How Much Work Does It Take To Straighten a Plane Graph Out?

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Abstract

We prove that if one wants to make a plane graph drawing straight-line then in the worst case one has to move almost all vertices.

The second version of this e-print includes literally the first version. In addition, Appendix [A] gives an explicit bound on the number of fixed vertices and Appendix [B] gives an overview of related work.

The final version appears as [KPRSV09].

We use the standard concepts of a plane graph and a plane embedding (or drawing) of an abstract planar graph (see, e.g., [1]). Given a plane graph $G$, we want to redraw it making all its edges straight line segments while keeping as many vertices on the spot as possible. Let $\text{shift}(G)$ denote the smallest $s$ such that we can do the job by shifting only $s$ vertices. We define $s(n)$ to be the maximum $\text{shift}(G)$ over all $G$ with $n$ vertices.

The function $s(n)$ can have another interpretation closely related to a nice web puzzle called Planarity Game [3]. At the start of the game, a player sees a straight line drawing of a planar graph with many edge crossings. In a move s/he is allowed to shift one vertex to a new position; the incident edges are redrawn correspondingly (being all the time straight line segments). The objective is to obtain a crossing-free drawing. Thus, $s(n)$ is equal to the number of moves that the player, playing optimally, is forced to make on an $n$-vertex game instance at the worst case.

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The Wagner-Fáry-Stein theorem (see, e.g., [2]) says that every $G$ has a straight line plane embedding and immediately implies an upper bound $s(n) \leq n - 3$. We here aim at proving a lower bound.

Given an abstract planar graph $G$, let $\text{shift}(G)$ denote the maximum $\text{shift}(G')$ over all plane embeddings $G'$ of $G$. Thus, we are seeking for $G$ with large $\text{shift}(G)$. Every 4-connected planar graph $G$ is Hamiltonian (Tutte [4]), therefore, has a matching of size at least $(n - 1)/2$ and, therefore, $\text{shift}(G) \geq (n - 3)/2$. An example of planar $G$ with $3k$ vertices and $\text{shift}(G) \geq 2k - 8$ is shown in [5], thereby giving us a bound $s(n) > \frac{2}{3}n - 10$. We now prove a much stronger bound.

**Theorem 1** $s(n) = n(1 - o(1))$.

**Proof.** The vertex set of a graph $G$ will be denoted by $V(G)$. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph induced by $G$ on $X$.

It suffices to prove that for every $k$ and every its multiple $n$ there is an $n$-vertex $G$ with $\text{shift}(G) > (1 - 1/k)n - k^2$.

**Construction of $G$.**

Let $n = k(s + k)$. Let $V(G) = \bigcup_{i=1}^{s+k} V_i$ with $|V_i| = k$ for all $i$. We will describe a plane embedding of $G$ (crossing-free, not necessary straight line). Let each $G[V_i]$ be an arbitrary maximal planar graph. Draw these $s + k$ fragments of $G$ so that they lie in the outer faces of each other (a very important condition!). Finally, add some edges to make $G$ 3-connected. Say, we can join each pair $G[V_i]$ and $G[V_{i+1}]$ by two non-adjacent edges and add yet another edge between $G[V_1]$ and $G[V_{s+k}]$.

This embedding is needed only to define $G$ as an abstract graph. Once this is done, we have to specify a “bad” drawing of $G$ which is far from any straight line drawing.

**“Bad” drawing of $G$.**

Let $C$ be a circle. Put each $V_i$ on $C$ at the vertices of some regular $k$-gon. The drawing is specified.

**Making it straight line, crossing-free: Analysis.**

Let $G'$ be an arbitrary straight line, crossing-free redrawing of $G$ in the same plane. We have to show that not many vertices of $G'$ keep the same

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1There is no need to describe edges; we can suppose either that the drawing is straight line with edge crossings as in the Planarity Game or that we have an arbitrary crossing-free drawing with edges of any shape.
location as they had in $G$. Let $V'_i$ denote the location of $V_i$ in $G'$. Denote the complement of the outer face of $G'[V'_i]$ by $T_i$. Since $G'[V'_i]$ is a triangulation, $T_i$ is a triangle containing this plane graph. Recall that $G$ is 3-connected. By the Whitney theorem (e.g. [1]), $G'$ is equivalent to the original (defining) plane version of $G$. That is, either these two embeddings are obtainable from one another by a plane homeomorphism or this is true after changing outer face in one of them. By construction, the regions occupied by the $G[V_i]$’s in the original embedding are pairwise disjoint. If we change outer face, this is still true possibly with one exception. It follows that all but one $T_i$’s are pairwise disjoint. Without loss of generality suppose that the possible exception is $T_{s+k}$.

Call $V'_i$ persistent if $i < s + k$ and $|V'_i \cap V_i| \geq 2$. Since all persistent $T_i$’s are pairwise disjoint and each of them contains a pair of vertices of some regular $k$-gon, there can be at most $k - 1$ persistent sets. It follows that the number of moved vertices is at least

$$\sum_{i=1}^{s+k-1} |V'_i \setminus V_i| \geq s(k - 1) = \left(\frac{n}{k} - k\right)(k - 1) = (1 - \frac{1}{k})n - k^2 + k,$$

as claimed. \hfill $\blacksquare$

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A An explicit bound for the number of fixed vertices

We now reprove Theorem 1, going over our original argument with somewhat more care and achieving two improvements. First, we obtain an explicit bound $s(n) \geq n - 2\sqrt{n} - 1$. Second, we show that this bound is attained by drawings with vertices occupying any prescribed set of $n$ points in weakly convex position.

By a drawing of a planar graph $G$ we mean an arbitrary injective map $\pi : V(G) \to \mathbb{R}^2$. Given a drawing $\pi$, we suppose that each edge $uv$ of $G$ is drawn as the straight line segment with endpoints $\pi(u)$ and $\pi(v)$. Due to possible edge crossings and even overlaps, $\pi$ may not be a plane drawing of $G$. Hence it is natural to consider a parameter

$$fix(G, \pi) = \max_{\pi'} \{ v \in V(G) : \pi'(v) = \pi(v) \}$$

where the maximum is taken over all plane straight line drawings $\pi'$ of $G$. Note a relation to our previous notation, namely $fix(G, \pi) = n - shift(\pi)$.

We will use some elementary combinatorics of integer sequences. A sequence identified with all its cyclic shifts will be referred to as circular. Subsequences of a circular sequence $S$ will be considered also circular sequences. Note that the set of all circular subsequences is the same for $S$ and any its shift. The length of a $S$ will be denoted by $|S|$.

**Lemma A.1** Let $k, s \geq 1$ and $S^{k,s}$ be the circular sequence consisting of $s$ successive blocks of the form $12\ldots k$. Suppose that $S$ is a subsequence of $S^{k,s}$ with no 4-subsubsequence of the form $xyxy$, where $x \neq y$. Then $|S| < k + s$.

**Proof.** We proceed by the double induction on $k$ and $s$. The base case where $k = 1$ and $s$ is arbitrary is trivial. Let $k \geq 2$ and consider an $S$ with no forbidden subsequence. If every of the $k$ elements occurs in $S$ at most once, then $|S| \leq k$ and the claimed bound is true. Otherwise, without loss of generality we suppose that $S$ contains $\ell \geq 2$ occurrences of $k$. Let $A_1, \ldots, A_\ell$ (resp. $B_1, \ldots, B_\ell$) denote the parts of $S$ (resp. $S^{k,s}$) between these $\ell$ elements. Thus, $|S| = \ell + \sum_{i=1}^{\ell} |A_i|$.
Denote the number of elements with at least one occurrence in $A_i$ by $k_i$. Each element $x$ occurs in at most one of the $A_i$’s because otherwise $S$ would contain a subsequence $xkkx$. It follows that $\sum_{i=1}^{\ell} k_i \leq k - 1$. Note that, if we append $B_i$ with an element $k$, it will consist of blocks $12\ldots k$. Denote the number of these blocks by $s_i$ and notice the equality $\sum_{i=1}^{\ell} s_i = s$. If $k_i$ has no forbidden subsequence, we have $|A_i| \leq k_i + s_i - 1$. If $k_i \geq 1$, this follows from the induction assumption because $A_i$ can be regarded as a subsequence of $S^{k_1,s_1}$. If $k_i = 0$, this is also true because then $|A_i| = 0$. Summarizing, we obtain $|S| \leq \ell + \sum_{i=1}^{\ell} (k_i + s_i - 1) \leq \ell + (k - 1) + s - \ell < k + s$.

**Theorem A.1** Let $k \geq 3$, $n = k^2$, and $H$ be a 3-connected plane graph with $n$ vertices having the following property: Its vertex set can be split into $k$ equal parts $V(H) = V_1 \cup \ldots \cup V_k$ so that each $H[V_i]$ is a triangulation and these $k$ triangulations lie in the outer faces of each other. Let $X$ be an arbitrary set of $n$ points on the boundary $\Gamma$ of a convex plane body. Then there is a drawing $\pi : V(H) \to X$ such that $\text{fix}(H, \pi) \leq 2\sqrt{n} + 1$.

**Proof.** Let $X = \{x_1, \ldots, x_n\}$, where the points in $X$ are numbered in the order of their appearance along $\Gamma$. Fix $\pi$ to be an arbitrary map such that $\pi(V_i) = \{x_i, x_{i+k}, x_{i+2k}, \ldots, x_{i+(k-1)k}\}$ for each $i \leq k$.

Let $\pi'$ be an arbitrary crossing-free straight line redrawing of $H$. We have to show that not many vertices of $H$ keep the same location in $\pi'$ as they had in $\pi$. Denote $A_i = \{\pi(v) : v \in V_i, \pi(v) = \pi'(v)\}$ and $A = \bigcup_{i=1}^{k} A_i$. The union $A$ consists of exactly those vertices that keep their position under transition from $\pi$ to $\pi'$. Thus, we have to bound the number of vertices in $A$ from above.

Denote the complement of the outer face of $H[V_i]$ in $\pi'$ by $T_i$. Since $H[V_i]$ is a triangulation, $T_i$ is a triangle containing all $\pi'(V_i)$. Recall that $H$ is 3-connected. By Whitney’s theorem (see, e.g., [1]), $\pi'$ is equivalent to the original plane embedding of $H$, which we denote by $\delta$. This means that one of the following two cases occurs:

A $\pi'$ is obtainable from $\delta$ by a plane homeomorphism.

B $\pi'$ is obtainable by a plane homeomorphism from $\delta_F$, where $F$ is an inner face of $\delta$ and $\delta_F$ is an embedding of $H$ obtained from $\delta$ by making the face $F$ outer.
By construction, the regions occupied by the $H[V_i]$’s are pairwise disjoint in $\delta$. For $\pi'$ this implies that, if we have Case A, then all $k$ triangles $T_i$ are pairwise disjoint. The same holds true in Case B if $F$ is not a face of any $H[V_j]$-fragment. If $F$ is a face of some $H[V_j]$-fragment, then the $T_i$’s are pairwise disjoint with one exception for the triangle $T_j$, which contains all the others.

Consider first the case that all the triangles are pairwise disjoint. Since $A_i \subset T_i$, the convex hulls of these sets of points are pairwise disjoint. Label each $x_j$ by the index $i$ for which $x_j \in \pi(V_i)$ and consider the circular sequence of these labels in the order of their appearance along $\Gamma$. This is exactly the sequence $S^{k-k}$ as in Lemma A.1. Let $S$ be the subsequence corresponding to the points in $A$. Since the points in $A_i$ are labeled by $i$, we see that $S$ has no subsequence of the form $xyxy$. By Lemma A.1, $|A| = |S| < 2k$.

Consider now the case that the triangles $T_i$ are pairwise disjoint with the exception, say, for $T_k$. Let $a, b, c \in V_k$ be the vertices on the boundary of the outer face of $H[V_k]$ in $\delta$. Let $T$ denote the geometric triangle with vertices $\pi'(a), \pi'(b)$, and $\pi'(c)$. Note that $A_i \subset T$ for all $i < k$. Note also that $A_k \subset \Gamma \setminus T$. The set $\Gamma \setminus T$ consists of at most three continuous components; denote the corresponding parts of $A_k$ by $A'_{k}, A'_{k+1}, A'_{k+2}$. Consider the circular sequence $S$ as above with the following modification: the vertices in $A'_{k+1}$ are relabeled with $k + 1$ and the vertices in $A'_{k+2}$ are relabeled with $k + 2$ (the vertices in $A'_{k}$ keep label $k$). This modification rules out any $xyxy$-subsequence. Note that the modified $S$ can be considered a subsequence of $S^{k+2-k}$. By Lemma A.1 we have $|A| \leq 2k + 1$. $\blacksquare$

## B Related work

Given a planar graph $G$, define $\text{fix}(G) = \min_{\pi} \text{fix}(G, \pi)$, where the minimum is taken over all drawings of $G$. In other words, $\text{fix}(G)$ is the maximum number of vertices which can be fixed in any drawing of $G$ while “untangling” it. Note that $\text{shift}(G) = n - \text{fix}(G)$.

The cycle (resp. path; empty graph) on $n$ vertices will be denoted by $C_n$ (resp. $P_n$; $E_n$). Recall that the join of vertex-disjoint graphs $G$ and $H$ is the graph $G \ast H$ consisting of the union of $G$ and $H$ and all edges between $V(G)$ and $V(H)$. The graphs $W_n = C_{n-1} \ast E_1$ (resp. $F_n = P_{n-1} \ast E_1$; $S_n = E_{n-1} \ast E_1$) are known as wheels (resp. fans; stars). By $kG$ we denote the disjoint union of $k$ copies of a graph $G$. 

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Pach and Tardos [PT02] were first who established a principal fact: Some graphs can be drawn so that, in order to untangle them, one has to shift almost all their vertices. In fact, this is already true for cycles. More precisely, Pach and Tardos [PT02] proved that
\[ \text{fix}(C_n) = O((n \log n)^{2/3}). \]
This bound is nearly optimal, as shown by Cibulka [C08].

The best known upper bounds are of the form \( \text{fix}(G) = O(\sqrt{n}) \). Goaoc et al. [GKOSW07] showed it for certain triangulations. More specifically, they proved that
\[ \text{fix}(P_{n-2} \star P_2) < \sqrt{n} + 2. \]

Shortly after [GKOSW07] and independently of it, there appeared the first version of the current e-print. We constructed 3-connected planar graphs \( H_n \) with \( \text{fix}(H_n) = o(n) \). Though no explicit bound was specified in that version, a simple analysis of our construction reveals that
\[ \text{fix}(H_n) \leq 2\sqrt{n} + 1, \]
see Appendix A. While \( H_n \) is not as simple as \( P_{n-2} \star P_2 \) and the subsequent examples in the literature, the construction of \( H_n \)'s has the advantage that it can ensure certain special properties of these graphs, as bounded vertex degrees. By a later result of Cibulka [C08], for graphs with bounded vertex degrees we have \( \text{fix}(G) = O(\sqrt{n}(\log n)^{3/2}) \) whenever their diameter is logarithmic. Note in this respect that \( H_n \) has diameter \( \Omega(\sqrt{n}) \).

In subsequent papers [SW07, BDHLMW07] examples of graphs with small \( \text{fix}(G) \) were found in special classes of planar graphs, as outerplanar and even acyclic graphs. Spillner and Wolff [SW07] showed for the fan graph that
\[ \text{fix}(F_n) < 2\sqrt{n} + 1 \]
and Bose et al. [BDHLMW07] established for the star forest with \( n = k^2 \) vertices that
\[ \text{fix}(kS_k) \leq 3\sqrt{n} - 3. \]
Finally, Cibulka [C08] proved that
\[ \text{fix}(G) = O((n \log n)^{2/3}) \]
for all 3-connected planar graphs.
Improving a result of Spillner and Wolff [SW07], Bose et al. [BDHLMW07] showed that
\[ \text{fix}(G) \geq \left(\frac{n}{3}\right)^{1/4} \]
for every planar graph \( G \). Better bounds on \( \text{fix}(G) \) are known for cycles [PT02], trees [GKOSW07, BDHLMW07] and, more generally, outer-planar graphs [SW07, RV08]. In all these cases it was shown that \( \text{fix}(G) = \Omega(n^{1/2}) \).

No efficient algorithm determining the parameter \( \text{fix}(G) \) is known. Moreover, computing \( \text{fix}(G, \pi) \) is known to be NP-hard [GKOSW07, V07].

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