Abstract. In this paper we consider conditional random quantities (c.r.q.’s) in the setting of coherence. Based on betting scheme, a c.r.q. $X|H$ is not looked at as a restriction but, in a more extended way, as $XH + \mathbb{P}(X|H)H^c$; in particular (the indicator of) a conditional event $E|H$ is looked at as $EH + P(E|H)H^c$. This extended notion of c.r.q. allows algebraic developments among c.r.q.’s even if the conditioning events are different; then, for instance, we can give a meaning to the sum $X|H + Y|K$ and we can define the iterated c.r.q. $(X|H)|K$. We analyze the conjunction of two conditional events, introduced by the authors in a recent work, in the setting of coherence. We show that the conjoined conditional is a conditional random quantity, which may be a conditional event when there are logical dependencies. Moreover, we introduce the negation of the conjunction and by applying De Morgan’s Law we obtain the disjoined conditional. Finally, we give the lower and upper bounds for the conjunction and disjunction of two conditional events, by showing that the usual probabilistic properties continue to hold.

Keywords: Conditional events, Conditional random quantities, Coherence, Iterated conditioning, Import–Export principle, Conjunction, Disjunction, Lower/upper prevision bounds.

1. Introduction

Probabilistic reasoning under coherence allows a consistent treatment of uncertainty in many applications of statistics, economy, decision theory and artificial intelligence; it allows one to manage incomplete probabilistic assignments in a situation of vague or partial knowledge (see e.g. [4,6–8,14,17,18,29,42]); in particular, it is useful for a flexible numerical approach to inference rules in nonmonotonic reasoning and for the psychology of uncertain reasoning (see, e.g., [5,24,26–28,34,35,47–50,56,57]). We recall that nonmonotonic reasoning and conditional logics are important topics in
artificial intelligence (see e.g. [2,21,37,38,53,55]). In probability theory and in probability logic a relevant problem, discussed by many authors working in artificial intelligence, is that of suitably defining the logical operations of conjunction and disjunction among conditional events. In this context we recall the pioneering paper written in 1935 by de Finetti [19], where it was proposed a three-valued logic for conditional events coinciding with that one of Lukasiewicz. An interesting survey of the contributions by different authors (such as Adams, Belnap, Calabrese, de Finetti, Dubois, van Fraassen, McGee, Goodmann, Lewis, Nguyen, Prade, Schay) to the study of three-valued logics and compounds of conditionals has been given in [45]; an extensive study of conditionals has been made in [22]; see also [44]. In the many works concerning logical operations among conditional events, the conjunction and disjunction have been usually defined as suitable conditionals; see e.g. [2,11–13,21,36,37,52]. Our perspective is different because we work in a quantitative framework rather than in a logical one: in our approach the conjunction and disjunction of two conditional events are (not conditional events but) conditional random quantities. We recall that usually in the literature a c.r.q. $X|H$ is looked at as a restriction of $X$ to $H$, with $X|H$ undefined when $H$ is false. In our approach, using the conditional prevision $\mathbb{P}(X|H)$, we look at $X|H$ as an extended quantity which coincides with the restriction when $H$ is true and is equal to $\mathbb{P}(X|H)$ when $H$ is false. In this way, based on the betting scheme of de Finetti ([20]), in agreement with [40,41] the c.r.q. $X|H$ may be interpreted as the amount that you receive in a bet on $X$ conditional on $H$, if you agree to pay $\mathbb{P}(X|H)$. In our paper we formally define the equality between two c.r.q.’s; in particular we show that $X|HK$ coincides with $[XH + \mathbb{P}(X|HK)H^c]|K$. Moreover, among other results, by the extended notion of c.r.q. we obtain a meaning for numerical operations, such as the sum $X|H + Y|K$, and a natural definition for a notion of iterated c.r.q. $(X|H)|K$, which coincides with $X|HK$ if $H \subseteq K$, or $K \subseteq H$. Then, we develop in the framework of coherence a theory for the compounds of conditionals connected with that ones proposed in [39,44]. We suitably define the conjunction $(A|H) \land (B|K)$ of two conditional events $A|H, B|K$, which in the case of some logical dependencies may be a conditional event. We introduce the negation for conjoined conditionals; then, by De Morgan’s Law, we define the disjunction of two conditional events. Finally, we obtain the lower and upper bounds for the coherent extensions of a probability assessment $(x, y)$ on $\{A|H, B|K\}$ to their conjunction $(A|H) \land (B|K)$ and their disjunction $(A|H) \lor (B|K)$. Interestingly, the usual probabilistic properties continue to hold in terms of previsions.
2. Preliminary Notions and Results

In this section we recall some basic notions and results on coherence for conditional prevision assessments. In our approach an event $A$ represents an uncertain fact described by a (non ambiguous) logical proposition; hence we look at $A$ as a two-valued logical entity which can be true ($T$), or false ($F$). The indicator of $A$, denoted by the same symbol, is a two-valued numerical quantity which is 1, or 0, according to whether $A$ is true, or false. The sure event is denoted by $\Omega$ and the impossible event is denoted by $\emptyset$. Moreover, we denote by $A \land B$, or simply $AB$, (resp., $A \lor B$) the logical conjunction (resp., logical disjunction). The negation of $A$ is denoted $A^c$. Given any events $A$ and $B$, we simply write $A \subseteq B$ to denote that $A$ logically implies $B$, that is $AB^c$ is the impossible event $\emptyset$; an equivalent notation is $A \vdash B$. We recall that $n$ events are logically independent when the number $m$ of constituents, or possible worlds, generated by them is $2^n$ (in general $m \leq 2^n$).

2.1. Coherent Conditional Prevision Assessments

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite c.r.q.'s, consider a finite subfamily $\mathcal{F}_n = \{X_i | H_i, i \in J_n\} \subseteq \mathcal{K}$, where $J_n = \{1, \ldots, n\}$, and the vector $\mathcal{M}_n = (\mu_i, i \in J_n)$, where $\mu_i = \mathbb{P}(X_i | H_i)$ is the assessed prevision for the c.r.q. $X_i | H_i$. With the pair $(\mathcal{F}_n, \mathcal{M}_n)$ we associate the random gain $G = \sum_{i \in J_n} s_i H_i (X_i - \mu_i)$; moreover, we set $\mathcal{H}_n = H_1 \lor \cdots \lor H_n$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$, the set of values of $G$ restricted to $\mathcal{H}_n$. Then, using the betting scheme of de Finetti, we have

**Definition 1.** The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1$, $\forall \mathcal{F}_n \subseteq \mathcal{K}$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, it holds that: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Given a family $\mathcal{F}_n = \{X_1 | H_1, \ldots, X_n | H_n\}$, for each $i \in J_n$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values for the restriction of $X_i$ to $H_i$; then, for each $i \in J_n$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. Of course, for each $i \in J_n$, the family $\{H_i^c, A_{ij} H_i, j = 1, \ldots, r_i\}$ is a partition of the sure event $\Omega$, with $A_{ij} H_i = A_{ij}, \bigvee_{j=1}^{r_i} A_{ij} = H_i$. Then, the constituents generated by the family $\mathcal{F}_n$ are (the elements of the partition of $\Omega$) obtained by expanding the expression $\bigwedge_{i \in J_n} (A_{i1} \lor \cdots \lor A_{ir_i} \lor H_i^c)$. We set $C_0 = H_1^c \cdots H_n^c$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \lor \cdots \lor H_n$. Hence $\bigvee_{i \in J_n} (A_{i1} \lor \cdots \lor A_{ir_i} \lor H_i^c) = \bigvee_{h=0}^{m} C_h$. With each $C_h$, $h \in J_m$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}, j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq H_i^c$; with $C_0$ it is associated $Q_0 = \mathcal{M}_n = (\mu_1, \ldots, \mu_n)$. We illustrate by an example how to compute the constituents $C_h$’s and the associated point $Q_h$’s for a pair of c.r.q.’s.
Example 1. Given any r.q. \(X \in \{1, \ldots, 6\}\), let be \(H = (X \in \{2, 4, 6\})\) and \(K = (X \in \{4, 5, 6\})\). Now, let us consider the family \(\{X_1|H_1, X_2|H_2\} = \{X|H, X|K\}\). We have: \(A_{11} = (X = 2), A_{12} = (X = 4), A_{13} = (X = 6), A_{21} = (X = 4), A_{22} = (X = 5), A_{23} = (X = 6)\). Then \((A_{11} \lor A_{12} \lor A_{13} \lor H^c) \land (A_{21} \lor A_{22} \lor A_{23} \lor H^c)\) is satisfied.

1. Deepenings on Conditional Random Quantities

In this section we deepen some aspects on c.r.q.'s. We consider an extended notion for c.r.q.'s; we define the equality between two c.r.q.'s and we give

**Theorem 1.** [Operative characterization of coherence] A conditional prevision assessment \(\mathcal{M}_n = (\mu_1, \ldots, \mu_n)\) on the family \(\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}\) is coherent if and only if the following conditions are satisfied:

(i) the system \((\Sigma)\) defined in (1) is solvable;

(ii) if \(I_0 \neq \emptyset\), then \(\mathcal{M}_0\) is coherent.

Coherent conditional previsions and non dominance w.r.t. proper scoring rules have been investigated in [9]. Comparative previsions, qualitative criteria of coherence and weak dominance have been studied in [46].
a condition under which two c.r.q.’s are equal. Then, we examine a non trivial case where such a condition is satisfied; moreover, by linearity of prevision, we give a simple proof of the general compound prevision theorem. We also recall a notion of iterated conditioning for c.r.q.’s, which has a rationale in terms of economic transactions [32]. We recall that, given an event \( H \neq \emptyset \) and a r.q. \( X \), based on the betting metaphor the conditional prevision \( \mathbb{P}(X|H) \) is defined as the amount you agree to pay, by knowing that you will receive the amount \( X \) if \( H \) is true, or you will receive back the amount you paid if \( H \) is false, because the bet is called off.

**Remark 1.** Usually, in literature (see e.g. [3]) the c.r.q. \( X|H \) is looked at as the *restriction* of \( X \) to \( H \); that is a r.q. whose possible values are that ones of \( X \) compatible with \( H \), while \( X|H \) is undefined when \( H \) is false. Thus, \( X|H \) cannot be interpreted as the amount you receive when you pay \( \mathbb{P}(X|H) \).

This interpretation is obtained by suitably defining \( X|H \) when \( H \) is false, as made by some authors; see e.g. [15,32,40,41]. In Lad ([40], [41, Section 3.2]) \( X|H \) is defined as the random quantity \( X|H = XH + \mathbb{P}(X|H)H^c \), by looking in this way at the conditional prevision \( \mathbb{P}(X|H) \) as the prevision of the ”new object” \( X|H \). As we can see, formally this extended notion of \( X|H \) also depends on the conditional prevision \( \mathbb{P}(X|H) \). In [15] a systematic study has been developed on conditional events, where among other things it has been shown that, for the indicator \( T(E|H) = EH + tH^c \) of a conditional event \( E|H \), the truth-value \( t \) satisfies all the properties of a conditional probability. This study has been extended in the setting of possibility theory in [10]. Thus, the third value of a conditional event is not strictly related to a particular measure of uncertainty (see [16]). Our semantics is probabilistic, then in our approach the third value is a conditional probability. A similar comment applies for c.r.q.’s studied in this paper; of course, our approach could be generalized by using other measures of uncertainty. In [15] also a generalization to a c.r.q. \( Y|K \) has been considered by setting \( Y|K = Y \cdot K + t^*K^c \), where \( Y \cdot K \) is looked at as a suitable restriction of \( Y \) to \( K \). Then, taking as \( Y \) the conditional event \( E|H \), the relation \( (E|H)|K = E|HK \) is obtained, which is in agreement with the so-called Import–Export Principle ([44]). Concerning the approach of Lad, we could observe that it may be ambiguous to use the same symbol \( X|H \) to denote both the restriction and the quantity \( XH + \mathbb{P}(X|H)H^c \). In Sect. 3.1 we analyze this aspect.

### 3.1. An Extended Notion for Conditional Random Quantities

We introduce below our extended notion of conditional random quantity.

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Then, we examine a non trivial case where such a condition is satisfied; moreover, by linearity of prevision, we give a simple proof of the general compound prevision theorem. We also recall a notion of iterated conditioning for c.r.q.’s, which has a rationale in terms of economic transactions [32]. We recall that, given an event \( H \neq \emptyset \) and a r.q. \( X \), based on the betting metaphor the conditional prevision \( \mathbb{P}(X|H) \) is defined as the amount you agree to pay, by knowing that you will receive the amount \( X \) if \( H \) is true, or you will receive back the amount you paid if \( H \) is false, because the bet is called off.

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### 3.1. An Extended Notion for Conditional Random Quantities

We introduce below our extended notion of conditional random quantity.
DEFINITION 2. Given an event $H \neq \emptyset$ and a r.q. $X$, let $\mathbb{P}(X|H)$ be the conditional prevision of $X|H$. We set $X|^*H = XH + \mathbb{P}(X|H)H^c$.

We observe that $X|^*H = X|H$, when $H$ is true, and $X|^*H = \mathbb{P}(X|H)$, when $H$ is false. In particular, for the indicator of a conditional event $A|H$ (usually denoted by the same symbol) we have $A|^*H = AH + P(A|H)H^c$. The choice of $P(A|H)$, as third value of the indicator when $H$ is false, has been considered in other works (see e.g. [17,25,38,40,41,44]). In the next result we show that $\mathbb{P}(X|^*H) = \mathbb{P}(X|H)$ and $(X|^*H)^*H = X|^*H$.

PROPOSITION 1. Given any event $H \neq \emptyset$, any r.q. $X$, and any coherent assessment $\mathbb{P}(X|H) = \mu$, we have:

(a) The extension $\mathbb{P}(X|^*H) = \mu^*$ is coherent if and only if $\mu^* = \mu$;
(b) $(X|^*H)|^*H = X|^*H$.

PROOF. (a) The random gain associated with the assessment $(\mu, \mu^*)$ is $G = s_1H(X - \mu) + s_2(X|^*H - \mu^*) = s_1H(X - \mu) + s_2(XH + \mu H^c - \mu^*)$, with $s_1, s_2$ arbitrary real numbers. By choosing $s_1 = 1, s_2 = -1$, we have

$$G = H(X - \mu) - (XH + \mu H^c - \mu^*) = -\mu H - \mu H^c + \mu^* = \mu^* - \mu.$$  

As $G$ is constant, by coherence, it must be $G = 0$, that is: $\mu^* = \mu$.

(b) We have $(X|^*H)|^*H = (XH + \mu H^c)|^*H = (XH + \mu H^c)H + \mu^* H^c = XH + \mu H^c = X|^*H$. \hfill \qed

REMARK 2. Given an event $H \neq \emptyset$ and a finite r.q. $X$, let $V_H = \{x_1, \ldots, x_r\}$ be the set of possible values of $X$ restricted to $H$. By Definition 1, the assessment $\mathbb{P}(X|H) = \mu$ is coherent if and only if $\min V_H = \mu \leq \max V_H$. In particular, if $V_H = \{c\}$, then by coherence it must be $\mu = c$ and hence $X|^*H = cH + c H^c = c$. Of course, for $X = H$ (resp. $X = H^c$) it holds that $\mu = P(H|H) = 1$ (resp. $\mu = P(H^c|H) = 0$); hence $H|^*H = 1$, $H^c|^*H = 0$.

We define below the notion of equality for two c.r.q.’s $X|H$ and $Y|K$.

DEFINITION 3. Given any events $H \neq \emptyset, K \neq \emptyset$, and any r.q.’s $X, Y$, let $\Pi$ be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$. We define $X|^*H$ and $Y|^*K$ equal if and only if

$$XH + \mu H^c = YK + \nu K^c,$$  for every $(\mu, \nu) \in \Pi$.  \hfill (2)

As shown in Theorem 4, $X|^*H = Y|^*K$ implies $\mu = \nu, \forall(\mu, \nu) \in \Pi$. 

EXAMPLE 2. We discuss a critical example\(^1\) on the equality of c.r.q.’s. Given a partition \(\{A_1, \ldots, A_5\}\) of \(\Omega\), we set \(H = A_1 \vee A_2 \vee A_3, K = A_2 \vee A_3 \vee A_4 \vee A_5,\) \(X = A_1 + 2A_2 - A_3 + xA_4 + yA_5\) and \(Y = zA_1 + 2A_2 - A_3 + A_4 + A_5,\) with \(x, y, z\) arbitrary real numbers. We observe that the c.r.q.’s \(X|H\) and \(Y|K,\) looked at as restrictions, do not coincide; moreover, \(X|^*H\) and \(Y|^*K\) do not coincide too. We remark that as specified in Definition 3, in order that \(X|^*H\) and \(Y|^*K\) coincide, the equality (2) must be satisfied for every \((\mu, \nu) \in \Pi.\) We have \(XH = A_1 + 2A_2 - A_3\) and \(YK = 2A_2 - A_3 + A_4 + A_5;\) then defining \(\mu = \mathbb{P}(X|H), \nu = \mathbb{P}(Y|K),\) with \((\mu, \nu)\) coherent, it follows \(X|^*H = A_1 + 2A_2 - A_3 + \mu(A_4 + A_5),\) \(Y|^*K = 2A_2 - A_3 + A_4 + A_5 + \nu A_1.\)

The assessment \((\mu, \nu) = (\frac{2}{3}, \frac{2}{3})\), being associated with the uniform distribution on the partition, is coherent and \(A_1 + 2A_2 - A_3 + \frac{2}{3}(A_4 + A_5) \neq 2A_2 - A_3 + A_4 + A_5 + \frac{2}{3}A_1.\) Therefore, by Definition 3, \(X|^*H \neq Y|^*K.\) Of course, as a kind of ‘local coincidence’, it may happen that \(A_1 + 2A_2 - A_3 + \mu(A_4 + A_5) = 2A_2 - A_3 + A_4 + A_5 + \nu A_1,\) for some coherent assessment \((\mu, \nu).\) For instance, this coincidence holds for \((\mu, \nu) = (1, 1),\) which is obtained by assessing \(P(A_i) = \frac{1}{5}, i = 1, 2, 4, P(A_3) = \frac{1}{10}, P(A_5) = \frac{3}{10}.\)

REMARK 3. As shown by Proposition 1, \(X|^*H\) and \(X|H\) coincide when \(H\) is true and have the same prevision, the unique difference being that \(X|^*H\) is defined when \(H\) is false. Moreover, defining \(\mathbb{P}(X|H) = \mu,\) the random gains \(G\) and \(G^*,\) associated with \(X|H\) and \(X|^*H,\) coincide because \(G^* = s(X|^*H - \mu) = s(XH + \mu H^c - \mu) = s(XH - \mu H) = sH(X - \mu) = G.\)

More in general, in Definition 1, for the random gain \(G\) it holds that \(G = \sum_{i \in J_n} s_i H_i (X_i - \mu_i) = \sum_{i \in J_n} s_i (X_i H_i + \mu_i H_i^c - \mu_i) = \sum_{i \in J_n} s_i (X_i|^*H_i - \mu_i) = G^*.\)

Then, in what follows we identify \(X|H\) and \(X|^*H;\) in other words, from now on the symbol \(X|H\) denotes the quantity \(XH + \mathbb{P}(X|H)H^c.\)

We also remark that, by this extended notion of c.r.q., given a prevision assessment \(\mathcal{M}_n = (\mu_1, \ldots, \mu_n)\) on a family \(\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\},\) the points \(Q_h\)’s associated with the constituents \(C_h\)’s represent the possible values of the random vector \(\{X_1|H_1, \ldots, X_n|H_n\}.\) Moreover, we can give a meaning, for instance, to the expression \(X|H + Y|K,\) which is identified with \(X|^*H + Y|^*K.\) As a consequence, we can look at the sum \(\mathbb{P}(X|H) + \mathbb{P}(Y|K)\) as the prevision of \(X|H + Y|K\) (see Theorem 2 below), and so on.

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\(^1\)We thank an anonymous referee for this example.
3.2. Some Algebraic Developments

The next result shows that the additivity of prevision for our extended notion of c.r.q. is preserved and that $X|H + Y|K = (X|H + Y|K)|(H \lor K)$.

**Theorem 2.** Given any events $H \neq \emptyset, K \neq \emptyset$ and any r.q.’s $X, Y$, we have:

(i) $\mathbb{P}(X|H + Y|K) = \mathbb{P}[(X|H + Y|K)|(H \lor K)] = \mathbb{P}(X|H) + \mathbb{P}(Y|K)$;

(ii) $X|H + Y|K = (X|H + Y|K)|(H \lor K)$.

**Proof.** (i) Given any coherent assessment $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$, we have $X|H + Y|K = XH + \mu Hc + YK + \nu Kc$. We set $\mathbb{P}(X|H + Y|K) = \eta$, $\mathbb{P}[(X|H + Y|K)|(H \lor K)] = \beta$; then, the random gain associated with the prevision assessment $(\mu, \nu, \eta)$ is

$$G = s_1 H(X - \mu) + s_2 K(Y - \nu) + s_3 (XH + \mu Hc + YK + \nu Kc - \eta),$$

with $s_1, s_2, s_3$ arbitrary real numbers, and for $s_1 = s_2 = 1, s_3 = -1$ we have $G = H(X - \mu) + K(Y - \nu) - (XH + \mu Hc + YK + \nu Kc - \eta) = \eta - \mu - \nu$.

As $G$ is constant, by coherence it must be $G = 0$, i.e. $\eta = \mu + \nu$. Moreover, the random gain associated with the assessment $(\mu, \nu, \beta)$ is

$$G = s_1 H(X - \mu) + s_2 K(Y - \nu) + s_3 (XH + \mu Hc + YK + \nu Kc - \beta),$$

and, denoting by $G_{H \lor K}$ the set of values of $G$ restricted to $H \lor K$, for $s_1 = s_2 = 1, s_3 = -1$ we have

$$G = H(X - \mu) + K(Y - \nu) - (XH + \mu Hc + YK + \nu Kc - \beta) = \beta - \mu - \nu[H \lor K].$$

As $G_{H \lor K} = \{\beta - \mu - \nu\}$, by coherence $\beta - \mu - \nu = 0$; i.e. $\beta = \mu + \nu = \eta$.

(ii) Given any coherent assessment $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$, we have

$$(X|H + Y|K)|(H \lor K) = (XH + \mu Hc + YK + \nu Kc)(H \lor K) + (\mu + \nu)HcKc = XH + \mu HcKc + HcKc + YK + \nu HcKc + \nu HcKc = X|H + Y|K.$$

Below, we consider linear combinations of c.r.q.’s.

**Theorem 3.** Given any real quantities $a, b$, any events $H \neq \emptyset, K \neq \emptyset$, any r.q.’s $X, Y$, and any coherent assessment $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$, we have $(aX)|H + (bY)|K = a(X|H) + b(Y|K) = aX|H + bY|K$.

**Proof.** By linearity of prevision, $\mathbb{P}[(aX)|H] = a\mu, \mathbb{P}[(bY)|K] = b\nu$; then, $(aX)|H = (aX)H + (a\mu)Hc = a(XH + \mu Hc) = a(X|H)$; moreover $(bY)|K = \cdots = b(Y|K)$ and we can simply write $aX|H$ and $bY|K$. By Theorem 2, we have $\mathbb{P}[(aX)|H + (bY)|K] = \mathbb{P}[(aX)|H] + \mathbb{P}[(bY)|K] = a\mu + b\nu$; then $(aX)|H + (bY)|K = (aXH + a\mu Hc + bYK + b\nu Kc)(H \lor K) + (a\mu + b\nu)HcKc = aXH + a\mu Hc + bYK + b\nu Kc = a(X|H) + b(Y|K) = aX|H + bY|K$. 

$\blacksquare$
In the next result we show that two c.r.q.’s $X|H$ and $Y|K$ are equal if, for each constituent contained in $H \lor K$, the values of $X|H$ and $Y|K$ coincide.

**Theorem 4.** Given any events $H \neq \emptyset$, $K \neq \emptyset$, and any r.q.’s $X$, $Y$, let $\Pi$ be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu$, $\mathbb{P}(Y|K) = \nu$. Then, the statements (i) and (ii) below hold.

(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \lor K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.

(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \lor K$ is true if and only if $X|H = Y|K$.

**Proof.** (i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ associated with the constituent $C_h$ coincide, for each $C_h \subseteq H \lor K$; then for each $(\mu, \nu) \in \Pi$, by choosing $s_1 = 1$, $s_2 = -1$ in the random gain, we have

$$G = H(X - \mu) - K(Y - \nu) = (X|H - \mu) - (Y|K - \nu) = (X|H - Y|K) + (\nu - \mu).$$

Then, by the hypothesis, $G_{H \lor K} = \{\nu - \mu\}$ and by coherence it must be $\nu - \mu = 0$; that is $\nu = \mu$, $\forall (\mu, \nu) \in \Pi$.

(ii) By hypothesis, we have $(XH + \mu H^c)(H \lor K) = (YK + \nu K^c)(H \lor K)$; moreover, from condition (i), $\mu = \nu$ for every $(\mu, \nu) \in \Pi$; then

$$X|H = XH + \mu H^c = (XH + \mu H^c)(H \lor K) + (XH + \mu H^c)H^c K^c = (YK + \nu K^c)(H \lor K) + \mu H^c K^c = YK + \nu HK^c + \nu H^c K^c = Y|K.$$

Vice versa, $X|H = Y|K$ trivially implies $X|H = Y|K$ when $H \lor K$ is true.

By Definition 3 and Theorem 4 it immediately follows

**Corollary 1.** Given any event $H \neq \emptyset$ and any r.q.’s $X$ and $Y$, assume that $XH = YH$. Then $X|H = Y|H$.

In Corollary 1 the values $\mathbb{P}(X|H)$ and $\mathbb{P}(Y|H)$ do not play any role. We illustrate now a non trivial case where two c.r.q.’s are equal; then, by linearity of prevision we obtain the formula $\mathbb{P}(XH|K) = P(H|K)\mathbb{P}(X|HK)$.

**Theorem 5.** Given any events $H \neq \emptyset, K \neq \emptyset$, and any r.q. $X$, we have:

(i) for all coherent assessments $\mathbb{P}(X|HK) = y$, $\mathbb{P}[(XH + y H^c)|K] = \mu$, it holds that $y = \mu$ and $X|HK = (XH + y H^c)|K$;

(ii) for all coherent assessments $P(H|K) = x$, $\mathbb{P}(X|HK) = y$, $\mathbb{P}(XH|K) = z$, it holds that $z = xy$.
PROOF. (i) Given any coherent assessment \((y, \mu)\), we have
\[
X|HK = XHK + y(HK)^c = XHK + yK^c + yH^cK = (XH + yH^c)K + yK^c
\]
and \((XH + yH^c)|K = (XH + yH^c)K + \mu K^c\). We observe that \(HK \vee K = K\) and that \(X|HK = (XH + yH^c)|K\) when \(K\) is true. Then, by Theorem 4, \(y = \mu\) for every \((y, \mu)\) coherent and \(X|HK = (XH + yH^c)|K\).

(ii) Given any coherent assessment \((x, y, z)\) on \(\{H|K, X|HK, XH|K\}\), by condition (i) and by linearity of prevision we have
\[
y = \mathbb{P}(X|HK) = \mathbb{P}((XH + yH^c)|K] = \mathbb{P}(XH|K) + yP(H^c|K) = z + y(1 - x),
\]
from which it follows: \(z = xy\), that is: \(\mathbb{P}(XH|K) = P(H|K)\mathbb{P}(X|HK)\), which represents the (general) compound prevision theorem. 

We illustrate condition (i) of Theorem 5 by the example below.

Example 3. Consider a r.q. \(X \in \{1, \ldots, 6\}\), with \(P(X = h) = p_h, h = 1, \ldots, 6\), with \(H = \{X \in \{2, 4, 6\}\}\) and \(K = \{X \in \{4, 5, 6\}\}\). We have \(HK = (X \in \{2, 4, 6\}\) and \(P(X|HK) = y = \mathbb{P}((XH + yH^c)|K] = \cdots = \frac{4p_4 + 6p_6}{p_4 + p_6}\), when \(p_4 + p_6 > 0\); moreover, when \(p_4 + p_6 = 0\), using coherence, we can still verify that \(\mathbb{P}(X|HK) = y = \mathbb{P}((XH + yH^c)|K] \in [4, 6]\). Finally, we remark that \(X|HK = (XH + yH^c)|K = X\), for \(X \in \{4, 6\}\); \(X|HK = (XH + yH^c)|K = y\), for \(X \in \{1, 2, 3, 5\}\); hence \(X|HK = (XH + yH^c)|K\). We observe that to check the equality \(X|HK = [XH + \mathbb{P}(X|HK) H^c]|K\) does not make sense if we look at the c.r.q.’s as restrictions.

3.3. Iterated Conditioning

We recall below the notion of iterated c.r.q. \((X|H)K\) given in [32]; such a notion is consistent with that one given for conditional events in [33] and will be used in Section 4.

Definition 4. Given any events \(H, K\), with \(H \neq \emptyset, K \neq \emptyset\), and a r.q. \(X\), we define \((X|H)|K = [XH + \mathbb{P}(X|H) H^c]|K\).

If \(X\) is any event \(A\), we have \((A|H)|K = [AH + P(A|H) H^c]|K\); then
\[
\mathbb{P}((A|H)|K = P(A|HK)P(H|K) + P(A|H)P(H^c|K),
\]
which coincides with a result given in [44]. In condition (i) of Theorem 5 the value \(y\) is the prevision of \(X|HK\); hence \(X|HK = [XH + \mathbb{P}(X|HK) H^c]|K \neq (X|H)|K = [XH + \mathbb{P}(X|H) H^c]|K, with (X|H)|K = X|HK = (X|K)|H,
if \( H \subseteq K \), or \( K \subseteq H \) (see [32, Proposition 1]); then, given any event \( A \) it holds that \((A|H)|(H \lor K) = A|H\). Moreover \( A|HK \neq (A|H)|K\); therefore, in agreement with [2,39], in our approach the Import–Export Principle ([44]) does not hold. For instance, assume that \( K = H^c \lor A \) (material conditional associated with \( A|H \)) \( H^c \lor A = \emptyset \), so that \( P(A|H) = 0 \). Then the Import–Export Principle cannot be applied because \( A|HK = A|AH = A|\emptyset \); on the contrary, as \( H^c \lor A = H^c \), by Definition 4 we have
\[
(A|H)|K = (A|H)|(H^c \lor A) = (A|H)|H^c = (AH + 0 \cdot H^c)|H^c = 0|H^c = 0;
\]
therefore \( P[(A|H)|K] = P(A|H) = 0 \), while \( P(H^c \lor A) \) could be high. A probabilistic analysis of weak and strong inferences from the material conditional \( A^c \lor B \) to the associated conditional \( B|A \) has been given in [28].

4. Conjunction of Conditional Events

Some authors look at the conditional “if \( A \) then \( C \)”, denoted \( A \rightarrow C \), as the event \( A^c \lor C \) (material conditional), but, since some years, many authors look at \( A \rightarrow C \) as the conditional event \( C|A \) (see e.g. [28,48,51]). Compounds of conditionals have been studied by many researchers in many fields, such as mathematics, philosophical logic, artificial intelligence, non-monotonic reasoning, psychology. A very general discussion of the different aspects which concern conditionals has been given in [22,45]. Recently, a probabilistic theory of conditionals has been proposed by Kaufmann in [39]. In such a paper the author obtains, by a complex procedure, probabilistic formulas which suggest how to assign values to conditionals. Given a conditional \( A \rightarrow C \) and the associated conditional event \( C|A \), with \( P(A) > 0 \), Kaufmann shows that, by defining the truth value of \( A \rightarrow C \) as:
\[
V(A \rightarrow C) = \begin{cases} 
1, & AC \text{ true} \\
0, & AC^c \text{ true} \\
P(C|A), & A^c \text{ true}
\end{cases}
\]
it follows: \( P(A \rightarrow C) = P(C|A) = \frac{P(AC)}{P(A)} \).

Moreover, assuming \( P(A \lor C) > 0 \), Kaufmann obtains for the conjoined conditional \( (A \rightarrow B) \land (C \rightarrow D) \) the formula
\[
P[(A \rightarrow B) \land (C \rightarrow D)] = \frac{P(ABCD)+P(B|A)P(A^cCD)+P(D|C)P(ABC^c)}{P(A \lor C)}.
\]
Based on this result, Kaufmann suggests a natural way of defining the values of conjoined conditionals. Another relevant theory of compounds of conditionals is given by McGee in [44], where also the problem of what would be
fair betting odds on conjunctions of conditionals is investigated. The papers of Kaufmann and McGee contain very nice results on compounds of conditionals. We obtain (and generalize) such results in a direct and simpler way in the setting of coherence; we also show that some well known probabilistic properties, which hold in the classical setting, are preserved.

A basic aspect: if we only assess \( P(B|A) = x, P(D|C) = y \), how can we check the consistency of the extension \( P[(A \rightarrow B) \land (C \rightarrow D)] = z ? \)

In our setting \((A \rightarrow B) \land (C \rightarrow D)\) is looked at as a conditional random quantity \((B|A) \land (D|C)\); hence, we assess the prevision (and not the probability) of the conjoined conditional. Moreover, we can manage without problems the case where the denominator \( P(A \lor C) \), in the formula of Kaufmann, is zero and, starting with the assessment \( P(B|A) = x, P(D|C) = y \), we can determine the values of \( z = \mathbb{P}[(B|A) \land (D|C)] \) which coherently extend the assessment \((x, y)\) on \( \{B|A, D|C\} \) to the c.r.q. \((B|A) \land (D|C)\).

Conjunction of conditionals in the setting of coherence. We introduce the notion of conjunction, by first giving some logical and probabilistic remarks. Given any events \( A, B, H \), with \( H \neq \emptyset \), let us consider the conjunctions \( AB \) and \( (A|H) \land (B|H) = AB|H \). We recall that the indicator \( AB \) coincides with both the minimum and the product of the indicators; i.e. \( AB = \min \{A, B\} = A \cdot B \). Moreover, \( AB|H = \min \{A, B\}|H = (A \cdot B)|H \). If we assess \( P(A|H) = x, P(B|H) = y \), then

\[
A|H = AH + xH^c = \begin{cases} 
A, & \text{if } H = 1, \\
x, & \text{if } H = 0,
\end{cases}
B|H = BH + yH^c = \begin{cases}
B, & \text{if } H = 1, \\
y, & \text{if } H = 0.
\end{cases}
\]

We set \( Z = \min \{A|H, B|H\} = \min \{AH + xH^c, BH + yH^c\} \); we have \( Z \in \{1, 0, x, y\} \) and, defining \( \mathbb{P}(Z|H) = z \), we have \( Z|H = ZH + zH^c \), with \( Z|H \in \{1, 0, z\} \). We observe that \( ZH = ABH \); then, by Corollary 1, we have \( Z|H = AB|H \). In other words, the c.r.q.’s \( \min \{A|H, B|H\}|H \) and \( AB|H \) are equal.\(^2\) Then, as \( H = H \lor H \), we have

\[
(A|H) \land (B|H) = \min \{A|H, B|H\} \mid H = \min \{A|H, B|H\} \mid (H \lor H). \quad (3)
\]

Formula (3) suggests how to define, still taking the minimum, the notion of conjunction (already given in [33]) for \( A|H \) and \( B|K \), with \( K \neq H \).

DEFINITION 5. (Conjunction) Given any pair of conditional events \( A|H \) and \( B|K \), and any coherent assessment \( P(A|H) = x, P(B|K) = y \), we define

\(^2\)In particular, for \( B = A \), in agreement with Definition 4 we have \( Z = A|H, Z|H = (A|H)|H = A|H, z = x \); the equality \( (A|H)|H = A|H \) still holds from the viewpoint of iterated conditionals introduced in [33].
\[(A|H) \land (B|K) = \min \{A|H, B|K \} \mid (H \lor K). \quad (4)\]

Notice that, defining \(Z = \min \{A|H, B|K \}\), the conjunction \((A|H) \land (B|K)\) is the c.r.q. \(Z \mid (H \lor K)\). Moreover, defining \(T = A|H \cdot B|K\) we have \(Z \neq T\), while by Corollary 1 it holds that \(Z \mid (H \lor K) = T \mid (H \lor K)\). Then
\[(A|H) \land (B|K) = (A|H \cdot B|K) \mid (H \lor K). \quad (5)\]

**Interpretation with the betting scheme.**\(^3\) By assessing \(\mathbb{P}[(A|H) \land (B|K)] = z\), you agree to pay the amount \(z\) and to receive the amount \(\min \{A|H, B|K\}\) when the disjunction \(H \lor K\) is true, or to receive back the amount \(z\) when the bet is called off \((H \lor K\) false). That is, you pay \(z\) and you receive
\[(A|H) \land (B|K) = \begin{cases} 
1, & AHBK \text{ true}, \\
0, & A^cH \lor B^cK \text{ true}, \\
x, & H^cBK \text{ true}, \\
y, & AHK^c \text{ true}, \\
z, & H^cK^c \text{ true};
\end{cases}\]

therefore, operatively, for \((A|H) \land (B|K)\) we obtain the representation
\[(A|H) \land (B|K) = 1 \cdot AHBK + x \cdot H^cBK + y \cdot AHK^c + z \cdot H^cK^c, \quad (6)\]

with \((x, y, z)\) coherent. Then, by linearity of prevision, it follows
\[
\mathbb{P}[(A|H) \land (B|K)] = z = P(AHBK) + xP(H^cBK) + yP(AHK^c) + zP(H^cK^c),
\]

and we obtain: \(zP(H \lor K) = P(AHBK) + xP(H^cBK) + yP(AHK^c)\).

In particular, if \(P(H \lor K) > 0\), as in [39, 44], we obtain
\[
\mathbb{P}[(A|H) \land (B|K)] = \frac{P(AHBK) + P(A|H)P(H^cBK) + P(B|K)P(AHK^c)}{P(H \lor K)}.
\]

**Remark 4.** We observe that, in case of some logical dependencies, the conjunction may be a conditional event; for instance, it may be verified that for \(K = AH\) we have \((A|H) \land (B|AH) = (A|H \cdot B|AH)\mid H = AB\mid H\); moreover, assuming \(A|H \subseteq B|K\), where \(\subseteq\) denotes the inclusion relation of Goodman and Nguyen, it holds that \((A|H) \land (B|K) = A\mid H\) (see [33]).

**Import–Export Principle and Lewis’ triviality result.** We observe that, as in our approach the Import–Export Principle does not hold, we avoid the counter-intuitive consequences related to Lewis’ triviality result ([43]; see also [54, Section 3.7], where Lewis’ triviality result and its consequences for

---

\(^3\)Adams [1] found problematic this interpretation for the conjunction of conditionals. Conjoined conditionals and conditional bets have been also investigated in [44].
The next theorem shows that the same result holds for logics for indicative conditionals have been discussed. We first prove that the probability of \( A|H \) can be disintegrated w.r.t. a partition \( \{B, B^c\} \).

**Theorem 6.** Given any events \( A, H, B \), with \( H \neq \emptyset, B \neq \emptyset \), it holds that

\[
P(A|H) = P((A|H)|B)P(B) + P((A|H)|B^c)P(B^c). \tag{7}
\]

**Proof.** For the sake of simplicity we set \( P(A|H) = x \). Then, by Definition 4 we have:

\[
\begin{align*}
P((A|H)|B)P(B) & = P((A|H)|B^c)P(B^c) = \\
P((AH + xH^c)|B)P(B) & = P((AH + xH^c)|B^c)P(B^c) = \\
P((AH|B)P(B) + xP(H^c|B)P(B) & = P((AH|B^c)P(B^c) + xP(H^c|B^c)P(B^c) = \\
P((AH + xP(H^c))P(B) & = P((AH)P(H) + P(A|H)P(H^c) = P(A|H).
\end{align*}
\]

Of course, the previous result still holds for a partition \( \{B_1, \ldots, B_n\} \). In particular, formula (7) holds for \( B = A \); but, as in our approach in general \( P((A|H)|A) \neq P(A|HA) = 1 \) and \( P((A|H)|A^c) \neq P(A|HA^c) = 0 \), we have

\[
P(A|H) = P((A|H)|A)P(A) + P((A|H)|A^c)P(A^c) \neq P(A).
\]

On the contrary, if the Import–Export Principle were valid, we would have \( P((A|H)|A)P(A) + P((A|H)|A^c)P(A^c) = P(A) \) (Lewis’ triviality result).

A critical analysis related to the disintegration of conditional probabilities is given in [23]. We recall that, given two conditional events \( A|H, B|K \) the iterated conditional \( (B|K)|(A|H) \) has been defined in [33] as

\[
(B|K)|(A|H) = (A|H) \wedge (B|K) + P((B|K)|(A|H)|A^c|H),
\]

where the prevision of \( (B|K)|(A|H) \) is defined in agreement with the betting metaphor. In the same paper it has been proved that

\[
P((B|K) \wedge (A|H)) = P((B|K)|(A|H))P(A|H).
\]

In particular, for \( K = \Omega \), we have \( P((A|H) \wedge B) = P((A|H)|B)P(B) \). Then, assuming \( P(A|H) > 0 \), we obtain the following Bayes formula for iterated conditionals:

\[
P(B|(A|H)) = \frac{P((A|H)\wedge B)}{P(A|H)} = \frac{P((A|H)|B)P(B)}{P((A|H)|B)P(B) + P((A|H)|B^c)P(B^c)}.
\]

5. **Lower and Upper Bounds for \( (A|H) \wedge (B|K) \)**

We will now determine the coherent extensions of the assessment \( (x, y) \) on \( \{A|H, B|K\} \) to the conjunction \( (A|H) \wedge (B|K) \). We recall that the extension \( z = P(AB|H) \) of the assessment \( (x, y) \) on \( \{A|H, B|H\} \), with \( A, B, H \) logically independent, is coherent if and only if

\[
\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}.
\]

The next theorem shows that the same result holds for \( (A|H) \wedge (B|K) \).
Theorem 7. Given any coherent assessment \((x, y)\) on \(\{A\mid H, B\mid K\}\), with \(A, H, B, K\) logically independent, and with \(H \neq \emptyset, K \neq \emptyset\), the extension 
\(z = \mathbb{P}(A\mid H) \land (B\mid K)\) is coherent if and only if the Fréchet-Hoeffding bounds are satisfied, that is
\[
\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}. \tag{8}
\]
Proof. First of all we observe that, by logical independence of \(A, H, B, K\), the assessment \((x, y)\) is coherent for every \((x, y) \in [0, 1]^2\). We will determine the values \(z', z''\) by the geometrical approach described in Section 2. The constituents associated with the family \(\mathcal{F} = \{A\mid H, B\mid K, (A\mid H) \land (B\mid K)\}\) and contained in \(H \lor K\) are: \(C_1 = AHBK, C_2 = AHB\lor K, C_3 = A\lor HBK, C_4 = A\lor HB\lor K, C_5 = AH\lor K, C_6 = A\lor H\lor K, C_7 = H\lor B\lor K, C_8 = H\lor B\lor K\). The associated points \(Q_h\)'s are: \(Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (0, 1, 0), Q_4 = (0, 0, 0), Q_5 = (1, y, y), Q_6 = (0, y, 0), Q_7 = (x, 1, x), Q_8 = (x, 0, 0)\). Considering the convex hull \(\mathcal{I}\) of \(Q_1, \ldots, Q_8\), the coherence of the prevision assessment \(\mathcal{M} = (x, y, z)\) on \(\mathcal{F}\) requires that the condition \(\mathcal{M} \subseteq \mathcal{I}\) be satisfied, which amounts to the solvability of the following system
\[
(\Sigma) \quad \mathcal{M} = \sum_{h=1}^{8} \lambda_h Q_h, \quad \sum_{h=1}^{8} \lambda_h = 1, \quad \lambda_h \geq 0, \quad \forall h.
\]
We observe that 
\(Q_5 = yQ_1 + (1 - y)Q_2, Q_6 = yQ_3 + (1 - y)Q_4, Q_7 = xQ_1 + (1 - x)Q_3, Q_8 = xQ_2 + (1 - x)Q_4;\) then, the convex hull \(\mathcal{I}\) is the tetrahedron with vertices \(Q_1, Q_2, Q_3, Q_4\). Thus, \((\Sigma)\) is equivalent to
\[
(\Sigma') \quad \mathcal{P} = \sum_{h=1}^{4} \lambda'_h Q_h, \quad \sum_{h=1}^{4} \lambda'_h = 1, \quad \lambda'_h \geq 0, \quad \forall h,
\]
with \(\lambda'_1 = 1 + y\lambda_5 + x\lambda_7, \lambda'_2 = 2 + (1 - y)\lambda_5 + x\lambda_8, \lambda'_3 = \lambda_3 + y\lambda_6 + (1 - x)\lambda_7, \lambda'_4 = \lambda_4 + (1 - y)\lambda_6 + (1 - x)\lambda_8\). Then, \(\mathcal{M} \subseteq \mathcal{I}\) if and only if \((\Sigma')\) is solvable. We observe that \((\Sigma')\) can be written as
\[
(\Sigma') \quad \lambda'_1 + \lambda'_2 = x, \quad \lambda'_1 + \lambda'_3 = y, \quad \lambda'_1 = z, \quad \lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 = 1, \quad \lambda'_h \geq 0, \quad \forall h; \quad \text{that is}
\]
\[
(\Sigma') \quad \lambda'_1 = z, \quad \lambda'_2 = x - z, \quad \lambda'_3 = y - z, \quad \lambda'_4 = z - (x + y - 1), \quad \lambda'_h \geq 0, \quad \forall h.
\]
\((\Sigma')\) is solvable if and only if 
\(\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}\). Moreover, the vector \((\lambda_1, \ldots, \lambda_8) = (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, 0, 0, 0, 0)\), where \((\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4)\) is any solution of \((\Sigma')\), is a solution of \((\Sigma)\) such that
\[
\sum_{r: C_r \subseteq H} \lambda_r = \sum_{r: C_r \subseteq K} \lambda_r = \sum_{r: C_r \subseteq H \lor K} \lambda_r = 1 > 0,
\]
and hence $I_0 = \emptyset$; then, by Theorem 1, the solvability of $(\Sigma)$ is also sufficient for the coherence of $\mathcal{M}$. Therefore, the extension $\mathbb{P}[(A|H) \land (B|K)] = z$ of the assessment $(x, y)$, with $(x, y) \in [0, 1]^2$, is coherent if and only if $z \in [z', z'']$, where $z' = \max \{x + y - 1, 0\}$ and $z'' = \min \{x, y\}$.

We recall that for the quasi conjunction of $A|H$ and $B|K$, defined as the conditional event $\mathcal{C}(A|H, B|K) = (AH \lor HC) \land (BK \lor KC)|(H \lor K)$, only the inequality on the lower bound holds; indeed, the extension $\gamma = P[\mathcal{C}(A|H, B|K)]$ of $(x, y)$ is coherent if and only if $\gamma' \leq \gamma \leq \gamma''$, where $\gamma' = z' = \max \{x + y - 1, 0\}$ and $\gamma'' = \frac{x + y - 2xy}{1 - xy}$ if $(x, y) \neq (1, 1)$; $\gamma'' = 1$ if $(x, y) = (1, 1)$. We observe that: $\gamma'' \geq \max \{x, y\} \geq \min \{x, y\} = z''$.

Quasi conjunction is a basic notion in nonmonotonic reasoning ([2, 21, 31, 34]). A probabilistic analysis of the lower and upper bounds for the quasi conjunction, in terms of t-norms and t-conorms, has been given in [30, 35].

6. Negation and Disjunction

Given any coherent assessment $(x, y, z)$ on $\{A|H, B|K, (A|H) \land (B|K)\}$, it holds that $(A|H) \land (B|K) \in \{1, 0, x, y, z\} \subset [0, 1]$. We recall that for conditional events the negation is usually defined as $(E|H)^c = E^c|H = (1 - E)|H$. In our approach we have $(1 - E)|H = 1 - E|H$; hence $(E|H)^c = 1 - E|H$. Then, for the negation of conjunction we give the definition below.

**Definition 6.** Given any conditional events $A|H, B|K$, the negation of $(A|H) \land (B|K)$ is defined as $[(A|H) \land (B|K)]^c = 1 - (A|H) \land (B|K)$.

We observe that $[(A|H) \land (B|K)]^c = 1 - \min \{A|H, B|K\} | (H \lor K) = (1 - \min \{A|H, B|K\}) | (H \lor K) = \max \{A^c|H, B^c|K\} | (H \lor K)$. Then, based on the relation $A \lor B = (A^cB^c)^c$ (De Morgan’s Law), for the notion of disjunction we give the definition below.

**Definition 7.** Given any conditional events $A|H, B|K$, the disjunction $(A|H) \lor (B|K)$ is defined as: $(A|H) \lor (B|K) = [(A^c|H) \lor (B^c|K)]^c$.

We observe that $(A|H) \lor (B|K) = 1 - \min \{A^c|H, B^c|K\} | (H \lor K) = (1 - \min \{A^c|H, B^c|K\}) | (H \lor K) = \max \{A|H, B|K\} | (H \lor K)$. If we assess $P(A|H) = x, P(B|K) = y, \mathbb{P}[(A|H) \lor (B|K)] = \gamma$, then $(A|H) \lor (B|K) = 1 \cdot (AH \lor BK) + x \cdot HC \lor KC + y \cdot A^cHK^c + \gamma \cdot HC^cK^c$.

*Prevision sum rule.* The classical formula $P(A \lor B) = P(A) + P(B) - P(AB)$ still holds for the conjunction and disjunction of conditional events. Indeed, by recalling Theorem 2 and the properties of iterated conditioning, we have
\[(A|H) \lor (B|K) + (A|H) \land (B|K) = [\max\{A|H,B|K\} + \min\{A|H,B|K\}](H \lor K) = (A|H + B|K)(H \lor K) = (A|H)(H \lor K) + (B|K)(H \lor K) = A|H + B|K. \] 

Thus 

\[\mathbb{P}[(A|H) \lor (B|K)] = \mathbb{P}(A|H) + \mathbb{P}(B|K) - \mathbb{P}((A|H) \land (B|K)).\]

Finally, assuming \(A, H, B, K\) logically independent, from (8), \(\gamma\) is a coherent extension of \((x, y)\) if and only if \(\max\{x, y\} \leq \gamma \leq \min\{x + y, 1\}\), i.e.

\[\max\{P(A|H), P(B|K)\} \leq \mathbb{P}[(A|H) \lor (B|K)] \leq \min\{P(A|H) + P(B|K), 1\}.\]

7. Conclusions

We have considered c.r.q.’s and conditional previsions in the setting of coherence. In agreement with the approach of other authors, a c.r.q. \(X|H\) has been looked at as the quantity \(XH + \mathbb{P}(X|H)H^c\), which assumes the value \(\mathbb{P}(X|H)\) when \(H\) is false. We have defined the equality among c.r.q.’s, by showing that \(X|HK\) coincides with \([XH + \mathbb{P}(X|HK)H^c]|K\); this equality, by linearity of prevision, allows to directly obtain the compound prevision theorem. By this extended notion of c.r.q., we have obtained some algebraic developments; for instance, we have given a meaning to the sum \(X|H + Y|K\); then, the sum of previsions \(\mathbb{P}(X|H) + \mathbb{P}(Y|K)\) can be read as the prevision of the sum, \(\mathbb{P}(X|H + Y|K)\). We have examined the notion of iterated c.r.q. \((X|H)|K\), which does not coincide with the c.r.q. \(X|HK\); in particular, given any event \(A\), the iterated conditional \((A|H)|K\) does not coincide with the conditional event \(A|HK\); thus, in our approach the Import–Export Principle is not valid and we avoid the counter-intuitive consequences related to Lewis’ triviality result. We have studied the conjunction of two conditional events and we have interpreted the prevision of a conjoined conditional by the betting scheme. Then, using iterated conditionals, we have obtained a disintegration formula for conditional probabilities and a kind of Bayes formula. We have defined the negation of the conjunction of conditionals and by De Morgan’s Law the associated disjunction. Differently from other authors, in our coherence-based approach, the result of the conjunction, or the disjunction, is (not a conditional, but) a c.r.q., for which we have determined the lower and upper prevision bounds. We have shown that the usual probabilistic properties continue to hold in terms of previsions.

Acknowledgements. The authors thank the editors and the anonymous referees for their valuable comments and criticisms. This work was partially supported by grant FFR 2012-ATE-0585 from the University of Palermo.
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