QUASI-EINSTEIN METRICS ON HYPERSURFACE FAMILIES

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Abstract. We construct quasi-Einstein metrics on some hypersurface families. The hypersurfaces are circle bundles over the product of Fano, Kähler-Einstein manifolds. The quasi-Einstein metrics are related to various gradient Kähler-Ricci solitons constructed by Dancer and Wang and some Hermitian, non-Kähler, Einstein metrics constructed by Wang and Wang on the same manifolds.

1. Introduction

1.1. Motivation and definitions. This article is concerned with a generalisation of Einstein metrics that in some sense interpolates between Einstein metrics and Ricci solitons, namely, quasi-Einstein metrics.

Definition 1.1. Let $M^n$ be a smooth manifold and $g$ be a complete Riemannian metric. The metric $g$ is called quasi-Einstein if it solves

$$\text{Ric}(g) + \text{Hess}(u) - \frac{1}{m} du \otimes du + \frac{\epsilon}{2} g = 0,$$

(1.1)

where $u \in C^\infty(M)$, $m \in (1, \infty]$ and $\epsilon$ is a constant.

It is clear that if $u$ is constant then we recover the notion of an Einstein metric; we will refer to these metrics as trivial quasi-Einstein metrics. By letting the constant $m$ go to infinity we can also recover the definition of a gradient Ricci soliton. In line with the terminology used for Ricci solitons, we will refer to the quasi-Einstein metrics with $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$ as shrinking, steady and expanding respectively.

There has been a great deal of effort invested in finding non-trivial examples of Ricci solitons on compact manifolds. However, they remain rare and the only known examples are Kähler. Due to work of Hamilton [13] and Perelman [19], non-trivial Ricci solitons on compact manifolds must be shrinking gradient Ricci solitons. The first non-trivial examples were constructed independently by Koiso and Cao on $\mathbb{CP}^1$-bundles over complex projective spaces [3, 15]. These examples were subsequently generalised by Chave and Valet [7] and Pedersen, Tonneson-Freidman and Valent [18] who found Kähler-Ricci solitons on the projectivisation of various line bundles over a Fano Kähler-Einstein base. The reader should note that what we call a Ricci soliton is referred to as a quasi-Einstein metrics in the papers [7] and
Recently Dancer and Wang generalised these examples by constructing some Kähler Ricci solitons on various hypersurface families where the hypersurface is a circle bundle over the product of Fano Kähler-Einstein manifolds. The solitons found by Dancer and Wang were also independently constructed by Apostolev, Calderbank, Gauduchon and Tønneson-Freidman. In the complete non-compact case Feldman, Ilmanen and Knopf found shrinking gradient Kähler-Ricci solitons on certain line bundles over $\mathbb{C} \mathbb{P}^n$. Steady gradient Kähler-Ricci solitons were first constructed on $\mathbb{C}^n$ by Cao (the $n = 1$ case was first found by Hamilton). Cao also found steady gradient Kähler-Ricci solitons on the blow up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. Expanding gradient Kähler-Ricci solitons have been found by Cao on $\mathbb{C}^n$ and by Feldman, Ilmanen and Knopf on the blow ups of $\mathbb{C}^n/\mathbb{Z}_k$ for $k = n + 1, n + 2, \ldots$. Examples were also found by Pedersen, Tønneson-Freidman and Valent on the total space of holomorphic line bundles over Kähler-Einstein manifolds with negative scalar curvature. As in the compact case, these examples have been generalised by Dancer and Wang who constructed shrinking, steady and expanding Kähler-Ricci solitons on various vector bundles over the product of Kähler-Einstein manifolds.

In the recent work Case suggested that there should be quasi-Einstein analogues of Dancer-Wang’s solitons. He points out that the quasi-Einstein analogue of Koiso-Cao, Chave-Valent and Pedersen-Tønneson-Freidman-Valent type solitons was already constructed by Lü, Page and Pope. The purpose of this article is to show that Dancer-Wang’s solitons indeed have quasi-Einstein analogues. However it is better to think of these metrics as quasi-Einstein analogues of various Hermitian, non-Kähler, Einstein metrics constructed by Wang and Wang on these spaces. The Wang-Wang Einstein metrics generalise a construction originating with Page and Berard-Bergery. We now state the precise results we wish to prove.

**Theorem 1.2.** Let $(V_i, J_i, h_i), 1 \leq i \leq r, r \geq 3,$ be Fano Kähler-Einstein manifolds with complex dimension $n_i$ and first Chern class $p_i a_i$ where $p_i > 0$ and $a_i$ are indivisible classes in $H^2(V_i, \mathbb{Z})$. Let $V_1$ be a complex projective space with normalised Fubini-Study metric i.e. $p_1 = (n_1 + 1)$. Let $P_q$ denote the principal $S^1$-bundle over $V_1 \times \ldots \times V_r$ with Euler class $\pm \pi_i^*(a_i) + \sum_{i=2}^{r} q_i \pi_i^*(a_i)$, i.e. $q_i^2 = 1$.

1. Suppose $(n_1 + 1)|q_i| < p_i$ for $2 \leq i \leq r$ then, for all $m > 1$, there exists a non-trivial, complete, steady, quasi-Einstein metric on the total space of the $\mathbb{C}^{n_1+1}$-bundle over $V_2 \times \ldots \times V_r$ corresponding to $P_q$. 

In the non-compact case we have the following which is the quasi-Einstein analogue of theorem 1.6 in [20]:

**Theorem 1.2.** Let $(V_i, J_i, h_i), 1 \leq i \leq r, r \geq 3,$ be Fano Kähler-Einstein manifolds with complex dimension $n_i$ and first Chern class $p_i a_i$ where $p_i > 0$ and $a_i$ are indivisible classes in $H^2(V_i, \mathbb{Z})$. Let $V_1$ be a complex projective space with normalised Fubini-Study metric i.e. $p_1 = (n_1 + 1)$. Let $P_q$ denote the principal $S^1$-bundle over $V_1 \times \ldots \times V_r$ with Euler class $\pm \pi_i^*(a_1) + \sum_{i=2}^{r} q_i \pi_i^*(a_i)$, i.e. $q_i^2 = 1$.

1. Suppose $(n_1 + 1)|q_i| < p_i$ for $2 \leq i \leq r$ then, for all $m > 1$, there exists a non-trivial, complete, steady, quasi-Einstein metric on the total space of the $\mathbb{C}^{n_1+1}$-bundle over $V_2 \times \ldots \times V_r$ corresponding to $P_q$. 

2.
(2) For all $m > 1$ there exists at least one one-parameter family of non-trivial, complete, expanding, quasi-Einstein metrics on the total space of the $\mathbb{C}^{n+1}$-bundle over $V_2 \times \ldots \times V_r$ corresponding to $P_q$.

For the compact case we have the following analogue of theorem 1.2 in [20].

**Theorem 1.3.** Let $(V_i, J_i, h_i), 1 \leq i \leq r, r \geq 3$, be Fano Kähler-Einstein manifolds with complex dimension $n_i$ and first Chern class $p_i a_i$ where $p_i > 0$ and $a_i$ are indivisible classes in $H^2(V_i, \mathbb{Z})$. Let $V_1$ and $V_r$ be a complex projective space with normalised Fubini-Study metrics. Let $P_q$ denote the principal $S^1$-bundle over $V_1 \times \ldots \times V_r$ with Euler class $\pm \pi_1^*((a_1) + \sum_{i=2}^{r-1} q_i \pi_i^*(a_i) \pm \pi^*(a_r)), \text{i.e. } |q_1| = |q_r| = 1$.

Suppose that $|q_i|(n_1 + 1) < p_i$ and $|q_i|(n_r + 1) < p_i$ for $2 \leq i \leq r - 1$ and that there exists $\chi = (\chi_1, \chi_2, \ldots, \chi_r)$ where $|\chi_i| = 1$, $\chi_1 = -\chi_r = 1$ such that

$$\int_{(n_1+1)}^{(n_r+1)} \left( \frac{p_1}{q_1} - x \right)^{n_1} \left( \frac{p_2}{q_2} - x \right)^{n_2} \ldots \left( \frac{p_r}{q_r} - x \right)^{n_r} xdx < 0,$$

(1.2)

then, for all $m > 1$ there exists a non-trivial, shrinking quasi-Einstein metric on $M_q$, the space obtained from $P_q \times_{S^1} \mathbb{C}P^1$ by blowing-down one end to $V_2 \times \ldots \times V_r$ and the other end to $V_1 \times \ldots \times V_{r-1}$.

We remark that the Futaki invariant (evaluated on the holomorphic vector field $f(t) \partial \bar{t}$ in the notation of the next section) is given by

$$\int_{-(n_1+1)}^{(n_r+1)} \left( \frac{p_1}{q_1} - x \right)^{n_1} \left( \frac{p_2}{q_2} - x \right)^{n_2} \ldots \left( \frac{p_r}{q_r} - x \right)^{n_r} xdx.$$

If this integral vanishes then Dancer-Wang construct a Kähler-Einstein metric on $M_q$.

Finally we note that none of the metrics we find are Kähler. Indeed there is a rigidity result due to Case-Shu-Wei [3] that says, on compact manifolds, Kähler-quasi-Einstein metrics are trivial i.e. Kähler-Einstein.

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### 2. Proof of main theorems

#### 2.1. Derivation of equations.** We use the same notation as above. We consider the manifold $M_0 = (0, l) \times P_q$. Let $\theta$ be the principal $U(1)$-connection on $P_q$ with curvature $\Omega = \sum_{i=1}^{r} q_i \pi^* \eta_i$ where $\eta_i$ is the Kähler form of the metric $h_i$. We form
the 1-parameter family of metrics on $P_g$

$$g_t = f^2(t)\theta \otimes \theta + \sum_{i=1}^{r} g_i(t)\pi_{i}^* h_i$$

and we then form the metric $\tilde{g} = dt^2 + g_t$ on $M_0$. The group $U(1)$ acts on $M_0$ by isometries and generates a Killing field $Z$. We define a complex structure $J$ on $M_0$ by $J(\partial_t) = -f^{-1}(t)Z$ on the vertical space of $\theta$ and by lifting the complex structure from the base on the horizontal spaces of $\theta$.

**Lemma 2.1.** Let $M_0$ be as above and let $v = e^{-\frac{u}{m}}$. Then the quasi-Einstein equations in this setting are given by:

$$\frac{\ddot{f}}{f} + \sum_{i=1}^{r} 2n_i \frac{\ddot{g}_i}{g_i} + m \frac{\dot{v}}{v} = \frac{\epsilon}{2},$$

$$\frac{\ddot{f}}{f} + \sum_{i=1}^{r} \left( 2n_i \frac{\ddot{g}_i}{g_i} - \frac{n_i q_i^2 f^2}{2 g_i^4} \right) + m \frac{\dot{v}}{v} = \frac{\epsilon}{2},$$

$$\frac{\ddot{g}_i}{g_i} - \left( \frac{\ddot{g}_i}{g_i} \right)^2 + \frac{\ddot{g}_i}{g_i} + \sum_{j=1}^{j=r} \left( 2n_j \frac{\ddot{g}_j g_j}{g_i} - \frac{p_i}{g_i^2} + \frac{q_i^2 f^2}{2 g_i^4} \right) + m \frac{\dot{g}_i \dot{v}}{g_i v} = \frac{\epsilon}{2}.$$  

In order that $(M, g, u)$ be a quasi-Einstein manifold, as well as equation (1.1), $u$ must also satisfy an integrability condition that essentially comes from the second Bianchi identity (c.f. Lemma 3.4 in [9]). The form we use here is given in Case [6] using the Bakry-Émery Laplacian:

$$\Delta_u := \Delta - \langle \nabla u, \cdot \rangle.$$

**Lemma 2.2** (Kim-Kim [14] Corollary 3). Let $(M, g, u)$ be a quasi-Einstein manifold then there exists a constant $\mu$ such that

$$\Delta_u \left( \frac{u}{m} \right) + \frac{\epsilon}{2} = -\mu e^{\frac{2u}{m}}.$$  

In the notation above (recalling $v = e^{-\frac{u}{m}}$) this condition becomes

$$\mu = \dot{v} \ddot{v} + \dot{v} \left( \frac{\ddot{f}}{f} + \sum_{i=1}^{r} 2n_i \frac{\dot{g}_i}{g_i} \right) + (m - 1) \dot{v}^2 - \frac{\epsilon}{2} v^2.$$  

The constant $\mu$ enters into the discussion of Einstein warped products when $m$ is an integer. If $(M, g, u)$ is a quasi-Einstein manifold with constant $\mu$ coming from (2.4) and $(F^m, h)$ is an Einstein manifold with constant $\mu$, then $(M \times F^m, g \oplus v^2 h)$ is an Einstein metric with constant $-\epsilon/2$ as in equation (1.1) (c.f. [14]).

Introducing the moment map change of variables as in [9] and [20] yields the following set of equations:
Proposition 2.3. Let \( s \) be the coordinate on \( I = (0, l) \) such that \( ds = f(t) dt \), \( \alpha(s) = f^2(t) \), \( \beta_i(s) = g_i^2(t) \), \( \phi(s) = v(t) \) and \( V = \prod_{i=1}^{r} g_i^{2n_i}(t) \). Then the equations (2.1), (2.2), (2.3) and (2.5) transform to the following:

\[
\frac{1}{2} \alpha'' + \frac{1}{2} \alpha' \log V' + \alpha \sum_{i=1}^{r} n_i \left( \frac{\beta_i''}{\beta_i} - \frac{1}{2} \left( \frac{\beta_i'}{\beta_i} \right)^2 \right) + m \left( \frac{\alpha \phi''}{\phi} + \frac{\alpha' \phi'}{2 \phi} \right) = \frac{\epsilon}{2},
\]

\[
\frac{1}{2} \alpha'' + \frac{1}{2} \alpha' \log V' - \alpha \sum_{i=1}^{r} n_i q_i^2 + m \frac{\alpha' \phi'}{2 \phi} = \frac{\epsilon}{2},
\]

\[
\frac{1}{2} \alpha' \beta_i' + \frac{1}{2} \alpha \left( \frac{\beta_i'}{\beta_i} - \left( \frac{\beta_i'}{\beta_i} \right)^2 \right) + \frac{1}{2} \alpha' \log V' - \frac{p_i}{\beta_i} + \frac{q_i^2 \alpha}{2 \beta_i^2} + m \frac{2 \beta_i' \phi}{\beta_i} = \frac{\epsilon}{2},
\]

\[
\phi \left( \phi'' \alpha + \frac{\phi' \alpha'}{2} \right) + \phi \phi' \left( \frac{\alpha'}{2} + \log V' \alpha \right) + (m - 1)(\phi')^2 \alpha - \frac{\epsilon}{2} \phi^2 = \mu.
\]

Equating (2.6) and (2.7) we obtain

\[
-m \frac{\phi''}{\phi} = \sum_{i=1}^{r} n_i \left( \frac{\beta_i''}{\beta_i} - \frac{1}{2} \left( \frac{\beta_i'}{\beta_i} \right)^2 + \frac{q_i^2}{2 \beta_i^2} \right)
\]

(2.10)

Following [9] [20] we look for solutions that satisfy

\[
\frac{\beta_i''}{\beta_i} - \frac{1}{2} \left( \frac{\beta_i'}{\beta_i} \right)^2 + \frac{1}{2} \frac{q_i^2}{\beta_i^2} = 0.
\]

This condition can be geometrically interpreted as saying that the curvature of \( \bar{g} \) is \( J \)-invariant in the sense that \( \text{Im}(J_1, J_2) = \text{Im}(\langle \cdot, \cdot \rangle) \) where \( J \) is the complex structure on \( M_0 \). Imposing this forces \( \phi \) to be a linear function of \( s \). We write \( \phi(s) = \kappa_1(s + \kappa_0) \) for some constants \( \kappa_0, \kappa_1 \in \mathbb{R} \). Hence (2.9) becomes

\[
\alpha' + \alpha ((\log V)' + \frac{(m - 1)(s + \kappa_0)}{(s + \kappa_0)} = \frac{\epsilon(s + \kappa_0)}{2} + \frac{\mu}{\kappa_1(s + \kappa_0)}.
\]

(2.11)

Accordingly there are two classes of solution \( \beta_i \):

\[
\beta_i = A_i(s + s_0)^2 - \frac{q_i^2}{4A_i}
\]

or

\[
\beta_i = \pm q_i(s + \sigma_i)
\]

where \( A_i \neq 0 \) and \( \sigma_i \) are constants. We note that the case \( \beta_i = -q_i(s + \sigma_i) \) corresponds to the metric \( \bar{g} \) being Kähler with respect to the complex structure. Hence the rigidity result of Case-Shu-Wei rules out having any solutions of this form (in fact choosing \( \beta_i \) of this form leads to inconsistency).
If we input $\beta_i = A_i(s + s_0)^2 - \frac{q_i^2}{4A_i}$ into (2.8) we see that
\[
\alpha' + \alpha \left((\log V)' + m(\log \phi)' - \frac{1}{(s + s_0)}\right) = \frac{\epsilon}{2}(s + \kappa_0) + \frac{E^*}{(s + \kappa_0)}
\]
where
\[
E^* := \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2}.
\]
Comparing with equation (2.11) we see that solutions are consistent providing $\kappa_0 = s_0$ and
\[
\frac{\mu}{\kappa_1^2} = E^* = \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2}.
\]
Solving gives
\[
\alpha(s) = V^{-1}(s + \kappa_0)^{1-m} \int_0^s V(s + \kappa_0)^{m-2} \left(E^* + \frac{\epsilon}{2}(s + \kappa_0)^2\right) ds. \quad (2.12)
\]

2.2. Compactifying $M_0$. We recall that $V_1 = \mathbb{CP}^{n_1}$ and we are adding in the manifold $V_2 \times \ldots \times V_r$ at the point $s = 0$. We refer the reader to the discussion immediately after equation (4.17) in [9]. In a nutshell, in order for the metric to extend smoothly we require that $\alpha(0) = 0$, $\alpha'(0) = 2$, $\beta_1(0) = 0$ and $\beta_1'(0) = 1$.

As we are using $\beta_1(s) = A_1(s + \kappa_0)^2 - \frac{q_1^2}{4A_1}$ we must have $A_1 = \frac{1}{2s_0}$ and $|q_1| = 1$. We also have normalised so that $p_1 = n_1 + 1$ hence the consistency conditions become
\[
E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2} (4(n_1 + 1) - \epsilon\kappa_0) = \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2} \quad \text{for } 2 \leq i \leq r.
\]

2.3. Steady quasi-Einstein metrics. In this case $\epsilon = 0$. Setting $V_1 = \mathbb{CP}^{n_r}$ and compactifying we obtain a $\mathbb{C}^{n_1+1}$-vector bundle over $V_2 \times \ldots \times V_r$.

In order that $\beta_i(0) > 0$ on $I = [0, \infty)$ we must have $A_i > 0$ and
\[
E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2} (4(n_1 + 1)) = \frac{p_i}{A_i} \quad \text{for } 2 \leq i \leq r.
\]

Hence $A_i = \frac{p_i}{E^*}$ and
\[
\beta_i(s) = \frac{p_i}{E^*}(s + \kappa_0)^2 - \frac{E^* q_i^2}{4p_i}.
\]
It is clear that in order for $\beta_i(0) > 0$ we must have
\[
(n_1 + 1)|q_i| < p_i \quad \text{for } 2 \leq i \leq r.
\]

In order to ensure the metrics are complete it is sufficient to check that the integral
\[
t = \int_0^s \frac{dx}{\sqrt{\alpha(x)}} \quad (2.13)
\]
diverges as $s \to \infty$ (this says that geodesics cannot reach the boundary at infinity and have finite length). If we compute the function $\alpha(s)$ we see that
it is asymptotic to a positive constant $K$. Hence the above integral diverges and we obtain a complete quasi-Einstein metric for all $m > 1$ generalising the non-Kähler, Ricci-flat ones constructed in [20]. Choosing a different value of $E^*$ simply varies the metric by homothety.

2.4. Expanding quasi-Einstein metrics. Here we take $\epsilon = 1$ to factor out homothety. Again the manifolds in question are $\mathbb{C}^{n_1+1}$-vector bundles over $V_2 \times \ldots \times V_r$. Here the consistency conditions become

$$E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2} (4(n_1 + 1) - \kappa_0) = \frac{8A_i p_i - q_i^2}{8A_i^2} \text{ for } 2 \leq i \leq r.$$  

If $|q_i|(n_1 + 1) \leq p_i$ then we choose $0 < E^* < 2(n_1 + 1)^2$,

$$\kappa_0 = 2(n_1 + 1) + 2 \sqrt{(n_1 + 1)^2 - \frac{E^*}{2}}$$

and

$$A_i = \frac{1}{2E^*} \left( p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}} \right).$$

In order that $\beta_i(0) > 0$ we require $2\kappa_0 A_i > |q_i|$ for $2 \leq i \leq r$. This can be seen as

$$2 \left( 2(n_1 + 1) + 2 \sqrt{(n_1 + 1)^2 - \frac{E^*}{2}} \right) \frac{1}{2E^*} \left( p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}} \right) > \frac{2(n_1 + 1)p_i}{E^*} > |q_i|.$$  

In the case that $|q_i|(n_1 + 1) < p_i$ we note also that

$$\left( 1 + \sqrt{1 - \frac{E^* q_i^2}{2p_i^2}} \right) > \left( 1 + \sqrt{1 - \frac{E^*}{2(n_1 + 1)^2}} \right),$$

hence,

$$2 \left( 2(n_1 + 1) - 2 \sqrt{(n_1 + 1)^2 - \frac{E^*}{2}} \right) \frac{1}{2E^*} \left( p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}} \right) > \frac{4p_i(n_1 + 1)}{2E^*} \left( 1 - \sqrt{1 - \frac{E^*}{2(n_1 + 1)^2}} \right) \left( 1 + \sqrt{1 + \frac{E^*}{2(n_1 + 1)^2}} \right) = \frac{p_i}{(n_1 + 1)} > |q_i|.$$  

Therefore if we have the strict inequality $|q_i|(n_1 + 1) < p_i$ then we can also choose

$$\kappa_0 = 2(n_1 + 1) - 2 \sqrt{(n_1 + 1)^2 - \frac{E^*}{2}}.$$  

If $|q_i|(n_1 + 1) > p_i$ then we can choose $0 < E^* < 2(n_1 + 1)^2 \min(p_i^2 / q_i^2, \ldots, p_r^2 / q_r^2)$. If we also choose

$$\kappa_0 = 2(n_1 + 1) + 2 \sqrt{(n_1 + 1)^2 - \frac{E^*}{2}}$$
and
\[ A_i = \frac{1}{2E^*} p_i + \sqrt{\frac{p_i^2 - E^*q_i^2}{2}}, \]
then \( \beta_i(0) > 0 \). We can also choose \( E^* < 0 \) in this case. Completeness follows as \( \alpha(s) \) is asymptotic to \( Ks^2 \) for a positive constant \( K \) and so the integral \((2.13)\) diverges. Hence we find complete, quasi-Einstein analogues of the non-Kahler, Einstein metrics constructed in [20].

2.5. Shrinking quasi-Einstein metrics. In order to factor out homothety we take \( \epsilon = -1 \) and so the consistency conditions are
\[ \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2} (4(n_1 + 1) + \kappa_0) = \frac{8A_i p_i + q_i^2}{8A_i^2} \text{ for } 2 \leq i \leq r. \]

We split the discussion into the compact case and the non-compact, complete case. For the compact case we consider \( I \) to be the finite interval \([0, s_*]\). We set \( V_r = \mathbb{C}P^{n_r} \) and at the point \( s = s_* \) we add in the manifold \( V_1 \times \ldots \times V_{r-1} \). For the metric to extend smoothly we require that \( q_r^2 = 1 \), \( p_r = n_r + 1 \), and \(-1 = 2A_r(s_* + \kappa_0)\). Putting these into the consistency conditions we see that
\[ \kappa_0 (4(n_1 + 1) + \kappa_0) = (s_* + \kappa_0)^2 - 4(n_r + 1)(s_* + \kappa_0) \]
and hence
\[ s_* = \sqrt{\kappa_0 (4(n_1 + 1) + \kappa_0) + 4(n_r + 1)^2} - \kappa_0 + 2(n_r + 1). \]

We note that if \( n_1 = n_r \) then \( s_* = 4(n_1 + 1) \). For the time being we note that \( s_* = s_*(E^*) \) and \( \beta_i \) is completely determined by \( E^* \) once we have chosen the value of \( q_i^2 \) and the sign of \( A_i \). The value \( A_i \) is given by
\[ A_i = \frac{1}{2E^*} \left( p_i + \chi_i \sqrt{p_i^2 + E^*q_i^2} \right) \]
where \( \chi_i = 1 \) if \( A_i > 0 \) and \( \chi_i = -1 \) if \( A_i < 0 \). In order to have a quasi-Einstein metric we must be able choose a value of \( E^* > 0 \) such that the integral
\[ \int_0^{s_*(E^*)} \prod_{i=0}^{i=r} \left[ \left( s + \kappa_0 \right)^2 - \frac{q_i^2}{4A_i^2} \right]^{n_i} \left( s + \kappa_0 \right)^{m-2} \left( E^* - \frac{1}{2}(s + \kappa_0)^2 \right) ds = 0. \]

Changing coordinates to
\[ x = \frac{1}{2}(s + \kappa_0) - ((n_1 + 1)^2 + \frac{E^*}{2})^{1/2}, \]
then the above integral becomes (ignoring constants)
\[ F(E^*) = \int_{-(n_1+1)}^{x_*(E^*)} \prod_{i=0}^{i=r} P_i(x) \left( x + ((n_1+1)^2 + \frac{E^*}{2})^{1/2} \right)^{m-2} \left( x^2 + 2x((n_1+1)^2 + \frac{E^*}{2})^{1/2} \right) \left( (n_1+1)^2 \right) dx \]
where

\[ P_i(x) = \left( x^2 + 2x((n_1 + 1)^2 + \frac{E^*}{2})^{1/2} + (n_1 + 1)^2 + \frac{2p_i(\chi_i\sqrt{p_i^2 + \frac{E^*}{2}q_i^2} - p_i)}{q_i^2} \right)^{n_i} \]

and

\[ x_*(E^*) = (n_r + 1) + \frac{E^*}{2} + (n_r + 1)^2 \left( \frac{E^*}{2} + (n_1 + 1)^2 \right)^{1/2}. \]

We will compute the limit \( \lim_{E^* \downarrow 0} F(E^*) \) and the limit \( \lim_{E^* \to \infty} F(E^*) \).

We begin with 0. We note that as \( m > 1 \) the function \( f(x) = (x + (n_1 + 1)^{n-2} \)

is integrable on \( -(n_1 + 1), x(E^*) \) so by the dominated convergence theorem

we can evaluate the integral of the limit. This is given by

\[ S \int_{-(n_1+1)}^{2(n_r+1)-(n_1+1)} \prod_{\chi_i=-1} \left[ x + (n_1 + 1) \right]^{2n_i} \prod_{\chi_j=1} \left[ \frac{4p_i^2}{q_i^2} - (x + (n_1 + 1))^2 \right]^{n_j} (x + (n_1 + 1))^{m} dx, \]

where

\[ S = (-1)^{\sum_{\chi_i=-1} n_i}. \]

The hypothesis on the \( p_i \) and \( q_i \) mean that the sign of \( \lim_{E^* \downarrow 0} F(E^*) \) is that of \( S \).

For \( E^* \to \infty \) we consider

\[ \lim_{E^* \to \infty} F(E^*)(E^*)^{\frac{1}{2}(1-m-\sum_{\chi_i=-1} n_i)} = K(-1)^{\sum_{\chi_i=-1} n_i} \int_{-(n_1+1)}^{(n_r+1)} \prod_{i=1}^{r} \left[ \chi_i x + \frac{p_i}{|q_i|} \right]^{n_i} x dx, \]

where \( K \) is a positive constant. Hence if we can choose \( \chi_i \) so that

\[ \int_{-(n_1+1)}^{(n_r+1)} \prod_{i=1}^{r} \left[ \chi_i x + \frac{p_i}{|q_i|} \right]^{n_i} x dx < 0, \]

we can find an \( E^* > 0 \) such that \( \alpha(s_*) = 0 \). A discussion similar to that in \([9]\) and \([20]\) shows that this is enough to ensure we have smooth quasi-Einstein metrics.

3. Examples and Future Work

We end with an example of theorem \([1,3]\) some discussion of the geometry of the quasi-Einstein metrics constructed and a discussion of possible sources future compact examples.

3.1. An example. We consider an example that is also considered in \([9]\). They consider a \(\mathbb{CP}^1\)-bundle over \(\mathbb{CP}^2 \times \mathbb{CP}^2\). In theorem \([1,3]\) this corresponds to taking \( r = 4, n_1 = n_4 = 0, n_2 = n_3 = 2 \) and \( p_2 = p_3 = 3 \). Hence to apply the theorem we must consider \(|q_2|, |q_3| < 3\). They take \((q_2, q_3) = (1, -2)\). The Futaki invariant is given by

\[ \int_{-1}^{1} (3 - x)^2 \left( \frac{3}{2} + x \right)^2 x dx \]
which they calculate is 7.8. This means that
\[ \int_{-1}^{1} (3 + x)^2 \left( \frac{3}{2} - x \right)^2 x dx = -7.8 < 0 \]
and we have non-trivial quasi-Einstein metrics on this space for all \( m > 1 \).

3.2. **Remarks on the geometry of the quasi-Einstein metrics.** In [9] section 4, the authors comment on the geometry at infinity of their examples of steady and expanding gradient Kähler-Ricci solitons. In particular they conclude that their steady examples are asymptotically parabolic and that the expanding examples are asymptotically conical. We recall that the examples of steady quasi-Einstein metrics constructed in theorem 1.2 have \( \alpha(s) \sim K \) for some positive constant \( K \) and so the following asymptotic behaviour holds (ignoring multiplicative constants)
\[ f(t) = O(1) \text{ and } g_i(t) \sim t. \]
In the expanding case we recall that \( \alpha(s) \sim K s^2 \) and so we have
\[ f(t) \sim e^t \text{ and } g_i(t) \sim e^t. \]

3.3. **Future families.** The space \( \mathbb{CP}^2 \sharp \mathbb{CP}^2 \) fits into the framework of theorem 1.3 as a non-trivial \( \mathbb{CP}^1 \)-bundle over \( \mathbb{CP}^1 \). On this space there is the Page metric, the Koiso-Cao soliton and the quasi-Einstein metrics of theorem 3 (originally due to Lü-Page-Pope). The space \( \mathbb{CP}^2 \sharp 2 \mathbb{CP}^2 \) also admits a non-Kähler, Einstein metric due to Chen, LeBrun and Weber [8] and a Kähler-Ricci soliton due to Wang and Zhu [21]. It would seem reasonable that there should be a family of quasi-Einstein analogues to these metrics. The metrics on \( \mathbb{CP}^2 \sharp 2 \mathbb{CP}^2 \) are not cohomogeneity-one but do have an isometric action by \( T^2 \). One observation is that the Lü-Page-Pope quasi-Einstein metrics are conformally Kähler (as any \( U(2) \)-invariant metric on \( \mathbb{CP}^2 \sharp 2 \mathbb{CP}^2 \) is). The Chen-LeBrun-Weber metric is also conformally Kähler (a fact originally proved by Derdzinski [10]) and so one might hope that the same would be true for analogues of the Lü-Page-Pope metrics on \( \mathbb{CP}^2 \sharp 2 \mathbb{CP}^2 \). Both the Page and Chen-LeBrun-Weber metrics are conformal to extremal Kähler metrics which satisfy a PDE that ‘occurs naturally’ in Kähler geometry. It would be an interesting first step to try and find an analogous PDE/ODE for the Kähler metrics that are conformal to the Lü-Page-Pope metrics. The author hopes to take up the existence questions in a future work.

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