Dynamic Programming for Discrete Memoryless Channel Quantization

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Abstract—In this paper, we present a general framework for applying dynamic programming (DP) to discrete memoryless channel (DMC) quantization. The DP has complexity \( O(q(N - M)^2 M) \), where \( q \), \( N \), and \( M \) are alphabet sizes of the DMC input, DMC output, and the quantizer output, respectively. Then, starting from the quadrangle inequality (QI), we apply two techniques to reduce the DP’s complexity. One technique makes use of the SMAWK algorithm with complexity \( O(q(N - M)^2 M) \), while the other technique is much easier to be implemented and has complexity \( O(q(N^2 - M^2)) \). Next, we give a sufficient condition on the channel transition probability, under which the two low-complexity techniques can be applied for designing quantizers that maximize the \( \alpha \)-mutual information, which is a generalized objective function for channel quantization. This condition works for the general \( q \)-ary input case, including the previous work for \( q = 2 \) as a subcase. Moreover, we propose a new idea, called iterative DP (IDP). Theoretical analysis and simulation results demonstrate that IDP can improve the quantizer design over the state-of-the-art methods in the literature.

Index Terms—\( \alpha \)-mutual information (\( \alpha \)-MI), discrete memoryless channel (DMC) quantization, dynamic programming (DP), \( q \)-ary input, quadrangle inequality (QI).

I. INTRODUCTION

Channel quantization is of great interest to many practical applications such as communication receivers, data recovery of non-volatile memories (NVMs) \([1], [2]\), look-up table (LUT) decoding of low-density parity-check (LDPC) codes \([3]–[6]\), and construction of polar codes \([7]\). In this paper, we consider the quantization problem for the \( q \)-ary input discrete memoryless channel (DMC) with \( q \geq 2 \), as shown in Fig. 1. The channel input \( X \) takes values from \( \mathcal{X} \),

\[
\mathcal{X} = \{x_1, x_2, \ldots, x_q\}
\]

with probability

\[
P_X(x_i) = \Pr(X = x_i) > 0, \quad i = 1, 2, \ldots, q.
\]

The channel output \( Y \) takes values from \( \mathcal{Y} \),

\[
\mathcal{Y} = \{y_1, y_2, \ldots, y_N\}
\]

with channel transition probability

\[
P_{Y|X}(y_j|x_i) = \Pr(Y = y_j|X = x_i) > 0, \quad i = 1, 2, \ldots, q, \quad j = 1, 2, \ldots, N.
\]

Here we restrict \( P_{Y|X} \) to be positive because it may be used as denominators in this paper. The channel output \( Y \) needs to be quantized to \( Z \) which takes values from \( M (M < N) \) is of interest) levels, i.e.,

\[
Z \in \mathcal{Z} = \{1, 2, \ldots, M\}.
\]

For binary-input DMC, dynamic programming (DP) \([8]\) was applied by Kurkoski and Yagi \([9]\) to design quantizers that maximize the mutual information (MI) between \( X \) and \( Z \), i.e., \( I(X; Z) \). The complexity of this DP method was reduced \([10], [11]\) by applying the SMAWK algorithm \([12]\). However, for general \( q \)-ary input DMC with \( q \geq 2 \), design of the optimal quantizers that maximize \( I(X; Z) \) is an NP-hard problem \([13], [14]\), and thus only suboptimal design methods \([3], [14]–[17]\) exist. Moreover, to the best of our knowledge, so far no work has been reported on applying DP to the general \( q \)-ary input DMC quantization. This motivates us to develop a general framework for applying DP to \( q \)-ary input DMC quantization.

A deterministic quantizer (DQ) \( Q : \mathcal{Y} \rightarrow \mathcal{Z} \) means that for each \( y \in \mathcal{Y} \), there exists a unique \( z' \in \mathcal{Z} \) such that

\[
P_{Z|Y}(z'|y) = \begin{cases} 1 & z = z', \\ 0 & \text{otherwise}, \end{cases}
\]

or equivalently, we say \( y \)'s quantization result \( Q(y) \) is a deterministic element in \( \mathcal{Z} \). In this paper, we focus only on DQs, since DQs generally work better than non-deterministic quantizers in practice \([1], [2], [9], [11], [14]–[18]\). For any DQ \( Q : \mathcal{Y} \rightarrow \mathcal{Z} \), denote \( Q^{-1}(z) \subset \mathcal{Y} \) as the preimage of \( z \in \mathcal{Z} \). Then, we have

\[
P_{Z|X}(z|x) = \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) P_{Z|Y}(z|y) = \sum_{y \in Q^{-1}(z)} P_{Y|X}(y|x). \tag{1}
\]

In order for DP to apply, we further restrict our focus on a specific type of DQ \( Q : \mathcal{Y} \rightarrow \mathcal{Z} \) satisfying

\[
Q^{-1}(1) = \{y_1, y_2, \ldots, y_{\lambda_1}\}, \\
Q^{-1}(2) = \{y_{\lambda_1+1}, y_{\lambda_1+2}, \ldots, y_{\lambda_2}\}, \\
\vdots \\
Q^{-1}(M) = \{y_{\lambda_{M-1}+1}, y_{\lambda_{M-1}+2}, \ldots, y_{\lambda_M}\}, \tag{2}
\]
where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{M-1} < \lambda_M = N$. We name this type of DQ sequential deterministic quantizer (SDQ). Based on (2), every SDQ $Q : \mathcal{Y} \to \mathcal{Z}$ can be equivalently described by the integer set $\Lambda = \{\lambda_0, 0, \lambda_1, \lambda_2, \ldots, \lambda_M = N\}$, in which each element is regarded as a quantization boundary/threshold. Thus, in the rest of this paper, we do not distinguish $Q$ and $\Lambda$ when referring to an SDQ. Note that most of the channel quantizers adopted by communication and data storage systems are SDQs, while the DQ can be used in the learning theory [14]. Moreover, any DQ $Q : \mathcal{Y} \to \mathcal{Z}$ can be made to be an SDQ by relabelling the elements in $\mathcal{Y}$.

The number of different SDQs quantizing $\mathcal{Y}$ to $\mathcal{Z}$ is $(N/M)^{M-1}$. In general, it is a prohibitive task to find an optimal SDQ (note that there may exist multiple optimal SDQs) by enumerating all these cases. Instead, this paper presents a general framework for applying DP to find an optimal SDQ. On the other hand, the computational complexity of DP is $O(q(N - M)^2 M)$. Starting from the quadrangle inequality (QI) [19], we apply two techniques to reduce the computational complexity of the DP method. One technique making use of the SMAWK algorithm [12] has complexity $O(q(N - M)M)$, while the other one is much easier to be implemented and has complexity $O(q(N^2 - M^2))$. Note that unlike the work in [10] and [11] where the SMAWK algorithm is used to design the quantizer for the binary-input DMC through maximizing the $\alpha$-mutual information ($\alpha$-MI) [20]–[22] between $X$ and $Z$, i.e., $I_\alpha(X; Z)$, the two techniques presented by this work can be applied to the general $q$-ary input DMC as well, and the cost function for channel quantization can be any general cost function instead of the $\alpha$-MI only.

We also investigate the design of a practical SDQ that maximizes $I_\alpha(X; Z)$, which is a generalized objective function for channel quantization. In particular, we give a sufficient condition on the channel transition probability $P_{Y|X}$, under which the aforementioned two techniques can be applied to reduce the complexity for the quantizer design using DP. This condition works for the general $q$-ary input case, including the previous work [10] and [11] for $q = 2$ as a subcase. Moreover, for the DMC satisfying this sufficient condition, the simulation results show that DP can outperform the prior art greedy combining (GC) algorithm [3], [15] and KL-means algorithm [16] in terms of the MI gap $\Delta l = I(X; Y) - I(X; Z)$.

Moreover, we propose a new idea, called iterative dynamic programming (IDP), to improve the quantizer design for DMCs with $q > 2$, such as the design in [3], [15]–[17]. An IDP algorithm works iteratively based on any initial DQ, and after each iteration the algorithm can design a DQ that is at least as good (such as in terms of the MI gap $\Delta l$) as the DQ obtained from the previous iteration. Simulation results show that IDP can improve the design of the prior art GC algorithm [3], [15] and KL-means algorithm [16].

The main contributions of this paper are summarized as follows:

- Present a general framework for applying DP to the general $q$-ary input DMC quantization, and apply two techniques to reduce the complexity of the DP method.
- Propose a sufficient condition on $P_{Y|X}$ such that the two low-complexity techniques can be applied to design quantizers that maximize $I_\alpha(X; Z)$.
- Propose the idea of IDP.

The remainder of this paper is organized as follows. Section II presents a general framework for applying DP to the general $q$-ary input DMC quantization, and also applies two techniques to reduce the DP’s complexity. Section III focuses on SDQ that maximizes $I_\alpha(X; Z)$, and proposes a sufficient condition on $P_{Y|X}$ for the two low-complexity techniques to apply. Section IV develops the idea of IDP and also investigates its performance. Finally, Section V concludes the paper.

II. DYNAMIC PROGRAMMING FOR FINDING AN OPTIMAL SEQUENTIAL DETERMINISTIC QUANTIZER

For $1 \leq l \leq r \leq N$ and $1 \leq m \leq r - l + 1$, let $P(l, r, m)$ denote the problem of finding an SDQ quantizing $\{y_i, y_{i+1}, \ldots, y_r\}$ to $m$ levels. $P(l', r', m')$ is called a subproblem of $P(l, r, m)$ if $l \leq l' \leq r'$ and $1 \leq m' < m$. Denote $\Lambda(l, r, m) = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$ with $\lambda_0 = l - 1 < \lambda_1 < \cdots < \lambda_m = r$ as a solution to $P(l, r, m)$, and denote the cost function of $\Lambda(l, r, m)$ by $C(\Lambda(l, r, m))$. Denote an optimal (minimizing cost function value) solution to $P(l, r, m)$ by $\Lambda^*(l, r, m) = \{\lambda_0^* = l - 1, \lambda_1^*, \ldots, \lambda_m^* = r\}$, whose cost function is denoted by $C^*(l, r, m) = C(\Lambda^*(l, r, m))$. Our task is to find an optimal solution to $P(1, N, M)$, i.e., to find $\Lambda^*(1, N, M)$.

Optimally solving $P(1, N, M)$ is essentially an optimization problem. In order for DP to apply, an optimization problem must have the two key ingredients below [8, Section 15.3].

- Optimal substructure: An optimal solution to the problem contains within it the optimal solutions to the corresponding subproblems.
- Overlapping subproblems: A recursive algorithm for the problem solves the same subproblems over and over, rather than always generating new problems.

The first ingredient ensures that DP provides the optimal solution, and the second ingredient ensures that DP works efficiently by solving each subproblem once and then storing the solution in a table where it can be looked up when needed [8, Section 15.3].

Note that for $1 \leq l \leq r \leq N$, the optimal solution to $P(l, r, 1)$ is $\Lambda^*(l, r, 1) = \{l - 1, r\}$, which is indeed the unique solution. For convenience, let

$$w(l, r) = C^*(l, r, 1).$$

In general, $w(l, r)$ can be computed based only on $P_X$ and $P_{Y|X}$ at a computational complexity linear to the input alphabet size $q$. We thus denote the complexity for computing $w(l, r)$ for any given pair $(l, r)$ by $O(q)$. The computational complexity of all algorithms discussed later in this paper is given without precomputing $w(\cdot, \cdot)$. The precomputing has computational complexity of $O(qN^2)$ and storage complexity of $O(N^2)$. With this precomputing, an algorithm may reduce its computational complexity by a factor of $q$ but at the cost of having an extra storage complexity of $O(N^2)$. 

Fig. 2. Subproblems of $\mathcal{P}(1, n, m)$, the problem of finding an SDQ that quantizes $\{y_1, y_2, \ldots, y_n\}$ to $m$ levels.

**Theorem 1:** For any $1 \leq m \leq n \leq N$ and any solution $\Lambda(1, n, m)$ to $\mathcal{P}(1, n, m)$, if

$$C(\Lambda(1, n, m)) = \sum_{i=1}^{m} w(\lambda_{i-1} + 1, \lambda_i)$$

and $w(\cdot, \cdot)$ can be computed independently from $m$, $n$, and $\Lambda(1, n, m)$, then, DP with computational complexity $O(q(N - M)^2M)$ can be used to find an optimal solution to $\mathcal{P}(1, N, M)$.

**Proof:** To prove Theorem 1 we first illustrate that $\mathcal{P}(1, n, m)$ has the aforementioned two ingredients, and then conduct a DP method with computational complexity $O(q(N - M)^2M)$ to find an optimal solution to $\mathcal{P}(1, N, M)$.

**Optimal substructure:** Let $m$ and $n$ be two arbitrary integers satisfying $1 < m \leq n \leq N$. For any optimal solution $\Lambda^*(1, n, m) = \{\lambda^*_0 = 0, \lambda^*_1, \ldots, \lambda^*_m = n\}$ to $\mathcal{P}(1, n, m)$, $\Lambda^*(1, n, m) \setminus \lambda^*_m$ is an optimal solution to $\mathcal{P}(1, \lambda^*_m - 1, m - 1)$.

Otherwise, we have

$$C^*(1, \lambda^*_m - 1, m - 1) < \sum_{i=1}^{m-1} w(\lambda^*_i + 1, \lambda^*_i);$$

and additionally, since $w(\cdot, \cdot)$ can be computed independently from $m$, $n$, and the solutions to $\mathcal{P}(1, n, m)$, we have

$$\sum_{i=1}^{m} w(\lambda^*_i + 1, \lambda^*_i) > C^*(1, \lambda^*_m - 1, m - 1) + w(\lambda^*_m - 1, n),$$

which would contradict to the optimality of $\Lambda^*(1, n, m)$. Thus, $\mathcal{P}(1, n, m)$ has the ingredient of optimal substructure.

**Overlapping subproblems:** Let $m$ and $n$ be two arbitrary integers satisfying $1 < m \leq n \leq N$. According to the discussion on optimal substructure, we have

$$C^*(1, n, m) = C^*(1, \lambda^*_m - 1, m - 1) + w(\lambda^*_m - 1, n) = \min_{m-1 \leq \lambda^*_m \leq n} C^*(1, \lambda^*_m - 1, m - 1) + w(\lambda^*_m - 1, n).$$

Thus, we can find the optimal solutions to $\mathcal{P}(1, n, m)$ by optimally solving its subproblems $\mathcal{P}(1, \lambda^*_m - 1, m - 1)$, $m \leq \lambda^*_m < n$, and we can then optimally solve these subproblems' subproblems in a recursive manner, as shown by Fig. 2. In this recursive process, the subproblems of $\mathcal{P}(1, n, m)$ need to be solved are $\mathcal{P}(1, n', m')$, $1 \leq m' < m$, $n' \leq n$, while each of them can be solved only once and then the solution is stored in a table where it can be looked up when needed. Consequently, $\mathcal{P}(1, n, m)$ has the ingredient of overlapping subproblems.

**DP for solving $\mathcal{P}(1, N, M)$:** For $1 \leq m \leq n \leq N$, denote $C^*(1, n, m)$ by $d_p(n, m)$ for brevity. For $m - 1 \leq t < n$, denote

$$d_p(t, m - 1) + w(t + 1, n).$$

In addition, denote

$$\text{sol}(n, m) = \arg \min_{m-1 \leq \lambda^*_m \leq n-1} d_p(\lambda^*_m, n)$$

as the solution for $\lambda^*_m$ involved in $\Lambda^*(1, n, m)$. An optimal solution to $\mathcal{P}(1, N, M)$ can be found by using the DP described in Algorithm 1 where line 9 is illustrated by (4). It is easy to check that the computational complexity of Algorithm 1 is $O(qN^2 + (N - M)^2M)$ (Recall that this complexity is given without precomputing $w(\cdot, \cdot)$. It becomes $O(qN^2 + (N - M)^2M)$ with the precomputing.)

Note that Algorithm 1 only outputs one optimal solution to $\mathcal{P}(1, N, M)$. However, if multiple or all optimal solutions are needed, we can use a tie-preserving implementation [9] at line 9 and recursively find all optimal solutions at the end of Algorithm 1.

Algorithm 1 Dynamic programming for solving $\mathcal{P}(1, N, M)$

**Input:** $P_X, P_{Y|X}, N, M$.

**Output:** $\Lambda^*(1, N, M)$.

1: //Initialization
2: for $n = 1$; $n \leq N$; $n + +$ do
3: $d_p(n, 1) = w(1, n)$.
4: $\text{sol}(n, 1) = 0$.
5: end for
6: //Compute $d_p(n, M)$
7: for $m = 2$; $m \leq M$; $m + +$ do
8: for $n = N - M + m$; $n \geq m$; $n - -$ do
9: $\text{sol}(n, m) = \arg \min_{m-1 \leq \lambda^*_m \leq n-1} d_p(n, m)$.
10: $d_p(n, m) = d_p(\text{sol}(n, m), n, m)$.
11: end for
12: end for
13: //Recursively to find $\Lambda^*(1, N, M)$
14: $\lambda^*_M = N$.
15: for $m = M$; $m > 0$; $m - -$ do
16: $\lambda^*_m = \text{sol}(\lambda^*_m, m)$.
17: end for
18: return $\Lambda^*(1, N, M)$. 

In certain cases $N$ can be very large, and hence Algorithm 1 may need to take a long time to find an optimal solution. For example, when we use Algorithm 1 to quantize the output of a continuous memoryless channel (CMC) to $M$ levels, we may need to first uniformly quantize the continuous output to $N$ levels, after which Algorithm 1 can be applied. Obviously increasing $N$ can reduce the loss due to uniform quantization. Thus, it is worth reducing the computational complexity of Algorithm 1 to make it work well for large $N$. Hence, the
rest of this section is devoted to reduce the complexity of Algorithm 1.

Definition 1 (Quadrangle inequality): $w(\cdot, \cdot)$ is said to satisfy the QI if it satisfies

$$w(i, k) + w(j, l) \leq w(i, l) + w(j, k) \quad (5)$$

for all $1 \leq i < j < k < l \leq N$.

The QI of (5) was first proposed by Yao [19] as a sufficient condition to reduce the complexity of a class of DP. Then, it was pointed out in [23] that Yao’s result can be achieved by using the SMAWK algorithm [12]. We first give the definition of totally monotone matrix, and then show how to use the SMAWK algorithm to reduce the complexity of Algorithm 1.

Definition 2 (Totally monotone matrix): A $2 \times 2$ matrix $A = [a_{ij}]_{0 \leq i, j \leq 1}$ is monotone if $a_{0,0} > a_{0,1}$ implies $a_{1,0} > a_{1,1}$. A matrix $D$ is totally monotone if every $2 \times 2$ submatrix (intersections of arbitrary two rows and two columns) of $D$ is monotone.

Inspired by the works of [23] and [10], for $2 \leq m \leq M$, we define $D^m = [d_{i,j}^m]_{0 \leq i, j \leq N-M}$ as a matrix with $d_{i,j}^m$ given by

$$d_{i,j}^m = \begin{cases} dp_{i-1,j} + (i + m, m) & i \geq j, \\ dp_{i,j-1} - m(i + m, k + m) + w(j + m, l + m) - w(j + m, k + m) - w(i + m, l + m) & i < j, \\ \infty & \text{otherwise} \end{cases} \quad \quad (6)$$

where $\infty$ indeed can be replaced by any constant larger than all $d_{i,j}^m$ for $i \geq j$.

We define $D^m$ because it can be computed in the order of $D^2, D^3, \ldots, D^M$ and $dp(n, m)$ is given by the minima of the $(n-m)$-th row of $D^m$. Then, the computation in lines 8 to 11 of Algorithm 1 can be replaced by solving the new problem: finding the minima of each row of $D^m$. The new problem is indeed the classical problem discussed in [12]. The SMAWK algorithm was proposed by [12] to solve this problem when $D^m$ is totally monotone. In our case, we have the following lemma.

Lemma 1: If $w(\cdot, \cdot)$ satisfies the QI, $D^m$ is totally monotone for $2 \leq m \leq M$.

Proof: Consider the $2 \times 2$ submatrix of $D^m$ consisting of the intersections of rows $k, l$ with $k < l$ and columns $i, j$ with $i < j$, denoted by $D_s$. If $j > k$, we have $d_{i,j}^m = \infty$, implying that $D_s$ is monotone. For $j \leq k$, we have $d_{i,j}^m = \min\{d_{i,j}^m, d_{i,k}^m + w(i + m, k + m) + w(j + m, l + m) - w(j + m, k + m) - w(i + m, l + m)\}$, indicating that $D_s$ is monotone. This completes the proof.

The property of $d_{i,j}^m = \min\{d_{i,j}^m, d_{i,k}^m + w(i + m, k + m) + w(j + m, l + m) - w(j + m, k + m) - w(i + m, l + m)\}$ is called the Monge property of $D^m$ [10], [23]. According to (6), any entry $d_{i,j}^m$ of $D^m$ can be computed with the same complexity as computing $w(j + m, i + m)$, i.e., $O(q)$ in general. Thus, generating $D^m$ by computing all its entries has complexity $O(q(N - M)^2)$. Fortunately, the SMAWK algorithm does not require $D^m$ to be precomputed. Instead, a specific entry of $D^m$ can be computed only when needed. This ensures that the SMAWK algorithm can find the minima of each row of $D^m$ with complexity $O(q(N - M))$ without precomputing $D^m$ when $D^m$ is totally monotone. Consequently, Lemma 1 implies that if $w(\cdot, \cdot)$ satisfies the QI, the complexity of Algorithm 1 can be reduced to $O(q(N - M)M)$ by applying the SMAWK algorithm.

To check whether $w(\cdot, \cdot)$ satisfies the QI it is vital for reducing the complexity of Algorithm 1. For $w(\cdot, \cdot)$ that cannot be determined analytically of whether it satisfies the QI or not, we can test it exhaustively by checking

$$w(r, s) + w(r + 1, s + 1) \leq w(r, s + 1) + w(r + 1, s) \quad (7)$$

for $1 \leq r < s < N$. It can be easily proved that (6) is equivalent to (7). Checking (7) has complexity $O(qN^2)$. It is worth doing the checking if $qN^2 < q(N - M)^2M$, i.e., the checking costs less complexity than Algorithm 1.

If $w(\cdot, \cdot)$ is verified by the exhaustive test to satisfy the QI, the SMAWK algorithm can be used to reduce the complexity of Algorithm 1 and hence its complexity approaches this lower-bound. Considering that implementing the SMAWK algorithm is tricky and sophisticated, in the following, we present another low-complexity DP algorithm which is much easier to be implemented, and its complexity also approaches this lower-bound. By simply modifying the upper and lower bounds of $t$ in line 9 of Algorithm 1 (i.e. the standard DP algorithm), it can reduce the complexity from $O(q(N - M)^2M)$ to $O(q(N^2 - M^2))$. The corresponding details are as follows.

Lemma 2: If $w(\cdot, \cdot)$ satisfies the QI, we then have

$$\text{sol}(n, m - 1) \leq \text{sol}(n, m) \leq \text{sol}(n + 1, m) \quad (8)$$

for $2 \leq m \leq n < N$.

Proof: See Appendix A.

The inequality of (8) was first proved by Yao as a consequence of the QI in order to reduce the complexity for solving the DP problem considered in [19]. Though our DP problem is different from that considered in [19], fortunately, (8) still holds as a consequence of the QI and can also be used to reduce the complexity for solving our DP problem. In particular, when (8) holds, for $n = N - M + m - 1, N - M + m - 2, \ldots, m$ in line 8 of Algorithm 1, we can conduct a low-complexity technique by enumerating $t$ in line 9 from $\max\{m - 1, \text{sol}(n, m - 1)\}$ to $\min\{n - 1, \text{sol}(n + 1, m)\}$ instead of from $m - 1$ to $n - 1$. Let $T(n, m)$ denote the complexity for enumerating $t$ in line 9 with respect to the $m$ in line 7 and the $n$ in line 8. Then, the total complexity for enumerating $t$, after applying this low complexity algorithm, is given by

$$\sum_{m=2}^{M} \sum_{n=m}^{N-M+m} T(n, m)$$

$$\leq \sum_{m=2}^{M} \sum_{n=m}^{N-M+m} (\text{sol}(n + 1, m) - \text{sol}(n, m - 1) + 1)$$

$$\leq M(N - M + 1) + \sum_{n=M+1}^{N} \text{sol}(n, M)$$

$$\leq (N + M)(N - M + 1)$$

Therefore, this low-complexity algorithm has complexity $O(q(N^2 - M^2))$. 
The two low-complexity techniques presented in this section can be used to reduce the complexity of Algorithm 1 once \( w(\cdot, \cdot) \) satisfies the QI, no matter this requirement is verified analytically or by exhaustive test. The technique making use of the SMAWK algorithm works faster, while being more complicated than the other one making use of \([8]\) in terms of the implementation complexity.

III. \( \alpha \)-Mutual Information Maximized Quantizer

Recall that \( P_{Z|X} \) is defined by \([1]\). Let \( P_{X,Z}(x,z) = \Pr(X = x, Z = z) = P_X(x)P_{Z|X}(z|x) \) and \( P_Z(z) = \Pr(Z = z) = \sum_{x \in X} P_{X,Z}(x,z) \). For \( \alpha > 0 \), the \( \alpha \)-MI between \( X \) and \( Z \) is defined by \([9]\) \([11]\) \([20]\) \([22]\). Note that \( I_1(X;Z) \) is equivalent to the standard MI between \( X \) and \( Z \), i.e., \( I(X;Z) \), and \( I_{1/2}(X;Z) \) is equivalent to the cutoff rate between \( X \) and \( Z \) \([22]\).

Many efforts \([1]\) \([2]\) \([9]\) \([11]\) \([14]\) \([18]\) have been devoted to design the quantizer \( Q^* : Y \rightarrow Z \) that maximizes \( I_\alpha(X;Z) \), i.e.

\[
Q^* = \arg \max_Q I_\alpha(X;Z),
\]

especially for \( \alpha = 1 \). Thus, \( I_\alpha(X;Z) \) is a generalized objective function for quantization. According to \([9]\) \([11]\), \( Q^* \) must be deterministic; meanwhile, for the binary-input case (i.e., \( q = 2 \)), \( Q^* \) is an optimal SDQ when elements in \( Y \) are relabelled to satisfy

\[
\frac{P_{Y|X}(y_1|x_1)}{P_{Y|X}(y_2|x_2)} \geq \frac{P_{Y|X}(y_3|x_3)}{P_{Y|X}(y_4|x_4)} \geq \ldots \geq \frac{P_{Y|X}(y_M|x_1)}{P_{Y|X}(y_M|x_M)}.
\]

(11)

Note that if needed, we can reduce \( |Y| \) and replace “\( \geq \)” with “\( > \)” in \([11]\) by recursively merging two elements \( y_i, y_j \in Y \) for \( \frac{P_{Y|X}(y_i|x_1)}{P_{Y|X}(y_j|x_2)} = \frac{P_{Y|X}(y_i|x_3)}{P_{Y|X}(y_j|x_4)} \). However, this operation does not affect the optimality of the design result, since any two elements \( y_i, y_j \in Y \) satisfying \( \frac{P_{Y|X}(y_i|x_1)}{P_{Y|X}(y_j|x_2)} = \frac{P_{Y|X}(y_i|x_3)}{P_{Y|X}(y_j|x_4)} \) will be quantized to a same level according to the necessary condition derived by \([24]\) for the optimal quantizer. Furthermore, according to \([10]\) \([11]\), an optimal SDQ can be found with complexity \( O((N - M)M) \) by applying the SMAWK algorithm when \( q = 2 \) and \([11]\) holds.

However, for \( q > 2 \), the problem of finding \( Q^* \) is NP-hard even for \( M = 2 \) \([13]\) \([14]\). Also, to the best of our knowledge, so far in the literature, no work has been reported on applying DP to the quantizer design for \( q > 2 \). In this section, we deal with the general \( q \)-ary input case by using DP. Given the labelling of elements in \( Y \), we first specify how to design an optimal SDQ \( \Lambda^* \) to maximize \( I_\alpha(X;Z) \) by using Algorithm \([11]\) Note that \( I_\alpha(X;Z) \) is maximized in the sense that the labelling of elements in \( Y \) is given and \( \Lambda^* \) is an optimal SDQ, while \( q^* \) in \([10]\) generally is a DQ. Then, we give a sufficient condition on the transition probability \( P_{Y|X} \) for the two low-complexity techniques discussed in Section I to apply.

To make Algorithm \([1]\) work for the design of \( \alpha \)-MI maximized quantizer, we only need to properly define \( w(\cdot, \cdot) \). Recall that \( w(l,r) \), \( 1 \leq l \leq r \leq N \) denotes the cost function for quantizing \( \{y_l, y_{l+1}, \ldots, y_r\} \) to a single level, say \( z \). Since \( w(\cdot, \cdot) \) differs when \( I_\alpha(X;Z) \) with different \( \alpha \) is used in \([10]\), we use \( w_{\alpha}(\cdot, \cdot) \) to replace \( w(\cdot, \cdot) \). We define \( w_{\alpha}(l,r) \) by

\[
w_{\alpha}(l,r) = \begin{cases} 
\sum_{x \in X} P_{X,Z}(x,z) \log \frac{P_{Z|x}(z|x)}{P_{Z|x}(z|x)} & \alpha = 1, \\
- \max_{x \in X} P_{Z|x}(z|x) & \alpha = \infty, \\
\left( \sum_{x \in X} P_X(x)P_{Z|x}^\alpha(z|x) \right)^{1/\alpha} & \alpha \in (0, 1), \\
\left( \sum_{x \in X} P_X(x)P_{Z|x}^\alpha(z|x) \right)^{1/\alpha} & \alpha \in (1, \infty),
\end{cases}
\]

(12)

where \( P_{Z|x}(z|x) = \sum_{j=r}^\infty P_{Y|X}(y_j|x) \) according to \([1]\).

According to \([9]\) \([12]\), for any SDQ \( \Lambda(1,N,M) \) quantizing \( Y \) to \( Z \) and having cost function \( C(\Lambda(1,N,M)) \) given by \([3]\), we have

\[
I_\alpha(X;Z) = \begin{cases} 
H(X) - C(\Lambda(1,N,M)) & \alpha = 1, \\
\log(-C(\Lambda(1,N,M))) & \alpha = \infty, \\
\frac{\alpha}{\alpha} \log(-C(\Lambda(1,N,M))) & \alpha \in (0, 1), \\
\frac{\alpha}{\alpha} \log(-C(\Lambda(1,N,M))) & \alpha \in (1, \infty),
\end{cases}
\]

(13)

where \( H(X) = \sum_{x \in X} (-P_X(x) \log P_X(x)) \) is a constant given \( P_X \). We define \( w_{\alpha}(l,r) \) by using \([12]\) since it results in a negative relationship between \( I_\alpha(X;Z) \) and \( C(\Lambda(1,N,M)) \) given by \([13]\). Note that \( w_{\alpha}(l,r) \) may be defined in other ways while maintaining a similar negative relationship between \( I_\alpha(X;Z) \) and \( C(\Lambda(1,N,M)) \).

According to \([13]\), we conclude that maximizing \( I_\alpha(X;Z) \) is equivalent to minimize \( C(\Lambda(1,N,M)) \). Theorem \([1]\) implies that the optimal SDQ \( \Lambda^*(1,N,M) \) having the minimum cost function value \( C^*(1,N,M) \) can be found by using Algorithm \([1]\) with complexity \( O(q(N - M)^2M) \).

Theorem 2: \( w_{\alpha}(\cdot, \cdot) \) defined in \([12]\) satisfies the QI if for any \( 1 \leq i < q \), \( Y \) satisfies

\[
\frac{P_{Y|X}(y_i|x_i)}{P_{Y|X}(y_i|x_{i+1})} \geq \frac{P_{Y|X}(y_2|x_i)}{P_{Y|X}(y_2|x_{i+1})} \geq \ldots \geq \frac{P_{Y|X}(y_M|x_i)}{P_{Y|X}(y_M|x_{i+1})}.
\]

(14)

Proof: See Appendix \([8]\).

The condition of \([14]\) works as a sufficient condition for \( w_{\alpha}(\cdot, \cdot) \) to satisfy the QI. If this condition holds, the two low-complexity algorithms can be applied to the design of \( \Lambda^*(1,N,M) \) according to the discussion in Section II. In addition, Theorem 2 works for \( q \geq 2 \), and hence applies to the work of \([10]\) and \([11]\) with \( q = 2 \) as a subcase.

For \( q = 2 \), given the condition of \([14]\), the SDQ \( \Lambda^*(1,N,M) \) designed by DP is global optimal among all the quantizers quantizing \( Y \) to \( Z \) (i.e., \( \Lambda^*(1,N,M) \) is equivalent to \( Q^* \) given by \([10]\)). However, for \( q > 2 \), the condition of \([14]\) still cannot ensure \( \Lambda^*(1,N,M) \) to be global optimal.

To verify this fact, we have tested more than \( 10^7 \) random cases satisfying \([14]\). In all cases, we use small \( q, N, M \) with \( 3 \leq q \leq 8, 2 \leq M < N \leq 8 \) such that \( Q^* \) can be found by brute-force. We observed that except for one counter example, \( Q^* \) is given by \( \Lambda^*(1,N,M) \) for most of the test cases.

Example: We give a practical case where the condition of \([14]\) holds. Consider a pulse-amplitude modulation (PAM) system, as shown in Fig. \([3]\) where symbols \( X = \{x_1, x_2, \ldots, x_q\} \) with small symbols \( \{x_1, x_2, \ldots, x_q\} \). The symbols are transmitted over an additive white Gaussian noise (AWGN)
channel with noise variance $\sigma^2$. Then, the probability density function (pdf) of the channel continuous output $\tilde{Y} = \tilde{y}$ conditioned on channel input $X = x_i$ is given by

$$f_{\tilde{Y}|X}(\tilde{y}|x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\tilde{y} - x_i)^2}{2\sigma^2}\right)$$

for $1 \leq i \leq q$ and $\tilde{y} \in \mathbb{R}$. Our goal is to quantize $\tilde{Y}$ to $Z \in Z = \{1, 2, \ldots, M\}$ that maximize $I_0(X; Z)$. The quantization should be done by finding $M + 1$ thresholds $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_M\}$, $\lambda_0 = -\infty < \lambda_1 < \cdots < \lambda_M < \infty$, such that for $1 \leq i \leq M$, $\tilde{Y} \in [\lambda_{i-1}, \lambda_i)$ is quantized to $Z = i$.

In order for DP to apply, we first convert the channel to a DMC, as shown in Fig. 3, with output $Y \in \mathbb{Y} = \{y_1, y_2, \ldots, y_N\}$, where $N > M$. More specifically, we create $N + 1$ candidate thresholds $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_N\}$, $\gamma_0 = -\infty < \gamma_1 < \cdots < \gamma_N < \infty$, such that the transition probability of the DMC is given by

$$P_{Y|X}(y_j|x_i) = \int_{\gamma_{j-1}}^{\gamma_j} f_{\tilde{Y}|X}(\tilde{y}|x_i) d\tilde{y}$$

for $1 \leq i \leq q$ and $1 \leq j \leq N$. In general, we can set $\gamma_1 = x_1 - 3\sigma$ and $\gamma_{N-1} = x_N + 3\sigma$, and set $\gamma_j = \gamma_{j-1} + (\gamma_N - \gamma_1)/(N - 2)$ for $j = 2, \ldots, N - 2$ to uniformly partition $[\gamma_1, \gamma_N]$ to $N - 2$ segments.

We can then use Algorithm 1 to find the optimal thresholds from the candidate thresholds $\Gamma$ according to Theorem 1. In particular, for $1 \leq i < q$ and $1 \leq j \leq N$, we have

$$\frac{P_{Y|X}(y_j|x_i)}{P_{Y|X}(y_j|x_{i+1})} = \frac{\int_{\gamma_{j-1}}^{\gamma_j} f_{\tilde{Y}|X}(\tilde{y}|x_i) d\tilde{y}}{\int_{\gamma_{j-1}}^{\gamma_{j+1}} f_{\tilde{Y}|X}(\tilde{y}|x_{i+1}) d\tilde{y}} = \frac{\int_{\gamma_{j-1}}^{\gamma_j} f_{\tilde{Y}|X}(\tilde{y}|x_i) f_{\tilde{Y}|X}(\tilde{y}|x_{i+1}) d\tilde{y}}{\int_{\gamma_{j-1}}^{\gamma_{j+1}} f_{\tilde{Y}|X}(\tilde{y}|x_{i+1}) d\tilde{y}},$$

implying the condition of (14) holds.

Next, for the DMC converted from this $q$-ary PAM system, we compare the quantization performance (in terms of the MI gap $\Delta I = I(X; Y) - I(X; Z)$) of the DP method with the prior art quantizer design algorithms proposed for the non-binary-input DMC, i.e., the GC algorithm [3], [15] and the KL-means algorithm [16]. In the simulations, for the $q$-ary PAM system, we set $\sigma = 1$, $x_i = 2i - q - 1$ for $1 \leq i \leq q$, $N = 128$, and $\gamma_j$ to uniformly partition $[x_1 - 3\sigma, x_q + 3\sigma]$ to $N - 2$ segments for $1 \leq j \leq N - 1$. $\tilde{Y}$ is quantized to $M$ levels, $M = 2, 3, \ldots, 20$. When KL-means is used, we randomly choose $M$ out of $N$ points as the initial means (see [16] for 100 times, and for each time KL-means runs for 100 iterations to find a DQ, and finally the best ($\Delta I$ is minimized) DQ among the 100 times is chosen. The simulation results are illustrated by Fig. 4.
Fig. 4 shows that DP performs better than both the GC and KL-means algorithms, for different values of \( q \). Moreover, since (14) holds, DP has complexity \( O(q(N - M)M) \) if applying the SMAWK algorithm and \( O(q(N^2 - M^2)) \) if applying the other low-complexity algorithm described in Section II. In contrast, greedy combining and KL-means algorithms have complexities \( O(qN^2(N - M)) \) and \( O(10^4qNM) \), respectively, and hence are much more complex than DP.

IV. ITERATIVE DYNAMIC PROGRAMMING FOR IMPROVING THE DESIGN OF DQ

Regarding the problem of finding optimal DQs (such as finding \( Q^* \) given by (10) for \( q > 2 \), due to its NP-hardness, the state-of-the-art practical methods \([3], [14]–[17]\) for solving it are suboptimal. In this section, we propose the new idea of IDP for improving the DQ design for \( q > 2 \).

An IDP algorithm works iteratively, where for \( t \geq 1 \), the \( t \)-th iteration contains the following two steps.

- Relabelling step: Elements in \( Y \) are relabelled.
- Update step: After relabelling, Algorithm 1 is applied to find a new optimal SDQ \( Q_t \).

Here the relabelling step is necessary because i) any DQ can be made as an SDQ by relabelling the elements in \( Y \) and ii) Algorithm 1 can only deal with SDQs.

One important principle for designing an IDP algorithm is to make the quality of \( Q_t \) (such as in terms of the MI gap \( \Delta I \)) better and better as \( t \) increases. Following this principle, we give a simple criterion on how to do the relabelling in the \( t \)-th iteration based on the DQ obtained from the previous iteration, i.e., based on \( Q_{t-1} \), where \( Q_0 \) can be any initial DQ. Denote a relabelling result by an \( N \)-tuple \((i_1, i_2, \ldots, i_N)\), which is a permutation of \( 1, 2, \ldots, N \) and means that \( y_i \) is relabelled with \( i \) (or relabelled as \( y_i \)) for \( 1 \leq i \leq N \). Recall that \( Q_{t-1}^{-1}(z) \) is the preimage of \( z \in Z \) with respect to \( Q_{t-1} : Y \to Z \). The relabelling criterion is given below.

- Relabelling criterion: For each \( z \in Z \), elements in \( Q_{t-1}^{-1}(z) \) are relabelled with continuous numbers.

Relabelling following this criterion can ensure \( Q_t \) to be at least as good as \( Q_{t-1} \), because after the relabelling, \( Q_{t-1} \) is made to be an SDQ while \( Q_t \) is an optimal SDQ found in the update step.

We give a practical way of relabeling which follows the aforementioned relabelling criterion. First, create an arbitrary permutation \((z_1, z_2, \ldots, z_M)\) of \( 1, 2, \ldots, M \). Then, randomly relabel elements in \( Q_{t-1}^{-1}(z_i) \) with \( S_i + 1, S_i + 2, \ldots, S_i + |Q_{t-1}^{-1}(z_i)| \), where

\[
S_i = \begin{cases} 
0 & i = 1, \\
\sum_{j=1}^{i-1} |Q_{t-1}^{-1}(z_j)| & i = 2, \ldots, M.
\end{cases}
\]

For example, assume \( N = 3, M = 2, Q_{t-1}^{-1}(1) = \{y_1, y_3\} \), and \( Q_{t-1}^{-1}(2) = \{y_2\} \). If \((z_1, z_2) = (1, 2)\), the relabelling result can be either \((1, 3, 2)\) or \((3, 1, 2)\); if \((z_1, z_2) = (2, 1)\), the relabelling result can be either \((2, 1, 3)\), or \((2, 3, 1)\). Each of the four relabelling result will make \( Q_{t-1} \) to be an SDQ.

If a DQ obtained by using the methods proposed in \([3], [14]–[17]\) is used as the initial quantizer \( Q_0 \), the IDP algorithm is able to improve the quality of \( Q_0 \) after running for some iterations. We verify this by simulations. In particular, we use \( q = 16, N = 100, \) and \( M = 16, 17, \ldots, 32 \). We assume a uniform input distribution, i.e., \( P_X(x_1) = \cdots = P_X(x_q) \), and \( P_{Y|X} \) is randomly generated. We investigate the performance of the five quantizers below in terms of the MI gap \( \Delta I = I(X;Y) - I(X;Z) \).

- KL-means \((Q_{KL})\): We randomly choose \( M \) out of \( N \) points as the initial means (see (16)) for 50 times, and for each time the KL-means algorithm \([3], [10]\) runs for 50 iterations to design a DQ. The final \( Q_{KL} \) is given by the best \( (\Delta I \) is minimized) DQ among the 50 times.
- KL-means & IDP: \( Q_{KL} \) is used as the initial DQ, and then the final DQ is designed by using the IDP algorithm with 50 iterations.
- Greedy Comb. \((Q_{GC})\): \( Q_{GC} \) is designed by using the GC algorithm \([16]\).
- Greedy Comb. & KL-means: \( Q_{GC} \) is used to compute the initial means, and then the final DQ is designed by using the KL-means algorithm with 50 iterations.
- Greedy Comb. & IDP \((Q_{GC})\): \( Q_{GC} \) is used as the initial DQ, and then the final DQ is designed by using the IDP algorithm with 50 iterations.

We test 50 random cases at each simulation point. The average performance is shown in Fig. 5. We can see that the KL-means algorithm performs the worst (it can outperform the GC algorithm at smaller \( q \) and \( M \)), and cannot achieve obvious improvement over \( Q_{GC} \) when using \( Q_{GC} \) to compute the initial means. In contrast, the IDP algorithm can achieve obvious improvement over \( Q_{KL} \) and \( Q_{GC} \) when using them as initial DQs.

When the GC algorithm is used for generating initial DQs for the IDP algorithm, the computational complexity for the quantizer (i.e., \( Q_{GI} \)) design is \( O(qN^2(N - M)) + q(N - M)^2MT_{max} \), where \( O(qN^2(N - M)) \) and \( O(q(N - M)^2MT_{max}) \) are the complexities of the GC algorithm and...
IDP algorithm with maximum iteration $T_{\text{max}}$, respectively. Thus, the computational complexity for designing $Q_{GC}$ is generally dominated by the complexity of the GC algorithm for a general situation where $N$ is large and $M$ is small. To make the design of $Q_{GC}$ more time-efficient, in the rest of this section, we discuss how to reduce of complexity of the GC algorithm.

Recall that the original GC algorithm [5, 15] works in $N - M$ stages. In each stage, the loss for combining each pair of elements of $\mathcal{Y}$ is computed and compared, and then the pair with the minimum loss is replaced by a new element combined from them ($|\mathcal{Y}|$ is reduced by 1 per stage). The term $qN^2$ in the complexity of the original GC algorithm is due to the computation of all pairs’ combining loss at each stage. We note that starting from the second stage, about $(|\mathcal{Y}| - 1)^2$ out of $|\mathcal{Y}|(|\mathcal{Y}| - 1)$ pairs’ combining loss is recomputed with respect to the computation at the previous stage. The computational complexity of the original GC algorithm can be reduced by removing these recomputation and by efficiently finding the pair with the minimum combing loss instead of comparing each pair with complexity $O(N^2)$.

To this end, we can use a data structure, called heap [8, Chapter 6], to store all pairs’ combining loss. A heap (min-heap) has an important property: it can be regarded as a complete binary tree where the combining loss stored at an inner node is not smaller than that stored at its parent node. Thus, the root always stores the pair with the minimum combining loss. The modified GC algorithm is illustrated by the two steps below.

- For the first stage, all pairs’ combining loss is used to build a heap. Then, the pair stored at the root, say $(y_1, y_2)$, is combined to form a new element, say $y$, and $\mathcal{Y}$ is updated to $\mathcal{Y} \cup \{y\} \setminus \{y_1, y_2\}$. Meanwhile, the root is deleted from the heap and the heap is updated to maintain the aforementioned heap property.
- Starting from the second stage, denote the new element inserted in $\mathcal{Y}$ at the previous stage by $y_p$. We compute the combining loss of each pair $(y, y')$, $y' \in \mathcal{Y} \setminus \{y\}$, and insert $(y_p, y')$ together with its combining loss to the heap. Then, we repeat to delete the root and update the heap until the pair stored at the root, say $(y_1', y_2')$, satisfies $y_1', y_2' \in \mathcal{Y}$. Next, $(y_1', y_2')$ is combined to a new element, say $y_n$, and $\mathcal{Y}$ is updated to $\mathcal{Y} \cup \{y_n\} \setminus \{y_1', y_2'\}$. Finally, the root (where $(y_1', y_2')$ is stored) is deleted from the heap and the heap is updated to maintain the aforementioned heap property.

We name the modified GC algorithm heap-optimized GC algorithm. The complexity for building the heap in the first stage is linear to the number of nodes in the heap, i.e., $O(N^2)$. Throughout the heap-optimized GC algorithm, less than $2N^2$ pairs’ combining loss is computed with complexity $O(q)$ per computation. In addition, the total number of insertion, deletion, and update operations over the heap is less than $6N^2$ with complexity $O(\log_2 N)$ per operation. Therefore, the heap-optimized GC algorithm has computational complexity of $O(qN^2 + N^2 \log_2 N)$, being much faster than the original GC algorithm with computational complexity $O(qN^2(N - M))$.

However, the heap-optimized GC algorithm has an extra storage complexity $O(N^2)$ due to the needs of storing all pairs’ combining loss.

V. Conclusion

In this paper, we have first presented a general framework for applying DP to DMC quantization. The DP has complexity $O(q(N - M)^2 M)$. Starting from the condition that $w(\cdot, \cdot)$ satisfies the QI, we then applied two efficient techniques to reduce the complexity of DP. One technique making use of the SMAWQ algorithm has complexity $O(q(N - M)M)$. The other one is much easier to be implemented but has complexity $O(q(N^2 - M^2))$. Next, for the SDQ that maximizes $I_\alpha(X; Z)$, we gave the sufficient condition of such that the aforementioned two low-complexity algorithms can be used to replace the original DP. This condition works for $q \geq 2$, covering the previous work [10] and [11] for $q = 2$ as a subcase. Simulation results show that when $\mathcal{Y}$ satisfies 14, DP can provide better quantization performance than the greedy combining [3, 15] and KL-means 16 algorithms in terms of the MI gap, with a much lower computational complexity. Moreover, we proposed a new idea called IDP for DQ design. Theoretical analysis and simulation results show that IDP can improve the prior art design presented in 3, 14–17.

Appendix A

Proof of Lemma 2

For $2 \leq m \leq n < N$, let $t = \text{sol}(n, m)$ for brevity. For any $m - 1 \leq k < t$, we have

$$
\begin{align*}
dp_t(n + 1, m) - \dp_k(n + 1, m) & = \dp_t(n, m) - w(t + 1, n) + w(t + 1, n + 1) - \dp_k(n, m) - w(k + 1, n) + w(k + 1, n + 1) \\
& \leq w(t + 1, n + 1) + w(k + 1, n) - w(t + 1, n) - w(k + 1, n + 1) \\
& \leq 0,
\end{align*}
$$

where the last inequality holds because $w(\cdot, \cdot)$ satisfies the QI. Therefore, we have $\text{sol}(n, m) = t \leq \text{sol}(n + 1, m)$.

We now continue to prove $\text{sol}(n, m) \geq \text{sol}(n, m - 1)$ for $m = 2$, we have $\text{sol}(n, m) \geq \text{sol}(n, m - 1) = 0$ trivially. For $m \geq 3$, let $t = \text{sol}(n, m - 1)$ for brevity. For any $m - 1 \leq k < t$, we have

$$
\begin{align*}
dp_t(n, m) - \dp_k(n, m) & = \dp(t, m - 1) + \dp_t(n, m - 1) - \dp(t, m - 2) - \dp(k, m - 1) + \dp_k(n, m - 1) - \dp(k, m - 2) \\
& \leq \dp(k, m - 2) + \dp(t, m - 1) - \dp(k, m - 1) - \dp(t, m - 2).
\end{align*}
$$

We continue the proof by first proving the following lemma.

**Lemma 3:** For $2 \leq m \leq i < j \leq N$, denoting $\dp(i, m - 1) + \dp(j, m) - \dp(i, m) - \dp(j, m - 1)$ by $\psi(i, j, m)$, we have $\psi(i, j, m) \leq 0$. 

Proof: Let \( t = \text{sol}(i, 2) \) for brevity. We have
\[
\psi(i, j, 2) \leq dp(i, 1) + dp_i(j, 2) - dp_i(j, 2) - dp(j, 1) \\
= w(1, i) + w(t + 1, j) - w(t + 1, i) - w(1, j) \\
\leq 0.
\]
We then inductively prove \( \psi(i, j, m) \leq 0 \) for \( m \geq 3 \) given \( \psi(i, j, m - 1) \leq 0 \) for \( m - 1 \leq i < j \).

Let \( a = \text{sol}(i, m) \) and \( b = \text{sol}(j, m - 1) \) for brevity. Note that \( m - 1 \leq a < i \). If \( a < b \), we have
\[
\psi(i, j, m) \\
\leq dp_i(i, m - 1) + dp_i(j, m) - dp_a(i, m) - dp_a(j, m - 1) \\
= w(1, i) + w(a + 1, j) - w(a + 1, i) - w(1, j) \\
\leq 0.
\]

If \( a \geq b \), we have
\[
\psi(i, j, m) \\
\leq dp_i(i, m - 1) + dp_i(j, m) - dp_a(i, m) - dp_a(j, m - 1) \\
= w(1, i) + w(a + 1, j) - w(a + 1, i) - w(1, j) \\
\leq 0.
\]
This completes the proof of Lemma 3.

Our goal is to prove
\[
w_\alpha(r, s) + w_\alpha(r', s') - w_\alpha(r', s) - w_\alpha(r, s') \leq 0 \tag{17}
\]
for \( \alpha \in (0, \infty) \) and for all \( 1 \leq r < r' \leq s < s' \leq N \) given the condition of (14). The proof is divided into four parts based on the four situations of \( \alpha = 1, \alpha = \infty, \alpha \in (0, 1), \) and \( \alpha \in (1, \infty) \). We first recall the fact below, as it will be frequently used in the proof of (17).

Fact 1: For four arbitrary positive real numbers \( u_1, u_2, v_1, \) and \( v_2 \), if \( u_1 / u_2 \geq v_1 / v_2 \), we have \( u_1 / u_2 \geq (u_1 + v_1) / (u_2 + v_2) \geq v_1 / v_2 \).

Proof: The proof is ignored since it is trivial.

Part 1: \( \alpha = 1 \)

For any three vectors \( a = (a_i)_{1 \leq i \leq q}, b = (b_i)_{1 \leq i \leq q}, \) and \( c = (c_i)_{1 \leq i \leq q} \) with \( a_i, b_i, c_i > 0 \), we use the following notations:

i) \( \|a\|_1 = \sum_{i=1}^{q} a_i \),
ii) \( g(a) = \sum_{i=1}^{q} a_i \log \frac{\|a\|_1}{a_i} \),
iii) \( a + b = (a_1 + b_1)_{1 \leq i \leq q} \),
iv) \( f(a, b, c) = g(a + b) + g(b + c) - g(a + b + c) - g(b) \).

Here \( g(a) \) uses the natural logarithm in base \( e \). For other bases, the following proof can be similarly carried out.

Set \( a, b, c \) with \( a_i, b_i, c_i \) given by
\[
a_i = \sum_{j=r}^{r'-1} PX(x_i)P_{Y|X}(y_j|x_i), \\
b_i = \sum_{j=r'}^{s} PX(x_i)P_{Y|X}(y_j|x_i), \\
c_i = \sum_{j=s+1}^{t} PX(x_i)P_{Y|X}(y_j|x_i).
\]

Given the condition of (14) and according to Fact 1, we have
\[
a_i / a_{i+1} \geq b_i / b_{i+1} \geq c_i / c_{i+1} \tag{18}
\]
for \( 1 \leq i < q \).

For \( j = 1, \ldots, q - 1 \), let
\[
\psi_j = (a_{j+1}b_j) / (a_j b_{j+1}) \tag{19}
\]
and
\[
\psi_j = (c_{j+1}b_j) / (c_j b_{j+1}) \tag{20}
\]
For \( j = 0, 1, \ldots, q - 1 \), let \( a^j = (a^j_i)_{1 \leq i \leq q} \) with
\[
a^j_i = \begin{cases} a_i & i > j, \\
\psi_j c^{j-1} & i \leq j; \end{cases} \tag{21}
\]
and
\[
\psi_j = (c^j_i)_{1 \leq i \leq q} \tag{22}
\]
According to (19)–(22), we have
\[
a^j_i / a^j_{i+1} = b_i / b_{i+1} = c^j_i / c^j_{i+1} \tag{23}
\]
for \( 1 \leq i \leq j < q \). In addition, we have
\[
f(a^j, b, c^0) = w_\alpha(r, s) + w_\alpha(r', s') - w_\alpha(r', s) - w_\alpha(r, s') \tag{24}
\]
Moreover, based on (23), we have
\[
\frac{a^j_i}{\|a^j\|_1} = \frac{b_i}{\|b\|_1} = \frac{c^j_i}{\|c^{j-1}\|_1} \tag{25}
\]
for \( 1 \leq i \leq q \). Then, based on (25) and Fact 1, it is easy to prove
\[
f(a^{q-1}, b, c^{q-1}) = 0. \tag{26}
\]
In the rest of this part, we finish the proof of (17) for \( \alpha = 1 \) by proving
\[
f(a^0, b, c^0) \leq f(a^1, b, c^1) \leq \cdots \leq f(a^{q-1}, b, c^{q-1}). \tag{27}
\]
According to (18)–(20), we have \( u_j \leq 1 \leq v_j \) for \( 1 \leq j < q \). For any given \( j, 1 \leq j < q \), let \( u \) and \( v \) be variables (real numbers) belonging to \([u_j, 1]\) and \([1, v_j]\), respectively. Denote \( m(u) = (m_i)_{1 \leq i \leq q} \) with
\[
m_i = \begin{cases} a_i & i > j, \\
\psi_j c^{j-1} & i \leq j. \end{cases}
\]
Denote \( n(v) = (n_i)_{1 \leq i \leq q} \) with

\[
    n_i = \begin{cases} 
      c_i & i > j, \\
      v c_i^{1-1} & i \leq j.
    \end{cases}
\]

According to (18) and (23), we can easily verify that

\[
    m_i \leq b_i \leq n_i
\]

holds for \( 1 \leq i \leq q \) and \( 1 \leq k \leq j \). We then have

\[
    \frac{\| m(u) \|}{m_k} \leq \frac{\| b \|}{b_k} \leq \frac{\| n(v) \|}{n_k}
\]

for \( 1 \leq k \leq j \). Let \( h(u,v) = f(m(u), b, n(v)) \). Then, according to (29) and Fact [1] we have

\[
    \frac{\partial h}{\partial u} = \sum_{i=1}^{j} \left( a_i^{j-1} \log \frac{\| m(u) + b \|_1}{m_i + b_i} \right) - \sum_{i=1}^{j} \left( a_i^{j-1} \log \frac{\| m(u) + b + n(v) \|_1}{m_i + b_i + n_i} \right)
\]

\[
    = \sum_{i=1}^{j} a_i^{j-1} \left( \log \frac{\| m(u) \|_1 + \| b \|_1}{m_i + b_i} - \log \frac{\| m(u) \|_1 + \| b + n(v) \|_1}{m_i + b_i + n_i} \right)
\]

\[
    \leq 0,
\]

implying \( h(u,v) \) keeps decreasing as \( u \) increases from \( u_j \) to 1; meanwhile, we similarly have

\[
    \frac{\partial h}{\partial v} = \sum_{i=1}^{j} \left( c_i^{j-1} \log \frac{\| b + n(u) \|_1}{b_i + n_i} \right) - \sum_{i=1}^{j} \left( c_i^{j-1} \log \frac{\| m(u) + b + n(v) \|_1}{m_i + b_i + n_i} \right)
\]

\[
    \geq 0,
\]

implying \( h(u,v) \) keeps increasing as \( v \) increases from 1 to \( v_j \). Therefore, we have

\[
    f(a_q^{-1}, b, c) = h(1, 1) \leq h(u_j, v_j) = f(a_j, b, c),
\]

implying (27) and (17) are correct.

Part II: \( \alpha = \infty \)

Set \( a, b, c \) with \( a_i, b_i, c_i \) given by

\[
    a_i = \sum_{j=r}^{r'-1} P_{Y_j | X}(y_j | x_i),
\]

\[
    b_i = \sum_{j=r'}^{s} P_{Y_j | X}(y_j | x_i),
\]

\[
    c_i = \sum_{j=s+1}^{s'} P_{Y_j | X}(y_j | x_i).
\]

In addition, let \( i = \arg \max_{1 \leq t \leq q} (a_t + b_t), j = \arg \max_{1 \leq t \leq q} (b_t + c_t), k = \arg \max_{1 \leq t \leq q} (a_t + b_t + c_t), \) and \( l = \arg \max_{1 \leq t \leq q} b_t \). Then, we have

\[
    w_{\alpha}(r, s) + w_{\alpha}(r', s') - w_{\alpha}(r, s') - w_{\alpha}(r', s) = -(a_i + b_i) - (b_j + c_j) + (a_k + b_k + c_k) + b_l
\]

\[
    = -(a_i + b_i) - (b_j + c_j) + (a_k + b_k) + (b_l + c_l)
\]

\[
    \leq c_k - c_l.
\]

Based on a similar deduction, we indeed have \( w_{\alpha}(r, s) + w_{\alpha}(r', s') - w_{\alpha}(r, s') - w_{\alpha}(r', s) \leq \min\{a_k - a_t, b_l - b_k, c_k - c_l\} \). Note that given the condition of (14) and according to Fact [1] (18) also holds in this case. If \( \min\{a_k - a_t, b_l - b_k, c_k - c_l\} > 0 \), we have \( a_k/a_t > 1, b_l/b_k < 1, \) and \( c_k/c_l > 1 \), leading to a contradiction to (18). Therefore, we have \( \min\{a_k - a_t, b_l - b_k, c_k - c_l\} \leq 0 \), implying (17) is correct.

Part III: \( \alpha \in (0, 1) \)

The proof in this part is similar to that in Part I for \( \alpha = 1 \). We may skip some details that have been given in Part I.

For any three vectors \( a = (a_i)_{1 \leq i \leq q}, b = (b_i)_{1 \leq i \leq q}, \) and \( c = (c_i)_{1 \leq i \leq q} \) with \( a_i, b_i, c_i > 0 \), we use the following notations:

i) \( \| a \|_1 = \sum_{i=1}^{q} a_i, \)

ii) \( g(a) = \sum_{i=1}^{q} P_X(x_i) a_i^\alpha, \)

iii) \( a + b = (a_i + b_i)_{1 \leq i \leq q}, \)

iv) \( f(a, b, c) = g^{1/\alpha}(a + b) + g^{1/\alpha}(b + c) - g^{1/\alpha}(a + b + c) - g^{1/\alpha}(b). \)

Set \( a, b, c \) in the same way as in Part II for \( \alpha = \infty \). In addition, for \( j = 0, 1, \ldots, q - 1 \), we define \( a^j \) and \( c^j \) in the same way as in (21) and (22), respectively. It can be easily verified that (23)–(25) hold in this case. Then, based on (25) and Fact [1], we have

\[
    g^{1/\alpha}(a^{q-1} + b) = g^{1/\alpha}(b + c^{q-1}) = g^{1/\alpha}(a^{q-1} + b + c^{q-1}) = g^{1/\alpha}(b).
\]

leading to

\[
    f(a^{q-1}, b, c^{q-1}) = 0.
\]

Thus, our task again becomes to prove (27) like in Part I.

Recall that (18) holds given the condition of (14) and according to Fact [1]. For a given \( j, 1 \leq j < q \), we define \( u, v, m(u), \) and \( n(v) \) in the same way as in Part I. It can be easily verified that (28) still holds in this case. Let
\[ h(u, v) = f(m(u), b, n(v)). \] Given (28) and Fact 1 we have
\[
\frac{\partial h}{\partial u} = \sum_{i=1}^{j} \left( P_X(x_i) a_i^{-1} \left( \frac{g(m(u) + b)}{(m_i + b_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right) - \sum_{i=1}^{j} \left( P_X(x_i) a_i^{-1} \left( \frac{g(m(u) + b + n(v))}{(m_i + b_i + n_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right)
\]
\[= \sum_{i=1}^{j} \left( P_X(x_i) a_i^{-1} \left( \frac{\sum_{k=1}^{q} P_X(x_k) (m_k + b_k)}{(m_i + b_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right) - \sum_{i=1}^{j} \left( P_X(x_i) a_i^{-1} \left( \frac{\sum_{k=1}^{q} P_X(x_k) (m_k + b_k + n_k)}{(m_i + b_i + n_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right) \]
\[\leq 0\]
and
\[
\frac{\partial h}{\partial v} = \sum_{i=1}^{j} \left( P_X(x_i) c_i^{-1} \left( \frac{g(b + n(v))}{(b_i + n_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right) - \sum_{i=1}^{j} \left( P_X(x_i) c_i^{-1} \left( \frac{g(m(u) + b + n(v))}{(m_i + b_i + n_i)^{\alpha}} \right)^{\frac{1}{1-\alpha}} \right) \]
\[\geq 0.\]

Then, we have
\[ f(a^{j-1}, b, c^{j-1}) = h(1, 1) \leq h(u_j, v_j) = f(a^j, b, c^j), \]
implying (27) and (17) are correct.

**Part IV:** \( \alpha \in (1, \infty) \)

We ignore the proof in this part since it can be carried out almost the same as that in Part III for \( \alpha \in (0, 1) \). Thus, we complete the proof of (17).

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