Gaussian noise removal with exponential functions and spectral norm of weighted Hankel matrices

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Abstract—Exponential functions are powerful tools to model signals in various scenarios, such as magnetic resonance spectroscopy/imaging, radar, and concatenative text-to-speech synthesis. Exponential signals, however, are usually corrupted by Gaussian noise in practice, raising difficulties in sequential analysis and quantification of the signals. In this work, we propose a denoising method based on low-rank Hankel matrices for exponential signals corrupted by Gaussian noise. An accurate estimate on the spectral norm of weighted Hankel matrices is provided as a theoretical guidance to set the regularization parameter. The bound can be efficiently calculated since it only depends on the standard deviation of the noise and a constant. Aided by the bound, one can easily obtain a good regularization parameter to produce promising denoised results. Our experiments on simulated and magnetic resonance spectroscopy data demonstrate a superior denoising performance of our proposed approach in comparison with the typical Cadzow and the state-of-the-art QR decomposition methods, especially in the low signal-to-noise ratio regime.

Index Terms—spectral denoising, exponential signals, Hankel matrix, signal reconstruction.

I. INTRODUCTION

Exponential functions are widely used to model signals in applications, such as concatenative, text-to-speech synthesis [1], radar scattering [2], Nuclear Magnetic Resonance (NMR) spectroscopy [3]–[9], and magnetic resonance spectroscopic imaging [10]–[12]. However, signals are often corrupted by noise during acquisition and/or transmission [13]. The noise problem turns out to be severe in the low Signal-to-Noise Ratio (SNR) regime [12], [14]. Therefore, there is a strong demand to denoise signals, particularly in the low SNR regime.

Gaussian noise is commonly encountered in denoising applications [7], [15]. One of the most effective and widely adopted approaches to suppress Gaussian noise is to average multiple signal acquisitions. However, the multiple acquisitions are not always available or too costly in real applications.

For this reason, effective denoising of the signals with a limited number of scans is favorable.

Numerous efforts have been made to denoise signals. The key idea of many existing methods is to map the corrupted signals into another space so that the signal and the noise can be well separated. Among them, the filter-based approach [16] is easy to apply but may lead to blurred results, with loss of details and smoothing of edges [17]. In 1990s, Donoho et al [18] proposed a wavelet thresholding technique for denoising, which was widely applied in various fields [19]. The wavelet-based technique preserves signal features well when the wavelet basis are properly chosen. Nevertheless, this denoising technique may introduce problematic characteristic artifacts [20].

The denoising methods mentioned above are not based on any signal model. For exponential signals, we expect to have a better denoising method if the signal characteristics are exploited. For instance, the exponential characteristic of NMR spectroscopy signals has been shown to be powerful in spectra denoising [21]–[25] and signal reconstruction [5], [7], [9]. In particular, the low-rank rank property of the Hankel matrix constructed from NMR spectroscopy signals was exploited in [24]–[26]. Such low-rank properties were also utilized in the field of magnetic resonance spectroscopic imaging [10]. The Cadzow enhancement approach is popular in spectra denoising with the exploitation of the low-rank property of exponentials [21]–[23]. However, it is a challenging task to choose a proper parameter \( R \) as the number of exponential components in practical applications, unless a priori information is given. Efforts have been made to estimate \( R \), such as the indicator function [27] and the significance level function [28], but the estimation on \( R \) may not be satisfactory enough to yield good results [29]. Another denoising method called random QR denoising method (rQRd) is based on an approximate low-rank decomposition, and accelerates the computation by avoiding the Singular Value Decomposition (SVD) in the Cadzow method [30]. It is, however, based on an accurate estimation of the rank \( R \). Developing reliable and easy-to-use denoising methods for exponential signals are still of great demand and challenging.

In this paper, we propose to exploit the low-rank property of the Hankel matrix constructed from exponential signals for denoising. An optimization problem is formulated for denoising with a regularization parameter \( \lambda \), which plays an important role in the results. We show that a good \( \lambda \) can be chosen according to the spectral norm of a weighted Hankel matrix, which is estimated by random matrix theory as a theoretical guideline for the selection of a proper \( \lambda \).
only needs to estimate the standard deviation of the noise to calculate this proper $\lambda$. Numerical experiments on both simulated and real NMR spectroscopy data show that noise can be effectively removed when the parameter is chosen according to our analysis.

The rest of the paper is organized as follows. Section II introduces the signal model and the proposed denoising method. Our theoretical estimate on the spectral norm of weighted Hankel matrices is presented in Section III. Section IV contains numerical results on simulated and real NMR spectroscopy data. Section V discusses the robustness to the estimate on the noise standard deviation. Finally, we conclude and discuss future works in Section VI.

Notations used in the paper are introduced below. We denote vectors through bold lowercase letters and matrices through bold uppercase letters. The entry in vectors and matrices is denoted by a normal letter with a subscript which stands for its location. For example, $x_n$ denotes the $n$th element of $x$, and $X_{m,n}$ denotes the $(m,n)$th entry of $X$. For any vector $x$, $\|x\|_2$ represents the $l_2$ norm. For any matrix $X$, $\|X\|_*$ and $\|X\|_F$ denote the nuclear norm and the spectral norm, respectively. The Hadamard product is denoted by $\circ$. We use superscript $T$ and $H$ to denote the transpose and the conjugate transpose of $x$ and $X$. Most of operators are denoted by calligraphic letters. We denote $\text{diag}$ as the operator transforming a sequence to a diagonal matrix whose diagonal entries are given by the sequence.

II. SIGNAL MODEL AND Denoising

A. Signal model

Signals of interest in many imaging and signal processing applications can be expressed as the sum of $R$ exponentials:

$$x_0(t_n) = \sum_{r=1}^{R} a_r e^{(j2\pi f_r - \tau_r) t_n}, \quad n = 0, \ldots, 2N$$

(1)

where $a_r$ denotes the signal amplitude, $f_r$ is the central frequency, and $\tau_r$ is the decay factor. The number of exponentials $R$ is usually small.

In practice, observations are often contaminated by noise and one receives

$$y = x_0 + z$$

where $x_0 = \{x_0(t_n)\}_{n=0}^{2N}$ is a noiseless signal and $z \in \mathbb{C}^{2N+1}$ is a random vector whose real and imaginary parts are i.i.d Gaussian with mean $0$ and variance $\sigma^2$.

Exponential signals can be transformed into Hankel matrices with a Vandermonde decomposition. Given $x_0$, one forms the square Hankel matrix

$$\mathcal{R}x_0 = \begin{bmatrix} x_0(t_0) & x_0(t_1) & \cdots & x_0(t_N) \\ x_0(t_1) & x_0(t_2) & \cdots & x_0(t_{N+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_0(t_N) & x_0(t_{N+1}) & \cdots & x_0(t_{2N}) \end{bmatrix},$$

where $\mathcal{R} : \mathbb{C}^{2N+1} \rightarrow \mathbb{C}^{(N+1) \times (N+1)}$ is the operator transforming a vector to the square Hankel matrix. It is well known that $\mathcal{R}x_0$ possesses a Vandermonde decomposition with rank $R$ [3], [4] such that

$$\mathcal{R}x_0 = \mathbf{E} \mathbf{A} \mathbf{E}^T,$$

where $\mathbf{A} = \text{diag}\left(\{a_r\}_{r=1}^{R}\right)$, and

$$\mathbf{E} = \begin{bmatrix} e^{(j2\pi f_1 - \tau_1)t_0} & e^{(j2\pi f_2 - \tau_2)t_0} & \cdots & e^{(j2\pi f_R - \tau_R)t_0} \\ e^{(j2\pi f_1 - \tau_1)t_1} & e^{(j2\pi f_2 - \tau_2)t_1} & \cdots & e^{(j2\pi f_R - \tau_R)t_1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(j2\pi f_1 - \tau_1)t_N} & e^{(j2\pi f_2 - \tau_2)t_N} & \cdots & e^{(j2\pi f_R - \tau_R)t_N} \end{bmatrix}.$$

This decomposition implies that $\mathcal{R}x_0$ is of rank $R$.

Our denoising method is based on the low-rank property of $\mathcal{R}x_0$, and called Convex Hankel Low rank matrix approximation for Denoising exponential signals (CHORD), where one solves the following optimization problem:

$$\hat{x} = \arg \min_{\mathbf{x} \in \mathbb{C}^{2N+1}} \|\mathcal{R}\mathbf{x}\|_* + \frac{\lambda}{2} \|y - \mathbf{x}\|_2^2,$$

(2)

where $\lambda$ denotes the regularization parameter. The nuclear norm $\|\cdot\|_*$ is a surrogate for the rank [31].

Alternating Direction Method of Multipliers (ADMM) [32] is a typical iterative algorithm, which can be used to solve (2). After introducing two variables $B, D \in \mathbb{C}^{(N+1) \times (N+1)}$, we can reformulate (2) as follows:

$$\max_{\mathbf{D}} \min_{\mathbf{B}} \|\mathbf{B}\|_* + \frac{\beta}{2} \|\mathcal{R}\mathbf{B} - \mathbf{D}\|_F^2 + (\mathcal{R}\mathbf{B} - \mathbf{D}, \mathbf{D}) + \frac{\lambda}{2} \|y - \mathbf{x}\|_2^2.$$  

(3)

is solved with the following iterative scheme:

$$\begin{align*}
\mathbf{x}_{k+1} &= (\beta \mathcal{R}^* \mathcal{R} + \lambda I)^{-1} [\beta \mathcal{R}^* (\mathbf{B}_k - \mathbf{D}_k / \beta) + \lambda y] \\
\mathbf{B}_{k+1} &= \mathcal{S}_{1/\beta}(\mathcal{R}\mathbf{x}_{k+1} + \mathbf{D}_k / \beta) \\
\mathbf{D}_{k+1} &= \mathbf{D}_k + \tau (\mathcal{R}\mathbf{x}_{k+1} - \mathbf{B}_{k+1})
\end{align*}$$

(4)

where $\mathbf{1} \in \mathbb{R}^{(N+1) \times (N+1)}$ denotes an identity matrix. The subscript $k$ denotes results in the $k$th iteration, $\mathcal{R}^* : \mathbb{C}^{(N+1) \times (N+1)} \rightarrow \mathbb{C}^{2N+1}$ is the adjoint operator of $\mathcal{R}$, which transforms a Hankel matrix into a vector through summing each anti-diagonal.

Let $\mathbf{X} \in \mathbb{C}^{(N+1) \times (N+1)}$ be with the SVD $\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^H$, where $\Sigma = \text{diag}\{\sigma_r\}_{r=1}^{R}$. The singular thresholding operator which applies in matrix $\mathbf{X}$ is $\mathcal{S}_{1/\beta}(\mathbf{X}) = \text{Udiag}\{\sigma_r - 1/\beta\} \mathbf{V}^H$, where $\mathbf{1} = \max(0, 1)\mathbf{1}$. In ADMM, $\beta$ and $\tau$ are two parameters and we set $\beta = 1$ and $\tau = 1$.

The optimization problem in (2) involves a single regularization parameter $\lambda$, and the denoised result crucially depends on the choice of $\lambda$. As an example, Fig. 1 shows the denoising results with different $\lambda$. If $\lambda$ is too large, the majority of the noise remains since the effect of the nuclear norm minimization is ignorable; if $\lambda$ too small, the spectral peaks are seriously distorted.

Therefore, automatically setting an appropriate $\lambda$ is a crucial issue in this denoising method. This paper provides a theory on the proper choice of $\lambda$, and validations by experimental results.
SVD of (2), i.e., \( \hat{\lambda} \left\| W \right\| \) where the matrices are from the SVD of \( R x_0 \) such that \( R x_0 = U_0 \Sigma_0 V^H_0 \) and \( W_0 \in \mathbb{C}^{(N+1) \times (N+1)} \) satisfies \( U_0^H W_0 = 0 \), \( W_0 V_0 = 0 \), and \( \left\| W_0 \right\|_2 \leq 1 \).

Denote \( C = R R^* : \mathbb{C}^{(N+1) \times (N+1)} \rightarrow \mathbb{C}^{(N+1) \times (N+1)} \), and \( w = [1 \ 2 \ \cdots \ N+1 \ \cdots \ 2 \ 1]^T \in \mathbb{R}^{2N+1} \). For any matrix \( X \in \mathbb{C}^{(N+1) \times (N+1)} \), \( C X = (R w) \circ X \) where \( \circ \) denotes Hadamard product. The adjoint operator of \( C \) is \( C^* : \mathbb{C}^{(N+1) \times (N+1)} \rightarrow \mathbb{C}^{(N+1) \times (N+1)} \), which is given by \( C^* \times = (R \frac{1}{w}) \circ X \) where \( \frac{1}{w} \) and \( k = 0, \ldots, 2N \). Then (8) can be simplified as follows:

\[
\lambda R z = R R^* (U_0 V_0^H + W_0) = (R w) \circ (U_0 V_0^H + W_0).
\]

Multiplying both sides by \( R \frac{1}{w} \) gives rise to

\[
\lambda \left( R \frac{1}{w} \right) \circ R z = U_0 V_0^H + W_0,
\]

which implies

\[
\left\| \lambda \left( R \frac{1}{w} \right) \circ R z \right\|_2 - \left\| U_0 V_0^H \right\|_2 \leq \left\| W_0 \right\|_2 \leq 1.
\]

Therefore,

\[
\lambda \leq \left. \frac{\left\| U_0 V_0^H \right\|_2 + 1}{\left\| \left(R \frac{1}{w} \right) \circ R z \right\|_2} \right| = 2 \left( \left\| \left(R \frac{1}{w} \right) \circ R z \right\|_2 \right)^2.
\]

We denote \( Z = \left(R \frac{1}{w} \right) \circ R z \) such that

\[
Z = \left(R \frac{1}{w} \right) \circ R z = \begin{pmatrix} z_1 & z_2 & \cdots & z_{2N+1} \\ \frac{z_1}{N+1} & \frac{z_2}{N+1} & \cdots & \frac{z_{2N+1}}{N+1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{z_1}{N+1} & \frac{z_2}{N+1} & \cdots & \frac{z_{2N+1}}{N+1} \end{pmatrix}.
\]

Fig. 2. The relation between \( \left\| Z \right\|_2 \) and the standard deviation \( \sigma \) of the Gaussian noise \( z \) in 100 Monte Carlo trials. The matrix \( Z \) is of size \((N+1) \times (N+1)\) with \((N+1) = 25, 50, 250, 500, 2500, 5000\), respectively. The curve represents the mean of \( \left\| Z \right\|_2 \) in 100 trials versus \( \sigma \), and the standard deviation of \( \left\| Z \right\|_2 \) in 100 trials is indicated by the vertical bar.

Eq. (10) shows that noise will be removed if

\[
\lambda \leq 2 \left\| Z \right\|_2.
\]

As \( z \in \mathbb{C}^{2N+1} \) is a random vector, \( Z \) is a random weighted Hankel matrix. According to the law of large numbers, \( \left\| Z \right\|_2 \)
will concentrate at $E\|Z\|_2$. In Fig. 2 we observe that the empirical mean of $\|Z\|_2$ is proportional to the standard deviation $\sigma$ of the noise and is almost independent of $N$.

In this paper, we propose to select the regularization parameter as

$$\lambda = \frac{2}{E\|Z\|_2}.$$  

III. SELECTION OF $\lambda$ BASED ON ESTIMATES ON $\|Z\|_2$

According to the analysis above, our choice of $\lambda$ depends on $E\|Z\|_2$. This section is devoted to an estimation of the lower and upper bounds of $E\|Z\|_2$.

A. A lower bound of $E\|Z\|_2$

**Theorem 1.** Suppose the real and imaginary parts of the entries in $z \in \mathbb{C}^{2N+1}$ are i.i.d. Gaussian random variables with mean 0 and variance $\sigma^2$. Define $R_N$ and $Q_N$ such that

$$R_N^2 = \sum_{k=0}^{2N} |d_k|^2 \quad \text{and} \quad Q_N^2 = \sum_{k=0}^{2N} |d_k|^4,$$

where $d_k = \left\{ \begin{array}{ll} 2 & \text{if } k+1 \leq N, \\ 0 & \text{otherwise} \end{array} \right.$

Then there exists a constant $C$ such that the matrix $Z$ defined in (11) satisfies

$$\mathbb{E}\|Z\|_2 \geq \sigma \frac{CN + 1}{2N + 1} \sqrt{\frac{R_N^2}{1 + \log \frac{R_N^4}{Q_N^4}}}. \quad (13)$$

The proof of Theorem 1 are based on the following lemmas:

**Lemma 1.** Suppose the real and imaginary parts of the entries in $z \in \mathbb{C}^{2N+1}$ are i.i.d. Gaussian random variables with mean 0 and variance $\sigma^2$. Define $\{d_k\}_{k=0}^{2N}$ as Theorem 1 and then

$$\|Z\|_2 \geq \sigma \frac{(N+1)}{2N+1} \sup_{0 \leq \omega \leq 1} \left| \sum_{k=0}^{2N} d_k p_k e^{i2\pi k \omega} \right|, \quad (14)$$

where $p_k \sim \mathcal{N}(0, 1), k = 0, 1, 2, \cdots, 2N$.

**Proof.** For any vectors $a, b \in \mathbb{C}^{N+1}$,

$$|\langle b, Za \rangle| = |\langle b^H, Za \rangle| \leq \|b\|_2 \|a\|_2 \|Z\|_2.$$  

We use the technique in [36] to derive the lower bound of $\mathbb{E}\|Z\|_2$ by choosing proper vectors $a$ and $b$. Let $a_{k_1} = \frac{1}{\sqrt{k_1+1}} e^{i2\pi k_1 \omega}$ and $b_{k_2} = \frac{1}{\sqrt{k_2+1}} e^{i2\pi k_2 \omega}$, where $\omega \in [0, 1]$ and $k_1, k_2 = 0, \ldots, N$. Then

$$\|Z\|_2 \geq \sigma \frac{(N+1)}{2N+1} \sup_{0 \leq \omega \leq 1} \left| \sum_{k=0}^{2N} d_k p_k e^{i2\pi k \omega} \right|,$$

where $C_N = \sum_{k=0}^{N} \frac{1}{(k+1)^2} \leq \frac{N+1}{N+1}$, which yields the conclusion in (14).

**Lemma 2.** Let $\{d_k\}_{k=0}^{2N}$ and $\{p_k\}_{k=0}^{2N}$ be the sequences defined in Theorem 1 and Lemma 1. If $R_N$ and $Q_N$ are defined as (12), then there exists a constant $C$ such that

$$\mathbb{E} \left( \sup_{0 \leq \omega \leq 1} \left| \sum_{k=0}^{2N} d_k p_k e^{i2\pi k \omega} \right|^2 \right) \geq C \frac{R_N^2}{1 + \log \frac{R_N^4}{Q_N^4}}. \quad (16)$$

**Proof.** We express $Z$ as a sum of independent matrices such that

$$Z = \sum_{k=0}^{2N} \frac{1}{w_k} z_k B_k,$$

where $B_k \in \mathbb{R}^{(N+1) \times (N+1)}$ has one on the $(k+1)^{th}$ skew diagonal and all other entries are 0. For example,

$$B_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

According to [38],

$$\mathbb{E} \|Z\|_2^2 \leq 2\nu^2(Z) \log (2N + 2), \quad (19)$$

where $\nu^2(Z) = \max \left\{ \mathbb{E}(\|Z^H Z\|_2), \mathbb{E}(\|Z Z^H\|_2) \right\}$. The parameter $\nu^2(Z)$ can be calculated as follows

$$\mathbb{E}(\|Z^H Z\|_2) = \mathbb{E}(\|Z Z^H\|_2)$$

$$= \mathbb{E} \left( \left\| \sum_{k=0}^{2N} \frac{z_k}{w_k} B_k \cdot \left( \sum_{m=0}^{2N} \frac{z_m}{w_m} B_m^H \right) \right\|_2 \right).$$

Denote the diagonal matrix $C_k = B_k B_k^H \in \mathbb{R}^{(N+1) \times (N+1)}$, $k = 0, 1, \cdots, 2N$. When $0 \leq k \leq N$, the first $k+1$ diagonal entries of $C_k$ are one, and others are zero. When $N+1 \leq k \leq 2N$, the last $2N+1-k$ diagonal entries of $C_k$ are one, and others are zero. For example,

$$C_1 = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad C_{N+1} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}.$$
Substituting (20) into the definition of $\nu^2 (Z)$ in (19) results in

$$\nu^2 (Z) = E \left( \sum_{k=0}^{2N} \left\| \frac{|z_k|^2}{w_k} - C_k \right\|_2 \right) = \sigma^2 C_w. \quad (21)$$

Finally we combine (21) and (19) to obtain Theorem 2.

C. Asymptotic analysis of the estimates and our suggested $\lambda$

The two subsections above provide the following upper and lower bounds of $E \|Z\|_2$:

$$\sigma \frac{C(N+1)}{2N+1} R_N^2 \left( 1 + \log \frac{R_N^4}{Q_N^4} \right) \leq E \|Z\|_2 \leq \sigma \sqrt{2C_w \log (2N+2)}.$$ \quad (22)

The upper bound scales as $\sigma \sqrt{\log N}$, while the lower bound depends on $R_N$ and $Q_N$. We next show that the lower bound scales like $C_l \sigma$ for some constant $C_l > 0$ when $N$ is sufficiently large.

**Theorem 3.** Let $R_N$ and $Q_N$ be defined as (12). Then there exists a constant $C_l > 0$ such that

$$\lim_{N \to +\infty} \frac{(N+1)C}{2N+1} R_N^2 \left( 1 + \log \frac{R_N^4}{Q_N^4} \right) = C_l. \quad (23)$$

The proof of Theorem 3 is based on two lemmas below which study the asymptotic of $R_N^2$ and $Q_N^4$ as $N \to \infty$.

**Lemma 3.** Let $R_N$ be defined as (12). Then there exists a constant $C_R > 0$ such that

$$\lim_{N \to +\infty} R_N^2 = C_R. \quad (24)$$

**Lemma 4.** Let $Q_N$ be defined as (12). Then there exists a constant $C_Q > 0$ such that

$$\lim_{N \to +\infty} Q_N^4 = C_Q. \quad (25)$$

Lemma 3 and Lemma 4 are proved in Appendix B and C. Combining Lemma 3 and Lemma 4 gives rise to

$$\lim_{N \to +\infty} \frac{(N+1)C}{2N+1} R_N^2 \left( 1 + \log \frac{R_N^4}{Q_N^4} \right) = C \sqrt{C_R \left( 1 + \log \frac{C_R^2}{C_Q} \right)} \lim_{N \to +\infty} \frac{(N+1)}{2N+1} \quad (26)$$

which gives rise to Theorem 3 with

$$C_l = \frac{C}{2} \sqrt{C_R \left( 1 + \log \frac{C_R^2}{C_Q} \right) \frac{4}{N+1}}.$$ \quad (27)

We next provide the empirical strategy to determine the constant $C$.

**D. The empirical $C$**

To evaluate the denoising performance, we define the following Relative Least Normalized Error (RLNE)

$$RLNE = \frac{\|\hat{x} - x_0\|_2}{\|x_0\|_2}, \quad (28)$$

where $\hat{x}$ and $x_0$ are the denoised signal and the noiseless signal respectively.

The existence of $C$ is proved in Theorem 3. However, Theorem 3 does not provide an explicit expression of $C$. We next find the empirical constant $C$ through repetitive experiments on simulated data. Our theory suggests $C > 0$ is a constant, which is independent of the signal length and the standard deviation $\sigma$. To confirm this conclusion, we perform experiments with different $N$, $\sigma$, signals and noises.

We generate a simulated data set, including 90 random damping complex exponential signals with $2N = 256$, 512, and 1024 respectively, and repeat 100 Monte Carlo trials to incorporate the randomness of Gaussian noise. Each signal in the data set has $3R + 1$ parameters, including $R$, $\alpha_r$, $f_r$, and $\tau_r$, where $r = 1, 2, \ldots, R$. The number of exponential components is $R = 4 + M_r$, where $M_r$ denotes a pseudo-random scalar integer of range $[1, 9]$. The amplitude $\alpha_r$ is uniformly sampled from $(0, 10)$. Each frequency $f_r$ is uniformly sampled from $(0, 1)$. The damping factor is $\tau_r = 4 + 60m_r$, where $m_r$ is uniformly sampled from $(0, 1)$.

1) Fixed Gaussian noise: We fix one Monte Carlo trial and evaluate the RLNEs of the denoised results with different noise standard deviation $\sigma$, the length $N$, and signals. Results in Fig.3(a)-(c) indicate that the empirical $C$ is independent of the length and signals.

2) Fixed noiseless data: We fix one noiseless signal to explore the effect of the randomness of the noises. The RLNEs of the denoised results have been presented in Fig.3(d)-(f), illustrating that the choice of $C$ is independent of the noises.

Fig.3 confirms our theory that $C$ is a constant which is independent of the length, the standard deviation of the noise and the randomness of the noises and signals. Moreover, we suggest $C = 1.2$ for denoising.

**IV. Numerical Experiments**

In this section, we evaluate the performance of CHORD with the suggested constant $C$ on the simulated data and a realistic NMR spectroscopy data set. The Relative Least Normalized Error, RLNE, defined in (28), is regarded as the objective criteria in the evaluation. The typical method, Cadzow [23], [30], and the state-of-the-art method, rQRd [30] are compared with our proposed method. For Cadzow, its key parameter is the rank of the Hankel matrix. For rQRd, its primary parameter is the number of the matrix $Q$'s column,
Fig. 3. The RLNEs of the denoised results with different length $N$, $\sigma$, signals and noises. (a)-(c) show the average RLNEs of a series of fixed Gaussian noises with $\sigma = 0.010, 0.0150, 0.020, 0.025, 0.030$ and $0.035$ respectively. (d)-(f) show the average RLNEs of a series of fixed random signals with $\sigma = 0.010, 0.0150, 0.020, 0.025, 0.030$ and $0.035$ respectively. The horizontal axis denotes the $C$ value, and the vertical axis denotes the RLNE of the denoised results. The vertical error bars in (a)-(c) come from the randomness of signals, and the vertical error bars in (d)-(f) represent the randomness of the noises.

denoted as $\text{rank}_Q$, in QR decomposition. For the rest of the manuscript, without explicit illustration, the main parameters in Cadzow and rQRd are chosen to be the ones yielding the lowest reconstruction error, RLNE.

A. Denoising of simulated complex data

We generated a simulated exponential complex data with five peaks (presented in Fig. 1(a)). In the following, the simulated data indicates the signal in Fig. 1(a). The denoising performance of three methods is tested through recovering the signal from complex Gaussian noise with different standard deviation ($\sigma = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, \text{ and } 0.08$, respectively). 100 Monte Carlo trials are done to avoid the randomness of noise.

In practice, we do not, in advance, know the standard deviation of the noise that corrupts the signal of interest. Here, we use the last 100 time-domain data points of the signal to estimate the standard deviation of the noise to mimic the real cases. Also, we compare the denoised performances of CHORD given the known standard deviation and the estimated standard deviation. For clarity, we name the CHORD using the known standard deviation CHORD$_\text{Prior}$ and the CHORD using estimated standard deviation CHORD$_\text{Esti}$, respectively.

Figure 4 shows the denoising performance under different noise levels. Under relatively weak noise ($\sigma \leq 0.03$), Cadzow achieves the lowest RLNE compared to other approaches. Under relatively high noise ($\sigma \geq 0.05$), however, the RLNEs of Cadzow increase faster than rQRd, particularly, CHORD, implying Cadzow is not robust to relatively high noise levels. The proposed method produces the lowest RLNE when the noise is higher than 0.04 and produces smallest variances. Furthermore, the results of CHORD with the estimated noise standard deviation are very close to that of CHORD with known noise standard deviation, indicating the feasibility of CHORD. In the following, without explicit illustration, the mentioned CHORD is CHORD$_\text{Esti}$.

![Fig. 4. The reconstruction error, RLNE, for synthetic data (Fig. 1(a)) under different noise levels. CHORD$_\text{Esti}$ and CHORD$_\text{Prior}$ denote denoised results of CHORD with estimated standard deviation and the known standard deviation, respectively. Cadzow and rQRd present the optimal (minimal RLNE) denoised results, respectively. The height height of columns shows the average of the RLNEs over 100 trials. The vertical bar comes from the randomness of noise.](image-url)
of the Hz, avoiding the ambiguity when spectrometers are at
shift is usually expressed in part per million (ppm) instead
the reference and add the Gaussian noise retrospectively.
NMR spectroscopy signals are commonly modeled
atively low SNR. Although multi-scanning improves the SNR
reasons that limits the widespread of this technology is the rel-
medicine, such as the diagnosis of diseases [10]. One of the
widely utilized in the study of chemistry, biology, and

B. Denoising of real NMR spectroscopy data
NMR spectroscopy, as a non-invaded technology, has been
widely utilized in the study of chemistry, biology, and
and medicine, such as the diagnosis of diseases [10]. One of the
reasons that limits the widespread of this technology is the rel-
low SNR. Although multi-scanning improves the SNR
of the spectroscopy, accordingly, the acquisition time multiply
increases. NMR spectroscopy signals are commonly modeled
as the sum of damped complex exponential signals as (1).
Therefore, we evaluate CHORD on the denoising of an NMR
spectroscopy data. We acquired the signal with high SNR as
the reference and add the Gaussian noise retrospectively.

In real NMR spectroscopy applications, the unit of chemical
shift is usually expressed in part per million (ppm) instead
of the Hz, avoiding the ambiguity when spectrometers are at
different magnet strengths. The definition of chemical shift is
given by

\[
\text{chemical shift(ppm)} = \frac{f_{\text{test}} - f_{\text{ref}}}{f_{\text{spec}}} \times 10^6, \tag{29}
\]

where \(f_{\text{test}}\) denotes the resonance frequency of the sample,
\(f_{\text{ref}}\) the absolute resonance frequency of a standard compound
measured in the same magnetic field, and \(f_{\text{spec}}\) the frequency of
the magnetic field strength of spectrometers.

The real data is a 1D \(^1\text{H}\) NMR spectrum that was acquired
at 298 K on a Varian 500 MHz NMR system (Agilent
Technologies, Santa Clara, CA, USA) equipped with a 5
mm indirect detection probe. A standard 1D pulse sequence
was used. An experiment time of \(2\) s (delay time \(1\) s and
acquisition time \(1\) s). The sample is a mixture consisting of
creatine, choline, magnesium citrate and calcium citrate. The
concentration of these metabolites is 2:2:1:1.

The denoised results of the metabolic spectrum are pre-
SENT in Figure. 7, which supports the conclusion made
on the simulated data. Under a relatively strong noise level
(\(\sigma = 0.035\)), Cadzow smooths the spectrum, which, on the
one side, offers a nice noise removing results, on the other
side, however, leads to the missing of some peaks (such as the
peaks at 6.8 ppm). rQRd provides a spectrum with obvious
noise (orange lines in Fig. 7(a)), and weakens low-intensity
peaks (such as the peaks at 6.8 and 3.7 ppm). CHORD is
capable of effectively removing noise and keeping more details
of peaks (see Fig. 7(a)). For the high SNR scenario, all the
three methods produce nice and comparable denoised results
(see Fig. 7(c)).

Experiments on synthetic complex exponential and realistic
NMR spectroscopy data demonstrate that CHORD with the
auto-setting parameter achieves more robust and accurate
results compared with Cadzow and rQRd method.

V. DISCUSSIONS
A. The estimate of noise
Ideally, the more data points used to estimate the noise, the
better estimation accuracy we can obtain. However, when the
noise is relatively large, it is difficult to distinguish signals
from noise. Thus, choosing a proper number of data points
for noise pose as a challenging task. In this subsection, we
discuss the effect of the number of data points used for noise
estimate on the denoised results of the simulated data in Fig.
3(a).

We performed experiments with different numbers of data
points from the end of the signal to estimate the noise. And then used the estimated standard deviation for spectrum
denoising (Figure. 8).

From the results in Fig. 8 (a), we observed that, using
fewer data points results in larger vertical bars, while using too
many data points causes a larger standard deviation estimate.
Besides, for the high SNR signal, the estimate is sensitive
to the selection of the number of data points (Fig. 8(a),
\(\sigma = 0.01, 0.02\)). Therefore, we recommend to use the last
100 data points. Notably, the results in Fig. 8(b) indicate that
the number of data points used for noise estimation makes no
distinction on the denoised results (in terms of RLNE) except in the high SNR case (red lines in Fig. 8(b)).

VI. CONCLUSION

Based on CHORD, a denoising method based on Hankel low-rankness of the complex exponential signals, we attempt to figure out the bound of the regularization parameter, determine the empirical optimal constant, and estimate the standard derivation of the noise, so that the users are able to apply CHORD with a automatically setting parameter. Experiments on simulated complex exponential and realistic NMR spectroscopy data demonstrate that CHORD with the auto-setting parameter achieves more robust and accurate results compared with Cadzow and rQRd method. 

For the future work, it is worthwhile to explore the probability distribution of the spectral norm instead of its expectation, and extend the 1-D model in (2) into the higher dimensionality distribution of the spectral norm instead of its expectation, with Cadzow and rQRd method.

APPENDIX A

THE PROOF OF LEMMA [2]

Here is the proof of Lemma [2]

Fig. 6. The typical denoising results comparison. (a) and (b) denote the simulated signals without and with noise (σ = 0.05) respectively. (c) is the denoising results of CHORD with the suggested parameter. (d)-(f) show denoised result of Cadzow with three different estimated ranks (small, optimal in terms of RLNE, and large). (g)-(i) are denoised results of rQRd with three different estimated ranks (small, optimal in terms of RLNE, and large).

Proposition 1. [37] For every $M < \infty$ there exists a constant $C(M) > 0$ such that, whenever $\{\psi_k\}_{k=0}^{2N}$ is a system of functions in an $L_2(\mu)$-space satisfying

$$(1^o) \|\psi_k\|_{L_2(\mu)} = 1 \text{ and } ||\phi_k||_{L_2(\mu)} \leq M \text{, for all } 0 \leq k \leq 2N,$$

$$(2^o) \left|\sum_{k=0}^{2N} d_k \psi_k\right|_{L_2(\mu)} \leq M \sqrt{\sum_{k=0}^{2N} |d_k|^2} \text{, for all } 0 \leq k \leq 2N,$$

and $\{p_k\}_{k=0}^{2N}$ are independent random variables over a probability space $(T, \mathcal{T}, \tau)$ with

$$(3^o) \mathbb{E} \left[p_k\right] = 0, \mathbb{E} |p_k|^2 = 1, \text{ and } 3\mathbb{E} |p_k|^3 \leq M \text{, for all } 0 \leq k \leq 2N,$$

then, for any choice of the coefficients of $\{d_k\}_{k=0}^{2N}$, we have

$$\mathbb{E} \left|\sum_{k=0}^{2N} d_k p_k \psi_k\right|_{L_\infty(\mu)} \geq C \sum_{k=0}^{2N} |d_k|^2 \left[1 + \log \left(\frac{\sum_{k=0}^{2N} |d_k|^2}{\sum_{k=0}^{2N} |d_k|^4}\right)\right].$$

Proof. According to (15), $\psi_k = e^{i2\pi k\omega}$. It is obvious that for all $0 \leq k \leq 2N$,

$$\|e^{i2\pi k\omega}\|_{L_2(\mu)} = 1 \text{ and } ||e^{i2\pi k\omega}\|_{L_3(\mu)} = 1. \quad (30)$$

According to the triangle inequality,

$$\left|\sum_{k=0}^{2N} d_k \psi_k\right|_{L_2(\mu)} \geq \sqrt{\int_0^1 \left|\sum_{k=0}^{2N} d_k e^{i2\pi k\omega}\right|^2 d\omega} = \sqrt{\sum_{k=0}^{2N} |d_k|^2}. \quad (31)$$

$p_k$ is a random variable which satisfies normal distribution, thus
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Fig. 7. Denoised results of a $^1$H spectrum of metabolites with $\sigma = 0.035$ (a), 0.020 (b), and 0.005 (c), respectively. The green lines denote the ground truth. The black lines indicate observation. The blue, orange and red line are denoised results of Cadzow, rQRd and CHORD, respectively. Note: The results of Cadzow and rQRd that enable the lowest RLNE are presented here.

Fig. 8. The effect of the number of data points. (a) denotes the standard deviation of the estimated noise with different data points. (b) denotes the RLNE of the CHORD denoising results with different estimated noise. The horizontal axis denotes the number of data points utilized in noise estimate. The vertical bars denote the standard deviation of the noise and RLNE, respectively.

$E(p_k) = 0$, $E|p_k|^2 = 1$ and $\sqrt{E|p_k|^2} = 0$. (32)

Combining (30), (31) and (32) yields Lemma 2.

APPENDIX B
THE PROOF OF LEMMA 3

According to the definition, $R_N^2$ is expressed as

$$R_N^2 = 4 \sum_{k=0}^{N} \left( \frac{1}{(k+1)(k+2)} \sum_{m=0}^{k} \frac{1}{m+1} \right)^2$$

$$+ 4 \sum_{k=N+1}^{2N} \left( \frac{1}{(2N-k+1)(k+2)} \sum_{m=k}^{2N} \frac{1}{m-N+1} \right)^2.$$

The sequence $R_N^{(1)}$ is positive, and increases as $N$ increases. It is straight forward

$$1 \leq \sum_{m=0}^{k} \frac{1}{m+1} \leq k + 1.$$  (34)

Let $F_1(N) = \int_0^N \frac{1}{(l+1)^2(l+2)^2} dl$. The sequence $R_N^{(1)}$ satisfies the following upper bound and lower bound

$$R_N^{(1)} \geq \sum_{k=0}^{N} \frac{1}{(k+1)^2(k+2)^2} > F_1(N)$$

$$= \frac{3}{2} - 2 \log 2 - \left( \frac{1}{N+1} + \frac{1}{N+2} \right)$$

$$+ 2 \log \left( 1 + \frac{1}{N+1} \right).$$  (35)
and

\[ R_N^{(1)} \leq \sum_{k=0}^{N} \frac{1}{(k+2)^2} < 1 - \frac{1}{N+2}. \]  

(36)

Combining (34) and (35) gives rise to

\[ \frac{3}{2} - 2 \log 2 < \lim_{N \to +\infty} R_N^{(1)} < 1. \]  

(37)

Since \( R_N^{(1)} \) increases as \( N \) increases,

\[ \lim_{N \to +\infty} R_N^{(1)} = C_{R_1}, \]  

(38)

where \( \frac{3}{2} - 2 \log 2 < C_{R_1} < 1. \)

The sequence \( R_N^{(2)} \) can be rewritten as

\[ R_N^{(2)} = \sum_{k=0}^{N-1} \left( \frac{1}{(N-k)(k+N+3)} \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^2. \]  

(39)

For \( \left( \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^2 \), where \( k = 0, 1, \ldots, N - 1 \), it is obvious that

\[ \left( \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^2 \leq \left( \sum_{m=0}^{N-1} \frac{1}{m+2} \right)^2. \]  

According to Cauchy-Buniakowsky-Schwarz inequality,

\[ \left( \sum_{m=0}^{N-1} \frac{1}{m+2} \right)^2 \leq \sum_{m=0}^{N-1} \left( \frac{1}{m+2} \right)^2 \sum_{m=0}^{1} \leq N \left( 1 - \frac{1}{N+1} \right) < N. \]  

(40)

We have

\[ 0 \leq \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} < \sqrt{N}. \]  

(41)

Denote \( g(k) = \frac{N}{(N-k)^2(k+N+3)^2} \) where \( k = 0, 1, 2, \ldots, N - 1 \). The limit of \( \sum_{k=0}^{N-1} g(k) \) can be calculated as

\[
\begin{align*}
\sum_{k=0}^{N-1} g(k) &= \sum_{k=0}^{N-1} \frac{1}{(N-k)^2(2N+3)^2(N-k)} \\
&= \sum_{k=0}^{N-1} \frac{2N}{(N-k)(2N+3)^4(N-k)} + \frac{2N}{(N-k)(2N+3)^4(N-k)(N-k)^2} + \frac{N}{(N-k)(2N+3)^4(k+N+3)^2} \\
&+ \frac{N}{(2N+3)^4(N-k)^2} + \frac{N}{(2N+3)^4(k+N+3)^2} < \frac{4N^2}{(2N+3)^3} + \frac{2N-1}{(2N+3)^3} + \frac{N^2}{(N+1)(N+2)(2N+3)^2} \\
&< \frac{2N}{(2N+3)^4} + \frac{2N-1}{(2N+3)^3} + \frac{2N-1}{(2N+3)^2} + \frac{2N-1}{N(2N+3)}. \\
\end{align*}
\]

(42)

Thus

\[ \lim_{N \to +\infty} \sum_{k=0}^{N-1} g(k) = 0. \]  

(43)

Since \( R_N^{(2)} \geq 0 \) and it satisfies \( R_N^{(2)} < \sum_{k=0}^{N-1} g(k) \),

\[ \lim_{N \to +\infty} R_N^{(2)} = 0. \]  

(44)

Combining (37) and (43) results in Lemma 3.

**Appendix C**

**The proof of Lemma 3**

According to the definition, \( Q_N^{(1)} \) is given by

\[
Q_N^{(1)} = 16 \sum_{k=0}^{N} \left( \frac{1}{(k+1)(k+2)} \sum_{m=0}^{k} \frac{1}{m+1} \right)^4 \]

\[
+ 16 \sum_{k=N+1}^{2N} \left( \frac{1}{(2N-k+1)(k+2)} \sum_{m=k}^{2N} \frac{1}{m-N+1} \right)^4 Q_{k-2N}^{(2)}.
\]

(45)

The same technique as Lemma 3 to prove (45)

\[ \frac{111}{8} - 20 \log 2 \leq \lim_{N \to +\infty} Q_N^{(1)} < \frac{1}{3}, \]  

(46)

and the details will not be shown here.

The sequence \( Q_N^{(2)} \) is restated as

\[
Q_N^{(2)} = \sum_{k=0}^{N-1} \left( \frac{1}{(N-k)(k+N+3)} \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^4.
\]  

(47)

According to (39),

\[
\left( \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^4 \leq \left( \sum_{m=0}^{N-1} \frac{1}{m+2} \right)^4 < N^2 \leq (k + N + 3)^3.
\]  

(48)

\[
0 \leq \left( \sum_{m=0}^{N-1-k} \frac{1}{m+k+2} \right)^4 < (k + N + 3)^3. \]  

(49)

Define \( h(k) = \frac{1}{(N-k)^4(k+N+3)} \) where \( k = 0, 1, 2, \ldots, N - 1 \), then the limit of \( \sum_{k=0}^{N-1} \frac{1}{(N-k)^4(k+N+3)} \) can be calculated as

\[
\begin{align*}
\sum_{k=0}^{N-1} \frac{1}{(N-k)^4(k+N+3)} &= \sum_{k=0}^{N-1} \frac{1}{(N-k)(2N+3)^4(N-k)} + \sum_{k=0}^{N-1} \frac{1}{(2N+3)^3(N-k)^2} \\
&+ \sum_{k=0}^{N-1} \frac{1}{(2N+3)^4(N-k)^2} + \sum_{k=0}^{N-1} \frac{1}{(2N+3)^4(k+N+3)^2} \\
&< \frac{2N}{(2N+3)^4} + \frac{2N-1}{(2N+3)^3} + \frac{2N-1}{(2N+3)^2} + \frac{2N-1}{N(2N+3)}. \\
\end{align*}
\]  

(50)
Thus
\[
\lim_{N \to +\infty} \sum_{k=0}^{N-1} h(k) = 0. \tag{51}
\]
Since sequence \(Q_n^2 \geq 0\) and it satisfies \(Q_n^2 < \sum_{k=0}^{N-1} h(k)\),
\[
\lim_{N \to +\infty} Q_n^2 = 0. \tag{52}
\]
Combining (46) and (52) yields Lemma 4.

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