Analytic Cauchy Problem for the $\mu$-Camassa-Holm Equation and Its Non-Quasilinear Version

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Abstract. We solve the Cauchy problems for the $\mu$-Camassa-Holm integro-partial differential equation of Khesin-Lenells-Misiołek and its non-quasilinear version introduced by Qu-Fu-Liu in the complex-analytic framework. These equations have nonlocal nature at two levels: they involve a pseudo-differential operator of negative order and this operator is defined in terms of an integral of the unknown function. We prove unique solvability of the Cauchy problems and provide an estimate of the lifespan of the solutions. Our method is the Ovsyannikov type argument by Batrostichi-Himonas-Petronilho about a scale of Banach spaces of analytic functions, but the non-quasilinearity is dealt with in a different way from theirs. Indeed, we use a simple reduction which is just a nonlocal version of a classical trick.

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1. Introduction

The Camassa-Holm equation

\[ u_t - u_{txx} = -3u u_x + 2u_x u_{xx} + u u_{xxx} \]

was introduced by [3] and [5] in different contexts. It is known to be completely integrable and admits peaked soliton (peakon) solutions. The Cauchy problem can be formulated for this equation by introducing a pseudo-differential operator. Indeed, it can be written in the form

\[ (1.1) \quad u_t = -\frac{1}{2}(u^2)_x - \partial_x(1 - \partial_x^2)^{-1}\left[u^2 + \frac{1}{2}u_x^2\right]. \]

There are a lot of works about well-posedness of its Cauchy problem in Sobolev spaces. See [1] and the references therein. Since (1.1) is Kowalevskian in a generalized sense, it is natural to solve this equation in the analytic setting as in the classical Cauchy-Kowalevsky theorem. However, in view of the nonlocal nature of the pseudo-differential operator $(1 - \partial_x^2)^{-1}$, the Cauchy problem must be global in space and local in time only. This is a great difference from the classical theorem. In [1], the authors introduced a kind of Sobolev spaces with exponential weights consisting of functions holomorphic in a strip of the type $|\text{Im} \ z| < \text{const}$. Since these spaces form a scale of Banach spaces, an Ovsyannikov type argument, which gives an abstract version of the Cauchy-Kowalevsky theorem, can be applicable. It leads to unique solvability and an estimate of the lifespan of the solution in the periodic and non-periodic cases. Continuous dependence of the solution on the Cauchy data follows as well.

Key words and phrases. Ovsyannikov theorem for nonlocal equations, Camassa-Holm equations, analytic Cauchy problem.

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In [6], a $\mu$-version of (1.1), namely
\begin{equation}
\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \ x \in S^1 = \mathbb{R}/\mathbb{Z},
\end{equation}
was introduced. Here $u = u(t,x)$ is a time-dependent function on the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{S^1} u \, dx$ is its mean. In [6], the authors call this equation $\mu$HS (HS is for Hunter-Saxton), while it is called $\mu$CH in [7]. We have $\mu(u_t) = 0$, but we keep $\mu(u_t)$ because this formulation facilitates later calculation. Set $A(\varphi) = \mu(\varphi) - \varphi_{xx}$. Then it is invertible for a suitable choice of function spaces and commutes with $\partial_x$. The equation (1.2) can be written in the following form ([6, (5.1)]):
\begin{equation}
\begin{align*}
  u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right] = 0.
\end{align*}
\end{equation}

There are similar $\mu$-equations. In [7, (5.3)], the following equation, called $\mu$DP (DP is for Degasperis-Procesi), was introduced:
\begin{equation}
\begin{align*}
  u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1} [3\mu(u)u] = 0.
\end{align*}
\end{equation}

In [8, (3.2)], the modified $\mu$-Camassa-Holm equation (modified $\mu$CH) with non-quasilinear terms
\begin{equation}
\begin{align*}
  u_t + \mu(u)(u^2)_x - \frac{1}{3}u_x^3 + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)u_x^2 \right] + \frac{1}{3}\mu(u_x^3) = 0
\end{align*}
\end{equation}
was introduced. All these $\mu$-equations are integrable and admit peakon solutions. Local well-posedness is known in Sobolev or Besov spaces. In the present paper, we study the Cauchy problems for (1.3), (1.4) and (1.5) in the analytic category by using the method of [1]. The difficulty of (1.5) lies in the presence of the non-quasilinear terms $-\frac{1}{3}u_x^3$ and $-\frac{1}{3}\mu(u_x^3)$. It seems that they are out of the framework of [1]. In [2], the authors employed the power series method to deal with the non-quasilinear term $au^k - 2u_x^3$ of the $k$-abc-equation. In the present paper, we overcome the difficulty of non-quasilinearity by a $\mu$-version of a classical trick used in the proof of the Cauchy-Kowalevsky theorem ([4]). We set $v = u_x$ and differentiate (1.5) in $x$. It can be proved that (1.5) is equivalent to the following quasilinear modified $\mu$CH system:
\begin{equation}
\begin{align*}
  \begin{cases}
    u_t + 2\mu(u)uv - \frac{v^3}{3} + \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 \right] + \frac{\mu(v^3)}{3} = 0, \\
    v_t + 2\mu(u)(uv)_x - \frac{(v^3)_x}{3} + \partial_x A^{-1} \left[ 2\mu^2(u)v + \mu(u)(v^2)_x \right] = 0.
  \end{cases}
\end{align*}
\end{equation}

Of course, this trick works for the the $k$-abc-equation as well. We can prove well-posedness of the Cauchy problem for (1.6) and (1.5), as well as for (1.3) and (1.4), and get an estimate of the lifespan of the solution.
2. Function spaces and operators

For a function on $S^1 = \mathbb{R}/\mathbb{Z}$, we set $\hat{\varphi}(k) = \int_{S^1} \varphi(x)e^{-2k\pi x} \, dx$. Following [1,2], we introduce a family of Hilbert spaces $G^{\delta,s}$ ($\delta > 0, s \geq 0$) by

$$
G^{\delta,s} = \{ \varphi \in L^2(S^1); \| \varphi \|_{\delta,s} < \infty \}, \quad \| \varphi \|_{\delta,s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{2\delta|k|} |\hat{\varphi}(k)|^2,
$$

where $\langle k \rangle = (1 + k^2)^{1/2}$. It is easy to see that we have continuous injections $G^{\delta,s} \to G^{\delta',s}$ and $G^{\delta,s} \to G^{\delta,s'}$ if $0 < \delta' < \delta, 0 < s' < s$. Their norms are 1. We recall some more facts in [1,2]. Since the base space is $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$ in these papers, the following statements have some minor differences from those in them.

**Lemma 1.** Let $\varphi \in G^{\delta,s}$. Then this periodic function on $\mathbb{R}_x$ has an analytic extension to $\{ x + iy \in \mathbb{C}; |y| < \delta/(2\pi) \}$.

**Lemma 2.** Assume $s > 1/2$. Then $G^{\delta,s}$ is closed under pointwise multiplication and we have

$$
\| \varphi \psi \|_{\delta,s} \leq c_s \| \varphi \|_{\delta,s} \| \psi \|_{\delta,s},
$$

where $c_s = \left[ 2(1 + s^{2s}) \sum_{k=0}^{\infty} \langle k \rangle^{-2s} \right]$. 

**Proof.** The proof is given in [2]. Although different formulations are used in [2] and the present article, the constant $c_s$ is the same. This is because $\varphi \psi(k) = \sum_n \hat{\varphi}(n) \hat{\psi}(k - n)$ holds in either situations. The difference of the base spaces are offset by that of conventions, namely the presence or the absence of the $1/(2\pi)$ factor in the definition of the Fourier coefficients. \qed

**Lemma 3.** If $0 < \delta' < \delta \leq 1, s \geq 0$ and $\varphi \in G^{\delta,s}$, then

$$
\| \varphi_x \|_{\delta',s} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \| \varphi \|_{\delta,s},
$$

$$
\| \varphi_x \|_{\delta,s} \leq 2\pi \| \varphi \|_{\delta,s+1}.
$$

We set $A(\varphi) = \mu(\varphi) - \varphi_{xx}$. For $\varphi = \sum_{k \in \mathbb{Z}} a_k e^{2k\pi i x} \in G^{\delta,s}$, we have

$$
\mu(\varphi) = a_0,
$$

$$
A(\varphi) = a_0 + \sum_{k \neq 0} 4\pi^2 k^2 a_k e^{2k\pi i x},
$$

$$
A^{-1}(\varphi) = a_0 + \sum_{k \neq 0} \frac{a_k}{4\pi^2 k^2} e^{2k\pi i x}.
$$

It follows that $A$ is a bounded operator from $G^{\delta,s+2}$ to $G^{\delta,s}$. It is a bijection and its inverse $A^{-1}$ is a pseudodifferential operator of order $-2$. Therefore it is bounded from $G^{\delta,s}$ to $G^{\delta,s+2}$. But in the present article we only need its boundedness as an operator from $G^{\delta,s}$ to $G^{\delta,s+1}$. Indeed, its property $\text{ord} A^{-1} \leq -1$ is enough to ensure that (1.3), (1.4), (1.5) and (1.6) are Kowalevskian in a generalized sense.

**Lemma 4.** We have $|\mu(\varphi)| \leq \| \varphi \|_{\delta,s}$ and $\| A^{-1}(\varphi) \|_{\delta,s+1} \leq \| \varphi \|_{\delta,s}$ for $\varphi \in G^{\delta,s}$. 

If $\varphi \in G^{\delta,s}$, then by Lemmas 3 and 4, we have

(2.1) $\|\partial_x A^{-1}(\varphi)\|_{\delta,s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \|\varphi\|_{\delta,s}$.

In later sections, we will use the following estimates repeatedly. Let $R > 0$ and $u_0 \in G^{\delta,s+1} \subset G^{\delta,s}$ be given. If $\|u_j - u_0\|_{\delta,s+1} < R$ for $u_j \in G^{\delta,s+1} \subset G^{\delta,s}$ ($j = 1, 2$), then we have

(2.2) $\|u_j\|_{\delta,s} \leq \|u_j\|_{\delta,s+1} \leq \|u_0\|_{\delta,s+1} + R$,
(2.3) $\|u_1 + u_2\|_{\delta,s} \leq \|u_1 + u_2\|_{\delta,s+1} \leq 2 (\|u_0\|_{\delta,s+1} + R)$.

3. Autonomous Ovsyannikov theorem

We recall some basic facts about the autonomous Ovsyannikov theorem. Among many versions, we adopt the one in [1, 2]. Let $\{X_\delta\}_{0 < \delta \leq 1}$ be a scale of Banach spaces, i.e. each $X_\delta$ is a Banach space and $X_\delta \subset X_{\delta'}$, $\|\cdot\|_{\delta'} \leq \|\cdot\|_{\delta}$ for any $0 < \delta' < \delta \leq 1$. (For $s$ fixed, $\{G^{\delta,s}\}_{0 < \delta \leq 1}$ is a scale of Banach spaces.) Assume that $F: X_\delta \to X_{\delta'}$ is a mapping satisfying the following conditions.

(a) For any $u_0 \in X_1$ and $R > 0$, there exist $L = L(u_0, R) > 0, M = M(u_0, R) > 0$ such that we have

(3.1) $\|F(u) - F(v)\|_{\delta'} \leq L \frac{\|u - v\|_{\delta}}{\delta - \delta'},$
(3.2) $\|F(u_0)\|_{\delta} \leq M \frac{1}{1 - \delta}$

for any $0 < \delta' < \delta \leq 1$ and any $u, v \in X_\delta$ with $\|u - u_0\|_{\delta} < R, \|v - u_0\|_{\delta} < R$.

(b) If $u(t)$ is a holomorphic function on the disk $D(0, a(1 - \delta)) = \{t \in \mathbb{C}: |t| < a(1 - \delta)\}$ with values in $X_\delta$ for $a > 0, 0 < \delta < 1$ satisfying $\sup_{|t| < a(1 - \delta)} \|u(t) - u_0\|_{\delta} < R$, then the composite function $F(u(t))$ is a holomorphic function on $D(0, a(1 - \delta))$ with values in $X_{\delta'}$ for any $0 < \delta' < \delta$.

The autonomous Ovsyannikov theorem below is our main tool. For the proof, see [1].

**Theorem 5.** Assume that the mapping $F$ satisfies the conditions (a) and (b). For any $u_0 \in X_1$ and $R > 0$, set

(3.3) $T = \frac{R}{16LR + 8M}$.

Then, for any $0 < \delta < 1$, the Cauchy problem

(3.4) $\frac{du}{dt} = F(u), \ u(0) = u_0$

has a unique holomorphic solution $u(t)$ in the disk $D(0, T(1 - \delta))$ with values in $X_\delta$ which satisfies

$$\sup_{|t| < T(1 - \delta)} \|u(t) - u_0\|_{\delta} < R.$$
4. \( \mu \)CH AND \( \mu \)DP EQUATIONS

First we consider an analytic Cauchy problem for the \( \mu \)CH equation (1.3), namely.

\[
\begin{aligned}
&u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right] = 0, \\
&u(0, x) = u_0(x).
\end{aligned}
\tag{4.1}
\]

**Theorem 6.** Let \( s > 1/2 \). If \( u_0 \in G^{1,s+1} \), then there exists a positive time \( T = T(u_0, s) \) such that for every \( \delta \in (0, 1) \), the Cauchy problem (4.1) has a unique solution which is a holomorphic function valued in \( G^{\delta,s+1} \) in the disk \( D(0, T(1 - \delta)) \). Furthermore, the analytic lifespan \( T \) satisfies the estimate

\[
T \approx \frac{1}{\|u_0\|_{1,s+1}}.
\]

**Proof.** Assume \( \|u - u_0\|_{\delta,s+1} < R, \|v - u_0\|_{\delta,s+1} < R \). By Lemma 2, the first inequality in Lemma 3 and (2.3),

\[
\| (u^2)_x - (v^2)_x \|_{\delta,s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \|u^2 - v^2\|_{\delta,s+1} \]

\[
\leq \frac{2\pi e^{-1}c_{s+1}}{\delta - \delta'} \|u + v\|_{\delta,s+1} \|u - v\|_{\delta,s+1} \]

\[
\leq \frac{4\pi e^{-1}c_{s+1}}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

On the other hand, since we have \( \mu(u)u - \mu(v)v = \mu(u - v)u + \mu(v)(u - v) \), Lemma 4 and (2.2) imply

\[
\| \mu(u)u - \mu(v)v \|_{\delta,s} \leq \| \mu(u - v)u \|_{\delta,s} + \| \mu(v)(u - v) \|_{\delta,s}
\]

\[
\leq (\|u\|_{\delta,s} + \|v\|_{\delta,s}) \|u - v\|_{\delta,s}
\]

\[
\leq 2 (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

By (2.1), we get

\[
\| \partial_x A^{-1} [\mu(u)u - \mu(v)v] \|_{\delta,s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \|\mu(u)u - \mu(v)v\|_{\delta,s}
\]

\[
\leq \frac{4\pi e^{-1}}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

Next, by Lemma 2 and the second inequality in Lemma 3

\[
\|u_x^2 - v_x^2\|_{\delta,s} \leq c_s \|u_x + v_x\|_{\delta,s} \|u_x - v_x\|_{\delta,s} \leq 4\pi^2 c_s \|u + v\|_{\delta,s+1} \|u - v\|_{\delta,s+1}
\]

\[
\leq 8\pi^2 c_s (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]

By using (2.1), we get

\[
\| \partial_x A^{-1} (u_x^2 - v_x^2) \|_{\delta,s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \|u_x^2 - v_x^2\|_{\delta,s}
\]

\[
\leq \frac{16\pi^3 e^{-1}c_s}{\delta - \delta'} (\|u_0\|_{\delta,s+1} + R) \|u - v\|_{\delta,s+1}.
\]
Now set 

\[ F(u) = -\frac{1}{2}(u^2)_x - \partial_x A^{-1} \left[ 2\mu(u)u + \frac{1}{2}u_x^2 \right]. \]

Then (4.2), (4.3) and (4.4) give the Lipschitz continuity of \( F \):

\[ \|F(u) - F(v)\|_{s',s+1} \leq \frac{L}{\delta - \delta'}\|u-v\|_{s,s+1}, \]

where \( L = C'(\|u_0\|_{1,s+1} + R), C = 2\pi e^{-1}(c_{s+1} + 4 + 4\pi^2 c_s) \). Simpler estimates give

\[ \|F(u_0)\|_{s,s+1} \leq \frac{M}{1 - \delta}, \quad M = \frac{C}{2}\|u_0\|_{1,s+1}^2. \]

We set

\[ T = \frac{R}{16LR + 8M} = \frac{R}{4C \left[ 4(\|u_0\|_{1,s+1} + R) R + \|u_0\|_{1,s+1}^2 \right]}. \]

Because of Theorem 4, there exists a unique solution \( u = u(t) \) to (4.1) which is a holomorphic mapping from \( D(0, T(1 - \delta)) \) to \( G_{\delta,s+1} \) and

\[ \sup_{|t| < T(1 - \delta)} \|u(t) - u_0\|_{\delta,s+1} < R. \]

If we set \( R = \|u_0\|_{\delta,s+1} \), we have

\[ T = \frac{1}{72\pi e^{-1}(c_{s+1} + 4 + 4\pi^2 c_s)\|u_0\|_{1,s+1}}. \]

We can study the following Cauchy problem for the \( \mu \)DP equation (1.3) by using the same estimates.

\[ \begin{cases} u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1} [3\mu(u)u] = 0, \\ u(0, x) = u_0(x). \end{cases} \]

**Theorem 7.** Let \( s > 1/2 \). If \( u_0 \in G^{1,s+1} \), then there exists a positive time \( T = T(u_0, s) \) such that for every \( \delta \in (0, 1) \), the Cauchy problem (4.1) has a unique solution which is a holomorphic function valued in \( G^{\delta,s+1} \) in the disk \( D(0, T(1 - \delta)) \). Furthermore, the analytic lifespan \( T \) satisfies the estimate

\[ T \approx \frac{1}{\|u_0\|_{1,s+1}}. \]

5. NON-QUASILINEAR MODIFIED \( \mu \)CH EQUATION

The Cauchy problem for the non-quasilinear modified \( \mu \)CH equation (1.5) can be written in the following form:

\[ \begin{cases} u_t + 2\mu(u)uv - \frac{1}{3}v^3 + \partial_x A^{-1} [2\mu^2(u)u + \mu(u)v^2] + \frac{1}{3}\mu(v^3) = 0, \\ v = u_x, \\ u(0, x) = u_0(x). \end{cases} \]
The non-quasilinear term $-\frac{1}{3}v^3 = -\frac{1}{3}u_x^3$ is not easy to deal with, at least by direct application of the method of [1]. In order to circumvent the difficulty, we employ a classical trick ([4]) used in the proof of the Cauchy-Kowalevsky theorem and introduce the system below.

$$
\begin{align*}
&u_t + 2\mu(u)uv - \frac{1}{3}v^3 + \partial_x A^{-1} [2\mu^2(u)u + \mu(u)v^2] + \frac{1}{3} \mu(v^3) = 0, \\
v_t + 2\mu(u)(uv)_x - \frac{1}{3}(v^3)_x + \partial_x A^{-1} [2\mu^2(u)v + \mu(u)(v^2)_x] = 0,
\end{align*}
$$

(5.2)

Theorem 8. The Cauchy problems (5.1) and (5.2) are equivalent to each other if $v_0(x) = u_0'(x)$.

Proof. By differentiation with respect to $x$, we get the second equation in (5.2) from (5.1). To show the converse, differentiate both sides of the first equation of (5.2) in $x$. Comparison with the second equation shows

$$(v - u_x)_t + \partial_x A^{-1} [2\mu^2(u)(v - u_x)] = 0.$$ 

We want to show $v = u_x$. It is enough to prove that $w_t + \partial_x A^{-1} [a(t)w] = 0$ and $w(0, x) = 0$ imply $w = 0$, where $a(t)$ is a function in $t$. Set $w = \sum_{k \in \mathbb{Z}} w_k(t)e^{2k\pi ix}$. Then we have

$$w'_k(t) = 0, \quad w'_k(t) - \frac{a(t)w_k(t)}{2k\pi i} = 0 \quad (k \neq 0).$$

Since $w_k(0) = 0$, we have $w_k(t) = 0 \quad (k \in \mathbb{Z})$ for any $t$. It implies $w = 0$ and $v = u_x$. \hfill \Box

Theorem 9. Let $s > 1/2$. If $u_0 \in G^{1,s+1}, v_0 \in G^{1,s+1}$. Then there exists a positive time $T = T(u_0, s)$ such that for every $\delta \in (0, 1)$, the Cauchy problem (5.2) has a unique solution which is a holomorphic function valued in $\oplus^2 G^{\delta,s+1}$ in the disk $D(0, T(1 - \delta))$. Furthermore, the analytic lifespan $T$ satisfies the estimate

$$T \approx \frac{1}{\|(u_0, v_0)\|_{1,s+1}^2}.$$ 

Proof. The norm on $\oplus^2 G^{\delta,s+1}$ is defined by $\|(u, v)\|_{\delta,s+1} = \|u\|_{\delta,s+1} + \|v\|_{\delta,s+1}$. We use the same notation for $G^{\delta,s+1}$ and $\oplus^2 G^{\delta,s+1}$. Assume $\|(u, v) - (u_0, v_0)\|_{\delta,s+1} < R, \|(u', v') - (u_0, v_0)\|_{\delta,s+1} < R$. Set $R_{s+1} = R + \max \left(\|u_0\|_{1,s+1}, \|v_0\|_{1,s+1}\right)$. Assume $0 < \delta' < \delta \leq 1$. Then by (2.2), we have $|\mu(u)| \leq \|u\|_{\delta',s+1} < R_{s+1}, \|v'\|_{\delta',s+1} < R_{s+1}$.

Let us consider differences related to $\mu(u)uv$ and $\mu(u)(uv)_x$. Since $uv - u'v' = u(v - v') + (u - u')v'$, we get

$$
\begin{align*}
\|uv - u'v'\|_{\delta',s+1} &\leq c_{s+1} (\|u\|_{\delta',s+1}\|v - v'\|_{\delta',s+1} + \|u - u'\|_{\delta',s+1}\|v'\|_{\delta',s+1}) \\
&\leq c_{s+1} R_{s+1}\|(u, v) - (u', v')\|_{\delta,s+1}.
\end{align*}
$$

(5.3)
Combining a similar calculation with Lemma 3, we get
\[
\|(uv)_x - (u'v')_x\|_{s',s+1}
\]
\[
\leq \frac{2\pi e^{-1}c_{s+1}}{\delta' - \delta'} (|u|\|\delta_{s+1}\|v - v'|_{\delta,s+1} + \|u - u'|_{\delta,s+1}\|v'|_{\delta,s+1})
\]
\[
\leq \frac{2\pi e^{-1}c_{s+1}R_{s+1}}{\delta - \delta'} \|(u, v) - (u', v')\|_{\delta,s+1}.
\]
On the other hand, we have
\[
\|(u'v)_x\|_{s',s+1} \leq \frac{2\pi e^{-1}}{\delta' - \delta'} \|u'v'|_{\delta,s+1} \leq \frac{2\pi e^{-1}R_{s+1}^2}{\delta - \delta'}.
\]
Combining
\[
\mu(u)uv - \mu(u'v) = \mu(u)(uv - u'v') + [\mu(u) - \mu(u')] u'v'
\]
with (5.3), we obtain
\[
\|\mu(u)(uv)_x - \mu(u')(u'v')_x = \mu(u) [(uv)_x - (u'v')_x] + [\mu(u) - \mu(u')] (u'v')_x
\]
with (5.4) and (5.5), we obtain
\[
\|\mu(u)(uv)_x - \mu(u')(u'v')_x\|_{s',s+1}
\]
\[
\leq |\mu(u)|\|uv - u'v'|_{\delta,s+1} + |\mu(u) - \mu(u')|\|u'v'|_{\delta,s+1}
\]
\[
\leq R_{s+1}^2 \|uv - u'v'|_{\delta',s+1} + c_{s+1}R_{s+1}^2 \|u - u'|_{\delta,s+1}
\]
\[
\leq \frac{2c_{s+1}R_{s+1}^2}{\delta - \delta'} \|(u, v) - (u', v')\|_{\delta,s+1}.
\]
Next, combining
\[
\mu(u)(uv)_x - \mu(u')(u'v')_x = \mu(u) [(uv)_x - (u'v')_x] + [\mu(u) - \mu(u')] (u'v')_x
\]
with (5.4) and (5.5), we obtain
\[
\|v^3 - v'^3\|_{s',s+1} \leq c_{s+1} \|v^2 + vv' + v'^2\|_{\delta',s+1} \|v - v'|_{\delta,s+1}.
\]
Here we have \|v^2 + vv' + v'^2\|_{\delta',s+1} \leq c_{s+1} \left( \|v\|^2_{\delta',s+1} + \|v'\|^2_{\delta',s+1} \right)^{\frac{1}{2}} \leq 4c_{s+1}R_{s+1}^2. Therefore
\[
\|v^3 - v'^3\|_{s',s+1} \leq 4c_{s+1}R_{s+1}^2 \|v - v'|_{\delta',s+1} \leq \frac{4c_{s+1}R_{s+1}^2}{\delta - \delta'} \|v - v'|_{\delta,s+1}.
\]
Combining a similar calculation with Lemma 3, we get
\[
\|(v^3)_x - (v'^3)_x\|_{s',s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'} \|v^3 - v'^3\|_{\delta,s+1}
\]
\[
\leq \frac{8\pi e^{-1}c_{s+1}R_{s+1}^2}{\delta - \delta'} \|v - v'|_{\delta,s+1}.
\]
An immediate consequence of (5.8) is
\[
\| \mu(v^3) - \mu(v'^3) \|_{s,s+1} = |\mu(v^3) - \mu(v'^3)| \leq \frac{4c_s^2 R_{s+1}^2}{\delta - \delta'} \| v - v' \|_{s,s+1}.
\]

The last step is to consider $\mu^2(u)v, \mu^2(u)v, \mu(u)v^2$ and $\mu(u)(v^2)_x$. Since
\[
\begin{align*}
\mu^2(u)u - \mu^2(u')u' &= \mu^2(u)(u - u') + [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] u', \\
\mu^2(u)v - \mu^2(u')v' &= \mu^2(u)(v - v') + [\mu(u) + \mu(u')] [\mu(u) - \mu(u')] v', \\
\mu(u)v^2 - \mu(u')v'^2 &= \mu(u)(v^2 - v'^2) + [\mu(u) - \mu(u')] v'^2, \\
\mu(u)(v^2)_x - \mu(u')(v'^2)_x &= \mu(u)(v^2 - v'^2)_x + [\mu(u) - \mu(u')] (v^2)_x,
\end{align*}
\]
we have, by $c_s \leq c_{s+1}$, $\| \bullet \|_{s,s} \leq \| \bullet \|_{s,s+1}$ and $R_s \leq R_{s+1}$,
\[
\begin{align*}
\| \mu^2(u)u - \mu^2(u')u' \|_{s,s} &\leq \| \mu^2(u) \|_{s,s} \| u - u' \|_{s,s} + |\mu(u) + \mu(u')| \| \mu(u - u') \|_{s,s} \\
&\leq 3R_{s+1}^2 \| u - u' \|_{s,s+1}, \\
\| \mu^2(u)v - \mu^2(u')v' \|_{s,s} &\leq \| \mu^2(u) \|_{s,s} \| v - v' \|_{s,s} + |\mu(u) + \mu(u')| \| \mu(u - u') \|_{s,s} \\
&\leq 3R_{s+1}^2 \| (u,v) - (u',v') \|_{s,s+1}, \\
\| \mu(u)v^2 - \mu(u')v'^2 \|_{s,s} &\leq |\mu(u)| \| v^2 - v'^2 \|_{s,s} + |\mu(u - u')| \| v^2 \|_{s,s} \\
&\leq 3c_{s+1} R_{s+1}^2 \| (u,v) - (u',v') \|_{s,s+1}, \\
\| \mu(u)(v^2)_x - \mu(u')(v'^2)_x \|_{s,s} &\leq 2\pi |\mu(u)| \| v^2 - v'^2 \|_{s,s+1} + 2\pi |\mu(u - u')| \| v^2 \|_{s,s+1} \\
&\leq 6\pi c_{s+1} R_{s+1}^2 \| (u,v) - (u',v') \|_{s,s+1}.
\end{align*}
\]
To derive the last one, we have used the second inequality of Lemma 3. We employ (2.1) to obtain
\[
\begin{align*}
\| \partial_x A^{-1} \left[ \mu^2(u)u - \mu^2(u')u' \right] \|_{s,s+1} &\leq \frac{6\pi e^{-1} R_{s+1}^2}{\delta - \delta'} \| u - u' \|_{s,s+1}, \\
\| \partial_x A^{-1} \left[ \mu^2(u)v - \mu^2(u')v' \right] \|_{s,s+1} &\leq \frac{6\pi e^{-1} R_{s+1}^2}{\delta - \delta'} \| (u,v) - (u',v') \|_{s,s+1}, \\
\| \partial_x A^{-1} \left[ \mu(u)v^2 - \mu(u')v'^2 \right] \|_{s,s+1} &\leq \frac{6\pi e^{-1} c_{s+1} R_{s+1}^2}{\delta - \delta'} \| (u,v) - (u',v') \|_{s,s+1}, \\
\| \partial_x A^{-1} \left[ \mu(u)(v^2)_x - \mu(u')(v'^2)_x \right] \|_{s,s+1} &\leq \frac{12\pi e^{-1} c_{s+1} R_{s+1}^2}{\delta - \delta'} \| (u,v) - (u',v') \|_{s,s+1}.
\end{align*}
\]
Set
\[
F_1(u,v) = -2\mu(u)uw + \frac{1}{3} v^3 - \partial_x A^{-1} \left[ 2\mu^2(u)u + \mu(u)v^2 \right] - \frac{1}{3} \mu(v^3), \\
F_2(u,v) = -2\mu(u)(uw)_x + \frac{1}{3} (v^3)_x - \partial_x A^{-1} \left[ 2\mu^2(u)v + \mu(u)(v^2)_x \right].
\]
Then by (5.6), (5.8), (5.11), (5.13) and (5.10), we have
\[
\begin{align*}
\| F_1(u,v) - F_1(u',v') \|_{s,s+1} &\leq \frac{2R_{s+1}^2 (6c_{s+1} + 2c_{s+1}^2 + 18\pi e^{-1} + 9\pi e^{-1} c_{s+1})}{3(\delta - \delta')} \| (u,v) - (u',v') \|_{s,s+1}.
\end{align*}
\]
By (5.7), (5.9), (5.12) and (5.14), we have
\[
\|F_2(u, v) - F_2(u', v')\|_{\delta', s+1} \\
\leq \frac{4\pi e^{-1}R_{s+1}^2(6c_{s+1} + 2c_{s+1}^2 + 9 + 9\pi c_{s+1})}{3(\delta - \delta')} \|(u, v) - (u', v')\|_{\delta, s+1}.
\]
We have obtained two inequalities of Lipschitz type.

Next we estimate $F_j(u_0, v_0)$ ($j = 1, 2$). We have
\[
\|\mu(u_0)u_0v_0\|_{\delta, s+1} \leq c_{s+1}\|u_0\|_{1,s+1}^2\|v_0\|_{1,s+1}^2,
\]
\[
\|v_0^3\|_{\delta, s+1} \leq c_{s+1}^2\|v_0\|_{1,s+1}^3,
\]
\[
\|\partial_x A^{-1} [\mu^2(u_0)u_0]\|_{\delta, s+1} \leq \frac{2\pi e^{-1}}{1 - \delta}\|u_0\|_{1,s+1}^3,
\]
\[
\|\partial_x A^{-1} [\mu(u_0)v_0^2]\|_{\delta, s+1} \leq \frac{2\pi e^{-1}}{1 - \delta}c_{s+1}\|u_0\|_{1,s+1}\|v_0\|_{1,s+1}^2,
\]
\[
\|\mu(v_0^3)\|_{\delta, s+1} \leq \|v_0\|_{1,s+1}^3.
\]
Since $X^2Y \leq (X + Y)^3/3$, $X^3 \leq (X + Y)^3$ for $X, Y \geq 0$, we get
\[
\|F_1(u_0, v_0)\|_{\delta, s+1} \leq \frac{2c_{s+1} + c_{s+1}^2 + 12\pi e^{-1} + 2\pi e^{-1}c_{s+1} + 1}{3(1 - \delta)}\|(u_0, v_0)\|_{1,s+1}^3.
\]
Similarly we have
\[
\|\mu(u_0)(u_0v_0)x\|_{\delta, s+1} \leq \frac{2\pi e^{-1}c_{s+1}}{1 - \delta}\|u_0\|_{1,s+1}^2\|v_0\|_{1,s+1},
\]
\[
\|v_0^2\|_{\delta, s+1} \leq \frac{2\pi e^{-1}c_{s+1}^2}{1 - \delta}\|v_0\|_{1,s+1}^3,
\]
\[
\|\partial_x A^{-1} [\mu^2(u_0)v_0]\|_{\delta, s+1} \leq \frac{2\pi e^{-1}}{1 - \delta}\|u_0\|_{1,s+1}^2\|v_0\|_{1,s+1},
\]
\[
\|\partial_x A^{-1} [\mu(u_0)v_0^2]x\|_{\delta, s+1} \leq \frac{4\pi^2 e^{-1}}{1 - \delta}c_{s+1}\|u_0\|_{1,s+1}\|v_0\|_{1,s+1}^2.
\]
Therefore
\[
\|F_2(u_0, v_0)\|_{\delta, s+1} \leq \frac{2(2\pi e^{-1}c_{s+1} + \pi e^{-1}c_{s+1}^2 + 2\pi e^{-1} + 2\pi^2 e^{-1}c_{s+1})}{3(1 - \delta)}\|(u_0, v_0)\|_{1,s+1}^3.
\]
If we set $R = \|(u_0, v_0)\|_{1,s+1}$, the constants corresponding to $L$ and $M$ in the general framework are of degrees 2 and 3 respectively. (They were 1 and 2 in the preceding section.) We get $T \approx \|(u_0, v_0)\|_{1,s+1}^{-2}$.

**Theorem 10.** Let $s > 1/2$. If $u_0 \in G^{1,s+2}$, then there exists a positive time $T = T(u_0, s)$ such that for every $\delta \in (0, 1)$, the Cauchy problem (5.1) has a unique solution which is a holomorphic function valued in $G^{6,s+2}$ in the disk $D(0, T(1-\delta))$. Furthermore, the analytic lifespan $T$ satisfies the estimate
\[
T \approx \frac{1}{\|(u_0, u_0')\|_{1,s+1}^2}.
\]
Proof. Set $v_0 = u_0' \in G^{1,s+1}$. Then by Theorem 9 we obtain a unique solution $(u, v) \oplus^2 G^{\delta,s+1}$ to (5.2). Since $v = u_x$ by Theorem 8 we have $u \in G^{\delta,s+2}$. □

Remark 11. The modified $\mu$CH equation with a lower-order term

$$m_t + [2\mu(u)u - u_x^2]_x + \gamma u_x = 0, m = \mu(u) - u_{xx},$$

in [8] (2.7) can be studied in the same fashion.

6. Continuous dependence on the initial data

The solutions in Theorems 6-10 depends continuously on the initial data. The proof is the same as in [1] and is omitted. In the case of the non-quasilinear modified $\mu$CH equation, we choose the initial data in $G^{1,s+2}$.

**Theorem 12.** Given $u_0 \in G^{1,s+2}, s > 1/2$ and $R > 0$, there exists $T = T(u_0) > 0$ such that the Cauchy problem (5.1) has a unique solution

$$u \in E_{T,R} := \left\{ u(t) \in \cap_{0<\delta<1} \mathcal{H}(D(0, T(1-\delta)); G^{\delta,s+2}); \sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, 0 < \delta < 1 \right\}.$$

Moreover the mapping $G^{1,s+2} \ni u_0 \mapsto u \in E_{T,R}$ is continuous in the following sense: for a given $u_\infty(0) \in G^{1,s+2}$, there exist $T = T(\|u_\infty(0)\|_{1,s+2}) > 0$ and $R > 0$ such that for any sequence $u_n(0) \in G^{1,s+2}$ converging to $u_\infty(0)$ in $G^{1,s+2}$, the corresponding solutions $u_n(t), u_\infty(t)$ belong to $E_{T,R}$ for all sufficiently large $n$ and $\|u - u_\infty\|_T \to 0$, where

$$\|u\|_T := \sup \left\{ \|u(t)\|_\delta(1-\delta) \sqrt{1 - \frac{|t|}{T(1-\delta)}}; 0 < \delta < 1 \text{ and } |t| < T(1-\delta) \right\}.$$

**References**

[1] R. F. Barostichi, A. A. Himonas and G. Petronilho, Autonomous Ovsyannikov theorem and applications to nonlocal evolution equations and systems, *J. Funct. Anal.*, 270 (2016), 330-358.

[2] R. F. Barostichi, A. A. Himonas and G. Petronilho, The power series method for nonlocal and nonlinear evolution equations, *J. Math. Anal. Appl.*, 443 (2016), 834-847.

[3] R. Camassa and d. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (11) (1993), 1661-1664.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics, II*, New York, N.Y.: Interscience Publishers, Inc., 1962.

[5] A. Fokas and B. Fuchssteiner, Symplectic structurces, their Bäcklund transformations and hereditary symmetries, Phys. D 4 (1) 1981/1982, 47-66.

[6] B. Khesin, J. Lenells and G. Misiołek, Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms, *Math. Ann.*, 342 (2008), 617-656.

[7] J. Lenells, G. Misiołek, F. Tiğlay, Integrable evolution equations on spaces of tensor densities and their peakon solutions, *Comm. Math. Phys.*, 299 (1) (2010) 129–161.

[8] C. Qu, Y. Fu and Y. Liu, Well-posedness, wave breaking and peakons for a modified $\mu$-Camassa-Holm equation, *J. Funct. Anal.*, 266 (2014), 433-477.

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