AF $C^*$-ALGEBRAS FROM NON AF GROUPOIDS
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Abstract. We construct ample groupoids from certain categories of paths, and prove that their $C^*$-algebras coincide with the continued fraction AF algebras of Effros and Shen. The proof relies on recent classification results for simple nuclear $C^*$-algebras. The groupoids are not principal. This provides examples of Cartan subalgebras in the continued fraction AF algebras that are isomorphic, but not conjugate, to the standard diagonal subalgebras.

1. Introduction

Approximately finite dimensional (AF) $C^*$-algebras were introduced by Bratteli, generalizing the uniformly hyperfinite (UHF) $C^*$-algebras of Glimm. They have become one of the most intensely studied classes of $C^*$-algebras. They are an integral part, and in fact were the beginning, of Elliott's classification program for (simple) separable nuclear $C^*$-algebras. Despite the many remarkable successes in the study of AF algebras, it can be difficult to decide if a given $C^*$-algebra is AF. Namely, while the definition requires that there be an increasing family of finite dimensional $C^*$-subalgebras whose union is dense, finding such a family is not always practical. The known examples of this problem are crossed products (or fixed point algebras) of an action of a finite group on a non AF algebra. In some instances ([11]) the approximating family of finite dimensional subalgebras is explicitly constructed, while in others ([2], [5]) the proof consists of showing that the algebra satisfies the hypotheses of a classification theorem, and then checking that the invariants match those of an AF algebra.

One context where the problem has been completely solved is that of the $C^*$-algebras of directed graphs. It is known that a graph algebra is AF if and only if the graph does not contain a cycle ([21, Section 5.4]). A bit more detail is known as well. Since AF algebras are always stably finite, the presence of an infinite projection (equivalently, a corner containing a nonunitary isometry) precludes the AF property. It is known that a graph algebra contains an infinite projection if and only if the graph contains a cycle with an entrance ([21 Proposition 5.4]). Moreover, it is also known that all separable AF algebras are Morita equivalent to graph algebras ([4], [31]).

A significant generalization of graph algebras was made by Kumjian and Pask with the definition of higher rank graphs ([12]). This vastly expanded the family of $C^*$-algebras represented beyond the graph algebras (which are the rank one case). However, the identification of AF algebras among the higher rank graph algebras is much harder. This problem was studied in detail by Evans and Sims ([9]). They

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gave many positive results. In particular they define generalized cycles that play much of the role that cycles play in the case of directed graphs. They proved that a higher rank graph containing a generalized cycle cannot have an AF $C^*$-algebra, and that if a higher rank graph contains a generalized cycle with an entrance then its $C^*$-algebra is infinite. However it is not true that absence of generalized cycles implies that the $C^*$-algebra is AF ([16], and Example 2.10 below). Moreover, they present an example of a rank two graph whose $C^*$-algebra seems in every way to be AF (even UHF), but were unable to prove this. Similar difficulties occur in other examples ([9, Subsection 6.1]). Evans and Sims point out that if their example is AF it would give an example of a non principal groupoid whose $C^*$-algebra is AF. (See [22, Definition III.1.1] for the definition of AF groupoid, and the proof that AF groupoids are principal.) Moreover this would present an explicit Cartan subalgebra that is not conjugate to the canonical diagonal subalgebra.

In this paper we give examples that are similar, but in a more general setting, and we are able to prove that they are actually AF algebras. One of the factors complicating the study of higher rank graph algebras is that the requirements for a higher rank graph are quite rigid. A directed graph can be thought of as an arbitrary collection of dots and arrows. Thus it is easy to give examples and to tailor them for certain purposes. By contrast, it is difficult to construct explicit higher rank graphs. The second author introduced categories of paths ([26], see also [27]) as a simultaneous generalization of higher rank graphs and of the quasi-lattice ordered groups of Nica. The main motivation, however, was to try to bring some of the flexibility of construction back to the setting of higher rank graphs. A category of paths can be thought of as a directed graph with identifications, but the requirements are much looser than those for higher rank graphs. The notions of generalized cycle and entrance are the same as for higher rank graphs. It is shown in [26] that the presence of a generalized cycle with an entrance implies that the $C^*$-algebra is infinite. However the presence of a generalized cycle without an entrance is not sufficient to preclude that the $C^*$-algebra is AF ([16], and Example 2.9 below). Categories of paths determine $C^*$-algebras very much in the same spirit as higher rank graphs, and these can be defined either via an étale groupoid or by generators and relations.

Our examples are constructed from categories of paths that are amalgamations of a 2-graph and a 1-graph. We identify the $C^*$-algebras as being Morita equivalent to the continued fraction AF algebras of [6]. Moreover there is a natural full corner in our example that is isomorphic to the Effros Shen algebra $A_\theta$, where $\theta \in (0,1) \setminus \mathbb{Q}$ is arbitrary. The proof relies on the remarkable recent classification results for separable simple nuclear $C^*$-algebras, for which we cite [29, Theorem D]. The classification is in terms of the Elliott invariant, which consists of the ordered $K$-theory of the algebra, the position of the unit in $K_0^+$, the space of tracial states, and the pairing between the traces and $K_0$. Thus we must compute the Elliott invariant of our examples, and also verify that they satisfy the hypotheses of the classification theorem. Sections 4 - 6 are devoted to calculating the Elliott invariant, as well as working out the fine structure of our examples. In section 7 we verify the hypotheses of the classification theorem and deduce our main theorem.

This provides examples of Cartan subalgebras of AF algebras that are isomorphic, but not conjugate, to the diagonal subalgebra defined by a dense increasing union of
finite dimensional subalgebras. We also identify a subalgebra that is the closure of a (nondense) increasing union of finite dimensional subalgebras, which is isomorphic to the whole algebra, and such that the inclusion induces an isomorphism of Elliott invariants.

We briefly describe the rest of the paper. Section 2 gives a short account of the basic facts about the construction of groupoids and $C^*$-algebras from categories of paths, and the two examples mentioned above. In section 3 we describe the categories of paths and groupoids that are the main subject of the paper. The groupoids are inductive limits, and in section 4 we compute the ordered $K$-theory of the $C^*$-algebras of the terms in this limit. In section 5 we compute the ordered $K$-theory of the $C^*$-algebra of the whole groupoid and identify it as that of a continued fraction AF algebra. This requires the use of a somewhat technical sequence of partitions refining the compact open subsets of the unit space. (We say that a partition refines a set $S$ if $S$ equals the union of a subcollection of the partition.) In section 6 we prove that there exists a unique invariant measure on the unit space of the groupoid. In section 7 we verify that the groupoid $C^*$-algebra is classifiable and complete the calculation of its Elliott invariant, proving that we have presented the Effros Shen algebras as $C^*$-algebras of non principal groupoids. In section 8 we show that our algebra contains a proper copy of itself in the usual form of the Effros Shen algebra. In section 9 we investigate the scale of the algebras and give conditions under which it equals the stabilization of the Effros Shen algebra.

We thank the referee for a suggestion that led to section 9.

2. Background on categories of paths

In this section we briefly recall the basic facts about categories of paths and their $C^*$-algebras. This generalizes the case of higher rank graphs. We refer to [20], [27].

**Definition 2.1.** A category of paths is a cancellative small category in which no nonunit has an inverse.

Let $\Lambda$ be a category of paths, with unit space $\Lambda^0$. We write $s, r : \Lambda \to \Lambda^0$ for the source and range maps. For $\mu \in \Lambda$ we write $\mu \Lambda = \{ \mu \nu : \nu \in \Lambda, r(\nu) = s(\mu) \}$, and similarly for $\lambda \mu$. We say that $\nu$ extends $\mu$ if $\nu \in \mu \Lambda$, and in this case we call $\mu$ a prefix of $\nu$. The set of prefixes of an element $\nu \in \Lambda$ is denoted $[\nu]$. For two elements $\mu, \nu \in \Lambda$ the common extensions of $\mu$ and $\nu$ are the elements of $\mu \Lambda \cap \nu \Lambda$. We write $\mu \cap \nu$ if $\mu \Lambda \cap \nu \Lambda \neq \emptyset$, and we say that $\mu$ meets $\nu$; otherwise we write $\mu \perp \nu$, and we say that $\mu$ and $\nu$ are disjoint. There is an important simplifying assumption that was given for higher rank graphs in [20].

**Definition 2.2.** The category of paths $\Lambda$ is finitely aligned if for all $\mu, \nu \in \Lambda$ there exists a (possibly empty) finite set $F \subseteq \Lambda$ such that $\mu \Lambda \cap \nu \Lambda = \bigcup_{\eta \in F} \eta \Lambda$.

In the finitely aligned case there is a unique minimal such set $F$, denoted $\mu \vee \nu$, the set of minimal common extensions of $\mu$ and $\nu$.

**Definition 2.3.** Let $\Lambda$ be a finitely aligned category of paths. A subset $x \subseteq \Lambda$ is directed if for all $\mu, \nu \in x$, $x \cap \mu \Lambda \cap \nu \Lambda \neq \emptyset$, hereditary if for all $\mu \in x$, $[\mu] \subseteq x$. 
All elements of a directed hereditary set have the same range, denoted \( r(x) \). We let \( \Lambda^* \) denote the set of all directed hereditary subsets of \( \Lambda \), and for \( u \in \Lambda^0 \) we write \( u^* = \{ x \in \Lambda^* : r(x) = u \} \). For \( \mu \in \Lambda \) we write \( Z(\mu) = \{ x \in \Lambda^* : \mu \in x \} \). We think of \( Z(\mu) \) as a generalized cylinder set.

**Theorem 2.4.** Let \( \Lambda \) be a finitely aligned category of paths. The collection \( \{ Z(\mu) : \mu \in \Lambda \} \) generates a totally disconnected locally compact Hausdorff topology on \( \Lambda^* \), and \( Z(\mu) \) is compact open in this topology. The collection \( B^* = \{ Z(\lambda) \setminus \bigcup_{i=1}^{n} Z(\mu_i) : \lambda, \mu_1, \ldots, \mu_k \in \Lambda \} \) is a base (of compact open sets) for this topology.

We let \( \Lambda^{**} \) denote the maximal elements of \( \Lambda^* \) with respect to inclusion, and put \( \partial \Lambda = \overline{\Lambda}^{**} \), the boundary of \( \Lambda \). Left cancellation in \( \Lambda \) implies that for \( \mu \in \Lambda \), the map \( \nu \in s(\mu) \Lambda \mapsto \mu \nu \in \mu \Lambda \) is bijective. It extends to a partial homeomorphism of \( \Lambda^* : x \in s(\mu) \Lambda^* \mapsto x \mu = \bigcup \{ [\mu \nu] : \nu \in x \} \). Moreover, \( \mu x \in \partial \Lambda \) if and only if \( x \in \partial \Lambda \).

We define a groupoid with unit space \( \Lambda^* \) as follows. Let \( \Lambda \ast \Lambda \ast \Lambda^* = \{ (\mu, \nu, x) \in \Lambda \times \Lambda \times \Lambda^* : s(\mu) = s(\nu) = r(x) \} \). Let \( \sim \) be the equivalence relation on \( \Lambda \ast \Lambda \ast \Lambda^* \) generated by \( (\mu, \nu, \eta x) \sim (\mu \eta, \nu \eta, x) \), and define \( G(\Lambda) = \Lambda \ast \Lambda \ast \Lambda^*/\sim \). The groupoid operations are given by

1. \((\mu, \nu, x) \cdot (\eta, \xi, y) = (\mu \eta, \nu \xi, x \eta y) \)
2. \([\mu, \nu, x]^{-1} = [\nu, \mu, x] \)
3. \([\mu, \nu, x] \cdot [\eta, \xi, y] = [\mu \eta, \nu \xi, x \eta y] \).

More precisely, we use [26] Lemma 4.12: if \( \mu, \mu' \in \Lambda \) and \( x, x' \in \Lambda^* \) are such that \( \mu x = \mu' x' \) then there exist \( \eta, \eta' \in \Lambda \) and \( y \in \Lambda^* \) such that \( x = \eta y, x' = \eta' y \), and \( \mu \eta = \mu' \eta' \). Then the composable pairs are given by \( G^{(2)} = \{ ([\mu, \nu, x], [\xi, \eta, y]) : \nu x = \xi y \} \), and multiplication is given as follows. Let \( ([\mu, \nu, x], [\xi, \eta, y]) \in G^{(2)} \). Then \( \nu x = \xi y \), so the lemma cited above yields \( z \in \Lambda^* \) and \( \delta, \epsilon \in \Lambda \) such that \( x = \delta z, y = \epsilon z \), and \( \nu \delta = \xi \epsilon \). Then

\[
[\mu, \nu, x] \cdot [\xi, \eta, y] = [\mu \delta, \nu \delta, z] \cdot [\xi \epsilon, \eta \epsilon, z] = [\mu \delta, \eta \epsilon, z].
\]

The unit space \( G(\Lambda)^{(0)} = \{ [r(x), r(x), x] : x \in \Lambda^* \} \) can be identified with \( \Lambda^* \). For a subset \( E \subseteq s(\nu) \Lambda^* \) we write \( [\mu, \nu, E] := \{ [\mu, \nu, x] : x \in E \} \).

**Theorem 2.5.** The collection of sets \( [\mu, \nu, E] \) for \( E \subseteq \Lambda^* \) compact open is a base for a topology making \( G(\Lambda) \) into an ample Hausdorff (étale) groupoid.

The boundary \( \partial \Lambda \) is a closed invariant subset of \( G(\Lambda)^{(0)} \equiv \Lambda^* \).

**Definition 2.6.** The Toeplitz algebra of \( \Lambda \) is \( TC^*(\Lambda) := C^*(G(\Lambda)) \). The Cuntz-Krieger algebra of \( \Lambda \) is \( C^*(\Lambda) := C^*(G(\Lambda)|_{\partial \Lambda}) \).

Generalized cycles and entrances are defined exactly as for higher rank graphs ([9]).

**Definition 2.7.** Let \( \Lambda \) be a category of paths. A **generalized cycle** in \( \Lambda \) is an ordered pair \( (\mu, \nu) \in \Lambda \times \Lambda \) such that \( \mu \neq \nu \), and

1. \( s(\mu) = s(\nu) \)
2. \( r(\mu) = r(\nu) \)
3. \( \eta \in \mu \Lambda, (\nu \cap \eta) \)

The generalized cycle \( (\mu, \nu) \) has an entrance if \( (\nu, \mu) \) is not a generalized cycle.

It is proved in [26] that if a finitely aligned category of paths contains a generalized cycle with an entrance then its Cuntz-Krieger algebra is infinite. In particular,
Theorem 2.8. If the finitely aligned category of paths \( \Lambda \) contains a generalized cycle with an entrance, then \( C^*(\Lambda) \) is not AF.

It is proved in [9] that the \( C^* \)-algebra of a higher rank graph that contains a generalized cycle without an entrance is not AF. However, this is not generally true for categories of paths. The following example is due to the first author ([16]).

Example 2.9. Let \( \Lambda \) be the category of paths given by the directed graph

\[
\begin{array}{ccc}
\beta_1 & -\alpha_1 & v_1 \\
\alpha_2 & v_2 \\
\beta_2 & -
\end{array}
\]

with commuting relations \( \alpha_1 \beta_2 = \beta_1 \alpha_2 \) and \( \alpha_1 \alpha_2 = \beta_1 \beta_2 \). Explicitly,

\[
\Lambda = \{v_1, v_2, v_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1 \alpha_2, \alpha_1 \beta_2\}
\]

where

\[
\Lambda^0 = \{v_1, v_2, v_3\}
\]

and

\[
\Lambda^2 = \{(\alpha_1, \alpha_2), (\alpha_1, \beta_2), (\beta_1, \alpha_2), (\beta_1, \beta_2)\}
\]

with the obvious range and source maps, and ranges for multiplication. It is clear that this is a small category, without inverses since the commuting relations preserve length, and there is no (non-trivial) cancellation to check. Note that \( (\alpha_1, \beta_1) \) is a generalized cycle without an entrance. Since the category is finite, \( G(\Lambda) \) is finite, and hence \( C^*(\Lambda) \) is finite dimensional.

We also note that even for higher rank graphs, the absence of generalized cycles is not sufficient for the \( C^* \)-algebra to be AF. We give the following example, due to the first author ([16]).

Example 2.10. Let \( \Lambda \) be the 2-graph which has skeleton

\[
\begin{array}{ccc}
\alpha_{-1} & \alpha_0' & \alpha_1' \\
\beta_{-1} & v_0 & v_1 \\
\beta_0' & \beta_1 & v_2 \ldots
\end{array}
\]

and factorization rules \( \sigma_i \tau_{i+1}' = \tau_i' \sigma_{i+1} \) where \( \sigma, \tau \in \{\alpha, \beta\} \). As in [9] Section 2.4, this defines a 2-graph. We will use the convention that \( d(\alpha_i) = d(\beta_i) = (1, 0) \) and \( d(\alpha_i') = d(\beta_i') = (0, 1) \). By [8] Proposition 3.16 (see also [7] Corollary 5.1) and the remarks following [7] Figure 4.3, \( K_1(C^*(\Lambda)) \cong \mathbb{Z}[\frac{1}{2}] \) so that \( C^*(\Lambda) \) is not an AF-algebra. We will show that \( \Lambda \) has no generalized cycles.

To facilitate the argument, we will drop subscripts when the range is understood. Thus we may write \( \alpha_i \beta_{i+1}' = \alpha_i \beta' = v_i \alpha \beta' \). We note the following notational device.
Let $\lambda \in v_i \Lambda$ with $d(\lambda) = (m, 0)$. Write $\lambda = \lambda(i) \lambda(i+1) \cdots \lambda(i+m-1)_{i+m-1}$, where $\lambda(j) \in \{\alpha, \beta\}$ for $i \leq j < i + m$. Then the factorization rules give
\[
\lambda \alpha'_{i+m} = \lambda(i) \lambda(i+1) \cdots \lambda(i+m-2)_{i+m-2} \alpha'_{i+m} = \cdots = \alpha'_{i} \lambda(i+1) \cdots \lambda(i+m-1)_{i+m} = \alpha' \lambda.
\]
Similarly, $\lambda \beta' = \beta' \lambda$, and in general, if $\lambda' \in v_{i+m} \Lambda$ with $d(\lambda') = (0, n)$ we have $\lambda \lambda' = \lambda' \lambda$.

Now fix $\mu \neq \nu \in \Lambda$ with $r(\mu) = r(\nu)$ and $s(\mu) = s(\nu)$. Write $\mu = \sigma \sigma', \nu = \tau \tau'$ where $d(\sigma), d(\tau) \in \mathbb{N} \times \{0\}$ and $d(\sigma'), d(\tau') \in \{0\} \times \mathbb{N}$. First assume $d(\sigma) \geq d(\tau)$ and suppose that $\sigma$ does not extend $\tau$. Then there is some $j$ such that $\sigma_j \neq \tau_j$. This implies that $\mu$ and $\nu$ have no common extension, since if $\lambda$ were such an extension, by factoring $\lambda = \sigma \lambda'$ we would have $\sigma_j = \lambda_j = \tau_j$.

Now suppose $\sigma$ extends $\tau$ and consider the case where $d(\sigma) > d(\tau)$. Let $v_j = s(\tau)$ and suppose $\sigma_j = \alpha_j$. Then we see that $\mu, \nu \beta$ have no common extension by the same reasoning as above, since $\nu \beta = \tau \beta_j \tau'$ while $\mu_j = \alpha_j$. Similarly, if $\sigma_j = \beta_j$, then $\mu, \nu \alpha$ have no common extension. If $d(\sigma) = d(\tau)$, then $\sigma = \tau$ and hence $\sigma' \neq \tau'$ so there is some $j$ with $\sigma'_j \neq \tau'_j$, and reasoning as above shows this precludes the existence of a common extension of $\mu$ and $\nu$.

If $d(\sigma) < d(\tau)$, we may write $\mu = \sigma' \sigma$ and $\nu = \tau' \tau$. Then the previous argument applies to $\sigma'$ and $\tau'$. Thus in all cases we can find $\eta \in \Lambda$ with $\mu, \nu \eta$ having no common extension, so $(\mu, \nu)$ is not a generalized cycle.

3. The examples

We will build a family of categories of paths as amalgamations of a 2-graph and a 1-graph ([26] Section 11). We begin with the following directed graph $E$:
\[
\begin{array}{c}
\text{w}_1 \leftarrow \text{a}_1 \rightarrow \text{w}_2 \leftarrow \text{a}_2 \rightarrow \text{w}_3 \leftarrow \text{a}_3 \rightarrow \text{w}_4 \rightarrow \ldots
\end{array}
\]
Thus $E^0 = \{w_1, w_2, \ldots\}$, $E^1 = \{a_1, a_2, \ldots\}$, and $s(a_i) = w_{i+1}, r(a_i) = w_j$. Let $E^*$ be the set of finite paths in $E$, that is, the 1-graph associated with $E$. Let $f : \mathbb{N}^2 \to \mathbb{N}$ be given by $f(n) = n_1 + n_2$. Using [12] Definition 1.9, set $\Lambda_1 = f^*(E^*)$; $\Lambda_1$ is a 2-graph. Then $\Lambda_1 = \{(\mu, n) \in E^* \times \mathbb{N}^2 : |\mu| = f(n)\}$. For $i \geq 1$ we let $\alpha_i = (a_i, e_1)$ and $\beta_i = (a_i, e_2)$; these are the edges in $\Lambda_1$:

\[
\begin{array}{c}
\text{w}_1 \xleftarrow{\alpha_1} \text{w}_2 \xrightarrow{\alpha_2} \text{w}_3 \xleftarrow{\alpha_3} \text{w}_4 \xrightarrow{\alpha_4} \ldots
\end{array}
\]
\[
\begin{array}{c}
\text{w}_1 \xrightarrow{\beta_1} \text{w}_2 \xleftarrow{\beta_2} \text{w}_3 \xrightarrow{\beta_3} \text{w}_4 \xleftarrow{\beta_4} \ldots
\end{array}
\]
The commuting squares in $\Lambda_1$ are $\alpha_i \beta_{i+1} = \beta_i \alpha_{i+1}$. We may write a typical element of $\Lambda_1$ as a product of edges: $(a_i a_{i+1} \cdots a_j, n) = \alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1} \beta_{i+n_1} \beta_{i+n_1+1} \cdots \beta_{i+n_1+n_2-1}$. We will often write this in the form $\alpha^n_1 \beta^n_2$ if the source (and range) are understood (otherwise we might write, for example, $w_i ^{\alpha_1^n \beta_2^n}$). By the factorization rule in
the edges may be “permuted” as desired; thus, for example, \( \alpha_i \alpha_{i+1} \beta_{i+2} \beta_{i+3} = \beta_i \alpha_{i+1} \beta_{i+2} \alpha_{i+3} \), etc.

Now choose nonnegative integers \( k_1, k_2, \ldots \), with \( k_i > 0 \) for infinitely many \( i \), and let \( \Lambda_2 \) be the 1-graph with vertex set \( \Lambda_0^0 = \{ u_1, u_2, \ldots \} \) and edge set \( \Lambda_1^0 = \{ \gamma_i^{(j)} : i \geq 1, 1 \leq j \leq k_i \} \), with \( s(\gamma_i^{(j)}) = u_{i+1} \), \( r(\gamma_i^{(j)}) = u_i \). (Note that if \( k_i = 0 \) then \( u_i \Lambda_2 u_{i+1} = \emptyset \).

\[ \Lambda_2: \]

\[ \gamma_1^{(1)} \]
\[ \vdots \]
\[ \gamma_1^{(2)} \]
\[ \gamma_1^{(k_1)} \]
\[ u_1 \]
\[ \gamma_2^{(1)} \]
\[ \gamma_2^{(2)} \]
\[ \gamma_2^{(k_2)} \]
\[ u_2 \]
\[ \gamma_3^{(1)} \]
\[ \gamma_3^{(2)} \]
\[ \gamma_3^{(k_3)} \]
\[ u_3 \]
\[ \vdots \]
\[ \gamma_4^{(1)} \]
\[ \gamma_4^{(2)} \]
\[ \gamma_4^{(k_4)} \]
\[ u_4 \]
\[ \ldots \]

**Definition 3.1.** Let \( \sim \) be the equivalence relation on \( \Lambda_0^0 \sqcup \Lambda_0^1 \) generated by \( w_i \sim u_i \) for \( i \geq 1 \). Let \( \Lambda \) be the category of paths given by the amalgamation of \( \Lambda_1 \) and \( \Lambda_2 \) over \( \sim \), as in [26, Definition 11.3]. Let \( v_i = [u_i] = [w_i] \) for \( i \geq 1 \). We will make a small notational abuse to write \( v_i \Lambda_1 \) for the set \( \{ v_i \alpha^m \beta^n : m, n \geq 0 \} \subseteq \Lambda \), and identify it with \( w_i \Lambda_1 \) (and similarly for \( v_i \Lambda_2 \)).

We may picture \( \Lambda \) as follows:

\[ \Lambda: \]

\[ \gamma_1^{(1)} \]
\[ \vdots \]
\[ \gamma_1^{(2)} \]
\[ \gamma_1^{(k_1)} \]
\[ v_1 \]
\[ \alpha_1 \]
\[ \beta_1 \]
\[ \gamma_2^{(1)} \]
\[ \gamma_2^{(2)} \]
\[ \gamma_2^{(k_2)} \]
\[ v_2 \]
\[ \alpha_2 \]
\[ \beta_2 \]
\[ \gamma_3^{(1)} \]
\[ \gamma_3^{(2)} \]
\[ \gamma_3^{(k_3)} \]
\[ v_3 \]
\[ \alpha_3 \]
\[ \beta_3 \]
\[ \vdots \]
\[ \gamma_4^{(1)} \]
\[ \gamma_4^{(2)} \]
\[ \gamma_4^{(k_4)} \]
\[ v_4 \]
\[ \ldots \]

We remark that the sequence \( (k_i) \) is a necessary part of the definition of \( \Lambda \). We will use it when working with \( \Lambda \) and other objects constructed from \( \Lambda \) (such as \( G(\Lambda) \)), understanding that a choice of this sequence has been made, even if this was not explicit.

**Remark 3.2.** It is fruitful to think of \( \Lambda_1 \) as the base of our construction, having a regular structure, while \( \Lambda_2 \) represents a sporadic introduction of “impurities” into that regular structure. (For example, in Theorem [4.6] we will show that \( C^*(\Lambda) \) is stable if and only if the terms of the sequence \( (k_i) \) are rarely nonzero.) A similar construction could be made with \( \Lambda_1 \) changed to an analogous higher rank graph with rank larger than two. Thus we trust that the reader will not be misled by the apparent mixup of rank and subscript.

**Lemma 3.3.** Let \( \lambda \in \Lambda \setminus \Lambda^0 \). There exist unique \( m \geq 1 \) and elements \( \theta_1, \ldots, \theta_m \in \Lambda_1 \cup \Lambda_2 \) such that

\[
\begin{align*}
&\cdot \theta_i \notin \Lambda_0^0 \cup \Lambda_2^0 \\
&\cdot s(\theta_i) \sim r(\theta_{i+1}) \\
&\cdot s(\theta_i) \neq r(\theta_{i+1})
\end{align*}
\]
\[ \lambda = \theta_1 \cdots \theta_m. \]

The decomposition \( \lambda = \theta_1 \cdots \theta_m \) is called the normal form of \( \lambda \).

**Proof.** This follows from [26, Lemma 11.2]. \( \square \)

Thus \( \theta_i \in \Lambda_1 \) if and only if \( \theta_{i+1} \in \Lambda_2 \). We let \( |\lambda| = \sum_{i=1}^m |\theta_i| \), where for \( \theta \in \Lambda_1 \cup \Lambda_2 \) we let \( |\theta| \) denote the number of edges in \( \theta \). If \( r(\lambda) = v_j \) and \( |\lambda| = n \), we may write \( \lambda = \lambda_j \lambda_{j+1} \cdots \lambda_{j+n-1} \), where \( \lambda_i \in \{ \alpha_i, \beta_i \} \cup \{ \gamma_i(t) : 1 \leq t \leq k_i \} \). Of course this representation of \( \lambda \) might not be unique, as a factor \( \theta_i \in \Lambda_1 \) may be written in several ways as a path.

**Lemma 3.4.** Let \( \lambda, \mu \in \Lambda \). Write \( \lambda = \theta_1 \cdots \theta_m, \mu = \phi_1 \cdots \phi_n \) in normal form. Then \( \lambda \) and \( \mu \) have a common extension if and only if \( \theta_i = \phi_i \) for \( i \) \( < \) \( \min\{m, n\} \), and one of the following conditions holds:

1. \( m \neq n \), and if, say, \( m < n \), then \( \phi_m \) extends \( \theta_m \). (Note that in this case, \( \mu \) extends \( \lambda \).)

2. \( m = n \), and \( \theta_m \) and \( \phi_m \) have a common extension in \( \Lambda_j \) (where \( j \) is such that \( \theta_m, \phi_m \in \Lambda_j \)). (Note that in this case, if \( j = 2 \) then one of \( \lambda, \mu \) extends the other.)

**Proof.** This follows from [26, Lemma 11.4]. \( \square \)

**Corollary 3.5.** Let \( \lambda, \mu \in \Lambda \) have a common extension, and suppose that neither extends the other. Then \( \lambda \) and \( \mu \) must be in situation (2) of Lemma 3.4, and with notation as in Lemma 3.4 we have that \( j = 1 \) and that \( d(\theta_m), d(\phi_m) \) are not comparable (in \( \mathbb{N}^2 \)).

It follows from Corollary 3.5 that \( \Lambda \) is finitely aligned. In fact, since both \( \Lambda_1 \) and \( \Lambda_2 \) are singly aligned (or right LCM) then so is \( \Lambda \); this means that if \( \lambda \) and \( \mu \) have a common extension then they have a unique minimal common extension.

**Proposition 3.6.** \( \Lambda \) contains no generalized cycles.

**Proof.** Let \( \mu, \nu \in \Lambda \) with \( r(\mu) = r(\nu), s(\mu) = s(\nu) \) and \( \mu \neq \nu \). Then neither of \( \mu, \nu \) can extend the other. If \( \mu \perp \nu \) then \( (\mu, \nu) \) is not a generalized cycle. So suppose that \( \mu \cong \nu \). By Corollary 3.5 \( \mu \) and \( \nu \) have normal forms \( \mu = \theta_1 \cdots \theta_{m-1} \phi \) and \( \nu = \theta_1 \cdots \theta_{m-1} \phi' \), where \( \phi, \phi' \in \Lambda_1 \) and \( d(\phi), d(\phi') \) are not comparable in \( \mathbb{N}^2 \). Letting \( \phi = \alpha^p \beta^q \) and \( \phi' = \alpha'^p \beta'^q \), we may assume without loss of generality that \( p < p' \) and \( q > q' \). Choose \( i \geq j \), where \( v_j = s(\mu) \), such that \( k_i \neq 0 \). Let \( \eta = \beta^{i-|\alpha|-1} \gamma_i \). Then Lemma 3.4 implies that \( \mu \eta \perp \nu \), and hence \( (\mu, \nu) \) is not a generalized cycle. \( \square \)

We will describe the elements of \( \Lambda^* \) explicitly. First note that the finite directed hereditary subsets of \( \Lambda \) are precisely those that contain a maximal element, and these are in one-to-one correspondence with the elements of \( \Lambda \), via \( \lambda \leftrightarrow [\lambda] \).

**Lemma 3.7.** The infinite directed hereditary subsets of \( \Lambda_1 \) are the following: for \( \ell \geq 1 \),

- \( w_\ell \alpha^m \beta^\infty := \{ w_\ell \alpha^i \beta^j : 0 \leq i \leq m, 0 \leq j \leq \ell \} \), \( m = 0, 1, 2, \ldots \)
- \( w_\ell \alpha^\infty \beta^m := \{ w_\ell \alpha^i \beta^j : 0 \leq i, 0 \leq j \leq n \} \), \( n = 0, 1, 2, \ldots \)
- \( w_\ell \alpha^\infty \beta^\infty := w_\ell \Lambda_1 \).

**Proof.** This follows from the commutation relations in \( \Lambda_1 \). \( \square \)
**Notation 3.8.** We write \( \nu \Lambda_1^\infty = \{ \nu \alpha^m \beta^n : m + n = \infty \} \) for the infinite elements of \( \Lambda_1^\ast \) with range \( \nu \). The infinite elements of \( \nu \Lambda_2^\ast \) can be identified with the infinite path space \( \nu \Lambda_2^\infty \) of the directed graph \( \Lambda_2 \). (We note that \( \Lambda_2^\infty = \emptyset \) unless \( k_i > 0 \) eventually.)

**Theorem 3.9.** Let \( x \) be an infinite element of \( \Lambda^* \). Then \( x \) has one of the following forms:

1. there is an infinite sequence \( \theta_1, \theta_2, \ldots \) such that \( \theta_1 \theta_2 \cdots \theta_m \) is in normal form for all \( m \), and \( x = \bigcup_{m=1}^\infty [\theta_1 \cdots \theta_m] \);
2. there is a finite normal form \( \theta_1 \cdots \theta_{M-1} \), with \( \theta_{M-1} \in \Lambda_2 \) if \( M > 1 \), and an infinite element \( y \in \Lambda_1^\infty \) such that \( x = \theta_1 \cdots \theta_{M-1} y := \bigcup \{ [\theta_1 \cdots \theta_{M-1} \eta] : \eta \in y \} \);
3. there is a finite normal form \( \theta_1 \cdots \theta_{M-1} \), with \( \theta_{M-1} \in \Lambda_1 \) if \( M > 1 \), and an element \( y \in \Lambda_2^\infty \) such that \( x = \theta_1 \cdots \theta_{M-1} y := \bigcup \{ [\theta_1 \cdots \theta_{M-1} \eta] : \eta \in y \} \).

**Proof.** Let \( x \in \Lambda^* \) be infinite. Let \( M = \sup \{ m : \text{there is } \lambda \in x \text{ such that } \lambda = \theta_1 \cdots \theta_m \text{ in normal form} \} \). We first consider the case that \( M = \infty \). For \( m \geq 1 \) let \( \lambda_m \in x \) have normal form \( \lambda_m = \theta_{m,1} \cdots \theta_{m,m} \). By Lemma 3.4, \( \theta_{m,i} = \theta_{n,i} \) for all \( i < \min \{ m, n \} \). Therefore \( \theta_i := \theta_{m,i} \) for any \( m > i \) is well defined, and \( \theta_1 \cdots \theta_m \in x \) for all \( m \). It follows from Lemma 3.4 that \( x = \bigcup_{m=1}^\infty [\theta_1 \cdots \theta_m] \).

Next suppose that \( M < \infty \) (still assuming that \( x \) is infinite). Let \( \theta_1 \cdots \theta_M \in x \) be in normal form. If \( \theta_M \in \Lambda_1 \) then there is a unique infinite element \( y \in \Lambda_1^\ast \) such that \( x = \bigcup \{ [\theta_1 \cdots \theta_{M-1} \eta] : \eta \in y \} \). Similarly, if \( \theta_M \in \Lambda_2 \) then there is a unique infinite element \( y \in \Lambda_2^\infty \) such that \( x = \bigcup \{ [\theta_1 \cdots \theta_{M-1} \eta] : \eta \in y \} \). \( \square \)

**Notation 3.10.** In case (1) of Theorem 3.9 we say that \( x = \theta_1 \theta_2 \cdots \) is in normal form. In cases (2) and (3) we say that \( x = \theta_1 \cdots \theta_{M-1} y \) is in normal form.

We may represent infinite directed hereditary subsets of \( \Lambda \) as infinite words as follows. If \( x \) is as in Theorem 3.9(1) for the sequence \( \theta_1, \theta_2, \ldots \), write \( \theta_i = \mu_{i,1} \cdots \mu_{i,n_i} \) as a sequence of edges in \( \Lambda_1 \) or \( \Lambda_2 \). Then we associate \( x \) with the infinite word

\[
\mu_{1,1} \cdots \mu_{1,n_1} \mu_{2,1} \cdots \mu_{2,n_2} \mu_{3,1} \cdots \mu_{3,n_3} \cdots
\]

This representation need not be unique, since if \( \theta_i \in \Lambda_1 \) there may be various expressions for \( \theta_i \) as a sequence of edges. However, if the infinite directed hereditary set \( x \) is represented by two infinite words, \( x_1 x_2 \cdots \) and \( x'_1 x'_2 \cdots \), then \( x_i \in \Lambda_1 \) if and only if \( x'_i \in \Lambda_1 \), and if, say, \( x_i, x'_i \in \Lambda_2 \) then \( x_i = x'_i \). It is easily seen that the same result holds if \( x \) is as in Theorem 3.9(2) or (3).

**Theorem 3.11.** Let \( x \in \Lambda^* \). Then \( x \in \Lambda^{**} \) if and only if \( x \) is infinite, and \( x \) has the form either of Theorem 3.9(1) or (3), or of Theorem 3.9(2) with \( y = \alpha^\infty \beta^\infty \).

**Proof.** Let \( x \in \Lambda^{**} \). It is clear that \( x \) must be infinite. Suppose that \( x \) has the form of Theorem 3.9(2). Let \( x = \theta_1 \cdots \theta_{M-1} y, y \in \Lambda_1^\infty \), as in Theorem 3.9(2). Then \( x \subseteq \theta_1 \cdots \theta_{m-1} \alpha^\infty \beta^\infty \). Since \( x \) is maximal, \( x = \theta_1 \cdots \theta_{m-1} \alpha^\infty \beta^\infty \).

Conversely, let \( x \in \Lambda^* \) be infinite. Suppose first that \( x \) is as in Theorem 3.9(1). Let \( \theta_1, \theta_2, \ldots \) be as in Theorem 3.9(1). We will show that \( x \) is maximal. Let
Let \( x' \in \Lambda^* \) with \( x \subseteq x' \). Let \( \mu \in x' \), and write \( \mu = \phi_1 \cdots \phi_n \) in normal form. Let \( \lambda = \theta_1 \cdots \theta_{n+1} \in x \subseteq x' \). Since \( \lambda, \mu \in x' \) and \( x' \) is directed, \( \lambda \) and \( \mu \) have a common extension. By Lemma 3.4 (1) we have that \( \mu \in [\lambda] \subseteq x \), since \( x \) is hereditary. Thus \( x' \subseteq x \). Now suppose that \( x \) is as in Theorem 3.9(2) with \( y = \alpha \gamma^\infty \beta^\infty \). Write \( x = \theta_1 \cdots \theta_{m-1} \alpha \gamma^\infty \beta^\infty \) as in Theorem 3.9(2). Again let \( x' \in \Lambda^* \) with \( x \subseteq x' \). Let \( \mu \in x' \). For each \( k \geq 0 \), let \( \lambda_k = \theta_1 \cdots \theta_{m-1} \alpha_k \beta^k \in x \subseteq x' \). Then \( \lambda_k \) and \( \mu \) have a common extension. Write \( \mu = \phi_1 \cdots \phi_n \) in normal form. If \( n < m \) then Lemma 3.4 (1) implies that \( \mu \in [\lambda_k] \subseteq x \). If \( n = m \) then \( \phi_m \in \Lambda_1 \) and hence \( \phi_m \in [\alpha_k \beta^k] \) for some \( k \). Then again \( \mu \in [\lambda_k] \subseteq x \). If \( n > m \) then by Lemma 3.4 (1), \( \phi_n \) extends \( \alpha_k \beta^k \) for all \( k \), which is impossible. Therefore \( x' \subseteq x \). Finally, suppose that \( x \) is as in Theorem 3.9(3). Write \( x = \theta_1 \cdots \theta_{m-1} y \), where \( y = \gamma_1(i_1) \gamma_2(i_2) \cdots \in v_t \Lambda_2^\infty \). Let \( x' \in \Lambda^* \) with \( x \subseteq x' \). Let \( \mu \in x' \). Write \( \mu = \phi_1 \cdots \phi_n \) in normal form. Choose \( p \geq 1 \) such that \( |\mu| < |\theta_1 \cdots \theta_{m-1} + p - \ell + 1 \). Then \( n \leq m \), and by Lemma 3.4 \( \mu \in [\theta_1 \cdots \theta_{m-1} \gamma(i_1) \cdots \gamma(i_p)] \subseteq x \). Therefore \( x' \subseteq x \). \( \square \)

**Proposition 3.12.** The closure of \( \Lambda^{**} \) in \( \Lambda^* \) equals the subset of infinite elements of \( \Lambda^* \).

**Proof.** Let \( x \in \Lambda^* \) be infinite. We may as well assume that \( x \notin \Lambda^{**} \). By Theorem 3.11 \( x = \theta_1 \cdots \theta_{m-1} y \), where \( y \in \Lambda_1^\infty \) and \( y \neq \alpha \gamma^\infty \beta^\infty \). By Lemma 3.7 we let, say, \( y = \alpha \gamma^\infty \beta^\infty \) for some \( 0 \leq \ell < \infty \). Let \( Z(\lambda) \setminus \bigcup_{i=1}^t Z(\mu_i) \subseteq B^x \) be a neighborhood of \( x \). Then \( \lambda \in x \) and for all \( i \), \( \mu_i \notin x \). Let \( \lambda' = \theta_1 \cdots \theta_{m-1} \alpha \gamma^\infty \beta^\infty \), where \( \ell \) is so large that \( |\lambda'| > |\mu_i| \) for all \( 1 \leq i \leq t \). Further, by increasing \( \ell \) if necessary, we choose \( \ell \) so that \( v_h = s(\lambda') \) gives \( k_{h+1} > 0 \). Then \( x \in Z(\lambda') \setminus \bigcup_{i=1}^t Z(\mu_i) \subseteq Z(\lambda) \setminus \bigcup_{i=1}^t Z(\mu_i) \). Fix \( i, 1 \leq i \leq t \). Since \( \mu_i \notin x \) there are two possibilities: either the normal form of \( \mu_i \) differs from that of \( \lambda \) in a term before the \( m \)th term, or \( \mu_i \) agrees with \( \lambda \) in the first \( m \) terms, and has \( m \)th term equal to \( \alpha \gamma^\infty \beta^\infty \) for some \( \delta \in \Lambda_1 \). Put \( \lambda'' = \lambda' \gamma(i-1) \). Then in all cases, \( \mu_i \) and \( \lambda'' \) have no common extension. Let \( x' = \lambda'' \alpha \gamma^\infty \beta^\infty \). Then \( x' \in \Lambda^{**} \cap (Z(\lambda') \setminus \bigcup_{i=1}^t Z(\mu_i)) \). Therefore \( x \in \Lambda^{**} \).

Now let \( x \in \Lambda^* \) be finite, so there is \( \lambda \in \Lambda \) with \( x = [\lambda] \). Let \( n = |\lambda| \). Then \( Z(\lambda) \setminus (Z(\lambda \alpha_n+1) \cup Z(\lambda \beta_n+1) \cup \bigcup_{i=1}^{k_{n+1}} Z(\lambda \gamma_n(i)) = \{x\} \). Therefore \( x \notin \Lambda^{**} \). \( \square \)

We will simplify the situation further by restricting to a transversal in \( \partial \Lambda \) (in the sense of [17] Example 2.7).

**Lemma 3.13.** Let \( X = v_1 \partial \Lambda \). Then \( X \) is a compact open transversal in \( G(\Lambda)|_{\partial \Lambda} \).

**Proof.** \( X \) is the intersection of \( \partial \Lambda \) with the compact open subset \( v_1 \Lambda^* \), hence \( X \) is compact open. To see that \( X \) is a transversal, let \( x \in \Lambda^* \), and let \( \mu \in v_1 \Lambda^r(x) \) be arbitrary. Then \( \mu x \in X \). Since \( G(\Lambda)|_{\partial \Lambda} \) is étale, \( r \) and \( s \) are open maps. Since \( \partial \Lambda \cdot G(\Lambda) \cdot X \) is open in \( G(\Lambda)|_{\partial \Lambda} \), the restrictions of \( r \) and \( s \) to it are open as well. This verifies the hypotheses of [17] Example 2.7. \( \square \)

It now follows from [17] Theorem 2.8] that \( C^*(\Lambda) \) is Morita equivalent to \( C^*(G(\Lambda)|_X) \).

**Definition 3.14.** We will write \( G := G(\Lambda)|_X \), the restriction of \( G(\Lambda) \) to the transversal \( X \).

For the rest of this paper (except for section 3.1) we will study \( G \) and \( C^*(G) \). Thus \( G = \{[\mu, v, x] \in G(\Lambda) : r(\mu) = r(v) = v_1, x \in s(\mu) \partial \Lambda \} \). Note that for \( [\mu, v, x] \in G \) it is necessarily the case that \( |\mu| = |v| \).
Definition 3.15. For $i \geq 1$ let $G_i$ be the subgroupoid of $G$ generated by elements of the form $[\mu, \nu, x] \in G$ such that $|\mu| = |\nu| \leq i$.

Definition 3.16. For $p \geq 1$, let $X_p = v_p \partial \Lambda$ (thus $X_1 = X$).

The following simple lemma will be useful several times.

Lemma 3.17. For $p \geq 1$ and $x \in X_{p+1}$,
\[
[\beta^p, \alpha^p, x] = \prod_{j=0}^{p-1} (\beta, \alpha, \alpha^j \beta^{p-j-1} x).
\]

Proof. The proof is by induction on $p$. This is clearly true when $p = 1$. Let $p \geq 1$ and suppose the formula is true for $p$. Then
\[
\prod_{j=0}^{p} (\beta, \alpha, \alpha^j \beta^{p-j} x) = \prod_{j=0}^{p-1} (\beta, \alpha, \alpha^j \beta^{p-j} \beta x)[\beta, \alpha, \alpha^p x]
\]
\[
= [\beta^p, \alpha^p, \beta x][\beta, \alpha, \alpha^p x], \text{ by the inductive hypothesis,}
\]
\[
= [\beta^{p+1}, \alpha^p \beta, x][\alpha^p \beta, \alpha^{p+1}, x]
\]
\[
= [\beta^{p+1}, \alpha^{p+1}, x],
\]
which is the formula for the case $p + 1$. \hfill $\Box$

Theorem 3.18. $G_i = \{[\mu \theta, \mu \theta', x] : |\mu| = |\mu'| \leq i$ and $\theta, \theta' \in \Lambda_1\}$.

Before beginning the proof we note the following. Let $[\mu \theta, \mu \theta', x]$ be an element of the righthand side in Theorem 3.18. We may write $\theta = \phi \eta$, $\theta' = \phi' \eta$ where $d(\phi) \perp d(\phi')$ in $\mathbb{N}^2$. Then $[\mu \theta, \mu \theta', x] = [\mu \phi, \mu \phi', \eta x]$. Thus when describing the elements of $G_i$ we may assume that $d(\theta) \perp d(\theta')$. (This means that one of $\theta, \theta'$ equals $\alpha^\ell$ and the other equals $\beta^\ell$.)

Proof. We begin with the proof of the containment “\(\supseteq\)”. Let $|\mu| = |\mu'| = m \leq i$ and let $\ell \geq 0$. Then for $x \in X_{|\mu|+\ell+1}$,
\[
[\mu \beta^\ell, \mu \alpha^\ell, x] = [\mu, \beta^m, \beta^\ell x][\beta^m + \ell, \alpha^{m+\ell}, x][\alpha^m, \mu, \alpha^\ell x]
\]
\[
= [\mu, \beta^m, \beta^\ell x](\prod_{j=0}^{m+\ell-1} (\beta, \alpha, \alpha^j \beta^{m+\ell-j-1} x)) [\alpha^m, \mu, \alpha^\ell x]
\]
(by Lemma 3.17), which is a product of generators of $G_i$. The case where the roles of $\alpha$ and $\beta$ are switched is obtained by taking inverses.

For the reverse containment, let us denote by $S$ the set on the righthand side of the statement. It is clear that all generators of $G_i$ belong to $S$. The proof will be finished if we show that the product of an element of $S$ and a generator is again an element of $S$. Note that if $[\mu, \mu', x]$ is a generator, and $m = |\mu| = |\mu'| < i$, then we may write $x = \epsilon y$ where $|\epsilon| = i - m$. Then $[\mu, \mu', x] = [\mu \epsilon, \mu \epsilon', y]$, and $|\mu \epsilon| = |\mu' \epsilon| = i$. Thus we may assume that the first two coordinates of a generator of $G_i$ have length $i$. Now let $|\mu| = |\mu'| = i$, let $\ell \geq 0$, and let $|\nu| = |\nu'| = i$. Then assuming that $x$,
and \( \phi \).

If \( \eta = \eta \), then it follows that \( \mu \beta^t \in \eta \eta \), and then

\[
\mu \alpha^t, \mu \beta^t, x]\in \eta, \nu', \eta, z) = [\mu \alpha^t, \mu \beta^t, z]\in \eta, \nu', \eta, z = [\mu \alpha^t, \eta', \nu, \eta, z].
\]

Write \( \mu' = \theta_1 \cdots \theta_p \) and \( \nu = \phi_1 \cdots \phi_q \) in normal form. We claim that \( p = q \). For suppose, e.g., that \( p < q \). We know that \( \mu' \beta^t \in \eta \eta \). Then by Lemma 3.4 we have that \( \mu' = \phi_1 \cdots \phi_{p-1} \) and that \( \phi_p \) extends \( \theta_p \). Since \( \mu' = \nu \) we have that \( |\theta_p| = |\phi_p\cdots \phi_q| \). But \( |\theta_p| \leq |\phi_p\cdots \phi_q| \), a contradiction. Thus \( p = q \). Then \( \theta_p \) and \( \phi_p \) have a common extension. If \( \theta_p, \phi_p \in \Lambda_2 \) then \( \theta_p = \phi_p \), hence \( \mu' = \nu \). In this case it follows that \( \eta = \beta^t \xi \), and hence that \( [\mu \alpha^t, \nu', \eta, z] = [\mu \alpha^t, \nu', \beta^t, \xi, z] \in \Omega \).

On the other hand, if \( \theta_p, \phi_p \in \Lambda_1 \), we know only that \( |\theta_p| = |\phi_p| \) and \( \theta_p \beta^t \xi = \phi_p \eta \). Write \( \xi = \xi'', \eta = \eta'' \), where \( \xi'', \eta'' \in \Lambda_1 \), and \( \xi'', \eta'' \) have normal forms with first factor in \( \Lambda_2 \) (if nontrivial). Then \( \theta_p \beta^t \xi'' \) and \( \phi_p \eta'' \) are the first factors in the normal forms of \( \theta_p \beta^t \xi \) and of \( \phi_p \eta \).

Then uniqueness of normal form, these first factors are equal, and hence also \( \xi'' = \eta'' \). Thus again we find that \( [\mu \alpha^t, \nu', \eta, z] = [\mu \alpha^t, \nu', \eta', \xi'' \eta', \xi'' \eta'' \xi'' z] \in \Omega \). A similar argument treats the case where an element of \( \Omega \) is multiplied on the left by a generator of \( G_1 \).

\[ \square \]

4. **K-theory of \( C^*(G_i) \)**

Let \( i \) be fixed throughout this section. We now begin the analysis of \( C^*(G_i) \). First we give a decomposition of \( X = G_i^{(0)} \).

**Definition 4.1.** Let

\[
U_i = \{ x = x_1 x_2 \cdots \in X : x_\ell \in \Lambda_2 \text{ for some } \ell > i \},
\]

\[
F_i = X \setminus U_i,
\]

and for \( \ell > i \), let

\[
E_\ell = \{ x \in X : x_\ell \in \Lambda_2, \text{ and } x_j \in \Lambda_1 \text{ for } i < j < \ell \},
\]

\[
\Omega_\ell = \{ \lambda \in \nu_1 \Lambda_\nu_\ell : \lambda_j \in \Lambda_1 \text{ for } i < j < \ell \}.
\]

(Note that \( E_\ell = \emptyset \) if \( k_\ell = 0 \).)

**Lemma 4.2.** Let \( \ell > i \) and let \( \mu, \nu \in \Omega_\ell \). Then \( \chi_{Z(\mu)} \) and \( \chi_{Z(\nu)} \) are equivalent projections, and \( \chi_{Z(\mu)Z(\nu)} \) and \( \chi_{Z(\nu)Z(\mu)} \) are equivalent projections, in \( C^*(G_i) \).

**Proof.** Write \( \mu = \mu_1 \mu_2 \) and \( \nu = \nu_1 \nu_2 \) where \( |\mu_1| = |\nu_1| = i \) and \( \mu_2, \nu_2 \in \Lambda_1 \). Without loss of generality we may suppose that \( \mu_2 = \theta \alpha^p \) and \( \nu_2 = \theta \beta^p \), with \( \theta \in \Lambda_1 \).

Now

\[
\nu, Z(v_\ell) = [\nu_1 \theta \beta^p, \alpha^i \theta \beta^p, Z(v_\ell)] \cdot [\beta^p, \alpha^p, Z(\alpha^i \theta)] \cdot [\alpha^i \theta \alpha^p, \mu_1 \theta \alpha^p, Z(v_\ell)]
\]

\[= [\nu_1 \theta \beta^p, \alpha^i \theta \beta^p, Z(v_\ell)] \cdot \prod_{j=0}^{p-1} [\beta, \alpha, Z(\beta^p - j \alpha^i \theta)] \cdot [\alpha^i \theta \alpha^p, \mu_1 \theta \alpha^p, Z(v_\ell)],
\]

by Lemma 3.17. Therefore \( u = \chi_{[\nu_1 \mu, Z(v_\ell)]} \in C^*(G_i) \), and \( u^* u = \chi_{[\nu_1, \nu_2, Z(\mu)]}, \) \( uu^* = \chi_{[\nu_1, \nu_2, Z(\nu)]} \). Now we may let \( v = \chi_{[\nu_1, \mu, Z(v_\ell)]} \), and we have that \( v^* v = \chi_{Z(\mu)Z(\mu)} \) and \( vv^* = \chi_{Z(\nu)Z(\nu)} \). \[ \square \]
Lemma 4.3. For each $\ell > i$, $E_\ell$ is an open $G_i$-invariant subset of $X$. Each point of $E_\ell$ has trivial isotropy, and has finite orbit equivalent to $\Omega_\ell$.

Proof. $E_\ell = \bigcup_{\lambda \in \Omega_\ell} \bigcup_{r=1}^{k_\ell} Z(\lambda \gamma_\ell^{(r)})$. Therefore $E_\ell$ is open. Now we show that $E_\ell$ is $G_i$-invariant. Let $x \in E_\ell$. Then $x = \lambda \phi \gamma_\ell^{(r)} x'$, where $|\lambda| \leq i$ and $\phi \in \Lambda_1$. Let $g \in G_i x$. By Theorem 3.18, $g = [\mu \theta, \mu' \theta', y]$, where $|\mu| = |\mu'| \leq i$, $\theta, \theta' \in \Lambda_1$, and $\mu' \theta' y = x$. Then $\mu' \theta' y = \lambda \phi \gamma_\ell^{(r)} x'$. It follows that $|\mu' \theta'| < \ell$, and that $y = \theta'' \gamma_\ell^{(r)} x'$, where $\theta'' \in \Lambda_1$. Then $r(g) = \mu \theta y = \mu \theta'' \gamma_\ell^{(r)} x' \in E_\ell$.

With $x$ and $g$ as above we see that $r(g) \in \Omega_\ell \gamma_\ell^{(r)} x'$, so the orbit of $x$ under $G_i$ is $\Omega_\ell \gamma_\ell^{(r)} x'$, which is equivalent to $\Omega_\ell$. If we assume that $r(g) = x$ (so that $g$ is in the isotropy subgroup of $G_i$ at $x$), then $\mu \theta'' \gamma_\ell^{(r)} x' = \mu \theta y = \mu' \theta' y = \mu' \theta'' \gamma_\ell^{(r)} x'$, and hence $\mu \theta = \mu' \theta'$. Therefore $g = [v_1, v_1, x] \in G(0)$.

Proposition 4.4. $U_i$ is an open $G_i$-invariant subset of $X$, and

$$C^*(G_i|U_i) \cong \bigoplus_{\ell > i} (M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1}))^{k_\ell}.$$ 

(Notice that if $k_\ell = 0$ then the $\ell$th summand is not present.)

Proof. Note that $U_i = \sqcup_{\ell > i} E_\ell$, and hence $U_i$ is open and $G_i$-invariant. Moreover, $C^*(G_i|U_i) = \bigoplus_{\ell > i} C^*(G_i|E_\ell)$. But $E_\ell = \bigcup_{r=1}^{k_\ell} \Omega_\ell \gamma_\ell^{(r)} X_{\ell+1}$, so $G_i|E_\ell = \bigcup_{r=1}^{k_\ell} (\Omega \times \Omega) \times \gamma_\ell^{(r)} X_{\ell+1}$. It follows that $C^*(G_i|E_\ell) \cong \bigoplus_{r=1}^{k_\ell} M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1}) = (M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1}))^{k_\ell}$. The isomorphism in the proposition now follows.

Remark 4.5. It follows from Lemma 4.3 that $G_i|U_i$ is an AF groupoid (see [22, Definition III.1.1]), and hence is amenable.

Remark 4.6. The isomorphism $C^*(G_i|E_\ell) \cong (M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1}))^{k_\ell}$ is given by

$$\chi_{[\mu, \nu, \gamma_\ell^{(r)}]} : (0, \ldots, e_{\mu, \nu} \otimes x_F, \ldots, 0),$$

(in the $r$th coordinate) for $F \subseteq X_{\ell+1}$ a compact open subset.

Notation 4.7. For $f \in C^*(G_i|U_i)$ we write $f = (f_{i+1}, f_{i+2}, \ldots)$ with $f_\ell = (f_{\ell,1}, f_{\ell,2}, \ldots)$, where $f_{\ell, r} \in M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1}) \cong C(X_{\ell+1}, M_{\Omega_\ell \times \Omega_\ell})$.

Proposition 4.8. $K_1(C^*(G_i|U_i)) = 0$, and $K_0(C^*(G_i|U_i)) \cong \bigoplus_{\ell > i} C(X_{\ell+1}, \mathbb{Z})^{k_\ell}$, with generators $[\chi_{[\ell-1, \ell-1, \gamma_\ell^{(r)}]}]_0$ for $F \subseteq X_{\ell+1}$ a compact open subset.

Proof. Since $X_{\ell+1}$ is totally disconnected, $C(X_{\ell+1})$ is AF. It follows that $C^*(G_i|U_i)$ is AF, and hence that $K_1(C^*(G_i|U_i)) = 0$. Moreover, $K_0(M_{\Omega_\ell \times \Omega_\ell} \otimes C(X_{\ell+1})) \cong C(X_{\ell+1}, \mathbb{Z})$. The description of $K_0(C^*(G_i|U_i))$ follows. The description of generators of the group follows from Remark 1.6.

The open invariant set $U_i$ determines an ideal in $C^*(G_i)$, and its complement $F_i$ is a closed invariant set determining the quotient $C^*$-algebra. There is thus a short exact sequence

$$0 \longrightarrow C^*(G_i|U_i) \longrightarrow C^*(G_i) \longrightarrow C^*(G_i|F_i) \longrightarrow 0.$$
We now study $C^*(G_i|_{F_i})$. Note that $F_i = \{ \lambda \alpha^m \beta^n : |\lambda| \leq i \text{ and } m + n = \infty \}$. Let
\[ F_i^\infty = \{ \lambda \alpha^\infty \beta^\infty : |\lambda| \leq i \} \]
\[ F_i^0 = F_i \setminus F_i^\infty. \]

Then $F_i^\infty$ is a finite invariant set, and hence is also closed. Therefore $F_i^0$ is a relatively open invariant subset of $F_i$. We have a further exact sequence
\[
0 \rightarrow C^*(G_i|_{F_i^0}) \xrightarrow{\iota} C^*(G_i|_{F_i}) \xrightarrow{\pi} C^*(G_i|_{F_i^\infty}) \rightarrow 0.
\]

We first analyze $G_i|_{F_i^\infty}$. The following definition will be convenient in several places.

**Definition 4.9.** Let $\Phi_i = \{ \lambda \in v_1 \Lambda : |\lambda| \leq i, \lambda|_\Lambda \in \Lambda_2 \}$.

**Lemma 4.10.** $F_i^\infty$ is a single orbit for $G_i$. The isotropy at points of $F_i^\infty$ is infinite cyclic. Then the map $\lambda \mapsto \lambda \alpha^\infty \beta^\infty$ is a bijection of $\Phi_i$ onto $F_i^\infty$.

**Proof.** Let $x, y \in F_i^\infty$. Then $x = \lambda \alpha^\infty \beta^\infty$, $y = \mu \alpha^\infty \beta^\infty$, where as in the proof of Theorem 3.18 we may assume that $|\lambda| = |\mu| = i$. Then $g = [\lambda, \mu, \alpha^\infty \beta^\infty] \in G_i$, and $s(g) = \mu \alpha^\infty \beta^\infty = y, r(g) = \lambda \alpha^\infty \beta^\infty = x$. This proves that $G_i|_{F_i^\infty}$ has a single orbit. Now let $g$ be in the isotropy group at, say, $v_1 \alpha^\infty \beta^\infty$. Then $g = [\lambda, \mu, \alpha^\infty \beta^\infty]$, with $|\lambda| = |\mu| \leq i$, and $\mu \alpha^\infty \beta^\infty = v_1 \alpha^\infty \beta^\infty = \lambda \alpha^\infty \beta^\infty$. Then $\lambda, \mu \in \Lambda_1$. We may assume without loss of generality that $\lambda = \alpha^p$ and $\mu = \beta^p$ for some $p \geq 0$. Then by Lemma 3.17 we have that $g = [\alpha^p, \beta^p, \alpha^\infty \beta^\infty] = [\alpha, \beta, \alpha^\infty \beta^\infty]^p$. The last statement of the lemma is clear. \[ \square \]

**Remark 4.11.** It is easy to verify that the range and source maps on $G_i|_{F_i^\infty}$ are open maps. It then follows from Lemma 4.10 and Corollary 9.78 that $G_i|_{F_i^\infty}$ is amenable.

**Corollary 4.12.** $C^*(G_i|_{F_i^\infty}) \cong M_{\Phi_i \times \Phi_i} \otimes C(\mathbb{T})$. The isomorphism is given by
\[
[\alpha, \beta, \alpha^\infty \beta^\infty] \mapsto e_{v_1, v_1} \otimes z
\]
\[
[\lambda, \alpha^{|\lambda|}, \alpha^\infty \beta^\infty] \mapsto e_{\lambda, v_1} \otimes 1.
\]

**Proof.** The isomorphism follows from, e.g., [17, Theorem 3.1]. The explicit version is clear from the proof of Lemma 4.10. \[ \square \]

Now we study $F_i^0 = \{ \lambda \alpha^m \beta^n : |\lambda| \leq i, \max\{m, n\} = \infty, \min\{m, n\} < \infty \}$.

**Lemma 4.13.** $G_i|_{F_i^0}$ is principal with two orbits:
\[ F_i^{0,1} = \{ \lambda \alpha^\infty \beta^n : |\lambda| \leq i, n \geq 0 \} \]
\[ F_i^{0,2} = \{ \lambda \alpha^m \beta^\infty : |\lambda| \leq i, m \geq 0 \}. \]

**Proof.** We first show that $G_i|_{F_i^0}$ is principal. Let $g = [\lambda, \mu, y] \in G_i|_{F_i^0}$ with $\lambda y = r(g) = s(g) = \mu y$. Write $\lambda = \lambda' \lambda''$ and $\mu = \mu' \mu''$, where $\lambda', \mu'$ have source-most edge in $\Lambda_2$ and $\lambda'', \mu'' \in \Lambda_1$. We may assume that, say, $\lambda'' = \alpha^p, \mu'' = \beta^q$, and $y = \alpha^\infty \beta^n$. Then our assumption implies that $\lambda' \alpha^\infty \beta^n = \mu' \alpha^\infty \beta^{q+n}$. It follows that $q = 0$ and $\lambda' = \mu'$. Then since $|\lambda| = |\mu|$ we have that $\lambda' \alpha^p = \mu'$. Since the source-most edge of $\mu'$ is in $\Lambda_2, p = 0$. Therefore $\lambda' = \mu'$, and hence $\lambda = \mu$. Therefore $g \in G^{(0)}$. \[ \square \]
Now consider $\lambda \in v_1\Lambda$ with $|\lambda| < i$, and $n \geq 0$. By Lemma 3.17 we have
\[
[\lambda \beta^n, \alpha^{[\lambda]+n}, \alpha^\infty] = [\lambda, \alpha^{[\lambda]}, \alpha^\infty \beta^n][\beta^n, \alpha^n, \alpha^\infty]
\]
\[
= [\lambda, \alpha^{[\lambda]}, \alpha^\infty \beta^n] \prod_{j=0}^{n-1} [\beta, \alpha, \alpha^\infty \beta^{n-j-1}]
\]
has source $\alpha^\infty$, and has range $\lambda \alpha^\infty \beta^n$, a typical element of $F_i^{0,1}$. Thus $F_i^{0,1}$ is contained in the orbit of $\alpha^\infty$. A similar argument shows that $F_i^{0,2}$ is contained in the orbit of $\beta^\infty$. To finish the proof we observe that $\alpha^\infty$ and $\beta^\infty$ are not in the same orbit. □

Remark 4.14. Note that the sets $F_i^{0,1}$ and $F_i^{0,2}$ of Lemma 4.13 are disjoint, open and invariant. Therefore $G_i|_{F_i^0} = G_i|_{F_i^{0,1}} \cup G_i|_{F_i^{0,2}}$. Again, it is easy to see that the range and source maps in $G_i|_{F_i^0}$ are open. Then [33, Corollary 9.78] implies that $G_i|_{F_i^0}$ is amenable.

Proposition 4.15. $G_i$ is amenable.

Proof. From Remarks 4.11 and 4.14 and [33, Proposition 9.83], it follows that $G_i|_{F_i^0}$ is amenable. Then it follows from Remark 4.13 and the same Proposition that $G_i$ is amenable. □

Corollary 4.16. $C^*(G_i|_{F_i^0}) \cong M_{F_i \times F_i} \otimes (\mathcal{K} \oplus \mathcal{K})$. The isomorphism is given by
\[
[\beta^n, \alpha^n, \alpha^\infty] \mapsto e_{v_1,v_1} \otimes (e_{n,0} \oplus 0)
\]
\[
[\lambda, \alpha^{[\lambda]}, \alpha^\infty] \mapsto e_{\lambda,v_1} \otimes (e_{0,0} \oplus 0)
\]
\[
[\alpha^n, \beta^n, \beta^\infty] \mapsto e_{v_1,v_1} \otimes (0 \oplus e_{n,0})
\]
\[
[\lambda, \beta^{[\lambda]}, \beta^\infty] \mapsto e_{\lambda,v_1} \otimes (0 \oplus e_{0,0}).
\]

Proof. The isomorphism follows from, e.g., [17, Theorem 3.1]. The explicit version is clear from the proof of Lemma 4.13. □

The exact sequence (4.12) becomes
\[
0 \to M_{F_i \times F_i} \otimes (\mathcal{K} \oplus \mathcal{K}) \xrightarrow{i} C^*(G_i|_{F_i}) \xrightarrow{\pi} M_{F_i \times F_i} \otimes C(\mathbb{T}) \to 0.
\]

The corresponding long exact sequence in $K$-theory is
\[
0 \xrightarrow{l_{*1}} K_1(C^*(G_i|_{F_i})) \xrightarrow{\pi_{*1}} \mathbb{Z} \xrightarrow{\partial_1} K_0(C^*(G_i|_{F_i})) \xrightarrow{l_{*0}} \mathbb{Z}^2
\]

In Corollaries 4.12 and 4.16 we have identified elements that serve as generators of the $K$-groups at the corners of this exact sequence:
\[
[\chi_{[\alpha,\beta,\alpha^\infty \beta^\infty]}]_1 \in K_1(C^*(G_i|_{F_i}))
\]
\[
[\chi_{[v_1,v_1,\alpha^\infty \beta^\infty]}]_0 \in K_0(C^*(G_i|_{F_i}))
\]
\[
[\chi_{[v_1,v_1,\alpha^\infty]}]_0, [\chi_{[v_1,v_1,\beta^\infty]}]_0 \in K_0(C^*(G_i|_{F_i^0})).
\]

We will compute the index map $\partial_1$. 
Lemma 4.17. The map $\partial_1$ in the above long exact sequence is given by $\partial_1(1) = (1, -1)$.

Proof. The unitary
$$u = x_{[\alpha, \beta, \alpha^\infty \beta^\infty]} + x_{[v_1, v_1, F_i] \setminus \{\alpha^\infty \beta^\infty\}} \in C^\ast(G_i|F_i)$$
lifts to the following partial isometry in $C^\ast(G_i|F_i)$:
$$w = x_{[\alpha, \beta, \Lambda^\infty \cap X_2]} + x_{[v_1, v_1, F_i \setminus v_1 \Lambda^\infty]} \in C^\ast(G_i|F_i).$$

Then
$$w^* w = x_{[\beta, \beta, \Lambda^\infty \cap X_2]} + x_{[v_1, v_1, F_i \setminus v_1 \Lambda^\infty]}$$
$$\quad = x_{[v_1, v_1, F_i] \setminus v_1 \Lambda^\infty} + x_{[v_1, v_1, F_i \setminus v_1 \Lambda^\infty]}$$
$$\quad = x_{[v_1, v_1, F_i \setminus \{\alpha^\infty\}]} \cap x_{[v_1, v_1, F_i \setminus \{\alpha^\infty\}]}.$$

and similarly,
$$w w^* = x_{[v_1, v_1, F_i \setminus \{\beta^\infty\}]}.$$

Therefore
$$\partial_1([u]_1) = [1 - w^* w]_0 - [1 - w w^*]_0$$
$$\quad = \left[ x_{[v_1, v_1, \alpha^\infty]} \right]_0 - \left[ x_{[v_1, v_1, \beta^\infty]} \right]_0$$
$$\quad \equiv (1, -1) \in \mathbb{Z}^2. \quad \Box$$

Proposition 4.18. $K_1(C^\ast(G_i|F_i)) = 0$ and $K_0(C^\ast(G_i|F_i)) \cong \mathbb{Z}^2$, with generators
$$\left[ x_{[v_1, v_1, \alpha^\infty]} \right]_0 \text{ and } \left[ x_{[v_1, v_1, \alpha^\infty \beta^\infty]} \right]_0.$$

Proof. Since $\partial_1$ is injective we know that $\pi_* \alpha = 0$, and hence that $K_1(C^\ast(G_i|F_i)) = 0$. We may choose $(1, 0)$ and $(1, -1)$ as basis for $\mathbb{Z}^2$. Then since $\partial_1(\mathbb{Z}) = \mathbb{Z}(1, -1)$ we obtain a short exact sequence
$$0 \to \mathbb{Z}(1, 0) \to K_0(C^\ast(G_i|F_i)) \to \mathbb{Z} \to 0.$$}

Therefore $K_0(C^\ast(G_i|F_i)) \cong \mathbb{Z}^2$. As generators we use $\iota_* (1, 0) = \left[ x_{[v_1, v_1, \alpha^\infty]} \right]_0$, and a lift of
$$\left[ x_{[v_1, v_1, \alpha^\infty \beta^\infty]} \right]_0.$$ In choosing the lift, we may choose any compact open subset of $F_i$ whose intersection with $F_i^\infty$ equals $\{\alpha^\infty \beta^\infty\}$. The set $Z(\beta^{i+1}) \cap F_i$ satisfies this requirement, so our second generator can be chosen to be
$$\left[ x_{[v_1, v_1, Z(\beta^{i+1}) \cap F_i]} \right]_0. \quad \Box$$

Now we describe the $K$-theory of $C^\ast(G_i)$.

Theorem 4.19. $K_1(C^\ast(G_i)) = 0$, and $K_0(C^\ast(G_i)) \cong \bigoplus_{k \geq 1} C(X_{k+1}, Z)^{k\ell} \oplus \mathbb{Z}^2$. The generators of the first term are as in Proposition 4.18, and the generators of the second term may be chosen to be
$$\left[ x_{[v_1, v_1, Z(\alpha^i \setminus \alpha^\ell \beta^\ell)]} \right]_0 \text{ and } \left[ x_{[v_1, v_1, Z(\beta^{i+1})]} \right]_0.$$

Proof. The long exact sequence in $K$-theory associated to the sequence (4.1) reduces to
$$0 \to K_0(C^\ast(G_i|F_i)) \to K_0(C^\ast(G_i)) \to K_0(C^\ast(G_i|F_i)) \to 0.$$ Since $K_0(C^\ast(G_i|F_i)) \cong \mathbb{Z}^2$ is free abelian, the central group is isomorphic to the direct sum of the other two groups. For generators we may use generators of the subgroup, together with lifts of the generators of the quotient group. Thus we must choose lifts for the generators given in Proposition 4.18. We note that $(Z(v_1 \alpha^i) \setminus Z(v_1 \alpha^i \beta^i)) \cap
$F_i = \{v_1^{α_i}\}$. Therefore $[X[v_1,v_1,Z(α^i)\cap Z(α^j)]]_0^0$ is a lift of $[X[v_1,v_1,α^i]]_0^0$. It is clear that $[X[v_1,v_1,Z(β^{i+1})]]_0^0$ is a lift of $[X[v_1,v_1,α^{i+1}]]_0^0$. □

**Corollary 4.20.** A typical element of $K_0(C^*(G_i))$ has the form

\[
(4.3) \quad a = \sum_{\ell \geq 1} \sum_{r=1}^{k_ℓ} \sum_{j=1}^{h_ℓ} c_{ℓ,r,j} [X_{α^{ℓ-1}γ^{r}_ℓ} F_{ℓ,j}]_0^0 + m[XZ(α^i)\cap Z(α^j)]_0^0 + n[XZ(β^{i+1})]_0^0,
\]

where for each $ℓ$, $\{F_{ℓ,j} : 1 \leq j \leq h_ℓ\}$ is a partition of $X_{ℓ+1}$ by compact open subsets (and the coefficients $c_{ℓ,r,j}$, $m$, $n$ are integers).

Now we characterize the positive elements of $K_0(C^*(G_i))$.

**Theorem 4.21.** Let $a \in K_0(C^*(G_i))$ be as in equation (4.3). Then $a \geq 0$ if and only if for all $ℓ > i$, $r$ and $j$, $c_{ℓ,r,j} + m + n(ℓ - i - 1) \geq 0$.

We give some lemmas before the proof.

**Lemma 4.22.** For $p \geq ℓ \geq 1$,

(i) $Z(α^ℓ) \setminus Z(α^j β) = (\bigcup_{q=ℓ+1}^{k_q} \bigcup_{r=1}^{l_q} Z(α^{q-1}γ_q^r)) \cup (Z(α^p) \setminus Z(α^j β))$

(ii) $Z(α^ℓ) = (\bigcup_{r=1}^{k_q} Z(α^γ^{r}_ℓ)) \cup Z(α^j β) \cup (Z(α^ℓ+1) \setminus Z(α^ℓ+β))$.

**Proof.** (i) $\subseteq$: Let $x \in Z(α^ℓ) \setminus Z(α^j β)$. Write the normal form of $x$ as $θ_1θ_2⋯$. Then $θ_1 = α^{q-1}$ for some $q > ℓ$. If $q ≤ p$ then $x \in Z(α^{q-1}γ_q^r)$ for some $r$. If $q > p$ then $x \in Z(α^p) \setminus Z(α^j β)$. The reverse containment is clear.

(ii) $\subseteq$: Let $x \in Z(α^ℓ)$. Write $x = θ_1θ_2⋯$ in normal form. Then $θ_1 \in α^{ℓ+1}$. Write $θ_1 = α^{ℓ+m}β^n$ with $m$, $n ≥ 0$. If $m = n = 0$ then $θ_2 \in γ^{r}_ℓ+1α^ℓ+1$ for some $r$, and then $x \in Z(α^γ^{r}_ℓ+1)$. If $n > 0$ then $θ_1 \in Z(α^j β)$. Finally, if $n = 0$ and $m > 0$ then $x \in Z(α^ℓ) \setminus Z(α^j β)$. The reverse containment is clear. □

**Lemma 4.23.** Let $p ≥ i$. Then

\[
[XZ(α^i)]_0^0 = \sum_{ℓ=i+1}^{p} (ℓ - i) \sum_{r=1}^{k_ℓ} [XZ(α^{ℓ-1}γ^{r}_ℓ)]_0^0 + (p - i)[XZ(α^p)\cap Z(α^j β)]_0^0 + [XZ(α^p)]_0^0.
\]

**Proof.** By Lemma 4.22 we have that $[XZ(α^i β)]_0^0 = [XZ(α^{i+1})]_0^0$. Then Lemma 4.22(ii) implies that

\[
[XZ(α^i)]_0^0 = \sum_{r=1}^{k_i+1} [XZ(α^{γ^{r}_{i+1}})]_0^0 + [XZ(α^{i+1})]_0^0 + [XZ(α^i+1)\setminus Z(α^{i+1} β)]_0^0.
\]

Now we apply this fact to the middle term, repeatedly, to obtain

\[
= \sum_{ℓ=i+1}^{p} \sum_{r=1}^{k_ℓ} [XZ(α^{ℓ-1}γ^{r}_ℓ)]_0^0 + \sum_{ℓ=i+1}^{p} [XZ(α^ℓ)\setminus Z(α^ℓ β)]_0^0 + [XZ(α^p)]_0^0.
\]

Now using Lemma 4.22(ii) we obtain

\[
= \sum_{ℓ=i+1}^{p} \sum_{r=1}^{k_ℓ} [XZ(α^{ℓ-1}γ^{r}_ℓ)]_0^0 + \sum_{ℓ=i+1}^{p} \left( \sum_{q=ℓ+1}^{k_q} [XZ(α^{q-1}γ_q^r)]_0^0 + [XZ(α^p)\setminus Z(α^p β)]_0^0 \right)
\]
Making this replacement in the previous calculation, we find that

\[ [\chi Z(\alpha^p)]_0 = 0 \]

\[ \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \alpha_{\ell} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + (p - i) \left[ \chi Z(\alpha^p) \right]_0 \]

\[ + [\chi Z(\alpha^p)]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ + (p - i) \left[ \chi Z(\alpha^p) \right]_0 + [\chi Z(\alpha^p)]_0. \]

The second term can be rewritten:

\[ \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 \]

\[ \cdot \left[ \chi Z(\alpha^p) \right]_0 + [\chi Z(\alpha^p)]_0. \]

Making this replacement in the previous calculation, we find that

\[ [\chi Z(\alpha^1)]_0 = \sum_{\ell = i+1}^p \sum_{r = 1}^{q-1} \sum_{k_q = 1}^{k_q} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + (p - i) \left[ \chi Z(\alpha^p) \right]_0 + [\chi Z(\alpha^p)]_0. \]

\[ \square \]

Proof. (of Theorem 4.21) First suppose that \( c_{r,\ell,j} + m + n(\ell - i - 1) \geq 0 \) for all \( \ell, r \) and \( j \). Choose \( p > i + 1 \) such that \( c_{r,\ell,j} = 0 \) for \( \ell > p \). Then for \( \ell > p \) we have \( m + n(\ell - i - 1) \geq 0 \), hence \( n \geq -\frac{m}{\ell - i - 1} \). Letting \( \ell \to \infty \) we see that \( n \geq 0 \). Now we use the fact that for each \( \ell \), \( \{F_{\ell,j} : 1 \leq j \leq h_\ell\} \) is a partition of \( X_{\ell+1} \), and also Lemmas 4.22 and 4.23, to obtain

\[ a = \sum_{\ell > i} \sum_{r = 1}^{k_\ell} \sum_{j = 1}^{h_\ell} \alpha_{\ell} \chi_{\ell} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + m \left[ \chi Z(\alpha^p \setminus \alpha^p) \right]_0 \]

\[ + n \left( \sum_{\ell = i+1}^p \sum_{r = 1}^{k_\ell} \sum_{j = 1}^{h_\ell} \alpha_{\ell} \chi_{\ell} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + [\chi Z(\alpha^p \setminus \alpha^p)]_0 \right) \]

\[ + m \left( \sum_{\ell = i+1}^p \sum_{r = 1}^{k_\ell} \sum_{j = 1}^{h_\ell} \alpha_{\ell} \chi_{\ell} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + [\chi Z(\alpha^p \setminus \alpha^p)]_0 \right) \]

\[ + n \left( \sum_{\ell = i+1}^p \sum_{r = 1}^{k_\ell} \sum_{j = 1}^{h_\ell} \alpha_{\ell} \chi_{\ell} \left[ \chi Z(\alpha^{q-1}_{\ell}) \right]_0 + [\chi Z(\alpha^p \setminus \alpha^p)]_0 \right) \]

\[ \geq 0. \]

Now suppose that \( a \geq 0. \) Fix \( x \in U_i \). There is \( \ell(x) > i \) such that \( x \in E_{\ell(x)} \). Then there are \( \mu(x) \in \Omega_{\ell(x)} \), \( 1 \leq t(x) \leq k_{\ell(x)} \), and \( x' \in X_{\ell(x)+1} \) such that \( x =
\[\mu(x)\gamma_{\ell(x)}^{(t(x))}x'.\] Let \(1 \leq j(x) \leq h_{\ell(x)}\) with \(x' \in F_{\ell(x),j(x)}\). Recalling Notation \[4.7\] define \(\pi_x : C^*(G_i|U_i) \to M_{\Omega(\ell) \times \Omega(\ell)}\) by \(\pi_x(f) = f_{\ell(x),t(x)}(x')\). Since \(C^*(G_i|U_i)\) is an ideal in \(C^*(G_i)\), \(\pi_x\) extends uniquely to a \(*\)-homomorphism \(\widehat{\pi}_x : C^*(G_i) \to M_{\Omega(\ell) \times \Omega(\ell)}\). For \(\nu \in \Omega_{\ell(x)}\) let \(p_\nu = \chi_{\nu_{\ell(x)}}X_{\ell(x)+1} \in C(X)\). Then \(\widehat{\pi}_x(p_\nu) = \pi_x(p_\nu) = e_{\nu,\nu} \in M_{\Omega(\ell) \times \Omega(\ell)}\).

We calculate \(\widehat{\pi}_x(a)\). First, for any \(\ell > i, r, j\),
\[\widehat{\pi}_x(\chi_{\ell-1,\ell}^{(r)}F_{\ell,j}) = \begin{cases} e_{\alpha(\ell)-1,\alpha(\ell)-1}, & \text{if } \ell = \ell(x), r = t(x), j = j(x) \\ 0, & \text{otherwise.} \end{cases}\]

Next we let \(\nu \in \Omega_{\ell(x)}\) and calculate (in \(C^*(G_i|U_i)\)).
\[\begin{align*}
p_\nu \cdot \chi Z(\alpha^i)\setminus Z(\alpha^i\beta) &= \chi_{\nu^{\ell(x)}}X_{\ell(x)+1} \cdot XZ(\alpha^i)\setminus Z(\alpha^i\beta) \\
&= \chi_{\nu^{\ell(x)}}X_{\ell(x)+1} \cap (Z(\alpha^i)\setminus Z(\alpha^i\beta)) \\
&= \delta_{\nu,\alpha(\ell)-1}X_{\alpha(\ell)-1,\ell(x)+1} \\
&= \delta_{\nu,\alpha(\ell)-1}p_{\alpha(\ell)-1}.\end{align*}\]

Now we have
\[\widehat{\pi}_x(\chi Z(\alpha^i)\setminus Z(\alpha^i\beta)) = \sum_{\nu \in \Omega_{\ell(x)}} \pi_x(p_\nu \cdot \chi Z(\alpha^i)\setminus Z(\alpha^i\beta))\]
\[= \pi_x(p_{\alpha(\ell)-1})\]
\[= e_{\alpha(\ell)-1,\alpha(\ell)-1}.\]

Next note that \(p_\nu \cdot \chi Z(\beta^{i+1}) = \chi_{\nu^{\ell(x)}}X_{\ell(x)+1} \setminus Z(\beta^{i+1})\). The intersection in the last expression is nonempty only if \(\nu = \beta^{i+1}\alpha^d\), where \(c + d = \ell(x) - i - 2\). There are \(\ell(x) - i - 1\) choices where this happens. Therefore \(\widehat{\pi}_x(\chi Z(\beta^i))\) has rank \(\ell(x) - i - 1\) as a projection in \(M_{\Omega_{\ell(x)},\Omega_{\ell(x)}}\).

Putting these three calculations together we obtain
\[0 \leq \widehat{\pi}_x(a) = (c_{\ell,r,j}(x) + m + n(\ell(x) - i - 1))e_{\alpha(\ell)-1,\alpha(\ell)-1} \cdot 0.\]

Since for any choice of \(\ell, r, j\) there exists \(x \in U_i\) with \(\ell = \ell(x), r = t(x)\) and \(j = j(x)\), it follows that \(c_{\ell,r,j} + m + n(\ell - i - 1) \geq 0\) for all \(\ell, r, j\).

Theorem \[4.21\] and its proof have a corollary which we will need later (namely, in the proof of Theorem \[5.9\]).

**Corollary 4.24.** Let \(a \in K_0(C^*(G_i))\) with
\[(4.4)\quad a = m[\chi Z(\alpha^i)\setminus Z(\alpha^i\beta)]_0 + n[\chi Z(\beta^{i+1})]_0.\]

Then \(a \geq 0\) if and only if \(m, n \geq 0\).

**Proof.** If \(m, n, \geq 0\), then it is clear from Theorem \[4.21\] that \(a \geq 0\) (noting that \(c_{\ell,r,j} = 0\) for all \(\ell, r, j\)). Conversely, suppose that \(a \geq 0\). In the first lines of the proof of Theorem \[4.21\] it is shown that \(n \geq 0\). Letting \(\ell = i + 1\), the theorem implies that \(m \geq 0\).
5. K-theory of $C^*(G)$

To analyze $C^*(G)$, we begin with the following observation:

**Theorem 5.1.** $C^*(G)$ is the limit of the inductive sequence

$$C^*(G_1) \rightarrow C^*(G_2) \rightarrow \cdots$$

where the connecting maps are induced from the inclusion maps $C_c(G_i) \hookrightarrow C_c(G_{i+1})$.

**Proof.** For $\mu, \nu \in \Lambda$ we say that $\mu$ and $\nu$ have a common suffix if $\mu = \mu'\theta$ and $\nu = \nu'\theta$, where $\theta \not\in \Lambda^0$. Let

$$S = \{ (\mu, \nu) \in v_1\Lambda \times v_1\Lambda : s(\mu) = s(\nu) \text{ and } \mu, \nu \text{ do not have a common suffix} \}.$$ 

Thus $G = \{ [\mu, \nu, x] : (\mu, \nu) \in S, x \in s(\mu)\partial\Lambda \}$. We define a subset $S_i \subseteq S$ as follows. Recall the set $\Phi_i$ from Definition 4.9. Let

$$S_i = \{ (\mu\theta, \nu\phi) : S, \theta, \phi \in \Lambda_i, d(\theta) \perp d(\phi) \}.$$ 

(Note that since a pair in $S$ does not have a common suffix, if $\phi = \theta = s(\mu)$ then $|\mu| = |\nu|$ and $\mu|_{|\mu|} \neq \nu|_{|\nu|}$.) Then $G_i = \{ [\xi, \eta, x] : (\xi, \eta) \in S_i \}$. Note that for each $(\mu, \nu) \in S$, the set $[\mu, \nu, s(\mu)\partial\Lambda]$ is a compact open subset of $G$. Therefore $G_i = \bigcup_{(\mu, \nu) \in S_i} [\mu, \nu, s(\mu)\partial\Lambda]$ is an open subset of $G_{i+1}$, and of $G$, and also $G_{i+1} \setminus G_i = \bigcup_{(\mu, \nu) \in S_{i+1} \setminus S_i} [\mu, \nu, s(\mu)\partial\Lambda]$ is an open subset of $G_{i+1}$, and similarly, $G \setminus G_i$ is an open subset of $G$. Therefore $G_i$ is a clopen subgroupoid of $G_{i+1}$ and of $G$. By (the proof of) [27 Theorem 14.2], the inclusion $C_c(G_i) \hookrightarrow C_c(G_{i+1})$ induces an injective $\ast$-homomorphism $C^*(G_i) \hookrightarrow C^*(G_{i+1})$. Since $C_c(G) = \bigcup_i C_c(G_i)$, the limit of the system $C^*(G_i) \rightarrow C^*(G_{i+1})$ equals $C^*(G)$. \hfill $\Box$

**Remark 5.2.** Since $G_i$ is a clopen subgroupoid of $G$ (as seen in the proof of Theorem 5.1) and is amenable by Proposition 4.15, it follows from [33 Proposition 9.84] that $G$ is amenable. In particular, all $C^*$-algebras we discuss are nuclear. Moreover since the groupoids we discuss are étale, it follows from [30] that the $C^*$-algebras satisfy the UCT.

We will see shortly that $K_0(C^*(G)) \cong \mathbb{Z}^2$ (Theorem 5.9), but to prove this we will need some preliminary results. (Recall that a partition refines a set $S$ if $S$ equals the union of a subcollection of the partition.)

**Lemma 5.3.** For $m \geq 1$ and $n \geq 0$, let $W_{m,n} = \bigcup_{p+q=n} Z(v_m\alpha^p\beta^q)$. Define collections of subsets of $W_{m,n}$ by

$P_{m,n}^{(1)} = \{ Z(v_m\alpha^{n+1}\beta^j) \setminus Z(v_m\alpha^{n+1}\beta^{j+1}) : j \leq n \}$

$P_{m,n}^{(2)} = \{ Z(v_m\alpha^j\beta^{n+1}) \setminus Z(v_m\alpha^{j+1}\beta^{n+1}) : i \leq n \}$

$P_{m,n}^{(3)} = \{ Z(v_m\alpha^i\beta^j\lambda) : i, j \leq n \leq i + j, \lambda \in v_{m+i+j}\Lambda_2 \}$

$P_{m,n}^{(4)} = \{ Z(v_m\alpha^{n+1}\beta^{n+1}) \}$

$P_{m,n} = \bigcup_{r=1}^4 P_{m,n}^{(r)}$.

Then $P_{m,n}$ is a partition of $W_{m,n}$ that refines $Z(\nu)$ and $Z(\nu) \setminus Z(\nu\lambda)$ for every $\nu \in v_m\Lambda_1$ with $|\nu| = n$ and every $\lambda \in s(\nu)\Lambda$ with $|\lambda| = 1$. 


Proof. First we show that $W_{m,n} = \bigcup P_{m,n}$. It is clear that $\bigcup P_{m,n} \subseteq W_{m,n}$. To show the other inclusion, fix $x \in W_{m,n}$ and let $p = \min\{h : x_h \in \Lambda_2\}$. Then $n+1 \leq p \leq \infty$. We have several cases to consider.

1. Suppose $p \leq 2n+1$. Then $x \in Z(v_m\alpha^i\beta^jx_p)$ for some $i$ and $j$ with $n \leq i+j \leq 2n$.
   
   (a) If $i > n$, then $j < n$. Then $Z(v_m\alpha^i\beta^jx_p) \subseteq Z(v_m\alpha^{n+1}\beta^j) \setminus Z(v_m\alpha^{n+1}\beta^{j+1})$, which is a set in $P_{m,n}^{(1)}$.
   
   (b) If $j > n$, then $i < n$. Then $Z(v_m\alpha^i\beta^jx_p) \subseteq Z(v_m\alpha^i\beta^{n+1}) \setminus Z(v_m\alpha^i\beta^{n+1+1})$, which is a set in $P_{m,n}^{(2)}$.
   
   (c) If $i,j \leq n$, then $Z(v_m\alpha^i\beta^jx_p) \in P_{m,n}^{(3)}$.

2. Suppose $p > 2n+1$. Note that $Z(v_m\alpha^{n+1}\beta^{n+1}) \in P_{m,n}^{(4)}$. If $x \notin Z(v_m\alpha^{n+1}\beta^{n+1})$ then there are two possibilities. We know that $x_1 \cdots x_{p-1} \in \Lambda_1$. Let $\ell = d(x_1 \cdots x_{p-1}) \in \mathbb{N}^2$.
   
   (a) If $\ell_2 \leq n$, then $x \in Z(v_m\alpha^{n+1}\beta^{\ell_2}) \setminus Z(v_m\alpha^{n+1}\beta^{\ell_2+1})$, a set in $P_{m,n}^{(1)}$.
   
   (b) If $\ell_1 \leq n$, then $x \in Z(v_m\alpha^{\ell_1}\beta^{n+1}) \setminus Z(v_m\alpha^{\ell_1+1}\beta^{n+1})$, a set in $P_{m,n}^{(2)}$.

Thus in all cases we have $x \in \bigcup P_{m,n}$ so $W_{m,n} = \bigcup P_{m,n}$.

Now we show that the elements of $P_{m,n}$ are pairwise disjoint. Consider the normal form of an element $x \in W_{m,n}$: $x = \theta_1\theta_2 \cdots$, where $\theta_1 \in \Lambda_1$ and $|\theta_1| \geq n$. Let $d(\theta_1) = \ell$.

From the definition of the sets $P_{m,n}^{(r)}$ we see that

- $x \in \bigcup P_{m,n}^{(1)}$ if and only if $\ell_1 \geq n + 1$ and $\ell_2 \leq n$.
- $x \in \bigcup P_{m,n}^{(2)}$ if and only if $\ell_1 \leq n$ and $\ell_2 \geq n + 1$.
- $x \in \bigcup P_{m,n}^{(3)}$ if and only if $\ell_1, \ell_2 \leq n \leq \ell_1 + \ell_2$.
- $x \in \bigcup P_{m,n}^{(4)}$ if and only if $\ell_1, \ell_2 \geq n + 1$.

It follows that for any $1 \leq r_1 < r_2 \leq 4$, each element of $P_{m,n}^{(r_2)}$ is disjoint from each element of $P_{m,n}^{(r_2)}$. The elements of $P_{m,n}^{(1)}$ with different values of $\ell_2$ are disjoint, and similarly the elements of $P_{m,n}^{(2)}$ with different values of $\ell_1$ are disjoint. The same is true for $P_{m,n}^{(3)}$, for the same reason. Therefore $P_{m,n}$ is a partition of $W_{m,n}$.

Next let $\nu = v_m\alpha^p\beta^q$ (with $p + q = n$), so that $Z(\nu) = Z(v_m\alpha^p) \cap Z(v_m\beta^q)$. We will show that $P_{m,n}$ refines $Z(\nu)$. Let $S \subseteq P_{m,n}$ be such that $S \cap Z(\nu) \neq \emptyset$. We will show that $S \subseteq Z(\nu)$.

First suppose $S \subseteq P_{m,n}^{(1)}$, so $S = Z(v_m\alpha^{n+1}\beta^j) \setminus Z(v_m\alpha^{n+1}\beta^{j+1})$ for some $j \leq n$. Then $S = Z(v_m\alpha^{n+1}) \cap (Z(v_m\beta^j) \setminus Z(v_m\beta^{j+1}))$. Since $Z(\nu) \cap S \neq \emptyset$, we know $Z(v_m\beta^j) \setminus Z(v_m\beta^{j+1}) \neq \emptyset$, hence $q < j$. Then $Z(v_m\beta^j) \setminus Z(v_m\beta^{j+1}) \subseteq Z(v_m\beta^q)$, and since $p \leq n$, $Z(v_m\alpha^{n+1}) \subseteq Z(v_m\alpha^p)$. Therefore $S = Z(v_m\alpha^{n+1}) \cap (Z(v_m\beta^j) \setminus Z(v_m\beta^{j+1})) \subseteq Z(v_m\alpha^p) \cap Z(v_m\beta^q) = Z(\nu)$. It follows that $Z(v_m\alpha^{n+1}\beta^j) \setminus Z(v_m\alpha^{n+1}\beta^{j+1})$ is contained in $Z(\nu)$ if $q \leq j$ and is disjoint from $Z(\nu)$ if $j < q$. If $S \subseteq P_{m,n}^{(2)}$, an analogous argument shows that $Z(v_m\alpha^{n+1}\beta^{n+1}) \setminus Z(v_m\alpha^{n+1+1}\beta^{n+1})$ is contained in $Z(\nu)$ if $p \leq i$ and is disjoint from $Z(\nu)$ if $i < p$.

Now suppose $S \subseteq P_{m,n}^{(3)}$ so $S = Z(v_m\alpha^i\beta^j\lambda)$ with $i,j \leq n \leq i+j$, and $\lambda \in v_{m+i+j}\Lambda_2^1$. Then $S \subseteq (Z(v_m\alpha^i) \cap Z(v_m\alpha^{i+1}))(Z(v_m\beta^j) \setminus Z(v_m\beta^{j+1}))$. Since $Z(\nu) \cap S \neq \emptyset$, $p \leq i$ and $q \leq j$, and in this case, $S \subseteq Z(\nu)$. Thus $Z(v_m\alpha^i\beta^j\lambda)$ is contained in $Z(\nu)$ if $p \leq i \leq n$ and $q \leq j \leq n$, and is disjoint from $Z(\nu)$ if $i < p$ or if $j < q$. 


Finally, since \( p, q < n + 1 \), we have \( Z(v_m \alpha^{n+1} \beta^{n+1}) \subseteq Z(\nu) \). Therefore \( P_{m,n} \) refines \( Z(\nu) \).

Lastly, let \( \nu = v_m \alpha^p \beta^q \) with \( p + q = n \) as above, and let \( \lambda \in s(\nu)A \) with \( |\lambda| = 1 \). We will show that \( P_{m,n} \) refines \( E = Z(\nu) \setminus Z(\nu \lambda) \). We first consider the case that \( \lambda = \alpha_{m+n} \). Then \( E = Z(v_m \alpha^p \beta^q) \setminus Z(v_m \alpha^{p+1} \beta^q) \). Since \( p \leq n \) we know that \( E \) is disjoint from every element of \( P_{m,n}^{(1)} \cup P_{m,n}^{(4)} \). Let \( S \in P_{m,n}^{(2)} \) with \( S \cap E \neq \emptyset \). Then it must be the case that \( S = Z(v_m \alpha^p \beta^{n+1}) \setminus Z(v_m \alpha^{p+1} \beta^{n+1}) \), and hence \( S \subseteq E \). Now we consider elements of \( P_{m,n}^{(3)} \). Since \( p, q \leq n = p + q \), \( S = Z(v_m \alpha^p \beta^{j \gamma_{m+n}}) \in P_{m,n}^{(3)} \) for \( q \leq j \leq n \) and \( 1 \leq r \leq k_{m+p+j} \). Moreover for such \( S, S \subseteq E \). These are the only elements \( S \in P_{m,n}^{(3)} \) with \( S \cap E \neq \emptyset \). Therefore \( P_{m,n} \) refines \( E \). The case where \( \lambda = \beta_{m+n} \) is analogous. Let \( \lambda = \beta_{m+n}^{(r)} \) for some \( 1 \leq r \leq k_{m+n} \). Then \( E = Z(v_m \alpha^p \beta^q) \setminus Z(v_m \alpha^p \beta^q) \cap \gamma_{m+n}^{(r)} \). Note that \( Z(v_m \alpha^p \beta^q) \cap \gamma_{m+n}^{(r)} \) is one of the sets from \( P_{m,n}^{(3)} \) contained in \( Z(\nu) \). Therefore \( E \) equals the union of the other sets from \( P_{m,n} \) that are contained in \( Z(\nu) \). Therefore in all cases, \( P_{m,n} \) refines \( Z(\nu) \setminus Z(\nu \lambda) \). □

**Remark 5.4.** It will be convenient to list the explicit refinements given in the proof of Lemma 5.3. Let \( p + q = n \).

\[
Z(v_m \alpha^p \beta^q) = \bigcup_{j=q}^{n} (Z(v_m \alpha^{n+1} \beta^j) \setminus Z(v_m \alpha^{n+1} \beta^{j+1})) \\
\cup \bigcup_{i=p}^{n} (Z(v_m \alpha^i \beta^{n+1}) \setminus Z(v_m \alpha^{i+1} \beta^{n+1})) \\
\cup \bigcap_{p \leq i \leq n} \bigcup_{q \leq j \leq n} \bigcup_{r=1}^{k_{m+i+j}} Z(v_m \alpha^i \beta^j \gamma_{m+i+j}^{(r)}) \\
\cup Z(v_m \alpha^{n+1} \beta^{n+1})
\]

\[
Z(v_m \alpha^p \beta^q) \setminus Z(v_m \alpha^{p+1} \beta^q) = (Z(v_m \alpha^p \beta^{n+1}) \setminus Z(v_m \alpha^{p+1} \beta^{n+1})) \\
\cup \bigcup_{j=q}^{n} \bigcup_{r=1}^{k_{m+p+j}} Z(v_m \alpha^p \beta^j \gamma_{m+p+j}^{(r)})
\]

\[
Z(v_m \alpha^p \beta^q) \setminus Z(v_m \alpha^p \beta^{q+1}) = (Z(v_m \alpha^{n+1} \beta^q) \setminus Z(v_m \alpha^{n+1} \beta^{q+1})) \\
\cup \bigcup_{i=p}^{n} \bigcup_{r=1}^{k_{m+i+q}} Z(v_m \alpha^i \beta^q \gamma_{m+i+q}^{(r)})
\]

\[
Z(v_m \alpha^p \beta^q) \setminus Z(v_m \alpha^p \beta^q \gamma_{m+n}^{(r)}) = \bigcup_{j=q}^{n} (Z(v_m \alpha^{n+1} \beta^j) \setminus Z(v_m \alpha^{n+1} \beta^{j+1})) \\
\cup \bigcup_{i=p}^{n} (Z(v_m \alpha^i \beta^{n+1}) \setminus Z(v_m \alpha^{i+1} \beta^{n+1}))
\]
where \( a \) is a typical generator of the \( \partial \Lambda \) then we write \( \lambda E := \{ \lambda x : x \in E \} \). If \( P \) is a family of subsets of \( s(\lambda) \partial \Lambda \) we write \( \lambda P := \{ \lambda E : E \in P \} \).

**Proposition 5.6.** For \( n \geq 1 \) let \( Q_n = \bigcup_{\mu \in \Phi_n} \mu P_{|\mu|+1,n,|\mu|} \), where \( P_{r,s} \) is as in Lemma 5.3 and \( \Phi_n \) is as in Definition 4.9. Then \( Q_n \) is a partition of \( X \) that refines \( Z(\nu) \) and \( Z(\nu) \setminus Z(\nu \lambda) \) for all \( \nu \in v_1 \Lambda \) with \( |\nu| \leq n \) and \( \lambda \in s(\nu) \Lambda \) with \( |\lambda| = 1 \).

**Proof.** Let \( n \geq 1 \). We first show that the sets in \( Q_n \) are pairwise disjoint. Let \( \mu, \nu \in \Phi_n \) with \( \mu \neq \nu \). Let \( x \in \bigcup_{\mu \in \Phi_n} \mu P_{|\mu|+1,n,|\mu|} \) and \( y \in \bigcup_{\nu P_{|\nu|+1,n,|\nu|}} \). Then \( \mu \in x \) and \( \nu \notin y \), and \( x_{|\mu|+1} \cdots x_n, y_{|\nu|+1} \cdots y_n \in \Lambda_1 \). Without loss of generality suppose that \( |\mu| < |\nu| \). Then \( y_{|\nu|} \in \Lambda_2 \) and \( y_{|\nu|} = \nu_{|\nu|} \). If \( |\mu| < |\nu| \) then \( x_{|\nu|} \in \Lambda_1 \), and hence \( x_{|\nu|} \neq \nu_{|\nu|} \), hence \( x \neq y \). If \( |\mu| = |\nu| \) and \( \mu_{|\nu|} = \nu_{|\nu|} \) then since \( \mu \neq \nu \), \( x \neq y \) by Lemma 3.4. Thus if \( S \in P_{|\mu|+1,n,|\mu|} \) and \( T \in P_{|\nu|+1,n,|\nu|} \) then \( \mu S \cap \nu T = \emptyset \). Since \( P_{|\mu|+1,n,|\mu|} \) is a pairwise disjoint family, so is \( \mu P_{|\mu|+1,n,|\mu|} \). Therefore \( Q_n \) is pairwise disjoint.

Next we show that \( \cup Q_n = X \). Let \( x \in X \). Write \( x = x_1 x_2 \cdots \) as an infinite word. Let \( p = \max \{ j \leq n : x_j \in \Lambda_2 \} \), and let \( \mu = x_1 \cdots x_p \). Then \( \mu \in \Phi_n \) and \( x_{p+1} \cdots x_n \in \Lambda_1 \), and hence \( x \in \bigcup_{\mu P_{p+1,n,|\mu|}} \).

Finally, let \( \nu \in v_1 \Lambda_1 \) with \( |\nu| \leq n \). Write \( \nu = \mu \theta \) where \( \mu \in \Phi_n \) and \( \theta \in \Lambda_1 \).

By Lemma 5.3 \( Z(v_{|\mu|+1,\theta}) \) and \( Z(v_{|\mu|+1,\theta}) \setminus Z(v_{|\mu|+1,\theta} \lambda) \) are refined by \( P_{|\mu|+1,|\theta|} \). Then \( Z(\nu) = Z(\mu \theta) \) and \( Z(\nu) \setminus Z(\nu \lambda) = Z(\mu \theta) \setminus Z(\mu \theta \lambda) \) are refined by \( \mu P_{|\mu|+1,|\theta|} \).

Recall from Theorem 4.19 and Corollary 4.20 that

\[
K_0(C^*(G_i)) \cong \bigoplus_{\ell \geq 1} C(X_{\ell+1}, \mathbb{Z})^{k_{\ell}} \oplus \mathbb{Z}^2
\]

where a typical generator of the \( \ell \)-th summand is \( [\chi_{\alpha^{\ell-1} \gamma \ell}]_0 \), where \( F \subseteq X_{\ell+1} \) is a compact open subset, and the generators of the right summand are \( [\chi_{Z(\alpha \gamma)}]_0 \) and \( [\chi_{Z(\gamma^{\ell+1})}]_0 \).

**Lemma 5.7.** The induced map \( K_0(C^*(G_i)) \to K_0(C^*(G_{i+1})) \) carries the summand \( \mathbb{Z}^2 \) of \( K_0(C^*(G_i)) \) into the summand \( \mathbb{Z}^2 \) of \( K_0(C^*(G_{i+1})) \). Using the generators chosen in Theorem 4.19 for the summand \( \mathbb{Z}^2 \), this restriction is implemented by the matrix \( B_i = \begin{pmatrix} k_{i+1} & 1 \\ k_{i+1} & 1 \end{pmatrix} \).
Proof. From Lemma 4.22 letting \( \ell = i \) and \( p = i + 1 \), we have
\[
[\chi Z(\alpha^i)\setminus Z(\alpha^i)]_0 = \left[\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1}) \right]_0 + \sum_{r=1}^{k_{i+1}} [\chi Z(\alpha^i, 1^r)]_0
\]
\[
= \left[\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1}) \right]_0 + \sum_{r=1}^{k_{i+1}} (\left[\chi Z(\alpha^i, 1^r)\setminus Z(\alpha^i, 1^r) \right]_0 + [\chi Z(\alpha^i, 1^r)]_0).
\]
By Lemma 4.2 we have
\[
[\chi Z(\alpha^i, 1^r)]_0 = [\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1})]_0
\]
and
\[
[\chi Z(\alpha^i, 1^r)]_0 = [\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1})]_0
\]
in \( K_0(C^*(G_{i+1})) \). Therefore
\[
[\chi Z(\alpha^i)\setminus Z(\alpha^i)]_0 \mapsto (k_{i+1} + 1) [\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1})]_0 + k_{i+1} [\chi Z(\beta^{i+2})]_0.
\]
Finally,
\[
[\chi Z(\beta^{i+1})]_0 = [\chi Z(\beta^{i+1})\setminus Z(\beta^{i+2})]_0 + [\chi Z(\beta^{i+2})]_0
\]
so that
\[
[\chi Z(\beta^{i+1})]_0 \mapsto [\chi Z(\alpha^{i+1})\setminus Z(\alpha^{i+1})]_0 + [\chi Z(\beta^{i+2})]_0
\]
again, using equivalences in \( C^*(G_{i+1}) \). The matrix of the restriction follows from these formulas. \( \square \)

**Lemma 5.8.** Let \( 1 \leq i < \ell, 1 \leq r \leq k_\ell, \) and \( F \subseteq X_{\ell+1} \) a compact open subset. There is \( i' > i \) such that the image of \( \chi_{\alpha^{i-1}_\ell} h_{i'})_F \) under the induced map \( K_0(C^*(G_i)) \rightarrow K_0(C^*(G_{i'})) \) lies in the \( \mathbb{Z}^2 \) summand (of \( K_0(C^*(G_{i'})) \)).

**Proof.** We may suppose that \( F = Z(\nu) \setminus \bigcup_{j=1}^m Z(\nu_j) \), where \( \nu \in v_{i+1}\Lambda \) and \( \nu_1, \ldots, \nu_m \) extend \( \nu \). Choose \( n \geq \ell + 1 + \max \{ |\nu_1|, \ldots, |\nu_m| \} \). By Proposition 5.6 we have
\[
Z(\alpha^{\ell-1}_\ell \gamma_{i')}) = \bigcup \{ w \in Q_n : w \subseteq Z(\alpha^{\ell-1}_\ell \gamma_{i')} \}
\]
and for each \( j \),
\[
Z(\alpha^{\ell-1}_\ell \gamma_{i')}) = \bigcup \{ w \in Q_n : w \subseteq Z(\alpha^{\ell-1}_\ell \gamma_{i'}) \}, \quad 1 \leq j \leq m.
\]
Hence
\[
\alpha^{\ell-1}_\ell h_{i')} F = Z(\alpha^{\ell-1}_\ell \gamma_{i'}) \setminus \bigcup_{j=1}^m Z(\alpha^{\ell-1}_\ell \gamma_{i'})
\]
\[
= \bigcup \{ w \in Q_n : w \subseteq \alpha^{\ell-1}_\ell h_{i'} F \}.
\]
Now fix \( w \in Q_n \) with \( w \subseteq \alpha^{\ell-1}_\ell h_{i'} F \). Then there is \( \mu \in \Phi_n \) such that \( w \in \mu P_{|\mu|+1, n-|\mu|} \). Therefore \( w \) has one of the following forms:

(i) \( Z(\mu a^{-|\mu|+1} b^t) \setminus Z(\mu a^{-|\mu|+1} b^{t+1}) \), where \( t \leq n - |\mu| \)
(ii) \( Z(\mu a^s b^{n-|\mu|+1}) \setminus Z(\mu a^{s} b^{n-|\mu|+1}) \), where \( s \leq n - |\mu| \)
(iii) \( Z(\mu a^s b^t \lambda) \), where \( s, t \leq n - |\mu| \leq s + t, \lambda \in v_{|\mu|+s+t+1} \)
(iv) \( Z(\mu a^{-|\mu|+1} b^{n-|\mu|+1}) \).
Using Lemma 4.2 we see that in case (i), since $|\mu| \leq n$, we have that $\chi_w$ is equivalent in $C^*(G_{n+\ell+1})$ to $\chi_{Z(\alpha^{n+\ell+1})} \setminus Z(\alpha^{n+\ell+1})$. Case (ii) is nearly identical. In case (iii) we have that $\chi_w$ is equivalent in $C^*(G_{|\mu|+s+\ell})$ to $\chi_{Z(\beta^{\mu+|\mu|+s+\ell})}$. In case (iv) we have that $\chi_w$ is equivalent in $C^*(G_{2n-|\mu|})$ to $\chi_{Z(\beta^{2n-|\mu|})}$. Choose $\ell'$ so large that the equivalences described above occur in $C^*(G_{1})$. By Lemma 5.7 for all $w \in Q_{n}$ with $w \preceq \alpha^{\ell-1} \gamma_{\ell}^{(r)} F$, $[\chi_w]_0$ lies in the summand $Z^2$ in $K_0(C^*(G_{1}))$. Since $\chi_{\alpha^{\ell-1} \gamma_{\ell}^{(r)} F} = \sum \{ \chi_w : w \in Q_{n}, w \preceq \alpha^{\ell-1} \gamma_{\ell}^{(r)} F \}$, the lemma follows. □

We can now give the ordered $K$-theory for $C^*(G)$.

**Theorem 5.9.** $K_1(C^*(G)) = 0$ and $K_0(C^*(G)) \cong \mathbb{Z}^2$. Moreover, $K_0(C^*(G))$ is realized as the limit of the inductive sequence $\mathbb{Z}^2 \to \mathbb{Z}^2 \to \cdots$, where the $i^{th}$ connecting map is given by the matrix $B_i$ from Lemma 5.7 and each term in the sequence has the standard positive cone $\mathbb{N}^2$.

**Proof.** By Theorem 4.19 we know that $K_1(C^*(G_i)) = 0$ for all $i$, and hence that $K_1(C^*(G)) = 0$. Let $\eta_{i',i} : C^*(G_i) \to C^*(G_{i'})$ and $\eta_i : C^*(G_i) \to C^*(G)$ denote the inclusions of Theorem 5.1. Let $a \in K_0(C^*(G))$. Then there is $i \geq 1$ and $b \in K_0(C^*(G_i))$ such that $a = \eta_{i,i+1}(b)$. Let

$$b = \sum_{\ell \geq 1} \sum_{r=1}^{k_{\ell}} \sum_{j=1}^{h_{\ell}} c_{\ell,r,j}[\chi_{\alpha^{\ell-1} \gamma_{\ell}^{(r)} F_{\ell,r,j}}]_0 + m[\chi_{Z(\alpha^{i'}) \setminus Z(\alpha^{i})}]_0 + n[\chi_{Z(\beta^{i+1})}]_0$$

as in (4.3). Note that this is a finite sum. By Lemmas 5.7 and 5.8, there is $i' > i$ such that $\eta_{i,i+1}(b)$ lies in the summand $Z^2$ of $K_0(C^*(G_{i'}))$. Let $H_i$ denote the $\mathbb{Z}^2$ summand of $K_0(C^*(G_i))$. Then $K_0(C^*(G))$ is the inductive limit of the sequence $(H_i, \eta_{i,i+1} | H_i)$. By Lemma 5.7, the connecting maps in this sequence are given by the matrices $B_i$, and hence are invertible. Therefore the limit $K_0(C^*(G))$ is isomorphic to $\mathbb{Z}^2$. The fact that each term in the sequence has standard positive cone follows from Corollary 4.24. □

Before determining the positive cone $K_0(C^*(G))_+$, we establish notation and give a theorem, both of which are taken from [6].

**Notation 5.10.** For $\sigma \in \mathbb{R}_+ \setminus \mathbb{Q}$, let

$$P_{\sigma} := \{(m, n) \in \mathbb{Z}^2 : \sigma m + n \geq 0\}$$

and $(\mathbb{Z}^2, P_{\sigma})$ be the Riesz group given by the (total) ordering of $\mathbb{Z}^2$ by $P_{\sigma}$. We write $\sigma = [c_0, c_1, c_2, \ldots]$ for a continued fraction expansion of $\sigma$,

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cdots}}$$

where $c_0 \in \mathbb{Z}$ and $c_i \in \mathbb{N}$ for $i > 0$. The continued fraction expansion is simple if $c_i > 0$ for $i \geq 1$.

It is shown in [6] Lemma 3.1 that the continued fraction expansion makes sense if

$c_1 > 0$ and if $c_i > 0$ for infinitely many even and infinitely many odd values of $i$. We will need this generality later. The following result is given in this context.
Theorem 5.11. ([6 Theorem 3.2]) Suppose that \( \sigma \in \mathbb{R}^+ \setminus \mathbb{Q} \) has continued fraction expansion \( \sigma = [c_0, c_1, \ldots] \) with \( c_1 > 0 \) and \( c_{2j} > 0, c_{2j+1} > 0 \) for infinitely many \( j \) and \( k \). Then

\[
(Z^2, P_\sigma) = \lim(Z^2, \phi_{m,n})
\]

where \( \phi_{m,m+1} = \left( \begin{smallmatrix} c_m & 1 \\ 0 & 1 \end{smallmatrix} \right) \) and the terms in the sequence \( (Z^2, \phi_{m,n}) \) have the standard positive cone.

We can now conclude the following:

Theorem 5.12. Let \( \sigma \in \mathbb{R}^+ \setminus \mathbb{Q} \). Let \( \sigma \) have simple continued fraction expansion \([c_0, c_1, \ldots]\). Define integers \( k_i \geq 0 \) as follows. Let \( k_1 \) be arbitrary, and for \( p \geq 0 \),

\[
k_i = \begin{cases} 0, & \text{for } c_1 + c_3 + \cdots + c_{2p-1} + 2 < i < c_1 + c_3 + \cdots + c_{2p+1} + 2 \\ c_{2p}, & \text{for } i = c_1 + c_3 + \cdots + c_{2p-1} + 2. \end{cases}
\]

We may indicate this visually as

\[
(k_i)_{i=1}^\infty = (k_1, c_0, 0, \ldots, 0, c_2, 0, \ldots, 0, c_4, 0, \ldots, 0, c_6, \ldots).
\]

Let \( \Lambda \) be the category of paths as in Definition 3.7. Then

\[
(K_0(C^*(G)), K_0(C^*(G))_+, [1])_0 \cong (Z^2, P_\sigma, (k_i+1))_0.
\]

Proof. Note that \( \left( \begin{smallmatrix} k_i+1 \\ k \end{smallmatrix} \right) \) = \( \left( \begin{smallmatrix} 1+1 \\ 1 \end{smallmatrix} \right) \). Note also that “collapsing” in the sequence \( Z^2 \xrightarrow{B_{1_1}} Z^2 \xrightarrow{B_{2_1}} \cdots \) yields

\[
Z^2 \xrightarrow{B_{1_1} \cdots B_{1_2}} Z^2 \xrightarrow{B_{2_1} \cdots B_{1_2} B_{1_3}} Z^2 \xrightarrow{\cdots} \ldots
\]

Let \( T = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \). When \( k_i = 0 \) we have \( \left( \begin{smallmatrix} k_i+1 \\ k \end{smallmatrix} \right) = T \), so the \( p \)th string of zeros in \( (k_i) \) collapses to \( Z^2 \xrightarrow{T_{c_2p-1}} Z^2 \). Then the portion \( Z^2 \xrightarrow{B_{c_2p}} Z^2 \xrightarrow{T_{c_2p-1}} Z^2 \) collapses to the single map

\[
T_{c_2p-1} B_{c_2p} = \left( \begin{smallmatrix} 1 & c_2p+1 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & c_2p+1 \\ 0 & 1 \end{smallmatrix} \right)
\]

Thus the sequence \( Z^2 \xrightarrow{B_{1_1}} Z^2 \xrightarrow{B_{2_1}} Z^2 \xrightarrow{\cdots} \) collapses to

\[
Z^2 \xrightarrow{(k_1+1)} Z^2 \xrightarrow{(c_1+1)} Z^2 \xrightarrow{\cdots} \ldots
\]

Now the fact that \( (K_0(C^*(G)), K_0(C^*(G))_+, [1])_0 \cong (Z^2, P_\sigma) \) follows from Theorems 5.9 and 5.11. To see that \( [1] = (k_i+1) \), first note that

\[
Z(v_1) = Z(\alpha) \cup Z(\beta) \cup (\bigcup_{j=1}^{k_1} Z(\gamma_1(1)))
\]

\[
= (Z(\alpha) \setminus Z(\alpha\beta)) \cup Z(\beta) \cup \left( \bigcup_{j=1}^{k_1} Z(\gamma_1(1)) \right)
\]

\[
= (Z(\alpha) \setminus Z(\alpha\beta)) \cup (Z(\beta) \setminus Z(\beta^2)) \cup Z(\beta^2)
\]

\[
\cup \left( \bigcup_{j=1}^{k_1} Z(\gamma_1(1)) \setminus Z(\gamma_1(1)\beta) \right) \cup \left( \bigcup_{j=1}^{k_1} Z(\gamma_1(1)\beta) \right).
\]
Now applying Lemma 4.2, we have
\[ [1]_0 = [\chi_{Z(v_1)}]_0 = (k_1 + 2)[\chi_{Z(\alpha)}Z(\alpha\beta)]_0 + (k_1 + 1)[\chi_{Z(\beta^2)}]_0. \]

6. Invariant measures on \( G^{(0)} \)

**Definition 6.1.** Suppose \( \Gamma \) is an étale groupoid and \( \mu \) is a measure on \( \Gamma^{(0)} \). We say \( \mu \) is invariant if \( \mu(s(E)) = \mu(r(E)) \) for every open bisection \( E \subset \Gamma \).

**Remark 6.2.** The groupoids studied in this paper are ample as well as étale, in that the unit space is totally disconnected. It follows that there is a base for the topology consisting of compact-open bisections. Then every open bisection is a countable disjoint union of compact-open bisections. Therefore in the condition in Definition 6.1 it suffices to consider only compact-open bisections.

Let \( \sigma \) and \( G \) be as in Theorem 5.12. We will show that there exists a unique invariant Borel probability measure on \( G^{(0)} \) (Theorems 6.6 and 6.10). Both theorems will require several lemmas of preparation. We mention that the existence can be established abstractly. (Namely, the ordered \( K \)-theory of \( C^*(G) \) has a state (the same one as for the corresponding continued fraction AF algebra). Since \( C^*(G) \) is nuclear (Remark 5.2), a quasitrace inducing this state must actually be a trace, which must arise from an invariant measure. See [1, section 6.9].) However the analysis used in the proof of uniqueness provides the basis for an explicit proof of existence, and we felt it worthwhile to give this proof.

It will be convenient to use the map from invertible integer matrices to fractional linear transformations, and the action of these on the extended real line, to extend the notation for finite (and infinite) continued fractions to include the case where some of the coefficients are 0, and the final coefficient is an extended real number. We identify \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in PGL(2, \mathbb{Z}) \) with the fractional linear transformation \( z \mapsto \frac{az + b}{cz + d} \). We have the quotient maps \( \pi : GL(2, \mathbb{Z}) \to PGL(2, \mathbb{Z}) \), and \( \nu : \mathbb{C}^2 \setminus \{(0, 0)\} \to \mathbb{CP}^1 = \mathbb{C} \) by \( \nu(z_1, z_2) = \frac{z_1}{z_2} \). Then for \( T \in GL(2, \mathbb{Z}) \) and \( z \in \mathbb{C} \), \( \pi(T)(z) = \nu(T(z)) \). Then we note that \( \pi(\left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right]) \) is \( \frac{a + 1}{a} = a + \frac{1}{a} \). For \( a_0, \ldots, a_n \in \mathbb{Z} \) nonegative, and \( \alpha \in \mathbb{R} \), we write \( [a_0, a_1, \ldots, a_n, \alpha] := \pi(\left( \begin{array}{c} a_0 \\ 0 \end{array} \right) \ldots \left( \begin{array}{c} a_n \\ 0 \end{array} \right)) \)(\( \alpha \)). Thus
\[
[a_0, a_1, \ldots, a_n, \alpha] = a_0 + \frac{1}{\frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n + \alpha}}}}.
\]

The usual arithmetic in \( \mathbb{R} \) manages the situations where some of the \( a_i \) equal 0. Also we have the usual identity (even allowing zero for coefficients):
\[
[a_0, \ldots, a_n] = \pi(\left( \begin{array}{c} a_0 \\ 1 \end{array} \right) \ldots \left( \begin{array}{c} a_n^{-1} \\ 1 \end{array} \right)) \)(\( a_n \))
\[
= \pi(\left( \begin{array}{c} a_0 \\ 1 \end{array} \right) \ldots \left( \begin{array}{c} a_i^{-1} \\ 1 \end{array} \right)) \)(\( \left( \begin{array}{c} a_i^{-1} \\ 1 \end{array} \right) \)(\( a_n \))
\[
= \pi(\left( \begin{array}{c} a_0 \\ 1 \end{array} \right) \ldots \left( \begin{array}{c} a_i^{-1} \\ 1 \end{array} \right)) \)(\( [a_i, \ldots, a_n] \))
\[
= [a_0, \ldots, a_{i-1}, [a_i, \ldots, a_n]] .
\]

Now we begin the preparation for the proof of uniqueness of invariant measures.
Lemma 6.3. Let \( c, d \geq 0, c, d \in \mathbb{R} \), with \( c + d = 1 \), and let \( k \in \mathbb{Z} \) with \( k \geq 0 \). Set \( B = \left( \begin{smallmatrix} k+1 \\ k \end{smallmatrix} \right) \). Suppose that \( e, f \geq 0, e, f \in \mathbb{R} \), are such that \( \left( \begin{smallmatrix} c \\ d \end{smallmatrix} \right) = B^t \left( \begin{smallmatrix} e \\ f \end{smallmatrix} \right) \). Then \( c < 1 \), \( e + f = 1 - c \), and \( [0, 1, k] \leq c \leq [0, 1, k, 1] \).

Proof. We have

\[
\left( \begin{smallmatrix} e \\ f \end{smallmatrix} \right) = \left( \begin{smallmatrix} k+1 \\ 1 \\ 1 \\ k+1 \end{smallmatrix} \right)^{-1} \left( \begin{smallmatrix} c \\ d \end{smallmatrix} \right) = \left( \begin{smallmatrix} 1 \\ -k \\ 1 \\ k+1 \end{smallmatrix} \right) \left( \begin{smallmatrix} c \\ 1-c \end{smallmatrix} \right) = \left( \begin{smallmatrix} (k+1)c-k \\ -(k+2)c+k+1 \end{smallmatrix} \right).
\]

Then \( e + f = ((k+1)c - k) + (-(k+2)c + k + 1) = 1 - c \). Moreover,

\[
e \leq (k+1)c - k,
\]

hence

\[
c \geq \frac{k}{k+1} = \frac{1}{1 + \frac{1}{k}} = [0, 1, k].
\]

Also

\[
0 \leq f = -(k+2)c + k + 1,
\]

hence

\[
c \leq \frac{k+1}{k+2} = \frac{1}{1 + \frac{1}{k+1}} = [0, 1, k, 1].
\]

Finally, since \( \frac{k+1}{k+2} < 1 \) we have that \( c < 1 \). \( \square \)

Lemma 6.4. In the context of Lemma 6.3, let \( \alpha, \beta \in \mathbb{R} \) with \( 0 \leq \alpha \leq \frac{e}{1-c} \leq \beta \leq 1 \). Then \( [0, 1, k + \alpha] \leq c \leq [0, 1, k + \beta] \).

Proof. Using the equation \( e = (k+1)c - k \) from the proof of Lemma 6.3, the hypothesis gives

\[
\alpha \leq \frac{(k+1)c - k}{1-c} = \frac{c}{1-c} - k \leq \beta
\]

\[
k + \alpha \leq \frac{c}{1-c} \leq k + \beta
\]

\[
\frac{1}{k + \alpha} \geq \frac{1}{c} - 1 \geq \frac{1}{k + \beta}
\]

\[
1 + \frac{1}{k + \alpha} \geq \frac{1}{c} \geq 1 + \frac{1}{k + \beta}
\]

\[
[0, 1, k + \alpha] = \frac{1}{1 + \frac{1}{k + \alpha}} \leq c \leq \frac{1}{1 + \frac{1}{k + \beta}} = [0, 1, k + \beta]. \quad \square
\]

Proposition 6.5. Let \( \sigma \in \mathbb{R}^+ \setminus \mathbb{Q} \) have simple continued fraction expansion \([c_0, c_1, \ldots] \) as in Theorem 5.12 and let \( (k_i)_{i=1}^\infty \) and \( \Lambda \) be as in that theorem. Suppose that \( \mu \) is an invariant Borel probability measure on \( \mathbb{G}^{(0)} \). Let \( a_0 = \mu(Z(\beta)^\sigma) \). Then for each \( n \geq 1 \),

\[
[0, 1, k_1, 1, k_2, \ldots, 1, k_n] \leq a_0 \leq [0, 1, k_1, 1, k_2, \ldots, 1, k_n, 1].
\]
Proof. Recall from the proof of Theorem 5.9 that \( H_i \) denotes the summand \( \mathbb{Z}^2 \) of \( K_0(C^*(G_i)) \), with generators \( p_i = [\chi_Z(\alpha^i)] \) and \( q_i = [\chi_Z(\beta^i)] \). Let \( a_i = \mu_*(p_i) = \mu(Z(\alpha^i) \setminus Z(\alpha^i)) \) and \( b_i = \mu_*(q_i) = \mu(Z(\beta^i)) \). Then \((a_i, b_i) \in \mathbb{R}^2 = \text{Hom}(H_i, \mathbb{R})\) represents \( \mu_3 \). It follows that \((a_{i+1}, b_{i+1})B_i \). We extend this to the case \( i = 0 \) using the same formulas.

We apply Lemma 6.3 with \((c, d) = (a_0, b_0)\), \((e, f) = (a_1, b_1)\), \( k = k_1\), and \( B = B_0\). In particular we have \( a_0 < 1 \). Let \((a_{n1}, b_{n1}) = (1 - a_0)^{-1}(a_n, b_n) \) for \( n \geq 1 \). Then
\[
(a_{n1}, b_{n1}) = (a_{n+11}, b_{n+11})B_n,
\]
for \( n \geq 1 \). Then \( a_{11} + b_{11} = 1 \), and Lemma 6.3 implies that \( a_{11} < 1 \), \( a_{21} + b_{21} = 1 - a_{11} \), and \([0, 1, k_2] \leq a_{11} \leq [0, 1, k_2, 1] \). Let \((a_{n2}, b_{n2}) = (1 - a_{11})^{-1}(a_{n1}, b_{n1}) \) for \( n \geq 2 \). Then
\[
(a_{n2}, b_{n2}) = (a_{n+12}, b_{n+12})B_n,
\]
for \( n \geq 2 \), and Lemma 6.3 implies that \( a_{22} < 1 \), \( a_{32} + b_{32} = 1 - a_{22} \), and \([0, 1, k_3] \leq a_{22} \leq [0, 1, k_3, 1] \). Continuing this process we define \( k_{n1}(m) \) and \( b_{n1}(m) \) for \( n \geq m \geq 1 \) so that \( a_{n1}(m), b_{n1}(m) \geq 0 \), \( a_{n1} + b_{n1} = 1 \), and
\[
(a_{n1}(m), b_{n1}(m)) = (a_{n+11}(m), b_{n+11}(m))B_n,
\]
for \( n \geq m \). For each \( n \), Lemma 6.3 implies that \([0, 1, k_n] \leq a_{n+11} \leq [0, 1, k_n, 1] \). Now we apply Lemma 6.4 repeatedly: since \( a_{n+11} = \frac{a_{n+12}}{1 - a_{n+12}} \) plays the role of \( \frac{e}{1 - c} \),
\[
[0, 1, k_{n-1} + [0, 1, k_n]] \leq a_{n+11} \leq [0, 1, k_{n-1} + [0, 1, k_n, 1]],
\]
that is,
\[
[0, 1, k_{n-1}, 1, k_n] \leq a_{n+11} \leq [0, 1, k_{n-1}, 1, k_n, 1];
\]
repeating, we get
\[
[0, 1, k_{n-2} + [0, 1, k_{n-1}, 1, k_n]] \leq a_{n+11} \leq [0, 1, k_{n-2} + [0, 1, k_{n-1}, 1, k_n, 1]],
\]
that is,
\[
[0, 1, k_{n-2}, 1, k_{n-1}, 1, k_n] \leq a_{n+11} \leq [0, 1, k_{n-2}, 1, k_{n-1}, 1, k_n, 1];
\]
\[
\cdots
\]
\[
[0, 1, k_1, 1, k_2, \ldots, 1, k_n] \leq a_{0} \leq [0, 1, k_1, 1, k_2, \ldots, 1, k_n, 1].
\]
\[ \square \]

**Theorem 6.6.** Let \( \sigma \in \mathbb{R}^+ \setminus \mathbb{Q} \) have simple continued fraction expansion \([c_0, c_1, \ldots] \) as in Theorem 5.12 and let \((k_i)_{i=1}^{\infty}\) and \( \Lambda \) be as in that theorem. If there exists an invariant Borel probability measure on \( G(0) \) it is unique.

Proof. Write
\[
[g_0, g_1, \ldots] = [0, 1, k_1, 1, k_2, \ldots] = [0, 1, k_1, 1, c_0, 1, 0, 1]^{c_1-1}, c_2, 1, 0, 1]^{c_2-1}, c_4, 1, \ldots].
\]
Then \( g_1 > 0 \), and the consecutive terms \( c_{2j}, 1 \) imply that \( g_{2j} > 0 \) and \( g_{2k+1} > 0 \) for infinitely many \( j \) and \( k \). By [6] Lemma 3.1, \( \theta = \lim_{i \to \infty} [g_0, \ldots, g_i] \) exists. It follows from Proposition 6.5 that \( a_0 = \theta \). Since \( b_0 = 1 - a_0 \), and \((a_i, b_i) = (a_{i+1}, b_{i+1})B_i \), all \( a_i \) and \( b_i \) are determined by \( a_0 \). Then the measure is determined on all sets of the form \( Z(\mu_1 \cdots \mu_n) \) and \( Z(\mu_1 \cdots \mu_n) \setminus Z(\mu_1 \cdots \mu_{n+1}) \). By Proposition 5.6 these sets form a base for the topology of \( G(0) \). Since Borel probability measures on a compact metrizable space are regular, the measure is completely determined. \[ \square \]
Now we begin the preparation for the proof of the existence of an invariant measure.

**Definition 6.7.** For $h \geq 0$ let $\mathcal{E}_h = \{Z(\nu), Z(\nu) \setminus Z(\nu \lambda) : \nu \in v_1 \Lambda, \lambda \in s(\nu) \Lambda, |\nu| = h, |\lambda| = 1\}$, and let $\mathcal{E} = \bigcup_h \mathcal{E}_h$.

**Lemma 6.8.** Let $a_i, b_i$ be as in the proof of Theorem 6.6. Define $\mu : \mathcal{E} \to \mathbb{R}$ by $\mu(Z(\lambda_1 \cdots \lambda_h)) = b_{h-1}$ and $\mu(Z(\lambda_1 \cdots \lambda_h) \setminus Z(\lambda_1 \cdots \lambda_{h+1})) = a_h$. Let $\nu \in v_1 \Lambda$ with $|\nu| = h$ and let $\lambda \in s(\nu) \Lambda$ with $|\lambda| = 1$. For $E = Z(\nu)$ or $Z(\nu) \setminus Z(\nu \lambda)$ we have that $\mu(E) = \sum \{\mu(S) : S \in Q_h, \ S \subseteq E\}$. (Recall the definition of $Q_h$ from Proposition 5.6.)

**Proof.** We may write $\nu = \eta \alpha \beta \eta \alpha$ where $\eta \in \Phi_h$ and $p + q = h - |\eta|$. Let $n = h - |\eta|$ and let $m = |\eta| + 1$, so that $s(\eta) = v_m$. Then $Z(\nu) = \eta Z(v_m \alpha \beta \eta \alpha)$. We will use equation (5.2) from Remark 5.4. For the moment we will write $[c, d] = \{c, c+1, \ldots, d\}$ for $c \leq d$ integers. Then we claim that

$$[p, n] \times [q, n] = \bigcup_{\ell = 0}^{\min\{p, q\}} (\{p + \ell, n\} \times \{q + \ell\}) \cup (\{p + \ell\} \times \{q + \ell + 1, n\}).$$

To prove this claim, we first check the disjointness. For fixed $\ell$, the elements of $[p + \ell, n] \times [q + \ell]$ and those of $[p + \ell] \times [q + \ell + 1, n]$ differ in their second coordinate. Let $\ell_1 < \ell_2$, and suppose that $(i, j)$ is in the $\ell_1$-term and in the $\ell_2$-term. If $j = q + \ell_1$ then $j < q + \ell_2$, contradicting the fact that $(i, j)$ is in the $\ell_2$-term. Therefore $j \geq q + \ell_1 + 1$. But then $i = p + \ell_1 < p + \ell_2$, again contradicting the fact that $(i, j)$ is in the $\ell_2$-term. Next we verify the equality. It is clear that the right-hand side is contained in the left. (We note that if $p \leq q$ then the second part of the last term will be empty.) For the reverse containment, let $(i, j) \in [p, n] \times [q, n]$. First suppose that $i - p < j - q$. Put $\ell = i - p$. Note that then $\ell \leq n - p = q$, and $j < q + \ell = \ell < \min\{p, q\}$. Thus $\ell \leq \min\{p, q\}$. Then $i = p + \ell$, and $j > q + (i - p) = q + \ell$. Therefore $(i, j) \in [p + \ell] \times [q + \ell + 1, n]$. Next suppose that $i - p \geq j - q$. Put $\ell = j - q$. Note that $\ell \leq n - q = p$, and $\ell \leq \min\{p, q\}$. Thus again we have that $\ell \leq \min\{p, q\}$. Then $j = q + \ell$, and $i \geq p + (j - q) = p + \ell$. Therefore $(i, j) \in [p + \ell, n] \times [q + \ell]$. This finishes the proof of the claim.

We use the above claim to rewrite (5.2), as the third term is indexed by $[p, n] \times [q, n]$. For definiteness we assume that $p \leq q$ (the other possibility is handled in a similar manner). Then we have

$$Z(v_m \alpha \beta \eta \alpha) = \bigcup_{\ell = 0}^{p} \left( \bigcup_{i = p + \ell}^{n} \bigcup_{r = 1}^{k_{m+i+q+\ell}} Z(v_m \alpha^{i+q+\ell} \gamma_{m+i+q+\ell}^{(r)}) \right)$$

$$\cup \left( Z(v_m \alpha^{n+1} \beta^{q+\ell}) \setminus Z(v_m \alpha^{n+1} \beta^{q+\ell+1}) \right)$$

$$\bigcup_{j = q + \ell + 1}^{n} \bigcup_{r = 1}^{k_{m+p+\ell+j}} Z(v_m \alpha^{p+\ell} \beta^{j} \gamma_{m+p+\ell+j}^{(r)})$$

$$\cup \left( Z(v_m \alpha^{p+\ell} \beta^{n+1}) \setminus Z(v_m \alpha^{p+\ell+1} \beta^{n+1}) \right)$$

$$\bigcup_{i = 2p+1}^{n} \left( Z(v_m \alpha^{i} \beta^{n+1}) \setminus Z(v_m \alpha^{i+1} \beta^{n+1}) \right)$$
Now we prepend $\eta$ to all of the sets in the above, and apply $\mu$ to each. The lefthand side becomes $\mu(Z(\eta a^n \beta^{n+1})).$ To compute the righthand side we first note that

$$m + i + q + \ell = |\eta| + 1 + i + q + \ell$$

$$= h - p - q + 1 + i + q + \ell$$

$$= h + i - p + \ell + 1;$$

$$|\eta a^i \beta^{q+\ell} \gamma_{m+i+q+\ell}^{(r)}| = m + i + q + \ell$$

$$= h + i - p + \ell + 1;$$

$$|\eta a^{n+1} \beta^{q+\ell}| = |\eta| + n + 1 + q + \ell$$

$$= h + q + \ell + 1;$$

$$m + p + \ell + j = |\eta| + 1 + p + \ell + j$$

$$= h - p - q + 1 + p + \ell + j$$

$$= h + \ell + j - q + 1;$$

$$|\eta a^{p+\ell} \beta^{j} \gamma_{m+p+\ell+j}^{(r)}| = m + p + \ell + j$$

$$= h + \ell + j - q + 1$$

$$|\eta a^{p+\ell} \beta^{n+1}| = |\eta| + p + \ell + n + 1$$

$$= h + p + \ell + 1;$$

$$|\eta a^{i} \beta^{n+1}| = |\eta| + i + n + 1$$

$$= h + i + 1;$$

$$|\eta a^{n+1} \beta^{n+1}| = |\eta| + 2n + 2$$

$$= h + n + 2.$$

Then the righthand side becomes

$$\sum_{\ell=0}^{p} \left( \sum_{i=p+\ell}^{n} k_{h+i-p+\ell+1} b_{h+i-p+\ell} + a_{h+q+\ell+1} \right.$$  

$$+ \sum_{j=q+\ell+1}^{n} k_{h+\ell+j-q+1} b_{h+\ell+j-q} + a_{h+p+\ell+1} \right)$$

$$+ \sum_{i=2p+1}^{n} a_{h+i+1} + b_{h+n+1}.$$

Recall that

$$(a_i b_i) = (a_{i+1} b_{i+1}) B_i = (a_{i+1} b_{i+1}) \left( \begin{array}{c} k_{i+1}+1 \\ k_{i+1} \end{array} \right) = (k_{i+1}+1 a_{i+1} + k_{i+1} b_{i+1} a_{i+1} + b_{i+1})$$

$$(a_{i+1} b_{i+1}) = (a_i b_i) B_i^{-1} = (a_i b_i) \left( \begin{array}{c} 1 \\ -k_{i+1} \end{array} \right) = (a_{i+1} b_{i+1} - a_i - (k_{i+1}+1) b_i).$$

From these equations we will need

$$b_i = a_{i+1} + b_{i+1}$$

$$k_{i+1} b_i = a_i - a_{i+1}.$$
Using (6.4) we have
\[
\sum_{i=p+\ell}^{n} k_{h+i-p+\ell+1}b_{h+i-p+\ell} + a_{h+q+\ell+1} = \sum_{i=p+\ell}^{n} (a_{h+i-p+\ell} - a_{h+i-p+\ell+1}) + a_{h+q+\ell+1} \\
= (a_{h+2\ell} - a_{h+n-p+\ell+1}) + a_{h+q+\ell+1} \\
= a_{h+2\ell}, \text{ since } n = p + q.
\]

Using (6.3) we have that the sum in (6.2) equals
\[
\sum_{j=q+\ell+1}^{n} k_{h+\ell+j-q+1}b_{h+\ell+j-q} + a_{h+p+\ell+1} = \sum_{j=q+\ell+1}^{n} (a_{h+\ell+j-q} - a_{h+\ell+j-q+1}) + a_{h+p+\ell+1} \\
= (a_{h+2\ell+1} - a_{h+\ell+n-q+1}) + a_{h+p+\ell+1} \\
= a_{h+2\ell+1}.
\]

Using (6.3) we have that the sum in (6.2) equals
\[
\sum_{\ell=0}^{p} (a_{h+2\ell} + a_{h+2\ell+1}) + \sum_{i=2p+1}^{n} a_{h+i+1} + b_{h+n+1} = \sum_{i=0}^{n+1} a_{h+i} + b_{h+n+1} \\
= \sum_{i=0}^{n} a_{h+i} + b_{h+n} \\
= \cdots \\
= a_{h} + b_{h} \\
= b_{h-1}.
\]

This proves that \( \mu(Z(\nu)) = \sum\{\mu(S) : S \in Q_{h}, S \subseteq Z(\nu)\} \).

Now we consider sets of the form \( Z(\nu) \setminus Z(\nu\lambda) \). First we let \( \lambda = \alpha \). Prepending \( \eta \) to the sets on the righthand side of equation (5.3) and applying \( \mu \) gives
\[
a_{h+p+1} + \sum_{j=q}^{n} k_{h+j-q+1}b_{h+j-q} = a_{h+p+1} + \sum_{j=q}^{n} (a_{h+j-q} - a_{h+j-q+1}) \\
= a_{h+p+1} + (a_{h} - a_{h+p+1}) \\
= a_{h} \\
= \mu(Z(\eta\alpha^p\beta^q) \setminus Z(\eta\alpha^{p+1}\beta^q)).
\]

The case where \( \lambda = \beta \) is similar. Finally, if \( \lambda = \gamma_{h+1}^{(r)} \) for some \( 1 \leq r \leq k_{h+1} \), the result of prepending \( \eta \), then applying \( \mu \), to the sets on the righthand side of equation (5.3) yields the corresponding result for equation (5.2) less the one term \( \mu(Z(\eta\alpha^p\beta^q\gamma_{h+1}^{(r)})) \). Using equation (6.3), this gives \( b_{h-1} - b_{h} = a_{h} = \mu(Z(\nu) \setminus Z(\nu\lambda)) \), as required. \( \square \)

**Lemma 6.9.** Let \( 0 \leq g \leq h \). Every set in \( E_{g} \) is refined by \( Q_{h} \). For \( E \in E_{g} \), \( \mu(E) = \sum\{\mu(S) : S \in Q_{h}, S \subseteq E\} \).

**Proof.** When \( g = h \) this follows from Proposition 5.6 and Lemma 6.8. We consider the case \( h = g + 1 \). Let \( \nu \in v_{1}\Lambda \) with \( |\nu| = g \). We have
\[
Z(\nu) = (Z(\nu\alpha) \setminus Z(\nu\alpha\beta)) \cup Z(\nu\beta) \cup \bigcup_{r=1}^{k_{g+1}} Z(\nu\gamma_{g+1}^{(r)}).
\]

Then clearly we have
We claim first that if \( h \in A \), then \( G \) of all compact-open subsets of \( Z(\nu) \) be constructed from the \( \mathbb{Z} \). Proof. Let \( E \) from \( \mu \) of the refining sets equals the \( \mu \) of the original set. Combining these facts finishes the proof of the lemma.

The remaining three equations all lead to the same equation involving \( \mu \)-values. For the first of these three equations, say, the righthand side gives
\[
a_{g+1} + k_{g+1} b_g = a_{g+1} + (a_g - a_{g+1})
= a_g
= \mu(Z(\nu) \setminus Z(\nu/\beta)).
\]
This proves the lemma when \( h = g + 1 \). For the general case, repeating the result just proved shows that each set in \( \mathcal{E}_g \) is refined by sets from \( \mathcal{E}_h \), and the sum of \( \mu \)-values of the refining sets equals the \( \mu \)-value of the original set. By Lemma 6.8 we know that each set in \( \mathcal{E}_h \) is refined by \( Q_h \), and the sum of the \( \mu \)-values of the refining sets equals the \( \mu \)-value of the original set. Combining these facts finishes the proof of the lemma.

Theorem 6.10. Let \( \sigma \in \mathbb{R}^+ \setminus \mathbb{Q} \) have continued fraction expansion \([c_0, c_1, \ldots]\) as in Theorem 5.12 and let \((k_i)_{i=1}^\infty\) and \( \Lambda \) be as in that theorem. There exists an invariant Borel probability measure on \( G(0) \).

Proof. Let \( \mathcal{A} \) be the algebra of subsets of \( G(0) \) generated by \( \mathcal{E} \). (Thus \( \mathcal{A} \) is the algebra of all compact-open subsets of \( G(0) \).) Let \( \mu : \mathcal{E} \to \mathbb{R} \) be defined as in Lemma 6.8. We claim first that if \( F = \bigsqcup_{i=1}^m E_i \) with \( F, E_i \in \mathcal{E} \), then \( \mu(F) = \sum_{i=1}^m \mu(E_i) \). To prove this, choose \( h \) such that \( Q_h \) refines \( F \) and all \( E_i \); \( h \) exists by Lemma 6.9. Then \( \{S \in Q_h : S \subseteq F\} = \bigsqcup_{i=1}^m \{S \in Q_h : S \subseteq E_i\} \). Then Lemma 6.9 implies that
\[
\mu(F) = \sum_{S \in Q_h} \mu(S) = \sum_{i=1}^m \sum_{S \in Q_h} \mu(S) = \sum_{i=1}^m \mu(E_i).
\]
Next, observe that every set in \( \mathcal{A} \) can be written as a (finite) disjoint union of sets from \( \mathcal{E} \). To see this, consider \( A \in \mathcal{A} \). There are \( \nu_1, \ldots, \nu_n \in v_1 \Lambda \) such that \( A \) can be constructed from the \( Z(\nu_i) \) by intersection, union and difference. By Lemma 6.9
there is \( h \) such that \( Z(\nu_1), \ldots, Z(\nu_\mu) \) are all refined by \( Q_h \). Then any combination of the \( Z(\nu_\mu) \) using intersection, union and difference, e.g., \( A \), will equal the union of a subcollection of \( Q_h \). It follows that \( A = \bigsqcup \{ S \in Q_h : S \subseteq A \} \). We claim that 
\( \sum \{ \mu(S) : S \in Q_h, S \subseteq A \} \) is independent of \( h \), for \( h \) large enough that \( Q_h \) refines \( A \). To see this, let \( h_1 \) and \( h_2 \) both be large, and let \( h_3 \geq \max\{h_1, h_2\} \). By Lemma 6.9, each set in \( Q_{h_1} \cup Q_{h_2} \) is refined by \( Q_{h_3} \), and the \( \mu \)-value of the set equals the sum of the \( \mu \)-values of the refining sets. We have

\[
\sum_{S \in Q_{h_3}} \mu(S) = \sum_{S \in Q_{h_1}} \sum_{T \subseteq S} \mu(T) = \sum_{T \subset Q_{h_1}} \mu(T) = \cdots = \sum_{S \in Q_{h_3}} \mu(S).
\]

Now for \( A \subseteq A \) we define \( \mu(A) = \sum \{ \mu(S) : S \in Q_h, S \subseteq A \} \) for any \( h \) large enough so that \( A \) is refined by \( Q_h \). The previous claim shows that this is independent of \( h \). Now we show that \( \mu \) is finitely additive on \( A \). Let \( A_1, \ldots, A_m \in A \) be pairwise disjoint. Choose \( h \) large enough that \( A_1, \ldots, A_m \) are all refined by \( Q_h \). Then

\[
\mu(\bigsqcup_{i=1}^m A_i) = \sum_{S \in Q_h} \mu(S) = \sum_{i=1}^m \sum_{S \in Q_h} \mu(S) = \sum_{i=1}^m \mu(A_i).
\]

Now, since all sets in \( A \) are compact, \( \mu \) is actually countably additive on \( A \), i.e. \( \mu \) is a premeasure on \( A \). By Caratheodory’s theorem, \( \mu \) extends (uniquely) to a measure on the Borel sets of \( G^{(0)} \).

Finally, we prove that \( \mu \) is invariant. Let \( \Delta \subseteq G \) be a compact-open bisection. For \( g \in \Delta \) there are \( \nu_1, \nu_2 \in v_1\Lambda \) with \( |\nu_1| = |\nu_2| \), and a compact-open set \( E \subseteq s(\nu_1)\Lambda \) such that \( \nu_2 E \subseteq s(\Delta) \) and \( g \in [\nu_1, \nu_2, E] \subseteq \Delta \). Choose \( h \) so that \( \nu_2 E \) is refined by \( Q_h \). If \( S \in Q_h \) with \( S \subseteq \nu_2 E \) then \( S \) is of one of the forms \( Z(\nu_2 \eta) \) or \( Z(\nu_2 \eta) \cap Z(\nu_2 \eta) = 1 \). Then \( \Delta S \Delta^{-1} \) has one of the forms \( Z(\nu_1 \eta) \), respectively \( Z(\nu_1 \eta) \cap Z(\nu_1 \eta) \). In either case, \( \mu(\Delta S \Delta^{-1} = \mu(S) \). Since \( \Delta \) is compact we may write \( \Delta = \bigsqcup_{i=1}^m [\nu_1, \nu_2, E] \) with each term as above. Then we may choose \( h \) so that all \( \nu_2 E_i \) are refined by \( Q_h \). It now follows from the above calculation that \( \mu(r(\Delta)) = \mu(s(\Delta)) \). Therefore \( \mu \) is invariant.

7. Identifying \( C^*(G) \)

In their fundamental paper \cite{16} introducing the continued fraction AF algebras, Effros and Shen do not actually identify a specific \( C^* \)-algebra corresponding to an irrational number. In \cite{16} Section VI.3], Davidson does make such an identification.

**Definition 7.1.** (\cite{16} Section 10], \cite{16} Section 3], \cite{16} Section VI.3])

Let \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) have simple continued fraction expansion \( \theta = [g_0, g_1, g_2, \ldots] \). Let the \( n \)th convergent be \( \frac{p_n}{q_n} \). (Thus \( p_n = g_0, p_1 = g_0 g_1 + 1, q_0 = 1, q_1 = g_1 \), and for \( n \geq 2, p_n = g_n p_{n-1} + p_{n-2} \) and \( q_n = g_n q_{n-1} + q_{n-2} \).) For \( n \geq 0 \) let \( A_n = M_{q_n} \oplus M_{q_{n-1}} \) (where we let \( A_0 = M_{q_0} \oplus 0 \), and for \( n \geq 1 \) we include \( A_{n-1} \hookrightarrow A_n \) with partial multiplicities given by \( \left( \begin{array}{c} g_n \\ 0 \\ 1 \end{array} \right) \)). The Effros Shen algebra \( A_\theta \) is the AF algebra \( \bigcup_{n \geq 1} A_n \).

It is proved in \cite{16} VI.3] that \( (K_0(A_\theta), K_0(A_\theta)_+, [1]_0) \cong (\mathbb{Z}\theta + \mathbb{Z}, P_\theta, (\theta)) \). Moreover, if \( \tau \) is the unique tracial state on \( A_\theta \), then \( \tau_n(K_0(A_\theta)_+) = (\mathbb{Z}\theta + \mathbb{Z})_+ = (\theta)P_\theta \).
Let $\theta$ be a positive irrational number, with simple continued fraction expansion $[c_0, c_1, \ldots]$. Let $(k_i)_{i \geq 1}$ be the sequence of nonnegative integers defined in Theorem 7.2. Let $A$ be the (non-simple) continued fraction expansion $[0, 1, k_1, 1, k_2, 1, \ldots]$.

Thus $A$ is isomorphic to the Effros Shen algebra $A_\theta$. Since the continued fraction expansion of $\theta$, the Bratteli diagram for $A$ equivalent to $A_\theta$, the Bratteli diagram for $A_\sigma$ is a tail of that of $\theta$, the Bratteli diagram for $A_\sigma$ is a tail of that for $A_\theta$, except for the dimensions of the full matrix algebras at the vertices. It follows that tensoring with the algebra of compact operators renders the diagram for $A_\sigma$ identical to the tail of the diagram for $A_\theta$.

Broadly speaking, the proof of Theorem 7.2 is given in two steps: first showing that $C^*(G)$ is classified by its Elliott invariant, $\text{Ell}(C^*(G))$, and then by completing the computation of $\text{Ell}(C^*(G))$. To show that it is classifiable we must show that it is infinite dimensional, separable, unital, simple, nuclear, satisfies the UCT, and has finite nuclear dimension.

**Lemma 7.4.** $C^*(G)$ has nuclear dimension at most three.

**Proof.** Recall the short exact sequences 4.1 and 4.2

$$
0 \rightarrow C^*(G_i|_{F_i}) \rightarrow C^*(G_i) \rightarrow C^*(G_i|_{F_i}) \rightarrow 0
$$

$$
0 \rightarrow C^*(G_i|_{F_i^0}) \rightarrow C^*(G_i|_{F_i^0}) \rightarrow C^*(G_i|_{F_i^0}) \rightarrow 0.
$$

By Corollary 4.16 $C^*(G_i|_{F_i^0})$ is AF and therefore 

$$\dim_{\text{nuc}} C^*(G_i|_{F_i^0}) = 0$$

by [34] Remark 2.2(iii)]. Corollary 4.12 shows $C^*(G_i|_{F_i^0}) \cong M_{\phi_i\times\psi_i} \otimes C(\mathbb{T})$ so that

$$\dim_{\text{nuc}} C^*(G_i|_{F_i^0}) = 1$$
by [34, Proposition 2.4 and Corollary 2.8(i)]. Then by [34, Proposition 2.9],
\[ \dim_{nuc} C^*(G_i|F_i) \leq 2. \]
As in the proof of Proposition 4.8, \( C^*(G_i|U_i) \) is AF and hence
\[ \dim_{nuc} C^*(G_i|U_i) = 0, \]
again by [34, Remark 2.2(iii)], so that
\[ \dim_{nuc} C^*(G_i) \leq 3 \]
by another application of [34, Proposition 2.9]. Since the above holds for all \( i > 0 \),
[34, Proposition 2.3(iii)] implies
\[ \dim_{nuc} C^*(G) \leq 3. \]
\[ \square \]

To show that \( C^*(G) \) is simple, we will show that \( G \) has the following two properties:

Definition 7.5. Let \( \Gamma \) be a groupoid. We say that \( \Gamma \) is topologically free if the set \( \{ x \in \Gamma^{(0)} : x \Gamma x = \{ x \} \} \) is dense in \( \Gamma^{(0)} \). We say that \( \Gamma \) is minimal if for every \( x \in \Gamma^{(0)} \), the orbit of \( x \) is dense in \( \Gamma^{(0)} \). We note that if \( \Gamma \) is minimal, and if there exists a point in \( \Gamma^{(0)} \) with trivial isotropy, then \( \Gamma \) is topologically free.

Lemma 7.6. \( G \) is topologically free and minimal.

Proof. We first show that \( G \) is minimal. Let \( x \in X \) and let \( U \) be a nonempty open subset of \( X \). We will find \( g \in G \) such that \( s(g) = x \) and \( r(g) \in U \). By Proposition 5.6 we may assume that \( U \) has one of the forms \( Z(\mu), Z(\mu) \setminus Z(\mu\gamma) \). We give the proof for the third of these; the proofs of the other two situations are similar. So we assume that \( U = Z(\mu) \setminus Z(\mu\beta) \). Let \( i \geq |\mu| \) be such that \( k_i > 0 \). Let \( g = [\mu\alpha^i|\mu\gamma_i(1), x_1x_2\cdots x_i, x_{i+1}x_{i+2}\cdots] \). Then \( s(g) = x \) and \( r(g) = \mu\alpha^i|\mu\gamma_i(1)x_{i+1}x_{i+2}\cdots \in Z(\mu) \setminus Z(\mu\beta) \).

Since \( G \) is minimal, to prove that \( G \) is topologically free it suffices to exhibit a point of \( x \) with trivial isotropy. Note that if \( x \) has nontrivial isotropy in \( G \) then for some \( i \), \( x \) has nontrivial isotropy in \( G_i \). It then follows from Lemmas 4.3, 4.10 and 4.13 that only the points of \( \bigcup_i F_i^\infty = \{ \mu\alpha^\infty\beta^\infty : \mu \in \Lambda \} \) have nontrivial isotropy. Therefore all other points of \( X \) have trivial isotropy.

Corollary 7.7. \( C^*(G) \) is a simple \( C^* \)-algebra.

Proof. This follows from Lemma 7.6 and [23, Corollary 4.6].

Lemma 7.8. \( C^*(G) \) is separable and unital.

Proof. Separability follows from the fact that \( \Lambda \) is countable. \( C^*(G) \) is unital since \( G^{(0)} \) is compact.

Corollary 7.9. \( C^*(G) \) is classifiable.

Proof. By Lemmas 7.4 and 7.8 Corollary 7.7 and Remark 5.2 we know that \( C^*(G) \) is separable, unital, simple, nuclear, satisfies the UCT, and has finite nuclear dimension. It is clearly infinite dimensional. By [29, Corollary D], \( C^*(G) \) is classified by its Elliott invariant.

Theorem 7.10. There exists a unique trace on \( C^*(G) \).
Proof. By Theorems 6.6 and 6.10 there exists a unique invariant Borel probability measure, \( \mu \), on \( G^{(0)} \). Let \( E : C^*(G) \to C_0(G^{(0)}) \) be the canonical conditional expectation. Then \( \mu \) gives rise to a state \( \phi \) on \( C^*(G) \) by taking
\[
\phi(f) = \int_{G^{(0)}} f \circ E \, d\mu
\]
for \( f \in C_c(G) \). Since \( \mu \) is invariant, \( \phi \) is tracial.

To see that \( \phi \) is the unique tracial state on \( C^*(G) \), we show that the points of \( G^{(0)} \) with non-trivial isotropy form a set of measure zero, and apply [18, Corollary 1.2]. Suppose \( x, y \in G^{(0)} \) have non-trivial isotropy. As in the proof of Lemma 7.6, \( x = \sigma \alpha^{\infty} \beta^{\infty} \) and \( y = \tau \alpha^{\infty} \beta^{\infty} \) for some \( \sigma, \tau \in \Lambda \). Supposing, say, \( |\tau| \geq |\sigma| \), the element \( [\sigma \alpha^{\infty} | - \sigma |, \tau, \alpha^{\infty} \beta^{\infty}] \) has range \( x \) and source \( y \). Thus the points of \( G^{(0)} \) having nontrivial isotropy form a single countably infinite orbit. By invariance \( \mu(\{x\}) = \mu(\{y\}) \), and since \( \mu \) is a finite measure it must be that \( \mu(\{x\}) = 0 \). \( \square \)

Proof. (of Theorem 7.2) By Corollary 7.9 we know that \( C^*(G) \) is in the class of \( C^* \)-algebras classified by [29, Corollary D]. By Theorem 5.12, \( C^*(G) \) has ordered \( K \)-theory with position of the unit \((K_0(C^*(G)), K_0(C^*(G))^+, [1_0]_0) \cong (\mathbb{Z}^2, P_\sigma, (k_{11} + 2)) \).

Recall from the proof of Theorem 5.9 the maps \( \eta_{i, i'} : C^*(G_i) \to C^*(G_{i'}) \) and \( \eta_i : C^*(G_i) \to C^*(G) \). We thus have the diagram
\[
\begin{array}{cccccc}
\mathbb{Z}^2 & \xrightarrow{T} & \mathbb{Z}^2 & \xrightarrow{B_0} & \mathbb{Z}^2 & \xrightarrow{B_1} & \mathbb{Z}^2 & \xrightarrow{B_2} & \cdots & \xrightarrow{\cong} & \mathbb{Z}^2,
\end{array}
\]

where \( T \) is the composition \( \mathbb{Z}^2 \xrightarrow{(0 \ 1)} \mathbb{Z}^2 \xrightarrow{(1 \ 1)} \mathbb{Z}^2 \xrightarrow{(k_{11} + 2)} \mathbb{Z}^2 \xrightarrow{(1 \ 1)} \mathbb{Z}^2 \xrightarrow{(k_{21} + 1)} \mathbb{Z}^2 \cdots \).

We claim that \( B_0 T \) is an isomorphism from \((\mathbb{Z}^2, P_\sigma, (\frac{1}{1} ))\) to \((\mathbb{Z}^2, P_\sigma, (k_{11} + 2))\). Since \( T, B_0 \in GL(2, \mathbb{Z}) \), \( B_0 T \) is a isomorphism on \( \mathbb{Z}^2 \), and an easy calculation shows that \( B_0 T(\frac{1}{1}) = (k_{11} + 2) \). To see that \( B_0 T(P_\theta) = P_\sigma \), we first note the following fact. Let \( s = [d_0, d_1, t] \), where \( d_0 \in \mathbb{Z} \), \( d_1 \in \mathbb{N} \), and \( t \in (0, \infty) \setminus \mathbb{Q} \). Let \( s' = [d_1, t] \). Then \( (\frac{d_0}{1} \ 0)(P_s) = P_{s'} \). For the proof, we have
\[
\begin{align*}
\left( \frac{d_0}{1} \ 0 \right) (P_s) &= \left\{ \left( \frac{m}{n} \right) : \left( s \frac{1}{1} \right) \left( \frac{m}{n} \right) \geq 0 \right\} \\
&= \left\{ \left( \frac{m}{n} \right) : \left( s \frac{1}{1-d_0} \right) \left( \frac{m}{n} \right) \geq 0 \right\} \\
&= \left\{ \left( \frac{m}{n} \right) : \left( s \frac{1}{s-d_0} \right) \left( \frac{m}{n} \right) \geq 0 \right\} \\
&= \left\{ \left( \frac{m}{n} \right) : \left( s \frac{1}{s-d_0} \right) \left( \frac{m}{n} \right) \geq 0 \right\} , \text{ since } s - d_0 > 0, \quad (\frac{1}{s-d_0})
\end{align*}
\]

Since \( s - d_0 = [0, d_1, t] \), we have \( \frac{1}{s-d_0} = [d_1, t] = s' \). With this fact in hand, we see that
\[
B_0 T(P_\theta) = \left( \frac{1}{1} \right) \left( \frac{k_{11} + 2}{1} \right) \left( \frac{1}{1} \right) \left( \frac{0 \ 1}{1 \ 0} \right) (P_{[0, 1, k_{11}, 1, \sigma]}) = P_\sigma.
\]
Moreover, \((\mathbb{Z}^2, P_\theta, (\xi))\) is a simple dimension group, with unique state \(\rho\) given by \(\rho((\xi)) = \theta\) and \(\rho((\xi)) = 1\). Since \(\tau_*\) does the same, \( \tau_* = \rho \). Thus \(C^*(G)\) and \(A_\theta\) have the same Elliott invariant. By [29, Theorem D] it follows that \(C^*(G) \cong A_\theta\). \(\square\)

**Remark 7.11.** We have that \(\theta = [0, 1, k_1, 1, k_2, \ldots] = [0, 1, k_1, 1, \sigma]\) in the notation introduced before Lemma 6.3. Since \(k_1 \in \mathbb{N}\), and \(\sigma \in (0, \infty) \setminus \mathbb{Q}\) are arbitrary, a short calculation with a finite continued fraction shows that \(\frac{k_1}{k_1 + 1} < \theta < \frac{k_1 + 1}{k_1 + 2}\). Thus \(A_\theta\) can be realized in the form \(C^*(G)\) for any \(\theta \in (0, 1) \setminus \mathbb{Q}\).

**Corollary 7.12.** For each irrational number \(\theta \in (0, 1)\), the Effros Shen algebra \(A_\theta\) contains a Cartan subalgebra that is not conjugate to the standard diagonal subalgebra \(D_\theta\) (as in [28]).

**Proof.** As shown in the proof of Lemma 7.6, \(G\) is not principal but is topologically free (or topologically principal). By [24, Proposition 5.11], \((C^*(G), C(G^{(0)}))\) is a Cartan pair not having the unique extension property, whereas \((A_\theta, D_\theta)\) is a Cartan pair that does have that property. Therefore \(C(G^{(0)})\) and \(D_\theta\) are nonconjugate Cartan subalgebras in \(A_\theta \cong C^*(G)\). \(\square\)

**Remark 7.13.** It is shown in [15] that every classifiable \(C^*\)-algebra contains infinitely many pairwise nonisomorphic Cartan subalgebras. The proof shows that all but one of these have spectra with positive dimension, and in fact, the dimension tends to infinity. The examples we give are of a second nonconjugate Cartan subalgebra having zero dimensional spectrum. Both are isomorphic to the continuous functions on a Cantor space, and hence are isomorphic. Thus we prove that the Effros Shen algebras contain two isomorphic but nonconjugate Cartan subalgebras.

## 8. A locally finite subalgebra of \(C^*(G)\)

In this section we show that \(C^*(G)\) has an isomorphic subalgebra presented as an AF algebra in the usual way. Recall the set \(\Phi_i\) from Definition 4.9. We note that the sets \(E_i\) and \(F_i\) in the next definition (and the rest of this section) are unrelated to the sets of Definition 4.11.

**Definition 8.1.** For \(i \geq 0\) let

\[
E_i = \{\mu \in v_1 \Lambda : |\mu| = i\}
\]

\[
F_i = \{\eta \beta^q : \eta \in \Phi_i, |\eta| + q = i + 1\}
\]

\[
A_i = \{Z(\mu) \setminus Z(\mu \beta) : \mu \in E_i\}
\]

\[
B_i = \{Z(\mu) : \mu \in F_i\}.
\]

**Lemma 8.2.** For each \(i \geq 0\),

1. \(A_i \cup B_i\) is a partition of \(X\),
2. \(A_{i+1} \cup B_{i+1}\) refines \(A_i \cup B_i\). (Here the word refine is used in the usual way for a pair of partitions of a set.)

**Proof.** Fix \(i \geq 0\). We first show that \(A_i \cup B_i\) is pairwise disjoint. Let \(\mu, \nu \in E_i\) be distinct. Write \(\mu = \xi \alpha^p \beta^{i-|\xi|-p}\) and \(\nu = \eta \alpha^q \beta^{i-|\eta|-q}\), where \(\xi, \eta \in \Phi_i\). If \(|\xi| = i\) or \(|\eta| = i\) then it is clear that \(Z(\mu) \cap Z(\nu) = \emptyset\). So we may suppose that \(|\xi|, |\eta| < i\).
Suppose there exists $x \in (Z(\mu) \setminus Z(\mu \beta)) \cap (Z(\nu) \setminus Z(\nu \beta))$. Letting $\xi = \theta_1 \cdots \theta_k$ and $\eta = \phi_1 \cdots \phi_\ell$ in normal form, we have that the normal form of $x$ is

$$
x = \theta_1 \cdots \theta_k \alpha^{p'} \beta^{i - \lfloor \xi \rfloor - p} \theta_{k+2} \cdots
= \phi_1 \cdots \phi_\ell \alpha^{q'} \beta^{i - \lfloor \eta \rfloor - q} \phi_{\ell+2} \cdots,
$$

where $p \leq p' \leq \infty$ and $q \leq q' \leq \infty$. Then the uniqueness of the normal form implies that $\xi \alpha^{p'} \beta^{i - \lfloor \xi \rfloor - p} = \eta \alpha^{q'} \beta^{i - \lfloor \eta \rfloor - q}$. Then $\xi = \eta$ and $i - \lfloor \xi \rfloor - p = i - \lfloor \eta \rfloor - q$, hence $p = q$, hence $\mu = \nu$. Therefore $A_i$ is a pairwise disjoint family. Now let $\mu, \nu \in F_i$. Write $\mu = \theta_1 \cdots \theta_m \beta^p$ and $\nu = \phi_1 \cdots \phi_n \beta^q$ in normal form, with $p, q \geq 1$.

Suppose that $Z(\mu) \cap Z(\nu) \neq \emptyset$, i.e. that $\mu \cap \nu$. Without loss of generality let $m \leq n$. By Lemma 3.3, $\phi_j = \theta_j$ for $j \leq m$, and $\beta^p \cap \phi_{m+1} \cdots \phi_n \beta^q$. If $m < n$ then we must have $\beta^p \in [\phi_{m+1}]$, and this contradicts the fact that $|\mu| = i = |\nu|$. Thus $m = n$. Then again since $|\mu| = |\nu|$ it follows that $p = q$, hence that $\mu = \nu$. Thus $B_i$ is pairwise disjoint. Finally, let $\mu \in E_i$ and $\nu \in F_i$. Suppose that there exists $x \in (Z(\mu) \setminus Z(\mu \beta)) \cap Z(\nu)$. Write $\nu = \eta \beta^q$ with $\eta \in \Phi_i$ and $q > 0$. Since $\mu \cap \nu$, and $|\mu| < |\nu|$, we must have $\mu = \eta \alpha \beta^{i - \lfloor \eta \rfloor - p}$, with $p \leq i - |\eta|$. Since $x \not\in Z(\mu \beta)$ we must have $x = \eta \alpha \beta^{i - \lfloor \eta \rfloor - p} y$, where $y \in \partial \Lambda$ either equals $\infty$ or begins in $A_2$, and $p \leq p'$. But $i - |\eta| - p \leq i - |\eta| = q - 1$, so $x \not\in Z(\nu)$, a contradiction. Thus no such $x$ can exist, and it follows that $A_i \cup B_i$ is a pairwise disjoint family.

Next we show that $X$ equals the union of the family of sets $A_i \cup B_i$. Let $x \in X$.

First we consider the case that the normal form of $x$ does not end with an element of $A_\infty^\infty$. Write $x = \eta \alpha \beta^q y$, where $\eta \in \Phi_i$, $|\eta| + p + q \geq i$, and $y$ begins in $A_2$. If $|\eta| + q \leq i$, then $\xi := \eta \alpha^{i - |\eta| - q} \beta^q \in E_i$, and $x \in Z(\xi) \setminus Z(\xi \beta)$, an element of $A_i$. Suppose that $q > i - |\eta|$. Then letting $\xi := \eta \beta^{i - |\eta| + 1} \in x$ we have $\xi \in F_i$. Then $x \in Z(\xi)$, an element of $B_i$. Secondly we consider the case that $x = \eta \alpha \beta^q y$, where $\eta \in \Phi_i$ and $p + q = \infty$. If $q \leq i - |\eta|$, let $\xi = \eta \alpha^{i - |\eta| - q} \beta^q \in E_i$. Then $x \in Z(\xi) \setminus Z(\xi \beta)$, an element of $A_i$. If $q > i - |\eta|$, let $\xi = \eta \beta^{i - |\eta| + 1} \in F_i$. Then $x \in Z(\xi)$, an element of $B_i$.

Finally we show that $A_{i+1} \cup B_{i+1}$ refines $A_i \cup B_i$. First let $\mu \in E_i$. For $1 \leq r \leq k_{i+1}$ we have that $\mu \gamma_{i+1}^{(r)} \in E_{i+1}$ and $\mu \gamma_{i+1}^{(r)} \beta \in F_{i+1}$. Then $Z(\mu \gamma_{i+1}^{(r)}) \setminus Z(\mu \gamma_{i+1}^{(r)} \beta) \subset A_{i+1}$, $Z(\mu \gamma_{i+1}^{(r)} \beta) \subset B_{i+1}$, and

$$
Z(\mu) \setminus Z(\mu \beta) = (Z(\mu \alpha) \setminus Z(\mu \alpha \beta))
\cup \left( \bigcup_{r=1}^{k_{i+1}} (Z(\mu \gamma_{i+1}^{(r)}) \setminus Z(\mu \gamma_{i+1}^{(r)} \beta)) \right)
\cup \left( \bigcup_{r=1}^{k_{i+1}} Z(\mu \gamma_{i+1}^{(r)} \beta) \right).
$$

Thus $A_{i+1} \cup B_{i+1}$ refines all elements of $A_i$. Next let $\mu \in F_i$. Then $\mu \beta \in F_{i+1}$, so $Z(\mu \beta) \subset B_{i+1}$. Moreover $\mu \in E_{i+1}$, so $Z(\mu) \setminus Z(\mu \beta) \subset A_{i+1}$. Since $Z(\mu) = (Z(\mu) \setminus Z(\mu \beta)) \cup Z(\mu \beta)$, we have that $A_{i+1} \cup B_{i+1}$ refines all elements of $B_i$. □
Lemma 8.3. For $i \geq 0$,

$$E_{i+1} = F_i \cup \left( \bigsqcup_{r=1}^{k_{i+1}} E_{i+1}^{(r)} \right) \cup E_i \alpha$$

$$F_{i+1} = F_i \beta \cup \left( \bigsqcup_{r=1}^{k_{i+1}} E_i \gamma_i^{(r)} \right).$$

Proof. Both containments of the right side in the left are clear. For the reverse containments, first let $\mu \in E_{i+1}$. Write $\mu = \eta \alpha^p \beta^{i+1-|\eta|} - p$, where $\eta \in \Phi_{i+1}$ and $p \leq i + 1 - |\eta|$. If $|\eta| = i + 1$ then the last edge in $\mu$ is in $A_2$, i.e. $\mu \in \bigcup_{r=1}^{k_{i+1}} E_i \gamma_i^{(r)}$. If $|\eta| \leq i$ and $p > 0$ then $\mu \in E_i \alpha$. If $|\eta| \leq i$ and $p = 0$ then $\mu \in F_i \beta$, proving the first equality. For the second, let $\mu \in F_{i+1}$. Then $\mu = \eta \beta^q$ with $\eta \in \Phi_{i+1}$ and $q = i + 2 - |\eta|$. If $q \geq 2$, then $\mu \in F_i \beta$. If $q = 1$, then $|\eta| = i + 1$. Therefore $\eta \in \bigcup_{r=1}^{k_{i+1}} E_i \gamma_i^{(r)}$, and hence $\mu \in \bigcup_{r=1}^{k_{i+1}} E_i \gamma_i^{(r)}$. \hfill \Box

Theorem 8.4. There is a sequence $C_0 \subseteq C_1 \subseteq \cdots$ of finite dimensional $C^*$-subalgebras of $C^*(G)$ with $C_i \cong M_{|E_i|} \oplus M_{|F_i|}$. $C_i := \bigcup_{\alpha=0}^{\infty} C_i$ is isomorphic to $C^*(G)$, and the inclusion $C \hookrightarrow C^*(G)$ induces an isomorphism of Elliott invariants.

Proof. Fix $i \geq 0$. Lemma 8.2 implies that for $\mu, \nu \in E_i$, $\chi_{[\nu, \mu, Z(\beta)]}$ is a partial isometry in $C^*(G)$ with initial and final projections equal to $\chi_{Z(\mu) \setminus Z(\mu \beta)}$ and $\chi_{Z(\nu) \setminus Z(\nu \beta)}$, and that for $\xi, \eta \in F_i$, $\chi_{[\nu, \xi, Z(\nu \beta)]}$ is a partial isometry in $C^*(G)$ with initial and final projections $\chi_{Z(\xi) \setminus Z(\xi \beta)}$. It is clear that the span of these partial isometries is a finite dimensional $C^*$-algebra $C_i \subseteq C^*(G)$, and that these partial isometries are matrix units defining an isomorphism of $C_i$ with $M_{|E_i|} \oplus M_{|F_i|}$. By Lemma 8.2 it follows that $C_i \subseteq C_{i+1}$. Lemma 8.3 implies that for $\xi \in E_i$, $Z(\xi) \setminus Z(\xi \beta)$ is the disjoint union of $k_{i+1} + 1$ sets of the form $Z(\mu) \setminus Z(\mu \beta)$ with $\mu \in E_{i+1}$ and one set of the form $Z(\nu)$ with $\nu \in F_{i+1}$. Similarly, for $\eta \in F_i$, $Z(\eta)$ is the disjoint union of $k_{i+1}$ sets of the form $Z(\mu) \setminus Z(\mu \beta)$ with $\mu \in E_{i+1}$ and one set of the form $Z(\nu)$ with $\nu \in F_{i+1}$. Thus the matrix of multiplicities of the embeddings of the summands of $C_i$ into the summands of $C_{i+1}$ are given by the matrix $B_i$ from Lemma 5.7. Therefore $C$ is isomorphic to the Effros Shen algebra $A_0$ from Theorem 7.2 and thus also to $C^*(G)$. Now it follows from Theorem 4.19 that the inclusion induces an isomorphism of Elliott invariants. \hfill \Box

Remark 8.5. The partitions defined in Lemma 8.2 are simpler than those defined in Lemma 5.3 and Proposition 5.6. However the more complicated partitions are necessary to prove the results of the earlier sections. The reason is that the partitions of Lemma 8.2 do not refine all cylinder sets. To see this we consider $Z(\alpha)$. For each $i$, $Z(\beta_i^{i+1}) \in B_i$. But $Z(\beta_i^{i+1}) \cap Z(\alpha) = Z(\alpha \beta_i^{i+1}) \neq \emptyset$, and also $Z(\beta_i^{i+1}) \setminus Z(\alpha)$ is nonempty, since for example $\beta_\infty$ is in it. Thus none of the partitions $A_i \cup B_i$ refines $Z(\alpha)$. One consequence of this is the following.

Theorem 8.6. The AF subalgebra $C$ in Theorem 8.4 is proper.

Proof. We will show that $\chi_{Z(\alpha)} \notin C$. Let $\text{diag} C_i$ denote the diagonal subalgebra of $C_i$, i.e. the span of $\{ \chi_D : D \in A_i \cup B_i \}$. Then the diagonal of $C$, $\text{diag} C$, is the closure of $\bigcup_i \text{diag} C_i$, and the closure occurs in $C(G^{(0)})$, with the uniform norm. Since $\text{diag} C$ is a masa in $C$, and commutes with $\chi_{Z(\alpha)}$, it suffices to show that $\chi_{Z(\alpha)} \notin \text{diag} C$.\hfill \Box
Suppose to the contrary that \( \chi_{Z(\alpha)} \in \text{diag} C \). Then there are \( i, D_1, \ldots, D_n \in A_i \cup B_i \), and \( c_1, \ldots, c_n \in \mathbb{C} \) such that \( f = \sum_{j=1}^{n} c_j \chi_{D_j} \in \text{diag} C \) satisfies \( \| \chi_{Z(\alpha)} - f \|_u < \frac{1}{2} \). If \( j \) is such that \( Z(\alpha) \cap D_j \neq \emptyset \), let \( x \in Z(\alpha) \cap D_j \). Since \( D_1, \ldots, D_n \) are pairwise disjoint we have \( x \not\in D_{\ell} \) for \( \ell \neq j \). Then

\[
\frac{1}{2} > |\chi_{Z(\alpha)}(x) - c_j \chi_{D_j}(x)| = |1 - c_j|.
\]

If \( j \) is such that \( D_j \setminus Z(\alpha) \neq \emptyset \), let \( x \in D_j \setminus Z(\alpha) \). Again, \( x \not\in D_{\ell} \) for \( \ell \neq j \), and we have

\[
\frac{1}{2} > |\chi_{Z(\alpha)}(x) - c_j \chi_{D_j}(x)| = |0 - c_j| = |c_j|.
\]

Thus the same value of \( j \) cannot satisfy both properties. It follows that for each \( j \) either \( D_j \subseteq Z(\alpha) \) or \( D_j \cap Z(\alpha) = \emptyset \). Since \( A_i \cup B_i \) does not refine \( Z(\alpha) \), Remark 8.5 implies that there exists \( x \in Z(\alpha) \setminus \cup\{D_j : D_j \subseteq Z(\alpha)\} \). Then \( \chi_{D_j}(x) = 0 \) for all \( j \), and hence

\[
\| \chi_{Z(\alpha)} - f \|_u \geq |\chi_{Z(\alpha)}(x) - \sum_{j=1}^{n} c_j \chi_{D_j}(x)| = |1 - 0| = 1,
\]
a contradiction. \( \square \)

9. Stability of \( C^*(G(\Lambda)) \)

In this section we fix the sequence \((k_i)_{i=1}^{\infty}\). We noted before Definition 6.14 that \( C^*(G) \) is Morita equivalent to \( C^*(\Lambda) \). In this section we investigate \( C^*(\Lambda) \). Since \( C^*(\Lambda) \) is an AF algebra, we identify it by means of the scale it determines in \( K_0(C^*(\Lambda))_+ = (\mathbb{Z}\theta + \mathbb{Z})_+ \). (The scale is the image in \( K_0 \) of the projections actually in the algebra \( \mathbb{I} \).) We know, for example, that the scale determined by \( C^*(G) \) is \((\mathbb{Z}\theta + \mathbb{Z}) \cap [0, 1]\). In \( C^*(\Lambda) \) the projections \( \chi_{[v_i, v_{i+2}, z(v_{i+1})]} \), \( n \geq 1 \), are pairwise orthogonal, and their partial sums form an approximate identity. Therefore the scale equals \( \{ t \in (\mathbb{Z}\theta + \mathbb{Z})_+ : t < \sum_{i=1}^{n} [\chi_{[v_i, v_{i+1}, z(v_{i+1})]}]_0 \}, \) for some \( n \).

**Lemma 9.1.** \( \chi_{[v_i, v_{i+2}, z(v_{i+1})]} \) and \( \chi_{[v_i, v_{i+1}, z(v_{i+1})]} \) are equivalent projections in \( C^*(\Lambda) \).

**Proof.** The partial isometry \( \chi_{[v_i, v_{i+2}, z(v_{i+1})]} \) implements the claimed equivalence. \( \square \)

Recall from the proof of Proposition 6.3 that we let \( b_i = |\mu(z^{i+1})| \in \mathbb{Z}\theta + \mathbb{Z} \). Thus Lemma 9.1 implies that \( \{ [\chi_{[v_i, v_{i+1}, z(v_{i+1})]}]_0 : n \geq 2, \} \) and \( \{ [\chi_{[v_i, v_{i+1}, z(v_{i+1})]}]_0 : n \geq 1 \} \).

**Theorem 9.2.** For \( i \geq 0 \) let \( \theta_i = [0, 1, k_{i+1}, 1, k_{i+2}, 1, \ldots] \). The scale of \( K_0(C^*(\Lambda)) \) equals the set \( \{ t \in (\mathbb{Z}\theta + \mathbb{Z}) : 0 \leq t < 1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} (1 - \theta_i) \} \).

**Proof.** From the above remarks, and Lemma 9.1 the theorem will follow if we show that \( b_n = \prod_{i=0}^{n-1} (1 - \theta_i) \). Recalling the proof of Proposition 6.3 we have \( a_0 = \theta_0 \), \( b_0 = 1 - \theta_0 \), and also \( a_i b_i = (a_{i+1} b_{i+1}) B_i \), \( i \geq 0 \), where \( B_i = \left( \begin{array}{c} k_{i+2} \ 1 \\ k_{i+1} \ 1 \end{array} \right) \). We defined \( a_i^{(1)} \) and \( b_i^{(1)} \), \( i \geq 1 \), by

\[
(a_{i+1} b_i) = (1 - a_0) (a_i^{(1)} b_i^{(1)})
\]

\[
(a_i^{(1)} b_i^{(1)}) = (a_{i+1}^{(1)} b_{i+1}^{(1)}) B_i.
\]

It follows from the proof of Proposition 6.3 that \( a_1^{(1)} = \theta_1 \) and \( b_1^{(1)} = 1 - \theta_1 \). Therefore \( b_1 = (1 - \theta_0)(1 - \theta_1) \). Inductively we have that \( b_n = \prod_{i=0}^{n-1} (1 - \theta_i) \). \( \square \)
Example 9.3. We consider the simplest example: \( k_i = 1 \) for \( i \geq 1 \). Then \( \theta_i = [0, 1, 1, 1, \ldots] = \tau - 1 = \tau^{-1} \) for all \( i \geq 0 \), where \( \tau = \frac{1}{2}(\sqrt{5} + 1) \) is the golden ratio. Then \( 1 - \theta_i = 1 - \tau^{-1} = 2 - \tau = \tau^{-2} \). Then the scale of \( K_0(C^*(\Lambda)) \) is defined (as in Theorem 9.2) by

\[
1 + \sum_{j=1}^{\infty} (\tau^{-2})^{j+1} = 1 + \sum_{j=2}^{\infty} \tau^{-2j}
\]

\[
= 1 + \frac{1}{\tau^4} \frac{1}{1 - \tau^{-2}}
\]

\[
= 1 + \frac{1}{\tau^2} \cdot \frac{1}{\tau^2 - 1}
\]

\[
= 1 + (2 - \tau)(\tau - 1)
\]

\[
= 2\tau - 2
\]

\[
= \frac{2}{\tau}.
\]

We now wish to give reasonable bounds on \( 1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j} (1 - \theta_{p_{j+1}}) \). Let \( (k_{p_j})_{j=1}^{\infty} \) be the subsequence of \( (k_j) \) consisting of those terms that are nonzero. Thus \( (k_j)_{j=1}^{\infty} = (0^{p_1-1}, k_{p_1}, 0^{p_2-p_1-1}, k_{p_2}, \ldots) \). We let \( q_i = p_i - p_{i-1} \) for \( i \geq 1 \) (with \( p_0 := 0 \)).

Lemma 9.4. For \( \ell \geq 0 \),

\[
\frac{1}{4}(q_{\ell+1} - 1) < \sum_{j=0}^{q_{\ell+1}-1} \prod_{i=0}^{j} (1 - \theta_{p_{j+i}}) < \frac{2}{3}q_{\ell+1}
\]

\[
\frac{e^{-1}e^{-\frac{1}{2}(k_{p_{\ell+1}}+3)}}{q_{\ell+1}} < \prod_{i=0}^{q_{\ell+1}-1} (1 - \theta_{p_{\ell+i}}) < \frac{2}{q_{\ell+1} + 2}.
\]

Proof. Recalling the remarks before Lemma 6.3 we have for \( p > 1 \) and \( y > 0 \),

\[
[(0,1)^p, y] = \pi\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^p\right)(y) = \pi\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)(y)
\]

\[
= \pi\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)(y) = \frac{y}{py + 1} = \frac{1}{p + \frac{1}{y}} = [0, p, y].
\]

It follows that for \( i > 1, k \geq 1, \) and \( x > 0 \), we have that

\[
\frac{1}{i + 1} < [(0, 1)^i, k, 1, x] < \frac{1}{i}.
\]

Note also that

\[
[0, 1, k, 1, x] < [0, 1, k, 1] = \frac{k + 1}{k + 2}.
\]
and therefore that $\frac{1}{2} < [0, 1, k, 1, x] < \frac{1}{2} + 1$. Recalling the definition of $\theta_i$ from Theorem 9.2 we note that $\theta_0 = [0, 1, (0, 1)^{q_{\ell+1}-1}, k_{p_{\ell+1}}, 1, (0, 1)^{q_{\ell+1}-1}, k_{p_{\ell+1}}, 1, \ldots]$. Then we have

$$\theta_{p_{\ell+1}+i} = [0, 1, (0, 1)^{q_{\ell+1}-i-1}, k_{p_{\ell+1}}, 1, \ldots]$$

$$= [(0, 1)^{q_{\ell+1}-i}, k_{p_{\ell+1}}, 1, \ldots] \in \left(\frac{1}{q_{\ell+1} - i + 1}, \frac{1}{q_{\ell+1} - i}\right), \text{ for } 0 \leq i < q_{\ell+1} - 1,$$

$$\theta_{p_{\ell+1}+1} = [0, 1, k_{p_{\ell+1}}, 1, \ldots] \in \left(\frac{1}{2}, k_{p_{\ell+1}} + 2\right).$$

We summarize this as

$$\begin{equation}
\frac{1}{q_{\ell+1} - i + 1} < \theta_{p_{\ell+1}+i} < \begin{cases} 
\frac{1}{q_{\ell+1} - i}, & \text{if } 0 \leq i < q_{\ell+1} - 1 \\
\frac{k_{p_{\ell+1}} + 1}{k_{p_{\ell+1}} + 2}, & \text{if } i = q_{\ell+1} - 1.
\end{cases}
\end{equation}
$$

We use the estimates

$$t < \sum_{n=1}^{\infty} \frac{1}{n} t^n < t + \frac{1}{2} \sum_{n=2}^{\infty} t^n = t + \frac{1}{2} \frac{t^2}{1 - t}, \text{ for } 0 < t < 1,$$

and for $1 \leq m \leq n$,

$$\int_m^{n+1} \frac{1}{u} du < \sum_{i=m}^{n} \frac{1}{i} \leq \begin{cases} 
\int_m^{n-1} \frac{1}{u} du, & \text{if } m \geq 2 \\
1 + \int_1^{n} \frac{1}{u} du, & \text{if } m = 1,
\end{cases}$$

or equivalently,

$$\begin{equation}
\log \frac{n+1}{m} < \sum_{i=m}^{n} \frac{1}{i} \leq \begin{cases} 
\log \frac{n}{m-1}, & \text{if } m \geq 2 \\
1 + \log n, & \text{if } m = 1.
\end{cases}
\end{equation}
$$

Since $\log(1 - t) = -\sum_{n=1}^{\infty} \frac{1}{n} t^n$ for $|t| < 1$, equation (9.2) implies

$$-t - \frac{1}{2} \frac{t^2}{1 - t} < \log(1 - t) < -t.$$
Now we establish the upper bounds. For $0 \leq j \leq q_{\ell+1} - 1$ we have

\[
\sum_{i=0}^{j} \log(1 - \theta_{p_{\ell+i}}) < -\sum_{i=0}^{j} \theta_{p_{\ell+i}}, \text{ by (9.4),}
\]

\[
< -\sum_{i=0}^{j} \frac{1}{q_{\ell+1} - i + 1}, \text{ by (9.1),}
\]

\[
= -\sum_{i=q_{\ell+1}+1-j}^{q_{\ell+1}+1} \frac{1}{i},
\]

\[
< -\log\frac{q_{\ell+1} + 2}{q_{\ell+1} + 1 - j}, \text{ by (9.3),}
\]

\[
= \log(1 - \frac{j + 1}{q_{\ell+1} + 2}),
\]

\[
\prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) < 1 - \frac{j + 1}{q_{\ell+1} + 2}.
\]

Then

\[
\prod_{i=0}^{q_{\ell+1} - 1} (1 - \theta_{p_{\ell+i}}) < \frac{2}{q_{\ell+1} + 2},
\]

and

\[
\sum_{j=0}^{q_{\ell+1} - 1} \prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) < q_{\ell+1} - \frac{q_{\ell+1}(q_{\ell+1} + 1)}{2(q_{\ell+1} + 2)}
\]

\[
= q_{\ell+1} - \frac{q_{\ell+1} + 3}{2(q_{\ell+1} + 2)}
\]

\[
\leq \frac{2}{3} q_{\ell+1},
\]

since $\frac{t+3}{t+2}$ decreases for $t \geq 1$.

For the lower bounds we first note that since $t + \frac{1}{2} \frac{t^2}{1-t} = t + \frac{1}{2} \sum_{i=2}^{\infty} t^i$ is increasing for $t \in (0, 1)$, then for $0 \leq i < q_{\ell+1} - 1$, (9.1) gives

\[
\theta_{p_{\ell+i}} + \frac{1}{2} \frac{\theta_{p_{\ell+i}}^2}{1 - \theta_{p_{\ell+i}}} < \frac{1}{q_{\ell+1} - i} + \frac{1}{2} \frac{1}{(q_{\ell+1} - i)^2} \left( \frac{1}{q_{\ell+1} - i} - \frac{1}{q_{\ell+1} - i} \right)
\]

\[
= \frac{1}{q_{\ell+1} - i} + \frac{1}{2} \left( \frac{1}{q_{\ell+1} - i - 1} - \frac{1}{q_{\ell+1} - i} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{q_{\ell+1} - i} + \frac{1}{q_{\ell+1} - i - 1} \right)
\]

\[
< \frac{1}{q_{\ell+1} - i - 1},
\]

(9.5)
while
\[
\theta_{p_{\ell+1}} + \frac{1}{2} \frac{\theta^2_{p_{\ell+1}}}{1 - \theta_{p_{\ell+1}} - 1} < \frac{k_{p_{\ell+1}} + 1}{k_{p_{\ell+1}} + 2} + \frac{1}{2} \frac{(k_{p_{\ell+1}} + 1)^2}{(k_{p_{\ell+1}} + 2)(1 - \frac{k_{p_{\ell+1}} + 1}{k_{p_{\ell+1}} + 2})}
\]
\[
< 1 + \frac{1}{2} \frac{(k_{p_{\ell+1}} + 1)^2}{k_{p_{\ell+1}} + 2}
\]
\[
< 1 + \frac{1}{2} (k_{p_{\ell+1}} + 1)
\]
\[
= \frac{1}{2} (k_{p_{\ell+1}} + 3).
\]
(9.6)

Now we have for \(0 \leq j < q_{\ell+1} - 2\) (in case \(q_{\ell+1} \geq 2\),
\[
\sum_{i=0}^{j} \log(1 - \theta_{p_{i+1}}) > - \sum_{i=0}^{j} \left( \theta_{p_{i+1}} + \frac{1}{2} \frac{\theta^2_{p_{i+1}}}{1 - \theta_{p_{i+1}} + 1} \right), \text{ by (9.4)},
\]
\[
> - \sum_{i=0}^{j} \frac{1}{q_{\ell+1} - i - 1}, \text{ by (9.5)},
\]
\[
= - \sum_{i=q_{\ell+1}-(j+1)}^{q_{\ell+1} - 1} \frac{1}{i},
\]
\[
> - \log \frac{q_{\ell+1} - 1}{q_{\ell+1} - 1 - (j + 1)}, \text{ by (9.3)},
\]
\[
= \log(1 - \frac{j + 1}{q_{\ell+1} - 1}).
\]
(9.7)

For \(j = q_{\ell+1} - 2\) (still in case \(q_{\ell+1} \geq 2\), we have
\[
\sum_{i=0}^{q_{\ell+1} - 2} \log(1 - \theta_{p_{i+1}}) > - \sum_{i=1}^{q_{\ell+1} - 1} \frac{1}{i}, \text{ by (9.7)}
\]
\[
\geq -1 - \log(q_{\ell+1} - 1), \text{ by (9.3)}
\]
(9.9)
\[
> -1 - \log q_{\ell+1}.
\]
(9.10)

Now suppose \(j = q_{\ell+1} - 1\). If \(q_{\ell+1} \geq 2\), by (9.9), (9.10) and (9.6) we have
\[
\sum_{i=0}^{q_{\ell+1} - 1} \log(1 - \theta_{p_{i+1}}) > -1 - \log(q_{\ell+1} - 1) - \frac{1}{2} (k_{p_{\ell+1}} + 3)
\]
(9.11)
\[
> -1 - \log q_{\ell+1} - \frac{1}{2} (k_{p_{\ell+1}} + 3).
\]
(9.12)

If \(q_{\ell+1} = 1\) then \(p_{\ell} = p_{\ell+1} - 1\), and we have
\[
\sum_{i=0}^{q_{\ell+1} - 1} \log(1 - \theta_{p_{i+1}}) = \log(1 - \theta_{p_{\ell+1} - 1})
\]
\[
> -(\theta_{p_{\ell+1} - 1} + \frac{1}{2} \frac{\theta^2_{p_{\ell+1} - 1}}{1 - \theta_{p_{\ell+1} - 1}})
\]
\[
> -\frac{1}{2} (k_{p_{\ell+1}} + 3), \text{ by (9.6)}.
\]
Now we exponentiate: for \( j < q_{\ell+1} - 2 \) we have
\[
\prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) > 1 - \frac{j + 1}{q_{\ell+1} - 1}, \text{ by (9.8)};
\]
for \( j = q_{\ell+1} - 2 \) we have, by (9.9) and (9.10),
\[
(1 - \theta_{p_{\ell+i}}) > e - \frac{1}{q_{\ell+1} - 1};
\]
and finally,
\[
(1 - \theta_{p_{\ell+i}}) > \begin{cases} 
\frac{e - 1}{(q_{\ell+1} - 1)} e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}, & \text{if } q_{\ell+1} > 1, \text{ by (9.11)}, \\
\frac{1}{2} e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}, & \text{if } q_{\ell+1} = 1.
\end{cases}
\]

Now summing gives
\[
\sum_{j=0}^{q_{\ell+1}-1} \prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) > \sum_{j=0}^{q_{\ell+1}-3} \left(1 - \frac{j + 1}{q_{\ell+1} - 1}\right) + \frac{e - 1}{q_{\ell+1} - 1} + \frac{e - 1}{q_{\ell+1} - 1} e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}
\]
\[
= q_{\ell+1} - 2 - \frac{1}{q_{\ell+1} - 1} \sum_{b=1}^{q_{\ell+1} - 2} b + \frac{e - 1}{q_{\ell+1} - 1} + \frac{e - 1}{q_{\ell+1} - 1} e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}
\]
\[
= \frac{1}{2}(q_{\ell+1} - 2) + \frac{e - 1}{q_{\ell+1} - 1}(1 + e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)})
\]
\[
> \frac{1}{2}(q_{\ell+1} - 2) + \frac{e - 1}{q_{\ell+1} - 1},
\]
in case \( q_{\ell+1} \geq 2 \), and in case \( q_{\ell+1} = 1 \),
\[
\sum_{j=0}^{q_{\ell+1}-1} \prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) = 1 - \theta_{p_{\ell}} > e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}.
\]

Note that for \( q \geq 2 \) we have \( \frac{1}{2}(q - 2) + \frac{e - 1}{q - 1} > \frac{1}{4}(q - 1) \), since this inequality reduces to \( e^{-1} > \frac{1}{4} \) when \( q = 2 \), and since \( \frac{1}{2}(q - 2) \geq \frac{1}{4}(q - 1) \) when \( q > 2 \). Thus we have
\[
\sum_{j=0}^{q_{\ell+1}-1} \prod_{i=0}^{j} (1 - \theta_{p_{\ell+i}}) \geq \frac{1}{4}(q_{\ell+1} - 1).
\]
Lemma 9.5. Define $U$ and $L$ by

$$U = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \prod_{\ell=0}^{n-1} \frac{2}{q_{\ell+1} + 2}$$

$$L = \sum_{n=0}^{\infty} \frac{1}{4} (q_{n+1} - 1) e^{-n} \prod_{\ell=0}^{n-1} \frac{1}{q_{\ell+1}} e^{-\frac{1}{2}(k_{p_{\ell+1}} + 3)}.$$ 

Then $L \leq 1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) \leq U + \theta_0$.

Proof. Let $a_i = 1 - \theta_i$. We have

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} a_i = \sum_{j=0}^{p_1-1} \prod_{i=0}^{j} a_i + \sum_{j=p_1}^{p_2-1} \prod_{i=0}^{j} a_i + \sum_{j=p_2}^{p_1-1} \prod_{i=0}^{j} a_i + \cdots$$

$$= \sum_{j=0}^{q_1-1} \prod_{i=0}^{j} a_i + \sum_{j=0}^{q_2-1} \prod_{i=0}^{j} a_i + \sum_{j=0}^{q_3-1} \prod_{i=0}^{j} a_i + \cdots$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{q_{n+1} - 1} \prod_{i=0}^{j} a_i$$

$$= \sum_{n=0}^{\infty} \prod_{i=0}^{p_{n-1}} a_i \left( \sum_{j=0}^{q_{n+1} - 1} \prod_{i=0}^{j} a_{p_n+i} \right)$$

$$= \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} \left( \prod_{i=0}^{q_{\ell+1}-1} a_{p_{\ell+1}+i} \right) \left( \prod_{j=0}^{q_{n+1} - 1} \prod_{i=0}^{j} a_{p_n+i} \right).$$

Thus we have

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) = \sum_{n=0}^{\infty} \left( \prod_{\ell=0}^{n-1} \left( \prod_{i=0}^{q_{\ell+1}-1} (1 - \theta_{p_{\ell+1}+i}) \right) \left( \prod_{j=0}^{q_{n+1} - 1} \prod_{i=0}^{j} (1 - \theta_{p_n+i}) \right) \right).$$

Since $1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) = \theta_0 + \sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i)$, we may use the estimates in Lemma 9.3 to finish the proof. \qed

Recall that the sequence $(q_j - 1)_{j=1}^{\infty}$ gives the lengths of the strings of zeros between consecutive nonzero terms of the sequence $(k_i)_{i=1}^{\infty}$. We use Lemma 9.5 to give some general situations when $C^*(\Lambda)$ is, and is not, stable.

Theorem 9.6. Let $\Lambda$ be defined by the sequence $(k_i)_{i=1}^{\infty}$ as in section 3. Let $(p_n)_{n=0}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ be as above.

1. If $(q_n)$ is a bounded sequence then $\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) < \infty$, and hence $C^*(\Lambda)$ is not stable.

2. If $q_n > 1 + e^{n-1} \prod_{i<n} q_i e^{\frac{1}{2}(k_{p_n}+3)}$ for all $n$, then $\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) = \infty$, and hence $C^*(\Lambda)$ is stable.
Proof. We first suppose that \((q_n)\) is a bounded sequence, say that \(q_n \leq C\) for all \(n\). Then
\[
\sum_{n=0}^{\infty} \frac{2}{3}q_{n+1} \prod_{\ell=0}^{n-1} \frac{2}{q_{\ell+1}+2} \leq \frac{2}{3}C \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n < \infty.
\]
By Lemma 9.5 it follows that \(\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) < \infty\).

Next we suppose that the condition in (2) holds. Then
\[
\sum_{n=0}^{\infty} \frac{1}{4}(q_{n+1} - 1) e^{-n} \prod_{\ell=0}^{n-1} \frac{1}{q_{\ell+1}} e^{-\frac{1}{2}(k_{\ell+1} + 3)} > \sum_{n=0}^{\infty} \frac{1}{4} = \infty.
\]
By Lemma 9.5 it follows that \(\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - \theta_i) = \infty\). \(\square\)

A loose interpretation of Theorem 9.6 is that \(C^*(\Lambda)\) is stable if and only if the sequence \((k_i)\) is rarely nonzero.

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