Invariant submodel of rank 1 and two families of exact solutions of gas dynamics equations with an equation of state of the special form

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Abstract. In this article, the gas dynamics equations with an equation of state of the special form are considered. The equation of state is the pressure which is equal to the sum of two functions, with one being a function of a density, and the other one being a function of an entropy. The system of equations is invariant under the action of 12-parameter transformations group. For three-dimensional subalgebra 3.32 of the 12-dimensional Lie algebra invariants are calculated, an invariant submodel of rank 1 is constructed, and two families of exact solutions are obtained. The obtained solutions specify the motion of particles in space with a linear velocity field with inhomogeneous deformation. The first family of solutions has two moments of time of particles collapse. The second family of solutions has one moment of time of particles collapse on the plane. In the simplest case of second family of solutions, a surface consisting of particle trajectories is constructed.

1. Introduction
The gas dynamics equations with an arbitrary equation of state using group analysis methods were first studied by L.V. Ovsyannikov [1]. He announced the ”Podmodeli” program for studying of the gas dynamics equations, which includes the following items:

• group classification by an arbitrary element;
• calculation of the admissible Lie algebra;
• computation of the optimal system of dissimilar subalgebras;
• tracing the embeddings of all subalgebras of the optimal system (hierarchy of submodels);
• construction of invariant, partially invariant submodels;
• analysis of submodels, including by symmetry methods;
• obtaining of group solutions;
• investigation of the behavior of particles and identification of the features of group solutions.

An 11–parameter transformations group are allowed by the gas dynamics equations with an arbitrary equation of state. L.V. Ovsyannikov singled out special forms of the equations of state, expanding the admissible transformations group, for instance, the equation of state of a polytropic gas; a monatomic gas; an equation of state with a separated density; the pressure being the sum of the power function of a density and the function of an entropy; etc. In this
article, we consider the gas dynamics equations with an equation of state of the special form — pressure is equal to the sum of two functions, one of which depends on a density, and the other depends on an entropy. In contrast to the gas dynamics equations with an arbitrary equation of state, the system is invariant under the action of a 12-parameter transformations group, which consists of translation of time, translations of space, Galilean translations, rotations around the coordinate axes, space-time uniform dilatation and pressure translation. The optimal system of dissimilar subalgebras of the admitted 12-dimensional Lie algebra is known [2]. Submodels for two-dimensional subalgebras were classified [3, 4, 5], exact solutions were obtained for an invariant submodel of rank 2 [6]. Some exact solutions were obtained for 3-dimensional subalgebras [7]. In this article, for the 3-dimensional subalgebra from 12-dimensional Lie algebra consisting of operators of space translations, Galilean translations and pressure translation, an invariant submodel of rank 1 is constructed and exact solutions are found that define particles motion in space with a linear velocity field with inhomogeneous deformation. The moments of collapse of particles for each solution are found.

2. The main formulas and definitions
The gas dynamics equations are as follows [8]:

\[ \begin{align*}
D\vec{u} + \rho^{-1}\nabla p &= 0, \\
D\rho + \rho \text{div}\vec{u} &= 0, \\
Dp + \rho f_{\rho} \text{div}\vec{u} &= 0,
\end{align*} \]

(1)

where the total differentiation operator has the form

\[ D = \partial_t + (\vec{u} \cdot \nabla); \]

t is the time, \( \nabla = \partial_x \) is the gradient with respect to the spatial independent variables \( \vec{x} \), \( \vec{u} \) is the velocity vector, \( \rho \) is the density, and \( p \) is the pressure.

In the Cartesian coordinate system we have [9]

\[ \begin{align*}
\vec{x} &= x\vec{i} + y\vec{j} + z\vec{k}, \\
\nabla &= \vec{i}\partial_x + \vec{j}\partial_y + \vec{k}\partial_z, \\
\vec{u} &= u\vec{i} + v\vec{j} + w\vec{k},
\end{align*} \]

where \( \vec{i}, \vec{j}, \) and \( \vec{k} \) is an orthonormal basis.

The equation of state has special form [1]

\[ p = f(\rho) + h(S), \quad f = \rho^2 F'(\rho), \]

(2)

where \( S \) is entropy. The thermodynamic parameters of an ideal medium, the specific internal energy and temperature, have the form

\[ \begin{align*}
T &= g'(S) - \rho^{-1}h'(S), \\
\varepsilon &= F(\rho) - \rho^{-1}h(S) + g(S).
\end{align*} \]

The system (1), (2) has an equivalence transformation for the function \( h(S) \). Let us \( h(S) \) be \( S \).

The last equation of system (1) can be replaced by the equation for entropy:

\[ DS = 0. \]
The equations (1) with the equation of state (2) are invariant under the action of the Galilean group extended by the space-time uniform dilatation and pressure translation:

1. \( \vec{x}' = \vec{x} + \vec{a} \) (space translations);
2. \( t' = t + a_0 \) (time translation);
3. \( \vec{x}' = O\vec{x}, \vec{u}' = O\vec{u}, OO^T = E, \det O = 1 \) (rotations);
4. \( \vec{x}' = \vec{x} + t\vec{b}, \vec{u}' = \vec{u} + \vec{b} \) (Galilean translations);
5. \( t' = ct, \vec{x}' = c\vec{x} \) (uniform dilatation);
6. \( p' = p + p_0 \) (pressure translation).

The transformations group (3) is associated with the 12-dimensional Lie algebra \( L_{12} \). The basis operators of Lie algebra \( L_{12} \) are written in the Cartesian coordinate system:

\[
\begin{align*}
X_1 &= \partial_x, & X_2 &= \partial_y, & X_3 &= \partial_z, \\
X_4 &= t\partial_x + \partial_u, & X_5 &= t\partial_y + \partial_v, \\
X_6 &= t\partial_z + \partial_w, \\
X_7 &= y\partial_z - z\partial_y + v\partial_w - w\partial_v, \\
X_8 &= z\partial_x - x\partial_z + w\partial_u - u\partial_w, \\
X_9 &= x\partial_y - y\partial_x + u\partial_v - v\partial_u, & X_{10} &= \partial_t, \\
X_{11} &= t\partial_t + x\partial_x + y\partial_y + z\partial_z, & Y_1 &= \partial_p.
\end{align*}
\]

A function that vanishes under the action of the operators of a subalgebra is called an invariant of the subalgebra [10]. Invariants can be chosen in an infinite number of ways, since any function of invariants is again an invariant. The representation of the invariant solution is selected from the invariants. In this case, some of the invariants (from 0 to 3) can only consist of independent variables. Each of the remaining invariants contains one gas-dynamic function. Such invariants are assigned as new functions of invariants consisting of independent variables (or assigned as constants). After substituting the representation of the solution into system (1), (2), an invariant submodel is obtained. The rank of a submodel is the number of independent variables.

3. Invariant submodel and exact solutions

The basis operators of the subalgebra \( 3.32 \) [2] in the Cartesian coordinate system have the form

\[
\begin{align*}
aX_1 + X_2 &= a\partial_x + \partial_y, & X_4 &= t\partial_x + \partial_u, \\
Y_1 + dX_1 + eX_3 + nX_5 + lX_6 &= \partial_p + d\partial_x + (e + lt)\partial_z + nt\partial_y + n\partial_v + l\partial_w, \\
d^2 + e^2 + n^2 + l^2 &= 1.
\end{align*}
\]

The invariants of subalgebra (4) for \( e^2 + l^2 \neq 0 \) have the form

\[
t, \quad u - \frac{x}{t} + a\frac{y}{t} + \frac{d - an}{t(e + lt)}z, \quad v - \frac{n}{e + lt}z, \quad w - \frac{l}{e + lt}z, \quad \rho, \quad p - \frac{z}{e + lt}.
\]
The representation of an invariant solution is

\[ u = u_1(t) + \frac{x}{t} - a \frac{y}{t^2} + \frac{atn - d}{t(e + lt)} z, \]
\[ v = v_1(t) + \frac{n}{e + lt} z, \]
\[ w = w_1(t) + \frac{e + lt}{l} z, \]
\[ \rho = \rho(t), \quad p = p_1(t) + \gamma \frac{z}{e + lt}, \]
\[ S = S_1(t) + \gamma \frac{z}{e + lt}, \]

where the coefficient \( \gamma \) was added for clear distinction between the 3-dimensional submodels of 11-dimensional and 12-dimensional Lie algebras, i.e. \( \gamma = 0 \) in the case of \( L_{11} \), while \( \gamma = 1 \) in the case of \( L_{12} \).

The invariant submodel is as follows

\[ u_{1t} = -\frac{u_1}{t} + \frac{a}{t} v_1 + d - atn \frac{w_1}{t(e + lt)}, \]
\[ v_{1t} = -\frac{n}{e + lt} w_1, \]
\[ w_{1t} = -\frac{l}{e + lt} w_1 - \gamma \frac{\rho}{e + lt}, \]
\[ \rho_t = -\frac{2lt}{(e + lt)} \rho, \]
\[ S_{1t} = -\frac{\gamma}{e + lt} w_1, \]
\[ p_1 = f(\rho) + S_1. \]  

(5)

For \( l \neq 0 \) solution of (5) is

\[ u = \frac{x + x_0}{t} - a \frac{y - v_0 t}{t^2} + \frac{an - d}{t(e + lt)} z - \frac{cen + dl}{l^2(e + lt)t} w_0 + \]
\[ + \frac{\gamma}{6\rho_0 l^3 t} \left[ \frac{ae^4 n - dl^4 l^3}{l(e + lt)} + de^2 + aln(lt - e)t^2 \right], \]
\[ v = \frac{n(lz + w_0)}{l(e + lt)} + \frac{\gamma n}{6\rho_0} \left[ \frac{l^3}{e + lt} - \frac{e^2}{l^3} \right] + v_0, \]
\[ w = \frac{lz + w_0}{e + lt} - \frac{t^2}{6\rho_0} \left[ 2 + \frac{e}{e + lt} \right], \]
\[ \rho = \frac{t(e + lt)}{l^3}, \]
\[ S = \frac{\gamma^2}{6\rho_0} \left[ \frac{l^3}{e + lt} - \frac{e^2}{l^3} \right] + \gamma \frac{w_0 + lz}{l^2(e + lt)} + S_0, \]
\[ p = S + f \left( \frac{\rho_0}{t(e + lt)} \right); \]

(6)
and for $l = 0$ the solution of (5) is

$$
\begin{align*}
&u = \frac{x + x_0}{t} - \frac{a y - v_0 t}{t} + \frac{ant - d}{et} z + \frac{\gamma}{6e^2 \rho_0} (ant - d) t^2 - \frac{ant - d}{e} w_0, \\
v = \frac{n}{e} (z - w_0 t) + \frac{\gamma n t^3}{6e^2 \rho_0} + v_0, \\
p = \frac{\rho_0}{t}, \\
S &= \frac{\gamma}{e} (z - w_0 t) + \frac{\gamma^2 t^3}{6e^2 \rho_0} + S_0, \\
p &= S + f \left( \frac{\rho_0}{t} \right).
\end{align*}
$$

(7)

In the solution (6) the inessential constants vanish

$$
x_0 = v_0 = w_0 = S_0 = 0
$$

under the action of space translations (3) with parameters

$$
\vec{a} = (x_0 - anw_0/l^2, 0, w_0/l),
$$

under the action of pressure translation (3) with $p_0 = -S_0$ and under the action of Galilean translations (3) with

$$
\vec{b} = (0, -v_0, 0).
$$

In the solution (7) the inessential constants vanish

$$
x_0 = v_0 = w_0 = S_0 = 0
$$

under the action of Galilean translations (3) with

$$
\vec{b} = (0, -v_0, -w_0)
$$

and under the action of the space translations (3) with parameters

$$
\vec{a} = (x_0, 0, 0).
$$

In the solutions (6), (7) in entropy $S$ the constant $S_0$ is zeroed out not by transformations (3), but by introducing the variable $\tilde{S} = S - S_0$.

The solution (6) has two times of particles collapse at $t = 0$ and at $t = -e/l$. The solution (7) has time of particles collapse at $t = 0$.

4. Particles motion

We investigate the motion of particles for solution (7). Particles motion is given by the equation [8]:

$$
\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}, t).
$$

(8)

The integral curves of the equation (8) are the world lines of particles in the space $\mathbb{R}^4(t, \vec{x})$, the projection of which to $\mathbb{R}^3(\vec{x})$ are particles trajectories. The world lines of particles for the solution (7) are given by the formulas:

$$
\begin{align*}
x &= C_1 t + a C_2 + \frac{d}{e} C_3, \\
y &= \frac{n}{e} C_3 t + C_2, \\
z &= -\gamma \frac{t^3}{6e \rho_0} + C_3.
\end{align*}
$$

(9)
where \( C_1, C_2, C_3 \) are global Lagrangian coordinates. \( C_1 \) is the particle velocity along the x axis, \( C_2 \) is the projection of the particle onto the y axis at \( t = 0 \), \( C_3 \) is the projection of the particle onto the z axis at \( t = 0 \).

Jacobian of transformation (9) is equal to \( t \). The rank of the Jacobian matrix at the time of particles collapse \( t = 0 \) is equal to 2. The surface of particles collapse is the plane \( x - ay - zd = 0 \).

The motion of particles throughout space is vortex motion since \( \vec{\omega} = \left( -\frac{a}{e}, \frac{ant - d}{et}, \frac{a}{t} \right) \).

Let (9) satisfy
\[
\begin{align*}
  x(t_0) &= x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0; \\
  x_0, y_0, z_0 \text{ are local Lagrangian coordinates, then}
\end{align*}
\]
\[
x = \left[ \frac{\gamma t_0^3}{6\rho_0 e^2} (ant_0 - d) + \frac{x_0 - ay_0}{t_0} + \frac{ant_0 - d}{et_0} z_0 \right] t - \frac{ant_0 - d}{6e^2\rho_0} \gamma t_0^3 + ay_0 - \frac{ant_0 - d}{e} z_0, \\
y = \left( \frac{\gamma nt_0^3}{6e^2\rho_0} + \frac{n z_0}{e} \right) t - \frac{nt_0^4}{6e^2\rho_0} \gamma - \frac{n z_0 t_0}{e} + y_0, \\
z = -\frac{\gamma t_0^3}{6e\rho_0} + \frac{\gamma t_0^3}{6e\rho_0} + z_0.
\]

Let the particles with trajectories (10) at the moment of time \( t_0 = 1 \) be on a circle \( x^2 + y^2 = 1 \) at the plane \( z = 0 \). The surface of particle trajectories (10) at \( t = 1 \ldots 2, x_0 = \cos \varphi, y_0 = \sin \varphi, z_0 = 0, \varphi = 0 \ldots 2\pi, a = e = n = 1, d = 0, \rho_0 = 1/6 \) are shown in figure 1.

\[\text{Figure 1. The surface of particle trajectories (10) at } t = 1 \ldots 2, \varphi = 0 \ldots 2\pi. \text{ The two curves are trajectories with initial conditions } x(1) = 0, y(1) = \pm 1.\]
5. Conclusion
Thus, for the three-dimensional subalgebra $3.32$ of the 12-dimensional Lie algebra admitted by the gas dynamics equations with pressure in the form of the sum of the density and entropy functions, invariants are calculated, an invariant submodel of rank 1 is constructed, and two families of exact solutions are obtained. The obtained solutions specify the motion of particles in space with a linear velocity field with inhomogeneous deformation. The first family of solutions has two moment of time of particles collapse. The second family of solutions has one moment of time of particles collapse on the plane. In the simplest case of second family of solutions, a surface consisting of particle trajectories is constructed.

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