Statistics of infinite dimensional random matrix ensembles

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1 Abstract

A quantum statistical system with energy dissipation is studied. Its statistics is governed by random complex-valued non-Hermitean Hamiltonians belonging to complex Ginibre ensemble. The eigenenergies are shown to form stable structure in thermodynamical limit (large matrix dimension limit). Analogy of Wigner and Dyson with system of electrical charges is drawn.

2 Summary

A complex quantum system with energy dissipation is considered. The quantum Hamiltonians $H$ belong to the complex Ginibre ensemble. The complex-valued eigenenergies $Z_i$ are random variables. The second differences $\Delta^1 Z_i$ are also complex-valued random variables. The second differences have their real and imaginary parts and also radii (moduli) and main arguments (angles). For $N=3$ dimensional Ginibre ensemble the distributions of above random variables are provided whereas for generic $N$-dimensional Ginibre ensemble second difference distribution is analytically calculated. The law of homogenization of eigenenergies is formulated. The analogy of Wigner and Dyson of Coulomb gas of electric charges is studied.

3 Introduction

We study generic quantum statistical systems with energy dissipation. The quantum Hamiltonian operator $\hat{H}$ is in given basis of Hilbert’s space a matrix with random elements $H_{ij}$ [1, 2, 3]. The Hamiltonian $\hat{H}$ is not hermitean operator, thus its eigenenergies $Z_i$ are complex-valued random variables. We assume that distribution of $H_{ij}$ is governed by Ginibre ensemble [1, 2, 4, 5]. $\hat{H}$ belongs to general linear Lie group $\text{GL}(N, \mathbb{C})$, where $N$ is dimension and $\mathbb{C}$ is complex numbers field. Since $\hat{H}$ is not hermitean, therefore quantum system is dissipative.
system. Ginibre ensemble of random matrices is one of many Gaussian Random Matrix en-
ssembles GRME. The above approach is an example of Random Matrix theory RMT \[1, 2, 3\].
The other RMT ensembles are for example Gaussian orthogonal ensemble GOE, unitary GUE, symplectic GSE, as well as circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE. The distributions of the eigenenergies \(Z_1, ..., Z_N\) for \(N \times N\) Hamiltonian matrices is given by Jean Ginibre’s formula \[1, 2, 4, 5\]:

\[
P(Z_1, ..., Z_N) = \prod_{j=1}^N \frac{1}{\pi \cdot j!} \cdot \prod_{i<j}^N |z_i - z_j|^2 \cdot \exp(-\sum_{j=1}^N |z_j|^2),
\]

where \(z_i\) are complex-valued sample points \((z_i \in \mathbb{C})\). For Ginibre ensemble we define complex-valued spacings \(\Delta^1 Z_i\) and second differences \(\Delta^2 Z_i\):

\[
\Delta^1 Z_i = Z_{i+1} - Z_i, i = 1, ..., (N - 1),
\]

\[
\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, i = 1, ..., (N - 2).
\]

The \(\Delta^2 Z_i\) are extensions of real-valued second differences

\[
\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, i = 1, ..., (N - 2),
\]

of adjacent ordered increasingly real-valued energies \(E_i\) defined for GOE, GUE, GSE, and Poisson ensemble PE (where Poisson ensemble is composed of uncorrelated randomly distributed eigenenergies) \[6, 7, 8, 9, 10, 11\].

There is an analogy of Coulomb gas of unit electric charges pointed out by Eugene Wigner and Freeman Dyson. A Coulomb gas of \(N\) unit charges moving on complex plane (Gauss’s plane) \(\mathbb{C}\) is considered. The vectors of positions of charges are \(z_i\) and potential energy of the system is:

\[
U(z_1, ..., z_N) = -\sum_{i<j} \ln |z_i - z_j| + \frac{1}{2} \sum_i |z_i|^2.
\]

If gas is in thermodynamical equilibrium at temperature \(T = \frac{1}{k_B} \) (\(\beta = \frac{1}{k_B T} = 2\), \(k_B\) is Boltzmann’s constant), then probability density function of vectors of positions is \(P(z_1, ..., z_N)\) Eq. (1). Complex eigenenergies \(Z_i\) of quantum system are analogous to vectors of positions of charges of Coulomb gas. Moreover, complex-valued spacings \(\Delta^1 Z_i\) are analogous to vectors of relative positions of electric charges. Finally, complex-valued second differences \(\Delta^2 Z_i\) are analogous to vectors of relative positions of vectors of relative positions of electric charges.

The \(\Delta^2 Z_i\) have their real parts \(\text{Re}\Delta^2 Z_i\), and imaginary parts \(\text{Im}\Delta^2 Z_i\), as well as radii (moduli) \(|\Delta^2 Z_i|\), and main arguments (angles) \(\text{Arg}\Delta^2 Z_i\).

4 Second Difference Distributions

We define following random variables for \(N=3\) dimensional Ginibre ensemble:

\[
Y_1 = \Delta^2 Z_1, A_1 = \text{Re} Y_1, B_1 = \text{Im} Y_1,
\]
\[ R_1 = |Y_1|, \Phi_1 = \text{Arg}Y_1, \] (7)

and for the generic \( N \)-dimensional Ginibre ensemble [12]: \( W_1 = \Delta^2 Z_1 \).

Their distributions for are given by following formulae [12]:

\[ f_Y(y_1) = f_{(A_1,B_1)}(a_1,b_1) = \frac{1}{576\pi}[(a_1^2 + b_1^2)^2 + 24] \cdot \exp\left(-\frac{1}{6}(a_1^2 + a_2^2)\right). \] (8)

\[ f_{A_1}(a_1) = \frac{\sqrt{6}}{576\sqrt{\pi}}(a_1^4 + 6a_1^2 + 51) \cdot \exp\left(-\frac{1}{6}a_1^2\right), \] (9)

\[ f_{B_1}(b_1) = \frac{\sqrt{6}}{576\sqrt{\pi}}(b_1^4 + 6b_1^2 + 51) \cdot \exp\left(-\frac{1}{6}b_1^2\right), \] (10)

\[ f_{R_1}(r_1) = \Theta(r_1) \frac{1}{288}r_1(r_1^4 + 24) \cdot \exp\left(-\frac{1}{6}r_1^2\right), \] (11)

\[ f_{\Phi_1}(\phi_1) = \frac{1}{2\pi}, \Phi_1 \in [0, 2\pi]. \]

\[ P_3(w_1) = \] (12)

\[ = \pi^{-3} \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} I_{j_1,j_2,j_3}(w_1), \]

\[ I_{j_1,j_2,j_3}(w_1) = \] (13)

\[ = 2^{-2j_2} \frac{\partial^{j_1+j_2+j_3}}{\partial^{j_1}\lambda_1\partial^{j_2}\lambda_2\partial^{j_3}\lambda_3} F(w_1, \lambda_1, \lambda_2, \lambda_3)|_{\lambda_i=0}, \]

\[ F(w_1, \lambda_1, \lambda_2, \lambda_3) = \] (14)

\[ = A(\lambda_1, \lambda_2, \lambda_3) \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2], \]

\[ A(\lambda_1, \lambda_2, \lambda_3) = \] (15)

\[ = \frac{(2\pi)^2}{(\lambda_1 + \lambda_2 - \frac{5}{4}) \cdot (\lambda_1 + \lambda_3 - \frac{5}{4}) \cdot (\lambda_1 - 1)^2}, \]

\[ B(\lambda_1, \lambda_2, \lambda_3) = \] (16)

\[ = (\lambda_1 - 1) \cdot \frac{2\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{2}}{2\lambda_1 + \lambda_2 + \lambda_3 - \frac{9}{2}}. \]
5 Conclusions

We compare second difference distributions for different ensembles by defining following dimensionless second differences:

\[ C_\beta = \frac{\Delta^2 E_1}{\langle S_\beta \rangle}, \quad (17) \]

\[ X_1 = \frac{A_1}{\langle R_1 \rangle}, \quad (18) \]

where \( \langle S_\beta \rangle \) are the mean values of spacings for GOE(3) (\( \beta = 1 \)), for GUE(3) (\( \beta = 2 \)), for GSE(3) (\( \beta = 4 \)), for PE (\( \beta = 0 \)) [6, 7, 8, 9, 10, 11], and \( \langle R_1 \rangle \) is mean value of radius \( R_1 \) for \( N=3 \) dimensional Ginibre ensemble [12].

On the basis of comparison of results for Gaussian ensembles, Poisson ensemble, and Ginibre ensemble we formulate homogenization law [6, 7, 8, 9, 10, 11, 12]: *Eigenenergy for Gaussian ensembles, for Poisson ensemble, and for Ginibre ensemble tend to be homogeneously distributed.* The second differences’ distributions assume global maxima at origin for above ensembles. For Coulomb gas the vectors of relative positions of vectors of relative positions of charges statistically vanish. It can be called stabilisation of structure of system of electric charges.

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