SHEAVES AND DUALITY IN THE TWO-VERTEX GRAPH
RIEMANN-ROCH THEOREM

NICOLAS FOLINSBEE AND JOEL FRIEDMAN

Abstract. For each graph on two vertices, and each divisor on the graph in the sense of Baker-Norine, we describe a sheaf of vector spaces on a finite category whose zeroth Betti number is the Baker-Norine “Graph Riemann-Roch” rank of the divisor plus one. We prove duality theorems that generalize the Baker-Norine “Graph Riemann-Roch” Theorem.

Contents

1. Introduction 2
2. Overview of Results 4
   2.1. The Main Modeling Result 5
   2.2. Cohomology Groups and the First Betti Number 5
   2.3. The Euler Characteristic Formula 5
   2.4. The Euler Characteristic and Graph Riemann-Roch Theorem 6
   2.5. \( \mathcal{O}_{k,r} \)-Modules and the Duality Theorems 6
   2.6. Proof of the Duality Theorems and the “Method of Grothendieck” 7
   2.7. Remarks on \( \mathcal{O}_{k,r} \), Remarks of De Shallit and Illusie 8
3. Foundations, Part 1: Bipartite Graphs, Categories, and Sheaves 8
   3.1. Bipartite Graphs 8
   3.2. Bipartite Categories 9
   3.3. Sheaves of \( k \)-Vector Spaces on Bipartite Categories 9
   3.4. Global Sections and the Zeroth Betti Number 10
4. Foundations, Part 1.5: Topological Sheaf Theory 10
5. Sheaves Modeling the Graph Riemann-Roch Rank for a Graph With Two Vertices 12
   5.1. The Definition of \( \mathcal{C}_{2V}, \mathcal{O}_{k,r}, \) and \( \mathcal{M}_{k,r,d} \) 13
   5.1.1. Definition of \( \mathcal{M}_{r,d} \) as Sheaves of Vector Spaces 13
   5.1.2. Intuitive Definition of \( \mathcal{M}_{k,r,d} \) Via \( \mathcal{O}_{k,r} \) 13
   5.2. The Graph Riemann-Roch Rank of Baker and Norine 14
   5.3. The Main Modeling Theorem 15
   5.4. A Formula for the Graph Riemann-Roch Rank for a Graph with Two Vertices 15
   5.5. Alternative Description of GRRR\(_r\)(d) 16
   5.6. Counting Lattice Points in the Down Cone 16
   5.7. Conclusion of the Proof of Theorem 5.5 18

Date: July 26, 2022.
2010 Mathematics Subject Classification. Primary: 05C38, 14H55. Secondary: 55N30.
Research supported in part by an NSERC grant.
Research supported in part by an NSERC grant.
1. INTRODUCTION

The main goals of this article are to give an algebraic model of the Graph Riemann-Roch Theorem of Baker-Norine [BN07] in the special case of a graph with two vertices, and—in our algebraic model—prove a duality theorem that generalizes the Graph Riemann-Roch Theorem. Our algebraic model was formulated
by a process of trial and error and seems a bit ad hoc to us; there may well be entirely different ways to algebraically model the Graph Riemann-Roch Theorem.

Perhaps the main result in this paper is our method of modeling the Graph Riemann-Roch theorem with sheaves and algebra. The machinery we use to prove more general duality theorems is based on standard tools used for modern, sheaf theoretic proofs of the classical Riemann-Roch theorem for algebraic curves and common ideas in topos theory. However, there is some work needed to formulate and verify the duality theorems we prove that generalize the Graph Riemann-Roch theorem.

We believe—for a number of reasons—that this article just scratches the surface of the connection between sheaves and the Baker-Norine theorem. First, we only study the Baker-Norine theorem for graphs with two vertices. Second, there are likely better ways to understand the sheaves we use in this article. Third, there are a number of aspects for further study of these sheaves; we mention some in the closing section of this article.

This is part of a general line of research ([Fri05, Fri06, Fri07, Fri15, FIS15, Izs15]) to investigate Grothendieck topologies on finite categories and their possible applications to problems in computer science theory, discrete mathematics, and linear algebra. The main inspiration of this line of research is based on Ben-Or’s suggestion—in view of work on algebraic computations [DL76, SY82, BO83]—that to settle problems in Boolean complexity theory one should search for cohomology theories connected to Boolean functions; according to Ben-Or’s work, even if the zeroth Betti number in such a cohomology theory is not particularly interesting, the higher Betti numbers may also be able to produce lower bounds in complexity theory. This investigation of Grothendieck topologies has uncovered a number of surprising connections to graph theory and linear algebra, notably a solution to the Hanna Neumann Conjecture of the 1950’s [Fri15]; another such connection is to linear algebra, namely a study of 2-independence in [Izs15]. Furthermore, there are other problems in computer science theory that can be stated in terms of sheaves on graphs, such as the construction of relative expanders [Fri03, FK14, MSS15a, MSS15b]. For the above reasons we believe that the application of Grothendieck topologies to these fields is likely to yield further results.

We believe that the model and duality theorems in this article may not only shed light on the work of Baker-Norine, but also adds to the foundational tools and examples of interesting sheaves based on finite categories.

The sheaves we study look similar to what one would get from a Čech cohomology computation of an algebraic curve, but seem fundamentally different. One reason for this is that the first Betti number of the structure sheaf is not generally finite.

Our methods are simple adaptations of known methods in algebraic topology and geometry that closely mimic standard methods and theorems for algebraic curves. The duality theorems we prove are akin to the Serre duality theorem, although only “half” of the duality theorem holds for sheaves of interest to us; strong duality seems to only hold for certain torsion skyscraper sheaves. We prove our duality theorems using the method of Grothendieck, whereby we prove the theorems for certain sheaves, and infer the rest by short exact sequences and the five-lemma.

While do not entirely understand our models in a simple way, we will make a number of remarks regarding our model. The category we use for the models in this article is a category with five objects, and is the opposite category of those used in
We discuss the possibility of local methods using such categories, and explain how this approach may be of interest to related discrete structures, such as more general graphs and graphs of groups ([Ser03]).

This version corrects the original version of this preprint. In the original version, an erroneous version of Serre duality was claimed for the $O$-modules in this paper that model the two-vertex Riemann-Roch rank. We believe that there is a version of Serre duality in this situation, although it is more involved and subtle. We hope to address this in a future paper. For ease of reading, we will make our corrections in RED PRINT. All the other results, including a number of interesting foundational remarks, hold.

The rest of this article is organized as follows. In Section 2 we give an overview of the main results in this paper, using terminology to be defined in later subsections; the reader familiar with sheaf theory and/or with the Baker-Norine Graph Riemann-Roch Theorem [BN07] will likely understand some of this terminology. In Section 3 we give some fundamental definitions regarding the categories and sheaves that we use. Section 4 discusses the connection of our notion of a sheaf to the classical notion of a sheaf on a topological space; it is not essential to the rest of the paper, but the reader familiar with sheaves on topological spaces will likely benefit from the discussion. The main modeling theorem in this paper is proven in Section 5, which expresses the Baker-Norine Graph Riemann-Roch rank plus one as the zeroth Betti number of a sheaf on a finite category. Sections 6 and 7 make some Betti number computations we use in Section 8, where we compute the Euler characteristics of our sheaves; this computation shows that Baker-Norine theorem is equivalent to a duality result regarding the zeroth and first Betti numbers of our sheaves that model the Graph Riemann-Roch rank. In Section 9 we give some general statements about tools that are useful in cohomology, and in Section 10 we describe the specific implications for our situation. In Sections 11 and 12 we prove partial results regarding duality and the Graph Riemann-Roch Theorem. Although our generalization regards “global sections” and “global Ext groups,” there are a number of indications that these are special cases of more general “local theorems” that are more general and—in a sense—easier to prove. We give a very concrete statement of a “local version” of some of our partial duality theorems in Section 13. In Section 14 we make a number of remarks regarding future directions of research, beginning with some longer remarks on “local methods” for finite categories—including graphs and graphs of groups—and ending with a few brief remarks.

We wish to thank Ehud De Shallit and Luc Illusie for discussions regarding our models; their observations will be described in Section 2.

Throughout this article, [SGA4] refers to [sga72]; [EGA1] refers to [GD71].

2. Overview of Results

In this section we summarize the main results of this paper, using terminology to be defined later in this paper. The reader familiar with sheaf theory or the Baker-Norine graph Riemann-Roch theorem will likely be familiar with some of this terminology already.
2.1. The Main Modeling Result. Our main “modeling” result is that for any field, $k$, integer $r \geq 1$, and $d \in \mathbb{Z}^2$, we prove that

\[(1)\quad b^0(\mathcal{M}_{k,r,d}) = \text{GRRR}_r(d) + 1,\]

where

1. $\text{GRRR}_r(d)$ denotes the graph Riemann-Roch rank—in the sense of Baker-Norine—of the divisor $d$ on a graph with two vertices joined by $r$ edges,
2. $b^0$ is the zeroth Betti number, i.e., the dimension of the $k$-vector space of global sections of a sheaf, and
3. $\mathcal{M}_{k,r,d}$ are sheaves of $k$-vector spaces on fixed finite category $\mathcal{C}_{2V}$ (see Definition 5.2 and Figures 4 and 5).

We emphasize that the above formula is valid for an arbitrary field $k$, and at times we omit the $k$ in our notation for brevity.

The category $\mathcal{C}_{2V}$ is category with five elements, endowed with its coarsest topology; a sheaf of $k$-vector spaces is therefore a contravariant functor from $\mathcal{C}_{2V}$ to the category of $k$-vector spaces; equivalently, we may think of $\mathcal{C}_{2V}$ as a topological space with five points, five irreducible open subsets (which are the “stalks” of the five points), and 13 open subsets (see Figure 3).

2.2. Cohomology Groups and the First Betti Number. For any sheaf of vector spaces, $\mathcal{F}$, on $\mathcal{C}_{2V}$, the cohomology groups of $\mathcal{F}$, denoted $H^0(\mathcal{F}), H^1(\mathcal{F})$, are the right derived functors of $\mathcal{F} \mapsto \Gamma(\mathcal{F})$, and can be computed as the kernel and cokernel (respectively) of

\[(2)\quad \mathcal{F}(B_1) \oplus \mathcal{F}(B_2) \oplus \mathcal{F}(B_3) \to \mathcal{F}(A_1) \oplus \mathcal{F}(A_2);\]

we set $b^i(\mathcal{F}) = \dim_k H^i(\mathcal{F})$. [The maps $\mathcal{F}(B_3) \to \mathcal{F}(A_i)$ are minus the restriction maps, but this doesn’t change the $b^i$.]

We can also see that to any short exact sequence of sheaves, there is a long exact sequence in cohomology, as usual.

We give a general method to compute the first Betti number, i.e., $b^1$, for many sheaves of interest to us, including sheaves of the form $\mathcal{O}_{k,r}, \mathcal{M}_{k,r,d}$.

2.3. The Euler Characteristic Formula. Next we derive the formula

\[(3)\quad \chi(\mathcal{M}_{r,d}) = d_1 + d_2 - (r - 1),\]

where $\chi(\mathcal{F}) = b_0(\mathcal{F}) - b_1(\mathcal{F})$. Our proof is to verify this formula for $d = 0$, and to use a short exact sequence

\[(4)\quad 0 \to \mathcal{M}_{r,d} \to \mathcal{M}_{r,d+1(0)} \to \text{Sky}(B_1, k) \to 0\]

where $\text{Sky}(B_1,k)$ is a skyscraper sheaf (this exact sequence mimicks the standard one for algebraic curves, e.g., [Har77], page 296, just above Remark IV.1.3.1). The long exact sequence for (4) then implies—after a simple computation of the Betti numbers of the above skyscraper sheaf—that if both Betti numbers of $\mathcal{M}_{r,d}$ are finite, or both of $\mathcal{M}_{r,d+1(0)}$ are finite, then all are finite and

$\chi(\mathcal{M}_{r,d+1(0)}) = \chi(\mathcal{M}_{r,d}) + 1$.

Doing the same for $(0,1)$ replacing $(1,0)$ and $B_2$ replacing $B_1$ in the skyscraper), we see that

$\chi(\mathcal{M}_{r,d}) = \chi(\mathcal{M}_{r,0}) + d_1 + d_2 = d_1 + d_2 - (r - 1)$

which implies (3).
2.4. The Euler Characteristic and Graph Riemann-Roch Theorem. The formula (3) says that
\[ b_0(M_{r,d}) - b_1(M_{r,d}) = d_1 + d_2 - (r - 1). \]
Using (1), we see that the Baker-Norine Graph Riemann-Roch Theorem is equivalent to the formula
\[ b_0(M_{r,d}) - b_0(M_{r,K_r - d}) = d_1 + d_2 - (r - 1) \]
for what Baker and Norine call the canonical divisor, \( K_r = (r - 2, r - 2) \). It follows that the Baker-Norine Theorem can be stated as
\[ b_1(M_{r,d}) = b_0(M_{r,K_r - d}) . \]
We know that (5) holds since the Baker-Norine Theorem is true; we imagine that it may not be difficult to prove (5) from scratch, using the formulas that we develop for \( b(M_{r,d}) \). The rest of this paper is mainly devoted to proving a generalization of (6), which gives another proof of (6). Our generalization relies on general facts (such as the Yoneda pairing and the Method of Grothendieck), but involves a smaller number of specific calculations.

2.5. \( \mathcal{O}_{k,r} \)-Modules and the Duality Theorems. For any sheaf of rings, \( \mathcal{O} \), on \( \mathcal{C}_{2V} \), there is a standard notion of a sheaf of \( \mathcal{O} \)-modules and the morphisms \( F \to G \) of any two \( \mathcal{O} \)-modules, which we denote \( \text{Hom}_\mathcal{O}(F, G) \). In the special case where \( \mathcal{O} \) is a sheaf of \( k \)-algebras, \( \text{Hom}_\mathcal{O}(F, G) \) has the natural structure of a \( k \)-vector space. In the further special case where \( \mathcal{O} = k \), the constant sheaf \( k \), the category of sheaves of \( k \)-modules is the same thing as the category of sheaves of \( k \)-vector spaces.

In this article we will define sheaves of rings \( \mathcal{O}_{k,r} \) on \( \mathcal{C}_{2V} \) for each integer \( r \geq 1 \) and field, \( k \). The sheaves \( M_{k,r,d} \) are sheaves of \( \mathcal{O}_{k,r} \)-modules.

Let \( \omega \) be a sheaf of \( k \)-vector spaces on \( \mathcal{C}_{2V} \) such that \( b^1(\omega) = 1 \). For any sheaf of \( k \)-algebras, \( \mathcal{O} \), and \( i = 0, 1 \), we define Duality \( \text{Duality}_{\mathcal{O}, i}(\omega) \) to be class of \( \mathcal{O} \)-modules, \( F \), for which duality holds for \( H^i(F) \), in the sense that the Yoneda pairing
\[ H^i(F) \times \text{Ext}^{1-i}_\mathcal{O}(F, \omega) \to H^1(\omega) \cong k \]
is a perfect pairing, where \( \text{Ext}^1_{\mathcal{O}} \) denotes the derived functors of \( \text{Hom}_\mathcal{O} \); we define Strong – Duality \( \text{Duality}_{\mathcal{O}}(\omega) \) to be the class of \( \mathcal{O} \)-modules, \( F \), for which strong duality holds (with respect to \( \mathcal{O} \) and \( \omega \), meaning that duality holds for all \( i \geq 0 \) (hence necessarily \( H^i(F) = \text{Ext}^1_{\mathcal{O}}(F, \omega) = 0 \) for \( i \geq 2 \).

Our generalization of (5) consists of the following two theorems: for any field \( k \) and integer \( r \geq 1 \), set \( \omega_{k,r} = M_{k,r,K_r} \) with \( K_r = (r - 2, r - 2) \); then
1. \( b^1(\omega_{k,r}) = 1 \) and \( \omega_{k,r} \) has an injective resolution of length two;
2. for \( i = 1, 2 \), Strong – Duality \( \text{Duality}_{\mathcal{O}_{k,r}}(\omega_{k,r}) \) contains the (torsion skyscraper) sheaf \( S_i = \text{Sky}(B_i, k[y_i]/y_i^2, k[y_i]) \);
3. a previous version claimed that \( M_{k,r,d} \in \text{Duality}_{\mathcal{O}_{k,r}}(\omega_{k,r}) \) for any \( d \in \mathbb{Z}^2 \), however this is not generally true;
4. for any \( d, d' \in \mathbb{Z}^2 \) there is a canonical morphism a previous version claimed that this was an isomorphism
\[ \text{Hom}_{\mathcal{O}_{k,r}}(M_{r,d}, M_{r,d'}) \to \Gamma(M_{r,d' - d}). \]
We will prove (8) by a simple computation: in the case where \( d = 0 \), the morphism is induced a natural inclusion \( \mathcal{O}_{k,r} \to M_{k,r,0} \); the case of general \( d \) is proved similarly. We will see that strong duality cannot hold for sheaves of the form...
More specifically, we define line bundles, \( L_{k,r,d} \), that are subsheaves of \( \mathcal{M}_{k,r,d} \), where \( L_{k,r,0} \) is just the structure sheaf (just as for algebraic curves). From the duality theorems and some easier computations, we derive a number of interesting formulas including

\[ L_{k,r,d} \otimes \mathcal{M}_{k,r,d'} \simeq \mathcal{M}_{k,r,d+d'} , \]

and the duality theorem

\[ H^1(\mathcal{M}_{k,r,d}) \simeq \text{Hom}_{\mathcal{O}_{k,r}}(\mathcal{M}_{k,r,d}, \omega_{k,r})^* , \quad \text{where} \quad \omega_{k,r} = \mathcal{M}_{k,r,K}. \]

A consequence of the above results is (the less interesting result) that

\[ H^0(\mathcal{M}_{k,r,d}) \simeq \text{Ext}^1_{\mathcal{O}_{k,r}}(L_{k,r,d}, \omega_{k,r})^* \]

(which is the correct \( H^0 \) statement resembling Serre duality, except that \( L_{k,r,d} \) must be used instead of \( \mathcal{M}_{k,r,d} \) for the right-hand-side); this result follows from the above since

\[ \text{Ext}^1_{\mathcal{O}}(L_{k,r,d}, \omega_{k,r}) \simeq \text{Ext}^1_{\mathcal{O}_{k,r}}(\mathcal{O}, \mathcal{M}_{k,r,K,-d}) \simeq H^1(\mathcal{M}_{k,r,K,-d}) \]

(with \( \mathcal{O} = \mathcal{O}_{k,r} \), whereupon we apply the above duality theorem and (8).

We will also prove that there is a morphism

\[ \text{Hom}(\mathcal{M}_{k,r,d}, \mathcal{M}_{k,r,d'} \to \mathcal{M}_{k,r,d'-d}, \]

but that

\[ \mathcal{M}_{k,r,d} \otimes \mathcal{M}_{k,r,d'} \]

has an interesting “twisted” structure.

2.6. **Proof of the Duality Theorems and the “Method of Grothendieck”**.

The “Method of Grothendieck” implies that for any sheaf of \( k \)-algebras \( \omega \) on \( \mathcal{C}_{2V} \) with \( b^1(\omega) = 1 \),

1. if any two elements of any short exact sequence are contained in Strong - Duality(\( \omega \)), then all three are; and
2. if \( 0 \to F_1 \to F_2 \to F_3 \to 0 \) is exact, \( \text{Ext}^2(F_1, \omega) = 0 \), and \( H^1 \) duality holds for \( F_1 \) and \( F_2 \), then \( H^1 \) duality holds for \( F_3 \).

We first verify that \( \text{Sky}(B_1, k) \in \text{Duality}(\omega) \), where when \( \text{Sky}(B_1, k) \) is viewed as an \( \mathcal{O}_{k,r} \)-module by the exact sequence (4) and the structure of the \( \mathcal{M}_{k,r,d} \) as \( \mathcal{O}_{k,r} \)-modules; this means that \( k \) is really

\[ \widetilde{k} \overset{\text{def}}{=} k[y_1]/(y_1) = k[y_1]/y_1k[y_1], \]

so that \( k \) is a set of representatives of the classes of \( \widetilde{k} \), and \( k[y_1] \) acts on \( \widetilde{k} \) in that \( y_1 \) annihilates \( \widetilde{k} \). We use \( \widetilde{k} \) instead of \( k \) to remind ourselves that \( \widetilde{k} \) is \( k \) with this particular structure of a \( k[y_1] \)-algebra. We similarly show that \( \text{Sky}(B_2, k) \in \text{Duality}(\omega) \) where this time \( \widetilde{k} \) really means \( k[y_2]/y_2k[y_2] \).

It then follows from the Method of Grothendieck that \( \mathcal{M}_{k,r,d} \in \text{Duality}(\omega) \) for all \( d \) provided that this holds for any one value of \( d \); we then verify that it holds for any \( d \) such that \( d_1 + d_2 \) is sufficiently smaller than zero. However \( \mathcal{M}_{k,r,d} \in \text{Duality}(\omega) \) can only hold in very simple cases, such as \( r = 1 \); hence a duality theory for \( \mathcal{M}_{k,r,d} \) needs to remedy this problem and is likely more subtle.
2.7. **Remarks on** $O_{k,r}$, **Remarks of De Shallit and Illusie.** For $r = 1$, the sheaf $O_{k,r}$ represents the algebraic curve of genus zero, where the sphere is covered by two “hemispheres,” one omitting $\infty$ and one omitting 0. For $r \geq 2$ the sheaf $O_{k,r}$ does not represent an algebraic curve; one way to see this is that $b^1(O_{k,r})$ is infinite for such $r$.

For $r \geq 2$, part of $O_{k,r}$ is identical to $O_{k,1}$ and represents two hemispheres. Ehud De Shallit has pointed out to us that for $r \geq 2$ one can view the hemispheres as being glued by a clutching map (or cylinder) that wraps $r$ times around in the clutching. Hence we get a topological idea (at least for $k = \mathbb{C}$) of how to think of $O_{k,r}$. The sheaves $M_{k,r}$ are standard line bundles away from this “clutching,” but are of rank $r$, rather than one, on the cylinder that performs this clutching.

Luc Illusie has made a number of interesting remarks and asked us some interesting questions regarding the $O_{k,r}$, such as asking whether they represent an algebraic curve (which they do not seem to, since $b^1(O_{k,r}) = \infty$). Regarding another remark of Illusie: the sheaves $M_{k,r,d}$ are all finitely generated $O_{k,r}$-modules; however the sheaves $M_{k,r,d}$ are not coherent $O_{k,r}$ because of the rank $r$ part of $M_{k,r}$. We remark that all values of $O_{k,r}$ are PID’s, and hence any finitely generated $O_{k,r}$ module has values that are finite sums of free modules and torsion modules. We also mention that at times $M_{k,r,0}$ seems like a “good substitute” for $O_{k,r}$; for example, $b^0(M_{k,r,0}) = 1$ and $b^1(M_{k,r,0}) = r$, which are the correct Betti numbers for a curve of genus $r - 1$ (which is the genus that Baker-Norine define for the graph on two vertices joined by $r$ edges). Another interesting question of Illusie is whether or not for genus $r - 1 \geq 1$ our construction chooses a particular curve (or algebraic space, etc.), since there are infinitely many such curves of fixed genus at least one over an algebraically closed field. Also the map

$$\left(\text{Hom}_{O}(F, G)\right)_x \to \text{Hom}_{O_x}(F_x, G_x),$$

(see [EGA1], Section 0.5.2.6 or [Har77], Proposition III.6.8) is not an isomorphism at $x = B_3$ (this corresponds to the clutching cylinder) for sheaves $F = M_{r,d}$ and $G = M_{r,d'}$ and $r \geq 2$.

3. **Foundations, Part 1: Bipartite Graphs, Categories, and Sheaves**

In this section we define the minimum regarding bipartite categories and sheaves in order to state our main modeling theorem, Theorem 5.5 in Section 5. Beyond this we give some common terminology regarding bipartite graphs, and give a few examples of sheaves on bipartite categories.

3.1. **Bipartite Graphs.**

**Definition 3.1.** By a bipartite graph we mean a triple $G = (A, B, E)$ where $A, B$ are finite sets—the sets of left vertices of $G$ and right vertices $G$ respectively—and $E \subset A \times B$, called the edges of $G$. We refer to the disjoint union $A \amalg B$ as the set of vertices of $G$.

In graph theory, this would be called, more precisely, a finite bipartite graph with a given bipartition and without multiple edges. One could allow for $A, B, E$...
to be arbitrary (possibly infinite) sets, but in this article all graphs are finite unless otherwise indicated.

**Definition 3.2.** Let $G = (A, B, E)$ be a bipartite graph.

1. By a *subgraph* of $G$ we mean a graph $G' = (A', B', E')$ such that $A' \subseteq A$, $B' \subseteq B$, and $E' \subseteq E$ (and therefore $E' \subseteq E \cap (A' \times B')$). In this case we write $G' \subseteq G$ and say that $G'$ is *included* in $G$.

2. We say that $a \in A$ and $b \in B$ are *adjacent* if $(a, b) \in E$.

3. We say that $G$ is *connected* if for any two vertices, $v, v' \in A \cup B$ there is a sequence $v = v_0, v_1, \ldots, v_k = v'$ of vertices such that $v_i$ and $v_{i+1}$ are adjacent for all $i = 0, \ldots, k - 1$.

4. By a *connected component* of $G$ we mean a connected subgraph $G' \subseteq G$ that is maximal with respect to inclusions (i.e., if $G' \subseteq G'' \subseteq G$ and $G''$ is connected, then $G'' = G'$).

### 3.2. Bipartite Categories.

**Definition 3.3.** Let $G = (A, B, E)$ be a bipartite graph. The *bipartite category associated to $G$* is the category $\mathcal{C}$ given as follows:

1. Its set of objects, $\text{Ob}(\mathcal{C})$, is the set $A \cup B$ of vertices of $G$;

2. Its morphisms, $\text{Fl}(\mathcal{C})$, consist of identity morphisms and $E$, where each $(a, b) \in E = \text{Fl}(\mathcal{C})$ has source $a$ and target $b$.

Note that there is no need to define composition for simple categories, since if two morphisms are composable then at least one morphism is an identity morphism.

**Definition 3.4.** By the *bipartite category* $(A, B, E)$ we mean the category, $\mathcal{C}$, associated to a bipartite graph $G = (A, B, E)$ as above. We refer to $A$ in $A \cup B$ as the *left objects* of $\mathcal{C}$, and similarly to $B$ as the *right objects*. When we speak of $(a, b) \in E$, we view $b \in B$ as the corresponding right object of $\mathcal{C}$, and $a \in A$ as the corresponding left object.

By our conventions $A, B$ are finite sets unless otherwise indicated.

### 3.3. Sheaves of $k$-Vector Spaces on Bipartite Categories.

**Definition 3.5.** Let $k$ be a field, $G = (A, B, E)$ a bipartite graph, and $\mathcal{C}$ the associated bipartite category. By a *sheaf (of $k$-vector spaces)* on $\mathcal{C}$ we mean the data, $\mathcal{F}$, consisting of:

1. A $k$-vector space $\mathcal{F}(P)$ for each object, $P \in \text{Ob}(\mathcal{C}) = A \cup B$, and

2. For each non-identity morphism of $\mathcal{C}$, i.e., each $e = (a, b) \in E$, a linear map $\mathcal{F}(e) : \mathcal{F}(b) \to \mathcal{F}(a)$.

**Example 3.6.** Consider any any $k, G, \mathcal{C}$ in Definition 3.5. If $M$ is a $k$-vector space, then we define the *constant sheaf* $M$, denoted $\underline{M}$, to be the sheaf all of whose values are $M$, and all of whose restrictions are the identity maps. In particular $k$ is sheaf whose values are all $k$ and whose restrictions are the identity map.

**Example 3.7.** Consider any any $k, G, \mathcal{C}$ in Definition 3.5, and consider a subgraph $G' = (A', B', E')$ of $G$. If $\mathcal{F}$ is a sheaf on $G$, then we use $\mathcal{F}_{G'}$ to denote the following sheaf: its values and restrictions are given by those of $\mathcal{F}$ for vertices and edges in $G'$, and its other values are 0, and other restriction maps the zero map. We call $\mathcal{F}_{G'}$ the *extension by zero of the restriction of $\mathcal{F}$ to $G'$*.
The above example will be elaborated upon in Definition 6.4. The sheaves $k_{G'}$ were a fundamental building block of the sheaves in [Fri15].

3.4. Global Sections and the Zeroth Betti Number.

**Definition 3.8.** Let $\mathcal{F}$ be a sheaf of $k$-vector spaces on $\mathcal{C}$, the bipartite category $(A, B, E)$. By *global section* of $\mathcal{F}$ we mean the data consisting of elements $f_P \in \mathcal{F}(P)$ for each $P \in \text{Ob}(\mathcal{C}) = A \cup B$ such that for each $(a, b) \in E$ we have $\mathcal{F}(e)F(b) = \mathcal{F}(a)$. We denote the set of global sections by $\Gamma(\mathcal{C}, \mathcal{F})$ or $\Gamma(\mathcal{F})$; this is a $k$-vector space in evident fashion, and we define the zero-th Betti number of $\mathcal{F}$, denoted $b^0(\mathcal{F})$, to be $b^0(\mathcal{F}) = \dim_k \Gamma(\mathcal{F})$.

**Example 3.9.** If $k$ is a field and $\mathcal{C}$ the bipartite category $(A, B, E)$, then $b^0(k)$ is the number of connected components of $G$. More generally, if $G' \subset G$ is a subgraph, then $b^0(k_{G'})$ is the number of connected components of $G'$.

4. Foundations, Part 1.5: Topological Sheaf Theory

The following section is independent of the rest of this article. However, this section will motivate some of the definitions in this paper, especially that of sheaf $\text{Hom}$ given in Section 9.

In this section we assume the reader is familiar with sheaves on topological spaces (e.g., [Har77], Section 2.1), and describe the fundamentals of bipartite graphs, categories, and sheaves from this point of view.

To fix ideas we first work with $\mathcal{C}_{2V}$, and then make some remarks on the more general case.

**Definition 4.1.** By the *two vertex graph Riemann-Roch topological space*, denoted $X_{2V}$, we mean the topological space on the five point set $X = X_{2V} = \{A_1, A_2, B_1, B_2, B_3\}$ with a basis for the topology given by the subsets:

1. For $i = 1, 2$
   
   $U_{A_i} = \{A_i\}, \quad U_{B_i} = \{A_i, B_i\},$

   and

2. $U_{B_3} = \{A_1, A_2, B_3\}$.

Figure 1 shows the five points and $X$ and the basis $\{U_x\}_{x \in X}$ for its topology. Figure 2 a diagram of this basis $\{U_x\}_{x \in X}$ and arrow representing the inclusions between these sets, and the diagram of the points of $X$ and arrows indicating which points are specializations of other points (these two diagrams are essentially the same).

These diagrams are bipartite graphs, and the associated bipartite category (which we define as $\mathcal{C}_{2V}$ in Section 5) will be the underlying category used for all the main results in this paper.

The category $\mathcal{C}_{2V}$ corresponds to either of these diagrams (that are essentially the same). Figure 3 depicts the entire collection of open subsets of $X$, with arrows representing inclusions.

It is not hard to check that (1) for each $x \in X$, $U_x$ is the smallest open subset of $X$ containing $x$ (so the value of a sheaf $\mathcal{F}$ on $U_x$ represents the stalk of $\mathcal{F}$ at $x$)
and (2) the $\{U_x\}_{x \in X}$ are precisely the open subsets that are irreducible in the sense that they cannot be written as a union of proper subsets.\footnote{The empty set is the empty union and hence is not irreducible; the reader may alternatively just accept that by definition the empty set is not irreducible.}
A sheaf, $F$, on $X$ in the classical sense (see Figure 3) gives rise to a sheaf $F'$ in our sense by restricting its values and restrictions to the subcategory of the open sets $\{U_x\}_{x \in X}$, which is the category $C_{2V}$. This gives a “restriction functor,” and we easily verify that this is an equivalence of categories as follows: given any sheaf $F'$ on $C_{2V}$ in our sense, we can extend $F'$ to a sheaf $F$ on $X$ as follows: (1) given any open subset $W \subset X$, let $C_{2V}|_W$ be the full subcategory of $C_{2V}$ on the objects $x \in W$; (2) define $F(W)$ to be the limit of the functor (as in [SGA4] Definition I.2.1) that is the restriction of $F'$ to $C_{2V}|_W$ (this limit exists and is unique up to unique isomorphism); (3) we check that $F$ is a sheaf on $X$ in the classical sense on $F'$; (4) clearly the restriction of this $F$ to $C_{2V}$ is just $F'$; (5) we check that the “restriction functor” is fully faithful.

Let us discuss the relative advantages of working with a classical sheaf $F$ as opposed with its restriction $F'$ to $C_{2V}$, which is the notion we use in this article (and used in [Fri15]). For one, $\Gamma(X, F) = \Gamma(C_{2V}, F')$ is simply $F(X)$, which provides useful intuition and explains the term global section. On the other hand, organizing the information as $F'$ seems simpler than the classical notion; also the construction of cokernels and other limits is done “value-by-value” in our notion; classically one needs to sheafify certain limits.

One pedagogical advantage of the classical notion of sheaf is that it is quite commonly used in many areas of mathematics, and there are many excellent expositions that describe a number of concepts that work well in practice. For example, we know what is good definition of sheaf Hom (see, e.g., [Har77], Exercise II.1.15) and what is meant by saying that to verify that an evident morphism $\text{Hom}(M_{k,r,d}, M_{k,r,d'}) \to M_{k,r,d'} - d$ is an isomorphism; it suffices to check this locally.

One advantage of our notion of sheaf, i.e., working with presheaves on a category $C$ endowed with the topologie grossi`ere, is that it is more general (than finite topological spaces). There are finite structures, such as graphs with whole-loops and half-loops, and graphs of groups, which are not topological spaces, where topos theory gives us generalizations of definitions that tend to work well; for example, with the topologie grossi`ere, the notion of locally at an object $P \in \text{Ob}(C)$ means we consider the slice category $C/P$ (i.e., of objects over $P$) and the functor $C/P \to C$. This tells us how to define sheaf Hom and related notions more generally for the aforementioned finite structures, and explains why locally these structures have the same structure as graphs; we shall make some more remarks on this in Section 14.

5. Sheaves Modeling the Graph Riemann-Roch Rank for a Graph With Two Vertices

In this section we will define a simple category $C_{2V}$ and a class of sheaves, $M_{k,r,d}$, on $C_{2V}$, depending on a field, $k$, an integer $r \geq 1$, and a $d = (d_1, d_2) \in \mathbb{Z}^2$. The main theorem of this section is that

$$b_0(M_{r,d}) - 1 = \text{GRRR}_r(d_1, d_2),$$

where the right-hand-side denotes the Baker-Norine graph Riemann-Roch rank of the divisor $d$ on a graph with two vertices. We begin by defining the $M_{k,r,d}$ in two ways, and then review the Baker-Norine notion of rank for a divisor on a graph.
5.1. The Definition of $C_{2V}$, $O_{k,r}$, and $M_{k,r,d}$. In this section we give two equivalent definitions of $M_{r,d}$.

5.1.1. Definition of $M_{r,d}$ as Sheaves of Vector Spaces.

**Definition 5.1.** Let $C_{2V}$ be the bipartite category $(A, B, E)$ where

$$A = \{A_1, A_2\}, \quad B = \{B_1, B_2, B_3\}$$

and

$$E = \{(A_1, B_1), (A_2, B_2), (A_1, B_3), (A_2, B_3)\}.$$ 

We depict $C_{2V}$ in in Figure 4.

Recall that if $k$ is a field and $x$ an indeterminate, then $k[x]$ denotes the polynomials in $x$ with coefficients in $k$; and similarly, $k[x, 1/x]$ denotes the polynomials in $x$ and $1/x$, i.e. the Laurent polynomials in $x$.

**Definition 5.2.** For a field, $k$, integer $r \geq 1$, and $d \in \mathbb{Z}^2$, we define a sheaf of $k$-vector spaces $\mathcal{M} = \mathcal{M}_{k,r,d}$ on $C_{2V}$ as follows:

1. its values are

$$\mathcal{M}(B_1) = k[y_1], \quad \mathcal{M}(B_2) = k[y_2], \quad \mathcal{M}(B_3) = k[v, 1/v]^{\otimes r}$$

(i.e., the direct sum of $r$ copies of the $k$-vector space $k[v, 1/v]$),

$$\mathcal{M}(A_1) = k[x_1, 1/x_1], \quad \mathcal{M}(A_2) = k[x_2, 1/x_2],$$

where $x_1, x_2, y_1, y_2, v$ are indeterminates;

2. its restriction maps are given as follows:

(a) for $i = 1, 2$, $\mathcal{M}(A_i, B_i)(p(y_i)) = x_i^d p(1/x_i)$;

(b) $\mathcal{M}(A_1, B_3)(q_1(v), \ldots, q_r(v)) = q_1(x_1^r) + x_1 q_2(x_1^r) + \cdots + x_1^{r-1} q_{r-1}(x_1^r)$;

(c) $\mathcal{M}(A_2, B_3)(q_1(v), \ldots, q_r(v)) = q_1(x_2^{-r}) + x_2 q_2(x_2^{-r}) + \cdots + x_2^{-r-1} q_{r-1}(x_2^{-r})$.

We will often suppress the field $k$ and the integer $r$ in the notation $\mathcal{M}_{k,r,d}$. All the results in this article are independent of $k$ (which is not assumed to be algebraically closed or even infinite).

5.1.2. Intuitive Definition of $\mathcal{M}_{k,r,d}$ Via $O_{k,r}$. Here we give a simpler and more intuitive way to describe the $\mathcal{M}_{k,r,d}$ in terminology that we now explain. We view $\mathcal{M}_{k,r,d}$ (in Figure 5) as a sheaf of $O_{k,r}$-modules for a sheaf of $k$-algebras $O_{k,r}$ (see Figure 4) that we now define.
\[ M(A_1) = k[x_1, 1/x_1] \leftarrow k[y_1] = M(B_1) \]
\[ M(A_2) = k[x_2, 1/x_2] \leftarrow k[y_2] = M(B_2) \]
\[ k[v, 1/v]^{gr} = M(B_3) \]

**Figure 5.** Depiction of \( M = M_{k,r,d} \) as an \( O = O_{k,r} \)-module.

**Definition 5.3.** For a field, \( k \) and an integer \( r \geq 1 \) we define \( O_{k,r} \) as follows: its values are the \( k \)-algebras

\[
O_{k,r}(B_1) = k[y_1], \quad O_{k,r}(B_2) = k[y_2], \quad O_{k,r}(B_3) = k[v, 1/v],
\]

\[
O_{k,r}(A_1) = k[x_1, 1/x_1], \quad O_{k,r}(A_2) = k[x_2, 1/x_2],
\]

where \( x_1, x_2, y_1, y_2, v \) are indeterminates and the restrictions maps are the unique morphisms of \( k \)-algebras for which we have

\[
y_1 \mapsto 1/x_1, \quad y_2 \mapsto 1/x_2, \quad v \mapsto x_1^r, \quad v \mapsto x_2^{-r}.
\]

Hence, the sheaf \( k \)-vector spaces \( O_{k,r} \) can be viewed as a *sheaf of rings* or as a *sheaf of \( k \)-algebras* in that its values and restriction maps can be viewed in the category of rings or of \( k \)-algebras.

In terms of \( M = M_{k,r,d} \), for each \( P \in \text{Ob}(\mathcal{C}) = \mathcal{A} \mathcal{I} \mathcal{B} \), \( M(P) \) is an \( \mathcal{O}(P) \)-module (free of rank \( r \) for \( P = B_3 \) and otherwise free of rank 1), and the restriction maps of \( M \) satisfy:

1. \( M(A_i, B_j)(1) = x_i^{d_i} \) for \( i = 1, 2 \);
2. \( M(A_i, B_3)(e_j) = x_i^{r-1} \) for \( i = 1, 2 \) and \( j = 1, \ldots, r \) where \( e_j \) is the standard basis vector of \( \mathcal{O}(B_3)^{gr} \) (that is 1 in the \( j \)-th component, and 0 elsewhere).

This information uniquely determines the restriction maps of \( M \) provided that we insist that \( M \) is a *sheaf of \( O_{k,r} \)-modules*, in a sense that we will formalize and discuss at length in Section 9. Formally this means that for each \( (a, b) \in E \), \( r \in \mathcal{O}(b) \) and \( m \in M(b) \),

\[
M(a, b)(rm) = (\mathcal{O}(a, b)r)(M(a, b)m).
\]

Intuitively this means that \( M(a, b) \) is compatible with the restriction map \( \mathcal{O}(a, b) \) and the module structures of the values of \( M \).

**5.2. The Graph Riemann-Roch Rank of Baker and Norine.** In this section we review the Baker-Norine notion of *Graph Riemann-Roch Rank*; see [BN07, Bak13] for details.

Let \( G = (V, E) \) be a graph (we allow multiple edges but not self-loops). The Laplacian of \( G \) is therefore a morphism \( \mathbb{Z}^V \to \mathbb{Z}^V \). By a *divisor* we mean an element of \( \mathbb{Z}^V \), and we say that two divisors are *equivalent* if their difference is in the image of the Laplacian. We say that a divisor is *effective* if all its components are non-negative. If \( \mathbf{d} \in \mathbb{Z}^V \) is a divisor, we define its *graph Riemann-Roch rank*,
GRRR\(_G(d)\), to be \(-1\) if \(d\) is not effective, and otherwise to be the largest non-negative integer, \(m\), such that for all \(m \geq 0, m \in \mathbb{Z}^V\), whose sum of components is \(m\), we have that \(d - m\) is effective.

In this article we restrict our attention to the case where \(V = \{v_1, v_2\}\) and \(E\) consists of \(r\) edges joining \(v_1\) and \(v_2\). In this case we identify \(\mathbb{Z}^V\) with \(\mathbb{Z}^2\) arbitrarily; by symmetry, the rank of \((d_1, d_2)\) is the same as that of \((d_2, d_1)\), so the rank is defined unambiguously.

**Definition 5.4.** Let \(r \geq 1\) be an integer and \(d = (d_1, d_2) \in \mathbb{Z}^2\). We use GRRR\(_G(d)\) to denote the graph Riemann-Roch rank of \(d\) as a divisor on a graph of two vertices joined by \(r\) edges (in the Baker-Norine [BN07] sense above).

Hence, GRRR\(_G(d)\) depends only on the class of \(d\) in \(\mathbb{Z}^2/(r, -r)\mathbb{Z}\).

We may equivalently define GRRR\(_G(d)\) inductively as:

1. \(-1\) if \(d_1 + d_2 \leq -1\), or if \(d_1 + d_2 = 0\) and \(r\) does not divide \(d_1 = -d_2\);
2. \(0\) if \(d_1 + d_2 = 0\) and \(r\) does divide \(d_1 = -d_2\);
3. \(1 + \min(GRRR_G(d_1 - 1, d_2), GRRR_G(d_1, d_2 - 1))\) inductively for \(d_1 + d_2 \geq 1\).

**5.3. The Main Modeling Theorem.** The following theorem is the main result of this section.

**Theorem 5.5.** Let \(k\) be any field, \(r \geq 1\) an integer, and \(d = (d_1, d_2) \in \mathbb{Z}^2\). Then

\[
b_0(M_{k,r,d}) = GRRR_r(d_1, d_2) + 1
\]

with \(M_{k,r,d}\) as in Definition 5.2, and GRRR as above.

We arrived at this theorem by ad hoc process of "trial and error," rather than a systematic method; there may be other ways to algebraically model graph Riemann-Roch questions.

The goal of the rest of this section is to prove Theorem 5.5.

**5.4. A Formula for the Graph Riemann-Roch Rank for a Graph with Two Vertices.** Let us give an explicit formula for the graph Riemann-Roch rank.

**Theorem 5.6.** With notation as above, let \(d \in \mathbb{Z}^2\) with \(0 \leq d_2 \leq r - 1\). Then

1. if \(d_1 \leq -1\), we have \(GRRR_r(d) = -1\);
2. if \(0 \leq d_1 \leq r - 1\), we have \(GRRR_r(d) = \min(d_1, d_2)\);
3. if \(r - 1 \leq d_1\), we have \(GRRR_r(d) = d_1 + d_2 - r + 1\).

We remark that any divisor, \(d\), is linearly equivalent to one where \(0 \leq d_2 \leq r - 1\) by adding the appropriate multiple of \((r, -r)\); for this reason the above theorem essentially determines all values of GRRR\(_r\).

**Proof.** The first claim is easy: if \((d_1, d_2) + m(r, -r)\) is effective, then we need \(m \geq 1\) in view of the first component, but then \(d_2 + m(-r) < 0\) so that \((d_1, d_2) + m(r, -r)\) can never be effective.

To prove the second claim, let \(k = \min(d_1, d_2) \geq 0\). To see that GRRR\(_r(d_1, d_2) \geq k\), we see that if \(m \geq 0, m_1 + m_2 = k\), then \(m_i \leq k\) for \(1, 2\), and hence \(d - m \geq 0\), which is effective; hence GRRR\(_r(d_1, d_2) \geq k\). Let us show that equality holds: by symmetry we may assume that \(d_1 \leq d_2\), and hence \(d_1 = k\); then \((d_1, d_2) - (k + 1, 0) = (-1, d_2)\), and by the first claim this divisor is not linearly equivalent to an effective
divisor. Since \((k + 1, 0)\) is effective and has sum of components equal to \(k + 1\), we have \(\text{GRRR}_r(d_1, d_2) < k + 1\). Hence \(\text{GRRR}_r(d_1, d_2) = k\).

To prove the third claim, let us first show that in this case
\[
\text{(11)} \quad \text{GRRR}_r(d_1, d_2) < d_1 + d_2 - r + 2.
\]
Consider \(m = (d_1 - r + 1, d_2 + 1)\), which is clearly effective; we have \(d - m = (r - 1, -1)\), which by the first claim is not linearly equivalent to an effective divisor, and hence (11). Let us now show that
\[
\text{(12)} \quad \text{GRRR}_r(d_1, d_2) \geq d_1 + d_2 - r + 1.
\]
It suffices to consider a divisor \(m \geq 0\) with \(m_1 + m_2 = d_1 + d_2 - r + 1\) and to show that \(d - m\) is equivalent to an effective divisor. But \(m\) is equivalent to a divisor \(m'\) with \(d_2 - r + 1 \leq m'_2 \leq d_2\), and in this case \(m_1 + m_2 = m'_1 + m'_2\) implies that \(m'_1 = d_1 + d_2 - r + 1 - m'_2\). Hence \(d - m' = (r + m_2' - 1 - d_2, d_2 - m'_2)\), whose second component is clearly non-negative, and whose first component is
\[
r + m'_2 - 1 - d_2 = r - 2 + (m'_2 - d_2) \geq r - 1 + (1 - r) \geq 0.
\]
Hence \(d - m'\) is effective, and it is equivalent to \(d - m\). This establishes (12). Together with (11) we have proven the third claim.

\[\square\]

5.5. **Alternative Description of GRRR\(_r(d)\).** Here is another description of \(\text{GRRR}_r(d)\) that is easier to visualize and may be simpler to work with when generalizing the methods in this article to graphs with more than two vertices.

**Theorem 5.7.** For an integer \(r \geq 1\) and \(i \geq 0\), let \(S_i\) denote the subset of points \((d_1, d_2) \in \mathbb{Z}^2\) for which \(\text{GRRR}_r(d_1, d_2) = i\). Then

1. For \(0 \leq d_2 \leq r - 1\), \(S_0\) consists of those \((d_1, d_2)\) with either \(d_1 = 0\) or \(d_2 = 0\);
2. if \(d\) lies “to the left of \(S_0\),” meaning that \((a, d_2) \in S_0\) for some \(a > d_1\) but \((d_1, d_2) \notin S_0\), then \(\text{GRRR}_r(d) = -1\); and
3. \(\text{GRRR}_r(d_1, d_2)\) is the \(L^1\) distance to \(S_0\) if \(d\) lies “to the right of \(S_0\).”

We illustrate this theorem in Figure 6; as above, we reduce to the case \(0 \leq d_2 \leq r - 1\), which in Figure 6 corresponds to the \(r\) (horizontal) rows beginning with the \(d_1\)-axis \(d_2 = 0\), and the \(r - 1\) rows above this axis.

**Proof.** Since \(\text{GRRR}_r(d)\) and therefore \(S_0\) are invariant under adding multiples of \((r, -r)\), it suffices to prove them in the case where \(0 \leq d_2 \leq r - 1\).

Claim (1) is immediate from claim (2) of Theorem 5.6. Claim (2) is immediate from claim (1) of Theorem 5.6. Claim (3) is immediate from the definition of the Graph Riemann-Roch Rank.

\[\square\]

5.6. **Counting Lattice Points in the Down Cone.** In this subsection we prove the following result.

**Theorem 5.8.** For an integer \(r \geq 1\), let
\[
L_r \overset{\text{def}}{=} \bigcup_{i=0}^{r-1} (\mathbb{Z}(r, -r) + (i, i)) ;
\]
for a \(d \in \mathbb{Z}^2\), let the down cone at \(d\) be
\[
\text{DownCone}(d) \overset{\text{def}}{=} \{ x \in \mathbb{Z}^2 \mid x \leq d \} ;
\]
Figure 6. The case $r = 4$ in the $(d_1, d_2)$-plane

and set

$$\text{LatCount}_r(d) \overset{\text{def}}{=} |L_r \cap \text{DownCone}(d)|.$$ 

Then for any $d \in \mathbb{Z}^2$, we have

$$(13) \quad \text{LatCount}_r(d) = \text{GRRR}_r(d) + 1.$$ 

Proof. Since the functions of $d$ in (13) are invariant under adding $(r, -r)\mathbb{Z}$, it suffices to prove (13) for $0 \leq d_2 \leq r - 1$. So assume this; we will discuss a number of cases.

If $d_1 < 0$, then $d$ is not $\geq (i, i)$ for any $i \geq 0$, and neither can $d + m(r, -r)$ for any $m \in \mathbb{Z}$, since if $m \leq 0$ then the first component of $d + m(r, -r)$ is negative, and otherwise the second component is. Hence \(\text{LatCount}_r(d) = 0\) in this case, and hence, using Theorem 5.6, (13) holds in this case.

Next consider the case where $0 \leq d_1 \leq r - 1$, and further assume that $d_2 \leq d_1$. The same argument as in the previous paragraph shows that if $d + m(r, -r)$ cannot be $\geq (i, i)$ for any $i \geq 0$ unless $m = 0$; if $m = 0$ and $0 \leq d_2 \leq d_1 \leq r - 1$, then $d \geq (i, i)$ iff $i \in [d_2]$. Hence

$$\text{LatCount}_r(d) = d_2 + 1 = \min(d_1, d_2) + 1.$$ 

We similarly see that this formula holds if $d_1 \leq d_2$. Theorem 5.6 now shows that (13) holds in this case.

Next consider the case where $r \leq d_1 \leq 2r - 1$. A similar argument as above shows that if $d + m(r, -r)$ cannot be $\geq (i, i)$ for any $i \geq 0$ unless $m = 0$ or $m = -1$. For $m = 0$, we have that $d \geq (i, i)$ for $i \in [d_2]$; for $m = 1$ we have that $d + (-r, r) \geq (i, i)$
iff $i \leq [d_1 - r]$. It follows that 

$$\text{LatCount}_r(d) = (d_2 + 1) + (d_1 - r + 1) = d_1 + d_2 - r + 2$$

in this case, and hence (13) holds in this case.

More generally, if $d_1 \geq 2r$ and $a \geq 2$ is the unique integer for which $ar \leq d_1 \leq (a + 1)r - 1$, then $d + m(r, -r) \geq (i, i)$ requires $m = 0, -1, \ldots, -a$, and the of such $i$ with $i \in [r - 1]$ is $d_2 + 1$ for $m = 0$, $d_1 - ar + 1$ for $m = -a$, and $r$ for $m = -1, \ldots, -(a - 1)$. Hence the number of such pairs $(m, i)$ is 

$$(d_2 + 1) + (d_1 - ar + 1) + (a - 1)r = d_1 + d_2 - r + 2,$$

and again (13) holds.

Hence (13) holds for all $d_1$ if $0 \leq d_2 \leq r - 1$, and hence it holds for all $d$. 

Let us sketch another proof of Theorem 5.8 based on Theorem 5.7. We may assume that $0 \leq d_2 \leq r - 1$. If $d$ is to the left of $S_0$, we see that there are no lattice points in the down cone of $d$. Similarly, if $d$ lies on $S_0$, then $(0, 0)$ is the single element of $L_r$ in the down cone of $d$. Now we argue that the theorem holds for any $d \in S_i$ for any $i$; the base case $i = 0$ is given above.

For the inductive argument, say that the theorem holds for all $d \in S_i$ for some $i \geq 0$, and let $d \in S_{i+1}$. We may assume $d_1 \geq d_2$, or else $0 \leq d_1 < d_2 \leq r - 1$ and we may use symmetry to exchange $d_1$ and $d_2$ and reduce to this case. Given that $d_2 \leq d_1$, then $d_2 \geq 1$ and $d - (0, 1)$ lies on $S_i$; if $d_1 \leq r - 1$, then $d$ contains exactly one more lattice point in its down cone than $d - (0, 1)$, namely $(d_2, d_2)$.

5.7. Conclusion of the Proof of Theorem 5.5.

Proof of Theorem 5.5. By Theorem 5.8 it suffices to show that 

$$b^0(M_{k,r}, d) = \dim_k H^0(M_{k,r}, d) = \text{LatCount}(d)$$

Each global section $\gamma \in \Gamma(M_{k,r}, d)$ has a value 

$$\gamma(B_3) = (f_1(v), \ldots, f_r(v)) \in (k[v, 1/v])^{\oplus r}$$

which uniquely determines $\gamma$, since $M_{k,r}, d(A_i)$ are isomorphic to $M_{k,r}, d(B_3)$ as vector spaces, and since the restriction maps $M_{k,r}, d(B_3) \to M_{k,r}, d(A_i)$ are injections. So it remains to check which tuples $(f_1, \ldots, f_r) \in M_{k,r}, d(B_3)$ extend to a global section. If for $i \in [r]$ we have 

$$f_i = \sum_n c_{i,n}v^n$$

(so only finitely many of the $c_{i,n}$ are nonzero), then the restriction maps map $(f_1, \ldots, f_r)$ to the values 

$$g_1(x_1) \overset{\text{def}}{=} \sum_{n,i} c_{i,n}x_1^{rn+i-1} \in M_{k,r}, d(A_1), \quad g_2(x_2) \overset{\text{def}}{=} \sum_{n,i} c_{i,n}x_2^{-rn+i-1} \in M_{k,r}, d(A_2),$$

and these restriction maps are isomorphisms; so this extends to $B_1, B_2$ and hence to a global section iff for $i = 1, 2, g_i(x_i)$ lies in the image of the restriction map to $A_i$ from $B_i$; in other words, iff 

$$(14) \quad rn + i - 1 \leq d_1, \quad \text{and} \quad -rn + i - 1 \leq d_2$$
whenever $c_{i,n} \neq 0$. It follows that a basis for $\Gamma(\mathcal{M}_{k,r,d})$ is indexed by the pairs $(i, n) \in \{1, \ldots, r\} \times \mathbb{Z}$ satisfying (14), i.e., for which
\begin{equation}
(15) \quad n(r, -r) + (i - 1, i - 1) \leq d.
\end{equation}
But the set of points $u(r, -r) + (i - 1, i - 1)$ with $i \in \{1, \ldots, r\}$ and $n \in \mathbb{Z}$ is precisely the set $L_r$; hence the number of pairs $(i, n)$ satisfying (15) is precisely $\text{LatCount}_r(d)$. \hfill \Box

6. Foundlations, Part 2: Morphisms of Sheaves, Short/Long Sequences

In this section we develop definitions and notions needed to prove our theorem regarding the Euler characteristic of the sheaves $\mathcal{M}_{k,r,d}$.

6.1. First Betti Number.

**Definition 6.1.** Let $\mathcal{F}$ be a sheaf of $k$-vector spaces on the category $C_{2V}$. We define $H^0(\mathcal{F})$ and $H^1(\mathcal{F})$ to be, respectively, the kernel and cokernel of the map
\begin{equation}
u: \bigoplus_{i=1}^{3} \mathcal{F}(A_i) \to \bigoplus_{j=1}^{2} \mathcal{F}(B_j)
\end{equation}
where $\nu$ is given by map
\begin{equation}u(m_1, m_2, m_3) = (\mathcal{F}(A_1, B_1)m_1 - \mathcal{F}(A_1, B_3)m_3, \mathcal{F}(A_2, B_2)m_2 - \mathcal{F}(A_2, B_3)m_3).
\end{equation}
We define the first Betti number of $\mathcal{F}$ to be
\begin{equation}b^1(\mathcal{F}) = \dim_k H^1(\mathcal{F}).
\end{equation}

Note that there is a simple one-to-one correspondence between $H^0(\mathcal{F})$ and $\Gamma(\mathcal{F})$, and hence we define $b^0(\mathcal{F})$ as
\begin{equation}b^0(\mathcal{F}) = \dim_k H^0(\mathcal{F}) = \Gamma(\mathcal{F}).
\end{equation}
In Section 9 we will show that the $H^i(\mathcal{F})$ above are the right derived functors of the functor $\mathcal{F} \mapsto \Gamma(\mathcal{F})$.

6.2. The Short/Long Exact Sequence. The main technique we will use to prove our Euler characteristic formula is a standard short/long exact sequence.

**Definition 6.2.** Let $\mathcal{F}, \mathcal{G}$ be sheaves of $k$-vector spaces on the category $C$, associated to the bipartite graph $G = (A, B, E)$. We define a morphism $\mathcal{F} \to \mathcal{G}$ to be a collection $u = \{u_P\}_{P \in \text{Ob}(C)}$ of $k$-linear maps $u_P: \mathcal{F}(P) \to \mathcal{G}(P)$ such that for each $(a, b) \in E$ we have $u_a \mathcal{F}(a, b) = \mathcal{G}(a, b)u_b$.

This notion of morphism turns the class of sheaves of $k$-vector spaces on a bipartite category, $C$, into a category. We readily verify that this category is abelian, and all projective and injective limits (e.g., products, kernels, cokernels) are computed “value-by-value;” for example the cokernel of a morphism $u: \mathcal{F} \to \mathcal{G}$ is simply the sheaf whose value at $P \in \text{Ob}(C)$ is $\mathcal{G}(P)/\mathcal{F}(P)$, and whose restriction maps are obtained from those of $\mathcal{F}$ and $\mathcal{G}$ in the natural fashion. For details, see [SGA4], Section I.3; our notion of sheaf is the notion of presheaf in [SGA4] (endowed with the coarsest topology, i.e., la topologie grossière).

**Theorem 6.3.** If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is any short exact sequence of sheaves, then we have a correspond long exact sequence of sheaves
\begin{equation}0 \to H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_3) \to H^1(\mathcal{F}_1) \to H^1(\mathcal{F}_2) \to H^1(\mathcal{F}_3) \to 0.
\end{equation}
The proof of this theorem is standard diagram chasing argument. A second proof of this theorem is given in Section 9: the theorem is a standard result once we have proven that the $H^i(F)$ are the right derived functors of $F \rightarrow \Gamma(F)$.

6.3. Restriction and Extension by Zero. Later we will need some operations on sheaves; the ones we describe here are exact, meaning they take short exact sequences to short exact sequences.

Definition 6.4. Let $G' = (A', B', E')$ be a subgraph of some bipartite graph, $G = (A, B, E)$, and let $C', C$ be the respective associated bipartite categories.

1. If $F$ is a sheaf on $C$, then there is an evident sheaf restriction of $F$ to $C'$, denoted $F|_{G'}$, which is the sheaf on $C'$ whose values and restrictions are taken from $C$.

2. If $F'$ is a sheaf on $C'$, we define the extension by zero of $F'$ to $C$ we mean the sheaf on $C$ whose values and restrictions are taken from $F'$ for objects and morphisms in $C'$ and whose remaining values and restrictions are zero. 

3. If $F$ is a sheaf on $C$, we use $F_{C'}$ to denote the sheaf on $C$ that is the extension by zero of the restriction of $F$ to $C'$.

We easily see that the three above operations each define functors that are exact in the sense that they preserve exactness of short exact sequences (i.e., sequences $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$). Note that the above definition of $F_{C'}$ is consistent with the notation in Example 3.7.

7. Some Betti Number Computations

In this section we give a few simple Betti number computations. First, we give a procedure to compute $H^1(F)$ for various sheaves $F$, including all the $M_{k,r,d}$, the $O_{k,r}$, and some line bundles $L_{k,r,d}$ that we define in this section. Second, we compute the Betti numbers of a simple type of skyscraper sheaf that we will use in the next section.

7.1. Partial-Line Bundles.

Definition 7.1. Let $k$ be a vector space and $d \in \mathbb{Z}^2$. By a $d$-twisted partial-$k$-line bundle (on $C_{2V}$) (or simply a partial-line bundle) we mean a sheaf of $k$-vector spaces on $C_{2V}$ such that

1. for $i = 1, 2$ we have
   
   $F(A_i) = k[x_i, 1/x_i], \quad F(B_i) = k[y_i]$ 

   for indeterminates $x_1, x_2, y_1, y_2$, and

2. for $i = 1, 2$ we have

   $F(A_i, B_i)(p(y_i)) = x_i^d p(1/x_i)$. 

3. there is a basis $\{m_j\}_{j \in J}$ for $F(B_3)$ such that there are injections $f, g$ from $J$ to $\mathbb{Z}$ such that for each $j \in J$

   $F(A_1, B_3)m_j = x_1^{f(j)}, \quad F(A_2, B_3)m_j = x_2^{g(j)}$.

In terms to be given in Section 9, such sheaves $F$ are examples of line bundles (or invertible sheaves) when restricted to $\{A_1, A_2, B_1, B_2\}$, and they have a particular form on $B_3$ and the restriction maps from $B_3$ to the $A_i$.

The sheaves $O_{k,r}$ and $M_{k,r,d}$ are all partial-line bundles. Let us describe another set of sheaves that will be of interest to us.
Definition 7.2. Let $k$ be a field, $r \geq 1$ an integer, and $d \in \mathbb{Z}^2$. We use $L = L_{k,r,d}$ to denote the partial $k$-line bundle with $d$ twists on $C_{2V}$ such that $L = k[v, 1/v]$ and

$$L(A_1, B_3)p(v) = p(x_1^{r}), \quad L(A_2, B_3)p(v) = p(x_2^{-r})$$

(setting $J = \mathbb{Z}$ and taking $\{v^j\}_{j \in J}$ as a basis for $L(B_3)$, we see that $L$ is indeed a partial-line bundle).

7.2. The First Cohomology Formula for Many Cases of Interest. In this subsection we describe a result that allows us to quickly compute $H^1(F)$ (and hence $b^1(F)$) for $F$ being any of $\mathcal{M}_{k,r,d}, \mathcal{O}_{k,r}$ and some other sheaves of interest to us in this article.

Lemma 7.3. Let $F$ be a partial $d$-twisted $O_{k,r}$-line bundle, and let $J, f, g$ be as in Definition 7.1. Then $H^1(F)$ (i.e., the cokernel of $u$ in (16)) has a basis consisting of coset representatives of the following elements of $F(A_1) \oplus F(A_2)$:

$$\{x_1^i\}_{i \in I_1} \cup \{x_2^i\}_{i \in I_2} \cup \{x_1^f(j)\}_{j \in J}$$

where

$I_1 \overset{\text{def}}{=} \{i \in \mathbb{Z} \mid i > d_1 \text{ and } i \notin \text{Im}(f)\}$, \quad $I_2 \overset{\text{def}}{=} \{i \in \mathbb{Z} \mid i > d_2 \text{ and } i \notin \text{Im}(g)\},$

and

$J_1 \overset{\text{def}}{=} \{j \in J \mid f(j) > d_1 \text{ and } g(j) > d_2\}.$

(For any $j \in J_1$, we could use $x_2^{g(j)}$ instead of $x_1^{f(j)}$ in the basis (17).)

Proof. A basis of $F(A_1) \oplus F(A_2)$ (see (16)) is given by all the $x_1^i$ and all the $x_2^i$ (with $i$ ranging over all $i \in \mathbb{Z}$); the image of $u$ consists of the sum of the image of $u$ restricted to $F(B_1)$, to $F(B_2)$, and to $F(B_3)$. The image of $u$ restricted to $F(B_1)$ is $x_1^i$ with $i \leq d_1$; hence $x_1^i \sim 0$ for $i \leq d_1$, where $\sim$ denotes equivalence modulo the image of $u$. Similarly for $F(B_2)$, so that $x_1^i \sim 0$ for $i \leq d_2$. Finally image of $u$ restricted to $F(B_3)$ implies that for each $j \in J$

$$x_1^{f(j)} \sim -x_2^{g(j)}.$$ 

It follows that any element of $F(A_1) \oplus F(A_2)$ is equivalent to an element of the form

$$\sum_{i \in I_1} \alpha_i x_1^i + \sum_{i \in I_2} \beta_i x_2^i + \sum_{j \in J_1} \gamma_j x_1^{f(j)}$$

with $\alpha_i, \beta_i, \gamma_j$ in $k$.

To finish the proof it suffices to show that the vectors in (17) are linearly independent modulo the image of $u$. In other words we must show that

$$\sum_{i \in I_1} \alpha_i x_1^i + \sum_{i \in I_2} \beta_i x_2^i + \sum_{j \in J_1} \gamma_j x_1^{f(j)} \sim 0$$

implies that all the $\alpha_i, \beta_i, \gamma_j$ are equal to zero. So consider an equation of the form (19); the condition $\sim 0$ is equivalent to belonging to the image of $u$, and therefore

$$\sum_{i \in I_1} \alpha_i x_1^i + \sum_{i \in I_2} \beta_i x_2^i + \sum_{j \in J_1} \gamma_j x_1^{f(j)} = \sum_{i \leq d_1} \mu_i x_1^i + \sum_{i \leq d_2} \mu_i x_2^i + \sum_{j \in J} x_1^{f(j)} + x_2^{g(j)}$$

\[\square\]
7.3. The First Betti Numbers of $\mathcal{M}_{r, d}, \mathcal{O}_{k, r}, \mathcal{L}_{k, r, d}$. Let us state some easy consequences of Lemma 7.3.

**Corollary 7.4.** Let $k$ be a field, $r \geq 1$ an integer, and $d \in \mathbb{Z}^2$. Then

(1) if $d_1, d_2 \geq 0$, then

$$b^1(\mathcal{M}_{k, r, d}) = \max(0, r - 1 - \max(d_1, d_2)).$$

(2) for any $r \geq 2$ we have that $H^1(\mathcal{L}_{k, r, d})$ (and therefore $H^1(\mathcal{O}_{k, r})$) is infinite dimensional.

**Proof.** For the first claim, i.e., for $\mathcal{M}_{k, r, d}$, we have that $J = \{1, \ldots, r\} \times \mathbb{Z}$, and $f(i, n) = i - 1 + rn$, $g(i, n) = i - 1 - rn$. It follows that (1) $f, g$ are bijections, (2) $f(i, 0) = g(i, 0) = i - 1$, (3) for $n \geq 1$, $f(i, n) > 0$ and $g(i, n) \leq 0$, and (4) for $n \leq -1$, $f(i, n) \leq 0$ and $g(i, n) > 0$. In particular, $f(i, n) > d_1 \geq 0$ and $g(i, n) > d_2 \geq 0$ can only occur for $n = 0$ and $1 \leq i \leq r$. Furthermore the number of $i$ between 1 and $r$ for which $i - 1 = f(i, 0) > d_1$ and $i - 1 = g(i, 0) > d_2$ is precisely the number of integers $i$ with $2 + \max(d_1, d_2) \leq i \leq r$, of which there are

$$\max(0, r - 1 - \max(d_1, d_2)).$$

For the second claim, if $r \geq 2$, then we get the same functions $f, g$ as before, but they are restricted to $J' = \{1\} \times \mathbb{Z}$ instead of $J = \{1, \ldots, r\} \times \mathbb{Z}$. In particular, the image of $f$ does not include $f(r, n) = r - 1 + rn$ for any value of $n$, i.e., does not include any number congruent to $r - 1$ modulo $r$. Since there are infinitely many such values that are greater than $d_1$, this means that the subset $I_1$ of Lemma 7.3 is infinite.

The first claim in Corollary 7.4 proves that $b^1(\mathcal{M}_{k, r, K_r}) = 1$, where $K_r = (r - 2, r - 2)$ is the canonical divisor of Baker-Norine. The second claim in the corollary indicates that one cannot expect a simple duality theory based on $\mathcal{O}_{k, r}$ or a line bundle such as $\mathcal{L}_{k, r, d}$ for any value of $d$.

7.4. The Betti Numbers of Certain Skyscrapers.

**Definition 7.5.** Let $k$ be a field. By the skyscraper at $B_1$ with value $k$, $\text{Sky}(B_1, k)$ we mean the sheaf of $k$-vector spaces whose only nonzero value is at $B_1$, with value $k$ (all restriction maps are therefore zero). We similarly define the same with $B_1$ replaced with $B_2$.

In Section 9 we will explain this terminology.

**Lemma 7.6.** We have that for $i = 1, 2$,

$$b^0(\text{Sky}(B_i, k)) = 1, \quad b^1(\text{Sky}(B_i, k)) = 0.$$  

**Proof.** This is immediate from the definition of the cohomology groups (16), since the map $u$ there,

$$u: \bigoplus_{j=1}^{3} \mathcal{F}(B_j) \rightarrow \bigoplus_{i=1}^{2} \mathcal{F}(A_i)$$

amounts to $u: k \rightarrow 0$. □
8. The Euler Characteristic and Riemann-Roch Theorem

Here we calculate the Euler characteristic of the sheaves $\mathcal{M}_{k,r,d}$, whereupon the Baker-Norine graph Riemann-Roch Theorem becomes equivalent to the formula

$$b_1(\mathcal{M}_{k,r,d}) = b_0(\mathcal{M}_{k,r,K^r-d}).$$

Our Euler characteristic calculation is simplified by the use of short/long exact sequences.

8.1. Euler Characteristic Computations.

**Definition 8.1.** Let $F$ be a sheaf of $k$-vector spaces on a simple category $C$ such that $b_i(F)$ are finite for $i = 0, 1$. We define

$$\chi(F) \overset{\text{def}}{=} b_0(F) - b_1(F).$$

**Theorem 8.2.** For an integer $r \geq 1$, field $k$, and $d \in \mathbb{Z}^2$, let $\mathcal{M}_{r,d}$ be as Definition 5.2. Then

$$\chi(\mathcal{M}_{r,d}) = d_1 + d_2 - (r - 2).$$

**Proof.** For $d = (0, 0)$, we have

$$\chi(\mathcal{M}_{r,0}) = b_0(\mathcal{M}_{r,0}) - b_1(\mathcal{M}_{r,0}) = 1 - (r - 1) = 2 - r,$$

which verifies (20) in this case.

We claim that if (20) holds for some $d$, then it also holds for $d + (1, 0)$. To see this, let us describe a morphism $\mu$ giving rise to an exact sequence of sheaves of $k$-vector spaces

$$0 \rightarrow \mathcal{M}_d \xrightarrow{\rho} \mathcal{M}_{d+1,0} \rightarrow \text{Sky}(B_1, k) \rightarrow 0 :$$

we simply take $\mu$ to be the identity map on $A_1, A_2, B_2, B_3$, and let $\mu(B_1)$ be the map $1 \mapsto y_1$ (this choice is forced when $\mu(A_1)$ is the identity map, given the restrictions of $\mathcal{M}_d, \mathcal{M}_{d+1,0}$ along $A_1 \rightarrow B_1$. Then $\mu$ is surjective everywhere except at $B_1$, where it is the map $k[y_1] \rightarrow k[y_1]$ of $k[y_1]$ modules taking 1 to $y_1$, whose cokernel is $k[y_1]/(y_1 k[y_1]) \simeq k$.

Hence the cokernel of $\mu$ is just the sheaf supported on $B_1$ (therefore a skyscraper sheaf at $B_1$) whose value there is $k$.

The exact sequence (21) is depicted in Figure 8.1.

![Diagram](image-url)
In view of Lemma 7.6 and the short exact sequence (21), the associated long exact sequence shows that if one of \(M_d, M_{d+(0,1)}\) has finite Betti numbers, then the other one does and

\[
\chi(M_{d+(1,0)}) = \chi(M_d) + \chi(S\kern-.25emky(B_1, k)) = \chi(M_d) + 1.
\]

It follows that if (20) holds for one of \(d\) or \(d+(1,0)\), then it holds for the other one as well. It follows similarly for \(d\) and \(d+(0,1)\). Hence if it holds for one value of \(d \in \mathbb{Z}^2\), then it holds for all elements of \(\mathbb{Z}^2\). Since we have verified (20) for \(d = (0,0)\), it holds for all \(d\).

8.2. Consequences of the Euler Characteristic Formula. Let us gather some consequences of Theorem 8.2, using results from Sections 5 and 7. The first consequence is a weaker form of the Riemann-Roch theorem for graphs.

Theorem 8.3. Let \(k\) be a field, and \(r \geq 1\) an integer. Then

1. For all \(d = (d_1, d_2)\) with \(d_1 + d_2 < 0\) we have

\[
b^0(M_{k,r,d}) = 0, \quad b^1(M_{k,r,d}) = 2(r - 2) - d_1 - d_2.
\]

and

2. For all \(d = (d_1, d_2)\) with \(d_1 + d_2 \geq 2(r - 2)\) we have

\[
b^1(M_{k,r,d}) = 0, \quad b^0(M_{k,r,d}) = d_1 + d_2 - 2(r - 2).
\]

Proof. The case where \(d_1 + d_2 < 0\) follows from Theorem 5.5 and the fact that if \(d_1 + d_2 < 0\), \(d\) cannot be equivalent to an effective divisor (since adding any multiple of \((r,-r)\) to \(d\) does not change the sum of the first and second components of the vector). Hence \(\text{GRRR}_r(d) = -1\), and by Theorem 5.5 we conclude the first equality in (22). The second equality in (22) then follows from the first equality and Theorem 20.

Similarly, by Theorem 5.6 parts (2) and (3), if \(0 \leq d_2 \leq r - 1\), then \(\text{GRRR}_r(d) = d_1 + d_2 - r + 1\) if \(d_1 \geq r - 1\); and \(d_1 \geq r - 1\) is certainly true if \(d_1 + d_2 \geq 2r - 2\), since \(d_2 \leq r - 1\). Hence \(\text{GRRR}_r(d) = d_1 + d_2 - r + 1\) whenever \(d_1 + d_2 \geq 2r - 2\) and \(0 \leq d_2 \leq r - 1\); but any vector \(d\) can be brought to a vector with \(0 \leq d_2 \leq r - 1\) by adding the appropriate multiple of \((r,-r)\), and this multiple does not change the sum of the first and second components.

Hence whenever \(d_1 + d_2 \geq 2r - 2\) we have that \(\text{GRRR}_r(d) = d_1 + d_2 - r + 1\), and this implies the first equality of (23), by Theorem 5.5. The second equality in (23) then follows from the first equality and Theorem 20.

9. Foundations, Part 3: \(O\)-modules and \(\text{Ext}\) Groups

In this section we discuss the notion of a \(O\)-module for a sheaf of rings, \(O\), on a category, \(C\). In specific computations we often assume that \(C\) is a bipartite category, or even just \(\mathcal{C}_{2V}\); however, it is often conceptually simpler to work with a general category, and working as such makes it easier to cite the literature.

9.1. \(O\)-Modules. Just after Definition 5.3 we explained that the sheaves \(O_{k,r}\) can be viewed a sheaves of \(k\)-algebras, and that the \(M_{k,r,d}\) are \(O_{k,r}\)-modules. Let us make this precise.

Definition 9.1. Let \(C\) be a category. By a sheaf of rings on \(C\) we mean a contravariant functor, \(O\), from \(C\) to the category of rings.
In other words, a sheaf of rings \( \mathcal{O} \) is the data consisting of

1. a ring, \( \mathcal{O}(P) \), for each \( P \in \text{Ob}(\mathcal{C}) \), and
2. a morphism of rings \( \mathcal{O}(\phi) : \mathcal{O}(Q) \to \mathcal{O}(P) \) for each morphism \( \phi : P \to Q \) in \( \mathcal{C} \)

such that \( \phi \to \mathcal{O}(\phi) \) respects composition (i.e., \( \mathcal{O}(\phi_2 \circ \phi_1) = \mathcal{O}(\phi_1) \circ \mathcal{O}(\phi_2) \)) and takes identity maps to identity maps. Hence, if \( \mathcal{C} \) is the bipartite category \( (A, B, E) \), a sheaf of rings, \( \mathcal{O} \), therefore consists of giving a ring, \( \mathcal{O}(P) \), for each \( P \in A \sqcup B \), and a morphism \( \mathcal{O}(\phi) : \mathcal{O}(B) \to \mathcal{O}(A) \) for each \( (A, B) \in E \); there are no requirements on the \( \mathcal{O}(A, B) \), since there are no nontrivial compositions in \( \mathcal{C} \) (i.e., if \( \phi_2 \circ \phi_1 \) is defined, then at least one of \( \phi_1, \phi_2 \) is an identity morphism).

For example, a sheaf of \( \mathcal{O} \)-modules on \( C_{2V} \) is illustrated in Figure 8; this consists of arbitrary rings \( R_1, R_2, R_3, S_1, S_2 \) and, for \( i = 1, 2 \), arbitrary morphisms of rings \( R_i \to S_i \) and \( R_3 \to S_1 \).

![Figure 8. A sheaf of rings \( \mathcal{O} \) and a sheaf of \( \mathcal{O} \)-modules, \( \mathcal{M} \) on \( C_{2V} \).](image)

**Definition 9.2.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. By a sheaf on \( \mathcal{C} \) with values in \( \mathcal{D} \) we mean a contravariant functor \( \mathcal{C} \to \mathcal{D} \).

This definition not only generalizes our definition of sheaves of \( k \)-vector spaces and of rings—making precise what we mean by a sheaf of \( k \)-algebras—but will also be useful later in this section.

**Definition 9.3.** If \( \mathcal{O} \) is a sheaf of rings on \( \mathcal{C} \), then a sheaf of \( \mathcal{O} \)-modules is a sheaf \( \mathcal{M} \) of abelian groups on \( \mathcal{C} \) such that

1. for each \( P \in \text{Ob}(\mathcal{C}) \), \( \mathcal{M}(P) \) is endowed with the structure of an \( \mathcal{O}(P) \)-module, and
2. the ring structure is respected under restriction, i.e., for each morphism \( \phi : P \to Q \) of \( \mathcal{C} \) we have

\[
\mathcal{M}(\phi)(r m) = (\mathcal{O}(\phi)r)(\mathcal{M}(\phi)m)
\]

for all \( r \in \mathcal{O}(Q) \) and \( m \in \mathcal{M}(Q) \).

Note that (24) is the same as (10). In this article all modules are assumed to be unital, i.e., the unit in the ring acts as the identity element on the module\(^3\).

**Example 9.4.** If \( k \) is a field, then a \( k \)-module is the same thing as a \( k \)-vector space. If \( \mathcal{C} \) is any category, then the constant sheaf \( \underline{k} \) is a sheaf of rings, and a \( \underline{k} \)-module is the same thing as a sheaf of \( k \)-vector spaces.

\(^3\)In [SGA4] this is not generally assumed; see, for example, [SGA4] Proposition II.6.7, which speaks of \( \text{modules unitaires} \); therefore, in [SGA4], a \( k \)-module \( k[t]/(t^2) \) could have 1 acting on it via \( p(t) \to (1 + t)p(t) \).
Example 9.5. If \( \mathcal{C} = \mathcal{C}_{2V} \) and \( k \) is a field, then \( \mathcal{O}_{k,r} \) (Definition 5.3) is a sheaf of rings and, moreover, a sheaf of \( k \)-algebras, and for each \( d \in \mathbb{Z}^2 \), \( \mathcal{M}_{k,r,d} \) is a sheaf of \( \mathcal{O}_{k,r,d} \)-modules.

Definition 9.6. A morphism \( u: \mathcal{F} \to \mathcal{G} \) of sheaves on \( \mathcal{C} \) with values in \( \mathcal{D} \) is a natural transformation of functors.

In other words, a morphism \( u: \mathcal{F} \to \mathcal{G} \) consists of the data \( u(P): \mathcal{F}(P) \to \mathcal{G}(P) \) for each \( P \in \text{Ob}(\mathcal{C}) \) such that for any \( \phi: P \to Q \) in \( \mathcal{C} \) we have \( u(P) \mathcal{F}(\phi) = \mathcal{G}(\phi)u(Q) \).

Definition 9.7. Let \( \mathcal{O} \) be a sheaf of rings on \( \mathcal{C} \). If \( \mathcal{M}, \mathcal{M}' \) are two sheaves of \( \mathcal{O} \)-modules, we use

\[
\text{Hom}_\mathcal{O}(\mathcal{M}, \mathcal{M}')
\]

to denote the set of morphisms \( u: \mathcal{M} \to \mathcal{M}' \) such that respect the \( \mathcal{O} \)-structure of \( \mathcal{M}, \mathcal{M}' \), i.e., such that for each \( P \in \text{Ob}(\mathcal{C}) \) we have \( u(P)(rm) = ru(P)m \) (this is an equation in \( \mathcal{M}'(P) \)) for all \( r \in \mathcal{O}(P) \) and \( m \in \mathcal{M}(P) \).

9.2. Examples of \( \text{Hom}_\mathcal{O} \).

Example 9.8. Let \( k \) be a field and \( r \geq 1 \) an integer, and let \( \mathcal{M} \) be a sheaf of \( \mathcal{O}_{k,r} \)-modules on \( \mathcal{C}_{2V} \). Then there are natural isomorphisms

\[
\Gamma(\mathcal{M}) \cong \text{Hom}_k(k, \mathcal{M}) \cong \text{Hom}_{\mathcal{O}_{k,r}}(\mathcal{O}_{k,r}, \mathcal{M}).
\]

Example 9.9. Let \( k \) be a field and \( y \) an indeterminate. Let \( \mathcal{C} = \Delta_0 \) be the category with one object, 0, and one morphism, i.e., the bipartite category \( \{0, 0, 0\} \). Then a sheaf of rings (of \( k \)-vectors spaces, etc.) on \( \Delta_0 \) is just a ring (a \( k \)-vector spaces, etc.), i.e., its value on \( 0 \in \text{Ob}(\Delta_0) \). Consider a

\[
\phi \in \text{Hom}_{k[y]}(k[y], k[y]),
\]

where \( \text{Hom}_{k[y]} \) means morphisms of \( k[y] \)-modules; then \( \phi \) is determined by the image of \( 1 \in k[y] \), i.e., by \( \phi(1) \in k[y] \). It follows that

\[
\text{Hom}_{k[y]}(k[y], k[y]) \cong k[y].
\]

There is a natural inclusion

\[
(25) \quad \text{Hom}_{k[y]}(k[y], k[y]) \to \text{Hom}_k(k[y], k[y]),
\]

where \( \text{Hom}_k \) means morphisms of \( k \)-algebras, i.e., of \( k \) vector spaces. However a linear transformation \( \phi: \text{Hom}_k(k[y], k[y]) \) is determined by \( \phi(1), \phi(y), \phi(y^2), \ldots \), each of which has no forced dependence on the others; hence

\[
\text{Hom}_k(k[y], k[y]) \cong (k[y])^\mathbb{Z}
\]

Hence the inclusion \((25)\) is strict. Notice that this inclusion is an inclusion of sets, but can be viewed as an inclusion in the category of \( k \)-vector spaces or even left or right \( k[y] \)-algebras (where \( k[y] \) acts on \( \phi \in \text{Hom}_k(k[y], k[y]) \) by post- or pre-multiplication).

More generally, for a category \( \mathcal{C} \) and a sheaf of \( k \)-algebras, \( \mathcal{O} \), there is an inclusion

\[
\text{Hom}_\mathcal{O}(\mathcal{M}, \mathcal{M}') \to \text{Hom}_k(\mathcal{M}, \mathcal{M}')
\]

which is often a strict inclusion.
Example 9.10. Let $k$ be a field and $r \geq 1$ an integer. We shall see that
\[ \text{Hom}_{\mathcal{O}_r,k}(\mathcal{M}_{k,r,d}, \mathcal{M}_{k,r,d'}) \]

is a finite dimensional $k$-vector space for any $d, d' \in \mathbb{Z}^2$, whereas
\[ \text{Hom}_k(\mathcal{M}_{k,r,d}, \mathcal{M}_{k,r,d'}) \]

for $d = d' = 0$ (and many other pairs $d, d'$) is infinite dimensional.

9.3. (Co)homology and Ext groups. Below we will see that for any sheaf of rings $\mathcal{O}$ on $\mathbb{C}_{2V}$, the category of $\mathcal{O}$-modules is an abelian category with enough projectives (and injectives).

Definition 9.11. If $\mathcal{O}$ is a sheaf of rings on a category $\mathcal{C}$, and $\mathcal{F}, \mathcal{G}$ are two $\mathcal{O}$-modules, we use $\text{Ext}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ to denote the derived bifunctors of
\[ (\mathcal{F}, \mathcal{G}) \mapsto \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}). \]

Recall that, in practice, this means that we can compute $\text{Ext}_\mathcal{O}(\mathcal{F}, \mathcal{G})$ from a projective resolution of $\mathcal{F}$ or an injective resolution of $\mathcal{G}$, or from the double complex involving both these resolutions (and the resulting Ext groups are isomorphic via a unique isomorphism that can be constructed from two different projective resolutions of $\mathcal{F}$ and/or two injective resolutions of $\mathcal{G}$). For a reference, see [Wei94, HS97] in the case of $R$-modules for a ring $R$; these facts are valid for arbitrary abelian category, for reasons discussed in [Har77], page 203, and in the discussion regarding Theorem 1.6.1 of [Wei94], i.e., regarding the Freyd-Mitchell Embedding Theorem (or [Fre64] or the original [Fre64]).

9.4. The Yoneda Pairing. Here is standard lemma regarding Ext groups that we will need, usually called the Yoneda product or Yoneda pairing for Ext groups, described in [HS97], Exercise IV.9.3. Let us summarize the result that we need.

Lemma 9.12 (Yoneda Product). Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be any three elements of an abelian category with either (1) enough projectives, or (2) enough injectives. Then for any $i, j$, there is a pairing
\[ \text{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) \times \text{Ext}^j(\mathcal{F}_2, \mathcal{F}_3) \to \text{Ext}^{i+j}(\mathcal{F}_1, \mathcal{F}_3) \]
such that

1. the pairing is natural (i.e., functorial) in $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, where naturality in $\mathcal{F}_2$ means that the morphism of $\mathcal{F}_2$-covariant functors
\[ \text{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) \to \text{Hom}_\mathbb{Z}(\text{Ext}^i(\mathcal{F}_2, \mathcal{F}_3), \text{Ext}^{i+j}(\mathcal{F}_1, \mathcal{F}_3)) \]
is natural in $\mathcal{F}_2$;

2. if $j = 0$ and $\mathcal{F}_2 = \mathcal{F}_3$, then the pairing (26) acts via the functoriality of $\text{Ext}^i(\mathcal{F}_1, \mathcal{F}_2)$ in the variable $\mathcal{F}_2$ (and similarly for $i = 0$ and $\mathcal{F}_1 = \mathcal{F}_2$); in particular, the image $\text{id}_{\mathcal{F}_1}$ in (27) is the identity morphism of $\text{Ext}^i(\mathcal{F}_1, \mathcal{F}_3)$; and

3. if $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ is any short exact sequence, then the resulting long exact sequence
\[ \cdots \to \text{Ext}^i(\mathcal{F}_1, \mathcal{G}_1) \to \text{Ext}^i(\mathcal{F}_1, \mathcal{G}_2) \to \text{Ext}^i(\mathcal{F}_1, \mathcal{G}_3) \to \text{Ext}^{i+1}(\mathcal{F}_1, \mathcal{G}_1) \to \cdots \]
adopts a chain map to the long exact sequence resulting from the right-hand-side of (27), which takes $\text{Ext}^i(\mathcal{F}_1, \mathcal{G}_i)$ to
\[ \text{Hom}_\mathbb{Z}(\text{Ext}^{k-i}(\mathcal{G}_i, \mathcal{F}_3), \text{Ext}^k(\mathcal{F}_1, \mathcal{F}_3)) \]
We remark that in this article, we only need the case $i + j = 1$, whereby one of $i, j$ must be zero. In this case one can verify that the pairing of an $\alpha \in \text{Ext}^1(F_1, F_2)$ with $\beta \in \text{Ext}^0(F_2, F_3)$ is just the map $\text{Ext}^1(F_1, F_2) \to \text{Ext}^1(F_1, F_3)$ induced by $\beta$ viewed as an element of $\text{Hom}(F_2, F_3)$ by the fact that $\text{Ext}^1(F_1, F_2)$ is natural (i.e., functorial) in $F_2$. A similar discussion holds for the pairing of $\text{Ext}^0(F_1, F_2)$ with $\text{Ext}^1(F_2, F_3)$.

9.5. Skyscrapers and Coskyscrapers. In this subsection we explain the construction of skyscraper and coskyscraper sheaves. We give these constructions for more general categories than $\mathcal{C}_{2V}$. The notion of a skyscraper sheaf and their use to construct injective resolutions is standard (see [Har77], Proposition III.2.2). The notion of a coskyscraper sheaf and their use to construct projective resolutions is less standard—since this construction doesn’t work for (infinite) topological spaces. However, for sheaves of $k$-vector spaces, or sheaves of $\mathcal{O}$-modules for any constant sheaf, $\mathcal{O}$, the construction is immediate from [SGA4] Proposition I.5.1, as we will explain at the end of this subsection; the adaptation to the more general situation is akin to the standard definition of the pullback $f^*$ of modules where $f$ is a morphism of ringed spaces (see [Har77], top of page 110).

Definition 9.13. We say that a category is topological all its Hom-sets are of size one or zero. We say that a category, $\mathcal{C}$, is semi-topological if for for $P \in \text{Ob}(\mathcal{C})$, $\text{Hom}(P, P)$ consists of only the identity map. If $\mathcal{C}$ is any category and $x, y \in \text{Ob}(\mathcal{C})$, we use $x \leq y$ to mean that $\text{Hom}(x, y)$ is nonempty.

The following properties are easy to verify: if $\mathcal{C}$ is semi-topological, then $x \leq y$ and $y \leq x$ implies that $x$ and $y$ are isomorphic; if $\mathcal{C}$ is a bipartite category, then $x, y$ are isomorphic iff $x = y$; if $\mathcal{C}$ is finite and topological, then the downsets of $\mathcal{C}$ are the open subsets of a topological space whose points are isomorphism classes of elements of $\mathcal{C}$. Semitopological categories have many properties in common with topological spaces (see [Fri05]).

Definition 9.14. Let $\mathcal{O}$ be a sheaf of rings on a topological category, $\mathcal{C}$. For $P \in \text{Ob}(\mathcal{C})$ and $M$ an $\mathcal{O}(P)$-module we define the skyscraper (sheaf) at $P$ with value $M$ to be the sheaf $\text{Sky} = \text{Sky}(P, M)$

1) whose values are

\[
\text{Sky}(Q) = \begin{cases} 
M & P \leq Q, \\
0 & \text{otherwise};
\end{cases}
\]

2) whose restriction maps are the identity map for all $Q_1, Q_2 \geq P$;

3) where the $\mathcal{O}(Q)$-module structure on $\mathcal{S}(Q)$ is given by the restriction map $\mathcal{O}(Q) \to \mathcal{O}(P)$ and the $\mathcal{O}(P)$-module structure of $M$.

Lemma 9.15. Let $\mathcal{O}$ be a sheaf of rings on a topological category $\mathcal{C}$, and let $M$ be an $\mathcal{O}(P)$-module. Then for any $\mathcal{F}$ there is a natural isomorphism

\[
\text{Hom}_{\mathcal{O}(P)}(\mathcal{F}(P), M) \cong \text{Hom}_\mathcal{O}(\mathcal{F}, \text{Sky}(P, M)).
\]

If $M$ is an injective $\mathcal{O}(P)$-module, then $\text{Sky}(P, M)$ is an injective $\mathcal{O}$-module.

The proof is straightforward; the reader is free to skip this general proof and just verify this for $\mathcal{C}_{2V}$ as indicated in the next section.

Similarly we have the construction of coskyscrapers.
Definition 9.16. Let \( \mathcal{O} \) be a sheaf of rings on a topological category, \( \mathcal{C} \). For \( P \in \text{Ob}(\mathcal{C}) \) and \( M \) an \( \mathcal{O}(P) \)-module we define the \textit{coskyscraper (sheaf)} at \( P \) with \textit{value} \( M \) to be the sheaf \( \text{Sky} = \text{Sky}(P,M) \)

1. whose values are
\[
\text{CoSky}(Q) = \begin{cases} 
M \otimes_{\mathcal{O}(P)} \mathcal{O}(Q) & \text{if } Q \leq P, \\
0 & \text{otherwise;}
\end{cases}
\]

2. whose restriction maps are induced by the map \( m \otimes 1 \mapsto m \otimes 1 \) for all \( Q_1, Q_2 \geq P \);

3. where the \( \mathcal{O}(Q) \)-module structure on \( S(Q) \) is given by acting on the second component in the tensor product.

Lemma 9.17. Let \( \mathcal{O} \) be a sheaf of rings on a topological category \( \mathcal{C} \), and let \( M \) be an \( \mathcal{O}(P) \)-module. Then for any \( F \) there is a natural isomorphism
\[
\text{Hom}_{\mathcal{O}(P)}(M, F(P)) \rightarrow \text{Hom}_{\mathcal{O}}(\text{CoSky}(P,M), F).
\]
If \( M \) is a projective \( \mathcal{O}(P) \)-module, then \( \text{CoSky}(P,M) \) is a projective \( \mathcal{O} \)-module.

It is a standard fact that for any ring, \( A \), the category of \( A \)-modules has enough injectives and projectives (see [Wei94]).

Lemma 9.18. Let \( \mathcal{O} \) be a sheaf of ring on any topological category \( \mathcal{C} \). Then the category of \( \mathcal{O} \)-modules has enough injectives and projectives.

Proof. Let \( \mathcal{M} \) be an \( \mathcal{O} \)-module, and for each \( x \in \text{Ob}(\mathcal{C}) \) let \( P(x) \) be a projective \( \mathcal{O}(x) \)-module such that there is a surjection \( P(x) \rightarrow \mathcal{M}(x) \). Then the natural map of
\[
\bigoplus_{x \in \text{Ob}(\mathcal{C})} \text{CoSky}(x, P(x))
\]
to \( \mathcal{M} \) is a surjection, and hence there are enough projectives. We argue similarly that there are enough injectives. \( \square \)

All of the above constructions work for semitopological categories. For example, \( \text{Sky}(P,M) \) is, more generally, constructed as having values
\[
Q \mapsto \mathcal{M}^{\text{Hom}(P,Q)}
\]
and there is a natural restriction map; similarly for \( \text{CoSky}(P,M) \). For the case where \( \mathcal{O} \) is a constant sheaf, this follows from the following discussion (the more general case is the above simple adaptation).

We remark that our construction of skyscrapers and coskyscrapers follows from the very general Proposition I.5.1 of [SGA4]; let us summarize the main points. If \( u: \mathcal{C} \rightarrow \mathcal{C}' \) is any functor, then if \( \mathcal{F}' \) is a sheaf on \( \mathcal{C}' \) with values in any category, \( \mathcal{D} \) (i.e., a presheaf in the sense of [SGA4]), then one sets \( u^* \mathcal{F}' \) to be the sheaf on \( \mathcal{C} \) with values in \( \mathcal{D} \) given by \( \mathcal{F} = \mathcal{F}' \circ u \) (so \( \mathcal{F}(P) = \mathcal{F}'(u(P)) \) for \( P \in \text{Ob}(\mathcal{C}) \)). The functor \( u^* \) has a left adjoint \( u_! \) and a right adjoint \( u_* \) described in [SGA4]. Proposition I.5.1, assuming that certain limits in \( \mathcal{D} \) are representable (i.e., exist). Consider the special case of the above, where \( \mathcal{C} = \Delta_0 \) (the terminal category, whose object is 0 and has only the identity morphism at 0), \( x \in \text{Ob}(\mathcal{C}') \), and \( u_x: \Delta_0 \rightarrow \mathcal{C}' \) is the morphism taking 0 to \( x \). Taking \( \mathcal{D} \) to be the category of \( k \)-vector spaces for a field \( k \) yields the skyscrapers and coskyscrapers, as \( (u_x)_* M \) and \( (u_x)_! M \), respectively in the case of sheaves with values in \( \mathcal{D} \), i.e., sheaves of \( k \)-modules.
10. Foundations, Part 4: \( \mathcal{O} \)-modules and Ext groups for \( C_{2V} \)

In this section we state the practical consequences of Section 9 regarding a projective resolution for \( \mathcal{O}_{k,r} \) and an injective resolution for \( \omega_{k,r} = \mathcal{M}_{k,r,d} \). These method are much more general, and immediately yield projective resolutions for any sheaf of the form \( L_{k,r,d} \) and injective resolutions for any sheaf of the form \( \mathcal{M}_{k,r,d} \).

10.1. The Standard Setup.

**Definition 10.1.** By a standard setup on \( C_{2V} \) we mean a sheaf of rings \( \mathcal{O} \) on \( C_{2V} \) all of whose restriction maps are injections. We use the notation \( S_i = \mathcal{O}(A_i) \) for \( i = 1, 2 \) and \( R_j = \mathcal{O}(B_j) \) for \( j = 1, 2, 3 \); similarly, we often discuss an \( \mathcal{O} \)-module \( \mathcal{M} \) with the notation \( N_i = \mathcal{O}(A_i) \) and \( M_j = \mathcal{M}(B_j) \).

We depict the standard setup as follows:

![Diagram of standard setup](image)

The skyscraper sheaves of Definition 9.14 can be depicted by the following diagrams:

![Diagrams of skyscraper sheaves](image)

where \( N \) is an \( S_1 \)-module, and \( M \) is an \( R_i \)-module in \( \text{Sky}(B_i, M) \); we have similar diagrams for \( \text{Sky}(A_2, N) \) and \( \text{Sky}(A_1, N) \). In \( \text{Sky}(A_1, N) \), the values at \( B_i \) is \( N \) for \( i = 1, 3 \), and for these \( i \) the set \( N \) is an \( R_i \)-module via the restriction maps \( R_i \to S_1 \) and the \( S_1 \)-module structure on \( N \). The importance of the skyscraper sheaves are due to the following two facts:

1. there is a canonical isomorphism:

   \[
   \text{Hom}_{\mathcal{O}}(\mathcal{F}, \text{Sky}(x, M)) \to \text{Hom}_{\mathcal{O}(x)}(\mathcal{F}(x), M)
   \]

   for any \( x \in \text{Ob}(C_{2V}) \) and \( \mathcal{O}(x) \)-module \( M \), and therefore

   \[
   \text{Sky}(x, I)
   \]

   is an injective \( \mathcal{O} \)-module for any injective \( \mathcal{O}(x) \)-module \( I \).

The second fact is an easy consequence of (28). To prove (28), the reader may either accept Lemma 9.15, or verify this lemma, or simply verify this lemma in the case of \( C_{2V} \).

For example, to check (28) for \( x = A_1 \), one checks that any morphism represented by the solid arrow in Figure 9 extends uniquely to the dashed arrows there to produce a morphism of \( \mathcal{O} \)-modules. The verification for \( x = A_2 \) is—by symmetry—the same, and the verification for \( x = B_1, B_2, B_3 \) is immediate.
Analogously we have coskyscraper sheaves of Definition 9.16 depicted for $x = A_1, B_1, B_3$ as follows:

| CoSky($A_1, N$) | CoSky($B_1, M$) | CoSky($B_3, M$) |
|----------------|----------------|----------------|
| $N$            | $M \otimes_{R_1} S_1$ | $M \otimes_{R_1} S_1$ |
| $0$            | $M \otimes_{R_1} S_1$ | $M \otimes_{R_1} S_2$ |
|                | $0$              | $0$            |

and the satisfy:

1. there is a canonical isomorphism:

\[(29) \quad \text{Hom}_{\mathcal{O}}(\text{CoSky}(x, M), \mathcal{F}) \to \text{Hom}_{\mathcal{O}(x)}(M, \mathcal{F}(x))\]

for any $x \in \text{Ob}(\mathcal{C}_{2V})$ and $\mathcal{O}(x)$-module $M$, and therefore

2. $\text{CoSky}(x, P)$ is a projective $\mathcal{O}$-module for any projective $\mathcal{O}(x)$-module $P$.

Again, to directly verify (29) for $x = B_3$, one checks that any morphism represented by the solid arrow in Figure 10 extends uniquely to the dashed arrows there to give a morphism of $\mathcal{O}$-modules. The verification of (29) for $x = B_1, B_2$ is analogous, and the verification for $x = A_1, A_2$ is easier.

10.2. Projective Resolutions of $\mathcal{O}$ and Beyond. One consequence of the above is that if the restriction maps of $\mathcal{O}$ are injections, then $S_i \otimes_{R_i} R_i \simeq S_i$ for all morphisms $(A_i, B_j)$ in $\mathcal{C}_{2V}$. This leads to a convenient projective resolution of $\mathcal{O}$ and similar sheaves.
Lemma 10.2. Let $\mathcal{O}$ be as in Definition 10.1 and assume that all restriction maps in $\mathcal{O}$ are injections. Then the $\mathcal{O}$-module $\mathcal{O}$ has a projective resolution (in the category of $\mathcal{O}$-modules) given by:

(30) \[ 0 \to P_1 \to P_0 \to \mathcal{O} \to 0, \]

where

\[ P_1 = \bigoplus_{i=1}^{2} \text{CoSky}(A_i, \tilde{S}_i), \quad P_0 = \bigoplus_{j=1}^{3} \text{CoSky}(B_j, R_j) \]

where $\tilde{S}_i$ is the $S_i$-submodule of $S_i \oplus S_i$ generated by $(1, -1)$.

Proof. First note that for any $i, j$ such that there is a morphism $A_i \to B_j$, we have that $S_i \otimes R_j = S_i$ since the $\mathcal{O}$ restriction map $R_j \to S_i$ is an injection. It follows that $P_0 \to \mathcal{O}$ is a surjection. Since $R_j$ is a free $R_j$-module, it follows that $P_0$ is projective. Next we verify (value-by-value) that the kernel of $P_0 \to \mathcal{O}$ is given by the sheaf in the diagram below:

\[ \begin{array}{ccc}
0 & \to & S_1 \oplus S_1 \\
\tilde{S}_1 & \to & S_1 \oplus S_1 \\
0 & \to & S_2 \oplus S_2 \\
\tilde{S}_2 & \to & S_2 \oplus S_2 \\
0 & \to & S_i \oplus S_i \\
\tilde{S}_i & \to & S_i \oplus S_i \\
0 & \to & R_1 \\
\tilde{R}_1 & \to & R_1 \\
0 & \to & R_2 \\
\tilde{R}_2 & \to & R_2 \\
0 & \to & R_3 \\
\tilde{R}_3 & \to & R_3 \\
\end{array} \]

Now we see that (1) $\tilde{S}_i$ is a free $S_i$-module, so that the kernel $P_1$ is a projective sheaf that is the sum of coskyscrapers as indicted above. \(\square\)

A similar projective resolution can be made for any sheaf, $\mathcal{L}$, such that its values are those of $\mathcal{O}$ and its restriction maps $\mathcal{O}(A_i, B_j): R_j \to S_i$ map 1 to a unit in $S_i$. These sheaves are what we call line bundles or invertible sheaves in Section 13, and they include the $\mathcal{O}_{k,r}$-modules $\mathcal{L}_{k,r,d}$ for any $k, r, d$.

As a consequence we may compute the cohomology groups $H^i(\mathcal{M}) = \text{Ext}^i(\mathcal{O}, \mathcal{M})$ by taking the above projective resolution of $\mathcal{O}$, which yields the following result.

Corollary 10.3. Let $\mathcal{O}$ be any sheaf of $k$-algebras on $\mathcal{C}_{2N}$ with injective restriction maps. Then, for any $\mathcal{O}$-module, $\mathcal{M}$, with notation as in Definition 10.1, we have that $H^i(\mathcal{M}) = \text{Ext}^i(\mathcal{O}, \mathcal{M})$ for all $i$ are the same groups as those defined in Definition 6.1.

Proof. By (29), we see that applying $\text{Hom}(\cdot, \mathcal{M})$ to (30) we get that the groups $\text{Ext}^i(\mathcal{O}, \mathcal{M})$ for $i = 0, 1$ are, respectively, the kernel and cokernel of the map

\[ \bigoplus_{j=1}^{3} \text{Hom}_{R_j}(R_j, M_j) \to \bigoplus_{i=1}^{2} \text{Hom}_{S_i}(\tilde{S}_i, N_i). \]

Since $\tilde{S}_i = (1, -1)S_i$, we may replace $\tilde{S}_i$ with $S_i$ in the above map and view it as the map

\[ \bigoplus_{j=1}^{3} \text{Hom}_{R_j}(R_j, M_j) \to \bigoplus_{i=1}^{2} \text{Hom}_{S_i}(S_i, N_i) \]

where we negate the natural maps for $j = 3$ and $i = 1, 2$. Lastly, for any $R$-module $M$ over a ring $R$, we may identify $\text{Hom}_R(R, M)$ with $M$ via $\phi \in \text{Hom}_R(R, M)$.
maps to $\phi 1 \in M$. Hence the groups $\text{Ext}^i(O, M)$ can be computed as the kernel and cokernel of
\[
\bigoplus_{j=1}^3 M_j \to \bigoplus_{i=1}^2 N_i
\]
with the above sign conventions; as such, these are the same groups as the $H^i(M)$ in Definition 6.1. □

10.3. **Injective Resolutions of $M_k,r,d$.** We can similarly use skyscraper sheaves to build injective resolutions. In the case of $M_k,r,d$ (or $O_k,r$), we can use the injective resolutions of $k[x_1, 1/x_1]$-modules
\[
k[x_1, 1/x_1] \to k(x_1) \to k(x_1)/k[x_1, 1/x_1]
\]
and similar resolutions to build injective resolutions of $M_k,r,d$.

In this article we prefer to use projective resolutions for all of our specific computations.

11. **Computation of $\text{Hom}(M_d, M_{d'})$**

**Remark 11.1.** A previous version of the articles erroneously claimed that $\text{Hom}(M_d, M_{d'}) \simeq \Gamma(M_{k,r,d'-d})$. This is in general not the case.

For the theorem will need the following result.

**Theorem 11.2.** Let $k$ be a field, $r \geq 1$ be an integer, and $d,d' \in \mathbb{Z}_2$. Then there is a canonical morphism (not an isomorphism)
\[
(31) \quad \text{Hom}(M_k,r,d, M_k,r,d') \to \Gamma(M_{k,r,d'-d}).
\]

In Section 13 we shall define sheaf Hom; it will be clear that the proof we now give of the above theorem is based on local considerations, and more generally gives a morphism
\[
\mathcal{H}om(M_d, M_{d'}) \to M_{d'-d}.
\]
In Section 13 we shall also discuss the line bundles $L_{k,r,d}$ defined in Definition 7.2; the formulas
\[
L_{k,r,d} \otimes M_k,r,d' \simeq M_{k,r,d+d'}
\]
to conceptually simplify matters a bit; indeed, such formulas imply that we need only consider the case $d = 0$ in Theorem 11.2.

The morphism (31) can also be obtained from the morphism $L_{k,r,d} \to M_{k,r,d}$; the above theorem is much easier since we are only claiming the existence of a morphism.

**Proof.** Let $S_1 = O(A_1) = k[x_1, 1/x_1]$, and consider the map taking
\[
\phi \in \text{Hom}_{O_k,r}(M_k,r,d, M_k,r,d')
\]
to
\[
\phi(A_1) \in \text{Hom}_{O(A_1)}(M_k,r,d(A_1), M_k,r,d'(A_1)) = \text{Hom}_{S_1}(S_1, S_1).
\]
Identifying $\text{Hom}_{S_1}(S_1, S_1)$ with $S_1$ in the usual fashion, we get a canonical morphism
\[
u: \text{Hom}_{O_k,r}(M_k,r,d, M_k,r,d') \to S_1.
\]
We similarly define a map
\[
v: \text{Hom}_{O_k,r}(O_k,r,0, M_k,r,d'-d) \to S_1
\]
(since $O_{k,r}(A_1) = S_1$). Clearly $u, v$ are $k$-linear maps. To prove the theorem it suffices to show that

1. $u$ and $v$ are injections, and
2. the image of $u$ lies in $v$ (it was previously claimed that the image of $u$ was equal to that of $v$).

To show that $u$ is an injection, say that $u\phi = 0$, i.e., $\phi(A_1) = 0$ Since the restriction maps

$$M_{k,r,d}(A_1, B_3), \ M_{k,r,d'}(A_1, B_3)$$

are isomorphisms, it follows $\phi(B_3) = 0$. Since the restriction maps

$$M_{k,r,d}(A_2, B_3), \ M_{k,r,d'}(A_2, B_3)$$

are isomorphisms, it follows $\phi(A_2) = 0$. Since the restriction maps

$$M_{k,r,d}(A_1, B_1), \ M_{k,r,d'}(A_1, B_1)$$

are injections, it follows that $\phi(B_1) = 0$; similarly $\phi(B_2) = 0$. Hence $\phi$ is zero at $A_1, A_2, B_1, B_2, B_3$.

To show that $v$ is an injection, say that $v\psi = 0$. Since $M_{k,r,d'-d}(A_1, B_3)$ is an isomorphism and $O_{k,r}(A_1, B_3)$ is an injection, it follows that $\psi(B_3) = 0$. It follows that $\psi(A_2)$ maps the image of $1 \in O(A_2, B_3)$ to zero; since $\psi$ is a map of $O_{r,K}$ modules, $\psi(A_2)$ is a map of $O_{r,K}(A_2)$-modules taking 1 to zero, and hence $\psi(A_2) = 0$. Similar to the last paragraph, we then have $\psi$ is zero at $B_1$ and $B_2$, and hence $\psi = 0$.

Now consider the image of $u$. Say that $u\phi = \gamma \in S_1$; then we may uniquely write $\gamma$ as

$$\phi(A_1) = \gamma = \sum_{i=1}^{r} p_i(x^i_r) x^i_1.$$  

This uniquely determines $\phi(B_3)$ and $\phi(A_2)$ to be

$$\phi(A_2) = \gamma' = \sum_{i=1}^{r} p_i(x^{r-i}_2) x^i_2$$

assuming they exist; the problem is that there are conditions on the $p_i$ above for $\phi(B_3)$ to exist as a morphism that intertwines the $O(B_3) = k[v, 1/v]$ action on $M_{k,r,d}(B_3)$ with the $O(A_2) = k[x_2, 1/x_2]$ action on the $A_2$ values; it is only for $r = 1$ that there are no extra conditions. We see that the condition that $\phi(A_1)$ extends to $\phi(B_1)$ is that that $\gamma$ have no terms in $x_1$ of degree $d'_1 - d'_1$; similarly for $\gamma'$, $x_2$, and $d'_2 - d_2$.

Now consider the image of $v$. If $v\psi = \psi(A_1) = \gamma_1$, then $\gamma_1$ determines the image of $1 \in O_{k,r}(A_2)$ in $M_{k,r,d'-d}$ under $\psi(A_2)$, which gives forces the relations (32) and (33) with $\psi$ replacing $\phi$ in both equations. The conditions that the value of $\psi(A_1)$ extend to $\psi(B_1)$ give the same degree conditions.

Hence any point in the image of $u$ lies in the image of $v$. $\Box$

It should be clear that the above argument is “local” in the sense that we have studied a global section by studying what happens at $A_1$, and then at $A_2$ by considering the “neighbourhood” $\{B_3, A_1, A_2\}$ of $B_3$, and then studying the neighbourhoods $\{B_1, A_1\}$ and $\{B_2, A_2\}$. For this reason the above proof really shows that
the relation (31) is really a “local” morphism (not generally an isomorphism)
\[ \text{Hom}(M_d, M_{d'}) \to M_{d'-d} \]
as we will describe in Section 13 (where we formalize the Hom above as sheaf Hom).

12. Proof of the Duality Theorems

In this section we show that skyscrapers satisfy a strong duality property; we
cannot expect that this duality holds for \( \omega = M_{k,r,K} \) or any other \( M_{k,r,d} \) when
\( r \geq 2 \); the older version claimed strong duality holds for the \( M_{k,r,d} \).

**Definition 12.1.** Let \( k \) be a field, and \( O \) a sheaf of \( k \)-algebras on a bipartite
category \( C \). Let \( \omega \) be a sheaf of \( k \)-on a bipartite category \( C \) and fix a morphism
\( \phi: H^1(\omega) \to k \) of \( k \)-vector spaces. For each sheaf \( F \) of \( O \)-algebras we get a pairing

\[ H^i(F) \times \text{Ext}^{1-i}(F, \omega) \to H^1(\omega) \xrightarrow{\phi} k. \]

For any \( i \) we say that \( H^i \)-duality holds for \( F \) with respect to \( \phi, \omega \) if (34) is a perfect
paring; for any \( i \) we use Duality\(_{\omega,i}(\phi, \omega) \) to denote the class of sheaves satisfying
\( H^i \)-duality; we say that strong duality holds for \( F \) with respect to \( \phi, \omega \) if (34)

1. is a perfect pairing for \( F \) and both \( i = 0, 1 \), and
2. \( H^i(F) \) and \( \text{Ext}^1(F, \omega) \) vanish for \( i \geq 2 \);

we use Strong - Duality\(_{\omega}(\phi, \omega) \) to denote the set of sheaves, \( F \), for which strong
duality holds. If \( H^1(\omega) \simeq k \), then the class of sheaves for which any of the above
notions of duality holds is the same for all isomorphisms \( \phi \), and we drop \( \phi \) in the
above notation (and assume that \( \phi \) is any isomorphism).

Of course, the zero sheaf always lies in Duality\(_{\omega}(\phi, \omega) \). We also drop the \( O \) from
the notation if \( O \) is understood.

Now we state the main theorem in this section; before doing so we remark that the
cokernel of

\[ k[y_1] \xrightarrow{p(y_1) + y_1} k[y_1] \]
is an \( k[y_1] \)-algebra \( \tilde{k} = k[y_1]/y_1k[y_1] \), which is as a set is isomorphic to \( k \) in on
which \( k[y_1] \) acts by the rule \( a_0 + y_1a_1 + \cdots + y_1a_m \); times \( b \) is taken to \( a_0b \) (for
\( a_0, \ldots, a_m, b \in k \)). We may write \( k \) for \( \tilde{k} \), but we often write \( k \) to remind ourselves
of this particular \( k[y_1] \)-module structure; similarly for 2 replacing 1 in the subscripts.

These are the main duality theorems that we prove in this article.

**Theorem 12.2.** Let \( k \) be a field and \( r \geq 1 \) an integer. Then \( \omega = \omega_{k,r} = M_{k,r,K} \)
has \( H^1(\omega) \simeq k \) and Strong - Duality\(_{\omega} \) contains

1. the skyscraper sheaf \( \text{Sky}(B_1, k) \), where \( k = k[y_1]/(y_1k[y_1]) \); and
2. the skyscraper sheaf \( \text{Sky}(B_2, k[y_2]/y_2k[y_2]) \).

An earlier version claimed that \( \omega \) satisfies \( H^1 \)-duality.

We have already computed \( H^1(\omega_{k,r}) \simeq k \); hence to prove the theorem above
it suffices to verify the duality statements. We shall divide this proof into several
subsections.
12.1. Proof of Theorem 12.2.

**Proof of Theorem 12.2.** By symmetry it suffices to verify the case $i = 1$, i.e., that

$$\mathcal{S}_1 = \text{Sky}(B_1, k[y_1]/y_1 k[y_1]) \in \text{Duality}(\omega_{k,r}).$$

To verify that (34) holds for $\mathcal{F} = \mathcal{S}_1$ and $i = 1$ is easy: since $\mathcal{S}_1$ is a skyscraper sheaf, $H^1(\mathcal{S}_1) = 0$. Furthermore any $\gamma \in \text{Hom}(\mathcal{S}_1, \omega)$ is determined by its only possible nonzero map $\gamma(B_1): \mathcal{S}_1(B_1) \to \omega(B_1)$; but this map must be zero, since $\omega(A_1, B_1)$ is an injection and $\mathcal{S}_1(A_1, B_1)$ is the zero map.

This shows that (34) holds for $\mathcal{F} = \mathcal{S}_1$ and $i = 1$; Lemma 7.6 also shows that $H^1(\mathcal{S}_1) = 0$ for $i \geq 2$ (which is true where $\mathcal{S}_1$ is replaced with any sheaf of $k$-vector spaces). Hence it remains to verify that (34) holds for $\mathcal{F} = \mathcal{S}_1$ and $i = 0$, and to verify that $\text{Ext}^1(\mathcal{F}, \omega_{k,r}) = 0$ for $i \geq 2$.

In the short exact sequence of $k$-vector spaces (21), the source and target of $\mu$ are $\mathcal{O}$-modules, and hence this becomes a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{M}_{d} \xrightarrow{\mu} \mathcal{M}_{d+(1,0)} \to \text{Sky}(B_1, \tilde{k}) \to 0$$

with $\tilde{k} = k[y_1]/y_1 k[y_1]$. Now take $d = \omega_{k,r}$, so that we get an exact sequence

$$H^0(\text{Sky}(B_1, k)) \to H^1(\mathcal{M}_{r,K}) \to H^1(\mathcal{M}_{r,K+(1,0)}) \to \text{Ext}^1(\text{Sky}(B_1, k), \omega_r)^* \to \text{Hom}(\mathcal{M}_{r,K}, \omega_r)^* \to \text{Hom}(\mathcal{M}_{r,K+(1,0)}, \omega_r)^*$$

Using Theorem 11.2, we now observe, that

1. $\text{Hom}(\mathcal{M}_{r,K}, \omega_r) = \text{Hom}(\mathcal{M}_{r,K}, \mathcal{M}_{r,K}) \simeq \Gamma(\mathcal{M}_{k,r,0})$ which is one-dimensional;
2. similarly $\text{Hom}(\mathcal{M}_{r,-K+(1,0)}, \omega_r) \simeq \Gamma(\mathcal{M}_{k,r,-1,0})$ which vanishes;
3. by Corollary 7.4, $H^1(\mathcal{M}_{k,r,K}) \simeq k$ and $H^1(\mathcal{M}_{k,r,K+(1,0)}) = 0$;
4. the middle downward arrow $H^1(\mathcal{M}_{k,r,K}) \to \text{Hom}(\mathcal{M}_{r,K}, \omega_r)^*$ is an isomorphism, since both these spaces are one-dimensional and the identity in $\text{Hom}(\mathcal{M}_{k,r,K}, \omega_r) = \text{Hom}(\omega_{k,r}, \omega_{k,r})$ induces, via the pairing (26), the identity map $H^1(\omega_{k,r}) \to H^1(\omega_{k,r})$.

Hence the above diagram amounts to

$$H^0(\text{Sky}(B_1, \tilde{k})) \to k \to 0$$

Furthermore $H^0(\text{Sky}(B_1, \tilde{k})) \simeq \tilde{k}$ by Lemma 7.6. Finally, consider the short exact sequence

$$0 \to \text{CoSky}(B_1, k[y_1]) \to \text{CoSky}(B_1, k[y_1]) \to \text{Sky}(B_1, \tilde{k}) \to 0$$

obtained from (21) by setting all values at $A_2, B_2, B_3$ (and appropriate restriction maps) to zero. Setting $\mathcal{F} = \text{CoSky}(B_1, k[y_1])$ we get a long exact sequence

$$0 \to \text{Ext}^0(\mathcal{S}_1, \omega) \to \text{Ext}^0(\mathcal{F}, \omega) \to \text{Ext}^0(\mathcal{F}, \omega) \to \text{Ext}^1(\mathcal{S}_1, \omega) \to 0$$
since $\text{Ext}^1(\mathcal{F}, \omega) = 0$ since $\mathcal{F}$ is a projective $\mathcal{O}_{k,r}$-module. We have seen above that $\text{Ext}^0(\mathcal{S}_1, \omega) = \text{Hom}(\mathcal{S}_1, \omega) = 0$, and (28) implies that
\[ \text{Ext}^0(\mathcal{F}, \omega) = \text{Hom}(\mathcal{F}, \omega) = \text{Hom}_{k[y_1]}(k[y_1], \omega(B_1)) \simeq \omega(B_1) = k[y_1], \]
so the long exact sequence becomes
\[ 0 \to k[y_1] \xrightarrow{\phi} k[y_1] \to \text{Ext}^1(\mathcal{S}_1, \omega) \to 0, \]
and the functoriality of the coskyscraper functor shows that $\phi$ is just multiplication by $y_1$. It follows that $\text{Ext}^1(\mathcal{S}_1, \omega) \simeq k[y_1]/y_1 k[y_1]$, which is one dimensional. Hence (36) becomes
\[ H^0(\text{Sky}(B_1, \hat{k})) \simeq k \xrightarrow{\sim} k \xrightarrow{\sim} k \xrightarrow{\sim} 0. \]
The exactness in the rows show that the upper left and lower left horizontal arrows are isomorphisms, and hence and leftmost downward arrow is an isomorphism.

This verifies (34) for $\mathcal{F} = \mathcal{S}_1$ and $i = 0$. But (37) shows that $\mathcal{S}_1$ has a projective resolution of length two, and hence $\text{Ext}^i(\mathcal{S}_1, \omega_{k,r}) = 0$ for $i \geq 2$. □

12.2. A Lemma Related to the Five-Lemma.

This lemma was used to study the strong duality of $\omega$; although this strong duality does not hold for $r \geq 2$, we anticipate that this lemma may be useful in fixing the duality result in future work.

**Lemma 12.3.** Let
\[ \begin{array}{ccc}
A_3 & \to & A_2 \\
\downarrow & & \downarrow \\
B_3 & \to & B_2 \\
\end{array} \]
be a morphism of exact chains of $k$-vector spaces such that the maps $A_3 \to B_3$ and $A_1 \to B_1$ are surjective. Then the map $A_2 \to B_2$ is surjective.

Let us make three remarks regarding this lemma. First, the proof we give below would still work if the top and bottom 0’s in the diagram were replaced by $A_0$ and $B_0$ and a downward isomorphism $A_0 \to B_0$; this generalization and its dual proves the five-lemma. Second, the special case where $A_2 = A_3 = B_3 = k$ shows that one cannot replace “surjective” with “injective” in this lemma. Third, any proof via “diagram chasing”—such as the one below—generalizes to the same statement in any abelian category, by the Freyd-Mitchell Embedding Theorem.

**Proof of Lemma 12.3.** This is an easy diagram chase: let $\beta_2 \in B_2$; we need to show that $\beta_2$ has a preimage in $A_2$.

Let $\beta_1$ be the image of $\beta_2$ in $B_1$; by surjectivity, $\beta_1$ has a preimage $\alpha_1$ in $A_1$; by the exactness of the rows, $\alpha_1$ has a preimage $\alpha_2 \in A_2$; choose any such $\alpha_2$, and let $\beta_2'$ be the image of $\alpha_2$. Since $\beta_2'$ and $\beta_2$ both map to $\beta_1$ ($\beta_2'$ by commutativity of the diagram, since $\alpha_2$ that maps to $\alpha_1$ that maps to $\beta_2'$), we have that $\beta_2' - \beta_2$ has a preimage $\beta_3$ in $B_3$. Since $A_3 \to B_3$ is surjective, $\beta_3$ has a preimage $\alpha_3$ in $A_3$, and
hence if $\alpha'_{2}$ is the image of $\alpha_{3}$ in $A_{2}$, then $\alpha_{3}$ maps to $\beta'_{2} - \beta_{2}$. Hence $\alpha_{2} - \alpha'_{2}$ maps to $\beta_{2}$ in $B_{2}$. Hence $\beta_{2}$ has a preimage in $A_{2}$.

**12.3. The Method of Grothendieck.**

We anticipate that this discussion will be useful in future work.

The following is a special case of what is sometimes called the “method of Grothendieck:”

**Lemma 12.4.** Let $k, \mathcal{O}, \mathcal{C}, \omega, \phi$ be as in Definition 12.1, and let

$$0 \to F_{1} \to F_{2} \to F_{3} \to 0$$

be a short exact sequence of $\mathcal{O}$-modules.

(1) Then if any two of $F_{1}, F_{2}, F_{3}$ lie in $\text{Strong} - \text{Duality}(\phi, \omega)$, then all three do; and

(2) If $F_{1}, F_{2}$ lie in $\text{Duality}_{\mathcal{O},i}(\phi, \omega)$ and $F_{1}$ lies in $\text{Duality}_{\mathcal{O},i+1}(\phi, \omega)$ with $H^{i+1}(F_{1}) = 0$, then $F_{3}$ lies in $\text{Duality}_{\mathcal{O},i}(\phi, \omega)$.

The proof is immediate from the five-lemma applied to

$$
\begin{array}{cccccccc}
0 & \to & H^{1}(F_{3})^{*} & \longrightarrow & H^{1}(F_{2})^{*} & \longrightarrow & H^{1}(F_{1})^{*} & \longrightarrow & H^{0}(F_{3})^{*} & \longrightarrow & H^{0}(F_{2})^{*} & \longrightarrow & H^{0}(F_{1})^{*} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}(F_{3}, \omega) & \to & \text{Hom}(F_{2}, \omega) & \to & \text{Hom}(F_{1}, \omega) & \to & \text{Ext}^{1}(F_{3}, \omega) & \to & \text{Ext}^{1}(F_{2}, \omega) & \to & \text{Ext}^{1}(F_{1}, \omega) & \to & 0
\end{array}
$$

13. Foundations, Part 5: Sheaf Hom and Local Considerations

It conceivable that there is a more general strong duality theorem that is even easier to prove, namely a local theorem whose global version Theorem 12.2. In this section we describe some “local considerations.” In particular our proof Theorem 11.2 really proves that there is an injection (not an isomorphism)

$$(38) \quad \mathcal{H}om_{\mathcal{O}_{r}}(M_{r,d}, M_{r,d'}) \to M_{r,d'-d},$$

which immediately implies that

$$(39) \quad \dim(\text{Hom}_{\mathcal{O}_{r}}(M_{r,d}, M_{r,d'})) \leq b_{0}(M_{r,d'-d})$$

by taking global sections. We will also define the tensor product of sheaves and derive some convenient formulas such as

$$\mathcal{H}om(\mathcal{L}_{k,r,d} \otimes \mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{L}_{k,r,-d} \otimes \mathcal{G}), \quad \mathcal{L}_{k,r,d} \otimes \mathcal{M}_{k,r,d'} \simeq \mathcal{L}_{k,r,d+d'},$$

$$\mathcal{L}_{k,r,d} \otimes \mathcal{M}_{k,r,d'} \simeq \mathcal{M}_{k,r,d+d'}.$$

We will also make some remarks about the curious nature of

$$M_{k,r,d} \otimes M_{k,r,d'}.$$

13.1. Localization.** Topos theory strongly suggests how we should define “localization,” and thereby a number of concepts such as sheaf $\text{Hom}$. Here is the upshot in our case.

**Definition 13.1.** Let $G = (\mathcal{A}, \mathcal{B}, E)$ be a bipartite graph. If $P \in \mathcal{A} \sqcup \mathcal{B}$, we define the restriction of $G$ to $P$ as follows:

1. if $P = A \in \mathcal{A}$, we define $G/A$ to be the bipartite graph $(\{A\}, \emptyset, \emptyset)$;
2. if $P = B \in \mathcal{B}$, we define $G/B$ to be the bipartite graph $(\mathcal{A}_{B}, \{B\}, \mathcal{A}_{B} \times \{B\})$

where $\mathcal{A}_{B}$ consists of those elements of $\mathcal{A}$ incident upon $B$. 
Furthermore, if $C$ is the bipartite category associated to $G = (A, B, E)$, for any $P \in \text{Ob}(C) = A \amalg B$, we define the restriction of $C$ to $P$, denoted $C/P$, to be the subcategory of $C$ associated to $G/P$.

More generally, if $C$ is any category endowed with its coarsest topology, then [SGA4], Exposé IV, gives a recipe where the points of the topos are naturally identified with $\text{Ob}(C)$, each of which is contained in a minimal open subtopos, which is precisely the “slice category” $C/P$, often called the category “of objects over $P$” or “of morphisms to $C$.”

**Definition 13.2.** If $F$ is a sheaf of $k$-vector spaces on a bipartite category, $C$, and $P \in \text{Ob}(C)$, then the restriction of $F$ to $P$, denoted $F|_P$, is the subcategory of $C$ associated to $G/P$. Similarly if $u: F \to G$ is a morphism, the localization of $u$ to $P$ is the evident restriction map $u|_P$ from $F|_P$ to $G|_P$.

This definition is valid for any category $C$ (viewed as a topos with the coarsest topology), although $C/P$ is not generally a subcategory of $C$, and the natural “localization map” $C/P \to C$ is not an inclusion; the exception is when $C$ is topological in the sense of Definition 9.13, which includes the case of bipartite categories.

In the following subsections we will see the importance of the notion of localization.

### 13.2. Sheaf Hom.

**Definition 13.3.** Let $C$ be the simple category associated to a bipartite graph, $G$, $O$ a sheaf of rings, and $F, G$ sheaves of $O$-modules. We define a sheaf $H = \mathcal{H}\hom(F, G)$ as the sheaf of $O$-modules whose value at $P$ is

$$H(P) = \text{Hom}_{O|_P}(F|_P, G|_P),$$

and whose restriction maps $H(B) \to H(A)$, for a morphism $A \to B$, is given by the restriction since $G_A$ is a subgraph of $G_B$.

In the above definition, if $(A, B)$ is an edge in $G$, then a morphism $F(B) \to G(B)$ does not necessarily extend to a morphism $F(A) \to G(A)$, and if it does then the extension is not necessarily unique. Hence we cannot define a good notion of sheaf Hom by looking “value-by-value.” Furthermore there is a canonical morphism

$$\text{Hom}_{O|_P}(F|_P, G|_P) \to \text{Hom}_{O(P)}(F(P), G(P))$$

(see [EGA1], Section 0.5.2.6 or [Har77], Proposition III.6.8), but this morphism is not an isomorphism when $F, G$ are of the form $M_{k,r,d}, M_{k,r,d'}$, since in this case, the left-hand-side $O(P)$ module is of rank $r$ for $P = B_3$, whereas the right-hand-side is of rank $r^2$.

**13.3. Proof of (38).** Let us briefly indicate a proof of (38).

Again, since we are only claiming the existence of a morphism, this result is far simpler. Yet, we expect that future work may use such local considerations. Again, the reason that we get a morphism and not an isomorphism is the problem with the $B_3$ neighbourhood; all other neighbourhoods are fine, but the $B_3$ neighbourhood is essential in connecting the points $B_1, A_1$ to $B_2, A_2$.

The proof of Theorem 11.2 really gives

1. a morphism

$$\mathcal{H}\hom_{O}(M_{r,d}, M_{r,d'})(B_3) \to M_{r,d'-d}(B_3)$$
(2) morphisms for \( i = 1, 2 \)
\[
\mathcal{H}om_{\mathcal{O}}(\mathcal{M}_{r,d}, \mathcal{M}_{r,d'})(B_i) \to \mathcal{M}_{r,d'-d}(B_i)
\]
which are both isomorphic to \( \text{Hom}_R(R_i, R_i) = R_i \) with \( R_i = k[y_i] \),
(3) such that for \( i = 1, 2 \), these isomorphisms at \( B_3 \) and \( B_i \) restrict to the same
isomorphism
\[
\mathcal{H}om_{\mathcal{O}}(\mathcal{M}_{r,d}, \mathcal{M}_{r,d'})(A_i) \to \mathcal{M}_{r,d'-d}(A_i)
\]
which are both isomorphic to \( \text{Hom}_{S_i}(S_i, S_i) \cong S_i \) with \( S_i = k[x_i, 1/x_i] \).
Hence Theorem 11.2 really gives morphisms for (38) restricted to open neighbourhoods of \( B_1, B_2, B_3 \) which (1) are isomorphisms in the neighbourhoods of \( B_1 \) and
of \( B_2 \), but only morphisms in that of \( B_3 \), and (2) agree on their overlap when restricted to \( A_1, A_2 \) (this overlap is small because of the problems at \( B_3 \)). Hence we
get a global morphism (38).

13.4. Tensor Product and Line Bundles.

**Definition 13.4.** Let \( \mathcal{F}, \mathcal{G} \) be sheaves of \( \mathcal{O} \)-modules for some sheaf of rings \( \mathcal{O} \) on
a bipartite category \( \mathcal{C} \). We define the **tensor product** of \( \mathcal{F}, \mathcal{G} \), denoted \( \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \), to be
the sheaf whose values are
\[
(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})(P) = \mathcal{F}(P) \otimes_{\mathcal{O}(P)} \mathcal{G}(P),
\]
and whose restriction maps are obtained as the tensor product of the restriction maps of \( \mathcal{F} \) and \( \mathcal{G} \).

Analogous to (38) we easily see that
\[
\mathcal{L}_{k,r,d} \otimes \mathcal{L}_{k,r,d'} \cong \mathcal{L}_{k,r,d+d'}
\]
and
\[
(40) \quad \mathcal{L}_{k,r,d} \otimes \mathcal{M}_{k,r,d'} \cong \mathcal{M}_{k,r,d+d'}.
\]
The fact that
\[
\mathcal{L}_{k,r,d} \otimes \mathcal{L}_{k,r,-d} \cong \mathcal{L}_{k,r,0} = \mathcal{O}_{k,r}
\]
justifies calling the \( \mathcal{L}_{k,r,d} \) **invertible sheaves**. Let us summarize this idea.

By a **vector bundle** we mean a sheaf \( \mathcal{F} \), such that for some integer \( n \geq 1 \), \( \mathcal{F} \) is
**locally isomorphic to** \( \mathcal{O}^n \), i.e., for any \( P \in \text{Ob}(\mathcal{C}) \) we have
\[
\mathcal{F}|_P \cong \mathcal{O}^n|_P.
\]
The case \( n = 1 \) is called a **line bundle**.

**Example 13.5.** Consider any sheaf \( \mathcal{F} \) on \( C_{2V} \) such that \( \mathcal{F}(P) = \mathcal{O}(P) \) for all
\( P \in \text{Ob}(\mathcal{C}_{2V}) \) and that \( \mathcal{F}(A_i, B_3) = \mathcal{O}(A_i, B_3) \) for \( i = 1, 2 \). Then for \( \mathcal{F} \) to be a line
bundle it is necessary that \( \mathcal{F}(A_1, B_1) \) be multiplicative unit in \( \mathcal{F}(A_1) = k[y_1, 1/y_1] \),
since if not we cannot have \( \mathcal{F}|_{B_1} \cong \mathcal{O}|_{B_1} \); we easily check that if \( \mathcal{F}(A_i, B_i) \) is a
multiplicative unit in \( \mathcal{F}(A_i) \) for \( i = 1, 2 \), then \( \mathcal{F} \) is indeed a line bundle. Hence for
\( i = 1, 2 \) we have \( \mathcal{F}(A_i, B_i) = x_i^d c_i \) for some nonzero \( c_i \in k \); our sheaves \( \mathcal{L}_{k,r,d} \) are
the special cases where \( c_1 = c_2 = 1 \), and for general nonzer \( c_1, c_2 \) the sheaf \( \mathcal{F} \) is
isomorphic to \( \mathcal{L}_{k,r,d} \).
There are a number of standard facts about line bundles and vector bundles: for example, any line bundle \(\mathcal{L}\) has an “inverse” line bundle \(\mathcal{L}^{-1}\) for which \(\mathcal{L} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}\), and
\[
\text{Hom}(\mathcal{L} \otimes \mathcal{F}, \mathcal{G}) \simeq \text{Hom}(\mathcal{F}, \mathcal{L}^{-1} \otimes \mathcal{G})
\]
for any \(\mathcal{O}\)-modules \(\mathcal{F}, \mathcal{G}\) (where we have dropped the subscripts of \(\mathcal{O}\) from \(\text{Hom}\) and \(\otimes\)). It follows immediately that
\[
\text{Hom}(\mathcal{M}_{k,r,d}, \mathcal{M}_{k,r,d'}) \simeq \text{Hom}(\mathcal{M}_{k,r,0}, \mathcal{M}_{k,r,-d})
\]
We easily verify a number of other standard formulas such as the existence of an isomorphism
\[
(41) \quad u: \text{Hom}_\mathcal{O}(\mathcal{F}, \text{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{H})) \to \text{Hom}_\mathcal{O}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H})
\]
that is natural in \(\mathcal{F}, \mathcal{G}, \mathcal{H}\). To describe this isomorphism we need to specify \(u\) at each \(P \in \text{Ob}(\mathcal{C})\), which amounts to a morphism
\[
u(P): \text{Hom}_\mathcal{O}_P(\mathcal{F}|_P, \text{Hom}_\mathcal{O}_P(\mathcal{G}|_P, \mathcal{H}|_P)) \to \text{Hom}_\mathcal{O}_P((\mathcal{F} \otimes \mathcal{G})|_P, \mathcal{H}|_P)
\]
which is straightforward to describe (see [Har77] Exercise II.5.1(c), or [SGA4] Section V.12.8 for the more general case). By Yoneda’s lemma, (41) uniquely determines \(\mathcal{F} \otimes \mathcal{G}\).

13.5. **Remarks on** \(\mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d}\). It is curious to note that
\[
(\mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d})(B_3)
\]
is of rank \(r^2\), while the inclusion
\[
\mathcal{O}_{k,r} \to \mathcal{M}_{k,r,0}
\]
sending \(\mathcal{O}_{k,r}(B_3)\) to the first component of \(\mathcal{M}_{k,r,0}(B_3)\) gives an inclusion
\[
\mathcal{O}_{k,r} \otimes \mathcal{M}_{k,r,d} \to \mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d}
\]
where the left-hand-side is just \(\mathcal{M}_{k,r,d}\). Note that this map takes \(e_1 \otimes e_j\) to \(e_j\) and \(e_j \otimes e_j\) to zero for \(j' \geq 2\); one could do the reverse map; more generally, for each \(j = 1, \ldots, r\) and each \(j', j''\) with \(j' + j'' = r\), one can map \(e_j \otimes e_{j'}\) to some multiple of \(e_{j'}\) and get a morphism
\[
\mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d} \to \mathcal{M}_{k,r,d}.
\]
We check that the cokernel of any such map (that is an inclusion) is supported at \(B_3\) and seems a bit strange: it contains the classes of elements
\[
(42) \quad (e_{i_1} \otimes e_{i_2}) - (e_{i_3} \otimes e_{i_4}) \quad \text{s.t.} \quad i_1 + i_2 = i_3 + i_4
\]
that restrict to zero; this gives a \(k[v, 1/v]\)-module of rank \(\binom{r}{2}\) of elements of
\[
(\mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d})(B_3)
\]
that map to zero along all restriction maps. However when \(i_1 + i_2 = i_3 + i_4 - r\), the class of
\[
(e_{i_1} \otimes e_{i_2})v - (e_{i_3} \otimes e_{i_4})
\]
is taken to 0 in the \((A_1, B_3)\) restriction but not in the \((A_2, B_3)\) restriction, and vice versa for the class of
\[
(e_{i_1} \otimes e_{i_2})v^{-1} - (e_{i_3} \otimes e_{i_4})
\]
similarly, this generates a module of rank \(\binom{r}{2}\) of
\[
(\mathcal{M}_{k,r,0} \otimes \mathcal{M}_{k,r,d})(B_3)
\]
with a curious “twisting” property. It follows that $M_{k,r,0} \otimes M_{k,r,d}$ is the direct sum of an module supported at $B_3$ generated by the elements of (42) and the rest, which admits an inclusion from $M_{k,r,d}$, but has some curious “twisting” elements.

Of course, the commutativity (up to isomorphism) of $\otimes$ and (40) shows that

$$M_d \otimes M_{d'} \cong M_0 \otimes M_{d+d'},$$

and so similar remarks hold for this more general left-hand-side tensor product.

14. Remarks for Future Research

A main goal of future research is to get a duality theorem for $O$-modules with $O = O_{k,r}$ that holds for the $M_{k,r,d}$. We hope to address this in future work.

In this section we make some remarks regarding future research. We will discuss “local methods” and some issues involving them, both for $C_{2V}$ and for more general categories including those of interest in previous works on sheaves on graphs.

We conclude with some brief remarks regarding other directions.

14.1. Local Duality Theorems. In this subsection we explore possible generalizations stronger duality theories.

It would seem simpler and more natural to prove that for $\text{Sky}(B_1, O_{k,r}(B_1)) = \text{Sky}(B_1, k[y_1])$ satisfies strong duality, and then infer this for $\text{Sky}(B_1, \tilde{k})$ from the method of Grothendieck and the exact sequence

$$0 \to \text{Sky}(B_1, k[y_1]) \to \text{Sky}(B_1, k[y_1]) \to \text{Sky}(B_1, \tilde{k}) \to 0$$

which is immediate from the exact sequence of $k[y_1]$-algebras

$$0 \to k[y_1] \to k[y_1] \to \tilde{k} \to 0.$$

The problem is that $\text{Sky}(B_1, k[y_1])$ does not have finite zeroth Betti number, and two infinite dimensional vector spaces cannot be perfectly paired with each other since their dual spaces are too large. Here we make a number of possible ways in which one can study strong duality of $\text{Sky}(B_1, k[y_1])$.

First, we have a canonical isomorphism

$$H^0\left(\text{Sky}(B_1, k[y_1])\right) \cong k[y_1]$$

and—in view of the projective resolution of $\omega_{k,r}$—a canonical isomorphism

$$\text{Ext}^1\left(\text{Sky}(B_1, k[y_1]), \omega_{k,r}\right) \cong k[x_1,1/x_1]/k[y_1] = k[x_1,1/x_1]/k[1/x_1];$$

with these identifications we could compute the pairing of these spaces and presumably infer an explicit formula for the pairing on $\text{Sky}(B_1, k)$. Second, one could also write

$$\text{Ext}^1\left(\text{Sky}(B_1, k[y_1]), \omega_{k,r}\right) \cong k[y_1,1/y_1]/k[y_1],$$

giving this group the structure of a finitely generated $k[y_1]$-algebra; the group (43) also has such a structure. Hence, although these two groups are infinite dimensional $k$-vector spaces, they are both finitely generated $k[y_1]$-algebras.

Third, although the dual $k$-vector space of (43) is too large to be isomorphic to (44), there is a natural subspace, $S$, of the dual space—which one could call the tame dual (perhaps finitely supported dual) of (43)—to which (44) appears to be
isomorphic: namely, define the \textit{tame dual} of \( k[y_1] \) to be the set of linear functionals \( \ell \colon k[y_1] \to k \) such that
\[
\ell(a_0 + a_1 y_1 + a_2 y_1^2 + \cdots + a_n y_1^n)
\]
depends only on finitely many of the \( a_i \)'s (independent of \( n \), of course).

More generally, if \( W \) is any \( k \)-vector space with a countable basis \( w_1, w_2, \ldots \), we define the \textit{tame dual} or \textit{finitely supported dual} to be the linear functionals that depend on only finitely many of the coefficients of \( w_i \) in the unique representation of any vector. Although this notion depends on the basis we choose for \( W \), it is pretty clear that the pairing
\[
\text{H}^0\left(\text{Sky}(B_1, k[y_1])\right) \times \text{Ext}^1\left(\text{Sky}(B_1, k[y_1]), \omega_{k,r}\right) \to k
\]
should be a tame pairing, i.e., the pairing of \( y_1^n \) in the first factor should depend on only finite many of the \( 1/y_1^m \) in the second. One possible problem with the tame dual is that if the graph Riemann-Roch or some other application requires us to model spheres with more than just 0 and \( \infty \) missing, i.e., three or more points missing, then it is not clear we can choose local bases for Hom and Ext groups that are globally compatible.

14.2. \textbf{Alternative Views of Sky}(B_1, k[y_1]/(y_1)) \textbf{Strong Duality.} There are two other methods that one could use to understand strong duality of \( \mathcal{S}_1 = \text{Sky}(B_1, k[y_1]/(y_1)) \) (and the same with 2 replacing 1 in the subscripts). First, one could compute the effect of \( \text{Hom}_O(O, \mathcal{S}_1) = k[y_1]/(y_1) \) on \( \text{Ext}^1(\mathcal{S}_1, \omega_{k,r}) \) using the factorial nature of Ext. Second, one could do this in terms of Yoneda ext groups. These two are essentially the same computation; however, either of these two methods may illuminate and provide an easier proof of the perfect pairing of \( \text{Hom}_O(O, \mathcal{S}_1) \) with \( \text{Ext}^1(\mathcal{S}_1, \omega_{k,r}) \).

14.3. \textbf{Localization for Graphs and Graphs of Groups.} The categories used in [Fri15] were based on oriented graphs, \( G \), that could have self-loops. In the event that such a \( G \) has no self-loops, then the resulting category is a bipartite category with left objects being \( V_G \), the vertices of the oriented graph, and the right objects were the edges of the oriented graph, \( E_G \). However, when such a \( G \) has self-loops (which would be \textit{whole-loops} in the sense of [Fri93]), one gets two morphisms from the vertex to its self-loop: this therefore a semi-topological category (Definition 9.13), but not a topological category, and the resulting topos is not a topological space.

One aspect of localization in general finite categories, \( \mathcal{C} \), is that when we work with the slice category \( \mathcal{C}/P \) as the “neighbourhood of \( P \),” we seem to get very reasonable definitions. One superficial aspect of this is that if we generalize our bipartite categories to include the above categories of [Fri15], then if \( e \) is a self-loop about a vertex \( v \) as above, then \( \mathcal{C}/e \) is the category with three objects, whose objects are the two morphisms to \( e \) from \( v \) and the identity morphism \( \text{id}_e \); hence \( \mathcal{C}/e \) is two objects mapping to one object, just as \( \mathcal{C}/e \) is when \( e \) is not a self-loop (i.e., when \( e \) has distinct endpoints).

We can see that this remark regarding \( \mathcal{C}/e \) also holds when a graph has \textit{half-loops} in the sense of [Fri93] (half-loops play an important role in defining regular random graphs of odd degree and other aspects of covering theory). A graph is defined as a directed graph with an orientation reversing involution; therefore if \( e \) is a half-loop,
i.e., an edge that is the fixed point of the involution, then in the corresponding category one would expect that its incident vertex, \( v \), has two morphisms to \( e \), but that \( \text{Hom}(e, e) \) is a cyclic group of order two. In this case the slice category \( \mathcal{C}/e \) has four objects, including the two elements of \( \text{Hom}(e, e) \) which are isomorphic (as objects of \( \mathcal{C}/e \)); hence the resulting category is still the above three-object category.

Similarly consider a graphs of groups, \((T, G)\), as in [Ser03] Section 4.4, Definition 8, with the notation there. It is natural associate to \((T, G)\) a category, \( \mathcal{C} \), whose objects are \( V \amalg E' \), where \( V \) is the vertex set of \( T \), \( E \) the edge set, and \( E' \) is the collection of unordered pairs \( \{e, \overline{e}\} \) with \( e \in E \); the morphisms of \( \mathcal{C} \) are as follows:

1. \( \text{Hom}(v, v) = G_v \) for \( v \in V \);
2. \( \text{Hom}(e', e') = G_e = G_{\overline{e}} \) for \( e' = \{e, \overline{e}\} \in E' \);
3. \( \text{we introduce a set of morphisms} \ t(e) \to \{e, \overline{e}\} \) for each \( e \in E \) that consists of \( G_v \).

The compositions with elements of \( \text{Hom}(t(e), \{e, \overline{e}\}) \) are multiplication in \( G_v \) either on the left with \( G_v \) (for \( \text{Hom}(v, v) \)) or on the right with \( G_t(e) \) acting via its image in \( G_e \) under \( G_e \to G_{t(e)} \). In this case, for \( e' = \{e, \overline{e}\} \in E' \) we have \( \mathcal{C}/e' \) consists of objects \( G_{t(e)} \amalg G_{t(\overline{e})} \amalg G_e \), where any two objects in any of these three summands are isomorphic; we therefore see that \( \mathcal{C}/e' \) is equivalent to the same three-object category as \( \mathcal{C}/e \) for graphs above. [Similarly, we see that \( \mathcal{C}/v \) for \( v \in V \) is equivalent to the category with one element and one morphism, just as it would be in a graph.]

From the above examples, the view of topos theory ([SGA4.IV]) that \( \mathcal{C}/P \) is the “smallest neighbourhood of \( P \) indicates that graphs, graphs with whole-loops and/or half-loops, and graphs of groups all have the same “local” structure. This seems quite promising when we try to combine the methods of this article with those of [Fri15], and to generalize these methods to graphs with half-loops and graphs of groups.

14.4. Other Remarks. For any integers \( r', r'' \geq 1 \) and field \( k \), there is a morphism \( \mathcal{O}_{k, r'} \to \mathcal{O}_{k, r'+r''} \) that maps the indeterminate \( v \in \mathcal{O}_{k, r'} \) to \( v^{r''} \in \mathcal{O}_{k, r'+r''} \) and is otherwise the identity. Hence there is a morphism of ringed spaces ([Har77], page 72) from \( (C_{2V}, \mathcal{O}_{k, r'}) \) to \( (C_{2V}, \mathcal{O}_{k, r''}) \). It would be interesting to study such morphisms; in particular, each \( \mathcal{O}_{k, r} \) can be considered as a (presumably \( r \)-to-1) covering space of \( \mathcal{O}_{k, 1} \), which is just the sphere.

Of course, ultimately we would like to generalize this theory to include the Riemann-Roch Graph Theorem on an arbitrary number of vertices, and, more generally, to study \( \mathbb{Z}^n/L \) where \( L \) is any lattice in \( \mathbb{Z}^n \) (for the Riemann-Roch Graph Theorem, \( n \) is the number of vertices and \( L \) is the image of the graph Laplacian).

One could also study for a covering map \( G \to G' \) of graphs, how the Graph Riemann-Roch Theorem and how our type of algebraic models behave.

References

[Bak13] Matthew Baker, Matt baker’s math blog: Riemann-roch for graphs and applications, 2013, Available as https://mattbakerblog.wordpress.com/2013/10/18/riemann-roch-for-graphs-and-applications/.

[BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744886

[BN07] Matthew Baker and Serguei Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph, Adv. Math. 215 (2007), no. 2, 766–788. MR 2355607

---

4See also [BH99], III.C.2.8 (page 538), which is presumably equivalent (we do not understand what \( t(\alpha) \) would mean for \( \alpha \in \mathcal{Y} \) if \( \alpha \in V(\mathcal{Y}) \) in the second paragraph of this section).
Michael Ben-Or, *Lower bounds for algebraic computation trees*, Proceedings of the Fifteenth Annual ACM Symposium on Theory of Computing (a.k.a. STOC 1983), 1983, pp. 80–86.

David Dobkin and Richard J. Lipton, *Multidimensional searching problems*, SIAM J. Comput. 5 (1976), no. 2, 181–186. MR MR0416099 (54 #4175)

Joel Friedman, Alice Izsak, and Lior Silberman, *Abelian girth and girth*, Available as http://arxiv.org/abs/1511.03878.

Joel Friedman and David-Emmanuel Kohler, *The relativized second eigenvalue conjecture of alon*, Available at http://arxiv.org/abs/1403.3462.

Peter Freyd, *Abelian categories. An introduction to the theory of functors*, Harper’s Series in Modern Mathematics, Harper & Row, Publishers, New York, 1964. MR 0166240

Peter J. Freyd, *Abelian categories [mr0166240]*, Repr. Theory Appl. Categ. (2003), no. 3, 1–190, available from http://www.tac.mta.ca/tac/reprints/index.html, specifically http://www.tac.mta.ca/tac/reprints/articles/3/tr3.pdf, see also http://www.tac.mta.ca/tac/reprints/articles/3/tr3abs.html. MR 2050440

Joel Friedman, *Some geometric aspects of graphs and their eigenfunctions*, Duke Math. J. 69 (1993), no. 3, 487–525. MR 94b:05134

______, *Relative expanders or weakly relatively Ramanujan graphs*, Duke Math. J. 118 (2003), no. 1, 19–35. MR 1978881

______, *Cohomology of Grothendieck topologies and lower bounds in Boolean complexity*, preprint, 70 pages. Available at http://arxiv.org/abs/cs/0512008 and http://www.math.ubc.ca/~jf, specifically http://www.math.ubc.ca/~jf/pubs/web_stuff/groth1.pdf http://arxiv.org/abs/cs/0512008.

______, *Cohomology of Grothendieck topologies and lower bounds in Boolean complexity ii*, preprint, 8 pages. Available at http://arxiv.org/abs/cs/0604024 and http://www.math.ubc.ca/~jf/pubs/web_stuff/groth2.pdf.

______, *Linear transformations in Boolean complexity theory*, CiE ’07: Proceedings of the 3rd conference on Computability in Europe (Berlin, Heidelberg), Springer-Verlag, 2007, pp. 307–315.

______, *Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks*, Mem. Amer. Math. Soc. 233 (2015), no. 1100, xii+106, With an appendix by Warren Dicks. MR 3289057

A. Grothendieck and J. A. Dieudonné, *Eléments de géométrie algébrique. I*, Grundllehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971. MR 3075000

Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 57 #3116

P. J. Hilton and U. Stammbach, *A course in homological algebra*, second ed., Graduate Texts in Mathematics, vol. 4, Springer-Verlag, New York, 1997. MR 1438546

Alice Izsak, *Abelian girth and gapped sheaves*, Doctoral thesis, Available as https://open.library.ubc.ca/circle/collections/ubctheses/24/items/1.0223486.

Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) 182 (2015), no. 1, 307–325. MR 3374962

______, *Interlacing families IV: Bipartite Ramanujan graphs of all sizes*, 2015 IEEE 56th Annual Symposium on Foundations of Computer Science—FOCS 2015, IEEE Computer Soc., Los Alamitos, CA, 2015, pp. 1358–1377. MR 3473375

Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. MR 1954121 (2003m:20032)

Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269. MR 50 #7130

J. Michael Steele and Andrew C. Yao, *Lower bounds for algebraic decision trees*, J. Algorithms 3 (1982), no. 1, 1–8. MR 646886 (83i:68076)
[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada.

Email address: nicolASFolinsbee@gmail.com

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, Canada.

Email address: jf@cs.ubc.ca