Sharp bounds for Hardy type operators on higher-dimensional product spaces

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Abstract  In this paper, we investigate a class of fractional Hardy type operators \( H_{\beta_1,\ldots,\beta_m} \) defined on higher-dimensional product spaces \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \). We use novel methods to obtain two main results. One is that the operator \( H_{\beta_1,\ldots,\beta_m} \) is bounded from \( L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\gamma) \) to \( L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\alpha) \) and the sharp bound of the operator \( H_{\beta_1,\ldots,\beta_m} \) is worked out. The other is that when \( \alpha = \gamma = (0,\ldots,0) \), the norm of the operator \( H_{\beta_1,\ldots,\beta_m} \) is obtained.

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1 Introduction
Let \( f \) be non-negative measurable function on the \( m \)-fold product space \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \), the Hardy type operator \( H_{\beta_1,\beta_2,\ldots,\beta_m} \) is defined by

\[
H_{\beta_1,\ldots,\beta_m}(f)(x) := \prod_{i=1}^{m} \frac{1}{|B(0,|x_i|)|^{\frac{\beta_i}{n_i}}} \int_{|y_1|<|x_1|} \cdots \int_{|y_m|<|x_m|} f(y_1,\ldots,y_m)dy_1 \cdots dy_m, \quad (1.1)
\]
where \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \) with \( \prod_{i=1}^{m} |x_i| \neq 0 \) and \( 0 \leq \beta_i < n_i \) for \( i = 1, 2, \ldots, m \).

Obviously, the operator \( H_{\beta_1,\beta_2,\ldots,\beta_m} \) is natural generalization of the classical Hardy operators, such as the operator \( H \) \[3, 4\], the fractional Hardy operator \( H_{\beta} \) \[7\], the Hardy operator \( \mathcal{H} \) on the \( n \)-dimensional product space \[11\] and the Hardy type operator \( \mathcal{H}_{m} \) on \( m \)-dimensional product spaces \[6\].

If \( f \) is a non-negative measurable function on \( G = (0,\infty) \), the classical Hardy operator defined as

\[
H(f)(x) := \frac{1}{x} \int_{0}^{x} f(t)dt, \quad x > 0.
\]

The following Theorem A due to Hardy \[3, 4\] is well known.

**Theorem A**  If \( f \) is a non-negative measurable function on \( G \), let \( 1 < p < \infty \) and \( \alpha < p - 1 \), then the following two inequalities

\[
\|H(f)\|_{L^p(G)} \leq \frac{p}{p-1} \|f\|_{L^p(G)} \quad \text{and} \quad \|H(f)\|_{L^p(G,x^\alpha)} \leq \frac{p}{p-1-\alpha} \|f\|_{L^p(G,x^\alpha)}
\]
If we refer to ([1],[2],[5],[8],[9]) and references therein.

weight on higher-dimensional product spaces. For more information about the Hardy type operator, the method in [10] to the case dimensional greater than two. In [11], Wang, Lu and Yan studied the weight functions needs to be proved. Our results in the following will show that sharp constant in (1.5) is not equal to holds, where the constant of inequality in (1.3) is sharp.

In 2015, Lu and Zhao considered the operator defined by (1.2) and obtained the following Theorem B.

Theorem B If \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \), let \( 0 < \beta < n \), \( 1 < p < q < \infty \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \), then the inequality

\[
\|H(f)\|_{L^q} \leq \left( \frac{p'}{q} \right)^{1/q} \left( \frac{n}{q\beta} \cdot B \left( \frac{n}{q\beta}, \frac{n}{q\beta} \right) \right)^{-\beta/n} \|f\|_{L^p}
\]

holds, where the constant of inequality in (1.3) is sharp.

For the weighted case, in 1985, Sawyer [10] considered the weighted Hardy inequalities with general weight functions \( u \) and \( v \) only on two-dimensional product space. However, it is much hard to apply the method in [10] to the case dimensional greater than two. In [11], Wang, Lu and Yan studied the Hardy operator \( H \) with power weight on the \( m \)-dimensional case, and obtained the following result.

Theorem C If \( f \) is a nonnegative measurable function on \( \mathbb{G}^m \) and \( x = (x_1, x_2, \cdots, x_m) \in \mathbb{G}^m \), let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \), \( \beta = (\beta_1, \beta_2, \cdots, \beta_m) \), \( x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} \), \( x^\beta = x_1^{\beta_1}x_2^{\beta_2}\cdots x_m^{\beta_m} \), \( 1 < p < q < \infty \), \( \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{\beta} \), \( \alpha_i < p - 1 \) and \( \beta_i + 1 - q = \frac{\alpha_i + 1}{p} \) with \( 1 \leq i \leq m \), then

\[
\|Hf\|_{L^p(G^m,x^\alpha)} \leq \left( \prod_{i=1}^{m} \frac{p}{p - \alpha_i - 1} \right) \|f\|_{L^p(G^m,x^\alpha)}
\]

and

\[
\|Hf\|_{L^q(G^m,x^\beta)} \leq \left( \prod_{i=1}^{m} \frac{q}{r(q - \beta_i - 1)} \right)^{1/r} \|f\|_{L^p(G^m,x^\alpha)},
\]

where the operator \( H \) is the fractional Hardy type operator \( H_{\beta_1, \beta_2, \cdots, \beta_m} \) with \( \beta_1 = \cdots = \beta_m = 0 \) and \( n_1 = \cdots = n_m = 1 \). The authors [11] not sure the constant \( \left( \prod_{i=1}^{m} \frac{q}{r(q - \beta_i - 1)} \right)^{1/r} \) is sharp. However, it needs to be proved. Our results in the following will show that sharp constant in (1.5) is not equal to \( \left( \prod_{i=1}^{m} \frac{q}{r(q - \beta_i - 1)} \right)^{1/r} \).

For another high dimensional case with power weight, Lu, Yan and Zhao in [9] studied the operator \( H_m \) is definition of \( H_{\beta_1, \beta_2, \cdots, \beta_m} \) with \( \beta_1 = \beta_2 = \cdots = \beta_m = 0 \) and obtained sharp constant \( \prod_{j=1}^{m} \frac{p}{p - 1 - \alpha_j/n_j} \).

A natural question is to consider the case of higher-dimensional product space for fractional Hardy type operator \( H_{\beta_1, \beta_2, \cdots, \beta_m} \). We will use novel methods and ideas to study Hardy operator with power weight on higher-dimensional product spaces. For more information about the Hardy type operator, we refer to ([1],[2],[3],[4],[5]) and references therein.

Now, we formulate our main results as follows.
Theorem 1.1 Suppose that $1 < p < q < \infty$, $0 < \beta_i < n_i$ and $\frac{1}{p} = \frac{1}{p} - \frac{\beta_i}{n_i}$ with $1 \leq i \leq m$. If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})$, then we have

$$\|\mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}(f)\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})} \leq C \|f\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})}. \quad (1.6)$$

Moreover,

$$\|\mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}) \rightarrow L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})} = C,$$

where

$$C = \prod_{i=1}^{m} \left(\frac{p}{q}^i\right)^{1/q} \left(\frac{n_i}{q\beta_i} \cdot B\left(\frac{n_i}{q\beta_i}, \frac{n_i}{q\beta_i}\right)\right)^{-\beta_i/n_i}.$$

For two differences power weight, we have following result.

Theorem 1.2 Suppose that $1 < p \leq q < \infty$, $m \in \mathbb{N}$, $n_i \in \mathbb{N}$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $x_i \in \mathbb{R}^{n_i}$, $0 \leq \beta_i < n_i$, $\gamma_i < n_i(p - 1)$ and $\beta_i + \alpha_i/n_i = \frac{\gamma_i + n_i}{p}$ with $1 \leq i \leq m$. If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|\gamma)$ with $|x|^\gamma = |x_1|\gamma_1 |x_2|\gamma_2 \cdots |x_m|\gamma_m$, then we have

$$\|\mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}(f)\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\gamma)} \leq C^* \|f\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\gamma)}, \quad (1.7)$$

Moreover,

$$\|\mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\gamma) \rightarrow L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^\gamma)} = C^*,$$

where $C^*$ is equal to

$$\prod_{i=1}^{m} \left|S_i^{n_i-1}\right|^{-\frac{1}{q} - \frac{1}{\gamma} + \frac{n_i}{q} - \frac{1}{q} - \frac{\beta_i}{n_i} } \left(\frac{n_i(p - 1)}{n_i(p - 1) - \gamma_i}\right)^{\frac{1}{p} + \frac{1}{q}} \left(\frac{p}{q}\right)^{\frac{1}{q}} \left(\frac{p}{q - p} \cdot B\left(\frac{p}{q - p}, \frac{pq}{q(q - p)}\right)\right)^{\frac{1}{q} - \frac{1}{p}}.$$

It is worth mentioning that that proof in [11] is not suitable to the operator $\mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}$. Although the idea in the paper is motivated by the reference [9], there are some essential differences. The difficulty is how to deal with the product space case. In this paper we will use the novel method to become as result in [7]. The reconstruct some auxiliary functions to achieve the sharp bounds, which is quite different from [9].

Throughout the note, we use the following notation. The definition of the usual beta function is defined by

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt,$$

where $z$ and $w$ are complex numbers with the positive real parts. The set $B(0, |x|)$ denotes a open ball with center at the original point and radius $|x|$, and $|B(0, |x|)|$ denotes the volume of the ball $B(0, |x|)$. For one $m$-dimensional vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $|x|^\alpha = |x_1|^{\alpha_1} |x_2|^{\alpha_2} \cdots |x_m|^{\alpha_m}$, $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}$. For a real number $p$, $1 < p < \infty$, $p'$ is the conjugate number of $p$, that is, $1/p + 1/p' = 1$.

2 Preliminaries

To reduce the dimension of function space, we need the following lemma which was obtained by some ideas and methods used in [6].
Lemma 2.1 Suppose that \( f \in L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_m, |x|) \) with \( |x| = |x_1|^{\gamma_1} |x_2|^{\gamma_2} \cdots |x_m|^{\gamma_m} \). Let
\[
gf(x) = \frac{1}{\omega_{n_1}} \frac{1}{\omega_{n_2}} \cdots \frac{1}{\omega_{n_m}} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \cdots \int_{|\xi_m|=1} f(|x_1|\xi_1, |x_2|\xi_2, \cdots, |x_m|\xi_m) d\xi_1 d\xi_2 \cdots d\xi_m,
\]
where \( x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_m \), \( \omega_n = 2\pi^{\frac{n}{2}}/\Gamma(n_i/2) \) with \( 1 \leq i \leq m \). Then
\[
\mathcal{H}_{\beta_1, \beta_2, \cdots, \beta_m}(|f|)(x) = \mathcal{H}_{\beta_1, \beta_2, \cdots, \beta_m}(gf)(x)
\]
and
\[
\|gf\|_{L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_m, |x|)} \leq \|f\|_{L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_m, |x|)}.
\]

Proof. We merely the proof with the case \( m = 2 \) for the sake of clarity in writing, and the same is true for the general case \( m > 2 \).

It follows that \( \mathcal{H}_{\beta_1, \beta_2}(gf)(x_1, x_2) \) is equal to
\[
\frac{1}{|B(0, |x_1|)|^{1-\frac{2\beta_1}{n_1}} |B(0, |x_2|)|^{1-\frac{2\beta_2}{n_2}}} \int_{|y_1|<|x_1|} \int_{|y_2|<|x_2|} \frac{1}{\omega_{n_1} \omega_{n_2}} \int_{|\xi_1|=1} \int_{|\xi_2|=1} f(|y_1|\xi_1, |y_2|\xi_2) d\xi_1 d\xi_2 dy_1 dy_2
\]
\[
= \frac{1}{\omega_{n_1} \omega_{n_2}} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \frac{1}{|B(0, |x_1|)|^{1-\frac{2\beta_1}{n_1}} |B(0, |x_2|)|^{1-\frac{2\beta_2}{n_2}}} \int_{|y_1|<|x_1|} \int_{|y_2|<|x_2|} f(|y_1|\xi_1, |y_2|\xi_2) dy_1 dy_2 d\xi_1 d\xi_2
\]
\[
= \int_{S^{n_1-1}} \int_{S^{n_2-1}} \frac{1}{|B(0, |x_1|)|^{1-\frac{2\beta_1}{n_1}} |B(0, |x_2|)|^{1-\frac{2\beta_2}{n_2}}} \int_{0}^{[x_1]} \int_{0}^{[x_2]} f(r_1\xi_1, r_2\xi_2) r_1^{\gamma_1-1} r_2^{\gamma_2-1} dr_1 dr_2 d\sigma(\xi_1) d\sigma(\xi_2)
\]
\[
= \mathcal{H}_{\beta_1, \beta_2}(f)(x_1, x_2).
\]

Using the generalized Minkowski's inequality and Hölder's inequality, we conclude that \( \|gf\|_{L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2, |x|)} \) is not greater than
\[
\frac{1}{\omega_{n_1} \omega_{n_2}} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \left( \int_{\mathbb{R}^n_1} \int_{\mathbb{R}^n_2} (f(x_1\xi_1, x_2\xi_2))^p |x_1|^{\gamma_1} |x_2|^{\gamma_2} dx_1 dx_2 \right)^\frac{1}{p} d\xi_1 d\xi_2
\]
\[
\leq \left( \frac{1}{\omega_{n_1} \omega_{n_2}} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \int_{\mathbb{R}^n_1} \int_{\mathbb{R}^n_2} (f(x_1\xi_1, x_2\xi_2))^p |x_1|^{\gamma_1} |x_2|^{\gamma_2} dx_1 dx_2 d\xi_1 d\xi_2 \right)^\frac{1}{p}
\]
\[
= \left( \int_{S^{n_1-1}} \int_{S^{n_2-1}} \int_{0}^{\infty} \int_{0}^{\infty} (f(r_1\xi_1, r_2\xi_2))^p r_1^{\gamma_1+n_1-1} r_2^{\gamma_2+n_2-1} dr_1 dr_2 d\sigma(\xi_1) d\sigma(\xi_2) \right)^\frac{1}{p}
\]
\[
= \|f\|_{L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2, |x|)}.
\]

This finishes the proof of the lemma. \( \square \)
Remark 2.2 It follows from Lemma 2.1 that
\[
\left\| \mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}(f) \right\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^{\gamma})} \leq \left\| \mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m}(g_f) \right\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^{\gamma})}.
\]
Therefore, the norm of the operator \( \mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m} \) from \( L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^{\gamma}) \) to \( L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^{\alpha}) \) is equal to the norm that \( \mathcal{H}_{\beta_1, \beta_2, \ldots, \beta_m} \) restricts to radial functions.

For two differences power weight of fractional Hardy operator \( \mathbb{H}_\beta \) with \( \beta = 0 \) (write as \( \mathbb{H} \)), we have following lemma, which can be found in the paper [9].

Lemma 2.3 Suppose that \( 1 < p < q < \infty \), \( n \in \mathbb{N} \), \( x \in \mathbb{R}^n \), \( \gamma < n(p-1) \) and \( \frac{n+n}{q} = \frac{\gamma+n}{p} \). If \( f \in L^p(\mathbb{R}^n, |x|^{\gamma}) \), then we have
\[
\| \mathbb{H}(f) \|_{L^q(\mathbb{R}^n, |x|^{\alpha})} \leq C_{pq}^* \| f \|_{L^p(\mathbb{R}^n, |x|^{\gamma})},
\]
where \( C_{pq}^* \) is sharp and equal to
\[
|S^{n-1}|^{\frac{1}{q} - 
frac{1}{p}} \frac{n}{p} \frac{\gamma}{q} \left( \frac{n(p-1)}{n(p-1) - \gamma} \right)^{\frac{1}{q} + \frac{1}{p}} \left( \frac{p'}{q} \right)^{1/q} \left( \frac{p}{q-p} \right)^{1/q} \left( \frac{p}{q-p} \right) \left( \frac{p}{q-p} \right) \left( \frac{pq}{q(q-p)} \right)^{1/q - \frac{1}{q}}.
\]

Remark 2.4 There holds
\[
C_{pq}^* \to \frac{p}{p-1 - \frac{n}{p}},
\]
as \( q \to p \).

In fact, by using Persson and Samko [9] result:
\[
\left( \frac{p'}{q} \right)^{1/q} \left( \frac{p}{q-p} \right) \left( \frac{pq}{q(q-p)} \right)^{1/q - \frac{1}{q}} \to \frac{p}{p-1} \quad \text{as} \quad q \to p,
\]
therefore, we find that
\[
C_{pq}^* \approx |S^{n-1}|^{\frac{1}{q} - 
frac{1}{p}} \frac{n}{p} \frac{\gamma}{q} \left( \frac{n(p-1)}{n(p-1) - \gamma} \right)^{\frac{1}{q} + \frac{1}{p}} \frac{p}{p-1} = \frac{p}{p-1 - \frac{n}{p}} \quad \text{as} \quad q \to p.
\]

With the help of previous consequences, we shall prove our main statements.

3 Proof of main results

First we use the Lu and Zhao [7] result to derive a new constant, which is sharp in (1.6) for each \( \gamma \in (1, q) \).

Proof of Theorem 1.1 We merely the proof with the case \( m = 2 \) for the sake of clarity in writing, and the same is true for the general case \( m > 2 \).

Without loss of generality, we suppose that \( f \) is an nonnegative integrable function. For fixed variable \( x_2 \) we denote
\[
F(y_1) := F(y_1, x_2) = \int_{|y_2| < |x_2|} f(y_1, y_2)dy_2.
\]
According to Lu and Zhao [7] estimate for the high dimensional fractional Hardy operator in the case \(1 < p < q < \infty\) we find that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{1}{|B(0, |x_1|)|^{1 - \frac{p}{n}}} \frac{1}{|B(0, |x_2|)|^{1 - \frac{q}{n}}} \right| f(y_1, y_2)dy_1dy_2 \, dx_1dx_2 = \int_{\mathbb{R}^n} \left[ \frac{1}{|B(0, |x_1|)|^{1 - \frac{p}{n}}} \int_{|y_1| < |x_1|} F(y_1)dy_1 \right]^q dx_2
\]

\[
= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left| \frac{1}{|B(0, |x_2|)|^{1 - \frac{q}{n}}} \right| f(y_1, y_2)dy_2 \right)^q \left( \int_{\mathbb{R}^n} f(y_1)dy_1 \right)^{\frac{q}{p}} dx_2
\]

\[
\leq A_1 \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y_1)dy_1 \right)^{\frac{q}{p}} dy_1
\]

where \(A_1\) is a constant in the inequality of \((1.3)\) with \(n = n_1\).

By applying the generality Minkowski’s inequality with the power \(\frac{q}{p}\), we obtain that

\[
II \leq A_1 \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y_1)dy_1 \right)^{\frac{q}{p}} dy_1 \right)^{\frac{p}{q}}
\]

\[
= A_1 \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y_1, y_2)dy_2 \right)^q dy_1 \right)^{\frac{p}{q}}
\]

\[
\leq A_1 A_2 \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y_1, y_2)dy_2 \right)^q dy_1 \right)^{\frac{p}{q}}
\]

where \(A_2\) is a constant in the inequality of \((1.3)\) with \(n = n_2\).

Therefore, it implies that

\[
\|\mathcal{H}_{\beta_1, \beta_2}(f)\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \prod_{i=1}^2 \left( \frac{p_i}{q} \right)^{1/q} \left( \frac{n_i}{q_i} \right)^{\frac{n_i}{q_i}} \left( \frac{n_i}{q_i} \right)^{-\beta_i/n_i} \|f\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})},
\]

\[(3.9)\]

Next, we need to proved the converse inequality.

It follows from Lemma [2.2] that the norm of the operator \(\mathcal{H}_{\beta_1, \beta_2}\) from \(L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) to \(L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\) is equal to the norm that \(\mathcal{H}_{\beta_1, \beta_2}\) restricts to radial functions. Consequently, without loss of generality, it suffices to carry out the proof the converse inequality by assuming that \(f\) is a nonnegative, radial, smooth function with compact support on \(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\).

Using the polar coordinate transformation, we can rewrite \((3.9)\) as

\[
n_1 \int_0^\infty n_2 \int_0^\infty \left( n_1 \int_0^{s_1} n_2 \int_0^{s_2} f(r_1, r_2)r_1^{n_1-1}r_2^{n_2-1}dr_1dr_2 \right)^q \left( \frac{1}{s_1^{q(\beta_1-n_1)}} s_1^{n_1-1} s_2^{q(\beta_2-n_2)} s_2^{n_2-1}ds_1ds_2 \right)^{\frac{p}{q}} \leq (A_1 A_2)^{\frac{p}{q}} \left( \int_0^\infty \int_0^\infty f^q(s_1, s_2)ds_1ds_2 \right)^{\frac{p}{q}}.
\]

\[(3.10)\]
For the purpose of getting the sharp bound, we take \( \tilde{f}(x_1, x_2) = \frac{1}{(1+|x_1|^{q\beta_1})^{1+\frac{m}{p}n_1}(1+|x_2|^{q\beta_2})^{\frac{m}{p}}}. \)

It follows from

\[
n_1 \int_0^{s_1} n_2 \int_0^{s_2} \tilde{f}(r_1, r_2) r_1^{n_1-1} r_2^{n_2-1} dr_1 dr_2 = \frac{s_1^{n_1}}{(1 + s_1^{q\beta_1})^{\frac{m}{p}n_1}} \frac{s_2^{n_2}}{(1 + s_2^{q\beta_2})^{\frac{m}{p}}},
\]

that the left side of (3.10) is

\[
\prod_{i=1}^{2} \frac{n_i}{q\beta_i} \cdot B \left( \frac{n_i}{q\beta_i} + 1, \frac{n_i}{q'\beta_i} - 1 \right).
\]

It is easy to verify that

\[
n_1 \int_0^{\infty} n_2 \int_0^{\infty} \tilde{f}^p(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 = \prod_{i=1}^{2} \frac{n_i}{q\beta_i} \cdot B \left( \frac{n_i}{q\beta_i}, \frac{n_i}{q'\beta_i} \right).
\]

Therefore,

\[
\| \mathcal{H}_{\beta_1, \beta_2} \|_{L^p(\mathbb{R}^n \times \mathbb{R}^n) \to L^q(\mathbb{R}^n \times \mathbb{R}^n)} = \sup_{\| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \neq 0} \frac{\| \mathcal{H}_{\beta_1, \beta_2}(f) \|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)}}{\| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}} \geq \frac{\| \mathcal{H}_{\beta_1, \beta_2}(\tilde{f}) \|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)}}{\| \tilde{f} \|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}} = A_1 A_2,
\]

This completes the proof of Theorem 1.11.

**Proof of Theorem 1.2** We merely the proof with the case \( m = 2 \) for the sake of clarity in writing, and the same is true for the general case \( m > 2 \).

Without loss of generality, it follows from Lemma 2.11 we can assuming that \( f \) is a nonnegative, radial, smooth function with compact support on \( \mathbb{R}^{n_1} \times \mathbb{R}^n_2 \).

Using the polar coordinate transformation, \( \| \mathcal{H}_{\beta_1, \beta_2}(f) \|_{L^q(\mathbb{R}^n_1 \times \mathbb{R}^n_2, |x|^{\alpha})} \) is equal to

\[
K_q \left( \int_0^{\infty} \int_0^{\infty} \left( \int_0^{\rho_1} \int_0^{\rho_2} f(t_1, t_2) t_1^{\alpha_1-1} t_2^{\alpha_2-1} dt_1 dt_2 \right)^{\frac{1}{q}} \right) \theta(t_1, t_2) \rho_1^{\beta_1-\alpha_1} \rho_2^{\beta_2-\alpha_2} \rho_1^{\alpha_1-1} \rho_2^{\alpha_2-1} d\rho_1 d\rho_2 \quad (3.11)
\]

and \( \| f \|_{L^p(\mathbb{R}^n_1 \times \mathbb{R}^n_2, |x|^{\gamma})} \) is equal to

\[
K_p \left( \int_0^{\infty} \int_0^{\infty} f^p(\rho_1, \rho_2) \rho_1^{n_1-1+\gamma_1} \rho_2^{n_2-1+\gamma_2} d\rho_1 d\rho_2 \right)^{\frac{1}{p}}, \quad (3.12)
\]

where two constants \( K_q = n_1^{1-\frac{\beta_1}{n_1}} n_2^{1-\frac{\beta_2}{n_2}} |S_1^{n_1-1}|^{\frac{1}{q} + \frac{\beta_1}{n_1}} |S_2^{n_2-1}|^{\frac{1}{q} + \frac{\beta_2}{n_2}} \) and \( K_p = |S_1^{n_1-1}|^{\frac{1}{p}} |S_2^{n_2-1}|^{\frac{1}{p}} \).

First we make a change of variables in (3.12) by putting

\[
\frac{s_1^{n_1}}{n_1} = s_1^{n_1}(\rho_1) = \int_0^{\rho_1} t_1^{n_1-1-\frac{\gamma_1}{p-1}} dt = \frac{p-1}{n_1(p-1)-\gamma_1} \rho_1^{n_1(p-1)-\gamma_1}, \quad (3.13)
\]
\[ \frac{s^{n_2}}{n_2} = \frac{s^{n_2}(\rho_2)}{n_2} = \int_{0}^{\rho_2} t_2^{n_2-1} \rho_2 \, dt = \frac{p - 1}{n_2(p - 1) - \gamma} \rho_2^{\frac{n_1(p-1) - \gamma_1}{n_1(p-1)}} \quad (3.14) \]

and define
\[ g(s_1, s_2) = g(s_1(\rho_1), s_2(\rho_2)) = f(\rho_1, \rho_2) \rho_1^{\frac{-\gamma_1}{p-1}} \rho_2^{\frac{-\gamma_2}{p-1}}. \quad (3.15) \]

Then
\[ \int_{0}^{\infty} \int_{0}^{\infty} f^p(\rho_1, \rho_2) \rho_1^{n_1-1+\gamma_1} \rho_2^{n_2-1+\gamma_2} \, d\rho_1 d\rho_2 = \int_{0}^{\infty} \int_{0}^{\infty} f^p(\rho_1, \rho_2) \rho_1^{\frac{n_1-1-\gamma_1}{p-1}} \rho_2^{\frac{n_2-1-\gamma_2}{p-1}} \, d\rho_1 d\rho_2 \quad (3.16) \]

and
\[ H_0 := \left( \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{0}^{\rho_1} \int_{0}^{\rho_2} f(t_1, t_2) t_1^{n_1-1} t_2^{n_2-1} \, dt_1 dt_2 \right)^q \times \rho_1^{q(\beta_1-n_1)+\alpha_1} \rho_1^{n_1-1} \rho_2^{q(\beta_2-n_2)+\alpha_2} \rho_2^{n_2-1} \, d\rho_1 d\rho_2 \right)^{\frac{1}{q}} \]
\[ = \left( \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{0}^{\rho_1} \int_{0}^{\rho_2} f(t_1, t_2) t_1^{n_1-1} t_2^{n_2-1} \, dt_1 dt_2 \right)^q \times \rho_1^{q(\beta_1-n_1)+\left(\frac{n_1+n_2}{p}-1\right)q-n_1} \rho_1^{n_1-1} \rho_2^{q(\beta_2-n_2)+\left(\frac{n_2}{p}-2\right)q-n_2} \rho_2^{n_2-1} \, d\rho_1 d\rho_2 \right)^{\frac{1}{q}}. \quad (3.17) \]

To obtain we desire result, we need to following form. And (3.17) is equal to
\[ H_0 = \left( \int_{0}^{\infty} \int_{0}^{\infty} \left( \int_{0}^{\rho_1} \int_{0}^{\rho_2} f(t_1, t_2) t_1^{n_1-1} t_2^{n_2-1} \, dt_1 dt_2 \right)^q \times \rho_1^{q(\beta_1-n_1)+\left(\frac{n_1+n_2}{p}-1\right)q-n_1} \rho_1^{n_1-1} \rho_2^{q(\beta_2-n_2)+\left(\frac{n_2}{p}-2\right)q-n_2} \rho_2^{n_2-1} \, d\rho_1 d\rho_2 \right)^{\frac{1}{q}}. \]

Hence, since
\[ f(t_1, t_2) t_1^{n_1-1} t_2^{n_2-1} \, dt_1 dt_2 = f(t_1, t_2) t_1^{\frac{n_1-1}{p-1}} t_2^{\frac{n_2-1}{p-1}} \, dt_1 t_2^{\frac{n_1}{p-1} - \frac{n_2}{p-1}} \, dt_2 \quad (3.18) \]
and
\[ \frac{s^{n_1}}{n_1} = \frac{p - 1}{n_1(p - 1) - \gamma_1} \rho_1^{\frac{n_1(p-1) - \gamma_1}{n_1(p-1)}} \Rightarrow \rho_1 = \left( \frac{n_1(p - 1) - \gamma_1}{n_1(p - 1) s_1^{n_1}} \right)^{\frac{p-1}{n_1(p-1)-\gamma_1}}, \quad (3.19) \]
similarly,
\[ \rho_2 = \left( \frac{n_2(p - 1) - \gamma_2}{n_2(p - 1) s_2^{n_2}} \right)^{\frac{p-1}{n_2(p-1)-\gamma_2}}. \quad (3.20) \]
Applying the (3.17), (3.18), (3.19), and (3.20) we have that

\[
IH_0 = \prod_{i=1}^{2} \left( \frac{n_i(p-1)}{n_i(p-1) - \gamma_i} \right)^{\frac{1}{p} + \frac{1}{q}} \left( \int_0^1 \int_0^1 \int_0^{s_1} \int_0^{s_2} g(r_1, r_2) r_1^{n_i-1} r_2^{n_i-1} dr_1 dr_2 \right)^{\frac{1}{q}}
\]

\[
\times \int_0^{\infty} \int_0^{\infty} g(r_1, r_2) r_1^{n_i-1} r_2^{n_i-1} ds_1 ds_2 \right)^{\frac{1}{q}}.
\]

Taking \( \beta_1 = n_1 \left( \frac{1}{p} - \frac{1}{q} \right) \) and \( \beta_2 = n_2 \left( \frac{1}{p} - \frac{1}{q} \right) \) in (3.10), we obtain that

\[
n_1 \int_0^{\infty} n_2 \int_0^{\infty} \left( n_1 \int_0^{s_1} n_2 \int_0^{s_2} f(r_1, r_2) r_1^{n_1-1} r_2^{n_2-1} dr_1 dr_2 \right)^{\frac{1}{q}} s_1^{\frac{n_1 q}{p'}} s_2^{\frac{n_2 q}{p'}} ds_1 ds_2
\]

\[
\leq (A_1 A_2)^{\frac{1}{q}} \left( \int_0^{\infty} \int_0^{\infty} f^p(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 \right)^{\frac{1}{p}}.
\]

Combining (3.21) and (3.22) we find that

\[
IH_0 \leq \prod_{i=1}^{2} \left( \frac{n_i(p-1)}{n_i(p-1) - \gamma_i} \right)^{\frac{1}{p} + \frac{1}{q}} n_i^{\frac{1}{p} - \frac{1}{q}} A_i \left( \int_0^{\infty} \int_0^{\infty} g^p(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 \right)^{\frac{1}{p}}.
\]

So using the Lemma 2.1 we implies that

\[
\frac{\|H_{\beta_1, \beta_2}(f)\|_{L^q(\mathbb{R}^n \times \mathbb{R}^m, |x|^\alpha)}}{\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m, |x|^\gamma)}} \leq \frac{\|H_{\beta_1, \beta_2}(gf)\|_{L^q(\mathbb{R}^n \times \mathbb{R}^m, |x|^\alpha)}}{\|gf\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m, |x|^\gamma)}}
\]

\[
= \frac{K_p \left( \int_0^{\infty} \int_0^{\infty} g^p(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 \right)^{\frac{1}{p}}}{K_q I_0}.
\]

Therefore, combining (3.23) and (3.24), we have \( \|H_{\beta_1, \beta_2}\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m, |x|^\gamma) \rightarrow L^q(\mathbb{R}^n \times \mathbb{R}^m, |x|^\alpha)} \) is not greater than

\[
\prod_{i=1}^{2} \left( \frac{n_i(p-1) - \gamma_i}{n_i(p-1) - \gamma_i} \right)^{\frac{1}{p} + \frac{1}{q}} n_i^{\frac{1}{p} - \frac{1}{q}} A_i \left( \int_0^{\infty} \int_0^{\infty} g^p(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 \right)^{\frac{1}{p}}.
\]

From the Theorem 1.1 it also follows that above constant is sharp when

\[
g(s_1, s_2) = \frac{1}{1 + s_1^{\frac{n_1 q}{p' - q}}(n_1(p-1) - \gamma_1)} \quad \frac{1}{1 + s_2^{\frac{n_2 q}{p' - q}}(n_2(p-1) - \gamma_2)},
\]

i.e. when (see 3.13, 3.14 and 3.15)

\[
f(x_1, x_2) = \frac{|x_1|^{-\frac{n_1 q}{p' - q}}}{\left( 1 + d_1 |x_1|^{n_1(p-1) - \gamma_1} \right)^{\frac{n_1 q}{p' - q}}} \quad \frac{|x_2|^{-\frac{n_2 q}{p' - q}}}{\left( 1 + d_2 |x_2|^{n_2(p-1) - \gamma_2} \right)^{\frac{n_2 q}{p' - q}}},
\]

where two constants \( d_1 \) and \( d_2 \) are \( \left( \frac{n_1(p-1)}{n_1(p-1) - \gamma_1} \right)^{1/n_1} \) and \( \left( \frac{n_2(p-1)}{n_2(p-1) - \gamma_2} \right)^{1/n_2} \), respectively.

This completes the proof of Theorem 1.2. \( \square \)
Remark 3.1 If $\beta_1 = \beta_2 = \cdots = \beta_m = 0$, the constant is also sharp in Theorem [1.2]. Moreover, the constant is equal to

$$
\prod_{i=1}^{m} \left( S_{n_i}^{n_i-1} \right)^{\frac{1}{q} - \frac{1}{p}} \frac{1}{n_i} \left( \frac{n_i(p-1)}{n_i(p-1) - \gamma_i} \right)^{\frac{1}{p'} + \frac{1}{q}} \left( \frac{p'}{q} \right)^{1/q} \left( \frac{p}{q - p} \cdot B \left( \frac{p}{q - p}, \frac{pq}{q(q - p)} \right) \right)^{\frac{1}{q} - \frac{1}{p}},
$$

and it shows that the constant in (1.5) is not really the best.

Using the Lemma [2.3] and Remark [2.4], we find that

$$
\prod_{i=1}^{m} \left| S_{n_i}^{n_i-1} \right|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{n_i} \left( \frac{n_i(p-1)}{n_i(p-1) - \gamma_i} \right)^{\frac{1}{p'} + \frac{1}{q}} \left( \frac{p'}{q} \right)^{1/q} \left( \frac{p}{q - p} \cdot B \left( \frac{p}{q - p}, \frac{pq}{q(q - p)} \right) \right)^{\frac{1}{q} - \frac{1}{p}}
$$

$$
\approx \prod_{i=1}^{m} \frac{p}{p - 1 - \alpha_i/n_i}
$$

as $p \to q$

and it remains to note that $\prod_{i=1}^{m} \frac{p}{p - 1 - \alpha_i/n_i}$ is the sharp constant in [6].

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