Numerical calculation method of aperture function for rough rock fracture flow

G Chen¹, L Ma²*, H S Gong¹ and F Q Luo³

Abstract: Fracture is the main path of rock seepage and plays a decisive role in the permeability of rock masses. The method of solving Navier-Stokes (NS) equation directly to study the fracture seepage flow has high accuracy and can well reflect the nonlinear flow characteristics in the fracture, but there is also the problem of excessive calculation. In this paper, the fracture aperture spatial data is substituted into the local cubic law as an aperture function to form a numerical calculation method for seepage in rough rock fractures, namely, the aperture function method(AFM). After comparing with the experimental calculation results of fracture seepage, it is found that the AFM calculation results are in good agreement. Using the method of solving the NS equation, the seepage calculations were performed for two typical fracture models with multiple boundary conditions. After comparison and analysis, a) the nonlinear term in the rough fracture flow mainly originates from the local pressure drop (LPD), and the generation of the recirculation zone(RZ) is the main reason for the formation of the LPD. b) The AFM calculation results are compared with the NS method under low hydraulic gradient conditions and its error is not significant (i. e., 0.21%, 0.81%). AFM is an excellent numerical calculation method when the condition of neglecting the effect of nonlinear items in fracture flow is allowed. c) The main advantage of AFM is that it has a very high numerical calculation efficiency, which makes the method available for seepage calculations in large-scale rough fractures.

Keywords: Rough rock fracture, Fracture flow, Local cubic law, Aperture function method (AFM), Local pressure drop (LPD)

1. Introduction
Rock fractures are widely distributed in subsurface rock masses, and for fluid transport, the permeability of rock fractures is significantly greater than that of the rock matrix[1]. The hydraulic
properties of rock fractures play an important role in assessing the performance of subsurface engineering, such as geothermal energy development, enhanced recovery, nuclear waste disposal, and groundwater pollution management[2][3]. Fluid flow in rock fractures can be calculated using the Navier-Stokes (NS) set of equations. Using the NS equation to solve the fracture flow, the calculation results are highly accurate, due to the fact that the fracture aperture is much smaller relative to the extensional direction size, resulting in difficult modeling and high computational effort.

Due to the complexity of solving the NS equation, a concise fracture seepage equation, cubic law, can be derived by neglecting the nonlinear terms in fluid flow[4]. The surface of natural rock fractures is rough, which restricts the flow of fluid in the fractures, and using the cubic law to calculate the fracture flow will overestimate the permeability of the fractures[5]. Through theoretical derivation and experimental studies, it has been proposed that the effect of inertial flow of the fluid can be neglected under low Reynolds number (Re<<1) conditions and the local cubic law (Reynolds equation) can be used to calculate the fracture seepage flow with little deviation[6][7]. Javadi et al.[8] found that there is a deviation between the calculated results of linear and nonlinear flow in the fracture, but the difference is not significant at low Reynolds number conditions.

Using the upper and lower surfaces of the fracture can generate a list of spatial data of the fracture aperture, the values in the list vary with the spatial displacement and the whole list can be used as a function of the aperture. Using a fracture wall as a geometric model and the aperture function as the aperture parameter of the fracture numerical model, the numerical calculation of rough fracture seepage is carried out based on the local cubic law, and the method is called the Aperture Function Method (AFM). Comparing the physical experiments of fracture seepage with the calculation results of AFM method, the two have a good agreement. The numerical models of fracture seepage based on NS equation are compared, and the reasons for the deviation of the calculation results of AFM method are further analyzed.

2. Methodology

2.1 Interpolation method of Aperture

The rough rock fracture surface morphology is complex, and the data obtained by using high-precision laser scanning technology can reflect the real fracture surface morphology[9]. The aperture between two fracture walls can be calculated by the following equation[10]:

\[
e(x, y) = \begin{cases} 
 g_u(x, y) - g_l(x, y) & (g_u(x, y) > g_l(x, y)) \\
 0 & (g_u(x, y) \leq g_l(x, y))
\end{cases}
\]

(1)

Where: \(e(x, y)\) is the fracture aperture, \(g_u(x, y)\) is the upper fracture surface space height; \(g_l(x, y)\) is the lower fracture surface space height.

When \(e(x, y)\) does not correspond exactly to the spatial coordinates \((x, y)\) of \(g_u(x, y)\) (or \(g_l(x, y)\)), it is necessary to use the interpolation method to find the aperture value at the desired spatial location. In Figure 1a, the \(g_{l,i}\) points on the geometric fracture surface correspond to the \(e'_{l,i}\) points in the aperture, but there is no aperture value at this point, so the interpolation method is used to obtain the aperture value at the \(e'_{l,i}\) point.
Figure 1. Schematic diagram of fracture aperture interpolation method; (a) schematic diagram of the relationship between the grid points of the fracture wall and the grid points of the aperture; (b) solve the value of the aperture at a certain position \((e(x, y))\).

The aperture is a function of the spatial coordinates \((x, y)\), so the four points can be expressed as: 
\[(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\], and if the aperture value \(e(x,y)\) at a position in the middle of the four points is to be obtained, it can be estimated by linear interpolation:

First, interpolate \(x\) to find \(e(x,y_1)\) and \(e(x,y_2)\):

\[
\begin{align*}
e(x, y_1) &= e(x_1, y_1) \cdot \frac{x - x_1}{x_2 - x_1} + e(x_2, y_1) \cdot \frac{x - x_1}{x_2 - x_1} \\
e(x, y_2) &= e(x_1, y_2) \cdot \frac{x - x_1}{x_2 - x_1} + e(x_2, y_2) \cdot \frac{x - x_1}{x_2 - x_1}
\end{align*}
\]

(2)

Then use the same idea to find \(e(x,y)\):

\[
e(x, y) = e(x_1, y_1) \cdot \frac{y - y_1}{y_2 - y_1} + e(x_2, y_2) \cdot \frac{y - y_1}{y_2 - y_1}
\]

(3)

The above equations give the value of the aperture at any position within the spatial distribution of the aperture. Each point \(g_{i,j}(x, y)\) on the fracture wall grid can also be used to find the required aperture value.

Figure 2 Discretizing the rock fracture to the volumetric elements, (a) triangular discrete grid; (b) quadrilateral discrete grid

When discretizing the fracture numerical model, each fracture wall grid point corresponds to a tiny geometric grid cell. Each grid cell can find the corresponding fracture aperture data, and within this cell, the aperture value is a given value, then the geometric grid cell and the aperture value can form a tiny virtual flat flow model. For each tiny virtual flat plate model the fracture flow calculation can be performed using the cubic law, and the overall view is an application of the local cubic law. Since the fracture surface and aperture values of each cell are a list of values that vary with space, a rough fracture model with two walls is formed. A triangular mesh can be used for discrete fracture data, or a
rectangular mesh can be used (Figure 2).

2.2 From the NS to the Local Cubic Law Equations

The fluid flow velocity in the fracture is very slow, and neglecting the effect of inertia forces, the equations can be derived from the NS equation (Zimmerman et al.[6], Lee et al.[11], Huang et al.[12]) for the fracture flow in the laminar flow state, i.e., the Reynolds equation, also called the local cubic law. Substituting the aperture function $e(x,y)$ into the Reynolds equation gives a clear expression [13]:

$$
\frac{\partial}{\partial x} \left( \rho g e^3 (x,y) \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho g e^3 (x,y) \frac{\partial h}{\partial y} \right) = 0
$$

where $h$ is the hydraulic head, $\mu$ is the dynamic viscosity coefficient of the fluid, $\rho$ is the density of the fluid, $g$ is the acceleration of gravity, and $e(x,y)$ is the fracture aperture function.

3. Results

To verify the rationality of the AFM, this paper compares the results of the rough fracture flow physics experiments done by Zhu H G[14]. The experiments adopt the typical definition of rock fracture roughness curve (JRC) in rock mechanics, and the 1st, 3rd, 6th and 9th curves are selected, and only the 3rd and 6th curves are shown and introduced in this paper due to the limitation of space. The 3rd and 6th curves are combined with a smooth flat plate to form 2 different fracture models with different roughness, each with 100mm fracture length and a minimum fracture width of 0.51mm. The model numbers are JRC3 and JRC6, respectively. Multiple experiments were conducted for each artificial fracture sample, and the pressure gradient and flow rate were averaged.

The numerical calculation of seepage flow for the 2 fracture models was completed using AFM. The numerical model of the fracture is completed and calculated by the finite element numerical calculation software COMSOL Multiphysics. Figure 3 shows the numerical model, aperture space distribution and fracture seepage field.

![Figure 3](image)

Figure 3  (a),(d) 2 fracture curf(Stretched properly for display) , (b), (e) Spatial distribution of fracture aperture and (c), (f) numerical model fracture surface velocity distribution.

As can be seen in Table 1, the results calculated by the average aperture and the overall cubic law deviate very much from the experimental measured results. In contrast, the results calculated by the numerical model using the AFM are in good agreement.
Table 1 Comparative analysis of test results according to average aperture and aperture interpolation function

| Specimen number | $P_s/Pa$ | $q_{c}/(10^{-4}m^3/s)$ | $e_p(mm)$ | $q_{o}(m^2/s)$ | $\varepsilon(q_p)/\%$ | $q_w/(10^{-4}m^3/s)$ | $e_n(mm)$ | $\varepsilon(q_n)/\%$ |
|-----------------|---------|------------------------|-----------|----------------|----------------------|-----------------------|-----------|----------------------|
| JRC3            | 147.50  | 0.097                  | 1.35      | 0.265          | 174                  | 0.106                 | 0.956     | 9                    |
| JRC6            | 98.20   | 0.034                  | 1.35      | 0.088          | 160                  | 0.035                 | 0.956     | 4                    |
|                 | 290.20  | 0.187                  | 3.30      | 7.623          | **3977**             | 0.217                 | 0.967     | 16                   |
|                 | 149.00  | 0.106                  | 3.30      | 3.914          | **3593**             | 0.112                 | 0.968     | 5                    |

Note: The subscript $s$ represents the actual measurement result of the physical experiment; the subscript $p$ represents the calculation result according to the average aperture and the cube law; the subscript $n$ represents the calculation results of AFM. $\varepsilon(q_p)$ and $\varepsilon(q_n)$ represent the error calculated by the flow rate.

4. Discussion

4.1 Error analysis and discussion

The AFM uses the local cubic law, which ignores the effect of inertial forces of the fluid and makes a linear relationship between the flow and the hydraulic gradient. Javadi [8] considered two types of pressure drop during fluid flow in a fracture: one is viscous pressure drop due to fluid viscosity (VPD: viscous pressure drop); and the other is the local pressure drop(LPD) due to the sudden change in aperture. For laminar flow in a smooth flat plate, the VPD can be calculated using the cubic law. For rough fractures, corrections for fracture wall roughness and fracture curvature are required.

LPD is the pressure drop (or energy loss) that occurs when the fluid flows through the part of the fracture where the aperture changes suddenly. The local cubic law cannot calculate the LPD of the fracture flow, which is the main reason for the deviation of the AFM calculation.

Lee et al.[11], Zhou et al.[15] found through physical and numerical experiments that in the steady laminar flow state, the fluid in the fracture generates a RZ when flowing through the abrupt change in the aperture, which forms at the irregular solid boundary and is separated from the main flow. The fluids in the RZs also consume part of the flow energy, and because the fluid in the RZ occurs on its own, it does not produce effective flow at the exit of the fracture. Therefore, the energy consumption caused by this part of the fluid in the RZs is the main source of LPD. The size of the RZ is nonlinearly related to the fluid flow rate and the fracture roughness procedure, and the size of its range cannot be calculated using simple equations.

Under the condition of constant geometry and aperture, a variety of pressure boundary conditions are set, and two methods, NS solution and AFM solution, are used to calculate and compare, respectively. The JRC3 fracture model inlet is applied with pressures from 10 Pa to 890 Pa ($-dh/dl = 0.01$–0.76), respectively, for a total of 36 sub-models. JRC6 fracture model inlet is applied 25 Pa–650 Pa pressure ($-dh/dl = 0.02$–0.55), respectively, for a total of 26 sub-models, and the outlet pressure is 0 Pa for all models. The fluid density in the model is 998.2 kg/m$^3$ and the dynamic viscosity coefficient is 0.0010093 Pa.s. The calculation software was carried out using COMSOL Multiphysics finite element method. The numerical model information and calculation results of the two fractures JRC3 and JRC6 are shown in Table 2.
Figure 4(c)–(g), showing the streamlines in the fracture under typical hydraulic gradient conditions of JRC6. Under the low hydraulic gradient (JRC6, \(-dh/dl=0.02\)) condition, there are no obvious RZs (Figure 4(c,d)). As the hydraulic gradient increases, the influence of fluid inertia force gradually emerges, not only to expand the range of RZs, but also the emergence of multiple RZs of different sizes, resulting in the narrowing and curving of the mainstream zones (Figure 4(e~h)). Figure 4(i,j) are a zoomed-in view of the local flow line, and the arrows on the flow line show that the fluid in the RZs significantly and did not enter the mainstream zone. The fluid in the RZs consumes part of the energy of the fracture flow and generates LPD, but does not generate effective flow at the fracture outlet.

Table 2 Basic information of JRC3, JRC6 numerical models

| Solution       | JRC3       | JRC6       |
|----------------|------------|------------|
| Number of elements | 743377   | 265772     |
| Mesh size/m     | 3.58×10\(^{-7}\)~1.82×10\(^{-5}\)~ | 9.83×10\(^{-7}\)~2.54×10\(^{-5}\)~ |
| Hydraulic gradient(\(-dh/dl\)) | 0.01~0.76 | 0.02~0.55  |
| Number of sub-models | 36      | 26         |
| Calculation duration /s | 176712  | 58665     |
| \(q/10^3m^2/s\) | 0.00363~0.242 | 0.00985~0.218 |
| \(ε(q_n)\)%       | 0.15~29.12 | 0.73~32.47 |
| \(Re\)            | 3.59~239.63 | 9.74~259.32 |

Figure 4(a) Discrete mesh (partial), (b) JRC6 model, (c)–(h) The streamline with different hydraulic gradient(\(-dh/dl\)). (i),(j) The streamlines in the local area are enlarged, and the recirculation zone and main flow zone are marked.

The results of the flow rate calculated by the AFM are compared with the results of the NS equation solution. We plotted the flow curves with hydraulic gradient from the two calculation methods and plotted the error curves for both flow rates in Figure 5. The error curves show that the AFM
calculations overestimate the permeability of the fracture, causing its calculated flow rate to be large, which is consistent with the results of Bauget et al. [5]. The errors of both NS and AFM calculations at low hydraulic gradients are small, with a minimum flow error of 0.21% (-dh/dl=0.01) for the JRC3 model and 0.81% (-dh/dl=0.02) for the JRC6 model. The error becomes progressively larger with increasing hydraulic gradient, which is consistent with the results of Zimmerman et al. [7]. The flow error variation also shows a non-linear variation pattern rather than a simple linear increase.

Therefore, under low hydraulic gradient and low Reynolds number conditions, the calculation using the AFM will only produce a small error. It also shows that the RZs in the fracture due to nonlinear flow is the main cause of LPD.

![Figure 5](image.png)

**Figure 5** Flow per unit width (q) ~ hydraulic gradient (dh/dl) curf. the solid circle indicates the NS calculation result curve, the solid square indicates the AFM calculation result curve, the hollow triangle indicates the error in the calculation result of NS vs AFM.

### 4.2 Advantages of AFM

For the calculation of rough fracture seepage flow, the main advantages of AFM are the simplicity of the model and high computational efficiency compared with the numerical calculation method of solving the NS equation directly (Table 2).

The substantial computational efficiency advantage of the AFM can complete more model calculations in a short time. Because the calculation time of a single rough fracture model is very short, the seepage calculation of a complex 3D spatial rough fracture network (e.g., a fracture network composed of thousands of rough fractures) can be completed by using only an ordinary computer, which is one of our future works.

### 5. Conclusions

From the above analysis and description, we can draw several conclusions as follows.

1) The numerical calculation of seepage flow in rough fractures can be done using the AFM under low Reynolds number conditions, but the calculation results will produce some small errors.

2) Using the local cubic law to solve the rough fracture seepage flow, the error of the calculation results mainly originates from the LPD caused by the RZs; the fracture flow error will become larger with the increase of hydraulic gradient.

3) The main advantage of the AFM is its high efficiency of numerical calculation. Based on this
advantage, AFM can quickly complete the numerical calculation of large-scale 3D rough fracture network.

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