On the Cappell-Lee-Miller glueing theorem

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Abstract

We formulate a more conceptual interpretation of the Cappell-Lee-Miller glueing/splitting theorem in terms of asymptotic maps and asymptotic exact sequences. Additionally, we show this gluing result is equivalent to a Mayer-Vietoris-type long exact sequence. We also present applications to eigenvalue estimates, approximation of obstruction bundles and glueing of determinant line bundles arising frequently in gauge theory.

All these results are true in a slightly more general context than in [6]. We work with operators which differ from translation invariant ones by exponentially decaying terms.

Introduction

Consider the following set-up. We are given two oriented, Riemann manifolds $M_i(\infty), i = 1, 2$. $M_1(\infty)$ has a (metrically) cylindrical end $\mathbb{R}_+ \times N$ while $M_2(\infty)$ has a cylindrical end $\mathbb{R}_- \times N$. Here $N$ is a closed, compact oriented Riemann manifold, not necessarily connected. $\hat{E}_i \to M_i(\infty)$ are bundles equipped with inner products along their fibers and $\hat{D}_i : C^\infty(\hat{E}) \to C^\infty(\hat{E}_i)$ are self-adjoint Dirac-type operators which along the necks have the form

$$\hat{D}_i = J(\nabla_t - D)$$

where $D$ is a selfadjoint Dirac-type operator on the $E := E_1 |_N \cong \hat{E}_2 |_N$ and $J$ denotes the Clifford multiplication by $dt$. We want to emphasize that $D$ is independent of the longitudinal coordinate $t$ along the necks. Consider now two smooth self-adjoint endomorphisms $\hat{B}_i$ of $\hat{E}_i$ and along the necks set

$$B_i(t) := \hat{B}_i |_{t \times N}, \quad A_i(t) := JB_i(t).$$

We assume the following about $A_i(t)$.

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• $A_i(t)$ anti-commutes with $J$

$$\{J, A_i(t)\} = JA_i(t) + A_i(t)J = 0.$$  

• There exist $C, \lambda > 0$ such that

$$\sup\{|\hat{A}_i(x)|; \ x \in [t, t + 1] \times N\} \leq C \exp(-\lambda|t|).$$  \hspace{1cm} (0.1)$$

Consider a smooth, decreasing, cut-off function $\eta: \mathbb{R}_+ \to [0, 1]$ such that

$$\eta(t) \equiv 1, \ t \in [0, 1/4]$$

$$\eta(t) \equiv 0, \ t \geq 3/4$$

and

$$\left|\frac{d\eta}{dt}\right| \leq 4, \ \forall t \geq 0.$$  

For each $r > 0$ set $\eta_r(t) := \eta(t - r)$. Now extend $\eta_r$ by symmetry to a function on $\mathbb{R}$ still denoted by $\eta_r$. We can regard $\eta_r(t), t \geq 0$, as a smooth function on $M_1(\infty)$ and $\eta_r(t), t \leq 0$, as a smooth function on $M_2(\infty)$ so we can form

$$\hat{D}_{i,r} := \hat{D}_i + \eta_r \hat{B}_i.$$  

Also define

$$\hat{D}_{i,\infty} := \hat{D}_i + \hat{B}_i.$$  

Note that via the obvious linear increasing diffeomorphism

$$\iota_r : [r + 1, r + 2] \to [-r - 2, -r - 1]$$

we have identifications

$$D_{1,r}|_{C(r+1)} = D_{2,r}|_{C(r-2)}, \ C(t) := [t, t + 1] \times N.$$  

Denote by $M_1(r)$ the manifold obtained from $M_1(\infty)$ by chopping off the cylinder $(r + 2, \infty) \times N$ and by $M_2(r)$ the manifold $M_2(\infty) \setminus (-\infty, -r - 2) \times N$. We glue $M_1(r)$ to $M_2(r)$ via $\iota_r$ to obtain a closed manifold $M(r)$; see Figure 1. Similarly the operators $\hat{D}_{i,r}$ can be glued together to produce a Dirac type operator $\hat{D}_r$ on the obvious glued bundle $\hat{E}_r \to M(r)$.

The operators $\hat{D}_i$ may have additional symmetries. We will be particularly interested in super-symmetric operators. This means the bundles $\hat{E}_i$ are equipped with orthogonal (unitary) decompositions

$$\hat{E}_i = \hat{E}_i^+ \oplus \hat{E}_i^-$$  \hspace{1cm} (0.2)$$

which determine the chiral operators $\hat{C}_i := \hat{P}_i^+ - \hat{P}_i^-$ where $\hat{P}_i^\pm$ denotes the orthogonal projection $\hat{E}_i \to \hat{E}_i^\pm$. The Dirac operator $\hat{D}_i$ is said to be super-symmetric if

$$\{C_i, \hat{D}_i\} = 0.$$  \hspace{1cm} (0.3)$$
Figure 1: Glueing two manifolds with cylindrical ends.

Equivalently, in terms of the splitting (0.2) it has the block decomposition

$$\hat{D}_i = \begin{bmatrix} 0 & \hat{D}_i^* \\ \hat{D}_i & 0 \end{bmatrix}$$

where $\hat{D}_i$ is a first order elliptic operator $C^\infty(\hat{E}_i^+) \to C^\infty(\hat{E}_i^-)$. The condition (0.3) implies that for any 1-form $\alpha$ on $M_i(\infty)$ the Clifford multiplication by $\alpha$ anti-commutes with $\hat{C}_i$

$$\{\hat{c}(\hat{\alpha}), \hat{C}_i\} = 0.$$  \hspace{1cm} (0.4)

Note that along the neck the operator $\hat{D}_i$ has the form

$$\hat{D}_i = G(\nabla_t - D)$$

where $G : E^+ \to E^-$ is the bundle isomorphism given by the Clifford multiplication by $dt$ and $D : C^\infty(E_i^+) \to C^\infty(E_i^+)$ is a self-adjoint, Dirac-type operator.

We will further assume that the two super-symmetries are compatible along the “boundary” $N$ i.e.

$$\hat{C}_1|_N = \hat{C}_2|_N =: C.$$

Thus the bundle $E$ is super-symmetric with chiral operator $C$. The conditions (0.3) and (0.4) imply that

$$[C, D] = CD - DC = 0.$$  \hspace{1cm} (0.5)
In this case we assume the perturbations $\hat{B}_i$ are compatible with the chiral operators in an obvious sense. Clearly the super-symmetry is transmitted to the glued bundle $\mathcal{E}(r)$ and the glued operator $\mathcal{D}_r$. The kernel $\mathcal{K}_r$ is naturally a finite dimensional $\mathbb{Z}_2$-graded space. In this paper we will address the following question.

**Main Problem** Understand the behavior of $\mathcal{K}_r$ as $r \to \infty$.

The kernel of an operator is a notoriously unstable object so it is unrealistic to be able to solve the Main Problem as stated. We need to “stabilize” $\mathcal{K}_r$ if we expect to say something of significance.

To formulate the main result we need to introduce some additional notions. We begin with the notions of asymptotic map and asymptotic exactness. An *asymptotic map* is a sequence $(U_r, V_r, f_r)_{r>0}$ with the following properties

(a) There exist Hilbert spaces $H_0$ and $H_1$ such that $U_r$ is a closed subspace of $H_0$ and $V_r$ is a closed subspace of $H_1$, $\forall r > 0$.

(b) $f_r$ is a densely defined linear map $f_r : U_r \to H_1$ with closed graph and range $R(f_r)$, $\forall r > 0$.

(c) $\lim_{r \to \infty} \hat{\delta}(R(f_r), V_r) = 0$ where, following [8], we set

$$
\hat{\delta}(U, V) = \sup\{\text{dist}(u, V) : u \in U, |u| = 1\}.
$$

We will denote asymptotic maps by $U_r \xrightarrow{f_r}^a V_r$. There is a super-version of this notion when $U_r$ and $V_r$ are $\mathbb{Z}_2$-graded and are closed subspaces in $\mathbb{Z}_2$-graded Hilbert spaces such that the natural inclusions are even.

The next result, proved in the Appendix, explains the motivation behind the above definition.

**Lemma 0.1** If

$$
\hat{\delta}(U, V) < 1
$$

then the orthogonal projection $P_V$ onto $V$ induces a one-to one map $U \to V$. If additionally

$$
\hat{\delta}(V, U) < 1
$$

then $P_V : U \to V$ is a linear isomorphism.

Define the gap between two closed subspaces $U, V$ in a Hilbert space $H$ by

$$
\delta(U, V) = \max(\hat{\delta}(U, V), \hat{\delta}(V, U)).
$$

The sequence of asymptotic maps

$$
U_r \xrightarrow{f_r}^a V_r \xrightarrow{g_r}^a W_r, \ r \to \infty
$$

is said to be *asymptotically exact* if

$$
\lim_{r \to \infty} \delta(R(f_r), \ker g_r) = 0.
$$

We have the following consequence of Lemma 0.1.
Lemma 0.2 If the sequence
\[ U_r \xrightarrow{f_r} V_r \xrightarrow{g_r} W_r, \quad r \to \infty \]
is asymptotically exact, \( P_r \) denotes the orthogonal projection onto \( \text{ker} \ g_r \) and \( Q_r \) the orthogonal projection onto \( W_r \) then there exists \( r_0 > 0 \) such that the sequence
\[ U_r \xrightarrow{P_r \circ f_r} V_r \xrightarrow{Q_r \circ g_r} W_r \]
is exact for all \( r > r_0 \).

If the spaces \( H_j \) are \( \mathbb{Z}_2 \)-graded \( H_j = H_j^+ \oplus H_j^- \) we say the sequence is \textit{super-symmetric} if the maps \( f_r \) and \( g_r \) are even i.e. are compatible with the splitting. In this case we get two asymptotically exact sequences
\[ U^\pm_r \to V^\pm_r \to W^\pm_r. \]

Next we need to introduce suitable functional spaces. For brevity we discuss only distributions on \( M_1(\infty) \). Define the \textit{extended} \( L^2 \)-space \( L^2_{\text{ex}}(\hat{E}_1) \) as the space of sections \( \hat{u} \in L^2_{\text{loc}}(\hat{E}_1) \) such that there exists \( u_\infty \in L^2(E) \) such that
\[ \hat{u} - \hat{u}_\infty \in L^2(\hat{E}_1). \]

Above, \( \hat{u}_\infty \) denotes the section in \( L^2_{\text{loc}}(\hat{E}_1) \) which is identically zero on \( M_1(0) \) and coincides with the translation invariant section \( u_\infty \) on the infinite cylinder \( \mathbb{R}_+ \times N \). \( u_\infty \) is uniquely determined by \( \hat{u} \) and thus we get well defined map
\[ T_\infty : L^2_{\text{ex}}(\hat{E}_1) \ni \hat{u} \mapsto u_\infty \in L^2(E) \]
called \textit{asymptotic limit (trace)} map. \( L^2_{\text{ex}}(\hat{E}_1) \) is naturally equipped with a norm
\[ \| \hat{u} \|^2 = \| \hat{u} - \hat{u}_\infty \|^2_{L^2(\hat{E}_1)} + \| u_\infty \|^2_{L^2(E)}. \]

Clearly \( L^2_{\text{ex}}(\hat{E}_1) \) with the above norm is a Hilbert space and we have a short exact sequence
\[ 0 \to L^2(\hat{E}_1) \to L^2_{\text{ex}}(\hat{E}_1) \xrightarrow{T_\infty} L^2(E) \to 0. \]

The map
\[ L^2(E) \ni u_\infty \mapsto \hat{u}_\infty \in L^2_{\text{ex}}(\hat{E}_1) \]
defines a splitting of this sequence.

Define the space of extended \( L^2 \)-solutions of \( \hat{D}_{i,\infty} \) as
\[ K_i := \ker \hat{D}_{i,\infty} \cap L^2_{\text{ex}}(\hat{E}_i). \]
The results of [2] and [9] show that these are finite dimensional spaces and the spaces of asymptotic traces $L_i := T_\infty(K_i)$ are subspaces in $\mathcal{H} := \ker D \subset L^2(E)$. We have a difference map

$$\Delta : K_1 \oplus K_2 \to \mathcal{H}, \quad \hat{u}_1 \oplus \hat{u}_2 \mapsto T_\infty(\hat{u}_1) - T_\infty(\hat{u}_2).$$

We denote its kernel by $K_\infty$. It is a finite dimensional subspace of $L^2_{ex}(\hat{E}_1) \oplus L^2_{ex}(\hat{E}_2)$.

Finally we define the splitting map

$$S_r : C^\infty(\mathcal{E}_r) \to L^2_{ex}(\hat{E}_1) \oplus L^2_{ex}(\hat{E}_2)$$

by $\psi \mapsto S^1_r \psi \oplus S^2_r \psi$ where

$$S^1_r \psi = \psi \text{ on } M_1(r) \subset M_1(\infty)$$

and

$$(S^1_r \psi)(t, x) = \psi(r, x), \quad \forall t \geq r, x \in N.$$

$S^2_r$ is defined similarly. We can now formulate the main result of this paper. When $\hat{B}_i \equiv 0$, i.e. the operators $\hat{D}_i, \infty$ are translation invariant along the neck, this result was proved by Cappell-Lee-Miller in [6] and is implicitly contained in [14]. The super-symmetric situation was discussed in [10] in the special case of anti-selfduality operators.

**Main Theorem** Fix a positive real number $\delta$ such that

$$\delta < \min(\gamma, \lambda)$$

where $\gamma$ is the smallest positive eigenvalue of $D$. For every function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$c(r) = o(1/r), \quad c(r) \geq C e^{-\delta r} \text{ as } r \to \infty$$

denote by $\tilde{K}_r(c)$ the subspace $L^2(\mathcal{E}_r)$ of spanned by the eigenvectors of $D_r$ corresponding to eigenvalues $|\lambda| \leq c(r)$. Then the following hold.

(a) The splitting maps define an asymptotic map

$$\tilde{K}_r(c(r)) \xrightarrow{\alpha} K_1 \oplus K_2.$$

(b) The sequence

$$0 \to \tilde{K}_r(c(r)) \xrightarrow{S_r} K_1 \oplus K_2 \xrightarrow{\Delta} H \to H \to H/(L_1 + L_2) \to 0$$

is asymptotically exact. Furthermore, if all the operators involved are super-symmetric, the above sequence is super-symmetric as well.

In the course of the proof we will construct an asymptotic inverse $\Psi_r$ to the splitting map which we call the **glueing map**. This is an asymptotic map $K_\infty \to \ldots$
such if \( P_r \) denotes the orthogonal projection onto \( \tilde{K}_r \) and \( P_\infty \) the orthogonal projection onto \( K_\infty \) then

\[
\|(P_\infty S_r) \circ (P_r \Psi_r) - \text{id}_{K_\infty}\| + \|(P_r \Psi_r) \circ (P_\infty S_r) - \text{id}_{\tilde{K}_r}\| = o(1) \quad \text{as} \quad r \to \infty.
\]

The rest of the paper is occupied with the proof and applications of the main theorem. It is divided as follow. In Section 1 we list some basic analytical facts about elliptic equations on manifolds with cylindrical ends. We mention in particular the **Key Estimate** which adds a bit of compactness to the situation. Its proof is deferred to the Appendix. Section 2 contains the proof of the Main Theorem itself. The strategy is similar to the approaches in [6] and [14] but the details are greatly simplified. Section 3 is devoted to a comparative study between the Main Theorem and the Mayer-Vietoris theorem for complexes of differential operators described in [1]. In the case of Dirac operators, we actually have a version of Poincaré lemma which follows from the existence results of [4]. To construct the connecting homomorphism we follow closely its description in the DeRham case contained in [5]. This leads to a natural asymptotic connecting morphism. The Main Theorem is equivalent with an asymptotic Mayer-Vietoris sequence; see (3.10).

In Section 4 we present two applications. The first one is concerned with small eigenvalues of selfadjoint elliptic operators on manifolds containing long necks of the type considered by W.Chen in [7]. He proved that if \( K_\infty = 0 \) then the operators \( D_r \) have no kernel for \( r \gg 0 \) the norms of their inverses are \( O(r) \). We consider next super-symmetric operators and we study what happens if the space \( K_\infty \) is purely even. We show that the component \( C^\infty(\mathcal{E}_r^+) \to C^\infty(\mathcal{E}_r^-) \) of \( D_r \) admits \( L^2 \)-bounded right inverses of norms \( O(r) \) as \( r \to \infty \). Such a situation is often encountered in gauge theoretic gluing problems over even dimensional manifolds.

The second application, also suggested by problems in gauge theory, has to do with families of operators. We describe ways to glue the indices of some families of elliptic problems.

We believe the asymptotic language will find applications in other problems involving adiabatic deformations. It is not difficult to introduce the notion of asymptotic (co)chain complexes and asymptotic cohomology. Many of the basic results in homological algebra have an asymptotic counterpart.

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1 First order elliptic equations on manifolds with cylindrical ends

In this section we will survey a few analytical facts which are needed in the proof of the Main Theorem. The adequate functional background will be that of the Sobolev spaces $L^{k,p}$ consisting of distributions $k$-times differentiable with derivatives in $L^p$.

For any $L^2_{loc}$ distribution $\hat{u} : t \mapsto u(t)$ on a cylinder $[0, L) \times N$ (where $L$ can be $\infty$) we denote by $\rho_t(\hat{u})$ the function $[0, L) \to \mathbb{R}^+$ defined by

$$t \mapsto \rho_t(\hat{u}) := \left( \int_{C(t)} |u|^2 d\text{vol} \right)^{1/2}, \quad C(t) = [t, t + 1] \times N.$$ 

Additionally, define

$$q : [0, L) \to [0, \infty], \quad t \mapsto q_{t, L}(\hat{u}) = \sup_{t < s < L} \rho_s(\hat{u}).$$

Note that if finite, $q_{t, L}$ is a decreasing function and thus belongs to $L^\infty_{loc}(0, L)$.

Now let us observe that the operator $J$ induces a symplectic structure on $L^2(E)$ defined by

$$\omega(u, v) := \int_N (Ju, v) d\text{vol}$$

The spectrum of $D$ is real and consists only of discrete eigenvalues with finite multiplicities. Set

$$\mathcal{H}_\mu := \ker(\mu - D).$$

and denote by $P_\mu$ the orthogonal projection onto $\mathcal{H}_\mu$. Since $\{J, D\} = 0$ we deduce $J\mathcal{H}_\mu = \mathcal{H}_{-\mu}$. The spectral gap of $D$ is the positive real number $\gamma = \gamma(D)$ defined as the smallest positive eigenvalue of $D$. Note that due to the spectral symmetry, $-\gamma(D)$ is also an eigenvalue of $D$. In particular, $\mathcal{H} = \mathcal{H}_0$ is $J$ invariant and thus has an induced symplectic structure. We have the following result (see [8], [11], [12]).

**Lemma 1.1** The spaces $L_i = T^\infty(K_i)$ of asymptotic traces of extended $L^2$ solutions are lagrangian subspaces of $\mathcal{H}$ i.e. $L_i^\perp = JL_i$. 

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In the super-symmetric case we have \( E = E^+ \oplus E^- \) and \( G^* G = 1_{E^+} \), \( GG^* = 1_{E^-} \)

\[
J = \begin{bmatrix}
0 & -G^* \\
G & 0
\end{bmatrix}.
\]

Then \( J(E^\pm) = JE^\mp \) and

\[
D = \begin{bmatrix}
D & 0 \\
0 & JDJ^{-1}
\end{bmatrix}.
\]

The space \( \mathcal{H} \) is \( \mathbb{Z}_2 \)-graded

\[ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \]

and \( G\mathcal{H}^+ = \mathcal{H}^- \). The asymptotic limit spaces \( L_i \) now have decompositions

\[ L_i = L_i^+ \oplus L_i^- \]

and the lagrangian condition translates into

\[
(L_i^+)\perp = G^* L_i^- \quad (L_i^-)\perp = GL_i^+ \quad (1.1)
\]

where \( \perp \) denotes the orthogonal complements in \( \mathcal{H}^\pm \).

Consider a cylinder \([0,L) \times N \). Denote by \( \hat{E} \) the pullback of \( E \to N \) to this cylinder and by \( \hat{D} \) the partial differential operator on \( C^\infty(\hat{E}) \)

\[
\hat{D} = \partial_t - D.
\]

For any eigenvalue \( \mu \) of \( D \) and any smooth section \( \hat{u} \) of \( \hat{E} \) define a new section \( \hat{u}_\mu \) by the condition

\[
\hat{u}_\mu |_{t \times N} = P_\mu u(t) \quad u(t) := \hat{u}|_{t \times N}.
\]

Clearly \( \hat{u}_\mu \) is a smooth section which we will regard as a smooth map

\[
\hat{u}_\mu = u_\mu(t) : [0, L) \to \mathcal{H}_\mu.
\]

Set

\[
u_\perp(t) = u(t) - u_0(t).
\]

**Proposition 1.2 (Key Estimate)** There exists a constant \( C > 0 \) depending (continuously) only on the geometry of \( N \) and the coefficients of \( D \) with the following property. For any smooth sections \( \hat{u}, \hat{f} \) of \( \hat{E} \) such that

\[
\hat{D}\hat{u} = \hat{f}
\]

and

\[
q_t(f) < \infty \quad (1.2)
\]
the following inequalities hold
\[
\|u_0(t) - u_0(t + n)\|_{L^2} \leq C \int_t^{t+n} q_s L(\hat{f}) ds, \quad \forall n \in \mathbb{Z} \cap [0, L - t) \tag{1.3}
\]
\[
\rho_{t+n}(\hat{u}^\perp) \leq C(e^{-\gamma n} \rho_t(\hat{u}^\perp) + e^{-\gamma n} \rho_{t+2n}(\hat{u}^\perp) + \frac{1}{\gamma} q_t L(\hat{f})), \quad \forall n \in \mathbb{Z} \cap [0, (L - t)/2). \tag{1.4}
\]

Above, \(\gamma\) denotes the spectral gap of \(D\).

We have the following immediate consequence whose proof is left to the reader.

**Corollary 1.3** Let \(L = \infty\) in the **Key Estimate**. If
\[
\hat{D} \hat{u} = \hat{f}
\]
where both \(\hat{u}\) and \(\hat{f}\) are smooth and satisfy
\[
\rho_t(\hat{u}) \in L^\infty(\mathbb{R}_+), \quad q_t(f) \in L^1(\mathbb{R}_+) \tag{1.5}
\]
then
\[
\hat{u} \in L^2_{ex}(\hat{E})
\]
and
\[
\|T_\infty \hat{u} - u_0(t)\|_{L^2} \leq C(\gamma^{-2} \int_t^\infty q_s(\hat{f}) ds + e^{-\gamma t} q_t(\hat{u})).
\]

Suppose \(\hat{A}\) is a smooth selfadjoint endomorphism of \(\hat{E} \to \mathbb{R}_+ \times N\) such that for some \(\lambda > 0\) we have
\[
\sup\{|A(t, x)| ; x \in N\} = O(e^{-\lambda t}). \tag{1.6}
\]
Set \(\hat{A}_r := \eta_r(t)\hat{A}\).

**Proposition 1.4** Suppose that we have a sequence of smooth sections \(\hat{u}_r\) satisfying the following conditions.
(a) There exists \(C > 0\) such that \(\rho_t(\hat{u}_r) < C\) for all \(t, r > 0\).
(b) The sections \(\hat{u}_r\) and their derivatives are uniformly bounded on \(C(0)\).
(c) There exists a sequence of smooth endomorphisms \(B_r\) of \(E\) such that
\[
m(r) := \sup\{|B_r(x)| ; x \in N\} = o(1/r) \quad \text{as } r \to \infty
\]
and \(\hat{D} - \hat{A}_r \hat{u}_r - B_r \hat{u}_r = 0\) on the cylinder \([0, r] \times N\).
(d) \(u_r(t) = u_r(r), \forall t \geq r \geq 0\).

Then a subsequence of \(\hat{u}_r\) converges **in the norm of** \(L^2_{ex}\) to a section \(\hat{u}\) satisfying \(\hat{D} - \hat{A} \hat{u} = 0\) on \(\mathbb{R}_+ \times N\). Moreover, on a subsequence
\[
u_r(r) \to T_\infty \hat{u} \quad \text{in the norm of } L^2(E). \tag{1.7}
\]
Proof In the sequel we will use the same symbol $C$ to denote positive constants independent of $t, r > 0$. Set $\hat{f}_r = A_r \hat{u}_r + B_r \hat{u}_r$. Then
\[ \hat{D} \hat{u}_r = \hat{f}_r \text{ on } [0, r] \times N. \] (1.8)
The conditions (1.6), (a) and (b) coupled with a standard bootstrap argument imply that there exists a constant $C > 0$ such that
\[ \sup \{ |\hat{u}_r(t, x)| ; (t, x) \in [0, r - 1] \times N \} \leq C, \quad \forall r > 0. \] (1.9)
This implies that a subsequence of $\hat{u}_r |_{[0, r] \times N}$ converges weakly in $L^2_{\text{loc}}$ to a section $\hat{u}$ defined over the entire cylinder. Clearly $B_r \hat{u}_r \to 0$ in $L^2_{\text{loc}}$ so that $\hat{u}$ is a weak solution of
\[ \hat{D} \hat{u} - \hat{A} \hat{u} = 0 \text{ on } \mathbb{R}_+ \times N. \]
We can now conclude via elliptic estimates that we can extract a subsequence which converges strongly in $L^2_{k,\text{loc}}$. Moreover, according to (1.9) we deduce $\hat{u} \in L^\infty$. If we now set $\hat{f} = A \hat{u}$ we deduce
\[ \rho_t(f) \leq \|\hat{u}\|_{\infty} \rho_t(\hat{A}) = O(e^{-\lambda t}). \]
Corollary 1.3 implies $\hat{u} \in L^2_{\text{ex}}$ and
\[ \|u(t) - T_\infty \hat{u}\|_2 \leq C(e^{-\gamma t} + e^{-\lambda t}). \] (1.10)
The Key Estimate for (1.8), where
\[ q_{t, r}(\hat{f}_r) \leq C(rm(r) + q_t(A_r)) \leq C(rm(r) + e^{-\lambda t}), \quad r \geq t \geq 0. \]
implies that for all $0 \leq t \leq r$ we have
\[ \|u_r(t) - u_r(r)\|_2 \leq C(rm(r) + e^{-\lambda t}). \] (1.11)
This proves (1.7) since $rm(r) = o(1)$. To show that the convergence $\hat{u}_r \to \hat{u}$ also takes place in the norm of $L^2_{\text{ex}}$ we only need to establish that on a subsequence
\[ \lim_{r \to \infty} \int_0^\infty dt \int_N \| (u_r(t) - u(t) - (u_r(r) - T_\infty \hat{u}))^2 dvol \to 0. \]
We extract the subsequence using the following argument. For every $n > 0$ pick $r = r_n > n$ such that the following inequalities hold.
\[ \int_0^n dt \int_N |u_r(t) - u(t)|^2 dvol \leq \frac{1}{n^2} \] (1.12)
\[ \int_N |u_{r_n}(n) - u_{r_n}(r_n)|^2 dvol \leq \frac{1}{n^2} \] (1.13)
\[ \int_n^\infty dt \int_N |u(t) - T_\infty \hat{u}|^2 dvol \leq \frac{1}{n^2} \] (1.14)
The choice (1.12) is possible because the sequence $\hat{u}_r$ converges to $\hat{u}$ in the norm $L^2([0, n] \times N)$. The choice (1.13) is possible because $rm(r) = o(1)$ and (1.11). Finally, the choice (1.14) is possible because of (1.10). The subsequence $\hat{u}_{r_n}$ chosen as above converges to $\hat{u}$ in the norm of $L^2_{\text{ex}}$. $\square$
2 Proof of the Main Theorem

To show that \( \lim_{r \to \infty} \delta(S_r(\tilde{K}_r), K_\infty) = 0 \) we will use the following elementary result which follows immediately from Lemma 1.1.

**Lemma 2.1** Suppose \( U \) is a finite dimensional subspace in a Hilbert space and \( U_r \) is a sequence of finite dimensional subspaces such that

\[
\lim_{r \to \infty} \hat{\delta}(U_r, U) = 0. \tag{2.1}
\]

and

\[
\liminf \dim U_r \geq \dim U. \tag{2.2}
\]

Then

\[
\lim_{r \to \infty} \delta(U_r, U) = 0.
\]

We will show that the two assumptions in the lemma are satisfied if \( U_r = S_r \tilde{K}_r \) and \( U = K_\infty \). The proof of the Main Theorem is thus divided in two steps.

**Step 1**

\[
\lim_{r \to \infty} \delta(S_r(\tilde{K}_r), K_\infty) = 0.
\]

We argue by contradiction. Thus we assume there exists a sequence \( \psi_r \in \tilde{K}_r \) such that

\[
\|S_r \psi_r\|_{ex} = O(1) \text{ as } r \to \infty \tag{2.3}
\]

and there exists \( d_0 > 0 \) such that

\[
\text{dist}(S_r \psi_r, K_\infty) > d_0, \; \forall r > 0. \tag{2.4}
\]

Set \( \psi_i^r := S_r^i \psi_r, \; i = 1, 2 \). We study only the behavior of \( \psi_1^r \). The sequence \( \psi_2^r \) behaves similarly. The condition (2.4) shows there exists a constant \( c > 0 \) such that

\[
\|\psi_1^r\|_{ex} \geq c, \; \forall r > 0.
\]

Thus we can normalize \( \psi_1^r \) so that \( \|\psi_1^r\|_{ex} = 1 \) and (2.4) continues to hold (with an eventually smaller \( d_0 > 0 \)).

Note first that using standard elliptic estimates and (2.3) we deduce that \( \psi_2^r \) and its derivatives are uniformly bounded on \( M_1(0) \). Thus a subsequence of \( \psi_1^r \) converges to a solution of \( \hat{D}_1,\infty \hat{u} = 0 \) on \( M_1(0) \). Using Proposition 1.4 we deduce that a further subsequence of the restriction of \( \psi_1^r \) to \( \mathbb{R}_+ \times N \) converges in the norm of \( L^2_{ex} \) to a solution of \( \hat{D}_1,\infty \hat{\psi} = 0 \) on this semi-infinite cylinder. Clearly we have produced a section \( \psi_1^\infty \in \ker \hat{D}_1,\infty \cap L^2_{ex} \) of norm 1. We proceed similarly with \( \psi_2^r \).

We now have a pair

\[
\Psi := \psi_1^\infty \oplus \psi_2^\infty \in K_1 \oplus K_2
\]
of norm 2 which according to (1.7) in Proposition 1.4 have the same asymptotic limit. Thus \( \Psi \in \mathcal{K}_\infty \). However, this contradicts (1.4). Step 1 is completed.

**Step 2** We will prove that

\[
\dim \mathcal{K}_\infty \leq \dim S_r \tilde{K}_r \quad \forall r \gg 0.
\]

We will rely on the following auxiliary result.

**Lemma 2.2** Suppose \( u \in L^{1,2}(\mathcal{E}_r) \) is such that

\[
\|D_r u\|_2 < (1 - \varepsilon)c(r)\|u\|_2.
\]

Then \( \text{dist} (u, \tilde{K}_r(c(r))) < (1 - \varepsilon)\|u\|_2 \).

**Proof of the lemma** Using the orthogonal decomposition

\[
L^2(\mathcal{E}_r) = \tilde{K}_r c(r) \oplus \tilde{K}_r(c(r))^\perp
\]

we can write

\[
u = v + v^\perp.
\]

Then \( \text{dist} (u, \tilde{K}_r) = \|v^\perp\|_2 \). On the other hand

\[
(1 - \varepsilon)^2c(r)^2\|u\|^2 > \|D_r u\|^2 \geq \|D_r v^\perp\|^2 \geq \Lambda^2\|v^\perp\|^2
\]

where \( \Lambda^2 > c(r)^2 \). The lemma is proved. \( \square \)

To conclude the proof of Step 2 we will construct for \( r \gg 0 \) a space \( V_r \subset L^2(\mathcal{E}_r) \) isomorphic to \( \mathcal{K}_\infty \) such that

\[
\hat{\delta}(V_r, \tilde{K}_r) < 1. \quad (2.5)
\]

According to Lemma 2.3, Chap. IV, §2 in [8] this means that the orthogonal projection onto \( \tilde{K}_r \) induces an injection \( V_r \to \tilde{K}_r \) so that

\[
\dim \mathcal{K}_\infty = \dim V_r \leq \dim \tilde{K}_r, \quad \forall r \gg 0.
\]

The condition (2.5) is satisfied provided \( \text{dist} (v, \tilde{K}_r) < v \), for all \( v \in V_r \setminus \{0\} \). According to Lemma 2.2 suffices to construct a subspace \( V_r \subset L^{1,2}(\mathcal{E}_r) \) isomorphic to \( \mathcal{K}_\infty \) such that

\[
\sup_{v \in V_r \setminus \{0\}} \frac{\|D_r v\|_2}{\|v\|_2} < c(r). \quad (2.6)
\]

Such a subspace is obtained via a simple glueing construction.

We construct a glueing map

\[
\Psi_r : \mathcal{K}_\infty \to L^{1,2}(\mathcal{E}_r), \quad \hat{u}_1 \oplus \hat{u}_2 \mapsto \Psi_r
\]

\[13\]
uniquely determined by the following conditions. Let \( u_\infty \) denote the common asymptotic limit of \( \hat{u}_i \). Now set
\[
\hat{v}_1 = \eta_r(t)\hat{u}_1 + (1 - \eta_r(t))u_\infty.
\]
Define \( \hat{v}_2 \) similarly. Clearly on the overlap
\[
\iota_r : [r + 1, r + 2] \times N \to [-r - 2, -r - 1] \times N
\]
we have
\[
\hat{v}_1 = \hat{v}_2 = u_\infty
\]
so we can glue these two sections on the overlap to produce a smooth section \( \Psi_r \in C^\infty(E_r) \). Clearly the map \( \Psi_r \) is linear. Set \( V_r := \Psi_r(K_r) \).

Note that \( \Psi_r \) is injective because if \( \Psi_r(\hat{u}_1, \hat{u}_2) \equiv 0 \) then both \( \hat{u}_i \) must vanish on \( M_1(0) \) and by unique continuation they must vanish everywhere. We claim \( V_r \) satisfies (2.4).

Clearly \( \mathcal{D}_r \Psi_r \equiv 0 \) on \( M_1(r - 1), M_2(-r + 1) \subset M(r) \) so we only need an estimate of \( \mathcal{D}_{1,r}\hat{v}_1 \) on the cylinder \([r - 1, r + 2] \times N\) and a similar one for \( \mathcal{D}_{2,r}\hat{v}_2 \).

On this cylinder we have \( \mathcal{D}_{1,r} = J(\partial_t - D - \eta_r\hat{A}_1) \) so that we have
\[
- J\mathcal{D}_{1,r}\hat{v}_1 = (\hat{D} - \eta_r\hat{A}_1)(\eta_r\hat{u}_1 + (1 - \eta_r)u_\infty)
\]
\[
= (\hat{D} - \hat{A}_1)(\eta_r\hat{u}_1 + (1 - \eta_r)u_\infty)
\]
\[
+ (1 - \eta_r)\hat{A}_1(\eta_r\hat{u}_1 + (1 - \eta_r)u_\infty).
\]

We examine the two terms separately. The first one can be rewritten as
\[
(\hat{D} - \hat{A}_1)(\eta_r\hat{u}_1 + (1 - \eta_r)u_\infty) = |\hat{D} - \hat{A}_1, \eta_r|\hat{u}_1 + |\hat{D} - \hat{A}_1, (1 - \eta_r)|u_\infty
\]
\[
+ \eta_r(\hat{D} - \hat{A}_1)\hat{u}_1 + (1 - \eta_r)(\hat{D} - \hat{A}_1)u_\infty
\]
\[
(\hat{=}\text{Clifford multiplication on } \mathbb{R}_+ \times N)
\]
\[
= \hat{c}(d\eta_r)(u_1(t) - u_\infty) - (1 - \eta_r)\hat{A}_1u_\infty.
\]

Thus
\[
- J\mathcal{D}_{1,r}\hat{v}_1 = \hat{c}(d\eta_r)(u_1(t) - u_\infty) + \eta_r(1 - \eta_r)\hat{A}_1(\hat{u}_1 - u_\infty).
\]

We can now use Corollary 1.3 for the equation
\[
\mathcal{D}\hat{u}_1 = \hat{f} = \hat{A}_1\hat{u}_1 \text{ on } \mathbb{R}_+ \times N.
\]

The decay rate of \( A_1(t) \) shows that \( \rho_t(\hat{f}) = O(e^{-\lambda t}\rho_t(\hat{u}_1)) \). Hence
\[
\|\mathcal{D}_r\hat{v}_1\|_{2,[r-1, r+2] \times N} \leq C(e^{-\lambda r} + e^{-\gamma r})q_1(\hat{v}_1). \tag{2.7}
\]

(2.7) implies (2.6) since \( q_1(u_1) \leq C\|\Psi_r\| \) (for \( r \gg 0 \)) and
\[
c(r) > C(e^{-\lambda r} + e^{-\gamma r}), \forall r \gg 0.
\]

The Main Theorem is proved \( \Box \)
Remark 2.3 The exponential decay condition (0.1) on the coefficients of $\hat{D}_{i,\infty}$ can be replaced by a milder one

$$\sup\{|\hat{A}_i(\hat{x})|; \hat{x} \in [t, t+1] \times N\} \leq C t^{-p}$$

where $p > 2$. The statement of the Main Theorem changes in an obvious way to take this decay into account.

Remark 2.4 We can rewrite the conclusion of the Main Theorem as a short asymptotically exact sequence

$$0 \to \tilde{K}_r \xrightarrow{S_r} K_1 \oplus K_2 \xrightarrow{\Delta} L_1 \oplus L_2 \to 0.$$  

The gluing map $\Psi_r$ is an asymptotic splitting of this sequence, in the sense described in the introduction.

Remark 2.5 The Main Theorem extends easily to families of operators. Suppose $X$ is a compact CW-complex and all the constructions in the introduction depend continuously on the parameter $x \in X$ such that the spectral gaps of the boundary operators $D_x$ are bounded from below

$$\gamma_0 := \inf_{x \in X} \gamma(D_x) > 0.$$  

Then $h(x) := \dim H_x$ is independent of $x$. We denote this common dimension by $h$. Assume also that the functions

$$\kappa_i : X \to \mathbb{Z}, \quad \kappa_i(x) := \dim K_i(x) \quad (i = 1, 2)$$

$$\ell : X \to \mathbb{Z}, \quad \ell(x) = \dim L_1(x) \cap L_2(x)$$

are constant, $k_i(x) \equiv \kappa_i, \ell(x) \equiv \ell$. One then can show that $K_i(x)$ and $L_1(x) \cap L_2(x)$ depend continuously upon $x$ in the gap topology. Thus they can be viewed as continuous maps in Grassmannians of finite dimensional subspaces in Hilbert spaces and as such they define vector bundles over $X$. The Main Theorem for families states that for $r \gg 0$ the spaces $\tilde{K}_{r,x}(c)$ form a vector bundle over $X$ and we have an exact sequence of vector bundles

$$0 \to \tilde{K}_r \xrightarrow{\Gamma} K_1 \oplus K_2 \xrightarrow{\Delta} H \to H/(L_1 + L_2) \to 0.$$  

Since $L_1, L_2$ are lagrangian then

$$H/(L_1 + L_2) \cong (L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp = J(L_1 \cap L_2).$$

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A similar statement is true in the super-symmetric case. In \[13\] we describe a general gluing formula for index of families when the functions \(h(x), \ell(x)\) and \(\kappa_i(x)\) are not necessarily constant.

In practice, one often encounters a fibered version of the problem. Suppose \(U, V\) and \(Z\) are compact CW complexes and \(\zeta_U : U \to Z, \zeta_V : V \to Z\) are continuous maps. We think of \(U\) as a parameter space for a continuous family of operators \(\hat{D}_1, \infty(u) = J(\partial_t - D_u - A_{1,u})\) on \(\hat{E}_1\) while \(V\) is a parameter space for a continuous family of operators \(\hat{D}_2, \infty(v) = J(\partial_t - D_v - A_{2,v})\) on \(\hat{E}_2\). \(Z\) parameterizes a continuous family of elliptic operators \(D_z\) on \(E\) such that \(D_z = D_u = z = D_v\) if \(z = \zeta(u) = \zeta(v)\). Form the fiber product

\[X = \{(u, v) \in U \times V ; \zeta(u) = \zeta(v)\} = \zeta^{-1}(\Delta_Z)\]

where \(\Delta_Z\) is the diagonal \(\{(z, z) \in Z \times Z ; z \in Z\}\). For each \(x = (u, v) \in X\) we set \(\hat{D}_{1,\infty}(x) := D_{2,\infty}(x)\). It is precisely this fibered context used in \[10\] to establish the super-symmetric version of the Main Theorem for families of anti-selfduality operators.

### 3 A Mayer-Vietoris interpretation

The Main Theorem has a Mayer-Vietoris flavor. One can formulate a genuine Mayer-Vietoris theorem as follows. Consider the super-symmetric Dirac operator \(D_r\) on \(\mathcal{E}_r\). For simplicity we drop the subscript \(r\) and we assume the operators \(\hat{D}_i, \infty\) are actually translation invariant. The results below do hold for exponentially decaying perturbations as well. The operator \(D = D_0\) has a block decomposition

\[
D = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix}
\]

Consider now the sheaf \(S\) on \(M = M(r)\) defined by

\[S(u) := \{u \in C^\infty(\mathcal{E}^+ |_U) ; D^* u = 0\}\]

for any open \(U \subset M\). Denote by \(\Gamma^\pm\) the sheaves of locally defined smooth sections of \(\mathcal{E}^\pm\). Then the sequence

\[0 \to S \to \Gamma^+ \xrightarrow{D} \Gamma^- \to 0 \quad (3.8)\]

is a fine resolution of \(S\). Indeed, the exactness at \(\Gamma^+\) is tautological. The exactness at \(\Gamma^-\) is equivalent to local existence of \(D^* u = f, \forall f \in \Gamma^-\). This can be proved easily.
using the existence and regularity results in [4]. Thus the Cech cohomology of the
sheaf $M$ on $M$ is given by the complex

$$(\Gamma, D) \quad 0 \to C^\infty(\mathcal{E}^+) \xrightarrow{\mathcal{D}} C^\infty(\mathcal{E}^-) \to 0$$

so that

$$H^0(M, S) \cong \ker \mathcal{D}, \quad H^1(M, S) \cong \mathrm{coker} \mathcal{D}, \quad H^k(M, S) = 0, \quad \forall k \geq 2. \quad (3.9)$$

In particular,

$$\chi(M, S) = \text{ind}(\mathcal{D}).$$

**Remark 3.1** The same techniques in [11] can be used to show that the element in
$K$-theory defined by the symbol of $\mathcal{D}$ is completely determined by the sheaf $S$. This
implies the index of $\mathcal{D}$ is completely determined by this sheaf. The equality (3.9)
explains how.

Denote by $U_i$ an open neighborhood of $M_i(r) \subset M(r)$, slightly larger that $M_i(r)$.
As in [1] or [5] one can show that we have a short Mayer-Vietoris exact sequence of
co-chain complexes

$$0 \to (\Gamma(M), \mathcal{D}) \to (\Gamma(U_1), \mathcal{D}) \oplus (\Gamma(U_2), \mathcal{D}) \xrightarrow{\Delta} \Gamma(U_1 \cap U_2) \to 0$$

from which we get a long Mayer-Vietoris sequence

$$0 \to \ker \mathcal{D} \to \mathcal{S}(U_1) \oplus \mathcal{S}(U_2) \to \mathcal{S}(U_1 \cap U_2) \xrightarrow{\partial} \mathcal{H}^1(U_1, \mathcal{S}) \oplus \mathcal{H}^1(U_2, \mathcal{S}) \to \mathcal{H}^1(U_1 \cap U_2, \mathcal{S}) \to 0.$$  

Imitating [5], we can provide quite an explicit description of the connecting morphism.
Choose a partition of unity subordinated to the cover $\{U_1, U_2\}$. Denote it by \{\omega_i \in C^\infty_0(U_i)\}. Let $u \in \mathcal{S}(U_1 \cap U_2)$. The section $\omega_2 u$, extended by zero outside $U_2$ will be regarded as a smooth section over $U_1$ which we denote by $u_1$. Similarly, we can regard $-\omega_1 u$ as a smooth section $u_2$ over $U_2$. Note that

$$\mathcal{D}(u_1 - u_2) = \mathcal{D} u = 0 \text{ on } U.$$  

Thus the two sections $\mathcal{D} u_i$ match-up to define a section $\partial u \in C^\infty(\mathcal{E}^-)$. We can give $\partial u$ a more suggestive description using the equalities

$$\mathcal{D} u_1 = [\mathcal{D}, \omega_2] u + \omega_2 \mathcal{D} u = c(d\omega_2) u$$

and similarly,

$$\mathcal{D} u_2 = -c(d\omega_1) u$$

where $c(\cdot)$ denotes the Clifford multiplication by a 1-form.
This suggests the existence of an asymptotic map
\[ \mathcal{H}^+ \rightarrow^a \ker \mathcal{D}_r^* \]
defined by
\[ \partial : u \mapsto c(d\omega_2)u. \]
To evaluate how far is it from being in the kernel of \( \mathcal{D}_r^* \) we use Proposition 3.45 in [3] and we deduce
\[ \mathcal{D}_r^*(c(d\omega_2)u) = -\ddot{\omega}_2 u \]
where we use dots to denote \( t \)-derivatives. The quantity
\[ \frac{||\mathcal{D}_r^*(c(d\omega_2)u)||^2}{||c(d\omega_2)u||^2} = \frac{||\ddot{\omega}_2||^2}{||\dot{\omega}_2||^2} \]
offers an indication about the distance between \( \partial u := c(d\omega_2)u \) and \( \tilde{K}_{r}^- \). It suggest choosing \( \omega_2 \) so that it depends only on the longitudinal coordinate \( t \) along the neck. In this case we deduce
\[ \mathcal{D}_r^*(c(d\omega_2)) = -\ddot{\omega}_2 u \]
where \( \ddot{\omega}_2 \) is close the first eigenvalue of the Dirichlet problem of the one dimensional Laplacian on an interval of length \( \sim r \). In such an instance, we see that the above quantity is \( O(r^{-2}) \). For example \( (L = L(r) = kr, \kappa \in (0,1)) \)
\[ \ddot{\omega}_2 = \frac{\pi}{2L} \alpha(t) \sin\left(\frac{\pi t}{L}\right) \]
where \( \alpha(t) \) is a nonnegative, smooth cut-off function supported in \([0,L]\) such that \( \alpha(t) \equiv 1 \) on \([1,L-1]\) and there exists a constant \( C > 0 \) independent of \( L \) such that \( \frac{d^k \alpha}{dt^k}(t) \leq 2, \forall t, k = 1, 2, 3 \). Now set
\[ \omega_2(t) = \frac{\pi}{2L} \int_0^t \alpha(s) \sin\left(\frac{\pi s}{L}\right) ds. \]
We denote by \( \partial_r u \) the orthogonal projection onto \( \tilde{K}_{r}^- \) of
\[ c(d\omega_2(t + L/2))u \in C^\infty(\mathcal{E}_r^-). \]

**Definition 3.2** The sequence of asymptotic maps
\[ U_r \xrightarrow{f_r^a} V_r \xrightarrow{g_r^a} W_r \]
is said to be *weakly asymptotically exact* if for \( r \gg 0 \) there exists an exact sequence
\[ U_r \xrightarrow{f_r^a} V_r \xrightarrow{g_r^a} W_r \]
such that
\[ \delta(\text{Graph}(f_r), \text{Graph}(f_r')) + \delta(\text{Graph}(g_r), \text{Graph}(g_r')) = o(1) \text{ as } r \to \infty. \]
Note that Lemma 0.1 (see also Remark A.2) shows that any asymptotically exact sequence is also weakly asymptotically exact.

One can show that we have a weakly asymptotically exact sequence

\[ 0 \rightarrow \tilde{K}_1^+ \oplus K_2^+ \Delta \rightarrow L_1^+ + L_2^+ \rightarrow a \]

\[ \rightarrow a \tilde{K}_r^- S_r^- \oplus K_2^- \Delta \rightarrow L_1^- \oplus L_2^- \rightarrow 0. \] (3.10)

Moreover, \( \ker S_r^- = 0, \forall r > 0 \). This sequence is only weakly asymptotically exact because the range of \( \partial_r \) is never trivial but nevertheless,

\[ \| \partial_r \|^2 = o(1). \] (3.11)

The above estimate follows from the fact that \( c(r) = o(r^{-1}), L \sim r \) and

\[ \| D_r \phi_r - \frac{\pi}{L} \phi_r \|_{L^2(M(r))}^2 = o(1) \| \phi_r \|^2_{L^2(M(r))} \]

where \( \phi_r = \tilde{\omega}_2(t + L/2)u \oplus \tilde{\omega}_2(t + L/2)Gu \in C^\infty(\mathcal{E}_r^+) \oplus C^\infty(\mathcal{E}_r^-) \).

We leave the details to the reader.

Formula (3.10) predicts

\[ \text{ind } (\mathcal{D}) = (\dim K_1^+ - \dim K_1^-) + (\dim K_2^+ - \dim K_2^-) \]

\[ - \dim(L_1^+ + L_2^+) + \dim(L_1^- + L_2^-). \] (3.12)

This equality can be alternatively established as follows.

Set \( h = \dim \mathcal{H}^\pm \) and

\[ U_i^+ = \ker \hat{D}_i \cap L^2(M_i(\infty; E_i^+)) \]

\[ U_i^- = \ker \hat{D}_i^* \cap L^2(M_i(\infty; E_i^-)), \ i = 1, 2. \]

We have short exact sequences

\[ 0 \rightarrow U_i^+ \rightarrow K_i^+ \rightarrow L_i^+ \rightarrow 0. \]

The Atiyah-Patodi-Singer index theorem (Thm. 3.10 in [2]) coupled with the Atiyah-Singer index theorem on closed manifolds implies immediately

\[ \text{ind } \mathcal{D} = (\dim U_1^+ - \dim K_1^-) + (\dim U_2^+ - \dim K_2^-) + h \]

\[ = (\dim K_1^+ - \dim K_1^-) + (\dim K_2^+ - \dim K_2^-) + h - (\dim L_1^+ + \dim L_2^+). \] (3.13)

Now observe that

\[ \dim L_1^+ + \dim L_2^+ = \dim(L_1^+ + L_2^+) + \dim(L_1^+ \cap L_2^+) \]
\[ = \dim(L_1^+ + L_2^+) + h - \dim((L_1^+)\perp + (L_2^+)\perp) \]

(the above \(\perp\) denotes the orthogonal complement in \(\mathcal{H}^+\))

\[ \dim(L_1^+ + L_2^+) + h - \dim(G^*(L_1^- + L_2^-)) \]

\[ = h + \dim(L_1^+ + L_2^+) - \dim(L_1^- + L_2^-). \]

Using this last equality in (3.13) we obtain (3.12).

### 4 Applications

As promised, we will include some simple applications of the Main Theorem.

Suppose for example that \(\mathcal{K}_\infty = 0\). This is possible if and only if

\[ L_1 \cap L_2 = 0 \]

and \(\ker(T_\infty : K_i \to L_i) = 0, \ i = 1, 2\). These kernels consist of the \(L^2\)-solutions of \(\hat{D}_{i,\infty}\). This shows that the operators \(D_r\) cannot have eigenvalues \(\lambda_r\) such that \(|\lambda_r| = o(1/r)\) as \(r \to \infty\). We have thus established the following result (proved for the first time in [7]).

**Corollary 4.1** Suppose that

\[ L_1 \cap L_2 = \{0\} \quad \text{and} \quad \ker(\hat{D}_{i,\infty} \cap L^2(\hat{E}_i)) = \{0\} \]

Then for \(r \gg 0\) the operator \(D_r\) has a bounded inverse

\[ D_r^{-1} : L^2(\mathcal{E}_r) \to L^2(\mathcal{E}_r) \]

and

\[ \|D_r^{-1}\| = O(r) \quad \text{as} \quad r \to \infty. \]

Suppose now the entire situation is super-symmetric. Thus, we have decompositions

\[ K_i = K_i^+ \oplus K_i^- , \quad \mathcal{K}_\infty = \mathcal{K}_\infty^+ \oplus \mathcal{K}_\infty^- . \]

In [11], \(\tilde{K}_r^-\) was called the *obstruction space*. We assume

\[ K_i^- = \{0\}, \ i = 1, 2. \quad (4.1) \]

This implies \(L_i^- = \{0\}\) and \(\mathcal{K}_\infty^- = \{0\}\). The equality (1.1) shows that \(L_1^+ = L_2^+ = \mathcal{H}^+\). We deduce

\[ \tilde{K}_r^- = \{0\}, \quad \forall r \gg 0 \]

while the even part \(\tilde{K}_r\) fits in an exact sequence

\[ 0 \to \tilde{K}_r \xrightarrow{\Gamma_r^+} K_1 \oplus K_2 \xrightarrow{\Delta^+} \mathcal{H}^+ \to 0. \]
The bundle $\mathcal{E}_r$ has a decomposition

$$
\mathcal{E}_r = \mathcal{E}_r^+ \oplus \mathcal{E}_r^-
$$

with respect to which $\mathcal{D}_r$ has the super-symmetric block decomposition

$$
\mathcal{D}_r = \begin{bmatrix}
0 & \mathcal{D}_r^* \\
\mathcal{D}_r & 0
\end{bmatrix}
$$

where $\mathcal{D}_r : C^\infty(\mathcal{E}_r^+) \to C^\infty(\mathcal{E}_r^-)$. The equality (4.2) implies that $\mathcal{D}_r$ is onto since

$$
\mathcal{K}_r = \ker \mathcal{D}_r^* \cong \coker \mathcal{D}_r.
$$

Thus $\mathcal{D}_r \mathcal{D}_r^*$ is one-to-one and onto and admits a bounded inverse $L^2(\mathcal{E}_r^-) \to L^2(\mathcal{E}_r^-)$.

We claim that

$$
\| (\mathcal{D}_r \mathcal{D}_r^*)^{-1} \| = O(r^2)
$$

To prove this claim we argue by contradiction.

Because $\mathcal{D}_r \mathcal{D}_r^*$ is self-adjoint, positive and has compact resolvent, the norm of its inverse is $m(r)^{-1}$ where

$$
m(r) = \inf \{ \langle \mathcal{D}_r \mathcal{D}_r^* u, u \rangle ; \| u \| = 1 \}.
$$

Suppose that for every $r \gg 0$ we can find $\phi_r \in L^2(\mathcal{E}_r^-)$ such that

$$
\| \phi_r \| = 1
$$

and

$$
m(r) = \| \mathcal{D}_r^* \phi_r \|^2 = \langle \mathcal{D}_r \mathcal{D}_r^* \phi_r, \phi_r \rangle = o(1/r^2) \text{ as } r \to \infty.
$$

Now pick $c(r) > \exp(-\delta(r))$ such that

$$
c(r) = o(1/r), \quad m(r) = o(c(r)^2) \quad \text{as } r \to \infty.
$$

The above $\delta$ is the same exponent as in the Main Theorem.

Now apply $\mathcal{D}_r$ to the vector $u_r := 0 \oplus \phi_r \in L^2(\mathcal{E}_r^+ \oplus \mathcal{E}_r^-)$. We deduce

$$
\| u_r \| = 1
$$

and

$$
\frac{\| \mathcal{D}_r u_r \|}{\| u_r \|} = \sqrt{m(r)}.
$$

Thus, according to Lemma 2.2 we can conclude

$$
\text{dist} (u_r, \mathcal{K}_r(c(r))) \leq \frac{\sqrt{m(r)}}{c(r)} = o(1).
$$

On the other hand, $u_r$ is purely odd which implies $\mathcal{K}_r^{-1}(c(r)) \neq 0$ for all $r \gg 0$. This contradicts (1.2) and thus proves (1.3). This estimate also shows that $\mathcal{D}_r$ has a right inverse $R_r : L^2(\mathcal{E}_r^-) \to L^2(\mathcal{E}_r^+)$ of norm $O(r)$. We can now state our next result.
Proposition 4.2 Suppose the condition (4.1) is satisfied. Then for \( r \gg 0 \) the operator \( \mathcal{D}_r \) is onto and admits a bounded right inverse of norm \( O(r) \). Moreover

\[
\ker \mathcal{D}_r = \tilde{K}_r^+ \quad \text{for} \quad r \gg 0. \tag{4.4}
\]

**Proof** The only thing left to prove is the equality (4.4) which follows immediately from the fact that the index of \( \mathcal{D}_r \) is independent of \( r \) and

\[
\dim \tilde{K}_r^+ - \dim \tilde{K}_r^- = \text{ind} \mathcal{D} = \dim \ker \mathcal{D}_r - \dim \ker \mathcal{D}_r^* . \tag*{\Box}
\]

Remark 4.3 Results of this type are needed in gauge theoretical gluing problems over (even dimensional) smooth manifolds. In such problems, the condition (4.1) appears in the following disguise.

The operators \( \hat{D}_{i,\infty} \) (defined in the introduction) have a super-symmetric decompositions

\[
\hat{D}_{i,\infty} = \begin{bmatrix} 0 & \hat{D}_{i,\infty}^* \\ \hat{D}_{i,\infty} & \hat{D}_{i,\infty} \end{bmatrix}.
\]

For simplicity, we will omit the subscripts \( i, \infty \). According to [9], the operator \( \hat{D} \) induces a Fredholm operator

\[
L^{1,2}_\delta(\hat{E}^+) \to L^{2,2}_\delta(\hat{E}^-)
\]

where \( \delta \) is a small positive number and the \( L^{k,2}_\delta \) norm is the \( L^{k,2} \) norm with respect to a measure on \( M_i(\infty) \) which along the neck has the form \( e^{\delta|t|} dt \wedge dvol_N \). The condition (4.1) signifies that \( \hat{D} \), in this functional set-up, is onto.

Suppose now that in Proposition 4.2 we have a family of operators, each satisfying (4.1) and subject to the restrictions listed in Remark 2.5. We deduce immediately the following consequence.

Corollary 4.4 Under the above assumptions, there exists an exact sequence of vector bundles

\[
0 \to \ker \mathcal{D}_r \xrightarrow{\Gamma_r} K_1 \oplus K_2 \xrightarrow{\Delta^+} \mathcal{H}^+ \to 0.
\]

In particular, by passing to determinant line bundles we deduce an isomorphism of line bundles over \( X \)

\[
det(\text{ind} (\mathcal{D}_r)) \cong \det K_1 \otimes \det K_2 \otimes (\det \mathcal{H}^+)^*.
\]

where in the right hand side \( \text{ind} (\mathcal{D}_r) \) is viewed as an element in an appropriate \( K \)-theory of the parameter space \( X \).
Remark 4.5 (a) The terms $\det K_i$ are also determinant line bundles associated to the indices of the families of Atiyah-Patodi-Singer problems determined by $D_i$, $i = 1, 2$.

(b) Corollary 4.4 is also useful in orientability issues involving various moduli spaces arising in gauge theory.

A Some technical proofs

The Key Estimate is a consequence of the following elementary result.

Lemma A.1 Fix $\mu \in \mathbb{R}$. Suppose $U$ is a finite dimensional Hilbert space and $u(t), f(t) : [0, L) \to U$ are two smooth functions satisfying the ordinary differential equation

$$\dot{u} = \mu u + f.$$ \hspace{1cm} (A.1)

Then there exists a constant $C > 0$ independent of $\mu$, $u$ and $f$ such that the following hold. (a) If $\mu = 0$ then

$$|u(t) - u(t + n)| \leq \int_t^{t+n} q_{t,L}(f)ds, \ \forall t \in [0, L - n).$$ \hspace{1cm} (A.2)

(b) If $\mu > 0$ then

$$|u(t)| \leq e^{-\mu t}|u(t + n)| + \frac{C}{\mu^2} q_{t,L}(f), \ \forall t \in [0, L - n).$$ \hspace{1cm} (A.3)

(c) If $\mu < 0$ then

$$|u(t + n)| \leq e^{-\mu t}|u(t)| + \frac{C}{\mu^2} q_{t,L}(f) \ \forall t \in [0, L - n).$$ \hspace{1cm} (A.4)

Proof We prove only (a) and (b). (c) follows from (b) by time reversal.

Proof of (a) We have

$$|u(t + 1) - u(t)| \leq \int_t^{t+1} |f(s)|ds \leq \rho_t(f).$$

Thus

$$|u(t + n) - u(t)| \leq \sum_{k=1}^{n} |u(t + k) - u(t + k - 1)| \leq \sum_{k=1}^{n} \rho_{t+k-1}(f) \leq q_{t,L}(f).$$

Proof of (b) Denote by $e^\mu$ the exponential function $e^{\mu t}$. We have

$$u(t + 1) = e^\mu + \int_0^1 e^\mu(1 - s)f(t + s)ds.$$
so that by Cauchy’s inequality

\[ |u(t + 1) - e^{\mu}u(t)| \leq \rho_0(e_{t})(p(f)). \]

Hence

\[ |u(t)| \leq e^{-\mu}|u(t + 1)| + e^{-\mu}\rho_0(e_{t})(p(f)). \]

Now observe that

\[ e^{-\mu}\rho_0(e_{t}) = e^{-\mu}e^{\mu} - \frac{1}{\mu} \leq \frac{1}{\mu}. \]

Set \( x_k := |u(t + k)| \). The sequence \( x_k \) satisfies the difference inequality

\[ x_k \leq e^{-\mu}x_{k+1} + \frac{1}{\mu}\rho_{t+k}(f). \]

Thus

\[
x_0 \leq e^{-\mu}x_n + \frac{1}{\mu}\sum_{k=0}^{n-1} \rho_{t+k} e^{-(n-1-k)\mu} \leq e^{-\mu} + \frac{1}{\mu}q_{n.L} \sum_{k=0}^{n-1} e^{-k\mu} \]
\[
\leq e^{-\mu} + \frac{1}{\mu(1 - e^{-\mu})}q_{n.L} \leq e^{-\mu} + \frac{C}{\mu^2q_{n.L}(f)}. \]

This proves \( (A.3) \) and the lemma. \( \square \)

**Proof of the Key Estimate.** Let \( \hat{u} \) and \( \hat{f} \) as in the statement of Proposition 1.2.

Using the spectral decomposition of \( D \) we obtain a family of ordinary differential equations of the type \( (A.1) \). The Key Estimate is now an immediate consequence of Lemma A.1. The details can be safely left to the reader. \( \square \)

**Proof of Lemma 0.1** Denote by \( P_U \) and \( P_V \) the orthogonal projections onto \( U \) and respectively \( V \).

Suppose \( \delta(U, V) = \sqrt{1 - a^2}, \ a \in (0, 1) \). This means that for every \( u \in U \) we have

\[ \|u - P_Vu\|^2 \leq (1 - a^2)\|u\|^2 \]

so that

\[ \|P_Vu\|^2 = \|u\|^2 - \|u - P_Vu\|^2 \geq a^2\|u\|^2 \]

i.e.

\[ \|P_Vu\| \geq a\|u\|, \ \forall u \in U. \] (A.5)

This shows \( P_V \) is one-to-one.

Suppose now that \( \delta(V, U) = \sqrt{1 - b^2}, \ b \in (0, 1) \) so that

\[ \delta(U, V) = \max(\sqrt{1 - a^2}, \sqrt{1 - b^2}) < 1. \]

We deduce similarly

\[ \|P_Uv\| \geq b\|v\|, \ \forall v \in V. \] (A.6)
Introduce the operators

\[ A : U \xrightarrow{P_V} V \xrightarrow{P_U} U, \quad B : V \xrightarrow{P_U} U \xrightarrow{P_V} V. \]

Note first that both \( A \) and \( B \) are selfadjoint operators. From (A.5) and (A.6) we deduce

\[ \|Au\| \geq ab\|u\|, \quad \|Bv\| \geq ab\|v\|, \quad \forall u \in U, \ v \in V. \]

Thus both \( A \) and \( B \) are linear isomorphisms which implies that the operators

\[ P_V : U \to V \quad \text{and} \quad P_U : V \to U \]

are bounded, one-to-one and onto. We conclude from the closed graph theorem that they must be linear isomorphisms. \( \square \)

**Remark A.2** A similar argument proves that if \( U_r, V_r \) is a family of closed subspaces of a Hilbert space \( H \) such that

\[ \delta(U_r, V_r) = o(1), \quad \text{as} \quad r \to \infty \]

then

\[ \|1_{U_r} - P_U, P_V\| + \|1_{V_r} - P_V, P_U\| = o(1) \quad \text{as} \quad r \to \infty. \]

**References**

[1] A. Andreotti: *Complexes of Partial Differential Operators*, Yale University Press, New Haven and London, 1975.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer: *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. 77(1975), 43-69.

[3] N. Berline, E.Getzler, M. Vergne: *Heat Kernels and Dirac Operators*, Springer Verlag, 1992.

[4] B. Booss-Bavnbek, K. Wojciechowski: *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, 1993.

[5] R. Bott, L. Tu: *Differential Forms in Algebraic Topology*, Springer Verlag, 1982.

[6] S. Cappell, R. Lee, E. Miller: *Self-adjoint elliptic operators and manifold decompositions. Part I: Low eigenmodes and stretching*, Comm. Pure Appl. Math., 49(1996), 825-866.
[7] W. Chen: A lower bound of the first eigenvalue of certain self-adjoint operators on manifolds containing long necks, Turkish J. of Math, 21(1997), 93-98.

[8] T. Kato: Perturbation Theory for Linear Operators, Springer Verlag, 1984.

[9] R. Lockart, R. McOwen: Elliptic operators on non-compact manifolds, Ann. Scuola Norm. Sup. Pisa 12(1985), 409-446.

[10] T. Mrowka: A local Mayer-Vietoris principle for Yang-Mills moduli spaces, PhD Thesis, 1988.

[11] L.I. Nicolaescu: Rigidity of generalized laplacians and some geometric applications, Aequationes Mathematicae, 48(1994), 143-162.

[12] L.I. Nicolaescu: The Maslov index, the spectral flow and decompositions of manifolds, Duke Math. J., 80(1995), 485-533.

[13] L.I. Nicolaescu: Generalized Symplectic Geometries and the Index of Families of Elliptic Problems, Mem. A.M.S., 128(1997), no. 609.

[14] T. Yosida: Floer homology and splittings of manifolds, Ann. of Math. 134(1991), 277-324.