Unstable modes of oscillations of a tangential discontinuity in partly ionized plasma with charge exchange: long waves

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Abstract. An analytical study of linear unstable modes of oscillations of a tangential velocity discontinuity in partly ionized space plasma with charge exchange is carried out. The long-wave approximation is considered. In this approximation, the scale of perturbations is much larger than a certain characteristic size, which is determined by the charge-exchange cross-section and number density of neutral atoms. It is shown that along with the classical Kelvin–Helmholtz instability, there exists an unstable mode whose increment does not depend explicitly on the relative velocity of the two media separated by the discontinuity. The presence of this mode is entirely determined by the presence of the charge-exchange process.

1. Introduction
The problem of instability of a tangential velocity discontinuity in hydrodynamics and gas dynamics has a long history. The instability of such discontinuity in an incompressible ideal fluid was first demonstrated in the works [1] and [2] about 150 years ago. Moreover, unstable perturbations propagate in the discontinuity plane in all directions. Much later, it was shown that the tangential discontinuity is unstable in a compressible gas as well [3, 4]. However, in the compressible case, the spatial region of instability decreases. For example, at high speeds of relative motion (high Mach numbers) of two media, unstable waves propagate in a narrow range of angles almost perpendicular to the velocity vector.

In space plasma, an important physical effect is the charge-exchange process. The charge exchange is the transfer of an electron from an atom to an ion (proton) so that the heavy charged particle becomes neutral and vice versa. Naturally, this process occurs in partly ionized plasma. As a result, it leads to the transfer of momentum and energy between two components – charged and neutral. In space, charge exchange is often the only possible mechanism of interaction between atoms and ions. An example is the electron-proton plasma of the solar wind and interstellar hydrogen atoms, which are relatively free to penetrate into the heliosphere due to the large mean free path. Thus, the question naturally arises of the effect of charge exchange on stability of shear flows in space plasma. These types of flows include, for example, the flow in the vicinity of the heliopause separating the solar wind and interstellar medium, and the flow near cometary ionopauses.

In the present paper we study instability of a velocity tangential discontinuity in a partly ionized hydrogen plasma with charge exchange. The study is carried out by the method of small perturbations. We consider a long-wave approximation, in which the scale of perturbations is much larger than a certain characteristic length, determined by the charge-exchange cross-section and number density of atoms. This paper is a continuation of work [5], where the short-wave approximation was considered.
in the same formulation. As already mentioned in [5] the main difficulty in the formulation of the problem of stability of a tangential discontinuity in a plasma with charge exchange is to construct the basic state. The friction force between atoms and ions leads to pressure and density gradients along or perpendicular to the discontinuity surface, so that the plasma state on both sides of the discontinuity is significantly nonuniform. There are articles [6, 7] in which Kelvin-Helmholtz instability is analytically investigated in relation to heliopause. However, these articles do not provide a clear description of the basic state. In addition, a “quasi-incompressibility” approximation of the medium is used, which has some justification for stationary flows with small Mach numbers and is not justified for the explicitly unsteady stability problem. The inconsistency of this approximation is discussed in paper[8].

2. Problem statement

2.1. Governing equations
Consider a flat surface separating two uniform electron-proton plasma with different densities \( \rho_{01}, \rho_{02} \) and temperatures \( T_{01}, T_{02} \). We introduce a Cartesian coordinate system in which the lower plasma is at rest, and the upper one moves at a speed \( U_1 \) parallel to the plane of discontinuity (xy plane) and constituting an angle \( \phi \) with the x axis (see figure 1). Thus, one can write:

\[
\begin{align*}
U_1 &= \{U \cos \phi, \sin \phi, 0\}\ldots z > 0 \\
U_2 &= 0\ldots z < 0
\end{align*}
\]  

(1)

![Figure 1](image_url)  

*Figure 1.*Plasma flow in the vicinity of the tangential discontinuity (xy plane). The angle \( \phi \) between the velocity vector and the x axis varies from 0 to \( \pi \).

If \( p_{0j} \) is the plasma pressure \( (j = 1, 2) \) and \( a_{0j}^2 = \gamma p_{0j} / \rho_{0j} \) is the square of the sound speed, then the following relations can be written from the condition of equality of pressures on both sides of the discontinuity:

\[
\frac{\rho_{02}}{\rho_{01}} = \frac{T_{01}}{T_{02}} = \frac{a_{01}^2}{a_{02}^2} = \kappa
\]  

(2)

Here \( \kappa \) is a free parameter. In addition to the electron-proton plasma, there is a collisionless atomic hydrogen gas. Hydrogen atoms occupy the entire space (\( n_H = \text{const} \) ), have a zero mean velocity
( \( U_H = 0 \) ), can freely penetrate through the discontinuity surface and are in thermodynamic equilibrium with the lower plasma (\( T_H = T_{02} \)). In the present paper we believe that the temperatures of electrons and protons are equal. The atoms interact with the lower plasma through charge exchange, while at the upper plasma the cross section of charge exchange is zero. Since the cross section drops sharply with energy at high temperatures, this assumption can be true for very hot plasma. In our notation, this means that \( \kappa \gg 1 \).

The charge-exchange terms in the basic equations for plasma, we write in the Holzer’s approximation [9]:

\[
\frac{\partial n_H}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{3}
\]

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho} \nabla p = -\sigma U^* n_H (u - U_H) \tag{4}
\]

\[
\frac{\partial p}{\partial t} + u \cdot \nabla p + \rho \nabla \cdot u = \sigma U^* n_H \left[ \frac{(\gamma - 1)(u - U_H)^2}{2} + \frac{kT_H}{m_p} - \frac{p}{2\rho} \right] \tag{5}
\]

In equations (3) – (5) \( \rho, u \) and \( p \) are the plasma density, velocity and pressure, \( \sigma \) is the cross section for charge-exchange collisions, \( m_p \) is the proton mass, and

\[
U^* = \left[ \left( u - U_H \right)^2 + \frac{64}{9\pi} \left( \frac{2kT_H}{m_p} + \frac{2kT}{m_p} \right)^{3/2} \right]^{1/2} \tag{6}
\]

Although we assume here that \( U_H = 0 \), we nevertheless keep \( U_H \) in equations (4) – (6) for clarity.

2.2. Small disturbances and dispersion relation

Consider small disturbances of the interface propagating along the \( x \) axis.

\[
\xi = \bar{\xi} \exp[i(k_x x - \omega t)] \tag{7}
\]

where \( \bar{\xi} \) is the complex amplitude and \( k_x > 0 \). The density, velocity and pressure can be written as (j=1, 2)

\[
\rho_j = \rho_{0j} + \tilde{\rho}_j(t, x, z) \]

\[
u_j = \tilde{U}_j(t, x, z) \]

\[
p_j = \tilde{p}_j(t, x, z) \tag{8}
\]

We look for solutions of the linearized version of equations (3) – (5) in the form

\[
\tilde{f}_j = \bar{f}_j \exp[i(k_x x + k_z z - \omega t)] \tag{9}
\]

From the condition of damping of the perturbations at a distance from the discontinuity we obtain

\[
\text{Im}(k_{z1}) > 0, \quad \text{Im}(k_{z2}) < 0. \tag{10}
\]
The expressions (8) and (9) are substituted in the linearized system of equations (3) – (6). From the condition of solvability of the resulting systems of linear equations for amplitudes we obtain in each semi-infinite space

\[
\left( \frac{k_{z1}}{k_x} \right)^2 = \left( \chi - M \cos \varphi \right)^2 - 1
\]  

(11)

\[
\left( \frac{k_{z2}}{k_x} \right)^2 = \frac{\kappa \chi \left( \chi + i \varepsilon M^* \right) \left( \chi + i \varepsilon M^*/2 \gamma \right)}{\chi + i \varepsilon M^*/2 \gamma} - 1
\]  

(12)

In expressions (11), (12) we have introduced the following notations:

\[
\chi = \frac{\omega}{k_x a_{01}}, \quad M = \frac{U}{a_{01}}, \quad M^* = \frac{U^*}{a_{01}}, \quad \varepsilon = \frac{\sigma n_H}{k_x}.
\]  

(13)

In notations (13) \( U_0^* = U_{02}^* \) and \( \sigma = \sigma_2 \). Equations (11) and (12) will be used to determine wave numbers \( k_{zj} \).

Finally, to obtain the dispersion relation for the interface oscillation frequency, we use boundary conditions that relate the perturbation amplitudes on both sides of the discontinuity. Namely, we use the equality of the plasma pressures and the equality of the normal plasma velocities and velocities of the interface:

\[
\frac{\partial \xi}{\partial t} + U \cos \varphi \frac{\partial \xi}{\partial x} = \tilde{u}_{z1}
\]  

(14)

\[
\frac{\partial \xi}{\partial t} = \tilde{u}_{z2}
\]  

(15)

\[
\tilde{p}_1 = \tilde{p}_2
\]  

(16)

As is usually the case in linear problems, the boundary conditions are set at \( z = 0 \). If we insert expression (7) in equations (14) – (15) we obtain after some algebra the following dispersion relation:

\[
\frac{k_{z1}}{(\chi - M \cos \varphi)^2} - \frac{k_{z2}}{\kappa \chi \left( \chi + i \varepsilon M^* \right)} = 0
\]  

(17)

The wave numbers \( k_{zj} \) are given by expressions (11) and (12). For the first time equation (17) was obtained in the work [5]. From equation (6) it is easy to deduce that

\[
M^* = \frac{16}{3\sqrt{2\pi} \kappa}
\]

Thus, \( M^* \) is a constant and, unlike \( M \), does not depend on the relative velocity \( U \).

3. Long-wave approximation

When charge exchange is absent (\( \varepsilon = 0 \)) dispersion relation takes the usual classic form. In paper [5] solutions of equation (17) were investigated in the short-wave approximation \( \varepsilon \ll 1 \). Here we consider the opposite case \( \varepsilon \gg 1 \). For convenience, we introduce a new variable \( \eta = 1/\varepsilon \ll 1 \).
3.1. Flank modes
Write the dimensionless frequency as the following expansion:

\[ \chi = \chi_0 + \chi_1 \eta + O(\eta^2) \]  \hfill (18)

Substitute (18) into expressions (11), (12) for \( k_{\perp} \) and then into dispersion relation (17). In the zero order approximation with respect to the small parameter we obtain

\[ \chi_0 = \beta \pm 1 \]  \hfill (19)

where the new variable \( \beta = M \cos \varphi \) is introduced. In the first order approximation with respect to \( \eta \) we obtain for the wave numbers \( k_{\perp} \):

\[ \frac{k_{\perp}}{k_x} = \sqrt{2(\chi_0 - \beta)} \chi_1 \eta^{1/2} \]  \hfill (20)

\[ \frac{k_{\perp}}{k_x} = \sqrt{2(\chi_0 - \beta)} \chi_1 \eta^{1/2} \]  \hfill (21)

Then from the dispersion relation (17) we find \( \chi_1 \). Finally, we have

\[ \chi_{\pm} = \beta \pm 1 \mp \frac{i\gamma}{2\kappa M^* (\beta \pm 1)} \eta \]  \hfill (22)

It is easy to show from expressions (20) – (22) that conditions (10) are satisfied everywhere. Expression (22) gives the value of the oscillation frequency of the tangential discontinuity surface in a partially ionized plasma taking into account charge exchange. The frequency is a complex quantity, the imaginary part of which is proportional to

\[ \text{Im}(\chi_{\pm}) \propto \frac{k_x}{\kappa \sigma n_H} \frac{1}{\beta \pm 1} \]  \hfill (23)

According to expressions (7) or (9) disturbances increase with time if \( \text{Im}(\chi_{\pm}) > 0 \). One can deduce from equation (23) that all disturbances are damped when \( |\beta| < 1 \), while at \( |\beta| > 1 \) frequencies (22) have both positive and negative imaginary parts (remind that \( k_{\perp} > 0 \)). In other words, in this range of values \( \beta \) the tangential discontinuity is unstable. Instability is qualitatively different from the classical Kelvin-Helmholtz instability, since it occurs only in the presence of the charge-exchange process. In addition, the instability increment does not depend essentially on the relative velocity of the two media. This type of instability was obtained also in a partially ionized plasma in the short-wave approximation [5].

It should be noted that expansion (22) is incorrect near points \( \beta = \pm 1 \). The behavior of the solution in a vicinity of these points requires special consideration.

3.2. Damping modes
Let us seek a solution for the frequency in the form of the following expansion:

\[ \chi = \eta^{-1}(\chi_0 + \chi_1 \eta + \ldots) \]  \hfill (24)

In this case we obtain from the dispersion relation in the zero order approximation:
\[ \chi_\pm = -\frac{i M^* S_\pm}{4(\kappa - 1)} \eta \] 

(25)

where

\[ S_\pm = \left( \gamma^{-1} + 2 \right) \kappa - 1 \pm \left[ \left( \gamma^{-1} - 2 \right) \kappa + 1 \right]^2 + 4 \kappa \gamma^{-1} \right]^{1/2} \] 

(26)

One can show that \( S_+ > 0 \) at \( \kappa > 1 \) (it is this case that we consider in the paper). Then, as it is easy to see from (25), all disturbances are damped. As for the wave numbers, they have the following form in the main approximation:

\[ \frac{k_{z1}}{k_x} = -\frac{i M^* S_\pm}{4(\kappa - 1)} \eta \] 

(27)

\[ \frac{k_{z2}}{k_x} = -\frac{i}{\eta} \left[ \frac{\kappa M^* S_\pm}{4(\kappa - 1)} \right] - \frac{S_\pm}{4(\kappa - 1)} - \frac{1}{2} \left[ \frac{S_\pm}{4(\kappa - 1)} - \frac{1}{2} \gamma \right]^{-1} \] 

(28)

As a result of elementary estimations it can be shown that

\[ \frac{S_+}{4(\kappa - 1)} - 1 > 0, \quad \frac{S_-}{4(\kappa - 1)} - \frac{1}{2} < 0, \quad \frac{S_-}{4(\kappa - 1)} - \frac{1}{2} \gamma > 0. \] 

(29)

From the relations (29) it follows that \( \text{Im}(k_{z1}) > 0 \) and \( \text{Im}(k_{z2}) < 0 \). Perturbations with frequencies (25) are strongly damped, since the small parameter in terms of which the expansion is carried out is in the denominator.

### 3.3. Kelvin – Helmholtz instability

The solution is sought in the form of the following expansion:

\[ \chi = \eta (\chi_0 + \eta \chi_1 + \ldots) \] 

(30)

It follows from the dispersion relation in the zero approximation that

\[ \chi_\pm = \frac{i \beta^2 Q_\pm}{2 \kappa M^* \left( 1 - \beta^2 \right)} \eta \] 

(31)

where

\[ Q_\pm = 1 \pm \left[ 1 + \frac{4 \left( 1 - \beta^2 \right)^2}{\gamma^2 \beta^4} \right]^{1/2} \] 

(32)

Here we are limited to only one principal term in expansion (30). Let us consider two cases separately:

1) \( |\beta| < 1 \)

In this case \( Q_+ > 0 \) and \( Q_- < 0 \). Then one of the frequencies in expression (31) has a positive imaginary part and the other negative. In other words, \( \chi_+ \) describes an unstable mode and \( \chi_- \) a damped mode. This is the classical Kelvin – Helmholtz instability modified by charge-exchange effects. In the classical theory, the dimensional frequency \( \omega \) is proportional to the wave number \( k_x \). Here it is
proportional to the square of the wave number. The frequency of the unstable mode depends essentially \(( \sim \beta^4 )\) on the relative velocity of the upper and lower plasmas.

One can show that for \( \chi \), the wave numbers \( k_{zj} \) are

\[
\frac{k_{z1}}{k_z} = i \sqrt{1 - \beta^2} + O(\eta), \tag{33}
\]

\[
\frac{k_{z2}}{k_z} = -i \left[ 1 + \frac{\chi^2 \beta^4 Q_\pm}{2(1 - \beta^2)} \right]^{1/2} + O(\eta). \tag{34}
\]

The wave numbers (33), (34) satisfy conditions (10) and the dispersion relation, but the wave numbers corresponding to \( \chi_\pm \), as it turns out, cannot simultaneously satisfy (10) and (17). Thus at \( |\beta| < 1 \) there is only one acceptable solution consistent with the expansion (30) and this solution describes the Kelvin–Helmholtz instability.

2) \( |\beta| > 1 \)

In this case \( Q_\pm > 0 \) and both modes describe damped disturbances. The wave number \( k_{z1} \) is a complex value while \( k_{z2} \) is real. In such a situation it is necessary to apply the Sommerfeld radiation condition: disturbances must be outgoing when \( z \to \pm \infty \) (in our specific case at \( z \to -\infty \)) in a reference frame moving with each medium separately [10]. Since the case of damped perturbations is of little interest and algebraic calculations are very cumbersome, we omit this case in the present paper.

**Conclusions**

Solutions of the dispersion equation for the frequency of small disturbances of the velocity tangential discontinuity in partly ionized hydrogen plasma are studied in the long-wave approximation. The wavelength of the disturbances is assumed to be large in comparison with some characteristic length determined by the charge-exchange cross-section and number density of neutral atoms. The atomic hydrogen gas is collisionless, uniformly fills the entire space and interacts with plasma by means of charge exchange. Due to difficulties in constructing a homogeneous basic state, we use several simplifying assumptions. First, we believe that the gas is in dynamic and thermodynamic equilibrium with the lower plasma. Secondly, we believe that the charge-exchange cross-section in the upper plasma is zero. It is possible if the upper plasma is very hot.

One of the main results of the paper is that, along with the Kelvin-Helmholtz instability, in partially ionized plasma with charge exchange a new unstable mode appears. The instability increment of the new mode does not depend essentially on the relative velocity of the two media. This instability can probably be attributed to the so-called dissipative instability. This type of instability exists also in partially ionized plasma in the short-wave approximation [5]. It is interesting to note that the instability condition with respect to the new mode is \( |\beta| > 1 \), while the Kelvin–Helmholtz instability condition is \( |\beta| < 1 \). Points \( \beta = \pm 1 \) and their close vicinities require additional separate consideration.

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**References**

[1] von Helmholtz H 1868 *Monatsschr. Preuss. Akad. Wiss.* (Berlin) **23** 215

[2] Kelvin Lord 1871 *Phil. Mag.* **42** 362
[3] Landau L D 1944 Dokl. Acad. Nauk USSR (in Russian) 44 151
[4] Syrovatskii S I 1954 Zh. Eksp. Teor. Fiz. (in Russian) 27 121
[5] Chalov S V 2019 Mon. Not. R. Astron. Soc. 482 1664
[6] Florinski V, Zank G P, Pogorelov N V 2005 J. Geophys. Res. 110 A07104
[7] Avinash K, Zank G P, Dasgupta B, Bhadoria S 2014 Astrophys. J. 791 102
[8] Belov N A 2010 Astron. Lett. 36 144–149
[9] Holzer T E 1972 J. Geophys. Res. 77 5407
[10] Gerwin R A 1968 Rev. Modern Phys. 40 652–658