Rotating Dilaton Solutions in 2+1 Dimensions

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Abstract

We report a three parameter family of solutions for dilaton gravity in 2+1 dimensions with finite mass and finite angular momentum. These solutions are obtained by a compactification of vacuum solutions in 3+1 dimensions with cylindrical symmetry. One class of solutions corresponds to conical singularities and the other leads to curvature singularities.

Keywords: 2+1 Dimensions, Dilaton, Cylindrical symmetry.
1 Introduction

Three dimensional gravity has provided us with many important clues about higher dimensional physics. String theory, which seems to be the best candidate available for a consistent theory of quantum gravity, requires studies of low dimensional string effective actions. In this respect, dilaton gravity in 2+1 dimensions deserve further attention since it arises from a low energy string effective theory.

In one of the earlier works, Shiraishi[1] found a family of static multi centered solutions for Einstein-Maxwell-dilaton gravity. Park and Kim[2] constructed general static axially symmetric solutions to the same model by dimensional reducing to two dimensions. In recent times, lot of attention has been given to gravity in 2+1 dimensions with a negative cosmological constant due to the existence of black hole solutions[3]. Modification of this black hole with a dilaton and Maxwell’s fields have lead to many interesting results. Static charged black holes and spinning black holes in anti-de Sitter space by Chan and Mann[4][5], spinning solutions with self dual electromagnetic fields by Fernando[6], black holes in generalized dilaton gravity with a Brans-Dicke type parameter by Sá et.al.[7], magnetic solutions by Koikawa et.al.[8] are some of the work related to dilaton gravity in anti-de Sitter space.

In this paper we present an interesting class of dilaton solutions arising from four dimensional gravity. This is achieved by a compactification of vacuum solutions with cylindrical symmetry. These new set of solutions are different from the ones presented above due its direct relation to four dimensions. Our final motivation in studying three dimensional gravity is to gain further understanding of higher dimensional world. Therefore these solutions are important since it provide us with a clear understanding of how the 4D gravity and 3D gravity are related to each other.

We have structured the paper as follows. In section 2 we will give a brief introduction to four dimensional vacuum solutions with cylindrical symmetry and compactify them to obtain dilaton solutions in three dimensions. In section 3 a prescription to compute the mass and the angular momentum is highlighted. In section 4 we will study static solutions. In section 5 we will study rotating solutions and finally we will conclude.

2 Vacuum Solutions with Cylindrical symmetry in Four dimensions and its Compactification

Cylindrical symmetrical space-times in four dimensions consists of isometries generated by two commuting space-like Killing vectors. If the solutions are stationary then the space-time admit another Killing vector along the time axis. General stationary cylindrical symmetric line element with three Killing vectors \( \partial_t, \partial_z, \) and \( \partial_\phi \) can be written as,

\[
ds^2 = e^{-2U} \left( e^{2K} (dr^2 + dz^2) + r^2 d\phi^2 \right) - e^{2U} (dt + Ad\phi)^2 \tag{1}
\]
where $U$, $K$ and $A$ are functions of $r$ only. The general stationary vacuum cylindrical solutions to pure Einstein action

$$S = \int d^4x \sqrt{-GR}$$

are given by,

$$e^{2U} = r(a_1r^n + a_2r^{-n})$$

$$e^{2K-2U} = r^{\frac{n^2-1}{2}}$$

$$A = \frac{c}{na_2 (a_1r^n + a_2r^{-n})} + b$$

$$e^2 = -n^2a_1a_2$$

(3)

The complex constants, $n, c, a_1, a_2, b$ have to be chosen such that the metric is real. In this paper, we will choose all these constants to be real.

The purpose of this paper is to dimensionally reduce four dimensional Einstein gravity to three dimensions to obtain dilaton gravity. The field content of the reduced theory would be gravity, dilaton $\phi$ and the gauge field $A_\mu$. However, if we pick the cylindrical solutions given above and treat the $z$ coordinate to be compact for the purposes of equations of motion and their symmetries, the reduction will yield a theory in three dimensions with only a dilaton field coupled to gravity. To perform compactification along the $z$ direction, let us rewrite the above metric in four dimensions as follows:

$$ds^2_{3+1} = G_{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + e^{-4\phi} dz^2$$

(4)

with

$$e^{-4\phi} = G_{zz} = r^{\frac{(n^2-1)}{2}}$$

(5)

Here $(\mu, \nu = 0,1,2,3)$ and $(a, b = 0,1,2)$ are four and three dimensional indices respectively. The dimensionally reduced action in three dimensions is given by,

$$S_{\text{string}} = \int d^3x \sqrt{-g^S} e^{-2\phi} R^S$$

(6)

Here, $g^S$ corresponds to $g_{ab}$ of the four dimensional metric and the metric can be treated to be in the “string frame”. One can perform a conformal transformation to bring the metric to Einstein frame as follows,

$$e^{-4\phi} g^S_{\mu\nu} = g^E_{\mu\nu}$$

(7)

This transformation will lead to the following action and the corresponding field equations,

$$S_{\text{Einstein}} = \int d^3x \sqrt{-g^E} (R^E - 8\nabla \phi \nabla \phi)$$

(8)
\[ R_{\mu\nu} = 8 \nabla_\mu \phi \nabla_\nu \phi \]
\[ \nabla_\mu \nabla_\nu \phi = 0 \]  
(9)  
(10)

Hence by starting from the metric given in eq.(1), one can obtain solutions to three dimensional space-time as,
\[ ds^2_{\text{Einstein}} = e^{-2U - 4\phi} \left( e^{2K} dr^2 + r^2 d\varphi^2 \right) - e^{2U - 4\phi} (dt + Ad\varphi)^2 \]  
(11)

with a dilaton given by,
\[ \phi = \frac{(-n^2 + 1)}{8} \ln(r) \]  
(12)

The above metric can be rewritten in the following form,
\[ ds^2 = g_{tt} dt^2 + g_{\varphi \varphi} d\varphi^2 + 2 g_{t\varphi} d\varphi dt + g_{rr} dr^2 \]  
(13)

with
\[ g_{rr} = r^{n^2 - 1} \]
\[ g_{tt} = - a_1 r^{(n+1)^2} - a_2 r^{(n-1)^2} \]
\[ g_{\varphi \varphi} = \left( \frac{1}{a_2} - 2b \sqrt{-a_1/a_2} \right) r^{(n+1)^2} - b^2 a_2 r^{(n-1)^2} \]
\[ g_{t\varphi} = (-ba_2) r^{(1-n)^2} + \left( -ba_1 - \sqrt{-a_1/a_2} \right) r^{(1+n)^2} \]  
(14)

This metric and the dilaton \( \phi \) satisfy the field equations of eqs.(9) and (10). In the discussion of the solutions we will pick \( a_2 > 0, a_1 = 0 \) and \( n \geq 1 \).

### 3 Mass and Angular Momentum of the Solutions

The mass and the angular momentum of the source of this solution is computed by following the prescription of Brown and York \[10\] which is briefly described as follows. If the metric in 2+1 dimensions is written in the following form,
\[ ds^2 = -N^2 dt^2 + \frac{dR^2}{f^2} + R^2 (d\varphi + N^\varphi dt)^2 \]  
(15)

the quasi-local mass \( m(R) \) and quasi-local angular momentum \( j(R) \) are given by,
\[ m(R) = 2N^4 (f_0(R) - f(R)) - j(R) N^\varphi \]  
(16)
\[ j(R) = \frac{f(R) \frac{dN^\varphi}{dR} R^3}{N^4(R)} \]  
(17)

Here, \( f_0 \) comes from a background metric which corresponds to the solution with zero mass. If \( R \) is a function of \( r \), as it is in the case of the solutions described in this paper, \( f(r) \) and \( g_{rr} \) are related to each other as \( f(r) = \sqrt{1/g_{rr} \frac{dr}{dr}} \). The mass and the angular momentum are computed as \( R \to \infty \),
\[ M = \lim_{R \to \infty} m(R); \quad J = \lim_{R \to \infty} j(R) \]  
(18)
4 Static Dilaton Solutions

In this section we will consider the static dilaton solutions corresponding to \( b = 0 \). Then the metric simplifies to,

\[
\begin{align*}
    ds^2 &= -a_2 r^{\frac{(-1+n)^2}{2}} dt^2 + \frac{dr^2}{r^{(1-n^2)}} + \frac{r^{\frac{(1+n)^2}{2}}}{a_2} d\varphi^2 \\
    \phi &= \frac{(-n^2 + 1)}{8} \ln(r)
\end{align*}
\]  
(19)

4.1 Static solutions for \( n = 1 \)

For \( n = 1 \), the dilaton vanishes leaving the following metric.

\[
    ds^2 = -a_2 dt^2 + dR^2 a_2 + R^2 d\varphi^2
\]  
(21)

Note that a coordinate transformation \( r = R \sqrt{a_2} \) has been performed. This is the well known metric of a point source in 2+1 dimensions which leads to a conical singularity at the origin [11] [12]. Let us describe how the conical singularity arises as follows:

By redefining \( t, R \) and \( \varphi \) as \( t' = \sqrt{a_2} t, R' = \sqrt{a_2} R, \varphi' = \varphi / \sqrt{a_2} \), the metric in eq.(21) becomes,

\[
    ds^2 = -dt'^2 + dR'^2 + R'^2 d\varphi'^2
\]  
(22)

Note that the former periodic coordinate \( \varphi \) has the range \( 0 \leq \varphi \leq 2\pi \) and the new period coordinate has the range \( 0 \leq \varphi' \leq 2\pi a_2^{-1/2} \). Hence there is a deficit angle \( D \) at the origin due to the presence of a massive source given by \( D = 2\pi (1 - a_2^{-1/2}) \) leading to the conical space-time. Now, to relate the parameter \( a_2 \) appearing in the metric to the mass of the source we will use the prescription given in the section (3). The zero mass metric is chosen to be the Minkowski space which corresponds to \( a_2 = 1 \), leading to \( f_0 = 1 \). In the presence of the source, \( f(R) = a_2^{-1/2} \) and \( N^t(R) = 1 \) from eq.(21). By the definition in eq.(16),

\[
    m(R) = M = 2(1 - a_2^{-1/2}).
\]  
(23)

Hence the mass of the source \( M \) and \( a_2 \) are related by \( a_2 = \left(1 - \frac{M}{2}\right)^{-2} \).

4.2 Static solutions for \( n > 1 \)

For \( n > 1 \), the dilaton has a non zero value and the metric can be written completely in terms of \( R \) as follows,

\[
    ds^2 = R^{2(\frac{n^2+1}{n^2-1})^2} \left( -a_2^{\frac{2(n^2+1)}{(n+1)^2}} dt^2 + 16a_2^{\frac{2(n^2+1)}{(n+1)^2}} (1 + n)^{-4} dR^2 \right) + R^2 d\varphi^2
\]  
(24)
By scaling the time as \(a_2^{(n^2+1)/(1+n^2)} t = t'\), the metric simplifies to,

\[
\frac{a_2^{(n^2+1)/(1+n^2)} dR^2}{(A(R)^{-1}a_n)^2} + \frac{dR^2}{R^2} + R^2 d\varphi^2
\]

where

\[
a_n = a_2^{-(n^2+1)/(1+n^2)} \frac{(1+n)^2}{4}; \quad A(R) = R^{(\frac{n-1}{n+1})^2}
\]

The functions \(f(R)\) and \(N^t(R)\) can be read from the metric as,

\[
f(R) = A(R)^{-1}a_n; \quad N^t(R) = A(R)
\]

To compute the mass, let the reference metric corresponds to \(a_2^2 = 1\) which leads to

\[
f_0 = A(R)^{-1}(1+n)^2 \frac{1}{4}
\]

With above preliminaries, one can compute the mass and the angular momentum to be,

\[
M_n = \frac{(n+1)^2}{2} \left(1 - a_2^{-((1+n)^2/4)}\right); \quad J_n = 0
\]

When \(n \to 1\), the mass \(M_n \to M\) as expected.

The Ricci scalar \(R_s\) and Kretschmann scalar \(K_r\) are computed for the above metric as follows:

\[
R_s = R_{\mu\nu}g^{\mu\nu} = \frac{(-n^2+1)}{8r^{(1+n^2)}}; \quad K_r = R_{\mu\nu\rho\gamma}R^{\mu\nu\rho\gamma} = \frac{3(-n^2+1)}{64r^{2(1+n^2)}}
\]

For \(n > 1\) these scalars have singularities at \(r = 0\) leading to a curvature singularity. Since there are no horizons, it is a naked singularity. For \(n = 1\) the scalar curvature is zero everywhere leading to the conical singularity at the origin as discussed in the previous section. Hence the conical singularity has turned into a curvature singularity due to the presence of the dilaton field. Since the scalar curvature \(R \to 0\) at large \(R\), the space-time is asymptotically flat.

5 Rotating Dilaton Solutions

In this section we will consider the solutions with the parameter \(b \neq 0\). Such solutions can be written as,

\[
ds^2 = -\frac{r^{(n^2+1)}}{R^2} dt^2 + \frac{dr^2}{r^{-(n^2+1)}} + R^2 \left(d\varphi - \frac{a_2 br^{(n-1)^2}}{R^2} dt\right)^2
\]

with

\[
R^2 = \frac{r^{(n+1)^2/2}}{a_2} - b^2 a_2 r^{-\frac{(n-1)^2}{2}}
\]
5.1 Rotating flat solutions with \( n = 1 \)

For \( n = 1 \), the dilaton field vanishes and the solutions correspond to the following metric.

\[
ds^2 = -\frac{r^2}{R^2} dt^2 + dr^2 + R^2 (d\varphi - \frac{a_2 b}{R^2} dt)^2
\]

where \( R^2 = (\frac{r^2}{a_2} - b^2 a_2) \). The conical singularity at \( r = 0 \) still exists. Also \( g_{\varphi\varphi} \) component of the metric becomes negative for \( r < \frac{b}{a_2} \) leading to closed time-like curves as described in Deser et.al.\[11\]. The space-time is flat everywhere since the scalar curvature vanishes similar to the static flat case.

To compute the mass, the metric can be rewritten completely in terms of \( R \) by a coordinate transformation \( r^2 = (R^2 + a_2 b^2) a_2 \) and \( t' = \sqrt{a_2} t \).

\[
ds^2 = -\frac{(R^2 + a_2 b^2)}{R^2} dt'^2 + \frac{R^2 a_2}{(R^2 + a_2 b^2)} dR^2 + R^2 (d\varphi - \frac{\sqrt{a_2 b}}{R^2} dt')^2
\]

Considering the Minkowski space-time as the reference one can compute the mass and the angular momentum as

\[
M = 1 - a_2^{-1/2}; \quad J = 2b.
\]

5.2 Rotating Dilaton Solutions with \( n > 1 \)

For \( n > 1 \), a non-zero dilaton field exists which modifies the flat space-time considerably. These solutions do have the same scalar invariants as computed for the static case in eq.(31) which signals curvature singularities at \( r = 0 \). Furthermore the metric function \( g_{\varphi\varphi} \) becomes negative for \( r < \frac{b}{a_2} \) leading to closed time like curves.

To calculate the mass and the angular momentum, one has to rewrite \( r \) as a function of \( R \). Due to the nature of the expression in eq.(32) it is not possible to find an exact expression for \( r \). Hence, by a binomial expansion of eq.(32) around \( R \to \infty \) followed by an inversion of the series lead to,

\[
\frac{1}{r} = \left( \frac{1}{R^2 a_2} \right)^\frac{2}{(1+n)^2} - \frac{2a_2 b}{(1+n)^2} \left( \frac{1}{R^2 a_2} \right)^\frac{2(2n+1)}{(1+n)^2} + ....
\]

Substitution of \( r \) into the functions \( N^t(r) \), \( f(r) \), \( N^\varphi(r) \) in eq.(31) to compute the quasi-local mass and angular momentum \( m(R) \), \( j(R) \) and taking the limit \( R \to \infty \) leads to following quantities,

\[
M = \frac{(n+1)^2}{2} \left( 1 - a_2^{-\frac{(1+n)^2}{(1+n)^2}} \right); \quad J = 2nb
\]

Note that in computing the mass we have chosen the reference metric as the one with \( a_2 = 1, b = 0 \) and \( n \neq 1 \) which gives \( f_0 = R^{-\frac{(1+n)^2}{2(1+n)^2}} \) as in the static case.
6 Conclusions

We have discussed the properties of a new three parameter family of solutions to Einstein-dilaton gravity in 2+1 dimensions obtained by a compactification of stationary cylindrical symmetrical solutions in 3+1 dimensions. The mass and the angular momentum of each solution are computed in terms of the parameters of the solution $a_2, n$ and $b$. For $n = 1$, the compactified solutions lead to the well known conical space-time with a mass deficit and time helical structure presented by Deser et.al.\cite{11} arising in pure Einstein gravity in 2+1 dimensions. For $n > 1$, a non zero dilaton field exists and the resulting space-time has curvature singularities. The rotating solutions also has closed time like curves.

It is natural to extend this work to compactify charged cylindrical solutions in four dimensions to see the relation to the existing charged solutions in 2+1 dimensional dilaton gravity. In this context, there are two solutions which would be interesting to study. One is the static cylindrical solutions by Safko \cite{14} and the other is the solutions of a charged line-mass by Muckherji \cite{13}.

It would be also interesting to embed the solutions discussed here in a supergravity theory arising from a low energy string theory along the lines of supersymmetric solutions to three dimensional heterotic string action considered by Bakas et.al.\cite{15}.

We hope to address these issues in the future.

**Note added in proof:** The authors were informed of related work mentioned in references \cite{16}, \cite{17}, \cite{18}, \cite{19}, \cite{20}.

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