Vortex stretching and a modified zeroth law for the incompressible
3D Navier-Stokes equations

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Abstract

We consider the 3D incompressible Navier-Stokes equations under the following 2 + 1/2-dimensional
situation: small-scale horizontal vortex blob being stretched by large-scale, anti-parallel pairs of vertical
vortex tubes. We prove a modified version of the zeroth law induced by such vortex-stretching.

1 Introduction

The zeroth law of turbulence states that, in the limit of vanishing viscosity, the rate of kinetic energy
dissipation for solutions to the incompressible Navier-Stokes equations becomes nonzero. This is one of the
central ansatz of Kolmogorov’s 1941 theory ([23]). To formulate this law, we recall the 3D incompressible
Navier-Stokes equations on $T^3_\ast := (\mathbb{R}/2\mathbb{Z})^3$:

$$
\begin{aligned}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p &= \nu \Delta u^\nu + f, \\
\nabla \cdot u^\nu &= 0, \\
u^\nu(t = 0) &= u_0^\nu
\end{aligned}
$$

(1.1)

where $\nu > 0$ is the viscosity and $u^\nu : T^3_\ast \to \mathbb{R}^3, p : T^3_\ast \to \mathbb{R}$ denote the velocity and pressure of the fluid,
respectively. Here $f : T^3_\ast \to \mathbb{R}^3$ is some external force. Assuming that the solution is sufficiently smooth,
taking the dot product of the equation with $u^\nu$ and integrating over $T^3_\ast$ gives the energy balance

$$
d \frac{1}{2} \| u^\nu(t) \|_{L^2}^2 = \int_{T^3_\ast} f(t) \cdot u^\nu(t) dx - \nu \| \nabla u^\nu(t) \|_{L^2}^2.
$$

The zeroth law then postulates that, under the normalization $\| u_0^\nu \|_{L^2} = 1$, the mean energy dissipation rate
does not vanish as $\nu \to 0^+$:

$$
\liminf_{\nu \to 0} \nu \langle \| \nabla u_0^\nu \|_{L^2}^2 \rangle > 0,
$$

where $\langle \cdot \rangle$ usually denotes some ensemble or long-time, space averages. Laboratory experiments and numerical
simulations of turbulence both confirm the above zeroth law ([5] [17] [21] [31]). See recent works of Drivas [11]
and Buckmaster-Vicol [5] for more precise formulation and developments related to the zeroth law. In this
paper, we take $\langle \cdot \rangle$ to be a short-time space average, and take sequences of smooth initial data $u_0^\nu \in C^\infty(T^3_\ast)$.
Hence we may take $f \equiv 0$, and the energy balance is justified. However, in the short-time, a trivial version
of zeroth law appears, and thus we need to avoid it carefully. We now explain it more precisely. Let $H^s$

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we obtain the 3D vorticity equations:

\[ \omega \]

of the zeroth law satisfying the above. We achieve this in the framework of 2 + \(1\)

The above condition can be interpreted as occurrence of strong “vortex-stretching”. We shall prove a version

\[ \omega \text{ smooth solution to the 3D Euler equations (also for the 3D Navier-Stokes) with initial data} \]

flow. Note that the data and solution are independent of \(x\)

The incompressible Euler equations are obtained by taking \(\nu = 0\) in [1.1]. Introducing the vorticity \(\omega = \nabla \times u\), we obtain the 3D vorticity equations:

\[ \partial_t \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u, \quad x \in T^3_\mathbb{Z} := (\mathbb{R}/2\mathbb{Z})^3 \]

where the velocity \(u\) is determined by the (periodic) 3D Biot-Savart law:

\[ u(t, x) = \int_{T^3_\mathbb{Z}} K_3(x-y)\omega(t, y) \, dy, \]

with

\[ K_3(x)v = \frac{1}{4\pi} \frac{x \times v}{|x|^3} \quad (\text{with reflections}). \]

The associated Lagrangian flow is then given by

\[ \partial_t \Phi(t, x) = u(t, \Phi(t, x)) \quad \text{with} \quad \Phi(0, x) = x \in T^3_\mathbb{Z}. \]

In this paper, we shall examine a sequence of smooth initial vorticity of the form

\[ \omega_{n,0} = \omega_{n,0}^L + \omega_{n,0}^S \]

and we restrict them to the following symmetry (with a slight abuse of notation):

\[ \omega_{n,0}^L = (0, 0, \omega_{n,0}^L(x_1, x_2))^T \quad \text{and} \quad \omega_{n,0}^S = (\omega_{n,0,1}^S(x_1, x_2), \omega_{n,0,2}^S(x_1, x_2), 0)^T. \]

The corresponding solution also keeps this symmetry, which is commonly referred as to the 2 + \(\frac{1}{2}\)-dimensional flow. Note that the data and solution are independent of \(x_3\), and in this setting there is a global unique smooth solution to the 3D Euler equations (also for the 3D Navier-Stokes) with initial data \(\omega_{n,0}\), which we shall denote by \(\omega_n(t)\). By the Biot-Savart law,

\[ u_n(t, x) = \int_{T^2_\mathbb{Z}} K_3(x-y)\omega_n(t, y) \, dy, \quad \partial_t \Phi_n(t, x) = u_n(t, \Phi_n(t, x)) \]

and then

\[ \omega_n(t, \Phi_n(t, x)) = D\Phi_n(t, x)\omega_n,0(x) = D\Phi_n(t, x)(\omega_{n,0}^L(x) + \omega_{n,0}^S(x)). \]
where $D\Phi_n = (\partial_j \Phi_{n,j})_{1 \leq j \leq 3}$. This is the famous Cauchy formula. Moreover, since there is no dependence on the third variable for the solution $u$, $\Phi_n$ is determined by the 2D flow arising from the solution of the 2D Euler equations with initial data $\omega_{n,0}^C(x_1, x_2)$. We denote the 2D flow map by $\eta_n$, and by trivially extending the 2D flow into 3D, some abuse of notation, the 3D flow map associated with the solution for $\omega_{0,n}^C$ can be written as

$$D\eta_n := \begin{pmatrix} \partial_1 \eta_{n,1} & \partial_2 \eta_{n,1} & 0 \\ \partial_1 \eta_{n,2} & \partial_2 \eta_{n,2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D\eta_n^{-1} := \begin{pmatrix} \partial_2 \eta_{n,2} & -\partial_1 \eta_{n,1} & 0 \\ -\partial_2 \eta_{n,1} & \partial_1 \eta_{n,1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

It is not difficult to verify that $\partial_j \Phi_{n,i} = \partial_j \eta_{n,i}$ for $1 \leq i, j \leq 2$ and we have the following explicit formulas:

$$\omega_{n}^C(t, \eta_n(t, x)) = \omega_{n,0}^C(x) \quad \text{and} \quad \omega_{n}^S(t, \eta_n(t, x)) = D\eta_n(t, x)\omega_{n,0}^S(x). \hspace{1cm} (1.4)$$

Again, by the Biot-Savart law, we can also recover the large-scale velocity:

$$u_{n}^L(t, x) = \int_{\mathbb{T}_2^2} K_2(x-y)\omega_{n}^C(t, y) \, dy$$

where

$$K_2(x) = \frac{1}{2\pi} \frac{x^+}{|x|^2} \quad \text{(with reflections)}$$

and also $u_{n}^S(t, x) = u_{n,0}^S(t, \eta_n(t, x))$ where $u_{n}^S(t, x)$ can be uniquely recovered from $\omega_{n}^S(t, x)$ by $\nabla \times u_{n,0}^S = \omega_{n,0}^S$ and $\nabla \cdot u_{n}^S(t, x) = 0$. Now note that since $\omega_{n}^S(t) \perp e_3$ and $\omega_{n}^C(t) \parallel e_3$, we have

$$\|\omega_n(t)\|^2_{L^2} = \|\omega_{n}^C(t)\|^2_{L^2} + \|\omega_{n}^S(t)\|^2_{L^2}.$$

### 1.2 Main results

To state our result, let us briefly explain the construction of the initial data sequence. We consider data independent of $x_3$, which allows us to treat them as functions defined on $\mathbb{T}_L^2 := (\mathbb{R}/(2\mathbb{Z}))^2$. We take the Bahouri-Chemin stationary solution introduced in [1] $\omega(x_1, x_2) = \text{sgn}(x_1)\text{sgn}(x_2)$ on $[-L, L]^2$ and smooth it out at scale $\ell \ll L$ to define $\omega_{n,0}^C(x)$. Then we place a small “bump” $\omega_{n,0}^S(x)$ in a ball of radius $\ell \ll \ell$ centered at the origin. Then the initial data sequence is simply given by $\omega_{n,0}^C = \omega_{n,0}^C + \omega_{n,0}^S$, where $\ell, \ell$ (and even $L$ in some cases) depend on $n$. Details of the construction will be explained in Section 2 for now see Figure 2. We now give the main theorem, which roughly states that the vortex-stretching in the $2 + \frac{1}{2}$-dimensional setup is enough to create vortex stretching of order $\nu^{-\epsilon_0}$ for some $\epsilon_0 > 0$ for the 3D Navier-Stokes equations in the limit $\nu \to 0^+$, with uniformly bounded (at least) in $L^2$ initial data. To motivate the statements, let us recall the energy identity for the Navier-Stokes equations:

$$\frac{1}{2} \|u''(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u'(s)\|^2_{L^2} \, ds = \frac{1}{2} \|u_0''\|^2_{L^2}.$$ 

From the symmetry in our initial data, the solution $u''(t)$ can be written as the sum $u^{L,v} + u^{S,v}$ where $u^{L,v}$ and $u^{S,v}$ are defined by

$$\begin{cases} 
\partial_t u^{L,v} + u^{L,v} \cdot \nabla u^{L,v} + \nabla p'' = \nu \Delta u^{S,v}, \quad \nabla \cdot u^{L,v} = 0, \\\n\partial_t u^{S,v} + u^{L,v} \cdot \nabla u^{S,v} = \nu \Delta u^{S,v}
\end{cases} \hspace{1cm} (1.5)$$

and we similarly have the following energy identity for the small-scale:

$$\frac{1}{2} \|u^{S,v}(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u^{S,v}(s)\|^2_{L^2} \, ds = \frac{1}{2} \|u_0^{S,v}\|^2_{L^2}.$$
Remark 1.1 (A modified zeroth law in a fixed time interval). We consider the torus \( T^3_n := (\mathbb{R}/(2L_n\mathbb{Z}))^3 \) for some \( L_n \leq 1 \). There exists some absolute constant \( \delta > 0 \) such that the following statements hold: for any \( 0 < a_0 < 1 \), there exist length scales \( L_n \leq 1 \), a sequence of \( C^\infty \)-smooth initial data \( u_{n,0} = u_{n,0}^L + u_{n,0}^S \) with uniform bounds
\[
\|u_{n,0}\|^2_{L^2(T^3_n)} = \|u_{n,0}^L\|^2_{L^2(T^3_n)} + \|u_{n,0}^S\|^2_{L^2(T^3_n)} \approx L_n^2,
\]
and viscosity constants \( \nu_n \to 0 \) such that the unique smooth solution \( u_n^\nu \) of the 3D Navier-Stokes equations with initial data \( u_{n,0} \) and viscosity \( \nu_n \) on \( T^3_n \) satisfies
\[
\liminf_{n \to \infty} \nu_n^{\bar{a}_0} \frac{1}{\delta} \int_0^\delta \|\nabla u_n^\nu(t)\|^2_{L^2(T^3_n)} dt \gtrsim \|u_{n,0}\|^2_{L^2(T^3_n)}.
\]
(1.6)

Remark 1.2. We remark that for \( a_0 \leq \frac{1}{2} \), we can take \( L = 1 \) for all \( n \), while for \( a_0 > \frac{1}{2} \), we need \( L \to 0 \). Of course, one can take \( L \) to be dyadic and still regard the data as being defined on the unit torus; see Figure 1 illustrating this point.

Remark 1.3. Given a sequence of initial data (normalized in \( L^2 \) norm by \( L_n^3 \)) and viscosity constants, it is reasonable to define the index \( 0 \leq \bar{b}_0 < 1 \)
\[
\bar{b}_0 := \inf \{ b_0 : \limsup_{n \to \infty} \nu_n^{b_0} \|\nabla u_{n,0}\|^2_{L^2(T^3_n)} \lesssim \|u_{n,0}\|^2_{L^2(T^3_n)} \}.
\]
In the above theorem, one can check from the proof (see (2.38)) that
\[
\bar{b}_0 = \bar{a}_0 - c_* \delta,
\]
where \( c_* > 0 \) is a constant depending only on \( \bar{a}_0 \) which possibly vanishes only when \( \bar{a}_0 \to 1 \) (this \( \bar{b}_0 \) consideration essentially comes from (1.3)). On the other hand, if one is interested only in the case of \( \bar{b}_0 = 0 \), we can take \( \bar{a}_0 = c_0 \delta \) where \( c_0 > 0 \) is an absolute constant, with initial data sequence \( \{u_{n,0}\} \) uniformly bounded in \( H^1(T^3_n) \) with \( T^3_n = (\mathbb{R}/(2\mathbb{Z}))^3 \). In this case, a recent result of Drivas and Eyink \[12\] puts a restriction that \( c_0 \delta < \frac{3}{2} \), where \( \delta > 0 \) is the same universal constant in the statement of Theorem 1.1. Let us explain it more precisely. They showed that if a sequence of Leray solutions \( \{u^\nu\} \), are uniformly bounded in \( L^3([0,\delta];B^{\sigma}_{3,\infty}(T^3_n)) \) for some \( \sigma \in (0,1) \), then the corresponding solutions satisfy
\[
\nu^{1/\delta} \int_0^\delta \int_{T^3_n} |\nabla u^\nu(t,x)|^2 dx dt \lesssim \nu^{\frac{2\sigma-1}{\sigma+1}}.
\]
(Note that the function space \( L^3([0,\delta];B^{\sigma}_{3,\infty}(T^3_n)) \) is physically natural; see Remark 1 in \[12\].) The estimate (1.7) gives an upper bound on the value of the constant \( \bar{a}_0 \) from (1.6): for \( \sigma > (2-\bar{a}_0)/(2+\bar{a}_0) \), the sequence of solutions \( \{u_n^\nu\}_n \) (the corresponding vorticities are \( \{\omega_n^\nu\}_n \) does not belong to \( L^3([0,\delta];B^{\sigma}_{3,\infty}(T^3_n)) \) uniformly in \( n \). The proof is the following: assume to the contrary that the sequence of solutions \( \{u_n^\nu\}_n \) belongs to \( L^3([0,\delta];B^{\sigma}_{3,\infty}(T^3_n)) \) uniformly in \( n \). By (1.6), we see
\[
\nu \int_0^\delta \int_{T^3_n} |\nabla u_n^\nu(t,x)|^2 dx dt \gtrsim \nu^{1-\bar{a}_0}.
\]
Thus, if \( \sigma \) satisfies \( 1-\bar{a}_0 > \frac{3\sigma-1}{\sigma+1} \), that is, \( \sigma > (2-\bar{a}_0)/(2+\bar{a}_0) \), then this contradicts (1.7) for sufficiently large \( n \). On the other hand, the sequence of solutions \( \{u_n^\nu\}_n \) belongs uniformly in \( L^3 B^{\sigma}_{3,\infty} \) with some \( \sigma \). To see this, one can directly estimate the equation
\[
\partial_t \omega_n^{S,\nu} + u_n^{L,\nu} \cdot \nabla \omega_n^{S,\nu} = \nabla u_n^{L,\nu} \omega_n^{S,\nu} + \nu \Delta \omega_n^{S,\nu}
\]
in $L^p$: \( \| \omega_n^{S,\nu}(t) \|_{L^p} \lesssim \| \omega_{n,0}^S \|_{L^p} \exp(\int_0^t |\nabla u_n^{S,\nu}(s)|_{L^\infty} ds) \), with an implicit constant independent of $\nu \geq 0$. From our choice of initial data and $|\nabla u_n^{S,\nu}(s)|_{L^\infty} \lesssim n$ (see Lemma \ref{lemma:initial} for details), it follows that the corresponding solution $\omega_n^{S,\nu}$ belongs to $L^\infty([0,t];L^p(t)(\mathbb{T}^2))$ with $p(t) = 2 - ct$ for some constant $c > 0$. This is due to the fact that $\| \omega_{n,0}^S \|_{L^p} \lesssim \| \nabla \omega_{n,0}^S \|_{L^\infty} ds \lesssim e^{n(1-2/p+t)}$ and to get the uniform bound, $1 - 2/p + t$ must be zero. Then at least for $t > 0$ sufficiently small, the velocity must be uniformly in $L^\infty W^{1,p(t)} \subset L^1_t B^{3,\infty}_2$ with $2 - 3/p(t) = \sigma$. This gives the restriction that $\sigma_0 \leq 2/3$.

**Remark 1.4.** In our result, the large-scale vorticity is uniformly bounded in $L^\infty$. Therefore it is tempting to approach the actual zeroth law using initial data which is singular, e.g. vorticities which are only $C^{-\alpha}$ and not better. We present a simple computation which illustrates that, at least in the setup of $2 + \frac{1}{2}$-dimensional flows, anomalous dissipation is not caused by vortex-stretching from velocity fields with exact $C^{1/3}$-regularity. To this end, recall that

\[
\begin{aligned}
\partial_t u_1 + u_1 \cdot \nabla u_1 + \nabla p &= 0, \\
\partial_t u_3 + u_1 \cdot \nabla u_3 &= 0,
\end{aligned}
\]  

(1.8)

where $u = (u_1, u_3)$ is a function of $(x_1, x_2)$ only. We take $\omega_h = \nabla \times u_h$, and consider the initial data

$\omega_{h,0} = r^{-\frac{5}{4}} \sin(2\theta),$

where $(r, \theta)$ is the usual polar coordinates. The corresponding velocity is

$u_{h,0} = \nabla^2 \Delta^{-1}(\omega_{h,0}) = c_{R^2} \left( r^{-2/3} - \frac{2}{3} r^{-2/3 - 2} x_1 x_2 \right).$

Note that $u_{h,0}$ belongs to exactly $C^{1/3}(\mathbb{R}^2)$ and not better. We may further set $u_{3,0}$ to behave like $|x_h|^{1/3}$ near the origin. However, the energy is conserved (at least at the initial time). Indeed,

\[
\frac{1}{2} \left| \frac{d}{dt} \int_{t=0} u_h^2 + u_3^2 \, dx_1 dx_2 \right| = - \int_{t=0} u_{h,0} \cdot (\nabla u_{h,0} \cdot u_{h,0} + \nabla u_{3,0} \cdot u_{3,0}) \, dx_1 dx_2
\]

\[
= \lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \frac{1}{2} u_{h,0} \cdot N ( |u_{h,0}|^2 + |u_{3,0}|^2 ) \, d\sigma
\]

\[
= \lim_{\epsilon \to 0} O(\epsilon^{1+3(1-2/3)}) = 0,
\]

where $N$ is the outwards unit normal vector on $\partial B(0,\epsilon)$ and $\sigma$ is the Lebesgue measure on $\partial B(0,\epsilon)$. We refer to recent works of Luo and Shvydkoy \cite{LuoShvydkoy24, LuoShvydkoy25, LuoShvydkoy30} which systematically studies the radially homogeneous solutions to 2D and 3D Euler equations and conclude absence of anomalous dissipation in that class of solutions.

### 1.3 Ideas and comments regarding the proofs

Regarding the well-posedness theory of the incompressible Euler equations, a recent breakthrough was made in the work of Bourgain-Li \cite{BourgainLi} (see also \cite{BourgainLi13, BourgainLi15, Li29, Li22}) where the authors have shown *ill-posedness* of the Euler equations in critical Sobolev spaces. In the case of 2D, the critical $L^2$-based Sobolev space is $H^1$ in terms of the vorticity. The strategy in \cite{BourgainLi} is to show that there exists *large Lagrangian deformation* for arbitrarily short time with initial vorticity uniformly bounded in $H^1$. This large Lagrangian deformation is responsible for the statement of Theorem 1.1 as the small-scale vorticity is being stretched by the deformation of the base large-scale flow. To achieve this we need to prove a sharp and quantitative bounds on the Lagrangian deformation, using smoothed-out Bahouri-Chemin solutions. This should be compared with previous results \cite{BourgainLi, Li13} where Lagrangian deformation and vorticity norm growth were obtained via a contradiction argument.

Another important breakthrough regarding the 2D Euler equations was the work of Kiselev-Sverak \cite{KiselevSverak22} on the double exponential growth of the vorticity gradient. The main tool was the so-called “Key Lemma”
which surprisingly gave an explicit integral representation for the main term in the velocity gradient for vorticity capped in $L^\infty$ and is odd with respect to both axes (i.e. anti-parallel). To calculate in a sharp way the velocity gradient in our setting, we adopt the Kiselev-Sverak approach, which then yields a quantitative large Lagrangian deformation with a careful ODE argument. We achieve this improvement only in the concrete setting of perturbed Bahouri-Chemin vorticities.

In view of the above, we would like to emphasize the following points:

- We prove sharp, quantitative bounds on the perturbed Bahouri-Chemin solutions.
- We obtain inviscid limit estimates which are quantitative in nature, which does not seem available in the literature.
- In the estimates we prove in Section 2 we have retained all physical parameters until the very end (before 2.3.5), and therefore the resulting estimate could be useful for the readers who would like to try out different scaling of physical quantities as $n \to +\infty$.

Remark 1.5. Let us mention a recent numerical simulation which have inspired the current work. Recently, using direct numerical simulations of the 3D Navier-Stokes equations, Goto, Saito, and Kawahara [20] have found that sustained turbulence consists of a hierarchy of antiparallel pairs of vortex tubes. Their main conclusions can be summarized as follows, which bear some similarity with our constructions:

- Turbulence, in the inertial length scales, is composed of hierarchy of vortex tubes with different sizes.
- At each hierarchy level, vortex tubes tend to form antiparallel pairs and they effectively stretch and create smaller-scale vortex tubes. Moreover, stretched vortex tubes tend to align in the direction perpendicular to larger-scale vortex tubes.
- Vortices at each hierarchical level are most likely to be stretched in strain fields around 2-8 times larger vortices.

It would be interesting to push our results further to be closer to the picture they have.
1.4 Organization of the paper

The rest of this paper is organized as follows: we first collect the notations and conventions that we use. The entire Section 2 is devoted to the proofs of the main results. In 2.1, we define the (sequence of) large-scale vorticity and obtain various sharp estimates. In particular, we prove creation of large Lagrangian deformation. Then in 2.2, we explain the setup for the (sequence of) small-scale vorticity and establish sharp upper bounds for them. Finally in 2.3, we perform inviscid limit computations and conclude the proof.

1.5 Notations

For the reader’s convenience, we collect the notations that will be used frequently in the paper.

- We shall work with the 2D domain $\mathbb{T}_L^2 = (\mathbb{R} / (2L\mathbb{Z}))^2$ and $\mathbb{T}_3^2 = ((\mathbb{R} / (2L\mathbb{Z}))^2 \times (\mathbb{R} / (2\mathbb{Z}))$ where $0 < L \leq 1$.
- Given a scalar-valued function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$, we define the $L^p$ norms by
  \[ \|f\|_{L^p}^p = \int_{\mathbb{T}_L^2} |f|^p \, dx, \quad 1 \leq p < +\infty. \]
  The case $p = +\infty$ is given by
  \[ \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{T}_L^2} |f(x)|. \]
- If $v$ is a vector-valued function $v = (v_1, \cdots, v_d)$,
  \[ \|v\|_{L^p}^p := \sum_{i=1}^d \|v_i\|_{L^p}^p, \quad 1 \leq p < +\infty, \quad \|v\|_{L^\infty} := \max_i \|v_i\|_{L^\infty}. \]
- The homogeneous Sobolev spaces are defined by
  \[ \|v\|_{\dot{H}^m} := \|\nabla^m v\|_{L^2} \]
  for integers $m \geq 1$, where $\nabla^m v$ is a vector consisting of all possible $m$-th order partial derivatives of $v$.
- The homogeneous Hölder norms are defined by
  \[ \|f\|_{C^\alpha} := \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\alpha} \]
  for $0 < \alpha \leq 1$.
- In this paper, $n \rightarrow +\infty$ is a large parameter. We shall use the notation $A \ll B$ (equivalently, $B \gg A$) if the ratio $A/B$ tends to 0 as $n \rightarrow +\infty$, where $A$ and $B$ are positive expressions depending on $n$. Moreover, we use $A \lesssim B$ (equivalently, $B \gtrsim A$) if there is an absolute constant $C > 0$ such that $A \leq CB$ uniformly for $n \rightarrow +\infty$. Then, we say $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Finally, we write $A \simeq B$ if $A/B \rightarrow 1$ as $n \rightarrow +\infty$.
- We shall consider the solutions defined on the time interval $[0, \delta]$, where we take $\delta > 0$ to be smaller whenever it becomes necessary, without explicitly mentioning it. We emphasize again that $\delta$ is independent of $n$.
- We comment on a few important parameters: $L$, $\ell$, $\tilde{\ell}$, and $\bar{\ell}$, all of which depend on $n$. We use $L \leq 1$ to denote the length-scale of the torus, which we also take to be the length-scale of the large-scale vorticity. The gradient of the large-scale vorticity is taken to be of order $\ell^{-1}$, where $\ell \ll L$. We introduce the convenient notation $\tilde{\ell} := \ell L^{-1}$, which is a non-dimensional parameter. One may simply fix it as $\tilde{\ell}_n = 2^{-n} \rightarrow 0$. Finally, $\tilde{\ell} := \ell^{1+\delta}$ (here $c > 0$ is some small absolute constant) is the length-scale of the small-scale vorticity.
- As it is usual, we use the letters $C, c$ to denote various absolute constants whose value can change from a line to another or even within a single line.
Figure 2: A diagram showing the support of $\omega^L_{n,0}$ (four large squares) and $\omega^S_{n,0}$ (circle in the center) in $T^2$.

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2 Proofs

Before we proceed to the description of the sequences of large and small scale vorticities, which will be denoted by $\omega^L_n$ and $\omega^S_n$, respectively.

2.1 Setup for the large-scale vorticity

2.1.1 Estimates for smoothed out Bahouri-Chemin solutions

Here, we precisely define the smoothed-out Bahouri-Chemin data and prove estimates for the corresponding solutions. For some length-scale $L > 0$, we set $T^2 := \mathbb{R}^2 / (2L\mathbb{Z})^2$, and recall that the Bahouri-Chemin solution can be written as $\text{sgn}(x_1)\text{sgn}(x_2)$ where $|x_1|, |x_2| \leq L$.

Given a length scale $\ell = \ell_n \ll L$, we cut the Bahouri-Chemin solution near the axes as follows:

$$\tilde{\omega}_n(x_1, x_2) := \text{sgn}(x_1)\text{sgn}(x_2)\chi_{\{\ell < |x_1|, |x_2| < L - \ell\}}$$

Now let $\varphi \in C^\infty_c(\mathbb{R}^2)$ be a standard mollifier; a radial function whose support is contained in the unit ball. With $\varphi_\ell(x) := \ell^{-2}\varphi(\ell^{-1}x)$, we define

$$\omega^L_n := \varphi_\kappa \ast \tilde{\omega}_n$$

for some $0 < \kappa \leq \frac{1}{2}$. In the following, we shall denote $\omega^L_n(t)$ be the unique solutions of the 2D Euler equation defined respectively on $T^2_L$ with initial data $\omega^L_n$.

We now recall a simple estimate of Yudovich (see e.g. [13] for a proof):
Lemma 2.1. Let $\omega(t) \in L^\infty([0, \infty) : L^\infty(T^2))$ be a solution of the 2D Euler equations, and $\eta(t)$ be the associated flow map. Then for some absolute constant $c > 0$, we have
\[
\left(\frac{|x - x'|}{L}\right)^{1+c\ell\|\omega_0\|_{L^\infty}} \leq \left|\eta(t, x) - \eta(t, x')\right| \leq \left(\frac{|x - x'|}{L}\right)^{1-c\ell\|\omega_0\|_{L^\infty}},
\] for all $0 \leq t$ and $|x - x'| \leq L/2$.

We now take a “small ball” region
\[
D = \{|x| < \tilde{\ell}\}
\]
where $0 < \tilde{\ell} \ll \ell$. The following lemma establishes a sharp estimate for the velocity gradient inside this region.

Lemma 2.2. Let $\omega_n(t)$ be the unique solution to the 2D Euler equations with initial data $\omega_{0,n}^\ell$ given in (2.1). We define the corresponding velocity field by $u_{n,1}^\ell(t)$. There exists some constant $c > 0$ such that for any $\delta > 0$, if $\ell$ satisfies $\ell \leq c\ell\ell^3$, then we have
\[
\partial_1 u_{n,1}^\ell(t, x) \geq \frac{2}{\pi}\|\omega_0\|_{L^\infty} \left(1 - C\delta - \epsilon_n\right) \ln \frac{1}{\ell},
\]
for
\[
(t, x) \in \left[0, \frac{\delta}{\|\omega_0\|_{L^\infty}}\right] \times D,
\]
where $\epsilon_n \to 0$ as $n \to +\infty$.

In the proof, we fix some $0 < \ell$ sufficiently smaller than $L$ and omit the indices $\mathcal{L}$ and $n$.

Proof. We begin by noting that $\omega_{0,n}$ is odd with respect to both axes, $\omega_{0,n} = 1$ on $[(1 + \kappa)|\ell, L - (1 + \kappa)|\ell]^2$, and vanishes on $[0, L]^2 \backslash [(1 - \kappa)|\ell, L - (1 - \kappa)|\ell]^2$. We claim that for small $\delta > 0$,
\[
\omega(t, x) \equiv \|\omega_0\|_{L^\infty} \quad \text{on} \quad (t, x) \in \left[0, \frac{\delta}{\|\omega_0\|_{L^\infty}}\right] \times \left[(1 + \kappa)|\ell, L\right] \frac{\ell}{2}^2.
\]
To show this, it suffices to observe that fluid particles starting from $\partial(1 + (1 + \kappa)|\ell, L - (1 + \kappa)|\ell]^2$ cannot reach the internal square $((1 + \kappa)|\ell, L - (1 + \kappa)|\ell]^2$ within time $\delta/\|\omega_n\|_{L^\infty}$. For this we need to consider four sides of this internal square. We shall only consider the left side, as the other sides can be treated in a similar way. To this end, take a point of the form $x = ((1 + \kappa)|\ell, a)$ for some $(1 + \kappa)|\ell \leq a \leq L - (1 + \kappa)|\ell$. Setting $x' = (0, a)$ and applying (2.2), we obtain
\[
|\eta_1(t, x) - \eta_1(t, x')| \leq L \left((1 + \kappa)|\ell\right)^{1-c\ell\delta},
\]
for all $0 \leq t \leq \delta/\|\omega_n\|_{L^\infty}$. Since $\eta_1(t, x') = 0$ (by odd symmetry) and $\eta_1(t, x) > 0$ for all $t$, we deduce that $\eta_1(t, x) \leq (1 + \kappa)|\ell\left((\frac{L}{\ell})^c\delta\right)$. Applying a similar argument to the other pieces of the boundary, we deduce (2.7).
A completely parallel argument, but instead using the lower bound in (2.2) rather than the upper bound, gives that
\[
\omega(t, x) \equiv 0 \quad \text{on} \quad (t, x) \in \left[0, \frac{\delta}{\|\omega_0\|_{L^\infty}}\right] \times [0, L]^2 \setminus \left[\left(1 - \kappa\right)l \left(\frac{L}{l}\right)^{-\epsilon\delta}, L - (1 - \kappa)l \left(\frac{L}{l}\right)^{-\epsilon\delta}\right]^2.
\] (2.8)

From now on we shall restrict to \(t \in \left[0, \delta/\|\omega_n\|_{L^\infty}\right]\), and recall explicit formulas
\[
\partial_1 u_1(t, x_1, x_2) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{(y_1 - x_1)(y_2 - x_2)}{|y - x|^4} \omega(y) \, dy = -\partial_2 u_2(t, x_1, x_2)
\]
where we have extended \(\omega\) to \(\mathbb{R}^2\) by periodicity and the integral is defined in the sense of principal value. Moreover, assuming for simplicity that \((x_1, x_2)\) does not belong to the support of \(\omega(t)\),
\[
\partial_1 u_2(t, x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|y - x|^4} \omega(y) \, dy = -\partial_2 u_1(t, x_1, x_2).
\]

We now take \(0 \leq x_1, x_2 \leq \frac{L}{2}\) and observe the uniform bounds
\[
\left|\int_{\mathbb{R}^2 \setminus [-L, L]^2} \frac{(y_1 - x_1)(y_2 - x_2)}{|y - x|^4} \omega(y) \, dy\right| + \left|\int_{\mathbb{R}^2 \setminus [-L, L]^2} \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|y - x|^4} \omega(y) \, dy\right| \lesssim \|\omega_0\|_{L^\infty}.
\]
(This is elementary but see for instance [32] for a proof.) We are ready to prove the claimed estimates. We proceed in several steps:

**Step 1. Lower bound of \(\partial_1 u_1\)**

We now estimate \(\partial_1 u_1(t)\). In view of the previous bound, we restrict the integral to \([-L, L]^2\) and then to \([0, L] \times [0, L]\) owing to the odd symmetry:
\[
\left|\partial_1 u_1(t, x)|_{x=0} - \frac{2}{\pi} \int_{[0, L] \times [0, L]} \frac{y_1 y_2}{(y_1^2 + y_2^2)^2} \omega(t, y) \, dy\right| \lesssim \|\omega_0\|_{L^\infty},
\]
where the constant is independent of \(L\). Let
\[
I := \frac{2}{\pi} \int_{[0, L] \times [0, L]} \frac{y_1 y_2}{(y_1^2 + y_2^2)^2} \omega(t, y) \, dy
\]
an and note that the integrand is non-negative. Using (2.7), we obtain a simple lower bound on \(I\):
\[
I \geq \frac{2}{\pi} \|\omega_0\|_{L^\infty} \int \left\{1 + \kappa\ell\left(\frac{L}{\ell}\right)\right\}^2 \left(\frac{y_1 y_2}{(y_1^2 + y_2^2)^2}\right) \, dy.
\]
This immediately gives
\[
\partial_1 u_1(t, x)|_{x=0} \geq \frac{2}{\pi} \|\omega_0\|_{L^\infty} \left(1 - \epsilon_n \ln \frac{1}{\ell}\right).
\] (2.9)

**Step 2. Upper bound of \(\partial_1 u_1\) along the \(x_1\)-axis.**

This time, we obtain an upper bound for \(\partial_1 u_1\) at the origin. We obtain a simple upper bound on the integrals \(I\) by replacing \(\omega(y)\) with \(\|\omega_0\|_{L^\infty}\) in the region \(\omega(y) > 0\) (\(\omega(y) < 0\), resp.). We obtain that
\[
\partial_1 u_1(t, 0, x_2) \leq \frac{2}{\pi} \|\omega_0\|_{L^\infty} \left(1 + \epsilon_n \ln \frac{1}{\ell}\right).
\] (2.10)
Step 3. Bounds on $\partial_1 u_1$ in the small ball region.

In order to estimate $\partial_1 u_1$ not only on the axis but also inside the small ball region, we shall use the classical estimates for the 2D Euler solutions:

$$1 + \log \left( 1 + \frac{L \|\omega(t)\|_{C^1}}{\|\omega_0\|_{L^\infty}} \right) \leq \left( 1 + \log \left( 1 + \frac{L \|\omega_0\|_{C^1}}{\|\omega_0\|_{L^\infty}} \right) \right) \exp(C \|\omega_0\|_{L^\infty} t)$$

(cf. [23, Theorem 2.1]). Since $\|\omega_0\|_{C^1} \lesssim (\kappa \ell)^{-1} \|\omega_0\|_{L^\infty}$, we obtain that

$$\log(1 + \frac{L \|\omega(t)\|_{C^1}}{\|\omega_0\|_{L^\infty}}) \leq \log(L(\kappa \ell)^{-1}) e^{C \delta} \leq \log(L(\kappa \ell)^{-1})^{1+C \delta}$$

for $\kappa \ell \ll 1$ and $\delta > 0$. Hence

$$\|\omega(t)\|_{C^1} \leq C \left( \frac{L}{k \ell} \right)^{1+C \delta} \|\omega_0\|_{L^\infty}. \tag{2.11}$$

Then we use the singular integral estimate

$$\|\nabla u(t)\|_{C^1} \lesssim \|\omega(t)\|_{C^1} \lesssim L^\frac{1}{2} \|\omega(t)\|_{C^1} \|\omega_0\|^{\frac{1}{2}}_{L^\infty} \lesssim L^{-\frac{1}{2}} \left( \frac{L}{k \ell} \right)^{\frac{1}{2}+C \delta} \|\omega_0\|_{L^\infty}. \tag{2.12}$$

We then obtain for $x = (x_1, x_2) \in D$ (recall the definition of $D$ from [23]),

$$|\nabla u(t, x) - \nabla u(t, 0)| \lesssim \left( \frac{\hat{\ell}}{L} \right)^{\frac{1}{2}} \left( \frac{L}{k \ell} \right)^{\frac{1}{2}+C \delta} \|\omega_0\|_{L^\infty}. \tag{2.12}$$

Therefore, we conclude that as long as $\hat{\ell}$ is chosen in a way that

$$\hat{\ell} \lesssim (k \ell)^{1+C \delta} L^{-C \delta}, \tag{2.13}$$

the same lower and upper bounds for $\partial_1 u_1(t)$ given in (2.9) and (2.10) holds for $x = (x_1, x_2)$ (possibly with larger $\epsilon_n > 0$).

Step 4. Bounds on $\partial_2 u_2$ and $\partial_2 u_1$ in the small ball region.

Along the axis $x_1 = 0$, we have vanishing of $u_1(t)$ from the odd symmetry for all $t$. In particular, taking a $x_2$-derivative, we also have that $\partial_2 u_1(t, 0, x_2) = \partial_1 u_2(t, 0, x_2) = 0$ for all $x_2$. Applying (2.12) under the condition (2.13) ensures that for $x \in D$,

$$|\partial_2 u_1(t, x)| + |\partial_1 u_2(t, x)| \leq C \|\omega_0\|_{L^\infty}. \tag{2.14}$$

The proof is now complete.

Lemma 2.3. Under the same assumptions in Lemma 2.2, we have

$$\|\partial_1 u_{t,n}^\ell(t)\|_{L^\infty} \leq \frac{2}{\pi} \|\omega_{n,0}^\ell\|_{L^\infty} \left( 1 + \epsilon_n \right) \ln \frac{1}{\ell} + C \ln \left( (1 + \kappa) \left( \frac{L}{\ell} \right)^{c \delta} \right) \tag{2.14}$$

Moreover,

$$\|\partial_2 u_{t,n}^\ell(t)\|_{L^\infty} + \|\partial_1 u_{2,n}^\ell(t)\|_{L^\infty} \leq C \delta \|\omega_{n,0}^\ell\|_{L^\infty} \ln \frac{1}{\ell}. \tag{2.15}$$
Proof. We first prove (2.14). From the explicit formula
\[
\partial_t u_1(t, x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}^2} \frac{(y_1 - x_1)(y_2 - x_2)}{|y - x|^4} \omega(t, y) dy,
\]
we divide the integral into three regions: (i) \(|x - y| < \|\omega_0\|_{L^\infty} \|\omega(t)\|_{C^1_1}^{-1}\), (ii) \(\frac{L}{2} \geq |x - y| \geq \|\omega_0\|_{L^\infty} \|\omega(t)\|_{C^1_1}^{-1}\), (iii) \(|x - y| > \frac{L}{2}\). In (iii), the integral can be estimated by \(C\|\omega_0\|_{L^\infty}\), and one estimates the integral in (ii) as in the proof of Lemma 2.2 which gives the expression in (2.14), recalling the bound
\[
\frac{L\|\omega(t)\|_{C^1_1}}{\|\omega_0\|_{L^\infty}} \leq C(\kappa \ell)^{-(1 + \delta_0)}.
\]
Lastly, in the region (i), we write
\[
\int_{|x - y| < \frac{\|\omega_0\|_{L^\infty}}{\|\omega(t)\|_{C^1_1}}} \frac{(y_1 - x_1)(y_2 - x_2)}{|y - x|^4} (\omega(t, y) - \omega(t, x)) dy \lesssim \|\omega(t)\|_{C^1_1} \int_{|x - y| < \frac{\|\omega_0\|_{L^\infty}}{\|\omega(t)\|_{C^1_1}}} \frac{1}{|x - y|} dy \lesssim C\|\omega_0\|_{L^\infty}.
\]
This concludes the proof.

Turning to (2.15), it suffices to show the estimate for \(\partial_x u_1\) only. We do this again by estimating the explicit form of the singular integral kernel. However, using the fact that \(\partial_x u_1\) is uniformly bounded when \(\omega\) is given exactly by the Bahouri-Chemin stationary solution (this can be shown using either Fourier series with Poisson summation formula or radial-angular decomposition; cf. [14, 9, 10]), we just need to estimate the part where \(\omega_n(t)\) is different from the Bahouri-Chemin solution. Moreover, without loss of generality we take \(x = (x_1, x_2)\) with \(0 \leq x_1 \leq x_2 \leq \frac{L}{2}\) and we need to show a bound on the following:
\[
\int_{\{\|\omega_n(t, y)\|_{L^\infty} \neq \|\omega_n\|_{L^\infty}\}} \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|y - x|^4} \omega_n(t, y) dy.
\]
From our assumption that \(x\) lies in the first quadrant, the main term in the integral comes from the strips \(S_1 = \{0 \leq x_1 \leq L, 0 \leq x_2 \leq L\ell_1^{-\eps_0}\}\) and \(S_2 = \{0 \leq x_2 \leq L, 0 \leq x_1 \leq L\ell_1^{-\eps_0}\}\). We shall further assume that \(x\) belongs to \(S_1\) since otherwise then the kernel becomes less singular (and a similar argument gives the same bound). Then, we estimate
\[
\int_{S_1} \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|y - x|^4} \omega_n(t, y) dy = \left[ \int_{|x - y| \leq L\ell_1^{-\eps_0}} + \int_{L\ell_1^{-\eps_0} \leq |x - y| \leq L\ell_1^{-\eps_0}} + \int_{S_1 \setminus \{y : |x - y| \leq L\ell_1^{-\eps_0}\}} \right] \frac{(y_1 - x_1)^2 - (y_2 - x_2)^2}{|y - x|^4} \omega_n(t, y) dy =: I + II + III
\]
and then it is straightforward to bound terms \(I\) and \(II\):
\[
|I| \lesssim \|\omega_0\|_{L^\infty}
\]
(proceeding as in region (i) from the proof of (2.14))
\[
|II| \lesssim \|\omega_0\|_{L^\infty} \int_{L\ell_1^{-\eps_0} \leq |x - y| \leq L\ell_1^{-\eps_0}} \frac{1}{|y - x|^2} dy \lesssim \delta \ln \frac{1}{\ell}.
\]
Finally, to estimate \(III\) it suffices to bound the following “rectangular” integral (note that \(y_2 \geq y_1\) in this region):
\[
\int_{[0, L\ell_1^{-\eps_0}] \times [L\ell_1^{-\eps_0}, L]} \frac{y_2^2 - y_1^2}{|y|^4} dy = \int_{[0, 1] \times [1, \ell_1^{-\eps_0}]} \frac{x_2^2 - x_1^2}{|z|^4} dz = \int_{1}^{\ell_1^{-\eps_0}} \frac{1}{1 + z_2^2} dz < +\infty.
\]
Indeed, this type of rectangular integral bound has appeared already in [22, 32]. The proof is complete. \(\Box\)
2.1.2 Estimates for trajectories and Lagrangian deformation

We keep working in the time interval \([0, \delta \|\omega_0\|_{L^\infty}^{-1}]\) and we shall first extract a smaller ball region \(D'_\delta\) such that \(\eta(t, D'_\delta) \subset D\) during this time interval. We then prove estimates regarding the Lagrangian deformation \(\nabla \eta(t, x)\) for \(x \in D'_\delta\).

First, it is not difficult to show that \(u_2(t, x_1, x_2) < 0\) when \(|x| \leq \hat{\ell}\). (For a proof, one can see the Key Lemma from [22] and [52]. This piece of information will not be essential in our arguments.) Next, we use that (assuming \(\eta_1(0) > 0\))

\[
\dot{\eta}_1(t) = u_1(t, \eta(t)) \leq \|\partial_1 u_1\|_{L^\infty(D)} \eta_1(t)
\]

which is valid as long as \(\eta(t) \in D\). We have used that \(u_1(t, 0, \eta_2) = 0\) holds in the above estimate. Assuming formally that \(\eta(t) \in D\), we have from (2.10) that

\[
\eta_1(t) \leq x_1 \exp \left( ct \|\omega_0\|_{L^\infty}(C + \ln \left( \frac{1 + \kappa A}{\ell} \right) \right) \leq x_1 (1 + \delta) \left( \frac{1 + \kappa A}{\ell} \right)^{c \delta} < 2x_1 \tilde{e}^{-(1+c\delta)c\delta} < x_1 \tilde{e}^{-c\delta}
\]

(by taking \(\delta > 0\) sufficiently small; recall that the value of \(c\) can change several times even within a single line). Hence we may define the region

\[
D'_\delta = \{ |x| < \hat{\ell} \cdot \tilde{e}^{c\delta} \} \tag{2.16}
\]

so that \(\eta(t, D'_\delta) \subset D\) for \(t \in [0, \delta \|\omega_0\|_{L^\infty}^{-1}]\). In the remainder of this section, we always take \(x \in D'_\delta\) and \(t \in [0, \delta \|\omega_0\|_{L^\infty}^{-1}]\).

**Lemma 2.4** (Creation of large Lagrangian deformation). Let us denote \(\eta(t) = \eta_n(t)\) to be the flow associated with \(\omega_n(t)\) from Lemma 2.2. For \(x \in D'_\delta\) and \(t \in [0, \delta \|\omega_0\|_{L^\infty}^{-1}]\), we have that

\[
\exp \left( \frac{2}{\pi} \|\omega^C_{n,0}\|_{L^\infty}(1 - \epsilon_n - C\delta - \ln \frac{1}{\ell}) \right) \leq \partial_1 \eta_1(t, x) < \exp \left( \frac{2}{\pi} \|\omega^C_{n,0}\|_{L^\infty}(1 + \epsilon_n + C\delta - \ln \frac{1}{\ell}) \right) \tag{2.17}
\]

and

\[
|\partial_1 \eta_2(t, x)| + |\partial_2 \eta_1(t, x)| \leq \epsilon_n |\partial_1 \eta_1(t, x) | \tag{2.18}
\]

where \(\epsilon_n \to 0\) as \(n \to +\infty\).

**Proof.** Now we consider the following system of ODEs: for each \(x\), denoting for simplicity \(\eta := \eta(t, x)\) and \(\partial_i \eta_j(t) := \partial_i \eta_j(t, x)\),

\[
\frac{d}{dt} \partial_1 \eta_1(t) = \partial_1 u_1(t, \eta) \partial_1 \eta_1(t) + \partial_2 u_1(t, \eta) \partial_1 \eta_2(t)
\]

\[
\frac{d}{dt} \partial_1 \eta_2(t) = -\partial_1 u_1(t, \eta) \partial_2 \eta_1(t) + (\partial_2 u_1(t, \eta) + \omega(t, \eta)) \partial_1 \eta_1(t) \tag{2.19}
\]

As long as \(\eta \in D\) we have that \(\omega(t, \eta) = 0\). We shall prove that for each fixed \(x\), we have both

\[
\partial_1 \eta_1(t) > 0, \quad \epsilon \partial_1 \eta_1(t) - |\partial_1 \eta_2(t)| > 0. \tag{2.20}
\]

Here \(\epsilon := \inf_{[0, \delta \|\omega_0\|_{L^\infty}^{-1}] \times D} \frac{\partial_2 u_1}{\partial_1 u_1} > 0\). Note that both inequalities are satisfied for some nonempty interval of time containing \(t = 0\), since \(\partial_1 \eta_1(t = 0) = 1\) and \(\partial_1 \eta_2(t = 0) = 0\). Multiplying the first equation of (2.19) by \(\epsilon\) and subtracting the second,

\[
\frac{d}{dt} \left( \epsilon \partial_1 \eta_1 - |\partial_1 \eta_2| \right) \geq \partial_1 u_1 (\epsilon \partial_1 \eta - |\partial_1 \eta_2|) - \epsilon |\partial_2 u_1| \partial_1 \eta_1 - |\partial_2 u_1| |\partial_1 \eta_1| \\
\geq \partial_1 u_1 (\epsilon \partial_1 \eta - |\partial_1 \eta_2|) + 2\partial_1 u_1 |\partial_1 \eta_2| - \epsilon |\partial_2 u_1| |\partial_1 \eta_1| \\
\geq (\partial_1 u_1 - \frac{1}{\epsilon |\partial_2 u_1|}) (\epsilon \partial_1 \eta - |\partial_1 \eta_2|) + \frac{1}{\epsilon} |\partial_2 u_1| (\epsilon \partial_1 \eta - |\partial_1 \eta_2|) + 2\partial_1 u_1 |\partial_1 \eta_2| - \epsilon |\partial_2 u_1| |\partial_1 \eta_1| - |\partial_2 u_1| |\partial_1 \eta_1| 
\]

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and assuming \( \partial_1 \eta_1 > 0 \),
\[
\frac{1}{\epsilon} |\partial_2 u_1| (\epsilon \partial_1 \eta_1 - |\partial_1 \eta_2|) + 2 \partial_1 u_1 |\partial_1 \eta_2| - \epsilon |\partial_2 u_1| \partial_1 \eta_2| - |\partial_2 u_1| |\partial_1 \eta_1| \\
= (2 \partial_1 u_1 - \frac{1}{\epsilon} |\partial_2 u_1| - \epsilon |\partial_2 u_1|) |\partial_1 \eta_2| > 0.
\]

Hence this shows that under the assumption \( \partial_1 \eta_1 > 0 \), we can propagate in time that \( \epsilon \partial_1 \eta_1 - |\partial_1 \eta_2| \). Of course the latter again implies \( \partial_1 \eta_1 > 0 \). Therefore a simple continuity argument establishes (2.20).

Returning to (2.19), we have that
\[
(1 - \epsilon^2) \partial_1 u_1(t, \eta) \partial_1 \eta_1(t) < \frac{d}{dt} \partial_1 \eta_1(t) < (1 + \epsilon^2) \partial_1 u_1(t, \eta) \partial_1 \eta_1(t)
\]
and integrating in time gives, with \( \epsilon = \epsilon_n \to 0 \) as \( n \to +\infty \),
\[
\exp \left( \frac{2}{\pi} \| \omega^{\mathcal{C}} \|_{L^\infty} t (1 - \epsilon_n - C \delta) \ln \frac{1}{T} \right) < \partial_1 \eta_1(t) < \exp \left( \frac{2}{\pi} \| \omega^{\mathcal{C}} \|_{L^\infty} t (1 + \epsilon_n + C \delta) \ln \frac{1}{T} \right)
\]
This finishes the proof. \( \square \)

### 2.1.3 Estimates for the gradient of the vorticity

In this section, we shall establish that for \( p = 2, +\infty \), we have the following sharp estimate on \( \nabla \omega_n^{\mathcal{C}} \):
\[
\| \nabla \omega_n^{\mathcal{C}} (t) \|_{L^p} \leq \| \nabla \omega_n^{\mathcal{C},0} \|_{L^p} \exp \left( \delta (1 + C \delta) \frac{\| \partial_1 u_{n,1}^{\mathcal{C}} \|_{L^\infty}}{\| \omega_n^{\mathcal{C},0} \|_{L^\infty}} \right) \tag{2.21}
\]
for \( t \in [0, \delta/\| \omega_n^{\mathcal{C},0} \|_{L^\infty}] \). The same estimate holds uniformly for \( \nabla \omega_n^{\mathcal{C},\nu} \) with any \( \nu > 0 \) (possibly with some different constant \( C > 0 \)). Here \( \omega_n^{\mathcal{C},\nu} \) is defined by the solution of 2D Navier-Stokes
\[
\partial_t \omega_n^{\mathcal{C},\nu} + u_n^{\mathcal{C},\nu} \cdot \nabla \omega_n^{\mathcal{C},\nu} = \nu \Delta \omega_n^{\mathcal{C},\nu}, \\
\nabla \cdot u_n^{\mathcal{C},\nu} = 0, \\
\omega_n^{\mathcal{C},\nu}(t = 0) = \omega_n^{\mathcal{C},0}.
\]

To see that (2.21) holds, simply take the gradient of the equation for \( \omega_n^{\mathcal{C}} \):
\[
\partial_t \nabla \omega_n^{\mathcal{C}} + u_n^{\mathcal{C}} \cdot \nabla (\nabla \omega_n^{\mathcal{C}}) = (\nabla u_n^{\mathcal{C}})^T \nabla \omega_n^{\mathcal{C}}.
\]

Taking the dot product with \( \nabla \omega_n^{\mathcal{C}} \) and integrating in space gives
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \omega_n^{\mathcal{C}} \|_{L^2}^2 \leq \left| \int \nabla \omega_n^{\mathcal{C}} \cdot (\nabla u_n^{\mathcal{C}})^T \nabla \omega_n^{\mathcal{C}} \right|.
\]

Recall from (2.14)–(2.15) that the \( 2 \times 2 \) matrix \( \nabla u_n^{\mathcal{C}} \) has the following structure:
\[
M = \begin{pmatrix} X & O(X \delta) \\ O(X \delta) & -X \end{pmatrix}
\]
where \( X \gg 1 \). Eigenvalues of \( M \), in absolute value, has size \( X(1 \pm O(\delta)) \). In particular we see that for any \( 2 \times 1 \) vector \( \nu \),
\[
|\nu^T M \nu| \leq X(1 \pm O(\delta))|\nu|^2.
\]
Applying this observation with $X = \|\partial_1 u_{n,1}^\ell\|_{L^\infty_r}$ gives (2.21) for $p = 2$, since we have the bound
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \omega_{n,1}^\ell\|_{L^2}^2 \leq (1 + C\delta) \|\partial_1 u_{n,1}^\ell\|_{L^\infty_r} \|\nabla \omega_{n,1}^\ell\|_{L^2}^2.
\]

The argument for $p = +\infty$ is similar. (Indeed the same estimate holds uniformly for $p$ in $1 \leq p \leq +\infty$.) We shall use this type of argument several times in the following.

Based on (2.21), let us obtain a sharp bound for the second gradient $\|\nabla^2 u_{n,1}^\ell(t)\|_{L^\infty_r}$. Note that each component of $\nabla^2 u_{n,1}^\ell(t)$ is a singular integral transform applied to a derivative of $\omega_{n,1}^\ell$, which vanishes both near and away from the axes. Proceeding similarly as in the proof of Lemma 2.3, we estimate for $x \in D$
\[
|\nabla^2 u_{n,1}^\ell(t,x)| \leq C \|\nabla \omega_{n,1}^\ell(t)\|_{L^\infty} \left(1 + \int_{\ell^{c-\epsilon_1-\rho_0}}^{\ell^{c+\rho_0+\rho_0}} \frac{dr}{r}\right) \leq C(1 + t \ln \frac{1}{\ell}) \|\nabla \omega_{n,1}^\ell(t)\|_{L^\infty} \exp\left(t(1 + C\delta) \|\partial_1 u_{n,1}^\ell\|_{L^\infty_r}\right). \tag{2.22}
\]

### 2.2 Setup for the small-scale vorticity

With a length scale $\ell = \ell_n \ll \ell$ and small $\delta > 0$, we recall the definition of $\mathcal{D}_0'$ from (2.16). Define $u_{n,0}^S \in C^\infty(\mathbb{T}^3)$ in a way that
\[
u_{n,0}(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ 0 \\ Mx_2 \end{pmatrix}
\]
on $\mathcal{D}_0' \times \mathbb{T}$ and $u_{n,0}^S \equiv 0$ on $(\mathbb{T}^3 \setminus D) \times \mathbb{T}$. We may arrange in addition that $u_{n,0}^S$ is only a function of $x_2$ and has vanishing first and second components. Therefore we shall identify $u_{n,0}^S$ with its third component with some abuse of notation. Note that $u_{n,0}^S$ is divergence-free. Note that taking the curl gives
\[
\omega_{n,0}^S := \nabla \times u_{n,0}^S = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \mathcal{D}_0' \times \mathbb{T}
\]
and we see that $M \leq \|\omega_{n,0}^S\|_{L^\infty} \leq 2M$ (by redefining $u_{n,0}^S$ outside $\mathcal{D}_0'$ if necessary).

**Remark 2.5.** We simply have $\|u_{n,0}^S\|_{L^2}^2 \approx M \int_0^{\ell} \int_0^{\ell} x_3^2 dx_2 dx_3 \approx M \ell^4 \approx \ell^2 \|\omega_{n,0}^S\|_{L^2}^2$.

#### 2.2.1 Estimates for the small-scale vorticity

We consider the equation
\[
\partial_t u_{n,0}^S + u_{n,0}^S \cdot \nabla u_{n,0}^S = 0.
\]
Since $u_{n,0}^S$ is divergence-free, we immediately have
\[
\|u_{n,0}^S\|_{L^p} = \|u_{n,0}^S\|_{L^p}
\]
for all $1 \leq p \leq +\infty$. Next, taking the curl gives
\[
\partial_t \omega_{n,0}^S + (u_{n,0}^S \cdot \nabla) \omega_{n,0}^S = \nabla u_{n,0}^S \omega_{n,0}^S, \tag{2.23}
\]

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and we obtain that
\[ \frac{1}{2} \frac{d}{dt} \| \omega_n^S \|_{L^2}^2 \leq \left| \int \omega_n^S \cdot \nabla u_n^C \cdot \omega_n^S \right| \leq (1 + C\delta) \| \partial_1 u_{n,1}^C \|_{L^\infty} \| \omega_n^S \|_{L^2}, \]
where we have used that
\[ \| \partial_2 u_{n,1}^C \|_{L^\infty} + \| \partial_1 u_{n,2}^C \|_{L^\infty} \lesssim \delta \| \partial_1 u_{n,1}^C \|_{L^\infty}. \]
We shall use this observation frequently in the following. Similarly, it is not difficult to see (repeating a bootstrap argument as in the proof of Lemma 2.4) that
\[ \frac{d}{dt} \| \omega_n^S \|_{L^\infty} \leq (1 + C\delta) \| \partial_1 u_{n,1}^C \|_{L^\infty} \| \omega_n^S \|_{L^\infty}. \]
Integrating in time, we see that for \( p = 2, +\infty \) (actually this holds uniformly for any \( 1 \leq p \leq +\infty \))
\[ \| \omega_n^S(t) \|_{L^p} \leq \| \omega_{0,n}^S \|_{L^p} e^{(1 + C\delta) \int_0^t \| \partial_1 u_{n,1}^C(\tau) \|_{L^\infty} d\tau} \leq \| \omega_{0,n}^S \|_{L^p} \exp \left( \delta (1 + C\delta) \frac{\| \partial_1 u_{n,1}^C \|_{L^\infty}}{\| \omega_{0,n}^S \|_{L^\infty}} \right). \] (2.24)
It is not difficult to see that
\[ \omega_{n,1}^S \leq C\delta |\omega_{n,1}^S| \] (2.25)
pointwise in space and time. This can be seen directly from (2.23) but it is easy to obtain from Lemma 2.4 and the following Cauchy formula
\[ \omega_n^S(t, \eta(t, x)) = D\eta(t, x)\omega_{n,0}^S(x) \]
since \( \omega_{n,1}^S(t = 0) = 0 \).
Similarly, from
\[ \partial_t \omega_n^{S,\nu} + (u_n^{C,\nu} \cdot \nabla) \omega_n^{S,\nu} = \nabla u_n^{C,\nu} \omega_n^{S,\nu} + \nu \Delta \omega_n^{S,\nu}, \]
one can obtain that the estimates (2.24) are valid for \( \omega_{n}^{S,\nu} \) uniformly for any \( \nu > 0 \), perhaps with some different absolute constant \( C > 0 \).

We shall need just one more estimate: take the first component of the equation for \( \omega_n^S \) and differentiating gives
\[ D_t \partial_1 \omega_n^S = -\partial_1 u_{n,2}^C \partial_2 \omega_n^S + \partial_2 u_{n,1}^C \partial_1 \omega_n^S + \partial_1 \partial_2 u_{n,2}^L \omega_n^S + \partial_1 \partial_2 u_{n,1}^L \omega_n^S \]
and
\[ D_t \partial_2 \omega_n^S = 2 \partial_1 u_{n,1}^L \partial_2 \omega_n^S - \partial_2 u_{n,1}^C \partial_1 \omega_n^S + \partial_2 u_{n,2}^L \partial_2 \omega_n^S + \partial_2 \partial_2 u_{n,1}^L \omega_n^S + \partial_2 \partial_1 u_{n,1}^L \omega_n^S \]
where we have written
\[ D_t = \partial_t + u_n^C \cdot \nabla \]
for simplicity. It is not difficult to see that we have
\[ \| \partial_1 \omega_{n,1}^S(t) \|_{L^\infty} + \| \partial_2 \omega_{n,2}^S(t) \|_{L^\infty} \leq C \| \omega_{n,1}^S(t) \|_{L^\infty}. \]
We then estimate using (2.25) that
\[ \frac{d}{dt} \| \omega_{n,1}^S(t) \|_{L^\infty} \leq 2(1 + C\delta) \| \partial_1 u_{n,1}^C \|_{L^\infty} \| \partial_2 \omega_{n,1}^S(t) \|_{L^\infty} + (1 + C\delta) \| \nabla^2 u_{n}^C(t) \|_{L^\infty} \| \omega_{n,1}^S(t) \|_{L^\infty}. \]
Using Gronwall’s inequality together with (2.22), we obtain from
\[
\|\partial_2 \omega_n^S(t)\|_{L^\infty} \leq \|\nabla \omega_n^S\|_{L^\infty} \exp \left(2t(1 + C\delta)\|\partial_1 u_n^L\|_{L^\infty}\right)
+ (1 + C\delta) \int_0^t \exp \left(2(t - s)(1 + C\delta)\|\partial_1 u_n^L\|_{L^\infty}\right) \|\nabla^2 u_n^L(s)\|_{L^\infty} \omega_n^S(s)\|_{L^\infty} ds
\]
that
\[
\|\partial_2 \omega_n^S(t)\|_{L^\infty} \leq \left(\|\nabla \omega_n^S\|_{L^\infty} + t(1 + \delta \ln \frac{1}{t})\|\nabla \omega_n^S\|_{L^\infty}\right) \|\omega_n^S\|_{L^\infty} \exp \left(2t(1 + C\delta)\|\partial_1 u_n^L\|_{L^\infty}\right).
\]
(2.26)

One can similarly estimate \(\partial_1 \omega_n^S\) and the gradient of the second component in a parallel manner; it turns out that \(\|\nabla \omega_n^S(t)\|_{L^\infty}\) satisfies the estimate (2.27) as well. We omit the details. Moreover, in \(L^2\) we can obtain a corresponding estimate:
\[
\|\nabla \omega_n^S(t)\|_{L^2} \leq \left(\|\nabla \omega_n^S\|_{L^2} + t(1 + \delta \ln \frac{1}{t})\|\nabla \omega_n^S\|_{L^2}\right) \|\omega_n^S\|_{L^2} \exp \left(2t(1 + C\delta)\|\partial_1 u_n^L\|_{L^\infty}\right).
\]
(2.28)

\section{Inviscid limits}

As before, we shall always take \(t \in [0, \delta/\|\omega_0\|_{L^\infty}]\) throughout this section. We obtain sharp upper bounds for the \(L^2\) and \(H^1\) differences between the Euler and Navier-Stokes velocities, both for the large and small scales.

\subsection{\(L^2\) for the large scale}

We define
\[
I_n^L(t) := \|u_n^L - u_n^S\|_{L^2}^2.
\]

We compare the 2D Euler and Navier-Stokes equations of the velocity:
\[
\partial_t u_n^L + u_n^L \cdot \nabla u_n^L + \nabla p_n^L = \nu \Delta u_n^L,
\]
\[
\partial_t u_n^S + u_n^S \cdot \nabla u_n^S + \nabla p_n^S = 0.
\]

Then, we see that
\[
\frac{1}{2} \frac{d}{dt} \|u_n^L - u_n^S\|_{L^2}^2 + \int (u_n^L - u_n^S) \cdot \nabla u_n^L \cdot (u_n^L - u_n^S) = \nu \int \Delta u_n^L \cdot (u_n^L - u_n^S).
\]

We handle the right hand side as follows:
\[
\nu \int \Delta u_n^L \cdot (u_n^L - u_n^S) = -\nu \int |\nabla u_n^L|^2 + \nu \int \nabla u_n^L : \nabla u_n^L \leq C\nu \|\nabla u_n^L\|_{L^2}^2.
\]

Moreover, inspecting the second term on the left hand side, we may bound
\[
\left|\int (u_n^L - u_n^S) \cdot \nabla u_n^L \cdot (u_n^L - u_n^S)\right| \leq \|\partial_1 u_n^L\|_{L^\infty} I_n^L + C(\|\partial_1 u_n^L\|_{L^\infty} + \|\partial_2 u_n^L\|_{L^\infty})I_n^L.
\]

Hence, appealing to the global bound (2.14)=(2.15),
\[
\frac{d}{dt} I_n^L \leq 2(1 + C\delta)\|\partial_1 u_n^L\|_{L^\infty} I_n^L + C\nu \|\nabla u_n^L\|_{L^2}^2.
\]

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Using that \( \| \nabla u^n \|_{L^2} \leq C \| \omega^n \|_{L^2} \leq C \| \omega^n_{0,0} \|_{L^2} \) and \( I^L(0) = 0 \), we arrive at
\[
I^L(t) \leq C \nu \| \omega^n \|_{L^2}^2 \exp \left( 2 \delta (1 + C \delta) \frac{\| \partial_1 u^L_{n,1} \|_{L^\infty}}{\| \omega^L_0 \|_{L^\infty}} \right). \tag{2.29}
\]
The exponential term on the right hand side will appear frequently, so we shall introduce notation
\[
E := \exp \left( (1 + C \delta) \frac{\| \partial_1 u^L_{n,1} \|_{L^\infty}}{\| \omega^L_0 \|_{L^\infty}} \right). \tag{2.30}
\]
Note that
\[
E \approx \bar{E}^{-\frac{2}{8} (1 + \delta)} \gg 1.
\]

### 2.3.2 \( L^2 \) for the small scale

Now we set
\[
I^S(t) := \| u^S_{\nu} - u^n_S \|_{L^2}^2
\]
and again note that \( I^S(0) = 0 \). Compare the equations satisfied by \( u^S_n \) and \( u^S_{\nu, n} \):
\[
\partial_t u^S_n + u^L_n \cdot \nabla u^S_n = 0,
\]
\[
\partial_t u^S_{\nu, n} + u^L_{\nu, n} \cdot \nabla u^S_{\nu, n} = \nu \Delta u^S_{\nu, n}.
\]
Proceeding similarly as in the above, we have
\[
\frac{1}{2} \frac{d}{dt} I^S \leq \| \omega^n_{S} \|_{L^\infty} (I^L)^{\frac{1}{2}} \left( I^S \right)^{\frac{1}{2}} + C \nu \| \nabla u^n_S \|_{L^2}^2.
\]
It was crucially used that
\[
\left| \int \nabla u^n_S \cdot (u^L_n - u^L_{\nu, n})(u^S_n - u^S_{\nu, n}) \right| \leq \| \omega^n_{S} \|_{L^\infty} (I^S I^L)^{\frac{1}{2}};
\]
recall that \( \omega^n_S \) is simply
\[
\omega^n = \left( \begin{array}{c}
\partial_1 u^n_S \\
-\partial_2 u^n_S \\
0
\end{array} \right).
\]
Using (2.29) we write for simplicity
\[
\frac{d}{dt} I^S \leq A(t I^S)^{\frac{1}{2}} + B
\]
where \( A \) and \( B \) are positive constants defined by
\[
A = \nu^2 \| \omega^n_{S,0} \|_{L^\infty} \| \omega^L_{0,0} \|_{L^2} \mathcal{E}^{2\delta}, \quad B = C \nu \| \omega^n_{S,0} \|_{L^2} \mathcal{E}^{2\delta}.
\]
We have used (2.24). To estimate \( I^S \), we instead estimate the solution of the ODE
\[
\frac{d}{dt} X = A(t X)^{\frac{1}{2}} + B, \quad X(0) = 0.
\]
We estimate $X(t)$ differently in $0 \leq t < t^*$ and $t > t^*$; here $t^* > 0$ is the solution to

$$A(t^* X(t^*))^{\frac{1}{2}} = B$$

which is uniquely well-defined since initially $B > A(tX)^{\frac{1}{2}}$ and $A(tX)^{\frac{1}{2}}$ is strictly increasing in time. Then, we have trivially

$$Bt^* \leq X(t^*) \leq 2Bt^*$$

so using the definition of $t^*$ above, we deduce

$$t^* = \frac{B^{\frac{1}{2}}}{A},$$

with some absolute constant $\frac{1}{2} \leq c \leq 2$. In turn, this implies that

$$X(t^*) \leq 4B^{\frac{3}{2}}A$$

Next, for $t > t^*$, we have

$$2X^{\frac{1}{2}} \frac{d}{dt} X^{\frac{1}{2}} = \frac{d}{dt} X \leq 2A(tX)^{\frac{1}{2}}$$

and integrating in time gives

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(t^*) + (t - t^*)^{\frac{3}{2}}A.$$ 

Using the above upper bound for $X(t^*)$ and squaring both sides, we conclude that

$$X(t) \leq C\left(\frac{B^{\frac{3}{2}}}{A} + t^3A^2\right).$$

Recalling the expressions for $A$ and $B$, we deduce that

$$I^S(t) \leq X(t) \leq C\left(\frac{\nu \|\omega^{S,0}_{n,0}\|_{L^2}^3}{\|\omega^{L,0}_{n,0}\|_{L^\infty} \|\omega^{L}_{n,0}\|_{L^2}} \mathcal{E}^{-3\delta} + \nu t^3 \|\omega^{S,0}_{n,0}\|_{L^\infty} \|\omega^{L,0}_{n,0}\|_{L^2}^2 \left(\|\omega^{S,0}_{n,0}\|_{L^\infty} \|\omega^{L,0}_{n,0}\|_{L^2} \mathcal{E}^{-\delta} + \delta \|\omega^{L,0}_{n,0}\|_{L^\infty} \|\omega^{L,0}_{n,0}\|_{L^2} \right) \right)^{\mathcal{E}^4 \delta}. \quad (2.31)$$

**2.3.3 $\dot{H}^1$ for the large scale**

We define

$$II^L = \|\omega^{L,0}_{n,0} - \omega^{L}_{n,0}\|_{L^2}^2.$$ 

From the equations

$$\partial_t \omega^{L,0}_{n,0} + u^{L,0}_{n,0} \cdot \nabla \omega^{L,0}_{n,0} = \nu \Delta \omega^{L,0}_{n,0},$$

$$\partial_t \omega^{L}_{n,0} + u^{L}_{n,0} \cdot \nabla \omega^{L}_{n,0} = 0,$$
we obtain
\[
\frac{1}{2} \frac{d}{dt} I^L \leq \| \nabla \omega_n^\nu \|_{L^\infty} (I^L I^L)^\frac{1}{2} + C \nu \| \nabla \omega_n^\nu \|_{L^2}^2.
\]
We use (2.29) and (2.21) to bound \( I^L \) and \( \| \nabla \omega_n^\nu \|_{L^p} \), respectively:
\[
\frac{d}{dt} I^L \leq C \left( (\nu t)^\frac{1}{2} \| \nabla \omega_n^\nu \|_{L^\infty} \| \omega_n^\nu \|_{L^2} (I^L)^\frac{1}{2} + \nu \| \nabla \omega_n^\nu \|_{L^2}^2 \right) \Delta t^\frac{1}{2}.
\]
Proceeding as in 2.3.2, we deduce that
\[
I^L \leq C \left( \frac{B \frac{3}{2}}{A} + t^3 A^2 \right)
\]
where this time,
\[
A = \nu \frac{1}{2} \| \nabla \omega_n^\nu \|_{L^\infty} \| \omega_n^\nu \|_{L^2} \Delta t^\frac{1}{2},
\]
\[
B = \nu \| \nabla \omega_n^\nu \|_{L^2} \Delta t^\frac{1}{2}.
\]
We arrive at
\[
I^L \leq C \frac{\nu}{\| \nabla \omega_n^\nu \|_{L^\infty} \| \omega_n^\nu \|_{L^2}} \left( \| \nabla \omega_n^\nu \|_{L^2} \Delta t^\delta + \delta \frac{\| \nabla \omega_n^\nu \|_{L^\infty} \| \omega_n^\nu \|_{L^2}}{\| \omega_n^\nu \|_{L^\infty}} \right)^3 \Delta t^\delta. \tag{2.32}
\]

### 2.3.4 \( \dot{H}^1 \) for the small scale

We now define
\[
I^S := \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2}^2.
\]
Recall that
\[
\partial_t \omega_n^{S,\nu} + (u_n^{\nu} \cdot \nabla) \omega_n^{S,\nu} = \nabla u_n^{\nu} \omega_n^{S,\nu} + \nu \Delta \omega_n^{S,\nu},
\]
\[
\partial_t \omega_n^{S} + (u_n^L \cdot \nabla) \omega_n^{S} = \nabla u_n^L \omega_n^{S}.
\]
We have
\[
\frac{1}{2} \frac{d}{dt} \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2}^2 + \int (u_n^L - u_n^{\nu}) \cdot \nabla \omega_n^S : (\omega_n^S - \omega_n^{S,\nu}) = \int \nabla u_n^{\nu} (\omega_n^S - \omega_n^{S,\nu}) : (\omega_n^S - \omega_n^{S,\nu}) + \int \nabla u_n^L \omega_n^{S,\nu} : (\omega_n^S - \omega_n^{S,\nu}) + \nu \int \Delta \omega_n^{S,\nu} \Delta \omega_n^{S,\nu}.
\]
After some routine massaging,
\[
\frac{1}{2} \frac{d}{dt} \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2}^2 \leq C \| \nabla \omega_n^S \|_{L^\infty} \| u_n^L - u_n^{\nu} \|_{L^2} \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2} + (1 + C \delta) \| \partial_t u_n^L \|_{L^\infty} \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2}^2 + C \| \omega_n^{S,\nu} \|_{L^\infty} \| u_n^L - u_n^{\nu} \|_{L^2} \| \omega_n^S - \omega_n^{S,\nu} \|_{L^2} + C \nu \| \nabla \omega_n^S \|_{L^2}^2
\]
and we rewrite the above as follows:
\[
\frac{d}{dt} I^S \leq 2(1 + C \delta) \| \partial_t u_n^L \|_{L^\infty} \| \omega_n^S \|_{L^2} + C \left( \| \nabla \omega_n^S \|_{L^\infty} (I^S)^\frac{1}{2} + \| \omega_n^{S,\nu} \|_{L^\infty} (I^S)^\frac{1}{2} \right) (I^S)^\frac{1}{2} + C \nu \| \nabla \omega_n^S \|_{L^2}^2.
\]
To simplify the estimate, we introduce
\[
\tilde{I}^S(t) := \exp(-2t(1 + C \delta) \| \partial_t u_n^L \|_{L^\infty} \| \omega_n^S \|_{L^2}) relaxed estimate, I^S(t)
\]

so that
\[
\frac{d}{dt} \tilde{I}^S(t) \lesssim \left( \| \nabla \omega_n^S \|_{L^\infty} (I^C)^{\frac{1}{2}} + \| \omega_n^{S,\nu} \|_{L^\infty} (II^C)^{\frac{1}{2}} \right) \left( \tilde{I}^{S}\right)^{\frac{1}{2}} t^{\| \omega_n^S \|_{L^\infty}} + \| \nabla \omega_n^S \|_{L^2}^2 e^{-2t \| \omega_n^S \|_{L^\infty}}.
\]

Now, from previous bounds, we estimate
\[
\left( \| \nabla \omega_n^S \|_{L^\infty} (I^C)^{\frac{1}{2}} + \| \omega_n^{S,\nu} \|_{L^\infty} (II^C)^{\frac{1}{2}} \right) \lesssim \nu^{\frac{1}{2}} e^{2\delta \left( \| \nabla \omega_n^S \|_{L^\infty} + \delta (1 + \delta \ln \frac{1}{\ell}) \frac{\| \nabla \omega_n^S \|_{L^\infty}}{\| \omega_n^S \|_{L^\infty}} \right) \frac{\delta^\frac{1}{2}}{\| \omega_n^S \|_{L^2}}^{\frac{1}{2}} \| \omega_n^S \|_{L^2}^2 + \frac{\| \omega_n^S \|_{L^\infty}}{\| \omega_n^S \|_{L^2}^2} \left( \| \nabla \omega_n^S \|_{L^2} e^{-\delta} + \delta \| \nabla \omega_n^S \|_{L^\infty} \| \omega_n^S \|_{L^2}^2 \right)^{\frac{1}{2}} : = \nu \frac{1}{2} A
\]
and
\[
\nu \| \nabla \omega_n^S \|_{L^2} e^{-2t \| \omega_n^S \|_{L^\infty}} \lesssim \nu \left( \| \nabla \omega_n^S \|_{L^2} + \delta (1 + \delta \ln \frac{1}{\ell}) \frac{\| \nabla \omega_n^S \|_{L^\infty}}{\| \omega_n^S \|_{L^\infty}} \| \omega_n^S \|_{L^2}^2 \right)^{2} e^{2\delta} : = \nu B.
\]

We compare \( \tilde{I}^S(t) \) with \( \nu X(t) \) solving
\[
\frac{d}{dt} X(t) = A(X(t))^{\frac{1}{2}} + B, \quad X(0) = 0.
\]

We easily obtain that
\[
\tilde{I}^S(t) \leq \nu X(t) \leq C \nu (\frac{B^2}{A^2} + A^2 \ell^2)
\]
and hence
\[
II^S(t) \leq C \nu (\frac{B}{A} + \frac{A^4}{\| \omega_n^S \|_{L^\infty}^2})^2 e^{2\delta}.
\]

We keep the expressions \( A, B \) as they are for now and simplify later with our explicit choice of parameters.

### 2.3.5 Final estimate and Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. We proceed in several steps.

1. Inviscid limit holds for the \( L^2 \) of the vorticity.

   We would like to have
   \[
   \frac{1}{t_n} \int_0^{t_n} \| \omega_n^S(t) \|_{L^2}^2 \, dt \gg \frac{1}{t_n} \int_0^{t_n} II^S(t) \, dt.
   \] (2.33)

   We bound the right hand side simply by
   \[
   \frac{1}{t_n} \int_0^{t_n} II^S(t) \, dt \leq \sup_{t \in [0,t_n]} II^S(t) \lesssim \nu \left( \frac{B}{A} + \frac{A^4}{\| \omega_n^S \|_{L^\infty}^2} \right)^2 e^{2\delta}.
   \]

   In this case, a lower bound on the left hand side is given by
   \[
   \frac{1}{t_n} \int_0^{t_n} \| \omega_n^{S,\nu}(t) \|_{L^2}^2 \, dt \geq \frac{1}{t_n} \int_0^{t_n} || \omega_n^{S,\nu}(t) ||_{L^2}^2 \, dt \geq e^{2\delta - \delta^2} \frac{|| \omega_n^S ||_{L^2}^2}{\delta}.
   \]

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The above bound follows from Lemma 2.4 and the Cauchy formula (applied to the first component, recalling that $\omega_{n,0,2} = 0$)

$$\omega_{n,1}^{S}(t, \eta(t, x)) = \partial_{1}\eta_{1}(t, x)\omega_{n,0,1}^{S}(x).$$

This determines the maximal value of $\nu = \nu_{n}$ which allows for the crucial estimate (2.33): namely,

$$\nu_{n} := \frac{c}{\delta} \left( \frac{A}{B} + \frac{\lambda}{\|\omega_{n,0}^{S}\|_{L^{\infty}}} \right)^{2} \|\omega_{n,0}^{S}\|_{L^{2}}^{2}. \quad (2.34)$$

2. The expression for $\nu_{n}$.

Let us now extract the main terms in (2.34), with our explicit choice of $\omega_{n,0}^{C}$ and $\omega_{n,0}^{S}$. We start by recalling that

$$A := E^{2\delta} \left( \left( \|\nabla\omega_{n,0}^{C}\|_{L^{\infty}} + \delta(1 + \delta \ln \frac{1}{\ell_{n}}) \|\nabla\omega_{n,0}^{C}\|_{L^{\infty}} \right) \|\omega_{n,0}^{C}\|_{L^{2}}^{2} + \|\omega_{n,0}^{S}\|_{L^{\infty}} \left( \|\nabla\omega_{n,0}^{C}\|_{L^{2}} \|\omega_{n,0}^{C}\|_{L^{2}} \right) \right).$$

We compute

$$\|\omega_{n,0}^{C}\|_{L^{\infty}} = 1, \quad \|\omega_{n,0}^{C}\|_{L^{2}} \approx \ell \approx \ell^{2}, \quad \|\nabla\omega_{n,0}^{C}\|_{L^{\infty}} \approx \ell^{-\frac{1}{2}} \ell^{\frac{1}{2}}, \quad \|\nabla\omega_{n,0}^{C}\|_{L^{2}} \approx \ell^{-1}, \quad \|\nabla\omega_{n,0}^{C}\|_{L^{2}} \approx L^{2}. \quad (2.35)$$

We have, with a free parameter $q \in \mathbb{R}$ to be determined,

$$\|\omega_{n,0}^{S}\|_{L^{2}} = \ell^{-\frac{1}{2} + \delta}, \quad \|\nabla\omega_{n,0}^{S}\|_{L^{2}} \approx \ell^{-1} \ell^{-\frac{1}{2}}, \quad \|\nabla\omega_{n,0}^{S}\|_{L^{\infty}} \approx \ell^{-1 - \frac{1}{2}}, \quad \|\nabla\omega_{n,0}^{S}\|_{L^{2}} \approx \ell^{-1 - \frac{1}{2}}, \quad \|\nabla\omega_{n,0}^{S}\|_{L^{\infty}} \approx \ell^{-2 - \frac{1}{2}}. \quad (2.36)$$

Recall that $\ell_{n} = \ell_{n}^{L+e}$ and $\ell_{n} = \ell L^{-1} := 2^{-n}$. Now we observe that

$$\ell^{-1} = \frac{\|\nabla\omega_{n,0}^{S}\|_{L^{\infty}}}{\|\omega_{n,0}^{S}\|_{L^{2}}} \gg \delta(1 + \delta \ln \frac{1}{\ell_{n}}) \|\nabla\omega_{n,0}^{S}\|_{L^{\infty}} \approx \delta(1 + \delta \ln \frac{1}{\ell_{n}}) \ell^{-1}$$

since $L^{-1} \geq 1 \gg \ell$. Note that the above estimate is independent of $L$. Next, it is not difficult to see that (recalling $E \gg 1$)

$$\|\nabla\omega_{n,0}^{C}\|_{L^{2}} \approx \delta \|\nabla\omega_{n,0}^{C}\|_{L^{\infty}}.$$

These observations simplify $A$ significantly:

$$A \approx E^{2\delta} \left( \frac{\ell^{2}}{\|\omega_{n,0}^{S}\|_{L^{2}}} \|\nabla\omega_{n,0}^{C}\|_{L^{2}}^{2} + \frac{\|\nabla\omega_{n,0}^{S}\|_{L^{\infty}}}{\|\omega_{n,0}^{C}\|_{L^{2}}} \left( \frac{\|\nabla\omega_{n,0}^{C}\|_{L^{\infty}} \|\omega_{n,0}^{C}\|_{L^{2}}}{\|\omega_{n,0}^{C}\|_{L^{\infty}}} \right)^{2} \right) \approx E^{2\delta} \left( \frac{\ell^{2}}{\|\omega_{n,0}^{S}\|_{L^{2}}} \|\nabla\omega_{n,0}^{C}\|_{L^{2}}^{2} + \frac{\|\nabla\omega_{n,0}^{S}\|_{L^{\infty}}}{\|\omega_{n,0}^{C}\|_{L^{2}}} \left( \frac{\|\nabla\omega_{n,0}^{C}\|_{L^{\infty}} \|\omega_{n,0}^{C}\|_{L^{2}}}{\|\omega_{n,0}^{C}\|_{L^{\infty}}} \right)^{2} \right) \approx \delta^{\frac{1}{2}} E^{2\delta} \ell^{-2 - \frac{1}{2}} L.$$

Next, we similarly obtain that

$$B \approx \ell^{-1 - \frac{1}{2}} E^{2\delta}.$$
Then we can see that
\[ \frac{A\delta}{\|\omega_{n,0}^S\|_{L^\infty}} \gg \frac{B}{A}. \]

Recalling that \( \mathcal{E} \approx \bar{\ell}^{-\frac{3}{2}(1+C \delta)} \), we have the following formula for \( \nu_n \):
\[ \nu_n = \frac{c}{\delta^4} \bar{\ell}^{4\alpha_0(1-C \delta)} \bar{\ell}^{4(1+C \delta)} L^{-2} \]
which is independent of \( q \). Rewriting in terms of \( \bar{\ell} \) and \( L \) using \( \ell = \bar{\ell} L \), it is easy to see that \( \nu_n \to 0 \) if \( L \leq 1 \).

3. Modified zeroth law.

Given our definition of \( \nu_n \), we would like to have, with \( t_n = \frac{\delta}{\|\omega_{n,0}^S\|_{L^\infty}} \),
\[ \nu_n \frac{1}{t_n} \int_0^{t_n} \|\omega_n^S(t)\|_{L^2}^2 dt \gtrsim \|u_{n,0}\|_{L^2}^2. \]

We compute:
\[ \nu_n \frac{1}{t_n} \int_0^{t_n} \|\omega_n^S(t)\|_{L^2}^2 dt \gtrsim \bar{\ell}^{4\alpha_0(1+C \delta)} L^{5\alpha_0(2+C \delta)} \bar{\ell}^{-2c_0\delta(1-C \delta)} \|\omega_{n,0}^S\|_{L^2}^2. \]

We require that the above satisfies \( \gtrsim \|u_{n,0}^S\|_{L^2}^2 \) and \( \gtrsim \|u_{n,0}^L\|_{L^2}^2 \). For the former, we need
\[ \bar{\ell}^{4\alpha_0(1+C \delta)} \bar{\ell}^{-2c_0\delta(1-C \delta)} L^{\alpha_0(2+C \delta)} \gtrsim \bar{\ell}^2 = \bar{\ell}^{2(1+C \delta)} L^{2(1+C \delta)}. \]

This requirement sets restriction on \( L \):
\[ \bar{\ell}^{2(2\alpha_0 - 1)(1+C \delta) - c_0\delta} \gtrsim L. \]

From the above, one sees that there are two cases: \( a_0 \leq \frac{1}{2} \) and \( 1 > a_0 > \frac{1}{2} \). In the former, the left hand side satisfies \( \gg 1 \), so we simply fix \( L = 1 \) for all \( n \). In the latter, we simply define
\[ L = \bar{\ell}, \quad \gamma := \frac{(2\alpha_0 - 1)(1 + C_1 \delta) - c_0\delta}{1 - \alpha_0(1 + C_2 \delta)} \]
where \( C_1, C_2, \) and \( c_0 \) are some positive absolute constants. Note that \( L \ll 1 \). In the following, we shall proceed with assuming \( 1 > a_0 > \frac{1}{2} \). Now, for \( \gtrsim \|u_{n,0}^S\|_{L^2}^2, \) we need
\[ \bar{\ell}^{4\alpha_0(1+C \delta)} \bar{\ell}^{-2c_0\delta(1-C \delta)} L^{\alpha_0(2+C \delta)} \bar{\ell}^{-\frac{\gamma}{2}} \gtrsim L^2 \]
and this determines the value of \( \gamma \). We can just require that \( \|u_{n,0}^S\|_{L^2} \approx L^2; \)
\[ \bar{\ell}^{1 + \gamma)(1+C \delta)(1-\frac{\gamma}{2})} = \bar{\ell}^{1-\frac{\gamma}{2}} \approx L = \bar{\ell} \]
which gives
\[ \frac{\gamma}{1 + \gamma)(1+C \delta)} = 1 - \frac{2}{q}. \]

Note that \( q = 2 \left(1 - \left(\frac{\gamma}{1 + \gamma)(1+C \delta)}\right)^{-1}\right) \) clearly satisfies the above, and in this case we have that \( \|u_{n,0}^S\|_{L^2} \approx L^2 \).

Finally, we note that when \( a_0 \leq \frac{1}{2}, \) we already have the lower bound on the energy from the large-scale:
\[ \|u_{n,0}^L\|_{L^2}^2 \gtrsim 1, \]
so that we can take \( q \) in a way that \( \|\omega_{n,0}^S\|_{L^2} \) is uniformly bounded in \( n \). The proof is now complete. \( \square \)
3 Conclusion

We prepared small-scale vortex blob and large-scale anti-parallel vortex tubes for the initial data, and showed that the corresponding 3D Navier-Stokes flow creates instantaneous vortex-stretching. In turn, using this stretching, we showed that the flows satisfy a modified version of the zeroth law in a uniform time interval.

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