PARTICLES IN SINGULAR MAGNETIC FIELD

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Abstract
An algebraic formalism for description of quantum states of charged particle with spin moving in two-dimensional space under influence of singular magnetic field is developed in terms of graded algebras. The fundamental assumption is that the particle is transformed into a composite system which consists quasiparticles, quasiholes and magnetic fluxes. Such system is endowed with generalized statistics determined by a grading group and a commutation factor on it. Composite systems corresponding to the quantum Hall effect and the electronic magnetotransport anomaly are described. The Fock space representation are also given.

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1 Introduction

In the last years charged particles moving under influence of a strong and singular magnetic field have been studied from a few different points of view by many authors. New and very specific effects in two dimensional systems of particles have been discovered, see Ref. [1, 2, 3, 4, 5, 6] for example. A fundamental quantity for the description of these new effects is the so–called filling factor $v$ which is defined as the ratio $\frac{p}{N}$ of the number of particles $p$ by the number of magnetic fluxes $N$. A magnetic flux is a magnetic field concentrated completely on a vertical line carrying an elementary flux $\Phi_0 := \frac{h}{e}$.

For integer–valued $v$ there is the well–known integer quantum Hall effect (IQHE). For fractional–valued $v$ with odd denominator there is the famous fractional quantum Hall effect (FQHE) [1]. It is very interesting and fascinating fact that for fractional–valued filling factor with even denominator the corresponding effect is completely different from the mentioned above quantum Hall effect. This is the so–called electronic magnetotransport anomaly (the $\frac{1}{2}$–anomaly). In this case the magnetic field is completely compensated such that particles move like in the the absence of the magnetic field! In this way we obtain the following scheme

$$v = \begin{cases} 1, 2, \ldots & \text{IQHE}, \\ \frac{1}{2}, \frac{1}{4}, \ldots & \text{the $\frac{1}{2}$ anomaly}, \\ \frac{1}{3}, \frac{1}{5}, \ldots & \text{FQHE}. \end{cases} \quad (1)$$

At present there is no satisfactory theory for unified description of all these effects. The most popular theory for the quantum Hall effect is the Chern–Simons-Landau-Ginzburg (CSLG) theory, see for example [7] and reference therein. In this theory electrons are described as bosons carrying odd number of flux quanta. Such composite system is said to be a composite boson. The $\frac{1}{2}$–anomaly can be explained in a similar way. In this case electrons can be considered as a degenerate system of new fermions in the absence of magnetic field. Such particles are called composite fermions [4, 5, 8]. They can be imagined as electrons carrying even number of magnetic fluxes.

An interesting concept for composite system which consist a particle with charge $e$ and a singular magnetic field concentrated completely on a vertical line has been introduced by Wilczek [9], see also [10]. The charge is moving
around the singular line in the flat region in which there is no influence of the magnetic field. Every rotation yields certain phase factor \(q\). Observe that the space on which the particle moving is \(\mathcal{M} \equiv \mathbb{R}^2 \setminus \{s\}\), where \(s \in \mathbb{R}^2\) is the point of intersection of the magnetic line with the plane \(\mathbb{R}^2\). The factor \(q\) can be in general a complex number of the following form: \(q := \exp(i\varphi)\), where \(0 \leq \varphi < 2\pi\) is the so-called statistics parameter. This means that the statistics of the particle is determined by the value of \(\varphi\). For \(\varphi = 0\) we have boson, for \(\varphi = \pm \pi\) – fermion. For arbitrary \(\varphi \in [0, 2\pi)\) we obtain anyons. Note that the name ”anyons” is sometimes reserved to the case when \(q\) is the \(m\)-th root of unity, see for example the papers of Majid \[11, 12\], where anyons are given in terms of certain \(\mathbb{Z}_m\)-graded structures. The description of the statistics corresponding to composite fermions or bosons is a problem. A few remarks on this problem has been given by Jacak, Sitko and Wieczorek in Ref.\[13\]. According to their suggestion the value of the statistics parameter not need to be restricted to the interval \([0, 2\pi)\), but it can be an arbitrary real number. If \(\varphi = (2k + 1)\pi, k = 1, 2, \ldots\) for example, then one can expect that we obtain nothing else but the usual fermion statistics. But according to Jacak, Sitko and Wieczorek in this case we obtain new kind of statistics, just the statistics of composite fermions. The case of \(\varphi = 2k\pi, k = 1, 2, \ldots\) corresponds to composite bosons. An algebraic formalism for particle system with generalized statistics has been considered by the author in Ref.\[14, 15\], see also \[18, 19, 20, 21\]. It is interesting that in this algebraic approach all commutation relations for particles equipped with arbitrary statistics can be described as a representation of the so-called quantum Weyl algebra \(\mathcal{W}\) (or Wick algebra) \[22, 23, 27, 38\]. Similar approach has been also considered by others authors, see \[24, 25, 26, 27\] and \[28, 29\]. In this attempt the creation and anihilation operators act on certain quadratic algebra \(\mathcal{A}\). The creation operators act as the multiplication in \(\mathcal{A}\) and the anihilation ones act as a noncommutative contraction (noncommutative partial derivatives). The algebra \(\mathcal{A}\) play the role of noncommutative Fock space. The applications for particles in singular magnetic field has been described in Ref.\[20, 21\]. As a result the fact that at \(v = \frac{1}{2}\) the system is degenerate and at \(v = \frac{1}{N}\) for \(N\) odd the Landau level is fractionally filled has been obtained in an obvious way.

In this paper an algebraic formalism for a system of charged particles with spin \(\frac{1}{2}\) in singular magnetic field is developed. We restrict our attention
to the system with the filling factor $v = \frac{1}{N}$ for simplicity. The generalization for arbitrary filling factor is possible but it need more complicated notation. The fundamental assumption of the paper is that the particle is transformed into a composite system which consists quasiparticles, quasiholes and magnetic fluxes. The transformation a result of interactions of charged particle with spin with magnetic field. If $N$ is even, then the system is identified as composite fermions and if $N$ is odd, then we obtain composite bosons. It is obvious the statistics of our composite systems can not be described by the scalar statistics parameter $q = \exp(i\varphi)$. We use here the concept of commutation factor [31] in order to describe the statistics. The paper is organized as follows. The fundamental concept of the paper based on Refers [13, 14, 21] is given In Section 2. In Section 3 the quantum Weyl algebra is described in details. It is shown that it is in fact a $\Gamma$–graded generalized (color) Lie algebra, [31]. The representation of the quantum Weyl algebra is considered in Section 4. The Fock space of quantum states for composite fermions and composite bosons is described in Section 5. The notion of commutation factors is shortly summarized in the Appendix. Note that the algebraic approach to flat particle system in singular magnetic field is not complete theory but it can be further developed. One can write down the Hamiltonian and study the corresponding field theory. One can use the formalism described by Liguori and Mintchev [30] for such study. We hope that the possibility for the unified description of the quantum Hall effect and the electronic magnetotransport anomaly based on the presented in this paper algebraic formalism should be clear.

2 Fundamental assumptions

Let us consider a system of charged particles with spin $\frac{1}{2}$ moving on a flat space $\mathbb{R}^2$ under influence of singular perpendicular magnetic field. The magnetic field is sufficiently strong for the polarization effects of the spin. The singularity of the magnetic field means here that the field is completely concentrated into fluxes. We assume that there is in average $N$ magnetic fluxes per particle. This means that the filling factor is $v = \frac{1}{N}$. Our fundamental assumption is that the problem of many particles interacting with the singular magnetic field can be reduced to the study of a system consisting just one particle and $N$ magnetic fluxes. In this way we can restrict our attention to
study of such system. It is known that quantum states of charged particle in
the magnetic field are described by Landau levels. Here is an additional de-
generacy of such states connected with the singular nature of the field. The
structure of such degeneration is determined by the change of phase of the
particle. Let us study the degeneration in more details. We assume that the
particle is in the Landau lowest level. The ”effective” space for the particle
in singular magnetic field is $M \equiv \mathbb{R}^2 \setminus \{s_1, \ldots, s_N\}$, where $s_i$ is the point of
intersection of the $i$-th magnetic flux with the plane. Let us denote by $P_{m_0}$
the space of all homotopy classes of paths which starts at $m_0$ and and end
at arbitrary point $m \in M$. It is known that the union $U_m P_{m_0} M = PM$ is a
covering space $P = (PM, \pi, M)$. The projection $\pi : PM \longrightarrow M$ is given by

$$\pi(\xi) = m \quad \text{iff} \quad \xi \in P_{m_0} M. \quad (2)$$

The homotopy class $P_{m_0} M$ of all paths which start at $m_0$ and end at the
same point $m_0$, (i.e. a loop space) can be naturally endowed with a group
structure. This group is known as the fundamental group of $M$ at $m_0$ and
is denoted by $\pi_1(M, m_0)$. Observe that two groups $\pi_1(M, m_0)$ and
$\pi_1(M, m)$ at two arbitrary points $m_0$ and $m$, respectively, are isomorphic.
Hence we can introduce the fundamental group $\pi_1(M)$ for the whole space $M$
in an obvious way. Note that in our case the fundamental group $\pi_1(M)$ for
the space $M \equiv \mathbb{R}^2 \setminus \{s_1, \ldots, s_N\}$ is a free group generated by $\tilde{\sigma}_i$, ($i = 1, \ldots, N$)
and is denoted by $G_N \equiv G_N(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N)$, where $\tilde{\sigma}_i$ is the homotopy class of
all paths which encloses the point $m_i$ and none of the remaining points $m_j$
for $j \neq i$, see Ref.\[35\] for example for more details. Next we assume that
for every class of loops the phase change of particle under consideration is
fixed. We introduce an equivalence relation in the fundamental group $G_N$.
Two homotopy classes of loops in $G_N$ are equivalent if and only if the phase
changes corresponding to this classes are equal. The equivalence class of $\tilde{\sigma}_i$
with respect to the above relation is denoted by $\sigma_i$. Let $\Gamma$ be a quotient group
of the fundamental group $G_N$ by the above relation. It is obvious that the
phase change corresponding for $\tilde{\sigma}_i \tilde{\sigma}_j$ and for $\tilde{\sigma}_j \tilde{\sigma}_i$ is the same for all $i \neq j$.
Hence we have the relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ in the quotient group $\Gamma$. Observe that
generators $\sigma_i$ of $\Gamma$ can be identified with rotations around axis at the point $s_i$.
This means that the group $\Gamma$ can be identified with the group $Z \oplus \ldots \oplus Z$ which
contains $N$ copies of the group of integers. Every rotation around the axis at
$s_i$ yields the same phase factor $q_i$. Two rotations around two different axis $s_i$
and $s_j$ are independent and they yield the phase factor $c_{ij}$. It is natural to assume that $c_{ij}c_{ji} = 1$ and $c_{ii} \equiv q_i$. It is obvious that there is a commutation factor $c : \Gamma \times \Gamma \rightarrow \mathcal{C} \setminus \{0\}$ on the group $\Gamma$ (see the Appendix) such that $c(\sigma_i, \sigma_j) \equiv c_{ij}$. The commutation factor $c$ generalizes the scalar statistics parameter $q = \exp(i\varphi)$, where $\varphi$ is an arbitrary real number. It follows from the above considerations that the structure of degenerations of Landau levels can be described by the commutation factor $c$ on $\Gamma$. Structures of different kind of degenerations correspond to nonequivalent classes of commutation factors on the group $\Gamma$. This means that we can use the concept of color statistics and related $\Gamma$–graded structures in order to obtain the algebraic formalism for the particle in the singular magnetic field \cite{[17]}. In is interesting that in certain particular cases, see the Appendix, the grading group can be reduced to the group $\Gamma = \mathbb{Z}_n^N \equiv \mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n$ for $n > 2$ or to $\mathbb{Z}_2^N \equiv \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$.

It is interesting that the structure of degeneration of the Landau levels can be described as quantum states of new particles equipped with generalized statistics. We assume that the particle under considerations is transformed into a composite system which consists quasiparticles, quasiholes and magnetic fluxes. Such composite system is said to be a composite particle or generalized quon. The transformation of the particle into the composite system is a result of interactions of charged particles with spin with magnetic field. Note that if the particle is coupled with certain flux in such a way that the magnetic field of this flux is canceled, then we say that we have a quasiparticle. In this case the particle is said to be bound to the flux. In the opposite case, i.e. when the magnetic field is not canceled we say on quasiholes. A quasihole is a ”free” flux which behave like particle endowed with fermion statistics. Hence the points of intersections of $N$ magnetic fluxes with the plane must be a set of $N$ different points $s_1, \ldots, s_N$. The number of quasiparticles and quasiholes is equal to the number of magnetic fluxes. This means that the filling factor for the transformed system is $v' = \frac{m+n}{N} \equiv 1$, where $m$ is the number of quasiparticles, and $n$ is the number of quasiholes.

The ”effective” magnetic field is

$$B_{\text{eff}} := B - m\Phi_0, \quad (3)$$

where $B$ is the external magnetic field. Observe that if $B = N\Phi_0$ and the number of quasiparticles is equal to $N$, then the magnetic field is completely canceled! A quasiparticle is in fact the charged particle bound to single magnetic flux. The particle bound to two different magnetic fluxes are understand
as a system of two different quasiparticles. Obviously the particle can not be bound to two fluxes at the same point! Quasiparticles as components of certain composite particle have also their own statistics. This statistics is determined by the commutation factor \( c \) on the group \( \Gamma \). For electron in singular magnetic field it is natural to assume that we have \( c_{ii} \equiv \exp(i\varphi) \), where \( \varphi = \pi(N + 1) \), i.e. \( c_{ii} = -(1)^N \) and \( c_{ij} \equiv \exp(i\pi N) \equiv (-1)^N \) for \( i \neq j \). The reason that in the formula for \( c_{ii} \) we have that \( \varphi \) is equal to the number of magnetic fluxes \( N \) plus 1 is the spin of electrons. The above expressions for the factor \( c \) means that two different quasiparticles for \( N \) even commute and for \( N \) odd – anticommute. This also means that the composite boson can contain only one quasiparticle. The number of quasiparticles can be equal to \( N \) for composite fermions only! In others words, our quasiparticles are examples of particles equipped with the so-called color statistics, i.e. the parastatistics in Green representation [17]. This means that one can use the concept of color statistics and related structures with abelian grading group in order to obtain an algebraic formalism for particles in singular magnetic field.

3 On quantum Weyl algebras

Let us denote by \( \Theta^i \) the quantum quasiparticle state coupled with the flux at \( s_i \). The corresponding conjugate state is denoted by \( \Theta^*_i \). It is natural to assume that there is a finite dimensional vector space \( E \) equipped with a basis \( \Theta^i, i = 1, ..., N = \text{dim} E \). The complex conjugate space \( E^* \) is endowed with a basis \( \Theta^*_i \) such that we have that \( < \Theta^*_i | \Theta^j > = \delta_{ij} \) for \( i, j = 1, ..., N \). Let us give the notion of quantum Weyl algebras \( W \equiv W_{\Gamma,c}(N) \) corresponding to an arbitrary grading group \( \Gamma \) and a commutation factor \( c \) on it.

**Definition:** A \((\Gamma, c)\)-quantum Weyl algebra is a quotient algebra

\[
W \equiv W_{\Gamma,c}(N) := T(E \oplus E^*)/I_{\Gamma,c},
\]

where \( I_{\Gamma,c} \) is an \(*\)-ideal in the tensor algebra \( T(E \oplus E^*) \) generated by the following elements

\[
\Theta^*_i \otimes \Theta^j - c_{ij} \Theta^j \otimes \Theta^*_i - \delta^*_i \Theta^j, \quad \text{for all} \ i, j,
\]

\[
\Theta^i \otimes \Theta^j - c_{ij} \Theta^j \otimes \Theta^i, \quad \Theta^*_j \otimes \Theta^*_i - c_{ij} \Theta^*_i \otimes \Theta^*_j, \quad \text{for} \ i \neq j,
\]

\[
(\Theta^i)^2, \quad (\Theta^*_j)^2 \quad \text{if} \ c_{ii} = -1,
\]

(5)
The above definition means that $W_{\Gamma,c}(N)$ is a $*$-algebra generated by $\Theta^i$ and $\Theta^*_j$; $(i, j = 1, ..., N)$ subject to relations

\begin{align*}
\Theta^*_j \Theta^i &= 1 + q_i \Theta^i \Theta^*_j \quad \text{for } i = j, \\
\Theta^i \Theta^j &= c_{ij} \Theta^j \Theta^*_i \quad \text{for } i \neq j,
\end{align*}

\begin{align*}
\Theta^i \Theta^j &= c_{ij} \Theta^j \Theta^i \\
(\Theta^i)^2 &= 0, \quad (\Theta^*_j)^2 = 0, \quad \text{if } c_{ii} = -1,
\end{align*}

where $q_i := c_{ii} : i = 1, \ldots, N$ are diagonal elements of $c$, note that the same symbols for elements in the tensor algebra $T(E \oplus E^*)$ and for generators of the algebra $W_{\Gamma,c}(N)$ have been used for simplicity. The algebra $W_{\Gamma,c}(N)$ should be also denoted by $W_{\Gamma,c} < \Theta^1, \ldots, \Theta^N, \Theta^*_1, \ldots, \Theta^*_N >$.

**Theorem:** The quantum Weyl algebra $\mathcal{W} \equiv \mathcal{W}(\Gamma, c)$ is a $\Gamma$–graded $c$–Lie algebra.

**Proof:** We introduce the $\Gamma$–gradation of the algebra $\mathcal{W}$ as follows: for generators of the algebra we define that grade($\Theta^i$) $\equiv |\Theta^i| := \sigma_i, \text{ and grade}(\Theta^*_i) \equiv |\Theta^*_i| := -\sigma_i$, where $\{\sigma_i\}_{i=1}^N$ is a set of generators of $\Gamma$. All monomials in $\Theta^i$ and $\Theta^*_j$ modulo generating relations are homogeneous elements of $\mathcal{W}$. We use the formula $|XY| = |X| + |Y|$ for the extension of gradation for arbitrary homogeneous elements of the algebra. The generalized bracket $[.,.]_c : \mathcal{W} \otimes \mathcal{W} \rightarrow \mathcal{W}$ is defined by the formula

\begin{equation}
[.,.]_c X \otimes Y \equiv [X, Y]_c := XY - c(|X|, |Y|) YX,
\end{equation}

where $X, Y$ are arbitrary homogeneous elements of the algebra $\mathcal{W}$. We can calculate the bracket as follows: for generators we obtain the relation

\begin{equation}
[\Theta^*_i, \Theta^j]_c := \Theta^*_i \Theta^j - c(|\Theta^*_i|, |\Theta^j|) \Theta^j \Theta^*_i = \delta^j_i,
\end{equation}

where $c(|\Theta^*_i|, |\Theta^j|) = c(-\sigma_i, \sigma_j) = c_{ij}$ and the relations $\{3\}$ have been used.

In the similar way we obtain the formula

\begin{align*}
[\Theta^i, \Theta^*_j]_c &= \Theta^i \Theta^*_j - c(|\Theta^i|, |\Theta^*_j|) \Theta^*_j \Theta^i \\
&= -c(|\Theta^i|, |\Theta^*_j|) (\Theta^*_j \Theta^i - c(|\Theta^*_j|, |\Theta^i|) \Theta^i \Theta^*_j) = -c_{ij} \delta^j_i.
\end{align*}

For the extension of our generalized bracket to the higher order homogeneous elements of the algebra we can use the following formulae

\begin{align*}
[\Theta^*_i, \Theta^j X]_c &= c_{ij} \Theta^j [\Theta^*_i, X] + \delta^j_i X, \\
[\Theta^*_i X, \Theta^j]_c &= \Theta^*_i [X, \Theta^j] - c_{ij} c(|X|, \sigma_j) \delta^j_i X,
\end{align*}

\begin{equation}
\text{(no sum), for } i, j = 1, ..., N, \text{ where } c_{ij} \text{ are coefficient of certain commutation factor on the grading group } \Gamma.
\end{equation}
for arbitrary homogeneous $X$ in $W$. One can prove these formulae by induction. One can see that the generalized bracket is $c$–anticommutative

$$[X, Y]_c = -c(|X|, |Y|) [Y, X]_c$$

for homogeneous $X$ and $Y$ in $W$. Finally, it not difficult to calculate the following generalized Jacobi identity

$$[X, [Y, Z]_c]_c = [[X, Y]_c, Z]_c + c(|X|, |Y|) [Y, [X, Z]_c]_c.$$ (12)

It is interesting that for arbitrary generalized Lie algebra $L \equiv L_{\Gamma, c}$ there is a corresponding Lie algebra or superalgebra $s(L)$, [31, 32]. This algebra is called the superisation of $L$ and we have the following crossed product

$$L := s(L) \# C_{\Gamma, b}(N),$$ (13)

where $C_{\Gamma, b}(N)$ is $\Gamma$–graded $b$–commutative $*$–bialgebra. This means that we have the relation $gh = b(|g|, |h|)hg$ for all homogeneous $g, h \in C_{\Gamma, b}(N)$ of grade $|g|$ and $|h|$, respectively. The factor $b$ is given by the formula (60). The bialgebra $C_{\Gamma, b}(N)$ as an unital, associative algebra is generated by $e^i, e^*_j$ and the following relations

$$e^i e^j = b_{ij} e^i e^j, \quad e^*_i e^*_j = b_{ij} e^*_j e^*_i, \quad \text{for } i \neq j,$$

$$e^*_i e^i = e^i e^*_i = \delta_i^j, \quad \text{for all } i, j,$$ (14)

(no sum). The gradation is given in the standard way. This means that $|e^i| = \sigma_i$ and $|e^*_i| = -\sigma_i$, $\sigma_i$ are generators of the group $\Gamma$. The $*$–operation and the comultiplication by the formulae

$$(e^i)^* := e^*_i, \quad (e^*_i)^* := e^i,$$ (15)

i. e. $(e^i)^* \equiv (e^i)^{-1}$ and

$$\Delta(e^i) := e^i \otimes 1 + 1 \otimes e^i,$$ (16)

respectively. The extension to the whole bialgebra is obvious [39]. For the tensor product we have here the relation

$$(g \otimes h)(k \otimes l) = b(|h|, |k|)gk \otimes hl$$ (17)
for homogeneous $g, h, k, l \in C_{\Gamma,b}(N)$. Observe that the bialgebra $C_{\Gamma,b}(N)$ is not an usual Hopf algebra, but a $\Gamma$–graded generalized Hopf algebra, see [32]. The relation (13) means that $L$ is a subalgebra of the tensor product $s(L) \otimes C_{\Gamma,b}(N)$ spanned by elements of the form $X = x \otimes g$ for homogeneous $X \in L$, $x \in s(L)$ $g \in C_{\Gamma,b}(N)$ of the same grade, i.e. $|X| = |x| = |g| = \alpha \in \Gamma$. The algebra $s(L)$ is also $\Gamma$–graded, but this gradation can be reduced to the group $\Gamma/\Gamma_0$. The reduction is given by the quotient map $\pi: \Gamma \rightarrow \Gamma/\Gamma_0$.

In this way we obtain that the gradation of the algebra $s(L)$ can be given by the group $\mathbb{Z}_2$ or is trivial. Let us describe the superisation $s(W_{\Gamma,c}(N))$ of the quantum Weyl algebra $W_{\Gamma,c}(N) \equiv W_{\pi(\Gamma),c}$ in terms of generators. The algebra $s(W_{\Gamma,c}(N))$ is generated by $x^i$ and $x_j^*$ ($i, j = 1, \ldots, N$) and relations

$$
\begin{align*}
x_i^* x_i &= 1 + q_i x_i x_i^* \quad \text{for } i = j, \\
x_i^* x_j &= c'_{ij} x_j^* x_i \quad \text{for } i \neq j, \\
x^i x^j &= c'_{ij} x^j x^i \quad \text{for all } i, j.
\end{align*}
$$

(18)

where $c'$ is given by the formula (61). This means that $s(W_{\Gamma,c}(N))$ is also a quantum Weyl algebra, $s(W_{\Gamma,c}(N)) \equiv W_{\pi(\Gamma),c'}$. The crossed product of algebras $s(W_{\Gamma,c}(N))$ and $C_{\Gamma,b}(N)$ is a subalgebra of the tensor product $s(W_{\Gamma,c}(N)) \otimes C_{\Gamma,b}(N)$ generated by elements

$$
\Theta^i := x^i \otimes e^i \quad \Theta_j^* := x_j^* \otimes e_j^* \quad \text{(no sum),}
$$

(19)

Note that the above expressions for $\Theta^i$ and $\Theta_j^*$ are not unique. There is a freedom of choosing of the generators of the bialgebra $C_{\Gamma,b}(N)$ described by the orthogonal group $O(N)$, [33]. Let us consider a few simple examples:

**Example 1** Assume that $\Gamma = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ ($N$–sumands) and the factor $c$ is given by the relation (63), where $\Omega_{ij} = 1$ for $i \neq j$. Hence the commutation rules (6) can be given in the following form

$$
\begin{align*}
\Theta^i \Theta^j &= 1 - q_i \Theta^i \Theta^j \quad \text{for } i = j, \\
\Theta^i \Theta^j &= \omega^{-1} \Theta^j \Theta^i \quad \text{for } i < j, \\
\Theta^i \Theta^j &= \omega \Theta^j \Theta^i \quad \text{for } i < j.
\end{align*}
$$

(20)

**Example 2** Let us consider the case corresponding for $\Gamma = \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$, in more detail. If we substitute the commutation factor $c$ of the form (67) into
the formulae (21), then we obtain the following relations

\[ \Theta_i^* \Theta^j + (-1)^N \Theta^i \Theta_i^* = 1 \quad \text{for } i = j, \]
\[ \Theta_i^* \Theta^j - (-1)^{\alpha_{ij}} \Theta^j \Theta_i^* = 0 \quad \text{for } i < j, \]
\[ \Theta^i \Theta^j - (-1)^{\alpha_{ij}} \Theta^j \Theta^i = 0 \quad \text{for } i < j. \]

(21)

It is interesting that in this case the bialgebra \( C_{\Gamma,b}(N) \) as algebra is reduced to the standard Clifford algebra \( C_N \). In fact we have \((e^i)^* \equiv e^i\) and we obtain the well-known relations \(e_i e_j + e_j e_i = 2\delta_{ij}\) for \(i, j = 1, \ldots, N\).

**Example 3** If \( N \) is even, we obtain the following relations for composite fermions

\[ \Theta_i^* \Theta^i - \Theta^i \Theta_i^* = 1 \quad \text{for } i = j, \]
\[ \Theta_i^* \Theta^j + \Theta^j \Theta_i^* = 0 \quad \text{for } i < j, \]
\[ \Theta^i \Theta^j + \Theta^j \Theta^i = 0 \quad \text{for } i < j. \]

(22)

The quantum Weyl algebra generated by these relations is denoted by \( \mathcal{W}_{cf}(N) \). We can see that the algebra \( s(\mathcal{W}_{cf}(N)) \) is generated by the usual canonical commutation relations for the system of \( N \) bosons.

**Example 4** If \( N \) is odd, then we obtain relations for composite bosons

\[ \Theta_i^* \Theta^i + \Theta^i \Theta_i^* = 1 \quad \text{for } i = j, \]
\[ \Theta_i^* \Theta^j - \Theta^j \Theta_i^* = 0 \quad \text{for } i < j, \]
\[ \Theta^i \Theta^j - \Theta^j \Theta^i = 0 \quad \text{for } i < j. \]

(23)

The corresponding quantum Weyl algebra is denoted by \( \mathcal{W}_{cb}(N) \). In this case the algebra \( s(\mathcal{W}_{cb}(N)) \) is generated by the usual canonical anticommutation relations for the system of \( N \) fermions.

**Example 5** Let \( C_N \) be the Clifford algebra. If we substitute \( \Theta^i \equiv \Theta_i^* \equiv \frac{1}{2} e_i \), and

\[ c_{ij} = \begin{cases} +1 & \text{for } i = j, \\ -1 & \text{for } i \neq j, \end{cases} \]

then we obtain a particular example of quantum Weyl algebra denoted by \( C_{\Gamma}(N) \).

### 4 Representation of the quantum Weyl algebra

In this section we are going to construct a representation of the quantum Weyl algebra \( \mathcal{W} \equiv \mathcal{W}_{\Gamma,c}(N) \) on the \( c \)-symmetric algebra \( \mathcal{A} \). An algebra
defined as the quotient $\mathcal{A} \equiv \mathcal{A}_c(E) := TE/I_c$ where $I_c$ is an ideal in $TE$ generated by the following relations

$$\Theta^i \Theta^j - c_{ij} \Theta^j \Theta^i, \quad \text{for } i \neq j,$$

$$\Theta^i)^2 = 0, \quad \text{if } c_{ii} = -1,$$

is said to be $c$-symmetric algebra over $E$ [32]. The algebra $\mathcal{A} \equiv \mathcal{A}_c(E)$ can be also denoted by $\mathcal{A}_c < \Theta^1, \ldots, \Theta^N >$, sometime. It is easy to see that $\mathcal{A}_c(E)$ is $\Gamma$-graded $c$-commutative algebra.

**Theorem:** Let $\mathcal{W}$ be a Clifford–Weyl algebra, $\mathcal{A}$ be a $c$-symmetric algebra, then there is a representation $a : \mathcal{W} \rightarrow \text{End}(\mathcal{A})$ of the quantum Weyl algebra $\mathcal{W} \equiv \mathcal{W}_{\Gamma,c}(N)$ on the $\Gamma$-graded $c$-commutative algebra $\mathcal{A}$.

**Proof:** We must prove that there is a well–defined homogeneous homomorphism $a : \mathcal{W} \rightarrow \text{End}(\mathcal{A})$ which transform the algebra $\mathcal{W}$ as generalized Lie algebra into the generalized Lie algebra $\text{End}(\mathcal{A})$ of linear endomorphisms of $\mathcal{A}$. This means that we have the relation

$$a_{[X,Y]}f = [a_X, a_Y]f$$

for every homogeneous $X, Y \in \mathcal{W}$ and $f \in \mathcal{A}$. The bracket in the right hand side of the above formula is defined by the relation

$$[a_X, a_Y] := a_X a_Y - c(|X|, |Y|) a_Y a_X.$$  \hfill (26)

For generators $\Theta^i$ and $\Theta^*_j$, $(i, j = 1, \ldots, N)$ we define

$$a_{\Theta^i}f := m(\Theta^i \otimes f) \equiv \Theta^i f, \quad a_{\Theta^*_j}f \equiv ev_k(\Theta^*_j \otimes f)$$

for every $f \in \mathcal{A}^k$, where $ev_k : E^* \otimes \mathcal{A}^\otimes k \rightarrow \mathcal{A}^\otimes k-1$ are a set of linear mappings defined by the following formulae

$$ev_1(\Theta^*_i \otimes \Theta^j) := \langle \Theta^*_i| \Theta^j \rangle = \delta^j_i,$$

$$ev_k(\Theta^*_i \otimes \Theta^j f) := ev_1(\Theta^*_i \otimes \Theta^j)f + c_{ij} \Theta^j ev_{k-1}(\Theta^*_i \otimes f),$$

for $f \in \mathcal{A}^k$. These mappings are said to be a right evaluation. Observe that we have the relation

$$ev_{k-1}(\Theta^*_i \otimes ev_k(\Theta^*_j \otimes f)) - c_{ij} ev_{k-1}(\Theta^*_j \otimes (ev_k(\Theta^*_i \otimes f))) = 0. \hfill (29)$$
on \( E^* \otimes E^* \otimes A^k \). We have for example
\[
ev_2 (\Theta_i^* \otimes \Theta^l \Theta^l) = \delta_i^k \Theta^l + c_{ik} \delta_{il} \Theta^k.
\] (30)

We can see that the evaluation \( \ev := \{ \ev_k : k = 1, \ldots \} \) can be well defined in a consistent way on the whole algebra \( A \), see [18, 22, 23]. We can also see that the definition (27) can be extended in a consistent way for arbitrary homogeneous element of the algebra \( \mathcal{W} \). Observe that we have here the following lemma:

\[ \square \]

**Lemma:** We have on the algebra \( A \) the following commutation relations for the representation \( a \) of algebra \( \mathcal{W}_{\Gamma, c}(N) \) on \( A \)
\[
[a_{\Theta^i}, a_{\Theta^j}]_c = \delta^j_i 1, \quad [a_{\Theta^i}, a_{\Theta^j}]_c = 0, \quad [a_{\Theta^i}, a_{\Theta^j}]_c = 0.
\] (31)

**Proof** Using the relations (27) for the first relation (31) we obtain
\[
(a_{\Theta^j} a_{\Theta^i} - c_{ij} a_{\Theta^i} a_{\Theta^j}) f = [\ev_1 (\Theta_i^* \otimes \Theta^j f) - c_{ij} \Theta^j \Theta^i \ev_1 (\Theta_i^* \otimes f)] = \ev_1 (\Theta_i^* \otimes \Theta^j f) = \delta_{ij} f,
\] (32)

where \( f \in A^l \) and the relation (28) has been used. The second relation (31) follows immediately from the \( c \)-commutativity of the algebra \( A \). The last relation follows from the equation (29). \( \square \)

Let \( a : \mathcal{W}_{\Gamma, c}(N) \rightarrow \text{End}(A) \) be a Fock representation of \( \mathcal{W}_{\Gamma, c}(N) \) on \( A \). Then there is a corresponding Fock representation \( \tilde{a} : s(\mathcal{W}_{\Gamma, c}(N)) \rightarrow \text{End}(\text{s}(A_c)) \) of the algebra \( s(\mathcal{W}_{\Gamma, c}(N)) \) on \( s(A) \), where \( s(A) \) is a commutative (or supercommutative) algebra, the superisation of \( A \). For the representation \( a \) we have the following decomposition
\[
a_{\Theta^i} \equiv a_{x^i \otimes e^i} := \tilde{a}_{\Theta^i} \otimes L_{e^i},
a_{\Theta^i}^* \equiv a_{x^i \otimes e^i}^* := \tilde{a}_{\Theta^i}^* \otimes L_{e^i},
\] (33)

where \( x^i, x^i \in s(\mathcal{W}_{\Gamma, c}(N)) \) are given by (19) \( e^i, e^i \in C_{\Gamma, b}(N) \), \( L : C_{\Gamma, b}(N) \rightarrow \text{End}(C_{\Gamma, b}(N)) \) is the left regular representation of \( C_{\Gamma, b}(N) \) on itself, i.e. \( L_g h = gh \) for arbitrary \( g, h \in C_{\Gamma, b}(N) \). Obviously the above decomposition is not unique.
5 Fock space for composite fermion and boson

Let us describe the Fock space for composite fermion and composite boson in details. The ground state vector $|0\rangle$ is defined as usual, i.e. $a_\Theta^+|0\rangle = \Theta_1$, \( (34) \)

and

$$ a_\alpha^+ |0\rangle = (a_1^+)^{\alpha_1} \cdots (a_N^+)^{\alpha_N} |0\rangle = \Theta^{\alpha_1} \cdots (\Theta^N)^{\alpha_N}. \quad (35) $$

We use the similar notation for the operators $\tilde{a}_\times$ and $L_e$. From relations \( (33) \) we obtain

$$ a_\alpha^+ |0\rangle = \tilde{a}_\alpha^+ \otimes L_\alpha |0\rangle = \Theta^{\alpha_1} \cdots (\Theta^N)^{\alpha_N} \otimes (e_1)^{\alpha_1} \cdots (e^N)^{\alpha_N}. \quad (36) $$

Let us consider an example of representation corresponding for an electron in singular magnetic field. We assume that the electron is represented by one grassmann variable $\Theta$, i.e. $\Theta^2 = 0$. For this representation we introduce here the following state vectors

$$ a_i^+ |0\rangle =: \begin{pmatrix} \Theta_\alpha \end{pmatrix}, \quad (37) $$

where $\Theta$ is on the $i$-th row. In this way an arbitrary state vector can be given in the following form

$$ a_\alpha^+ |0\rangle =: \begin{pmatrix} \Theta^{\alpha_1} \\ \vdots \\ \Theta^{\alpha_N} \end{pmatrix}, \quad (38) $$

for $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i = 0$ or $1$ for $i = 1, \ldots, N$. Now let us study this representation in more details. Observe that for even $N$ we obtain

$$ \Theta_i \Theta_j = \Theta_j \Theta_i, \quad \Theta_i^2 = 0. \quad (39) $$
We have
\[ \Theta_i \Theta_j = \begin{pmatrix} 0 \\ \vdots \\ \Theta \\ \vdots \\ 0 \end{pmatrix} = \Theta_j \Theta_i. \] (40)

and
\[ \Theta_i^2 = \begin{pmatrix} 0 \\ \vdots \\ \Theta^2 \\ \vdots \\ 0 \end{pmatrix} = 0. \] (41)

Let us consider the case of \( N = 2 \) in more details. In this case we have the following states
\[ \Theta^1 = \begin{pmatrix} \Theta \\ 0 \end{pmatrix}, \quad \Theta^2 = \begin{pmatrix} 0 \\ \Theta \end{pmatrix}, \] (42)

and
\[ \Theta^1 \Theta^2 = \begin{pmatrix} \Theta \\ \Theta \end{pmatrix}. \] (43)

The filling factor for all these states (42) and (43) is \( v = \frac{1}{2} \). These two states (42) contain quasiholes. Observe that the state (43) describe electron bond to two fluxes which is transformed into a system containing two quasiparticles and two fluxes. This system is nonlocal due to the anticommutativity of fluxes. Note that the state (43) is the unique state corresponding to the filling factor \( v = \frac{1}{2} \) without quasiholes. Observe that this state is completely filled by quasiparticles. This means that for \( B = 2 \Phi_0 \) we obtain that \( B_{\text{eff}} = 0 \). In this way the state (43) corespond to the \( \frac{1}{2} \)-anomaly.

For odd \( N \) we obtain
\[ \Theta_i \Theta_j = -\Theta_j \Theta_i \quad \text{for } i \neq j. \] (44)
For $i = j$ we obtain the identity $\Theta^i \Theta^j = \Theta^i \Theta^j$. In this case the space of states can also be represented by the variable $\Theta$ such that we have

$$\Theta^i = \begin{pmatrix} 0 \\ \vdots \\ \Theta \\ \vdots \\ 0 \end{pmatrix},$$

where $\Theta$ is on the $i$-th row. We have

$$\Theta^i \Theta^j = \begin{pmatrix} 0 \\ \vdots \\ \Theta \\ \vdots \\ 0 \end{pmatrix} = -\Theta^j \Theta^i.$$  \hspace{1cm} (46)

This means that

$$\Theta^i \Theta^j = 0$$ \hspace{1cm} (47)

for all $i \neq j$. Observe that the quantum state $x^i x^j, \ (i \neq j)$ corresponding to particle coupled to magnetic fluxes at two different points disappear

$$\Theta^i \Theta^j = x^i x^j \otimes e^i e^j = 0, \hspace{1cm} (48)$$

and the state describing the particle coupled to a few fluxes at the same point is also impossible. In fact we have

$$(\Theta^i)^2 = (x_i)^2 \otimes (e_i)^2 = (x_i)^2 \otimes 1 = (x_i)^2.$$ \hspace{1cm} (49)

This means that the state $(\Theta^i)^2$ is not equipped with a flux. Let us consider as an example the case of $N = 3$ i.e. the filling factor is $v = \frac{1}{3}$. In this case we have the states

$$\Theta^1 = \begin{pmatrix} \Theta \\ 0 \\ 0 \end{pmatrix}, \quad \Theta^2 = \begin{pmatrix} 0 \\ \Theta \\ 0 \end{pmatrix}, \quad \Theta^3 = \begin{pmatrix} 0 \\ 0 \\ \Theta \end{pmatrix},$$

\hspace{1cm} (50)
which contain two quasiholes. Observe that the following states

\[ \Theta_1 \Theta_2, \quad \Theta_1 \Theta_3, \quad \Theta_2 \Theta_3. \tag{51} \]

which contain one quasihole and the state

\[ \Theta_1 \Theta_2 \Theta_3 \tag{52} \]

which not contain quasiholes are impossible. Hence in this case the single
quasiparticle states with two quasiholes are possible! This also means that the Landau lowest level can not be completely filled by quasiparticles but fractionally filled only! This is just the FQHE.

We can see that there is a well defined scalar product on \( A \)

\[ < \Theta^\sigma |\Theta^\tau >_{q,b}= \sum_{\sigma \in S_n} \chi_n(\pi) < \Theta^{i_1} |\Theta^{\pi(j_1)} > \ldots < \Theta^{i_n} |\Theta^{\pi(j_n)} >, \tag{53} \]

where \( \Theta^\sigma, \Theta^\tau \in A^n, \Theta^\sigma = \Theta^{i_1} \ldots \Theta^{i_n}, \Theta^\tau = \Theta^{j_1} \ldots \Theta^{j_n}, \) and

\[ \chi(\pi) := \Pi_{(i,j) \in J(\pi)} c_{ij} = \Pi_{(i,j) \in J(\pi)} s_{ij} b_{ij}, \tag{54} \]

\( J(\pi) := \{(i, j) : 1 \leq i \leq j \leq n\}, \pi(i) > \pi(j), n_i \) is the number of elements of the set \( K(\pi) := \{(i, j) \in J(\pi) : i = j\}. \) It follows from the theorem of Bożejko and Speicher [40] that the corresponding scalar product is positive definite.

6 Appendix

Let us shortly recall the concept of commutation factors on an arbitrary abelian group \( \Gamma, \) [31]. A mapping \( c : \Gamma \times \Gamma \longrightarrow \mathbb{C} \) such that

\[ c(\alpha + \beta, \gamma) = c(\alpha, \gamma)c(\alpha, \gamma), \quad c(\alpha, \beta + \gamma) = c(\alpha, \beta)c(\beta, \gamma), \tag{55} \]

is said to be a bicharacter on \( \Gamma. \) If in addition we have the relation

\[ c(\alpha, \beta)c(\beta, \alpha) = 1, \tag{56} \]

then \( c \) is said to be a commutation factor on \( \Gamma. \) We use here the following notation \( c(\sigma_i, \sigma_i) \equiv c_{ii} \equiv q_i \) and \( c(\sigma_i, \sigma_j) = c_{ij} \) for \( i \neq j, \) where \( q_i = \pm 1 \) or \( -1 \) is said to be a parity of \( c, c_{ij} \in \mathbb{C} \setminus \{0\} \) are deformation parameters such that \( c_{ij}c_{ji} = 1; \alpha = \Sigma_i \alpha^i \sigma_i, \beta = \Sigma_j \alpha^j \sigma_j, \sigma_i \) are generators of \( \Gamma. \) Note that
we have in general the following formula for the commutation factor $c$ on the group $\Gamma$

$$c(\alpha, \beta) = \Pi_{i,j}(c_{ij})^{\alpha_i \beta_j} = \Pi_i(c_{ii})^{\alpha_i \beta_i} \Pi_{i<j}(c_{ij})^{\alpha_i \beta_j - \alpha_j \beta_i}. \quad (57)$$

It is obvious that the set $\Gamma_0 := \{\alpha \in \Gamma : c(\alpha, \alpha) = 1\}$ is a subgroup of $\Gamma$ of index at most 2. This means that the quotient $\Gamma / \Gamma_0$ is isomorphic to the group $\mathbb{Z}_2$ or is trivial. The commutation factor $c$ can be given in the following form

$$c(\alpha, \beta) = c'(\alpha, \beta) b(\alpha, \beta), \quad (58)$$

where $b$ and $c'$ are two new commutation factors on $\Gamma$. The factor $b$ is defined by the relations

$$b(\sigma_i, \sigma_j) \equiv b_{ij} := \begin{cases} +1 & \text{for;} \ i = j \\ -c_{ij} & \text{for } i \neq j \text{ if } q_i = q_j = -1 \\ c_{ij} & \text{in the remaining cases} \end{cases} \quad (59)$$

It follows immediately from the formula (57) that

$$b(\alpha, \beta) = \Pi_{i<j}(b_{ij})^{\alpha_i \beta_j - \alpha_j \beta_i}. \quad (60)$$

The factor $c'$ is defined by the formula $c'_{ii} \equiv c_{ii} \equiv q_i$ and

$$c'(\sigma_i, \sigma_j) \equiv c'_{ij} = \begin{cases} -1 & \text{if } q_i = q_j = -1, \\ +1 & \text{otherwise} \end{cases} \quad (61)$$

for $i \neq j$. It is interesting that the factor $c'$ can be reduced to the group $\pi(\Gamma) \equiv \Gamma / \Gamma_0$, where $\pi : \Gamma \rightarrow \Gamma / \Gamma_0$ is the quotient map. We can see that there is the relation

$$c'(\alpha, \beta) = (-1)^{\pi(\alpha) \pi(\beta)}. \quad (62)$$

For $\Gamma = \mathbb{Z}^N \equiv Z \oplus \ldots \oplus Z$ ($N$-sumands) we have

$$q_i = (-1)^{S_i}, \quad b_{ij} = \omega^{\Omega_{ij}}, \quad (63)$$

where $S_i = 0, 1$, and $\Omega_{ij}$ are elements of a skew-symmetric integer-valued, matrix, and $\omega \neq -1$ is some complex parameter. If $\omega = exp\left(\frac{2\pi i}{n}\right)$, $n \geq 3$ and $S_i \equiv 0$, then the grading group $G \equiv \mathbb{Z}^N$ can be reduced to $\Gamma = \mathbb{Z}_n^N \equiv \ldots \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$.
If $\omega = -1$, then the grading group $\Gamma$ can be reduced to the group

$$G \equiv Z_2^N := Z_2 \oplus \ldots \oplus Z_2 \quad (N \text{sumands}).$$

(64)

In follows from our considerations in the Section 2 that for the charged particle in the singular magnetic field we have $S_i = N + 1 \pmod{2}$, and

$$\Omega_{ij} := \begin{cases} 0 & \text{for} \quad i = j, \\ 1 & \text{for} \quad i \neq j. \end{cases}$$

(65)

This means that

$$b_{ij} := \begin{cases} +1 & \text{for} \quad i = j, \\ -1 & \text{for} \quad i \neq j. \end{cases}$$

(66)

and the factor $c$ is given by the formula

$$c_{ij} = -(-1)^N(-1)^{\Omega_{ij}}.$$  

(67)

In such case if the number $N$ of fluxes is even (i.e., for composite fermions) we obtain

$$c_{ij} = -(-1)^{\Omega_{ij}} = \begin{cases} -1 & \text{for} \quad i = j, \\ +1 & \text{for} \quad i \neq j. \end{cases}$$

(68)

If $N$ is odd (composite bosons), then we obtain

$$c_{ij} = (-1)^{\Omega_{ij}} := \begin{cases} +1 & \text{for} \quad i = j, \\ -1 & \text{for} \quad i \neq j. \end{cases}$$

(69)

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