A Direct Coupling Coherent Quantum Observer

Ian R. Petersen

Abstract—This paper considers the problem of constructing a direct coupling quantum observer for a closed linear quantum system. The proposed observer is shown to be able to estimate some but not all of the plant variables in a time averaged sense. A simple example and simulations are included to illustrate the properties of the observer.

I. INTRODUCTION

A number of papers have recently considered the problem of constructing a coherent quantum observer for a quantum system; see [1]–[3]. In the coherent quantum observer problem, a quantum plant is coupled to a quantum observer which is also a quantum system. The quantum observer is constructed to be a physically realizable quantum system so that the system variables of the quantum observer converge in some suitable sense to the system variables of the quantum plant.

In the papers [1], [2], the quantum plant under consideration is a linear quantum system. In recent years, there has been considerable interest in the modeling and feedback control of linear quantum systems; e.g., see [4]–[6]. Such linear quantum systems commonly arise in the area of quantum optics; e.g., see [7], [8]. For such linear quantum system models an important class of quantum control problems are referred to as coherent quantum feedback control problems; e.g., see [4], [5], [9]–[14]. In these coherent quantum feedback control problems, both the plant and the controller are quantum systems and the controller is typically to be designed to optimize some performance index. The coherent quantum observer problem can be regarded as a special case of the coherent quantum feedback control problem in which the objective of the observer is to estimate the system variables of the quantum plant.

In the previous papers on quantum observers such as [1]–[3], the coupling between the plant and the observer is via a field coupling. This leads to an observer structure of the form shown in Figure 1. This enables a one way connection between the quantum plant and the quantum observer. Also, since both the quantum plant and the quantum observer are open quantum systems, they are both subject to quantum noise.

However in the paper [11], a coherent quantum control problem is considered in which both field coupling and direct coupling is considered between the quantum plant and the quantum controller. In this paper, we explore the construction of a coherent quantum observer in which there is only direct coupling between quantum plant and the quantum observer. Furthermore, both the quantum plant and the quantum observer are assumed to be closed quantum systems which means that they are not subject to quantum noise and are purely deterministic systems. This leads to an observer structure of the form shown in Figure 2. It is shown that for the case being considered, a quantum observer can be constructed to estimate some but not all of the system variables of the quantum plant. Also, the observer variables converge to the plant variables in a time averaged sense rather than a quantum expectation sense such as considered in the papers [1], [2].

![Fig. 1. Coherent Observer Structure with Field Coupling.](image1)

II. QUANTUM LINEAR SYSTEMS

In this section, we describe the class of closed linear quantum systems under consideration; see also [4], [11], [15]. We consider linear non-commutative systems of the form

\[
\dot{x}(t) = Ax(t); \quad x(0) = x_0
\]

where \( A \) is a real matrix in \( \mathbb{R}^{n \times n} \), and \( x(t) = [x_1(t) \ldots x_n(t)]^T \) is a vector of self-adjoint possibly non-commutative system variables; e.g., see [4]. Here \( n \) is assumed to be an even number and \( \frac{n}{2} \) is the number of modes in the quantum system.

The initial system variables \( x(0) = x_0 \) are assumed to satisfy the commutation relations

\[
[x_j(0), x_k(0)] = 2i\Theta_{jk}, \quad j, k = 1, \ldots, n,
\]

where \( \Theta \) is a real antisymmetric matrix with components \( \Theta_{jk} \). Here, the commutator is defined by \([A, B] = AB - BA\). In the case of a single degree of freedom quantum particle, \( x = (x_1, x_2)^T \) where \( x_1 = q \) is the position operator, and \( x_2 = p \) is the momentum operator. The commutation relations are \([q, p] = 2i\). Here, the matrix \( \Theta \) is assumed to be of the form \( \Theta = \text{diag}(J, J, \ldots, J) \) where \( J \) denotes the real skew-symmetric \( 2 \times 2 \) matrix

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
A linear quantum system \( [1] \) is said to be **physically realizable** if it ensures the preservation of the canonical commutation relations (CCRs):

\[
x(t)x(t)^T - (x(t)x(t)^T)^T = 2i\Theta \text{ for all } t \geq 0.
\]

This holds when the system \( [1] \) corresponds to a collection of **closed quantum harmonic oscillators**; see [4]. Such quantum harmonic oscillators are described by a quadratic Hamiltonian \( \mathcal{H} = \frac{1}{2}x^T Rx(0) \), where \( R \) is a real symmetric matrix.

**Theorem 1 ([4]):** The system \( [1] \) is physically realizable if and only if:

\[
A\Theta + \Theta A^T = 0.
\]

In this case, the corresponding Hamiltonian matrix \( R \) is given by \( R = \frac{1}{2}(-\Theta A + A^T \Theta) \). In addition, for a given Hamiltonian matrix \( R \), the corresponding matrix \( A \) in \( [1] \) is given by

\[
A = 2\Theta R.
\]

**Remark 1:** Note that the system \( [1] \) cannot be asymptotically stable if it is physically realizable. To see this, first suppose \( R \neq 0 \). Then, observe that the Hamiltonian is preserved in time. Indeed, \( \mathcal{H} = \frac{1}{2}x^T Rx + \frac{1}{2}x^T Rx = -x^T R\Theta Rx + x^T R\Theta R\Theta = 0 \) since \( R \) is symmetric and \( \Theta \) is skew-symmetric. However, if the system were asymptotically stable, then \( x(t) \to 0 \) as \( t \to \infty \) which would contradict this fact. Also, if \( R = 0 \), then \( A = 0 \) which is again not asymptotically stable. A similar conclusion can also be drawn from the fact that the CCRs are preserved in time.

Since it is not possible for a physically realizable quantum system of the form \( [1] \) to be asymptotically stable, we will need a new notion of convergence for our direct coupled quantum observer.

### III. Direct Coupling Coherent Quantum Observers

We first consider general **closed linear quantum plants** described by non-commutative models of the following form:

\[
\begin{align*}
\dot{x}_p(t) &= A_p x_p(t); \quad x_p(0) = x_{0p};
\dot{z}_p(t) &= C_p x_p(t)
\end{align*}
\]

where \( z_p \) denotes the vector of system variables to be estimated by the observer and \( A_p \in \mathbb{R}^{n_p \times n_p}, C_p \in \mathbb{R}^{m_p \times n_p} \). It is assumed that this quantum plant is physically realizable and corresponds to a plant Hamiltonian \( \mathcal{H}_p = \frac{1}{2}x_p^T R_p x_p(0) \) where the symmetric matrix \( R_p \) is given by \( R_p = \frac{1}{2}(-A_p + A_p^T \Theta) \).

Also, we consider a direct coupled linear quantum observer defined by a symmetric matrix \( R_o \in \mathbb{R}^{n_o \times n_o} \), and matrices \( R_c \in \mathbb{R}^{n_p \times n_o}, C_o \in \mathbb{R}^{m_p \times n_o} \). These matrices define an observer Hamiltonian

\[
\mathcal{H}_o = \frac{1}{2}x_o^T R_o x_o(0),
\]

and a coupling Hamiltonian

\[
\mathcal{H}_c = \frac{1}{2}x_p^T R_c x_o(0) + \frac{1}{2}x_o^T R_c^T x_p(0).
\]

The matrix \( C_o \) also defines the vector of estimated variables for the observer as \( z_o(t) = C_o x_o(t) \).

The augmented quantum linear system consisting of the quantum plant and the direct coupled quantum observer is then a quantum system of the form \( [1] \) described by the total Hamiltonian

\[
\mathcal{H}_a = \mathcal{H}_p + \mathcal{H}_c + \mathcal{H}_o = \frac{1}{2}x_a^T R_a x_a(0)
\]

where \( x_a = [x_p' \ x_o'] \) and \( R_a = \begin{bmatrix} R_p & R_c \\ R_c^T & R_o \end{bmatrix} \).

### IV. Constructing a Direct Coupling Coherent Quantum Observer

We now describe the construction of a direct coupled linear quantum observer. In this section, we assume that \( A_p = 0 \) in \( [5] \). This corresponds to \( R_p = 0 \) in the plant Hamiltonian. It follows from \( [5] \) that the plant system variables \( x_p(t) \) will remain fixed if the plant is not coupled to the observer. However, when the plant is coupled to the quantum observer this will no longer be the case. We will show that if the quantum observer is suitably designed, the plant quantity to be estimated \( z_p(t) \) will remain fixed and the condition \( [10] \) will be satisfied.

We also assume that \( m_p = \frac{n_p}{2} \) and the matrix \( C_p \) is of the form \( C_p = \beta^T \) where

\[
\beta = \begin{bmatrix}
\beta_1 & 0 & 0 & \ldots & 0 \\
0 & \beta_2 & 0 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & \ldots & \beta_{np} & 0
\end{bmatrix} \in \mathbb{R}^{n_p \times \frac{n_p}{2}}
\]
and $\beta_i \in \mathbb{R}^{2 \times 1}$ for $i = 1, 2, \ldots, n_p$. This assumption means that the plant variables to be estimated include only one quadrature for each mode of the plant.

We now suppose that the matrices $R_o$, $C_o$, $S_o$ are such that $R_o = \beta \Theta^T$, $\alpha \in \mathbb{R}^{n_x \times n_p}$ and the matrix $R_o$ is positive definite. Also, we write $\Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}$ where $\Theta_1 \in \mathbb{R}^{n_x \times n_r}$ and $\Theta_2 \in \mathbb{R}^{n_o \times n_r}$.

and hence,

and

and $A_o = 2\Theta R_o = \begin{bmatrix} 0 & 0 \\ 2\Theta_2 \beta^T & 2\Theta_2 R_o \end{bmatrix}$. Hence, the augmented system equations describing the combined plant-observer system become

$$
\begin{align*}
\dot{x}_p(t) &= 2\Theta_1 \beta^T x_o(t); \\
\dot{x}_o(t) &= 2\Theta_2 \alpha^T x_p(t) + 2\Theta_2 R_o x_o(t); \\
\dot{z}_p(t) &= C_p x_p(t); \\
\dot{z}_o(t) &= C_o x_o(t). 
\end{align*}
$$

We now use Laplace Transforms to solve these equations. It follows that

and hence,

and

and hence,

$$
\begin{align*}
x_p(s) &= 2\Theta_1 \beta^T X_o(s) + x_p(0); \\
x_o(s) &= 2\Theta_2 \alpha^T X_p(s) + 2\Theta_2 R_o X_o(s) + x_o(0). 
\end{align*}
$$

and

and hence,

and hence,

Taking the inverse Laplace Transform of this equation, we obtain

and hence,

Also, if we substitute (14) into (13), we obtain

Taking the inverse Laplace Transform of this equation, we obtain

We now choose the parameters of the quantum observer so that $C_R R_o^{-1} \alpha = -I$. It follows from (15) and (16) that the quantities $z_p(t) = C_p x_p(t)$ and $z_o(t) = C_o x_o(t)$ are given by

where we have used the fact that $C_p \Theta_1 \beta = \beta^T \Theta_1 \beta = 0$. That is, the quantity $z_p(t)$ remains constant and is not affected by the coupling to the coherent quantum observer.

Note that the equation (18) can be derived directly since

$$
\begin{align*}
X_o(s) &= (sI - 2\Theta_2 R_o)^{-1} \left( \frac{2}{s} \Theta_2 \alpha^T x_p(0) + x_o(0) \right).
\end{align*}
$$

Taking the inverse Laplace Transform of this equation, we obtain

where

$$
\begin{align*}
x_o(t) &= e^{2\Theta_2 R_o t} x_o(0) + 2 \int_0^t e^{2\Theta_1 R_o (t-\tau)} d\tau \Theta_2 \alpha^T x_p(0) \\
&= e^{2\Theta_2 R_o t} x_o(0) \\
&\quad - e^{2\Theta_2 R_o t} (e^{-2\Theta_2 R_o t} - I) R_o^{-1} \Theta_2 \alpha^T x_p(0) \\
&= e^{2\Theta_2 R_o t} (x_o(0) + R_o^{-1} \Theta_2 \alpha^T x_p(0)) \\
&\quad - R_o^{-1} \Theta_2 \alpha^T x_p(0).
\end{align*}
$$

Also, if we substitute (14) into (13), we obtain

$$
\begin{align*}
x_p(t) &= 4 \Theta_1 \beta^T (sI - 2\Theta_2 R_o)^{-1} \Theta_2 \alpha^T x_p(0) \\
&\quad + \frac{2}{s} \Theta_1 \beta^T (sI - 2\Theta_2 R_o)^{-1} x_o(0) \\
&\quad + \frac{1}{s} x_p(0).
\end{align*}
$$

Taking the inverse Laplace Transform of this equation, we obtain

We now suppose the vector $\begin{bmatrix} x_p \\ x_o \end{bmatrix}$ is an eigenvector of $A_o$ with corresponding eigenvector $\lambda$. Hence,

and hence

and

and

where we have used the fact that $C_p \Theta_1 \beta = \beta^T \Theta_1 \beta = 0$. That is, the quantity $z_p(t)$ remains constant and is not affected by the coupling to the coherent quantum observer.

Note that the equation (18) can be derived directly since
We now premultiply (19) by $\beta^T$ and use the fact that $\beta^T \Theta_1 \beta = 0$ to obtain

$$\lambda \beta^T \lambda = 0.$$  

Hence, either $\lambda = 0$ which means that the eigenvalue is purely imaginary or $\beta^T x_p = 0$. If $\lambda \neq 0$ the condition $\beta^T x_p = 0$ is substituted into (20) to obtain

$$2 \Theta_2 R_o \alpha_o = \lambda x_o.$$  

Furthermore, if $x_o = 0$, it follows from (19) that $\lambda x_o = 0$ and hence, $x_p = 0$ since $\lambda \neq 0$. However, this contradicts the fact that $\left[ \begin{array}{c} x_p \\ x_o \end{array} \right]$ is an eigenvector of $A_o$. Thus, we must have $x_o \neq 0$. Thus, we can now conclude that $\lambda$ is an eigenvalue of $2 \Theta_2 R_o$ which we have already established has only purely imaginary eigenvalues. Thus, $\lambda$ must be purely imaginary in this case as well.

We now verify that the condition (10) is satisfied for this quantum observer. We recall from Remark 1 that the quantity $\frac{1}{\lambda} x^T R_o x$ remains constant in time for the linear system:

$$\dot{x} = 2 \Theta_2 R_o x; \quad x(0) = x_0.$$  

That is

$$\frac{1}{2} x(t)^T R_o x(t) = \frac{1}{2} x_0^T R_o x_0 \quad \forall t \geq 0. \quad (21)$$

However, $x(t) = e^{2 \Theta_2 R_o t} x_0$ and $R_o > 0$. Therefore, it follows from (21) that

$$\sqrt{\lambda_{\text{min}}(R_o)} \| e^{2 \Theta_2 R_o t} x_0 \| \leq \sqrt{\lambda_{\text{max}}(R_o)} \| x_0 \|$$

for all $x_0$ and $t \geq 0$. Hence,

$$\| e^{2 \Theta_2 R_o t} \| \leq \frac{\lambda_{\text{max}}(R_o)}{\lambda_{\text{min}}(R_o)}$$  

(22)

for all $t \geq 0$.

Now since $\Theta_2$ and $R_o$ are non-singular,

$$\int_0^T e^{2 \Theta_2 R_o t} dt = \frac{1}{2} e^{2 \Theta_2 R_o T} R_o^{-1} \Theta_2^{-1} - \frac{1}{2} R_o^{-1} \Theta_2^{-1}$$

and therefore, it follows from (22) that

$$\int_0^T \left\| e^{2 \Theta_2 R_o t} \right\| dt = \frac{1}{T} \int_0^T e^{2 \Theta_2 R_o t} dt \leq \frac{1}{T} \left( \frac{1}{2} \right) \left( \frac{\lambda_{\text{max}}(R_o)}{\lambda_{\text{min}}(R_o)} \right)^{\frac{1}{2}}$$

as $T \to \infty$. Hence, (17) implies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_o(t) dt = z_p(0).$$

Also, (18) implies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_p(t) dt = z_p(0).$$

Therefore, condition (10) is satisfied. Thus, we have established the following theorem.

**Theorem 2:** Consider a quantum plant of the form (5) where $A_p = 0$, $C_p = \beta^T$ and $\beta$ is as defined in (11). Then the matrices $R_o > 0$, $R_c$, $C_o$ will define direct coupled quantum observer for this quantum plant if $R_c$ is of the form $R_c = C_p^T \alpha T$ where $\alpha \in \mathbb{R}^{n \times \frac{1}{2}}$ and $C_p R_o^{-1} \alpha = -I$.

**Remark 3:** We consider the above result for the single mode case with $n_o = 2$, $m_p = 1$, in which $C_p = [1 \ 0]$. This means that the variable to be estimated by the quantum observer is the position operator of the quantum plant; i.e., $z_p(t) = q_p(t)$ where $x_p(t) = \left[ \begin{array}{c} q_p(t) \\ p_p(t) \end{array} \right]$. By choosing $n_o = 2$, $R_o = I$, $C_o = [1 \ 0]$, $\beta = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\alpha = \left[ \begin{array}{c} -1 \\ 0 \end{array} \right]$, the conditions of Theorem 2 will be satisfied and the observer output variable will be the position operator of the quantum observer $q_o(t)$; i.e., $z_o(t) = q_o(t)$ where $x_o(t) = \left[ \begin{array}{c} q_o(t) \\ q_o(t) \end{array} \right]$. Before the quantum observer is connected to the quantum plant, the quantities $q_p(t)$ and $p_p(t)$ will remain constant since we have assumed that $A_p = 0$. Now suppose that the quantum observer is connected to the quantum plant at time $t = 0$. According to (18), the plant position operator $q_p(t)$ will remain constant at its initial value $q_p(0) = q_p(0)$ but the plant momentum operator $p_p(t)$ will evolve in an time varying and oscillatory way as defined by (16). In addition, the observer position operator $q_o(t)$ will evolve in an oscillatory way as defined by (17) but its time average will converge to $q_p(0)$ according to (18).

Now suppose that after a sufficiently long time $T$ such that the time average of $q_o(t)$ has essentially converged to $q_p(0)$, the observer is disconnected from the quantum plant. Then, the plant position operator $q_p(t)$ will remain constant at $q_p(t) = q_p(0)$ and the plant momentum operator $p_p(t)$ will remain constant at a value $p_p(T)$ which is determined by the formula (16) in terms of $x_p(0)$, $x_o(0)$ and the time $T$. This will be an essentially random value. If at a later time an observer with the same parameters as above is connected to the quantum plant, then time average of its output $z_o(t) = q_o(t)$ will again converge to $q_p(0)$ and $q_o(t)$ will remain constant at $q_o(t) = q_p(0)$. However, suppose that instead an observer with different parameters $R_o = I$, $C_o = [0 \ 1]$ and $\alpha = \left[ \begin{array}{c} 0 \\ -1 \end{array} \right]$ is used. This observer is designed so that the time average of the observer output $z_o(t) = p_o(t)$ converges to the momentum operator of the quantum plant $p_p(t)$. This quantity is the essentially random value $p_p(T)$ mentioned above. In addition, the previously constant value of $q_p(t) = q_p(0)$ will now be destroyed and will evolve to another essential random value. This behavior of the quantum observer is similar to the behavior of quantum measurements; e.g., see [21]. This is not surprising since the behavior of the direct coupled quantum observers considered
in this paper and the behavior of quantum measurements are both determined by the quantum commutation relations which are fundamental to the theory of quantum mechanics.

V. NUMERICAL SIMULATIONS OF A QUANTUM OBSERVER FOR A ONE MODE PLANT

We now present some numerical simulations to illustrate the direct coupled quantum observer described in the previous section. We consider the quantum observer considered in Remark 3 above where $n_p = 2$, $n_o = 1$, $n_o = 2$, $A_p = 0$, $C_p = [1 0]$, $R_o = I$, $C_o = [1 0]$, $\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\alpha = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. As described in Remark 3 the variable to be estimated by the quantum observer is the position operator of the quantum plant; i.e., $z_p(t) = q_p(t)$ where $x_p(t) = \begin{bmatrix} q_p(t) \\ p_p(t) \end{bmatrix}$. Also, the observer output variable will be the position operator of the quantum observer $q_o(t)$; i.e., $z_o(t) = q_o(t)$ where $x_o(t) = \begin{bmatrix} q_o(t) \\ p_o(t) \end{bmatrix}$. Then the augmented plant-observer system is described by the equations

$$\begin{bmatrix} \dot{q}_p(t) \\ \dot{p}_p(t) \\ \dot{q}_o(t) \\ \dot{p}_o(t) \end{bmatrix} = A_o \begin{bmatrix} q_p(t) \\ p_p(t) \\ q_o(t) \\ p_o(t) \end{bmatrix},$$

where

$$A_o = \begin{bmatrix} 0 & 2J_0\alpha^T & 2J_0\beta^T \\ 2J_0\beta^T & 2J_0R_o \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2J_0\beta^T & 2J_0R_o \end{bmatrix}.$$

Then, we can write

$$\begin{bmatrix} q_p(t) \\ p_p(t) \\ q_o(t) \\ p_o(t) \end{bmatrix} = \Phi(t) \begin{bmatrix} q_p(0) \\ p_p(0) \\ q_o(0) \\ p_o(0) \end{bmatrix},$$

where

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) & \phi_{14}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \phi_{23}(t) & \phi_{24}(t) \\ \phi_{31}(t) & \phi_{32}(t) & \phi_{33}(t) & \phi_{34}(t) \\ \phi_{41}(t) & \phi_{42}(t) & \phi_{43}(t) & \phi_{44}(t) \end{bmatrix} = e^{A_o t}.$$

Thus, the plant variable to be estimated $q_p(t)$ is given by

$$q_p(t) = \phi_{11}(t)q_p(0)+\phi_{12}(t)p_p(0)+\phi_{13}(t)q_o(0)+\phi_{14}(t)p_o(0)$$

and we plot the functions $\phi_{11}(t)$, $\phi_{12}(t)$, $\phi_{13}(t)$, $\phi_{14}(t)$ in Figure 3. From this figure, we can see that $\phi_{11}(t) \equiv 1$, $\phi_{12}(t) \equiv 0$, $\phi_{13}(t) \equiv 0$, $\phi_{14}(t) \equiv 0$, and $q_p(t)$ will remain constant at $q_p(0)$ for all $t \geq 0$.

Also, the other plant variable $p_p(t)$ is given by

$$p_p(t) = \phi_{21}(t)q_p(0)+\phi_{22}(t)p_p(0)+\phi_{23}(t)q_o(0)+\phi_{24}(t)p_o(0)$$

and we plot the functions $\phi_{21}(t)$, $\phi_{22}(t)$, $\phi_{23}(t)$, $\phi_{24}(t)$ in Figure 4. From this figure, we can see that $p_p(t)$ evolves in a time-varying and oscillatory way when the quantum plant is connected to the quantum observer.

We now consider the output variable of the quantum observer $q_o(t)$ which is given by

$$q_o(t) = \phi_{31}(t)q_p(0)+\phi_{32}(t)p_p(0)+\phi_{33}(t)q_o(0)+\phi_{34}(t)p_o(0)$$

and we plot the functions $\phi_{31}(t)$, $\phi_{32}(t)$, $\phi_{33}(t)$, $\phi_{34}(t)$ in Figure 5. To illustrate the time average convergence property of the quantum observer in Figure 5, we now plot the quantities

$$\phi_{31}^{ave}(T) = \frac{1}{T} \int_0^T \phi_{31}(t)dt$$

$$\phi_{32}^{ave}(T) = \frac{1}{T} \int_0^T \phi_{32}(t)dt$$

$$\phi_{33}^{ave}(T) = \frac{1}{T} \int_0^T \phi_{33}(t)dt$$

$$\phi_{34}^{ave}(T) = \frac{1}{T} \int_0^T \phi_{34}(t)dt$$

in Figure 6. From this figure, we can see that the time average of $q_o(t)$ converges to $q_p(0)$ as $t \to \infty$. Note that the effect of time averaging can be regarded as a low pass filtering effect which removes the sinusoidal oscillations but retains the DC component which represents the estimate of the plant variable.
We now consider the other variable of the quantum observer \( p_o(t) \) which is given by

\[
p_o(t) = \phi_{41}(t)q_p(0)+\phi_{42}(t)p_o(0)+\phi_{43}(t)q_o(0)+\phi_{44}(t)p_o(0)
\]
and we plot the functions \( \phi_{41}, \phi_{42}(t), \phi_{43}(t), \phi_{44}(t) \) in Figure 7.

To investigate the time average property of the other quantum observer variable, we now plot the quantities

\[
\begin{align*}
\bar{\phi}^{ave}_{41}(T) &= \frac{1}{T} \int_{0}^{T} \phi_{41}(t)dt \\
\bar{\phi}^{ave}_{42}(T) &= \frac{1}{T} \int_{0}^{T} \phi_{42}(t)dt \\
\bar{\phi}^{ave}_{43}(T) &= \frac{1}{T} \int_{0}^{T} \phi_{43}(t)dt \\
\bar{\phi}^{ave}_{44}(T) &= \frac{1}{T} \int_{0}^{T} \phi_{44}(t)dt
\end{align*}
\]

in Figure 8.

We now illustrate the comments in Remark 3 by supposing that the above quantum observer is applied to the quantum plant for the time interval \( t \in [0, 20] \). Then, the quantum observer is disconnected from the quantum plant for the time interval \( t \in [20, 25] \). During this time interval, the quantum plant can be regarded to be connected to a null quantum observer so that \( A_\alpha = 0 \) in this time interval. At time \( t = 25 \), the quantum plant is then connected to a different quantum observer defined by the parameters \( R_o = I, C_o = [0 1], \beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( \alpha = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \). We write

\[
A_{a1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad A_{a2} = 0, \\
A_{a3} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}
\]

so that the matrix \( A_{a1} \) defines the dynamics of the augmented plant-observer system in the time interval \( t \in [0, 20] \), the matrix \( A_{a2} \) defines the dynamics of the augmented plant-observer system in the time interval \( t \in [20, 25] \), and the matrix \( A_{a3} \) defines the dynamics of the augmented plant-observer system for \( t \geq 25 \). Then, we can write

\[
x_a(t) = \tilde{\Phi}(t)x_a(0)
\]
where

\[
\hat{\Phi}(t) = \begin{cases} 
  e^{A_{11}t} & \text{for } t \in [0, 20], \\
  e^{A_{22}(t-20)}e^{A_{11}20} & \text{for } t \in [20, 25], \\
  e^{A_{23}(t-25)}e^{A_{11}20} & \text{for } t \geq 25 
\end{cases}
\]

Now in a similar fashion to Figure 8 we plot the quantities \(\hat{\phi}_{11}(t), \hat{\phi}_{12}(t), \hat{\phi}_{13}(t), \) and \(\hat{\phi}_{14}(t)\) in Figure 9.

Also, in a similar fashion to Figure 7 we plot the quantities \(\hat{\phi}_{21}(t), \hat{\phi}_{22}(t), \hat{\phi}_{23}(t), \) and \(\hat{\phi}_{24}(t)\) in Figure 10.

Moreover, in a similar fashion to Figure 5 we plot the quantities \(\hat{\phi}_{31}(t), \hat{\phi}_{32}(t), \hat{\phi}_{33}(t), \) and \(\hat{\phi}_{34}(t)\) in Figure 11.

In addition, in a similar fashion to Figure 6 we now plot the quantities

\[
\hat{\phi}_{\text{ave}}^{31}(T) = \frac{1}{T} \int_0^T \hat{\phi}_{31}(t)dt \\
\hat{\phi}_{\text{ave}}^{32}(T) = \frac{1}{T} \int_0^T \hat{\phi}_{32}(t)dt \\
\hat{\phi}_{\text{ave}}^{33}(T) = \frac{1}{T} \int_0^T \hat{\phi}_{33}(t)dt \\
\hat{\phi}_{\text{ave}}^{34}(T) = \frac{1}{T} \int_0^T \hat{\phi}_{34}(t)dt
\]
the quantities

\[
\bar{\phi}_{41}^{\text{ave}}(T) = \frac{1}{T} \int_{0}^{T} \phi_{41}(t) dt
\]

\[
\bar{\phi}_{42}^{\text{ave}}(T) = \frac{1}{T} \int_{0}^{T} \phi_{42}(t) dt
\]

\[
\bar{\phi}_{43}^{\text{ave}}(T) = \frac{1}{T} \int_{0}^{T} \phi_{43}(t) dt
\]

\[
\bar{\phi}_{44}^{\text{ave}}(T) = \frac{1}{T} \int_{0}^{T} \phi_{44}(t) dt
\]

in Figure 14.

In this paper we have introduced a notion of a direct coupling observer for closed quantum linear systems and given a result which shows how such an observer can be constructed. The main result shows the time average convergence properties of the direct coupling observer. We have also presented an illustrative example along with simulations to investigate the behavior of a direct coupling observer

when applied to a simple one mode quantum linear system. Future research in this area might involve extending the class of quantum linear systems for which a direct coupling observer can be designed and also considering the problem of constructing an observer which is optimal in some sense. Also, future research could investigate the role of direct coupling observers in the design of direct coupling coherent quantum feedback control systems.

**ACKNOWLEDGEMENT**

The author would like to thank Dr. Igor Vladimirov for his useful comments.

**REFERENCES**

[1] Z. Miao and M. R. James, “Quantum observer for linear quantum stochastic systems,” in *Proceedings of the 51st IEEE Conference on Decision and Control*, Maui, December 2012.

[2] I. Vladimirov and I. R. Petersen, “Coherent quantum filtering for physically realizable linear quantum plants,” in *Proceedings of the 2013 European Control Conference*, Zurich, Switzerland, July 2013, arXiv:1307.3154.

[3] Z. Miao, L. A. D. Espinosa, I. R. Petersen, V. Ugrinovskii, and M. R. James, “Coherent quantum observers for finite level quantum systems,” in *Australian Control Conference*, Perth, Australia, November 2013.

[4] M. R. James, H. I. Nurdin, and I. R. Petersen, “H∞ control of linear quantum stochastic systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008, arXiv:quant-ph/0703150.

[5] H. I. Nurdin, M. R. James, and I. R. Petersen, “Coherent quantum LQG control,” *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009, arXiv:0711.2551.

[6] A. J. Shaiju and I. R. Petersen, “A frequency domain condition for the physical realizability of linear quantum systems,” *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2033 – 2044, 2012.

[7] C. Gardiner and P. Zoller, *Quantum Noise*. Berlin: Springer, 2000.

[8] H. Bachor and T. Ralph, *A Guide to Experiments in Quantum Optics*, 2nd ed. Weinheim, Germany: Wiley-VCH, 2004.

[9] A. I. Maalouf and I. R. Petersen, “Bounded real properties for a class of linear complex quantum systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 786 – 801, 2011.

[10] H. Mabuchi, “Coherent-feedback quantum control with a dynamic compensator,” *Physical Review A*, vol. 78, p. 032323, 2008.

[11] G. Zhang and M. James, “Direct and indirect couplings in coherent feedback control of linear quantum systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1535–1550, 2011.

[12] I. G. Vladimirov and I. R. Petersen, “A quasi-separation principle and Newton-like scheme for coherent quantum LQG control,” *Systems & Control Letters*, vol. 62, no. 7, pp. 550–559, 2013, arXiv:1105.3185.

[13] I. Vladimirov and I. R. Petersen, “A dynamic programming approach to finite-horizon coherent quantum LQG control,” in *Proceedings of the 2011 Australian Control Conference*, Melbourne, November 2011, arXiv:1105.1574.

[14] R. Hamerly and H. Mabuchi, “Advantages of coherent feedback for cooling quantum oscillators,” *Physical Review Letters*, vol. 109, p. 173602, 2012.

[15] J. Gough and M. R. James, “The series product and its application to quantum feedforward and feedback networks,” *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530–2544, 2009.

[16] H. Nurdin, “Synthesis of linear quantum stochastic systems via quantum feedback networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1008 –1013, April 2010.

[17] H. I. Nurdin, M. R. James, and A. C. Doherty, “Network synthesis of linear dynamical quantum stochastic systems,” *SIAM Journal on Control and Optimization*, vol. 48, no. 4, pp. 2686–2718, 2009.

[18] H. Nurdin, “On synthesis of linear quantum stochastic systems by pure cascading,” *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2439 –2444, October 2010.

[19] I. R. Petersen, “Realization of single mode quantum linear systems using static and dynamic squeezers,” in *Proceedings of the 8th Asian Control Conference*, Kaohsiung, Taiwan, May 2011.
[20] ——, "Cascade cavity realization for a class of complex transfer functions arising in coherent quantum feedback control," *Automatica*, vol. 47, no. 8, pp. 1757–1763, 2011.

[21] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge University Press, 2010.