Diffraction propagation of vortex diffractals

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Abstract. A brief retrospective analysis of studies of fractal light radiation is carried out. To assess the prospects of this scientific direction, new original results of studying the diffraction propagation of vortex wave beams with a fractal structure (vortex diffractals) are also presented. For this purpose, computational algorithms and related software have been developed. In calculating the amplitude-phase and scaling characteristics of diffractals two-dimensional Weierstrass functions and multistructures of Gaussian beams were used. The results indicate a high information capacity of vortex diffractals and explain their resistance to the influence of turbulence in the propagation medium.

1. Introduction
The results of studying the characteristics of light beams with an initially complex amplitude-phase profile significantly improved the characteristics of laser information systems. Thus, beams with a vortex wavefront structure have a number of unique properties that provide an increased degree of stability of characteristics when propagating in a turbulent atmosphere [1, 2]. Waves with a fractal wavefront structure (diffractals) have also attracted the attention of researchers [3]. Their valuable quality is the manifestation of structural self-similarity during the propagation process. However, in the literature, the case when radiation has both vortex and fractal properties is insufficiently considered [4]. The aim of this work is to develop and use specially developed algorithms and computational programs to determine the specific characteristics of vortex fractal beams. Particular attention is paid to assessing the practical significance of the properties of such beams.

2. Retrospective analysis of fractal beam studies
There are several ways to create fractal beams. The most promising and widespread of them were covered in the review [3]. Fractal wave beams can be obtained by passing a plane wave through various kinds of masks, plates and screens with a fractal configuration. The fields of these beams have a number of remarkable properties. Thus, annular axisymmetric masks, obtained by rotating the one-dimensional Cantor system, form self-similar distributions in both the transverse and longitudinal directions. Moreover, a whole system of radiation focusing regions is formed along the optical axis, the distance between which obeys the scaling dependence [6]. Fractal wave structures can also be obtained as a result of the passage of a plane wave through apertures, the boundaries of which are described by fractal curves. In particular, the diffraction pattern at the aperture described by the Koch curve has a self-similar form, and the fractal dimensions of the diffracting and initial fields are interrelated [7]. This makes it possible to determine the fractality of an object based on the properties of a particular diffraction pattern. When analyzing the transmission of radiation through the plates, the
transmission of which is determined by the two-dimensional Weierstrass function, a high correlation was established between the fractal properties of the Fourier transforms of the plate structures and images of light beams. This result is also important for improving methods for diagnosing objects with fractal features, as well as for interpreting the aesthetic properties of fractal images [5]. The disadvantage of the method for producing fractal beams using specially made plate masks is their low radiation strength. In this regard, the possibility of generating waves with a fractal structure directly in lasers with unstable cavities has acquired a topical character [8].

In a number of works on the use of fractal radiation in communication systems, fractal vortex beams with a helicoid structure of the wave front were studied. These tube-shaped beams producing a circular energy flow are characterized by the quantum orbital momentum and its related order (topological charge) of the screw axis dislocation [3, 9].

Waves with a vortex structure have become the subject of much investigation due to the variety of their possible practical applications, including optical communication lines. Their valuable property is rather high stability against the effect of atmospheric inhomogeneities [1, 10]. The degree of stability increases with increasing angular momentum and dislocation order (topological charge). In this relation, of particular interest is the work [4] where it is proposed to form a fractal vortex field using superposition of Gaussian beams. Varying their phase relations, one can widely vary the effective orbital momentum of the total beam. Fractality of the latter manifests itself in transverse distributions of both the intensity and the phase.

There is another method for obtaining a fractal vortex beam, which uses a special type of fractal phase plate based on properties of the Cantor set [11]. This plate, unlike an ordinary spiral zone plate producing one vortex, gives rise to a whole system of vortex axis formations after the passage of a plane wave.

To characterize fractal properties of radiation more completely, it should be mentioned that evolution of illumination along the propagation axis reproduces the fractality of the lens, and diameters of partial vortices are proportional to the topological charge. The authors believe that their developed system can be used for manipulation and three-dimensional monitoring of microobjects and for improvement of lithographic processes and quality of multifocal contact lenses.

The results obtained in [11] confirm the data of the extensive theoretical and experimental investigations with various types of aperiodic fractal plates [12]. It is stressed that plates with an aperiodic structure admit of fractional topological charges which break symmetry of screw phase dislocations and form chains of anisotropic vortices.

3. Results of application of the two-dimensional Weierstrass function

Two approaches were used to analyze the characteristics of vortex fractal beams. The first approach was based on the properties of the two-dimensional Weierstrass function. In that case, the following expression was used:

\[
W_{k,m} = \sqrt{2\sigma} \frac{(1-b^{2D-4})^{1/2} \sum_{v=0}^{V} \sum_{n=0}^{N} \left( b^{(2D-2)n} \sin(2\pi b^{v} (k \sin(\alpha v) + m \cos(\alpha v))) \right) e^{i\varphi}}{(1-b^{(2D-4)(N+1)})^{1/2}}. \tag{1}
\]

Formula (1) is the result of the rotation of the coordinate system in which the one-dimensional Weierstrass function was previously specified. Here \(\alpha\) is the angle of single rotation, \(V\) is the number of repetitions, \(\sigma\) is the standard deviation, \(N+1\) is the number of harmonics, \(D\) is the fractal dimensionality, \(b\) and \(s\) are the scaling parameters, \(k\) and \(m\) are the transverse indices that correspond to mutually perpendicular coordinates. If \(v = 0\), the wavefront does not have a helical structure and the beam is not vortex. Let us compare this case with the case \(v = 1\), where there is a smooth azimuthal phase distribution, which is characteristic of an optical vortex. The initial amplitude-phase structure of the beam if parameter \(v\) in exponent indicator is equal to zero is shown on figure 1a, \(b\). In the numerical simulations the parameters were set as follows: \(D = 1.25\), \(\sigma = 3.3\), \(V=2,7\), \(N=5\), \(b = 2\) and
s = 0.03. The range was \( k \times m = 128 \times 128 \). Amplitude \( W \) and phase \( \Phi \) were determined from the relations: 
\[
W_{k,m} = |w_{k,m}|, \quad \Phi_{k,m} = \arg(w_{k,m}).
\]

![Figure 1](image1.png)

**Figure 1.** Distribution of the amplitude \( W_{k,m} \) (a) and phase \( \Phi_{k,m} \) (b) in the cross section of the fractal beam \( v = 0 \). (c) is palette for phase and intensity range.

The amplitude and phase distributions obey the same scaling law. The scaling factor \( \eta \) is equal to the parameter \( b = 2 \). It is clearly seen that the amplitude maxima form segments of circles, the radii of which differ by a factor of 2. In particular, the radius of the circle of segment 1 is 2 times less than the radius of circle of segment 2. The phase distribution has a different form (Figure 1b). The phase values are distributed according to a binary law, since the Weierstrass function is an alternating function. Circles with the same phase values obey the same scaling law. In the figure, the radius of circle 3 is 2 times less than radius of circle 4. Circular zones with identical phases are separated from each other by edge dislocations, where the phase abruptly changes by \( \pi \).

Figure 2a,b corresponds to \( v = 1 \). In this case, the amplitude distribution is axisymmetric (Figure 2a), and the phase distribution is characterized by smooth azimuthal changes by \( 2\pi \) (Figure 2b). In the center there is a helical dislocation of the wave front. This distribution of amplitude and phase gives the beam a vortex tubular appearance.

![Figure 2](image2.png)

**Figure 2.** Distribution of the amplitude \( W_{k,m} \) (a) and phase \( \Phi_{k,m} \) (b) in the cross section of the fractal beam \( v = 1 \). (c) is palette for phase and intensity range.
Calculations show that the tubular shape of the beam is retained in the far propagation zone. This form of the light field provides a high resistance of radiation to the influence of turbulent formations on atmospheric optical paths [2].

4. Properties of beams with a screw system dislocation

Another approach was used to obtain additional information on the properties of vortex fractal beams. The calculations were carried out based on the approach based on the addition of the light fields of a fractal system of Gaussian beams with a stepwise changing configuration [2,4]. A step-by-step calculation of the change in the shape of the distribution of the amplitude and phase of the light field was carried out by the iteration method using the following expression for the initial Gaussian beam:

$$g^{cl}(x, y) = \exp \left( -\frac{(x-c_x)^2 + (y-c_y)^2}{w^2} \right),$$  \hspace{1cm} (2)

where $x, y$ - transverse coordinates, $w$ is the width of the beam centered at the point $(c_x, c_y)$. Amplitude $W$ and phase $\Phi$ were determined from the relations:

Each of these beams decayed into $N$ daughter beams, thus

$$g^{<n+1>}(x, y) = \sum_{k=0}^{N-1} g^{<n>}(x + \frac{R}{\tau^n} \cos \left( \frac{2\pi k}{N} \right), y + \frac{R}{\tau^n} \sin \left( \frac{2\pi k}{N} \right)).$$ \hspace{1cm} (3)

Here $\frac{R}{\tau^n}$ characterizes the distance of the displacement of the daughter beams from the parent at the next iteration ($\tau$ is the scaling coefficient), $\cos \left( \frac{2\pi k}{N} \right)$ and $\sin \left( \frac{2\pi k}{N} \right)$ are the components of the displacement vector, the exponent gives the azimuthal phase incursion in the case fractal helical beams. In the calculations, the values $\tau = 2$ and $N = 3, 4, 5, 6$ were used. The $R$ value was considered a constant that could vary.

The propagation of the beams was simulated by the method of decomposing the field into plane waves. In the paraxial approximation, the phase incursion of the wave with indices $p, q$ at a distance $z$ was

$$\varphi_{p,q} = \frac{2\pi z}{T} \left( f(p)^2 + f(q)^2 \right).$$ \hspace{1cm} (4)

Here $T = 2a^2/\lambda$ is the Talbot distance, $a$ is the full size of the grid on which the field is specified, $\lambda$ is the wavelength, $f(p) = \text{mod} \left( p + K / 2, K \right) - K / 2$ is an auxiliary function that allows one to take into account the positive and negative harmonics defined from 0 up to $K - 1$. The amplitudes of plane waves at distance $z$, thus, had the form $S_{p,q} \exp \left( i\varphi_{p,q} \right)$. To obtain the final field distribution, an inverse FFT was used.

This algorithm made it possible to simulate fractal structures with several screw dislocations of the wavefront. As examples, two such structures with a scaling coefficient $\eta = 3$ are shown in Figures 3a, b and 4a, b.

Figures 3a and 4a show the images of the beams in the initial plane, and figures 3b and 4b - in the far propagation zone. Cross dots mark the position of dislocations with a topological charge of +1, and triangular ones - with a topological charge of -1. In the distant zones, the images are morphologically similar to the initial ones. They are characterized by one scaling factor $\eta = 3$. For clarity, in Figure 3b, numbers 1 - 4 and 5 - 8 mark the location of the amplitude maxima, which corresponds to the geometry of similar elements. The sizes of the marked fragments differ in accordance with the scaling factor by a factor of 3. The presence of dislocations leads to local “vortices” of the field. The dislocation present in the center of the image determines the general vortex type of the beam.
Despite the structural difference, the beam in Figure 4a is transformed in a similar way. However, in the process of propagation, the central dislocation, in contrast to the previous case, changes its sign (figure 4b). This feature is explained by the diffraction transformation of the amplitude-phase profile of the fractal beam during its propagation. Thus, the method of forming fractal waves in the form of a superposition of Gaussian beams can significantly expand the understanding of the properties of fractal vortex beams.
5. Conclusions
The performed analysis of studies of the propagation of radiation with a fractal and vortex structure shows that the problem of determining the properties of light beams, which simultaneously have a fractal and vortex form, is urgent. Such beams have a high information resistance to the influence of inhomogeneities of the propagation medium. The used method of transforming the two-dimensional Weierstrass fractal structure made it possible to determine the effect of the vortex shape of the radiation on its amplitude and phase characteristics. Characteristics indicate that the vortex shape of the beam provides a practically valuable tubular type of radiation while maintaining the scaling parameters. The use of an alternative method for calculating the structure of fractal vortex beams with a system of screw dislocations in the cross section showed that in this case the morphological features of the amplitude-phase distributions are preserved. The developed and applied design scheme, which has proven its effectiveness, can be used to calculate the characteristics of other types of waves. The significance of the results obtained is largely determined by the fact that they were obtained for the practically important case of beam propagation in a turbulent atmosphere. The discovered structural stability of fractal vortex beams can play a positive role when used in atmospheric communication lines. The number of dislocations at the wave front at different distances from the initial plane remains constant. In this case, the balance between the number of positive and negative dislocations doesn't change. Their location determines the general morphology and symmetry in the distribution of the field amplitude.

The obtained results will be useful in the development of optical information systems based on fractal vortex beams.

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