Merging Process Algebra and Action-based Computation Tree Logic

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Abstract

Process algebra and temporal logic are two popular paradigms for the specification, verification and systematic development of reactive and concurrent systems. These two approaches take different standpoint for looking at specifications and verifications, and offer complementary advantages. In order to mix algebraic and logic styles of specification in a uniform framework, the notion of a logic labelled transition system (LLTS) has been presented and explored by Lütten and Vogler. This paper intends to propose a LLTS-oriented process calculus which, in addition to usual process-algebraic operators, involves logic connectives (conjunction and disjunction) and standard temporal operators (always and unless). This calculus preserves usual properties of these logic operators, allows one to freely mix operational and logic operators, and supports compositional reasoning. Moreover, the links between this calculus and Action-based Computation Tree Logic (ACTL) including characteristic formulae of process terms, characteristic processes of ACTL formulae and Galois connection are explored.

Key Words Process Calculus Action-based Computation Tree Logic Ready Simulation Logic Labelled Transition System Galois Connection

1 Introduction

1.1 Two popular paradigms in formal method

The dominant approaches for the specification, verification and systematic development of reactive and concurrent systems are based on either states or actions. For state-based approaches, an execution of a system is viewed as a sequence of states, while another approach regards an execution as a sequence of actions.

State-based approaches devote themselves to specifying and verifying abstract properties of systems, which often involve formalisms in logic style. Since the seminal work of Pnueli [53], logics have been adopted to serve as useful tools for specifying and verifying of reactive and concurrent systems. In such framework, a
specification is expressed by a set of formulae in some logic system and verification is a deductive or model-checking activity.

Action-based approaches put attention to behavior of systems, which have tended to use formalisms in algebraic style. These formalisms are referred to as process algebra or process calculus [45, 36, 35, 10]. In such paradigm, a specification and its implementation usually are formulated by the same notations, which are terms (expressions) of a formal language built from a number of operators, and the underlying semantics are often assigned operationally. Intuitively, a specification describes the desired high-level behavior, and an implementation provides lower-level details indicating how this behavior is to be achieved. The verification amounts to compare terms, which is often referred to as implementation verification or equivalence checking [2]. The comparison of a specification to an implementation is based on behavioral relations. Such relations depend on particular observation criterions, and are typically equivalences (or preorders), which capture a notion of “having the same observation” (respectively, “refinement”). At the present time, due to lack of consensus on what constitutes an appropriate notion of observable behavior, a variety of observation criterions and behavioral relations have been proposed [25]. The correctness of an implementation may be verified in a proof–theory oriented manner or in a semantics oriented manner. The former is rooted in an axiomatization of the behavioral relation, while the later appeals to coinduction technology which is considered as one of the most important contributions of concurrency theory to computer science [55].

Since logic and algebraic frameworks take different standpoint for looking at specifications and verifications, they offer complementary advantages:

On the logic side, there exist a number of logic systems, e.g., Linear temporal logic [53], Computation tree logic [17], $\mu$–calculus [38] and so on, in which the most common reasonable property of concurrent systems, such as invariance (safety), liveness, etc., can be formulated without referring operational details (see, e.g., [16, 57]). Moreover, one of inherent advantage of logic approach is that it is ability to deal with partial specifications: one can establish that a given system realizes a particular property without involving its full specification. On the other hand, the inclusion of classes of models is a natural refinement preorder on logic specifications, hence refining a logic specification amounts to enrich original one by adding new formulas consistently. However, logic approach has been criticized for being global, non-modular and non-compositional. In other words, we often are required to consider a given system as a whole whenever formulating and verifying a logic property. For instance, it always lacks a natural way to combine temporal properties, which are required separately for subsystem $P_1$ and $P_2$, into a temporal specification for $P_1 \parallel P_2$. Such deficiency has been indicated by Pnueli in [53] where temporal logic is described as being endogenous, that is, assuming the complete program as fixed context. Summarizing, a variety of logics may serve as powerful tools for expressing and verifying a wide spectrum of properties of concurrent systems, but, due to their global perspective and abstract nature, it is difficult for them to describe the link between the structure of implementation and that of specification, and hence logic approaches often give little support for systematic development of concurrent systems.

On the algebraic side, since systems are represented by terms in some algebras, complex systems may be built up from existent systems using algebraic operators. Moreover, the observable behavior of the complex system does not change if an subsystem is replaced by one with the same behavior, which is granted by the fact that behavioral relations considered in process algebras are often required to be compatible with process operators, in other words, these relations are (pre)congruence over terms. These features cause the main advantage of algebraic paradigm, that is, it always supports compositional constructing and reasoning. Such compositionality
brings us advantages in developing systems, such as, supporting modular design and verification, avoiding verifying the whole system from scratch when its parts are modified, allowing reusability of proofs and so on [5]. Thus algebraic approaches offer significant support for rigorous systematic development of reactive and concurrent systems. However, since algebraic approaches specify a system by means of prescribing in detail how the system should behave, it is often difficult for them to describe abstract properties of systems, which is a major disadvantage of such approaches.

1.2 Connections between process algebras and logics

It is natural to wonder what the connection between the algebraic approach and logic approach is. Based on structural operational semantics (SOS) in Plotkin-style, terms in process algebras can be “transformed” into labelled transition systems. The latter may be viewed as models (in the model-theoretic sense) for suitable logic language. Hence this induces the satisfiability relation $|=\,|$ between process terms and formulas. Given such satisfiability relation, three connections between a process algebra and a logic deserve special mention, which are considered by Pnueli in [54] and recalled in the following. Let $P$ be a process algebra equipped with a behavioral relation $\triangleright$, and $L$ a logic language associated with a satisfiability relation $|=\,|$. 

- **Adequacy of** $(L, |=) \text{ w.r.t } (P, \triangleright)$

  The logic $L$ is said to be adequate w.r.t $(P, \triangleright)$ if for any process $p$ and $q$, either

  \[
  p \triangleright q \text{ iff } \forall \alpha \in L (p |= \alpha \iff q |= \alpha) \text{ (if } \triangleright \text{ is an equivalence)}
  \]

  or

  \[
  p \triangleright q \text{ iff } \forall \alpha \in L (q |= \alpha \Rightarrow p |= \alpha) \text{ (if } \triangleright \text{ is a preorder)}
  \]

  This notion is considered by Hennesy and Milner in [34], where they prove that Hennesy-Milner logic (HML) is adequate w.r.t bisimilarity for image finite CCS terms. It is one of key events that make Milner think that CCS is definitely interesting enough [1]. Following their work, the literature on concurrency theory offers a wealth of modal characterizations for various behavioral relations. A good overview on this subject may be found in [9]. In the realm of modal logic, more generalized results concerning Hennesy-Milner property (class) have been established [8, 11, 27, 28, 37]. Recently, such issue is also considered in depth in the framework of coalgebras [see, e.g., 47, 51].

  As pointed out by Pnueli in [54], the requirement of adequacy is the weakest one of compatibility between a process algebra and a logic. A symptom of its weakness is that the same logic may be adequate for some process languages with very different expressivity [54]. For instance, HML is adequate w.r.t bisimilarity for both CCS and the fragment of CCS consisting of recursion-free terms. Moreover, the Hennesy-Milner characterization is less useful if one intend to check the equivalence of process terms using model checking [2].

  Stronger associations between processes algebras and logic systems involve translating between them: characterizing a given process in terms of logic formulae, and graphical representing a given logic formula by means of process terms. Next we recall them in turns.

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1 See: [http://www.sussex.ac.uk/Users/mfb21/interviews/milner/](http://www.sussex.ac.uk/Users/mfb21/interviews/milner/)

2 In this realm, a coalgebraic modal logic for $F$–coalgebras is said to be adequate if behavioral equivalence implies logical equivalence, and it is said to be expressive if the converse holds.
• **Expressivity** of \((L, \models)\) w.r.t \((P, \bowtie)\)

A stronger compatibility requirement involves expressivity. The logic \(L\) is said to be expressivity w.r.t \((P, \bowtie)\) if for any process \(p\) in \(P\), there exists a formula \(L(p)\) \(\in L\) such that

(E1) \(q \models L(p)\) iff \(q \bowtie p\) for any process \(q\) in \(P\), and
(E2) \(p \models \varphi\) iff \(L(p) \rightarrow \varphi\) for any formula \(\varphi\) in \(L\).

Clearly, if such formula for a process can be algorithmically constructed, implementation verification can be reduced to model checking according to (E1), and the verification of an assertion \(p \models \varphi\) can be transformed into the validity problem within \(L\) by (E2). Graf and Sifakis were probably the first to develop logics which are expressive for process algebras. In [29], they present Synchronization Tree Logic (STL, for short) for a process algebra with a congruence relation \(\approx\). STL contains process terms as formulae, and its semantic is defined so that both (E1) and (E2) hold with the function \(L = \lambda x.x\).

Given a process \(p\), a formula \(\phi_p\) is said to be a characteristic formula of \(p\) if it satisfies (E1). Such notion also provides a very elegant link between process algebra and logic, and between implementation verification and model checking [2]. Graf and Sifakis provide a method of constructing characteristic formula modulo observational congruence for any recursion-free CCS term [30]. Hitherto, over different structures, e.g., finite LTS, Kripke structures, time automata and so on, a number of examples of characteristic-formula constructions for various behavioral relations have been reported in the literature [4, 15, 20, 23, 39, 40, 46, 56, 59]. The underlying structures of these constructions are identical, that is, characteristic formulae often are defined as fixed points of some functions. Recently, ground on this phenomenon, L.Aceto et al. offer a general framework for the constructions of characteristic formulae [2, 3].

• **Expressivity** of \((P, \bowtie)\) w.r.t \((L, \models)\)

Another stronger association between process algebras and logics involves an inverse translation, which associates with each formula \(\varphi \in L\) a set \(P(\varphi)\) that consists of all the processes satisfying \(\varphi\). A process language is said to be expressive for \(L\) if such translation is given in a syntactic manner. In order to obtain such expressivity, additional operators that construct process sets are often needed.

In a classic paper [14], Boudol and Larsen offer a process language \(\Theta\) and a translation \(\zeta(.)\) in a syntactic manner, and show that any HML formula \(\phi\) is representable by a finite set \(\zeta(\phi)\) of terms in \(\Theta\). In particular, \(\zeta(\phi)\) can be reduced to a singleton, say \{\(\phi^*\)\}, if and only if the given formula \(\phi\) is consistent and prime. Moreover, such term \(\phi^*\) satisfies the property below

\[
t \models \phi \iff \phi^* \sqsubseteq t \text{ for any term } t \text{ in } \Theta.
\]

Here \(\sqsubseteq\) is a behavioral relation considered in [14]. In such situation, the model checking problem can be reduced to implementation verification. Clearly, \(\phi^*\) plays an analogous role of characteristic formula in a contrary way. In fact, characteristic-formula construction and \(\zeta(.)\) indeed induce a Galois connection between \((\Theta, \sqsubseteq)\) and the set of consistent prime formulae augment with some preorder [14]. In [1], L.Aceto et al. address the same issue, and show that, modulo the covariant-contravariant simulation preorder, any consistent and prime formula in the covariant-contravariant modal logic also admits a representation by means of process terms.

### 1.3 Background and motivation

As mentioned above, logic approaches and algebraic approaches offer complementary advantages when specifying systems. The former is good at specifying abstract
properties of systems, while the latter is applicable if we intend to specify the system itself through describing its behavioral and structural properties.

Impelled by taking advantage of both approaches when designing systems, so-called heterogeneous specifications have been proposed, which uniformly integrate these two specification styles. Among them, based on Büchi automata and LTS augmented with a predicate, Cleaveland and Lüttgen provide a semantic framework for heterogenous system design [18, 19], where must-testing preorder offered by Nicola and Hennessy [48] is adopted to describe refinement relation. In addition to usual operational operators, such framework also involves logic connectives. However, since must-testing preorder is not a precongruence in such situation, this setting does not support compositional reasoning. Moreover, the logic connective conjunction in this framework lacks the desired property that \( r \) is an implementation of the specification \( p \land q \) if and only if \( r \) implements both \( p \) and \( q \).

Recently, Lüttgen and Vogler introduce the notion of a Logic LTS (LLTS, for short), which combines operational and logic styles of specification in one unified framework [42, 43]. In order to handle logic conjunctions of specifications, LLTS involves consideration of inconsistencies, which, compared with usual LTS, is one distinguishing feature of it. Two kinds of constructors over LLTSs are considered in [42, 43]: operational constructors, e.g., CSP-style parallel composition, hiding and so on, and logic connectives including conjunction and disjunction. Such framework allows one to freely mix these two kinds of constructors, while most early theories couple them loosely and do not allow for mixed specification. Moreover, the drawbacks in [19, 18] mentioned above have been remedied by adopting ready-tree semantics [42]. In order to support compositional reasoning in the presence of the parallel constructor, a variant of the usual notion of ready simulation is employed to characterize the refinement relation [43]. Some standard modal operators in temporal logics, such as \textit{always} and \textit{unless}, are also integrated into this framework [44].

Along the direction suggested by Lüttgen and Vogler in [43], we propose a process calculus called CLL in [60], which reconstructs their setting in process algebraic style. In addition to prefix \( a() \), external choice \( □ \) and parallel operator \( \parallel \), CLL contains logic operators \( \land \) and \( \lor \) over process terms, which correspond to the constructors conjunction and disjunction over LLTSs respectively. The language CLL is explored in detail from two different but equivalent angles. Based on behavioral view, the notion of ready simulation is adopted to formalize the refinement relation, and the behavioral theory is developed. Based on proof-theoretic view, a sound and ground-complete axiomatic system for CLL is provided. In effect, it gives an axiomatization of ready simulation in the presence of logic operators.

However, due to lack of modal operators, CLL still does not afford describing abstract properties of systems. This paper intends to enrich CLL with temporal operators \textit{always} and \textit{unless} by two distinct approaches. One approach is to introduce nonstandard process-algebraic operators \( \sharp \), \( \varpi \), \( \vartriangle \) and \( \odot \) to capture Lüttgen and Vogler’s constructions in [44] directly. The other is to provide graphical representing of temporal operators \textit{always} and \textit{unless} in recursive manner. The latter is independent of Lüttgen and Vogler’s constructions but depends on the greatest fixed-point characterization obtained in this paper. Moreover, the connections between the resulting calculus (that we call CLLT) and ACTL [49] are explored from angles recalled in the preceding subsection. These connections include characteristic formulae of process terms, characteristic processes of formulae and Galois connection.

The remainder of this paper is organized as follows. The next section presents some preliminaries. In Section 3, SOS rules of CLLT are introduced, the existence and uniqueness of stable transition model for CLLT is demonstrated, and a few of basic properties of the LTS associated with CLLT are given. Section 4 and 5 are
Section 6 establishes a fixed-point characterization of the operator $\Box$ and $\Rightarrow$ respectively. Section 7 provides a recursive approach to dealing with the temporal operator $\Box$. Section 8 explores the links between CLLT and ACTL. Finally, a brief conclusion and discussion are given in Section 9.

2 Preliminaries

In this section, we shall set up notation and terminology and briefly sketch the process calculus CLL.

2.1 Logic LTS

This subsection will introduce some useful notations and recall the notion of a Logic LTS. Here we do not give examples motivating and illustrating the use of such notation, which may be found in [43, 44].

Let $\text{Act}$ be a set of visible actions ranged over by letters $a$, $b$, etc., and let $\text{Act}_{\tau}$ denote $\text{Act} \cup \{\tau\}$ ranged over by $\alpha$ and $\beta$, where $\tau$ represents invisible actions. An LTS with a predicate $F$ is a quadruple $(P, \text{Act}_{\tau}, \rightarrow, F)$, where $P$ is a set of states, $\rightarrow \subseteq P \times \text{Act}_{\tau} \times P$ is the transition relation and $F \subseteq P$. As usual, we write $p \xrightarrow{\alpha} q$ if $(p, \alpha, q) \in \rightarrow$. A state $q$ is said to be an $\alpha$-derivative of $p$ if $p \xrightarrow{\alpha} q$. The assertion $p \xrightarrow{\alpha} n$ holds if $p$ has an $\alpha$-derivative, otherwise $p \not\xrightarrow{\alpha}$. Given a state $p$, the ready set of $p$, denoted by $I(p)$, is defined as $\{\alpha \in \text{Act}_{\tau} : p \xrightarrow{\alpha} \}$. A state $p$ is said to be stable if it can not engage in any $\tau$-transition, i.e., $p \not\xrightarrow{\tau}$. Some useful decorated transition relations are listed below.

- $p \xrightarrow{\alpha} q \text{ iff } p \xrightarrow{\alpha} q \text{ and } p, q \notin F$.
- $p \Rightarrow q \text{ iff } p(\xrightarrow{\tau})^* q$, where $(\xrightarrow{\tau})^*$ is the transitive and reflexive closure of $\xrightarrow{\tau}$.
- $p \Rightarrow q \text{ iff } p \Rightarrow r \xrightarrow{\alpha} s \Rightarrow q \text{ for some } r, s \in P$.
- $p \Rightarrow q \text{ (or, } p \Rightarrow q \text{ ) iff } p \Rightarrow q \not\Rightarrow (p \Rightarrow q \not\Rightarrow$, respectively).
- $p \Rightarrow_F q \text{ if there exists a sequence of } \tau\text{-labelled transitions from } p \text{ to } q \text{ such that all states along this sequence, including } p \text{ and } q, \text{ are not in } F$. The decorated transition $p \Rightarrow_F q \text{ may be defined similarly}$.

\textbf{Remark 2.1} Notice that some notations above are slightly different from ones adopted by Lüttgen and Vogler. In [43, 44], the notation $p \Rightarrow_F |q \text{ (or, } p \Rightarrow_F |q \text{ has the same meaning as } p \Rightarrow_F |q \text{ (respectively, } p \Rightarrow_F |q \text{ in this paper.}$

\textbf{Definition 2.1} ([43]) An LTS $(P, \text{Act}_{\tau}, \rightarrow, F)$ is said to be a LLTS if, for each $p \in P$,

\textbf{(LTS1)} $p \in F$ if $\exists \alpha \in I(p) \forall q \in P (p \xrightarrow{\alpha} q \text{ implies } q \in F)$,

\textbf{(LTS2)} $p \in F$ if $\neg \exists q \in P, p \Rightarrow_F |q$.

A LLTS $(P, \text{Act}_{\tau}, \rightarrow, F)$ is said to be $\tau$-pure if, for each $p \in P$, $p \Rightarrow$ implies $\neg \exists q \in \text{Act}. \ x \rightarrow$. Hence, for any state $p$ in a $\tau$-pure LTS, either $I(p) = \{\tau\}$ or $I(p) \subseteq \text{Act}$.

Here the predicate $F$ is used to denote the set of all inconsistent states. Compared with usual LTSs, it is one distinguishing feature of LLTS that it involves consideration of inconsistencies. Roughly speaking, the motivation behind such
consideration lies in dealing with inconsistencies caused by conjunctive composition. In the sequel, we shall use the phrase "inconsistency predicate" to refer to $F$. The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence (i.e., infinite sequences of $\tau$-transitions) should be viewed as catastrophic. For more intuitive idea about inconsistency and motivation behind (LTS1) and (LTS2), the reader may refer to [43, 44].

2.2 A variant of ready simulation

In [43, 44], the notion of ready simulation below is adopted to formalize the refinement relation, which is a modified version of the usual notion of ready simulation (see, e.g., [25]).

**Definition 2.2** ([43, 44]) Given a LLTS $(P, \text{Act}_\tau, \rightarrow, F)$, a relation $R \subseteq P \times P$ is said to be a stable ready simulation relation if, for any $(t, s) \in R$ and $a \in \text{Act}$, the following conditions hold

(RS1) Both $t$ and $s$ are stable;

(RS2) $t \notin F$ implies $s \notin F$;

(RS3) $t \overset{a}{\Rightarrow} F \mid u$ implies $\exists v, s \overset{a}{\Rightarrow} F \mid v$ and $(u, v) \in R$;

(RS4) $t \notin F$ implies $I(t) = I(s)$.

We say that $t$ is stable ready simulated by $s$, in symbols $t \sqsubseteq_{RS} s$, if there exists a stable ready simulation relation $R$ with $(t, s) \in R$. Further, $t$ is said to be ready simulated by $s$, written $t \sqsubseteq_{RS} s$, if

$$\forall u (t \overset{a}{\Rightarrow} F \mid u \text{ implies } \exists v, s \overset{a}{\Rightarrow} F \mid v \text{ and } u \sqsubseteq_{RS} v).$$

It is easy to see that both $\sqsubseteq_{RS}$ and $\sqsubseteq_{RS}$ are pre-order (i.e., reflexive and transitive). The equivalence relations induced by them are denoted by $\approx_{RS}$ and $=_{RS}$, respectively, that is

$$\approx_{RS} \triangleq \sqsubseteq_{RS} \cap (\sqsubseteq_{RS})^{-1} \text{ and } =_{RS} \triangleq \sqsubseteq_{RS} \cap (\sqsubseteq_{RS})^{-1}.$$

The notion of ready simulation presented in Def. 2.2 is a central notion in [43, 44, 60] and this paper. It is natural to wonder why such notion is adopted to formalize the refinement relation. From our point of view, whenever we try to mix process-algebraic and logic styles of specification in a uniform framework, the requirements below should be met by such framework.

- It is well known that parallel composition and conjunction are two fundamental ways of combining specifications: the former is adopted to structurally compose two or more subsystems, and the latter is used to combine specifications expressed by logic formulae. Thus such uniform framework should include these two constructors.

- Since such framework involves specifications in logic style, we should take account of the consistency of specifications. A trivial and desired property is that an inconsistent specification can only be refined by inconsistent ones.

- Such uniform framework should support compositional reasoning. Hence the behavior relation adopted in this framework need to be (pre)congruent w.r.t all operators within it.
Consequently, the result below reveals that it is reasonable to adopt the notion of ready simulation in Def. 2.2 as behavior relation when we intend to explore such uniform framework.

**Theorem 2.1** ([43]) The ready simulation \( \sqsubseteq_{RS} \) exactly is the largest precongruence \( \preceq \) w.r.t parallel composition and conjunction such that \( p \preceq q \) and \( q \in F \) implies \( p \in F \).

**Proof.** See Theorem 21 in [43]. \( \square \)

### 2.3 Transition system specifications

Structural Operational Semantics (SOS) is a logic method of giving operational semantics, which provides a syntax oriented view on operational semantics [52]. Transition System Specifications (TSSs), as presented by Groote and Vaandrager in [31], are formalizations of SOS. This subsection recalls basic concepts related to TSS. Further information on this issue may be found in [9, 13, 31].

Given an infinite set \( V \) of variables and a signature \( \Sigma \), we assume that the resulting notions of term, closed (ground) terms, substitution and closed (ground) substitution are already familiar to the reader. Following standard usage, the set of resulting notions of term, closed (ground) terms, substitution and closed (ground) substitution are already familiar to the reader. Following standard usage, the set of resulting notions of term, closed (ground) terms, substitution and closed (ground)

Transition System Specifications (TSSs) is a quadruple \( \Gamma = (\Sigma, A, \Lambda, \Xi) \), where \( \Sigma \) is a signature, \( A \) is a set of labels, \( \Lambda \) is a set of predicate symbols and \( \Xi \) is a set of rules. Positive literals are all expressions of the form \( \psi \), where \( \psi \) is a positive closed literal. A rule \( r \in \Xi \) has the form like \( \psi r \), where \( \psi r \) is the conclusion of the rule \( r \), and \( \psi r \) is a positive literal. Given a rule \( r \), the set of positive premises (or, negative premises) of \( r \) is denoted by \( \text{pprem}(r) \) (respectively, \( \text{nprem}(r) \)), moreover, \( r \) is said to be positive if \( \text{nprem}(r) = \emptyset \). A TSS is said to be positive if it has only positive rules. Given a substitution \( \sigma \) and a rule \( r \in \Xi \), \( r \sigma \) is the rule obtained from \( r \) by replacing each variable in \( r \) by its \( \sigma \)-image, that is, \( r \sigma = \ell r | \ell \in \text{prem}(r) \). Moreover, if \( \sigma \) is closed then \( r \sigma \) is said to be a ground instance of \( r \).

**Definition 2.3** (Proof in Positive TSS) Let \( \Gamma = (\Sigma, A, \Lambda, \Xi) \) be a positive TSS. A proof of a closed positive literal \( \psi \) from \( \Gamma \) is a well-founded, upwardly branching tree, whose nodes are labelled by closed literals, such that

- the root is labelled with \( \psi \),

- if \( \chi \) is the label of a node \( q \) and \( \{ \chi_i : i \in I \} \) is the set of labels of the nodes directly above \( q \), then there is a rule \( \{ \varphi_i : i \in I \} \) in \( \Xi \) and a closed substitution \( \sigma \) such that \( \chi = \varphi \sigma \) and \( \chi_i = \varphi_i \sigma \) for each \( i \in I \).

If a proof of \( \psi \) from \( \Gamma \) exists, then \( \psi \) is said to be provable from \( \Gamma \), in symbols \( \Gamma \vdash \psi \).

Given a TSS \( \Gamma = (\Sigma, A, \Lambda, \Xi) \), a transition model \( M \) is a subset of \( Tr(\Sigma, A) \cup \text{Pred}(\Sigma, \Lambda) \), where \( Tr(\Sigma, A) = T(\Sigma) \times A \times T(\Sigma) \) and \( \text{Pred}(\Sigma, \Lambda) = T(\Sigma) \times \Lambda \). Following standard usage, elements \( (t, a, s) \) and \( (t, P) \) in \( M \) are written as \( t \stackrel{a}{\rightarrow} s \) and \( t \stackrel{P}{\rightarrow} \) respectively. A positive closed literal \( \psi \) is said to be valid in \( M \), in symbols \( M \models \psi \), if \( \psi \in M \). A negative closed literal \( t \not\stackrel{a}{\rightarrow} \) (or, \( t \not\stackrel{P}{\rightarrow} \)) holds in \( M \), in symbols \( M \models t \not\stackrel{a}{\rightarrow} \) (\( M \models t \not\stackrel{P}{\rightarrow} \), respectively), if there is no \( s \) such that \( t \stackrel{a}{\rightarrow} s \in M \) (\( t \not\stackrel{P}{\rightarrow} \not\in M \), respectively).
respective). As usual, for a set $\Psi$ of closed literals, $M \models \Psi$ iff $M \models \psi$ for each $\psi \in \Psi$.

**Definition 2.4** Let $\Gamma = (\Sigma, A, \Lambda, \Xi)$ be a TSS and $M$ a transition model. $M$ is said to be a model of $\Gamma$ if, for each $r \in \Xi$ and $\sigma : V \rightarrow T(\Sigma)$ such that $M \models \text{prem}(r \sigma)$, we have $M \models \text{conc}(r \sigma)$. $M$ is said to be supported by $\Gamma$ if, for each $\psi \in M$, there exist $r \in \Xi$ and $\sigma : V \rightarrow T(\Sigma)$ such that $M \models \text{prem}(r \sigma)$ and $\text{conc}(r \sigma) = \psi$. $M$ is said to be a supported model of $\Gamma$ if $M$ is supported by $\Gamma$ and $M$ is a model of $\Gamma$.

A natural and simple method of describing the operational nature of processes is in terms of LTSs. Given a TSS, an important problem is how to associate LTSs with process terms. For positive TSS, the answer is straightforward. It is well known that every positive TSS $\Gamma$ has a least transition model, which exactly consists of provable transitions of $\Gamma$ and induces a LTS naturally. However, since it is not immediately clear what can be considered as a “proof” for a negative formula, it is much less trivial to associate a model with a TSS containing negative premises [32]. The first generic answer to this question is formulated in [32, 12], where the above notion of supported model is introduced. However, this notion doesn’t always work well. Several alternatives have been proposed, and a good overview on this issue is provided in [26]. In the following, we recall the notions of stratification and stable transition model, which play an important role in this field.

**Definition 2.5** (Stratification [13]) Let $\Gamma = (\Sigma, A, \Lambda, \Xi)$ be a TSS and $\zeta$ an ordinal number. A function $S : \text{Tr}(\Sigma, A) \cup \text{Pred}(\Sigma, \Lambda) \rightarrow \zeta$ is said to be a stratification of $\Gamma$ if, for every rule $r \in \Xi$ and every substitution $\sigma : V \rightarrow T(\Sigma)$, the following conditions hold.

- $S(\psi) \leq S(\text{conc}(r \sigma))$ for each $\psi \in \text{pprem}(r \sigma)$,
- $S(tP) < S(\text{conc}(r \sigma))$ for each $t \neg P \in \text{nprem}(r \sigma)$, and
- $S(t \xrightarrow{\alpha} s) < S(\text{conc}(r \sigma))$ for each $s \in T(\Sigma)$ and $t \xrightarrow{\alpha} \notin \text{nprem}(r \sigma)$.

A TSS is said to be stratified iff there exists a stratification function for it.

**Definition 2.6** (Stable Transition Model [13, 24]) Let $\Gamma = (\Sigma, A, \Lambda, \Xi)$ be a TSS and $M$ a transition model. $M$ is said to be a stable transition model for $\Gamma$ if

$M = M_{\text{Strip}(\Gamma, M)}$,

where $\text{Strip}(\Gamma, M)$ is the TSS $(\Sigma, A, \Lambda, \text{Strip}(\Xi, M))$ with

$\text{Strip}(\Xi, M) \triangleq \left\{ \frac{\text{prem}(r)}{\text{conc}(r)} : r \text{ is a ground instance of some rule in } \Xi \text{ and } M \models \text{prem}(r) \right\}$,

and $M_{\text{Strip}(\Gamma, M)}$ is the least transition model of the positive TSS $\text{Strip}(\Gamma, M)$.

As is well known, stable models are supported models, and each stratified TSS $\Gamma$ has a unique stable model [13], moreover, such stable model does not depend on particular stratification function [32].
2.4 Process calculus CLL

For the convenience of the reader this subsection will briefly sketch the process calculus CLL proposed in [60], thus making our exposition self-contained. The processes in CLL are given by BNF below, where \( \alpha \in \text{Act}_r \) and \( A \subseteq \text{Act} \).

\[
p ::= 0 \mid \bot \mid (\alpha.p) \mid (p \sqcap p) \mid (p ||_A p) \mid (p \lor p) \mid (p \land p).
\]

As usual, 0 is a process that can do nothing. The prefix \( \alpha.t \) has a single capability expressed by \( \alpha \), and the process \( t \) cannot proceed until \( \alpha \) has been exercised. \( \bot \) is an external choice operator. \( ||_A \) is a CSP-style parallel operator, \( t_1 ||_A t_2 \) represents a process that behaves as \( t_1 \) in parallel with \( t_2 \) under the synchronization set \( A \). \( \bot \) represents an inconsistent process which cannot engage in any transition. \( \lor \) and \( \land \) are logic operators, which are intended for describing logic combinations of processes. In addition to operators over processes, CLL also contains predicate symbols \( F \) and \( F_\alpha \) for each \( \alpha \in \text{Act}_r \). Intuitively, given a process \( p, pF \) says that \( p \) is inconsistent, and \( pF_\alpha \) says that \( p \) has a consistent \( \alpha \)-derivative, which is useful when describing (LTS1) (see, Def. 2.1) in terms of SOS rules. The SOS rules of CLL are divided into two parts: transition rules and predicate rules, which are given below.

\[
(Ra_1) \frac{\alpha.p \to t}{\alpha.p \to p} \quad (Ra_2) \frac{p_1 t \not\to \gamma}{p_1 \sqcap p_2 \to p_2 \not\to t} \quad (Ra_3) \frac{p_1 t \not\to \gamma}{p_1 \sqcap p_2 \to p_1 \not\to t}
\]

\[
(Ra_4) \frac{p_1 t \not\to \gamma}{p_1 \sqcap p_2 \to t \sqcap p_2} \quad (Ra_5) \frac{p_1 t \not\to \gamma}{p_1 \sqcap p_2 \to p_1 \not\to t} \quad (Ra_6) \frac{p_1 t \not\to \gamma}{p_1 \sqcap p_2 \to t \sqcap p_2 \not\to t_2}
\]

\[
(Ra_7) \frac{p_1 t \not\to \gamma}{p_1 \land p_2 \to t \land p_2} \quad (Ra_8) \frac{p_1 t \not\to \gamma}{p_1 \land p_2 \to p_1 \not\to t} \quad (Ra_9) \frac{p_1 t \not\to \gamma}{p_1 \land p_2 \to t \land p_2}
\]

\[
(Ra_{10}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to t \lor p_2} \quad (Ra_{11}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to p_1 \not\to t} \quad (Ra_{12}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to p_1 \not\to t}
\]

\[
(Ra_{13}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to t \lor p_2 \not\to a \notin A} \quad (Ra_{14}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to p_1 \not\to t \lor p_2 \not\to a \notin A}
\]

\[
(Ra_{15}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to t_1 \lor p_2 \not\to a \notin A} \quad (Ra_{16}) \frac{p_1 t \not\to \gamma}{p_1 \lor p_2 \to p_1 \not\to t_1 \lor p_2 \not\to a \notin A}
\]

\[
Table 1 The transition rules of CLL
\]

\[
(Rp_1) \frac{\bot \not\to F}{F} \quad (Rp_2) \frac{p \not\to F}{\alpha.p} \quad (Rp_3) \frac{p \not\to F}{p \lor q \not\to F} \quad (Rp_4) \frac{p \not\to F}{p \land q \not\to F}
\]

\[
(Rp_5) \frac{q \not\to F}{p \sqcap q \not\to F} \quad (Rp_6) \frac{p \not\to F}{p \lor q \not\to F} \quad (Rp_7) \frac{q \not\to F}{p \lor q \not\to F} \quad (Rp_8) \frac{p \not\to F}{p \land q \not\to F}
\]

\[
(Rp_9) \frac{q \not\to F}{p \land q \not\to F} \quad (Rp_{10}) \frac{p \not\to q_1, q \not\to r, p \land q \not\to F}{p \land q \not\to F} \quad (Rp_{11}) \frac{p \not\to q_1, q \not\to r, p \land q \not\to F}{p \land q \not\to F}
\]

\[
Table 2 The predicate rules about \( F \)
\]

\[
(Rp_{CLL12}) \frac{p \land q \not\to r}{p \land q \not\to F_{\alpha}} \quad (Rp_{CLL13}) \frac{p \land q \not\to r}{p \land q \not\to F_{\alpha}}
\]

\[
Table 3 The predicate rules about \( F_{\alpha} \)
\]
Table 1 consists of transition rules $Ra_i (1 \leq i \leq 15)$, where $a \in \text{Act}$, $\alpha \in \text{Act}_\tau$ and $A \subseteq \text{Act}$. Negative premises in rules $Ra_2$, $Ra_3$, $Ra_{13}$ and $Ra_{14}$ give $\tau$-transition precedence over transitions labelled by visible actions, which guarantees that the transition model of CLL is $\tau$-pure. Rules $Ra_9$ and $Ra_{10}$ illustrate that the operational aspect of $t_1 \vee t_2$ is same as internal choice in usual process calculus. The rule $Ra_6$ reflects that the conjunction operator $\wedge$ is a synchronous product for visible transitions.

Table 2 contains predicate rules about the inconsistency predicate $F$. Although both $0$ and $\bot$ have empty behavior, they represent different processes. The rule $Rp_1$ says that $\bot$ is inconsistent, but $0$ is consistent as there is no proof of $0F$. The rule $Rp_3$ reflects that if both two disjunctive parts are inconsistent then so is the disjunction. Rules $Rp_4 - Rp_9$ describe the system design strategy that if one part is inconsistent, then so is the whole composition. The rules $Rp_{10}$ and $Rp_{11}$ reveal that a stable conjunction is inconsistent if its conjuncts have distinct ready sets.

Table 3 contains predicate rules ($Rp_{CLL12}$) and ($Rp_{CLL13}$) which formalize (LTS1) in Def. 2.1 for processes with the format $p \wedge q$.

Following [43], the notion of ready simulation (see, Def. 2.2) is adopted to formalize the refinement relation in [60]. Moreover, a sound and ground-complete axiomatic system is provided to characterize the operators within CLL in terms of (in)equational laws in [60].

3 Process calculus CLLT

This section will introduce the process calculus CLLT, which is obtained by enriching CLL with two temporal operators and two useful auxiliary operators, but omitting all predicate symbols $F_\alpha$ with $\alpha \in \text{Act}_\tau$. In the following, we will give syntax and SOS rules of CLLT, and demonstrate that CLLT has a unique stable model. Moreover, a number of simple but useful properties of such model are given.

3.1 Syntax and SOS rules of CLLT

In addition to operators in CLL, new process operators $true$, $\sharp$ and $\bar{\flat}$, and auxiliary operators $\triangle$ and $\odot$ are added to CLLT. Before describing their behavior formally in terms of SOS rules, we give a brief, informal account of the intended interpretation of these operators. The constant (i.e., 0-ary operator) $true$ represents the “loosest” specification: it does not require anything except consistency, while admitting any possible move. The operators $\sharp$ and $\bar{\flat}$ are intended to capture modal operators always and unless respectively through providing graphical representations of logic specifications “always $p$” and “$p$ unless $q$”. They turn out to be suitable in describing the “loosest” implementations that realize these two logic specifications respectively. Auxiliary operators $\triangle$ and $\odot$ themselves have little computational (or logic) meaning, but they are useful stepping-stones when we assign operational semantics to operators $\sharp$ and $\bar{\flat}$ by means of SOS rules. Roughly speaking, the whole point of using $\triangle$ (or, $\odot$) is to record the evolving paths of processes with the format $\sharp p$ ($p \bar{\flat} q$, respectively).

**Definition 3.1** The processes in CLLT are defined by BNF below

\[
p ::= q \mid true \mid (\sharp p) \mid (p \bar{\flat} q) \mid (p \odot (p \bar{\flat} p)) \mid (p \triangle p) \text{ with } q \in T(\Sigma_{CLL}).
\]

Here $T(\Sigma_{CLL})$ is the set of all processes in CLL. In the remainder, we shall always use $t_1 \equiv t_2$ to mean that the expressions $t_1$ and $t_2$ are syntactically identi-
cal, and use the notation □ \_i<n t_i for a generalized external choice, which is defined formally below.

**Definition 3.2** Let \(< t_0, t_1, \ldots, t_{n-1} >\) be a finite sequence of process terms with \(n \geq 0\). The generalized external choice \(\square\ i<n t_i\) is defined recursively as

1. \(\square\ i<0\ t_i = 0\),
2. \(\square\ i<1\ t_i = t_0\),
3. \(\square\ i<k+1\ t_i = (\square\ i<k t_i)\square t_k\) for \(k \geq 1\).

In fact, modulo \(=_{RS}\), the order and grouping of terms in \(\square\ i<n t_i\) may be ignored by virtue of the commutative and associative laws [60]. Therefore we also often use the notation \(\square\ i\in I t_i\) to denote generalized external choice, where \(I\) is an arbitrary finite indexed set.

\[\begin{array}{ll}
\text{(Ra}_{16}\text{)} & \text{true } \xrightarrow{\tau} \square a \rightarrow s \\
\text{(Ra}_{17}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{18}\text{)} & p \xrightarrow{a} q \\
\text{(Ra}_{19}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{20}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{21}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{22}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{23}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{24}\text{)} & p \xrightarrow{\tau} q \\
\text{(Ra}_{25}\text{)} & p \xrightarrow{\tau} q \\
\end{array}\]

**Table 4 Additional transition rules**

\[\begin{array}{ll}
\text{(Rp}_{12}\text{)} & qF \\
\text{(Rp}_{13}\text{)} & p \xrightarrow{\tau} q \\
\text{(Rp}_{14}\text{)} & p \xrightarrow{\tau} q \\
\text{(Rp}_{15}\text{)} & p \xrightarrow{\tau} q \\
\text{(Rp}_{16}\text{)} & p \xrightarrow{\tau} q \\
\end{array}\]

**Table 5 Additional predicate rules**

Similar to CLL, the SOS rules of CLLT are divided into two parts: transition rules and predicate rules. In effect these rules capture Lüttgen and Vogler's construction in process algebraic style.

---

3 In particular, \(\text{true } \xrightarrow{\tau} 0\) by setting \(A = \emptyset\).

4 That is, \(p\) has one of the formats: \(r \land t, \sharp t, t \triangle s\) and \(t \odot (p \odot q)\).

5 See Definition 9 and 10 in [44].
On the side of transition rules, in addition to all transition rules of CLL (i.e., rules in Table 1), rules in Table 4 are adopted to describe the behavior of \( \text{true}, \sharp, \varpi, \Delta \) and \( \odot \), where \( a \in \mathit{Act} \) and \( A \) is any finite subset of \( \mathit{Act} \).

On the side of rules concerning inconsistency predicate \( F \), rules in Table 2 are preserved, and rules in Table 5 are added to CLLT. Notice that the rules \( \langle \mathcal{R} \mathcal{P}_{CLLT12} \rangle \) and \( \langle \mathcal{R} \mathcal{P}_{CLLT13} \rangle \) in Table 3 are replaced by the rule \( \langle \mathcal{R} \mathcal{P}_{16} \rangle \). The motivation behind this modification may be found in the next subsection (see, Remark 3.1).

Summarizing, the TSS for CLLT is \( \Gamma_{CLLT} = (\Sigma_{CLLT}, \mathit{Act}_\tau, \Lambda_{CLLT}, \Xi_{CLLT}) \), where

- \( \Sigma_{CLLT} = \{ \Box, \land, \lor, 0, \bot \} \cup \{ \alpha.() | \alpha \in \mathit{Act}_\tau \} \cup \{ \| A \ | A \subseteq \mathit{Act} \} \cup \{ \text{true}, \sharp, \varpi, \Delta, \odot \} \),
- \( \Lambda_{CLLT} = \{ F \} \), and
- \( \Xi_{CLLT} = \{ Ra_1, \ldots, Ra_{23} \} \cup \{ Rp_1, \ldots, Rp_{16} \} \).

### 3.2 Stable transition model of CLLT

This subsection will illustrate that \( \Gamma_{CLLT} \) has a unique stable model. To this end, a few preliminary definitions are needed.

**Definition 3.3** The degree of terms is defined inductively below

\[
|0| = |\bot| = |\text{true}| = 1 \\
|t| = |\alpha.t| = |t| + 1 \\
|t_1 \spadesuit t_2| = |t_1| + |t_2| + 1 \text{ for } \spadesuit \in \{ \varpi, \land, \lor, ||, \Box \} \\
|t_1 \triangle t_2| = |t_1 \circ (t_2 \varpi t_3)| = |t_1|
\]

**Definition 3.4** The function \( S \) from \( Tr(\Sigma_{CLLT}, \mathit{Act}_\tau) \cup \mathit{Pred}(\Sigma_{CLLT}, \Lambda_{CLLT}) \) to \( \omega+1 \) is defined as: \( S(t \rightarrow r) = |t| \) for any \( t \rightarrow r \in Tr(\Sigma_{CLLT}, \mathit{Act}_\tau) \), and \( S(tF) = \omega \) for any \( tF \in \mathit{Pred}(\Sigma_{CLLT}, \Lambda_{CLLT}) \), where \( \omega \) is the initial limit ordinal.

It is easy to check that this function \( S \) is a stratification of \( \Gamma_{CLLT} \). Thus \( \Gamma_{CLLT} \) has a unique stable transition model. Henceforward such model is denoted by \( M_{CLLT} \). As usual, the LTS associated with CLLT is defined below.

**Definition 3.5** The LTS associated with CLLT, in symbols \( \mathit{LTS}(CLLT) \), is the quadruple \( (T(\Sigma_{CLLT}), \mathit{Act}_\tau, \rightarrow_{CLLT}, F_{CLLT}) \) such that for any \( t, s \in T(\Sigma_{CLLT}) \) and \( \alpha \in \mathit{Act}_\tau \), \( t \overset{\alpha}{\rightarrow}_{CLLT} s \) iff \( t \overset{\alpha}{\rightarrow} s \in M_{CLLT} \), and \( t \in F_{CLLT} \) iff \( tF \in M_{CLLT} \).

Since \( M_{CLLT} \) is a stable transition model, which exactly consists of provable transitions of the positive TSS \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \), the result below follows immediately.

**Theorem 3.1** For any \( t, t_1, t_2 \in T(\Sigma_{CLLT}) \) and \( \alpha \in \mathit{Act}_\tau \), we have

1. \( t_1 \overset{\alpha}{\rightarrow}_{CLLT} t_2 \) iff \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash t_1 \overset{\alpha}{\rightarrow} t_2 \).
2. \( t \in F_{CLLT} \) iff \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash tF \).

**Proof.** Straightforward. \( \square \)

This theorem is trivial but useful. It provides a way to establish the properties of \( \mathit{LTS}(CLLT) \). That is, we can demonstrate some conclusions by proceeding induction on the depth of inferences in the positive TSS \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \). In the remainder of this paper, we will apply this theorem without any reference.
Remark 3.1 Although the universal quantifier symbol does not occur in (Rp_{16}) explicitly, it is not difficult to see that the premise of (Rp_{16}) involves universal quantifier in spirit. Analogous to CLL, we may adopt the method given in [58] to avoid this. In detail, for each $\alpha \in Act_r$, the auxiliary predicates $F_\alpha$ is added to CLLT and the rule (Rp_{16}) is replaced by two rules below, where the topmost operator of $p$ is in $\{\wedge, \triangledown, \bigtriangleup, \circ\}$.

\[
\frac{p \alpha \rightarrow q}{pF_\alpha} \quad \frac{p \rightarrow \neg F_\alpha}{pF}
\]

Similar to (Rp_{16}), these two rules also capture (LTS1). However, it is easy to see that, due to two rules above, the stratifying function does not exist for resulting calculus. By means of technique so-called positive after reduction [13, 26], we can also get its stable transition model as done in [60]. Moreover, such stable transition model coincides with $M_{CLLT}$. To avoid cumbersome reduction procedure, our current system employs (Rp_{16}) instead of (Rp_{16-1}) and (Rp_{16-2}).

Convention 3.1 For the sake of convenience, in the remainder of this paper, we shall omit the subscript in labelled transition relations $\alpha^\dagger_{CLLT}$, that is, we shall use $\alpha \rightarrow$ to denote transition relations within $LTS(CLLT)$. Thus, the notation $\alpha \rightarrow$ has double utility: predicate symbol in the TSS $\Gamma_{CLLT}$ and labelled transition relation on processes in $LTS(CLLT)$. However, it usually does not lead to confusion in a given context. Similarly, the notation $F_{CLLT}$ is abbreviated to $F$. Hence the symbol $F$ is overloaded, predicate symbol in the TSS $\Gamma_{CLLT}$ and the set of all inconsistent processes in $LTS(CLLT)$, in each case the context of use will allow us to make the distinction.

### 3.3 Basic properties of $LTS(CLLT)$

This subsection will provide a number of simple properties of $LTS(CLLT)$. In particular, we will show that $LTS(CLLT)$ is indeed a $\tau$–pure LTS. We begin with listing a few simple properties in the next three lemmas, which will be frequently used in subsequent sections.

**Lemma 3.1** Let $t, p, q \in T(\Sigma_{CLLT})$ and $\alpha, \beta \in Act_r$.

1. $\alpha.t \rightarrow r$ iff $\alpha = \beta$ and $r \equiv t$.
2. $p \lor q \rightarrow r$ iff $\beta = \tau$ and either $p \equiv r$ or $q \equiv r$.
3. $\text{true} \rightarrow r$ iff $\beta = \tau$ and either $r \equiv 0$ or $r \equiv \bigtriangleup a.\text{true}$ for some nonempty finite set $A \subseteq Act$.
4. $\triangledown p \rightarrow r$ iff $r \equiv p_1 \bigtriangleup p$ for some $p_1$ with $p \triangledown \rightarrow p_1$.
5. $p \bigtriangleup q \rightarrow r$ iff $r \equiv p_1 \bigtriangleup q$ for some $p_1$ with $p \rightarrow \rightarrow p_1$.
6. $p \triangledown q \rightarrow r$ iff $\beta = \tau$ and either $r \equiv q$ or $r \equiv p \bigtriangledown (p \triangledown q)$.
7. $t \circ (p \triangledown q) \rightarrow r$ iff $r \equiv t_1 \triangledown (p \triangledown q)$ for some $t_1$ with $t \rightarrow \rightarrow t_1$.
8. $p \triangledown q \rightarrow r$ iff either $(r \equiv s \triangledown q$ and $p \rightarrow s)$ or $(r \equiv p \bigtriangledown s$ and $q \rightarrow s)$ for some $s$, where $\bigtriangledown \in \{\wedge, \bigtriangleup, \|_A\}$.

**Proof.** For each item, the implication from right to left is obvious. The proof of converse implication is a routine case analysis on the last rule applied in the inference. As a sample case, we consider (6), the remainder may be handled in the similar manner and omitted. It follows from $p \triangledown q \rightarrow r$ that $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash p \triangledown q \rightarrow r$. Clearly, the last rule applied in the inference has the format below.
Let $t,p,q \in T(\Sigma_{CLLT})$ and $a \in Act$.

1. $p \sqcup q \Rightarrow r$ iff either $p \Rightarrow r$ and $q \not\Rightarrow r$, or $q \Rightarrow r$ and $p \not\Rightarrow r$.
2. $p \land q \Rightarrow r$ iff $p \Rightarrow r_1$, $q \Rightarrow r_2$ and $r \equiv r_1 \land r_2$ for some $r_1, r_2$.
3. $\sharp p \Rightarrow r$ iff $r \equiv (q \land p) \triangle p$ for some $q$ with $p \Rightarrow q$.
4. $p \triangle q \Rightarrow r$ iff $r \equiv (s \land q) \triangle q$ for some $s$ with $p \Rightarrow s$.
5. $t \circ (p \triangledown q) \Rightarrow r$ iff there exists $s$ such that $t \Rightarrow s$ and either $r \equiv s \land q$ or $r \equiv (s \land p) \circ (p \triangledown q)$.
6. If $a \notin A$ then, $p||_A q \Rightarrow r$ iff either $(r \equiv s||_A q, q \not\Rightarrow r)$ and $p \Rightarrow s$) or $(r \equiv p||_A s, q \not\Rightarrow r)$ and $q \not\Rightarrow s$ for some $s$. 
7. If $a \in A$ then, $p||_A q \Rightarrow r$ iff $r \equiv s||_A t$, $p \Rightarrow s$ and $q \Rightarrow t$ for some $s,t$.

Proof. Analogous to Lemma 3.1, omitted. □

Lemma 3.3 Suppose $p,q,r \in T(\Sigma_{CLLT})$.

1. $p \lor q \in F$ iff $p, q \in F$.
2. $\alpha.p \in F$ iff $p \in F$.
3. $p \triangledown q \in F$ iff either $p \in F$ or $q \in F$ for $\triangledown \in \{\sqcup, ||_A\}$.
4. Either $p \in F$ or $q \in F$ implies $p \land q \in F$.
5. $p \triangledown q \in F$ iff $q, p \circ (p \triangledown q) \in F$.
6. $r \in F$ implies $r \triangle p, \sharp r, r \circ (p \triangledown q) \in F$.
7. $0 \notin F$, $true \notin F$ and $\bot \in F$.

Proof. Straightforward. □

Lemma 3.4 LTS($CLLT$) is $\tau - pure$.

Proof. Let $t \Rightarrow s$. It is enough to show that $t \not\Rightarrow s$. The proof is done by induction on the inference $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash t \Rightarrow s$, which is a long but routine case analysis based on the last rule applied in the inference, omitted. □

Lemma 3.5 LTS($CLLT$) satisfies (LTS1).

Proof. Let $t$ be any process and assume that $\forall s(t \Rightarrow s$ implies $s \in F$) for some $\alpha \in I(t)$. We intend to verify $t \in F$ by induction on $t$.

- $t \equiv 0, \bot, true, \alpha.p, p \lor q$ or $p \triangledown q$

Follows from Lemma 3.1 and 3.3. In particular, for $t \equiv 0, \bot$ or $true$, since the premise does not hold at all, it holds trivially.

- $t \equiv p \land q, \sharp p, p \triangle q$ or $p \circ (r \triangledown q)$.

Immediately follows from the rule (Rp16).

- $t \equiv p \square q$

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By Lemma 3.3(3), it suffices to show that either $p \in F$ or $q \in F$. Conversely, suppose that $p \notin F$ and $q \notin F$. By Lemma 3.2 (1) and 3.1(8), we have either $\alpha \in I(p)$ or $\alpha \in I(q)$. W.l.o.g, we consider the first alternative. If $\alpha \in \text{Act}$, then, by Lemma 3.2(1), we get

$$\left\{ s : p \overset{\alpha}{\rightarrow} s \right\} \subseteq \left\{ s : p \square q \overset{\alpha}{\rightarrow} s \right\} \subseteq F.$$ 

Hence, by induction hypothesis (IH, for short), we have $p \in F$, a contradiction.

If $\alpha = \tau$, then, by Lemma 3.1(8), it follows that

$$\left\{ s \square q : p \overset{\tau}{\rightarrow} s \right\} \subseteq \left\{ s : p \square q \overset{\tau}{\rightarrow} s \right\} \subseteq F.$$ 

Further, by Lemma 3.3(3), it follows from $q \notin F$ that $\left\{ s : p \overset{\tau}{\rightarrow} s \right\} \subseteq F$. Then, by IH, we also obtain $p \in F$, a contradiction.

Case 1 $\alpha \notin A$.

Then either $\alpha \in I(p)$ or $\alpha \in I(q)$ by Lemma 3.1(8) and 3.2(6). W.l.o.g, we handle the first alternative. Hence

$$\left\{ s \| A q : p \overset{\alpha}{\rightarrow} s \right\} \subseteq \left\{ s : p \| A q \overset{\alpha}{\rightarrow} s \right\} \subseteq F.$$ 

Further, by Lemma 3.3(3), it follows from $q \notin F$ that $\left\{ s : p \overset{\tau}{\rightarrow} s \right\} \subseteq F$. Then $p \in F$ due to IH, a contradiction.

Case 2 $\alpha \in A$.

In such situation, we get $\alpha \in I(p)$ and $\alpha \in I(q)$. Then, by IH, it follows from $p \notin F$ and $q \notin F$ that there exist $p_1$ and $q_1$ such that

$$p \overset{\alpha}{\rightarrow} p_1, q \overset{\alpha}{\rightarrow} q_1, p_1 \notin F \text{ and } q_1 \notin F.$$ 

Thus $p \| A q \overset{\alpha}{\rightarrow} p_1 \| A q_1$ and $p_1 \| A q_1 \notin F$ by Lemma 3.2(7) and 3.3(3), a contradiction. □

A simple but useful result is given below, which provides a necessary and sufficient condition for a non-stable process to be inconsistent. An analogous result have been obtained for CLL in [60].

**Lemma 3.6** For any $t \in T(\Sigma_{CLLT})$, we have

1. $t \in F$ iff $\forall s(t \overset{\tau}{\rightarrow} s \text{ implies } s \in F)$ whenever $\tau \in I(t)$.
2. If $t \overset{\triangleright}{\rightarrow} s$ and $s \notin F$ then $t \notin F$ and $t \overset{\triangleright}{\rightarrow} F$.

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Proof. Clearly, (2) immediately follows from (1). In the following, we consider (1). Assume that \( \tau \in I(t) \). Then, by Lemma 3.5, we need only show that the left implies the right. We can prove it by induction on the inference \( Strip(\Gamma_{CLLT}, M_{CLLT}) \vdash t F \), which is a case analysis based on the format of \( t \). As an instance, we shall deal with the case \( t \equiv p \triangle q \), the remainder may be handled in a similar way and omitted.

Since \( t \equiv p \triangle q \), the last rule applied in the inference has the format

\[
\text{either } \frac{p \triangle q \xrightarrow{\alpha} r}{p \triangle q F} \quad \text{or} \quad \frac{p F}{p \triangle q F}.
\]

For the first alternative, since \( \tau \in I(p \triangle q) \), we get \( \alpha = \tau \) by Lemma 3.4. Then it immediately follows that \( \{ r : p \triangle q \xrightarrow{\tau} r \} \subseteq F \). For the second alternative, we have \( p \in F \). Moreover, by Lemma 3.1(5), we get \( \tau \in I(p) \) because of \( \tau \in I(p \triangle q) \). Hence, by IH, it follows that \( \{ r : p \xrightarrow{\tau} r \} \subseteq F \). Further, since \( \{ r : p \triangle q \xrightarrow{\tau} r \} = \{ r \triangle q : p \xrightarrow{\tau} r \} \), we obtain \( \{ r : p \triangle q \xrightarrow{\tau} r \} \subseteq F \) by Lemma 3.3(6), as desired. □

In order to show that \( LTS(\text{CLLT}) \) satisfies (LTS2), we introduce the notion of \( \tau \)-degree as follows, which measures processes’s capability of executing successive \( \tau \) actions.

**Definition 3.6** The \( \tau \)-degree of processes is defined inductively below
\[
d(\text{true}) = 1 \\
d(0) = d(\bot) = d(a.t) = 0 \text{ whenever } a \in \text{Act} \\
d(\tau.t) = d(t) + 1 \\
d(t_1 \odot t_2) = d(t_1 \lor t_2) = \max\{d(t_1), d(t_2)\} + 1 \\
d(t_1 \land t_2) = d(t_1 \parallel t_2) = d(t_1 \sqcup t_2) = d(t_1) + d(t_2) \\
d(\#t) = d(t \triangle t_1) = d(t \circ (t_1 \odot t_2)) = d(t)
\]

**Lemma 3.7** If \( t \xrightarrow{\tau} r \) then \( d(r) < d(t) \) for any \( t, r \in T(\Sigma_{CLLT}) \).

**Proof.** Proceeding by induction on the inference \( Strip(\Gamma_{CLLT}, M_{CLLT}) \vdash t \xrightarrow{\tau} r \), which is a routine case analysis on the last rule applied in the inference. □

This elementary property makes it effective to apply the induction on the \( \tau \)-degree in the next proof.

**Lemma 3.8** \( LTS(\text{CLLT}) \) satisfies (LTS2).

**Proof.** Let \( t \in T(\Sigma_{CLLT}) \) with \( t \notin F \). It suffices to find \( p \) such that \( t \xrightarrow{\tau} \sigma F \). We prove it by induction on the \( \tau \)-degree of \( t \). Assume that it holds for all \( p \) with \( d(p) < d(t) \). If \( t \) is stable, then \( t \xrightarrow{\tau} \sigma F \) follows from \( t \notin F \). Next we consider another case where \( \tau \in I(t) \). Since \( t \notin F \) and \( \tau \in I(t) \), by Lemma 3.6(1), we have \( t \xrightarrow{\tau F} s \) for some \( s \). Hence \( d(s) < d(t) \) by Lemma 3.7. Thus \( s \xrightarrow{\tau F} \sigma F \) for some \( r \) due to IH. Then \( t \xrightarrow{\tau F} \sigma F \), as desired. □

Now we get the main result of this section as follows.

**Theorem 3.2** \( LTS(\text{CLLT}) \) is a \( \tau - \) pure LLTS.
Proof. Obvious from Lemma 3.4, 3.5 and 3.8. □

In contrast with usual process calculuses, one of features of LLTS-oriented process calculuses is that these calculuses take into account consistency of processes. The inconsistency predicate is central to the description of behavior. We often need to prove that a given process \( p \) is consistent, which boils down to show that there is no inference for \( pF \) in \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \). To this end, we introduce the notion below, which is useful for demonstrating the consistency of processes. The motivation behind this notion is that we intend to establish the consistency of a given process based on the well-foundedness of proof trees.

**Definition 3.7** \((F−\text{hole})\) A set \( \Omega \) of processes is said to be a \( F−\text{hole} \) if, for each \( q \in \Omega \), any proof tree of \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \) \( \vdash qF \) has a proper subtree with the root labelled with \( uF \) for some \( u \in \Omega \).

As the name suggests, each process in a \( F−\text{hole} \) is not in \( F \). Formally, we have the result below.

**Lemma 3.9** If \( \Omega \) is a \( F−\text{hole} \) then \( \Omega \cap F = \emptyset \).

**Proof.** Conversely, suppose that \( \Omega \cap F \neq \emptyset \), say, \( q \in \Omega \cap F \). Thus there exists a proof tree of \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \) \( \vdash qF \). However, by Definition 3.7, such proof tree is not well-founded, which contradicts Def. 2.3. □

Therefore, in order to verify that a given process \( p \) is consistent, it suffices to provide a \( F−\text{hole} \) including \( p \). The next lemma has been showed for CLL in pure process-algebraic style in [60], where the proof essentially depends on the fact that, for any process \( t \) within CLL and \( \alpha \in \text{Act} \), \( t \) is of more complex structure than its \( \alpha \)-derivatives. Unfortunately, such property does not always hold for CLLT. For instance, consider processes \( \text{true}, p \triangleq r \) and \( r \sqcap (p \varpi q) \). Here we give an alternative proof for it and indicate how the notion of \( F−\text{hole} \) may be used to show the consistency of a given process.

**Lemma 3.10** If \( s \sqsubseteq_p \sim_{RS} r, s \sqsubseteq_{RS} t \) and \( s \notin F \) then \( r \wedge t \notin F \).

**Proof.** Put

\[
\Omega = \left\{ p_1 \wedge p_2 : q \sqsubseteq_p \sim_{RS} p_1, q \sqsubseteq_{RS} p_2 \text{ and } q \notin F \right\}.
\]

It is enough to show that \( \Omega \) is a \( F−\text{hole} \). Let \( p_1 \wedge p_2 \in \Omega \) and \( \exists \) be any proof tree of \( p_1 \wedge p_2F \). Thus \( q \sqsubseteq_{RS} p_1, q \sqsubseteq_{RS} p_2 \) and \( q \notin F \) for some \( q \). So, it follows that \( I(p_1) = I(p_2) = I(q) \), \( p_1 \notin F \) and \( p_2 \notin F \). Further, since \( p_1 \wedge p_2 \not\rightarrow^* \), the last rule applied in \( \exists \) has the format below

\[
\frac{p_1 \wedge p_2 \rightarrow^a u, \left\{ \tau F : p_1 \wedge p_2 \rightarrow^a \tau \right\}}{p_1 \wedge p_2F} \text{ for some } a \in \text{Act}. \quad (3.10.1)
\]

Hence \( a \in I(q) \). Moreover, by Lemma 3.5 and 3.8, it follows from \( q \notin F \) that \( q \rightarrow^a_F | q_1 \) for some \( q_1 \). Since \( q \sqsubseteq_{RS} p_1 \) and \( q \sqsubseteq_{RS} p_2 \), there exist \( r_i, t_j \) with \( i \leq n \) and \( j \leq m \) such that \( p_1 \rightarrow^a_F r_1 \rightarrow^F r_2 \rightarrow^F \cdots \rightarrow^F r_n \rightarrow^F t_1 \rightarrow^F t_2 \rightarrow^F \cdots \rightarrow^F t_m \) and

\[
q_1 \sqsubseteq_{RS} r_n \text{ and } q_1 \sqsubseteq_{RS} t_m. \quad (3.10.2)
\]
Moreover, we also have \( p_1 \land p_2 \xrightarrow{a} r_1 \land t_1 \) and
\[
  r_1 \land t_1 \xrightarrow{a} ... \xrightarrow{a} r_n \land t_1 \xrightarrow{a} ... \xrightarrow{a} r_{n} \land t_{m}.
\] (3.10.3)

Then, by (3.10.1), \( \mathcal{S} \) contains a proper subtree \( \mathcal{S}_1 \) with the root labelled with \( r_1 \land t_1 F \). On the other hand, it follows from (3.10.2) that \( r_n \land t_m \in \Omega \). Thus, to complete the proof, it suffices to show that \( \mathcal{S} \) contains a proper subtree with the root labelled with \( r_n \land t_m F \). If \( m = n = 1 \), this holds obviously due to \( r_1 \land t_1 \equiv r_n \land t_m \).

Otherwise, w.l.o.g, we assume \( n > 1 \). Hence \( r_1 \land t_1 \) is not stable. Moreover, since \( r_1 \land t_1 / \in F \), the last rule applied in \( \mathcal{S}_1 \) is \( r_1 \land t_1 \xrightarrow{a} s \), \( \{ rF : r_1 \land t_1 \xrightarrow{a} r \} \). Thus \( \mathcal{S}_1 \) contains a node labelled with \( r_2 \land t_1 F \). By repeating this procedure along (3.10.3), it is easily seen that \( \mathcal{S} \) contains a proper subtree with the root labelled with \( r_n \land t_m F \), as desired. □

We end this section with recalling some useful properties of the operator \( \land \), which has been obtained in [43] and [60] in different style.

**Lemma 3.11** For any process \( p_1, p_2 \) and \( q \), we have

1. \( p_1 \land p_2 \sqsubseteq \sim RS p_i \) for \( i = 1, 2 \) whenever \( p_1 \not\xrightarrow{a} \) and \( p_2 \not\xrightarrow{a} \),
2. if \( q \sqsubseteq \sim RS p_1 \) and \( q \sqsubseteq \sim RS p_2 \) then \( q \sqsubseteq \sim RS p_1 \land p_2 \),
3. \( p_1 \land p_2 \sqsupseteq \sim RS p_i \) for \( i = 1, 2 \), and
4. if \( q \sqsubseteq RS p_1 \) and \( q \sqsubseteq RS p_2 \) then \( q \sqsubseteq RS p_1 \land p_2 \).

**Proof.** A proof in pure process-algebraic style has been given in [60]. Here we only draw the outline of its proof. Item (3) and (4) follow from (1) and (2), respectively. For (1) and (2), we set
\[
R_1 = \left\{ (s \land t, s) : s \land t \xrightarrow{a} \right\} \quad \text{and} \quad R_2 = \left\{ (s, r \land t) : s \sqsubseteq \sim RS r \text{ and } s \sqsubseteq \sim RS t \right\}.
\]

It suffices to show that these two relations are stable ready simulation relations. Notice that Lemma 3.10 is used to prove that \( R_2 \) satisfies (RS2) and (RS3). For more details we refer the reader to [60]. □

It is an immediate consequence of the above lemma that, modulo \( = RS \) (or, \( \approx RS \)), the operator \( \land \) satisfies the idempotent, commutative and associative laws.

### 4 The operator \( \sharp \)

This section aims to explore properties of the operator \( \sharp \). In particular, we shall characterize processes that refine processes with the format \( \sharp p \). This result supports the claim that the operator \( \sharp \) captures the modal operator \( \text{always} \). Since the behavior of \( \sharp \) is described in terms of \( \triangle \), we will study the latter firstly.

**Lemma 4.1** If \( p \sqsubseteq \sim RS r \triangle t \) and \( p \xrightarrow{a} F | p_1 \) then \( p_1 \sqsubseteq \sim RS (s \land u) \triangle t \) for some \( s \) and \( u \) such that \( r \xrightarrow{a} F | s \) and \( t \xrightarrow{a} F | u \).
Proof. Since \( p \sqsubseteq_{RS} r \triangle t \) and \( p \xrightarrow{a,F} p_1, p_1 \sqsubseteq_{RS} q \) for some \( q \) with \( r \triangle t \xrightarrow{a,F} |q| \).

Then it follows from \( r \triangle t \not\sim \) that \( r \triangle t \xrightarrow{a,F} q_1 \xrightarrow{a,F} |q| \) for some \( q_1 \). Further, by Lemma 3.1(5)(8) and 3.2 (4), there exist \( r_1, r_2 \) and \( t_1 \) such that \( q_1 \equiv (r_1 \wedge t_1) \triangle t \) with \( r \xrightarrow{a,F} r_1 \), and \( q \equiv (r_2 \wedge t_1) \triangle t \) with \( t \xrightarrow{a,F} |t_1| \) and \( r_1 \xrightarrow{a,F} |r_2| \). Hence \( p_1 \sqsubseteq_{RS} (r_2 \wedge t_1) \triangle t \) with \( t \xrightarrow{a,F} |t_1| \) and \( r \xrightarrow{a,F} |r_2| \), as desired. \( \Box \)

A simple method for showing that one process simulates another one is to find a stable ready simulation relating them. It is well known that up-to technique is a tractable way for such coinduction proof. Here we introduce the notion of a stable ready simulation up to \( \sqsubseteq_{RS} \) as follows.

**Definition 4.1** (stable ready simulation up to \( \sqsubseteq_{RS} \)) A binary relation \( R \subseteq T(\Sigma_{CLLT}) \times T(\Sigma_{CLLT}) \) is said to be a stable ready simulation relation up to \( \sqsubseteq_{RS} \) if for any \( \langle t, s \rangle \in R \), it satisfies (RS1), (RS2), (RS4) in Def. 2.2 and

\[(RS3-upto) \quad t \xrightarrow{a,F} |t_1| \text{ implies } \exists s_1(s \xrightarrow{a,F} |s_1| \text{ and } \langle t_1, s_1 \rangle \in R \circ \sqsubseteq_{RS}) \text{ for any } a \in Act.\]

As usual, given a relation \( R \) satisfying the above conditions, \( R \) itself is not in general a stable ready simulation relation. But the simple result below ensures that up-to technique based on the above notion is sound.

**Lemma 4.2** If \( R \) is a stable ready simulation relation up to \( \sqsubseteq_{RS} \), then \( R \subseteq \sqsubseteq_{RS} \).

**Proof.** Due to the reflexivity of \( \sqsubseteq_{RS} \), we have \( R \subseteq R \circ \sqsubseteq_{RS} \). Thus it suffices to show that \( R \circ \sqsubseteq_{RS} \) is a stable ready simulation. For any pair \( \langle s, t \rangle \in R \circ \sqsubseteq_{RS} \), based on Def. 4.1 and the transitivity of \( \sqsubseteq_{RS} \), it is straightforward to check that \( \langle s, t \rangle \) satisfies four conditions in Def. 2.2. \( \Box \)

**Lemma 4.3** If \( p \sqsubseteq_{RS} u \triangle t \) then \( p \sqsubseteq_{RS} u \). Hence \( u \triangle t \sqsubseteq_{RS} u \) for any \( u \) and \( t \).

**Proof.** Set \( R = \{ \langle q, s \rangle : q \sqsubseteq_{RS} s \triangle r \text{ for some } r \} \).

We wish to prove that \( R \) is a stable ready simulation relation up to \( \sqsubseteq_{RS} \). Let \( \langle q, s \rangle \in R \). Then \( q \sqsubseteq_{RS} s \triangle r \) for some \( r \). Thus both \( q \) and \( s \triangle r \) are stable. By item (5) in Lemma 3.1, so is \( s \). Hence (RS1) holds.

(RS2) Suppose \( q \not\in F \). Due to \( q \sqsubseteq_{RS} s \triangle r \), we get \( s \triangle r \not\in F \), which implies \( s \not\in F \) by Lemma 3.3 (6).

(RS3-up to) Let \( q \xrightarrow{a,F} q_1 \). Since \( q \sqsubseteq_{RS} s \triangle r \), by Lemma 4.1, \( q_1 \sqsubseteq_{RS} (s_1 \wedge r_1) \triangle r \) for some \( r_1 \) and \( s_1 \) such that \( r \xrightarrow{a,F} |r_1| \) and \( s \xrightarrow{a,F} |s_1| \). Thus \( \langle q_1, s_1 \wedge r_1 \rangle \in R \). On the other hand, by item (1) in Lemma 3.11, we get \( s_1 \wedge r_1 \sqsubseteq_{RS} s_1 \). Hence \( \langle q_1, s_1 \rangle \in R \circ \sqsubseteq_{RS} \) and \( s \xrightarrow{a,F} |s_1| \), as desired.

(RS4) If \( q \not\in F \) then it follows from \( q \sqsubseteq_{RS} s \triangle r \) that \( I(q) = I(s \triangle r) \), and hence \( I(q) = I(s) \) by Lemma 3.2 (4). \( \Box \)
**Notation 4.1** For a more convenient notation, we introduce the notations below.

1. Following [44], the notation $\equiv_{\mathcal{F}}$ is used to stand for $\bigcup_{a \in \text{Act}} \mathcal{A}_F^a$.

2. The notation $p \trianglerighteq_{RS} t$ means that $\forall n \in \omega \forall p_0, p_1, \ldots, p_n (p \trianglerighteq_{\mathcal{F}} | p_0 \trianglerighteq_{\mathcal{F}} | p_1 \ldots \trianglerighteq_{\mathcal{F}} | p_n \text{ implies } p_n \sqsubseteq_{RS} t)$.

3. The notation $p \trianglerighteq_{RS} t$ means that $\forall n \in \omega \forall p_0, p_1, \ldots, p_n (p \trianglerighteq_{\mathcal{F}} | p_0 \trianglerighteq_{\mathcal{F}} | p_1 \ldots \trianglerighteq_{\mathcal{F}} | p_n \text{ implies } p_n \sqsubseteq_{RS} t)$.

The next two results provide a necessary condition for a process to refine $\trianglerighteq$, where the refinement relation is captured by $\sqsubseteq$ and $\sqsubseteq_{RS}$ respectively.

**Lemma 4.4** If $p \sqsubseteq_{RS} \trianglerighteq$ then $p \trianglerighteq_{RS} t$.

**Proof.** Assume that $p \trianglerighteq_{\mathcal{F}} | p_0 \trianglerighteq_{\mathcal{F}} | p_1 \ldots \trianglerighteq_{\mathcal{F}} | p_n$. If it were true that

$$p_n \sqsubseteq_{RS} (r \land t) \trianglerighteq t \quad (4.4.1)$$

we would have $p_n \sqsubseteq_{RS} t$ by Lemma 4.3 and 3.11 (1), and hence the proof would be complete. In the following, we thus intend to prove (4.4.1) by induction on $n$.

For the induction basis $n = 0$, we have $p \trianglerighteq_{\mathcal{F}} | p_0$ for some $a \in \text{Act}$. It follows from $p \sqsubseteq_{RS} \trianglerighteq$ that $p_0 \sqsubseteq_{RS} t_1$ for some $t_1$ with $\trianglerighteq \trianglerighteq_{\mathcal{F}} | t_1$. Due to the stableness of $\trianglerighteq$, we get $\trianglerighteq \triangleright_{\mathcal{F}} | t_2 \trianglerighteq_{\mathcal{F}} | t_1$ for some $t_2$. Then, by Lemma 3.2 (3), $t_2 \equiv (s \land t) \trianglerighteq t$ for some $s$. Further, by Lemma 3.1 (5) (8) and 3.3 (6), it follows from $(s \land t) \trianglerighteq_{\mathcal{F}} | t_1$ that there exist $t_3$ and $s_1$ such that $t_3 \trianglerighteq_{\mathcal{F}} | t_3$, $s \trianglerighteq_{\mathcal{F}} | s_1$ and $t_1 \equiv (s_1 \land t_3) \trianglerighteq$. Since $t$ is stable, we get $t \equiv t_3$. Thus $p_0 \sqsubseteq_{RS} (s_1 \land t) \trianglerighteq t$, as desired.

For the induction step $n = k + 1$, suppose that $p \trianglerighteq_{\mathcal{F}} | p_0 \trianglerighteq_{\mathcal{F}} | p_1 \ldots \trianglerighteq_{\mathcal{F}} | p_k \trianglerighteq_{\mathcal{F}} | p_{k+1}$. By IH, $p_k \sqsubseteq_{RS} (s \land t) \trianglerighteq t$ for some $s$. Then, by Lemma 4.1, it follows from $p_k \trianglerighteq_{\mathcal{F}} | p_{k+1}$ and $t \trianglerighteq$ that $p_{k+1} \sqsubseteq_{RS} (r \land t) \trianglerighteq t$ for some $r$. □

This result is of independent interest, but its principal significance is that it will serve as a stepping stone in demonstrating the next lemma.

**Lemma 4.5** $p \sqsubseteq_{RS} \trianglerighteq$ implies $p \trianglerighteq_{RS} t$.

**Proof.** Assume that $p \trianglerighteq_{\mathcal{F}} | p_0 \trianglerighteq_{\mathcal{F}} | p_1 \ldots \trianglerighteq_{\mathcal{F}} | p_n$. We intend to prove that $p_n \sqsubseteq_{RS} t$. The argument splits into two cases depending on whether $t$ is stable.

Case 1 $t \not\trianglerighteq$.

So, $\trianglerighteq \not\trianglerighteq$. Then it follows from $p \sqsubseteq_{RS} \trianglerighteq$ and $p \trianglerighteq_{\mathcal{F}} | p_0$ that $p_0 \sqsubseteq_{RS} \trianglerighteq$. Hence $p_n \sqsubseteq_{RS} t$ by Lemma 4.4. Consequently, $p_n \sqsubseteq_{RS} t$ holds due to $t \not\trianglerighteq$ and $p_n \not\trianglerighteq$.

Case 2 $t \trianglerighteq$.

Since $p \trianglerighteq_{\mathcal{F}} | p_0$ and $p \sqsubseteq_{RS} \trianglerighteq$, it follows that $p_0 \sqsubseteq_{RS} t_0$ for some $t_0$ with $\trianglerighteq_{\mathcal{F}} | t_0$. By Lemma 3.1(4) and 3.3(6), there exists $r$ such that $t_0 \equiv r \trianglerighteq t$ and $t \trianglerighteq | r$. 21
Thus, by Lemma 4.3, we have \( p_0 \sim_{RS} r \). If \( n = 0 \) then \( p_0 \sim_{RS} t \) comes from \( p_0 \overset{\tau}{\to} t \) and \( t \overset{s}{\to} F \mid r \). We now turn to the case \( n \geq 1 \). Due to \( p_0 \sim_{RS} r \triangle t \), applying Lemma 4.1 repeatedly, it may be proved without any difficulty that \( p_n \sim_{RS} \triangle t \) for some \( s \) and \( t \). Further, by Lemma 4.3 and 3.1(5), we have \( p_n \sim_{RS} s \land t_1 \sim_{RS} t_1 \). Then it follows from \( p_n \overset{\tau}{\to} t_1 \) that \( p_n \sim_{RS} t \). □

The converse of the above lemma also holds. However, its proof is far from straightforward. A few of preliminary results are needed. Two results concerning consistency are given firstly.

**Lemma 4.6** Let \( p \) and \( t \) be any process such that \( p \sim_{RS} t \), and put

\[
\Omega = \left\{ r \triangle t : \exists q_0, q_1, ..., q_n \left( p \overset{s}{\to} F | q_0 \Rightarrow_{RS} F | q_1 \Rightarrow_{RS} F | q_n \sim_{RS} r \right) \right\}.
\]

Then \( \Omega \) is a \( F \)-hole.

**Proof.** Let \( r \triangle t \in \Omega \). Then there exist \( p_0, p_1, p_2, ..., p_n \) such that \( p \overset{s}{\to} F | p_0 \Rightarrow_{RS} F | p_1 \equiv_{RS} F | p_n \sim_{RS} r \). Let \( \exists \) be any proof tree of \( Strip(\Gamma_{CLLT}, M_{CLLT}) \vdash r \triangle tF \). Since \( p_n \sim_{RS} r \) and \( p_n \notin F \), we get \( r \notin F \). Moreover, \( r \triangle t \) is stable due to \( r \not\overset{\tau}{\to} t \) and Lemma 3.1(5). Thus the last rule applied in \( \exists \) is of the format below

\[
\frac{r \triangle t \overset{a}{\to} u, \left\{ qF : r \triangle t \overset{a}{\to} q \right\}}{r \triangle tF}
\]  

(4.6.1)

Due to \( p_n \sim_{RS} r \) and \( p_n \notin F \), we have \( I(p_n) = I(r) = I(r \triangle t) \) by Lemma 3.2(4). Hence \( a \in I(p_n) \). Moreover, by Lemma 3.5 and 3.8, it follows from \( p_n \notin F \) that \( p_n \Rightarrow_{RS} p_{n+1} \) for some \( p_{n+1} \). Then, due to \( p_n \sim_{RS} r \), there exist \( r_1 \) and \( r_2 \) with

\[
r \overset{\alpha}{\to} F r_1 \overset{\tau}{\to} F | r_2 \text{ and } p_{n+1} \sim_{RS} r_2.
\]

Moreover, \( p_{n+1} \sim_{RS} t \) because of \( p \sim_{RS} t \). Hence \( p_{n+1} \sim_{RS} t_1 \) for some \( t_1 \) with \( t \overset{\tau}{\to} F | t_1 \). Thus, by Lemma 3.11(2), it follows that \( p_{n+1} \sim_{RS} \triangle t_1 \). Hence \( (r_2 \land t_1) \triangle t \in \Omega \). Consequently, in order to complete the proof, it is enough to show that \( \exists \) contains a proper subtree with the root labelled with \( (r_2 \land t_1) \triangle tF \). Next we shall prove this.

By Lemma 3.2(4) and 3.1(5), it follows from \( r \overset{a}{\to} F r_1 \overset{\tau}{\to} F | r_2 \) and \( t \overset{s}{\to} F | t_1 \) that

\[
r \triangle t \overset{a}{\to} (r_1 \land t) \overset{\tau}{\to} (r_2 \land t_1) \triangle t.
\]

Hence, by (4.6.1), \( \exists \) contains a proper subtree with the root labelled with \( (r_1 \land t) \triangle tF \). Obviously, if \( (r_1 \land t) \triangle t \) is stable then \( (r_1 \land t) \triangle t \equiv (r_2 \land t_1) \triangle t \), and hence \( \exists \) contains a node labelled with \( (r_2 \land t_1) \triangle tF \), as desired. In the following, we handle the nontrivial case \( (r_1 \land t) \triangle t \overset{s}{\to} \). In such situation, there exist \( s_1, s_2, ..., s_m \) such that

\[
r_1 \land t \overset{s}{\to} s_1 \overset{s}{\to} s_2 \overset{s}{\to} ... \overset{s}{\to} s_m \overset{s}{\to} | r_2 \land t_1 \text{, and } \quad (4.6.2)
\]

\[
(r_1 \land t) \triangle t \overset{s}{\to} s_1 \triangle t \overset{s}{\to} s_2 \triangle t \overset{s}{\to} ... \overset{s}{\to} s_m \triangle t \overset{s}{\to} | (r_2 \land t_1) \triangle t. \quad (4.6.3)
\]
On the other hand, due to $p_{n+1} \not\in F$ and $p_{n+1} \sqsubseteq_{RS} r_2 \land t_1$, we get $r_2 \land t_1 \not\in F$. Then, by Lemma 3.6(1) and (4.6.2), it is easy to see that

$$r_1 \land t \not\in F$$

Thus, for each $u \in \{r_1 \land t\} \cup \{s_i : 1 \leq i \leq m\}$, the last rule applied in any proof tree of $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash u \triangle tF$ must be of the format below

$$\frac{u \triangle t \rightarrow \mathcal{R}}{u \triangle tF}$$

Therefore, by (4.6.3), it follows that $\exists$ contains a proper subtree with the root labelled with $(r_2 \land t_1) \triangle tF$, as desired. □

With the help of this result, we shall prove the assertion below, which is a crucial part of the proof for the converse of Lemma 4.5.

**Lemma 4.7** If $p \sqsubseteq_{RS} t$ and $p \not\in F$ then $\sharp t \not\in F$.

**Proof.** Since $p \not\in F$, by Lemma 3.8, there exists $q_0$ such that $p \Rightarrow_F |q_0$. Then $q_0 \sqsubseteq_{RS} t$ due to $p \sqsubseteq_{RS} t$. We distinguish two cases depending on whether $t$ is stable.

**Case 1** $t \Rightarrow_F$.

In such situation, since $q_0 \sqsubseteq_{RS} t$, there exists $t_1$ such that $q_0 \sqsubseteq_{RS} t_1$ and $t \Rightarrow_F |t_1$. Then $\sharp t \Rightarrow |t_1 \triangle t$ by Lemma 3.1(4)(5). On the other hand, by Lemma 3.9 and 4.6, it follows from $p \Rightarrow_F |q_0 \sqsubseteq_{RS} t_1$ that $t_1 \triangle t \not\in F$. Thus $\sharp t \not\in F$ by Lemma 3.6(2).

**Case 2** $t \not\Rightarrow_F$.

Assume that $\sharp t \in F$ and let $\exists$ be any proof tree of $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash \sharp t F$. Since $t$ is stable, so is $\sharp t$ by Lemma 3.1(4). Moreover, it follows from $t \not\Rightarrow_F$, $p \Rightarrow_F |q_0$ and $q_0 \sqsubseteq_{RS} t$ that $q_0 \sqsubseteq_{RS} t$ and $t \not\in F$. Thus the last rule applied in $\exists$ is of the format below

$$\frac{\sharp t \Rightarrow_F u, \left\{qF : \sharp t \Rightarrow_F q\right\}}{\sharp t F}$$

for some $a \in Act$. (4.7.1)

Since $q_0 \sqsubseteq_{RS} t$ and $q_0 \not\in F$, by Lemma 3.2(3), we have $I(q_0) = I(t) = I(\sharp t)$. Hence $a \in I(q_0)$. Further, by Lemma 3.5 and 3.8, it follows from $q_0 \not\in F$ that $q_0 \Rightarrow_F |q_1$ for some $q_1$. Thus there exist $t_1$ and $t_2$ such that

$$t \Rightarrow_F t_1 \Rightarrow_F |t_2 \sqsubseteq_{RS} t_2.$$ (4.7.2)

Clearly, we also have

$$\sharp t \Rightarrow (t_1 \land t) \triangle t \Rightarrow |(t_2 \land t) \triangle t.$$ (4.7.3)
Since \( p \xrightarrow{\tau} F \mid q_0 \xrightarrow{\tau} F \mid q_1 \) and \( p \sqsubseteq_{RS} t \), we obtain \( q_1 \sqsubseteq_{RS} t \). Then \( q_1 \sim_{RS} t \) because of \( t \not\sim \), which, together with \((4.7.2)\), implies that \( q_1 \sqsubseteq_{RS} t \sqcap t \) by Lemma 3.11(2). From this and \( p \xrightarrow{\tau} F \mid q_0 \xrightarrow{\tau} F \mid q_1 \), we conclude \((t_2 \sqcap t) \triangle t \notin F \) by Lemma 3.9 and 4.6. Then, by Lemma 3.6(2), it follows from \((4.7.3)\) that \((t_1 \sqcap t) \triangle t \notin F \). But we also have \((t_1 \sqcap t) \triangle t \in F \) due to \((4.7.1)\) and \((4.7.3)\), a contradiction. \( \square \)

In addition to preceding two lemmas, the next result will be applied in demonstrating the converse of Lemma 4.5.

**Lemma 4.8** Let \( p \) and \( t \) be any process such that \( p \sqsubseteq_{RS} t \). For any process \( u \) and \( v \), if \( \exists u_0, u_1, u_2, \ldots, u_{n-1} \ (p \xrightarrow{\sigma} F \mid u_0 \xrightarrow{\sigma} F \mid u_1 \xrightarrow{\sigma} F \mid u_2 \cdots \mid u_{n-1} \xrightarrow{\sigma} F \mid u) ^6 \) and \( u \sqsubseteq_{RS} v \) then \( u \sqsubseteq_{RS} v \triangle t \).

**Proof.** Set

\[
R = \left\{ (q,r \triangle t) : q \sqsubseteq_{RS} r \text{ and } \begin{array}{c} q \sim_{RS} r \text{ and } \vphantom{\neg}\vphantom{\neg} p \xrightarrow{\tau} F \mid q_0 \xrightarrow{\tau} F \mid q_1 \cdots \xrightarrow{\tau} F \mid q_n \xrightarrow{\tau} F \mid q \end{array} \right\}.
\]

Obviously, it suffices to show that \( R \) is a stable ready simulation relation. Suppose \((q,r \triangle t) \in R \). Then it is easy to see that both \( q \) and \( r \triangle t \) are stable, and \( r \triangle t \notin F \) by Lemma 3.9 and 4.6. Moreover, by Lemma 3.2(4), since \( q \notin F \) and \( q \sqsubseteq_{RS} r \), we also have \( I(q) = I(r) = I(r \triangle t) \). Thus it remains only to prove that the pair \((q,r \triangle t)\) satisfies (RS3).

Let \( q \xrightarrow{\sigma} F \mid s \). Then \( s \sqsubseteq_{RS} t \) due to \( p \sqsubseteq_{RS} t \). Moreover, it follows from \( q \sqsubseteq_{RS} r \) that \( s \sqsubseteq_{RS} r_1 \) for some \( r_1 \) with \( r \xrightarrow{\sigma} F \mid r_1 \). On the other hand, since \( s \sqsubseteq_{RS} t \) and \( s \xrightarrow{\sigma} F \mid s \), we have \( s \sqsubseteq_{RS} t_1 \) for some \( t_1 \) such that \( t \xrightarrow{\sigma} F \mid t_1 \). Hence \( s \sqsubseteq_{RS} r_1 \sqcap t_1 \) by Lemma 3.11(2). Thus \((s,(r_1 \sqcap t_1) \triangle t) \in R \).

Next we shall show that \( r \triangle t \xrightarrow{\sigma} F \mid (r_1 \sqcap t_1) \triangle t \). Since \( r \xrightarrow{\sigma} F \mid r_1 \) and \( r \not\sim \), we have \( r \xrightarrow{\sigma} F \mid v \xrightarrow{\sigma} F \mid r_1 \) for some \( v \). Then, by Lemma 3.2(4), it follows that \( r \triangle t \xrightarrow{\sigma} F \mid (v \sqcap t) \triangle t \). Further, by Lemma 3.1(5), it follows from \( t \xrightarrow{\sigma} F \mid t_1 \) and \( v \xrightarrow{\sigma} F \mid r_1 \) that

\[
r \triangle t \xrightarrow{\sigma} F \mid (v \sqcap t) \triangle t \xrightarrow{\sigma} F \mid (r_1 \sqcap t_1) \triangle t. \tag{4.8.1}
\]

Moreover, since \( q \xrightarrow{\sigma} F \mid s \sqsubseteq_{RS} r_1 \sqcap t_1 \), by Lemma 3.9 and 4.6, we get \((r_1 \sqcap t_1) \triangle t \notin F \). Then, by Lemma 3.6(2), it follows from \( r \triangle t \notin F \) and \((4.8.1)\) that \( r \triangle t \xrightarrow{\sigma} F \mid (r_1 \sqcap t_1) \triangle t \), as desired. \( \square \)

We are now ready to prove the converse of Lemma 4.5.

**Lemma 4.9** \( p \sqsubseteq_{RS} t \) implies \( p \sqsubseteq_{RS} \not\sim t \).

**Proof.** Let \( p \xrightarrow{\tau} F \mid s \). It is enough to find \( q \) such that \( \not\sim t \xrightarrow{\tau} F \mid q \) and \( s \sqsubseteq_{RS} q \). We consider two cases below.

**Case 1** \( t \not\sim \).

---

^6It means \( p \xrightarrow{\tau} F \mid u \) whenever \( n = 0 \).
Since \( s \sqsubseteq_{RS} t \) and \( s \Rightarrow_{F} |s| \), there exists \( r \) such that \( s \sqsubseteq_{RS} r \) and \( t \Rightarrow_{F} |r| \). Then \( s \sqsubseteq_{RS} r \triangle t \) by Lemma 4.8. On the other hand, by Lemma 3.1(4)(5), it follows from \( t \Rightarrow_{F} |r| \) that \( \nexists t \Rightarrow_{F} |r| \triangle t \). Moreover, \( r \triangle t \notin F \) due to \( s \sqsubseteq_{RS} r \triangle t \) and \( s \notin F \). Hence \( \nexists t \Rightarrow_{F} |r| \triangle t \) by Lemma 3.6(2). Consequently, \( r \triangle t \) is exactly one that we seek.

Case 2 \( t \not\Rightarrow_{F} \).

In such situation, since \( s \subseteq_{RS} t \) and \( s \Rightarrow_{F} |s| \), we have \( s \subseteq_{RS} t \). Moreover, \( \nexists t \) is stable because of \( t \not\Rightarrow_{F} \). To complete the proof, it suffices to prove that \( s \subseteq_{RS} \nexists t \).

Put

\[
R = \{(s, \nexists t)\} \bigcup \sim_{RS}.
\]

We intend to show that \( R \) is a stable ready simulation relation. Clearly, since both \( s \) and \( \nexists t \) are stable, it is enough to prove that the pair \( (s, \nexists t) \) satisfies (RS2)-(RS4). By Lemma 4.7, we have \( \nexists t \notin F \). So, \( (s, \nexists t) \) satisfies (RS2). Moreover, by Lemma 3.2(3), it follows from \( s \notin F \) and \( s \subseteq_{RS} t \) that \( I(s) = I(t) = I(\nexists t) \), that is, such pair satisfies (RS4). The remaining work has then to be spent on checking (RS3).

Let \( s \Rightarrow_{F} |q| \). Clearly, it suffices to find a process \( w \) such that \( \nexists t \Rightarrow_{F} |w| \) and \( q \subseteq_{RS} w \). It follows from \( s \subseteq_{RS} t \) that \( q \subseteq_{RS} t_{1} \) for some \( t_{1} \) such that \( t \Rightarrow_{F} |t_{1}| \).

Then, due to \( t \not\Rightarrow_{F} \), we have \( t \Rightarrow_{F} v \Rightarrow_{F} |t_{1}| \) for some \( v \). Hence, by Lemma 3.2(3) and 3.1(5), it follows that

\[
\nexists t \Rightarrow_{F} (v \land t) \triangle t \Rightarrow_{F} |(t_{1} \land t) \triangle t |. \quad (4.9.1)
\]

On the other hand, since \( p \Rightarrow_{F} |s| \Rightarrow_{F} |q| \), \( p \subseteq_{RS} t \) and \( t \not\Rightarrow_{F} \), we get \( q \subseteq_{RS} t \).

Thus \( q \subseteq_{RS} t_{1} \land t \) by Lemma 3.11(2). Further, by Lemma 4.8, it follows that \( q \subseteq_{RS} (t_{1} \land t) \triangle t \), and hence \( (t_{1} \land t) \triangle t \notin F \). Then, by Lemma 3.6(2), it comes from \( \nexists t \notin F \) and (4.9.1) that \( \nexists t \Rightarrow_{F} |(t_{1} \land t) \triangle t | \). Consequently, the process \( (t_{1} \land t) \triangle t \) is one that we need. \( \square \)

The development so far can be summarized in the following theorem, which provides a natural and intrinsic characterization of processes that refine ones with the format \( \nexists t \).

**Theorem 4.1** For any process \( p \) and \( t \), we have

1. \( p \subseteq_{RS} \nexists t \iff p \subseteq_{RS} t \).
2. \( p \subseteq_{RS} \nexists t \iff p \subseteq_{RS} t \) whenever \( p \) and \( t \) are stable.

**Proof.** Immediately follows from Lemma 4.9, 4.5 and 4.4. In particular, by Lemma 4.9, it is a simple matter to verify the implication from right to left in item (2). \( \square \)

As an immediate consequence of the above theorem, we have the result below, which reveals that both \( \subseteq_{RS} \) and \( \subseteq_{RS} \) are precongruent w.r.t the operator \( \nexists \).

**Corollary 4.1** (Monotonicity Law for \( \nexists \)) \( t \subseteq_{RS} s \) implies \( \nexists t \subseteq_{RS} \nexists s \). Hence \( \nexists t \subseteq_{RS} \nexists s \) whenever \( t \subseteq_{RS} s \).
Proof. Suppose that \( t \sqsubseteq_{RS} s \). Then it follows from Theorem 4.1 and the transitivity of \( \sqsubseteq_{RS} \) that, for any process \( p, p \sqsubseteq_{RS} \not\in t \implies p \sqsubseteq_{RS} \not\in s \). Further, due to the reflexivity of \( \sqsubseteq_{RS} \), we have \( \not\in t \sqsubseteq_{RS} \not\in s \). \( \square \)

We conclude this section with proving that \( \sqsubseteq_{RS} \) is also precongruent w.r.t the operator \( \triangle \). To this end, a preliminary result concerning inconsistency predicate is given below. Although it can be proved by an analogous argument of Lemma 4.6, for the sake of integrality, we still show it in detail.

**Lemma 4.10** The set \( \Omega \) is a \( F \)-hole, where \( \Omega \) is given as

\[
\Omega = \left\{ r \triangle t : \exists p, u \left( p \sqsubseteq_{RS} r, u \sqsubseteq_{RS} t \text{ and } p \triangle u \not\in F \right) \right\}.
\]

**Proof.** Suppose \( r \triangle t \in \Omega \). Then there exist \( p \) and \( u \) such that \( p \triangle u \not\in F \), \( u \sqsubseteq_{RS} t \) and \( p \sqsubseteq_{RS} r \). Let \( \exists t \) be any proof tree of \( \text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash r \triangle t \). Since \( p \triangle u \not\in F \), we have \( p \not\in F \) by Lemma 3.3(6). Hence \( r \not\in F \) due to \( p \sqsubseteq_{RS} r \).

Moreover, by Lemma 3.1.1(5), \( r \triangle t \) is stable because of \( r \not\in F \). Thus the last rule applied in \( \exists t \) has the format below

\[
\begin{align*}
\exists \Delta t \to w, \quad \left\{ qF : r \Delta t \to a \right\} & \quad \text{for some } a \in \text{Act}. \quad (4.10.1)\\
\end{align*}
\]

By Lemma 3.2(4), since \( p \sqsubseteq_{RS} r \) and \( p \not\in F \), we get \( I(p) = I(r) = I(r \triangle t) \).

Hence \( a \in I(p) \). Moreover, by Lemma 3.5 and 3.8, it follows from \( p \triangle u \not\in F \) that \( p \triangle u \not\in F \) due to \( p \sqsubseteq_{RS} t \). Further, by Lemma 3.2(4), we obtain \( s \equiv (p_1 \wedge u) \triangle u \) and \( v \equiv (p_2 \wedge u_1) \triangle u \) for some \( p_1, p_2 \) and \( u_1 \) with

\[
p \sim_{RS} a p_1 \Rightarrow_F p_2 \text{ and } u \Rightarrow_F u_1.
\]

Then it follows from \( p \sqsubseteq_{RS} r \) and \( u \sqsubseteq_{RS} t \) that there exist \( t_1, r_1 \) and \( r_2 \) such that \( p_2 \sqsubseteq_{RS} r_2 \) with \( r \sim_{RS} a r_1 \Rightarrow_F r_2 \), and \( u_1 \sqsubseteq_{RS} t_1 \) with \( \Rightarrow_F t_1 \). By Lemma 3.11(1)(2), 3.2(4) and 3.1(5), this clearly forces \( p_2 \wedge u_1 \sqsubseteq_{RS} r_2 \wedge t_1 \) and

\[
r \triangle t \sim_{RS} (r_1 \wedge t) \triangle t \Rightarrow_{RS} ((r_2 \wedge t_1) \triangle t) \quad (4.10.2)
\]

Further, due to \( v \equiv (p_2 \wedge u_1) \triangle u \not\in F \) and \( u \sqsubseteq_{RS} t \), we have

\[
(r_2 \wedge t_1) \triangle t \in \Omega.
\]

Then it remains to show that \( \exists t \) contains a proper subtree with the root labelled with \( (r_2 \wedge t_1) \triangle t \). By (4.10.1) and (4.10.2), \( \exists t \) contains a proper subtree with the root labelled with \( (r_1 \wedge t) \triangle t \). If \( (r_1 \wedge t) \triangle t \) is stable then \( \exists t \) contains a node labelled with \( (r_2 \wedge t_1) \triangle t \) because of \( (r_1 \wedge t) \triangle t \equiv (r_2 \wedge t_1) \triangle t \). In the following, we consider another case \( (r_1 \wedge t) \triangle t \not\in F \). In such situation, there exist \( s_1, s_2, ..., s_m \) such that

\[
\begin{align*}
v_1 \wedge t & \sim_{RS} s_1 \not\in s_2 \sim_{RS} ... \sim_{RS} s_m \not\in |r_2 \wedge t_1, \text{ and} \\
(r_1 \wedge t) \sim_{RS} |s_1 \triangle t \sim_{RS} ... \sim_{RS} s_m \triangle t \sim_{RS} |(r_2 \wedge t_1) \triangle t. \quad (4.10.3) \end{align*}
\]

Since \( v \equiv (p_2 \wedge u_1) \triangle u \not\in F \), by Lemma 3.3(6), we get \( p_2 \wedge u_1 \not\in F \). Then \( r_2 \wedge t_1 \not\in F \) due to \( p_2 \wedge u_1 \sqsubseteq_{RS} r_2 \wedge t_1 \). Hence, by Lemma 3.6 (1) and (4.10.3), it is evident that
Thus, for each $w \in \{r_1 \wedge t\} \cup \{s_i : 1 \leq i \leq m\}$, the last rule applied in any proof tree of $\text{Strip}(I_{r_1 \wedge t}, M_{r_1 \wedge t}) \vdash w \triangleright tF$ must be of the format below:
\[
\frac{w \triangleright t \Rightarrow u, \left\{qF : w \triangleright t \Rightarrow q\right\}}{w \triangleright tF}.
\]

Consequently, by (4.10.4), it is not difficult to see that $\exists$ contains a proper subtree with the root labelled with $(r_2 \wedge t_1) \triangleright tF$, as desired. \hfill \Box

**Theorem 4.2** (Monotonicity Law for $\triangleright$) For any process $t_i$, $p_i$ $(i = 1, 2)$, we have:

1. If $p_1 \triangleright_{RS} p_2$ and $t_1 \triangleright_{RS} t_2$ then $p_1 \triangleright_{RS} t_1 \triangleright_{RS} p_2 \triangleright_{RS} t_2$.
2. If $p_1 \triangleright_{RS} p_2$ and $t_1 \triangleright_{RS} t_2$ then $p_1 \triangleright_{RS} p_2 \triangleright_{RS} t_2$. Hence $\triangleright_{RS}$ is a precongruence w.r.t the operator $\triangleright$.

**Proof.** (1) Set
\[
R = \left\{(p \triangleright t, q \triangle w) : p \triangleright_{RS} q \text{ and } t \triangleright_{RS} w\right\}.
\]

It suffices to show that $R$ is a stable ready simulation relation. Let $(p \triangleright t, q \triangle w) \in R$. Hence $p \triangleright_{RS} q$ and $t \triangleright_{RS} w$. Then, by item (5) in Lemma 3.1, both $p \triangleright t$ and $q \triangle w$ are stable, that is, (RS1) holds. Moreover, it immediately follows from Lemma 3.9 and 4.10 that (RS2) holds.

(RS3) Let $p \triangleright t \Rightarrow_{F} |u$. Hence $p \triangleright t \Rightarrow_{F} s \Rightarrow_{F} |u$ for some $s$ due to (RS1). Further, by Lemma 3.2(4), we get $s \equiv (p_1 \wedge t) \triangleright t$ and $u \equiv (p_2 \wedge t_1) \triangleright t$ for some $p_1, p_2$ and $t_1$ such that
\[
p \Rightarrow_{F} p_1 \Rightarrow_{F} |p_2 \text{ and } t \Rightarrow_{F} |t_1.
\]

Then it follows from $p \triangleright_{RS} q$ and $t \triangleright_{RS} w$ that there exist $w_1, q_1$ and $q_2$ such that $t_1 \triangleright_{RS} w_1$ with $w \Rightarrow_{F} |w_1$, and $p_2 \triangleright_{RS} q_2$ with $q \Rightarrow_{F} q_1 \Rightarrow_{F} q_2$. Hence, by Lemma 3.2(4) and 3.1(5), we obtain
\[
q \triangle w \Rightarrow (q_1 \wedge w) \triangleright (q_2 \wedge w_1) \triangleright w. \quad (4.2.1)
\]

Moreover, by Lemma 3.11 (1)(2), we have $p_2 \wedge t_1 \triangleright_{RS} q_2 \wedge w_1$. Combining this with $t \triangleright_{RS} w$ we conclude that
\[
(p_2 \wedge t_1) \triangleright (q_2 \wedge w_1) \triangle w) \in R.
\]

On the other hand, by Lemma 3.9 and 4.10, it follows from $p \triangleright t \notin F$ and $(p \triangleright t, q \triangle w) \in R$ that $q \triangle w \notin F$. Similarly, we also have $(q_2 \wedge w_1) \triangle w \notin F$. Further, by (RS2), (4.2.1) and Lemma 3.6 (2), it follows that
\[
q \triangle w \Rightarrow_{F} (q_1 \wedge w) \triangleright (q_2 \wedge w_1) \triangle w.
\]

(RS4) Assume that $p \triangleright t \notin F$. Then $p \notin F$ by Lemma 3.3(6). Thus it follows from $p \triangleright_{RS} q$ that $I(p) = I(q)$. Further, by Lemma 3.2 (4), we get $I(p \triangleright t) = I(p) = I(q \triangle w)$, as desired.

(2) Suppose $p_1 \triangleright_{RS} t \Rightarrow_{F} |u$. The task is now to seek $t$ such that $p_2 \triangleright_{RS} |t$ and $u \triangleright_{RS} t$. By Lemma 3.1(5), we get $u \equiv s \triangleright t_1$ for some $s$ with $p_1 \Rightarrow_{F} |s$. \hfill 27
Moreover, since \( p_1 \sqsubseteq_{RS} p_2 \), we have \( s \sqsubseteq_{\sim_{RS}} w \), for some \( w \) with \( p_2 \xrightarrow{F} w \). Then \( p_2 \triangle t_2 \xrightarrow{\sim_{RS}} w \triangle t_2 \) by Lemma 3.1(5). On the other hand, by Lemma 3.9 and 4.10, it follows from \( s \sqsubseteq_{\sim_{RS}} w \), \( t_1 \sqsubseteq_{RS} t_2 \) and \( u \equiv s \triangle t_1 \not\in F \) that \( w \triangle t_2 \not\in F \). So, by Lemma 3.6, we have \( p_2 \triangle t_2 \xrightarrow{\sim_{RS}} w \triangle t_2 \). Moreover, by item (1) in this lemma, it follows from \( s \sqsubseteq_{\sim_{RS}} w \) and \( t_1 \sqsubseteq_{RS} t_2 \) that \( u \equiv s \triangle t_1 \sqsubseteq_{\sim_{RS}} w \triangle t_2 \). Therefore \( w \triangle t_2 \) indeed is one that we need. \( \square \)

By the way, according to item (1) in the above theorem, it is obvious that \( \sqsubseteq_{\sim_{RS}} \) is also a precongruence w.r.t the operator \( \triangle \), that is, \( p_1 \triangle t_1 \sqsubseteq_{\sim_{RS}} p_2 \triangle t_2 \) holds whenever \( p_1 \sqsubseteq_{\sim_{RS}} p_2 \) and \( t_1 \sqsubseteq_{RS} t_2 \).

5 The operator \( \varpi \)

This section will focus on the temporal operator \( \varpi \), and characterize processes that refine processes with the topmost operator \( \varpi \). Since the auxiliary operator \( \odot \) plays an important role in describing the behavior of \( \varpi \), we begin with exploring the properties of it. We first want to indicate some simple properties.

**Lemma 5.1** For any process \( s, t, p \) and \( q \), we have

1. If \( s \sqsubseteq_{\sim_{RS}} t \odot (p \varpi q) \) then \( s \sqsubseteq_{\sim_{RS}} t \).
2. \( t \odot (p \varpi q) \sqsubseteq_{\sim_{RS}} t \) whenever \( t \not\rightarrow \).
3. \( t \odot (p \varpi q) \sqsubseteq_{RS} t \).

**Proof.** (1) Set

\[
R = \left\{ (u, v) : u \sqsubseteq_{\sim_{RS}} v \odot (r \varpi w) \right\} \bigcup \sqsubseteq_{\sim_{RS}}.
\]

We intend to show that \( R \) is a stable ready simulation up to \( \sqsubseteq_{\sim_{RS}} \). Suppose that \( u \sqsubseteq_{\sim_{RS}} v \odot (r \varpi w) \). It is straightforward to verify that the pair \( (u, v) \) satisfies (RS1), (RS2) and (RS4). To deal with (RS3-upto), we suppose \( u \xrightarrow{F} v \). It suffices to find \( v_1 \) such that \( v \xrightarrow{a} v_1 \) and \( R \odot (r \varpi w) \sqsubseteq_{\sim_{RS}} u \).

Clearly, it follows from \( u \sqsubseteq_{\sim_{RS}} v \odot (r \varpi w) \) that \( u_1 \sqsubseteq_{\sim_{RS}} t \) for some \( t \) with \( v \odot (r \varpi w) \xrightarrow{a} F \). Since \( v \odot (r \varpi w) \) is stable, there exists \( t_1 \) such that \( v \odot (r \varpi w) \xrightarrow{a} F t_1 \). We proceed by considering two cases depending on the last rule applied in the proof tree of \( \text{Strip}(\Gamma_{C_{\text{CLLT}}, M_{\text{CLLT}}}) \vdash v \odot (r \varpi w) \xrightarrow{a} t_1 \).

**Case 1**

\[
\begin{align*}
v \xrightarrow{a} s \\
v \odot (r \varpi w) \xrightarrow{a} s \land w
\end{align*}
\]

Then \( t_1 \equiv s \land w \). Moreover, by Lemma 3.1(8), \( t \equiv s_1 \land w_1 \) for some \( s_1, w_1 \) such that \( s \xrightarrow{F} s_1 \) and \( w \xrightarrow{F} w_1 \). Thus \( v \xrightarrow{a} F | s_1 \). On the other hand, by Lemma 3.11(1), it follows that \( u_1 \sqsubseteq_{\sim_{RS}} t \equiv s_1 \land w_1 \sqsubseteq_{\sim_{RS}} s_1 \). Then \( (u_1, s_1) \in R \sqsubseteq_{\sim_{RS}} \) due to \( \sqsubseteq_{\sim_{RS}} \sqsubseteq R \).

**Case 2**

\[
\begin{align*}
v \xrightarrow{a} s \\
v \odot (r \varpi w) \xrightarrow{a} (s \land v) \odot (r \varpi w)
\end{align*}
\]
Hence $t_1 \equiv (s \land r) \circ (r \bowtie \omega w)$. By Lemma 3.1(7) and (8), $t \equiv (s_1 \land r_1) \circ (r_\bowtie \omega w)$ for some $s_1, r_1$ such that $s \overset{\epsilon}{\Rightarrow}_F s_1$ and $r \overset{\epsilon}{\Rightarrow}_F r_1$. Thus it follows from $u_1 \perp_{RS} t \equiv (s_1 \land r_1) \circ (r_\bowtie \omega w)$ that $\langle u_1, (s_1 \land r_1) \rangle \in R$. Moreover, by Lemma 3.1(1), we also have $s_1 \land r_1 \perp_{RS} s_1$. Hence $\langle u_1, s_1 \rangle \in R \circ \perp_{RS}$ and $v \overset{\delta}{\Rightarrow}_F s_1$, as desired.

(2) Immediately follows from the item (1) and $t \circ (p \bowtie \omega q) \perp_{RS} t \circ (p \bowtie \omega q)$.

(3) Let $t \circ (p \bowtie \omega q) \overset{\epsilon}{\Rightarrow}_F s$. By Lemma 3.1(7) and 3.3(6), $s \equiv r \circ (p \bowtie \omega q)$ for some $r$ such that $t \overset{\epsilon}{\Rightarrow}_F r$. Moreover, by item (2) in this lemma, we have $s \equiv r \bowtie (p \bowtie \omega q) \perp_{RS} r$. □

The next result provides a necessary condition for a process to refine $t_1 \bowtie t_2$.

Before giving it, for the sake of convenience, we introduce the notation below.

**Notation 5.1** For any process $p$, $t_1$ and $t_2$, the notation $p \equiv_{RS} t_1 \uparrow t_2$ is used to stand for for any $\forall n \in \omega \forall p_0, p_1, \ldots, p_n \ (p \overset{\epsilon}{\Rightarrow}_F \ | p_0 \overset{\epsilon}{\Rightarrow}_F \ | p_1 \overset{\epsilon}{\Rightarrow}_F \ | p_2 \overset{\epsilon}{\Rightarrow}_F \ | \ldots \ | p_n \overset{\epsilon}{\Rightarrow}_F \ | p_n \ implies \ p_n \equiv_{RS} t_1$ or $\exists i \leq n (p_i \equiv_{RS} t_2)$).

**Lemma 5.2** If $p \equiv_{RS} t_1 \bowtie t_2$ then $p \equiv_{RS} t_1 \uparrow t_2$.

**Proof.** Assume that $p \overset{\epsilon}{\Rightarrow}_F \ | p_0 \overset{\epsilon}{\Rightarrow}_F \ | p_1 \overset{\epsilon}{\Rightarrow}_F \ | p_2 \overset{\epsilon}{\Rightarrow}_F \ | \ldots \ | p_n \overset{\epsilon}{\Rightarrow}_F \ | p_n \ implies \ p_n \equiv_{RS} t_1$ or $\exists i \leq n (p_i \equiv_{RS} t_2)$.

For the induction basis $n = 0$, since $p \equiv_{RS} t_1 \bowtie t_2$ and $t_1 \bowtie t_2$ is not stable, there exist $s$ and $s_1$ such that $t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F s \overset{\epsilon}{\Rightarrow}_F s_1$ and $p_0 \equiv_{RS} s_1$. The argument splits into two cases based on the last rule applied in the inference $Strip(\Gamma_{CLLT}, MC_{LLT}) \vdash t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F s$.

Case 1

$\hline$

$t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F t_2$

Thus $s \equiv t_2$ and $t_2 \overset{\epsilon}{\Rightarrow}_F s_1$. Then it follows from $p_0 \perp_{RS} s_1$ that $p_0 \equiv_{RS} t_2$.

Case 2

$t_1 \bowtie t_2 \rightarrow t_1 \circ (t_1 \bowtie t_2)$

Then $s \equiv t_1 \circ (t_1 \bowtie t_2)$. Moreover, by Lemma 3.1(7), $s_1 \equiv u \circ (t_1 \bowtie t_2)$ for some $u$ with $t_1 \overset{\epsilon}{\Rightarrow}_F | u$. By Lemma 5.1(1), it follows from $p_0 \perp_{RS} s_1 \equiv u \circ (t_1 \bowtie t_2)$ that $p_0 \perp_{RS} u$. Hence $p_0 \equiv_{RS} t_1$.

For the induction step $n = k + 1$, by IH, we have either $\exists i \leq k (p_i \equiv_{RS} t_2)$ or $p_k \equiv_{RS} t_1$. If the former holds, then we get $\exists i \leq k + 1 (p_i \equiv_{RS} t_2)$ immediately. In the following, we consider another case where $\exists i \leq k (p_i \equiv_{RS} t_2)$.

Since $p \equiv_{RS} t_1 \bowtie t_2$ and $p_i \overset{\epsilon}{\Rightarrow}_F | p_{i+1}$ for any $i \leq k + 1$, there exist $r_0, r_1, \ldots, r_{k+1}$ such that $t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F r_0, r_{i} \overset{\epsilon}{\Rightarrow}_F r_{i+1}$ and $p_i \equiv_{RS} r_i$ for each $i \leq k + 1$. To conclude the proof, we need the claim below.

**Claim 1** For each $j \leq k$, $r_j \equiv v \circ (t_1 \bowtie t_2)$ for some $v$.

We proceed by induction on $j$. For the induction basis $j = 0$, due to $t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F$, we obtain $t_1 \bowtie t_2 \overset{\epsilon}{\Rightarrow}_F s \overset{\epsilon}{\Rightarrow}_F | r_0$ for some $s$. It is easy to see that either $s \equiv t_2$ or $s \equiv t_1 \circ (t_1 \bowtie t_2)$ If the first alternative holds, then $p_0 \equiv_{RS} t_2$ by the similar argument applied to Case 1 in the above. This contradicts the assumption that $\exists j \leq k (p_j \equiv_{RS} t_2)$. Hence $s \equiv t_1 \circ (t_1 \bowtie t_2)$. Then it immediately follows that $r_0 \equiv v_1 \circ (t_1 \bowtie t_2)$ for some $v_1$ with $t_1 \overset{\epsilon}{\Rightarrow}_F | v_1$.
For the induction step \( j = i + 1 \leq k \), we assume that \( r_i \equiv v_i \circ (t_1 \varpi t_2) \) for some \( v_i \). Since \( r_i \overset{a_{i+1}}{\rightarrow}_F \) \( r_{i+1} \) and \( r_i \) is stable, we obtain \( r_i \overset{a_{i+1}}{\rightarrow}_F s \overset{\circ}{\Rightarrow} F | r_{i+1} \) for some \( s \).

Clearly, the last rule applied in the inference \( \text{Strip}(\Gamma_{\text{CLLT}}, M_{\text{CLLT}}) \vdash r_i \overset{a_{i+1}}{\rightarrow}_F s \) is

either \[
\frac{v_i \overset{a_{i+1}}{\rightarrow}_F u}{v_i \circ (t_1 \varpi t_2) \overset{a_{i+1}}{\rightarrow}_F (u \land t_2)}
\]
or

either \[
\frac{v_i \overset{a_{i+1}}{\rightarrow}_F u}{v_i \circ (t_1 \varpi t_2) \overset{a_{i+1}}{\rightarrow}_F (u \land t_1) \circ (t_1 \varpi t_2)}.
\]

For the first alternative, we get \( s \equiv u \land t_2 \) and \( r_{i+1} \equiv q \land w \) for some \( q \) and \( w \) such that \( u \overset{\circ}{\Rightarrow} F | q \) and \( t_2 \overset{\circ}{\Rightarrow} F | w \). On the other hand, since \( p_{i+1} \sqsubseteq_{RS} \sim t_2 \), by Lemma 3.11(1), we have \( p_{i+1} \sqsubseteq_{RS} w \). Further, it follows from \( t_2 \overset{\circ}{\Rightarrow} F | w \) that \( p_{i+1} \sqsubseteq_{RS} t_2 \), which, due to \( i < k \), contradicts the assumption \( \neg \exists j \leq k(p_j \sqsubseteq_{RS} t_2) \).

Thus we can conclude that the last rule applied in the inference is the second alternative. Then it is clear that \( r_{i+1} \equiv v_{i+1} \circ (t_1 \varpi t_2) \) for some \( v_{i+1} \) as the operator \( \circ \) is static w.r.t the \( \tau \)-labelled transition relation \( \sqsubseteq_{RS} \).

Returning now to the proof of the lemma, by the above claim, we may assume that \( r_k \equiv t \circ (t_1 \varpi t_2) \) for some \( t \). Since \( r_k \overset{a_{k+1}}{\rightarrow}_F r_{k+1} \) and \( r_k \) is stable, we obtain \( r_k \overset{a_{k+1}}{\rightarrow}_F s \overset{\circ}{\Rightarrow} F | r_{k+1} \) for some \( s \). The last rule applied in the inference \( \text{Strip}(\Gamma_{\text{CLLT}}, M_{\text{CLLT}}) \vdash r_k \overset{a_{k+1}}{\rightarrow}_F s \) is

either \[
\frac{t \overset{a_{k+1}}{\rightarrow}_F u}{t \circ (t_1 \varpi t_2) \overset{a_{k+1}}{\rightarrow}_F (u \land t_2)}
\]
or

either \[
\frac{t \overset{a_{k+1}}{\rightarrow}_F u}{t \circ (t_1 \varpi t_2) \overset{a_{k+1}}{\rightarrow}_F (u \land t_1) \circ (t_1 \varpi t_2)}.
\]

For the former, \( p_{k+1} \sqsubseteq_{RS} t_2 \) follows by the argument similar to that in the proof of the induction step in Claim 1. For the latter, we have \( s \equiv (u \land t_1) \circ (t_1 \varpi t_2) \), and \( r_{k+1} \equiv (w \land q) \circ (t_1 \varpi t_2) \) for some \( w, q \) such that \( u \overset{\circ}{\Rightarrow} F | w \) and \( t_1 \overset{\circ}{\Rightarrow} F | q \). Moreover, by Lemma 5.1, it follows from \( p_{k+1} \sqsubseteq_{RS} r_{k+1} \equiv (w \land q) \circ (t_1 \varpi t_2) \) that \( p_{k+1} \sqsubseteq_{RS} w \land q \). Then \( p_{k+1} \sqsubseteq_{RS} q \) by Lemma 3.11(1). Further, due to \( t_1 \overset{\circ}{\Rightarrow} F | q \), we get \( p_{k+1} \sqsubseteq_{RS} t_1 \), as desired. \( \square \)

In order to establish the converse of the above lemma, we need the following two results which concern themselves with inconsistency predicate.

**Lemma 5.3** If \( u \overset{\circ}{\Rightarrow} | u_1, p \sqsubseteq_{RS} u_1, p \notin F \) and \( p \sqsubseteq_{RS} t \) then \( u \land t \notin F \).

**Proof.** Since \( p \sqsubseteq_{RS} t \), it follows from \( p \notin F \) and \( \overset{\tau}{\Rightarrow} t \) that \( p \sqsubseteq_{RS} t_1 \) for some \( t_1 \) with \( t \overset{\circ}{\Rightarrow} t_1 \). Moreover, we have \( p \sqsubseteq_{RS} u_1 \land t_1 \) due to \( p \sqsubseteq_{RS} u_1 \) and Lemma 3.11(2). Then \( u \land t_1 \notin F \) because of \( p \notin F \). Thus, by Lemma 3.6(2), it follows from \( u \land t \overset{\circ}{\Rightarrow} | u_1 \land t_1 \) that \( u \land t \notin F \). \( \square \)

**Lemma 5.4** Let \( p, t_1 \) and \( t_2 \) be any process such that \( p \overset{\downarrow}_{RS} t_1 \uparrow t_2 \), and set

\[
\Omega = \left\{ t \circ (t_1 \varpi t_2) : \exists p_0, p_1, \ldots p_l \left( p \overset{\downarrow}_F | p_0 \overset{\downarrow}_F | p_1 \ldots \overset{\downarrow}_F | p_l \quad p_l \sqsubseteq_{RS} t \text{ and } \neg \exists j \leq l (p_j \sqsubseteq_{RS} t_2) \right) \right\}.
\]

Then \( \Omega \) is a \( F \)-hole.

\footnote{That is, the structure that \( \circ \) represents is preserved under \( \tau \)-transitions.}
Proof. Let \( t \odot (t_1 \varpi t_2) \in \Omega \) and \( \exists \) be any proof tree of \( Strip(\Gamma_{\text{CLLT}}, M_{\text{CLLT}}) \vdash t \odot (t_1 \varpi t_2) F \). Hence there exist \( p_0, p_1, ..., p_n, a_1, a_2, ... a_n \) such that

(a) \( p \xrightarrow{\exists} F | p_0 \xrightarrow{a} F | p_1 \xrightarrow{a} F | p_n \),

(b) \( p_n \sqsubseteq_{RS} t \), and

(c) \( \neg \exists j \leq n(p_j \sqsubseteq_{RS} t_2) \).

Since \( p_n \sqsubseteq_{RS} t \) and \( p_n \notin F \), we get \( t \notin F \). Moreover, it follows from \( t \xrightarrow{\exists} \) that the last rule applied in \( \exists \) is

\[
\frac{t \odot (t_1 \varpi t_2) \xrightarrow{a} w, \{qF : t \odot (t_1 \varpi t_2) \xrightarrow{a} q\}}{t \odot (t_1 \varpi t_2)F} \quad \text{for some } a \in \text{Act.} \quad (5.4.1)
\]

Since \( p_n \sqsubseteq_{RS} t \) and \( p_n \notin F \), we have \( I(p_n) = I(t) = I(t \odot (t_1 \varpi t_2)) \). Hence \( a \in I(p_n) \). Moreover, by Lemma 3.5 and 3.8, it follows from \( p_n \notin F \) that \( p_n \xrightarrow{a} F | p_{n+1} \) for some \( p_{n+1} \). Due to \( p_n \sqsubseteq_{RS} t \), there exist \( s_0, s_1 \) such that

\[
t \xrightarrow{a} F \quad s_0 \xrightarrow{a} F | s_1 \quad \text{and} \quad p_{n+1} \sqsubseteq_{RS} s_1.
\]

Then \( t \odot (t_1 \varpi t_2) \xrightarrow{a} s_0 \land t_2 \) and \( t \odot (t_1 \varpi t_2) \xrightarrow{a} (s_0 \land t_1) \odot (t_1 \varpi t_2) \) are two \( a \)-labelled transitions from \( t \odot (t_1 \varpi t_2) \). Thus, by (5.4.1), we get

\[
s_0 \land t_2 \in F \quad \text{and} \quad (s_0 \land t_1) \odot (t_1 \varpi t_2) \in F.
\]

In particular, \( \exists \) contains a proper subtree with the root labelled with \( (s_0 \land t_1) \odot (t_1 \varpi t_2) F \). Clearly, to complete the proof, it is enough to show that either \( (s_0 \land t_1) \odot (t_1 \varpi t_2) \in \Omega \) or any proof tree of \( (s_0 \land t_1) \odot (t_1 \varpi t_2) F \) must contain a proper subtree with the root labelled with \( uF \) for some \( u \in \Omega \). In the following, we intend to prove this.

Since \( p \xrightarrow{\exists} F | p_0 \xrightarrow{a} F | p_1 \xrightarrow{a} F | p_n \xrightarrow{a} F | p_{n+1} \) and \( p \sqsubseteq_{RS} t_1 \uparrow t_2 \), we get

\[
either \quad p_{n+1} \sqsubseteq_{RS} t_1 \quad \text{or} \quad \exists i \leq n + 1(p_i \sqsubseteq_{RS} t_2). \quad (5.4.2)
\]

On the other hand, by Lemma 5.3, it follows from \( s_0 \xrightarrow{a} F | s_1, p_{n+1} \sqsubseteq_{RS} s_1 \) and \( s_0 \land t_2 \in F \) that

\[
p_{n+1} \sqsubseteq_{RS} t_2.
\]

Further, due to (5.4.2) and (c) (i.e., \( \neg \exists j \leq n(p_j \sqsubseteq_{RS} t_2) \)), we have

\[
p_{n+1} \sqsubseteq_{RS} t_1 \quad \text{and} \quad \neg \exists i \leq n + 1(p_i \sqsubseteq_{RS} t_2). \quad (5.4.3)
\]

Since \( p_{n+1} \xrightarrow{\exists} F \) and \( p_{n+1} \sqsubseteq_{RS} t_1 \), there exists \( v \) such that \( p_{n+1} \sqsubseteq_{RS} v \) and \( t_1 \xrightarrow{a} F \mid v \). Then, by Lemma 3.11(2) and 3.1(8), it follows from \( p_{n+1} \sqsubseteq_{RS} s_1 \) and \( s_0 \xrightarrow{a} F \mid s_1 \) that

\[
p_{n+1} \sqsubseteq_{RS} s_1 \land v \quad \text{and} \quad s_0 \land t_1 \xrightarrow{a} s_1 \land v. \quad (5.4.4)
\]

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If \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2)\) is stable, then so are \(s_0\) and \(t_1\). Thus \(s_0 \equiv s_1\) and \(t_1 \equiv v\). Further, by (5.4.4) and (5.4.3), we get \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \in \Omega\), as desired. In the following, we deal with another case \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \in \Omega\).

Due to (5.4.4) and \(p_{n+1} \notin F\), we obtain \(s_1 \land v \notin F\), moreover, by Lemma 3.6 (2), it follows that \(s_0 \land t_1 \vartriangleright_F |s_1 \land v\). Since \(s_0 \land t_1 \notin F\) and \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \rightarrow\), the last rule applied in the proof tree of \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2)F\) is

\[
\frac{(s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \rightarrow w, \ qF : (s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \rightarrow q}{(s_0 \land t_1) \circ (t_1 \vartriangleright t_2)F}.
\]

(5.4.5)

On the other hand, since \(s_0 \land t_1 \vartriangleright_F |s_1 \land v\), there exists \(r_i (1 \leq i \leq m)\) such that \(s_0 \land t_1 \rightarrow_F r_1 \rightarrow_F r_2 \rightarrow_F \ldots \rightarrow_F r_m \rightarrow_F |s_1 \land v\). Thus

\[
(s_0 \land t_1) \circ (t_1 \vartriangleright t_2) \rightarrow r_1 \circ (t_1 \vartriangleright t_2) \ldots \rightarrow r_m \circ (t_1 \vartriangleright t_2) \rightarrow |(s_1 \land v) \circ (t_1 \vartriangleright t_2)\).
\]

(5.4.6)

For each \(i \leq m\), due to \(r_i \notin F\), the last rule applied in any proof tree of \(r_i \circ (t_1 \vartriangleright t_2)F\) has the format below

\[
\frac{r_i \circ (t_1 \vartriangleright t_2) \rightarrow w, \ qF : r_i \circ (t_1 \vartriangleright t_2) \rightarrow q}{r_i \circ (t_1 \vartriangleright t_2)F}.
\]

Then, by (5.4.6) and (5.4.5), it is obvious that any proof tree of \((s_0 \land t_1) \circ (t_1 \vartriangleright t_2)F\) must contain a proper subtree with the root labelled with \((s_1 \land v) \circ (t_1 \vartriangleright t_2)F\). Moreover, by (5.4.4) and (5.4.3), we also have \((s_1 \land v) \circ (t_1 \vartriangleright t_2) \in \Omega\), as desired. □

We are now in a position to show the converse of Lemma 5.2.

Lemma 5.5 If \(p \vartriangleright_R \subseteq t_1 \uparrow t_2\) then \(p \subseteq_R \vartriangleright t_1 \vartriangleright t_2\).

Proof. Let \(p \vartriangleright_F |q_0\). The task is to find \(r\) such that \(q_0 \subseteq_R r\) and \(t_1 \vartriangleright t_2 \vartriangleright_F |r\).

If \(q_0 \subseteq_R t_2\), then, due to \(q_0 \not\rightarrow\) and \(q_0 \notin F\), we get \(q_0 \subseteq_R v\) for some \(v\) with \(t_2 \vartriangleright_F |v\). Moreover, by Lemma 3.1(6) and 3.6, it follows from \(t_2 \notin F\) that \(t_1 \vartriangleright t_2 \vartriangleright_F t_2 \vartriangleright_F |v\), as desired.

We now turn to another case \(q_0 \nsubseteq_R t_2\). Set \(R = R_0 \cup \subseteq_R\) with

\[
R_0 = \{ \langle q, t \circ (t_1 \vartriangleright t_2) \rangle : \exists p_0, p_1, \ldots, p_i \left( p \vartriangleright_F |p_0 \vartriangleright_F |p_1 \vartriangleright_F \ldots \vartriangleright_F |p_i \equiv q, \right. \left. q \subseteq_R t \text{ and } \neg \exists i \leq l(p_i \subseteq_R t_2) \right) \}.
\]

The rest of the proof is based on the following claim.

Claim 1 \(R\) is a stable ready simulation relation.

Clearly, it suffices to prove that each pair in \(R_0\) satisfies (RS1)-(RS4). Let \(\langle q, t \circ (t_1 \vartriangleright t_2) \rangle \in R_0\). Thus there exist \(p_0, p_1, \ldots, p_n, a_1, a_2, \ldots, a_n\) such that

(a) \(p \vartriangleright_F |p_0 \vartriangleright_F |p_1 \vartriangleright_F \ldots \vartriangleright_F |p_n \equiv q\),

(b) \(p_n \subseteq_R t\), and

(c) \(\neg \exists i \leq n(p_i \subseteq_R t_2)\).
Then (RS1) immediately follows from (b), and (RS2) is guaranteed by Lemma 3.9 and 5.4. By Lemma 3.2(5) and (b), it follows from $p_n \not\in F$ that $I(p_n) = I(t) = I(t \circ (t_1 \ll t_2))$, and hence (RS4) holds. We next verify (RS3).

Suppose $q \equiv p_n \not\rightarrow F |p_{n+1}$. Then, due to (b), there exist $w$ and $u$ such that $p_{n+1} \subseteq_{RS} w \Rightarrow |u$ and $t \not\rightarrow_F w \Rightarrow |u$. Moreover, it follows from (a), (c), $p_n \not\rightarrow_F |p_{n+1}$ and $p \subseteq_{RS} t_1 \uparrow t_2$ that

$$
either p_{n+1} \subseteq_{RS} t_1 \quad \text{or} \quad p_{n+1} \subseteq_{RS} t_2 \quad (5.5.1)$$

The argument splits into two cases depending on whether it holds that $p_{n+1} \subseteq_{RS} t_2$.

Case 1 $p_{n+1} \subseteq_{RS} t_2$.

Due to $p_{n+1} \not\rightarrow_F$ and $p_{n+1} \not\in F$, we get $p_{n+1} \subseteq_{RS} v$ for some $v$ with $t_2 \not\rightarrow_F |v$. By Lemma 3.11(2), it follows from $p_{n+1} \subseteq_{RS} v$ and $p_{n+1} \subseteq_{RS} u$ that

$$p_{n+1} \subseteq_{RS} u \land v. \quad (5.5.2)$$

On the other hand, by Lemma 3.2(5) and 3.1(8), we have

$$\not\rightarrow_F (t \circ (t_1 \ll t_2)) \not\rightarrow_F w \land t_2 \not\rightarrow_F |u \land v. \quad (5.5.4)$$

Moreover, by Lemma 3.9 and 5.4, $t \circ (t_1 \ll t_2) \not\in F$. By (5.5.2) and $p_{n+1} \not\in F$, we also have $u \land v \not\in F$. Then, by Lemma 3.6 (2), it follows that

$$t \circ (t_1 \ll t_2) \not\rightarrow_F w \land t_2 \not\rightarrow_F |u \land v. \quad (5.5.3)$$

On account of (5.5.2) and (5.5.3), we have the diagram below, as desired.

$$q \equiv p_n \quad R_0 \quad t \circ (t_1 \ll t_2)$$

$$\not\rightarrow_F \quad \not\rightarrow_F \quad \not\rightarrow_F$$

$$p_{n+1} \subseteq_{RS} u \land v$$

Case 2 $p_{n+1} \not\subseteq_{RS} t_2$.

Hence $p_{n+1} \subseteq_{RS} t_1$ by (5.5.1). Then it follows from $p_{n+1} \not\rightarrow_F$ and $p_{n+1} \not\in F$ that $p_{n+1} \subseteq_{RS} v$ for some $v$ with $t_1 \not\rightarrow_F |v$. Moreover, by Lemma 3.11(2) and $p_{n+1} \subseteq_{RS} u$, we have

$$p_{n+1} \subseteq_{RS} u \land v.$$

Further, due to $p_{n+1} \not\subseteq_{RS} t_2$ and $\not\exists i \leq n(p_i \subseteq_{RS} t_2)$ (i.e., (c)), we get

$$\langle p_{n+1}, (u \land v) \circ (t_1 \ll t_2) \rangle \in R_0. \quad (5.5.4)$$

By Lemma 3.2(5) and 3.1(7)(8), it follows that

$$t \circ (t_1 \ll t_2) \not\rightarrow_F (w \land t_1) \circ (t_1 \ll t_2) \not\rightarrow_F |(u \land v) \circ (t_1 \ll t_2).$$

Moreover, by Lemma 3.9 and 5.4, $t \circ (t_1 \ll t_2) \not\in F$ and $(u \land v) \circ (t_1 \ll t_2) \not\in F$. Then, by Lemma 3.6(2), we obtain
\[ t \circ (t_1 \uplus t_2) \Rightarrow_F |(u \land v) \circ (t_1 \uplus t_2). \quad (5.5.5) \]

According to (5.5.4) and (5.5.5), we get the diagram below, as desired.

\[
\begin{array}{ccc}
q \equiv p_n & R_0 & t \circ (t_1 \uplus t_2) \\
a & \downarrow F & a \\
p_{n+1} & R_0 & (u \land v) \circ (t_1 \uplus t_2)
\end{array}
\]

From the arguments applied to two cases above, it may be concluded that \( \langle q, t \circ (t_1 \uplus t_2) \rangle \) satisfies (RS3). Therefore, the binary relation \( R \) is indeed a stable ready simulation relation.

We now return to the proof of the lemma itself. Since \( p \sqsubseteq_{RS} t_1 \uparrow t_2 \) and \( p \Rightarrow_F |q_0 \), it follows from \( q_0 \sqsubseteq_{RS} t_2 \) that \( q_0 \sqsubseteq_{RS} t_1 \). Then \( q_0 \sqsubseteq_{RS} u \) for some \( u \) with \( t_1 \Rightarrow_F |u \). Thus \( \langle q_0, u \circ (t_1 \uplus t_2) \rangle \in R_0 \). By Claim 1, this clearly forces
\[ q_0 \sqsubseteq_{RS} u \circ (t_1 \uplus t_2). \quad (5.5.6) \]

On the other hand, by Lemma 3.1 (6) and (7), it holds that
\[ t_1 \uplus t_2 \Rightarrow t_1 \circ (t_1 \uplus t_2) \Rightarrow |u \circ (t_1 \uplus t_2). \]

Moreover, it follows from \( q_0 \notin F \) and (5.5.6) that \( u \circ (t_1 \uplus t_2) \notin F \). Hence, by Lemma 3.6(2), we have
\[ t_1 \uplus t_2 \Rightarrow |u \circ (t_1 \uplus t_2). \quad (5.5.7) \]

Consequently, by (5.5.6) and (5.5.7), the process \( u \circ (t_1 \uplus t_2) \) is indeed the one that we seek. \( \Box \)

Now the main theorem of this section is stated below, which, together with Theorem 5.1, gives a bridge from CLLT to the action-based CTL that will be considered in Section 8.

**Theorem 5.1** For any process \( p, t_1 \) and \( t_2, p \sqsubseteq_{RS} t_1 \uplus t_2 \) iff \( p \sqsubseteq_{RS} t_1 \uparrow t_2. \)

**Proof.** Immediately follows from Lemma 5.2 and 5.5. \( \Box \)

Let us mention two important consequences of the above theorem:

**Corollary 5.1** Suppose \( p \sqsubseteq_{RS} t_1 \uplus t_2 \) and \( p \Rightarrow_F |p_0 \Rightarrow_F |p_1 \Rightarrow_F \ldots \Rightarrow_F |p_k \). If \( \neg \exists i \leq k(p_i \sqsubseteq_{RS} t_2) \) then \( p_k \sqsubseteq_{RS} t_1 \uplus t_2. \)

**Proof.** Straightforward. \( \Box \)

**Corollary 5.2** (Monotonicity Law of \( \uplus \)) If \( t_1 \sqsubseteq_{RS} s_1 \) and \( t_2 \sqsubseteq_{RS} s_2 \) then \( t_1 \uplus t_2 \sqsubseteq_{RS} s_1 \uplus s_2. \) Hence \( \sqsubseteq_{RS} \) is a precongruence w.r.t the operator \( \uplus. \)
Proof. Since $\sqsubseteq_{RS}$ is reflexive, it is enough to prove that, for any $p, p \sqsubseteq_{RS} t_1 \sqsubseteq_{RS} t_2$ implies $p \sqsubseteq_{RS} s_1 \sqsubseteq_{RS} s_2$. This immediately follows from Theorem 5.1 and the transitivity of $\sqsubseteq_{RS}$. □

The remainder of this section will be devoted to the proof of that $\sqsubseteq_{RS}$ is also precongruent w.r.t. the operator $\circ$. To this end, the following preliminary result concerning inconsistency predicate is needed.

Lemma 5.6 The set $\Omega$ given below is a $F$-hole.

$$\Omega = \left\{ u \circ (p \varpi q) : \exists u_1, p_1, q_1 \left( u_1 \sqsubseteq_{RS} u, p_1 \sqsubseteq_{RS} p, q_1 \sqsubseteq_{RS} q \land (p_1 \varpi q_1) \notin F \right) \right\}.$$  

Proof. Suppose $t_1 \circ (p_1 \varpi q_1) \in \Omega$. That is, there exist $t_2, p_2$ and $q_2$ such that $t_2 \sqsubseteq_{RS} t_1, p_2 \sqsubseteq_{RS} p_1, q_2 \sqsubseteq_{RS} q_1$ and $t_2 \circ (p_2 \varpi q_2) \notin F$.

Let $\exists$ be any proof tree of $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash t_1 \circ (p_1 \varpi q_1) F$. Since $t_2 \circ (p_2 \varpi q_2) \notin F$, $t_2 \notin F$ by Lemma 3.3(6). Then, due to $t_2 \sqsubseteq_{RS} t_1$, we also get $t_1 \notin F$.

Moreover, by Lemma 3.1(7), $t_1 \circ (p_1 \varpi q_1)$ is stable because of $t_1 \not\vdash$. Thus the last rule applied in $\exists$ is

$$t_1 \circ (p_1 \varpi q_1) \xrightarrow{a} w, \left\{ tF : t_1 \circ (p_1 \varpi q_1) \xrightarrow{a} r \right\}$$

for some $a \in \text{Act}$. (5.6.1)

Then $a \in I(t_1)$ by Lemma 3.2(5). Since $t_2 \sqsubseteq_{RS} t_1$ and $t_2 \notin F$, we get $I(t_1) = I(t_2) = I(t_2 \circ (p_2 \varpi q_2))$ by Lemma 3.2(5). Hence $a \in I(t_2 \circ (p_2 \varpi q_2))$. Moreover, by Lemma 3.5, it follows from $t_2 \circ (p_2 \varpi q_2) \notin F$ that $t_2 \circ (p_2 \varpi q_2) \xrightarrow{a} s$ for some $s$.

The remaining proof depends on the claim below, which yields information about the format of $s$.

Claim 1 $s \equiv (s_1 \land p_2) \circ (p_2 \varpi q_2)$ for some $s_1$ with $t_2 \xrightarrow{a} F$ $s_1$.

Since $t_2 \circ (p_2 \varpi q_2) \xrightarrow{a} F$ $s$, by Lemma 3.2(5) and 3.3(4)(6), there exists $s_1$ such that $t_2 \xrightarrow{a} F$ $s_1$ and

either $s \equiv (s_1 \land p_2) \circ (p_2 \varpi q_2)$ or $s \equiv s_1 \land q_2$.

Thus it is enough to show that $s \notin s_1 \land q_2$. Conversely, suppose that $s \equiv s_1 \land q_2$. Due to $s_1 \land q_2 \notin F$, by Lemma 3.1(8) and 3.8, $s_1 \land q_2 \xrightarrow{a} F$ $s_2 \land q_3$ for some $s_2, q_3$ with $s_1 \xrightarrow{a} F | s_2$ and $q_2 \xrightarrow{a} F | q_3$. Since $t_2 \sqsubseteq_{RS} t_1$ and $t_2 \xrightarrow{a} F$ $s_1 \xrightarrow{a} F | s_2$, $s_2 \sqsubseteq_{RS} u$ for some $u$ with $t_1 \xrightarrow{a} F v \xrightarrow{a} F | u$. Moreover, it follows from $q_2 \sqsubseteq_{RS} q_1$ and $q_2 \xrightarrow{a} F | q_3$ that $q_3 \sqsubseteq_{RS} q_4$ for some $q_4$ with $q_1 \xrightarrow{a} F | q_4$. Hence $s_2 \land q_3 \sqsubseteq_{RS} u \land q_4$ by Lemma 3.11(1)(2). Then $u \land q_4 \notin F$ because of $s_2 \land q_3 \notin F$. Further, by Lemma 3.6(2), it follows from $v \land q_1 \xrightarrow{a} F | u \land q_4$ that $v \land q_1 \notin F$. However, due to $t_1 \circ (p_1 \varpi q_1) \xrightarrow{a} v \land q_1$ and (5.6.1), we have $v \land q_1 \in F$. Thus a contradiction arises, as desired.

Now we return to the proof of the lemma. Since $s \equiv (s_1 \land p_2) \circ (p_2 \varpi q_2) \notin F$, by Lemma 3.8 and 3.1(7)(8), $(s_1 \land p_2) \circ (p_2 \varpi q_2) \xrightarrow{a} F | (s_1 \land p_3) \circ (p_2 \varpi q_2)$ for some $s_3, p_3$ such that $s_1 \xrightarrow{a} F | s_3$ and $p_2 \xrightarrow{a} F | p_3$. Moreover, it follows from $t_2 \sqsubseteq_{RS} t_1$ and $t_2 \xrightarrow{a} F s_1 \xrightarrow{a} F | s_3$ that $s_3 \sqsubseteq_{RS} u_1$ for some $u_1, v_1$ with $t_1 \xrightarrow{a} F v_1 \xrightarrow{a} F | u_1$.
Due to $p_2 \subseteq_{RS} p_1$ and $p_2 \Rightarrow_F p_3$, we also have $p_3 \subseteq_{RS} p_4$ for some $p_4$ with $p_1 \Rightarrow_{F} p_4$. Then $s_3 \land p_3 \subseteq_{RS} u_1 \land p_4$ by Lemma 3.11(1)(2). Combining this with $(s_3 \land p_3) \circ (p_2 \varpi q_2) \notin F$, we get

$$(u_1 \land p_4) \circ (p_1 \varpi q_1) \in \Omega.$$ 

Clearly, in order to complete the proof, it suffices to prove that $\exists$ contains a proper subtree with the root labelled with $(u_1 \land p_4) \circ (p_1 \varpi q_1) F$. Since $t_1 \Rightarrow_{F} v_1$, we have $t_1 \circ (p_1 \varpi q_1) \Rightarrow_{F} (v_1 \land p_4) \circ (p_1 \varpi q_1)$. Thus, by (5.6.1), $\exists$ contains a proper subtree with the root labelled with $(v_1 \land p_1) \circ (p_1 \varpi q_1) F$. If $v_1 \land p_1$ is stable then $\exists$ contains a node labelled with $(u_1 \land p_4) \circ (p_1 \varpi q_1) F$ because of $(v_1 \land p_1) \circ (p_1 \varpi q_1) \equiv (u_1 \land p_4) \circ (p_1 \varpi q_1)$, as desired. We next manage another case $v_1 \land p_1 \not\Rightarrow_{F}$.

Since $v_1 \land p_1 \Rightarrow_{F} |u_1 \land p_4, s_3 \land p_3 \subseteq_{RS} u_1 \land p_4$ and $s_3 \land p_3 \notin F$, we get $v_1 \land p_1 \Rightarrow_{F} |u_1 \land p_4$ by Lemma 3.6(2). Hence there exist $r_1, r_2, ..., r_m$ ($m \geq 1$) such that

$$v_1 \land p_1 \Rightarrow_{F} r_1 \Rightarrow_{F} r_2 \Rightarrow_{F} ... \Rightarrow_{F} r_m \Rightarrow_{F} |u_1 \land p_4. \quad (5.6.3)$$

By Lemma 3.1(7), we also have

$$(v_1 \land p_1) \circ (p_1 \varpi q_1) \Rightarrow_{F} r_1 \circ (p_1 \varpi q_1) ... \Rightarrow_{F} r_m \circ (p_1 \varpi q_1) \Rightarrow_{F} |(u_1 \land p_4) \circ (p_1 \varpi q_1). \quad (5.6.4)$$

Then, by (5.6.3), the last rule applied in any proof tree of $w \circ (p_1 \varpi q_1) F$ with $w \in \{v_1 \land p_1, r_i : 1 \leq i \leq m\}$ must be of the format below

$$w \circ (p_1 \varpi q_1) \Rightarrow_{F} r, \quad \{r F : w \circ (p_1 \varpi q_1) \Rightarrow_{F} r\}.$$ 

Consequently, by (5.6.4), it is immediate that $\exists$ contains a proper subtree with the root labelled with $(u_1 \land p_4) \circ (p_1 \varpi q_1) F$, as desired. \(\Box\)

Having disposed of this preliminary step, we can now establish the monotonicity laws of the operator $\circ$, which will be useful in the sequel.

**Theorem 5.2** For any process $u_i, r_i$ and $s_i$ ($1 \leq i \leq 2$), we have

1. If $u_2 \not\subseteq_{RS} u_1, r_2 \subseteq_{RS} r_1, s_2 \subseteq_{RS} s_1$ then $u_2 \circ (r_2 \varpi s_2) \not\subseteq_{RS} u_1 \circ (r_1 \varpi s_1)$.
2. If $u_2 \subseteq_{RS} u_1, r_2 \subseteq_{RS} r_1, s_2 \subseteq_{RS} s_1$ then $u_2 \circ (r_2 \varpi s_2) \subseteq_{RS} u_1 \circ (r_1 \varpi s_1)$.

**Proof.** Clearly, (2) immediately follows from (1). In the following, we shall prove (1). Put

$$R = \left\{(t_2 \circ (p_2 \varpi q_2), t_1 \circ (p_1 \varpi q_1)) : t_2 \subseteq_{RS} t_1, p_2 \subseteq_{RS} p_1, q_2 \subseteq_{RS} q_1\right\} \cup \subseteq_{RS}.$$ 

We wish to demonstrate that $R$ is a stable ready simulation. Suppose that $t_2 \subseteq_{RS} t_1, p_2 \subseteq_{RS} p_1$ and $q_2 \subseteq_{RS} q_1$. By Lemma 3.1(7), 3.9, 5.6 and 3.2(5), it is easy to verify that the pair $(t_2 \circ (p_2 \varpi q_2), t_1 \circ (p_1 \varpi q_1))$ satisfies (RS1), (RS2) and (RS4). It remains to prove that such pair satisfies (RS3). Suppose $t_2 \circ (p_2 \varpi q_2) \Rightarrow_{F} u$. It is enough to find $s$ such that

$$t_1 \circ (p_1 \varpi q_1) \Rightarrow_{F} s \ land (u, s) \in R.$$ 

Since $t_2 \circ (p_2 \varpi q_2) \notin F$, by Lemma 3.9 and 5.6, we have

$$t_1 \circ (p_1 \varpi q_1) \notin F. \quad (5.2.1)$$
Moreover, due to $t_2 \odot (p_2 \varpi) \not\to \tau$, $t_2 \odot (p_2 \varpi) \not\to_F v \vartriangleright_F |u$ for some $v$. The argument splits into two cases based on the last rule applied in the inference $\text{Strip}(\Gamma_{CLLT}, M_{CLLT}) \vdash t_2 \odot (p_2 \varpi) \not\to v$. Clearly, the last rule is

$$
\begin{align*}
\text{either } & \quad t_2 \not\to s \quad \text{or} \quad t_2 \not\to (s \land q_2).
\end{align*}
$$

These two cases may be handled in a similar way. Here we consider only the second alternative. In such situation, we get $v \equiv (s \land p_2) \odot (p_2 \varpi)$ with $t_2 \not\to_F s$, and $u \equiv (s_1 \land p_3) \odot (p_2 \varpi)$ for some $s_1$ and $p_3$ with $s_1 \not\to_F |s_1$ and $p_2 \not\to_F |p_3$. Then it follows from $t_2 \not\subseteq \mathcal{R} S \mathcal{R}$ that there exist $t_3, t_4$ and $p_3$ such that $t_1 \not\to_F t_3 \not\to_F |t_4, p_1 \not\to_F |p_4, s_1 \not\subseteq \mathcal{R} S \mathcal{R} t_4$ and $p_3 \subseteq \mathcal{R} S \mathcal{R} p_4$. Thus

$$
t_1 \odot (p_1 \varpi q_1) \not\to (t_3 \land p_1) \odot (p_1 \varpi q_1) \not\to_F |(t_4 \land p_4) \odot (p_1 \varpi q_1). \quad (5.2.2)
$$

By Lemma 3.11, we also have $s_1 \land p_3 \subseteq \mathcal{R} S \mathcal{R} t_4 \land p_4$. Hence $(u, (t_4 \land p_4) \odot (p_1 \varpi q_1)) \in R$. Moreover, by Lemma 3.9 and 5.6, it follows from $u \equiv (s_1 \land p_3) \odot (p_2 \varpi) \not\in F$ that $(t_4 \land p_4) \odot (p_1 \varpi q_1) \not\in F$. Then $t_1 \odot (p_1 \varpi q_1) \not\to_F |(t_4 \land p_4) \odot (p_1 \varpi q_1)$ due to (5.2.1), (5.2.2) and Lemma 3.6(2). Therefore, the process $(t_4 \land p_4) \odot (p_1 \varpi q_1)$ is exactly one that we seek. \( \square \)

Hitherto we have showed that $\subseteq \mathcal{R} S \mathcal{R}$ is precongruent w.r.t the operators $\land, \land, \odot$ and $\land$. For the remainder operators (i.e., operators in CLLT), such property has been established in [60]. Consequently, $\subseteq \mathcal{R} S \mathcal{R}$ is precongruent w.r.t all operators involved in CLLT.

### 6 Fixed-point characterization of the operator $\varpi$

From now on we make the assumption: the set $\mathcal{A}ct$ is finite. The motivation behind this assumption will be given in Remark 6.1. This section is devoted to a few further properties of the operator $\varpi$ including fixed point characterization and approximation. These properties will serve as a stepping stone in giving a graphical representation of the temporal operator unless in a recursive manner. We begin with introducing some preliminary notions.

**Definition 6.1** Given a finite sequence of processes $< t_0, t_1, \ldots, t_{n-1} >$ with $n \geq 0$, the generalized disjunction $\bigvee_{i<n} t_i$ is defined inductively as

$$
\begin{align*}
(1) \quad & \bigvee_{i<1} t_i = t_0, \\
(2) \quad & \bigvee_{i<k+1} t_i = (\bigvee_{i<k} t_i) \lor t_k \text{ for } k \geq 1.
\end{align*}
$$

Moreover, for any nonempty subset $S \subseteq \{ t_0, \ldots, t_{n-1} \}$, the generalized disjunction $\bigvee S$ is defined as $\bigvee_{i<|S|} t_i$, where the sequence $< t'_0, \ldots, t'_{|S|-1} >$ is the restriction of $< t_0, \ldots, t_{n-1} >$ to $S$. Similar to generalized external choice, modulo $= \mathcal{R} S \mathcal{R}$, the order and grouping of processes in $\bigvee S$ may be ignored due to the commutative and associative laws [43, 60].

Analogously, the notion of a generalized conjunction $\bigwedge S$ is defined in the same manner, and the order and grouping of processes in $\bigwedge S$ may also be ignored by the same reason. It should be pointed out that such generalized conjunction $\bigwedge S$
preserves usual logic laws of the connective conjunction only if \( S \) is finite (see, Remark 6.1). For the sake of simplicity we also introduce the notions below.

**Definition 6.2** Given any process \( p \) and \( t \), \( \delta_{p,t} \) is a function assigning to each visible action a process, which is given by

\[
\delta_{p,t}(a) = \begin{cases} 
\square & \text{if } a \notin I(p) \\
\beta \in I(p) & \text{otherwise}
\end{cases}
\]

for any \( a \in Act \).

Given an action \( a \in Act \), the operator \([a]\) is introduced below. This operator will be used to explore the fixed point characterization of \( p \). Moreover, itself is also of logic meaning, that is, it captures the modal operator “along \( a \) labelled transitions, it is necessary that . . .” in a sense.

**Definition 6.3** For any \( a \in Act \), the operator \([a]\) over processes is defined by

\[
[a] = \lambda X. \left( \bigvee_{a \in A \subseteq Act} \left( (\square b.true) \square a.X \right) \right) \bigvee_{a \notin A \subseteq Act} \left( (\square b.true) \right).
\]

By the way, since \([a] p \nrightarrow \bigvee_{a \notin A \subseteq Act} (\square b.true) \notin F \), it is easy to see that \( [a] p \notin F \) for any \( a \) and \( p \). A simple but useful result is given below.

**Lemma 6.1** \( p \sqsupseteq \bigvee_{RS} a.true \) whenever \( p \nrightarrow \).

**Proof.** Put

\[
R = \left\{ \left\langle q, \bigvee_{a \in I(q)} a.true \right\rangle : q \nrightarrow \right\}^{\text{8}}
\]

We only need to show that \( R \) is a stable ready simulation relation, which is routine and is left to the reader. \( \square \)

In the following, we shall give some basic properties of the operator \([a]\). The theorem below characterizes processes that refine ones with the format \([a]t\).

**Theorem 6.1** \( p \sqsubseteq RS [a] t \) iff \( \forall p_0, p_1 \left( p \nrightarrow F \mid p_0 \nrightarrow F \mid p_1 \text{ implies } p_1 \sqsubseteq RS t \right) \).

**Proof.** (Left implies Right) Assume that \( p \nrightarrow F \mid p_0 \nrightarrow F \mid p_1 \). Then it follows from \( p \sqsubseteq RS [a] t \) that there exists \( r \) such that \( p_0 \sqsubseteq RS r \) and \( [a] t \nrightarrow F \mid r \). Moreover, since \( a \in I(p_0) = I(r) \), we get \( r \equiv \delta_{p_0,t}(a) \). On the other hand, due to \( p_0 \sqsubseteq RS r \) and \( p_0 \nrightarrow F \mid p_1 \), we have \( p_1 \sqsubseteq RS q \) for some \( q \) with \( r \equiv \delta_{p_0,t}(a) \nrightarrow F \mid q \). Further, by Lemma 3.1 (8) and 3.2(1), since \( \delta_{p_0,t}(a) \nrightarrow F \) and \( \left( \bigvee_{b \in I(p_0)} (\square b.true) \right) \nrightarrow F \), we obtain \( a.t \nrightarrow F \mid q \). Hence \( t \nrightarrow F \mid q \). Then \( p_1 \sqsubseteq RS t \) follows from \( p_1 \sqsubseteq RS q \), as desired.

(Right implies Left) Let \( p \nrightarrow F \mid p_0 \). It suffices to prove that \( p_0 \sqsubseteq RS q \) for some \( q \) with \([a] t \nrightarrow F \mid q \). If \( a \notin I(p_0) \), then \([a] t \nrightarrow F \mid b \in I(p_0) \).

\[\text{8} \quad \text{Notice that, if } I(q) = \emptyset \text{ then } a.true \text{ is defined as } 0, \text{ see Def. 3.2.} \]
\[ a \in \{ p_0, \delta_{p_0,t}(a) \} \]; \[ b.true \) due to Lemma 6.1. In the following, we consider another case \[ a \in I(p_0). \]

In such situation, by Lemma 3.5 and 3.8, it follows from \( p_0 \notin F \) that there exists \( p_1 \) such that \( p \Rightarrow_F [p_0] \Rightarrow_F p_1 \). Hence \( p_1 \subseteq_{RS} t \). Then \( t \notin F \) because of \( p_1 \notin F \). Further, by Lemma 3.3 (2)(3)(7), it follows that \( \delta_{p_0,t}(a) \notin F \). Thus, by Def. 6.3 and Lemma 3.6(2), we obtain \([a] t \Rightarrow_F [\delta_{p_0,t}(a)]\). Clearly, in order to complete the proof, it is enough to show that \( p_0 \subseteq_{RS} \delta_{p_0,t}(a) \). To do this, we intend to prove that \( R \) given below is a stable ready simulation relation.

\[ R = \{ [p_0, \delta_{p_0,t}(a)] \} \subseteq_{RS} \]

It is straightforward to verify that \( R \) satisfies (RS1), (RS2) and (RS4). For (RS3), suppose \( p_0 \Rightarrow_F [p_1] \). If \( c \neq a \), we have \( p_1 \subseteq_{RS} \delta_{p_0,t}(a) \) by Lemma 6.1, and \( \delta_{p_0,t}(a) \Rightarrow_F t \). If \( c = a \), then \( \delta_{p_0,t}(a) \Rightarrow_F t \), and it follows from \( p_1 \subseteq_{RS} t \) and \( p_1 \Rightarrow_F [p_1] \) for some \( t_1 \) with \( t \Rightarrow_F t_1 \). Summarizing, we can conclude that there exists \( r \) such that \( p_1 \subseteq_{RS} r \) and \( \delta_{p_0,t}(a) \Rightarrow_F r \). Hence (RS3) holds, as desired. \[ \square \]

**Corollary 6.1** (Monotonicity Law of \([a]\)) If \( t \subseteq_{RS} s \) then \([a] t \subseteq_{RS} [a] s \) for each \( a \in Act \). Hence \( \subseteq_{RS} \) is a precongruence w.r.t the operator \([a] \).

**Proof.** Analogous to that of Corollary 5.2, but using Theorem 6.1 instead of Theorem 5.1. \[ \square \]

Now we are ready to discuss the fixed-point characterization of \( \varpi \). For this purpose, a series of functions \( \eta_{p,q} \) is introduced below.

**Definition 6.4** For any process \( p \) and \( q \), the function \( \eta_{p,q} \) over processes is defined by

\[ \eta_{p,q} = \lambda X. q \lor (p \land (\bigwedge_{a \in Act} [a] X)). \]

Obviously, as all operators involved in \( \eta_{p,q} \) are monotonic w.r.t \( \subseteq_{RS} \), the function \( \eta_{p,q} \) itself is also monotonic. In the following, we intend to show that \( p \varpi q \) is the largest fixed point of \( \eta_{p,q} \). We begin with arguing that \( p \varpi q \) is a post-fixed point of \( \eta_{p,q} \).

**Lemma 6.2** For any process \( p \) and \( q \), \( p \varpi q \subseteq_{RS} \eta_{p,q} (p \varpi q) \).

**Proof.** If \( p \varpi q \in F \) then it holds trivially. In the following, we consider the nontrivial case \( p \varpi q \notin F \). For simplicity of notation, we shall omit the subscript in \( \eta_{p,q} \). Clearly, it is enough to show that, for any process \( v \),

\[ v \subseteq_{RS} p \varpi q \text{ implies } v \subseteq_{RS} \eta(p \varpi q). \]

Assume that \( t \) is any process such that \( t \subseteq_{RS} p \varpi q \). Let \( t \Rightarrow_F [t_0] \). We want to find \( s \) such that \( \eta(p \varpi q) \Rightarrow_F [s] \) and \( t_0 \subseteq_{RS} s \). It proceeds by distinguishing two cases below.

Case 1 \( t_0 \subseteq_{RS} q \).

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Thus $t_0 \sqsubset_{RS} \eta q_0$ for some $q_0$ with $q \Rightarrow_F |q_0$. Easily, $\eta(p\varpi q) \Rightarrow q \Rightarrow |q_0$. Then, by $q_0 \notin F$ and Lemma 3.6 (1)(2), it follows that $\eta(p\varpi q) \Rightarrow q \Rightarrow |q_0$. Hence $q_0$ is indeed the one that we seek.

Case 2 $t_0 \not\sqsubseteq_{RS} q$.

In this case, by Theorem 5.1, it follows from $t \sqsubseteq_{RS} p\varpi q$ and $t \Rightarrow_F |t_0$ that $t_0 \not\sqsubseteq_{RS} p$. Then $t_0 \sqsubseteq_{RS} p_0$ for some $p_0$ with $p \Rightarrow_F |p_0$. The claim below is needed to complete the proof.

Claim 1 $[a]p\varpi q \Rightarrow_F |\delta_{t_0,p\varpi q}(a)$ and $t_0 \sqsubseteq_{RS} \delta_{t_0,p\varpi q}(a)$ for each $a \in Act$.

For any $a \in Act$, due to $p\varpi q \notin F$, it is easy to see that $\delta_{t_0,p\varpi q}(a) \notin F$. Further, by Lemma 3.6 and Def. 6.3, it follows that $[a]p\varpi q \Rightarrow_F |\delta_{t_0,p\varpi q}(a)$. We next prove that $t_0 \sqsubseteq_{RS} \delta_{t_0,p\varpi q}(a)$. By Lemma 6.1, this is immediate whenever $a \notin I(t_0)$. In the following, we consider the case $a \in I(t_0)$. Put

$$R = \{(t_0,\delta_{t_0,p\varpi q}(a))\} \cup \sqsubseteq_{RS} \delta_{t_0,p\varpi q}(a).$$

We want to show that $R$ is a stable ready simulation relation. Since it can be checked without any difficulty that the pair $\langle t_0,\delta_{t_0,p\varpi q}(a) \rangle$ satisfies (RS1), (RS2) and (RS4), we put attention to verify that such pair satisfies (RS3).

Assume $t_0 \not\Rightarrow_F |t_1$. Then $b \in I(t_0)$ because of $t_0 \not\in F$. If $b \neq a$ then $\delta_{t_0,p\varpi q}(a) \not\Rightarrow_F |btrue \Rightarrow_F |ctrue$, and $t_1 \sqsubseteq_{RS} ctrue$ by Lemma 6.1, as desired. We next handle another case $a = b$. In such situation, we get $\delta_{t_0,p\varpi q}(a) \not\Rightarrow_F |p\varpi q$. If $t_1 \sqsubseteq_{RS} q$ then $t_1 \sqsubseteq_{RS} q_1$ for some $q_1$ with $\delta_{t_0,p\varpi q}(a) \not\Rightarrow_F |p\varpi q, q \Rightarrow_F |q_1$. If $t_1 \not\sqsubseteq_{RS} q$ q then, by Corollary 5.1, it follows from $t_0 \not\sqsubseteq_{RS} q, t \sqsubseteq_{RS} p\varpi q$ and $t \Rightarrow_F |t_0 \Rightarrow_F |t_1$ that $t_1 \sqsubseteq_{RS} p\varpi q$, moreover, due to $t_1 \Rightarrow_F |t_1$, we have $t_1 \sqsubseteq_{RS} v$ for some $v$ with $\delta_{t_0,p\varpi q}(a) \not\Rightarrow_F |p\varpi q \Rightarrow_F |v$, as desired.

Now we return to the proof of the lemma. From Claim 1 and $t_0 \sqsubseteq_{RS} p_0$, by Lemma 3.11(2), it follows that

$$t_0 \sqsubseteq_{RS} p_0 \wedge \left( \bigwedge_{a \in Act} \delta_{t_0,p\varpi q}(a) \right). \quad (6.2.1)$$

Moreover, it is obvious that

$$\eta(p\varpi q) \Rightarrow_F |p_0 \wedge \left( \bigwedge_{a \in Act} \delta_{t_0,p\varpi q}(a) \right). \quad (6.2.2)$$

Further, by Lemma 3.6(2), it follows from $t_0 \notin F$, (6.2.1) and (6.2.2) that

$$\eta(p\varpi q) \Rightarrow_F |p_0 \wedge \left( \bigwedge_{a \in Act} \delta_{t_0,p\varpi q}(a) \right).$$

Hence the process $p_0 \wedge \left( \bigwedge_{a \in Act} \delta_{t_0,p\varpi q}(a) \right)$ is exactly one that we seek. \square

We are almost ready now to establish the fixed point characterization of the operator $\varpi$. The following lemma is instrumental in doing this.
Lemma 6.3 For any $k < \omega$, if $t \subseteq_{RS} \eta_{p,q}^{k+1}(u)$, $t \overset{\tau}{\Rightarrow}_F | t_0 \overset{\tau}{\Rightarrow}_F | t_1 \ldots \overset{\tau}{\Rightarrow}_F | t_k$ and $\neg \exists i \leq k(t_i \subseteq_{RS} q)$ then $t_k \subseteq_{RS} w \wedge \left( \bigwedge_{a \in Act} \delta_{w,u}(a) \right)$ for some $w$ with $p \overset{\tau}{\Rightarrow}_F | w$, and hence $t_k \subseteq_{RS} p$.

Proof. Prove it by induction on $k$. For the induction basis $k = 0$, since $t \subseteq_{RS} \eta_{p,q}(u)$, we get $t_0 \subseteq_{RS} r$ for some $r$ with $\eta_{p,q}(u) \overset{\tau}{\Rightarrow}_F | r$. Then it immediately follows from $t_0 \not\subseteq_{RS} q$ that

$$\eta_{p,q}(u) \overset{\tau}{\Rightarrow}_F | p \wedge \left( \bigwedge_{a \in Act} [a] \ u \right) \overset{\tau}{\Rightarrow}_F | r.$$  

Thus $r \equiv w \wedge s$ for some $w$ and $s$ with $p \overset{\tau}{\Rightarrow}_F | w$ and $\bigwedge_{a \in Act} [a] \ u \overset{\tau}{\Rightarrow}_F | s$. Due to $w \wedge s \not\equiv F$ and $w \wedge s \not\overset{\tau}{\Rightarrow}$, we get $I(w) = I(s)$. Further, by Def. 6.3, it is easy to see that $s \equiv \bigwedge_{a \in Act} \delta_{w,u}(a)$, as desired.

For the induction step $k = n + 1$, since $t \subseteq_{RS} \eta_{p,q}^{n+1}(u) = \eta_{p,q}^{n+1}(\eta_{p,q}(u))$, by IH, there exists $w$ such that $p \overset{\tau}{\Rightarrow}_F | w$ and

$$t_n \subseteq_{RS} w \wedge \left( \bigwedge_{a \in Act} \delta_{w,n_{p,q}(u)}(a) \right). \quad (6.3.1)$$

Since $t_n \overset{a_{n+1}}{\Rightarrow}_F | t_{n+1}$ and $t_n \not\overset{\tau}{\Rightarrow}$, we have $a_{n+1} \in I(t_n)$. Then $a_{n+1} \in I(w)$ because of $t_n \not\subseteq F$ and (6.3.1). Hence

$$\delta_{w,n_{p,q}(u)}(a_{n+1}) \equiv \left( \bigwedge_{a \in Act} [a] \ u \right) \overset{\tau}{\Rightarrow}_F | a_{n+1} \cdot \eta_{p,q}(u).$$

By (6.3.1) and $t_n \overset{a_{n+1}}{\Rightarrow}_F | t_{n+1}$, we get $t_{n+1} \subseteq_{RS} r$ for some $r$ with

$$w \wedge \left( \bigwedge_{a \in Act} \delta_{w,n_{p,q}(u)}(a) \right) \overset{a_{n+1}}{\Rightarrow}_F | r.$$  

Further, due to commutative and associative laws of $\wedge$, it is not difficult to see that $r \overset{\approx_{RS}}{\equiv} v \wedge s$ for some $v$ and $s$ with $\delta_{w,n_{p,q}(u)}(a_{n+1}) \overset{a_{n+1}}{\Rightarrow}_F \eta_{p,q}(u) \overset{\tau}{\Rightarrow}_F | s$. Moreover, it follows from $t_{n+1} \subseteq_{RS} r$ and $t_k \equiv t_{n+1} \not\subseteq_{RS} q$ that

$$\eta_{p,q}(u) \overset{\tau}{\Rightarrow}_F | p \wedge \left( \bigwedge_{a \in Act} [a] \ u \right) \overset{\tau}{\Rightarrow}_F | s.$$  

Then, analogous to the induction basis, we have $s \equiv s_1 \wedge s_2$ for some $s_1$ and $s_2$ with $p \overset{\tau}{\Rightarrow}_F | s_1$ and $s_2 \equiv \bigwedge_{a \in Act} \delta_{s_1,u}(a)$. Hence $t_{n+1} \subseteq_{RS} r \overset{\approx_{RS}}{\equiv} v \wedge s \subseteq_{RS} s \equiv s_1 \wedge s_2$, as desired. □

We are thus led to the following strengthening of Lemma 6.2.

Lemma 6.4 For any process $p$ and $q$, $p \equiv q$ is the greatest (w.r.t. $\subseteq_{RS}$) post-fixed point of $\eta_{p,q}$.
The next theorem constitutes one of the two main theorems of this section.

**Theorem 6.2** (Fixed-point characterization of \( \sqcap \)) For any process \( p \) and \( q \), \( p \sqcap q \) is the greatest (w.r.t \( \sqsubseteq_{RS} \)) fixed point of \( \eta_{p,q} \).

**Proof.** By Lemma 6.4 and 6.2, we only need to show that \( \eta_{p,q} (p \sqcap q) \sqsubseteq_{RS} p \sqcap q \). It follows from \( p \sqcap q \sqsubseteq_{RS} \eta_{p,q} (p \sqcap q) \) that \( \eta_{p,q} (p \sqcap q) \sqsubseteq_{RS} \eta_{p,q} (p \sqcap q) \). Then, by Lemma 6.4, we have \( \eta_{p,q} (p \sqcap q) \sqsubseteq_{RS} p \sqcap q \), as desired. □

It is well known that, for any continuous function \( \Phi \) over a complete lattice with the top element \( \top \), its greatest fixed-point is exactly the largest lower bound of the decreasing sequence \( \{ \Phi^i(\top) \}_{i \in \omega} \) (i.e., \( \nu Z. \Phi = \bigcap_{i \in \omega} \Phi^i(\top) \)) (see for instance [21]). The next theorem gives an analogous result for \( p \sqcap q \).

**Theorem 6.3** (Approximation of \( \sqcap \)) For any process \( p \) and \( q \), \( p \sqcap q \) is the greatest lower bound of the decreasing (w.r.t \( \sqsubseteq_{RS} \)) sequence \( \{ \eta_{p,q}(\text{true}) \}_{i \in \omega} \).

**Proof.** Since \( p \sqcap q \sqsubseteq_{RS} \text{true} \), by Lemma 6.2, it is obvious that \( p \sqcap q \) is a lower bound of \( \{ \eta_{p,q}(\text{true}) \}_{i \in \omega} \). Let \( t \) be any lower bound of \( \{ \eta_{p,q}(\text{true}) \}_{i \in \omega} \). We intend to show \( t \sqsubseteq_{RS} p \sqcap q \). Assume that \( t \not\sqsubseteq_{F} \{ t_0 \sqsubseteq_{F} t_1 \ldots \sqsubseteq_{F} t_k \} \) with \( k \geq 0 \) and \( \neg \exists i \leq k (t_i \sqsubseteq_{RS} q) \). Clearly, we have \( t \sqsubseteq_{RS} \eta_{p,q}^{k+1}(\text{true}) \). Then \( t_k \sqsubseteq_{RS} p \) due to Lemma 6.3. Consequently, \( t \sqsubseteq_{RS} p \sqcap q \) follows from Theorem 5.1. □

**Remark 6.1** It has been established in [43, 60] (see also Lemma 3.11 in this paper) that, for any process \( q \), \( p_{1} \text{ and } p_{2} \), (i) \( p_{1} \land p_{2} \sqsubseteq_{RS} p_{i} \) (i = 1, 2) and (ii) if \( q \sqsubseteq_{RS} p_{1} \text{ and } q \sqsubseteq_{RS} p_{2} \) then \( q \sqsubseteq_{RS} p_{1} \land p_{2} \). That is, \( p_{1} \land p_{2} \) is the largest lower bound of \( \{ p_{1}, p_{2} \} \) w.r.t \( \sqsubseteq_{RS} \). Inspired by this, someone may try to introduce the notion of generalized conjunction in a natural way to express the largest lower bound of \( \{ \eta_{p,q}(\text{true}) \}_{i \in \omega} \) by the term \( \bigwedge_{i \in \omega} \eta_{p,q}(\text{true}) \). The rule below is one of potential candidate rules that generalize the rules (Ra-7) and (Ra-8) to the generalized conjunction.

\[
\frac{p_{k} \rightarrow t_{i}}{\bigwedge_{i \in I} p_{i} \rightarrow \bigwedge_{i \in I} t_{i}} \quad \text{with } k \in I.
\]

\((GC)\)

Here \( I \) is an arbitrary indexed set, and for \( i \in I \), if \( i \neq k \) then \( t_{i} \equiv p_{i} \) else \( t_{i} \equiv t_{k} \). Unfortunately, it would be an unsuccessful attempt if the rule \((GC)\) is adopted as the only rule concerning \( \tau \)-transition for such generalized conjunction. By this rule, \( \bigwedge_{i \in \omega} \eta_{p,q}(\text{true}) \) can not arrive at any stable state within finitely many \( \tau \)-transitions. Thus \( \bigwedge_{i \in \omega} \eta_{p,q}(\text{true}) \) is inconsistent and \( \bigwedge_{i \in \omega} \eta_{p,q}(\text{true}) =_{RS} \bot \). In fact, the conjunction \( \land \) in the framework of LLTS can not be generalized in the above manner to capture the generalized conjunction in usual logics. For instance, by \((GC)\), it is easy to see that the (generalized) idempotent law \( \bigwedge_{i \in I} p =_{RS} p \) does not
always hold, e.g., consider \( p_i \equiv a.0 \lor b.0 \) with \( i \in \omega \), then we have \( \bigwedge_{i \in \omega} p_i \in F \) but \( a.0 \lor b.0 \not\in F \), and hence \( \bigwedge_{i \in \omega} p_i \not\equiv_{RS} a.0 \lor b.0 \). By the way, since the definition of the function \( \eta_{p,q} \) refers to the term \( \bigwedge_{a \in \Act} [a] X \), we assume that \( \Act \) is finite in this and the next two sections.

Analogous to [44], some basic laws concerning \( \sharp \) and \( \lceil a \rceil \) are listed below, which reveals that a few of standard temporal laws hold in CLLT.

**Corollary 6.2** For any process \( p \) and \( q \), we have

1. \( \lceil a \rceil \text{true} =_{RS} \text{true} =_{RS} \bigvee_{A \subseteq \Act} \left( \Box_a \text{true} \right) \)
2. \( \lceil a \rceil (p \land q) =_{RS} \lceil a \rceil p \land \lceil a \rceil q \)
3. \( \sharp(p \land q) =_{RS} \sharp p \land \sharp q \)
4. \( \sharp p =_{RS} p \text{true} \)
5. \( \sharp p =_{RS} p \land \left( \bigwedge_{a \in \Act} [a] \sharp p \right) \)
6. \( \text{true} \land p =_{RS} p \land \text{true} =_{RS} p \)

**Proof.** (1) Obvious. (2) is implied by Lemma 3.11(3) and (4), Theorem 6.1 and Corollary 6.1. (3) follows from Lemma 3.11(3) and (4), Theorem 4.1 and Corollary 4.1. (4) It is enough to show that \( t \sqsubseteq_{RS} \sharp p \) iff \( t \sqsubseteq_{RS} p \text{true} \) for any \( t \), which is implied by Theorem 4.1 and 5.1. (5) follows from the item (4) in this lemma and Theorem 6.2. (6) is implied by \( p \sqsubseteq_{RS} \text{true} \) and Lemma 3.11(3)(4). □

As an easy consequence, we also obtain the following fixed point characterization and approximation of \( \sharp \).

**Corollary 6.3** (Fixed-point characterization of \( \sharp \))

1. \( \lceil \sharp p \rceil =_{RS} \eta_{p,\bot}(\lceil \sharp p \rceil) \).
2. \( \lceil \sharp p \rceil \) is the greatest (post-)fixed point of \( \eta_{p,\bot} \).
3. \( \lceil \sharp p \rceil \) is the greatest lower bound of the decreasing sequence \( \{ \eta_{p,\bot}(\text{true}) \}_{i \in \omega} \).

**Proof.** Follows from Theorem 6.2 and 6.3 and Corollary 6.2 (4). □

We conclude this section with providing some sound inference rules concerning \( \ominus \) w.r.t \( \sqsubseteq_{RS} \). As an immediate consequence of Lemma 6.4 and Theorem 6.3, it is obvious that the rules below are sound provided that \( \leq \) is interpreted as \( \sqsubseteq_{RS} \). Moreover, by Corollary 6.3, similar rules also exist for \( \lceil \sharp p \rceil \).

\[
\begin{align*}
t \leq \eta_{p,q}(t) \quad &\text{(GPF)} \\
t \leq p \ominus q \quad &\text{(APP)}
\end{align*}
\]

Clearly, since the premise in (APP) may be proved by induction on natural numbers, we also have the rule below. Notice that, since it always holds that \( t \sqsubseteq_{RS} \text{true} \equiv_{RS} \eta^0_{p,q}(\text{true}) \), the premise \( t \leq \eta^0_{p,q}(\text{true}) \) in (INAPP) may be omitted.

\[
\begin{align*}
t \leq \eta^0_{p,q}(\text{true}), \forall i < \omega(t \leq \eta^i_{p,q}(\text{true}) \rightarrow t \leq \eta^{i+1}_{p,q}(\text{true})) \quad &\text{(INAPP)}
\end{align*}
\]
7 Graphically representing \textit{unless} by recursion

In the light of the greatest fixed-point characterization of \( \varpi \), this section will consider an alternative approach to giving a graphical representation of the temporal operator \textit{unless} in pure process-algebraic style. Following Milner [45], for any process \( p \) and \( q \), we introduce the constant \( \mathcal{F}_{p\varpi q} \), which is defined by the equation below

\[
\mathcal{F}_{p\varpi q} = \eta_{p,q}(\mathcal{F}_{p\varpi q}).
\]

Formally, two rules below are added into \textit{CLLT}, which are usual rules about recursion.

\[
(Ra_{\eta}) \quad \eta_{p,q}(\mathcal{F}_{p\varpi q}) \xrightarrow{\alpha} t \quad \quad (Rp_{\eta}) \quad \eta_{p,q}(\mathcal{F}_{p\varpi q})F \xrightarrow{\tau} t
\]

The resulting calculus is denoted by \textit{CLLT}_{\eta}. \textit{CLLT}_{\eta} inherits the notion of the degree of a process (see, Def. 3.3) with adding the clause \( |\mathcal{F}_{p\varpi q}| = 1 \) for each \( p \) and \( q \). Then it is easy to check that the function \( S_{\eta} \) is a stratification of \textit{CLLT}_{\eta}, where

\[
S_{\eta}(t) = G(t) \times \omega + |t| \quad \text{for any literal } t \xrightarrow{\alpha} r, \text{ and}
\]

\[
S_{\eta}(tF) = \omega \times 2 \quad \text{for any process } t.
\]

Here \( G(t) \) is the number of \textit{unguarded} occurrences of constants with the format \( \tau\varpi s \) in \( t \). For instance, \( G(p\varpi q \tau\varpi r) = 2 \) and \( G(p\varpi q \tau\varpi r \lor s) = 0 \). Obviously, the function \( G \) can be defined inductively, and we leave it to the reader.

Therefore \textit{CLLT}_{\eta} has a unique stable transition model, and the LTS associated with \textit{CLLT}_{\eta}, denoted by \textit{LTS(CLLT}_{\eta}), may be defined as usual. Moreover, all results obtained in Subsection 3.3 still hold for \textit{LTS(CLLT}_{\eta}) and will be used in the remainder of this section. Here we do not verify them in full detail and only illustrate that \textit{LTS(CLLT}_{\eta}) is a LLTS. To this end, the notion of \( \tau-\text{degree} \) (see, Def. 3.6) is enriched by adding the clause below, for any process \( p \) and \( q \),

\[
d(\mathcal{F}_{p\varpi q}) = \max\{d(q), d(p \land \bigwedge_{a \in \text{Act}} [a] \mathcal{F}_{p\varpi q})\} + 1.
\]

Clearly, \( \tau \) is the only action enabled from \( \mathcal{F}_{p\varpi q} \) and the target state of such \( \tau \)-transition is either \( q \) or \( p \land \bigwedge_{a \in \text{Act}} [a] \mathcal{F}_{p\varpi q} \). Thus the above clause also appropriately measures \( \mathcal{F}_{p\varpi q} \)'s capability of executing successive \( \tau \) actions. Moreover, since \( d(a,\mathcal{F}_{p\varpi q}) = 0 \) for each \( a \in \text{Act} \) (see also, Def. 3.6), the definition above is well defined.

By (Ra_{\eta}) and Def. 6.4, it is obvious that Lemma 3.7 still holds for \( \mathcal{F}_{p\varpi q} \). Then, analogous to Lemma 3.8, we can prove that the condition (LTS2) holds for \textit{LTS(CLLT}_{\eta}). Moreover, by (Ra_{\eta}) and (Rp_{\eta}), it can be showed without any difficulty that \textit{LTS(CLLT}_{\eta}) is \( \tau \)-pure and satisfies (LTS1). Summarizing, \textit{LTS(CLLT}_{\eta}) is a \( \tau \)-pure LLTS.

As mentioned above, this section aims to capture the temporal operator \textit{unless} in the recursive manner. Thus we need to show an analogue of Theorem 5.1 for any constant \( \mathcal{F}_{p\varpi q} \). We do not intend to prove such result from scratch. The remaining work will attend to proving that \( \mathcal{F}_{p\varpi q} \) is equivalent to \( pq \varpi q \) modulo \( =_{RS} \), which implies one that we desire.

9 Notice that, since the ‘first move’ of \( r \lor s \) is independent of \( r \) and \( s \), the occurrence of \( r \) and \( s \) are (weakly) guarded in \( r \lor s \).
Although the equivalence between $\text{p} \text{w} q$ and $p \text{w} q$ seems straightforward, its proof is far from trivial and requires a solid effort. In fact, if we neglect the requirement on the consistency in the notion of ready simulation (see, Def. 2.2), it is trivial to show that $\text{p} \text{w} q$ and $p \text{w} q$ are matching on actions. However, everything becomes quite troublesome when the predicate $F$ is involved. The main difficulty in carrying out such proof is that we need to prove that $\text{p} \text{w} q \in F$ implies $p \text{w} q \in F$. This requires a sequence of auxiliary propositions about proof trees. Before giving these propositions, we introduce the notion below.

**Definition 7.1** Given processes $p_i$ with $i \leq n$, a process $u$ is said to be a conjunction of these $p_i$ if each $p_i$ occurs in $u$ and $u$ is obtained from these $p_i$ by using only the operator $\land$ in arbitrary order and grouping. Similarly, we can define the analogous notion for disjunction.

**Lemma 7.1** Given processes $p_i$ and $p^*_j$ such that $p_i \overset{r}{\Rightarrow}_F p^*_j$ with $i \leq n$, and let $p$ be a conjunction of these $p_i$. If $\exists$ is a proof tree of $pF$ then there exists a nonempty set $K \subseteq \{0, 1, 2 \ldots n\}$ such that $\exists$ contains a subtree with the root labelled with $wF$, in particular, such subtree is proper provided that $p_{i_0} \overset{r}{\Rightarrow} p^*_{i_0}$ for some $i_0 \leq n$, where $w$ is a conjunction of $p^*_i$ with $i \in K$.

**Proof.** The proof will be done by induction on the depth of inference by which $pF$ is inferred. We denote $[p(p^*_1/p_1 \cdots p_n/p_n)]$ briefly by $p^*$. Then $p(\overset{r}{\Rightarrow})^m p^*$ for some $m$. If $m = 0$ then the conclusion holds trivially due to $p^* \equiv p$. Next we consider the case $m > 0$. Then $p \overset{r}{\Rightarrow} s (\overset{r}{\Rightarrow})^m-1 p^*$ for some $s$. Moreover, since $p_i \not\in F$ for each $i$ and $p \in F$, we get $n > 0$. Hence $p \equiv w_1 \land w_2$ for some $w_1$ and $w_2$. Thus the last rule applied in $\exists$ is

\[
\text{either } w_i F \text{ with } i \in \{1, 2\} \text{ or } \frac{p \overset{r}{\Rightarrow} u, \{tF : p \overset{r}{\Rightarrow} t\}}{pF}.
\]

For the first alternative, w.l.o.g, we assume $i = 1$. Then $\exists$ contains a proper subtree $\exists_1$ with the root labelled with $w_1 F$. Clearly, there exists a nonempty set $N \subseteq \{0, 1, 2 \ldots n\}$ such that $w_1$ is a conjunction of $p_i$ with $i \in N$. Thus, by IH, it follows that there exists a nonempty set $K \subseteq N \subseteq \{0, 1, 2 \ldots n\}$ such that $\exists_1$ contains a node labelled with $wF$, where $w$ is a conjunction of all $p^*_i$ with $i \in K$.

For the second alternative, since $p \overset{r}{\Rightarrow}$, there exists $k \leq n$ and $p_k'$ such that $p_k \overset{r}{\Rightarrow} p_k' \overset{r}{\Rightarrow} p^*_k$. Then $\exists$ contains a proper subtree $\exists_1$ with the root labelled with $sF$, where $s$ is a conjunction of $p_k'$ and $p_i$ with $k \neq i \leq n$. Further, by IH, it follows that there exists a nonempty set $K \subseteq \{0, 1, 2 \ldots n\}$ such that $\exists_1$ contains a node labelled with $wF$ for some conjunction $w$ of all $p^*_i$ with $i \in K$. □

**Lemma 7.2** For any nonempty $A \subseteq \text{Act}$ and processes $r$ and $t$, let $p$ be any conjunction of $\delta_{r,t}(a)$ with $a \in A$, then each proof tree of $pF$ must contain a proper subtree with the root labelled with $uF$, where $u \equiv t$ or $u$ is a conjunction of $t$ and true.

**Proof.** Prove it by induction on the depth of inference. Let $\exists$ be any proof tree of $pF$. Since $\text{Act}$ is finite, so is $A$. If $|A| = 1$ then $p \equiv \delta_{r,t}(a)$ for some $a$. Hence it follows from $\delta_{r,t}(a) \in F$ that $a \in I(r)$ and $\delta_{r,t}(a) \equiv (b \in I(r) \rightarrow \square \ b \text{true}) \square a.t$. Moreover, since $b \in I(r) \rightarrow \square \ b \text{true} \notin F$, it is easy to see that $\exists$ contains a proper subtree with the root labelled with $tF$. In the following, we consider the case where $|A| > 1$. In such situation, $p \equiv p_1 \land p_2$ for some $p_1$, $p_2$. Since $I(p_1) = I(p_2) = I(r)$, the last rule applied in $\exists$ is
either $\frac{p_kF}{pF}$ with $k \in \{1, 2\}$ or $\frac{p \rightarrow s, \left\{ wF : p \rightarrow w \right\}}{pF}$ for some $\alpha$.

The proof for the first case is immediate by applying IH. For the second one, we must have $\alpha \neq \tau$, for otherwise it immediately follows that $\{\tau\} = I(r)$ and $p =RS$ true $\not\in F$, a contradiction. Moreover, we also have $\alpha \in A$, for otherwise a contradiction arises as $p \rightarrow p' =RS$ true $\not\in F$ for some $p'$. Then it follows from $\delta_{r,t}(wF \rightarrow t)$ and $\delta_{r,t}(b) \rightarrow$ true with $b(\neq \alpha) \in A$ that the only $\alpha$--labelled transition from $p$ is $p \rightarrow u \equiv p[t/\delta_{r,t}(\alpha), \text{true}/\delta_{r,t}(b_1), \ldots, \text{true}/\delta_{r,t}(b_n)]$ with $\{b_1, \ldots, b_n\} = A - \{\alpha\}$. Clearly, $u$ is a conjunction of $t$ and a number of true, and $\exists$ contains a proper subtree with the root labelled with $uF$. □

By the lemma above, it is obvious that $\bigwedge_{a \in A} \delta_{r,t}(a) \in F$ implies $t \in F$. In fact, the converse also holds if $I(r) \cap A \neq \emptyset$. The result below is analogous to the well-known fact that the sentence $\bigwedge_{i \leq n} (\bigvee \beta_{i,j})$ is inconsistent in classical logics if and only if, for any set $\{\beta_{j_0}, \beta_{j_1}, \ldots, \beta_{j_n}\}$ with $j_i \leq m_i$ for each $i \leq n$, there exists a nonempty set $N \subseteq \{0, 1, 2, \ldots, n\}$ such that $\bigwedge_{k \in N} \beta_{k_j}$ is inconsistent.

**Lemma 7.3** Assume that $p$ is a conjunction of $p_i$ with $0 \leq i \leq n$ and for any $i \leq n$, there exist $p_{j_i}$ with $j_i \leq m_i$ such that $p_i$ is a disjunction of $p_{j_i}$.\footnote{Notice that if $m_i = 0$ then $p_i \equiv p_{j_0}$.} Then, for any proof tree $\exists$ of $pF$ and $n + 1$--tuple $\overrightarrow{p_{ik}}$ such that $\forall i \leq n(k_i \leq m_i)$, there exists a nonempty set $K \subseteq \{0, 1, 2, \ldots, n\}$ such that $\exists$ contains a subtree with the root labelled with $wF$ for some conjunction $w$ of $p_{ik}$ with $i \in K$, in particular, such subtree is proper whenever $\exists i \leq n(m_i > 0)$.

**Proof.** Proceeding by induction on the depth of $\exists$. Suppose that $\overrightarrow{p_{ik}}$ is any $n + 1$--tuple such that $\forall i \leq n(k_i \leq m_i)$. If $m_i = 0$ for each $i \leq n$ then there exists exactly one such $n + 1$--tuple and $p$ is a conjunction of $p_{ik}$. Hence the conclusion holds trivially. In the following, we consider the case where $\exists i \leq n(m_i > 0)$.

If $n = 0$ then $p \equiv p_0$, and hence $p$ is a disjunction of $p_{j_0}$ with $j_0 \leq m_0$. Moreover, due to $m_0 > 0$, it is obvious that $\exists$ contains a proper subtree with the root labelled with $pF$. We next consider the case where $n > 0$. In such situation, we may assume that $p \equiv w_1 \wedge w_2$ for some $w_1$ and $w_2$. Moreover, it is not difficult to see that the last rule applied in $\exists$ is

\[
\text{either } \frac{w_iF}{pF} \text{ with } i \in \{1, 2\} \text{ or } \frac{p \rightarrow s, \left\{ wF : p \rightarrow w \right\}}{pF}.
\]

For the first alternative, w.l.o.g, we assume $i = 1$. Thus $\exists$ contains a proper subtree $\exists_1$ with the root labelled with $w_1F$. Clearly, there exists a nonempty set $N \subseteq \{0, 1, 2, \ldots, n\}$ such that $w_1$ is a conjunction of $p_i$ with $i \in N$. For $|N|$--tuple $\overrightarrow{p_{ik}}$ with $i \in N$, by IH, there exists a nonempty set $K \subseteq N \subseteq \{0, 1, 2, \ldots, n\}$ such that $\exists_1$ contains a node labelled with $wF$, where $w$ is a conjunction of $p_{ik}$ with $i \in K$, as desired.

For the second alternative, it follows from $\exists i \leq n(m_i > 0)$ that $p \rightarrow s \Rightarrow p[\overrightarrow{p_{ik}}/\overrightarrow{p_i}]$ for some $s$. Thus $\exists$ contains a proper subtree $\exists_1$ with the root labelled with $sF$. Obviously, for some $j_0 \leq n$ and $p_{j_0}$ with $p_{j_0} \rightarrow p_{j_0}$, $s$ is a conjunction of $p_{j_0}$ and $p_i$ with $j_0 \neq i \leq n$. Moreover, there exists a nonempty set $N \subseteq \{0, 1, 2, \ldots, m_{j_0}\}$ such that $p_{j_0}$ is a disjunction of $p_{j_0i}$ with $i \in N$. In particular, $p_{j_0k_{j_0}} \equiv p_{j_0l}$ for some $l \in N$ due to $p \rightarrow s \Rightarrow p[\overrightarrow{p_{ik}}/\overrightarrow{p_i}]$. Then, by IH, there exists a nonempty
set \( K \subseteq \{0, 1, 2, \ldots, n\} \) such that \( \exists \) contains a node labelled with \( wF \), where \( w \) is a conjunction of \( p_{ik} \), with \( i \in K \). □

Now we are ready to show that \( p_\sigma q \in F \) implies \( p_\sigma q \in F \) by induction on inference. The lemma below contains four assertions which state the links between consistency of some processes in the transition system generated by \( p_\sigma q \) and consistency of corresponding processes in the transition system generated by \( p_\sigma q \).

Lemma 7.4 Assume that \( u \in F \). Then

1. If \( u \equiv p_\sigma q \) or \( u \) is a conjunction of \( p_\sigma q \) and a number of \emph{true} then \( p_\sigma q \in F \).
2. If \( u \) is a conjunction of \( p_\land \bigwedge_{a \in Act} [a] p_\sigma q \) and \( p_i \) with \( i < n \) then \( t \circ (p_\sigma q) \in F \) for any conjunction \( t \) of \( p \) and \( p_i \)'s.
3. If \( u \) is a conjunction of \( \delta_{p_\sigma q}(a) \) and stable \( p_i \) with \( a \in A \) and \( i \leq n \), then \( t \circ (p_\sigma q) \in F \) for any conjunction \( t \) of \( p_i \)'s, where \( \emptyset \neq A \subseteq Act \).
4. If \( u \) is a conjunction of \( p_i \) with \( i \leq n \), \( p_\sigma q \) and a number of \emph{true} then \( (t \land p) \circ (p_\sigma q) \in F \) for any conjunction \( t \) of \( p_i \)'s.

Proof. Let \( \exists \) be any proof tree of \( uF \). We will prove item (1)-(4) simultaneously by induction on the depth of \( \exists \). The argument splits into five cases based on the format of \( u \).

Case 1 \( u \equiv p_\sigma q \).

It is obvious that the last two inference steps in \( \exists \) are

\[
\begin{align*}
qF, \ (p_\land \bigwedge_{a \in Act} [a] p_\sigma q)F & \\
\eta_{p,q}(p_\sigma q)F & \quad (p_\sigma q)F
\end{align*}
\]

Thus \( q \in F \). Moreover, by IH about item (2) with \( n = 0 \), we also get \( p \circ (p_\sigma q) \in F \). Then \( p_\sigma q \in F \) by Lemma 3.3 (5).

Case 2 \( u \) is a conjunction of \( p_\sigma q \) and \( p_i \) \((\equiv \text{true})\) with \( i \leq n \).

In this situation, we may assume that \( u \equiv u_1 \land u_2 \). Since \( u \vdash r \), the last rule applied in \( \exists \) is

\[
\begin{align*}
\text{either} & \quad u_1F \quad \text{with} \quad i \in \{1, 2\} \quad \text{or} \\
\quad \text{or} \quad u \vdash w, \ \{rF : u \vdash r\} \quad \quad \text{if} \quad uF
\end{align*}
\]

For the first alternative, w.l.o.g, we assume \( i = 1 \). Since \( u_1 \in F \), the process \( p_\sigma q \) must occur in \( u_1 \). So, by IH about item (1), we have \( p \circ (p_\sigma q) \in F \), as desired.

For the second alternative, since \( p_\sigma q \vdash q \) and \( p_\sigma q \vdash p_\land \bigwedge_{a \in Act} [a] p_\sigma q \), there exist two proper subtrees of \( \exists \) whose roots are labelled with \( v_1F \) and \( v_2F \) respectively, where \( v_1 \) (or, \( v_2 \)) is a conjunction of \( q \) (respectively, \( p_\land \bigwedge_{a \in Act} [a] p_\sigma q \)) and \( p_i \)'s. Then \( q \in F \) by Corollary 6.2(6). Moreover, by IH about item (2), Corollary 6.2(6) and Theorem 5.2, we also get \( p \circ (p_\sigma q) \in F \). Hence \( p_\sigma q \in F \).

Case 3 \( u \) is a conjunction of \( p_\land \bigwedge_{a \in Act} [a] p_\sigma q \) and \( p_i \) with \( i < n \).
Let $t$ be any conjunction of $p$ and $p_i$’s. If $t \in F$ then it immediately follows that $t \circ (p \bowtie q) \in F$. In the following, we consider the nontrivial case where $t \notin F$. If it were true that $s \circ (p \bowtie q) \in F$ for any $s$ with $t \xrightarrow{F} s$, we would have $t \circ (p \bowtie q) \in F$ by Lemma 3.1(7), 3.3(6) and 3.8. Thus we assume that $t \nrightarrow F$ and intend to prove $t \circ (p \bowtie q) \in F$. Clearly, $\bigwedge_{a \in Act} [a] p \bowtie q \notin F$\footnote{This follows from $\bigwedge_{a \in Act} [a] p \bowtie q \xrightarrow{F} \bigwedge_{a \in Act} 0 \notin F$ and Lemma 3.6.} and there exist $p^*$ and $w_i$ ($i < n$) with properties below:

$t_0$ is a conjunction of $p^*$ and $w_i$, $p \xrightarrow{F} p^*$ and $p_i \xrightarrow{F} w_i$ for each $i < n$.

Then, by Lemma 7.1, there is a nonempty set $\Gamma \subseteq \{p^*, w_i, \bigwedge_{a \in Act} [a] p \bowtie q : i < n\}$ such that $\exists$ contains a node labelled with $wF$, where $w$ is a conjunction of processes within $\Gamma$. Moreover, due to $\bigwedge_{a \in Act} [a] p \bowtie q \notin F$ and $t_0 \notin F$, we have $\bigwedge_{a \in Act} [a] p \bowtie q \in \Gamma$ and $\Gamma \cap \{p^*, w_i : i < n\} \neq \emptyset$.

On the other hand, by Def. 6.3, for each $a \in Act$, the process $[a] p \bowtie q$ is a disjunction of processes $S_A$ with $A \subseteq Act$, where

$$S_A \equiv \begin{cases} 
\square \text{b.true} & \text{if } a \notin A \\
(\square \text{b.true}) \square a.p \bowtie q & \text{otherwise}
\end{cases}$$

In particular, by setting $A = I(p^*)$ for each $a \in Act$, we get a tuple $\overline{\delta_{p^*, p \bowtie q}}(a)$ with $a \in Act$. Moreover, each process in $\Gamma \cap \{p^*, w_i : i < n\}$ may be regarded as a disjunction of itself. Thus, by Lemma 7.3, there exists a nonempty set $\Theta \subseteq (\Gamma \cap \{p^*, w_i : i < n\}) \cup \{\delta_{p^*, p \bowtie q}(a) : a \in Act\}$ such that $\exists$ contains a proper subtree $\exists_1$ with the root labelled with $sF$ for some conjunction $s$ of all processes in $\Theta$. Due to $t_0 \notin F$, $\Theta$ must contain $\delta_{p^*, p \bowtie q}(a)$ for some $a \in Act$. We distinguish two cases below.

Case 3.1 $\Theta \subseteq \{\delta_{p^*, p \bowtie q}(a) : a \in Act\}$.

Then $s$ is a conjunction of some processes with the format $\delta_{p^*, p \bowtie q}(a)$. By Lemma 7.2, $\exists_1$ contains a proper subtree with the root labelled with $rF$, where either $r \equiv p \bowtie q$ or $r$ is a conjunction of $p \bowtie q$ and a number of $\text{true}$. So, by IH about item (1), we have $p \bowtie q \in F$. Hence $p \circ (p \bowtie q) \in F$ by Lemma 3.3(5). Moreover, by Lemma 3.6 (2), it follows from $p \circ (p \bowtie q) \xrightarrow{F} s \circ (p \bowtie q)$ that $p^* \circ (p \bowtie q) \in F$. Further, by Theorem 5.2 and $t_0 \sqsubseteq_{RS} p^*$, we get $t_0 \circ (p \bowtie q) \in F$, as desired.

Case 3.2 $\Theta \not\subseteq \{\delta_{p^*, p \bowtie q}(a) : a \in Act\}$.

In such situation, $\Theta \cap \{p^*, w_i : i < n\} \neq \emptyset$. Let $t_1$ be any conjunction of processes within $\Theta \cap \{p^*, w_i : i < n\}$. Then $t_1 \circ (p \bowtie q) \in F$ due to IH about item (3)\footnote{Notice that, due to $t_0 \notin F$, $I(p^*) = I(w_i)$ for each $i < n$. Hence, for any $i < n$ and $a \in Act$, $\delta_{p^*, p \bowtie q}(a)$ is identical with $\delta_{w_i, p \bowtie q}(a)$.}. Further, by Theorem 5.2 and $t_0 \sqsubseteq_{RS} t_1$, we get $t_0 \circ (p \bowtie q) \in F$.

Case 4 $u$ is a conjunction of stable processes $p_i$ and $\delta_{po, p \bowtie q}(a)$ with $i \leq n$ and $a \in A \neq \emptyset$. 

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Let $t$ be any conjunction of $p_i$ with $i \leq n$. Clearly, $u = u_1 \land u_2$ for some $u_1$ and $u_2$. In the following, we consider only the nontrivial case $t \notin F$. In such situation, it is obvious that all $p_i$ have the same ready set. Since $u$ is stable and $I(u_1) = I(u_2) = I(p_i)$ for each $i \leq n$, we may distinguish two cases based on the last rule applied in $\Xi$.

Case 4.1 $\frac{u_i \in F}{u_i \in F}$ with $i \in \{1, 2\}$.

W.l.o.g, we assume $i = 1$. Since $t \notin F$, $\delta_{p_i, \overline{v}\overline{w}q}(a)$ occurs in $u_1$ for some $a \in A$. If $u_1$ also contains $p_i$ for some $i \leq n$ then, by IH about item (3), we have $t \circ (p\overline{w}q) \in F$, where $t_1$ is any conjunction of all $p_i$ occurring in $u_1$. By $t \subseteq_R t_1$ and Theorem 5.2, we have $t \circ (p\overline{w}q) \subseteq_R t_1 \circ (p\overline{w}q)$. Then it follows from $t_1 \circ (p\overline{w}q) \in F$ that $t \circ (p\overline{w}q) \in F$, as desired.

Next we consider another case where none of $p_i$ ($i \leq n$) occurs in $u_1$. Then there exists a nonempty set $B \subseteq A$ such that $u_1$ is a conjunction of $\delta_{p_i, \overline{v}\overline{w}q}(a)$ with $a \in B$. Thus, by Lemma 7.2, $\Xi$ contains a proper subtree with the root labelled with $wF$, where $w$ is either $\overline{v}\overline{w}q$ or a conjunction of $\overline{v}\overline{w}q$ and a number of $\text{true}$. Hence $p\overline{w}q \in F$ due to IH about item (1). Further, by Lemma 3.3(5), we obtain

$$q \in F \land p \circ (p\overline{w}q) \in F.$$ 

On the other hand, since $u_1$ is a conjunction of $\delta_{p_i, \overline{v}\overline{w}q}(a)$ with $a \in B$, we must have $I(p_0) \neq \emptyset$, for otherwise a contradiction arises due to $u_1 =_{RS} \bigwedge_{a \in B} \delta_{p_i, \overline{v}\overline{w}q}(a) \equiv \bigwedge_{a \in B} 0 \notin F$. Let $b$ be any action in $I(p_0)$. Since all $p_i$ have the same ready set and $t$ is a conjunction of $p_i$'s, we have $b \in I(t) = I(t \circ (p\overline{w}q))$. In order to prove that $t \circ (p\overline{w}q) \in F$, by Lemma 3.5, it is enough to show that $v \in F$ for each $v$ with $t \circ (p\overline{w}q) \xrightarrow{b} v$. Let $r$ be any target state of $b$-labelled transitions from $t \circ (p\overline{w}q)$. Then $r \equiv t_1 \land q$ or $r \equiv (t_1 \land p) \circ (p\overline{w}q)$ for some $t_1$ with $t \xrightarrow{b} t_1$. For the former, it follows from $q \in F$ that $r \in F$. For the latter, by Lemma 3.11(3) and Theorem 5.2, we have $r \equiv (t_1 \land p) \circ (p\overline{w}q) \subseteq_R p \circ (p\overline{w}q)$, and hence $r \in F$ because of $p \circ (p\overline{w}q) \in F$.

$$u \xrightarrow{b} s, \left\{rf : u \xrightarrow{b} r\right\}_{RF \in F}$$

for some $b \in \text{Act}$.

Since $u$ is a conjunction of $p_i$ and $\delta_{p_i, \overline{v}\overline{w}q}(a)$ with $i \leq n$ and $a \in A$, we get $b \in I(p_i)$ for each $i \leq n$. Thus $b \in I(t) = I(t \circ (p\overline{w}q))$. Analogous to Case 4.1, in order to prove that $t \circ (p\overline{w}q) \in F$, it is enough to show that each $b$-derivative of $t \circ (p\overline{w}q)$ is inconsistent. Let $r$ be any process such that $t \circ (p\overline{w}q) \xrightarrow{b} r$. Then $r \equiv t_1 \land q$ or $r \equiv (t_1 \land p) \circ (p\overline{w}q)$ for some $t_1$ with $t \xrightarrow{b} t_1$. In the following, we intend to prove that both $t_1 \land q$ and $(t_1 \land p) \circ (p\overline{w}q)$ are inconsistent.

We first prove that $b \in A$. On the contrary, suppose that $b \notin A$. Hence $\delta_{p_i, \overline{v}\overline{w}q}(a) \xrightarrow{b} \text{true}$ for each $a \in A$. Moreover, since $\bigwedge_{i \leq n} p_i =_{RS} t \notin F$, by Lemma 3.5, $\bigwedge_{i \leq n} p_i \xrightarrow{b} s$ for some $s \notin F$. Then a contradiction arises as $s =_{RS} r$ for some $r$ with $u \xrightarrow{b} r$.

Since $t \xrightarrow{b} t_1$, there exist $w_i$ with $p_i \xrightarrow{b} w_i$ for each $i \leq n$ and $t_1$ is a conjunction of these $w_i$. Moreover, since $b \in A$ and $u$ is a conjunction of $p_i$ and $\delta_{p_i, \overline{v}\overline{w}q}(a)$ with $i \leq n$ and $a \in A$, there is a process $w$ such that $u \xrightarrow{b} w$ and $w$ is a conjunction of $w_i$ with $i \leq n, \overline{v}\overline{w}q$ and a number of $\text{true}$. Thus $\Xi$ contains a proper subtree with the root labelled with $wF$. Hence $(t_1 \land p) \circ (p\overline{w}q) \in F$ due to IH about item (4).
On the other hand, since \( \overline{pq} \rightarrow q \), there exists \( v \) such that \( w \rightarrow v \) and \( v \) is a conjunction of \( w_i \) with \( i \leq n \), \( q \) and a number of \textit{true}. So, by Lemma 3.6 (1), it follows from \( w \in F \) that \( v \in F \). Moreover, by Corollary 6.2 (6) and the idempotent, commutative and associative laws of \( \land \), it is easy to see that \( v =_{RS} t_1 \land q \). Hence \( t_1 \land q \in F \).

Case 5 \( u \) is a conjunction of processes \( p_i, \overline{pq} \) and \( t_j \) \((= \text{true})\) with \( i \leq n \) and \( j < m \).

Let \( t \) be any conjunction of \( p_i \) with \( i \leq n \). Similarly, we consider only the nontrivial case \( t \land p \not\in F \), and assume that \( u \equiv u_1 \land u_2 \). Since \( u \rightarrow s \), we may distinguish two cases based on the last rule applied in \( s \).

Case 5.1 \( uF \) with \( i \in \{1, 2\} \).

W.l.o.g, we assume \( i = 1 \). Since \( t \land p \not\in F \), \( \overline{pq} \) must occur in \( u_1 \). We consider two cases below.

If \( p_i \) does not occur in \( u_1 \) for each \( i \leq n \), then \( u_1 \) is a conjunction of \( \overline{pq} \) and a number of \textit{true}. Thus \( p \overline{pq} \in F \) by applying IH about item (1). So, by Lemma 3.3(5), we have \( p \circ (p \overline{pq}) \in F \). On the other hand, by Theorem 5.2(2), it follows from \( t \land p \subseteq_R S p \) that \((t \land p) \circ (p \overline{pq}) \in F \). Hence \((t \land p) \circ (p \overline{pq}) \in F \).

If there exist some \( p_i \) occurring in \( u_1 \), then \((t_1 \land p) \circ (p \overline{pq}) \in F \) by IH about item (4), where \( t_1 \) is any conjunction of all \( p_i \) occurring in \( u_1 \). Similarly, by Theorem 5.2(2) and \( t \land p \subseteq_R S t_1 \land p \), it follows that \((t \land p) \circ (p \overline{pq}) \in F \).

The preceding result guarantees that a series of processes are consistent under certain circumstance. We will encounter such processes and circumstance in the next lemma, which will be used in demonstrating the main result of this section.

**Lemma 7.5** Suppose that \( v \subseteq_R S p \overline{pq} \) and the relation \( R \) exactly consists of all pairs \( < t, w \land ( \bigwedge_{a \in Act} \delta_w, \overline{pq}(a) > ) \) such that there exist \( n < \omega \), \( p_i, v_i, a_j \) and \( u_j \) with \( i \leq n \) and \( j \leq n - 1 \) satisfying the conditions below

\[(a) \ p \Rightarrow_F |p_0 \text{ and } v \Rightarrow_F |v_0, \]
\[(b) \text{ for each } i \text{ with } 0 \leq i \leq n - 1, v_i \Rightarrow_F |v_{i+1}, p_i \Rightarrow_F |u_j \text{ and } u_i \land p \Rightarrow_F |p_{i+1}, \]
\[(c) \text{ for each } i \text{ with } 0 \leq i \leq n, v_i \subseteq_R S p_i \circ (p \overline{pq}) \text{ and } v_i \not\subseteq_R S q, \]
\[(d) \ t \equiv v_n \text{ and } w \equiv p_n. \]

Then \( R \cup \subseteq_R S \) is a stable ready simulation relation up to \( \subseteq_R S \).
Proof. Let \( r, s \land (\bigwedge_{a \in Act} \delta_s, p \to p_q(a)) > b \) be any pair in \( R \). Thus there exist \( p_i, v_i \), \( a_j \) and \( u_j \) with \( i \leq n \) and \( j \leq n - 1 \) satisfying the conditions (a)-(d). In particular, \( r \equiv v_n \) and \( s \equiv p_n \). We intend to check that this pair satisfies four conditions in Def. 4.1. Amongst, it is straightforward for (RS1) and (RS4). Moreover, due to \( v_n \sqsubseteq_{RS} p_n \circ (p \bowtie q) \) and \( v_n \not\in F \), we have \( p_n \circ (p \bowtie q) \not\in F \). Then, by Lemma 7.4 (3), it follows that

\[
p_n \land (\bigwedge_{a \in Act} \delta_{p_n, p \to p_q}(a)) \not\in F. \quad (7.5.1)
\]

Hence (RS2) holds. Next we verify (RS3-upto). Let \( r \equiv v_n \Rightarrow F_1 | v_{n+1} \). Since \( v \sqsubseteq_{RS} p \bowtie q \) and \( v \not\in F \), by Lemma 7.4(1), it follows that \( p \bowtie q \not\in F \). Then, due to \( b \in I(v_n) = I(p_n \circ (p \bowtie q)) = I(p_n) \), we have, for any \( a \in Act \),

\[
\delta_{p_n, p \bowtie q}(a) \stackrel{b}{\to}_{F} \begin{cases} 
\text{true} & \text{if } a \neq b \\
p \bowtie q & \text{if } a = b
\end{cases}. \quad (7.5.2)
\]

To complete the proof, we want to find \( t \) such that \( p_n \land (\bigwedge_{a \in Act} \delta_{p_n, p \bowtie q}(a)) \stackrel{b}{\Rightarrow} F | t \) and \( \langle v_{n+1}, t \rangle \in (R \cup \sqsubseteq_{RS}) \circ \sqsubseteq_{RS} \). We distinguish two cases below.

Case 1 \( v_{n+1} \sqsubseteq_{RS} q \).

Due to \( v_n \sqsubseteq_{RS} p_n \circ (p \bowtie q) \) and Lemma 5.1 (1), we have \( v_n \sqsubseteq_{RS} p_n \). Further, we get \( v_{n+1} \sqsubseteq_{RS} p_n \)' for some \( p_n \)' with \( p_n \stackrel{b}{\Rightarrow}_{F} u_n \). On the other hand, it follows from \( v_{n+1} \sqsubseteq_{RS} q_1 \) for some \( q_1 \) with \( q \Rightarrow F | q_1 \). By Lemma 3.11(2), since \( v_{n+1} \sqsubseteq_{RS} p_n \)' and \( v_{n+1} \sqsubseteq_{RS} q_1 \), we obtain

\[
v_{n+1} \sqsubseteq_{RS} p_n \land q_1. \quad (7.5.3)
\]

Hence \( p_n \land q_1 \not\in F \) because of \( v_{n+1} \not\in F \). Further, by Lemma 3.6 (2) and 3.1(8), it follows that

\[
u_n \land p \bowtie q \stackrel{u_n \land q}{\Rightarrow}_{F} u_n \land q \Rightarrow F | p_n \land q_1.
\]

By (7.5.2), Lemma 3.2(2), Corollary 6.2(6) and the idempotent, commutative and associative laws of \( \land \), we obtain

\[
p_n \land (\bigwedge_{a \in Act} \delta_{p_n, p \bowtie q}(a)) \stackrel{b}{\Rightarrow}_{F} u =_{RS} u_n \land p \bowtie q \text{ for some } u.
\]

Hence there exists \( t \) such that

\[
p_n \land (\bigwedge_{a \in Act} \delta_{p_n, p \bowtie q}(a)) \stackrel{b}{\Rightarrow}_{F} u \Rightarrow F | t \text{ and } p_n \land q_1 \sqsubseteq_{RS} t.
\]

Further, by (7.5.3), we have \( v_{n+1} \sqsubseteq_{RS} p_n \land q_1 \sqsubseteq_{RS} t \). Thus the process \( t \) is exactly the one that we seek.

Case 2 \( v_{n+1} \not\sqsubseteq_{RS} q \).
Since \( v_n \subseteq_{RS} p_n \circ (p \circ q) \) and \( v_n \Rightarrow F | v_{n+1} \), there exists \( u \) such that \( v_{n+1} \subseteq_{RS} u \) and \( p_0 \circ (p \circ q) \Rightarrow F | u \). Further, due to \( v_{n+1} \not\subseteq_{RS} q \), it is not difficult to see that there exist \( p_{n+1}, u, p' \) and \( p_n \) such that

\[
p_n \Rightarrow F \quad u_n \Rightarrow F \quad | p' := p \quad p_{n+1} = p_n \land p' \quad \text{and}
\]

\[
p_n \circ (p \circ q) \Rightarrow F \quad (u_n \land p) \circ (p \circ q) \Rightarrow F \quad | p_{n+1} \circ (p \circ q) \equiv u.
\]

Then it follows that

\[
\langle v_{n+1}, p_{n+1} \land (\bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)) \rangle \in R.
\]

(7.5.4)

On the other hand, by Lemma 3.2 (2), Corollary 6.2(6) and the idempotent, commutative and associative laws of \( \land \), it follows from (7.5.2) that there exists \( t \) such that

\[
p_n \land (\bigwedge_{a \in \text{Act}} \delta_{p_n \circ p \circ q} (a)) \Rightarrow F t =_{RS} u_n \land p \circ q.
\]

(7.5.5)

Moreover, it is obvious that

\[
u_n \land p \circ q \Rightarrow F | p' \land (p' \land \bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)) \Rightarrow_{RS} p_{n+1} \land (\bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)).
\]

By Lemma 7.4 (3), it follows from \( p_{n+1} \circ (p \circ q) \not\in F \) that

\[
p_{n+1} \land (\bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)) \not\in F.
\]

Hence \( p' \land (p' \land \bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)) \not\in F \). Thus, by Lemma 3.6 (2), we have

\[
u_n \land p \circ q \Rightarrow F | p' \land (p' \land \bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)).
\]

So, it follows from (7.5.1) and (7.5.5) that there exists \( w \) such that

\[
p_n \land (\bigwedge_{a \in \text{Act}} \delta_{p_n \circ p \circ q} (a)) \Rightarrow F t \Rightarrow_F | w \quad \text{and} \quad p_{n+1} \land (\bigwedge_{a \in \text{Act}} \delta_{p_{n+1} \circ p \circ q} (a)) \subseteq_{RS} w.
\]

Moreover, due to (7.5.4), we get \( \langle v_{n+1}, w \rangle \in (R \bigcup \subseteq_{RS}) \circ \subseteq_{RS} \), as desired. \( \square \)

We now have the below assertion of the equivalence of \( p \circ q \) and \( p \circ \overline{q} \).

**Theorem 7.1** \( p \circ q =_{RS} p \circ \overline{q} \) for any process \( p \) and \( q \).

**Proof.** Since \( p \circ q =_{RS} \eta_{p,q}(p \circ q) \), by Theorem 6.2, it is enough to prove that \( p \circ q \subseteq_{RS} p \circ \overline{q} \). To this end, we intend to show that \( v \subseteq_{RS} p \circ \overline{q} \) for any \( v \) such that \( v \subseteq_{RS} p \circ q \). Assume that \( v \subseteq_{RS} p \circ q \) and \( v \Rightarrow_F | v_0 \). Then \( p \circ q \not\in F \). By Lemma 7.4(1), we have \( p \circ q \not\in F \). In the following, we want to find \( s \) such that \( p \circ q \Rightarrow_F | s \) and \( v_0 \subseteq_{RS} s \). In the situation that \( v_0 \subseteq_{RS} q \), this is straightforward. We next consider the case where \( v_0 \not\subseteq_{RS} q \). In such case, it follows from \( v \subseteq_{RS} p \circ q \) and \( v \Rightarrow_F | v_0 \) that \( v_0 \subseteq_{RS} p_0 \circ (p \circ q) \) for some \( p_0 \) such that \( p \Rightarrow_F | p_0 \). Hence, by Lemma 7.4 (3), we have \( p_0 \land (\bigwedge_{a \in \text{Act}} \delta_{p_0 \circ p \circ q} (a)) \not\in F \). Then, by the rule (Ra_\eta) and Lemma 3.6(2), it follows that \( p \circ q \Rightarrow_F | p_0 \land (\bigwedge_{a \in \text{Act}} \delta_{p_0 \circ p \circ q} (a)) \). Moreover, by Lemma 7.5,
we also have \( v_0 \sqsubseteq _{RS} p_0 \wedge ( \bigwedge_{a \in \text{Act}} \delta_{p_0, p_\eta}(a)) \). Thus the process \( p_0 \wedge ( \bigwedge_{a \in \text{Act}} \delta_{p_0, p_\eta}(a)) \) is indeed the one that we need. □

It is obvious that the temporal operator \( always \) can also be handled in the recursive manner. Formally, we have the result below.

**Corollary 7.1** \( \#p = _{RS} p \sqsubseteq _{\bot} \) for any process \( p \).

**Proof.** Immediately follows from Corollary 6.2(4) and Theorem 7.1. □

Hitherto this paper has provided two approaches to dealing with the temporal modal operator \( unless \) in pure process algebraic style. One approach is to introduce the operators \( \boxdot \) and \( \odot \), and provide SOS rules to describe their behavior. The other is to define constants \( p_\eta q \) in terms of \( \eta_{p,q} \). The latter resorts to only usual rules about recursion, but depends on the finiteness of \( \text{Act} \) as the definition of \( \eta_{p,q} \) refers to the process having the format \( \bigwedge_{a \in \text{Act}} [a] p \), which can not be generalized smoothly to the situation involving infinitely many actions (see, Remark 6.1).

### 8 Connections between CLLT and ACTL

As mentioned in Section 1, the links between process algebras and (modal) logics have been of concern in the literature. Amongst, Pnueli points out that \([54]\), given a logic language and a process algebra, interesting connections between them at least include (see, Section 1):

- Hennessy-Milner-style characterization
- expressivity of the logic language w.r.t the process algebra
- expressivity of the process algebra w.r.t the logic language

This section will study the links between two specification formalisms, namely CLLT and a fragment of ACTL\([49]\), from these three angles. Following \([44]\), the fragment of ACTL considered in this section, denoted by \( \ell \), consists of all formulas generated by BNF below

\[
\phi ::= tt | ff | en(a) | dis(a) | \phi \lor \phi | [a] \phi | [\boxdot] \phi | [\odot] \phi | \phi W \phi, \text{ where } a \in \text{Act}.
\]

As noticed by Lüttgen and Vogler, \( \ell \) contains essentially the safety properties of the universal fragment of ACTL \([44]\). The satisfaction relation \( p \models \phi \), to be read as “the process \( p \) satisfies the formula \( \phi \)”, is given as follows.

**Definition 8.1**\([44]\) The satisfaction relation \( \models \subseteq T(\Sigma_{CLLT}) \times \ell \) is defined inductively by:

- \( p \models ff \) if and only if \( p \in F \).
- \( p \models en(a) \) if and only if \( \forall p_0 (p \Rightarrow_F p_0 \Rightarrow a \in I(p_0)) \).
- \( p \models dis(a) \) if and only if \( \forall p_0 (p \Rightarrow_F p_0 \Rightarrow a \notin I(p_0)) \).
- \( p \models \phi \lor \varphi \) if and only if \( \forall p_0 (p \Rightarrow_F p_0 \models \phi \lor \varphi) \).

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\[ p \models \phi \land \varphi \quad \text{iff} \quad \forall p_0 (p \xrightarrow{F} p_0 \Rightarrow p_0 \models \phi \text{ and } p_0 \models \varphi). \]

\[ p \models [a] \phi \quad \text{iff} \quad \forall p_0, p_1 (p \xrightarrow{F} p_0 \xrightarrow{a} p_1 \Rightarrow p_1 \models \phi). \]

\[ p \models \Box \phi \quad \text{iff} \quad \forall p_0, p_1, \ldots, p_k (p \xrightarrow{F} p_0 \xrightarrow{\text{Act}} F \mid p_1 \ldots \xrightarrow{\text{Act}} F \mid p_k \Rightarrow p_k \models \phi). \]

\[ p \models \phi W \varphi \quad \text{iff} \quad \forall p_0, p_1, \ldots, p_k \left( p \xrightarrow{F} p_0 \xrightarrow{\text{Act}} F \mid p_1 \ldots \xrightarrow{\text{Act}} F \mid p_k \Rightarrow \right). \]

Two simple results immediately follows from the above definition:

**Lemma 8.1** For any \( p \in T(\Sigma_{\text{CLLT}}) \) and \( \phi \in \ell \), \( p \models \phi \) if and only if \( \forall p_0 (p \xrightarrow{F} p_0 \Rightarrow p_0 \models \phi) \). In particular, \( p \models \phi \) whenever \( p \in F \).

**Proof.** Easily by induction on \( \phi \). \( \square \)

**Lemma 8.2** If \( p \sqsubseteq_{RS} q \) then \( q \models \phi \) implies \( p \models \phi \) for each \( \phi \in \ell \).

**Proof.** Straightforward by induction on \( \phi \). \( \square \)

The converse of Lemma 8.2 can be proved in the standard manner. Hence we can get a Hennessy-Milner-style characterization of \( \sqsubseteq_{RS} \). In fact, to obtain such characterization, a fragment of \( \ell \) is enough [44].

As argued by Pnueli, Hennessy-Milner-style characterization presents only the weakest requirement of compatibility between a process calculus and a logic [54].

The remainder of this section will devote itself to explore stronger associations between \((T(\Sigma_{\text{CLLT}}), \sqsubseteq_{RS})\) and \((\ell, \models)\). Firstly, we consider the expressivity of \((T(\Sigma_{\text{CLLT}}), \sqsubseteq_{RS})\) w.r.t \((\ell, \models)\). The starting point of our discussion is the notion of a characteristic process.

**Definition 8.2** Given a formula \( \phi \in \ell \), a process \( t_\phi \in T(\Sigma_{\text{CLLT}}) \) is said to be a characteristic process for \( \phi \) if \( \forall p \in T(\Sigma_{\text{CLLT}}) (p \models \phi \Leftrightarrow p \sqsubseteq_{RS} t_\phi) \). Moreover, \((T(\Sigma_{\text{CLLT}}), \sqsubseteq_{RS})\) is said to be expressive w.r.t \((\ell, \models)\) if there exists a translation function from \( \ell \) to \( T(\Sigma_{\text{CLLT}}) \) which associates each formula \( \phi \in \ell \) with a characteristic process \( t_\phi \) in syntactic manner.

Intuitively, the characteristic process \( t_\phi \) represents the most loose process that realizes \( \phi \). If such \( t_\phi \) exists, verifying the validity of an assertion \( p \models \phi \) may be reduced to the implementation verification of \( p \sqsubseteq_{RS} t_\phi \). It can be showed without any difficulty that, for any \( \phi \), it has at most one characteristic process modulo \( =_{RS} \).

In the following, a function \([\cdot] : \ell \rightarrow T(\Sigma_{\text{CLLT}})\) is provided, which associates each formula \( \phi \in \ell \) with a characteristic process \([\phi]\).

**Definition 8.3** The translation function \([\cdot] : \ell \rightarrow T(\Sigma_{\text{CLLT}})\) is defined by

\[
[ff] = \bot \quad [tt] = \text{true} \quad [\phi \land \varphi] = [\phi] \land [\varphi] \quad [\phi \lor \varphi] = [\phi] \lor [\varphi]
\]

\[
[en(a)] = \bigvee_{a \in A_{\subseteq \text{Act}}} (\Box b.\text{true}) \quad [\text{dis}(a)] = \bigvee_{a \notin A_{\subseteq \text{Act}}} (\Box b.\text{true})
\]

\[
[a] [\phi] = [a] [\phi] \quad [\Box \phi] = [\phi] \quad [\phi W \varphi] = [\phi] \supseteq [\varphi]
\]

The above definition is motivated by Lüttgen and Vogler’s construction. In the framework of LLTS, they have given the method of embedding of formulas (in \( \ell \)) into LLTS [44].
Lemma 8.3 If \( p \sqsubseteq_{RS} p_i \) for some \( i \in \{1, 2\} \) then \( p \sqsubseteq_{RS} p_1 \lor p_2 \). Moreover, the converse also holds whenever \( p \) is stable.

Proof. Straightforward. \( \Box \)

Notice that the assumption that \( p \) is stable is necessary for the converse implication in the above. For instance, \( a.0 \lor b.0 \sqsubseteq_{RS} a.0 \lor b.0 \) but neither \( a.0 \lor b.0 \sqsubseteq_{RS} a.0 \lor b.0 \sqsubseteq_{RS} a.0 \lor b.0 \). Next we intend to show that, given a \( \varphi \in \ell \), \([\varphi]\) indeed is the characteristic process of \( \varphi \), which, as the most important result in [44], have been obtained by Lüttgen and Vogler in the framework of LLTS.

Lemma 8.4 For any \( \varphi \in \ell \), \([\varphi]\) is the characteristic process of \( \varphi \).

Proof. It is enough to prove that, \( p \models \varphi \) if and only if \( p \sqsubseteq_{RS} [\varphi] \) for any \( p \in T(\Sigma_{CLLT}) \) and \( \varphi \in \ell \). This can be proved by induction on \( \varphi \). Here we do not present them in full detail but handle three cases as samples. In particular, for the case where \( \varphi \) has one of formats \([a] \phi \), \( \Box \phi \) and \( \phi_1 W \phi_2 \), the proof is straightforward by applying Theorem 6.1, 4.1 and 5.1 respectively.

\[ \varphi \equiv t \]

The implication from right to left follows trivially from Definition 8.1. For the converse implication, it suffices to prove \( p \sqsubseteq_{RS} \text{true} \). Let \( p \overset{\text{true}}{\Rightarrow} [p_0] \). Clearly, \( \text{true} \overset{\text{true}}{\Rightarrow} a.\text{true} \), moreover, we also have \( p_0 \sqsubseteq_{RS} a.\text{true} \) by Lemma 6.1.

\[ \varphi \equiv \text{en}(a) \]

(Left implies Right) Let \( p \overset{\text{true}}{\Rightarrow} [p_0] \). Then it follows from \( p \models \text{en}(a) \) that \( a \in I(p_0) \). Thus \( [\text{en}(a)] \equiv \bigvee_{a \in A_{\text{Act}}} \big([\Box b.\text{true}] \overset{\text{true}}{\Rightarrow} [b.\text{true}] \big) \overset{\text{true}}{\Rightarrow} [b.\text{true}] \). Moreover, by Lemma 6.1, \( p_0 \sqsubseteq_{RS} a.\text{true} \).

(Right implies Left) Let \( p \overset{\text{true}}{\Rightarrow} [p_0] \). It suffices to show that \( a \in I(p_0) \). Since \( p \sqsubseteq_{RS} [\text{en}(a)] \equiv \bigvee_{a \in A_{\text{Act}}} \big([\Box b.\text{true}] \big) \), we get \( p_0 \sqsubseteq_{RS} a.\text{true} \) for some \( A_0 \) with \( a \in A_0 \). Then, due to \( p_0 \not\models F \), we have \( I(p_0) = I(a.\text{true}) = A_0 \). Hence \( a \in I(p_0) \).

\[ \varphi \equiv \phi_1 \lor \phi_2 \]

\[ p \models \phi_1 \lor \phi_2 \]

\[ \iff \forall p_0(p \overset{\text{true}}{\Rightarrow} [p_0] \models \phi_1 \lor \phi_2) \]

\[ \iff \forall p_0(p \overset{\text{true}}{\Rightarrow} [p_0] \sqsubseteq_{RS} [\phi_1] \lor [\phi_2]) \] (by IH)

\[ \iff \forall p_0(p \overset{\text{true}}{\Rightarrow} [p_0] \sqsubseteq_{RS} [\phi_1] \lor [\phi_2]) \]

\[ \iff \forall p_0(p \overset{\text{true}}{\Rightarrow} [p_0] \sqsubseteq_{RS} [\phi_1] \lor [\phi_2]) \]

\[ \iff p \sqsubseteq_{RS} [\phi_1 \lor \phi_2]. \]

As usual, for any formula \( \phi \) and \( \varphi \), \( \varphi \) is said to be a logic consequence of \( \phi \), in symbols \( p \models \varphi \), if for any process \( p \), \( p \models \phi \) implies \( p \models \varphi \). Moreover, \( \phi \) and \( \varphi \) are said to be logic equivalent if \( \phi \models \varphi \) and \( \varphi \models \phi \). As an immediate consequence of the above theorem, we have the result below.
Corollary 8.1 For any formula φ and ϕ in ℓ,

1. [φ] ⊑ [φ].
2. φ |= ϕ if and only if [φ] ⊑ RS [ϕ].

Proof. (1) immediately follows from [φ] ⊑ RS [φ] and Lemma 8.4. (2) follows from (1), the transitivity of ⊑ RS and Lemma 8.4. □

Moreover, since the function [·] : ℓ → T(ΣCLLT) is given in syntactic manner, we have the result below.

Theorem 8.1 (T(ΣCLLT), ⊑ RS) is expressive w.r.t (ℓ, |=).

We next deal with another stronger connection between CLLT and (ℓ, |=), which involves the fragment T(ΣCLLT)− of T(ΣCLLT) defined below.

Definition 8.4 T(ΣCLLT)− consists of processes generated by BNF below, where A ⊆ Act and a ∈ Act.

\[ p ::= 0 | \bot | true | a.p | p \lor p | p \land p | \square b.true | \neg p \land \neg p | (\square a.p \land \neg b.true) \]

In the following, we intend to prove that (ℓ, |=) is expressive w.r.t (T(ΣCLLT)−, ⊑ RS). Analogous to [54], such notion is defined formally as follows.

Definition 8.5 (ℓ, |=) is said to be expressive w.r.t (T(ΣCLLT)−, ⊑ RS) if for any process p in T(ΣCLLT)−, there exists a formula φp in ℓ such that

(E1) ∀q ∈ T(ΣCLLT)( q ⊑ RS p ⇔ q |= φp), and

(E2) ∀φ ∈ ℓ(p |= φ ⇔ φp |= φ).

Obviously, given a process p, φp (if it exists) is a characteristic formula for p due to (E1), moreover, it is the strongest logic formula φ in ℓ such that p |= φ due to (E2). In order to prove that (ℓ, |=) is expressive w.r.t (T(ΣCLLT)−, ⊑ RS), we will introduce the function * below, and show that it is exactly the lower adjoint of the function [·] and associates each process p in T(ΣCLLT)− with a characteristic formula p*.

Definition 8.6 The translation function * : T(ΣCLLT)− → ℓ is defined inductively by

\[ \bot^* = ff \quad true^* = tt \quad (a.p)^* = en(a) \land [a]p^* \land \bigwedge_{a \not\in b \in Act} dis(b) \]

\[ 0^* = \bigwedge_{a \in Act} dis(a) \quad (\square b.true)^* = (\bigwedge_{b \in A} en(b)) \land (\bigwedge_{a \not\in b \in Act - A} dis(a)) \]

\[ (p \land q)^* = p^* \land q^* \quad (p \lor q)^* = p^* \lor q^* \quad (\neg p)^* = \square p^* \quad (p \lor p)^* = p^* W q^* \]

\[ (\square a.p)^* = (\bigwedge_{a \not\in b \in A} b.true)^* \land [a]p^* \]

Lemma 8.5 p ⊑ RS q if and only if p |= q* for any p ∈ T(ΣCLLT) and q ∈ T(ΣCLLT)−.
Proof. Clearly, it holds trivially whenever \( p \in F \). In the following, we consider the nontrivial case \( p \notin F \), and proceed by induction on \( q \).

- \( q \equiv \bot \)

It follows from \( p \notin F \) that \( p \not\subseteq RS \bot \) and \( p \not\models ff \). Hence \( p \subseteq RS \bot \iff p \models ff \).

- \( q \equiv true \)

Immediately follows from \( p \subseteq RS true \) and \( p \models tt \) for each \( p \).

- \( q \equiv 0 \)

\[
p \subseteq RS 0 \\
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow I(p_0) = \emptyset).
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow \forall a \in Act (a \notin I(p_0))) \quad \text{ (due to } p_0 \not\Rightarrow \text{ )}
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow \forall a \in Act (p_0 \models dis(a)))
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow p_0 \models \bigwedge_{a \in Act} dis(a))
\]

\[
\iff p \models \bigwedge_{a \in Act} dis(a) \quad \text{ (by Lemma 8.1)}
\]

- \( q \equiv \square_b true \)

\[
p \subseteq RS \square_b true
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow I(p_0) = A)
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow A \subseteq I(p_0) \text{ and } (Act - A) \cap I(p_0) = \emptyset)
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow p_0 \models \bigwedge_{a \in A} en(a) \text{ and } p_0 \models \bigwedge_{b \in Act - A} dis(b))
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow p_0 \models \bigwedge_{a \in A} en(a) \land \bigwedge_{b \in Act - A} dis(b)) \quad \text{ (by Lemma 8.1)}
\]

- \( q \equiv a.q_1 \)

\[
p \subseteq RS a.q_1
\]

\[
\iff \forall p_0 (p \not\Rightarrow F | p_0 \Rightarrow p_0 \subseteq_{RS} a.q_1)
\]

\[
(\blacklozenge) \iff \forall p_0 \left( p \not\Rightarrow F | p_0 \Rightarrow \left( \begin{array}{l} a \in I(p_0) \text{ and } \forall b \in Act (a \neq b \Rightarrow b \notin I(p_0)) \\text{ and } \forall p_1 (p_0 \not\Rightarrow F | p_1 \Rightarrow p_1 \subseteq_{RS} q_1) \end{array} \right) \right)
\]

\[
\iff \forall p_0 \left( p \not\Rightarrow F | p_0 \Rightarrow \left( \begin{array}{l} p_0 \models en(a) \text{ and } p_0 \models \bigwedge_{a \neq b \in Act} dis(b) \\text{ and } \forall p_1 (p_0 \not\Rightarrow F | p_1 \Rightarrow p_1 \models q_1^*) \end{array} \right) \right)
\]

\[
\iff \forall p_0 \left( p \not\Rightarrow F | p_0 \Rightarrow \left( \begin{array}{l} p_0 \models en(a) \text{ and } p_0 \models \bigwedge_{a \neq b \in Act} dis(b) \\text{ and } p_0 \models \left[ a \right]q_1^* \end{array} \right) \right)
\]

\[
\iff \forall p_0 \left( p \not\Rightarrow F | p_0 \Rightarrow \left( \begin{array}{l} p_0 \models en(a) \land \left[ a \right]q_1^* \land \bigwedge_{a \neq b \in Act} dis(b) \end{array} \right) \right) \quad \text{ (by Lemma 8.1)}
\]

\[
(\blacklozenge) \text{ For the implication from right to left, we need to show that } a.q_1 \notin F \text{ under the assumption } p \not\Rightarrow F | p_0. \text{ By Lemma 3.8 and 3.5, it follows from } p_0 \notin F \text{ and } a \in I(p_0) \text{ that } p_0 \not\Rightarrow F | p_1 \text{ for some } p_1. \text{ Hence } p_1 \subseteq RS q_1. \text{ Then } q_1 \notin F \text{ because of } p_1 \notin F. \text{ Thus } a.q_1 \notin F \text{ by Lemma 3.3(2).}
\]

- \( q \equiv \exists q_1 \) or \( q_1 \equiv q_2 \)

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Immediately follows from Theorem 4.1, 5.1 and IH.

- **q \equiv q_1 \land q_2**

\[ p \subseteq_{RS} q_1 \land q_2 \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} q_1 \land q_2) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} q_1 \text{ or } p_0 \subseteq_{RS} q_2) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \models q_1^* \text{ or } p_0 \models q_2^* \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \models (q_1 \land q_2)^*) \]
\[ \iff p \models (q_1 \land q_2)^*. \]

(by Lemma 8.1)

- **q \equiv q_1 \lor q_2**

\[ p \subseteq_{RS} q_1 \lor q_2 \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} q_1 \lor q_2) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} q_1 \text{ or } p_0 \subseteq_{RS} q_2) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \models q_1^* \lor q_2^*) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \models (q_1 \lor q_2)^*) \]
\[ \iff p \models (q_1 \lor q_2)^*. \]

(by Lemma 8.1)

- **q \equiv \square \not\in a \cdot q_1**

\[ p \subseteq_{RS} \square \not\in a \cdot q_1 \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} \square \not\in a \cdot q_1) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} A \cup \{a\} \not\in a \cdot q_1) \]
\[ \iff \forall p_0 (p \Rightarrow_{F} p_0 \Rightarrow p_0 \subseteq_{RS} A \cup \{a\} \not\in a \cdot q_1^*) \]
\[ \iff p \models (\not\in a \cdot q_1^*) \]

(\(\not\in\)) For the implication from right to left, it is required to verify \(\not\in b \cdot q_1^* \) \(\not\in F\) under the assumption \(p \Rightarrow_{F} p_0\). Clearly, it suffices to prove that \(q_1 \not\in F\), which can be proved analogously to (\(\not\in\)).

As an immediate consequence of the above result, we have

**Corollary 8.2** For any process \(p \text{ and } q \text{ in } T(\Sigma_{CLLT})^\sim\),

1. \(p \models p^*\)
2. \(p \subseteq_{RS} q\) if and only if \(p^* \models q^*\).

**Proof.** (1) immediately follows from \(p \subseteq_{RS} p\) and Lemma 8.5. (2) follows from (1), the transitivity of \(\subseteq_{RS}\) and Lemma 8.5.

In order to prove that \((\ell, \models)\) is expressive w.r.t \((T(\Sigma_{CLLT})^\sim, \subseteq_{RS})\), the only point remaining concerns \((E2)\), that is, \(p \models \varphi\) if \(p^* \models \varphi\) for any \(p \in T(\Sigma_{CLLT})^\sim\) and \(\varphi \in \ell\). Before proving it, let we recall the well-known notion of a Galois connection between two preordered sets.
Definition 8.7 A Galois connection between two preordered sets \((A, \preceq_A)\) and \((B, \preceq_B)\) is a pair of function \(F : B \to A\) and \(G : A \to B\) satisfying that, for any \(x \in B\) and \(y \in A\), \(F(x) \preceq_A y\) if and only if \(x \preceq_B G(y)\).

It is well known that \((F, G)\) is a Galois connection if and only if \(F\) and \(G\) are monotonic and satisfy the cancellation laws below (see for instance [6])

\[
\begin{align*}
(C1) \quad & x \preceq_B G(F(x)) \quad \text{for all } x \in B, \text{ and} \\
(C2) \quad & F(G(y)) \preceq_A y \quad \text{for all } y \in A.
\end{align*}
\]

Using Lemma 8.4, it is easy to see that (E2) holds if and only if the pair \((*, \cdot]\) is a Galois connection between preordered sets \((\ell, \models)\) and \((T(\Sigma_{CLLT})^-, \sqsubseteq_{RS})\). Next we shall prove the latter.

Theorem 8.2 (Galois connection) The pair of functions \(* : T(\Sigma_{CLLT})^- \to \ell\) and \(\cdot : \ell \to T(\Sigma_{CLLT})^-\) is a Galois connection between preordered sets \((\ell, \models)\) and \((T(\Sigma_{CLLT})^-, \sqsubseteq_{RS})\). That is, \(p^* \models \phi\) if and only if \(p \sqsubseteq_{RS} [\phi]\) for any \(p \in T(\Sigma_{CLLT})^-\) and \(\phi \in \ell\).

Proof. By Definition 8.3, 8.4 and 6.3, it is easy to check that \([\phi] \in T(\Sigma_{CLLT})^-\) for any \(\phi \in \ell\). Thus the function \([\cdot]\) may be regarded as a function from \(\ell\) to \(T(\Sigma_{CLLT})^\). On the other hand, by Corollary 8.1 and 8.2, both the function \(*\) and \(\cdot\) are monotonic. Thus it suffices to prove that cancellation laws (C1) and (C2) hold.

For (C1), suppose \(p \in T(\Sigma_{CLLT})^\). By Corollary 8.2, we get \(p \models p^*\). Then \(p \sqsubseteq_{RS} [p^*]\) by Lemma 8.4. Hence (C1) holds.

For (C2), let \(\phi \in \ell\). We intend to prove that \([\phi]^* \models \phi\). Let \(q\) be any process such that \(q \models [\phi]^*\). To complete the proof, it is enough to verify that \(q \models \phi\). By Corollary 8.1, we obtain \([\phi] \models \phi\). Moreover, by Lemma 8.5, it follows from \(q \models [\phi]^*\) that \(q \sqsubseteq_{RS} [\phi]\). Hence \(q \models \phi\) by Lemma 8.2, as desired. \(\square\)

Roughly speaking, the above theorem says that the function \(*\) is exactly the lower adjoint of the function \([\cdot]\). That is, for each process \(p \in T(\Sigma_{CLLT})^-\), \(p^*\) is the strongest logic formula \(\phi\) in \(\ell\) that \(p \sqsubseteq_{RS} [\phi]\). Dually, the function \([\cdot]\) associates with each formula \(\phi\) in \(\ell\) the most loose process \(p \in T(\Sigma_{CLLT})^-\) such that \(p^* \models \phi\).

As an immediate consequence of Theorem 8.2, we obtain the assertion below.

Theorem 8.3 \((\ell, \models)\) is expressive w.r.t \((T(\Sigma_{CLLT})^-, \sqsubseteq_{RS})\).

Proof. Let \(p \in T(\Sigma_{CLLT})^-\). It suffices to illustrate that \(p^*\) satisfies (E1) and (E2) in Definition 8.5. Clearly, (E1) holds due to Lemma 8.5, and (E2) comes from Theorem 8.2 and Lemma 8.4. \(\square\)

By the way, it is obvious that, for \(CLLT_n\), all results obtained in this section also hold by making a few slight modifications.

9 Conclusions and future work

This paper gives two distinct methods of representing the loosest (modulo \(\sqsubseteq_{RS}\)) implementations that realize logic specifications “always \(p\)" or “\(p\) unless \(q\)" in terms of algebraic expressions. One method is to introduce nonstandard process-algebraic operators \(\mathcal{L}, \mathcal{C}\), and \(\mathcal{D}\) to capture Lüttingen and Vogler’s constructions in [44] directly. The other is to apply the greatest fixed-point characterizations of \(\mathcal{C}\) and
obtained in this paper (see, Theorem 6.2 and Corollary 6.3) and provide graphical representing of temporal operators always and unless in a recursive manner. The latter is independent of Lüttgen and Vogler’s constructions, and its advantage lies in the fact that it makes no appeal to any nonstandard operational operators, but it depends on the mild assumption that $Act$ is finite. In a word, this paper not only lifts Lüttgen and Vogler’s work in [44] to a pure process algebraic setting but also provides another more succinct method to realize their intention.

This work brings the process calculuses CLLT in which usual operational operators (prefix, external choice and parallel operator), logic connectives (conjunction and disjunction) and standard temporal operators (always and unless) may be freely mixed without any restriction, and compositional reasoning is admitted. Such calculus allows one to capture desired operational behavior and describe intended safety properties in the same framework. Moreover, the links between CLLT and the fragment $\ell$ of ACTL are explored from angles suggested by Pnueli in [54]. These links reveal that there exist intimate relationships among distinct verification activities including model checking, implementation verification and validity problem within $\ell$. We summarize the reductions among these verification activities in Fig.1, where dashed lines are used to indicate that the process term involved in the corresponding reduction is required to be in $T(\Sigma_{CLLT})^-$.

In the literature, various work on combining operational operators with logic operators have been reported [33, 41, 50]. Olderog provides a framework in which operational operators may be combined with trace formula [50]. But such framework does not allow one to freely mix operational and logic specifications. Guerra and Costa enrich a simple process algebra with a modal operator which can express some liveness property [33]. However, due to adopting trace semantics, this system is not deadlock-sensitive, and hence it is inadequate in the situation where concurrency is involved. In [41], based on the notion of modal LTS, Larsen et al. consider the operator conjunction over independent processes and obtain the result analogous to Lemma 3.11. Moreover, in such framework, it is shown that conjunction may distribute over parallel composition. However, an algebraic theory of mixing operational and logic operators is not considered in [41]. There also exist investigations of operational behavior involving logic ingredient but without admitting the free mixing of operational and logic operators, see, e.g., [7, 22].

We conclude this paper with giving several possible avenues for further work. Firstly, finding a complete proof system for CLLT would be the next task. Secondly, although this paper provides recursive constants to represent the “loosest” implementations realizing logic specifications “always $p$” or “$p$ unless $q$”, no attempt has made here to develop general theory concerning recursion for LLTS and a few fundamental problems are still open. For instance, whether $\sqsubseteq_{RS}$ is precongruent in the presence of (nested) recursive operator? Under usual conditions (see, e.g.,
whether equations containing (nested) recursive operators still have a unique solution? Notice that, since LLTS involve consideration of inconsistencies, the answers for these questions can not be trivially inferred from existent results in the literature. Thirdly, it would also be interesting to develop a general view of the connections between process algebras and modal logics. We leave these further developments for further work.

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