Brane inflation and the WMAP data: a Bayesian analysis

Larissa Lorenz\textsuperscript{1}, Jérôme Martin\textsuperscript{1} and Christophe Ringeval\textsuperscript{2}

\textsuperscript{1} Institut d’Astrophysique de Paris, UMR 7095-CNRS, Université Pierre et Marie Curie, 98bis boulevard Arago, 75014 Paris, France
\textsuperscript{2} Theoretical and Mathematical Physics Group, Centre for Particle Physics and Phenomenology, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium
E-mail: lorenz@iap.fr, jmartin@iap.fr and ringeval@fyma.ucl.ac.be

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Abstract. The Wilkinson Microwave Anisotropy Probe (WMAP) constraints on string inspired ‘brane inflation’ are investigated. Here, the inflaton field is interpreted as the distance between two branes placed in a flux-enriched background geometry and has a Dirac–Born–Infeld (DBI) kinetic term. Our method relies on an exact numerical integration of the inflationary power spectra coupled to a Markov chain Monte Carlo exploration of the parameter space. This analysis is valid for any perturbative value of the string coupling constant and of the string length, and includes a phenomenological modelling of the reheating era to describe the post-inflationary evolution. It is found that the data favour a scenario where inflation stops by violation of the slow-roll conditions well before brane annihilation, rather than by tachyonic instability. As regards the background geometry, it is established that \( \log v > -10 \) at 95\% confidence level (CL), where \( v \) is the dimensionless ratio of the five-dimensional sub-manifold at the base of the six-dimensional warped conifold geometry to the volume of the unit 5-sphere. The reheating energy scale remains poorly constrained, \( T_{\text{reh}} > 20 \) GeV at 95\% CL, for an extreme equation of state \( (w_{\text{reh}} \gtrsim -1/3) \) only. Assuming that the string length is known, the favoured values of the string coupling and of the Ramond–Ramond total background charge appear to be correlated. Finally, the stochastic regime (without and with volume effects) is studied using a perturbative treatment of the Langevin equation. The validity of such an approximate scheme is discussed and shown to be too limited for a full characterization of the quantum effects.

Keywords: CMBR theory, string theory and cosmology, physics of the early universe, cosmological applications of theories with extra dimensions
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1. Introduction

Inflation is currently considered our best description of the early Universe, since it solves the puzzles of the standard hot big bang model [1]–[6] and, in addition, provides a very convincing mechanism for structure formation leading to an almost scale invariant power spectrum [7]–[10]. The small deviations from scale invariance are connected to the microphysics of inflation and are, therefore, of utmost importance when investigating the physical conditions in the very early Universe [11]–[13]. In practice, inflation is usually driven by a scalar field (or possibly several fields) and for this type of matter, the effective pressure can become negative, if the scalar field’s potential is sufficiently flat. This condition is necessary for producing a phase of accelerated expansion (i.e. inflation) in general relativity. In this context, one of the main open issues is the physical nature of the inflaton field. Since it is clear that this question should be addressed in the framework of extensions of the standard model of particle physics, string theory seems to be a promising place to look for a physically well-motivated candidate field. In recent years, this issue has given rise to several studies; for reviews see [14]–[19]. In particular, string inspired brane world scenarios have turned out to be especially fruitful. In this context, the inflaton field is usually interpreted as the distance between two branes moving relative to each other in the extra dimensions [20]–[22]. The corresponding potential generically contains a Coulomb term of the form $\sim C - \phi^{-4}$, where $C$ is a constant and $\phi$ the inflaton field [22]–[29]. The end of inflation can then either occur by violation of the slow-roll conditions or by a mechanism of tachyonic instability, as in hybrid inflation. In either case, when the distance between the branes becomes of the order of the string length, the simple single-field description breaks down and a more refined (fully stringy) treatment is necessary. The main goal of the present article is to carry out a detailed comparison of the inflationary predictions derived from one particular class of brane inflation scenarios, namely the Kachru–Kallosh–Linde–Maldacena–McAllister–Trivedi (KKLMMT) models, with the cosmic microwave background (CMB) data currently available.

Analysing this class of potentials is further motivated by two rather different features. Firstly, although it is possible (for instance by using the slow-roll approximation) to span the space of (single-field) inflationary models without relying on a specific form of the inflaton potential [30]–[33], it is also interesting to not use any such approximation except, of course, the linear theory for the cosmological perturbations [34]–[36]. Let us also mention that with the incoming flow of more accurate data from the Planck satellite [37,38] or from the future arcminute resolution CMB experiments [39]–[41], the errors linked with the current approximation schemes may become a problem. The no-approximation approach often requires numerical computations. As a consequence, one has to pay the price of choosing a particular form for the inflaton potential, which implies some loss of generality. A possible way out and a systematic strategy for spanning the
space of inflationary models is by choosing potentials that are representative examples of a whole class of scenarios. For instance, a scenario where inflation takes place in the regime of large (in Planck mass units) vacuum expectation value (vev) for the field, and where the potential goes to infinity as the vev of the field increases towards infinity, is illustrated by chaotic and/or hybrid inflation. The difference between these two models consists in the way inflation stops. In contrast, a model where inflation occurs for small values of the inflaton vev is well represented by models in which the potential has a symmetry breaking shape. These classes of models were studied in great detail and compared to the WMAP third-year data in [34]. From this general perspective, the KKLMMT models with their above-mentioned Coulomb term generate inflation when the vev of the field is large, but with a potential bounded at infinity.

Further motivation, different in spirit from the above, lies in model building issues. The Coulomb-type potential naturally arises in string inspired scenarios of inflation. Moreover, this type of potential has been known for a long time, even without relying on stringy inspiration, since it can occur in models of hybrid inflation from dynamical supersymmetry breaking [42]. Recently, in the context of string theory, corrections to the Coulomb potential associated with the problem of moduli stabilization have been explored [17], [24]–[27], [29]. Although interesting from the theoretical point of view, they are not considered in the following. Including these effects in the analysis would greatly increase the number of free parameters relevant for the description of the inflationary part of the model and their degeneracy by opening the parameter space. As a result, given the current data, no constraint could be obtained (as is the case, for instance, for running mass inflation [34]). We have therefore chosen to perform an exhaustive Bayesian analysis of the CMB data by considering only the generic Coulomb contribution to the potential in the context of string inspired brane inflation.

Compared to the existing literature [43]–[46], our study uses a complete Markov chain Monte Carlo (MCMC) analysis based on an exact numerical integration of this type of potential. In this respect, we do not consider particular values of the string length $\sqrt{\alpha'}$ and the string coupling $g_s$ but, on the contrary, develop a parameter scanning strategy which allows us to leave these parameters free. However, we will see in the following that, already with only the Coulomb part of the potential, degeneracies in the parameter space do not permit the WMAP data to fully constrain these stringy quantities. Nevertheless, the throat geometry and some combinations of the KKLMMT model parameters are constrained. Finally, we present new approximate solutions for the field evolution in the DBI regime and refine the analysis of inflationary quantum effects. In particular, it is shown that the approximate solutions usually considered are not sufficient to fully characterize this regime.

This paper is organized as follows. In section 2, we briefly recall some basic facts about the KKLMMT scenario of brane inflation, deriving the governing equations from the effective four-dimensional action based on type IIB string theory. In particular, we discuss the intrinsic string features of the model, which among others manifest themselves through an unusual kinetic behaviour as well as a restricted domain of validity for the model. In section 3, we briefly consider the impact of quantum fluctuations that may affect the classical trajectory in certain regions. Section 4 is devoted to the slow-roll phase of

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3 Some conditions on these parameters must be nevertheless fulfilled, for instance, $g_s < 1$, in order to be in the perturbative regime.
the scenario, which can be treated analogously to small field models. However, in the case of the KKLMMT scenario, the slow-roll discussion has to be complemented by a study of the stringy aspects: this is presented in section 5. After deriving the restrictions on the parameter space from model-intrinsic arguments in section 6, we present our results from the WMAP third-year data in sections 7 and 8. Conclusions are drawn in the final section 9. Finally, in the appendix, we give details on the stochastic regime.

2. Brane inflation

In the following, we discuss the KKLMMT model of brane inflation, first proposed in [22] as a realization of inflation within string theory. Our goal here is to consider all aspects (string related or not) at play during the inflationary phase. We begin with a intuitive description of how the KKLMMT model arises in string theory.

2.1. Qualitative description

The model considers a D3 and an anti-D3 brane in a ten-dimensional supergravity background whose world volume (time axis included) is aligned with the $x^\mu$ coordinate axes, $\mu$ varying from 0 to 3. The six extra dimensions $y^A$, with $A = 4, \ldots, 9$, are compactified such that the ‘radial’ distance separating brane and anti-brane is $y^4 = r$, and their distance in the other coordinates vanishes. While inflationary models involving D3 (or higher dimensional) branes have been considered in the literature before, the KKLMMT model assumes that the six-dimensional section forms a so-called Klebanov–Strassler (KS) throat [47]–[51], i.e. a warped deformed conifold with background fluxes for certain background fields.

Intuitively, this background can be thought of as populated by $N$ heavy D3 branes with Ramond–Ramond (RR) charge. The anti-D3 brane is embedded at a fixed position $r_0$ (the bottom of the KS throat), representing an additional source of RR field strength that can be described as a small perturbation. The D3 brane is inserted into this perturbed background at $r_1 \gg r_0$ (far from the bottom of the throat) and is considered as a probe: it does not affect the geometry itself, but experiences the forces due to gravity and RR interaction, brought about by the exchange of light closed string modes between the branes. The distance $r = r_1 - r_0$ between the test D3 and the fixed anti-D3 brane corresponds (up to normalization) to the inflaton field $\phi$ whose potential $V(\phi)$ can be calculated from the forces experienced by the test D3 brane in the limit $r \gg \ell_s$, $\ell_s = \sqrt{\alpha'}$ being the string length scale. Inflation takes place while $\phi$ ‘rolls down’ along a flat interaction potential in the direction of decreasing $\phi$. When the branes become too close, in the sense that their proper distance approaches the string scale, a tachyon, the lightest open string mode stretching from one brane to the other, appears, and the long distance potential $V(\phi)$ is no longer valid. When the D3 brane reaches the bottom of the throat $r_0$, the two branes annihilate in a complex process whose details are beyond the scope of the present paper. We will, however, introduce a phenomenological model which assumes that the brane annihilation triggers a reheating era.

2.2. The inflaton effective action

In this section, we use an effective field theory representation of the KKLMMT model, and our starting point will be the effective action of the inflaton field $\phi$. Considering that
the inflaton $\phi$ is itself an open string mode, it is related to the distance $r$ between the brane and the anti-brane by

$$\phi = \sqrt{T_3} r, \quad (1)$$

and its effective four-dimensional action reads

$$S = -\frac{1}{2\kappa} \int R\sqrt{-g} \, d^4x - T_3 \int \frac{1}{h(\phi)} \sqrt{-g} \, d^4x - T_3 \int \frac{1}{h(\phi)} \sqrt{\frac{\hat{h}(\phi)}{T_3} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - g} \, d^4x, \quad (2)$$

where the quantity $\kappa$ is defined by

$$\kappa \equiv \frac{8\pi}{m_{P_1}^2}, \quad (3)$$

and

$$T_3 = \frac{1}{(2\pi)^3 g_s \alpha'^2} \quad (4)$$

is the D3 brane tension, $g_s$ being the string coupling constant. The quantity $\hat{h}(\phi)$ is the (perturbed) warp factor of the ten-dimensional metric whose form can be calculated from the full Einstein equations [51]. To simplify the notation, one can introduce the effective brane tension $T(\phi)$

$$T(\phi) \equiv \frac{T_3}{\hat{h}(\phi)}, \quad (5)$$

allowing us to rewrite equation (2) as [52]

$$S = S_{\text{grav}} + S_{\text{DBI}}$$

$$= -\frac{1}{2\kappa} \int R\sqrt{-g} \, d^4x - \int \left[ T(\phi) \sqrt{\frac{\hat{h}(\phi)}{T_3} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + T(\phi)} \right] \sqrt{-g} \, d^4x. \quad (6)$$

It is clear that, for a slowly varying scalar field, the terms in $(\partial \phi)^2$ are small and the square root in $S_{\text{DBI}}$ can be Taylor expanded in the field derivatives. The resulting action at leading order identifies with that of a canonically normalized scalar field evolving in a self-interaction potential.

### 2.3. The kinetic term

On a flat Friedmann–Lemaître–Robertson–Walker (FLRW) brane, the metric reads

$$ds^2 = a(\eta)^2 \left( -d\eta^2 + \delta_{ij} \, dx^i \, dx^j \right), \quad (7)$$

where $\eta$ and $x^i$ are the conformal time and spatial coordinates. In terms of the cosmic time $dt = a \, d\eta$, the metric tensor reduces to $\text{diag}[-1, a^2(t), a^2(t), a^2(t)]$. For such a homogeneous metric, $\phi$ can only depend on time and keeping the first non-trivial term in
the square root expansion reproduces a standard kinetic term, namely
\[ S_{\text{DBI}} = \int \left[ \frac{1}{2} \dot{\phi}^2 - 2T(\phi) \right] a^3(t) \, d^4x, \] (8)
where a dot denotes a derivative with respect to the cosmic time. However, this remains true as long as higher order terms in the expansion can be neglected, i.e. for
\[ \dot{\phi}^2 \lesssim T(\phi). \] (9)
Let us denote by \( \phi_{\text{DBI}} \) the field value from which onwards the field evolution would be dominated by the string corrections to the standard kinetic term. Notice that equation (9) is also the condition required to have a positive square root argument in equation (6). In this sense, \( \sqrt{T(\phi)} \) can be understood as the field maximum speed in the warped throat, and one can, by relativistic analogy, define a Lorentz factor \( \gamma \) such that [53]
\[ \gamma(\dot{\phi}, \phi) \equiv \frac{1}{\sqrt{1 - \dot{\phi}^2/T(\phi)}}. \] (10)
When \( \gamma \approx O(1) \), the field \( \phi \) follows the dynamics of a canonically normalized scalar field which can slow-roll (and hence, produce inflation) on a sufficiently flat potential. However, there also exists an ‘ultra-relativistic’ regime, \( \gamma \gg 1 \), where the string-intrinsic DBI form of the kinetic term is important. In this regime, inflation may be possible even if the scalar field potential is not flat.

2.4. The inflaton potential
Having established the inflaton dynamics, we now identify the potential \( V(\phi) \). There is only one term in the expansion of equation (6) which does not contain field derivatives and hence we conclude that
\[ V(\phi) = 2T(\phi) = \frac{2T_3}{\tilde{h}(\phi)}. \] (11)
So far we have not specified the form of the warp factor \( \tilde{h}(\phi) \). In the case of a KS background perturbed by one anti-D3 brane at the bottom of the throat, we have [22, 51]
\[ \tilde{h}(\phi) = \frac{2T_3}{M^4} \left[ 1 + \left( \frac{\mu}{\phi} \right)^4 \right], \] (12)
leading to the potential
\[ V(\phi) = \frac{M^4}{1 + (\mu/\phi)^4}. \] (13)
In the case where \( \phi \gg \mu \), the first-order expansion in \( \mu/\phi \) reads
\[ V(\phi) \simeq M^4 \left[ 1 - \left( \frac{\mu}{\phi} \right)^4 \right], \] (14)
which is the most commonly used form of the KKLMMT Coulomb-type potential. The full expression for the potential as well as its approximate expression are represented in figure 1.
Figure 1. Left panel: the dotted blue line represents the potential given by equation (13). The solid green line shows the approximate potential for $\phi \gg \mu$; see equation (14). The conventional slow-roll phase occurs while the field is rolling on the extremely flat region to the right of the plot. Right panel: same as the left panel, but in logarithmic units for the potential.

2.5. Parameter sets

From the string theory point of view, a complete description of the KKLMMT model is given by the parameter set

$$(g_s, \alpha', v, M, K),$$

(15)

where we recall that $g_s$ and $\sqrt{\alpha'}$ are the string coupling and length, respectively. The quantity $v$ is the (dimensionless) volume ratio of the five-dimensional sub-manifold forming the basis of the six-dimensional conifold geometry to the volume of the 5-sphere

$$v = \frac{\text{Vol}(X_5)}{\text{Vol}(S_5)}. $$

(16)

For instance, the base of the KS throat is the Einstein space $T_{1,1}$; hence in this case one has $v = 16/27$ [54]. The quantities $M$ and $K$ are two positive integers associated with the separately quantized background fluxes. They satisfy

$$\mathcal{N} = K M, $$

(17)

where $\mathcal{N}$ is a positive integer representing the total background RR charge.

Another set of parameters equivalent to (15) is given in terms of the extra dimensions’ geometry. The two natural length scales are the location of the bottom of the throat $r_0$, which corresponds to the field value

$$\phi_0 = \sqrt{\mathcal{T}_3} r_0,$$

(18)

and the throat edge $r_{\text{UV}}$. The position $r = r_{\text{UV}}$ where the throat is smoothly glued into the rest of the extra dimensional bulk. Provided the depth of the throat is comparable to its width, the edge field value can be approximated by [24]

$$\phi_{\text{UV}} = \sqrt{\mathcal{T}_3} r_{\text{UV}}, $$

(19)
with
\[ r_{UV}^4 = 4\pi g_s \alpha' r^2 \frac{N}{v}. \] (20)

Consequently, one can deduce the maximum warp factor between the edge and the bottom of the throat \[55\]
\[ \left( \frac{\phi_{UV}}{\phi_0} \right)^4 = \left( \frac{r_{UV}}{r_0} \right)^4 \simeq \exp \left( \frac{8\pi K}{3g_s M} \right), \] (21)

where the anti-D3’s small perturbation to the KS warp factor has been neglected. The set of geometrical parameters is therefore given by \( (g_s, \alpha', \phi_0, \phi_{UV}, N) \). (22)

The potential expression in equation (13) contains the parameters \( M \) and \( \mu \), which are defined from the string parameters as
\[ M^4 = \frac{4\pi^2 v \phi_0^4}{N}, \quad \mu^4 = \frac{\phi_0^4}{4\pi^2 v}. \] (23)

Note that \( v \) can be expressed through \( M \) and \( \mu \),
\[ v = \frac{M^4}{4\pi^2 \mu^4}. \] (24)

Since both \( M \) and \( \mu \) are proportional to the ratio \( \phi_0^4/N \), one has to provide another parameter to maintain equivalence with the set of equations (15) and (22), for instance \( N \). In that case, equation (23) can be inverted for \( \phi_0 \):
\[ \phi_0 = \mu N^{1/4} = \frac{M}{\sqrt{2\pi}} \left( \frac{N}{v} \right)^{1/4}. \] (25)

The model is thus equally well described by the ‘cosmological’ parameter set \( (g_s, \alpha', M, \mu, N) \). (26)

2.6. Domain of validity

We move on to discuss the domain of validity of the effective field theory description. It turns out that in order for equation (2) to be justified, one has to impose some constraints on the model parameters and the accessible field values \( \phi \).

2.6.1. Throat within the overall six-dimensional volume. In equation (3), the four-dimensional gravitational coupling constant \( \kappa \) is expressed as a function of the four-dimensional Planck mass, which, in a ten-dimensional geometry, depends on the volume of the compactified extra dimensions,
\[ m_{Pl}^2 = 8\pi \frac{V_6^{\text{tot}}}{\kappa_{10}}, \] (27)

where \( \kappa_{10} \) is related to the string parameters by
\[ \kappa_{10} = \frac{1}{2(2\pi)^7 g_s^2 \alpha'}. \] (28)
and $V_{\text{tot}}^6$ is the total volume of the six compactified extra dimensions [56, 57]. Now remembering that the total six-dimensional volume consists of the volume $V_{\text{throat}}^6$ of the KS throat, where inflation takes place, plus the bulk volume $V_{\text{bulk}}^6$, into which this throat is glued at $r = r_{\text{UV}}$, one has

$$V_{\text{tot}}^6 = V_{\text{bulk}}^6 + V_{\text{throat}}^6,$$

(29)

implying $V_{\text{tot}}^6 / V_{\text{throat}}^6 > 1$. Although we ignore the precise form of the six-dimensional bulk, the throat volume can be calculated from [24]

$$V_{\text{throat}}^6 = 2\pi^2 g_s N_{\alpha'}^2 r_{\text{UV}}^2,$$

(30)

with $r_{\text{UV}}$ given by equation (19). Then, rewriting equation (27) as

$$m_{\text{Pl}}^2 = 8\pi \frac{V_{\text{throat}}^6}{\kappa_{10}} \frac{V_{\text{tot}}^6}{V_{\text{throat}}^6},$$

(31)

gives the total to throat volume ratio

$$\frac{V_{\text{tot}}^6}{V_{\text{throat}}^6} = \alpha' m_{\text{Pl}}^2 \sqrt{\frac{4\pi^3 g_s v}{N^3}}.$$  

(32)

Since this ratio must exceed 1, we obtain a condition on the possible combinations of $N$ and $v$, depending on the choice of $g_s$ and $\alpha'$:

$$N^{3/2} v^{-1/2} < 2\pi^{3/2} \alpha' m_{\text{Pl}}^2 g_s^{1/2}.$$  

(33)

Note that this condition heavily depends on the choice of $g_s$ and $\alpha'$, the value of which can vary significantly according to the specific string theory considered. It is also convenient to recast equation (33) in terms of $\phi_{\text{UV}}$. From equations (19) and (32), one has

$$\kappa \phi_{\text{UV}}^2 = \frac{4}{N} \frac{V_{\text{throat}}^6}{N^3 V_{\text{tot}}^6},$$

(34)

and the volume ratio limit (33) implies

$$\phi_{\text{UV}} < \frac{m_{\text{Pl}}}{\sqrt{2\pi N}}.$$  

(35)

Let us also remark that, for the supergravity effective description to be valid, the volume of the extra dimensions, and therefore of the throat, should be larger than $\alpha'^3$, even when the volume of the five-dimensional conifold basis is very small. However, one can show that this condition does not play a role in the following analysis.

2.6.2. Inflation within a single throat. We will restrict our attention to the case where the D3 brane begins its journey towards the anti-D3 brane inside the KS throat, i.e. for

$$\phi < \phi_{\text{UV}},$$

(36)

where $\phi_{\text{UV}}$ is the field value on the throat edge (see equation (19)). In cosmological terms, this means that all the e-folds required to solve the standard big bang model problems occur inside one and the same throat. We will discuss in section 6 in which sense equation (36) allows us to impose restrictions on the parameter values.
2.6.3. End of brane motion by an instability-like mechanism. The derivation of (13) assumes that brane interaction is due to the exchange of the lightest closed string modes only. This is only approximately true: firstly, one could include the contribution of heavier modes (whose propagation through the bulk is Yukawa suppressed) as well as considering more general effects such as the Kähler potential necessary for stabilizing the moduli. In this case, deriving the full brane–anti-brane interaction potential is highly non-trivial. A phenomenological approach is to assume that these effects can be summarized in an additional potential term (e.g. of the form $V(\phi) = m^2 \phi^2$) which can be added to equation (13). This route has been explored in the literature; see e.g. [45]. Very recently, the theoretical underpinning of the form and origin of these terms has also received much attention [24]–[28]. For our cosmological purposes, these effects have not been considered, following the argumentation outlined earlier, based on the accuracy of the present data. Secondly, in addition to the above-mentioned corrections to the potential, the effective field model intrinsically has a restricted domain of validity. When the two branes come too close in terms of their proper distance $s$ (with respect to the string scale $\ell_s$), an open string can stretch between them and a tachyon, being the lightest excitation of this open string, appears. Let us determine the value $\phi_{\text{strg}}$ for which the proper distance between the branes $s_{\text{end}} \simeq \ell_s$, where $\ell_s = \sqrt{\alpha'}$. It is solution of the following equation [45]:

$$\int_{s_0}^{s_{\text{end}}} ds = \int_{r_0}^{r_{\text{strg}}} h^{1/4}(r) dr,$$

where the origin $s_0$ can be chosen at the brane annihilation location $r_0$, i.e. $s_0 = 0$, with $r_0$ the radial coordinate of the bottom of the throat. Using the warp factor of the unperturbed background, $h(r) = r_{\text{UV}}^4/r^4$, one gets $s_{\text{end}} = r_{\text{UV}} \ln(r_{\text{strg}}/r_0)$. Since $s_{\text{end}} = \sqrt{\alpha'}$, we have $r_{\text{strg}} = r_0 e^{\sqrt{\alpha'}/r_{\text{UV}}}$, which after normalization gives

$$\phi_{\text{strg}} = \phi_0 e^{\sqrt{\alpha'}/r_{\text{UV}}} = \mu N^{1/4} \exp \left[ \left(4\pi g_s \frac{N}{v}\right)^{-1/4}\right],$$

where use has been made of equations (19) and (25). Notice that $\phi_{\text{strg}}$ explicitly depends on the background flux $N$. As a consequence, another equivalent formulation of the parameter set (26) is therefore given by

$$(g_s, \alpha', M, \mu, \phi_{\text{strg}}).$$

From equation (38), one immediately gets

$$\phi_{\text{strg}} > \mu,$$

since the background flux number $N \geq 1$. Although the string theory details of the tachyon appearance and brane annihilation are beyond the scope of this work, we will assume that these processes trigger a reheating era which precedes the cosmological radiation dominated era (see section 8.1).
2.7. Equations of motion

The equations of motion for the inflaton in a FLRW metric on the brane can be obtained from the total action (6). From the stress tensor associated with the scalar field \( \phi \), the energy density and pressure read

\[
\rho = (\gamma - 1) T(\phi) + V(\phi), \quad P = \frac{\gamma - 1}{\gamma} T(\phi) - V(\phi),
\]

from which one obtains the Friedmann–Lemaître equations

\[
H^2 = \frac{\kappa}{3} \left[ T(\phi) (\gamma - 1) + V(\phi) \right], \quad \dot{H} = \frac{\kappa}{2} T(\phi) \left( \frac{1}{\gamma} - \gamma \right).
\]

When the velocity of the field is small, the Lorentz factor \( \gamma \) defined in (10) is close to 1 and its square root can be expanded. Likewise, varying the action with respect to the inflaton field leads to the following ‘Klein–Gordon-like’ equation of motion

\[
\frac{d^2 \phi}{dN^2} + \left( \frac{3}{\gamma^2} + \frac{d \ln H}{dN} \right) \frac{d \phi}{dN} + \left( \frac{3}{\gamma^2} - 1 \right) \frac{T'(\phi)}{2H^2} + \frac{V'(\phi) - T'(\phi)}{H^2\gamma^3} = 0,
\]

where, in the present context, a prime denotes a derivative with respect to \( \phi \). Note that \( N \) here is the number of e-folds (not to be confused with the background flux \( N \)), which will be used as a convenient measure of time. In the limit \( \gamma \to 1 \), equations (42) and (43) reduce to the standard Friedmann–Lemaître and Klein–Gordon equations. It will be important to establish in which regime the DBI corrections to the kinetic term (and therefore, deviation from the standard slow-roll dynamics) may be important.

2.8. Field evolution overview

Brane inflation according to the KKLMMT scenario incorporates both features of usual slow-roll inflation as well as intrinsic string theory effects. Let us sketch the successive phases of the field evolution as \( \phi \) rolls down the potential \( V(\phi) \) towards its small values.

In figure 2 (left panel), one can distinguish a region (for very large field values, dark green shaded) in which the quantum character of the inflaton field cannot be neglected: quantum fluctuations are of the same order as the classical ones where the potential is extremely flat, i.e. for \( \phi \) exceeding a certain \( \phi_{\text{fluct}} \). This is notably of interest if the brane starts its journey above this limit. The peculiar properties of this regime will be discussed in section 3 and the appendix.

For the moment, let us assume that the brane motion starts at \( \phi_{\text{in}} < \phi_{\text{fluct}} \). In this region, the potential (13) is still very flat since \( \phi \gg \mu \), allowing for most of the inflationary expansion to take place. One may therefore expect the usual slow-roll approximation to be valid and it will be used in section 4 to derive the shape of the induced primordial power spectra.

Rolling further to the left, the field \( \phi \) eventually enters a region where the slope of the potential becomes noticeable (light green shaded region). Therefore, the conventional slow-roll approximations are certainly no longer sufficient for describing this evolution, and, since the field velocity should increase, one may expect the DBI dynamics to be the driving force for \( \phi < \phi_{\text{DBI}} \). In addition, \( \phi_{\text{strg}} \) is also located around this potential domain. The precise order of events in this region will be discussed in section 5.
Brane inflation and the WMAP data

Figure 2. Left panel: sketch of the expected dynamical regimes according to the vev of the inflaton field for the potential (13). The field starts out in the flat (no hatching) region of the potential and rolls towards smaller field values. Eventually, it will reach the light green hatched region where the slope of the potential becomes noticeable and where the DBI effects can no longer be neglected. This regime may or may not be reached according to the value of $\phi_{\text{strg}}$ for which the derivation of equation (13) breaks down. On the far right (dark green hatched region), the potential is extremely flat and quantum fluctuations are expected to dominate over the field classical evolution. Right panel: the slow-roll parameters $\epsilon_1$ (green line) and $\epsilon_2$ (blue line) for the KKLMMT potential. The solid curves have been obtained for $\mu/m_{\text{Pl}} = 0$ whereas the dotted ones have been obtained for $\mu/m_{\text{Pl}} = 0.01$. Decreasing $\mu/m_{\text{Pl}}$ increases the differences between $\phi_{\epsilon_1}$ and $\phi_{\epsilon_2}$, the field values for which $\epsilon_1 = 1$ and $\epsilon_2 = 1$, respectively. Notice that we always have $\phi_{\epsilon_2} > \phi_{\epsilon_1}$.

3. Stochastic inflation

In this section, we will denote the classically predicted field value by $\phi_{\text{cl}}$, and consider the effect of the quantum fluctuations on the classical trajectory [58]–[61]. The field values for which quantum fluctuations are comparable to the classical ones can be obtained by first estimating the size of the classical fluctuations in the slow-roll regime. Denoting as $V_{\phi_{\text{cl}}}$ the potential derivative with respect to the field, one gets

$$\Delta \phi_{\text{cl}} \simeq - \frac{V_{\phi_{\text{cl}}}}{3H(\phi_{\text{cl}})} \Delta t,$$

which comes from the Klein–Gordon equation on neglecting $\ddot{\phi}$. For $\Delta t$ it is convenient to take the typical timescale at work during expansion, namely one Hubble time, $\Delta t = 1/H$.

Concerning the quantum fluctuations, one has for a test field in de Sitter space–time,

$$\Delta \phi_{\text{qu}} \simeq \frac{H(\phi_{\text{cl}})}{2\pi}.$$

The field value $\phi_{\text{fluct}}$, for which classical and quantum fluctuations are of equal amplitude, verifies $\Delta \phi_{\text{cl}}(\phi_{\text{fluct}}) = \Delta \phi_{\text{qu}}(\phi_{\text{fluct}})$. From equations (44) and (45) it follows that

$$\frac{\phi_{\text{fluct}}}{\mu} \simeq \left[ \frac{3}{8\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left( \frac{m_{\text{Pl}}}{M} \right)^4 \right]^{1/10}.$$

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Using equation (23), we can express (46) in terms of $v$ and the overall scale of inflation $M/m_{\text{Pl}}$. According to their value, $\phi_{\text{fluct}}$ can generically be much larger than the minimal initial field value needed to produce $N_T \simeq 10^2$ e-folds of inflation. To ensure the consistency of the semi-classical approach, we require

$$\phi_{\text{in}} < \phi_{\text{fluct}},$$

in the following. Quantum effects can be nevertheless described using the stochastic approach [60,61] which is presented in detail in the appendix. Since the initial value of the field is also limited by the requirement of having the D3 brane inside the throat, the effective semi-classical model remains valid only if $\phi_{\text{in}} < \min(\phi_{\text{fluct}}, \phi_{\text{UV}})$.

### 4. The slow-roll regime

In this section we analyse the KKLMMT potential according to its slow-roll characteristics. When denoted as in (13), the analogy between $V(\phi)$ and small field models (investigated in detail in [34]) is evident. Notice, however, that for small field inflation, $\phi$ is increasing while inflation is under way, whereas KKLMMT inflation proceeds from large to small field values (see figure 1).

#### 4.1. Slow-roll parameters

Among the several definitions of the slow-roll parameters, we here use the Hubble-flow functions $\epsilon_n$ defined by [62]–[64]

$$\epsilon_{n+1} \equiv \frac{d \ln |\epsilon_n|}{dN}, \quad n \geq 0,$$

where $N$ is the number of e-folds since some initial time $\eta_{\text{in}}$. The above hierarchy starts from $\epsilon_0 = H_{\text{in}}/H$. Slow-roll inflation proceeds as long as $|\epsilon_n| \ll 1$, for all $n > 0$, while inflation takes place if the scale factor is accelerating, i.e. $\epsilon_1 < 1$.

From the potential (13), the first two Hubble-flow parameters in their slow-roll formulation read

$$\epsilon_1 = \frac{1}{\pi} \left(\frac{m_{\text{Pl}}}{\mu}\right)^2 \frac{(\phi/\mu)^{10}}{\left[1 + (\phi/\mu)^{-4}\right]^2},$$

$$\epsilon_2 = \frac{1}{\pi} \left(\frac{m_{\text{Pl}}}{\mu}\right)^2 \frac{(\phi/\mu)^{-6}}{\left[1 + (\phi/\mu)^{-4}\right]^2},$$

and they are represented in figure 2 (right panel). They depend on the ratio $\phi/\mu$ and the vev energy scale $\mu/m_{\text{Pl}}$ (or $M/m_{\text{Pl}}$, by virtue of equation (23)) only. As mentioned above, inflation ends when $\epsilon_1 = 1$, while the slow-roll approximation itself breaks down when $\epsilon_2 = 1$ (and/or $\epsilon_1 = 1$). Let us stress again that the field theory description of the brane motion fails at $\phi = \phi_{\text{strg}}$. 
4.2. Classical field trajectory

The next step is to obtain the classical field trajectory; this is possible while the slow-roll approximation is valid but even in this case, the trajectory is found only implicitly. The total number of e-folds $N(\phi)$ reads

$$N(\phi) = 2\pi \frac{\mu^2}{m_{Pl}^2} \left[ \frac{1}{6} \left( \frac{\phi_{in}}{\mu} \right)^6 + \frac{1}{2} \left( \frac{\phi_{in}}{\mu} \right)^2 - \frac{1}{6} \left( \frac{\phi}{\mu} \right)^6 - \frac{1}{2} \left( \frac{\phi}{\mu} \right)^2 \right].$$

(51)

For the generic situation where $\phi/\mu \gg 1$, the two quadratic terms are sub-dominant. Neglecting them yields the following explicit solution:

$$\frac{\phi(N)}{\mu} \simeq \left( \frac{\phi_{in}}{\mu} \right)^6 - \frac{3}{\pi} \left( \frac{m_{Pl}}{\mu} \right)^2 N^{1/6}.$$

(52)

As seen in figure 2, given that the field evolution starts on the very flat part of the potential (but outside the domain of quantum fluctuations), there will surely be an initial phase of evolution during which the slow-roll approximation, and hence the trajectory (52), is valid. However, moving towards smaller field values, this trajectory will eventually become inaccurate and one has to determine at which field value the slow-roll approximation breaks down and how this field value compares to the end of inflation and/or to $\phi_{strg}$.

4.3. When does the slow-roll regime break down?

The slow-roll approximation breaks down when one of the slow-roll parameters becomes of order 1, which need not coincide with the end of accelerated expansion ($\epsilon_1 = 1$). If slow-roll breaks down first, inflation can still proceed (typically during a very small number of e-folds), but the evolution of the field is no longer described by the slow-roll trajectory (52). Let us define the field values $\phi_{\epsilon_1}$ and $\phi_{\epsilon_2}$ by

$$\epsilon_1(\phi_{\epsilon_1}) = 1, \quad \epsilon_2(\phi_{\epsilon_2}) = 1.$$

(53)

From equations (49) and (50), one obtains the algebraic equations

$$1 + 2 \left( \frac{\mu}{\phi_{\epsilon_1}} \right)^4 + \left( \frac{\mu}{\phi_{\epsilon_1}} \right)^8 - \frac{1}{\pi} \left( \frac{m_{Pl}}{\mu} \right)^2 \left( \frac{\mu}{\phi_{\epsilon_1}} \right)^{10} = 0,$$

(54)

and

$$1 + 2 \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^4 + \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^8 - \frac{5}{\pi} \left( \frac{m_{Pl}}{\mu} \right)^2 \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^6 - \frac{1}{\pi} \left( \frac{m_{Pl}}{\mu} \right)^2 \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^{10} = 0.$$

(55)

These equations cannot be solved explicitly (except numerically). Some approximate analytical solutions are derived in the following.

4.3.1. Case $\mu > m_{Pl}$ with $\phi_{\epsilon_1, \epsilon_2} < \mu$. In this case, keeping only the two dominant terms in equation (54) gives

$$\left( \frac{\mu}{\phi_{\epsilon_1}} \right)^8 - \frac{1}{\pi} \left( \frac{m_{Pl}}{\mu} \right)^2 \left( \frac{\mu}{\phi_{\epsilon_1}} \right)^{10} \simeq 0,$$

(56)
whose solution is
\[ \frac{\phi_{\epsilon_1}}{\mu} \simeq \frac{m_{\text{Pl}}}{\mu} \frac{1}{\sqrt{\pi}}. \]  
(57)

This expression is consistent with our assumptions: \( \mu > m_{\text{Pl}} \) does indeed imply \( \phi_{\epsilon_1} < \mu \).

Still in the same limit, the three remaining terms in equation (55) are
\[ \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^8 - \frac{5}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left( \frac{1}{\phi_{\epsilon_2}} \right)^6 - \frac{1}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left( \frac{\mu}{\phi_{\epsilon_2}} \right)^{10} \simeq 0. \]  
(58)

This equation has an exact solution with two acceptable roots
\[ \left( \frac{\phi_{\epsilon_2}}{\mu} \right) \pm \frac{m_{\text{Pl}}}{\mu} \sqrt{\frac{2}{\pi}} \left[ 1 \pm \sqrt{1 - \frac{20}{\pi^2} \left( \frac{m_{\text{Pl}}}{\mu} \right)^4} \right]^{-1/2}. \]  
(59)

For \( \mu > m_{\text{Pl}} \), slow-roll inflation can proceed until the field \( \phi \) reaches \( \max(\phi_{\epsilon_1}, \phi_{\epsilon_2}) < \mu \). By comparing equations (57) and (59), we further see that \( \phi_{\epsilon_2} > \phi_{\epsilon_1} \), i.e. the slow-roll approximation breaks down before the end of inflation.

However, as already shown in section 5, the KKLMMT model itself is no longer well defined for \( \phi < \phi_{\text{strg}} \) due to the stringy origin of the potential (13). The value \( \phi_{\text{strg}} \) given in equation (38) is always greater than \( \mu \). Hence, in the case where \( \mu > m_{\text{Pl}} \), inflation will definitely come to an end at \( \phi_{\text{strg}} > \max(\phi_{\epsilon_1}, \phi_{\epsilon_2}) \). As a result, the entire field evolution occurs in the slow-roll regime and, as will be shown in the following, we do not need to worry about DBI effects which remain negligible in that case.

4.3.2. Case \( \mu < m_{\text{Pl}} \) with \( \phi_{\epsilon_1,\epsilon_2} > \mu \). In this limit, the second and third terms of equation (54) are small compared to 1, which leads to
\[ \frac{\phi_{\epsilon_1}}{\mu} \simeq \left( \frac{1}{\sqrt{\pi}} \frac{m_{\text{Pl}}}{\mu} \right)^{1/5}. \]  
(60)

Similarly, in equation (55), the second and third terms are small compared to 1, while the last term is small compared to the fourth, so by keeping only the two dominant terms we obtain
\[ \frac{\phi_{\epsilon_2}}{\mu} \simeq \left[ \frac{5}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \right]^{1/6}. \]  
(61)

Comparing equation (61) and (60) shows that one still has \( \phi_{\epsilon_2} > \phi_{\epsilon_1} \), and the slow-roll approximation breaks down before inflation stops. This is confirmed in figure 2. However, this time, the field does not necessarily reach \( \phi_{\text{strg}} \) first and non-slow-roll inflation may continue between \( \phi_{\epsilon_2} \) and \( \phi_{\epsilon_1} \). Moreover, from \( \phi_{\epsilon_1} \) to \( \phi_{\text{strg}} \), the field encounters a transitory regime interpolating between the conventional slow-roll and DBI dynamics. As discussed above, all the slow-roll formulae are strictly speaking no longer valid after \( \epsilon_2 = 1 \). As a result, the region \( \phi < \phi_{\epsilon_2} \) is only accessible numerically and we will show in this way that the number of e-folds occurring between \( \phi_{\epsilon_2} \) and \( \phi_{\epsilon_1} \) (or \( \phi_{\text{strg}} \)) is actually small. The key lesson to be learnt from \( \phi_{\epsilon_2} > \phi_{\epsilon_1} \) is that whenever we want to rely on analytic results of the slow-roll calculus, we have to confine the field to values \( \phi > \phi_{\epsilon_2} \) for consistency and
keep in mind that the results will be applicable only if the number of e-folds occurring after $\phi_{e2}$ remains negligible.

To proceed further, let us summarize the possible field evolution given that $\mu < m_{Pl}$: the first option is that, also in this case, the field reaches the value $\phi_{strg}$ first, i.e. $\phi_{strg} > \phi_{e2}$; this is possible depending on the values of $N$ and $v$ for the chosen background geometry (see equation (38)). In that case, the entire field evolution corresponds to slow-roll inflation and is described by the analytical trajectory (52).

If $\phi_{strg} < \phi_{e2}$, then the situation is more complex and depends on the exact order of $\phi_{e1}$, $\phi_{DBI}$ and $\phi_{strg}$. For instance, if $\phi_{e1}$ is the next field value to be reached after $\phi_{e2}$, the accelerated expansion would come to a halt here for a field $\phi$ with standard dynamics, rendering the number of e-folds between $\phi_{e2}$ and $\phi_{e1}$ extremely small. However, the subsequent evolution for $\phi < \phi_{e1}$ could bring the field into the regime of DBI dominance and inflation may restart. At the same time, $\phi_{strg}$ could be located between $\phi_{e1}$ and $\phi_{DBI}$ and, as a consequence, $\phi_{DBI}$ would never be reached. All other combinations are of course a priori possible. Which one is realized in practice depends on the value of the free parameters characterizing the model.

5. String-intrinsic aspects

We now move on to investigating those aspects of the KKLMMT model due to its stringy origin. In particular, two questions remain to be properly addressed: the unusual DBI dynamics, and the appearance of a tachyon once the proper brane distance approaches the string scale.

5.1. When does inflation end?

The potential (13) results from string theory under the assumption that only the lightest closed string modes (gravitons and the RR particles) contribute to the brane interaction. When the proper distance $s$ between the branes becomes comparable to the string length scale $\ell_s$, the exchange of heavier closed modes become relevant, and an open string can stretch from one brane to the other. The appearance of a tachyon triggers the brane annihilation process, or reheating from a cosmological point of view.

We already calculated the field value $\phi_{strg}$ at which the tachyon appears in equation (38). The crucial question is, does the field first reach the value $\phi_{strg}$ or $\phi_{e2}$? If $\phi_{e2} < \phi_{strg}$, inflation ends at brane annihilation for $\phi_{end} = \phi_{strg}$ and the field $\phi$ spends its entire ‘lifetime’ in the usual slow-roll regime. Such is the course of events when $\mu > m_{Pl}$. For $\mu < m_{Pl}$, one has to compare $\phi_{e2}$ given in equation (61), with $\phi_{strg}$ given by equation (38). Their ratio reads

$$\frac{\phi_{e2}}{\phi_{strg}} = \left( \frac{M}{m_{Pl}} \right)^{-1/3} \mathcal{N}^{-1/4} v^{1/12} 10^{1/6} \exp \left[ - \left( 4 \pi g_s \mathcal{N} \right)^{-1/4} \right].$$

This expression involves the energy scale $M/m_{Pl}$ of the potential because $\phi_{e2}$ and $\phi_{strg}$ both depend on $M/m_{Pl}$, but not in the same way. However, from an observational point of view, this scale is involved in the amplitude of the cosmological perturbations and certainly fixed by the CMB normalization. Therefore, we may go further and access the above ratio using the current WMAP measurement of the CMB quadrupole.
Let us call the field value at the time when the wavelength of the cosmological perturbations of physical interest today left the Hubble radius during inflation \( \phi_s \). The slow-roll field trajectory allows us to express \( \phi_s \) in terms of \( N_\ast \), the number of e-folds between the time of Hubble exit and the end of inflation at \( \phi_{\text{end}} \). Using the approximated classical trajectory (52), this leads to

\[
\left( \frac{\phi_s}{\mu} \right)^6 \simeq \left( \frac{\phi_{\text{end}}}{\mu} \right)^6 + \frac{3}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 N_\ast.
\]

(63)

To proceed further, we have to specify when exactly inflation ends. But, at the same time, this is also the question that we try to address. Changing the mechanism which stops inflation will change the scale \( M/m_{\text{Pl}} \) and, hence, will affect the ratio (62).

It is convenient to determine the frontier \( \phi_{\epsilon_2} = \phi_{\text{strg}} \) in the parameter space. Setting for convenience \( \phi_{\text{end}} = \phi_{\epsilon_2} \) in equation (63) gives

\[
\frac{\phi_s}{\mu} \simeq \left[ \frac{3}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left( N_\ast + \frac{5}{3} \right) \right]^{1/6}.
\]

(64)

Note that, had we used \( \phi_{\text{end}} = \phi_{\epsilon_1} \) instead, the result would be (64) up to the replacement \( N_\ast \to N_\ast \). Therefore, ending inflation at \( \phi_{\epsilon_2} \) instead of \( \phi_{\epsilon_1} \) only causes a small shift in \( N_\ast \). This is expected since, as soon as \( \epsilon_2 > 1 \), slow-roll is violated and the field starts to evolve rapidly: even if the precise evolution can only be probed numerically (see section 5.2), the number of e-folds spent in this regime is generically small. Inserting the result (64) into the expressions (49) and (50) for the slow-roll parameters, one arrives at

\[
\epsilon_1 \simeq \frac{1}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left[ \frac{3}{\pi} \left( \frac{m_{\text{Pl}}}{\mu} \right)^2 \left( N_\ast + \frac{5}{3} \right) \right]^{-5/3},
\]

(65)

\[
\epsilon_2 \simeq \frac{5}{3N_\ast} \left( 1 + \frac{5}{3N_\ast} \right)^{-1} \simeq \frac{5}{3N_\ast}.
\]

(66)

As can be seen in these equations, the value of \( \epsilon_1 \) is generically much smaller than \( \epsilon_2 \). Moreover, the quantity \( \epsilon_2 \) does not depend on the scale \( \mu/m_{\text{Pl}} \), at first order. The CMB quadrupole normalization gives the additional relation [34]

\[
\frac{V_s}{m_{\text{Pl}}^4} \simeq \frac{45\epsilon_1 Q_{\text{rms-PS}}^2}{2 T^2},
\]

(67)

where, for \( \phi_s \gg \mu \), one may approximate \( V_s \simeq M^4 \). Using the above expression (65) for \( \epsilon_1 \), we find the relations

\[
\frac{M}{m_{\text{Pl}}} \simeq \left( \frac{45 Q_{\text{rms-PS}}^2}{T^2} \right)^{3/8} (6N_\ast + 10)^{-5/8} v^{-1/8},
\]

(68)

\[
\frac{\mu}{m_{\text{Pl}}} \simeq \left( \frac{45 Q_{\text{rms-PS}}^2}{T^2} \right)^{3/8} (6N_\ast + 10)^{-5/8} (2\pi)^{-1/2} v^{-3/8}.
\]

(69)

The numerical value of the WMAP quadrupole is [66]

\[
\frac{Q_{\text{rms-PS}}}{T} \simeq 6 \times 10^{-6}.
\]

(70)
Figure 3. Left panel: the contour $\phi_{\epsilon} = \phi_{\text{strg}}$ in the plane $(\ln v, \ln N)$, obtained from equations (62) and (71) using the normalization given by the CMB quadrupole with $N_s = 50$ (see equation (68)). The dotted line corresponds to $g_s = 0.1$, the dashed line to $g_s = 10^{-3}$ and the dotted-dashed one to $g_s = 10^{-5}$. In each case, the area enclosed by the contour is the part of the parameter space where the slow-roll conditions are violated before brane annihilation and $\phi_{\text{strg}}$ does not play an important role for the end of inflation. It is clear from this plot that the contour sensitively depends on the value of $g_s$. However, such a dependence can be absorbed by an appropriate rescaling of the parameters as shown in the right panel. The same contour $\phi_{\epsilon} = \phi_{\text{strg}}$ is represented in the plane $(\ln x, \ln \bar{v})$, these parameters being defined in equation (73). It is universal for all values of the string coupling $g_s$.

Inserting this expression into equation (62), the ratio $\phi_{\epsilon}/\phi_{\text{strg}}$ becomes a function of $N$ and $v$ only, at fixed string coupling $g_s$. Explicitly, one obtains

$$\frac{\phi_{\epsilon}}{\phi_{\text{strg}}} = C N^{-1/4} v^{1/8} 10^{1/6} \exp \left[ - \left( 4\pi g_s \frac{N}{v} \right)^{-1/4} \right], \quad (71)$$

where the constant $C$ reads

$$C = 10^{1/6} (6N_s + 10)^{5/24} \left( \frac{45 Q_{\text{rms}}^2}{T^2} \right)^{-1/8}. \quad (72)$$

In the plane $(v, N)$, the condition $\phi_{\epsilon} = \phi_{\text{strg}}$ is a curve separating the plane into the two domains $\phi_{\text{strg}} < \phi_{\epsilon}$ and $\phi_{\text{strg}} > \phi_{\epsilon}$. Notice that the shape and position of the frontier depend on the value of $g_s$; this is illustrated in the left panel of figure 3.

It is convenient to rescale the parameters $g_s$, $N$ and $v$ to minimize the dependence on $g_s$ of the contour $\phi_{\epsilon} = \phi_{\text{strg}}$. For this purpose, we define the new variables

$$x \equiv 4\pi g_s \frac{N}{v}, \quad \bar{v} \equiv \frac{v}{(4\pi g_s)^2}. \quad (73)$$

Usually, the analysis is performed with some specific values of $g_s$. In this paper, our strategy is different and more general. With the help of the above rescaling, our results will be valid for any values of the coupling constant. In terms of these new parameters,
the condition \( \phi_{\text{strg}} = \phi_{\epsilon_2} \) from equation (71) now reads, in logarithmic units,
\[
\ln \bar{v} = 8 \ln C - 2 \ln x - 8x^{-1/4}.
\]
(74)
This contour is represented in figure 3 (right panel). All dependence of this line on \( g_b \) has been absorbed in the rescaling, so one can now state universally that inside the contour, slow-roll breaks down before the model-intrinsic instability is reached \( (\phi_{\text{strg}} < \phi_{\epsilon_2}) \), whereas outside, slow-roll inflation proceeds all the way until brane annihilation \( (\phi_{\text{strg}} > \phi_{\epsilon_2}) \). In fact, this result is more general and we demonstrate in the following that the rescaling permits us to absorb the \( g_b \) dependence in all the equations expressing a physically relevant condition for the model.

Note that in the above discussion, we have used \( \phi_{\text{end}} = \phi_{\epsilon_2} \) to derive the normalization (68). In doing so, we have in fact ignored the other ‘stringy’ characteristic of the KKLMMT model, namely the DBI kinetic term. While brane evolution definitely ends at \( \phi_{\text{strg}} \), it is possible that, due to the unusual dynamics, a phase of ‘DBI inflation’ occurs even after violation of the slow-roll conditions. If a considerable number of e-folds could be produced in the DBI regime, the value of \( N_\ast \) used in equation (68) would no longer be correct. In the following, we discuss for which field value \( \phi_{\text{DBI}} \) those effects will be important and how much expansion may occur outside the slow-roll regime.

5.2. When is the DBI regime important?

An order of magnitude of the field value \( \phi_{\text{DBI}} \) for which the DBI regime is relevant can be obtained from the potential (11) by use of the condition (9):
\[
\dot{\phi}_{\text{DBI}}^2 \simeq V(\phi_{\text{DBI}}).
\]
(75)
For a standard kinetic term \( (\gamma = 1) \), this value would precisely coincide with \( \epsilon_1 = 1 \). This suggest that the DBI regime appears when standard inflation ends. Of course, given the increasingly non-canonical dynamics as the field is approaching \( \phi_{\text{DBI}} \), the standard slow-roll formula is no longer valid and should be replaced by its generalized form
\[
\epsilon_1 \equiv -\frac{\ln H}{dN} = \frac{1}{2} \kappa \gamma \left( \frac{d\phi}{dN} \right)^2 = \frac{2}{\kappa \gamma} \left( \frac{d\ln H}{d\phi} \right)^2,
\]
(76)
where use has been made of equation (10) and of the exact expression (42). Therefore, in the ultra-relativistic limit \( \gamma \gg 1 \), equation (76) suggests that the expansion of the universe may accelerate even with a steep potential. If such a second phase of inflation occurs for more than typically 60 e-folds, then the slow-roll phase becomes observationally irrelevant and the DBI regime crucial. We address this issue by analysing the DBI regime using both analytical and numerical methods.

The above expression for \( \epsilon_1 \) can be further simplified by using equation (10) to express the field derivatives in terms of \( \gamma \), \( H \) and \( T \) solely. The Friedmann–Lemaître equation (42) can then be used to remove any explicit dependence in the Hubble parameter in favour of \( \gamma \):
\[
H^2 = \frac{\kappa V}{3 - 2\epsilon_1/(1 + \gamma^{-1})},
\]
(77)
As a result, the first slow-roll parameter reads, in terms of $\gamma$, $V$ and $T$,

$$\epsilon_1 = \frac{3}{2} \frac{1 - \gamma^{-2}}{1 + \gamma^{-1} (V/T - 1)}. \quad (78)$$

Equations (77) and (78) are exact and clearly enhance the effect of $\gamma$. In the limit $\gamma \to 1$, one recovers the usual slow-roll formulae obtained with a standard kinetic term, whereas an approximate analytic solution can be derived in the ultra-relativistic limit $\gamma \gg 1$. Under the assumption $V/T = \mathcal{O}(1)$, at leading order in $\gamma^{-1}$, one gets

$$\epsilon_1 = \frac{3}{2} + \mathcal{O}(\gamma^{-1}), \quad (79)$$

showing that deep in the DBI regime the universe is not inflating but expands as in a matter dominated era. In our case $V/T = 2$ which ensures the validity of the above expansion. Notice, however, that this relationship does no longer hold if one considers an additional term in the potential, e.g. $m^2 \phi^2$. In that case the $V/T$ term in equation (78) can no longer be neglected and inflation may indeed proceed in the DBI regime [45, 46].

Although the universe is not inflating for $\gamma \gg 1$, we still have to check that the number of e-folds that the field spends in the DBI ‘matter dominated’ era remains small. From equations (76) and (79), the Hubble trajectory at leading order in $\gamma^{-1}$ reads

$$H(N) = H_1 \exp \left[ -\frac{3}{2} (N - N_1) \right], \quad (80)$$

where $H_1$ and $N_1$ respectively denote the Hubble parameter and the e-fold at which the limit $\gamma \gg 1$ becomes relevant. They could, for instance, be defined as the ones corresponding to the end of the slow-rolling phase provided that the transitory regime occurring between the conventional slow-roll and the region of DBI dominance is short. The field evolution at leading order is given in terms of $\gamma$ by equation (76) and reads

$$\kappa \left( \frac{d\phi}{dN} \right)^2 = 3\gamma^{-1} + \mathcal{O}(\gamma^{-2}). \quad (81)$$

The system of equations is closed by using expression (42) for the Hubble parameter

$$H^2 = \frac{\kappa T}{3} (\gamma + 1), \quad (82)$$

which is exact since, in our case, one has $V/T = 2$. As a consequence, one gets the field trajectory (neglecting one compared to $\gamma \gg 1$)

$$\frac{d\phi}{dN} \simeq -\frac{\sqrt{T(\dot{\phi})}}{H_1} \exp \left[ \frac{3}{2} (N - N_1) \right]. \quad (83)$$

This expression can be implicitly integrated in terms of the Gauss hypergeometric function through use of equations (11) and (13), yielding

$$N = N_1 + \frac{2}{3} \ln \left( 1 + \frac{3}{\sqrt{2}} \frac{\mu}{M} \frac{H_1}{M} \left\{ \sqrt{1 + \frac{x^4}{4}} - \frac{\sqrt{1 + x^4}}{x_1} \right\} ight.
\left. - \frac{2}{x_1} \frac{\sqrt{1 + \frac{x^4}{4}}}{1 + \frac{x^4}{16}} \right) + \frac{2}{x_1} \frac{\sqrt{1 + \frac{x^4}{4}}}{1 + \frac{x^4}{16}} \right) \right), \quad (84)$$
where $x \equiv \mu/\phi$. A more illuminating form can be obtained in both limits $x \gg 1$ and $x \ll 1$. In the limit $x \ll 1$, that is to say $\phi \gg \mu$, using the Taylor series defining the hypergeometric function, equation (84) becomes

$$N \simeq N_1 + \frac{2}{3} \ln \left[ 1 + \frac{3 H_1 \mu}{\sqrt{2} M M} \left( \frac{1}{x} - \frac{1}{x_1} \right) \right],$$

which gives for the field

$$\frac{\dot{\phi}}{\mu} \simeq \frac{\dot{\phi}_1}{\mu} - \frac{\exp \left[ (3/2) (N - N_1) \right] - 1}{(3/\sqrt{2})(H_1/M)(\mu/M)}.$$

Similarly, in the limit $x \gg 1$ or $\phi \ll \mu$, using the linear transformation formulae associated with the Gauss hypergeometric function [67], equation (84) yields

$$N \simeq N_1 + \frac{2}{3} \ln \left[ 1 + \frac{3 \sqrt{2} H_1 \mu}{2 M M} (x - x_1) \right],$$

and, therefore, inverting the previous relation, one obtains

$$\frac{\dot{\phi}}{\mu} \simeq \left\{ \frac{\mu}{\dot{\phi}_1} + \frac{\exp \left[ (3/2) (N - N_1) \right] - 1}{(3/\sqrt{2})(H_1/M)(\mu/M)} \right\}^{-1}.$$

In fact, the limit $x \gg 1$ ($\phi \ll \mu$) cannot be reached in the model under scrutiny since the tachyonic instability occurs for $\phi \simeq \phi_{\text{strg}} \geq \mu$. As a result, the ultra-relativistic DBI regime could only occur for $\phi > \mu$. As can be seen in equation (86), the corresponding field evolution is exponentially fast implying that the number of e-folds during which $\phi > \phi_{\text{strg}}$ and $\gamma \gg 1$ is negligible, provided $\dot{\phi}_1/\mu$ and the denominator are not too large. This is indeed the case for the following reasons. Firstly, as already mentioned, equation (78) shows that $\gamma$ can deviate from unity only if $\epsilon_1$ is also of order unity, i.e. the field should be in the non-flat region of the potential and therefore $\dot{\phi}_1 \gtrsim \mu$. As regards the denominator, as can be seen from equation (77), $H_1/M^2 \lesssim 1/m_{\text{Pl}}^2$ and this term is at most of order $\mu/m_{\text{Pl}}$. The constraint (35) coming from the size of the throat, together with the fact that brane annihilation occurs at $\phi_{\text{strg}} > \mu$, requires that

$$\mu < \frac{m_{\text{Pl}}}{\sqrt{2\pi}},$$

ensuring that $H_1 \mu/M^2$ cannot be large.

To end this section, we have numerically checked that the previous analysis was still qualitatively valid during the intermediate regime in which neither the slow-roll nor the ultra-relativistic approximations can be used. Figure 4 shows the last e-foldings of evolution for an extreme model, close to the limit (89), for which the effects of the DBI regime can be seen. The differences between the slow-roll approximation (under the standard kinetic term hypothesis) and the exact DBI integration appear only during less than an e-fold and close to the region where $\epsilon_1 = 1$. Remembering that the effective field description of the brane motion physically ends at $\phi_{\text{strg}}$, we conclude that for all practical purposes, the DBI regime has no cosmological observable effects for the model based on the potential (13). The situation, however, can be entirely different in models that include additional terms (e.g. $m^2 \phi^2$) in the potential, as shown in [45].
Figure 4. Evolution of the field $\phi$, the Hubble-flow functions $\epsilon_1$, $\epsilon_2$ and the DBI parameter $\gamma$ in the last e-foldings of an extreme model living close to the throat edge ($\mu = m_{Pl}/\sqrt{32\pi}$). This model has been chosen to emphasize the DBI effects: the solid lines correspond to an exact numerical integration of the full action (6) whereas the dashed lines are obtained by using the slow-roll approximations (49)–(51). The DBI regime smoothly connects the slow-roll evolution ($\gamma \simeq 1$) to the ultra-relativistic matter-like expansion ($\gamma \gg 1$). Note that in the model at hand, brane annihilation occurs at $\phi_{str} > \mu$ preventing any observable effects coming from the DBI evolution.

6. Theoretical restrictions on the parameter space

We are ready to conclude our analytical investigation of the KKLMMT inflationary model by deriving restrictions on its free parameters. For our numerical calculations and for comparison with the WMAP3 data, we use the observable parameter set $(M, \mu, N)$, where $M$ and $\mu$ are related by equation (24). There are essentially three theoretical consistency relations to be satisfied.

6.1. Constraints from the size of the throat

For the model to be consistent, the warped throat in which the D3 and anti-D3 branes are located should be smaller than the total size of the compactified sub-manifold spanned by the extra dimensions. Earlier, we derived the restriction (33) from this condition. It turns out that this condition can be conveniently rewritten in terms of our rescaled parameters $(x, \bar{v})$ defined in equation (73) as

$$\ln \bar{v} < \ln \left(\alpha' m_{Pl}^2 / \pi \right) - \frac{3}{2} \ln x.$$  \hspace{1cm} (90)
Figure 5. Allowed regions (ticks in) in the rescaled parameter plane \((\ln x, \ln \bar{v})\) for \(\alpha' m_P^2 = 1000\) (left panel) and \(\alpha' m_P^2 = 10\) (right panel) with the fiducial values \(N_s = 50\) and \(N_T = 60\). The black dotted curve is the contour \(\phi_{\epsilon_2} = \phi_{\text{strg}}\) of figure 3. For all points located inside this contour, the slow-roll conditions are violated before brane annihilation. The green dashed line (with ticks down) represents the volume ratio constraint \((90)\), whose slope is universal but whose offset depends on \(\alpha' m_P^2\). Regions above this line are therefore excluded. The solid green curve (with ticks up) represents the condition that all of the brane evolution occurs within one throat, and has been obtained through a numerical integration. All points below that curve would not satisfy this condition. Its shape can be piecewise analysed. In the region \(\phi_{\text{strg}} < \phi_{\epsilon_2}\) (inside the black dotted contour), this is a straight line given by equation \((92)\). The slope of this line is universal, but the offset again depends on \(\alpha' m_P^2\). Outside the dotted contour, namely for \(\phi_{\text{strg}} > \phi_{\epsilon_2}\) and in the limit \(\phi_{\text{strg}} \gg \phi_{\epsilon_2}\), this boundary is described by equation \((101)\). Combined, these pieces lead to the solid green curve with ticks up; to make the shapes of the two pieces visible individually, they have been extended outside their respective domains of validity (dotted–dashed black curve).

This is a straight line in the plane \((\ln x, \ln \bar{v})\) whose offset depends on \(\alpha'\) (see figure 5). A significant fraction of the parameter space is cut out by this requirement. Notice that, as announced above, the rescaling \((73)\) has removed any dependence of this bound on \(g_s\) and it is universal in the sense that it does not involve the precise inflationary trajectory, nor the mechanism that ends inflation.

6.2. Constraints from starting inflation inside the throat

For all the e-folds to occur inside one throat, we enforce \(\phi < \phi_{\text{UV}}\). Since \(\phi\) decreases during inflation, this restriction applies to the initial field value \(\phi_{\text{in}}\). On the other hand, there is a lower bound on \(\phi_{\text{in}}\) such that at least \(N_T\) e-folds are produced. This bound in turn depends on the mechanism ending the inflationary expansion.

Let us start with the case in which the slow-roll approximation is violated before brane annihilation. Making use of the classical trajectory \((52)\), the initial field value necessary to produce \(N_T\) e-folds before \(\phi_{\epsilon_2}\) can be derived from equation \((61)\) and reads

\[
\frac{\phi_{\text{in,\epsilon_2}}}{\mu} \simeq \left\{ \frac{3}{\pi} \left( \frac{m_P}{\mu} \right)^2 \left[ \frac{5}{3} + N_T \right] \right\}^{1/6}.
\]
For the KKLMMT case, imposing the CMB normalization (69) calculated earlier and using equation (19), one can recast the bound $\phi_{m,2} < \phi_{UV}$ in terms of the rescaled variables

$$\ln \tilde{v} = - \ln x - 4 \ln D,$$

where the constant $D$ reads

$$D = (\pi \alpha' m_{Pl}^2)^{-1/2} (6N_s + 10)^{5/12} (6N_T + 10)^{-1/6} \left(45 \frac{Q_{rms-PS}}{T^2}\right)^{-1/4}.\quad (93)$$

This is also a straight line in the plane $(x, \tilde{v})$ and further constrains from below the parameter space (see figure 5). As before, all the $g_s$ dependence has been removed and only the $\alpha'$ one remains in the expression for the offset. This offset also depends on $N_s$ and $N_T$ but only weakly because these two quantities appear in a logarithm. In the following, the fiducial values $N_s = 50$ and $N_T = 60$ will be used.

Notice that by using the normalization (69) derived from $\phi_{end} = \phi_{e2}$, the previous result is valid only inside the contour (74). Therefore, we still have to treat the case where inflation ends prematurely at $\phi_{strg} > \phi_{e2}$. Since the entire field evolution occurs in the slow-roll regime, the trajectory (52) remains valid all the time. The minimal initial field value $\phi_{m,\text{strg}}$ leading to $N_T$ e-folds of inflation is therefore obtained by setting $\phi_{end} = \phi_{\text{strg}}$.

This leads to

$$\frac{\phi_{m,\text{strg}}}{\mu} = \left[3N_T \left(\frac{m_{Pl}}{\mu}\right)^2 + \left(\frac{\phi_{\text{strg}}}{\mu}\right)^6\right]^{1/6},\quad (94)$$

where $\phi_{\text{strg}}$ is given by equation (38). In order to use the CMB normalization, the new value of $\phi_s$ has to be derived from equation (63) where, now, $\phi_{end} = \phi_{\text{strg}}$. One obtains

$$\frac{\phi_s}{\mu} = \left[\left(\frac{\phi_{\text{strg}}}{\mu}\right)^6 + \frac{3}{\pi} \left(\frac{\mu}{m_{Pl}}\right)^{-2} N_s\right]^{1/6}.\quad (95)$$

The slow-roll parameter associated with this field value reads

$$\epsilon_1 \simeq \frac{1}{\pi} \left(\frac{\mu}{m_{Pl}}\right)^{-2} \left(\frac{\phi_s}{\mu}\right)^{-10},\quad (96)$$

and can be plugged into equation (67) to give the CMB normalization condition

$$\left(\frac{M}{m_{Pl}}\right)^6 = 45 \frac{Q_{rms-PS}^2}{T^2} \nu^{1/2} \left[\left(\frac{\phi_{\text{strg}}}{\mu}\right)^6 + 6N_s \left(\frac{M}{m_{Pl}}\right)^{-2} \nu^{1/2}\right]^{-5/3}.\quad (97)$$

Unlike for the case $\phi_{end} = \phi_{e2}$, this equation for $M/m_{Pl}$ cannot be made explicit but one can use equation (61) to replace the last term in the square brackets and obtain

$$\left(\frac{M}{m_{Pl}}\right)^6 = 45 \frac{Q_{rms-PS}^2}{T^2} \nu^{1/2} \left[\left(\frac{\phi_{\text{strg}}}{\mu}\right)^6 + \frac{3N_s}{5} \left(\frac{\phi_{e2}}{\mu}\right)^6\right]^{-5/3}.\quad (98)$$

An explicit analytical solution can be derived in the limit $\phi_{\text{strg}} \gg \phi_{e2}$. Due to the presence of the $N_s$ term, this approximation may be violated only for large $N_s$ values. But if we
are not in this extreme situation, in the limit $\phi_{\text{strg}} \gg \phi_{\text{UV}}$, one gets

$$\frac{M}{m_{\text{Pl}}} \approx \left(\frac{45 Q_{\text{rms-PS}}^2}{T^2}\right)^{1/6} v^{1/2} \left(\frac{\phi_{\text{strg}}}{\mu}\right)^{-5/3},$$

(99)

$$\frac{\mu}{m_{\text{Pl}}} \approx \left(\frac{45 Q_{\text{rms-PS}}^2}{T^2}\right)^{1/6} \left(2\pi\right)^{-1/2} v^{-1/6} \left(\frac{\phi_{\text{strg}}}{\mu}\right)^{-5/3}.$$  

(100)

Enforcing $\phi_{\text{m, strg}} < \phi_{\text{UV}}$ from equations (19) and (94) gives, in terms of the rescaled parameters,

$$\ln \left[ 1 + 6N_T \left(\frac{45 Q_{\text{rms-PS}}^2}{T^2}\right)^{-1/3} x^{-2/3} \bar{v}^{-1/3} \exp \left(\frac{8}{3} x^{-1/4}\right) \right]$$

$$< 4x^{-1/4} + 2 \ln \bar{v} + \frac{5}{2} \ln x - 3 \ln \left(\pi \alpha' m_{\text{Pl}}^2\right) - \ln \left(\frac{45 Q_{\text{rms-PS}}^2}{T^2}\right).$$  

(101)

Again, there is no trace of $g_s$ left after rescaling, but instead of a straight line as was the case for equations (90) and (92), this is an implicit contour in the $(\ln x, \ln \bar{v})$ plane. It is represented in figure 5. Let us recall that equation (101) is only valid far enough from the contour $\phi_{\text{strg}} = \phi_{\text{end}}$, in such a way that $\phi_{\text{strg}} \gg \phi_{\text{end}}$. In the intermediate regime where inflation does end by instability at $\phi_{\text{strg}}$, but the two terms in the square brackets of equation (98) are of comparable value, the correct normalization of $M/m_{\text{Pl}}$ to the CMB quadrupole is only accessible numerically and has also been represented in figure 5 for convenience. It is remarkable that when one normalizes with either (68) (obtained from $\phi_{\text{end}} = \phi_{\epsilon_2}$) or with (99) (assuming $\phi_{\text{strg}} \gg \phi_{\epsilon_2}$), the $g_s$ dependence can be absorbed into the unique rescaling given in equation (73).
Finally, in figure 6, various concrete models for different values of $v$, $N$ and $g_s$ are compared with the consistency conditions derived above. In particular, one notices that the volume ratio constraint is quite restrictive and that many possible models are already ruled out by this condition.

6.3. Constraints from stochastic inflation

If the field $\phi$ starts out at a value comparable to $\phi_{\text{fluct}}$ given in equation (46), then stochastic effects will be important and the classical field trajectory no longer suffices for describing the field evolution. As a minimum requirement, in the case where brane annihilation occurs after $\phi_{\epsilon_2}$, one has to impose

$$\phi_{\text{in},\epsilon_2} \ll \phi_{\text{fluct}},$$

(102)

where $\phi_{\text{in},\epsilon_2}$ is given by equation (91). Comparing equation (91) with equation (46), and using the correct normalization (68), we notice that the dependence on $N$ and $v$ disappears. As a result, the restriction coming from (102) concerns the maximal total number of e-folds $N_T$ achievable,

$$N_T \leq -\frac{5}{3} + (3^2 2^{11})^{-1/5} (6N_\ast + 10) \left(45 \frac{Q_{\text{rms-PS}}^2}{T^2}\right)^{-3/5},$$

which gives $\max(N_T) \simeq 10^7$ for $N_\ast \simeq 50$. Similarly, if inflation ends at $\phi_{\text{strg}} > \phi_{\epsilon_2}$, one imposes

$$\phi_{\text{in},\text{strg}} \ll \phi_{\text{fluct}},$$

(103)

to avoid quantum fluctuation dominance. In this case, using equations (46) and (94), together with the normalization given in equation (99), gives

$$\ln \left[1 + 6N_T \left(45 \frac{Q_{\text{rms-PS}}^2}{T^2}\right)^{-1/3} x^{-2/3} v^{-1/3} \exp \left(-\frac{8}{3} x^{-1/4}\right)\right] < \frac{3}{5} \ln \frac{3}{4} - \frac{3}{5} \ln \left(45 \frac{Q_{\text{rms-PS}}^2}{T^2}\right),$$

(104)

As usual, the rescaling allows us to get rid of any explicit dependence in $g_s$. We do not pursue this issue further here. For a detailed study of the presence of quantum fluctuation effects in the throat, see the appendix.

7. Implications of the WMAP slow-roll bounds

In this section, the theoretically predicted values of the KKLMMT slow-roll parameters are confronted with the third-year WMAP data (WMAP3). The WMAP3 bounds on the Hubble-flow parameters $\epsilon_1$ and $\epsilon_2$ have been discussed in [34, 33, 68] for various prior choices and hypotheses. For the sake of robustness, in the following we use the constraints derived in [34] on the first-order Hubble-flow parameters obtained by marginalizing over the second-order slow-roll parameter $\epsilon_3$ and for a uniform prior choice for $\log(\epsilon_1)$ in $[-5, 0]$ and for $\epsilon_2$ in $[-0.2, 0.2]$. The resulting one-and two-sigma contour intervals of the two-dimensional marginalized posteriors are represented in figure 7.
Let us first consider the situation where violation of the slow-roll conditions occurs before brane annihilation. In this case, if one chooses for instance $N_* = 50$, the CMB normalization from equations (68) and (69) yields

$$\frac{\mu}{m_{Pl}} \simeq 5.6 \times 10^{-6} v^{-3/8}, \quad \frac{M}{m_{Pl}} \simeq 1.4 \times 10^{-5} v^{-1/8}.$$  

Hence these two parameters become functions of $v$ only. The resulting numerical values for $\epsilon_1$ and $\epsilon_2$ are given by equations (65) and (66) and read

$$\epsilon_1 \simeq 4.8 \times 10^{-11} v^{-1/2}, \quad \epsilon_2 \simeq 0.03.$$  

This has two very important consequences: it is clear that except for very small values of the volume ratio $v < 10^{-18}$, $\epsilon_1$ remains negligible compared to $\epsilon_2$. In particular, since the spectral indices of the scalar and tensor primordial power spectra, at first order in the Hubble-flow parameters, read

$$n_S - 1 = -2\epsilon_1 - \epsilon_2, \quad n_T = -2\epsilon_1,$$

we have unobservably small tensor modes and a scalar spectral index

$$n_S \simeq 0.97.$$  

In figure 7, we have studied the slow-roll predictions in more detail. The slow-roll parameters have been calculated exactly in the sense that equation (55) has been solved numerically, that is to say we have obtained the exact value of $\phi_{\epsilon_2}$. The same has been done for $\phi_*$. Then, the corresponding values of the Hubble-flow parameters have been derived with the help of equations (65) and (66) for different $N_*$ such that $N_* \in [40,60]$. For each value of $N_*$, this gives a point in the space ($\epsilon_1, \epsilon_2$) and for the range under consideration, a segment line spanning twenty e-folds (see figure 7). The ‘left’ end of this segment line corresponds to the largest value of $N_*$, namely $N_* = 60$, and the ‘right’ end corresponds to $N_* = 40$. The left panel assumes $v = 1$, while the right one was plotted for $v = 10^{-6}$. The dotted–dashed blue contours are the WMAP3 constraints on...
the slow-roll parameters and correspond to the 68% and 95% confidence intervals of the two-dimensional marginalized posteriors. The dotted black lines are the lines of constant $n_S$, the red one tracing the scale invariant case $n_S = 1$. The figure confirms the previous analysis. The gravitational waves level is generically very low and the spectral index is slightly red in full agreement with the WMAP3 data. Moreover, changing the size of the extra dimension changes $\epsilon_1$ without affecting $\epsilon_2$. However, for reasonable values of $v$, $\epsilon_1$ remains so small that there is no hope to detect primordial gravitational waves in the future.

Let us now turn to the situation where the parameters of the model are such that inflation ends by instability at $\phi_{\text{strg}}$. In this case, the WMAP normalization leads to the scale $M/m_{\text{Pl}}$ given by equation (99). Then, straightforward calculations lead to the following expressions for the Hubble-flow parameters:

\begin{align}
\epsilon_1 &= \frac{v^{1/3}}{2\pi^2} \left( 45 \frac{Q^2_{\text{rms} - \text{PS}}}{T^2} \right)^{-1/3} \left( \frac{\phi_{\text{strg}}}{\mu} \right)^{-20/3}, \\
\epsilon_2 &= 5 \frac{v^{1/3}}{2\pi^2} \left( 45 \frac{Q^2_{\text{rms} - \text{PS}}}{T^2} \right)^{-1/3} \left( \frac{\phi_{\text{strg}}}{\mu} \right)^{-8/3}.
\end{align}

In the limit $\phi_{\text{strg}}/\mu \gg 1$, one has $\epsilon_1 \ll \epsilon_2 \ll 1$ except perhaps for extreme values of $v$. Therefore, $n_S$ is now pushed towards 1, while $n_T$ becomes even smaller for all values of $N_*$. As a consequence, one expects this scenario to be disfavoured by the data. Indeed, as can be noticed in figure 7, a scale invariant power spectrum is not likely if the gravitational waves level is low (this is no longer true if $\epsilon_1 > 10^{-2}$).

To conclude this section, let us recap what can be deduced from the slow-roll analysis. Firstly, the situation where inflation ends by violation of the slow-roll condition is favoured by the data as compared to the case where inflation stops by brane annihilation. Secondly, one expects the level of gravitational waves to be extremely low. The stringy interpretation of this result is linked to the so-called Lyth bound [69,24] which relates the tensor to scalar ratio to the total variation of the inflaton field value. In our case, this gives $\Delta \phi = \sqrt{T_3 (r_{\text{UV}} - r_0)} \simeq \sqrt{T_3 r_{\text{UV}}} = \phi_{\text{UV}}$. Therefore, given equation (35), the volume bound on the throat limits the gravitational wave contribution, as recovered here. Thirdly, the constraint $\epsilon_1 \lesssim 3 \times 10^{-2}$ implies the limit

$$\log \left( 4\pi^2 v \right) \gtrsim -16.$$ 

This is clearly not a very stringent constraint. In the next section, we will see that, combined with the constraint on the size of the throat, the data can lead to a better limit.

8. WMAP constraints in the exact numerical approach

In this section, the cosmological consequences of the KKLMMT model are investigated using a numerical integration of the brane motion up to its linear perturbations to extract the exact scalar and tensor primordial power spectra. The tachyonic instability occurring at the bottom of the throat triggering a reheating era is considered through a simple phenomenological model. The predicted CMB power spectra can then be computed by integrating the seeded cosmological perturbations through the radiation and matter era and we have used a modified version of the CAMB code for this purpose [70]. This method
allows a Markov chain Monte Carlo (MCMC) analysis of the WMAP third-year data involving the usual cosmological parameters together with the KKLMMT parameters. A modified version of the \texttt{COSMOMC} code [71] has been used to derive the probability density distributions of the cosmological, KKLMMT and reheating parameter values given the data.

8.1. Method and hypotheses

The numerical method used has been introduced in [34,35] and consists in the mode by mode integration of both the background and the perturbed quantities up to the end of inflation (see also [72]–[78]). As shown in section 5.2, the DBI regime is not relevant for the model under scrutiny and we have chosen to integrate the background equations stemming from the action (6) in the standard kinetic term limit. The resulting equations of motion are obtained from equations (43), (76) and (77) in the $\gamma = 1$ limit and for the KKLMMT potential (13).

In the longitudinal gauge, we consider the scalar and tensor linear perturbations to the flat FLRW metric (7),
\begin{equation}
    ds^2 = a^2 \left\{ -(1 + 2\Phi) d\eta^2 + [(1 - 2\Psi)\delta_{ij} + h_{ij}] dx^i dx^j \right\},
\end{equation}
where $\Phi$ and $\Psi$ are the Bardeen potentials [79] and $h_{ij}$ is a transverse and traceless tensor. From the perturbed Einstein and Klein–Gordon equations, the dynamics of the scalar and tensor modes, in Fourier space, reduces to the equations of motion of two uncoupled parametric oscillators [7], [80]–[82]
\begin{equation}
    \frac{d^2 \mu_{S,T}}{d\eta^2} + \omega_{S,T}^2 (k, \eta) \mu_{S,T} = 0,
\end{equation}
where the frequencies are respectively
\begin{equation}
    \omega_S^2 (k, \eta) = k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}, \quad \omega_T^2 (k, \eta) = k^2 - \frac{a''}{a},
\end{equation}
the prime denoting differentiation with respect to conformal time. The scalar and tensor mode functions are gauge invariant and read [7]
\begin{equation}
    \mu_S = a\sqrt{2\kappa} \left( \delta\phi + \frac{d\phi}{d\eta} \Phi \right), \quad \mu_T = ah_{ij},
\end{equation}
where $\delta\phi$ stands for the linear perturbations, in the longitudinal gauge, of the inflaton field $\phi$ (see equation (1)). The initial conditions for the perturbed quantities $\mu_{S,T}$ are set in the decoupling limit $k \gg H$ where the deep sub-Hubble modes behave as free quantum fields. Assuming their initial state to be the usual Bunch–Davies vacuum, one gets [83]
\begin{equation}
    \lim_{k/H \to \infty} \mu_{S,T} = \sqrt{\kappa} e^{-ik(\eta - \eta_i)} \frac{e^{-ik(\eta - \eta_i)}}{\sqrt{k}},
\end{equation}
with $\eta_i$ some (arbitrary) initial conformal time. These initial conditions uniquely determine the solutions of equation (113) which can be numerically integrated along the lines described in [35]. From such a numerical integration, the primordial scalar and tensor power spectra can be evaluated at the end of inflation for all the relevant
observable wavenumbers

\[ P_\zeta \equiv k^3 P_\zeta(k) = \frac{k^3}{8\pi^2} \frac{\mu_S}{a\sqrt{\epsilon_1}} \left| a \right|^2, \quad P_h \equiv k^3 P_h(k) = \frac{2k^3}{\pi^2} \frac{\mu_T}{a} \left| a \right|^2. \] (117)

As regards the background quantities, the attractor mechanism during inflation ensures that the initial field value \( \phi_{in} \) has no direct observable effects since it determines only the total number of e-folds \( N_T \). Its value can therefore be, \textit{a priori}, arbitrarily chosen provided \( N_T \) is big enough to solve the problems of the standard hot big bang phase and allows the observable perturbations to be sub-Hubble initially. As thoroughly discussed in sections 5 and 6, the brane motion ends by the appearance of a tachyon from its coalescence with the anti-brane at the bottom of the throat, and this determines the field value at which the above classical evolution can no longer be used, namely \( \phi \simeq \phi_{strg} \). The initial field value \( \phi_{in} \) can therefore be chosen in such a way that the brane motion generates \( N_T \) e-folds of inflation before reaching \( \phi_{strg} \). However, one still has to impose that the brane motion starts inside the throat for the model to be consistent, i.e. \( \phi_{in} < \phi_{UV} \).

According to the previous discussion, it is natural to assume that the reheating era starts when the brane alights on the anti-brane. Although, strictly speaking, brane annihilation occurs at the bottom of the throat \( r_0 \), the string theory details of the evolution for \( \phi < \phi_{strg} \) are beyond the scope of our simple approach [84]–[86]. As a result, we have adopted the phenomenological reheating model discussed in [34,35]. It assumes that the reheating can only influence the observed perturbations through its effects on the cosmological redshift \( z_{strg} \), which will be identified with the beginning of the reheating era. Such an assumption is motivated by the fact that adiabatic super-Hubble perturbation modes should not be significantly modified during reheating, at least in the absence of entropy modes (here, we deal with a single-field model). The influence of \( z_{strg} \) can be seen by rewriting equation (113) in terms of the number of e-folds as a time variable. The wavenumbers to be considered during inflation always appear in the ratio \( k/H \) which should be determined from the observable wavenumbers \( k/a_0 \) measured today, typically three decades around \( k^* = 0.05 \text{ Mpc}^{-1} \). At a given e-fold \( N \) during inflation,

\[ \frac{k}{H} = \frac{k}{a_0} \left( 1 + z_{strg} \right) \frac{e^{N_T-N}}{H(N)}. \] (118)

Assuming instantaneous transitions between the successive expansion eras, the redshift \( z_{strg} \) is given by

\[ \ln(1 + z_{strg}) = \frac{1}{4} \ln(k^2 \rho_{reh}) - \ln \frac{a_{strg}}{a_{reh}} - \frac{1}{2} \ln \left( \sqrt{3\Omega_{\text{rad}} H_0} \right), \] (119)

where \( \rho_{reh} \) is the total energy density at the end of the reheating era, and \( \Omega_{\text{rad}}, H_0 \) are the density parameter of radiation and the Hubble parameter today. The first two terms in equation (119) are clearly reheating dependent and in the absence of a microscopic model we can define a phenomenological reheating parameter \( R_{rad} \) such that

\[ \ln R_{rad} \equiv -\frac{1}{4} \ln \left( \frac{\rho_{reh}}{\rho_{strg}} \right) + \ln \frac{a_{strg}}{a_{reh}}. \] (120)

This parameter has a simple physical interpretation: it encodes the global deviation that the dynamics of the reheating era may have with respect to a pure radiation-like era.
(for which \( R_{\text{rad}} \) vanishes). Notice that, from the numerical integration, the energy density \( \rho_{\text{strg}} \) when \( \phi = \phi_{\text{strg}} \) is known and only depends on \( \phi_{\text{strg}} \) and the potential parameters.

Under the previous assumptions, the KKLMMT inflation era requires the knowledge of five primordial parameters: the KKLMMT potential parameters, namely \( M \) and \( \mu \), the reheating parameter \( R_{\text{rad}} \) and the field values \( \phi_{\text{UV}} \) and \( \phi_{\text{strg}} \) encoding the observable properties of the throat and the branes. The resulting CMB anisotropies are uniquely determined once the ‘low energy’ cosmological model is fixed. We are considering the \( \Lambda \)CDM flat universe model which adds four cosmological parameters: the number density of baryons \( \Omega_b \), of cold dark matter \( \Omega_{\text{dm}} \), the reduced Hubble parameter today \( h \), and the redshift of reionization \( z_{\text{re}} \).

### 8.2. MCMC parameters and priors

Due to parameter degeneracies with respect to the CMB temperature and polarization angular power spectra, some parameter combinations are more appropriate for an efficient MCMC exploration. Concerning the \( \Lambda \)CDM parameters, instead of directly sampling models according to the values of the set \( (\Omega_b, \Omega_{\text{dm}}, z_{\text{re}}, H_0) \), it is more convenient to use the equivalent set \( (\Omega_b h^2, \Omega_{\text{dm}} h^2, \tau, \theta) \), where \( \tau \) is the optical depth and \( \theta \) measures the ratio of the sound horizon to the angular diameter distance \([71]\).

Similarly, it is more convenient to use an optimal derived set for the primordial parameters. The amplitude of the scalar primordial power spectrum at a given observable wavenumber \( P_s = P_\zeta(k_s) \) is a well measured quantity (see equation (67)). It is therefore more convenient to directly sample the models according to the values of \( P_s \) rather than the potential energy scale \( M/m_{\text{Pl}} \). This can be done by integrating the perturbations with the artificial value \( M/m_{\text{Pl}} = 1 \) and then performing a rescaling of \( M/m_{\text{Pl}} \) from unity to its physical value that would be associated with the wanted \( P_s \). As shown in \([34]\), under the rescaling \( M/m_{\text{Pl}} \rightarrow sM/m_{\text{Pl}} \), the power spectra become \( P(k) \rightarrow s P(s^{1/2}k) \). As a result, the value of \( s \) required is the ratio \( P_s/P_0(M=1) \), where \( P_0(M=1) \) is the amplitude of the scalar power spectrum obtained with \( M/m_{\text{Pl}} = 1 \) and evaluated at \( k_s = k_s s^{-1/2} \). In fact, to circumvent such a rescaling on the wavenumbers, it is more convenient to introduce the rescaled reheating parameter \( R \) defined by

\[
\ln R \equiv \ln R_{\text{rad}} + \frac{1}{4} \ln (\kappa^2 \rho_{\text{strg}}) .
\] (121)

The quantity \( R \) represents an effective energy, in Planck units, at the time of brane merging, which is exactly equal to \( \sqrt{\kappa} \rho_{\text{strg}}^{1/4} \) for a radiation-like reheating. The advantage of \( R \) with respect to \( R_{\text{rad}} \) is that, at fixed \( R \), one has \( k_o = k_s \) \([34]\). Finally, as shown in section 5, one may expect some observable effects coming from the situations in which there is violation of the slow-roll conditions before brane annihilation, i.e. according to the value of \( \phi_{\text{strg}}/\phi_{\epsilon_2} \). Since \( \phi_{\epsilon_2} \) is determined by \( \mu/m_{\text{Pl}} \) only (see equation (61)), we have chosen to perform the MCMC exploration on the parameter \( \phi_{\text{strg}}/\mu \) instead of \( \phi_{\text{strg}} \).

Finally, the nine MCMC parameters are: \( \Omega_b h^2, \Omega_{\text{dm}} h^2, \tau, \theta \) on the cosmological side, \( P_s, \mu/m_{\text{Pl}}, \phi_{\text{UV}}, \phi_{\text{strg}}/\mu \) for the primordial parameters and \( R \) for the reheating era. We still have to specify their prior probability distributions according to the theoretical constraints and our best knowledge on their value. For the base cosmological parameters \( \Omega_b h^2, \Omega_{\text{dm}} h^2, \tau \) and \( \theta \), wide top hat uniform distributions have been chosen centred over their preferred value from the previous analysis of the CMB data \([87, 71]\). For the primordial parameters,
$P_*$ fixes the amplitude of the cosmological perturbations and we have chosen a uniform prior on the logarithm compatible with the amplitude of the CMB fluctuations:

$$2.7 \leq \ln \left(10^{10}P_*\right) \leq 4.0.$$  \hfill (122)

The parameter $\mu/m_{Pl}$ is related to the underlying string theory model by equation (23) and we will assume that the order of magnitude of the volume ratio $v$ is not known. As a result, it is natural to also use a uniform prior probability distribution on $\log(\mu/m_{Pl})$ to ensure its conformal invariance. The same considerations hold for the two other primordial parameters, $\phi_{\text{UV}}$ and $\phi_{\text{strg}}$, whose dependence with respect to the fundamental string theory parameters is given in equations (19) and (38). Our ignorance on the values of $N$, $g_s$ and $\alpha'$ motivates the use of uninformative uniform logarithmic priors. However, the analysis of section 6 imposes various bounds. The maximal size of the throat in equation (35) and the requirement $N \geq 1$ give the upper limit

$$\log(\sqrt{\kappa} \phi_{\text{UV}}) \leq \log 2.$$  \hfill (123)

Similarly, the definition (38) of $\phi_{\text{strg}}$ ensures that $\phi_{\text{strg}} \geq \mu$ implying the lower limit

$$\log(\phi_{\text{strg}}/\mu) \geq 0.$$  \hfill (124)

Since $\phi_{\text{strg}} < \phi_{\text{UV}}$, these conditions impose

$$\log(\sqrt{\kappa} \mu) < \log 2,$$  \hfill (125)

which will be our upper prior limit for the parameter $\mu/m_{Pl}$. However, we should ensure that there are enough e-folds of inflation to solve the flatness problem and to set the sub-Hubble initial conditions (116) for the observable perturbations. As shown in [88], only for some extreme reheating models may the values of $N_T$ exceed $10^2$. In order to include all the models, we have implemented a ‘hard prior’ which, during the MCMC exploration, rejects any model that does not support at least 110 e-folds of inflation inside the throat. This is implemented in the following way. Once $\phi_{\text{strg}}$ is known (from $\log(\phi_{\text{strg}}/\mu)$ and $\log(\sqrt{\kappa} \mu)$), one can determine $\phi_{\text{in}}$ such that there are 110 e-folds of inflation in between: the model is accepted if $\phi_{\text{in}} < \phi_{\text{UV}}$ and rejected otherwise. The prior limits on $\ln R$ are determined by requiring that the end of reheating occurs before nucleosynthesis, characterized by $\rho_{\text{nuc}}$, and that the instantaneous equation of state parameter $w_{\text{reh}}$ satisfies the strong and dominant energy conditions $-1/3 < w_{\text{reh}} < 1$. Under these assumptions, one gets [34]

$$-\frac{1}{4} \ln \left(\kappa^2 \rho_{\text{nuc}}\right) < \ln R < -\frac{1}{12} \ln \left(\kappa^2 \rho_{\text{nuc}}\right) + \frac{1}{3} \ln \left(\kappa^2 \rho_{\text{strg}}\right).$$  \hfill (126)

This equation clearly involves $\rho_{\text{strg}}$ and the upper prior limit on $\ln R$ depends on the other primordial parameters. We have therefore coded another ‘hard prior’ checked during the MCMC exploration to dismiss any model violating this limit. The energy density $\rho_{\text{nuc}}$ at nucleosynthesis time has been quite extremely set around the MeV scale: $\ln R > -46$. Finally, we have to set a lower limit on the $\log(\sqrt{\kappa} \mu)$ prior. According to the values of $v$, $\mu/m_{Pl}$ may be extremely small. Our choice has been motivated by numerical convenience and we have chosen $\log(\sqrt{\kappa} \mu) > -3$. Indeed, it turns out that, for smaller values, one runs into tricky numerical difficulties. The dependence of the results with respect to this choice will be carefully discussed in the following. Notice that the lower limit of the $\log(\sqrt{\kappa} \phi_{\text{UV}})$ prior is implicitly set by the others since $\phi_{\text{UV}} > \phi_{\text{in}}$ and we ensure that there are at least 110 e-folds of inflation in between $\phi_{\text{in}}$ and $\phi_{\text{strg}}$. 

8.3. Data used

As already mentioned, the CMB measurements used are the WMAP third-year data [89, 87, 66, 90]. The degeneracies of some cosmological parameters have been reduced by adding the Hubble Space Telescope (HST) constraint $H_0 = 72 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [91] and a uniform top hat prior on the age of the universe between 10 and 20 Gyr [71]. The likelihood estimator is provided by the WMAP team\footnote{http://lambda.gsfc.nasa.gov} and we have used the current 2.2.2 version. Use has been made of the eigenvalue compression option at low multipoles to decrease the computing time. The convergence of the Markov chains has been monitored from the Gelman and Rubin $R$-test implemented in COSMOMC [92]. The MCMC exploration was stopped once the ratio of the variance of the means to the mean of the variances between the different chains was less than $R - 1 < 0.2\%$. The total number of samples obtained from this convergence criterion and drawing the posterior distributions is around 1.5 millions. In the following, we present and interpret the posterior marginalized probability distributions for the nine model parameters obtained from the previous priors and data.

8.4. Cosmological parameters

The marginalized posteriors and mean likelihoods of the cosmological parameters and power spectra amplitude are the same as those found in previous analysis of the same data but that were using other primordial power spectra, such as the first-and second-order slow-roll spectra in [34] or the phenomenological power law models in [87]. This is expected since these parameters are rather well constrained and supported by the slow-roll analysis of section 7 showing that the KKLMMT model can indeed produce the preferred value of the power spectra amplitude and spectral indices. The best fit model is found with a $\chi^2 \simeq 3538.1$ for nine parameters (or 3517 degrees of freedom after eigenvalue compression). This may be compared with the WMAP power law best fit model having $\chi^2 \simeq 3540.8$ with three parameters less. Although it shows that the KKLMMT model can provide a good fit to the data, it is not currently favoured because of its intrinsic numbers of parameters.

8.5. Primordial parameters

The marginalized posteriors and mean likelihoods for the sampled primordial parameters $\log(\sqrt{\kappa} \mu)$, $\log(\sqrt{\kappa} \phi_{\text{UV}})$, $\log(\phi_{\text{strg}}/\mu)$ and $\ln R$ are represented in figure 8. The fact that the mean likelihood (dotted curves) remains uniform for $\log(\sqrt{\kappa} \mu)$ and $\log(\sqrt{\kappa} \phi_{\text{UV}})$ shows that, on the prior range explored, these parameters do not help to improve the fit to the data. On the other hand, their probability distributions are not flat and show that large values of $\mu/m_{Pl}$ and small values of $\phi_{\text{UV}}$ are strongly disfavoured: $\log(\sqrt{\kappa} \mu) < -1.1$ at 95\% confidence level. As discussed in [93], discrepancies between the marginalized probability and the mean likelihood occur when ‘volume effects’ are induced by strong correlations between the parameters. The full likelihood in the multi-dimensional parameter space can be uniform along peculiar regions whose volume may be parameter dependent. The shapes of the $\mu/m_{Pl}$ and $\phi_{\text{UV}}$ probabilities come from the amount of fine-tuning required.
to have a successful model of inflation when $\mu/m_{Pl}$ is close to the throat edge. In that case, there is indeed not too much parameter space available for the brane motion since $\phi_{UV}$ has to saturate its maximal value to allow both $\phi_{strg} > \mu$ and $\phi_{strg} < \phi_{in} < \phi_{UV}$. As a result of these correlations, the marginalized probabilities for $\mu/m_{Pl}$ and $\phi_{UV}$ penalize the values for which strong fine-tunings are required to satisfy the theoretical priors. Let us mention again that these priors are imposed by the consistency of the model.

On the other hand, the marginalized posteriors and mean likelihoods associated with $R$ and $\phi_{strg}/\mu$ show that these parameters are directly constrained by the data. Clearly, $\phi_{strg} \gg \mu$ is disfavoured and at 95% confidence level one has $\log(\phi_{strg}/\mu) < 1.4$. This result can be understood from section 7. Large values of $\phi_{strg}/\mu$ correspond to models in which brane annihilation occurs well inside the slow-roll regime, i.e. $\phi_{strg} > \phi_{x2}$. As a result, the observable perturbation modes are generated during the brane motion in the flat part of the potential (see figure 2) and both the slow-roll parameters are small: the spectral index is close to $n_S \simeq 1$ while the amplitude of the tensor modes is negligible with respect to the scalar modes. As already mentioned in section 7, this situation is disfavoured by the WMAP data. The upper bound on $\phi_{strg}/\mu$ is in fact slightly dependent on the lower limit of the $\mu$ prior due to the presence of some correlations with the other parameters.

Figure 8. Marginalized posterior probability distributions (solid lines) and mean likelihoods (dotted lines) for the sampled primordial parameters of the ΛCDM–KKLMPT model.
As discussed in the following, a \( \mu \)-prior independent upper limit can be more conveniently given for the rescaled parameter \( \phi_{\text{strg}}/\mu^{2/3} \) and one gets, at 95% confidence level,

\[
\log \left( \frac{\kappa^{1/6} \phi_{\text{strg}}}{\mu^{2/3}} \right) < 0.52.
\] (127)

The marginalized probability distribution for the reheating parameter is peaked around \( \ln R \simeq -22 \). Its behaviour at large values directly comes from the prior (126) which is a function of \( \rho_{\text{strg}} \). As will be discussed in the following, \( \rho_{\text{strg}} \) is related to the energy scale of inflation which is bounded from above by the amplitude of the cosmological perturbations. Here again, the differences between the marginalized probability and the mean likelihood trace the correlations induced by the prior hypothesis between these two parameters, namely that reheating occurs after inflation. The lower tail of the distribution is driven by the data and falls off until the prior lower bound is saturated at \( \ln R = -46 \) (nucleosynthesis limit). This posterior is very similar to the one derived for the small field models in [34] where it has been shown that the CMB data were disfavouring a low energy scale reheating for these models. In the present case, the \( \ln R \) posterior of figure 8 does not fall off to zero in its lower part but still yields a limit

\[
\ln R > -38,
\] (128)

at a 95% confidence level.

The physical interpretation of this limit comes from the influence of \( R \) on \( N_\ast \), the e-fold at which an observable perturbation mode with wavenumber \( k_\ast \) crossed the Hubble radius during inflation. Indeed, one has \( a_\ast/a_0 = a_\ast/a_{\text{strg}} \times a_{\text{strg}}/a_0 \), with, by definition, \( a_\ast/a_0 \simeq k_\ast/(a_0\kappa^{1/2}V^{1/2}) \), where we have expressed the Hubble parameter in terms of the potential. The quantity \( a_\ast/a_{\text{strg}} \) can be expressed in terms of \( N_\ast \) and \( a_{\text{strg}}/a_0 = (1+z_{\text{strg}})^{-1} \) is given by equations (119)–(121), expressing \( \rho_{\text{strg}} \) in terms of the potential evaluated at \( \phi_{\text{strg}} \). One gets [34]

\[
N_\ast \simeq 58 - \ln \left[ \frac{k_\ast}{a_0} \text{ (Mpc}^{-1}) \right] + \ln R + \frac{1}{2} \ln \frac{V_\ast}{V_{\text{strg}}}.
\] (129)

As shown in section 7, since the WMAP data prefer a slightly red-tilted spectral index in the absence of tensor modes, it is not surprising that, for a given model of inflation, some values of \( N_\ast \), and therefore \( R \), end up being favoured. This interpretation can be further explored by using the slow-roll approximation. As already mentioned, the KKLMMT model generically leads to observable \( \epsilon_1 \) values much smaller than \( \epsilon_2 \). Let us first determine \( N_\ast \) such that \( \epsilon_2(\phi_\ast) = \epsilon_{\text{2obs}} \), where \( \phi_\ast = \phi(N_\ast) \) and \( \epsilon_{\text{2obs}} \simeq 0.05 \) is the preferred observed value (see figure 7). Using the classical trajectory given by equation (51), one finds that

\[
N_\ast \simeq \frac{\kappa \mu^2}{24} \left[ \left( \frac{\phi_\ast}{\mu} \right)^6 - \left( \frac{\phi_{\text{end}}}{\mu} \right)^6 \right],
\] (130)

and using the expression for the second horizon flow parameter (50), one arrives at

\[
\epsilon_2 \simeq \frac{40}{\kappa \mu^2} \left( \frac{\phi_\ast}{\mu} \right)^{-6}.
\] (131)
Combining these two last equations, one obtains

\[
N_* = \frac{5}{3\epsilon_{\text{obs}}} \left[ 1 - \frac{\kappa \mu^2 \epsilon_{\text{obs}}}{40} \left( \frac{\phi_{\text{end}}}{\mu} \right)^6 \right].
\]  

(132)

As already discussed, if the slow-roll is violated before brane annihilation, we can use the slow-roll trajectory with the approximation \( \phi_{\text{end}} \simeq \phi_\epsilon = \left[ \frac{40}{(\kappa \mu^2)} \right]^{1/6} \), from which one deduces that

\[
N_* \approx \frac{5}{3\epsilon_{\text{obs}}}. 
\]  

(133)

Inserting this value into formula (129) with \( k_s/a_0 \approx 0.05 \, \text{Mpc}^{-1} \) and neglecting the potential term gives \( \ln R \approx -28 \). This is the preferred value of \( \ln R \) that would give an observable spectral index at \( k_s \) compatible with the favoured value and in agreement with the shape of the marginalized posterior of figure 8. Let us notice that the two-sigma limit (128) is the best that can be extracted from the data. Indeed, the three-sigma lower limit matches the nucleosynthesis prior bound. This differs from the results obtained in [34] for the small field models and is due to the marginalized posterior which does not vanish for low values of \( \ln R \). Moreover, the mean likelihood appears to increase again in that region. This effect comes from correlations between parameters and suggests that some, but quite fine-tuned, models provide a good fit to the data even for \( \ln R \) small. To help with understanding this property, we have represented in figure 9 the two-dimensional marginalized posterior and mean likelihood in the plane \([\ln R, \log(\phi_{\text{strg}}/\mu)]\) (left panel). One recovers the favoured value for \( \ln R \approx -22 \) as the central vertical broad region (one-sigma contour), and as shown above, this domain is associated with the models in which slow-roll inflation ends before the branes collide. The two-sigma contour extends towards the left side of the plot but only for \( 1 < \log(\phi_{\text{strg}}/\mu) < 1.5 \). This tail is associated with the models in which brane annihilation occurs before the slow-roll conditions are violated. To see why such models are not disfavoured we can still use equations (129) and (132) but, now, with \( \phi_{\text{end}} = \phi_{\text{strg}} \) to get

\[
\ln R \approx -61 + \frac{5}{3\epsilon_{\text{obs}}} \left[ 1 - \frac{\kappa \mu^2 \epsilon_{\text{obs}}}{40} \left( \frac{\phi_{\text{strg}}}{\mu} \right)^6 \right].
\]  

(134)

The \( \ln R \) contour plot coming from this function is represented on the right panel of figure 9 for different values of \( \phi_{\text{strg}} \) and \( \mu/m_{\text{Pl}} \). One sees that it is indeed possible to get \( \epsilon_{\text{obs}} \approx 0.05 \) for \( \phi_{\text{strg}} > \phi_\epsilon \) provided \( \ln R \approx -40 \), in agreement with the plot. The physical interpretation is that a low energy reheating allows the observable window of wavenumbers to be shifted towards the end of brane evolution in such a way that, although brane annihilation occurs before violation of the slow-roll conditions, it is possible to get not so small values for \( \epsilon_2 \). Of course, this requires some amount of fine-tuning between \( \ln R, \mu/m_{\text{Pl}} \) and \( \phi_{\text{strg}} \), which is statistically penalized and explains why these models are outside of the one-sigma contour.

### 8.6. Derived primordial parameters

As described in section 8.1, the Markov chain simulations have been performed on the power spectra amplitude \( P_* \) and its posterior probability distribution is plotted in figure 8.
Figure 9. The left panel shows the one- and two-sigma confidence intervals of the two-dimensional marginalized posterior as a function of the parameters $\ln R$ and $\log(\phi_{\text{strg}}/\mu)$. The shading traces the corresponding two-dimensional mean likelihood. The right panel is a contour plot of $\ln R$ as a function of $\log(\sqrt{\kappa \mu})$ and $\log(\phi_{\text{strg}}/\mu)$ obtained from the slow-roll approximation by assuming that $\epsilon_2 = \epsilon_{2\text{obs}} \approx 0.05$. The dashed line marks the limit $\phi_{\text{strg}} = \phi_2$. This plot shows that, although $\phi_{\text{strg}} > \mu$ is generically disfavoured (see figure 8), it is still possible to obtain a good fit to the data provided some fine-tuning is performed between $\phi_{\text{strg}}/\mu$, $\mu/m_{\text{Pl}}$ and $\ln R$: a low energy reheating allows for some models where brane annihilation occurs before slow-roll violations to have red-tilted power spectra.

Since $P_*$ and $M/m_{\text{Pl}}$ are in direct relation (see section 8.2), it is possible to derive the marginalized probability distribution of $M/m_{\text{Pl}}$ from that of $P_*$. The same considerations apply to the volume ratio $v$ which is given by equation (24); using importance sampling, one can extract its posterior distribution from those of $\mu/m_{\text{Pl}}$ and $M/m_{\text{Pl}}$ [71]. Both probability distributions are plotted in figure 10 and seem to favour some peculiar values of $M/m_{\text{Pl}}$ and $v$. Once again, the discrepancies between the posteriors and the mean likelihoods come from the correlations with other parameters and may hide some effects coming from the priors. The right panel of figure 10 represents the two-dimensional probability distribution (point density), as well as its one- and two-sigma confidence level regions, obtained without marginalizing over the parameter $\log(\sqrt{\kappa \mu})$. As can be seen on these plots, the narrow highly probable regions trace the strong correlations between $M/m_{\text{Pl}}$ and $\mu/m_{\text{Pl}}$, and between $v$ and $\mu/m_{\text{Pl}}$. Remembering that the lower limit on the $\mu/m_{\text{Pl}}$ parameter comes from our numerically convenient prior $\sqrt{\kappa \mu} > 10^{-3}$, one immediately sees that this choice has a direct influence on the upper limit and the lower limit of, respectively, the $\log(4\pi^2 v)$ and $\log(\sqrt{\kappa M})$ probability distributions. If we had chosen a smaller limit for $\mu/m_{\text{Pl}}$, the $v$ posterior would have been shifted towards larger values, whereas the $M/m_{\text{Pl}}$ one would have been shifted towards the smaller values. As a result, we conclude that there are no upper and lower constraints on, respectively, $v$ and $M/m_{\text{Pl}}$. 
Figure 10. Marginalized posterior probability distributions for the $M/m_{Pl}$ and $v$ parameters (solid curves) and their associated mean likelihoods (dotted curves). The right panel shows the corresponding one- and two-sigma contours of the two-dimensional posteriors obtained without marginalizing over $\log(\sqrt{\kappa\mu})$. The two-dimensional probability is proportional to the point density while the colour map traces correlations with a third parameter.

The same correlations are at work for the other tails of the probability distribution associated with $v$ and $M/m_{Pl}$. However, this time, we are probing the highest allowed values for $\mu/m_{Pl}$ which are disfavoured due to the amount of fine-tuning required for a successful inflation near to the throat edge (see the previous section). As a result, we obtain the 95% confidence levels

$$\log(\sqrt{\kappa M}) < -2.9, \quad \log(4\pi^2 v) > -8.5.$$  \hspace{1cm} (135)

Let us stress that these bounds come from both the data and the requirement that inflation proceeds inside the throat, as imposed by the self-consistency of the model used.

To understand how the data lead to the correlations observed, between the parameters $\mu/m_{Pl}$, $M/m_{Pl}$ and $v$, we have plotted in figure 11 the one- and two-sigma contours (solid curves) of the two-dimensional posteriors associated with various pairs of primordial parameters. The corresponding two-dimensional mean likelihood is traced by the shaded areas. The highly probable region spanning the plane $[\log(\sqrt{\kappa M}), \log(\sqrt{\kappa\mu})]$ directly comes from the constraints on the power spectrum amplitude and spectral index. This
can be shown by using the slow-roll approximations. Indeed, if one assumes that the slow-roll conditions are violated before brane annihilation, namely $\phi_{\text{strg}} < \phi_{\epsilon_2}$, then the WMAP normalization through equations (68) and (69) leads to

$$
\log (\sqrt{\kappa} M) = \log \left[ \left( \frac{O_{\text{rms}}^{2}}{T^{2}} \right)^{1/4} (6N_{s} + 10)^{-5/12} \left( 4\pi^{2} \right)^{1/12} (8\pi)^{1/3} \right] + \frac{1}{3} \log (\sqrt{\kappa} \mu). 
$$

(136)

This is a straight line with slope equal to $1/3$, exactly as observed for the highly probable regions in figure 11. Using a fiducial value for the quadrupole and $N_{s} = 50$, the offset is
around $-2.6$. Consequently, for $\log(\sqrt{\kappa\mu}) \simeq -3$ one gets $\log(\sqrt{\kappa M}) \simeq -3.6$, as confirmed by the plot. The effect of the spectral index is indirect and, as previously discussed, tends to favour the models in which $\phi_{\text{strg}} < \phi_{e_2}$. This suggests that the less probable and more widely shaded region in the plane $[\log(\sqrt{\kappa M}), \log(\sqrt{\kappa\mu})]$ corresponds to the cases where $\phi_{\text{strg}} > \phi_{e_2}$. Again, the slow-roll approximation in this limit gives the relation

$$\left(\frac{M}{m_{\text{Pl}}}\right)^6 \simeq 45 \frac{Q_{\text{rms-PS}}}{T^2} \frac{1}{\sqrt{\pi}} \left(\frac{\phi_{\text{strg}}}{\mu}\right)^{-10}. \tag{137}$$

Since $\nu$ is also a function of $M/m_{\text{Pl}}$ and $\mu/m_{\text{Pl}}$, this is a three-parameter relation, or a surface in the plane $[\log(\sqrt{\kappa M}), \log(\sqrt{\kappa\mu})]$. We can, however, derive the slope associated with the lower boundary of the blurred region (beyond which there are no acceptable models). This boundary is reached for the largest $\phi_{\text{strg}}$ compatible with the throat size, i.e. for $\phi_{\text{in, strg}} = \max(\phi_{UV}) = m_{\text{Pl}}/\sqrt{2\pi}$ (see equation (35)). This limit is reached when the ratio of throat to bulk volume is maximal. Using the expression for $\phi_{\text{in, strg}}$ given by equation (94) and solving for $\phi_{\text{strg}}$ leads to

$$\phi_{\text{strg}} = \left[\left(\frac{m_{\text{Pl}}}{\sqrt{2\pi}\mu}\right)^6 - \frac{3N_T}{\pi} \left(\frac{m_{\text{Pl}}}{\mu}\right)^2\right]^{1/6}. \tag{138}$$

This expression can now be inserted into equation (137) and one gets

$$\log(\sqrt{\kappa M}) \simeq \log \left[\left(45 \frac{Q_{\text{rms-PS}}}{T^2} \frac{1}{\sqrt{\pi}}\right)^{1/4} 4\sqrt{\pi} - \frac{1}{2} \log(\sqrt{\kappa\mu}) - \frac{5}{12} \log \left[\frac{64}{(\sqrt{\kappa\mu})^6} - \frac{24N_T}{\sqrt{\kappa\mu}}\right]\right]. \tag{139}$$

For not too large values of $\sqrt{\kappa\mu}$ (in practice $\sqrt{\kappa\mu} \lesssim 0.1$), the term proportional to $N_T$ can be neglected and one obtains

$$\log(\sqrt{\kappa M}) \simeq \log \left[\left(45 \frac{Q_{\text{rms-PS}}}{T^2} \frac{1}{\sqrt{\pi}}\right)^{1/4} 4\sqrt{\pi} - \frac{5}{12} \log(64) + 2 \log(\sqrt{\kappa\mu})\right]. \tag{140}$$

This is a straight line in the plane $[\log(\sqrt{\kappa M}), \log(\sqrt{\kappa\mu})]$, the slope of which is $+2$: this matches with the observed lower boundary of the blurred region in figure 11. The offset is around $-2.1$ giving $\log(\sqrt{\kappa M}) \simeq -4.1$ for $\log(\sqrt{\kappa\mu}) \simeq -1$, again in agreement with the figure. One might also notice that the boundary curve in figure 11 slightly bends over for small values of $\mu/m_{\text{Pl}}$ as the effects of the terms proportional to $N_T$ start to appear. The previous interpretation is confirmed by the left bottom panel of figure 11. The point density clearly decreases in the domain of low likelihood which appears blurred in the upper left panel. As traced by the colour map, this region does indeed correspond to large values of $\phi_{\text{strg}}/\mu$. Let us notice that, although this region is outside of the 95% confidence contour, it remains inside the three-sigma one.

The previous discussion also applies to the right plots of figure 10. The above-described relations between $M/m_{\text{Pl}}$, $\mu/m_{\text{Pl}}$ and $\phi_{\text{strg}}$ are directly converted into correlations between $\nu$ and the other parameters through equation (24). As a result, the main degeneracy seen in the plane $[\log(\sqrt{\kappa\mu}), \log(4\pi^2\nu)]$ is also a consequence of both the power spectra normalization and spectral index constraints. The bottom right panel illustrates the degeneracy of $M/m_{\text{Pl}}$ and $\mu/m_{\text{Pl}}$ coming from the data and differs from the
one associated with equation (24): such a difference explains the two-sigma lower limit on \(\log(4\pi^2 v)\). Let us elaborate on this point. The limit on \(v\) can be understood from the slow-roll result even if, at first sight, there is a mismatch between the constraint obtained above and equation (111). In fact, equation (61) implies
\[
\log (4\pi^2 v) \approx -\frac{8}{3} \log (\sqrt{\kappa \mu}) - 9.
\]
This relation is consistent with the top right panel in figure 10. Let us notice that, in this context, it is relevant to use equation (105), which is derived under the assumption that inflation stops by violation of the slow-roll conditions, since one observes in figure 10 that the two-sigma contour corresponds to ‘small’ values of \(\phi_{\text{strg}}\). Moreover, if one inserts the limit (89), namely \(\sqrt{\kappa \mu} < 2\), in the above relation, then the constraint (135) is reproduced. This constraint is different from equation (111) because it has a different origin. The limit (111) assumes that the maximum allowed contribution of tensor modes to the observed CMB data can indeed be generated during KKLMMT inflation. However, the requirement that the brane motion proceeds inside the throat strongly limits the generation of tensor modes and leads to the stronger bound of equation (135). Therefore, the two limits are consistent and the stronger bound is given by equation (135).

Finally, we have plotted in the right hand panels of figure 11 the two-dimensional probability distribution and mean likelihood in the plane \([\log(\sqrt{\kappa \mu}), \log(\phi_{\text{strg}}/\mu)]\), as well as the effect of \(\phi_{\text{UV}}\). These plots exhibit the volume effects associated with the large values of the parameter \(\mu/m_{\text{Pl}}\). Clearly, the allowed ranges of \(\phi_{\text{strg}}/\mu\) and \(\phi_{\text{UV}}\) are all the more reduced as \(\mu/m_{\text{Pl}}\) increases, thereby decreasing the statistical weight of these domains in the marginalized posterior of \(\mu/m_{\text{Pl}}\) (see figure 8). The two-sigma contours are found to follow the high likelihood region, and as before, it corresponds to models in which brane annihilation occurs after violation of the slow-roll conditions \((\phi_{\text{strg}} < \phi_{\text{ez}})\). Indeed, the edge between the dark and the blurred regions is just given by the equation \(\phi_{\text{strg}} = \phi_{\text{ez}}\). Using equation (61), which gives \(\phi_{\text{ez}}\) when \(\sqrt{\kappa \mu}\) is small, one gets
\[
\log \left( \frac{\phi_{\text{strg}}}{\mu} \right) \approx \frac{1}{6} \log(40) - \frac{1}{3} \log (\sqrt{\kappa \mu}) .
\]
(142)
This is a straight line in the plane \([\log(\sqrt{\kappa \mu}), \log(\phi_{\text{strg}}/\mu)]\) with a \(-1/3\) slope, as observed in the figure. The offset is close to 0.3, implying that for \(\log(\sqrt{\kappa \mu}) = -3\) one would have \(\log(\phi_{\text{strg}}/\mu) \approx 1.3\). There is a slight difference since the contour seems to intercept the vertical axis at \(\log(\phi_{\text{strg}}/\mu) \approx 1.5\). Such a difference may be due to the fact that \(\phi_{\text{strg}}\) must significantly deviate from \(\phi_{\text{ez}}\) for the effect to be visible. Moreover, the reheating effects already mentioned are not considered in the derivation of equation (142). Equation (142) renders explicit the \(\mu\)-prior dependence previously noted on the upper two-sigma limit associated with the marginalized probability distribution of \(\phi_{\text{strg}}/\mu\) (see figure 8). The \(-1/3\) slope explains why the prior independent limit of equation (127) is more conveniently expressed in terms of \(\phi/\mu^{2/3}\). Finally, the upper edge of the blurred region is, as before, given by the condition \(\phi_{\text{in, strg}} = \max(\phi_{\text{UV}})\) (see the lower right panel of figure 11). Using equation (138), one finds
\[
\log \left( \frac{\phi_{\text{strg}}}{\mu} \right) \approx \log (2) - \log (\sqrt{\kappa \mu}) .
\]
(143)
Again, one can check that the slope and the offset are consistent with what is observed in figure 11. Beyond that limit, inflation would not take place in the throat and those models are rejected.
8.7. Fundamental parameters

Our choice of MCMC parameters is minimal as far as observable effects are concerned. The cosmological consequences of the KKLMMT model are described by the set of four parameters \((M, \mu, \phi_{\text{strg}}, \phi_{\text{UV}})\) whereas, as discussed in section 2.5, the underlying string theory model involves five parameters \((g_s, \alpha', M, v, \mathcal{N})\). As a result, it is impossible without additional assumptions to extract more information for the string parameters. As found in the previous section, the CMB data, however, allow some constraints to be derived for the \(M/m_{\text{Pl}}\) and \(v\) parameters. Their probability distributions do not assume anything on the value of \(g_s\), \(\alpha'\) or \(\mathcal{N}\) (apart from consistency requirements) and are also robust against the reheating as long as it can be described by our phenomenological model. If one wants to go further on the fundamental theory parameters, additional assumptions have to be made. In the following, such a step is performed first on the reheating model by assuming that it proceeds with a constant equation of state \(P = w_{\text{reh}}\rho\). Then, a similar approach is adopted to derive some posterior probability distributions on the remaining string parameters by assuming that the value of \(\alpha'\) is known. As a working example, \(\alpha' = 10^3 m_{\text{Pl}}^{-2}\) and \(\alpha' = 10^5 m_{\text{Pl}}^{-2}\) are considered.

For a constant equation of state parameter during reheating, \(\rho \propto a^{-3(1+w_{\text{reh}})}\) and equation (121) can be further simplified to

\[
\ln R = \frac{1 - 3w_{\text{reh}}}{12 + 12w_{\text{reh}}} \ln (\kappa^2 \rho_{\text{reh}}) + \frac{1 + 3w_{\text{reh}}}{6 + 6w_{\text{reh}}} \ln (\kappa^2 \rho_{\text{strg}}).
\]  

(144)

From the MCMC analysis performed in the previous section, samples on \(R\) and \(\rho_{\text{strg}}\) can be used to extract by importance sampling the probability distribution associated with \(\rho_{\text{reh}}\) (the prior choice remains that the reheating occurs before nucleosynthesis). The resulting posteriors are plotted in figure 12 for four values of \(w_{\text{reh}}\) spanning the range allowed by the dominant and strong energy conditions in general relativity. For an extreme equation of state with \(w_{\text{reh}} \gtrsim -1/3\), i.e. which is on the verge of an accelerated expansion, one has the 95% confidence limit

\[
\rho_{\text{reh}}^{1/4} > 20 \text{ GeV}.
\]  

(145)

Although such models are certainly already ruled out by particle physics experiments, let us notice that this limit comes from CMB data only. As can be seen in figure 12, all the other cases associated with more reasonable values of the equation of state parameter are not constrained.

A similar method has been applied to the parameters \(g_s\) and \(\mathcal{N}\) by assuming that the value of \(\alpha'\) is known. In figure 13, we have plotted the two-dimensional marginalized probability distributions, as well as their one- and two-sigma contours, in the plane \((\log \mathcal{N}, \log g_s)\) and for the two fiducial values of \(\alpha'\). As can be seen in these plots (right panels), there is basically no constraint coming from the data for the case \(\alpha' = 10^5 m_{\text{Pl}}^{-2}\). Indeed, on one hand, the two-dimensional posterior is limited by the consistency conditions \(\mathcal{N} \geq 1\) and \(g_s < 1\) to satisfy flux quantization and perturbative treatment. On the other hand, the colour bar shows that the two-sigma contour simply traces the high values of \(v\) which have been shown in section 8.6 to be essentially related to our lower prior on \(\mu/m_{\text{Pl}}\). The case \(\alpha' = 10^3 m_{\text{Pl}}^{-2}\) is slightly different. As can be seen in the lower right panel of figure 13, the two-sigma contour is detached from the axes and therefore the data give slightly more information than the prior consistency limit. Again, the right end of the
Figure 12. Marginalized probability distributions on the reheating energy scale derived from the WMAP data under the assumption that the reheating proceeds with a constant equation of state: $P = w_{\text{reh}} \rho$. For an extreme equation of state $w_{\text{reh}} \gtrsim -1/3$, one finds a weak (non-trivial) lower bound $\rho_{\text{reh}}^{1/4} > 20$ GeV at the two-sigma level. For all the other cases, the reheating can occur at any energy scale higher than nucleosynthesis (from the CMB point of view).

contour is not physical since it comes only from our $\mu/m_{\text{Pl}}$ prior, as before. However, the observed degeneracy of $\log N$ and $\log g_s$ is a non-trivial information. This means that for a given flux number, a value of $g_s$ is favoured by the data. We have not represented any one-dimensional posterior for $g_s$ or $N$ since there is typically no more information than the above-mentioned degeneracy. To do so, another assumption would have to be made, such as some prior knowledge on the flux number or on the coupling constant. In the left panels of figure 13, we have plotted the corresponding two-dimensional posteriors in the plane $\{\log(4\pi g_s N/v), \log[v/(4\pi g_s)^2]\}$ to allow a comparison with the slow-roll results of section 6 (see also figure 5). As expected, all the models lie in the range predicted by the slow-roll analysis, up to the prior limit fixed by the numerically convenient lower limit on $\mu/m_{\text{Pl}}$. The associated $g_s$ values are traced by the colour map.
Figure 13. Two-dimensional posteriors of the string parameters and their one- and two-sigma contours obtained by importance sampling for two fiducial values of $\alpha'$: $\alpha' = 10^3 m_{Pl}^{-2}$ and $\alpha' = 10^5 m_{Pl}^{-2}$. Correlations with a third parameter are traced by the colour map. The CMB data give some non-trivial constraints only for large values of $\alpha'$. This appears as the degeneracy observed for $N$ and $g_s$ in the bottom right panel. The left panels show that the models lie in the regions predicted by the slow-roll analysis in section 6.

9. Conclusions

To conclude, let us summarize and discuss the main results obtained in this paper. Our goal was to compare a typical model of string inspired brane inflation to the CMB data. In this scenario, the inflaton field is interpreted as the distance between two branes moving in six dimensions along a warped throat. The end of inflation either occurs by violation of the slow-roll conditions or by tachyonic instability at brane annihilation. Our MCMC analysis was carried out by using an exact numerical approach in which the only approximation used is the linear theory of cosmological perturbations. Moreover, various effects have been considered, for instance, the presence of a DBI kinetic term...
or the possible quantum effects. The MCMC exploration has also been performed for arbitrary values of $g_s$ and $\alpha'$. From this analysis, we have obtained the following results. Firstly, the WMAP data favour a scenario where inflation ends by violation of the slow-roll conditions before brane annihilation rather than by the tachyonic instability brought about by the annihilation. In other words, $\phi_{\text{strg}}$ cannot be too close to the edge of the throat and one finds

$$\log \left( \kappa^{1/6} \phi_{\text{strg}} \right) < 0.52,$$

at 95% confidence level. This constraint originates from the fact that ending inflation by instability pushes the spectral index towards 1 while preserving a low level of gravitational waves, a situation which is disfavoured by the data.

Secondly, one has obtained a limit on $v$, the volume ratio of the five-dimensional submanifold forming the basis of the six-dimensional conifold to the volume of the 5-sphere. This limit reads

$$\log v > -10,$$

at 95% confidence level. This constraint comes both from the data and a requirement of self-consistency of the model, namely that inflation proceeds inside the throat.

Thirdly, we find that the reheating period is slightly constrained. Although we have not considered the detailed process of brane annihilation, the total number of e-folds for which the universe reheats may change the part of the inflaton potential probed by the observations. Combined with the power spectra generated during brane inflation, the WMAP data provide the 95% confidence limit

$$\ln R > -38.$$

In the case where the reheating proceeds with a constant equation of state parameter, we find that if $w_{\text{reh}} \gtrsim -1/3$, then $\rho_{\text{reh}}^{1/4} > 20$ GeV (at two-sigma level). This limit is certainly already ruled out by particle physics experiments, but it is worth recalling that it has been obtained from the CMB data only. With more accurate data, it will certainly be possible to improve this bound, and, one hopes, for other values of the equation of state parameter.

Fourthly, on the theoretical side, we have obtained approximate solutions in the case where the DBI kinetic term does not reduce to the standard one. The regime where the initial conditions are such that the quantum effects are important has also been considered in detail; see the appendix. In this case, using a perturbative treatment, we have computed how the trajectory (with or without volume effects) can deviate from the classical motion. We have also shown that this approximative scheme breaks down when the brane approaches the bottom of the throat. In this regime, only a numerical integration of the Langevin equation could allow us to go further and, for instance, to see whether the field really starts to climb the potential as indicated by the calculation presented in the appendix. Clearly, this discussion is relevant for knowing whether a regime of eternal inflation can be established in this model.

Finally, let us discuss how the present work could be improved. Recently, various works have been devoted to the type of model studied here. In particular, special attention has been paid to the general set-up and exactly calculable corrections to
the potential (13) have been proposed. It was also shown that, in some situations, 
these corrections play a crucial role [24]–[28]. The next step would therefore be to 
include them in our analysis. Since they introduce new parameters into the problem, 
and given the weakness of the constraints obtained here, this would probably be 
meaningful only at the expense of including other more accurate data sets to break 
degeneracies. Time dependent KS-like compactifications have also been discussed 
in the literature to provide a late-time acceleration of the universe [94,95]. These models 
may provide an alternative to the standard ΛCDM universe that we have considered 
at low energy. As regards the approach of the present paper, more precise data would 
certainly help to disentangle the correlations between \( N \) and \( g_s \), and might therefore 
allow us to use astrophysical data to constrain string theory, at least under the strong 
thoretical prejudice that the relevant model of inflation is indeed the one described 
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Appendix. Stochastic inflation

A.1. Regime of quantum fluctuations within the throat

We previously stated (section 2.6) that all e-folds of inflation should occur within a single 
throat and hence \( \phi < \phi_{UV} \) always, \( \phi_{UV} \) being the field value at the edge of the throat. In 
section 3, we further calculated the field value \( \phi_{\text{fluct}} \) above which quantum fluctuations have 
an impact on the classical trajectory. It seems reasonable to ask under which conditions, if 
any, the field value \( \phi_{\text{fluct}} \) is located within the throat and hence stochastic inflation effects 
can, at least in principle, affect the field evolution. Let us try to derive the parameter 
restriction resulting from

\[ \phi_{\text{fluct}} < \phi_{UV}. \] (A.1)

Again, it will be important to decide which normalization is used to express \( \phi_{\text{fluct}} \) given by 
equation (46). With \( \phi_{UV} \) from equation (19), consider first the normalization (68) valid 
for \( \phi_{\text{end}} = \phi_{x2} \). In this case, the rescaled parameters \((x, \bar{v})\) of equation (73) are again a 
useful set for expressing (A.1) in a simple form,

\[ \ln \bar{v} > 4 \ln F - \ln x, \] (A.2)

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\(^5\) http://www.idris.fr

\(^6\) http://www.planck.fr
Figure A.1. The panels illustrate the parameter regions allowing the stochastic regime to take place inside the throat, for $\alpha' m^2_{\text{Pl}} = 1000$ (left panel) and $\alpha' m^2_{\text{Pl}} = 10$ (right panel). All curves are the same as in figure 5, but in addition a solid green curve illustrates the condition $\phi_{\text{fluct}} = \phi_{\text{UV}}$. Stochastic inflation may occur inside the throat only for the parameter values lying above the solid green curve, and below the volume constraint (dashed green curve with ticks down).

where the constant $\mathcal{F}$ can be expressed as

$$\mathcal{F} = 3^{1/10} 2^{-1/5} \left( 45 \frac{Q^2_{\text{rms-PS}}}{T^2} \right)^{3/20} (6N_* + 10)^{-1/4} \left( \pi \alpha' m^2_{\text{Pl}} \right)^{1/2}. \quad (A.3)$$

This is a condition of the same type as the one given by equation (92) (from $\phi_{\text{in}, \epsilon^2} < \phi_{\text{UV}}$), but more restrictive. The parameter region in the $(\ln x, \ln \bar{v})$ plane for which stochastic effects can occur within the throat is therefore smaller than the one obtained from equation (92) (see figure A.1).

Now let us investigate the case of equations (99) and (100) where the WMAP normalization is done using $\phi_{\text{end}} = \phi_{\text{strg}} \gg \phi_{\epsilon^2}$. Inserting this normalization into equation (46) for $\phi_{\text{fluct}}$, we find the following inequality in terms of $(\ln x, \ln \bar{v})$:

$$\ln \mathcal{G} < \frac{5}{12} \ln x + \frac{1}{3} \ln \bar{v} + \frac{2}{3} x^{-1/4}, \quad (A.4)$$

where

$$\mathcal{G} = 3^{1/10} 2^{-1/5} \left( \pi \alpha' m^2_{\text{Pl}} \right)^{1/2} \left( 45 \frac{Q^2_{\text{rms-PS}}}{T^2} \right)^{1/15}. \quad (A.5)$$

This is another curve in the plane $(\ln x, \ln \bar{v})$, analogous to (101) though slightly simpler. Again, equation (A.4) constrains the possible parameter space outside the contour $\phi_{\epsilon^2} = \phi_{\text{strg}}$ in the same manner, but somewhat more tightly than equation (101). Note that inside and outside of the contour $\phi_{\epsilon^2} = \phi_{\text{strg}}$, the requirements $\phi_{\text{in}, \epsilon^2} < \phi_{\text{UV}}$ and $\phi_{\text{fluct}} < \phi_{\text{UV}}$ are consistent with each other. Both are represented in figure A.1. We have therefore shown that the condition (A.1) can be met for sufficiently generic parameter combinations, and hence stochastic effects may occur inside the throat.
A.2. Detailed analysis of the stochastic regime

In the following, we apply the treatment established in [60] to the case of the brane inflationary potential (13). The underlying approach of stochastic inflation consists in describing the evolution of a coarse-grained field $\phi$ corresponding to the original scalar field $\varphi$ averaged over e.g. the Hubble patch. The quantum fluctuations correspond in this picture to stochastic noise due to small-scale Fourier modes. Therefore, the dynamics of the coarse-grained field are controlled by a Langevin equation, whose solution enables one to calculate the probability density function of the field $\varphi$. In the slow-roll approximation, this coarse-grained field obeys the equation

$$\dot{\varphi} + \frac{V_{\varphi}}{3H} = \frac{H^{3/2}}{2\pi} \xi(t),$$

(A.6)

where the Hubble parameter $H$ is entirely described by the coarse-grained field as well, i.e. one has (in the slow-roll approximation)

$$H^2 \simeq \frac{\kappa}{3} V(\varphi).$$

(A.7)

In equation (A.6), $\xi(t)$ is the noise field and its mean and two-point correlation function satisfy

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \delta(t - t'),$$

(A.8)

where $\delta$ is the Dirac delta function. The approach in [60] has been used to solve this system of equations using a perturbative expansion in the noise, that is, one writes $\varphi$ as

$$\varphi(t) = \phi_{cl}(t) + \delta\varphi_1(t) + \delta\varphi_2(t) + \cdots,$$

(A.9)

where $\phi_{cl}(t)$ is the solution of the Langevin equation without the noise, and $\delta\varphi_1(t)$ is linear in $\xi(t)$, $\delta\varphi_2(t)$ quadratic and so on. To avoid notational confusion, the classical field value, which is denoted by $\varphi$ in the rest of this paper, is called $\phi_{cl}$ in this appendix. As discussed in [60], there are, up to second order, two resulting probability distributions for $\varphi$. The first of these, $P_c(\varphi, t)$, is the probability of the stochastic process assuming a given field value at a given time in a single coarse-grained domain. The second distribution, $P_v(\varphi, t)$, takes into account volume effects, namely the dependence of the volume size on the field value within this domain. These two distributions are given by

$$P_c(\varphi, t) \equiv \langle \delta(\varphi - \varphi[\xi]) \rangle = \frac{1}{\sqrt{2\pi \langle \delta\varphi_1^2 \rangle}} \exp \left[ -\frac{(\varphi - \langle \varphi \rangle)^2}{2 \langle \delta\varphi_1^2 \rangle} \right],$$

(A.10)

and

$$P_v(\varphi, t) = \frac{\langle \delta(\varphi - \varphi[\xi]) \exp \left[ 3 \int d\tau H(\varphi[\xi]) \right] \rangle}{\langle \exp \left[ 3 \int d\tau H(\varphi[\xi]) \right] \rangle} = \frac{1}{\sqrt{2\pi \langle \delta\varphi_1^2 \rangle}} \exp \left[ -\frac{(\varphi - \langle \varphi \rangle - 3I^T J)^2}{2 \langle \delta\varphi_1^2 \rangle} \right],$$

(A.11)

where the mean field value is given by $\langle \varphi \rangle \approx \phi_{cl} + \langle \delta\varphi_2 \rangle$ and where $I$ and $J$ are two continuous vectors [60]. The influence of stochastic effects therefore requires three quantities to be determined for any given inflationary potential, namely $\langle \delta\varphi_1^2 \rangle$, $\langle \delta\varphi_2 \rangle$, and
$I^T J$. These quantities have been calculated in [60] as functions of the Hubble parameter (in the slow-roll approximation) and read

$$\langle \delta \varphi_1^2 \rangle = \frac{\kappa}{2} \left( \frac{H'}{2\pi} \right)^2 \int_{\phi_{cl}}^{\phi_m} \left( \frac{H}{H'} \right)^3 d\phi,$$  \hspace{1cm} (A.12)

$$\langle \delta \varphi_2 \rangle = \frac{H''}{2H'} \langle \delta \varphi_1^2 \rangle + \frac{H'}{4\pi m_{Pl}^2} \left[ \frac{H_{in}^3}{H_{in}^2} - \frac{H^3}{H'^2} \right],$$  \hspace{1cm} (A.13)

where, in the present context, a prime denotes a derivative with respect to $\varphi$. In the slow-roll approximation, the Hubble parameter can be directly replaced by $\sqrt{\kappa V/3}$ and the correction due to volume effects can be expressed as

$$3I^T J = \frac{12H'}{m_{Pl}} \int_{\phi_{cl}}^{\phi_m} \frac{H^4}{H'^2} d\phi - 12\pi \frac{H}{H'} \langle \delta \varphi_1^2 \rangle.$$  \hspace{1cm} (A.14)

In the stochastic inflation regime, we are dealing with very large field values $\phi \gg \mu$ and one can use the expansion (14) of the potential. Equation (A.12) then yields

$$\frac{\langle \delta \varphi_1^2 \rangle}{\mu^2} = \frac{\kappa^2 M^4}{48\pi^2} \left( \frac{\mu}{\phi_{cl}} \right)^6 \left[ \left( \frac{\phi_{cl}}{\mu} \right)^4 - 1 \right]^{-1} \left\{ \frac{1}{16} \left( \frac{\phi_{in}}{\mu} \right)^{16} - \left( \frac{\phi_{cl}}{\mu} \right)^{16} \right\}$$

$$- \frac{3}{12} \left[ \left( \frac{\phi_{in}}{\mu} \right)^{12} - \left( \frac{\phi_{cl}}{\mu} \right)^{12} \right] + \frac{3}{8} \left[ \left( \frac{\phi_{in}}{\mu} \right)^8 - \left( \frac{\phi_{cl}}{\mu} \right)^8 \right]$$

$$- \frac{1}{4} \left[ \left( \frac{\phi_{in}}{\mu} \right)^4 - \left( \frac{\phi_{cl}}{\mu} \right)^4 \right] \right\}. \hspace{1cm} (A.15)$$

Since the brane motion in the throat occurs only for $\phi_{cl} > \mu$ (see section 4), one gets in the regime $\phi_{cl} \ll \phi_{in}$

$$\frac{\langle \delta \varphi_1^2 \rangle}{\mu^2} \approx \frac{\kappa^2 M^4}{768\pi^2} \left( \frac{\phi_{cl}}{\mu} \right)^{10} \left( \frac{\phi_{in}}{\mu} \right)^{16}. \hspace{1cm} (A.16)$$

Since $\phi_{cl}/\mu$ is decreasing, the variance is, as expected, increasing as inflation proceeds.

Let us now determine the correction to the mean value. As can be seen from (A.13), no additional integration is necessary for calculating $\langle \delta \varphi_2 \rangle$, and after some algebra one finds

$$\frac{\langle \delta \varphi_2 \rangle}{\mu} = \frac{1}{2} \left[ 3 - 5 \left( \frac{\phi_{cl}}{\mu} \right)^4 \right] \left( \frac{\phi_{cl}}{\mu} \right)^{-1} \left[ \left( \frac{\phi_{cl}}{\mu} \right)^4 - 1 \right]^{-1} \frac{\langle \delta \varphi_1^2 \rangle}{\mu^2}$$

$$+ \frac{\kappa^2 M^4}{192\pi^2} \left( \frac{\mu}{\phi_{cl}} \right)^5 \left[ 1 - \left( \frac{\mu}{\phi_{cl}} \right)^4 \right]^{-1/2} \left\{ \left( \frac{\phi_{in}}{\mu} \right)^{10} \right\}$$

$$\times \left[ 1 - \left( \frac{\mu}{\phi_{in}} \right)^{4/5} \right]^{-2/5} \left( \frac{\phi_{cl}}{\mu} \right)^{10} \left[ 1 - \left( \frac{\mu}{\phi_{cl}} \right)^{4/5} \right]^{5/2} \right\}. \hspace{1cm} (A.17)$$

Finally, in order to evaluate the volume effects characterized by the term $3I^T J$, one needs to calculate the integral in equation (A.14). For the potential (14), this can still be
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Figure A.2. Left panel: evolution of the inflaton field when the quantum effects (with or without volume effects) are taken into account. The underlying model is such that ‘classical’ inflation stops by violation of the slow-roll conditions. Hence, the WMAP normalization is calculated with the help of equation (68). With $v = 1$ and $N_\ast = 50$, this leads to $\phi_{\text{fluct}}/\mu \sim 793.5$. The initial condition is chosen to be $\phi_{\text{in}}/\mu = 700$. The solid line represents the classical trajectory, the dotted line the mean value without the volume effects and the dotted–dashed line the mean value with the volume effects. Right panel: evolution of the probability distribution $P_c(\varphi)$ (solid lines) and $P_v(\varphi)$ (dashed lines). As in figure A.2, the case displayed here corresponds to $v = 1$ and $N_\ast = 50$, leading to $\phi_{\text{fluct}}/\mu \sim 793.5$. The initial shape of the probability density function has been chosen to be $\delta(\varphi - \phi_{\text{in}})$ with $\phi_{\text{in}}/\mu = 700$. The vertical dashed lines represent the location of the field at successive points of the classical trajectory (the field moves ‘from right to left’).

solved exactly. One obtains

$$
\frac{3 I T J}{\mu} = \frac{\kappa^3 M^4 \mu^2}{128 \pi^2} \left( \frac{\mu}{\phi_{\text{cl}}} \right)^5 \left[ 1 - \left( \frac{\mu}{\phi_{\text{cl}}} \right)^4 \right]^{-1/2} \times \left[ S \left( \frac{\phi_{\text{in}}}{\mu} \right) - S \left( \frac{\phi_{\text{cl}}}{\mu} \right) \right] - \frac{3 \kappa \mu^2}{4} \left[ 1 - \left( \frac{\mu}{\phi_{\text{cl}}} \right)^4 \right] \left( \frac{\phi_{\text{cl}}}{\mu} \right)^5 \frac{\langle \delta\varphi_1^2 \rangle}{\mu^2},
$$

(A.18)

where the function $S$ is defined by

$$
S(x) \equiv \frac{1}{383} \left[ x^2 \sqrt{x^4 - 1} \left( -279 + 326 x^4 - 200 x^8 + 48 x^{12} \right) + 105 \ln \left( x^2 + \sqrt{x^4 - 1} \right) \right].
$$

(A.19)

The corresponding trajectory is plotted in figure A.2 (left panel). The solid line represents the classical trajectory without the quantum effects. Since we are using $\phi_{\text{cl}}$ as a clock, this line is just the straight line ‘$y = x$’. The dotted line gives $\langle \varphi \rangle = \phi_{\text{cl}} + \langle \delta\varphi_2 \rangle$, i.e. the mean value of the field in the situation where the volume effects are ignored. The two dashed lines above and below this line are $\phi_{\text{cl}} + \langle \delta\varphi_2 \rangle \pm \sqrt{\langle (\delta\varphi_1^2) \rangle}$. As discussed at length in [60], the quantum effects accumulate and manifest themselves only once the field is approaching the end of inflation. As clearly shown in figure A.2, the quantum effects are such that the field rolls down the potential more quickly than its classical counterpart. However,
the story is different when volume effects are taken into account. The corresponding trajectory is represented by the dotted–dashed line, the two dashed lines above and below being the limits of the one-sigma interval. In this case, the field is first slowed down and starts moving back to even climb up the potential. Then, at some point, it stops and rolls down the potential again. However, as we will see in the next subsection, this last part of the evolution cannot be tracked within our approximation since it is outside its domain of validity (represented by the shaded regions).

In figure A.2 (right panel), we have represented the corresponding distributions without and with volume effects as given by equations (A.10) and (A.11) respectively. The vertical dotted lines corresponds to the classical values of the field at three successive times (denoted as ‘1’, ‘2’ and ‘3’), namely $\phi^{(1)}/\mu = 300$, $\phi^{(2)}/\mu = 200$ and $\phi^{(3)}/\mu = 150$. The solid lines are the three distributions $P_c$ corresponding to the three classical field values while the dashed lines are the three distributions $P_v$. This plot essentially confirms the above analysis. In particular, one notices that the maximum of $P_c$ is ahead of the classical vacuum expectation value as discussed before. In contrast, the volume effects ‘retain’ the field and thus one can even change the direction of its motion (from right to left to left to right). Finally, the two distributions strongly spread out while inflation is proceeding.

### A.3. Reliability of the perturbative treatment

In this section, we discuss in which range of field values $\langle \phi \rangle \in [\phi_{cl} - |\Delta \varphi_{\text{min}}(\phi_{cl})|, \phi_{cl} + \Delta \varphi_{\text{max}}(\phi_{cl})]$ the previous analysis is valid. Indeed, as discussed above, in order to estimate the mean value of the field with or without the volume effects, a perturbative expansion in the noise has been used which may break down at some point. It has been shown in [61] that $\Delta \varphi_{\text{min}}$ and $\Delta \varphi_{\text{max}}$ can be found from requiring that the two conditions

\[
\max_{x \in [\phi_{cl}, \phi_{cl} + \Delta \varphi_{\text{max}}(\phi_{cl})]} \left| \frac{H^{(4)}(x)}{6} \Delta \varphi^3 \right| \ll \left| \frac{H'''}{2} \right| \Delta \varphi^2, \tag{A.20}
\]

\[
\max_{x \in [\phi_{cl}, \phi_{cl} + \Delta \varphi_{\text{max}}(\phi_{cl})]} \left| \frac{[H^{3/2}(x)]''}{2} \right| \Delta \varphi^2 \ll \left| \frac{H^{3/2}}{2} \right| \Delta \varphi \tag{A.21}
\]

are simultaneously satisfied. The strongest bounds on $\Delta \varphi_{\text{min}}$ and $\Delta \varphi_{\text{max}}$ that follow from (A.20) and (A.21) then give the reliability interval for the distributions $P_c(\varphi, t)$ and $P_v(\varphi, t)$. In the present case, a straightforward calculation leads to

\[
\frac{\Delta \varphi_{\text{min}}}{\mu} \approx -\frac{2 \phi_{cl}}{5}, \quad \frac{\Delta \varphi_{\text{max}}}{\mu} = \frac{2 \phi_{cl}}{5}. \tag{A.22}
\]

The resulting reliability region is plotted in figure A.2 and corresponds to the purple hatched region. We see that the approximation breaks down more quickly for $P_v(\varphi, t)$ than for $P_c(\varphi, t)$. In particular, the conclusion that, due to volume effects, the field reverses its motion comes from a regime which is at the border of the reliability interval. We conclude that, in order to have access to this phenomenon and, in particular, to discuss eternal inflation in this model, a more powerful method is needed, for instance, a full numerical integration of the Langevin equation (A.6).
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