GRADIENT ESTIMATES FOR THE ALLEN-CAHN EQUATION ON RIEMANNIAN MANIFOLDS

SONGBO HOU

Abstract. In this paper, we consider bounded positive solutions to the Allen-Cahn equation on complete noncompact Riemannian manifolds without boundary. We derive gradient estimates for those solutions. As an application, we get a Liouville type theorem on manifolds with nonnegative Ricci curvature.

1. Introduction

The Allen-Cahn equation
\[ \Delta u + (1 - u^2)u = 0, \] (1.1)
has its origin in the gradient theory of phase transitions [1], and has attracted a lot of attentions in the last decades. The famous De Giorgi conjecture states that for \( n \leq 8 \), any entire solution to (1.1) in \( \mathbb{R}^n \) with \(|u| < 1\) which is monotone in one direction should be one-dimensional [6]. The conjecture was proved in dimension 2 by Ghoussoub-Gui [8] and in dimension 3 by Ambrosio-Cabré [2], and in dimensions \( 4 \leq n \leq 8 \) by Savin [16], under an extra assumption. For \( n \geq 9 \), the conjecture is false [7].

Solutions to the Allen-Cahn equation have the intricate connection to the minimal surface theory. There are many results in the literature, such as, solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [14]. So the equation is also an interesting topic for geometry.

The gradient estimate is a useful method in the study of elliptic and parabolic equations. It was originated by Yau [20], Cheng-Yau [5], and Li-Yau [11], and was extended by many authors, say Li [9], Negrin [13], Souplet-Zhang [17], Ma [10], Yang [18,19], Cao [4] for various purposes. In this paper, we consider bounded positive solutions to Eq.(1.1) and get the following theorem.

Theorem 1.1. Let \( M \) be a complete noncompact \( n \)-dimensional Riemannian manifold without boundary. Denote by \( B_{2R}(2R) \) the geodesic ball of radius 2R around

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$P \in M$. Suppose $\text{Ric} \geq -K(2R)$ in $B_p(2R)$ with $K(2R) \geq 0$, $u$ is a bounded positive smooth solution of (1.1) on $M$ $u \leq C$ where $C$ is a positive constant.

(1) If $C \leq 1$, then we have

$$\frac{|\nabla u|^2}{u^2} + \frac{2}{3}(1 - u^2) \leq \frac{n}{1 - \varepsilon} \left( \frac{2C_1^2 + (n - 1)C_1^2(1 + R\sqrt{K(2R)}) + C_2}{R^2} + \frac{2n}{(1 - \varepsilon) R^2} + 2K(2R) \right)$$

on $B_p(R)$, where $C_1, C_2$ are positive constants, $0 < \varepsilon < 1$.

(2) If $C > 1$, then we have

$$\frac{|\nabla u|^2}{u^2} + s(1 - u^2) \leq \frac{ns^2}{2(1 - \varepsilon)} \left( \frac{n}{4(1 - \varepsilon) (sq + s - 1)} \frac{s^2}{R^2} \frac{C_1^2 2C_1^2 + (n - 1)C_1^2(1 + R\sqrt{K(2R)}) + C_2}{R^2} \right) + \frac{ns^2}{2(1 - \varepsilon)(s - 1)} K(2R) + \frac{s}{q} \sqrt{\frac{n}{2(1 - \varepsilon)}} C^2$$

on $B_p(R)$, where $C_1, C_2$ are positive constants; $0 < \varepsilon < 1$, $s > 1$, $q > 0$ such that $\frac{2(1 - \varepsilon)}{n} \frac{s - 1}{sq} \geq \frac{1}{\varepsilon} - 1 + \frac{(3s - 1)^2}{2}$. In particular, we can choose $q = \frac{2(1 - \varepsilon)(s - 1)}{ns \left[ \frac{1}{\varepsilon} - 1 + \frac{(3s - 1)^2}{2} \right]}$. Taking $s = 2$ and $\varepsilon = 1/2$, we get

$$\frac{|\nabla u|^2}{u^2} + 2(1 - u^2) \leq 4n \left( \frac{54n^2}{27n + 2} \frac{C_1^2}{R^2} + \frac{2C_1^2 + (n - 1)C_1^2(1 + R\sqrt{K(2R)}) + C_2}{R^2} \right) + 4nK(2R) + 54n\sqrt{n}C^2.$$

As a consequence of Theorem 1.1, we have the following:

**Corollary 1.1.** Let $M$ be a complete noncompact $n$-dimensional Riemannian manifold with Ricci tensor $\text{Ric} \geq -k$ ($k \geq 0$). Suppose $u$ is a positive solution of (1.1) and $u \leq C$.

(1) If $C \leq 1$, we have

$$\frac{|\nabla u|^2}{u^2} + \frac{2}{3}(1 - u^2) \leq \frac{2nk}{1 - \varepsilon}.$$

Letting $\varepsilon$ approach zero, we get

$$\frac{|\nabla u|^2}{u^2} + \frac{2}{3}(1 - u^2) \leq 2nk.$$

Furthermore,

$$\frac{|\nabla u|^2}{u^2} \leq 2nk.$$
(2) If $C > 1$, we have
\[
\frac{|\nabla u|^2}{u^2} + s(1 - u^2) \leq \frac{ns^2k}{2(1 - \varepsilon)(s - 1)} + \frac{s}{q} \sqrt{\frac{n}{2(1 - \varepsilon)}} C^2.
\]
In particular, choosing $s = 2$ and $\varepsilon = 1/2$, we have
\[
\frac{|\nabla u|^2}{u^2} \leq 4nk + (54n\sqrt{n} + 2)C^2.
\]
Furthermore,
\[
|\nabla u|^2 \leq (4nk + (54n\sqrt{n} + 2)C^2)C^2.
\]

For an application of Corollary 1.1, we get the following Liouville type theorem:

**Theorem 1.2.** Let $M$ be a complete noncompact $n$-dimensional Riemannian manifold with nonnegative Ricci curvature. If $u$ is a solution of (1.1) with $0 < u \leq 1$, then $u$ is equal to 1 identically on $M$.

In general, let $F \in C^2(\mathbb{R})$ be a nonnegative function and $u \in C^3(\mathbb{R}^n)$ a bounded entire solution in $\mathbb{R}^n$ of the equation
\[
\Delta u = f(u),
\]
where $f = F'$ is the first derivative of $F$. L. Modica [12] proved that $|\nabla u|^2(x) \leq 2F(u(x))$ for every $x \in \mathbb{R}^n$. Later Ratto-Rigoli [15] extended Modica’s result to manifolds with nonnegative Ricci curvature. Also the conclusion of Theorem 1.2 can be deduced from the result of Ratto and Rigoli by setting $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2 + \frac{1}{4}$. However our result gives an explicit bound of $|\nabla u|$ in the case $k \neq 0$. In addition, Corollary 1.1 implies that the equation (1.1) does not admit an entire solution with values in $(0, 1)$ on manifolds with nonnegative Ricci curvature. The method in this paper can be also applied to the equation
\[
\Delta u + u^p - u^q = 0,
\]
where $p, q \in \mathbb{R}$.

The rest of the paper is arranged as follows. In Section 2, we prove a basic lemma. In Section 3, we prove the main results.

2. Basic Lemma

We consider
\[
W(x) = u^{-q},
\]
as the one defined in [9], where $q$ is a positive constant to be chosen later. A straightforward computation shows that
\[ \nabla W = -qu^{-q-1}\nabla u, \]
\[ |\nabla W|^2 = q^2u^{-2q-2}|\nabla u|^2, \]
\[ \frac{|\nabla W|^2}{W^2} = q^2u^{-2}|\nabla u|^2, \quad (2.1) \]

\[ \Delta W = q(q+1)u^{-q-2}|\nabla u|^2 - qu^{-q-1}\Delta u \]
\[ = \frac{q+1}{q} \frac{|\nabla W|^2}{W} + qW - qW^{\frac{q+2}{q}}. \quad (2.2) \]

We introduce the function
\[ F(x) = \frac{|\nabla W|^2}{W^2} + \alpha(1 - W^{-2/q}), \quad (2.3) \]
where \( \alpha \) is a positive constant to be fixed later.

Now we calculate
\[ \nabla F(x) = \frac{\nabla |\nabla W|^2}{W^2} - 2\frac{|\nabla W|^2 \nabla W}{W^3} + \frac{2\alpha}{q} W^{-(q+2)/q} \nabla W, \quad (2.4) \]
\[ \Delta F(x) = \frac{2|\nabla^2 W|^2}{W^2} + \frac{2\langle \nabla W, \Delta \nabla W \rangle}{W^3} - 8\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} \]
\[ + 6\frac{|\nabla W|^4}{W^4} - 2\frac{|\nabla W|^2 \Delta W}{W^3} - \frac{2\alpha(q+2)}{q^2} W^{-2/q} \frac{|\nabla W|^2}{W} \]
\[ + \frac{2\alpha}{q} W^{-(q+2)/q} \Delta W. \quad (2.5) \]

Noting (2.2) we have
\[ \frac{2\langle \nabla W, \Delta \nabla W \rangle}{W^2} = \frac{2\langle \nabla W, \nabla \Delta W \rangle}{W^2} + 2\text{Ric} \langle \nabla W, \nabla W \rangle \]
\[ = \frac{4(q+1) }{q} \frac{\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} - \frac{2(q+1)}{q} \frac{|\nabla W|^4}{W^4} \]
\[ + \frac{2|\nabla W|^2}{W^2} \left[ q - (q-2)W^{-2/q} \right] + \frac{2\text{Ric} \langle \nabla W, \nabla W \rangle}{W^2}, \quad (2.6) \]
\[ - \frac{2|\nabla W|^2 \Delta W}{W^3} = \frac{2(q+1)}{q} \frac{|\nabla W|^4}{W^4} - 2q \frac{|\nabla W|^2}{W^2} + 2qW^{-\frac{2}{q}} \frac{|\nabla W|^2}{W^2}, \quad (2.7) \]
\[ \frac{2\alpha}{q} W^{-(q+2)/q} \Delta W = \frac{2\alpha(q+1)}{q^2} W^{-2/q} \frac{|\nabla W|^2}{W^2} + 2\alpha W^{-2/q} - 2\alpha W^{-4/q}. \quad (2.8) \]

By the Hölder inequality, we have
\[ \frac{2\epsilon |\nabla^2 W|^2}{W^2} + \frac{2}{\epsilon} \frac{|\nabla W|^4}{W^4} \geq 4 \frac{\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3}. \]
Hence
\[
2|\nabla^2 W|^2 - \frac{8\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} + 6\frac{|\nabla W|^4}{W^4} \geq \frac{2(1 - \varepsilon)|\nabla^2 W|^2}{W^2} - 4\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} + \left(6 - \frac{2}{\varepsilon}\right)\frac{|\nabla W|^4}{W^4},
\]
where \(0 < \varepsilon < 1\).

Using the fact \(|\nabla^2 W|^2 \geq \frac{1}{n}(\Delta W)^2\), we get
\[
\frac{2|\nabla^2 W|^2}{W^2} - \frac{8\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} + 6\frac{|\nabla W|^4}{W^4} \geq \frac{2(1 - \varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 4\left(\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} - \frac{|\nabla W|^4}{W^4}\right) \quad (2.9)
\]
By (2.4),
\[
\nabla F \cdot \nabla \log W = \frac{2\langle \nabla^2 W, \nabla W \otimes \nabla W \rangle}{W^3} - \frac{2|\nabla W|^4}{W^4} + \frac{2\alpha}{q} W^{-\frac{2}{q}} |\nabla W|^2. \quad (2.10)
\]
From (2.5) to (2.10), we obtain
\[
\Delta F \geq \frac{2(1 - \varepsilon)}{n} \left(\frac{\Delta W}{W}\right)^2 - 2\left(\frac{1}{\varepsilon} - 1\right)\frac{|\nabla W|^4}{W^4} + \frac{2}{q} \langle \nabla F, \nabla \log W \rangle + \left(4 - \frac{6\alpha}{q^2}\right) \frac{|\nabla W|^2}{W^2} W^{-\frac{2}{q}} + 2\text{Ric} \langle \nabla W, \nabla W \rangle W^2 + 2\alpha W^{-\frac{2}{q}} (1 - W^{-\frac{2}{q}}). \quad (2.11)
\]
It follows from (2.2) and (2.3) that
\[
\frac{\Delta W}{W} = \frac{q}{\alpha} F + \left(\frac{q + 1}{q} - \frac{q}{\alpha}\right) \frac{|\nabla W|^2}{W^2}. \quad (2.12)
\]
Set \(\alpha = sq^2\), then
\[
\frac{\Delta W}{W} = \frac{1}{sq} F + \left(\frac{q + 1}{q} - \frac{1}{sq}\right) \frac{|\nabla W|^2}{W^2} = \frac{1}{sq} F + \left(\frac{q + 1 - 1/s}{q}\right) \frac{|\nabla W|^2}{W^2}. \quad (2.13)
\]
Substituting (2.13) into (2.11) gives
\[
\Delta F \geq \frac{2(1 - \varepsilon)}{n} \frac{1}{s^2 q^2} F^2 + \left[ \frac{2(1 - \varepsilon)}{n} \frac{(s q + s - 1)^2}{s^2 q^2} - 2 \left( \frac{1}{\varepsilon} - 1 \right) \right] \frac{|\nabla W|^4}{W^4} \\
+ \frac{4(1 - \varepsilon)}{n} \frac{(s q + s - 1)}{s^2 q^2} F \frac{|\nabla W|^2}{W^2} + \frac{2}{q} \frac{\langle \nabla F, \nabla \log W \rangle}{W} \\
+ (4 - 6 s) \frac{|\nabla W|^2}{W^2} W^{-\frac{2}{q}} \\
+ \frac{2 \text{Ric} \langle \nabla W, \nabla W \rangle}{W^2} + 2 s q W^{-\frac{2}{q}} (1 - W^{-\frac{2}{q}}).
\]

We get the following lemma.

**Lemma 2.1.** Let \( M \) be a complete noncompact \( n \)-dimensional Riemannian manifold without boundary. If \( F \) is defined by (2.3) where \( \alpha = s q^2 \), then we have
\[
\Delta F \geq \frac{2(1 - \varepsilon)}{n} \frac{1}{s^2 q^2} F^2 + \left[ \frac{2(1 - \varepsilon)}{n} \frac{(s q + s - 1)^2}{s^2 q^2} - 2 \left( \frac{1}{\varepsilon} - 1 \right) \right] \frac{|\nabla W|^4}{W^4} \\
+ \frac{4(1 - \varepsilon)}{n} \frac{(s q + s - 1)}{s^2 q^2} F \frac{|\nabla W|^2}{W^2} + \frac{2}{q} \frac{\langle \nabla F, \log W \rangle}{W} \\
+ (4 - 6 s) \frac{|\nabla W|^2}{W^2} W^{-\frac{2}{q}} + 2 s q W^{-\frac{2}{q}} (1 - W^{-\frac{2}{q}}).
\]

\( (2.14) \)

**3. Proof of Main Results**

*Proof of Theorem 1.1.* Chose a cut-off function \( \chi \in C^2[0, +\infty) \) such that \( \chi(r) = 1 \) for \( r \leq 1 \), \( \chi(r) = 0 \) for \( r > 2 \), and \( 0 \leq \chi(r) \leq 1 \). In addition, we require \( \chi \) satisfies
\[
-C_1 \leq \chi^{-1/2}(r) \chi'(r) \leq 0 \quad \text{and} \quad \chi''(r) \geq -C_2,
\]
where \( C_1, C_2 \) are positive constants.

For a fixed point \( p \), denote by \( r(x) \) the geodesic distance between \( x \) and \( P \). Define
\[
\phi(x) = \chi \left( \frac{r(x)}{R} \right).
\]

It is clear that
\[
|\nabla \phi|^2 \leq \frac{C_1^2}{R^2} \phi.
\]

By the Laplacian comparison theorem, we get
\[
\Delta \phi \geq -\frac{(n - 1) C_1^2 (1 + R \sqrt{K(2R)}) + C_2}{R^2}.
\]
Now we consider the function $\phi(x)F(x)$. By the argument of Calabi [3], we assume that the function $\phi(x)F(x)$ is smooth in $B_P(2R)$. Let $z$ be the point where $\phi F$ achieves its maximum in $B_P(2R)$. We can assume that $\lambda := \phi(z)F(z) > 0$ since the theorem is obviously true if $\lambda \leq 0$. Then we have

$$\nabla(\phi F) = \nabla \phi F + \phi \nabla F = 0 \quad (3.1)$$

and

$$\Delta(\phi F) \leq 0 \quad (3.2)$$

at the point $z$.

Using Eq. (3.1), we have

$$\nabla F = -\frac{\nabla \phi}{\phi} F.$$

By (3.2), we have

$$\Delta \phi \cdot F + 2\nabla \phi \cdot \nabla F + \phi \Delta F \leq 0.$$

Thus we obtain

$$F \Delta \phi + \phi \Delta F - 2F \phi^{-1} |\nabla \phi|^2 \leq 0$$

at $z$.

Then for

$$B = \frac{2C_1^2 + (n-1)C_1^2(1 + R\sqrt{K(2R)}) + C_2}{R^2},$$

we have

$$\phi \Delta F \leq BF.$$

Multiplying both sides of (2.14) by $\phi^2$, we obtain at $z$,

$$B\phi F \geq \frac{2(1-\varepsilon)}{n} \frac{1}{s^2q^2} (\phi F)^2 + \phi^2 \left[ \frac{2(1-\varepsilon)}{n} \frac{(sq + s - 1)^2}{s^2q^2} - 2 \left( \frac{1}{\varepsilon} - 1 \right) \right] \frac{|\nabla W|^4}{W^4} + \phi^2 \frac{4(1-\varepsilon)}{n} \frac{(sq + s - 1)}{s^2q^2} F \frac{|\nabla W|^2}{W^2} + \frac{2}{q} \phi^2 \langle \nabla F, \log W \rangle + 2W^{-\frac{2}{s}} \phi^2 F + \frac{2\text{Ric} \langle \nabla W, \nabla W \rangle}{W^2} \phi^2 - (6s - 2) \phi^2 \frac{|\nabla W|^2}{W^2} W^{-\frac{2}{q}}. \quad (3.3)$$

We consider two cases: (1) $C \leq 1$ and (2) $C > 1$.

(1) Since $u \leq 1$, it is easy to see that

$$\frac{2\text{Ric} \langle \nabla W, \nabla W \rangle}{W^2} \phi^2 \geq -2K(2R) \frac{|\nabla W|^2}{W^2} \phi^2 \geq -2K(2R)\phi F \quad (3.4)$$
and
\[ 2W^{-\frac{2}{q}} \phi^2 F - (6s - 2) \phi^2 \frac{|\nabla W|^2}{W^2} W^{-\frac{2}{q}} \geq -(6s - 4) \phi F \] (3.5)

if \( s \geq \frac{2}{3} \).

Substituting (3.4), (3.5) into (3.3), and choosing \( s = \frac{2}{3} \) and \( q > 0 \) small enough such that
\[ \frac{(1-\varepsilon)(eq+\delta-1)^2}{s^q} \geq \frac{1}{\varepsilon} - 1, \]
then we have
\[ B\phi F \geq \frac{9(1-\varepsilon)}{2nq^2} (\phi F)^2 - \frac{3(1-\varepsilon)}{nq^2} (\phi F)^2 - \frac{2}{q} F\phi (\nabla \phi, \frac{\nabla W}{W}) - 2K(2R)\phi F. \] (3.6)

We take the similar technique as in [10]. Clearly,
\[ -\frac{2}{q} F\phi (\nabla \phi, \frac{\nabla W}{W}) \geq -\frac{2C_1}{qR} (\phi F)^{3/2}. \] (3.7)

Combining (3.6) and (3.7), we arrive at
\[ B\phi F \geq \frac{3(1-\varepsilon)}{2nq^2} (\phi F)^2 - \frac{2C_1}{qR} (\phi F)^{3/2} - 2K(2R)\phi F. \]

It follows that
\[ B + \frac{2C_1}{qR} (\phi F)^{1/2} + 2K(2R) \geq \frac{3(1-\varepsilon)}{2nq^2} (\phi F). \]

In other words, we get
\[ B + \frac{2C_1}{qR} \lambda^{1/2} + 2K(2R) \geq \frac{3(1-\varepsilon)}{2nq^2} \lambda. \] (3.8)

Note that
\[ \frac{2C_1}{qR} \lambda^{1/2} \leq \frac{(1-\varepsilon)}{2nq^2} \lambda + \frac{2n}{(1-\varepsilon)} \frac{C_1^2}{R^2}. \] (3.9)

Substituting (3.9) into (3.8), we get
\[ B + \frac{2n}{(1-\varepsilon)} \frac{C_1^2}{R^2} + 2K(2R) \geq \frac{1-\varepsilon}{nq^2} \lambda. \]

Then we get
\[ \lambda \leq \frac{nq^2}{1-\varepsilon} \left( B + \frac{2n}{(1-\varepsilon)} \frac{C_1^2}{R^2} + 2K(2R) \right) \]
\[ = \frac{nq^2}{1-\varepsilon} \left( 2C_1^2 + (n-1)C_1^2(1 + R\sqrt{K(2R)}) + C_2 \right. \]
\[ + \left. \frac{2n}{(1-\varepsilon)} \frac{C_1^2}{R^2} + 2K(2R) \right). \] (3.10)
(2) By the condition on Ricci curvature, we derive
\[ \frac{2\text{Ric} \langle \nabla W, \nabla W \rangle}{W^2} \phi^2 \geq -2K(2R)\frac{\vert \nabla W \vert^2}{W^2} \phi^2. \]

By Hölder’s inequality, we get
\[ 2K(2R)\phi^2 \frac{\vert \nabla W \vert^2}{W^2} \leq \frac{2}{n} \frac{(1 - \varepsilon)(s - 1)^2}{s^2 q^2} \frac{\vert \nabla W \vert^4}{W^4} \phi^2 + \frac{n s^2 q^2}{2(1 - \varepsilon)(s - 1)^2} K^2(2R) \phi^2 \]
and
\[ (6s - 2) \phi^2 \frac{\vert \nabla W \vert^2}{W^2} W^{-2/q} \leq \frac{(6s - 2)^2}{4} \frac{\vert \nabla W \vert^4}{W^4} \phi^2 + C^4 \phi^2. \]

By (3.4),
\[ \frac{2}{q} \phi^2 \langle \nabla F, \log W \rangle = -\frac{2}{q} \phi F \langle \nabla \phi, \nabla W \rangle. \]

Using Hölder’s inequality again gives
\[ \frac{2}{q} \phi F \langle \nabla \phi, \nabla W \rangle \leq \frac{4}{n} (1 - \varepsilon) \frac{(sq + s - 1)}{s^2 q^2} \frac{\vert \nabla W \vert^2}{W^2} F \phi^2 + \frac{n s^2 \phi F}{4 (1 - \varepsilon) (sq + s - 1)} \frac{\vert \nabla \phi \vert^2}{\phi}. \]

Choose \( s > 1 \) and \( q > 0 \) such that \( \frac{2(1 - \varepsilon)}{n} \frac{s - 1}{sq} \geq \frac{1}{\varepsilon} - 1 + \frac{(3s - 1)^2}{2} \). Then (3.3) becomes
\[ B \phi F \geq \frac{2(1 - \varepsilon)}{n} \frac{1}{s^2 q^2} (\phi F)^2 - \frac{n s^2 \phi F}{4 (1 - \varepsilon) (sq + s - 1)} \frac{C^2}{R^2} \phi F \\
- \frac{n s^2 q^2}{2 (1 - \varepsilon) (s - 1)^2} K^2(2R) - C^4, \]
whence
\[ 0 \geq \frac{2(1 - \varepsilon)}{n} \frac{1}{s^2 q^2} \lambda^2 - \left( \frac{n s^2}{4 (1 - \varepsilon) (sq + s - 1)} \frac{C^2}{R^2} + B \right) \lambda \\
- \frac{n s^2 q^2}{2 (1 - \varepsilon) (s - 1)^2} K^2(2R) - C^4. \]

Thus
\[ \lambda \leq \frac{n s^2 q^2}{2(1 - \varepsilon)} \left( \frac{n s^2}{4 (1 - \varepsilon) (sq + s - 1)} \frac{C^2}{R^2} + \frac{2C^2}{4} + (n - 1)C^2 \right) \\
+ \frac{n s^2 q^2}{2(1 - \varepsilon)(s - 1)} K^2(2R) + sq \sqrt{\frac{n s^2 q^2}{2(1 - \varepsilon)} C^2}. \] (3.11)

Combining (3.10) and (3.11), we conclude the theorem. \( \square \)

**Proof of Corollary 1.1.** Passing to the limit \( R \to +\infty \) in the estimates of Theorem 1.1, we get the desired results. \( \square \)
Proof of Theorem 1.2. Suppose that $M$ is a complete noncompact Riemannian manifold with nonnegative Ricci curvature. If $u$ is a solution of (1.1) on $M$ and $0 < u \leq 1$, then by Corollary 1.1, we get
\[ \frac{\| \nabla u \|^2}{u^2} + \frac{2}{3}(1 - u^2) \leq 0. \]
It follows that $|\nabla u| \equiv 0$ and $u \equiv 1$. This concludes Theorem 1.2. \hfill $\square$

References

[1] S. M. Allen, J. W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall. 27 (1979) 1085-1095.
[2] L. Ambrosio, X. Cabré, Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi, J. Amer. Math. Soc. 13(4) (2000) 725-739 (electronic).
[3] E. Calabi, An extension of E. Hopf’s maximum principle with application to Riemannian geometry, Duke Math. J. 25 (1958) 45-46.
[4] X. Cao, B. Fayyazuddin Ljungberg, B. Liu, Differential Harnack estimates for a nonlinear heat equation, J. Funct. Anal. 265 (2013) 312-2330.
[5] S. Y. Cheng, S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28(3) (1975) 333-354.
[6] E. De Giorgi, Convergence problems for functionals and operators, In Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pages 131-188. Pitagora, Bologna, 1979.
[7] M. del Pino, M. Kowalczyk, J. Wei, On De Giorgi’s conjecture in dimension $N \geq 9$, Ann. of Math. (2), 174(3) (2011) 1485-1569.
[8] N. Ghoussoub, C. Gui, On a conjecture of De Giorgi and some related problems, Math. Ann., 311(3) (1998) 481-491.
[9] Jiayu Li, Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds, J. Funct. Anal. 100 (1991) 233-256.
[10] L. Ma, Gradient estimates for a simple elliptic equation on non-compact Riemannian manifolds, J. Funct. Anal. 241 (2006) 374-382.
[11] P. Li, S. T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156 (1986) 153-201.
[12] L. Modica, A gradient bound and a liouville theorem for nonlinear poisson equations, Commun. Pure Appl. Math. 38 (1985) 679-684.
[13] E. Negrin, Gradient estimates and a Liouville type theorem for the Schrödinger operator, J. Funct. Anal. 127 (1995) 198-203.
[14] F. Pacard, M. Ritoré, From the constant mean curvature hypersurfaces to the gradient theory of phase transitions, J. Differential Geom. 64(3) (2003) 356-423.
[15] A. Ratto, M. Rigoli, Gradient bounds and Liouville’s type theorems for the Poisson equation on complete Riemannian manifolds, Tohoku Mathematical Journal 47(4) (1995) 509-519.
[16] O. Savin. Regularity of flat level sets in phase transitions, Ann. of Math. (2) 169 (1) (2009) 41-78.
[17] P. Souplet, Q. S. Zhang, Sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds. Bull. London Math. Soc. 38 (2006) 1045-1053.
[18] Y. Yang, Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds, Proc. Amer. Math. Soc. 136 (2008) 4095-4102.
[19] Y. Yang, Gradient estimates for the equation $\Delta u + cu^{-\alpha} = 0$ on Riemannian manifolds, Acta Math. Sin. (Engl. Ser.) 26 (2010) 1177-1182.
[20] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975) 201-228.

DEPARTMENT OF APPLIED MATHEMATICS, COLLEGE OF SCIENCE, CHINA AGRICULTURAL UNIVERSITY, BEIJING, 100083, P.R. CHINA

E-mail address: housb10@163.com