Motion of a rough disc in Newtonian aerodynamics

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Abstract. Dynamics of rough discs in a zero-temperature rarified medium (Newtonian aerodynamics) is considered. New mathematical models for such motions are constructed. We write down differential equations that approximately describe motions of rough discs i.e. bodies close to balls in Hausdorff metric. Estimates for solutions of those equations are provided. We study possible trajectories of centers of discs. It is proved that any rectifiable curve of finite length can be approximated, in the Hausdorff metric, by trajectories of centers of rough discs (that is, $C^0$-small perturbations of regular discs), provided that the parameters of the system are carefully chosen. To control the dynamics of the disc, we use the so-called inverse Magnus effect, which causes deviation of the trajectory of a spinning body. We use the shape of the perturbed disc as a parameter to control the Magnus effect. First of all, we study the so-called response laws for scattering billiards, corresponding to domains with openings on the plain. We construct a special family of such billiards whose response laws are dense in the set of symmetric Borel measures.

Keywords: billiards; shape optimization; Magnus effect; Newtonian aerodynamics; retroreflectors.

Mathematics subject classifications: 37D50, 49Q10, 70Q05, 93B05.

1 Introduction

Consider a body moving in a two-dimensional rarefied homogeneous medium. The body is a bounded connected domain with piecewise smooth boundary. It is assumed that the density of the medium is much smaller than one of the body. The particles composing this medium are initially at rest. The medium is rarified so those particles do not interact with each other. They collide with the body in the perfectly elastic fashion and move freely between consecutive reflections from the boundary. All reflections obey the billiard law; see Section 2 for details.

This simple aerodynamic model was first introduced by Newton in his Principia (1687) [1]. He studied a particular case of this model where a convex axially symmetric body translates in the medium along its axis of symmetry. Due to collisions of the body with particles that form the medium, the force of resistance acts on the body and slows down its motion. Newton studied the problem of finding the shape of the body, from the class of bodies with fixed length (along the direction of motion) and width, that minimizes the force of resistance. Newton’s solution looks like a truncated cone with slightly inflated lateral surface. Several generalizations of Newton’s problem related to (generally) nonconvex and/or non-symmetric bodies have been studied in 1990s and 2000s by various authors [2–14]. The general problem setting is as in the above problem: a body translates through a medium (or, equivalently, there is a parallel flow of particles incident on a body at rest), and the body’s length along the direction of motion and the maximum orthogonal cross section are fixed. It is required to find the shape of the body

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that minimizes the resistance. There are open problems in this area; for instance, the shape of the convex and (generally) non-symmetric body of least resistance is not fully understood.

These investigations are closely related to the so-called problem of invisibility. Roughly speaking, these are attempts to construct a system of mirrors, which is invisible for an observer (observers), placed in a fixed point (points) or looking from a fixed direction (directions). This problem can be considered in any dimension. Of course, the most interesting cases correspond to dimensions 2 and 3. Some of related problems, for example, invisibility from one point or invisibility from one direction have been already solved [15–17]. However, it is easy to see that the perfect invisibility is never possible.

Dynamics of a rarified gas is a well-studied problem see [18–21] and references therein for a review. There exist distinct models describing dynamics of a rarified gas. The key point where the majority of authors agree, is that the medium cannot be always treated as continuous any more. Many problems of dynamics of a rarified gas can be reduced to well-known Boltzmann equation. This model is good to describe dynamics of a hot gas where the particles do collide.

**Boltzmann equation.** Usually, the dynamics of the rarified gas is reduced to Boltzmann equation:

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{p}{m} + \frac{\partial f}{\partial p} F = \frac{\partial f}{\partial t} \bigg|_{\text{coll}}.
\]

Here \(f(x, p, t)\) is the density (proportion) of particles at the time instant \(t\), \(x\) is the coordinate, \(p\) is the impulse, \(F\) is the field of forces, acting on particles, \(m\) is a mass of a single particle,

\[\frac{\partial f}{\partial t} \bigg|_{\text{coll}}\]

is the so-called integral of collisions, which is zero if there is no threshold and particles do not interact between themselves. We use classical notions just for this case of Boltzmann equation and would not remember them for future needs. Boltzmann equation is precise but quite difficult to solve.

Forces, acting on a body moving in a rarified medium are estimated in [22–25]. We pay a special attention to the paper [25] where the so-called inverse Magnus effect has been described. This effect will pay an important role in methods, we are going to use in this paper.

Even more difficult and diverse are problems related to combined translational and rotational motion of bodies in a rarified medium. Some of these problems are addressed in [24–32] under the assumption that the rotational motion is much slower than the translational one. In this case interaction of each individual particle with the body occurs as if there were no rotation at all: the turn of the body during the time of interaction can be neglected. It is shown, in particular, that the resistance of a convex body, in Euclidean space of arbitrary dimension, can be both increased and decreased by roughening its surface. The rates of maximum increase and decrease are found to depend only on the dimension and not on the original convex body; in the 3D case they are equal, respectively, to 2 and (approx.) 0.969445. However, even having precise formulae for resistance forces, one cannot trivially derive dynamics of the moving body. The main difficulty is that, similarly to classical aerodynamical problems, one must take into account the related dynamics of the medium, not only velocities of the moving body.

In this article we study a simplified model of Newtonian dynamics, i.e. we assume that the temperature of the rarified gas is zero. However, the threshold (moving body) will be mobile, so we would need another model. The model we are going to present is really distinct from other ones which take into account Brownian motion and/or interactions between particles. Non-formally we deal with a ”naïve model” of gas dynamics, which is however applicable for rarified media of zero temperature.
Dynamics, in the framework of Newtonian model, of a body that performs both translational and rotational motion is a very intriguing and completely unexplored subject even in the 2D case. For such a motion, general results of existence and uniqueness have not yet been obtained before. In this paper, we are going to provide a mathematical model of motion of a body and a medium, for which existence of solution follows from standard methods of Variational problems. The question on uniqueness of solutions is still open. Even attempts to study dynamics of very simple shapes, like a rod, not to say about an ellipse or a triangle, meet serious difficulties. The only exception is a circle, whose dynamics is trivial: the path of its center is just a straight line. There is a way to avoid some of mentioned difficulties by concentrating on the dynamics of rough discs. Namely, fix $r > 0$, take a regular $n_0$-gon ($n_0 \geq 3$) inscribed in a circle with radius $r$ (let its center be $O$), and substitute each side of the $n_0$-gon with a curve joining its endpoints. Each curve is piecewise smooth, has no self-intersections, and is contained in the circular sector with vertices at $O$ and at the endpoints of the corresponding side. In addition, all the curves are congruent: each curve can be obtained from another one by rotation around $O$ by $2\pi k/n_0$. For each integer $n > n_0$ make a similar procedure: take a regular $n$-gon inscribed in the same circle and substitute its sides for curves, so that the obtained sequence of sets tends to the circle in Hausdorff metrics. The union of the curves in each $n$-gon bounds a domain $B_n$. Later on, we deal with a special types of domains $B_n$, namely such that their convex hulls are unions of $n$ congruent arcs of the circle and $n$ segments.

**Definition 1.** The sequence of domains $B_n$, $n \geq n_0$ is called a **rough disc**.

Thus, a rough disc is an idealized object. It is not a domain, but rather it can be informally viewed as the "limit" of a sequence of domains $B_n$. Its "boundary" is obtained by repetition of identical infinitesimal curves similar to the original one. They are interpreted as infinitesimal hollows on the disc boundary. The billiard scattering by the rough disc is uniquely defined by the shape of the curve.

The force of resistance of the medium acting on the disc and the moment of this force are defined as limits, as $n \to \infty$, of the force and the moment of force acting on $B_n$. Using these values, we derive the equations of motion of a rough disc on the plane. These equations, and therefore the trajectory of the disc, depend on the shape of the infinitesimal curve forming its boundary. The natural question arises: what curves can be traversed by the disc center? There are three principal goals of the article.

1. Basing on general principles such as Conservation of Energy, Impulse and Angular Impulse and a variational principle, similar in a sense to Principle of the Least Action, we derive equations of motions for bodies in a rough medium.

2. For rough bodies we compare the obtained model with one, studied by A. Plakhov [16]. We estimate possible difference between solutions of those two models.

3. We show that any rectified plane curve of finite length can be approximated, in the Hausdorff sense, by trajectories of centers of rough discs, provided parameters of the system are carefully chosen.

Constructions of first two parts are quite standard: we construct a "Lagrangian", select an appropriate functional space and prove that there exists an element of that space, minimizing the "action". The "Lagrangian", we consider, is just the overall kinetic energy of the medium. However, unlike the case of classical Hamiltonian dynamics, the considered system does not include any potential energy, and we cannot derive Conservation Laws from the variational principle. So, we have to make additional assumptions that these laws take place.

Using variational principle we can write down differential equations for which the minimizing function is a generalized solution. These differential equations are very special. However, for some special rough
discs we may construct their approximate solutions, for which the corresponding system of o.d.es is much simpler.

The proof of the last statement is based on the following idea. In the typical case, if a disc rapidly rotates, say, counterclockwise, then the velocity vector of its center of mass will change in the clockwise direction. This phenomenon is called the inverse Magnus effect, see Figure 1. The word "inverse" means this effect is inverse to the effect proper for classical gas dynamics and well-known for soccer or ping-pong players where a ball deviates at the direction of rotation. There is no contradiction: influence of a classic gas is very different from one of rarified media.

![Figure 1. Inverse Magnus effect.](image)

The magnitude of the effect depends on the disc roughness (that is, on the shape of cavities on its boundary) as well as on the relative angular velocity of the disc. The proof of Theorem 3 given below is based on this effect. We construct a very special cavity in such a way that (i) the relative angular velocity (let it be \( \lambda \)) monotonously increases and (ii) the magnitude of the effect is nearly zero for all values of \( \lambda \) except for several (relatively small) intervals of values. On these intervals the effect is adjusted so as to ensure right turn of the velocity vector to a certain angle. So the principal idea of the proof of Theorem 3 is to use shapes of cavities to control inverse Magnus effect.

We believe that our construction can be generalized to three dimensions, but postpone the 3D study to the future.

There are two optimization problems, we solve in this paper. The first one corresponds to optimal approximation of the dynamics of the rough body by ordinary differential equations; observe that the mentioned dynamics itself cannot be generally described by ODEs. The second one is selection of the best shape to control the Magnus effect and, therefore, Newtonian dynamics of rotating body.

**Structure of the paper.** The paper consists of 14 sections. In Section 2, we give a model of interaction between a particle of non-zero mass and the disc. In Section 3, we consider an immobile scattering billiard which gives a simplified model for dynamics of a particle inside a hollow. In the next section we formulate physical assumptions on the moving body and medium and introduce some notions. In Section 5 we introduce a general model for motions of two-dimensional bodies in rarified media (up to that section dimension of the phase space of the system is not principle for us). Starting from Section 5, we proceed to the model of (initially) continuous media where the mass is initially uniformly distributed. Section 6 is technical. There, we introduce the concept of \( \delta \) – pseudotrajectory, corresponding to immobile billiard system. We show that, for sufficiently precise approximations to so-called perfect rough discs, scattering billiard model gives a good approximation for relative motions of particles inside cavities. Then we can apply the model for motions of rough bodies, introduced in [16]. This is done in Section 7. We study some special types of cavities and related reflection laws (an example of the so-called perfect billiards, Sections 8–10). In some special cases we can estimate the error, given by a simplified model (Section 11). In that
section we compare precise motions of a rough disc with approximate ones, introduced in Section 7. We estimate errors for solutions of the simplified model with respect to solutions to precise one. Then we are prepared to formulate the main result of the paper (Section 12) and prove it (Section 13). We main idea of the proof is approximation of a curve with broken lines, for which we can write down equations of motions and shapes of cavities explicitly. Finally Conclusion and Discussion are provided in Section 14.

2 Interaction with a single particle of a non-zero mass

We start with considering an impact interaction between a single point particle of the mass $m > 0$ and the disc of the mass $M$ and momentum of inertia $I$. Let $v_-$ and $v_+$ be velocities of the particle before and after impact, $V_-$ and $V_+$ be corresponding velocities of the center of the disc, $\omega_-$ and $\omega_+$ be angular velocities of the disc before and after impact. Let $n$ be the dimensionless outer unit normal vector at the point of interaction (suppose it is correctly defined), $r$ be the unit vector of the direction from the center of the disc to that point; $\Theta$ be the angle from $n$ to $r$ calculated in the direction of rotation of the disc. Let $r$ be the distance between the center and the point of interaction. Write down the conservation laws for energy, impulse and angular momentum taking into account that the velocity changes in the normal direction to the point of impact:

$$mv_-^2 + I\omega_-^2 + MV_-^2 = mv_+^2 + I\omega_+^2 + MV_+^2;$$

$$MV_- + mw_- = MV_+ + mw_+;$$

$$\Delta v = v_+ - v_- \parallel n;$$

$$I\omega_+ + m(r v_+ - V_+ r^\perp) = I\omega_+ + m(r v_+ - V_+ r^\perp).$$

Here and in what follows $\langle \cdot, \cdot \rangle$ indicates the scalar product; $r^\perp$ is the unit vector obtained by rotating $r$ by the angle $\pi/2$ in the direction of rotation of the disc. Introduce the pre-impact and post-impact relative velocities of the particle by the formulae

$$v_{e\pm} = v_\pm - V_\pm - \frac{M + m}{M} \omega_\pm r \sin \Theta n.$$  

Then we obtain from (1) after some algebra that

$$\Delta v = -\frac{2\langle v_- n, n \rangle n M^2 I}{M^2 I + mMI + m r^2 (M + m)^2 \sin^2 \Theta};$$

$$\Delta V = V_+ - V_- = -\frac{m}{M} \Delta v;$$

$$\Delta \omega = \omega_+ - \omega_- = -\frac{m(M + m) r \sin \Theta}{MI} |\Delta v| = -\frac{m(M + m) r}{MI} (\Delta v, r^\perp)$$

and

$$v_{e+} = v_{e-} - 2\langle v_{e-}, n \rangle n.$$  

Later on, we always assume that the mass of a single particle is much smaller than one of the body so, for example we replace $(M + m)/M$ with 1. Rewrite (4) in order to have explicit formula for a post-impact velocity. We obtain

$$v_+ = v_- - \langle 2\langle v_- - V_-, n \rangle - \omega_- r \sin \Theta \rangle n =$$

$$v_- - 2 \langle v_- - V_- - \omega_- r^\perp, n \rangle n$$
and, consequently,
\[
v^2_+ = v^2_- - 4 \langle v_- - V_- - \omega_- r^\perp, n \rangle \langle V_+ - \omega_+ r^\perp, n \rangle.
\] (5)

If energy and impulses of the system body-particle are not conserved during impact interaction, namely if the following equalities take place:
\[
\begin{align*}
 mv^2_- + I\omega^2_- + MV_-^2 + \delta E &= mv^2_+ + I\omega^2_+ + MV_+^2; \\
 MV_- + mv_- + \delta P_V &= MV_+ + mv_+;
\end{align*}
\]
\[
\Delta v = v_+ - v_- \parallel n;
\]
\[
I\omega_- + mr\langle v_- - V_-, r^\perp \rangle + \delta P_\omega = I\omega_+ + mr\langle v_+ - V_+, r^\perp \rangle,
\]
we have
\[
v_{e+} - v_{e-} - 2\langle v_{e-}, n \rangle n = O(|\delta E/|MV_-| + |\delta P_V/M| + |\delta P_\omega r/I|). \quad (6)
\]
Here \(\delta E, \delta P_V\) and \(\delta P_\omega\) are eventually non-zero variations of energy, momentum and momentum of inertia during the impact.

This can be deduced similarly to Eq. (4) or via Implicit Function Theorem.

Now we study another problem. A particle, initially immobile, hits the disc several times. All these impact interactions take place in a very short interval of time and within a single cavity of the disc. We neglect that duration and linear sizes of the disc cavities. In this section, we treat cavities as black boxes. We know input and output data and do not take care on what happens inside. We would like to know the overall influence of the particle on the dynamics of the disc.

Consider a disc, initially moving with the velocity \(V_- = (V, 0)\) and rotating counterclockwise with the angular velocity \(\omega\). We assume that an immobile particle hits the disc at a point \(p\) whose current radius vector forms the angle \(\alpha \in [-\pi/2, \pi/2]\) with the velocity of the center of the disc. We assume that the angle of reflection or, equivalently, direction of the post-impact velocity of the particle with respect to \(V\) is \(\beta\). Let \(v = (\cos \beta, \sin \beta)v = (v_x, v_y)\) be the post-impact velocity of the particle. Introduce the notion: \(V_+ = (V_{x+}, V_{y+}), |V_+| = V_+, \) and \(\omega_+\) for the post-impact angular velocity:
\[
\Delta V = (\Delta V_x, \Delta V_y) = V_+ - V; \quad \Delta \omega = \omega_+ - \omega.
\]
Notice that \(\Delta V_x/\Delta V_\perp = \tan \beta\).

Rewrite the conservation laws (1) for this case:
\[
\begin{align*}
 I\omega^2 + MV^2 &= mv^2 + I\omega^2 + MV^2; \\
 MV &= MV_{x+} + mv_x; \\
 0 &= MV_{y+} + mv_y; \\
 I\omega + mrV \sin \alpha &= I\omega_+ + mr(V_{x+} - v_x) \sin \alpha - mr(V_{y+} - v_y) \cos \alpha.
\end{align*}
\] (7)

Also, we know that \(\Delta V_x = -\Delta V \cos \beta, \Delta V_\perp = -\Delta V \sin \beta\) if \(\Delta V = |\Delta V|\). Equations (7) imply that
\[
I\Delta \omega = -(M + m)r \Delta V_x \sin \alpha + (M + m)r \Delta V_y \cos \alpha.
\]

Substituting this to the first equation of (7) and neglecting small terms we obtain
\[
\Delta V = \frac{2m}{M}(V \cos \beta + r \omega \sin(\beta - \alpha)) + o \left(\frac{m}{M}\right),
\]

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which means that

\[
\Delta V_x = -\frac{2m \cos \beta}{M} (V \cos \beta + r \omega \sin(\beta - \alpha)) + o\left(\frac{m}{M}\right);
\]

\[
\Delta V_y = -\frac{2m \sin \beta}{M} (V \cos \beta + r \omega \sin(\beta - \alpha)) + o\left(\frac{m}{M}\right);
\]

\[
\Delta \omega = -\frac{2m \sin(\beta - \alpha)}{\kappa Mr} (V \cos \beta + r \omega \sin(\beta - \alpha)) + o\left(\frac{m}{M}\right).
\]

Starting from here, we proceed to a continuous medium, which is initially uniformly distributed out of the initial position of the body. We use the obtained formulae (1)–(7) to have a description for interaction between an infinitesimal element of the medium (still called particle) and the body.

### 3 Law of scattering for immobile billiards

Description of scattering by a rough disc and derivation of equations of motion can be found in [32], as well as in chapters 4 and 7 of the book [16]. For the reader’s convenience, we partly reproduce them here.

**Definition 2.** A hollow is a piecewise smooth non self-intersecting curve with finite length contained in a closed isosceles triangle whose base is the segment joining the endpoints of the curve. The segment is called the opening of the hollow.

We use the notion $\Omega$ for a hollow and $I$ for its opening. Introduce the uniform coordinate $\xi \in [0, 1]$ on the opening $I$; the values $\xi = 0$ and $\xi = 1$ correspond to its endpoints. Recall that $n$ is the unit outer normal vector to $I$. Consider a billiard particle that comes to a hollow $\Omega$ through its opening $I$. Fix the point $\xi$ where it intersects the opening and the angle $\varphi \in (-\pi/2, \pi/2)$ formed by $-n$ and the velocity of incidence $v$. If the particle makes a finite number of reflections from regular points of $\Omega$ and then intersects $I$ again and goes away, we denote by $\xi^+ = \xi^+_{\Omega}(\varphi, \xi)$ the point of the second intersection and by $\varphi^+ = \varphi^+_{\Omega}(\varphi, \xi)$ the angle formed by $n$ and the velocity $v^+$. Again, the angle is positive, if it is counted counterclockwise from $n$ to $v^+$, and negative otherwise.

Almost all particles leave the hollow after a finite number of reflections. This follows from the measure-preserving property of billiard and from Poincarié’s recurrence theorem. Thus for almost all initial data $(\varphi, \xi) \in [-\pi/2, \pi/2] \times [0, 1]$ the values $\varphi^+_{\Omega}(\varphi, \xi)$ and $\xi^+_{\Omega}(\varphi, \xi)$ are well-defined. Introduce the probability measure $\mu$ on $[-\pi/2, \pi/2] \times [0, 1]$ according to the equality $d\mu(\varphi, \xi) = \frac{1}{2} \cos \varphi \, d\varphi \, d\xi$. The map

\[
T_\Omega : (\varphi, \xi) \mapsto (\varphi^+_{\Omega}(\varphi, \xi), \xi^+_{\Omega}(\varphi, \xi))
\]

is defined on a full-measure subset of $[-\pi/2, \pi/2] \times [0, 1]$ and maps it bijectively onto itself. Moreover, it conserves the measure $\mu$ and is involutive, $T_\Omega^{-1} = T_\Omega$.

Next introduce the Borel measure $\eta_\Omega$ on the square $\square := [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ as follows: for any Borel set $A \subset \square$, $\eta_\Omega(A) = \mu\left(\{(\varphi, \xi) : (\varphi, \varphi^+_{\Omega}(\varphi, \xi)) \in A\}\right)$. This measure can be defined in a different way: let $\sigma_\Omega$ be the mapping $(\varphi, \xi) \mapsto (\varphi, \varphi^+_{\Omega}(\varphi, \xi))$ from $[-\pi/2, \pi/2] \times [0, 1]$ to $\square$; then $\eta_\Omega$ is the push-forward measure $\eta_\Omega = \sigma_\Omega \mu$.

**Definition 3.** $\eta_\Omega$ is called the measure induced by the hollow $\Omega$.

**Remark 1.** The measure $\eta_\Omega$ describes the distribution of the pair $(\varphi, \varphi^+)$, with $\varphi$ being the angle of entrance and $\varphi^+$ the angle of exit for a randomly chosen particle incident in $\Omega$.

Define the probability measure $\gamma$ on $[-\pi/2, \pi/2]$ by $d\gamma(\varphi) = \frac{1}{2} \cos \varphi \, d\varphi$. For a set $A \subset \square$, denote by $A^*$ the set symmetric to $A$ with respect to the straight line $\varphi = \varphi^+$, $A^* = \{(\varphi, \varphi^+) : (\varphi^+, \varphi) \in A\}$.

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Denote by $\Upsilon$ the set of Borel measures $\eta$ on $\square$ such that for all Borel sets $A \subset \square$ and $I \subset [-\pi/2, \pi/2]$ one has $\eta(A) = \eta(A^\ast)$ and $\eta(I \times [-\pi/2, \pi/2]) = \gamma(I) = \eta([[-\pi/2, \pi/2] \times I])$.

In other words, all measures from $\Upsilon$ are symmetric with respect to the diagonal $\varphi = \varphi^+$, and both its marginal measures coincide with $\gamma$. One always has $\eta_0 \in \Upsilon$; this can be easily deduced from the measure preserving and involutive properties of the map $T_\Omega$; see [16] or [18] for details. The following important theorem states that, inversely, the set of measures induced by hollows is weakly dense in $\Upsilon$.

**Theorem 1.** The set $\{\eta_0 : \Omega \text{ is a hollow} \}$ is weakly dense in $\Upsilon$. In other words, for any $\eta \in \Upsilon$ there exists a sequence of hollows $\Omega_k$ such that

$$
\lim_{k \to \infty} \int\int_\square f(\varphi, \varphi^+) \, d\eta_k(\varphi, \varphi^+) = \int\int_\square f(\varphi, \varphi^+) \, d\eta(\varphi, \varphi^+)
$$

for any continuous function $f : \square \to \mathbb{R}$.

The proof of this theorem can be found in [16] or in [19].

## 4 Assumptions on the medium and the body and some notions

We assume that we have a medium where the mass is distributed according to locally finite measure $m$, assume that it is weakly close on compact sets to a uniform distribution $\rho \, dS$, where $S$ is the Lebesgue measure in $\mathbb{R}^2$. We treat particles as infinitesimal parts of the medium. Formally speaking, we identify particles with continuous functions $x(t, x_0)$, supposing that $x(0, x_0) = x_0$ if $x_0 \notin B$ where $B$ is the initial position of the moving body. We say that this particle hits the moving body at the instant $t_0$ if $x(t_0, x_0) \in \partial B(t_0)$ where $B(t_0)$ is the current position of the moving body.

Always, we neglect Brownian motion of particles (the zero-temperature assumption) and interaction between particles (the rarefied medium assumption).

We apply this model to bodies with piecewise smooth boundaries which are $h$-perturbations of a ball in the Hausdorff metric (a ball with small cavities).

We suppose that the body rotates and moves in the considered medium. We use the following notion. Let $X = (X, Y)$ be the current position of the center of the body, $\phi$ be the current angle of rotation of the body with respect to its initial position. Let $V$ be the vector velocity of the center of mass of the disc, $|V| = V$. Denote by $\omega$ the angular velocity, by $r$ be the radius of the disc and by $I = \kappa Mr^2$ the moment of inertia. For an ordinary disc with uniformly distributed mass we have $I = Mr^2/3$, and in any case $\kappa \in [0, 1]$. We introduce the angular coordinate $\xi$ on the boundary of the ordinary disc representing the smooth approximation of the moving body, identifying this boundary with the unit circle $S^1 = [-\pi, \pi]/\{\pi = -\pi\}$. Recall the notion for the dimensionless relative angular velocity $\lambda = \omega r/V$.

In next sections, we are going to provide model for motion of a rough disc which is introduced in the following sense.

We take a sequence of bodies approximating the rough disc and proceed to the limit in approximating bodies $B_n$, $n \to \infty$.

We say that $(X(t), \varphi(t))$ is a motion of the rough disc corresponding to initial conditions $X(t_0) = X_0$, $X(t_0) = V_0$, $\varphi(t) = \varphi_0$, $\varphi(t) = \varphi_0$, $\varphi(t) = \omega_0$ if $X(t)$ is a limit of trajectories of centers of masses for approximating bodies and $\varphi(t)$ is the limit of rotational angles for these bodies, corresponding to the same initial conditions (provided that these limits are correctly defined).
5 Mathematical models for motion of a body in a rarified medium

In this section we use a variational principle, similar to Hamilton’s principle of least action to provide mathematical models for Newtonian aerodynamics which will be valid to describe any motion of the considered system. Also, we study some simple properties for the constructed model, for instance, we prove existence for their solutions.

Consider kinetic energy of the moving body:

\[ -U = \frac{MV^2}{2} + \frac{I\omega^2}{2}, \]

and the kinetic energy be one of particles that form the medium i.e.

\[ K = \frac{1}{2} \int_{\mathbb{R}^2 \setminus B} v^2(t, x_0) \, dm(x_0). \]

Here \( B \) is the initial position of the moving body which is a simply connected domain, \( x_0 \) is the initial position of the point of the medium; \( v(t, x_0) \) is the velocity of this point for given value of \( t \). The value \( K \) is always finite.

Let \( m \) be the initial distribution of mass in the medium. We can write down following conservation laws

\[ E := K - U = \text{const}; \]
\[ P := MV + \int_{\mathbb{R}^2 \setminus B} v(t, x_0) \, dm(x_0) = \text{const}; \]
\[ L := I\omega + \int_{\mathbb{R}^2 \setminus B} x(t, x_0) \times v(t, x_0) \, dm(x_0) = \text{const}. \]

Let us clarify the last equality. We treat angular impulse \( L \) as a 3-dimensional vector, orthogonal to the plane where everything moves. Here we use a canonical embedding of \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) and consider the vector product in the sense of this embedding.

It is important to say that none of the particles can ever be inside the current position of the body.

In order to formalize all this information, we consider the following model: take \( B(t) = X(t) + A(t)B \) – the current position of the body (a simply connected domain), where \( X(t) \) is the position of the center of mass of the body,

\[ A(t) = \begin{pmatrix} \cos \varphi(t) & \sin \varphi(t) \\ -\sin \varphi(t) & \cos \varphi(t) \end{pmatrix} \]

Let \( S = \partial B \), consider two copies \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) of \( \mathbb{R}^2 \setminus B \) and identify their boundaries \( S \). We obtain a piecewise smooth cylinder \( \mathcal{M} \) that can be made smooth if \( S \) is smooth (Figure 2). There is a natural projection \( \pi : \mathcal{M} \to \mathbb{R}^2 \setminus B \). Tangent spaces to all points of \( \mathcal{M} \) except ones of \( S \) can be canonically identified with \( \mathbb{R}^2 \).

Let us define how can we add a vector \( v \in \mathbb{R}^2 \) to a point \( x \in \mathcal{M} \). Suppose, without loss of generality that \( x \in \mathcal{M}_+ \) that can be identified with \( \mathbb{R}^2 \setminus B \). If the vector \( v \) is such that

\[ x + \alpha v \notin B \]

for all \( \alpha \in [0, 1] \), we just set \( x + v \) to be the corresponding point of \( \mathcal{M}_+ \). If Eq. (9) is violated for some values of \( \alpha \), we take the infimum \( \alpha_1 \) for such values \( \alpha \). Set \( x_1 = x + \alpha_1 v \in S \) and define

\[ v_1 = (1 - \alpha_1)v + 2((1 - \alpha_1)v \cdot n)n. \]
Figure 2. Manifold $\mathcal{M}$.

Here $\mathbf{n}$ is the outer unit normal vector to $S$ (if the normal vector is not defined, the sum is neither). We repeat a similar procedure taking $\mathcal{M}_-$ instead of $\mathcal{M}_+$ and $x_1 + v_1$ instead of $x + v$. We say that the sum $x + v$ is correctly defined if such procedure takes a finite number of steps (this is always true if the curve $S$ is piecewise analytical). We use this construction later when we study variations of functions, acting to $\mathcal{M}$.

Now we are ready to define the solution (motion of the considered mechanical system) on the segment as a set of functions:

$$\Xi = (X(t), V(t), \varphi(t), \omega(t), \bar{x}(t, x_0), \bar{v}(t, x_0))$$

where $X \in H^1([0, T] \rightarrow \mathbb{R}^2)$, $\varphi \in H^1([0, T] \rightarrow S^1)$, $V \in L^2([0, T] \rightarrow \mathbb{R}^2)$, $\omega \in L^2([0, T] \rightarrow S^1)$, $\bar{v} \in L^2([0, T] \times \mathbb{R} \rightarrow \mathbb{R}^2)$, $\bar{x} - \text{id} \in H^1 \cap L^2_0([0, T] \times \mathcal{M} \rightarrow \mathcal{M})$; $H^1$ is the corresponding Sobolev space (we consider $\mathbb{R}^2$ endowed with measure $m$), $S^1$ is the unit circle in $\mathbb{R}^2$. Let $\Xi \in Y$ where $Y$ is the product of all mentioned Banach spaces that is again Banach space. Notice that we consider the following norm in $L^2_0([0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2)$:

$$\|v\|^2 = \int_0^T \int_{\mathbb{R}^2} \langle v, v \rangle \, dm(x_0) \, dt.$$ 

Movements of particles of the medium are related with functions $\bar{x}$ and $\bar{v}$ (so-called relative positions and velocities) by formulae:

$$x(t, x_0) = A(t) \pi \bar{x}(t, x_0) + X(t); \quad v(t, x_0) = A(t) \bar{v}(t, x_0) + A'(t) \omega(t) \pi \bar{x}(t, x_0) + V(t).$$

Here

$$A'(t) = \begin{pmatrix} -\sin \varphi(t) & \cos \varphi(t) \\ -\cos \varphi(t) & -\sin \varphi(t) \end{pmatrix}.$$ 

Notice that $v$ belongs to the same Banach space as $\bar{v}$. 
We are going to find motions of the considered system, using the following variational principle:

$$\mathcal{L} \Xi := \int_0^T K(t) \, dt \rightarrow \min.$$  \hspace{1cm} (10)

Corollaries and remarks to Lemma 1 (see the end of this section) demonstrate why it is natural to use such a variational principle. We accept conservation laws Eq. (8) and variational principle (10) as axioms from which we deduce dynamics of the system. First of all we consider solutions for boundary value problems, corresponding to the problem, we are studying.

Fix \( t_1 < t_2 \) and consider a set of boundary value conditions

$$\begin{align*}
X(t_1) &= X_1, \quad \bar{x}(t_1) = \bar{x}_1, \quad \varphi(t_1) = \varphi_1; \\
X(t_2) &= X_2, \quad \bar{x}(t_2) = \bar{x}_2, \quad \varphi(t_2) = \varphi_2.
\end{align*}$$  \hspace{1cm} (11)

For such motions we fix levels of the energy \( E \) impulse \( P \) and angular impulse \( L \) in Eq. (8) and take a set \( Z \) of all motions \( \Xi \in Y \), satisfying (8) and (11).

**Lemma 1.** If the set \( Z \) is non-empty, there exists a motion \( \Xi_* \in Z \) which gives a solution of (10) in \( Z \).

**Proof.** Clearly, \( \mathcal{L} \) satisfies the following property: if \( \Xi_k \) weakly converges to \( \Xi_* \) in \( Y \) then

$$\mathcal{L} \Xi_* \leq \lim \inf \mathcal{L} \Xi_k.$$  \hspace{1cm} (11)

This is true because the same is satisfied for the square of the norm in \( L^2 \). Functional \( \mathcal{L} \) is non-negative, so it has an infimum on the set of all eventual trajectories \( \Xi \), corresponding to fixed initial conditions. Consider the sequence \( \Xi_k \), approximating that infimum. The set of all \( \Xi \) for which \( \mathcal{K}(t) \leq E \), is bounded and closed in \( Y \) and, due to Banach–Alaoglu theorem, is compact in the *-weak topology. So, we may assume that the sequence \( \Xi_k \) weakly converges in \( Y \). Let us demonstrate that \( \Xi_* \in Z \). Indeed, for any \( t \in (t_1, t_2) \) we have

$$\int_{t_1}^{t} (K_k(s) - U_k(s)) \, ds = E(t - t_1)$$

for all \( k \) where \( K_k \) and \( U_k \) are energies, corresponding to motions \( \Xi_k \). So, taking a *-weak limit, we obtain similar equalities for \( \Xi_* \) which demonstrate that the energy conserves. Similarly, we prove conservation of impulse and angular impulse as we proceed to *-weak limit.

Also, *-weak convergence of velocities \( \mathbf{V}_k \), \( \bar{v}_k \) and \( \omega_k \) implies uniform convergence of \( \mathbf{X}_k \), \( \bar{x}_k \) and \( \varphi_k \), so boundary values are conserved for \( \Xi_* \).

So, \( \Xi_* \) indeed realizes the minimum of \( \mathcal{L} \). \( \square \)

Now we are studying initial value problem for considered motion. We fix initial value of time, for instance \( t = 0 \) and initial conditions \( \Xi(0) \). This value uniquely defines the initial value of energy \( E \). Since velocities for considered motions belong to \( L^2 \), it is not mathematically correct to fix a value of such functions at \( t = 0 \). So, we use the following idea to define solutions for Cauchy problems.

**Definition 4.** Let \( \mathbf{X}_0 \), \( \mathbf{V}_0 \), \( \varphi_0 \) and \( \omega_0 \) be initial conditions for the moving body; assume that the medium is initially uniformly distributed and immobile. Let \( K(0) = E_0 \).

We take a sequence of smooth functions \( \Xi_k(t) \), corresponding to the fixed level of energy \( E_0 \) satisfying considered initial conditions and approximating infimum in (8). Then a *-weak limit \( \Xi_* \) of such system will be called solution for the considered initial value problem, even though the initial conditions corresponding to velocities may fail for \( \Xi_* \). Observe that for all \( t_2 > t_1 > 0 \) the reduction of the constructed motion \( \Xi_* \) to the segment \( [t_1, t_2] \) coincides with the solution of boundary value problem, corresponding to \( (\mathbf{X}(t_1), \bar{x}(t_1), \varphi(t_1)) \) and \( (\mathbf{X}(t_2), \bar{x}(t_2), \varphi(t_2)) \).
Remark 2. In general, we have nothing to say about uniqueness of solutions of the considered system. However, if we are able to write down equations of motion for the considered system. As usually, we use variational methods, i.e. we have \( \delta L(\Xi, \omega) = 0 \) where \( \delta L \) is variation of the functional \( L \), which does not hurt the energy preservation. Let \( \xi = (X, \varphi, x) \). Respectively, \( \tilde{\xi} = (V, \omega, v) \); \( \Xi = (\xi, \tilde{\xi}) \). We get following Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} - \frac{\partial L}{\partial \xi} = \frac{\partial L}{\partial \dot{\tilde{\xi}}} - \frac{\partial L}{\partial \tilde{\xi}}.
\]

This is not ordinary differential equations, since their left and right hand sides are distributions and, moreover, the system is infinite-dimensional; derivatives are treated in Frechet sense. We can rewrite obtained equation as follows:

\[
m\ddot{X} = \partial K/\partial X; \\
I\ddot{\varphi} = \partial K/\partial \varphi; \\
\ddot{x} = \partial K/\partial x. \tag{12}
\]

Right hand sides of Eq. (11) are, generally speaking, distributions, not regular functions. Actually, Eq. (12) imply that

\[
M \int_0^T (\dot{X}(t), \dot{H}(t)) \, dt = - \int_0^T \langle \partial K/\partial X(t), H(t) \rangle \, dt, \quad H \in C^\infty(\mathbb{R} \to \mathbb{R}^2);
\]

\[
I \int_0^T \dot{\varphi}(t, \eta(t)) \, dt = - \int_0^T \partial K/\partial \varphi(t), \eta(t) \rangle \, dt, \quad \eta \in C^\infty(\mathbb{R} \to S^1);
\]

\[
\int_0^T \int_{\mathbb{R}^2 \setminus \partial B} (\dot{x}(t, x_0), h(t, x_0)) \, dm(x_0) \, dt = \\
- \int_0^T \int_{\mathbb{R}^2 \setminus \partial B} (\partial K/\partial x(t, x_0), h(t, x_0)) \, dm(x_0) \, dt, \quad h \in C^\infty(\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2).
\]

We say that set \( \Xi \) defines a motion of a body in a rarified medium if it is a generalized solution of Equations (12). Let us write down right hand sides of Eq. (12) explicitly. We start with the distribution \( \partial K/\partial x \).

For any particle, the set of instants of time, when the particle hits the body and Eq. (4) is not satisfied, is, at most, countable (in that case either the pre-impact or the post-impact velocity is non-zero). Therefore, for almost all particles (with respect to measure \( m \)), there is a countable set of time values \( \Theta(x_0) \subset [0, T] \) for which \( x(t, x_0) \in \partial B(t) \) and either pre-impact or post-impact velocity is non-zero.

Given a perturbation \( h \in C^\infty([0, T] \times \mathbb{R}^2 \to \mathbb{R}^2) \), we replace \( x(t, x_0) \) with \( x(t, x_0) + h(t, x_0) \) in the manifold \( M \). So, some points \( x(t, x_0) \) which lay near the boundary \( S \) move through it as we change the function \( x \). The function \( v(t, x_0) \) is not continuous at points of \( S \), it changes according to impact law \( v_- \rightarrow v_+ \). So, the distribution \( \partial K/\partial x \) satisfies the following equality:

\[
\int_0^T \frac{\partial K}{\partial x} h \, dt = \int_{\mathbb{R}^2 \setminus \partial B} \sum_{t \in \Theta(x_0)} \langle n(x(t, x_0)), h(t, x_0) \rangle (v_+(t, x_0) - v_-(t, x_0)) \, dm(x_0).
\]

Similarly, we can write down formulae for \( \partial K/\partial X \) and \( \partial K/\partial \varphi \). Namely,

\[
\int_0^T \frac{\partial K}{\partial X} H \, dt = \int_{\mathbb{R}^2 \setminus \partial B} \sum_{t \in \Theta(x_0)} \langle n(x(t, x_0)), H(t) \rangle (v_+(t, x_0) - v_-(t, x_0)) \, dm(x_0);
\]

\[
\int_0^T \frac{\partial K}{\partial \varphi} \eta \, dt = \int_{\mathbb{R}^2 \setminus \partial B} \sum_{t \in \Theta(x_0)} \langle n(x(t, x_0)), \eta(t) r^\perp(x(t, x_0)) \rangle (v_+(t, x_0) - v_-(t, x_0)) \, dm(x_0). \tag{13}
\]

Finally, we introduce a notion for infinitesimal impulse acting on a point of a boundary of a moving body. Let \( \xi \in S^1 \) be an immobile coordinate on the boundary of the moving body. This boundary is
rectifiable, so we can select the parametrization so that the measure engendered by Lebesgue measure on $S^1$ and the length on $\partial B$ are proportional. Introduce the distribution of impulses by formula

$$\int_0^T \int_{S^1} \Phi(t, \xi) \mathrm{d}P(t, \xi) = \int_{\mathbb{R}^2 \setminus B} \sum_{t \in \Theta(x_0)} \Phi(t, \xi(x(t, x_0)))(v_+(t, x_0) - v_-(t, x_0)) \mathrm{d}m(x_0).$$

for any continuous function $\Phi$. Here $\xi(x(t, x_0))$ is the $\xi$ coordinate of the point $x(t, x_0)$, correctly defined if this point belongs to the boundary of $B(t)$. Later on, in Section 11, we use the notion

$$dP(t, \xi) = p(t, \xi) \mathrm{d}t \mathrm{d}\xi$$

where $p(t, \xi)$ is an element of the space $Z$, dual to $C^0([0, T] \times S^1 \rightarrow \mathbb{R}^2)$.

To finish this section, we demonstrate that for some trivial cases, our approach gives expected dynamics. We start with two simple corollaries of Lemma 1. In both of these cases, we do not assume that the medium is initially immobile.

**Corollary 1.** For almost all particles with respect to initial distribution of $x_0$ the following is true: if a particle does not have impacts on an interval $(t_1, t_2)$, its motion is constant at that interval.

**Proof.** Here we may assume, without loss of generality that all particles move in the free flight regime i.e. there is no moving body. In this case constant motion for all particles easily follows from Eq. (12) and Eq. (13). □

**Corollary 2.** For the limit case of the considered system, when the moving body is replaced with an immobile threshold, whose mass is infinite, interaction is described by billiard laws.

**Proof.** Lemma 1 implies that for any particle, its motion between neighbor impacts is constant. Impact laws are given by the limit form of Eq. (4) where we just replace relative velocities with absolute ones. □

**Remark 3.** Also, it is easy to see that for some trivial cases of dynamics, for instance when the moving body is a ball or if it is a non-rotating rod, Eq. (12) coincides with previously described models. The corresponding proofs are quite trivial and not principal in the rest of the paper, we leave them to readers.

We return to this model of motion in Section 11, where it will be compared with another model, which is much simpler, but, generally speaking, not precise. Now we start constructing this simplified model and discussing when it is applicable.

6 Pseudotrajectories

Given a $\delta > 0$, we introduce the notion of $\delta$-pseudotrajectory for a billiard. Later on, we will use this notion to describe properties of trajectories of billiards corresponding to moving domains.

**Definition 5.** We say that a piecewise $C^1$ smooth curve $x(t) : t \in [t_0, \hat{t}_0]$ is a $\delta$-pseudotrajectory for the exterior billiard corresponding to the immobile body $A$ if the following statements are true.

1. $x(t) \notin \text{int} A$ for all $t \in [t_0, \hat{t}_0]$.
2. The set of $t$ such that $x(t) \in \partial A$ is finite. Let it be $\{t_1, \ldots, t_N\} : N \in \mathbb{N} \cup \{0\}$. We also use the notion $t_{N+1} = \hat{t}_0$.
3. For all $k \in \{1, \ldots, N\}$ the velocities $v_{r+} = v(t_k + 0) = \dot{x}(t_k + 0)$ and $v_{r-} = v(t_k - 0) = \dot{x}(t_k - 0)$ for the corresponding impacts satisfy inequality

$$|v_{r+} - v_{r-} - 2\langle v_{r-}, n \rangle n| \leq \delta$$

(14)
(for singularity points we select one of the possible values for normal vectors).

4. The function

\[ v(t) = \frac{d}{dt} x(t) \]

is piecewise smooth and for any \( k \in \{0, \ldots, N\} \) and any \( t \in (t_k, t_{k+1}) \)

\[ |v(t) - v(t_k + 0)| \leq \delta. \]

We use this notion to describe trajectories of particles of non-zero mass that interact with a moving and rotating body.

**Definition 6.** We say that a rough disc defined by a sequence \( B_n \) is perfect if there exist \( n_0 \in \mathbb{N}, \lambda_0 > 0, 0 < \delta < \pi/2 \) and \( k > 0 \) such that for any \( n \geq n_0 \) almost all billiard \( \delta \)-pseudotrajectories, entering a hollow of this body with the angle \( \geq \lambda_0 \) with respect to the normal vector of the opening of the hollow, have at most \( k \) impacts before they leave the hollow.

**Remark 4.** In particular, for a body \( B_n \) (with \( n \) sufficiently large) approximating a perfect rough disc moving with a big relative angular velocity, any particle has a finite number of interactions and leaves a hollow after a time period proportional to diameter of the hollow.

**Remark 5.** One easily sees that all rough bodies formed by smooth hollows are not perfect. Later (starting from Section 7) we give an example of a perfect rough body, which is quite exotic. There are simpler examples; for example, a rough disc with triangular hollows is perfect. We leave the corresponding proof to the reader.

Dynamics of rough bodies (even of perfect ones) is quite sophisticated. It suffices to say that particles may be reflected from the body and hit it once again after a long period of time. So, even in the limiting case, the current state of the body without taking into account the state of the medium does not uniquely define its further dynamics and that dynamics cannot be described by a finite system of ordinary differential equations. However, first of all we introduce a simplified mathematical model for dynamics of approximations for rough bodies. We make the following artificial assumptions.

1. We assume that all particles which collide once with a point of the boundary of the body out of any hollow do not interact with the body anymore (assume that they "disappear").

2. We assume that there is a number \( k > 0 \) such that all particles interacting with a fixed hollow of the body, have at most \( k \) impacts and, after leaving the hollow, do not interact with the body anymore. In other words, we say that the sum \( \sum_{t \in \Theta(x_0)} \) consists of exactly one term if a particle, initially placed at \( x_0 \) hits the disc at a point out of a cavity and consists of at most \( k \) terms if this particle hits the disc at a point of a cavity.

3. We assume that the moving body does always meet a medium uniformly filled with immobile particles, so that \( dm = \rho dS, \rho = \text{const} \) and the same is true for all \( t \). In reality this is incorrect, because the moving body cleans the trajectory behind it and the trajectory may have self-intersections. Here we assume that particles in the "cleaned" part of the plane "appear" from nowhere. In other words, we suppose that we can assume without any influence to dynamics of the disc that \( dm(x_0) \) is constant everywhere but in current positions of cavities.

**Lemma 2.** For any \( \delta > 0, \lambda_1 > 0, R > 1, V_0, V_1 \) and any perfect rough disc \( \{B_n\} \) there exist \( n_0, \rho_0 \) and \( \lambda_0 < \lambda_1 / R \) such that the following is true. Let a disc \( B_n \) (\( n > n_0 \)) be moving during a period \( [t_1, t_2] \) with the relative angular velocity \( \lambda \in [\lambda_0, \lambda_1] \), with velocity \( V \in [V_0, V_1] \) and angular velocity \( \omega \in [\omega_0, \omega_1] \)
where \( V_0, \omega_0 > 0 \). Let assumptions 1–3 be satisfied, and the density of the medium be \( \leq \rho_0 \). Then in the coordinate system attached to the disc any trajectory of a particle inside a hollow is a billiard \( \delta \)-pseudotrajectory.

**Proof.** Motion of almost any particle between neighbor impacts is constant in the immobile coordinate system (Corollary 1). Let us estimate parameters of motion of the coordinate system attached to the disc. The absolute acceleration of any point of the disc between two hits is bounded by \( \omega^2 r \) where \( \omega \) is the current value of angular velocity, since the motion of the center is constant. The overall impulse of particles that could hit the body during the time period of the length \( \tau \) is not greater than \( 2\rho r (V_1 + \omega r)^2 \tau \).

Then the variation of the velocity for the considered point during that time interval could be estimated as \( \frac{2\rho r (V_1 + \omega r)^3 \tau}{(MV_0)} + \omega \tau \) and one of the angular velocity can be estimated as \( \frac{2\rho (V_1 + \omega r)^3 \tau}{(I\omega_0)} \).

If \( h \) is the diameter of a hollow, due to item 2 of assumptions of this section, the time of residence of a particle inside a selected hollow is not greater than \( \tau_0 = \frac{(K + 2) h}{V_0} \) (the absolute value of relative velocity of a particle with respect to a point of the hit does not change during the impact, see [3]). So, if \((t_k, t_{k+1})\) is an interval between two neighbor impacts of a particle, and \( v \) is the velocity of the particle in coordinate system, associated with the disc, we have

\[
|v(t) - v(t_k + 0)| \leq \delta_0 := \omega^2 r \tau_0 + 2\rho r V_2^2 \tau_0/(MV_0) + 2\rho(V_1 + \omega r)^3 \tau/(I\omega_0) = O(\tau_0) = O(h)
\]

for any \( t \in (t_k, t_{k+1}) \). This finishes the proof.

In general, it may happen that a non-zero mass of particles interacts with the body at a fixed instant of time, so Eq. (4) may be wrong. However, the total impulse of those particles does not exceed \( C \rho (V + \omega r) \). This is the total impulse of particles that reside in hollows at the fixed instant of time, \( C \) is a constant. A similar estimate would be true for kinetic energy of those particles and for the instantaneous change of rotational impulse of the body (called \( \delta P_\omega \) in Section 1). So, due to estimate (6), we have (14) provided \( h \) is small. □

**Remark 6.** Parameters of cavities including their maximal diameter (which is \( \sim h \)) will be selected as small as we need, which would not hurt other parameters of the system.

**Remark 7.** As we already noticed, some trajectories of the disc may be non-unique. Anyway, statement of Lemma 2 is true for any selection of a possible trajectory.

### 7 Dynamics of perfect rough bodies: a simplified model

In this section we write down the simplified model for dynamics of a rough body. Since the considered medium is uniform (respecting important remarks at the end of the section) and since hollows are uniformly distributed on the boundary of the rough disc, the dynamics of the rough disc is described by the following system of ordinary differential equations (here we use the result of [16, Theorem 7.1, P. 203]):

\[
\begin{align*}
M \frac{dV}{dt} &= R(\eta, \omega, V) = \frac{8}{3} \rho V^2 \bar{R}(\eta, \lambda); \\
I \frac{d\omega}{dt} &= R_I(\eta, \omega, V) = \frac{8}{3} \rho V^2 \bar{R}_I(\eta, \lambda).
\end{align*}
\]

(15)

Here \( \eta \) is the billiard law corresponding to the selected rough disc. Formulae for dimensionless resistances \( \bar{R} \) depend on \( \lambda \). There are three cases \( \lambda > 1, \lambda = 1 \) and \( \lambda < 1 \). In this paper, we always consider the first case only. Functions \( \bar{R} \) and \( R_I \) can be found from the following formulae (notice that we always consider
the coordinate system composed of the vector $\mathbf{V}$ and the vector $\mathbf{V}^\perp$, orthogonal to $\mathbf{V}$:

$$\mathbf{R}(\eta, \lambda) = \begin{pmatrix} R_T(\eta, \lambda) \\ R_L(\eta, \lambda) \end{pmatrix}, \quad R_T(\eta, \lambda) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} c_T(x, y, \lambda) \, d\eta(x, y);$$

$$R_L(\eta, \lambda) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} c_L(x, y, \lambda) \, d\eta(x, y);$$

$$R_I(\eta, \lambda) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} c_I(x, y, \lambda) \, d\eta(x, y).$$

Here $\mathbf{col}(a_1, a_2)$ is the vector consisting of scalar values $a_1$ and $a_2$;

$$c_T(x, y, \lambda) = \frac{3\cos \frac{x-y}{2}}{\sin \zeta} \left( (\lambda^3 \sin^3 x + 3\lambda \sin x \sin^2 \zeta) \cos \zeta \cos \frac{x-y}{2} - (3\lambda^2 \sin^2 x \sin \zeta + \sin^3 \zeta \sin \frac{x-y}{2}) \chi_{x \geq x_0}(x, y); \right)$$

$$c_L(x, y, \lambda) = -\frac{3\cos \frac{x-y}{2}}{\sin \zeta} \left( (\lambda^3 \sin^3 x + 3\lambda \sin x \sin^2 \zeta) \cos \zeta \sin \frac{x-y}{2} + (3\lambda^2 \sin^2 x \sin \zeta + \sin^3 \zeta) \sin \zeta \cos \frac{x-y}{2} \chi_{x \geq x_0}(x, y); \right)$$

$$c_I(x, y, \lambda) = -\frac{3\lambda^3 \sin^3 x + 3\lambda \sin x \sin^2 \zeta}{\sin \zeta} \chi_{x + \sin y}(x, y) \chi_{x \geq x_0}(x, y);$$

$$\zeta = \arcsin \sqrt{1 - \lambda^2 \cos^2 x}, \ x_0 = \arccos(1/\lambda); \ \chi \text{ stands for the characteristic function.}$$

Make a transformation of variables in equations (15). First of all, change the independent variable, supposing

$$d\tau = \frac{8\rho \rho V}{3M} \, dt.$$

Then we define $\theta$ so that $\mathbf{V} = V \mathbf{col}(\cos \theta, \sin \theta)$. Let $\beta = Mr^2/I$ be the inverse relative moment of inertia of the rough disc. It follows from equations (15) that

$$\frac{d\lambda}{d\tau} = \beta R_T(\lambda) - \lambda R_L(\lambda),$$

$$\frac{dV}{d\tau} = -V R_L(\lambda),$$

$$\frac{d\theta}{d\tau} = -R_T(\lambda).$$

Notice that the variable $\tau$ is a natural parametrization of the trajectory of the center of the disc. Namely, is $S(t)$ is the overall way passed by the center of the disc by the moment $t$ then $dS/d\tau = 3M/(8\rho r) = \text{const}.$

In what follows we use equations (19) to describe dynamics of a rough disc.

In the following sections we provide a family of specially selected roughnesses and justify that the proposed model of dynamics is applicable for rough discs with such shapes of cavities. First of all we need two types of auxiliary scattering billiards.

### 8 Bunimovich mushrooms

Later on, we need the so-called retroreflectors. These are families of scattering billiards with the following properties. Consider a family of domains $\Theta_h \subset \mathbb{R}^2$ ($h$ is a small positive parameter) with a
piecewise smooth boundary $\partial \Theta_h$ which can be represented as a disjoint union $\partial \Theta_h = \Omega_h \cup I_h$ where $\Omega_h$ and $I_h$ satisfy following properties.

1. The arc $\Omega_h$ is a hollow with opening $I_h$.

2. Consider a uniform (with respect to points of entrance inside $\partial \Theta_h$ and angles of entrance) distribution of incoming particles. For any finite $T > 0$ the proportion of particles which spend more than $T$ units of time inside the domain $\Omega$ tends to zero as $h \to 0$.

3. Let $\nu_-$ be the entrance angle of a particle and $\nu_+$ be the exit angle, then for any $\sigma > 0$ the proportion of particles such that $|\nu_+ + \nu_- - \pi| > \sigma$ tends to zero as $h \to 0$.

In this paper we consider so-called "Bunimovich mushroom" [33–35], Figure 3. There are other patterns of retroreflectors (see [16, Chapter 9] for more examples and for more properties of Bunimovich mushrooms).

The pattern of the mushroom, we are going to use in this article is the following: a domain $\Theta_h$ which is a union of two domains: $\Theta_{h1}$ and $\Theta_{h2}$. The first one ("pileus" of the mushroom) is a strictly convex domain which is the upper part of an ellipse, whose principal axis is horizontal.

The second part of the mushroom (call it stipe) is a $b_{12} \times b_{13}$ rectangle. Let $b_{11}$ be the length of the long axis of the ellipse $\Theta_{h1}$. We assume that the center of the bigger edge of the stipe coincides with one of the ellipse and the corresponding tops are foci of the ellipse. Suppose that

$$\frac{b_{12}}{b_{11}} = 2h; \quad \frac{b_{13}}{b_{12}} = h.$$  \hspace{1cm} (20)

We call the value $h$, for which Eq. (20) is satisfied, imperfectness of the mushroom.

We claim that $\Omega_h = (P_L P_R)$ on Figure 3 is the entrance for the considered scattering billiard. Consequently, we suppose that $I_h$ is the rest of the boundary of the "mushroom".

![Figure 3. Mushroom billiard.](image)

Notice that this mushroom is a retroreflector. In other words, if a particle crosses the pileus of the mushroom without colliding with its edge, it leaves the mushroom with almost the same angle since the excentricity of the corresponding ellipse is small. After reflection, it does necessarily cross the segment between foci of the ellipse. All such particles have only one interaction with the boundary of the mushroom.
For a fixed $\sigma > 0$ we define sets $\Sigma_\sigma = \{(x_-, v_-) \in X : |x_+ - x_-| \leq \sigma b_{22}, \ |v_+ + v_-| \leq \sigma\}$.

The following statement has been proved in [16, Lemma 4.1, p. 1.15].

**Lemma 3.** There exist constants $h_0 > 0$ and $C_m > 0$ such that for any $\sigma > 0$ there is $h \in (0, h_0)$ such that if parameters of a mushroom billiard satisfy conditions (20), measure of the set $\Sigma_\sigma$ is greater than $1 - C_m \sigma$.

## 9 Amphora billiards: an example of quasi-elastic hollow

In this section we provide an example of hollows, corresponding to quasi-elastic interaction with particles that enter those hollows.

Select a small positive parameter $h$ (imperfectness of the billiard). Consider two arcs of confocal parabolas with the focus at the origin $0$ e.g. ones given by equations $x = \pm (1 - y^2)/2$ and corresponding to $y \in [0, 1]$. Link the lower ends of these arcs by a segment. We obtain a curve triangle. Cut the middle part $F_L F_R$ of the base of this triangle, corresponding to $x \in [-h, h]$ and construct two segments $A_L F_L$ and $A_R F_R$ of the length $h^2$ at ends of the obtained gap, which make angles $\pm \pi/4$ with the axis $Ox$, so that these ends are linked with the axis $Ox$ (Figure 4(a)). The domain for the amphora billiard is now constructed.

![Figure 4. Amphora billiard (a) and its modification (b).](image)

Later on, we deal with magnifications of these amphora billiards. We take the principal parameter $b_{21}$ so that

$$b_{21} = h; \quad b_{22}/b_{21} = 2h; \quad b_{23}/b_{22} = h$$

where $h$ is the imperfectness, $b_{22}$ is the width of the entrance corridor of the billiard domain, call it "neck", $b_{23}$ is the length if this corridor.

Let $X = [-b_{22}, b_{22}] \times (-\pi, 0)$ be endowed with the smooth measure $\nu$ with the density $-\sin v_+/(4b_{22})$. 

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Next lemma demonstrates that this amphora hollow works like a smooth mirror i.e. for "almost" all particles the angle of incidence "almost" equals to the angle of reflection. Let \( N_\sigma \) be the set of initial conditions \( (x_-, v_-) \in X \) of the entrance which correspond to billiard trajectories with two impacts such that \( |x_+ + x_-| \leq \sigma b_{22} \), \( |v_+ - Rv_-| \leq \sigma \). Here \( R(v_x, v_y) = (v_x, -v_y) \) and, as usually, unit vectors \( v_\pm \) correspond to angles \( v_\pm \).

**Lemma 4.** There exist constants \( h_0 > 0 \) and \( C_\sigma > 0 \) such that for any \( \sigma > 0 \) there is \( h \in (0, h_0) \) such that if parameters of a amphora billiard satisfy conditions (21), the measure \( \nu \) of the set \( N_\sigma \) is greater than \( 1 - C_\sigma \).

**Proof.** Let \((x_-, v_-)\) be the initial position and the angle of the initial velocity of the particle \((x_-, 0)\) corresponds to the center of the segment, linking foci). For the amphora billiard we assume that the initial position \( x_- \) is always placed at the axis \( Ox \), this is why we set this parameter scalar. The value \( v_- \) is the angle of the vector of initial velocity; we identify it with a point of the lower semicircle. The exit data \((x_+, v_+)\) will be, respectively, the pair of a value \( x_+ \), which is the \( x \) coordinate of the point of exit (the \( y \) coordinate is zero) and the angle of the unit vector \( v_+ \) from the upper semicircle.

Observe two important properties of the amphora billiard. Any particle, corresponding to initial data \((0, v_-), |\tan v_-| > 2h\), is reflected back to the same point after two impacts (unless the particle is moving strictly down). Moreover, after the first impact the motion of the particle is strictly parallel to the axis \( Ox \). Also, this billiard is symmetric with respect to the vertical axis.

Then for an amphora billiard the condition of two reflections is open for particles which start moving from a position near the focus. Let \( v_0 \) and \( v_1 \) be such that \( \tan v_0 = -2h \), \( \tan v_1 = 2h \). Then every trajectory of the amphora billiard, corresponding to initial data \((0, v): v \in (v_-, v_+), v \neq -\pi/2\) has exactly two impacts and both of them correspond to points of "sides of the amphora" i.e. parabolas. It suffices to prove that

\[
D_v = \frac{\partial(x_+, v_+)}{\partial(x_-, v_-)}(0, v) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

for any \( v \in (\Theta_-, \Theta_+) \). The sign of the element \( d_{11} \) is not important for us.

Since every trajectory, which passes via the focus, comes back to the focus after two reflections, we have \( d_{12} = 0 \). Grace to symmetry of such trajectories, we have \( d_{22} = -1 \).

Let \( n_- \) and \( n_+ \) be unit normal vectors for points of the first and the second impact respectively. Fix the angle \( v_- \) of the initial velocity. Then \( n_\pm \) are functions of \( x_- \) and, moreover, grace to the structure of the considered domain, the vector \( n_- \) uniquely defines the point of the first impact and, consequently, uniquely defines the vector \( n_+ \). Let \( n_\pm \) be the corresponding angles with respect to the axis \( Ox \). Consider the angle \( \alpha \) between the axis \( Ox \) and the trajectory of the particle after the first impact. Clearly, \( \alpha = 0 \) for all solutions, passing via the focus. Also, due to reflection law, \( \alpha = v_- - 2n_- \). Comparing the trajectory of a particle with one, obtained by reversion of time, one gets \( v_- = 2n_- + v_+ - 2n_+ = \pi \).

On the other hand, for all solutions, passing via the focus, one can easily see that \( d_{11} = -1 \). This implies

\[
\left( \frac{\partial n_+}{\partial x_-} + \frac{\partial n_-}{\partial x_-} \right) \bigg|_{x_-=0} = 0
\]

and, consequently,

\[
\left( \frac{\partial v_+}{\partial x_-} + \frac{\partial v_-}{\partial x_-} \right) \bigg|_{x_-=0} = 0 \Rightarrow d_{12} = 0.
\]

The equality \( d_{11} = \pm 1 \) easily follows from the symmetry of the considered billiard. \( \square \)
Notice also, that if a trajectory meets the neck of the amphora so that the absolute value of the direction of the entrance velocity is less than $\pi/4$, it is reflected upwards and does not interact with the boundary of the amphora any more.

Amphora billiards have a property, similar to one of mushrooms: particles can get stuck there, having a big number of impacts until they leave the amphora domain. We modify the amphora in the following way. Attach two triangles $B_LC_LF_L$ and $B_RC_RF_R$ to horizontal parts of the boundary of the billiard (Figure 4(b)). We do it so that the following facts would be true:

1. $|B_LF_L| = |B_RF_R| = h^{5/4} (= o(F_LF_R))$,
2. $\angle F_LB LC_L = \angle F_RB RC_R = \pi/6$,
3. $\angle F_LB LC_L = \angle F_RB RC_R = \pi/4$.

On ”boards” of the amphora, we mark a coordinate $\Phi \in [-\pi/2, \pi/2]$. Notice that $\Phi = 0$ corresponds to the vertical direction. This coordinate corresponds to the inclination of the line, passing through the origin and the selected point.

Consider two symmetric points $O_L$ and $O_R$ which are centers of segments $[B_LF_L]$ and $[B_RF_R]$ respectively. Replace parts of parabolas, corresponding to $\Phi \in [-\pi/4, \pi/4]$ with arcs of ellipses $E_L$ and $E_R$ such that one focus for both of these ellipses is $O$ and another one is $O_L$ for $E_R$ and $O_R$ for $E_L$. The modified amphora domain is constructed (see Figure 4(b)).

Now we study billiard trajectories for the modified amphora billiard. Suppose that the angle between the initial velocity and the line of the opening belongs to $(-\pi, -6\pi/7) \cup (-\pi/7, 0)$.

If such trajectory meets a point of one of segments $[A_LB_L]$ or $[A_RB_R]$ it is reflected upwards and does not have any other impacts. Otherwise, it interacts twice with arcs of parabolas. After that, due to Lemma 4 there exist following three alternatives (Figure 4).

1. The trajectory leaves the amphora domain forever without having any more impacts. These trajectories are the same for the amphora and its modification.
2. The trajectory hits one of segments $[A_RB_R]$ and leaves the amphora domain.
3. The trajectory hits one of segments $[B_LC_L]$ or $[B_RC_R]$ then reflects to a point of the ellipse $E_L$ if the previous impact took place at a point of $[B_RC_R]$ or $E_R$ if the previous impact took place at a point of $[B_LC_L]$. After that, the trajectory crosses the $2h^{5/4}$ - neighborhood of the origin and leaves amphora domain forever.

Lemma 4 guarantees that the ”majority” of trajectories behave according to the first scenario. Notice that for any initial data of the considered type the number of impacts cannot exceed 4.

10 Hybrid hollows

Now we are ready to construct a rough element, i.e. the hollow, corresponding to the rough disc with a prescribed law of reflection.

We modify the constructed amphora billiard so that for some selected directions of billiard trajectories and pseudotrajectories it would work as a retroreflector and for some others it would work as a quasielastic reflector. Select two symmetric sets of non-intersecting segments $J_{kL}$ and $J_{kR}$ $(k = 1, \ldots m)$ given by

$$J_{kR} = [\Phi_k^0, \Phi_k^1], \quad J_{kL} = [-\Phi_k^1, -\Phi_k^0]$$
(assume that $5\pi/14 = \pi/2 - \pi/7 < \Phi_1^0 < \Phi_1^1 < \ldots < \Phi_m^0 < \Phi_m^1 < \pi/2$).

For $\Phi \geq \Phi_m^1$ we leave the arc of the parabola same as it was. Then we attach an arc of ellipse with foci at $F_L$ and $F_R$ and corresponding to $\Phi \in (\Phi_m^0, \Phi_m^1)$. This arc is defined uniquely. Then we draw an arc of the parabola with a focus at the origin and the vertical axis of symmetry through the free end of the arc of the ellipse. We make it corresponding to the segment $(\Phi_m^1, \Phi_m^0)$. We repeat similar constructions of arcs of ellipses with same foci and parabolas with the same focus until we reach $\Phi = \Phi_1^1$. Then we attach the last arc of parabola, corresponding to $\Phi \in (\pi/4, \Phi_1^1)$. To finish the construction we attach an arc of an ellipse, corresponding to $(\pi/4, \pi/2)$ similarly as we did it for modified amphora billiards (Figures 4, 5).

The majority of billiard trajectories corresponding to hybrid hollows, belong to one of two types: either they have one hit with "elliptic" part of the hollow of the boundary or two hits with opposite "parabolic" parts.

However, it may happen that a trajectory or a pseudotrajectory which first hits the parabolic part of the boundary of the hollow near its junction point with the elliptic part, is reflected to an elliptic part on the opposite side of the hollow, not to parabolic one. Generally, this means, since the junction is non-smooth, that the corresponding billiard trajectory hits one of segments $\Sigma_L = [D_{LL}, D_{LR}]$ or $\Sigma_R = [D_{RL}, D_{RR}]$ on the upper part of the boundary of the hollow (Figure 5). Let $G_{1L}, \ldots, G_{mL}$ and $G_{1R}, \ldots, G_{mR}$ be junction points between elliptic and parabolic sectors; consider $H_1, \ldots, H_{2m} -$ points on the union $\Sigma_L \cup \Sigma_R$, corresponding to "parabolic+elliptic" reflections from points $G_{kL}$ and $G_{kR}$ or vice versa. We put a system of flat mirrors (segments) of sizes $h^{5/4}$ centered at $H_j$ ($j = 1, \ldots, 2m$) so that all trajectories and $h^{3/2}$ pseudotrajectories, hitting first parabolic, then elliptic sectors, are reflected via these mirrors to $h^{9/8}$ neighborhoods of points $H_1', \ldots, H_{2m}'$ such that $H_j' \in (-\pi/4, -\pi/7) \cup (\pi/7, \pi/4)$ for all $j$. We put flat mirrors of lengths $h^{9/8}$, centered at points $H_j'$ so that all considered trajectories and pseudotrajectories are reflected by these mirrors to the $h^{17/16}$ neighborhood of the center of the entrance of the hollow (Figure 5). That size is still much less than length of the entrance, equal to $h$. Trajectories and pseudotrajectories for this hybrid billiard, corresponding to initial angles $\psi_\pm \in (-\pi, -6\pi/7) \cup (-\pi/7, 0)$ are the following.

1. If a pseudotrajectory does not hit points, corresponding to one of segments $J_{kL}$ or $J_{kR}$, the behavior
is the same as for the modified amphora billiard.

2. If it hits one of the mentioned segments, it reflects "almost back" (similarly to what happens for Bunimovich mushrooms). Then the pseudotrajectory leaves the domain without further interactions with walls.

3. A small proportion of particles (which tends to 0 as $h \to 0$) has a distinct behavior. However, all such particles leave the hollow, having at most 4 impacts.

Now we describe how it is possible to cover almost all segment $I \in Ox$ (we may also do the same if $I$ is an arc of the circle) with tops of hybrid billiards. Cut the middle part of $I$ of the length $2h|I|$ and insert there a hybrid billiard of imperfectness $h$ and with the basis of the neck equal to $2h|I|$. Call this hollow one of the first generation. Let $b_1$ be the corresponding magnification. Then all other parameters of the hybrid billiard are uniquely defined. Take $b_2 = \varrho h^2 b_1$. Here $\varrho < 1$ is the principle magnification for smaller mushrooms of the "second generation" (see Figure 5); the imperfectness will always be the same. Then we put $N \sim h^{-1} |I|$ non-intersecting (and not intersecting with the first generation hollow) hollows of the second generation whose tops correspond to subsegments of $I$. We repeat this procedure, creating hollows of the third level (the value $\varrho$ is always the same) and so on. On the step number $L$, the measure of the part of the segment $I$, not covered by openings of already constructed hollows can be estimated by the value $|I|(1 - \tilde{h}/2)^{L}$. In the limit, we get a Cantor set. However, we do not need a perfect construction so we stop after finitely many steps which depends on perfectness of approximation, we need.

Later on, we consider billiards with roughnesses of the described type.

11 Comparison between motions of a perfect rough body: the precise model and approximations

In this section, we are going to provide a precise model for motion of a rough disc. We do this using a method of successive approximations. Consider a motion, given by Eq. (15) as an initial approximation for dynamics of the disc. Every step of our approximation will contain following stages.

1. First of all, we take into account the fact that the moving disc cleans a neighborhood of its trajectory. If the trajectory of the center has a transverse self-intersection, any approximating curve must have self-intersections, as well. So, sometimes the disc must pass through void domains where there is no particles any more. We take into account absence of particles in that domains.

2. Some particles, reflected by the disc, could interact with this body once or even more times again. However, the number of those interactions must be finite which follows from definition of perfect rough disc. For the fixed motion of disc, we estimate the influences of all particles, that hit the disc several times, without residing inside a single hollow.

3. All those additional impulses give corrections to differential equations, describing dynamics of the disc. So, we rewrite Eq. (15) and then proceed to the next step.

So, first of all we study dynamics of the disc in the "immobile" medium. Then we make corrections to dynamics of the medium, caused by motion of the disc. This correction changes dynamics of the disc. Taking into account new dynamics of the disc we make correction to motion of the medium and so on...
In this section we always assume that cavities of the considered billiard are of the hybrid type, constructed at Section 10. We study trajectories of the body such that all points of self-intersections are not triple. Also, we assume that angles between tangent directions at all points of self-intersections are greater than π/4, Figure 6.

![Figure 6. Self-intersections: a sample of a curve Γ.](image)

Consider a curve $G_0$, defined by a function $X_0(t) : t \in [0, T]$. Let $V_0(t) = X_0(t)$. Suppose, that there is a smooth function $\omega_0(t)$ such that Equations (15) are satisfied, so we can write down the zero approximation for the motion. We consider curves which are magnifications of a fixed one, where the overall length $L$ of the curve is a big parameter.

Denote $\lambda_0(t) = \omega_0(t)r/V(0)$. Let $p_{X_0,\omega_0}(t, \xi)$ be the impulse of the infinitesimal element of the media acting during the time interval $[t, t + dt]$ on the part of the boundary, corresponding to the interval $[\xi, \xi + d\xi]$ (see last paragraph of Section 5). The value $\xi \in S^1$ is the normal parametrization for the boundary of the considered body. For every $\xi$ and $t$ there is a uniquely defined angle $\phi = \phi(t, \xi)$ between the radius vector of the corresponding point and velocity of the center of the body. Suppose $p_{X_0,\omega_0}(t, \xi) = -\rho r \cos^2 \phi V(t)\dot{V}(t) + \dot{p}_{X_0,\omega_0}(t, \xi)$. Notice that $-\rho r \cos^2 \phi V(t)\dot{V}(t) dt d\xi$ is the infinitesimal impulse corresponding to the dynamics, described by Equations (15), so $\dot{p}$ is the element of correction of the impulse during the first step.

We are going to estimate norm of the function $p_{X_0,\omega_0}$ at the space $Z = (C([0, T] \times S^1 \to \mathbb{R}^2))^*$. Represent $p_{X_0,\omega_0}(t, \xi) = p_{X_0,\omega_0}'(t, \xi) + p_{X_0,\omega_0}^2(t, \xi)$ where $p_{X_0,\omega_0}'$ is the infinitesimal impulse, corresponding to taking self-interactions into account and $p_{X_0,\omega_0}^2$ is one, corresponding to multiple intersections between particles and the body. We start with an estimate of $p_{X_0,\omega_0}'$.

Split the curve $G_0$ into $N$ segments $G_j^0$ $(j = 1, \ldots, N)$ so that overall rotation of velocity vector $V$ is at most $\pi/4$ on every segment $G_j^0$. Then these segments cannot be self-intersecting and the overall number of self-intersections for the curve $G_0$ cannot exceed $N(N - 1)/2$. So, there are $\nu \leq N(N - 1)/2$ disjoint time intervals $[t_j^-, t_j^+] \subset [0, T]$ such that $|I_j| = t_j^+ - t_j^- \leq 2\rho r V_0(t_j^-)$ for all $j$ (here we notice that $V_0 = |V_0|$) and

$$p_{X_0,\omega_0}'(t, \xi) = 0 \quad \text{if} \ t \notin \bigcup_{j=1}^{\nu} [t_j^-, t_j^+];$$

$$|p_{X_0,\omega_0}'(t, \xi)| \leq 4\rho r V_0^2(t_j^-) \quad \text{if} \ t \in \bigcup_{j=1}^{\nu} [t_j^-, t_j^+].$$

This means that

$$\|p_{X_0,\omega_0}'\|_Z \leq 8N(N - 1)\pi \rho r V(0). \tag{22}$$

Now we estimate the function $p_{X_0,\omega_0}^2$. First of all introduce a function $p_{X_0,\omega_0}^3(t, \xi)$ satisfying following properties:

1. $p_{X_0,\omega_0}^3(t, \xi) = 0$ if the particle reflected at the instant $t$ from the point of the surface of the disc, corresponding to $\xi$, does not hit the disc once more and
2. $p_{X_0,\omega_0}^3(t, \xi) \, dt \, d\xi$ is the infinitesimal post-impact impulse of the reflected particle else.

Since velocities of particles do not change between neighbor impacts, we have

$$||p_{X_0,\omega_0}^2||_Z = ||p_{X_0,\omega_0}^3||_Z.$$  

In other words, $p_{X_0,\omega_0}^2$ and $p_{X_0,\omega_0}^3$ are same impulses calculated at distinct instants of time: $p_{X_0,\omega_0}^2$ is the impulse just after an impact and $p_{X_0,\omega_0}^3$ is the impulse before the next impact.

Let $p_{X_0,\omega_0}^3(t, \varphi) \neq 0$. So, the corresponding particle hits the disc at the instant $s \in [t, T]$. Notice that $t$ and $s$ cannot belong to the same segment $G^j_0$ or even to neighbor ones. Let $j$ be such that $t \in G_{j-1}^0$ and $L_j$ be length of the segment $G^j_0$. Since the velocity of the particle between two interactions is constant, it equals to

$$V_p := \frac{X_s - X_t}{s - t}$$

where $X_s$ and $X_t$ are points of neighbor impact interactions for the particle on the surface of the moving disc.

It follows from (19) that there exists a dimensional constant $c > 0$, which only depends on the parameters of the system and such that $||X_s - X_t|| \leq c \log(1 + V(t)(s - t)/c)$. Respectively

$$\frac{V_p}{V(t)} := \frac{|V_p|}{V(t)} \leq \frac{\log(1 + V(t)(s - t)/c)}{V(t)(s - t)/c} \leq \frac{\log(1 + c_1 L_j)}{c_1 L_j} = l_j, \quad (23)$$

where $c_1$ is another positive constant.

Therefore, if $L$ is big and, respectively, all $l_j$ are small, $p_{X_0,\omega_0}^2(t, \xi) = 0$ if $\cos \phi \geq l_j$. Due to (23) in this case reflected particles are "too fast" and cannot hit the moving disc any more. Then $p_{X_0,\omega_0}^2(t, \xi) \, dt \, d\xi = V_p \, dt \, d\xi = V_p \, V(t) \, d\rho \cos \phi \, dt \, d\xi$.

Summarizing all the information above, we obtain that

$$||p_{X_0,\omega_0}^2||_Z \leq \frac{C}{\min_j l_j} \leq \frac{CC_1 \log L/C_1}{L} = l, \quad (24)$$

where $C$, $C_1$ are positive constants.

Infinitesimal impulses $p_{X_0,\omega_0}^2(t, \varphi) \, dt \, d\varphi$ cause infinitesimal accelerations $w_{X_0,\omega_0}(t, \xi) \, dt \, d\xi$ and angular accelerations $\epsilon_{X_0,\omega_0}(t, \xi) \, dt \, d\xi$ of the disc, see (3) and (13).

Let

$$W_{X_0,\omega_0}(t) = \int_{s_1}^t w_{X_0,\omega_0}(t, \xi) \, d\xi,$$

$$E_{X_0,\omega_0}(t) = \int_{s_1}^t \epsilon_{X_0,\omega_0}(t, \xi) \, d\xi,$$

$$\dot{V}_{X_0,\omega_0}(t) = \int_0^t \dot{W}_{X_0,\omega_0}(s) \, ds,$$

$$\dot{\omega}_{X_0,\omega_0}(t) = \int_0^t \dot{E}_{X_0,\omega_0}(s) \, ds,$$

$$V_0 + \dot{V}_{X_0,\omega_0} = |V_0 + \dot{V}_{X_0,\omega_0}|,$$

$$\lambda_0 \dot{X}_{X_0,\omega_0} = (\omega_0 + \dot{\omega}_{X_0,\omega_0}) r/(V_0 + \dot{V}_{X_0,\omega_0}).$$

Also, we use the notion $(\dot{V}_{X_0,\omega_0}, \dot{\omega}_{X_0,\omega_0}) = \mathcal{H}(X_0, \omega_0)$. It follows from (22) and (24) that

$$\max_t \max_r (|\dot{V}_{X_0,\omega_0}(t)|, |\dot{\omega}_{X_0,\omega_0}(t)|) \leq C_2 \rho',$$

where $C_2$ is a positive constant, $\rho' = \rho + l$.

Now we study how do functions $\dot{V}_{X_0,\omega_0}$ and $\dot{\omega}_{X_0,\omega_0}$ depend on $X_0$ and $\omega_0$. 

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Let a pair \( X'_0, \omega'_0 \) be close to \( X_0, \omega_0 \) in \( C^1([0, T+1] \to \mathbb{R}^n) \) (we can always extend considered functions to that segment). Denote \( \Delta = \max(\|X'_0 - X_0\|_{C^1}, \|\omega'_0 - \omega_0\|_{C^1}) \).

Due to Implicit Function Theorem graph of \( X'_0(t) \) has points of self-intersections, which are \( C_3 \Delta \) close to ones of the function \( X_0(t) \). Here \( C_3 \) is a positive constant. The angles of these self-intersections are also close (a similar estimate is true). Then there exists a \( C^1 \) smooth transformation \( \Phi_1^0 : [0, \infty)_t \times [0, \infty)_\varphi \) such that

\[
\frac{\partial \Phi_1^0}{\partial t}(t, \xi) > 0, \quad \frac{\partial \Phi_1^0}{\partial \varphi}(t, \xi) > 0 \quad \text{for all } t, \xi; \quad \Phi_1^0(0, \cdot) = \text{id} \tag{26}
\]

and \( \|\Phi_1^0 - \text{id}\|_{C^1} \leq C_4 \Delta, \|p_{X_0, \omega_0}(t, \xi) - p'_{X_0, \omega_0}(\Phi_1^0(t, \xi))\| \leq C_4 \rho' \Delta. \)

Similarly, using Implicit Function Theorem, one can demonstrate that there exists a smooth transformation \( \Phi_2^0 \), satisfying an analogue of (26) and such that

\[
\frac{\partial \Phi_2^0}{\partial t}(t, \xi) > 0, \quad \frac{\partial \Phi_2^0}{\partial \varphi}(t, \xi) > 0 \quad \text{for all } t, \xi; \quad \Phi_2^0(0, \cdot) = \text{id} \tag{26'}
\]

and \( \|\Phi_2^0 - \text{id}\|_{C^1} \leq C_5 \Delta, \|p_{X_0, \omega_0}(t, \varphi) - p'_{X_0, \omega_0}(\Phi_2^0(t, \varphi))\| \leq C_5 \rho' \Delta. \)

This means that

\[
\max_t \max(\|\dot{V}_{X_0, \omega_0}(t) - \dot{V}_{X_0, \omega_0}(t)\|, \|\bar{\omega}_{X_0, \omega_0}(t) - \bar{\omega}_{X_0, \omega_0}(t)\|) \leq C_6 \rho'. \tag{27}
\]

For fixed initial conditions \( V(0), \omega(0), \) we define

\[
V_1(t) = V_1(t) + V_{X_0, \omega_0}(t); \quad \omega_1(t) = \omega_1(t) + \omega_{X_0, \omega_0}(t)
\]

where \( \dot{V}_1(t) \) and \( \dot{\omega}_1(t) \) are solutions of the system

\[
M \frac{dV}{dt} = \frac{8}{3} \rho(V + \dot{V}_{X_0, \omega_0}(t))^2 \overline{R}(\eta, r(V + \dot{V}_{X_0, \omega_0}(t))/\omega + \dot{\omega}_{X_0, \omega_0}(t));
\]

\[
I \frac{d\omega}{dt} = \frac{8}{3} \rho r(V + \dot{V}_{X_0, \omega_0}(t))^2 \overline{R}_I(\eta, r(V + \dot{V}_{X_0, \omega_0}(t))/\omega + \dot{\omega}_{X_0, \omega_0}(t)). \tag{28}
\]

(see (15) for definition of \( \overline{R} \) and \( \overline{R}_I \)).

Consider (28) as a perturbation of (15) and apply Theorem on Differentiability of Solutions with respect to Parameters. We obtain that

\[
\max_t \max(\|V_1(t) - X_0(t)\|, \|\omega_1(t) - \omega_0(t)\|) \leq C_7 \rho'.
\]

First of all, integrating \( V_1(t) \), we can find \( X_1(t) \). Then we can use a procedure, similar to what we have already done, to define functions \( \dot{V}_{X_1, \omega_1}, \dot{\omega}_{X_1, \omega_1} \). These functions satisfy analogs of (25) and (27) with other transformations \( \Phi_1^0 \) and \( \Phi_2^0 \).

In order to find dynamics of the disc consider the following method of successive approximations: find sequences of functions \( V_k(t), \omega_k(t) \) such that for any \( k \in \mathbb{N} \)

\[
X_k(0) = X(0), \quad V_k(0) = V(0), \quad \omega_k(0) = \omega(0);
\]

\[
V_k(t) = \dot{V}_k(t) + \dot{V}_{X_{k-1, \omega_{k-1}}}(t); \quad \omega_k(t) = \omega_k(t) + \dot{\omega}_{X_{k-1, \omega_{k-1}}}(t);
\]

\[
M \frac{d\dot{V}_k}{dt} = \frac{8}{3} \rho^{(\dot{V}_k + \dot{V}_{X_{k-1, \omega_{k-1}}}(t))^2 \overline{R}(\eta, r(\dot{V}_k + \dot{V}_{X_{k-1, \omega_{k-1}}}(t))/\omega_k + \dot{\omega}_{X_{k-1, \omega_{k-1}}}(t));
\]

\[
I \frac{d\dot{\omega}_k}{dt} = \frac{8}{3} \rho r(\dot{V}_k + \dot{V}_{X_{k-1, \omega_{k-1}}}(t))^2 \overline{R}_I(\eta, r(\dot{V}_k + \dot{V}_{X_{k-1, \omega_{k-1}}}(t))/\omega_k + \dot{\omega}_{X_{k-1, \omega_{k-1}}}(t)). \tag{29}
\]

\[
(\dot{V}_{X_{k-1, \omega_{k-1}}}, \dot{\omega}_{X_{k-1, \omega_{k-1}}}) = H(X_k, \omega_k).
\]
Let us prove that these approximations converge. Establish the following estimate

$$\max_t \max_{k} \left( |V_{k+1}(t) - V_k(t)| + |\omega_{k+1}(t) - \omega_k(t)| \right) \leq \frac{1}{2} \max_t \max_{k} \left( |V_k(t) - V_{k-1}(t)| + |\omega_k(t) - \omega_{k-1}(t)| \right).$$

Observe that taking a big value of \( L \) we can set here any small positive value \( \kappa \) instead of \( 1/2 \).

We do it by induction. On every step we obtain, similarly to (27) that

$$\max_t \max_{k} \left( |\tilde{V}_{k+1}(t) - \tilde{V}_k(t)|, |\tilde{\omega}_{k+1}(t) - \tilde{\omega}_k(t)| \right) \leq C_s \rho' \max_t \max_{k} \left( |V_k(t) - V_{k-1}(t)| + |\omega_k(t) - \omega_{k-1}(t)| \right).$$

Then (30) follows (if \( \rho \) and \( l' \) are selected sufficiently small) from Theorem on Differentiability of Solutions with respect to Parameters and (31). So the method of successive approximations converges. Let \( X_k \to X_* \) in \( C^1([0,T] \to \mathbb{R}^2) \), \( \omega_k \to \omega_* \) in \( C^0([0,T] \to \mathbb{R}^2) \). Proceeding to limit in (29) we obtain equations

$$X_*(0) = X(0), \quad V_*(0) = V(0), \quad \omega_*(0) = \omega(0);$$

$$\langle \tilde{V}_{X_*, \omega_*}, \tilde{\omega}_{X_*, \omega_*} \rangle = \mathcal{H}(X_*, \omega_*);$$

$$V_*(t) = \tilde{V}_*(t) + \tilde{V}_{X_*, \omega_*}(t); \quad \omega_*(t) = \tilde{\omega}_* + \tilde{\omega}_{X_*, \omega_*}(t);$$

$$M \frac{d \tilde{V}_*}{dt} = \frac{8}{3} r \rho (\tilde{V}_* + \tilde{V}_{X_*, \omega_*}(t))^2 \mathcal{R}(\eta, r (\tilde{V}_* + \tilde{V}_{X_*, \omega_*}(t)) / (\tilde{\omega}_* + \tilde{\omega}_{X_*, \omega_*}(t)));$$

$$I \frac{d \tilde{\omega}_*}{dt} = \frac{8}{3} r \rho (\tilde{V}_* + \tilde{V}_{X_*, \omega_*}(t))^2 \mathcal{R}(\eta, r (\tilde{V}_* + \tilde{V}_{X_*, \omega_*}(t)) / (\tilde{\omega}_* + \tilde{\omega}_{X_*, \omega_*}(t))).$$

This model takes into account self-intersections of trajectory of the disc and double interaction with some particles. For the solution \((X_*, \omega_*)\) of Eq. (32), that can be obtained by successive approximations, the following estimate is true:

$$\max_t \max_{k} \left( |X_*(t) - X_0(t)|, |\omega_*(t) - \omega_0(t)| \right) \leq C_* \rho'$$

where \( C_* \) is another positive constant. This follows from (30).

Reformulating the result of this section in a few words: we have demonstrated that Equations (15) give a good (in the sense of Eq. (33)) approximation for a trajectory of a rough disc if the corresponding solution of these equations is a curve of a small curvature which does not self-intersect at a small angle. Particularly, this works for magnifications of a fixed curve with transverse self-intersections. In this case, one can still use a model, given by Equations (15) to model dynamics of the disc.

### 12 Universal boomerang

We are going to prove that any rectifiable curve on the plane can be approximated in \( C^0 \) by trajectories of rough discs (see next section for a precise model of their dynamics). The following theorem takes place.

**Theorem 2.** Let \( g : [a, b] \to \mathbb{R}^2 \) be a continuous rectifiable curve. Then for any \( \varsigma > 0 \) there exists a motion of a rough disc with a radius \( r > 0 \) and with the coordinate of center \( X(\tau) \) such that after a continuous and monotone increasing change of parameter \( t = \tau(t) \), \( t \in [a, b] \) one has

$$|g(t) - X(\tau(t))| < \varsigma.$$
This means, in particular, that the path of the disc is contained in the $\varepsilon$-neighborhood of the curve. Obviously, the disc radius $r$ must be smaller than $\varepsilon$.

Notice that we the curve $g$ is not necessarily injective: self-intersections and even coincidence of some fragments of the curve are allowed.

The following auxiliary theorem states that any broken line can be approximated by trajectories of rough discs. Namely, let $G(t), t \in [a, b]$ be a parameterized broken line with a finite number of segments, $\Gamma = \{G(t) : t \in [a, b]\}$, Figures 6, 7. The left-hand side of this figure represents the curve $g$, we are approximating; the right hand side represents broken line $\Gamma$.

Self-intersections are allowed, but we require that no vertex of the broken line is a point of intersection. Moreover, we approximate broken lines so that inclinations of every segment with respect to the previous one varies from $-\pi/4$ to $0$. For instance, instead a rotation by the angle $\pi/4$, we apply seven rotations by $-\pi/4$. Here all angles are counted in counterclockwise direction, Figure 7.

![Approximated curves and broken lines. Turns to the left.](image)

**Theorem 3.** For any $\varsigma > 0$ and any broken line, satisfying mentioned properties of self-intersection and parameterized by $G(t)$, there exists a motion of a rough disc with a radius $r > 0$ and with the coordinate of center $X(\tau)$ such that after a continuous and monotone increasing change of parameter $\tau = \tau(t), t \in [a, b]$ an analog inequality (34) is satisfied:

$$|G(t) - X(\tau(t))| < \varsigma. \tag{35}$$

Theorem 2 is an obvious consequence of Theorem 3. Indeed, each rectifiable curve can be uniformly approximated by broken lines, and each broken line can be uniformly approximated by trajectories of rough discs. Then we use the method of diagonal sequence to choose a sequence of trajectories converging to the curve. Later on, we suppose without loss of generality that $a > 0$.

13 Proof of Theorem 3

First we notice that a curve homothetic to the trajectory of a rough disc is also the trajectory of a rough disc. More precisely, let $X(t)$ be the motion of the center of a rough disc of radius $r$, let $\omega(t)$ be its angular velocity and $\kappa$ a positive constant. Then the coordinate of the center of a disc of radius $kr$ homothetic to the original one moving in the same medium, with the initial velocity $\kappa X'(0)$ and initial angular velocity $\kappa \omega(0)$, is given by $\kappa X(t)$, and its angular velocity is $\kappa \omega(t)$.

This scaling argument allows one to reduce Theorem 3 to the problem of approximation of the family of broken lines $\frac{\varepsilon}{2}g(t)$ as $\varepsilon \to 0$. Here $\varepsilon > 0$ is a small parameter which later on will be related with the...
parameter $\varsigma$ from the condition of Theorem 2 and with the mentioned parameter $\kappa$. Select a splitting of the broken line into segments with ends, corresponding to $a = T_0 < T_1 < \ldots < T_{m-1} < T_m = b$.

Take a disc $B_{n_\varepsilon}$ with the roughness of the considered form and assume that the measure in $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ associated with the cavity which has the density

$$\frac{1}{2} \cos x \{ \delta(x - y) \cdot \chi_{J \cup J'}(x) + \delta(x + y) \cdot [1 - \chi_{J \cup J'}(x)] \} \, dx \, dy. \quad (36)$$

if $|x|, |y| \leq 5\pi/14$. Here

$$J = \bigcup_{i=1}^m J_{iR} = \bigcup_{i=1}^m \left[ \pi/2 - e^{-T_i/\varepsilon}, \pi/2 - e^{-(T_i + \Delta T_i)/\varepsilon} \right] \quad \text{and} \quad J' = -J, \quad (37)$$

$i = 1, 2, \ldots, m$ is a finite set of indices. We select $J_{iR}$ as "elliptic" segments on the boundary of a cavity (see Section 10 and Figure 5). The initial angular velocity $\lambda(0) = \omega(0)/rV(0)$ is taken to be $\lambda(0) = e^{T_0/\varepsilon}$, $T_i - T_{i-1}$ is the length of the $i$-th segment of the broken line, $\Delta T_i = \varphi_i e^{-T_i/\varepsilon}$, and $\varphi_i$ are parameters, close to angles $\varphi_i^0 \in [-\pi/4, 0]$ (counted clockwise) between the $i$-th and $(i+1)$-th segments of the broken line, Figure 8. We specify these values later. Now we notice that $\varepsilon > 0$ is taken so small that all segments $J_i$ are disjoint.

The left hand side of Figure 8 represents the support of the joint distribution $\nu_B$, corresponding to the selected hollow and the right-hand side gives a sample of corresponding motion. Parts, corresponding to rotations, are drawn in bold. The approximated broken line is dashed. As soon as the disc moves, the relative angular velocity increases and less of the part of the cavity is "observable" by particles. Depending on the value of $\lambda$ either $J \cup J'$ or completion of this set dominate in the "observable" part. Respectively, we have rotation or "almost straight forward" motion. A small part $\varepsilon$ of the boundary is filled with the cavities, and the rest, $1 - \varepsilon$, of the boundary is not filled, that is, is just a union of arcs of the unit circumference. Both parts are uniformly distributed along the boundary.

Figure 8. Billiard responses, broken line and trajectory of the disc.

Consider the natural parametrization $G(\tau)$, $\tau \in [T_0, T_m]$, where the intervals $[T_{j-1}, T_j]$ parameterize segments of the broken line. We are going to find a motion of a rough disc of unit radius with the position of the center $X(\tau)$, and values $S_j$ ($j = 0, \ldots, m$), $\tau \in [S_0, S_m]$ such that

$$|G(\tau) / \varepsilon - X(\tau / \varepsilon)| < (m + 1) / \sqrt{\varepsilon} \quad (37)$$

or, equivalently, $|G(\tau) - \varepsilon X(\tau / \varepsilon)| < (m + 1) \sqrt{\varepsilon}$. Given $\varsigma$, we take $\varepsilon$ so that $\varsigma = (m + 1) \sqrt{\varepsilon}$. Then we take the magnification parameter $\kappa = \varepsilon$ and easily obtain inequality (35).
The motion of the disc will be described in terms of the parameter \( \tau \) proportional to the natural one (see (18)). It can be deduced from equations (15) and (19) and from equations defining the measure (36), (37) that the differential equation for \( \lambda(\tau) \) reduces to the form

\[
\lambda' = \lambda u(\lambda, \varepsilon, \tau)
\]

where \( u(\lambda, \varepsilon, \tau) \Rightarrow 1 \) as \( \varepsilon \to 0 \), whence \( \lambda = e^{w(\varepsilon, \tau)} \) where \( w \) is increasing with respect to \( \tau \) and \( w(\varepsilon, \tau)/\tau \Rightarrow 1 \). This is true since dependence of solutions on initial data is continuous uniformly with respect to time. Consider values \( S_j \) defined by equalities \( w(\varepsilon, S_j) = T_j/\varepsilon \).

Using equations (16), (17) and (19), introduce the notion \( x_0 = x_0(\lambda) = \arccos(1/\lambda) \), and obtain the equality

\[
1/2 \int_{x_0}^{\pi/2} c_T(x, -x, \lambda) \cos x \, dx = 0
\]

(recall that the function \( c_T \) is defined by (17)). This means that the component, orthogonal to the current velocity, of the force acting on a smooth (without roughness) disc is zero. So we obtain

\[
\theta'(\tau) = -\varepsilon R_T(\lambda(\tau)),
\]

where

\[
R_T(\lambda) = 1/2 \int_{|x_0, \pi/2| \cap \tilde{J}} (c_T(x, x, \lambda) - c_T(x, -x, \lambda)) \cos x \, dx,
\]

with

\[
c_T(x, x, \lambda) - c_T(x, -x, \lambda) = \frac{3 \sin x}{\sin \zeta} \left\{ (\lambda^3 \sin^2 x + 3 \lambda \sin x \sin^2 \zeta) \cos \zeta \sin x + (3 \lambda^2 \sin^2 x \sin \zeta + \sin^3 \zeta) \sin \zeta \cos x \right\}
\]

and \( \zeta = \zeta(x) = \arccos(\lambda \cos x) \). After some algebra we get

\[
c_T(x, x, \lambda) - c_T(x, -x, \lambda) = \frac{3 \sin x \cos \zeta}{\lambda \sin \zeta} \left\{ (\lambda^2 - \cos^2 \zeta)^2 + 6 \sin^2 \zeta (\lambda^2 - \cos^2 \zeta) + \sin^4 \zeta \right\}.
\]

Making the change of variable \( x \to \zeta \) in the integral (39), we obtain

\[
R_T(\lambda) = \int_{[0, \pi/2] \cap \tilde{J}} \frac{3}{2 \lambda^3} \left\{ (\lambda^2 - \cos^2 \zeta)^2 + 6 \sin^2 \zeta (\lambda^2 - \cos^2 \zeta) + \sin^4 \zeta \right\} \cos^2 \zeta \, d\zeta,
\]

where

\[
\tilde{J} = \bigcup_{j=0}^{m-1} [\zeta_j, \zeta_j + \Delta \zeta_j],
\]

with \( \zeta_j = \arccos(\lambda e^{-S_j}) \), \( \zeta_j + \Delta \zeta_j = \arccos(\lambda e^{-w_{-1}(T_j + \Delta_j)/\varepsilon}) \). Notice that the expression \( \{ \ldots \} \) in the integral in the right hand side of (40) can be estimated as \( \{ \ldots \} = \lambda^4 + O(\lambda^3) \) for large values of \( \lambda \).

Substituting \( \lambda = e^{w(\varepsilon, \tau)} \), one obtains

\[
\zeta_j = \arccos(e^{w(\varepsilon, \tau) - T_j/\varepsilon}) \quad \text{and} \quad \Delta \zeta_j = \frac{e^{w(\varepsilon, \tau) - T_j/\varepsilon} - 1}{e^{2w(\varepsilon, \tau) - 2T_j/\varepsilon} - 1} \frac{\Delta j}{\varepsilon} (1 + o(1))
\]
where \( o_\varepsilon(1) \to 0 \) as \( \varepsilon \to 0 \). The value of \( \varepsilon R_T(\lambda) \) can now be evaluated as

\[
\varepsilon R_T(\lambda) = \varepsilon \frac{3}{2\lambda^3} (\lambda^4 \cos^2 \frac{\zeta_j}{\varepsilon} \Delta \xi + \hat{R}_j^0(\lambda, \varepsilon)) = \\
\varepsilon \frac{3\lambda}{2} e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon \\
\frac{e^{\omega(\tau, \tau)} - T_j/\varepsilon}{\sqrt{1 - e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon}} \Delta \xi + \hat{R}_j^0(\tau, \varepsilon) \frac{3\varphi_j}{2} \\
\varepsilon \frac{e^{4\omega(\tau, \tau)} - 4T_j/\varepsilon}{\sqrt{1 - e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon}} + \hat{R}_j^1(\tau, \varepsilon).
\]

Here \(|\hat{R}_j^0(\lambda, \varepsilon)| \leq C\lambda^3\) where \( C \) is a constant; \( \hat{R}_j^1(\tau, \varepsilon) \) tends to zero as \( \lambda(\tau) \to \infty, \varepsilon \to 0 \). Thus, we come to the following differential equation for \( \theta(\tau) \),

\[
\frac{d\theta}{d\tau} = \frac{3\varphi_j}{2} e^{\omega(\tau, \tau)} - 4T_j/\varepsilon \\
\varphi_j \frac{e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon}{\sqrt{1 - e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon}} + \hat{R}_j^1(\tau, \varepsilon), \quad \text{if} \quad \tau \in [S_j, S_{j+1} - 1/\sqrt{\varepsilon}].
\]

Solutions for this equation are

\[
\theta(\tau) = \theta(S_j) + \varphi_j [1 - \sqrt{1 - e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon} \left(1 + \frac{1}{2} e^{2\omega(\tau, \tau)} - 2T_j/\varepsilon\right)] + \hat{R}(\xi, \tau), \quad (41)
\]

\[
\text{if} \quad \tau \in [S_j, S_{j+1}]; \quad j = 0, \ldots, m - 1.
\]

Here \(|\hat{R}(\xi, \tau)| \leq \sqrt{\varepsilon}\) if \( \varepsilon \) is sufficiently small. The function \( \theta \) is increasing with respect to \( \tau \) and with respect to each parameter \( \varphi_j \). So, we can select all \( \varphi_j \) so that \( \theta(S_{j+1}) - \theta(S_j) = \varphi_j^0 \). Thus, any part of the trajectory \( X([S_j, S_{j+1} - 1/\sqrt{\varepsilon}]) \) \((j \geq 1)\) is an arc, close to a line segment of length

\[
\frac{T_{j+1} - T_j}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}}.
\]

Let \( L \) be the length of the curve \( g, \theta_0(\tau) \) be the piecewise constant function, equal to 0 on \([S_0, S_1]\) and equal to \( \varphi_1^0 + \ldots + \varphi_{j-1}^0 \) on \([S_{j-1}, S_j]\). Then for any \( \tau \in [0, L] \) we have

\[
\frac{X(\tau/\varepsilon) - \frac{1}{\varepsilon}g(\tau)}{\varepsilon} \leq \int_{S_0}^{\tau/\varepsilon} |\varphi(s) - \varphi_0(s)| ds. \quad (42)
\]

For \( \tau \in [\varepsilon S_j, \varepsilon S_{j+1} - \sqrt{\varepsilon}] \) the velocity vector \( X'(t/\varepsilon) \) forms an angle \( O(e^{-1/\sqrt{\varepsilon}}) \) with the \( j \)-th segment of the broken line. This follows from representations (41). On the other hand, contributions of any segment \([\varepsilon S_j - \sqrt{\varepsilon}, \varepsilon S_{j+1}]\) to the right hand side of (42) are estimated by \( \varepsilon^{-1/2} \). So, we have inequality (37) satisfied if \( \varepsilon \) is small. \( \square \)

14 Conclusion and discussion

The main results of this paper are the following.

1. We have described motions of bodies in rarified media without Brownian motion. In a certain sense, the introduced models generalize ones given at [16]. Constructed models are very general. Particularly, they work for any dimension.
Two-dimensional trajectories of bodies, whose boundaries are close to circles, may have (up to magnification) any shape. The same statement is true for flat curves in the three dimensional real space. The question of approximation of rectifiable three dimensional curves still remains open because there the rotational velocity is a vector value which is much harder to be controlled.

Apart from this the following results of the paper are principally novel.

1. A description of amphora billiard (quasi-elastic reflector) and its modifications with a wide variety of response functions have been given (Sections 9 and 10).

2. Simplified models for dynamics of some classes of rough bodies (so-called perfect discs) has been described and asymptotic estimates for its principal characteristics have been written down (Sections 11 and 13).

However, we would like to notice that our construction, being mathematically correct, cannot be implemented in practice. First, we make the (impossible) assumptions that the medium temperature is absolute zero, the particles of the medium do not collide, and (even worse) the collisions of the particles with the body boundary are perfectly elastic. Second, even if all these assumptions are satisfied, each cavity should be fabricated with exceptionally high precision, the scale of precision being much smaller than the size of atoms. Third, the path traversed by a disc is proportional to the logarithm of time. Roughly speaking, it may happen that the first meter of the trajectory is traversed in a second, the second meter in a minute, the third meter in an hour, ..., the tenth meter in a billion of years. The experimenter may just not survive the end of the experiment.

Imagine a football player who wants to send the ball so that the trajectory goes round all the players of the rival team and finally gets into the gate. He can indeed do so making use of our results, but the ball surface should be very special; the pressure of the atmosphere should be very low; the Earth gravitation should be negligible; the rival players should be asked not to prevent the (eventually very slow) motion of the ball. And it remains to wait. Oh, forgot to say that all this should happen in two dimensions.

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