Extending the controlled SWAP test to higher dimensions

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Driven by the value of quantum entanglement as a resource in quantum information, great interest has been shown in methods of detecting and measuring entanglement. An efficient test for entanglement in pure qubit states has recently been explored [Foulds et al, QST 6 035002 (2021)] based on an adaptation of the widely used controlled SWAP test for equivalence. We provide a range of extensions to both the controlled SWAP test for equivalence and the controlled SWAP test for entanglement, including an extension to optical states and qudits, as well as a modification to test for bipartite entanglement exclusively. We compare results from these cases with those from Foulds et al. and so show the test for equivalence can be successfully applied to these states and the test for entanglement can be applied in certain cases. We conclude that, while the test in qudits is successful in general, the extension to optical states has limited applicability, relevant only for entangled coherent states and similar systems. The test for bipartite entanglement yields useful results for bipartite cuts across a multi-qubit state, but fails to correctly detect entanglement between two qubits in a larger system, a failure which points to the limitations of the test when dealing with mixed states.

I. INTRODUCTION

Entanglement is considered an essential resource in the field of quantum information [1]. Its importance has generated great interest in methods for practical detection of entanglement, of which the most common currently are entanglement witnesses and quantum state tomography [1, 2]. The latter requires many measurements on a large ensemble of identical states which scales exponentially with system size and therefore the search for more feasible schemes is ongoing. Entanglement witnesses for an n-qubit state require far fewer measurements, but must be optimised for the state under consideration [3].

The controlled SWAP test can evidence entanglement with fewer test state copies required than for quantum state tomography for states with large numbers of qubits n; furthermore, the test can be arbitrarily applied to any n-qubit pure state so long as a source of identical copies is available [4, 5]. The setup is similar to the controlled SWAP test for equivalence which compares two input states with the aid of an ancilla qubit [6]. Given the success of this family of tests in qubits, here we extend the tests to a range of states, namely optical states and qudits, and investigate how well they perform. Optical states are important in the fields of quantum metrology [7], imaging and computing [8], and qudits have the potential for increased quantum computing power and fault tolerance [9]. We also characterise the performance of the SWAP test for entanglement for detecting bipartite entanglement.

The paper proceeds as follows. In Section II, we provide a brief introduction to pure qubit and qudit systems, followed by a more detailed discussion on entanglement quantification and detection. We then describe both controlled SWAP tests as they have been proposed in the past. An overview of photon states with non-classical properties relevant for this work is then provided. In the next two sections, we consider a series of extensions to the test for entanglement: a modified version of the qubit test with bipartite entanglement detection in Section III and entangled qudit states in section IV. In Section V, we adapt the SWAP test for equivalence to a selection of photonic states, and adapt the SWAP test for entanglement to entangled coherent states in Section VI. We summarise and conclude in section VII. Our ultimate goal is twofold. Firstly, we aim to identify promising avenues for further applications of both these tests, supported by preliminary results. Secondly, we provide a detailed investigation of the test for entanglement to determine its action and limitations in a wider range of scenarios.

II. BACKGROUND

A. Qubits, qudits and entanglement

A qubit is analogous to a classical binary bit but can also be in a superposition of the computational basis states |0⟩ and |1⟩, with the general form

|ψ⟩ = A₀ |0⟩ + A₁ |1⟩

for A₀, A₁ ∈ C, where the probability of measuring state |i⟩ is P(|i⟩) = |Aᵢ|². The general two-qubit state is [10]

|ψ⟩ = A₀₀ |00⟩ + A₀₁ |01⟩ + A₁₀ |10⟩ + A₁₁ |11⟩  

(1)

where |j⟩ ⊗ |k⟩ = |jk⟩. A multiple qubit system that cannot be expressed as a tensor product of its composite single-qubit states is said to be entangled. The class of

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maximally bipartite entangled states are the Bell states:

\[ |\psi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \tag{2} \]

For states with a larger number of qubits \( n \), the classification of entangled states is richer than in the bipartite case, and multiple distinct classes of entanglement exist \cite{11}. One class of maximally multipartite entangled states are GHZ states, for example \cite{12}

\[ |\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|00\rangle^n + |11\rangle^n). \tag{3} \]

where \( |0\rangle^n \) indicates \( n \) qubits all in state \( |0\rangle \). Under reversible local operations and classical communication (LOCC), n-qubit GHZ states can be transformed into one another, but cannot be transformed into a state with less than maximal entanglement. For each \( n \), the states in the set \( \text{GHZ}_n \) are considered equivalent to one another and form a unique class \cite{13}.

In the quantum circuit model of computation, reversible state transformations are represented as quantum logic gates. Relevant examples include the single-qubit Hadamard gate \( H \), which operates on the computational basis states in the following way:

\[ H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{4} \]

The two-qubit CNOT gate flips the target qubit if the control qubit is in state \( |1\rangle \) \cite{10}. It can be represented by the matrix

\[ \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5} \]

where the first qubit is the control and the second is the target. The three-qubit Toffoli gate flips the target qubit if and only if the two control qubits are both in state \( |1\rangle \) \cite{10}. It can be represented by the matrix

\[ T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6} \]

where the first two qubits are the controls and the third qubit is the target.

Qudits behave very similarly to qubits, but are of a higher dimension and are therefore not restricted to superpositions of the 0 and 1 binary states. A general one-qudit state is of the form

\[ |\psi^D_1\rangle = \sum_{j=0}^{D-1} A_j |j\rangle \tag{7} \]

where \( D > 2 \) and \( D \in \mathbb{Z}^+ \) is the dimension of the qudit. The \( D = 3 \) case is known as a qutrit \cite{14,15}. Higher dimensions can reduce experimental complexity and could be used for more efficient and error-tolerant quantum computation \cite{16}. Entanglement in qudits is defined similarly to the qubit case \cite{13}; for example \( |\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \) is a maximally entangled two-qutrit state.

Entanglement in a bipartite pure state is now well understood as the degree of mixedness of each subsystem, where the ‘mixedness’ characterises a lack of information about the state of a quantum system, such that it must be represented using a density matrix \( \rho \) rather than a vector in the Hilbert space \cite{17}. Accordingly, the entanglement in a pure state is quantified by the entropy of entanglement, given by the von Neumann entropy \( S_V(\rho) \) of the reduced density matrix representing each subsystem, where

\[ S_V(\rho) = -\text{Tr}[\rho \ln \rho] \tag{8} \]

and \( \text{Tr}[A] \) indicates the trace of a matrix \( A \). For two-qubit pure states as in equation (1) the concurrence \( C_2 \) quantifies entanglement and is given by

\[ C_2 = 2|A_{00}A_{11} - A_{01}A_{10}| \tag{9} \]

with \( 0 \leq C_2 \leq 1 \). A concurrence of \( C_2 = 0 \) indicates that no entanglement is present in the system and \( C_2 = 1 \) corresponds to a maximally entangled state \cite{18}.

The most general method for experimentally determining entanglement is quantum state tomography (QST). QST involves performing a large number of measurements on an ensemble of identical quantum states, using the results to reconstruct the system’s density matrix \( \rho \). The entanglement of the system can then be quantified using the entropy of entanglement from equation (8). For \( n \) qubits, QST requires at least \( 3^n \) measurements and as many copies of the state \( |\psi\rangle \), an exponential scaling in \( n \) which renders the method impractical for large numbers of qubits.

A method with more efficient scaling uses entanglement witnesses. An entanglement witness \( W \) is an observable such that \( \text{Tr}(W\rho_s) \geq 0 \) for all separable \( \rho_s \), and \( \text{Tr}(W\rho_e) < 0 \) for the entangled state \( \rho_e \) it detects. It can be shown that for any pure state there exists a witness that requires only \( 2n - 1 \) measurements, far fewer than the number needed for QST \cite{2}. Nevertheless, as suggested by the definition, entanglement witnesses must be optimised for the state considered and are therefore not a general method.
B. Controlled SWAP tests in qubits

The SWAP test is a widely used procedure for state comparison \[20\], first experimentally demonstrated in \[21\]. The name derives from the controlled SWAP (c-SWAP) gate used to perform the test, which has input systems \(A\) and \(B\) controlled on a control qubit \(C\). If the control qubit is in state \(|1\rangle_C\), the gate swaps the states of \(A\) and \(B\), and if in state \(|0\rangle_C\), states \(A\) and \(B\) are unchanged.

Three states are prepared in the initial composite state \(|\Psi\rangle = |\psi\rangle_A |\phi\rangle_B |0\rangle_C\), where \(|\psi\rangle\) and \(|\phi\rangle\) are pure qubit states to be compared \[20\]. First, a Hadamard gate (equation (4)) is applied to the control qubit. A c-SWAP gate is then performed on \(|\psi\rangle_A\) and \(|\phi\rangle_B\) controlled on \(C\), followed by a second Hadamard gate applied to the control qubit, giving the composite state: \[2\]

\[
|\Psi\rangle = \frac{1}{2}[(|\phi\rangle_A |\psi\rangle_B + |\psi\rangle_A |\phi\rangle_B)]|0\rangle_C \\
+ [(|\phi\rangle_A |\psi\rangle_B - |\phi\rangle_A |\phi\rangle_B)]|1\rangle_C. \tag{10}
\]

This sequence of gates is shown in figure 1. Clearly, the probability of measuring state \(|1\rangle_C\) in the control is zero if the two states \(|\psi\rangle_A\) and \(|\phi\rangle_B\) are identical. A measurement of \(|1\rangle_C\) therefore proves the two states are different, while multiple measurements of \(|0\rangle_C\) are required to achieve confidence of equivalence. It can be shown that in general the probability of measuring \(|1\rangle_C\) is given by

\[
P(|1\rangle_C) = \frac{1}{2} - \frac{1}{2} |\langle \phi|\psi\rangle|^2 \tag{11}
\]

giving a maximum probability of \(\frac{1}{2}\) for orthogonal states. Further, the overlap can be calculated from obtaining the probability from repeat measurements \[20\].

There also exists a modified version of the c-SWAP test for equivalence designed to test for entanglement \[4\,5\]. The two-qubit case is outlined in \[5\] and replicated here, but the method can be generalised to higher numbers of qubits \[4\]. To determine the entanglement of test state \(|A\rangle\) the test requires also a copy state \(|B\rangle\). Here we consider the copy state to be an exact replica of state \(|A\rangle\) but error cases are considered in \[4\]; the initial composite state is therefore \(|\Psi\rangle = |A\rangle_A |A\rangle_B |00\rangle_C\). The entanglement test follows a similar procedure to the test for equivalence, but with a c-SWAP operation performed on each qubit in state \(|A\rangle\) and its corresponding qubit in state \(|B\rangle\), controlled on the corresponding qubit in \(C\).

The circuit is shown in figure 2.

After the test, the system is in the state

\[
|\Psi\rangle = \frac{1}{2} \sum_{jks} |j\rangle_A |s\rangle_B [(A_{jk}A_{rs} + A_{js}A_{rk}) |00\rangle_C \\
+ (A_{jk}A_{rs} - A_{js}A_{rk}) |11\rangle_C].
\]

The probability of the control qubit being in state \(|01\rangle_C\) or \(|10\rangle_C\) is zero, and one can see that if the concurrence from equation (9) is zero, so is the probability of measuring the \(|11\rangle_C\) control state. Any measurement of this state therefore evidences the presence of entanglement by implying a non-zero concurrence. Furthermore, the value of the concurrence – and therefore the entanglement of the system as described in section II A – can be determined from the probability distribution of the control qubit states.

The extension to higher numbers of qubits simply requires a c-SWAP gate applied to each of the \(n\) sets of qubits and therefore the control state is initialised in \(|00\rangle_C\). Any measurement of an even number of \(|1\rangle\)'s in the control state evidences entanglement (and therefore are referred to as entanglement signatures). For more than two qubits the concurrence is no longer directly related to the entropy of entanglement \[11\]; nevertheless, the final state probabilities obtained from the c-SWAP test can still be used as a degree of entanglement as in the two-qubit case. Foulds et al. \[4\] propose the following relation for the degree of multiparticle entanglement, \(C_n\):

\[
C_n = 2P(|\text{even no. of 1s}\rangle_C) \tag{12}
\]

which reduces to the concurrence for the two-qubit case, and where \(0 \leq C_n \leq 2(\frac{1}{2} - \frac{1}{n^2})\). Foulds et al. prove computationally that this quantity satisfies the basic requirements for a degree of entanglement in \(n\)-qubit GHZ-like states and in any three-qubit pure state. They thus conjecture that the requirements are fulfilled for any \(n\)-qubit pure state \[4\].

Beckey et al. \[22\] prove this conjecture and expand the
If the test operator wants to detect genuine \( n \)-qubit entanglement, the 'Concentratable Entanglement' is:

\[
C_{|A\rangle}(s) = \sum_{z \in Z_{n}^{\text{even}}} P(z) \tag{13}
\]

where \( P(z) \) is the probability of measuring \( z \) in the control state at the end of the test and \( Z_{n}^{\text{even}} \) is the set of bit strings with even Hamming weight and with at least a 1 in an index in \( s \). For \( s = S \), \( C_{|\psi\rangle} = C_{n}^{2} / 4 \). This is in fact a specific case of Concentratable Entanglement, the general form being:

\[
C_{|A\rangle}(s) = 1 - \frac{1}{2^{c(s)}} \sum_{\alpha \in P(s)} \text{Tr} \rho_{\alpha}^{2} \tag{14}
\]

where \( c(s) \) is the cardinality of the sets, \( \rho_{\alpha} \) the joint reduced state, associated to \( |A\rangle \), of the subsystems labeled by the elements in \( \alpha \) [22].

The efficiency of the test depends on the application. Should one only be interested in evidencing any entanglement in the state, only a single observation of a control state at the end of the test and \( Z_{n}^{\text{even}} \) is required. Although in this case the c-SWAP test performs better than the 3\(^{n} \) required for QST for \( C_{n} \) values in the regime:

\[
\left( \frac{8}{3^{n}} \right)^{1/2} < C_{n} \leq 2 \left( \frac{1}{2} - \frac{1}{2^{n}} \right)^{1/2}.
\]

If the test operator wants to detect genuine \( n \)-qubit entanglement, significantly more measurements are required. Although in this case the c-SWAP test performs worse, for small highly entangled systems it remains favourable when compared to QST. The test also exhibits robustness against a variety of errors [4].

C. Optical States

Given the usefulness and robustness of this family of tests, we look to extend them to non-qubit states relevant to quantum information, starting with optical states. For Gaussian states, a family of Gaussian entanglement measures has been introduced, which for pure states reduces to the entropy of entanglement [23], as in the case of the concurrence [18]. In particular, we will consider coherent states and squeezed states. A coherent state \( |\alpha\rangle \) is the unique eigenstate of the annihilation operator \( \hat{a} \) with complex eigenvalue \( \alpha \) in a quantum harmonic oscillator [24]:

\[
\hat{a} |\alpha\rangle = \alpha |\alpha\rangle.
\]

Coherent states follow a Poisson number distribution when represented in the basis of photon number states or Fock states \( |n\rangle \) as seen in the following expression:

\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{16}
\]

where \( |\alpha|^2 = \mu \) is the average number of photons. It follows that the probability of finding \( m \) photons is

\[
P(m|\mu) = \langle m | \alpha^* \alpha | m \rangle = \mu^m e^{-\mu}/m!.
\]

A coherent state can also be thought of as the vacuum state \( |0\rangle \) displaced to a location \( \alpha \) in phase space, due to the action of a displacement operator \( D(\alpha) \) such that [24]

\[
|\alpha\rangle = e^{\alpha \hat{a}^* - \alpha^* \hat{a}} |0\rangle = D(\alpha) |0\rangle. \tag{17}
\]

In contrast to the photon number states, these states are not orthogonal, and form an overcomplete basis. The inner product between coherent states \( |\alpha\rangle \) and \( |\beta\rangle \) is given by [24]

\[
\langle \alpha | \beta \rangle = e^{-\frac{1}{2} |\alpha|^2 - |\beta|^2 / 2 + \beta^* \alpha}. \tag{18}
\]
We will consider several kinds of coherent states: squeezed coherent states, cat states, and entangled coherent states.

The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are defined as

$$\hat{X}_1 = \frac{1}{2}(\hat{a}^\dagger + \hat{a}), \quad \hat{X}_2 = \frac{i}{2}(\hat{a}^\dagger - \hat{a}).$$

These two operators obey the following uncertainty relation:

$$\langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq \frac{1}{16};$$

Coherent states minimise this relation and so are known as minimum uncertainty states, for example with $\langle (\Delta \hat{X}_1)^2 \rangle_\alpha = \langle (\Delta \hat{X}_2)^2 \rangle_\alpha = \frac{1}{4}$. Squeezed coherent states are also minimum uncertainty states but with one quadrature ‘squeezed’: a state is said to be squeezed whenever $|\alpha| > 1$.

$$\langle (\Delta \hat{X}_1)^2 \rangle < \frac{1}{4} \text{ or } \langle (\Delta \hat{X}_2)^2 \rangle < \frac{1}{4}.$$  

Mathematically, a single mode squeezed state can be generated using the squeeze operator $\hat{S}(\xi)$, defined as

$$\hat{S}(\xi) = \exp\left\{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^\dagger \hat{a})\right\}$$

for $\xi = r e^{i\theta}$ where $r$ is known as the squeeze parameter and $\theta$ indicates the direction of squeezing [24]. Therefore, a general squeezed coherent state $|\alpha, \xi\rangle$ can be written as

$$|\alpha, \xi\rangle = D(\alpha)\hat{S}(\xi)|0\rangle$$

using the displacement operator from equation (17). The general squeezed state can also be expanded in the basis of Fock states $|n\rangle$ to give

$$|\alpha, \xi\rangle = \frac{1}{\sqrt{\cosh r}} \exp\left[ -\left( (|\alpha|^2 + \alpha^* e^{i\theta} \tanh r)/2 \right) \right]$$

$$+ \sum_{n=0}^{\infty} \frac{[\frac{1}{2} e^{i\theta} \tanh r]^n/2}{\sqrt{n!}} H_n(\gamma e^{i\theta} (\sinh(2r)-\frac{1}{2})) |n\rangle$$

where $H_n(x)$ are the Hermite polynomials and $\gamma = \alpha \cosh r + \alpha^* e^{i\theta} \sinh r$ [24].

Squeezing can also be performed over two modes, with even greater prospects for nonclassical results [24]. In this case, the squeeze operator can be expressed as

$$\hat{S}_2(\xi) = \exp(\xi^* \hat{a}^\dagger \hat{b} - \xi \hat{a} \hat{b}^\dagger).$$

Since $\hat{S}_2(\xi)$ cannot be factored as a product of two single-mode squeeze operators as given in equation (19), this squeezing entangles the two modes. It can be shown that the von Neumann entropy, related to the measure of entanglement outlined in equation (8), increases with the squeeze parameter $r$ [24, 25].

Cat states are linear superpositions of coherent states with equal amplitude but a phase difference. Their name reflects the fact that one can macroscopically distinguish the states in the superposition, similar to the states of the cat in Schrödinger’s thought experiment [26]. A general cat state is of the form [24]

$$|\psi\rangle = N(|\alpha\rangle + e^{i\phi}|-\alpha\rangle)$$

for relative phase angle $0 \leq \phi \leq 2\pi$ and where we have assumed $\alpha$ to be real. These states also exhibit highly nonclassical properties and are of particular interest in quantum optics and information, both due to their fundamental intrigue as superpositions of the most classical states available in these fields, but also thanks to their applicability in quantum computing [8] and as the building blocks for entangled coherent states [8, 27].

Entangled coherent states (ECS) have importance across a range of fields, including quantum optics [8], quantum information processing [27], and quantum metrology [7]. ECS are also fundamentally intriguing as entangled macroscopic states with minimised uncertainty. We will primarily consider entangled coherent states of the form

$$|ECS\rangle = \frac{N_{ECS}}{\sqrt{2}}(|\alpha\rangle |\alpha\rangle \pm (-\alpha\rangle |-\alpha\rangle)$$

analogous to the qubit Bell states in equation (2), and where $N_{ECS} = (1+\exp[-4|\alpha|^2])^{-1/2}$. Instead of between qubits, however, ECS exhibit entanglement between two modes of the electromagnetic field. These states can be realised [27] by combining and operating on cat states of the form given in equation (23).

III. THE CONTROLLED SWAP TEST FOR BIPARTITE ENTANGLEMENT

A. Entanglement across a bipartite cut

In large qubit states, one may only be interested in determining entanglement across a single cut that divides the state into two subsystems. This form of entanglement is of particular interest in building robust entangled states for any purpose, given that the entanglement between two large subsystems is less susceptible to environmental influences than entanglement between two individual qubits [28]. Note that Beckey et al. [22] consider bipartite cuts but includes the entanglement within one subsystem as well as across the cut, whereas here we consider only the entanglement across the cut.

The test for bipartite entanglement can be carried out on a qubit state with an arbitrarily large number of qubits and only two control qubits. Instead of swap-
ping each corresponding qubit in states $|A\rangle$ and $|B\rangle$, the c-SWAP gate will swap two sets of qubits between A and B. The first set of qubits in A and B is controlled on the first qubit in C, and the second set is controlled on the second qubit in C. Therefore the test characterises the entanglement between the two sets.

Consider a four-qubit state of the form

$$|\Phi_4\rangle = \frac{1}{2} (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle). \quad (25)$$

This state is clearly partially separable, with entanglement present between the first and second as well as the third and fourth qubits in the state. However, the state exhibits no entanglement between the first pair and the second pair. The numerical results of the bipartite test for entanglement applied on this state for each possible cut are presented in Table I. Note that the differences in the probabilities for cuts between one qubit and the three others is determined by the order of the swaps. However the probability of relevance, $P(|11\rangle_C)$, remains constant.

As expected, the state $|11\rangle_C$ only has non-zero probability if entanglement is present across that cut. Additionally, $P(|11\rangle_C)$ is highest for the maximally entangled cuts across the two Bell states with 0.375. Unfortunately, for the non-symmetrical cuts the probabilities of $|01\rangle_C$ and $|10\rangle_C$ are non-zero and so in this case measuring odd numbers of ones does not evidence $|A\rangle \neq |B\rangle$, as with the typical case. In this case the test does not act as an equivalence test as the asymmetry prevents the amplitudes from cancelling.

Performing a bipartite c-SWAP test across every possible cut should detect any entanglement in the system. Given that this procedure requires only two controlled SWAP operations and two control qubits, it is of interest to determine its efficiency compared to the general c-SWAP test for entanglement. Consider every possible bipartite cut in a four-qubit system (of which there are seven) and compare the total probability of determining entanglement from these seven cuts with the total for the same state using seven trials of the original multipartite c-SWAP test.

We make this comparison in a large random selection of qubit states with a broad range of entanglements. A general four-qubit state can be written in the form $|\Psi_4\rangle = e^{i\xi_0} \cos(\theta_0) |0000\rangle + e^{i\xi_1} \sin(\theta_0) \cos(\theta_1) |0001\rangle + e^{i\xi_2} \sin(\theta_0) \sin(\theta_1) \cos(\theta_2) |0010\rangle + \ldots + e^{i\xi_{14}} \sin(\theta_0) \sin(\theta_1) \ldots \sin(\theta_5) \cos(\theta_{14}) |1111\rangle$ for $0 \leq \xi_j < 2\pi$ and $0 \leq \theta_j < \pi/2$, and can be randomly selected if we uniformly sample $\xi_j$, $\theta_j$ according to the Haar measure [30]. This method is equivalent to randomly choosing points on the surface of a hypersphere, and in general provides a wide range of entanglements in the four-qubit system [31]. The results of this comparison are presented in figure 3.

As seen from figure 3 for all $C_n$ the general test is more efficient, with a higher probability of detecting entanglement with the same number of trials and (therefore) test state copies. This efficiency gap is greater for larger entanglements, and is larger for the four qubit case than when considering a three qubit system. From this result, in terms of the number of copies required, we conjecture that the greater the number of qubits, the worse the test for bipartite entanglement performs when compared with the general multipartite c-SWAP test.

Experimentally, this reduction in copy state efficiency may be tolerable in exchange for a smaller number of c-SWAP gates and control qubits required for each trial. Given that the practical implementation of three-qubit gates such as the c-SWAP is in its infancy [32], this test...
may prove easier to set up and operate, even if a higher number of trials is required to detect entanglement.

B. Entanglement between two qubits in a multi-qubit state

Next we consider applying the test to determining entanglement between two qubits within a larger state. Many quantum communication protocols, such as multiparty quantum secret sharing schemes, involve distributing qubits to several parties such that they retain their entanglement and can be used to obtain a quantum communication advantage. It is therefore of interest to detect and measure the entanglement between states given to just two of the parties out of a larger group of shared states.

A c-SWAP gate is applied to each qubit of interest, controlled on the two control qubits respectively. The test therefore reproduces the action of the two-qubit circuit from figure 2 with no gates acting on the remainder of the state. Consider a partially separable four-qubit state of the form:

\[ |\chi_4\rangle = |\phi_2\rangle \otimes |\Phi^+\rangle \]

where \( |\phi_2\rangle \) is a separable two-qubit state and \( |\Phi^+\rangle \) is the Bell state from equation (2). Applying the test across all possible qubit pairs gives the results in table I.

| 1 and 2 | 1 and 3 | 2 and 3 | 3 and 4 |
|---------|---------|---------|---------|
| \( P(|00\rangle_C) \) | \( 3/4 \) | \( 3/4 \) | \( 3/4 \) |
| \( P(|01\rangle_C) \) | 0       | \( 1/4 \) | \( 1/4 \) | 0       |
| \( P(|10\rangle_C) \) | 0       | 0       | 0       | 0       |
| \( P(|11\rangle_C) \) | 0       | 0       | 0       | \( 1/4 \) |

Table II. Control state probabilities for the test for bipartite entanglement executed across different pairs of qubits within the state \( |\chi_4\rangle \) in equation (27). For instance, the first column refers to a controlled SWAP between the first qubit of input state \( |A\rangle \) and \( |B\rangle \) followed by controlled SWAP between the second qubit of \( |A\rangle \) and \( |B\rangle \), ignoring all other qubits. Entangled pairs are shown in orange.

As expected, except for a test between qubits 3 and 4 the probability of \( |11\rangle_C \) is zero. Contrary to the multi-qubit test however, we find non-zero probabilities for \( |01\rangle_C \) and \( |10\rangle_C \) for pairs with one entangled and one separate qubit. This is because when tracing out the extra qubits, the remaining state with only the qubits of interest is mixed.

We now apply this extension to the more complex case of state \( |\Phi_4\rangle \) from equation (25). In this case, we expect to observe entanglement only between the first two and last two qubits. Unlike in the previous example with state \( |\chi_4\rangle \) in equation (27), no qubit is entirely separable from the rest of this state. The test gives results presented in table III.

These results indicate a failure of the test. Even for qubits which share no entanglement, the test returns a non-zero probability of observing \( |11\rangle \) in the control, meaning one cannot use this procedure to detect entanglement. One possible physical explanation becomes apparent when considering, for example, the states of the second and third qubit once all the other qubits have been traced out. Although these qubits share no entanglement, they are each maximally entangled with one other qubit, meaning a partial trace on each returns a maximally mixed state. This highlights a fundamental limitation of the test: it only works on pure states, and when operating on a mixed state cannot reliably detect entanglement, given the lack of knowledge about the exact state of a mixed system.

IV. THE CONTROLLED SWAP TEST FOR ENTANGLEMENT IN QUUDITS

In this section, we consider testing for entanglement in a two-qudit state where each qudit is of the form given in equation (7). The control state remains qubit and so the test’s operation is unchanged. The distinction in the results is due to the higher level of possible entanglement as the qudit dimension increases. In this section, we restrict investigation to the two-qudit case, but in principle the test can be performed on states with any number of qudits.

The maximally entangled qudit states of each dimension \( D \) provide an insight into the increase in entanglement provided by higher dimensions. We consider maximally entangled 2-qudit states of the form (14):

\[ |\Phi_D^+\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |j\rangle |j\rangle \]

analogous to Bell states in qubits, which gives

\[ P(|11\rangle_C) = \frac{1}{2} - \frac{1}{2D}. \]

Note the similarity to the probability results for a n-qubit GHZ state, \( P(\text{even no. of 1s}) = \frac{1}{2} - \frac{1}{2n} \). Figure 4 shows the c-SWAP probability results for these states along with the GHZ qubit state results for comparison.

The greater the number of qubits \( n \) the greater the probability of measuring \( \text{even no. of 1s} \) in the control.
due to the increased amount of entanglement. Similarly, in the Bell state case the greater $D$ and therefore the greater the amount of entanglement, the larger $P(|11\rangle_C)$ (tending to $\frac{1}{2}$). Increasing $n$ however has a greater effect on $P(|$even no. of 1$\rangle_C)$ than increasing $D$, and in fact they are related by $P(n = n', D = 2) = P(n = 2, D = 2^{n'-1})$. Perhaps most importantly, these results suggest the possibility of a relation for a measure of qudit entanglement similar to the measure of entanglement $C_n$ from equation (12), such as an extension of the Concentratable Entanglements [22] to qudit states.

In general, if $|A\rangle = |B\rangle = \sum_{j=0}^{D-1} A_j |jj\rangle$:

$$P(|11\rangle_{C}) = \frac{1}{2} \sum_{j=0}^{D-1} \sum_{k=0, k\neq j}^{D-1} |A_j|^2 |A_k|^2. \quad (30)$$

V. THE CONTROLLED SWAP TEST FOR EQUIVALENCE IN OPTICAL STATES

We now consider the test for equivalence with an extension to optical systems with behaviour outside the qubit framework. To perform the controlled SWAP operation and other key elements of the circuit in figure 4 optically, we consider a set up such as that shown in figure 5.

The two states to be compared are input as two modes into the circuit, dual rail encoded such that the initial state is $|A\rangle_a |B\rangle_b$ where $a$ and $b$ are two spatial paths [34] [35]. The control qubit is then encoded as the polarization of the beam. This allows for a controlled swap, dependant on the polarization, with the SWAP gate itself implemented by crossing the spatial paths.

A detector is then placed at the $|0\rangle_{C}$ output (denoted by C in figure 5) which will have an amplitude proportional to that of $|A\rangle_a |B\rangle_b - |B\rangle_a |A\rangle_b$. Aspects of this set-up are similar to that implemented by Goswami et al. [36]. We now consider this test for a selection of states important in quantum optics and information.

A. Squeezed coherent states

Consider the controlled SWAP test for equivalence applied to a coherent state of the form equation (17) and squeezed coherent state of the form equation (20), both with amplitude $\alpha$, such that

$$|\psi\rangle = |\alpha\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle, \quad (31)$$

$$|\phi\rangle = |\alpha, \xi\rangle = e^{\frac{i}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^\dagger \hat{a})} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle$$

where $\xi = re^{i\theta}$. This gives a probability of observing the control in state $|1\rangle$ according to equation (11) equal to

$$P(|1\rangle_C) = \frac{1}{2} - \frac{1}{2} |\langle \alpha | \alpha, \xi \rangle|^2. \quad (32)$$

Using the Fock state expansions in equations (16) and (21) this probability is

$$P(|1\rangle_C) = \frac{1}{2} - \frac{1}{2 \cosh(r)}$$

dependant only on the squeeze parameter $r$. As $r$ tends to infinity, $P(|1\rangle_C)$ tends to $\frac{1}{2}$ as the two states become closer to orthogonal. When $r = 0$ the two states are equivalent and therefore $P(|1\rangle_C) = 0$, as expected.
B. Cat states and squeezed cat states

A similar process can be performed with cat states such as in equation (23). First we consider using the c-SWAP test to compare cat states with differing phases: $|\psi\rangle = N_1(|\alpha\rangle + e^{i\phi_1}|\alpha\rangle)$ and $|\phi\rangle = N_2(|\alpha\rangle + e^{i\phi_2}|\alpha\rangle)$. From equation (11) one obtains the result

$$P(|1\rangle_C) = \frac{1}{2} - \frac{1}{2}(N_1N_2)^2(e^{-4\alpha^2} - 1)(\cos(\phi_1 - \phi_2) - 1)$$

(33)

shown in figure 6 for $\phi_1 = 0$. The maximum value of $P(|1\rangle_C)$ is at $\phi = \phi_2 - \phi_1 = \pi$, for which $|\phi\rangle = N_2(|\alpha\rangle - |\alpha\rangle)$, the most dissimilar to $|\psi\rangle$. We therefore conclude that the value of $P(|1\rangle)$ behaves as a measure of similarity between the two states.

We then similarly consider the comparison between a normal cat state as in equation (23) and a squeezed cat state. Let

$$|\psi\rangle = N_1(|\alpha\rangle + e^{i\phi_1}|\alpha\rangle)$$

(34)

and

$$|\phi\rangle = N_2(|\alpha\rangle + e^{i\phi_2}|\alpha\rangle).$$

(35)

In this case, using the decompositions in (16) and (21), equation (11) becomes

$$P(|1\rangle_C) = \frac{1}{2} - \frac{1}{2}(N_1N_2)^2\left[\cos(\phi_-)^2 + 2\exp(-2\alpha^2(1+\tanh(r))\cos\theta)\cos(-2\alpha^2\tanh(r)\sin(\theta))\cos(\phi_-)\cos(\phi_+\right. + \exp\left.-4\alpha^2(1+\tanh(r))\cos\theta\cos(\phi_+)^2\right],$$

(36)

where $\phi_+ = \phi_1 + \phi_2$ and $\phi_- = \phi_1 - \phi_2$, and which for squeezing along the $\theta = 0$ axis reduces to

$$P(|1\rangle_C) = \frac{1}{2} - \frac{2(N_1N_2)^2}{\cosh(r)}\cos(\phi_-)^2 + e^{-2\alpha^2(1+\tanh(r))\cos(\phi_+)^2}. \quad (37)$$

This is plotted in figure 7. As seen from figure 7(a) max($P(|1\rangle_C)$) is at $\phi = \phi_2 - \phi_1 = \pi$, where $|\psi\rangle$ and $|\phi\rangle$ are most dissimilar. Further, the greater $r$ and therefore the more squeezed $|\phi\rangle$, the greater $P(|1\rangle_C)$ as the states diverge. The equivalence test therefore functions as expected on this states. Figure 7(b) is interesting as despite

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The probability of observing the control qubit in state $|1\rangle$ for an equivalence test between the even cat state ($\phi_1 = 0$) and a general cat state of superposition phase $\phi_2$, based on the analytic result given in equation (33), for various coherent state amplitudes $\alpha$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The probability of observing the control qubit in state $|1\rangle$ for an equivalence test between a normal and a squeezed cat state, based on the analytic result given in equation (37) where $\phi_1 = 0$. (a) shows probability against relative superposition $\phi = \phi_2 - \phi_1$ for various squeezing parameters $r$, with coherent state amplitude $\alpha = 1$. (b) shows probability against coherent state amplitude $\alpha$ for various squeezing parameters $r$, with relative superposition phase $\phi = 0$.

$\alpha$ being the same in each input state, for $1 < r < 4$ the probability results are influenced by the coherent state amplitude in the approximate region $0 < \alpha < 2$, with a maximum at around $\alpha = 1$.

As can be seen from equation (16), when $\alpha < 1$ the corresponding coherent state has on average less than one photon. Such states are known as weak coherent...}
states and can be used to generate single photon pulses [24].

Equations (33) and (37) are dependant on \( \alpha \) with the terms \( e^{-4\alpha^2} - 1 \) and \( (e^{-2\alpha} + ...)^2 \) respectively. It is clear that this has arisen from the non-zero inner product \( \langle \alpha | -\alpha \rangle = e^{-2\alpha^2} \). However this inner product is negligible for \( \alpha > 2 \) and therefore large amplitudes give probabilities independent of \( \alpha \). Specifically this gives rise to the turning points in figure 4 for small \( \alpha \), the greater \( r \) the more orthogonal the two states, whereas for large \( \alpha \) this effect becomes negligible as the states are nearly orthogonal already.

Overall, this extension to the test for equivalence produces a useful method for distinguishing coherent states and coherent state superpositions, but requires a potentially difficult practical implementation of the controlled SWAP gate capable of acting on optical states.

VI. THE CONTROLLED SWAP TEST FOR ENTANGLEMENT IN OPTICAL STATES

The c-SWAP test for entanglement differs from the test for equivalence in that there is a control qubit for each subsystem in the test state \( |A\rangle \). Therefore for optical states, the setup is that in figure 5 for each of the \( n \) subsystems.

A. Entangled coherent states

The test for entanglement relies on being able to swap sub-systems within the test state. An extension of the test to optical states therefore requires a system with entanglement between two or more subcomponents, analogous to the qubit case, restricting the types of states to which the test can be applied. Entangled coherent states, such as equation (24), fulfil this criterion, exhibiting entanglement between two modes in a structure similar to the two-qubit Bell states. The key difference between ECS and Bell states is that the basis states are not orthogonal. For \( \alpha < 1.5 \) however, the entanglement detection probability falls off rapidly. It is possible that the test is less effective for low coherent state amplitudes, but we hypothesise that the lower probabilities reflect less entanglement in the system for low \( \alpha \) states; maximum entanglement increases with system size and coherent states with low amplitudes have fewer degrees of freedom. In the next section, [?], we consider a qudit approximation to ECS that supports this.

B. Entangled coherent states as high dimensional qudits

High dimensional qudits can be used to approximate coherent states, due to the qudit-like form of the number state expansion in equation (14). Although mathematically we require a qudit of infinite dimension to replicate the summation over every possible number state, for low amplitude coherent states - those of greatest interest - the higher order terms quickly become negligible. Therefore consider

\[
|\alpha_1^{\text{qudit}}\rangle = e^{-|\alpha|^2} \sum_{j=0}^{D-1} \frac{\alpha^j}{\sqrt{j!}} |j\rangle .
\] (40)

These coherent state approximations can in turn be used to approximate ECS such as those given in equation (24),

Figure 8. Graph showing the probability of observing the entanglement signature \( |11\rangle_C \) for the c-SWAP test in two-mode coherent state superpositions of the form given in equation (38), plotted against coherent state amplitude \( \alpha \) for various values of \( C' \) as defined in equation (39).

From this figure we see that for high coherent state amplitudes we revert back to the results for the qubit case. For amplitudes greater than \( \alpha = 2 \) the inner product \( \langle \alpha | -\alpha \rangle \) given by equation (18) tends to zero as the states \( |\alpha\rangle \) and \( |-\alpha\rangle \) become approximately orthogonal. For \( \alpha < 1.5 \) however, the entanglement detection probability falls off rapidly. It is possible that the test is less effective for low coherent state amplitudes, but we hypothesise that the lower probabilities reflect less entanglement in the system for low \( \alpha \) states; maximum entanglement increases with system size and coherent states with low amplitudes have fewer degrees of freedom. In the next section, [?], we consider a qudit approximation to ECS that supports this.

\[
|\psi_2^{\alpha}\rangle = N_2^{\alpha} (A_{++} |\alpha\rangle |\alpha\rangle + A_{+-} |\alpha\rangle |-\alpha\rangle + A_{-+} |-\alpha\rangle |\alpha\rangle + A_{--} |-\alpha\rangle |-\alpha\rangle) \] (38)

where \( N_2^{\alpha} \) is a normalisation constant. Such states show similarities in form to those used when implementing ECS through parametric amplification and photodetection [27].

Given the state’s similarity to a qubit Bell state, we characterise the amplitudes’ contribution to the state’s entanglement with the equation

\[
C' = 2 (N_2^{\alpha})^2 |A_{++} A_{--} - A_{+-} A_{-+}| \] (39)

as an analogue to the concurrence in equation (9). The values of \( P(|11\rangle_C) \) for test state \( |\psi_2^{\alpha}\rangle \) for varying \( C' \) and coherent state amplitude \( \alpha \) are shown in figure 8.
and so

\[ |ECS_{\text{qudit}}\rangle = e^{-|\alpha|^2} \sum_{j,k=0}^{D-1} \frac{\alpha^j \alpha^k}{\sqrt{j! \sqrt{k!}}} |j \rangle |k\rangle. \]

(41)

We choose \( D = 15 \) – see figure 9(b) – so that \( |ECS_{\text{qudit}}\rangle \) is approximately normalised in the range \( 0 < \alpha < 3 \) and therefore models a coherent state for this range. The c-SWAP probabilities for this state are presented in figure 9(a).

Figure 9. (a) shows a comparison of probabilities against coherent state amplitude \( \alpha \) between ECS results from equation (38), with \( C_2' = 1 \) as in equation (40), and a qudit approximation of ECS as in equation (41) with \( D = 15 \). (b) shows the norm of the qudit approximated ECS state as a heatmap across different qudit dimensions and coherent state amplitudes. This was used to determine a value of \( D \) for which the qudit state accurately models the ECS across the coherent state amplitude \( \alpha \) range considered.

Assuming that our hypothesis that an increase of \( P(|11\rangle_C) \) is indicative of increased entanglement in the qudit test state, the results presented in figure 9(a) suggest that for \( \alpha < 2 \) entanglement increases with \( \alpha \), as we proposed in section VII.

Figure 9(a) also shows the probability results for a general ECS-like state (equation (38) where \( C_2' = 1 \) considered in section VII. Although they are strikingly similar in form, the difference is not simply due to the finite dimension approximation made in equation (40). Instead, we suggest it relates to the basis used, and the associated general state on which the test is applied. For the optical extension, we apply the test to a general state of the form given in equation (38) where the set of possible states is spanned by a basis consisting of states \( |\alpha\rangle \) and \( |\alpha\rangle \) analogous to the two-qubit case. For the qudit extension, we execute the c-SWAP test on a general two qudit state with dimension \( D \), where the basis which spans the set of possible states is \( \{|0\rangle, |1\rangle, |2\rangle, ..., |D-1\rangle\} \). This difference in the c-SWAP test action may account for the discrepancy observed. In any case, the similarity of the results for these two different methods remains noteworthy.

Approximating two-mode entangled optical states as qudits provides a valuable theoretical insight into the entanglement in the systems using this test. However, a practical implementation capable of swapping states in the Fock basis remains implausible.

C. Entangled coherent and vacuum state

Another form of entangled coherent state is a coherent state mixed with a vacuum state, which can be created with high fidelity and is reminiscent of a NOON state (38):

\[ |\psi^{ECS}_{2}\rangle = N_\alpha (|\alpha\rangle |0\rangle + |0\rangle |\alpha\rangle) \]

(42)

where \( N_\alpha = 1/\sqrt{2(1+e^{-|\alpha|^2})} \). Applying the entanglement c-SWAP test to this state gives

\[ P(|11\rangle_C) = \frac{(N_\alpha)^4}{2} (1 + e^{-|\alpha|^2}) \]

(43)

\[ = \frac{1}{8(\text{sech}(4|\alpha|^2) + 1)} \]

(44)

shown in figure 10. Compare this with the entangled coherent state \( |\psi^{ECS}_{2}\rangle \) in equation (38). The probability result is then:

\[ P(|11\rangle_C) = \frac{1}{8(\text{sech}(4|\alpha|^2) + 1)} \]

(45)

which also tends to \( \frac{1}{2} \) as \( \alpha \) increases, but at a faster rate. This is due to \( \langle \alpha|\alpha\rangle = e^{-2|\alpha|^2} \) and \( \langle \alpha|\alpha\rangle = e^{-\frac{1}{4}|\alpha|^2} \).

Again we see an increase of \( P(|11\rangle_C) \) as \( \alpha \) increases.

D. Two-mode squeezed states as high dimensional qudits

A two-mode squeezed vacuum state \( |\psi_{TMSV}^{2}\rangle = S_2(\xi) |0,0\rangle \), where \( S_2 \) is the two-mode squeeze operator defined in equation (23), has been used to demonstrate
the EPR paradox experiment with continuous position and momentum variables [39]. In the photon number basis this is [40]

\[ |\psi_{\text{TMSV}}\rangle = \frac{1}{\cosh r} \sum_{j=0}^{D-1} (-e^{i\theta} \tanh r)^j |jj\rangle. \] (46)

the normalisation values of which are shown in figure 11b. After applying the test for entanglement to this state the final entanglement signature probability is

\[ P(|11\rangle_C) = \frac{1}{2 \cosh^4 r} \sum_{j=0}^{D-1} \sum_{k=0}^{D-1} (\tanh r)^{2(j+k)} \] (47)

shown in figure 11[a] where \( D = 250 \).

VII. CONCLUSIONS AND FURTHER WORK

The c-SWAP test for entanglement has been shown to be an efficient and effective procedure for detecting entanglement in qubit systems, potentially surpassing quantum state tomography, especially for large numbers of qubits [4]. It is based on a similar circuit to the widely used SWAP test for equivalence. The success of this family of tests motivates an investigation into their limits and possible extensions. Accordingly, this paper explores a range of extensions to the SWAP tests for equivalence and entanglement and evaluates their success, applicability, and whether they warrant further work and more detailed examination.

The test for equivalence can trivially be applied to optical states. The results largely follow expected outcomes, although with several features of interest; the results differ particularly from the qubit case for low coherent state amplitudes, where the states exhibit interesting properties.

While the test for equivalence can be used to great effect in optical states, the entanglement test is limited in its applicability to certain systems. In general, the c-SWAP test for entanglement performs best on a state which is built up from well defined subsystems which can easily be swapped using a quantum circuit. The test succeeds for (two-mode) entangled coherent states, entangled coherent and vacuum states, and squeezed vacuum states, with increased coherent state amplitudes evidencing increased amounts of entanglement in the states.

We explore two forms of a bipartite entanglement test. If characterising entanglement across a bipartite cut, the extension is successful, and despite being in general less efficient than the multipartite test at capturing all entanglement in a system, requires only two c-SWAP operations and control qubits. In practice, this variant of the c-SWAP test for entanglement may prove easier to implement, given the current difficulty in realising c-SWAP operations.

Conversely, if investigating entanglement between two qubits within a larger system, the entanglement test functions in a limited capacity: the test breaks down when the subsystem of interested is mixed. The c-SWAP test’s inability to handle mixed systems is its main limitation.
As the results of the qubit entanglement test are directly related to Concentratable Entanglements as defined in [22], it should be possible to extend this family of entanglement measures to qudit states. Further, if the results of the entanglement test on optical states can be related to Gaussian entanglement measures, Concentratable Entanglements could be extended to optical states.

It remains an open question whether the c-SWAP test’s shortcoming with regards to mixed states can be circumvented in some way or whether it is a fundamental limitation.

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