Bounds for the Rainbow Disconnection Numbers of Graphs

Xu Qing BAI        Zhong HUANG        Xue Liang LI¹)

Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, P. R. China
E-mail: baixqing@163.com  2120150001@mail.nankai.edu.cn  lxl@nankai.edu.cn

Abstract    An edge-cut of an edge-colored connected graph is called a rainbow cut if no two edges in
the edge-cut are colored the same. An edge-colored graph is rainbow disconnected if for any two distinct
vertices u and v of the graph, there exists a rainbow cut separating u and v. For a connected graph G,
the rainbow disconnection number of G, denoted by rd(G), is defined as the smallest number of colors
required to make G rainbow disconnected. In this paper, we first give some upper bounds for rd(G),
and moreover, we completely characterize the graphs which meet the upper bounds of the Nordhaus–
Gaddum type result obtained early by us. Secondly, we propose a conjecture that for any connected
graph G, either rd(G) = λ+(G) or rd(G) = λ+(G) + 1, where λ+(G) is the upper edge-connectivity, and
prove that the conjecture holds for many classes of graphs, which supports this conjecture. Moreover,
we prove that for an odd integer k, if G is a k-edge-connected k-regular graph, then χ′(G) = k if and
only if rd(G) = k. It implies that there are infinitely many k-edge-connected k-regular graphs G for
which rd(G) = λ+(G) for odd k, and also there are infinitely many k-edge-connected k-regular graphs
G for which rd(G) = λ+(G) + 1 for odd k. For k = 3, the result gives rise to an interesting result,
which is equivalent to the famous Four-Color Problem. Finally, we give the relationship between rd(G)
of a graph G and the rainbow vertex-disconnection number rvd(L(G)) of the line graph L(G) of G.

Keywords    Edge-coloring, edge-connectivity, rainbow disconnection coloring (number), line graph

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1 Introduction

All graphs considered in this paper are finite and undirected, and all graphs are simple unless
specifically stated. Let G = (V(G), E(G)) be a nontrivial connected graph with vertex-set
V(G) and edge-set E(G). For v ∈ V(G), let dG(v) and NG(v) (NG[v]) denote the degree and
the open (closed) neighborhood of v in G (or simply d(v) and N(v) (N[v]) respectively, when
the graph G is clear from the context). We use δ(G) and Δ(G) to denote the minimum and
maximum degree of G, respectively. For S ⊆ V(G), the subgraph of G induced by S is denoted
by G[S]. A u-v-path is a path with ends u and v. For a positive integer k, we always write [k]
for the set {1, 2, . . . , k} of integers. For any notation or terminology not defined here, we follow
those used in [8, 9].

Let G be a graph with an edge-coloring c: E(G) → [k], where adjacent edges may be colored
the same. When adjacent edges of G receive different colors by c, the edge-coloring c is called

¹) Corresponding author

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proper. The chromatic index of \( G \), denoted by \( \chi'(G) \), is the minimum number of colors needed in a proper edge-coloring of \( G \). A famous theorem due to Vizing [23] asserts that for any simple graph \( G \), either \( \chi'(G) = \Delta(G) \) or \( \chi'(G) = \Delta(G) + 1 \). If \( \chi'(G) = \Delta(G) \), then \( G \) is said to be in Class 1; if \( \chi'(G) = \Delta(G) + 1 \), then \( G \) is said to be in Class 2.

We know that there are two ways to study the connectivity of graphs, one is to use paths, and the other is to use cuts. The rainbow connection using paths has been studied extensively; see for examples, papers [11, 16, 18] and book [17] and the references therein. So, it is natural to consider rainbow edge-cuts for the colored connectivity in edge-colored graphs. In [10], Chartrand et al. first discussed the rainbow edge-cuts by introducing the concept of rainbow disconnection of graphs. In [4] we call all of them global colorings of graphs since they relate to global structural property: connectivity of graphs.

Recall that an edge-cut of a connected graph \( G \) is a set \( F \) of edges such that \( G - F \) is disconnected. For two distinct vertices \( u \) and \( v \) of \( G \), let \( \lambda_G(u, v) \) (or simply \( \lambda(u, v) \)) when the graph \( G \) is clear from the context) denote the minimum number of edges in an edge-cut \( F \) such that \( u \) and \( v \) lie in different components of \( G - F \). The minimum cardinality of an edge-cut of \( G \) is the edge-connectivity of \( G \), denoted by \( \lambda(G) \) (i.e., \( \lambda(G) \)) is the minimum value of \( \lambda_G(u, v) \) taken over all pairs of distinct vertices \( u, v \); whereas the maximum value of \( \lambda_G(u, v) \) taken over all pairs of distinct vertices \( u, v \) is the upper edge-connectivity of \( G \), denoted by \( \lambda^+(G) \). This graph parameter \( \lambda^+(G) \) was introduced and extensively studied in [6, 7]. The following proposition presents another interpretation for \( \lambda(u, v) \).

**Proposition 1.1** ([12, 13]) For every two vertices \( u \) and \( v \) in a graph \( G \), \( \lambda(u, v) \) is equal to the maximum number of pairwise edge-disjoint \( u \)-\( v \)-paths in \( G \).

The following concept of rainbow disconnection of graphs was introduced by Chartrand et al. in [11]. An edge-cut \( R \) of an edge-colored connected graph \( G \) is called a rainbow cut if no two edges in \( R \) are colored the same. Let \( u \) and \( v \) be two distinct vertices of \( G \). A rainbow edge-cut \( R \) of \( G \) is called a \( u \)-\( v \)-rainbow cut if \( u \) and \( v \) belong to different components of \( G - R \). An edge-colored graph \( G \) is called rainbow disconnected if for every two distinct vertices \( u \) and \( v \) of \( G \), there exists a \( u \)-\( v \)-rainbow cut in \( G \). In this case, the edge-coloring is called a rainbow disconnection coloring (or rd-coloring for short) of \( G \). For a connected graph \( G \), the rainbow disconnection number of \( G \), denoted by \( \text{rd}(G) \), is defined as the smallest number of colors required to make \( G \) rainbow disconnected. An optimal rd-coloring of \( G \) is an rd-coloring with \( \text{rd}(G) \) colors.

Similarly, in [3, 19] we introduce the concept of rainbow vertex-disconnection of graphs. For a connected and vertex-colored graph \( G \), let \( x \) and \( y \) be two vertices of \( G \). If \( x \) and \( y \) are nonadjacent, then an \( x \)-\( y \)-vertex-cut is a subset \( S \) of \( V(G) \) such that \( x \) and \( y \) belong to different components of \( G - S \). If \( x \) and \( y \) are adjacent, then an \( x \)-\( y \)-vertex-cut is a subset \( S \) of \( V(G) \) such that \( x \) and \( y \) belong to different components of \( (G - xy) - S \). A vertex subset \( S \) of \( G \) is rainbow if no two vertices of \( S \) have the same color. An \( x \)-\( y \)-rainbow vertex-cut is an \( x \)-\( y \)-vertex-cut \( S \) such that if \( x \) and \( y \) are nonadjacent, then \( S \) is rainbow; if \( x \) and \( y \) are adjacent, then \( S + x \) or \( S + y \) is rainbow. Here one can see that if \( x \) and \( y \) are adjacent, this really causes some inconvenience for the definition of rainbow \( x \)-\( y \)-vertex-cuts. This is done just for in accordance with the common sense connectivity of graphs.
A vertex-colored graph $G$ is called rainbow vertex-disconnected if for any two distinct vertices $x$ and $y$ of $G$, there exists an $x$-$y$-rainbow vertex-cut. In this case, the vertex-coloring $c$ is called a rainbow vertex-disconnection coloring (or rvd-coloring for short) of $G$. For a connected graph $G$, the rainbow vertex-disconnection number of $G$, denoted by $\text{rvd}(G)$, is the minimum number of colors required to make $G$ rainbow vertex-disconnected. An optimal rvd-coloring of $G$ is an rvd-coloring with $\text{rvd}(G)$ colors.

This paper is organized as follows. In Section 2, we obtain some upper bounds for $\text{rd}(G)$, and moreover, we completely characterize the graphs which meet the upper bound of the Nordhaus–Gaddum type result obtained early by us. In Section 3, we propose a conjecture that for any connected graph $G$, $\lambda^+(G) \leq \text{rd}(G) \leq \lambda^+(G)+1$, and prove that the conjecture holds for many classes of graphs, which supports this conjecture. Furthermore, for all odd $k \geq 1$, we give a sufficient and necessary condition for a $k$-edge-connected $k$-regular graph $G$ with $\text{rd}(G) = k$, from which we get that there are infinitely many $k$-edge-connected $k$-regular graphs $G$ for which $\text{rd}(G) = \lambda^+(G)$ for odd $k$, and also there are infinitely many $k$-edge-connected $k$-regular graphs $G$ for which $\text{rd}(G) = \lambda^+(G)+1$ for odd $k$. Moreover, we get a relationship between face-colorings and rainbow disconnection colorings of 3-connected cubic plane graphs. Finally, we give the relationship between $\text{rd}(G)$ of $G$ and $\text{rvd}(L(G))$ of the line graph $L(G)$ of $G$.

2 Some Upper Bounds for $\text{rd}(G)$

In this section, we obtain some upper bounds for the rainbow disconnection number of a graph $G$. Let $G$ be a graph and $X$ a proper subset of $V(G)$. To shrink $X$ is to delete all the edges between vertices of $X$ and then identify the vertices of $X$ into a single vertex. We denote the resulting graph by $G/X$. For each vertex $x$ of $G$, let $E_x$ be a set of all edges incident with $x$ in $G$. For an edge-colored graph $G$, a vertex $v$ of $G$ is proper if the colors of edges incident with $v$ are distinct in $G$. Now we give some upper bounds for $\text{rd}(G)$ in terms of the upper edge-connectivity. First, we give some useful lemmas and introduce a shrinking operation.

**Lemma 2.1** ([10]) If $G$ is a nontrivial connected graph, then

$$\lambda(G) \leq \lambda^+(G) \leq \text{rd}(G) \leq \chi'(G) \leq \Delta(G)+1.$$  

**Lemma 2.2** ([10])

(i) If $G$ is the Petersen graph, then $\text{rd}(G) = 4$.

(ii) If $W_n = C_{n-1} \cup K_1$ is the wheel of order $n \geq 4$, then $\text{rd}(W_n) = 3$.

**Lemma 2.3** ([2]) For a graph $G$, the following results hold.

(i) For any vertex $u$ of $G$, let $H = G - u$. Then $\text{rd}(G) \leq \Delta(H) + 1$.

(ii) If there exists a vertex $u$ of $G$ such that $H = G - u$ is in Class 1 and $d_H(x) \leq \Delta(H) - 1$ for each $x \in N_G(u)$, then $\text{rd}(G) \leq \Delta(H)$.

**Remark** 1 From the proof of Lemma 2.3 (i), we know that there exists an $\text{rd}$-coloring of $G$ using colors in $[\Delta(H)+1]$ such that each vertex is proper except the vertex $u$.

**Lemma 2.4** ([21]) Let $G$ be a loopless multigraph with maximum degree $\Delta(G)$. Then $\chi'(G) \leq \lfloor \frac{3}{2} \Delta(G) \rfloor$.

**Lemma 2.5** Let $G$ be a loopless multigraph and $H$ be a graph by shrinking a vertex subset of $G$ to a single vertex $h$. If $C_H(u,v)$ is a $u$-$v$-edge-cut in $H$, where $u,v \in V(H) \setminus h$, then it is also a $u$-$v$-edge-cut in $G$. 
Proof Let $H = G / Y$, where $Y \subseteq V(G)$. Assume that $C_H(u, v)$ is not a $u$-$v$-edge-cut in $G$, namely, there exists a $u$-$v$-path $P$ avoiding $C_H(u, v)$ in $G$. Then $P / Y$ is still a $u$-$v$-path in $H$ avoiding $C_H(u, v)$, which is a contradiction.

We define a shrinking operation on a graph $G$ as follows.

For a given graph $G$, let $\lambda^+(G) = k$ and $S = \{x \mid d(x) \geq k + 1\}$. For fixed $k$ and $S$, suppose $|S| \geq 2$. Let $u, v$ be two vertices of $S$. Then we can find a minimum $u$-$v$-edge-cut $C(u, v)$ such that $|C(u, v)| \leq \lambda^+(G)$ and $G \setminus C(u, v) = C_1 \cup C_2$. Then we define the two operations $o$ and $O$ as follows:

$$o(\{G\}) = \begin{cases} \{G / V(C_1), G / V(C_2)\}, & \text{if } |G \cap S| \geq 2, \\
\{G\}, & \text{otherwise.} \end{cases}$$

$$O(\{G_1, G_2, \ldots, G_p\}) = \bigcup_{i=1}^{p} o(\{G_i\}).$$

We keep the multiple edges in each operation. Since the graph is split into two pieces when we do the operation, the operation cannot last endlessly. Hence, there exists an integer $r$ such that $O^r(\{G\}) = O^{r+1}(\{G\})$. Finally, we get a finite set of connected graphs, where each graph has exactly one vertex with degree at least $\lambda^+(G) + 1$. We call this procedure of making a graph $G$ into such pieces the shrinking operation on $G$.

Then we derive the following theorem by the shrinking operation and Lemmas 2.4 and 2.5.

**Theorem 2.6** Let $G$ be a loopless multigraph with upper edge-connectivity $\lambda^+(G)$. Then $\text{rd}(G) \leq \left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor$.

**Proof** If $\Delta(G) = \lambda^+(G)$, then the result holds by Lemma 2.1. Otherwise, suppose that we get a family of graphs $H = \{H_1, H_2, \ldots, H_t\}$ by the shrinking operation on $G$ (if only one vertex has degree at least $\lambda^+(G) + 1$, then let $H = \{G\}$). Let $h_i$ be the unique vertex of $H_i$ with $d_{H_i}(h_i) \geq \lambda^+(G) + 1$ and $H'_i = H_i - h_i$ for each $i \in [t]$. Then $\Delta(H'_i) \leq \lambda^+(G)$ for each $i \in [t]$. It follows from Lemma 2.4 that $\chi'(H'_i) \leq \frac{3}{2} \Delta(H'_i) \leq \left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor$ for each $i \in [t]$. Then there exists a proper edge-coloring $f'_i$ for $H'_i$ using colors from $\left[\frac{3}{2} \lambda^+(G)\right]$ for each $i \in [t]$. For each graph $H_i$ ($i \in [t]$), we now define a coloring $f_i$ of $H_i$ as follows. Let $f_i(e) = f'_i(e)$ for each $e \in H'_i$, where $i \in [t]$. For each $u \in N_{H_i}(h_i)$ ($i \in [t]$), since $d_{H'_i}(u) \leq \Delta(H'_i) < \left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor$, there is an $a_u \in \left[\frac{3}{2} \lambda^+(G)\right]$ such that the color $a_u$ is not assigned to any edge incident with $u$ in $H_i$. Define $f_i(h_i, u) = a_u$ for each $i \in [t]$. We show that $f_i$ is an rd-coloring of $H_i$ for each $i \in [t]$. Let $w$ and $z$ be two distinct vertices of $H_i$ for some $i \in [t]$. Then at least one of the vertices $w$ and $z$ belongs to $H'_i$ ($i \in [t]$), say $w \in V(H'_i)$. Since the set $E_w$ separates $w$ and $z$ and is rainbow under $f_i$ in $H_i$ ($i \in [t]$), then $f_i$ is an rd-coloring of $H_i$ using colors from $\left[\frac{3}{2} \lambda^+(G)\right]$, and each vertex of $H_i$ is proper except vertex $h_i$ under $f_i$. Namely, $\text{rd}(H_i) \leq \left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor$ for each $i \in [t]$.

Now we claim that we can obtain an rd-coloring of $G$ using colors from $\left[\frac{3}{2} \lambda^+(G)\right]$ by adjusting colorings of shrunk graphs. Suppose that $F_1$ and $F_2$ are two graphs obtained by one operation for $F$ in terms of two vertices $x_1, x_2$ of $F$, where $d(x_1), d(x_2) \geq \lambda^+(G) + 1$ in $F$.

Without loss of generality, let $F_i = F / V(F_i)$ and $x_i \in F_i$ ($i \in [2]$). Suppose that for each $i \in [2], F_i$ has an rd-coloring $f_i$ using colors from $\left[\frac{3}{2} \lambda^+(G)\right]$. Let $y_i$ be the vertex by shrinking vertex-set $V(F_i)$ in $F$ ($i \in [2]$). Note that $d(y_i) \leq \lambda^+(G)$ in $F_i$ for each $i \in [2]$. So, $y_i \neq x_i$ and the vertex $y_i$ is proper in $F_i$. We can exchange the colors of $f_2$ for $F_2$ such that $c(e)|_{F_2} = c(e)|_{F_1}$
for each $e \in C(x_1, x_2)$ in $F$ and the new coloring $f_2$ is still an rd-coloring of $F_2$ using colors from $\left[\left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor \right]$. Then we obtain a coloring $f$ of $F$ using colors from $\left[\left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor \right]$ by letting $f(e) = f_1(e)$ if $e \in F_1$ and $f(e) = f_2(e)$ if $e \in F_2 \setminus F_1$. We now verify that the coloring $f$ is an rd-coloring of $F$. For any two vertices $p, q$ of $F$, if $p \in V(F_1)$ and $q \in V(F_2)$, then $C_{F}(x_1, x_2)$ is a $p-q$-rainbow cut in $F$; if $p, q$ belong to one of $V(F_1), V(F_2)$, without loss of generality, say $p, q \in V(F_1)$, then there exists a $p-q$-rainbow cut $C_{F}(p, q)$ in $F_1$ that is also a $p-q$-rainbow cut in $F$ by Lemma 2.5. Repeating the above inverse shrinking procedure, we finally get an rd-coloring of $G$ using colors from $\left[\left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor \right]$. Hence, $rd(G) \leq \left\lfloor \frac{3}{2} \lambda^+(G) \right\rfloor$. Moreover, for the Petersen graph $P$, we have $rd(P) = 4 = \left\lfloor \frac{3}{2} \lambda^+(P) \right\rfloor$ since $\lambda^+(P) = 3$. Thus, the upper bound is sharp in some sense.

Next we obtain another bound for $rd(G)$.

**Theorem 2.7** Let $G$ be a graph of order $n$ with maximum degree $\Delta(G)$ and upper edge-connectivity $\lambda^+(G)$. Then $rd(G) \leq \min\{n + \lambda^+(G) - \Delta(G) - 1, \Delta(G) + 1\}$. Furthermore, the bound is sharp in some sense.

**Proof** Let $v$ be a vertex with degree $\Delta(G)$ and $S = V(G) \setminus N[v]$. Then there exist at most $\lambda^+(G)$ edges from $x$ to $N[v]$ for each vertex $x$ of $S \cup N(v)$ in $G$ by the definition of upper edge-connectivity. Let $G' = G - v$. Observe that $\Delta(G') \leq \min\{n + \lambda^+(G) - \Delta(G) - 2, \Delta(G)\}$. So, $rd(G) \leq \Delta(G') + 1 = \min\{n + \lambda^+(G) - \Delta(G) - 1, \Delta(G) + 1\}$ by Lemma 2.3. Furthermore, it follows from Lemma 2.2 that $rd(G) = n + \lambda^+(G) - \Delta(G) - 1$ if $G$ is $K_{1, n-1}$ or $W_n$ and $rd(G) = \Delta(G) + 1$ if $G$ is the Petersen graph. The upper bound is sharp in some sense. □

In the rest of this section, we always assume that all graphs have at least four vertices, and that both $G$ and $\overline{G}$ are connected. For any vertex $u \in V(G)$, let $\bar{u}$ denote the vertex in $\overline{G}$ corresponding to the vertex $u$. We then characterize the graphs which meet the upper bounds of the Nordhaus–Gaddum type result obtained early by us. The following several lemmas will be used.

**Lemma 2.8** ([1]) Let $G$ be a connected graph and let $G_\Delta$ denote the subgraph of $G$ induced by the vertices of maximum degree $\Delta(G)$. If every connected component of $G_\Delta$ is a unicyclic graph or a tree, and $G_\Delta$ is not a disjoint union of cycles, then $G$ is in Class 1.

**Lemma 2.9** ([2]) Let $G$ be a connected graph of order $n$. If $rd(G) \geq n - 2$, then $G$ has at least two vertices of degree at least $n - 2$.

**Lemma 2.10** ([10]) If $H$ is a connected subgraph of a graph $G$, then $rd(H) \leq rd(G)$.

**Lemma 2.11** ([10]) Let $G$ be a connected graph, and let $B$ be a block of $G$ such that $rd(B)$ is maximum among all the blocks of $G$. Then $rd(G) = rd(B)$.

**Lemma 2.12** ([10]) Let $G$ be a connected graph of order $n \geq 2$. Then $rd(G) = n - 1$ if and only if $G$ has at least two vertices of degree $n - 1$.

**Lemma 2.13** Let $G$ be a graph of order $n \geq 3$. Then $rd(G) = n - 2$ if and only if one of the following conditions holds.

(i) $G$ has exactly one vertex of degree $n - 1$ and another vertex of degree $n - 2$.

(ii) $G$ is a graph with $\Delta(G) = n - 2$ and there are two nonadjacent vertices of degree $n - 2$ in $G$. 
(iii) $G$ is a graph not in (ii) with $\Delta(G) = n - 2$ and at least two maximum degree vertices. In addition, for some pair of vertices $u, v$ of degree $n - 2$, there is a vertex $z$ of $G$ such that $z \notin N(u) \cup N(v)$ or there are two distinct vertices $x, y$ such that $x \in N(u) \setminus N[v]$ and $y \in N(v) \setminus N[u]$ and $x, y$ belong to the same component of $G[V \setminus \{u, v\}]$.

Proof. For any graph that satisfies one of the conditions (i), (ii) and (iii), we first get that $rd(G) \leq n - 2$ by Lemma 2.12. Furthermore, we find that $\lambda^+(G) \geq n - 2$, so $rd(G) \geq n - 2$ by Lemma 2.1.

We now verify the converse. Assume, to the contrary, that there exists a graph $G$ with $rd(G) = n - 2$ but it does not satisfy any of the conditions (i), (ii) and (iii). Note that $G$ has at least two vertices of degree at least $n - 2$ by Lemma 2.9 and $G$ does not have two vertices of degree $n - 1$. Therefore $G$ satisfies the following two conditions:

1. $\Delta(G) = n - 2$ and there exists an edge for any two vertices of degree $n - 2$ in $G$.
2. $G$ has two distinct vertices $x, y$ such that $x \in N(u) \setminus N[v]$ and $y \in N(v) \setminus N[u]$ for any pair vertices $u, v$ of degree $n - 2$, and $x, y$ belong to different components of $G[V \setminus \{u, v\}]$.

We will show that the rainbow disconnection number of a graph $G$ satisfying conditions (1) and (2) is at most $n - 3$. We first present a claim as follows.

Claim 1. If $G$ be a graph satisfying conditions (1) and (2), then $d_G(a) \leq n - 3$ for each $a \in V(G) \setminus \{u, v\}$.

Proof of Claim 1. If $G[V \setminus \{u, v\}]$ has at least three components or two components where each part has at least two vertices, then $d_G(a) \leq n - 3$ for each $a \in V(G) \setminus \{u, v\}$. If $G[V \setminus \{u, v\}]$ has two components, one of which has exactly one vertex, then the vertex of the single vertex component is $x$ or $y$. Without loss of generality, let $x$ be the vertex in this single vertex component. Assume, to the contrary, that there exists a vertex $w$ of $V(G) \setminus \{u, v\}$ with $d_G(w) = n - 2$. Then $v, w$ are two vertices of degree $n - 2$ that do not meet condition (2) since $x \notin N(v) \cup N(w)$. This is a contradiction.

If $x$ or $y$ is a pendent vertex in $G$, without loss of generality, say $x$, then let $G' = G - x$. Then $d_G' \setminus \{a\} \leq n - 4$ for each $a \in V(G') \setminus v$. Thus, we have $rd(G') \leq n - 3$ by Lemma 2.3 (i). Furthermore, $rd(G) = rd(G') \leq n - 3$ by Lemma 2.11. Otherwise, suppose that $G_1$ is the component in $G[V \setminus \{u, v\}]$ that contains vertex $x$ and $G_2$ is the remaining components in $G[V \setminus \{u, v\}]$, and $|G_1|, |G_2| \geq 2$. Let $G_1 = G[V(G_1) \cup \{u, v\}]$ and $G_2 = G[V(G_2) \cup \{u, v\}]$. Observe that $\Delta(G_1), \Delta(G_2) \leq n - 3$ and $\Delta(G_1 \setminus u), \Delta(G_2 \setminus v) \leq n - 4$. Therefore, there is an rd-coloring $c_1$ ($c_2$) of $G_1$ ($G_2$) using colors from $[n - 3]$ such that each vertex of $G_1$ ($G_2$) is proper except for the vertex $u$ ($v$) by Remark 1. Then we can exchange the colors of $c_2$ for $G_2$ such that the colors of edges incident with $u$ in $G_2 \setminus v$ are different from the colors of edges incident with $v$ in $G_1$, and color the edge $uv$ in $G_2$ with the same color as $uv$ under $c_1$. The new coloring $c'_2$ is still an rd-coloring of $G_2$ using colors from $[n - 3]$ such that each vertex of $G_2$ is proper except for the vertex $v$. Then we get a coloring $c$ of $G$ by letting $c(e) = c_1(e)$ if $e \in G_1$ and $c(e) = c'_2(e)$ if $e \in G_2$ and $|c| = n - 3$. We can verify that coloring $c$ is an rd-coloring of $G$. Let $p, q$ be two vertices of $G$. If there exists a vertex of $\{p, q\}$ that does not belong to $\{u, v\}$, without loss of generality, say $p$, then the set $E_p$ is a $p$-$q$-rainbow cut in $G$. If $\{p, q\} = \{u, v\}$, then the set $E^1_u \cup E^2_u$ is a $p$-$q$-rainbow cut in $G$, where $E^1_u$ is the set of edges incident with vertex $v$ in $G_1$ and $E^2_u$ is the set of edges incident with vertex $u$ in $G_2$. Hence, $rd(G) \leq n - 3$. This is
a contradiction with our assumption. \qed

In [2], we obtained the Nordhaus–Gaddum type result for \( \text{rd}(G) \), and examples were given to show that the upper and lower bounds are sharp. However, we are not satisfied with these examples because they are special graphs. We restate it as follows.

**Lemma 2.14** ([2]) If \( G \) is a connected graph such that \( \overline{G} \) is also connected, then \( n - 2 \leq \text{rd}(G) + \text{rd}(\overline{G}) \leq 2n - 5 \) and \( n - 3 \leq \text{rd}(G) \cdot \text{rd}(\overline{G}) \leq (n - 2)(n - 3) \). Furthermore, these bounds are sharp.

Next we will completely characterize the graphs which meet the upper bounds in the above Nordhaus–Gaddum type result combining Lemma 2.13.

**Theorem 2.15** Let \( G \) be a graph of order \( n \geq 4 \). Then \( \text{rd}(G) + \text{rd}(\overline{G}) = 2n - 5 \) (or \( \text{rd}(G) \cdot \text{rd}(\overline{G}) = (n - 2)(n - 3) \)) if and only if one of \( G \) and \( \overline{G} \) satisfies the following three conditions:

(i) condition (ii) or (iii) in Lemma 2.13 holds;

(ii) it has exactly two vertices of degree \( n - 2 \), say \( u, v \);

(iii) it has at least two vertices of degree 2 except \( \{x, y\} \) or \( \{z\} \), where \( x \in N(u) \setminus N[v] \), \( y \in N(v) \setminus N[u] \) and \( z \notin N(u) \cup N(v) \).

**Proof** Without loss of generality, suppose that \( G \) satisfies the above three conditions. Obviously, \( \text{rd}(G) = n - 2 \) by Lemma 2.13. Since \( G \) has at least two vertices of degree 2 except \( \{x, y\} \) or \( \{z\} \), the graph \( \overline{G} \setminus \{u, v\} \) is of order \( n - 2 \) and has at least two vertices of degree \( n - 3 \). So, \( \text{rd}(\overline{G}) = \text{rd}(\overline{G} \setminus \{u, v\}) = n - 3 \) by Lemmas 2.11 and 2.12.

Conversely, assume that there exists a graph \( G \) with \( \text{rd}(G) + \text{rd}(\overline{G}) = 2n - 5 \). Since \( \overline{G} \) is connected, we have \( \text{rd}(\overline{G}) \leq n - 2 \) by Lemma 2.12. Therefore, it remains to consider the case \( \text{rd}(G) = n - 2 \), \( \text{rd}(\overline{G}) = n - 3 \) by symmetry. Similarly, if \( \text{rd}(G) \cdot \text{rd}(\overline{G}) = (n - 2)(n - 3) \), we only need to consider the case \( \text{rd}(G) = n - 2 \), \( \text{rd}(\overline{G}) = n - 3 \) by symmetry. Obviously, \( G \) satisfies (ii) or (iii) of Lemma 2.13. If \( G \) has more than 2 vertices with degree \( n - 2 \), then \( \overline{G} \) has at least 3 vertices with degree 1. Then \( \text{rd}(\overline{G}) \leq n - 4 \) by Lemmas 2.10 and 2.11, this is a contradiction. Thus, condition (ii) holds. To prove (iii), assume, to the contrary, that \( G \) has at most one vertex of degree 2 except \( \{x, y\} \) or \( \{z\} \). Then \( \overline{G} \setminus \{u, v\} \) has at most one vertex of degree at least \( n - 3 \). Since \( d_{\overline{G}}(u) = d_{\overline{G}}(v) = 1 \), we have \( \text{rd}(\overline{G}) = \text{rd}(\overline{G} \setminus \{u, v\}) \leq n - 4 \) by Lemmas 2.11 and 2.12. This is a contradiction with our condition. \qed

3 Graphs with \( \text{rd}(G) \leq \lambda^+(G) + 1 \)

First, we recall some known results.

**Lemma 3.1** ([2]) If \( G \) is a connected \( k \)-regular graph, then \( k \leq \text{rd}(G) \leq k + 1 \).

**Lemma 3.2** ([2]) If \( G = K_{n_1, n_2, \ldots, n_k} \) is a complete \( k \)-partite graph of order \( n \), where \( k \geq 2 \) and \( n_1 \leq n_2 \leq \cdots \leq n_k \), then

\[
\text{rd}(K_{n_1, n_2, \ldots, n_k}) = \begin{cases} n - n_2, & \text{if } n_1 = 1, \\ n - n_1, & \text{if } n_1 \geq 2. \end{cases}
\]

**Lemma 3.3** ([10]) The rainbow disconnection number of the grid graph \( G_{m, n} \) is as follows.

(i) For all \( n \geq 2 \), \( \text{rd}(G_{1, n}) = \text{rd}(P_n) = 1 \).

(ii) For all \( n \geq 3 \), \( \text{rd}(G_{2, n}) = 3 \).
Lemma 3.6 is a tree.

Observe that \( \text{rd}(G) \leq \lambda^+(G) + 1 \) for all connected regular graphs, complete multipartite graphs and grid graphs. Therefore, we propose the following conjecture.

**Conjecture 3.4** Let \( G \) be a connected graph with upper edge-connectivity \( \lambda^+(G) \). Then \( \lambda^+(G) \leq \text{rd}(G) \leq \lambda^+(G) + 1 \).

Obviously, the lower bound is always true by Lemma 2.1. Furthermore, we give some classes of graphs that support the upper bound of the conjecture. The following are some useful lemmas which will be used in the sequel.

**Lemma 3.5** ([10]) Let \( G \) be a nontrivial connected graph. Then \( \text{rd}(G) = 1 \) if and only if \( G \) is a tree.

**Lemma 3.6** ([10]) Let \( G \) be a nontrivial connected graph. Then \( \text{rd}(G) = 2 \) if and only if each block of \( G \) is either \( K_2 \) or a cycle and at least one block of \( G \) is a cycle.

**Lemma 3.7** ([20]) Let \( G \) be a graph of order \( n \) \((n \geq k + 2 \geq 3)\). If \( |E(G)| > \frac{k+1}{2}(n-1) - \frac{1}{2}\sigma_k(G) \), where \( \sigma_k(G) = \sum_{x \in V(G), d(x) \leq k} (k - d(x)) \), then \( \lambda^+(G) \geq k + 1 \).

We recall some notions of graphs from [14]. A simple graph \( G \) is overfull if \( |E(G)| > \frac{|V(G)|}{2}\Delta(G) \). A graph \( G \) is subgraph-overfull if it has an overfull subgraph \( H \) with \( \Delta(H) = \Delta(G) \). Obviously, every overfull graph is subgraph-overfull. For subgraph-overfull graphs and graphs whose maximum degree does not exceed 3, we have the following results, which support Conjecture 3.4.

**Theorem 3.8** Let \( G \) be a subgraph-overfull graph with upper edge-connectivity \( \lambda^+(G) \). Then \( \text{rd}(G) \leq \lambda^+(G) + 1 \).

**Proof** Let \( H \) be an overfull subgraph of \( G \) with \( \Delta(H) = \Delta(G) \). Then \( |E(H)| > \frac{|V(H)|}{2} \Delta(H) \geq \frac{|V(H)|}{2} \Delta(H) \). Combining Lemma 3.7, we have \( \Delta(G) \geq \lambda^+(G) \geq \lambda^+(H) \geq \Delta(H) = \Delta(G) \). So, \( \lambda^+(G) = \Delta(G) \). It follows from Lemma 2.1 that \( \text{rd}(G) \leq \lambda^+(G) + 1 \).

**Theorem 3.9** Let \( G \) be a graph with \( \Delta(G) \leq 3 \). Then \( \text{rd}(G) \leq \lambda^+(G) + 1 \).

**Proof** Obviously, \( \lambda^+(G) \leq \Delta(G) \leq 3 \). If \( \lambda^+(G) = 1 \), we get that \( G \) is a tree. It follows from Lemma 3.5 that \( \text{rd}(G) = 1 \). If \( \lambda^+(G) = 2 \), \( G \) must contain a cycle and any cycle of \( G \) does not have a chord, namely, \( G \) is a graph that each block of \( G \) is a cycle or \( K_2 \) and at least one block of \( G \) is a cycle. Then \( \text{rd}(G) = 2 \) by Lemma 3.6. If \( \lambda^+(G) = 3 \), we have \( \text{rd}(G) \leq \Delta(G) + 1 = 4 \) by Lemma 2.1.

For a graph with a large maximum degree, we get the following result.

**Theorem 3.10** Let \( G \) be a graph with order \( n \) and \( \Delta(G) \geq n - 3 \). Then \( \text{rd}(G) \leq \lambda^+(G) + 1 \).

**Proof** Let \( d(u) = \Delta(G) \) and \( G' = G - u \). Suppose \( \lambda^+(G) = k \). If \( \Delta(G) \geq n - 2 \), we have \( \Delta(G') \leq k \); otherwise, let \( v \) be a vertex with \( d_{G'}(v) \geq k + 1 \). Then we have \( \lambda^+(u, v) \geq k + 1 \), which is a contradiction. Thus, \( \text{rd}(G) \leq \Delta(G') + 1 \leq k + 1 \) by Lemma 2.3.

If \( \Delta(G) = n - 3 \), let \( d(u) = n - 3 \) and let \( p, q \) be two vertices which are not adjacent to \( u \), i.e., \( V(G) = N[u] \cup \{p, q\} \). Note that \( d_G(x) \leq k + 2 \) for each \( x \in N(u) \) and \( d_G(p), d_G(q) \leq k + 1 \) since \( \lambda^+(G) = k \). Thus, \( \Delta(G') \leq k + 1 \). We distinguish the following cases to discuss.
Case 1 $\Delta(G') \leq k$.

It follows from Lemma 2.3 that $\text{rd}(G) \leq \Delta(G') + 1 \leq k + 1$.

Case 2 $\Delta(G') = k + 1$.

Let $D = \{x \mid x \in V(G') \text{ and } d_{G'}(x) = k + 1\}$. If $D \subseteq \{p, q\}$, then $G'[D]$ is $K_1$ (otherwise, $\lambda(p, q) = k + 1$, a contradiction). Thus, it follows from Lemma 2.8 that $G'$ is in Class 1. Moreover, $d_{G'}(x) \leq \Delta(G') - 1$ for each $x \in N_G(u)$. So, $\text{rd}(G) \leq \Delta(G') = k + 1$ by Lemma 2.3.

Suppose $D \cap N(u) \neq \emptyset$. We claim $|D \cap N(u)| = 1$. Assume that there are at least two vertices in $D \cap N(u)$, say $x_1, x_2$. Note that $d_{N(u)}(x_1) = d_{N(u)}(x_2) = k - 1$. So, $\{p, q\} \subseteq N(x_i)$ for each $i \in [2]$. Then we find $\lambda_G(x_1, x_2) \geq k + 1$, which is a contradiction. Let $D \cap N(u) = \{a\}$. Then $\{p, q\} \subseteq N(a)$ since $d_{N(u)}(a) \leq k - 1$. Let $R = N(u) \setminus N[a]$, $T = N(p) \cup N(q)$. Note that $R \cap T = \emptyset$ and there is no edge between $R$ and $T \cup \{p, q\}$. Assume that $R \cap T \neq \emptyset$ or there exists a vertex of $R$ adjacent to a vertex of $T \cup \{p, q\}$. Then we have $\lambda^+(u, a) \geq k + 1$, which is a contradiction. Thus, $T \subseteq N[a]$. Let $S = N[a] \setminus T$. If there exists a vertex $s \in S$ such that $s$ belongs to a component with a vertex of $R$ in $G[R \cup S]$, then let $s \in S_1$ and $S_2 = S \setminus S_1$. Let $G_1 = G[R \cup S_1 \cup \{u, a\}] - ua$ and $G_2 = G[T \cup S_2 \cup \{u, p, q\}]$. Write $G'_1 = G_1 - u$ and $G'_2 = G_2 - a$. Observe that $\Delta(G'_1), \Delta(G'_2) \leq k$. By Lemma 2.3 and Remark 1, there exists an rd-coloring $c_i$ of $G_i$ ($i \in [2]$) using colors from $[k + 1]$, moreover, $x$ is proper for each $x \in V(G_1) \setminus \{u\}$ ($x \in V(G_2) \setminus \{a\}$) in coloring $c_1$ of $G_1$ ($c_2$ of $G_2$). Since $|E(u, S_2 \cup T) \cup E(S_1, a)| = k$, we can exchange colors of $c_2$ such that $E(u, S_2 \cup T) \cup E(S_1, a)$ have distinct colors. Then we get a coloring $c$ of $G$ by identifying the graphs $G_1$ and $G_2$ using colors from $[k + 1]$.

Furthermore, we can verify that the coloring $c$ is an rd-coloring of $G$. For any two vertices $w, z$ of $G$, if there exists a vertex not in $\{u, a\}$, say $w$, then the set $E_w$ is a $w-z$-rainbow cut in $G$; if $\{w, z\} = \{u, a\}$, then $E(u, S_2 \cup T) \cup E(S_1, a)$ is a $u-a$-rainbow cut in $G$. Hence, $\text{rd}(G) \leq k + 1$. \hfill $\square$

By Theorem 3.9 and Theorem 3.10, we know that Conjecture 3.4 holds for small graphs.

Corollary 3.11 Let $G$ be a graph of order $n \leq 7$. Then $\text{rd}(G) \leq \lambda^+(G) + 1$.

For a $k$-regular graph $G$, we know that $\lambda^+(G) = k$ by Lemma 3.7, and then the conjecture holds by Lemma 3.1. In particular, we want to further know which $k$-regular graphs satisfy $\text{rd}(G) = k$. In [2], we presented some results for this question. For all odd $k \geq 1$, we now deduce the following result for $k$-edge-connected $k$-regular graphs.

Theorem 3.12 Let $k$ be an odd integer, and $G$ be a $k$-edge-connected $k$-regular graph. Then $\chi'(G) = k$ if and only if $\text{rd}(G) = k$.

Proof Suppose, first, that $\chi'(G) = k$. By Lemma 2.1, we have $k = \lambda(G) \leq \text{rd}(G) \leq \chi'(G) = k$. Thus, $\text{rd}(G) = k$.

Conversely, suppose $\text{rd}(G) = k$ and let $c$ be an optimal rd-coloring of $G$. If $G$ has a $k$-rainbow cut $T$ such that $G \setminus T$ has two non-trivial components, say $G_1, G_2$, then we do an operation $f$, i.e., the graph $G$ shrinks $V(G_1), V(G_2)$ to vertices $x_1, x_2$, respectively. The resulting edge-colored graphs are denoted by $G/V(G_1), G/V(G_2)$, respectively. Furthermore, the obtained edge-colored graphs $G/V(G_1)$ and $G/V(G_2)$ are both $k$-edge-connected $k$-regular. Assume, to the contrary, that there exists a $u-v$-edge-cut $V$ in $G/V(G_1)$, where $u, v \in G/V(G_1)$ and
$|V| < k$. If $x_1 \notin \{u, v\}$, then $V$ is also a $u$-$v$-edge-cut in $G$ by Lemma 2.5, which contradicts the condition; if one of $\{u, v\}$ is $x_1$, without loss of generality, say $u = x_1$, then $V$ is a $w$-$v$-edge-cut in $G$, where $w$ is any vertex of $G(V_1)$. Similarly, this is a contradiction.

Claim 1 The coloring $c$ of $G$ restricted to $G/V(G_1)$ is an rd-coloring of $G/V(G_1)$.

Proof of Claim 1 Note that $V(G/V(G_1)) = V(G_2) \cup \{x_1\}$. Let $u, v$ be two vertices of $G/V(G_1)$. Suppose $u, v \in V(G_2)$. Let $W$ be a $u$-$v$-rainbow cut in $G$ and let $W_H$ be the set of edges of $W \cap H$ for any subgraph $H$ of $G$. Since $G_1, G_2$ are both $\lceil \frac{k}{2} \rceil$-connected, we have $|W_{G_2}| \geq \lceil \frac{k}{2} \rceil$.

Now we show $W \subseteq G_2 \cup T$. If the remaining edges of $W$ are all in $G_1$, then there is still a $u$-$v$-path in $G \setminus W$ since $G_1$ is $\lceil \frac{k}{2} \rceil$-connected and $|W_{G_1}| \leq \lfloor \frac{k}{2} \rfloor < \lfloor \frac{k}{2} \rfloor$ for $k$ odd, which is a contradiction. If $G_1$ and $T$ both have edges of $W$, without loss of generality, suppose that $|W_{G_1}| = s$, $|W_T| = t$ and $|W_{G_2}| = r$, where $0 < t, s < \lceil \frac{k}{2} \rceil$, $s + t \leq \lfloor \frac{k}{2} \rfloor$ and $r + s + t = k$. When we remove the set $W_T$ from $G$, at most $t$ $u$-$v$-paths that go through $T$ are destroyed. However, there are $s + t$ $u$-$v$-paths going through $T$ in $G$, and so at least one $u$-$v$-path goes through $T \setminus W$ in $G$ since $s \geq 1$. Moreover, $G_1 \setminus W$ is connected since $|W_{G_1}| < \lfloor \frac{k}{2} \rfloor$ and $G_1$ is $\lceil \frac{k}{2} \rceil$-connected.

So, there is at least one $u$-$v$-path in $G \setminus W$, which is a contradiction. Therefore, $W \subseteq G_2 \cup T$. Then $W$ is a $u$-$v$-rainbow cut of $G/V(G_1)$, otherwise, if $G/V(G_1)$ has a $u$-$v$-path avoiding the set $W$, then there exists a $u$-$v$-path in $G \setminus W$ since $G_1$ is connected, which is a contradiction. If $x_1 \in \{u, v\}$, then the set $E_{x_1}$ is a $u$-$v$-rainbow cut of $G/V(G_1)$. The proof is complete. □

Repeating the operation $f$ until the obtained edge-colored graphs do not satisfy the condition of operation $f$, the resulting edge-colored $k$-edge-connected $k$-regular graphs are denoted by $\mathcal{F} = \{F_i \mid i \in [\ell]\}$.

Claim 2 The coloring of the graph $F_i$ in $\mathcal{F}$ is a proper coloring of $F_i$ for each $i \in [\ell]$.

Proof of Claim 2 Assume that there exists a graph $F_i$ for some $i \in [\ell]$ whose coloring is not proper. If $F_i$ $(i \in [\ell])$ has two vertices, say $p, q$, which are not proper, then there exists a $p$-$q$-rainbow cut $Z$ in $F_i$ that are not $E_p$ or $E_q$. Thus, we get that $Z$ is a rainbow cut in $F_i$ such that $F_1 \setminus Z$ has two non-trivial components, which is a contradiction to our operation. Hence, $F_i$ has at most one vertex, say $b_i$, which is not proper for each $i \in [\ell]$. Given an $i \in [\ell]$, let $k_i (t \in [k])$ be the number of edges incident with vertex $b_i$ and with color $t$ in $F_i$, and moreover, let $F_{i,A_j}$ be a subgraph of $F_i$ induced by the set of edges with colors in $A_j$, where $A_j$ is the color set $[k] \setminus \{j\}$ for some $j \in [k]$. Then for the graph $F_i (i \in [\ell])$, we get $(k - 1)(|F_i| - 1) + \sum_{t \in A_j} k_t \equiv 0 \pmod 2$ since the sum of degrees of vertices in $F_{i,A_j}$ is even for each $j \in [k]$. Therefore, we have $\sum_{t \in A_1} k_t \equiv \sum_{t \in A_2} k_t \equiv \cdots \equiv \sum_{t \in A_k} k_t \equiv 0 \pmod 2$ since $(k - 1)(|F_i| - 1)$ is even, which gives $k_1 \equiv k_2 \equiv \cdots \equiv k_k \equiv 0 \pmod 2$. Since $\sum_{i=1}^k k_i = k$ is odd, we obtain that $k_1 \equiv k_2 \equiv \cdots \equiv k_k \equiv 1 \pmod 2$, and then $k_1 = k_2 = \cdots = k_k = 1$. So, the vertex $b_i$ is also proper in $F_i$ for each $i \in [\ell]$.

For each vertex $x$ of $G$, the colors of edges incident with vertex $x$ are not changed in each operation $f$. Thus, the optimal rd-coloring $c$ of $G$ is a proper coloring of $G$, i.e., $\chi'(G) \leq k$. Combining Lemma 2.1, we get $\chi'(G) = k$.

Remark 2 For odd $k$, any optimal rd-coloring of a $k$-edge-connected $k$-regular graph $G$ is a proper coloring of $G$. However, for even $k$ the argument does not follow; see the following example. Let $G$ be a $k$-edge-connected $k$-regular graph with $V(G) = \{u_i, v_i \mid i \in [2r]\} \cup \{p_i, q_i \mid i \in [s]\}$, where $r, s < \lfloor \frac{k}{2} \rfloor$ and $r + s = \lfloor \frac{k}{2} \rfloor$. The coloring of the graph $G$ restricted to $G/V(G_1)$ is an rd-coloring of $G/V(G_1)$.
\[2r-2\] \text{and } E(G) = \{p_iu_j, q_iv_j, u_jv_i | i \in [2r-2], j \in [2r]\} \cup \{u_iu_{i+r}, v_iv_{i+r} | i \in [r]\}.\] Observe that \(k = 2r\). We give a coloring \(c\) of \(G\) as follows.

\* \(c(p_iu_j) = c(p_iu_{i+r}) = i, i \in [r]\);
\* \(c(u_iu_{i+r}) = r + i, i \in [r]\);
\* \(c(u_iv_i) \equiv i + 1 \pmod{2r}, i \in [2r]\);
\* \(c(u_ip_j) \equiv i + j \pmod{2r}, 2 \leq j \leq r - 1\);
\* \(c(u_ip_j) \equiv i + j + 1 \pmod{2r}, r \leq j \leq 2r - 2\);
\* the coloring of the remaining edges can be obtained symmetrically.

Note that \(|c| = 2r\) and each vertex of \(G\) is proper except vertices \(p_i, q_i\) under the coloring \(c\). Therefore, it is easy to verify that \(c\) is an optimal \(rd\)-coloring of \(G\), but it is not a proper coloring of \(G\).

\textbf{Remark 3} From Vizing’s theorem we know that for a \(k\)-regular graph \(G, \chi'(G) = k\) or \(k + 1\). Since for \(k = 3\) determining \(\chi'(G) = 3\) or 4 is NP-complete [15], there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(\chi'(G) = k\) for odd \(k\), and also there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(\chi'(G) = k + 1\) for odd \(k\). So, from our Theorem 3.12 one can get that there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(rd(G) = k\) for odd \(k\), and also there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(rd(G) = k + 1\) for odd \(k\). Since for \(k\)-edge-connected \(k\)-regular graphs \(G\) one has \(\lambda^+(G) = k\), we then get that there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(rd(G) = \lambda^+(G)\) for odd \(k\), and also there are infinitely many \(k\)-edge-connected \(k\)-regular graphs \(G\) for which \(rd(G) = \lambda^+(G) + 1\) for odd \(k\).

Obviously, for \(k = 3\), we get the following corollary from Theorem 3.12.

\textbf{Corollary 3.13} ([2]) \textit{Let }\(G\) \textit{be a }\(3\)-\textit{edge-connected cubic graph. Then }\(\chi'(G) = 3\) \textit{if and only if }\(rd(G) = 3\).

For planar graphs, Corollary 3.13 gives rise to an interesting result, which is equivalent to the famous Four-Color Problem. We know the Four-Color Problem is equivalent to the following statement (For a history of the Four-Color Problem, see Biggs et al. [5] or Wilson [24].)

\textbf{Problem 3.14} ([5, 8, 24]) \textit{Every plane graph without cut edges is }\(4\)-\textit{face-colorable.}

The Four-Color Problem is equivalent to the assertion that every \(3\)-connected cubic plane graph is \(4\)-face-colorable. Tait [22] found a surprising relationship between face-colorings and edge-colorings of \(3\)-connected cubic plane graphs.

\textbf{Theorem 3.15} ([22]) \textit{A }\(3\)-\textit{connected cubic plane graph }\(G\) \textit{is }\(4\)-\textit{face-colorable if and only it is }\(3\)-\textit{edge-colorable}, i.e., \(\chi'(G) = 3\).

Moreover, we prove that a cube graph is \(3\)-connected is equivalent to that it is \(3\)-edge-connected.

\textbf{Theorem 3.16} \textit{A cube graph }\(G\) \textit{is }\(3\)-\textit{connected if and only if }\(G\) \textit{is }\(3\)-\textit{edge-connected.}

\textbf{Proof} First, let \(G\) be a \(3\)-connected cube graph. Obviously, the \(G\) is \(3\)-edge-connected. Conversely, suppose that \(G\) is a \(3\)-edge-connected cube graph. We now prove that \(G\) is \(3\)-connected. Assume, to the contrary, that \(G\) has a \(2\)-vertex-cut \(\{u, v\}\). Since \(G\) is a \(3\)-regular graph, we can find a \(2\)-edge-cut incident with vertex \(\{u, v\}\). This is a contradiction with our assumption. \(\square\)
Therefore, we get the following result.

**Lemma 3.17** Let \( G \) be a 3-connected cubic graph. Then \( \chi'(G) = 3 \) if and only if \( \text{rd}(G) = 3 \).

Combining Theorem 3.15 and Lemma 3.17, we get a relationship between face-colorings and rainbow disconnection colorings of 3-connected cubic plane graphs.

**Theorem 3.18** A 3-connected cubic plane graph \( G \) is 4-face-colorable if and only if \( \text{rd}(G) = 3 \).

## 4 Relationship of \( \text{rd}(G) \) and \( \text{rvd}(L(G)) \)

The line graph \( L(G) \) of a graph \( G \) has the edges of \( G \) as its vertices, and two distinct edges of \( G \) are adjacent in \( L(G) \) if and only if they share a common vertex in \( G \). Now, we study the relationship between \( \text{rd}(G) \) and \( \text{rvd}(L(G)) \).

**Lemma 4.1** ([3]) For an integer \( n \geq 2 \),

\[
\text{rvd}(K_n) = \begin{cases} 
\ n - 1, & \text{if } n = 2, 3, \\
\ n, & \text{if } n \geq 4.
\end{cases}
\]

**Theorem 4.2** Let \( G \) be a graph and \( L(G) \) be the line graph of \( G \). Then \( \text{rd}(G) \leq \text{rvd}(L(G)) \).

**Proof** Let \( c_0 \) be an optimal \( \text{rvd} \)-coloring of the line graph \( L(G) \). Then we get an edge-coloring \( c \) of \( G \) corresponding to the vertex-coloring \( c_0 \) of \( L(G) \). We can verify that \( c \) is an \( \text{rd} \)-coloring of \( G \). For any two vertices \( u, v \) of \( G \), if \( uv \) is not a pendent edge, we can find two edges \( e_1, e_2 \) incident with vertices \( u, v \), respectively, and the edge \( e_1 \) (or \( e_2 \)) does not have two ends as \( u, v \).

Suppose that \( e_1 = ux \) and \( e_2 = vy \), where \( x, y \in V(G) \setminus \{u, v\} \) and \( x, y \) could be the same vertex.

We know that \( e_1, e_2 \) correspond to two vertices of \( L(G) \), denoted by \( a \) and \( b \). We claim that the edge-set \( S \) of \( G \) which corresponds to an \( a-b \)-rainbow vertex-cut \( S' \) in \( L(G) \) is a \( u-v \)-rainbow cut in \( G \). Assume that there still exists a \( u-v \)-path \( P \) in \( G \) which avoids the edge-set \( S \) of \( G \). Then the \( u-v \)-path \( P \) in \( G \) corresponds to an \( a-b \)-path \( P' \) which avoids the vertex set \( S' \) in \( L(G) \).

This is a contradiction. If \( uv \) is a pendent edge of \( G \), then \( uv \) is a \( u-v \)-rainbow cut in \( G \). \( \square \)

We know that the chromatic index of \( G \) is equal to the chromatic number of \( L(G) \). Similarly, we want to know whether \( \text{rd}(G) = \text{rvd}(L(G)) \) for any graph \( G \). However, the equality is not always true. For the moment we obtain the following necessary condition for the equality.

**Theorem 4.3** Let \( G \) be a graph with \( \delta(G) \geq 4 \) and \( L(G) \) be the line graph of \( G \). If \( \text{rd}(G) = \text{rvd}(L(G)) \), then \( \text{rd}(G) = \chi'(G) \).

**Proof** Assume, to the contrary, that \( \text{rd}(G) < \chi'(G) \). Then \( \text{rvd}(L(G)) < \chi(L(G)) \) since \( \chi'(G) = \chi(L(G)) \). Let \( \text{rvd}(L(G)) = t \) and \( c \) be any vertex-coloring of \( L(G) \) using colors from \([t]\). Then there exist two adjacent vertices \( u, v \) which have the same color in \( L(G) \). Observe that \( u, v \) must be contained in some \( K_\ell \) \( (\ell \geq 4) \) of \( L(G) \) since \( \delta(G) \geq 4 \), and the \( K_\ell \) has at most \( \ell - 1 \) colors in \( L(G) \). Thus, any \( t \)-vertex-coloring \( c \) of \( L(G) \) is not an \( \text{rvd} \)-coloring of \( L(G) \) by Lemma 4.1. This contradicts \( \text{rvd}(L(G)) = t \). \( \square \)

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