q - Ising model on a duplex and a partially duplex clique

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We analyze a modified kinetic Ising model, so called q-neighbor Ising model, with Metropolis dynamics, [Phys. Rev. E 92 052105] on a duplex clique and a partially duplex clique. In the q-Ising model each spin interacts only with q spins randomly chosen from the whole neighborhood. In the case of a duplex clique the change of a spin is allowed only if both levels simultaneously induce this change. Due to the mean-field like nature of the model we are able to derive the analytic form of transition probabilities and solve the corresponding master equation. The existence of the second level changes dramatically the character of the phase transition. In the case of the monoplex clique, the q-neighbor Ising model exhibits continuous phase transition for q = 3, discontinuous phase transition for q ≥ 4 and for q = 1 and q = 2 the phase transition is not observed. On the other hand, in the case of the duplex clique continuous phase transitions are observed for all values of q, even for q = 1 and q = 2. Subsequently we introduce a partially duplex clique, parametrized by r ∈ [0, 1], which allows us to tune the network from monoplex (r = 0) to duplex (r = 1). Such a generalized topology, in which a fraction r of all nodes appear on both levels, allows to estimate a critical value of r = r∗(q) at which a switch from continuous to discontinuous phase transition is observed.

I. INTRODUCTION

Multiplex networks have become one of the most active area of recent network research [1, 2] mainly due to the fact that many real-world systems like public transport or social networks consists of many layers. After six years of studies we know much more about the structure and the function of multiplex networks [3–5]. A lot of attention has been devoted to the analysis of various dynamics on multiplex networks, including diffusion processes [6], epidemic spreading [7, 8] and voter dynamics [9, 10].

However, there is still an open question how the number of levels influence the macroscopic properties of the system, such as dynamics of global variables, phase transitions or other emerging patterns. In order to find a general answer to this question systematic studies of various models are needed. However, one of the fundamental problems when dealing with models on the multiplex network is how to define the model on such a network, because usually there are several possibilities. For example, in case of models defined by Hamiltonians [12] inter-layer interactions can be introduced by the relations between the coupling constants. However, the results of the model can strongly depend on this inter-layer interaction. Jang et al thoroughly analyzed the behavior of the Ashkin-Teller model in various types of inter-layer interaction, and presented a rich phase diagram containing three types of phase transitions – 1st order, 2nd order, and mixed order. For the models defined by the transition probabilities, like the voter model, the threshold model [14] or the q-Ising model [16] one way of defining interactions between layers is the adoption of so called AND and OR rules, proposed for the first time by Brunnett et al. in [13] where the generalized threshold cascade model [14] on multiplex networks has been studied. Authors introduced two kinds on nodes: (1) an OR node was activated as soon as a sufficiently large fraction of its neighbors were active in at least one level, (2) an AND node was activated only if in each and every layer a sufficiently large fraction of its neighbors were active.

The concept of AND and OR rules has been recently adopted in the q-voter model with independence [11]. For systems consisting only AND nodes the phase transition becomes discontinuous for q = 4 if the number of layers L ≤ 3, q = 5 for duplex, whereas for a monoplex such behavior is observed for q = 6.

The Ising model has always played a very special role in the statistical physics. For the original Ising model a continuous phase transition is observed for both the regular lattices as well as monoplex complex networks [18, 21]. On the other hand, in the case of network of network topology Suchecki and Holyst observed a discontinuous phase transition [21]. Moreover, in [16] it has been shown that a seemingly small modification of the kinetic Ising model – in which a randomly chosen spin interacts only with its q neighbors – leads to the surprising result on monoplex complete graph, i.e. a switch from a continuous to a discontinuous phase transition at q = 4.

In this paper we ask the same question that has been asked in [11], but this time within the q-neighbor Ising model, namely “How the additional level will introduce the type of the phase transition?” Analogously, as in [11] we focus on trivial topology duplex clique. The experience gained from the q-voter model allows to predict that the switch from a continuous to a discontinuous phase transition will appear for a smaller value of q in case of a duplex clique than for a monoplex. Such a result would be also expected from the theory of equilibrium phase transitions, if the additional layer could be treated analogously to the additional dimension [11]. However, we will show that our simple prediction fails in case of the q-neighbor Ising model and results are exactly oppo-
site. Adding a second layer causes that only continuous phase transitions are observed for all values of \( q \). Because for a monoplex clique a discontinuous phase transition is observed for \( q \geq 4 \) and for a duplex cliques the phase transition is continuous, we expect that there is an intermediate topology for which a switch from a discontinuous to a continuous phase transition appears. Therefore we investigate the \( q \)-neighbor Ising model on a generalized topology of a partially duplex clique, parametrized by \( r \in [0, 1] \), which allows us to tune the network from monoplex (\( r = 0 \)) to duplex (\( r = 1 \)). Such a generalized topology, in which a fraction \( r \) of all nodes appear on both levels, allows to estimate a critical value of \( r = r^*(q) \) at which a switch from continuous to discontinuous phase transition is observed.

### II. Duplex Clique and Partially Duplex Clique

Multiplex networks consist of distinct levels (layers) and the interconnections between levels are only between a node and its counterpart in the other layer (i.e. the same node). A duplex clique is a particular case of a multiplex networks, which consists of two distinct levels, each of which is represented by a complete graph (i.e. a clique) of size \( N \). The same topology was consider to analyze the \( q \)-voter model with independence \[8\]. Levels can represent two different communities (e.g. Facebook and school class) and are composed of exactly the same people – each node possesses a counterpart node in the second layer. Analogously as in \[11\], we assume that each node possesses the same state on each level.

In the partially duplex clique only the fraction \( r \) of \( N \) nodes have a counterpart in the other layer, remaining nodes belong only to one community (layer). This means that at each level we have \( Nr \) duplex-type nodes and \( N_m = N(1 - r) \) monoplex-type nodes, which means that in total the system consists of \( Nr + 2N(1 - r) \) distinguishable nodes; an example of such a topology is shown in Fig. 1. The fraction \( r \) of individuals who are active in both layers was introduced in \[7\]. It has been suggested that the parameter \( r \) can be interpreted as inter-layer connectivity or the degree of structural multiplexity of the system \[22\].

### III. The q-Neighbor Ising Model on Duplex Clique

In \[16\] we have modified the kinetic Ising model with Metropolis dynamics allowing each spin to interact with \( q \) spins randomly chosen from the whole system, (i.e complete graph). Here we consider a set of \( N \) spin described by the binary variables \( S_i = \pm 1 \), which are duplicated on the second level. The algorithm of a single update of the \( q \)-neighbor Ising model on a duplex clique consists of eight consecutive steps:

1. Randomly choose a spin \( S_i \)
2. From all neighbors of \( S_i \) choose a subset of \( q \) neighbors on the first level: \( nm_1 \)
3. Calculate the value of the following function:
   \[
   E_1(S_i) = -S_i \sum_{j \in nn_1} S_j,
   \]
   and the value of the same function for the flipped spin, i.e. \( E_1(-S_i) \), finally calculate \( \Delta E_1 = E_1(-S_i) - E_1(S_i) \)
4. Select randomly a real number \( p_1 \in U[0, 1] \) and if \( p_1 < \min[1, e^{-\Delta E_1/T}] \) then set flag \( f_1 = 1 \) else \( f_1 = 0 \)
5. From all neighbors of \( S_i \) choose a subset of \( q \) neighbors on the second level: \( nm_2 \)
6. Calculate the value of the following function:
   \[
   E_2(S_i) = -S_i \sum_{j \in nn_2} S_j,
   \]
   and the value of the same function for the flipped spin, i.e. \( E_2(-S_i) \), finally calculate \( \Delta E_2 = E_2(-S_i) - E_2(S_i) \)
7. Select randomly a real number \( p_2 \in U[0, 1] \) and if \( p_2 < \min[1, e^{-\Delta E_2/T}] \) then set flag \( f_2 = 1 \) else \( f_2 = 0 \)
8. If \( f_1 = 1 \) and \( f_2 = 1 \) then set \( S_i = -S_i \) else do nothing
and the value of the same function for the flipped spin, i.e. $E_2(-S_i)$, finally calculate $\Delta E_2 = E_2(-S_i) - E_2(S_i)$.

7. Select randomly a real number $p_2 \in U[0,1]$ and if $p_2 < \min[1,e^{-\Delta E_2/T}]$ then set flag $f_2 = 1$ else $f_2 = 0$.

8. If $f_1 = 1$ and $f_2 = 1$ then flip the spin $S_i \rightarrow -S_i$ and its counterpart in the second layer.

As usual, a single time step consists of $N$ elementary updates, i.e. $\Delta t = 1/N$, which means that one time unit corresponds to the mean update time of a single individual. As an order parameter we choose magnetization:

$$m = \frac{1}{N} \sum_{i=1}^{N} S_i.$$  \hspace{1cm} (3)

In a single update the number of spins ‘up’ $N^\uparrow$ can change according to the following process:

$$N^\uparrow(t + \Delta t) = \begin{cases} N^\uparrow(t) + 1 & \text{with prob } \gamma^+, \\ N^\uparrow(t) - 1 & \text{with prob } \gamma^-, \\ N^\uparrow(t) & \text{with prob } 1 - (\gamma^+ + \gamma^-). \end{cases}$$  \hspace{1cm} (4)

Simultaneously with $N^\uparrow$, magnetization $m$ increases/decreases by $2/N$ or remains constant with the above probabilities.

To calculate transition probabilities $\gamma^+$ and $\gamma^-$ for $N \rightarrow \infty$ it is convenient to use the concentration of ‘up’ spins, which is related to the magnetization by the simple formula:

$$c = \frac{N^\uparrow}{N} = \frac{m + 1}{2}.$$  \hspace{1cm} (5)

The transition probabilities as a function of $c$ and model’s parameters $T$ and $q$ have the following form:

$$\gamma^+(c,T,q) = (1 - c) \sum_{k=0}^{k=q} \binom{q}{k} c^k (1 - c)^{q-k} E(q,k)^2,$$

$$\gamma^-(c,T,q) = c \sum_{k=0}^{k=q} \binom{q}{k} (1 - c)^{q-k} c^k E(q,k)^2,$$  \hspace{1cm} (6)

where

$$E(q,k) = \min \left[ 1, \exp \left( \frac{2(q - 2k)}{T} \right) \right].$$  \hspace{1cm} (7)

For the average values of concentration we can also write the rate equation [23], which has the following form in the rescaled time $t$:

$$\langle c(t+1) \rangle = \langle c(t) \rangle + (\gamma^+(c,T) - \gamma^-(c,T)).$$  \hspace{1cm} (8)

In the stationary state $\langle c(t+1) \rangle = \langle c(t) \rangle$, which is equivalent to the condition that the effective force [26]:

$$F(c,T,q) = \gamma^+(c,T,q) - \gamma^-(c,T,q) = 0.$$  \hspace{1cm} (9)

In Fig. 2 we compare the results obtained from the Monte Carlo simulations and numerical solutions of Eq. [23] [recovered by plotting the implicit function $F(c,T,q) = 0$]. Both methods give consistent results and continuous phase transition is visible for all values of $q$. In order to find analytically the value of the critical temperature we can use the method proposed in [17], namely we calculate $T$ for which the following condition is fulfilled:

$$\left. \frac{\partial F}{\partial c} \right|_{c=0.5} = 0,$$  \hspace{1cm} (10)

which in our case takes the form of

$$\sum_{k=0}^{k=q} \binom{q}{k} (2q - 4k - 1) E(q,k) = 0.$$  \hspace{1cm} (11)

For certain small values of $q$ it is possible to show $T_c$ in a compact form (see Table I) while for $q \gg 1$ the solution can be well approximated by $T_c \approx 2q$.

| $q$ | $T_c$ value |
|-----|-------------|
| 1   | 1.82        |
| 2   | 2.49        |
| 3   | 4.86        |
| 4   | 6.22        |
| 5   | 8.35        |
| 6   | 9.90        |

TABLE I. Critical temperature for duplex cliques for first values of $q$.  

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Let us recall here that for the $q$-neighbor Ising model on a monoplex clique the behavior is much richer: for $q = 1$ and $q = 2$ there is no phase transition, for $q = 3$ the phase transition is continuous and for $q \geq 4$ discontinuous [16]. Moreover, for $q > 3$ the hysteresis exhibits oscillatory behavior — expanding for even values of $q$ and shrinking for odd values of $q$. It is worth to stress that the rule used here corresponds to the ANO dynamics [11, 13] which means that a change of a spin is possible only if both layers indicate the change. In case of the second rule, a so-called OR dynamics, where for the spins's flip indication from only one level is sufficient, there is no phase transition for the $q$-neighbor Ising model on a duplex clique, regardless of the value of $q$.  


IV. THE $q$-NEIGHBOR ISING MODEL ON A PARTIALLY DUPLEX CLIQUE

As it has been already mentioned in Sec. I in the partially duplex clique only the fraction $r$ of $N$ nodes has a counterpart in the other layer, remaining nodes belong only to one level (see Fig. 1). Therefore for intermediate topologies, i.e. $r \in (0,1)$ there are two types of nodes at each level: we have $N_d = Nr$ duplex-type nodes (the state is the same on both levels) and $N_m = N(1 - r)$ monoplex-type nodes. The algorithm of a single update of the $q$-neighbor Ising model on a partially duplex clique can be described as follows:

1. Choose randomly a level; the first or the second with equal probability 1/2.

2. From $N$ spins on the selected level choose randomly a single spin $S_i$.

3. If the chosen spin belongs to the subset of duplex nodes, the algorithm looks the same as for the duplex clique.

4. If the spin belongs to the subset of monoplex nodes then:

   (a) From all the neighbors of $S_i$ choose a subset of $q$ neighbors, $nm$ (monoplex nodes have neighbors only on one level).

   (b) Calculate the value of the following function for the original state of the $i$-th spin:

   $$ E(S_i) = -S_i \sum_{j \in \text{nn}} S_j $$

   and the value of the same function for the flipped $i$-th spin, i.e. $E(-S_i)$.

   (c) Flip the $i$-th spin with probability $\min[1, e^{-\Delta E/T}]$, where $\Delta E = E(-S_i) - E(S_i)$

We calculate separately concentration of ‘up’-spins on a single layer for the monoplex nodes ($c_m$) and duplex nodes ($c_d$):

$$ c_m = \frac{N_m^\uparrow}{N_m} $$

$$ c_d = \frac{N_d^\uparrow}{N_d}. $$

Since the average magnetization on both layers is the same we can restrict ourselves to analysis of concentrations on one level.

The transition probabilities for the monoplex nodes that $c_m \to c_m \pm 1/N$ are given by:

$$ \beta_m^+(c, c_m, T, q) = (1 - r)(1 - c_m) \times $$

$$ \times \sum_{k=0}^{\infty} \left( \frac{q}{k} \right) c^{q-k}(1-c)^k E(q, k), $$

$$ \beta_m^-(c, c_m, T, q) = (1 - r)c_m \times $$

$$ \times \sum_{k=0}^{\infty} \left( \frac{q}{k} \right) (1-c)^{q-k} c^k E(q, k) $$

and the corresponding rate equation in the rescaled time $t$:

$$ \langle c_m(t + 1) \rangle = \langle c_m(t) \rangle + \left( \beta_m^+(c) - \beta_m^-(c) \right). $$

The transition probabilities for the duplex nodes that $c_d \to c_d \pm 1/N$ are given by:

$$ \beta_d^+(c, c_d, T, q) = r(1 - c_d) \left[ \sum_{k=0}^{\infty} \left( \frac{q}{k} \right) c^{q-k}(1-c)^k E(q, k) \right]^2 $$

$$ \beta_d^-(c, c_d, T, q) = rc_d \left[ \sum_{k=0}^{\infty} \left( \frac{q}{k} \right) (1-c)^{q-k} c^k E(q, k) \right]^2 $$

and the corresponding rate equation in the rescaled time $t$:

$$ \langle c_d(t + 1) \rangle = \langle c_d(t) \rangle + \left( \beta_d^+(c) - \beta_d^-(c) \right). $$

Finally the total number of ‘up’ spins in the single layer is equal to $N^\uparrow(t) = N_m^\uparrow + N_d^\uparrow$ which after dividing by $N$ gives

$$ \langle c(t) \rangle = (1 - r) \langle c_m(t) \rangle + r \langle c_d(t) \rangle. $$

A. Results for $q \leq 2$

For a monoplex network the phase transition is not present for $q = 1$ and $q = 2$. We solve analytically Eq. [19], for an arbitrary value of $r$ in case of $q = 1$ and $q = 2$; for $q > 2$ we can find solutions numerically applying Runge-Kutta IV fourth-order method. For $q = 1$ we obtain the following relation between the critical temperature $T_c$ and $r$ (see Appendix A for details):

$$ T_c(r) = \frac{2}{\ln \frac{1 + r}{r}}. $$

Phase transition appears for $r > 0$ and it is continuous (when $r \to 0$ we have $T_c \to 0$ and no phase transition). For $r = 1$ we have $T_c = \frac{4}{\ln 2}$ confirming the result from Table I.
We also have two stable (symmetric) solutions which create shapes of two tubes.

For $q = 2$ adding a fraction of duplex nodes causes the appearance of a phase transition for $r > r_c = 2(3\sqrt{2} - 4) \approx 0.485281$. For the range $r \in (r_c; \frac{1}{2})$ we observe an interesting setting with seven real solutions (see Fig. 3). The simplest one $m_1 = 0$ is stable (attracting) for the whole regime of temperature while next two ($m_{2,3} = \pm 1$) are unstable (repellent) for $T \in [0; 0.0992]$. We also have two stable ($m_{4,5}$) and two unstable ($m_{6,7}$) symmetric solutions which create shapes of two tubes. Analyzing solutions for $r > 0.486$ we observe an increase of the tube width: in Fig. 4 we show specific solutions for $r = 0.486$ and $r = 0.495$. Larger tube width corresponds directly to larger area of initial conditions which lead to stable solutions different than $m = 0$. Solutions $m = 0$ and $m = \pm 1$ overlap for any value of $r \in (r_c; \frac{1}{2})$, e.g., $r = 0.485$ lines cover $r = 0.495$ ones in Fig. 4. Finally for $r = \frac{1}{2}$ tubes cover the whole space of solutions (see Fig. 5a) and $m = \pm 1$ solution is absorbed by the tube and becomes stable.

For $r > \frac{1}{2}$ we observe a standard schema for a discontinuous phase transition. The point where two unstable solutions disappear and $m = 0$ becomes a stable one is clearly seen for $r = 0.51$ in Fig. 3. With increasing $r$ this transition becomes weakly discontinuous — stable solutions dominate while the unstable one is visible only for a very small range of $r$ (see consecutive panels in Fig. 5a-f).

For further analysis we define the width of the hysteresis as $T_1^* - T_2^*$ where $T_1^*$ is the temperature for which stable solutions $m \neq 0$ disappear whereas $T_2^*$ is the temperature for which $m = 0$ becomes a stable solution and two unstable solutions disappear. Similarly the jump of the order parameter $m$ is defined as the value of magnetization for stable solution $\Delta m = |m(T_1^*)|$. Figure 6 shows how the value of the jump of magnetization and the width of the hysteresis depend on $r$. Both relations are decreasing and for $r = 1$ both $\Delta m$ as well as $T_1^* - T_2^*$ are equal to zero. These plots confirm the above presented results that the phase transition becomes continuous only for $r = 1$.

**B. Results for $q > 2$**

In the case of $q = 3$ the phase transition is continuous for all values of $r$. This observation is confirmed by two different methods: numerical solutions of Eq. (19) using Runge-Kutta method \cite{32} and Monte Carlo simulations.
FIG. 6. Left panel: the jump of order parameter versus $r$ for $q = 2$. Right panel: the width of the hysteresis $T_1^* - T_2^*$ in a semi-logarithmic scale as a function of $r$ for $q = 2$.

FIG. 7. (Color online) The average magnetization $\langle m \rangle$ as a function of $T$ for $q = 3$ (left panel) and $q = 4$ (right panel). Results were obtained from Monte Carlo simulations (denoted with ‘MC’) with system size $N = 10^4$ thermalization $M = 10^3$ and averaged over $R = 500$ realizations as well as from numerical solutions of Eq. (19) by Runge-Kutta method (denoted with ‘RK’).

FIG. 8. (Color online) The average magnetization $\langle m \rangle$ as a function of $T$ for $q = 6$. All points were obtained from numerical solutions using Runge-Kutta except for the $r = 0$ curve recovered by plotting the implicit function $F(c, T, q) = 0$.

FIG. 9. (Color online) The Dependence between temperatures $T_1^*$, $T_2^*$ and $r$ for $q = 4$ and $q = 6$. All points were obtained from numerical solutions using Runge-Kutta method.

FIG. 10. (Color online) Left panel: the jump of magnetization $\Delta m$ as a function of $r$ for $q = 4$ and $q = 6$. Right panel: the width of the hysteresis $T_1^* - T_2^*$ in a semi-logarithmic scale for $q = 4$ and $q = 6$ versus $r$. All points were obtained from numerical solutions using Runge-Kutta method.

of the model (see left panel in Fig. 7). This is an expected result since for $q = 3$ continuous phase transitions have already been observed for both monoplex [16] and duplex cliques (see Fig. 3). On the other hand we have different expectations for $q = 4$ where for a monoplex clique the phase transition is discontinuous [16] while for the duplex case its character is continuous (Fig. 2). In Fig. 7 (right panel) we show magnetization versus temperature: for $r = 0$ the order parameter jumps, whereas it behaves continuously for $r = 1$. Although these results are obtained numerically, they are consistent with our expectations based on analytic results for monoplex and duplex cliques: as there is a first order phase transition for $r = 0$ and continuous for $r = 1$ there needs to be a threshold value of $r = r^*$ which separates these two regimes. A natural question that emerges here can be formed in the following way “for which value of $r^*$ the transition becomes continuous?” The direct answer might be troublesome especially that one can expect a weak first order phase transition on a border between a first order and continu-
ous phase transitions, analogously to the two-dimensional $q$-state Potts models [27, 28]. However we can measure simultaneously two quantities — the jump of the order parameter and the width of the hysteresis (defined as a distance between spinodal line $T^*_1$ and $T^*_2$ [16]) as a function of $r$ — and then estimate $r^*$. Taking into account fine agreement between Monte Carlo results and numerical solutions of Eq. [11] by Runge-Kutta method we further use the second approach as being faster as well as more precise.

To illustrate the phenomenon of the hysteresis width decreasing and moving we plot Fig. 8 where we present average magnetization ($m$) for $q = 6$ as a function of temperature $T$ for several values of inter-layer connectivity $r = 0, r = 0.06$, $r = 0.1$ and $r = 0.12$. We also compare a semi-analytic result for $r = 0$ with numerical results obtained by Runge-Kutta method — full compliance among the results is visible in Fig. 8.

To calculate the width of the hysteresis $T^*_1 - T^*_2$ we measure two quantities: $T^*_1$ — the temperature for which magnetization is equal zero when we start from ordered initial conditions $m(0) = 1$ and $T^*_2$ — the temperature for which magnetization is equal zero when we start from random initial conditions $m(0) = 0$ (see Fig. 9). Both quantities increase with $r$, moreover the difference between them, i.e., the width of the hysteresis $T^*_1 - T^*_2$, also decays with $r$. A more detailed analysis of the size of the hysteresis is show in the right panel of Fig. 10. In both cases we observe a quadratic decay with $r$ up to $r = 0.21$ and $q = 4$ while it is equal to $10^{-5}$ (i.e., the sampling step of $T$) for $r = 0.19$ and $q = 6$. After crossing these values the width of hysteresis drops down to zero for $r = 0.21$ and $q = 4$ while it is equal to $10^{-5}$ (i.e., the sampling step of $T$) for $r = 0.19$ and $q = 6$. In the case of the jump of magnetization (see left panel in Fig. 10) we also observe a decrease in $\Delta m$ with growing $r$ until reaching $r = 0.21$ and $r = 0.19$, respectively for $q = 4$ and $q = 6$.

To sum it up: for $q = 4$ phase transition is discontinuous for $r \in [0, 0.2]$, and $r \in [0, 0.18]$ for $q = 6$. After crossing these values the transitions become continuous.

In the case of $q = 5$ even for $r = 0$ the transition is weakly discontinuous — the width of the hysteresis equals to $T^*_1 - T^*_2 = 0.0091$, decreasing very fast with $r$ and for $r = 0.015$ dropping down to $10^{-5}$, i.e., temperature sampling step. The jump of the order parameter decays with $r$ as long as $r < 0.016$. Thus we conclude that for small values of $r$ we observe a weakly discontinuous phase transition which changes its character at $r^*(q = 5) = 0.016$ becoming continuous. It is essential to notice here that in this case the range of $r$ where the phase transition changes its character is very short. Therefore we need to stress that our result about the specific value of $r$ for which the we observe a switch of phase transition character should be treated as approximate.

All the above discussed observations regarding the character of phase transitions for $q = 1, \ldots, 6$ are gathered in a $(q, r)$-space phase diagram presented in Fig. 12 where show different behavior of the model: discontinuous phase transition (marked by ‘DC’), continuous phase transition, and a no transition phase. The phase diagram underlines differences between even and odd values of $q$. In particular for odd values continuous phase transition dominates: for $q = 1$ phase transition is continuous for all $r > 0$, in case of $q = 3$ — for the whole regime of $r$ while for $q = 5$ we have discontinuous phase transition only for a small range of $r \in [0, 0.016]$.

In case of even values of $q$ the situation looks strikingly different. Firstly $q = 2$ is a special case with a special phase transition occurring for $r \in (2(3\sqrt{2} - 4), \frac{1}{2})$, classic discontinuous phase for $r \in (\frac{1}{2}, 1)$ and finally continuous phase transition for $r = 1$. In the case of $q = 4$ and $q = 6$ we have discontinuous phase transitions for $r < r^*$ with $r^* = 0.21$ for $q = 4$ and $r^* = 0.19$ for $q = 6$; for $r \geq r^*$ the transition is continuous.

V. CONCLUSIONS

In this study we analyzed modified kinetic Ising model on duplex clique. Adding second level radically
changes the behavior observed in the model — whereas for a monoplex network model \( q \geq 4 \) exhibits discontinuous phase transition, for duplex network we have continuous phase transitions for all value of \( q \).

The mechanism that leads to a discontinuous transition is usually related to fluctuations and discontinuous phase transition is usually observed above a certain critical dimension. Switch from a continuous to a discontinuous phase transition is observed also for larger number of states (like in the Potts model [24] or in a model of tactical voting [25]) or even for a larger number of neighbors (like in the q-voter model with independence on a complete graph [26]). In [11] authors suggest that an additional level plays a similar role to an extra dimension and therefore an increase in the number of levels supports discontinuity. Here for the q-neighbor Ising model we observed opposite situation. It might suggest that the role of another layer is not trivial and specific for a non-equilibrium model like q-Ising and q-voter or threshold model and thus it cannot be easily predicted on the basis of the observation of monoplex network behavior.

Analyzing the dynamics of the q-Ising model on a partially duplex clique we observed very smooth change of the character of phase transition — from discontinuous to continuous with increasing inter-layer connectivity \( r \). Similar like in the q-neighbor Ising model on monoplex topology we observed different behavior for even and odd values of \( q \). We need to underline that as the parameter \( r \) is not discrete like in Potts model [24], the continuous character of \( r \) significantly impedes analysis of model by numerical methods.

In this paper we deliberately focused on a basic topology of duplex cliques. A possible and open future task is finding answers to the question How topology of network influence on phase diagram for q-Ising model ? If we will also observed a changing character of phase transition from discontinuous to continuous one with adding second layer of network ?

### Appendix A: Derivation of analytic equations for \( q = 1 \) and \( q = 2 \) cases with arbitrary \( r \)

In the case of \( q = 1 \) combining the set of equations (19) with \( \beta_0^+ (c) - \beta_0^- (c) = 0 \) and \( \beta_0^+ (c) - \beta_0^- (c) = 0 \) leads to the following three solutions

\[
c_1 = \frac{1}{2}
\]

\[
c_{2,3} = \frac{1}{2} \pm \frac{1 - e^{-2} - \frac{2}{r} - e^{-2} - \frac{2}{r}}{1 + e^{-2} - \frac{2}{r} - e^{-2} - \frac{2}{r}}
\]

(A1)

The nominator under the radical allows us to get the formula (20) from the main text.

[1] S. Boccaletti, G. Bianconi, R. Criado, C.I. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, and M. Zanin, The structure and dynamics of multilayer networks, Phys. Rep. 544, 1-122 (2014).

[2] M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, Multilayer networks, Journal of Complex Networks 2, 203 (2014).

[3] F. Battiston, V. Nicosia, V. Latora, Structural measures for multiplex networks, Phys. Rev. E 89, 032804 (2014).

[4] V. Nicosia, V. Latora, Measuring and modeling correlations in multiplex networks, Phys. Rev. E 90, 032805 (2015).

[5] M. De Domenico, V. Nicosia, A. Arenas, V. Latora, Structural reducibility of multilayer networks, Nature communications 6 6864 (2015).

[6] S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, Diffusion dynamics on multiplex networks, Phys. Rev. Lett. 110, 028701 (2013).

[7] C. Buono, L. G. Alvarez-Zuzek, P. A. Macri, and L. A. Braunstein, Epidemics in Partially Overlapped Multiplex Networks, PLoS ONE 9, e92200 (2014).

[8] C. Granell, S. Gómez, and A. Arenas, Dynamical interplay between awareness and epidemic spreading in multiplex networks, Phys. Rev. Lett. 111, 128701 (2013).

[9] J. Sanz, Ch.-Y. Xia, S. Meloni, and Y. Moreno, Dynamics of Infectious Diseases, Phys. Rev. X 4, 041005 (2014).

[10] M. Diakonova, M. San Miguel, and V. Eguíluz, Absorbing and Shattered Fragmentation Transitions in Multilayer Coevolution, Phys. Rev. E 89, 06218 (2014).

[11] A. Chmiel, K. Sznajd-Weron Phase transitions in the q-voter model with independence on a duplex clique, Phys. Rev. E 92, 052812 (2015).

[12] S. Jang, J.S. Lee, S. Hwang, B. Khang Ashkin-Teller model and diverse opinion phase transitions on multiplex network, Phys. Rev. E 92, 022110 (2015).

[13] C. D. Brummitt, K-M Lee, and K.-I. Goh, Multiplexity-facilitated cascades in network, Phys. Rev. E 85, 045102 (2012).

[14] D. J. Watts, A Simple Model of Global Cascades on Random Networks, Proc. Natl. Acad. Sci. USA 99, 5766 (2002).

[15] K.-M. Lee, C. D. Brummitt, and K.-I. Goh, Threshold cascades with response heterogeneity in multiplex networks, Phys. Rev. E 90, 062816 (2014).

[16] A. Jedrzejewski, A. Chmiel, K. Sznajd-Weron Oscillating hysteresis in the q-neighbor Ising model, Phys. Rev. E 92, 052105 (2015).

[17] P. Nyczka, K. Sznajd-Weron, Anticonformity or Independence? Insights from Statistical Physics, J Stat Phys 151, 174-202 (2013).

[18] A. Pekalski Ising model on small world networks Phys. Rev. E 64.057104 (2001).

[19] A. Aleksiejuk, J. A. Hołyst, D.Stauffer, Ferromagnetic phase transition in Barabasi-Albert networks, Physica A 310 260 - 266 (2002)

[20] S.N Dorogovtsev, A.V. Goltsev , J.F.F Mendes Ising model on networks with an arbitrary distribution of connectionsPhys. Rev. E 66,016104 (2002).
[21] K. Suchecki, Janusz A. Ho/ls Bistable-monostable transi-
tion in the Ising model on two connected complex net-
works Phys. Rev. E 80, 031110 (2009).
[22] M. Diakonova, V. Nicosia, V. Latora, M. San-
Miguel, Irreducibility of multilayer network dynamics: the
case of the voter model New J. Phys 18, 023010 (2016).
[23] V. Spirin, P.L. Krapivsky, S. Redner, Freezing in Ising ferromag-
nets, Physical Review E 65, 016119 (2001).
[24] F. Y. Wu, The Potts model, Rev. Mod. Phys. 54, 235 (1982).
[25] N. A. M. Araujo, J. S. Andrade Jr., and H. J. Her-
rmann, Tactical voting in plurality elections, PLoS ONE 5, e12446 (2010).
[26] P. Nyczka, K. Sznajd-Weron, J. Cislo, Phase transitions in the q-voter model with two types of stochastic driving, Phys. Rev. E 86, 011105 (2012).

[27] L.A. Fernández, J.J. Ruiz-Lorenzo, M.P. Lombardo, A. Tarancón, Weak first order transitions. The two-
dimensional Potts model, Phys. Lett. B 44 485-490 (1992)
[28] L. Schülke and B. Zheng, Dynamic approach to weak first-
order phase transitions, Phys. Rev. E 62 7482 (2000)
[29] P. Grassberger, C. Christensen, G. Bizhani, S. Son nd M. Paczuski, Explosive Percolation is Continuous, but with Unusual Finite Size Behavior, PRL 106, 225701 (2011).
[30] J. Lee and J. M. Kosterlitz Finite-size scaling and Monte Carlo simulations of first-order phase transitions Phys. Rev. B 43, NUMBER 4 3265 43, (1991)
[31] K. Binder Theory of first-order phase transitionsRep. Prog. Phys. 50 783-859 (1987).
[32] https://en.wikipedia.org/wiki/Runge
