Kaluza–Klein models with spherical compactification: observational constraints and possible examples

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Abstract

We consider Kaluza–Klein models with background matter in the form of a multicomponent perfect fluid. This matter provides spherical compactification of the internal space with an arbitrary number of dimensions. The gravitating source has the dust-like equation of state in the external/our space and an arbitrary equation of state (with the parameter $\Omega_1$) in the internal space. In the single-component case, tension ($\Omega_1 = -\frac{1}{2}$) is the necessary condition to satisfy both the gravitational tests in the solar system and the thermodynamical observations. In the multicomponent case, we propose two models satisfying both of these observations. One of them also requires tension $\Omega_1 = -\frac{1}{2}$, but the second one is of special interest because it is free of tension, i.e. $\Omega_1 = 0$. To obtain this result, we need to impose certain conditions.

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1. Introduction

The modern theories of unification such as superstrings, supergravity and M-theory have the most self-consistent formulation in spacetime with extra dimensions. Different aspects of the idea of multidimensionality are intensively used in numerous modern articles. Therefore, it is important to construct viable multidimensional models which are in good agreement with physical experiments. The known gravitational experiments (the deflection of light and the time delay of radar echoes) in the solar system are good filters to screen out non-physical theories. It is well known that the weak-field approximation is enough to calculate the corresponding formulas for these experiments \cite{1}. For example, in the case of general relativity, these formulas demonstrate excellent agreement with the experimental data.
In our previous papers, we investigated the popular Kaluza–Klein (KK) models with toroidal compactification of the internal spaces. As we have shown, to be at the same level of agreement with the gravitational tests as general relativity, the gravitating masses should have tension (i.e. negative pressure) in the internal spaces. This is true for both linear [2, 3] and nonlinear \( f(R) \) [4, 5] models. At first glance, it looks unexpectedly, since the dust-like equation of state \( p = 0 \) in all spatial dimensions is the most natural one for nonrelativistic objects because the energy density for them is much bigger than the pressure. Such approach for gravitating matter sources works very well in general relativity in the three-dimensional space to describe the gravitational tests in the solar system [1]. Therefore, we expected the same in KK models. Unfortunately, the physically reasonable models with the dust-like equation of state \( p = 0 \) in both external and internal spaces contradict these tests [6]. It happens because of the fifth force generated by the variations of the internal space volume [7]. For the proper value of tension, the internal space volume is fixed and the fifth force is absent. It takes place, e.g., for the black strings/branes which have the equation of state in the internal space \( p = \frac{-1}{2} \). For black strings and black branes, the notion of tension is defined, e.g., in [8], and it follows from the first law for black hole spacetimes [9–11]. However, up to now we are not aware of the reasonable physical explanation for ordinary nonrelativistic objects, possessing such relativistic tension. Moreover, in the recent paper [12], it was shown that in the case of non-zero tension there is a problem to formulate a many-body theory for such models. Black strings/branes are compactified on tori. However, it is well known that superstring/supergravity models can be compactified also on Ricci-flat compact spaces (e.g., Calabi–Yau) and spheres. For instance, 11-dimensional supergravity can be compactified on a torus \( T^7 \) and a sphere \( S^7 \) (see, e.g., [13, 14]). The Freund–Rubin mechanism [15] is the most famous example of spontaneous compactification on \( S^7 \) where this is achieved with the help of the form fields. So, setting a goal to find viable models without tension, we considered in [16, 17] Kaluza–Klein models where the internal space is a two-dimensional sphere and the tension of the gravitating mass is absent: \( \Omega = 0 \). Here, we have shown that the conformal variation of the volume of the 2-sphere generates the Yukawa-type admixture to the metric coefficients and the non-relativistic gravitational potential. We estimated this admixture and found that it has a negligible value in the solar system. Therefore, this model (with \( \Omega = 0 \)) satisfies the above-mentioned gravitational tests. However, our subsequent research [18] indicates that tension (\( \Omega = -1/2 \)) plays a crucial role for the considered model because the non-relativistic gravitating masses acquire effective relativistic pressure in the external space if \( \Omega = 0 \). Obviously, such pressure contradicts the observations. The equality \( \Omega = -1/2 \) is the only possibility of preserving the dust-like equation of state in the external space. So, we need tension again! How general is this result? Is it possible, in principle, to construct a model without tension, satisfying the observations (from both gravitational and thermodynamical points of view)? In this paper, we give an affirmative answer to this question and show what is the price for it.

Here, we consider a model where background matter is taken in the form of a multicomponent perfect fluid. This matter provides spherical compactification of the internal space with arbitrary number of dimensions. The multicomponent approach is quite natural. For example, because of the nontrivial topology of our multidimensional spacetime (i.e. compactness of the internal space), vacuum fluctuations of quantized fields result in the nonzero energy density (the Casimir effect) [19]. Various variants of the string theory contain real-valued form fields [20]. We mentioned above that these fields may result in spontaneous compactification due to the Freund–Rubin mechanism. The model may also include other types of fluids. We perturb this background by a compact (usually, point-like) gravitating mass with the dust-like equation of state in the external/our space and an arbitrary equation of state
(with the constant parameter $\Omega$) in the internal space. In the single-component case, we prove that to satisfy both the gravitational tests and thermodynamical observations, gravitating masses should have tension $\Omega = -1/2$. However, in the multicomponent case, there is a possibility of constructing a model which satisfies both the gravitational tests and the thermodynamical properties due to the multicomponent nature of the background matter. It takes place for any reasonable value of $\Omega$ including the most interesting dust-like case $\Omega = 0$. However, to achieve it we need a fine-tuning condition.

The paper is organized as follows. In section 2, we define the background solution in the case of spherical compactification of the internal space with an arbitrary number of dimensions. Here, the background matter takes the form of the multicomponent perfect fluid. Then, we perturb this solution by a gravitating mass with an arbitrary equation of state in the internal space and obtain equations for the metric perturbations. In section 3, we investigate in detail the single-component case and prove the necessity of tension ($\Omega = -1/2$) for such model. Section 4 is devoted to the general multicomponent case. Here, we solve the equations for metric perturbations for two particular examples and demonstrate a principal possibility for the considered model to satisfy the gravitational and thermodynamical observations in the case $\Omega = 0$. The main results are shortly summarized in the concluding section 5.

2. Multicomponent background solution and perturbations

Before the consideration of the gravitational field produced by the gravitating mass, we need to create an appropriate background metric. Such metric is defined on the product manifold $M = M_4 \times M_d$, where $M_4$ describes external four-dimensional flat spacetime and $M_d$ corresponds to the $d$-dimensional internal space which is a sphere with the radius (the internal space scale factor) $a$, and should have the form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 + \sum_{\mu=4}^{D} g_{\mu \mu} d\xi_\mu^2,$$

where

$$g_{DD} = -a^2, \quad g_{\mu \mu} = -a^2 \prod_{\nu=\mu+1}^{D} \sin^2 \xi_\nu, \quad \mu = 4, \ldots, D-1,$$

and $D = 3 + d$ is the total number of spatial dimensions. To create such metric with the curved internal space, we have to introduce background matter with the energy–momentum tensor

$$\bar{T}_{ik} = \begin{cases} \frac{(d-1)}{2a^2} - \Lambda D \end{cases} g_{ik} \quad \text{for } i, k = 0, \ldots, 3;$$

$$\bar{T}_{ik} = \begin{cases} \frac{2(d-2)}{a^2} - \Lambda D \end{cases} g_{ik} \quad \text{for } i, k = 4, 5, \ldots, D.$$

These components of the energy–momentum tensor can be easily obtained from the Einstein equation

$$\kappa \bar{T}_{ik} = R_{ik} - \frac{1}{2} R g_{ik} - \kappa \Lambda D g_{ik}$$

for the background metric (2.1). Here, $\kappa = 2S_D \bar{G}_D/c^4$, $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the total solid angle (the surface area of the $(D-1)$-dimensional sphere of a unit radius) and $\bar{G}_D$ is the gravitational constant in $(D = D + 1)$-dimensional spacetime. We also include in the model a bare multidimensional cosmological constant $\Lambda D$. To obtain these components of the energy–momentum tensor, we took into account the fact that the only nonzero Ricci-tensor components are $R_{\mu \mu} = -[(d-1)/a^2]g_{\mu \mu}$, $\mu = 4, \ldots, D$, and the scalar curvature is
\[ R = -d(d-1)/a^2. \] The minus sign in these formulas follows from adopted metric and curvature sign conventions (as in the book [1]).

Now, we suppose that the energy–momentum tensor (2.3) corresponds to the \( N \)-component perfect fluid:

\[ \tilde{T}_0^0 = \tilde{\xi} = \sum_{q=1}^{N} \tilde{\xi}_q, \quad (2.5) \]

\[ \tilde{T}_{\alpha}^\alpha = -\tilde{P}_0 = - \sum_{q=1}^{N} \tilde{p}_{(0)q} = - \sum_{q=1}^{N} \omega_{(0)q} \tilde{\xi}_q, \quad \alpha = 1, 2, 3, \quad (2.6) \]

\[ \tilde{T}_\mu^\mu = -\tilde{P}_1 = - \sum_{q=1}^{N} \tilde{p}_{(1)q} = - \sum_{q=1}^{N} \omega_{(1)q} \tilde{\xi}_q, \quad \mu = 4, 5, \ldots, D, \quad (2.7) \]

where \( \omega_{(0)q} \) and \( \omega_{(1)q} \) are the parameters of equations of state of the \( q \)-th perfect fluid in the external and internal spaces, respectively. It can be easily seen that the following relations take place:

\[ \tilde{P}_0 = (\text{hypothesis}) \Rightarrow \sum_{q=1}^{N} (1 + \omega_{(0)q}) \tilde{\xi}_q = 0 \quad (2.8) \]

and

\[ \frac{\tilde{P}_1}{\tilde{\xi}} = \frac{2 \Lambda_{D} \kappa a^2 - (d - 1)(d - 2)}{d(d - 1) - 2 \Lambda_{D} \kappa a^2} \]

\[ \Rightarrow \sum_{q=1}^{N} \left[ \omega_{(1)q} - \frac{2 \Lambda_{D} \kappa a^2 - (d - 1)(d - 2)}{d(d - 1) - 2 \Lambda_{D} \kappa a^2} \right] \tilde{\xi}_q = 0, \quad (2.9) \]

or, equivalently,

\[ \Lambda_{D} = \frac{d - 1}{\kappa a^2} \sum_{q=1}^{N} \frac{d \omega_{(1)q} + d - 2}{2 \sum_{q=1}^{N} (\omega_{(1)q} + 1) \tilde{\xi}_q}. \quad (2.10) \]

There is also another useful relation:

\[ \Lambda_{D} = \frac{1}{2} \sum_{q=1}^{N} (d \omega_{(1)q} + d - 2) \tilde{\xi}_q, \quad (2.11) \]

which together with (2.10) gives

\[ \frac{d - 1}{\kappa a^2} = \sum_{q=1}^{N} (\omega_{(1)q} + 1) \tilde{\xi}_q. \quad (2.12) \]

According to equations (2.10) and (2.12), in what follows we consider models that satisfy the inequality \( \sum_{q=1}^{N} (\omega_{(1)q} + 1) \tilde{\xi}_q \neq 0. \)

Now, we perturb our background ansatz by a static point-like massive source with the nonrelativistic rest mass density \( \tilde{\rho} \). We suppose that the matter source is uniformly smeared over the internal space [21]. Hence, multidimensional \( \tilde{\rho} \) and three-dimensional \( \tilde{\rho}_3 \) rest mass densities are connected as follows:

\[ \tilde{\rho} = \tilde{\rho}_3(r_3)/V_{\text{int}}, \]

where \( V_{\text{int}} = [2\pi^{(d+1)/2}/\Gamma((d+1)/2)]a^d \) is the surface area of the \( d \)-dimensional sphere of the radius \( a \) (see, e.g., [6, 21]). In the case of a point-like mass \( m \), \( \tilde{\rho}_3(r_3) = m \delta(r_3), \) where \( r_3 = |r_3| = \sqrt{x^2 + y^2 + z^2}. \) In the nonrelativistic approximation, the energy density of the point-like mass is \( \tilde{T}_{00}^{\text{p}} \approx \tilde{\rho} c^2 \) and up to linear in perturbation terms \( \tilde{T}_{00} \approx \tilde{\rho} c^2. \) Inasmuch as the gravitating mass is at rest in the external space, it has the dust-like equation of state \( \tilde{p}_0 = 0 \Rightarrow \tilde{P}_1 = \tilde{T}_2^2 = \tilde{T}_3^3 = 0 \) in our dimensions. However,
it may have the nonzero equation of state \( \hat{\rho}_1 \approx \Omega \hat{\rho} c^2 \Rightarrow \hat{T}_\mu^\nu \approx -\Omega \hat{\rho} c^2 \), \( \mu = 4, 5, \ldots, D \) in the internal space. All other components of the energy–momentum tensor of the gravitating mass are equal to zero.

Concerning the energy–momentum tensor of the background matter, we suppose that perturbation does not change the equations of state of the multicomponent perfect fluid in the external and internal spaces, i.e. \( \omega_{(0)q} \) and \( \omega_{(1)q} \) are constants. Therefore, the energy–momentum tensor of the perturbed background is

\[
\hat{T}_{00} \approx \left( \sum_{q=1}^{N} \epsilon_q + \epsilon_q^1 \right) g_{00},
\]

\[
\hat{T}_{\alpha\alpha} \approx - \left( \sum_{q=1}^{N} \omega_{(0)q} (\epsilon_q + \epsilon_q^1) \right) g_{\alpha\alpha}, \quad \alpha = 1, 2, 3,
\]

\[
\hat{T}_{\mu\nu} \approx - \left( \sum_{q=1}^{N} \omega_{(1)q} (\epsilon_q + \epsilon_q^1) \right) g_{\mu\nu}, \quad \mu = 4, 5, \ldots, D,
\]

where the corrections \( \epsilon_q^1 \) are of the same order of magnitude as the perturbation \( \hat{\rho} c^2 \).

We suppose that the perturbed metric preserves its diagonal form. Obviously, the off-diagonal coefficients \( g_{\alpha\nu} \), \( \alpha = 1, \ldots, D \), are absent for the static metric. It is also clear that in the case of the uniformly smeared (over the internal space) gravitating mass, the perturbed metric coefficients (see functions \( A, B, C, D \) and \( G \) below) depend only on \( x, y, z \) [21], and the metric structure of the internal space does not change, i.e. \( g_{\mu\nu} = g_{DD} \prod_{\nu=\mu+1}^{D} \sin^2 \xi_{\nu}, \mu = 4, \ldots, D - 1 \). The latter statement can be proved, e.g., in the weak-field approximation from the Einstein equations (see appendix B in [17]). It is also easy to show that in this case the spatial part of the external metric can be diagonalized by coordinate transformations. Therefore, the perturbed metric reads

\[
dx^2 = A c^2 \, \, dx^2 + B \, \, dx^2 + C \, \, dy^2 + D \, \, dz^2 + 2 \, \, \, \, \, \, G \left[ \sum_{\mu=1}^{D-1} \left( \prod_{\nu=\mu+1}^{D} \sin^2 \xi_{\nu} \right) \, \, dx^2 + dx^2 \right]
\]

with

\[
\begin{align*}
A & \approx 1 + A^1 (r_3), \\
B & \approx -1 + B^1 (r_3), \\
C & \approx -1 + C^1 (r_3), \\
D & \approx -1 + D^1 (r_3), \\
G & \approx -a^2 + G^1 (r_3).
\end{align*}
\]

All metric perturbations \( A^1, B^1, C^1, D^1 \) and \( G^1 \) are of the order of quantities \( \epsilon_q^1 \). To find these corrections as well as the background matter perturbations \( \epsilon_q^1 \), we should solve the Einstein equation

\[
R_{ik} = \kappa \left( T_{ik} - \frac{1}{2+d} \, T \, g_{ik} - \frac{2}{2+d} \, \Lambda \, g_{ik} \right),
\]

where the energy–momentum tensor \( T_{ik} \) is the sum of the perturbed background \( \bar{T}_{ik} \) (2.13)–(2.15) and the energy–momentum tensor of the perturbation \( \hat{T}_{ik} \). First, we would like to note

\[ It is worth noting that, in KK models with toroidal compactification, the presence of nonzero pressure/tension in the extra dimensions results in uniform smearing of the gravitating mass over the internal space [12].]
that the diagonal components of the Ricci tensor for the metric (2.16) up to linear terms \(A^1, B^1, C^1, D^1\) and \(G^1\) are
\[
R_{00} \approx \frac{1}{2} \Delta_3 A^1, \\
R_{11} \approx \frac{1}{2} \Delta_3 B^1 + \frac{1}{2} \left( -A^1 - B^1 + C^1 + D^1 + \frac{G^1}{a^2} \right)_{xx}, \\
R_{22} \approx \frac{1}{2} \Delta_3 C^1 + \frac{1}{2} \left( -A^1 + B^1 - C^1 + D^1 + \frac{G^1}{a^2} \right)_{yy}, \\
R_{33} \approx \frac{1}{2} \Delta_3 D^1 + \frac{1}{2} \left( -A^1 + B^1 + C^1 - D^1 + \frac{G^1}{a^2} \right)_{zz},
\]
\[
R_{DD} \approx d - 1 + \frac{1}{2} \Delta_3 G^1, \tag{2.19}
\]
where \(\Delta_3 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2\) is the three-dimensional Laplace operator. Additionally, for the static metric (2.16), where the coefficients \(A, B, C, D\) and \(G\) depend only on \(x, y\) and \(z\), we have the following relation:
\[
R_{\mu\mu} = R_{DD} \prod_{\nu=\mu+1}^{d} \sin^2 \xi_\nu, \quad \mu = 4, \ldots, D - 1. \tag{2.20}
\]
Concerning the off-diagonal components of the Ricci tensor, they should be equal to zero according to the Einstein equation (2.18). Taking into account the relations \(g_{\mu\nu} = g_{DD} \prod_{\nu=\mu+1}^{d} \sin^2 \xi_\nu, \mu = 4, \ldots, D - 1\), and that the metric coefficients \(A, B, C, D\) and \(G\) depend only on \(x, y\) and \(z\), we can easily verify that all off-diagonal components are identically equal to zero except the components \(R_{12}, R_{13}\) and \(R_{23}\). Equating these components to zero, we obtain the following relations between the metric coefficients:
\[
B^1 = C^1 = D^1 \tag{2.21}
\]
and
\[
A^1 - B^1 - \frac{d}{a^2} G^1 = 0, \tag{2.22}
\]
which demonstrate that the expressions in brackets for 11, 22 and 33 components in (2.19) vanish. Therefore, in the weak-field limit, the Einstein equation (2.18) is reduced to the following system of equations:
\[
\Delta_3 A^1 = \frac{1 + d + d \Omega}{1 + d/2} \kappa \hat{\rho} c^2 + \frac{\kappa}{1 + d/2} \sum_{q=1}^{N} (d + 1 + 3 \omega_{(0)q} + d \omega_{(1)q}) \epsilon^1_q, \tag{2.23}
\]
\[
\Delta_3 B^1 = \frac{1 - d \Omega}{1 + d/2} \kappa \hat{\rho} c^2 + \frac{\kappa}{1 + d/2} \sum_{q=1}^{N} \left( 1 - \omega_{(0)q} + d \omega_{(1)q} - d \omega_{(0)q} \right) \epsilon^1_q, \tag{2.24}
\]
\[
\Delta_3 G^1 = \frac{1 + 2 \Omega}{1 + d/2} \kappa a^2 \hat{\rho} c^2 - \frac{2(d - 1)}{a^2} G^1 + \frac{\kappa a^2}{1 + d/2} \sum_{q=1}^{N} \left( 1 - 3 \omega_{(0)q} + 2 \omega_{(1)q} \right) \epsilon^1_q, \tag{2.25}
\]
where we have used relations (2.11) and (2.12). From these equations and the condition (2.22), we obtain the connection between perturbations \(G^1\) and \(\epsilon^1_q\):
\[
G^1 = -\frac{2 \kappa a^2}{d(d - 1)} \sum_{q=1}^{N} \omega_{(0)q} \epsilon^1_q. \tag{2.26}
\]
We have four equations (2.23)–(2.26) to define $3 + N$ unknown functions $A^1$, $B^1$, $G^1$ and $\varepsilon^1_q$. This is impossible in the general case $N > 1$ without some additional assumptions. However, for $N = 1$, we can solve these equations exactly and this is the subject of the following section.

3. Single-component background ($N = 1$). Effective energy–momentum tensor for gravitating mass

Before we consider the general case $N > 1$, it is instructive to explore in detail an exactly solvable single-component model $N = 1$. For the single-component background, it makes sense to drop the index $q$. Therefore, $\bar{E} = \bar{\varepsilon}_1 \equiv \bar{\varepsilon}$, $\bar{P}_0 = \bar{p}_0 \equiv \bar{\rho}_0$, $\bar{P}_1 = \bar{p}_1 \equiv \bar{\rho}_1$, $\omega_{(0)1} \equiv \omega_0$ and $\omega_{(1)1} \equiv \omega_1$. Equation (2.8) demonstrates that the single-component background matter has the vacuum-like equation of state in the external space:

$$\bar{\rho}_0 = \omega_0 \bar{\varepsilon}, \quad \omega_0 = -1,$$

(3.1)

but the equation of state in the internal space is not fixed:

$$\omega_1 = \frac{2 \Lambda_D \kappa a^2 - (d - 1)(d - 2)}{d(d - 1) - 2 \Lambda_D \kappa a^2},$$

(3.2)
i.e. $\omega_1$ is arbitrary. Choosing different values of $\omega_1$ (with fixed $\omega_0 = -1$), we can simulate different forms of matter. For example, $\omega_1 = 1$ corresponds to the monopole form-fields (the Freund–Rubin scheme of compactification [15])3, and for the Casimir effect we have $\omega_1 = 4/d$ [22]. It is worth noting that the parameter $\omega_1$ can be positive only in the presence of a positive bare cosmological constant $\Lambda_D$. Moreover, it takes place only if $d - 2 < 2 \Lambda_D \kappa a^2 / (d - 1) < d$. In contrast to the model with the two-dimensional ($d = 2$) sphere in [17], the parameter $\omega_1$ does not disappear in the case of vanishing $\Lambda_D$ for $d \geq 3$.

Taking into account that the relation (2.26) has now the form

$$\kappa \varepsilon^1 = \frac{d(d - 1)}{2 d^4} G^1,$$

(3.3)
the system of equations (2.23)–(2.25) reads

$$\Delta_3 \left( A^1 - \frac{d}{2d^2} G^1 \right) = \kappa \hat{\rho} c^2 = \frac{8 \pi G_N}{c^2} \hat{\rho}_3,$$

(3.4)

$$\Delta_3 \left( B^1 + \frac{d}{2d^2} G^1 \right) = \kappa \hat{\rho} c^2 = \frac{8 \pi G_N}{c^2} \hat{\rho}_3,$$

(3.5)

$$\Delta_3 G^1 - \lambda^{-2} c^2 G^1 = \frac{2(1 + 2 \Omega)}{2 + d} \kappa a^2 \hat{\rho} c^2 = \frac{2(1 + 2 \Omega)}{2 + d} \frac{8 \pi G_N}{c^2} \hat{\rho}_3,$$

(3.6)

where

$$\lambda^2 = \frac{(d + 2)a^2}{2(d - 1)(d - 2 + d \omega_1)},$$

(3.7)
and we introduced the Newton gravitational constant

$$4 \pi G_N = \frac{S_D \hat{G}_D}{V_{int}}.$$

(3.8)

5 In our case, they are $d$-forms (see, e.g., equations (2.9) and (5.1) in [23]).
Let us consider now the point-like (in the external space) approximation for the gravitating objects: \( \rho_1(r) = m \delta(r) \). The generalization of the obtained results to the case of extended compact objects is obvious. It is well known that to obtain the physically reasonable solution of (3.6) with the boundary condition \( G^1 \to 0 \) for \( r \to +\infty \), the parameter \( \lambda^2 \) should be positive, i.e. the equation of state parameter \( \omega_1 \) should satisfy the condition

\[
\omega_1 > -1 + \frac{2}{d}. \tag{3.9}
\]

which allows also negative values of \( \omega_1 \). From equation (3.2), we can obtain the corresponding restrictions for the bare cosmological constant:

\[
0 < \frac{2 \Lambda_{D} a^2}{d - 1} < d. \tag{3.10}
\]

This inequality relaxes the condition of the positiveness of \( \omega_1 \). Then, equations (3.4)–(3.6) have solutions

\[
A^1 = \frac{2 \varphi_N}{c^2} + \frac{d}{2a^2} G^1, \tag{3.11}
\]

\[
B^1 = \frac{2 \varphi_N}{c^2} - \frac{d}{2a^2} G^1, \tag{3.12}
\]

\[
G^1 = a^2 \frac{4 \varphi_N}{(2 + d)c^2} (1 + 2 \Omega) \exp \left( -\frac{r_1}{\lambda} \right), \tag{3.13}
\]

where the Newtonian potential \( \varphi_N = -G_N m/r_3 \). It is well known that the metric correction term \( A^1 \sim O(1/c^2) \) describes the non-relativistic gravitational potential: \( A^1 = 2 \varphi/c^2 \). Therefore, this potential acquires the Yukawa correction term:

\[
\varphi = \varphi_N \left[ 1 + \frac{d}{2 + d} (1 + 2 \Omega) \exp \left( -\frac{r_1}{\lambda} \right) \right]. \tag{3.14}
\]

The inequalities (3.9) and (3.10) provide the condition of the internal space stabilization. Obviously, the monopole form-field ansatz with \( \omega_1 = 1 \) satisfies this condition. Let us consider this example in more detail. From the fine-tuning relation (3.2), we obtain \( 2 \Lambda_{D} a^2 = (d - 1)^2 \). Precisely, this quantity provides in an effective dimensionally reduced model a zero value of the effective four-dimensional cosmological constant (see equation (5.7) in [23]) where we should make the substitutions \( \Lambda_D \to \Lambda_{D}\kappa \), \( d_1 \to d \), \( D \to 4 + d \) and where \( \tilde{R}_1 = d(d - 1)/a^2 \), \( d_0 = 3 \), \( D_0 = 4 \). It is clear that in our case with flat background external space-time, the effective four-dimensional cosmological constant should vanish. Additionally, this value of \( \Lambda_D \) satisfies the stability condition (5.15) in [23]. Moreover, the gravexciton/radion mass squared (5.12) (with substitution (5.11)) exactly coincides with the Yukawa mass squared \( m_{\text{Yuk}}^2 = \lambda^{-2} = 4(d - 1)^2/(d + 2)a^2 \equiv m_{\text{Yuk}}^{2\text{exc}} \). Therefore, we arrived at the very natural and important conclusion that the Yukawa mass is defined by the mass of the gravexcitons/radions.

It is of interest to estimate the Yukawa correction term in formula (3.14). It is worth mentioning that in our KK model, all gravitating masses (e.g., the balls in the inverse square law experiment or the Sun and planets in the solar system) have in the non-relativistic limit the gravitational potential of the form (3.14). For reasonable values of the equation of state parameter \( \Omega \sim O(1) \), the Yukawa parameter \( \alpha \) (this is the standard notation for the prefactor in front of the exponential function in the Yukawa potential) is also of the order of 1. Then, the inverse square law experiments restrict the characteristic range of the Yukawa interaction [24]: \( \lambda \lesssim 10^{-3} \) cm. Obviously, for the above-mentioned gravitational experiments in the solar system, \( r_1 \gtrsim r_0 \sim 7 \times 10^{10} \) cm and the ratio \( r_1/\lambda \gtrsim 10^{14} \). On the other hand, the collider experiments also restrict the sizes of the extra dimensions for KK models [25]: \( a \lesssim 10^{-12} \) cm.

\[^6\] We mean the KK models without branes. Precisely, such models are considered in our paper.
Since $\lambda \sim a$ (see equation (3.7)), then for considered gravitational experiments in the solar system, $r_3/\lambda \gtrsim 10^{28}$. It is clear that for such ratios we can drop the Yukawa correction terms in equations (3.11) and (3.12). Therefore, the post-Newtonian parameter $\gamma = B_1/A_1$ is equal to 1 with extremely high accuracy for any value of $\Omega$ including the most physically reasonable case of the dust-like value $\Omega = 0$. The case $\Omega = -1/2$ is a special one, and we consider it below. Thus, our model satisfies the gravitational tests (the deflection of light and the time delay of radar echoes) for any value of the equation of state parameter $\Omega$ in the internal space.

So, at first glance, it seems that we have found a model, which, on the one hand, satisfies the gravitational experiments and, on the other hand, may not contain tension (i.e. $\Omega = 0$) in the internal space. However, let us examine in detail the energy–momentum tensor of the gravitating mass. As follows from equations (3.3) and (3.13), the background perturbation $\varepsilon_1$ is localized around the gravitating object and falls exponentially with the distance $r_3$ from this mass. Therefore, the bare gravitating mass is covered by this ‘coat’. For an external observer, this coated gravitating mass is characterized by the effective energy–momentum tensor with the following nonzero components:

$$T_0^{(\text{eff})} \approx \varepsilon_1 + \hat{\rho}(r_3)c^2 = -(1 + 2\Omega)\frac{d(d-1)mc^2}{4(2+d)\pi V_{\text{int}}\alpha(r_3)} \cdot \frac{1}{r_3} \exp\left(-\frac{r_3}{\lambda}\right) \cdot \frac{mc^2\delta(r_3)}{V_{\text{int}}},$$  \hspace{1cm} (3.15)

$$T_{\mu}^{\alpha(\text{eff})} \approx \varepsilon_1 = -(1 + 2\Omega)\frac{d(d-1)mc^2}{4(2+d)\pi V_{\text{int}}\alpha(r_3)} \cdot \frac{1}{r_3} \exp\left(-\frac{r_3}{\lambda}\right), \quad \alpha = 1, 2, 3,$$  \hspace{1cm} (3.16)

$$T_{\mu}^{\mu(\text{eff})} \approx -\omega_1\varepsilon_1 - \Omega\hat{\rho}(r_3)c^2 = \omega_1(1 + 2\Omega)\frac{d(d-1)mc^2}{4(2+d)\pi V_{\text{int}}\alpha(r_3)} \cdot \frac{1}{r_3} \exp\left(-\frac{r_3}{\lambda}\right) - \Omega mc^2\delta(r_3),$$

$$\mu = 4, \ldots, D,$$  \hspace{1cm} (3.17)

which, for illustrative purposes, can be presented as follows:

$$T_0^{(\text{eff})} \approx \frac{mc^2\delta(r_3)}{V_{\text{int}}} \left[1 - \frac{d(1 + 2\Omega)}{2(d - 2 + d\omega_1)}\right],$$  \hspace{1cm} (3.18)

$$T_{\mu}^{\alpha(\text{eff})} \approx -\frac{mc^2\delta(r_3)}{V_{\text{int}}} \cdot \frac{d(1 + 2\Omega)}{2(d - 2 + d\omega_1)}, \quad \alpha = 1, 2, 3,$$  \hspace{1cm} (3.19)

$$T_{\mu}^{\mu(\text{eff})} \approx \frac{mc^2\delta(r_3)\omega_1}{V_{\text{int}}} \cdot \frac{d(1 + 2\Omega)}{2(d - 2 + d\omega_1)}, \quad \mu = 4, \ldots, D,$$  \hspace{1cm} (3.20)

where we have replaced the rapidly decreasing exponential function by the delta function

$$\frac{1}{r_3} \exp\left(-\frac{r_3}{\lambda}\right) \int \frac{1}{r_3} \exp\left(-\frac{r_3}{\lambda}\right) \cdot V_{\text{int}3} \cdot \delta(r_3) = 4\pi \lambda^2\delta(r_3).$$  \hspace{1cm} (3.21)

The less the parameter $\lambda$ is, the better this transformation is executed\(^7\). On the other hand, the smaller the characteristic scale $\lambda$ of the Yukawa interaction is, the better concordance

\(^7\) By definition of the delta-function

$$\int f(r_3)\delta(r_3) \cdot dr_3 = f(0),$$

which holds for any function $f(r_3)$ continuous at $r_3 = 0$. It can be easily seen that

$$\lim_{\lambda \to 0} \int f(r_3) \frac{1}{4\pi \lambda^2 r_3} \exp\left(-\frac{r_3}{\lambda}\right) \cdot dr_3 = f(0).$$

Therefore,

$$\lim_{\lambda \to 0} \frac{1}{4\pi \lambda^2 r_3} \exp\left(-\frac{r_3}{\lambda}\right) = \delta(r_3).$$
with the gravitational tests takes place for our model. Obviously, taking into account the limitation \( \lambda \lesssim 10^{-17} \) cm and the characteristic sizes of massive bodies, we can conclude that all corrections connected with \( G \) are negligible for the gravitational potentials outside of these bodies including atomic nuclei and up to astrophysical objects (see, e.g., equations (3.11)–(3.13)). Unfortunately, this model faces the problem from the thermodynamical point of view, and this problem does not disappear, even in the case of perfect concordance with the direction of the Lorentz transformation, we can rewrite them in a laboratory system of coordinates (we shall consider them as a system of point-like particles. This approach is well grounded in statistical physics (see, e.g., chapters 1 and 10 in [26]) Then, as we have shown above, each of these particles is covered by the coat consisting of two parts: \( \hat{T}^{\alpha \text{(eff)}} = T^{\alpha \text{(eff)}} + T^{\alpha \text{(bare)}} \). The first part coincides with the 33 component in the co-moving frame: \( T^{\alpha \text{(eff)}} = T^{\alpha \text{(eff)}} \sim mc^2 \) up to the transformation of \( r_3 \), i.e. it has the relativistic nature. The second part is connected with the non-relativistic motion of the particle: \( T^{\lambda \text{(bare)}} \sim mv^2 \). These molecules move through the membrane to an empty part of the box. It is well known that momentum crossing the elementary spatial area \( d\vec{v} \wedge d\vec{x} \) per unit time is given by \( \int_{\omega} d\vec{k} \delta(\vec{k}) \delta(\vec{x}) d\vec{v} \wedge d\vec{x} \). In the case of our membrane, we obtain

\[
T^{\lambda}_3 d\vec{x} \wedge d\vec{y} = (T^{\lambda \text{(eff)}} + T^{\lambda \text{(bare)}}) d\vec{x} \wedge d\vec{y}.
\]

Therefore, the non-relativistic particles have relativistic momentum crossing the membrane and resulting in relativistic pressure.

We can also obtain the relativistic pressure by averaging \( T^{\alpha \text{(eff)}} \) over a three-dimensional volume \( V \):

\[
T^{\alpha \text{(eff)}} \approx -(1 + 2\Omega) \frac{d(d - 1)mc^2}{4(2 + d)\pi V_{\text{int}}a^2} \times \frac{1}{V} \int_{r_3} \exp \left( -\frac{r_3}{\lambda} \right) dV \\
\approx -(1 + 2\Omega) \frac{d(d - 1)mc^2}{4(2 + d)\pi V_{\text{int}}a^2} \frac{4\pi \lambda^2}{V} = -\frac{d(1 + 2\Omega)c^2}{2(d + 2 + d\omega_1)} \frac{m}{V_{\text{int}}a^2},
\]

where we dropped the exponentially small terms. To obtain usual three-dimensional quantities, e.g., the pressure measured as erg cm\(^{-3}\), we should multiply equation (3.23) by \( V_{\text{int}} \). Taking this remark into account and multiplying both sides of this equation by the total number \( N \) of
particles in the volume $V$, we obtain the expression for the effective pressure of these particles in $V$

$$
\hat{p}_0^{(\text{eff})} \sim \rho c^2,
$$

(3.24)

where $\rho = Nm/V$ is the rest-mass density and we did not take into account the part of the pressure connected with the non-relativistic motion of these particles. Of course, such relativistic pressure contradicts the observations. It can be easily seen that the equality $\Omega = -1/2$ is the only possibility of achieving $\hat{p}_0^{(\text{eff})} = 0$ for our model. It means that the bare gravitating mass should have tension with the equation of state $\hat{p}_1 = -\dot{\varepsilon}/2$ in the internal space. Then, the effective and bare energy densities coincide with each other and the gravitating mass remains pressureless in our space. In the internal space, the gravitating mass still has tension with the parameter of state $-1/2$. Therefore, to be in agreement with observations, the presence of tension is a necessary condition for the considered model. However, we still do not know a physically reasonable explanation for the origin of relativistic tension ($\Omega = -1/2$) for non-relativistic gravitating objects such as molecules, massive balls or our Sun. Moreover, as it was shown in [12], the presence of tension may result in difficulties in the many-body problem for KK models.

4. Multicomponent background $N > 1$. Particular examples

Let us turn now to the general case $N > 1$. As we have already mentioned at the end of section 2, we need to assume some additional constraints to define perturbations $A^1, B^1, G^1$ and $\varepsilon_1^q$ which satisfy equations (2.23)–(2.26). In this section, we provide two exactly solvable examples. Here, the additional constraint (the fine-tuning condition) is chosen in such a way that, for the first example, the metric coefficients and the non-relativistic gravitational potential acquire corrections in the form of the Yukawa potential, as in the previous section. Then, this example satisfies the gravitational tests in the solar system if the Yukawa potential is negligible. For the second example, the fine-tuning condition will provide the condition of the constant/unperturbed internal space: $G^1 = 0$, as it takes place for black string/branes with toroidal compactification. Such example satisfies the gravitational tests at the same level of accuracy as general relativity.

4.1. Yukawa corrections

In the first example, we generalize the solutions from the previous section, i.e. we shall find the Yukawa correction term to the nonrelativistic gravitational potential. Above, we have shown that such term arises in the case of stable compactification of the internal spaces (this is equivalent to the conditions (3.9) and (3.10) in the single-component case), and the mass of the Yukawa interaction is defined by the mass of gravexciton/radion. A perfect fluid stabilizes the internal space in the case of the vacuum-like equation of state in the external space [2, 23]. Therefore, in this subsection we assume that

$$
\omega_{0q} = -1, \quad q = 1, \ldots, N.
$$

(4.1)

Note that with this choice of the parameters $\omega_{0q}$, equation (2.8) is satisfied automatically. Hence, equation (2.26) is reduced to

$$
G^1 = \frac{2\kappa d^4}{d(d - 1)} \sum_{q=1}^N \varepsilon_q^1 \neq 0.
$$

(4.2)
The case of the zero sum in (4.2) (i.e. the zero perturbation $G^1$) will be considered in the following subsection. Additionally, we assume the following fine-tuning condition:

$$\sum_{q,p=1, q\neq p}^{N} \omega_{(1)q} \epsilon_p^1 = 0. \quad (4.3)$$

Together with the relation (4.2), this is equivalent to the following condition:

$$\left(\sum_{q=1}^{N} \omega_{(1)q}\right) G^1 = \frac{2\kappa a^2}{d(d-1)} \sum_{q=1}^{N} \omega_{(1)q} \epsilon_q^1. \quad (4.4)$$

Then, equation (2.25) is exactly reduced to (3.6) where

$$\lambda^{-2} \equiv \frac{1}{a^2(2 + d)} \left[ 2(d - 1) \left( d - 2 + d \sum_{q=1}^{N} \omega_{(1)q} \right) \right] \quad (4.5)$$

generates the definition (3.7). The positiveness of $\lambda^2$ is the necessary condition of the internal space stabilization. For positive $\lambda^2$ that takes place if the equation of state parameters $\omega_{(1)q}$ satisfy the condition

$$\sum_{q=1}^{N} \omega_{(1)q} > -1 + \frac{2}{d}, \quad d \geq 2, \quad (4.6)$$

the solution $G^1$ has the form (3.13) (in the case of the point-like gravitating mass in the external space: $\hat{\rho}_3(r_3) = m \delta(r_3)$).

It can be easily verified that, provided that equations (4.1)–(4.4) hold, equations (2.23) and (2.24) take the form of (3.4) and (3.5) with solutions (3.11) and (3.12). Therefore, the nonrelativistic gravitational potential $\phi$ is given by formula (3.14), i.e. it has the Yukawa correction term.

As a particular example, we consider the two-component model ($N = 2$) with the monopole form field $\omega_{(1)1} = 1$ and vacuum fluctuations of quantized fields (Casimir effect) $\omega_{(1)2} = 4/d$ [22, 23]. For these perfect fluids $\omega_{(0)1} = \omega_{(0)2} = -1$. Real-valued solitonic/monopole form fields naturally arise as Ramond–Ramond form fields in the type II string theory and M-theory [20]. It is also well known that because of the nontrivial topology of our multidimensional spacetime (i.e. compactness of the internal space), vacuum fluctuations of quantized fields result in the nonzero energy density (the Casimir effect) [19]. Therefore, from equations (4.2) and (4.3), we obtain

$$\epsilon_{1}^1 = -\frac{d}{4} \epsilon_{2}^1 = \frac{d^2(d-1)}{2(d-4)\kappa a^2} G^1. \quad (4.7)$$

The case $d = 4$ is excluded for this particular example because it contradicts the inequality (4.2) $G^1 \neq 0$.

We have shown in the previous section that the Yukawa correction terms to the metric coefficients are negligible for gravitating masses in the solar system. Therefore, examples considered in this subsection satisfy the gravitational tests for any reasonable values $\Omega$ including the most interesting dust-like case $\Omega = 0$. Unfortunately, the nonrelativistic gravitating matter source acquires effective relativistic pressure in the external/our space (remind that $\omega_{(0)q} = -1$, $\forall q$):

$$\hat{p}_0^{(\text{eff})} = -T_q^{(\text{eff})} \approx -\sum_{q=1}^{N} \epsilon_q^1 \sim mc^2, \quad \alpha = 1, 2, 3. \quad (4.8)$$
which, certainly, is unacceptable from the thermodynamical point of view. The value \( \Omega = -1/2 \) is the only possibility of avoiding it, i.e. the gravitating source should have tension in the internal space. However, this is not the desired result. Therefore, in the following subsection we construct a model without tension, satisfying observations from both gravitational and thermodynamical points of view.

4.2. À la black brane with zero tension

Let us assume now the following fine-tuning condition:

\[
\sum_{q=1}^{N} \omega_{0,q} \epsilon_{q}^1 = 0 \quad \Leftrightarrow \quad G^1 = 0,
\]

where the latter equation follows from equation (2.26). If all parameters of the equations of state in the external space are equal to each other \( \omega_{0,1} = \cdots = \omega_{0,N} \), then equation (4.9) is reduced to

\[
\sum_{q=1}^{N} \epsilon_{q}^1 = 0,
\]

but we do not specially impose this condition. Taking into account equation (4.9), we obtain from (2.25) the following additional relation:

\[
\sum_{q=1}^{N} \omega_{1,q} \epsilon_{q}^1 = -\frac{1}{2} (1 + 2\Omega) \hat{\rho} c^2 - \frac{1}{2} \sum_{q=1}^{N} \epsilon_{q}^1.
\]

With the help of this relation and the condition (4.9), equations (2.23) and (2.24) read

\[
\Delta_3 A^1 = \kappa \hat{\rho} c^2 + \kappa \sum_{q=1}^{N} \epsilon_{q}^1,
\]

\[
\Delta_3 B^1 = \kappa \hat{\rho} c^2 + \kappa \sum_{q=1}^{N} \epsilon_{q}^1.
\]

We may conclude from these equations that metric coefficients \( A^1 = B^1 \) for any value of \( \Omega \) including the dust-like case \( \Omega = 0 \). Therefore, the PPN parameter \( \gamma = 1 \) in full analogy with general relativity. So, we achieved in this model agreement with the gravitational tests (the deflection of light, the Shapiro effect) in the solar system. To obtain the Newtonian limit, we should either consider the case (4.10), or assume that \( \sum_{q=1}^{N} \epsilon_{q}^1 \sim \hat{\rho} c^2 \). The latter case results in the renormalization of the multidimensional gravitational constant \( \kappa \). The Newton gravitational constant is defined by equation (3.8) and the nonrelativistic gravitational potential coincides exactly with the Newtonian expression

\[
A^1 = \frac{2\phi_{N}}{c^2}.
\]

One of the main features of this model consists in the constant/unperturbed internal space because of \( G^1 = 0 \). It means that the conformal prefactor for the internal space metric was not changed (up to \( O(1/c^2) \)): \( G = -\alpha^2 = \text{const.} \). A similar situation takes place for black string/branes. Usually, they have the unperturbed toroidal internal space. However, there is also generalization to unperturbed spherical compactification [27]. For the model considered in this subsection, we have shown (up to \( O(1/c^2) \)) that external spacetime metric is the Schwarzschild one and the internal space has the unperturbed spherical metric. The main
advantage of this model with respect to the black branes is a possibility of eliminating the
tension of the gravitating mass due to the dust-like choice $\Omega = 0$. Because of this difference
we call this case ‘à la black brane’.

This model is also satisfactory from the thermodynamical point of view because due to the
condition (4.9) the effective relativistic pressure (which is related to the excitation of the
background matter in equation (2.14)) in the external/our space is absent:

$$p_0^{(\text{eff})} = -T_\mu^{(\text{eff})} \approx \sum_{q=1}^{N} \omega_0(q) \delta_{\mu}^1 = 0, \quad \alpha = 1, 2, 3. \quad (4.15)$$

As a particular example, we consider the two-component perfect fluid from the previous
subsection where the monopole form field is characterized by the following parameters
of equations of state: $\omega_{(0)1} = -1$, $\omega_{(1)1} = 1$, and for the Casimir effect we have
$\omega_{(0)2} = -1$, $\omega_{(1)2} = 4/d$. Hence, equation (2.8) is automatically satisfied. Then, for the most
interesting dust-like case $\Omega = 0$, we obtain from equations (4.10) and (4.11) the following
relations:

$$\varepsilon_1 = -\varepsilon_2 = -\frac{d}{2(d-4)} \hat{\rho} c^2 = -\frac{d}{2(d-4)} \frac{mc^2 \delta(r)}{V_{\text{int}}}. \quad (4.16)$$

The case $d = 4$ is excluded for this particular example because it contradicts equation (4.11)
(here, $\omega_{(1)1} = \omega_{(1)2}$, and from equation (4.11), we obtain a non-physical result $\hat{\rho} = 0$).

To conclude our investigations, we want to make the following remark. At the very end of
section 3, we wrote that we do not know a physically reasonable explanation for the origin of
relativistic tension ($\Omega = -1/2$) in the internal space for a non-relativistic gravitating object.
However, our last example demonstrates how such an object can acquire effective tension in
the internal space. Let us consider the case where the bare parameter $\Omega$ for this gravitating
mass is zero: $\Omega = 0$. Then, from equations (2.15) and (4.11), we obtain the expression for the
effective pressure of this object in the internal space:

$$p_1^{(\text{eff})} = -T_\mu^{(\text{eff})} \approx \sum_{q=1}^{N} \omega_{(1)q} \delta_1 = -\frac{1}{2} \hat{\rho} c^2, \quad \mu = 4, 5, \ldots, D, \quad (4.17)$$

where we took into account the relation (4.10). That is we obtain an effective tension
($\Omega^{(\text{eff})} = -1/2$). Hence, this is one more reason to call the considered model à la black
brane. It is worth noting that this effective tension arises not because of the bare tension of the
gravitating mass, but because of the perturbation of the background matter, and such procedure
is physically well motivated.

5. Conclusion

In this paper, we investigated the viability of Kaluza–Klein models with spherical compactification of the internal space. To achieve such compactification, we introduced background matter in the form of a multicomponent perfect fluid. The multicomponent approach is quite natural. For example, because of the nontrivial topology of our multidimensional spacetime (i.e. compactness of the internal space), vacuum fluctuations of quantized fields result in the nonzero energy density (the Casimir effect). Various variants of the string theory contain real-valued form fields. The model may also include other types of fluids. This matter provides spherical compactification of the internal space with an arbitrary number of dimensions $d$. We perturbed this background by a compact (in our three-dimensional space) gravitating source with the dust-like equation of state in the external/our space $\hat{\rho}_0 = 0$ and an arbitrary equation of state $\hat{\rho}_1 \approx \Omega \hat{\rho} c^2$ (with the constant parameter $\Omega$) in the internal
space. We assumed that this matter source is uniformly smeared over the internal space. In the weak-field limit (up to the order $1/c^2$), we obtained equations for the perturbations of the metric and the background matter. It is impossible to solve these equations in the multicomponent case $N > 1$ because the number of equations is less than the number of unknowns. Hence, we need to introduce some additional fine-tuning relations.

However, a single-component case ($N = 1$) is exactly solvable without additional constraints. Hence, we investigated this model in detail. The perturbed (up to $O(1/c^2)$) metric coefficients were found from the Einstein equations. For the external space, these coefficients consist of two parts: the standard general relativity expressions plus the admixture of the Yukawa interaction. The Yukawa interaction arises only in the case when the background matter satisfies some condition (see the inequality (3.9)) for the parameter of the equation of state in the internal space which is equivalent to the condition of the internal space stabilization. From the cosmological point of view, such stabilization was considered in [22, 23] (see also the appendix in [2]). The stabilization takes place if the conformal excitations of the internal spaces (referred to as gravexcitons [22] or radions) acquire the positive mass squared. In our paper, we have obtained an important result that the mass of the Yukawa interaction is exactly defined by the mass of the gravexciton/radion. In the solar system, the Yukawa mass is big enough for dropping the admixture of this interaction and getting very good agreement with the gravitational tests for any value of $\Omega$.

Nevertheless, our subsequent investigation of this single-component model showed that the gravitating body acquires the effective relativistic pressure in the external/our space. Clearly, it is hardly possible to detect such pressure for the astrophysical objects (we cannot place a membrane or a wall in the path of these objects), but any system of nonrelativistic particles such as molecules of a liquid or a gas may have the relativistic momentum crossing any spatial area. Of course, it contradicts the observations. We have demonstrated that the value $\Omega = -1/2$ (i.e. tension!) is the only possibility to avoid this problem for the single-component case. It is worth noting that, in multidimensional models, the standard approach implies that to match the observations it is enough to stabilize the internal space. To achieve it, the radion should have a relatively large mass. However, the single-component case demonstrates that this is not sufficient. In other words, the big enough Yukawa/radion mass (for models with the arbitrary physically reasonable parameter $\Omega$ in the equation of state in the internal space) is not sufficient to have a good theory. From the standard point of view, this result is fresh and unexpected. The models should also possess the thermodynamical properties which do not contradict the observations. This natural demand leads to the unique value $\Omega = -1/2$ in the case $N = 1$, i.e. the gravitating mass should have tension in the internal space (the $d$-dimensional sphere). We have presented the detailed investigation of this problem. Such investigation was absent before.

Then, we turned to the multicomponent case $N > 1$. Here, we need to assume some additional constraints (fine tuning conditions) to solve the Einstein equations. We provided two exactly solvable examples.

In the first example (with the fine tuning condition (4.3)), we arrived at the model with the Yukawa correction terms for the metric perturbations, by analogy with the single-component case. For the sufficiently large Yukawa mass, this model satisfies the gravitational tests. Unfortunately, because of the same reasoning as in the case $N = 1$, the nonrelativistic gravitating matter source acquires the effective relativistic pressure in the external/our space which, certainly, is unacceptable from the thermodynamical point of view. The value $\Omega = -1/2$ is the only possibility to avoid it, i.e. the gravitating source should have tension in the internal space.
In the following subsection, we propose the other model (with the fine tuning condition \((4.9)\)). Here, the PPN parameter \(\gamma\) is exactly equal to 1 similar to general relativity. Therefore, this model satisfies the gravitational tests (the deflection of light, the Shapiro effect) for any reasonable value of \(\Omega\). Moreover, the nonrelativistic gravitational potential exactly coincides with the Newtonian one. One of the main features of this model consists in the constant/ unperturbed internal space. It means that the conformal prefactor for the internal space metric was not perturbed by the gravitating mass. A similar situation takes place for black string/branes. The main advantage of this model with respect to the black branes is a possibility of eliminating tension due to the dust-like choice \(\Omega = 0\). This model is also satisfactory from the thermodynamical point of view because the effective relativistic pressure in the external/our space is absent: \(\hat{p}^{(\text{eff})}_0 = 0\). Therefore, we demonstrated a principal possibility for the considered model to satisfy the gravitational and thermodynamical observations in the case of vanishing tension in the internal space for the gravitating mass. Certainly, it happens due to the multicomponent nature of the background matter. Although the fine-tuning conditions impose strong constraints on the model and look artificial, however, we think that they look less artificial than the bare relativistic tension \(\Omega = -1/2\). As an additional bonus for this model, we found a physically reasonable mechanism to generate an effective tension for a gravitating mass. In this particular model, we demonstrated that a non-relativistic gravitating mass with zero bare tension \(\Omega = 0\) acquires an effective tension in the internal space \(\Omega^{(\text{eff})} = -1/2\) due to the perturbation of the background matter, and such procedure is physically well motivated.

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