ON KOSTANT’S THEOREM FOR THE LIE SUPERALGEBRA $Q(n)$

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Abstract. In this paper we study finite $W$-algebras for basic classical superalgebras and $Q(n)$ associated with the regular even nilpotent coadjoint orbits. We prove that this algebra satisfies the Amitsur-Levitzki identity and therefore all its irreducible representations are finite-dimensional. In the case of $Q(n)$ we give an explicit description of the $W$-algebra in terms of generators and relations and realize it as a quotient of the super-Yangian of $Q(1)$.

1. Introduction

A finite $W$-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element. Geometrically a finite $W$-algebra is a quantization of the Poisson structure on the so-called Slodowy slice (a transversal slice to the orbit of $e$ in the adjoint representation). In the case when $e = 0$ the finite $W$-algebra coincides with the universal enveloping algebra $U(\mathfrak{g})$ and in the case when $e$ is a regular nilpotent element, the corresponding $W$-algebra coincides with the center of $U(\mathfrak{g})$. The latter case was studied by B. Kostant [15] who was motivated by applications to generalized Toda lattices. The general definition of a finite $W$-algebra was given by A. Premet in [25]. I. Losev used the machinery of Fedosov quantization to prove important results relating representations of $W$-algebras and primitive ideals of $U(\mathfrak{g})$ [16, 17, 18] (see also [26, 27, 28]). He used this result to prove long standing conjectures of A. Joseph and others concerning primitive ideals in $U(\mathfrak{g})$, [11].

On the other hand, affine $W$-algebras were first constructed by physicists [8, 9]. The role of the Slodowy slice in $W$-algebras in the principal case was recognized in [2]. A. De Sole and V.G. Kac in [7] established the relation between affine and finite $W$-algebras.

Let us mention an important discovery of physicists, [29], that for $\mathfrak{g} = \mathfrak{sl}(n)$ finite $W$-algebras are closely related to Yangians. This connection was further studied in [4] and [6].

It is interesting to generalize all above applications to Lie superalgebras. Finite $W$-algebras for Lie superalgebras have been extensively studied by C. Briot, E. Ragoucy, J. Brunet, J. Brown, S. Goodwin, W. Wang, L. Zhao and other mathematicians and physicists [3, 5, 32, 33, 34]. Analogues of finite $W$-algebras for Lie superalgebras in terms of BRST cohomology were defined in by A. De Sole and V.G. Kac in [7].
In [3] C. Briot and E. Ragoucy observed that finite W-algebras associated with certain nilpotent orbits in \( \mathfrak{gl}(pm|pn) \) can be realized as truncations of the super-Yangian of \( \mathfrak{gl}(m|n) \), see [19] for definition.

The principal finite W-algebras for \( \mathfrak{gl}(m|n) \) associated to regular (principal) nilpotent elements were described as certain truncations of a shifted version of the super-Yangian \( Y(\mathfrak{gl}(1|1)) \) in [5]. It is also proven there that all irreducible modules over principal finite W-algebras are finite-dimensional for \( \mathfrak{gl}(m|n) \). Furthermore, [5] contains a classification of irreducible modules using highest weight theory.

In [33] L. Zhao generalized certain results about finite W-algebras to the case of Lie superalgebras. In particular he has proved that the definition of a finite W-algebra does not depend on a choice of an isotropic subspace \( l \) and a good \( Z \)-grading. He has also proved an analogue of the Skryabin theorem establishing equivalence between the category of modules over a finite W-algebra and the category of generalized Whittaker \( g \)-modules. He also gave a definition of a finite W-algebra for the queer Lie superalgebra \( Q(n) \).

In [23, 24] we described the finite W-algebras in the regular case for some classical and exceptional Lie superalgebras of defect one.

In this paper we are interested in the finite W-algebra associated with a regular nilpotent element \( \chi \in g^\ast_0 \) for a Lie superalgebra \( g \) with reductive even part \( g^\ast \). (Since not all such superalgebras admit an even invariant form, we can not identify \( g \) with \( g^\ast \), and we use the notation \( W_\chi \) instead of \( W_e \).) We prove that for basic classical \( g \) or \( Q(n) \) and the regular \( \chi \) the algebra \( W_\chi \) satisfies the Amitsur-Levitzki identity ([1]) (Corollary 3.6). In the proof we use some sort of reduction by constructing an injective homomorphism \( \vartheta : W_\chi \to \bar{W}_\chi^s \), where \( s \) is the reductive part of some parabolic subalgebra \( p \subset g \), and \( \bar{W}_\chi^s \) is an analogue of \( W_\chi \) for \( s \). As a corollary we obtain that all irreducible representations of \( W_\chi \) are finite-dimensional (Proposition 3.7).

We study in detail the case when \( g = Q(n) \) and \( \chi \) is regular. In this case, \( p \) is a Borel subalgebra and \( s \) is a Cartan subalgebra. We obtain results about the image of \( \vartheta \) in this case, which imply, in particular, that the center of \( W_\chi \) coincides with the center of \( U(Q(n)) \) (Corollary 5.10).

Using Sergeev’s construction of certain elements in the universal enveloping algebra \( U(Q(n)) \) ([24]), we construct generators of \( W_\chi \). Using these generators, we prove that the associated graded algebra \( Gr_K W_\chi \) with respect to the Kazhdan filtration is isomorphic to \( S(g^{\chi}) \) (the symmetric algebra of the annihilator \( g^{\chi} \) of \( \chi \) in \( g \)) (Conjecture 2.8 and Corollary 4.9). Furthermore, we prove that \( W_\chi \) is isomorphic to a quotient of the super-Yangian of \( Q(1) \) defined by M. Nazarov ([20]) and studied by M. Nazarov and A. Sergeev ([21, 22]) (Theorem 6.2). We prove that certain even generators of the super-Yangian of \( Q(1) \) commute (Proposition 6.4). Using the latter result, we construct \( n \) even and \( n \) odd generators in \( W_\chi \), such that all even generators commute.
and generate the polynomial subalgebra of rank \( n \) in \( W_\chi \), and the commutators of odd generators lie in the center of \( W_\chi \) (Theorem 6.6).

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### 2. Finite \( W \)-algebras for Lie superalgebras

#### 2.1. Definitions.

Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra with reductive even part \( g_0 \). Let \( \chi \in g_0^* \subset g^* \) be an even nilpotent element in the coadjoint representation. \(^1\) By \( g^\chi \) we denote the annihilator of \( \chi \) in \( g \). By definition

\[
g^\chi = \{ x \in g \mid \chi([x, g]) = 0 \}.
\]

A good \( \mathbb{Z} \)-grading for \( \chi \) is a \( \mathbb{Z} \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) satisfying the following two conditions

1. \( \chi(g_j) = 0 \) if \( j \neq -2 \);
2. \( g^\chi \) belongs to \( \bigoplus_{j \geq 0} g_j \).

Note that \( \chi([\cdot, \cdot]) : g_{-1} \times g_{-1} \to \mathbb{C} \) is a non-degenerate skew-symmetric even bilinear form on \( g_{-1} \). Let \( I \) be a maximal isotropic subspace with respect to this form. We consider a nilpotent subalgebra \( m = \left( \bigoplus_{j \leq -2} g_j \right) \oplus I \) of \( g \). The restriction of \( \chi \) to \( m \)

\[
\chi : m \longrightarrow \mathbb{C}
\]

defines a one-dimensional representation \( C_\chi = \langle v \rangle \) of \( m \).

**Definition 2.1.** The induced \( g \)-module

\[
Q_\chi := U(g) \otimes_{U(m)} C_\chi \cong U(g)/I_\chi,
\]

where \( I_\chi \) is the left ideal of \( U(g) \) generated by \( a - \chi(a) \) for all \( a \in m \), is called the generalized Whittaker module.

**Definition 2.2.** \([25]\). Define the finite \( W \)-algebra associated to the nilpotent element \( \chi \) to be

\[
W_\chi := \text{End}_{U(g)}(Q_\chi)^{op}.
\]

\(^1\)Denote by \( G_0 \) the algebraic reductive group of \( g_0 \). Then \( \chi \) is nilpotent if the closure of the \( G_0 \)-orbit of \( \chi \) in \( g_0^* \) contains zero.
As in the Lie algebra case, the superalgebras $W_\chi$ are all isomorphic for different choices of good gradings and maximal isotropic subspaces $l$ [33]. If $g$ admits an even non-degenerate invariant supersymmetric bilinear form, then $g \simeq g^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in g_0$. By the Jacobson–Morozov theorem $e$ can be included in $\mathfrak{sl}(2) = \langle e, h, f \rangle$. As in the Lie algebra case, the linear operator $\text{ad}h$ defines a Dynkin $\mathbb{Z}$-grading $g = \bigoplus_{j \in \mathbb{Z}} g_j$, where

$$g_j = \{ x \in g \mid \text{ad}(x) = jx \}.$$ 

As follows from representation theory of $\mathfrak{sl}(2)$, the Dynkin $\mathbb{Z}$-grading is good for $\chi$. Let $g^c := \text{Ker}(\text{ad}e)$. Note that as in the Lie algebra case, $\dim g^c = \dim g_0 + \dim g_1$ and $g^c \subseteq \bigoplus_{j \geq 0} g_j$.

Most results of this paper concern the case when $g$ admits an odd non-degenerate invariant supersymmetric bilinear form. In this case $g \simeq \Pi g^*$ and $\chi(x) = (E|x)$ for some nilpotent $E \in g_1$. Among classical Lie superalgebras only $Q(n)$ or $PSQ(n)$ admit an odd non-degenerate invariant supersymmetric bilinear form. We will see that in this case there is an analogue of the Dynkin $\mathbb{Z}$-grading.

Note that by Frobenius reciprocity

$$\text{End}_{U(g)}(Q_\chi) = \text{Hom}_{U(m)}(C_\chi, Q_\chi).$$

That defines an identification of $W_\chi$ with the subspace

$$Q^m_\chi = \{ u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in m \}.$$

In what follows we denote by $\pi : U(g) \to U(g)/I_\chi$ the natural projection. By above (2.1) $W_\chi = \{ \pi(y) \in U(g)/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in m \}$, or, equivalently,

(2.2) $W_\chi = \{ \pi(y) \in U(g)/I_\chi \mid \text{ad}(a)y \in I_\chi \text{ for all } a \in m \}$.

The algebra structure on $W_\chi$ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1y_2)$$

for $y_i \in U(g)$ such that $\text{ad}(a)y_i \in I_\chi$ for all $a \in m$ and $i = 1, 2$.

**Definition 2.3.** A $\mathbb{Z}$-grading $g = \bigoplus_{j \in \mathbb{Z}} g_j$ is called *even*, if $g_j = 0$ unless $j$ is an even integer.

The definition of $W_\chi$ for an even good $\mathbb{Z}$-grading is simpler, since in this case $g_{-1} = 0$. Hence there is no complications of choice of a Lagrangian subspace $I$ and $m = \bigoplus_{j \geq 1} g_{-2j}$.
Let $\mathfrak{p} := \bigoplus_{j \geq 0} \mathfrak{g}_{2j}$. It follows directly from definition that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. From the PBW theorem,

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_{\chi}.$$  

The projection $pr : U(\mathfrak{g}) \rightarrow U(\mathfrak{p})$ along this direct sum decomposition induces an isomorphism: $U(\mathfrak{g})/I_{\chi} \simeq U(\mathfrak{p})$. Thus, the algebra $W_{\chi}$ can be regarded as a subalgebra of $U(\mathfrak{p})$.

2.2. Kazhdan filtration on $W_{\chi}$. Define a $\mathbb{Z}$-grading on $T(\mathfrak{g})$ by shifting by 2 the fixed good $\mathbb{Z}$-grading. In other words, we set the degree of $X \in \mathfrak{g}_{j}$ to be $j + 2$. This grading induces a filtration on $U(\mathfrak{g})$ and therefore on $U(\mathfrak{g})/I_{\chi}$, which is called the Kazhdan filtration. We will denote by $Gr_{K}$ the corresponding graded algebras. Recall that by (2.1) $W_{\chi} \subset U(\mathfrak{g})/I_{\chi}$, therefore the Kazhdan filtration is defined on $W_{\chi}$. It is not hard to see that $Gr_{K}U(\mathfrak{g})$ is supercommutative and therefore $Gr_{K}W_{\chi}$ is also supercommutative. For any $X \in W_{\chi}$ we denote by $Gr_{K}X$ the corresponding element in $Gr_{K}W_{\chi}$. The following result is very important.

**Theorem 2.4.** A. Premet [25]. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the associated graded algebra $Gr_{K}W_{\chi}$ is isomorphic to $S(\mathfrak{g}^{\chi})$.

We believe that the above theorem holds for basic classical Lie superalgebras if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. In fact, for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and regular $\chi$ it is proven in [5]. In this paper we prove the analogous result for regular $\chi$ and $\mathfrak{g} = Q(n)$ (see Corollary 4.9).

We will prove now a weaker general result. Let $\mathfrak{l}'$ be some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties

- $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$;
- $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form $\chi([\cdot, \cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $\mathfrak{l}'$ is a maximal isotropic subspace. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $\mathfrak{l}' \cap \mathfrak{l}'$ is one-dimensional and we fix $\theta \in \mathfrak{l}' \cap \mathfrak{l}'$ such that $\chi([\theta, \theta]) = 2$. It is clear that $\pi(\theta) \in W_{\chi}$ and $\pi(\theta)^{2} = 1$.

Let $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_{j}$. By the PBW theorem, $U(\mathfrak{g})/I_{\chi} \simeq S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space.

Therefore $Gr_{K}(U(\mathfrak{g})/I_{\chi})$ is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. The good grading on $\mathfrak{g}$ induces the grading on $S(\mathfrak{p} \oplus \mathfrak{l}')$. For any $X \in S(\mathfrak{p} \oplus \mathfrak{l}')$ we denote by $\bar{X}$ the element of highest degree in this grading. Following the original Premet’s proof we will prove now the following statement.

**Theorem 2.5.** (a) Assume that $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. If $X \in Gr_{K}W_{\chi}$, then $\bar{X} \in S(\mathfrak{g}^{\chi})$.
(b) Assume that $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd. If $X \in Gr_{K}W_{\chi}$, then $\bar{X} \in S(\mathfrak{g}^{\chi} \oplus \mathbb{C}\theta)$.

**Proof.** We start with the following simple observation.
Lemma 2.6. Let \( x \in p \oplus l' \). Then \( \chi([m, x]) = 0 \) if and only if \( x \in g^x \) for even \( \dim(g_{-1})_1 \) and \( x \in g^x \oplus C\theta \) for odd \( \dim(g_{-1})_1 \).

Proof. Note that if \( x \in g_i \) and \( Y \in g_j \), then \( \chi([Y, x]) \neq 0 \) implies \( i + j = -2 \). Therefore if \( x \in p \), the condition \( \chi([m, x]) = 0 \) implies the condition \( \chi([g, x]) = 0 \), and thus \( x \in g^x \). If \( x \in l' \), then the condition \( \chi([m, x]) = 0 \) is equivalent to the condition \( \chi([l, x]) = 0 \). Therefore \( x \in l' \cap l' = C\theta \). □

Let \( X \in Gr_K W_\chi \). Passing to the graded version of (2.2) we obtain that for any \( Y \in m \) we have

\[
\pi(\text{ad}Y(X)) = 0.
\]

Define \( \gamma : m \otimes S(p \oplus l') \rightarrow S(p \oplus l') \) by putting

\[
\gamma(Y, Z) = \pi(\text{ad}Y(Z))
\]

for all \( Y \in m, Z \in S(p \oplus l') \). It is easy to see that if \( Y \in g_{-i} \), where \( i > 0 \), and \( Z \in S(p \oplus l')_j \), then \( \gamma(Y, Z) \in S(p \oplus l')_{j-i} \oplus S(p \oplus l')_{j-i+2} \). Hence we can write \( \gamma = \gamma_0 + \gamma_2 \) where \( \gamma_0(Y, Z) \) is the projection on \( S(p \oplus l')_{j-i} \) and \( \gamma_2(Y, Z) \) is the projection on \( S(p \oplus l')_{j-i+2} \). The condition (2.3) implies that for any \( \bar{X} \in Gr_K W_\chi \)

\[
\gamma_2(m, \bar{X}) = 0.
\]

On the other hand, \( \gamma_2 : m \times S(p \oplus l') \rightarrow S(p \oplus l') \) is a derivation with respect to the second argument defined by the condition

\[
\gamma_2(Y, Z) = \chi([Y, Z])
\]

for any \( Y \in m, Z \in p \oplus l' \). Now by induction on the polynomial degree of \( \bar{X} \) in \( S(p \oplus l') \), using Lemma 2.6, one can show that (2.4) implies \( \bar{X} \in S(g^x) \) (respectively, \( \bar{X} \in S(g^x \oplus C\theta) \)). □

Proposition 2.7. Assume that \( \dim(g_{-1})_1 \) is even (respectively, odd). Let \( y_1, \ldots, y_p \) be a basis in \( g^x \) homogeneous in the good \( Z \)-grading. Assume that there exist \( Y_1, \ldots, Y_p \in W_\chi \) such that \( \text{Gr}_K Y_i = y_i \) for all \( i = 1, \ldots, p \).

(a) \( Y_1, \ldots, Y_p \) generate \( W_\chi \) (respectively, \( Y_1, \ldots, Y_p \) and \( \pi(\theta) \) generate \( W_\chi \));

(b) \( Gr_K W_\chi \simeq S(g^x) \) (respectively, \( Gr_K W_\chi \simeq S(g^x) \otimes \mathbb{C}[\xi] \), where \( \mathbb{C}[\xi] \) is the exterior algebra generated by one element \( \xi \)).

Proof. We will give a proof in the case when \( \dim(g_{-1})_1 \) is even. The odd case is analogous and we leave it to the reader. Let us first prove (a) by contradiction. Assume that \( X \in W_\chi \) is an element of minimal Kazhdan degree such that it does not lie in the subalgebra generated by \( Y_1, \ldots, Y_p \). By Theorem 2.5 we have

\[
\text{Gr}_K \bar{X} = \sum c(a_1, \ldots, a_p) g_1^{a_1} \cdots g_p^{a_p}.
\]

Let

\[
Z = X - \sum c(a_1, \ldots, a_p) Y_1^{a_1} \cdots Y_p^{a_p}.
\]
Then Kazhdan degree of $Z$ is less than that of $X$. By minimality of degree of $X$ we conclude that $Z = 0$. That contradicts our assumption.

To prove (b) write $p = g^\chi \oplus r$, where $r$ is some graded subspace complementary to $g^\chi$. Let $\gamma : S(p \oplus r') \to S(g^\chi)$ denote the natural projection with kernel $(r \oplus r').S(p \oplus r')$. By (a) and Theorem 2.5 the restriction $\gamma : Gr_K W_\chi \to S(g^\chi)$ is an isomorphism of rings.

**Conjecture 2.8.** Assume that $g$ is a Lie superalgebra with reductive even part $g_0$. If $\dim(g_{-1})_1$ is even, then $Gr_K W_\chi \simeq S(g^\chi)$ and if $\dim(g_{-1})_1$ is odd, then $Gr_K W_\chi \simeq S(g^\chi) \otimes C[\xi]$, where $C[\xi]$ is the exterior algebra generated by one element $\xi$.

The above conjecture is proved in [34] for basic classical superalgebras excluding the case of $D(2, 1; a)$ when $a \notin Z$. The proof is a generalization of original Premet’s proof, [25], which uses reduction modulo $p$.

### 2.3. Kostant’s theorem and the regular case for Lie superalgebras.

A nilpotent $\chi \in g_0$ is called *regular* if $G_0$-orbit of $\chi$ has maximal dimension, i.e. the dimension of $g_0^\chi$ is minimal. Let us recall that for a regular nilpotent $\chi$ and a reductive Lie algebra $g$ the algebra $W_\chi$ is isomorphic to the center $Z(g)$ of $U(g)$, see [15].

It is not hard to see that this result of B. Kostant does not hold for Lie superalgebras. In Section 3 we will prove that for regular $\chi$, $W_\chi$ satisfies the Amitsur–Levitzki identity and all irreducible representations of $W_\chi$ are finite-dimensional with dimension not greater than $2^{k+1}$, where $k$ is the constant depending on defect of $g$ and the parity of $\dim(g^\chi_1)$. Recall that for a contragredient $g$ the defect of $g$ is the maximal number of mutually orthogonal linearly independent isotropic roots, [14].

### 2.4. Good $Z$-gradings for superalgebras in the regular case.

Good $Z$-gradings for basic classical superalgebras are classified in [12]. In the case when $\chi$ is regular and $g$ is of type II (i.e. $g_0$ is semisimple and $g_1$ is a simple $g_0^\chi$-module), the only good $Z$-grading is the Dynkin $Z$-grading, and it is never even. If $g$ is of type I, i.e. $g_0$ has a non-trivial center, we can choose an even good $Z$-grading for any $\chi$. For the Lie superalgebra $Q(n)$ any Dynkin $Z$-grading is even for any $\chi$ and a good $Z$-grading is Dynkin for a regular $\chi$.

Let us concentrate on the case of basic classical or exceptional Lie superalgebras of type II and regular $\chi$. In this case $\chi(\cdot) = (\cdot|\cdot)$ for some principal nilpotent element $e \in g_0$. We are going to describe the Dynkin $Z$-grading on $g$ in terms of a specific Borel subalgebra. Let $b_0 \subseteq g_0$ be the Borel subalgebra containing $e$. Since $e$ is principal, this Borel subalgebra is unique. Let $\Pi_0$ denote the set of simple roots of $b_0$.

**Lemma 2.9.** Let $g$ be a basic classical or exceptional Lie superalgebra of type II.

(a) There exists a Borel subalgebra $b_0 \subseteq b \subseteq g$ with the set of simple roots $\Pi$ such that for any root $\beta \in \Pi_0$ either $\beta \in \Pi$ or $\beta = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Pi$. 
(b) Let \( d \) denote the defect of \( g \). Then the number of odd roots in \( \Pi \) equals 2\( d \) if \( g = \mathfrak{osp}(2m + 1|2n) \) for \( m \geq n \), \( \mathfrak{osp}(2m|2n) \) for \( m \leq n \) or \( G_3 \), and the number of odd roots in \( \Pi \) equals 2\( d + 1 \) if \( g = \mathfrak{osp}(2m + 1|2n) \) for \( m < n \), \( \mathfrak{osp}(2m|2n) \) for \( m > n \), \( D(2,1;\alpha) \) or \( F_4 \).

(c) Let \( e, h, f \) be the \( \mathfrak{sl}(2) \)-triple such that \( h \in \mathfrak{h} \). Then \( \alpha(h) = 2 \) for any even \( \alpha \in \Pi \) and \( \alpha(h) = 1 \) for any odd \( \alpha \in \Pi \), i.e. the Dynkin \( Z \)-grading is consistent.

**Proof.** (a) Among all Borel subalgebras containing \( \mathfrak{b}_0 \) pick up the one that has maximal number of odd roots and contains an odd non-isotropic root if such roots exist. For ortho-symplectic superalgebra those Borel subalgebras are listed in [10].

For the exceptional superalgebras we list the simple roots using the roots description in [13]. If \( g = G_3 \), the set of simple roots is \( \{\delta, \gamma_1 - \delta, \gamma_2\} \), where \( \gamma_1 \) is the short and \( \gamma_2 \) is the long simple root of \( G_2 \). If \( g = F_4 \), then the set of simple roots is \( \{\varepsilon_1 - \varepsilon_2, \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \delta), \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \delta)\} \).

(b) follows by direct inspection.

(c) follows from the condition \([h,e] = 2e\) and (a). \( \Box \)

**Corollary 2.10.** Let \( g \) be a basic classical or exceptional Lie superalgebra of type II, and \( d \) be its defect. If \( g = \mathfrak{osp}(2m + 1|2n) \) for \( m \geq n \), \( \mathfrak{osp}(2m|2n) \) for \( m \leq n \) or \( G_3 \), then \( \dim(g^-) = (0|2d) \). If \( g = \mathfrak{osp}(2m + 1|2n) \) for \( m < n \), \( \mathfrak{osp}(2m|2n) \) for \( m > n \), \( D(2,1;\alpha) \) or \( F_4 \), then \( \dim(g^-) = (0|2d + 1) \). By Lemma 2.9 (c) \( \dim(g^-) \) equals the number of irreducible \( \mathfrak{sl}(2) \)-components in \( g_{\bar{1}} \). Therefore \( \dim(g_{\bar{1}}) = 2d \) or \( 2d + 1 \).

**Corollary 2.11.** Let \( g \) satisfy the assumptions of Corollary 2.10.

(a) One can choose a maximal isotropic subspace \( l \subset g^- \) such that \( l = g_{-\alpha_1} \oplus \cdots \oplus g_{-\alpha_d} \) for some isotropic mutually orthogonal roots \( \alpha_1, \ldots, \alpha_d \in \Pi \). In particular, \( [l,l] = 0 \).

(b) There exists a parabolic subalgebra \( p \subset g \) with Levi subalgebra \( s \) such that \( m \cap s \) is an even one-dimensional subspace, and if \( n^- \) denotes the nil radical of the opposite parabolic \( p^- \), then \( n^- \subset m \).

(c) If \( g \) does not have non-isotropic roots (i.e. \( g = \mathfrak{osp}(2m|2n), D(2,1;\alpha) \) or \( F_4 \)), then \( [s,s] \) is isomorphic to a direct sum of several copies of \( \mathfrak{sl}(1|1) \) and one copy of \( \mathfrak{sl}(1|2) \). If \( g \) has non-isotropic roots (i.e. \( g = \mathfrak{osp}(2m + 1|2n) \) or \( G_3 \)), then \( [s,s] \) is isomorphic to a direct sum of several copies of \( \mathfrak{sl}(1|1) \) and one copy of \( \mathfrak{osp}(1|2) \).

**Proof.** Let \( \Gamma \) denote the Dynkin diagram of \( \Pi \). For any subset \( C \subset \Pi \) we denote by \( \Gamma_C \) the corresponding subdiagram of \( \Gamma \). Let \( \Pi' \) denote the set of all odd roots of \( \Pi \), the subgraph \( \Gamma_{\Pi'} \) is connected and \( \Pi' \) has at most one non-isotropic root. Let us choose a subset \( A = \{\alpha_1, \ldots, \alpha_d\} \subset \Pi' \) of mutually orthogonal isotropic roots such that the subgraph \( \Gamma_{\Pi'\setminus A} \) has maximal number of connected components. If \( \Pi' \) contains a non-isotropic root, then \( \Gamma_{\Pi'\setminus A} \) is a disjoint union of single vertex diagrams. If all roots of \( \Pi' \) are isotropic, then \( \Gamma_{\Pi'\setminus A} \) is a disjoint union of several single vertex
diagrams and one diagram consisting of two connected isotropic vertices. The latter is the diagram of $\mathfrak{sl}(1|2)$.

Now we set $\mathfrak{s}$ to be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and $\mathfrak{g}_{\pm\beta}$ for all $\beta \in \Pi' \setminus A$ and let $\mathfrak{p} = \mathfrak{b} + \mathfrak{s}$. We leave to the reader to check that all requirements of the corollary are true for this choice. □

**Example 2.12.** Let $\mathfrak{g} = \mathfrak{osp}(3|4)$. Then $\Pi$ has the Dynkin diagram

$$\otimes - \otimes \Rightarrow \bullet,$$

and $A$ consists of one middle vertex. In this case $[\mathfrak{s}, \mathfrak{s}] \simeq \mathfrak{sl}(1|1) \oplus \mathfrak{osp}(1|2)$.

**Example 2.13.** Let $\mathfrak{g} = \mathfrak{g}_3$. Then $\Pi$ has the Dynkin diagram

$$\bullet \Leftarrow \otimes \Leftarrow \circ,$$

and $A$ again coincides with the middle vertex. In this case we also have $[\mathfrak{s}, \mathfrak{s}] \simeq \mathfrak{sl}(1|1) \oplus \mathfrak{osp}(1|2)$.

2.5. **The queer superalgebra $Q(n)$**. Recall that the queer Lie superalgebra is defined as follows

$$Q(n) := \{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \text{ are } n \times n \text{ matrices} \}.$$

Let $\text{otr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{tr}B$.

**Remark 2.14.** $Q(n)$ has one-dimensional center $< z >$, where $z = 1_{2n}$. Let

$$SQ(n) = \{ X \in Q(n) \mid \text{otr}X = 0 \}.$$

The Lie superalgebra $\tilde{Q}(n) := SQ(n)/< z >$ is simple for $n \geq 3$, see [13].

Note that $\mathfrak{g} = Q(n)$ admits an odd non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form

$$(x|y) := \text{otr}(xy) \text{ for } x, y \in \mathfrak{g}.$$

Therefore, we identify the coadjoint module $\mathfrak{g}^*$ with $\Pi(\mathfrak{g})$, where $\Pi$ is the change of parity functor.

Let $e_{i,j}$ and $f_{i,j}$ be standard bases in $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively:

$$e_{i,j} = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}, \quad f_{i,j} = \begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix},$$

where $E_{ij}$ are elementary $n \times n$ matrices.

Let $\mathfrak{sl}(2) = < e, h, f >$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \text{diag}(n-1, n-3, \ldots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i)e_{i+1,i}.$$
Note that $e$ is a regular nilpotent element, $h$ defines an even Dynkin $\mathbb{Z}$-grading of $g$ whose degrees on the elementary matrices are

$$
\begin{pmatrix}
0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 \\
\vdots & \vdots & \ddots & \vdots \\
2-2n & \cdots & \cdots & 0 \\
0 & 2 & \cdots & 2n-2 \\
-2 & 0 & \cdots & 2n-4 \\
\vdots & \vdots & \ddots & \vdots \\
2-2n & \cdots & \cdots & 0 \\
\end{pmatrix}
$$

Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$. Since we have an isomorphism $g^* \simeq \Pi(g)$, an even regular nilpotent $\chi \in g^*$ can be defined by $\chi(x) := (x|E)$ for $x \in g$. Note that the Dynkin $\mathbb{Z}$-grading is good for $\chi$. We have that

$$(2.5) \quad g^\chi = g^E = \{z, e, e^2, \ldots, e^n \mid H_0, H_1, \ldots, H_{n-1}\}, \quad \dim(g^E) = (n|n),$$

where $H_k = \sum_{i=1}^{n-k} (-1)^{i+k-1} f_{i,i+k}$ for $k = 0, \ldots, n-1$. Let

$$m = \bigoplus_{j=1}^{n-1} g_{-2j}.$$

Note that $m$ is generated by $e_{i+1,i}$ and $f_{i+1,i}$, where $i = 1, \ldots, n-1$, and

$$(2.6) \quad \chi(e_{i+1,i}) = 1, \quad \chi(e_{i,k+i}) = 0 \text{ if } k \geq 2, \quad \chi(f_{i,k,i}) = 0 \text{ if } k \geq 1.$$

The left ideal $I_\chi$ and $W_\chi$ are defined now as usual. Moreover,

$$b := \bigoplus_{j=0}^{n-1} g_{2j}$$

is a Borel subalgebra of $g$, $\mathfrak{h} := g_0$ is a Cartan subalgebra, and $b = \mathfrak{h} \oplus \mathfrak{n}$, where

$$n := \bigoplus_{j=1}^{n-1} g_{2j}.$$

Note that the algebra $W_\chi$ can be regarded as a subalgebra of $U(b)$.

3. Some general results

3.1. The Harish-Chandra homomorphism. In this section we assume that $g$ is a basic classical Lie superalgebra or $Q(n)$. Let $p \subset g$ be a parabolic subalgebra such that $n^- \subset m \subset p^-$, where $n^-$ denotes the nilradical of the opposite parabolic $p^-$. Let $s$ be the Levi subalgebra of $p$, $n$ be its nilradical and $m^s = m \cap s$. Note that
m = n^- ⊕ m^s. We denote by Q^s_χ the induced module U(s) ⊗_{U(m^s)} C_χ, where by χ we understand the restriction of χ on s. Let

W^s_χ = \text{End}_{U(s)}(Q^s_χ)^{op} = (Q^s_χ)^m^s.

Let J_χ (respectively J^s_χ) be the left ideal in U(p) (respectively in U(s)) generated by a - χ(a) for all a ∈ m^s.

Finally, let \vartheta : U(p) \to U(s) denote the projection with the kernel nU(p). Note that \vartheta(J_χ) = J^s_χ. Thus, the projection \vartheta' : U(p)/J_χ \to U(s)/J^s_χ is well defined.

Note that we have an isomorphism of vector spaces Q_χ ≃ U(p)/J_χ, hence W_χ can be identified with a subspace in (U(p)/J_χ)^m^s. On the other hand, W^s_χ can be identified with the subspace (U(s)/J^s_χ)^m^s. Consider a map \vartheta : W_χ \to U(s)/J^s_χ obtained by the restriction of \vartheta' to W_χ. Since adm^s(n) ⊂ n, \vartheta maps adm^s-invariants to adm^s-invariants. In other words, \vartheta(W_χ) ⊂ W^s_χ. Furthermore, one can easily see that \vartheta : W_χ \to W^s_χ is a homomorphism of algebras.

An important example is as follows. Assume that g admits an even good Z-grading with respect to χ. Then we can set p = \bigoplus_{i\geq 0} g_i. Then s = g_0, m^s = 0 and \vartheta is a homomorphism W_χ \to U(s).

**Theorem 3.1.** The homomorphism \vartheta : W_χ \to W^s_χ is injective.

*Proof.* We consider a new grading g = \bigoplus_{i\in\mathbb{Z}} g_{(i)} such that p = \bigoplus_{i\geq 0} g_{(i)}. Note that J_χ is a graded ideal and hence Q_χ is also a graded vector space with respect to this new grading. Note that (Q_χ)_{(0)} = Q^s_χ. For any t ∈ \mathbb{C} \setminus \{0\} let \phi_t denote the automorphism of g that multiplies elements of g_{(j)} by t^j. Let X ∈ W_χ = Q^m_χ. Write

X = \sum_{i=0}^s X_{(i)},

where X_{(i)} ∈ (Q_χ)_{(i)} and X_{(s)} \neq 0. Our goal is to show that d = 0. Let

χ_0 = \lim_{t\to 0} \phi_t(χ).

Then χ_0(n^-) = 0, χ_0|m^s = χ|m^s. Note that

t^{-d}\phi_t(X) ∈ W_{\phi_t(χ)}

and hence by the standard continuity argument X_{(d)} ∈ (U(g) ⊗_{U(m)} C_{χ_0})^m. Note that

U(g) ⊗_{U(m)} C_{χ_0} = U(g) ⊗_{U(p^-)} U(p^-) ⊗_{U(m)} C_{χ_0}.

Furthermore, U(p^-) ⊗_{U(m)} C_{χ_0} has the trivial action of n^- and is isomorphic to Q^s_χ as an s-module. Thus, X_{(d)} ∈ (U(g) ⊗_{U(p^-)} Q^s_χ)^m.

We need now the following Lemma.

**Lemma 3.2.** (U(g) ⊗_{U(p^-)} Q^s_χ)^n^- = Q^s_χ.
Proof. Let ζ be a generic central character of U(s) and S be a quotient of $Q_λ^s$ admitting this central character. Consider the parabolically induced module $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} S$ (here we assume that $\mathfrak{n}^-$ acts trivially on S). We will prove that $M^{n^\ominus} = S$.

Let $γ : Z(\mathfrak{g}) \to Z(\mathfrak{s})$ be the restriction of the Harish-Chandra projection $U(\mathfrak{g}) \to U(\mathfrak{s})$ with kernel $\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-$. Note that $M$ admits central character $γ^*(ζ)$. Any simple $\mathfrak{s}$-submodule $N \subset M^{n^\ominus}$ that admits central character $ζ'$ generates in $M$ a submodule admitting central character $γ^*(ζ')$. Hence we have $γ^*(ζ) = γ^*(ζ')$.

Recall the correspondence between central characters and weights. One chooses a typical $W$-orbit of center of $U(\mathfrak{s})$ and set $ζ_λ$ to be the central character of the Verma module over $\mathfrak{g}$ with highest weight $λ$. Furthermore $\mathfrak{b}^s \oplus \mathfrak{n}^-$ is a Borel subalgebra in $\mathfrak{g}$ and we define the $U(\mathfrak{g})$-central character $\tilde{ζ}_λ$ to be the central character of the Verma module over $\mathfrak{g}$ with highest weight $λ$. Obviously, $γ^*(ζ_λ) = \tilde{ζ}_λ$. Moreover, all simple $\mathfrak{s}$-subquotients of $M$ admit central character $ζ_μ$ for some $μ \in λ + R(\mathfrak{n})$ where $R(\mathfrak{n})$ is the set of weights of $U(\mathfrak{n})$. Recall that if $λ$ is typical, then $\tilde{ζ}_λ = \tilde{ζ}_ν$ implies that $ν$ is obtained from $λ$ by the shifted action of the Weyl group of $\mathfrak{g}_0$.

Let us choose a typical $λ$ such that the intersection of the orbit of $λ$ and $λ + R(\mathfrak{n})$ equals $λ$. Suppose that there exists a simple $N \subset M^{n^\ominus} \cap \mathfrak{n}M$. Then $N$ admits $U(\mathfrak{g})$-central character $ζ_μ$ for some $μ \in λ + R(\mathfrak{n})$, $μ \neq λ$. But then $\tilde{ζ}_μ \neq \tilde{ζ}_λ$. A contradiction.

Since $S$ is generic, the above argument implies $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^-)} Q_λ^s)^{n^\ominus} = Q_λ^s$. That implies $d = 0$. □

3.2. The case of a regular $χ$. If $χ$ is regular and admits an even good $\mathbb{Z}$-grading, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(m|n), \mathfrak{osp}(2|2n)$ or $\mathfrak{Q}(n)$. In this case we set $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$. If $\mathfrak{g}$ is of type II, then we define $\mathfrak{p}$ as in Corollary 2.11.

If $\mathfrak{g} = \mathfrak{Q}(n)$ we set $k = \frac{n}{2}$ if $n$ is even and $\frac{n-1}{2}$ if $n$ is odd. In other cases we set $k = \text{defect of } \mathfrak{g}$ if $\mathfrak{g}$ is of type I or $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_k^\chi)$ is even. If $\mathfrak{g}$ is of type II and $\dim(\mathfrak{g}_k^\chi)$ is odd, then we set $k = d + 1$.

Proposition 3.3. $\bar{W}_X^s$ satisfies Amitsur–Levitzki identity, i.e. for any $u_1, \ldots, u_{2k+1} \in \bar{W}_X^s$

$$\sum_{σ ∈ S_{2k+1}} \text{sgn}(σ)u_{σ(1)} \cdots u_{σ(2k+1)} = 0. \quad (3.1)$$

Proof. We first consider the case of an even $\mathbb{Z}$-grading. Then $\bar{W}_X^s = U(\mathfrak{g})$. Let us assume first that $\mathfrak{g} = \mathfrak{Q}(n)$. Then the even good $\mathbb{Z}$-grading coincides with the Dynkin
ON KOSTANT’S THEOREM FOR THE LIE SUPERALGEBRA $Q(n)$

$\mathbb{Z}$-grading and $s = g_0 = h$ is a Cartan subalgebra of $g$. Denote

$$x_i = e_{i,i}, \quad \xi_i = (-1)^{i+1}f_{i,i}.$$  

Then $x_i$ lie in the center of $U(h)$ and we have $[f_{i,i}, f_{i,i}] = 2x_i$. From this it is easy to see that $U(h_0) = \mathbb{C}[x_1, \ldots, x_n]$ coincides with the center of $U(h)$.

Let $F$ denote the algebraic closure of the field of fractions of $U(h_0)$ and let $U(h) F = F \otimes_{U(h_0)} U(h)$. Then $U(h)_F$ is isomorphic to the Clifford algebra associated with a non-degenerate symmetric form on an $n$-dimensional space. Thus, $U(h)_F \simeq M_{2k}(F)$ for even $n$ and $U(h)_F \simeq M_{2k}(F) \times M_{2k}(F)$ for odd $n$, where by $M_s(F)$ we denote the algebra of $s \times s$ matrices over $F$. Thus, by the Amitsur–Levitzki theorem (see [1]), $U(h)_F$ satisfies (3.1). Since $U(h)$ is a subalgebra of $U(h)_F$, it also satisfies (3.1).

Now let $g = sl(m/n)$ or $osp(2|2n)$. Then the even part of $s$ coincides with the Cartan subalgebra $h$, which is abelian. The basis of the odd part consists of root elements $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ such that $[X_i, Y_j] = 0$ if $i \neq j$, $[X_i, X_j] = [Y_i, Y_j] = 0$ for all $i, j \leq k$. Thus, $s$ has a triangular decomposition $s = s^- \oplus h \oplus s^+$, with $s^+$ spanned by $X_1, \ldots, X_k$ and $s^-$ spanned by $Y_1, \ldots, Y_k$. Let $\lambda \in h^*$ and $M_\lambda = U(s) \otimes_{U(h \oplus s^+)} \mathbb{C}_\lambda$ denote the Verma module over $s$. The dimension of $M_\lambda$ equals $2^k$.

An easy calculation shows that $\prod_{\lambda \in h^*} M_\lambda$ is a faithful $U(s)$-module. Therefore $U(s)$ is isomorphic to a subalgebra in $\prod_{\lambda \in h^*} \text{End}_\mathbb{C}(M_\lambda)$. Since $\prod_{\lambda \in h^*} \text{End}_\mathbb{C}(M_\lambda)$ satisfies the Amitsur–Levitzki identity, $U(s)$ must satisfy it as well.

Finally, let us consider the case when $g$ is of type II. Here we are going to consider two subcases. We will use notations of the proof of Corollary 2.11.

First, let us assume that $\Pi$ contains an odd non-isotropic root $\beta$. Then $\Pi' \setminus A = \{\beta_1, \ldots, \beta_{k-1}, \beta = \beta_k\}$. Then $[s, s]$ is a direct sum of $k - 1$ copies of $sl(1|1)$ generated by the root spaces $g_{\pm \beta_i}, i = 1, \ldots, k - 1$ and one copy of $osp(1|2)$ generated by $g_{\pm \beta_k}$. Furthermore, $m^s \subset osp(1|2)$ is generated by $g_{-23}$. Let us write $s = s' \oplus r$, where $r = osp(1|2)$. Then $W^r = U(s') \otimes W^r_x$, where $W^r_x$ is the usual $W$-algebra for the regular $r$ and $r = osp(1|2)$. In the following example we give an explicit description of $W$-algebra for $osp(1|2)$.

Let $r = osp(1|2) = \langle X, Y, H \mid \theta, r \rangle$, where

$$X = E_{23}, Y = E_{32}, H = E_{22} - E_{33}, \theta = E_{12} - E_{31}, r = E_{13} + E_{21}.$$  

Let $sl(2) = \langle e, h, f \rangle$, where $e = X, h = H, f = Y$. The element $h$ defines a $\mathbb{Z}$-grading on $r$:

$$\mathfrak{r} = \mathfrak{r}_{-2} \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_{2},$$

where

$$\mathfrak{r}_{-2} = \langle Y \rangle, \quad \mathfrak{r}_{-1} = \langle \theta \rangle, \quad \mathfrak{r}_0 = \langle H \rangle, \quad \mathfrak{r}_1 = \langle r \rangle, \quad \mathfrak{r}_2 = \langle X \rangle.$$  

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = 1/2 \text{str}(ab)$ on $r$: $\langle \theta | r \rangle = 1$, $\langle X | Y \rangle = -1/2$, $\langle H | H \rangle = -1$. Let $\chi(x) = (e|x)$ for $x \in \mathfrak{r}$ and let $W_\chi$ be the corresponding $W$-algebra. Note that $g^\chi = g^e = \langle X \mid r \rangle$, $m = \mathfrak{r}_{-2}$,
and \( \chi(Y) = -\frac{1}{2} \). We have that \( \pi(\theta) \in W_\chi \), and \( \pi(\theta)^2 = \frac{1}{2} \). Let \( \Omega \) be the Casimir element of \( \mathfrak{r} \). Then

\[
\pi(\Omega) = \pi(2X + H - H^2 + 2r\theta).
\]

Let

\[
R = \pi(r - H\theta).
\]

Note that \( \pi(\Omega) \) and \( R \) belong to \( W_\chi \).

**Lemma 3.4.**

a) \( W_\chi \) is generated by \( \pi(\Omega) \), \( \pi(\theta) \) and \( R \). The defining relations are

\[
[\pi(\Omega), R] = [\pi(\Omega), \pi(\theta)] = 0, \\
[R, R] = \pi(\Omega), \\
[R, \pi(\theta)] = -\frac{1}{2}, \\
[\pi(\theta), \pi(\theta)] = 1.
\]

b) For any \( c \in \mathbb{C} \), \( W_\chi/(\Omega - c) \) is isomorphic to a Clifford algebra with two generators and it has a unique irreducible representation \( M_c \) of dimension 2.

**Proof.** Since \( \text{Gr}_K(\pi(\Omega)) = 2X \), \( \text{Gr}_K(R) = r \), then (a) follows from Proposition 2.7 (a).

The proof of (b) is straightforward. \( \square \)

We use Lemma 3.4 (b) to prove the Amitsur–Levitzki identity in the latter case. We again consider the family \( M_\lambda \otimes M_c \), where \( M_\lambda \) is the Verma module over \( \mathfrak{s}' \) and \( M_c \) is as in Lemma 3.4 (b). Then

\[
\prod_{\lambda \in \mathfrak{h}^*, c \in \mathbb{C}} (M_\lambda \otimes M_c)
\]

is a faithful \( \bar{W}_\chi \)-module. Therefore \( \bar{W}_\chi \) is isomorphic to a subalgebra in

\[
\prod_{\lambda \in \mathfrak{h}^*, c \in \mathbb{C}} \text{End}_\mathbb{C}(M_\lambda \otimes M_c).
\]

Since \( \prod_{\lambda \in \mathfrak{h}^*, c \in \mathbb{C}} \text{End}_\mathbb{C}(M_\lambda \otimes M_c) \) satisfies the Amitsur–Levitzki identity, \( \bar{W}_\chi \) must satisfy it as well.

Finally we assume that all odd roots in \( \Pi \) are isotropic. Then

\[
\Pi^\prime \setminus A = \{\beta_1, \ldots, \beta_{k-1}, \beta_k\}
\]

with the only non-orthogonal pair \( \beta_{k-1}, \beta_k \). In this case \( [\mathfrak{s}, \mathfrak{s}] \) is a direct sum of \( k - 2 \) copies of \( \mathfrak{sl}(1|1) \) generated by the root spaces \( \mathfrak{g}_{\pm \beta_i}, \ i = 1, \ldots, k - 2 \) and one copy of \( \mathfrak{sl}(1|2) \) generated by \( \mathfrak{g}_{\pm \beta_{k-1}}, \mathfrak{g}_{\pm \beta_k} \). Furthermore, \( \mathfrak{m}^s \subset \mathfrak{sl}(1|2) \) is generated by \( f \in \mathfrak{g}_{-\beta_{k-1}-\beta_k} \). As in the previous case we write \( \mathfrak{h} = \mathfrak{s}' \oplus \mathfrak{r} \), where \( \mathfrak{r} = \mathfrak{sl}(1|2) \). Then

\[
\bar{W}_\chi = U(\mathfrak{s}') \otimes \bar{W}_{\chi^r},
\]

where \( \chi(f) = 1 \).

We realize \( \mathfrak{r} \) in the standard matrix form and introduce the following notations:

\[
\begin{align*}
h_1 &= E_{11} + E_{33}, \\
h_2 &= E_{11} + E_{22}, \\
f &= E_{32}, \\
e &= E_{23}, \\
e^+ &= E_{13}, \\
e^- &= E_{12}, \\
f^- &= E_{31}, \\
f^+ &= E_{21}, \\
C &= h_1 + h_2.
\end{align*}
\]
Let $\pi : U(\mathfrak{r}) \to U(\mathfrak{r})/U(\mathfrak{r})(f - 1)$ be the natural projection. We denote by $\Omega$ the quadratic Casimir element of $\mathfrak{r}$ and set

$$a = [f^-, e^+e^-], \quad b = [e^-, f^+f^-].$$

Then $\pi(a) = \pi(h_1)\pi(e^-) - \pi(e^+), \pi(b) = \pi(h_2)\pi(f^-) - \pi(f^+)$. The reader can easily check that $\pi(C), \pi(\Omega), \pi(e^-), \pi(f^-), \pi(a)$ and $\pi(b)$ belong to $\tilde{W}^s_\chi$.

**Lemma 3.5.** a) $\tilde{W}^s_\chi$ is generated by $\pi(C), \pi(\Omega), \pi(e^-), \pi(f^-), \pi(a)$ and $\pi(b)$. It is clear that $\pi(\Omega)$ lies in the center of $\tilde{W}^s_\chi$. The other defining relations are

$$[\pi(C), \pi(e^-)] = \pi(e^-), \quad [\pi(C), \pi(a)] = \pi(a),$$

$$[\pi(C), \pi(f^-)] = -\pi(f^-), \quad [\pi(C), \pi(b)] = -\pi(b),$$

$$[\pi(e^-), \pi(f^-)] = 1, \quad [\pi(a), \pi(b)] = \pi(\Omega),$$

and the commutators of all other odd generators are zero.

b) Let $c, d \in \mathbb{C}, c \neq 0$, $U$ be the subalgebra in $\tilde{W}^s_\chi$ generated by $\pi(C), \pi(\Omega), \pi(a)$ and $\pi(e^-)$, $U_{c,d} = U/(\pi(\Omega) - c, \pi(C) - d, \pi(a), \pi(e^-))$ be the one-dimensional $U$-module. The induced module $M_{c,d} = \tilde{W}^s_\chi \otimes_U U_{c,d}$ is simple and has dimension 4. The product $\prod_{c,d \in \mathbb{C}} M_{c,d}$ is a faithful $\tilde{W}^s_\chi$-module.

**Proof.** We leave the proof to the reader. For assertion (a) one should use a suitable modification of Proposition 2.7 (a).

We also leave to the reader the proof of Proposition 3.3 in the last case since it is completely similar to the previous case.

In what follows we denote by $\mathcal{A}$ the image $\vartheta(W_\chi)$ of $W_\chi$ in $\tilde{W}^s_\chi$.

**Corollary 3.6.** $W_\chi$ satisfies (3.1).

**Proof.** By Proposition 3.1, $\mathcal{A} \simeq W_\chi$. By Proposition 3.3, $\mathcal{A}$ satisfies (3.1).

**Proposition 3.7.** Let $M$ be a simple $W_\chi$-module. Then $\dim M \leq 2^{k+1}$.

**Proof.** Consider $M$ as a module over the associative algebra $W_\chi$, forgetting the $\mathbb{Z}_2$-grading. Then either $M$ is simple or $M$ is a direct sum of two non-homogeneous simple submodules: $M = M_1 \oplus M_2$.

In the former case we claim that $\dim M \leq 2^k$. Indeed, assume $\dim M > 2^k$. Let $V$ be a subspace of dimension $2^k + 1$. By the density theorem for any $X_1, \ldots, X_{2^k+1} \in \text{End}_C(V)$ one can find $u_1, \ldots, u_{2^k+1}$ in $W_\chi$ such that $(u_i)_V = X_i$ for all $i = 1, \ldots, 2^k+1$. Since $\text{End}_C(V)$ does not satisfy (3.1), we obtain a contradiction with Corollary 3.6.

In the latter case, we can prove in the same way that $\dim M_1 \leq 2^k$ and $\dim M_2 \leq 2^k$. Therefore $\dim M \leq 2^{k+1}$.

**Conjecture 3.8.** Every irreducible representation of $\mathcal{A} \simeq W_\chi$ is isomorphic to a subquotient of some irreducible representation of $\tilde{W}^s_\chi$ restricted to $\mathcal{A}$. 
4. Generators of $W_\chi$ for the queer Lie superalgebra $Q(n)$

In the rest of the paper we study in detail the case when $\chi$ is regular and $\mathfrak{g} = Q(n)$.

In this section we construct some generators of $W_\chi$. In particular, we will prove that $W_\chi$ is finitely generated. We use the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ of $U(Q(n))$ defined in [30] recursively:

\begin{align}
(4.1) \quad e_{i,j}^{(m)} &= \sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} f_{k,j}^{(m-1)}, \\
& \quad f_{i,j}^{(m)} &= \sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} e_{k,j}^{(m-1)}.
\end{align}

Then

\begin{align}
(4.2) \quad [e_{i,j}, e_{k,l}^{(m)}] &= \delta_{j,k} e_{i,l}^{(m)} - \delta_{i,k} e_{k,l}^{(m)}, \quad [e_{i,j}, f_{k,l}^{(m)}] = \delta_{j,k} f_{i,l}^{(m)} - \delta_{i,k} f_{k,l}^{(m)} , \\
& \quad [f_{i,j}, e_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} e_{i,l}^{(m)} - \delta_{i,k} e_{k,l}^{(m)}, \\
& \quad [f_{i,j}, f_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} f_{i,l}^{(m)} + \delta_{i,k} f_{k,l}^{(m)} .
\end{align}

**Proposition 4.1.** A. Sergeev [30].
The elements $\sum_{i=1}^{n} e_{i,i}^{(2m+1)}$ generate $Z(Q(n))$.

**Remark 4.2.** In contrast with the Lie algebra case the center $Z(Q(n))$ is not Noetherian, in particular, it is not finitely generated.

**Lemma 4.3.** $\pi(e_{n,1}^{(m)})$ and $\pi(f_{n,1}^{(m)})$ belong to $W_\chi$.

**Proof.** By (4.2) we have that

\[ [e_{i,j}, e_{n,1}^{(m)}] = [f_{i,j}, e_{n,1}^{(m)}] = [e_{i,j}, f_{n,1}^{(m)}] = [f_{i,j}, f_{n,1}^{(m)}] = 0 \]

for all $i > j$. In other words, $e_{n,1}^{(m)}, f_{n,1}^{(m)} \in U(\mathfrak{g})^{ad}$. Hence $\pi(e_{n,1}^{(m)}), \pi(f_{n,1}^{(m)}) \in W_\chi$. \(\square\)

**Lemma 4.4.** Let $1 \leq l \leq n - 1$. Then

\begin{align}
(4.3) \quad \pi(e_{m,1}^{(l)}) &= \begin{cases} 1 & \text{if } m = l + 1, \\
0 & \text{if } l + 2 \leq m \leq n, \end{cases} \quad \pi(f_{m,1}^{(l)}) = 0, \text{ if } l + 1 \leq m \leq n.
\end{align}

**Proof.** We will prove the statement by induction in $l$. For $l = 1$ we have that

\begin{align}
(4.4) \quad \pi(e_{m,1}^{(1)}) &= \pi(e_{m,1}), \quad \pi(f_{m,1}^{(1)}) = \pi(f_{m,1}).
\end{align}

Then (4.3) follows from (2.6). Assume that (4.3) holds for $l$. From (4.1) we have that

\begin{align}
(4.5) \quad e_{m,1}^{(l+1)} &= \sum_{k=1}^{n} e_{m,k} e_{k,1}^{(l)} + (-1)^{l} \sum_{k=1}^{n} f_{m,k} f_{k,1}^{(l)}, \\
& \quad f_{m,1}^{(l+1)} &= \sum_{k=1}^{n} e_{m,k} f_{k,1}^{(l)} + (-1)^{l} \sum_{k=1}^{n} f_{m,k} e_{k,1}^{(l)}.
\end{align}
Note that
\begin{equation}
[e_{m,k}, e_{k,1}^{(l)}] = e_{m,1}^{(l)}, \quad [e_{m,k}, f_{k,1}^{(l)}] = f_{m,1}^{(l)}, \\
[f_{m,k}, e_{k,1}^{(l)}] = (-1)^{l+1} f_{m,1}^{(l)}, \quad [f_{m,k}, f_{k,1}^{(l)}] = (-1)^{l+1} e_{m,1}^{(l)}.
\end{equation}

Hence
\begin{equation}
\begin{aligned}
\epsilon_{m,1}^{(l+1)} &= \sum_{k=1}^{m-1} (e_{k,1}^{(l)} e_{m,k}^{(l)} + e_{m,k}^{(l)} e_{k,1}^{(l)}) + \sum_{k=m}^{n} e_{m,k} e_{k,1}^{(l)} + (-1)^l \left( \sum_{k=1}^{m-1} (-f_{k,1}^{(l)} f_{m,k}^{(l)} + (-1)^{l+1} e_{m,1}^{(l)}) + \sum_{k=m}^{n} f_{m,k} f_{k,1}^{(l)} \right), \\
f_{m,1}^{(l+1)} &= \sum_{k=1}^{m-1} (f_{k,1}^{(l)} e_{m,k}^{(l)} + f_{m,k}^{(l)} e_{k,1}^{(l)}) + \sum_{k=m}^{n} e_{m,k} f_{k,1}^{(l)} + (-1)^l \left( \sum_{k=1}^{m-1} (e_{k,1}^{(l)} f_{m,k}^{(l)} + (-1)^{l+1} f_{m,1}^{(l)}) + \sum_{k=m}^{n} f_{m,k} e_{k,1}^{(l)} \right).
\end{aligned}
\end{equation}

Then
\begin{equation}
\begin{aligned}
\pi(e_{m,1}^{(l+1)}) &= \sum_{k=1}^{m-1} \pi(e_{k,1}^{(l)}) \pi(e_{m,k}) + \sum_{k=m}^{n} \pi(e_{m,k}) \pi(e_{k,1}^{(l)}) + (-1)^l \pi(f_{m,k}) \pi(f_{k,1}^{(l)}) + \sum_{k=1}^{m-1} \pi(f_{k,1}^{(l)}) \pi(f_{m,k}), \\
\pi(f_{m,1}^{(l+1)}) &= \sum_{k=1}^{m-1} \pi(f_{k,1}^{(l)}) \pi(e_{m,k}) + \sum_{k=m}^{n} \pi(e_{m,k}) \pi(f_{k,1}^{(l)}) + (-1)^l \pi(f_{m,k}) \pi(f_{k,1}^{(l)}) + \sum_{k=1}^{m-1} \pi(e_{k,1}^{(l)}) \pi(f_{m,k}).
\end{aligned}
\end{equation}

Then by (2.6)
\begin{equation}
\begin{aligned}
\pi(e_{m,1}^{(l+1)}) &= \pi(e_{m-1,1}^{(l)}) + \sum_{k=m}^{n} \left( \pi(e_{m,k}) \pi(e_{k,1}^{(l)}) + (-1)^l \pi(f_{m,k}) \pi(f_{k,1}^{(l)}) \right), \\
\pi(f_{m,1}^{(l+1)}) &= \pi(f_{m-1,1}^{(l)}) + \sum_{k=m}^{n} \left( \pi(e_{m,k}) \pi(f_{k,1}^{(l)}) + (-1)^l \pi(f_{m,k}) \pi(e_{k,1}^{(l)}) \right).
\end{aligned}
\end{equation}

Let \( m \geq l + 2 \). Then by induction hypothesis,
\begin{equation}
\pi(e_{k,1}^{(l)}) = \pi(f_{k,1}^{(l)}) = 0 \text{ for } k = m, \ldots, n.
\end{equation}

If \( m = l + 2 \), then \( \pi(e_{m,1}^{(l+1)}) = \pi(e_{l+1,1}^{(l)}) = 1 \), and if \( m \geq l + 3 \), then \( \pi(e_{m,1}^{(l+1)}) = \pi(e_{m-1,1}^{(l)}) = 0 \). Also, if \( m \geq l + 2 \), then \( \pi(f_{m,1}^{(l+1)}) = \pi(f_{m-1,1}^{(l)}) = 0 \). Hence (4.3) holds for \( l + 1 \).

**Corollary 4.5.**
\begin{equation}
\pi(e_{n,1}^{(m)}) = 0 \text{ for } m \leq n - 2, \quad \pi(e_{n,1}^{(n-1)}) = 1; \quad \pi(f_{n,1}^{(m)}) = 0 \text{ for } m \leq n - 1,
\end{equation}
Lemma 4.6.
\[\pi(e_{n,1}^{(n)}) = \pi(z), \quad \pi(f_{n,1}^{(n)}) = \pi(H_0).\]

Proof. Let \(1 \leq m \leq n\). We will show that
\[\pi(e_{m,1}^{(m)}) = \sum_{k=1}^{m} \pi(e_{k,k}), \quad \pi(f_{m,1}^{(m)}) = \sum_{k=1}^{m} (-1)^{k-1} \pi(f_{k,k}).\]

Again we proceed by induction on \(m\). If \(m = 1\), then (4.11) obviously holds by (4.4). Assume that (4.11) holds for \(m\). From (4.1) and (4.2) we have that
\[e_{m+1,1}^{(m+1)} = \sum_{k=1}^{n} e_{m+1,k} e_{k,1}^{(m)} + (-1)^m \sum_{k=1}^{n} f_{m+1,k} f_{k,1}^{(m)} = \sum_{k=1}^{m} \left( e_{k,1}^{(m)} e_{m+1,k} + e_{m+1,k} e_{m+1,k}^{(m)} \right) + \sum_{k=m+1}^{n} e_{m+1,k} e_{k,1}^{(m)} + (-1)^m \left( \sum_{k=1}^{m} (-1)^m f_{k,1}^{(m)} f_{m+1,k} + (-1)^{m+1} f_{m+1,k}^{(m)} \right),\]
\[f_{m+1,1}^{(m+1)} = \sum_{k=1}^{n} e_{m+1,k} f_{k,1}^{(m)} + (-1)^m \sum_{k=1}^{n} f_{m+1,k} e_{k,1}^{(m)} = \sum_{k=1}^{m} \left( f_{k,1}^{(m)} e_{m+1,k} + f_{m+1,k}^{(m)} \right) + \sum_{k=m+1}^{n} e_{m+1,k} f_{k,1}^{(m)} + (-1)^m \left( \sum_{k=1}^{m} e_{k,1}^{(m)} f_{m+1,k} + (-1)^{m+1} f_{m+1,k}^{(m)} \right).\]

Using (2.6) and (4.3) we obtain
\[\pi(e_{m+1,1}^{(m+1)}) = \pi(e_{m,1}^{(m)}) + \pi(e_{m+1,m+1}),\]
\[\pi(f_{m+1,1}^{(m+1)}) = \pi(f_{m,1}^{(m)}) + (-1)^m \pi(f_{m+1,m+1}).\]

By induction hypothesis we have
\[\pi(e_{m,1}^{(m)}) = \sum_{k=1}^{m} \pi(e_{k,k}) + \pi(e_{m+1,m+1}) = \sum_{k=1}^{m+1} \pi(e_{k,k}),\]
\[\pi(f_{m,1}^{(m)}) = \sum_{k=1}^{m} (-1)^{k-1} \pi(f_{k,k}) + (-1)^m \pi(f_{m+1,m+1}) = \sum_{k=1}^{m+1} (-1)^{k-1} \pi(f_{k,k}).\]

Hence \(\pi(e_{n,1}^{(n)}) = \pi(z)\) and \(\pi(f_{n,1}^{(n)}) = \pi(H_0)\). \(\square\)

Consider the Kazhdan filtration on \(U(b)\). By definition, the graded algebra \(Gr_K U(b)\) is isomorphic to \(S(b)\). Moreover, \(Gr_K U(b) \simeq S(b)\) is a commutative graded ring, where the grading is induced from the Dynkin \(\mathbb{Z}\)-grading of \(g\). For any \(X \in U(b)\) let \(Gr_K(X)\) denote the corresponding element in \(Gr_K U(b)\) and \(P(X)\) denote the highest weight component of \(Gr_K(X)\) in the Dynkin \(\mathbb{Z}\)-grading. For \(X \in U(b)\), we denote by \(\text{deg} P(X)\) the Kazhdan degree of \(Gr_K(X)\) and by \(\text{wt} P(X)\) the weight of the highest weight component of \(Gr_K(X)\).
Lemma 4.7. \( P(\pi(e_{n,1}^{(n)})) = z, \)
\[
(4.12) \quad P(\pi(e_{n,1}^{(n-1+k)})) = e^{k-1}, k = 2, \ldots, n,
\]
\[
P(\pi(f_{n,1}^{(n-1+k)})) = H_{k-1}, k = 1, \ldots, n.
\]

Proof. We will prove a more general statement. We claim that for 0 ≤ \( l \leq n - 1 \) and \( 1 \leq p \leq n \)
\[
(4.13) \quad P(\pi(e_{p,1}^{(p+l)})) = \sum_{i=1}^{r} e_{i, i+l}^{(p+l)},
\]
\[
P(\pi(f_{p,1}^{(p+l)})) = \sum_{i=1}^{r} (-1)^{l+1-i} f_{i, i+l}, r = \min\{p, n-l\}.
\]

In particular,
\[
\deg P(\pi(e_{p,1}^{(p+l)})) = \deg P(\pi(f_{p,1}^{(p+l)})) = 2l + 2,
\]
\[
\wt P(\pi(e_{p,1}^{(p+l)})) = \wt P(\pi(f_{p,1}^{(p+l)})) = 2l.
\]

We proceed to the proof of (4.13) by induction on \( l \) and \( p \). Note that if \( l = 0 \), then (4.13) holds for any 1 ≤ \( p \leq n \) by (4.11). Assume that if \( l \leq k - 1 \), then (4.13) holds for any 1 ≤ \( p \leq n \). Let \( l = k \). Show that (4.13) holds for \( p = 1 \). Note that
\[
e_{1,1}^{(1+k)} = \left( \sum_{i=1}^{n} e_{1, i}^{(1)} e_{i,1}^{(k)} \right) + (-1)^{k} \left( \sum_{i=1}^{n} f_{1, i} f_{i,1}^{(k)} \right).
\]

Let \( X = \pi(e_{1,1}^{(1+k)}), Y = \pi(f_{1,1} f_{1,1}^{(k)}) \) where \( i = 1, \ldots, k \). Note that
\[
\deg P(\pi(e_{1,1}^{(1)})) = \deg P(\pi(f_{1,1}^{(k)})) = 2i,
\]
\[
\wt P(\pi(e_{1,1}^{(1)})) = \wt P(\pi(f_{1,1}^{(k)})) = 2i - 2.
\]

By induction hypothesis,
\[
\deg P(\pi(e_{1,1}^{(1+k)})) = \deg P(\pi(f_{1,1}^{(k)})) = 2(k - i) + 2,
\]
\[
\wt P(\pi(e_{1,1}^{(1+k)})) = \wt P(\pi(f_{1,1}^{(k)})) = 2(k - i).
\]

Then
\[
\deg P(X) = 2k + 2, \quad \wt P(X) = 2k - 2,
\]
\[
\deg P(Y) = 2k + 2, \quad \wt P(Y) = 2k - 2.
\]

Let \( X = \pi(e_{1,k+1} e_{k+1,1}^{(k)}) \). Then by (4.3) \( X = \pi(e_{1,k+1}) \). Hence
\[
\deg P(X) = 2k + 2, \quad \wt P(X) = 2k.
\]

Finally, by (4.3)
\[
\pi(e_{k+1,1}^{(k)}) = 0 \text{ for } i = 2, \ldots, n - k, \quad \pi(f_{k+1,1}^{(k)}) = 0 \text{ for } i = 1, \ldots, n - k.
\]
Hence
\[ P(\pi(L^{(1+k)})) = e_{1,k+1}. \]

Let \( l = k \) and assume that (4.13) holds for \( p \leq m \). Show that it holds for \( p = m + 1 \).

Note that
\[ e_{m+1,1}^{(m+1+k)} = \left( \sum_{i=1}^{n} e_{m+1,i} e_{i,1}^{(m+k)} \right) + (-1)^{m+k} \left( \sum_{i=1}^{n} f_{m+1,i} f_{i,1}^{(m+k)} \right). \]

Thus
\[ \pi(e_{m+1,1}^{(m+1+k)}) = \left( \sum_{i=1}^{m-1} \pi(e_{m+1,i} e_{i,1}^{(m+k)}) + \pi(e_{m+1,m} e_{m,1}^{(m+k)}) + \sum_{i=1}^{k} \pi(e_{m+1,m+i} e_{m+i,1}^{(m+k)}) + \sum_{i=2}^{n-m-k} \pi(e_{m+1,m+k+i} e_{m+k+i,1}^{(m+k)}) + \sum_{i=1}^{n-m-k} \pi(f_{m+1,m+k+i} f_{m+k+i,1}^{(m+k)}) \right). \]

Let \( X = \pi(e_{m+1,i} e_{i,1}^{(m+k)}) \), where \( i = 1, \ldots, m - 1 \), and \( Y = \pi(f_{m+1,i} f_{i,1}^{(m+k)}) \), where \( i = 1, \ldots, m \). Then by (4.2) and (2.6)
\[ X = \pi(e_{m+1,i} e_{m+1,i} + e_{m+1,1} e_{m+1,1}^{(m+k)}) = \pi(e_{m+1,1}^{(m+k)}), \]
\[ Y = \pi(-f_{i,1}^{(m+k)} f_{m+1,i} + (-1)^{m+k+1} e_{m+1,1}^{(m+k)}) = \pi((-1)^{m+k+1} e_{m+1,1}^{(m+k)}). \]

By induction hypothesis
\[ \deg P(X) = \deg P(Y) = 2k, \]
\[ \wt P(X) = \wt P(Y) = 2k - 2. \]

Let \( X = \pi(e_{m+1,m} e_{m,1}^{(m+k)}) \). Then by (4.2) and (2.6)
\[ X = \pi(e_{m+1,m} e_{m+1,m} + e_{m+1,1}^{(m+k)}) = \pi(e_{m+1,m} + e_{m+1,1}^{(m+k)}). \]

By induction hypothesis
\[ \deg P(\pi(e_{m+1,m}^{(m+k)})) = 2k, \]
\[ \wt P(\pi(e_{m+1,m}^{(m+k)})) = 2k - 2, \]
\[ \deg P(\pi(e_{m+1,m}^{(m+k)})) = 2k + 2, \]
\[ \wt P(\pi(e_{m+1,m}^{(m+k)})) = 2k. \]
Let \( X = \pi(e_{m+1,m+1}e^{(m+k)}_{m,i}) \), \( Y = \pi(f_{m+1,m+1}f^{(m+k)}_{m,i}) \) for \( i = 1, \ldots, k \). Then by induction hypothesis

\[
\text{deg} P(X) = \text{deg} P(Y) = 2k + 2, \\
\text{wt} P(X) = \text{wt} P(Y) = 2k - 2.
\]

Let \( X = \pi(e_{m+1,m+k+1}e^{(m+k)}_{m+k+1,i}) \). Hence by (4.3) \( X = \pi(e_{m+1,m+k+1}) \). Then

\[
\text{deg} P(X) = 2k + 2, \\
\text{wt} P(X) = 2k.
\]

Finally, by (4.3) \( \pi(e_{m+1,m+k+1}e^{(m+k)}_{m+k+1,i}) = 0 \) for \( i = 2, \ldots, n - m - k \) and \( \pi(f_{m+1,m+k+1}f^{(m+k)}_{m+k+1,i}) = 0 \) for \( i = 1, \ldots, n - m - k \). From (4.14)-(4.18) one can see that the highest degree component in \( \pi(e^{(m+1+k)}_{m+1,1}) \) has degree \( 2k + 2 \), and its highest weight component has weight \( 2k \). In fact, if \( m \geq n - k \), then by (4.16) this component is \( P(\pi(e^{(m+k)}_{m,1})) \). By induction hypothesis \( P(\pi(e^{(m+k)}_{m,1})) = \sum_{i=1}^{n-k} e_{i,i+k} \). If \( m < n - k \), then \( P(\pi(e^{(m+k)}_{m,1})) = \sum_{i=1}^{m} e_{i,i+k} \). Note that in this case \( \pi(e^{(m+1+k)}_{m+1,1}) \) has an additional element \( \pi(e_{m+1,m+k+1}) \) of degree \( 2k + 2 \) and weight \( 2k \) according to (4.18). Clearly, \( P(\pi(e_{m+1,m+k+1})) = e_{m+1,m+k+1} \) and \( P(\pi(e^{(m+k)}_{m,1})) + P(\pi(e_{m+1,m+k+1})) \neq 0 \). Hence

\[
P(\pi(e^{(m+1+k)}_{m+1,1})) = P(\pi(e^{(m+k)}_{m,1})) + P(\pi(e_{m+1,m+k+1})) = \sum_{i=1}^{m+1} e_{i,i+k}.
\]

Then in either case,

\[
P(\pi(e^{(m+1+k)}_{m+1,1})) = \sum_{i=1}^{r} e_{i,i+k}, \text{ where } r = \min\{m + 1, n - k\}.
\]

Thus if \( 0 \leq l \leq n - 1 \) and \( 1 \leq p \leq n \), then

\[
P(\pi(e^{(p+l)}_{p,1})) = \sum_{i=1}^{r} e_{i,i+l}, \text{ where } r = \min\{p, n - l\}.
\]

Similarly, one can prove that

\[
P(\pi(f^{(p+l)}_{p,1})) = \sum_{i=1}^{r} (-1)^{l+1+i} f_{i,i+l}, \text{ where } r = \min\{p, n - l\}.
\]

In particular, if \( p = n \) and \( l = k \), where \( k = 0, \ldots, n - 1 \), we have

\[
P(\pi(e^{(n+k)}_{n,1})) = \sum_{i=1}^{n-k} e_{i,i+k} = e^{k},
\]

\[
P(\pi(f^{(n+k)}_{n,1})) = \sum_{i=1}^{n-k} (-1)^{k+1+i} f_{i,i+k} = H_{k}.
\]
Proposition 4.8. $\pi(e_{n,1}^{(m)})$ and $\pi(f_{n,1}^{(m)})$ for $m = n, \ldots, 2n - 1$ generate $W_\chi$.

Proof. The statement follows from Lemma 4.7 and Proposition 2.7 (a). □

Corollary 4.9. Lemma 4.7 and Proposition 2.7 (b) imply that Conjecture 2.8 is true for $g = Q(n)$ and regular $\chi$.

Corollary 4.10. The natural homomorphism $U(g)^{adm} \rightarrow W_\chi$ is surjective.

Proof. Since $e_{n,1}^{(m)}, f_{n,1}^{(m)} \in U(g)^{adm}$, the statement follows from Proposition 4.8. □

5. Further results about the structure of $W_\chi$ for $g = Q(n)$

5.1. The Harish-Chandra homomorphism for $Q(n)$. Recall that for $g = Q(n)$ and regular $\chi$ we have $p = b$. We study in detail the restriction of the Harish-Chandra homomorphism $\vartheta : U(b) \rightarrow U(h)$ to $W_\chi$. We start with calculating the images of the generators.

Proposition 5.1.

\begin{equation}
\vartheta(\pi(e_{n,1}^{(n+k-1)})) = \left[ \sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{even}},
\end{equation}

\begin{equation}
\vartheta(\pi(f_{n,1}^{(n+k-1)})) = \left[ \sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{odd}}.
\end{equation}

Proof. We will prove by induction on $l$ and $p$ that if $l \geq 0$ and $1 \leq p \leq n$ then

\begin{equation}
\vartheta(\pi(e_{p,1}^{(p+l)})) + \vartheta(\pi(f_{p,1}^{(p+l)})) = \sum_{p \geq i_1 \geq i_2 \geq \ldots \geq i_k \geq 1} (x_{i_1} + (-1)^l \xi_{i_1}) \ldots (x_{i_p} - \xi_{i_p})(x_{i_{p+1}} + \xi_{i_{p+1}}).
\end{equation}

Note that if $l = 0$, then (5.2) holds for any $1 \leq p \leq n$ since by (4.11)

\begin{equation}
\vartheta(\pi(e_{p,1}^{(p)})) + \vartheta(\pi(f_{p,1}^{(p)})) = \vartheta(\sum_{i=1}^p \pi(e_{i,i}^{(i)})) + \vartheta(\sum_{i=1}^p (-1)^{i-1}\pi(f_{i,i})) = \sum_{i=1}^p (x_i + \xi_i).
\end{equation}
Assume that if \( l \leq k - 1 \), then (5.2) holds for any \( 1 \leq p \leq n \). Let \( l = k \), show that (5.2) holds for \( p = 1 \). We have

\[
\vartheta(\pi(e^{(1+k)})) + \vartheta(\pi(f^{(1+k)})) = \\
\vartheta(\pi(e_{1,1})\pi(e^{(k)})) + \vartheta(\pi(e_{1,1})\pi(f^{(k)})) + \vartheta(\pi(e_{1,1})\pi(e^{(k)})) = \\
e_{1,1} + (1)k f_{1,1})(\vartheta(e^{(k)}) + \vartheta(f^{(k)}) = \\
(x_1 + (-1)k \xi_1) \sum_{i_1 = i_2 = \ldots = i_k = 1} \ldots (x_{i_k} + (-1)k i_k)(x_{i_k + 1} + \xi_{i_k + 1}).
\]

Let \( l = k \) and assume that (5.2) holds for \( p \leq m \). Show that it holds for \( p = m + 1 \). By induction hypothesis we have

\[
\vartheta(\pi(e^{(m+1+k)})) + \vartheta(\pi(f^{(m+1+k)})) = \vartheta(\pi(e^{(m+1)})) + \vartheta(\pi(f^{(m+1)})) + \\
(e_{m+1,m+1} + (-1)^{m+k} f_{m+1,m+1})\vartheta(\pi(e^{(m+1)})) + \vartheta(\pi(f^{(m+1)})) = \\
\sum_{m+1 \geq i_1 \geq i_2 \geq \ldots \geq i_{k+1} \geq 1} (x_{i_1} + (1)k \xi_{i_1}) \ldots (x_{i_k} + (1)k \xi_{i_k})(x_{i_k + 1} + \xi_{i_k + 1}) + \\
\sum_{m+1 \geq i_1 \geq i_2 \geq \ldots \geq i_{k+1} \geq 1} (x_{i_1} + (1)k \xi_{i_1}) \ldots (x_{i_k} + (1)k \xi_{i_k})(x_{i_k + 1} + \xi_{i_k + 1}).
\]

Thus (5.2) is proven. In particular, if \( p = n \) we obtain (5.1). □

**Proposition 5.2.**

(5.3) \[
\pi(e^{(n+1)}_{n,1}) = \pi\left(\frac{1}{2} \sum_{i=1}^{n} e_{i,i}^2 + \sum_{i=1}^{n-1} e_{i,i+1} + \sum_{i<j} (-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2} z^2 - z\right),
\]

and

(5.4) \[
\vartheta(\pi(e^{(n+1)}_{n,1})) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \sum_{i<j} \xi_i \xi_j + \frac{1}{2} z^2 - z.
\]

**Proof.** We will prove by induction on \( m \) that for \( 1 \leq m \leq n \)

(5.5) \[
\pi(e^{(m+1)}_{m,1}) = \pi\left(\frac{1}{2} \sum_{i=1}^{m} e_{i,i}^2 + \sum_{i=1}^{\min(m,n-1)} e_{i,i+1} + \sum_{1 \leq i < j \leq m} (-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2} (\sum_{i=1}^{m} e_{i,i})^2 - \sum_{i=1}^{m} e_{i,i}\right).
\]

If \( m = 1 \), then

\[
\pi(e_{1,1}^{(2)}) = \pi(e_{1,1}^2 + e_{1,2}^2 - f_{1,1}^2) = \pi(e_{1,1}^2 + e_{1,2} - e_{1,1}).
\]
Lemma 5.3. Define odd elements $U$ on the center of $W_{\chi}$. By (5.2), we obtain (5.3). □

Then (2.6) and (4.3)

$$\pi(e_{m+1,1}^{(m+2)}) = \pi(e_{m,1}^{(m+1)}) + \pi(e_{m+1,m+1})\pi(e_{m+1,1}^{(m+1)}) + \pi(e_{m+1,m+2})\pi(e_{m+2,1}^{(m+1)}) + (-1)^m\pi(f_{m+1,m+1})\pi(f_{m+1,1}^{(m+1)})$$

By induction hypothesis and using (4.11) we have

$$\pi(e_{m+1,1}^{(m+2)}) = \pi(e_{m,1}^{(m+1)}) + \sum_{i=1}^{m} e_{i,i} + \sum_{1\leq i\neq j\leq m} (-1)^{i-j}f_{i,j} + \frac{1}{2}(\sum_{i=1}^{m} e_{i,i})^2 - \sum_{i=1}^{m} e_{i,i}$$

and

$$\pi(e_{m+1,m+1}^{(m+1)}\sum_{i=1}^{m} e_{i,i} + e_{m+1,m+2} - (-1)^m f_{m+1,m+1}\sum_{i=1}^{m+1} (-1)^{i-1}f_{i,i}) =$$

$$\pi\left(\frac{1}{2}\sum_{i=1}^{m+1} e_{i,i}^2 + \sum_{i=1}^{m+1} e_{i,i+1} + \sum_{1\leq i\neq j\leq m+1} (-1)^{i-j}f_{i,j} + \frac{1}{2}(\sum_{i=1}^{m+1} e_{i,i})^2 - \sum_{i=1}^{m+1} e_{i,i}\right).$$

Thus (5.5) is proven. In particular, if $m = n$ we obtain (5.3). Finally, applying \( \vartheta \) to (5.3) we obtain (5.4).

5.2. On the center of $W_{\chi}$. Recall that we denote by $A$ the image $\vartheta(W_{\chi})$ of $W_{\chi}$ in $U(\mathfrak{h})$. Set $A^0 = A \cap U(\mathfrak{h}_0)$.

Lemma 5.3. Define odd elements $\Phi_k (k \geq 0)$ of $W_{\chi}$ as follows:

$$\Phi_0 = \pi(f_{n,1}^{(n)}) = \pi(H_0),$$

$$\Phi_k = \left(\frac{1}{2} \operatorname{ad}(\pi(e_{n,1}^{(n+1)}))\right)^k(\Phi_0).$$

Then (a) for $0 \leq k \leq n-1$, we have that $P(\Phi_k) = H_k$,

(b) $[\Phi_m, \Phi_p] = 0$, if $m + p$ is odd,

(c) there exist $z_0, z_2, \ldots \in \pi(Z(Q(n)))$ such that

$$[\Phi_m, \Phi_p] = (-1)^m z_{m+p}$$

if $m + p$ is even.

Proof. Let $X, Y \in W_{\chi}$. To prove (a) observe that if $P(X), P(Y) \in \mathfrak{g}^\chi$ and $[P(X), P(Y)] \neq 0$, then $P([X,Y]) = [P(X), P(Y)]$. Since $P(\pi(e_{n,1}^{(n+1)})) = e$ and $P(\Phi_0) = H_0$, the statement follows from the relation

$$H_k = \left(\frac{1}{2} \operatorname{ad}(e)\right)^k(H_0).$$
To prove (b) and (c) we use \( \vartheta \). We first notice that (5.4) implies
\[
\vartheta(\Phi_k) = \sum_{j=1}^{n} \phi_j^{(k)} \xi_j
\]
for some polynomial \( \phi_j^{(k)} \in \mathbb{C}[x_1, \ldots, x_n] \) of degree \( k \). Hence \([\vartheta(\Phi_m), \vartheta(\Phi_p)]\in \mathbb{C}[x_1, \ldots, x_n]\). Since \( x_i \) lie in the center of \( \mathfrak{h} \), we get
\[
[\vartheta(\Phi_{m+1}), \vartheta(\Phi_p)] = \frac{1}{2}[[\vartheta(\pi(e_{n,1}^{(n+1)})), \vartheta(\Phi_m)], \vartheta(\Phi_p)] =
-\frac{1}{2}[\vartheta(\Phi_m), [\vartheta(\pi(e_{n,1}^{(n+1)})), \vartheta(\Phi_p)]] = -[\vartheta(\Phi_m), \vartheta(\Phi_{p+1})].
\]
Since \( \vartheta \) is injective, that implies
\[
[\Phi_p, \Phi_q] = (-1)^{q-p}[\Phi_r, \Phi_s],
\]
if \( p + q = r + s \).

In particular, if \( p + q \) is odd we have
\[
[\Phi_p, \Phi_q] = (-1)^{q-p}[\Phi_q, \Phi_p] = 0.
\]
This implies (b).

To prove (c) we set
\[
z_i = [\Phi_0, \Phi_i] \quad \text{for even } i \geq 0.
\]
Since \( \vartheta(z_i) \in \mathcal{A}^0 \), (c) follows from Lemma 5.4.

**Lemma 5.4.** \( \mathcal{A}^0 = \vartheta(\pi(Z(\mathfrak{g}))) \).

**Proof.** It is not hard to see that the restriction of \( \vartheta \) on \( Z(\mathfrak{g}) \) coincides with the standard Harish-Chandra homomorphism. Thus, from Sergeev’s result, [31], we know that \( \vartheta(Z(\mathfrak{g})) \) coincides with the space of symmetric polynomials \( p \) in \( x_1, \ldots, x_n \) satisfying the additional condition
\[
(5.6) \quad \frac{\partial p}{\partial x_i} - \frac{\partial p}{\partial x_j} \in (x_i + x_j)U(\mathfrak{h}_0)
\]
for all \( i < j \leq n \). In view of Proposition 3.1, it is sufficient to prove that if \( p \in \mathcal{A}^0 \), then \( p \) is symmetric and satisfies (5.6).

First, we will prove the last assertion in the case when \( n = 2 \). It follows from Lemma 5.3 and Theorem 5.1 that \( \mathcal{A} \) is generated by \( z_0 = 2x_1 + 2x_2 \), \( \phi_0 = \xi_1 + \xi_2 \), \( \phi_1 = x_2 \xi_1 - x_1 \xi_2 \) and \( z_1 = -\vartheta(\pi(e_{2,1}^{3})) + \frac{1}{2}z_0^2 - \frac{1}{2}z_0 = x_1 x_2 - \xi_1 \xi_2 \). By direct calculation we can check that
\[
\phi_0^2 = \frac{1}{2}z_0, \phi_0 \phi_1 = -\frac{1}{2}z_0 \xi_1 \xi_2, \phi_1^2 = \frac{1}{2}z_0 x_1 x_2, [z_1, \phi_0] = -2\phi_1, [z_1, \phi_1] = 2x_1 x_2 \phi_0.
\]
Let \( \mathcal{A}_0 \) denote the even part of \( \mathcal{A} \). The above relations imply that \( \mathcal{A}_0 \) is a subring in \( \mathbb{C}[z_0, x_1 x_2] \oplus \mathbb{C}[z_0, x_1 x_2][\xi_1 \xi_2] \). Moreover, \( \mathcal{A}_0/(z_0 \mathcal{A}_0) = \mathbb{C}[z_1] \). Therefore \( \mathcal{A}^0 = \mathbb{C} \oplus z_0 \mathbb{C}[z_0, x_1 x_2] \), i.e. \( \mathcal{A}^0 \) consists of symmetric polynomials satisfying (5.6).
Let $p = \vartheta(u)$ for some $u \in W_\chi \subset U(b)$. Then we have
\begin{equation}
\pi(\text{ad} \, e_{i+1, i}(u)) = 0
\end{equation}
and
\begin{equation}
\pi(\text{ad} \, f_{i+1, i}(u)) = 0
\end{equation}
for all $i = 1, \ldots, n - 1$.

Let $\mathfrak{s}_i$ be the subalgebra in $\mathfrak{g}$ generated by $e_{i, i+1}, e_{i, i}, e_{i+1, i}, e_{i+1, i+1}, f_{i, i+1}, f_{i, i}, f_{i+1, i}, f_{i+1, i+1}$. Clearly, $\mathfrak{s}_i$ is isomorphic to $Q(2)$. Note that the orthogonal complement $\mathfrak{s}_i^\perp$ (with respect to the invariant form) is $\text{ad} \, \mathfrak{s}_i$-invariant, $b \cap \mathfrak{s}_i^\perp$ is a Lie subalgebra and, moreover,
\[\pi(\text{ad} \, e_{i+1, i}(u)) = 0, \quad \pi(\text{ad} \, f_{i+1, i}(u)) = 0\]
whenever $u \in U(b \cap \mathfrak{s}_i^\perp)$.

Therefore any $u \in W_\chi$ satisfying (5.7) and (5.8) for a given $i$ can be written in the form $u = \sum u_j v_j$ for some $u_j \in U(\mathfrak{s}_i \cap b)$ satisfying (5.7) and (5.8) and arbitrary $v_j \in U(\mathfrak{s}_i^\perp \cap b)$.

Thus, (5.7) and (5.8) can be checked locally for $\mathfrak{s}_i$. Indeed, if $\vartheta(u) \in U(\mathfrak{h}_0)$, then
\[\vartheta(u) = \sum \vartheta(u_j) \partial(v_j),\]
where $\vartheta(v_j) \in U(\mathfrak{h}_0 \cap \mathfrak{s}_i^\perp) = \mathbb{C}[x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n]$ and $\vartheta(u_j) \in U(\mathfrak{h}_0 \cap \mathfrak{s}_i) = \mathbb{C}[x_i, x_{i+1}]$. Since we already know the result for $Q(2)$, we obtain $\vartheta(u_j)(x_{i+1}, x_i) = \vartheta(u_j)(x_i, x_{i+1})$ and $\partial\vartheta(u_j)/\partial x_i - \partial\vartheta(u_j)/\partial x_{i+1} \in (x_i + x_{i+1})U(\mathfrak{h}_0 \cap \mathfrak{s}_i)$. Therefore $\vartheta(u)$ is invariant under all adjacent transpositions and therefore is symmetric. Moreover,
\[\frac{\partial \vartheta(u)}{\partial x_i} - \frac{\partial \vartheta(u)}{\partial x_{i+1}} \in (x_i + x_{i+1})U(\mathfrak{h}_0).\]
Since $\vartheta(u)$ is symmetric, the last condition implies (5.6) for $\vartheta(u)$. \hfill \Box

Let $\phi_k := \vartheta(\Phi_k)$. Consider $U(\mathfrak{h})$ as a free $U(\mathfrak{h}_0)$-module and let $V$ denote the free submodule generated by $\xi_1, \ldots, \xi_n$. Then $V$ is equipped with $U(\mathfrak{h}_0)$-valued bilinear symmetric form $B(x, y) = [x, y]$. If $\omega = \vartheta(\pi(\sum_{i \leq j} e_{i, j})), \text{ then } T = \text{ad} \, \omega$ is an $U(\mathfrak{h}_0)$-linear operator. As we have seen in the proof of Lemma 5.3, $T$ is skew-symmetric with respect to the form $B$, i.e.
\[B(Tv, w) + B(v, Tw) = 0.\]
Furthermore, in these terms $\phi_k = T^k(\phi_0)$. The matrix of $T$ in the standard basis $\xi_1, \ldots, \xi_n$ has 0 on the diagonal and
\begin{equation}
t_{ij} = \begin{cases} x_j & \text{if } i < j, \\ -x_j & \text{if } i > j. \end{cases}
\end{equation}
Lemma 5.5. The characteristic polynomial $\det(\lambda \text{Id} - T)$ of $T$ equals

$$
\lambda^n + \sigma_2 \lambda^{n-2} + \cdots + \sigma_{2k} \lambda^{n-2k},
$$

where $\sigma_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$ are the elementary symmetric functions.

Proof. Let

$$
p_n(x_1, \ldots, x_n; \lambda) = \det(\lambda \text{Id} - T) = \lambda^n + \sum_{i=1}^n f_n,i(x_1, \ldots, x_n) \lambda^{n-i}.
$$

Note that $f_n,i(x_1, \ldots, x_n)$ is a symmetric polynomial, since the substitutions $x_i \mapsto x_j, x_j \mapsto x_i$ preserves the determinant of $\lambda \text{Id} - T$. It is also easy to calculate that $\det T = x_1 \cdots x_n$ if $n$ is even. If $n$ is odd, then $\det T = 0$, since $T$ is skew-symmetric with respect to $B$. Finally, if $x_n = 0$ we have a relation

$$
p_n(x_1, \ldots, x_{n-1}, 0; \lambda) = \lambda p_{n-1}(x_1, \ldots, x_{n-1}; \lambda).
$$

That implies

$$
f_{n,i}(x_1, \ldots, x_{n-1}, 0) = f_{n-1,i}(x_1, \ldots, x_{n-1}),
$$

for $i \leq n - 1$. Since it is also easy to show that the degree of $f_{n,i}$ is $i$, we can finish the proof by induction in $n$. \hfill \Box

Corollary 5.6. There exists $s = (s_1, \ldots, s_n) \in \mathbb{R}_{>0}^n$ such that the specialization of $\det(\lambda \text{Id} - T)$ at the point $x_1 = s_1, \ldots, x_n = s_n$ has distinct eigenvalues.

Proof. Assume that $n = 2k$ is even. Let $\text{Pol}_{n\text{ev}}^n$ denote the set of monic even polynomials in $\mathbb{C}[\lambda]$ of degree $n$ and $\text{Pol}_{n\text{ev}}^{n+}$ denote the subset of polynomials with real positive coefficients. Let $\varphi : \mathbb{R}_{>0}^n \to \text{Pol}_{n\text{ev}}^{n+}$ be the specialization map, i.e. $\varphi(s)$ is the specialization of $\det(\lambda \text{Id} - T)$ at $s \in \mathbb{R}_{>0}^n$. From the above Lemma, $d\varphi(s)$ is surjective for generic $s \in \mathbb{R}_{>0}^n$. Therefore $\text{Im}\varphi$ contains a non-empty open subset in $\mathbb{R}_{>0}^n$.

Define the map $\rho : \mathbb{C}^n \to \text{Pol}_{n\text{ev}}^n$ by the formula

$$
\rho(t_1, \ldots, t_k) = \prod_{i=1}^k (\lambda^2 - t_i^2).
$$

Obviously, $\rho$ is surjective. Set

$$
\mathcal{U} = \{(t_1, \ldots, t_k) \in \mathbb{C}^k \mid t_i \neq \pm t_j \text{ for all } i \neq j\}.
$$

Then $\rho(\mathcal{U})$ is Zariski open in $\text{Pol}_{n\text{ev}}^n$. Therefore the intersection $\text{Pol}_{n\text{ev}}^{n+} \cap \rho(\mathcal{U})$ is a non-empty Zariski open subset in $\text{Pol}_{n\text{ev}}^{n+}$. Hence $\text{Pol}_{n\text{ev}}^{n+} \cap \rho(\mathcal{U})$ is dense in $\text{Pol}_{n\text{ev}}^{n+}$ in the usual topology and the intersection $\text{Im}\varphi \cap \rho(\mathcal{U})$ is not empty. This implies the statement for even $n$.

For odd $n$ the proof is similar and we leave it to the reader. \hfill \Box

Lemma 5.7. $\phi_0, \ldots, \phi_{n-1}$ are linearly independent over $U(\mathfrak{h}_0)$. 
Proof. For each \( s = (s_1, \ldots, s_n) \in \mathbb{C}^n \) consider the ideal \( I_s = (x_1 - s_1, \ldots, x_n - s_n) \subset U(\mathfrak{h}_0) \). Let \( V_s = V/I_s V \) and \( T_s, B_s \) and \((\phi_i)_s\) denote the corresponding operator, form and vector in \( V_s \). It suffices to show that \((\phi_0)_s, \ldots, (\phi_{n-1})_s\) are linearly independent for some \( s \). By Corollary 5.6 we can find \( s \in \mathbb{R}^n_{>0} \) such that all eigenvalues of \( T_s \) are distinct. Let \( v_1, \ldots, v_n \) denote an eigenbasis for \( T_s \), and let \( H_s \) denote the Hermitian form such that \( H_s(\xi_i, \xi_j) = B_s(\xi_i, \xi_j) \) for all \( i, j = 1, \ldots, n \). Then \( H_s \) is positive definite and \( T_s \) is skew-hermitian with respect to \( H_s \). Hence all eigenvalues of \( T_s \) are purely imaginary and \( H_s(v_i, v_j) = 0 \) if \( i \neq j \). Let \( (\phi_0)_s = \sum_{i=1}^n a_i v_i \). Since all eigenvalues of \( H_s \) are distinct and \((\phi_i)_s = T^i(\phi_0)_s\), linear independence of \((\phi_0)_s, \ldots, (\phi_{n-1})_s\) is equivalent to the fact that \( a_i \) are not zero for all \( i = 1, \ldots, n \).

Assume that some \( a_i = 0 \). Since \( a_i = \frac{H_s(v_i, (\phi_0)_s)}{H_s(v_i, v_i)} \), that implies \( H_s(v_i, (\phi_0)_s) = 0 \). Let \( v_i = t_i \xi_1 + \cdots + t_n \xi_n \). Then the last condition implies \( \sum_{i=1}^n s_i t_i = 0 \). But then the first coordinate of \( T_s v_i \) equals \( s_2 t_2 + \cdots + s_n t_n = -s_1 t_1 \). Since \( T_s v_i = a v_i \) for some purely imaginary \( a \), we obtain \( t_1 = 0 \). Repeating this argument we can prove by induction that all \( t_i \) are zero and obtain a contradiction. \( \square \)

**Problem.** Calculate \( \partial(\Phi_i) \) and \( \partial(z_i) \).

**Lemma 5.8.** The centralizer of \( \mathcal{A} \) in \( U(\mathfrak{h}) \) coincides with \( U(\mathfrak{h}_0) \).

Proof. Suppose that \( u \) lies in the centralizer of \( \mathcal{A} \). Recall that \( F \) denotes the field of fractions of \( U(\mathfrak{h}_0) \). Then since \( U(\mathfrak{h}) \) is a free \( U(\mathfrak{h}_0) \)-module, \( U(\mathfrak{h}) \subset U(\mathfrak{h})_F \). By Lemma 5.7, \( \mathcal{A}_F \) contains \( \xi_1, \ldots, \xi_n \). Hence we have \([\xi_i, u] = 0 \) for all \( i = 1, \ldots, n \). Therefore \( u \) lies in the center of \( U(\mathfrak{h}) \), which coincides with \( U(\mathfrak{h}_0) \). \( \square \)

**Corollary 5.9.** The center of \( \mathcal{A} \) coincides with \( \mathcal{A}^0 \).

**Proposition 3.1, Lemma 5.4 and Corollary 5.9 imply.**

**Corollary 5.10.** The center of \( W_x \) coincides with \( \pi(Z(Q(n))) \).

6. **Connection with Super-Yangians**

6.1. **Super-Yangian of \( Q(N) \).** Super-Yangian \( Y(Q(N)) \) was introduced by M. Nazarov in [20]. Recall that \( Y(Q(N)) \) is the associative unital superalgebra over \( \mathbb{C} \) with the countable set of generators

\[
T^{(m)}_{i,j} \quad \text{where} \quad m = 1, 2, \ldots \quad \text{and} \quad i, j = \pm 1, \pm 2, \ldots, \pm N.
\]

The \( \mathbb{Z}_2 \)-grading of the algebra \( Y(Q(N)) \) is defined as follows:

\[
p(T^{(m)}_{i,j}) = p(i) + p(j), \quad \text{where} \quad p(i) = 0 \text{ if } i > 0, \quad \text{and} \quad p(i) = 1 \text{ if } i < 0.
\]

To write down defining relations for these generators we employ the formal series in \( Y(Q(N))[u^{-1}] \):

\[
T_{i,j}(u) = \delta_{ij} \cdot 1 + T^{(1)}_{i,j} u^{-1} + T^{(2)}_{i,j} u^{-2} + \ldots
\]

Then for all possible indices \( i, j, k, l \) we have the relations
One can easily verify the following recursive relations:

\[ (u^2 - v^2) [T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) - (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}, \]

where \( v \) is a formal parameter independent of \( u \), so that (6.2) is an equality in the algebra of formal Laurent series in \( u^{-1}, v^{-1} \) with coefficients in \( Y(Q(N)) \).

For all indices \( i, j \) we also have the relations

\[ T_{i,j}(-u) = T_{-i,-j}(u). \]

Note that the relations (6.2) and (6.3) are equivalent to the following defining relations:

\[
\begin{align*}
(6.4) \quad & \left( [T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]\right) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = \\
& T_{k,j}^{(m)}T_{i,l}^{(r-1)} - T_{i,j}^{(m-1)}T_{k,l}^{(r)} - T_{k,j}^{(m-1)}T_{i,l}^{(r-1)} - T_{i,j}^{(m)}T_{k,l}^{(r)} + (-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)}T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)}T_{-i,l}^{(r)} + T_{-k,-j}T_{i,-l}^{(r-1)} - T_{k,-j}T_{i,-l}^{(m-1)}),
\end{align*}
\]

\[ (6.5) \quad T_{-i,-j}^{(m)} = (-1)^m T_{i,j}^{(m)}, \]

where \( m, r = 1, \ldots \) and \( T_{i,j}^{(0)} = \delta_{i,j} \).

Recall that \( Y(Q(N)) \) is a Hopf superalgebra, see [21], with comultiplication given by the formula

\[
\Delta(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T_{i,k}^{(s)} \otimes T_{k,j}^{(r-s)}.
\]

We will use the opposite comultiplication

\[
\Delta_{op}(T_{i,j}^{(r)}) = \sum_{s=0}^{r} \sum_{k} T_{k,j}^{(r-s)} \otimes T_{i,k}^{(s)}.
\]

Observe that the map \( \Delta_{op} : Y(Q(N)) \rightarrow Y(Q(N))^\otimes n \)

\[
\Delta_{op} := \Delta_{op}^{n=1,n} \circ \cdots \circ \Delta_{op}^{n=2,3} \circ \Delta_{op}
\]

is a homomorphism of associative algebras.

At the moment we are interested in the case \( N = 1 \). Set

\[ T^{(r)} := T^{(r)}_{1,1} + T^{(r)}_{-1,1}. \]

One can easily verify the following recursive relations

\[
(6.6) \quad \Delta_{op}^{n}(T^{(r)}) = \sum_{s=0}^{r} \left( T_{1,1}^{(r-s)} + (-1)^s T_{-1,1}^{(r-s)} \right) \otimes \Delta_{op}^{n-1}(T^{(s)}).
\]
It follows from [22] that one can define a homomorphism \( U : Y(Q(1)) \to U(Q(1)) \) by
\[
T^{(r)}_{1,1} \mapsto (-1)^r e^{(r)}_{1,1}, \quad T^{(r)}_{-1,1} \mapsto (-1)^r f^{(r)}_{1,1}.
\]
As a direct consequence of (4.1) we obtain
\[
(6.7) \quad U(T^{(r)}) = (-1)^r \prod_{i=1}^r (e_{1,1} + (-1)^{i-1} f_{1,1}).
\]

Lemma 6.1. Identify \( U(\mathfrak{h}) \subset U(Q(n)) \) with \( U(Q(1))^{\otimes n} \) by setting
\[
x_i \mapsto 1^{\otimes n-i} \otimes e_{1,1} \otimes 1^{\otimes i-1}, \quad \xi_i \mapsto 1^{\otimes n-i} \otimes f_{1,1} \otimes 1^{\otimes i-1}.
\]
Then
\[
U^{\otimes n} \circ \Delta^\text{op} n(T^{(k)}) = (-1)^k \sum_{i_1 \geq i_2 \geq \ldots \geq i_k} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \ldots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})
\]

Proof. Follows from (6.7) and (6.6).

Theorem 6.2. There exists a surjective homomorphism:
\[
\varphi : Y(Q(1)) \to W_x
\]
defined as follows:
\[
(6.8) \quad \varphi(T^{(k)}_{1,1}) = (-1)^k \pi(e_{n,1}^{(n+k-1)}), \quad \varphi(T^{(k)}_{-1,1}) = (-1)^k \pi(f_{n,1}^{(n+k-1)}), \quad \text{for } k = 1, 2, \ldots.
\]

Proof. Recall that the Harish-Chandra homomorphism \( \vartheta : W_x \to U(\mathfrak{h}) \) is injective. Note that by Lemma 6.1 and Proposition 5.1 we have
\[
(-1)^k \vartheta(\pi(e_{n,1}^{(n+k-1)})) = U^{\otimes n} \circ \Delta^\text{op} n(T^{(k)}_{1,1}), \quad (-1)^k \vartheta(\pi(f_{n,1}^{(n+k-1)})) = U^{\otimes n} \circ \Delta^\text{op} n(T^{(k)}_{-1,1}).
\]
Hence \( \varphi = \vartheta^{-1} \circ U^{\otimes n} \circ \Delta^\text{op} n \) is a surjective homomorphism \( \varphi : Y(Q(1)) \to W_x \).

Remark 6.3. The proof of above theorem uses the same method as in [6] but we start with \( U : Y(Q(1)) \to U(Q(1)) \) instead of the evaluation map. It is desirable to construct a homomorphism \( Y(Q(1)) \to W_x \) using the evaluation map, but at the moment we do not have suitable generators in \( W_x \).

Proposition 6.4. If \( k + m \) is even, then
\[
(6.9) \quad [T^{(k)}_{1,1}, T^{(m)}_{1,1}] = 0.
\]

Proof. According to (6.4) and (6.5)
\[
(6.10) \quad [T^{(1)}_{1,1}, T^{(k)}_{1,1}] = 0, \quad [T^{(k)}_{1,1}, T^{(k)}_{-1,1}] = ((-1)^k - 1)T^{(k)}_{1,1}.
\]
It follows from (6.4) by induction that if \( m \geq 2 \), then
\[
(6.11) \quad [T^{(m)}_{1,1}, T^{(k)}_{1,1}] = - \sum_{r=1}^{m-1} [T^{(k+r-1)}_{1,1}, T^{(m-r)}_{1,1}] + \sum_{r=1}^{m-1} (-1)^r ((-1)^k T^{(k+r-1)}_{-1,1} T^{(m-r)}_{-1,1} + T^{(m-r)}_{-1,1} T^{(k+r-1)}_{-1,1}).
\]
It follows from (6.10) and (6.11) that if $m$ is even, then $[T_{1,1}^{(2)}, T_{1,1}^{(m)}] = 0$.

We will prove (6.9) by induction on $l = k + m$. Obviously, if $l = 2$, then (6.9) is true. Assume that the statement is true if $k + m$ is even and $k + m \leq l - 2$.

Assume that $k$ and $m$ are even and $k, m \geq 4$, or $k$ and $m$ are odd and $k, m \geq 3$.

Denote by $[T_{1,1}^{(m)}, T_{1,1}^{(k)}]_{ev}$ and by $[T_{1,1}^{(m)}, T_{1,1}^{(k)}]_{odd}$ the first and the second sum in (6.11), respectively. Thus

$$[T_{1,1}^{(m)}, T_{1,1}^{(k)}] = [T_{1,1}^{(m)}, T_{1,1}^{(k)}]_{ev} + [T_{1,1}^{(m)}, T_{1,1}^{(k)}]_{odd}.$$ 

Also, denote by $[T_{-1,1}^{(m)}, T_{-1,1}^{(k)}]_{odd}$ and by $[T_{-1,1}^{(m)}, T_{-1,1}^{(k)}]_{ev}$ the first and the second sum in (6.12), respectively. Thus

$$[T_{-1,1}^{(m)}, T_{-1,1}^{(k)}] = [T_{-1,1}^{(m)}, T_{-1,1}^{(k)}]_{odd} + [T_{-1,1}^{(m)}, T_{-1,1}^{(k)}]_{ev}.$$ 

From (6.10) and (6.11)

$$[T_{1,1}^{(k)}, T_{1,1}^{(m)}]_{ev} = -([T_{1,1}^{(m)}, T_{1,1}^{(k-1)}] + [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}] + \ldots + [T_{1,1}^{(m+k-3)}, T_{1,1}^{(2)}]),$$

(6.13)

$$[T_{1,1}^{(k)}, T_{1,1}^{(m)}]_{odd} = -([T_{-1,1}^{(m)}, T_{-1,1}^{(k-1)}] - [T_{-1,1}^{(m+1)}, T_{-1,1}^{(k-2)}] + \ldots - (-1)^k[T_{-1,1}^{(m+k-3)}, T_{1,1}^{(2)}].$$

(6.14)

Let

$$A_n = [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}] + [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}] + [T_{-1,1}^{(m)}, T_{-1,1}^{(k-1)}] - [T_{-1,1}^{(m+1)}, T_{-1,1}^{(k-2)}].$$

We claim that $A_n = 0$. Note that $A_n = A_{ev} + A_{odd}$, where

$$A_{ev} = [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{ev} + [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{ev} + [T_{-1,1}^{(m)}, T_{-1,1}^{(k-1)}]_{ev} - [T_{-1,1}^{(m+1)}, T_{-1,1}^{(k-2)}]_{ev}.$$ 

$$A_{odd} = [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} + [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} + [T_{-1,1}^{(m)}, T_{-1,1}^{(k-1)}]_{odd} - [T_{-1,1}^{(m+1)}, T_{-1,1}^{(k-2)}]_{odd}.$$ 

Let us show that $A_{ev} = 0$. Note that

$$[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{ev} = -[T_{1,1}^{(k-1)}, T_{1,1}^{(m-1)}] - [T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] - [T_{1,1}^{(k+1)}, T_{1,1}^{(m-3)}] - [T_{1,1}^{(k+2)}, T_{1,1}^{(m-4)}] - \ldots;$$

$$[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{ev} = [T_{1,1}^{(k-1)}, T_{1,1}^{(m-1)}] - [T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] + [T_{1,1}^{(k+1)}, T_{1,1}^{(m-3)}] - [T_{1,1}^{(k+2)}, T_{1,1}^{(m-4)}] + \ldots.$$ 

Thus

$$[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{ev} + [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{ev} = -2[T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] - 2[T_{1,1}^{(k+2)}, T_{1,1}^{(m-4)}] - \ldots.$$
By induction hypothesis
\[
[T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] = [T_{1,1}^{(k+2)}, T_{1,1}^{(m-4)}] = \ldots = 0.
\]

Similarly
\[
[T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{ev} - [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{ev} = -2[T_{1,1}^{(k-2)}, T_{1,1}^{(m)}] - 2[T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] = \ldots.
\]

By induction hypothesis
\[
[T_{1,1}^{(k-2)}, T_{1,1}^{(m)}] = [T_{1,1}^{(k)}, T_{1,1}^{(m-2)}] = \ldots = 0.
\]

Hence \(A_{ev}^m = 0\). Let us show that \(A_{odd}^m = 0\). Note that
\[
[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} = (T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) - (T_{1,1}^{(k)} T_{1,1}^{(m-2)} - T_{1,1}^{(m-2)} T_{1,1}^{(k)}) + \ldots
\]
\[
+ (-1)^k T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)},
\]
\[
[T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} = (T_{1,1}^{(k-2)} T_{1,1}^{(m)} - T_{1,1}^{(m)} T_{1,1}^{(k-2)}) - (T_{1,1}^{(k)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
+ (-1)^{k+1} T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)},
\]
\[
[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} = (T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + (T_{1,1}^{(k)} T_{1,1}^{(m-2)} - T_{1,1}^{(m-2)} T_{1,1}^{(k)}) + \ldots
\]
\[
+ (T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} - T_{1,1}^{(k+1)} T_{1,1}^{(k+m-3)}),
\]
\[
[T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} = (T_{1,1}^{(k-2)} T_{1,1}^{(m)} - T_{1,1}^{(m)} T_{1,1}^{(k-2)}) - (T_{1,1}^{(k)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
- (T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} - T_{1,1}^{(k+1)} T_{1,1}^{(k+m-3)}).
\]

Then if \(k\) and \(m\) are even, we have that
\[
[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} + [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} = 2(T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
2(T_{1,1}^{(k+m-3)} T_{1,1}^{(k)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)}),
\]
\[
[T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} - [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} = -2(T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
+ 2(T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)}).
\]

Hence \(A_{odd}^m = 0\). Then if \(k\) and \(m\) are odd, we have that
\[
[T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} + [T_{1,1}^{(m)}, T_{1,1}^{(k-1)}]_{odd} = 2(T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
2(T_{1,1}^{(k-1)} T_{1,1}^{(m-3)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)}) + \ldots + 2(T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(k-1)} T_{1,1}^{(k-1)}),
\]
\[
[T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} - [T_{1,1}^{(m+1)}, T_{1,1}^{(k-2)}]_{odd} = -2(T_{1,1}^{(k-1)} T_{1,1}^{(m-1)} - T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}) + \ldots
\]
\[
+ 2(T_{1,1}^{(k+m-3)} T_{1,1}^{(k+1)} T_{1,1}^{(m-1)} T_{1,1}^{(k-1)}).
\]

Hence \(A_{odd}^m = 0\). Then \(A^m = 0\). Similarly,
\[
A^{m+2} = A^{m+4} = \ldots = 0.
\]
Note that each of the sums in (6.13) and (6.14) has \( k - 2 \) terms. Then if \( k \) and \( m \) are even, we have that
\[
[T_1^{(k)}, T_1^{(m)}] = -(A^m + A^{m+2} + \ldots + A^{m+k-4}) = 0.
\]
If \( k \) and \( m \) are odd and \( k \geq 5 \), then
\[
[T_1^{(k)}, T_1^{(m)}] = -(A^m + A^{m+2} + \ldots + A^{m+k-5}) - [T_1^{(m+k-3)}, T_1^{(2)}] - [T_1^{(m+k-3)}, T_1^{(2)}] - [T_1^{(m+k-3)}, T_1^{(2)}].
\]
If \( m \) is odd and \( k = 3 \), then
\[
[T_1^{(k)}, T_1^{(m)}] = -[T_1^{(m+k-3)}, T_1^{(2)}] - [T_1^{(m+k-3)}, T_1^{(2)}].
\]
Finally, let us show that
\[
(6.15) [T_1^{(m+k-3)}, T_1^{(2)}] + [T_1^{(m+k-3)}, T_1^{(2)}] = 0.
\]
From (6.11)
\[
[T_1^{(m+k-3)}, T_1^{(2)}]_{ev} = -[T_1^{(2)}, T_1^{(k+m-4)}] - [T_1^{(3)}, T_1^{(k+m-5)}] - \ldots,
\]
and from (6.12)
\[
[T_1^{(m+k-3)}, T_1^{(2)}]_{ev} = [T_1^{(2)}, T_1^{(k+m-4)}] - [T_1^{(3)}, T_1^{(k+m-5)}] + \ldots.
\]
Hence
\[
(6.16) [T_1^{(m+k-3)}, T_1^{(2)}]_{ev} + [T_1^{(m+k-3)}, T_1^{(2)}]_{ev} = -2[T_1^{(3)}, T_1^{(k+m-5)}] - 2[T_1^{(5)}, T_1^{(k+m-7)}] - \ldots = 0,
\]
since all terms are zero by induction hypothesis. From (6.11)
\[
[T_1^{(m+k-3)}, T_1^{(2)}]_{odd} = (T_1^{(2)} T_1^{(k+m-4)} - T_1^{(k+m-4)} T_1^{(2)}) - (T_1^{(3)} T_1^{(k+m-5)} - T_1^{(k+m-5)} T_1^{(3)}) +
(T_1^{(4)} T_1^{(k+m-6)} - T_1^{(k+m-6)} T_1^{(4)}) - (T_1^{(5)} T_1^{(k+m-7)} - T_1^{(k+m-7)} T_1^{(5)}) + \ldots +
(T_1^{(k+m-4)} T_1^{(2)} - T_1^{(2)} T_1^{(k+m-4)}) - (T_1^{(k+m-3)} T_1^{(1)} - T_1^{(1)} T_1^{(k+m-3)}),
\]
and from (6.12)
\[
[T_1^{(m+k-3)}, T_1^{(2)}]_{odd} = (T_1^{(2)} T_1^{(k+m-4)} - T_1^{(k+m-4)} T_1^{(2)}) + (T_1^{(3)} T_1^{(k+m-5)} - T_1^{(k+m-5)} T_1^{(3)}) +
(T_1^{(4)} T_1^{(k+m-6)} - T_1^{(k+m-6)} T_1^{(4)}) + (T_1^{(5)} T_1^{(k+m-7)} - T_1^{(k+m-7)} T_1^{(5)}) + \ldots +
(T_1^{(k+m-4)} T_1^{(2)} - T_1^{(2)} T_1^{(k+m-4)}) + (T_1^{(1)} T_1^{(k+m-3)} - T_1^{(k+m-3)} T_1^{(1)}).
\]
Hence
\[
(6.17) [T_1^{(m+k-3)}, T_1^{(2)}]_{odd} + [T_1^{(m+k-3)}, T_1^{(2)}]_{odd} = 2(T_1^{(2)} T_1^{(k+m-4)} - T_1^{(k+m-4)} T_1^{(2)}) +
2(T_1^{(4)} T_1^{(k+m-6)} - T_1^{(k+m-6)} T_1^{(4)}) + \ldots + 2(T_1^{(k+m-4)} T_1^{(2)} - T_1^{(2)} T_1^{(k+m-4)}) = 0.
\]
Then (6.15) follows from (6.16) and (6.17).
Remark 6.5. If $k + m$ is odd, then $T_{1,1}^{(k)}$ and $T_{1,1}^{(m)}$ might not commute. For example, one can check that $[T_{1,1}^{(2)}, T_{1,1}^{(3)}] = T_{-1,1}^{(3)}T_{1,1}^{(2)} - T_{-1,1}^{(2)}T_{1,1}^{(3)} \neq 0$, since $U^{\otimes 3} \circ \Delta^e_0 \left( T_{1,1}^{(3)}T_{1,1}^{(2)} - T_{-1,1}^{(2)}T_{1,1}^{(3)} \right) \neq 0$.

6.2. New generators and relations. Recall that in Lemma 5.3 (c) we defined elements $z_i \in \pi(Z(Q(n)))$ for even $i \geq 0$. We set $z_i = \pi(e^{(n+i)}_{n,1})$ for odd $i$.

Theorem 6.6. Elements $z_0, \ldots, z_{n-1}$ are algebraically independent in $W_\chi$ and they commute with each other. Together with $\Phi_0, \ldots, \Phi_{n-1}$ they form a complete set of generators in $W_\chi$.

Proof. It follows from (6.9) and (6.8) that if $k + m$ is even, then $[\pi(e^{(n+k)}_{n,1}), \pi(e^{(n+m)}_{n,1})] = 0$.

Hence $[z_i, z_j] = 0$ for odd $i, j \geq 1$. By Lemma 5.3, $z_i \in \pi(Z(Q(n)))$ for even $i \geq 0$, and we have that $P(\Phi_i) = H_i$ for $0 \leq i \leq n - 1$, $P(z_i) = e^i$ for even $0 < i \leq n - 1$ and $P(z_0) = z$. By Lemma 4.7, $P(z_i) = e^i$ for odd $i \leq n - 1$. Therefore the second assertion follows from Proposition 2.7. The algebraic independence of $z_0, \ldots, z_{n-1}$ follows from algebraic independence of the corresponding elements in $S(g^\chi)$. \hfill \Box

Conjecture 6.7. Let $g$ be a basic classical Lie superalgebra and $\chi$ be regular. Then it is possible to find a set of generators of $W_\chi$ such that even generators commute, and the commutators of odd generators are in $\pi(Z(g))$.

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