Entanglement entropy and the Ricci flow

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Abstract

We analyze the Ricci flow of a noncompact metric that describes a two-dimensional black hole. We consider entanglement entropy of a 2d black hole which is due to the quantum correlations between two subsystems: one is inside and the other is outside the black hole horizon. It is demonstrated that the entanglement entropy is monotonic along the Ricci flow.
1 Introduction

The irreversibility of the observable phenomena has always been an intriguing fact that needed an explanation. In quantum field theory the irreversibility is usually associated with the Renormalization Group (RG) flow. Although it is very difficult to give a general proof of the irreversibility this proof exists in two space-time dimensions and is known as the c-theorem [1]. For 2d $\sigma$-models, as those arising in string theory, this proof assumes compactness of the target space [2], [3]. Generalization of the c-theorem to higher dimensions and more general class of theories is not straightforward. In general, a rather complicated and even chaotic behavior of the RG flow is possible [4].

The irreversibility of time evolution is usually associated with the notion of entropy in a coarse-grained description of system when the detail knowledge about the individual states is lost for the advantage of having an effective description of a huge number of such states. Yet the entropy can be associated with geometry in the case the coarse-grained states are residing entirely inside a compact surface. This entropy is known as entanglement (or geometric) entropy and is due to the quantum correlations between the subsystems divided by the surface. Entanglement entropy is determined by geometry of the surface and, in this sense, reminds the Bekenstein-Hawking entropy of black holes. In fact, the study of entanglement entropy in the past was mainly motivated by the hope to find a statistical-mechanical explanation to the black hole entropy [5]-[16]. Recently, entanglement entropy has been in the focus of studies in condensed matter as an appropriate quantity that measures quantum entanglement in the quantum-mechanical systems [17]-[19]. Although a great deal is known about entanglement entropy of various systems its irreversibility was not fully revealed. A natural guess is that the irreversibility should be understood with respect to the RG flow. A relation between entanglement entropy in two-dimensional field theory and the c-theorem was discussed in [20]. On the other hand, the recent work [19] investigates how entanglement entropy of certain two-dimensional spin chain models changes under the RG flow. It is found that the entanglement entropy is monotonically decreasing along the RG trajectory in this case. This is interpreted as loss of entanglement in the infra-red regime. The RG flow is not directly relevant to entanglement entropy of black holes since the entropy is entirely due to the presence of the black hole and is nontrivial even in the case when interactions, and, hence, the non-trivial renormalization group flow are absent in the field system.

In this note we suggest that entanglement entropy of a black hole decreases monotonically under the Ricci flow. The latter is analogous to the RG flow. These two flows are, in fact, related for the two-dimensional $\sigma$-models with the Riemann manifold as target space [21]. The Ricci flow [22], [23] plays an important role in the problem of geometrization of 3-dimensional manifolds. The recent enthusiasm regarding the Ricci flow has been sparked of course by works of Perelman [24]. For elementary review on the Ricci flow see [25], [26], a review on the recent development is [27]. Some applications of the Ricci flow and, in particular, of Perelman’s results to the physics models are discussed in [28]-[35]. Our main focus in this note is on the two-dimensional case when the metric evolving under the Ricci flow describes a 2d black hole. We first remind the reader some facts about entanglement entropy and present an exact expression for the entropy of 2d black hole which is due to a two-dimensional conformal field. We then discuss the properties of the Ricci flow in two dimensions putting focus on whether the property of metric to describe black hole changes under the flow. We finally show that entanglement entropy is monotonically decreasing along the Ricci flow provided the scalar curvature of the initial metric is positive. Our result looks similar to the one obtained in [19] for the RG flow and, perhaps, there is a reason why the entanglement entropy should decrease.
2 Entanglement entropy of 2d black holes

Entanglement entropy is defined for a system divided into two subsystems (L and R) and is due to the short-range correlations that are present in the system. The notion of entanglement entropy is ideally suited for black holes since the black hole horizon naturally divides the space-time on two subsystems so that an observer outside a black hole does not have access to excitations propagating inside the horizon. Assume, that the whole system of a quantum field $\phi$, that takes values $\phi_L$ and $\phi_R$ in the subsystems L and R respectively, is in a pure ground state

$$\Psi_0 = \Psi_0(\phi_R, \phi_L) \tag{2.1}$$

that is the functional of both $\phi_R$ and $\phi_L$ modes. For an observer who has access only to one subsystem, say R, it is more reasonable to introduce a density matrix

$$\rho(\phi^1_R, \phi^2_R) = \int [D\phi_L] \Psi_0^*(\phi^1_R, \phi_L) \Psi_0(\phi^2_R, \phi_L) \tag{2.2}$$

where one traces over all modes $\phi_L$. Entanglement entropy is defined as

$$S = -\text{Tr } \hat{\rho} \ln \hat{\rho} = (-\alpha \partial_{\alpha} + 1)\text{Tr } \rho^\alpha|_{\alpha=1}, \quad \hat{\rho} = \frac{\rho}{\text{Tr } \rho} \tag{2.3}$$

Applying this construction to a 2d black hole (for a general case see [11]), we identify all modes inside the black hole horizon with L-modes that are to be traced over. The ground state of the black hole is given by the Euclidean functional integral [12] over fields defined on a half of the Euclidean black hole space-time

$$ds^2_{E'} = \beta_H^2 f(x) d\varphi^2 + \frac{1}{f(x)} dx^2 \tag{2.4}$$

defined for values $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ of angular coordinate $\varphi$. The inverse Hawking temperature $\beta_H = T_H^{-1}$ is determined by the derivative of the metric function $f(x)$ on the horizon ($f(x_+) = 0$, $\beta_H = \frac{4\pi}{\int f(x) dx}$). Functions $\phi_R$ and $\phi_L$ that enter as arguments in (2.1) are the fixed values at the boundaries $\phi_R = \phi(\varphi = \frac{\pi}{2})$; $\phi_L = \phi(\varphi = -\frac{\pi}{2})$, giving the boundary condition in the path integral. The density matrix $\rho(\phi^1_R, \phi^2_R)$, obtained by tracing over $\phi_L$-modes, is defined by the path integral over fields on the full black hole instanton $E$, ($-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$), with a cut along the $\varphi = \frac{\pi}{2}$ axis and taking values $\phi_{R,1,2}$ above and below the cut. The trace $\text{Tr } \rho$ is obtained by equating the fields across the cut and doing the unrestricted Euclidean path integral on the complete black hole instanton $E$. Analogously, $\text{Tr } \rho^n$ is given by the path integral over fields defined on $E_n$, a $n$-fold cover of $E$. Thus, $E_n$ is the manifold with an abelian isometry (generated by vector $\partial_\varphi$) with horizon $\Sigma$ as a stationary point. Near $\Sigma$ the $E_n$ looks as a cone $E_n = C_n$ with tip at $\Sigma$ and the angle deficit $\delta = 2\pi(1-n)$. This construction can be analytically continued to arbitrary (non-integer) $n \rightarrow \alpha = \frac{n}{\beta_H}$.

The calculation of entanglement entropy thus reduces to a calculation of the functional integral on a gravitational background with a conical singularity. In two dimensions, for a conformal field, the result of the functional integration is the non-local Polyakov action. The entropy calculation thus can be carried out explicitly. For a black hole metric written in the conformal form $g_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$ entanglement entropy takes the form [13], [14] $S = \frac{\epsilon}{6}(x_+) + \frac{\epsilon}{6} \ln(\Lambda/\epsilon)$, where $x_+$ is the location of the horizon and $\epsilon$ is an UV regulator. The complete black hole geometry can be described by a 2d metric in the Schwarzschild like form

$$ds^2_{bh} = f(x) d\tau^2 + \frac{1}{f(x)} dx^2 \tag{2.5}$$
where the metric function $f(x)$ has a simple zero in $x = x_+; x_+ \leq x \leq L$, $0 \leq \tau \leq \beta_H$, $\beta_H = \frac{4\pi}{f'(x_+)}$. It is easy to see that (2.5) is conformal to the flat disk of radius $z_0$ ($\ln z = \frac{2\pi}{\beta_H} \int f(x) dx$):

$$ds^2_{bh} = e^{2\sigma} z_0^2(dx^2 + z^2d\tilde{\tau}^2),$$

$$\sigma = \frac{1}{2} \ln f(x) - \frac{2\pi}{\beta_H} \int f(x) dx + \ln \frac{\beta_H}{2\pi z_0},$$

where $\tilde{\tau} = \frac{2\pi \tau}{\beta_H}$ ($0 \leq \tilde{\tau} \leq 2\pi$), $0 \leq z \leq 1$. So that entanglement entropy of a 2d black hole takes the form [16] (see also [10] and [15])

$$S = \frac{c}{12} \int_{x_+}^L dx \left( \frac{4\pi}{\beta_H^2} f' - f' \right) + \frac{c}{6} \ln \left( \frac{\beta_H f^{1/2}(L)}{\epsilon} \right),$$

where we omitted the irrelevant term which is function of $(\Lambda, z_0)$ but not of the parameters of the black hole and have retained dependence on UV regulator $\epsilon$.

In the above analysis it is assumed that the black hole resides inside a finite size box and $L$ is the value of $x$-coordinate of the boundary of the box. The coordinate invariant size of the subsystem outside the black hole is $L_{inv} = \int_{x_+}^L dx/\sqrt{f(x)}$. Entanglement entropy is a monotonic function of the size of the box,

$$\partial_{L_{inv}} S = \frac{c}{12} \frac{4\pi}{\beta_H^2} \frac{1}{\sqrt{f(L)}} > 0.$$

For small $L_{inv}$ one finds an expansion [36]

$$S = \frac{c}{6} \left( \ln \frac{L_{inv}}{\epsilon} - \frac{R(x_+)}{36} L_{inv}^2 \right),$$

where $R(x_+)$ is the value of the Ricci scalar at the horizon. If the black hole space-time is asymptotically flat then entanglement entropy approaches the entropy of the two-dimensional thermal gas, $S_{th} = \frac{c\pi}{2} L_{inv} T_H$, in the limit of large $L_{inv}$. Thus, entropy (2.7), in spite the fact that it is essentially due to the presence of the black hole horizon, contains an extensive component that carries information about the global structure of the black hole geometry.

### 3 The Ricci flow in two dimensions

**The Ricci flow of static metric.** In two dimensions the Ricci flow takes the form

$$\partial_\lambda g_{ij} = -R g_{ij},$$

where $R$ is the Ricci scalar of metric $g_{ij}$. We choose that initial (at $\lambda = 0$) metric describes a static black hole in the Schwarzschild like form (2.5)

$$ds^2 = g(x) d\tau^2 + \frac{1}{g(x)} dx^2.$$

The metric function $g(x)$ is positive everywhere except the horizon where it becomes zero. If black hole is non-degenerate than near this point, say $x = x_+$, one has that $g(x) = b_0(x-x_+) + O((x-x_+)^2)$ so that the Hawking temperature $T_H = b_0/4\pi$ is non-vanishing. Metric (3.2) has a
Killing vector $\partial_\tau$ that generates the shifts along the time direction. In the Euclidean case the $\tau$ coordinate should be identified with period $1/T_H$ so that the Killing vector generates rotations. Asymptotically flat ($g \to 1$ at infinity) metric (3.2) is topologically disk. Geometrically it looks like a cigar with a tip (where metric is completely regular) at the horizon.

For $\lambda > 0$ the rotational symmetry is preserved so that the appropriate form for the metric is

$$ds^2 = g(x, \lambda)d\tau^2 + \frac{e^{2\phi(x, \lambda)}}{g(x, \lambda)}dx^2 \quad ,$$

where $g(x, \lambda)$ and $\phi(x, \lambda)$ are now functions of both $x$ and $\lambda$. The “initial” value for $\phi$ is $\phi(x, \lambda = 0) = 0$. The Ricci scalar for metric (3.3) reads

$$R = e^{-2\phi}(g'' + \phi' g') \quad .$$

The Ricci flow (3.1) for metric (3.3) reduces to a set of equations

$$\partial_{\lambda} g = -g e^{-2\phi}(-g'' + \phi' g') \quad ,$$

$$\partial_{\lambda} e^{2\phi} = 2(g'' - \phi' g') \quad .$$

Combining these two equations one finds that

$$\frac{\partial_{\lambda} g}{g} = \partial_{\lambda} \phi \quad .$$

Taking into account the “initial” condition for $\phi(x, \lambda)$ one has

$$e^{\phi(x, \lambda)} = \frac{g(x, \lambda)}{g(x)} \quad .$$

So that the metric (3.3) takes the form

$$ds^2 = g(x, \lambda)\left(d\tau^2 + \frac{1}{g^2(x)}dx^2\right) \quad .$$

The function $g(x, \lambda)$ satisfies equation

$$\partial_{\lambda} g = g_0 \left(\frac{g_0 g'_x}{g}\right)'_x \quad ,$$

where we denoted $g_0(x) \equiv g(x)$.

**Transformation to the Schwarzschild like form. The Ricci-DeTurck flow.** For $\lambda \neq 0$ metric (3.8) does not take the Schwarzschild like form (2.5) anymore. But it can be brought to this form by a $\lambda$-dependent coordinate transformation. Introduce a new coordinate $y$,

$$y = \int_{x+}^{x} \frac{g(x, \lambda)}{g_0(x)} dx \quad , \quad \partial_{\lambda} y = (g'_y - b_0) \quad .$$

We choose limits of integration in (3.10) so that $y = 0$ corresponds to $x = x_+$, the location of the horizon in the initial metric (3.2). In terms of the coordinate $y$ metric (3.8) takes the Schwarzschild like form (2.5)

$$ds^2 = g(y, \lambda)d\tau^2 + \frac{1}{g(y, \lambda)}dy^2 \quad .$$
The flow equation for $g(y, \lambda)$ reads
\[ \partial_\lambda g = gg'' - g'^2 - g^'b_0, \tag{3.12} \]
where $b_0 = g'_0(x)|_{x=x_+}$.

Since we applied a $\lambda$-dependent coordinate transformation (3.10) when derived metric (3.11) the components of the transformed metric satisfy a modified flow equation (3.12) known as the Ricci-DeTurck flow,
\[ \partial_\lambda g_{ij} = -Rg_{ij} - \nabla_i \xi_j - \nabla_j \xi_i, \tag{3.13} \]
where $\xi_i$ is the deformation vector. In the case at hand we have
\[ \xi_\tau = 0, \quad \xi_y = \frac{1}{g(y, \lambda)}(g'_y - b_0). \tag{3.14} \]

Note that this is a gradient vector,
\[ \xi_t = -\partial_\tau \psi(y, \lambda), \quad \xi_y = -\partial_y \psi(y, \lambda), \quad \psi(y, \lambda) = \int_y^\infty \frac{dy}{g(y', \lambda)}(b_0 - g'_y). \tag{3.15} \]

In fact, $\psi$ is a rotation invariant solution to equation [10], [15]
\[ \Delta \psi = R, \quad R = -g'' \tag{3.16} \]
and is a non-local object, $\psi = \frac{1}{\lambda^2} R$. Choosing the integration constant in (3.15) appropriately, one can express entanglement entropy (2.7) in terms of $\psi(y)$ [15],
\[ S = -\frac{c}{12} \psi(0) + \frac{c}{6} \ln(1/\epsilon). \tag{3.17} \]

For metric (3.8) we have that $\psi(x, \lambda) = \ln \frac{g(x)}{g(x, \lambda)} + \psi_0(x)$, where $\psi_0(x)$ is a solution to equation (3.16) for metric $g_0(x)$.

**The near-horizon analysis.** Assume that near $y = 0$ the function $g(y, \lambda)$ can be presented in the form of the Taylor series
\[ g(y, \lambda) = \sum_{n=0}^\infty a_n(\lambda)y^n, \tag{3.18} \]
where we included the constant term with $n = 0$. This term is absent in the initial metric, $a_0 = 0$ at $\lambda = 0$, but may appear for some $\lambda > 0$. If it appears then this means that the point $y = 0$ is not a horizon anymore. As we show below, this does not happen and the point $y = 0$ stays to be horizon for all $\lambda > 0$.

Substituting expansion (3.18) into equation (3.12) we get the flow equations for the coefficients in the Taylor series
\[
\begin{align*}
n &= 0 : & \partial_\lambda a_0 &= (b_0 - a_1)a_1 + 2a_0a_2 \\
n &= 1 : & \partial_\lambda a_1 &= 2(b_0 - a_1)a_2 + 6a_3a_0 \\
n &> 1 : & \partial_\lambda a_n &= (n + 2)(n + 1)a_0a_{n+2} + (n + 1)(b_0 - a_1)a_{n+1} \\
& & & + (n^2 - 1)a_1a_{n+1} + \sum_{m=2}^n m(2m - n - 3)a_ma_{n-m+2}
\end{align*} \tag{3.19}
\]
At $\lambda = 0$ we have that $a_0 = 0$ and $a_1 = b_0$. Substituting this into the first two equations in (3.19) and after a number of differentiations we find that $\partial_\lambda^{(k)} a_0 = 0$ and $\partial_\lambda^{(k)} a_1 = 0$, $k \geq 1$ at $\lambda = 0$. From this we conclude that $a_0 = 0$ and $a_1 = b_0$ for all $\lambda$ (expanding $a_0$ and $a_1$ in Taylor series in $\lambda$ one finds that all terms, except first, in the series vanish). The value of $y$-derivative of metric function in (3.11) at $y = 0$ gives the Hawking temperature $T_H = \frac{1}{4\pi} g'_y(y = 0)$ of the horizon. The fact that this value does not change means that the Hawking temperature remains constant under the Ricci flow, $T_H(\lambda) = T_H(\lambda = 0)$.

The maximum principles. The Ricci scalar $R$ is an important quantity to look at since its evolution preserves certain bounds. Under the Ricci-DeTurck flow (3.13) the Ricci scalar changes according to equation

$$\partial_\lambda R = -\xi^k \partial_k R + \Delta R + R^2 \quad .$$

(3.20)

By applying the maximum principle to this equation one obtains important bounds on the curvature of the space-time under the flow. In terms of metric (3.11) and vector field (3.14) equation (3.20) takes the form

$$\partial_\lambda R = b_0 R'_y + g R''_y + R^2 \quad .$$

(3.21)

At a point in which $R(y, \lambda)$ takes a local minimum one has $R'_y = 0$ and $R''_y > 0$ and hence $\partial_\lambda R > 0$. The minimal value of scalar curvature is thus increasing with $\lambda$. Therefore, if the initial metric (at $\lambda = 0$) has everywhere positive curvature $R(y) > 0$, $y > 0$ then for any $\lambda > 0$ we have that $R_{\text{min}} > 0$. It does not immediately follow that the the Ricci scalar is everywhere positive for $\lambda > 0$ since equation (3.21) is defined on a half-line and hence one should check that values of $R$ at the boundaries do not become negative. We assume that metric is asymptotically flat so that $R = 0$ at $y = +\infty$. On the other boundary, at $y = 0$, where $g(y)$ vanishes, one finds that

$$\partial_\lambda R(0) = b_0 R'_y(0) + R^2(0) \quad .$$

(3.22)

There are two cases to consider. 1) $R'_y(0, \lambda) < 0$, then $R(0, \lambda) > R_{\text{min}} > 0$ if there is a local minimum at some point $y > 0$ or $R(0, y) > R(y = \infty) = 0$ if no such point exists. 2) $R'_y(0) > 0$, then $R(0, \lambda)$ is increasing with $\lambda$ and hence $R(0, \lambda) > 0$ as well. Thus, the value of $R$ at $y = 0$ remains positive in any case. The property of the Ricci flow to preserve the positive sign of curvature is important for our discussion in the next section.

The maximum principle is an useful tool. It can be used to prove that there can be no more horizons than in the initial metric. Suppose that the initial metric function $g_0(y)$ has a horizon at $y = 0$ and is asymptotically flat. So it changes from zero at $y = 0$ to $1$ at $y = +\infty$ and is positive everywhere in between. Can there, for some $\lambda > 0$, appear another point $y = y_1 > 0$ at which $g(y_1, \lambda) = 0$? If there is such a point it forms from a minimum of the function $g(y, \lambda)$. At a local minimum one has that $g'_y = 0$ and $g''_y > 0$. Then from equation (3.12) one finds that $\partial_\lambda g = gg''_y > 0$ and hence the value of function $g(y, \lambda)$ at the minimum is increasing. This means that it never can reach zero so that no new horizon will be formed.

On the other hand, if $g(y, \lambda)$ has a local maximum then $g'_y = 0$ and $g''_y < 0$ at that point so that $\partial_\lambda g = gg''_y < 0$. Thus, the maximal value of the function $g(y, \lambda)$ is decreasing with $\lambda$. The Ricci flow thus tends to smear out the metric function $g(y, \lambda)$ over the half line $y > 0$.

The tendency of the Ricci flow to smooth out the initial geometry works in an interesting way. We could have started (at $\lambda = 0$) with a black hole at a temperature $\beta^{-1}$ different from...
the Hawking temperature, $\beta \neq \beta_H$. The initial geometry then would have a conical singularity at the horizon. As the example of the “decaying cone” solution found in [32], [33] shows, the geometry evolving under the Ricci flow then would be completely regular for any $\lambda > 0$ so that the temperature is immediately switched to the Hawking temperature under the flow.

**Asymptotically flat metrics.** At infinity of asymptotically flat space-time one has $g = 1 + h$, $h \ll 1$. $h$ satisfies the heat type equation

$$\partial_\lambda h = h''_y + b_0 h'_y .$$

It is a linear equation that is also satisfied for the derivatives of function $h(y, \lambda)$. Suppose that the initial data for $h$ falls off by a power law, $h_0(y) \sim \frac{1}{y^k}$, $k \geq 1$ for large $y$. Then the term $h''_y$ in (3.23) can be neglected as small so we have that $h(y, \lambda)$ satisfies equation

$$\partial_\lambda h = b_0 h'_y .$$

Taking into account the initial condition we get

$$h(y, \lambda) = h_0(y + b_0 \lambda) \sim \frac{1}{(y + b_0 \lambda)^k}$$

for the solution. Clearly, it decays both for large $y$ and large $\lambda > 0$. For $\lambda > 0$ the bound $|h(y, \lambda)| \leq |h_0(y)|$ is satisfied. Same is true for the scalar curvature $R = -h''_y(y, \lambda)$. Thus, the initially asymptotically flat metric remains to be asymptotically flat under the Ricci flow.

**The stationary point (cigar soliton).** Summarizing our analysis so far we have found that

1. the black hole metric at $\lambda = 0$ remains to be a black hole for $\lambda > 0$;
2. the Hawking temperature $T_H$ does not change under the Ricci flow;
3. asymptotically flat metric at $\lambda = 0$ remains to be asymptotically flat for $\lambda > 0$;
4. no new horizons are formed.

We can draw an important conclusion from these observations. The end point (if there is one) of the Ricci flow that started with an asymptotically flat 2d black hole metric is a non-constant curvature space-time that has same temperature as the initial geometry and is asymptotically flat. This is different from how the Ricci flow behaves in the case of compact manifold. In the latter case the Ricci flow ends on a geometry with constant (positive or negative) scalar curvature [23].

The stationary point of equation (3.12) is easy to find. The right hand side of equations (3.12) vanishes identically for a metric function

$$g_{st}(y) = \frac{b_0}{\Lambda} (1 - e^{-\Lambda y}) ,$$

where $\Lambda$ is integration constant. Demanding $g_{st} = 1$ for $y = +\infty$ we have that $\Lambda = b_0$. This metric is the so-called Ricci soliton. It was found in [23]. In physics this metric is known as 2d black hole that appears in the context of two-dimensional string theory [37], [38]. It solves the one-loop renormalization group equations in certain $\sigma$-model with non-compact target space. The function $\psi = \Lambda y$ (3.16) is related to the linear dilaton. Note that this metric has the everywhere positive curvature and is sometimes called the cigar soliton. It is asymptotic to a flat cylinder at infinity and has maximal curvature at the origin. Depending on the initial metric the flow may or may not approach the stationary point. An example of the flow that approaches the stationary point is given in [34].
It should be noted that (3.26) is a stationary point of the Ricci-DeTruick flow (3.13) but not of the original Ricci flow equation (3.1), (3.9). Indeed, solving the equation (3.10) we find a relation between coordinates \( y \) and \( x \), so that at \( \lambda = 0 \) we have that \( y = x \). The corresponding solution to equation (3.9) is given by function \( g(x, \lambda) = g_{st}(y(x, \lambda)) \).

\[
g(x, \lambda) = \frac{1 - e^{-b_0x}}{1 + (e^{b_0\lambda} - 1)e^{-b_0x}}. \tag{3.27}
\]

The metric (3.8)

\[
ds^2 = \frac{1}{1 + (e^{b_0\lambda} - 1)e^{-b_0x}} \left( (1 - e^{-b_0x})d\tau^2 + \frac{dx^2}{(1 - e^{-b_0x})} \right) \tag{3.28}
\]
then solves the Ricci flow equation (3.1) and is a conformal, \( \lambda \)-dependent, deformation of the stationary metric (3.11), (3.26). This metric is eternal solution to the Ricci flow equations. It extrapolates between flat metric at \( \lambda = -\infty \) and a constant curvature \( (R = b_0^2) \) metric at \( \lambda = +\infty \). In these limits one focuses on various regions of the cigar soliton: asymptotic infinity or the near horizon (tip of cigar) region.

**Hamilton’s entropy.** It was noted in [39] and [40] that the 2d metric (3.11), (3.26) that describes a black hole in the string inspired \( \sigma \)-model is also a solution to the field equations that follow from the action

\[
N = \int_M R \ln R, \tag{3.29}
\]
considered on a non-compact spacetime. This functional plays an important role in the Hamilton’s analysis of the Ricci flow on compact two-dimensional manifold. It is a monotonic function along the flow on manifold with positive curvature, \( R > 0 \). A simple proof of this statement in the case of compact manifold was suggested by Chow [41]. Below we slightly modify this proof for the case of non-compact asymptotically flat manifold.

Define a symmetric, trace free tensor \( M_{ij} = \nabla_i \nabla_j \psi - \frac{1}{2} R g_{ij} \), where \( \Delta \psi = R \), and a 1-form \( X_i = \nabla_i R + R \nabla_i \psi \). They are related as \( X_i = 2 \nabla^j M_{ij} \). The direct calculation shows that under the Ricci flow the functional (3.29) evolves as follows

\[
\frac{dN}{d\lambda} = -2 \int_M |M_{ij}|^2 - \int_M \frac{|X|^2}{R} \leq 0. \tag{3.30}
\]
A number of integrations by parts and the identity \( \Delta \nabla_i f = \nabla_i \Delta f + \frac{1}{2} R \nabla_i f \) are needed in order to get (3.30). The right hand side of (3.30) vanishes when \( M_{ij} = 0 \), and, hence, \( X_i = 0 \) (that gives relation \( \psi = -\ln R \)), that is exactly the field equations that follow from (3.29) by variational principle. This is also condition for the steady solitonic solution to the Ricci flow equation [23].

Analyzing the convergence of the integrals let’s focus on the class of static metrics (3.11) with \( g(y) = 1 + O(1/y^k) \) with \( k > 0 \) for large \( y \). Then we have that \( \psi_y^\prime = b_0 + O(1/y^k) \) (as follows from (3.15)), \( R = O(1/y^{k+2}) \) and, hence, \( X_y = b_0 R + O(1/y^{k+3}) \). It follows that for large \( y \) we have that \( R \ln R = O(y^{-k-2} \ln y) \), \( |X|^2/R \sim b_0^2 R \sim O(1/y^{k+2}) \) and \( |M_{ij}|^2 \sim O(1/y^{2(k+1)}) \). Thus, all integrals in (3.29) and (3.30) perfectly converge.
4 The monotonicity of entanglement entropy

In order to show that the Ricci flow is irreversible in the mathematics literature were introduced various definitions of entropy [23], [24] that have the property to change monotonically under the flow. On the other hand, if the evolving metric describes a black hole then this metric may have an intrinsic gravitational entropy that measures the loss of information due to the presence of the black hole horizon. So it is a natural question how this entropy changes under the Ricci flow. It should be noted that the Bekenstein-Hawking entropy, that is standardly associated with a black hole, is defined “on-shell” for a metric that satisfies certain gravitational equations obtained by variational principle from a gravitational action. Entanglement entropy, discussed in section 2, is yet another entropy naturally defined for a black hole. Its advantage is that it does not require knowledge of any particular gravitational action and is defined “off-shell”, i.e. for any metric that has properties of black hole.

For the black hole geometry, at any given \( \lambda \), entanglement entropy can be defined by first transforming metric to the Schwarzschild like form (3.11) and then applying the general formula (2.7),

\[
S(\lambda) = \frac{c}{12} \left( \int_0^{L_y} \frac{dy}{g(y, \lambda)} (b_0 - g'_y) + \ln g(L_y, \lambda) - 2 \ln(b_0 \epsilon) \right),
\]

where we take into account that \( g'_y(0) = b_0 \) for any \( \lambda \). \( L_y \) is the value of \( y \)-coordinate at the boundary of the box. We find that

\[
L_y = \int_0^L \frac{g(x, \lambda)}{g_0(x)} \, dx, \quad \partial_\lambda L_y = (g'_y(L_y) - b_0).
\]

In order to check whether the quantity (4.1) is monotonic along the Ricci flow we have to calculate its derivative with respect to \( \lambda \) and look at the sign of this derivative. Before doing this it is convenient to reshuffle a bit the terms in (4.1). Integrating by parts in (4.1) we get

\[
S(\lambda) = \frac{c}{12} \left( b_0 \int_\delta^{L_y} \frac{dy}{g(y, \lambda)} + \ln g(\delta) - 2 \ln(b_0 \epsilon) \right),
\]

where, for convenience, we introduced a small quantity \( \delta \), (4.3) should be understood as limit \( \delta \to 0 \). Calculating now the derivative of (4.3) with respect to \( \lambda \), using equations (3.12) and (4.2) and integrating by parts we find that

\[
\partial_\lambda S(\lambda) = \frac{c}{12g(\delta)} \left( \partial_\lambda g(\delta) + b_0(g'_y(\delta) - b_0) \right).
\]

Using once again equation (3.12) at \( y = \delta \) we get that

\[
\partial_\lambda S(\lambda) = \frac{c}{12} \left( g''_y(\delta) - \frac{(g'_y(\delta) - b_0)^2}{g(\delta)} \right).
\]

The second term in (4.5) is of order \( \delta \) and vanishes in the limit \( \delta \to 0 \). We finally obtain that

\[
\partial_\lambda S(\lambda) = - \frac{c}{12} R(0, \lambda),
\]

where \( R(0) = -g''_y(0) \) is the value of the Ricci scalar at horizon.
As we discussed in section 3 the everywhere positive Ricci scalar remains positive under the Ricci flow. Thus, if the initial black hole metric has \( R(x) > 0 \) for all \( x > x_+ \) then entanglement entropy is monotonically decreasing under the Ricci flow. Note that the Ricci scalar is positive in the case of important 2d metrics: the Schwarzschild metric with \( g_0(x) = 1 - \frac{a}{x} \) and the string-inspired black hole metric with \( g_0(x) = 1 - ae^{-\Lambda x} \). Since for the monotonicty of entanglement entropy we need the Ricci scalar to be positive only at the horizon it is possible that one can relax the condition on the Ricci scalar of the initial metric and admit metrics with \( R \) changing the sign far from the horizon. It would be interesting to analyze this possibility.

It should be noted that the evolution of the entanglement entropy does not stop when the metric reaches the stationary point (3.26). Although the metric components do not depend on \( \lambda \) in this case, the size \( L_y \) of the box in metric (3.11) changes monotonically with \( \lambda \), \( \partial_\lambda L_y = b_0(e^{-b_0L_y} - 1) < 0 \). The monotonicity of the entropy for the stationary metric,

\[
\partial_\lambda S_{\text{st}} = -\frac{c}{12}b_0^2 < 0 ,
\]

then is entirely due to the monotonicity of the entropy as function of the size (2.8).

5 Conclusions

With almost every known definition of entropy it is closely associated a property to change monotonically under the change of certain parameters. This property indicates the underlying irreversibility and motivates the actual usage of word “entropy” in each case. Entanglement entropy is an interesting quantity that has various applications. Defined in two dimensions for the quantum mechanical non-gravitational systems it was shown to be monotonic under the renormalization group flow [19]. The gravitational analog of the RG flow is the Ricci flow. In fact, the both flows are related for the two-dimensional \( \sigma \)-models whose target space is the Riemann manifold. In particular, this manifold may be noncompact and may describe a black hole as it was demonstrated in [37]. The usual \( c \)-theorem does not immediately apply to the noncompact case [3]. Thus, it is desirable to identify a quantity that changes monotonically under the RG flow in this case. In this note we have shown that entanglement entropy, defined for a two-dimensional black hole metric, is monotonically decreasing under the Ricci flow. The necessary condition for the monotonicity is the positiveness of the Ricci scalar at the black hole horizon. In the case of the \( \sigma \)-model this allows us to assign with the evolving metric an important quantity that characterizes both the irreversibility of the RG flow and the irreversible loss of information due to the black hole.

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References

[1] A. B. Zamolodchikov, JETP Lett. 43 (1986) 730 [Pisma Zh. Eksp. Teor. Fiz. 43 (1986) 565].
[2] A. A. Tseytlin, Phys. Lett. B 194, 63 (1987).
[3] J. Polchinski, Nucl. Phys. B 303, 226 (1998).
[4] A. Morozov and A. J. Niemi, Nucl. Phys. B 666, 311 (2003).
[5] L. Bombelli, R. K. Koul, J. H. Lee and R. D. Sorkin, Phys. Rev. D 34, 373 (1986).
[6] M. Srednicki, Phys. Rev. Lett. 71, 666 (1993).
[7] V. P. Frolov and I. Novikov, Phys. Rev. D 48, 4545 (1993).
[8] C. G. Callan and F. Wilczek, Phys. Lett. B 333, 55 (1994).
[9] C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B 424, 443 (1994).
[10] S. N. Solodukhin, Phys. Rev. D 51, 609 (1995).
[11] S. N. Solodukhin, Phys. Rev. D 52, 7046 (1995).
[12] A. O. Barvinsky, V. P. Frolov and A. I. Zelnikov, Phys. Rev. D 51, 1741 (1995).
[13] R. C. Myers, Phys. Rev. D 50, 6412 (1994).
[14] D. V. Fursaev and S. N. Solodukhin, Phys. Rev. D 52, 2133 (1995).
[15] S. N. Solodukhin, Phys. Rev. D 53, 824 (1996).
[16] V. P. Frolov, W. Israel and S. N. Solodukhin, Phys. Rev. D 54, 2732 (1996).
[17] V. Korepin, Phys. Rev. Lett. 92, 096402 (2004).
[18] P. Calabrese and J. Cardy, J. Stat. Mech. 0406, P002 (2004).
[19] J. I. Latorre, C. A. Lutken, E. Rico and G. Vidal, Phys. Rev. A 71, 034301 (2005).
[20] H. Casini and M. Huerta, Phys. Lett. B 600, 142 (2004).
[21] D. Friedan, Phys. Rev. Lett. 45, 1057 (1980).
[22] R. S. Hamilton, J. Diff. Geom. 17 (1982) 255-306.
[23] R. S. Hamilton, Contemporary Mathematics 71 (1988), 237-261.
[24] G. Perelman, “The entropy formula for the Ricci flow and its geometric applications,” math.DG/0211159.
[25] H.-D. Cao and B. Chow, “Recent developments on the Ricci flow”, math.DG/9811123.
[26] B. Chow, “Ricci flow I, II, III”, Lectures at Clay Math. Institute, June 2005.
[27] M. T. Anderson, “Geometrization of 3-manifolds via the Ricci flow”, Notices of the AMS, 51, 184 (2004).

[28] I. Bakas, Comptes Rendus Physique 6, 175 (2005).

[29] T. Oliynyk, V. Suneeta and E. Woolgar, Phys. Lett. B 610, 115 (2005).

[30] T. Oliynyk, V. Suneeta and E. Woolgar, Nucl. Phys. B 739, 441 (2006).

[31] T. A. Oliynyk and E. Woolgar, “Asymptotically Flat Ricci Flows,” arXiv:math.dg/0607438.

[32] A. Adams, J. Polchinski and E. Silverstein, JHEP 0110, 029 (2001).

[33] M. Gutperle, M. Headrick, S. Minwalla and V. Schomerus, JHEP 0301, 073 (2003).

[34] K. Hori and A. Kapustin, JHEP 0108, 045 (2001).

[35] M. Headrick and T. Wiseman, “Ricci flow and black holes,” arXiv:hep-th/0606086.

[36] S. N. Solodukhin, “Entanglement entropy of black holes and AdS/CFT correspondence,” arXiv:hep-th/0606205.

[37] E. Witten, Phys. Rev. D 44, 314 (1991).

[38] G. Mandal, A. M. Sengupta and S. R. Wadia, Mod. Phys. Lett. A 6, 1685 (1991).

[39] V. P. Frolov, Phys. Rev. D 46, 5383 (1992).

[40] S. N. Solodukhin, Phys. Rev. D 51, 591 (1995).

[41] B. Chow, J. Diff. Geom. 33, 597 (1991).