THE PETERSEN–WILHELM CONJECTURE ON PRINCIPAL BUNDLES

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ABSTRACT. This paper studies Cheeger deformations on $S^3$, $SO(3)$ principal bundles to obtain conditions for positive sectional curvature submersion metrics. We conclude, in particular, a stronger version of the Petersen–Wilhelm fiber dimension conjecture to the class of principal bundles. We prove any $\pi: SO(3) \rightarrow \mathcal{S} \rightarrow \mathcal{P} \rightarrow B$ principal bundle over a positively curved base admits a metric of positive sectional curvature if, and only if, the submersion is fat, in particular, $\dim B \geq 4$. The proof combines the concept of “good triples” due to Munteanu and Tapp [MT11], with a Chaves–Derdzinski–Rigas type condition to nonnegative curvature. Additionally, the conjecture is verified for other classes of submersions.

CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

1. INTRODUCTION

Riemannian Submersions play an essential role in Riemannian Geometry with a lower curvature bound. The Horizontal Curvature Equation ([O’N66, Gra67]) highlights a key reason for this: if $\pi: M \rightarrow B$ is a Riemannian submersion, $B$ has a lower sectional curvature bound at least as large as any lower sectional curvature bound for $M$. One can often use it as a practical tool to construct new examples of Riemannian manifolds, especially of non-negative or positive sectional curvature. Indeed, we remark that apart from spheres and the recent constructions in [PW09, Dea11, GVZ11], all known examples of closed manifolds with positive sectional curvature arise via a Riemannian submersion ([Ber61, Wal72, Esc82, Baz96, EH98, Zil14, L14]).

Still, positive sectional curvature appears to be a highly restrictive condition. Excluding the rank one symmetric space, examples of compact manifolds admitting positive curvature have only been found in dimensions 6, 7, 12, 13, and 24 ([Zil14]). Surprisingly, it remains an open and fundamental problem of Riemannian geometry: finding properties that distinguish the class of compact simply connected manifolds admitting nonnegative sectional curvature from its sub-class of positive sectional curvature. Seeking such distinguishment conditions, in the realm of Riemannian submersions, Petersen and Wilhelm conjectured dimension restrictions on the fibers of a positively curved submersion:
Conjecture A (Petersen–Wilhelm Fiber dimension conjecture). If $F \to M \to B$ is a Riemannian submersion from a positively curved closed manifold $M$, then
\[ \dim F < \dim B. \]

Several works already appear seeking to solve Conjecture A for instance: [GAR18, Spe17a, Spe18]. Of particular interest to the results here obtained is [Spe18]. More precisely, we often consider dual-holonomy fields. Let $(M,g)$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$. Considering the natural decomposition $T M = \mathcal{V} \oplus \mathcal{H}$ where $\mathcal{V}$ stands for the sub-bundle of $V$ containing the vectors tangent to the leaves of $\mathcal{F}$, if $c : I \to M$ is a geodesic such that $\dot{c}(s) \in \mathcal{H}_{c(s)}$ for any $s \in I \subset \mathbb{R}$ a vector field $v$ such that $v(s) \in \mathcal{V}_{c(s)}$ for every $s \in I$ is named a dual-holonomy field if
\[ \nabla_{\dot{c}} v = S_{\dot{c}} v - A_{\dot{c}}^v. \]

Analogously, basic horizontal fields along a vertical curve $\gamma$ are horizontal solutions of
\[ \nabla_{\gamma} X = -A_{\gamma}^X \gamma - S_{\gamma} X. \]

In the above equations, $S_X$ always stands for the second fundamental form of a leaf. The tensor $A_{\gamma}^X$ is $g$-dual of the O’Neill tensor $A_X Y = \frac{1}{2}[X, Y]^*, X, Y \in \mathcal{H}$. Additionally, the former equations are equivalent definitions of holonomy and basic horizontal fields, up to changing the sign $+ \to -$ in the term $S_X v$ on Equation 1. Moreover, in the totally geodesic case, both dual-holonomy and holonomy fields along horizontal geodesics and basic horizontal fields along vertical geodesics are the Jacobi fields induced by, respectively, local horizontal lifting and holonomy transport.

We state our main result, which settles Conjecture A to the class of principal bundles:

Theorem B (Strong Petersen–Wilhelm’s conjecture for principal fiber bundles). Any $SO(3), S^3$ principal bundle $\pi : \mathcal{P} \to B$ over a positively curved manifold admits a metric of positive sectional curvature if, and only if, $\pi$ is fat. In particular, $\dim B \equiv 0 \pmod 4$.

We recall that for a Riemannian submersion, being fat is an intrinsic property of its horizontal distribution $\mathcal{H}$. Namely, a Riemannian submersion $\pi : M \to B$ is said to be fat if, for every non-zero $X \in \mathcal{H}$, $\mathcal{V} = [X, \mathcal{H}]^*$ for some horizontal extension of $X$, where we denote by $\mathcal{V}$ the vertical distribution associated to $\pi$. We must take some care here: in the literature, the terminology for fatness is often widely employed once assumed the preexistence of a Riemannian metric with totally geodesic leaves.

It is worth noting that verifying Petersen–Wilhelm conjecture on principal bundles is reduced to the case of $SO(3), S^3$-principal bundles. Assuming the total space of a principal bundle has a metric of positive sectional curvature implies that the structure group $G$ acting by isometries has rank 1, see Theorem 2.1 and Proposition 4.1 in [Wal72], what reduces $G$ to either $SO(3), S^3, S^1$ but the long exact homotopy sequence for fibrations implies $\dim B \geq 2$ otherwise $B$ is diffeomorphic to a circle (assuming connectedness), and $\mathcal{P}$ has infinite fundamental group, a contradiction with the Bonnet–Myers Theorem.

It is also remarkable that Theorem B can be seen as the analogous can one lift positive sectional curvature? to the question raised by Searle and Wilhelm in [SW15].

To accomplish the proof of Theorem B we base our computations on the concept of “good triples” introduced by Munteanu and Tapp in [MT11]. Namely, a triple $(X, V, \alpha')$ is said to be a good triple provided if $\exp(tV(s)) = \exp(sX(t))$ for all $s, t \in \mathbb{R}$, where $V(s), X(t)$ denote the Jacobi fields along $\exp(sV)$ and $\exp(tX)$, respectively, that satisfy $V(0) = V, X(0) = X$ and $V'(0) = \alpha' = X'(0)$. On the one hand, these are very rare in the sense that, given $(X, V)$, there are only finitely many choices of $\alpha'$ making $(X, V, \alpha')$ to be a good triple provided if $X$ or $V$ satisfies a generic property named “weakly regular”, Definition 9.1 in [MT11]. On the other hand, good triples occur naturally in totally geodesic Riemannian foliations by a horizontal $X$ and a vertical $V$ (or vice-versa). In this case, $\alpha' = -A_X^V$, see for instance, Proposition 1.4 in [MT11].

A key curvature identity provided in Proposition 1.6 in [MT11] allows us to approach the problem of producing and/or obstructing positive sectional curvature for some Riemannian foliations. We prove a sort of generalization to it under the assumption of totally geodesic leaves:

Theorem C (= Theorem 5.3). Let $(M, g)$ be a compact Riemannian manifold with a Riemannian foliation $\mathcal{F}$ with totally geodesic leaves and bounded holonomy. Then $A_X^V \gamma$ is basic along the geodesic $\gamma(s) := \exp(sV)$, where $V \in \mathcal{V}$ and $X \in \mathcal{H}$ is basic. Moreover,
\[ \langle R_g(X, V)(A_X^V V), A_X^V V \rangle = 0, \]
where $R_g$ denotes the Riemann curvature tensor associated with the Levi-Civita connection related to $g$. 

(2) If \( Y \) is a basic vector field then
\[
\langle R_g(X,V)A_X^i V, Y \rangle = \langle \nabla_X A_V^i V, A_X^i V \rangle = \langle A_{\nabla_X A_V^i V} V, V \rangle.
\]
In particular, \( R_g(X,V)A_X^i V = 0 \) if, and only if, \( \nabla_X A_V^i V = A_{\nabla_X A_V^i V} V \).

(3) \( R_g(X,V)A_X^i V = 0 \) if, and only if, \( (\nabla_X A^i V) V = 0 \). Such equality is always true in locally symmetric spaces.

(4) If the foliation is fat then \( \langle R(X,V)A_X^i V, X \rangle = 0 \).

We remark that Theorem C should be clear from the results in the first version of the paper [Spe17b]. Moreover, the bounded holonomy condition is translated in the case of foliations generated by the connected components of the fibers of submersion, requiring that its structure group is compact. See [Spe18] Definition 1.3 for further details.

The proof of Theorem B is obtained by looking carefully at the case of bundles with totally geodesic fibers. Theorem A in [SSW15] implies that it is always possible to get, using a Cheeger deformation, a metric with totally geodesic leaves. Additionally, for the general case, one notes that assuming the existence of a non-fat point, in particular, the presence of an orthonormal set \( X \in \mathcal{H}, V \in \mathcal{Y} \) with \( A_X^i V = 0 \) recovers \( R(X,V)A_X^i V = 0 \), indicating that the proof to this case could be derived from the employed technique used in the totally geodesic setting, this is what we explore (see the proof of Corollary 4.6).

We digress a bit illustrating how the proof of Theorem B can be obtained in the totally geodesic case by simply combining Theorem C with the classical condition of Chaves–Derdzinski–Rigas.

**A sketch of the proof of Theorem B**. Recall the Chaves–Derdzinski–Rigas result:

**Theorem 1.1** (Chaves–Derdzinski–Rigas, [CDR92]). Let \( \pi : \mathcal{P} \rightarrow B \) to be a principal bundle with \( G = S^3, SO(3) \) and \( \dim B \geq 2 \). Fix a Riemannian metric \( h \) at \( B \) and a bi-invariant inner product \( \langle \cdot, \cdot \rangle \) in \( \mathfrak{g} \). Then there is \( \varepsilon > 0 \) such that \( g_{\mathcal{P}} := \pi^* h + \varepsilon \langle \omega, \omega \rangle \) has positive sectional curvature if, and only if, for any point \( p \in \mathcal{P} \) and any mutually orthonormal vectors \( X, Y \in T_{\pi(p)} B \) and any non-zero element \( u \in \text{Ad}(\mathcal{P}) := \mathcal{P} \times \text{Ad} \mathcal{g} \) over \( \pi(p) \) it holds that

\[
R^h(X, Y, X, Y) \sum_{i=1}^n \langle u, \Omega(X, X_i) \rangle^2 > \langle u, (\nabla_X \Omega)(X, Y) \rangle^2,
\]

where \( R^h \) is the curvature tensor of \( (B, h) \), \( \Omega \) is the curvature of the connection form \( \omega \), \( \nabla \) is the covariant derivative induced by \( \omega \) and \( \{X_1, \ldots, X_n\} \) is an arbitrary orthonormal basis to \( T_{\pi(p)} B \).

**Remark 1.** We often name inequality (3) as the CDR condition.

We first remark that there is a hidden Cheeger deformation (Section 2) on Theorem 1.1: note that \( \text{Ad} \) corresponds to the representation \( \text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}) \) so the following action

\[
g \in G, \quad g \ast (p, V) := (gp, \text{Ad}_g(V))
\]
defines a free and isometric action in \( \mathcal{P} \times \mathfrak{g} \) that recovers \( \text{Ad}(\mathcal{P}) \) as quotient. Also, observe that \( \Omega \) can be identified with the usual O‘Neill tensor via \( \Omega(X,Y)^* = [X,Y]^* = -2A_X Y \), where the superscript \( * \) stands to the corresponding action vector obtained from \( \Omega(X,Y) \in \mathfrak{g} \). Equation (3) becomes

\[
K_B(X,Y) \sum_{i=1}^n \langle U^*, A_X X_i \rangle^2 > \langle U^*, (\nabla_X A^i) Y \rangle^2
\]

where \( U^* \) is an action vector (see Section 2, particularly Equation (5) for needed definitions).

Since \( \langle U^*, A_X X_i \rangle = (A_X^i U^*, X_i) \) one gets that

\[
\sum_{i=1}^n \langle U^*, A_X X_i \rangle^2 = |A_X^i U^*|^2
\]
thus translating the CDR condition in

\[(4) \quad K_B(X,Y)|A_X^*U'|^2 > \langle U^*, (\nabla_X A)X Y \rangle^2.\]

We sketch how to show that if $\pi$ is fat, then $\langle U^*, (\nabla_X A)X Y \rangle = 0$, concluding the proof since inequality \((4)\) is naturally satisfied. Thus Theorem \([B]\) is verified under totally geodesic fibers’ assumption.

To conclude that $\langle U^*, (\nabla_X A)X Y \rangle$ we observe that, as consequence of Theorem \([C]\) in the totally geodesic setting (see Corollary \([4.4]\))

\[(a) \quad R_g(X,Y,Y,V) = 0 \quad \text{for every} \quad Y \in \ker A_X : \mathcal{H} \rightarrow \mathcal{V}, \quad \text{for every} \quad V \in \mathcal{V};\]

\[(b) \quad \text{if the foliation is fat then} \quad R_g(X,Y,Y,V) = 0 \quad \text{for every basic vectors} \quad X,Y \in \mathcal{H} \quad \text{and every} \quad V \in \mathcal{V}.\]

From Theorem 1.5.1 in \([\text{GW09}]\) one gets $R_g(X,Y,Y,V) = -g((\nabla_X A)X Y, V)$, therefore, it suffices to show $R_g(X,Y,Y,V) = 0$ for any basic vectors $X,Y \in \mathcal{H}$ and every $V \in \mathcal{V}$.

Under the fatness assumption, any horizontal vector $Y$ can be uniquely written as $Y = aA_X^* V + Z$ where $a \in \mathbb{R}$ and $Z \in \ker A_X$. This holds since the fatness condition ensures that if $X \neq 0$ then $A_X^* : \mathcal{V} \rightarrow \mathcal{H} \cap \{X\}^\perp$ defines an injection. It is straightforward to check that the elements not attained by the image of this operator are the ones in $\ker A_X$.

We now have

\[R_g(X,Y,Y,V) = R_g(X,aA_X^* V + Z, aA_X^* V + Z, V)\]
\[= a^2 R_g(X,A_X^* V, A_X^* V) + 2a R_g(X, A_X^* V, Z, V) + R_g(X, Z, Z, V).\]

Item \([1]\) guarantees that $R_g(X,Z,Z,V) = 0$. Item \([1]\) in Theorem \([C]\) ensures that $R_g(X,A_X^* V, A_X^* V, V) = 0$. Finally, item \([2]\) implies that $R_g(X, A_X^* V, Z, V) = 0$. Indeed, once $R_g$ is a tensor, we can extend $A_X^* V$ via horizontal lifts to the computations ensuring that $\nabla_X^h A_X^* V = 0$, and so

\[R_g(X,A_X^* V, Z, V) = X(A_X^* V, A_X^* V).\]

The proof is done once one checks that the fatness assumption ensures $A_X^* Z \parallel V$ is parallel to $Z$ (see the proof of Theorem \([C]\) since then there exists $c \in \mathbb{R}$ such that $\langle A_X^* V, A_X^* V \rangle = c \langle Z, A_X Z \rangle = 0$.

**Further rigidity results.** Theorem \([B]\) has the following interesting consequence:

**Corollary E.** No Riemannian submersion metric on $S^3 \times S^2 \rightarrow S^2$ has positive sectional curvature. In particular, $S^3 \times S^2$ has no $S^3$-invariant metrics of positive sectional curvature. Moreover, $S^2 \times S^2 \rightarrow S^2$ carries no Riemannian submersion metric of positive sectional curvature.

As far as the authors know, the former corollary was not known to be true. We want to thank Kris Tapp for pointing it out.

Following \([\text{Zilb}]\) Proposition 2.2, p.16], a sphere bundle $S^2 \hookrightarrow M \rightarrow B$ is fat if, and only if, the respective associated $\text{SO}(3)$-principal bundle is fat. Considering this, Corollary \([F]\) below responds affirmatively to a case of Problem 9 in \([\text{Zilb}]\), that is:

**Corollary F.** (1) if the associated principal bundle $\pi : \text{SO}(3) \hookrightarrow \mathcal{P} \rightarrow B$ to some sphere bundle $\mathcal{P} : S^2 \hookrightarrow M \rightarrow B$ admits a metric of positive sectional curvature then $\mathcal{P}$ admits a metric of nonnegative sectional curvature and positive vertizontal curvature. In particular, $\mathcal{P}$ is fat

(2) if $\pi : \text{SO}(3) \hookrightarrow \mathcal{P} \rightarrow B$ is a fat bundle over a positively curved manifold $B$, then any oriented Euclidean $\mathbb{R}^3$-vector bundle associated to $\pi$ has a metric of non-negative sectional curvature.

Another direct consequence of Theorem \([B]\) is

**Theorem G.** The only $S^3$-principal bundle over $S^4$ with a Riemannian submersion metric of positive sectional curvature is the Hopf bundle $^3$

\[S^3 \hookrightarrow S^7 \rightarrow S^4.\]

The former is obtained once one observes that the only fat $S^3$-principal bundle over $S^4$ is the Hopf fibration (\([\text{DR01}]\)).

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1 We thank W. Ziller for bringing our attention to this fact
Homogeneous bundles and weakly non-negatively curved submersions. We say that a Riemannian metric $g$ on $M$ is adapted to $(\mathcal{H}, b)$ if $g(\mathcal{V}, \mathcal{H}) = 0$ and $\pi(\mathcal{H}) : \mathcal{H} \to TB$ is an isometry. Since it is of interest to understand examples of manifolds in which obstructions to positive curvature hold, we prove:

**Theorem H** (=Theorem [3.1]). Let $K < H < G$ be compact Lie groups and consider the submersion $\pi : (G/K, \mathcal{H}) \to (G/H, b)$ defined by $\pi(gK) = gH$. Suppose that $(G/H, b)$ is normal homogeneous and $g : \mathcal{H}_p = \mathcal{H}_{g_0}$ for all $g \in G$. Then $G/K$ has an adapted metric with positive sectional curvature only if $\mathcal{H}$ is fat.

We recall that if $\pi : (M, \mathcal{H}) \to (B, b)$ is a submersion, then an existing adapted metric $g$ to $\pi$ is called weakly non-negatively curved (WNN) if every $p \in M$ has a neighborhood $U$ where

$$\tau|X|^2A^*_X\xi^2 \geq \langle (\nabla A^*)_{X}\xi + A^*_S S_X \xi, A^*_\xi \rangle,$$

for some $\tau > 0$ and all $X \in \mathcal{H}|_U$, $\xi \in \mathcal{V}|_U$. Theorem H is obtained as a corollary of the following result:

**Theorem I** (=Theorem [3.3]-Proposition [4.3]). Let $\pi : (M, \mathcal{H}) \to (B, b)$ be a submersion and $g$ be adapted to $(\mathcal{H}, b)$. Then,

1. $g$ is weakly non-negatively curved if and only if every metric adapted to $(\mathcal{H}, b)$ is weakly non-negatively curved
2. if $g$ is non-negatively curved and the fibers of $\pi$ are totally geodesic fibers, then $g$ is weakly non-negatively curved
3. if $(M, g)$ is complete, weakly non-negatively curved, has positive sectional curvature, and $\pi$ has a compact structure group, then $\mathcal{H}$ is fat
4. if $\pi$ has compact structure group, then the WNN condition is satisfied if $\langle R(X, V)A^*_XV, X \rangle = 0$ for every $X \in \mathcal{H}, V \in \mathcal{V}$.

As a last result directly related to the Conjecture A, we prove:

**Corollary J** (=Corollary [4.3]). Let $(M, g)$ be a compact manifold with a Riemannian foliation $\mathcal{F}$ with totally geodesic leaves and bounded holonomy. Then $g$ has positive sectional curvature only if

A. $(M, g)$ is a locally symmetric space
B. $(\nabla A^*V)_{X}V = 0$ for every basic field $X \in \mathcal{H}$ and every vertical field $V \in \mathcal{V}$.

We believe Corollary J provides a natural path to pursue the complete solution to Conjecture A. The authors have made some nontrivial progress in this direction, which will be the subject of a future project. We also wonder whether the converse to the second part of its statement could hold, namely, if fatness plus $(\nabla A^*)_{X}V = 0$ could guarantee the existence of a positively curved metric on $M$.

**Notation and conventions.** We denote by $R_g$ the Riemannian tensor of the metric $g$:

$$R_g(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where $\nabla$ stands for the Levi-Civita connection of $g$. We denote either by $K_g(X, Y) = g(R_g(X, Y)Y, X)$ or by $R_g(X, Y) = R_g(X, Y, Y, X)$, making it clear in the context, the unreduced sectional curvature of $g$. We adopt the standard decomposition notation $TM = \mathcal{V} \oplus \mathcal{H}$ for any foliated manifold $M$, being $\mathcal{V}$ the subbundle collecting tangent spaces to the leaves of the foliation $\mathcal{F}$ and $\mathcal{H}$ to some subbundle complementary choice in $TM$.

The superscripts $\mathfrak{v}$ or $\mathfrak{h}$ denote the projection of the underlined quantities on such subbundles, $\mathcal{V}$ and $\mathcal{H}$, respectively. Whenever we say we have a Riemannian principal bundle, we mean that the principal bundle is considered with a Riemannian submersion metric.

2. A review on group actions and Cheeger deformations

Let $(M, g)$ be a Riemannian manifold with an isometric free $G$-action, where $G$ is a compact Lie group, and $Q$ is a bi-invariant metric on $g$, the Lie algebra of $G$. As usual, the tangent space to the orbit $Gp$ is named the vertical space at $p$, denoted by $\mathcal{V}_p$. Its $g$-orthogonal complement, denoted by $\mathcal{H}_p$, is named the horizontal space at $p$. Any tangent vector $X \in T_pM$ can be uniquely decomposed as $X = X + U^*(p)$, where $X$ is horizontal and $U \in \mathfrak{g}$. Moreover,

$$U^*(p) := \frac{d}{dt}_{t=0} \exp(tU)p$$

where $\exp : \mathfrak{g} \to G$ denotes the Lie group exponential.

The vector described by Equation (5) corresponds to an action vector, obtained via the linearization of the existing diffeomorphism between $Gp$ and $G$. 


In this paper, we make use of the classical Cheeger deformations (see [Che73, M87, Zila]). It consists in inducing on \( M \) one-parameter family of metrics via recovering \( M \) through an associated bundle construction. Precisely, it is considered the product manifold \( M \times G \) with the \( \ast \)-action

\[
(r, p, g) := (rp, rg)
\]

which is free and isometric for the product metric \( g + t^{-1}Q, t > 0 \). Hence, its quotient space is diffeomorphic to \( M \) via the projection

\[
\pi' : M \times G \to M
\]

\[
(p, g) \mapsto g^{-1} p
\]

which defines a principal bundle for the \( \ast \)-action (6). This manner, \( g \) is induced via \( \pi' \).

We summarize useful tensors associated with Cheeger deformations (see either [M87] or [Zila]):

1. The orbit tensor at \( p \) is the linear map \( P : g \to g \) defined by

\[
g(U^*, V^*) = Q(P_U, V), \quad \forall U^*, V^* \in \mathcal{V}_p
\]

2. For each \( t > 0 \) we define \( P_t : g \to g \) as

\[
g_t(U^*, V^*) = Q(P_t U, V), \quad \forall U^*, V^* \in \mathcal{V}_p
\]

3. The metric tensor of \( g_t, \mathcal{C}_t : T_p M \to T_p M \) is defined as

\[
g_t(X, Y) = g(C_t X, Y), \quad \forall X, Y \in T_p M
\]

All three tensors above are symmetric and positive definite. The next proposition shows how these quantities are related to the original metric quantities.

**Proposition 2.1** (Proposition 1.1 in [Zila]). The tensors above satisfy:

1. \( P_t = (P^{-1} + t1)^{-1} = P(1 + tP)^{-1}, \)
2. If \( \mathcal{X} = X + U^* \) then \( \mathcal{C}_t(\mathcal{X}) = X + ((1 + tP)^{-1}U)^* \).

As observed by Cheeger and crucial in the work of M. Mütter [M87], the metric tensor \( C_t^{-1} \) can be used to obtain a more suitable expression to the sectional curvature of a Cheeger deformation.

**Theorem 2.2.** Let \( \mathcal{X} = X + U^* \), \( \mathcal{Y} = Y + V^* \) be tangent vectors. Then \( \kappa_t(\mathcal{X}, \mathcal{Y}) := R_{g_t}(C_t^{-1}X, C_t^{-1}Y, C_t^{-1}Y, C_t^{-1}X) \) satisfies

\[
\kappa_t(\mathcal{X}, \mathcal{Y}) = \kappa_0(\mathcal{X}, \mathcal{Y}) + \frac{t^3}{4}||P_U, PV||_Q^2 + z_t(\mathcal{X}, \mathcal{Y}),
\]

where \( z_t \) is nonnegative.

We refer to either [Zila] Proposition 1.3 or [CSS18] Lemma 3.5 for the details on the proof and more references.

A final preliminary result to this paper provides a more concrete expression to \( z_t \) in the case of free actions, for a proof see [CGS22] Lemma 2, p.6.

**Lemma 1.** It holds that

\[
z_t(\mathcal{X}, \mathcal{Y}) = 3t \max_{Z \in \mathcal{H}, \underline{Q} = 1} \frac{dw_Z(\mathcal{X}, \mathcal{Y}) + \frac{t}{2}Q([P_U, PV], Z)}{tg(Z^*, Z^*) + 1},
\]

where \( dw_Z \) is obtained from

\[
w_Z : TM \to \mathbb{R}
\]

\[
\mathcal{X} \mapsto \frac{1}{2}g(\mathcal{X}, Z^*)
\]

where \( Z^* \) is the action vector associated to \( Z \in g \).

Moreover, if \( q \in M^{fr}, X, Y \in \mathcal{H}_q \) and \( U \in g \), then

\[
dw_Z(U^*, X) = \frac{1}{2}Xg(U^*, Z^*)
\]

\[
dw_Z(X, Y) = -\frac{1}{2}g([X, Y]^Y, Z^*) = -g(A_X Y, Z^*).
Therefore,

\[ z_t(X, Y) = 3t \left( (1 + tP)^{-1/2} \Pi^t_X Y - (1 + tP)^{-1/2} \frac{1}{2} [P, P] \right)^2. \]

3. An intrinsic curvature condition for submersions over Riemannian manifolds

In this section, we prove Theorems 1 and 1. Let \( \pi: (M, \mathcal{H}) \rightarrow (B, b) \) be a submersion equipped with a horizontal connection \( \mathcal{H} \), i.e., a distribution \( \mathcal{H} \) complementary to \( \mathcal{V} = \ker \pi \). We call a Riemannian metric \( g \) on \( M \) adapted to \( (\mathcal{H}, b) \) if \( g(\mathcal{V}, \mathcal{H}) = 0 \) and \( d\pi|_\mathcal{H}: \mathcal{H} \rightarrow TB \) is an isometry. We get the following result about homogeneous connections over homogeneous submersions.

**Theorem 3.1.** Let \( K < H < G \) be compact Lie groups and consider the submersion \( \pi: (G/K, \mathcal{H}) \rightarrow (G/H, b) \) defined by \( \pi(gK) = gH \). Suppose that \( (G/H, b) \) is normal homogeneous and \( g: \mathcal{H}_p = \mathcal{H}_g \) for all \( g \in G \). Then \( G/K \) has an adapted metric with positive sectional curvature on every \( \mathcal{H} \) if \( \mathcal{H} \) is fat.

We call a horizontal distribution fat if, for every non-zero \( X \in \mathcal{H} \), \( \mathcal{V} = [\tilde{X}, \mathcal{H}]^\mathcal{V} \) for some horizontal extension of \( X \). Theorem 3.1 in its context, alludes to the fact that a connection on a positively curved submersion must be fat. For a general submersion, being fat at a point solves Petersen–Wilhelm’s conjecture (see, e.g., [AK15, Che16, GAG16, Jim9]), which we restate here:

**Conjecture 3.2.** (Petersen–Wilhelm). Let \( \pi: (M^n \times k, g) \rightarrow (B^n, b) \) be a Riemannian submersion. If \( (M, g) \) is compact and has positive sectional curvature, then \( k < n \).

González and Radeschi [GAR18] considered submersions from spaces homotopically equivalent to the known examples with positive sectional curvature. They prove that such submersions satisfy Petersen–Wilhelm’s Conjecture conclusion without curvature assumptions. The other hand, Theorem 3.1 assumes a hypothesis on curvature, but it might cover different spaces. For instance, let \( G = \text{Spin}(5) \), \( H = \text{Spin}(3) \) and \( G/H \) be the Berger space \( B^7 \) (Ber61). Then Theorem 3.1 guarantees that \( \text{Spin}(5) \) does not have a positively curved metric adapted to \( (\mathcal{H}, b) \), the usual horizontal distribution and metric induced by the bi-invariant metric. Theorem 3.3 below covers a broader class of submersions, including the non-standard submersions constructed in [KST72].

**Definition 1.** Let \( \pi: (M, \mathcal{H}) \rightarrow (B, b) \) be a submersion. An adapted metric \( g \) is called weakly non-negatively curved (WNN) if every \( p \in M \) has a neighborhood \( U \) where

\[ \tau |X|^2 |A^*_X \xi|^2 \geq \langle (\nabla_X A^*) X \xi + A^*_X S_X \xi, A^*_X \xi \rangle, \]

for some \( \tau > 0 \) and all \( X \in \mathcal{H}|_U \), \( \xi \in \mathcal{V}|_U \).

We call a pair \((\mathcal{H}, b)\) weakly non-negatively curved if there is a WNN metric \( g \) adapted to \((\mathcal{H}, b)\). Inequality (9) holds on submersions with totally geodesic fibers from a non-negatively curved manifold, being a condition strictly weaker than nonnegative sectional curvature – observe that (9) does not give any estimate on the curvature of the base in contrast to nonnegative curvature (see [ON66]). Furthermore, the WNN property is intrinsic to \((\mathcal{H}, b)\) (item (1) below). Theorem 3.3 also solves Petersen–Wilhelm’s conjecture for a particular class of metrics.

**Theorem 3.3.** Let \( \pi: (M, \mathcal{H}) \rightarrow (B, b) \) be a submersion and \( g \) be adapted to \((\mathcal{H}, b)\). Then,

1. \( g \) is weakly non-negatively curved if and only if every metric adapted to \((\mathcal{H}, b)\) is weakly non-negatively curved.
2. \( g \) is weakly non-negatively curved and the fibers of \( \pi \) are totally geodesic fibers, then \( g \) is weakly non-negatively curved.
3. \( (M, g) \) is complete, weakly non-negatively curved, has positive sectional curvature, and \( \pi \) has a compact structure group, then \( \mathcal{H} \) is fat.

Theorem 3.3 follows from Theorem 3.1 by observing that \( G/K \rightarrow G/H \) has a metric as in the item (2) on Theorem 3.1. Also, observe that the combination of the items (2), (3) of Theorem 3.3 corresponds to part of Theorem D.

In what follows, we shall explore the behavior of the set of vertizontal planes that vanishes under Gray–O’Neill’s A-tensor under the curvature condition given in Definition 1. We first remark that, given \( X, Y \), horizontal vectors at \( p \in M \), one computes the O’Neill tensor \( \tilde{A}: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{V} \) as

\[ A_X Y = \frac{1}{2} [\tilde{X}, \tilde{Y}]^\mathcal{V}, \]

where \( \tilde{X}, \tilde{Y} \in \mathcal{H} \) are horizontal extensions of \( X, Y \) and \( \mathcal{V} \) (respectively \( \mathcal{H} \)-component) stands for the projection onto the \( \mathcal{V} \)-component (respectively, onto \( \mathcal{H} \)-component). Therein \( \tilde{A}^* : \mathcal{H} \times \mathcal{V} \rightarrow \mathcal{V} \) stands for the dual of \( A \), defined as \( \langle A_X \xi, Y \rangle = \langle A_X Y, \xi \rangle \) for all \( X, Y \in \mathcal{H} \) and \( \xi \in \mathcal{V} \).
3.0.1. Duonholonomy fields. The main tool used here is duonholonomy fields (as introduced in [Spe18]). Let \( c \) be a horizontal curve, i.e., \( \dot{c} \in \mathcal{H} \), recall that a duonholonomy field \( \xi \) along \( c \) is a vertical field satisfying
\[
\nabla_c \xi = -A^* c \xi - S c \xi,
\]
with \( \tilde{X} \) any horizontal extension of \( X \). A vertical field \( v \) along \( c \) is called a duonholonomy field if
\[
\nabla_c v = -A c v + S c v.
\]

As we shall see, duonholonomy fields naturally appear when dealing with adapted metrics. Lemmas 2 and Equation (15) stress their role.

Given a horizontal geodesic \( c \), we define its infinitesimal duonholonomy transformation \( \dot{c}(t) : \gamma_{c(0)} \to \gamma_{c(t)} \) by setting \( \dot{c}(t) \xi_0 = \xi(t) \), where \( \xi \) is a duonholonomy field along \( c \). We recall that, given an adapted metric \( g \), the duonholonomy field \( v \) along \( c \) satisfies ([Spe18, Proposition 4.1]):
\[
v(t) = \dot{c}(t)^{-1} v(0),
\]
where \( \dot{c}(t)^{-1} \) is the inverse of the \( g \)-dual of \( \dot{c}(t) \). Let \( g, g' \) be metrics adapted to \( (\mathcal{H}, b) \) and let \( P_p : \gamma_p \to \gamma_p \) be the tensor defined by
\[
g'(\xi, \eta) = g(P_p \xi, \eta).
\]

Lemma 2. If \( v(t) \) is a duonholonomy field on \( (M, g') \), then \( v(t) = \dot{c}(t)^{-1} v(0) \), where \( v(t) \) is the duonholonomy field on \( (M, g) \) with initial condition \( v(0) = P_{c(0)} v(0) \). In particular, for every \( t \),
\[
A^1 c v(t) = A c v(t).
\]

Proof. According to the discussion above,
\[
v'(t) = \dot{c}(t)^{-1} v(0) = (\dot{c}(t)^{-1}) v(0) = \dot{c}(t)^{-1} v(0) = \dot{c}(t)^{-1} v(0) = P_{c(0)} v(0) = P_{c(0)} v(0).
\]

Equation (14) follows since \( A^1 c \xi = A c P_p \xi \) for all \( X \in \mathcal{H}_p, \xi \in \gamma_p \). □

Lemma 3. Let \( g \) be a WNN metric adapted to \( (\mathcal{H}, b) \). If \( v \) is a duonholonomy field along a horizontal geodesic \( c \) satisfying \( A c v(0) = 0 \), then \( A c v(t) = 0 \) for all \( t \).

Proof. The proof goes like Proposition 5.2 in [Spe17b]. Take \( |c| = 1 \) and denote \( u(t) = |A c v(t)|^2 \). From (12), we have
\[
(|A c v|)^2 v = |A c v|^2 v = |A c v|^2 v = |A c v|^2 v = |A c v|^2 v.
\]

Thus, Equation (9) gives \( 2 \tau u(t) \geq u(t) \). Grönwall’s inequality now implies that \( u(t) \leq u(0) e^{2 \tau t} \) for all \( t \). Replacing \( c \) by \( \dot{c}(t) = c(-t) \) proves the assertion. □

As a last observation, we recall an identity in [Spe18]. Given a Riemannian metric \( g \), let \( K_g(X, v) = R_g(X, v, v, X) \) be the unreduced sectional curvature of \( X \wedge v \). We recall from [Spe18, Proposition 4.2] that a duonholonomy field \( v \) satisfies
\[
K_g(\dot{c}, v(t)) = \frac{1}{2} \frac{d^2}{dt^2} |v(t)|^2 - 3 |c| |v(t)|^2 + |A c v(t)|^2.
\]

3.0.2. Proof of Theorem 3.3

Proof of (1). Let \( g, g' \) be adapted to \( (\mathcal{H}, b) \) and assume that \( g \) satisfies the condition WNN. Given \( p \in M \), there is a neighborhood \( U \) of \( p \) and a \( t > 0 \) such that (9) holds. For any metric \( g \) adapted to \( \mathcal{H} \), \( X \in \mathcal{H}_q \) and \( v_0 \in \gamma_q \), denote \( u(g, v_0, X, t) = |A c v(t)|^2 \) where \( c(t) = \exp(tX) \), \( \mathcal{V} \) is the duonholonomy field along \( c \) with \( v(0) = v_0 \) and \( A^v \) is the \( g \)-dual of \( A \). Lemma 2 gives \( u(g', v_0, X, t) = u(g, P q v_0, X, t) \) for every \( X \in \mathcal{H}_q, v_0 \in \gamma_q \) and \( t \). In particular,
\[
2 \tau u(g', v_0, X, t) = 2 \tau u(g, P q v_0, X, t) \geq u(g, P q v_0, X, t) = u'(g', v_0, X, t).
\]

Since \( u(g', v_0, X, t) = |A c v(t)|^2 \) and \( u'(g', v_0, X, t) \) is precisely the right-hand-side of equation (9) for \( g' \) one concludes the proof. □
Proof of (2). Item (2) follows by analyzing the discriminant of the polynomial
\[ K_g(X, \lambda A_X^* \xi + \xi) = K_g(\xi, X) + 2\lambda R_g(X, A_X^* \xi, \xi, X) + \lambda^2 K_g(X, A_X^* \xi). \]

The above equation is a polynomial in \( \lambda \) for fixed vectors \( X, \xi, A_X^* \xi \). Since \( (M, g) \) has positive sectional curvature, such a polynomial is everywhere positive, and its discriminant is negative. Hence,
\[ K_g(\xi, A_X^* \xi) K_g(X, A_X^* \xi) \geq R_g(X, A_X^* \xi, \xi, X)^2. \]

On the other hand, if \( (M, g) \) has totally geodesic fibers, O'Neill’s equation (see [O’N66] or [GW09, p. 44]) gives \( R_g(X, A_X^* \xi, \xi, X) = (\langle X_{\xi} \rangle X, A_X^* \xi) \) and \( K_g(\xi, X) = |A_X^* \xi|^2 \). Moreover, taking a relatively compact neighborhood on \( M \), there is a \( \tau > 0 \) such that \( K_g(\xi, A_X^* \xi) K_g(X, A_X^* \xi) \leq \tau |X|^2 |A_X^* \xi|^2 \).

Proof of (3). We argue by contradiction. Suppose there are non-zero \( X \in \mathcal{H}, \nu_0 \in \mathcal{V} \) such that \( A_X^* \nu_0 = 0 \). On one hand, Lemma guarantees that \( A_X^* \nu(t) = 0 \) for the dual holonomy field \( \nu \) along \( v \) with \( v(0) = \nu_0 \). On the other hand, holonomy fields have bounded norm whenever \( \tau \) admits a compact structure group, i.e., there is a constant \( L \) such that \( |\nu(t)|^2 \leq L |\nu(0)|^2 \) for every \( t \) (see [Spe18 Proposition 3.4]). Applying Equation (15), since \( g \) has positive vertizontal curvature, \( K_g \) is a tensor and \( \dot{c} \) has constant unit norm; there exists a constant \( \kappa > 0 \) for which
\[ \frac{d^2}{dt^2} |\nu(t)|^2 \geq 2 K_g(\dot{c}, \nu) \geq \kappa |\nu(t)|^2 \]
which contradicts the boundedness of \( |\nu(t)|^2 \).

4. Nonnegative and positive sectional curvature on some principal bundles

In this section, we prove the remaining results. Before proceeding, let us recall that Cheeger deformations have strong regularization properties. Namely, let \( (M, g) \) be a compact Riemannian manifold with an effective isometric action by a compact Lie group \( G \). Suppose \( g \) denotes a one-parameter metric family obtained from \( g \) via Cheeger deformations. In that case, we can check from the expression to \( P_t = P(1 + tP)^{-1} \), see Equation (1), that as \( t \to \infty \) the Riemannian metric \( g_t \) degenerates. Considering this, in [SSW15], it is shown that fixed any nonnegative integer \( p \), for any compact subset of the regular stratum of the \( G \)-action on \( M \), re-scaling the fibers of the fiber bundle \( \pi : (M^{reg}, g_t) \to M^{reg}/G \) with the same parameter \( t \) guarantees the convergence of \( \overline{g}_t := t g_t|_V + g|_{\mathcal{V}} \), in the \( C^p \)-topology, to a Riemannian metric with totally geodesic fibers. It is worth pointing out that Searle–Solórzano–Wilhelm’s approach in [SSW15] is curvature independent, but since we are primarily interested in curvature properties, we prove:

Theorem 4.1. Let \( G \to (\mathcal{P}, g) \to B \) be a compact Riemannian principal bundle with \( G = S^3, SO(3) \). If \( B \) has positive sectional curvature and the submersion is fat, then for any integer \( p \geq 2 \), the limit metric \( \overline{g}_\infty := \lim_{t \to \infty} \overline{g}_t := t g_t|_V + g|_{\mathcal{V}} \) has: totally geodesic fibers, positive vertizontal curvature, and the Riemannian metric induced on \( B \) has positive sectional curvature.

Proof. Fixed an integer \( p \geq 2 \), we first observe that the limit metric \( \overline{g}_\infty \), which exists due to Searle–Solórzano–Wilhelm’s result ([SSW15 Theorem A]), has positive vertizontal curvature. Indeed, since the O’Neill tensor is metric independent, to compute the \( g |_V \)-dual to it regarding the corresponding \( g \)-dual one proceeds as:
\[ \overline{g}_t(X,Y,V) = g(A_X Y, t A_V Y) \]
\[ = g(A_X Y, t(t+1)^{-1} Y) \]
\[ = g(Y, A_X^* (t(t+1)^{-1} Y)). \]

But since
\[ \overline{g}_t(X,Y,V) = \overline{g}_t(Y,(A^*_X) Y) = g(Y,(A^*_X) Y) \]
one concludes that
\[ (A^*_X) Y = A_X^*(t(t+1)^{-1} Y). \]
So it follows that, as operators,
\[ \lim_{t \to \infty} (A^*_X) Y = A_X^* P^{-1}. \]

Then, since the vertizontal curvature of a Riemannian metric with totally geodesic fibers is always positive under the assumption of fatness ([GW09 Corollary 1.5.1, p.28]), the first part of the claim follows. It is left to prove that the Riemannian metric induced on \( B \) has positive sectional curvature.
Let $Q$ be any bi-invariant metric on $G$ and $P$ be such that $g\bigg|_\gamma (\cdot,\cdot) = Q(P,\cdot)$. Note that according to Equation 19 in the appendix [A], the O’Neill’s submersion formula [GW09] Corollary 1.5.1, p. 28 and the definition of $C_t$

$$K_{g_t}(X,Y) = (1-t)K_{\delta}^g + tK_{g_t}(X,Y)$$

$$= (1-t)\left\{K_{\delta}(X,Y) + 3[(1+tp)^{-1/2}A_X Y_\delta^2] + tK_{g_t}(X,Y)\right\}$$

$$= 3[(1+tp)^{-1/2}A_X Y_\delta^2 + K_{g_t}(X,Y) - 3t(1+tp)^{-1/2}A_X Y_\delta^2].$$

Moreover, according to Equation (7)

$$K_{g_t}(X,Y) = K_g(X,Y) + 3t(1+tp)^{-1/2}PA_X Y_\delta^2.$$ 

Therefore,

$$K_{g_t}(X,Y) = K_g(X,Y) + 3t\left\{[(1+tp)^{-1/2}PA_X Y_\delta^2 - 3[(1+tp)^{-1/2}A_X Y_\delta^2]\right\} + 3[(1+tp)^{-1/2}A_X Y_\delta^2].$$

Since the induced metric $\tilde{h}$ on the base via $g_t$ has sectional curvature given by

$$K_{\tilde{h}}^t = K_{g_t}(X,Y) + 3t(1+tp)^{-1/2}A_X Y_\delta^2$$

one gets that

$$K_{\tilde{h}}^t = K_g(X,Y) + 3t(1+tp)^{-1/2}PA_X Y_\delta^2 + 3[(1+tp)^{-1/2}A_X Y_\delta^2]$$

and letting $t \to \infty$ yields $K_{\tilde{h}}^t(X,Y) = K_g(X,Y) > 0$. 

\[\square\]

**Theorem 4.2.** Let $\pi: G \to (\mathcal{S}, g) \to (B, h)$ be a Riemannian principal bundle with $G = S^3, SO(3)$. If $g$ has positive sectional curvature, then for any integer $p \geq 2$, the bundle $\pi$ admits a $C^p$-Riemannian submersion metric with positive sectional curvature and totally geodesic leaves if, and only if, $\pi$ is fat.

Observe that Theorem 4.2 generalizes the results due to Chaves–Derdzinski–Rigas and Yang, [Zilb] Theorem 5.1, p. 27], [CDR92], that are restricted to connection metrics. Also, see a different form given by Theorem G in [CSS18]. Related to it, observe that we verify Conjecture 4.3 in the same reference under the assumption of fatness.

Next, we prove Theorem 4.2 (Theorem 4.3 below). Aiming to, we recall the concept of “doubly ruled surface” in a Riemannian manifold $M$, as introduced in [MT11]. A doubly ruled surface in a Riemannian manifold $(M,g)$ consists of a surface $S$ which admits a transversal pair of smooth foliations by geodesics of $M$. By a “doubly ruled parameterized surface” we mean a special type of doubly ruled surface; to know, a doubly ruled surface whose geodesic foliations can be obtained from taking constant parameters of a single parameterization $(s,t) \mapsto f(s,t) \in M$.

Proposition 1.4 in [MT11] ensures that if $X$ is a basic horizontal vector at a point $p \in M$ and $V$ is a vertical vector at $p$, then $\{X,V,-A_X^* V\}$ consists of a good triple. In this manner, if $V(t)$ denotes the Jacobi field along $t \mapsto \exp(tX)$ with $V(0) = V$ and $V'(0) = -A_X^* V$, and $X(s)$ denotes the Jacobi field along $s \mapsto \exp(sV)$ with $X(0) = X$ and $X'(0) = -A_X^* V$ then $\exp(s \cdot V(t)) = \exp(t \cdot X(s))$, $\forall s,t \in \mathbb{R}$ and $f(s,t) := \exp(t \cdot V(s)) = \exp(s \cdot X(t))$ is a doubly ruled parameterized surface. As in [MT11], such a doubly ruled parameterized surface is the vertical surface induced by lifting a single geodesic in the base space to points along a geodesic in a fiber.

Items (2) and (3) on the theorem below shall follow via exploiting the derivatives of the doubly ruled parameterized surface described before.

**Theorem 4.3.** Let $(M,g)$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ with bounded holonomy (Definition 1.3 in [Spe18]). Alternatively, for simplicity, since it suffices to this paper, the foliation $\mathcal{F}$ is induced by the fibers of a Riemannian submersion with a compact structure group. If the foliation is totally geodesic for $g$, then

1. $A_X^* V$ is basic along the geodesic generated by $V$ provided if $X$ is basic
2. $R_g(X,A_X^* V,A_X^* V,V) = 0$. Moreover, $R_g (X,V,A_X^* V,Y) = \langle \nabla_X A_X^* V, A_X^* V \rangle$
3. If the foliation is fat then $R_g(X,V,A_X^* V, X) = 0$.

**Proof.** (1) Recall that under the assumption of totally geodesic leaves, a basic vector field $Z$ can be characterized by $\nabla_V Z = -A_Z^* V$. Moreover, since the fibers are totally geodesic, we have that $\nabla_V$ preserves both $\mathcal{H}$ and $\mathcal{Y}$. 

On the other hand, if $X$ and $Y$ are basic horizontal fields, then $-A_X^*V$ is a horizontal vector and

$$
-\langle \nabla_V A_X^*V, Y \rangle = -V\langle A_X^*V, Y \rangle - \langle A_X^*V, A_Y^*V \rangle
= -V\langle V, A_XY \rangle - \langle A_X^*V, A_Y^*V \rangle
= -\langle V, \nabla_V (A_XY) \rangle - \langle A_X^*V, A_Y^*V \rangle
= -\langle V, \nabla_V (A_XY) \rangle + A_XA_Y^*V.
$$

Finally, since $A_XY$ is a Killing field whenever $X$ and $Y$ are basic it holds that

$$
-\langle V, \nabla_V (A_XY) \rangle = \langle V, \nabla_V (A_XY) \rangle
$$

hence $\langle V, \nabla_V (A_XY) \rangle = 0$ and

$$
-\langle \nabla_V A_X^*V, Y \rangle = -\langle A_X^*V, A_Y^*V \rangle
= -\langle A_Y(A_X^*V), V \rangle
= \langle A_{A_X^*V}V, V \rangle
= (Y, A_{A_X^*V}V).
$$

Hence, we conclude that $\nabla_V (A_X^*V) = -A_{A_X^*V}V$, i.e., $A_X^*V$ is a horizontal field satisfying the equation that characterizes basic fields.

(2) Note that if $X$ is basic and $[X, V] = 0$ then assuming (1), denote by $s \mapsto \gamma(s)$ the geodesic generated by $V$ and by $t \mapsto c(t)$ the geodesic generated by $X$. Let $X(s)$ the Jacobi field along $\gamma$ which satisfies $\frac{\nabla}{ds}X(s) = -A_{X(s)}^*(\gamma(s))$ and $V(t)$ be the dual-holonomy (hence Jacobi) field along $c$ which satisfies $\frac{\nabla}{dt}V(t) = -A_{c(t)}^*(V(t))$. As discussed earlier, and following Proposition 1.4 in [MT11], one recalls that $f(s, t) = \exp(t \cdot X(s)) = \exp(s \cdot V(t))$ is a doubly ruled parameterized surface so

$$
0 = \frac{\partial^2}{\partial t \partial s} |A_X^*V|^2_g = \frac{\partial^2}{\partial s \partial t} |A_X^*V|^2_g
= 2\frac{\partial}{\partial s} \langle \nabla A_X^*V, A_X^*V \rangle
= 2\langle \nabla \frac{\partial}{\partial s} A_X^*V, A_X^*V \rangle + 2\langle \frac{\partial}{\partial t} A_X^*V, \nabla A_X^*V \rangle
= 2\langle \nabla \frac{\partial}{\partial s} A_X^*V, A_X^*V \rangle - 2\langle \frac{\partial}{\partial t} A_X^*V, A_X^*V \rangle
= 2\langle -R(X, V)A_X^*V + \frac{\nabla}{dt} A_X^*V, A_X^*V \rangle - 2\langle \frac{\partial}{\partial t} A_X^*V, A_X^*V \rangle
= 2\langle -R(X, V)A_X^*V - \frac{\nabla}{dt} A_{A_X^*V}V, A_X^*V \rangle - 2\langle \frac{\partial}{\partial t} A_X^*V, A_X^*V \rangle
= -2\langle R(X, V)A_X^*V, A_X^*V \rangle - 2\langle \nabla A_{A_X^*V}V, A_X^*V \rangle - 2\langle \frac{\partial}{\partial t} A_X^*V, A_X^*V \rangle
= -2\langle R(X, V)A_X^*V, A_X^*V \rangle - 2\frac{d}{dt} \langle A_{A_X^*V}V, A_X^*V \rangle
= -2\langle R(X, V)A_X^*V, A_X^*V \rangle - 2\frac{d}{dt} \langle V, A_{A_X^*V}A_X^*V \rangle
= -2\langle R(X, V)A_X^*V, A_X^*V \rangle - 2\frac{d}{dt} 0
= -2\langle R(X, V)A_X^*V, A_X^*V \rangle.
$$
We now prove that \( \langle R(X, V)A^*_X V, Y \rangle = \langle \nabla_X A^*_X V, A^*_X V \rangle \). This follows easily since
\[
0 = \frac{\partial}{ds} \left( \frac{\partial}{dt} A^*_X V, Y \right) \\
= \left< \frac{\partial}{ds} \frac{\partial}{dt} A^*_X V, Y \right> - \left< \frac{\partial}{dt} A^*_X V, \frac{\partial}{ds} A^*_X V \right> \\
= - \langle R(X, V)A^*_X V, Y \rangle + \left< \frac{\partial}{dt} \frac{\partial}{ds} A^*_X V, Y \right> - \left< \frac{\partial}{dt} A^*_X V, \frac{\partial}{ds} A^*_X V \right>
\]
\[
= \langle R(X, V)A^*_X V, Y \rangle + X[(V, A_Y A^*_X V)] - \left< \frac{\partial}{dt} A^*_X V, A^*_X V \right>
\]
\[
= - \langle R(X, V)A^*_X V, Y \rangle + X(A^*_X V, A^*_X V) - \langle \nabla_X A^*_X V, A^*_X V \rangle
\]
\[
= - \langle R(X, V)A^*_X V, Y \rangle + \langle \nabla X A^*_Y V, A^*_X V \rangle.
\]
So if \( X = Y \) then \( \langle R(X, V)A^*_X V, A^*_X V \rangle = \frac{1}{2} \text{trace}(A^*_X V) \).

(3) We show that if the foliation is fat, it holds that for any unit vector \( g \), one can obtain an orthogonal basis to \( \mathcal{X}_p \) for which \( -a X = A^*_X V, a \in \mathbb{R} \), concluding the proof since in this case \( R_g(X, V, A^*_X V, X) = -a R(X, V, X) = 0 \). So let us prove the claim:

Due to the fatness assumption, fixing any vertical vector \( V \in \mathcal{V}_p \), it is possible to define a non-degenerate 2-form
\[
\omega_V : \mathcal{X}_p \times \mathcal{X}_p \to \mathbb{R}
\]
by
\[
\omega_V(X, Y) := g(A_X Y, V).
\]
Hence, pointwise, we see that
\[
\omega_V(X, A^*_X V) := g(A^*_X V, A^*_X V) = g(V, A^*_X V(A^*_X V)) = 0.
\]

But since \( \omega_V(X, X) = 0 \) it follows that both \( X, A^*_X V \) lie in \( \ker \omega_V(X, \cdot) \). Finally, if \( [\omega_V] \) represents the matrix of \( \omega_V \) seen as the linear map \( \omega_V : X \to ([\omega_V(X, \cdot)]^T) \), using that \( [\omega_V]^2 \) is symmetric, hence, diagonalizable by an orthonormal basis of eigenvectors, considering \( X \) one of such eigenvectors, one has
\[
\lambda X = [\omega_V(X, \cdot)]^T \omega_V(X, \cdot)
\]
\[
= [\omega_V(g(A_X \cdot, V)]^T \omega_V(X, \cdot)
\]
\[
= [g(\cdot, g(A_X \cdot, V)]^T \omega_V(X, \cdot)
\]
what finishes the proof.

**Corollary 4.4.** Let \( \langle M, g \rangle \) be a Riemannian manifold with a Riemannian foliation \( \mathcal{F} \) induced by the fibers of a Riemannian submersion with a compact structure group. If the foliation is totally geodesic for \( g \), then

(1) \( R_g(X, Y, V) = 0 \) for every \( Y \in \ker A_X : \mathcal{X} \to \mathcal{V} \)

(2) if the foliation is fat then \( R_g(X, Y, V) = 0 \) for every basic vectors \( X, Y \in \mathcal{X} \) and every \( V \in \mathcal{V} \).

**Proof.** (1) This follows from applying the Koszul formula for the Levi-Civita connection: One arrives at
\[
3g(\nabla_X (A_X Y), V) = X g(Y, A_X V) - g(\nabla_X V, A_X Y).
\]

Since we can extend \( X, Y, V \) by \( \nabla^H X = \nabla^H Y = \nabla^H V = 0 \) and \( V \) by a holonomy field along the geodesic generated by \( X \) one gets that \( g(\nabla_X (A_X Y), V) = R_g(X, Y, V) \). Due to the extension of \( V \) to holonomy fields one gets \( g(\nabla_X V, A_X Y) = g(S_X V, A_X Y) = 0 \). Moreover, if \( Y \in \ker A_X \) then \( Y \perp A_X V \) and the claim follows.

(2) If \( X = 0 \) or \( Y = 0 \) there is nothing to prove. Moreover, if \( X \) is parallel to \( Y \), then the result holds triviality due to the anti-symmetries of the Riemannian tensor. Therefore, without loss of generality, we can assume \( X, Y \neq 0 \) and \( X \perp Y \). We also assume that \( V \neq 0 \); otherwise the claim is trivially true.

Due to the fatness assumption, one derives that the map \( A^*_X : \mathcal{V} \to \mathcal{X}_p \cap (\mathbb{R} \{X\})^\perp \) is injective. Moreover, that \( Z \perp \text{im} A^*_X \) if and only if, \( Z \in \ker \{ A_X : \mathcal{X}_p \cap (\mathbb{R} \{X\})^\perp \to \mathcal{V} \} \). We thus have a unique decomposition \( Y = Ya^*_X V + Z \) for \( Z \in \ker \{ A_X : \mathcal{X}_p \cap (\mathbb{R} \{X\})^\perp \to \mathcal{V} \} \).
Let \( x \in \mathbb{R}^n \) for instance, that: fix \( p \) and extend \( \mathbf{R}^n \). Equation (9) (the WNN condition), computed extending \( \mathbf{R}^n \) gives us a special case:

\[
R_g(X,Y,V) = R_g(X,yA^*_X V + Z, yA^*_X V + Z) \\
= y^2 R_g(X,A^*_X V, A^*_X V) + R_g(X,Z,Z,V) + 2y R_g(X,A^*_X V, Z,V) \\
= 0 + R_g(X,Z,Z,V) + 2y R_g(X,A^*_X V, Z,V),
\]

where the last equality comes from item (2). Also observe that item (3) ensures \( R_g(X,Z,Z,V) = 0 \). Furthermore, item (2) once more guarantees

\[
R_g(X,A^*_X V, Z,V) = -\langle \nabla_X A^*_X V, A^*_X V \rangle.
\]

Since \( R_g \) is a tensor we can extend \( A^*_X V \) via horizontal lifts to the computations ensuring that \( \nabla_X A^*_X V = 0 \), and so

\[
R_g(X,A^*_X V, Z,V) = X\langle A^*_X V, A^*_X V \rangle.
\]

Now the proof is done since, due to the flatness assumption, fixing any vertical vector \( V \in \mathcal{V}_p \), we can write

\[ aZ = A^*_X V \]

for some \( a \in \mathbb{R} \) (recall the proof of item \( 3 \) of Theorem \( 4.3 \) and so \( R_g(X,A^*_X V, Z,V) = X\langle aZ, V \rangle \).

**Remark 2** Corollary \( 4.4 \) is local. Corollary \( 4.4 \) is local. The hypotheses and conclusions work pointwise, meaning, for instance, that: fix \( p \in M \). Then

1. \( R_g(X,Y,Y,V) = 0 \) for every \( Y \in \ker X : \mathcal{H}_p \to \mathcal{V}_p \)
2. if the foliation is fat at \( p \in M \) then \( R_g(X,Y,Y,V) = 0 \) for every basic vectors \( X,Y \in \mathcal{H}_p \) and every \( V \in \mathcal{V}_p \)
3. previous item holds in a more general aspect: fixed a vertical vector \( V \in \mathcal{V}_p \), if \( A^*_X V \neq 0 \) for every horizontal vector \( X \), then \( R_g(X,Y,Y,V) = 0 \) for every \( X,Y \in \mathcal{H}_p \). See, for instance, the proof of Item \( 3 \) in Theorem \( 4.3 \). The vector \( V \) is named a fat vector – [GW09, p. 64].

Next, we achieve the proof of the final claim in Theorem \( 4.1 \) through the following proposition.

**Proposition 4.5.** Let \( \pi : (M,\mathcal{H}) \to (B,\mathcal{B}) \) be a submersion with an adapted metric \( g \) such that \( \langle R(X,V)A^*_X V, X \rangle = 0 \). Suppose \( \pi \) has a compact structure group. Then the WNN condition is satisfied.

**Proof.** According to Theorem 1.5.1 in [GW09], extending \( V \) as a holonomy field along the geodesic generated by \( X \) for the computations we have

\[
\langle R(X,V)A^*_X V, X \rangle = \langle (\nabla_X A)_X A^*_X V, V \rangle - \langle S_X A_X A^*_X V, V \rangle - \langle V, S_X A_X A^*_X V \rangle \\
= \langle \nabla_X (A_X A^*_X V), V \rangle - \langle A_X (\nabla_X (A^*_X V)), V \rangle - 2\langle A_X A^*_X V, S_X V \rangle \\
= X\langle A^*_X V, A^*_X V \rangle - \langle A_X (\nabla_X (A^*_X V)), V \rangle - 2\langle A_X A^*_X V, S_X V \rangle \\
= X\langle A^*_X V, A^*_X V \rangle - \langle A_X (\nabla_X (A^*_X V)), V \rangle - 2\langle A_X A^*_X V, S_X V \rangle \\
= X\langle A^*_X V, A^*_X V \rangle - \langle A_X (\nabla_X (A^*_X V)), V \rangle - 2\langle A_X A^*_X V, S_X V \rangle \\
= X\langle A^*_X V, A^*_X V \rangle - \langle A_X (\nabla_X (A^*_X V)), V \rangle - 2\langle A_X A^*_X V, S_X V \rangle \\
= \frac{1}{2} X\langle A^*_X V, A^*_X V \rangle
\]

So \( \langle R(X,V)A^*_X V, X \rangle = 0 \) if, and only if, \( \frac{1}{2}X\langle A^*_X V, A^*_X V \rangle = \langle A_X (\nabla_X (A^*_X V)), V \rangle \). On the other hand, the right-hand-side of Equation \( 9 \) (the WNN condition), computed extending \( V \) by holonomy fields along the geodesic generated by \( X \), is given by

\[
\langle (\nabla_X A)_X V + A_X S_X V, A^*_X V \rangle = \langle \nabla_X (A_X V), A^*_X V \rangle = \langle \nabla_X (A_X V), A^*_X V \rangle + 2\langle A_X (S_X V), A^*_X V \rangle = \langle \nabla_X (A_X V), A^*_X V \rangle + 2\langle A_X (S_X V), A^*_X V \rangle
\]

Since by assumption we have \( \frac{1}{2}X\langle A^*_X V, A^*_X V \rangle = \langle A_X (\nabla_X (A^*_X V)), V \rangle \) we get that the right-hand-side of Equation \( 9 \) is given by \( \frac{1}{2}X\langle A^*_X V, A^*_X V \rangle \). The proof is finished by contradiction: Suppose we can find a point \( p \in M \), \( \tau > 0 \) and unit vectors \( X \in \mathcal{H}_p, V \in \mathcal{V}_p \) for which

\[
\tau\langle A^*_X V, A^*_X V \rangle < \frac{3}{2} X\langle A^*_X V, A^*_X V \rangle.
\]
Computing the former along the geodesic \( c \) generated by \( X \) and extending \( V \) along it by a holonomy field \( \xi \), let us denote \( u(t) := |A^*_\xi \xi|^2 \). We have
\[
\frac{2}{3} \tau u(t) < u'(t)
\]
Thus \( u(0) \neq 0 \) and \( u(t) \) is increasing along \( X \), so \( A^*_\xi \xi \) is unbounded along \( c \). We derive a contradiction since \( \pi \) is supposed to have a compact structure group, and so there is a constant \( L \) such that \( |\xi(t)|^2 \leq L |\xi(0)|^2 \) for every \( t \) (see [Spe18, Proposition 3.4]).

We finally prove Theorem 4.2.

**Proof of Theorem 4.2.** In what follows, assume that \( X, Y \in \mathcal{H} \) are \( g \)-orthogonal and unit, the same for \( V \in \mathcal{Y} \). The pairs \( X, Y + V \) are referred to as CDR-pairs (see [CDR92, Lemma 1, p. 156]).

We now prove that if \((\mathcal{P}, g)\) has positive sectional curvature, then the regularized metric \( \tilde{g}_0 \) obtained from Theorem 4.1 has positive sectional curvature after a finite Cheeger deformation if, and only if, the submersion \( \pi : (G, Q) \to (\mathcal{P}, g) \to (B, h) \) is fat, where \( G = S^3 \) or \( SO(3) \). This statement holds more generally: positive sectional curvature for \( h \) is sufficient instead of positive sectional curvature for \( g \).

Recalling the formulas given by Equations (7) and (8),
\[
(17) \quad \kappa_i(X + U, Y + V) = \kappa_0(X + U, Y + V) + \frac{3}{4} |[PU, PV]|_Q^2 + \zeta_i(X + U, Y + V),
\]
where
\[
\zeta_i(X + U, Y + V) = 3 \left( (1 + t^P)^{-1/2}P\nabla^X + U \bigg| (1 + t^P)^{-1/2} \right) \frac{1}{Q} [PU, PV]_Q^2,
\]
we first observe that it is necessary and sufficient for \( \tilde{g}_0 \) to admit positive sectional curvature after a finite Cheeger deformation that, for any \( \lambda \in \mathbb{R} \), \( q_i(\lambda) = \kappa_i(X, \lambda Y + V) > 0 \) for \( t \) sufficiently large and uniformly for the \( \tilde{g}_0 \)-orthonormal \( X, Y \in \mathcal{H}, V \in \mathcal{Y} \), for any \( p \in \mathcal{P} \), see, for instance, the proof of Theorem 1.1 in [CGS22] or obtained from Theorem 11 in the Introduction (both results are equivalent under totally geodesic fibers).

Since
\[
q_\infty(\lambda) = \lim_{t \to \infty} q_i(\lambda) := \lim_{t \to \infty} \kappa_i(X, \lambda Y + V) = K^\perp_{\tilde{g}_0}(X, \lambda Y + V) + 3 |\nabla^X \lambda Y + V|_{\tilde{g}}^2,
\]
one can check that
\[
(18) \quad q_\infty(\lambda) = \lambda^2 K_{\tilde{g}_0}(X, Y) + 2 \lambda R^\perp_{\tilde{g}_0}(X, Y, V, X) + |A^X_\lambda P^{-1} V|_{\tilde{g}}^2,
\]
since \( \nabla^X Y = A_X Y \) and \( \nabla^X V = 0 \) because such a derivative is taken concerning the Levi-Civita connection of \( \tilde{g}_0 \), and for such a metric, the foliation is totally geodesic.

In what follows, we prove that if the submersion \( \pi \) is fat, then \( p_i(\lambda) > 0 \) for some \( t \gg 1 \) for any \( \lambda \in \mathbb{R} \), concluding the proof since we would have built a metric of positive sectional curvature with totally geodesic leaves, so fatness is also necessary.

Now recall that the dual to \( A_X \) in the metric \( \tilde{g}_0 \) is \( A_X^\perp P^{-1} \), that is, it is the \( g \)-dual of \( A_X \), composed with the isomorphism \( P^{-1} \). Since \( A_X^\perp \) has no kernel, so has no kernel \( A_X \). In this manner, Item 2 in Corollary 4.4 ensures that \( \lambda R^\perp_{\tilde{g}_0}(X, Y, V, X) = 0 \) (interchanging the roles of \( X \) and \( Y \) in such statement). So assume by contradiction that for each \( t > 0 \), there is \( \lambda > 0 \) and a \( \tilde{g}_0 \)-orthonormal plane \( X + V \wedge Y \) for which \( q_i(\lambda) \leq 0 \). Equation (18) implies that \( R^\perp_{\tilde{g}_0} \neq 0 \), a contradiction.

As a consequence of Theorem 4.3, one has:

**Corollary 4.6.** A \( SO(3), S^3 \)-principal Riemannian bundle \( \mathcal{P} \to B \) over a positively curved manifold carries a metric of positive sectional curvature if and only if it is fat.

**Proof.** Suppose a given \( SO(3), S^3 \)-principal Riemannian bundle \( \pi : \mathcal{P} \to B \) over a positively curved Riemannian manifold \( (B, h) \) is fat. Considering \( \pi \) a connection metric (hence having totally geodesic fibers), as in [GW09, Proposition 2.7.1, p. 97], we can start with such a metric to apply the arguments in the proof of Theorem 4.2, obtaining that there exists a Riemannian submersion metric on \( \pi \) with positive sectional curvature. Or we could follow the “Sketch of the proof of Theorem 3” in the Introduction.

For the converse, let \( \pi : \mathcal{P} \to B \) be a \( SO(3), S^3 \)-principal Riemannian bundle over a positively curved Riemannian manifold \( (B, h) \). Assume a Riemannian submersion metric \( g \) exists on \( \pi \) with positive sectional curvature. Let \( \tilde{g}_0 \) be the Searle–Solórzano–Wilhelm metric deformation used in Theorem 4.1. We know it converges to a metric with totally positive sectional curvature.
geodesic fibers as $t \to \infty$. Moreover, either due to Theorem [14] or Theorem 1.1 in [CGS22], there is $T > 0$ such that for any $t > T$ it holds that $\tilde{g}_t$ has positive sectional curvature after a finite Cheeger deformation if, and only if, there is $t \gg 1$ such that $K_{\tilde{g}_t}(X, Y + \lambda V) > 0$ for any $\lambda \in \mathbb{R}$ and uniformly for non-zero $X \perp Y \in \mathcal{H}$, $V \in \mathcal{Y}$ (recall the proof of Theorem 4.1). We show that the submersion $\pi$ is fat by analyzing the long-time behavior of a Cheeger deformation of a metric $\tilde{g}_t$ arbitrarily close to the metric with totally geodesic leaves given by Theorem 4.1. More precisely, we shall verify, due to the maintenance of positive sectional curvature on the base manifold $B$ given by Theorem 4.1 that the sectional curvature of vertical planes for $g$ is preserved by $\tilde{g}_t$ for $t$ sufficiently large, thus ensuring fatness.

Fixed any $p \geq 2$ take $t \gg 1$ such that $\tilde{g}_t$ is arbitrarily close in the $C^p$-topology to a metric with totally geodesic fibers. Since the curvature on horizontal planes for the metric $\tilde{g}_\infty$ is kept the same, such planes have positive sectional curvature. Moreover, since such a metric is arbitrarily close to the metric $\tilde{g}_\infty$ then Equation (13) rules the sign of the sectional curvature of $\tilde{g}_t$. This control is given by the polynomial $q(\lambda) = K_B(X, Y) + 2\lambda R_{\tilde{g}_t}(X, Y, V, X) + \lambda^2 K_{\tilde{g}_t}(X, V)$ where $K_B(X, V) > 0$, $K_{\tilde{g}_t}(X, V) \geq 0$. Given $\varepsilon > 0$ arbitrarily small take $t > 0$ such that considering the $C^p$-norm in the Grassmanian bundle of planes of the form $X \wedge Y + V$ for CDR-pairs we have (see Remark 3 for further details)

$$\|K_{\tilde{g}_t}(X, V) - |A^\varepsilon P^{-1}V|^2_{\tilde{g}_t}\|_\infty < \varepsilon$$

$$\|K_{\tilde{g}_t}(X, Y) - K_B(X, Y)\|_\infty < \varepsilon$$

$$\|R_{\tilde{g}_t}(X, Y, V, X) - R_{\tilde{g}_t}(X, Y, V, X)\|_\infty < \varepsilon.$$

Now assume by contradiction that there are $\lambda > 0$ and a $\tilde{g}_\infty$-orthonormal plane $X + V \wedge Y$ for which $q(\lambda) \leq 0$. We first check that $K_{\tilde{g}_t}(X, V) = |A^\varepsilon P^{-1}V|^2_{\tilde{g}_t} = 0$.

Since $q(\lambda) \leq 0$ then once $K_B(X, Y) > 0$ and $|A^\varepsilon P^{-1}V| \geq 0$ one arrives at $R_{\tilde{g}_t}(X, Y, V, X) \neq 0$. In this manner, Item 3 in Theorem 4.4 ensures that $A^\varepsilon P^{-1}V = 0$; – Following Item 3 in Remark 2 we have that $P^{-1}V$ is not a fat vector. Hence, $A^\varepsilon P^{-1}V = 0$ and $K_B(X, Y) + 2\lambda R_{\tilde{g}_t}(X, Y, V, X) \leq 0$. But this is another contradiction since Theorem 1.5.1 in [GW09] gives

$$R_{\tilde{g}_t}(X, Y, V, X) = \langle \langle \nabla_X A \rangle_X Y, V \rangle$$

$$= \langle \nabla_X (A_{X}Y), V \rangle - (A_{\nabla_X X} Y, V) - (A_{X} \nabla_X Y, V)$$

$$= \langle A_X Y, V \rangle - (A_{X} Y, \nabla_X V) - (A_{\nabla_X X} Y, V) - (A_{X} \nabla_X Y, V)$$

$$= \langle X Y, A^\varepsilon P^{-1}V \rangle - (A_{X} Y, \nabla_X V) - (A_{\nabla_X X} Y, V) - (A_{X} \nabla_X Y, V)$$

and the last quality is 0 due to the tensoriality of $R_{\tilde{g}_t}$ since we can take $\nabla^h X = \nabla_X Y = 0$ and extend $V$ by holonomy fields for the computations and $A^\varepsilon P^{-1}V = 0$. □

**Remark 3.** According to Section 2 in [SSW15], two smooth maps $\Phi, \Psi : M \to N$ are close in the weak $C^p$-topology if all of their values and partials up to order $p$ is close for fixed atlases for $M$ and $N$. If the atlases are both finite, this leads to a notion of $C^p$-distance (that may depend on the atlases but causes no problem here).

Given vector bundles $E_1$ and $E_2$ with Euclidean metrics and a bundle map $\Phi : E_1 \to E_2$, the $C^p$-norm $|\Phi|_{C^p}$ of $\Phi$ is obtained in the following way: Let $E_1^1$ be the unit sphere bundle of $E_1$. The quantity $|\Phi|_{C^p}$ is defined to be the $C^p$-distance from $\Phi|_{E_1^1}$ to the zero bundle map. In particular, the $C^p$-norm of a tensor corresponds to its $C^p$-distance to the zero section.
(B) \((\nabla_X A^*) X V = 0\) for every basic field \(X \in \mathcal{H}\) and every vertical field \(V \in \mathcal{V}\).

**Proof.** The main idea is to observe if \((M, g)\) is regarded with a Riemannian foliation \(\mathcal{F}\) with totally geodesic leaves, according to Munteanu and Tapp, Proposition 1.6 in [MT11], if \((M, g)\) is a locally symmetric space, then \(R_g(X, V)(-A^*_X V) = 0\) so the WNN condition is satisfied. Analogously, Theorem C implies that if \((\nabla_X A^*) X V = 0\), then it also holds that \(R_g(X, V)(-A^*_X V) = 0\) and the WNN condition is once more satisfied (Proposition 4.8). The proof is finished due to Item (1) of Theorem 3.3. □

We finish this paper proving Corollaries (E) and (F). The proof for Corollary (E) is obvious from the fact that \(S^3 \times S^2 \rightarrow S^2\) defines a trivial principal bundle with structure group \(S^3\). Since it is trivial, its connection is flat, so it cannot admit any fat connection (Corollary 2.10 in [Zilb]), and hence no \(S^3\)-invariant metric of positive sectional curvature. The same argumentation adapts to show that the trivial bundle \(S^2 \times S^2 \rightarrow S^2\) carries no Riemannian submersion metric with positive sectional curvature considering \(SO(3)\) as its structure group.

Finally, Item (1) of Corollary (F) follows once one observes that any \(S^2\)-bundle \(\pi\) is fat if, and only if, the corresponding principal bundle \(\pi: SO(3) \rightarrow \mathcal{P} \rightarrow B\) is fat (Proposition 2.2 in [Zilb]). Theorem 3.1 in [CGS22] guarantees that \(\pi\) has a metric of nonnegative sectional curvature if \(\pi\) has positive sectional curvature. Since the former condition is equivalent to the fatness of \(\pi\) (hence of \(\pi\)), we conclude the result. Item (2) follows similarly once positive curvature of \(\mathcal{P}\) ensures a metric of nonnegative sectional curvature on any associated fiber bundle with fibers of nonnegative sectional curvature; see the proof of Theorem 4.3 in [CGS22].

**APPENDIX A. GENERAL VERTICAL WARPLING FORMULAE**

The following formulae are presented in [GW09] Section 2.1.3, p. 52 with misprints. Since we shall use these, we restate them here with appropriate errata.

**Proposition A.1** (The sectional curvature of a General Vertical Warping). Let \((M, g)\) be a Riemannian manifold with a Riemannian foliation induced by a Riemannian submersion and \(\bar{g}\) be a metric of general vertical warping by a smooth and basic function \(h^{-1}\). Fix \(p \in M\). Below, let \(\sigma\) denote the \(g\)-dual to a given leaf’s second fundamental form \(S_X\). Then the following formulae for the sectional curvature \(\bar{K}\) of \(\bar{g}\) hold:

\[
\bar{K}(X, Y) = (1 - h^{-1})K_B(X, Y) + h^{-1}K(X, Y), \quad \forall X, Y \in \mathcal{H}_p;
\]

\[
\bar{K}(V_1, V_2) = (h^{-1} - h^{-2})K_F(V_1, V_2) + h^{-2}K_g(V_1, V_2) - \frac{1}{4}h^{-4}|V_1|^2|V_2|^2|\nabla h|^2 - \frac{1}{2}h^{-3}d\mu(V_1, V_2)|V_2|^2 - \frac{1}{2}h^{-3}d\mu(V_2, V_2)|V_1|^2, \quad \forall V_1, V_2 \in \mathcal{V}_p
\]

\[
g(V_1, V_2) = 0;
\]

\[
\bar{K}(X, V) = K_g(X, V)h^{-1} - h^{-1}(1 - h^{-1})|A^*_X V|^2 - h^{-2}d\mu(X)g(S_X V, V) - \frac{1}{4}\left( -2\text{Hess } h(X, X) + 3d\mu(X)\right)h^{-2}|V|^2, \quad \forall X \in \mathcal{H}_p, \forall V \in \mathcal{V}_p.
\]

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