Variants of theorems of Baer and Hall on finite-by-hypercentral groups

Carlo Casolo - Ulderico Dardano - Silvana Rinauro

dedicated to the memory of Guido Zappa

Abstract We show that if a group $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral, then the index of the hypercenter of $G$ is bounded by a function of the order of $L$. This completes recent results generalizing classical theorems by R. Baer and P. Hall. Then we apply our results to groups of automorphisms of a group $G$ acting in a restricted way on an ascending normal series of $G$.

1 Introduction

A classical theorem by R. Baer states that, if the $m$-th term $Z_m(G)$ of the upper central series a group $G$ has finite index $t$ in $G$ for some positive integer $m$, then there is a finite normal subgroup $L$ of $G$ such that $G/L$ is nilpotent of class at most $m$, that is $G/L = Z_m(G/L)$ (see 14.5.1 in [7], which shall be the reference for undefined notation). Recently, in [6] it has been shown that there is such an $L$ with finite order $d$ bounded by a function of $t$ and $m$.

In the opposite direction, P. Hall showed that, if there is a normal subgroup $L$ with finite order $d$ such that $G/L$ is nilpotent of class at most $m$, then $G/Z_{2m}(G)$ has finite order bounded by a function of $d$ and $m$ (see [7], page 118).

Recently, in [2] it has been shown that the hypercenter of $G$ has finite index $t$ if and only if there is a finite normal subgroup $L$ with order $d$ such that $G/L$ is hypercentral, that is coincides with its hypercenter. Recall that the hypercenter of a group $G$ is the last term of the upper central series of $G$ (see details below). Then in [5] it has been shown that $d$ may be bounded by a function of $t$, namely $t^{(1+\log_2 t)/2}$. Here we complete the picture by showing that $t$ in turn may be bounded by a function of $d$.

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Theorem 1 If a group $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral, then the hypercenter of $G$ has index bounded by $|\text{Aut}(L)| \cdot |Z(L)|$.

Corollary 1 If a group $G$ has a finite normal subgroup $L$ such that $G/L$ is nilpotent of class $m$, then $|G/Z_{zm}(G)|$ is bounded by a function of $d := |L|$.

There are many generalizations and variants of Baer and Hall theorems. By applying Theorem 1 above, we improve the results in [3] which are concerned with possibly non-inner automorphisms.

Before stating our Theorem 2 we recall some definitions. As usual, we say that the group $A$ acts on a group $G$ if and only if there is a homomorphism $\tilde{\cdot} : A \to \text{Aut}(G)$ (called action). We will regard both $G$ and $\tilde{A}$ as subgroups of the holomorph group $G \rtimes \text{Aut}(G)$ of $G$. In particular, we will denote by a bar $\overline{\cdot}$ the action of a group $G$ on itself by conjugation, that is the natural $\text{Aut}(G)$-homomorphism $G \to \overline{G} \leq \text{Aut}(G)$. If an action is such that its image $\tilde{A}$ is normalized by $\overline{G} = \text{Inn}(G)$, we define by recursion an ascending $G$-series $Z_\alpha(G, A)$ (with $\alpha$ ordinal number) by $Z_0(G, A) := 1$, $Z_{\alpha+1}(G, A)/Z_\alpha(G, A) := C_{G/Z_\alpha(G, A)}(A)$ and $Z_{\lambda}(G, A) := \cup_{\alpha<\lambda} Z_\alpha(G, A)$ when $\lambda$ is a limit ordinal. We call $Z_\alpha(G, A)$ the $\alpha$th $A$-center of $G$. Recall that an ascending $G$-series is a well ordered (by inclusion) set of normal subgroups of $G$. Clearly the series $Z_\alpha(G, A)$ is stabilized by $A$, in the sense that $A$ acts trivially on the factors between consecutive terms. The last term $Z_\infty(G, A)$ of this series is called $A$-hypercenter of $G$.

We say that $G$ is $A$-hypercentral with (ordinal) type at most $\alpha$ if and only if $G = Z_\alpha(G, A)$. Clearly $Z_\alpha(G) := Z_\alpha(G, \overline{G})$ is the usual $\alpha$th center of $G$ and if $G = Z_\alpha(G)$, then $G$ is hypercentral of type at most $\alpha$.

Now we are in a position to state our second result, which consists in two parts that refer to theorems of Baer and Hall, respectively. In fact, if $A = \text{Inn}(G)$, then part (B) reduces to Theorem B in [5] and part (H) to Theorem 1 above.

Theorem 2 Let $G$ be a group and $A$ be a subgroup of $\text{Aut}(G)$ such that $A^{\text{Inn}(G)} = A$ and the hypercenter of $A/(A \cap \text{Inn}(G))$ has finite index $k$.

(B) If the $A$-hypercenter of $G$ has finite index $t$, then there is a finite normal $A$-subgroup $L$ with order bounded by a function of $(t, k)$ such that $G/L$ is $A$-hypercentral.

(H) If there is a finite normal $A$-subgroup $L$ with order $d$ such that $G/L$ is $A$-hypercentral, then the $A$-hypercenter of $G$ has finite index bounded by a function of $(d, k)$.
Remark that this theorem generalizes Theorems 4 and 3 of [3] where the same picture is considered, but with more restrictive conditions, that is $A$ contains $\text{Inn}(G)$, the factor $A/\text{Inn}(G)$ is finite and the involved series which are stabilized by $A$ are finite. Clearly, our bounding functions do not depend on the length of the considered series.

Finally note that the hypothesis that $A$ is normalized by $\text{Inn}(G)$ is necessary, as shown by Example in Sect. 2 below.

2 Proof of Theorem 1

To prove Theorem 1 we use a key lemma. Recall that we denote the hypercenter of a group $G$ by $Z_\infty(G)$.

**Lemma 1** Let $A \leq H$ be normal subgroups of a group $G$ with $A$ finite and $A \leq Z(H)$. If $G/C_G(H)$ is locally nilpotent and $H/A \leq Z_\infty(G/A)$, then $H \leq Z_\infty(G)A$.

**Proof.** Arguing by induction on the order of $A$, we may assume that $A$ is minimal normal in $G$. Then $A$ is an elementary abelian $p$-group for some prime $p$. If $A \cap Z(G) \neq 1$, then $A \leq Z(G)$ by minimality of $A$ and so we have $H \leq Z_\infty(G)A$.

Suppose then $A \cap Z(G) = 1$ (and so $A \cap Z_\infty(G) = 1$) and let $N := Z_\infty(G) \cap H$. Note that the hypotheses hold for the subgroups $\bar{A} := AN/N$, $\bar{H} := H/N$ of the group $\bar{G} := G/N$. Since from $\bar{H} \leq Z_\infty(\bar{G})\bar{A}$ it follows $H \leq Z_\infty(G)A$, we may assume $Z_\infty(G) \cap H = 1$.

We claim that $H = A$ (note that $H \leq Z_\infty(G)A$ if and only if $H = H \cap Z_\infty(G)A = (H \cap Z_\infty(G))A = A$). Suppose, by contradiction, $H > A$ and let $X/A \neq 1$ be either infinite cyclic or of prime order $r$ and contained in $(H/A) \cap Z(G/A)$. Since by hypotheses $A \leq Z(H)$, then $X$ is abelian and $X < G$, clearly.

Let us show now that $X$ is a $p$-group. If, by contradiction, $X/A$ is infinite or $r \neq p$, then $X^p \neq 1$ and $X^p \cap A = 1$. Thus $X^p$ is $G$-isomorphic to $X^pA/A \leq Z_\infty(G/A)$. Hence $X^p \leq H \cap Z_\infty(G) = 1$, a contradiction. So $X/A$ has order $p$.

Assume, again by contradiction, $X^p \neq 1$. By minimality of $A$, we have $X^p = A = [G, X]$ and so $[G, A] = [G, X^p] = [G, X]^p = A^p = 1$, a contradiction.

Then $X$ is a finite elementary abelian $p$-group. Since $[G, A] = A \leq X$, the subgroup $X \rtimes (G/C_G(X))$ of the holomorph of $X$ is not nilpotent, and so
\(G/C_G(X)\) is not a \(p\)-group. Hence there are a prime \(q \neq p\) and a normal non-trivial \(q\)-subgroup \(Q/C_G(X)\) of \(G/C_G(X)\). Since \(Q \not\leq C_G(X)\), then \([X, Q] \neq 1\). Thus \([X, Q] = A\), as \([X, Q] \leq A\) and by minimality of \(A\).

By a standard argument on coprime actions (see for example Exercise 4.1 in [1]), we have

\[ X = [X, Q] \times C_X(Q) = A \times C_X(Q), \]

therefore \(C_X(Q) \neq 1\). On the other hand, \(C_X(Q)\) is a normal subgroup of \(G\) and so \(C_X(Q) \leq Z_\infty(G) \cap H = 1\), a contradiction which gives the claim \(H = A\).

\(\square\)

**Proof of Theorem 1.** Let us apply Lemma 1 with \(A := Z(L)\) and \(H := C_G(L)\). In fact on one hand \(H/A = H/(H \cap L) \simeq_G LH/L\), then \(H/A \leq Z_\infty(G/A)\). On the other hand \(L \leq C_G(H)\) and so \(G/C_G(H)\) is hypercentral, since it is an image of \(G/L\). Therefore \(H \leq Z_\infty(G)A\). Hence

\[ |H/(H \cap Z_\infty(G))| = |A(Z_\infty(G) \cap H)/(Z_\infty(G) \cap H)| \leq |A| = |Z(L)|. \]

Since \(H = C_G(L)\), then \(|G/H| \leq |\text{Aut}(L)|\). Thus

\[ |G/Z_\infty(G)| \leq |G/H| \cdot |H/(H \cap Z_\infty(G))| \leq |\text{Aut}(L)| \cdot |Z(L)|. \]

\(\square\)

**Proof of Corollary 1.** Note that \(Z_{d+m}(G) = Z_\infty(G)\) has finite index. Thus if \(d \leq m\), the statement follows directly from Theorem 1. Otherwise, \(|G/Z_{2m}(G)|\) is bounded by the maximum of the \(h(d, i)\) with \(i = 1, \ldots, d\), where \(h(d, m)\) is the bounding function in Hall Theorem.

\(\square\)

From Theorem 1 and the above quoted result from [3] we deduce a corollary which gives a rather complete picture of finite-by-hypercentral groups.

**Corollary 2** If \(G\) is a group with a (finite) normal series

\[ G = G_0 \geq F_1 \geq G_1 \geq \ldots \geq F_n \geq G_n = 1 \]

where
- each factor \(F_i/G_i\) is finite with order \(t_i > 1\),
- each factor \(G_{i-1}/F_i\) is contained in the hypercenter of \(G/F_i\),
then there is a normal subgroup \(L\) with finite order bounded by a function of \(t = t_1 \cdot \ldots \cdot t_n\) such that \(G/L\) is hypercentral.

Moreover the hypercenter of \(G\) has finite index bounded by a function of \(t\).
Proof. Define recursively a function \( f : \mathbb{N} \to \mathbb{N} \) by means of \( f(1) = 1 \) and \( f(t + 1) = (t + 1)g(g(f(t))) \) for each \( t \in \mathbb{N} \), where \( g(t) := t^{1 + \log_2 t} \).

We show that there is \( L \triangleleft G \) such that \( |L| \leq f(t) \) and \( G/L = Z_\alpha(G/L) \) for \( \alpha := \alpha_n + \ldots + \alpha_1 + m' \), where the \( \alpha_i \)'s are ordinal numbers such that \( G_{i-1}/F_i \leq Z_{\alpha_i}(G/F_i) \) for each \( i \) and \( m' \in \mathbb{N} \) may be bounded by a function of \( t \) and of the \( \alpha_i \)'s which are finite. Since \( f(t) \geq t \) for each \( t \), the statement is trivial if \( n = 1 \).

Assume then by induction on \( n \) that there is a normal series
\[
G \geq F_{n-1} \geq G_{n-1} \geq F_n \geq G_n = 1
\]
such that \( G/F_{n-1} \) is hypercentral of type \( \alpha' = \alpha_{n-1} + \ldots + \alpha_1 + m'' \), with \( m'' \in \mathbb{N} \) and \( |F_{n-1}/G_{n-1}| \leq f(t_*) \) with \( t_* = t_1 \cdot \ldots \cdot t_{n-1} \). Applying Theorem B in \([4]\) to \( G/G_{n-1} \), if \( Z/G_{n-1} := Z_{\lceil \log_2 f(t_*) \rceil + \alpha'}(G/G_{n-1}) \), then \( |G/Z| \leq g(f(t_*)) \).

Thus, applying Theorem B of \([KOS]\) to \( G/F_n \), we have that there is a normal subgroup \( L \) such that \( G/L \) is hypercentral with ordinal type at most \( \alpha_1 + \lceil \log_2 f(t_*) \rceil + \alpha' + \lceil \log_2 g(f(t_*)) \rceil \) and \( |L/F_n| \leq g(g(f(t_*))) \). We have: \(|L| \leq t_1 g(g(f(t_*))) \leq tg(g(f(t - 1))) = f(t)\), as wished. \( \square \)

Remark: In the above proof, if \( \alpha \) is infinite, then clearly \( G/Z_\alpha(G) \) is finite. Otherwise, if \( G_{i-1}/F_i \leq Z_{m_i}(G/F_i) \) for each \( i \) with \( m_i \in \mathbb{N} \), then there is a finite normal subgroup \( L \) such that \( G/L = Z_m(G/L) \) with \( m := m_1 + m_2 + \ldots + m_n \), by Theorem B in \([4]\). Hence, in this case, \( G/Z_2m(G) \) is finite.

3 Proof of Theorem 2

Proof of Theorem 2. Let \( \alpha' \) such that \( B/(A \cap \text{Inn}(G)) := Z_{\alpha'}(A/(A \cap \text{Inn}(G)) \) has finite index in \( A/(A \cap \text{Inn}(G)) \). Consider the subgroup \( S := G \times A \) of the holomorph group of \( G \).

Assume first \( A \geq \text{Inn}(G) \). Let \( G_\delta := Z_\delta(G, A) \) for any ordinal \( \delta \). We claim:

\[(*) \quad \forall \delta \quad S_\delta := G_\delta G_{\delta} \leq Z_\delta(S).\]

By induction, suppose true for \( \delta \). Note that \( G \leq A \) acts by conjugation on \( G \) the same way as \( G \). We have \([S_{\delta+1}, S] = [G_{\delta+1} \bar{G}_{\delta+1}, GA] \cdot [G_{\delta+1}, G]^A \leq G_\delta \). On the other hand, \([G_{\delta+1}, GA] \leq [\bar{G}_{\delta+1}, A] \cdot [G_{\delta+1}, G]^A \leq G_\delta = S_\delta \). It follows \( S_{\delta+1} \leq Z_{\delta+1}(S) \) and the claim is proved since the limit ordinal step is trivial.
To prove (B) in the case \( A \geq \text{Inn}(G) \), let \( \alpha \) be such that \( Z_\alpha(G, A) \) has finite index in \( G \) and note that in the normal series

\[
S = GA \geq GB \geq G\bar{G} \geq G_{\alpha}\bar{G}_{\alpha} \geq 1
\]

the factors \( GA/GB \) and \( G\bar{G}/G_{\alpha}\bar{G}_{\alpha} \) are finite with order \( k \) and \( t^2 \), respectively. Moreover, by \((*)\), factors \( GB/G\bar{G} \) and \( G_{\alpha}\bar{G}_{\alpha} \) are contained in the \( \alpha' \)th and \( \alpha \)th center of \( S/G\bar{G} \) and \( S \), respectively. Thus we apply Corollary 2 to the group \( S = GA \). Then the statement (for the group \( G \)) follows easily.

Concerning part (H) in the case \( A \geq \text{Inn}(G) \), consider the normal series

\[
S = GA \geq GB \geq G\bar{G} \geq L\bar{L} \geq 1.
\]

Note that \( GA/GB \) and \( L\bar{L} \) are finite with order \( k \) and \( d^2 \), respectively. Moreover, if \( \alpha_1 \) is such that \( Z_{\alpha_1}(G/L, A) \) has finite index in \( G/L \), then by \((*)\) we have that \( GB/L\bar{L} \) is contained in the \( (\alpha_1 + \alpha') \)th \( A \)-center of \( S/L\bar{L} \). We may apply Corollary 2 and get the statement.

To deal with the more general case, let \( \bar{N} := A \cap \text{Inn}(G) \) such that \( Z(G) \leq N \leq G \). Note that \( [G,A] \leq N \), as \( [g,\gamma] = [\bar{g},\gamma] \in A \cap \text{Inn}(G) \) \( \forall \gamma \in A \) since \( A^{\text{Inn}(G)} = A \). Thus \( A \) acts trivially on \( G/N \). Moreover the group \( \bar{A} := A/C_A(N) \) may be considered as a group of automorphisms on \( N \) containing \( \text{Inn}(N) \). Thus, to prove (H), one may apply the above case to \( N \) and \( \bar{A} := A/C_A(N) \).

To prove (B) in the general case note that, by the above, the subgroup \( Z := Z_\infty(N, A) \) has finite index in \( N \), bounded by a function of \( |L \cap N| \leq |L| \). Let \( K/Z \) be the \( A \)-hypercenter of \( G/Z \). Clearly, \( K \cap N = Z \). Moreover \( K/Z = Z(G/Z, A) \). Consider then \( C/Z := C_{G/Z}([G,A]Z/Z) \) and note that \( C \) has finite index in \( G \), since \( [G,A] \leq N \). By applying the Three Subgroup Lemma to \( A, C/Z, C/Z \), we have that \( A \) acts trivially on the derived subgroup \( C/Z \). Thus \( C'/Z \leq C_{G/Z}(A) \leq K/Z \). Therefore \( CK/K \) is abelian. We consider the series

\[
G \geq CK \geq K \geq Z \geq 1.
\]

The index of \( CK \) in \( G \) is finite and bounded by a function of \( d = |L| \), as \( |N/Z| \) is. Then consider the action of \( A \) on the abelian group \( \hat{G} := CK/K \). Since \( K \cap N = Z \), we have that \( |NK/K| \) is bounded by a function of \( d \). Thus the image of \( A \cap \text{Inn}(G) \) in \( \hat{A} := A/C_A(\hat{G}) \) is finite with order bounded by a function of \( d \). By Corollary 2, \( Z_{\alpha'}(\hat{A}) \) has finite index \( q \) in \( \hat{A} \), bounded by a function of \( d \) and \( k \). Recall that \( \hat{G} \) is abelian and \( [\hat{G}, \hat{A}] \) is finite, as \( [G,A] \) is finite modulo \( K \). Let \( \hat{S} := \hat{G} \rtimes \hat{A} \). Then \( Z_{1+\alpha'}(\hat{S}/[\hat{G}, \hat{A}]) \) has finite index at
most $q$. By Theorem 1, the index of $Z_{1+\alpha'}(\hat{S})$ in $\hat{S}$ is finite and bounded by a function of $d$ and $q$. Thus the $A$-hypercenter of $\hat{G} := CK/K$ has finite index and bounded by a function of $d$ and $k$, as wished. \hfill \Box

**Remark:** in the case $A \geq \text{Inn}(G)$ of the above proof, if $\alpha$, $\alpha_1$ and $\alpha'$ are finite, we have that:
- in case (B), the $2(\alpha + \alpha')$th $A$-center has finite index in $G$, by the above quoted result in \cite{4}. In particular, for $\alpha' = 0$ we have Theorem 3 of \cite{3}.
- in case (H), there is a boundedly finite normal $A$-subgroup $L$ such that $G/L$ coincides with its $(\alpha_1 + \alpha')$th $A$-center. This follows by applying the remarks after Corollary 2 to the group $S$. In particular, for $\alpha' = 0$ we have Theorem 2 and 4 of \cite{3}.

Let us see that the condition that $A$ is normalized by $\text{Inn}(G)$ is necessary.

**Example** There is an elementary abelian group $G$ and a bounded abelian group $A \leq \text{Aut}(G)$ such that $G/Z_{\omega}(G, A)$ is finite (of prime order), while $G/L$ is not $A$-hypercentral, for any finite $A$-subgroup $L \leq G$.

**Proof.** Let $G := Dr_{i<\omega}\langle a_i \rangle$ be an elementary abelian $p$-group, where $p$ is an odd prime and let $Z := Dr_{0<i<\omega}\langle a_i \rangle$. For any $i > 0$, consider $\gamma_i \in \text{Aut}(G)$ centralizing $Z$, and such that $a_0^{\gamma_i} := a_0 a_i$. Let $\tau \in \text{Aut}(G)$ centralizing $Z$ and such that $a_0^{\tau} := a_0^2$. Let $A$ be the subgroup of $\text{Aut}(G)$ generated by $\tau$ and all the $\gamma_i$'s. Then $Z = Z_1(G, A)$ has index $p$ in $G$, while if $K$ is a proper $A$-subgroup of $G$, then $a_0 \notin K$, as $a_0^A = G$. Clearly $\tau$ does not centralizes $a_0$ mod $K$. Thus $G/K$ is not $A$-hypercentral, for any proper $A$-subgroup $K$ of $G$ and in particular for any finite $A$-subgroup $L \leq G$. \hfill \Box

We finish by noticing that Theorem 2 may be formulated in a different way. Recall that the factor of two consecutive terms of a series is called just factor.

**Corollary 3** Let $A$ be a finite-by-hypercentral group of automorphisms of a group $G$ such that $A^{\text{Inn}(G)} = A$. If there is an ascending normal series in $G$ with a finite number of finite factors and such that $A$ acts trivially on all other factors, then:

$i)$ there is a finite index normal $A$-subgroup $G_0$ of $G$ such that $A$ stabilizes an ascending $G$-series of $G_0$;

$ii)$ there is a finite normal $A$-subgroup $L$ such that $A$ stabilizes an ascending $G$-series of $G/L$. 

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Carlo Casolo, Dipartimento di Matematica U. Dini, Università di Firenze, Viale Morgagni 67A, I-50134 Firenze, Italy.
email: casolo@math.unifi.it

Ulderico Dardano, Dipartimento di Matematica e Applicazioni “R.Caccioppoli”, Università di Napoli “Federico II”, Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy.
email: dardano@unina.it

Silvana Rinauro, Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell’Ateneo Lucano 10 - Contrada Macchia Romana, I-85100 Potenza, Italy.
email: silvana.rinauro@unibas.it