A COMBINATORIAL IDENTITY

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Let \( q < p < k \) and \( \nu \) be positive integers, \( n \) be a nonnegative integer, \( v_0 = 1 \) and \( \{ v_1, v_2, \ldots \} \) be a sequence of marks. Further let \( T_{k,j} \) be the Stirling numbers of the first kind defined as the coefficients of

\[
T(x) = \sum_{j=1}^{k} T_{k,j} x^j = x(x-1)(x-2) \cdots (x-k+1)
\]

and let

\[
L(v, p, q) = \sum_{j=1}^{\nu} \sum_{r_1, r_2, \ldots, r_\nu} r_1 r_2 \cdots r_\nu q_{r_1} q_{r_2} \cdots q_{r_\nu},
\]

where the summation is over all the sequences of integers \( r_1, r_2, \ldots, r_\nu \) satisfying

\[
p = r_\nu > r_{\nu-1} > \cdots > r_2 > r_1 = p - q, \quad \text{and} \quad d_i = r_{i+1} - r_i.
\]

In connection with integration of differential equations of a group, A Ran proved in his thesis [1], using analytical methods, that

\[
\sum_{j=1}^{k} T_{k,j} L(j+n, p, q) = 0
\]

identically, i.e., that on the left side of (3) the coefficient of every product \( \Pi q_i \) equals zero.

Here the proof of (3) is given by combinatorial methods. To begin we write (2) in the form

\[
L(v, p, q) = \sum_{a=0}^{q} R(v, p, q, a, \pi v_i^{a_i}) \Pi v_i^{a_i},
\]

where the summation \( \sum_{a} \) is over all sequences of nonnegative integers \( a_1, a_2, \ldots, a_q \) satisfying \( \sum a_i = q \), and

\[
a = \sum a_i,
\]

and prove the following

**Lemma.**

(6)

\[
R(v, p, q, a, \pi v_i^{a_i}) = \sum_{h=0}^{q} c_h (p-h)^v,
\]

where the coefficients \( c_h \) do not depend on \( v \) (but may depend on \( p, q, a \) and \( \pi v_i^{a_i} \)) and are such that

\[
\sum_{h=0}^{q} c_h (p-h)^t = 0, \quad t = 0, 1, \ldots, a-1.
\]

**Proof.** The proof is given by induction on \( a \). For \( a = 1 \) we have

\[
R(v, p, q, 1, v_0) = (p-q) \sum_{j=0}^{v-1} p^j (p-q)^{v-j-1} = \frac{p-q}{q} (p^v - (p-q)^v),
\]

49
which satisfies both (6) and (7).

Suppose now that (6) and (7) are satisfied for \( a = b - 1 \). It is easily seen that

\[
R(v, p, q, b, \pi_{i}^{Qj}) = \sum_{\eta} (p - \eta) \sum_{\beta=0}^{v-b} \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)^{v-b-1}}{h} R(v - \beta - 1, p - \eta, q - \eta, b - 1, \pi_{i}^{Qj}/\eta),
\]

where \( \eta \) obtains the values of \( i \) for which \( c_{ij} > 1 \). We make use of (6) with \( a = b - 1 \) and in order to stress that the coefficients \( c_{\eta,h} \) depend on \( \eta \) we write them in the form \( c_{\eta,h} \). We have

\[
R(v, p, q, b, \pi_{i}^{Qj}) = \sum_{\eta} (p - \eta) \sum_{\beta=0}^{v-b} \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)^{v-b-1}}{h} R(v - \beta - 1, p - \eta, q - \eta, b - 1, \pi_{i}^{Qj}/\eta),
\]

\[
= \sum_{\eta} (p - \eta) \left[ \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)}{h} (p^{v} - (p-h)^{v}) - \sum_{\beta=0}^{v-b+1} \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)}{h} (p^{v} - (p-h)^{v}) \right].
\]

By (7) follows that

\[
\sum_{h=0}^{q} c_{\eta,h}(p-h)^{v-b-1} = 0
\]

for every \( \eta \) and for \( 0 < v - \beta - 1 < b - 2 \), i.e., for \( v - b + 1 < \beta < v - 1 \) and consequently

\[
R(v, p, q, b, \pi_{i}^{Qj}) = \sum_{\eta} (p - \eta) \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)}{h} (p^{v} - (p-h)^{v})
\]

which proves (6) for \( a = b \).

To prove (7) let us denote for every \( \eta \)

\[
D_{\eta}(t) = \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)}{h} (p^{t} - (p-h)^{t}).
\]

Evidently \( D_{\eta}(0) = 0 \). For \( t > 1 \) we have

\[
D_{\eta}(t) = \sum_{h=0}^{q} \frac{c_{\eta,h}(p-h)}{h} \cdot h \sum_{i=0}^{t-1} (p-h)^{t-i} = \sum_{h=0}^{q} \sum_{h=0}^{t-1} c_{\eta,h}(p-h)^{t-i-i}.
\]

By (7) with \( a = b - 1 \),

\[
\sum_{h=0}^{q} c_{\eta,h}(p-h)^{t-i-1} = 0
\]

for \( t = 1, 2, \cdots, b - 1 \) and \( 0 < i < t - 1 \) and consequently \( D_{\eta}(t) = 0 \) for \( 0 < t < b - 1 \). By (6), (8) and (9),

\[
\sum_{h=0}^{q} c_{\eta,h}(p-h)^{t} = R(t, p, q, b, \pi_{i}^{Qj}) = \sum_{\eta} (p - \eta) D_{\eta}(t) = 0, \quad t = 0, 1, \cdots, b - 1
\]

which proves (7) with \( a = b \).

Theorem.

\[
\sum_{j=1}^{k} T_{k,j} L(j + n, p, q) = 0.
\]
Proof. By (4), (6) and (1) we have

\[
\sum_{j=1}^{k} T_{k,j} L(j + n, p, q) = \sum_{j=1}^{k} T_{k,j} \prod_{i=1}^{q} \sum_{h=0}^{q} c_{h}(p - h)^{j+n}
\]

\[
= \sum_{j=1}^{k} \prod_{i=1}^{q} \sum_{h=0}^{q} c_{h}(p - h)^{j} T_{k,j}(p - h)^{j} = \sum_{j=1}^{k} \prod_{i=1}^{q} \sum_{h=0}^{q} c_{h}(p - h)^{j} f(p-h).
\]

By definition \( p - h \) is an integer satisfying \( 1 < p - h < p < k - 1 \) and consequently by (1), \( f(p-h) = 0 \) which proves the theorem.

**REFERENCE**

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[Continued from Page 48.]

Much more recently (1973), Jakobczyk [6] has given new iterative procedures for determining answers to both:

(a) for each \( k \), \( 1 < k < N \), which will be the \( k^{th} \) place to be cast out?

(b) for each \( k \), \( 1 < k < N \), when will the \( k^{th} \) place be cast out?

(The "Oberreihen" methods described by Ahrens also provide answers to both questions.)

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