Directed Ramsey and Anti-Ramsey Algebras
and the Flexible Atom Conjecture

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Abstract

We generalize the notion of a Ramsey algebra to directed graphs
(asymmetric relations). We construct \( m \)-color Directed Anti-Ramsey
algebras for \( 2 \leq m \leq 1000 \) (except for \( m = 8 \)). We also construct
\( m \)-color Directed Ramsey algebras for various \( m < 500 \). As a corol-
larly, we give finite representations for relation algebras 35, 37,
78, 83, 80, 82, 1310, 1316, 1313, 1316, and 1315, none of which were
previously known to be finitely representable.

1 Introduction

A Ramsey algebra in \( m \) colors is a partition of a set \( U \times U \) into disjoint
binary relations \( \text{Id}, R_0, \ldots, R_{m-1} \) such that

(I.) \( R_i^{-1} = R_i \);

(II.) \( R_i \circ R_i = R_i^c \);

(III.) for \( i \neq j \), \( R_i \circ R_j = \text{Id}^c \).

Here, \( \text{Id} = \{(x, x) : x \in U\} \) is the identity over \( U \), \( \circ \) is relational composition,
\( -1 \) is relational inverse, and \( c \) is complementation with respect to \( U \times U \).
Ramsey algebras are representations of relation algebras first defined (but not named) in a 1982 paper by Maddux [7]. Kowalski later called the (abstract) relation algebras “Ramsey Relation Algebras” [6].

The usual method of attempting to construct the relations $R_0, \ldots, R_{m-1}$ is a “guess-and-check” prime field method due to Comer [4], as follows: Fix $m \in \mathbb{Z}^+$, and let $X_0 = H$ be a multiplicative subgroup of $\mathbb{F}_p$ of order $(p - 1)/m$, where $p \equiv 1 \pmod{2m}$. Let $X_1, \ldots, X_{m-1}$ be its cosets; specifically, let $X_i = g^i X_0 = \{g^{am+i} \pmod{p} : a \in \mathbb{Z}^+\}$, where $g$ is a generator of $\mathbb{F}_p^\times$.

Suppose the following conditions obtain:

(i.) $-X_i = X_i$;

(ii.) $X_i + X_i = \mathbb{F}_p \setminus X_i$,

(iii.) for $i \neq j$, $X_i + X_j = \mathbb{F}_p \setminus \{0\}$.

Then define $A_i = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : x - y \in X_i\}$. It is easy to check that (i.)-(iii.) imply (I.)-(III.), and we get a Ramsey algebra. Condition (ii.) implies that all the $X_i$s are sum-free. The fact that $p \equiv 1 \pmod{2m}$ implies that the order of $X_0$ is even, which implies $-X_0 = X_0$, i.e., $X_0$ is symmetric. In [7], Ramsey algebras were constructed for all $m \leq 2000$, except for $m = 8, 13$, and it was shown that if $p > m^4 + 5$, then $X_0$ contains a solution to $x + y = z$, hence $\mathbb{F}_p$ does not give an $m$-color Ramsey algebra via Comer’s construction.

A natural generalization is to consider $X_0$ of odd order, so that $-X_0 \cap X_0 = \emptyset$; this gives relations $R_i$ that are antisymmetric. This motivates the following definition.

**Definition 1.** Let $k$ be odd and $n$ be even, with $p = nk + 1$ prime. Let $m = \frac{n}{2}$. Let $g$ be a primitive root modulo $p$. Let $X_0 = \{g^{\alpha n} : 0 \leq \alpha < k\}$ be the multiplicative subgroup of $\mathbb{F}_p$ of index $n$, and let $X_i = \{g^{\alpha n+i} : 0 \leq \alpha < k\}$ be its cosets $(1 \leq i < n)$. Suppose the following conditions obtain:

1. $\forall i, X_i + X_i = \bigcup_{j \neq i} X_j$

2. $\forall i, X_i + X_{i+m} = \{0\} \cup \bigcup_{j \neq i \neq i+m} X_j$

3. $\forall i, j$ with $i \neq j \neq i + m$, $X_i + X_j = \bigcup_{\ell} X_\ell = \mathbb{F}_p^\times$.

Then the $X_i$s are called a (prime field) Directed Ramsey Algebra in $m$ colors.

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Condition 1.-3. make each $X_i$ sum-free, but otherwise all sumsets are as large as possible. One might think that the exclusions in 2. are unnecessary, but one would be mistaken; they are implied by the exclusion $X_i + X_i \not\supseteq X_i$, the “triangle symmetry”

$$X_i + X_j \supseteq X_\ell \iff X_j + X_{\ell+m} \supseteq X_{i+m}$$

and the “rotational symmetry”

$$X_i + X_j \supseteq X_\ell \iff X_{i+a} + X_{j+a} \supseteq X_{\ell+a}.$$

(Proofs of these symmetries are left to the reader.)

We say $m$ colors rather than $n$ colors since the $n$ cosets come in $m$ pairs, where $X_i = -X_{i+m}$. One can give a directed coloring of the edges of $K_p$ in $m$ colors by labelling the vertices with the elements of $\mathbb{F}_p$ with color classes $C_i = \{(x, y) : x - y \in X_i\}$ for $0 \leq i < m$. (So if $X_0$ is “directed red”, then $X_m$ would just be the “backward” red color. The first $m$ cosets $X_0, \ldots, X_{m-1}$ are sufficient to determine the directed edge-coloring of $K_p$.)

In such a directed coloring, the monochromatic “transitive” triangles (i.e., triples $(x, y), (y, z), (x, z)$) are forbidden. This suggests a different generalization, where we forbid monochromatic “intransitive” triangles (i.e., triples $(x, y), (y, z), (z, x)$).

**Definition 2.** Let $k$ be odd and $n$ be even, with $p = nk + 1$ prime. Let $m = \frac{n}{2}$. Let $g$ be a primitive root modulo $p$. Let $X_0 = \{g^{an} : 0 \leq \alpha < k\}$ be the multiplicative subgroup of $\mathbb{F}_p$ of index $n$, and let $X_i = \{g^{an+i} : 0 \leq \alpha < k\}$ be its cosets (1 $\leq i < n$). Suppose the following conditions obtain:

1. $\forall i, X_i + X_i = \bigcup_{j \neq i+m} X_j$

2. $\forall i, X_i + X_{i+m} = \{0\} \cup \bigcup_\ell X_\ell = \mathbb{F}_p$

3. $\forall i, j$ with $i \neq j \neq i + m, X_i + X_j = \bigcup_\ell X_\ell = (\mathbb{F}_p)^\times$.

Then the $X_i$s are called a (prime field) Directed Anti-Ramsey Algebra in $m$ colors.

So we ask: for which $m > 1$ is there a prime $p$ so that there exists a Directed Ramsey algebra (resp., a Directed Anti-Ramsey Algebra) over $\mathbb{F}_p$?
(Note to experts in relation algebras: we refer to the explicit combinatorial objects of Definitions 1 and 2 as Directed (Anti-)Ramsey algebras, and to the associated abstract algebras as Directed (Anti-)Ramsey Relation algebras, following Kowalski and Maddux. We observe, for example, that the 1-color Directed Anti-Ramsey Relation algebra is the so-called point algebra, and is not finitely representable.)

**Lemma 3.** Let \( p = nk + 1 \) be prime, with \( k \) odd and \( n = 2m \). If \( p > n^4 + 5 \), then \( \mathbb{F}_p \) does not give either an \( m \)-color Directed Ramsey algebra nor an \( m \)-color Directed Anti-Ramsey algebra via Comer’s construction.

**Proof.** For Ramsey algebras, the proof of Theorem 4 from [1] goes through with “\( m \)” replaced by “\( n \)”, showing that \( X_0 \) is not sum-free. For Anti-Ramsey algebras, one also replaces the equation \( x+y = z \) by \( x+y = -z \); the argument goes through, *mutatis mutandis*.

(Lemma 3 also follows from [3], a paper of which the author was unaware when proving Theorem 4 from [1].)

**2 Data**

We found \( m \)-color Directed Ramsey algebras for 37 values of \( m < 500 \); see Table 1. We found \( m \)-color Directed Anti-Ramsey algebras for all values of \( 1 < m \leq 1000 \) except for \( m = 8 \). All data collected for this paper made use of the fast algorithm for computing cycle structures of Comer’s algebras over \( \mathbb{F}_p \) given in [2]. The data for Directed Anti-Ramsey algebras are available at https://oeis.org/A294615. See Figure 1.
| $m$ | $p$ | $m$ | $p$ | $m$ | $p$ |
|-----|-----|-----|-----|-----|-----|
| 1   | 3   | 106 | 497141 | 198 | 2192653 |
| 10  | 3221| 111 | 559219 | 206 | 2020861 |
| 15  | 4231| 116 | 679993 | 210 | 2728741 |
| 17  | 11527| 122 | 814717 | 213 | 2420959 |
| 23  | 15319| 129 | 764971 | 295 | 4017311 |
| 35  | 38011| 132 | 1118569 | 311 | 4618351 |
| 48  | 91873| 141 | 1043683 | 394 | 8881549 |
| 59  | 135347| 176 | 1946209 | 419 | 9071351 |
| 66  | 209221| 177 | 1470871 | 447 | 11279599 |
| 67  | 228203| 179 | 1521859 | 474 | 12190333 |
| 74  | 309173| 186 | 1514413 | 482 | 11383877 |
| 89  | 476863| 188 | 1968361 | 483 | 12390883 |
|     |      |     |        | 495 | 15468751 |

Table 1: Smallest modulus $p$ for $m$-color Directed Ramsey algebras

![Graph](#)

Figure 1: Smallest modulus $p$ for $m$-color Directed (Anti-)Ramsey algebras
2.1 Asymmetry between Directed Ramsey and Anti-Ramsey Algebras

The difference in ease in finding Directed Ramsey and Anti-Ramsey algebras was at first surprising to the author. But there is a fundamental asymmetry that is Ramsey-theoretic in nature. For consider the subgraph of the coloring described in section 1 induced by any one color. With Directed Anti-Ramsey algebras, this subgraph will consist of a bunch of transitive triangles pasted together, and can contain large cliques. With Directed Ramsey algebras, however, any such color class contains only intransitive triangles. This forces the color class to be sparse in the sense that the only cliques are triangles, since any tournament (directed clique) on 4 or more vertices contains a transitive triple \((a, b), (b, c), (a, c)\). This is a plausible explanation for the greater difficulty in finding Directed Ramsey algebras.

3 Some finite Relation algebras

This section is directed at experts in relation algebras. We will give some explicit finite representations of relation algebras. Many of these algebras contain flexible atoms. Thus we eliminate several potential small counterexamples to the Flexible Atom Conjecture:

**Conjecture 4.** Every finite integral relation algebra with a flexible atom, i.e., an atom that does not participate in any forbidden diversity cycles, is representable over a finite set.

Since all representations considered here are group representations, throughout we let \(\rho\) be a representation that maps into the powerset of \(\mathbb{F}_p\) (although the base set for the representation is actually \(\mathbb{F}_p \times \mathbb{F}_p\), where \(p\) is given by context. We use the numbering system and notation given in Maddux’s book [8]. Unless otherwise noted, all algebras in this section were not previously known to be finitely representable. In most cases, we use the fact that the algebra in question embeds in some Directed Ramsey or Anti-Ramsey algebra.
3.1 The class $a, r, \check{r}$

3.1.1 $33_{37}$
Relation algebra $33_{37}$ has atoms $1', a, r, \check{r}$. The forbidden cycle is $rr\check{r}$. (See Figure 2) Let $p = 29$ and $m = 2$. Then

$$\rho(a) = X_1 \cup X_3$$
$$\rho(r) = X_0$$
$$\rho(\check{r}) = X_2$$

is a representation over $\mathbb{F}_p$. Notice that $33_{37}$ is a subalgebra of the 2-color Directed Anti-Ramsey algebra.

Relation algebra $33_{37}$ was previously shown to be finitely representable by J. Manske and the author, but the representation given here is smaller.

![Figure 2: Cycle structure of $33_{37}$](image)

3.1.2 $35_{37}$
Relation algebra $35_{37}$ has atoms $1', a, r, \check{r}$. The forbidden cycle is $rrr$. (See Figure 3) Let $p = 3221$ and $m = 10$. Then

$$\rho(a) = \bigcup_{0 \neq i \neq 10} X_i$$
$$\rho(r) = X_0$$
$$\rho(\check{r}) = X_{10}$$

is a representation over $\mathbb{F}_p$. Notice that $35_{37}$ is a subalgebra of the 10-color Directed Ramsey algebra.
3.2 The class $r, \bar{r}, s, \bar{s}$

3.2.1 77$_{83}$

Relation algebra 77$_{83}$ is the 2-color Directed Anti-Ramsey algebra. The forbidden cycles are $rr\bar{r}$ and $ss\bar{s}$. Let $p = 29$ and $m = 2$. Then

$$\rho(r) = X_0$$
$$\rho(\bar{r}) = X_2$$
$$\rho(s) = X_1$$
$$\rho(\bar{s}) = X_3$$

is a representation over $\mathbb{F}_p$.

3.2.2 78$_{83}$

Relation algebra 78$_{83}$ has forbidden cycles $ss\bar{s}$ and $ss\bar{s}$. The atoms $r$ and $\bar{r}$ are flexible. Let $p = 33791$ and $m = 31$. Then

$$\rho(r) = \bigcup_{0<i<31} X_i$$
$$\rho(\bar{r}) = \bigcup_{31<i<62} X_i$$
$$\rho(s) = X_0$$
$$\rho(\bar{s}) = X_{31}$$

is a representation over $\mathbb{F}_p$. In this case, $\mathbb{F}_p$ admits a 31-color (symmetric) Ramsey algebra, but the symmetric colors are splittable into asymmetric pairs.
3.2.3 80_{83}

Relation algebra 80_{83} has forbidden cycle ss\bar{s}. The atoms \( r \) and \( \bar{r} \) are flexible. Let \( p = 67 \) and \( m = 3 \). Then

\[
\begin{align*}
\rho(r) &= X_1 \cup X_2 \\
\rho(\bar{r}) &= X_4 \cup X_5 \\
\rho(s) &= X_0 \\
\rho(\bar{s}) &= X_3
\end{align*}
\]

is a representation over \( \mathbb{F}_p \). 80_{83} embeds into the 3-color Directed Anti-Ramsey algebra.

3.2.4 82_{83}

Relation algebra 82_{83} has forbidden cycle sss. The atoms \( r \) and \( \bar{r} \) are flexible. Let \( p = 3221 \) and \( m = 10 \). Then

\[
\begin{align*}
\rho(r) &= \bigcup_{0<i<10} X_i \\
\rho(\bar{r}) &= \bigcup_{10<i<20} X_i \\
\rho(s) &= X_0 \\
\rho(\bar{s}) &= X_{10}
\end{align*}
\]

is a representation over \( \mathbb{F}_p \). 82_{83} embeds into the 10-color Directed Ramsey algebra.

3.2.5 83_{83}

Relation algebra 83_{83} has no forbidden cycles; all diversity atoms are flexible. 82_{83} embeds in the 4-color Directed Anti-Ramsey algebra, hence is representable over \( \mathbb{F}_p \) for \( p = 233 \). There is a smaller “direct” representation over \( \mathbb{F}_p \), \( p = 37 \) and \( m = 2 \), where the images of the four diversity atoms are just the four cosets of the multiplicative subgroup of index 4. This algebra was previously known to be finitely representable, by a slight generalization of the probabilistic argument in [5]. The representation given here is probably the smallest known.
3.3  The class $a$, $b$ $r$, $\bar{r}$

3.3.1  $1310_{1316}$

Relation algebra $1310_{1316}$ has forbidden cycle $rr\bar{r}$. The atoms $a$ and $b$ are flexible. Let $p = 67$ and $m = 3$. Then
\[
\begin{align*}
\rho(a) &= X_1 \cup X_4 \\
\rho(b) &= X_2 \cup X_5 \\
\rho(r) &= X_0 \\
\rho(\bar{r}) &= X_3
\end{align*}
\]

is a representation over $\mathbb{F}_p$. $1310_{1316}$ embeds into the 3-color Directed Anti-Ramsey algebra.

3.3.2  $1313_{1316}$

Relation algebra $1313_{1316}$ has forbidden cycle $rrr$. The atoms $a$ and $b$ are flexible. Let $p = 3221$ and $m = 10$. Then
\[
\begin{align*}
\rho(a) &= (\cup_{0<i<6} X_i) \cup (\cup_{11<i<16} X_i) \\
\rho(b) &= (\cup_{5<i<10} X_i) \cup (\cup_{15<i<20} X_i) \\
\rho(r) &= X_0 \\
\rho(\bar{r}) &= X_{10}
\end{align*}
\]

is a representation over $\mathbb{F}_p$. $1313_{1316}$ embeds into the 10-color Directed Ramsey algebra.

3.3.3  $1315_{1316}$

Relation algebra $1315_{1316}$ has forbidden cycle $bbb$. The atoms $a$, $r$, and $\bar{r}$ are flexible. Let $p = 33791$ and $m = 31$. Then
\[
\begin{align*}
\rho(a) &= (\cup_{2<i<31} X_i) \cup (\cup_{33<i<62} X_i) \\
\rho(b) &= X_0 \cup X_{31} \\
\rho(r) &= X_1 \cup X_2 \\
\rho(\bar{r}) &= X_{32} \cup X_{33}
\end{align*}
\]

is a representation over $\mathbb{F}_p$. 

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3.3.4 1316_{1316}

Relation algebra 1316_{1316} has no forbidden cycles; all diversity atoms are flexible. 1316_{1316} Let $p = 73$ and $m = 4$. Then

$$\rho(a) = X_2 \cup X_6$$

$$\rho(b) = X_3 \cup X_7$$

$$\rho(r) = X_0 \cup X_1$$

$$\rho(\bar{r}) = X_4 \cup X_5$$

is a representation over $\mathbb{F}_p$. This algebra was previously known to be finitely representable, by a slight generalization of the probabilistic argument in \[5\].

4 Problems

**Problem 1.** Consider the finite integral algebra with diversity atoms $a$, $r$, $\bar{r}$, $s$, $\bar{s}$, with only $rrr$ and $ss\bar{s}$ forbidden. Then $a$ is flexible, so a representation exists over a countable set. Can this algebra be finitely represented?

**Problem 2.** The (abstract) 2-color Directed Ramsey Relation algebra is $81_{83}$, and is not representable; it fails to satisfy the equation that Maddux calls $(J)$. We showed above that the 10-color Directed Ramsey algebra is representable. What is the smallest $m > 2$ such that the $m$-color Directed Ramsey algebra is representable? (The 1-color Directed Ramsey algebra is representable over $\mathbb{Z}/3\mathbb{Z}$.)

**Problem 3.** Are all sufficiently large Directed Ramsey and Directed Anti-Ramsey algebras constructible (i.e., are the associated abstract algebras representable)? Clearly the computational approach taken here is of no avail.

**Problem 4** (due to Peter Jipsen). Is the smallest set over which relation algebra 1316_{1316} has a representation smaller than the corresponding smallest set for 83_{83}? Both have four flexible diversity atoms; 83_{83} has four asymmetric diversity atoms, while 1316_{1316} has two symmetric and two asymmetric.

The author’s guess is that the answer to Problem 4 is “No”. 83_{83} has only two “colors” — in this case, two pairs of asymmetric diversity atoms — while 1316_{1316} has three colors.
References

[1] Jeremy F. Alm. 401 and beyond: improved bounds and algorithms for the Ramsey algebra search. *J. Integer Seq.*, 20(8):Art. 17.8.4, 10, 2017.

[2] Jeremy F. Alm and A. Ylvisaker. A fast coset-translation algorithm for computing the cycle structure of Comer relation algebras over \( \mathbb{Z}/p\mathbb{Z} \). To Appear.

[3] Noga Alon and Jean Bourgain. Additive patterns in multiplicative subgroups. *Geom. Funct. Anal.*, 24(3):721–739, 2014.

[4] S. D. Comer. Color schemes forbidding monochrome triangles. In *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983)*, volume 39, pages 231–236, 1983.

[5] P. Jipsen, R. D. Maddux, and Z. Tuza. Small representations of the relation algebra \( E_{n+1}(1, 2, 3) \). *Algebra Universalis*, 33(1):136–139, 1995.

[6] Tomasz Kowalski. Representability of Ramsey relation algebras. *Algebra Universalis*, 74(3-4):265–275, 2015.

[7] R. Maddux. Some varieties containing relation algebras. *Trans. Amer. Math. Soc.*, 272(2):501–526, 1982.

[8] R. Maddux. *Relation algebras*, volume 150 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2006.