Exact Edgeworth expansion for a Lévy process

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Abstract

The one dimensional distribution of a Lévy process is not known in general even though its characteristic function is given by the famous Lévy-Khinchine theorem. This article gives an exact series representation for the one dimensional distribution of a Lévy process satisfying certain moment conditions. Moreover, this work clarifies an old result by Cramér on Edgeworth expansions for the distribution function of a Lévy process.

Keywords: Asymptotic expansions, Cramér’s condition, cumulants, Edgeworth approximation, Lévy process

AMS subject classification: 60G51, 60E07, 60G50.
1 Introduction

The Lévy-Khinchine theorem gives the characteristic function of a Lévy process. In spite of this, the distribution of a Lévy process is not analytically known, except in few special cases such as the Brownian motion, the Poisson process and the gamma process. For example, the distribution function of the compound Poisson process is not known in general despite its popularity as a risk process in insurance applications.

This article has two contributions. First of all, this article introduces some sufficient extra conditions to get an exact Edgeworth type series representation for the one dimensional distribution of a Lévy process in the presence of all moments. Secondly, this paper goes beyond an old result on Edgeworth approximation introduced by Cramér (1962) as an analogue to the i.i.d. sum case. This article clarifies the connection between the distribution functions of Lévy processes and classical approximation results of sums of independent random variables. As a consequence, we will give an approximation method for the distribution of spectrally positive (negative) Lévy processes. This kind of processes are widely used in modern insurance models, see e.g. Klüppelberg and Kyprianou (2006).

Besides the insurance applications, the results of this article could be applicable in the simulations of Lévy processes. In fact, the classical Edgeworth approximation has been used for getting error estimates for simulations of the small jumps of a Lévy process in Asmussen and Rosiński (2001). Moreover, the exact series representation might be useful tool for the study of theoretical properties of Lévy processes.

There are lots of approximation results in the literature. The normal approximation approximates well asymptotically the distribution function of a Lévy process when $t \to \infty$ if the third moment exists, see for instance Valkeila (1995). Several authors have considered asymptotic expansions in the central limit theorem (Edgeworth approximation) for the sums of independent random variables to improve the normal approximation, see e.g. Petrov (1995) or Cramér (1962). These approximation methods are also well known in statistics and insurance mathematics (see (Beard et al., 1977; Kolassa, 2006)). Another approximation result (Theorem 3.2 here) is introduced for the distribution function of a Lévy process by Cramér (1962) as an analogue to the i.i.d. sum result. This is the starting point of the research presented in this article.

2 Definitions

In this section, we define the concepts needed in the rest of the article.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X$ be a real valued random variable defined on this space. Let $\psi_X(s) = \mathbb{E}e^{isX}$ denote the characteristic function of $X$.

Definition 2.1 (Cramér’s condition). A random variable $X$ is said to satisfy
Cramér’s condition if
\[
\limsup_{|s| \to \infty} |v_X(s)| < 1.
\]

Remark 2.4 characterises Cramér’s condition in the case of Lévy processes.

**Definition 2.2 (Cumulants).** Let \( k \in \mathbb{N} = \{1, 2, \ldots\} \). The cumulant of order \( k \) of a random variable \( X \) is defined as
\[
\gamma^X_k = \frac{1}{k!} \left[ \frac{d^k}{ds^k} \log v_X(s) \right]_{s=0}.
\]

Note that the cumulant of \( X \) of order \( k \) is finite if we have \( E|X|^k < \infty \).

We use the following definition for the (non-normalised) Hermite polynomial of order \( n \in \mathbb{N} \)
\[
H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]

This choice of the definition makes the series representation much simpler than the normalised one. The same choice is done e.g. by Petrov (1995), Kolassa (2006). With this definition one gets the identities
\[
H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),
\]
\[
H'_n(x) = nH_{n-1}(x),
\]
\[
H_n(-x) = (-1)^n H_n(x)
\]

analogous to those in Nualart (1995).

We set \( V^2_X = EX^2 \). Let \( \nu \in \mathbb{N} \) s.t. \( E|X|^\nu < \infty \). We are now ready to define the approximating function \( Q^X_\nu \) to be used in the series approximations. We set
\[
Q^X_\nu(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma^X_{m+2}}{(m+2)!V^m_X} \right)^{k_m},
\]
where the summation is extended over the non-negative integer solutions \((k_1, \ldots, k_\nu)\) of the equation \( k_1 + 2k_2 + \cdots + \nu k_\nu = \nu \). Here we have \( l = \sum_{j=1}^{\nu} k_j \). The first few of these functions are
\[
Q^X_1(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^2 - 1) \frac{\gamma^X_3}{6V^3_X},
\]
\[
Q^X_2(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ((x^5 - 10x^3 + 15x) \frac{(\gamma^X_3)^2}{72V^6_X} + (x^3 - 3x) \frac{\gamma^X_4}{24V^4_X}),
\]
\[
Q^X_3(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ((x^8 - 28x^6 + 210x^4 - 420x^2 + 105) \frac{(\gamma^X_3)^3}{1296V^9_X} + (x^6 - 15x^4 + 45x^2 - 15) \frac{\gamma^X_4}{44V^6_X} + (x^4 - 6x^2 + 3) \frac{\gamma^X_5}{120V^8_X}).
\]
The approximating function of order zero is the cumulative distribution function of the standard normal distribution \( \Phi(x) \).

In the remaining of this article the process \( X = (X_t)_{t \geq 0} \) is assumed to be a Lévy process on \( \mathbb{R} \). The standard definition for Lévy processes can be found for instance from \[ \text{Bertoin (1996)} \] or \[ \text{Kyprion} \ (2006) \]. The approximation results are written for centered processes i.e. \( \mathbb{E}X_1 = 0 \).

We use the following version of the Lévy-Khinchine theorem to represent the characteristic function \( v_X(s) \). The theorem can be found in one form or another for example in \[ \text{Bertoin (1996)}; \ \text{Cont and Tankov (2004)}; \ \text{Sato (1999)} \].

**Theorem 2.3** (Lévy-Khinchine). There are unique \( \sigma^2 \geq 0, \rho \in \mathbb{R} \) and a Radon measure \( \mu \) on \( \mathbb{R} \setminus \{0\} \) satisfying

\[
\int_{\mathbb{R} \setminus \{0\}} \min(u^2,1) d\mu(u) < \infty
\]

such that

\[
\psi(s) = -\frac{1}{2}\sigma^2 s^2 + i\rho s + \int_{\mathbb{R} \setminus \{0\}} (e^{isu} - 1 - isu 1_{\{|u| \leq 1\}}) \mu(du)
\]

and

\[
v_X(s) = e^{t\psi(s)}.
\]

The measure \( \mu \) is called the Lévy measure of \( X \) and \( (\sigma^2, \rho, \mu) \) is the characteristic triplet of \( X \).

**Remark 2.4.** The random variable \( X_1 \) satisfies Cramér’s condition if the law of \( X_1 \) has absolutely continuous component w.r.t. Lebesgue measure. This follows e.g. if \( \sigma^2 \neq 0 \) or \( \mu \) has absolutely continuous component w.r.t. Lebesgue measure.

Moreover, if \( X_1 \) satisfies Cramér’s condition for some \( t > 0 \), then \( X_t \) satisfies the same condition for all \( t > 0 \) by the Lévy-Khinchine theorem.

### 3 Approximation results

In the literature, there are lots of classical asymptotic expansion results for the i.i.d. sum case. I.i.d. sums are in some sense the discrete time analogues of Lévy processes. The following theorem is presented in \[ \text{Petrov (1995)} \].

**Theorem 3.1.** Let \( \{Y_j\}_{j=1}^n \) be a sequence of i.i.d. random variables satisfying Cramér’s condition s.t. \( \mathbb{E}Y_1 = 0 \) and \( \mathbb{E}|Y_1|^k < \infty \) for some integer \( k \geq 3 \). Then

\[
P\left( \sum_{j=1}^n Y_j < \sqrt{n}\sigma Y_1 x \right) = \Phi(x) + \sum_{\nu=1}^{k-2} Q^Y_\nu(x) n^{-\frac{k-\nu}{2}} + o\left(n^{-\frac{k-2}{2}}\right)
\]

uniformly in \( x \in \mathbb{R} \).
This kind of results are presented also in Petrov (1973); Kolassa (2006); Cramér (1962). Generalisation of Theorem 3.1 is presented by Cramér (1962) as an analogue to corresponding i.i.d. sum result:

**Theorem 3.2.** Let $X_1$ satisfy Cramér’s condition, $EX_1 = 0$ and $k \geq 3$ be such an integer that $EX_1^k < \infty$. Then

$$P(X_t < xV_{X_t}) = \Phi(x) + \sum_{\nu=1}^{k-3} Q_{\nu}^{X_1}(x)t^{-\frac{k-3}{2}} + O\left(t^{-\frac{k-2}{2}}\right).$$

In fact, Cramér (1962) introduces the form for the function $Q_{\nu}^{X_1}(x)$ only implicitly. See Cramér (1962) pages 72, 98 and 99.

Next we are going to present some lemmata to scale the approximating functions $Q_{\nu}^{X_1}(x)$ with respect to $t$. The first of them is well-known but it is included here for convenience.

**Lemma 3.3.** Let $k \in \mathbb{N}$ be s.t. $EX_1^k < \infty$. Then

$$\gamma_{X_t}^k = t\gamma_{X_1}^k.$$

**Proof.** Take $q \in \mathbb{Q}^+$. Now $q = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ and

$$\gamma_{X_t}^k = \frac{1}{t^k} \left[ \frac{d^k}{ds^k} \log \nu_{X_1}(s)^{\frac{1}{n}} \right]_{s=0} = \frac{1}{n} \frac{1}{t^k} \left[ \frac{d^k}{ds^k} \log \nu_{X_1}(s) \right]_{s=0} = \frac{1}{n} \gamma_{X_1}^k.$$

By repeating the previous argument we get

$$\gamma_{X_t}^m = m \gamma_{X_t}^k = m \frac{1}{n} \gamma_{X_1}^k = q \gamma_{X_1}^k.$$

The general claim follows now by a simple density argument. □

**Lemma 3.4.** Let $\nu \in \mathbb{N}$ be s.t. $EX_1^{\nu+2} < \infty$, then

$$Q_{\nu}^{X_1}(x) = t^{-\frac{\nu}{2}} Q_{\nu}^{X_1}(x), \text{ for } x \in \mathbb{R}.$$

**Proof.** By definition,

$$Q_{\nu}^{X_1}(x) = f(x) \sum H_{\nu+2l-1}(x) \prod_{m=1}^{l} \frac{1}{k_m! \left((m+2)! E^{m+2}_{X_1}\right)^{k_m}},$$

where $f(x) = -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ and the summation is extended over all non-negative
integer solutions of the equation $\sum_{j=1}^{\nu} jk_j = \nu$, and we have $l = \sum_{j=1}^{\nu} k_j$.

$$Q_{\nu}^{X_1}(x) = f(x) \sum_{H_{\nu+2l-1}} \left( \prod_{m=1}^{\nu} \frac{t^{\gamma_{m+2}}_X}{(m+2)!V_{X_1}^{m+2}} \right)^{k_m}$$

$$= f(x) \sum_{H_{\nu+2l-1}} \left( \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_1}^{m+2}} \right)^{k_m} \right)$$

$$= f(x) \sum_{H_{\nu+2l-1}} \left( \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_1}^{m+2}} \right)^{k_m} \right)$$

$$= t^{-\frac{1}{2}} Q_{\nu}^{X_1}(x).$$

In the last step, we used the fact that $\nu = k_1 + 2k_2 + \cdots + \nu k_\nu$. 

We get the following result by combining the classical results to the previous lemmata and using some continuity arguments.

**Corollary 3.5.** Let $k \geq 3$ be integers s.t. $E|X_1|^k < \infty$ and let $X_1$ satisfy Cramér's condition and $E X_1 = 0$. Then

$$\mathbb{P}(X_t < xV_{X_1}) = \Phi(x) + \sum_{\nu=1}^{k-2} Q_{\nu}^{X_1}(x) t^{-\frac{k-3}{2}} + o \left( t^{-\frac{k-3}{2}} \right)$$

$$= \Phi(x) + \sum_{\nu=1}^{k-2} Q_{\nu}^{X_1}(x) + o \left( t^{-\frac{k-3}{2}} \right), \quad \text{uniformly in } x \in \mathbb{R}.$$

From now on in this paper, we assume (if not otherwise stated) that $E X_1 = 0$, $X_1$ satisfies Cramér’s condition and has moments of all orders i.e.

$$E|X_1|^\nu < \infty, \quad \text{for } \nu \in \mathbb{N}.$$ 

Now we have everything ready for introducing the main results of the article to get exact series representations. The proofs are in Section 5. In the following Theorems 3.6, 3.7 and 3.8, $\mu$ is assumed to be the Lévy measure of process $X$.

**Theorem 3.6.** Let the Lévy measure of $X$ have bounded support, then we get for $x_1 < x_2$ points of continuity of $\mathbb{P}(X_t < xV_{X_1})$ that

$$\mathbb{P} \left( x_1 < \frac{X_t}{V_{X_1}} < x_2 \right) = \mathbb{P}(X_t < x_2 V_{X_1}) - \mathbb{P}(X_t < x_1 V_{X_1})$$

$$= \Phi(x_2) - \Phi(x_1) + \sum_{\nu=1}^{\infty} (Q_{\nu}^{X_1}(x_2) - Q_{\nu}^{X_1}(x_1))$$

$$= \Phi(x_2) - \Phi(x_1) + \sum_{\nu=1}^{\infty} (Q_{\nu}^{X_1}(x_2) - Q_{\nu}^{X_1}(x_1)) t^{-\frac{1}{2}}.$$
There is some discussion about the Lévy measures with bounded support for example in Sato (1999). In fact, this is a reasonable class to be considered in the simulations because of the practical limitations.

Nevertheless, the result of Theorem 3.6 is true with more general conditions:

**Theorem 3.7.** Let \( \mu \) be s.t. for some \( a \geq 0 \), \( \mu(x)1_{\{|x| > a\}} \) is absolutely continuous with respect to Lebesgue measure and for some \( C, \epsilon > 0 \)
\[
\frac{d\mu(x)}{dx} \leq C \exp\{-|x|^{1+\epsilon}\}, \text{ for } |x| \geq a.
\]

Then the assertion of Theorem 3.6 holds.

And even more generally we get the following:

**Theorem 3.8.** Assume that there are \( a \geq 0 \) and \( C, \epsilon > 0 \) s.t.
\[
\mu((-x-1,-x],[x,x+1)) \leq C \exp\{-x^{1+\epsilon}\}, \text{ for } x \geq a.
\]

Then the representation of Theorem 3.6 holds.

**Remark 3.9.** In the cases of Theorems 3.6, 3.7 and 3.8, we get some series representation also for other finite dimensional distributions since the series representation can be written for all increments separately.

Moreover, we get a representation for the distribution function of the absolute value of a Lévy process as follows:

**Corollary 3.10.** Assume that the assumptions of 3.6, 3.7 or 3.8 hold. Then we get for \( x > 0 \) and \(-x\) points of continuity of \( \mathbb{P}(X_t < \cdot V_{X_t}) \) that
\[
\mathbb{P}(|X_t| < xV_{X_t}) = 2\Phi(x) - 1 + 2 \sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x) = 2\Phi(x) - 1 + 2 \sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x)t^{-\nu} \quad (2)
\]
and
\[
\mathbb{P}(|X_t| > xV_{X_t}) = 2 - 2\Phi(x) - 2 \sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x) = 2 - 2\Phi(x) - 2 \sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x)t^{-\nu}. \quad (3)
\]

**Proof.**
\[
\mathbb{P}(|X_t| < xV_{X_t}) = \mathbb{P}(X_t < xV_{X_t}) - \mathbb{P}(X_t < -xV_{X_t})
\]
\[
= \Phi(x) - \Phi(-x) + \sum_{\nu=1}^{\infty} (Q_{2\nu}^{X_t}(x) - Q_{2\nu}^{X_t}(-x))
\]
\[
= 2\Phi(x) - 1 + \sum_{\nu=1}^{\infty} (Q_{2\nu}^{X_t}(x) - Q_{2\nu}^{X_t}(-x)).
\]
We use the symmetry condition for Hermite polynomials and get
\[ Q_{\nu}^{X_t}(x) - Q_{\nu}^{X_t}(-x) \]
\[ = - \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \sum (H_{\nu+2l-1}(x) - H_{\nu+2l-1}(-x)) \prod_{m=1}^{\nu} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right) \]
\[ = 2Q_{\nu}^{X_t}(x) 1_{\{\nu=2p|p\in\mathbb{N}\}}. \]

Equation (3) is a direct consequence of (2).

If \( X_t \) has density function for all \( t > 0 \), we get the following:

**Corollary 3.11.** Assume besides the assumptions of 3.6, 3.7 or 3.8 that \( X_t \) has density function \( g_{X_t}(s) \) for all \( t > 0 \). Then

\[ g_{X_t}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{\nu=1}^{\infty} \frac{d}{dx}Q_{\nu}^{X_t}(x). \]

Corollary 3.11 gives us together with the following lemma an exact series representation for the density function.

**Lemma 3.12.** For \( \nu \in \mathbb{N} \) we have

\[ \frac{d}{dx}Q_{\nu}^{X_t}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right)^{k_m}, \]

with the notation of (7).

**Proof.**

\[ \frac{d}{dx}Q_{\nu}^{X_t}(x) = \left( \frac{d}{dx} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \right) \sum H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right)^{k_m} \]
\[ - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum \frac{d}{dx}H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right)^{k_m} \]
\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum (xH_{\nu+2l-1}(x) - (\nu + 2l - 1)H_{\nu+2l-2}(x)) \times \]
\[ \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right)^{k_m} \]
\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!V_{X_t}^{m+2}} \right)^{k_m}. \]

In the last step, we used the recursion formula for the Hermite polynomials. \( \square \)
Remark 3.13. The approximation results of this section such as Theorem 3.6 give exact series representation also for any infinitely divisible distribution satisfying the conditions for $X_1$ in each theorem, since any infinitely divisible distribution can be considered as the one dimensional distribution of some Lévy process at time 1.

4 Insurance Applications

Let us consider briefly Lévy processes with only positive (respectively negative) jumps and drift term. This is a reasonable class for risk processes, more precisely claim surplus processes in the sense of [Asmussen (2000)]. This class includes spectrally positive (negative) Lévy processes without gaussian component in the sense of [Kyprianou (2006)].

Remark 4.1 (Risk process). Consider a Lévy process $X$ satisfying conditions of Theorem 3.7, 3.7 or 3.8. Furthermore, assume that its Lévy measure is concentrated on positive reals and satisfies $\int_{\mathbb{R} \setminus \{0\}} |x| \mu(dx) < \infty$. Then there is some $x_1 \in \mathbb{R}$ s.t. $P(X_t < x_1 V_{X_t}) = 0$ for all $t > 0$ and thus we get easily a series representation for $P(X_t < x_2 V_{X_t})$ alone.

Remark 4.2. Throughout the paper we have assumed that the process is centered. This assumption is only technical since we can write for non-centered $X_t$ that $$P(X_t < x) = P(X_t - \mathbb{E}X_t < y V_{X_t} - \mathbb{E}X_t),$$ where $$y = \frac{x - \mathbb{E}X_t}{V_{X_t} - \mathbb{E}X_t} = \frac{x - \mathbb{E}X_t}{\sqrt{\text{Var} X_t}}.$$

The Edgeworth approximation is widely used in insurance applications [Beard et al. (1977), Asmussen (2000)]. Present results justify the use of Edgeworth expansion of any order to approximate the claim surplus process in a Lévy driven model. By increasing the order of the approximation we will asymptotically get rid of the error term. The series representation can be written for all $t > 0$. Naturally we will have to take more correction terms $Q^{X_t}_x(x)$ into account if $t$ is small or $|x|$ is large to get sharp estimates.

If there exist only $k$ first moments (the heavy tailed case), Corollary 3.5 tells to what extent one can refine the approximation.

Even the most restrictive case of the main result of this article, Theorem 3.6 can be justified by actuarial reasoning. The bounded support of the Lévy measure corresponds to the case that the insurer has arranged an excess-of-loss reinsurance [Asmussen (2000)].

5 Proofs

The following lemma gives us a representation formula for the cumulants of a Lévy process. The result may be well known but it is included in this pa-
per for convenience. It is worth mentioning that Cramér’s condition is not assumed in the following lemma. The condition is used in the literature e.g. by Nualart and Schoutens (2000). This condition is enough to guarantee the existence of all moments. On the other hand, processes satisfying the assumptions of Theorem 3.6, 3.7 or 3.8 also satisfy condition (4).

**Lemma 5.1.** Let $(\sigma^2, \rho, \mu)$ be the characteristic triplet of $X$. Furthermore, assume that for some $\lambda > 0$ and for all $\delta > 0$

$$\int_{\mathbb{R}\setminus (-\delta, \delta)} e^{\lambda|x|} \mu(dx) < \infty. \tag{4}$$

Then

$$\gamma^X_1 = \int_{\mathbb{R}\setminus \{0\}} x^\nu \mu(dx), \quad \nu \geq 3,$$

and

$$\gamma^X_2 = \int_{\mathbb{R}\setminus \{0\}} x^2 \mu(dx) + \sigma^2.$$

The proof is a straightforward computation using Lévy-Itô decomposition and it is omitted. The next lemma gives us another characterisation of the condition on the Lévy measure in Theorem 3.6. From now on in this article, we will use the following notation of scaled cumulants $\lambda^X_{\nu} = \frac{\gamma^X_{\nu}}{V^X_1}$, for $\nu \in \mathbb{N}$.

**Lemma 5.2.** The Lévy measure of process $X$ is concentrated on some bounded interval is equivalent to the condition that there exists some $C > 0$ s.t. $\lambda^X_{\nu} \leq C^\nu$, for all $\nu \in \mathbb{N}$.

**Proof.** Let us first assume that such $C$ exists. Now we can use Lemma 5.1 and we get for $\nu \geq 3$ that

$$\int_{\mathbb{R}\setminus \{0\}} x^\nu \mu(dx) \leq C^\nu V^X_1.$$

For even $\nu$, $|\gamma_{\nu}| = \gamma_{\nu}$. We know also by Rudin (1987) page 71 that it holds for $L^p(\mu)$ norms that

$$||x||_{2n+1} \leq \max(||x||_{2n}, ||x||_{2n+2}), \quad \text{for } n \geq 1.$$

Hence there is some $D > 0$ s.t. $\int_{\mathbb{R}\setminus \{0\}} |x|^\nu \mu(dx) \leq D^\nu$ for all $\nu \geq 4$. Moreover, we get

$$D \geq ||x||_\nu \rightarrow ||x||_\infty \quad \text{as } \nu \rightarrow \infty.$$

Now $||\frac{x}{v^X_1}||_\infty \leq 1$ with respect to $\mu$. In other words, $\mu$ is concentrated on some bounded interval.

The other way is even simpler. Because $\mu$ is concentrated on some bounded interval, it follows that $||x||_\infty < \infty$. We can choose $C = \frac{1}{V^X_1} \sup_\nu ||x||_\nu$.  

Now we have everything ready for the proofs of the main results.
In the last step, we used the characterisation of Lemma 5.2. Now this series converges to an analytic function of $t > \epsilon$ which is bounded when $t > \epsilon > 0$. Let us first work out the representation for the logarithm of the characteristic function i.e. the characteristic exponent of the Lévy process.

$$
\sum_{\nu=2}^{\infty} \left| \frac{\lambda X_1}{\nu} \frac{1}{\nu!} (is)^\nu \right| = \sum_{\nu=2}^{\infty} \left| \frac{1}{\nu!} \frac{t \gamma_x}{\nu} V^{\nu}_{X_1} (is)^\nu \right|
$$

$$
= \sum_{\nu=2}^{\infty} \left| \frac{1}{\nu!} \frac{t^{\nu - 2}}{\nu} \frac{1}{\nu!} V^{\nu}_{X_1} (is)^\nu \right|
$$

$$
= \sum_{\nu=2}^{\infty} \left| \frac{\lambda X_1}{\nu} \frac{1}{\nu!} \frac{(is)^\nu}{\sqrt{t}} \right| \leq t \sum_{\nu=2}^{\infty} \frac{1}{\nu!} \frac{C s^\nu}{\sqrt{t}},
$$

which is bounded when $t > \epsilon > 0$ and $|s| < K < \infty$ for arbitrary $\epsilon, K \in (0, \infty)$. In the last step, we used the characterisation of Lemma 5.2. Now this series is dominated by the series expansion of the exponential function and thus the series

$$
\sum_{\nu=2}^{\infty} \frac{\lambda X_1}{\nu} (is)^\nu
$$

converges to an analytic function of $s$ when $t > 0$ is fixed. Now, define

$$
f_{X_t}(s) = v_{X_t} \left( \frac{s}{V_{X_t}} \right).
$$

By computing the cumulants, this notation gives for $n \in \mathbb{N}$

$$
\left[ \frac{d^n}{ds^n} \log f_{X_t}(s) \right]_{s=0} = \left[ \frac{d^n}{ds^n} \log v_{X_t} \left( \frac{s}{\sqrt{t V_{X_t}}} \right) \right]_{s=0}
$$

$$
= t \left( \frac{1}{\sqrt{t V_{X_t}}} \right) \left[ \frac{d^n}{ds^n} \log v_{X_t}(s) \right]_{s=0}
$$

$$
= t^{\frac{n - 2}{2}} \frac{n \gamma_x}{V_{X_1}} = t^{\frac{n - 2}{2}} \frac{n \lambda_n}{V_{X_1}} = t^{\frac{n - 2}{2}} \lambda_n X_1.
$$

Now

$$
\log f_{X_t}(s) = \sum_{\nu=2}^{\infty} \frac{\lambda X_1}{\nu} \frac{t^{-\frac{\nu - 2}{2}} (is)^\nu}{\nu!}.
$$

We observe that $\lambda_n X_1 = 1$ for all $t > 0$. So we obtain

$$
f_{X_t}(s) = e^{-\frac{t}{2}} \exp \left( \sum_{j=1}^{\infty} \frac{\lambda X_1}{(j + 2)!} t^{-\frac{j}{2}} (is)^{j+2} \right).
$$

Next, consider a more general form

$$
\exp \left( \sum_{j=1}^{\infty} \frac{\lambda X_1}{(j + 2)!} z^{j+2} \right).
$$
With fixed \( u \), this series converges absolutely, uniformly in any compact set with respect to the parameter \( z \). Thus in every compact set with respect to \( z \), we rearrange the series of the exponential function and get a series representation with respect to \( z \). Hence,

\[
\exp \left( \sum_{j=1}^{\infty} \frac{\lambda_j X_1}{(j+2)!} z^{j+2} u^{j+2} \right) = 1 + \sum_{\nu=1}^{\infty} P_{\nu}(u) z^{\nu}
\]

for some polynomials \( (P_{\nu})_{\nu=1}^{\infty} \) that can be computed formally by compounding these two series, which is possible due to the absolute convergence. Now

\[
f_{X_1}(s) = e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty} P_{\nu}(is) e^{-\frac{t^2}{2} t^{-\frac{s^2}{2}}}.
\]

By the inversion formula of the characteristic function (Petrov, 1995), we get

\[
P(X_t < x_2 V_{X_t}) - P(X_t < x_1 V_{X_t}) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-i s x_2} - e^{-i s x_1}}{-i s} \left( e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty} P_{\nu}(is) e^{-\frac{t^2}{2} t^{-\frac{s^2}{2}}} \right) ds.
\]

With fixed \( t > 0 \), the series inside the integral is absolutely convergent uniformly in compact sets with respect to \( s \). Thus the integral is always well-defined and can be computed term-wise. Moreover, the limit exists since

\[
\int_{-\infty}^{\infty} e^{-i s x_2} - e^{-i s x_1} f_{X_1}(s) ds
\]

$$
- \int_{-T}^{T} e^{-i s x_2} - e^{-i s x_1} \left( e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty} P_{\nu}(is) e^{-\frac{t^2}{2} t^{-\frac{s^2}{2}}} \right) ds
$$

$$
= \int_{|s| > T} e^{-i s x_2} - e^{-i s x_1} f_{X_1}(s) ds \to 0, \quad \text{when } T \to \infty,
$$

since \( f_{X_1} \) is characteristic function of \( \frac{X_1}{\Sigma_{X_1}} \).

Hence, there are such functions \( (R_{\nu})_{\nu=1}^{\infty} \) that we can write

\[
P(X_t < x_2 V_{X_t}) - P(X_t < x_1 V_{X_t}) = \Phi(x_2) - \Phi(x_1) + \sum_{\nu=1}^{\infty} (R_{\nu}(x_2) - R_{\nu}(x_1)) t^{-\frac{s^2}{2}}.
\]

We use the classical Theorem 3.1 and the scaling Lemma 3.4 and find out that for all \( \nu = 1, 2, \ldots \)

\[
R_{\nu}(x) = Q_{\nu}^{X_1}(x) = t^{\frac{s^2}{2}} Q_{\nu}^{X_1}(x).
\]
Proof. (Theorem 3.7)

The proof proceeds analogously to the proof of Theorem 3.6 but we have to argue why we can rearrange the series of

\[ f_X(s) = e^{-\frac{s^2}{2}} \exp \left( \sum_{j=1}^{\infty} \frac{\lambda^{X_i}}{(j+2)!} t^{-\frac{j}{2}} (is)^{j+2} \right). \]  

(5)

With present assumptions on the Lévy measure \( \mu \), we can use the representation Lemma 5.1 for the cumulants. Let \( m \in \mathbb{N} \) be such that \( \frac{1}{m} \leq \epsilon \). Observe now that

\[
\int_0^\infty x^n e^{-x^{1+\frac{1}{m}}} \, dx = \int_0^\infty -\frac{m}{m+1} x^n \left( -\frac{m}{m+1} e^{-x^{1+\frac{1}{m}}} \right) \, dx \\
= -\frac{m}{m+1} \left[ x^n \left( -\frac{m}{m+1} e^{-x^{1+\frac{1}{m}}} \right) \right]_0^\infty + \int_0^\infty \frac{m}{m+1} \left( n-1 \right) x^{n-2} \frac{2}{m} e^{-x^{1+\frac{1}{m}}} \, dx \\
= \left( \frac{m}{m+1} \right)^{n-m} \prod_{j=1}^{n-m} \left( n+1-j \left( 1+ \frac{1}{m} \right) \right) \int_0^\infty x^{n-\left[ \frac{n-m}{m+1} \right]} e^{-x^{1+\frac{1}{m}}} \, dx \\
\leq \prod_{j=1}^{n-m} \left( n+1-j \left( 1+ \frac{1}{m} \right) \right) \times D,
\]

where

\[ D = \max_{t=0,\ldots,m} \int_0^\infty x^{t-\left[ \frac{n-m}{m+1} \right]} e^{-x^{1+\frac{1}{m}}} \, dx. \]

Note that the constant \( D \) is finite and does not depend on \( n \). Without loss of generality, we can assume \( X \) to be compensated compound Poisson process with \( a = 0 \), since we can express general \( X \) as a sum of this kind of process and a process satisfying the conditions of Theorem 3.6. Then we get a bound for (5) by the additivity of cumulants.

Note that this decomposition can be made such a way that Cramér’s condition does not fail here if the Lévy measure has unbounded support. This is due to the fact that the tail of the Lévy measure is absolutely continuous with respect to Lebesgue measure. Now we have

\[
\sum_{\nu=2}^{\infty} \frac{\gamma_i^X}{V_i^X \nu!} (is)^\nu \\
= \sum_{\nu=2}^{m} \frac{\gamma_i^X}{V_i^X \nu!} (is)^\nu + t \sum_{\nu=m+1}^{\infty} \frac{1}{\nu!} \left| \frac{s}{\sqrt{V_X}} \right| \gamma_i^X \left( \frac{m+1}{m+1} \right)^\nu \\
= \sum_{\nu=2}^{m} \frac{\gamma_i^X}{V_i^X \nu!} (is)^\nu + t \sum_{j=1}^{\infty} \sum_{k=0}^{m} \frac{1}{(m+1)\nu!} \left| \frac{s}{\sqrt{V_X}} \right| \gamma_i^X \left( \frac{m+1}{m+1} \right)^{(m+1)j+k}.
\]

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The first term is a finite sum of finite summands if $0 < t < \infty$. We get an estimate for the other sum as follows

$$t \sum_{j=1}^{\infty} \sum_{k=0}^{m} \frac{1}{((m+1)j+k)!} \left| \gamma_{(m+1)j+k} \right| \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{(m+1)j+k} \leq t \sum_{j=1}^{\infty} \sum_{k=0}^{m} \frac{1}{((m+1)j+k)!} \times \\
2CD \prod_{l=1}^{\left\lfloor \frac{((m+1)j+k)}{m+1} \right\rfloor} ((m+1)j+k+1-l) \left( 1 + \frac{1}{m} \right) \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{(m+1)j+k}.$$ 

Now define

$$g(l) = (m+1)j+k+1-l - \left\lfloor \frac{l}{m} \right\rfloor, \quad l = 1, \ldots, \left\lfloor \frac{(m+1)j+k}{m+1} \right\rfloor.$$ 

We observe that $g(l) > g(l+1)$ and the values of $g$ are integers from 1 to $(m+1)j+k$. Nevertheless, $g$ does not take every $(m+1)$th integer value. This fact is due to the jump of the floor function. So there is at least $j$ terms missing in the product. By assuming them to be the $j$ smallest ones, we get a rough estimate

$$\prod_{l=1}^{\left\lfloor \frac{((m+1)j+k)}{m+1} \right\rfloor} ((m+1)j+k+1-l) \left( 1 + \frac{1}{m} \right) \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{(m+1)j+k} \leq \frac{(m+1)j+k)!}{j!}.$$ 

And finally

$$t \sum_{j=1}^{\infty} \sum_{k=0}^{m} \frac{1}{((m+1)j+k)!} \left| \gamma_{(m+1)j+k} \right| \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{(m+1)j+k} \leq t \sum_{j=1}^{\infty} \sum_{k=0}^{m} \frac{1}{j!} 2CD \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{(m+1)j+k} \leq 2CDt \left( \sum_{k=0}^{m} \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^k \right) \sum_{j=1}^{\infty} \frac{1}{j!} \left( \left( \frac{|s|}{\sqrt{tV_{X_1}}} \right)^{m} \right)^j < \infty,$$

as an exponential series when $0 < t < \infty$. The last part of the proof is analogous to the proof of Theorem 3.6. \[\square\]

Proof. (Theorem 3.8)

We have to get a suitable estimate for the cumulants from above to be able to continue as in the proof of Theorem 3.7. Let us define function $\eta$ on positive reals as follows

$$\eta(x) = \mu((-x - 1, -x], [x, x + 1)).$$

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We can easily represent the growing condition for the Lévy measure using this function. Now we can estimate the cumulants in the spirit of Lemma 5.1. Without loss of generality, we can assume that \( a \geq 1 \). We get

\[
\int_{-\infty}^{\infty} |x|^\nu \mu(dx) \leq \int_{a}^{a} |x|^\nu \mu(dx) + \sum_{j=0}^{\infty} |j + a + 1|^\nu \eta(a + j).
\]

For \( \nu \geq 2 \), the first term is bounded by \( D^\nu \) for some \( D > 0 \). For the second term we get

\[
\sum_{j=0}^{\infty} |j + a + 1|^\nu \eta(a + j) \leq \int_{a}^{\infty} (x + 2)^\nu Ce^{-x^{1+\epsilon}} dx \leq C \int_{a}^{\infty} (3x)^\nu e^{-x^{1+\epsilon}} dx.
\]

The rest of the proof is analogous to the proof of Theorem 3.7.

**Proof.** (Corollary 3.11)

Let \( (P_\nu)_{\nu=1}^{\infty} \) be the same polynomials as in the proof of Theorem 3.6. We can use the series representation for characteristic function of \( \frac{X}{V_t} \) and get

\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} f_{X_t}(s) ds = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} \left( e^{-s^2/2} + \sum_{\nu=1}^{\infty} P_\nu(is)e^{-s^2/2} t^{-\frac{\nu}{2}} \right) ds.
\]

With fixed \( t > 0 \), the absolute convergence is uniform in compact sets with respect to \( s \), as in the preceding proofs. Thus, the integral is well-defined and can be computed term-wise. Moreover, with fixed \( x \in \mathbb{R} \)

\[
\int_{|s|>T} e^{-isx} f_{X_t}(s) ds \to 0, \quad \text{as} \; T \to \infty,
\]

since \( f_{X_t} \) is a characteristic function of some random variable with density function. Hence,

\[
\lim_{T \to \infty} \left| g_{X_t}(x) - \frac{1}{2\pi} \int_{-T}^{T} e^{-isx} \left( e^{-s^2/2} + \sum_{\nu=1}^{\infty} P_\nu(is)e^{-s^2/2} t^{-\frac{\nu}{2}} \right) ds \right| = \lim_{T \to \infty} \left| \frac{1}{2\pi} \int_{|s|>T} e^{-isx} f_{X_t}(s) ds \right| = 0.
\]

We have shown that there is some series representation but we still have to show that the limit equals to what is claimed. We have

\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} P_\nu(is)e^{-s^2/2} t^{-\frac{\nu}{2}} ds = \frac{1}{2\pi} \int_{\mathbb{R}} i s e^{-is\nu} P_\nu(is)e^{-s^2/2} t^{-\frac{\nu}{2}} ds = \frac{d}{dx} Q_{\nu,X_t}(x).
\]

\( \square \)
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