CHARACTERIZING C*-ALGEBRAS OF COMPACT OPERATORS BY GENERIC CATEGORICAL PROPERTIES OF HILBERT C*-MODULES

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Dedicated to the memory of Yu. P. Solovyov

Abstract. B. Magajna and J. Schweizer showed in 1997 and 1999, respectively, that C*-algebras of compact operators can be characterized by the property that every norm-closed (and coinciding with its biorthogonal complement, resp.) submodule of every Hilbert C*-module over them is automatically an orthogonal summand. We find out further generic properties of the category of Hilbert C*-modules over C*-algebras which characterize precisely the C*-algebras of compact operators.

In 1997 B. Magajna obtained the equivalence of the property of the category of Hilbert C*-modules over a certain C*-algebra $A$ that any Hilbert C*-submodule is automatically an orthogonal summand with the property of the C*-algebra $A$ of coefficients to admit a faithful $\ast$-representation in some C*-algebra of compact operators on some Hilbert space, cf. Theorem 2.1. In 1999 J. Schweizer was able to sharpen the argument replacing the Hilbert C*-module property of B. Magajna by the property of the category of Hilbert C*-modules over a certain C*-algebra $A$ that any Hilbert C*-submodule which coincides with its biorthogonal complement is automatically an orthogonal summand, cf. Theorem 2.1. Later on in 2003 M. Kusuda published further results which indicate that in the majority of situations the Hilbert C*-module property can be weakened merely requiring the $K_A(\mathcal{M})$-subbimodules of the Hilbert C*-modules $\mathcal{M}$ to be always orthogonal summands, see [11, 12] for the details.

Studying the work of B. Magajna and J. Schweizer C*-algebras $A$ of the form $A = c_0-\sum_{\alpha} \oplus K(H_\alpha)$ become of special interest, where the symbol $K(H_\alpha)$ denotes the C*-algebra of all compact operators on some Hilbert space $H_\alpha$, and the $c_0$-sum is either a finite block-diagonal sum or a block-diagonal sum with a $c_0$-convergence condition on the C*-algebra.

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components $K(H_α)$. The $c_0$-sum may possess arbitrary cardinality. This kind of $C^*$-algebras has been precisely characterized by W. Arveson [11 §I.4, Th. I.4.5] as the $C^*$-subalgebras of (full) $C^*$-algebras of compact operators on Hilbert spaces. Throughout the present paper we refer to these $C^*$-algebras as to $C^*$-algebras of compact operators on certain Hilbert spaces.

In 1998-99 V. I. Paulsen and the author investigated injective and projective objects in categories of Hilbert $C^*$-modules over certain $C^*$-algebras, cf. [7]. The research work revealed a number of further properties every Hilbert $C^*$-submodule of any Hilbert $C^*$-module over a fixed $C^*$-algebras $A$ might admit. The goal of the present paper is to present such results which all lead to further equivalent characterizations of the $C^*$-algebras of compact operators. Together with results of D. Bakic and B. Guljas (2) on the existence of modular normalized tight frames in Hilbert $C^*$-modules over $C^*$-algebras of compact operators we collect a large number of useful properties of these classes of Hilbert $C^*$-modules. (For a concise introduction to modular frame theory we refer to [3].)

The paper consists of a section with introductory material and of a second section which contains the results.

1. Preliminaries

We give definitions and basic facts of Hilbert $C^*$-module theory needed for our investigations. The papers [18, 10, 3, 14, 15, 6], some chapters in [9, 22], and the books by E. C. Lance [13] and by I. Raeburn and D. P. Williams [20] are used as standard sources of reference. We make the convention that all $C^*$-modules of the present paper are left modules by definition. A pre-Hilbert $A$-module over a $C^*$-algebra $A$ is an $A$-module $M$ equipped with an $A$-valued mapping $\langle ., . \rangle : M \times M \to A$ which is $A$-linear in the first argument and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^* , \quad \langle x, x \rangle \geq 0 \quad \text{with equality iff} \quad x = 0.$$  

The mapping $\langle ., . \rangle$ is said to be the $A$-valued inner product on $M$. A pre-Hilbert $A$-module $\{M, \langle ., . \rangle\}$ is Hilbert if and only if it is complete with respect to the norm $\| . \| = \| \langle ., . \rangle \|_A^{1/2}$. We always assume that the linear structures of $A$ and $M$ are compatible. Two Hilbert $A$-modules are isomorphic if they are isometrically isomorphic as Banach $A$-modules, if and only if they are unitarily isomorphic, [13]. We
would like to point out that Banach $A$-modules can carry unitarily non-isomorphic $A$-valued inner products which induce equivalent norms to the given one, nevertheless, [6].

Hilbert $C^*$-submodules of Hilbert $C^*$-modules might not be direct summands, and if they are direct summands then they can be topological, but not orthogonal summands. We say that a Hilbert $C^*$-module $N$ is a topological summand of a Hilbert $C^*$-module $M$ which contains $N$ as a Banach $C^*$-submodule in case $M$ can be decomposed into the direct sum of the Banach $C^*$-submodule $N$ and of another Banach $C^*$-submodule $K$. The denotation is $M = N \oplus K$. If, moreover, the decomposition can be arranged as an orthogonal one (i.e. $N \perp K$) then the Hilbert $C^*$-submodule $N \subseteq M$ is an orthogonal summand of the Hilbert $C^*$-module $M$. Examples of any kind of appearing situations can be found in [6].

Finally, we are going to consider various bounded $C^*$-linear operators $T$ between Hilbert $C^*$-modules $M$, $N$ with one and the same $C^*$-algebra of coefficients. Quite regularly those operators $T$ may not admit an adjoint bounded $C^*$-linear operator $T^* : N \to M$ fulfilling the equality $\langle T(x), y \rangle_N = \langle x, T^*(y) \rangle_M$ for any $x \in M$, any $y \in N$. We denote the $C^*$-algebra of all bounded $C^*$-linear adjointable operators on a given Hilbert $A$-module $M$ by $\text{End}_{A}^{*}(M)$. The Banach algebra of all bounded $A$-linear operators on $M$ is denoted by $\text{End}_{A}(M)$. For more detailed information on such situations we refer to [6].

2. $C^*$-algebras of compact operators and the Magajna-Schweizer theorem

This section aims to characterize the class of $C^*$-algebras of compact operators on certain Hilbert spaces and their $C^*$-subalgebras by the appearance of certain properties common to all Hilbert $C^*$-modules over them. Our starting point is the following result by B. Magajna and J. Schweizer and some of its immediate consequences:

**Theorem 2.1.** (B. Magajna, J. Schweizer [16, 21])

Let $A$ be a $C^*$-algebra. The following three conditions are equivalent:

(i) $A$ is of $c_0-\sum_i \oplus K(H_i)$-type, i.e. it has a faithful $*$-representation as a $C^*$-algebra of compact operators on some Hilbert space.
(ii) For every Hilbert $A$-module $M$ every Hilbert $A$-submodule $N \subseteq M$ is automatically orthogonally complemented in $M$, i.e. $N$ is an orthogonal summand of $M$.
(iii) For every Hilbert $A$-module $M$ every Hilbert $A$-submodule $N \subseteq M$ that coincides with its bi-orthogonal complement $N^{\perp \perp} \subseteq M$ is automatically orthogonally complemented in $M$. 


Corollary 2.2. Let $A$ be a $C^*$-algebra. The following three conditions are equivalent:

(i) $A$ is of $c_0\oplus\bigoplus_i K(H_i)$-type, i.e. it has a faithful $\ast$-representation as a $C^*$-algebra of compact operators on some Hilbert space.

(iv) For every Hilbert $A$-module $M$ and every bounded $A$-linear map $T : M \to M$ there exists an adjoint bounded $A$-linear map $T^* : M \to M$.

(v) For every pair of Hilbert $A$-modules $M, N$ and every bounded $A$-linear map $T : M \to N$ there exists an adjoint bounded $A$-linear map $T^* : N \to M$.

Proof. Condition (i) is equivalent to the assertion that every Hilbert $A$-module $M$ is orthogonally complemented as a Hilbert $A$-submodule of arbitrary Hilbert $A$-modules by Theorem 2.1. The latter condition on a certain $M$ is equivalent to the assertion (iv) of the corollary by [6, Th. 6.3] - in case the range of the $C^*$-valued inner product is dense in the $C^*$-algebra of coefficients. However, the range of the $C^*$-valued inner product of Hilbert $C^*$-modules over $C^*$-algebras of type (i) gives always rise to an ideal of them, and these norm-closed two-sided ideals are of type (i) again. So the restriction does not matter, and the equivalence of (i) and (iv) follows.

To establish the equivalence of (i) with the last condition consider the operator $T$ described at (v) as a bounded $A$-linear operator on the Hilbert $A$-module $M \oplus N$ mapping pairs $(x, y)$ to pairs $(0, T(x))$. Then the equivalence (i)$\iff$(iv) applied to this particular situation implies condition (v) as a simple calculation shows. The converse conclusion becomes trivial resorting to the situation $M = N$. \hfill $\Box$

The next group of facts is concerned with particular Hilbert $C^*$-modules, bounded module operators on them and their properties. We aim to describe $C^*$-algebras $A$ of compact operators on Hilbert spaces by general properties of kernels of bounded module operators on Hilbert $A$-modules and of images of images of bounded module operators on Hilbert $C^*$-modules that admit a closed range. By the way we prove the closed graph theorem for bounded module operators on Hilbert $C^*$-modules.

Proposition 2.3. (N. E. Wegge-Olsen [22, Th. 15.3.8])
Let $A$ be a $C^*$-algebra, $\{M, \langle , \rangle\}$ be a Hilbert $A$-module and $T$ be an adjointable bounded module operator on $M$. If $T$ has closed range then $T^*, (T^*T)^{1/2}$ and $(TT^*)^{1/2}$ have also closed ranges and

$M = \text{Ker}(T) \oplus T^*(M) = \text{Ker}(T^*) \oplus T(M)$

$= \text{Ker}(|T|) \oplus |T|(M) = \text{Ker}(|T^*|) \oplus |T^*|(M).$
In particular, each orthogonal summand appearing on the right is automatically norm-closed and coincides with its bi-orthogonal complement inside \( \mathcal{M} \). Moreover, \( T \) and \( T^* \) have polar decomposition.

**Corollary 2.4.** (bounded closed graph theorem)

Let \( A \) be a C*-algebra and \( \{ \mathcal{M}, \langle \cdot, \cdot \rangle \}, \{ \mathcal{N}, \langle \cdot, \cdot \rangle \} \) be two Hilbert \( A \)-modules. A bounded \( A \)-linear operator \( T: \mathcal{M} \rightarrow \mathcal{N} \) possesses an adjoint operator \( T^*: \mathcal{N} \rightarrow \mathcal{M} \) if and only if the graph of \( T \) is an orthogonal summand of the Hilbert \( A \)-module \( \mathcal{M} \oplus \mathcal{N} \). Beside this equivalence, the graph of every bounded \( A \)-linear operator \( T \) coincides with its bi-orthogonal complement in \( \mathcal{M} \oplus \mathcal{N} \), and it is always a topological summand with topological complement \( \{ (0, z) : z \in \mathcal{N} \} \). By a counterexample due to E. C. Lance ([13, pp. 102-104]) this fails for some closed, self-adjoint, densely defined, unbounded module operators on certain Hilbert C*-modules.

**Proof.** Since the inequality \( \|T(x)\| \leq \|T\|\|x\| \) is valid for every \( x \in \mathcal{M} \) the graph of \( T \) is a norm-closed Hilbert \( A \)-submodule of the Hilbert \( A \)-module \( \mathcal{M} \oplus \mathcal{N} \). Moreover, since the graph of \( T \) is the kernel of the bounded module operator \( S: (x, y) \rightarrow (0, T(x) - y) \) on \( \mathcal{M} \oplus \mathcal{N} \) it coincides with its bi-orthogonal complement there, [5, Cor. 2.7.2]. If \( T \) has an adjoint then the operator \( T': (x, y) \rightarrow (x, T(x)) \) is adjointable on \( \mathcal{M} \oplus \mathcal{N} \). By Proposition 2.3 the graph of \( T \) is an orthogonal summand.

Conversely, if the graph of \( T \) is an orthogonal summand of \( \mathcal{M} \oplus \mathcal{N} \) then its orthogonal complement consists precisely of the pairs of elements \( \{(x, y) : x = -T^*(y), y \in \mathcal{N} \} \) since \( \langle z, x \rangle_M + \langle T(z), y \rangle_N = 0 \) forces \( T^*(y)(z) = \langle z, (-x) \rangle_M \) for any \( z \in \mathcal{M} \) and \( T^*: \mathcal{N} \rightarrow \mathcal{M}' \). So \( T^* \) is everywhere defined on \( \mathcal{N} \) taking values exclusively in \( \mathcal{M} \subseteq \mathcal{M}' \). This shows the existence of the adjoint operator \( T^* \) of \( T \) in the sense of its definition.

The property of the graph of a bounded module operator to be a topological summand with topological complement \( \{(0, z) : z \in \mathcal{N} \} \) follows from the decomposition \( (x, y) = (x, T(x)) + (0, y - T(x)) \) for every \( x \in \mathcal{M}, y \in \mathcal{N} \). Since \( T(0) = 0 \) for any linear operator \( T \) the intersection of the graph with the \( A-B \) submodule \( \{(0, z) : z \in \mathcal{N} \} \) is always trivial. □

**Problem 2.5.** It would be highly interesting to know whether the kernel of every surjective bounded module operator would be merely a topological summand, or whether there are counterexamples.

The observation above gives us the opportunity to add two further equivalent conditions to Theorem 2.4. Much less obvious is the fact that only topological summands have to be considered to characterize
the same class of coefficient C*-algebras as a distinguished one in the research field of Hilbert C*-modules. This is established by the next theorem which also demonstrates the missing in [4] self-duality assumption to be inevitable for obtaining the results of [4, §1] (cf. [17]).

**Theorem 2.6.** There are further equivalent conditions to the conditions listed in Theorem 2.1 and Corollary 2.2:

(vi) The kernels of all bounded $A$-linear operators between arbitrary Hilbert $A$-modules are orthogonal summands.

(vii) The images of all bounded $A$-linear operators with norm-closed range between arbitrary Hilbert $A$-modules are orthogonal summands.

(viii) For every Hilbert $A$-module every Hilbert $A$-submodule is automatically topologically complemented there, i.e. it is a topological summand.

(ix) For every (maximal) norm-closed left ideal $I$ of $A$ the corresponding open projection $p \in A^{**}$ is an element of the multiplier C*-algebra $M(A)$ of $A$.

**Proof.** By Theorem 2.1 (ii) condition (i) on $A$ implies condition (viii). Since norm-closed one-sided ideals $I$ of $A$ and open projections $p \in A^{**}$ are in one-to-one correspondence by [19] a close examination of the norm-closed left ideals of C*-algebras $A$ with property (i) shows assertion (ix) to be valid.

If condition (ix) on $A$ holds then the multiplier C*-algebra $M(A)$ is an atomic type I von Neumann algebra, cf. [15, Lemma 2]. It can be represented as a direct sum over a discrete measure space of a certain cardinality, say $l_\infty \sum_i B(H_i)$ for Hilbert spaces $H_i$ by the direct integral decomposition theory of von Neumann algebras. If one of the Hilbert spaces $H_i$ is infinite-dimensional the appropriate $i$-th block in the direct sum decomposition of $A$ above cannot coincide with the entire set $B(H_i)$ since the latter contains maximal one-sided ideals not supported by projections of $B(H_i)$ itself. So the $i$-th block has to be *-isomorphic to $K(H_i)$ for every Hilbert space $H_i$. Considering the two-sided ideal $c_0A$ in $A$ the corresponding carrier projection is central in $A$, and so we can resort to the center of $A$ to continue our considerations. The center of $A$ cannot be equal to $l_\infty$ since the norm-closed ideal $c_0$ of $l_\infty$ would not be supported by a projection inside $l_\infty$ as supposed. So the center of $A$ has to be either finite-dimensional or a $c_0$-space of arbitrary cardinality. This implies condition (i).

Furthermore, suppose the C*-algebra $A$ fulfills condition (viii) and $A \not\equiv \mathbb{C}$. Consider $A$ and an arbitrary maximal left ideal $I$ of $A$ as Hilbert $A$-modules equipped with the standard $A$-valued inner product
of the C*-algebra $A$. Since $I$ is a Hilbert $A$-submodule of $A$ and all bounded module maps on $A$ can be identified with right multipliers $\text{RM}(A)$ of $A$ by [6] there exists an idempotent right multiplier $p \neq 1_A$ of $A$ such that $x = xp$ for any $x \in I$ by assumption. Indeed, the element $t = pp^* + (1-p)(1-p)$ is invertible in $M(A)$, and $q = t^{-1}pp^* = pp^*t^{-1} \in M(A)$ is the orthogonal projection with the same range as $p$. Note, that $p$ and $q$ commute. Since the orthogonal complement of the (positive) carrier projection $q \in A^{**}$ of $I$ is a non-trivial minimal projection of $A^{**}$ and $p$ and $q$ commute in $A^{**}$, the coincidence $p = q$ and the inclusion $q \in M(A)$ turn out to hold. We arrive at assertion (ix) because the maximal ideal $I$ of $A$ was assumed to be an arbitrary one and any other norm-closed left ideal equals to the intersection of maximal left ideals. The established equivalence of (ix) and (i) completes the argument.

Consider the situation of a graph of a bounded $A$-linear operator $T : \mathcal{M} \to \mathcal{N}$ between two Hilbert $A$-modules. By Theorem 2.1, Corollary 2.2 and Corollary 2.4 the operator $T$ is adjointable if and only if its graph is an orthogonal summand. However, in any case its graph is the kernel of the bounded $A$-linear operator $S : (x, y) \to (0, T(x) - y)$ that acts on $\mathcal{M} \oplus \mathcal{N}$. Consequently, condition (vi) is equivalent to the adjointability of all bounded module operators between Hilbert $A$-modules. Resorting to the special case of canonical embeddings of maximal ideals $I$ of $A$ into $A$ we obtain the equivalence of (vi) with (ix) and hence, with Theorem 2.1 (i).

If a bounded $A$-linear operator $T : \mathcal{M} \to \mathcal{N}$ between two Hilbert $A$-modules admits an image which is an orthogonal summand then the graph of $T$ is the image of the bounded module operator $(\text{id}_\mathcal{M} \oplus (0_N + T))$ on $\mathcal{M} \oplus \mathcal{N}$. Moreover this set is an orthogonal summand of $\mathcal{M} \oplus \mathcal{N}$, and it is the graph of $T$. Again, the graph of $T$ is an orthogonal summand of $\mathcal{M} \oplus \mathcal{N}$ if and only if $T$ is adjointable, and the arguments above can be repeated demonstrating the equivalence of (vii) and Theorem 2.1 (i).

Problem 2.7. Is any Hilbert C*-submodule of a any Hilbert C*-module $\mathcal{H}$ that coincides with its bi-orthogonal complement inside $\mathcal{H}$, the kernel of a bounded $A$-linear operator mapping $\mathcal{H}$ into itself (or, alternatively, to other Hilbert C*-modules)?

Problem 2.8. Characterize those C*-algebras $A$ for which the following condition holds: For every Hilbert C*-module over $A$ every Hilbert $A$-submodule which coincides with its bi-orthogonal complement is automatically topologically complemented there.

Problem 2.8 remodels the difference between B. Magajna’s theorem and J. Schweizer’s theorem on the level of topological summands. The
results of M. Kusuda [12] indicate that the solution has to be similar to that one given on the level of orthogonal summands. However, research has to be continued.

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