Geometric quantum discord through the Schatten 1-norm

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Quantum discord is an information-theoretic measure of non-classical correlations, initially proposed by Ollivier and Zurek [1], which goes beyond entanglement (i.e., separable states can have nonzero discord) and whose characterization has attracted much attention during the last decade (see Ref. [2] for a review and Ref. [3] for an operational interpretation). From an analytical point of view, the evaluation of quantum discord is a difficult task, even for (general) two-qubit states, since an optimization procedure is required for the conditional entropy over all local generalized measurements. In this scenario, closed expressions are known only for classes of states [4–9].

The difficulty of extracting analytical solutions for quantum discord led Dakić, Vedral, and Brukner to propose a geometric measure of quantum discord [10], which quantifies the amount of quantum correlations of a state in terms of its minimal Hilbert-Schmidt distance from the set of classical states. The calculation of this alternative measure requires a simpler minimization process, which is realizably available for general two-qubit states [11] as well as for arbitrary bipartite states [12, 13]. Moreover, it has been shown to exhibit operational significance in specific quantum protocols (see, e.g., Ref. [14]). Despite those remarkable features, geometric discord is known to be sensitive to the choice of distance measures (see, e.g., Ref. [11]). In turn, as recently pointed out [12, 14], the geometric discord as proposed in Ref. [10] cannot be regarded as a good measure for the quantumness of correlations, since it may increase under local operations on the unmeasured subsystem. In particular, it has explicitly been shown by Piani [14] that the simple introduction of a factorized local ancillary state on the unmeasured party changes the geometric discord by a factor given by the lack of purity of the ancilla. This is in contrast with the entropic quantum discord, which does not suffer this problem. From a technical point of view, the root of this drawback is the lack of contractivity of geometric discord under trace-preserving quantum channels. Remarkably, this is strongly connected with the norm adopted to define distance in the state space.

Most recently, Tufarelli et al. [13] have introduced a modified version of geometric discord that is immune to the particular ancilla considered in Ref. [10]. However, since this measure is also based on Hilbert-Schmidt distance, it inherits the noncontractivity problem (see, e.g., examples in Ref. [12]). A way to circumvent this issue is to employ the trace distance in place of the Hilbert-Schmidt norm [12, 16, 17]. In this direction, we consider the generalization of the geometric discord in terms of Schatten p-norms. More specifically, we show that the geometric discord as defined by the 1-norm is the only p-norm geometric discord invariant under the class of channels considered in Ref. [12]. Furthermore, by restricting the minimization to states in the Bell-diagonal form, we analytically evaluate the 1-norm geometric discord for arbitrary Bell-diagonal two-qubit states. As an illustration, we compare our result with the entropic quantum discord and the 2-norm geometric discord, analyzing its monotonicity properties as a function of the correlation functions.

Entropic and geometric measures of quantum discord.
Quantum discord has been introduced as an entropic measure of quantum correlation in a quantum state. For a bipartite system described by the density matrix \( \rho \), it is defined by the difference \( Q(\rho) = I(\rho) - J(\rho) \), where \( I(\rho) \) is the quantum mutual information, which represents the total correlation in \( \rho \) [18], and \( J(\rho) \) is the measurement-based mutual information, which can be interpreted as the classical correlation in \( \rho \) [18]. These quantities are given by \( I(\rho) = S(\rho_a) + S(\rho_b) - S(\rho) \) and \( J(\rho) = S(\rho_b) - \min\{E_k\} [\sum_{k} p_k S(\rho_{b|k})] \). In these expressions, \( S(\rho) = -\text{tr} [\rho \log \rho] \) denotes the von Neumann entropy, \( p_{a(b)} \) is the reduced density matrix of the subsystem \( a(b) \), and the minimum is taken over all positive operator-valued measures (POVMs) \( \{E_k\} \) on subsystem \( a \), where \( \rho_{b|k} = \text{tr}_a [E_k \rho] / p_k \) is the post-measurement state of \( b \) after the outcome \( k \) on \( a \) is obtained with probability \( p_k = \text{tr} [E_k \rho] \).

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Bell diagonal states, whose density operator presents the four Bell states. Quantum discord is a maximum (Q = 1 and $D_G = 1/2$) in these vertices and vanishing ($Q = D_G = 0$) over the perpendicular axes $c_1$, $c_2$, and $c_3$ (dashed lines).

The analytical minimization over POVMs involved in $J(\rho)$ constitutes a hard task, even for two-qubit systems in a general state. This motivated the introduction of an alternative measure $\mathcal{B}$, which was named geometric quantum discord. Such a geometric measure is based on the distance between the given quantum state $\rho$ and the closest classical-quantum state $\rho_c$, reading

$$D_G(\rho) = \min_{\Pi_c} \| \rho - \rho_c \|^2_2,$$

where $\|X\|_2 = \sqrt{\text{tr}[X^\dagger X]}$ is the Hilbert-Schmidt norm (2-norm) and $\Omega_b$ is the set of classical-quantum states, whose general form is given by

$$\rho_c = \sum_k p_k \Pi_k^a \otimes \rho_k^b,$$

with $0 \leq p_k \leq 1$ ($\sum_k p_k = 1$), $(\Pi_k^a)$ denoting a set of orthogonal projectors for subsystem $a$, and $\rho_k^b$ being a general reduced density operator for subsystem $b$. Note that extremization here is over a distance measure rather than POVMs, as in $J(\rho)$. In terms of the entropic quantum discord $Q(\rho)$ and of the negativity of entanglement $N(\rho) = \| \rho^{sa} \|_1 - 1$, where $\rho^{sa}$ denotes partial transposition of $\rho$ with respect to subsystem $a$ and $\|X\|_1 = \text{tr} \left[ \sqrt{X^\dagger X} \right]$ is the trace norm, the geometric discord presents the following bound for two-qubit states $\mathcal{B}$:

$$2D_G(\rho) \geq Q^2(\rho), N^2(\rho).$$

The inequality $2D_G \geq N^2$ is not universal, with counterexamples in spaces of dimension higher than $2 \times 2$.

We will focus here in the particular case of two-qubit Bell diagonal states, whose density operator presents the form

$$\rho = \frac{1}{4} [I \otimes I + \hat{c} \cdot (\hat{\sigma} \otimes \hat{\sigma})],$$

where $I$ is the identity matrix, $\hat{c} = (c_1, c_2, c_3)$ is a three-dimensional vector and $\hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector formed by Pauli matrices. In this case, the entropic quantum discord and the geometric discord are given by

$$Q = \log_2 \frac{4\lambda_0 \lambda_1 \lambda_{10} \lambda_{11}}{(1 - c_+) \frac{1}{c_+} (1 + c_+)}$$

and

$$D_G = \frac{1}{4} (c_-^2 + c_0^2),$$

where $\lambda_{ij} = \frac{1}{4} [1 + (-1)^i c_1 + (-1)^{i+j} c_2 + (-1)^j c_3] / 4$ are the eigenvalues of the density operator $\rho$, whereas $c_+ = \max \{ |c_1|, |c_2|, |c_3| \}$, $c_0 = \text{int} \{ |c_1|, |c_2|, |c_3| \}$, and $c_- = \min \{ |c_1|, |c_2|, |c_3| \}$ represent the maximum, intermediate, and minimum among the absolute values of the correlation functions $c_1, c_2,$ and $c_3$, respectively. If $\rho$ describes a physical state, then $0 \leq \lambda_{ij} \leq 1$ and $\sum_{ij} \lambda_{ij} = 1$. In this condition, the vector $\hat{c}$ must be restricted to the tetrahedron whose vertices situated on the points $(1,1,-1)$, $(1,-1,1)$, and $(-1,1,1)$ represent the Bell states (see Fig. 1). Quantum discord is a maximum ($Q = 1$ and $D_G = 1/2$) in these vertices and minimum ($Q = D_G = 0$) over the perpendicular axis $c_1, c_2,$ and $c_3$ (dashed lines).

Geometric quantum discord and Schatten $p$-norms. Despite being easier to compute and exhibiting an interesting geometric interpretation, the measure $D_G$ fails as a rigorous quantifier of quantum correlation, since it may increase under local reversible operations on the unmeasured subsystem. Explicitly, by assuming the map $\Gamma_{\sigma} : X \rightarrow X \otimes \sigma$, a channel that introduces a noisy ancillary state, Piani has recently shown that $D_G(\Gamma_{\sigma}^{a} [\rho]) = D_G(\rho) \| \sigma \|_p^2$. This means that the geometric discord may increase under local operations on the unmeasured subsystem $b$, because $\| \sigma \|_p^2 \leq 1$ in general. Indeed, by considering the coupling of $b$ with an arbitrary auxiliary system in a mixed state $\sigma$, we obtain that $D_G$ increases by the simple reversible removal of $\sigma$. The origin of this problem is the Hilbert-Schmidt norm, which is not an appropriate choice for geometrically quantifying the quantumness of correlations (for a similar analysis in the case of entanglement, see Ref. [23]).

Let us then consider the geometric discord based on a more general norm, defined by

$$D_p(\rho) = \min_{\Pi_c} \| \rho - \rho_c \|_p^p,$$

where $\|X\|_p = \text{tr} \left[ (X^\dagger X)^{\frac{p}{2}} \right]^p$ is the Schatten $p$-norm, with $p$ denoting a positive integer number. In this notation, the geometric discord is simply obtained by taking $p = 2$, namely, $D_G = D_2$. Since the $p$-norm is multiplicative under tensor products, it is then easy to see that $\| X \|_p \rightarrow \| \Gamma_{b}^{a} [X] \|_p = \| X \|_p \| \sigma \|_p$. Thus,

$$D_p(\Gamma_{b}^{a} [\rho]) = D_p(\rho) \| \sigma \|_p^p.$$
Note that $||\sigma||_p = 1$ if and only if $p = 1$, since $||\sigma||_1 = \text{tr}|\sigma| = 1$ for a general state $\sigma$. Therefore, the geometric discord based on the 1-norm is the only possible Schatten $p$-norm able to consistently quantify non-classical correlations. Indeed, one can show that $D_1(\rho)$ is non-increasing under general local operations on $b$ (see also Ref. [14]). Due to the properties of the trace distance, the 1-norm geometric discord is contractive under trace-preserving quantum channels (2) (13), i.e. $||\rho - \rho_c||_1 \geq ||\varepsilon(\rho) - \varepsilon(\rho_c)||_1$, where $\varepsilon$ is a general trace-preserving quantum operation. Then, let us consider a quantum operation $\varepsilon_b$, which acts only over subsystem $b$. By denoting as $\rho_c$ the closest classical state to a given quantum state $\rho$, we can write $D_1(\rho) = ||\rho - \rho_c||_1 \geq ||\varepsilon_b(\rho) - \varepsilon_b(\rho_c)||_1$. Note that $\varepsilon_b(\rho_c)$ is still a classical state, but it is not necessarily the closest classical state to $\varepsilon_b(\rho)$, then, $||\varepsilon_b(\rho) - \varepsilon_b(\rho_c)||_1 \geq D_1(\varepsilon_b(\rho))$. Hence it follows that $D_1(\rho) \geq D_1(\varepsilon_b(\rho))$ (2), which implies that $D_1(\rho)$ cannot increase under operations over subsystem $b$.

1-norm geometric quantum discord for Bell-diagonal states. In order to obtain the 1-norm geometric discord for two-qubit systems described by Bell-diagonal states given by Eq. (6), let us start from the expression

$$D_1(\rho) = \min_{\Omega_0} ||\rho - \rho_c||_1,$$

where $||X||_1 = \text{tr}(\sqrt{X^\dagger X})$ is the 1-norm, $\rho$ is given by Eq. (4) and $\rho_c$ is an arbitrary classical-quantum state given by Eq. (5). The minimization over the whole set of classical states was obtained for the 2-norm (15) and the relative entropy (2), where it can be proved that the minimal state is the measured original state. We will make a similar hypothesis and assume that the minimal state preserves the Bell-diagonal form of the original state. This has been numerically checked for a number of Bell-diagonal states, as will be discussed below. Therefore, we assume that the minimization in Eq. (6) is achieved by a Bell-diagonal classical state $\rho_c^{(BD)}$, which is denoted by

$$\rho_c^{(BD)} = \frac{1}{4} \left[ I \otimes I + \mathbf{1} \cdot (\mathbf{\sigma} \otimes \mathbf{\sigma}) \right],$$

with $\mathbf{1}$ representing a vector over the perpendicular classical axes in the tetrahedron of Bell-diagonal states (dashed lines in Fig. 1). Then, $\mathbf{1}$ has the form $\mathbf{1}_1 = (l_1, 0, 0)$, $\mathbf{1}_2 = (0, l_2, 0)$, or $\mathbf{1}_3 = (0, 0, l_3)$, with $l_i \in \mathbb{R}$ and $-1 \leq l_i \leq 1$. From Eqs. (4) and (10), we can then write

$$D_1 = \min \left[ \min_{l_1} f_1(l_1), \min_{l_2} f_2(l_2), \min_{l_3} f_3(l_3) \right],$$

where

$$f_i(l_i) = \left| \frac{1}{4} (\mathbf{e} - \mathbf{1}_i) \cdot (\mathbf{\sigma} \otimes \mathbf{\sigma}) \right|_1 = \sum_{p=0}^{1} \sum_{q=0}^{1} |T_{pq,i}|,$$

with $T_{pq,i} = \frac{1}{4} (-c_{i} - d_{i} - (c_{i} + d_{i})/4 (i \neq j \neq k)$ denoting the eigenvalues of the operator $(\mathbf{e} - \mathbf{1}_i) \cdot (\mathbf{\sigma} \otimes \mathbf{\sigma})/4$. Now, by defining $d_i = l_i - c_i$ and $d_{\pm} = c_k \pm c_i$, we find $f_i(l_i) = (|d_i| + d_{+}| + |d_i| - d_{-}| + |d_i| + d_{-}|) / 4$. Because $f_i(l_i)$ reaches its minimum value when $d_i = 0$, then $\min_{l_i} f_i(l_i) = \max_{c_i} (|c_j|, |c_k|)$. By using this result in Eq. (11), we then obtain

$$D_1 = c_0 = \int \left[ |c_1|, |c_2|, |c_3| \right].$$

The same result encapsulated by Eq. (13) was obtained in the context of the study of the negativity of quantumness, which is a measure of nonclassicality recently introduced in Refs. [27, 28] and experimentally discussed in Ref. [29]. In such a case, the 1-norm distance is computed with respect to the decohered (measured) state $\rho'' = \sum_{i} \Pi_i \rho \Pi_i$.

For a finite subset of classical states $\Omega_0$, the equivalence between Eqs. (4) and (13) is numerically supported by the condition

$$\delta = \min_{\Omega_0} ||\rho - \rho_c||_1 - c_0 \geq 0,$$

with the equality expected after minimization over all classical states $\rho_c$, i.e. $\Omega_0' = \Omega_0$. In Fig. 2, we present a numerical analysis of Eq. (14) through a histogram of $\delta$. This has been obtained for $N = 10^3$ Bell-diagonal states $\rho$ randomly generated inside of the tetrahedron (Fig. 1). For each $\rho$, we have performed the minimization in Eq. (14) with $N_c = 10^6$ classical states $\rho_c$ randomly chosen from Eq. (6). Note that $\delta \geq 0$, with an average value $\delta = 0.06$. In the inset, we have investigated the behavior of $\delta$ as we increase the number of classical states $N_c$ in $\Omega_0'$. For each value of $\log_{10} N_c$ (data point), we compute $\log_{10} \delta$ by randomly selecting $10^6$ independent states $\rho$. By a linear fit (solid line) we obtain that $\delta$ decreases to zero for $N_c \to \infty$, according to the power law $\delta = 0.56 \times N_c^{-0.16}$ (23).

**FIG. 2.** (Color online) Histogram of $\delta$ for $N = 10^3$ Bell-diagonal states and $N_c = 10^6$ classical states. In the inset, we show the decreasing behavior of $\log_{10} \delta \times \log_{10} N_c$ for $N_c = 10^2$, $10^3$, $10^4$, $10^5$, and $10^6$.

**Monotonicity with other quantum discord measures.** Let us now apply Eq. (13) to investigate the monotonicity of $D_1(\rho)$ with the quantum correlation measures $Q(\rho)$...
and \( D_2(\rho) \), which are given by Eqs. (1) and (2). First of all, we readily conclude that \( D_1 = 0 \) over the orthogonal axes \( c_1 \), \( c_2 \), and \( c_3 \), and is maximal \( (D_1 = 1) \) for the four Bell states, as it occurs for \( Q \) and \( D_G \). Moreover, since \( 0 \leq c_0 \leq 1 \) and \( c_- \leq c_0 \), it follows that
\[
c_0^2 \geq (c_2^2 + c_3^2)/2 \implies D_G^2 \geq 2D_G.
\]
From this inequality and from Eq. (3), we can find the following hierarchy for two-qubit Bell-diagonal states:
\[
D_1^2 \geq 2D_G \geq Q^2, N^2. \tag{15}
\]
The inequality \( D_1 \geq N \) that emerges from Eq. (14) has also been proposed for arbitrary bipartite states in Ref. [24], but counterexamples have subsequently pointed out in Ref. [17].

![FIG. 3](image)

**FIG. 3:** (Color online) Plots of \( Q \) (solid line), \( 2D_G \) (dashed line) and \( D_1 \) (dotted line) for SU(2)-symmetric states \((c_1 = c_2 = c_3)\).

Concerning mononticity relationships, the symmetry exhibited by the quantum state plays a fundamental role. For instance, in the case of SU(2) symmetry, i.e., \( c_1 = c_2 = c_3 \), the three measures of discord maintain the ordering of states throughout the physical region \((0 \leq c_1 \leq 1/3)\), as we can observe in Fig. 3. However, this does not occur for more general classes of states. For instance, the triangle shown in Fig. 4 represents the set of physical states corresponding to the class of \( U(1) \)-symmetric states, i.e., \( c_1 = c_2 \neq c_3 \). Inside the triangle, the shaded region and the dashed lines indicate the points where \( Q \) is monotonically related along the \( c_3 \) direction with \( D_G \) and \( D_1 \), respectively. In this situation, note that the ordering of states between \( Q \) and the geometric measures \( D_1 \) and \( D_G \) is strongly violated. As the shaded region and the dashed lines do not cover the same space (a situation that occurs only when \( c_3 = -c_2^2 \) and \( c_1 = 0 \)), we also concluded that \( D_G \) and \( D_1 \) are not monotonic between themselves in general.

In conclusion, the 1-norm geometric discord has by itself a conceptual importance since it is the only \( p \)-norm able to yield a well-defined quantum correlation measure. Moreover, it exhibits remarkable properties under decoherence for simple Bell diagonal states as, for instance, freezing and double sudden change [17]. As a future challenge, it would be useful to investigate its relevance for the advantage quantum protocols.

![FIG. 4](image)

**FIG. 4:** (Color online) Triangle representing \( U(1) \)-symmetric states \((c_1 = c_2 \neq c_3)\). Shaded regions and dashed lines indicate the points for which \( Q \) is monotonically related along the \( c_3 \) direction with \( D_G \) and \( D_1 \), respectively.

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[32] After the completion of this work, an analytical discussion of Eq. (14) has appeared in a revised version of
    Ref. [28], which is in agreement with our numerical results. Moreover, it has been shown in Ref. [31] that $D_1$
    can be computed for generic X states.