The stability of a bouncing universe.

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We investigate the stability of a spatially homogeneous and isotropic non-singular cosmological model. We show that the complete set of independent perturbations (the electric part of the perturbed Weyl tensor and the perturbed shear) are regular and well behaved functions which have no divergences, contrary to previous claims in the literature.

I. INTRODUCTION

The existence of singularities appears to be a property inherent to most of the physically relevant solutions of Einstein equations, in particular to all known up-to-date black hole and conventional cosmological solutions (11). In the case of black holes, to avoid the singularity some models have been proposed (2, 3, 4, 5). These models nonetheless are not exact solutions of Einstein equations since there are no physical sources associated to them. Many attempts try to solve this problem by modifying general relativity (6, 7, 8). More recently it has been shown that in the framework of standard general relativity it is possible to find spherically symmetric singularity-free solutions of the Einstein field equations that describe a regular black hole. The source of these solutions are generated by suitable nonlinear vector field Lagrangians, which in the weak field approximation become the usual linear Maxwell theory (9, 10, 11). Similarly in Cosmology many non-singular cosmological models with bounce were constructed where the energy conditions or the validity of Einstein gravity were violated. Such models are based on a variety of distinct mechanisms, such as a cosmological constant (12), non-minimal couplings (13), nonlinear Lagrangians involving quadratics terms in the curvature (14), modifications of the geometric structure of space-time (15), quantum gravity (16), and nonequilibrium thermodynamics (17), among others to restrict ourselves to homogeneous and isotropic solutions. Further investigations on regular cosmological solutions can be found in (18).

In a previous paper (19) we have investigated a cosmological model with a source produced by a nonlinear generalization of electrodynamics and succeeded to obtain a regular cosmological model. The Lagrangian of our model is a function of the field invariants up to second order. This modification is expected to be relevant when the fields reach large values, as is the case in the primeval era of our universe. The model is in the framework of the Einstein field equations and the bounce is possible because the singularity theorems (20) are circumvented by the appearance of a negative pressure (although the energy density is positive definite). Recently some papers started a detailed investigation of the transition from contraction to expansion in the bounce of several models (21). In particular, in Einstein general relativity, models with stress-energy sources constituted by a collection of perfect fluids and Friedmann-Robertson-Walker like geometry were examined (22). The claim in that paper is that a generic result about the behavior of scalar adiabatic perturbations was obtained. The result is the following: scalar adiabatic perturbations can grow without limit in two situations represented by the points where the scale factor attains its minimum value and where $\rho + p = 0$. The first point corresponds to the moment in which the Universe passes through the bounce; the second corresponds to the transition from the region where the Null Energy Condition (NEC) is violated to the region where it is not. We will show that these instabilities are not an intrinsic property of a model with bounce as claimed in reference [24] but a consequence of the existence of a divergence already appearing in the background solution if the description of the source is made in terms of a perfect fluid. We will present a specific example of a model with bounce, generated by a source representing two non-interacting perfect fluids, that has regular perturbations in the situations described on reference [24].

II. THE MODEL

We limit our analysis to a model (19) in which the singularity of the Friedmann-Robertson-Walker (FRW) geometry is avoided by the introduction of nonlinear corrections to Maxwell electrodynamics. We will consider modifications up to second order terms in the field invariants

$$ L = -\frac{1}{4} F + \alpha F^2 + \beta G^2, \quad (1) $$

where $F = F_{\mu\nu} F^{\mu\nu}$, $G = \frac{1}{2} \epsilon_{\mu\nu\beta\gamma} F^{\mu\nu} F^{\beta\gamma}$, $\alpha$ and $\beta$ are arbitrary constants (20). The term $FG$ will not be included in order to preserve parity. The energy-momentum tensor for nonlinear electrodynamics theories reads

$$ T_{\mu\nu} = -4 L F_{\mu}^{\phantom{\mu}\alpha} F_{\alpha\nu} + (GL - L) g_{\mu\nu}, \quad (2) $$
where \( L_F \) represents the partial derivative of the Lagrangian with respect to the invariant \( F \) and similarly for the invariant \( G \). In the early universe, matter should be identified with a primordial hot plasma \([28, 30]\) and as a consequence we are led to the case in which only the average of the squared magnetic field survives (see \([19]\) and references cited there for details). Since the average procedure is independent of the equations of the electromagnetic field we can use it in the generic expression of the energy-momentum tensor to obtain

\[
T_{\mu\nu} = (\rho + p) v_{\mu} v_{\nu} - p g_{\mu\nu},
\]

(3)

where

\[
\rho_\gamma = \frac{1}{2} H^2 (1 - 8 \alpha H^2),
\]

(4)

\[
p_\gamma = \frac{1}{6} H^2 (1 - 40 \alpha H^2).
\]

(5)

The standard result of the linear Maxwell theory can be recovered by setting \( \alpha = \beta = 0 \).

We set for the fundamental line element

\[
ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right],
\]

(6)

where \( \epsilon = -1, 0, +1 \) hold for the open, flat (or Euclidean) and closed cases, respectively. The Einstein’s equations and the equation of energy conservation written for this metric become:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{\epsilon}{a^2} - \frac{1}{3} \rho_\gamma = 0,
\]

(7)

\[-2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\epsilon}{a^2} - p_\gamma = 0,
\]

(8)

\[
\dot{\rho}_\gamma + 3(\rho_\gamma + p_\gamma) \frac{\dot{a}}{a} = 0,
\]

(9)

Inserting (4) and (5) in (8) yields for the magnetic field:

\[
H = H_0 a^{-2},
\]

(10)

where \( H_0 \) is an arbitrary constant. With this result the equation (7) can be integrated. For the case \( \epsilon = 0 \) the solution is:

\[
a(t)^2 = H_0 \left[ \frac{2}{3} (kc^2 t^2 + 12 \alpha) \right]^{1/2}.
\]

(11)

The interpretation of the source as a one component perfect fluid in an adiabatic regime has some difficulties that are at the origin of the instabilities found in \([24]\). The sound velocity of the fluid in this case is given by (57)

\[
\left( \frac{\partial p_{\gamma}}{\partial \rho_{\gamma}} \right) = \frac{\dot{p}_{\gamma}}{\rho_{\gamma}} = -\frac{\dot{\rho}_{\gamma}}{\theta (\rho_{\gamma} + p_{\gamma})}
\]

(12)

This expression, involving only the background, is not defined at the points where the energy density attains an extremum given by \( \theta = 0 \) and \( \rho_\gamma + p_\gamma = 0 \). In terms of the cosmological time they are determined by \( t = 0 \) and \( t = \pm t_c = 12 \alpha / k c^2 \). These points are well-behaved regular points of the geometry indicating that the description of the source is not appropriate.

This difficulty can be circumvented if we adopt another description for the source of the model. This can be achieved if we separate the part of the source related to Maxwell dynamics from the additional non-linear \( \alpha \)-dependent term on the Lagrangian. By doing this the source automatically splits in two non-interacting perfect fluids:

\[
T_{\mu\nu}^1 = T_{\mu\nu}^1 + T_{\mu\nu}^2,
\]

(13)

where,

\[
T_{\mu\nu}^1 = (\rho_1 + p_1) v_{\mu} v_{\nu} - p_1 g_{\mu\nu},
\]

(14)

\[
T_{\mu\nu}^2 = (\rho_2 + p_2) v_{\mu} v_{\nu} - p_2 g_{\mu\nu},
\]

(15)

and

\[
\rho_1 = \frac{1}{2} H^2
\]

(16)

\[
p_1 = \frac{1}{6} H^2.
\]

(17)

\[
\rho_2 = -4 \alpha H^4.
\]

(18)

\[
p_2 = -\frac{20}{3} \alpha H^4.
\]

(19)

Using the above decomposition it follows that each one of the two components of the fluid satisfy independently equation (6). This indicates that the source can be described by two non-interacting perfect fluids with equations of state \( p_1 = 1/3 \rho_1 \) and \( p_2 = 5/3 \rho_2 \). The equation of state for the second fluid should be understood only formally as a mathematical device to allow for a fluid description.

III. GAUGE INVARIANT TREATMENT OF PERTURBATION

In a series of papers \([31, 32, 33]\) we have established a complete and self-consistent theory to deal with the problem of perturbations of the FRW cosmology. The very well-known problem of the gauge dependence of the perturbations was addressed and solved by the introduction of a complete set of gauge invariants variables that represents direct observable quantities. We present here a summary of this formalism in order to fix the notation and to aim for the self consistence of this paper.

The source of the background geometry is represented by two-fluids, each one having an independent equation
of state relating the pressure and the energy density \( p_i = \lambda_i \rho_i \), where \( i = 1, 2 \). Following the standard procedure we consider arbitrary perturbations that preserve each equation of state. Thus the general form of the perturbed energy-momentum tensor is written as

\[
\delta T^\mu_\nu = (1 + \lambda_1) \delta (\rho_1 v_\mu v_\nu) - \lambda_1 \delta (\rho_1 g_{\mu\nu}).
\]  

(20)

The background geometry is conformally flat. Thus any perturbation of the Weyl tensor is a true perturbation of the gravitational field. It is convenient to represent the Weyl tensor \( W_{\alpha\beta\gamma\delta} \) in terms of its corresponding electric and magnetic parts (these names come from the analogy with the electromagnetic field):

\[
E_{\alpha\beta} = -W_{\alpha\beta\gamma\delta} v^\gamma v^\delta
\]  

(21)

\[
H_{\alpha\beta} = -W^{*}_{\alpha\beta\gamma\delta} v^\gamma v^\delta.
\]  

(22)

These variables have the advantage that since they are null in the background, their perturbations are gauge invariant quantities. These definitions imply for the tensors \( E_{\mu\nu} \) and \( H_{\mu\nu} \) the following properties:

\[
E_{\mu\nu} = E_{\nu\mu}, \quad E_{\mu\nu} g^{\mu\nu} = 0,
\]

and for the magnetic tensor:

\[
H_{\mu\nu} = H_{\nu\mu}, \quad H_{\mu\nu} g^{\mu\nu} = 0.
\]

A. Some mathematical machinery

The metric \( g_{\mu\nu} \) and the vector \( v^\mu \) (tangent to a timelike congruence of curves \( \Gamma \) ) induce a projector tensor \( h_{\mu\nu} \) which separates any tensor in terms of quantities perpendicular and parallel to \( v^\mu \). The projector is defined as

\[
h_{\mu}^\nu \equiv \delta_{\mu}^\nu - v_{\mu} v^{\nu},
\]  

(23)

and has the property that

\[
h_{\mu\nu} h^{\nu\lambda} = h_{\mu}^\lambda.
\]  

(24)

The equations of motion for the first order perturbations are linear so it is useful to develop all perturbed quantities in the spherical harmonics basis. In this paper we will limit our analysis to perturbations represented in the scalar base defined by the equation:

\[
h^{\nu\lambda} \tilde{\nabla}_\nu \nabla_\lambda Q_{(n)} = -m n^2 \frac{1}{\alpha^2} Q_{(n)}, \]

(25)

where \( m \) is the wave number and \( \tilde{\nabla}_\mu \) is the covariant derivative in the hypersurface with normal \( v^\mu \) and metric \( h_{\mu\nu} \). From now on we will suppress the index \( n \).

The scalar \( Q \) allows us to define the associated vector \( \pi_\mu \) and traceless tensor \( F_{\mu\nu} \):

\[
\pi_\mu = \frac{\alpha^2}{m^2} h_{\mu}^\nu \tilde{\nabla}_\nu Q,
\]  

(26)

\[
P_{\mu\nu} = \tilde{\nabla}_\mu \pi_\nu - \frac{1}{3} h_{\mu\nu} Q.
\]  

(27)

In the case of scalar perturbations the fundamental set of equations, determining the dynamics of the perturbations are (see the appendix):

\[
(\delta E_{\mu\nu}^\alpha)^* h_{\mu}^\alpha h_{\nu}^\beta + (\delta E_{2\mu}^\alpha)^* h_{\mu}^\alpha h_{\nu}^\beta + \Theta (\delta E_{1\alpha}^\beta + \delta E_{2\alpha}^\beta)
\]

\[
= -\frac{1}{2} (\rho_1 + \rho_2) \delta_{\alpha}^\beta
\]

\[
- \frac{1}{2} (\rho_2 + \rho_2) \delta_{\alpha}^\beta
\]  

(28)

\[
(\delta E_{1\alpha\mu} + \delta E_{2\alpha\mu}) \nabla_{\alpha} h^{\alpha\nu} h_{\mu\nu} = \frac{1}{3} (\delta \rho_1 + \delta \rho_2) h_{\alpha\nu}^\mu
\]

\[
- \frac{1}{3} \rho_1 \delta v_1^\alpha - \frac{1}{3} \rho_2 \delta v_2^\alpha
\]  

(29)

\[
(\delta \sigma_{1\mu})^* + (\delta \sigma_{2\mu})^* + \frac{1}{3} \delta_{\mu\nu} (\delta a_{1\alpha\beta} + \delta a_{2\alpha\beta})
\]

\[
- \frac{1}{3} (\delta a_{1\alpha\beta} + \delta a_{2\alpha\beta}) h_{\mu}^\alpha h_{\nu}^\beta
\]

\[
+ \frac{2}{3} \Theta (\delta \sigma_{1\mu} + \delta \sigma_{2\mu}) = -\delta E_{1\mu}^\nu - \delta E_{2\mu}^\nu,
\]

(30)

\[
- \lambda_1 \delta (\rho\beta h_{\beta}^\mu) + (1 + \lambda_1) \rho \delta a_{\mu}^\beta = 0
\]

(31)

\[
- \lambda_2 \delta (\rho h_{\beta}^\mu) + (1 + \lambda_2) \rho \delta a_{\mu}^\beta = 0.
\]  

(32)

The acceleration \( a^\mu \), the expansion \( \Theta \) and the shear \( \sigma_{\mu\nu} \) that appear in the above equations are parts of the irreducible components of the covariant derivative of the velocity field defined as:

\[
a^\mu = v^\mu_{,\nu} v^\nu,
\]  

(33)

\[
\Theta = v^\mu_{,\mu},
\]  

(34)

\[
\sigma_{\alpha\beta} = \frac{1}{2} h^{\mu\nu} (h_{\alpha\beta})_{\mu\nu} - \frac{1}{3} \Theta h_{\alpha\beta}.
\]  

(35)

The expansion of the perturbations in terms of the spherical harmonic basis is

\[
\delta \rho = N(t) Q,
\]  

(36)

\[
\delta v^\mu = V(t) h^{\alpha\mu} Q_{(1)},
\]  

(37)

\[
\delta a^\mu = \tilde{V} h^{\alpha\mu} Q_{(1)},
\]  

(38)

\[
\delta E_{\mu\nu} = E(t) P^{\mu\nu},
\]  

(39)

\[
\delta \sigma^{\mu\nu} = \Sigma(t) P^{\mu\nu},
\]  

(40)
B. Perturbation of a bouncing universe

After presenting the necessary formalism we shall to start the analysis of the perturbations of the bouncing cosmological model displayed in previous section. Using the above expansion into the equations (cf. appendix) \[100, 103, 105, 111\] and \[32\] we obtain:

\[E_1 + E_2 = \frac{a^2}{6\epsilon + k^2}(N_1 + N_2 - \rho_1 V_1 - \rho_2 V_2), \quad (41)\]

\[\dot{E}_1 + \dot{E}_2 + \frac{1}{3} \Theta (E_1 + E_2) = - \left( \frac{1 + \lambda_1}{2} \right) \rho_1 \Sigma_1 - \left( \frac{1 + \lambda_1}{2} \right) \rho_2 \Sigma_2, \quad (42)\]

\[\Sigma_1 + \Sigma_2 - \dot{V}_1 - \dot{V}_2 = -E_1 - E_2, \quad (43)\]

\[-\lambda_1 (N_1 - \dot{\rho}_1 V_1) + (1 + \lambda_1) \rho_1 \dot{V}_1 = 0, \quad (44)\]

\[-\lambda_2 (N_2 - \dot{\rho}_2 V_2) + (1 + \lambda_2) \rho_2 \dot{V}_2 = 0, \quad (45)\]

These equations can be rewritten in a more convenient way as:

\[\dot{\Sigma}_1 = - \left( \frac{2 \lambda_1 (3 \epsilon + k^2)}{a^2 (1 + \lambda_1) \rho_1} + 1 \right) E_1, \quad (46)\]

\[\dot{\Sigma}_2 = - \left( \frac{2 \lambda_1 (3 \epsilon + k^2)}{a^2 (1 + \lambda_2) \rho_2} + 1 \right) E_2, \quad (47)\]

\[\dot{E}_1 + \frac{1}{3} \Theta E_1 = - \frac{1}{2} (1 + \lambda_1) \rho_1 L \Sigma_1, \quad (48)\]

\[\dot{E}_2 + \frac{1}{3} \Theta E_2 = - \frac{1}{2} (1 + \lambda_2) \rho_2 \Sigma_2, \quad (49)\]

As we have shown in \([31]\), the whole set of scalar perturbations can be expressed in terms of the two basic variables: \(E_i\) and \(\Sigma_i\). The corresponding equations can be decoupled. The result in terms of variables \(E_i\) is the following:

\[\dot{E}_i + \frac{4 + 3 \lambda_i}{3} \Theta E_i + \left( \frac{2 + 3 \lambda_i}{9} \right) \Theta^2 \]

\[- \frac{2}{3} (1 + \lambda_i) \rho_i \]

\[- \frac{1}{6} (1 + 3 \lambda_i) \rho_j - \frac{3 + k^2) \lambda_i}{a^2} \right) E_i = 0. \quad (50)\]

Note that there is no sum in indices and \(j \neq i\) in this expression. In our case the values of \(\lambda_i\) are \(\lambda_i = \left( \frac{4}{3}, \frac{5}{3} \right)\). In the first case the equation for the variable \(E_1\) became:

\[\dot{E}_1 + \frac{5}{3} \Theta E_1 + \left[ \frac{1}{3} \Theta^2 - \rho_1 - \rho_2 - \frac{5 k^2}{3 a^2} \right] E_1 = 0 \quad (51)\]

We should analyze the behavior of these perturbations in the neighborhood of the points where the energy density attain an extremum. This means not only the bouncing point but also the point in which \(p + \dot{p}\) vanishes. Let us start the examination at the bouncing point \(t = 0\.

The expansion of the equation of \(E_1\) in the neighborhood of the bouncing, up to second order, is given by:

\[\dot{E}_1 + a \dot{E}_1 + (b + b_1 t^2) E_1 = 0 \quad (52)\]

The constant \(a\) and the parameter \(b\) are defined as follows

\[a = \frac{5}{2 \epsilon c} \quad (53)\]

\[b = \frac{m^2}{\sqrt{6} H_0 t_c} \quad (54)\]

\[b_1 = \frac{b}{2 \epsilon c} - \frac{3}{4 t_c} \quad (55)\]

Defining a new variable \(f\) as

\[f(t) = E_1(t) \exp \left( \frac{a}{4} - \frac{i}{2} \sqrt{b_1 - \frac{a^2}{4}} \right) t^2, \quad (56)\]

doing a coordinate transformation for time as indicated below

\[\xi = -i \sqrt{b_1 - \frac{a^2}{4}} t^2, \quad (57)\]

We obtain the following confluent hypergeometric equation \([35]\)

\[\xi \ddot{f} + (1/2 - \xi) \dot{f} + e f = 0, \quad (58)\]

where

\[e = \frac{i (b - a/2)}{4 (b_1 - a^2/4)^{1/2} - 1/2} \quad (59)\]

The solution of this equation is given by:

\[f(t) = A M(d, 1/2, -i \sqrt{b_1 - \frac{a^2}{4}} t^2), \quad (60)\]

where \(A\) is an arbitrary constant and \(M(d, 1/2, \xi)\) confluent hypergeometric function. The confluent hypergeometric function is well behaved in this neighborhood and so also is the perturbation \(E_1(t)\) given by:

\[E_1(t) = Re[A M(d, 1/2, -i \sqrt{b_1 - \frac{a^2}{4}} t^2) \]

\[* exp \left( -\frac{a}{4} + \frac{i}{2} \sqrt{b_1 - \frac{a^2}{4}} t^2 \right)] \quad (61)\]
The perturbation $E_2$ at this same neighborhood, after the same procedure we did before result in the following equation:

$$\dot{E}_2 + a\dot{E}_2 + (b + b_1 t^2)E_2 = 0$$  \hspace{1cm} (62)

This is the same equation we obtained for $E_1$, they differ only by the values of the parameters $a, b$ and $b_1$ that in this case are:

$$a = \frac{9}{2t_c^2}$$  \hspace{1cm} (63)

$$b = \frac{3}{2t_c^2} - \frac{5}{\sqrt{6}H_0 t_c}$$  \hspace{1cm} (64)

$$b_1 = \frac{5m^2}{t_c^3 H_0 \sqrt{6}} - \frac{5}{t_c^2}$$  \hspace{1cm} (65)

Then the solution in this case is:

$$E_2(t) = \exp[A M \left( d, 1/2, -i(\sqrt{b_1 - a^2/4})t^2 \right) \times \exp\left( -\frac{a}{4} - \frac{i}{2} \sqrt{b_1 - a^2/4}t^2 \right)],$$  \hspace{1cm} (66)

Again the confluent hypergeometric function is well behaved in this neighborhood and so also is the perturbation $E_2(t)$. At the neighborhood of the point $t = t_c$ the equation for the perturbation $E_1$ is given by

$$\dot{E}_1 + a\dot{E}_1 + (b + b_1 t)E_1 = 0$$  \hspace{1cm} (67)

where the parameters $a, b, b_1$ in this case are given by

$$a = \frac{5}{4t_c}$$  \hspace{1cm} (68)

$$b = \frac{3}{4t_c^2} - \frac{\sqrt{3}m^2}{6H_0 t_c}$$  \hspace{1cm} (69)

$$b_1 = \frac{\sqrt{3}}{4t_c^2} \left( \frac{m^2}{3H_0} - \frac{3}{2t_c} \right)$$  \hspace{1cm} (70)

We would like to remark that this equation is different from the equations (62) obtained in the neighborhood of $t = 0$. We proceed doing the following variable transformation:

$$E_1(t) = \exp\left( \frac{at}{2} \right)w(t)$$  \hspace{1cm} (71)

The differential equation for this new variable is

$$\ddot{w} + (b - (a/2)^2 + b_1 t)w = 0$$  \hspace{1cm} (72)

The solution for this equation is

$$w(t) = \left[ w_0 \text{AiryAi} \left( \frac{b - (a/2)^2 + b_1 t}{b_1^{2/3}} \right) \right]$$  \hspace{1cm} (73)

The AiryAi are regular well behaved functions in this neighborhood and so also the perturbations $E_1$. Finally we look for the equation of $E_2$ at the neighborhood of $t = t_0$, it becomes

$$\ddot{E}_2 + a\dot{E}_2 + (b + b_1 t)E_2 = 0$$  \hspace{1cm} (74)

where the parameters $a, b, b_1$ in this case are given by

$$a = \frac{9}{4t_c}$$  \hspace{1cm} (75)

$$b = \frac{5}{t_c} \left( \frac{5}{4t_c} - \frac{\sqrt{3}m^2}{6H_0} \right)$$  \hspace{1cm} (76)

$$b_1 = \frac{5\sqrt{3}}{2t_c^2} \left( \frac{1}{t_c} - \frac{m^2}{6H_0} \right)$$  \hspace{1cm} (77)

This equation differ from eq.(67) only by the numerical values of the parameters $a, b, b_1$ so we obtain the same regular solution

$$E_2 = \exp\left( \frac{at}{2} \right)w_0 \text{AiryAi} \left( \frac{b - (a/2)^2 + b_1 t}{b_1^{2/3}} \right)$$  \hspace{1cm} (78)

IV. CONCLUSION

Recently there has been a renewed interest on nonsingular cosmology. As a direct consequence of this some authors have argued against these models based on instability reasons. In (24) it has been argued that a rather general analysis shows that there are instabilities associated to some special points of the geometrical configuration. They correspond to the points of bouncing of the model and maxima of the energy density, where the description of the matter content in terms of a single perfect fluid does not apply. In the present paper we have shown, by a direct analysis of a specific nonsingular universe, that the result claimed in the quoted paper does not apply to our model. We took the example from a recent paper (13) in which the avoidance of the singularity comes from a non linear electrodynamic theory. We used the quasi Maxwellian equations of motion (31, 32, 33) in order to undertake the analysis of the perturbed set of Einstein equations of motion. We showed that in the neighborhood of the special points in which a change of regime occurs, all independent perturbed quantities are well behaved. Consequently the model does not present any difficulty concerning its instability. This paves the way to investigate models with bounce in more detail and to consider them as good candidates to describe the evolution of the Universe.
V. APPENDIX A: COMPARISON WITH OTHERS GAUGE-ININVARIANT VARIABLES

FRW cosmology is characterized by the homogeneity of the fundamental variables that specify its kinematics (the expansion factor Θ), its dynamics (the energy density ρ) and its associated geometry (the scalar of curvature R). This means that these three quantities depend only on the global time t, characterized by the hypersurfaces of homogeneity. We can thus use this fact to define in a trivial way 3-tensor associated quantities, which vanish in this geometry, and look for its corresponding non-identically vanishing perturbation. The simplest way to do this is just to let U be a homogeneous variable (in the present case, it can be any one of the quantities ρ, Θ or R), that is U = U(t). Then use the 3-gradient operator

\[ (3)^\mu \nabla \equiv h_\mu^\lambda \nabla_\lambda \]  

(79)
to produce the desired associated variable

\[ U_\mu = h_\mu^\lambda \nabla_\lambda U. \]  

(80)

In [36] these quantities were discussed and its associated evolution analysed. In the present section we will exhibit the relation of these variables to our fundamental ones. We shall see that under the conditions of our analysis [39] these quantities are functionals of our basic variables (E and Σ) and of the background ones.

A. The matter variable χi

It seems useful to define the fractional gradient of the energy density χα as [36]

\[ \chi_\alpha \equiv \frac{1}{\rho} (3)^\alpha \nabla_\alpha \rho. \]  

(81)

Such quantity χα is nothing but a combination of the acceleration and the divergence of the anisotropic stress. Indeed, from the above equations it follows (in the frame in which there is no heat flux)

\[ \delta \chi_i = \frac{(1 + \lambda)}{\lambda} \delta u_i + \frac{1}{\lambda \rho} \delta \Pi_i^{\beta;\beta} \]  

(82)

\[ \delta \chi_i = -2 \left( 1 - \frac{3K}{m} \right) \frac{1}{\rho A^2} \left( E - \frac{\xi}{2} \Sigma \right) Q_i. \]  

(83)

B. The kinematical variable δη

The only non-vanishing quantity of the kinematics of the cosmic background fluid is the (Hubble) expansion factor Θ. This allows us to define the quantity ηα as:

\[ \eta_\alpha = h_\alpha^\beta \Theta_{,\beta}. \]  

(84)

Using the constraint relation equation (97) we can relate this quantity to the basic ones:

\[ \delta \eta_i = -\frac{\Sigma}{A^2} \left( 1 - \frac{3K}{m} \right) Q_i. \]  

(85)

C. The geometrical variable τ

We can choose the scalar of curvature R which depends only on the cosmical time t like ρ and Θ to be the U-geometrical variable. However it seems more appealing to use a combined expression τ involving R, ρ and Θ given by

\[ \tau = R + (1 + 3\lambda) \rho - \frac{2}{3} \Theta^2. \]  

(86)

In the unperturbed FRW background this quantity is defined in terms of the curvature scalar of the 3-dimensional space and the scale factor A(t):

\[ \frac{(3)^R}{A^2}. \]

We define then the new associated variable τα as

\[ \tau_\alpha = h_\alpha^\beta \tau_{,\beta}. \]  

(87)

This quantity τα vanishes in the background. Its perturbation can be written in terms of the previous variations, since Einstein’s equations give

\[ \tau = 2 \left( \rho - \frac{1}{3} \Theta^2 \right). \]

VI. APPENDIX B: QUASI-MAXWELLIAN EQUATIONS

We list below the quasi-Maxwellian equations of gravity. They are obtained from Bianchi identities as true dynamical equations which describe the propagation of gravitational disturbances. Making use of Einstein’s equations and the definition of Weyl tensor, Bianchi identities can be written in an equivalent form as

\[ W^{\alpha\beta\nu} = \frac{1}{2} R^{\alpha[\alpha;\beta]} - \frac{1}{12} g^{[\alpha} R^{\beta]. \]  

Using the decomposition of Weyl tensor in terms of Eαβ and Hαβ (see Section [31]) and projecting appropriately, Einstein’s equations can be written in a form
which is similar to Maxwell’s equations. There are 4 independent projections for the divergence of Weyl tensor, namely:
\[
W^{\alpha\beta\mu\nu} V_{\beta} V_{\mu} h_{\alpha}^{\sigma}, \\
W^{\alpha\beta\mu\nu} \eta^{\beta\gamma} \alpha_{\beta} V_{\mu} V_{\lambda}, \\
W^{\alpha\beta\mu\nu} h_{\mu}^{\sigma} \eta^{\gamma} \alpha_{\beta} V_{\lambda}, \\
W^{\alpha\beta\mu\nu} V_{\beta} h_{\mu}(\tau h_{\sigma})_{\alpha}.
\]

The unperturbed quasi-Maxwellian equations are thus given by:
\[
\begin{align*}
\eta^{\alpha} h^{\lambda\gamma} E_{\alpha\lambda;\gamma} + \eta^{\beta\mu} V^{\beta} H^{\mu\lambda} \sigma_{\mu \lambda} + 3 H^{\mu\nu} \omega_{\mu
u} \\
= \frac{1}{3} h^{\rho \alpha} \rho_{\alpha} + \frac{\Theta}{3} \equiv - \frac{1}{2} (\sigma^{\gamma} \nu - 3 \omega^{\gamma} \nu) q^{\nu} \\
+ \frac{1}{2} \pi^{\alpha} \alpha_{\mu} + \frac{1}{2} h^{\alpha} \pi_{\alpha}^{\nu} \quad (88)
\end{align*}
\]

\[
\begin{align*}
\eta^{\alpha} h^{\lambda\gamma} H_{\alpha\lambda;\gamma} - \eta^{\beta\mu} V^{\beta} E^{\mu\lambda} \sigma^{\mu \lambda} - 3 E^{\mu\nu} \omega_{\mu
u} \\
= (\rho + p) \omega_{\omega} - \frac{1}{2} \eta^{\alpha\beta} \xi_{\alpha\beta} V_{\lambda} q_{\alpha\beta} \\
+ \frac{1}{2} \eta^{\alpha\beta\lambda}(\sigma_{\mu \beta} + \omega_{\mu \beta}) \pi_{\mu}^{\alpha} V_{\lambda} \quad (89)
\end{align*}
\]

\[
\begin{align*}
\nu^{\alpha} h^{\lambda\nu} H^{\mu\nu} + \Theta H^{\lambda\nu} - \frac{1}{2} H^{\nu}(\epsilon h^{\lambda}) \mu V^{\mu\nu} \\
+ \eta^{\lambda\nu\gamma} \eta^{\beta \gamma \alpha \beta} V_{\mu} V_{\gamma} H_{\alpha\beta;\gamma} \Theta_{\nu\beta} \\
- a_{\alpha} E_{\beta}^{\lambda}(\epsilon h_{\gamma})^{\gamma} \alpha \beta V_{\gamma} \\
+ \frac{1}{2} E_{\mu}^{\beta} h_{\alpha}^{\epsilon}(\epsilon h_{\gamma})^{\gamma} \alpha \beta V_{\gamma} \\
- \frac{3}{4} q(\epsilon \omega_{\gamma}) + \frac{1}{2} h^{\lambda\nu} q^{\mu \omega_{\mu}} \\
+ \frac{1}{4} \sigma_{\alpha}^{\beta}(\epsilon h_{\gamma})^{\gamma} \alpha \beta V_{\gamma} q_{\alpha} \\
+ \frac{1}{4} h^{\nu}(\epsilon h_{\gamma})^{\gamma} \alpha \beta V_{\gamma} \pi_{\mu \nu \alpha \beta} 
\end{align*}
\]

The contracted Bianchi identities and Einstein’s equations give the conservation law
\[
T^{\mu\nu}_{\nu} = 0.
\]

Projecting it both in the parallel and the orthogonal subspaces we obtain:
\[
T^{\mu\nu}_{\nu} V_{\mu} = 0,
\]
\[
T^{\mu\nu} h_{\mu}^{\alpha} = 0,
\]

which give the following equations:
\[
\begin{align*}
\dot{\rho} + (\rho + p) \Theta + \dot{q}^{\mu} V_{\mu} + q^{\mu\alpha} - \pi^{\mu\nu} \Theta_{\mu\nu} &= 0, \quad (92) \\
(\rho + p)a_{\alpha} - p_{\mu} h_{\mu}^{\alpha} + q_{\mu} h_{\mu}^{\alpha} + \Theta q_{\alpha} \\
+ \dot{q}_{\alpha}^{\nu} \Theta_{\alpha\nu} + q^{\mu} \omega_{\alpha \nu} + \pi_{\alpha}^{\mu} \nu \\
+ \pi^{\mu\nu} \Theta_{\mu\nu} V_{\alpha} &= 0, \quad (93)
\end{align*}
\]

and, from the definition of Riemann curvature tensor
\[
V_{\mu;\alpha\beta} - V_{\mu;\beta\alpha} = R_{\mu\alpha\beta\lambda} V_{\lambda},
\]

we obtain the equations of motion for the unperturbed kinematical quantities as:
\[
\dot{\Theta} + \Theta^{2} + 2a^{2} + 2a^{\alpha a} = R_{\mu\nu} V^{\mu} V^{\nu}, \quad (94)
\]
\[
\begin{align*}
\dot{h}_{\alpha}^{\beta} h_{\beta}^{\nu} & \sigma_{\mu\nu} + \frac{1}{3} h_{\alpha\beta}(-2 \omega_{\alpha\beta} - 2a^{\alpha a} + 2a_{\alpha \beta}) \\
+ (\sigma_{\alpha\beta} + \omega_{\alpha\beta} + a_{\alpha \beta}) \\
+ q_{\mu\nu} q^{\nu} q_{\mu} + a_{\alpha} a_{\beta} \\
+ \frac{1}{3} R_{\mu\nu} V^{\mu} V^{\nu} - \frac{1}{3} R_{\mu\nu} V^{\mu} V^{\nu} h_{\alpha\beta}. \quad (95)
\end{align*}
\]

We also obtain from the definition of $R_{\alpha\beta\mu\nu}$ three constraint equations:
\[
\begin{align*}
\frac{2}{3} \Theta_{\mu} h_{\lambda}^{\mu} -(\sigma_{\alpha \gamma} + \omega_{\alpha \gamma}) \alpha \gamma h^{\lambda \gamma} - a^{\nu}(\sigma_{\lambda\nu} + \omega_{\lambda\nu}) &= R_{\mu\nu} V^{\mu} V^{\nu}, \quad (97) \\
\omega^{\alpha \alpha} + 2a^{\alpha a} a_{\alpha} &= 0, \quad (98)
\end{align*}
\]

\[
\frac{1}{2} h_{\zeta}^{\epsilon} h_{\lambda}^{\alpha} \eta_{\beta}^{\beta \gamma \nu} V_{\nu} (\sigma_{\alpha \beta} + \omega_{\alpha \beta}) \gamma + a_{\epsilon} (\omega_{\lambda}) = H_{\epsilon \lambda}. \quad (99)
\]

These results constitute a set of 12 equations which will be used to describe the evolution of small perturbations in FRW background. Writing all the perturbed quantities in the form
\[
X_{(perturbed)} = X_{(background)} + \delta X
\]
and after straightforward manipulations we finally obtain the perturbed equations from the set of equations as:

\[(\delta E_{\mu}^\nu)^* h_\mu^\alpha h_\nu^\beta + \Theta (\delta E_{\nu}^\alpha) - \frac{1}{2} (\delta E_{\nu}^\alpha h_\beta^\mu) V_{\mu}^\nu \]

\[+ \frac{\Theta}{3} \eta^{\beta\mu\nu} \eta^{\alpha\gamma\tau} V_\mu V_\tau (\delta E_{\nu}^\alpha h_\gamma^\mu) h_{\gamma\nu} \]

\[- \frac{1}{2} (\delta H_{\lambda}^\mu)_{\gamma\nu} h_\mu^{(\alpha} \eta^{\beta)} \gamma\lambda V_\tau \]

\[= - \frac{1}{2} (\rho + p) (\delta \sigma^{\alpha\beta}) \]

\[+ \frac{1}{6} h^{\alpha\beta} (\delta q^\mu;\mu) - \frac{1}{4} h^{\mu(\alpha} h^{\beta)\nu} (\delta q_\mu)_{;\nu} \]

\[+ \frac{1}{2} h^{\mu(\alpha} h^{\beta)\nu} (\delta \Pi_{\mu\nu})^* + \frac{1}{6} \Theta (\delta \Pi^{\alpha\beta}) \]

\[(\delta H_{\mu\nu})^* h_\mu^\alpha h_\nu^\beta + \Theta (\delta H_{\nu}^\alpha) - \frac{1}{2} (\delta H_{\nu}^\alpha h_\beta^\mu) V_{\mu}^\nu \]

\[+ \frac{\Theta}{3} \eta^{\beta\mu\nu} \eta^{\alpha\lambda\gamma} V_\mu V_\tau (\delta H_{\nu}^\alpha h_\gamma^\mu) h_{\gamma\nu} \]

\[- \frac{1}{2} (\delta E_{\lambda}^\mu)_{\tau\nu} h_\mu^{(\alpha} \eta^{\beta)} \gamma\lambda V_\tau \]

\[= \frac{1}{4} h^{\mu(\alpha} h^{\beta)\nu} (\delta \Pi_{\mu\nu})_{;\tau} \]

\[(\delta H_{\lambda\nu};\nu h^{\alpha\epsilon} h_{\mu\nu} = (\rho + p) (\delta \omega^\epsilon) - \frac{1}{2} \eta^{\epsilon\alpha\beta\mu} V_\mu (\delta q_\alpha);\beta \]

\[(\delta E_{\alpha\mu})_{;\mu} h^{\alpha\epsilon} h_{\mu\nu} = \frac{1}{3} (\delta \rho);_\alpha h^{\alpha\epsilon} - \frac{1}{3} \delta (\delta V^\epsilon) \]

\[- \frac{1}{3} \rho,^0 \delta (V^0) V^\epsilon \]

\[+ \frac{1}{2} h^{\epsilon\alpha} (\delta \Pi^{\alpha\mu})_{;\mu} + \frac{\Theta}{3} \delta q^\epsilon \]

\[(\delta \Theta)^* + \frac{2}{3} \Theta (\delta \Theta) - (\delta a^\alpha)_{;\alpha} = - \frac{1}{2} (\delta \rho) \]

\[(\delta \sigma_{\mu\nu})^* + \frac{1}{3} h_{\mu\nu}(\delta a^\alpha)_{;\alpha} - \frac{1}{2} (\delta a_{(\alpha})_{;\beta} h_{\mu}^\alpha h_{\nu}^\beta \]

\[+ \frac{2}{3} \Theta (\delta \sigma_{\mu\nu}) = -(\delta E_{\mu\nu}) - \frac{1}{2} (\delta \Pi_{\mu\nu}) \]

\[(\delta \omega^\mu)^* + \frac{2}{3} \Theta (\delta \omega^\mu) = \frac{1}{2} \eta^{\mu\beta\gamma} (\delta a_{\beta})_{;\gamma} V_\alpha \]

\[\frac{2}{3} (\delta \Theta),\lambda h_{\lambda}^\mu - \frac{2}{3} \delta \Theta (\delta V_\mu) + \frac{2}{3} \delta \Theta (\delta V^0) \delta^0_\mu \]

\[- (\delta \sigma^\alpha_{\beta} + \delta \omega^\alpha_{\beta})_{;\alpha} h_{\beta}^\mu = - (\delta q_\mu) \]

\[(\delta \omega^\alpha)_{;\alpha} = 0 \]
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