EQUALITY OF LYAPUNOV AND STABILITY EXPONENTS FOR PRODUCTS OF ISOTROPIC RANDOM MATRICES

NANDA KISHORE REDDY

Abstract. We show that Lyapunov exponents and stability exponents are equal in the case of product of i.i.d isotropic(also known as bi-unitarily invariant) random matrices. We also derive asymptotic distribution of singular values and eigenvalues of these product random matrices. Moreover, Lyapunov exponents are distinct, unless the random matrices are random scalar multiples of Haar unitary matrices or orthogonal matrices. As a corollary of above result, we show probability that product of \( n \) i.i.d real isotropic random matrices has all eigenvalues real goes to one as \( n \to \infty \). Also, in the proof of a lemma, we observe that a real (complex) Ginibre matrix can be written as product of a random lower triangular matrix and an independent truncated Haar orthogonal (unitary) matrix.

1. Definitions and introduction

Let \( M_1, M_2, \ldots \) be sequence of i.i.d random matrices of order \( d \). Define \( \sigma_n \) to be diagonal matrix with singular values of product matrix \( P_n = M_1M_2\ldots M_n \) in the diagonal in decreasing order and similarly \( \lambda_n \) to be diagonal matrix with eigenvalues of \( P_n \) in the diagonal in decreasing order of absolute values, for \( n = 1, 2, \ldots \). Let \( |\lambda_n|^{\frac{1}{n}} \) and \( |\sigma_n|^{\frac{1}{n}} \) denote diagonal matrices with non-negative \( n \)-th roots of absolute values of diagonal entries of \( \lambda_n \) and \( \sigma_n \) in the diagonal, respectively.

Define \( \sigma := \lim_{n \to \infty} |\sigma_n|^{\frac{1}{n}} \) and \( \lambda := \lim_{n \to \infty} |\lambda_n|^{\frac{1}{n}} \), if the limits exist. Then diagonal elements of \( \ln \sigma \) and \( \ln \lambda \) are called Lyapunov exponents and stability exponents for products of i.i.d random matrices, respectively. In other words, they are rates of exponential growth(or decay) of singular values and eigenvalues of product matrices \( P_n \), respectively as \( n \to \infty \).

We consider both real and complex random matrices in this paper. For the sake of simplicity, we restrict ourselves mostly to complex random matrices. But all the definitions and statements, along with proofs, carry over immediately to real case (by replacing everywhere unitary matrices by orthogonal matrices).

Definition 1. A random matrix \( M \) is said to be isotropic if probability distribution of \( UMV \) is same as that of \( M \), for all unitary matrices \( U, V \).

They also go by the names of bi-unitarily invariant and rotation invariant random matrices. It follows from definition that distribution of \( UMV \) is same as that of \( M \), if \( U, V \) are Haar distributed random unitary matrices independent of \( M \) and each other. \( M = PDQ \) be the singular value decomposition of \( M \), then \( UMV = \ldots \)

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Since $U, V$ are Haar unitary matrices independent of $D$, then $UP$ and $QV$ are also independent Haar unitary matrices, independent of $D$.

So, it follows that $N = UDV$, where $D$ is diagonal matrix of singular values of rotation invariant random matrix $M$ and $U, V$ are independent Haar unitary matrices independent of $D$, has same distribution as that of $M$. So, $N$ is also isotropic and this can be taken as alternate definition of isotropic random matrices.

**Definition 2.** A product matrix $M = UDV$ is said to be isotropic if $D$ is random diagonal matrix with non-negative real diagonal entries and $U, V$ are Haar unitary matrices independent of $D$ and each other.

**Definition 3.** A product matrix $M = UDV$ is said to be right isotropic or right rotation invariant if $U$ is random unitary matrix, $D$ is random diagonal matrix with non-negative real diagonal entries and $V$ is Haar unitary matrix independent of $D$ and $U$.

Lyapunov exponents for products of i.i.d random matrices and criteria for their distinctness have been discussed in [5]. Exact expression for Lyapunov exponents in case of isotropic random matrices has been derived in [20]. In the same paper, explicit calculations and asymptotics have been done in case of Gaussian real random matrices, also called real Ginibre ensemble. Explicit calculations of Lyapunov exponents for modified Complex Ginibre ensembles have been done in [7]. Stability exponents have been considered first in the setting of dynamical systems in [13] and therein equality of Lyapunov and stability exponents has been conjectured based upon plausible arguments and numerical results. Recent comparative studies ([2], [13]) of Lyapunov exponents and stability exponents in the case of Ginibre matrix ensembles have verified the conjecture to be true in the respective cases. And also ([9]) mentions this in the case of random truncated unitary matrices. For a summary of results on this topic, we refer reader to [4]. In section 3, we give a proof of this conjecture in the case of $2 \times 2$ isotropic random matrices has been given. In section 3, we give a proof of this conjecture for isotropic random matrices of any order.

Another phenomenon of interest in products of random matrices is convergence of probability, that all eigenvalues of real random product matrix $M_1M_2...M_n$ are real, to one as $n \to \infty$. This phenomenon has been first studied extensively numerically in case of real Gaussian matrices in [18] and rigorous proofs of those results have been obtained in [8]. Numerical study of the same phenomenon for random matrices with i.i.d elements from various other distributions uniform, Laplace and Cauchy has been done in [11]. In section 5, we prove the same in case of real isotropic random matrices, as a corollary to existence and distinctness of stability exponents, generalizing the case of real Gaussian matrices in the direction of isotropic property.

Asymptotic distribution of first order fluctuations of action of product random matrices is known to be Gaussian, see Chapter V of part A in [5] for related central limit theorems. But the knowledge of variances of these distributions has been very limited until recently. In [2], asymptotic distributions of first order fluctuations of (logarithm of) both singular values and moduli of eigenvalues for product of large number of Ginibre matrices have been computed and shown to be equal. In [9], variances of asymptotic distributions of first order fluctuations of singular values in cases of generalised Ginibre matrices and truncated Haar unitary matrices have been computed. In section 4, we compute first order asymptotic joint probability
density of singular values of $\mathcal{P}_n$, also that of moduli of eigenvalues of $\mathcal{P}_n$ and show them to be equal.

The paper is organised as follows. In section 2 we give an algorithm for generating products of isotropic random matrices, using singular value decompositions, which would help us to see the relation between eigenvalues and singular values of product matrices very clearly. Thereafter it becomes very easy to deduce the equality of Lyapunov and stability exponents which is done in section 3. In section 4 we derive first order asymptotic distribution of singular values and eigenvalues of $\mathcal{P}_n$. In final section 5 probability of event that all eigenvalues of product $M_1M_2...M_n$ of $i.i.d$ isotropic random matrices are real is shown to converge to one as $n \to \infty$.

2. PRODUCTS OF RIGHT ISOTROPIC RANDOM MATRICES

Let $\mathcal{P}_n = M_1M_2...M_n$ be product of $i.i.d$ random matrices. $\mathcal{P}_n = U_n\boldsymbol{\sigma}_nV_n$ be singular value decomposition of $\mathcal{P}_n$ with diagonal entries of $\boldsymbol{\sigma}_n$ in descending order. From multiplicative ergodic theorem of Oseledec [21], [22] or Corollary 1.3 on page 79 of [5], we know that under certain conditions on distribution of $M_1, U_n$ converges to a fixed unitary matrix $U_\infty$ and $|\boldsymbol{\sigma}_n|^2$ converges to a fixed diagonal matrix $\boldsymbol{\sigma}$. But knowledge of matrices $\{V_n\}_{n=1}^\infty$ is also required in order to study the limiting behavior of eigenvalues of $\mathcal{P}_n$. The advantage with considering isotropic or just right rotation invariant random matrices is the full knowledge of distribution of $\{V_n\}_{n=1}^\infty$. It is a sequence of independent Haar unitary matrices. We refer the reader to a recent paper [6] to appreciate and understand the role of Haar unitary matrices in random matrix theory.

Let $M_i = U_iD_iV_i$ for $i = 1, 2, \ldots$ be sequence of $i.i.d$ right rotation invariant random matrices where $V_1, V_2 \ldots$ are all independent Haar unitary matrices, independent of positive diagonal matrices $D_1, D_2 \ldots$ and unitary matrices $U_1, U_2 \ldots$.

$$\mathcal{P}_{n+1} = \mathcal{P}_nM_{n+1} = U_n\boldsymbol{\sigma}_nV_nU_{n+1}D_{n+1}V_{n+1}. $$

Let $\boldsymbol{\sigma}_nV_nU_{n+1}D_{n+1}$ be singular value decomposition of $\mathcal{P}_n$, $\mathcal{P}_{n+1} = U_{n+1}\boldsymbol{\sigma}_{n+1}V_{n+1}D_{n+1}$ be singular value decomposition of $\mathcal{P}_{n+1}$ with diagonal entries of $\boldsymbol{\sigma}_{n+1}$ in descending order. Therefore

$$\mathcal{P}_{n+1} = U_nR_{n+1}\boldsymbol{\sigma}_{n+1}S_{n+1}V_{n+1} = U_{n+1}\boldsymbol{\sigma}_{n+1}V_{n+1}$$

where $U_{n+1} = U_nR_{n+1}$, $V_{n+1} = S_{n+1}V_{n+1}$. Since $V_{n+1}$ is Haar unitary matrix independent of preceding matrices $\{U_k, \sigma_k, U_k, D_k\}_{k=1}^{n+1}$ and $\{V_k\}_{k=1}^n$, we have that $V_{n+1}$ is also Haar distributed and independent of $\{U_k, \sigma_k, U_k, D_k\}_{k=1}^{n+1}$ and $\{V_k\}_{k=1}^n$.

This gives us the key idea that $V_{n+1}U_{n+1}$ is Haar distributed and independent of $\{V_kU_k\}_{k=1}^n$.

Remark 4. For $\{M_n\}_{n=1}^\infty$, a sequence of $i.i.d$ right rotation invariant random matrices, $\{V_nU_n\}_{n=1}^\infty$ is a sequence of independent Haar unitary matrices.

Observe that $\sigma_{n+1}$ is diagonal matrix of singular values of $\sigma_nV_nU_{n+1}D_{n+1}$. Since $V_n$ is Haar unitary matrix independent of $\{U_k, D_k\}_{k=1}^{n+1}$, $\{V_k\}_{k=1}^n$ and preceding diagonal matrices $\{\sigma_k\}_{k=1}^n$, we have that $V_nU_{n+1}$ is Haar unitary independent of $\{\sigma_k\}_{k=1}^n$, $\{V_kU_k\}_{k=1}^{n+1}$ and $\{D_k\}_{k=1}^{n+1}$. Therefore $\{V_nU_{n+1}\}_{n=1}^\infty$ is a sequence of $i.i.d$ Haar unitary matrices generated independently of $i.i.d$ sequence of random diagonal matrices $\{D_n\}_{n=1}^\infty$. The sequence of singular value diagonal matrices $\{\sigma_n\}_{n=1}^\infty$ is defined recursively by setting $\sigma_1 = D_1$ and taking $\sigma_{n+1}$ to be the diagonal matrix $\cdots D_{n+1}$. For $i = 1, 2, \ldots$
with singular values of $\sigma_n V_n U_{n+1} D_{n+1}$ in the diagonal, in descending order, for $n = 1, 2, \ldots$.

**Remark 5.** The probability distribution of $\{\sigma_n\}_{n=1}^{\infty}$ doesn’t depend on the distribution of $U_1$, left singular vectors of $M_1$. In other words, the probability distribution of $\{\sigma_n\}_{n=1}^{\infty}$ remains the same in the case of $i.i.d$ isotropic random matrices whose distribution of singular values is same as that of $M_1$.

Using these observations, we have an alternate algorithm of generating products of right rotation invariant random matrices $P_1, P_2, \ldots, P_n, \ldots$ sequentially.

At the first step, generate $M_1 = U_1 D_1 V_1$. Call $D_1$ to be $\sigma_1$ and $M_1$ to be $P_1$.

For $n \geq 1$, at $n+1$-step, we have $P_n = U_n \sigma_n V_n$ and

1. generate a diagonal matrix $D_{n+1}$ from the distribution of $D_1$ independently. Compute singular value decomposition of $\sigma_n V_n D_{n+1}$ to get $\sigma_n V_n D_{n+1} = R_{n+1} \sigma_{n+1} S_{n+1}$, with diagonal entries of diagonal matrix $\sigma_{n+1}$ in descending order. Call $U_{n+1} R_{n+1}$ to be $U_{n+1}$ and

2. generate a Haar unitary matrix $V_{n+1}$ independently. Set $U_{n+1} \sigma_{n+1} V_{n+1}$ to be $P_{n+1}$.

This recursive nature of singular values of isotropic random matrix products has been used in [17] to derive singular value statistics of isotropic random matrix products with singular values of repulsive(Vandermonde) nature. Also, in a very recent study [16], an exact relation between singular value and eigenvalue statistics of isotropic random matrices with repulsive(Vandermonde) singular values and eigenvalues has been established.

### 3. ASYMPTOTIC RELATION BETWEEN EIGENVALUES AND SINGULAR VALUES

Asymptotic behaviour of singular values of product random matrices has been discussed in detail in [5]. We combine Proposition 5.6 from Chapter III, Theorem 1.2 and Proposition 2.5 from Chapter IV of part A in [5] to arrive at the following statement (though part A of the book deals with real random matrices mainly, all the statements are true in complex case also, says the author in Introduction).

**Fact 6.** Let $M_1, M_2 \ldots$ be sequence of $i.i.d$ invertible random matrices of order $d$ with common distribution $\mu$, such that $\mathbb{E}(\log^+ \|M_1\|)$ is finite (spectral norm is used). If there exists a matrix $A$ such that $\|A\|^{-1} A$ is not unitary and $U A U^{-1}$ belongs to support of measure $\mu$ for every unitary matrix $U$, then $|\sigma_n|^\frac{1}{n}$ converges almost surely to a deterministic diagonal matrix $\sigma$ and non-zero diagonal elements of $\sigma$ are finite and distinct. Log values of diagonal elements of $\sigma$ are called Lyapunov exponents.

Additionally if we have that $\mathbb{E}(\log |\det(M_1)|)$ exists and is finite, then as $\det(\sigma_n) = \prod_{k=1}^{n} |\det(M_k)|$, by strong law of large numbers, we get that $\log(\det |\sigma_n|^\frac{1}{n})$ converges to $\mathbb{E}(\log |\det(M_1)|)$ almost surely and all diagonal elements of $\sigma$ are non-zero i.e Lyapunov exponents are all finite.

We use a simple idea of comparing coefficients of characteristic polynomial written in terms of both singular values and eigenvalues, Horn inequalities [12] and Fact 6 regarding convergence of singular values, to prove the main theorem of this paper, which states that as $n \to \infty$, eigenvalues of random isotropic product matrix $P_n$
grow (or decay) exponentially at the same rate as that of corresponding singular values.

**Theorem 7.** Let $M_1, M_2, \ldots$ be sequence of i.i.d invertible isotropic random matrices of order $d$, such that $E(\log^+ \|M_1\|)$ is finite. Then, both $|\sigma_n|^{\frac{1}{d}}$ and $|\lambda_n|^{\frac{1}{d}}$ converge almost surely to the same deterministic diagonal matrix $\sigma$, whose non-zero diagonal elements are all distinct. Additionally if $E(\log |\det (M_1)|)$ exists and is finite, then all diagonal elements of $\sigma$ are non-zero.

**Proof.** If $\text{Prob} \left( \frac{M^T M}{\|M\|^2} = I \right) = 1$, then $M_1$ is random scalar multiple of a Haar unitary matrix. In that case both $|\sigma_n|^{\frac{1}{d}}$ and $|\lambda_n|^{\frac{1}{d}}$ are equal to $(\prod_{k=1}^{n} \|M_k\|)^{\frac{1}{d}} I$ which converges to $e^{\sqrt{\log \|M_1\|}} I$ as $n \to \infty$, by strong law of large numbers. Hence the theorem is true in this case.

Suppose $\text{Prob} \left( \frac{M^T M}{\|M\|^2} = I \right) < 1$. Since $M_1 = U_1 D_1 V_1$, the above assumption ensures that, with positive probability, $D_1$ is not a scalar multiple of identity matrix. So there exists a diagonal matrix $A$ in the support of measure of $D_1$ such that $|A|^{-1} A$ is not unitary. By the definition of isotropic random matrices, $UAU^{-1}$ belongs to the support of measure of $M_1$ for every unitary matrix $U$. Therefore by Fact 6 $|\sigma_n|^{\frac{1}{d}}$ converges almost surely to a diagonal matrix $\sigma$ whose non-zero diagonal elements are finite and distinct. Let's say that the first $r$ diagonal entries of $\sigma$ are non-zero, $1 \leq r \leq d$.

Eigenvalues of $P_n = U_n \sigma_n V_n$ are same as that of $V_n U_n \sigma_n$. For the sake of simplicity of notation, we write $W_n$ in place of $V_n U_n$. By remark $\overline{3}$, $\{W_n\}_{n=1}^{\infty}$ is a sequence of independent Haar unitary matrices.

For $J \subseteq \{1, 2, \ldots, d\}$, $|M_J|$ denotes the determinant of the matrix formed from matrix $M$ by deleting rows and columns whose indices are not in $J$. $|J|$ denotes cardinality of $J$. From here onwards, $J$ is always a subset of $\{1, 2, \ldots, d\}$.

$$\text{det}(zI - \lambda_n) = \text{det}(zI - W_n \sigma_n)$$

By comparing coefficients of $z^{d-i}$ for $i \leq r$ in the above equation, we get

$$|J|=i \sum |J|=i (\lambda_n)_{J} = \sum |J|=i (W_n \sigma_n)_{J} = \sum |J|=i (W_n)_{J} (\sigma_n)_{J}.$$  

(1)

We can see that $|W_n| \leq 1$ and also, $|W_n|^{\frac{1}{d}} \to 1$ as $n \to \infty$ for every $J \subseteq \{1, 2, \ldots, d\}$ almost surely, from Lemma $\overline{5}$ stated and proved above. Also, we have noted earlier in this proof, $|\sigma_n|^{\frac{1}{d}}$ converges almost surely to a diagonal matrix $\sigma$ whose non-zero diagonal elements are finite and distinct, so we have $|\sigma| \leq |\sigma|_{\{1, 2, \ldots, i\}}$ for $|J| = i$, $J \neq \{1, 2, \ldots, i\}$. Therefore $|\sigma_n|^{\frac{1}{d}} \to 0$ for any $|J| = i$ and $J \neq \{1, 2, \ldots, i\}$ and $|\sigma_n|_{\{1, 2, \ldots, i\}}^{\frac{1}{d}} \to |\sigma|_{\{1, 2, \ldots, i\}}$ almost surely as $n \to \infty$, giving us almost surely

$$\lim_{n \to \infty} \left| \sum |J|=i (\lambda_n)_{J} \right|^{\frac{1}{d}} = \lim_{n \to \infty} \left| \sum |J|=i (W_n)_{J} (\sigma_n)_{J} \right|^{\frac{1}{d}} = |\sigma|_{\{1, 2, \ldots, i\}}.$$

Notice that

$$\left| \sum |J|=i (\lambda_n)_{J} \right| \leq \left( \frac{d}{i} \right) |\lambda_n|_{\{1, 2, \ldots, i\}} \leq \left( \frac{d}{i} \right) |\sigma_n|_{\{1, 2, \ldots, i\}}.$$
as the diagonal entries of $A_n$ are in descending order of absolute values and second inequality follows from Horn’s inequalities \[12\]. From the above expressions and sandwich theorem, we have that $\lim_{n \to \infty} \left| A_{n,i} \right|^\frac{1}{n} = |\sigma|_{1,2,i}$ for all $i \leq r$. For $i > r$, $\lim_{n \to \infty} \left| A_{n,i} \right|^\frac{1}{n} = |\sigma|_{1,2,i} = 0$, follows directly from Horn’s inequalities. Therefore, $\lim_{n \to \infty} \left| A_{n,i} \right|^\frac{1}{n} = |\sigma|_{1,2,i}$ for all $1 \leq i \leq d$. It implies that both $|\sigma_n|^\frac{1}{n}$ and $|\lambda_n|^\frac{1}{n}$ converge to the same diagonal matrix $\sigma$ almost surely.

**Remark 8.** We can observe from the proof of above Theorem that, for a sequence of product random matrices $\{ \mathcal{P}_n = U_n \sigma_n V_n \}_{n=1}^\infty$ which satisfy remark 4 existence and distinctness (of non-zero ones) of Lyapunov exponents implies existence of stability exponents and also their equality with Lyapunov exponents. As remark 5 implies that Lyapunov exponents of right isotropic random matrices are equal to that of isotropic random matrices with the same singular value distribution, the above Theorem holds for right isotropic random matrices also.

Now we proceed to prove the lemma used in the above proof.

**Lemma 9.** Let $\{ W_n \}_{n=1}^\infty$ be sequence of independent Haar unitary matrices (or orthogonal matrices) of order $d$. Then $\| W_n \|_J \to 1$ and $\| W_n \|_J \sqrt{n} \to 1$ as $n \to \infty$ for every $J \subseteq \{ 1,2,...d \}$ almost surely.

**Proof.** As there are only finitely many $J \subseteq \{ 1,2,...d \}$, it is enough to prove for every particular $J \subseteq \{ 1,2,...d \}$ that $\| W_n \|_J \to 1$ and $\| W_n \|_J \sqrt{n} \to 1$ as $n \to \infty$ almost surely. For any particular $J (|J| = i)$, since $\| W_n \|_J$ is determinant of sub-block of Haar unitary matrix (or orthogonal matrix), using invariance of measure of $W_n$ under permutations of rows and columns, we can assume without of loss of generality that $J = \{ 1,2,...i \}$.

By Borel-cantelli lemma, we know that for a sequence of i.i.d random variables $\{ Y_n \}_{n=1}^\infty$, $\frac{1}{n} \to 0$ almost surely if and only if $\mathbb{E} |Y_1|$ is finite. Therefore $\frac{\log \| W_n \|_J}{n} \to 0$ and $\frac{\log \| W_n \|_J}{\sqrt{n}} \to 0$ almost surely as $n \to \infty$ if and only if $\mathbb{E} \log \| W_1 \|_J$ and $\mathbb{E} \log \| W_1 \|_J^2$ are finite. Notice that $\mathbb{E} \log \| W_1 \|_J = -\mathbb{E} \log \| W_1 \|_J$. All we need to show now is that logarithm of absolute value of determinant of a truncated Haar unitary (or orthogonal) matrix has finite expected value. This can be done by using density of eigenvalues of truncated unitary (or orthogonal) matrices, see \[15\], \[23\].

We propose another way of doing this which also sheds some new light on many other aspects of truncated Haar Unitary or orthogonal matrices. We do it for orthogonal case. And unitary case would be a straightforward generalization of that. The idea is to write Ginibre matrix as product of a random lower triangular matrix and an independent truncated Haar orthogonal matrix.

Let $S$ be $i \times d$ real Ginibre matrix i.e probability density of $S$ is proportional to $e^{-\frac{1}{2} \text{tr}(S^T S)} dS$, where $dS$ denotes Lebesgue measure on $S$. By QR decomposition of $S$ i.e Gram-Schmidt orthogonalization of rows from top to bottom, we get $S = TO$ where $T$ is lower triangular matrix with non-negative diagonal entries and $O$ is $i \times d$ matrix with orthonormal rows. Lebesgue measure on $S$ can be written in terms of new variables as $dS = \prod_{j=1}^i T_{jj} dT d\mathcal{H}(O)$, where $dT$ denotes Lebesgue measure on non-zero entries of $T$ and $d\mathcal{H}(O)$ denotes probability measure of first $i$ rows of a $d \times d$ Haar orthogonal matrix (see \[19\], p. 63). For a geometric derivation of Jacobian of QR decomposition, we refer reader to \[14\] and also to appendix of
(though there the argument is for complex case, it easily carries over to real case). The joint probability density of $T, O$ is $(\prod_{j=1}^d T_{jj}^{-1/d}) e^{-\frac{1}{2}T^T d T d \mathcal{H}(O)}$, up to the normalizing constant. $T$ and $O$ are independent. Let $S = [S_1 S_2]$ and $O = [O_1 O_2]$ with $S_1, O_1$ being square matrices. Then $S_1 = T O_1$, where $S_1$ is real Ginibre matrix, $O_1$ is $i \times i$ subblock of a $d \times d$ Haar orthogonal matrix independent of $T$. All Non-zero entries of $T$ are independent random variables with non-diagonal ones distributed as standard normal random variables and diagonal elements $T_{jj}$ distributed as square-root of $\chi^2_{(d-j+1)}$ Chi-square random variable for $j = 1, 2 \ldots, i$.

Notice that if $d = i$ we have the usual QR decomposition of $S_1$. Therefore

$$
\mathbb{E} \log \| [W_1]_j \| = \mathbb{E} \log | \det(O_1)| \\
= \mathbb{E} \log | \det(S_1)| - \mathbb{E} \log | \det(T)| \\
= \frac{1}{2} \sum_{j=1}^i \mathbb{E} \log \chi^2_{(i-j+1)} - \frac{1}{2} \sum_{j=1}^i \mathbb{E} \log \chi^2_{(d-j+1)}
$$

and similarly

$$
\mathbb{E}(\log \| [W_1]_j \|)^2 = \mathbb{E}(\log | \det(O_1)|)^2 \\
= \mathbb{E}(\log | \det(S_1)| - \log | \det(T)|)^2 \\
\leq 2\mathbb{E}(\log | \det(S_1)|)^2 + 2\mathbb{E}(\log | \det(T)|)^2 \\
= 2\mathbb{E}(\frac{1}{2} \sum_{j=1}^i \log \chi^2_{(i-j+1)})^2 + 2\mathbb{E}(\frac{1}{2} \sum_{j=1}^i \log \chi^2_{(d-j+1)})^2 \\
\leq \frac{i}{2} \sum_{j=1}^i \mathbb{E}(\log \chi^2_{(i-j+1)}^2) + \frac{i}{2} \sum_{j=1}^i \mathbb{E}(\log \chi^2_{(d-j+1)}^2)
$$

Since $\mathbb{E} \log \chi^2_n$ and $\mathbb{E}(\log \chi^2_n)^2$ are finite for all $n = 1, 2 \ldots$, we have that $\mathbb{E} | \log \| [W_1]_j \|$ and $\mathbb{E}(\log \| [W_1]_j \|)^2$ are finite, which proves the lemma.

**Remark 10.** In the above lemma, $M_1 = LO_1$ where, since $\frac{\chi^2_n}{n} \to 1$ in distribution as $n \to \infty$, $\frac{L}{\sqrt{d}} \to I$ in distribution as $d \to \infty$. So, $\sqrt{d} O_1 \to M_1$ in distribution as $d \to \infty$.

4. **Fluctuations of singular values and eigenvalues**

Let $\lambda_n = \text{diag}(\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,d})$ and $\sigma_n = \text{diag}(\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,d})$ be as defined earlier for all $n = 1, 2 \ldots$. Assume that $\mathbb{E}(\log^+ \| M_1 \|)$ and $\mathbb{E}(\log | \det(M_1)|)$ are finite, so that Theorem 7 holds. Since $M_1 = \mathcal{P}_1 = \mathcal{U} \sigma_1 \mathcal{V}_1$, the above assumptions imply that $\mathbb{E}(\log^+ \sigma_{1,1})$ and $\mathbb{E}(\log \prod_{i=1}^d \sigma_{1,i})$ are finite. Observe that $-\infty < \frac{1}{2} \mathbb{E}(\log \prod_{i=1}^d \sigma_{1,i}) \leq \mathbb{E}(\log \sigma_{1,i}) \leq \mathbb{E}(\log^+ \sigma_{1,1}) < \infty$ and $-\infty < \mathbb{E}(\log \prod_{i=1}^d \sigma_{1,i}) - (d-1) \mathbb{E}(\log \sigma_{1,1}) \leq \mathbb{E}(\log \sigma_{1,d}) \leq \mathbb{E}(\log \sigma_{1,1}) < \infty$, which means that all the singular values of $M_1$ have finite log-moment. Theorem 7 gives us $\sigma$ to have all diagonal entries positive and distinct.

Using recursive structure of singular values $\sigma_n$ and the fact of almost sure convergence of $\sigma_n^{1/n}$ to $\sigma$, we can approximate $\log(\sigma_n)$ by a sum of $n \ i.i.d$ random variables,
to which we can apply central limit theorem to derive the Gaussian nature of first order fluctuations of $\frac{\log(\sigma_n)}{n}$. First of all, notice that

$$\det(zI - \sigma_n^2) = \det(zI - \mathcal{P}_n \mathcal{P}_n^*) = \det(zI - \mathcal{P}_{n-1} M_n M_n^* \mathcal{P}_{n-1}^*) = \det(zI - \sigma_{n-1} V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^*)$$

By comparing coefficients of $z^{d-i}$ for $i \leq r$ in the above equation, we get

$$\sum_{|J|=i} [\sigma_n^2 |J|] = \sum_{|J|=i} [\sigma_{n-1} V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^* |J|] = \sum_{|J|=i} [V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^* |J|] \sigma_{n-1}^2 |J|.$$ 

$\mathcal{V}_n$ is Haar unitary matrix independent of $\mathcal{V}_1, \mathcal{V}_2...\mathcal{V}_{n-1}$ and $M_1, M_2...M_{n-1}$. So, $\mathcal{V}_n$ is independent of $\mathcal{P}_n$. Even though $\mathcal{V}_n$ is not independent of $\mathcal{P}_n$, but being the matrix of right singular vectors of a isotropic matrix $\mathcal{P}_n$, it is independent of $\mathcal{P}_n \mathcal{P}_n^*$. So, $\mathcal{V}_n$ is independent of $M_n M_n^* = \mathcal{P}_{n-1}\mathcal{P}_n \mathcal{P}_n^*(\mathcal{P}_{n-1})^*$ and subsequently $V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^*$. This means that $\mathcal{V}_n M_{n+1} M_{n+1}^* \mathcal{V}_n^*$ is independent of $V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^*$ and the preceding matrices in the sequence. So, $\{V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^* \mathcal{V}_n\}_{n=1}^{\infty}$ forms a sequence of i.i.d isotropic random matrices with distribution same as that of $\mathcal{V}_0 M_1 M_1^* \mathcal{V}_0^*$ or $\mathcal{V}_0 D_1^2 \mathcal{V}_0^*$ or $\mathcal{V}_0 \sigma_1^2 \mathcal{V}_0^*$. For the sake of simplicity of notation, let us write $V_{n-1} M_n M_n^* \mathcal{P}_{n-1}^*$ as $X_n$ for all $n$.

$$\sum_{|J|=i} [\sigma_n^2 |J|] = \sum_{|J|=i} [X_n |J| \sigma_{n-1}^2 |J|].$$

This equation gives us the required recursive structure,

$$\sum_{|J|=i} [\sigma_n^2 |J|] = \prod_{k=1}^{n} \sum_{|J|=i} [X_k |J| \sigma_{k-1}^2 |J|].$$

Notice that $k$-th term in the product is convex combination of $i$-th order principal minors of matrix $X_k$. Because of the almost sure convergence of $|\sigma_n| \overset{p}{\to} \sigma$ (a diagonal matrix with distinct diagonal entries), $k$-th term in the product is approximately $|X_k|_{1,2,i}$ for all large values of $k$, almost surely. Using the above equalities, positivity of principal minors of hermitian matrices and the inequality $\log(a + x) \leq \log(a) + \frac{x}{a}$ for $x, a \geq 0$, we can get lower and upper bounds for logarithm of singular values $\sigma_n$ with both bounds being very close to same sums of i.i.d random variables for large values of $n$. We get, for all $n$

(2) \[
\sum_{k=1}^{n} \log(|X_k|_{1,2,i}) + E_{n,i} \leq \log(\sum_{|J|=i} [\sigma_n^2 |J|]) \leq \sum_{k=1}^{n} \log(|X_k|_{1,2,i}) + E_{n,i} + F_{n,i}
\]

where

$$E_{n,i} = \log \left( \prod_{k=1}^{n} \frac{|X_k|_{1,2,i}}{|X_k|_{1,2,i} |\sigma_{k-1}|_{1,2,i}} \right), F_{n,i} = \sum_{k=1}^{n} \sum_{|J|=i, J \neq \{1,2,i\}} \frac{|X_k|_{1,2,i} |J| \sigma_{k-1}^2 |J|}{|X_k|_{1,2,i} |J| |\sigma_{k-1}|_{1,2,i}}.$$ 

By using the inequality, $|\sigma_n_{1,2,i}| \leq \sum_{|J|=i} |\sigma_n^2 |J| \leq \left( \frac{i}{i} \right) |\sigma_n^2 |_{1,2,i}$, and using the inequalities of singular values, we get that for all $i = 1, 2, d$ and $n = 1, 2, ..$

(3) \[
\log(\sigma_n^2 i) \geq \sum_{k=1}^{n} \log \left( \frac{|X_k|_{1,2,i-1}}{|X_k|_{1,2,i-1}} \right) + E_{n,i} - E_{n,i-1} - F_{n,i-1} - \log \left( \frac{i}{i} \right) \\
\log(\sigma_n^2 i) \leq \sum_{k=1}^{n} \log \left( \frac{|X_k|_{1,2,i-1}}{|X_k|_{1,2,i-1}} \right) + E_{n,i} - E_{n,i-1} + F_{n,i} + \log \left( \frac{i}{i} \right).
\]
(say $E_{n,0} = F_{n,0} = 0$ and $[X_n]_0 = 1$ for all $n = 1, 2, ..$).

The similar bounds can be obtained for moduli of eigenvalues also. We obtain them by showing closeness between moduli of eigenvalues and singular values. From Horn’s inequalities [12], we have $|\lambda_n|_{(1,2,i)} \leq |\sigma_n|_{(1,2,i)}$ for $i = 1, 2, d$. Using the equation [1] we can get the following inequalities for $i = 1, 2, d$ and $n = 1, 2, ..$

$$\log|\sum_{|J|=i} [W_n]_j [\sigma_n]_j| - \log \left(\frac{d}{i}\right) \leq \log(|\lambda_n|_{(1,2,i)}) \leq \log(|\sigma_n|_{(1,2,i)})$$

Writing the difference between upper bound and lower bound as $H_{n,i}$ and using the above inequalities, we get

$$\log(\sigma_{n,i}) - H_{n,i} \leq \log(|\lambda_{n,i}|) \leq \log(\sigma_{n,i}) + H_{n,i-1}.$$ 

where $H_{n,0} = 0, H_{n,i} = \log \left(\frac{d}{i}\right) - \log(|\lambda_{n,i}|) - \log(\sum_{|J|=i} [W_n]_j [\sigma_n]_j)$, for all $i = 1, 2, d, n = 1, 2, ..$

Now we shall see the limiting behavior of $E_{n,i}, F_{n,i}$ and $H_{n,i}$. Since $|\sigma_n|^\frac{1}{2}$ converges almost surely to $\sigma$, a diagonal matrix with distinct positive diagonal entries, as $n \to \infty$, it follows from basic calculus that almost surely $\frac{1}{\sum_{|J|=i} |\sigma_n|^{|J|}}$ converges to one with error term going down to zero exponentially fast and so $E_{n,i}$ converges almost surely to a finite random variable $E_i$ as $n \to \infty$, for $i = 1, 2, d$.

If $\limsup_{n \to \infty} \frac{|X_n|_j [\sigma_n^{-1}]_j}{|X_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}} \frac{1}{\pi} < 1$ almost surely, then by the root test for convergence of a series, $\sum_{k=1}^n \frac{|X_n|_j [\sigma_n^{-1}]_j}{|X_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}}$ converges almost surely as $n \to \infty$. We already know that $\lim_{n \to \infty} \frac{|X_n|_j [\sigma_n^{-1}]_j}{|X_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}} \frac{1}{\pi} = \frac{|\sigma_j|^2}{|\sigma|^2_{(1,2,i)}} < 1$ for $J \neq \{1, 2, i\}$. Since $\{X_n\}_{n=1}^\infty$ is a sequence of i.i.d isotropic random matrices and $|X_n|_j$ has same distribution for all $J$ such that $|J| = i$, we have by triangle inequality that $E[\log(|X_n|_j)] \leq 2E[\log(|X|_{(1,2,i)})] \leq 2E[\log(|\sigma|_{(1,2,i)})] < \infty$ (because all the singular values of $M_1(\sigma)$ have finite log-moments). Therefore by Borel-Cantelli lemma $\frac{1}{\pi} \log\left(\frac{|X_n|_j}{|X_n|_{(1,2,i)}}\right) \to 0$ or $\frac{|X_n|_j}{|X_n|_{(1,2,i)}} \to 1$ almost surely as $n \to \infty$. So, for $J \neq \{1, 2, i\}$, $\limsup_{n \to \infty} \frac{|X_n|_j [\sigma_n^{-1}]_j}{|X_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}} \frac{1}{\pi} = \frac{|\sigma_j|^2}{|\sigma|^2_{(1,2,i)}} < 1$ which implies $\sum_{k=1}^n \frac{|X_n|_j [\sigma_n^{-1}]_j}{|X_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}}$ converges almost surely as $n \to \infty$. Therefore $F_{n,i}$ converges almost surely to a finite random variable $F_i$ as $n \to \infty$ for all $i = 1, 2, ..d$.

It follows from Lemma 9 that $\frac{\log(|W_n|_{(1,2,i)})}{\sqrt{n}} \to 0$ almost surely as $n \to \infty$. Again from Lemma 9 we get that $\lim_{n \to \infty} \frac{|W_n|_j [\sigma_n^{-1}]_j}{|W_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}} \frac{1}{\pi} = \frac{|\sigma_j|}{|\sigma|_{(1,2,i)}}$. So, for $J \neq \{1, 2, i\}$, $\lim_{n \to \infty} \frac{|W_n|_j [\sigma_n^{-1}]_j}{|W_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}} = 0$, which implies that $\log\left(\frac{|W_n|_{(1,2,i)} [\sigma_n^{-1}]_j}{|W_n|_{(1,2,i)} [\sigma_n^{-1}]_{(1,2,i)}}\right) \to 0$ almost surely as $n \to \infty$. Therefore $H_{n,i} \frac{1}{\sqrt{n}} \to 0$ almost surely as $n \to \infty$ for all $i = 1, 2, ..d$. 

For simple notation, denote $\frac{1}{2}(E_{n,i} - E_{n,i-1} - F_{n,i-1} - \log (d_i))$, $\frac{1}{2}(E_{n,i} - E_{n,i-1} + F_{n,i} + \log (d_i))$ by $\xi_{n,i}$ and $\tau_{n,i}$ respectively. $\xi_n$, $\tau_n$ be the vectors $(\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,d})$, $(\tau_{n,1}, \tau_{n,2}, \ldots, \tau_{n,d})$ respectively. Denote the vectors $(H_{n,0}, H_{n,1}, H_{n,2}, \ldots, H_{n,d-1})$ and $(H_{n,1}, H_{n,2}, \ldots, H_{n,d})$ by $\mathcal{P}_n$ and $H_n$ respectively. Observe that $\lim_{n \to \infty} \xi_n$ and $\lim_{n \to \infty} \tau_n$ are finite and $\lim_{n \to \infty} \frac{\mathcal{P}_n}{\sqrt{n}} = \lim_{n \to \infty} \frac{H_n}{\sqrt{n}} = 0$ almost surely.

Denote $\frac{[X_n}_{1212\ldots1]}{[X_n}_{1212\ldots1]}$ by $L_{n,i}$ and $\mathcal{L}_n$ be the vector $(L_{n,1}, L_{n,2}, \ldots, L_{n,d})$. We can see that $\{\mathcal{L}_n\}_{n=1}^{\infty}$ is a sequence of i.i.d random vectors, whose distribution is same as that of $\mathcal{L} = (L_1, L_2, \ldots, L_d) := \left(\sqrt{|V^* D^2 V|_{11}}, \sqrt{|V^* D^2 V|_{12}}, \ldots, \sqrt{|V^* D^2 V|_{1d}}\right)$, where $V$ is Haar unitary matrix independent of random diagonal matrix $D$ and $D$ is distributed like diagonal matrix $D_1$ of singular values of $M_1$. With a slight abuse of notation, $\sigma_n$ be the vector $(\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,d})$ and let $\lambda_n$ be the vector $(\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,d})$ and $|\lambda_n|$ be $(|\lambda_{n,1}|, |\lambda_{n,2}|, \ldots, |\lambda_{n,d}|)$ for all $n = 1, 2, \ldots$.

The inequalities [3] and [4] can be written in vector notation as

\[
\frac{\xi_n}{\sqrt{n}} \leq \sqrt{n} \left(\frac{\log(\sigma_n)}{n} - E \log \mathcal{L}\right) - \sqrt{n} \left(\frac{\sum_{k=1}^{n} \log(\mathcal{L}_k)}{n} - E \log \mathcal{L}\right) \leq \frac{\tau_n}{\sqrt{n}},
\]

\[
\frac{H_n}{\sqrt{n}} \leq \sqrt{n} \left(\frac{\log(\sigma_n)}{n} - E \log \mathcal{L}\right) - \sqrt{n} \left(\frac{\log(|\lambda_n|)}{n} - E \log \mathcal{L}\right) \leq \frac{\mathcal{P}_n}{\sqrt{n}}.
\]

Since $\lim_{n \to \infty} \frac{\mathcal{P}_n}{\sqrt{n}} = \lim_{n \to \infty} \frac{H_n}{\sqrt{n}} = 0$ and $\lim_{n \to \infty} \xi_n$, $\lim_{n \to \infty} \tau_n$ are finite almost surely, $\sqrt{n} \left(\frac{\log(\sigma_n)}{n} - E \log \mathcal{L}\right) - \sqrt{n} \left(\frac{\sum_{k=1}^{n} \log(\mathcal{L}_k)}{n} - E \log \mathcal{L}\right) \to 0$ and $\sqrt{n} \left(\frac{\log(|\lambda_n|)}{n} - E \log \mathcal{L}\right) - \sqrt{n} \left(\frac{\sum_{k=1}^{n} \log(\mathcal{L}_k)}{n} - E \log \mathcal{L}\right) \to 0$ almost surely as $n \to \infty$. By multivariate central limit theorem for sum of i.i.d random vectors, $\sqrt{n} \left(\frac{\sum_{k=1}^{n} \log(\mathcal{L}_k)}{n} - E \log \mathcal{L}\right)$ converges in distribution to a Gaussian random vector, whose co-variance matrix is same as that of $\log \mathcal{L}$.

To understand the random vector $\mathcal{L}$, observe that by applying Gram-Schmidt orthogonalization process to the columns of $DV (= M)$ from left to right, we can write $DV$ as product of a unitary matrix $Q$ and a upper triangular matrix $R$ with non-negative diagonal entries and $L_1, L_2, \ldots, L_d$ are going to be the diagonal entries (from left top to right bottom) of the upper triangular matrix $R$. Geometrically speaking, $L_1$ is length of the first column vector of $DV$, $L_2$ is length of the projection of second column vector onto the orthogonal complement of the space generated by the first column vector and similarly $L_i$ is length of the projection of $i$-th column vector onto the orthogonal complement of the subspace generated by the first $i-1$ column vectors of $DV$ (or $M$). Since $D$ is almost surely non-singular, $L_1, L_2, \ldots, L_d$ are positive almost surely.

For $j > i$, let $M_{j \perp (i)}$ denote the projection of $j$-th column of $M (= D V)$ onto the orthogonal complement of subspace generated by first $i$ columns of $M$. Because of right rotation invariance of distribution of $M$, the marginal probability distribution of $M_{k \perp (i)}$ is same as that of $M_{j \perp (i)}$ for any $k > i$. From QR decomposition of $M$, we can see that the length of $M_{i \perp (i-1)}$ is $L_i$ and the length of $M_{i+1 \perp (i-1)}$
is $\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}$. So, $L_i$ and $\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}$ have the same probability distribution. Therefore, $L_i$’s are stochastically decreasing in order i.e $P(L_1 > t) \geq P(L_2 > t) \geq \ldots \geq P(L_d > t)$ for any real $t$.

Again by right rotation invariance, since $R_{i,j}$ (for $j > i$) is inner product of $j$-th column of $DV$ with $i$-th column of $Q$, all $R_{i,j}$ for $j = i + 1, i + 2, \ldots$ have the same conditional distribution given first $i$ columns of $DV$, so same marginal distributions also. $R_{1,2}$ is inner product of second column of $DV$ and unit vector along first column of $DV$, so it is non-zero with positive probability unless $D$ is random scalar multiple of identity matrix. For $k > j \geq i$, the inner product of $M_{j \perp (i-1)}$ and $M_{k \perp (i-1)}$ is $\sum_{t=1}^{j-1} R_{t,j}R_{t,k} + L_jR_{j,k}$. For a fixed $i$, the inner product of $M_{i \perp (i-1)}$ with unit vector along $M_{j \perp (i-1)}$ has same distribution for any $k > j \geq i$. By taking $j = i, k = i + 1$ first and then $j = i + 1, k = i + 2$, we get that $R_{i,i+1}$ and $\frac{R_{i+1,i+1}R_{i+2,i+2} + L_{i+1}R_{i+1,i+2}}{\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}}$ have same distribution. If $R_{i,i+1}$ is non-zero with positive probability, then $R_{i+1,i+2}$ is also non-zero with positive probability. Otherwise $R_{i+1,i+2} = 0$ almost surely which implies that $R_{i,i+1}$ and $\frac{R_{i+1,i+1}R_{i+2,i+2} + L_{i+1}R_{i+1,i+2}}{\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}}$ have same distribution. But we already know that $R_{i,i+1}$ and $R_{i,i+2}$ have same distribution. So, it implies that $R_{i,i+2}$ and $\frac{R_{i+1,i+1}R_{i+2,i+2} + L_{i+1}R_{i+1,i+2}}{\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}}$ have same distribution, which leads to contradiction, because $\frac{R_{i+1,i+1}R_{i+2,i+2} + L_{i+1}R_{i+1,i+2}}{\sqrt{L_{(i+1)}^2 + |R_{i,i+1}|^2}}$ is strictly stochastically less than $R_{i,i+2}$ as $L_{(i+1)}$ is positive almost surely. Therefore, if $D$ is not random scalar multiple of identity matrix, by induction $R_{i,i+1}$ is non-zero with positive probability for all $i = 1, 2, \ldots d - 1$. This implies that $L_i$’s are stochastically strictly decreasing in order i.e $P(L_1 > t) \geq P(L_2 > t) \geq \ldots \geq P(L_d > t)$ for any real $t$ and set of $t$’s where strict inequalities hold between any two terms is of non-zero measure. As logarithm is a increasing function, this ensures distinctness and decreasing order of Lyapunov exponents i.e $E \log L_1 > E \log L_2 \ldots > E \log L_d$. We summarize the so far of this section into a theorem as follows.

**Theorem 11.** Let $M_1, M_2, \ldots$ be sequence of i.i.d right isotropic random matrices of order $d$, such that all singular values of $M_1$ have finite log-moments. $M_1 = QR$ be QR decomposition of $M_1$ i.e $Q$ is unitary matrix and $R$ is upper triangular matrix with non-negative diagonal entries $L_1, L_2, \ldots L_d$. Let $L$ be the vector $(L_1, L_2, \ldots L_d)$ and $\sigma_n, \chi_n$ be vectors of singular values and eigenvalues (in decreasing order of their absolute values) of $M_1M_2\ldots M_n$, respectively for all $n = 1, 2, \ldots$ Then both $\sqrt{n} \left( \frac{\log(\sigma_n)}{n} - E \log \sigma \right)$ and $\sqrt{n} \left( \frac{\log(\chi_n)}{n} - E \log \chi \right)$ converge in distribution to zero-mean Gaussian random vector whose covariance matrix is the same as that of log $L$.

In the case of Ginibre matrices, $L_1, L_2, \ldots L_d$ are independent. $L_i$ is $\chi^2_{d-i+1}$ random variable in case of real Ginibre matrix and $\chi^2_{2(d-i+1)}$ random variable in case of complex Ginibre matrix for all $i = 1, 2, \ldots d$. This means that in the case of products of Ginibre matrices the first order fluctuations of ordered log-singular values(also log-eigenvalues) are independent and Gaussian, implying the permanental nature of density of unordered log-singular values(also log-eigenvalues). This result has already been obtained in $[2]$ using the exact densities of singular values and eigenvalues.
In the case of truncated Haar unitary(orthogonal) matrices also, \( L_1, L_2, \ldots L_d \) are independent. By applying QR decomposition to \( d \times d \) left uppermost sub-block of \( m \times m \) Haar unitary(orthogonal) matrix and integrating out all the variables except \( L_1, L_2, \ldots L_d \), we get the density of \( L_1, L_2, \ldots L_d \). Integrating out the auxiliary variables here is the same as in when Schur decomposition is applied to the \( d \times d \) sub-block to get the eigenvalue density of truncated Haar unitary matrices(see [1], [3]). In the case of truncated Haar unitary matrices, \( L_1, L_2, \ldots L_d \) are independent and distribution of \( L_i^2 \) is Beta\((d - i + 1, m - d)\) for all \( i = 1, 2 \ldots d \). In the case of truncated Haar orthogonal matrices, \( L_1, L_2, \ldots L_d \) are independent and distribution of \( L_i^2 \) is Beta\((\frac{d - i + 1}{2}, \frac{m - d}{2})\) for all \( i = 1, 2 \ldots d \). This result about Lyapunov exponents of truncated Haar unitary(orthogonal) matrices has already been obtained in [9] using the exact density of truncated Haar unitary(orthogonal) matrices.

5. Towards the End All Eigenvalues are Real

Eigenvalues of a real matrix are either real or appear as complex conjugate pairs. The following theorem says, for sufficiently large \( n \), with high probability all eigenvalues of \( n \)-th real product matrix are real.

**Theorem 12.** Let \( M_1, M_2 \ldots \) be sequence of i.i.d real invertible isotropic random matrices of order \( d \), such that all singular values of \( M_1 \) have finite log-moment. Then, Probability of the event, that all eigenvalues of product matrix \( M_1 M_2 \ldots M_n \) are real, goes to one as \( n \to \infty \).

**Proof.** Since all singular values of \( M_1 \) have finite log-moment, both \( \mathbb{E}(\log^+ ||M_1||) \) and \( \mathbb{E}(\log |\det(M_1)|) \) are finite. So, from Theorem 7 we have that \( |\lambda_n|^\frac{d}{2} \) converges almost surely to a constant diagonal matrix \( \sigma \) with non-zero distinct diagonal entries. It implies that moduli of eigenvalues of product matrix \( \mathcal{P}_n \) grow(or decay) exponentially at distinct rates as \( n \to \infty \). For sufficiently large \( n \) the moduli of eigenvalues are distinct which means no two eigenvalues are complex conjugate of each other. Let \( \lambda_n = \text{diag}(\lambda_{n,1}, \lambda_{n,2} \ldots \lambda_{n,d}) \) and \( \sigma = \text{diag}(\sigma_1, \sigma_2 \ldots \sigma_d) \). We have \( |\lambda_{n,i}|^\frac{d}{2} \to \sigma_i \) for \( 1 \leq i \leq d \) almost surely. Almost sure convergence implies convergence in probability. Therefore

\[
\text{Prob} \left( \frac{1}{d} \sum_{i=1}^{d} |\lambda_{n,i}|^\frac{d}{2} - \sigma_i \right) < \epsilon \forall 1 \leq i \leq d \to 1 \text{ as } n \to \infty \text{ for any } \epsilon > 0.
\]

Since \( \sigma_i \)'s are distinct, it is possible to choose \( \epsilon \) such that none of the intervals \((\sigma_i - \epsilon, \sigma_i + \epsilon)\), \( 1 \leq i \leq d \) intersect. Then moduli of eigenvalues are distinct,

\[
\text{Prob} \left( \frac{1}{d} \sum_{i=1}^{d} |\lambda_{n,i}|^\frac{d}{2} \in (\sigma_i - \epsilon, \sigma_i + \epsilon) \right) \forall 1 \leq i \leq d \leq \text{Prob} \left( \lambda_{n,1}, \lambda_{n,2} \ldots \lambda_{n,d} \text{ are all real} \right).
\]

Finally

\[
\text{Prob} \left( \lambda_{n,1}, \lambda_{n,2} \ldots \lambda_{n,d} \text{ are all real} \right) \to 1 \text{ as } n \to \infty.
\]

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**References**

[1] Kartick Adhikari, Nanda Kishore Reddy, Tulasi Ram Reddy, and Koushik Saha, *Determinantal point processes in the plane from products of random matrices*, Ann. Inst. H. Poincar Probab. Statist. 52 (2016), no. 1, 16–46.
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[2] Gernot Akemann, Zdzislaw Burda, and Mario Kieburg, Universal distribution of lyapunov exponents for products of ginibre matrices, Journal of Physics A: Mathematical and Theoretical 47 (2014), no. 39, 395202.

[3] Gernot Akemann, Zdzislaw Burda, Mario Kieburg, and Taro Nagao, Universal microscopic correlation functions for products of truncated unitary matrices, Journal of Physics: A Mathematical and Theoretical 47 (2014), no. 25.

[4] Gernot Akemann and Jesper R. I. Ipsen, Recent exact and asymptotic results for products of independent random matrices, Acta Physica Polonica B 46 (2015), no. 9, 1747–1784.

[5] Philippe Bougerol and Jean Lacroix, Products of random matrices with applications to Schrödinger operators, Progress in Probability and Statistics, vol. 8, Birkhäuser Boston, Inc., Boston, MA, 1985. MR 886674 (88f:60013)

[6] Persi Diaconis and Peter J. Forrester, Hurwitz and the origins of random matrix theory in mathematics, [arXiv:1512.09229v2] (2016).

[7] Peter J. Forrester, Lyapunov exponents for products of complex Gaussian random matrices, J. Stat. Phys. 151 (2013), no. 5, 796–808. MR 3055376

[8] Peter J Forrester, Probability of all eigenvalues real for products of standard gaussian matrices, Journal of Physics A: Mathematical and Theoretical 47 (2014), no. 6, 065202.

[9] Jesper R. I. Ipsen, Lyapunov exponents for products of rectangular real, complex and quaternionic ginibre matrices, Journal of Physics A: Mathematical and Theoretical 48 (2015), no. 15, 155204.

[10] Boris A. Khoruzhenko, Hans-Jürgen Sommers, and Karol Życzkowski, Truncations of random orthogonal matrices, Phys. Rev. E 82 (2010), 040106.

[11] Arno B. J. Kuijlaars and Dries Stivigny, Singular values of products of random matrices and polynomial ensembles, Random Matrices: Theory and Applications 03 (2014), no. 03, 1450011.

[12] Arul Lakshminarayan, On the number of real eigenvalues of products of random matrices and an application to quantum entanglement, Journal of Physics A: Mathematical and Theoretical 46 (2013), no. 15, 152003.

[13] Charles M. Newman, The distribution of Lyapunov exponents: exact results for random matrices, Comm. Math. Phys. 103 (1986), no. 1, 121–126. MR 826860 (87h:60119)

[14] A. N. V. Oseledec, A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems, Trudy Moskov. Mat. Obščec. 19 (1968), 179–210. MR 0240280 (39 #1629)

[15] Karol Zyczkowski and Hans-Jürgen Sommers, Truncations of random unitary matrices, Journal of Physics A: Mathematical and General 33 (2000), no. 10, 2045.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: kishore11@math.iisc.ernet.in