Abstract

The constraint algebra is derived in the 2nd order tetrad Hamiltonian formalism of the bigravity. This is done by a straightforward calculation without involving any insights, implicit functions, and Dirac brackets. The tetrad approach is the only way to present the bigravity action as a linear functional of lapses and shifts, and the Hassan-Rosen transform (characterized as “complicated redefinition of the shift variable” according to the authors) appears here not as an ansatz but as a fixing of a Lagrange multiplier. A comparison of this approach with the others is provided.
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1 Introduction

The theory of bigravity based on de Rham, Gabadadze, Tolley (dRGT) potential [1, 2] is under active study about 10 years, so its general structure is widely known, and the detailed explanation may be found in reviews, see for instance [3]. The prehistory of it will be reminded in Section 3. Subject of this article is limited by the Hamiltonian approach to this theory and the constraint algebra of it. This subject was also discussed in many publications where various methods and variables were used. Here we are trying to combine them and show that the most transparent picture seems to appear in tetrad variables in the second order formalism.

The main problem of the Hamiltonian approach to massive gravity and bigravity was to prove the absence of ghosts in the theory. A scheme of such a proof by non-perturbative method in the metric Hamiltonian formalism was proposed for the first time in a set of articles by Hassan, Rosen et al [4, 5, 6, 7]. They found a transform which provided linearity of the action in both the two lapse functions denoted here as $N, \bar{N}$ and in one shift vector $N^i$. But this transform was reasonably characterized by one of its authors as complicated redefinition of the shift variable [8].

As it was uncovered by Dirac [9] and stressed by Arnowitt, Deser, and Misner (ADM) [10] in order to make the mechanical action invariant under time reparametrizations $t \to t' = f(t)$ one should provide the Lagrangian having the 1st order homogeneity in all its velocities, i.e. satisfying the condition $L(q, c\dot{q}) = cL(q, \dot{q})$, where the physical time should become one of the coordinates. This requires introduction of the lapse function as a Lagrange multiplier $N(t) = \dot{f} \equiv df/dt$, and the Hamiltonian acquires a form $H = NR(q, p)$ where $R$ occurs a constraint. In the field theory one needs to add an invariance under the spatial coordinate transformations and then additionally appears a shift vector $N^i = dx^i/dt$ accompanied by new constraints $R_i$. The form of Hamiltonian then becomes $H = \int d^nx (NR + N^iR_i)$.

In bigravity we have two metric tensors with the signature $(-, +, +, +)$ defined on the same 4-dimensional manifold, and the action is a sum of the three terms: two copies of the General Relativity (GR) action $I_f, I_g$, and a term providing the interaction $I_{\text{int}} = -\frac{2m^2}{\kappa} \int d^4x \ U_{\text{dRGT}}$. Here both $I_f, I_g$ are linear functionals of the corresponding lapse and shift functions $N, N^i, \bar{N}, \bar{N}^i$, whereas the potential does not have this property. Nevertheless this potential should be homogeneous of the 1st order under simultaneous multiplication of all functions $N, N^i, \bar{N}, \bar{N}^i$ on the same constant multiplier. The
Hassan-Rosen transform [4, 5, 6, 7] in particular exploited this homogeneity in the proof of the appearance of a new primary constraint which excluded the superficial degree of freedom as accompanied by a secondary constraint noncommuting with it. This pair of constraints is necessary to avoid of the ghost.

Next step forward was done in the work [8] where the preference of the tetrad approach over the metric one which is coming from the linearity of dRGT potential in all the lapses and shifts was demonstrated. This privilege was elaborated further in publications [11, 12] where the 1st order formalism was used, and so much more variables and 2nd class constraints appeared. On the base of the intermediate Dirac brackets the authors succeeded in deriving the full constraint algebra. But this approach [11, 12] unfortunately has not get a further application till now and no detailed comparison of its results with the metric approach has not been provided. In due term the 2nd order tetrad formalism for the bigravity has been developed in the article [13]. In the first version of it the author declared his disagreement with the conclusions of the predecessors, but later agreed with them.

In our previous work we considered the Hamiltonian approach to bigravity in metric variables [14, 15], in tetrad ones [16, 17], and in the minisuperspace [18]. The peculiarities of the different formalisms were that in the metric variables we did not use any explicit expression of the potential, and worked with some equations this potential should fulfil, whereas in tetrad variables the calculations were limited to the minimal potential case $\beta_1 \neq 0$. Only in the minisuperspace all the derivations were completed up to an explicit formula for the key Lagrange multiplier $u = \bar{N}/N$. In all these cases the presence of the pair of 2nd class constraints was demonstrated. The coefficients in the constraint algebra were also derived. Now we will see that they are independent of the formalism, and their explicit meaning is transparent from this work. We are sorry to acknowledge some misprints in the published [18] formula for the dRGT potential in tetrad variables. This work is to complete the analysis of the problem, and to compare the results with the ones obtained by other authors.

A recent publication by Kocic [20] provided an essential progress in calculations dealing with the dRGT potential and its derivatives and in deriving the Hamiltonian equations by $3 + 1$ decomposition of the Lagrangian ones. The key point of the article [20] is in defining and exploiting the geometric mean of the two metrics. Whereas it is very interesting mathematically, this is not necessary for the bigravity potential calculations. In a difference
to this work there were no Poisson brackets defined and constraint algebra calculated in [20]. This algebra has been recently considered by Hassan and Lundkvist [19] in the metric formalism. The constraint algebra itself was known before, see articles [11, 12, 15, 16], but in the metric formalism the calculations were provided with the general implicit formula of the potential [15], and in the 2nd order tetrad approach it was done only for the minimal potential case [16] \( \beta_1 \neq 0 \). The algebra obtained in the work [19] coincides with results of works [15, 16]. Another motivation of the article [19] was in extracting from the algebra of the 1st class constraints a new spatial metric for the bigravity. But this problem has not been solved there because it is found that both two induced metrics can play this role.

The most technically difficult problem in all the mentioned approaches was a derivation of the secondary constraint denoted here as \( \Omega \). In massive gravity it was done for the first time by Hassan and Rosen [7], see Eq. (3.32) there.

In the bigravity the calculation of \( \Omega \) has been completed in the articles (given in the chronological order):

- Alexandrov, Krason and Speciale [11], Eqs. (42)-(44), without any derivatives, as the 1st order formalism is used;
- Soloviev and Tchichikina [15], Eq. (55);
- Alexandrov [12], Eq. (3.14), without any derivatives, as the 1st order formalism is used;
- Soloviev [16], Eqs. (30), (39)-(41); see also [17], Eqs. (38), (47)-(49);
- Hassan and Lundkvist [19], Eqs. (3.18), (3.22);
- Kocic [20], Eq. (3.8);
- Soloviev (this work): Eq. (152).

This work is organized as follows. Section 2 contains the basics of the Kuchař formalism for the bigravity, this formalism enjoys the explicit covariance under spacetime diffeomorphisms. In Section 3 we remind the prehistory of the bigravity theory and the construction of the dRGT potential. In Section 4 we compare the approach of this work with the pioneer method by Hassan and Rosen. Section 5 contains a definition of the tetrad variables,
the symmetry conditions and their consequence: the Hassan-Rosen transform. In Section 6 the canonical variables are defined, and the Hamiltonian and primary constraints are provided. Section 7 is devoted to a study of the conditions for compatibility of the primary constraints with dynamics, i.e. to a derivation of the secondary constraints and to fixing of some Lagrange multipliers. In Section 8 we formulate our conclusions. Appendix A contains expressions for the dRGT potential and for the secondary constraint \( \Omega \). In Appendix B we compare our notations with notations of related works.

We prefer to use the same notations as in Refs. [16, 17, 18]. In particular, for spacetime coordinate indices running from 0 to 3 we use small Greek letters; for internal indices running from 1 to 3 we use small Latin letters from the beginning of the alphabet, for spatial indices small letters from the middle of the alphabet are used, for internal indices running from 0 to 3 the capital Latin letters are used. Of course, we consider only such couples of metrics that have common timelike and spacelike vectors. When the same letter is used for analogous quantities constructed with the first \( f_{\mu \nu} \) or with the second \( g_{\mu \nu} \) metric, then an upper bar refers to the second one. Some additional notations are explained in Appendix A.

## 2 Kuchař’s formalism in bigravity

In dealing with two metrics it is suitable to replace the ADM coordinate system where \( X^0 \) is not arbitrary but fixed by the given spacetime foliation. Let us instead take an arbitrary frame in order to save the explicate diffeomorphism invariance, so a foliation will be prescribed by four embedding variables \( X^\mu = e^\mu(\tau, x^i) \). Then the ADM frame [10] will be only one of the all possible embedding variables choices

\[
X^0 = \tau, \quad X^i = x^i. \tag{1}
\]

This approach was developed by Kuchař [23, 24, 25, 26], then analogous formalism was exploited by York [27]. The lapse and shift variables \( N, N^i \) here can not be expressed through \( f^{00}, f_{0i} \) components of the metric tensor \( f_{\mu \nu} \), now they are determined by the following equations

\[
N = -\dot{e}^\alpha n_\alpha, \quad N^i = e^\alpha e^i_\alpha. \tag{2}
\]
where
\[
e^\alpha_i = \frac{\partial e^\alpha}{\partial x^i}, \quad e_i^j = f_{\alpha\beta} e^\beta_j \eta^{ij}, \quad e^\alpha = \frac{\partial e^\alpha}{\partial \tau},
\]  
(3)
\[
\eta_{ij} = f_{\mu\nu} \epsilon^\mu_i \epsilon^\nu_j, \quad \eta_{ij} \eta^{jk} = \delta^k_i,
\]  
(4)
\[
n^\alpha e^\alpha_i = 0, \quad f_{\alpha\beta} n^\alpha n^\beta = -1, \quad n^\alpha = f_{\alpha\beta} n^\beta.
\]  
(5)

We use Kuchař’s approach in bigravity in order to obtain an explicitly covariant \((3 + 1)\)-decomposition of the matrix \(Y^\alpha_\beta = g^{\alpha\mu} f_{\mu\beta}\), because the invariant potentials are constructed as functions of this matrix. First we are to make a choice for the basis as there are two different normals to the given hypersurface for the two metric tensors. Without losing any generality we choose here the basis formed of \(f_{\mu\nu}\), and apply a notation \((n^\alpha, e^\alpha_i)\) for it. Then the corresponding metric is decomposed as follows
\[
f_{\mu\beta} = -n^\mu n^\beta + \eta_{ij} e^i_\mu e^j_\beta,
\]  
(6)
where \(\eta_{ij}\) is the spatial metric induced on the hypersurface, and \(\gamma_{ij}\) is the inverse matrix to it. In the full analogy we introduce the basis \((\bar{n}^\alpha, \bar{e}^\alpha_i)\) constructed of metric \(g_{\mu\nu}\), then
\[
g_{\mu\beta} = -\bar{n}^\mu \bar{n}^\beta + \gamma_{ij} \bar{e}^i_\mu \bar{e}^j_\beta, \quad \gamma_{ij} = g_{\mu\nu} \bar{e}^i_\mu \bar{e}^j_\nu, \quad \gamma_{ij} \gamma_{jk} = \delta^k_i,
\]  
(7)
\[
e^\alpha_i = \gamma_{ij} \bar{e}^j_\alpha, \quad \bar{n}^\alpha e^\alpha_i = 0, \quad g^{\alpha\beta} \bar{n}^\alpha \bar{n}^\beta = -1, \quad \bar{n}^\alpha = g^{\alpha\beta} \bar{n}^\beta.
\]  
(8)

Now we should calculate coefficients in the decomposition of matrix \(Y^\alpha_\beta\)
\[
Y^\alpha_\beta \equiv g^{\alpha\mu} f_{\mu\beta} = Y^\perp_\perp n^\alpha n^\beta + Y^\perp_\perp n^\alpha e^\beta_i + Y^\perp_i e^\alpha_i n^\beta + Y^\perp_k e^\alpha_k e^\beta_i,
\]  
(10)
they are the following
\[
Y^\perp_\perp = Y^\alpha_\beta n^\alpha n^\beta, \quad Y^\perp_i = -Y^\alpha_\beta n^\alpha e^\beta_i, \quad Y^\perp_k = -Y^\alpha_\beta e^\alpha_k n^\beta, \quad Y^\perp_i = Y^\alpha_\beta e^\alpha_k e^\beta_i.
\]  
(11)

When decomposing \(g^{\alpha\mu}\) in the basis \((n^\mu, e^\mu_i)\) we get
\[
g^{\alpha\mu} = g^{\perp\perp} n^\alpha n^\mu + g^{\perp k} n^\alpha e^\mu_k + g^{i\perp} e^\alpha_i n^\mu + g^{i k} e^\alpha_i e^\beta_k,
\]  
(12)
and for \(f_{\mu\beta}\), given Eq. (6), we obtain
\[
f^\perp\perp = -1, \quad f^\perp_i = 0 = f^\perp k, \quad f^i_k = \eta^i_k,
\]  
(13)
as basis $n^\mu, e_i^\mu$ is constructed of this metric.

Let us introduce new variables $u, u^i$, that have at least three meanings:

1) they appear in the formulas relating two pairs of lapse and shift functions:
   \[ u = \frac{\bar{N}}{N}, \quad u^i = \frac{\bar{N}^i - N^i}{N}, \]  

2) they appear in projecting tensor $g^{\mu\nu}$ onto basis $(n_\alpha, e_i^\alpha)$
   \[ u = \frac{1}{\sqrt{-g^{\perp\perp}}} \equiv \frac{1}{\sqrt{-g^{\mu\nu}n_\mu n_\nu}}, \quad u^i = -\frac{g^{\perp i}}{g^{\perp\perp}} \equiv \frac{g^{\mu\nu}n_\mu e_i^\nu}{g^{\alpha\beta}n_\alpha n_\beta}, \]  

3) they are coefficients of transformation between the two bases $(\bar{n}_\alpha, \bar{e}_i^\alpha)$ and $(n_\alpha, e_i^\alpha)$:
   \[ \bar{n}_\mu = u n_\mu, \quad \bar{e}_i^\mu = e_i^\mu - u^i n_\mu, \quad \bar{n}^\mu = \frac{1}{u} n^\mu - \frac{u^i}{u} e_i^\mu. \]

These variables allow to write Eq. (12) in the following form
\[ g^{\alpha\mu} = -u^{-2} n^\alpha n^\mu + u^{-2} u^k n^\alpha e_k^\mu + u^{-2} u^i e_i^\alpha n^\mu + (\gamma^{ij} - u^{-2} u_i^k u^k) e_i^\alpha e_j^\mu, \]

By contracting expressions from Eqs. (6), (12) we get
\[ Y = \begin{pmatrix} -u^{-2}[n^\mu n_\nu] & u^{-2}u^i[n^\mu e_i^\nu] \\ u^{-2}u^j[e_j^\mu n_\nu] & (\gamma^{ij} - u^{-2} u_i^k u^k) [e_j^\mu e_{\nu j}] \end{pmatrix}, \]  

or
\[ Y^{\perp} = -u^{-2}, \quad Y^{\perp i} = u^{-2} u_i, \]  
\[ Y^{k} = u^{-2} u^k, \quad Y^{k i} = \gamma^{kj} n_{ji} - u^{-2} u^k u_i. \]

3  **Bigravity and the dRGT potential**

The bimetric theory seems to appear for the first time in two articles by Rosen [28, 29]. Rosen’s motivation was to define the energy-momentum tensor for the gravitational field. The second metric was fixed and even flat. Many years later spin-2 fields had been introduced in particle physics, and they were already treated as dynamical ones [30, 31, 32, 34]. The renewed interest in multi-dimensional Kaluza-Klein models and the new problems of
dark energy and dark matter created bigravity [35] in the form close to the present. It was proposed to take two GR Lagrangians with the minimally coupled matter fields and to organize their coupling by means of a potential constructed as a scalar density constructed of the two metric tensors without any derivatives. Then the dynamical equations for both metrics are of the GR form where the sources are both the matter energy-momentum tensors, and the new tensors formed algebraically of the metrics. The last ones are obtained as variational derivatives of the potential with respect to the corresponding metric tensor. The energy-momentum conservation law is fulfilled separately for each source.

An obstacle to apply these theories to physics was in appearance of the ghost degree of freedom [36]. But soon it becomes clear that there are potentials free of this difficulty.

The dRGT potential is formed as a linear combination of the symmetric polynomials of matrix $X^\mu_\nu = \sqrt{Y^\mu_\nu} \equiv \left( \sqrt{g^{-1} f} \right)_\mu^\nu$,

$$ U = \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(X) = \beta_0 \sqrt{-g} + \ldots + \beta_4 \sqrt{-f}, $$

where

$$
\begin{align*}
  e_0 &= 1, \\
  e_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\
  e_2 &= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_4 + \lambda_4 \lambda_1 + \lambda_1 \lambda_3 + \lambda_2 \lambda_4, \\
  e_3 &= \lambda_1 \lambda_2 \lambda_3 + \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_4, \\
  e_4 &= \lambda_1 \lambda_2 \lambda_3 \lambda_4,
\end{align*}
$$

and $\lambda_i$ are eigenvalues of $X$. These symmetric polynomials may be also
expressed through traces of $X$ and of its degrees

$$e_0 = 1,$$

$$e_1 = \text{Tr}X,$$

$$e_2 = \frac{1}{2} \left( (\text{Tr}X)^2 - \text{Tr}X^2 \right),$$

$$e_3 = \frac{1}{6} \left( (\text{Tr}X)^3 - 3\text{Tr}X\text{Tr}X^2 + 2\text{Tr}X^3 \right),$$

$$e_4 = \frac{1}{24} \left( (\text{Tr}X)^4 - 6(\text{Tr}X)^2\text{Tr}X^2 + 3(\text{Tr}X^2)^2 + 8\text{Tr}X\text{Tr}X^3 - 6\text{Tr}X^4 \right) =$$

$$= \det X = \frac{\det ||F_{\mu\alpha}||}{\det ||E_{\mu\alpha}||} \equiv \frac{\sqrt{-f}}{\sqrt{-g}}.$$  \hspace{1cm} (22)

After 3+1-decomposition of both metrics based on the ADM [10], Kuchař [23, 24, 25, 26] and York [27] methods the potential can be expressed in the following form

$$U = N\tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij}).$$  \hspace{1cm} (23)

Below we follow notations of works [14, 15] which are as follows

$$V = \frac{\partial \tilde{U}}{\partial u},$$  \hspace{1cm} (24)

$$V_i = \frac{\partial \tilde{U}}{\partial u^i},$$  \hspace{1cm} (25)

$$W = \tilde{U} - u\frac{\partial \tilde{U}}{\partial u} - u^i \frac{\partial \tilde{U}}{\partial u^i}.$$  \hspace{1cm} (26)

Unfortunately it was possible to get the explicite form of function $\tilde{U}$ in metric approach only for 1+1 spacetime dimension. But in the massive gravity case Comelli et al. [37, 38, 39] working with an implicit potential function have succeeded to show$^1$ that if the potential $\tilde{U}$ is a solution of the homogeneous Monge-Ampere equation

$$\text{Det}||\frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}|| = 0,$$  \hspace{1cm} (27)

then the Hamiltonian formalism contains two second class constraints excluding the ghost degree of freedom. In bigravity the analogous result was

$^1$They applied the mathematical results by Leznov and Fairlie [40]
obtained in works [14, 15], the constraint algebra was derived there on the
base of the Dirac brackets. This algebra has been confirmed later in works
based on the tetrad approach [16, 19]. Here we come to the same algebra,
but now uncovering the meaning of all its coefficients.

4 The Hassan-Rosen transform

The first proof for the absence of ghost in bigravity with the dRGT potential
has been given by the authors of works [4, 5, 6, 7], who proposed a special
transform of the variables. But an implicit function was present in this
method also. The results were based on the properties of this function to
fulfill some equation and to have some symmetry.

Let us remind the idea. In our notations the Hassan-Rosen transform is
as follows
\[ u^i = v^i + uD^i_j v^j, \]  
(28)

here the mentioned implicit matrix function is \( D^i_j \) and the new variable is
\( v^i \). One easily obtain the conditions for it when requiring a fulfillment of the
following matrix equation
\[ X = \sqrt{Y}, \]  
(29)

it is supposed that
\[ X = \left( \begin{array}{c}
\left( \frac{-\varepsilon}{u} \right) [n^\mu n_\nu] \\
\frac{-\varepsilon v^i v^j}{u} + \frac{1}{\varepsilon} D^{ij} \left[ e^\mu_i e^\nu_j \right] \end{array} \right), \]  
(30)

where \( \varepsilon = 1/\sqrt{1 - \eta_{ij} v^i v^j} \) and \( Y \) is given by Eq. (18). The necessary conditions are
\[ D^{ij} = D^{ji}, \quad \gamma^{ij} = D^k_i v^k D^j_m v^m + \varepsilon^{-2} D^{ik} D^j_k. \]  
(31)
The indices here are moved up and down by the spatial metric \( \eta_{ij} \) and its
inverse. Below we obtain an explicit form of the matrix \( D^i_j \) in the tetrad
formalism.

5 The tetrads in GR and in bigravity

The description of the gravitational field in metric terms is not the only one
possible. The metric can be replaced, for example, by the matrix of the
tetrad variables $E^A_\mu$ that taken together with its inverse $E^\mu_A$ allows to use an orthonormalized basis required for coupling gravity to fermions. The metric and the tetrads are related by the following equations

$$g_{\mu\nu} = E^\mu_A E^\nu_B h_{AB}, \quad h_{AB} = \text{diag}(-1, 1, 1, 1) \quad (32)$$

$$g^{\mu\nu} = E^\mu_A E^\nu_B h^{AB}, \quad E^\mu_A E^\nu_B = \delta^\mu_\nu, \quad E^\mu_A E^\mu_B = \delta^B_A. \quad (33)$$

The Lorentz transformations (4-rotations) of tetrads

$$\Lambda^A_B E^B_\mu = E^\prime_\mu$, \quad E^\mu_A \Lambda^C_D = E^\prime_D, \quad \Lambda^C_A h_{CD} \Lambda^D_B = h_{AB}, \quad (34)$$
do not change the metric tensor. By using this freedom of 4-rotations we can chose one of the tetrad covectors (the timelike one) to coincide with the unit normal covector of the hypersurface, for example,

$$E_{0\mu} = \bar{n}_\mu, \quad (35)$$

then $E_{a\mu}$ will be tangential to the hypersurface. It is suitable to construct a triad $e^a_i = E^a_\mu e^\mu_i$ related to the induced metric by the following formulas

$$\gamma_{ij} = e^a_i e^b_j \delta_{ab}, \quad \delta_{ab} = \text{diag}(1, 1, 1), \quad (36)$$

$$\gamma^{ij} = e^a_i e^a_j, \quad e^a_i e^b = \delta_{ab}, \quad e^a_i e^j_a = \delta^j_i. \quad (37)$$

This is usually called as a choice of the suited tetrads $[47]$.

$$E^0_\mu = -\bar{n}_\mu, \quad E^a_\mu = \bar{e}^i_\mu e^a_i. \quad (38)$$

One may argue in the opposite direction: let us first introduce triad representation for the induced metric and then lift these triad 3-vectors from the hypersurface to the spacetime 4-vectors as follows

$$E^\mu_a = e^a_i e^\mu_i. \quad (39)$$

As the bigravity potential expressed in tetrads is invariant only under the diagonal rotations of the two tetrads $E^a_{\mu A}, F^\mu_A$ we can not take the second tetrad $F$ a suited one also. Instead we parametrize$^2$ it as a product of an arbitrary Lorentz boost

$$\Lambda^A_B = \begin{pmatrix} \varepsilon & p_b \\ p^a & \varepsilon p^a \end{pmatrix}, \quad \varepsilon = \sqrt{1 + p^a p_a}, \quad \mathcal{P}^a_b = \delta^a_b + \frac{1}{\varepsilon + 1} p^a p_b, \quad (40)$$

$^2$Below we will see that $\varepsilon$ introduced here coincide with that introduced in Eq. (30)
on a suited tetrad $\mathcal{F}$

\[
\mathcal{F}^0_\mu = -n_\mu, \\
\mathcal{F}^a_\mu = \epsilon^j_\mu f^a_{ja}, \quad \epsilon^j_\mu = f_{\mu\nu} e^i_\nu \eta^{ij}, \\
F^A_\nu = \Lambda^A_B F^B_\nu.
\] (41) (42) (43)

The parameter of this boost, $p_a$ or $v_a = p_a/\varepsilon$, therefore will be a dynamical variable of the bigravity. We express all the four vectors of the second tetrad ($F^0_i$, $F^a_i$) by means of the three spatial vectors of a triad (formed of the suited tetrad $\mathcal{F}$) and of this parameter. As a result, in the new notations

\[
F^a_i = \tilde{f}^a_i \equiv \mathcal{P}_{ab} f_{bi}, \\
F^0_i = \tilde{v}_i \equiv v_a \tilde{f}^a_i \equiv p_a \tilde{f}^a_i \equiv p_i,
\] (44) (45)

and the corresponding induced metric is as follows

\[
\eta_{ij} = F^A_i F^A_j = -\tilde{v}_i \tilde{v}_j + \tilde{f}^a_i \tilde{f}^a_j \equiv -p_i p_j + \tilde{f}^a_i \tilde{f}^a_j.
\] (46)

The Poisson brackets are canonical

\[
\{ F^0_i, \Pi^j_0 \} = 0, \quad \{ F^0_i, \Pi^j_0 \} = \delta^j_i \delta(x, y), \\
\{ F^a_i, \Pi^j_0 \} = 0, \quad \{ F^a_i, \Pi^j_0 \} = \delta^j_i \delta_{ab} \delta(x, y).
\] (47) (48)

In the practical calculations sometimes it is suitable to use noncanonical variables $v_a = p_a/\varepsilon$, $f_{ai}$ instead of the canonical ones $F^0_i = \tilde{v}_i \equiv p_i$, $F^a_i = \tilde{f}^a_i$, and the following relations derived from Eqs. (47), (48)

\[
\{ v_a(x), \Pi^0_b(y) \} = \tilde{f}^a_i \delta(x, y), \quad \{ v_a(x), \Pi^j_b(y) \} = -\tilde{f}^a_i v_b \delta(x, y), \\
\{ f_{ai}(x), \Pi^j_b(y) \} = \left( \delta^j_i \delta_{ab} + \frac{p_i p_b \tilde{f}^a_j}{\varepsilon + 1} \right) \delta(x, y), \\
\{ f_{ai}(x), \Pi^0_j(y) \} = -\frac{p_a \delta^j_i + \varepsilon \tilde{f}^{ja} p_i}{\varepsilon + 1} \delta(x, y).
\] (49) (50) (51)

where a new notation

\[
\tilde{f}^{ia} = (\mathcal{P})^{-1}_{ab} f^{ib}.
\] (52)

is introduced.
It was remarked in the work [8] that after replacing metric variables by
tetrads it is easy to obtain an explicit expression of the dRGT potential and
it is linear in all lapses and shifts of the two metrics\(^3\). In fact, matrix
\[ X_\nu^\mu = E^{\mu A} F_{\nu A}, \]  \hspace{1cm} (53)
occurring a square root of matrix \[ Y_\nu^\mu = g^{\mu \alpha} f_{\alpha \nu}, \]
it the symmetry conditions
\[ E_A^\mu F_B^\mu - E_B^\mu F_A^\mu = 0. \] \hspace{1cm} (54)
are fulfilled.

With the given formulas for tetrads we can calculate matrix \( X_\nu^\mu \) defined
by Eq. (53) and obtain the following
\[ X_\nu^\mu = \begin{pmatrix} A[n^\mu n_\nu] & B^j [n^\mu e_{\nu j}] \\ C^i [e^\mu_i n_\nu] & D^{ij} [e^\mu_i e_{\nu j}] \end{pmatrix}, \] \hspace{1cm} (55)
where
\[ A = -\frac{\varepsilon}{u}, \] \hspace{1cm} (56)
\[ B^j = \frac{p_{ja}}{u} \equiv \frac{\varepsilon v^j}{u}, \] \hspace{1cm} (57)
\[ C^i = \frac{\varepsilon u^i}{u} - p_{ia} e^a \equiv \frac{\varepsilon(u^i - u\bar{v}^j)}{u}, \] \hspace{1cm} (58)
\[ D^{ij} = -\frac{u^i p^a f_{ja}}{u} + f_{ja} p_{ab} e^b \equiv -\frac{\varepsilon u^i v^j}{u} + f_{ja} p_{ab} e^b. \] \hspace{1cm} (59)
In order to calculate the symmetric polynomials of matrix \( X \), at first we
estimate the traces\(^4\)
\[ \text{Tr} X = -A + D, \] \hspace{1cm} (60)
\[ \text{Tr} X^2 = A^2 - 2(BC) + \text{Tr} D^2, \] \hspace{1cm} (61)
\[ \text{Tr} X^3 = -A^3 + 3A(BC) - 3(BDC) + \text{Tr} D^3. \] \hspace{1cm} (62)

Given expressions for all the symmetric polynomials Eq. (22) we can obtain
an explicit formula for the dRGT potential that occurs linear in variables
\( u, u^i \)
\[ \tilde{U} = uV + u^i V_i + W. \] \hspace{1cm} (63)
\(^3\)the celebrated transform Eq. (28) permits to exclude one of the shifts
\(^4\)There was a misprint in the last sign of Eq. (62) in the published text of our work [16]
The formulas for $V$, $V_i$, and $W$ are given in Appendix A. These expressions depend on canonical variables $e_{ia}$, $f_{ia}$, $p_i$. As the potential does not contain velocities, it does not change in the course of transformation from the Lagrangian variables to the Hamiltonian ones.

At last, we should pay attention to the symmetry conditions Eq. (54). In the Hamiltonian variables they take the following form

$$G_a \equiv p_a + up_a f_{ibj} e^{ja} - u^i P_{ab} f_{bj} = 0, \quad (64)$$
$$G_{ab} \equiv f_{ci} P_{c[ab]} e^{i} \equiv P_{[ac} x_{eb]} \equiv z_{ab} - z_{ba} = 0. \quad (65)$$

Given (65) Eqs. (64) can be solved for $u^i$

$$u^i = f^{ib} \left( \frac{p_b}{\varepsilon} + up_a f_{ajb} e^{ja} P_{cb}^{-1} \right), \quad (66)$$
where

$$P_{cb}^{-1} = \delta_{cb} - \frac{p_c p_b}{\varepsilon (\varepsilon + 1)}. \quad (67)$$

The functions $u^i$ are possible to express\(^5\) from Eqs. (65), (66) as follows

$$u_i = v^i + u\bar{v}^i, \quad (68)$$
and this allows to present the tetrad symmetry condition in the simplest form

$$u^i = v^i + u\bar{v}^i, \quad \bar{v}^i = \frac{e^{ia} p_a}{\varepsilon} \equiv e^{ia} v_a, \quad (69)$$
$$z_{ij} = z_{ji}, \quad z_{ij} = e_{ai} f_{ajb} \equiv e_{ai} z_{ab} e_{bj}. \quad (70)$$

Given these results matrix $X$ takes the following form

$$X^\mu_{\nu} = \begin{pmatrix} -\frac{\xi}{u} [n^\mu n_\nu] & \frac{\varepsilon v_i}{u} [n^\mu e_{ij}] \\ \frac{\varepsilon v_i}{u} [e_i^\mu n_\nu] & \left( z_{ij} - \frac{\varepsilon v_i v_j}{u} \right) [e_i^\mu e_{ij}] \end{pmatrix}, \quad (71)$$
where the inverse of matrix $z_{ij}$ appears

$$z^{ij} = f^{ia} e^{ja} = z_{ji}. \quad (72)$$

If one compare the derived expression of matrix $X$ with Eq. (30) it is easy to see that they are equivalent if we take

$$D^{ij} = \varepsilon z^{ij} \equiv \varepsilon f^{ia} e^{ja}. \quad (73)$$

\(^5\)It was shown for the first time in the work [13]
Then Eq. (31) is satisfied as
\[ D^i_j v^j = \bar{v}^j, \quad D^i_k D^j_k = \varepsilon^{ia} P_{ab}^{-1} f^{k} \eta_{kt} \varepsilon P_{cd}^{-1} f^{ld} e^{jd} = e^{ia} e^{ja} - \bar{v}^i \bar{v}^j. \] (74)

The similar formulas were obtained in Ref. [22], see Eqs. (3.10), (3.11) there, but without application to the Hamiltonian formalism.

6 The constraints

The momenta do not depend on the potential therefore we can rewrite Hamiltonians \( H_g \) and \( H_f \) in the tetrad variables and their conjugate momenta as calculated before adding the interaction between \( f_{\mu\nu} \) and \( g_{\mu\nu} \). These Hamiltonians are the same as they were in the GR [45].

In the metric approach the Hamiltonians for \( f_{\mu\nu} \) and \( g_{\mu\nu} \) are as follows
\[
H_f = \int d^3x (N \mathcal{H} + N^i \mathcal{H}_i), \tag{75}
\]
\[
H_g = \int d^3x (\bar{N} \bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i), \tag{76}
\]
where
\[
\mathcal{H} = \mathcal{H}_M - \frac{\sqrt{\eta}}{\kappa(f)} \left( R^{(\eta)} - 2\Lambda^{(f)} \right) - \frac{\kappa(f)}{\sqrt{\eta}} \left( \frac{\Pi^2}{2} - \text{Tr}\Pi^2 \right), \tag{77}
\]
\[
\bar{\mathcal{H}} = \bar{\mathcal{H}}_M - \frac{\sqrt{\gamma}}{\kappa(g)} \gamma \left( R^{(\gamma)} - 2\Lambda^{(g)} \right) - \frac{\kappa(g)}{\sqrt{\gamma}} \left( \frac{\pi^2}{2} - \text{Tr}\pi^2 \right), \tag{78}
\]
and
\[
\mathcal{H}_i = \mathcal{H}_{iM} - 2\Pi_{ij}^i, \tag{79}
\]
\[
\bar{\mathcal{H}}_i = \bar{\mathcal{H}}_{iM} - 2\pi_{ij}^i. \tag{80}
\]

\( \mathcal{H}_M, \mathcal{H}_{iM}, \bar{\mathcal{H}}_M, \bar{\mathcal{H}}_{iM} \) are the matter contributions with the minimal interaction to the corresponding metrics \( f_{\mu\nu} \) and \( g_{\mu\nu} \), \( \kappa(f), \kappa(g) \) are coupling constants of both metrics to the corresponding matter, \( R^{(\eta)}, R^{(\gamma)} \) – the scalar curvatures of the two metrics \( \eta_{ij}, \gamma_{ij} \) induced on a hypersurface, \( \eta = \det||\eta_{ij}||, \gamma = \det||\gamma_{ij}||, \Pi = \eta_{ij} \Pi^{ij}, \pi = \gamma_{ij} \pi^{ij}, \text{Tr}\Pi^2 = \Pi^{ij} \Pi_{ij}, \text{Tr}\pi^2 = \pi^{ij} \pi_{ij}. \)
Next we are to express the canonical variables of the metric formalism in the tetrad variables. The formulas for coordinates, i.e. the induced metrics, were already given in Eqs. (36), (46). The momenta are expressed as follows
\[ \Pi^{ij} = \frac{1}{4} \left( p^i \Pi^j_0 + p^j \Pi^i_0 + \mathcal{P}_{ab} (f^{ia} \Pi^j_b + f^{ja} \Pi^i_b) \right), \] (81)
\[ \pi^{ij} = \frac{1}{4} \left( e^{ia} \pi^j_a + e^{ja} \pi^i_a \right). \] (82)

Then the canonical variables will be projections of the tetrads \(E, F\) on a spacelike hypersurface \(X^\mu = e^\mu(\tau, x^i)\), i.e. \(e_{ai}, p_i, \tilde{f}_{ai}\), and their conjugate momenta \(\pi^i_a, \Pi^i_0, \Pi^i_a\). As the number of variables increases in comparison to the metric approach, there are new constraint equations that will be generators of tetrad rotations leaving metrics invariant
\[ L_{AB} = F_{iA} \Pi^i_B - F_{iB} \Pi^i_A = 0, \] (83)
\[ L_{ab} = e_{ai} \pi^j_b - e_{ib} \pi^j_a = 0. \] (84)

We may divide 6 constraints (83) in two sets:
\[ L_{ab} = \tilde{f}_{ia} \Pi^i_b - \tilde{f}_{ib} \Pi^i_a, \] (85)
\[ L_{a0} = \tilde{f}_{ia} \Pi^i_0 + \tilde{v}_i \Pi^i_a. \] (86)

The number of constraints for \(L_g\) is less than for \(L_f\), because of using suited tetrads \(E^i_0 = 0\) there. Then the Poisson brackets for momenta of the metric formalism appear nonzero outside the constraints surface
\[ \{ \Pi^{ij}(x), \Pi^{kl}(y) \} = \frac{1}{4} \left( \eta^{ij} M^{kl} + \eta^{kl} M^{ij} + \eta^{jk} M^{il} + \eta^{jl} M^{ik} \right), \] (87)
\[ \{ \pi^{ij}(x), \pi^{kl}(y) \} = \frac{1}{4} \left( \gamma^{ij} \bar{M}^{kl} + \gamma^{kl} \bar{M}^{ij} + \gamma^{jk} \bar{M}^{il} + \gamma^{jl} \bar{M}^{ik} \right). \] (88)

Here
\[ M^{ij} = \frac{1}{4} \left( p^i \Pi^j_0 - p^j \Pi^i_0 + \mathcal{P}_{ab} (f^{ia} \Pi^j_b - f^{ja} \Pi^i_b) \right), \] (89)
\[ \bar{M}^{ij} = \frac{1}{4} \left( e^{ia} \pi^j_a - e^{ja} \pi^i_a \right). \] (90)

Also the following relations are valid:
\[ M^{ij} = \frac{1}{4} L_{AB} F^{jA} F^{iB} = 0, \] (91)
\[ \bar{M}^{ij} = \frac{1}{4} L_{ab} e^{ja} e^{ib} = 0. \] (92)
Given the modification of Poisson brackets (87), (88) the constraint algebra of GR in tetrad formalism differs in a presence of algebraic constraints. Therefore it is suitable [48] to modify $\mathcal{H}_i$ and $\mathcal{H}_i$ by changing Eqs. (79), (80).

Below we take

$$
\mathcal{H}_i = \mathcal{H}_{iM} + \Pi^k_0 P_{k,i} + \Pi^k_a \tilde{f}_{ak,i} - \left( \Pi^k_0 \dot{P}_{k} \right)_k - \left( \Pi^k_a \dot{f}_{ai} \right)_k, \\
\overline{\mathcal{H}_i} = \overline{\mathcal{H}_{iM}} + \pi^k_a \bar{e}_{ak,i} - \left( \pi^k_0 \dot{e}_{i} \right)_k.
$$

Then we get

$$
\{ \mathcal{H}(x), \mathcal{H}(y) \} = \eta^{ik}(x) \mathcal{H}_k(x) \delta_k(x,y) - \eta^{ik}(y) \mathcal{H}_k(y) \delta_k(y,x), \\
\{ \mathcal{H}_k(x), \mathcal{H}(y) \} = \mathcal{H}(x) \delta_k(x,y), \\
\{ \mathcal{H}_i(x), \mathcal{H}_j(y) \} = \mathcal{H}_j(x) \delta_i(x,y) - \mathcal{H}_i(y) \delta_j(y,x), \\
\{ \mathcal{H}_k(x), L_{AB}(y) \} = L_{AB}(x) \delta_k(x,y),
$$

and also,

$$
\{ \overline{\mathcal{H}}(x), \overline{\mathcal{H}}(y) \} = \gamma^{ik}(x) \overline{\mathcal{H}}_k(x) \delta_k(x,y) - \gamma^{ik}(y) \overline{\mathcal{H}}_k(y) \delta_k(y,x), \\
\{ \overline{\mathcal{H}}_k(x), \overline{\mathcal{H}}(y) \} = \overline{\mathcal{H}}(x) \delta_k(x,y), \\
\{ \overline{\mathcal{H}}_i(x), \overline{\mathcal{H}}_j(y) \} = \overline{\mathcal{H}}_j(x) \delta_i(x,y) - \overline{\mathcal{H}}_i(y) \delta_j(y,x), \\
\{ \overline{\mathcal{H}}_k(x), \overline{L}_{ab}(y) \} = \overline{L}_{ab}(x) \delta_k(x,y),
$$

The common Hamiltonian evidently include both constraints of the metric formalism $\mathcal{H}$, $\mathcal{H}_i$, $\mathcal{H}$, $\mathcal{H}_i$ and the new ones (83), (84)

$$
H_{g+f} = H_g + H_f \\
= \int d^3x \left( \bar{N} \bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i + \bar{\lambda}^{ab} \bar{L}_{ab} \right) \\
+ \int d^3x \left( N \mathcal{H} + N^i \mathcal{H}_i + \lambda^{AB} L_{AB} \right).
$$

Without the potential all the constraints are first class and all the Lagrange multipliers are arbitrary. But given the potential the bigravity action is invariant only under diagonal rotations of the tetrads and diagonal spacetime diffeomorphisms. In particular, as we will see below, only symmetric combinations of $\bar{L}_{ab}$, $L_{ab}$

$$
L^+_{ab} = \bar{L}_{ab} + L_{ab} = 0,
$$

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stay first class, whereas the antisymmetric ones
\begin{align}
L_{ab} \equiv \bar{L}_{ab} - L_{ab} &= 0, \\
L_{a} &\equiv 0,
\end{align}
become second class. Given the primary constraints (104) – (106) independent of the potential and the form of potential given in Eqs. (23) – (26) the complete bigravity Hamiltonian is as follows
\begin{align}
H &= H_g + f + \frac{2m^2}{\kappa} \int d^3 x U = \int d^3 x \left[ N^i (\mathcal{H}_i + \bar{\mathcal{H}}_i) \\
&+ N \left( (\mathcal{H} + \frac{2m^2}{\kappa} W) + u(\bar{\mathcal{H}} + \frac{2m^2}{\kappa} V) + u^i (\bar{\mathcal{H}}_i + \frac{2m^2}{\kappa} V_i) \right) \\
&+ \lambda^+_{ab} L^+_{ab} + \lambda^-_{ab} L^-_{ab} + \lambda^a L_a \right],
\end{align}
This Hamiltonian depends first on 9 Lagrange multipliers \( \lambda^+_ {ab}, \lambda^-_{ab}, \lambda^a \), next on the canonical variables, both on 42 gravitational ones, \( f_{ai}, \bar{v}_i \equiv p_i, e_{ai}, \Pi^a_i, \Pi^0_i, \pi^i_a \), and on the matter coordinates and momenta, at last it is linearly dependent on 8 Lagrange multipliers \( u, u^i, N, N^i \). The variation over \( u, u^i, N, N^i \) gives the following equations
\begin{align}
S' &\equiv \bar{\mathcal{H}} + \frac{2m^2}{\kappa} V = 0, \\
S_i &\equiv \mathcal{H}_i + \frac{2m^2}{\kappa} V_i = 0, \\
\mathcal{R} &\equiv \mathcal{R}'' + u S' + u^i S_i = 0, \\
\mathcal{R}_i &\equiv \mathcal{H}_i + \bar{\mathcal{H}}_i = 0,
\end{align}
where
\begin{align}
\mathcal{R}'' &= \mathcal{H} + \frac{2m^2}{\kappa} W.
\end{align}
It follows from Eqs. (108), (109), (110) that
\begin{align}
\mathcal{R}'' &= 0.
\end{align}
Eqs. (108), (109), (111), (113) are constraints, and they supplements Eqs. (104)–(106) to form the full set of 17 primary constraints. The Hamiltonian is zero on the surface of these constraints, this is a necessary condition for invariance of a theory under spacetime diffeomorphisms.
7 The algebra of constraints

According to the standard Dirac procedure in order to obtain a full set of constraints and to determine the Lagrange multipliers standing at the second class constraints it is necessary to calculate the Poisson brackets of primary constraints with the Hamiltonian. To determine where a constraint is of the first or second class one should estimate Poisson brackets between the constraints.

Thus to satisfy equations of primary constraints (105), (106) in every moment of evolution it is necessary to fulfil some equations, and some of these equations fortunately occur equivalent to the conditions of symmetry (64), (65). It is enough to demonstrate this for the case of minimal potential \( \beta_1 \neq 0, \beta_2 = \beta_3 = 0 \):

\[
\dot{L}_{ab} = \{L_{ab}, H\} \approx \{L_{ab}, \frac{2m^2}{k} \int d^3x \hat{N} \hat{U}\} = \frac{4m^2}{k} \beta_1 N e_u (z_{ba} - z_{ab}), \quad (114)
\]

\[
\dot{L}_{a0} = \{L_{a0}, H\} \approx \frac{2m^2}{k} \beta_1 N e \left( u^{\hat{f}_{ia}} - \varepsilon (v_a + uv_{j} e^{ja}) \right). \quad (115)
\]

Eqs. (114) are equivalent to the first group of the symmetry conditions (64), and therefore Eqs. (64) in fact are secondary constraints of the Hamiltonian formalism. Eqs. (115) which are equivalent to Eqs. (65) determine Lagrange multipliers \( u^i \) as functions of canonical variables, and look like the Hassan-Rosen transform

\[
u^i = v^i + u \hat{v}^i. \quad (116)
\]

In the works [4, 19] the analog of \( v^i \) has been denoted as \( n^i \) and treated as a new variable replacing \( (\hat{N}^i - N^i)/N \equiv u^i \), whereas \( \hat{v}^i \) corresponds to \( D^i, n^j \).

Two sets of constraints have nonzero Poisson brackets on the constraint surface, and therefore they are second class

\[
\{L_{ab}(x), G_{cd}(y)\} = \left[ \delta_{ac} z_{(bd)} - \delta_{ad} z_{(cb)} - \delta_{bc} z_{(ad)} + \delta_{bd} z_{(ca)} \right] \delta(x, y) \neq 0, \quad (117)
\]

\[
\{L_{a0}(x), S_i(y)\} = e^{\hat{f}_{ia}} \left[ \beta_1 \delta_{ba} e_0(z) + \beta_2 (e_1(z) - z_{ba}) + \beta_3 (e_2(z) + z_{bc} z_{ca} - z_{ba}) \right] \delta(x, y) \neq 0. \quad (118)
\]

All these constraints, besides \( S_i \), have no analogs in the metric approach. They serve to manage the variables those are necessary for the tetrad approach but absent in the metric one. The special role of constraints \( S_i \) in the metric approach is that the Lagrange multipliers \( u_i \) are determined from
them (or with the help of them the Dirac brackets are defined), and these $u_i$ are supposed to be in one-to-one correspondence with Hassan-Rosen new variables $n^i$ (i.e. $v^i$ in this work).

Next we exclude $u^i$ from the Hamiltonian with the help of Eq. (116). Also we omit second class constraints specific to tetrads (105), (106) from the Hamiltonian, then the Hamiltonian may be represented in a symmetric (with respect to both metrics) form as follows

$$H = \int d^3x \left( N \mathcal{R}' + N \mathcal{S} + N^i \mathcal{R}_i + \lambda^+_{ab} L^+_{ab} \right).$$  

(119)

Here the constraints standing at the lapse functions are the following

$$\mathcal{R}' = \mathcal{R}'' + v^i \mathcal{S}_i, \quad \mathcal{S} = \mathcal{S}' + \bar{v}^i \mathcal{S}_i.$$  

(120)

In the form preferring metric $f_{\mu\nu}$ used in our spacetime basis $(n^\alpha, e^\alpha_i)$ the Hamiltonian is as follows

$$H = \int d^3x \left( N \mathcal{R} + N^i \mathcal{R}_i + \lambda^+_{ab} L^+_{ab} \right),$$  

(121)

One may represent constraints (120) in other form$^6$

$$\begin{align*}
\mathcal{R}' &= \mathcal{H} + v^i \mathcal{H}_i + \frac{2m^2}{\kappa} W', \quad W' = W + v^i V_i, \\
\mathcal{S} &= \mathcal{H} + \bar{v}^i \mathcal{H}_i + \frac{2m^2}{\kappa} V', \quad V' = V + \bar{v}^i V_i.
\end{align*}$$  

(122, 123)

By straightforward calculation of Poisson brackets one can check that algebra (124) – (134) written below is valid for the constraints that compose Hamiltonian (121). In particular, functions $\mathcal{H}_k$ generate spatial coordinate transformations for the expressions constructed of metric $f_{\mu\nu}$,

$$\begin{align*}
\{ \mathcal{R}_k(x), L_{a0}(y) \} &\equiv \{ \mathcal{H}_k(x), L_{a0}(y) \} = L_{a0}(x) \delta_{,k}(x,y) \approx 0, \\
\{ \mathcal{R}_k(x), L_{ab}(y) \} &\equiv \{ \mathcal{H}_k(x), L_{ab}(y) \} = L_{ab}(x) \delta_{,k}(x,y) \approx 0, \\
\{ \mathcal{R}_k(x), \mathcal{H}(y) \} &\equiv \{ \mathcal{H}_k(x), \mathcal{H}(y) \} = \mathcal{H}(x) \delta_{,k}(x,y).
\end{align*}$$  

(124, 125, 126)

Functions $\bar{\mathcal{H}}_k$ do the same for the corresponding expressions constructed of $g_{\mu\nu}$,

$$\begin{align*}
\{ \mathcal{R}_k(x), L_{ab}(y) \} &\equiv \{ \bar{\mathcal{H}}_k(x), L_{ab}(y) \} = L_{ab}(x) \delta_{,k}(x,y), \\
\{ \mathcal{R}_k(x), \mathcal{H}(y) \} &\equiv \{ \bar{\mathcal{H}}_k(x), \mathcal{H}(y) \} = \bar{\mathcal{H}}(x) \delta_{,k}(x,y).
\end{align*}$$  

(127, 128)

$^6$By putting second class constraints equal to zero we get $\mathcal{R} = \mathcal{R}' = \mathcal{R}''$. 

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The Poisson brackets between $\mathcal{R}$ and $\mathcal{R}_i$ give the standard algebra of hypersurface deformations when the second class constraint $\mathcal{S}$ is taken into account

$$\{\mathcal{R}(x), \mathcal{R}(y)\} = (\eta^{ik}\mathcal{R}_k + uu^i\mathcal{S})(x)\delta_i(x, y) - (\eta^{ik}\mathcal{R}_k + uu^i\mathcal{S})(y)\delta_i(y, x),$$

(129)

$$\{\mathcal{R}_i(x), \mathcal{R}(y)\} = \mathcal{R}(x)\delta_i(x, y) + u_i\mathcal{S}\delta(x, y),$$

(130)

$$\{\mathcal{R}_i(x), \mathcal{R}_j(y)\} = \mathcal{R}_j(x)\delta_i(x, y) - \mathcal{R}_i(y)\delta_j(y, x),$$

(131)

Poisson brackets of the second class constraints $\mathcal{S}, \Omega$ provide the conditions for cancellation of the ghost degree of freedom

$$\{\mathcal{S}(x), \mathcal{S}(y)\} = \bar{v}^i\mathcal{S}(x)\delta_i(x, y) - \bar{v}^i\mathcal{S}(y)\delta_i(y, x),$$

(132)

$$\{\mathcal{R}(x), \mathcal{S}(y)\} = (u^i + u\bar{v}^i)\mathcal{S}(x)\delta_i(x, y) + (u(\bar{v}^i\mathcal{S})_i - \Omega)\delta(x, y),$$

(133)

$$\{\mathcal{S}(x), \Omega(y)\} \neq 0.$$  

(134)

Eqs. (132) – (134) give the most important set of the second class constraint algebra. The primary constraint $\mathcal{S}$ should commute with itself for appearance of the secondary constraint $\Omega$ from the compatibility condition $\dot{\mathcal{S}} = \{\mathcal{S}(x), H\} \approx 0$. The only one of the Poisson brackets (132) – (134) is nonzero on the constraints surface. It is necessary for the constraints $\mathcal{S}$ and $\Omega$ to form a pair of second class constraints. As $\Omega$ appears in Eq. (133) beeing multiplied on the $\delta$-function the lapse function $N$ appears in the evolutionary equation that is necessary to preserve the constraint $\mathcal{S}$ as a multiplier of constraints

$$\dot{\mathcal{S}} = \{\mathcal{S}(x), H\} = N(x)\Omega(x) + \left( (N^i + N\bar{v}^i)(x)\mathcal{S}(x) \right)_i \approx 0,$$

(135)

and therefore stays arbitrary.

It is interesting to compare Eqs. (129) – (133) with the similar relations from the work [15], derived without using any explicite expression of the dRGT potential, and based on the Dirac brackets: see Eqs. (48) – (51) of that work and also the nonnumbered equation preceeding to Eq. (55). The results are in full agreement.

The Lagrange multipliers standing at the first class constraints, $N, N^i, \lambda_{ab}^+$, are arbitrary functions of time $\tau$ and of spatial coordinates $x^i$. The variable $u$ can be determined from the following equation

$$\dot{\Omega} = 0 \approx \int d^3xN (\{\Omega, \mathcal{R}'\} + u\{\Omega, \mathcal{S}\}),$$

(136)
which is linear in $u$.

The Hassan-Rosen transform takes in the tetrad formalism the following form

$$u^i = v^i + u \bar{v}^i,$$

and arises simply as a solution for a Lagrange multiplier, this helps to avoid any need for the Dirac brackets.

## 8 Conclusion

In the tetrad Hamiltonian formalism of bigravity we have $n = 21 \times 2$ canonical variables, $(e_{ai}, \pi_a^i)$, $(p_i, \Pi_0^i)$, $(\bar{f}_{ai}, \Pi_a^i)$, $n_{fc} = 7$ first class constraints, $\mathcal{R}$, $\mathcal{R}_i$, $L_{ab}^+$, and $n_{sc} = 14$ second class constraints, $\mathcal{S}$, $\Omega$, $L_{ab}$, $G_{ab}$, $L_{ab}$, $S_i$. The calculation of the gravitational degrees of freedom gives us the well-known number

$$n_{DOF} = \frac{1}{2}(n - 2n_{fc} - n_{sc}) = 7.$$  

(138)

Dynamics of bigravity is expressed through the algebra of its constraints. Comparing our results with the other calculations we see that this algebra does not depend on the approach and on the choice of variables, in particular this is true for the key pair of second class constraints $\mathcal{S}$, $\Omega$, modulo other second class constraints. These derivations do not require using Dirac brackets, they are based on standard Poisson brackets, after their calculation the second class constraints are taken into account, especially $z_{ab} = z_{ba}$, or equivalently, $z_{ij} = z_{ji}$, and

$$\frac{\delta H}{\delta u^i} = \mathcal{S}_i \equiv \bar{\mathcal{H}}_i + \frac{2m^2}{\kappa} V_i = 0.$$  

(139)

Here we emphasize the difference from the approach of works [4, 19], as there the change of variable $u^i = n^i + uD_{ji}n^j$ is done before, and so instead, of equating $\frac{\delta H}{\delta u}$ to zero, whereas Eq. (139) is argued there to be a consequence of the variation in a new variable $\frac{\delta H}{\delta u} = 0$.

As it was mentioned long ago in studying the Hamiltonian structure of the GR [49, 50] the constraint algebra gives an important information on the theory. Any reasonable modification of the GR preserving the general covariance should give the first class constraints satisfying the same algebra. The appearance of canonical variables in the coefficients of the algebra allows to get information on the Hamiltonian.
With the second class constraints coming into play the algebra becomes more involved, but these constraints may be put to zero after the calculation of Poisson brackets. For the first class quantities, here all of them are first class constraints, one may use the Poisson brackets and take into account the second class constraints after this calculation. Also it is possible to make calculations step by step when first one defines intermediary Dirac brackets made of the subset of the second class constraints. In a sense, a similar procedure is used here, as we preserve the couple of second class constraints $S, \Omega$ in Eqs. (129) – (133) and at the same time treat other second class constraints $L_{ab}, G_{ab}, L_{a0}, S_i$ as equal to zero. This method for derivation the algebra of the most essential for the bigravity constraints seems the most natural and simple among the proposed before. As it was shown for the first time in the work [8], the tetrad variables allow to express the dRGT potential as a linear combination of all nonzero external products of the tetrad 1-forms. It was also proved there that one needs the symmetry conditions for the equivalence between the metric and the tetrad approaches. Here the parameter $p_a = \tilde{v}_a$ introduced for the nonsuited tetrad is treated as a function of canonical variables and has nonzero Poisson brackets. This considerably simplifies the calculations.

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Appendix A

The same problems were already considered in our previous work [16], but published [16, 17] results were limited to the case of minimal potential ($\beta_1 \neq 0$, $\beta_2 = \beta_3 = 0$) as that method of Poisson brackets calculations was more involved. The notations and canonical variables applied here simplify the work. The Appendix B is added in order to compare our notations with the ones used by other authors.

The potential that couples two metrics in the tetrad formalism given the symmetry of tetrads conditions $z_{ab} = z_{ba}$ following from Eq. (70) is as follows:

$$\tilde{U} = uV + u'V_i + W,$$

$$V = e(\beta_0e_0(z) + \beta_1e_1(z) + \beta_2e_2(z) + \beta_3e_3(z)),$$

$$V_i = -f_{ia}C_{ab}p_b,$$

$$W = e(\beta_0e_0(w) + \beta_1e_1(w) + \beta_2e_2(w) + \beta_3e_3(w)),$$

where $e = \det(e_{ai})$, $f = \det(f_{ai})$, $u = \frac{\bar{N}}{N}$, $u^i = \frac{\bar{N} - N^i}{N}$, $e_i$ are symmetric polynomials of $(3 \times 3)$-matrices $z_{ab}$, $w_{ab}$, $x_{ab}$ given below:

$$z_{ab} = \mathcal{P}_{ac}x_{cb} \equiv f_{ia}e^{ib}, \quad \mathcal{P}_{ac} = \delta_{ac} + \frac{p_ap_c}{\varepsilon + 1},$$

$$w_{ab} = \mathcal{P}^{-1}_{ac}x_{cb} \equiv \tilde{f}_{ia}\eta_{ij}e^{jb}, \quad \mathcal{P}^{-1}_{ac} = \delta_{ac} - \frac{p_ap_c}{\varepsilon(\varepsilon + 1)},$$

$$x_{cb} = f_{ic}e^{ib}, \quad \tilde{f}_{ia} = \mathcal{P}_{ac}f_{ic}, \quad \tilde{f}^{ia} = \mathcal{P}^{-1}_{ac}f^{ic},$$

$$C_{ab} = e[\beta_1\delta_{ba}e_0(x) + \beta_2(\delta_{ba}e_1(x) - x_{ba}) + \beta_3(\delta_{ba}e_2(x) + x_{bc}x_{ca} - xx_{ba})].$$

After substitution of the expression found for the Lagrange multiplier $u^i = v^i + u\tilde{v}^i$ which also follows from Eq. (69) the potential simplifies and becomes the following:

$$\tilde{U} = uV' + W',$$

where

$$V' = e\left(\beta_1e_1(w) + \beta_2e_2(w) + \beta_3e_3(w)\right) + \beta_0e,$$

$$W' = e\left(\beta_1e_0(z) + \beta_2e_1(z) + \beta_3e_2(z)\right) + \beta_4f.$$
By estimating Poisson brackets in Eq. (133) one obtains the secondary constraint

$$\Omega = \frac{2m^2}{\kappa} \left[ \frac{\partial H}{\partial \Pi_a} \frac{\partial V'}{\partial f_{ai}} - \frac{\partial \bar{H}}{\partial \bar{\pi}_a} \frac{\partial W'}{\partial e_{ai}} - v^i \bar{f}_{ak,i} \frac{\partial V'}{\partial f_{ak}} - \bar{v}^i e_{ak,i} \frac{\partial W'}{\partial e_{ak}} + v^i_k \left( e_{ai} \frac{\partial}{\partial e_{ak}} - \delta^k_i \right) V' + \bar{v}^i \left( e_{ai} \frac{\partial W''}{\partial e_{ak}} \right)_k \right]. \quad (152)$$

which forms a pair of second class constraints together with $S$. This pair just excludes the ghost degree of freedom.

**Appendix B**

As each research group exploits a lot of special notations we hope it would be useful to add dictionaries for translations between them. We include this Appendix to make reading of this paper easier for someone who is familiar with the others.
| variables                        | this work                  | Alexandrov |
|---------------------------------|----------------------------|------------|
| space-time coordinate 4-indices | $\alpha, \beta, \ldots, \mu, \nu$ | $\mu, \nu$ |
| spatial coordinate 3-indices    | $i, j, k \ldots$           | $a, b, c$  |
| internal 4-indices             | $A, B, C \ldots$           | $I, J, K \ldots$ |
| internal 3-indices             | $a, b, c \ldots$           | $i, j, k \ldots$ |
| 1st space-time tetrad          | $F^A_\mu$                  | $e^K_+\,$   |
| 2nd space-time tetrad          | $E^A_\mu$                  | $e^K_-\,$   |
| internal 4-metric              | $\eta_{AB}$                | $\eta_{IJ}$ |
| 1st tetrad spatial components  | $F^A_i = (p_i, \tilde{f}_{ai})$ | $e^I_+ = (E^j_+a\chi_j, E^i_+a)$ |
| 2nd tetrad spatial components  | $E^A_i = (0, e_{ai})$       | $e^I_- = (E^j_-a\chi_j, E^i_-a)$ |
| tetrad normal components       | $(F^A_+, E^A_+)$            | $(X^I_+, X^I_-)$ |
| Lorentz boost parameters       | $(p_a, 0)$                  | $(\chi_+, \chi_-)$ |
| 1st induced spatial metric     | $\eta_{ij} = f_{ia}f_{aj}$  | not defined |
| 2nd induced spatial metric     | $\gamma_{ij} = e_{ia}e_{aj}$| not defined |
| hybrid spatial metric          | $z_{ij} = \tilde{f}_{ia}e_{aj}$ | $g_{ab} = \eta_{IJ}e^I_+,e^I_-b$ |
| and its determinant            | $\det |z| = \varepsilon \det |f| \det |e|$ | $\det |g| = \eta_{IJ}X^I_+X^I_-$ |
| hybrid inverse spatial metric   | $z^{ij} = e^{ia}\tilde{f}^{aj}$ | $g^{ab} = E^{a}_{-i}E^{b}_{+j}\left(\delta^{ij} + \frac{x^{k}_+x^{k}_-}{1-x^{k}_+x^{k}_-}\right)$ |

Table 1: Dictionary to translate variables between this work notations and notations of Ref. [12]
| variables | this work | Kocic |
|-----------|-----------|-------|
| 1st spatial triad | $f_{ai}$ | $m = \|m_{ai}\|$ |
| 2nd spatial triad | $e_{ai}$ | $e = \|e_{ai}\|$ |
| 2nd induced spatial metric | $\gamma_{ij} = e_{ia}e_{aj}$ | $\gamma = e^T\delta e$ |
| internal 4-metric | $\eta_{AB}$ | |
| Lorentz boost parameters | $p_a$ | $p$ |
| Lorentz boost parameters | $v_a = \frac{p_a}{\lambda}$ | $v = \frac{p}{\lambda}$ |
| Hassan-Rosen variable $n^i$ | $v^i = f^{ia}v_a \equiv \tilde{f}^{ia}p_a$ | $\tilde{n} = m^{-1}v$ |
| Hassan-Rosen $D^i\;j^j$ | $\tilde{v}^i = e^{ia}v_a$ | $n$ |
| nonsymmetric internal hybrid | $x_{ab} = f_{ai}e^{ib}$ | $me^{-1}$ |
| symmetric internal hybrid | $z_{ab} = \tilde{f}^{ia}e^{ib}$ | $\hat{\Lambda}me^{-1}$ |
| one more internal hybrid | $w_{ab} = \tilde{f}^{ia}\eta_{ij}e^{ib}$ | $\hat{\Lambda}^{-1}me^{-1}$ |
| hybrid spatial metric | $z_{ij} = \tilde{f}^{ia}e_{aj}$ | $m^T\hat{\Lambda}e$ |
| its determinant | $\det |z| = \varepsilon \det |f| \det |e|$ | |
| hybrid inverse spatial metric | $z^{ij} = e^{ia}\tilde{f}^{aj}$ | $e^{-1}\hat{\Lambda}^{-1}(m^{-1})^T$ |
| Hassan-Rosen matrix $D^i_\;j$ | $z^{ik}\eta_{kj}$ | $e^{-1}\hat{\Lambda}^{-1}m$ |
| its symmetric polynomials | $\varepsilon^k e_k(w)$ | $e_k(D)$ |
| 1st space-time tetrad | $F^A_{\;\mu}$ | |
| 2nd space-time tetrad | $E^A_{\;\mu}$ | |
| 1st induced spatial metric | $\eta_{ij} = f_{ia}f_{aj}$ | $\phi = m^T\delta m$ |
| 1st tetrad spatial components | $F^A_{\;i} = (p_i, \tilde{f}_{ai})$ | |
| 2nd tetrad spatial components | $E^A_{\;i} = (0, e_{ai})$ | |
| tetrad normal components | $(F^A_{\perp i}, E^A_{\perp})$ | |

Table 2: Dictionary to translate variables between this work notations and notations of Ref. [20]
| variables                                      | this work     | Hassan – Lundkvist |
|-----------------------------------------------|---------------|--------------------|
| 1st induced spatial metric                    | $\eta_{ij} = f_{ia} f_{aj}$ | $\phi_{ij}$        |
| 2nd induced spatial metric                    | $\gamma_{ij} = e_{ia} e_{aj}$ | $\gamma_{ij}$      |
| 1st metric lapse and shift                    | $(N, N^i)$    | $(L, L^i)$         |
| 2nd metric lapse and shift                    | $(\bar{N}, \bar{N}^i)$ | $(N, N^i)$         |
| difference of shifts                          | $\bar{N}^i - N^i = N v^i + \bar{N} \bar{v}^i$ | $N^i - L^i = L n^i + N D^i_j n^j$ |
| Hassan-Rosen variable                         | $v^i = \bar{f}^{ia} p_a \equiv \bar{f}^{ia} v_a$ | $n^i$               |
| Hassan-Rosen matrix                           | $\varepsilon \bar{f}^{ia} \eta_{jk} e^{k a}$ | $D^i_j$             |
| symmetric polynomials                         | $\varepsilon^k e_k(w)$ | $e_k(D)$           |
| Lorentz factor                                | $\varepsilon = \frac{1}{\sqrt{1-v^i \eta_{ij} v^j}}$ | $x = 1 - n^i \phi_{ij} n^j$ |
| $(v \cdot x v) = v_a x_{ab} v_b$              | $n^i \phi_{ij} D^j_k n^k$ |                  |
| $(v^T \cdot x v) = v_a x_{ba} x_{bc} v_c$     | $D^i_k n^k \phi_{ij} D^j_i n^\ell$ |                  |

Table 3: Dictionary for translation of notations of this work and Ref. [19]
| constraints & multipliers | this work | Alexandrov | Hassan-Lundkvist |
|--------------------------|-----------|------------|-----------------|
| Hamiltonian constraint (1st class) | $N^i R'_i$ | $N^a H$ | $L_i \tilde{R}^0$ |
| 3-diff generator (1st class) | $N^a D_a$ | $L_i \tilde{R}_i$ | |
| diagonal Gauss (1st class) | $\lambda^+ L_{ab}^+$ | $n^{IJ} \tilde{G}_{IJ}$ | not defined |
| off-diagonal Gauss (2nd class) | $\lambda^a L_{ab}^-$ | $\hat{n}^{IJ} \hat{G}_{IJ}$ | not defined |
| off-diagonal Gauss (2nd class) | $\lambda^a L_{a0}$ | $\hat{n}^{IJ} \hat{G}_{IJ}$ | not defined |
| 2nd class | $u^i S_i$ | $\hat{N}^a \hat{D}_a$ | $C_k$ |
| 2nd class | $u S$ | $\hat{N} \hat{H}$ | $NC$ |
| 2nd class (secondary) | $G_{ab}$ or $z_{ij} = z_{ji}$ | $S^a$ | not defined |
| 2nd class (secondary) | $\Omega$ | $\Psi$ | $C_2$ |

Table 4: Dictionary to translate the 1st and 2nd class constraints between this work notations and notations of Refs. [12, 19]