On diagonal lower bound of Markov kernel from $L^2$ analyticity.

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Abstract

Let $\Gamma$ be a graph and $P$ be a reversible random walk on $\Gamma$. From the $L^2$ analyticity of the Markov operator $P$, we deduce an on-diagonal lower bound of an iterate of odd number of $P$. The proof does not require the doubling property on $\Gamma$ but only a polynomial control of the volume.

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3 Application to the $L^p$-boundedness of the Riesz transform for $p \in (1,2)$ under subgaussian estimates 8
We use the following notations. If \( x \in E \), \( A(x) \lesssim B(x) \) means that there exists \( C \) independent of \( x \in E \) such that \( A(x) \leq C B(x) \) for all \( x \in E \). \( A(x) \approx B(x) \) means that \( A(x) \lesssim B(x) \) and \( B(x) \lesssim A(x) \). The dependance of the constant \( C \) on the parameters will be either obvious or recalled.

1 Introduction and statement of the results

Let \((X, m)\) be a measured space and \( L \) be an unbounded operator on \( L^2(X, m) \). The analyticity of the semigroup \((e^{-tL})_{t>0}\) on \( L^2(X) \) can be characterized by: there exists \( C > 0 \) such that, for all \( t > 0 \),

\[
\|Le^{-tL}\|_{2 \rightarrow 2} \leq \frac{C}{t},
\]

where \( \|T\|_{p \rightarrow q} \) denotes the operator norm \( \sup_{f} \|Tf\|_{L^q} \). By the spectral theorem, one can check that \( L \) is analytic whenever \( L \) is a selfadjoint operator whose spectrum \( \text{Sp}_{L^2}(L) \) is included \([0, +\infty)\).

Let \( \Gamma \) be a graph (see section 1.1 for precise definitions). We consider a reversible random walk \( P \) on \( \Gamma \). Thus \( P \) is self-adjoint and power-bounded, that is \( \|P^n\|_{q \rightarrow q} \leq 1 \) for all \( n \in \mathbb{N} \) and \( q \in [1, +\infty] \). Hence, for all \( q \in [1, +\infty] \), \( \text{Sp}_{L^q}(P) \subset [-1, 1] \). We say that \((P^n)_{n \in \mathbb{N}} \) is analytic on \( L^2(\Gamma) \) if, there exists \( C > 0 \) such that for all \( n \in \mathbb{N}^* \),

\[
\|(I - P)P^{n-1}\|_{2 \rightarrow 2} \leq \frac{C}{n}.
\]

Since \( \Delta := I - P \) is the discrete Laplacian associated to \( P \), assumption \((2)\) can be seen as a discrete analogue of assumption \((11)\). However, even if \( \text{Sp}(\Delta) \subset [0, 2] \subset [0, +\infty) \), the spectral theorem yields that the additional assumption

\[
-1 \notin \text{Sp}_{L^2}(P),
\]

is required for \((P^n)_{n \in \mathbb{N}} \) to be analytic on \( L^2 \). Indeed, if we choose the uniform weight on \( Z \) and if \( P \) is the Markov operator where the probability to move one step forward and backward is \( \frac{1}{2} \), then \( \|(I - P)P^n\|_{2 \rightarrow 2} = 1 \) for all \( n \in \mathbb{N} \).

Another assumption is frequently used (see \([6, 5, 10, 9, 11, 12, 7, 13, 2, 1]\) and \([4, \text{Remark p.698}]\)): there exists \( \varepsilon > 0 \) such that for all \( x \in \Gamma \)

\[
p(x, x)m(x) \geq \varepsilon,
\]

where \( p(x, x)m(x) \) denotes the transition probability from \( x \) to \( x \). The condition \((4)\) implies \((3)\) (see \([13, 8]\]), but the converse implication remains unclear.

The aim of this article is to provide a converse to the implication \((1) \Rightarrow (4)\). We prove, under an at most polynomial growth of the volume, that \((3)\) causes an iterate of odd number of \( P \) to satisfy \((4)\). As an application, we give an improved version of a result in \((11)\), which establishes, for \( p \in (1, 2) \), the \( L^p \) boundedness of the Riesz transform under subgaussian estimates.

The plan of the paper is as follows. The remaining of the first chapter is devoted to, in order, the presentation of the graph structure, the introduction of the assumptions used in this article and the accurate statement of our main result. The second section is dedicated to the proof of the main result. Finally, in the last section, we present an application of our main result to weaken the assumptions of \((11)\) Corollary 1.30] on disret setting.

1.1 The discrete setting

Let \( \Gamma \) be an denombrable (infinite) set and \( \mu_{xy} = \mu_{yx} \geq 0 \) a symmetric weight on \( \Gamma \times \Gamma \).

We define the weight \( m(x) \) of a vertex \( x \in \Gamma \) by \( m(x) = \sum_{x \sim y} \mu_{xy} \). More generally, the volume (or measure) of a subset \( E \subset \Gamma \) is defined as \( m(E) := \sum_{x \in E} m(x) \).

The \( L^p(\Gamma) \) spaces are defined as follows: for all \( 1 \leq p < +\infty \), a function \( f \) on \( \Gamma \) belongs to \( L^p(\Gamma, m) \) (or \( L^p(\Gamma) \)) if

\[
\|f\|_p := \left( \sum_{x \in \Gamma} |f(x)|^p m(x) \right)^{\frac{1}{p}} < +\infty.
\]

Let us define for all \( x, y \in \Gamma \) the discrete-time reversible Markov kernel \( p \) associated with the measure \( m \) by \( p(x, y) = \frac{\mu_{xy}}{m(x)m(y)} \). The discrete kernel \( p_k(x, y) \) is then defined recursively for all \( k \geq 0 \) by

\[
\begin{cases}
p_0(x, y) = \frac{\delta(x, y)}{m(y)}, \\
p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z)p_k(z, y)m(z).
\end{cases}
\]
Notice that for all \( k \geq 1 \), we have for all \( x \in \Gamma \)
\[
\|p_k(x, \cdot)\|_{L^1(\Gamma)} = \sum_{y \in \Gamma} p_k(x, y)m(y),
\]
and that the kernel is symmetric, that is \( p_k(x, y) = p_k(y, x) \) for all \( x, y \in \Gamma \).

For all functions \( f \) on \( \Gamma \), we define \( P \) as the operator with kernel \( p_k \), i.e.
\[
P f(x) = \sum_{y \in \Gamma} p(x, y)f(y)m(y) \quad \forall x \in \Gamma.
\]

It is easily checked that \( P^k \) is the operator with kernel \( p_k \).

Since \( p(x, y) \geq 0 \) and (6) holds, one has, for all \( k \in \mathbb{N} \) and all \( p \in [1, +\infty] \),
\[
\|P^k\|_{p \rightarrow p} \leq 1.
\]

Let us define now the metric on the graph. In the following, \( \epsilon \) denotes a non-negative real number (\( \epsilon \geq 0 \)).

We say that \( x, y \in \Gamma \) are neighbors if \( p(x, y) > 0 \). The notation \( x \sim y \) is used when \( x \) and \( y \) are neighbors. In the same way, we say that \( x, y \in \Gamma \) are \( \epsilon \)-neighbors if \( p(x, y) \min\{m(y), m(x)\} > \epsilon \). If \( x \) and \( y \) are neighbors, we write \( x \sim_{\epsilon} y \), otherwise, we write \( x \not\sim_{\epsilon} y \).

We use \( \epsilon \)-neighbors to define paths in the following way: an \( \epsilon \)-safe path (joining \( x \) to \( y \), of length \( n \)) in \( \Gamma \) is a sequence \( x = x_0, x_1, \ldots, x_n = y \) of vertices of \( \Gamma \) such that \( x_{i-1} \sim_{\epsilon} x_i \) for all \( i \in [1, n] \). A closed \( \epsilon \)-safe path is a path \( x = x_0, x_1, \ldots, x_n = x \) with the same vertex \( x \) at the beginning and the end of the sequence.

The distance \( d_{\epsilon}(x, y) \) between \( x \) and \( y \) is defined as the length of the shortest \( \epsilon \)-safe path linking \( x \) to \( y \). We write \( d_{\epsilon}(x, y) = +\infty \) if no \( \epsilon \)-safe path linking \( x \) to \( y \) exists. Note that \( d(x, y) := d_0(x, y) \) is the classical distance on graphs.

Then an \( \epsilon \)-ball \( B_{\epsilon} = B_{\epsilon}(x_0, r) \) is defined as the set containing all vertices \( x \in \Gamma \) such that \( d_{\epsilon}(x_0, x) < r \). An \( \epsilon \)-ball \( B_{\epsilon} \) is said to be of radius \( r \) if there exists \( x_0 \in \Gamma \) such that \( B_{\epsilon} = B_{\epsilon}(x_0, r) \). Finally, we use the notation \( V(x_0, r) \) for \( m(B_0(x_0, r)) \).

**Remark 1.1.** Note that \( B_{\epsilon}(x, r) \supset B_{\eta}(x, r) \) if \( \epsilon \leq \eta \). Consequently, one has \( m(B_{\epsilon}(x_0, r)) \leq V(x_0, r) \).

We assume in all the article that \( \Gamma \) is locally uniformly finite, that is there exists \( N_{\text{dvl}} \in \mathbb{N} \) such that, for all \( x \in \Gamma \),
\[
\#B_0(x, 2) = \# \{ \{y \in \Gamma, y \sim x\} \cup \{x\}\} \leq N_{\text{dvl}}.
\]
At last, let us state the definition of the boundary of a set: the \( \epsilon \)-boundary of \( E \subset \Gamma \) is the set
\[
\partial_{\epsilon}E = \{ x \in E, d_{\epsilon}(x, E^c) = 1 \}.
\]

### 1.2 Statement of the assumptions

**Definition 1.2.** We say that \( (\Gamma, \mu) \) (or \( P \)) satisfies (LB) if there exists \( \epsilon = \epsilon_{LB} > 0 \) such that
\[
p(x, x)m(x) \geq \epsilon \quad \forall x \in \Gamma.
\]

**Definition 1.3.** We say that \( (\Gamma, \mu) \) satisfies (PDV) if there exists \( C_{pdv}, d > 0 \) such that
\[
V(x, r) \leq C_{pdv}r^d m(x) \quad \forall x \in \Gamma, \forall r \in \mathbb{N}^*.
\]

**Remark 1.4.** Define the doubling property \( (DV) \): there exists \( C_{dv} > 0 \) such that
\[
V(x, 2r) \leq C_{dv}V(x, r) \quad \forall x \in \Gamma, \forall r \in \mathbb{N}^*.
\]

Define also the local doubling property \( (DVL) \): there exists \( C_{dvl} > 0 \) such that
\[
V(x, 2) \leq C_{dvl}m(x) \quad \forall x \in \Gamma.
\]

We have the following implications: \( (DV) \Rightarrow (PDV) \Rightarrow (DVL) \).

**Remark 1.5.** Note that if \( (\Gamma, \mu) \) satisfies \( (DVL) \), then \( P^2 \) satisfies (LB) (see for example [5, Lemma 3.2]).

**Definition 1.6.** Let \( \epsilon \geq 0 \). A subset \( E \subset (\Gamma, \mu) \) is said to be \( \epsilon \)-bipartite if all closed \( \epsilon \)-safe paths in \( E \) are of even length.

**Remark 1.7.** If \( E \) is \( \epsilon \)-bipartite, then for all \( x \in E \), \( p(x, x)m(x) \leq \epsilon \).

Let us state the following result, proven in subsection (2.1). Note that, here and after, if \( E \subset \Gamma \), by “partition” of \( E \), we mean a collection of pairwise (possibly empty) disjoint subsets of \( E \).

**Proposition 1.8.** Let \( \epsilon \geq 0 \) and \( E \subset \Gamma \). The following conditions are equivalent

(i) \( E \) is \( \epsilon \)-bipartite.

(ii) There exists a partition \( (E_0, E_1) \) of \( E \) such that if \( i \in \{0, 1\} \) then
\[
x, y \in E_i \Rightarrow x \not\sim_{\epsilon} y.
\]
1.3 Main result

Theorem 1.9. Let $(\Gamma, \mu)$ be a graph satisfying [PDV]. The following conditions are equivalent:

(i) $-1 \in S_{PL}(P)$ for all $q \in [1, +\infty)$,

(ii) $-1 \in S_{PL}(P)$,

(iii) for all $k \in \mathbb{N}$, $P^{2k+1}$ does not satisfy [LB], that is for all $\epsilon > 0$, there exists $x_0 \in \Gamma$ such that

$$p_{2k+1}(x_0, x_0)m(x_0) \leq \epsilon.$$

(iv) for all $\epsilon > 0$ and all $r \in \mathbb{N}^*$, there exists an $\epsilon$-ball $B_r$ of radius $r$ that is $\epsilon$-bipartite.

2 Proofs

2.1 Proof of Proposition 1.8

Proof: Let us prove (i) $\Rightarrow$ (ii). We assume first that $E$ is $\epsilon$-connected, that is, for all $x, y \in E$, $d_r(x, y) < +\infty$. Indeed, let $A, B \subset \Gamma$ such that $(x, y) \notin A \times B$ implies $x \sim_r y$. If $(E_0^A, E_1^A)$ and $(E_0^B, E_1^B)$ are two partitions of respectively $A$ and $B$ satisfying $[\overline{B}]$, then $(E_0^A \cup E_0^B, E_1^A \cup E_1^B)$ and $(E_0^B \cup E_1^A, E_1^B \cup E_1^A)$ are both a partition of $A \cup B$ satisfying $[\overline{B}]$.

Let now $E$ be $\epsilon$-connected. Fix $e \in E$. Define now

$$E_0 = \{x \in E, \text{ there exists an } \epsilon\text{-safe path of even length in } E \text{ linking } e \text{ to } x\}$$
and

$$E_1 = \{x \in E, \text{ there exists an } \epsilon\text{-safe path of odd length in } E \text{ linking } e \text{ to } x\}.$$

- Since $E$ is $\epsilon$-connected, one has $E_0 \cup E_1 = E$.
- Check that $E_0 \cap E_1 = \emptyset$. Assume the contrary and let $f \in E_0 \cap E_1$. Then, since $f \in E_0$, there exists an integer $n$ and an $\epsilon$-safe path in $E$ given by:

$$e = x_0, \ldots, x_{2n} = f,$$

and since $f \in E_1$, there exists an integer $m$ and an $\epsilon$-safe path in $E$ given by:

$$e = y_0, \ldots, y_{2m+1} = f.$$

Thus $x_0, \ldots, x_{2n}, y_{2m}, \ldots, y_0$ is a closed $\epsilon$-safe path in $E$ of odd length, which is a contradiction.

- Let $x, y \in E$ such that $x \sim_r y$. If $x \in E_1$, then there exist an integer $n$ and an $\epsilon$-safe path $e = x_0, \ldots, x_{2n+1} = x$. Consequently, $e = x_0, \ldots, x_{2n+1}$ is an $\epsilon$-safe path of length $2n + 1 + i$, and thus $y \in E_{1-i}$.

Assume now that $E$ is not $\epsilon$-connected. Let $(E_1^1, \ldots, E_1^n)$ be the $\epsilon$-connex components of $E$ (that is, $(E_1^1, \ldots, E_1^n)$ is a partition of $E$ such that for all $x \in E_j$ and $y \in E_k$, $d(x, y) < +\infty$ if $j = k$ and $d_r(x, y) = +\infty$ otherwise). According to the above proof, for all $k \in \{1, \ldots, n\}$, there exists a partition $(E_0^k, E_1^k)$ of $E^n$ satisfying $[\overline{B}]$. Since no edges are linking $E_j$ to $E_k$ whenever $j \neq k$, the partition $(E_0^k, E_1^k)$ of $E$, where for any $i \in \{0, 1\}$

$$E_i := \bigcup_{k=1}^n E_i^k$$
satisfies $[\overline{B}]$.

Let us now turn to the proof of (ii) $\Rightarrow$ (i). There exists a partition $(E_0, E_1)$ of $E$ satisfying $[\overline{B}]$. Let $e = x_0, \ldots, x_n = e$ be a closed $\epsilon$-safe path in $E$. Without loss of generality, we can assume $e \in E_0$. Then, for all $k \in \{0, n\}$, $x_k \in E_i$ if and only if $k-i$ is even. As a consequence, since $x_n = e \in E_0$, the integer $n$ is even. As a conclusion, any closed $\epsilon$-safe path in $E$ is of even length. $\square$
2.2 Theorem 1.9

We split the proof in three parts. First, notice that two of the implications are easy to prove.

Proof: [Theorem 1.9 (i) implies (ii) and (ii) implies (iii)].

(i) implies (ii) is immediate. For the second implication, notice that

\[ -1 \in Sp_{P^2}(P) \implies \forall k \in \mathbb{N}, \ -1 \in Sp_{P^{2k+1}}(P^2) \implies \forall k \in \mathbb{N}, \ P^{2k+1} \text{ does not satisfy } [\text{LB}]. \]

The next implication make a link between the spectre of \( P \) and the geometry of the graph.

Proof: [Theorem 1.9 (iii) implies (iv)].

Fix \( \epsilon > 0 \) and \( r \in \mathbb{N}^* \).

**First step** We claim that there exists \( x_0 \in \Gamma \) such that there doesn’t exist any \( \epsilon \)-safe path of length \( 2r - 1 \) starting from \( x_0 \).

Indeed, since \( P^{2r-1} \) does not satisfy \([\text{LB}]\), there exists \( x_0 \in \Gamma \) such that \( p_{2r-1}(x_0, x_0)m(x_0) \leq \epsilon^{2r-1} \).

Let \( x_0, x_1, \ldots, x_{2r-1} = x_0 \) be a path of length \( 2r - 1 \). One has

\[
\prod_{i \in [1, 2r-1]} p(x_{i-1}, x_i)m(x_i) \leq p_{2r-1}(x_0, x_0)m(x_0) \leq \epsilon^{2r-1}.
\]

Then there exists \( i \in [1, 2r-1] \) such that \( p(x_{i-1}, x_i)m(x_i) \leq (\epsilon^{2r-1})^{\frac{1}{2r-1}} = \epsilon \) and the path is not \( \epsilon \)-safe.

**Second step**

Define now

\[ E_0 = \{ x \in B_e(x_0, r), \ d_e(x_0, x) \text{ is even} \} \]

and

\[ E_1 = \{ x \in B_e(x_0, r), \ d_e(x_0, x) \text{ is odd} \}. \]

The couple \((E_0, E_1)\) forms a partition of \( B(x_0, r) \). Let \( x, y \in B_e(x_0, r) \) such that \( x \sim \epsilon y \). Then \( |d_e(x_0, x) - d_e(x_0, y)| \leq 1 \).

We claim that \( |d_e(x_0, x) - d_e(x_0, y)| = 1 \). Assume on the contrary that \( n := d_e(x_0, x) = d_e(x_0, y) < r \). Then there exist two \( \epsilon \)-safe paths \( x_0, \ldots, x_n = x \) and \( x_0, y_1, \ldots, y_n = y \). Thus \( x_0, \ldots, x_n, y_n, \ldots, y_1, x_0 \) is an \( \epsilon \)-safe path of length \( 2n + 1 \). We want to extend this \( \epsilon \)-safe closed path to a \( \epsilon \)-safe closed path of length \( 2r - 1 \). Two cases may happen. Either \( n = 0 \) and then \( x_0, x_0 \) is an \( \epsilon \)-safe path of length 1, since \( x_0 = x = y \), and thus

\[
\underbrace{x_0, x_0, \ldots, x_0}_{2r \text{ times}}
\]

is an \( \epsilon \)-safe path of length \( 2r - 1 \). Or \( n > 0 \) and \( x_0, x_1, x_0 \) is an \( \epsilon \)-safe path of length 2 and thus

\[
\underbrace{x_0, x_1, \ldots, x_0, x_1, x_0, \ldots, x_n, y_n, \ldots, y_1, x_0}_{r-n-1 \text{ times}}
\]

is an \( \epsilon \)-safe path of length \( 2r - 1 \).

By contradiction with the first step, one has then

\[
|d_e(x_0, x) - d_e(x_0, y)| = 1. \tag{10}
\]

The identity \([\text{LB}]\) yields that \( d_e(x_0, x_0) \) and \( d_e(x_0, y) \) don’t have the same parity, and thus that \( x \) and \( y \) do not belong to the same \( E_i \).

**Conclusion**

A way to rephrase the second step is: \((E_0, E_1)\) is a partition of \( B(x_0, r) \) such that if \( x, y \in E_i \) for some \( i \in \{0, 1\} \), then \( x \not\sim \epsilon y \).

Implication (ii) \( \implies \) (i) in Proposition \([1, 8]\) yields then that \( B(x_0, r) \) is \( \epsilon \)-bipartite.

\[ \square \]
Note that we didn’t use the assumption $\text{(PDV)}$ for the proof of the previous implications of Theorem 1.9. We begin with a definition.

**Definition 2.1.** Let $\epsilon \geq 0$. Say that a collection of $\epsilon$-balls $(B_p)_{p \in \mathbb{N}^*}$ has the property $\text{(NB)}$ if and only if, for all $p \geq 1$,

$$m(\partial_p B_p) \leq \frac{1}{p} m(B_p).$$

**(NB)**

**Proposition 2.2.** Let $(\Gamma, m)$ be a weighted graph satisfying $\text{(PDV)}$ and $\epsilon \geq 0$. Let $(\bar{B}_n)_{n \in \mathbb{N}^*}$ a collection of $\epsilon$-balls such that, for all $n \geq 1$, the radius of $\bar{B}_n$ is equal to $n$. Then there exists a collection $(B_p)_{p \in \mathbb{N}^*}$ of $\epsilon$-balls that satisfies:

(i) For all $p \in \mathbb{N}^*$, there exists $n \in \mathbb{N}^*$ such that $B_p \subset \bar{B}_n$.

(ii) $(B_p)_{p \in \mathbb{N}^*}$ has the property $(\text{NB})$.

**Proof:** Define $x_n$ as the center of the $\epsilon$-ball $\bar{B}_n$. Assume that there exists $K > 0$ such that for all $n \in \mathbb{N}^*$ and all $l \in [1, n-1]$,

$$m(\partial_l B_l(x_n, l)) \geq K m(B_l(x_n, l)) \tag{11}$$

Since we have the inclusion

$$B_l(x_n, l+1) \supset \partial_l B_l(x_n, l+1) \cup B_l(x_n, l)$$

one has for all $n \in \mathbb{N}$ and all $l \in [1, n-1],

$$m(B_l(x_n, l+1)) \leq (1 + K)m(B_l(x_n, l)),$$

and thus by induction on $l$, one obtains for all $n \in \mathbb{N}^*$

$$m(\bar{B}_n) = m(B_l(x_n, n)) \geq (1 + K)^{n-1} m(x_n).$$

Yet, assumption $\text{(PDV)}$ and Remark 1.1 yield, for all $n \in \mathbb{N}^*$,

$$m(B_n) \lesssim n^d m(x_n).$$

Hence, we obtain for all $n \in \mathbb{N}^*$

$$(1 + K)^{n-1} \lesssim n^d,$$

which is impossible. This yields that $(\text{11})$ is false. That is, for all $p \in \mathbb{N}^*$, there exist $n_p \in \mathbb{N}^*$, $l_p \in [1, n_p - 1]$ such that

$$m(\partial_l B_l(x_{n_p}, l_p)) \leq \frac{1}{p} m(B_l(x_{n_p}, l_p)).$$

The collection $(B_p)_{p \in \mathbb{N}^*}$ where $B_p = B_l(x_{n_p}, l_p) \subset \bar{B}_{n_p}$ satisfies (i) and (ii). \hfill $\square$

We are now ready to prove the last part of Theorem 1.9.

**Proof:** [Theorem 1.9 (iv) implies (i)].

Assume (iv). Fix $\eta > 0$ and $q \in [1, +\infty)$. We want to prove that there exists a nonzero function $f_\eta$ on $\Gamma$ such that

$$\|(I + p) f_\eta \|_{L^q} \lesssim \eta \|f_0\|_{L^q}. \tag{12}$$

Let $p \in \mathbb{N}^*$ and $\epsilon > 0$ to be fixed later. Then there exists a collection of $\epsilon$-balls $(\bar{B}_r^*)_{r \in \mathbb{N}^*}$ such that for all $r \in \mathbb{N}^*$, $\bar{B}_r^*$ is of radius $r$ and $\bar{B}_r^*$ is $\epsilon$-bipartite.

According to Proposition 2.2 there exists then a collection $(B_p^*)_{p \in \mathbb{N}^*}$ of $\epsilon$-balls that satisfies

1. $m(\partial_p B_p^*) \leq \frac{1}{p} m(B_p^*),$

2. for all $p \in \mathbb{N}^*$, there exists $r_p \in \mathbb{N}^*$ such that $B_p^* \subset \bar{B}_{r_p}$. Hence, for all $p \in \mathbb{N}^*$, $B_p^*$ is $\epsilon$-bipartite.

Since $B_p^*$ is $\epsilon$-bipartite, Proposition 1.8 provides a partition $(E_0, E_1)$ of $B_p^*$. Define $f_\eta : \Gamma \to \mathbb{R}$ by

$$f_\eta(x) = \begin{cases} 
0 & \text{if } x \notin B_p^* \\
1 & \text{if } x \in E_0 \\
-1 & \text{if } x \in E_1
\end{cases}.$$
One has then
\[\|(I + P)f_\eta\|_{L^q}^q = \sum_{x \in \Gamma} \left| \sum_{y \in \Gamma} \left[ f_\eta(x) + f_\eta(y) \right] p(x, y) m(y) \right|^q m(x) \]
\[\leq \sum_{x \in \Gamma} \sum_{y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[\leq \sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) + \sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]

Let us estimate the first term. Since \( \left| f_\eta(x) + f_\eta(y) \right| = 0 \) when \( x, y \in B_p^\epsilon \) or \( x, y \in \left(B_p^\epsilon \right)^c \) (remember that, since \( x \sim y \), either \( x \in E_0 \) and \( y \in E_1 \), or \( x \in E_1 \) and \( y \in E_0 \)) and \( \left| f_\eta(x) + f_\eta(y) \right| = 1 \) when \( (x, y) \in (B_p^\epsilon)^c \times B_p^\epsilon \), we have
\[\sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[\leq \sum_{y \in \partial B_p^\epsilon} \sum_{x \in \left(B_p^\epsilon \right)^c} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[= 2 \sum_{y \in \partial B_p^\epsilon} \sum_{x \in \left(B_p^\epsilon \right)^c} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[\leq 2 \sum_{y \in \partial B_p^\epsilon} \sum_{x \in \left(B_p^\epsilon \right)^c} p(x, y) m(y) m(x) \]
\[\leq 2 \sum_{y \in \partial B_p^\epsilon} \left( \sum_{x \in \Gamma} p(x, y) m(x) \right) m(y) \]
\[= 2 \sum_{y \in \partial B_p^\epsilon} m(y) \]
\[\leq 2 \sum_{y \in \partial B_p^\epsilon} m(y) = 2 m(B_p^\epsilon). \]

We turn now to the estimate of the second term
\[\sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[\leq \sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[+ \sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q p(x, y) m(y) m(x) \]
\[\leq \epsilon \sum_{x, y \in \Gamma} \left| f_\eta(x) + f_\eta(y) \right|^q [m(x) + m(y)] \]
\[\leq 2^q \epsilon \left( \sum_{x \in B_p^\epsilon} \sum_{y \sim x} [m(x) + m(y)] + \sum_{y \in B_p^\epsilon} \sum_{x \sim y} [m(x) + m(y)] \right) \]
\[= 2^{q+1} \epsilon \sum_{x \in B_p^\epsilon} \sum_{y \sim x} [m(x) + m(y)] \]
\[\leq 2^{q+1} \epsilon \sum_{x \in B_p^\epsilon} [C_{det} + N_{det}] m(x) \]
\[\leq 2^{q+1} \epsilon [C_{det} + N_{det}] m(B_p^\epsilon). \]

As a consequence, since \( \|f_\eta\|_{L^q}^q = m(B_p^\epsilon) \), we obtain
\[\|(I + P)f_\eta\|_{L^q}^q \leq \left( 2 \frac{1}{p} + 2^{q+1} \epsilon [C_{det} + N_{det}] \right) \|f\|_{L^p}^q \]

We fix \( p \) big enough and \( \epsilon \) small enough (both depend of \( \eta \) and \( q \)) such that \( \frac{2}{p} + 2^{q+1} \epsilon [C_{det} + N_{det}] \leq \eta \), which gives (12).
We conclude by noticing that (12) yields that \(-1 \in \text{S}_{P\mathbb{L}_3}(P)\). \qed

3 Application to the $L^p$-boundedness of the Riesz transform for $p \in (1, 2)$ under subgaussian estimates

To give a symmetric weight $\mu$ is equivalent to give a measure $m$ and a Markov operator $P$ reversible with respect to $m$. For convenience, in this section, a weighted graph is written $(\Gamma, m, P)$.

**Definition 3.1.** Let $(\Gamma, m, P)$ be a weighted graph. A mapping $\rho$ from $\Gamma \times \Gamma$ to $\mathbb{R}_+$ is called a quasidistance (on $\Gamma$) if, and only if there exists $C_\rho > 0$ such that, for all $x, y, z \in \Gamma$:

(i) $\rho(x, y) = 0$ if, and only if, $x = y$,

(ii) $\rho(x, y) = \rho(y, x)$,

(iii) we have the inequality

$$\rho(x, y) \leq C_\rho (\rho(x, z) + \rho(z, y)).$$

(13)

The two next conditions are specific to our context (of graphs):

(iv) $\rho$ takes values in $\mathbb{N}$, that is $\rho(x, y) \in \mathbb{N}$ for all $x, y \in \Gamma$,

(v) $\rho(x, y) = 1$ whenever $\rho(x, y) > 0$ (i.e. $x \sim y$).

**Balls and volumes will be indexed by $\rho$ when they are defined through the quasidistance $\rho$.**

**Definition 3.2.** Let $\rho$ be a quasidistance. We say that $(\Gamma, m, \rho)$ satisfies $(\text{DV}_p)$ if the measure $m$ is doubling with respect to the quasidistance $\rho$, that is if there exists $C > 0$ such that

$$V_\rho(x, 2k) \leq CV_\rho(x,k) \quad \forall x \in \Gamma, \forall k \in \mathbb{N}^*.$$  

(DV$_\rho$)

**Proposition 3.3.** Let $(\Gamma, m, P, \rho)$ satisfying $(\text{DV}_p)$. Then there exists $D_\rho > 0$ such that

$$V_\rho(x, \lambda k) \leq \lambda^{D_\rho} V_\rho(x,k) \quad \forall x \in \Gamma, k \in \mathbb{N} \text{ and } \lambda \geq 1.$$  

(14)

**Definition 3.4.** Let $\rho$ be a quasidistance. We say that $(\Gamma, m, P, \rho)$ satisfies $(\text{UE}_\rho)$ if there exist three constants $c_{ue}, C_{ue} > 0$ and $\eta \in (0, 1]$ such that $p_k$ satisfies the subgaussian estimates

$$p_{k-1}(x, y) \leq \frac{C_{ue}}{V_\rho(x, k)} \exp \left[ -c_{ue} \left( \frac{\rho(x, y)}{k} \right)^\eta \right], \forall x, y \in \Gamma, \forall k \in \mathbb{N}^*.$$  

(UE$_\rho$)

**Remark 3.5.** We can recover the classical gaussian estimates by taking $\rho = d^2$ in $(\text{UE}_\rho)$. The choose of $\rho = |d^m|$ with $m > 2$ provides some subgaussian estimates, satisfied for example, with $m = \log_2 5$, by the Sierpinski gasket (see [13]).

We begin with the following lemma.

**Lemma 3.6.** Let $(\Gamma, m, P, \rho)$ satisfying $(\text{DV}_p)$. Then $(\Gamma, m, P)$ satisfies $(\text{PDV})$.

**Proof:** First, we prove by induction on $n$ that

$$d(x, y) \leq 2^n \Rightarrow \rho(x, y) \leq (2C_\rho)^n.$$  

(15)

- When $n = 0$, (15) is a consequence of (v) of the Definition 3.1
- Assume (15) is true for some $n \in \mathbb{N}$. Let $x, y \in \Gamma$ such that $d_0(x, y) \leq 2^{n+1}$. We can find $z$ such that $d_0(x, z) \leq 2^n$ and $d_0(z, y) \leq 2^n$. Property (15) provides $\rho(x, z) \leq (2C_\rho)^n$ and $\rho(z, y) \leq (2C_\rho)^n$. Thus, (13) yields

$$\rho(x, y) \leq C_\rho (\rho(x, z) + \rho(z, y)) \leq (2C_\rho)^{n+1}.$$
Let $\beta := \log_2(2C_\rho) = 1 + \log_2 C_\rho$. Then (15) implies
\[
\rho(x, y) \leq (2d_0(x, y))^\beta,
\]
whence we obtain $B_{d_0}(x, r) \subset B_{\rho}(x, (2r)^\beta)$.

Proposition 3.8 yields then
\[
V(x, r) = V_{d_0}(x, r) \leq V_\rho(x, (2r)^\beta) \leq (2r)^{\beta D_\rho} V_\rho(x, 1) \leq r^{\beta D_\rho} m(x),
\]
which is the desired result. \hfill \Box

Recall the definition of the positive Laplacian $\Delta := I - P$. We also introduce the length of the gradient
\[
\nabla f(x) := \left( \sum_{y \in \Gamma} p(x, y) |f(x) - f(y)|^2 m(y) \right)^{\frac{1}{2}}.
\]

**Theorem 3.7.** Let $(\Gamma, m, P, \rho)$ satisfying (DV$_\rho$) and (UE$_\rho$). Assume that $-1 \notin \text{Sp}_{L^2}(P)$. For all $p \in (1, 2)$, the Riesz transform $\nabla \Delta^{-\frac{\beta}{2}}$ is bounded on $L^p(\Gamma)$, that is, there exists $C_p > 0$ such that
\[
\|\nabla \Delta^{-\frac{\beta}{2}} f\|_p \leq C_p \|f\|_p \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma).
\]

**Proof:** The proof of Theorem 3.7 is identical to the one of [11, Corollary 1.30] once we have the following improved version of [11, Proposition 2.7]. \hfill \Box

**Proposition 3.8.** Let $(\Gamma, m, P, \rho)$ satisfying (DV$_\rho$) and (UE$_\rho$). Assume that $-1 \notin \text{Sp}_{L^2}(P)$. If $p \in (1, 2)$, there exist
\[C, c > 0\]
such that, for all subsets $E, F \subset \Gamma$, there holds
\[
\|\nabla P^{k-1}[f 1_F]\|_{L^p(E)} \leq \frac{C}{\sqrt{k}} \exp \left(-c \left[\frac{\rho(E, F)}{k}\right]^\eta\right) \|f\|_{L^p} \quad \forall k \in \mathbb{N}^*, \forall f \in L^p
\]
where $\eta$ is the exponent in (UE$_\rho$).

**Proof:** According to Theorem 1.9 and Lemma 3.6, there exists $l \in \mathbb{N}$, $l$ odd, such that $Q := P^l$ satisfies (LB). Define, if $\beta := 1 + \log_2(C_\rho)$ as the proof of Lemma 3.6
\[
\sigma := \left[\frac{\rho}{(2l)^\beta}\right].
\]

Note, with (14), that $\sigma(x, y) = 1$ whenever $p_l(x, y) > 0$. Hence $\sigma$ is a quasidistance on $(\Gamma, m, Q)$. Check now that assumptions (DV$_\rho$) and (UE$_\rho$) on $(\Gamma, m, P, \rho)$ yield (DV$_\sigma$) and (UE$_\sigma$) on $(\Gamma, m, Q, \sigma)$. As a consequence, $(\Gamma, m, Q, \sigma)$ satisfies (LB), (DV$_\sigma$) and (UE$_\sigma$) and Proposition 2.7 in [11] implies, for all $p \in (1, 2)$, the existence of
\[C, c > 0\]
such that, for all $E, F \subset \Gamma$, all $k \in \mathbb{N}^*$ and all $f \in L^p$
\[
\|\nabla Q^{k-1}[f 1_F]\|_{L^p(E)} \leq \frac{C}{\sqrt{k}} \exp \left(-c \left[\frac{\sigma(E, F)}{k}\right]^\eta\right) \|f\|_{L^p}
\]
where $\nabla$ is defined by
\[
\nabla f(x) := \left( \sum_{y \in \Gamma} p_l(x, y) |f(x) - f(y)|^2 m(y) \right)^{\frac{1}{2}}.
\]

Let $n \in \mathbb{N}^*$. We set $k = k(n) \in \mathbb{N}$ and $a = a(n) \in [0, l - 1]$ such that $n - 1 = (k - 1)l + a$. Therefore,
\[
\|\nabla P^{n-1}[f 1_F]\|_{L^p(E)} \leq \|\nabla Q^{k-1} P^a [f 1_F]\|_{L^p(E)} \leq \frac{C}{\sqrt{k}} \exp \left(-c \left[\frac{\sigma(E, F_{a+1})}{k}\right]^\eta\right) \|f\|_{L^p} \quad \forall k \in \mathbb{N}^*, \forall f \in L^p
\]
where $F_{a+1} := \{x \in \Gamma, \sigma(x, F) \leq 1\} \supset \text{Supp} P^a[f 1_F]$. We claim that
\[
\|\nabla P^{n-1}[f 1_F]\|_{L^p(E)} \leq \frac{C'}{l} \exp \left(-c' \left[\frac{\sigma(E, F)}{l}\right]^\eta\right) \|f\|_{L^p}.
\]
We are going to distinguish two cases. First, in the case \( \sigma(E, F) \geq 2C_\rho \), one has \( \sigma(E, F+1) \geq \sigma(E, F) + C_\rho - 1 \geq \sigma(E, F) - 2C_\rho \). Consequently, (17) becomes
\[
\| \nabla P^{n-1}[1 f] \|_{L^p(E)} \leq \frac{C}{\sqrt{k}} \exp \left( -c' \left[ \frac{\sigma(E, F)}{k} \right]^\eta \right) \| f \|_{L^p}.
\]
Second, in the case \( \sigma(E, F) \leq 2C_\rho \), (17) implies
\[
\| \nabla P^{n-1}[1 f] \|_{L^p(E)} \leq \frac{C'}{\sqrt{k}} \| f \|_{L^p}
\leq \frac{C''}{\sqrt{n}} \exp \left( -c'' \left[ \frac{\rho(E, F)}{n} \right]^\eta \right) \| f \|_{L^p}.
\]
Note then that \( \sigma \approx \rho \) and \( n \approx k \). Therefore (18) can be rewritten as
\[
\| \nabla P^{n-1}[1 f] \|_{L^p(E)} \leq \frac{C''}{\sqrt{n}} \exp \left( -c'' \left[ \frac{\rho(E, F)}{n} \right]^\eta \right) \| f \|_{L^p}.
\]
In order to conclude the proof of Proposition 3.8, it suffices now to check that \( \nabla f \preceq \nabla f \) for all functions \( f \), which is equivalent to prove \( p(x, y) \lesssim p_l(x, y) \) for all \( x, y \in \Gamma, x \neq y \). Indeed, since \( P^2 \) satisfies (LB) (see Remark 1.5) and \( l-1 \) is even, \( P^{l-1} \) also satisfies (LB) and
\[
p(x, y) \lesssim p_{l-1}(x, x)m(x)p(x, y) \leq p_l(x, y).
\]
□

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