BAYESIAN METRIC RECONSTRUCTION WITH GRAVITATIONAL WAVE OBSERVATIONS

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OVERVIEW

1 INTRODUCTION

2 METHODS

3 RESULTS

4 DISCUSSION

5 CONCLUSIONS
Can one hear the shape of a black hole?
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**BLACK HOLE RINGDOWN**

**FIGURE 1**: Time evolution of a perturbation that scattered with a black hole as seen by an observer far away (left normal scale, right log scale).
Black Holes

gravitational theory

grav. wave detections

background metric

quasi-normal modes (QNMs)

parameter estimation

perturbation equations
**Figure 2:** **Left:** Reconstructed final mass $M_f$ and final spin $a_f$ using different parts of the signal. **Right:** Reconstructed $l = m = 2, n = 0$ QNM frequency and damping time using different starting times as well as IMR result. Taken from B.P. Abbott et al. (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. 116, 221101, 2016, [1]

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**Figure 3**: Reconstructed final mass $M_f$ and final spin magnitude $\chi_f$. **Left**: using different overtone numbers to fit the ringdown, as well as IMR result. **Right**: as function of different starting times and overtone numbers, as well as IMR. Taken from Maximiliano Isi, Matthew Giesler, Will M. Farr, Mark A. Scheel, and Saul A. Teukolsky, Phys. Rev. Lett. 123, 111102, 2019, https://doi.org/10.1103/PhysRevLett.123.111102,[2]
**Figure 4:** **Left:** Combined relative errors of the $l = 2, n = 0$ QNM for different combinations of $\{M, a_0, b_0\}$ and Schwarzschild $M = 1$ as reference. **Right:** Some of the corresponding perturbation potentials. Taken from SHV and Kostas D. Kokkotas, Phys. Rev. D 100, 044026 2019, https://doi.org/10.1103/PhysRevD.100.044026, [3].
Our framework has several “building blocks”: 

- parametrized black hole space-time (Rezzolla-Zhidenko metric)
- perturbation equations (\(d R_{mn} = 0\))
- computation of QNMs (higher order WKB method)
- RZ models, QNM subsets, QNM precision
- Bayesian analysis via Markov chain Monte Carlo (PYMC3)

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We use the Rezzolla-Zhidenko (RZ) metric\textsuperscript{1}

- parametrization for spherically symmetric and static black holes
- continued fraction expansion for $\tilde{A}(x)$ and $\tilde{B}(x)$
- relation to PPN parameters $\beta$ and $\gamma$ possible

\[ ds^2 = -N^2(r)\,dt^2 + \frac{B^2(r)}{N^2(r)}\,dr^2 + r^2\,d\Omega^2, \quad x \equiv 1 - \frac{r_0}{r}, \quad N^2 = xA(x), \quad (1) \]

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\[
A(x) = 1 - \varepsilon (1 - x) + (a_0 - \varepsilon)(1 - x)^2 + \tilde{A}(x)(1 - x)^3, \quad (2)
\]

\[
B(x) = 1 + b_0(1 - x) + \tilde{B}(x)(1 - x)^2. \quad (3)
\]

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**METHODS**

**BACKGROUND Metric**

We use the Rezzolla-Zhidenko (RZ) metric\(^1\)

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\[
\begin{align*}
\mathrm{d}s^2 &= -N^2(r)\mathrm{d}t^2 + \frac{B^2(r)}{N^2(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \quad x \equiv 1 - \frac{r_0}{r}, \quad N^2 = xA(x), \quad (1) \\
A(x) &= 1 - \varepsilon(1 - x) + (a_0 - \varepsilon)(1 - x)^2 + \tilde{A}(x)(1 - x)^3, \quad (2) \\
B(x) &= 1 + b_0(1 - x) + \tilde{B}(x)(1 - x)^2. \quad (3) \\
\varepsilon &= -\left(1 - \frac{2M}{r_0}\right), \quad a_0 = \frac{(\beta - \gamma)(1 + \varepsilon)^2}{2}, \quad b_0 = \frac{(\gamma - 1)(1 + \varepsilon)}{2}. \quad (4)
\end{align*}
\]

\(^1\)Rezzolla and Zhidenko [4], Phys. Rev. D 90, 084009, 2014
We study “theory agnostic” gravitational axial perturbations

- we consider $\delta R_{\mu\nu} = 0$ for the RZ metric

- corresponds to GR, but also holds for some scalar tensor theories

$$\frac{d^2}{dr^*} Z + \left[ \omega^2 - V_1(r) \right] Z = 0,$$  \hspace{1cm} (5)
We study “theory agnostic” gravitational axial perturbations

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\[
\frac{d^2}{d r^*^2} Z + \left[ \omega^2 - V_l(r) \right] Z = 0, \quad (5)
\]

- also include parametrized modification of the potential ($K$)

\[
V_l(r) = \frac{l(l + 1)}{r^2} N^2(r) - \frac{K}{r} \frac{d}{dr^*} \frac{N^2(r)}{B(r)}, \quad (6)
\]
Quasi-Normal Modes (QNMs) describe ringdown of black holes

- defined by purely outgoing \((r \to \infty)\) and ingoing \((r \to r_0)\) waves
- computation of QNMs via higher order WKB method

\[
\frac{iQ_0}{\sqrt{2Q_0''}} - \Lambda_2 - \Lambda_3 - \Lambda_4 - \Lambda_5 - \Lambda_6 = n + \frac{1}{2},
\]

(7)

with \(Q(r^*) \equiv \omega_n^2 - V_l(r^*)\) evaluated at the maximum of potential\(^2\).

\(^2\)R. A. Konoplya, Phys. Rev. D 68, 024018, 2003, [5]
**Methods**

**Quasi-Normal Modes / WKB**

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with \(Q(r^*) \equiv \omega_n^2 - V_l(r^*)\) evaluated at the maximum of potential\(^2\).

\[
\Lambda_2(n) = \left(\left((-11(V^{(3)})^2 + 9V^{(2)}V^{(4)} - (30(V^{(3)})^2)n + 18V^{(2)}V^{(4)}n - (30(V^{(3)})^2)n^2 + 18V^{(2)}V^{(4)}n^2)\right)/144(V^{(2)})^2\right)
\]  

(8)

\[
\]  

(9)

\(^2\)R. A. Konoplya, Phys. Rev. D 68, 024018, 2003, [5]
Choose finite number of RZ parameters and available QNMs

RZ models

- model\(_1\) \equiv \{M, \varepsilon\}
- model\(_2\) \equiv \{M, \varepsilon, a_0, b_0\}
- model\(_3\) \equiv \{M, \varepsilon, a_1, b_1\}
- model\(_{K1}\) \equiv \{M, \varepsilon, K\}
- model\(_{K2}\) \equiv \{M, \varepsilon, a_0, b_0, K\}
Choose finite number of RZ parameters and available QNMs

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• model$_{K1} \equiv \{M, \varepsilon, K\}$
• model$_{K2} \equiv \{M, \varepsilon, a_0, b_0, K\}$

QNM spectra

• spectrum$_1 \equiv \{l = 2, n = [0, 1]\}$
• spectrum$_2 \equiv \{l = [2, 3], n = [0, 1]\}$
Choose finite number of RZ parameters and available QNMs

RZ models

- model\(_1\) \(\equiv \{M, \varepsilon\}\)
- model\(_2\) \(\equiv \{M, \varepsilon, a_0, b_0\}\)
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QNM spectra

- spectrum\(_1\) \(\equiv \{l = 2, n = [0, 1]\}\)
- spectrum\(_2\) \(\equiv \{l = [2, 3], n = [0, 1]\}\)

We also consider two different precision for both spectra

- relative errors for real and imaginary part of \(\pm 10\%\) or \(\pm 1\%\).
Bayesian Analysis

Bayes theorem:

\[ P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \]  

(10)

with

- \( \theta \) parameters of a model
- \( D \) observed data
Bayes theorem:

\[ P(\theta|D) = \frac{P(D|\theta) P(\theta)}{P(D)} \quad (10) \]

with

- \( \theta \) parameters of a model
- \( D \) observed data

and

- **posterior** \( P(\theta|D) \): probability of parameters given the data
- **likelihood** \( P(D|\theta) \): probability of data given the parameters
- **prior** \( P(\theta) \): probability of parameters before looking at data
- **evidence** \( P(D) \): probability of Data
Markov chain Monte Carlo (MCMC) are a class of algorithms

- directly sample from the posterior distribution
- computationally expensive and details complicated
- many toolkits available, pure application can be easy
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We use Python based framework PyMC3

- combine it with external C++ code
- use standard Metropolis Hastings algorithm
Starting Point
Perturbation Equations and QNM Data

MCMC Step 1
Choose Params

MCMC Step 2
Compute QNMs

MCMC Step 3
Compare current & previous Likelihood

Reject Params

Accept Params
Results for model 1 obtained by using spectrum 1 with ±1% relative error. Left: MCMC parameter estimation. Right top: Exact (black lines) and reconstructed (color lines) potentials $V_2(r)$ and $V_3(r)$. Right bottom: Exact (black lines) and reconstructed (color lines) metric functions $g_{tt}(r)$ and $g_{rr}(r)$. 
**Results**

**MODEL\textsubscript{2} WITH SPECTRUM\textsubscript{2} AT 1\%**

**Figure 5:** Results for model\textsubscript{2} obtained by using spectrum\textsubscript{2} with ±1\% relative error. Left: MCMC parameter estimation. Right top: Exact (black lines) and reconstructed (color lines) potentials $V_2(r)$ and $V_3(r)$. Right bottom: Exact (black lines) and reconstructed (color lines) metric functions $g_{tt}(r)$ and $g_{rr}(r)$. 
**RESULTS**

**MODEL3 WITH SPECTRUM2 AT 1 %**

![Graphs and plots showing results for model3](image)

**FIGURE 6:** Results for model3 obtained by using spectrum2 with ±1 % relative error. Left: MCMC parameter estimation. Right top: Exact (black lines) and reconstructed (color lines) potentials $V_2(r)$ and $V_3(r)$. Right bottom: Exact (black lines) and reconstructed (color lines) metric functions $g_{tt}(r)$ and $g_{rr}(r)$.
**RESULTS**

**MODEL\(_{K1}\) WITH SPECTRUM\(_1\) AT 1\%**

**FIGURE 7:** Results for model\(_{K1}\) obtained by using spectrum\(_1\) with ±1\% relative error. Left: MCMC parameter estimation. Right top: Exact (black lines) and reconstructed (color lines) potentials \(V_2(r)\) and \(V_3(r)\). Right bottom: Exact (black lines) and reconstructed (color lines) metric functions \(g_{tt}(r)\) and \(g_{rr}(r)\).
Results

Model $K_2$ with spectrum $\lambda_2$ at 1% relative error.

**Figure 8:** Results for model $K_2$ obtained by using spectrum $\lambda_2$ with ±1% relative error. Left: MCMC parameter estimation. Right top: Exact (black lines) and reconstructed (color lines) potentials $V_2(r)$ and $V_3(r)$. Right bottom: Exact (black lines) and reconstructed (color lines) metric functions $g_{tt}(r)$ and $g_{rr}(r)$. 
General observation for models, spectra and precision
Overall Findings

General observation for models, spectra and precision

- low dimensional models work well with spectrum $1$ ($l = 2$, $n = [0, 1]$)
- higher dimensional models need spectrum $2$ ($l = [2, 3]$, $n = [0, 1]$)
- benefits from 1% precise QNMs outperform additional $l = 3$ QNMs
- modified perturbation potential decreases reconstruction mildly
- sampled potentials are suitable for WKB treatment
**Left:** Axial perturbations ultra compact stars, V. Ferrari and K. D. Kokkotas, Phys. Rev. D 62, 107504, 2000.

**Right:** Echoes from the abyss: Tentative evidence for Planck-scale structure at black hole horizons, Abedi, Dykaar and Afshordi, Phys. Rev. D 96, 082004 2017.
- Inverse problem for horizonless ultra compact objects is different

- **WKB** method and **Bohr-Sommerfeld** rules\(^3\) are powerful here (approximate, but easier to invert)

\[^3\text{here } E_n \equiv \omega_n^2\]
Inverse problem for horizonless ultra compact objects is different

- WKB method and Bohr-Sommerfeld rules\(^3\) are powerful here (approximate, but easier to invert)

\[
\int_{x_0}^{x_1} \sqrt{E_n - V(x)} \, dx = \pi \left( n + \frac{1}{2} \right) - \frac{i}{4} \exp \left( 2i \int_{x_1}^{x_2} \sqrt{E_n - V(x)} \, dx \right)
\]

\(^3\)here \(E_n \equiv \omega_n^2\)
Discussion

Inverting Bohr-Sommerfeld Rules

- Known for **single** wells or **single** barriers (classical BS rule)\(^4\)

- Extended to **quasi-stationary states** (3 or 4 turning points) \(^5\)

- **Neutron star potentials** with discontinuity (1 turning point) \(^6\)

\[
\mathcal{L}_1(E) = x_1 - x_0 = 2 \frac{\partial}{\partial E} \int_{E_{\text{min}}}^{E} \frac{n(E') + 1/2}{\sqrt{E - E'}} \, dE' \tag{12}
\]

\[
\mathcal{L}_2(E) = x_2 - x_1 = -\frac{1}{\pi} \int_{E}^{E_{\text{max}}} \frac{(dT(E')/dE')}{T(E')\sqrt{E' - E}} \, dE' \tag{13}
\]

\(^4\)Wheeler [6]; Cole and Good [7]
\(^5\)Völkel and Kokkotas [8]; Völkel [9]; Völkel and Kokkotas [10]
\(^6\)Völkel and Kokkotas [11]
\section*{Discussion}

\section*{Inverting Bohr-Sommerfeld Rules}

\begin{align*}
V(x) &= V_{0}(E) + V_{1}(E) + V_{2}(E) + V_{3}(E) \\
L_{1}(E) &= L_{1}(E) + L_{2}(E)
\end{align*}

\begin{align*}
V(r) &= V_{0}(r) + V_{1}(r) + V_{2}(r) + V_{3}(r) \\
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\end{align*}
Several extensions planned:

- inverse problem for non GR QNMs
- address the rotating case
- combine with other approaches (BH shadows)
As a proof of principle addressed in this work
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- Bayesian framework to construct BH metrics from QNMs via MCMC
- allows to quantify constraints on multiple parameters simultaneously
- framework will be extended in multiple directions
- even under idealistic conditions, reconstruction is non-trivial!
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Black hole spectroscopy can be used for inverse spectrum problems!
[1] B. P. Abbott et al. Tests of general relativity with gw150914. Phys. Rev. Lett., 116:221101, May 2016. doi: 10.1103/PhysRevLett.116.221101.

[2] Maximiliano Isi, Matthew Giesler, Will M. Farr, Mark A. Scheel, and Saul A. Teukolsky. Testing the no-hair theorem with GW150914. Phys. Rev. Lett., 123(11):111102, 2019. doi: 10.1103/PhysRevLett.123.111102.

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[9] S. H. Völkel. Inverse spectrum problem for quasi-stationary states. *J. Phys. Commun.*, 2(2):025029, 2018. doi: 10.1088/2399-6528/aaaee2.

[10] S. H. Völkel and K. D. Kokkotas. Wormhole potentials and throats from quasi-normal modes. *Class. Quantum Grav.*, 35(10):105018, May 2018. doi: 10.1088/1361-6382/aabce6.

[11] S. H. Völkel and K. D. Kokkotas. On the inverse spectrum problem of neutron stars. *Class. Quantum Grav.*, 36(11):115002, Jun 2019. doi: 10.1088/1361-6382/ab186e.

[12] Kazunori Akiyama et al. First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *Astrophys. J.*, 875(1):L1, 2019. doi: 10.3847/2041-8213/ab0ec7.

[13] Dimitrios Psaltis et al. *in preparation (2020).*
\[ \tilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{1 + \cdots}}} \],

(14)

\[ \tilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{1 + \cdots}}} \].

(15)