INVARIANT THEORY OF RELATIVELY FREE
RIGHT-SYMMETRIC AND NOVIKOV ALGEBRAS

VESSELIN DRENSKY

Dedicated to Askar Dzhumadil’daev on the occasion of his 60th birthday

Abstract. Algebras with the polynomial identity 
\[(x_1, x_2, x_3) = (x_1, x_3, x_2),\]
where \((x_1, x_2, x_3) = x_1(2x_3) - (x_1x_2)x_3\) is the associator, are called right-symmetric. Novikov algebras are right-symmetric algebras satisfying additionally the polynomial identity 
\[x_1(x_2x_3) = x_3(x_1x_2).\]
We consider the free right-symmetric algebra \(F_d(\mathcal{R})\) and the free Novikov algebra \(F_d(\mathcal{N})\) freely generated by \(X_d = \{x_1, \ldots, x_d\}\) over a field \(K\) of characteristic 0. The general linear group \(GL_d(K)\) with its canonical action on the \(d\)-dimensional vector space \(KX_d\) acts on \(F_d(\mathcal{R})\) and \(F_d(\mathcal{N})\) as a group of linear automorphisms. For a subgroup \(G\) of \(GL_d(K)\) we study the algebras of \(G\)-invariants \(F_d(\mathcal{R})^G\) and \(F_d(\mathcal{N})^G\). For a large class of groups \(G\) we show that the algebras \(F_d(\mathcal{R})^G\) and \(F_d(\mathcal{N})^G\) are never finitely generated. The same result holds for any subvariety of the variety \(\mathcal{N}\) of right-symmetric algebras which contains the subvariety \(\Delta\) of left-nilpotent of class 3 algebras in \(\mathcal{N}\).

Introduction

In this paper we fix a field \(K\) of characteristic 0 and consider nonassociative \(K\)-algebras. An algebra \(A\) is called right-symmetric if it satisfies the polynomial identity 
\[(x_1, x_2, x_3) = (x_1, x_3, x_2),\]
where \((x_1, x_2, x_3) = x_1(x_2x_3) - (x_1x_2)x_3\) is the associator, i.e., 
\[(a_1, a_2, a_3) = (a_1, a_3, a_2) \text{ for all } a_1, a_2, a_3 \in A.\]
A right-symmetric algebra is Novikov if it satisfies additionally the polynomial identity of left-commutativity 
\[x_1(x_2x_3) = x_2(x_1x_3).\]
We denote by \(\mathfrak{R}\) and \(\mathfrak{N}\) the varieties of all right-symmetric algebras and all Novikov algebras, respectively. For details on the history of right-symmetric and Novikov algebras we refer to the introductions of the paper by Dzhumadil’daev and Löffwall [20] and the recent preprint by Bokut, Chen, and Zhang [4]. The origins of the right-symmetric algebras can be traced back till the paper by Cayley [6] in 1857. Translated in modern language, Cayley mentioned an identity which implies the

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right-symmetric identity for the associators and holds for the right-symmetric Witt algebra in \(d\) variables

\[
W_d^{rsym} = \left\{ \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} \mid f_i \in K[x_d] \right\}
\]
equipped with the multiplication

\[
\left( f_i \frac{\partial}{\partial x_i} \right) \ast \left( f_j \frac{\partial}{\partial x_j} \right) = \left( f_i \frac{\partial f_j}{\partial x_j} \right) \frac{\partial}{\partial x_i}.
\]

Cayley also considered the realization of \(W_d^{rsym}\) in terms of rooted trees. Later right-symmetric algebras were studied under different names: Vinberg, Koszul, Gerstenhaber, and pre-Lie algebras, see the references in [20]. The opposite algebras (satisfying the left-symmetric identity for the associators and right-commutativity) appeared in the paper by Gel’fand and Dorfman [22]. There the authors gave an algebraic approach to the notion of Hamiltonian operator in finite-dimensional mechanics and the formal calculus of variations. Independently, later Novikov algebras were rediscovered by Balinskii and Novikov in the study of equations of hydrodynamics [2], see also the survey article by Novikov [31]. (Due later Novikov algebras were rediscovered by Balinskii and Novikov in the study of equations of hydrodynamics [21], see also the survey article by Novikov [31].) An example of a Novikov algebra is the right-symmetric Witt algebra in \(d\) variables

\[
W_d^{rsym} \subseteq \{ \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} \mid f_i \in K[x_d] \}
\]
equipped with the multiplication

\[
\left( f_i \frac{\partial}{\partial x_i} \right) \ast \left( f_j \frac{\partial}{\partial x_j} \right) = \left( f_i \frac{\partial f_j}{\partial x_j} \right) \frac{\partial}{\partial x_i}.
\]

In commutative invariant theory one usually considers the general linear group \(GL_d(K)\) with its canonical action on the \(d\)-dimensional vector space \(V_d\) with basis \(\{e_1, \ldots, e_d\}\). This induces an action on the polynomial algebra \(K[X_d] = K[x_1, \ldots, x_d]\) in \(d\) variables

\[
g(f(v)) = f(g^{-1}(v)), \quad g \in GL_d(K), v \in V_d,
\]
where the linear functions \(x_i : V_d \to K\) are defined by

\[
x_i(e_j) = \delta_{ij}, \quad i, j = 1, \ldots, d,
\]
and \(\delta_{ij}\) is the Kronecker symbol. For our noncommutative considerations it is more convenient to suppress one step and, replacing \(V\) with its dual space \(V^*\), to assume that \(GL_d(K)\) acts canonically on the vector space \(KX_d\) with basis \(X_d = \{x_1, \ldots, x_d\}\). Then, identifying the polynomial algebra \(K[X_d]\) with the symmetric algebra of \(KX_d\), we extend diagonally this action of \(GL_d(K)\) on \(K[X_d]\):

\[
g(f(X_d)) = g(f(x_1, \ldots, x_d)) = f(g(x_1), \ldots, g(x_d)), \quad g \in GL_d(K), f(X_d) \in K[X_d].
\]

In this way \(GL_d(K)\) acts as the group of linear automorphisms of \(K[X_d]\). For a subgroup \(G\) of \(GL_d(K)\) the algebra of \(G\)-invariants is

\[
K[X_d]^G = \{ f \in K[X_d] \mid g(f) = f \text{ for all } g \in G \}.
\]

This is a \(\mathbb{Z}\)-graded vector space and its Hilbert (or Poincaré) series is the formal power series

\[
H(K[X_d]^G, z) = \sum_{n \geq 0} \dim(K[X_d]^G)_n z^n,
\]
where \((K[X_d]^G)_n\) is the homogeneous component of degree \(n\) in \(K[X_d]^G\). The following are among the main problems related with the description of the algebra \(K[X_d]^G\) for different groups or classes of groups \(G\). For details concerning also computational and algorithmic problems see [9] or [33].

- **Is the algebra \(K[X_d]^G\) finitely generated?** This problem was the main motivation for the Hilbert 14th problem in his famous lecture “Mathematische Probleme” given at the International Congress of Mathematicians held in 1900 in Paris [25]. It is known that \(K[X_d]^G\) is finitely generated for finite groups (the theorem of Emmy Noether [30]), for reductive groups (the Hilbert-Nagata theorem, see e.g., [11]), and for groups close to reductive (see e.g., Grosshans [23] and Hadžiev [24]). The first example of an algebra of invariants \(K[X_d]^G\) which is not finitely generated is due to Nagata [29].

- **If \(K[X_d]^G\) is finitely generated, describe it in terms of generators and defining relations.** In different degree of generality this problem is solved for classes of groups. For example, the theorem of Emmy Noether [30] gives that for finite groups the algebra \(K[X_d]^G\) is generated by invariants of degree \(\leq |G|\). Also for finite groups, the Chevalley-Shephard-Todd theorem [7, 33] states that the algebra \(K[X_d]^G\) is isomorphic to the polynomial algebra in \(d\) variables (i.e., it is generated by a set of \(d\) algebraically independent invariants) if and only if \(G\) is generated by pseudo-reflections.

- **Calculate the Hilbert series \(H(K[X_d]^G, z)\).** For finite groups the answer is given by the Molien formula [28]

\[
H(K[X_d]^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gz)}.
\]

The analogue for reductive and close to them groups is the Molien-Weyl integral formula [37], see also [38].

In noncommutative invariant theory one replaces the polynomial algebras \(K[X_d]\) with other noncommutative or nonassociative algebras still keeping some of the typical features of polynomial algebras. One such feature is the universal property that for an arbitrary commutative algebra \(A\) every mapping \(X_d \to A\) is extended to a homomorphism \(K[X_d] \to A\). In the noncommutative set-up the class of commutative algebras is replaced by an arbitrary variety of algebras \(\mathfrak{V}\) and instead on \(K[X_d]^G\) one studies the algebra of \(G\)-invariants \(F_d(\mathfrak{V})^G\) of the \(d\)-generated relatively free algebra \(F_d(\mathfrak{V})\) in \(\mathfrak{V}\), \(d \geq 2\). For a background see the surveys [21, 13]. Comparing with commutative invariant theory, when \(K[X_d]^G\) is finitely generated for all “nice” groups, the main difference in the noncommutative case is that \(F_d(\mathfrak{V})^G\) is finitely generated quite rarely. For a survey on invariants of finite groups \(G\) acting on relatively free associative algebras see [21, 13] and [26]. For finite groups \(G \neq (1)\) and varieties of Lie algebras \(F_d(\mathfrak{V})^G\) is finitely generated if and only if \(\mathfrak{V}\) is nilpotent, see [3, 12].

Concerning the Hilbert series of \(F_d(\mathfrak{V})^G\), for \(G\) finite there is an analogue of the Molien formula, see Formanek [21]. Let

\[
H(F_d(\mathfrak{V}), z_1, \ldots, z_d) = \sum_{n_i \geq 0} \dim F_d(\mathfrak{V})_{(n_1, \ldots, n_d)} z_1^{n_1} \cdots z_d^{n_d}
\]

be the Hilbert series of \(F_d(\mathfrak{V})\) as a multigraded vector space. It is equal to the generating function of the dimensions of the vector spaces \(F_d(\mathfrak{V})_{(n_1, \ldots, n_d)}\) of the
elements in $F_d(\mathfrak{W})$ which are homogeneous of degree $n_i$ in $x_i$. If $\xi_1(g), \ldots, \xi_d(g)$ are the eigenvalues of $g \in G$, then the Hilbert series of the algebra of invariants $F_d(\mathfrak{W})^G$ is
\[ H(F_d(\mathfrak{W})^G; z) = \frac{1}{|G|} \sum_{g \in G} H(F_d(\mathfrak{W}); \xi_1(g)z, \ldots, \xi_d(g)z). \]

There is also an analogue of the Molien-Weyl formula for the Hilbert series of $F_d(\mathfrak{W})^G$ which combines ideas of De Concini, Eisenbud, and Procesi [3] and Almkvist, Dicks, and Formanek [1]. Evaluating the corresponding multiple integral one uses the Hilbert series of $F_d(\mathfrak{W})$ instead of the Hilbert series of $K[X_d]$
\[ H(K[X_d], z_1, \ldots, z_d) = \prod_{i=1}^{d} \frac{1}{1 - z_i}. \]

We refer to [3] for other methods for computing the Hilbert series of $F_d(\mathfrak{W})^G$ when $G$ is isomorphic to the special linear group $SL_m(K)$ or to the group $UT_m(K)$ of the $m \times m$ unitriangular matrices.

In this paper we study invariant theory of relatively free right-symmetric and Novikov algebras. Let $\mathfrak{E}$ be the variety of right-symmetric algebras which are left-nilpotent of class 3, i.e., $\mathfrak{E}$ is the subvariety of $\mathfrak{N}$ satisfying the polynomial identity
\[ x_1(x_2x_3) = 0. \] (4)

For a large class of subgroups $G$ of $GL_d(K)$, $G \neq \{1\}$, $d > 1$, we show that $F_d(\mathfrak{W})^G$ is not finitely generated for any variety $\mathfrak{W}$ containing $\mathfrak{E}$. More precisely, let $A_d = K[X_d]_+$ be the algebra of polynomials without constant term and let $(A_d)^G = (KX_d)^G$ be the vector space of linear polynomials fixed by $G$. Clearly, $(A_d^2)^G$ is a $K[(A_d)^G]$-module. If $(A_d^2)^G$ is not finitely generated as a $K[(A_d)^G]$-module, then $F_d(\mathfrak{W})^G$ is not finitely generated for any $\mathfrak{W}$ containing $\mathfrak{E}$. The class of such groups $G$ contains all finite groups. It contains also the classical and close to them groups under some natural restrictions on the embedding into $GL_d(K)$. In particular, if $(A_d^2)^G = 0$ and $(A_d^2)^G \neq 0$, then $F_d(\mathfrak{W})^G$ is not finitely generated. Results in the same spirit hold if we replace the polynomial algebra $K[X_d]$ with the free metabelian Lie algebra $F_d(\mathfrak{W}) = L_d/L'_d$, where $L_d$ is the free Lie algebra freely generated by $X_d$ and $\mathfrak{W}$ is the variety of all metabelian (solvable of class 2) Lie algebras. If $(KX_d)^G = (A_d)^G = 0$ and $\dim F_d(\mathfrak{W})^G = \infty$, then again $F_d(\mathfrak{W})^G$ is not finitely generated.

1. Preliminaries

We fix a field $K$ of characteristic 0. All vector spaces and algebras will be over $K$. Let
\[ F(X) = K\{X\} = K\{x_1, x_2, \ldots\} \]
be the (absolutely) free nonassociative algebra freely generated by the countable set $X = \{x_1, x_2, \ldots\}$. Recall that the polynomial $f(x_1, \ldots, x_m) \in K\{X\}$ is a polynomial identity for the algebra $A$ if $f(a_1, \ldots, a_m) = 0$ for all $a_1, \ldots, a_m \in A$. The class of all algebras satisfying a given set $U \subset K\{X\}$ of polynomial identities is called the variety of associative algebras defined by the system $U$. If $\mathfrak{W}$ is a variety, then $T(\mathfrak{W})$ is the ideal of $K\{X\}$ consisting of all polynomial identities of $\mathfrak{W}$. Let $X_d = \{x_1, \ldots, x_d\} \subset X$. Then the algebra
\[ F_d(\mathfrak{W}) = K\{x_1, \ldots, x_d\}/(K\{x_1, \ldots, x_d\} \cap T(\mathfrak{W})) = K\{X_d\}/(K\{X_d\} \cap T(\mathfrak{W})) \]
is the relatively free algebra of rank \( d \) in \( \mathfrak{U} \). We shall denote the generators of \( F_d(\mathfrak{U}) \) with the same symbols \( X_d \). The ideals \( K\{X_d\} \cap T(\mathfrak{U}) \) of \( K\{X_d\} \) are preserved by all endomorphisms \( \varphi \) of \( K\{X_d\} \), i.e., \( \varphi(K\{X_d\} \cap T(\mathfrak{U})) \subseteq K\{X_d\} \cap T(\mathfrak{U}) \). In particular, \( GL_d(K)(K\{X_d\} \cap T(\mathfrak{U})) = K\{X_d\} \cap T(\mathfrak{U}) \). Here the general linear group \( GL_d(K) \) acts canonically on the vector space \( KX_d \) with basis \( X_d \) and this action is extended diagonally on the whole \( F_d(\mathfrak{U}) \) as in \( [3] \). Hence \( F_d(\mathfrak{U}) \) has a natural structure of a \( GL_d(K) \)-module. For a background on representation theory of \( GL_d(K) \) see, e.g., \([27, 38]\). Since \( \text{char}(K) = 0 \), the algebra \( F_d(\mathfrak{U}) \) is a direct sum of irreducible \( GL_d(K) \)-modules and

\[
F_d(\mathfrak{U}) = \sum m_{\lambda}(\mathfrak{U}) W_d(\lambda),
\]

where \( W_d(\lambda) \) is the irreducible polynomial \( GL_d(K) \)-module corresponding to the partition \( \lambda = (\lambda_1, \ldots, \lambda_d) \), \( \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \), and \( m_{\lambda}(\mathfrak{U}) \) is the multiplicity of \( W_d(\lambda) \) in the decomposition of \( F_d(\mathfrak{U}) \). Then the Hilbert series of \( F_d(\mathfrak{U}) \) is

\[
H(F_d(\mathfrak{U}), z_1, \ldots, z_d) = \sum m_{\lambda}(\mathfrak{U}) S_\lambda(z_1, \ldots, z_d),
\]

where \( S_\lambda(z_1, \ldots, z_d) \) is the Schur function corresponding to \( \lambda \). Since the Schur functions form a basis of the vector space \( K[X_d]^{S_n} \) of symmetric polynomials in \( d \) variables, the Hilbert series \( H(F_d(\mathfrak{U}), z_1, \ldots, z_d) \) determines the \( GL_d(K) \)-module structure of \( F_d(\mathfrak{U}) \).

In the sequel we shall need some well known information for two relatively free algebras: the polynomial algebra \( K[X_d] \) and the free metabelian Lie algebra \( F_d(\mathfrak{A}^2) = L_d/L''_d \).

**Lemma 1.1.** (i) The \( GL_d(K) \)-module structure of the polynomial algebra \( K[X_d] \) is

\[
K[X_d] = \sum_{n \geq 0} W_d(n).
\]

(ii) The free metabelian Lie algebra \( F_d(\mathfrak{A}^2) \) has a basis

\[
\{ x_{i_1}, \ldots, x_{i_n} | i_1 > i_2 > \cdots > i_n \}.
\]

The \( GL_d(K) \)-module structure of \( F_d(\mathfrak{A}^2) \) is

\[
F_d(\mathfrak{A}^2) = W_d(1) + \sum_{n \geq 2} W_d(n - 1, 1).
\]

Part (i) of the lemma is well known. Part (ii) is also well known, see e.g., \([31, 52, \text{pp. 274-276 of the English translation}] \) for the basis of \( F_d(\mathfrak{A}^2) \) and \([17, \text{the proof of Lemma 2.5}] \) for its \( GL_d(K) \)-module structure.

The product of two Schur functions \( S_\lambda(z_1, \ldots, z_d)S_\mu(z_1, \ldots, z_d) \) can be expressed as a sum of Schur functions using the Littlewood-Richardson rule. A very special case of this rule is the Branching theorem, when \( \mu = (1) \). It states that

\[
S_\lambda(z_1, \ldots, z_d)S_1(z_1, \ldots, z_d) = \sum S_\nu(z_1, \ldots, z_d),
\]

where the sum runs on all partitions \( \nu = (\nu_1, \ldots, \nu_d) \) obtained by adding 1 to one of the components \( \nu_i \) of \( \lambda = (\lambda_1, \ldots, \lambda_d) \). In other words, the Young diagram of \( \nu \) is obtained by adding a box to the diagram of \( \lambda \). Since the product of two Schur functions corresponds to the tensor product of the corresponding irreducible \( GL_d(K) \)-modules, we obtain equivalently

\[
W_d(\lambda) \otimes_K W_d(1) = \sum W_d(\nu),
\]
with the same summation on $\nu$ as in \cite{Drensky1}.

If $G$ is a subgroup of $GL_d(K)$, then the $GL_d(K)$-action on the irreducible $GL_d(K)$-module $W_d(\lambda)$ induces a $G$-action on $W_d(\lambda)$. Let $W_d(\lambda)^G$ be the vector space of the elements of $W_d(\lambda)$ fixed by $G$, i.e., of the $G$-invariants of $W_d(\lambda)$. If $W$ is a graded $GL_d(K)$-module with polynomial homogeneous components,

$$W = \bigoplus_{k \geq 0} W_k, \quad W_k = \sum_{\lambda} m_\lambda(k) W_d(\lambda),$$  \hspace{1cm} (7)

then its Hilbert series is

$$H(W, z_1, \ldots, z_d, z) = \sum_{k \geq 0} \left( \sum_{n_i \geq 0} \dim(W_k)_{(n_1, \ldots, n_d)} z_1^{n_1} \cdots z_d^{n_d} \right) z^k$$  \hspace{1cm} (8)

**Lemma 1.2.** Let $W$ be a graded $GL_d(K)$-module with polynomial homogeneous components, as in \cite{Drensky2}, and let $G$ be a subgroup of $GL_d(K)$. Then the Hilbert series of the $G$-invariants of $W$

$$H(W^G, z) = \sum_{k \geq 0} \dim W_k^G z^k$$

is determined from the Hilbert series \cite{Drensky3} of $W$.

**Proof.** We follow the main ideas of the recent preprint \cite{Drensky4} which contains more applications in the spirit of the lemma. Since the dimension of $W_d(\lambda)^G$ depends on $W_d(\lambda)$ only, and the Schur functions $S_\lambda(z_1, \ldots, z_d)$ are in 1-1 correspondence with the modules $W_d(\lambda)$, we conclude that $\dim W_d(\lambda)^G$ is a function of $S_\lambda(z_1, \ldots, z_d)$. This immediately completes the proof because

$$H(W^G, z) = \sum_{k \geq 0} \left( \sum_{\lambda} m_\lambda(k) \dim W_d(\lambda)^G \right) z^k.$$

$\square$

The proof of the next statement can be found in \cite{Drensky5} Proposition 4.2 in the case of homomorphic images of the free associative algebra $K(X_d)$. The proof in the case below is exactly the same.

**Proposition 1.3.** Let $I$ be an ideal of the relatively free algebra $F_d(\mathfrak{W})$ of the variety $\mathfrak{W}$ and let $I$ be preserved under the $GL_d(K)$-action on $F_d(\mathfrak{W})$. If $G$ is a subgroup of $GL_d(K)$, then every $G$-invariant of the factor algebra $F_d(\mathfrak{W})/I$ can be lifted to a $G$-invariant of $F_d(\mathfrak{W})$, i.e., under the canonical homomorphism

$$\pi : F_d(\mathfrak{W}) \rightarrow F_d(\mathfrak{W})/I$$

$F_d(\mathfrak{W})^G$ maps onto $(F_d(\mathfrak{W})/I)^G$. In particular, if $\mathfrak{V}$ is a subvariety of the variety $\mathfrak{W}$ and $\pi : F_d(\mathfrak{W}) \rightarrow F_d(\mathfrak{W})$, then

$$\pi(F_d(\mathfrak{W})^G) = F_d(\mathfrak{V})^G.$$

For more details on varieties of algebras (in the associative case) and the applications of representation theory of $GL_d(K)$ to PI-algebras we refer to the book \cite{Drensky6}.
2. The main result

Let $\mathfrak{L}$ be the subvariety of the variety of right-symmetric algebras $\mathfrak{R}$ defined by the identity (4) of left-nilpotency of class 3. Since the identity (2) of left-commutativity is a consequence of (4), $\mathfrak{L}$ is also a subvariety of the variety $\mathfrak{R}$ of Novikov algebras. Working in $\mathfrak{L}$, the only nonzero products are left-normed. We shall omit the parentheses and shall write $a_1a_2\cdots a_n$ instead of $(\cdots (a_1a_2)\cdots)a_n$ and $a_1a_2^k$ instead of $a_1a_2\cdots a_2$.

Lemma 2.1. (i) The relatively free algebra $F_d(\mathfrak{L})$ has a basis

\[ \{x_{i_1}x_{i_2}\cdots x_{i_n} \mid i_1 = 1, \ldots, d, \ 1 \leq i_2 \leq \cdots \leq i_n \leq d\}. \tag{9} \]

(ii) The $GL_d(K)$-module structure of $F_d(\mathfrak{L})$ is

\[ F_d(\mathfrak{L}) = W_d(1) + \sum_{n \geq 2} (W_d(n) + W_d(n-1,1)). \tag{10} \]

Proof. (i) Modulo the identity (4) the right-symmetric identity (1) reduces to

\[ x_1x_2x_3 = x_1x_3x_2. \tag{11} \]

Hence $\mathfrak{L}$ satisfies the identity

\[ x_1x_2\cdots x_n = x_1x_{\sigma(2)}\cdots x_{\sigma(n)}, \quad \sigma \in S_n, \sigma(1) = 1, \]

and the algebra $F_d(\mathfrak{L})$ is spanned as a vector space on the elements (9). In order to show that (9) is a basis of $F_d(\mathfrak{L})$ it is sufficient to construct an algebra $A$ in $\mathfrak{L}$ which is generated by $a_1, \ldots, a_d$ and has a basis

\[ \{a_ia_1^{n_1}\cdots a_d^{n_d} \mid i_1 = 1, \ldots, d, \ n_j \geq 0\}. \tag{12} \]

Since $A$ is a homomorphic image of $F_d(\mathfrak{L})$, this would imply that (9) is a basis of $F_d(\mathfrak{L})$. Consider the vector space $A$ with basis (12) and define a multiplication there by the rule

\[ (a_ia_1^{n_1}\cdots a_d^{n_d}) * a_j = a_ia_1^{n_1}\cdots a_j^{n_j+1}\cdots a_d^{n_d}, \]

\[ (a_ia_1^{n_1}\cdots a_d^{n_d}) * (a_ia_1^{m_1}\cdots a_d^{m_d}) = 0, \text{ if } m_j > 0 \text{ for some } j. \]

Obviously $A$ satisfies the identities (1) and (11), and hence belongs to $\mathfrak{L}$.

(ii) For $n \geq 2$ we divide the basis elements from (9) in two groups. The first group contains the monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$ with $i_1 \leq i_2$ and the second group the monomials with $i_1 > i_2$. Obviously, the monomials in the first group are in 1-1 correspondence with the monomials of degree $\geq 2$ in $K[X_d]$. By Lemma 1.1 (ii), the same holds for the monomials from the second group and the elements of degree $\geq 2$ in $F_d(\mathfrak{R})$. Hence the Hilbert series of $F_d(\mathfrak{L})$ is a sum of the Hilbert series of the algebra of polynomials without constant term and the commutator ideal of the Lie algebra $F_d(\mathfrak{R})$. Now the proof follows from Lemma 1.1. $\square$

The construction in the proof of Lemma 2.1 (i) suggests that the algebra $F_d(\mathfrak{L})$ has the structure of a right $K[X_d]$-module with action defined by

\[ (x_px_1^{n_1}\cdots x_d^{n_d}) \circ (x_1^{m_1}\cdots x_d^{m_d}) = x_px_1^{n_1+m_1}\cdots x_d^{n_d+m_d}, \quad p = 1, \ldots, d, n_j, m_j \geq 0. \]

Clearly, the ideal $F^2_d(\mathfrak{L})$ of the elements in $F_d(\mathfrak{L})$ without linear term is a $K[X_d]$-submodule. We shall denote by $(A_d)_1$ the vector space $KX_d$ and shall identify $K[X_d]$ and $K[(A_d)_1]$. 
Theorem 2.2. Let $\mathfrak{V}$ be a subvariety of the variety $\mathfrak{R}$ of all right-symmetric algebras and let $\mathfrak{V}$ contain the variety $\mathfrak{L}$ of left-nilpotent of class 3 algebras in $\mathfrak{R}$. If $G \neq \{1\}$ is a subgroup of $GL_d(K)$ such that the ideal $F_d^2(\mathfrak{L})^G$ of the algebra of invariants $F_d(\mathfrak{L})^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, then the algebra of $G$-invariants $F_d(\mathfrak{V})^G$ is not finitely generated.

Proof. By Proposition 1.3 the canonical homomorphism $F_d(\mathfrak{V}) \rightarrow F_d(\mathfrak{L})$ maps $F_d(\mathfrak{V})^G$ onto $F_d(\mathfrak{L})^G$ and if $F_d(\mathfrak{V})^G$ is finitely generated, the same is $F_d(\mathfrak{L})^G$. Hence it is sufficient to show that $F_d(\mathfrak{L})^G$ is not finitely generated. Therefore we may work in $F_d(\mathfrak{L})$ and assume that $F_d(\mathfrak{L})^G$ is finitely generated. As a vector space $F_d(\mathfrak{L})^G$ is a direct sum of the invariants of first degree $(KX_d)^G = (A_d)_1^G$ and the invariants $F_d^2(\mathfrak{L})^G$ without linear term. We may assume that $F_d(\mathfrak{L})^G$ is generated by $U = \{u_1, \ldots, u_k\} \subset (A_d)_1^G$ and $W = \{w_1, \ldots, w_l\} \subset F_d^2(\mathfrak{L})^G$. Since $F_d(\mathfrak{L})^G F_d^2(\mathfrak{L})^G = 0$, the only nonzero products of the generators of $F_d(\mathfrak{L})^G$ are $u_pu_{i_1}\cdots u_{i_m}$ and $w_qu_{i_1}\cdots u_{i_m}$, $m \geq 0$. Hence $KU = (A_d)_1^G$.

$$F_d^2(\mathfrak{L})^G = \sum_{i=1}^k u_p u_{p_2} \circ K[(A_d)_1^G] + \sum_{j=1}^l w_q \circ K[(A_d)_1^G]$$

and $F_d^2(\mathfrak{L})^G$ is a finitely generated $K[(A_d)_1^G]$-module which is a contradiction. \qed

Corollary 2.3. Let $A_d = K[X_d]_+$ be the algebra of polynomials without constant term and let $G$ be a subgroup of $GL_d(K)$. If $(A_d^2)^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, then $F_d(\mathfrak{V})^G$ is not finitely generated for any variety $\mathfrak{V}$ containing $\mathfrak{L}$.

Proof. By the Branching theorem (6)

$$W_d(n-1, 1) \otimes_K W_d(1) = W_d(n, 1) \oplus W_d(n-1, 2) \oplus W_d(n-1, 1, 1). \quad (13)$$

Consider the $GL_d(K)$-module decomposition of $F_d(\mathfrak{L})$ given in Lemma 2.1 (ii). Since $F_d(\mathfrak{L}) F_d^2(\mathfrak{L}) = 0$, the only nonzero products $W_d(\lambda) W_d(\mu)$ with $\lambda$ or $\mu$ equal to $(n-1, 1)$, $n \geq 2$, come from $W_d(n-1, 1) W_d(1) = W_d(n-1, 1, 1) F_d(\mathfrak{L}) = W_d(n-1, 1)(KX_d) = W_d(n-1, 1)(A_d)_1$. This is a homomorphic image in $F_d(\mathfrak{L})$ of $W_d(n-1, 1) \otimes_K W_d(1)$. By (13) we derive that $W_d(n-1, 1) F_d(\mathfrak{L}) \subset W_d(n, 1)$. This implies that

$$I = \sum_{n \geq 2} W_d(n-1, 1) \subset F_d(\mathfrak{L})$$

is an ideal of $F_d(\mathfrak{L})$ and the $GL_d(K)$-module structure of the factor algebra is

$$F_d(\mathfrak{L})/I = \sum_{n \geq 1} W_d(n) \cong A_d.$$

Hence the algebras $F_d(\mathfrak{L})/I$ and $A_d$ have the same Hilbert series and by Lemma 1.2 the same holds for their algebras of invariants. Since $(A_d^2)^G$ is not finitely generated as a $K[(A_d)_1^G]$-module, the same is true for the $K[(A_d)_1^G]$-module $F_d^2(\mathfrak{L})/I$. By Proposition 1.3 the $K[(A_d)_1^G]$-module $F_d^2(\mathfrak{L})$ is not finitely generated and the application of Theorem 2.2 completes the proof. \qed
Corollary 2.4. Let $F_d(\mathfrak{A}^2)$ be the free metabelian Lie algebra and let $G$ be a subgroup of $GL_d(K)$. If $(KX_d)^G = (A_d)^G = 0$ and $\dim F_d(\mathfrak{A}^2)^G = \infty$, then $F_d(\mathfrak{A})^G$ is not finitely generated for any variety $\mathfrak{B}$ containing $\mathfrak{L}$.

Proof. Since $(KX_d)^G = (A_d)^G = 0$ we obtain that $F_d(\mathfrak{L})^G = F_d^2(\mathfrak{L})^G$. Hence the algebra $F_d(\mathfrak{L})^G$ is with trivial multiplication and the finite generation is equivalent to the finite dimensionality. As a $GL_d(K)$-module $F_d(\mathfrak{A}^2)$ is a homomorphic image of $F_d(\mathfrak{L})$. Hence the vector space $F_d(\mathfrak{A}^2)^G$ is a homomorphic image of $F_d(\mathfrak{L})^G$. This implies that $\dim F_d^2(\mathfrak{L})^G = \infty$, i.e., both the algebras $F_d(\mathfrak{L})^G$ and $F_d(\mathfrak{A})^G$ are not finitely generated. □

Remark 2.5. In Corollary 2.4 we cannot remove directly the restriction $(KX_d)^G = 0$, as in Corollary 2.3 because the $GL_d(K)$-submodule $I = \sum_{n \geq 2} W_d(n)$ of $F_d(\mathfrak{L})$ is not an ideal. For example, one can show that $W_d(2)(KX_d) = W_d(3) \oplus W_d(2,1)$. Hence we cannot use the property that the Lie algebra $F_d(\mathfrak{A}^2)^G$ is not finitely generated to show that the algebra $F_d(\mathfrak{L})$ is also not finitely generated. On the other hand, we do not know examples of groups $G$ when $(KX_d)^G = 0$, $K[X_d]^G = K$, and $\dim F_d(\mathfrak{A}^2)^G = \infty$. Such an example would show that we may apply Corollary 2.4 when we cannot apply Corollary 2.3.

3. Examples

All examples in this section use the following statement which is a consequence of Corollary 2.3.

Proposition 3.1. If for a subgroup $G$ of $GL_d(K)$

$$\text{transcend. deg}(K[X_d]^G) > \dim(KX_d)^G,$$

then the algebra $F_d(\mathfrak{A})^G$ is not finitely generated for any variety $\mathfrak{B}$ containing $\mathfrak{L}$.

Proof. Let $t = \text{transcend. deg}(K[X_d]^G)$. Since $K[X_d]^G$ is graded, we may choose $t$ algebraically independent homogeneous elements in $A_d^G = (K[X_d]^G)^2$. If $m = \dim(KX_d)^G$, changing linearly the variables $X_d$ we assume that $(KX_d)^G$ has a basis $X_m = \{x_1, \ldots, x_m\}$ and $K[(A_d)^G] = K[X_m]$. Since $t > m$, we obtain that $(A_d^2)^G$ contains an element $f(X_d)$, such that the system $X_m \cup \{f(X_d)\}$ is algebraically independent. Hence the $K[X_m]$-module generated by the powers $f^k(X_d)$, $k = 1, 2, \ldots$, is not finitely generated. Now the proof follows from Corollary 2.3. □

3.1. Finite groups.

Theorem 3.2. Let $G$ be a finite subgroup of $GL_d(K)$ and $G \neq \langle 1 \rangle$. Then the algebra $F_d(\mathfrak{A})^G$ is not finitely generated for any variety $\mathfrak{B}$ containing $\mathfrak{L}$.

Proof. It is well known that for a finite group $G$

$$\text{transcend. deg}(K[X_d]^G) = \text{transcend. deg}(K[X_d]) = d.$$  \hspace{1cm} (14)

For self-containedness of the exposition, every element $f(X_d) \in K[X_d]$ satisfies the equation

$$u_f(z) = \prod_{g \in G} (z - g(f(X_d))) = z^{|G|} - c_1z^{|G|-1} + c_2z^{|G|-2} - \cdots \pm c_{|G|}.$$
where the coefficients $c_k$ are equal to the elementary symmetric polynomials in 
\{g(f(X_d)) \mid g \in G\}. Hence $c_k \in K[X_d]^G$ and as a $K[X_d]^G$-module $K[X_d]$ is 
generated by 
\[ x_1^{a_1} \cdots x_d^{a_d}, \quad 0 \leq a_i < |G|. \]
The finite generation of the $K[X_d]^G$-module $K[X_d]$ implies (14) and the theorem 
follows from Proposition 3.1.

3.2. Reductive groups. If $G \subset GL_d(K)$ is a reductive group then there exists a 
$G$-submodule $W$ of $KX_d$ such that $KX_d = (KX_d)^G \oplus W$.

**Proposition 3.3.** In the above notation, if $K[W]^G \neq K$, then $F_d(\mathfrak{W})^G$ is not 
finitely generated for all $\mathfrak{W}$ containing $\mathfrak{L}$.

**Proof.** Since the elements of $K[W]$ cannot be expressed as polynomials in $(KX_d)^G$, 
the condition $K[W]^G \neq K$ implies that 
\[ \text{transcend.deg}(K[X_d]^G) = \dim(KX_d)^G + \text{transcend.deg}(K[W]^G) > \dim(KX_d)^G \]
and this completes the proof in virtue of Proposition 3.1.

**Example 3.4.** For each $k \geq 1$ there is a unique irreducible rational $k$-dimensional 
$SL_2(K)$-module $W_k$. Let the subgroup $G$ of $GL_d(K)$ be isomorphic to $SL_2(K)$ and 
\[ KX_d \cong W_{k_1} \oplus \cdots \oplus W_{k_p} \]
as an $SL_2(K)$-module. It is well known that if $k_1 \geq 3$, then $K[W_{k_1}]$ contains 
nontrivial $SL_2(K)$-invariants. Similarly, $K[W_2 \oplus W_2]^{SL_2(K)} \neq K$. Hence the only 
cases when $K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}]$ are $k_1 = 2$, $k_2 = \cdots = k_p = 1$ when 
$K[X_d]^{SL_2(K)} = K[(KX_d)^{SL_2(K)}] \cong K[X_d-1]$ and $k_1 = \cdots = k_p = 1$ with the trivial 
action of $SL_2(K)$ on $KX_d$ (and the latter case is impossible because $G \cong SL_2(K)$ 
is a nontrivial subgroup of $GL_d(K)$).

3.3. Weitzenböck derivations. A linear operator $\delta$ of an algebra $A$ is a derivation if 
\[ \delta(uv) = \delta(u)v + u\delta(v), \quad u, v \in A. \]
If $\mathfrak{W}$ is a variety of algebras, then every mapping $\delta : X_d \rightarrow F_d(\mathfrak{W})$ can be uniquely 
extended to a derivation of $F_d(\mathfrak{W})$ which we shall denote by the same symbol $\delta$.

If $\delta$ is a nilpotent linear operator on $KX_d$, then the induced derivation is called a 
Weitzenböck derivation. Weitzenböck [36] proved that in the case of polynomial 
algebras the algebra of constants 
\[ K[X_d]^\delta = \{ f(X_d) \in K[X_d] \mid \delta(f(X_d)) = 0 \} \]
is finitely generated. Details on the algebra of constants $K[X_d]^\delta$ can be found in the book by Nowicki [32]. For varieties $\mathfrak{W}$ of unitary associative algebras (and $\delta \neq 0$) the algebra $F_d(\mathfrak{W})^\delta$ is finitely generated if and only if $\mathfrak{W}$ does not contain the algebra 
$T_2(K)$ of $2 \times 2$ upper triangular matrices, see [15, 16]. Up to a change of the basis 
of $KX_d$ the Weitzenböck derivation $\delta$ is determined by the Jordan normal form 
$J(\delta)$ of the linear operator $\delta$ acting on $KX_d$. Since $\delta$ acts nilpotently on $KX_d$, the 
matrix $J(\delta)$ consists of Jordan blocks with zero diagonals.

**Proposition 3.5.** If $d > 2$ and the Jordan normal form $J(\delta)$ of the Weitzenböck 
derivation consists of less than $d-1$ blocks, then the algebra $F_d(\mathfrak{W})^\delta$ is not finitely 
generated for any variety $\mathfrak{W}$ containing the variety $\mathfrak{L}$. 

Proof. Since \( \alpha \delta \), \( \alpha \in K \), is nilpotent on \( KX_d \), it is a locally nilpotent derivation of \( F_d(\mathfrak{g}) \), i.e., for every \( f(X_d) \in F_d(\mathfrak{g}) \) there exists an \( n \geq 1 \) such that \( (\alpha \delta)^n(f(X_d)) = 0 \). Hence
\[
\exp(\alpha \delta) = 1 + \frac{\alpha \delta}{1!} + \frac{(\alpha \delta)^2}{2!} + \cdots
\]
is a well defined linear automorphism of \( F_d(\mathfrak{g}) \). It is well known that the group
\[
\{ \exp(\alpha \delta) \mid \alpha \in K \}
\]
is isomorphic to the unipotent group \( UT_2(K) \) and
\[
F_d(\mathfrak{g})^\delta = F_d(\mathfrak{g})^{UT_2(K)}.
\]
If the matrix \( J(\delta) \) consists of \( p \) blocks, then the dimension of the vector space \( (KX_d)^\delta \) of the linear constants is equal to the number of the blocks \( p \). Reading carefully [32, Proposition 6.5.1, p. 65] we can see that
\[
\text{transcend.deg}(K[X_d]^\delta) = d - 1
\]
which is larger than \( p = \dim(KX_d)^\delta \). Now the proof follows from Proposition 3.1 applied for \( UT_2(K) \subset GL_d(K) \).

Remark 3.6. If in Proposition 3.5 the Jordan normal form of \( \delta \) consists of \( d - 1 \) blocks, then the algebra of constants \( K[X_d]^\delta \) is generated by linear constants. In this case we may assume that \( \delta(x_1) = x_2 \) and \( \delta(x_i) = 0 \) for \( i = 2, \ldots, d \). It is easy to see that \( F_d(\mathfrak{g})^\delta \) is generated by \( x_1x_2 - x_2x_1, x_2, \ldots, x_d \). We do not know how far can be lifted to \( F_d(\mathfrak{g}) \) the finite generation property of the algebra of constants and do not have a description of the varieties \( \mathfrak{g} \) containing \( \mathfrak{L} \) such that the algebra \( F_d(\mathfrak{g})^\delta \) is finitely generated.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

E-mail address: drensky@math.bas.bg