THIRD ORDER OPERATOR WITH SMALL PERIODIC COEFFICIENTS

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Abstract. We consider the third order operator with small 1-periodic coefficients on the real line. The spectrum of the operator is absolutely continuous and covers all real line. Under the minimal conditions on the coefficients we show that there are two possibilities: 1) The spectrum has multiplicity one except for a small spectral nonempty interval with multiplicity three. Moreover, the asymptotics of the small interval is determined. 2) All spectrum has multiplicity one only.

1. Introduction and main results

We consider the third order operator acting in and given by

\[ H_\varepsilon = i\partial^3 + \varepsilon V, \quad V = ip\partial + i\partial p + q, \]

where \( \varepsilon \in \mathbb{R} \) is a small coupling constant and the real 1-periodic coefficients \( p, q \) satisfy

\[ 0 < \int_0^1 |p(t)|dt + \int_0^1 |q(t)|dt < \infty, \quad \int_0^1 p(t)dt = \int_0^1 q(t)dt = 0. \]

Due to [BK3] the operator \( H_\varepsilon \) is self-adjoint on the domain

\[ \mathcal{D}(H) = \left\{ f \in L^2(\mathbb{R}) : i(f'' + pf)' + ipf' + qf \in L^2(\mathbb{R}), \ f'', f'' + pf' \in L^1_{loc}(\mathbb{R}) \right\}, \]

the spectrum of \( H_\varepsilon \) is absolutely continuous and covers all real line. Note that the case \( p, q \in C^\infty(\mathbb{R}) \) was considered by McGarvey [McG]. Introduce the value

\[ h = \frac{1}{3} \sum_{n \in \mathbb{Z}} \left( \frac{|\hat{p}_n|^2}{(2\pi n)^2} - \frac{3|\hat{q}_n|^2}{(2\pi n)^4} \right), \]

where \( \hat{f}_n = \int_0^1 f(t)e^{-i2\pi nt}dt, n \in \mathbb{Z} \). We present our main result.

Theorem 1.1. i) Let \( h > 0 \). Then there exist two functions \( r^\pm(\varepsilon) \), which are real analytic in the disk \( \{ |\varepsilon| < c \} \subset \mathbb{C} \) for some \( c > 0 \), \( r^\pm(0) = 0 \) and satisfy

\[ r^+(\varepsilon) - r^-(\varepsilon) = 4h^2\varepsilon^3 + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0. \]

Moreover, the spectrum of \( H_\varepsilon \) has multiplicity one except for a small spectral nonempty interval \((r^-(\varepsilon), r^+(\varepsilon))\) (or \((r^+(\varepsilon), r^-(\varepsilon))\)) with multiplicity three for any \( \varepsilon \in (-c, c) \setminus \{0\} \).

ii) Let \( h < 0 \). Then all spectrum of \( H_\varepsilon \) has multiplicity one for any \( \varepsilon \in (-c, c) \).
Remark. 1) Consider the Hill operator $\mathcal{H}_\varepsilon = -\partial^2 + \varepsilon q$ on the real line. The spectrum of $\mathcal{H}_\varepsilon$ is absolutely continuous and consists of spectral bands separated by gaps. Recall that in the case of the "generic" potential all gaps in the spectrum of $\mathcal{H}_\varepsilon$ are open, see [K] (see also [MO] for the case $q \in L^2(0,1)$).

2) Consider a fourth order operator $\mathcal{H}_\varepsilon = \partial^4 + \varepsilon (\partial p \partial + q)$ on the real line. Here the real 1-periodic coefficients $p,q$ satisfy (1.2) and $\varepsilon \neq 0$ is small enough. The spectrum of $\mathcal{H}_\varepsilon$ is absolutely continuous and consists of spectral bands separated by gaps. The authors proved that if either $p = 0, q \neq 0$ [BK1] or $q = 0, p \neq 0$ [BK2], then there exists a small non-empty spectral interval with the spectrum of multiplicity 4 and all other spectral spectrum have multiplicity 2.

Note that the result for the operator $\mathcal{H}_\varepsilon$ in the general case $p \neq 0, q \neq 0, \varepsilon \to 0$ is more complicated. One can choose $p,q$ such that all the spectrum has multiplicity two. It follows from the Papanicolaou result [P] that any point of the spectrum of the Euler-Bernoulli equation $(\xi y\eta')' = \lambda \eta y$ with any periodic $\xi > 0, \eta > 0$, has multiplicity 2.

3) If $q = 0$, then for real $\varepsilon \neq 0$ small enough there exists a small non-empty spectral band in $\sigma(\mathcal{H}_\varepsilon)$ with the spectrum of multiplicity 3. It is similar to the case of the operator $\mathcal{H}_\varepsilon$.

4) If $p = 0$, then for $\varepsilon \neq 0$ small enough all spectrum of $H_\varepsilon$ has multiplicity one. This is in contrast with the case of the fourth order operator $\mathcal{H}_\varepsilon$.

5) For the operator $H_\varepsilon$ the endpoints of the spectral interval with multiplicity 3 are branch points of the multipliers, see (2.23). For the operator $\mathcal{H}_\varepsilon$ one endpoint of the spectral interval with multiplicity 4 is a branch point and another endpoint is a periodic eigenvalue.

6) The proof of the theorem is based on the analysis of the monodromy matrix as $\varepsilon \to 0$ and identities (2.22), (2.24). In order to show (1.5) we determine the asymptotics of $r^\pm(\varepsilon)$ in the form $r^\pm(\varepsilon) = r(\varepsilon) \pm 2h^2 \varepsilon^3 + O(\varepsilon^4)$ as $\varepsilon \to 0$, where $r$ is some function. This gives the asymptotics of $r^+(\varepsilon) - r^-(\varepsilon)$.

The operator $i\partial^3 + ip\partial + i\partial p + q$ is used in the inverse problem method of integration of the non-linear evolution Boussinesq equation, see, e.g., [McK]. The scattering and inverse scattering theory for operators $i\partial^3 + ip\partial + i\partial p + q$ with decreasing potentials was developed in [DTT]. The work of McKean [McK] gives the numerous results in the inverse spectral theory for the non-self-adjoint operator $\partial^3 + p\partial + \partial p + q$ (and also for the multi-point Dirichlet problem), where $p$ and $q$ are smooth and sufficiently small. The third order operator on the bounded interval was considered in [A1], [A2]. The spectral properties of the arbitrary order operator with smooth periodic coefficients are studied in [DS]. Note that the fourth order operator $\partial^4 + \partial p\partial + q$ is also used for the integration of some non-linear evolution equation, see [HLO].

In [BK3] we study the basic spectral properties of the third order operator $H_\varepsilon$ at $\varepsilon = 1$. We introduce the Lyapunov function $\Delta$, which is analytic on a 3-sheeted Riemann surface $\mathcal{R}$ and satisfies the usual identity $\Delta = \cos k$, where $k$ is a quasimomentum. This important property essentially complicates the spectral analysis of the operator $H_\varepsilon$. Recall that the Lyapunov function for the second order operator $\mathcal{H}_\varepsilon$ is entire and the Riemann surface for the fourth order operator $\mathcal{H}_\varepsilon$ is 2-sheeted, see [BK1], [BK2]. Using the direct integral decomposition we prove that the spectrum of $H_\varepsilon$ is absolutely continuous and covers the whole real line. Moreover, we describe the spectrum, counting with multiplicity, in terms of the Lyapunov function similarly to the case of the Hill operator. In [BK4] we determine the high energy asymptotics of the periodic and antiperiodic spectrum and the branch points of the Lyapunov function.
2. MONODROMY MATRIX

We consider the equation
\[ iy''' + \varepsilon Vy = \lambda y, \quad \lambda \in \mathbb{C}. \quad (2.1) \]

We introduce the \( 3 \times 3 \) matrix-valued function \( M(t, \lambda, \varepsilon), (t, \lambda, \varepsilon) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R} \), by
\[
M(t, \lambda, \varepsilon) = \begin{pmatrix}
\varphi_1 & \varphi_2 & \varphi_3 \\
\varphi_1' & \varphi_2' & \varphi_3' \\
\varphi_1'' + \varepsilon p\varphi_1 & \varphi_2'' + \varepsilon p\varphi_2 & \varphi_3'' + \varepsilon p\varphi_3
\end{pmatrix} (t, \lambda, \varepsilon),
\quad (2.2)
\]
where \( \varphi_j, j = 1, 2, 3 \), are the fundamental solutions of equation \( (2.1) \) satisfying the conditions
\[
M(0, \lambda, \varepsilon) = \mathbb{I}_3,
\quad (2.3)
\]
and \( \mathbb{I}_3 \) is the \( 3 \times 3 \) identity matrix. Due to \( (2.1), (2.2) \) the matrix-valued function \( M(t, \lambda, \varepsilon) \) satisfies the matrix equation
\[
M' - P(\lambda)M = \varepsilon Q(t)M, \quad (t, \lambda, \varepsilon) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R},
\quad (2.4)
\]
where the \( 3 \times 3 \) matrices \( P \) and \( Q \) are given by
\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -i\lambda & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & 0 \\ iq & -p & 0 \end{pmatrix}.
\quad (2.5)
\]

Due to \( Q \in L^1(\mathbb{T}) \) there exists absolutely continuous in \( t \in \mathbb{R} \) matrix-valued solution \( M(t, \lambda, \varepsilon) \) of the problem \( (2.4), (2.5) \) and the monodromy matrix \( M(1, \lambda, \varepsilon), (\lambda, \varepsilon) \in \mathbb{C} \times \mathbb{R} \), is well-defined.

Consider the case \( \varepsilon = 0 \). The matrix-valued solution \( M_0(t, \lambda) = M(t, \lambda, 0) \) of the problem \( (2.4), (2.5) \) has the form \( M_0 = e^{tP} \). Each function \( M_0(t, \cdot), t \in \mathbb{R} \), is entire. Eigenvalues of the matrix \( P \) are given by \( iz, i\omega z, i\omega^2 z \), henceforth
\[
\omega = e^{\frac{iz\pi}{2}}, \quad z = \lambda^\frac{1}{2}, \quad \arg \lambda \in \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right], \quad \arg z \in \left( -\frac{\pi}{6}, \frac{\pi}{2} \right].
\]

Then eigenvalues of the matrix \( M_0 \) have the form \( e^{izt}, e^{i\omega z t}, e^{i\omega^2 z t} \). Estimates \( |e^{iz\omega t}| \leq e^{z_0|t|} \) imply
\[
|M_0(t, \lambda)| \leq e^{z_0|t|}, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad \text{where } z_0 = \max_{j=0,1,2} \text{Re}(iz\omega^j) = \text{Re}(iz\omega^2), \quad (2.6)
\]
and henceforth a matrix \( A \) has the norm given by
\[
|A| = \max \{ \sqrt{T} : f \text{ is an eigenvalue of the matrix } A^*A \}.
\]

Consider the case \( \varepsilon \neq 0 \). Using the standard arguments we deduce that the function \( M(t, \lambda, \varepsilon) \) satisfies the integral equation
\[
M(t, \lambda, \varepsilon) = M_0(t, \lambda) + \varepsilon \int_0^t M_0(t - s, \lambda)Q(s)M(s, \lambda, \varepsilon)ds.
\quad (2.7)
\]
The standard iterations in \( (2.7) \) lead to the formal series
\[
M(t, \lambda, \varepsilon) = \sum_{n \geq 0} \varepsilon^n M_n(t, \lambda), \quad M_n(t, \lambda) = \int_0^t M_0(t - s, \lambda)Q(s)M_{n-1}(s, \lambda)ds, \quad n \geq 1.
\quad (2.8)
\]

In the following Lemma we will prove estimates of the monodromy matrix \( M(1, \lambda, \varepsilon) \).
Lemma 2.1. Each matrix-valued function $M(t, \cdot, \cdot), t \in [0, 1]$ is entire in $(\lambda, \varepsilon) \in \mathbb{C}^2$ and satisfies:

$$|M(1, \lambda, \varepsilon)| \leq e^{\varepsilon_0 + \kappa}, \quad |M(1, \lambda, \varepsilon) - \sum_{n=0}^{N-1} M_n(1, \lambda, \varepsilon)| \leq |\varepsilon|^N e^{\varepsilon_0 + |\varepsilon|\kappa},$$

(2.9)

for all $(N, \lambda, \varepsilon) \in \mathbb{N} \times \mathbb{C}^2$, where

$$\kappa = \int_0^1 (|p(t)| + |q(t)|) dt.$$

Proof. Identity (2.8) gives

$$M_n(t, \lambda) = \int_{0 \leq t_1 < \ldots < t_n \leq t} \prod_{k=1}^{n} \left( M_0(t_{k+1} - t_k, \lambda)Q(t_k) \right) M_0(t_1, \lambda) dt_1 dt_2 \ldots dt_n,$$

(2.10)

the factors are ordering from right to left. Substituting estimates (2.6) into identities (2.10) we obtain

$$|M_n(t, \lambda)| \leq e^{2\varepsilon_0 t} \left( \int_0^t |Q(s)| ds \right)^n, \quad \text{all } (n, t, \lambda) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{C}.\quad (2.11)$$

These estimates show that for each fixed $t \geq 0$ the formal series (2.8) converges absolutely and uniformly on bounded subset of $\mathbb{C}^2$. Each term of this series is an entire function of $(\lambda, \varepsilon)$. Hence the sum is an entire function. Summing the majorants we get

$$|M(t, \lambda, \varepsilon) - \sum_{n=0}^{N-1} \varepsilon^n M_n(t, \lambda)| \leq \left( |\varepsilon| \int_0^t |Q(s)| ds \right)^N e^{2\varepsilon_0 t + |\varepsilon| \int_0^t |Q(s)| ds}$$

for all $(t, \lambda, \varepsilon) \in \mathbb{R}_+ \times \mathbb{C}^2$, which yields (2.9). \[ \blacksquare \]

The characteristic polynomial $D$ of the monodromy matrix $M(1, \lambda, \varepsilon)$ is given by

$$D(\tau, \lambda, \varepsilon) = \text{det}(M(1, \lambda, \varepsilon) - \tau \mathbb{I}_3), \quad (\tau, \lambda, \varepsilon) \in \mathbb{C}^3.\quad (2.12)$$

An eigenvalue of $M(1, \lambda, \varepsilon)$ is called a multiplier, it is a zero of the algebraic equation $D(\cdot, \lambda, \varepsilon) = 0$. Each $M(1, \lambda, \varepsilon), (\lambda, \varepsilon) \in \mathbb{C} \times \mathbb{R}$, has exactly 3 (counted with multiplicities) multipliers $\tau_j(\lambda, \varepsilon), j = 1, 2, 3.$ We need following results.

Lemma 2.2. i) The function $D$ satisfies

$$D(\tau, \lambda, \varepsilon) = -\tau^3 + \tau^2 T(\lambda, \varepsilon) - \tau \overline{T(\lambda, \varepsilon)} + 1, \quad \text{all } (\tau, \lambda, \varepsilon) \in \mathbb{C}^2 \times \mathbb{R},$$

(2.13)

where

$$T(\lambda, \varepsilon) = \text{Tr} M(1, \lambda, \varepsilon).$$

(2.14)

If $\tau(\lambda, \varepsilon), (\lambda, \varepsilon) \in \mathbb{C} \times \mathbb{R}$, is a multiplier, then $\tau^{-1}(\bar{\lambda}, \varepsilon)$ is also a multiplier.

ii) Let $(\lambda, \varepsilon) \in \mathbb{R}^2$. Two cases are possible only:

a) all three multipliers belong to the unit circle;

b) exactly one simple multiplier belongs to the unit circle.

In the case b) the multipliers have the form

$$e^{ik}, \quad e^{ik}, \quad e^{-i2\text{Re}k}, \quad \text{some } k \in \mathbb{C} : \text{Im} k \neq 0.$$\quad (2.15)

iii) The spectrum $\sigma(H_\varepsilon)$ of the operator $H_\varepsilon, \varepsilon \in \mathbb{R}$, is absolutely continuous and satisfies

$$\sigma(H_\varepsilon) = \{ \lambda \in \mathbb{R} : |\tau_j(\lambda, \varepsilon)| = 1, \text{ some } j = 1, 2, 3 \}.\quad (2.16)$$
The spectrum has multiplicity 3 (in the case ii a) or 1 (in the case ii b).

**Proof.** Lemma have been proved in [BK3], we give the sketch of the proof for completeness.

i) Identity (2.4) yields \( J M' = V M \) and \(- (M^*)' J = M^* V \) for \( \lambda \in \mathbb{R} \), where

\[
J = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}, \quad V = J(P + Q) = \begin{pmatrix} \lambda - q & -ip & 0 \\ ip & 0 & -i \\ 0 & i & 0 \end{pmatrix}.
\]

Then \( (M^* J M)' = -M^* V M + M^* V M = 0 \), which yields

\[
M^*(1, \overline{\lambda}) J M(1, \lambda) = J, \tag{2.17}
\]

here and below in this proof \( M(t, \lambda) = M(t, \lambda, \varepsilon) \), ... Identity (2.2) and equation (2.1) give \((\det M)' = 0\). Then \( \det M(t, \lambda) = \det M(0, \lambda) = 1 \), for all \((t, \lambda) \in \mathbb{R} \times \mathbb{C} \), which yields \( \det M(1, \lambda) = 1 \). Direct calculations show that

\[
D(\tau, \lambda) = \det(M(1, \lambda) - \tau \mathbb{I}_3) = -\tau^3 + \tau^2 \text{Tr} M(1, \lambda) + B(\lambda) \tau - 1 \quad \text{all} \quad (\tau, \lambda) \in \mathbb{C}^2,
\]

where \( B(\lambda) = \partial_\tau D(0, \lambda) \). The standard formula from the matrix theory gives

\[
\partial_\tau D(\tau, \lambda) = -D(\tau, \lambda) \text{Tr}(M(1, \lambda) - \tau \mathbb{I}_3)^{-1}.
\]

Using the identity \( D(0, \lambda) = 1 \), we obtain \( B(\lambda) = -\text{Tr} M^{-1}(1, \lambda) \). Identity (2.17) gives \( M^{-1}(1, \lambda) = -JM^*(1, \overline{\lambda}) J \), which implies \( \text{Tr} M^{-1}(1, \lambda) = \text{Tr} M^*(1, \overline{\lambda}) \), for all \( \lambda \in \mathbb{R} \). Then \( B(\lambda) = -\text{Tr} M^*(1, \overline{\lambda}) \), which yields (2.13).

ii) Identity (2.13) implies

\[
D(\tau, \lambda) = -\tau^3 D(\overline{\tau}^{-1}, \lambda) \quad (\tau, \lambda) \in \mathbb{C} \times \mathbb{R}, \quad \tau \neq 0. \tag{2.18}
\]

Then if \( \tau(\lambda) \) is a zero of \( D(\tau, \lambda) \) for some \( \lambda \in \mathbb{R} \), then \( \overline{\tau}^{-1}(\lambda) \) is also a zero. Using the identity \( \tau_1 \tau_2 \tau_3 = \det M(1, \cdot) = 1 \), we obtain the needed statements.

iii) The proof is standard and uses the direct integral decomposition of the operator \( H_\varepsilon \), see [RS].

Fix \( \varepsilon \in \mathbb{R} \). The coefficients of the polynomial \( D(\cdot, \lambda, \varepsilon) \) are entire functions in \( \lambda \). It is known (see, e.g., [Fo], Ch. 8) that the roots \( \tau_j(\cdot, \varepsilon), j = 1, 2, 3 \), constitute one, two or three branches of one, two or three analytic functions that have only algebraic singularities in \( \mathbb{C} \). The simple analysis (see [McK]) shows that the functions \( \tau_j(\cdot, \varepsilon) \) constitute three branches of one function \( \tau(\cdot, \varepsilon) \) analytic on a 3-sheeted Riemann surface. Asymptotics (2.23), proved in the following Lemma, shows that the functions \( \tau_j(\cdot, \varepsilon) \) are all distinct, that is if \( k \neq \ell \), then \( \tau_j(\lambda, \varepsilon) \neq \tau_\ell(\lambda, \varepsilon) \) for all \( \lambda \in \mathbb{C} \) with the exception of some special values of \( \lambda \). There are only a finite number of such exceptional points in any bounded domain.

We introduce the discriminant \( \rho(\lambda, \varepsilon), (\lambda, \varepsilon) \in \mathbb{C}^2 \), of the polynomial \( D(\cdot, \lambda, \varepsilon) \) by

\[
\rho = (\tau_1 - \tau_2)^2(\tau_1 - \tau_3)^2(\tau_2 - \tau_3)^2. \tag{2.19}
\]

Consider the case \( \varepsilon = 0 \). The multipliers \( \tau^0_1 \), the trace \( T_0 \) of the monodromy matrix and the discriminant \( \rho_0(\lambda) = \rho(\lambda, 0) \) are given by

\[
\begin{align*}
\tau^0_1 &= e^{iz}, \quad \tau^0_2 = e^{iwz}, \quad \tau^0_3 = e^{iw^2z}, \quad \omega = e^{\frac{2\pi i}{3}}, \\
T_0(\lambda) &= \text{Tr} M_0(1, \lambda) = e^{iz} + e^{iwz} + e^{iw^2z},
\end{align*}
\tag{2.20}
\]

\[
\rho_0(\lambda) = (\tau^0_1 - \tau^0_2)^2(\tau^0_1 - \tau^0_3)^2(\tau^0_2 - \tau^0_3)^2.
\]
\[ \rho_0 = (e^{iz} - e^{i\omega z})^2(e^{i\omega z} - e^{i\omega^2 z})^2 = 64 \sinh^2 \frac{\sqrt{3}z}{2} \sinh^2 \frac{\sqrt{3}\omega z}{2} \sinh^2 \frac{\sqrt{3}\omega^2 z}{2}. \] (2.21)

**Lemma 2.3.** i) The function \( \rho(\lambda, \varepsilon) \) is entire, real on \((\lambda, \varepsilon) \in \mathbb{R}^2\), and satisfies:

\[ \rho(\lambda, \varepsilon) = |T(\lambda, \varepsilon)|^4 - 8 \Re T^3(\lambda, \varepsilon) + 18|T(\lambda, \varepsilon)|^2 - 27, \quad \text{all} \quad (\lambda, \varepsilon) \in \mathbb{R}^2, \] (2.22)

\[ \rho(\lambda, \varepsilon) = \rho_0(\lambda)(1 + O(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0. \] (2.23)

uniformly in \( \lambda \) on any bounded subset of \( D = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \{ |\lambda - i(\frac{2\pi n}{\sqrt{3}})| \leq 1 \} \).

ii) Let \( \varepsilon \in \mathbb{R} \) and let \( \mathcal{S}_3 \) be the part of the spectrum of \( H_\varepsilon \) having the multiplicity 3. Then

\[ \mathcal{S}_3 = \{ \lambda \in \mathbb{R} : |\tau_j(\lambda, \varepsilon)| = 1, \text{ for all } j = 1, 2, 3 \} = \{ \lambda \in \mathbb{R} : \rho(\lambda, \varepsilon) \leq 0 \}. \] (2.24)

iii) Let \( |\varepsilon| < c \) for some \( c > 0 \) small enough. Then the function \( \rho(\cdot, \varepsilon) \) has exactly two zeros, counted with multiplicities, in each domain \( |\lambda - i(\frac{2\pi n}{\sqrt{3}})| < 1, n \in \mathbb{Z} \). There are no other zeros. In particular, the function \( \rho(\cdot, \varepsilon) \) has no any real zeros in the domain \( |\lambda| \geq 1 \).

**Proof.** i) The function \( \rho \) is a discriminant of the cubic polynomial \eqref{cubic_poly} with entire coefficients, then \( \rho \) is entire. The standard formula for the discriminant of the polynomial \( -\tau^3 + a\tau^2 - b\tau + 1 \) gives \( d = (ab)^2 - 4(a^3 + b^3) + 18ab - 27 \), which yields \eqref{rho_0_expression}.

We will show \eqref{rho_0_expression}. Estimates \eqref{asymptotics} give \( M(1, \lambda, \varepsilon) = M_0(1, \lambda) + O(\varepsilon) \) as \( \varepsilon \to 0 \), uniformly in \( \lambda \) on any compact in \( \mathbb{C} \). Let \( \lambda \in D \). Then all eigenvalues \( e^{iz}, e^{i\omega z}, e^{i\omega^2 z} \) of the matrix \( M_0(1, \lambda) \) are simple and the standard matrix perturbation theory gives

\[ \tau_j(\lambda, \varepsilon) = \tau_j^0(\lambda)(1 + O(\varepsilon)), \quad \text{as} \quad \varepsilon \to 0, \]

uniformly in \( \lambda \) on any bounded subset of \( D \). Substituting these asymptotics into \eqref{rho_def} we obtain \eqref{rho_0_expression}.

ii) Lemma 2.2 iii) implies the first identity in \eqref{rho_0_expression}. Let \( \varepsilon \in \mathbb{R} \) and let \( k_j, j = 1, 2, 3 \), be given by \( \tau_j = e^{ik_j} \). If \( k \neq \ell \), then \( k_j(\lambda, \varepsilon) \neq k_\ell(\lambda, \varepsilon) \) for all non-exceptional \( \lambda \in \mathbb{C} \). Identity \( \tau_1\tau_2\tau_3 = 1 \) yields \( k_1 + k_2 + k_3 = 0 \). This identity and \eqref{rho_def} give

\[ \rho = (e^{ik_1} - e^{ik_2})^2(e^{ik_1} - e^{ik_3})^2(e^{ik_2} - e^{ik_3})^2 = -64 \sin^2 \frac{k_1 - k_2}{2} \sin^2 \frac{k_1 - k_3}{2} \sin^2 \frac{k_2 - k_3}{2}. \] (2.25)

If \( \lambda \in \mathcal{S}_3 \), then, due to the first identity in \eqref{rho_0_expression}, \( k_j(\lambda, \varepsilon) \in \mathbb{R} \) for all \( j = 1, 2, 3 \), and \eqref{rho_0_expression} yields \( \rho(\lambda, \varepsilon) \leq 0 \). If \( \lambda \in \sigma(H_\varepsilon) \setminus \mathcal{S}_3 \), then exactly one \( k_j \), say \( k_1 \), is real and \( k_2 = k_3 \) are non-real, see \eqref{lambda_in_domain}. Identity \eqref{rho_0_expression} implies \( \rho(\lambda, \varepsilon) > 0 \), which yields the second identity in \eqref{rho_0_expression}.

iii) Let \( \lambda \) belong to the contours \( |\lambda - i(\frac{2\pi n}{\sqrt{3}})| = 1, n \in \mathbb{Z} \). Asymptotics \eqref{rho_0_expression} yields

\[ |\rho(\lambda, \varepsilon) - \rho_0(\lambda)| = |\rho_0(\lambda)| \left| \frac{\rho(\lambda, \varepsilon)}{\rho_0(\lambda)} - 1 \right| = |\rho_0(\lambda)| O(\varepsilon) < |\rho_0(\lambda)| \]
on all contours. Hence, by Rouché’s theorem, \( \rho \) has as many zeros, as \( \rho_0 \) in each of the bounded domains and the remaining unbounded domain. Since \( \rho_0 \) has exactly one zero of multiplicity two at each \( i(\frac{2\pi n}{\sqrt{3}}), n \in \mathbb{Z} \), the statement follows. ■
3. Proof of the main theorem

Introduce the entire functions $T_n = \text{Tr} M_n(1, \cdot), n \geq 0$. Estimates (2.9) imply

$$T(\lambda, \varepsilon) = T_0(\lambda) + \varepsilon T_1(\lambda) + \varepsilon^2 T_2(\lambda) + \varepsilon^3 T_3(\lambda) + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0$$

(3.1)

uniformly in $\lambda$ on any compact in $\mathbb{C}$.

**Lemma 3.1.** Let $h$ be given by (1.4). Then the following relations hold true:

$$T_1 = 0,$$

(3.2)

$$\text{Re} T_2(\lambda) = -3h(1 + O(\lambda)) \quad \text{as} \quad \lambda \to 0.$$  

(3.3)

**Proof.** Identity (2.8) and conditions (1.2) imply

$$T_1 = \text{Tr} M_1(1, \cdot) = \text{Tr} \int_0^1 e^{(1-t)P} Q(t) e^{tP} ds = \text{Tr} e^P \int_0^1 Q(t) dt = 0,$$

which yields (3.2). Moreover,

$$T_2 = \text{Tr} M_2(1, \cdot) = \text{Tr} \int_0^1 \int_0^t e^{(1-t+s)P} Q(t) e^{(t-s)P} Q(s) ds dt.$$  

(3.4)

We rewrite identities (2.4) for the matrices $P, Q$ in the form

$$P = P_0 - i\lambda P_1, \quad Q = -pP_0^* + iqP_1, \quad P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  

Using the identities

$$e^{tP_0} = \mathbb{I}_3 + tP_0 + \frac{t^2}{2} P_0^2 = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$e^{tP(\lambda)} Q(s) = e^{tP_0} Q(s)(\mathbb{I}_3 + O(\lambda)) = (-p(s) K_1(t) + iq(s) K_2(t))(\mathbb{I}_3 + O(\lambda))$$

as $\lambda \to 0$, uniformly on $(t, s) \in [0, 1]^2$, where

$$K_1 = e^{tP_0} P_0^* = \begin{pmatrix} t & \frac{t^2}{2} & 0 \\ 1 & t & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad K_2 = e^{tP_0} P_1 = \begin{pmatrix} \frac{t^2}{2} & 0 & 0 \\ t & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  

Using the identities

$$\text{Tr} K_1(1 - u) K_1(u) = 2(1 - u) u + \frac{(1 - u)^2}{2} + \frac{u^2}{2} = \frac{1}{2} + u(1 - u),$$

$$\text{Tr} K_2(1 - u) K_2(u) = \frac{(1 - u)^2 u^2}{4},$$

we obtain

$$\text{Re} \text{Tr} e^{(1-t+s)P} Q(t) e^{(t-s)P} Q(s) = \Phi(t, s)(1 + O(\lambda)),$$

as $\lambda \to 0$, uniformly on $(t, s) \in [0, 1]^2$, where

$$\Phi(t, s) = p(t)p(s) \left( \frac{1}{2} + u(1 - u) \right) - q(t)q(s) \frac{(1 - u)^2 u^2}{4}, \quad u = t - s.$$
Substituting this asymptotics into (3.1) we obtain
\[ \text{Re} T_2(\lambda) = \int_0^1 dt \int_0^t \Phi(t, s)ds(1 + O(\lambda)) = \frac{1}{2} \int_0^1 dt \int_{t-1}^t \Phi(t, s)ds(1 + O(\lambda)) \] (3.5)
as \( \lambda \to 0 \), since \( \Phi(t, s) = \Phi(s, t - 1) \). Using the simple identity
\[ \int_0^1 dt \int_{t-1}^t g(t-s)f(s)ds = \sum_{k \in \mathbb{Z}} |\hat{f}_k|^2 \hat{g}_k, \]
for all \( f \in L^1(0, 1), g, g' \in L^2(0, 1), g(0) = g(1) = 0 \), and identities
\[ \int_0^1 u(1-u)e^{-i2\pi nu}du = -\frac{1}{2(\pi n)^2}, \quad \int_0^1 u^2(1-u)^2e^{-i2\pi nu}du = -\frac{3}{2(\pi n)^4}, \]
for all \( n \neq 0 \), identities (3.5) imply (3.3).

**Proof of Theorem**

i) Identities (2.20) imply
\[ T_0(\lambda) = e^{iz} + e^{iz} + e^{iz} = 3 - \frac{i\lambda}{2} - \frac{\lambda^2}{240} + O(\lambda^3) \quad \text{as} \quad \lambda \to 0. \] (3.6)

Recall that the functions \( T(\lambda, \varepsilon), \rho(\lambda, \varepsilon) \) are entire in \((\lambda, \varepsilon) \in \mathbb{C}^2\). Let the entire functions \( a(\lambda, \varepsilon), b(\lambda, \varepsilon) \) and the numbers \( b_j, j = 2, 3 \), be given by
\[ T(\lambda, \varepsilon) = 3 + a(\lambda, \varepsilon) + ib(\lambda, \varepsilon), \quad b(\lambda, \varepsilon) = \text{Im} T(\lambda, \varepsilon), \quad \text{all} \quad (\lambda, \varepsilon) \in \mathbb{R}^2, \quad b_j = \text{Im} T_j(0). \]

Substituting relations (3.2), (3.3), (3.4) into (3.1) we obtain
\[ T(\lambda, \varepsilon) = 3 - \frac{i\lambda}{2} - \frac{\lambda^2}{240} - \frac{2h - h_2}{2} b_2 \varepsilon^2 + \varepsilon^3 T_3(0) + O(\lambda^3) + O(\lambda \varepsilon^2) + O(\varepsilon^4) \] (3.7)
as \((\lambda, \varepsilon) \to (0, 0)\). Asymptotics (3.7) gives
\[ a(\lambda, \varepsilon) = -3h \varepsilon^2 + O(\lambda^3) + O(\lambda \varepsilon^2) + O(\varepsilon^3), \quad b(\lambda, \varepsilon) = \mu(\lambda, \varepsilon) + O(\lambda^3) + O(\lambda \varepsilon^2) + O(\varepsilon^4), \] (3.8)
as \((\lambda, \varepsilon) \to (0, 0)\), where
\[ \mu = -\frac{\lambda}{2} + \frac{r}{2}, \quad r = 2b_2 \varepsilon^2 + 2b_3 \varepsilon^3. \] (3.9)

Identity (2.22) gives
\[ \rho = a^3(a + 4) + b^2 \left( 108 + 2(a + 18)a + b^2 \right). \] (3.10)

Substituting asymptotics (3.8) into (3.10) we obtain
\[ \rho(\lambda, \varepsilon) = f(\mu, \varepsilon) = 108 \left( \mu^2 - h^2 \varepsilon^6 + O(\mu^4) + O(\mu^2 \varepsilon^2) + O(\mu \varepsilon^4) + O(\varepsilon^7) \right), \]
as \((\mu, \varepsilon) \to (0, 0)\), where
\[ \lambda = r - 2\mu, \] (3.11)
and \( f(\mu, \varepsilon) \) is entire in \((\mu, \varepsilon)\). Introduce the new variable \( u \) by \( \mu = u \varepsilon^3 \). Then
\[ f(u \varepsilon^3, \varepsilon) = 108 \varepsilon^6 E(u, \varepsilon), \]
where \( E(u, \varepsilon) \) is entire in \((u, \varepsilon)\) and
\[ E(u, \varepsilon) = u^2 - h^3 + O(\varepsilon), \]
as \( \varepsilon \to 0 \) uniformly in \( u \).

Consider the equation \( E(u, \varepsilon) = 0 \). Let \( h \neq 0 \). Using \( \frac{\partial}{\partial u} E(\pm h^2, 0) \neq 0 \) and the Implicit Function Theorem we deduce that there exist two real analytic functions \( u^\pm(\varepsilon) \) in the disk
\{ |\varepsilon| < c \} \text{ for some } c > 0 \text{ such that each } u^\pm(\varepsilon) \text{ is a zero of the function } E(\cdot, \varepsilon). \text{ These functions satisfy}

\[ u^\pm(\varepsilon) = \pm h^2 + O(\varepsilon) \quad \text{as } \varepsilon \to 0. \]

Substituting \( \mu = u^\pm(\varepsilon)\varepsilon^3 \) into the identity (3.11) we deduce that there exist two real analytic functions \( r^\pm(\varepsilon) \) in the disk \( \{ |\varepsilon| < c \} \) such that

\[ r^\pm(\varepsilon) = r(\varepsilon) \pm 2h^2\varepsilon^3 + O(\varepsilon^4), \quad \text{as } \varepsilon \to 0, \]

which yields (1.5).

Consider \( h > 0 \). Let \( \varepsilon > 0 \) be small enough. Then \( r^-(\varepsilon) < r(\varepsilon) < r^+(\varepsilon) \). Asymptotics (3.11) shows that \( \rho(r(\varepsilon), \varepsilon) = f(0, \varepsilon) < 0 \). Then \( \rho(\cdot, \varepsilon) < 0 \) on the whole interval \( (r^-(\varepsilon), r^+(\varepsilon)) \) and, by Lemma 2.3 iii), \( \rho(\cdot, \varepsilon) \geq 0 \) out of this interval. Then, due to (2.24), the spectrum of \( H_\varepsilon \) in this interval has multiplicity 3 and the other spectrum has multiplicity 1. The proof for the case \( \varepsilon < 0 \) is similar.

ii) If \( h < 0 \), then, by Lemma 2.3 iii), the function \( \rho \) has no any real zeros and \( \rho(\cdot, \varepsilon) > 0 \) on the whole real axis. Hence all the spectrum has multiplicity 1.

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