ENERGY-MOMENTUM CONSERVATION LAW
IN HAMILTONIAN FIELD THEORY

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Abstract
In the Lagrangian field theory, one gets different identities for different stress
energy-momentum tensors, e.g., canonical energy-momentum tensors. Moreover,
these identities are not conservation laws of the above-mentioned energy-momentum
tensors in general. In the framework of the multimomentum Hamiltonian formalism,
we have the fundamental identity whose restriction to a constraint space can be
treated the energy-momentum conservation law. In standard field models, this
appears the metric energy-momentum conservation law.

1 Introduction

We follow the generally accepted geometric description of classical fields by sections of
fibred manifolds $Y \to X$. Their dynamics is phrased in terms of jet spaces (see [2, 3, 4, 5
for the bibliography). Given a fibred manifold $Y \to X$, the $k$-order jet space $J^k Y$ of
$Y$ comprises the equivalence classes $j^k_x s$, $x \in X$, of sections $s$ of $Y$ identified by the first
$(k+1)$ terms of their Taylor series at a point $x$. One exploits the well-known facts that: (i)
the $k$-jet space of sections of a fibred manifold $Y$ is a finite-dimensional smooth manifold
and (ii) a $k$-order differential operator on sections of a fibred manifold $Y$ can be described
as a morphism of $J^k Y$ to a vector bundle over $X$. As a consequence, the dynamics of field
systems is played out on finite-dimensional configuration and phase spaces. Moreover, this
dynamics is phrased in the geometric terms due to the 1:1 correspondence between the
sections of the jet bundle $J^1 Y \to Y$ and the connections on the fibred manifold $Y \to X$
[6, 7, 8].

In field theory, we can restrict ourselves to the first order Lagrangian formalism when
the configuration space is $J^1 Y$. Given fibred coordinates $(x^\mu, y^i)$ of $Y$, the jet space $J^1 Y$
is endowed with the adapted coordinates $(x^\mu, y^i, y^i_\mu)$. A first order Lagrangian density on
the configuration space $J^1 Y$ is represented by a horizontal exterior density

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu) \omega, \quad \omega = dx^1 \wedge \ldots \wedge dx^n, \quad n = \text{dim } X.$$  

The corresponding first order Euler-Lagrange equations for sections $\vec{s}$ of the fibred jet
manifold $J^1 Y \to X$ read

$$\partial_\lambda \vec{s}^j = \vec{s}^j_\lambda,$$
$$\partial_i \mathcal{L} - (\partial_\lambda + \vec{s}^i_\lambda \partial_j + \partial_\lambda \vec{s}^i_\mu \partial_j^\mu) \partial_\lambda^j \mathcal{L} = 0.$$  (1)
Conservation laws are usually related with symmetries. There are several approaches to examine symmetries in the Lagrangian formalism, in particular, in jet terms [3, 4, 12]. We consider the Lie derivatives of Lagrangian densities in order to obtain differential conservation laws. These are conservation laws of Noether currents and the energy-momentum conservation laws.

If the Lie derivative of a Lagrangian density $L$ along a vertical vector field $u_G$ on a bundle $Y$ is equal to zero, the current conservation law on solutions of the first order Euler-Lagrange equations (1) takes place. In this case, the vertical vector field $u_G$ plays the role of a generator of internal symmetries.

In case of the energy-momentum conservation law, a vector field on $Y$ is not vertical and the Lie derivative of $L$ does not vanish in general. Therefore, this conservation law is not related with a symmetry. Moreover, one can not say a priori what is conserved.

In gravitation theory, the first integral of gravitational equations is the conservation law of the metric energy-momentum tensor of matter, but only in the presence of the gravitational field generated by this matter. In other models, the metric energy-momentum tensor holds a posteriori.

Let

$$u = u^\mu \partial_\mu + u^i \partial_i$$

be a vector field on a fibred manifold $Y$ and $\pi$ its jet lift onto the fibred jet manifold $J^1Y \to X$. Given a Lagrangian density $L$ on $J^1Y$, let us computer the Lie derivative $L_{\pi} L$. On solutions $s$ of the first order Euler-Lagrange equations (1), we have the equality

$$s^* L_{\pi} L = \frac{d}{dx}[\pi^\lambda (s)(u^i - u^\mu \pi^i_{\mu}) + u^\lambda \mathcal{L}(s)] \omega, \quad \pi^\mu_i = \partial^\mu_i \mathcal{L}. \quad (2)$$

In particular, if $u$ is a vertical vector field such that

$$L_{\pi} L = 0,$$

the equality (2) takes the form of the current conservation law

$$\frac{d}{dx}[u^i \pi^\lambda (s)] = 0. \quad (3)$$

In gauge theory, this conservation law is exemplified by the Noether identities.

Let

$$\tau = \tau^\lambda \partial_\lambda$$

be a vector field on $X$ and

$$u = \tau_\Gamma = \tau^\mu (\partial_\mu + \Gamma^i_{\mu} \partial_i)$$

its horizontal lift onto the fibred manifold $Y$ by a connection $\Gamma$ on $Y$. In this case, the equality (2) takes the form

$$s^* L_{\tau_\Gamma} L = -\frac{d}{dx}[\tau^\mu T^\lambda_{\Gamma_{\mu}} (s)] \omega \quad (4)$$
where
\[ T^\lambda_\mu(\pi) = \pi^\lambda_i (\pi^i_\mu - \Gamma^i_\mu) - \delta^\lambda_\mu \mathcal{L} \] (5)
is the canonical energy-momentum tensor of a field \( \pi \) with respect to the connection \( \Gamma \) on \( Y \). The tensor (3) is the particular case of the stress energy-momentum tensors [1, 3, 5].

Note that, in comparison with the current conservation laws, the Lie derivative \( L_\tau \) fails to be equal to zero as a rule, and the equality (4) is not the conservation law of any canonical energy-momentum tensor in general. It does not look fundamental, otherwise its Hamiltonian counterpart. In the framework of the multimomentum Hamiltonian formalism, we get the fundamental identity (52) whose restriction to a constraint space can be treated the energy-momentum conservation law. In standard field models, this appears the metric energy-momentum conservation law in the presence of a background world metric.

Lagrangian densities of field models are almost always degenerate and the corresponding Euler-Lagrange equations are underdetermined. To describe constraint field systems, the multimomentum Hamiltonian formalism can be utilized [8, 10, 11]. In the framework of this formalism, the phinite-dimensional phase space of fields is the Legendre bundle
\[ \Pi = \wedge^n T^* X \otimes T X \otimes V^* Y \] (6)
over \( Y \) into which the Legendre morphism \( \hat{L} \) associated with a Lagrangian density \( L \) on \( J^1 Y \) takes its values. This phase space is provided with the fibred coordinates \( (x^\lambda, y^i, p^\lambda_i) \) such that
\[ (x^\mu, y^i, p^\mu_i) \circ \hat{L} = (x^\mu, y^i, \pi^\mu_i). \]
The Legendre bundle (3) carries the multisymplectic form
\[ \Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda. \] (7)
In case of \( X = \mathbb{R} \), the form \( \Omega \) recovers the standard symplectic form in analytical mechanics.

Building on the multisymplectic form \( \Omega \), one can develop the so-called multimomentum Hamiltonian formalism of field theory where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the temporal one. On the mathematical level, this is the straightforward multisymplectic generalization of the standard Hamiltonian formalism in analytical mechanics to fibred manifolds over an \( n \)-dimensional base \( X \), not only \( \mathbb{R} \).

Note that the Hamiltonian approach to field theory was called into play mainly for canonical quantization of fields by analogy with quantum mechanics. The major goal of this approach has consisted in establishing simultaneous commutation relations of quantum fields in models with degenerate Lagrangian densities, e.g., gauge theories. In classical field theory, the conventional Hamiltonian formalism fails to be so successful. In the straightforward manner, it takes the form of the instantaneous Hamiltonian formalism when canonical variables are field functions at a given instant of time. The corresponding
phase space is infinite-dimensional. Hamiltonian dynamics played out on this phase space
is far from to be a partner of the usual Lagrangian dynamics of field systems. In partic-
ular, there are no Hamilton equations in the bracket form which would be adequate to
Euler-Lagrange field equations, otherwise in the multumomentum Hamiltonian formalism.

We say that a connection $\gamma$ on the fibred Legendre manifold $\Pi \rightarrow X$ is a Hamiltonian
connection if the form $\gamma|\Omega$ is closed. Then, a Hamiltonian form $H$ on $\Pi$ is defined to be
an exterior form such that

$$dH = \gamma|\Omega$$

for some Hamiltonian connection $\gamma$. Every Hamiltonian form admits splitting

$$H = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Gamma^i_\lambda \omega - \tilde{\mathcal{H}}_\Gamma \omega = p^i_\lambda dy^i \wedge \omega_\lambda - \mathcal{H} \omega, \quad \omega_\lambda = \partial_\lambda |\omega,$$

where $\Gamma$ is a connection on $Y \rightarrow X$. Given the Hamiltonian form $H$ (9), the equality (8)
comes to the Hamilton equations

$$\partial_\lambda r^i(x) = \partial_i^\lambda \mathcal{H}, \quad \partial_\lambda r^i_\lambda(x) = -\partial_i \mathcal{H}$$

for sections $r$ of the fibred Legendre manifold $\Pi \rightarrow X$.

The Hamilton equations (10) are the multimomentum generalization of the standard
Hamilton equations in mechanics when $X = \mathbb{R}$. The energy-momentum conservation law
(52) which we suggest is accordingly the multimomentum generalization of the familiar
energy conservation law

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

in analytical mechanics.

2 Technical preliminary

A fibred manifold

$$\pi : Y \rightarrow X$$

is provided with fibred coordinates $(x^\lambda, y^i)$ where $x^\lambda$ are coordinates of the base $X$. A
locally trivial fibred manifold is termed the bundle. We denote by $VY$ and $V^*Y$ the
vertical tangent bundle and the vertical cotangent bundle of $Y$ respectively. For the
sake of simplicity, the pullbacks $Y \times X T X$ and $Y \times X T^*X$ are denoted by $T X$ and $T^*X$
respectively.

On fibred manifolds, we consider the following types of differential forms:

(i) exterior horizontal forms $Y \rightarrow \wedge T^*X$,

(ii) tangent-valued horizontal forms $Y \rightarrow \wedge T^*X \otimes TY$ and, in particular, soldering
forms $Y \rightarrow T^*X \otimes VY$, 
(iii) pullback-valued forms

\[ Y \rightarrow \tilde{\wedge} T^*Y \otimes TX, \]
\[ Y \rightarrow \tilde{\wedge} T^*Y \otimes T^*X. \]

Horizontal \( n \)-forms are called horizontal densities.

Given a fibred manifold \( Y \rightarrow X \), the first order jet manifold \( J^1Y \) of \( Y \) is both the fibred manifold \( J^1Y \rightarrow X \) and the affine bundle \( J^1Y \rightarrow Y \) modelled on the vector bundle \( T^*X \otimes_Y VY \). It is endowed with the adapted coordinates \( (x^\lambda, y^i, y^i_\lambda) \):

\[ y^i_\lambda = \frac{\partial y^i}{\partial y^j} y^j_\lambda + \frac{\partial y^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\lambda}. \]

We identify \( J^1Y \) to its image under the canonical bundle monomorphism

\[ \lambda : J^1Y \rightarrow T^*X \otimes TY, \]
\[ \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \]  

(12)

Given a fibred morphism of \( \Phi : Y \rightarrow Y' \) over a diffeomorphism of \( X \), its jet prolongation \( J^1\Phi : J^1Y \rightarrow J^1Y' \) reads

\[ y^i_\mu \circ J^1\Phi = (\partial_\lambda \Phi^i + \partial_j \Phi^i y^j_\lambda) \frac{\partial x^\lambda}{\partial x'^\mu}. \]

Every vector field

\[ u = u^\lambda \partial_\lambda + u^i \partial_i \]

on a fibred manifold \( Y \rightarrow X \) gives rise to the projectable vector field

\[ \pi = r_1 \circ J^1u : J^1Y \rightarrow J^1TY \rightarrow T J^1Y, \]
\[ \bar{u} = u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i + y^j_\lambda \partial_j u^i - y^i_\lambda \partial_\lambda u^\mu) \partial_i^\lambda, \]  

(13)

on the fibred jet manifold \( J^1Y \rightarrow X \) where \( J^1TY \) is the jet manifold of the fibred manifold \( TY \rightarrow X \).

The canonical morphism \((12)\) gives rise to the bundle monomorphism

\[ \hat{\lambda} : J^1Y \times_X TX \ni \partial_\lambda \mapsto \hat{\partial}_\lambda = \partial_\lambda \mid \lambda \in J^1Y \times TY, \]
\[ \hat{\partial}_\lambda = \partial_\lambda + y^i_\lambda \partial_i. \]

This morphism determines the canonical horizontal splitting of the pullback

\[ J^1Y \times_X TY = \hat{\lambda}(TX) \oplus J^1Y VY, \]

(14)

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\[ \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i = \dot{x}^\lambda (\partial_\lambda + y^i_\lambda \partial_i) + (\dot{y}^i - \dot{x}^\lambda y^i_\lambda) \partial_i. \]

In other words, over \( J^1Y \), we have the canonical horizontal splitting of the tangent bundle \( TY \).

Building on the canonical splitting \( \text{(14)} \), one gets the corresponding horizontal splittings of a projectable vector field \( u = u^\lambda \partial_\lambda + u^i \partial_i \) on a fibred manifold \( Y \to X \).

\[ u = u^\lambda \partial_\lambda + u^i \partial_i = u_H + u_V = u^\lambda (\partial_\lambda + y^i_\lambda \partial_i) + (u^i - u^\lambda y^i_\lambda) \partial_i \] \( \text{(15)} \)

Given a fibred manifold \( Y \to X \), there is the 1:1 correspondence between the connections on \( Y \to X \) and global sections \( \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \) of the affine jet bundle \( J^1Y \to Y \). Substitution of such a global section \( \Gamma \) into the canonical horizontal splitting \( \text{(14)} \) recovers the familiar horizontal splitting of the tangent bundle \( T Y \) with respect to the connection \( \Gamma \) on \( Y \). These global sections form the affine space modelled on the linear space of soldering forms on \( Y \).

Every connection \( \Gamma \) on \( Y \to X \) yields the first order differential operator

\[ D_\Gamma : J^1Y \to T^* X \otimes VY, \]
\[ D_\Gamma = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i, \]

on \( Y \) which is called the covariant differential relative to the connection \( \Gamma \).

The repeated jet manifold \( J^1J^1Y \), by definition, is the first order jet manifold of \( J^1Y \to X \). It is provided with the adapted coordinates \( (x^\lambda, y^i, y^i_\lambda, y^i_\mu, y^i_\mu_\lambda) \). Its subbundle \( \hat{J}^2Y \) with \( y^i_\lambda = y^i_\lambda \) is called the sesquiholonomic jet manifold. The second order jet manifold \( J^2Y \) of \( Y \) is the subbundle of \( \hat{J}^2Y \) with \( y^i_\lambda = y^i_\mu_\lambda \).

There exists the following generalizations of the contact map \( \text{(12)} \) to the second order jet manifold \( J^2Y \):

\[ \lambda : J^2Y \to T^* X \otimes T^1J^1Y, \]
\[ \lambda = dx^\lambda \otimes \tilde{\partial}_\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i + y^i_\mu_\lambda \partial^\mu_i), \] \( \text{(16)} \)

The contact map \( \text{(16)} \) defines the canonical horizontal splitting of the tangent bundle \( TJ^1Y \) and the corresponding horizontal splitting of a projectable vector field \( \overline{u} \) on \( J^1Y \) over \( J^2Y \):

\[ \overline{u} = u_H + u_V = u^\lambda (\partial_\lambda + y^i_\lambda \partial_i + y^i_\mu_\lambda \partial^\mu_i) + [(u^i - y^i_\lambda u^\lambda) \partial_i + (u^i - y^i_\mu_\lambda u^\lambda) \partial^\mu_i]. \] \( \text{(17)} \)
3 Conservation laws in the Lagrangian formalism

Let \( Y \rightarrow X \) be a fibred manifold and \( L = \mathcal{L}_\omega \) a Lagrangian density on \( J^1 Y \). With \( L \), the jet manifold \( J^1 Y \) carries the unique associated Poincaré-Cartan form

\[
\Xi_L = \pi_i^\lambda dy^i \wedge \omega_\lambda - \pi_i^\lambda y^i_\lambda \omega + \mathcal{L}_\omega
\]  

and the Lagrangian multisymplectic form

\[
\Omega_L = (\partial_j \pi_i^\lambda dy^j + \partial^\mu_i \pi_i^\lambda dy^j_\mu) \wedge dy^i \wedge \omega \otimes \partial_\lambda.
\]

Using the pullback of these forms onto the repeated jet manifold \( J^1 J^1 Y \), one can construct the exterior form

\[
\Lambda_L = d\Xi_L - \lambda [\Omega_L = [y^i_\lambda) - y^i_\lambda) d\pi_i^\lambda + (\partial_i - \hat{\partial}_\lambda \partial_\lambda^i) \mathcal{L} dy^i] \wedge \omega,
\]

\( \lambda = dx^\lambda \otimes \hat{\partial}_\lambda, \quad \hat{\partial}_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_{\mu \lambda} \partial_\mu, \)

on \( J^1 J^1 Y \). Its restriction to the second order jet manifold \( J^2 Y \) of \( Y \) reproduces the familiar variational Euler-Lagrange operator

\[
\mathcal{E}_L = [\partial_i (\partial_\lambda + y^j_\lambda \partial_i + y^j_{\mu \lambda} \partial_\mu) \partial_\lambda^j \mathcal{L} dy^j \wedge \omega.
\]  

The restriction of the form (19) to the sesquiholonomic jet manifold \( \hat{J}^2 Y \) defines the sesquiholonomic extension \( \mathcal{E}'_L \) of the Euler-Lagrange operator (20). It is given by the expression (21), but with nonsymmetric coordinates \( y^i_{\mu \lambda} \).

Let \( s \) be a section of the fibred jet manifold \( J^1 Y \rightarrow X \) such that its first order jet prolongation \( J^1 s \) takes its values into Ker \( \mathcal{E}'_L \). Then, \( s \) satisfies the first order differential Euler-Lagrange equations (1). They are equivalent to the second order Euler-Lagrange equations

\[
\partial_i \mathcal{L} - (\partial_\lambda + y^j_\lambda \partial_i + y^j_{\mu \lambda} \partial_\mu) \partial_\lambda^j \mathcal{L} = 0.
\]  

for sections \( s \) of \( Y \) where \( s = J^1 s \).

We have the following conservation laws on solutions of the first order Euler-Lagrange equations.

Let

\[
u = u^\mu \partial_\mu + u^i \partial_i
\]

be a vector field on a fibred manifold \( Y \) and \( \overline{s} \) its jet lift (13) onto the fibred jet manifold \( J^1 Y \rightarrow X \). Given a Lagrangian density \( L \) on \( J^1 Y \), let us compute the Lie derivative \( \mathcal{L}_{\overline{s}} L \).

We have

\[
\mathcal{L}_{\overline{s}} L = [\hat{\partial}_\lambda (\pi_i^\lambda (u^i - u^\mu y^i_\mu) + u^\lambda \mathcal{L}) + (u^i - u^\mu y^i_\mu)(\partial_i - \hat{\partial}_\lambda \partial_\lambda^i) \mathcal{L}] \omega,
\]

\[
\hat{\partial}_\lambda = \partial_\lambda + y^j_\lambda \partial_j + y^j_{\mu \lambda} \partial_\mu.
\]

On solutions \( s \) of the first order Euler-Lagrange equations, the equality (22) comes to the conservation law (2).
To calculate the Lie derivative (22), one should single out the vertical component $\pi_V$ of the vector field $\pi$ (13) on $J^1Y$. Its horizontal splitting (17) over $J^2Y$ reads

$$\pi_H = u^\lambda (\partial_\lambda + y^i_\lambda \partial_i) = u^\lambda \partial_\lambda, \tag{23}$$

$$\pi_V = (u^i - y^i_\lambda u^\lambda) \partial_i + (\partial_\lambda u^i + y^j_\lambda \partial_j u^i - y^i_\mu \partial_\lambda u^\mu - y^i_\lambda u^\mu) \partial_i. \tag{24}$$

Given the splitting (24), we have

$$L_{\pi_V}L = (L_{\pi_V}L)\omega + \hat{\partial}_\lambda (u^\lambda L)\omega. \tag{25}$$

In particular, if

$$(L_{\pi_V}L) = 0, \tag{26}$$

the Lie derivative (25) comes to the total differential.

If $u$ is a vertical vector field such that the relation (26) holds, we have the conservation law (2) of the current $u^i\pi_\mu$ associated with the vertical field $u$.

If the vector field $u$ is not vertical, the equation (26) where $\pi_V$ is given by the expression (24) is formal because of the factor $y^i_\lambda u^\lambda$. To overcome this difficulty, one can construct the lift $u_\infty$ of a projectable vector field $u$ on the bundle $Y$ onto the infinite order jet space $J^\infty Y$. Moreover, the vertical part of $u_\infty$ must take a certain form in order that the equation (26) makes sense. We have

$$u_\infty = u^\lambda (\partial_\lambda + y^i_\lambda \partial_i + \ldots) + j^\infty u_G, \tag{27}$$

$$j^\infty u_G = \sum_{k=0}^\infty \hat{\partial}_{\lambda_k} \ldots \hat{\partial}_{\lambda_1} u^i \partial^{\lambda_1 \ldots \lambda_k},$$

where $u_G$ is a certain vertical vector field on $Y$. In gauge theory, $u_G$ are principal vector fields on a principal bundle $P$ which are associated with isomorphisms of $P$. Next Section covers this case.

Now, let

$$u = \tau_T = \tau^\mu (\partial_\mu + \Gamma^i_\mu \partial_i)$$

be the horizontal lift of a vector field

$$\tau = \tau^\lambda \partial_\lambda$$
on $X$ onto the fibred manifold $Y$ by a connection $\Gamma$ on $Y$. In this case, the equality (2) takes the form (4) where $T_{\Gamma^\lambda_\mu}(\pi)$ (4) are coefficients of the $T^*X$-valued form

$$T_{\Gamma}(\pi) = - (\Gamma) \Xi_L) \odot \pi = [\pi^i (\Gamma - \Gamma^i_\mu) - \delta^i_\mu L] dx^\mu \otimes \omega_\lambda \tag{28}$$
on $X$. One can think on this form as being the canonical energy-momentum tensor of a field $\pi$ with respect to the connection $\Gamma$ on $Y$. In particular, when the fibration $Y \to X$ is trivial, one can choose the trivial connection $\Gamma = \theta_X$. In this case, the form (28) is precisely the standard canonical energy-momentum tensor. If

$$L_\tau L = 0$$

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for all vector fields $\tau$ on $X$, the conservation law (4) comes to the familiar conservation law
\[
\frac{d}{dx^\lambda} T^\lambda{}_{\mu}(\tau) = 0
\]
of the canonical energy-momentum tensor. In general, the Lie derivative $L_{\tau^\Gamma} L$ is not equal to zero and the equality (4) is not the conservation law of the canonical energy-momentum tensor.

Note that if the above-mentioned field $\tau$ on $X$ is associated with a diffeomorphism of the manifold $X$, its horizontal lift $\tau^\Gamma$ corresponds to the lift of this diffeomorphism to the bundle isomorphism of $Y$ by the connection $\Gamma$ on $Y$. But the equality (4) fails to be a symmetry condition if the corresponding Lie derivative is not equal to zero.

4 Gauge symmetries

In gauge theory, several types of gauge transformations are considered. We follow the definition of gauge transformations as isomorphisms of a principal bundle [7, 14].

Let
\[
\pi_P : P \to X
\]
be a principal bundle with a structure Lie group $G$ which acts freely and transitively on $P$ on the right:
\[
r_g : p \mapsto pg, \quad p \in P, \quad g \in G. \tag{29}
\]

A principal bundle $P$ is also the general affine bundle modelled on the left on the associated group bundle $\tilde{P}$ with the standard fibre $G$ on which the structure group $G$ acts by the adjoint representation. The corresponding bundle morphism reads
\[
\tilde{P} \times P \ni (\tilde{p}, p) \mapsto \tilde{p}p \in P.
\]

Note that the standard fibre of the group bundle $\tilde{P}$ is the group $G$, while that of the principal bundle $P$ is the group space of $G$ on which the structure group $G$ acts on the left.

A principal connection $A$ on a principal bundle $P \to X$ is defined to be a $G$-equivariant connection on $P$ such that
\[
J^1 r_g \circ A = A \circ r_g
\]
for each canonical morphism (29). There is the 1:1 correspondence between the principal connections on a principal bundle $P \to X$ and the global sections of the quotient
\[
C = J^1 P/G \tag{30}
\]
of the jet bundle $J^1P \to P$ by the first order jet prolongations of the canonical morphisms \(29\). We shall call $C$ the principal connection bundle. It is an affine bundle modelled on the vector bundle

$$\overline{C} = T^*X \otimes V^G P$$

where

$$V^G P = VP/G$$

is the quotient of the vertical tangent bundle $VP$ of $P$ by the canonical action \(29\) of $G$ on $P$. Its standard fibre is the right Lie algebra $G_r$ of the right-invariant vector fields on the group $G$. The group $G$ acts on this standard fibre by the adjoint representation.

Given a bundle atlas $Ψ^P$ of $P$, the bundle $C$ is provided with the fibred coordinates $(x^µ, k^m_µ)$ so that

$$(k^m_µ \circ A)(x) = A^m_µ(x)$$

are coefficients of the local connection 1-form of a principal connection $A$ with respect to the atlas $Ψ^P$. The first order jet manifold $J^1C$ of the bundle $C$ is provided with the adapted coordinates $(x^µ, k^m_µ, k^m_µλ)$.

Let $Y \to X$ be a bundle associated with a principal bundle $P \to X$. The structure group $G$ of $P$ acts freely on the standard fibre $V$ of $Y$ on the left. The total space of the $P$-associated bundle $Y$, by definition, is the quotient

$$Y = (P \times V)/G$$

of the product $P \times V$ by identification of its elements $(pg \times gv)$ for all $g \in G$.

Every principal connection $A$ on a principal bundle $P$ yields the associated connection

$$\Gamma = dx^λ \otimes [\partial_λ + A^m_µ(x)I_m^i j y^j i^j λ]$$

(31)

where $A^m_µ(x)$ are coefficients of the local connection 1-form and $I_m$ are generators of the structure group $G$ on the standard fibre $V$ of the bundle $Y$.

Since only Lie derivatives along vertical vector fields lead to conservation laws, we here restrict our consideration to isomorphisms of a principal bundle $P$ over the identity morphism of its base $X$. Given a principal bundle $P \to X$, by a principal morphism is meant its $G$-equivariant isomorphism $Φ_P$ over $X$ together with the first order jet prolongations $J^1Φ_P$. Whenever $g \in G$, we have

$$r_g \circ Φ_P = Φ_P \circ r_g.$$ 

Every such isomorphism $Φ_P$ is brought into the form

$$Φ_P(p) = pf_s(p), \quad p \in P,$$

(32)

where $f_s$ is a $G$-valued equivariant function on $P$:

$$f_s(qg) = g^{-1}f_s(q)g, \quad g \in G.$$
There is the 1:1 correspondence between these functions and the global section \( s \) of the group bundle \( \tilde{P} \):

\[ s(\pi(p))p = pf_s(p). \]

For each \( P \)-associated bundle \( Y \), there exists the fibre-preserving representation morphism

\[ \tilde{P} \times Y \ni (\tilde{p}, y) \mapsto \tilde{p}y \in Y \]

where \( \tilde{P} \) is the \( P \)-associated group bundle. Building on this representation morphism, one can induce principal morphisms of \( Y \):

\[ \Phi_s : Y \ni y \mapsto (s \circ \pi)(y)y \in Y \]

where \( s \) is a global section of \( \tilde{P} \).

Principal morphisms \( \Phi_P \) constitute the gauge group which is isomorphic to the group of global sections of the \( P \)-associated group bundle \( \tilde{P} \). The Sobolev completion of the gauge group is a Banach Lie group. Its Lie algebra in turn is the Sobolev completion of the algebra of generators of infinitesimal principal morphisms. These generators are represented by the corresponding vertical vector fields \( u_G \) on a \( P \)-associated bundle \( Y \) which carries representation of the gauge group. We call them principal vector fields.

It is readily observed that a Lagrangian density \( L \) on the configuration space \( J^1Y \) is gauge invariant iff, whenever local principal vector field \( u_G \) on \( Y \to X \),

\[ L_{u_G}L = 0. \quad (33) \]

In case of unbroken symmetries, the total configuration space of gauge theory is the product

\[ J^1Y \times J^1C \]

where \( Y \) is a \( P \)-associated vector bundle whose sections describe matter fields.

Local principal vector fields on the \( P \)-associated vector bundle \( Y \to X \) read

\[ u_G = \alpha^m(x)I_m \, i_j y^j \partial_i \]

where \( I_m \) are generators of the structure group \( G \) acting on \( V \) and \( \alpha^m(x) \) are arbitrary local functions on \( X \). Local principal vector fields on the principal connection bundle \( C \) are written

\[ u_G = (\partial_\mu \alpha^m + c_m^l k_\mu \alpha^n)\partial_\mu. \]

Then, a local principal vector field on the product \( C \times Y \) takes the form

\[ u_G = (u_m^A \alpha^m + u_m^{A\lambda} \partial_\lambda \alpha^m)\partial_A = (\partial_\mu \alpha^m + c_m^l k_\mu \alpha^n)\partial_\mu + \alpha^m(x)I_m \, i_j y^j \partial_i \]

where the collective index \( A \) is utilized. Substituting this expression into the equality (33), one recovers the familiar Noether identities for a gauge invariant Lagrangian density.
5 Multimomentum Hamiltonian formalism

Let Π be the Legendre bundle over a fibred manifold \( Y \to X \). It is provided with the fibred coordinates \((x^\lambda, y^i, p^\lambda_i)\):

\[
p_i' = J \frac{\partial y^j}{\partial x^\mu} \frac{\partial x^\lambda}{\partial y^i} p_j^\mu, \quad J^{-1} = \det\frac{\partial x^\lambda}{\partial y^i}.
\]

By \( J^1 \Pi \) is meant the first order jet manifold of \( \Pi \) over \( Y \). It is coordinatized by \((x^\lambda, y^i, p^\lambda_i, y^{i(\mu)}, p^{\lambda i}_\mu)\).

We call by a momentum morphism any bundle morphism \( \Phi : \Pi \to J^1 Y \) over \( Y \). Given a momentum morphism \( \Phi \), its composition with the monomorphism \( (12) \) is represented by the horizontal pullback-valued 1-form

\[
\Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda \partial_i)
\]
on \( \Pi \to Y \). For instance, let \( \Gamma \) be a connection on \( Y \). Then, the composition \( \hat{\Gamma} = \Gamma \circ \pi_\Pi \) is a momentum morphism. Conversely, every momentum morphism \( \Phi \) determines the associated connection \( \Gamma_\Phi = \Phi \circ \hat{0}_\Pi \) on \( Y \to X \) where \( \hat{0}_\Pi \) is the global zero section of \( \Pi \to Y \). Every connection \( \Gamma \) on \( Y \) gives rise to the connection

\[
\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda(y) \partial_i + (-\partial_j \Gamma^i_\lambda(y) p^\mu_i - K^\mu_\nu \lambda(x) p^\nu_j + K^\alpha_\alpha \lambda(x) p^\mu_j) \partial_\mu] \tag{34}
\]
on \( \Pi \to X \) where \( K \) is a linear symmetric connection on \( T^*X \).

The Legendre manifold \( \Pi \) carries the multimomentum Liouville form

\[
\theta = -p^\lambda_i dy^i \wedge \omega \otimes \partial_\lambda
\]
and the multisymplectic form \( \Omega \) \( (7) \).

The Hamiltonian formalism in fibred manifolds is formulated intrinsically in terms of Hamiltonian connections which play the role similar to that of Hamiltonian vector fields in the symplectic geometry.

We say that a connection \( \gamma \) on the fibred Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the exterior form \( \gamma \wedge \Omega \) is closed. An exterior \( n \)-form \( H \) on the Legendre manifold \( \Pi \) is called a Hamiltonian form if there exists a Hamiltonian connection satisfying the equation \( (8) \).
Let $H$ be a Hamiltonian form. For any exterior horizontal density $\tilde{H} = \tilde{\mathcal{H}}\omega$ on $\Pi \to X$, the form $H - \tilde{H}$ is a Hamiltonian form. Conversely, if $H$ and $H'$ are Hamiltonian forms, their difference $H - H'$ is an exterior horizontal density on $\Pi \to X$. Thus, Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities on $\Pi \to X$.

Let $\Gamma$ be a connection on $Y \to X$ and $\tilde{\Gamma}$ its lift (34) onto $\Pi \to X$. We have the equality $\tilde{\Gamma}\rfloor\Omega = d(\tilde{\Gamma}\rfloor\theta)$.

A glance at this equality shows that $\tilde{\Gamma}$ is a Hamiltonian connection and

$$H_\Gamma = \tilde{\Gamma}\rfloor\theta = p_\lambda^i dy^i \wedge \omega_\lambda - p_\lambda^i \Gamma_\lambda^i \omega$$

is a Hamiltonian form. It follows that every Hamiltonian form on $\Pi$ can be given by the expression (9) where $\Gamma$ is some connection on $Y \to X$. Moreover, a Hamiltonian form has the canonical splitting (9) as follows. Given a Hamiltonian form $H$, the vertical tangent morphism $VH$ yields the momentum morphism

$$\hat{H} : \Pi \to J^1 Y, \quad y^i_\lambda \circ \hat{H} = \partial^i_\lambda \mathcal{H},$$

and the associated connection $\Gamma_H = \hat{H} \circ \hat{0}$ on $Y$. As a consequence, we have the canonical splitting

$$H = H_{\Gamma_H} - \tilde{H}.$$

The Hamilton operator $\mathcal{E}_H$ for a Hamiltonian form $H$ is defined to be the first order differential operator

$$\mathcal{E}_H = dH - \hat{\Omega} = [(y^i_\lambda - \partial^i_\lambda \mathcal{H})dp_\lambda^i - (p_\lambda^i + \partial_\lambda \mathcal{H})dy^i] \wedge \omega,$$

where $\hat{\Omega}$ is the pullback of the multisymplectic form $\Omega$ onto $J^1 \Pi$.

For any connection $\gamma$ on $\Pi \to X$, we have

$$\mathcal{E}_H \circ \gamma = dH - \gamma\rfloor\Omega.$$
Given a Lagrangian density $L$, the vertical tangent morphism $VL$ of $L$ yields the Legendre morphism

$$
\hat{L} : J^1Y \to \Pi,
$$

$$
p_i^\lambda \circ \hat{L} = \pi_i^\lambda.
$$

We say that a Hamiltonian form $H$ is associated with a Lagrangian density $L$ if $H$ satisfies the relations

$$
\hat{L} \circ \hat{H} \mid_Q = \text{Id}_Q, \quad Q = \hat{L}(J^1Y),
$$

$$
H = H_{\hat{H}} + L \circ \hat{H}.
$$

(37a) (37b)

Note that different Hamiltonian forms can be associated with the same Lagrangian density.

Let a section $r$ of $\Pi \to X$ be a solution of the Hamilton equations (10) for a Hamiltonian form $H$ associated with a semiregular Lagrangian density $L$. If $r$ lives on the constraint space $Q$, the section $\tilde{\sigma} = \hat{H} \circ r$ of $J^1Y \to X$ satisfies the first order Euler-Lagrange equations (11). Conversely, given a semiregular Lagrangian density $L$, let $\tilde{\sigma}$ be a solution of the first order Euler-Lagrange equations (11). Let $H$ be a Hamiltonian form associated with $L$ so that

$$
\hat{H} \circ \hat{L} \circ \tilde{\sigma} = \tilde{\sigma}.
$$

(38)

Then, the section $r = \hat{L} \circ \tilde{\sigma}$ of $\Pi \to X$ is a solution of the Hamilton equations (10) for $H$. For sections $\tilde{\sigma}$ and $r$, we have the relations

$$
\tilde{\sigma} = J^1s, \quad s = \pi_{\Pi Y} \circ r
$$

where $s$ is a solution of the second order Euler-Lagrange equations (21).

We shall say that a family of Hamiltonian forms $H$ associated with a semiregular Lagrangian density $L$ is complete if, for each solution $\tilde{\sigma}$ of the first order Euler-Lagrange equations (11), there exists a solution $r$ of the Hamilton equations (10) for some Hamiltonian form $H$ from this family so that

$$
r = \hat{L} \circ \tilde{\sigma}, \quad \tilde{\sigma} = \hat{H} \circ r, \quad \tilde{\sigma} = J^1(\pi_{\Pi Y} \circ r).
$$

(39)

Such a complete family exists iff, for each solution $\tilde{\sigma}$ of the Euler-Lagrange equations for $L$, there exists a Hamiltonian form $H$ from this family so that the condition (38) holds.

The most of field models possesses affine and quadratic Lagrangian densities. Complete families of Hamiltonian forms associated with such Lagrangian densities always exist [9, 10].

6 Hamiltonian gauge theory

Let us consider the gauge theory of principal connections treated the gauge potentials.
In the rest of the article, the manifold $X$ is assumed to be oriented and provided with a nondegenerate fibre metric $g_{\mu\nu}$ in the tangent bundle of $X$. We denote $g = \det(g_{\mu\nu})$.

Let $P \to X$ be a principal bundle with a structure Lie group $G$ which acts on $P$ on the right. There is the 1:1 correspondence between the principal connections $A$ on $P$ and the global sections of the bundle $C = J^1 P/G$. The finite-dimensional configuration space of principal connections is the jet manifold $J^1 C$ coordinatized by $(x^\mu, k^m_{\mu\lambda})$.

There exists the canonical splitting

$$J^1 C = C_+ \oplus C_- = (J^2 P/G) \oplus (\hat{T}^* X \oplus V^G P),$$

over $C$. There are the corresponding canonical surjections:

$$S : J^1 C \to C_+, \quad S^m_{\lambda\mu} = k^m_{\mu\lambda} + k^m_{\lambda\mu} + c^m_{nl} k^n_{\lambda} k^l_{\mu},$$

$$F : J^1 C \to C_-, \quad F^m_{\lambda\mu} = k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl} k^n_{\lambda} k^l_{\mu}.$$}

The Legendre bundle over the bundle $C$ is

$$\Pi = \hat{T}^* X \otimes TX \otimes [C \times \overline{C}]^*.$$}

It is coordinatized by $(x^\mu, k^m_{\mu\lambda}, p^\mu_{\lambda m})$.

On the configuration space $\Pi$, the conventional Yang-Mills Lagrangian density $L_{YM}$ is given by the expression

$$L_{YM} = \frac{1}{4 \varepsilon^2} a^G_{mn} g^{\lambda\mu} g^{\beta\nu} F^m_{\lambda\mu} F^n_{\beta\nu} \sqrt{|g|} \omega,$$

(41)

where $a^G$ is a nondegenerate $G$-invariant metric in the Lie algebra of $G$. The Legendre morphism associated with the Lagrangian density $\Pi$ takes the form

$$p^{(\mu\lambda)}_m \circ \hat{L}_{YM} = 0,$$

(42a)

$$p^{[\mu|\lambda]}_m \circ \hat{L}_{YM} = \varepsilon^{-2} a^G_{mn} g^{\lambda\alpha} g^{\mu\beta} F^m_{\alpha\beta} \sqrt{|g|}.$$

(42b)

Let us consider connections on the bundle $C$ which take their values into $\Ker \hat{L}_{YM}$:

$$S : C \to C_+, \quad S^m_{\lambda\mu} - S^m_{\mu\lambda} - c^m_{nl} k^n_{\lambda} k^l_{\mu} = 0.$$

(43)

For all these connections, the Hamiltonian forms

$$H = p^{\mu\lambda}_m dk^m_{\mu} \wedge \omega_{\lambda} - p^{\mu\lambda}_m S^m_{\mu\lambda} \omega - \hat{H}_{YM} \omega,$$

$$\hat{H}_{YM} = \frac{\varepsilon^2}{4} a^G_{mn} g_{\mu\nu} g_{\lambda\beta} p^{[\mu\lambda]}_m p^{[\nu\beta]}_n |g|^{-1/2},$$

(44)
are associated with the Lagrangian density $L_{YM}$ and constitute the complete family. Moreover, we can minimize this complete family if we restrict our consideration to connections \((43)\) of the following type. Given a symmetric linear connection $K$ on the cotangent bundle $T^*X$ of $X$, every principal connection $B$ on $P$ gives rise to the connection $S_B$ (43) such that

$$S_B \circ B = S \circ J^1B,$$

$$S_B^m_{\mu\lambda} = \frac{1}{2}[c^m_{nl}B^l_{\mu\lambda} + \partial_\mu B^m_{\lambda} + \partial_\lambda B^m_{\mu} - c^m_{nl}(k^l_{\mu}\beta + \pi^l_{\mu\lambda}) - K_\mu^\beta \pi^m_{\mu\lambda}(B^m_{\beta} - k^m_{\beta}).$$

The corresponding Hamilton equations for sections $r$ of $\Pi \to X$ read

$$\partial_\lambda p^m_{\mu\lambda} = -c^m_{nl}k^l_{\mu\nu}p^l_{\mu\nu} + c^m_{nl}B^l_{\nu\mu}p^i_{\mu\lambda} - K_\mu^\nu \pi^m_{\mu\lambda},$$

(45)

$$\partial_\lambda k^m_{\mu} + \partial_\mu k^m_{\lambda} = 2S_B^m_{(\mu\lambda)},$$

(46)

plus the equation (12b). The equations (12a) and (15) restricted to the constraint space (12a) are the familiar Yang-Mills equations for $A = \pi_{\Pi^C} \circ r$. Different Hamiltonian forms (14) lead to different equations (19) which play the role of the gauge-type condition.

In gauge theory, matter fields possessing only internal symmetries are described by sections of a vector bundle

$$Y = (P \times V)/G$$

associated with a principal bundle $P$. It is provided with a $G$-invariant fibre metric $a^Y$. Because of the canonical vertical splitting

$$VY = Y \times Y,$$

the metric $a^Y$ is a fibre metric in the vertical tangent bundle $VY \to X$. A linear connection $\Gamma$ on $Y$ is assumed to be associated with a principal connection on $P$. It takes the form (31).

On the configuration space $J^1Y$, the Lagrangian density of matter fields in the presence of a background connection $\Gamma$ on $Y$ reads

$$L_{(m)} = \frac{1}{2}a^Y_i[f^\mu_{\nu}(y^i_{\mu} - \Gamma^i_{\mu})(y^j_{\nu} - \Gamma^j_{\nu}) - m^2y^iy^j]\sqrt{g}\omega.$$

(47)

The Legendre bundle of the vector bundle $Y$ is

$$\Pi = n^{\alpha}T^*X \otimes TX \otimes Y^*.$$

The unique Hamiltonian form on $\Pi$ associated with the Lagrangian density $L_{(m)}$ (17) reads

$$H_{(m)} = p_i^\lambda dy^i \wedge \omega^\lambda - p_i^\lambda \Gamma^i_{\lambda} \omega - \frac{1}{2}(a^i_{ij}g_{\mu\nu}p_i^\mu p_j^\nu \sqrt{g}^{-1}g^{-1} + m^2a^Y_i y^i y^j)\sqrt{g}\omega.$$

(48)
where \( a_Y \) is the fibre metric in \( V^*Y \) dual to \( a^Y \). There is the 1:1 correspondence between the solutions of the first order Euler-Lagrange equations for the regular Lagrangian density (17) and the solutions of the Hamilton equations for the Hamiltonian form (18).

In the case of unbroken symmetries, the total Lagrangian density of gauge potentials and matter fields is defined on the configuration space

\[ J^1Y \times J^1C. \]

It is the sum of the Yang-Mills Lagrangian density (11) and the Lagrangian density (17) where

\[ \Gamma^i = k^m_\lambda I^m_i y^j. \] (49)

The associated Hamiltonian forms are the sum of the Hamiltonian forms (14) where \( S = S_B \) and the Hamiltonian form (18) where \( \Gamma \) is given by the expression (49). In this case, the Hamilton equation (15) contains the familiar matter source \( p^\mu_i I^m_i j y^j \) of gauge potentials.

7 Energy-momentum conservation laws

In the framework of the multimomentum Hamiltonian formalism, we get the fundamental identity whose restriction to the Lagrangian constraint space recovers the familiar energy-momentum conservation law, without appealing to any symmetry condition.

Let \( H \) be a Hamiltonian form on the Legendre bundle \( \Pi \) over a fibred manifold \( Y \to X \). Let \( r \) be a section of of the fibred Legendre manifold \( \Pi \to X \) and \( (y^i(x), p^\lambda_i(x)) \) its local components. Given a connection \( \Gamma \) on \( Y \to X \), we consider the following \( T^*X \)-valued \((n - 1)\)-form on \( X \):

\[
T_\Gamma(r) = - \left( \Gamma \right) H \circ r,
\]

\[
T_\Gamma(r) = \left[ p^\mu_i (y^j_/ - \Gamma^i_j) - \left( \gamma^i_j (y^i_/ - \Gamma^i_/) - \tilde{H}_\Gamma \right) \right] dx^\mu \otimes \omega_\lambda, \quad \quad (50)
\]

where \( \tilde{H}_\Gamma \) is the Hamiltonian density in the splitting (13) of \( H \) with respect to the connection \( \Gamma \).

Let

\[ \tau = \tau^\lambda \partial_\lambda \]

be a vector field on \( X \). Given a connection \( \Gamma \) on \( Y \to X \), it gives rise to the vector field

\[
\tilde{\tau}_\Gamma = \tau^\lambda \partial_\lambda + \tau^i \Gamma^i_/ \partial_i + \left( - \tau^\mu p^\lambda_/ \partial_i \Gamma^_/ \mu - p^\lambda_/ \partial_\mu \tau^\mu + p^\lambda_/ \partial_\mu \tau^\lambda \right) \partial_/ \lambda
\]

on the Legendre bundle \( \Pi \). Let us calculate the Lie derivative \( \mathbf{L}_{\tau_\Gamma} \tilde{H}_\Gamma \) on a section \( r \). We have

\[
( \mathbf{L}_{\tau_\Gamma} \tilde{H}_\Gamma ) \circ r = p^\lambda_/ \gamma^_/ \mu + \left( \tau^\mu T^_/ \gamma^_/ \lambda \right) - \left( \tilde{\tau}_\Gamma \right) E_H \circ r \quad \quad (51)
\]
where
\[ R = \frac{1}{2} R_{\lambda\mu} d x^\lambda \wedge d x^\mu \otimes \partial_i = \]
\[ \frac{1}{2} (\partial_i \Gamma^i_{\mu} - \partial_\mu \Gamma^i_{\lambda} - \Gamma^j_{\lambda} \partial_j \Gamma^i_{\mu} - \Gamma^j_{\mu} \partial_j \Gamma^i_{\lambda}) d x^\lambda \wedge d x^\mu \otimes \partial_i; \]
of the connection \( \Gamma \), \( \mathcal{E}_H \) is the Hamilton operator (35) and
\[ \tilde{\tau}_V \Gamma = \tau_{\lambda}(\Gamma^i_{\lambda} - y^i_{\lambda}) \partial_i + (-\tau^i \partial_i \Gamma^j_{\mu} - p^i_{\lambda} \partial_\mu \tau^i + p^i_{\mu} \partial_\lambda \tau^i - \tau^i p^i_{\mu} \delta^i_{\lambda}) \partial_i \]
is the vertical part of the canonical horizontal splitting of the vector field \( \tilde{\tau}_V \) on \( \Pi \) over \( J^1 \Pi \). If \( r \) is a solution of the Hamilton equations, the equality (51) comes to the identity
\[ (\partial_\mu + \Gamma^i_{\mu} \partial_i - \partial_i \Gamma^j_{\mu} p^i_{\lambda} \partial^j_{\lambda}) \tilde{H}_\Gamma - \frac{d}{d x^\lambda} T^\lambda_{\mu}(r) = p^i_{\lambda} R^i_{\lambda\mu}. \] (52)
On solutions of the Hamilton equations, the form (53) reads
\[ T_\Gamma(r) = [p^i_{\lambda} \partial^j_{\mu} \tilde{H}_\Gamma - \delta^\lambda_{\mu}(p^i_{\lambda} \partial^j_{\mu} \tilde{H}_\Gamma - \tilde{H}_\Gamma)] d x^\mu \otimes \omega_{\lambda}. \] (53)
One can verify that the identity (52) does not depend upon choice of the connection \( \Gamma \).

For instance, if \( X = \mathbb{R} \) and \( \Gamma \) is the trivial connection, then
\[ T_\Gamma(r) = \tilde{H}_0 dt \]
where \( \tilde{H}_0 \) is a Hamiltonian and the identity (52) consists with the familiar energy conservation law.

Unless \( n = 1 \), the identity (52) can not be regarded directly as the energy-momentum conservation law. To clarify its physical meaning, we turn to the Lagrangian formalism. Let a multimomentum Hamiltonian form \( H \) be associated with a semiregular Lagrangian density \( L \). Let \( r \) be a solution of the Hamilton equations for \( H \) which lives on the Lagrangian constraint space \( Q \) and \( \tilde{s} \) the associated solution of the first order Euler-Lagrange equations for \( L \) so that they satisfy the conditions (39). Then, we have
\[ T_\Gamma(r) = T_\Gamma(\tilde{s}) \]
where is the Lagrangian canonical energy-momentum tensor (28). It follows that the form (53) may be treated as a Hamiltonian canonical energy-momentum tensor with respect to a background connection \( \Gamma \) on the fibred manifold \( Y \to X \) (or a Hamiltonian stress-energy-momentum tensor). At the same time, the examples below will show that, in field models, the identity (52) is precisely the energy-momentum conservation law for the metric energy-momentum tensor, not the canonical one.

In the Lagrangian formalism, the metric energy-momentum tensor is defined to be
\[ \sqrt{-g} t_{\alpha\beta} = 2 \frac{\partial L}{\partial g^{\alpha\beta}}. \]
In case of a background world metric \( g \), this object is well-behaved. In the framework of the multimomentum Hamiltonian formalism, one can introduce the similar tensor

\[
\sqrt{-g} t^\alpha_\beta H = 2 \frac{\partial H}{\partial g^\alpha_\beta}. \tag{54}
\]

Recall the useful relation

\[
\frac{\partial}{\partial g^\alpha_\beta} = -g^\alpha_\mu g^\beta_\nu \frac{dr}{g^\mu_\nu}.
\]

If a multimomentum Hamiltonian form \( H \) is associated with a semiregular Lagrangian density \( L \), we have the equalities

\[
t^\alpha_\beta(q) = -g^{\alpha\mu} g^{\beta\nu} t_{\mu\nu}(x^\lambda, y^i, \partial_i \lambda H(q)),
\]

\[
t^\alpha_\beta(x^\lambda, y^i, \pi_\lambda^i(z)) = -g^{\alpha\mu} g^{\beta\nu} t_{\mu\nu}(z)
\]

where \( q \in Q \) and

\[
\tilde{H} \circ \tilde{L}(z) = z.
\]

In view of these equalities, we can think of the tensor (54) restricted to the Lagrangian constraint space \( Q \) as being the Hamiltonian metric energy-momentum tensor. On \( Q \), the tensor (54) does not depend upon choice of a Hamiltonian form \( H \) associated with \( L \). Therefore, we shall denote it by the common symbol \( t \). Set

\[
t^\lambda_\alpha = g_{\alpha\nu} t^{\lambda\nu}.
\]

In the presence of a background world metric \( g \), the identity (52) takes the form

\[
t^\lambda_\alpha \{^\alpha_\lambda\mu\} \sqrt{-g} + (\Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^\lambda_\mu t_{\lambda ji}) \tilde{H}_\Gamma = \frac{d}{d x^\lambda} T^\lambda_\mu + p^\lambda_\nu R^i_\lambda_\mu \tag{55}
\]

where

\[
\frac{d}{d x^\lambda} = \partial_\lambda + \partial_\lambda y^i \partial_i + \partial_\lambda p^i_\mu \partial^i_\mu
\]

and by \( \{^\alpha_\lambda\mu\} \) are meant the Cristoffel symbols of the world metric \( g \).

For instance, let us examine matter fields in the presence of a background gauge potential \( A \) which are described by the Hamiltonian form (48). In this case, we have the equality

\[
t^\lambda_\mu \sqrt{-g} = T^\lambda_\mu = \left[ a_{ij} g_{\mu\nu} p^\lambda_\nu (-g)^{-1}
\right.
\]

\[
-\delta^\lambda_\mu \frac{1}{2} \left( a_{ij} g_{\alpha\nu} p^\alpha_\nu (-g)^{-1} + m^2 a^Y_{ij} y^i y^j \right) \sqrt{-g}
\]

and the gauge invariance condition

\[
I^j_\lambda p^\lambda_\mu \partial^\mu_\lambda \tilde{H} = 0.
\]
The identity (55) then takes the form of the energy-momentum conservation law

\[ \sqrt{-g} \nabla_\lambda t^\lambda_{\mu} = -p^1 F^m_{\lambda\mu} I^i_j y^j \]

where \( \nabla_\lambda \) is the covariant derivative relative to the Levi-Civita connection and \( F \) is the strength of the background gauge potential \( A \).

Let us consider gauge potentials \( A \) described by the complete family of the multi-momentum Hamiltonian forms (44) where \( S = S_B \) and \( K^\beta_{\mu\lambda} = \{^\beta_{\mu\lambda}\} \). On the solution \( k = B \), the curvature of the connection \( S_B \) is reduced to

\[ R^m_{\lambda\alpha\mu} = \frac{1}{2} \left( \partial_\lambda F^m_{\alpha\mu} - c_{q^n k^\lambda} F^m_{\alpha\mu} - \{^\beta_{\alpha\lambda}\} F^m_{\beta\mu} - \{^\beta_{\mu\lambda}\} F^m_{\alpha\beta} \right) = \frac{1}{2} \left[ \left( \partial_\alpha F^m_{\lambda\mu} - c_{q^n k^\alpha} F^m_{\lambda\mu} - \{^\beta_{\lambda\alpha}\} F^m_{\beta\mu} \right) - \left( \partial_\mu F^m_{\lambda\alpha} - c_{q^n k^\mu} F^m_{\lambda\alpha} - \{^\beta_{\lambda\mu}\} F^m_{\alpha\beta} \right) \right]. \]

Set

\[ S^\lambda_{\mu} = p^{[\alpha\lambda]}_{\mu} \partial^{m}_{\alpha\mu} \tilde{H}_{YM} = \frac{e^2}{2 \sqrt{-g}} a^m_{\alpha\nu} g_{\alpha\beta} p^{[\alpha\lambda]}_{m} [\beta^n]. \]

We have

\[ S^\lambda_{\mu} = \frac{1}{2} p^{[\alpha\lambda]} F^m_{\mu\alpha}, \quad \tilde{H}_{YM} = \frac{1}{2} S^\alpha_{\alpha}. \]

In virtue of Eqs. (42a), (42b) and (43), we obtain the relations

\[ p^{[\lambda\alpha]}_{m} R^m_{\lambda\alpha\mu} = \partial_\lambda S^\lambda_{\mu} - \{^\beta_{\mu\lambda}\} S^\lambda_{\beta}, \]

\[ \partial_\alpha F^m_{\mu\lambda} p^{\alpha\lambda}_{\mu} c_{\beta\lambda} \tilde{H}_{YM} = \{^\beta_{\alpha\mu}\} S^\alpha_{\beta}, \]

\[ t^\lambda_{\mu} \sqrt{-g} = 2 S^\lambda_{\mu} - \frac{1}{2} S^\lambda_{\mu} S^\alpha_{\alpha} \]

and

\[ t^\lambda_{\mu} \sqrt{-g} = T^\lambda_{\mu} + S^\lambda_{\mu}. \]

The identity (55) then takes the form of the energy-momentum conservation law

\[ \nabla_\lambda t^\lambda_{\mu} = 0 \]

in the presence of a background world metric.

Note the identity (52) remains true also in gravitation theory. But its treatment as an energy-momentum conservation law is under discussion. The key point consists in the feature of a gravitational field as a Higgs field whose canonical momenta on the constraint space are equal to zero.
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