Calculation of processes involving many particles at the kinematical threshold

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Abstract

A diagrammatic technique is derived for calculating processes involving the production of large numbers of particles. As an example, the amplitude of three particles scattering into \( n \) particles at the kinematical threshold in \( \lambda \phi^4 \) theory is calculated. From the \( 3 \rightarrow n \) amplitude, it is seen that the one-loop amplitude for \( 2 \rightarrow n \) processes is suppressed by a factor of \( 1/n \).
It has been shown that one can study multiscalar production at the tree-level by using recursion relations\(^1\),\(^2\) or the reduction formula method\(^3\). These techniques enable one to calculate the tree-level amplitude of one off-shell scalar particle decaying into many scalar particles at rest.

These methods have been shown to be applicable to other problems. The one-loop correction was found in $\lambda \phi^4$ theory for both the case of unbroken\(^4\) and broken\(^6\) reflection symmetry. In finding these corrections the amplitude of 2 on-shell scalar particles producing $n$ particles at the kinematical threshold was discovered to vanish for $n > 4$ in the case of unbroken symmetry and $n > 2$ in the case of broken symmetry (for a discussion on this phenomena see\(^5\)). This effect was further discussed by Argyres, Kleiss, and Papadopoulos\(^7\) and by Brown and Zhai\(^8\). Argyres, Kleiss, and Papadopolous have discussed the cross channel of this process (i.e. one off-shell particle decaying into $n$ particles at rest and one particle above the kinematic threshold).

In this paper, we discuss a diagrammatic technique for calculating these amplitudes. We show how to build a set of Feynman rules for the generating functions of processes involving multiparticle production. With these rules, one can consider processes involving $M$ scalars scattering into $N$ scalars with none of the $M$ or $N$ particles necessarily being at rest. However, instead of allowing one to calculate the $M \rightarrow N$ amplitude as in normal diagrammatic techniques, this technique allows one to calculate the generating function for $M \rightarrow N + n$ processes where the $n$ particles are at rest.

It must be emphasized that this procedure does not allow the study of many particles above the kinematical threshold. While large numbers of particles at rest may be produced, the calculations become more and more complicated as the number of particles above the threshold is increased. This approach does provide a compact method to calculate the amplitude for a few particles scattering to produce a few particles above the threshold and many particles at the kinematical threshold.

We shall demonstrate this technique by calculating the $3 \rightarrow n$ amplitude in $\lambda \phi^4$ theory. Through the unitarity relations, this amplitude provides an easy way to study the one-loop properties of $2 \rightarrow n$ processes.

We will be discussing $\lambda \phi^4$ theory with the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4}\phi^4.$$ (1)
We will primarily be deriving Feynman rules for the case of unbroken reflection symmetry \((m^2 > 0)\). Most of what is discussed may be trivially extended to the case of broken symmetry as well as theories discussed in [9], [10] and [8].

We wish to consider the amplitude for an interaction involving \(N\) particles with arbitrary four-momenta. During this interaction, \(n\) particles will be produced at the kinematic threshold. The particles produced at the threshold will have a four-momentum denoted as \(q \equiv (1, 0)\). The generating functions for these amplitudes can be denoted as the \(N\)-point Green function, \(G_n^{(N)}(p_1, ..., p_n)\). To calculate this function, it is easiest to consider the amplitude of processes involving one particle decaying into \(N - 1\) particles above the threshold and \(n\) particles at the threshold. This amplitude is written as

\[
A_n^{(N)}(p_1, ..., p_n) \equiv G_n^{(N+1)}(-P - nq, p_1, ..., p_n),
\]

where \(P \equiv p_1 + ... + p_n\).

The amplitude \(A_n^{(N)}\) can be considered recursively as in Fig. 1. We can write the recursion relation

\[
A_n^{(N)}(p_1, ..., p_n) = \frac{\lambda}{(P + nq)^2 - 1} \sum_{n_1, n_2, n_3} \delta_{n, n_1 + n_2 + n_3} \delta_{N, N_1 + N_2 + N_3} \frac{n!}{n_1! n_2! n_3!} \sum_{\text{perms.}} A_{n_1}^{(N_1)}(\{P_{n_1}\}) A_{n_2}^{(N_2)}(\{P_{n_2}\}) A_{n_3}^{(N_3)}(\{P_{n_3}\}),
\]

where \(\{P_m\}\) is a set of \(m\) of the original momenta and the second sum runs over all possible permutations of the momenta.

Equation (3) can be simplified by using the function,

\[
a_n^{(N)}(p_1, ..., p_n) \equiv -i \left( \frac{8}{\lambda} \right)^{n/2} \frac{A_n^{(N)}(p_1, ..., p_n)}{n!}.
\]

With this substitution, we can construct a differential equation for the generating function \(g\),

\[
g^{(N)}(p_1, ..., p_N; x) \equiv \sum_{n=1}^{\infty} a_n^{(N)}(p_1, ..., p_N)x^n.
\]

It can be verified that for all \(N > 1\), the generating function must satisfy,

\[
\left[ x^2 \frac{d^2}{dx^2} + (2P_0 + 1)x \frac{d}{dx} + P^2 - 1 \right] g^{(N)}(x) = \lambda \sum_{N_1, N_2, N_3} \delta_{N, N_1 + N_2 + N_3} g^{(N_1)}(x)g^{(N_2)}(x)g^{(N_3)}(x),
\]
where the momentum arguments have been omitted for simplicity and the sum is implied over all permutations of the momenta.

The homogeneous part of equation (6) contains \( g(0) \). This coefficient is the generating function for multiparticle production at the threshold, i.e. the amplitude for \( 1 \rightarrow n \) processes as found in previous works [1, 2]. Using the fact that

\[
\langle n | \phi(0) | 0 \rangle = (2k + 1)! \left( \frac{\lambda}{8} \right)^{(n-1)/2}, \tag{7}
\]

g(0) is found to be,

\[
g(0) = \sqrt{\frac{8}{\lambda}} \frac{x^2}{(1-x^2)^2}. \tag{8}
\]

To separate the homogeneous part of equation (6), one must simply extract the terms in the sum with \( N_1, N_2, \) or \( N_3 = N \). Equation (6) can now be recast in the form of the homogeneous differential equation,

\[
\left[ x^2 \frac{d^2}{dx^2} + (2P_0 + 1)x \frac{d}{dx} + P^2 - 1 - \frac{24x^2}{(1-x^2)^2} \right] g^{(N)}(x) = \lambda \sum_{N_1,N_2,N_3 < N} \delta_{N,N_1+N_2+N_3} g^{(N_1)}(x) g^{(N_2)}(x) g^{(N_3)}(x). \tag{9}
\]

As was noted by Voloshin in [3], the operator on the left hand side of equation (9) can be restated in a familiar form by making the substitutions,

\[
x \rightarrow ie^t \\
g^{(N)}(p_1,...,p_N;x) \rightarrow e^{-P_0t} y^N(p_1,...,p_N;t). \tag{10}
\]

This substitution transforms the homogeneous operator in equation (9) into the Schrödinger operator for an exactly solvable potential. The generating function can now be written as the solution of

\[
\left[ \frac{d^2}{dt^2} - P^2 - 1 + \frac{6}{\cosh^2(t)} \right] y^N(t) = \lambda \sum_{N_1,N_2,N_3 < N} \delta_{N,N_1+N_2+N_3} y^{N_1}(t) y^{N_2}(t) y^{N_3}(t), \tag{11}
\]

where \( P^2 \) is square of the magnitude of the spatial part of \( P \).

The operator in equation (12) has been discussed in [4]. With \( \omega \equiv \sqrt{P^2 + 1} \), the Green function for the operator in equation (12) was found to be

\[
G_{\omega}(t,t') = \frac{f^{+\omega}_i(t) f^{+\omega}_i(t') \theta(t-t') + f^{-\omega}_i(t') f^{-\omega}_i(t) \theta(t'-t)}{2\omega}, \tag{12}
\]
where
\[ f_{\pm}(t) = e^{\mp \omega t} F(-2, 3; 1 \pm \omega; \frac{1}{1 + e^{2\theta}}), \quad (13) \]
and \( F(a, b; c; x) \) is the hypergeometric function.

With the aid of this Green function, the solution of equation (12) is found to be,
\[ g^{(N)}(t) = \lambda e^{-\epsilon t} \int_{-\infty}^{+\infty} G_{\omega}(t, t') \sum_{N_1+N_2=N_3<0} \delta_{N_1+N_2+N_3} y^{N_1}(t') y^{N_2}(t') y^{N_3}(t') dt'. \quad (14) \]

Equation (14) allows one to recursively calculate any \( g^{(N)} \) if one knows all \( g^{(M)} \) for all \( M < N \). Equation (14) can be summarized by a set of Feynman rules for calculating amplitudes with multiparticle production in the background.

1. Draw all possible diagrams with the external legs labelled. Do not include any propagators that represent particles at rest.

2. Assign a unique time label to every vertex and to the endpoint of every external leg.

3. For every external leg, multiply the amplitude by a factor of \( e^{\epsilon t} \), where \( \epsilon \) is the energy of the particle at the endpoint of the leg, and \( t \) is the time label at the endpoint of the leg.

4. For every line, multiply by a factor of \( G_{\omega}(t_1, t_2) \) where \( t_1 \) and \( t_2 \) are the time labels of the two endpoints of the line, and \( \omega \) is the expected on-shell energy of the momenta flowing through the line, i.e. \( \omega = \sqrt{k^2 + 1} \).

5. For every vertex, multiply by a factor if \( \lambda \). In addition, for every three-point vertex multiply by a factor of \( g^{(0)}(t) \). This last factor is present to account for the production of particles at the vertex.

6. Integrate over all time labels except for one.

7. Multiply by a factor of \( e^{-\sum \epsilon t} \), where \( t \) is the time label that was not integrated over and the sum includes the label at every external leg.

8. Each diagram must be multiplied by an appropriate symmetry factor.

Note that spatial momenta is conserved at every vertex and over all. Energy, however, is not conserved. This basic procedure can be used to construct Feynman rules in a number of theories including \( \lambda \phi^4 \) in both the case of broken and unbroken symmetry.

For the case of unbroken symmetry, the Green function for the propagator is given by equation (12), and the vertex amplitude is given by equation (8). The rules will generate \( g^{(N)}(t) \). To find the generating function of the amplitudes, one simply has
to make the substitution,
\[ z(t) \equiv i \sqrt{\frac{8}{\lambda}} e^{t}. \] (15)

The amplitude is given by the \( n \)th derivative of \( g^{(N)}(z) \),
\[ A_{n}^{(N)}(p_1, \ldots, p_N) = \frac{d^n}{dz^n} g^{(N)}(p_1, \ldots, p_N; z). \] (16)

In theories with a broken symmetry, the propagator is given by\[F\],
\[ G_{\omega}(t, t') = f_{\omega}^{-}(t)f_{\omega}^{+}(t')\theta(t - t') + f_{\omega}^{-}(t')f_{\omega}^{+}(t)\theta(t' - t) \over 2\omega, \] (17)

where
\[ f_{\omega}^{\pm}(t) = e^{\mp\omega t} F(-2, 3; 1 \pm 2\omega; {1 \over 1 + e^t}). \] (18)

The generating function for the amplitude of one particle production is given by,
\[ g^{(0)}(t) = \sqrt{\frac{1}{2\lambda}} \left(1 - e^t\right) \over 1 + e^t}. \] (19)

The amplitudes are found in the same manner as in the case of unbroken symmetry. However, in the case of broken symmetry, one must make the substitution,
\[ z(t) \equiv - \sqrt{\frac{2}{\lambda}} e^t. \] (20)

Rules for other theories may be constructed if one can calculate the propagator and the generating function for \( 1 \to n \) processes.

As an example, we shall calculate the tree-level amplitude for threshold production of \( n \) particle by three on-shell particles in a theory with spontaneously broken symmetry. This amplitude is divergent when the incoming particles are on-mass-shell.

To consider the amplitude of the real process, one must take the residue of the triple pole as the energy of all three incoming particles tends to its mass-shell value. The generating functional can be seen from the diagram in figure 2 to be,
\[ g^{(3)}(t) = 6\lambda \int e^{\epsilon_1 t_1} G_{\omega_1}(t, t_1) e^{\epsilon_2 t_2} G_{\omega_2}(t, t_2) e^{\epsilon_3 t_3} G_{\omega_3}(t, t_3) \sqrt{\frac{1}{2\lambda}} \left(1 - e^t\right) \over 1 + e^t} dt_1 dt_2 dt_3. \] (21)

Taking the on-shell limit corresponds to the limit \( \epsilon_i \to \omega_i \). Inspection of equation (17) reveals that in order to have poles in the amplitude, the time ordering of the three propagators is uniquely determined, \( t_1, t_2, t_3 < t \). Causality considerations would also
lead one to believe that this is the only time ordering capable of producing a physical amplitude.

Equation (21) contains three integrals over hypergeometric functions. These are easiest to handle by writing the hypergeometric function in terms of the standard hypergeometric series. The integral of interest in equation (21) is

$$\int_{-\infty}^{t} e^{(\omega + \epsilon)t} F(-2, 3; 1 - 2\omega; \frac{1}{1 + e^{t'}})dt' = \int_{-\infty}^{t} e^{(\omega + \epsilon)t'} \sum_{i=1}^{3} \frac{(-2)^{n}(3)_{n}}{(1 + 2\omega)_{i} 1 + e^{t'}}dt',$$

where \((a)_{n}\) is the Pochhammer symbol,

$$\Gamma(a + n + 1)$$

$$\Gamma(a + 1).$$

(23)

When the integrand in equation (22) is expanded in a series in \(e^{t'}\), it is easy to see that only the first term in the series shall contribute to the pole as \(\epsilon \to -\omega\). The residue of the pole in equation (22) is the simple expression \(F(-2, 3; 1 - 2\omega; 1)\). We can now recast the integrals in equation (21) in the simple form,

$$\lim_{\epsilon \to -\omega} (\epsilon^{2} - \omega^{2}) \int_{-\infty}^{\infty} e^{t'} G_{\omega}(t, t')dt' = \frac{1}{(1 + e^{t'})^{2}} F(-2, -2; 1 - 2\omega; -\epsilon').$$

(24)

Using the definition of \(z\) in equation (20), we can write an expression for the amplitude in equation (21),

$$g^{(3)}(z) = 6 \sqrt{\frac{\lambda}{2}} \left(1 + \sqrt{\lambda/2z}\right)^{-1} \left(1 - 4 \sqrt{\frac{\lambda}{2}} \frac{1 + \omega_{1}}{1 + 2\omega_{1}} z + \frac{\lambda(1 + \omega_{1})}{2\omega_{1}} z^{2}\right)$$

$$\left(1 - 4 \sqrt{\frac{\lambda}{2}} \frac{1 + \omega_{2}}{1 + 2\omega_{2}} z + \frac{\lambda(1 + \omega_{2})}{2\omega_{2}} z^{2}\right)\left(1 - 4 \sqrt{\frac{\lambda}{2}} \frac{1 + \omega_{3}}{1 + 2\omega_{3}} z + \frac{\lambda(1 + \omega_{3})}{2\omega_{3}} z^{2}\right)$$

(25)

Equation (25) is the generating function for the tree-level amplitudes of \(3 \to n\) processes at the kinematical threshold. In the calculation of an amplitude, \(\omega_{1}, \omega_{2}, \omega_{3}\), and the number of particles are not four independent parameters. These four parameters can be reduced to two independent parameters through the application of momenta and energy conservation principles.

The \(3 \to n\) amplitudes contained in the generating function in equation (25) can be written in a compact form in the asymptotic limit, i.e. when \(\omega_{1}, \omega_{2}, \omega_{3}\), and \(n \gg 1\). The tree-level amplitude for \(3 \to n\) production at the kinematic threshold in the large \(n\) approximation in the theory with broken symmetry is

$$A^{0}(3 \to n) = \frac{2}{5!} \left(\frac{\lambda}{2}\right)^{(n+1)/2} \frac{n!n^{6}}{(2\omega_{1})^{2}(2\omega_{2})^{2}(2\omega_{3})^{2}}.$$

(26)
Voloshin has noted the $2 \rightarrow n$ at the one-loop level can be related to the tree-level $3 \rightarrow n$ amplitude. This reasoning is based on the argument that in the spontaneously broken theory, the imaginary part of $1 \rightarrow n$ processes must vanish to all orders in perturbation theory. One can consider the unitary cuts on the one-loop $1 \rightarrow n$ amplitude to derive the condition,

$$2ImA^1(1 \rightarrow n) = \int A^1(1 \rightarrow 2)A^0(2 \rightarrow n)d\tau_2 + \int A^0(1 \rightarrow 2)A^1(2 \rightarrow n)d\tau_2 + \int A^0(1 \rightarrow 3)A^0(3 \rightarrow n)d\tau_3,$$

(27)

where the integration is done over the phase space of the intermediate particles, and the superscripts denote the number of loops in the amplitude. The first term on the right hand side of equation (27) is zero due to the vanishing of $A^0(2 \rightarrow n)$ for all $n > 2$ as noted in [6]. Since the left hand side of equation (27) is zero, the one-loop $2 \rightarrow n$ amplitude may be related to the tree-level $3 \rightarrow n$ amplitude in equation (25). The integration over the phase space of equation (27) is non-trivial, but can determine whether the $2 \rightarrow n$ nullification noted in [6] will continue at the one-loop level.

The expression in equation (26) can now be combined with the $1 \rightarrow 3$ amplitude to find the contribution of the second term in the unitary relation, equation (27). This contribution is of the order of $\lambda^{(n+1)/2}n!n^2$. The order of magnitude of the one-loop correction to the $2 \rightarrow n$ amplitude is, therefore,

$$A^1(2 \rightarrow n) = \lambda^{(n+2)/2}n!n^2.$$  

(28)

Although the tree-level contribution vanishes in $\lambda\phi^4$ theory, the one-loop contribution can be compared with the tree-level amplitude for $2\chi \rightarrow n\phi$ in an interactive theory as calculated by Brown and Zhai [8]. They found that the tree level $2 \rightarrow n$ amplitude should grow like $\lambda^{n^2/n!}$. The above arguments indicate that although the $2 \rightarrow n$ tree-level threshold amplitude may vanish, diagrams at the one-loop level provide a correction of order $\lambda n$. If the loop expansion were a series expansion in powers of $\lambda n^2$, the results of [8] would indicate that the one-loop $2 \rightarrow n$ correction should grow like $\lambda^{(n+1)/2}n!n^3$. There is a $1/n$ suppression in the one-loop correction to threshold $2 \rightarrow n$ processes.

The rules discussed in this paper provide a compact method for calculating many processes involving large numbers of particles at the kinematic threshold. They provide a useful tool for investigating suppression of amplitudes in multiparticle pro-
cesses. They may prove to be of further assistance in studying multiparticle phenomena such as the vanishing of $2 \rightarrow n$ amplitudes.

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Figure 1 - Symbolic representation of recursion relation
Figure 2 - Feynman diagram for $3 \to n$ processes