Primordial Magnetic Field Limits from CMB Trispectrum - Scalar Modes and Planck Constraints

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Abstract

Cosmic magnetic fields are observed to be coherent on large scales and could have a primordial origin. Non-Gaussian signals in the cosmic microwave background (CMB) are generated by primordial magnetic fields as the magnetic stresses and temperature anisotropy they induce depend quadratically on the magnetic field. We compute the CMB scalar trispectrum on large angular scales, for nearly scale-invariant magnetic fields, sourced via the Sachs-Wolfe effect. The trispectra induced by magnetic energy density and by magnetic scalar anisotropic stress are found to have typical magnitudes of approximately $10^{-29}$ and $10^{-19}$, respectively. The scalar anisotropic stress trispectrum is also calculated in the flat-sky approximation and yields a similar result.

Observational limits on CMB non-Gaussianity from the Planck mission data allow us to set upper limits of $B_0 \lesssim 0.6$ nG on the present value of the primordial cosmic magnetic field. Considering the inflationary magnetic curvature mode in the trispectrum can further tighten the magnetic field upper limit to $B_0 \lesssim 0.05$ nG. These sub-nanoGauss constraints from the magnetic trispectrum are the most stringent limits so far on the strength of primordial magnetic fields, on megaparsec scales, significantly better than the limits obtained from the CMB bispectrum and the CMB power spectrum.

I. INTRODUCTION

Magnetic fields have been observed throughout the Universe, on all scales probed so far, from planets and stars to the large-scale magnetic fields detected in galaxies and galaxy clusters [1–8]. Both large-scale as well as stochastic components are present in magnetic fields observed in galaxies with magnitudes from a few to tens of microGauss. Coherent magnetic fields of a similar strength are also observed in higher redshift galaxies [8, 10]. In clusters of galaxies, stochastic magnetic fields of a few microGauss strength are present, correlated on ten kiloparsec scales [3, 4]. Moreover, there is circumstantial evidence of an intergalactic magnetic field that is present over most of the cosmic volume, even in the voids of large scale structure. A lower bound of $10^{-16} - 10^{-15}$ Gauss for such a pervasive intergalactic magnetic field has been derived from gamma-ray observations of blazars [11–13].

The origin as well as evolution of such large-scale magnetic fields remains an outstanding problem. Magnetic fields in collapsed structures can arise from dynamo amplification of seed magnetic fields [6–8]. The seed field could in turn be generated in astrophysical batteries [14–17] or due to processes in the early universe [18–28]. Indeed, the recent gamma-ray observations suggesting a lower limit to an all-pervasive intergalactic magnetic field [11, 13], would perhaps favour a primordial origin. A primordial magnetic field can be generated at inflation [8, 18, 23], or arise out of other phase transitions in the early Universe [24–28]. As yet there is no compelling mechanism which produces strong coherent primordial fields. Equally, the dynamo paradigm is not without its own challenges in producing sufficiently coherent fields and sufficiently rapidly [6–8]. Therefore, it is useful to keep open the possibility that primordial magnetic fields originating in the early universe play a crucial role in explaining the observed cosmic magnetism.

In this context it is important to investigate every possible observable signature of the putative primordial magnetic field. Magnetic fields give rise to scalar, vector and tensor metric perturbations as well as fluid perturbations via the Lorentz force. Constraints on large scale primordial magnetic fields have already been derived using the CMB temperature and polarization power spectra [29–36] and Faraday rotation [37–39]. However, the effects of a primordial magnetic field on the CMB are relatively more pronounced in its non-Gaussian correlations. This arises due to the fact that magnetic fields induce non-Gaussian signals at lowest order as the magnetic energy density and stress are quadratic in the field. In contrast, the standard inflationary perturbations, dominated by their linear component, can source non-Gaussian correlations only with higher order perturbations and thus necessarily can only produce a small amplitude of CMB non-Gaussianity (cf. [40–48]). Primordial magnetic fields have been shown capable of inducing appreciable CMB non-Gaussianity when considering the bispectrum [49–59]. Our earlier calculation of the magnetic CMB bispectrum sourced by scalar anisotropic stress led to a $\sim 2$ nG upper limit on the primordial magnetic field’s amplitude on megaparsec scales [59]. However, higher-order measures of non-Gaussianity like the trispectrum have been less investigated and as we show here, are very useful to
set further constraints on primordial magnetic fields.

In this article we present in detail the primordial magnetic field contribution to the CMB scalar mode trispectrum. The principal results were summarized in our earlier Letter [63], where WMAP5 and WMAP7 constraints on non-Gaussianity were utilized to derive magnetic field constraints. Here we present the full trispectrum calculations as well as an additional flat-sky calculation for the scalar anisotropic stress trispectrum. Furthermore, the new constraints on non-Gaussianity from the Planck mission 2013 data release [61] are utilized to obtain improved magnetic field constraints. We find that the trispectrum does better than the bispectrum at probing magnetic fields on large scales. We also show that even stronger constraints can be imposed on magnetic fields by considering the recently discussed magnetic inflationary curvature mode [62].

In the next section we describe the properties of the stochastic primordial magnetic field assumed for our calculations. The Sachs-Wolfe effect sourcing by the magnetic energy density of a stochastic primordial magnetic field is presented in Sec. III. The full mode-coupling calculations are then presented for the four-point correlation of magnetic energy density. In Sec. IV we present the Sachs-Wolfe effect and four-point calculation for magnetic scalar anisotropic stress. The magnetic CMB trispectrum is then calculated for energy density and scalar anisotropic stress in Sec. V. Additionally, in Sec. VI the trispectrum sourced by magnetic scalar anisotropic stress is also calculated using the flat-sky approximation. Finally, in Sec. VII the Planck 2013 data release constraints on CMB non-Gaussianity [61] are used to place improved upper limits on the strength of primordial magnetic fields.

II. PRIMORDIAL MAGNETIC FIELD

We consider a Gaussian random stochastic magnetic field \( \mathbf{B} \) characterized and completely specified by its power spectrum \( M(k) \). We further assume that the magnetic field is non-helical. On scales that are galactic and larger, any velocity induced by Lorentz forces is generally too small to appreciably distort the initial magnetic field \([63,64]\). Therefore, the magnetic field simply redshifts away as \( \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0(\mathbf{x})/a^2 \), where, \( \mathbf{B}_0 \) is the magnetic field at the present epoch (i.e. at \( z = 0 \) or \( a = 1 \)). We define \( b(k) \) as the Fourier transform of the magnetic field \( \mathbf{b}_0(\mathbf{x}) \). The magnetic field power spectrum is defined as

\[
\langle b_i(k)b_j^*(q) \rangle = (2\pi)^3 \delta(k-q) P_{ij}(k)M(k) \tag{1}
\]

where \( P_{ij}(k) = (\delta_{ij} - k_i k_j / k^2) \) is the projection operator ensuring \( \nabla \cdot \mathbf{b}_0 = 0 \). This gives \( b_0^2 = 2 \int (dk/k) \Delta^2(k) \), where \( \Delta^2(k) = k^3 M(k)/(2\pi^2) \) is the power per logarithmic interval in \( k \)-space present in the stochastic magnetic field. We also assume a power-law magnetic power spectrum, \( M(k) = Ak^n \) that is cutoff at \( k = k_c \), where \( k_c \) is the Alfvén-wave damping length-scale \([63,64]\). We then fix the normalization \( A \) by setting the variance of the magnetic field to be \( B_0 \), smoothed using a sharp \( k \)-space filter, over a ‘galactic’ scale \( k_G = 1h \) Mpc\(^{-1} \). This gives, (for \( n \gtrsim -3 \) and for \( k < k_c \))

\[
\Delta^2(k) = \frac{k^3 M(k)}{2\pi^2} = \frac{B_0^2}{2}(n+3) \left( \frac{k}{k_G} \right)^{3+n}. \tag{2}
\]

We restrict the magnetic spectral index to values near and above -3, i.e an inflation-generated field, as causal generation mechanisms necessarily produce much bluer magnetic power spectra \([65]\). Furthermore, blue spectral indices, on large scales, are strongly disfavoured by many observational constraints on primordial magnetic fields like the CMB power spectra \([29,33]\).

III. CMB ANISOTROPY FROM MAGNETIC ENERGY DENSITY AND FOUR-POINT CORRELATION

The Sachs-Wolfe type of contribution to the CMB temperature anisotropy sourced by the energy density of magnetic fields \([65,68]\), can be written as

\[
\frac{\Delta T}{T}(n) = \Omega_B(x_0 - nD^*). \tag{3}
\]

Here, \( \Omega_B(x) = B^2(x,t)/(8\pi \rho_r(t)) = b_0^2(x)/(8\pi \rho_0) \), where \( \rho_r(t) \) and \( \rho_0 \) are the CMB energy densities at times \( t \) and at the present epoch, respectively. Like the usual Sachs-Wolfe effect, the \( \Delta T/T \) given above is for large-angular scales. For calculating numerical values we adopt the \( \Omega_B \) value estimated by Bonvin and Caprini (Eq. 6.12 of \[68]\) which is expressed according to our definitions as \( \Omega_B = -R_x/15 \sim -0.04 \), where \( R_x \sim 0.6 \) is the fractional contribution of radiation energy density towards the total energy density of the relativistic component. The unit vector \( \mathbf{n} \) is defined along the direction of observation from the observer at position \( x_0 \) and \( D^* \) is the (comoving angular diameter) distance to the surface of last scattering. We have assumed instantaneous recombination which is a good approximation for large angular scales.

The temperature fluctuations of the CMB can be expanded in terms of spherical harmonics to give \( \Delta T(n)/T = \sum_{l_m} a_{lm} Y_{lm}(n) \), where

\[
a_{lm} = \frac{4\pi}{l^2} \int \frac{d^3k}{(2\pi)^3} \Omega_B(k) j_l(kD^*) Y_{lm}^*(\hat{k}). \tag{4}
\]

Note that \( \Omega_B(k) \) is the Fourier transform of \( \Omega_B(x) \). As \( \Omega_B(x) \) is quadratic in \( b_0(x) \), \( \Omega_B(k) \) is given by the convolution integral

\[
\Omega_B(k) = \left(1/(2\pi)^3\right) \int d^3s \ b_s(k+s) b_s^*(s)/(8\pi \rho_0). \tag{5}
\]

The trispectrum \( T^{m_1 m_2 m_3 m_4}_{l_1 l_2 l_3 l_4} \), or the four-point correlation function of the CMB temperature anisotropy in harmonic space, in terms of the \( a_{lm} \)'s is

\[
T^{m_1 m_2 m_3 m_4}_{l_1 l_2 l_3 l_4} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \rangle. \tag{6}
\]
From Eq. (3) we can express $T_{l_1, l_2, l_3, l_4}^{m_1, m_2, m_3, m_4}$ as

$$T_{l_1, l_2, l_3, l_4}^{m_1, m_2, m_3, m_4} = \left( \frac{R}{2\pi^2} \right)^4 \int \prod_{i=1}^{4} \frac{d^3k_i}{(2\pi)^3} J_i(k_i D^*) Y_{l_i m_i}^*(\hat{k}_i) \zeta_{1234}$$

(7)

with

$$\zeta_{1234} = \langle \Omega_B(k_1) \Omega_B(k_2) \Omega_B(k_3) \Omega_B(k_4) \rangle.$$  

(8)

The four-point correlation function of $\Omega_B(k)$ involves an eight-point correlation function of the magnetic fields. Using Wick’s Theorem, for Gaussian magnetic fields, we can express the magnetic eight-point correlation as a sum of 105 terms containing the magnetic two-point correlation. Neglecting 45 terms proportional to $\delta(k)$ that vanish and 12 terms proportional to $\delta(k_1 + k_2)$ that are the unconnected part of the four-point correlation, 48 terms remain. A long calculation using the relevant projection operators gives $\zeta_{1234} = \delta(k_1 + k_2 + k_3 + k_4) \psi_{1234}$, where $\psi_{1234}$ is a mode-coupling integral over a variable $s$ and also contains angular terms.

The full expression for $\psi_{1234}$ involving angular terms in the mode-coupling integral is

$$\psi_{1234} = \frac{8}{(8\pi\rho_0)^2} \int d^3s M(s) M(|k_1 + s|) \left[ M(|k_1 + k_3 + s|) (M(|k_2 - s|) \mathcal{F}_{(1)} + M(|k_4 - s|) \mathcal{F}_{(2)}) + M(|k_1 + k_2 + s|) (M(|k_3 - s|) \mathcal{F}_{(3)} + M(|k_4 - s|) \mathcal{F}_{(4)}) + M(|k_1 + k_4 + s|) (M(|k_2 - s|) \mathcal{F}_{(5)} + M(|k_3 - s|) \mathcal{F}_{(6)}) \right]$$

(9)

with

$$\mathcal{F}_{(1)} = -1 + (a_1^2 + a_2^2 + a_3^2 + a_4^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2) - (a_1 a_2 b_2 + a_1 a_3 b_3 + a_1 a_4 b_4 + b_1 b_2 b_3 + b_1 b_2 b_4) + a_1 a_2 b_3 b_4$$

$$\mathcal{F}_{(2)} = -1 + (a_2^2 + a_3^2 + a_4^2 + b_2^2 + b_3^2 + b_4^2 + c_2^2) - (a_1 a_3 b_3 + a_1 a_4 b_4 + a_1 a_2 b_5 + a_2 a_3 b_5 + a_2 a_4 b_5) + a_1 a_2 a_3 b_5$$

$$\mathcal{F}_{(3)} = -1 + (a_3^2 + a_4^2 + a_5^2 + a_6^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2) - (a_1 a_3 b_3 + a_2 b_5 b_6 + a_2 a_3 b_6 + a_2 a_4 b_6 + a_3 a_4 b_6) + a_1 a_2 b_5 b_6$$

$$\mathcal{F}_{(4)} = -1 + (a_4^2 + a_5^2 + a_6^2 + b_4^2 + b_5^2 + b_6^2 + c_4^2) - (a_1 a_4 b_4 + a_2 b_5 b_6 + a_2 a_4 b_6 + a_3 a_4 b_6) + a_1 a_2 a_3 b_6$$

$$\mathcal{F}_{(5)} = -1 + (a_5^2 + a_6^2 + a_7^2 + a_8^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2) - (a_1 a_5 b_5 + a_2 b_6 b_8 + a_2 a_5 b_8 + a_3 a_5 b_8 + a_4 a_5 b_8) + a_1 a_2 a_3 b_8$$

$$\mathcal{F}_{(6)} = -1 + (a_6^2 + a_7^2 + a_8^2 + b_6^2 + b_7^2 + b_8^2 + d_6^2) - (a_1 a_6 b_6 + a_2 b_7 b_8 + a_2 a_6 b_8 + a_3 a_6 b_8 + a_4 a_6 b_8 + a_5 a_6 b_8) + a_1 a_2 a_3 b_8.$$  

(10)

The angular terms $\mathcal{F}$ contain angles defined according to

$$\phi_1 = \hat{\omega} \cdot \hat{k}_1 + s, \quad \phi_2 = \hat{\omega} \cdot \hat{k}_2 - s, \quad \phi_3 = \hat{\omega} \cdot \hat{k}_3 - s,$$

$$\phi_4 = \hat{\omega} \cdot \hat{k}_1 - s, \quad \phi_5 = \hat{\omega} \cdot \hat{k}_1 + \hat{k}_2 + s,$$

$$\phi_6 = \hat{\omega} \cdot \hat{k}_1 + \hat{k}_3 + s, \quad \phi_7 = \hat{\omega} \cdot \hat{k}_1 + \hat{k}_4 + s,$$

where $\hat{k}_1 + s$ is a unit vector in the direction of $(k_1 + s)$ and the angle $\phi$ denotes different angles for different values of the
unit vector $\hat{\omega}$

\[
\phi = \alpha \text{ for } \hat{\omega} = \hat{s}, \quad \phi = \beta \text{ for } \hat{\omega} = \hat{k}_1 + \hat{s}, \\
\phi = \gamma \text{ for } \hat{\omega} = \hat{k}_2 - \hat{s}, \quad \phi = \delta \text{ for } \hat{\omega} = \hat{k}_3 - \hat{s}, \\
\phi = \epsilon \text{ for } \hat{\omega} = \hat{k}_4 - \hat{s}, \quad \phi = \kappa \text{ for } \hat{\omega} = \hat{k}_1 + \hat{k}_2 + \hat{s}, \\
\phi = \lambda \text{ for } \hat{\omega} = \hat{k}_1 + \hat{k}_3 + \hat{s}.
\]

For simplicity of calculation we evaluate the mode-coupling integral $\psi_{1234}$ in two cases: (I) considering only $s$-independent angular terms for all equal-sided configurations and (II) taking all angular terms for the collinear configuration.

A. Case I - $s$-independent terms for equal-sided configurations

Considering only $s$-independent angular terms, for a general configuration, we find $\psi_{1234} = -8/(8\pi \rho_0)^4 I$ where

\[
I = \int d^3s \, M(s) \, M(|k_1 + s|) \times \\
\left[ M(|k_1 + k_3 + s|) \left( M(|k_2 - s|) + M(|k_4 - s|) \right) \right. \\
+ M(|k_1 + k_2 + s|) \left( M(|k_3 - s|) + M(|k_4 - s|) \right) \\
+ M(|k_1 + k_4 + s|) \left( M(|k_2 - s|) + M(|k_3 - s|) \right) \\
= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6.
\]

We perform the mode-coupling integral employing the technique and approximations discussed in [52, 69, 71], while adopting the mean (zero) value of $\hat{k}_1 \cdot \hat{k}_3$, to find, for the first term,

\[
\mathcal{I}_1 \simeq 4\pi A^4 k_1^{2n+3} k_2^n k_3^{n/2} \left[ \frac{2n/2}{n + 3} - \frac{1}{4n + 3} \right].
\]

The value of each of the $\mathcal{I}_{i,j}$ integrals for $j = 1$ to 6 is the same when all the $k_i$ wavevectors are of equal magnitude $|k_i| = k$. We perform the $s$-independent (case I) trispectrum evaluation for such equal-sided quadrilateral configurations. Hence, $I = \sum_{j=1}^{6} \mathcal{I}_j = 6 \mathcal{I}_{11}$, and we obtain

\[
\zeta_{1234} = \frac{\delta(k_1 + k_2 + k_3 + k_4) \times}{-8 (2\pi)^4 A^4 k_1^{2n+3} k_2^n k_3^{n/2} (8\pi \rho_0)^4} \left[ \frac{(2n/2)(4n + 3) - (n + 3)}{(4n + 3)(n + 3)} \right].
\]

B. Case II - Equal-Sided Collinear Configuration

We calculate the full mode-coupling integral $\psi_{1234}$ (Eq. 12) (over all angular terms for each $F$ expression) for the case of the equal-sided collinear configuration. All the four wavevectors are of equal magnitude with configuration $k_1 = k_2 = -k_3 = -k_4$. We find that the 28 independent angles defined by Equations (11-12) reduce to just 6 independent angles $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and $\gamma_5$. The angular expressions $F$ also reduce in size from a total of 72 to 19 angular terms:

\[
\psi_{1234}^{\text{coll}} = \frac{8}{(8\pi \rho_0)^4} F_{1234}^{\text{coll}}
\]

where

\[
F_{1234}^{\text{coll}} \simeq 2 \int d^3s \, M(s) \, M(|k + s|) \times \\
\left[ M(s)M(|k - s|) \left( \alpha_1^2 + \alpha_2^2 + \beta_1^2 - 2\alpha_1\alpha_2\beta_2 + \alpha_2^2 \right) \right. \\
+ M(s)M(|k + s|) \left( 1 + \alpha_2^2 \right) \\
+ M(|2k + s|)M(|k + s|) \left( \alpha_1^2 + \alpha_2^2 + \beta_1^2 - \alpha_1\alpha_5\beta_5 \right. \\
\left. \left. + \frac{1}{2} \left\{ \delta_3^2 + \epsilon_5 + (\beta_5 + \alpha_1\alpha_5 - \alpha_2^2\beta_5) (\delta_5 + \epsilon_5) \right\} \right] \right) \\
\frac{1}{2} \left\{ \delta_3^2 + \epsilon_5 + (\beta_5 + \alpha_1\alpha_5 - \alpha_2^2\beta_5) (\delta_5 + \epsilon_5) \right\}.
\]

Using the same technique of evaluating the mode-coupling integrals as used earlier in Case I, we calculate the integrals for each of the 19 angular terms that sum together to give

\[
F_{1234}^{\text{coll}} \simeq 4\pi A^4 k_1^{2n+3} k_2^n k_3^{n/2} \left[ \frac{8 \cdot 2^{n/2}(4n + 3) - (12)(n + 3)}{(4n + 3)(n + 3)} \right].
\]

IV. CMB ANISOTROPY FROM MAGNETIC SCALAR ANISOTROPIC STRESS AND FOUR-POINT CORRELATION

The scalar anisotropic stress that is associated with a primordial magnetic field, in addition to its energy density, will also act as a separate source for CMB fluctuations - the passive mode scalar anisotropic stress generates a temperature anisotropy is again via the magnetic Sachs-Wolfe effect [72]. As we saw in our previous work [59], the magnetic scalar anisotropic stress generates $\sim 10^6$ times larger contribution to the CMB bispectrum compared to magnetic energy density. With this motivation in mind and employing the magnetic CMB trispectrum technique developed above, we carry out a longer calculation for the scalar anisotropic stress trispectrum.

On large angular scales, the magnetic contribution to the temperature anisotropy is again via the magnetic Sachs-Wolfe effect

\[
\frac{\Delta T}{T}(\theta) = \frac{1}{3} \Phi_0 \left[ 3 + 3 \Pi_B \ln \left( \frac{\tau_0}{\tau_B} \right) \right] \\
\zeta \simeq \frac{1}{3} R_\Pi \ln \left( \frac{\tau_0}{\tau_B} \right).
\]
to obtain temperature anisotropy, sourced by magnetic scalar anisotropic stress $\Pi_B$

$$\frac{\Delta T}{T}(n) = R_p \Pi_B(x_0 - nD^*), \quad (22)$$

where $R_p = R \ln (\tau_\nu / \tau_B) = [-R_\gamma/15] \ln (T_B/T_\nu)$ and $\tau_B$ as well as $T_\nu$ and $T_B$ as well as $T_\nu$ are the conformal time and temperatures at the epochs of magnetic field generation and neutrino decoupling, respectively. None of the details of the magnetic scalar anisotropic stress calculation were included in our letter [60] and they are presented below.

The CMB temperature fluctuations can be expanded in terms of spherical harmonics to give $\Delta T(n)/T = \sum_{lm} a_{lm} Y_{lm}(n)$, where

$$a_{lm} = \frac{4\pi}{i} \int \frac{d^3k}{(2\pi)^3} R_p \Pi_B(k) j_i(kD^*) Y_{lm}^*(\hat{k}). \quad (23)$$

Here, $\Pi_B(k)$ is the Fourier transform of $\Pi_B(x)$ and we recall the operator that projects out the scalar anisotropic stress from the full magnetic stress $\Pi_B^I(k)$

$$\Pi_B(k) = \frac{1}{2} \left( \delta_{ij} - 3k_i k_j \right) \Pi_B^I(k) \quad (24)$$

Since $\Pi_B(x)$ is quadratic in $b_0(x)$, we have a convolution of magnetic fields

$$\Pi_B(k) = \frac{1}{2} \left( \delta_{ij} - 3k_i k_j \right) \frac{1}{4\pi p^2} \int \frac{d^3s}{(2\pi)^3} b_i^*(s)b_j(k + s) \quad (25)$$

The trispectrum is $T^{m_1 m_2 m_3 m_4}_{l_1 l_2 l_3 l_4} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \rangle$, is then given by

$$T^{m_1 m_2 m_3 m_4}_{l_1 l_2 l_3 l_4} = \left( \frac{R_p}{2\pi^2} \right)^4 \int \prod_{i=1}^4 \frac{d^3k_i}{(2\pi)^3} j_i(k, D^*) Y_{l_i m_i}^*(\hat{k}_i) \left[ \zeta_{1234} \right]_\Pi \quad (26)$$

with $\left[ \zeta_{1234} \right]_\Pi$ defined as

$$\left[ \zeta_{1234} \right]_\Pi = \langle \Pi_B(k_1) \Pi_B(k_2) \Pi_B(k_3) \Pi_B(k_4) \rangle. \quad (27)$$

The four-point correlation function of $\Pi_B(k)$, like that of $\Omega_B(k)$, also involves an eight-point correlation function of the fields. In similar fashion, using Wick’s Theorem, for Gaussian magnetic fields, we express the magnetic eight-point correlation as a sum of 105 terms involving the magnetic two-point correlation function. Then 45 terms proportional to $\delta(k)$ vanish and we neglect the 12 terms proportional to $\delta(k_1 + k_2)$ that represent the unconnected part of the four-point correlation, to leave 48 terms. A long calculation involving the relevant projection operators in these terms gives $\left[ \zeta_{1234} \right]_\Pi = \delta(k_1 + k_2 + k_3 + k_4)$ $\left[ \psi_{1234} \right]_\Pi$, where $\left[ \psi_{1234} \right]_\Pi$ is a mode-coupling integral over a variable $s$ and also involves angular terms. The key difference between the $\Omega_B$ and the $\Pi_B$ four-point correlations is the number and type of operators acting on the magnetic field eight-point correlation. In the case of energy density $\Omega_B$, the operator $\delta_{ab} \delta_{cd} \delta_{ef} \delta_{gh}$ acted on

$$\langle b_a(-s)b_b(k_1 + s)b_c(-r)b_d(k_2 + r)b_e(-t)b_f(k_3 + t)b_g(-w)b_h(k_4 + w) \rangle. \quad (28)$$

However, in the case of scalar anisotropic stress $\Pi_B$, there are 16 operator terms

$$\left( \delta_{ab} - 3 k_{1a} k_{1b} \right) \left( \delta_{cd} - 3 k_{2c} k_{2d} \right) \left( \delta_{ef} - 3 k_{3e} k_{3f} \right) \left( \delta_{gh} - 3 k_{4g} k_{4h} \right)$$

$$= \delta_{ab} \delta_{cd} \delta_{ef} \delta_{gh} - 3 \left[ \delta_{ab} \delta_{cd} \delta_{ef} k_{4h} + \delta_{ab} \delta_{cd} \delta_{ef} k_{4h} + \delta_{ab} \delta_{cd} \delta_{ef} k_{4h} + \delta_{ab} \delta_{cd} \delta_{ef} k_{4h} + k_{1a} k_{1b} \delta_{de} \delta_{ef} \delta_{gh} \right]$$

$$+ 9 \left[ \delta_{ab} \delta_{cd} \delta_{ef} k_{3e} k_{3f} k_{4h} + \delta_{ab} \delta_{cd} \delta_{ef} k_{3e} k_{3f} k_{4h} + \delta_{ab} \delta_{cd} \delta_{ef} k_{3e} k_{3f} k_{4h} + k_{1a} k_{1b} \delta_{de} \delta_{ef} \delta_{gh} \right]$$

$$+ 27 \left[ \delta_{ab} k_{2e} k_{2f} k_{3e} k_{4h} + k_{1a} k_{1b} \delta_{de} \delta_{ef} k_{4h} + k_{1a} k_{1b} \delta_{de} \delta_{ef} k_{4h} + k_{1a} k_{1b} \delta_{de} \delta_{ef} k_{4h} \right]$$

$$+ 81 k_{1a} k_{1b} k_{2e} k_{2f} k_{3e} k_{3f} k_{4h} k_{4h} = 1 + 2 + ... + 16 \quad (29)$$

Each operator term $X$ from 1 to 16 generates its own separate angular term expression $\mathcal{F}_I$. When summed over all $X$ this yields the angular term expression $\mathcal{F}_I$, where $I$ takes values 1 to 6 in the six term mode-coupling integral $\left[ \psi_{1234} \right]_\Pi$. As operator $\mathbf{1}$ is identical to the operator for the $\Omega_B$ four-point correlation, the angular terms $\mathcal{F}$ for it are just given by Equation (10). We give below the expressions for $\left[ \psi_{1234} \right]_\Pi$ and the angular terms $\mathcal{F}$ generated by operators $\mathbf{2}$ and $\mathbf{16}$ suppressing the II subscript. The complete expression for the full set of over 1500 angular terms generated by all sixteen operators $\mathbf{1}$ through $\mathbf{16}$
with those combinations of angular terms for operator $\hat{s}$ that are constant for a given $(k_1, k_2, k_3, k_4)$
configuration also appear. In total, as pointed out above, over 1500 angular terms are present in all the $F$ expressions for $\psi_{1234}$, many more than the 72 terms for $\omega_{1234}$. To arrive at a representative estimate for $F$, we consider only the $s$-independent angular terms and restrict ourselves to equal-sided trispectrum configurations i.e. all $|k_i| \approx k$. The $s$-independent terms are

$$F_{s-indep} = 6 \left[ -13 + 9 (\theta_{12}^2 + \theta_{13}^2 + \theta_{14}^2 + \theta_{23}^2 + \theta_{24}^2 + \theta_{34}^2) ight. \
- 27 (\theta_{12}\theta_{13}\theta_{23} + \theta_{12}\theta_{14}\theta_{24} + \theta_{13}\theta_{14}\theta_{34} + \theta_{23}\theta_{24}\theta_{34}) \
+ 27 (\theta_{12}\theta_{13}\theta_{24}\theta_{34} + \theta_{12}\theta_{14}\theta_{23}\theta_{34} + \theta_{13}\theta_{14}\theta_{23}\theta_{24}) \right].$$

(33)

We evaluate $F_{s-indep}$ for specific equal-sided trispectrum configurations: collinear, square, rhombus and tetrahedral. Table II lists the values of $F_{s-indep}$ for the specific configurations $(k_1, k_2, k_3, k_4)$, showing that the greatest contribution to $\psi_{1234}$ and therefore to the scalar anisotropic stress trispectrum arises from the collinear configuration. The values for $F_{s-indep}$ range from $\sim -2$ to 14. We adopt a value of 10 as a typical value for the sum of all $s$-independent terms and denote it by $\xi$. We get a mode-coupling integral with an integrand that matches the $\psi_{1234}$ for the $\Omega_B$ $s$-independent equal-sided configuration case I Equation (13)

$$\psi_{1234} = \frac{8 \xi}{(8\pi\rho_0)^4} I = \frac{8 (3^4) \xi}{(8\pi\rho_0)^4} I$$

(34)

where

$$I = \int d^3s \ M(s) \ M(|k_1 + s|) \times \left( M(|k_2 + s|) + M(|k_4 - s|) \right) \times \left( M(|k_2 - s|) + M(|k_4 - s|) \right)$$

(35)

The integral $I$ is evaluated as earlier to yield the four-point correlation of the scalar anisotropic stress to be

$$[\zeta_{1234}] = \delta(k_1 + k_2 + k_3 + k_4) \times$$

$$3^4 \xi \frac{8 (24\pi A^4 k_1^{2n+3} k_2^{2n} k_3^{2n} k_4^{2n} \left[ (2n)^2 (4n + 3) - (n + 3) \right])}{(8\pi\rho_0)^4} (4n + 3)(n + 3)$$

(36)

or simply expressed, in relation to the four-point correlation of energy density,

$$[\zeta_{1234}] = 3^4 \xi [-\zeta_{1234}]$$

(37)

V. MAGNETIC CMB TRISPECTRUM

Having calculated the four-point correlations, in Fourier space, of energy density $[\zeta_{1234}]$ and scalar anisotropic stress $[\zeta_{1234}]$, we can now calculate the CMB trispectrum sourced by each.

A. CMB Trispectrum from Magnetic Energy Density

For the trispectrum sourced by magnetic energy density $\Omega_B$, we insert Eq. (15) into Eq. (7) for the trispectrum and following the approach of [73,74], we decompose our delta function as $\delta(k_1 + k_2 + k_3 + k_4) = \int d^3K \delta(k_1 + k_2 + K) \delta(k_3 + k_4 - K)$. We can then write the trispectrum as

$$T_{1234} = (4\pi)^4 \left( \sum_{\Omega_B} \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \right)$$

$$\times \int \left[ \frac{8(\xi)}{(8\pi\rho_0)^4} \right] \times \left[ -\left(192\pi A^4 k_1^{2n+3} k_2^{2n} k_3^{2n} k_4^{2n} \left[ (2n)^2 (4n + 3) - (n + 3) \right] \right) \right]$$

$$\times \int d^3K \delta(k_1 + k_2 + K) \delta(k_3 + k_4 - K).$$

(38)
Using the integral form of the delta functions
\[
\int d^3K \delta(k_1 + k_2 + K) \delta(k_3 + k_4 - K) = \oint \frac{d^3K}{(2\pi)^3} d^3r_1 \int d^3r_2 e^{i(k_1 + k_2 + K) \cdot r_1} e^{i(k_3 + k_4 - K) \cdot r_2},
\]
and the spherical wave expansion
\[
e^{i\mathbf{k} \cdot \mathbf{r}} = 4\pi \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} j_{l'}(k r) Y_{l'm'}^{*}(\hat{r}) Y_{l'm'}(\hat{r}),
\]
we perform the integrals over the angular parts of \((k_1, k_2, k_3, k_4, K)\), with algebra similar to \([49, 59, 75, 76]\), to give
\[
T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4} = \left[-768 \frac{R^4}{\pi} \right] \left( \frac{A}{8\pi \rho_0} \right)^4 \times \int \left[ \left( \frac{2^{n/2}}{(4n + 3) - (n + 3)} \right) \right] \times \int dr_1 r_1^2 \int dr_2 r_2^2 \int dk_1 k_1^4 \int dk_2 k_2^{6n+3} \int dk_3 k_3^{2\ell_3} \int dk_4 k_4^{2\ell_4} \times \int \left[ -1 \right]^{LM} L M \int \frac{d\Omega_{\mathbf{k}}}{4\pi} Y_{l_1 m_1}(\hat{\mathbf{k}}_1) Y_{l_2 m_2}(\hat{\mathbf{k}}_2) Y_{l_3 m_3}(\hat{\mathbf{k}}_3) Y_{l_4 m_4}(\hat{\mathbf{k}}_4),
\]
we separate out the reduced trispectrum \(T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L)\) (referred to as the angular averaged trispectrum in \([78]\)), from the full trispectrum. We again use the spherical Bessel function closure relation to perform the \(k_4\)-integral that yields \(\delta(r_1 - D^*) (\pi/2r_1^4)\). This facilitates the \(r_1\)-integral that results in \(r_1 \rightarrow D^*\) in the arguments of \(j_1, j_2\) and \(j_3\). The \(k_1, k_2\) and \(k_3\)-integrals containing a product of a power-law and \(J^2\) can be evaluated in terms of Gamma functions (e.g. Eq. 6.574.2 of \([79]\)). For a scale-invariant magnetic index \(n \rightarrow -3\), we get
\[
T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L) \Omega \approx -5.8 \times 10^{-29} \left( \frac{n + 3}{0.2} \right)^3 \left( \frac{B_{-9}}{0.3} \right)^8 \times \frac{A}{8\pi \rho_0} \frac{h_{l_1} l_1 l_2 l_3 l_4}{l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)}.\]
This equation gives us the amplitude of the magnetic CMB trispectrum sourced by the energy density \(\Omega_B\) of a primordial magnetic field, where we have used \(R \sim -0.04\) \([68]\). A factor of \(1/(D^* k_G)^{4(n+3)}\) also appears here and it approaches unity for the case \(n \rightarrow -3\) (a scale-invariant magnetic field index). When we evaluate the magnetic trispectrum for a near scale-invariant index \(n = -2.8\), this factor has a value \(\sim 1/1500\). It then turns out that this factor is almost entirely canceled by the simultaneous increase in the value of the \(k\)-integrals when evaluated for \(n = -2.8\) rather than \(n = -3\).

For the case II - collinear configuration case, proceeding from Eq. \([19]\) in exactly the same way as case I, we find that the amplitude of the collinear configuration trispectrum is
\[
T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L) \Omega \approx 3.9 \times 10^{-29} \left( \frac{n + 3}{0.2} \right)^3 \left( \frac{B_{-9}}{0.3} \right)^8 \times \frac{A}{8\pi \rho_0} \frac{h_{l_1} l_1 l_2 l_3 l_4}{l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)},\]
which is similar in magnitude to the case I trispectrum, but of positive sign.

\section{CMB Trispectrum from Magnetic Scalar Anisotropic Stress}

The scalar anisotropic stress trispectrum \(T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L)\) can be calculated in an analogous manner to the calculation presented above for case I \(s\)-independent \(T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L)\). Using

\[
V_A = B_0/ (16\pi \rho_0/3)^{1/2} \approx 3.8 \times 10^{-4} B_{-9},
\]
with \(B_{-9} \equiv (B_0/10^{-9}\text{Gauss})\). From the definition of the rotationally invariant angle-averaged trispectrum \([78]\),
\[
T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L) \Omega = \sum_{LM} \left(-1\right)^L \frac{l_1 l_2 L}{m_1 m_2 M} \left( l_3 l_4 L \right) T_{l_1\ell_1},l_2\ell_2,l_3\ell_3,l_4\ell_4}(L),
\]
with \(A/8\pi \rho_0)^4 = (2/3)^4 \pi/k_G)^8 ((n + 3)/k_G^{n+1})^4 V_A^8\),
\[
(2l_1 + 1)(2l_2 + 1)(2L + 1) \left(l_1 l_2 L \right) \left( m_1 m_2 M \right) \equiv h_{l_1} l_1 l_2 \left( l_1 l_2 L \right),
\]
where we have defined \(h_{l_1} l_1 l_2 \) above, in the same convention as \([73,74]\). We use the relation
\[
(2/3)^4 \pi/k_G)^8 ((n + 3)/k_G^{n+1})^4 V_A^8,
\]
Equations (26) and (36) we obtain
\[
\left[T_{\ell_1\ell_2}^{l_1l_2}(L)\right]_\Pi \approx \left(3 \frac{\mathcal{R}_p}{R}\right)^4 \xi \left[-T_{\ell_1\ell_2}^{l_1l_2}(L)\right]_\Omega \\
\approx 1.1 \times 10^{-19} \left(\frac{\xi}{10}\right) \left(\frac{n+3}{0.2}\right)^3 \left(\frac{B_{-9}}{3}\right)^8 \\
\times \frac{h_{11}l_1l_2 h_{1\ell_1} l_1}{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}.
\] (48)

We see that the amplitude of the trispectrum sourced by \(\Pi_B\) for equal-sided quadrilateral configurations is approximately \(10^{10}\) times larger than that sourced by \(\Omega_B\). Here, we have used \(T_B \approx 10^{14}\) GeV (corresponding to the reheating temperature) and \(T_\nu \approx 10^{-3}\) GeV.

VI. FLAT-SKY CALCULATION OF SCALAR ANISOTROPIC STRESS CMB TRISPECTRUM

We now consider a flat-sky analysis of the trispectrum. The flat-sky limit allows us to avoid the approximate treatment of the angular terms involving \(k_i\) while performing the \(k\) angular integrals that led to Equation (41). Therefore, to get a more accurate estimate of the \(s\)-independent anisotropic stress trispectrum, we now adopt the flat-sky limit for the CMB temperature anisotropy and recompute the trispectrum.

In the flat-sky limit [81,82], the CMB temperature fluctuations on the sky are expanded in terms of plane waves using a Fourier basis rather than a spherical harmonic basis,
\[
\frac{\Delta T}{T}(n) = \int \frac{d^2 l}{(2\pi)^2} a_i e^{i \ell \cdot n},
\]
\[
a_i = \int d^2 n \frac{\Delta T}{T}(n) e^{-i \ell \cdot n}.
\] (49)

In the flat-sky co-ordinates, \(\ell = (\ell_x, \ell_y)\) is a two-dimensional vector on the plane of the sky and \(n_z\) is a constant equal to unity at linear order. In order to check the validity of our flat-sky technique, we first computed the magnetic energy density for equal-sided quadrilateral configurations is approximately \(10^{10}\) times larger than that sourced by \(\Omega_B\). Here, we have used \(T_B \approx 10^{14}\) GeV (corresponding to the reheating temperature) and \(T_\nu \approx 10^{-3}\) GeV.

The magnetic Sachs-Wolfe effect for scalar anisotropic stress is given by
\[
\frac{\Delta T}{T}(n) = \mathcal{R}_p \Pi_B (x_0 - n D^*)
\]
\[
= \int \frac{d^3 k}{(2\pi)^3} \mathcal{R}_p \Pi_B (k) e^{i k \cdot (x_0 - n D^*)}
\]
\[
= \mathcal{R}_p \int \frac{d^3 k}{(2\pi)^3} \Pi_B (k) e^{-i (k \cdot n) D^*}.
\] (51)

where in the last line we set the observer’s position \(x_0\) to the origin.

The flat-sky limit is accurate for \(\ell \gtrsim 40\) [81,82] whereas the Sachs-Wolfe contribution is appreciable for \(\ell \lesssim 100\) (but dominant only till \(\ell \lesssim 50\)) [83]. Therefore, there exists an appreciable range of overlap \(40 \lesssim \ell \lesssim 100\) in harmonic space, where we can treat the Sachs-Wolfe contribution to the CMB temperature anisotropy in the flat-sky limit.

In the flat-sky limit, \(n_z\) is constant and is unity to linear order hence \(n \cdot k \rightarrow m \cdot k \perp + k_z\) which gives
\[
a_i = \int d^2 n \left(\frac{\Delta T}{T}\right)_{\text{flat sky}} e^{-i \ell \cdot n}
\]
\[
= \mathcal{R}_p \int \frac{d^3 k}{(2\pi)^3} \Pi_B (k) e^{-i k_z D^*} \int d^2 m e^{-i m \cdot (\ell + k \perp D^*)}.
\] (52)

The \(m\)-integral gives a delta function for \(k_\perp\)
\[
\int d^2 m e^{-i m \cdot (\ell + k_\perp D^*)} = (2\pi)^2 \delta^2 (\ell + k \perp D^*)
\]
\[
= \left(\frac{2\pi}{D^*}\right)^2 \delta^2 (\ell + k_\perp D^*)
\] (53)

to yield
\[
a_i = \frac{\mathcal{R}_p}{(D^*)^2} \int d^2 k \Pi_B (k_\perp = -\ell \perp, k_z) e^{-i k_z D^*}.
\] (54)

This flat-sky \(a_i\) for magnetic scalar anisotropic stress can then be used to calculate the corresponding trispectrum in the flat-sky limit
\[
\langle a_{i1} a_{i2} a_{i3} a_{i4} \rangle = \left(\frac{\mathcal{R}_p}{(D^*)^2} \int d^2 k \Pi_B (k_\perp = -\ell \perp, k_z) \right)^{fs}_{1234},
\]
where \(\zeta_{1234}^{fs}\) is the four-point correlation of magnetic scalar anisotropic stress in the flat-sky limit
\[
\zeta_{1234}^{fs} = \left(\frac{4}{\pi} \prod_{i=1}^4 \int d^2 k_i \right)^{fs}_{123}.
\] (56)

As before in the full-sky for \(\zeta_{1234}\) (Eqs. 27, 56), a four-point correlation of \(\Pi_B\) produces delta functions times a mode coupling integral \(\psi\).
\[
\zeta_{1234}^{fs} = \delta (k_1 + k_2 + k_3 + k_4) \times \delta^2 (\ell_1 D_1^* + \ell_2 D_2^* + \ell_3 D_3^* + \ell_4 D_4^*) \left[\psi_{1234}\right]_\Pi^{fs}
\] (57)

If we take the \(D_i^*\)’s to be similar, we find
\[
\zeta_{1234}^{fs} = \delta (k_1 + k_2 + k_3 + k_4) \times (D^*)^2 \delta^2 (\ell_1 + \ell_2 + \ell_3 + \ell_4) \left[\psi_{1234}\right]_\Pi^{fs}
\] (58)

Here the mode-coupling integral \(\psi\) is
\[
\left[\psi_{1234}\right]_\Pi^{fs} = \frac{8 \mathcal{R}_{p\text{-indep}}}{(8\pi)^2} \left[\mathcal{I}\right]
\] (59)

where the integral \(\mathcal{I}\) is the same as the one given by Eq. 33 and the \(s\)-independent angular terms for \(\Pi_B\) are denoted by \(\mathcal{R}_{p\text{-indep}}\) given by Eq. 33. In the flat-sky approach we perform the mode-coupling integral for general values of \(\ell_i \cdot \ell_j\)
TABLE III. The value of $\sigma$ [the product of the integral $I$ Eq. (55) and $k_{i_z}$ integrals Eq. (55)] for the three different trispectrum configurations (shown in Fig. 3) considered for the flat-sky magnetic scalar anisotropic stress trispectrum (Eq. (62)).

| Configuration | $(q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34})$ | $(\ell_1 \cdot \ell_2, \ell_1 \cdot \ell_3, \ell_1 \cdot \ell_4, \ell_2 \cdot \ell_3, \ell_2 \cdot \ell_4, \ell_3 \cdot \ell_4)$ | $\sigma$ |
|---------------|-----------------------------------------------|-------------------------------------------------|----------|
| kite          | $(\sqrt{3}, \sqrt{3}, 1, 1, 1/\sqrt{3}, 1/\sqrt{3})$ | $(0, -\sqrt{3}/2, 1/2, -\sqrt{3}/2, 0)$          | -15.2    |
| trapezium     | $(2, 2/3, 2, 1/3, 1, 3)$                       | $(1/2, -1, 1/2, -1/2, -1/2)$                    | -84.6    |
| scalene       | $(1/3, 2/3, 0.4406, 2, 1.322, 0.6609)$          | $(0, -\sqrt{3}/2, 0.1317, -1/2, -0.9912, 0.3815)$| -14.2    |

FIG. 3. The three specific $\ell$ wavevector configurations (i) kite, (ii) trapezium (both cyclic quadrilaterals) and (iii) scalene (an irregular convex quadrilateral) used to evaluate the flat-sky magnetic scalar anisotropic trispectrum. Trispectrum configuration shapes (i) and (ii) are also discussed in [84, 85].

...and later evaluate the trispectrum for particular configurations that are not necessarily equal-sided. The first term (out of six terms) of integral $I$ is

$$ I^{(1)} \approx 4\pi A^4 k_1^{2n+3} k_n \left[ \frac{k_1^2 + 2k_1k_3\xi k_3 + k_3^2}{n+3} - \frac{k_3^2}{4n+3} \right] $$

(60)

Whereas, in the full-sky $\Pi_{12}$ calculation we chose a representative value $\xi$ for $I_{\Pi_{12}}^{\text{indep}}$, we now integrate over all 14 terms of $I_{\Pi_{12}}^{\text{indep}}$ in the $k_{i_z}$ integrals.

For each of the 6 terms of $I$, the delta function of $k_{i_z}$ is used to perform that particular $k_{i_z}$ integral (one out of four) for which the variable $k_i$ that does not appear in the arguments of the magnetic spectrum $M$. This introduces substitutions in the angular structure $T_{\Pi_{12}}^{\text{indep}}$. Then the remaining three $k_{i_z}$ integrals are performed numerically and evaluated for several types of configurations. We use the relation for the flat-sky trispectrum (connected part $[73, 74])$

$$ \langle a_{\ell_1} a_{\ell_2} a_{\ell_3} a_{\ell_4} \rangle = (2\pi)^2 \delta^2(\ell_1 + \ell_2 + \ell_3 + \ell_4) T_{(\ell_1, \ell_2)}^{(\ell_3, \ell_4)}(L) $$

(61)

to get $[T_{(\ell_1, \ell_2)}^{(\ell_3, \ell_4)}(L)]_{\Pi_{12}}$ from the four-point correlation of $a_{\ell_4}$. The product of the mode coupling integral $I$ and the three $k_{i_z}$ integrals is denoted by $\sigma$. Table III shows different values of $\sigma$ for different $\ell$-space configurations with parameters $q_{ab} = l_a/l_b$ (ratio of different sides) and $\ell_i \cdot \ell_j$ (cosine of the angle between sides). We note that all the configurations thus evaluated in the flat-sky approach (for all $s$-independent terms) give a negative $\sigma$ that lead to a negative value of the trispectrum.

$$ [T_{(\ell_1, \ell_2)}^{(\ell_3, \ell_4)}(L)]_{\Pi_{12}} \approx 3.94 \times 10^{-19} \left( \frac{\sigma}{10} \right) \left( \frac{n+3}{0.2} \right)^3 \left( \frac{B_{\ell_3}}{3} \right)^8 $$

(62)

We see that the flat-sky evaluation of the scalar anisotropic stress trispectrum with $s$-independent terms results in trispectra that are negative and roughly an order of magnitude larger in absolute magnitude than the corresponding full-sky trispectrum with $s$-independent terms (with $\xi \approx 10$). The flat-sky and full-sky trispectra are related by

$$ T_{(\ell_1, \ell_2)}^{(\ell_3, \ell_4)}(L) \left[ h_{\ell_1L} L_{\ell_2} h_{\ell_3L} L_{\ell_4} \approx T_{(\ell_1, \ell_2)}^{(\ell_3, \ell_4)}(L) \right. $$

(63)

This allows us to compare the flat-sky trispectrum directly to the full-sky trispectrum form given in Eq. (48).

---

1 For some highly symmetrical configurations which have two $\ell$ vectors exactly anti-parallel and of equal magnitude, the $k_{i_z}$ integral becomes singular in the flat-sky limit. However, this is due to the exact $k_{i_z} = -\ell_{i_z}$ map which is enforced in this limit. If this were relaxed then we expect this mathematical pathology to be just an integrable singularity. The measure of such configurations in $d^3k$ is expected to go to zero faster than the reciprocal of the integrand.
VII. PRIMORDIAL MAGNETIC FIELD CONSTRAINTS

We can now compare our magnetic trispectra with the Sachs-Wolfe contribution to the standard CMB trispectrum sourced by non-linear terms in the inflationary perturbations calculated by Okamoto & Hu [73] and Kogo & Komatsu [74] (also see [47]).

\[ T_{l_1 l_2}^{C_{SW} L}(L) \approx 9 \, C_{l_1}^{SW} C_{l_2}^{SW} \left[ (25/9) \, \tau_{NL} C_L^{SW} + 6 \, g_{NL} (C_{l_1}^{SW} + C_{l_2}^{SW}) \right] h_{l_1 L l_2} h_{l_2 L l_4} \, q \]

We neglect the \( g_{NL} \) term that places far weaker constraints on the trispectrum compared to the \( \tau_{NL} \) term considering the current limits on \( g_{NL} \) from WMAP [88] and current limits on \( \tau_{NL} \) from Planck [61]. The CMB angular power spectrum \( C_L^{SW} \) in the Sachs-Wolfe approximation for a scale-invariant primordial power spectrum is

\[ C_L^{SW} = \frac{2}{9 \pi} \int k^2 dkP(k) j^2(kr_*) = \frac{A_\Phi}{l(l+1)}, \]

where \( A_\Phi \) is the amplitude of scalar potential perturbations.

\[ T_{l_1 l_2}^{C_{SW} L}(L) \approx 25 \, C_{l_1}^{SW} C_{l_2}^{SW} \tau_{NL} h_{l_1 L l_2} h_{l_2 L l_4} \approx 25 \, A_\Phi \tau_{NL} \frac{h_{l_1 L l_2} h_{l_2 L l_4}}{l_2(l_2 + 1)l_4(l_4 + 1) L(L + 1)} \]

\[ \approx 25 \, A_\Phi \tau_{NL} \frac{h_{l_1 L l_2} h_{l_2 L l_4}}{l_1(l_1 + 1)l_3(l_3 + 1)} \frac{l_1(l_1 + 1)l_3(l_3 + 1)}{l_4(l_4 + 1) L(L + 1)} \approx 25 \, A_\Phi \tau_{NL} \frac{h_{l_1 L l_2} h_{l_2 L l_4}}{l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)} \]

where we also define a factor \( q = [l_1(l_1 + 1)l_3(l_3 + 1)]/[l_4(l_4 + 1) L (L + 1)] \) which is of order unity for many configurations. To calculate the value of \( A_\Phi \) we begin with the most recent Planck 2013 data release value for the amplitude of scalar curvature perturbations on \( l = 20 \), \( A_\Phi = 2.2 \times 10^{-9} \) at a pivot scale \( k_0 = 0.05 \) Mpc\(^{-1}\). For the purpose of the Sachs-Wolfe contribution we then calculate the scalar amplitude at the larger scale of \( k_0 = 0.002 \) Mpc\(^{-1}\) using the Planck 2013 value for the scalar spectral index \( n_s = 0.96 \). After converting from curvature to potential we get \( A_\Phi = 6.96 \times 10^{-10} \). Hence, we find the amplitude for the Sachs-Wolfe contribution to the standard CMB trispectrum sourced by inflationary perturbations to be

\[ T_{l_1 l_2}^{C_{SW} L}(L) \approx 8.4 \times 10^{-27} \tau_{NL} \frac{h_{l_1 L l_2} h_{l_2 L l_4}}{l_1(l_1 + 1)l_2(l_2 + 1)l_3(l_3 + 1)} \]

Equation (67) is of the same form as Eq. (46) and Eq. (48) for the magnetic field-induced trispectra, facilitating direct comparison of trispectra values.

A. Limits from Magnetic Energy Density - Case I

We can put upper limits on the primordial magnetic field by comparing the magnetic energy density trispectrum Eq. (46) with the inflationary trispectrum Eq. (67), although stronger constraints follow from magnetic anisotropic stress. We take the two-sigma upper limit value on \( \tau_{NL} \) reported in the Planck 2013 data release: \( \tau_{NL} < 2,800 \) [61] and use it also as a lower limit for possible negative values of \( \tau_{NL} \) i.e. \( |\tau_{NL}| < 2,800 \). This is tighter than the \( \tau_{NL} > -6,000 \) negative-sided limit from WMAP5 data [88] that we employed in [60]. Magnetic field limits are obtained by taking the one-eighth power of the appropriate ratio of trispectra, which gives \( B_0 \lesssim 19 \) nG at a scale of \( k_0 = 1h \) Mpc\(^{-1}\) for a magnetic spectral index of \( n = -2.8 \). This trispectrum limit is almost a factor of 2 stronger than the bispectrum upper limit \( B_0 \lesssim 35 \) nG found for magnetic energy density [49] for the same scale and magnetic index.

We note that if we update the value of \( \tau \) used in the earlier bispectrum calculation [49] to the currently adopted value of \( \tau \) [68] then the magnetic energy density bispectrum yields a tighter upper limit of \( B_0 \lesssim 30 \) nG. The trispectrum constraint we calculated above, \( B_0 \lesssim 19 \) nG, seems significantly stronger than the bispectrum constraint (by a factor of 1.6). However, since the energy density bispectrum calculation [49] was performed, the \( f_{NL}^B \) two-sigma upper limit has tightened from \( \approx 100 \) (WMAP5) [87] to 74 (WMAP7) [88] to 14.3 (Planck 2013) [61]. Recalculation of the magnetic field constraint from the magnetic energy density bispectrum, now using \( f_{NL}^B \lesssim 14.3 \), yields \( B_0 \lesssim 22 \) nG. We see that the corresponding magnetic energy density trispectrum limit (19 nG) found in this work is, nevertheless, slightly stronger than the updated bispectrum limit.

B. Limits from Magnetic Energy Density - Collinear Configuration

We have also calculated the magnetic energy density trispectrum considering all the angular terms that appear for the collinear configuration (case II). Comparing the collinear configuration energy density trispectrum Eq. (67) to the inflationary trispectrum Eq. (67) leads to upper limits on the primordial magnetic field of \( B_0 \lesssim 20 \) nG, having employed the positive-sided limit \( \tau_{NL} < 2,800 \) [61]. This \( B_0 \) limit from the collinear configuration trispectrum that considers the full mode-coupling integral over all angular terms is similar to the limit above from case I: only \( \Phi \)-independent angular terms for any equal-sided configuration.

C. Limits from Scalar Anisotropic Stress

The trispectrum from magnetic scalar anisotropic stress Eq. (48) was found to be \( 10^{10} \) times larger than the trispectrum from magnetic energy density. Comparing it with the trispectrum from inflationary perturbations (Eq. (67)) gives a much stronger magnetic field constraint of

\[ B_0 \lesssim 0.9 \text{ nG}, \]

using the positive-sided limit \( \tau_{NL} < 2,800 \) from the Planck 2013 data release [61].
This $B_0 \lesssim 0.9 \text{nG}$ limit is over two and a half times as strong as the $B_0$ limit (2.4 nG) obtained from the $\Pi_B$ bispectrum \[59\]. In addition, for those theories of inflation, which lead to $\tau_{NL} = (6/5,f_{NL})^2$ we could perhaps use the relatively tighter limits on $f_{NL}$. The two-sigma limits on $f_{NL}$ are $-8.9 < f_{NL} < 14.3$, obtained from searching for the CMB primordial bispectrum signal in Planck 2013 data \[61\]. This gives a primordial magnetic field limit of

$$B_0 \lesssim 0.7 \text{nG}, \quad (69)$$

for both the negative and positive $f_{NL}^{\text{loc}}$ limits separately. We employ the local configuration $f_{NL}$ limits as the uncertainties $\sigma_{f_{NL}}$ in the other orthogonal and equilateral configurations are about an order of magnitude larger.

D. Limits from Scalar Anisotropic Stress - Flat-Sky

We can also compare the flat-sky calculation of the scalar anisotropic stress trispectrum to the trispectrum from inflationary perturbations (Eq. \[67\]) and obtain magnetic field limits using the negative-sided limit of $|\tau_{NL}| < 2.800$ to get

$$B_0 \lesssim 0.6 - 0.8 \text{nG}. \quad (70)$$

The range of magnetic field upper limits reflects the range of $\sigma$ values (-84.6 to -14.2) in Table (III) for different configurations of the flat-sky trispectrum. As before, we may again consider those theories of inflation which lead to $\tau_{NL} = (6/5,f_{NL})^2$ and use the relatively tighter limits on $f_{NL}$, i.e. $-8.9 < f_{NL}^{\text{loc}} < 14.3$ \[61\] to place magnetic field upper limits of

$$B_0 \lesssim 0.4 - 0.6 \text{nG}, \quad (71)$$

where we take the combined effect of the slightly different (positive and negative) limits for $f_{NL}$ as well as the range of values of $\sigma$ to arrive at the range of $B_0$ upper limits.

For magnetic scalar anisotropic stress, the flat-sky trispectra values give magnetic field upper limits that are slightly stronger but consistent with the sub-nanoGauss values derived from the full-sky trispectrum.

E. Limits from Inflationary Magnetic Curvature Mode

Recently, Bonvin et al. \[62, 89\] have found a magnetic mode in the curvature perturbation that is present only when magnetic fields are generated at inflation. This magnetic mode is always scale-invariant and is absent when magnetogenesis occurs causally e.g. via a phase transition. This inflationary magnetic mode is seen to exist in addition to the compensated and passive modes and dominates over them in the CMB anisotropy. The ratio of the passive mode power spectrum to the new inflationary magnetic mode power spectrum is proportional to $e^2$ where $e \sim 10^{-2}$ is the inflationary slow-roll parameter. We calculate the passive to inflationary power spectrum ratio using the relation given between Equations (45) and (46) in Bonvin et al. \[62\], for $n \rightarrow -3$,

$$\frac{C_{l}^{\text{passive}}}{C_{l}^{\text{infl. mag.}}} \simeq e^2 \ln^2 \frac{\eta_s}{\eta_0} \left( \frac{\eta_s}{\eta_0} \right)^{2n+6} \frac{\Gamma \left( -n - 2 \right)}{\Gamma \left( -n - \frac{3}{2} \right)}$$

$$\times l^{2n+6} \ln^2 \left( \frac{\eta_s}{\eta_0} \right), \quad (72)$$

to find

$$\frac{C_{l}^{\text{passive}}}{C_{l}^{\text{infl. mag.}}} \simeq 4.7 \times 10^{-5}. \quad (73)$$

Now consider the magnetic CMB trispectrum sourced by this inflationary magnetic mode. We assume the trispectra ratio scales approximately as the power spectrum ratio squared and magnetic field constraint will come from one-eighth power of trispectra ratio. The magnetic field constraint is then found to be significantly stronger than from magnetic passive modes (i.e. scalar anisotropic stress $\Pi_B$) roughly by a factor $\approx (4.7 \times 10^{-5})^{-0.25} \approx 12$. The magnetic field upper limit from the inflationary magnetic mode CMB trispectrum is then

$$B_0 \lesssim 0.05 \text{nG} \quad \text{i.e.} \quad B_0 \lesssim 50 \text{picoGauss}. \quad (74)$$

For this inflationary magnetic mode, the trispectrum, as well as other CMB correlations, give magnetic field upper limits that are an order of magnitude stronger than those derived from the magnetic passive mode (scalar anisotropic stress) alone. Clearly, the new inflationary magnetic mode presented by Bonvin et al. \[62\] seems to place stronger constraints on primordial magnetic fields from its CMB correlations and we hope to return to this in greater detail in future work.

 VIII. CONCLUSIONS

We have presented the full calculation for the CMB trispectrum sourced by primordial magnetic field scalar modes, first reported in our Letter \[61\]. In addition, we have calculated the scalar anisotropic stress trispectrum in the flat-sky limit. Together with recent improved observational constraints on primordial non-Gaussianity from the Planck mission 2013 data, the magnetic scalar trispectrum enables us to place sub-nanoGauss upper limits on the strength primordial magnetic fields.

Magnetic energy density gives rise to a trispectrum of magnitude $\approx 10^{-20}$, for $s$-independent terms. Also, the collinear configuration trispectrum for energy density, including all angular terms, gives a result that is very similar to the case of $s$-independent terms for energy density.
TABLE IV. Comparison of upper limits on primordial magnetic fields from magnetic mode contributions to the CMB power spectra, bispectra and trispectra (this work). We quote limits derived for close to scale-invariant magnetic fields and an early generation epoch ($10^{14}$ GeV) for magnetic passive modes.

| CMB Probe | Magnetic modes | Magnetic field upper limit $B_0$ (nG) | Reference |
|-----------|----------------|--------------------------------------|-----------|
| Power Spectrum | scalar, vector & tensor | 3.4 | [29] |
| Bispectrum | energy density | $22^a$ | [49] |
| Bispectrum | scalar anisotropic stress | 2.4 | [59] |
| Bispectrum | vector | 10 | [52] |
| Bispectrum | tensor | 3.2 | [55] |
| Trispectrum | energy density | 19 | this work |
| Trispectrum | scalar anisotropic stress | 0.6 | this work |
| Trispectrum | magnetic inflationary mode | 0.05 | this work; using [62] |

$^a$ The magnetic field upper limit from [49] has been updated with the current values for $R$ and current upper limit for $f_{NL}$.

For magnetic scalar anisotropic stress, we find a trispectrum of magnitude $\approx 10^{-19}$, which is ten orders of magnitude larger than the magnetic energy density trispectrum. We also present an independent flat-sky limit calculation of this trispectrum with its angular structure that yields a slightly larger trispectrum of magnitude $\approx 10^{-18}$.

The magnetic energy density trispectrum allows us to place stronger upper limits on the primordial magnetic field compared to a similar calculation with the magnetic energy density bispectrum [49–51]. Further, the much larger trispectrum due to magnetic scalar anisotropic stress leads to the tightest constraint so far on large scale magnetic fields of $\sim 0.6$ nG. This is approximately four times as strong as the corresponding upper limit from our previous bispectrum calculation ($\sim 2.4$ nG) [59]. We note that the vector and tensor mode bispectra have been calculated numerically [52, 53, 55] and give magnetic field limits of $\sim 3-10$ nG. Recently, polarization bispectra [56] constraints on magnetic fields have been forecast to be $\sim 2-3$ nG from expected Planck mission CMB polarization data. However, the scalar temperature trispectrum calculated in this work gives stronger magnetic fields constraints compared to the various kinds of bispectra that have been calculated (see Table IV). The trispectrum’s sensitivity can be illustrated by the magnetic to inflationary scalar trispectrum ratio, which is $\sim 10^2$ compared to $\sim 0.1$ for the ratio of magnetic to inflationary scalar bispectra (taking $f_{NL} \sim 10$ and $B_0 \sim 3$ nG).

We also note that the magnetic field upper limit at megaparsec scales derived from just the scalar mode magnetic CMB trispectrum is already several times better than the upper limit from the magnetic CMB power spectrum combining scalar, vector and tensor modes: $3.4$ nG from Planck mission 2013 data [29] and ($\sim 2-6$ nG) from WMAP data [30–33]. Non-Gaussian correlations like the bispectrum and especially the trispectrum are better able to constrain primordial cosmological magnetic fields than the CMB power spectrum.

Finally, we have utilized the recently uncovered magnetic inflationary mode [62] as a source for the CMB trispectrum. This new magnetic mode dominates over both energy density and scalar anisotropic stress and leads to an order of magnitude stronger constraint on the primordial magnetic field of $\sim 0.05$ nG. Further detailed investigation of the role this magnetic mode can play in sourcing various CMB correlations will be important.

Table IV summarizes the current constraints on primordial magnetic fields derived from various probes using CMB anisotropies. Thus, the CMB trispectrum is a new and more powerful probe of large scale primordial magnetic fields in the Universe.

Future consideration of magnetic vector and tensor modes in the trispectrum is likely to give additional constraints on primordial magnetic fields. Further improvement in magnetic field constraints is also possible from better $\tau_{NL}$ constraints that may emerge from a detailed analysis of the full Planck mission data.
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Appendix: A

In this Appendix we present the complete expressions for all angular terms generated by the sixteen operators present in the four-point correlation of magnetic anisotropic stress \( \langle \Pi_B(k_1)\Pi_B(k_2)\Pi_B(k_3)\Pi_B(k_4) \rangle \) (Eq. 29). The extensive angular term expressions presented below have also been checked by taking an alternative order of contraction while calculating angular terms.

Each operator term \( X \) from 1 to 16 generates its own separate angular term expression \( \mathcal{F}(X) \). When summed over all \( X \) this yields the angular term expression \( \mathcal{F}(J) \), where \( J \) takes values 1 to 6 in the six term mode-coupling integral \( \langle \psi_{1234} \rangle_{\Pi} \) below.

\[
\psi_{1234} = \frac{8}{(8\pi\rho_0)^2} \int d^3 s M(s)M(|k_1 + s|)\left[M(|k_1 + k_3 + s|)M(|k_2 - s|)\mathcal{F}(1) + M(|k_1 + k_2 + s|)M(|k_3 - s|)\mathcal{F}(3) + M(|k_1 + k_4 + s|)M(|k_3 - s|)\mathcal{F}(5)\right].
\]

As seen in Eq. (29), the angular term expressions \( \mathcal{F} \) generated by operators 2 to 5 will carry a prefactor of (-3), angular term expressions generated by 6 to 11 will carry a prefactor of (9), angular term expressions generated by 12 to 15 will carry a prefactor of (-27) and the angular term expressions generated by 16 will have a prefactor of (81). For clarity, we suppress these prefactors while writing out the full angular term expressions below. The angles involved in these expressions have been defined earlier in Equations (11), (12) and in Table I.

The angular terms for operator 1 are

\[
\mathcal{F}(1) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_6^2 + \beta_2^2 + \beta_6^2 + \gamma_6^2) - (\alpha_1\alpha_2\beta_2 + \alpha_1\alpha_6\beta_6 + \alpha_2\alpha_6\gamma_6 + \beta_2\beta_6\gamma_6) + \alpha_1\alpha_2\beta_6\gamma_6
\]

\[
\mathcal{F}(2) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_6^2 + \beta_2^2 + \beta_6^2 + \epsilon_6^2) - (\alpha_1\alpha_4\beta_4 + \alpha_1\alpha_6\beta_6 + \alpha_4\alpha_6\epsilon_6 + \beta_4\beta_6\epsilon_6) + \alpha_1\alpha_4\beta_6\epsilon_6
\]

\[
\mathcal{F}(3) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_6^2 + \beta_2^2 + \beta_6^2 + \epsilon_6^2) - (\alpha_1\alpha_3\beta_3 + \alpha_1\alpha_5\beta_5 + \alpha_3\alpha_5\epsilon_5 + \beta_3\beta_5\epsilon_5) + \alpha_1\alpha_3\beta_5\epsilon_5
\]

\[
\mathcal{F}(4) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_6^2 + \beta_2^2 + \beta_6^2 + \epsilon_6^2) - (\alpha_1\alpha_4\beta_4 + \alpha_1\alpha_5\beta_5 + \alpha_4\alpha_5\epsilon_5 + \beta_4\beta_5\epsilon_5) + \alpha_1\alpha_4\beta_5\epsilon_5
\]

\[
\mathcal{F}(5) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_6^2 + \beta_2^2 + \beta_6^2 + \gamma_7^2) - (\alpha_1\alpha_2\beta_2 + \alpha_1\alpha_7\beta_7 + \alpha_2\alpha_7\gamma_7 + \beta_2\beta_7\gamma_7) + \alpha_1\alpha_2\beta_7\gamma_7
\]

\[
\mathcal{F}(6) = -1 + (\alpha_1^2 + \alpha_4^2 + \alpha_7^2 + \beta_2^2 + \beta_7^2 + \epsilon_7^2) - (\alpha_1\alpha_3\beta_3 + \alpha_1\alpha_7\beta_7 + \alpha_3\alpha_7\epsilon_7 + \beta_3\beta_7\epsilon_7) + \alpha_1\alpha_3\beta_7\epsilon_7.
\]
The angular terms for operator $\mathbf{2}$ are

\[ F^{(1)}_2 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 - \alpha_1 \beta_3 \beta_4 - \alpha_4 [\gamma_4 - \alpha_4 \alpha_2] - \delta_3 \gamma_4 \beta_4 \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(2)}_2 = \begin{array}{c}
1 - \beta_4^2 - \alpha_2 \beta_4 \beta_4 - \delta_4 [\delta_4 - \alpha_2 \alpha_3 - \beta_4 \beta_3 + \alpha_1 \beta_4 \beta_3] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(3)}_2 = \begin{array}{c}
1 - \alpha_4^2 - \beta_4^2 - \alpha_1 \alpha_4 \beta_4 - \alpha_4 [\gamma_4 - \alpha_4 \alpha_6 + \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(4)}_2 = \begin{array}{c}
1 - \beta_4^2 - \alpha_2 \beta_4 \beta_4 - \delta_4 [\delta_4 - \alpha_2 \alpha_3 - \beta_4 \beta_3 + \alpha_1 \beta_4 \beta_3] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(5)}_2 = \begin{array}{c}
1 - \alpha_4^2 - \beta_4^2 - \alpha_1 \alpha_4 \beta_4 - \alpha_4 [\gamma_4 - \alpha_4 \alpha_6 + \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(6)}_2 = \begin{array}{c}
1 - \beta_4^2 - \alpha_2 \beta_4 \beta_4 - \delta_4 [\delta_4 - \alpha_2 \alpha_3 - \beta_4 \beta_3 + \alpha_1 \beta_4 \beta_3] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  
\[ F^{(A.2)}_2 = \begin{array}{c}
1 - \beta_4^2 - \alpha_2 \beta_4 \beta_4 - \delta_4 [\delta_4 - \alpha_2 \alpha_3 - \beta_4 \beta_3 + \alpha_1 \beta_4 \beta_3] \\
- \chi_4 [\chi_4 - \alpha_4 \alpha_6 - \beta_4 \beta_6 + \alpha_1 \beta_4 \beta_6] + \gamma_4 \chi_4 [\gamma_6 - \alpha_2 \alpha_6 - \beta_2 \beta_6 + \alpha_1 \alpha_2 \beta_6] \\
\end{array} \]  

The angular terms for operator $\mathbf{3}$ are

\[ F^{(1)}_3 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 + \alpha_1 \alpha_3 \beta_3 - \beta_3 [\beta_3 - \alpha_3 \alpha_1 - \beta_3 \beta_2 + \alpha_2 \gamma_3 \alpha_1] \\
- \chi_3 [\chi_3 - \alpha_3 \alpha_6 - \beta_3 \beta_6 + \alpha_1 \beta_3 \beta_6] + \gamma_3 \chi_3 [\alpha_1 \alpha_6 - \beta_2 \beta_6 + \alpha_2 \alpha_6 \beta_6] \\
\end{array} \]  
\[ F^{(2)}_3 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 + \alpha_1 \alpha_3 \beta_3 - \beta_3 [\beta_3 - \alpha_3 \alpha_1 - \beta_3 \beta_2 + \alpha_2 \gamma_3 \alpha_1] \\
- \chi_3 [\chi_3 - \alpha_3 \alpha_6 - \beta_3 \beta_6 + \alpha_1 \beta_3 \beta_6] + \gamma_3 \chi_3 [\alpha_1 \alpha_6 - \beta_2 \beta_6 + \alpha_2 \alpha_6 \beta_6] \\
\end{array} \]  
\[ F^{(3)}_3 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 + \alpha_1 \alpha_3 \beta_3 - \beta_3 [\beta_3 - \alpha_3 \alpha_1 - \beta_3 \beta_2 + \alpha_2 \gamma_3 \alpha_1] \\
- \chi_3 [\chi_3 - \alpha_3 \alpha_6 - \beta_3 \beta_6 + \alpha_1 \beta_3 \beta_6] + \gamma_3 \chi_3 [\alpha_1 \alpha_6 - \beta_2 \beta_6 + \alpha_2 \alpha_6 \beta_6] \\
\end{array} \]  
\[ F^{(4)}_3 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 + \alpha_1 \alpha_3 \beta_3 - \beta_3 [\beta_3 - \alpha_3 \alpha_1 - \beta_3 \beta_2 + \alpha_2 \gamma_3 \alpha_1] \\
- \chi_3 [\chi_3 - \alpha_3 \alpha_6 - \beta_3 \beta_6 + \alpha_1 \beta_3 \beta_6] + \gamma_3 \chi_3 [\alpha_1 \alpha_6 - \beta_2 \beta_6 + \alpha_2 \alpha_6 \beta_6] \\
\end{array} \]  
\[ F^{(5)}_3 = \begin{array}{c}
1 - \alpha_3^2 - \beta_3^2 + \alpha_1 \alpha_3 \beta_3 - \beta_3 [\beta_3 - \alpha_3 \alpha_1 - \beta_3 \beta_2 + \alpha_2 \gamma_3 \alpha_1] \\
- \chi_3 [\chi_3 - \alpha_3 \alpha_6 - \beta_3 \beta_6 + \alpha_1 \beta_3 \beta_6] + \gamma_3 \chi_3 [\alpha_1 \alpha_6 - \beta_2 \beta_6 + \alpha_2 \alpha_6 \beta_6] \\
\end{array} \]  
\[ F^{(6)}_3 = \begin{array}{c}
1 - \beta_3^2 - \alpha_3 \beta_3 \beta_3 - \alpha_3 [\alpha_3 - \alpha_1 \beta_3 - \alpha_7 \beta_7 + \alpha_8 \gamma_3 \alpha_1] \\
- \delta_3 [\delta_3 - \beta_3 \beta_3 - \delta_3 \delta_7 + \delta_7 \delta_3 \delta_7] + \alpha_3 \delta_3 [\alpha_3 - \alpha_1 \beta_3 - \alpha_7 \beta_7 + \alpha_8 \gamma_3 \alpha_1] \\
\end{array} \]  
\[ F^{(A.3)}_3 = \begin{array}{c}
1 - \beta_3^2 - \alpha_3 \beta_3 \beta_3 - \alpha_3 [\alpha_3 - \alpha_1 \beta_3 - \alpha_7 \beta_7 + \alpha_8 \gamma_3 \alpha_1] \\
- \delta_3 [\delta_3 - \beta_3 \beta_3 - \delta_3 \delta_7 + \delta_7 \delta_3 \delta_7] + \alpha_3 \delta_3 [\alpha_3 - \alpha_1 \beta_3 - \alpha_7 \beta_7 + \alpha_8 \gamma_3 \alpha_1] \\
\end{array} \]
The angular terms for operator $\mathbf{4}$ are

\[
\begin{align*}
\mathcal{F}_{(1)}^4 &= 1 - \beta_2^2 - \frac{\lambda_2}{\lambda_1} + \lambda_1 \beta_2 - \lambda_2 \beta_1 - \lambda_1 \beta_6 + \lambda_2 \beta_6, \\
\mathcal{F}_{(2)}^4 &= 1 - \alpha_2^2 - \beta_2^2 + \alpha_1 \lambda_2 \beta_2 - \alpha_1 \lambda_2 \beta_6 + \alpha_1 \lambda_2 \beta_6, \\
\mathcal{F}_{(3)}^4 &= 1 - \alpha_2^2 - \beta_2^2 - \beta_2 \beta_6 + \alpha_1 \lambda_2 \beta_6, \\
\mathcal{F}_{(4)}^4 &= 1 - \alpha_2^2 - \beta_2^2 + \alpha_1 \lambda_2 \beta_6, \\
\mathcal{F}_{(5)}^4 &= 1 - \beta_2^2 - \lambda_2 \beta_6 + \beta_2 \beta_6, \\
\mathcal{F}_{(6)}^4 &= 1 - \beta_2^2 - \lambda_2 \beta_6 + \beta_2 \beta_6.
\end{align*}
\]

The angular terms for operator $\mathbf{5}$ are

\[
\begin{align*}
\mathcal{F}_{(1)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1, \\
\mathcal{F}_{(2)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1, \\
\mathcal{F}_{(3)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1, \\
\mathcal{F}_{(4)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1, \\
\mathcal{F}_{(5)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1, \\
\mathcal{F}_{(6)}^5 &= 1 - \gamma_1^2 - \gamma_1^2 + \gamma_1 \gamma_1 \gamma_1.
\end{align*}
\]

The angular terms for operator $\mathbf{6}$ are

\[
\begin{align*}
\mathcal{F}_{(1)}^6 &= \theta_{34} - \beta_3 \beta_4 - \gamma_3 \gamma_4 (\gamma_3 - \beta_3 \beta_4), \\
\mathcal{F}_{(2)}^6 &= \theta_{34} - \alpha_3 \alpha_4 - \beta_3 \beta_4 + \alpha_4 \alpha_4 \beta_3, \\
\mathcal{F}_{(3)}^6 &= \theta_{34} - \alpha_3 \alpha_4 - \beta_4 \beta_4 - \alpha_4 \alpha_4 \beta_3, \\
\mathcal{F}_{(4)}^6 &= \theta_{34} - \alpha_3 \alpha_4 - \beta_4 \beta_4 - \alpha_4 \alpha_4 \beta_3, \\
\mathcal{F}_{(5)}^6 &= \theta_{34} - \alpha_3 \alpha_4 - \beta_4 \beta_4 - \alpha_4 \alpha_4 \beta_3, \\
\mathcal{F}_{(6)}^6 &= \theta_{34} - \alpha_3 \alpha_4 - \beta_4 \beta_4 - \alpha_4 \alpha_4 \beta_3.
\end{align*}
\]
The angular terms for operator \( F \) are

\[ F_{(1)} = \theta_{24} - \alpha_2 \alpha_4 - \chi_4 (\lambda_2 - \alpha_6 \lambda_2) - \beta_4 (\gamma_2 - \alpha_1 \gamma_2) + \lambda_4 \beta_6 (\gamma_2 - \alpha_1 \gamma_2) \] (\theta_{24} - \gamma_2 \gamma_4) \]

\[ F_{(2)} = \theta_{24} - \alpha_2 \alpha_4 - \chi_4 (\lambda_2 - \alpha_6 \lambda_2) - \beta_4 (\gamma_2 - \alpha_1 \gamma_2) + \lambda_4 \beta_6 (\gamma_2 - \alpha_1 \gamma_2) \] (\theta_{24} - \gamma_2 \gamma_4) \]

\[ F_{(3)} = \theta_{24} - \beta_2 \beta_4 - \delta_4 (\delta_2 - \beta_3 \beta_4) - \alpha_4 (\omega_2 - \alpha_1 \omega_2) + \delta_4 \alpha_3 (\omega_2 - \alpha_1 \omega_2) \] (\theta_{24} - \pi_2 \pi_4) \]

\[ F_{(4)} = (\theta_{24} - \alpha_2 \alpha_4 - \beta_2 \beta_4 + \alpha_1 \pi_2 \pi_4) \] (\theta_{24} - \gamma_2 \gamma_4 - \chi_2 \chi_4 + \gamma \gamma \chi_4) \]

\[ F_{(5)} = (\theta_{24} - \alpha_2 \alpha_4 - \beta_2 \beta_4 + \alpha_1 \pi_2 \pi_4) \] (\theta_{24} - \gamma_2 \gamma_4 - \chi_2 \chi_4 + \gamma \gamma \chi_4) \]

\[ F_{(6)} = \theta_{24} - \beta_2 \beta_4 - \delta_4 (\delta_2 - \beta_3 \beta_4) - \alpha_4 (\omega_2 - \alpha_1 \omega_2) + \delta_4 \alpha_3 (\omega_2 - \alpha_1 \omega_2) \] (\theta_{24} - \chi_2 \chi_4). \] (A.7)

The angular terms for operator \( G \) are

\[ G_{(1)} = (\theta_{23} - \alpha_2 \alpha_3 - \beta_2 \beta_3 + \alpha_1 \omega_2 \omega_3) \] \( \theta_{23} - \gamma_2 \gamma_3 - \lambda_2 \lambda_3 + \gamma \omega \gamma \lambda_3 \)

\[ G_{(2)} = (\theta_{23} - \beta_2 \beta_3 - \tau_2 (\tau_1 - \beta_3 \beta_2) - \alpha_4 (\omega_4 - \alpha_1 \omega_4) + \tau_4 \alpha_4 (\omega_4 - \alpha_1 \omega_4) \] \( \theta_{23} - \lambda_2 \lambda_3 \)

\[ G_{(3)} = (\theta_{23} - \tau_2 \omega_2 - \alpha_3 \omega_3 + \alpha_1 \omega_2 \omega_3) \] \( \theta_{23} - \delta_3 \delta_3 - \alpha_3 \omega_3 + \alpha_3 \omega_3 \)

\[ G_{(4)} = (\theta_{23} - \tau_2 \omega_2 - \alpha_3 \omega_3 + \alpha_1 \omega_2 \omega_3) \] \( \theta_{23} - \delta_3 \delta_3 - \alpha_3 \omega_3 + \alpha_3 \omega_3 \)

\[ G_{(5)} = (\theta_{23} - \tau_2 \omega_2 - \alpha_3 \omega_3 + \alpha_1 \omega_2 \omega_3) \] \( \theta_{23} - \delta_3 \delta_3 - \alpha_3 \omega_3 + \alpha_3 \omega_3 \)

\[ G_{(6)} = \theta_{23} - \tau_2 \omega_2 - \alpha_3 \omega_3 + \alpha_1 \omega_2 \omega_3 \] \( \theta_{23} - \delta_3 \delta_3 - \alpha_3 \omega_3 + \alpha_3 \omega_3 \). \] (A.8)

The angular terms for operator \( H \) are

\[ H_{(1)} = (\theta_{14} - \alpha_1 \alpha_4 - \gamma_1 \gamma_4 + \alpha_2 \alpha_1 \gamma_4) \] \( \theta_{14} - \beta_1 \beta_4 - \lambda_1 \lambda_4 + \alpha_6 \beta_1 \lambda_4 \)

\[ H_{(2)} = (\theta_{14} - \beta_1 \beta_4 - \tau_4 (\tau_1 - \beta_4 \beta_1) - \chi_4 (\lambda_1 - \beta_4 \beta_1) + \tau_4 \epsilon_6 (\lambda_1 - \beta_4 \beta_1) \) \( \theta_{14} - \alpha_1 \alpha_4 \)

\[ H_{(3)} = (\theta_{14} - \alpha_1 \alpha_4 - \delta_1 \delta_4 + \alpha_2 \alpha_1 \alpha_4) \] \( \theta_{14} - \beta_1 \beta_4 - \alpha_1 \alpha_4 + \beta_5 \beta_1 \alpha_4 \)

\[ H_{(4)} = (\theta_{14} - \beta_1 \beta_4 - \tau_4 (\tau_1 - \beta_4 \beta_1) - \chi_4 (\lambda_1 - \beta_4 \beta_1) + \tau_4 \epsilon_5 (\lambda_1 - \beta_4 \beta_1) \) \( \theta_{14} - \alpha_1 \alpha_4 \)

\[ H_{(5)} = \theta_{14} - \alpha_1 \alpha_4 - \chi_4 (\chi_1 - \alpha_7 \alpha_1) - \tau_4 (\tau_1 - \alpha_2 \alpha_1) + \chi_4 \alpha_7 (\tau_1 - \alpha_2 \alpha_1) \] \( \theta_{14} - \tau_1 \tau_4 \) \]

\[ H_{(6)} = \theta_{14} - \alpha_1 \alpha_4 - \chi_4 (\chi_1 - \alpha_7 \alpha_1) - \tau_4 (\tau_1 - \alpha_2 \alpha_1) + \chi_4 \alpha_7 (\tau_1 - \alpha_2 \alpha_1) \] \( \theta_{14} - \tau_1 \tau_4 \). \] (A.9)

The angular terms for operator \( I \) are

\[ I_{(1)} = \theta_{13} - \alpha_3 \alpha_3 - \lambda_3 (\lambda_1 - \alpha_6 \lambda_1) - \tau_3 (\tau_1 - \alpha_2 \tau_1) + \lambda_3 \gamma_6 (\tau_1 - \alpha_2 \tau_1) \] \( \theta_{13} - \beta_1 \beta_3 \)

\[ I_{(2)} = \theta_{13} - \alpha_3 \alpha_3 - \lambda_3 (\lambda_1 - \alpha_6 \lambda_1) - \tau_3 (\tau_1 - \alpha_2 \tau_1) + \lambda_3 \gamma_6 (\tau_1 - \alpha_2 \tau_1) \] \( \theta_{13} - \beta_1 \beta_3 \)

\[ I_{(3)} = \theta_{13} - \beta_1 \beta_3 - \delta_3 (\beta_1 - \beta_3 \beta_1) - \alpha_3 \omega_3 (\omega_1 - \alpha_4 \omega_1) \] \( \theta_{13} - \tau_1 \tau_3 \)

\[ I_{(4)} = \theta_{13} - \alpha_3 \alpha_3 - \tau_3 (\tau_1 - \alpha_2 \tau_1) + \lambda_3 \gamma_6 (\tau_1 - \alpha_2 \tau_1) \] \( \theta_{13} - \beta_1 \beta_3 \)

\[ I_{(5)} = \theta_{13} - \alpha_3 \alpha_3 - \tau_3 (\tau_1 - \alpha_2 \tau_1) + \lambda_3 \gamma_6 (\tau_1 - \alpha_2 \tau_1) \] \( \theta_{13} - \beta_1 \beta_3 \)

\[ I_{(6)} = \theta_{13} - \beta_1 \beta_3 - \delta_3 (\beta_1 - \beta_3 \beta_1) - \alpha_3 \omega_3 (\omega_1 - \alpha_4 \omega_1) \] \( \theta_{13} - \tau_1 \tau_3 \). \] (A.10)
The angular terms for operator $\mathbb{11}$ are

$$
\mathcal{F}_{11}^{(1)} = \left[ \theta_{12} - \beta_1 \beta_2 - \gamma_1 (\beta_1 - \beta_2) + \lambda_2 (\lambda_1 - \lambda_2) \right] (\theta_{12} - \alpha_1 \alpha_2)
$$

$$
\mathcal{F}_{11}^{(2)} = (\theta_{12} - \alpha_1 \alpha_2 - \epsilon_1 \epsilon_2 + \alpha_4 \alpha_1 \epsilon_2) (\theta_{12} - \beta_1 \beta_2 - \lambda_1 \lambda_2 + \beta_6 \beta_1 \lambda_2)
$$

$$
\mathcal{F}_{11}^{(3)} = \left[ \theta_{12} - \alpha_1 \alpha_2 - \beta_2 (\beta_1 - \beta_2) + \lambda_2 (\lambda_1 - \lambda_2) \right] (\theta_{12} - \beta_1 \beta_2)
$$

$$
\mathcal{F}_{11}^{(4)} = (\theta_{12} - \alpha_1 \alpha_2 - \beta_2 (\beta_1 - \beta_2) + \alpha_4 \alpha_1 \beta_2) (\theta_{12} - \beta_3 \beta_2 - \lambda_1 \lambda_2 + \beta_6 \beta_1 \lambda_2)
$$

$$
\mathcal{F}_{11}^{(5)} = \left[ \theta_{12} - \beta_1 \beta_2 - \gamma_1 (\beta_1 - \beta_2) + \lambda_2 (\lambda_1 - \lambda_2) \right] (\theta_{12} - \alpha_1 \alpha_2)
$$

$$
\mathcal{F}_{11}^{(6)} = (\theta_{12} - \alpha_1 \alpha_2 - \delta_1 \delta_2 + \alpha_3 \alpha_1 \delta_2) (\theta_{12} - \beta_1 \beta_2 - \lambda_1 \lambda_2 + \beta_7 \beta_1 \lambda_2).
$$

(A.11)

The angular terms for operator $\mathbb{12}$ are

$$
\mathcal{F}_{12}^{(1)} = (\theta_{12} - \alpha_2 \alpha_3 - \beta_2 \beta_3 + \alpha_1 \alpha_2 \beta_3) (\theta_{24} - \gamma_2 \gamma_4) (\theta_{34} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{12}^{(2)} = (\theta_{34} - \alpha_4 \lambda_3 - \beta_3 \lambda_4 + \alpha_1 \alpha_4 \beta_3) (\theta_{24} - \epsilon_2 \epsilon_4) (\theta_{23} - \lambda_2 \lambda_3)
$$

$$
\mathcal{F}_{12}^{(3)} = (\theta_{12} - \alpha_1 \alpha_2 - \beta_2 \beta_3 + \alpha_1 \alpha_2 \beta_3) (\theta_{24} - \delta_0 \delta_4) (\theta_{23} - \lambda_2 \lambda_3)
$$

$$
\mathcal{F}_{12}^{(4)} = (\theta_{24} - \alpha_1 \alpha_2 - \beta_2 \beta_3 + \alpha_1 \alpha_2 \beta_3) (\theta_{24} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{12}^{(5)} = (\theta_{24} - \alpha_1 \alpha_2 - \beta_2 \beta_3 + \alpha_1 \alpha_2 \beta_3) (\theta_{24} - \gamma_2 \gamma_4) (\theta_{34} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{12}^{(6)} = (\theta_{34} - \alpha_3 \alpha_4 - \beta_3 \beta_4 + \alpha_1 \alpha_3 \beta_4) (\theta_{25} - \delta_2 \delta_3) (\theta_{24} - \lambda_2 \lambda_3).
$$

(A.12)

The angular terms for operator $\mathbb{13}$ are

$$
\mathcal{F}_{13}^{(1)} = (\theta_{14} - \alpha_1 \alpha_4 - \gamma_1 \gamma_4 + \alpha_2 \alpha_1 \gamma_4) (\theta_{13} - \beta_1 \beta_4) (\theta_{34} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{13}^{(2)} = (\theta_{14} - \alpha_1 \alpha_4 - \gamma_1 \gamma_4 + \alpha_2 \alpha_1 \gamma_4) (\theta_{13} - \beta_1 \beta_4) (\theta_{34} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{13}^{(3)} = (\theta_{14} - \beta_1 \beta_4 - \gamma_1 \gamma_4 + \beta_0 \beta_1 \gamma_4) (\theta_{13} - \alpha_1 \alpha_4) (\theta_{34} - \delta_3 \delta_4)
$$

$$
\mathcal{F}_{13}^{(4)} = (\theta_{13} - \beta_1 \beta_4 - \gamma_1 \gamma_4 + \beta_0 \beta_1 \gamma_4) (\theta_{14} - \alpha_1 \alpha_4) (\theta_{34} - \delta_3 \delta_4)
$$

$$
\mathcal{F}_{13}^{(5)} = (\theta_{13} - \alpha_1 \alpha_4 - \gamma_1 \gamma_4 + \alpha_2 \alpha_1 \gamma_4) (\theta_{14} - \beta_1 \beta_4) (\theta_{34} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{13}^{(6)} = (\theta_{34} - \gamma_3 \gamma_4 - \alpha_2 \alpha_1 \gamma_4 + \delta_0 \delta_3 \delta_4) (\theta_{13} - \alpha_1 \alpha_4) (\theta_{14} - \beta_1 \beta_4).
$$

(A.13)

The angular terms for operator $\mathbb{14}$ are

$$
\mathcal{F}_{14}^{(1)} = (\theta_{14} - \beta_1 \beta_4 - \gamma_1 \gamma_4 + \beta_0 \beta_1 \gamma_4) (\theta_{12} - \alpha_1 \alpha_2) (\theta_{24} - \gamma_2 \gamma_4)
$$

$$
\mathcal{F}_{14}^{(2)} = (\theta_{12} - \beta_1 \beta_2 - \lambda_1 \lambda_2 + \beta_6 \beta_1 \lambda_2) (\theta_{14} - \alpha_1 \alpha_4) (\theta_{24} - \epsilon_2 \epsilon_4)
$$

$$
\mathcal{F}_{14}^{(3)} = (\theta_{14} - \alpha_1 \alpha_4 - \delta_1 \delta_4 + \alpha_3 \alpha_1 \delta_4) (\theta_{12} - \beta_1 \beta_2) (\theta_{24} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{14}^{(4)} = (\theta_{24} - \epsilon_2 \epsilon_4 - \gamma_2 \gamma_4 + \epsilon_0 \epsilon_4 \epsilon_2) (\theta_{12} - \beta_1 \beta_2) (\theta_{24} - \lambda_3 \lambda_4)
$$

$$
\mathcal{F}_{14}^{(5)} = (\theta_{24} - \gamma_2 \gamma_4 - \lambda_2 \lambda_4 + \gamma_7 \gamma_2 \lambda_4) (\theta_{12} - \alpha_1 \alpha_4) (\theta_{14} - \beta_1 \beta_4)
$$

$$
\mathcal{F}_{14}^{(6)} = (\theta_{12} - \alpha_1 \alpha_2 - \delta_1 \delta_2 + \alpha_3 \alpha_1 \delta_2) (\theta_{14} - \beta_1 \beta_4) (\theta_{24} - \lambda_2 \lambda_4).
$$

(A.14)

The angular terms for operator $\mathbb{15}$ are
Finally, the angular terms for operator $\mathcal{L}$ are

\[
\mathcal{F}_{\mathcal{L}}^{(1)} = (\theta_{23} - \bar{\nu}_3 \bar{\nu}_3 - \bar{\tau}_2 \bar{\tau}_3 + \gamma_6 \bar{\tau}_2 \bar{\tau}_4) (\theta_{12} - \bar{\alpha}_1 \bar{\alpha}_2) (\theta_{13} - \bar{\beta}_1 \bar{\beta}_3)
\]
\[
\mathcal{F}_{\mathcal{L}}^{(2)} = (\theta_{14} - \bar{\alpha}_4 \bar{\alpha}_4) (\theta_{13} - \bar{\beta}_1 \bar{\beta}_2) (\theta_{24} - \bar{\tau}_2 \bar{\tau}_4) (\theta_{23} - \bar{\nu}_2 \bar{\nu}_3)
\]
\[
\mathcal{F}_{\mathcal{L}}^{(3)} = (\theta_{13} - \bar{\alpha}_1 \bar{\alpha}_3) (\theta_{12} - \bar{\beta}_1 \bar{\beta}_3) (\theta_{24} - \bar{\tau}_2 \bar{\tau}_3) (\theta_{34} - \bar{\tau}_3 \bar{\tau}_4)
\]
\[
\mathcal{F}_{\mathcal{L}}^{(4)} = (\theta_{14} - \bar{\alpha}_4 \bar{\alpha}_3) (\theta_{13} - \bar{\beta}_1 \bar{\beta}_2) (\theta_{24} - \bar{\tau}_2 \bar{\tau}_3) (\theta_{34} - \bar{\tau}_3 \bar{\tau}_4)
\]
\[
\mathcal{F}_{\mathcal{L}}^{(5)} = (\theta_{13} - \bar{\alpha}_1 \bar{\alpha}_3) (\theta_{12} - \bar{\beta}_1 \bar{\beta}_3) (\theta_{23} - \bar{\tau}_2 \bar{\tau}_3) (\theta_{34} - \bar{\tau}_3 \bar{\tau}_4)
\]
\[
\mathcal{F}_{\mathcal{L}}^{(6)} = (\theta_{13} - \bar{\alpha}_1 \bar{\alpha}_3) (\theta_{12} - \bar{\beta}_1 \bar{\beta}_3) (\theta_{23} - \bar{\tau}_2 \bar{\tau}_3) (\theta_{24} - \bar{\tau}_2 \bar{\tau}_4)
\]  

(A.15)

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