DEFORMATION OF DIRAC OPERATOR ALONG ORBITS AND QUANTIZATION OF NON-COMPACT HAMILTONIAN TORUS MANIFOLDS

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Abstract. We give a formulation of a deformation of Dirac operator along orbits of a group action on a possibly non-compact manifold to get an equivariant index and a K-homology cycle representing the index. We apply this framework to proper Hamiltonian torus manifolds to define geometric quantization from the viewpoint of index theory. We give two applications. The first one is a proof of a $[Q,R]=0$ type theorem, which can be regarded as a proof of the Vergne conjecture for torus actions. The other is a Danilov-type formula for toric case in the non-compact setting, which shows that this geometric quantization is independent of the choice of polarization. The proofs are based on the localization of index to lattice points.

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1. Introduction

In the present paper we study the following two topics. Firstly, we give a formula- tion of a deformation of Dirac operator along orbits on a possibly non-compact man- ifold equipped with a group action to get an equivariant index and a K-homology cycle representing the index. Secondly, we apply this framework to Hamiltonian torus manifolds to define geometric quantization from the viewpoint of index theory. In particular we give proofs of a $[Q,R]=0$ type theorem and a Danilov-type formula for the toric case in the possibly non-compact setting. The proofs are based on the same perspective, taken in [9] and [11] by the author and joint works with Furuta and Yoshida, namely, the localization of index to lattice points. These results give a simplification and a generalization of [9] and [11]. They also make more clear the relation with a similar construction in [6].

Geometric quantization of symplectic manifolds originates from ideas in physics. However, nowadays it is related to several topics in various branches of mathematics. One of them is the index theory of Dirac operator. In fact, in some cases, the quantization can be regarded as an index of the spin$^c$ Dirac operator associated with a compatible almost complex structure. This approach is called spin$^c$ quantization. Studying quantization from the viewpoint of index theory, K-theory, K-homology and KK-theory is an active area of research.

Geometric quantization in the compact setting has been extensively studied. The non-compact case has also been studied to some extent. For example, such a generalization is important for quantization of Hamiltonian loop group space in [20]. In addition, the non-compact setting plays an essential role to obtain localization phenomena in geometric quantization as below. On the other hand, unlike the compact manifold case, the index of Dirac operator on a non-compact or open manifold is not well-defined in a straightforward way. To get the index in a possibly generalized sense, it is necessary to take an appropriate boundary condition or to consider additional structure such as a fiber bundle structure or a nice group action.

In [6], Braverman gave a formulation to define an equivariant index in a non-compact setting. This framework originates in a proof of $[Q,R]=0$ in [26] and was applied to a solution of the Verge conjecture in [21]. He used a deformation of the Dirac operator by the Clifford action of the vector field generated by the moment map$^1$. On the other hand in a series of papers [7–9] with Furuta and Yoshida the

\footnote{In [6] the formulation is established in a more general category which is not necessarily symplectic. In fact, an equivariant map which is called a taming map is used.}
author developed an index theory on open manifolds using a family of partly defined fiber bundle structures and a deformation of Dirac operator. The deformation in [7–9] is given by first-order differential operators, a family of Dirac operators along fibers, which need not use a group action essentially. We call it FFY’s deformation for short. Both Braverman’s and FFY’s deformation are motivated by Witten’s pioneering work [27], and in the equivariant case, these deformations have the same nature, that is, deformations along the orbits. Both of the resulting indices satisfy the excision formula, which leads us to the localization of index. Here we summarize the differences between Braverman’s and FFY’s deformation.

• Braverman’s deformation:
  (1) can be applied to compact group actions (not necessarily Abelian), and
  (2) realizes a localization of index to the zero level set of the moment map and fixed points (or critical points of the moment map).

• FFY’s deformation:
  (1) can be applied to torus fibrations (e.g., Lagrangian torus fibrations), and
  (2) realizes a localization of index to the inverse images of the lattice points (or Bohr-Sommerfeld fibers).

As an application of the FFY’s point (2) above, a geometric proof of $[Q,R]=0$ for the torus action case based on the localization of index is obtained in [9]. There is another application in [11] which gives a proof of Danilov’s formula. Danilov’s formula can be regarded as a localization of the geometric quantization of toric manifolds to lattice points in the momentum polytope. The proof in [11] realizes such a localization picture faithfully.

In the present paper we give a framework of a deformation of Dirac operator in a similar manner as in the equivariant setting for FFY’s deformation. We use a single Dirac operator along orbits for the deformation, which satisfies some acyclicity and boundedness condition. We call it an acyclic orbital Dirac operator (Definition 2.1). Though it is similar to the acyclic compatible system in [7] or [8], the definition of the acyclic orbital Dirac operator is much simpler due to the presence of the global group action and the isotypic component decomposition of the space of sections. We summarize our first main results:

**Theorem 1.** (Corollary 2.4, Definition 2.5 and Proposition 2.6) The deformation by an acyclic orbital Dirac operator gives an equivariant index valued in the formal completion of the representation ring and a natural $K$-homology cycle representing the index.

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Some generalizations to proper actions of non-compact Lie groups are established in [13] for example.
We can construct an acyclic orbital Dirac operator for torus actions on Hermitian manifold (Definition 3.1). It can be regarded as a combination of Kasparov’s orbital Dirac operator [15] and Braverman’s deformation, which in fact becomes the Braverman type Clifford action shifted by a weight when it is restricted to each isotypic component. The second main result is the following.

**Theorem 2. (Theorem 4.2)** In the torus action case with some technical conditions, the equivariant index defined by the acyclic orbital Dirac operator coincides with the equivariant index defined by Braverman’s deformation.

As a corollary of Braverman’s index theorem, our equivariant index is also equal to Atiyah’s transverse index [3] under the same assumptions.

Finally we apply the above construction to the setting of proper Hamiltonian torus manifolds with possibly non-compact fixed point sets, allowing us to define the spin$^c$ quantization of it as an equivariant index (Definition 7.2). Our quantization has a localization property to integral lattice points due to its origin. The third main result is the following.

**Theorem 3. (Theorem 7.4 and Theorem 7.6)** For proper Hamiltonian torus manifolds, the quantization defined by an acyclic orbital Dirac operator satisfies the following:

1. $[Q, R] = 0$ theorem for the symplectic reduction at the integral regular value, and
2. a Danilov-type formula for toric case.

The proofs of the above theorems apply also to the compact case, giving simple alternative proofs for [9] and [11]. Since our equivariant index can be identified with Atiyah’s transverse index, the proof of statement (1) in the above Theorem 3 gives an alternative proof of the Vergne conjecture in [21]. In the toric case, the lattice points in the momentum polytope are closely related to the geometric quantization obtained by a real polarization. There are several results concerning the coincidence between the spin$^c$ (or Kähler) quantization and real quantization from the viewpoint of the index theory. For example see [1], [7] and [16]. Theorem 7.6 can be regarded as such a coincidence in the non-compact setting.

This paper is organized as follows. In Section 2 we first give the set-up and definition of $K$-acyclic orbital Dirac operator for a compact Lie group $K$ (Definition 2.1). We show that a deformation by a $K$-acyclic orbital Dirac operator has a compact resolvent on each isotypic component of the space of $L^2$-sections (Corollary 2.4), and hence, it gives an equivariant ($K$-Fredholm) index and a K-homology cycle in a natural way (Definition 2.5). One of a key points in the proof is the presence of

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3In fact in [11] the author showed a Danilov-type formula for toric origami manifolds, which are a generalization of symplectic toric manifolds. It would be possible to give a proof of a similar formula for non-compact toric origami manifolds by modifying the proof in this paper.
a proper function in the deformation. We also show that the resulting Fredholm index is equal to that obtained from a deformation without a proper function (Theorem 2.7). This deformation is closer to the deformation studied in [7,8]. In Section 3 we construct a $K$-acyclic orbital Dirac operator on Hermitian manifolds with torus action (Definition 3.1 and Proposition 3.4). This example arises naturally in the situation of Hamiltonian torus manifolds. Although the abstract framework of the $K$-acyclic orbital Dirac operator can be established for general compact Lie groups $K$, we do not have any examples for non-Abelian $K$. In Section 4 we show that the our equivariant index is equal to the equivariant index obtained by Braverman’s deformation (Theorem 4.2). In Section 5 we summarize the product formula in a useful two ways (Proposition 5.3 and Proposition 5.10). Since the product formula itself can be obtained in the abstract framework of index theory of Fredholm operators we just confirm our set-up and statements. We also present two practical formulas which have key roles in Section 7. In Section 6 we show a vanishing formula for fixed point subsets (Theorem 6.1), which is also important in the construction in Section 7. In Section 7, by using the constructions and discussions in the previous sections we define quantization of proper Hamiltonian torus manifolds as an equivariant index (Definition 7.2). For our quantization we show $[Q,R]=0$ theorem (Theorem 7.3) and a Danilov-type formula for toric case (Theorem 7.6). The proofs are straightforward from the localization property of our index to lattice points and product formulas. In Section 8 we explain some future problems concerning quantization of Hamiltonian loop group spaces and a relation between the deformation and KK-product.

1.1. Notations. We fix some notations.

For a compact Lie group $K$ let $\text{Irr}(K)$ be the set of all isomorphism classes of finite dimensional irreducible unitary representations of $K$. We frequently do not distinguish an element $\rho \in \text{Irr}(K)$ and its corresponding representation space. Each unitary representation $\mathcal{H}$ of $K$ has the $K$-isotypic component decomposition

$$\mathcal{H} = \bigoplus_{\rho \in \text{Irr}(K)} \mathcal{H}^{(\rho)},$$

where each isotypic component $\mathcal{H}^{(\rho)}$ is defined by

$$\mathcal{H}^{(\rho)} = \text{Hom}_K(\rho, \mathcal{H}) \otimes \rho.$$ 

We also use the similar notation $A^{(\rho)}$ for the restriction of a $K$-equivariant linear map $A$ to the isotypic component. The representation ring of $K$ is denoted by $R(K)$, which is generated by $\text{Irr}(K)$. We denote its formal completion by $R^{-\infty}(K)$, namely

$$R^{-\infty}(K) := \text{Hom}(R(K), \mathbb{Z}).$$

Note that $R(K)$ can be identified with the subgroup consisting of finite support elements in $R^{-\infty}(K)$ by taking the coefficients in each irreducible representation.
Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$, $A$ and $B$ self-adjoint operators on $\mathcal{H}$ which have common domain. We write $A \geq B$ if
$$(Au, u) \geq (Bu, u)$$
for all $u \in \mathcal{H}$ in the domain of $A$. If $\mathcal{H}$ has a $\mathbb{Z}/2$-grading and $A$ is an odd Fredholm operator with the decomposition
$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}$$
according to the grading, then its $\mathbb{Z}/2$-graded Fredholm index is defined as the super dimension of $\ker(A)$:
$$\text{index}(A) := \dim(\ker A^+) - \dim(\ker A^-) \in \mathbb{Z}.$$

Let $M$ be a Riemannian manifold and $W \to M$ a vector bundle over $M$ equipped with a Hermitian metric $\langle \cdot, \cdot \rangle_W = \langle \cdot, \cdot \rangle$. This metric gives rise to a $L^2$-inner product on the space of compactly supported sections $\Gamma_c(W)$ of $W$ which is denoted by $(\cdot, \cdot)_W = \langle \cdot, \cdot \rangle$. The associated $L^2$-norm and $L^2$-completion are denoted by $\| \cdot \|_W = \| \cdot \|$ and $L^2(W)$ respectively.

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2. $K$-acyclic orbital Dirac operator

2.1. Set-up and definition. Let $M$ be a complete Riemannian manifold and $W \to M$ a $\mathbb{Z}/2$-graded $\text{Cl}(TM)$-module bundle with the Clifford multiplication $c : TM \cong T^*M \to \text{End}(W)$. Let $K$ be a compact Lie group acting on $M$ in an isometric way. We assume that the $K$-action lifts to a unitary action of $W$. Take and fix a $K$-invariant Dirac-type operator $D : \Gamma_c(W) \to \Gamma_c(W)$.

Definition 2.1 ($K$-acyclic orbital Dirac operator). Under the above set-up a collection of data
$$(D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)})$$
is called a $K$-acyclic orbital Dirac operator on $(M, W)$ if the following conditions are satisfied.

(1) $D_K : \Gamma_c(W) \to \Gamma_c(W)$ is a $K$-invariant differential operator such that :
(a) $D_K$ contains only differentials along $K$-orbits.
(b) The restriction of $D_K$ to each $K$-orbit is a Dirac-type operator on the orbit.
(c) $D_K$ anti-commutes with the Clifford multiplication of the transverse direction to orbits. Namely for any $K$-invariant function $h$ on $M$ one has
\[ D_K c(dh) + c(dh) D_K = 0. \]

(d) For each $\rho \in \text{Irr}(K)$, the isotypic component $D_{\rho}^{(K)}$ is a bounded operator on $\Gamma_c(W)^{(\rho)}$.

(2) For each $\rho \in \text{Irr}(K)$, $V_\rho$ is an open subset of $M$ such that $M \setminus V_\rho$ is compact.

(3) For each $\rho \in \text{Irr}(K)$, we have
\[ \ker(D_K|_{V_\rho})^{(\rho)} = 0. \]

(4) For each $\rho \in \text{Irr}(K)$, there exists a constant $C_\rho > 0$ such that
\[ |((DD_K + D_K D)s, s)_W| \leq C_\rho(D_K^2 s, s)_W \]
and
\[ |(D_K s, s)_W| \leq C_\rho(D_K^2 s, s)_W \]
hold for any $s \in \Gamma_c(W|_{V_\rho})^{(\rho)}$.

(5) For each $\rho \in \text{Irr}(K)$, there exists a constant $\kappa_\rho > 0$ such that
\[ \kappa_\rho(s, s)_W \leq (D_K^2 s, s)_W \]
holds for any $s \in \Gamma_c(W|_{V_\rho})^{(\rho)}$.

The completeness of $M$ implies that there exists a $K$-invariant smooth proper function $f : M \to [1, \infty)$ such that
\[ \|df\|_\infty := \sup_{x \in M} |df|_x < \infty. \]
We take and fix such $f$. For each $\rho \in \text{Irr}(K)$ we take and fix a $K$-invariant cut-off function
\[ (2.1) \quad \varphi_\rho : M \to [0, 1] \]
such that
\[ \varphi_\rho \equiv 0 \text{ on a sufficiently small compact neighborhood of } M \setminus V_\rho \]
and
\[ \varphi_\rho \equiv 1 \text{ on the complement of a relatively compact neighborhood of } M \setminus V_\rho. \]
We put $f_\rho := \varphi_\rho f$. For each $\rho \in \text{Irr}(K)$ consider the deformation of $D$ defined by
\[ \hat{D}_\rho := D + f_\rho D_K. \]

\[ \text{This condition implies that the anti-commutator } DD_K + D_K D \text{ contains only differentials along } K\text{-orbits.} \]

\[ \text{The condition } (\text{3}) \text{ implies that } (D_K^2)^{(\rho)} \text{ is a strictly positive operator on each } K\text{-orbit. On the other hand since } DD_K + D_K D \text{ and } D_K \text{ are differential operators on the orbits, we can take such a constant } C_\rho \text{ for each orbit. This condition means that we can take such constants uniformly on } V_\rho. \]
One can see that $\hat{D}_\rho$ is an elliptic operator by taking the square of the symbol. Since $D$ and $D_K$ has finite propagation speed, $\hat{D}_\rho$ gives an essentially self-adjoint operator on $L^2(W)$.

Hereafter we mainly consider the isotypic component $\hat{D}_\rho^{(\rho)}$. Even if so we often omit the superscript $(\cdot)^{(\rho)}$ of the isotypic component for simplicity and use the notation as $\hat{D}_\rho: L^2(W)^{(\rho)} \rightarrow L^2(W)^{(\rho)}$ and so on.

**Remark 2.2.** The Clifford module structure and Dirac-type condition are not so essential. In fact we can establish almost all propositions, definitions, etc., below for more general vector bundles and elliptic operators with finite propagation speed. However we do not have applications of such generalizations we only handle with Clifford module bundles and Dirac-type operators in the present paper.

### 2.2. Compactness and $K$-Fredholmness

**Proposition 2.3.** For each $\rho \in \text{Irr}(K)$ there exists a smooth $K$-invariant proper function $\Phi_\rho: M \rightarrow \mathbb{R}$ such that $\Phi_\rho$ is bounded below and we have

$$ (\hat{D}_\rho^{(\rho)})^2 + 1 \geq (D^{(\rho)})^2 + \Phi_\rho $$

as self-adjoint operators on $L^2(W)^{(\rho)}$.

**Proof.** Since $f_\rho$ is $K$-invariant we have an equality on $\Gamma_c(W)^{(\rho)}$:

$$ \hat{D}_\rho^{(\rho)} = D^2 + (Df_\rho^4D_K + f_\rho^4D_KD) + f_\rho^6D_K^2 $$

$$ = D^2 + (DD_K + D_KD)f_\rho^2 + c(df_\rho^2)D_Kf_\rho^2 - D_Kf_\rho^2c(df_\rho^2) + f_\rho^6D_K^2 $$

$$ = D^2 + f_\rho^2(DD_K + D_KD)f_\rho^2 + 2c(df_\rho^2)D_Kf_\rho^2 + f_\rho^6D_K^2. $$

Now for any $s \in \Gamma_c(W)^{(\rho)}$ we have

$$ |(f_\rho^2(DD_K + D_KD)f_\rho^2 s, s)_W| = |((DD_K + D_KD)f_\rho^2 s, f_\rho^2 s)_W| $$

$$ \leq C_\rho(D_K^2f_\rho^2 s, f_\rho^2 s)_W $$

$$ = C_\rho(f_\rho^4D_K^2 s, s)_W $$

and

$$ |(c(df_\rho^2)D_Kf_\rho^2 s, s)_W| = |(c(df_\rho^2)D_Kf_\rho s, f_\rho s)_W| $$

$$ = |(2f_\rho c(df_\rho)D_Kf_\rho s, f_\rho s)_W| $$

$$ \leq 2\|df_\rho\|_\infty |(D_K(f_\rho)^{3/2} s, (f_\rho)^{3/2} s)_W| $$

$$ = 2C_\rho\|df_\rho\|_\infty (f_\rho^3D_K^2 s, s)_W. $$

Summarizing the above inequalities we have

$$ \hat{D}_\rho^{(\rho)} \geq D^2 + (-C_\rho f_\rho^4 - 4C_\rho\|df_\rho\|_\infty f_\rho^3 + f_\rho^6)D_K^2 $$

$$ \geq D^2 + \frac{\kappa_\rho f_\rho^8}{2} + \left(-C_\rho f_\rho^4 - 4C_\rho\|df_\rho\|_\infty f_\rho^3 + f_\rho^6\right)D_K^2. $$
Now put \( g_\rho := \frac{f_\rho^8}{2} - C_\rho f_\rho^4 - 4C_\rho \|df_\rho\|_\infty f_\rho^3 : M \to \mathbb{R} \).

Since \( f_\rho \) is proper and bounded below the function \( g_\rho \) is also proper and bounded below. Note that \( M_- := g_\rho^{-1}((-\infty,0]) \) is a compact subset of \( M \), and hence, by the boundedness of \( D_K \) (condition (1d)) there exists a constant \( C_{\rho,M_-} > 0 \) such that we have

\[
\int_{M_-} (D_K^2 s, s)_W \leq C_{\rho,M_-} \int_{M_-} (s, s)_W
\]

and

\[
\langle g_\rho D_K^2 s, s \rangle_W = \left( \int_{M_-} + \int_{M\setminus M_-} \right) \langle g_\rho D_K^2 s, s \rangle_W \\
\geq \int_{M_-} \langle g_\rho D_K^2 s, s \rangle_W \\
\geq \min_{M_-} (g_\rho) C_{\rho,M_-} (s, s)_W.
\]

As a consequence we have

\[
\hat{D}_\rho^2 + 1 \geq D^2 + \Phi_\rho
\]

for

\[
\Phi_\rho := \frac{\kappa \rho^8}{2} + \min_{M_-} (g_\rho) C_{\rho,M_-} + 1
\]

which is \( K \)-invariant, proper and bounded below. \( \square \)

As a corollary we have the following compactness by [20, Proposition B.1].

**Corollary 2.4.** For any \( \rho \in \text{Irr}(K) \), a bounded operator \( ((\hat{D}_\rho^2)^{(\rho)} + 1)^{-1} \) on \( L^2(W)^{(\rho)} \) is a compact operator. In particular \( (\hat{D}_\rho)^{(\rho)} \) is a Fredholm operator on \( L^2(W)^{(\rho)} \).

**Definition 2.5.** Define an element \([\hat{D}] \in R^{-\infty}(K)\) by

\[
[\hat{D}](\rho) := \text{index}(\hat{D}_\rho)^{(\rho)}) \in \mathbb{Z}
\]

for each \( \rho \in \text{Irr}(K) \). We also use the notations

\[
[\hat{D}] = [M, W, D_K] = [M, W] = [M].
\]

Hereafter we often write \( \text{index}(\hat{D}_\rho) \in \mathbb{Z} \) instead of \( \text{index}(\hat{D}_\rho)^{(\rho)} \).

In general a \( K \)-equivariant operator \( A \) on a \( \mathbb{Z}/2 \)-graded Hilbert space \( \mathcal{H} \) with isometric \( K \)-action is called \( K \)-Fredholm if each isotypic component \( A^{(\rho)} : \mathcal{H}^{(\rho)} \to \mathcal{H}^{(\rho)} \) is Fredholm. Such a \( K \)-Fredholm operator \( A \) defines an element in \( R^{-\infty}(K) \) denoted by a formal expression:

\[
\text{index}_K(A) = \sum_{\rho \in \text{Irr}(K)} \text{index}(A^{(\rho)})\rho.
\]

Corollary 2.4 and Definition 2.5 imply that

\[
\bigoplus_{\rho \in \text{Irr}(K)} \hat{D}_\rho : L^2(W) \to L^2(W)
\]

is a \( K \)-Fredholm operator and \([\hat{D}]\) is its index in \( R^{-\infty}(K) \).
2.3. K-homology cycle representing the class $[\tilde{D}]$. For each $\rho \in \text{Irr}(K)$ we put

$$F_\rho := \frac{\hat{D}_\rho}{\sqrt{1 + (\hat{D}_\rho)^2}}$$

which is a bounded operator acting on $L^2(W)^{(\rho)}$ with $\|F_\rho\| = 1$. We can see that

$$F := \bigoplus_{\rho \in \text{Irr}(K)} F_\rho$$

gives a bounded operator on $L^2(W) = \bigoplus_{\rho \in \text{Irr}(K)} L^2(W)^{(\rho)}$.

It is known that the formal completion $R^{-\infty}(K)$ can be identified with the K-homology group of the group C*-algebra $K^0(C^*(K))$, which is also identified with the KK-group $KK(C^*(K), \mathbb{C})$. These groups are generated by triples consisting of a Hilbert space, a C*-representation of $C^*(K)$ and a bounded operator on the Hilbert space satisfying certain boundedness and compactness. See [5,12,15] for basic definitions on K-homology or KK-theory. The above Corollary 2.4 implies the following.

**Proposition 2.6.** The bounded operator $F$ together with the natural representation of $C^*(K)$ on $L^2(W)$ gives a K-homology cycle which represents $[\hat{D}]$;

$$[(L^2(W), F)] = [\hat{D}] \in KK(C^*(K), \mathbb{C}) = K^0(C^*(K)) = R^{-\infty}(K).$$

2.4. Relation with Fujita-Furuta-Yoshida type deformation. In this section we consider another deformation of the form

$$D_{\rho,t} := D + t\varphi^4_\rho D_K \quad (t \geq 0)$$

for $\rho \in \text{Irr}(K)$ using a $K$-acyclic orbital Dirac operator $(D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)})$, where $\varphi_\rho$ is the cut-off function as in [23]. This type of deformation was studied for an acyclic compatible system in a series of papers [7,9]. The difference between the above deformation and $\hat{D}_\rho$ is the presence of a proper function $f$. To compare them we introduce a 1-parameter family

$$D_\epsilon = D + (1 - \epsilon)f^4_\rho D_K + \epsilon t\varphi^4_\rho D_K = D + ((1 - \epsilon)f^4 + \epsilon t)\varphi^4_\rho D_K \quad (\epsilon \in [0,1])$$

which acts on $L^2(W)$. We show the following.

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4In fact the acyclic compatible system is a family of Dirac-type operators along the fibers which is defined on a family of open subsets. The deformation is given by the sum of them by using a partition of unity. It is one remarkable feature that the acyclic compatible system do not rely on a group action. Though in this paper we do not investigate any relation between the equivariant acyclic compatible system and the $K$-acyclic orbital Dirac operator we believe that they give the same index under a suitable assumptions.
Theorem 2.7. For each \( \rho \in \text{Irr}(K) \) there exists \( t_\rho > 0 \) such that \( \{D_\epsilon\}_{\epsilon \in [0,1]} \) gives a family of Fredholm operator on \( L^2(W)^{(\rho)} \) for any \( t > t_\rho \) and its Fredholm index does not depend on \( \epsilon \) and \( t \). In particular we have
\[
\text{index}((D_\rho,t)^{(\rho)}) = \text{index}((D_\rho)^{(\rho)}) \in \mathbb{Z}.
\]

Corollary 2.8. Define \([D_t] \in \mathbb{R}^{-\infty}(K)\) by
\[
[D_t](\rho) := \text{index}((D_{\rho,t})^{(\rho)}) \quad (t > t_\rho)
\]
for each \( \rho \in \text{Irr}(K) \). Then we have
\[
[D_t] = [D] \in \mathbb{R}^{-\infty}(K).
\]

Note that since \( D_K \) becomes a bounded operator of order 0 on \( L^2(W)^{(\rho)} \), the principal symbol of \( D_\epsilon \) is equal to that of \( D \), and hence, \( D_\epsilon \) has finite propagation speed on \( L^2(W)^{(\rho)} \). Theorem 2.7 follows from the following estimate, which is also known as the coercivity in [2]. In fact, as in [3], the \( \mathbb{Z}/2 \)-graded Fredholm index of a coercive family with finite propagation speed does not depend on a parameter of the family.

Proposition 2.9. There exists an open subset \( U_\rho \) and a constant \( t_\rho > 0 \) such that \( M \setminus U_\rho \) is compact and
\[
\|D_s\|^2_W \geq t_\rho \kappa_\rho \|s\|^2_W
\]
holds for any \( s \in \Gamma_c(W)^{(\rho)} \) with \( \text{supp}(s) \subset U_\rho \), \( \epsilon \in [0,1] \) and \( t > t_\rho \), where \( \kappa_\rho > 0 \) is the constant as in (2.4) of Definition 2.1.

Proof. We take \( U_\rho' \) to be the interior of \( \varphi_\rho^{-1}(1) \) and put \( h := (1 - \epsilon)f^4 + ct \). On \( U_\rho' \) consider the square
\[
(D + hD_h)^2 = D^2 + (DhD_K + hD_KD) + h^2D_K^2 = D^2 + c(dh)D_K + h(DD_K + D_KD) + h^2D_K^2.
\]
For any \( s \in \Gamma_c(W)^{(\rho)} \) with \( \text{supp}(s) \subset U_\rho' \) we have
\[
|\langle c(Dh)D_K s, s \rangle_W| = |\langle 4(1 - \epsilon)f^3c(df)D_K s, s \rangle_W| \\
\leq 4(1 - \epsilon)\|df\|_\infty \|\langle f^3D_K s, s \rangle_W| \\
\leq 4\|df\|_\infty C_\rho(hD_K^2s, s)_W
\]
and
\[
|\langle h(DD_K + D_KD)s, s \rangle_W| \leq C_\rho(hD_K^2s, s)_W.
\]
It implies
\[
\|D_\epsilon s\|^2_W = \langle (D + hD_K)^2 s, s \rangle_W \\
\geq \langle (c(dh)D_K + h(DD_K + D_KD) + h^2D_K^2)s, s \rangle_W \\
\geq \langle (-4\|df\|_\infty C_\rho - C_\rho + h)hD_K^2s, s \rangle_W.
\]
Now put \( t_\rho := 4\|df\|_\infty C_\rho + C_\rho + 1 \) and define \( U_\rho \) by
\[
U_\rho := \{ x \in U'_\rho \mid f(x)^4 > t_\rho \}.
\]
Then on \( U_\rho \) when \( t > t_\rho \) we have \((-4\|df\|_\infty C_\rho - C_\rho + h)h > t_\rho\). Finally we have\(^5\)
\[
\|\mathcal{D}_s\|_W^2 \geq (t_\rho D^2 K s, s)_W \geq t_\rho \kappa_\rho(s, s)_W = t_\rho \kappa_\rho\|s\|_W^2.
\]

\[\square\]

Hereafter we often use the deformation
\[
D + t\varphi_\rho^4 D_K \quad (t \gg 0)
\]
without the proper function \( f \) to discuss the equivariant index \([\hat{D}](\rho) = [M](\rho)\).

Theorem \(\text{2.7}\) implies\(^6\) that index\((\hat{D}_\rho)\) satisfies the excision formula, sum formula and product formula as stated in \([8\text{, Section 3]}\). In particular if there are two data \((M, W, D, D_K, V_\rho)\) and \((M', W', D', D'_K, V'_\rho)\) for the same \(K\) and \(\rho \in \text{Irr}(K)\) which are isomorphic on neighborhoods of compact subsets \(M \setminus V_\rho\) and \(M' \setminus V'_\rho\), then the excision formula implies that the resulting indices coincide:
\[
(2.3) \quad [M](\rho) = \text{index}(D + \varphi_\rho^4 D_K) = \text{index}(D' + \varphi_\rho^4 D'_K) = [M'](\rho).
\]

It ensures us to define the index starting from a non-complete manifold by taking an appropriate completion, for instance a cylindrical end as in \([8\text{, Section 7.1]}\) or \([20\text{, Section 4.7]}\). We will use such a construction in Section \(\text{7}\).

3. Acyclic orbital Dirac operator for torus action

We construct a prototypical example of \(D_K\) in some set-up which is extracted from Hamiltonian torus actions on prequantized symplectic manifold.

Suppose that \(K\) is an \(n\)-dimensional torus with Lie algebra \(\mathfrak{k}\). We identify as \(\text{Irr}(K) = \Lambda^*\), where we put \(\Lambda := \ker(\exp : \mathfrak{k} \to K)\). We fix an inner product on \(\mathfrak{k}\) and identify \(\mathfrak{k}^* = \mathfrak{k}\) so that a set of generators of \(\Lambda\) becomes an orthonormal basis.

Suppose that \(M\) is equipped with a \(K\)-invariant Hermitian structure \((g, J)\) and there is a \(K\)-equivariant Hermitian line bundle with Hemitian connection \((L, \nabla^L)\) over \(M\). For \(\xi \in \mathfrak{k}\) we denote the induced infinitesimal action of \(\xi\) on \(M\) by \(L^M_\xi\) and the induced Lie derivative by \(L^M_\xi\). Let \(\mu : M \to \mathfrak{k}^* = \mathfrak{k}\) be the map which is characterized by the Kostant’s formula:
\[
(3.1) \quad L^M_\xi - \nabla^L_\xi = \sqrt{-1}\mu(\xi) = \sqrt{-1}\mu_\xi \quad (\xi \in \mathfrak{k}),
\]

\(^5\)This argument shows that by taking \(t_\rho\) large enough and \(U_\rho = (f^4)^{-1}((t_\rho, \infty))\) we can refine the estimate as \(\|\mathcal{D}_s\|_W^2 \geq \|s\|_W^2\) for any \(s \in \Gamma_c(W)^{\mathfrak{g}}\) with \(\text{supp}(s) \subset U_\rho\).

\(^6\)We can apply the argument in \([8\text{, Section 3]}\) for \(D_\rho\) directly without using the finite propagation speed condition. In fact by taking a family of cut-off function \(\varphi_{a, s}\) in \([8\text{, Lemma A.1]}\) in a \(K\)-invariant way the arguments in \([8]\) can still work for \(D_\rho\).
where $\mathcal{L}_\xi^L : \Gamma(L) \to \Gamma(L)$ is the induced derivative defined by
$$\mathcal{L}_\xi^L s : x \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi)s(\exp(-t\xi)x)$$
for $s \in \Gamma(L)$.

We define a Hermitian vector bundle
$$W := \wedge^\bullet T\mathcal{C}M \otimes L$$
over $M$, where $T\mathcal{C}M = TM$ is the vector bundle regarded as a complex vector bundle by $J$. It is known that $W$ carries a structure of $\mathbb{Z}/2\mathbb{Z}$-module bundle with the Clifford multiplication $c : TM \to \text{End}(W)$ defined by the exterior product and its adjoint. Let $V$ be the complement of the fixed point set, $V := M \setminus M^K$. For each $x \in V$ there exists a subtorus $H_x$ in $K$ such that $K \cdot x = H_x \cdot x$. We can take a $K$-invariant small open neighborhood $V_x$ of $x$ in $V$ such that $H_x$ acts on $V_x$ with at most finite stabilizer subgroup. Then we have a natural projection $q : V_x \to V_x/H_x$ between smooth manifolds (or orbifolds) and an exact sequence of vector bundles

$$T_{H_x}V_x \to TV_x \to q^*T(V_x/H_x),$$

where $T_{H_x}V_x$ is the tangent bundle along $H_x$-orbits. Let $E(x)$ be the orthogonal complement of $T_{H_x}V_x \oplus J(T_{H_x}V_x)$ in $TV_x$. Then we have an isomorphism $J(T_{H_x}V_x) \oplus E(x) \cong q^*T(V_x/H_x)$. For each $x \in V$ we have

$$TV_x|_{K\cdot x} = T(K \cdot x) \oplus J(T(K \cdot x)) \oplus E(x)|_{K\cdot x} \cong (T(K \cdot x) \otimes \mathbb{C}) \oplus E(x)|_{K\cdot x}.$$  

Note that $E(x)|_{K\cdot x}$ has a natural trivialization using the $H_x$-action. In particular we have

$$W|_{K\cdot x} = \wedge^\bullet E(x)|_{K\cdot x} \otimes \wedge^\bullet T(K \cdot x) \otimes L|_{K\cdot x}.$$  

Now consider the induced operator $\mathcal{L}_\xi$ on $\Gamma(W)$ for each $\xi \in \mathfrak{k}$ defined by

$$\mathcal{L}_\xi := \mathcal{L}_\xi^M \otimes \text{id} + \text{id} \otimes \mathcal{L}_\xi^L.$$  

Note that $\mathcal{L}_\xi$ is a first order differential operator which only contains differential along the orbit. Fix an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ of $\mathfrak{k}$.

**Definition 3.1.** We define $D_K : \Gamma_c(W) \to \Gamma_c(W)$ by

$$D_K := \sum_{i=1}^n \sum_{\mu \in \mathbb{Z}} c(\mu \xi_i)(\mathcal{L}_{\xi_i} - \sqrt{-1}\mu \xi_i).$$  

For $x \in V$ and a non-zero section $s \in \Gamma(W|_{K\cdot x})$ if $D_Ks = 0$, then by using $\{\xi_i\}$, as a $K$-invariant frame of $T(K \cdot x)$ one can see that $s$ produces a non-zero section $s'$ of $L|_{K\cdot x}$ such that

$$\nabla_{\xi_i}^L s' = (\mathcal{L}_{\xi_i} - \sqrt{-1}\mu \xi_i)s' = 0$$  

for all $i$, in other words, a non-trivial global parallel section of $L|_{K\cdot x}$. Conversely if there exists a non-trivial global parallel section of $L|_{K\cdot x}$, then one can construct a
non-zero element \( s \in \ker(D_K|_{K \cdot x}) \). Moreover since \( \mathcal{L}_\xi = \sqrt{-1}\rho(\xi) \) on \( \Gamma(W|_{K \cdot x})^{(\rho)} \) for each \( \rho \in \text{Irr}(K) \) and \( \xi \in \mathfrak{k} \) one has the following:

**Proposition 3.2.** For \( x \in V \) and \( \rho \in \text{Irr}(K) \) we have

\[
\ker(D_K|_{K \cdot x})^{(\rho)} \neq 0 \iff H^0(K \cdot x; L|_{K \cdot x})^{(\rho)} \neq 0,
\]

where \( H^0(K \cdot x; L|_{K \cdot x}) \) is the space of global parallel sections on \( L|_{K \cdot x} \), and if the above non-vanishing occurs, then we have \( \rho = \mu(x) \in \text{Irr}(K) = \Lambda^* \).

**Remark 3.3.** If \( M = (M, \omega) \) is a symplectic manifold of dimension \( 2n \), the \( K \)-action is an effective Hamiltonian torus action and \( (L, \nabla_L) \) is a prequantizing line bundle, i.e., the curvature form of \( \nabla_L \) is equal to \( -\sqrt{-1}\omega \), then the condition \( H^0(K \cdot x; L|_{K \cdot x}) \neq 0 \) is equivalent to the Bohr-Sommerfeld condition for the orbit \( K \cdot x \), which is essential in the geometric quantization by the real polarization.

For each \( \rho \in \text{Irr}(K) \) we put

\[
V_\rho := (M \setminus \mu^{-1}(\rho)) \cap V.
\]

Then since \( D_K^2|_{K \cdot x} \) is a strictly positive operator on \( \Gamma_c(W_L|_{K \cdot x})^{(\rho)} \) for any \( x \in V_\rho \) there exists a constant \( C_{\rho,x} \) such that

\[
|((DD_K + D_K D)s, s)_W| \leq C_{\rho,x}(D_K^2 s, s)_W
\]

and

\[
|((D_K s, s)_W| \leq C_{\rho,x}(D_K^2 s, s)_W
\]

hold for any \( s \in \Gamma_c(W_L|_{K \cdot x})^{(\rho)} \).

**Proposition 3.4.** If the following conditions are satisfied then \( (D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)}) \) is a \( K \)-acyclic orbital Dirac operator on \((M, W)\).

1. The fixed point set \( M^K \) is compact.
2. For each \( \rho \in \text{Irr}(K) \), \( \mu^{-1}(\rho) \) is compact.
3. The metric \( g \) on \( M \) is complete.
4. There exists \( C > 0 \) such that

\[
C^{-1} < \sum_{i=1}^n |\xi_i|^2 \leq C
\]

on the outside of some compact set in \( M \).
5. For each \( \rho \in \text{Irr}(K) \),

\[
\sup\{C_{\rho,x} \mid x \in V_\rho\} < \infty.
\]
6. For each \( \rho \in \text{Irr}(K) \),

\[
\liminf_{x \in V_\rho} |\kappa| \mid \kappa \text{ is the minimum eigenvalue of } (D_K|_{K \cdot x})^2 \text{ on } L^2(W|_{K \cdot x})^{(\rho)} \} > 0.
\]
In particular if $M$ has a cylindrical (resp. periodic) end and all the data have translationally invariance (resp. periodicity), then all the conditions (3) $\sim$ (7) are satisfied. Moreover if there are two such data, then the product of them satisfies the same condition.

As we noted in the end of Subsection 2.4, the index associated with the $K$-acyclic orbital Dirac operator can be defined for a non-complete situation. In particular if the conditions in Proposition 3.4 are satisfied, then by taking a relatively small compact neighborhood of $\mu^{-1}(\rho) \cup M^K$ and its completion we can define the Fredholm index using the restriction of $D_K$ to the neighborhood. Though we agree that it is a little bit strange notation, we denote the index by

$$[\mu^{-1}(\rho) \cup M^K] \in \mathbb{Z}. \quad (3.2)$$

By the Kostant formula one can see that $M^K \subset \mu^{-1}(\Lambda)$ we have the disjoint union

$$M^K = \bigcup_{\rho' \in \text{Irr}(K)} \mu^{-1}(\rho') \cap M^K.$$ 

This description enable us to get more refined decomposition of (3.2) into the summation of local contributions from each $\mu^{-1}(\rho') \cap M^K$, which we denote by

$$[\mu^{-1}(\rho) \cup M^K] = [\mu^{-1}(\rho)] + \sum_{\rho \neq \rho' \in \text{Irr}(K)} [\mu^{-1}(\rho') \cap M^K].$$

The excision formula implies the following localization formula.

**Theorem 3.5.** If the conditions in Proposition 3.4 are satisfied, then the index $[D] = [M] \in R^{-\infty}(K)$ defined by the $K$-acyclic orbital Dirac operator $D_K$ satisfies

$$[M](\rho) = [\mu^{-1}(\rho)] + \sum_{\rho \neq \rho' \in \text{Irr}(K)} [\mu^{-1}(\rho') \cap M^K]$$

for each $\rho \in \text{Irr}(K)$.

**Remark 3.6.** The differential term $\sum_{i=1}^{n} c(\xi^M_{\xi_i})\mathcal{L}_{\xi_i}$ in $D_K$ is the orbital Dirac operator in the sense of Kasparov [15]. On the other hand the multiplication term $\sum_{i=1}^{n} c(\xi^M_{\xi_i})\mu_{\xi_i}$ is equal to $c(\mu)$ for $\mu := \sum_{i=1}^{n} \xi^M_{\mu_{\xi_i}}$, which gives the deformation studied by Braverman [6]. On each isotypic component $L^2(W_L)(\rho)$ one has $\mathcal{L}_{\xi_i} = \sqrt{-1}\rho(\xi_i)$, and hence,

$$D_K = \sqrt{-1}\sum_{i=1}^{n} c(\xi^M_{\rho(\xi_i) - \mu_{\xi_i}}) = \sqrt{-1}c(\rho - \mu).$$

In other words $D_K$ gives a kind of shift of Braverman’s deformation. We investigate the relation between our deformation and Braverman’s deformation in the next section.

$^9$The excision formula guarantees that this index around $\mu^{-1}(\rho) \cup M^K$ does not depend on a choice of the neighborhood.
4. Relation with Braverman type deformation

In [6] Braverman studied a Witten-type deformation of the Dirac operator and its equivariant index on non-compact $K$-manifold. In a symplectic geometric setting Braverman’s deformation is given by the Clifford multiplication of the Hamiltonian vector field of the norm square of the moment map. In particular in the setting in Section 3 (not necessarily $K$ is a torus) we can consider the Braverman’s deformation as

$$D_{\mu} := D - h\sqrt{-1}c(\mu),$$

where $h : M \to \mathbb{R}$ is a $K$-invariant function called an admissible function which satisfies a suitable growth condition. Braverman showed several fundamental properties of $D_{\mu}$. In particular he showed that $D_{\mu}$ is a $K$-Fredholm operator and the resulting index in $R^{-\infty}(K)$ is independent of a choice of the admissible function. Moreover the index is equal to Atiyah’s transverse index. After that his equivariant index has been applied in several directions, for instance, a solution to Vergne’s conjecture by Ma-Zhang [21].

In this section we assume the followings to make the situation simple.

Assumption 4.1. We assume the conditions in Proposition 3.4 are satisfied together with the cylindrical end condition and ;

- The moment map $\mu : M \to \mathfrak{k}$ defined by Kostant’s formula is proper in the sense that each inverse image $\mu^{-1}(Z)$ of a compact subset $Z$ of $\mathfrak{k}$ is compact.
- The differential $d\mu : TM \to \mathfrak{k}$ is $L^\infty$ bounded.

The second condition is satisfied for the symplectic setting and the genuine moment map $\mu$ by taking $J$ as an $\omega$-compatible almost complex structure.

Theorem 4.2. Under the Assumption 4.1 we have

$$\text{index}_K(D_{\mu}) = [\hat{D}] \in R^{-\infty}(K).$$

Remark 4.3. As it is noted in [10, Example 5.2] the above equality does not hold in general without properness of $\mu$ or completeness of $M$.

As a corollary of Braverman’s index theorem ([6, Theorem 5.5]) we also have the following.

Corollary 4.4. Under the Assumption 4.1 $[\hat{D}] \in R^{-\infty}(K)$ is equal to the transverse index in the sense of Atiyah [3].

---

10 The cylindrical end condition is used to have a uniform estimate on the end. It is possible to put weaker assumptions to have the uniform estimate. For example we can handle with products of manifolds with cylindrical end.
We first note that under the Assumption 4.1 we can take $f$ as in Section 2 so that $f = |\mu|$ on the outside of a compact neighborhood of the compact subset $\mu^{-1}(0)$. Moreover we can take an admissible function $h$ to be $f^1_{\rho} = \varphi^1_{\rho} f^4$ for each $\rho \in \text{Irr}(K)$, where $\varphi_{\rho}$ is the cut-off function for $V_{\rho} = M \setminus (\mu^{-1}(\rho) \cup M^K)$ as in (2.1).

Fix $\rho \in \text{Irr}(K)$ and consider the following 1-parameter family in the setting in Section 3:

$$D_\epsilon := D + \epsilon f^1_{\rho} D_K - (1 - \epsilon)\sqrt{-1} f^4_{\rho} c(\mu)$$

for $\epsilon \in [0, 1]$. We show that for each $\rho$ an unbounded operator $D_\epsilon$ on $L^2(W_{\rho})$ gives a norm-continuous family of the bounded transformation such as $\sqrt{1 + \epsilon^2}$, and hence, the equality $\text{index}_K(D_{\rho})(\rho) = \text{index}((D_\epsilon)^{(\rho)}) = \text{index}(\hat{D}_\rho)$ holds. We use the following criteria.

**Lemma 4.5** (Proposition 1.6 in [22]). Let $A_0$ and $A$ be unbounded self-adjoint operators on a Hilbert space such that $\text{dom}(A_0) \cap \text{dom}(A)$ is dense. Suppose that the family of operators $A_\epsilon = A_0 + \epsilon A$ ($\epsilon \geq 0$) is essentially self-adjoint and for each $\epsilon \geq 0$ the following conditions hold:

1. $A_\epsilon$ has a gap in its spectrum.
2. $\text{dom}(A_\epsilon) \subset \text{dom}(A)$
3. There exists constants $C, C' > 0$ such that $C' A^2 \leq A_\epsilon^2 + C$.

Then the family of bounded transforms $\epsilon \mapsto A_\epsilon / \sqrt{1 + \epsilon^2}$ is norm-continuous.

As in [19, Remark 4.10] it suffices to show the condition (3) in Lemma 4.5 in our situation.

As we noted in Remark 3.6 one can write as $D_K = \sqrt{-1} c(\rho - \mu)$ on $L^2(W_{\rho})$, and hence, we have

$$D_\epsilon = D + f^1_{\rho} \sqrt{-1} (c(\rho - \mu) - (1 - \epsilon)c(\mu))$$

$$= D + f^1_{\rho} \sqrt{-1} c(\rho - \mu)$$

$$= D - f^1_{\rho} \sqrt{-1} c(\mu) + \epsilon f^4_{\rho} \sqrt{-1} c(\mu).$$

Then the condition (3) in Lemma 4.5 is equivalent to

$$C' \left( f^4_{\rho} \sqrt{-1} c(\rho) \right)^2 \leq (D_\epsilon)^2 + C.$$
Proof. On $L^2(W_L)^{(\rho)}$ we have
\[
(\mathbb{D}_{\rho})^2 = D^2 + \sqrt{-1} (Df_\rho^3 c(\epsilon \rho - \mu) + f_\rho^4 c(\epsilon \rho - \mu)D) + f_\rho^8 |\epsilon \rho - \mu|^2 \]
\[
= D^2 + \sqrt{-1} (4f_\rho^3 c(\epsilon \rho - \mu) + f_\rho^4 (Dc(\epsilon \rho - \mu) + c(\epsilon \rho - \mu)D) f_\rho^2 + f_\rho^8 |\epsilon \rho - \mu|^2
\]

On the other hand there exist constants $C_1 > 0$ and $C_2 > 0$ such that
\[
|c(df_\rho)c(\epsilon \rho - \mu)| \leq \|df_\rho\| |\epsilon \rho - \mu| \leq C_1 |\epsilon \rho - \mu|
\]
and
\[
|f_\rho^2 (Dc(\epsilon \rho - \mu) + c(\epsilon \rho - \mu)D) f_\rho^2| \leq C_2 |\epsilon \rho - \mu|^3 f_\rho^4 D^2 = C_2 f_\rho^4 |\epsilon \rho - \mu|^3,
\]
where we get the inequality in a similar way as the proof of Proposition 2.3 and we use the assumption on cylindrical end so that we can take $C_2$ uniformly. So we have
\[
(\mathbb{D}_{\rho})^2 \geq -4C_1 f_\rho^3 |\epsilon \rho - \mu| - C_2 f_\rho^4 |\epsilon \rho - \mu|^3 + f_\rho^8 |\epsilon \rho - \mu|^2.
\]
On the other hand since $\mu$ is proper and $M^K$ is compact $|\epsilon \rho - \mu|$ is uniformly positive on the outside of a compact subset, and hence, there exists $C' > 0$ such that
\[
f_\rho^8 |\epsilon \rho - \mu| + 1 > 2C' f_\rho^8.
\]
Since $f_\rho = |\mu|$ on the outside of a compact subset there exists $C > 0$ independent from $\epsilon \in [0, 1]$ such that
\[
-4C_1 f_\rho^3 |\epsilon \rho - \mu| - C_2 f_\rho^4 |\epsilon \rho - \mu|^3 - 1 + C' f_\rho^8 > -C.
\]
Finally we have
\[
(\mathbb{D}_{\rho})^2 > (1 - C' f_\rho^8 - C) + (2C' f_\rho^8 - 1) = C' f_\rho^8 - C
\]
and hence, $(\mathbb{D}_{\rho})^2 + C > C' f_\rho^8$. □

5. Product formula

For later convenience we summarize the product formula for our index and some useful formulas derived from it. Instead of giving full general setting we explain typical two situations which will be used in the subsequent sections. We follow the basic formulation of the product formula of indices as in [4], and we give a formulation to adapt that in [3] Section 3.3. For simplicity we consider torus action and acyclic orbital Dirac operators constructed as in Section 3 and Proposition 3.4.
5.1. **Direct product.** For $i = 0, 1$ let $K_i$ be a torus. Let $M_i$ be a complete Riemannian manifold and $W_i \to M_i$ a $\mathbb{Z}/2$-graded Clifford module bundle on which $K_i$ acts in an isometric way. Suppose that there exists a $K_i$-acyclic orbital Dirac operator $(D_{K_i}, \{V_{\rho, \iota}, \rho \in \text{Irr}(K_i)\})$ on $(M_i, W_i)$. Put $M := M_0 \times M_1$ and define a Clifford module bundle $W$ over $M$ by the outer tensor product

$$W := W_0 \otimes W_1,$$

for the projections onto the first and second factor of $M$. For $\rho = (\rho_0, \rho_1) \in \text{Irr}(K_0) \times \text{Irr}(K_1) = \text{Irr}(K)$ we define $V_\rho$ by

$$V_\rho := V_{0, \rho_0} \times V_{1, \rho_1},$$

whose complement in $M$ is compact. Let $D_K : \Gamma(W) \to \Gamma(W)$ be an operator defined by

$$D_K := D_{K_0} \otimes \text{id} + \alpha_{W_0} \otimes D_{K_1} = D_{K_0} + \alpha_{W_0}D_{K_1},$$

where $\alpha_{W_0} : W_0 \to W_0$ is the grading operator on $W_0$. Since $D_{K_0}(\alpha_{W_0}D_{K_1}) + (\alpha_{W_0}D_{K_1})D_{K_0} = 0$ one has the following.

**Lemma 5.1.** $(D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)})$ is a $K$-acyclic orbital Dirac operator on $(M, W)$.

Dirac operators $D_i$ on $W_i$ give rise the Dirac operator $D$ on $W$:

$$D := D_0 \otimes \text{id} + \alpha_{W_0} \otimes D_1 = D_0 + \alpha_{W_0}D_1.$$

For each $\rho_i \in \text{Irr}(K_i)$ we take a $K_i$-invariant cut-off function $\varphi_{i, \rho_i}$ on $M_i$ with $\varphi_{i, \rho_i} |_{M_i \setminus V_{i, \rho_i}} \equiv 0$ as in (2.4). For $\rho = (\rho_1, \rho_2) \in \text{Irr}(K)$ define a function $\varphi_{\rho} : M \to [0, 1]$ by $\varphi_{\rho} := \varphi_{0, \rho_0} \varphi_{1, \rho_1}$, which gives a cut-off function with $\varphi_{\rho} |_{M \setminus V_\rho} \equiv 0$. Then we have a Fredholm operator on $L^2(W)^{(\rho)}$ as the deformation

$$\hat{D}_\rho = D + t\varphi_{\rho}^4D_K \quad (t \gg 0).$$

In particular we have the index

$$\text{index}(\hat{D}_\rho) = [M](\rho) \in \mathbb{Z}.$$

On the other hand we have the sum of the deformations

$$\hat{D}'_\rho = (D_0 + t\varphi_{0, \rho_0}^4D_{K_0}) + \alpha_{W_0}(D_1 + t\varphi_{1, \rho_1}^4D_{K_1}) = D + t(\varphi_{0, \rho_0}^4D_{K_0} + \alpha_{W_0}\varphi_{1, \rho_1}^4D_{K_1}),$$

which is also Fredholm on $L^2(W)^{(\rho)}$. In fact by using the similar estimate in the proof of Proposition 2.3, one can see that $\hat{D}'_\rho$ is coercive on the outside of a compact subset containing $\varphi_{0, \rho_0}^{-1}(0) \cap \varphi_{1, \rho_1}^{-1}(0) = \varphi_{\rho}^{-1}(0)$.

**Lemma 5.2.** $\text{index}(\hat{D}'_\rho) = \text{index}(\hat{D}_\rho) = [M](\rho)$.

**Proof.** This follows from the fact that the deformation of $D$ by

$$\varphi_{0, \rho_0}^4\varphi_{1, \rho_1}^4D_{K_0} + \alpha_{W_0}\varphi_{0, \rho_0}^4\varphi_{1, \rho_1}^4D_{K_1} \quad (0 \leq \epsilon \leq 1)$$

gives a family of coercive operators by using the similar argument in the proof of Proposition 2.3. \qed
Now consider the Fredholm operator \( D_1 + t\varphi^4_{1,\rho_1} D_{K_1} \) on \( L^2(W_1) \) and we put its kernel
\[
E_{\rho_1} := \ker(D_1 + t\varphi^4_{1,\rho_1} D_{K_1}) = E^+_{\rho_1} \oplus E^-_{\rho_1}
\]
as the \( \mathbb{Z}/2 \)-graded finite dimensional vector space. Then there is a natural embedding
\[
L^2(W_0 \otimes E_{\rho_1}) \to L^2(W)^{\rho_1}
\]
whose image is preserved by \( (D_{\rho_0} + t\varphi^4_{\rho_0} D_{K_0}) \otimes \text{id} \). Let \( D_{\rho_0, E_{\rho_1}} \) be the restriction of \( (D_{\rho_0} + t\varphi^4_{\rho_0} D_{K_0}) \otimes \text{id} \) on this image, which gives a Fredholm operator on \( L^2(W_0 \otimes E_{\rho_0})^{\rho_1} \).

**Proposition 5.3.** We have
\[
[M](\rho) = \text{index}(D_{\rho_0, E_{\rho_1}}).
\]
If we write \( \text{index}(D_{\rho_0} + t\varphi^4_{\rho_0} D_{K_0}) = E^+_{\rho_0} - E^-_{\rho_0} \) as an element in the \( K \)-group \( K(pt) \cong \mathbb{Z} \), then we have
\[
[M](\rho) = (E^+_{\rho_0} - E^-_{\rho_0}) \otimes (E^+_{\rho_1} - E^-_{\rho_1}).
\]

**Proof.** This follows from Lemma 5.2 and the fact that the above construction satisfies [8, Assumption 3.14]. \( \square \)

Hereafter we exhibit examples and useful formulas. These examples give local models in the computation in Section 7.

**Example 5.4** (Cylinder). Let \( M_1 \) be the cotangent bundle of the circle \( T^*S^1 \cong \mathbb{R} \times S^1 \) equipped with the standard symplectic structure, almost complex structure and the natural \( S^1 \)-action on the \( S^1 \)-factor. Let \((r, \theta)\) be the coordinate on \( M_1 \). Fix \( \rho \in \text{Irr}(S^1) \cong \mathbb{Z} \) and put
\[
L_{\rho} := M_1 \times \mathbb{C}_\rho,
\]
where \( \mathbb{C}_\rho \) is the one dimensional Hermitian vector space with \( S^1 \)-action of weight \( \rho \). We take a connection \( \nabla \) on \( L_{\rho} \) defined by
\[
\nabla = d - 2\pi \sqrt{-1} \mu(r) dr,
\]
where \( \mu : \mathbb{R} \to \mathbb{R} \) is a smooth non-decreasing \( S^1 \)-invariant function such that
\[
\mu(r) = \begin{cases} r + \rho & (|r| < \frac{1}{4}) \\ \frac{1}{2} + \rho & (|r| > \frac{3}{4}) \end{cases}.
\]
We take a Clifford module bundle \( W_{1,\rho} \) as
\[
W_{1,\rho} = \wedge^* T^*M_1 \otimes L_{\rho} = (\mathbb{C} \oplus \mathbb{C}) \otimes L_{\rho},
\]
with the Clifford action \( c : T^*M_1 \to \text{End}(W_{1,\rho}) \) given by
\[
c(dr) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad c(d\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
These structures give rise a Dolbeault-Dirac operator $D$ and an $S^1$-acyclic orbital Dirac operator $(D_{1,\rho}, \{V_{1,\rho,\tau}\}_{\tau})$ with

$$V_{1,\rho,\tau} = \begin{cases} M_1 \setminus \{0\} \times S^1 & (\tau = \rho) \\ M_1 & (\tau \neq \rho) \end{cases}$$

and all the data satisfy the condition in Proposition 3.4. In particular we have the resulting index as an element in $R^{-\infty}(S^1)$. We denote it by $[M_{1,\rho}]$. By the direct computation one has the following.

**Proposition 5.5.** $[M_{1,\rho}]$ is the delta function supported at $\rho \in \text{Irr}(S^1)$. Namely we have

$$[M_{1,\rho}] : R(S^1) \to \mathbb{Z}, \quad \tau \mapsto \delta_{\rho\tau}.$$  

**Example 5.6** (Vector space). Consider $M_2 = \mathbb{C}$ with the standard $S^1$-action. Let $B_2(0)$ be the open disc centered at the origin with radius $\delta > 0$. Here we take an $S^1$-invariant metric on $M_2$ so that it is standard on $B_2(0)$ and isometric on the outside of $B_2(0)$ to that on the subset of $M_1$ with $r \geq \frac{3}{4}$. Put

$$L_\rho := M_2 \times \mathbb{C}_\rho.$$  

We take a connection $\nabla$ on $L_\rho$ and a Clifford module bundle $W_{2,\rho}$ so that they are standard on $B_2(0)$ and isomorphic to those on $\{r > \frac{3}{4}\} \times S^1 \subset M_1$ in Example 5.4 under the identification between $M_2 \setminus B_2(0)$. These structures give rise a Dirac operator $D$ and an $S^1$-acyclic orbital Dirac operator $(D_{2,\rho}, \{V_{2,\rho,\tau}\}_{\tau})$ with

$$V_{2,\rho,\tau} = \mathbb{C} \setminus \{0\}$$

and all the data satisfy the condition in Proposition 3.4. We denote the resulting index by $[M_{2,\rho}]$. By the direct computation one has the following.

**Proposition 5.7.** $[M_{2,\rho}]$ is the delta function supported at $\rho \in \text{Irr}(S^1)$. Namely we have

$$[M_{2,\rho}] : R(S^1) \to \mathbb{Z}, \quad \tau \mapsto \delta_{\rho\tau}.$$  

**Example 5.8** (Product of cylinders and discs). Let $l, m$ be non-negative integers and $M$ the product of $l$ cylinders and $m$ discs in the previous examples;

$$M := M_1 \times \cdots \times M_1 \times M_2 \times \cdots \times M_2 = (M_1)^l \times (M_2)^m.$$  

There is the natural induced action of $K := (S^1)^{l+m}$ on $M$. We use the natural identifications

$$\text{Irr}(K) = (\text{Irr}(S^1))^{l+m},$$

and

$$R(K) = R(S^1)^{\otimes (l+m)}.$$  

Take $\rho = (\rho_1, \ldots, \rho_l, \rho'_1, \ldots, \rho'_m) \in \text{Irr}(K)$ and consider the corresponding structures $(M_1, W_{1,\rho}, D_{1,\rho}, \{V_{1,\rho,\tau}\}_{\tau \in \text{Irr}(S^1)})$ and $(M_2, W_{2,\rho'}, D_{2,\rho'}, \{V_{2,\rho',\tau}\}_{\tau \in \text{Irr}(S^1)})$. 

Using the outer tensor product we can define the product of the Clifford module bundle
\[ W_\rho := W_{1,\rho_1} \boxtimes \cdots \boxtimes W_{1,\rho_l} \boxtimes W_{2,\rho_1'} \boxtimes \cdots \boxtimes W_{2,\rho_m'} \]
which is a Clifford module bundle over \( M \). The products of them
\[ D_K := D_{1,\rho_1} \boxtimes \cdots \boxtimes D_{1,\rho_l} \boxtimes D_{2,\rho_1'} \boxtimes \cdots \boxtimes D_{2,\rho_m'} \]
and
\[ V_\tau := V_{1,\rho_1,\tau_1} \times \cdots \times V_{1,\rho_l,\tau_l} \times V_{2,\rho_1',\tau_1'} \times \cdots \times V_{2,\rho_m',\tau_m'} \quad (\tau = (\tau_1, \ldots, \tau_l, \tau_1', \ldots, \tau_m') \in \text{Irr}(K)) \]
induce a \( K \)-acyclic orbital Dirac operator on \( M \), where for operators \( A : \mathcal{H}_0 \to \mathcal{H}_0 \) and \( B : \mathcal{H}_1 \to \mathcal{H}_1 \) on \( \mathbb{Z}/2 \)-graded Hilbert spaces their product \( A \boxtimes B : \mathcal{H}_0 \otimes \mathcal{H}_1 \to \mathcal{H}_0 \otimes \mathcal{H}_1 \) is defined by
\[ A \boxtimes B := A \otimes \text{id} + \alpha_0 \otimes B \]
with the grading operator \( \alpha_0 \) of \( \mathcal{H}_0 \). This structure \( (D_K, \{V_\tau\}) \) also satisfies the condition in Proposition 3.4, in particular we have the resulting index \( [M_\rho] \in R^{-\infty}(K) \). The product formula (Proposition 5.3) implies the following equality.

**Proposition 5.9.** We have
\[ [M_\rho] = [M_{1,\rho_1}] \otimes \cdots \otimes [M_{1,\rho_l}] \otimes [M_{2,\rho_1'}] \otimes \cdots \otimes [M_{2,\rho_m'}]. \]
Namely \( [M_\rho] \) is the delta function supported at \( \rho \in \text{Irr}(K) \).

This structure serves as a local model of a neighborhood of the fiber of symplectic toric manifold in Section 7.2.

**5.2. Fiber bundle over a closed manifold.** Let \( X \) be a closed Riemannian manifold, \( E \to X \) a \( \mathbb{Z}/2 \)-graded Clifford module bundle over \( X \) and \( P \to X \) a principal \( G \)-bundle for a compact Lie group \( G \). Consider a \( K \)-acyclic orbital Dirac operator \( (D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)}) \) on \( (M, W) \) as in Proposition 3.4. Suppose that \( G \times K \) acts on \( W \to M \) in an isometric way and \( (D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)}) \) is \( G \)-invariant. Consider the diagonal action of \( G \) on \( P \times M \) and the quotient manifold
\[ \tilde{M} := (P \times M)/G, \]
which has a structure of \( M \)-bundle \( \pi : \tilde{M} \to X \). Let \( \tilde{W} \to \tilde{M} \) be the vector bundle defined by
\[ \tilde{W} := \pi^* E \otimes ((P \times W)/G), \]
which has a structure of a Clifford module bundle over \( \tilde{M} \) by using an appropriate connection of \( P \). One can define operators \( \tilde{D}_W \) and \( \tilde{D}_E \) on \( \tilde{W} \) as lifts (by using a trivialization of \( P \) and a partition of unity if necessary) of Dirac operators \( D_W \) on \( W \) and \( D_E \) on \( E \). Then
\[ \tilde{D} := \tilde{D}_E + \tilde{D}_W \]
is a Dirac operator on \( \tilde{W} \).
For \( \rho \in \text{Irr}(K) \) let \( \tilde{V}_\rho \) be the open subset defined by
\[
\tilde{V}_\rho := (P \times V_\rho)/G
\]
whose complement in \( \tilde{M} \) is compact. \( D_K \) induces an operator \( \tilde{D}_K \) on \( \tilde{W} \). One can see that \((\tilde{D}_K, \{\tilde{V}_\rho\}_{\rho \in \text{Irr}(K)})\) is a \( K \)-acyclic orbital Dirac operator on \((\tilde{M}, \tilde{W})\). In particular for \( \rho \in \text{Irr}(K) \) we have a Fredholm operator
\[
\tilde{D}_\rho = \tilde{D} + \tilde{t}\tilde{\varphi}_\rho^4 \tilde{D}_K
\]
on \( L^2(\tilde{W})^{(\rho)} \), where \( \tilde{\varphi}_\rho : \tilde{M} \to [0, 1] \) is the cut-off function induced from the cut-off function \( \varphi_\rho \) on \( M \) as in \((2.1)\). In this way we have an element \([\tilde{M}] \in R^{-\infty}(K)\) defined by
\[
[\tilde{M}](\rho) := \text{index}(\tilde{D}_\rho).
\]

Now consider the Fredholm operator \( D_W + t\varphi_\rho^4 D_K \) on \( L^2(W)^{(\rho)} \) and we put its kernel
\[
E_\rho := \ker(D_W + t\varphi_\rho^4 D_K) = E_\rho^+ \oplus E_\rho^-
\]
as the \( \mathbb{Z}/2 \)-graded finite dimensional vector space. Then there is a natural embedding
\[
L^2(E \otimes E_\rho) \to L^2(\tilde{W})^{(\rho)}
\]
whose image is preserved by \( \tilde{D}_E \). Let \( D_{E,\rho} \) be the restriction of \( \tilde{D}_E \) on this image, which gives a Fredholm operator on \( L^2(E \otimes E_\rho) \) because the symbol of \( D_{E,\rho} \) is equal to the tensor product of \( \text{id}_{E_\rho} \) and the symbol of \( D_E \), in particular it is an elliptic operator on the closed manifold \( X \).

**Proposition 5.10.** For each \( \rho \in \text{Irr}(K) \) we have
\[
[\tilde{M}](\rho) = \text{index}(D_{E,\rho}).
\]
If we write \( \text{index}(D_E) = E_0^+ - E_0^- \) as an element in the \( K \)-group \( K(pt) \cong \mathbb{Z} \), then we have
\[
[\tilde{M}](\rho) = (E_0^+ - E_0^-) \otimes (E_\rho^+ - E_\rho^-).
\]

**Proof.** This follows from the fact that the above construction satisfies [3] Assumption 3.14. \( \square \)

**Example 5.11.** Let \( K \) be a torus. Consider \( M = T^*K \) with the \( K \)-acyclic orbital Dirac operator \((D_K, \{V_\rho\}_{\rho \in \text{Irr}(K)})\) defined as the product of Example 5.4. Suppose that we take a Clifford module bundle by using \( \mathbb{C}_\rho \) for a fixed \( \rho \in \text{Irr}(K) \). Then we have
\[
[M] : R(K) \to \mathbb{Z}, \quad \rho' \mapsto \delta_{\rho\rho'}.
\]

Let \( X \) be a closed Riemannian manifold, \( E \to X \) a Clifford module bundle and \( P \to X \) a principal \( K \)-bundle. Let \( \tilde{M} \) be the \( M \)-bundle over \( X \) defined by
\[
\tilde{M} = (P \times M)/K.
\]
Proposition 5.10 ensures us that

\[ [\tilde{M}] : R(K) \to \mathbb{Z}, \quad \rho' \mapsto \text{index}(E)\delta_{\rho \rho'}, \]

where \( \text{index}(E) \) is the index of a Dirac operator on \( E \). This example serves as a local model of a neighborhood of the inverse image of the moment map of Hamiltonian torus action in Section 7.1.

6. Vanishing theorem for fixed points

In this subsection we consider a \( K \)-acyclic orbital Dirac operator on a Hermitian manifold \( M \) with a \( K \)-equivariant line bundle \( L \to M \) as in Section 3. We put the following additional assumptions.

- For a compact Lie group \( H \) there exists a \( H \times K \)-action on \( M \) and all the additional data are \( H \times K \)-equivariant.
- The fixed point set \( M^K \) is a closed connected submanifold of \( M \).
- The fixed point set \( L^K \) is equal to the image of \( M^K \) in \( L|_{M^K} \) by the zero section.

We show the following vanishing theorem for our index, which is a modification of [9, Theorem 6.1] and plays an important role in the subsequent section.

We use the notation \( L_\rho := L \otimes \mathbb{C}_\rho \) for \( \rho \in \text{Irr}(K) \), where \( \mathbb{C}_\rho \) is the one-dimensional irreducible representation of \( K \) with weight \( \rho \).

**Theorem 6.1.** We take the Clifford module bundle \( W_\rho = \wedge^\bullet T_0 M \otimes L_\rho \) for \( \rho \in \text{Irr}(K) \). If \( \mu^{-1}(\rho) = \emptyset \), and hence \( V_\rho = M \setminus M^K \), then we have

\[ \text{index}_H(\hat{D}_\rho) = 0 \in R(H). \]

To show it we show a rank reducing lemma. Suppose that there exists a subtorus \( K' \) of \( K \) and \( \rho' \in \text{Irr}(K') \) such that the following conditions are satisfied.

- \( M^{K'} \) is compact.
- \( (\iota_{K'}^* \circ \mu)^{-1}(\rho') = \emptyset \), where \( \iota_{K'}^* : \mathfrak{k}^* \to (\mathfrak{t}')^* \) is the dual of the inclusion map.
- The restriction of \( \rho \) to \( K' \)-action is \( \rho' \), i.e., \( \iota_{K'}^*(\rho) = \rho' \).
- The differential operator

\[ D_{K'} = \sum_{i=1}^{\dim K'} c(\xi_i^{K'}) (\mathcal{L}_{\xi_i} - \sqrt{-1}\mu_i) \]

satisfies all the conditions in Definition 2.1 on \( V_{\rho'} := M \setminus M^{K'} \).

The deformation \( \hat{D}_{\rho'} = D + t\varphi_{\rho'}^* D_{K'} \) gives a Fredholm operator on the isotypic component \( L^2(W)^{(\rho')} \) for \( t \gg 0 \). On the other hand the condition \( \iota_{K'}^*(\rho) = \rho' \) implies that \( L^2(W)^{(\rho)} \) is a subspace of \( L^2(W)^{(\rho')} \) and \( \hat{D}_{\rho'}^{(\rho')} \) preserves it. We define \( \text{index}(\hat{D}_{\rho', \rho}) \) as its Fredholm index;

\[ \text{index}(\hat{D}_{\rho', \rho}) := \text{index}(\hat{D}_{\rho'}^{(\rho')} : L^2(W)^{(\rho)} \to L^2(W)^{(\rho')}). \]
Note that the facts $V_{\rho'} \subset V_{\rho}$ and the excision formula (2.3) imply that the index $\text{index}(\hat{D}_\rho) = \text{index}((\hat{D}_\rho)^{(\rho)} : L^2(W)^{(\rho)} \to L^2(W)^{(\rho)})$ does not change when we take $V_{\rho'}$ instead of $V_{\rho}$. We can incorporate $H$-action and regard them as $H$-equivariant indices $\text{index}_H(\cdot)$.

**Lemma 6.2.** $\text{index}_H(\hat{D}_\rho) = \text{index}_H(\hat{D}_{\rho',\rho}) \in R(H)$.

*Proof.* Note that for $K'$, $\rho'$ and $\rho$ satisfying the assumption in this lemma any subtorus $K''$ of $K$ and $\rho'' \in \text{Irr}(K')$ with $K' \subset K''$ and $\iota'_{K'}(\rho'') = \rho'$ satisfy the same assumption. In particular we may assume that $n = \dim(K) = \dim(K') + 1$.

In this case we take

$$D_K = \sum_{i=1}^{n} c(\xi_i^M)(\mathcal{L}_{\xi_i} - \sqrt{-1}\mu_i), \quad D_{K'} = \sum_{i=1}^{n-1} c(\xi_i^M)(\mathcal{L}_{\xi_i} - \sqrt{-1}\mu_i).$$

It suffices to show that

$$\mathbb{D}_c := D + t\varphi^{A}_{\rho}(D_{K'} + cc(\xi^M_n)(\mathcal{L}_{\xi_n} - \sqrt{-1}\mu_n))$$

satisfies the coercivity on $L^2(W)^{(\rho)}$, that is,

$$\|\mathbb{D}_c s\|_W \geq \|s\|_W$$

holds for any $s \in \Gamma_c(W)^{(\rho)}$ with $\text{supp}(s) \subset \text{int}(\varphi^{-1}_\rho(1)) \cap V_{\rho'}$, $c \in [0,1]$ and sufficiently large $t > 0$. To see it one has

$$\mathbb{D}_c^2 = D^2 + \sqrt{-1}t \left( \sum_{i=1}^{n} \left[ D, c(\xi_i^M)(\rho(\xi_i) - \mu_i) \right] + \epsilon \left[ D, c(\xi_i^M)(\rho(\xi_n) - \mu_n) \right] \right)$$

$$+ t^2 \left( \sum_{i=1}^{n-1} |\xi_i^M|^2(\rho(\xi_i) - \mu_i)^2 + c^2|\xi_n^M|^2(\rho(\xi_n) - \mu_n)^2 \right)$$

on $\text{int}(\varphi^{-1}_\rho(1))$, where $[A,B] := AB + BA$ is the commutator. Since one has

$$\sum_{i=1}^{n-1} |\xi_i^M|^2(\rho(\xi_i) - \mu_i)^2 + c^2|\xi_n^M|^2(\rho(\xi_n) - \mu_n)^2 \geq \sum_{i=1}^{n-1} |\xi_i^M|^2(\rho(\xi_i) - \mu_i)^2$$

and the right hand side is uniformly positive outside a compact subset, the almost same argument in the proof of Theorem 6.1 shows that

$$\|\mathbb{D}_c s\|_W^2 \geq \|s\|_W^2$$

for $t$ large enough. \hfill \Box

**Proposition 6.3.** Theorem 6.1 is true when $M$ is a small open disc around the origin of a Hermitian vector space on which $K$ acts in an isometric way with $M^K$ consists of the origin.

*Proof.* By considering the tensor product it suffices to prove in the case that $\rho$ is the trivial representation $0$. We can choose an appropriate circle subgroup $K_1$ of $K$ so that the $K_1$ acts on $M$ with $M^{K_1} = \{0\}$ and the $K_1$-action on $L|_0$ is nontrivial. In fact let $\rho_1, \ldots, \rho_{\dim M} \in \text{Irr}(K)$ be the weights appeared in $M$, all of
which are non-zero by the assumption $M^K = \{0\}$, then we can take a splitting of the differential of the representation $K \to U(1)$ on $L|_0$ such that the image of the splitting in $\mathfrak{k}$ is rational and is not perpendicular to any $\rho_i$. The subgroup of the image gives the desired circle subgroup. By Lemma 6.2 we have

$$\text{index}(\hat{D}_0) = \text{index}(D + t\varphi^A_0 D_K) = \text{index}(D + t\varphi^A_0 D_{K_i}) \in \mathbb{Z}.$$ 

On the other hand [9, Proposition 6.8] and Theorem 2.7 imply

$$\text{index}(D + t\varphi^A_0 D_{K_i}) = 0,$$

and we complete the proof. □

**Proof of Theorem 6.1.** The claim follows from Proposition 6.3 and the product formula (Proposition 5.10) with the same argument in [9, Section 6.4]. □

### 7. Quantization of proper Hamiltonian torus manifolds

In this section by using the ingredients established in the previous sections we define quantization of proper Hamiltonian torus manifolds possibly with non-compact fixed point sets.

Let $K$ be a torus and $M$ be a Hamiltonian $K$-manifolds with $K$-equivariant prequantizing line bundle $(L, \nabla)$. We assume that the moment map $\mu : M \to \mathfrak{k}$ is proper. Take a regular value $r > 0$ of $|\mu|^2$ and put

$$X_r := (|\mu|^2)^{-1}([0, r]) = \{ x \in M \mid |\mu(x)|^2 \leq r \}.$$

Since $\mu$ is proper $X_r$ is a $K$-invariant compact manifold with $K$-invariant boundary $\partial X_r$. We may assume that $\mu(\partial X_r)$ does not contain integral lattice points, in particular $(\partial X_r)^K = \emptyset$. By taking a completion $\tilde{X}_r$ of $X_r$ so that it has cylindrical end $\partial X_r \times \mathbb{R}_{>0}$. Note that we use a $K$-invariant compatible almost complex structure on the outside of a small neighborhood of the boundary $\partial X_r$.

We also have data associated with $(L, \nabla)$, $W_L$ and $\mu$ which have translationally invariance on $\partial X_r \times \mathbb{R}_{>0}$, which will be denoted by the notations as $\tilde{\cdot}$.

Since $(\tilde{X}_r)^K$ is compact we have a $K$-homology cycle and its equivariant index

$$(7.1) \quad [\tilde{X}_r] := [L^2(\tilde{W}_L), F] = [\hat{D}] \in K^0(C^*(K)) = R^{-\infty}(K),$$

where $F$ is the bounded transformation defined as in (2.2).

**Proposition 7.1.** We have the following.

1. For $\rho \in \text{Irr}(K)$ with $|\rho| > r$ we have $[\tilde{X}_r](\rho) = 0$. In particular $[\tilde{X}_r]$ is an element in $R(K) \subset R^{-\infty}(K)$.

2. For $r' > r > 0$ and $\rho \in \text{Irr}(K)$ with $|\rho| < r$ we have

$$[\tilde{X}_{r'}](\rho) = [\tilde{X}_r](\rho) \in \mathbb{Z}.$$
Proof. (1) Since we define $[\tilde{X}_r]$ by the $K$-acyclic orbital Dirac-type operator $D_K$ with open subsets 
$$
\tilde{V}_\rho := \tilde{X}_r \setminus (\tilde{\mu}^{-1}(\rho) \cup (\tilde{X}_r)^K),
$$
the localization formula (Theorem 3.5) implies that each index $[\tilde{X}_r](\rho) = \text{index}((\tilde{D} + \tilde{j}_D D_K)(\rho))$ has a localized description on the compact set $\tilde{\mu}^{-1}(\rho) \cup (\tilde{X}_r)^K = \tilde{\mu}^{-1}(\rho) \cup (X_r)^K$. We denote it by

$$
(7.2) \quad [\tilde{X}_r](\rho) = [\tilde{\mu}^{-1}(\rho)] + \sum_{\rho \neq \rho' \in \text{Irr}(K)} [(X_r)^{K\rho}] \in \mathbb{Z},
$$

where $[\tilde{\mu}^{-1}(\rho)]$ and $[(X_r)^{K\rho}]$ are indices defined by the deformation by the restriction of $D_K$ to small neighborhoods of $\tilde{\mu}^{-1}(\rho)$ and $(X_r)^K \cap \tilde{\mu}^{-1}(\rho')$ respectively. Note that if $|\rho| > r$ then we may take $\tilde{\mu} : \tilde{X}_r \to \mathfrak{t}^*$ so that $\tilde{\mu}^{-1}(\rho) = \emptyset$. In particular we have

$$
[\tilde{X}_r](\rho) = \sum_{|\rho'| \leq r} [(X_r)^{K\rho'}].
$$

On the other hand since for $x \in M^K$ it is known that $\mu(x) \in \Lambda^*$ is equal to the 1-dimensional representation $L_x$ Theorem 6.1 imply that $[(X_r)^{K\rho}] = 0$, and hence, we have $[\tilde{X}_r](\rho) = 0$. (2) follows from the localization formula (Theorem 3.5) and (1).

Proposition 7.1 enable us to give the following definition.

**Definition 7.2.** We define the quantization

$$
Q_K(M) = \lim_{r \to \infty} [\tilde{X}_r] \in R^- \infty(K)
$$

of $M$ by

$$
Q_K(M)(\rho) := [\tilde{X}_r](\rho) \in \mathbb{Z} \quad (r > |\rho|)
$$

for $\rho \in \text{Irr}(K)$. We may write this element as

$$
Q_K(M) = \sum_{\rho \in \mu(M)^-} [\mu^{-1}(\rho)]^*,
$$

where $\mu(M)^- := \mu(M) \cap \Lambda^*$ is the set of all integral lattice points in $\mu(M)$.

Our quantization $Q_K(M)$ is a generalization of $K$-equivariant spin$^c$ quantization using the index of Dirac operator in the compact case, which is often denoted by $RR_K(M)$ and called the equivariant Riemann-Roch number or Riemann-Roch character.

**Remark 7.3.** We use the properness of $\mu$ to take an increasing family of compact manifolds with boundary $\{X_r\}_{r>0}$. In fact even though $\mu$ is not proper if $\mu^{-1}(\rho)$ is compact for each $\rho \in \text{Irr}(K)$, then by taking a relatively compact neighborhood $Y_\rho$ of $\mu^{-1}(\rho)$ we can define an element $[Y_\rho] \in R(K) \subset R^- \infty(K)$. One can see that $[Y_\rho]$ has its support only at $\rho$, that is, $[Y_\rho](\rho') = 0 \in \mathbb{Z}$ for any $\rho' \neq \rho$ by the
same argument in the proof of Proposition 7.1. In this case we can also define the element such as

$$Q_K(M) = \sum_{\rho \in \mu(M)_{T^*}} [\mu^{-1}(\rho)]^* \in \mathbb{R}^{\infty}(K).$$

Essentially the author constructed this element for $K = S^1$ in [10].

7.1. $[Q,R]=0$ for proper Hamiltonian torus manifolds. Our equivariant index $[M]$ has a localization property due to its origin. It enables us to give a natural proof of the $[Q,R]=0$ type theorem for our quantization.

For a regular value $\xi \in \mathfrak{k}^*$ of $\mu : M \rightarrow \mathfrak{k}^*$ let $M_\xi$ be the symplectic quotient at $\xi$:

$$M_\xi := \mu^{-1}(\xi)/K,$$

which is a closed symplectic manifold (orbifold) by the properness of $\mu$. Moreover if $\rho \in \text{Irr}(K)$ is a regular value, then there exists a natural prequantizing line bundle over $M_\rho$, and hence, one can define the Riemann-Roch number $RR(M_\rho)$ as the index of the Dolbeault-Dirac operator associated with a $K$-invariant compatible almost complex structure.

Theorem 7.4. Suppose that $\rho \in \text{Irr}(K)$ is a regular value of the moment map $\mu : M \rightarrow \mathfrak{k}^*$. Then we have

$$Q_K(M)(\rho) = RR(M_\rho).$$

Proof. Note that

$$Q_K(M)(\rho) = [\mu^{-1}(\rho)]$$

and a neighborhood of $\mu^{-1}(\rho)$ in $M$ can be identified with the product

$$(T^*K \times \mu^{-1}(\rho))/K$$

by the Darboux-type theorem (see [13] Lemma 7.1 for example), which has a structure of $T^*K$-bundle over $M_\rho$. By applying the product formula in Example 5.11 we have

$$[\mu^{-1}(\rho)] = RR(M_\rho).$$

□

Remark 7.5. Due to Corollary 4.4 the quantization $Q_K(M)$ can be identified with Atiyah’s transverse index. Theorem 7.4 gives an alternative proof of Vergne’s conjecture to Ma-Zhang proof in [21] which uses Braverman’s deformation.

7.2. A Danilov-type formula for proper toric manifolds. Now we focus on the symplectic toric case. Namely we assume that $2 \dim(K) = \dim(M)$. In this case the localization property of $Q_K(M)$ leads us to a non-compact version of Danilov’s theorem, which asserts that the quantization of symplectic toric manifold is localized at the lattice points in the momentum polytope.
Theorem 7.6.

\[ \mathcal{Q}_K(M) = \sum_{\rho \in \mu(M)\mathbb{Z}} \rho^*, \]

where the right hand side is an element in \( R^{-\infty}(K) \) which is characterized by

\[ \text{Irr}(K) \ni \rho' \mapsto \begin{cases} 
1 & (\rho' \in \mu(M)\mathbb{Z}) \\
0 & (\rho' \notin \mu(M)\mathbb{Z}).
\end{cases} \]

**Remark 7.7.** In a general framework of geometric quantization one uses an additional structure called a polarization, which is an integrable Lagrangian distribution of the complexification of the tangent bundle. One typical example is a Kähler polarization which is defined as a compatible complex structure. Our quantization is based on the spin\(^c\) quantization, which is a polarization relaxed the integrality condition in the Kähler polarization. The quantization is given by the Fredholm index of the Dolbeault-Dirac operator. The other example is a real polarization, which is defined by the tangent bundle along fibers of the Lagrangian fibration. In the real polarization case it is known that the quantization can be described by Bohr-Sommerfeld fibers, which are characterized by the existence of non-trivial global parallel sections of the prequantizing line bundle on the orbits. The moment map of toric manifolds can be regarded as a real polarization with singular fibers. In the toric case, the Bohr-Sommerfeld fibers are nothing other than the inverse images of the integral lattice points in the momentum polytope. One important topic in geometric quantization is the problem of independence from the polarizations. There are several results supporting the coincidence between the quantizations obtained by the spin\(^c\) polarization and the real polarization from the view point of index theory, such as [1], [7] and [16]. Theorem 7.6 can be considered as a non-compact version of the above results.

Theorem 7.6 follows from the computation of the local contribution \([\mu^{-1}(\rho)]^*\).

**Proposition 7.8.** For each \( \rho \in \text{Irr}(K) \) we have

\[ \mathcal{Q}_K(M)(\rho) = \begin{cases} 
1 & (\rho \in \mu(M)\mathbb{Z}) \\
0 & (\rho \notin \mu(M)\mathbb{Z}).
\end{cases} \]

**Proof.** This follows from the local uniqueness of the neighborhood of \( \mu^{-1}(\rho) \) (see [14, Appendix B] for example), the formula (Proposition 5.9) derived from the product formula. These arguments are same as those in [11, Section 6.1]. \( \square \)

**Remark 7.9.** In [11] we gave a proof of Danilov’s formula for compact symplectic toric manifolds (or more generally for toric origami manifolds) using a localization formula based on the theory of the acyclic compatible system developed in [3]. Since one can see that the acyclic compatible system constructed in a natural way on a given toric manifold does not have a product structure in general, we cannot apply
the product formula directly and have to compare the resulting index with the
index of the product. It is one remarkable difference in the proof of Proposition 7.8
that our deformation by $D_K$ fits into the local product structure of a neighborhood
of $\mu^{-1}(\rho)$. In particular we can apply the product formula directly.

8. Comments and further discussions

8.1. Application to quantization of Hamiltonian loop group spaces. Quan-
tization of Hamiltonian loop group spaces is studied in various directions. In par-
ticular Loizides-Song [20] studied it from the viewpoint of index theory and KK-
theory. Their construction is based on their previous work [17] with Meinrenken in
which they constructed a spinor bundle over a proper Hamiltonian loop group space
and a nice finite dimensional non-compact submanifold in it, which is transverse
to the orbits of the loop group action. One key ingredient in [20] is to associate
a K-homology cycle to such a non-compact manifold. They established an index
theory using the $C^*$-algebraic condition which they call the $(\Gamma, K)$-admissibility,
where $K$ is a compact Lie group and $\Gamma$ is a countable discrete group with proper
length function. They showed that in the proper Hamiltonian loop group space case
the $(\Lambda, T)$-admissibility is satisfied for a maximal torus $T$ of $K$, and the resulting
K-homology class has an anti-symmetric property with respect to some Weyl group
action of $K$, which gives rise quantization as an element in the fusion ring of $K$.

In this paper we constructed a similar K-homology class without using $(\Gamma, K)$-
admissibility. In the subsequent research we will investigate an approach of quanti-
zation of Hamiltonian loop group spaces by incorporating the action of the integral
lattice $\Lambda$ in our construction appropriately. In such an approach it would be inter-
esting to understand how the localization phenomenon of our index is reflected in
the quantization of loop group spaces.

There is another related work by Takata. In [25] an $LS^1$-equivariant index
is constructed as an element in the fusion ring from the viewpoint of KK-theory
and non-commutative geometry. He also developed an index theorem in infinite
dimensional setting in [23,24]. It would be also interesting to investigate how our
construction is positioned in Takata’s theory.

8.2. Deformation as KK-products. Motivated by the pioneering work by Kasp-
parov [15], Loizides-Rodsphon-Song showed in [18] that the K-homology class ob-
tained by Braverman’s deformation factors as a KK-product between the Dirac
class and a KK-class arising from the deformation. It is desirable to understand
our deformation by the acyclic orbital Dirac operator as a KK-product.

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