}(R)$ is feasible, then there exists such a distribution and its optimal value is 0, and in this case the optimal value of $\text{Dual}(R)$ is also 0. On the other hand, if there is some solution $(\alpha, \mu)$ to $\text{Dual}(R)$ with a positive objective value, then since the constraints of $\text{Dual}(R)$ are scale-invariant, $\text{Dual}(R)$ is unbounded and thus $\text{Primal}(R)$ is infeasible. We define another polytope $Q(R)$ that is also directed related to the feasibility of $\text{Primal}(R)$.

\[ Q(R) = \left\{ (\alpha, \mu) \in \mathbb{R}_{\geq 0}^{\mathcal{C}} \times \mathbb{R} \ \bigg| \ \sum_{j \in \mathcal{C}} \alpha_j \cdot e_j \geq \mu + 1, \sum_{j \in \mathcal{C}} \alpha_j \cdot \text{range}_S(j, 5R, r_j) \leq \mu, \forall S \in \mathcal{F}(R) \right\}. \]

And we immediately have the following lemma,

**Lemma 4.** $Q(R)$ is empty if and only if $\text{Primal}(R)$ is feasible.

The core lemma is given as follows, which we prove in Section 4.2 We then show how this lemma enables us to prove the main theorem.

**Lemma 5.** There is a polynomial time algorithm that, given some radius $R \geq 0$ and $(\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{\mathcal{C}} \times \mathbb{Q}$ that satisfies $\sum_{j \in \mathcal{C}} \alpha_j \cdot e_j \geq \mu + 1$, either certifies that $(\alpha, \mu) \in Q(R)$ or returns a set $S \in \mathcal{F}(5R)$ with $\sum_{j \in \mathcal{C}} \alpha_j \cdot \text{range}_S(j, 5R, r_j) > \mu$.

**Theorem 4.** There is a 5-approximation algorithm for $\text{FairRFT-}k\text{Center}$ that runs in polynomial time.

**Proof.** Let us start with an arbitrary set of values $(\alpha, \mu)$. First, if there is some violated constraint $\alpha_j < 0$ or $\sum_{j \in \mathcal{C}} \alpha_j \cdot e_j < \mu + 1$, the corresponding constraint is a hyperplane that separates $(\alpha, \mu)$ from $Q(5R)$, and we use the ellipsoid method to iterate for another $(\alpha, \mu)$. Otherwise, using Lemma 5, we can determine, in polynomial time, that either $(\alpha, \mu) \in Q(R)$, in which case we conclude that $Q(R)$ is non-empty, $\text{Primal}(R)$ is infeasible according to Lemma 1, and we abort from the choice $R$ since the optimal solution is strictly larger than $R$: otherwise, we get some set $S \in \mathcal{F}(5R)$ that satisfies $\sum_{j \in \mathcal{C}} \alpha_j \cdot \text{range}_S(j, 5R, r_j) > \mu$, which is also a hyperplane that separates $(\alpha, \mu)$ from $Q(5R)$.

Let us further assume that we always get some subset from $\mathcal{F}(5R)$, since otherwise we simply abort from $R$. Using the ellipsoid method (see e.g. Theorem 6.4.9 of 9), we can determine that $Q(5R) = 0$ in polynomial time. Indeed, let $\mathcal{H}$ denote all the subsets from $\mathcal{F}(5R)$ that are returned during the process, and the ellipsoid algorithm states that the following polytope

\[ Q_{\mathcal{H}}(5R) = \left\{ (\alpha, \mu) \in \mathbb{R}_{\geq 0}^{\mathcal{C}} \times \mathbb{R} \ \bigg| \ \sum_{j \in \mathcal{C}} \alpha_j \cdot e_j \geq \mu + 1, \sum_{j \in \mathcal{C}} \alpha_j \cdot \text{range}_S(j, 5R, r_j) \leq \mu, \forall S \in \mathcal{H} \right\}, \]
which contains $Q(5R)$, is empty. Using Lemma 4, again, now we know $\text{Primal}(5R)$ is feasible. More specifically, in the definition of $\text{Dual}(5R)$, if we replace the whole collection $F(5R)$ with $\mathcal{H}$, the resulting program $\text{Dual}_\mathcal{H}(5R)$ has objective value 0 due to the emptiness of $Q_\mathcal{H}(5R)$, hence its corresponding primal program is also feasible.

Now we have a polynomial-size linear program $\text{Primal}_\mathcal{H}(5R)$, which is exactly the result by replacing the whole collection $F(5R)$ in $\text{Primal}(5R)$ with $\mathcal{H}$. Since it is feasible, we directly solve for $\lambda_S, S \in \mathcal{H}$ and obtain the corresponding distribution, and the fairness constraint is satisfied.

4.2 Proof of Lemma 5

First, we define the set of violated subsets for notation simplicity,

$$F^{\alpha,\mu}(R) = \left\{ S \in F(R) \left| \sum_{j \in C} \alpha_j \cdot \text{range}_S(j, R, r_j) > \mu \right. \right\},$$

and we easily see that, if $(\alpha, \mu)$ satisfies $\sum_{j \in C} \alpha_j \cdot \epsilon_j \geq \mu + 1$ and $F^{\alpha,\mu}(R)$ is empty, then $(\alpha, \mu) \in Q(R)$.

Therefore, to prove Lemma 5 for any given $(\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{|C|} \times \mathbb{Q}$, we need to either certify that $F^{\alpha,\mu}(R)$ is empty, or find a subset $S \in F^{\alpha,\mu}(5R)$. The following lemma encodes the strict inequality as an equivalent non-strict one, and is directly obtained from [11], thus we omit the proof here.

**Lemma 6.** ([11]) Let $(\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{|C|} \times \mathbb{Q}$ with representation length $L$. Then one can efficiently compute an $\epsilon > 0$ with representation length $O(L)$, such that for any $S \in F(R)$, we have $\sum_{j \in C} \alpha_j \cdot \text{range}_S(j, R, r_j) > \mu$ if and only if $\sum_{j \in C} \alpha_j \cdot \text{range}_S(j, R, r_j) \geq \mu + \epsilon$.

We fix such an $\epsilon > 0$ from now on, and further define the modified polytope $P^{\alpha,\mu}$ for the given $(\alpha, \mu)$.

$$P^{\alpha,\mu} = \left\{ (u, x, y) \in P_{0,k}(R) \left| \sum_{j \in C} \alpha_j \cdot x_j \geq \mu + \epsilon \right. \right\}.$$

Let $P^{\alpha,\mu}_I = \text{conv}(P^{\alpha,\mu} \cap \mathbb{Z}^{d_P})$ be the convex hull of the integral points in $P^{\alpha,\mu}$, where $d_P$ corresponds to the dimensionality of $(u, x, y)$. By definition, if $(\alpha, \mu)$ satisfies $\sum_{j \in C} \alpha_j \cdot \epsilon_j \geq \mu + 1$ but $(\alpha, \mu) \notin Q(R)$, then there exists some subset $S \in F^{\alpha,\mu}(R)$, hence $S$ induces an integral solution to $P^{\alpha,\mu}$, by letting $x_j = \text{range}_S(j, R, r_j)$ and $P^{\alpha,\mu}_I$ is thus non-empty. The contrapositive of this observation tells us that if $(\alpha, \mu)$ satisfies $\sum_{j \in C} \alpha_j \cdot \epsilon_j \geq \mu + 1$ and $P^{\alpha,\mu}_I$ is empty, then $(\alpha, \mu) \in Q(R)$. Given $(\alpha, \mu)$ and $(u, x, y) \in P^{\alpha,\mu}$, we run $\text{ALG-Filter}(x, y, 4R)$ and obtain $C', \{D_j : j \in C'\}, \{e_j : j \in C'\}$. For convenience, we let the union of balls $\text{Ball}_R(C', R) = \bigcup_{j \in C'} \text{Ball}_R(j, R)$. We note that the parameter used in $\text{ALG-Filter}$ is $4R$ instead of $2R$, which proves to be necessary for later analysis.

**Lemma 7.** Let $(\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{|C|} \times \mathbb{Q}$, and $(u, x, y) \in P^{\alpha,\mu}$. If $y(\text{Ball}_R(C', R)) \leq k - 1$, then one can efficiently find $S \in F^{\alpha,\mu}(5R)$.

**Proof.** Notice that $\text{Ball}_R(j, R) \cap \text{Ball}_R(j_1, R) = \emptyset$ for any $j \neq j_1$ in $C'$, since $d(j, j_1) > 4R$. For any $j' \in D_j$, we have $x_{j'} \leq x_j$, and because $d(j, j') \leq 4R$, it is sufficient to serve $j'$ to an extent of at least $x_{j'}$, all using the (fractional) facilities within $\text{Ball}_R(j, R)$, within maximum distance of $4R + R = 5R$. Therefore, if we only open a total amount of $x_j$ (fractional) facilities within $\text{Ball}_R(j, R)$, we are still able to satisfy the current extents $\{x_j : j \in C\}$. Define $y'_i = u_{ij}$ for $i \in \text{Ball}_R(j, R), j \in C'$ and 0 otherwise, and $m_j = \sum_{j' \in D_j} \min\{r_{j'}, [x_{j'}]\}$ for $j \in C'$. Using Lemma 2 we obtain

$$\sum_{j \in C'} (c_j \cdot (y'_i(\text{Ball}_R(j, R)) - [x_{j'}]) + m_j) \geq m.$$
Next, in the same fashion as Lemma \[2\] we define $\beta_j = \sum_{j' \in D_j} r_{j', j} \alpha_{j'}$ and $\mu_j = \sum_{j' \in D_j} \alpha_{j'} \cdot \min\{r_{j', j}, [x_j]\}$ for $j \in C'$, and obtain the inequality
\[
\sum_{j \in C'} (\beta_j \cdot (y'('Ball_F(j, R)) - [x_j]) + \mu_j) \geq \mu + \epsilon.
\]

Now we consider the following auxiliary LP derived from $P^{\alpha,\mu}$ and $(u, x, y)$,
\[
\begin{align*}
\text{max:} & \quad \sum_{j \in C'} (\beta_j \cdot (z('Ball_F(j, R)) - [x_j]) + \mu_j) \\
\text{s.t.} & \quad \sum_{j \in C'} (c_j \cdot (z('Ball_F(j, R)) - [x_j]) + m_j) \geq m \\
& \quad \quad \quad \quad z('Ball_F(j, R)) \in ([x_j], [x_j]) \quad \forall j \in C' \\
& \quad \quad \quad \quad z = [0, 1] \quad \forall i \in F \\
& \quad \quad \quad \quad z('Ball_F(C', R)) \leq k - 1 \\
& \quad \quad \quad \quad z('Ball_F(C', R)) = 0.
\end{align*}
\]

Since $y'$ satisfies all the constraints, and all but one constraints form a laminar family on $F$, the optimal solution $z^*$ is almost-integral with at most 2 fractional variables, and the objective value associated with $z^*$ is at least $\mu + \epsilon$.

If we have any fractions in $z^*$, we round them to 1. Since we have at most two fractions in $z^*$, the resulting $z$ certainly has objective value at least $\mu + \epsilon$, satisfies the coverage constraint $m$ and $z(F) \leq k$ (indeed, if we have one fraction, we round it to 1; if we have two of them, the number of 1s is at most $k-2$ since the total value is $\leq k-1$, and by rounding the two fractions to 1, the number of 1s is at most $k$). Finally, we define $S = \{i \in F : z_i = 1\}$, $x_j = \min\{'Ball_S(j, R), r_j'\}$ for $j' \in D_j$ and $u_j$, accordingly. It is not hard to check that $(\bar{u}, \bar{x}, \bar{z})$ is an integral solution to $P^{\alpha,\mu}(5R)$ by definition of $Aux^{\alpha,\mu}_k(R)$. Therefore, $S \in \mathcal{F}^{\alpha,\mu}(5R)$.

Assume that the condition in Lemma \[7\] does not hold and $y('Ball_F(C', R)) > k-1$, we then try to determine whether there exists $S \in \mathcal{F}^{\alpha,\mu}(R)$ such that $|S \cap Ball_F(C', R)| = k$, in other words, $S \subset Ball_F(C', R)$. If so, we also find some $S' \in \mathcal{F}^{\alpha,\mu}(3R) \subset \mathcal{F}^{\alpha,\mu}(5R)$ and output it; otherwise, we know that $y('Ball_F(C', R)) \leq k-1$ is a hyperplane separating $(u, x, y)$ from $P^{\alpha,\mu}_k$.

**Lemma 8.** Let $(\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{|C|} \times \mathbb{Q}$, and $(u, x, y) \in \mathcal{P}^{\alpha,\mu}$. If there exists $S \in \mathcal{F}^{\alpha,\mu}(R)$ such that $|S \cap Ball_F(C', R)| = k$, then one can efficiently find $S' \in \mathcal{F}^{\alpha,\mu}(3R)$ with $|S' \cap Ball_F(C', R)| = k$.

**Proof.** We prove this with dynamic programming. Given $(u, x, y) \in \mathcal{P}^{\alpha,\mu}$ and ALG-Filter’s output $C'$, \(\{D_j : j \in C'\}\), denote $\alpha_{j'} = \arg \min_{j' \in C'} d(j, j')$ the closest neighbor of $j'$ in $C'$ and cluster $E_j = \{j' \in C : f(j') = j, d(j, j') \leq 2R\}$ for $j \in C'$. Since $S \in \mathcal{F}^{\alpha,\mu}(R)$ is feasible, we only consider clients $E = \bigcup_{j \in C'} E_j$, and every client $j'$ in $C \setminus E$ must have $l_{j'} = 0$, because for each $i \in S$, $d(j', i) \geq \min_{j \in C'} d(j', j) - \min_{j \in C'} d(i, j) \geq 2R - R = R$, and $j'$ cannot have any open facility within radius $R$ from $S$ anyway. We build a DP table for the desired open subset $T \subset \mathcal{P}^{\alpha,\mu}(C', R)$ with the recursion objective as,
\[
\sum_{j \in C'} \sum_{j' \in E_j} \alpha_{j'} \cdot \min\{r_{j', |Ball_T(j, R)|}\}.
\]
We note that for $j \neq j_1$ in $C'$, $E_j \cap E_{j_1} = \emptyset$, otherwise $d(j, j_1) \leq 4R$, which is a contradiction with ALG-Filter$(x, y, 4R)$. So with the solution $S$, we have $Ball_S(j, R) \supset Ball_S(j', R)$, therefore the objective with $T = S$ is at least $\mu + \epsilon$, according to the definition of $\mathcal{F}^{\alpha,\mu}(R)$.

Let $C' = \{j_1, \ldots, j_q\}$. For any $j \in C'$, we set $t_j = \max_{j' \in E_j} l_{j'}$ and arbitrarily pre-select $t_j$ facilities in $Ball_F(j, R)$ for $T$. The DP table has entries indexed as $A[p, M, t]$ where $p \in [q]$, $M \in [m]$ and $t \in [k]$, and represents the maximum objective value, given that the target set $T \subset Ball_F(C', R)$ is chosen such that,
• \( \text{Ball}_R(j_i, R) - t_{j_i} = 0 \) for \( l \geq p + 1 \),
• \( |T| \leq t \),
• \( \sum_{j \in C'} \sum_{j' \in E_j} \min \{ r_{j'}, |\text{Ball}_R(j, R)| \} \geq M \),

therefore the table is actually used to find the optimal way to distribute the extra \( k - \sum_{j \in C'} t_j \) facilities, in order to maximize the objective, while maintaining the coverage constraint of \( m \). The DP table obviously has a polynomial-bounded size, and for each iteration in which we calculate \( A[p, M, t] \), the number of options we need to consider is \( O(k) \), all taking the form of \( A[p - 1, M - h(\Delta), t - \Delta] \). Here \( \Delta \in [k] \) denotes the number of extra facilities we open in \( \text{Ball}_R(j_p, R) \), in addition to the previously chosen \( t_{j_p} \) facilities, and \( h(\Delta) \) is an easy-to-compute function that represents the number of extra connections we can establish via these \( \Delta \) facilities in \( \text{Ball}_R(j_p, R) \). The calculation of objective values follows the same idea, so we simply take the optimal option and store it in another auxiliary DP table \( B[p, M, t] \) for backtracking.

We fill the whole DP table, and conclude no such \( S \in \mathcal{F}^{\alpha, \mu}(R) \) exists if \( A[q, m, k] < \mu + \epsilon \). Assume otherwise and we use table \( B \) to backtrack and output \( S' \). By the conditions of the lemma, the solution has \( |S' \cap \text{Ball}_R(C', R)| = k \) and induces an integral solution that satisfies all the constraints of \( \mathcal{P}^{\alpha, \mu}(3R) \), hence \( S' \in \mathcal{F}^{\alpha, \mu}(3R) \).

**Lemma 9.** Let \( (\alpha, \mu) \in \mathbb{Q}_{\geq 0}^{[C]} \times \mathbb{Q} \). There is a polynomial-time algorithm that, given \( (u, x, y) \in \mathbb{R}^{[C]} \times \mathbb{R}^{[F]} \), either returns a set \( S \in \mathcal{F}^{\alpha, \mu}(5R) \) or returns a hyperplane separating \( (u, x, y) \) from \( \mathcal{P}^{\alpha, \mu}(3R) \).

**Proof.** First, we check whether \( (u, x, y) \in \mathcal{P}^{\alpha, \mu} \), if not, we return some constraint that is violated, which is also a hyperplane separating \( (u, x, y) \) from \( \mathcal{P}^{\alpha, \mu} \). Assume \( (u, x, y) \in \mathcal{P}^{\alpha, \mu} \), and we use ALG-Filter\((x, y, 4R)\) to get the filtered clients \( C' \) and \( \{ D_j : j \in C' \} \). If \( y(\text{Ball}_R(C', R)) \leq k - 1 \), we use Lemma 7 to get \( S \in \mathcal{F}^{\alpha, \mu}(5R) \); if \( y(\text{Ball}_R(C', R)) > k - 1 \), we use Lemma 8 to check whether there exists \( S \in \mathcal{F}^{\alpha, \mu}(3R) \) with \( S \cap \text{Ball}_R(C', R) = k \). If this is the case, we return such \( S \) since \( \mathcal{F}^{\alpha, \mu}(3R) \subseteq \mathcal{F}^{\alpha, \mu}(5R) \); otherwise, we conclude that every integral solution \( (u', x', y') \in \mathcal{P}^{\alpha, \mu}(3R) \) satisfies \( y'(\text{Ball}_R(C', R)) \leq k - 1 \), and this is indeed a hyperplane separating \( (u, x, y) \) from \( \mathcal{P}^{\alpha, \mu}(3R) \).

**Proof of Lemma 8** We use the algorithm \( A \) in Lemma 9 as part of the separation hyperplane oracle, in order to check whether \( \mathcal{P}^{\alpha, \mu}(3R) \) is empty. Fix \( (\alpha, \mu) \), and we start with any \( (u, x, y) \). Whenever the algorithm \( A \) is called, if it ever returns some \( S \in \mathcal{F}^{\alpha, \mu}(5R) \), then we are finished; assume that it always returns a separating hyperplane, then the ellipsoid algorithm verifies the emptiness of \( \mathcal{P}^{\alpha, \mu}(3R) \) in polynomial time, and using previous observations, we have \( (\alpha, \mu) \in \mathbb{Q}(R) \).

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A  Proof of Lemma 1

Proof. We provide a proof based on induction of $|E|$. When $|E|$ is 1 the result is trivial. Suppose the theorem holds true for $|E| = 1, \ldots, n$. Consider the case of $|E| = n + 1$. Let $x$ be a vertex of $\mathcal{P}$, and we consider the following three cases.

(1) When $\exists i, x_i = 0$. Delete $i$ from $\mathcal{M}$ and remove its corresponding column from $A_1, A_2$, also remove the $i$-th entry in $x$ and get $\hat{x}$. Obviously $\hat{x}$ still satisfies $A_1' \hat{x} \leq b_1$ and $A_2' \hat{x} \geq b_2$, and $\hat{x}$ is still in the matroid polytope of $\mathcal{M}/i$, therefore from induction, $\hat{x}$ is integral.

(2) When $\exists i, x_i = 1$. We contract $\mathcal{M}$ by $i$, remove its corresponding column from $A_1, A_2$, subtract 1 from corresponding entries in $b_1, b_2$ if the value removed is equal to 1, and remove the $i$-th entry in $x$ to obtain $\tilde{x}$. It is easy to see that $\tilde{x}$ still satisfies $A_1' \tilde{x} \leq b_1$ and $A_2' \tilde{x} \geq b_2$, and for any $S \subset E \setminus \{i\}$, we have $\tilde{x}(S) = x(S + i) - x_i \leq r(S + i) - 1 = r'(S)$, where $r'$ is the rank function of $\mathcal{M}/i$. Therefore, using the induction hypothesis, $\tilde{x}$ is integral.

(3) For any $i, x_i \in (0, 1)$. Let $\mathcal{T}_1 = \{L \in \mathcal{L} : x$ is tight at constraint of $\chi_L\}$ and $\mathcal{T}_2 = \{S \subseteq E : x(S) = r(S)\}$. It is well-known that there exists a maximal chain $C_2 = \{C_1, \ldots, C_k\} \subseteq \mathcal{T}_2$ with $\emptyset \subset C_1 \subset \cdots \subset C_k$ such that $\text{span}(\chi_S : S \in C_2) = \text{span}(\mathcal{T}_2)$.

Now since every constraint of $x_i \geq 0$ is not tight, the maximum number of linearly-independent tight constraints corresponding to $\mathcal{T}_1 \cup C_2$ must be at least $|E| = n + 1$. On the other hand, since $x_i \in (0, 1) \forall i$ and every tight constraint is of the form $(\chi_S, x) = b$ with $b \in \mathbb{Z}$ and $\chi_S$ being the characteristic vector of $S \subseteq E$, the size of maximum linearly-independent vectors in $\{\chi_L : L \in \mathcal{T}_1\}$ is at most $(n + 1)/2$, since $\mathcal{L}$ is laminar in the first place. The same argument applies to $C_2$, and we see that by combining the arguments above, the sizes of maximum linearly-independent subsets of $\mathcal{T}_1$ and $C_2$ are both exactly $(n + 1)/2$, but this puts $E$ in both subsets, and the two subsets cannot be combined into another larger linearly-independent subset of size exactly $n + 1$. Therefore $x$ is not a vertex solution, contradiction! \hfill \square

B  Missing Algorithms

Algorithm 1 ALG-Filter($x, y, R$)

1: $C' \leftarrow \emptyset$
2: Unmark every client $j \in C$
3: for Unmarked $j \in C$ in non-decreasing order of $x_j$ do
4:  Let $D_j$ be the set of unmarked clients $j' \in C$ (including $j$ itself) that satisfies $d(j, j') \leq R$
5:  Make all clients in $D_j$ marked. Place $C' \leftarrow C' \cup \{j\}$
6:  Let $c_j = \{|j' \in D_j : r_j' \geq |x_j| + 1\}$
7: end for
8: return $C', \{D_j : j \in C', \} \{c_j : j \in C'\}$

C  Robust Fault-Tolerant Center for $\ell_p$-norm, $p \in \mathbb{Z}_+$

We turn to the $\ell_p$-norm versions of robust fault-tolerant center problems and discuss the corresponding algorithms. We assume that the largest connection distance $R$ is known and $\text{OPT}$ is the (unknown) optimum value. It is easy to see that $\text{OPT} \geq (R^n)^{1/p} \geq R$.

We use the algorithm ALG-Bundle for processing the LP solution $(u, x, y)$, with the simpler versions of the algorithm in [10, 19]. We assume that given $x$ and $y$, $u_{ij}$ is the best (nearest) assignment possible and any facility location $i$ can be split into multiple co-located copies (see, e.g., [5, 10]). Using the split technique, we further assume that $u_{ij} \in \{0, y_i\}$ and define $F_j = \{i \in F : u_{ij} > 0\}$, where we let $\mathcal{F}'$ be the set of facility locations after the split. Define $g : \mathcal{F}' \rightarrow \mathcal{F}$, which takes the copy to the original facility location. For example, if the original facility location $i \in \mathcal{F}$ is split into 3 copies $i_1, i_2, i_3$, all in $\mathcal{F}'$, then
Algorithm 2 ALG-Bundle\((u, x, y)\)

1: \(U_1 \leftarrow \emptyset, U_2 \leftarrow \emptyset\)
2: \(\forall j \in C, F'_j \leftarrow F_j, \text{queue}_j \leftarrow \emptyset\)
3: \(\textbf{while exists } j \in C \text{ s.t. } |\text{queue}_j| < [x_j] \textbf{ do}\)
4: \(\text{Choose such } j, \text{ so that if } U' \text{ is the nearest unit volume of facilities in } F'_j, \max_{i \in U} d(i, j) \text{ is minimized}\) (split if necessary)
5: \(\textbf{if exists } U' \in U_1, U \cap U' \neq \emptyset \textbf{ then}\)
6: \(\text{queue}_j \leftarrow \text{queue}_j \cup \{U\}, \text{ remove } U' \cap F'_j \text{ from } F'_j\)
7: \(\textbf{else}\)
8: \(\text{queue}_j \leftarrow \text{queue}_j \cup \{U\}, \text{ remove } U \text{ from } F'_j, U_1 \leftarrow U_1 \cup \{U\}\)
9: \(\textbf{end if}\)
10: \(\textbf{end while}\)
11: \(\textbf{while exists } j \in C \text{ s.t. } |\text{queue}_j| < [x_j] \textbf{ do}\)
12: \(\text{Choose such } j, \text{ so that if } U' \text{ is the nearest } at \text{ most unit } \text{volume of facilities in } F'_j, y(U) \text{ is maximized}\) (split if necessary)
13: \(\textbf{if exists } U' \in U_1, U \cap U' \neq \emptyset \textbf{ then}\)
14: \(\text{queue}_j \leftarrow \text{queue}_j \cup \{U\}, F'_j \leftarrow \emptyset\)
15: \(\textbf{else if exists } U' \in U_2, U \cap U' \neq \emptyset \textbf{ then}\)
16: \(\text{queue}_j \leftarrow \text{queue}_j \cup \{U\}, F'_j \leftarrow \emptyset, n_U, \leftarrow n_U + 1, U_2 \leftarrow U_2 \cup \{U\}\) and set \(n_U = 1\)
17: \(\textbf{else}\)
18: \(\text{queue}_j \leftarrow \text{queue}_j \cup \{U\}, F'_j \leftarrow \emptyset, U_1 \leftarrow U_1 \cup \{U\}\)
19: \(\textbf{end if}\)
20: \(\textbf{end while}\)
21: \(\textbf{return } U_1, U_2, \{n_U : U \in U_2\}, \text{queue}_j = \{U_{j, l} : l = 1, \ldots, [x_j]\} \text{ for } j \in C\)

\(g(i_1) = g(i_2) = g(i_3) = i\). We note that by defining \(g^{-1} : \mathcal{F} \to 2^{\mathcal{F}'}\), \(g^{-1}(i) = \{i_1, i_2, i_3\} \subset \mathcal{F}'\) is the set of all copies of \(i\).

### C.1 The \(\ell_p\)-norm Base Case: RFT-\(k\)-Center\(_p\)

We first write the LP relaxation. Recall that \(\mathcal{P}_0\) is the feasible polytope with no constraints on facility locations or objective functions.

\[
\begin{align*}
\min: & \quad s \\
\text{s.t.} & \quad (u, x, y) \in \mathcal{P}_0 \\
& \quad \sum_{i \in \mathcal{F}} u_{ij} : d^p(i, j) \leq s \\
& \quad y(\mathcal{F}) \leq k.
\end{align*}
\]

Since \(R\) is known, \([\ell_p\text{-LP}_k]\) has an integral solution with objective at most \(\text{OPT}^p\), therefore with a slight abuse of notation, the optimal solution \((u, x, y)\) has objective at most \(\text{OPT}^p\). We run ALG-Bundle on \((u, x, y)\) and obtain the output \(U_1\) (full bundles), \(U_2\) (partial bundles), \(\{n_U : U \in U_2\}\) (profitable factors) and queues for every \(j \in C\). Define \(d_{\max}(j, S) = \max_{i \in S} d(i, j)\), and we have the following lemma.

**Lemma 10.** ([12]) Let \(U_{j, t}\) be the \(t\)-th bundle added to \(\text{queue}_j\), then if \(t \leq [x_j]\) and \(V_{j, t}\) is the \(t\)-th closest unit mass of facilities in \(F_j\) (split if necessary), we have \(d_{\max}(j, U_{j, t}) \leq 3d_{\max}(j, V_{j, t})\); if \(t = [x_j] + 1\), then \(d_{\max}(j, U_{j, t}) \leq 3R\).
Next we define the auxiliary LP for rounding.

$$\text{max: } \sum_{U \in \mathcal{U}_2} n_U \cdot z(U) + \sum_{j \in C} \sum_{U \in \text{queue}_j} 1 [U \in \mathcal{U}_1] \quad (\ell_p^{*}-\text{Aux}_k)$$

s.t. 
$$z(U) = 1 \quad \forall U \in \mathcal{U}_1$$
$$z(U) \leq 1 \quad \forall U \in \mathcal{U}_2$$
$$z(g^{-1}(i')) \leq 1 \quad \forall i' \in \mathcal{F}$$
$$z(\mathcal{F}) \leq k$$
$$z_i \geq 0 \quad \forall i \in \mathcal{F}'.$$

**Lemma 11.** $\ell_p^{*}-\text{Aux}_k$ has integral extreme points with optimum at least $m$.

**Proof.** We first recall that $g : \mathcal{F}' \to \mathcal{F}$ takes any split copy to its original location in $\mathcal{F}$, therefore, the constraints of $\ell_p^{*}-\text{Aux}_k$ contain two laminar families, $\mathcal{U}_1 \cup \mathcal{U}_2$ and $\{g^{-1}(i') : i' \in \mathcal{F} \cup \{\mathcal{F}'\}$. It is known that the extreme points of such an LP are integral. We also know that $y$ is feasible to this LP, and we only need to prove that the objective value corresponding to $y$ is at least $m$.

Fix $j$ and consider the contribution of $\text{queue}_j$ to the objective. The first $\lfloor x_j \rfloor$ bundles added to $\text{queue}_j$ are all full bundles, so they all contribute to the second sum. If $x_j \neq \lfloor x_j \rfloor$, then there is an additional bundle $U$ added. If $U$ is full, the contribution of $U$ in $\text{queue}_j$ is also included in the second sum; otherwise, the contribution $y(U)$ of the partial bundle $U$ is counted in the first sum, because in $\text{ALG-Bundle}$, whenever we add a partial bundle to $\text{queue}_j$, we make sure to increase the counter $n_U$ by 1.

Hence the objective associated to $y$ is at least

$$\sum_{j \in C} \sum_{U \in \text{queue}_j} y(U) \geq \sum_{j \in C} (\lfloor x_j \rfloor + (x_j - \lfloor x_j \rfloor)) = \sum_{j \in C} x_j - \lfloor x_j \rfloor,$$

where the first inequality is because in the second loop of $\text{ALG-Bundle}$, the partial bundles are sorted by their total mass in non-increasing order, hence any one added for $j$ has size at least $x_j - \lfloor x_j \rfloor$. The last inequality is due to $\mathcal{P}_0$.

Using Lemma 11, we simply compute the integral optimal solution $z^*$ and define the open facility set $F = \{g(i) : z^*_i = 1\}$.

**Theorem 5.** $F$ is a 9-approximation to $RFT-k\text{-Center}_p$.

**Proof.** Since $z^*$ has objective value at least $m$ according to Lemma 11 for any $j$ we assign all the open facilities in $\text{queue}_j$ to it, and the total number of connections we make is at least $m$.

For any $j \in C$, consider all the bundles in its queue. The first $\lfloor x_j \rfloor$ bundles are full, and by definition of $\ell_p^{*}-\text{Aux}_k$ there is a distinct open facility in each of them. It is possible that there exists another open facility in the last bundle. Using Lemma 10 the sum of $p$-th powers of all the distances is at most

$$\sum_{i=1}^{\lfloor x_j \rfloor} (3d_{\max}(j, V_{j,i}))^p + (3R)^p,$$

and via using a similar argument as Lemma 6 of [10], the first $(\lfloor x_j \rfloor - 1)$ values in the above can be bounded by the average $p$-th power of the unit mass following it, and the $\lfloor x_j \rfloor$-th term is bounded using $R$, therefore the sum is at most

$$3^p \cdot \sum_{i=1}^{\lfloor x_j \rfloor - 1} \left( \sum_{i \in V_{j,i+1}} u_{ij} \cdot d_p(i, j) \right) + 2(3R)^p \leq 3^p \cdot \text{OPT}^p + 2 \cdot 3^p \cdot R^p,$$

and finally we take the $(1/p)$-th power and obtain

$$(3^p \cdot \text{OPT}^p + 2 \cdot 3^p \cdot R^p)^{1/p} \leq 3 \cdot \text{OPT} + 6R \leq 9 \cdot \text{OPT},$$

from the fact that the function $x^{1/p}$ is concave in $x$ for $p \geq 1$. \qed
C.2 The $\ell_p$-norm Matroid Case

As a first extension, we consider RFT-MatCenter. We address the issue between splitting facility locations and the original matroid $M = (F, I)$ with rank function $r$. After the split is over, we define a new matroid $M' = (F', I')$ as follows: $S \in F'$ is independent if and only if $g$ restricted to $S$ is injective and $g(S)$ is independent in $M$. In other words, if and only if $g$ maps $S$ to an independent set in $M$ and $S$ does not contain two copies of the same facility location in $F$. It is not hard to verify that $M'$ is a matroid, and we denote its rank function by $r'$.

We use the same process as in the previous section, and we only need to replace the last constraint in $\ell_p^{p-1}$-LP, $y(F) \leq k$ with $y(S) \leq r(S) \forall S \subset F$, and the second to last constraint in $\ell_p^{p-1}$-Aux, $z(F') \leq k$ with $z(S) \leq r'(S) \forall S \subset F'$. From the definition of $M'$ and its rank function $r'$, we notice the following inequalities for $y$ and any $S \subset F'$,

$$\sum_{i \in S} y_i \leq \sum_{i' \in g(S)} \sum_{i \in g^{-1}(i')} y_i = \sum_{i' \in g(S)} y_{i'},$$

where we abuse the notation for $i' \in F$ and use $y_{i'}$ to represent its value before split. By definition, we have $r'(S) = r(g(S))$, hence this LP is satisfied by $y$.

On the other hand, the constraints $z(g^{-1}(i')) \leq 1$ is actually subsumed by the matroid constraints, because for any $i' \in F$, we have $r'(g^{-1}(i')) = r(i') \leq 1$, so the constraints form a laminar and a matroid. According to Lemma 1, the LP has integral extreme points. We choose one optimal integral solution as $z^*$ and let $F = \{g(i) : i \in F', z^*_i = 1\}$. The following is a simple corollary of Theorem 5.

**Corollary 3.** $F$ is a 9-approximation to RFT-MatCenter.

C.3 The $\ell_p$-norm Knapsack Case

In this section, we consider RFT-KnapCenter. Using the same process, we replace the last constraint in $\ell_p^{p-1}$-LP, with the knapsack constraint $\sum_{i \in F} w_i \cdot y_i \leq W$, and the second to last constraint in $\ell_p^{p-1}$-Aux, with $\sum_{i \in F} w_i \cdot z_i \leq W$, where all copies of the same facility $i$ inherit the same weight $w_i$. The constraints in the auxiliary LP are now the intersection of two laminar families, namely $U_1 \cup U_2$ and $\{g^{-1}(i') : i' \in F\}$, as well as a knapsack constraint. We have to drop the constraints corresponding to the second laminar family, and only consider the remaining ones, which is known to produce extreme points that have at most two fractional entries.

We choose one such optimal solution $z^*$, let $F = \{g(i) : i \in F', z^*_i > 0\}$ and allow it to choose multiple copies of the same $i' \in F$, i.e., $F$ is a multi-set. The standard rounding step gives the following result.

**Lemma 12.** $F$ is a 9-approximation to RFT-KnapCenter, which has total weight at most $W + 2 \max_{i \in F} w_i$ and can place multiple facilities at the same location.

**Proof.** The proof of the approximation factor is the same as the $k$-cardinality case, thus omitted here. Since there are at most two fractional values in $z^*$, $F$ contains at most two such facilities, therefore the total weight of $F$ is at most $w(F) \leq \sum_{i=1}^{2} w_i + 2 \max_{i \in F} w_i \leq W + 2 \max_{i \in F} w_i$.

Finally, because we drop the constraints $z(g^{-1}(i')) \leq 1$, the resulting solution $z^*$ may violate these constraints and place multiple facilities at different copies of the same location $i'$.

**Corollary 4.** For any $\epsilon > 0$, there exists a $(10 + \epsilon)$-approximation algorithm for RFT-KnapCenter that violates the knapsack constraint by a multiplicative factor $\epsilon$, is allowed to open multiple facilities at the same location, and runs in time $n^{O(1/\epsilon)}$, where $n$ is the input size.

**Proof.** We use the standard guessing technique and try to guess exactly the set of facilities having weight $\geq \epsilon \cdot W$ in the optimal solution $F^*$, and define it as $F_0 = \{i \in F^* : w_i \geq \epsilon \cdot W\}$. It is obvious that $|F_0| \leq 1/\epsilon$, and guessing it takes at most $n^{1/\epsilon}$ possible tries. Assume that we now know $F_0$.

Define $F_{\epsilon} = \{i \in F : w_i < \epsilon \cdot W\}$, and we further roughly guess the optimum value $OPT$. More specifically, recall that the largest distance in the optimal solution $R$ is also known to us and $OPT \geq R$,
and on the other hand, we have \( \text{OPT} \leq (n \cdot R^p)^{1/p} = n^{1/p} \cdot R \), so it takes \( O(\log_1 + \varepsilon n) \) guesses to find \( U \in [\text{OPT}, (1 + \varepsilon)\text{OPT}] \).

Next, for any \( j \in C \) and consider the number of nearby pre-selected facilities, namely \( \{i \in F_0 : d(i, j) \leq R\} \). From near to far, connect \( j \) to as many of them as possible, as long as the connection number is at most \( r_j \), and the connection distances have \( \ell_p \)-norm at most \( U \). Reduce \( l_j, r_j \) and \( m \) accordingly.

In the optimal solution, the overall \( \ell_p \)-norm of connections between \( j \) and \( F_0, F_{\leq \varepsilon} \) is at most \( \text{OPT} \leq U \), therefore by prioritizing the facilities in \( F_0 \), the new instance also has optimal solution at most \( \text{OPT} \leq U \), largest connection at most \( R \) and the set of facilities is now \( F_{\leq \varepsilon} \). Using Lemma 12 we can obtain a \( 9 \)-approximation \( F_1 \) violating the knapsack constraint \( W - w(F_0) \) by at most \( 2 \max_{i \in F_{\leq \varepsilon}} w_i \leq 2 \epsilon \cdot W \), so the total weight is at most,

\[
w(F_0 \cup F_1) = w(F_0) + w(F_1) \leq w(F_0) + W - w(F_0) + 2 \epsilon \cdot W = (1 + 2 \epsilon)W,
\]

and the connection cost for every \( j \) is at most, by adding those in \( F_0 \) and \( F_{\leq \varepsilon} \),

\[
((9 \cdot \text{OPT})^p + U^p)^{1/p} \leq 9 \cdot \text{OPT} + (1 + \epsilon) \text{OPT} = (10 + \epsilon) \text{OPT}.
\]

\( \square \)

**D Fair Robust Fault-tolerant Knapsack/Matroid Center**

**Corollary 5.** For any fixed \( \varepsilon > 0 \), there is a \( 5 \)-approximation algorithm for \( \text{FairRFT-KnapCenter} \) that violates the knapsack constraint by a multiplicative factor \( \varepsilon \) and runs in time \( n^{O(1/\varepsilon)} \).

**Corollary 6.** ([11]) There is a polynomial-time \( 3 \)-approximation algorithm for \( \text{FairRFT-MatCenter} \) that violates the matroid rank function by at most one.

**D.1 A Proof Sketch for Corollary 5**

The algorithm for \( \text{FairRFT-kCenter} \) also provides a method for obtaining similar results for \( \text{FairRFT-KnapCenter} \), albeit still unable to avoid the slight violation of the knapsack constraint. Given some fixed \( \varepsilon > 0 \), the proof can be modified such that in an algorithm \( A' \) very similar to that of the core Lemma 5, \( A' \) either certifies that \( (\alpha, \mu) \in Q(R) \), or returns a set \( S \in F'\cup(5R) \) that separates \( (\alpha, \mu) \) from \( Q'(5R) \), where the definition of \( F' \) and \( Q' \) are the same as before, except that we change the knapsack constraint from \( W \) to \( (1 + \varepsilon)W \). The modified lemma is easier to prove this time, since we are willing to violate the knapsack constraint and no dynamic programming is needed. To be more specific, for any given \( (\alpha, \mu) \in Q'_{\geq 0} \times Q \), we no longer need to test the emptiness of \( P_{\mu, \varepsilon}^{\alpha, \mu} \) but \( P_{\mu, \varepsilon}^{\alpha, \mu} \) is good enough. Whenever \( P_{\mu, \varepsilon}^{\alpha, \mu} \) is non-empty, we can construct some \( S \in F'\cup(5R) \) following an almost identical procedure as Lemma 7 (this time we have the intersection of a laminar family with \( 2 \) knapsack constraints, hence we have at most \( 4 \) fractional variables); otherwise, the emptiness of \( P_{\mu, \varepsilon}^{\alpha, \mu} \subseteq P^{\alpha, \mu} \) is guaranteed, and we know that \( (\alpha, \mu) \in Q(R) \).

We also use the standard guessing technique to fix at most \( 1/\varepsilon \) open facilities that have weights \( \geq \varepsilon \cdot W \) and truncate the demands \( (l_j, r_j, \varepsilon_j) \) associated with some of the clients in \( C \), which puts the running time of the algorithm at \( n^{O(1/\varepsilon)} \). The remainder of this proof is standard, hence omitted here.

**D.2 A Proof Sketch for Corollary 6**

Consider the problem \( \text{FairRFT-MatCenter} \). The iterative rounding algorithm in [11] for \( \text{fair robust matroid center} \) can be easily generalized to solve \( \text{FairRFT-MatCenter} \), and we obtain a polynomial-time randomized algorithm that returns a solution \( S \subseteq F \) which satisfies

- \( S \) is the union of some basis of \( M \) with at most one extra facility location,
- The total amount of demand \( S \) can satisfy within radius \( 3R \) is at least \( m \), with \( R \) being the optimum,
• $\mathbb{E}_S[\text{range}_S(j, 3R, r_j)] \geq e_j$ for all $j \in C$.

Roughly speaking, in the rounding algorithm of [11], each cluster $F_j$ is rounded so that $y(F_j) \in [0, 1]$ in the end, and we only need to change it to $y(F_j) \in [[x_j], [x_j]]$, meanwhile defining the net gaining factor $c_j$ for each $j \in C'$ like those in Lemma [2] which are already returned by the filtering ALG-Filter.

Indeed, all the subroutines in [11] works perfectly to guarantee that $y(F_j) \in [[x_j], [x_j]]$ and $\mathbb{E}[y(F_j)] \geq x_j$. More specifically, by starting with $y'(F_j) = x_j \forall j \in C'$ and considering in priority order:

(I) When the loop rounds a cycle, any $y'(F_j)$ is unchanged;

(II) When the loop rounds a single path that starts from some $F_{j}, j \in C'$ and ends in some $O_i$, we see that $O_i$ cannot be one of the constraints that characterizes the face of matroid polytope tight at $y'$ and $F_j$ only has one fraction, hence the rounding is feasible and $y'(F_j)$ cannot decrease after the rounding, and it cannot exceed $[x_j]$ either;

(III) When the loop rounds two paths that both have ends in $C'$, because we consider the options in order and there are no cycles, all the ends have exactly one fractional facilities within it and all others in the same $F_j$ must be either 0 or 1. The expectation $\mathbb{E}[y'(F_j)]$ is unchanged due to the selection of parameters in stochastic rounding;

(IV) When the loop rounds a single path that have both ends in $C'$, we note that since it is possible that $y'(F_j) > 1$, the argument in [11] on matching between the left and right banks is no longer valid. Let the path chosen be $v_1, \ldots, v_{2l}$, and we change the rounding procedure to the following:

(i) If for any $t = 1, \ldots, l - 1$, we have $y'_{v_{2t}} + y'_{v_{2t+1}} \leq 1$, then by using the matching argument in [11] and adding at most 1 facility, the coverage requirement is satisfied with probability 1. Terminate after this step;

(ii) Otherwise, when there exists $t \in \{l - 1\}$ such that $y'_{v_{2t}} + y'_{v_{2t+1}} > 1$, then obviously $l \geq 2$ and we can break down the path into two sub-paths, namely $v^{(1)} = v_1, \ldots, v_{2t}$ and $v^{(2)} = v_{2t+1}, \ldots, v_{2l}$.

Using the same method in (III), we perform one step of stochastic rounding with possibly sign-reversed directions on $v^{(1)}$, $v^{(2)}$, but with a slight modification by adding an additional constraint to make sure that $y'_{v_{2t}} + y'_{v_{2t+1}} \in [1, 2]$ at all times (notice that the same inequality holds automatically elsewhere). After the stochastic rounding, either the number of fractional edges is reduced, or we have one more pair $y'_{v_{2t}} + y'_{v_{2t+1}} = 1$. It is easy to see that by repeating (III) or (IV), the algorithm ends in polynomial time, and the coverage guarantee is always satisfied.

**Example 1.** To illustrate our modification to the original algorithm, consider a single path $v_1, v_2, v_3, v_4$ which has both ends in $C'$, by letting $v_1 \in F_{j_1}, v_2, v_3 \in F_{j_2}, v_4 \in F_{j_3}$ with $c_{j_1} = 3, c_{j_2} = 9$ and $c_{j_3} = 21$. Suppose we currently have $y' = (1/3, 2/3, 1/3, 1/3)$, in other words, $y'_{v_1} = 1/3, y'_{v_2} = 2/3, y'_{v_3} = 2/3$ and $y'_{v_4} = 1/3$, with $v_1, v_2 \in O_1$ and $v_3, v_4 \in O_2$.

Now we know $y'_{v_2} + y'_{v_3} = 4/3 > 1$, so we break down the path into two sub-paths $v^{(1)} = v_1, v_2$ and $v^{(2)} = v_3, v_4$. The first path has $j_1, j_2$ as its two ends and $\Delta_1 = c_{j_2} - c_{j_1} = 6$, and the second path has $j_2, j_3$ as its two ends and $\Delta_2 = c_{j_3} - c_{j_2} = 12$, so the perturbation vector $\rho$ is defined as $\rho = (2, -2, -1, 1)$, in order for $y' + \delta\rho$ to still satisfy the coverage constraint. By guaranteeing $y'_{v_2} + y'_{v_3} \geq 1$, it is not hard to see that, $\delta_1 = 1/9$ with $y_1 = y' + \delta_1 = (5/9, 4/9, 5/9, 4/9)$ and $\delta_2 = 1/6$ with $y_2 = y' - \rho/6 = (0, 1, 5/6, 1/6)$. Therefore, we return $y_1$ with prob. $(1/6)/(1/9 + 1/6) = 3/5$, and return $y_2$ with prob. $2/5$.

At the same time, the coverage constraint is preserved in both cases, since

$$
\frac{1}{3} \cdot 3 + \left(\frac{2}{3} \cdot \frac{2}{3} + \frac{2}{3}\right) \cdot 9 + \frac{1}{3} \cdot 21 = 20,
$$

$$
\frac{5}{9} \cdot 3 + \left(\frac{4}{9} \cdot \frac{5}{9} + \frac{5}{9}\right) \cdot 9 + \frac{4}{9} \cdot 21 = 20,
$$

$$
0 \cdot 3 + \left(1 + \frac{5}{6}\right) \cdot 9 + \frac{1}{6} \cdot 21 = 20.
$$

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