POSITIVE LYAPUNOV EXPONENT FOR A CLASS OF QUASI-PERIODIC COCYCLES

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Abstract. Young [17] proved the positivity of Lyapunov exponent in a large set of the energies for some quasi-periodic cocycles. Her result is also proved to be applicable for some quasi-periodic Schrödinger cocycles by Zhang [18]. However, her result cannot be applied to the Schrödinger cocycles with the potential \( v = \cos(4\pi x) + w(x) \), where \( w \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \) is a small perturbation. In this paper, we will improve her result such that it can be applied to more cocycles.

1. Introduction. Positivity of Lyapunov exponent (short for LE) has attracted great attention in the study of quasi-periodic Schrödinger operators (cocycles). It is often taken as an implicit definition of localization in physics literature, and the LE is often called the inverse localization length. Although positivity of LE does not imply localization in the precise mathematical definition, it is usually treated as a precondition for localization. For example, Bourgain-Goldstein [5] showed Anderson Localization for the analytic quasi-periodic Schrödinger operators provided positivity of LE. Positivity of LE is also closely related to the spectral properties of the corresponding operators. It was proved by Ishii [10] and Pastur [13] that if the LE is positive for all energies, then there is no absolutely continuous component in the spectrum.

Proving positivity of LE is one of the central topics in this field. The most intensively studied cases are the quasi-periodic Schrödinger cocycles with real analytic potentials. A first proof is given by Herman [9] for the operators with trigonometric polynomial potentials by applying subharmonicity. Sorets-Plomer [15] further developed Herman’s technique to prove positivity of LE for one-frequency non-constant real analytic potentials with large disorders. This result was later extended to the Diophantine multi-frequency case by Bourgain-Goldstein [5] and any rational independent multi-frequency case by Bourgain [4]. It is worth to mention that Zhang [18] gave a different proof of the result in [15] based on the techniques in [1] and [17].

Besides analytic cases, positivity of LE also holds for some Gevrey classes of potentials and some finitely smooth potentials. The former is given with some strong Diophantine conditions by Eliason [7] and Klein [11]. For the so-called \( C^2 \) cos-type potentials, the positivity of LE (and even Anderson Localization) with

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Diophantine frequencies was proved by Sinai [14] and Fröhlich-Spencer-Wittwer [8]. Similar results were obtained by different methods in [3] and [16]. The positivity of LE for some $C^3$ potentials was also established in [6] by excluding a positive measure of frequencies.

To prove the positivity of LE for smooth cocycles, an effective method was established without the nice properties of subharmonicity. In [17], Young developed a dynamical technique, which is close in spirit to the techniques in [2], to prove positivity of LE for a class of $C^1$ smooth SL(2, $\mathbb{R}$) cocycles with Brjuno frequencies under a nonresonant condition. This result was applied to the Schrödinger cocycles with some $C^1$ potentials by excluding a small set of energies in [18]. Later, Wang-Zhang [16] developed a method from [17, 18] to overcome the difficulties of resonances. And they gave the positivity of LE for the quasi-periodic Schrödinger cocycles with $C^2$ cos-type potentials, Diophantine frequencies and all energies.

In this paper, we aim to improve [17]'s result and then apply it to the quasi-periodic Schrödinger cocycles. We focus on a class of $C^1$ smooth cocycles in the following. For $C, \lambda \geq 1$, we let

$$A_{C, \lambda} = \{ A \in C^1(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{R})) : C^{-1} \lambda \leq \|A(x)\| \leq C\lambda, \quad \|\frac{dA^{n+1}}{dx}(x)\| \leq C\lambda, \quad \forall x \in \mathbb{R}/\mathbb{Z} \},$$

where $C$ is a constant and $\lambda$ is sufficiently large. Let $A \in A_{C, \lambda}$. The map

$$(x, w) \mapsto (x + \alpha, A(x)w)$$

defines a family of dynamic systems on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$, called a cocycle and denoted for simplicity by $(\alpha, A)$. The $n$th iteration of the cocycle is denoted by

$$(\alpha, A)^n = (n\alpha, A_n),$$

where

$$A_n(x) = \begin{cases} A(x + (n-1)\alpha) \cdots A(x), & n \geq 1; \\ I_2, & n = 0; \\ (A_n(x + n\alpha))^{-1}, & n \leq -1. \end{cases}$$

Define the LE of the cocycle $(\alpha, A)$ as

$$L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx.$$ 

This limit always exists by Kingman’s subadditive ergodic theorem. By the polar decomposition, we can rewrite $A$ as

$$A = R_u \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} R_{\pi/2-s},$$

where $R_\theta$ is a rotation by $\theta$ and $u, s \in \mathbb{RP}^1$. Now we are ready to introduce the result in [17].

**Theorem 1.1. (Theorem 2 in [17])** Let $A_t \in A_{C, \lambda}$ and $A_t$ $C^1$-depend on $t \in [0, 1]$. Define $g(x, t) = s(x, t) - u(x - \alpha, t)$. Let $\{p_n/q_n\}$ be the fraction approximant of $\alpha$. Assume that $\alpha$ satisfies the Brjuno condition

$$\sum q_n^{-1} \ln q_{n+1} < \infty.$$

And assume that $g(x, t)$ satisfies the following conditions. For $t \in (0, 1]$ outside a finite set,

**(T0):** Let $C(t) := \{ x : g(x, t) = 0 \}$. Then $\#C(t)$ is finite.
(T1): The map $x \mapsto g(x, t)$ is transversal to $\{\theta = 0\}$;
(T2): $\frac{\partial g}{\partial t}/(\partial g/\partial x)$ takes different values at different points in $C(t)$.

For any $\varepsilon > 0$, let
\[
\triangle(\lambda) = \{ t \in [0, 1] : L(\alpha, A_t) > (1 - \varepsilon) \ln \lambda \}.
\]
Then \(\text{Leb}(\triangle(\lambda)) \to 1\) as $\lambda \to \infty$.

Though Theorem 1.1 is a bit different from Theorem 2 in [17], it is not hard to see that they are equivalent. Theorem 1.1 proves positivity of LE for a large class of $C^1$ smooth SL(2, $\mathbb{R}$) cocycles. But since it needs $\|A\|$ to be uniformly large, one cannot directly apply it to the Schrödinger cocycles. Zhang [18] found a conjugation of the Schrödinger cocycles to make the norm of the matrix sufficiently large, and then successfully applied Theorem 1.1 to the Schrödinger cocycles with some $C^1$ smooth potentials. In fact, he proved

**Theorem 1.2.** (Theorem B’ in [18]) Let $(\alpha, A)$ be a Schrödinger cocycle with
\[
A(x) = \begin{pmatrix}
E - \lambda v(x) & -1 \\
1 & 0
\end{pmatrix}.
\]
Let $t = E/\lambda \in v(\mathbb{R}/\mathbb{Z})$. Assume that $v \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and has finitely many critical points, and for $t \in v(\mathbb{R}/\mathbb{Z})$ outside a finite set, $v'(x)$ takes different values at different $x \in v^{-1}(t)$. Let $\varepsilon > 0$ and
\[
\tilde{\triangle}(\lambda) = \{ t \in v(\mathbb{R}/\mathbb{Z}) : L(\alpha, A) > (1 - \varepsilon) \ln \lambda \}.
\]
Then for each Brjuno frequency $\alpha$, we have
\[
\lim_{\lambda \to \infty} \text{Leb}(\tilde{\triangle}(\lambda)) = \text{Leb}(v(\mathbb{R}/\mathbb{Z})).
\]
Positivity of LE is established in Theorem 1.2 for the Schrödinger cocycle with a large subset of the potentials in $C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. However, the theorem cannot be applied to the simple potential:
\[
v(x) = \cos(4\pi x) + w(x)
\]
with small perturbation $w \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. The reason that Theorem 1.2 cannot be applied is that the corresponding cocycle does not satisfy the Condition (T2) in Theorem 1.1. Hence this condition seems unessential. In fact, Young [17] used Condition (T2) to exclude all the energies with which resonances may happen, and to prove positivity of LE with no resonances. But the occurrence of resonances does not mean the vanishing of LE. It was proved in [16] that positivity of LE holds with some particular resonances, which are produced by only two points. This idea enlightens us to weaken Condition (T2).

In this paper, we will prove the positivity of LE with the conditions weaker than (T2). These conditions permit us to deal with some special resonances as in [16]. Here is our main theorem.

**Theorem 1.3.** Assume that $\lambda \in A_{C, \lambda}$, $A_t C^1$-depends on $t \in [0, 1]$, and $\alpha$ is a Diophantine number, i.e. $\exists \gamma > 0$ and $\tau > 1$ such that
\[
\|k\alpha\| \geq \gamma|k|^{-\tau}, \quad k \in \mathbb{Z}\setminus\{0\},
\]
where $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$. If $g(x, t)$ satisfies Condition (T0), (T1) and one of the following conditions.
(T3): The map \( x \mapsto A_t(x) \) is \( C^2 \) smooth with the estimate \( \| \frac{d^2}{dx^2} A_t^{\pm}(x) \| \leq C \). And for \( t \) outside a finite set, \( (\partial g/\partial t)/(\partial g/\partial x) \) takes the same value at no more than two points in \( C \);

(T4): For almost every \( t \), \( (\partial g/\partial t)/(\partial g/\partial x) \) takes different values at different points in \( C \).

Then
\[
\lim_{\lambda \to \infty} \text{Leb}(\triangle(\lambda)) = 1. \quad (1.2)
\]

**Remark 1.4.** The frequency in Theorem 1.3 is not necessary to be Diophantine. One may use the technique in [12] to obtain the result for the Liouvillean frequency, which is defined as the irrational number satisfying
\[
\limsup_{n \to \infty} q_n^{-1} \ln q_{n+1} < \infty.
\]
(Clearly, a Brjuno number is Liouvillean.) However, we still use the Diophantine condition here to simplify the proof.

Theorem 1.3 can be applied to more cocycles than Theorem 1.1. The most important example is for the Schrödinger cocycles.

**Theorem 1.5.** Let \((\alpha, A)\) be a Schrödinger cocycle as stated in Theorem 1.2. Assume that \( v \) is \( C^1 \) smooth and has finitely many critical points, and for \( t \in v(\mathbb{R}/\mathbb{Z}) \) outside a finite set, \( v'(x) \) takes the same value at most two points \( x_1, x_2 \in v^{-1}(t) \). Let \( t = E/\lambda \in v(\mathbb{R}/\mathbb{Z}) \). Then for each Diophantine \( \alpha \),
\[
\lim_{\lambda \to \infty} \text{Leb}(\tilde{\triangle}(\lambda)) = \text{Leb}(v(\mathbb{R}/\mathbb{Z})).
\]

**Remark 1.6.** For the Schrödinger cocycles, we only need \( x \mapsto g(x, t) \) to be \( C^1 \) smooth, instead of \( C^2 \) smooth for general cocycles. This will be illustrated in Section 5.

Theorem 1.5 can be applied to a larger subset of the \( C^1 \) smooth potentials. Here we give an example that Theorem 1.2 does not hold but Theorem 1.5 holds. Let \( F \) be a subset of \( C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \) satisfying the following properties.

- \( v \in F \) has only four critical points.
- These four critical points divide \( \mathbb{R}/\mathbb{Z} \) into 4 intervals, which are successively denoted by \( J_1, J_2, J_3, J_4 \). Then \( v'(x) \geq 0 \) in \( J_1 \) and \( J_3 \), and \( v'(x) \leq 0 \) in \( J_2 \) and \( J_4 \). See Figure 1.
Clearly, if \( v(x) = \cos(4\pi x) + w(x) \) with small perturbation \( w \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \), then \( v \in \mathcal{F} \). It is also clear that if \( v \in \mathcal{F} \), \( v \) may not satisfy the conditions in Theorem 1.2. But \( v \) satisfies the conditions in Theorem 1.5. And the conclusion follows.

**Corollary 1.7.** Let \((\alpha, A)\) be the Schrödinger cocycle as stated in Theorem 1.2. Assume \( v \in \mathcal{F} \) and \( t = E/\lambda \in \mathbb{v}^{(\mathbb{R}/\mathbb{Z})} \). Then for each Diophantine \( \alpha \),

\[
\lim_{\lambda \to \infty} \text{Leb}(\tilde{\Delta}(\lambda)) = \text{Leb}(v(\mathbb{R}/\mathbb{Z})).
\]

As another example, we will show the positivity of LE for the analytic quasi-periodic Schrödinger cocycles. Although our result is weaker than [15] and [18], it gives another way to deal with the analytic case.

**Corollary 1.8.** Let \((\alpha, A)\) be the Schrödinger cocycle as stated in Theorem 1.2. Assume that \( v \) is real analytic and \( t = E/\lambda \in \mathbb{v}^{(\mathbb{R}/\mathbb{Z})} \). Then for each Diophantine \( \alpha \),

\[
\lim_{\lambda \to \infty} \text{Leb}(\tilde{\Delta}(\lambda)) = \text{Leb}(v(\mathbb{R}/\mathbb{Z})).
\]

The rest of this paper is organized as follows. In Section 2 we will present some technical computations of product of two matrices. In Section 3 we will establish an induction theorem to deal with the resonances produced by only two critical points. In Section 4 we will apply the induction theorem to prove positivity of LE. In Section 5 we will estimate the measure of the parameter for which the resonances occur between only two critical points, and then finish the proof of Theorem 1.3, Theorem 1.5 and Corollary 1.8.

### 2. Estimates for the Product of the Matrices

In this section, we consider the product of two \( \text{SL}(2, \mathbb{R}) \) matrices. We will compute the angle between two matrices to obtain the norm of the product. Most of the lemmas are proved in [16].

Let \( A \in \text{SL}(2, \mathbb{R}) \) satisfy \( \|A\| > 1 \). Define the map \( s, u : \text{SL}(2, \mathbb{R}) \to \mathbb{RP}^1 := \mathbb{R}/(\pi \mathbb{Z}) \), such that \( s(A) \) is the most contraction direction of \( A \) and \( u(A) = s(A^{-1}) \).

Then for \( A \in \text{SL}(2, \mathbb{R}) \) with \( \|A\| > 1 \), we have the polar decomposition

\[
A = R_u \left( \begin{array}{cc} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{array} \right) R^{2\pi - s}, \tag{2.1}
\]

where \( s, u \in [0, 2\pi) \) are some suitable choices of angles corresponding to the directions \( s(A), u(A) \in \mathbb{RP}^1 \). For convenience, we also use \( s(A) \) and \( u(A) \) for \( s \) and \( u \).

Let \( I \) be an interval and \( x \in I \). Assume that \( x \mapsto E_j(x) \) is \( C^2 \) smooth for \( j = 1, 2 \).

Consider the product \( E_3(x) = E_2(x) \cdot E_1(x) \). By (2.1), we have

\[
E_3 = R_{u(E_2)} \left( \begin{array}{cc} \|E_2\| & 0 \\ 0 & \|E_2\|^{-1} \end{array} \right) R^{2\pi - s(E_2) + u(E_1)} \left( \begin{array}{cc} \|E_1\| & 0 \\ 0 & \|E_1\|^{-1} \end{array} \right) R^{2\pi - s(E_1)}.
\]

We say \( |s(E_2) - u(E_1)| \) is the angle between \( E_2 \) and \( E_1 \). If \( \|E_1\|, \|E_2\| \gg 1 \), then it is not hard to see that \( \|E_2 E_1\| \) approximately equals to \( C \cdot \|E_2\| \cdot \|E_1\| \) (\( C > 0 \)) unless the angle tends to 0. We say it is nonresonant if

\[
|s(E_2) - u(E_1)|^{-1} \ll \min \{\|E_1\|, \|E_2\|\}.
\]

Otherwise, we say it is resonant. The following lemma gives a method to compute \( E_3 \).
Lemma 2.1. Denote
\[ e_j = \|E_j\|, \quad s_j = s(E_j), \quad u_j = u(E_j), \quad e_0 = \min\{e_1, e_2\}, \quad \theta = s(E_2) - u(E_1), \]
for \( j = 1, 2, 3 \). Let \( e_0 \gg 1, \) \( 0 < \eta \ll 1 \) and \( 0 < \beta \ll 1 \). We assume that for \( x \in I \) and \( j, m = 1, 2 \),
\[ \frac{d^m e_j}{dx^m} < C e_j^{1+m\eta}, \quad \frac{d^m \theta}{dx^m} < C e_0^{\eta}. \]
Then the following holds.

**Nonresonant case:** If \( |\theta| > C e_0^{\eta} \), then for \( m = 1, 2 \),
\[ \|u_3 - u_2\|_{C^2} < C e_2^{-2(5\eta)}, \quad \|s_3 - s_1\|_{C^2} < C e_1^{-2(5\eta)}; \]
\[ \frac{d^m e_3}{dx^m} < C e_3^{1+m\eta}, \quad e_3 \geq e_1 e_2 \cdot |s_2 - u_1|. \]

**Resonant case:** If \( e_1 \leq e_2^{\frac{1}{2}} \), then for \( m = 1, 2 \),
\[ \frac{d^m s_3}{dx^m} < C e_1^{1+2\eta}, \quad \|u_3 - u_2\|_{C^2} < C e_3^{-\frac{3}{2}}; \]
\[ \frac{d^m e_3}{dx^m} < C e_3^{1+m\eta+2m\beta}, \quad e_3 \geq e_1^{1-\beta}. \]
Let \( s'_3 = \arctan(e_1^2 \tan \theta) - \frac{\pi}{2} + s_1 \). Then for \( m = 1, 2 \),
\[ C \frac{d^m s_3}{dx^m} < \frac{d^m s_3}{dx^m} < C' \left| \frac{d^m s_3}{dx^m} \right|. \]

Remark 2.2. For the resonant case, we can see that \( s_3 \) and \( s'_3 \) are nearly the same in \( C^2 \) sense. For convenience we may take \( s_3 = \arctan(e_1^2 \tan \theta) - \frac{\pi}{2} + s_1 \).

Remark 2.3. If \( e_2 \leq e_1^{\frac{1}{2}} \), then the similar estimates hold as the following. For \( m = 1, 2 \),
\[ \frac{d^m u_3}{dx^m} < C e_2^{4+2\eta}, \quad \|s_3 - s_1\|_{C^2} < C e_1^{-\frac{3}{2}}; \]
\[ \frac{d^m e_3}{dx^m} < C e_3^{1+m\eta+2m\beta}, \quad e_3 \geq e_1^{1-\beta}. \]
Let \( u'_3 = \arctan(e_2^2 \tan \theta) - \frac{\pi}{2} + u_2 \). Then for \( m = 1, 2 \),
\[ C \frac{d^m u'_3}{dx^m} < \frac{d^m u_3}{dx^m} < C' \left| \frac{d^m u'_3}{dx^m} \right|. \]

Remark 2.4. If \( x \mapsto E_j(x) \) is only \( C^1 \) smooth, this lemma also holds by removing the estimates for the second derivative.

Proof. This lemma is in fact a combination of Lemma 2, Lemma 3 and Lemma 5 in [16].

Lemma 2.1 tells us that if the angle between \( E_1 \) and \( E_2 \) is not too small, then \( \|E_2 E_1\| \) approximately equals to \( C \|E_2\| \|E_1\| \), and if the angle between \( E_1 \) and \( E_2 \) is too small and we further assume that \( \|E_2\| \gg \|E_1\| \), then \( \|E_2 E_1\| \) also has an acceptable lower bound. It inspires us to repeatedly apply the lemma to estimate the lower bound of \( \|A_n(x)\| \), which may lead to the lower bound of the Lyapunov exponent. This idea will be implemented in Section 3 and Section 4.
As we see in Lemma 2.1, the angle function $\theta(x) = s_2(x) - u_1(x)$ takes an important role in the product of two matrices. In order to study the function, we introduce some types of the angle functions.

**Definition 2.1.** Let $B(x, r) \subset \mathbb{R}/\mathbb{Z}$ be the ball centered around $x \in \mathbb{R}/\mathbb{Z}$ with a radius of $r$. For a connected interval $J \subset \mathbb{R}/\mathbb{Z}$ and a constant $0 < a \leq 1$, let $aJ$ be the subinterval of $J$ with the same center and whose length is $a|J|$. Let $I = B(c, r)$ with $c \in \mathbb{R}/\mathbb{Z}$ and $1 \ll r^{-1} \ll l_0$. Let $f \in C^2(I, \mathbb{R}^1)$. We define the following types of the angle functions, see Figure 2.

- **f is of type I.** If we have the following:
  - $\|f\|_{C^2} < C$ and $f(x) = 0$ has only one solution, say $x_0$, which is contained in $\pi$;
  - $\frac{df}{dx} = 0$ has at most one solution on $I$ while $\left|\frac{df}{dx}\right| > r^2$ for all $x \in B(x_0, \frac{\pi}{2})$;
  - let $J \subset I$ be the subinterval such that $\frac{df}{dx}(J) \frac{df}{dx}(x_0) \leq 0$, then $|f(x)| > cr^3$ for all $x \in J$.

- **f is of type III.** If for $l : I \to \mathbb{R}_+$ such that
  $$l(x) > l_0 \gg 1, \quad \left|\frac{d^m l(x)}{dx^m}\right| < l(x)^{1+\beta}, \quad \text{for } x \in I, \quad m = 1, 2,$$
  $$f(x) = \arctan (l(x)^2 \tan f_1(x)) - \frac{\pi}{2} + f_2(x).$$

Here $f_1$ and $f_2$ are of type I.

**Remark 2.5.** The type III function in [16] always satisfies $f'_1(c_1)f'_2(c_2) < 0$, where $c_j$ is the zero of $f_j$ in $I$. In our setting, we also consider the case with $f'_1(c_1)f'_2(c_2) > 0$. See Figure 2. In the induction process, we will prove that the sign of $f'_1(c_1)f'_2(c_2)$ is invariant in the iteration.

**Remark 2.6.** In [16], the authors also consider the so-called "type II" angle function, which behaves as a quadratic function. However, in our setting the energies which lead to the case with type II angle function have arbitrary small measure. Since our result concerns about the measure estimate, excluding a small set is allowed. For simplicity, we do not consider the "type II" angle function.

We care about the non-degenerate property of the angle functions. The function of type I behaves as an affine function and is non-degenerate near the zeros. For the functions of type III, they can be divided into two parts: $\arctan (t^2 \tan f_1(x)) - \frac{\pi}{2}$ and $f_2(x)$. If $x$ locates far away from the zero of $f_1$, then the first part is so small.
to be neglected and therefore $f(x)$ approximately equals to $f_2(x)$. If $x$ varies near
the zero of $f_1$, the first part has a drastic change from $-\pi$ to 0. Then the number of
zeros depends on the sign of $f_1'(c_1)f_2'(c_2)$. If $f_1'(c_1)f_2'(c_2) > 0$, then $f$ is
monotone in $I$ and has exactly two zeros. If $f_1'(c_1)f_2'(c_2) < 0$, then $f$ is not monotone
and may has no zero, one zero or two zeros. See Figure 2. In the appendix, Lemma A.1
gives a careful description of the functions of type III. And Lemma A.3 shows that
the type III functions are also non-degenerate.

3. The induction process. In this section we will establish an induction theorem
similar to [16] to deal with the resonances, which are produced by only two critical
points. More precisely, we will iterate to show $\|A_{r_n}(x)\|$ has a large lower bound
with $r_n \to \infty$, provided some assumptions on the critical points and the coupling
constant $\lambda$ sufficiently large. Moreover, the largeness of $\lambda$ does not depend on the
choice of $t$.

We first explain the idea of the induction process. To estimate $\|A_{r_n}(x)\|$ with
some suitable $r_n$, we start with the product

$$A_{r_1}(x) = A(x + (r_1 - 1)\alpha) \cdots A(x + 2\alpha) \cdot A(x + \alpha) \cdot A(x).$$

Before repeatedly applying Lemma 2.1 to the above product, we have to confirm
whether the angle between $A(x + j\alpha)$ and $A(x + (j - 1)\alpha)$ is not too small. The
non-degenerate property of the angle function is needed here. And we require $\alpha$ be
Diophantine and $x$ locate near a zero of the angle function such that $r_1$ can be large
enough. Then Lemma 2.1 gives the estimates of $\|A_{r_1}(x)\|$, $s(A_{r_1}(x))$ and $u(A_{r_1}(x))$.
Proceed to the next step, we consider the product

$$A_{r_2}(x) = A_{r_1}(x + (r_2 - r_1)\alpha) \cdots A_{r_1}(x + 2r_1\alpha) \cdot A_{r_1}(x + r_1\alpha) \cdot A_{r_1}(x).$$

We will use the estimates of $\|A_{r_1}(x)\|$, $s(A_{r_1}(x))$ and $u(A_{r_1}(x))$ to verify the conditions
of Lemma 2.1 and again repeatedly apply Lemma 2.1 to get the lower bound of $\|A_{r_2}(x)\|$.
Inductively, the estimates of $\|A_{r_n}(x)\|$, $s(A_{r_n}(x))$ and $u(A_{r_n}(x))$ can be
obtained in the same way.

In the process of estimating the angles, we are afraid that $x + j\alpha$ locates near one
zero of the angle function provided that $j$ is small and $x$ locates near another zero.
More precisely, if $x$ locates near one zero, then the trajectory $\{x + j\alpha : 0 < j < q_n\}$ ($p_n/q_n$ is the continued fraction approximant of $\alpha$, see Appendix B for more
properties) may goes into the neighbourhoods of other zeros $k$ times. We say this
case is a resonant case. If $k > 1$, then it is hard to get a desired lower bound of
$\|A_{q_n}(x)\|$ since we cannot verify the condition $\"e_1 \leq e_2^3\"$ in Lemma 2.1. Moreover,
the resonant case causes a great change of the angle function (see Resonant case in
Lemma 2.1) and brings a lot of troubles in the estimation of the angles. Due to the
complexity of the resonance case, Young [17] excluded all the energies which may lead
to the resonant cases. However, not all the resonances are hard to be dealt
with. In [16], the authors found that if the original angle function has only two
zeros, then the trajectory $\{x + j\alpha : 0 < j < q_n\}$ goes into the neighbourhoods of
other zeros at most $k = 1$ time. And then one can use Lemma 2.1 to deal with the
resonance with $k = 1$ by verifying the conditions with the Diophantine condition
on the frequency. In our setting, the angle function may have many zeros and $k$
may larger than 1. However, we can exclude all the "bad" energies which leads to
$k > 1$ such that we only need to consider the resonance with $k = 1$. In this section,
we are going to establish an induction theorem, which is similar to the induction
theorem in [16], to deal with the resonance with \( k = 1 \). In Section 5, the estimate of the measure of the "bad" energies will be obtained.

Now we go into the detail. Condition (T0) says that \( \#C(t) \) is finite. Let
\[
C(t) = \{ c_1(t), c_2(t), \ldots, c_m(t) \} \subset \mathbb{R}/\mathbb{Z}.
\]
Condition (T1) says that \( x \mapsto g(x, t) \) is transversal to \( \{ \theta = 0 \} \) for \( t \in [0, 1] \) outside a finite set. Since our result concerns about the measure estimate, excluding a small measure set of parameters is allowed. And one can replace the interval \([0, 1]\) with any connected interval in Theorem 1.3. Hence we can, for simplicity, assume that Condition (T1) holds for all \( t \in [0, 1] \). Then there exists a constant \( \xi \) such that for any \( 1 \leq j \leq m \),
\[
\min_{t \in [0, 1]} |\partial_x g(c_j(t), t)| > \xi > 0.
\]
For any \( c_j \in C(t) \), let \( I_{N,j} = B(c_j, q_n^{-3\tau}) \) and \( I_N = \cup_j I_{N,j} \). We choose \( N \) large enough such that
\[
\min_{x \notin I_N} |g(x,t)| > \xi/2 > 0,
\]
\[
\min_{x \in I_N} |\partial_x g(x,t)| > \xi/2 > 0.
\]
Hence \( g \) is a type I function in each \( I_{N,j} \). Note that \( N \) can be chosen independent of \( t \).

Consider the sequence \( \{ \lambda_n : n \geq N \} \) defined by
\[
\lambda_N = \lambda, \quad \ln \lambda_n = (1 - Cq_n^{-1} \ln q_n) \ln \lambda_{n-1}.
\]
Let \( \eta_n = \eta_{n-1}(1 + 8q_n^{-1}) (n \geq N) \) with \( \eta_{N-1} = 10^{-2} \varepsilon \). Since \( \alpha \) is Diophantine and \( N \) is sufficiently large, it is easy to see that \( \lambda_n \) decreases to some \( \lambda_\infty > 0 \) and \( \eta_n \) increases to some \( \eta_\infty \). And we have \( \lambda_\infty > \lambda^{1-\varepsilon} \) and \( \eta_\infty < \varepsilon \).

Fix any \( t \in [0, 1] \) below. Let
\[
s_n(x) = s(A_n(x)), \quad u_n(x) = s(A_{-n}(x)).
\]
We define the initial angle function \( g_N \) as
\[
g_N(x) := g(x,t) = s_1(x) - u_1(x),
\]
and the initial critical points \( c_{N,j} \) as
\[
C_N = \{ c_{N,1}, c_{N,2}, \ldots, c_{N,m} \} := C = \{ y : |g_N(y)| = \min_{x \in \mathbb{R}/\mathbb{Z}} |g_N(x)| \}.
\]
For \( n \geq N \), we inductively define the \((n)\)th critical interval, the \((n)\)th return time, the \((n+1)\)th angle function and the \((n+1)\)th critical points as the following.

- The critical interval \( I_{n,j} \) centers at the critical point \( c_{n,j} \), with radius of \( q_n^{-3\tau} \) on \( \mathbb{R}/\mathbb{Z} \).
- The return time \( r_{n,j}^+(x) : I_{n,j} \to \mathbb{Z}_+ \) is the first return time to \( I_n \) after \( q_n \), where \( r_{n,j}^+(x) \) is the forward return time and \( r_{n,j}^-(x) \) backward. That is
\[
r_{n,j}^+(x) = \min \{ l : x + l\alpha \in I_n, l \geq q_n \}, \quad x \in I_{n,j}.
\]
Let \( r_{n,j}^+ = \min_{x \in I_{n,j}} r_{n,j}^+(x) \) and \( r_{n,j}^- = \min \{ r_{n,j}^+, r_{n,j}^- \} \). And let \( r_{N-1,j} = 0 \).
- The angle function \( g_{n+1} : I_n \to \mathbb{R}^1 \) is defined by \( g_{n+1}(x) := s_{r_{n,j}^+}(x) - u_{r_{n,j}^-}(x) \). Moreover, the critical point \( c_{n+1,j} \) is the minimal point of the angle function \( g_{n+1} \) as the following
\[
C_{n+1} = \{ c_{n+1,j} : 1 \leq j \leq m \} \subset \{ y : |g_{n+1}(y)| = \min_{x \in I_{n,j}} |g_{n+1}(x)| \},
\]

with
\[ |c_{n+1,j} - c_{n,j}| < C\lambda^{-\frac{2}{3}}r^{-1} \quad 1 \leq j \leq m. \]

Let \( h \in [1, \frac{m}{2}] \cap \mathbb{Z} \) and
\[ P = \{(i_s, j_s) : 1 \leq s \leq h\}, \]
such that \( i_s \neq j_s, 1 \leq i_s, j_s \leq m, \{i_s, j_s\} \cap \{i_t, j_t\} = \emptyset \) if \( s \neq r \). Moreover, we let
\[ D^{(n)} = \{\{c_{n,i}, c_{n,j} : \{i, j\} \in P\}. \]

We assume the following properties about \( C^{(n)} \).

**Assumption (n).** \( \forall a, b \in C^{(n)}, if \{a, b\} \notin D^{(n)}, then \) for \( 0 < |k| < q_n \),
\[ \|a + k\alpha - b\| > q_n^{-3r}. \]

Now we are ready to give the induction theorem.

**Theorem 3.1.** Let \( \varepsilon \) be small enough. Let \( \lambda \) be large enough such that
\[ \lambda > \lambda_0(\alpha, \varepsilon) := \max\{C(\varepsilon, q_N), C(\gamma, \tau)\}. \quad (3.1) \]
Assume that \( C^{(n)} \) satisfies Assumption (\( \tilde{n} \)) for \( N \leq \tilde{n} \leq n + 1 \). We also assume that the following properties holds in the \( n \)th step.

- We have the estimates of the norm of \( A_{r_n}^{\pm} (x) \) and its derivatives. For \( x \in I_{n,i}, 1 \leq i \leq m, \nu = 1, 2, \)
\[ \|A_{r_n}^{\pm} (x)\| \geq \lambda_n^{r_n}, \quad \left| \frac{d^\nu}{dx^\nu} A_{r_n}^{\pm} (x) \right| \leq C\|A_{r_n}^{\pm} (x)\|^{1+\nu} \lambda_n. \]

- Depending on the positions of the critical intervals (points), it is divided into the nonresonant case and the resonant case, which leads to different types of the angle functions.
  - In the nonresonant case, the angle function \( g_{n+1} \) is of type I in \( I_{n,i} \). And we have
\[ \|g_{n+1} - g_n\|_{C^2} \leq C\lambda^{-\frac{2}{3}}r^{-1} \quad \text{on} \quad I_{n,i}. \]
Moreover, it satisfies \( \|g_{n+1}\|_{C^2} \leq C \).
  - In the resonant case, the angle function \( g_{n+1} \) is of type III in \( I_{n,i} \) (\( i = 1, 2 \)). And it satisfies \( \|g_{n+1}\|_{C^2} \leq C\lambda^5q_n \). Moreover, the function \( g_{n+1} \) satisfies the following properties:
    * If \( \frac{d}{dx} g_n(c_{n,i}) \frac{d}{dx} g_n(c_{n,j}) > 0 \), then \( g_{n+1} \) is monotone and has exactly 2 zeros in \( I_{n,i} \). The same is true for \( g_{n+1} \) in \( I_{n,j} \). And we also have
\[ \frac{d}{dx} g_{n+1}(c_{n+1,i}) \cdot \frac{d}{dx} g_{n+1}(c_{n+1,j}) > 0. \]
    * If \( \frac{d}{dx} g_n(c_{n,i}) \frac{d}{dx} g_n(c_{n,j}) \leq 0 \), then \( g_{n+1} \) may have 0, 1 or 2 zeros in \( I_{n,i} \). And we also have
\[ \frac{d}{dx} g_{n+1}(c_{n+1,i}) \cdot \frac{d}{dx} g_{n+1}(c_{n+1,j}) \leq 0. \]
Furthermore, \( g_{n+1} \) has one more minimal point \( c'_{n+1,i} \) on \( I_{n+1,i} \) such that \( g_{n+1}(c'_{n+1,i}) = g_{n+1}(c_{n+1,i}), \) and \( g_{n+1} \) has one more minimal point \( c'_{n+1,j} \) on \( I_{n+1,j} \) such that \( g_{n+1}(c'_{n+1,j}) = g_{n+1}(c_{n+1,j}), \) and it holds that
* if \( |g_n(c_{n,i})|, |g_n(c_{n,j})| > C\lambda^{-\frac{1}{3}}r^{n-1}, \)
\[ |g_{n+1}(c_{n+1,i})|, |g_{n+1}(c_{n+1,j})| > C\lambda^{-\frac{1}{3}}r^{n-1}; \]
Let
\[ \text{Theorem 4.1.} \]

4. Estimating the Lyapunov exponent.

**Theorem 4.1.** Let \( \varepsilon > 0 \) and \( \lambda_0 \) be as in Theorem 3.1. Assume \( \lambda > \lambda_0 \). If \( C^{(n)} \) satisfies Assumption (n) for all \( n \geq N \), then the LE is positive and satisfies
\[ L(\alpha, A) > (1 - \varepsilon) \ln \lambda. \]

**Proof.** Our idea is to construct an exceptional set \( B \subseteq S^1 \) with small measure, and to estimate the lower bound of \( \|A_t(x)\| \) outside \( B \) by using the induction theorem.

Let \( q_n = q_N \) and \( q_{nk+1} = \min\{q_l : q_l > q_{4r}\} \). Let \( i = q_{nk+1}^2 \) and
\[ B_{nk} := \sum_{i=1}^{i} \left( B(C^{(nk+1)}, q_{nk+1}^{-3r}) - l\alpha \right) \]

It is straightforward to compute that
\[ \text{Leb}(\bigcup_{k \geq 1} B_{nk}) \leq \sum_{k \geq 1} q_{nk+1}^{-r} \leq \sum_{n \geq N} q_n^{-r} \leq \varepsilon, \]
provided sufficiently large \( N \).

Let \( i_0 \) be the first time such that \( x + i_0\alpha \in I_{nk} \). By Lemma B.2, we have \( i_0 \leq q_{nk+1} \ll i \). Then we can assume that \( x \in I_{nk} \). Similarly, we can also assume that \( x + ia \in I_{nk} \). Since \( x \not\in B_{nk} \), the trajectory \( \{x + ja : 1 \leq j \leq i\} \) has not entered \( I_{nk+1} \), but has entered \( I_{nk} \). Let \( 0 = j_0 < j_1 < \cdots < j_p = i \) be the return times to \( I_{nk} \) such that \( j_{l+1} - j_l \geq q_{nk} \). Since the trajectory has not entered \( I_{nk+1} \), then
\[ \text{dist}\left(x + j_l\alpha, C^{(nk+1)}\right) > q_{nk+1}^{-3r}. \]

By the definition of \( n_{k+1} \), we have
\[ q_{nk+1} \leq q_{nk+1}^{-r} \leq q_{nk}^{-12r^3}. \]

Then
\[ \text{dist}\left(x + j_l\alpha, C^{(nk+1)}\right) > q_{nk}^{-12r^3}. \]
In the nonresonant case, since there are no extra zeros of $g_{n+1}$, then
\[ |g_{n+1}(x + j\alpha)| \geq C \cdot \text{dist}(x + j\alpha, C(n+1)) \]
\[ \geq C \cdot \text{dist}(x + j\alpha, C(n+1)) - C\lambda^{-\frac{3}{2}q_k} \]
\[ > Cq_{n+1}^{-12\tau^3}. \]

In the resonant case, we are worried about the possibility that
\[ |x + jl\alpha - c_{n+2}'| < Cq_{n+1}^{-3\tau}. \]
However, either $|c_{n+2} - c_{n+1}| < \lambda^{-\frac{3}{2}q_k}$ so that $|x + j\alpha - c_{n,k}| < Cq_{n+1}^{-3\tau}$, contradicting that the trajectory has not entered $I_{n+1}$. Or for some $|k| < q_{n,k}$,
\[ |c_{n+2} - c_{n+2}' - k\alpha| < \lambda^{-\frac{3}{2}q_k}. \]
This implies
\[ |x + j\alpha + k\alpha - c_{n+2}| < Cq_{n+1}^{-3\tau}. \]
This also contradicts that the trajectory has not entered $I_{n+1}$. It concludes by Lemma A.3 that
\[ |g_{n+1}(x + j\alpha)| \geq Cq_{n+1}^{-36\tau^3}. \]
Then by Lemma 2.1, we have
\[ \|A_i(x)\| \geq \prod_{1 \leq i \leq p} \|A_{ji-j_{i-1}}(x + j_{i-1}\alpha)\| \cdot q_{n+1}^{-36\tau^3p} \]
\[ \geq \lambda^{(1-\frac{3}{2})i} \cdot \lambda^{-36\tau^3\ln q_{n+1} - \frac{q_{n+1}}{q_{n,k}}} \geq \lambda^{(1-\epsilon)q_{n+1}}, \]
where we use $p \cdot q_{n,k} \leq i$ and $\ln \lambda > 1$ in the second inequality. Then
\[ i^{-1} \ln \|A_i(x)\| \geq (1-\epsilon) \ln \lambda, \quad i = q_{n+1} + q_{n+1}, \]
for all $k \geq 1$ and $x \in (\mathbb{R}/\mathbb{Z}) \setminus \sum_{k \geq 1} B_{n,k}$. By Kingman’s Subadditive Ergodic Theorem, the LE always exists and it holds that
\[ L(\alpha, A) = \lim_{i \to \infty} i^{-1} \ln \|A_i(x)\|, \quad \text{a.e. } x \in \mathbb{R}/\mathbb{Z}. \]
Hence we have
\[ L(\alpha, A) \geq (1-\epsilon) \ln \lambda. \]

5. Estimate the measure of the good parameter. We will mainly follow the idea in [17] to estimate the measure of the good parameter. However, since the elements in $C(t)$ do not need to take different derivatives with respect to $t$, the derivative of the difference of two critical points has no uniformly lower bound. In this section, we will obtain a lower bound depending on the contraction of every connected component of $t$. By controlling $\lambda$ to be sufficiently large, we can ensure that the contraction is small enough. And then we can use the lower bound to exclude the bad parameter.

Let
\[ T = \{ t \in [0,1] : \exists c, d \in C(t) \text{ s.t. } c'(t) - d'(t) = 0 \}. \]
We will divide the proof into two cases: inside $T$ and outside $T$. 

\[ \square \]
5.1. **Outside $\mathcal{T}$**. It is obvious that $\mathcal{T}$ is a closed set in $[0, 1]$. Then $(0, 1)\setminus\mathcal{T}$ consists of a sequence of open intervals. Let $J$ be one of those intervals. Let $t \in (1 - \frac{\varepsilon}{100})J$. Then there exists $\gamma_0 = \gamma_0(\varepsilon)$ such that for any $a, b \in C$,

$$|a'(t) - b'(t)| > \gamma_0.$$ 

Choose $N$ sufficiently large such that

$$C\gamma_0^{-1} \sum_{n \geq N} 2^n q_n^{-3\gamma_0^{-1}} < \frac{\varepsilon}{10}|J|.$$ 

Now we define

**Assumption (n').** \(\forall a, b \in C^{(n)}, \) for \(0 < |k| < q_n\),

$$\|a + k\alpha - b\| > q_n^{-3\gamma_0^{-1}}.$$ 

If $C^{(n)}$ satisfies Assumption (n') for all $n \geq N$, then $C^{(n)}$ satisfies Assumption (n) for all $n \geq N$. By Theorem 3.1, then there exists a $\lambda_J$ such that for $\lambda > \lambda_J$ the induction process in Section 3 applies. In fact, each step in the induction process is nonresonant. Hence we obtain the positivity of LE by Theorem 4.1. It remains to estimate the measure of $t$ with which $C^{(n)}(t)$ satisfies Assumption (n') for all $n \geq N$.

**Lemma 5.1.** There exist $\gamma_1, \gamma_2 > 0$ such that the following holds. (In fact, $\gamma_1 = \gamma_0/2$.) Let $\omega$ be any subinterval of $J$ on which $C^{(n)}$ is defined and $\gamma^\pm_{n-1,j}$ is independent of $t$ for every $j$. Let $a \neq b \in C$, and let $a^{(n)}(t), b^{(n)}(t) \in C^{(n)}(t)$ denote the corresponding critical points. Then for $t \in \omega$,

$$\gamma_1 \leq \left| \frac{d}{dt}(b^{(n)}(t) - a^{(n)}(t)) \right| \leq \gamma_2.$$ 

**Proof.** This lemma follows easily from lemma 2.1 and the induction theorem. □

On the interval $(1 - \frac{\varepsilon}{100})J$, let $\mathcal{P}_N$ be a partition into subintervals of length $\approx q_N^{-3\gamma_0^{-1}}$. Let

$$\Delta_N = \bigcup \{\omega \in \mathcal{P}_N : \forall a, b \in C(t), t \in \omega \text{ and } 0 < |j| < q_N, |a + j\alpha - b| \geq q_N^{-3\gamma_0^{-1}}\}.$$ 

For $\omega \in \mathcal{P}_N$, we let

$$t^\pm_{N,j}(\omega) = \min\{t^\pm_{N,j}(t) : t \in \omega\}.$$ 

And then we let

$$\hat{g}_{N+1} = s_{\hat{t}^+_{N,j}} - u_{\hat{t}^-_{N,j}},$$ 

and use it to define $C^{(N+1)}$. It is easy to see $\hat{g}_{N+1}$ satisfies all the estimates of $g_{N+1}$ in Section 3.

Moving on to the next step, we let $\mathcal{P}_{N+1}$ be a refinement of $\mathcal{P}_N|\Delta_N$, subdividing $\Delta_N$ into intervals of length $\approx q_N^{-3\gamma_0^{-1}}$. We define

$$\Delta_{N+1} = \bigcup \{\omega \in \mathcal{P}_{N+1} : \forall a, b \in C^{(N+1)}(t), t \in \omega \text{ and } 0 < |j| < q_{N+1}, |a + j\alpha - b| \geq q_{N+1}^{-3\gamma_0}\}.$$ 

For each $\omega \subset \Delta_{N+1}$, $\hat{t}^\pm_{N+1,j}$ and $C^{(N+2)}$ are also defined similarly.

This process gives a decreasing sequence of sets

$$(1 - \frac{\varepsilon}{100})J \supset \Delta_N \supset \Delta_{N+1} \supset \cdots$$ 

such that for any $t \in \Delta_n$, $C^{(l)}$ satisfies Assumption (l') for $N \leq l \leq n$. Moreover, there is an increasing sequence of partitions $\{P_n\}$, defined on $\Delta_n$, such that the definition of $C^{(n)}$ is consistent on each element of $P_n$. Hence Lemma 5.1 applies.
Consider $a, b \in C$. Define the function $\tau_N : (1 - \frac{\varepsilon}{100})J \to \mathbb{R}$ by $\tau_N(t) = b(t) - a(t)$. Since $\tau'_N(t) > \gamma_1$, we have

$$\text{Leb}\{t \in (1 - \frac{\varepsilon}{100})J : \|\tau_N(t) - 2j\pi\alpha\| < q_n^{-3\tau}\} < 2\gamma_1^{-1}q_n^{-3\tau}.$$ 

Then

$$\text{Leb}(\Delta_N) > (1 - \frac{\varepsilon}{50})|J| - m^\varepsilon \gamma_1^{-1}q_n^{-3\tau+1}.$$ 

For $n > N$, let $\tau_n : \Delta_{n-1} \to \mathbb{R}$ defined by $\tau_n(t) = b^{(n)}(t) - a^{(n)}(t)$.

**Lemma 5.2.** $\tau_n : \Delta_{n-1} \to \mathbb{R}$ is at most $2^{n-N}$ to 1.

**Proof.** Let $\omega \in \mathcal{P}_{n-1}$. If $\omega', \omega'' \subset \omega$ are two non-adjacent elements of $\mathcal{P}_n|\Delta_n$, $\omega'$ to the left of $\omega''$, then

$$\sup\{\tau_n(t) : t \in \omega' \cap \Delta_n, n_1 > n\} < \inf\{\tau_n(t) : t \in \omega'' \cap \Delta_n, n_1 > n\}.$$ 

This comes from

$$\inf\{\tau_n(t) : t \in \omega''\} - \sup\{\tau_n(t) : t \in \omega'\} > \gamma_1q_n^{-3\tau} \gg \lambda^{-\frac{3}{3}q_n} - |c_{n_1,i} - c_{n,i}|.$$ 

\[\square\]

By Lemma 5.1 and Lemma 5.2,

$$\text{Leb}\{t \in \Delta_{n-1} : \|\tau_n(t) - 2j\pi\alpha\| < q_n^{-3\tau}\} < 2^{n-N+1}\gamma_1^{-1}q_n^{-3\tau+1}.$$ 

And thus

$$\text{Leb}(\cap_{n\geq N}\Delta_n) \geq (1 - \frac{\varepsilon}{50})|J| - C\gamma_1^{-1}\sum_{n\geq N} 2^nq_n^{-3\tau+1} \geq (1 - \frac{\varepsilon}{3})|J|.$$ 

Let $M$ be a set of finitely many $J$s such that

$$\sum_{J \in M} |J| \geq (1 - \frac{\varepsilon}{3})\text{Leb}((0,1)\setminus\mathcal{T}).$$ 

Then

$$\text{Leb}(\cup_{J \in M} \cap_{n\geq N}\Delta_n(J)) \geq (1 - \varepsilon)\text{Leb}((0,1)\setminus\mathcal{T}).$$ 

Let $\lambda_T := \max\{\lambda_J : J \in M\}$ and $\lambda > \lambda_T$. For each $t \in \cup_{J \in M} \cap_{n\geq N}\Delta_n(J)$, there exists a $J \in M$ such that $t \in \cap_{n\geq N}\Delta_n(J)$. And the corresponding cocycle satisfies Assumption (a') for $n \geq N$. This implies that $C^{(n)}$ satisfies Assumption (n) for all $n \geq N$. Then by Theorem 4.1 we obtain the positivity of LE. Hence

$$\text{Leb}((0,1)\setminus\mathcal{T}) \cap \{t : L(\alpha, A) > (1 - \varepsilon)\ln\lambda\} \geq \text{Leb}(\cup_{J \in M} \cap_{n\geq N}\Delta_n(J)) \geq (1 - \varepsilon)\text{Leb}((0,1)\setminus\mathcal{T}). \quad (5.1)$$

5.2. **Inside $\mathcal{T}$.** If $\text{Leb}(\mathcal{T}) = 0$, we have nothing to say. Otherwise, $\text{Leb}(\mathcal{T}) > 0$. Then we choose $\varepsilon < 2^{-2m}(\text{Leb}(\mathcal{T}))^2$.

Let $c_i, c_j \in C$. We consider

$$E_{i,j} := \{t \in [0,1] : c_i(t) - c_j(t) = 0\}.$$ 

Let $F_{i,j} = (0,1)\setminus E_{i,j}$. Note that $F_{i,j}$ is an open set. Then $\bigcap_{i \neq j} F_{i,j}$ consists of a sequence of intervals. Let $J$ be one of those intervals. Let $t \in (1 - \frac{\varepsilon}{100})J$. Then there exists $\gamma_0 = \gamma_0(\varepsilon)$ such that for any $a, b \in C \setminus \{c_i, c_j\}$,

$$|a'(t) - b'(t)| > \gamma_0.$$
Choose $N$ sufficiently large such that
\[ C_{10}^{-1} \sum_{n \geq N} q_n^{-3\gamma + 1} < \varepsilon |J|. \]

We define

**Assumption (n′_{i,j}),** ∀ $a, b \in C^{(n)} \setminus \{ e_i^{(n)}, e_j^{(n)} \}$, for $0 < |k| < q_n$,
\[ \|a + k\alpha - b\| > q_n^{-3\gamma}. \]

If $t \in E_{i,j}$ and $C^{(n)}(t)$ satisfies Assumption (n′_{i,j}) for all $n \geq N$, then $C^{(n)}$ satisfies Assumption (n) for all $n \geq N$ by Condition (T3). By Theorem 3.1, there exists a $\lambda_j$ such that for $\lambda > \lambda_j$ the induction process in Section 3 applies. By Theorem 4.1 we obtain the positivity of LE. It remains to estimate the measure of such $t$.

Similar to in the previous subsection, we can define a sequence of $\Delta_n$ such that
\[ (1 - \frac{\varepsilon}{100})J \supset \Delta_N \supset \Delta_{N+1} \supset \cdots \]
and for any $t \in \Delta_n$, $C^{(l)}$ satisfies Assumption (l'_{i,j}) for $N \leq l \leq n$. Moreover, there is an increasing sequence of partitions $\{P_n\}$, defined on $\Delta_n$, such that the definition of $C^{(n+1)}$ is consistent on each element of $P_n$. Hence Lemma 5.1 applies. Using the same argument in the previous subsection, we can obtain
\[ \text{Leb}(\cap_{n \geq N} \Delta_n(J)) \geq (1 - \frac{\varepsilon}{3})|J|. \]

Let $M$ be a set of finitely many $J$s such that
\[ \sum_{J \in M} |J| \geq (1 - \frac{\varepsilon}{3})\text{Leb}(\cap_{i,j} \neq (i,j) F_{i,j}). \]

Then it holds that
\[ \text{Leb}(\cup_{J \in M} \cap_{n \geq N} \Delta_n(J)) \geq (1 - \varepsilon)\text{Leb}(\cap_{i,j} \neq (i,j) F_{i,j}). \]

Let
\[ G_{i,j} := \cup_{J \in M} \cap_{n \geq N} \Delta_n(J). \]

Then
\[
\text{Leb}(E_{i,j} \cap G_{i,j}) + \text{Leb}(F_{i,j} \cap G_{i,j}) \\
\geq (1 - \varepsilon)\text{Leb}(E_{i,j} \cap (\cap_{i,j} \neq (i,j) F_{i,j})) + (1 - \varepsilon)\text{Leb}(F_{i,j} \cap (\cap_{i,j} \neq (i,j) F_{i,j})).
\]

Since $G_{i,j} \subset \cap_{i,j} \neq (i,j) F_{i,j}$ and $\text{Leb}(F_{i,j} \cap G_{i,j}) \leq 1$,
\[ \text{Leb}(E_{i,j} \cap G_{i,j}) + \varepsilon \geq (1 - \varepsilon)\text{Leb}(E_{i,j} \cap (\cap_{i,j} \neq (i,j) F_{i,j})). \]

Let $\lambda > \lambda_{i,j} := \max\{\lambda_j : J \in M\}$. If $t \in E_{i,j} \cap G_{i,j}$, then $C^{(n)}$ satisfies Assumption (n′_{i,j}) for all $n \geq N$. And hence $C^{(n)}$ satisfies Assumption (n) by Condition (T3). By Theorem 4.1 we obtain the positivity of LE. Therefore, we have
\[ \text{Leb}(E_{i,j} \cap \{t : L(\alpha, A) > (1 - \varepsilon)\ln\lambda\}) + \varepsilon \geq \text{Leb}(E_{i,j} \cap G_{i,j}) + \varepsilon \geq (1 - \varepsilon)\text{Leb}(E_{i,j} \cap (\cap_{i,j} \neq (i,j) F_{i,j})). \]

Similarly, we consider $\cap_{i,j} \in K E_{i,j}$ for arbitrary nonempty subset $K \subset \{\{i, j\} : 1 \leq i, j \leq m, i \neq j\}$. Note $\cap_{i,j} \in K F_{i,j}$ consists of a sequence of open intervals. As before, there exists $\lambda_K$ sufficiently large such that if $\lambda > \lambda_K$, the following
holds. There exists $G_K$ satisfying $G_K \subset \bigcap_{i,j} \notin K F_{i,j}$ such that for $t \in G_K$, the corresponding cocycle satisfies

**Assumption ($n'_K$).** If

$$\{a, b\} \in \{\{a, b\} : a, b \in C^{(n)}, a \neq b\} \backslash \{\{c_n, i, c_n,j\} : \{i, j\} \in K\},$$

then for $0 < |k| < q_n$,

$$\|a + k\alpha - b\| > q_n^{-3\tau}.$$  

Moreover, we have

$$\text{Leb}(\bigcap_{i,j} E_{i,j} \cap G_K) + \varepsilon \geq (1 - \varepsilon)\text{Leb}(\bigcap_{i,j} E_{i,j} \cap \bigcap_{i,j} \notin K F_{i,j}).$$

Note that Condition (T3) implies that for $t \in G_K$, the cocycle satisfies Assumption (n). By Theorem 4.1, we obtain

$$\text{Leb}(\bigcap_{i,j} E_{i,j} \cap \{t : L(\alpha, A) > (1 - \varepsilon) \ln \lambda\}) + \varepsilon \geq (1 - \varepsilon)\text{Leb}(\bigcap_{i,j} E_{i,j} \cap \bigcap_{i,j} \notin K F_{i,j}).$$

(5.2)

Note that every two $(\bigcap_{i,j} E_{i,j}) \cap (\bigcap_{i,j} \notin K F_{i,j})$ do not intersect with each other and

$$\mathcal{T} = \bigcup_{K} [(\bigcap_{i,j} E_{i,j}) \cap (\bigcap_{i,j} \notin K F_{i,j})].$$

Take summations on both sides of (5.2)

$$\text{Leb}(\mathcal{T} \cap \{t : L(\alpha, A) > (1 - \varepsilon) \ln \lambda\}) + (2^m - 1)\varepsilon \geq (1 - \varepsilon)\text{Leb}(\mathcal{T}).$$

Due to the choice of $\varepsilon$,

$$\text{Leb}(\mathcal{T} \cap \{t : L(\alpha, A) > (1 - \varepsilon) \ln \lambda\}) \geq (1 - 2\sqrt{\varepsilon})\text{Leb}(\mathcal{T}).$$

(5.3)

5.3. **Proof of Theorem 1.3.** If Condition (T3) holds, then for each $t$, at most 2 elements in $C$ have the same derivative with respect to $t$. By (5.1) and (5.3), there exists $\lambda_*$ such that if $\lambda > \lambda_*$, we have

$$\text{Leb}(\{t : L(\alpha, A) > (1 - \varepsilon) \ln \lambda\}) \geq 1 - 2\sqrt{\varepsilon}.$$  

This implies (1.2).

If Condition (T4) holds, then for almost every $t$, each element in $C$ has different derivatives with respect to $t$. In other words, Leb($\mathcal{T}$) = 0. By (5.1), we have (1.2). Note that in this case, we do not deal with the resonant case in Section 3. Hence we do not need the cocycle to be $C^2$ smooth.

5.4. **Proof of Theorem 1.5.** Now we consider the Schrödinger cocycle with the form stated in Theorem 1.2. Assume that $v$ is $C^1$ smooth and has finitely many critical points. By Appendix A.1 in [16] or Section 5.3 in [18], we can choose

$$g_N(x, t) = t - v(x) + \chi(x, t, \lambda),$$

where $\chi$ is $C^1$ smooth on $x$ and $t$, and is continuous on $\lambda$. And it holds that for any $x \in \mathbb{R}/\mathbb{Z}$ and $t \in v(\mathbb{R}/\mathbb{Z})$,

$$|\chi(x, t, \lambda)|, |\partial_t \chi(x, t, \lambda)|, |\partial_x \chi(x, t, \lambda)| < C_1 \lambda^{-1},$$

where $C_1$ is a constant. If $(x, t)$ satisfies

$$\left\{\begin{array}{l}
g_N(x, t) = t - v(x) + \chi = 0, \\
\partial_x g_N(x, t) = -v'(x) + \partial_x \chi = 0,
\end{array}\right.$$

then such $t$ has small measure. In fact, since $v$ has finitely many critical points,

$$\{v(x) : |v'(x)| < \zeta_0\}$$
is contained in finitely many intervals, which are extremely small for \( \zeta \) small enough. Let \( \lambda \) satisfy \( \lambda > C_1\zeta_0^{-1} \). Then the measure of \( v(x) \) satisfies the second equality must be contained in finitely many small intervals. Then by the first equality, \( t \) must be contained in finitely many small intervals. Therefore Condition (T0) and Condition (T1) hold for \( t \) outside finitely many small intervals. Since our result concerns about the measure estimate, excluding a small set is allowed.

Now we are going to verify Condition (T3). As above, we only show it for \( t \) outside a set of small measure. For \( x \) satisfying \( t - v(x) = 0 \), we have a solution \( c = c(t) \) of \( g_N(c, t) = 0 \) near \( x \) for large \( \lambda \). Let \( c = c_j(t) \) be the solution of \( g_N(c, t) = 0 \) and \( x = x_j(t) \) be the solution of \( t - v(x) = 0 \) near \( c_j(t) \) for \( j \in [0, m] \cap \mathbb{Z} \). Let \( K \) be any subset of \( \{ \{i, j\} : 1 \leq i, j \leq m, \ i \neq j \} \). And define

\[
W_K = \left\{ \{ t \in [0, 1]: x'_i(t) = x'_j(t), \ {i, j} \in K \}, \ K \neq 0; \{ t \in [0, 1]: x'_i(t) \neq x'_j(t), \ \forall i, j \}, \ K = 0; \right\}
\]

\[
U_K = \left\{ \{ t \in [0, 1]: |x'_i(t) - x'_j(t)| > \zeta, \ {i, j} \notin K, \ i \text{ or } j \in \cup K \}, \ K \neq 0; \{ t \in [0, 1]: |x'_i(t) - x'_j(t)| > \zeta, \ \forall i, j \}, \ K = 0. \right\}
\]

Recall the condition that \( v'(x) \) takes the same value at most two points in \( v^{-1}(t) \) and \( v'(x(t)) \cdot x'(t) = 1 \). If \( v'(x_{i_0}) = v'(x_{j_0}) \), then \( x'_{i_0}(t) = x'_{j_0}(t) \neq x'_i(t) \) for \( i \in [1, m] \setminus \{i_0, j_0\} \). This implies that

\[
\text{Leb}(W_K \cap U_K(\zeta)) \rightarrow \text{Leb}(W_K)
\]

as \( \zeta \) goes to zero. In fact, since \( W_K \setminus U_K(\zeta') \subset W_K \setminus U_K(\zeta'') \) for \( \zeta' < \zeta'' \), we have

\[
\lim_{\zeta \to 0} \text{Leb}(W_K \setminus U_K(\zeta)) \rightarrow \text{Leb}(\cap_{\zeta > 0} W_K \setminus U_K(\zeta)) = 0.
\]

Note that \( \cup K W_K = [0, 1] \). Hence

\[
\text{Leb}(\cup_K (W_K \cap U_K(\zeta))) \rightarrow 1 \tag{5.4}
\]

as \( \zeta \) goes to zero. To obtain Condition (T3), we need the following lemma.

**Lemma 5.3.** Assume that \( |x'_i(t) - x'_j(t)| > \zeta \). Then for \( \lambda \) sufficiently large depending on \( \zeta \) and \( \zeta_0 \),

\[
|c'_i(t) - c'_j(t)| > \frac{\zeta}{4}.
\]

**Proof.** Note that \( v'(x) > \zeta_0 \). Then the function \( y = v(x) \) has \( C^1 \) smooth inverse function \( x = v^{-1}(y) \). And the derivative of \( v^{-1} \) is \( \frac{1}{v'(x)} \). Note that \( |v(x_i) - v(c_i)| = |\chi| < C_1 \lambda^{-1} \). Then for \( \lambda \) sufficiently large depending on \( \zeta \),

\[
|\frac{1}{v'(x_i)} - \frac{1}{v'(c_i)}| < \frac{\zeta}{4}.
\]

Similar estimate holds for \( j \). Then

\[
|\frac{1}{v'(c_i)} - \frac{1}{v'(c_j)}| = \left| \frac{1}{v'(x_i)} - \frac{1}{v'(x_j)} \right| - \left| \frac{1}{v'(c_i)} - \frac{1}{v'(c_j)} \right| < \frac{\zeta}{2} \geq \frac{\zeta}{2}.
\]

We also have \( c'_i(t)(v'(c_i) + \partial_x \chi) = -1 - \partial_t \chi \), which implies that \( c'_i(t)v'(c_i) = 1 + O(\lambda^{-1}) \) for large \( \lambda \) depending on \( \zeta_0 \). Then the lemma is proved for sufficiently large \( \lambda \) depending on \( \zeta \) and \( \zeta_0 \). \( \square \)
Combining this lemma and (5.4), we obtain that if $\lambda$ sufficiently large, then Condition (T3) holds for $t$ to a set of small measure.

What remains to do is to show we only need $v$ to be $C^1$ smooth in the induction process. For any $c_i(t) \in C(t)$, we have $g_N(c_i(t), t) = 0$. Take derivative with respect to $t$ on both sides, we get

$$\partial_x g_N(c_i(t), t)c'_i(t) = -1 - \partial_x \chi < -\frac{1}{2},$$

for $\lambda$ sufficiently large. Hence it follows that $\partial_x g_N(c_i(t), t)$ and $c'_i(t)$ have different signs. If $c'_i(t) \cdot c'_j(t) > 0$, then

$$\partial_x g_N(c_i(t), t) \cdot \partial_x g_N(c_j(t), t) > 0.$$  \hspace{0.5cm} (5.5)

Now consider $\cap_{(i,j) \in K} E_{i,j}$ for arbitrary $K \subset \{(i, j) : 1 \leq i, j \leq m, i \neq j\}$. Define

$$\tilde{E}_{i,j} = \{t \in [0, 1] : c'_i(t) \cdot c'_j(t) > 0\}.$$  \hspace{0.5cm} (5.6)

Since $c'_i(t) = -\partial_t g_N(\partial_x g_N)^{-1}$ and $\partial_t g_N \neq 0$, we have $c'_i(t) \cdot c'_j(t) \neq 0$. Then $E_{i,j} \subset \tilde{E}_{i,j}$. Note that $\tilde{E}_{i,j}$ is an open set. Then

$$\tilde{E}_{i,j} \cap \left( \bigcap_{(i,j) \notin K} F_{i,j} \right)$$

is also an open set. Now we follow the procedure in Subsection 5.2 by replacing $\cap_{(i,j) \notin K} F_{i,j}$ by (5.6). Note that for each $t \in \tilde{E}_{i,j} \cap (\bigcap_{(i,j) \notin K} F_{i,j})$, we have (5.5). Then we can inductively redefine $C(n)$ with

$$\partial_x g_n(c_{n,i}(t), t) \cdot \partial_x g_n(c_{n,j}(t), t) > 0.$$ \hspace{0.5cm} (5.7)

By Remark 2.4 and Remark A.2, we only need the $C^1$ estimates in the induction process. Then we can follow Subsection 5.2 to prove Theorem 1.5.

5.5. Proof of corollary 1.8. Now we consider the Schrödinger cocycle with the form stated in Theorem 1.2. Assume that $v$ is real analytic on $\{|x| < h\}$. We will show that the cocycle satisfies Condition (T4), i.e. $\text{Leb}(T) = 0$.

By Appendix A.1 in [16] or Section 5.3 in [18], it is not hard to see that

$$g_N(x, t) = t - v(x) + \chi(x, t, \lambda),$$

where $\chi$ is analytic on $x$ and $t$, and is continuous on $\lambda > 0$. Moreover, it holds that for any $x \in \{|x| < h\}$ and $t \in v(\mathbb{R}/\mathbb{Z})$,

$$|\chi(x, t, \lambda)| < C_2 \lambda^{-1},$$

where $C_2$ does not depend on $x$, $t$ and $\lambda$. Due the analyticity of $g_N$ on $x$, it is easy to see that Condition (T0) holds and Condition (T1) holds except for finitely many $t$. Moreover, since $v$ is analytic, $\{x : v'(x) = 0\}$ is a finite set. There exist an open interval $J$ and a positive number $\zeta$ such that for each $t \in J$, $\exists x$ satisfying $t = v(x)$ and $v'(x) > \zeta > 0$. Let $\lambda$ be large enough such that $C_2 \lambda^{-1} < \zeta$. By the Implicit Function Theorem, we can obtain $x = x(t)$ is analytic with respect to $t$. Hence for $c \in C$, $c(t)$ is analytic on $J$. Moreover, since $c(t)$ is monotone on $J$, $c(J)$ is also an open interval.

Suppose $\text{Leb}(T) > 0$. Note that Condition (T0) gives that $|C|$ is finite. Then there exists $i, j$ such that $\text{Leb}(E_{i,j}) > 0$. Hence $E_{i,j}$ has an accumulation point. Since $c'_i(t) \cdot c'_j(t)$ is analytic, we get that $c'_i(t) - c'_j(t) = 0$ for all $t \in J$. Then there exists a constant $T_1$ satisfying $0 < T_1 < 1$ and

$$c_i(t) = c_j(t) + T_1, \quad t \in J.$$
Now we consider the function
\[ h(x) = v(x) - v(x + T_1). \]
We will prove \( h(x) \equiv 0 \) on \( x \in c_j(J) \). If not, then \( h \) has finitely many zeros in \( \{ x : \Re x \in c_j(J), |\Re x| < \epsilon \} \). Denote the set of the zeros by \( X \). Let \( K \) be a compact subset of \( \{ x : \Re x \in c_j(J), |\Re x| < \epsilon \} \). By Lojasiewicz inequality, there exist positive constants \( \gamma \) and \( C_3 \) such that
\[ |h(z)| \geq C_3 (\text{dist}(z, X))^\gamma, \quad z \in K. \]
Let \( f(x) = \chi(x, t, \lambda) - \chi(x + T_1, t, \lambda) \). Then we have
\[ C_2 (\text{dist}(c_j(t), X))^{\gamma} \leq |h(c_j(t))| = |f(c_j(t))| < 2C_2\lambda^{-1}. \]
Hence
\[ \text{Leb}(c_j(J)) < \# X \cdot \left( \frac{2C_2}{C_3\lambda} \right)^{-\frac{1}{\gamma}}. \]
This leads to contradiction since the left hand side of the above inequality is a positive constant and the right hand side can be extremely small by choosing \( \lambda \) sufficiently large. Hence we have \( h(x) \equiv 0 \) on \( x \in c_j(J) \). By the analyticity of \( h \), it holds that
\[ v(x + T_1) \equiv v(x), \quad x \in \mathbb{R}/\mathbb{Z}. \]
This implies that \( v \) has a period \( T_1 \).

Note that \( v \) cannot have infinitely many periods which are smaller than 1. Otherwise, \( v \) must be constant. Assume that \( v \) has a minimal period \( T_0 > 0 \). Let \( w(x) := v(T_0x) \). Then \( w \) has a minimal period 1. Let
\[ B(x) = \begin{pmatrix} E - \lambda w(x) & -1 \\ 1 & 0 \end{pmatrix}. \]
We consider the cocycle \((\alpha, B)\). Then the corresponding \( \mathcal{T} \) satisfies \( \text{Leb}(\mathcal{T}) = 0 \). In fact, if \( \text{Leb}(\mathcal{T}) > 0 \), then by the above discussion we get that \( w \) has another period \( 0 < T_2 < 1 \). Then \( v \) has a period \( T_0 \cdot T_2 \), contradicting the minimality of \( T_0 \). Hence the cocycle \((\alpha, B)\) satisfies Condition (T4). And by Theorem 1.3, we have (1.2) for \((\alpha, B)\). Moreover, direct computation shows that
\[ L(\alpha, B) = L(\alpha, A). \]
This finishes the proof of Corollary 1.8.

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Appendix A: The properties of angle functions. The following lemma gives a delicate geometric description of the type III function.

Lemma A.1. Assume that the function \( f \) is of type III. Let \( c_j \) be the zero of \( f_j \). Then \( f \) has at most 2 zeros in \( I \). And the minimal points of \( f \), denoted by \( d_1 \) and \( d_2 \) (probably \( d_1 = d_2 \)), satisfy
\[ |d_j - c_j| < l_0^{-\frac{1}{2}}, \quad j = 1, 2, \]
\[ |f'(d_2)| > l_0^{\frac{1}{2}}. \]
Moreover,
\[ \| f - f_1 \|_{C^2} < Cl_0^{-\frac{3}{2}}, \quad \text{on} \ B(c_1, \frac{r}{4}) \setminus B(c_2, l_0^{-\frac{3}{2}}). \quad (A.1) \]
Furthermore,  

- If \( f'_1(c_1)f'_2(c_2) > 0 \), then \( f \) is monotone with  
  \[
  f'(x) \cdot f'_1(c_1) > 0, \quad x \in I
  \]  
  and has exactly 2 zeros in \( I \).

- If \( f'_1(c_1)f'_2(c_2) < 0 \), then we have the following bifurcation, see Figure 3. There exists \( \eta_0 \) satisfies \( r^2 l_0^{-1} < \eta_0 < r^{-2} l_0^{-1} \) such that
  - if \( |c_1 - c_2| > \eta_0 \), then \( f \) has two zeros.
  - if \( |c_1 - c_2| = \eta_0 \), then \( f \) has exactly one zero \( d_1 = d_2 \) and \( g'(d_1) = 0 \).
  - if \( 0 \leq |c_1 - c_2| < \eta_0 \), then \( f \) has no zeros and
  \[
  \min_{x \in I} |f(x)| \geq r^{-2} l_0^{-1} + r^{-2}|c_1 - c_2|.
  \]

**Remark A.2.** Note that the angle function \( f \) discussed above is \( C^2 \) smooth. However, if \( f \in C^1(I, \mathbb{R}P^1) \), then we can also define type I function and type III function as in Definition 2.1 by removing the estimate of \( |f''| \) and \( |l''| \). In Lemma A.1, we need \( f \) to be \( C^2 \) smooth only in the case with \( f'_1(c_1)f'_2(c_2) < 0 \). In other words, if \( f \) is \( C^1 \) smooth and \( f'_1(c_1)f'_2(c_2) > 0 \), then Lemma A.1 also holds.

**Proof.** Most of the results are proved in Lemma 6 in [16] except for the case with \( f'_1(c_1)f'_2(c_2) > 0 \). Now we assume \( f'_1(c_1)f'_2(c_2) > 0 \). Note  
\[
  f' = (1 + l^4 \tan^2(f_1))^{-1}(2l/l \tan(f_1) + l^2 \cos^{-2}(f_1)f'_1) + f'_2.
\]

If \( x \) locates far from \( c_1 \), then the first item on the right-hand side is very small and \( f'(x) \approx f'_2(x) \). If \( x \) locates near \( c_1 \), then \( |2l/l \tan(f_1)| < l^2 \cos^{-2}(f_1)f'_1 | \) This implies that \( f' \) and \( f'_1 \) have the same sign. Hence in both cases, we obtain that \( f \) is monotone with (A.2). Moreover, the function \( \arctan(l^2 \tan(f_1)) \) maps \((c_1 - \delta_1, c_1 + \delta_1)\) onto \((-\frac{\pi}{2} + \delta_2, \frac{\pi}{2} - \delta_2)\), where \( \delta_1 \gg l_0^{-1} \) and \( 0 < \delta_2 < l_0^{-1} \). This implies that \( f \) has exactly 2 zeros in \( I \). \( \square \)

No matter which type of the angle function is, the angle function satisfies the following non-degenerate property.

**Lemma A.3.** Let \( f : I \to \mathbb{R}P^1 \) be of type I or III. Define  
\[
  X = \left\{ x \in I : |f(x)| = \min_{y \in I} |f(y)| \right\} = \left\{ \begin{array}{ll}
  \{x_0\}, & \text{if } f \text{ is of type I;} \\
  \{x_1, x_2\}, & \text{if } f \text{ is of type III.}
  \end{array} \right.
\]

In case \( f \) is of type III, we further assume \( d := |x_1 - x_2| < \frac{\pi}{3} \). Then for any \( 0 < r' < r \), we have that  
\[
  |f(x)| > cr'^3, \quad \text{for } x \notin B(X, r').
\]
For the case that $f$ is of type III, we have the same estimate for $C\lambda^{-\frac{1}{2}} < \rho' < \rho$ if $d \geq \frac{1}{2}$.

Proof. Most of the results are proved in Corollary 3 in [16] except for the type III function $f$ with $f_1^*(c_1)f_2^*(c_2) > 0$. And this is obvious from Lemma A.1. \hfill \Box

Appendix B: Arithmetic properties of the frequencies. Let $\{p_n/q_n\}$ be the continued fraction approximant of $\alpha$. It is well known that $\{p_n/q_n\}$ satisfies

$$\frac{1}{q_n(q_{n+1}+q_n)} \leq \left| \frac{\alpha - p_n}{q_n} \right| \leq \frac{1}{q_nq_{n+1}}. \quad (B.1)$$

Lemma B.1. Let $\alpha$ be a Diophantine number as in (1.1), $c \in \mathbb{R}/\mathbb{Z}$ and $x \in B(c,q^{-3r})$. Then for $k$ satisfying $0 < |k| < q_n^2$, we have $x + k\alpha \notin B(c,q^{-3r})$.

Proof. It holds that $\|k\alpha\| \geq \gamma |k|^{-r} \geq \gamma q_n^{-2r}$ for $0 < |k| < q_n^2$. Then

$$|x - c + k\alpha| \geq \|k\alpha\| - |x - c| \geq \gamma q_n^{-2r} - q_n^{-3r} > q_n^{-3r}.$$ 

This implies the result. \hfill \Box

Lemma B.2. Let $q_l = \min\{q_l : q_l > q_n^{4r}\}$. Denote $I = B(c,q^{-3r})$. Let $x$ be an arbitrary point in $\mathbb{R}/\mathbb{Z}$. Then

$$\min\{i \in \mathbb{N} : x + i\alpha \in I\} < q_l.$$ 

Proof. By (B.1), it holds that

$$\left\| i\alpha - \frac{ip_l}{q_l} \right\| < \frac{i}{q_lq_{l+1}} < \frac{1}{q_{l+1}} < \frac{1}{q_l}, \quad 0 < i < q_l.$$ 

Since $q_l^{-1} < q_n^{-4r} \ll q_n^{-3r}$, there exists $i_0$ such that $0 < i_0 < q_l$ and

$$x + i\alpha \in \left( x + \frac{i_0p_l}{q_l} - \frac{1}{q_l} , x + \frac{i_0p_l}{q_l} + \frac{1}{q_l} \right) \subseteq I.$$ 

This implies the result. \hfill \Box

Appendix C: Proof of theorem 3.1. In this section, we will give a detailed proof of Theorem 3.1.

C.1. The started step. Consider the forward return time $r^+_N(x)$. If $r^+_N(x)$ is the actual first return time to $I_{N,i}$, then $r^+_N(x) \geq q_N$. Otherwise, $r^+_N(x)$ is the second return time to $I_{N,i}$ of $x$. By Lemma B.1, $r^+_N(x) \geq q_N^2$. In this case there exists a $k \in \mathbb{Z}$ such that $0 < k < q_N$ and $x + ka \in I_N$. Due to Assumption (N), there exists $j$ such that $(i,j) \in P$ and $(I_{N,i} + ka) \cap I_{N,j} \neq \emptyset$. Similar discussion for $r^+_N(x)$. We call the case is nonresonant, if for all $x \in I_{N,i}$, both $r^+_N(x)$ and $r^-_N(x)$ are the actual first return times. Otherwise, we call the case is resonant.

For $x \in I_N$, assume that $x + ja \notin I_N$ for $0 < j < l$. Now we are going to verify the conditions of Lemma 2.1. Since $A$ is $C^2$ smooth on $x$, then $g_N$ is $C^2$ smooth. We can choose $\lambda$ large enough such that

$$|s_1(x + ja) - u_1(x + ja)| = |g_N(x + ja)| \geq \xi/2 \gg \lambda^{-1},$$

$$\left\| \frac{d^\nu}{dx^\nu} g_N(x) \right\| \leq C \left\| \frac{d^\nu}{dx^\nu} A(x) \right\| \leq C \ll \lambda, \quad \nu = 1, 2.$$
By Lemma 2.1, one can obtain that
\[\|s_l - s_1\|_{C^2} < \lambda^{-2-\eta(N)}, \quad \|u_l - u_1\|_{C^2} < \lambda^{-2-\eta(N)},\]
\[\|A_l\| \geq (\xi\lambda/2)^l, \quad \left|\frac{d^{\nu}}{dx^\nu} A_l(x)\right| \leq \|A_l\|^{1+\nu\eta N^{-1}}, \quad \nu = 1, 2.\]

C.1.1. The nonresonant case. Suppose that for all \(x \in I_{N,i}\), both \(r^+_N(x)\) and \(r^-_N(x)\) are the actual first return times. Then \(r^+_N, r^-_N \geq q_N\). And we have
\[\|s_{r^+_N} - s_1\|_{C^2} < \lambda^{-\frac{\pi}{2}}, \quad \|u_{r^+_N} - u_1\|_{C^2} < \lambda^{-\frac{\pi}{2}},\]
\[\|A_{r^+_N}(x)\| \geq \lambda^{r^+_N+1}, \quad \left|\frac{d^{\nu}}{dx^\nu} A_{r^+_N}(x)\right| \leq \|A_{r^+_N}\|^{1+\nu\eta N^{-1}}, \quad \nu = 1, 2.\]

Here we need \(\lambda > C(\varepsilon, q_N)\). Define the \((N+1)\)th angle function as
\[g_{N+1}(x) := s_{r^+_N}(x) - u_{r^-_N}(x), \quad \text{on } I_{N,i}.\]

Hence we have
\[\|g_{N+1} - g_N\|_{C^2} < \lambda^{-\frac{\pi}{2}}, \quad \text{on } I_{N,i}.\]
Since \(g_N\) is a type I function, \(g_{N+1}\) is also a type I function. Denote the zero of \(g_{N+1}\) on \(I_{N,i}\) by \(c_{N+1,i}\). Then one has
\[|c_{N+1,i} - c_i| < \lambda^{-\frac{\pi}{2}}.\]

C.1.2. The resonant case. Suppose that there exists \(x \in I_{N,i}\) such that \(\exists k\) satisfying
\[0 < |k| < q_N\] and \(x + k\alpha \in I_N\). Then there exist \(j\) and \(k\) such that \(\{i, j\} \in P\)
\[0 < |k| < q_N\] and \((I_{N,i} + k\alpha) \cap I_{N,j} \neq \emptyset\). Without loss of generality, we assume \(k > 0\). We are going to define and estimate \(g_{N+1}\) on \(I_{N,i}\) and \(I_{N,j}\).

By the nonresonant case, we have for \(l = k, r^+_N, r^-_N,\)
\[\|s_l - s_1\|_{C^2} < \lambda^{-\frac{\pi}{2}}, \quad \|u_l - u_1\|_{C^2} < \lambda^{-\frac{\pi}{2}},\]
\[\|A_l\| \geq (\xi\lambda/2)^l, \quad \left|\frac{d^{\nu}}{dx^\nu} A_l(x)\right| \leq \|A_l\|^{1+\nu\eta N^{-1}}, \quad \nu = 1, 2.\]

Then by the resonant case in Lemma 2.1 we have
\[\|A_{r^+_N}(x)\| \geq (\xi\lambda/2)^{r^+_N}, \quad (C\lambda)^{-k} \geq \lambda^{r^+_N+1},\]
\[\left|\frac{d^{\nu}}{dx^\nu} A_{r^+_N}(x)\right| \leq \|A_{r^+_N}\|^{1+\nu\eta N^{-1}}, \quad \nu = 1, 2.\]

Define the angle function as
\[g_{N+1}(x) := s_{r^+_N}(x) - u_{r^-_N}(x), \quad \text{on } I_{N,i}.\]

By Lemma 2.1 and its remark, one may take
\[s_{r^+_N}(x) = \arctan(\|A_k(x)\|^2 (s_{r^-_N}(x + k\alpha) - u_k(x + k\alpha))) - \frac{\pi}{2} + s_k(x).\]
And hence
\[g_{N+1}(x) = \arctan(g'_N(x + k\alpha)) - \frac{\pi}{2} + g''_N(x), \quad x \in I_{N,i},\]
where \(l_k = \|A_k(x)\|, \quad g'_N = s_{r^-_N}(x + k\alpha) - u_k(x + k\alpha)\) and \(g''_N = s_k - u_{r^-_N}(x)\). Similarly,
\[g_{N+1}(x) = \arctan((l'_k g'_N(x + k\alpha)) - \frac{\pi}{2} + g''_N(x), \quad x \in I_{N,j}.\]

Note that \(\|g'_N - g_N\|_{C^2}, \|g''_N - g_N\|_{C^2} < \lambda^{-\frac{\pi}{2}}.\) Then \(g'_N\) and \(g''_N\) are type I functions. Hence \(g_{N+1}\) is a type III function. By Lemma 2.1, we have
\[\|g_{N+1}\|_{C^2} \leq \lambda^{5q_N}.\]
And by Lemma A.1, we have the properties of $g_{N+1}$ in the following.

If $\frac{d}{dx}g_N(c_{N,i}) \frac{d}{dx}g_N(c_{N,j}) > 0$, then $g_{N+1}$ is monotone and has exactly 2 zeros in $I_{N,i}$. This is also true for $I_{N,j}$. Since $g_{N+1}$ is monotone with

$$\frac{d}{dx}g_{N+1}(x) \frac{d}{dx}g_N(c_{N,i}) > 0,$$

then we also have

$$\frac{d}{dx}g_{N+1}(c_{N+1,i}) \frac{d}{dx}g_N(c_{N+1,j}) > 0.$$

Similarly, if $\frac{d}{dx}g_N(c_{N,i}) \frac{d}{dx}g_N(c_{N,j}) < 0$, then $g_{N+1}$ may has 0, 1 or 2 zeros in $I_{N,i}$. This is also true if we replace $i$ by $j$ and replace $j$ by $i$ simultaneously. Moreover, if $g_{n+1}$ has 0 or 1 zero in $I_{N,i}$, then

$$\frac{d}{dx}g_{N+1}(c_{N+1,i}) = 0.$$

Otherwise, $g_{n+1}$ has 2 zeros in $I_{N,i}$. By (A.1) and the choice of $c_{n,i}$ we have

$$\frac{d}{dx}g_{N+1}(c_{N+1,i}) \frac{d}{dx}g_N(c_{N+1,j}) \leq 0.$$

Denote the zeros of $g_{N+1}$ in $I_{N,i}$ by $c_{N+1,i}$ and $c'_{N+1,j}$ such that $|c_{N+1,i} - c_{N,i}| < \lambda^{-\frac{1}{2}}$. Similarly, denote the zeros of $g_{N+1}$ in $I_{N,j}$ by $c_{N+1,j}$ and $c'_{N+1,i}$ such that $|c_{N+1,j} - c_{N,i}| < \lambda^{-\frac{1}{2}}$. We say $c'_{N+1,i}$ and $c'_{N+1,j}$ are the extra critical points of $g_{N+1}$. Then we have the following important property.

**Lemma C.1.** If $|g_N(c_{N,i})| \leq CL^{-\frac{1}{3}}r_{N+1,i}$ or $|g_N(c_{N,j})| \leq CL^{-\frac{1}{3}}r_{N+1,i}$, then the extra critical points are nearly on the orbit of the critical points such that

$$|c_{N+1,i} + k\alpha - c'_{N+1,i}|, |c_{N+1,j} - k\alpha - c'_{N+1,j}| < \lambda^{-\frac{1}{2}}r_{N+1,i}.$$

**Proof.** This proof is the same as the proof of Lemma 7 in [16].

With this lemma, we do not need to worry about the extra critical points. In fact, we only consider the original critical points and regard the extra critical points as the $k$-orbit of the original critical points. Therefore we can keep a fixed number of the critical points as in [17].

C.2. The induction step. Suppose we have finished step $n - 1 \geq N$. We are going to push the induction forward to $n$. Similar to the $N$th step. We call the case is nonresonant, if for all $x \in I_{n,i}$, both $r^+_{n,i}(x)$ and $r^-_{n,i}(x)$ are the actual first return time. In this case, $r_{n,i} \geq q_n$. Otherwise, we call the case is resonant. Then there exist $j$ and $k$ such that $\{i, j\} \in P, 0 < |k| < q_n$ and $(I_{n,i} + k\alpha) \cap I_{n,j} \neq \emptyset$. In this case, $r_{n,i} \geq q_n^2$.

To study the $(n+1)$th angle function $g_{n+1}$, we need to go back to consider the $(n - 1)$th induction step. It is divided into two cases: the nonresonant case and the resonant case.

C.2.1. The $(n - 1)$th nonresonant case. First we deal with the $n$th nonresonant case. Suppose that for all $x \in I_{n,i}$, both $r^+_{n,i}(x)$ and $r^-_{n,i}(x)$ are the actual first return times. Let $0 = j_0 < j_1 < j_2 < \cdots < j_p = r^+_{n,i}$ be the return times of $x$ to $I_{n-1}$. Consider the decomposition of $A^+_{n,i}(x)$ in the following.

$$A^+_{r^+_{n,i}}(x) = A_{j_p-j_{p-1}}(x + j_{p-1}\alpha) \cdots A_{j_2-j_1}(x + j_1\alpha)A_{j_1}(x).$$
Note that in the previous step, we have the following
\[
\|s_{j+1 - j} - s_{r_{n-1}^{-}}\| C^2, \|u_{j+1 - j} - u_{r_{n-1}^{-}}\| C^2 < \lambda^{-\frac{2}{3}q_{n-1}},
\]
\[
\|A_{j+1 - j} \| \geq \lambda^{\frac{2}{3}q_{n-1}}, \quad \left| \frac{d^\nu}{dx^\nu} \right| A_{j+1 - j} \| \| < \|A_{j+1 - j} \|^{1+\nu q_{n-1}}, \quad \nu = 1, 2,
\]
where \( h = h(l) \) satisfies \( x + j\alpha \in I_{n-1,h} \). Then
\[
|s_{j+1 - j}(x + j\alpha) - u_{j+1 - j}(x + j\alpha)| \geq g_n(x + j\alpha) - \lambda^{-\frac{2}{3}q_{n-1}}
\geq Cq_n^{-\nu r}
\geq \lambda^{-q_{n-1}}.
\]
In the second inequality we used \( \lambda^{Cq_{n-1}} > q_n \), which can be derived from the
Diophantine property of \( \alpha \). Hence by Lemma 2.1, we have
\[
\|s_{r_{n,i}} - s_{j\alpha}\| C^2 < \lambda^{-\frac{2}{3}r_{n,1,i}},
\]
\[
\|A_{r_{n,i}}(x)\| \geq (Cq_n^{-\nu r} \lambda^{q_{n-1}}) r_{n,1,i} \geq \lambda^{r_{n,1,i}},
\]
\[
\left| \frac{d^\nu}{dx^\nu} \right| A_{r_{n,i}} \| \| < \|A_{r_{n,i}} \|^{1+\nu q_{n-1}}, \quad \nu = 1, 2.
\]
Then
\[
\|s_{r_{n,i}} - s_{r_{n,1,i}}\| C^2 < \lambda^{-\frac{2}{3}r_{n,1,i}}.
\]
Similarly, if we consider \( A_{-r_{n,i}}(x) \), we can obtain
\[
\|u_{r_{n,i}} - u_{r_{n,1,i}}\| C^2 < \lambda^{-\frac{2}{3}r_{n,1,i}}.
\]
Define \( g_{n+1} := s_{r_{n,i}} - u_{r_{n,i}} \) on \( I_{n,i} \). Then it holds that
\[
\|g_{n+1} - g_{n}\| C^2 < \lambda^{-\frac{2}{3}r_{n,1,i}}, \quad \text{on } I_{n,i}.
\]
Since \( g_n \) is a type I function, \( g_{n+1} \) is also a type I function. Denote the zero of \( g_{n+1} \)
in \( I_{n,i} \) by \( c_{n+1,i} \). And one has
\[
|c_{n+1,i} - c_{n,i}| < \lambda^{-\frac{2}{3}r_{n,1,i}}.
\]
Now we consider the \( n \)th resonant case. It is similar to the \( N \)th resonant case.
For \( x \in I_{n,i} \), there exist \( j \) and \( k \) such that \( \{i, j\} \in P, \; q_{n-1} < |k| < q_n \) and
\( (I_{n,i} + k\alpha) \cap I_{n,j} \neq \emptyset \). By the same argument above, we have for \( l = k, r_{n,i}^{-}, k, r_{n,j}^{-}, \)
\[
\|s_{j} - s_{r_{n-1,i}^{-}}\| C^2 < \lambda^{-\frac{2}{3}r_{n-1,i}}, \quad \|u_{j} - u_{r_{n-1,i}^{-}}\| C^2 < \lambda^{-\frac{2}{3}r_{n-1,i}},
\]
\[
\|A_{l} \| \geq (Cq_n^{-\nu r} \lambda^{q_{n-1}}) l, \quad \left| \frac{d^\nu}{dx^\nu} \right| A_{l} \| \| \leq \|A_{l} \|^{1+\nu q_{n-1}}, \quad \nu = 1, 2.
\]
Then we have
\[
\|A_{r_{n,i}}(x)\| \geq (Cq_n^{-\nu r} \lambda^{q_{n-1}}) r_{n,i}^{-}, \quad \left| \frac{d^\nu}{dx^\nu} \right| A_{r_{n,i}}(x) \| \| \leq \|A_{r_{n,i}}(x) \|^{1+\nu q_{n}}, \quad \nu = 1, 2.
\]
And
\[
\left| \frac{d^\nu}{dx^\nu} \right| A_{r_{n,i}}(x) \| \| \leq \|A_{r_{n,i}}(x) \|^{1+\nu q_{n}}, \quad \nu = 1, 2.
\]
Define the angle function as
\[
g_{n+1}(x) := s_{r_{n,i}}(x) - u_{r_{n,i}}(x), \quad \text{on } I_{n,i}.
\]
Again by Lemma 2.1 and its remark, one may take
\[
s_{r_{n,i}}(x) = \arctan(\|A_k(x)\|^{2}\|g_n\| x + k\alpha) - \frac{\pi}{2} + s_k(x), \quad x \in I_{n,i},
\]
and then one has
\[ g_{n+1}(x) := \arctan(\frac{l_k^2}{k} g_n'(x + k\alpha)) - \frac{\pi}{2} + g_n'(x), \quad x \in I_{n,i}, \]
where \( l_k = \| A_k(x) \| \), \( g_n' = s_{r_n-i} - u_k \) and \( g_n'' = s_k - u_{r_n-i} \). Similarly,
\[ g_{n+1}(x) = \arctan(\| A_k(x) \|^2 g_n''(x - k\alpha)) - \frac{\pi}{2} + g_n'(x), \quad x \in I_{n,j}. \]
Hence \( g_{n+1} \) is type III function. And by Lemma A.1, we obtain the corresponding estimates stated in the induction theorem.

Denote the zeros of \( g_{n+1} \) in \( I_{n,i} \) by \( c_{n+1,i,i} \) and \( c_{n+1,i,j} \) such that \( |c_{n+1,i,i} - c_{n,i}| < \lambda^{-\frac{3}{2}r_n-1} \). Similarly, denote the zeros of \( g_{n+1} \) in \( I_{n,j} \) by \( c_{n+1,j,i} \) and \( c_{n+1,j,j} \) such that \( |c_{n+1,j,j} - c_{n,j}| < \lambda^{-\frac{3}{2}r_n-1} \). We say \( c_{n+1,i,i} \) and \( c_{n+1,j,j} \) are the extra critical points of \( g_{n+1} \). Then we have the following important property.

**Lemma C.2.** If \( |g_{n+1}(c_{n+1,i,i})| \leq C\lambda^{-\frac{3}{2}r_n+1} \) or \( |g_{n+1}(c_{n+1,j,j})| \leq C\lambda^{-\frac{3}{2}r_n+1} \), then the extra critical points are nearly on the orbit of the critical points, i.e.
\[ |c_{n+1,i,i} + k\alpha - c_{n+1,j,j}|, |c_{n+1,j,j} - k\alpha - c_{n+1,i,i}| < \lambda^{-\frac{3}{2}r_n+1}. \]

**Proof.** This proof is the same as the proof of Lemma 7 in [16].

C.2.2. The \((n-1)\)th resonant case. For \( x \in I_{n-1,i} \), there exists \( j \) and \( k \) such that \( \{i, j\} \in P \), \( 0 < |k| < q_n-1 \) and \( (I_{n-1,i} + k\alpha) \cap I_{n-1,j} \neq \emptyset \).

We first deal with the \( n \)th nonresonant case. It is similar to the previous subsection. For any \( j \neq i \) and \( 1 < |l| < q_n \), we have \((I_{n,i} + l\alpha) \cap I_{n,j} = \emptyset \). Note that the great change of \( g_{n-1}(x) \) locates in \( I_{n-1,i} + k\alpha \), and hence in \( I_{n,i} + k\alpha \). Then \( g_{n-1} \) is monotone and has only one zero in \( I_{n,i} \), i.e. \( g_{n-1} \) is a type I function in \( I_{n,i} \).

Let \( 0 = j_0 < j_1 < j_2 < \cdots < j_p = r_n \) be the return times of \( x \) to \( I_{n-1} \) such that \( j_t - j_{t-1} \geq q_n \). Consider the decomposition of \( A_{r_n,i}(x) \) in the following.
\[ A_{r_n,i}(x) = A_{j_p-j_{p-1}}(x + j_p\alpha) \cdots A_{j_2-j_1}(x + j_1\alpha)A_{j_1}(x). \]
Then
\[ |s_{j_{i+1-j}}(x + j_i\alpha) - u_{j_{i+1-j}}(x + j_i\alpha)| \geq g_n(x + j_i\alpha) - \lambda^{-\frac{3}{2}q_n}. \]
We are afraid of the possibility that
\[ |x + j_i\alpha - c_{n,h}| < Cq_n^{-3\tau}, \]
for some \( h = h(l) \). Let \( x + j_i\alpha \in I_{n,h} \). Either \( |c_{n,h} - c_{n,h'}| < \lambda^{-\frac{3}{2}q_n} \) so that \( |x + j_i\alpha - c_{n,h'}| < Cq_n^{-3\tau} \), contradicting the trajectory has not entered \( I_n \). Or for some \(|k| < q_n-1 \),
\[ |c'_{n,h} - c_{n,h} - k\alpha| < \lambda^{-\frac{3}{2}q_n}. \]
This implies
\[ |x + j_i\alpha + k\alpha - c_{n,h}| < Cq_n^{-3\tau}. \]
This also contradicts that the trajectory has not entered \( I_n \). Hence by Lemma A.3 we have
\[ |s_{j_{i+1-j}}(x + j_i\alpha) - u_{j_{i+1-j}}(x + j_i\alpha)| \geq g_n(x + j_i\alpha) - \lambda^{-\frac{3}{2}q_n} \geq Cq_n^{-3\tau}. \]
Then the nonresonant case in Lemma 2.1 applies. And the estimates can be obtained similar to the \((n-1)\)th nonresonant case.
Similarly, we can obtain (C.1) on $r_{n-1,i}$. Using the same argument in the $n$th nonresonant case, one obtains

$$
\|A_{r_{n,i}}(x) - A_{r_{n,i} + r_{n-1,i}}(x + r_{n-1,i})\| \geq \lambda_{n,i}^{r_{n-1,i}} \geq \lambda_{n}^{q_{n-1}},
$$

and then

$$
|s_{r_{n,i}}(x + r_{n-1,i}) - u_{r_{n,i}}(x + r_{n-1,i})| \geq \left| g_{n}(x + r_{n-1,i}) \right| - \lambda_{n}^{q_{n-1}} \geq Cq_{n}^{-9r_{n-1,i}} \geq Cq_{n}^{-9r_{n-1,i}} \gg \lambda_{n}^{q_{n-1}}.
$$

By Lemma 2.1, we get

$$
\|s_{r_{n-1,i}} - u_{r_{n,i}}\| < \lambda_{n}^{q_{n-1,i}} \text{ on } I_{n,i}.
$$

Similarly, we have

$$
\|u_{r_{n,i}} - u_{r_{n,i}}\| < \lambda_{n}^{q_{n-1,i}} \text{ on } I_{n,i}.
$$

Define $g_{n+1} = s_{r_{n,i}} - u_{r_{n,i}}$. Hence

$$
\|g_{n+1} - g_{n}\| < \lambda_{n}^{q_{n-1,i}} \text{ on } I_{n,i}.
$$

Similarly, we can obtain (C.1) on $I_{n,j}$. If $|g_{n}(c_{n,i})|, |g_{n}(c_{n,j})| > C\lambda^{-\frac{1}{16}r_{n,i}}$, then by (C.1)

$$
|g_{n+1}(c_{n+1,i})|, |g_{n+1}(c_{n+1,j})| > C\lambda^{-\frac{1}{16}r_{n+1}}.
$$

Moreover, since $g_{n}$ is a type III function, $g_{n+1}$ is also a type III function. Then by Lemma A.1, $g_{n+1}$ satisfies the properties stated in the induction theorem. Moreover, we also have Lemma C.2 in this case.

Until now we have finished the proof of Theorem 3.1.

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