Stochastic force in gravitational systems.

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Abstract. In this paper I study the probability distribution of the gravitational force in gravitational systems through numerical experiments. I show that Kandrup’s (1980) and Antonuccio-Delogu & Atrio-Barandela’s (1992) theories describe correctly the stochastic force probability distribution respectively in inhomogeneous and clustered systems. I find equations for the probability distribution of stochastic forces in finite systems, both homogeneous and clustered, which I use to compare the theoretical predictions with Monte Carlo simulations of spherically symmetric systems. The agreement between theoretical predictions and simulations proves to be quite satisfactory.

Key words: stars: statistics

1. Introduction.

One of the main contributions to the understanding of the statistical mechanics of stellar system was provided by Chandrasekhar & von Neumann (1942) in a paper where they studied the probability force distribution for test-points in gravitational systems. They showed that under conditions typically found in astrophysical systems like globular clusters and clusters of galaxies, the average force $F_{tot}$, acting on a test star can be naturally decomposed as:

$$F_{tot} = F_{pot} + F_{stock}$$

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The first term on the right hand side of Eq. 1 is the mean field force produced by
the smoothed distribution of stars in the system and can be obtained from a potential
function, \( \Phi(r) \), which is also connected, by the Poisson equation, to the system mass
density, \( \rho(r) \). In the case of a truncated system having a power-law density profile:

\[
\rho(r) = \rho_0 \left( \frac{r_0}{r} \right)^p, \quad 0 \leq r \leq R
\]

the mean internal gravitational force is given by:

\[
F_{\text{pot}} = -\frac{GM_{\text{tot}}}{R^{3-p}} r^{1-p}
\]

where \( G \) is the Gravitational constant, \( M_{\text{tot}} \) is the total mass of the system and \( R \) is its
radius. The second term on the right hand side of equation Eq. 1 is a stochastic variable
describing the effects of the fluctuating part of the gravitational field. This component
arises because of the discreteness of the mass distribution: it would be null in a continuous
system. For a homogeneous system the average value of the stochastic force is given by:

\[
\langle F_{\text{stoch}} \rangle = 8.879 Gmn^{2/3}
\]

\( (\text{Kandrup}, 1980) \) where \( m \) and \( n \) are the average stellar mass and number density,
respectively.

The timescales of these two force components are very different. The stochastic force
is rapidly varying being produced by the fluctuations of the neighbouring stars number
density, while the mean force is slowly varying because is produced by the overall mass
distribution of the system, which changes on a dynamical timescale \( t \approx (G\rho)^{-1/2} \). The
stochastic force, just like any stochastic variable, may be described through the assigne-
ment of a density probability function \( W(F) \).

The function \( W(F) \) for a homogeneous system was obtained for the first time by Chan-
drasekhar & von Neumann (1942) under the hypotheses that the stars are distributed
with uniform density in a spherical system, that there are no correlations and that

\[
\frac{N}{4\pi R^3} = \text{constant}
\]

when \( N \to \infty \) and \( R \to \infty \), where \( N \) is the total number of stars and \( R \) the radius of the stellar system. They also showed that the force probability distribution
is given by Holtsmark’s law\(^1\) Spherical symmetry is used so that at the center of the
system \( F_{\text{pot}} = 0 \) and \( F_{\text{tot}} = F_{\text{stoch}} \), while the absence of correlation is required in order
to be able to decompose the gravitational field into a mean and a stochastic component.

The calculation of the random force probability distribution \( W(F) \) in inhomogeneous

\(^1\)Historically Holtsmark’s law was calculated to obtain the probability of a given electric
field strength at a point in a gas composed of ions.
systems, with particles distributed with probability density \( \tau(r) = \frac{a}{r^p} \), was performed by Kandrup (1980) under the same hypotheses of Chandrasekhar’s model. He showed that the theoretical probability distribution of random force is a generalization of Holtsmark’s law (that I hereafter call Kandrup’s law).

Finally Antonuccio-Delogu & Atrio-Barandela (1992) (hereafter AA92) obtained an equation, that I hereafter call AA(92) law, describing the probability distribution of stochastic forces in weakly clustered systems. The latter assumptions necessary to keep a meaning to the decomposition of the gravitational field into a mean and a fluctuating field component.

Numerical experiments attempting to verify these theoretical predictions were performed by Hunger et al. 1965 for plasma systems and by Ahmad & Cohen (1973) for homogeneous stellar systems. The latter authors verified the Chandrasekhar & von Neumann’s result using a series of numerical experiments that showed a good agreement between Chandrasekhar’s theoretical calculation and the numerical experiments.

Ahmad & Cohen’s numerical experiments regard only homogeneous systems while the theory of stochastic forces in inhomogeneous (Kandrup 1980) and in clustered systems (AA92) has never been tested.

In this paper I use numerical experiments to show that Kandrup’s and AA92 calculations give a good description of random forces in inhomogeneous and clustered systems, respectively.

The plan of this work is as follows: in Sect. 2 I review the fundamental equations. Section 3 and 4 are devoted to probe Kandrup’s and AA(92) laws. Finally in Appendix 1 I calculate the distribution of stochastic forces for a finite inhomogeneous system and in Appendix 2 the same distribution for a clustered system.

2. Stochastic force distribution law.

The force per unit mass acting on a test star of a finite N-body gravitating system is given by the usual formula:

\[
F_{\text{tot}}(r) = -G \sum_{i=1}^{N} \frac{m_i}{r_i - r^3} (r - r_i) \tag{5}
\]

where the sum is extended to the N stars, \( m_i \) is the mass of the i-th field star and \( r_i \) is its distance relative to the origin.

The value of \( F \) at a given time depends on the instantaneous positions of all the other stars and therefore is subject to fluctuations as these position change. Even if the stellar distribution were constant and homogeneous the total force experienced by a star, \( F_{\text{tot}} \),
would fluctuate around an average value due to local Poisson fluctuations in the number density of neighbouring stars. The fluctuating part of \( F_{\text{tot}} \) that I have previously indicated \( F_{\text{stoch}} \) can be studied by probabilistic methods defining a probability distribution of stochastic force \( W(F_{\text{stoch}}) \). In the case of an infinite homogeneous system, under the other hypotheses given in Sect. 1, the random force distribution law is given by Holtsmark’s distribution:

\[
W(F_{\text{stoch}}) = \frac{2F_{\text{stoch}}}{\pi} \int_0^\infty t dt \sin(tF_{\text{stoch}}) \exp \left[ -n(Gmt)^{3/2}(2\pi)^{3/2} \frac{4}{15} \right]
\]

(Chandrasekhar & von Neumann 1942), where \( m \) is the mass of a field star, \( n \) the mean density and \( G \) the gravitational constant. This equation shows that the stochastic force probability distribution depends only on the mean density \( n \) or, equivalently, on the static configuration of the system.

The stochastic force probability distribution in inhomogeneous systems can be calculated in the same way and under the same hypotheses as that in homogeneous systems. Kandrup (1980) obtained an equation for the distribution of random force of an infinite inhomogeneous system with probability density \( \tau(r) = \frac{a}{R^p} \). The resulting equation is a generalization of the Holtsmark law:

\[
W(F_{\text{stoch}}) = \frac{2F_{\text{stoch}}}{\pi} \int_0^\infty t dt \sin(tF_{\text{stoch}}) \exp \left[ -\frac{\alpha}{2}(Gmt)^{(3-p)/2} \int_0^\infty \frac{dz}{z^{(7-p)/2}} \sin(z) z^{(7-p)/2} \right]
\]

where \( \alpha = \frac{(3-p)N(R)}{R^{(3-p)}} = 4\pi a \) (because the total number of stars at a distance \( R \) is given by: \( N(R) = 4\pi \int_0^\infty dr \tau(r) = 4\pi a R^{(3-p)/(3-p)} \)). Equation (7) reduces to the Holtsmark distribution for a uniform system in the case \( p = 0 \). The probability density must be chosen in such a way to ensure a convergent integral in Eq. (7); this restricts the choice to \( p < 3 \). If the system is finite, as e.g. is the case for globular clusters, the Holtsmark and Kandrup laws must be substituted by a law in which \( N \) is finite. The stochastic force for a finite system in which particles are distributed with probability density \( \tau(r) = \frac{a}{R^p} \) is calculated in Appendix 1 (Eq. (20)):

\[
W_N(F_{\text{stoch}}) = \frac{2F_{\text{stoch}}}{\pi} \int_0^\infty t dt \sin(tF_{\text{stoch}}) \left[ \frac{(3-p)(Gmt)^{(3-p)/2}}{2R^{(3-p)}} \int_0^\infty \frac{\sin z z^{(7-p)/2} dz}{z^{(7-p)/2}} \right]^N
\]

As Ahmad & Cohen (1973) showed, the stochastic force probability distribution for an infinite homogeneous system (Eq. (3)) and that for a finite one (Eq. (8) with \( p = 0 \)) almost coincide for \( N \simeq 1000 \). In Fig. 1 I show that the same result holds in the inhomogeneous case (\( p = 0.5 \)). The different curves are obtained from Eq. (8), describing the stochastic force distribution in a finite inhomogeneous system, for increasing value of \( N \) (\( N = 50, 100, 1000 \)) and from Eq. (3), which gives the stochastic force distribution
Fig. 1. The theoretical probability distribution of stochastic force in an inhomogeneous system for different values of $N$. The solid line is Kandrup’s distribution for an infinite inhomogeneous system, the long dashed line is Kandrup’s distribution for a finite inhomogeneous system with $N = 50$, the dotted line is the same distribution for $N = 100$. The Kandrup’s distribution for a finite inhomogeneous system with $N = 1000$ is undistinguishable from that for the infinite system. The force is measured in units of $G m^2 (3-p)/(3-p)$.

in an infinite inhomogeneous system. When $N \approx 1000$ the two distributions are indistinguishable, meaning that for $N \approx 1000$ the stochastic force in an inhomogeneous system are equivalently described by the Kandrup’s law for an infinite (Eq. 7) or finite system (Eq. 8).

3. Test of Kandrup’s law.

In order to test the validity of Kandrup’s law (Eqs. 7 and 8) I operated in the following manner: I used a system of 10000 particles distributed in a spherical system of radius $R$ with a density probability $\tau(r) = \alpha r^p$, where $\alpha$ and $p$ are two constants. Then I computed the stochastic force using 2000 points randomly distributed within a concentric spherical region of radius $0.01R$. The test points were set in the central region of the system of particles in order to avoid boundary effects which are not reproduced by Kandrup’s distribution (which was deduced for a spherically symmetric system). The stochastic force was obtained subtracting the mean from the total force at each point.

I then normalized the stochastic force obtained by the numerical experiments and the theoretical force (Eq. 8) dividing by $GM R^2$ and finally I compared Kandrup’s distribution for a finite system of 10000 particles with the experimental stochastic force distribution obtained from the system as described. This procedure was repeated for increasing values of the inhomogeneity parameter, $p$, which is the index of the power law $\tau(r) = \alpha r^p$. I found a good agreement between the stochastic force distribution obtained from our system of particles and Kandrup’s equation for a finite system given in Sect. 2. I show this result in Figs. 2, 3, 4, 5, 6. In Fig. 2 I compare Kandrup’s distribution for a finite inhomogeneous system ($p = 0.01$) with the histogram of forces obtained from a system of 10000 particles. In Figs. 3, 4, 5, 6 I show the same comparison for systems with $p = 0.1, 0.2, 0.5, 1$ respectively.
Fig. 2. Experimental distribution of the stochastic force in an inhomogeneous system. The solid line is Kandrup’s distribution for a finite inhomogeneous system (\( p = 0.01 \)) of 10000 particles, the histogram is the experimental distribution of stochastic force obtained from a system of 10000 particles as described in the text. The force is measured in units of \( \frac{GM}{R^2} \).

Fig. 3. Same as figure 2 but with \( p = 0.1 \).

A final remark concerns the fact that I have not evolved the particles system.

Fig. 4. Same as figure 2 but with \( p = 0.2 \).

Firstly, as I have stressed Holtsmark’s and Kandrup’s distribution depends only on the system’s configuration and it is not necessary to evolve the system to verify them. Secondly, to verify that the theoretical distribution (Kandrup’s law) describes correctly the stochastic force in the particles system it is necessary that the inhomogeneity parameter, \( p \), has the same value in Kandrup’s law and in the generated system. If I evolve the system, its original configuration will change because of correlation effects induced during the evolution and as a result the quoted condition shall no more be verified. So I would compare an experimentally obtained stochastic force distribution having an inhomogeneity, \( p \), different from that of Kandrup’s law.

4. Test of the AA(92) distribution.

Chandrasekhar & von Neumann (1942) and Kandrup’s (1980) distributions can be used to describe the stochastic force probability in homogeneous and inhomogeneous systems, respectively, under the hypothesis of no correlations among stellar positions in a material system. Correlation effects in gravitational systems were investigated by Prigogine & Sev-
erne (1966), Gilbert (1970), Lerche (1971) with the conclusion that positive correlations should cause an increase of the probability of having larger forces, compared to the Holtsmark distribution. This is in agreement with Kandrup’s (1980) analysis showing that the most probable contribution to $W(F)$ comes from random displacements of the nearest neighbours. In fact positive spatial correlations cause an enhancement of the probability of finding a particle near a given one and so, in agreement with Kandrup’s conclusion, the probability force distribution shall be characterized by an increase of higher forces.

Antonuccio-Delogu & Atrio-Barandela (1992) obtained an equation describing the stochastic force distribution in weakly clustered systems that is a generalization of the Holtsmark distribution. This last distribution is given by:

\[
W(F) = \frac{2F}{\pi} \int_0^\infty t dt \sin(tF) A_f(t) \tag{9}
\]

(AA92) where $A_f(t)$ is a rather complicated function and is given in the quoted paper (AA92, Eq. 36). For a finite clustered system Eq. (9) must be modified as I have done in Appendix 2. I verify that this equation describes correctly the stochastic force distribution in a stellar system with correlations using a Montecarlo simulated clustered system generated as described below.

In a clustered system the probability to find a star at a distance $r$ from another star is given by:

\[
\delta P = n \delta V [1 + \xi(r)] \tag{10}
\]

where $n$ is the mean number density and $\xi(r)$ is the two-point correlation function. The total number of stars up to distance $r$ in the system is given by:
\[ N(r) = \int_0^\infty n \delta V [1 + \xi(r)] \]  

(11)

where the correlation function \( \xi(r) \) must verify the condition:

\[ \int_0^\infty r^2 \xi(r) dr = 0 \]  

(12)

The last equation derives from the mass conservation law, and tells us us that if there are regions of the system where \( \xi > 0 \) there must also necessarily be regions with \( \xi < 0 \) because the total mass of system cannot be affected by the correlations. I obtained a clustered system using a correlation function \( \xi(r) \) given by Peebles (1980):

\[ \xi(r) = A \frac{\text{arctan}(r/\lambda_0)}{r[\lambda_0^2 + r^2]^2} \]  

(13)

where \( \lambda_0 \) is a constant. I generated a system of 10000 particles distributed in a set of spherical shells having constant total number of particles, whose inner and outer inner radii \( r_i \) and outer radius \( r_f \) are then given by the solution of the equation:

\[ N_{\text{shell}} = \frac{3N_{\text{tot}}}{R^3} \int_{r_i}^{r_f} r^2 [1 + \xi(r)] dr \]  

(14)

where \( N_{\text{shell}} \) is the (constant) number of particles per shell, \( N_{\text{tot}} \) is its total number and \( R \) the radius of the system. Furthermore I also adopt the hypothesis of weak clustering \( \xi(r) << 1 \). I calculated the stochastic force on 2000 test points randomly distributed within a spherical region of radius 0.01\( R \) subtracting the mean force from the total force on each point. The system was not evolved for the same reason given in the previous section. In fact the evolution induces correlation effects changing the original correlation function, \( \xi(r) \), and also the density distribution of the system (the parameter \( p \)). In this way the condition of equality between the value of \( p \) in AA(92) law and that of the system, necessary to be sure I am comparing a theoretical law and an experimental distribution having the same density distribution, would be no more verified. In Fig. 7 I show that,

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**Fig. 7.** Experimental distribution of stochastic force in a clustered system. The solid line is the AA(92) distribution for a clustered system of 10000 particles. The histogram is the experimental distribution of stochastic force obtained from a clustered system of 10000 particles as described in the text. The dot dashed line and histogram are a Holtsmark’s distribution and an experimental distribution for an homogeneous system of 10000 particles.
as expected, there is an increase in higher force probability with respect to homogeneous and inhomogeneous systems.

5. Conclusions

In this paper I showed, through numerical experiments, that the stochastic force probability distribution, induced by the graininess of configuration space of stellar systems, can be described correctly by some theoretical distributions. In particular I showed that Kandrup’s (1980) theory, describing the stochastic force probability distribution in infinite inhomogeneous systems, and AA92 theory, giving the stochastic force probability distribution in weakly clustered systems, describe correctly the observed behaviour. Furthermore I showed that for $N > 1000$ Kandrup’s theory can be applied to finite systems of particles.

Acknowledgements.

I thank Vincenzo Antonuccio-Delogu for helpful and stimulating discussions during the period in which this work was performed.
Appendix A.

In this Appendix I derive the probability distribution of force per unit mass for a finite inhomogeneous system starting from the calculations of stochastic force for a homogeneous system given in Appendix B of Ahmad & Cohen's (1973) paper.

Suppose to have a cluster of radius R containing N stars of mass \( m \) with probability distribution law given by \( \tau(r) = \frac{a}{r^p} \). I define the characteristic function of the probability distribution \( W(F) \):

\[
C(t) = \int \exp(itF) W(F) dF
\]  
(15)

Suppose now that the system contains only one star. If \( W(F_i) \) is the probability distribution of the force due to the star calculated at the origin one verifies that:

\[
W(F_i) dF_i = \tau(r) dr
\]  
(16)

Using the usual equation \( F_i = Gm_i r_i \), I obtain the probability distribution for one star:

\[
W_1(F_i) = \frac{a(Gm)^{3-p}/2}{2|F_i|^{9-p}/2} \frac{GM}{R^2} < F_i < \infty
\]  
(17)

Introducing Eq. 17 into Eq. 15 the characteristic function for the force due to one star turns out to be given by:

\[
M(t) = C_1(t) = \frac{(3-p)(Gm)^{3-p}/2}{2R^{3-p}} \int_0^{\infty} \frac{\sin z}{z^{7-p}/2} \frac{GM}{R^2} dz
\]  
(18)

where \( z = |t \cdot F| \). Finally the probability distribution for \( F \) is:

\[
W_N(F) = \frac{1}{2\pi^2} \int_0^{\infty} \frac{\sin(t|F|)}{t|F|} M_N(t) t^2 dt
\]  
(19)

and the probability distribution of the modulus of the force is:

\[
W_N(F) = \frac{2F}{\pi} \int_0^{\infty} t dt \sin(t|F|) \left[ \frac{(3-p)(Gm)^{3-p}/2}{2R^{3-p}} \int_0^{\infty} \frac{\sin z}{z^{7-p}/2} dz \right]^N
\]  
(20)

that reduces to Holtsmark’s law for \( N \to \infty \). In terms of \( \beta = \frac{F}{Gma^{1/(7-p)}} \) and \( y = ta^{1/(3-p)} \) Eq. 21 can be written as:

\[
W_N(\beta) = \frac{2\beta}{\pi} \int_0^{\infty} dy \sin(\beta y) \left[ \frac{y^{(3-p)/2}}{2N} \int_0^{\infty} \frac{y}{z{(7-p)/2}} dz \right]^N
\]  
(21)

Appendix B.

In this Appendix I calculate the probability distribution of stochastic force in a finite clustered system.

Following AA92 I suppose that particles have the density distribution:
\[ \tau(r) = \frac{a}{r^p} \exp \left( -\frac{r^2}{r_0^2} \right) \]  

(22)

where \( a \) and \( p \) and \( r_0 \) are three constants. The stochastic force distribution is given by:

\[ W_N(F) = \int \frac{d^3t}{(2\pi)^3} M_N(t) \exp(-itF) \]  

(23)

The term \( M_N \) is given by:

\[ M_N(t) = \frac{A_n(t)}{N} \left[ 1 + \frac{1}{2} \left( 1 - \frac{1}{N} \right) \Sigma(t) \right] \]  

(24)

(AA92), where \( A_n(t) \) is given by:

\[ A_n(t) = \left[ \frac{\alpha (Gmt)^{(3-p)/2}}{2} \int_{Gmt}^\infty \frac{dz}{z^{(7-p)/2}} \right]^N \]  

(25)

\( A_2(t) \) is given by Eq. 25 with \( n = 2 \) while \( \Sigma(t) \) is given in the quoted paper (Eq. 34) and \( \alpha = \frac{N}{2\pi r_0} \Gamma[(3-p)/2] \). Integrating equation Eq. 23 I finally obtain:

\[ W_N(F) = 4\pi^2 |F|^2 W_N(F) = \frac{2F}{\pi} \int_0^\infty t \sin(tF) \left[ \frac{\alpha}{N} (Gmt)^{(3-p)/2} \int_{Gmt}^\infty \exp\left( -\frac{Gmt}{r_0^2 z} \right) \frac{\sin z}{z^{(7-p)/2}} \right] \left[ 1 + \frac{1}{2} \left( 1 - \frac{1}{N} \right) \Sigma(t) \right] \]  

(26)

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Figure caption

**Figure 1** The theoretical probability distribution of stochastic force in an inhomogeneous system for different values of $N$. The solid line is Kandrup’s distribution for an infinite inhomogeneous system, the long dashed line is Kandrup’s distribution for a finite inhomogeneous system with $N = 50$, the dashed line is the same distribution for $N = 100$. The Kandrup’s distribution for a finite inhomogeneous system with $N = 1000$ is undistinguishable from that for the infinite system. The force is measured in units of $Gm\alpha^{2/(3-p)}$.

**Figure 2** Experimental distribution of the stochastic force in a inhomogeneous system. The solid line is Kandrup’s distribution for a finite inhomogeneous system ($p = 0.01$) of 10000 particles, the histogram is the experimental distribution of stochastic force obtained from a system of 10000 particles as described in the text. The force is measured in units of $\frac{GM}{R}$.

**Figure 3** Same as figure 4 but with $p = 0.1$.

**Figure 4** Same as figure 4 but with $p = 0.2$.

**Figure 5** Same as figure 4 but with $p = 0.5$.

**Figure 6** Same as figure 4 but with $p = 1$.

**Figure 7** Experimental distribution of stochastic force in a clustered system. The solid line is AA92 distribution for a clustered system of 10000 particles. The histogram is the experimental distribution of stochastic force obtained from a clustered system of 10000 particles as described in the text. The dot dashed line and histogram is a Holtsmark’s distribution for an homogeneous system of 10000 particles. It is introduced for a visual comparison. The force is measured in units of $\frac{GM}{R}$.
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