Reconstruction of late-time cosmology using Principal Component Analysis

Ranbir Sharma1a, Ankan Mukherjee2b, H. K. Jassal1c

1 Indian Institute of Science Education and Research Mohali, SAS Nagar, Punjab 140306, India
2 Centre for Theoretical Physics, Jamia Millia Islamia, New Delhi 110025, India.

Abstract. We reconstruct late-time cosmology using the technique of Principal Component Analysis (PCA). In particular, we focus on the reconstruction of the dark energy equation of state from two different observational data-sets, Supernovae type Ia data, and Hubble parameter data. The analysis is carried out in two different approaches. The first one is a derived approach, where we reconstruct the observable quantity using PCA and subsequently construct the equation of state parameter. The other approach is the direct reconstruction of the equation of state from the data. A combination of PCA algorithm and calculation of correlation coefficients are used as prime tools of reconstruction. We carry out the analysis with simulated data as well as with real data. The derived approach is found to be statistically preferable over the direct approach. The reconstructed equation of state indicates a slowly varying equation of state of dark energy.

PACS. Cosmology, dark energy equation of state reconstruction, Principal Component Analysis, correlation coefficient.

1 Introduction

Cosmological parameters are now constrained to much better precision than before [1]. This has been facilitated with significant improvements in observational techniques and the accessibility of various observational data. Data in the last two decades has confirmed that the dynamics of the Universe is dominated by a negative pressure component, known as dark energy. Dark energy is understood to be the component driving the observed accelerated expansion of the present Universe. One of the prime endeavors of modern cosmological research is to reveal the fundamental identity of dark energy, its exact nature and evolution.

It is not clear from the present observations whether dark energy is a cosmological constant [2,3,4,5] or a time-evolving entity [6,7]. The dark energy can be described by equation of state parameter \( w = -P_{de}/\rho_{de} \), where \( \rho_{de} \) is the energy density and \( P_{de} \) is its pressure contribution. The form of the equation of state parameter \( w \) of dark energy depends on the theoretical scenario being considered. A constant value \( w = -1 \) corresponds to the \( \Lambda \)CDM (cosmological constant with cold dark matter) model, whereas in case of time-evolving dark energy, the equation of state parameter can have different values [2,5,6]. 

Since the approach to understanding dark energy is primarily phenomenological, it is necessary to determine the model parameters to constrain and rule out models that are not consistent with data. Likelihood analysis is the most commonly used technique in cosmological parameter estimation and model fitting [2,5,6,8,9,10,11,12,13,14,15,16,17,18]. It is based on Bayesian statistical inference, where the posterior probability distribution of a parameter is determined with a uniform or variant prior function and the likelihood function. A combination of different data-sets, with likelihood regions complementary to each other allows a very narrow range for cosmological parameters. Thus the combination of various data-sets tightens the constraints on the parameters. This parametric reconstruction [31,32,33,34,35,36,37] assumes a functional form of the equation of state parameter of dark energy.
energy. However, it induces the possibility of bias in the parameter values due to the assumption of the parametric form. One elegant way to constrain the equation of state parameter is by the differential age method of cosmic chronometer technique, which can be used to find out the evolution of the Universe without any inference from a particular cosmological model [16, 17, 58, 49].

An alternative method is to reconstruct the evolution of cosmological quantities in a non-parametric fashion. Various statistical techniques have been adopted for non-parametric reconstruction of cosmological quantities [50, 51, 52, 53, 54, 55]. One of the promising techniques in non-parametric approaches is the Gaussian Process (GP) [49, 40, 57, 68, 59, 60, 61]. We can create a multivariate Gaussian function with a determined mean and corresponding covariance function from any finite number of collections of random variables.

To accomplish the reconstruction of cosmological quantities, we use the Principal Component Analysis (PCA). A comparative study of different model-independent methods can be found in [62]. PCA is a multivariate analysis and is usually employed to predict the form of cosmological quantities in a model-independent, non-parametric manner [63, 64, 65, 66, 67]. In [68] and [69], different variants of PCA techniques have been adopted. Reference [68] uses an error model and then creates different set of simulated Hubble data to construct the covariance matrix while [69] used the weighted least square method and combines it with PCA.

From the viewpoint of the application of PCA, there exist two distinct methodologies. These methods differ mainly in the way the covariance matrix is calculated, which is the first step of any PCA technique. One way to implement PCA is through the computation of Fisher matrix. [62, 65, 66, 67, 70, 12, 23, 24, 75]. One can bin the redshift range and assume a constant value for the quantity to be reconstructed in that redshift bin. These constant values are the initial parameters of the PCA. Fisher Matrix quantifies the correlation and uncertainties of these parameters. Therefore, by deriving these constants in different bins using PCA, we can reproduce our targeted quantity in terms of redshift. [67]. Alternatively, a polynomial expression for the dynamical quantity can be assumed. In this case, the coefficients of the polynomial are the initial parameters for PCA, and the analysis gives the final values of these coefficients, which eventually gives the dynamical quantity [64].

PCA is independent of any prior biases and it is also helpful in comparing the quality of different data-sets [70, 76]. It is an application of linear algebra, which makes the linearly correlated data points uncorrelated. The correlated data points of the data-set used in PCA are transformed by rotating the axes, where the angle of rotation of these axes is such that linear-correlations between data-points are the smallest compared to any other orientation. The new axes are the principal component (PCs) of the data points, and these PCs are orthogonal to each other. In terms of information in the data-set, PCA creates a hierarchy of priority between these PCs. The first PC contains information of the signal the most and hence has the smallest dispersion of data-points about it. The second PC contains less information than the first PC and therefore provides a higher dispersion of the data-points as compared to the first PC. Higher-order PCs have the least priority as these correspond to noise and we can drop them. The reduction of dimensions is a distinctive feature of PCA. Therefore the final reconstructed curve in the lower dimension corresponds predominantly to the signal of the data-set. The PCA method also differs from the regression algorithms, which can not distinguish between signal and the noise. PCA can omit the features coming from the noise part and can pick the actual trend of the data-points [77, 78, 79]. In our case, the data-points on which we apply PCA are created in the process of reconstruction, see sect(2).

We assume polynomial expressions for the observational quantities, namely the Hubble parameter $H(z)$, and the distance modulus ($\mu(z)$). The polynomial form is then modified and predicted solely by PCA reconstruction algorithm. The only assumption we make is that it is possible to expand the reconstructed quantity in a polynomial. The dark energy equation of state is constructed in the subsequent steps after the reconstruction of the Hubble parameter and distance modulus.

The two-step process of reconstruction is termed as the derived approach. Further, in a direct approach, we apply the PCA technique to the polynomial expression of the equation of state parameter $w(z)$. For the application of our methodology, we require a tabulated dataset of the observational quantity, which can schematically be represented as, $(\text{independent variable})-(\text{dependent variable})-(\text{error bar of dependent variable})$. In the direct approach, described below, we do not have the tabulated data-set of $w(z)$ and we derive the functional form of $w(z)$ from the tabulated data-set of $z - H(z)$ and $z - \mu(z)$. In the case of the derived approach, PCA first reconstructs the functional form of $H(z)$ and $\mu(z)$ from the Hubble parameter and Supernovae data-set respectively without any cosmological model. In the absence of cosmological model, $H(z)$ and $\mu(z)$ have no any parameter dependencies. However to construct the final form of equation of state of dark energy, $w(z)$ we need $\Omega_m$, as input. Application of PCA on polynomial expressions to reconstruct cosmological quantities have been employed earlier in [67, 68, 69].

For a further check on the robustness of our results; we compute the correlations of the coefficients, present in the polynomial expressions of the reconstructed quantities. We apply PCA in the coefficient space, which is created by the polynomial. The PCA rearranges the correlation of these coefficients. Calculation of correlation-coefficients help to identify the linear and non-linear correlations of the components in the reconstructed quantities. We show that using the correlation coefficients calculation, we can restrict the allowed terms in the polynomial.

This paper is structured as follows. In sect(2) we describe the reconstruction algorithm and the use of correlation tests in these reconstructions. In sect(3), we describe the two approaches of reconstruction we follow. We...
present our results in sect[4]. In sect[5], we conclude by summarizing the results.

2 Methodology

We begin with an initial basis, \( g_i = f(x)^i - 1 \), where \( i = 1, 2, ..., N \) through which we can express the quantity to be reconstructed as,

\[
\xi(x) = \sum_{i=1}^{N} b_i f(x)^{(i-1)} \tag{1}
\]

The initial basis can be written in matrix form as, \( G = (f_1(x), f_2(x), ..., f_N(x)) \). Coefficients \( b_i \) create a coefficient space of dimension \( N \). Each point in the coefficient space gives a realization of \( \xi(x) \), including a constant \( \xi(x) \).

PCA modifies the coefficient space and chooses a single realization. One can consider different kinds of functions as well as different combinations of polynomials as initial bases. Choosing one initial basis function over the other as well as different combinations of polynomials as initial basis, which means a higher value of \( \rho \) more non-linear contributions to the correlation of the quantity we want to reconstruct. They are also sensitive towards the data-set that we use in reconstruction. For instance, reconstruction of a fast varying function, which has non-zero higher order derivatives will introduce more non-linear contributions to the correlation of \( b_i \) than linear contributions, and we need a greater set of initial basis, which means a higher value of \( N \). We take different values of \( N \) and check the linear and non-linear correlation coefficients to fix the value of \( N \). Though a large value would help, we can not, however, fix \( N \) to any arbitrarily significant number as it makes the analysis computationally expensive.

To compute the covariance matrix, we define the coefficient matrix \( Y \), by selecting different patches from the coefficient space,

\[
Y = [(b_1^j)]_{i=1,n} \tag{5}
\]

where \( n \) is the number of patches that we have taken into account and \( N \) is the total number of initial basis defined in eq[1]; therefore, \( Y \) is a matrix of dimension \( N \times n \) and \( b_i(N) \) being the value of \( N \)th coefficient in \( n \)th patch.

In the present analysis, we have taken \( n \) to be the order of 10^3. We estimate the best-fit values of the coefficients at each patch by \( \chi^2 \) minimization, where \( \chi^2 \) is defined as

\[
\chi^2 = \sum_{j=1}^{k} \frac{(\xi(x)_{\text{data}} - \xi(b_i, x))^2}{\sigma_j^2} \tag{6}
\]

\( k \) is the total number of points in the data-sets. If the observational data-set have significant non-diagonal elements in the data covariance matrix(\( C_{\text{data}} \)), we have to
incorporate $C_{\text{data}}$ in eq[6], as $\chi^2 = \Delta^T C^{-1} \Delta$, where $\Delta = \xi(x)_{\text{data}} - \xi((b), x)$ and $x$, $\xi(x)_{\text{data}}$ vary for each data-points. Calculation of $\chi^2$ in all the patches gives us the variation of the $N$ coefficients and finally gives $n$ number of points in the coefficient space, over which we apply PCA. We calculate the covariance matrix and correlations of the coefficients for these $n$ points. In this analysis, each patch contains the origin of the multi-dimensional coefficient space.

The covariance matrix of the coefficients, $C$ is written as,

$$C = \frac{1}{n} YY^T$$

Eigenvector matrix, $E$ of this covariance matrix will rotate the initial basis of the coefficient space to a position where the patch-points will be uncorrelated. We organize the eigenvectors in the eigenvector matrix $E$ in the increasing order of eigenvalues. Eigenvalues of the Covariance matrix quantifies the error associated with each principal component $[63,67,82,83]$. If $U = (u_1(x), u_2(x), ..., u_N(x))$ the final basis is given by,

$$U = GE$$

The final reconstructed form of $\xi(x)$ is,

$$\xi(x) = \sum_{i=1}^{M} \beta_i u_i(x)$$

where $M \leq N$ and the $\beta_i$s are the uncorrelated coefficients associated with the final basis. The coefficients $\beta_i$ are re-calculated by using $\chi^2$ minimization, for the same $n$ patches considered earlier to create initial coefficient matrix $[5]$. The value of $M$ can be determined by correlation coefficient calculation discussed above. As PCA only breaks the linear correlation, we select $M$ for which PCA is able to break Pearson correlation coefficient to the largest extent. Eigenvalues of the covariance matrix $(e_1, e_2, e_3, ..., e_M)$ indicate how well PCA can pick the best patch-point in the coefficient space. We look for the lowest number of final basis which can break the linear correlation to the greatest extent. We can start with smaller value of $N$ which eventually influence the value of $M$, but it will pose the risk of losing essential features from PCA data-set.

3 Reconstruction of dark energy equation of state

The Hubble parameter $(H(z))$ for a spatially flat Universe, composed of dark energy and non-relativistic matter is given by,

$$H^2(z) = H_0^2[\Omega_m(1+z)^3 + \Omega_x e^{3f_0\int_0^{z} \frac{dt}{1+w(t')}}]$$

Here we have assumed that the contributions to the energy density is only due to the non-relativistic dark matter and dark energy. The density parameters for non-relativistic matter and dark energy are given by $\Omega_m$ and $\Omega_x$. The quantity $H_0$ denotes the present-day value of the Hubble parameter, namely the Hubble constant and $w(z)$ is the dark energy equation of state parameter $(w(z))$.

We assume no interaction between matter and dark energy in the present analysis. In the following subsections, we discuss the derived and direct approach to reconstruct $w(z)$ using PCA.

3.1 Derived Approach

The derived approach is a two-step process in the reconstruction of dark energy Equation of State(EdS). In the first step, we reconstruct the observable, namely the Hubble parameter using the $H(z)$ data and the distance modulus $(\mu(z))$ using the type Ia supernova data. Subsequently, we reconstruct $w(z)$ as a derived quantity from these two different physical quantities. Similar sequence of reconstructions have already been discussed in $[63,65,67,76]$. We follow the approach mentioned in section 2 to reconstruct the curve of $H(z)$ and $\mu(z)$. Differentiating eq[10] with redshift $z$ as the argument and rearranging the terms we can express $w(z)$ as,

$$w(z) = \frac{3H^2 - 2(1+z)HH'}{3H_0^2(1+z)^2\Omega_M - 3H^2}$$

Here, $H'$ is the derivative of Hubble parameter with respect to redshift $z$. Since $w(z)$ is related to $H(z)$ through eq[10] by the zeroth and the first order differentiation of $H(z)$, the small difference in the actual and the reconstructed curve of $H(z)$ is amplified by the $H'$ term. This process of amplification of the deviation from actual nature becomes more severe with subsequent higher-order differentiation of the reconstructed quantity.

The luminosity distance $d_L(z)$ is given by,

$$d_L(z) = \frac{e}{H_0} (1+z) \int_0^z d_H(z')dz'$$

where $d_H$, in terms of eq[9] is,

$$d_H(z) = \left( \Omega_m (1+z)^3 + \Omega_x e^{3f_0\int_0^{z} \frac{dt}{1+w(t')}} \right)^{-1/2}$$

and is related to the distance modulus as

$$\mu(z) = 5 \log \left( \frac{d_L}{1Mpc} \right) + 25$$

which is the dependent-variable in the type Ia supernovae data. From PCA, we determine the form of $\mu(z)$ directly from data and then from eq[11] and eq[12] find the expression of $d_L$. From eq[13], we trace back to eq[12] and find an expression which gives the $w(z)$ in terms of the distance modulus.

Since $D(z) = (H_0/c)(1+z)^{-1}d_L(z)$, the equation of state parameter is given by

$$w(z) = \frac{2(1+z)D'' + 3D'}{3D''\Omega_m(1+z)^3 - 3D'}$$
The second order derivative in eq(14) makes the reconstruction of \( w(z) \) through that of distance modulus unstable. For instance, if the reconstruction fails to pick some of the minute difference in the observational curve, then that difference will be amplified twice in the final calculation of the EoS. Therefore, the reconstruction of \( \mu(z) \) should be more accurate in picking up approximately all the features of \( w(z) \) which may be hidden within the supernovae data [47,64,84,85].

In the reconstruction of \( H(z) \) and \( \mu(z) \), we begin with polynomial expansions in terms of the different variables \( z, a \) and \( (1-a) \) where \( z \) is the red-shift and \( a \) is the scale factor. The analysis is carried out with seven terms in initial basis, which means creating a coefficient space of \( N = 7 \) dimensions (eq(1)). We test our algorithm on a simulated data-set of \( \Lambda \) CDM cosmology. We generate the data-points of \( H(z) \) and \( \mu(z) \) for \( w(z) = -1 \) and the values of cosmological parameters \( \Omega_m \) and \( H_0 \) are fixed at Planck 2018 values [1]. Fig(2) and table 1 show that our algorithm can predict the simulated data.

### 3.2 Direct Approach

For the direct reconstruction approach, we begin with a polynomial form of \( w(z) \) itself. In eq(9), the quantity \( w(z) \) is in the exponent, and considering a polynomial form for \( w(z) \) implies addition of some non-linear components to our linear analysis in coefficient space. Again we fix the dimension of the initial basis \( N \) by computing the correlation coefficients. Here we need to balance available computational power as well as the non-linearity we introduce, with the accuracy we demand to choose the value of \( N \).

In this case too the independent variables are taken to be \( (1-a), a \) and \( z \). Here we introduce non-linear terms in the initial coefficients of PCA; here correlation coefficients calculation is not of assistance as in the case of derived approach [5] to select \( M \). Due to the risk of compromising different features of the data-set and also due to the complex dynamics of correlation coefficients, we do not reduce any terms in the final basis (\( N = M \)) of the direct approach.

To test the effectiveness of both the approaches described above, we first work with simulated data-sets for specific models. We create simulated data-points for \( w = -1 \) (\( \Lambda \) CDM) and \( w(z) = -\tanh (1/z) \) [8] where the values of \( \Omega_m \) and \( H_0 \) are fixed at Planck 2018 values [1]. We use eq(9) to calculate \( H(z) \) at the same redshift value as in the real Hubble parameter vs redshift data-set [50, 57, 58, 59, 60].

Similarly, the distance modulus data points are simulated using equations (11), (12) and (13). Here we evaluate distance modulus \( \mu(z) \) at the same redshift values as are there in type Ia supernovae (SNe) Pantheon data-set [91]. We later utilize the observational measurements of Hubble parameter at different redshift [50, 57, 58, 59, 60] and the distance modulus measurement of type Ia supernovae (SNe) data [91].

| Variable | Reduced Hubble constant \( h_0 \) | Simulated | Observed |
|----------|----------------------------------|-----------|----------|
| \((1-a)\) | 0.674 ± 0.118                    | 0.784 ± 0.01157 | |
| \(a\)      | 0.664 ± 0.443                    | 0.739 ± 0.103    | |

Table 1. The value of reduced Hubble constant estimated for the simulated as well as real date-set using the derived approach. For generation of the simulated data, \( h_0 \) is fixed at 0.685. Error associated with the estimation of reduced Hubble constant are also given.

### 4 Results

#### 4.1 Derived approach

We first reconstruct \( H(z) \) and \( \mu(z) \) using simulated dataset. The reconstruction with simulated data is a check on the viability of the method. Fig (1) shows the reconstructed curves of the reduced Hubble parameter \( h(z) \) and distance modulus \( \mu(z) \) obtained for the simulated data. The reconstructed curves are for three different reconstruction variables \( (1-a), \) the scale factor \( a \) and the redshift \( z \). It is clear from the plot [1] that the PCA reconstruction produces a consistent result when \( (1-a) \) is chosen as the independent variable. We also plot the difference between the fiducial model and the reconstructed curves along for a comparison. The plots of the residues clearly show that the reconstruction of both \( h(z) \) and \( \mu(z) \) validates the reconstruction approach. The number of terms on the initial basis is fixed at \( N = 7 \). We also check our results for \( N = 8 \) and \( N = 10 \), and find that the results do not vary significantly. Here, we present the reconstruction curves from the variable \( z \) only for the simulated data-set. Since in this case, we do not achieve a viable reconstruction, we do not analyse this further for real data.

From table 1 we see that both \((1-a)\) and \(a\) variables are able to predict the value of \( h_0 \) closer to the assumed value of \( h_0 \) in simulated data. We calculate the error in the prediction of \( h_0 \) from the Covariance matrix of the PCA data-set, where we give a cut off \( \chi^2 \) value, \( \chi^2_{\text{cutoff}} \) to mask out some patch points which lead to under-fitting [63, 92, 93].

Further, the correlation coefficient calculation suggests that the best choice of number of terms for variable \((1-a)\) in the final polynomial (15) is \( M = N - 1 \), reduction of one term from the initial polynomial expression (1). For variable \( a \) and \( z \), it is \( M = N \), hence there is no reduction of terms. See (5).

Fig(2) shows that the variable \((1-a)\) reproduces \( w = -1 \) behaviour when the simulated data is used. Shaded region of fig(2) represents all possible \( w(z) \) vs \( z \) curves produced by PCA for simulated data with a variation of \( \Omega_m \) from 0.1 to 0.5. \( \Omega_m \) is one of the free parameters in our methodology, and the methodology does not pick any one curve over another. The polynomial expression of \( H(z) \) in \((1-a)\) is effectively an infinite series in terms of \( z \). As the independent variable of the data-sets is \( z \), therefore, \((1-a)\) and \( a \) can capture more features than the initial basis \( z \). When the observational data-set is used in this
polynomial expression, it indicates a time evolving $w(z)$, fig 7. The other two variables, namely $a$ and $z$, could not successfully reproduce the $w = -1$ nature while using the simulated data. In the case of simulated supernovae data-set we see that all the $w(z)$ reconstructions for $h_0 = 0.685$ and $\Omega_m$ varying from 0.1 to 0.4 follows a similar trend after we choose a specific basis function. For all the three reconstruction variable $w(z)$ vs $z$ curves fluctuate between phantom and non-phantom regime.

We now construct the equation of state parameter $w(z)$ from the reconstructed $H(z)$ and $\mu(z)$. It is clear from fig 2 that the reconstruction by derived approach for the variable $(1 - a)$ successfully reproduces the $w(z)$ assumed earlier. On the other hand, the reconstruction variable $a$ and $z$ do not reproduce the $w(z)$ which has been assumed to simulate the data.

Using the reconstructed analytic form of $H(z)$ from derived approach, we estimate the present day value of the Hubble parameter $H_0$. The EoS parameter $w(z)$ calculated by the derived approach has the free parameter $\Omega_m$, which we vary in the reconstructed $w(z)$ curves, fig 7. We present the estimated values of $H_0$, scaled by $100 \text{ km sec}^{-1} \text{ Mpc}^{-1}$, for the analysis with simulated and the observational data in table 1. We also calculate the present day value of $w(z)$.

We calculate the range of $w(z)$ from eqn 9. We assume $w(z)$ to be constant ($w$) and find out the allowed range of $w$ for the range of $h(z)$ determined by PCA. This range of $w(z)$ is dependent on $\Omega_m$ and we do the
Fig. 2. The plots in this figure show reconstruction of \( w(z) \) for the simulated data. Red, blue and green curves are for the reconstruction with no reduction and with reduction of one and two terms respectively. In these curves, the value of \( \Omega_m \) is fixed at 0.364, the value of \( \Omega_m \) for which the reconstructed curve with no reduction and reduction of one term of the initial basis are the closest to the underlying \( w(z) = -1 \) for most of the low-redshift range. In the figure there are three patches; grey, yellow and brown, corresponding to all the reconstruction with \( \Omega_m \) vary from 0.1 to 0.5 with no reduction of final terms and with reduction of one and two terms respectively. Here we fix the reduced Hubble constant at the value predicted by PCA for the simulated data-set, \( h_0 = 0.674 \). The black line is for comparison and it corresponds to \( w(z) = -1 \).

Fig. 3. The plots show the reconstruction of \( h(z) \) and \( \mu(z) \) from the simulated Hubble parameter and supernovae data-set with allowed range \( \Lambda CDM \) cosmology with the initial basis function \((1-a)\). The brown patch is obtained for the variation of \( w \) and \( \Omega_m \) in the range [-1.5 to -0.5] and [0.2 to 0.4] for \( \Lambda CDM \) cosmology. Solid blue, green and red line are for PCA reconstruction with no-reduction and with one and two terms respectively.

| Variable | PCA | \( \Lambda CDM \) | wCDM | Planck(\( \Lambda CDM \)) |
|----------|-----|-----------------|-------|------------------------|
| \( 1-a \) | 78.4001 ± 1.157 | 67.94 ± 5.15 | 68.07 ± 1.63 | 67.9 ± 2.6 |
| \( a \) | 73.9271 ± 10.3 | 69.85 ± 4.44 | 68.19 ± 1.33 | 67.39 ± 0.54 |

Table 2. This is the comparison table of the values of Hubble constant in standard units \((km \ s^{-1} \ Mpc^{-1})\), obtained in the present analysis (PCA) and obtained from other model-dependent estimations.

analysis for \( \Omega_m = 0.2, 0.3, 0.4 \), as shown in fig(8). The allowed range of \( w \) for \( \Omega_m = 0.2, 0.3, 0.4 \) are \([-2.73, -0.71]\), \([-3.33, -0.71]\) and \([-4.14, -0.91]\) respectively. We do the analysis for \((1-a)\) as it is the selected variable by correlation test calculation. For supernovae data-set we find out the allowed range of \( w(z) \), which lies within the error bar of the data-set by considering a polynomial with variable \((1-a)\). Range of \( w(0) \) for \( \Omega_m = 0.2, 0.3, 0.4 \) are \([-1.2112, -0.7431]\), \([-1.224, -0.751]\) and \([-1.215, -0.7496]\) respectively.

Fig(3) indicates that the PCA reconstructions of \( h(z) \) is well within the range of \( wCDM \) parameters for the simulated data. We vary \( \Omega_m \) and equation of state parameter in \( wCDM \) cosmology \((w)\) in the range \([0.2, 0.4]\) and \([-1.5, -0.5]\) respectively. The ability of our methodology to reconstruct \( h(z) \) and \( \mu(z) \) reflects in the reconstruction of \( w(z) \) for the simulated data.

To quantify the efficiency of our algorithm in picking up the underlying theory, we create an error function by the method of interpolation with the errors of observa-
Fig. 4. Plots of $w(z)$ in the direct approach with simulated data-sets. The left panels are for simulated data with $w(z) = -1$ and the right panels are for $w(z) = -\tanh(1/z)$. The first row is the reconstruction by variable $(1 - a)$, second and the third row is the reconstruction by variable $a$ and $z$, respectively. Black solid line corresponds to the minimum $\chi^2$ and the red shaded region is the reconstruction using the PCA algorithm allowing a deviation of 0.3 from the minimum $\chi^2$ curve.

We now do the same analysis using observed data. Fig.[5] shows the reconstructed curves of reduced Hubble parameter $h(z)$ and $\mu(z)$ obtained for the real Hubble parameter and supernovae data-sets. It is evident from the plot that PCA reconstruction with $(1 - a)$ variable produces consistent results for observational data-set also. Fig.[7] shows the final reconstruction of $w(z)$ from the observed Hubble parameter data-set with the reduced Hubble constant fixed at the value predicted by PCA. Reconstruction of the functional form of $h(z)$ by choosing a point from coefficient space (eq 8) is the first step of derived approach, which eventually gives us the value of $h_0$ from the same point. We do not vary the value of $h_0$ in the $w(z)$ reconstruction curves to ensure that both $h(z)$ and $h_0$ are from the same point of coefficient space. The values obtained from the observational data-set are higher as compared to the other model-dependent estimations. Table 2 presents the values of $h_0$, obtained, along with model-dependent estimations of $h_0$ from other studies [1, 94, 95] for a comparison.
Fig. 5. The plots in the figure show the reconstructed reduced Hubble parameter $h(z)$ and distance modulus $\mu(z)$ for observed data-sets along with their residues. $\delta h(z)$ and $\delta \mu(z)$ are the residue in the hubble parameter and distance modulus respectively. Residues are calculated as the difference between PCA reconstruction and the corresponding $h(z)$ and $\mu(z)$ from the observational data. The left column is for the independent variable $(1 - a)$, and the right column is for $a$. For the blue curves, there is no reduction, that is, $M = N = 7$. The green and red curves are obtained by the reduction of the highest and second-highest Principal Components, respectively.
4.2 Direct approach

For the direct approach, we have considered only the Hubble parameter data-set. Reconstruction is carried out for all the three independent variables $(1 - a)$, $a$ and $z$. As in the case of derived approach, we first use simulated data-set. The results obtained for the simulated data-sets are shown in fig(4). In this case, all three independent variables reproduce the nature of the equation of state parameter. In the direct approach, using the correlation test calculation, we find $N = 5$ to be the best choice for the number of terms in the initial polynomial eq(1). As the number of terms on initial basis is comparatively low and we assume $M = N$ as our final basis number, the uncertainties which PCA poses in predicting the best patch-point is relatively low. The tiny non-linearity we introduced in the $H(z)$ by considering a polynomial form of $w(z)$ will be amplified in the case of supernovae data-set. Hence it makes the reconstruction much more unstable in the case of reconstruction of $w(z)$ through distance modulus calculation.

For the observed Hubble parameter data-set too we reproduce the curves using the same algorithm. In the
case of direct approach, PCA cannot predict the value of $H_0$ and $\Omega_m$; therefore, these two parameters have to be fixed prior to the analysis.

For real data-set, all the $w(z)$ plots for the variation of $\Omega_m$ from 0.2 to 0.4 after fixing reduced Hubble constant at $h_0 = 0.685$ have the similar trend. We also check the $w(z)$ plot for the variation of $h_0$ from 0.6 to 0.8 after fixing $\Omega_m$ at 0.30. Direct approach is more susceptible to model biasing as in the case of direct approach we have to select the value of $\Omega_m$ and $h_0$. Correlation coefficient calculation also indicates that derived approach have more potential of breaking the correlations of the coefficients of the polynomial than the direct approach.

In the direct approach, though the reconstruction of the fiducial $w(z)$ is consistent (fig[1]), the correlation test calculation for the direct approach shows that the algorithm is not able to break the Pearson Correlation as it breaks down in the case of the derived approach. In the case of direct approach, for $(1-a)$ reconstruction, the magnitude of Pearson correlation coefficients decreases after applying PCA but changes signs for the first two principal components, whereas Kendall and Spearman correlation coefficient of $(1-a)$ for these two principal components assume higher negative value. Again, for reconstruction by the variable $a$, the Pearson Correlation decreases, though both Spearman and Kendall Correlation coefficients decrease in magnitude, it changes sign for the first two principal components. For the variable $z$, up to the first two principal components, Pearson correlation coefficients decrease, but Spearman and Kendall correlation coefficients assume large negative value. From the correlation coefficient calculation the derived approach is selected over the direct approach and selects the reconstruction by the independent variable $(1-a)$ as compared to variables $a$ and $z$.

5 Summary and Conclusion

In this paper, we reconstruct late-time cosmology using the Principal Component Analysis. There are very few prior assumptions about nature and distribution of different components contributing to the energy of the Universe. Observational Hubble parameter and distance modulus measurements of type Ia supernovae are the observable quantities that are taken into account in the present analysis. We proceed in two different ways to do the reconstruction. The first one is a derived approach where the observable quantities are reconstructed from the data using PCA, and then $w(z)$ is obtained from the reconstructed quantities using Friedman equation. The other approach is a direct one. In this case, $w(z)$ is reconstructed directly from the observational data using PCA without any intermediate reconstruction. Based on the efficiency of the method to break the correlation among the coefficients we can select one reconstruction curve over the other. We achieve a better reconstruction in the derived approach as compared to the direct approach, even though the direct approach has lesser uncertainties than the derived approach in predicting the patch of the best coefficient point from the $N$ dimensional coefficient space due to the lower number of initial and final bases. In the derived approach, though, we need no cosmological theory or model to determine the functional form of $H(z)$ and $\mu(z)$; however for the calculation of $w(z)$, we need to input a cosmological model. On the other hand, the direct approach starts with using a polynomial form of $w(z)$ in Friedman equation. Therefore, in the case of direct approach, the dependencies of $w(z)$ on the cosmological parameters are greater.

We have adopted the simulated as well as observed data-sets for our analysis. Simulated data-sets are used to check the efficiency. For the reconstruction of $w(z)$ the analysis produces consistent result only for the Hubble parameter data. Though the reconstruction of $\mu(z)$ through the derived approach is consistent and reconstruction of $\mu(z)$ is within the error bars of the data-set, the result for the reconstructed $w(z)$ deviates drastically from the physically acceptable range. The increase in the order of differentiation to connect $w(z)$ with $\mu(z)$ is a possible reason for this inconsistency. Here we are focusing on the $w(z)$ reconstruction only from $H(z)-z$ data-set. The reconstructed $w(z)$ by the variable $(1-a)$, obtained in the derived approach for $H(z)-z$ data-set shows a phantom like nature, that is $w(z) < -1$ at present and a non-phantom nature in the past for most of the values of $\Omega_m$. In the case of direct approach, $w(z)$ curves show oscillations in the phantom and non-phantom regime. The calculation of the correlation coefficients clearly shows a preference for the derived approach. PCA lacks the efficiency of breaking the correlation in the initial basis in case of the direct approach. This probably causes the inconsistency between the results obtained in the derived and direct approaches.

The other important factor is the variable of reconstruction. In the present analysis, we have adopted three different reconstruction variables, namely $(1-a)$, $a$ and $z$, in both direct and derived approaches. Values of correlation coefficients after PCA select the reconstruction variable $(1-a)$ over the other two. We should emphasize the result obtained for variable $(1-a)$ by derived approach. Since we have a finite number of terms in the initial polynomial, due to the constraints set by computational power, to some extent the results depends on the assumption of polynomial expression of the observables. One of our future plans is to develop an algorithm which can inclusively select the most suitable initial basis form for a fix computational power and a given observational data-set. The reconstructed curves, obtained for $(1-a)$, show that $w(z)$ shows a phantom nature at present epoch and it was in non-phantom nature in the past. PCA reconstruction indicates an evolution of the dark energy equation of state parameter.

Appendices A

The correlation coefficients of the first three coefficients of the polynomial expression eq[1] and eq[2] of the reconstructed quantity is shown in table 3. PCA breaks the linear correlation of the coefficients, which is evident.
from table 3. Presence of the non-linear correlation in the initial coefficients complicates the process. As the first three terms of the ultimate expression of the reconstructed quantity contains the dominant trend of the data-points, we only mention the correlation coefficients only for the first three parameters. The reconstruction which can break Pearson Correlation to a greater extent as well as have lesser Spearman and Kendall Correlation coefficients is selected. We can see from table 3 that the reconstruction by $(1 - a)$, breaks the correlation to a greater extent than in the case of $a$ and $z$. Variation of correlation coefficients before and after the application of PCA is similar for both the simulated as well as the real data-set. Difference of Pearson correlation coefficients for simulated and real data-set in case of $(1 - a)$ and $a$ is of the order of $10^{-6}$ or less. For the variable $z$, the difference of Pearson correlation coefficients for real and simulated data is of the order of $10^{-4}$ or less.
Table 3. This table shows Pearson, Spearman and Kendall correlation coefficients between the coefficients of the first three terms of the series expansion of the reconstructed quantity for the reconstruction variables \((1 - a), a\) and \(z\) respectively. This is in the derived approach for simulated Hubble parameter data. The first, third and fifth rows (pre-PCA), in the table, shows the correlation coefficients of the first three coefficients of the initial polynomial we start with, viz \(b_1, b_2\) and \(b_3\), eq(1) for the variables \((1 - a), a\) and \(z\) respectively. The correlation coefficients of the first three coefficients of the final polynomial, viz \(\beta_1, \beta_2\) and \(\beta_3\), eq(8) given by the PCA algorithm is given in the second, fourth, sixth rows (post-PCA) for the variables \((1 - a), a\) and \(z\) respectively. Since the correlation matrix is symmetric, here we only mention the upper diagonal terms, whereas corresponding lower diagonal terms are replaced by *.

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### Data Availability

The observational data-sets used in the analysis are publicly available and duly referred in the text. The simulated data-sets can be provided on request with appropriate justification.

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