Many-body Dynamics with Time-dependent Interaction

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Recent advances in optical Feshbach resonance technique have enabled the experimental investigation of atomic gases with time-dependent interaction \cite{1, 2}. In this work, we study the many-body dynamics of weakly interacting bosons subject with an arbitrary time varying scattering length. By employing a variational ansatz, we derive an effective Hamiltonian that governs the dynamics of thermal particles. Crucially, we show that there exists a hidden symmetry in this Hamiltonian that can map the many-body dynamics to the precession of an SU(1,1) “spin”. As a demonstration, we calculate the situation where the scattering length is sinusoidally modulated. We show that the non-compactness of the SU(1,1) group naturally leads to solutions with exponentially growth of Bogoliubov modes and causes instabilities.

The ability to accurately control various parameters in cold atomic gases allows the investigation of quantum matter under extreme conditions that are beyond reach in other physical systems. Among these parameters, the tunable interaction strength is a key ingredient for many fascinating quantum phenomena such as the physics of BEC-BCS crossover \cite{3–6}, superfluid to Mott insulator transition \cite{7–10} and the few-body Efimov effect \cite{11–15}.

One of the recent progress in controlling the interatomic interaction strength is the development of the optical Feshbach resonance technique \cite{1, 2, 16}. Comparing to the traditional magnetic Feshbach resonance which relies on tuning the magnetic field, the optical Feshbach resonance controls the interatomic interaction via changing the detuning and intensity of the optical field. Such difference allows the rapid and spatial modulations of the scattering length between atoms and thus enables the experimental investigation of a variety of exotic many-body dynamic systems. For example, the bose fireworks experiment recently carried out by the Chicago group shows that a bose condensate emits matter-wave jets and form striking fireworks patterns while subject to periodic modulated interactions \cite{1, 17, 18}.

In this work, we focus on the dynamic problem of weakly interacting bose gas subject with an arbitrary time varying scattering length. In the low temperature limit, one might naively anticipate that the dynamics of the system could be described by a mean-field level Gross-Pitaevskii (GP) equation with time varying coupling constant $g(t)$. However, it is straightforward to show that the solution to the time-dependent GP equation is trivial as long as the system is initially in the ground state. This is closely related to the fact that the ground state solution (i.e. the saddle point) of a time-independent GP equation does not rely on the interaction strength $g$. Thus, it is necessary to go beyond the mean-field theory and consider the role of quantum fluctuation. In the corresponding static problem, the next order correction is known as the Lee-Huang-Yang correction which can be obtained by the Bogoliubov theory. Inspired by this correspondence, we propose a variational ansatz which accounts the next order quantum correction to the dynamic problem. We show that the dynamics of the variational wave function is governed by a Bogoliubov-like Hamiltonian. Crucially, we find that the Hamiltonian possesses a hidden SU(1,1) symmetry which allows an exact solution to the time-dependent Schrödinger equation but also maps the dynamic problem to an SU(1,1) “spin” moving in a time-varying magnetic field. The SU(1,1) “spin” model closely resembles a normal SU(2) spin in an external field as its dynamics can be viewed as a point moving on a hyperboloid which resembles an SU(2) Bloch sphere. To further demonstrate our method, we also calculate the dynamics of a system with periodically modulated scattering length.

\textbf{Model.}— We consider a Hamiltonian which describes a...
system of bosons interacting via short-range interaction,
\[ H(t) = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2} g(t) \sum_{k,q} a_{q+k}^\dagger a_q a_k^\dagger a_{q+k} + \frac{1}{2} g(t) \]

Here \( a_k^\dagger (a_k) \) are the bosonic creation (annihilation) operator with momentum \( k \) and mass \( m; g(t) \) is an arbitrary time-varying coupling constant which is related to the \( s \)-wave scattering length \( a_s \) by \( g(t) = 4\pi a_s(t)/m \) (we set \( h \) and the volume of the system to 1). The dynamic theory we develop in this work does not rely on the specific form of the dispersion \( \epsilon_k \) as long as the system has an inversion symmetry \( i.e. \epsilon_k = \epsilon_{-k} \), and without loss of generality, we set \( \epsilon_0 = 0 \).

To proceed, we assume that the system is weakly interacting, such that during the dynamic process the majority of the bosons still condense in the zero-momentum state, \( i.e. N_0(t) \approx N \gg 1 \). Therefore, one may approximate the time-dependent wave function by,
\[ \Psi(t) = \langle \psi(t) \rangle = e^{\sum_k \langle a_k^\dagger a_k \rangle} |0\rangle, \]

where the wave function \( \langle \psi(t) \rangle \) is decomposed into a product state of \( |\psi(t)\rangle \neq 0 \) which represents the state of non-condensed thermal bosons and a coherent state of \( N_0 \) condensed particles.

To determine the “best” variational wave function \( \langle \psi(t) \rangle \), we use Frenkel’s least action principle \[ 19, 20 \] for dynamic systems and minimize the action \( S = \int dt \langle \Psi(t) | i \partial_t - H(t) | \Psi(t) \rangle \) \[ 21 \]. This leads to a time-dependent Schrödinger equation
\[ i \partial_t |\psi(t)\rangle = H_B |\psi(t)\rangle, \]
with
\[ H_B(t) = \sum_{k \neq 0} (\epsilon_k + g(t)n) a_k^\dagger a_k + \frac{g(t)n}{2} (a_k^\dagger a_{-k}^\dagger + h.c.) + \frac{g(t)nN}{2} + O(N^{-1/2}). \]

We see that the dynamics of thermal particles are governed by a Bogoliubov-like Hamiltonian \( H_B(t) \).

It is worth noting that simply diagonalizing \( H_B(t) \) via Bogoliubov transformation does not solve the dynamic problem as its instantaneous eigen-state is not the solution to Eq. (3). The solution to the dynamic problem actually relies on the hidden dynamic symmetry of Hamiltonian \( H_B \).

Note that the \( k \) part in \( H_B \) only couples to \(-k\), and it can be rewritten as
\[ H_B = \sum_{k \neq 0} [g(t)n A_k^\dagger + (\epsilon_k + g(t)n) A_k^\dagger] + E_0. \]

Here \( E_0 = -\sum_k (\epsilon_k + gn) + g(t)nN/2 \), \( A_k^\dagger \) and \( A_k^\dagger \) are defined as \( A_k^\dagger = \frac{1}{2}(a_k^\dagger a_{-k}^\dagger + h.c.) \) and \( A_k^\dagger = \frac{1}{2}(a_k^\dagger a_{-k}^\dagger + a_{-k}^\dagger a_k^\dagger) \).

It was pointed out by Chen et al. \[ 22 \] that \( A_k^\dagger \) and \( A_k^\dagger \) can fit into an su(1,1) algebra by including an extra operator \( A_k^3 = \frac{1}{2}(a_k^\dagger - a_{-k}^\dagger - h.c.) \). Together with this operator, their commutators form a close algebra,
\[ [A_k^\dagger, A_k^\dagger] = -iA_k^3, \quad [A_k^\dagger, A_k^\dagger] = iA_k^\dagger, \quad [A_k^\dagger, A_k^\dagger] = iA_k^\dagger. \]

Note that Eq. (6) differ from the common su(2) algebra of a spin system by a minus sign. As we will see in the following, there is a close resemblance between the dynamics of Bogoliubov systems and the dynamics of an SU(2) spin in a time-dependent magnetic field.

**SU(1,1) spin model.** Note that all the \((k, -k)\) subspaces with different \( k \) are decoupled, which allows us dealing with a pair of momenta at one time. For generality, in the following we will consider a model that consists of all the \( A_i \) components,
\[ H_h = h \cdot A^k = h_1 A_1^k + h_2 A_2^k + h_3 A_3^k. \]

Here \( h = (h_1, h_2, h_3)^T \) is an arbitrary time-dependent vector and \( H_h \) can be reduced to \( H_B \) by letting \( h_1 = 2g(t)n, h_2 = 0 \) and \( h_3 = 2(\epsilon_1 + g(t)n) \).

The SU(1,1) symmetry leads to three time-dependent invariants for \( H_h \). To see this, we consider operator \( S \) in the form of \( S = \sum_i u_i(t) A_i^k \). In order to make \( S \) a time-dependent invariant under \( H_h \), we have
\[ \frac{d}{dt} S = i[H_h, S] + \frac{\partial S}{\partial h} = 0. \]

Thanks to the closed commutation relations, the above equation leads to a set of linear equations for \( u_i \),
\[ \dot{u}_i = \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ h_2 & -h_1 & 0 \end{pmatrix} u_i, \]
with \( u = (u_1, u_2, u_3)^T \).

Any \( u(t) \) satisfies Eq. (9) represents an invariant \( u \cdot A^k \) for \( H_h \). While there are three linear-independent solutions of this differential equation, which correspond to three independent invariants.

**Remark on \( H_h \).** From Eq. (9), one can prove that \( |u(t)|^2 \equiv -u_2^2 - u_3^2 + u_4^2 \) is a constant by showing that \( \frac{d}{dt} |u(t)|^2 = 0 \). This means that the three-dimensional vector \( u(t) \) is restricted on the surface of a hyperboloid defined by \( -u_2^2 - u_3^2 + u_4^2 = \text{const.} \). This may be viewed as the SU(1,1) analogue of the Bloch sphere in SU(2) spin case.

Without loss of generality, we consider the solution of \( u(t) \) on the upper sheet of the hyperboloid as shown in Fig. 1. The corresponding invariant can be parametrized as \( S(t) = u \cdot A^k = \sinh \theta \cos \phi A_1^k + \sinh \theta \sin \phi A_2^k + \cosh \theta A_3^k \). Using the commutation relations in Eq. (6), we can diagonalize it via the SU(1,1) rotation,
\[ e^{iA_3^\phi} e^{-iA_2^\theta} S(t) e^{-iA_3^\phi} e^{-iA_1^\theta} S(t) e^{-iA_3^\phi} = A_3^k. \]
Since $A_k^o = \frac{1}{2}(n_k + n_{-k} + 1)$, the eigenstates of $S(t)$ are thus parametrized by two integers $n = (n_+, n_-)$ with $n_k$ the number of bosons in $k$ and $-k$ states. They are given by $|n\rangle = |n_+, n_-\rangle = \frac{1}{\sqrt{n_+!n_-!}}e^{iA_k^o\phi}a_k^{n_k}a_{-k}^{n_{-k}}|0\rangle$.

The instantaneous eigenstates of invariant $S(t)$ are useful because they are proportional to the solution to the Schrödinger equation $|\Phi\rangle$. According to Lewis’s theory for time-dependent invariants [23, 24], we have

$$|\Phi(t)\rangle = e^{-i\phi(t)}|n\rangle.$$  \hspace{1cm} (11)

Here $|\Phi\rangle$ satisfies $[i\partial_t - H(t)]|\Phi(t)\rangle = 0$. The phase $\phi(t)$ contains a dynamical phase and a geometric phase with $\phi(t) = \phi_{\text{dyn}}(t) - \phi_{\text{g}}(t)$,

$$\phi_{\text{dyn}} = \int_{t_0}^t d\tau \langle n|H(t)|n\rangle, \quad \phi_{\text{g}} = i\int_{t_0}^t d\tau \langle n|\partial_\tau|n\rangle.$$  \hspace{1cm} (12)

Suppose the initial state of the system is the ground state of $\mathbf{h}_0 \cdot \mathbf{A}^o$, the initial condition for Eq. (9) is then set as $u(0) = \mathbf{h}_0$. We can then obtain the solution of the time-dependent Schrödinger equation by solving $u(t)$ and substitute it into Eq. (11) with $n_+ = n_- = 0$.

Remarks on $\phi_{\text{g}}$. By changing variable $t$ to $u$, we can show that the geometric phase $\phi_{\text{g}}$ depends only on the trajectory of $u$,

$$\phi_{\text{g}} = i\int_{\Gamma} d\mathbf{u} \cdot (\mathbf{n}|\nabla_u|\mathbf{n}) = \int_{\Gamma} \mathbf{A}_{\theta}d\theta + \mathbf{A}_\phi d\phi,$$  \hspace{1cm} (13)

where $\Gamma$ is the trajectory of $u$ on the hyperboloid as shown in Fig. 1. The Berry connection $\mathbf{A}_i$ is

$$\mathbf{A}_\theta = i\langle \mathbf{n}|\partial_\theta|\mathbf{n}\rangle = 0,$$

$$\mathbf{A}_\phi = i\langle \mathbf{n}|\partial_\phi|\mathbf{n}\rangle = -C_n \cosh \theta,$$  \hspace{1cm} (14, 15)

with charge $C_n = (n_+ + n_- + 1)/2$.

As it is well known that the Berry curvature of an SU(2) spin is identical to the field of a Dirac monopole positioned at the center of the Bloch sphere. In the SU(1,1) dynamic theory, the Berry curvature in $u$-space is given by $\nabla \times \mathbf{A} = C_n \frac{\hat{e}_\rho}{\rho}$ with $\rho = \sqrt{u_1^2 + u_2^2}$ the radial coordinate and $\hat{e}_\rho = (u_1 \hat{e}_1 + u_2 \hat{e}_2)/\rho$ is the unit vector along radial direction. This Berry curvature is equal to the field of a line of Dirac monopoles positioned on the $u_3$-axis with uniform linear density $d = C_n/2$ as shown in Fig. 1. The fact that the monopole line is infinitely long is a consequence of the non-compactness of the SU(1,1) group [25].

Bose gas with periodically driven $g(t)$. – In the following, we consider a specific form of time-varying interaction strength with $g(t) = g_0 + \delta g \sin \omega t$ and focus on the long-term behavior of the system. Such sinusoidal modulation is probably the most simple case and has already been implemented in several cold atom experiments [1, 2, 26].

In the case of the weakly-interacting Bose gas $h_1 = 2g(t)n$, $h_2 = 0$ and $h_3 = 2(\epsilon_k + g(t)n)$, the coupled linear equations (9) can be further simplified into a single differential equation for $u_{31} \equiv u_3 - u_1$ [27],

$$\ddot{u}_{31} + 4\epsilon_k (\epsilon_k + 2g(t)n)u_{31} + 4\epsilon_k \dot{g}(t)nu_{31} = 0.$$  \hspace{1cm} (16)

One may check that the above equation is equivalent to the coupled equations (9).

For the periodic driven case, the Floquet theorem asserts that the solution to Eq. (9) must take the form of $u(t) = e^{-iE_F^t}p(t)$ with $E_F$ the quasi-energy and $p(t)$ a periodic function in $t$.

The quasi-energy $E_F = \alpha + i\lambda$ is in general complex and its imaginary part controls the stability of the system. For a real quasi-energy, i.e., $\lambda = 0$, the vector $u$ is always bounded, meaning the condensate only emits a finite number of thermal particles with momentum $\pm k$.

On the other hand, if the quasi-energy $E_F$ is complex, the $u$ grows exponentially in the long term, meaning the condensate will keep emitting thermal particles until the

![FIG. 2. Solid lines: Typical long-term behavior for $u_3(t)$. The time $t$ is plotted in unit of $1/\omega$. For both plots, we have $g_0 = 0$, $\delta g = 0.1\omega/n$. In plot (a), we set $\epsilon_k = 1.2\omega$, which leads to a bounded oscillating behavior. In (b), we set $\epsilon_k = 0.5\omega$, and find that $u_3$ grows exponentially in the long term (The $y$-axes is in log-scale.). Dashed line: $e^{\lambda t}$ with $\lambda$ the Lyapunov exponent calculated by the Floquet theory. One can see its long-term trend nicely agrees with $u_{31}(t)$. The insets show actual trajectories of $u(t)$ in both cases.](image-url)
FIG. 3. Stability diagram for Bose gas with oscillating interaction strength \( g(t) = g_0 + \delta g \sin \omega t \). From left to right: \( g_0 = -0.5\omega/n, g_0 = 0, g_0 = 0.5\omega/n \). The white area marks the stable region with vanishing Lyapunov exponent. Colored area marks the unstable regions. Dashed lines represent the transition curves that separate two regions. Lyapunov exponent is shown via the colormap. 'B's and 'R's in the instability lobes stand for Bogoliubov and resonance, which categorizes two different origins of the instabilities.

One can see, the imaginary part \( \lambda \) plays an important role of controlling the growth speed of the thermal modes, which can thus be interpreted as the Lyapunov exponent of the system.

In Fig. 2, we plot \( u_3(t) \) for both cases by solving Eq. (9) and show that \( u_3 \) indeed grows in the form of \( e^{\lambda t} \). This is in contrast to the dynamics of an SU(2) spin as all the components of the SU(2) spin is bounded. As one can see from the insets, the exponentially growing solutions are related to the non-compactness of the SU(1,1) group, which is the main difference between the SU(1,1) and SU(2) groups.

To calculate \( \lambda \), we further show that the third order equation (16) is related to a second order one,

\[
\ddot{\nu} + \epsilon_k (\epsilon_k + 2g(t)n) \nu = 0.
\]

(17)

Namely, if \( v_1, v_2 \) are the two solutions of Eq. (17), \( u = v_1 v_2 \) is then the solution of Eq. (16). Thus the three linear independent solutions for Eq. (16) are given by \( v_1^2, v_1 v_2 \) and \( v_2^2 \), with \( v_1, v_2 \) the linear independent solutions of Eq. (17) [28, 29]. For \( g(t) = g_0 + \delta g \sin \omega t \), Eq. (17) reduces to a Mathieu’s equation. The Mathieu’s equation can be used to describe the classical dynamics of a parametric oscillator, whose long-term Lyapunov exponent [30] may be calculated by the standard Whittaker-Hill’s formula [31].

We plot the Lyapunov exponent \( \lambda \) as a function of \( \epsilon_k \) and \( \delta g \) in Fig. 3. One can see that the system develops several instability lobes while turning on the modulation \( \delta g \). These lobes are caused by two types of instability – the resonance instability and the Bogoliubov instability. The resonance instability lobes emerge from \( \sqrt{\epsilon_k (\epsilon_k + 2g_0 n)} = n\omega/2 \) for small modulation strength \( \delta g \) and keeps growing while increasing \( \delta g \). Note that \( \sqrt{\epsilon_k (\epsilon_k + 2g_0 n)} \) is the energy for Bogoliubov mode in the unperturbed system. This indicates that those instability appears because of the driven frequency is in resonance with two Bogoliubov excitations (one \( \mathbf{k} \) and one \( -\mathbf{k} \)) of the unperturbed system. The Bogoliubov instability lobes exist even when there is no interaction strength modulation and shrink with increasing \( \delta g \). They appear when \( \epsilon_k (\epsilon_k + 2g_0 n) < 0 \), corresponding to the system has imaginary energy for Bogoliubov mode. Such instability is an intrinsic instability of the unperturbed system and hence be named Bogoliubov instability. The fact that the Bogoliubov instability lobes shrink with increasing \( \delta g \) suggests that we may actually use the temporally modulated interaction to stabilize condensates that are originally unstable with static interactions (e.g. bosons with attractive interaction).

To conclude, we have developed a beyond mean-field theory to describe the dynamics of weakly interacting bosons with time-varying interaction strength. By assuming the majority of the bosons is condensed in the ground state, we found that the non-condensate part of the system can be well described by a Bogoliubov-like Hamiltonian. Furthermore, by identifying a hidden SU(1,1) symmetry of the system, we show that the dynamic problem of bosons can be mapped to the problem of an SU(1,1) spin in a time-varying magnetic field. We explicitly constructed the time-dependent invariants of this system which gives the exact solution to the original time-dependent Schrödinger equation. Interestingly, the Berry curvature of the SU(1,1) spin is found to be identical to the field of a line of Dirac monopoles. Experiments that can generate such gauge field in a BEC has been proposed for years but not yet realized [32]. Thus the model we described in this work might provide an alternative and feasible method to create and simulate such a novel configuration of gauge fields.

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[1] L. W. Clark, A. Gaj, L. Feng, and C. Chin, Collective emission of matter-wave jets from driven Bose–Einstein condensates, Nature 551, 356 (2017).
[2] L. Feng, J. Hu, L. W. Clark, and C. Chin, Correlations in high-harmonic generation of matter-wave jets revealed by pattern recognition, Science 363, 521 (2019).
[3] T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. Kokkelmans, and C. Salomon, Experimental study of the BEC-BCS crossover region in lithium 6, Physical Review Letters 93, 050401 (2004).
[4] M. Zwierlein, C. Stan, C. Schunck, S. Raupach, A. Kerman, and W. Ketterle, Condensation of pairs of fermionic atoms near a Feshbach resonance, Physical Review Letters 92, 120403 (2004).
[5] Q. Chen, J. Stajic, S. Tan, and K. Levin, BCS–BEC crossover: From high temperature superconductors to ultracold superfluids, Physics Reports 412, 1 (2005).
[6] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of ultracold atomic Fermi gases, Reviews of Modern Physics 80, 1215 (2008).
[7] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, nature 415, 39 (2002).
[8] T. Stöferle, H. Moritz, C. Schori, M. Köhl, and T. Esslinger, Transition from a strongly interacting 1D superfluid to a Mott insulator, Physical review letters 92, 130403 (2004).
[9] S. Fölling, A. Widera, T. Müller, F. Gerbier, and I. Bloch, Formation of spatial shell structure in the superfluid to Mott insulator transition, Physical Review Letters 97, 060403 (2006).
[10] W. S. Bakr, A. Peng, M. E. Tai, R. Ma, J. Simon, J. I. Gillen, S. Foellinger, L. Pollet, and M. Greiner, Probing the superfluid–to–Mott insulator transition at the single-atom level, Science 329, 547 (2010).
[11] V. Efimov, Weakly-bound states of three resonantly-interacting particles, Sov. J. Nucl. Phys. 12, 101 (1971).
[12] V. Efimov, Energy levels arising from resonant two-body forces in a three-body system, Phys. Lett. B 33, 563 (1970).
[13] T. Kraemer, M. Mark, P. Waldbuerger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, et al., Evidence for Efimov quantum states in an ultracold gas of caesium atoms, Nature 440, 315 (2006).
[14] E. Braaten and H.-W. Hammer, Universality in few-body systems with large scattering length, Phys. Rep. 428, 259 (2006).
[15] P. Naidon and S. Endo, Efimov physics: a review, Rep. Prog. Phys. 80, 056001 (2017).
[16] M. Theis, G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, R. Grimm, and J. H. Denschlag, Tuning the scattering length with an optically induced Feshbach resonance, Physical Review Letters 93, 123001 (2004).
[17] H. Fu, L. Feng, B. M. Anderson, L. W. Clark, J. Hu, J. W. Andrade, C. Chin, and K. Levin, Density waves and jet emission asymmetry in Bose Fireworks, Physical review letters 121, 243001 (2018).
[18] Z. Wu and H. Zhai, Dynamics and density correlations in matter-wave jet emission of a driven condensate, Physical Review A 99, 063624 (2019).
[19] A. I. Frenkel, Wave Mechanics; Advanced General Theory, Bull. Amer. Math. Soc 41, 776 (1935).
[20] A. McLachlan, A variational solution of the time-dependent Schrödinger equation, Molecular Physics 8, 39 (1964).
[21] Note that the least action principle leads to the exact time-dependent Schrödinger equation if we put no restriction on the wave function ($\Psi(t)$).
[22] Y.-Y. Chen, P. Zhang, W. Zheng, Z. Wu, and H. Zhai, Many-Body Echo, arXiv:1909.05183.
[23] H. R. Lewis Jr and W. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, Journal of Mathematical Physics 10, 1458 (1969).
[24] H. R. Lewis Jr, Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians, Physical Review Letters 18, 510 (1967).
[25] The parametrization of $u$ can be viewed as a U(1) fibration of SU(1,1). The non-compactness of SU(1,1) naturally leads to a non-compact base space (the “Bloch” hyperboloid shown in Fig. 1). As a consequence, the corresponding Berry curvature remains finite on the base space, which means the monopole line has to be infinitely long.
[26] J. Nguyen, M. Tsatsos, D. Luo, A. Lode, G. D. Telles, V. S. Bagnato, and R. Hulet, Parametric excitation of a Bose–Einstein condensate: from Faraday waves to gravulation, Physical Review X 9, 011052 (2019).
[27] Label the three equations in Eq. (9) by 1, 2, and 3, they can be reduced to a single equation by considering $rac{d^2}{dr^2} (1 - 3) - 2a \frac{d}{dr} + 4a gn(1 - 4a(ek + gn))$.
[28] S. Deng, Z.-Y. Shi, P. Diao, Q. Yu, H. Zhai, R. Qi, and H. Wu, Observation of the Efimovian expansion in scale-invariant Fermi gases, Science 353, 371 (2016).
[29] Z.-Y. Shi, R. Qi, H. Zhai, and Z. Yu, Dynamic super Efimov effect, Physical Review A 96, 050702 (2017).
[30] Note that the Lyapunov exponent for Eq. (17) is half the exponent for Eq. (16).
[31] N. W. McLachlan, Theory and application of Mathieu functions, (1951).
[32] G. Conduit, Line of Dirac monopoles embedded in a Bose–Einstein condensate, Physical Review A 86, 021605 (2012).