THE NEUMANN PROBLEM FOR A CLASS OF FULLY NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish a global $C^2$ estimates to the Neumann problem for a class of fully nonlinear elliptic equations. By the method of continuity, we establish the existence theorem of $k$-admissible solutions of the Neumann problems.

1. Introduction

In this paper, we consider the $k$-admissible solutions of the Neumann problem of the fully nonlinear equations

$$S_k(W) = f(x), \quad \text{in} \quad \Omega,$$

where the matrix $W = (w_{\alpha_1 \cdots \alpha_m, \beta_1 \cdots \beta_m})_{C_n \times C_n}$, for $2 \leq m \leq n - 1$ and $C_n^m = \frac{n!}{m!(n-m)!}$, with the elements as follows,

$$w_{\alpha_1 \cdots \alpha_m, \beta_1 \cdots \beta_m} = \sum_{i=1}^n \sum_{j=1}^n u_{\alpha_1 \cdots \alpha_i - 1 \gamma, \beta_1 \cdots \beta_{i-1} \beta_i \beta_{i+1} \cdots \beta_m},$$

a linear combination of $u_{ij}$, where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\delta_{\alpha_1 \cdots \alpha_i - 1 \gamma, \beta_1 \cdots \beta_{i-1} \beta_i \beta_{i+1} \cdots \beta_m}$ is the generalized Kronecker symbol. All indexes $i, j, \alpha, \beta, \cdots$ come from $1$ to $n$. $f \in C^\infty(\Omega)$ is a positive function. And for any $k = 1, 2, \cdots, C_n^m$, 

$$S_k(W) = S_k(\lambda(W)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq C_n^m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},$$

where $\lambda(W) = (\lambda_1, \lambda_2, \cdots, \lambda_{C_n^m})$ is the eigenvalues of $W$. We also set $S_0(W) = 1$.

In fact, the matrix $W$ comes from the following operator $U^{[m]}$ as in [4] and [14]. First, we note that $(u_{ij})_{n \times n}$ induces an operator $U$ on $\mathbb{R}^n$ by

$$U(e_i) = \sum_{j=1}^n u_{ij} e_j, \quad \forall 1 \leq i \leq n,$$

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where \( \{e_1, e_2, \ldots, e_n\} \) is the standard basis of \( \mathbb{R}^n \). We further extend \( U \) to acting on the real vector space \( \wedge^m \mathbb{R}^n \) by

\[
U^{[m]}(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_m}) = \sum_{i=1}^{m} e_{\alpha_i} \wedge \cdots \wedge U(e_{\alpha_i}) \wedge \cdots \wedge e_{\alpha_m},
\]

where \( \{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_m} | 1 \leq \alpha_1 < \cdots < \alpha_m \leq n\} \) is the standard basis for \( \wedge^m \mathbb{R}^n \). Then \( W \) is the matrix of \( U^{[m]} \) under this standard basis. It is convenient to denote the multi-index by \( \overline{\alpha} = (\alpha_1 \cdots \alpha_m) \). We only consider the admissible multi-index, that is, \( 1 \leq \alpha_1 < \alpha_2, \cdots < \alpha_m \leq n \). By the dictionary arrangement, we can arrange all admissible multi-indexes from 1 to \( C_n^m \), and use \( N_{\overline{\alpha}} \) denote the order number of the multi-index \( \overline{\alpha} = (\alpha_1 \cdots \alpha_n) \), i.e., \( N_{\overline{\alpha}} = 1 \) for \( \overline{\alpha} = (12 \cdots m) \), \cdots. We also use \( \overline{\alpha} \) denote the index set \( \{\alpha_1, \cdots, \alpha_n\} \). It is not hard to see that

\[
W_{N_{\overline{\alpha}}N_{\overline{\beta}}} = w_{\overline{\alpha\beta}} = \sum_{i=1}^{m} u_{\alpha_i\alpha_i},
\]

and

\[
W_{N_{\overline{\alpha}}N_{\overline{\beta}}} = w_{\overline{\alpha\beta}} = (-1)^{|i-j|} u_{\alpha_i\beta_j},
\]

if the index set \( \{\alpha_1, \cdots, \alpha_m\} \setminus \{\alpha_i\} \) equals to the index set \( \{\beta_1, \cdots, \beta_m\} \setminus \{\beta_j\} \) but \( \alpha_i \neq \beta_j \); and also

\[
W_{N_{\overline{\alpha}}N_{\overline{\beta}}} = w_{\overline{\alpha\beta}} = 0,
\]

if the index sets \( \{\alpha_1, \cdots, \alpha_m\} \) and \( \{\beta_1, \cdots, \beta_m\} \) are differed by more than one elements. It follows that \( W \) is symmetric and is diagonal if \( (u_{ij})_{n \times n} \) is diagonal. The eigenvalues of \( W \) are the sums of eigenvalues of \( (u_{ij})_{n \times n} \).

Define the Garding’s cone in \( \mathbb{R}^n \) as

\[
\Gamma_k = \{ \mu \in \mathbb{R}^n | S_i(\mu) > 0, \forall 1 \leq i \leq k \}.
\]

Then we define the generalized Garding’s cone as, \( 1 \leq m \leq n, 1 \leq k \leq C_n^m \),

\[
\Gamma_k^{(m)} = \{ \mu \in \mathbb{R}^n | \{ \mu_{i_1} + \cdots + \mu_{i_m} | 1 \leq i_1 < \cdots < i_m \leq n \} \in \Gamma_k \in \mathbb{R}^{C_n^m} \}.
\]

Obviously, \( \Gamma_k = \Gamma_k^{(1)} \) and \( \Gamma_n \subset \Gamma_k^{(m)} \subset \Gamma_1 \). If the eigenvalues of \( D^2u \), denoted by \( \mu(D^2u) \), is contained in \( \Gamma_k^{(m)} \) for any \( x \in \Omega \), then equivalently \( \lambda(W) \in \Gamma_k \), such that the equation (1.1) is elliptic (see [4] or [18]). It is naturally to define \( k \)-admissible solution as follows.

**Definition 1.1.** We say \( u \) is \( k \)-admissible if \( \mu(D^2u) \in \Gamma_k^{(m)} \). In addition, if \( u \) is a solution of (1.1), we say \( u \) is a \( k \)-admissible solution.

If \( m = 1 \), (1.1) is known as the k-Hessian equation. In particular, (1.1) is the Poisson equation if \( k = 1 \), and the Monge-Ampère equation if \( k = n, m = 1 \).
For the Dirichlet problem in \( \mathbb{R}^n \), many results are known. For example, the Dirichlet problem of Laplace equation is studied in [9]. Caffarelli-Nirenberg-Spruck [13] and Ivochkina [16] solved the Dirichlet problem of Monge-Ampère equation, and Caffarelli-Nirenberg-Spruck [4] solved the Dirichlet problem of general Hessian equations even including the case considered here. For the general Hessian quotient equation, the Dirichlet problem is solved by Trudinger in [29]. Finally, Guan [8] treated the Dirichlet problem for general fully nonlinear elliptic equation on the Riemannian manifolds without any geometric restrictions to the boundary.

Also, the Neumann or oblique derivative problem of partial differential equations was widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book [9]. Also, we can see the book written by Lieberman [17] for the Neumann or oblique derivative problem of linear and quasilinear elliptic equations. In 1987, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equation in the celebrated paper [20]. For the the Neumann problem of k-Hessian equations, Trudinger [30] established the existence theorem when the domain is a ball, and he conjectured (in [30], page 305) that one can solve the problem in sufficiently smooth uniformly convex domains. Recently, Ma and Qiu [22] gave a positive answer to this problem and solved the the Neumann problem of k-Hessian equations in uniformly convex domains. After their work, the research on the Neumann problem of other equations has made many progresses (see [21] [6] [2] [33]).

For general \( m \), the \( W \)-matrix is quite related to the “\( m \)-convexity” or “\( m \)-positivity” in differential geometry and partial differential equations. We say a \( C^2 \) function \( u \) is \( m \)-convex if the sum of any \( m \) eigenvalues of its Hessian is nonnegative, equivalently, \( \mu(D^2u) \in \Gamma^{(m)}_{C^m} \) or \( \lambda(W) \in \Gamma^{(m)}_{C^m} \). Similarly, we can formulate the notion of \( m \)-convexity for curvature operator and second fundamental forms of hypersurfaces. There are large amount literature in differential geometry on this subject. For example, Sha [27] and Wu [34] introduced the \( m \)-convexity of the sectional curvature of Riemannian manifolds and studied the topology for these manifolds. In a series interesting papers, Harvey and Lawson [10] [11] [12] introduce some generally convexity on the solutions of the nonlinear elliptic Dirichlet problem, \( m \)-convexity is a special case. Han-Ma-Wu [14] obtained an existence theorem of \( m \)-convex starshaped hypersurface with prescribed mean curvature. More recently, in the complex space \( \mathbb{C}^n \) case, Tosatti and Weinkove [31] [32] solved the Monge-Ampère equation for \((n-1)\)-plurisubharmonic functions on a compact Kähler manifold, where the \((n-1)\)-plurisubharmonicity means the sum of any \( n-1 \) eigenvalues of the complex Hessian is nonnegative.
From the above geometry and analysis reasons, it is naturally to study the Neumann problem for general equation (1.1).

The methods of Ma and Qiu [22] for the problem with \( m = 1 \) can be generalized to our case. The key ingredient in the present paper is to understand the structure of \( W \), precisely, to replace the eigenvalues of \( D^2 u \) by the sums of them. For \( k \leq C_{n-1}^{m-1} = \frac{m}{n} C_n^m \), we obtain an existence theorem of the \( k \)-admissible solution with less geometric restrictions to the boundary. For \( m < \frac{n}{2} \) and \( k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m \), we can obtain an existence theorem if \( \Omega \) is strictly \((m, k_0)\)-convex (see Definition 1.2). It seems that as the degree of nonlinearity of the equation (1.1) increases, i.e., \( k \) becomes larger, the problem becomes more difficult to solve. Particularly, for \( m = n - 1 \), we get the existence of the \( k \)-admissible solution for \( k \leq n - 1 \) only except that of the strictly \((n - 1)\)-convex solution for \( k = n \). The author will continue to study this case in [7].

A \( C^2 \) domain \( \Omega \subset \mathbb{R}^n \) is convex, that is, \( \kappa_i(x) \geq 0 \) for any \( x \in \partial \Omega \) and \( i = 1, \ldots, n - 1 \), or equivalently, \( \kappa(x) \in \Gamma_{n-1} \) for any \( x \in \partial \Omega \), where \( \kappa(x) = (\kappa_1, \ldots, \kappa_{n-1}) \) denote the principal curvatures of \( \partial \Omega \) with respect to its inner normal \( -\nu \). Then, we say \( \Omega \) is a strictly \( k \)-convex domain if \( \kappa(x) \in \Gamma_k \). To state the results in precise way, we need a definition of \((m, k_0)\)-convexity as follows.

**Definition 1.2.** We say \( \Omega \) is a strictly \((m, k_0)\)-convex if \( \kappa(x) = (\kappa_1, \ldots, \kappa_{n-1}) \in \Gamma_{k_0}^{(m)} \) for any \( x \in \partial \Omega \). Obviously, \( \Gamma_{n-1} \subset \Gamma_{k_0}^{(m)} \) in \( \mathbb{R}^{n-1} \), if \( k_0 \leq n \).

We now state the main results of this paper as follows. The case \( k \leq C_{n-1}^{m-1} \) is easy to treat so we consider that first.

**Theorem 1.3.** Suppose \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\) is a bounded domain with \( C^4 \) boundary, \( 2 \leq m \leq n - 1 \) and \( 2 \leq k \leq C_{n-1}^{m-1} \). Denote \( \nu(x) \) the outer unit normal vector, and \( \kappa_{\min}(x) \) the minimum principal curvature at \( x \in \partial \Omega \). Let \( f \in C^2(\Omega) \) is a positive function, and \( a, b \in C^3(\partial \Omega) \) with \( a > 0, a + 2\kappa_{\min} > 0 \). Then there exists a unique \( k \)-admissible solution \( u \in C^{3,\alpha}(\bar{\Omega}) \) of the Neumann problem

\[
\begin{align*}
S_k(W) &= f(x), \quad \text{in } \Omega, \\
-\nu \cdot \nabla u &= -a(x)u + b(x), \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.6)

For \( k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m \), we can settle more cases if \( \Omega \) is strictly \((m, k_0)\)-convex as in the following theorem.

**Theorem 1.4.** Suppose \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\) is a strictly \((m, k_0)\)-convex bounded domain with \( C^4 \) boundary, \( 2 \leq m \leq \frac{n}{2} \) and \( k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m \). Denote \( \nu(x) \) the outer unit normal vector, and \( \kappa_{\min}(x) \) the minimum principal curvature at \( x \in \partial \Omega \). Let \( f \in C^2(\Omega) \)
is a positive function, and \( a, b \in C^3(\partial \Omega) \) with \( a > 0, \ a + 2\kappa_{\text{min}} > 0 \). Then there exists a unique \( k \)-admissible solution \( u \in C^{3,\alpha}(\Omega) \) of the Neumann problem

\[
\begin{aligned}
S_k(W) &= f(x), \quad \text{in } \Omega, \\
u &= -a(x)u + b(x), \quad \text{on } \partial \Omega.
\end{aligned}
\]

The rest of this paper is arranged as follows. In section 2, we give some basic properties of the elementary symmetric functions. In section 3 and section 4, we establish \( C^0 \) estimates and the gradient estimates, interior and global. Specifically, we extend the interior gradient estimates in Chou and Wang [5] to our cases. In section 5, we show the proof of the global estimates of second order derivatives. Finally, we can prove the existence theorem by the method of continuity in section 6.

2. Preliminary

In this section, we give some basic properties of elementary symmetric functions.

First, we denote by \( S_k(\lambda|\cdot) \) the symmetric function with \( \lambda_i = 0 \) and \( S_k(\lambda|i) \) the symmetric function with \( \lambda_i = \lambda_j = 0 \).

**Proposition 2.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( k = 1, \ldots, n \), then

\[
\begin{aligned}
\sigma_k(\lambda) &= \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n, \\
\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda|i) &= k \sigma_k(\lambda), \\
\sum_{i=1}^{n} \sigma_k(\lambda|i) &= (n-k) \sigma_k(\lambda).
\end{aligned}
\]

We denote by \( S_k(W|i) \) the symmetric function with \( W \) deleting the \( i \)-row and \( i \)-column and \( S_k(W|ij) \) the symmetric function with \( W \) deleting the \( i,j \)-rows and \( i,j \)-columns. We also define the mixed symmetric functions as follows, for \( A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}, 0 \leq l \leq k \leq n \),

\[
S_{k,l}(A, B) = \frac{1}{k!} \sum_{i,l} \delta_{j_1\cdots j_k-l}^{i_1\cdots i_k-l+1\cdots i_k} a_{i_1j_1} \cdots a_{i_k-j_k-l} b_{i_k-l+1\cdots j_k},
\]

where \( \delta_{j_1\cdots j_k-l}^{i_1\cdots i_k-l+1\cdots i_k} \) is the Kronecker symbol. It is easy to see that

\[
S_k(A + B) = \sum_{i=0}^{k} C_i^k S_{k,i}(A, B),
\]

where \( C_i^k = \frac{k!}{i!(k-i)!} \). Then we have the following identities.
Proposition 2.2. Suppose $A = (a_{ij})_{n \times n}$ is diagonal, and $k$ is a positive integer, then

$$\frac{\partial S_k(A)}{\partial a_{ij}} = \begin{cases} S_{k-1}(A|i), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Furthermore, suppose $W = (w_{\alpha\beta})_{C_m \times C_m}$ defined as in (1.2) is diagonal, then

$$\frac{\partial S_k(W)}{\partial u_{ij}} = \begin{cases} \sum_{i \in \alpha} S_{k-1}(W|N_{\alpha}), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof. For (2.5), see a proof in [18].

Note that

$$\frac{\partial S_k(W)}{\partial u_{ij}} = \sum_{\alpha, \beta} \frac{\partial S_k(W)}{\partial u_{ij}},$$

Using (1.3), (1.4), and (1.5), (2.6) is immediately a consequence of (2.5). \qed

Recall that the Garding’s cone is defined as

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n | S_i(\lambda) > 0, \forall 1 \leq i \leq k \}.$$

Proposition 2.3. Let $\lambda \in \Gamma_k$ and $k \in \{1, 2, \cdots, n\}$. Suppose that

$$\lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n,$$

then we have

$$S_{k-1}(\lambda|n) \geq \cdots \geq S_{k-1}(\lambda|k) \geq \cdots \geq S_{k-1}(\lambda|1) > 0,$$

$$\lambda_1 \geq \cdots \geq \lambda_k > 0, \quad S_{k-1}(\lambda|k) \geq C(n, k)S_k(\lambda),$$

$$\lambda_1S_{k-1}(\lambda|1) \geq \frac{k}{n}S_k(\lambda),$$

$$S_k^+(\lambda) \text{ is concave in } \Gamma_k.$$\hspace{1cm}(2.11)

where $C_n^k = \frac{n!}{k! (n-k)!}$ and $C(n, k)$ is a positive constant depends only on $n$ and $k$.

Proof. All the properties are well known. For example, see [18] or [15] for a proof of (2.8), [21] for (2.9), [5] or [13] for (2.10) and [4] for (2.11). \qed

The Newton-Maclaurin inequality is as follows,

Proposition 2.4. For $\lambda \in \Gamma_k$ and $k > l \geq 0$, we have

$$\left( \frac{S_k(\lambda)}{C_n^k} \right)^\frac{1}{t} \leq \left( \frac{S_l(\lambda)}{C_n^l} \right)^\frac{1}{t},$$

(2.12)
where $C_{nk}^k = \frac{n!}{k!(n-k)!}$. Furthermore we have

\[(2.13)\]
\[
\sum_{i=1}^{n} \frac{\partial S_k^i}{\partial \lambda_i} \geq \frac{1}{k} S_k^{k-1} \sum_{i=1}^{n} S_{k-1}(\lambda | i) = \frac{n-k+1}{k} S_k^{k-1} S_{k-1}(\lambda) \geq [C_{nk}^k]^\frac{1}{n}.
\]

Proof. See [25] for a proof of (2.12). For (2.13), we use (2.12) and Proposition 2.1 to get

\[
\sum_{i=1}^{n} \frac{\partial S_k^i}{\partial \lambda_i} \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\partial S_k}{\partial \lambda_i} \geq [C_{nk}^k]^\frac{1}{n}.
\]

Then we give some useful inequalities of elementary symmetric functions.

**Proposition 2.5.** Suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, $k \geq 1$, satisfies $\lambda_1 < 0$. Then we have

\[(2.14)\]
\[
\frac{\partial S_k(\lambda)}{\partial \lambda_1} \geq \frac{1}{n-k+1} \sum_{i=1}^{n} \frac{\partial S_k}{\partial \lambda_i}.
\]

and

\[(2.15)\]
\[
\sum_{i=1}^{n} \frac{\partial S_k(\lambda)}{\partial \lambda_i} \geq (-\lambda_1)^{k-1}, \quad \forall 1 \leq k \leq n.
\]

Proof. See Lemma 3.9 in [1] for the proof of (2.14), and [6] or [5] for (2.15).

The following proposition is useful to establishments of gradient estimates(for $f = f(x, u, Du)$) and double normal estimates(for $m \leq \frac{n}{2}$). This proposition also indicates the major difference between our cases($m \geq 2$) and the $k$-Hessian($m=1$).

**Proposition 2.6.** Let $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \geq \cdots \geq \mu_n$, $\lambda = \{\mu_1 + \mu_2 + \cdots + \mu_m | 1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq n\}$ and $2 \leq k \leq \frac{n-m}{n} C_n^m$. If $\mu \in \Gamma_k^{(m)}$ and $\mu_n < -\delta L < 0$, where $\delta$ is a small positive constant, then there exits a constant $\theta_1 = (\frac{\delta}{C_n^m})^{k-1}$ such that

\[(2.16)\]
\[
\sum_{i=1}^{C_n^m} \frac{\partial S_k(\lambda)}{\partial \lambda_i} \geq \theta_1 L^{k-1}.
\]

Furthermore, if in addition that $-\delta_1 L \leq \lambda_i \leq mL$, $\forall 1 \leq i \leq C_n^m$, with $\delta_1 = \frac{\delta}{C_n^m}^k$, then there exists a constant $\theta_2 = \frac{\delta_1^{k-1}}{2^{C_n^m} C_n^m}$, such that, for $1 \leq j \leq C_n^m$

\[(2.17)\]
\[
\frac{\partial S_k(\lambda)}{\partial \lambda_i} \geq \theta_2 \sum_{j=1}^{C_n^m} \frac{\partial S_k(\lambda)}{\partial \lambda_j}.
\]
Proof. Let \( \lambda_1 \geq \cdots \geq \lambda_{C_n^m} \). We consider the following two cases.

**Case 1.** \( \lambda_{C_n^m} < -\delta_1 L \), where \( \delta_1 = \frac{\delta}{(C_n^m)^4} \).

It is exactly the case in Proposition 2.5, so we have

\[
\sum_{i=1}^{C_n^m} \frac{\partial S_k(\lambda)}{\partial \lambda_i} \geq (\delta_1 L)^{k-1}. \tag{2.18}
\]

**Case 2.** \( \lambda_{C_n^m} \geq -\delta_1 L \).

We see that

\[
\lambda_{C_n^m} = \sum_{i=n-m+1}^{n-1} \mu_i + \mu_n \geq -\delta_1 L.
\]

Since \( \mu_n < -\delta L \) and \( \delta_1 < \frac{\delta}{2} \), we obtain

\[
\sum_{i=n-m+1}^{n-1} \mu_i \geq \frac{\delta}{2} L, \quad \mu_{n-m+1} > 0.
\]

It follows that

\[
\lambda_{C_n^m - C_{n-1}^m} \geq \sum_{i=n-m+1}^{n-1} \mu_i + \mu_{n-m} > \frac{\delta}{2} L. \tag{2.19}
\]

Now we can write

\[
\lambda_1 \geq \cdots \geq \lambda_p \geq \frac{\delta}{2} L \geq \lambda_{p+1} \geq \cdots \geq \lambda_q \geq 0 \geq \lambda_{q+1} \geq \cdots \lambda_{C_n^m} \geq -\delta_1 L.
\]

Denote \( \lambda' = (\lambda_1, \cdots, \lambda_p) \), \( \lambda'' = (\lambda_1, \cdots, \lambda_q) \), and \( \lambda''' = (\lambda_{q+1}, \cdots, \lambda_{C_n^m}) \). We point out that \( \lambda''' \) may be empty. From (2.19) we see that

\[
p \geq C_n^m - C_{n-1}^m \geq k,
\]

and, use \( \lambda_1 \leq mL \) (only for the second inequality of (2.20)) to get

\[
C_p^{k-1}(\frac{\delta}{2})^{k-1}L^{k-1} \leq S_{k-1}(\lambda') \leq C_p^{k-1}m^{k-1}L^{k-1}. \tag{2.20}
\]

We also have

\[
S_{k-1}(\lambda') \leq S_{k-1}(\lambda'') \leq (C_n^m - 1)S_{k-1}(\lambda'), \tag{2.21}
\]

since every element of \( \lambda'' \) is positive.

By Proposition 2.2 and 2.4, we have

\[
\sum_{i=1}^{C_n^m} \frac{\partial S_k(\lambda)}{\partial \lambda_i} = \sum_{i=1}^{C_n^m} S_{k-1}(\lambda'|i) = (C_n^m - k + 1)S_{k-1}(\lambda)
\]

\[
= (C_n^m - k + 1)[S_{k-1}(\lambda'') + \sum_{i=1}^{k-1} C_i^{k-1}S_{k-1,i}(\lambda'', \lambda''')), \tag{2.22}
\]
where $S_{k-1,i}(\lambda'', \lambda''')$ is the mixed symmetric function. Recall $\delta_1 = \frac{\delta^k}{(C^m_n)^4}$ and (2.20), such that:

\[
(2.23) \quad \left| \sum_{i=1}^{k-1} C_{k}^i S_{k-1,i}(\lambda'', \lambda''') \right| \leq \left( \frac{\delta}{2} \right)^k L^{k-1} \leq \frac{1}{2} S_{k-1}(\lambda') .
\]

Plug (2.21) and (2.23) into (2.22),

\[
(2.24) \quad \frac{(C^m_n - k + 1)}{2} S_{k-1}(\lambda') \leq \sum_{i=1}^{C^m_n} \frac{\partial S_k(\lambda)}{\partial \lambda_i} \leq C^m_n (C^m_n - k + 1) S_{k-1}(\lambda') .
\]

Note that we don’t need $\lambda_i \leq mL$ in the first inequality. Combining (2.18), (2.20) and (2.24), we prove the (2.16).

We also have

\[
(2.25) \quad S_{k-1}(\lambda|1) \geq S_{k-1}(\lambda'|1) + \sum_{i=1}^{k-1} C_{k}^i S_{k-1,i}(\lambda''|1, \lambda''').
\]

Due to $p \geq k$, $\delta_1 = \frac{\delta^{k-1}}{(C^m_n)^4}$, (2.20) and (2.24), we have

\[
(2.26) \quad S_{k-1}(\lambda|1) \geq \frac{1}{2} S_{k-1}(\lambda'|1) \geq \frac{\delta^{k-1}}{2^{k} m^{k-1} C^m_n} S_{k-1}(\lambda') \geq \frac{\delta^{k-1}}{2^{k} m^{k-1} (C^m_n)^3} \sum_{i=1}^{C^m_n} \frac{\partial S_k(\lambda)}{\partial \lambda_i} .
\]

Then we proved the (2.17) since $S_{k-1}(\lambda|i) \geq S_{k-1}(\lambda|1)$ for $1 \leq i \leq C^m_n$.

Finally, we give a key inequality which plays an important role in the establishment of the double normal derivative estimate (see Theorem 5.4).

**Proposition 2.7.** Suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, $k \geq 2$, and $\lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda_1 > 0$, $\lambda_1 \geq \delta \lambda_2$, and $\lambda_n \leq -\varepsilon \lambda_1$ for small positive constants $\delta$ and $\varepsilon$, then we have

\[
(2.26) \quad S_l(\lambda|1) \geq c_0 S_l(\lambda), \quad \forall l = 0, 1, \cdots, k - 1,
\]

where $c_0 = \min\left\{ \frac{\varepsilon^2 \delta^2}{2(n-2)(n-1)}, \frac{\varepsilon^2 \delta^2}{4(n-1)} \right\}$.

One can find a generalized inequality and the proof in [6]. For completeness we give a proof for our case as same as in [22].

**Proof.** For $l = 0$, (2.26) holds directly. In the following, we assume $1 \leq l \leq k - 1$.

Firstly, if $\lambda_1 \geq \lambda_2$, we have from (2.10)

\[
(2.27) \quad \lambda_1 S_{l-1}(\lambda|1n) \geq \frac{l}{n-1} S_l(\lambda|n).
\]
If $\lambda_1 < \lambda_2$, use $\lambda_1 \geq \delta \lambda_2$ and (2.8) to get

$$\lambda_1 S_{l-1}(\lambda|1n) \geq \delta \lambda_2 S_{l-1}(\lambda|2n) \geq \delta \frac{l}{n-1} S_l(\lambda|n).$$  \(2.28\)

It follows from (2.27) and (2.28) that

$$(-\lambda_n) S_{l-1}(\lambda|1n) \geq \delta \epsilon \frac{l}{n-1} S_l(\lambda|n) \geq \delta \epsilon \frac{l}{n-1} S_l(\lambda).$$  \(2.29\)

We use $S_l(\lambda) = S_l(\lambda|n) + \lambda_n S_{l-1}(\lambda|n) \leq S_l(\lambda|n)$, for $\lambda_n < 0$, in the second inequality. Then we consider the following two cases.

**Case 1.** $S_l(\lambda|1) \geq \theta (-\lambda_n) S_{l-1}(\lambda|1n)$, $\theta$ is a small positive number to be determined.

Use (2.29) directly to obtain

$$S_l(\lambda|1) \geq \theta \delta \epsilon \frac{l}{n-1} S_l(\lambda).$$  \(2.30\)

**Case 2.** $S_l(\lambda|1) < \theta (-\lambda_n) S_{l-1}(\lambda|1n)$.

From proposition 2.1 we have

$$(l + 1) S_{l+1}(\lambda|1) = \sum_{i=2}^n \lambda_i S_l(\lambda|1i) = \sum_{i=2}^n \lambda_i [S_l(\lambda|1) - \lambda_i S_{l-1}(\lambda|1i)]$$  \(\sum_{i=2}^n \lambda_i S_l(\lambda|1) - \sum_{i=2}^n \lambda_i^2 S_{l-1}(\lambda|1i)$$

$$\leq (n-2) \lambda_2 S_l(\lambda|1) - \lambda_n^2 S_{l-1}(\lambda|1n)$$

$$\leq \frac{n-2}{\delta} \theta - \epsilon \lambda_1 (-\lambda_n) S_{l-1}(\lambda|1n) = -\frac{\epsilon}{2} \lambda_1 (-\lambda_n) S_{l-1}(\lambda|n),$$

if we choose $\theta = \frac{\epsilon \delta}{2(n-2)}$ in the last equality. From (2.29), we have

$$S_{l+1}(\lambda|1) \leq -\frac{\epsilon^2 \delta m}{2(n-1)(l+1)} \lambda_1 S_l(\lambda),$$  \(2.32\)

then

$$S_l(\lambda|1) = \frac{S_{l+1}(\lambda) - S_{l+1}(\lambda|1)}{\lambda_1} \geq -\frac{S_{l+1}(\lambda|1)}{\lambda_1}$$

$$\geq \frac{\epsilon^2 \delta}{2(l+1)} \frac{l}{n-1} S_l(\lambda) \geq \frac{\epsilon^2 \delta}{4(n-1)} S_l(\lambda).$$  \(2.33\)

Hence (2.26) holds. \(\square\)

3. $C^0$ Estimate

Following the idea of Lions-Trudinger-Urbas [20], we prove the following theorem.
Theorem 3.1. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with $C^1$ boundary, and $\nu$ be the unit outer normal vector of $\partial \Omega$. Suppose that $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ is an $k$-admissible solution of the following Neumann boundary problem,

$$
\begin{cases}
S_k(W) = f(x), & \text{in } \Omega, \\
u = -a(x)u + b(x), & \text{on } \partial \Omega.
\end{cases}
$$

where $f > 0$ and $a, b \in C^3(\partial \Omega)$ with $\inf_{\partial \Omega} a(x) > \sigma$. Then

$$
\sup_{\Omega} |u| \leq \frac{C}{\sigma},
$$

where $C$ depends on $k$, $n$, $a$, $b$, $f$ and $\text{diam}(\Omega)$.

Proof. Because $f > 0$, the comparison principle tells us that $u$ attains its maximum on the boundary. At the maximum point $x_0 \in \partial \Omega$ we have

$$
0 \leq u_\nu(x_0) = (-au + b)(x_0).
$$

It implies that

$$
 u(x) \leq u(x_0) \leq \frac{\sup_{\partial \Omega} b}{\inf_{\partial \Omega} a}.
$$

Assume $0 \in \Omega$ and let $w = u - A|x|^2$. We obtain

$$
F[A|x|^2] \geq f = F[u],
$$

if we choose $A$ large enough depends on $k$, $n$ and $\sup f$. Similarly $w$ attains its minimum on the boundary by comparison principle. At the minimum point $x_1 \in \partial \Omega$ we have

$$
0 \geq w_\nu(x_1) = (-au + b)(x_1) - 2Ax_0 \cdot \nu.
$$

We use $w(x) \geq w(x_1)$ to get

$$
 u(x) \geq -\frac{\inf_{\partial \Omega} b - 2AL(L + 1)}{\sup_{\partial \Omega} a} \geq -\frac{\inf_{\partial \Omega} b - 2AL(L + 1)}{\inf_{\partial \Omega} a},
$$

where $L = \text{diam}(\Omega)$. Then we complete the proof of Theorem 3.1. \qed

4. Global gradient estimate

Throughout the rest of this paper, we always admit the Einstein’s summation convention. All repeated indices come from 1 to $n$. We will denote $F(D^2 u) = S_k(W)$ and

$$
F_{ij} = \frac{\partial F(D^2 u)}{\partial u_{ij}} = \frac{\partial S_k(W)}{\partial w_{\alpha\beta}} \frac{\partial w_{\alpha\beta}}{\partial u_{ij}}.
$$
From (1.3) and (2.5) we have, for any $1 \leq j \leq n$,
\[
F^i_i = \sum_{i \in \mathcal{A}} \frac{\partial S_k(W)}{\partial w_\alpha}.
\]

Throughout the rest of the paper, we will denote $F = \sum_{i=1}^n F^i_i = m \sum_{N_i=1}^{C^m_n} S_{k-1}(W|N_i)$ for simplicity.

4.1. **Interior gradient estimate.** Chou-Wang [5] gave the interior gradient estimates for $k$-Hessian equations. In a similar way, we will prove the following theorem.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain and $2 \leq k \leq \frac{n-m}{n} C^m_n$. Suppose that $u \in C^3(\Omega)$ is a $k$-admissible solution of the following equation,
\[
S_k(W) = f(x,u,Du), \quad \text{in } \Omega,
\]
where $f(x,z,p) \in C^1(\Omega \times [-M_0,M_0] \times \mathbb{R}^n)$ is a nonnegative function, $M_0 = \sup_{\Omega} |u|$. We also assume that
\[
|f|^C_0 + \sum_{i=1}^n |f_x|^C_0 + |f_z|^C_0 + \sum_{i=1}^n |f_{p_i}|^C_0 |Du|^C_0 \leq L_1(1 + |Du|^{2k-1})^2,
\]
for some constant $L_1$ independent of $|Du|^C_0$. For any $B_r(y) \subset \Omega$, we have
\[
\sup_{B_r(y)} |Du| \leq C_1 + C_2 \frac{M_0}{r},
\]
where $C_1$ depends only on $M_0$, $L_1$, $n$, $m$, and $k$, and $C_2$ depends only on $L_1$, $n$, $m$, and $k$. Moreover, if $f \equiv \text{constant}$, then $C_1 = 0$.

**Proof.** Assume $y = 0 \in \Omega$ and $B_r(0) \subset \Omega$. Choose the auxiliary function as
\[
G(x) = \rho(x)\varphi(u)|Du|^2,
\]
where $\rho(x) = (1 - \frac{x^2}{r^2})^2$ such that $|D\rho| \leq b_0 \frac{1}{r^2}$ and $|\nabla^2 \rho| \leq b_0^2$, with $b_0 = \frac{1}{r}$, and $\varphi(u) = (M - u)^{-\frac{1}{2}}$ with $M = 4M_0$. It is easy to see that
\[
\varphi^\prime - \frac{2(\varphi^\prime)^2}{\varphi} \geq \frac{1}{16}M^{-\frac{3}{2}}.
\]

Suppose $G$ attains its maximum at the point $x_0 \in \Omega = B_r(0)$. In the following, all the calculations are at $x_0$. First, we have
\[
0 = G_i(x_0) = \rho_i \varphi |Du|^2 + \rho u_i \varphi^\prime |Du|^2 + 2\rho \varphi u_k u_{ki}, \quad i = 1, \ldots, n.
\]
After a rotation of the coordinates, we may assume that the matrix $(u_{ij})_{n \times n}$ is diagonal at $x_0$, so are $W$ and $(F^{ij})_{n \times n}$. The above identity can be rewrote as
\[
u_i u_{ii} = -\frac{1}{2\rho \varphi} (\varphi \rho_i + \rho \varphi^\prime u_i)|Du|^2, \quad i = 1, \ldots, n.
\]
We also have
\begin{align}
G_{ij}(x_0) &= 2\rho \varphi u_k u_{kij} + 2\rho \varphi u_k u_{kij} + 2\rho \varphi'(u_i u_k u_{kj} + u_j u_k u_{ki}) \\
& \quad + 2\varphi(\rho_i u_k u_{kj} + \rho_j u_k u_{ki}) + \rho u_{ij}\varphi'|Du|^2 + \rho \varphi''|Du|^2 u_i u_j \\
& \quad + \varphi''|Du|^2(\rho_i u_j + \rho_j u_i) + \rho_{ij}\varphi|Du|^2.
\end{align}

Use the maximum principle to get
\begin{align}
0 & \geq F^{ij}G_{ij} = F^{ii}G_{ii} \\
& = 2\rho \varphi u_k F^{ii}u_{iik} + 2\rho \varphi F^{ii}u_{i}^2 + 4\rho \varphi' F^{ii}u_{i}^2 u_{ii} + 4\varphi F^{ii}\rho_i u_{ii} \\
& \quad + \rho \varphi'|Du|^2 F^{ii}u_{i}^2 + \rho \varphi''|Du|^2 F^{ii}u_{i}^2 + 2\varphi'|Du|^2 F^{ii}\rho_i u_{ii} + F^{ii}\rho_{ii}\varphi|Du|^2.
\end{align}

From the facts that
\begin{align}
F^{ii}u_{i} = kf, \quad F^{ii}u_{il} = f_{x_l} + f_{z_l} u_{ll},
\end{align}
we have
\begin{align}
0 & \geq 2\rho \varphi u_{l}(f_{l} + f_{z_l} u_{ll}) + 2\rho \varphi f_{p_l} u_{l} u_{ll} + 2\rho \varphi F^{ii}u_{i}^2 \\
& \quad + 4\rho \varphi' F^{ii}u_{i}^2 u_{ii} + 4\varphi F^{ii}\rho_i u_{ii} + mf \rho \varphi'|Du|^2 + \rho \varphi''|Du|^2 F^{ii}u_{i}^2 \\
& \quad + 2\varphi'|Du|^2 F^{ii}\rho_i u_{ii} + F^{ii}\rho_{ii}\varphi|Du|^2.
\end{align}

Assume \(|Du|(x_0) \geq b_0\), otherwise we have (4.4). By (4.3) and (4.7), which used to deal with the second, fourth and fifth terms, then
\begin{align}
0 & \geq -4L_1(\varphi + \varphi')|Du|^{2k+1} + 2\rho \varphi F^{ii}u_{i}^2 - 2\varphi'|Du|^2 F^{ii}u_{i}\rho_i - \frac{2\varphi|Du|^2}{\rho}F^{ii}\rho_i^2 \\
& \quad + (\varphi'' - \frac{2\varphi^2}{\varphi})\rho |Du|^2 F^{ii}u_{i}^2 + \varphi|Du|^2 F^{ii}\rho_i.
\end{align}

By (4.6) and properties of \(\rho\) we have
\begin{align}
0 & \geq 2\rho \varphi F^{ii}u_{i}^2 - 2b_0\varphi'\rho\varphi^2|Du|^3\mathcal{F} - 3b_0^2\varphi|Du|^2\mathcal{F} \\
& \quad - 4L_1(\varphi + \varphi')|Du|^{2k+1}.
\end{align}

Assume \(G(x_0) \geq 20n b_0^2 M^\frac{3}{2}\), otherwise we have (4.4), which implies that \(|Du| \geq \frac{2\sqrt{5n b_0 M^\frac{3}{2}}}{\rho\varphi^2}\) at \(x_0\). There exists at least one index \(i_0\) such that \(|u_{i_0}| \geq \frac{|Du|}{\sqrt{n}}\). By (4.7), it is not hard to get
\begin{align}
|u_{i_0}| = & \left(-\frac{\varphi'}{2\varphi} + \frac{\rho_{i_0}}{2\rho u_{i_0}}\right)|Du|^2 \\
\quad \leq & \left(-\frac{\varphi'}{2\varphi} - \frac{1}{20M}\right)|Du|^2 \\
\quad \leq & \frac{|Du|^2}{4\varphi}.
\end{align}
Let $u_{11} \geq \cdots \geq u_{nn}$, from (2.8) and (4.12) we have

$$u_{nn} \leq -\frac{\varphi''|Du|^2}{4\varphi}, \quad F^{11} \leq \cdots \leq F^{nn}. \tag{4.13}$$

The second part implies that $F^{nn} \geq \frac{1}{n} F$. Returning to (4.11) we have

$$0 \geq 2\rho \varphi F^{nn}u_{nn}^2 - 2b_0 \varphi' \rho \frac{1}{n} |Du|^2 F - 4L_1(\varphi + \varphi') |Du|^{2k+1}

\quad \geq \frac{\rho \varphi^2}{8n\varphi} |Du|^4 F - 2b_0 \varphi' \rho \frac{1}{n} |Du|^2 F - 4L_1(\varphi + \varphi') |Du|^{2k+1}. \tag{4.14}$$

Both sides of (4.14) multiplied by $\rho \varphi^3$, then we have

$$0 \geq \frac{2G^2}{125nM^3} - \frac{8b_0 G^2}{3M^2} - \frac{6b_0^2 G}{M^2} F - 4L_1(\frac{1}{M^2} + \frac{1}{M^2}) |Du|^{2k-2} G^\frac{1}{2}. \tag{4.15}$$

By (4.13), we can choose $\delta = \frac{\varphi'}{4\varphi}, L = |Du|^2$ and $\theta_1 = \left(\frac{(\varphi')^k}{C_m^\varphi}\right)^{k-1}$ in the Proposition 2.6 such that

$$F \geq \theta_1 |Du|^{2k-2}.$$

Then,

$$0 \geq \frac{2G}{125nM^3} - \frac{8b_0 G^2}{3M^2} - \frac{6b_0^2 G}{M^2} - 4\theta_1 L_1(\frac{1}{M^2} + \frac{1}{M^2}) G^\frac{1}{2}.$$

It follows that

$$G^{\frac{1}{2}}(x_0) \leq C_1 + C_2 \frac{M^\frac{1}{2}}{r}.$$

Thus

$$\sup_{B_x} |Du| \leq C_1 + C_2 \frac{M}{r}, \tag{4.16}$$

where $C_1$ depends only on $M$, $L_1$, $n$, $m$, and $k$, and $C_2$ depends only on $L_1$, $n$, $m$, and $k$. It is not hard to see that $C_1 = 0$ when $f \equiv$ constant. \quad \Box

In fact, if we only consider for $f = f(x,u) > 0$ in the equation (4.2), we could remove the restriction to $k$ in Theorem 4.1 and the following Theorem 4.3. Precisely, we have

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $2 \leq k \leq C^m_n$. Suppose that $u \in C^3(\Omega)$ is a $k$-admissible solution of the following equation,

$$S_k(W) = f(x,u), \quad \text{in } \Omega, \tag{4.17}$$

where $f(x,z) \in C^1(\overline{\Omega} \times [-M_0,M_0] \times \mathbb{R}^n)$ is a positive function, $M_0 = \sup_{\overline{\Omega}} |u|$. We also assume that

$$|f|_{C^1(\overline{\Omega} \times [-M_0,M_0] \times \mathbb{R}^n)} \leq L_1, \tag{4.18}$$
for some constant \( L_1 \). For any \( B_r(y) \subset \Omega \), we have

\[
\sup_{B_r^2(y)} |Du| \leq C_1 + C_2 \frac{M_0}{r},
\]

where \( C_1 \) depends only on \( M_0, L_1, \min_f, n, m, \) and \( k \), and \( C_2 \) depends only on \( L_1, \min_f, n, m, \) and \( k \). Moreover, if \( f \equiv \text{constant} \), then \( C_1 = 0 \).

Proof. The proof of this result is essentially the same as the proof of Theorem 4.1, the only difference being that we cannot apply Proposition 2.6 to give a lower bound to \( F \). Instead, we use the Newton-Maclaurin inequality. From (4.12) we still have

\[
(4.20) \quad u_{nn} \leq -\frac{\varphi' |Du|^2}{4\varphi}, \quad F^{11} \leq \ldots \leq F^{nn}.
\]

The second part implies that

\[
F^{mn} \geq \frac{1}{n}F = \frac{m}{n} \sum_{i=1}^{C_m n} S_{k-1}(\lambda|i) = \frac{m(C_m^n - k + 1)}{n} S_{k-1}(\lambda).
\]

By the Newton-Maclaurin inequality, we have

\[
(4.21) \quad F^{nn} \geq \frac{1}{n}F \geq c S_k^\frac{1}{k}(\lambda) \geq c(\min f)^{\frac{1}{k}},
\]

where \( c = c(n, m, k) \) a universal constant. It is not hard to see, a different version of (4.15), that

\[
(4.22) \quad 0 \geq \left( \frac{2G^2}{125nM^3} - \frac{8b_0G^2}{3M^2} - \frac{6b_0^2G}{M^2} \right) F - 4L_1(\frac{1}{M^2} + \frac{1}{M^4})G.
\]

Plug (4.21) into (4.22), then

\[
0 \geq \frac{2G}{125nM^3} - \frac{8b_0G^2}{3M^2} - \frac{6b_0^2G}{M^2} - 4c^{-1}(\min f)^{-\frac{1}{k}} L_1(\frac{1}{M^2} + \frac{1}{M^4}).
\]

Thus we have

\[
(4.23) \quad \sup_{B_r^2} |Du| \leq C_1 + C_2 \frac{M}{r},
\]

where \( C_1 \) depends only on \( M_0, L_1, \min_f, n, m, \) and \( k \), and \( C_2 \) depends only on \( L_1, \min_f, n, m, \) and \( k \). It is not hard to see that \( C_1 = 0 \) when \( f \equiv \text{constant} \).
4.2. Gradient estimate near boundary. In this subsection, we will establish a gradient estimate in the small neighborhood near boundary. We use a similar method as in Ma-Qiu [22] with minor changes. We define

\[ d(x) = \text{dist}(x, \partial \Omega), \]
\[ \Omega_\mu = \{ x \in \Omega \mid d(x) < \mu \}. \]  (4.24)

It is well known that there exists a small positive universal constant \( \mu_0 \) such that \( d(x) \in C^k(\Omega_\mu) \), \( 0 < \mu \leq \mu_0 \), provided \( \partial \Omega \in C^k \). As in Simon-Spruck [26] or Lieberman [17] (in page 331), we can extend \( \nu \) by \( \nu = -Dd \) in \( \Omega_\mu \) and note that \( \nu \) is a \( C^2(\Omega_\mu) \) vector field. As mentioned in the book [17], we also have the following formulas

\[ |D\nu| + |D^2 \nu| \leq C(n, \Omega), \quad \text{in } \Omega_\mu, \]  (4.25)
\[ \sum_{i=1}^n \nu^i D_j \nu^j = \sum_{i=1}^n \nu^i D_i \nu^j = \sum_{i=1}^n d_i d_{ij} = 0, \quad |\nu| = |Dd| = 1, \quad \text{in } \Omega_\mu. \]

**Theorem 4.3.** Suppose \( \Omega \subset \mathbb{R}^n \) (\( n \geq 3 \)) is a bounded domain with \( C^3 \) boundary, and \( 2 \leq k \leq \frac{n-\mu}{n} C^m_n \), Let \( f(x, z, p) \in C^1(\overline{\Omega} \times [-M_0, M_0] \times \mathbb{R}^n) \) is a nonnegative function and \( \phi \in C^3(\overline{\Omega} \times [-M_0, M_0]) \), \( M_0 = \sup_{\overline{\Omega}} |u| \). We also assume that there exists constants \( L_1 \) (independent of \( |Du|_{C^0} \)) and \( L_2 \) such that

\[ |f|_{C^0} + \sum_{i=1}^n |f_{z_i}|_{C^0} + |f_z|_{C^0} + \sum_{i=1}^n |f_{p_i}|_{C^0} |Du|_{C^0} \leq L_1 (1 + |Du|_{C^0}^{2k-2}), \]  (4.26)
\[ |\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \leq L_2. \]  (4.27)

If \( u \in C^3(\Omega) \cap C^1(\overline{\Omega}) \) is a k-admissible solution of equation

\[ \begin{cases} S_k(W) = f(x, u, Du), & \text{in } \Omega, \\ u_\nu = \phi(x, u), & \text{on } \partial \Omega. \end{cases} \]  (4.28)

Then we have

\[ \sup_{\Omega_\mu} |Du| \leq C, \]  (4.29)

where \( C \) is a constant depends only on \( n, k, m, \mu, M_0, L_1, L_2 \) and \( \Omega \).

**Proof.** Let

\[ G(x) := \log |Dw|^2 + h(u) + \alpha_0 d(x), \quad \text{in } \Omega_\mu \forall 0 < \mu \leq \mu_0 \]  (4.30)

where

\[ w(x) = u(x) + \phi(x, u)d(x), \]  (4.31)
\[ h(x) = -\frac{1}{2} \log(1 + 4M_0 - u), \quad h'' - 2h^2 = 0, \]  (4.32)
and $\alpha_0$ is a constant to be determined.

Above and throughout the text, we always denote $C$ a positive constant depends on some known data.

**Case 1**: $G$ attains its maximum on the boundary $\partial \Omega$.

If we assume that $|Du| > 8nL^2$ and $\mu \leq \frac{1}{2L^2}$, it follows from (4.38) that

$$\frac{1}{4} |Du| \leq |Dw| \leq 2|Du|. \tag{4.33}$$

Assume $x_0$ is the maximum point of $G$, then we have

$$0 \leq G_\nu(x_0) = \frac{D(|Dw|^2) \cdot \nu}{|Dw|^2} + h' u_\nu + \alpha_0 Dd \cdot \nu \tag{4.34}$$

since $\nu = -Dd$.

On the boundary $\partial \Omega$, by the Neumann condition, we have

$$D(|Dw|^2) \cdot \nu = -w_i w_{ij} d_j = -(u_i + \phi d_i)(u_{ij} + D_{ij} \phi d_j + D_j \phi d_i + \phi d_{ij})d_j$$

$$= (u_i + \phi d_i)(u_{ij}d_j - \phi u_j d_i d_j - \phi x_j d_i d_j - \phi z u_j d_i d_j) \leq C(|Dw|^2 + |Dw|). \tag{4.35}$$

where $C = C(|d|_{C^2}, |\phi|_{C^1})$. Plug (4.35) into (4.34) to get

$$0 \leq G_\nu \leq C + \frac{C}{|Dw|} + h'|\phi| - \alpha_0 \tag{4.36}$$

provided $\alpha_0 = 2C + \frac{2L^2}{1 + M} + 1$. Thus we have $|Dw|(x_0) \leq 1$, and $G(x_0) \leq \alpha_0$.

**Case 2**: $G$ attains its maximum on the interior boundary $\partial \Omega \cap \Omega$. It follows from the interior gradient estimate (4.4) that

$$\sup_{\partial \Omega \cap \Omega} |Dw|(x_0) \leq C, \tag{4.37}$$

where $C$ depends only on $M$, $L_1$, $\mu$, $n$, $m$, and $k$. Thus we also have an upper bound for $G(x_0)$.

**Case 3**: $G$ attains its maximum at some point $x_0 \in \Omega_\mu$. 

We have
\begin{equation}
(4.38) \quad w_i = (1 + \phi_z d) u_i + R_i,
\end{equation}
and the second derivatives
\begin{equation}
(4.39) \quad w_{ij} = (1 + \phi_z d) u_{ij} + R_{ij},
\end{equation}
with
\begin{equation}
(4.40) \quad R_{ij} = d \phi_{zz} u_i u_j + (d \phi_{iz} u_j + d \phi_{zj} u_i + d_i \phi_z u_j + d_j \phi_z u_i)
\end{equation}
\begin{equation}
+ (d \phi_{ij} + d_i \phi_j + d_j \phi_i + d_{ij} \phi).
\end{equation}
It is easy to see that
\begin{equation}
(4.41) \quad |R_i| \leq 2L_2, \quad |R_{ij}| \leq C(\mu |Du|^2 + |Du| + 1),
\end{equation}
where \( C = C(L_2, n, |d|_{C^3}) \). The third derivatives are more complicated,
\begin{equation}
(4.42) \quad w_{ijl} = (1 + \phi_z d) u_{ijl} + d \phi_{zzz} u_i u_j u_l + R_{ijl}
\end{equation}
\begin{equation}
+ (d \phi_{zzi} u_l u_j + d \phi_{zzj} u_i u_l + d \phi_{zzz} u_i u_j u_l + R_{ijl})
\end{equation}
\begin{equation}
+ (d \phi_{ijz} u_l + d \phi_{iiz} u_j + d \phi_{zij} u_l + d \phi_{zzj} u_i u_l + d \phi_{zzz} u_i u_j u_l)
\end{equation}
\begin{equation}
+ (d \phi_{lji} + d \phi_{iji} + d \phi_{lij} + d \phi_{ijl} + d \phi_{lji} + d \phi_{lij} + d_{ijl} \phi + d_{ijl} \phi).
\end{equation}
So we have \(|R_{ijl}| \leq C(|Du|^2 + |Du| + 1)\) with \( C = C(|d|_{C^3}, L_2) \).

We compute at the maximum point \( x_0 \in \Omega_\mu \),
\begin{equation}
(4.43) \quad 0 = G_i(x_0) = \frac{2w_i w_{ij}}{|Dw|^2} + \alpha_0 d_i + h' u_i, \quad i = 1, \ldots, n,
\end{equation}
and
\begin{equation}
G_{ij}(x_0) = \frac{2w_i w_{ij}}{|Dw|^2} + \frac{2w_{ij} w_{ijkl}}{|Dw|^2} - \frac{4w_i w_{ij} w_{ijkl}}{|Dw|^4} + \alpha_0 d_{ij} + h'' u_i u_j + h' u_{ij}.
\end{equation}
By the maximum principle we have
\begin{equation}
(4.44) \quad 0 \geq F^{ij} G_{ij} = F^{ii} G_{ii}
\end{equation}
\begin{equation}
= \frac{2F^{ii} u^2_{ij}}{|Dw|^2} + \frac{2w_i F^{ii} w_{ijl}}{|Dw|^2} - \frac{4F^{ii} (w_i w_{ij})^2}{|Dw|^4} + \alpha_0 F^{ii} d_{ii}
\end{equation}
\begin{equation}
+ h'' F^{ii} u^2_i + h' F^{ii} u_{ii}.
\end{equation}
The (4.43) implies that $2w_l w_{li} = -(\alpha_0 d_i + h'u_i)|Dw|^2$, by the Cauchy-Schwarz inequality, then

$$\frac{4F^{ii}(w_l w_{ji})^2}{|Dw|^4} = \alpha_0 F^{ii} d_i^2 + 2\alpha_0 h' F^{ii} u_i d_i + h'^2 F^{ii} u_i^2$$

(4.45)

$$\leq 2h'^2 F^{ii} u_i^2 + C \mathcal{F},$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3})$. Combining (4.10), (4.32), (4.45) with (4.44), we get

$$0 \geq \frac{2F^{ii} w_l^2}{|Dw|^2} + \frac{2w_l F^{iii} w_{iii}}{|Dw|^2} - C \mathcal{F}.$$  

(4.46)

We may assume that $\mu \leq \frac{1}{2L_2}$ and $|Du|(x_0) \geq 16nL_2 + 1$, so that $\frac{1}{2} \leq 1 + \phi_z d \leq 1$ and $\frac{1}{2} |Du|^2 \leq |Dw|^2 \leq \frac{3}{2} |Du|^2$. By (4.42), we have

$$\frac{2w_l F^{iii} w_{iii}}{|Dw|^2} = \frac{1}{|Dw|^2} \left(2(1 + \phi_z d) w_i D_1 f + 2d\phi_{zz} w_l w_{ji} F^{ii} u_j^2 + 4d\phi_{zz} F^{ii} u_i u_j w_i w_j \right)$$

$$+ (2d\phi_{zz} w_l w_i + 2d\phi_{z} w_i + \phi_z d w_l) F^{ii} u_i$$

$$+ 4d\phi_{zz} F^{ii} u_i u_j w_l + 4\phi_z d F^{ii} u_i u_j w_i + 2F^{ii} R_{iii} w_l w_i)$$

$$\geq - \frac{C}{|Dw|^2} (\mu |Dw|^4 + |Dw|^3 + \left(\mu + \frac{1}{\mu}\right)|Du|^2 + |Dw|) \mathcal{F} + \mu F^{ii} u_i^2$$

(4.47)

$$+ \frac{2(1 + \phi_z d)}{|Dw|^2} w_l D_1 f,$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2)$. Here we use the Cauchy inequality and the fact that $|R_{iii}| \leq C(|Du|^2 + |Du| + 1)$. Now we deal with the last term. By (4.39) and (4.43), we have

$$\frac{2(1 + \phi_z d)}{|Dw|^2} |w_l D_1 f| = \frac{2(1 + \phi_z d)}{|Dw|^2} w_l (f_{x_i} + f_{z u_i}) - f_{p_i}(\alpha_0 d_i + h'u_i + \frac{2w_l R_{lj}}{|Du|^2})$$

(4.48)

$$\leq C(1 + |Du|^{2k-1}),$$

here we use the fact that $|R_{lj}| \leq C(\mu |Du|^2 + |Du| + 1)$. Put (4.47) and (4.48) into (4.46), we have

$$0 \geq \frac{2F^{ii} w_l^2}{|Dw|^2} - \frac{C\mu F^{ii} u_i^2}{|Dw|^2} - C\mu |Dw|^2 \mathcal{F} - C(|Du| \mathcal{F} + 1)$$

(4.49)

$$- C(1 + |Du|^{2k-1}),$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2, \mu)$.

By (4.39), (4.41) and the inequality (see [13])

$$(a + b)^2 \geq a^2 - \frac{\epsilon}{1 - \epsilon} b^2,$$
choose $\epsilon = \frac{1}{2}$, we obtain
\[
\begin{align*}
    w_{ii}^2 & \geq \frac{1}{4} u_{ii}^2 - R_{ii}^2 \\
           & \geq \frac{1}{4} u_{ii}^2 - C(\mu^2 |D\omega|^4 + |D\omega|^2 + 1).
\end{align*}
\]
It follows that
\[
\begin{align*}
    0 & \geq \left( \frac{1}{8} - C\mu \right) \frac{F_{ii}u_{ii}^2}{|D\omega|^2} - C\mu |D\omega|^2 F - C(|D\omega| + 1)F \\
& \quad - C(1 + |D\omega|^{2k-1}).
\end{align*}
\]
(4.50)

There exists at least an index $l_0$ such that $u_{l_0} \geq \sqrt{n}\mu |D\omega|$. We rewrite the (4.43) as
\[
\begin{align*}
    2w_{l_0}w_{l_0} + 2 \sum_{q \neq l_0} w_q w_{ql_0} = -(\alpha_0 d_{l_0} + h' u_{l_0})|D\omega|^2.
\end{align*}
\]
From (4.39) we have
\[
\begin{align*}
    2(1 + \phi_0 d)w_{l_0}u_{l_0} & = -(\alpha_0 d_{l_0} + h' u_{l_0})|D\omega|^2 - 2w_{l_0}R_{ql_0}.
\end{align*}
\]
(4.51)

Since $|R_{l}| \leq 2L \leq \frac{\mu}{4}$, from (4.38), we have $w_l \geq \frac{\mu}{4}$. If we assume that $|D\omega| \geq \frac{2\sqrt{n}a_0 |Dd|}{\mu}$, and use the facts that $1 + \phi_0 \geq \frac{1}{2}$ and $|R_{ij}| \leq C(\mu |D\omega|^2 + |D\omega| + 1)$, then
\[
    u_{l_0} \leq -2h' |D\omega|^2 + 12\sqrt{n}C(\mu |D\omega|^2 + |D\omega|).
\]
If we assume that $|D\omega| \geq \frac{2}{\mu} \geq 10M + 2$ and $\mu \leq \frac{h'}{12\sqrt{n}C}$, then
\[
    u_{l_0} \leq -\frac{h'}{2} |D\omega|^2.
\]
(4.52)

Denote $u_{11} \geq \cdots \geq u_{nn}$. By (4.52), we can choose $\delta = \frac{h'}{2}$, $L = |D\omega|^2$ and $\theta_1 = \frac{C_{nn}(h')^{k-1}}{4^{k-1}}$ in the Proposition 2.6, such that
\[
    u_{nn} \leq -\frac{h'}{2} |D\omega|^2, \quad F_{nn} \geq \frac{1}{n} F \geq \frac{1}{n} \theta_1 |D\omega|^{2k-2}.
\]
(4.53)

We assume that $\mu \leq \min \{ \frac{1}{10c}, \frac{h^2}{128nc} \}$. By (4.50) we obtain
\[
\begin{align*}
    0 & \geq \frac{h^2}{128n} |D\omega|^2 F - C(|D\omega| + 1)F - C(1 + |D\omega|^{2k-1}).
\end{align*}
\]
(4.54)

By (4.53), we have
\[
\begin{align*}
    0 & \geq \frac{h^2}{128n} |D\omega|^2 - C|D\omega| - C.
\end{align*}
\]
(4.55)

It is easy to get a bound for $|D\omega|(x_0)$, then a bound for $G(x_0)$.

Anyway we have the bound
\[
G(x_0) = \sup_{\Omega} G(x) \leq C,
\]
(4.56)
where \( C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2, \mu) \). Thus we obtain
\[
\sup_{\Omega} |Du| \leq C + \log(1 + 2M) + \alpha_0\mu. \tag{4.56}
\]

By the same reason for Theorem 4.2, we have the following boundary gradient estimate when \( f = f(x, u) \).

**Theorem 4.4.** Suppose \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\) is a bounded domain with \( C^3 \) boundary, and \( 2 \leq k \leq C_m \). Let \( f(x, z) \in C^1(\overline{\Omega} \times [-M_0, M_0]) \) is a nonnegative function and \( \phi \in C^3(\overline{\Omega} \times [-M_0, M_0]), M_0 = \sup_{\Omega} |u| \). We also assume that there exists constants \( L_1 \) and \( L_2 \) such that
\[
|f|_{C^1(\overline{\Omega} \times [-M_0, M_0])} \leq L_1, \tag{4.57}
\]
\[
|\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \leq L_2. \tag{4.58}
\]

If \( u \in C^3(\Omega) \cap C^1(\overline{\Omega}) \) is a \( k \)-admissible solution of the equation
\[
\begin{align*}
S_k(W) &= f(x, u), & \text{in } \Omega, \\
u_\nu &= \phi(x, u), & \text{on } \partial\Omega.
\end{align*}
\]

Then we have
\[
\sup_{\Omega} |Du| \leq C, \tag{4.60}
\]
where \( C \) is a constant depends only on \( n, k, m, \mu, M_0, L_1, L_2 \) and \( \Omega \).

**Proof.** By the same auxiliary function and the same computations as in the proof above, now we deal with terms in (4.48) as follows
\[
\frac{2(1 + \phi_2 d)}{|Dw|^2} |w_i D_i f| = \frac{2(1 + \phi_2 d)}{|Dw|^2} w_i (f_{x_i} + f_z u_i) \leq 4L_1(1 + |Du|^{-1}). \tag{4.61}
\]

It is not hard to get, a different version of (4.54),
\[
0 \geq \frac{h'^2}{128n} |Dw|^2 \mathcal{F} - C(|Dw| + 1) \mathcal{F} - C(1 + |Du|^{-1}). \tag{4.62}
\]

From (4.53), we still have
\[
u_{nn} \leq -\frac{h'}{2} |Dw|^2, \quad F^{nn} \geq \frac{1}{n} \mathcal{F}. \]

By the Newton-Maclaurin inequality, we have
\[
F^{nn} \geq \frac{1}{n} \mathcal{F} \geq cS_k^\frac{1}{k}(\lambda) \geq c(\min f)^{\frac{1}{k}}, \tag{4.63}
\]
where \( c = c(n, m, k) \) a universal constant. Then we also have

\[
0 \geq \frac{k^2}{128n}|Dw|^2 - C|Dw| - C.
\]

It is also give a bound for \( |Dw| \) at interior maximum point of \( G \). Through the same discussion as before, we have

\[
\sup_{\Omega_\mu}|Du| \leq C_0 + \log(1 + 2M) + \alpha_0\mu.
\]

\[
\square
\]

5. Global Second Order Derivatives Estimates

5.1. Reduce the global second derivative estimates into double normal derivatives estimates on boundary. Using the method of Lions-Trudinger-Urbas [20], we can reduce the second derivative estimates of the solution into the boundary double normal estimates.

**Lemma 5.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^4 \) boundary. Assume \( f(x, z) \in C^2(\overline{\Omega} \times \mathbb{R}) \) is positive and \( \phi(x, z) \in C^3(\overline{\Omega} \times \mathbb{R}) \) with \( \phi_x - 2\kappa_{\min} < 0 \). If \( u \in C^4(\Omega) \cap C^3(\overline{\Omega}) \) is a \( k \)-admissible solution of the Neumann problem

\[
\begin{cases}
S_k(W) = f(x, u), & \text{in } \Omega, \\
u = \phi(x, u), & \text{on } \partial\Omega.
\end{cases}
\]

Denote \( N = \sup_{\partial\Omega}|u_{\nu\nu}| \), then

\[
\sup_{\overline{\Omega}}|D^2u| \leq C_0(1 + N).
\]

where \( C_0 \) depends on \( n, m, k, |u|_{C^1(\overline{\Omega})}, |f|_{C^2(\overline{\Omega} \times [-M_0, M_0])}, \min f, |\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \) and \( \Omega \).

Here \( M_0 = \sup_{\overline{\Omega}}|u| \).

**Proof.** Write equation (5.1) in the form of

\[
\begin{cases}
S_k(W) = \tilde{f}(x, u), & \text{in } \Omega, \\
u = \phi(x, u), & \text{on } \partial\Omega.
\end{cases}
\]

where \( \tilde{f} = f^{\frac{1}{k}} \). Since \( \lambda(W) \in \Gamma_k \subset \Gamma_2 \) in \( \mathbb{R}^{C^m} \), we have

\[
\sum_{i \neq j} |u_{ij}| \leq c(n, m)S_1(W) = mc(n, m)S_1(D^2u),
\]
where $c(n,m)$ is a universal number independent of $u$. Thus, it is sufficiently to prove \((5.2)\) for any direction $\xi \in S^{n-1}$, that is

\[(5.5)\]

\[u_{\xi \xi} \leq C_0 (1 + N).\]

We consider the following auxiliary function in $\Omega \times S^{n-1}$,

\[(5.6)\]

\[v(x, \xi) := u_{\xi \xi} - v'(x, \xi) + K_1 |x|^2 + K_2 |Du|^2,\]

where $v'(x, \xi) = a^t u_t + b := 2(\xi \cdot \nu) \xi \cdot (\phi_{x_t} + \phi_z u_t - u_t D\nu_t)$, with $\xi' = \xi - (\xi \cdot \nu) \nu$ and $a^t = 2(\xi \cdot \nu)(\xi^a \phi_z - \xi^b D\nu)$. $K_1$, $K_2$ are positive constants to be determined. By a direct computation, we have

\[(5.7)\]

\[v_i = u_{\xi \xi i} - D_i a^t u_t - a^t u_{ii} - D_i b + 2K_1 x_i + 2K_2 u_{ii},\]

\[(5.8)\]

\[v_{ij} = u_{\xi \xi ij} - D_{ij} a^t u_t - D_i a^t u_{ij} - D_j a^t u_{ii} - a^t u_{lij} - D_{ij} b + 2K_1 \delta_{ij} + 2K_2 u_{ii} u_{ij},\]

Denote $\tilde{F}(D^2 u) = S_k^\frac{1}{2}(W)$, and

\[(5.9)\]

\[\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial u_{ij}} = \frac{\partial S_k^\frac{1}{2}(W)}{\partial w_{ij}} \frac{\partial w_{ij}}{\partial u_{ij}},\]

and

\[(5.10)\]

\[\tilde{F}^{pq,rs} = \frac{\partial^2 \tilde{F}}{\partial u_{pq} \partial u_{rs}} = \frac{\partial^2 S_k^\frac{1}{2}(W)}{\partial w_{pq}} \frac{\partial w_{rs}}{\partial u_{pq}} \frac{\partial w_{pq}}{\partial u_{rs}},\]

since $w_{ij}$ is a linear combination of $u_{ij}$, $1 \leq i, j \leq n$. Differentiating the equation \((5.3)\) twice, we have

\[(5.11)\]

\[\tilde{F}^{ij} u_{ij} = D_1 \tilde{f},\]

and

\[(5.12)\]

\[\tilde{F}^{pq,rs} u_{pq} u_{rs} + \tilde{F}^{ij} u_{ij} \xi \xi = D_{\xi \xi} \tilde{f}.\]

By the concavity of $S_k^\frac{1}{2}(W)$ operator with respect to $W$, we have

\[(5.13)\]

\[D_{\xi \xi} \tilde{f} = \tilde{F}^{pq,rs} u_{pq} u_{rs} + \tilde{F}^{ij} u_{ij} \xi \xi \leq \tilde{F}^{ij} u_{ij} \xi \xi .\]
Now we contract (5.8) with $\tilde{F}^{ij}$ to get, using (5.11)-(5.13),
\[
\tilde{F}^{ij} u_{ij} = \tilde{F}^{ij} u_{ij\xi\xi} - \tilde{F}^{ij} D_{ij} a^l u_l - 2 \tilde{F}^{ij} D_{ij} a^l u_l - \tilde{F}^{ij} u_{ijl} a^l - \tilde{F}^{ij} D_{ij} b + 2K_1 \tilde{F} + 2K_2 \tilde{F}^{ij} u_{ijl} u_l + 2K_2 \tilde{F}^{ij} u_{ijl} u_l - D_{ij} \tilde{f} - \tilde{F}^{ij} D_{ij} a^l u_l - a^l D_{ij} \tilde{f} - \tilde{F}^{ij} D_{ij} b + 2K_1 \tilde{F} + 2K_2 \tilde{F}^{ij} u_{ijl} u_l + 2K_2 u_l D_{ij} \tilde{f}.
\]
(5.14)

where $\tilde{F} = \sum_{i=1}^{n} \tilde{F}^{ii}$. Note that
\[
D_{ij} \tilde{f} = \tilde{f}_{\xi\xi} + 2f_{\xi z} u_{\xi} + \tilde{f}_{z} u_{\xi \xi},
\]
\[
D_{ij} a^l = 2(\xi \cdot \nu)\xi^{\prime} \phi_{z z} u_{ij} + r_{ij},
\]
\[
D_{ij} b = 2(\xi \cdot \nu)\xi^{\prime} \phi_{xz} u_{ij} + r_{ij},
\]
with $|r_{ij}|, |r_{ij}| \leq C(|u|_{C^1}, |\phi|_{C^3}, |\partial \Omega|_{C^4})$. At the maximum point $x_0 \in \Omega$ of $v$, we can assume $(u_{ij})_{n \times n}$ is diagonal. It follows that, by the Cauchy-Schwartz inequality,
\[
\tilde{F}^{ij} v_{ij} \geq -C(\tilde{F} + K_2 + 1) - C\tilde{F}^{ii} |u_{ii}| + \tilde{f}_{z} u_{\xi \xi} + 2K_1 \tilde{F} + 2K_2 \tilde{F}^{ii} u_{ii}^2
\]
(5.15)
\[
\geq -C(\tilde{F} + K_2 + 1) + \tilde{f}_{z} u_{\xi \xi} + 2K_1 \tilde{F} + (2K_2 - 1) \tilde{F}^{ii} u_{ii}^2,
\]
where $C = C(|u|_{C^1}, |\phi|_{C^3}, |\partial \Omega|_{C^4}, |f|_{C^2})$.

Assume $u_{11} \geq u_{22} \cdots \geq u_{nn}$, and denote $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{C_n}$ the eigenvalues of the matrix $(u_{ij})_{C_n \times C_n}$. It is easy to see $\lambda_1 = u_{11} + \sum_{i=2}^{m} u_{ii} \leq m u_{11}$. Then we have, by (2.6) in Proposition 2.2 and (2.11) in Proposition 2.3,
\[
\tilde{F}^{ii} u_{11}^2 = \sum_{k=1}^{n} \frac{1}{k} S_{k}^{k-1} S_{k-1}(\lambda |N_{\eta^2}) u_{11}^2 \geq \frac{1}{m k} S_{k}^{k-1} S_{k-1}(\lambda |1) \lambda_1 u_{11} \geq \frac{1}{m C_n} S_{k}^{k-1} u_{11} = \frac{\tilde{f}}{m C_n} u_{11}.
\]
(5.16)

We can assume $u_{\xi \xi} \geq 0$, otherwise we have (5.5). Plug (5.16) into (5.15) and use the Cauchy-Schwartz inequality, then
\[
\tilde{F}^{ii} v_{ii} \geq (K_2 - 1) \sum_{i=1}^{n} \tilde{F}^{ii} u_{ii}^2 + \left( \frac{K_2 \tilde{f}}{m C_n} + \tilde{f}_{z} u_{\xi \xi} + (2K_1 - C) \tilde{F} \right) \geq (K_2 - 1) - C(K_2 + 1).
\]
(5.17)

Choose $K_2 = \frac{m C_n |\max f_{\xi}|}{n_{mn} f_{\xi}} + 1$ and $K_1 = C(K_2 + 2) + 1$. It follows that
\[
\tilde{F}^{ii} v_{ii} \geq (2K_1 - C) \tilde{F} - C(K_2 + 1) > 0,
\]
(5.18)
since we have \( \tilde{\mathcal{F}} \geq 1 \) from [2,13]. This implies that \( v(x, \xi) \) attains its maximum on the boundary by the maximum principle. Now we assume \((x_0, \xi_0) \in \partial \Omega \times \mathbb{S}^{n-1}\) is the maximum point of \( v(x, \xi) \) in \( \overline{\Omega} \times \mathbb{S}^{n-1}\). Then we consider two cases as follows,

**Case 1.** \( \xi_0 \) is a tangential vector at \( x_0 \in \partial \Omega \).

We directly have \( \xi_0 \cdot \nu = 0 \), \( \nu = -Dd, v'(x_0, \xi_0) = 0 \), and \( u_{\xi_0, \xi_0}(x_0) > 0 \). As in [17], we define \( c_{ij} = \delta_{ij} - \nu^i \nu^j \), in \( \Omega_{\mu} \), (5.19)
and it is easy to see that \( c_{ij} D_j \) is a tangential direction on \( \partial \Omega \). We compute at \((x_0, \xi_0)\).

From the boundary condition, we have

\[
\phi_z u_{ij} \nu^l + c_{ij} \phi_{x^l j} = -c_{ij} u_l D_j \nu^l + \nu^i \nu^j \nu^l u_{ij}.
\]

It follows that

\[
u_{lip} \nu^l = [c^{pq} + \nu^p \nu^q] u_{ijq} \nu^l \\
= c^{pq} D_q (c_{ij} u_j \phi_z + c_{ij} \phi_{x^l j} - c_{ij} u_l D_j \nu^l + \nu^i \nu^j \nu^l u_{ij}) - c^{pq} u_l D_q \nu^l + \nu^p \nu^q \nu^l u_{iq},
\]

then we obtain

\[
u_{\xi_0, \xi_0} = \sum_{i=1}^{n} c_{i}^{ \xi_0, \xi_0} u_{lip} \nu^l \\
= \sum_{i=1}^{n} \xi_0 c_{ij} \phi_{x^l j} - c_{ij} u_l D_j \nu^l + \nu^i \nu^j \nu^l u_{ij} - u_l D_q \nu^l \\
\leq \phi_z u_{\xi_0, \xi_0} - 2c_{i}^{ \xi_0, \xi_0} u_l D_q \nu^l + C(1 + |u_{\nu\nu}|).
\]

We assume \( \xi_0 = e_1 \), it is easy to get the bound for \( u_{1i}(x_0) \) for \( i > 1 \) from the maximum of \( v(x, \xi) \) in the \( \xi_0 \) direction. In fact, we can assume \( \xi(t) = \frac{1,1,0,\ldots,0}{\sqrt{1+t^2}} \). Then we have

\[
0 = \frac{dv(x_0, \xi(t))}{dt} \bigg|_{t=0} = 2u_{12}(x_0) - 2\nu^2 (\phi_z v_1 - u_t D_t \nu^l),
\]

so

\[
u_{12}(x_0) \leq C + C|Du|.
\]

Similarly, we have for \( \forall i > 1 \),

\[
u_{1i}(x_0) \leq C + C|Du|.
\]
Thus we have, by $D_1 \nu^1 \geq \kappa_{\min}$,
\[
\begin{align*}
    u_{\xi_0 \xi_0 \nu} &\leq \phi_z u_{\xi_0 \xi_0} - 2D_1 \nu^1 u_{11} + C(1 + |u_{\nu \nu}|) \\
    &\leq (\phi_z - 2\kappa_{\min}) u_{\xi_0 \xi_0} + C(1 + |u_{\nu \nu}|).
\end{align*}
\]

On the other hand, we have from the Hopf lemma, (5.7) and (5.23),
\[
0 \leq \nu(x_0, \xi_0) = u_{\xi_0 \xi_0 \nu} - D_\nu a' u_t - a' u_{\nu \nu} - D_\nu b + 2K_1 x_i \nu^i + 2K_2 u_t u_{\nu}
\leq (\phi_z - 2\kappa_{\min}) u_{\xi_0 \xi_0} + C(1 + |u_{\nu \nu}|).
\]

Then we get, since $2\kappa_{\min} - \phi_z \geq c > 0$,
\[
(5.24) \quad u_{\xi_0 \xi_0}(x_0) \leq C(1 + |u_{\nu \nu}|).
\]

**Case 2.** $\xi_0$ is non-tangential.

We can find a tangential vector $\tau$, such that $\xi_0 = \alpha \tau + \beta \nu$, with $\alpha^2 + \beta^2 = 1$. Then we have
\[
\begin{align*}
    u_{\xi_0 \xi_0}(x_0) &= \alpha^2 u_{\tau \tau}(x_0) + \beta^2 u_{\nu \nu}(x_0) + 2\alpha \beta u_{\tau \nu}(x_0) \\
    &= \alpha^2 u_{\tau \tau}(x_0) + \beta^2 u_{\nu \nu}(x_0) + 2(\xi_0 \cdot \nu) \xi_0' \cdot (\phi_z Du - u_t D\nu).
\end{align*}
\]

By the definition of $v(x_0, \xi_0)$,
\[
\begin{align*}
    v(x_0, \xi_0) &= \alpha^2 v(x_0, \tau) + \beta^2 v(x_0, \nu) \\
    &\leq \alpha^2 v(x_0, \xi_0) + \beta^2 v(x_0, \nu).
\end{align*}
\]

Thus,
\[
    v(x_0, \xi_0) = v(x_0, \nu),
\]

and
\[
(5.25) \quad u_{\xi_0 \xi_0}(x_0) \leq |u_{\nu \nu}| + C.
\]

In conclusion, we have (5.5) in both cases. $\square$

### 5.2. Global second order estimates by double normal estimates on boundary.

Generally, the double normal estimates are the most important and hardest parts for the Neumann problem. As in [20] and [22], we construct sub and super barrier function to give lower and upper bounds for $u_{\nu \nu}$ on the boundary. Then we give the global second order estimates.
5.2.1. Global second order estimate for Theorem 5.3. In this subsection, we establish the following global second order estimate.

**Theorem 5.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^4 \) boundary, \( 2 \leq m \leq n-1 \), and \( 2 \leq k \leq C_{n-1}^m \). Assume \( f(x, z) \in C^2(\overline{\Omega} \times \mathbb{R}) \) is positive and \( \phi(x, z) \in C^3(\overline{\Omega} \times \mathbb{R}) \) with \( \phi_z - 2\kappa_{\text{min}} < 0 \). If \( u \in C^4(\Omega) \cap C^3(\overline{\Omega}) \) is a \( k \)-admissible solution of the Neumann problem \((5.7)\). Then we have

\[
(5.26) \quad \sup_{\Omega} |D^2 u| \leq C,
\]

where \( C \) depends only on \( n, m, k, |u|_{C^1(\overline{\Omega})}, |f|_{C^2(\overline{\Omega} \times [-M_0, M_0])}, \min f, |\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \) and \( \Omega \), where \( M_0 = \sup_{\Omega} |u| \).

First, we denote \( d(x) = \text{dist}(x, \partial \Omega) \), and define

\[
(5.27) \quad h(x) := -d(x) + K_3 d^2(x).
\]

where \( K_3 \) is large constant to be determined later. Then we give the following key Lemma.

**Lemma 5.3.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^2 \) boundary, \( 2 \leq m \leq n-1 \) and \( 2 \leq k \leq C_{n-1}^m \). Let \( u \in C^2(\overline{\Omega}) \) is a \( k \)-admissible solution of the equation \((1.1)\) and \( h \) is defined as in \((5.27)\). Then, there exists \( K^* \), a sufficiently large number depends only on \( n, m, k, \min f \) and \( \Omega \), such that,

\[
(5.28) \quad F^{ij} h_{ij} \geq K^*_3 \left(1 + F\right), \quad \text{in } \Omega_\mu \ (0 < \mu \leq \bar{\mu}),
\]

for any \( K_3 \geq K^* \), where \( \bar{\mu} = \min\left\{ \frac{1}{K_3}, \mu_0 \right\} \), \( \mu_0 \) is mentioned in \((4.25)\).

**Proof.** For \( x_0 \in \Omega_\mu \), there exists \( y_0 \in \partial \Omega \) such that \( |x_0 - y_0| = d(x_0) \). Then, in terms of a principal coordinate system at \( y_0 \), we have (see [9], Lemma 14.17),

\[
(5.29) \quad [D^2d(x_0)] = -\text{diag}[\frac{\kappa_1}{1-\kappa_1d}, \cdots, \frac{\kappa_{n-1}}{1-\kappa_{n-1}d}, 0],
\]

and

\[
(5.30) \quad Dd(x_0) = -\nu(x_0) = (0, \cdots, 0, -1).
\]

Observe that

\[
(5.31) \quad [D^2h(x_0)] = \text{diag}[\frac{(1-2K_3d)\kappa_1}{1-\kappa_1d}, \cdots, \frac{(1-2K_3d)\kappa_{n-1}}{1-\kappa_{n-1}d}, 2K_3^3].
\]

Denote \( \mu_i = \frac{(1-K_3d)\kappa_i}{1-\kappa_id} \), \( \forall 1 \leq i \leq n-1 \), and \( \mu_n = 2K_3 \) for simplicity. Then we define \( \lambda(D^2h) = \{\mu_1 + \cdots + \mu_{i_1}, \cdots, 1 \leq i_1 < \cdots < i_m \leq n\} \) and assume \( \lambda_1 \geq \cdots \geq \lambda_{C_m^2} \), it is easy to see

\[
\lambda_k \geq \lambda_{C_{n-1}^m} \geq 2K_3 + \sum_{i=1}^{m-1} \mu_i \geq K_3,
\]
if we choose $K_3$ sufficiently large and $\mu \leq \frac{1}{4K_3}$. It follows that, for $\forall 1 \leq l \leq k$,

$$
S_l(\lambda) \geq K_3^l - C(n, m, \kappa)K_3^{l-1}
$$

(5.32)

such that $h$ is $k$-admissible. Similarly, $w = h - \frac{K_3}{2n}|x|^2$ is also $k$-admissible if we choose $K_3$ sufficiently large. By the concavity of $\tilde{F}$, we have

$$
\tilde{F}^{ij} w_{ij} \geq \tilde{F}[D^2u + D^2w] - \tilde{F}[D^2u]
$$

$$
\geq \tilde{F}[D^2w]
$$

$$
\geq \frac{K_3}{4}.
$$

(5.33)

Then we have

$$
\tilde{F}^{ij} h_{ij} = \tilde{F}^{ij}(h - \frac{K_3}{2n}|x|^2 + \frac{K_3}{2n}|x|^2)_{ij} \geq \frac{K_3}{4n}(1 + \tilde{F}).
$$

(5.34)

If we choose $K_3 \geq \left(\frac{4n}{k_{\min}f_{\max}}\right)^2$, then we have

$$
F^{ij} h_{ij} \geq K_3^{\frac{1}{2}}(1 + \mathcal{F}).
$$

(5.35)

□

Now we can use Lemma 5.3 to prove Theorem 5.2

**Proof of Theorem 5.2.**

We define

$$
P(x) = Du \cdot \nu - \phi(x, u),
$$

with $\nu = -Dd$. Differentiate $P$ twice to obtain

$$
P_{ij} = -u_{rij}d_r - u_r d_{rij} - u_r d_{rij} - D_{ij} \phi.
$$

(5.36)

Then we obtain

$$
F^{ij} P_{ij} = -F^{ij}(u_{rij}d_r + 2u_r d_{rij} + u_r d_{rij} - D_{ij} \phi)
$$

$$
\leq -F^{ii} u_{ii} d_{ii} + C_1(1 + \mathcal{F}),
$$

where $C_1 = C_1(|u|_{C^1}, |\partial \Omega|_{C^3}, |\phi|_{C^2}, |f|_{C^1}, n)$. From (5.2) in Lemma 5.1 we have $|u_{ii}| \leq C_0(1 + N)$.

It follows that

$$
F^{ij} P_{ij} \leq C_2(1 + N)(1 + \mathcal{F}),
$$

(5.37)

where $C_2 = C_1 + C_0|d|_{C^2}$. 


On the other hand, using Lemma 5.3, we have

\[(A + \frac{1}{2}N)F^{ij}h_{ij} \geq (A + \frac{1}{2}N)K_3^\frac{1}{2}(1 + F) \geq C_2(1 + N)(1 + F) \geq F^{ij}P_{ij},\]

(5.39)

if we choose $K_3 = K^* + (2C_2)^2 + 1$ and $A \geq C_2 + 1$. On $\partial \Omega$, it is easy to see

\[P = 0.\]

(5.40)

On $\partial \Omega \cap \Omega$, we have

\[|P| \leq C_3(|u|_{C^1}, |\phi|_{C^0}) \leq (A + \frac{1}{2}N)\frac{\mu}{2},\]

(5.41)

if we take $A = \max\{\frac{2C_3}{\mu}, C_2 + 1\}$.

Finally the maximum principle tells us that

\[-(A + \frac{1}{2}N)h(x) \leq P(x) \leq (A + \frac{1}{2}N)h(x), \text{ in } \Omega_\mu.\]

(5.42)

Suppose $u_{\nu\nu}(y_0) = \sup_{\partial \Omega} u_{\nu\nu} > 0$, we have

\[0 \geq P_{\nu}(y_0) - (A + \frac{1}{2}N)h_{\nu} = u_{\nu\nu} - D_{\nu}\phi - (A + \frac{1}{2}N) \geq u_{\nu\nu}(y_0) - C(|u|_{C^1}, |\partial \Omega|_{C^2}, |\phi|_{C^2}) - (A + \frac{1}{2}N).\]

Then we get

\[\sup_{\partial \Omega} u_{\nu\nu} \leq C + \frac{1}{2}N.\]

(5.43)

Similarly, doing this at the minimum point of $u_{\nu\nu}$, we have

\[\inf_{\partial \Omega} u_{\nu\nu} \leq C + \frac{1}{2}N.\]

(5.44)

It follows that

\[\sup_{\partial \Omega} |u_{\nu\nu}| \leq C.\]

(5.45)

Combining (5.45) with (5.2) in Lemma 5.1, we obtain

\[\sup_{\Omega} |D^2u| \leq C.\]

(5.46)
5.2.2. Global second order estimate for Theorem 5.4. In this subsection we give a global second order estimate for the cases that \( m \leq \frac{n}{2} \). We can settle more cases for \( k \geq C_{n-1}^{m-1} \) than before, if \( \Omega \) is strictly \((m, k_0)\)-convex.

**Theorem 5.4.** Let \( \Omega \subset \mathbb{R}^n \) be a strictly \((m, k_0)\)-convex domain with \( C^4 \) boundary, \( 2 \leq m \leq n \), and \( k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m \). Assume \( f(x, z) \in C^2(\overline{\Omega} \times \mathbb{R}) \) is positive and \( f(x, z) \in C^2(\overline{\Omega} \times \mathbb{R}) \) with \( f_i \geq 2\kappa_{\min} < 0 \). If \( u \in C^1(\Omega) \cap C^3(\overline{\Omega}) \) is a k-admissible solution of the equation (1.1) and \( \lambda \) is defined as in (5.27). Then, there exists \( K_3 \), a sufficiently large number depends only on \( n, m, k, \min f \) and \( \Omega \), such that,

\[
(5.47) \quad \sup_{\Omega} |D^2 u| \leq C,
\]

where \( C \) depends only on \( n, m, k, \min f|_{C^1(\overline{\Omega})} \), \( \min f \), \( \phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \) and \( \Omega \), where \( M_0 = \sup_{\Omega} |u| \).

First, we prove the following Lemma.

**Lemma 5.5.** Let \( \Omega \subset \mathbb{R}^n \) be a strictly \((m, k_0)\)-convex domain with \( C^4 \) boundary, \( 2 \leq m \leq n-1 \), and \( k = C_{n-1}^{m-1} + k_0 \leq \frac{n-m}{n} C_n^m \), \( k_0 \) a positive integer. Assume \( u \in C^2(\overline{\Omega}) \) is a k-admissible solution of the equation (1.1) and \( h \) is defined as in (5.27). Then, there exists \( K_3 \), a sufficiently large number depends only on \( n, m, k, \min f \) and \( \Omega \), such that,

\[
(5.48) \quad F^{ij} h_{ij} \geq k_3 (1 + F), \quad \text{in } \Omega_{\mu} \quad (0 < \mu \leq \bar{\mu}),
\]

for \( k_3 \), a sufficiently small number depends only on \( n, m, k, \) and \( \Omega \). Here \( \bar{\mu} = \min \{ \frac{1}{4K_3}, \mu_0 \} \).

**Proof.** For \( x_0 \in \Omega_{\mu} \), there exists \( y_0 \in \partial \Omega \) such that \( |x_0 - y_0| = d(x_0) \). As before, in terms of a principal coordinate system at \( y_0 \), we have,

\[
(5.49) \quad [D^2 h(x_0)] = \text{diag}\left[ \frac{(1 - 2K_3d)\kappa_1}{1 - \kappa_1 d}, \ldots, \frac{(1 - 2K_3d)\kappa_{n-1}}{1 - \kappa_{n-1} d}, 2K_3 \right].
\]

Denote \( \mu_i = \frac{(1 - K_3d)\kappa_i}{1 - K_3d} \), \( \forall 1 \leq i \leq n-1 \), and \( \mu_n = 2K_3 \) for simplicity. Then we define \( \lambda(D^2 h) = \{ \mu_1 + \cdots + \mu_m \mid 1 \leq i_1 < \cdots < i_m \leq n \} \) and assume \( \lambda_1 \geq \cdots \geq \lambda_{C_n^m} \), it is easy to see

\[
\lambda_{C_{n-1}^{m-1}} \geq 2K_3 + \frac{1}{2} \sum_{i=1}^{m-1} \mu_i \geq \frac{3}{2} K_3,
\]

if we choose \( K_3 \) sufficiently large and \( \mu \leq \frac{1}{4K_3} \). Then we denote \( \lambda' = (\lambda_1, \cdots, \lambda_{C_{n-1}^{m-1}}) \) and \( \lambda(\kappa) = (\lambda_{C_{n-1}^{m-1} + 1}, \cdots, \lambda_{C_n^m}) \). Since \( \kappa \in \Gamma_{k_0}^{(m)} \), we have \( \lambda(\kappa) \in \Gamma_{k_0} \) and \( S_{k_0}(\lambda(\kappa)) \geq b_0 > 0 \). Then for \( \forall 1 \leq l \leq C_{n-1}^{m-1} \), we have

\[
(5.50) \quad S_l(\lambda) \geq S_l(\lambda') - c(n, m, k, \kappa) K_3^{l-1} \geq K_3^l > 0,
\]
and, for $\forall \ l = C_{n-1}^{m-1} + l_0 \leq k$, $l_0 \leq k_0$,

$$
S_l(\lambda) \geq \left(\frac{3K_3}{2}\right)^{\frac{m-1}{n-1}} S_l(\lambda(\kappa)) - c(n, m, k, \kappa)K_3^{\frac{m-1}{n-1}} - 1
$$

(5.51)

$$
\geq \frac{b_1^0}{b_0^0} \left(3K_3\right)^{\frac{m-1}{n-1}} > 0,
$$

if we choose $K_3$ sufficiently large. It implies that $h$ is $k$-admissible. Similarly, $w = h - k_3|x|^2$ is also $k$-admissible if $k_3$ sufficiently small. By the concavity of $\tilde{F}$, we have

$$
\tilde{F}^{ij} w_{ij} \geq \tilde{F}[D^2u + D^2w] - \tilde{F}[D^2u]
$$

$$
\geq \tilde{F}[D^2w]
$$

$$
\geq \frac{1}{2} b_0^0 \left(\frac{3K_3}{4}\right)^\gamma,
$$

(5.52)

where $\gamma = \frac{C_{n-1}^{m-1}}{k} \leq 1$.

Then we have

$$
\tilde{F}^{ij} h_{ij} = \tilde{F}^{ij} (h - k_3|x|^2 + k_3|x|^2)_{ij} \geq \frac{1}{2} b_0^0 \left(\frac{3K_3}{4}\right)^\gamma + k_3 \tilde{F},
$$

(5.53)

for a large $K_3$. If we choose $K_3 \geq 2\left(\frac{k_3 \max_h f_{\min}}{\|Dh\|_{\infty}}\right)^\gamma$, then we have

$$
F^{ij} h_{ij} \geq k_3(1 + \mathcal{F}).
$$

(5.54)

Following the line of Qiu and Ma [22], we construct the sub barrier function as

$$
P(x) := g(x)(Du \cdot Dh(x) - \psi(x)) - G(x).
$$

(5.55)

with

$$
g(x) := 1 - \beta h(x),
$$

$$
G(x) := (A + \sigma N)h(x),
$$

$$
\psi(x) := \phi(x, u)|Dh|(x),
$$

where $K_3$ is the constant in the following Lemma 5.6, and $A$, $\sigma$, $\beta$ are positive constants to be determined. We have the following lemma.

**Lemma 5.6.** Fix $\sigma$, if we select $\beta$ large, $\mu$ small, and $A$ large, then

$$
P \geq 0, \quad \text{in} \quad \Omega_\mu.
$$

Furthermore, we have

$$
\sup_{\partial \Omega} u_{\nu\nu} \leq C + \sigma N,
$$

(5.57)

where constant $C$ depends only on $|u|_{C^1}$, $|\partial \Omega|_{C^2}$ $|f|_{C^2}$ and $|\phi|_{C^2}$. 
Proof. We assume \( P(x) \) attains its minimum point \( x_0 \) in the interior of \( \Omega_\mu \). Differentiate \( P \) twice to obtain

\[
P_i = g_i(u_r h_r - \psi) + g(u_{ri} h_r + u_r h_{ri} - \psi_i) - G_i,
\]
and

\[
P_{ij} = g_{ij}(u_r h_r - \psi) + g_i(u_{rj} h_r + u_r h_{rj} - \psi_j) + g_j(u_{rj} h_r + u_r h_{rj} - \psi_i) + g(u_{rj} h_r + u_r h_{rj}) + u_{rj} h_r + u_r h_{rj} - \psi_{ij} - G_{ij}.
\]

By a rotation of coordinates, we may assume that \((u_{ij})_{n \times n}\) is diagonal at \( x_0 \), so are \( W \) and \((F_{ij})_{n \times n}\). Denote \( F = \sum_{i=1}^{n} F^{ii} \) the trace of \((F_{ij})_{n \times n}\). We choose \( \mu < \min\left\{ \frac{1}{K_3}, \frac{1}{\beta} \right\} \) so that \( |\beta h| \leq \beta \frac{\mu}{2} \leq \frac{1}{2} \). It follows that

\[
1 \leq g \leq \frac{3}{2},
\]
By a straight computation we obtain

\[
F_{ij} P_{ij} = F^{ii} g_{ij}(u_r h_r - \psi) + 2F^{ii} g_i(u_{ij} h_i + u_r h_{ri} - \psi_i) + g F^{ii} (u_{rij} h_r + 2u_{ii} h_{ii} + u_r h_{rj} - \psi_i) - (A + \sigma N) F^{ii} h_{ii}
\]

\[
\leq (\beta C_1 - (A + \sigma N) k_3)(F + 1)
\]

\[
-2\beta F^{ii} u_{ii} h_i^2 + 2g F^{ii} u_{ii} h_{ii},
\]

where \( C_1 = C_1(|u|_{C^1}, |\partial \Omega|_{C^1}, |\phi|_{C^2}, |f|_{C^1}, n) \).

We divide indexes \( I = \{1, 2, \cdots, n\} \) into two sets in the following way,

\[
B = \{ i \in I | |\beta h_i^2| < \frac{k_1}{4} \},
\]

\[
G = I \setminus B = \{ i \in I | |\beta h_i^2| \geq \frac{k_1}{4} \},
\]

where \( k_1 \) is a positive number depends on \( |\partial \Omega|_{C^2} \) and \( K_3 \) such that \( |D^2 h|_{C^0} \leq \frac{k_1}{2} \). For \( i \in G \), by \( P_i(x_0) = 0 \), we get

\[
u_{ii} = \frac{A + \sigma N}{g} + \frac{\beta(u_r h_r - \psi)}{g} - \frac{u_r h_{ri} - \psi_i}{h_i}.
\]

Because \( |h_i^2| \geq \frac{k_1}{4} \) and \( 1 \leq g \leq \frac{3}{2} \), we have

\[
\frac{\beta(u_r h_r - \psi)}{g} - \frac{u_r h_{ri} - \psi_i}{h_i} \leq \beta C_2 (k_2, |u|_{C^1}, |\partial \Omega|_{C^2}, |\psi|_{C^1}).
\]

Then let \( A \geq 3\beta C_2 \), we have

\[
\frac{A}{3} + 2\sigma N \leq u_{ii} \leq \frac{4A}{3} + \sigma N,
\]
for \( \forall i \in G \). We choose \( \beta \geq 4nk_1 + 1 \) to let \( |h_i^2| \leq \frac{1}{4n} \) for \( i \in B \). Because \( \frac{1}{2} \leq |Dh| \leq 2 \), there is a \( i_0 \in G \), say \( i_0 = 1 \), such that

(5.64) \[ h_1^2 \geq \frac{1}{4n}. \]

We have

(5.65) \[ -2\beta \sum_{i \in I} F_{ii} u_{ii} h_i^2 = -2\beta \sum_{i \in G} F_{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in B} F_{ii} u_{ii} h_i^2 \leq -2\beta F_{11} u_{11}^2 - 2\beta \sum_{u_{ii} < 0} F_{ii} u_{ii} h_i^2 \leq -\beta F_{11} u_{11} - \frac{k_1}{2} \sum_{u_{ii} < 0} F_{ii} u_{ii}. \]

and

(5.66) \[ 2g \sum_{i \in I} F_{ii} u_{ii} h_{ii} = 2g \sum_{u_{ii} \geq 0} F_{ii} u_{ii} h_{ii} + 2g \sum_{u_{ii} < 0} F_{ii} u_{ii} h_{ii} \leq k_1 \sum_{u_{ii} \geq 0} F_{ii} u_{ii} - \frac{k_1}{2} \sum_{u_{ii} < 0} F_{ii} u_{ii}. \]

Plug (5.65) and (5.66) into (5.61) to get

(5.67) \[ F_{ii} P_{ij} \leq (\beta C_1 - (A + \sigma N)k_3) (F + 1) - \frac{\beta}{2n} F_{11} u_{11} - k_1 \sum_{u_{ii} < 0} F_{ii} u_{ii} + k_1 \sum_{u_{ii} \geq 0} F_{ii} u_{ii}. \]

Denote \( u_{22} \geq \cdots \geq u_{nn} \), and \( \mu_i = u_{ii} (1 \leq i \leq n) \) for simplicity. We also denote

\[ \lambda_1 = \max_{1 \leq i \leq n}\{w_{ini}\} = \mu_1 + \sum_{i=2}^{m} \mu_i, \]

\[ \lambda_{m_1} = \min_{1 \leq i \leq n}\{w_{ini}\} = \mu_1 + \sum_{i=n-m+2}^{n} \mu_i, \]

and \( \lambda_2 \geq \cdots \geq \lambda_{C_m} \) the eigenvalues of the matrix \( W \). We may assume \( N > 1 \), then from (5.2) we see that

(5.68) \[ |u_{ii}| \leq 2C_0N, \quad \forall i \in I. \]

Then

(5.69) \[ \lambda_i \leq 2mC_0N \leq \frac{3mC_0}{\sigma} u_{11}, \quad \forall 1 \leq i \leq C_m^* \cdot \]

We will consider the following cases.

**Case 1.** \( \lambda_{m_1} \leq 0 \).
It follows from (2.14) that
\[ F_{11}^1 > S_{k-1}(\lambda|m_1) \]
\[ \geq \frac{1}{C^m_n - k + 1} \sum_{i=1}^{m} S_{k-1}(\lambda|i) = \frac{1}{m(C^m_n - k + 1)}F. \]

Then we have
\[ F_{ij}P_{ij} \leq \left( \beta C_1 - (A + \sigma N)k_3 \right)(F + 1) + 2C_0k_1NF \]
\[ - \frac{\beta}{2nm}(C^m_n - k + 1)\left( \frac{A}{3} + \frac{2\sigma N}{3} \right)F \]
\[ < 0. \]

if we choose \( \beta > \frac{6nmk_1C_0(C^m_n - k + 1)}{\sigma} \) and \( A > \frac{\beta C_1}{k_3} \).

**Case 2.** \( \lambda_{m_1} > 0, u_{nn} \geq 0. \)

It follows from
\[ kf = \sum_{i=1}^{n} F_{ii}u_{ii} = \sum_{u_{ii} \geq 0} F_{ii}u_{ii} \]
and (5.67) that
\[ F_{ij}P_{ij} \leq \left( \beta C_1 - (A + \sigma N)k_3 \right)(F + 1) + k_1kf < 0, \]
if we choose \( A > \frac{3\beta C_1 + k_1 \max f}{k_3} \).

**Case 3.** \( \lambda_{m_1} > 0, -\frac{k_3}{4k_1}u_{11} \leq u_{nn} < 0. \)

It follows from
\[ \sum_{u_{ii} \geq 0} F_{ii}u_{ii} + \sum_{u_{ii} < 0} F_{ii}u_{ii} = kf \]
that
\[ -k_1 \sum_{u_{ii} < 0} F_{ii}u_{ii} + k_1 \sum_{u_{ii} \geq 0} F_{ii}u_{ii} = k_1(kf - 2 \sum_{u_{ii} < 0} F_{ii}u_{ii}) \]
\[ \leq k_1kf - 2k_1u_{nn}F \]
\[ \leq k_1kf + \left( \frac{2A}{3} + \frac{\sigma N}{2} \right)k_3F \]
\[ (5.72) \]

Similarly we choose \( A > \frac{3(\beta C_1 + k_1 \max f)}{k_3} \) to get
\[ F_{ij}P_{ij} < 0. \]

**Case 4.** \( \lambda_{m_1} > 0, u_{nn} < -\frac{k_3}{4k_1}u_{11}, \lambda C^m_n \leq -\delta_1' u_{11}, \delta_1' \) a small positive constant to be determined later.
Obviously, we have \( \lambda_1 \geq \lambda_{m_1} > 0 \). If \( u_{11} \geq u_{22} \), then it is easy to see \( \lambda_1 \geq \lambda_2 \). Otherwise, \( u_{11} < u_{22} \), since \( 2 \leq m \leq \frac{n}{2} \), then we have
\[
\lambda_1 = \mu_1 + \sum_{i=2}^{m} \mu_i \\
\geq \lambda_{m_1} + \mu_2 - \mu_n \\
> u_{11} \geq \frac{\sigma}{3mC_0} \lambda_2.
\]
(5.74)

Here we use (5.69) in the last inequality. Again we use (5.69) to have
\[
\lambda_{C_m} \leq - \frac{\sigma \delta'_1}{3mC_0} \lambda_1.
\]
(5.75)

Now (5.74) and (5.75) permit us to choose \( \delta = \min\{1, \frac{\sigma}{3mC_0}\} = \frac{\sigma}{3mC_0} \) and \( \varepsilon = \frac{\sigma \delta'_1}{3mC_0} \) in Proposition 2.7 to give
\[
F^{ij} P_{ij} \geq \frac{c_0}{(C_m - k + 1) F}.
\]
(5.76)

where \( c_0 = \min\{\frac{\sigma^4 \delta^2_1}{162m^4(n-2)(n-1)C_0^4}, \frac{\sigma^4 \delta^2_1}{108m^4(n-1)C_0^2}\} \). Similar to the Case 1 we have
\[
F^{ij} P_{ij} \leq \beta C_1 - (A + \sigma N)k_3)(F + 1) + 2C_0k_1N F - \frac{c_0 \beta}{2n(C_m - k + 1)} (A + \frac{2\sigma N}{3}) F < 0,
\]
if we choose \( \beta > \frac{6nk_1C_0(C_m - k + 1)}{c_0 \sigma} \) and \( A > \frac{\beta C_1}{k_3} \).

**Case 5.** \( \lambda_{m_1} > 0, u_{m_1} < - \frac{k_1}{k_3} u_{11}, \lambda_{C_m} \geq - \delta'_1 u_{11} \).

Note that, by (5.69),
\[
\lambda_1 \leq \frac{3mC_0}{\sigma} u_{11}.
\]

Let \( \delta'_1 = \frac{3C_0 k_{1}^{k - 1}}{(C_m)^{(k - 1)} k_{1}^{k - 1}} \), now we can choose \( \delta = \frac{k_3}{4k_1} \) and \( \theta_2 = \frac{k_{1}^{k - 1}}{4^{k-1} m^{k-1} C_m} \) in the Proposition 2.6 such that
\[
F^{11} \geq S_{k-1}(\lambda|m_1) \geq \theta_2 \sum_{i=1}^{C_m} S_k(\lambda|i) = \frac{\theta_2}{m} F.
\]
(5.77)

Similarly we choose \( \beta > \frac{6nmC_0k_1}{\sigma \theta_2} \) and \( A > \frac{\beta C_1}{k_3} \) to get
\[
F^{ij} P_{ij} < 0.
\]
(5.78)

In conclusion, we choose
\[
\beta = \max\{2nk_2 + 1, \frac{6nmk_1C_0C_m}{\sigma}, \frac{6nk_1C_0C_m}{c_0 \sigma}, \frac{6nmC_0k_1}{\sigma \theta_2}\}.
\]
Taking $\mu = \min\{\mu_0, \frac{1}{4k_3}, \frac{1}{2}\}$ and $A > \max\{3\beta C_2, \frac{3(\beta C_1 + k_1 k_{\max} f)}{k_3}\}$, we obtain $F_{ii} P_{ij} < 0$, which contradicts to that $P$ attains its minimum in the interior of $\Omega_\mu$. This implies that $P$ attains its minimum on the boundary $\partial \Omega_\mu$.

On $\partial \Omega$, it is easy to see $P = 0$. (5.79)

On $\partial \Omega_\mu \cap \Omega$, we have
\begin{equation}
P \geq -C_3(|u|_{C^1}, |\psi|_{C^0}) + (A + \sigma N) \frac{\mu}{2} \geq 0,
\end{equation}
if we take $A = \max\{\frac{2C_1}{\mu}, 3\beta C_2, \frac{3(\beta C_1 + k_1 k_{\max} f)}{k_3}\}$. Finally the maximum principle tells us that
\begin{equation}
P \geq 0, \quad \text{in} \quad \Omega_\mu.
\end{equation}
Suppose $u_{\nu\nu}(y_0) = \sup_{\partial \Omega} u_{\nu\nu} > 0$, we have
\begin{align*}
0 & \geq P_\nu(y_0) \\
& \geq (u_{r\nu} h_r + u_{r\nu} h_{r\nu} - \psi_\nu) - (A + \sigma N) h_\nu \\
& \geq u_{\nu\nu}(y_0) - C(|u|_{C^1}, |\partial \Omega|_{C^2}, |\psi|_{C^2}) - (A + \sigma N).
\end{align*}
Then we get
\begin{equation}
\sup_{\partial \Omega} u_{\nu\nu} \leq C + \sigma N.
\end{equation}
\hfill\Box

In a similar way, we construct the super barrier function as
\begin{equation}
P(x) := g(x)(Du \cdot Dh(x) - \psi(x)) + G(x).
\end{equation}
We have the following lemma.

**Lemma 5.7.** Fix $\sigma$, if we select $\beta$ large, $\mu$ small, and $A$ large, then
\begin{equation}
P \leq 0, \quad \text{in} \quad \Omega_\mu.
\end{equation}
Furthermore, we have
\begin{equation}
\inf_{\partial \Omega} u_{\nu\nu} \geq -C - \sigma N,
\end{equation}
where constant $C$ depends on $|u|_{C^1}$, $|\partial \Omega|_{C^2}$, $|f|_{C^2}$ and $|\phi|_{C^2}$.

**Proof.** We assume $P(x)$ attains its maximum point $x_0$ in the interior of $\Omega_\mu$. Differentiate $P$ twice to obtain
\begin{equation}
P_i = g_i(u_r h_r - \psi) + g(u_{ri} h_r + u_r h_{ri} - \psi_i) + G_i,
\end{equation}
and

\begin{equation}
\overline{P}_{ij} = g_{ij}(u_r h_r - \psi) + g_i(u_{rj} h_r + u_r h_{rj} - \psi_j) + g_j(u_{rij} h_r + u_r h_{rij} - \psi_i) + g(h_{rrij} h_r + u_{rrij} h_r - \psi_{ij}) + G_{ij}.
\end{equation}

As before we assume that \((u_{ij})\) is diagonal at \(x_0\), so are \(W\) and \((F_{ij})\). We choose 

\[ \mu = \min\left\{ \frac{1}{4n}, \frac{1}{3} \right\} \]

so that \(|\beta h| \leq \frac{\mu^2}{2} \leq \frac{1}{2}\). By a straight computation we obtain

\begin{equation}
F_{ij}P_{ij} = F_{ii} g_{ii}(u_r h_r - \psi) + 2F_{ii} g_i(u_{ii} h_i + u_r h_{ri} - \psi_i) + gF_{ii} u_{ii} h_i^2 + 2gF_{ii} u_{ii} h_i + (A + \sigma N)F_{ii} h_i + \frac{A}{3} \sigma N \frac{1}{h_i}.
\end{equation}

where \(C_1 = C_1(|u|_{C^1}, |\partial \Omega|_{C^3}, |\varphi|_{C^2}, |f|_{C^1}, n)\).

We divide indexes \(I = \{1,2, \cdots, n\}\) into two sets in the following way,

\[ B = \{i \in I | \beta h_i^2 < \frac{k_1}{2}\}, \]

\[ G = I \setminus B = \{i \in I | \beta h_i^2 \geq \frac{k_1}{2}\}, \]

where \(k_1\) is a positive number depends on \(|\partial \Omega|_{C^2}\) and \(K_3\) such that \(|D^2 h|_{C^0} \leq \frac{k_1}{2}\).

For \(i \in G\), by \(\overline{P}_i(x_0) = 0\), we get

\begin{equation}
u_{ii} = -\frac{A + \sigma N}{g} + \frac{\beta(u_r h_r - \psi)}{g} - \frac{u_r h_{ri} - \psi_i}{h_i}.
\end{equation}

Because \(|h_i^2| \geq \frac{k_1}{2}\), we have

\[ |\frac{\beta(u_r h_r - \psi)}{g} - \frac{u_r h_{ri} - \psi_i}{h_i}| \leq \beta C_2(k_1, |u|_{C^1}, |\partial \Omega|_{C^2}, |\psi|_{C^1}). \]

Then let \(A \geq 3\beta C_2\), we have

\begin{equation}
-\frac{4A}{3} - \sigma N \leq u_{ii} \leq -\frac{A}{3} - \frac{2\sigma N}{3}, \quad \forall i \in G.
\end{equation}

We choose \(\beta \geq 2nk_1 + 1\) to let \(|h_i^2| \leq \frac{1}{4n}\) for \(i \in B\). Because \(\frac{1}{2} \leq |Dh| \leq 2\), there is a \(i_0 \in G\), say \(i_0 = 1\), such that

\[ h_1^2 \geq \frac{1}{4n} \]
It follows that

\[-2\beta \sum_{i \in I} F^{ii} u_{ii} h_i^2 = -2\beta \sum_{i \in G} F^{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in B} F^{ii} u_{ii} h_i^2 \geq -2\beta F^{11} u_{11} h_1^2 - 2\beta \sum_{u_{ii} \geq 0} F^{ii} u_{ii} h_i^2 \]

\[\geq -\frac{\beta F^{11} u_{11}}{2n} - k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii}.\]  

(5.91)

and

\[2g \sum_{i \in I} F^{ii} u_{ii} h_{ii} = 2g \sum_{u_{ii} \geq 0} F^{ii} u_{ii} h_{ii} + 2g \sum_{u_{ii} < 0} F^{ii} u_{ii} h_{ii} \geq -k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + 2k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii}.\]  

(5.92)

Plug (5.91) and (5.92) into (5.88) to get

\[F^{ii} P_{ij} \geq \left( (A + \sigma N)k_3 - \beta C_1 \right)(F + 1) - \frac{\beta}{2n} F^{11} u_{11} \]

\[-2k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + 2k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii}.\]  

(5.93)

Denote \(u_{22} \geq \cdots \geq u_{nn},\) and \(\mu_i = u_{ii} \) \((1 \leq i \leq n)\) for simplicity. We also denote \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{C_n^m}\) the eigenvalues of the matrix \(W,\) and

\[\lambda_k = \sum_{l=1}^m \mu_{il},\]

\[\geq \mu_{i_1} + \sum_{i=n-m+2}^n \mu_i, \quad \text{for } \mu_{i_1} \geq \cdots \geq \mu_{im},\]

\[\lambda_{m_1} = \min_{1 \in \pi} \{u_{m_1}\} = \mu_1 + \sum_{i=n-m+2}^n \mu_i.\]

As before, assume \(N \geq 1,\) from (5.2) we have

\[|u_{ii}| \leq 2C_0 N, \quad \forall i \in I.\]  

(5.94)

Because \(u_{11} < 0,\) and from (2.9) in Proposition (2.3) we have \(\lambda_k > 0,\) then \(\mu_{i_1} > 0.\) It follows that \(\lambda_k \geq \lambda_{m_1}.\) Using (2.8) and (2.9) again, we obtain

\[F^{11} > S_{k-1}(\lambda|m_1) \geq S_{k-1}(\lambda|k) \geq C(n, m, k)F.\]  

(5.95)

Similarly we choose \(\beta = \frac{6nk_1 C_0}{\sigma C(n, m, k)} + 2nk_1 + 1\) and \(A > \frac{\beta C_1}{k_3}\) to get

\[F^{ii} P_{ij} > 0.\]  

(5.96)
This contradicts to that \( P \) attains its maximum in the interior of \( \Omega_\mu \). This contradiction implies that \( P \) attains its maximum on the boundary \( \partial \Omega_\mu \).

On \( \partial \Omega \), it is easy to see

\[
\overline{P} = 0.
\]

On \( \partial \Omega_\mu \cap \Omega \), we have

\[
\overline{P} \leq C_3(|u|_{C^1}, |\psi|_{C^0}) - (A + \sigma N)\frac{\mu}{2} \leq 0,
\]

if we take \( A = \frac{2C_3}{\mu} + \frac{\beta C_1}{K_3} + 1 \). Finally the maximum principle tells us that

\[
(5.97) \quad \overline{P} \leq 0, \quad \text{in } \Omega_\mu.
\]

Suppose \( u_{\nu\nu}(y_0) = \inf_{\partial \Omega} u_{\nu\nu} \), we have

\[
0 \leq P_\nu(y_0) \leq (u_{\nu\nu}h_r + u_r h_{\nu\nu} - \psi_\nu) + (A + \sigma N)h_\nu
\]

\[
(5.98) \quad \leq u_{\nu\nu}(y_0) + C(|u|_{C^1}, |\partial \Omega|_{C^2}, |\psi|_{C^2}) + (A + \sigma N).
\]

Then we get

\[
(5.99) \quad \inf_{\partial \Omega} u_{\nu\nu} \geq -C - \sigma N.
\]

Then we prove Theorem 5.4 immediately.

**Proof of Theorem 5.4.** We choose \( \sigma = \frac{1}{2} \) in Lemma 5.6 and 5.7, then

\[
(5.100) \quad \sup_{\partial \Omega} |u_{\nu\nu}| \leq C.
\]

Combining (5.100) with (5.2) in Lemma 5.1, we obtain

\[
(5.101) \quad \sup_{\Omega} |D^2 u| \leq C.
\]

\[
\Box
\]

6. Existence of the Neumann boundary problem

We use the method of continuity to prove the existence theorem for the Neumann problem (1.6) and (1.7).

**Proof of Theorem 1.3 and 1.4.** Consider a family of equations with parameter \( t \),

\[
(6.1) \begin{cases} 
S_k(W) = tf + (1 - t) \frac{(C^m_n)!m^k}{(C^m_n - k)!k!}, \text{ in } \Omega, \\
\quad u_\nu = -au + tb + (1 - t)(x \cdot \nu + \frac{a}{2}x^2), \text{ on } \partial \Omega.
\end{cases}
\]
From Theorem 3.1, 4.2, 4.4, 5.2 and 5.4, we get a global $C^2$ estimate independent of $t$ for the equation (6.1) in both cases of Theorem 1.3 and Theorem 1.4. It follows that the equation (6.1) is uniformly elliptic. Due to the concavity of $S_k^*(W)$ with respect to $D^2u$ (see [4]), we can get the global H"older estimates of second derivatives following the arguments in [19], that is, we can get

$|u|_{C^{2,\alpha}} \leq C$,

(6.2)

where $C$ depends only on $n$, $m$, $k$, $|u|_{C_1}, |f|_{C_2}, \min f$, $|\phi|_{C_3}$ and $\Omega$. It is easy to see that $\frac{1}{2}x^2$ is a $k$-admissible solution to (6.1) for $t = 0$. Applying the method of continuity (see [9], Theorem 17.28), the existence of the classical solution holds for $t = 1$. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity.$\blacksquare$

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