WELL-POSEDNESS FOR A HIGHER-ORDER, NONLINEAR, DISPERSIVE EQUATION ON A QUARTER PLANE

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Abstract. The focus of the current paper is the higher order nonlinear dispersive equation which models unidirectional propagation of small amplitude long waves in dispersive media. The specific interest is in the initial-boundary value problem where spatial variable lies in \( \mathbb{R}^+ \), namely, quarter plane problem. With proper requirement on initial and boundary condition, we show local and global well posedness.

1. Introduction. A class of higher-order models for unidirectional water wave of the form

\[
\eta_t + \eta_x - \gamma_1 \beta \eta_{xxx} + \gamma_2 \beta \eta_{xxxx} + \delta_1 \beta^2 \eta_{xxxxxt} + \delta_2 \beta^2 \eta_{xxxxxx} + \frac{3}{4} \alpha (\eta^2)_x + \alpha \beta \left( \gamma (\eta^2)_{xx} - \frac{7}{48} \eta^2 \right)_x - \frac{1}{8} \alpha^2 (\eta^3)_x = 0
\]

(1)

has recently been derived by Bona, Carvajal, Panthee and Scialom [4]. With appropriate choices of the parameters \( \gamma_1, \gamma_2, \delta_1, \delta_2 \) and \( \gamma \), this equation serves as a model for the propagation of small-amplitude, long-crested surface waves moving to the direction of increasing values of the spatial variable \( x \). Here \( \alpha \) is a typical ratio of wave amplitude to depth, \( \beta \) is a representative value of the square of the depth to wavelength and \( t \) is proportional to elapsed time. The dependent variable \( \eta = \eta(x, t) \) is a real-valued function of \( x \in \mathbb{R}, t \geq 0 \) representing the deviation of the free surface from its undisturbed position at the point corresponding to \( x \) at time \( t \).

This model subsists on the assumptions that \( \alpha \) and \( \beta \) are comparably-sized small quantities while \( \eta \) and its first few partial derivatives are of order one. Moreover, \( \gamma_1 \) and \( \gamma_2 \) are restricted by \( \gamma_1 + \gamma_2 = \frac{1}{6} \), see Bona et al [4].

If the terms quadratic in \( \alpha \) and \( \beta \) moved to the right-hand side, this model takes the form

\[
\eta_t + \eta_x - \gamma_1 \beta \eta_{xxx} + \gamma_2 \beta \eta_{xxxx} + \frac{3}{4} \alpha (\eta^2)_x = O(\beta^2, \alpha \beta, \alpha^2) = O(\alpha^2) = O(\beta^2).
\]

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If the higher-order terms are neglected entirely, we obtain the well-known BBM-KdV equation

$$\eta_t + \eta_x - \gamma_1 \beta \eta_{xxx} + \gamma_2 \beta \eta_{xxx} + \frac{3}{4} \alpha (\eta^2)_x = 0.$$ 

Using the relation

$$\eta_t + \eta_x = O(\alpha, \beta),$$

hence

$$\eta_{xxx} = -\eta_{xxx} + O(\alpha, \beta),$$

after neglecting quadratic terms in $$\alpha$$ and $$\beta$$, we obtain the well-known KdV-equation

$$\eta_t + \eta_x + \frac{1}{6} \beta \eta_{xxx} + \frac{3}{4} \alpha (\eta^2)_x = 0 \quad (2)$$

and the BBM-equation

$$\eta_t + \eta_x - \frac{1}{6} \beta \eta_{xxx} + \frac{3}{4} \alpha (\eta^2)_x = 0, \quad (3)$$

respectively, (see [1], [13]). At the formal level, the error involved in using (2) or (3) to approximate the full, inviscid water-wave problem is locally of order $$\alpha^2 \sim \beta^2$$ while the error in the higher-order model (1) is of order $$\alpha^3 \sim \beta^3$$.

Bona and el. [4] investigated the pure initial-value problem of (1). They showed that if the coefficients $$\gamma_1$$ and $$\delta_1$$ appearing in front of $$\eta_{xxx}$$ and $$\eta_{xxxxx}$$-terms are positive, then it is locally well-posed in $$H^s(\mathbb{R})$$ for $$s \geq 1$$. However, the problem is linearly ill-posed in Sobolev classes if $$\delta_1 < 0$$. (The case $$\delta_1 = 0$$ is also locally well-posed, but suffers from other modeling issues, hence not being considered.) Furthermore, if $$\gamma = \frac{7}{48}$$, then it is globally well-posed in $$H^s(\mathbb{R})$$ for $$s \geq \frac{3}{2}$$.

If we make use of the lowest-order relation $$\eta_x = -\eta_t + O(\alpha, \beta)$$, then formally $$\eta_{xxxxx} = -\eta_{xxx} + O(\alpha, \beta)$$. Replacing the term $$\eta_{xxxxx}$$ with $$-\eta_{xxx} + O(\alpha, \beta)$$ in (1) gives

$$\eta_t + \eta_x - \gamma_1 \beta \eta_{xxx} + \gamma_2 (\delta_1 - \delta_2) \beta^2 \eta_{xxxxx} + \frac{3}{4} \alpha (\eta^2)_x$$

$$+ \alpha \beta \left( \gamma (\eta^2)_x - \frac{7}{48} \eta_x^2 \right)_x - \frac{1}{8} \alpha^2 (\eta^3)_x = O(\alpha^3, \beta^3). \quad (4)$$

As the additional error committed is cubic in $$\alpha$$ and $$\beta$$, the order of accuracy of the approximation is not affected. Consistent with this remark, the linear dispersion relation for (1) and (4) with the terms cubic in $$\alpha$$ and $$\beta$$ ignored are

$$\omega = \frac{k - \gamma_2 \beta k^3 + \delta_2 \beta^2 k^5}{1 + \gamma_1 \beta k^2 + \delta_1 \beta^2 k^4}$$

and

$$\omega = \frac{k - \gamma_2 \beta k^3}{1 + \gamma_1 \beta k^2 + (\delta_1 - \delta_2) \beta^2 k^4}$$

respectively. In this scaling, the wavenumber $$k$$ is of order 1, so the two dispersive relation agree to order $$\beta^3$$.

As [4] pointed out that boundary-value problem may be the most practically interesting, in some circumstances, see also [1], [2], [3], [6], [7], [8], [9] and the references in them. So it is our purpose here to address this issue.
Considered in the current study is the equation
\[
\eta_t + \eta_x - \gamma_1 \beta \eta_{xx} + \gamma_2 \beta \eta_{xxx} + (\delta_1 - \delta_2) \beta^2 \eta_{xxxx} + \frac{3}{4} \alpha (\eta^2)_x + \alpha \beta \left( (\eta^2)_{xx} - \frac{7}{48} \eta^2_x \right) - \frac{1}{8} \alpha^2 (\eta'^3)_x = 0
\]  \tag{5}
posed for \( x, t \geq 0 \), namely, quarter plane problem. This system describes model for long water waves of small but finite amplitude, generated by a wave-maker at the left-hand end and propagating to the right in a uniform, open channel. The channel is taken to be infinitely long to avoid dealing with a boundary condition at the right. Indeed, the model equation (5) is only valid for waves moving to the right. In a real channel, as soon as the wave motion reaches the right-hand end of the channel, comparison between theory and experiment needs to cease because the reflected waves are not described by this one-way equation (see [9] and references in [9]). So it is sometime called wave maker problem.

When the BBM-equation (3) is used to model the wave-maker problem, a single boundary data \( \eta(0, t) \), which can be measured by right equipments, in addition to an initial data \( \eta(x, 0) \), is sufficient. However, the equation (5) has fourth order derivative with respect to spatial variable \( x \), an additional boundary data is needed. Here is the proposed initial-boundary condition
\[
\begin{cases}
\eta(x, 0) = \eta_0(x) & \text{for } x \geq 0, \\
\eta(0, t) = G(t), & \eta_{xx}(0, t) = J(t) \quad \text{for } t \geq 0
\end{cases}
\]  \tag{6}
with the compatibility condition
\[
\eta_0(0) = G(0), \quad \eta_0''(0) = J(0),
\]  \tag{7}
giving that \( G \) and \( J \) are order one functions, independent of parameters \( \alpha \) and \( \beta \).

As mentioned earlier, Bona et al. [4] showed that the problem (1) with pure initial condition is linearly ill posed when \( \delta_1 \), the coefficient of the \( \eta_{xx} \) term, is negative (whilst \( \delta_1 = 0 \) is not relative to the modeling issue,) hence, it is naturally assumed \( \delta = \delta_1 - \delta_2 \), the coefficient of \( \eta_{xxxx} \) term in (5), to be positive throughout the current paper. Our attention now turns to the initial-boundary-value problem (5)-(6)-(7).

For the analysis that follows, make variable rescaling: \( \tilde{x} = \beta^{-\frac{1}{2}} x \), \( \tilde{t} = \beta^{-\frac{1}{2}} t \) and \( u(\tilde{x}, \tilde{t}) = \alpha \eta(x, t) \). Under these new variable, the problem (5)-(6) reduces to the following non-dimensional 5th order, nonlinear, dispersive equation
\[
\begin{cases}
u_t + u_x - \gamma_1 \beta \nu_{xx} + \gamma_2 \beta \nu_{xxx} + \frac{3}{4} \alpha (\nu'^2)_x + \alpha \beta \left( (\nu'^2)_{xx} - \frac{7}{48} \nu'^2_x \right) - \frac{1}{8} \alpha^2 (\nu'''^3)_x = 0, \\
u(\tilde{x}, 0) = \alpha \eta_0(\beta^{\frac{1}{2}} \tilde{x}), \quad u(0, \tilde{t}) = \alpha G(\beta^{\frac{1}{2}} \tilde{t}), \quad u_{xx}(0, \tilde{t}) = \alpha \beta J(\beta^{\frac{3}{2}} \tilde{t})
\end{cases}
\]  \tag{8}
for \( \tilde{x}, \tilde{t} \geq 0 \). Notice that, \( \alpha, \beta \) only appear in the auxiliary data. Drop the tildes to obtain the BBM-type equation
\[
\begin{cases}
u_t + u_x - \gamma_1 u_{xx} + \gamma_2 u_{xxx} + \frac{3}{4} \alpha (u'^2)_x + \alpha \beta \left( (u'^2)_{xx} - \frac{7}{48} (u'^2)_x \right) - \frac{1}{8} \alpha^2 (u'''^3)_x = 0, \\
u(x, 0) = \alpha \eta_0(\beta^{\frac{1}{2}} x), \quad u(0, t) = \alpha G(\beta^{\frac{1}{2}} t), \quad u_{xx}(0, t) = \alpha \beta J(\beta^{\frac{3}{2}} t)
\end{cases}
\]  \tag{9}
for \( x, t \geq 0 \).

**Remark 1.** As \( \eta_0 \) is order 1 function and independent of the physical parameters \( \alpha, \beta \) which are very small, so that the initial condition \( \alpha \eta_0(\beta^{\frac{1}{2}}x) \) is a wave form of small amplitude with long wave length.

We remind that the current work is motivated by many people’s work, (see [5], [7], [8], [10], [11] and [12]), there are also many further references in the works just cited.

Since (5)-(6) and (9) are equivalent, the equation in (9) has less number of parameters, we aim to bring the theory for (9) more closely into line with that appearing in [4] for the pure initial-value problem. The main result in this paper is as follows.

**Theorem 1.1.** Problem (9) is well posed locally in time if the boundary data \( G, J \in C([0, \infty)) \), the initial data \( \eta_0 \) lies in Sobolev space \( H^s(\mathbb{R}^+) \) for some \( s \geq 1 \) and is twice continuously differentiable locally at \( x = 0 \), and \( G, J \) and \( \eta_0 \) satisfy the compatibility condition (7). Moreover, if \( \gamma = \frac{7}{38} \) and \( s \geq 2 \), \( G, J \in L_1(\mathbb{R}^+) \cap L_3(\mathbb{R}^+) \) and the value \( \|\eta_0\|_{H^s(\mathbb{R}^+)} + \|G\|_{L_1(\mathbb{R}^+)} + \|G\|_{L_3(\mathbb{R}^+)} + \|J\|_{L_1(\mathbb{R}^+)} + \|J\|_{L_3(\mathbb{R}^+)} \) is of order 1, then (9) is well posed globally in time, the solution \( u \) lies in space \( C([0, \infty); H^s(\mathbb{R}^+)) \), its \( H^s(\mathbb{R}^+) \)-norm has the following growth bound in \( t \),

\[
\|u(\cdot, t)\|_s \leq c\alpha(\|G(\beta^{\frac{1}{2}}t)\| + \|J(\beta^{\frac{1}{2}}t)\|) + P_{\lfloor s \rfloor - 2}(t),
\]

where \( c = O(1) \) is a constant independent of \( t \), \( P_k \) is polynomial function of degree \( k \), \( \lfloor s \rfloor = \inf\{n \in \mathbb{N} : n \geq s \} \).

The next section is to derive an integral equation equivalent to (9), and provides a detailed statement of the results in view. Analyses leading to the conclusions advertised in the main result are presented in Sections 3 and 4. Section 3 consists of preliminaries and local well-posedness, while Section 4 provides conditions and proof of global well-posedness.

2. Derivation of the integral equation from (9).

2.1. Preliminary calculations. We commence this section with the following boundary value problem of a second order ordinary differential equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\rho^2 - \partial_{xx})v(x) = f(x), \quad &\text{for } x \geq 0, \\
v(0) = v_0, \quad &\text{if } x \text{ is finite as } x \to \infty,
\end{array} \right.
\end{aligned}
\]

(10)

where \( \rho \) is a positive number. It is straightforward calculation that the problem has a unique solution

\[
v(x) = v_0e^{-\rho x} + \int_0^\infty G_{\rho}(x, z)f(z)dz,
\]

(11)

where

\[
G_{\rho}(x, z) = \frac{1}{2\rho} \left( e^{-\rho|x-z|} - e^{-\rho(x+z)} \right).
\]

(12)

**Proposition 1.** For any \( \rho_1, \rho_2 > 0 \), and \( \rho_1 \neq \rho_2 \), the following fourth order ordinary differential equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\rho_1^2 - \partial_{xx})(\rho_2^2 - \partial_{xx})v(x) = f(x), \quad &\text{for } x \geq 0, \\
v(0) = v_0, \quad &v_{xx}(0) = v_2, \quad &\text{if } x \text{ is finite as } x \to \infty
\end{array} \right.
\end{aligned}
\]

(13)
has a unique solution
\[ v(x) = v_0 \frac{\rho_2^2 e^{-\rho_1 x} - \rho_1^2 e^{-\rho_2 x}}{\rho_2^2 - \rho_1^2} + v_2 \frac{e^{-\rho_2 x} - e^{-\rho_1 x}}{\rho_2^2 - \rho_1^2} + \int_0^\infty G_1(x, z) f(z) \, dz \]  
(14)

where
\[ G_1(x, z) = \frac{\rho_1 e^{-\rho_2 (x+z)} - \rho_2 e^{-\rho_1 (x+z)} - \rho_1 e^{-\rho_2 |x-z|} + \rho_2 e^{-\rho_1 |x-z|}}{2\rho_1 \rho_2 (\rho_2^2 - \rho_1^2)}. \]  
(15)

Proof. Let \( \tilde{v} = (\rho_2^2 - \partial_{xx}) v \), then (13) reduces to
\[
\begin{cases}
(\rho_1^2 - \partial_{xx}) \tilde{v}(x) = f(x), & x > 0, \\
\tilde{v}(0) = \rho_2^2 v_0 - v_2, & \tilde{v}(x) \text{ is finite as } x \to \infty.
\end{cases}
\]  
(16)

Hence,
\[ \tilde{v}(x) = (\rho_2^2 v_0 - v_2) e^{-\rho_1 x} + \frac{1}{2\rho_1} \int_0^\infty \left( e^{-\rho_1 |x-z|} - e^{-\rho_1 (x+z)} \right) f(z) \, dz. \]

Solve
\[
\begin{cases}
(\rho_2^2 - \partial_{xx}) v(x) = \tilde{v}(x), & x \geq 0, \\
v(0) = v_0, & v(x) \text{ is finite as } x \to \infty
\end{cases}
\]  
(17)

to get the solution
\[ v(x) = v_0 e^{-\rho_2 x} + \frac{1}{2\rho_2} \int_0^\infty \left( e^{-\rho_2 |x-z|} - e^{-\rho_2 (x+z)} \right) \tilde{v}(z) \, dz. \]

Elementary calculus reduces the last integral to the desired result in (14). \( \square \)

Proposition 2. For any \( \rho > 0 \), the following fourth order equation
\[
\begin{cases}
(\rho^2 - \partial_{xx})^2 v(x) = f(x), & x \geq 0, \\
v(0) = v_0, & v_{xx}(0) = v_2, & v(x) \text{ is finite as } x \to \infty
\end{cases}
\]  
(18)

has a unique solution
\[ v(x) = v_0 e^{-\rho x} + \frac{\rho^2 v_0 - v_2}{2\rho} xe^{-\rho x} + \int_0^\infty G_2(x, z) f(z) \, dz \]
\[ = v_0 (1 + \frac{1}{2\rho} \rho x e^{-\rho x} - \frac{v_2}{2\rho} xe^{-\rho x}) + \int_0^\infty G_2(x, z) f(z) \, dz \]

where
\[ G_2(x, z) = \frac{1}{4\rho^3} \left( e^{-\rho |x-z|} + \rho |x-z| e^{-\rho |x-z|} - e^{-\rho (x+z)} - \rho (x+z) e^{-\rho (x+z)} \right). \]  
(19)

The proof is similar to that of Proposition 1.

Proposition 3. For any \( \rho > 0 \) and any \( \theta \in (0, \frac{\pi}{2}) \), the following equation
\[
\begin{cases}
(\rho^2 e^{2\theta} - \partial_{xx}) (\rho^2 e^{-2\theta} - \partial_{xx}) v(x) = f(x), & x \geq 0, \\
v(0) = v_0, & v_{xx}(0) = v_2, & v(x) \text{ is finite as } x \to \infty
\end{cases}
\]  
(20)

has a unique solution
\[
\begin{align*}
v(x) &= \frac{v_0}{\sin 2\theta} e^{-\rho x \cos \theta} \sin (2\theta + \rho x \sin \theta) - \frac{v_2}{\rho^2 \sin 2\theta} e^{-\rho x \cos \theta} \sin (\rho x \sin \theta) \\
&\quad + \int_0^\infty G_3(x, z) f(z) \, dz
\end{align*}
\]  
(21)
where
\[ G_3(x, z) = \frac{1}{2\rho^3 \sin 2\theta} \left( e^{-\rho|x-z|\cos \theta} \sin (\theta + \rho|x - z| \sin \theta) - e^{-\rho(x+z)\cos \theta} \sin (\theta + \rho(x + z) \sin \theta) \right). \] (22)

Proof. In (14), replace \( \rho_1 \) and \( \rho_2 \) with \( \rho e^{-i\theta} \) and \( \rho e^{i\theta} \) respectively, upon simplification, it follows the solution described in (21). \( \square \)

2.2. Derivation of integral equation of problem (9). The problem under consideration is the initial-boundary value problem (9) with the compatibility conditions (7).

In this section, the letter \( f \) represents the following
\[ f(x,t) = f(u,u_x,u_{xx}) = u + \gamma_2 u_{xx} + \frac{3}{4} u^2 + \gamma(u^2)_{xx} - \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \] (23)
and denote
\[ u_0(x) = \alpha_0 \beta \frac{1}{(\beta^2 + 1)} \gamma \] to reduce messy form involving \( \alpha \) and \( \beta \). Then the differential equation in (9) is briefed as
\[ \begin{cases} (I - \gamma_1 \partial_{xx} + \delta \partial_{xxxx}) u_t = -\partial_x f \\ u(x,0) = u_0(x), \quad u(0,t) = g(t), \quad u_{xx}(0,t) = j(t). \end{cases} \] (25)

The operator \( I - \gamma_1 \partial_{xx} + \delta \partial_{xxxx} \) is going to be inverted based on values \( \gamma_1 \) and \( \delta \) taken in following three cases.

Case I. It is assumed that \( \delta \in (0, \frac{\gamma_1^2}{4}) \), or \( \gamma_1 > 2\sqrt{\delta} \), so (25) is written alternatively as
\[ (\rho_1^2 - \partial_{xx})(\rho_2^2 - \partial_{xx}) u_t = -\frac{1}{\delta} \partial_x f(u,u_x,u_{xx}) \] (26)
where \( \rho_1, \rho_2 > 0 \) are given as
\[ \rho_1^2 = \frac{\gamma_1 - \sqrt{\gamma_1^2 - 4\delta}}{2\delta}, \quad \rho_2^2 = \frac{\gamma_1 + \sqrt{\gamma_1^2 - 4\delta}}{2\delta}. \] (27)

Apply Proposition 1 to (26) coupled with the auxiliary data in (25), it obtains the following differential-integral equation
\[ u_t = u_t(x,t) = \frac{\rho_2^2 e^{-\rho_1 x} - \rho_1^2 e^{-\rho_2 x}}{\rho_2^2 - \rho_1^2} g'(t) + \frac{e^{-\rho_2 x} - e^{-\rho_1 x}}{\rho_2^2 - \rho_1^2} h'(t) \]
\[ -\frac{1}{\delta} \int_0^\infty G_1(x,z) f_z \, dz, \]
where \( G_1 \) is defined in (15). Take (27) into consideration, then
\[ \frac{1}{\delta} G_1(x,z) = \frac{\sqrt{\delta}}{2\sqrt{\gamma_1^2 - 4\delta}} \left( \rho_1 e^{-\rho_2 (x+z)} - \rho_2 e^{-\rho_1 (x+z)} - \rho_1 e^{-\rho_2 |x-z|} + \rho_2 e^{-\rho_1 |x-z|} \right). \]

Hence,
\[ u_t(x,t) = \frac{\delta(\rho_2^2 e^{-\rho_1 x} - \rho_1^2 e^{-\rho_2 x})}{\sqrt{\gamma_1^2 - 4\delta}} g'(t) + \frac{\delta(e^{-\rho_2 x} - e^{-\rho_1 x})}{\sqrt{\gamma_1^2 - 4\delta}} j'(t) \]
\[ -\frac{\sqrt{\delta}}{2\sqrt{\gamma_1^2 - 4\delta}} \int_0^\infty \left( \rho_1 e^{-\rho_2 (x+z)} - \rho_2 e^{-\rho_1 (x+z)} - \rho_1 e^{-\rho_2 |x-z|} + \rho_2 e^{-\rho_1 |x-z|} \right) \partial_z f(z,t) \, dz. \]
Integrations by parts with respect to $z$ yield the following equation
\[ u_t(x,t) = \frac{\delta(\rho_0^2 e^{-\rho_1 x} - \rho_1^2 e^{-\rho_2 x})}{\sqrt{\gamma_1^2 - 4\delta}} g'(t) + \frac{\delta(e^{-\rho_2 x} - e^{-\rho_1 x})}{\sqrt{\gamma_1^2 - 4\delta}} h'(t) \]
\[ + \int_0^\infty K_1(x,z) f \, dz \]  
(28)

where
\[ K_1(x,z) = \frac{1}{2\sqrt{\gamma_1^2 - 4\delta}} \left( - e^{-\rho_2(x+z)} + e^{-\rho_1(x+z)} - \text{sgn}(x-z)e^{-\rho_2|x-z|} + \text{sgn}(x-z)e^{-\rho_1|x-z|} \right). \]  
(29)

Integration with respect to $t$ yields the following integral equation,
\[ u(x,t) = u_0(x) + \frac{\delta(\rho_0^2 e^{-\rho_1 x} - \rho_1^2 e^{-\rho_2 x})}{\sqrt{\gamma_1^2 - 4\delta}} (g(t) - g(0)) \]
\[ + \frac{\delta(e^{-\rho_2 x} - e^{-\rho_1 x})}{\sqrt{\gamma_1^2 - 4\delta}} (j(t) - j(0)) + \int_0^t \int_0^\infty K_1(x,z) f(z,s) \, dz \, ds. \]  
(30)

Case II. When $\delta = \frac{\gamma_1^2}{4}$, or $\gamma_1 = 2\sqrt{\delta}$, equation (25) can be written as
\[ (\rho^2 - \partial_{xx}) u_t = -\frac{1}{\delta} \partial_x f \]  
(31)

where
\[ \rho = \delta^{-\frac{1}{4}} = \frac{\sqrt{2}}{\sqrt{\gamma_1}}. \]  
(32)

By Proposition 2,
\[ u_t(x,t) = \left( 1 + \frac{\rho}{2} x e^{-\rho x} g'(t) - \frac{1}{2\rho} x e^{-\rho x} h'(t) - \frac{1}{\delta} \int_0^\infty G_2(x,z) \partial_x f \, dz \right) \]  
(33)

where $G_2$ is given in (19). Hence
\[ \frac{1}{\delta} G_2(x,z) = \frac{\rho}{4} (e^{-\rho|x-z|} + \rho |x-z| e^{-\rho|x-z|} - e^{-\rho(x+z)} - \rho (x+z) e^{-\rho(x+z)}). \]

Integration by parts in (33) with respect to $z$ and then integrating with respect to $t$, it follows
\[ u(x,t) = u_0(x) + \left( 1 + \frac{\rho}{2} x e^{-\rho x} (g(t) - g(0)) - \frac{1}{2\rho} x e^{-\rho x} (j(t) - j(0)) \right) \]
\[ + \int_0^t \int_0^\infty K_2(x,z) f(z,s) \, dz \, ds \]  
(34)

where
\[ K_2(x,z) = \frac{\rho^3}{4} \left( (x+z)e^{-\rho(x+z)} + (x-z)e^{-\rho|x-z|} \right). \]  
(35)

Case III. The last consideration is for the value of $\delta$ strictly greater than $\frac{\gamma_1^2}{4}$, or $0 < \gamma_1 < 2\sqrt{\delta}$, so equation (25) is same as
\[ (\rho^2 e^{2i\theta} - \partial_{xx})(\rho^2 e^{-2i\theta} - \partial_{xx}) u_t = -\frac{1}{\delta} \partial_x f \]  
(36)

with
\[ \rho = \delta^{-\frac{1}{4}} \quad \text{and} \quad \theta = \frac{1}{2} \arcsin \frac{\sqrt{4\delta - \gamma_1^2}}{2\sqrt{\delta}}. \]  
(37)
Again, application of Proposition 3 to (25) leads to

\[ u_t(x, t) = \frac{e^{-\rho x \cos \theta} \sin (2 \theta + \rho x \sin \theta)}{\sin 2 \theta} g'(t) - \frac{e^{-\rho x \cos \theta} \sin (\rho x \sin \theta)}{\rho^2 \sin 2 \theta} j'(t) \]

\[ - \frac{1}{\delta} \int_0^\infty G_3(x, z) \partial_z f \, dz \]

\[ = \frac{2\sqrt{3} e^{-\rho x \cos \theta} \sin (2 \theta + \rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} g'(t) - \frac{2\delta e^{-\rho x \cos \theta} \sin (\rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} j'(t) \]

\[ - \frac{1}{\delta} \int_0^\infty G_3(x, z) \partial_z f \, dz \]

where

\[ \frac{1}{\delta} G_3(x, z) = \frac{1}{2\delta \rho^4 \sin 2 \theta} \left( e^{-\rho |x-z| \cos \theta} \sin \left( \theta + \rho |x-z| \sin \theta \right) \right. \]

\[ - e^{-\rho |x+z| \cos \theta} \sin \left( \theta + \rho (x+z) \sin \theta \right) \right) \]

\[ = \frac{\delta \rho}{\sqrt{4\delta - \gamma_1^2}} \left( e^{-\rho |x-z| \cos \theta} \sin \left( \theta + \rho |x-z| \sin \theta \right) \right. \]

\[ - e^{-\rho |x+z| \cos \theta} \sin \left( \theta + \rho (x+z) \sin \theta \right) \right). \]  

(38)

In (38), integrations by parts with respect to \( z \) first and then integration with \( t \) together with (37) gives the following form

\[ u(x, t) = u_0(x) + \frac{2\sqrt{3} e^{-\rho x \cos \theta} \sin (2 \theta + \rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} \left( g(t) - g(0) \right) \]

\[ - \frac{2\delta e^{-\rho x \cos \theta} \sin (\rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} (j(t) - j(0)) + \int_0^\infty K_3(x, z) f \, dz, \]  

(40)

where

\[ K_3(x, z) = \frac{1}{\sqrt{4\delta - \gamma_1^2}} \left\{ e^{-\rho (x+z) \cos \theta} \left( \cos \theta \sin (\theta + \rho (x+z) \sin \theta) \right) \right. \]

\[ - \sin \theta \cos (\theta + \rho (x+z) \sin \theta) \right) \]

\[ + \sign(x-z) e^{-\rho |x-z| \cos \theta} \left( \cos \theta \sin (\theta + \rho |x-z| \sin \theta) \right) \]

\[ - \sin \theta \cos (\theta + \rho |x-z| \sin \theta) \left\} \right. \right) \]

\[ = \frac{1}{\sqrt{4\delta - \gamma_1^2}} \left\{ e^{-\rho (x+z) \cos \theta} \sin (\rho (x+z) \sin \theta) \right. \]

\[ + \sign(x-z) e^{-\rho |x-z| \cos \theta} \sin (\rho |x-z| \sin \theta) \right\}. \]  

(41)

Summarize the three cases just discussed, the quarter plane problem (9) now is converted to the following integral equation,

\[ u(x, t) = u_0(x) + \mu(x, t) + \int_0^t K f(u, u_x, u_{xx})(x, s) \, ds \]

\[ =: A u, \]  

(42)
where

\[ \mu(x, t) = \phi_1(x)(g(t) - g(0)) + \phi_2(x)(j(t) - j(0)), \]

\[ \phi_1(x) = \begin{cases} 
\frac{\delta(e^{\rho_2 x} - e^{\rho_1 x})}{\sqrt{\gamma_1^2 - 4\delta}} & \text{for } \delta \in (0, \frac{\gamma_1}{4}) \text{ or } \gamma_1 > 2\sqrt{\delta}, \\
(1 + \frac{\rho}{2} x)e^{-\rho x} & \text{for } \delta = \frac{\gamma_1}{4} \text{ or } \gamma_1 = 2\sqrt{\delta}, \\
\frac{2\sqrt{\delta} e^{-\rho x} \cos \theta \sin (2\theta + \rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} & \text{for } \delta > \frac{\gamma_1}{4} \text{ or } \gamma_1 < 2\sqrt{\delta},
\end{cases} \]

\[ \phi_2(x) = \begin{cases} 
\frac{\delta(e^{-\rho_2 x} - e^{-\rho_1 x})}{\sqrt{\gamma_1^2 - 4\delta}} & \text{for } \delta \in (0, \frac{\gamma_1}{4}) \text{ or } \gamma_1 > 2\sqrt{\delta}, \\
-\frac{1}{2\rho} x e^{-\rho x} & \text{for } \delta = \frac{\gamma_1}{4} \text{ or } \gamma_1 = 2\sqrt{\delta}, \\
-\frac{2\delta e^{-\rho x} \cos \theta \sin (\rho x \sin \theta)}{\sqrt{4\delta - \gamma_1^2}} & \text{for } \delta > \frac{\gamma_1}{4} \text{ or } \gamma_1 < 2\sqrt{\delta},
\end{cases} \]

and the operator \( K \) is defined as

\[ K_v(x) = \int_0^\infty K(x, z)v(z) \, dz \]

with the integral kernel

\[ K(x, z) = \begin{cases} 
\frac{1}{2\sqrt{\gamma_1^2 - 4\delta}} \left\{ e^{-\rho_2(x + z)} + e^{-\rho_1(x + z)} - \text{sgn}(x - z)e^{-\rho_2|x - z|} \right. \\
+ \text{sgn}(x - z)e^{-\rho_1|x - z|} \left. \right\} & \text{for } \delta \in (0, \frac{\gamma_1}{4}) \text{ or } \gamma_1 > 2\sqrt{\delta}, \\
\frac{1}{4\delta^2} \left( (x + z)e^{-\rho(x + z)} + (x - z)e^{-\rho|x - z|} \right) & \text{for } \delta = \frac{\gamma_1}{4} \text{ or } \gamma_1 = 2\sqrt{\delta}, \\
\frac{1}{\sqrt{4\delta - \gamma_1^2}} \left\{ e^{-\rho(x + z) \cos \theta} \sin (\rho(x + z) \sin \theta) \right. \\
+ \text{sgn}(x - z)e^{-\rho|x - z| \cos \theta} \sin (\rho|x - z| \sin \theta) \left. \right\} & \text{for } \delta > \frac{\gamma_1}{4} \text{ or } \gamma_1 < 2\sqrt{\delta}
\end{cases} \]

in which \( \rho = \delta^{-\frac{1}{2}}, \rho_1, \rho_2, \theta \) dependent on \( \gamma_1, \delta \) are given in the corresponding context in (27) and (37), respectively.

**Remark 2.** \( \phi_1 \) and \( \phi_2 \) satisfy the following fourth order ordinary differential equations, respectively,

\[ \begin{cases} 
\phi_1(x) - \gamma_1 \phi_1''(x) + \delta \phi_1'''(x) = 0 & \text{for } x > 0 \\
\phi_1(0) = 1, \quad \phi_1'(0) = 0 & \text{\( \phi_1(x) \) is finite as } x \to \infty
\end{cases} \]

and

\[ \begin{cases} 
\phi_2(x) - \gamma_1 \phi_2''(x) + \delta \phi_2'''(x) = 0, & \text{for } x > 0 \\
\phi_2(0) = 0, \quad \phi_2''(0) = 1 & \text{\( \phi_2(x) \) is finite as } x \to \infty.
\end{cases} \]
3. Function classes and local well-posedness. The following notation for function classes is used throughout the discussion. All functions are real-valued and if a spatial domain is not specified for a function class, it is presumed to be $\mathbb{R}^+ = [0, \infty)$. Notation $C(X)$ is a standard continuous function space. However, its standard Fréchet-space topologies will not intervene in the analysis. Similarly, noted below.

Consists of all functions in $C(X)$ if a spatial domain is not specified for a function class, it is presumed to be $\mathbb{R}$. Function classes is used throughout the discussion. All functions are real-valued and $3.406$ HONGQIU CHEN

1. The Banach space $C_b^m(\mathbb{R}^+)$ consists of all continuous and bounded functions defined on $\mathbb{R}^+$ whose derivatives up to order $m$ are also continuous and bounded. The norm on this space is

$$
\|u\|_{C_b^m(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} \{|u(x)|, \cdots, |u^{(m)}(x)|\}.
$$

2. The Banach space $L_p = L_p(\mathbb{R}^+)$, for $1 \leq p < \infty$, is the standard $p$th-power Lebesgue-integrable functions with the usual modifications for the case $p = \infty$. The norm of a function $f \in L_p$ with $1 \leq p \leq \infty$ is written $\|f\|_p$. The Sobolev space of $L_2$-functions whose distributional derivatives up to order $m \geq 0$ also lie in $L_2$ is denoted simply $H^m$. These spaces carry their standard Hilbert-space structures with its norm $\| \cdot \|_m$ defined as

$$
\|f\|^2_m = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \int_{0}^{\infty} |f^{(j)}(x)|^2 \, dx.
$$

When $m = 0$, $H^0 = L_2$, its norm is simply denoted $\| \cdot \|_0$.

3. For $s > 0$ to be an non-integer, the Sobolev space $H^s(\mathbb{R}^+)$ is the set of restriction of functions from $H^s(\mathbb{R})$ to $\mathbb{R}^+$ equipped with the norm

$$
\|f\|_s = \|f\|_{H^s(\mathbb{R}^+)} = \inf \{\|g\|_{H^s(\mathbb{R})} : g \in H^s(\mathbb{R}), g|_{\mathbb{R}^+} = f\},
$$

where

$$
H^s(\mathbb{R}) = \{f \in L_2(\mathbb{R}) : \int_{-\infty}^{\infty} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty\}
$$

with the norm

$$
\|f\|^2_{H^s(\mathbb{R})} = \int_{-\infty}^{\infty} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, d\xi.
$$

When $s = m$ is a positive integer, the norm

$$
\|f\|^2_{H^s(\mathbb{R})} = \int_{-\infty}^{\infty} (1 + \xi^2)^{m} |\hat{f}(\xi)|^2 \, d\xi = \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \int_{-\infty}^{\infty} |f^{(j)}(x)|^2 \, dx.
$$

4. If $X$ is a Banach space, the space $C(0, T; X)$ of continuous functions from $[0, T]$ to $X$ will be found useful. The norm on this space is the usual one, viz.

$$
\|f\|_{C(0, T; X)} = \max_{0 \leq t \leq T} \|f(\cdot, t)\|_X.
$$

5. Triple

$$(\psi, g, j) : \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$
is said to be admissible if all three functions $\psi, g, j$ are continuous locally at the origin 0, and moreover, $\psi$ is second order differentiable locally at 0, they satisfy

$$\psi(0) = g(0), \quad \psi''(0) = j(0).$$

(See (7) as how the class from which the auxiliary data is drawn, $\psi, g$ and $j$ are not required to be continuous everywhere.)

Our attention now turns to the integral equation (42), in which we begin with properties of the operator $K$ given in (46).

**Proposition 4.** The function $K = K(x, z)$ defined in (47) is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$, it is infinitely smooth for $x \neq z$. Moreover, $K_z, K_x, \partial_x^{2k+1} K$ and $\partial_x^{2k+1} K$, for $k = 1, 2, \ldots$, are continuous functions on $\mathbb{R}^+ \times \mathbb{R}^+$ in the sense that the discontinuity at $x = z$ is removable. However, $\partial_x^{2k} K = \partial_x^{2k}$, for $k = 1, 2, \ldots$, have jump discontinuity at $x = z$ as follows

$$K_{xx}(x, x-) - K_{xx}(x, x+) = K_{zz}(x, x-) - K_{zz}(x, x+) = -\delta^{-1}$$

and

$$\partial_x^{2k} K(x, x-) - \partial_x^{2k} K(x, x+) = \partial_x^{2k} K(x, x-) - \partial_x^{2k} K(x, x+) = P_k(\delta^{-1})$$

where $P_k$ is a $k^{th}$ degree polynomial function.

**Proof.** It is straightforward calculations that $K_z, K_x, \partial_x^{2k+1} K$ and $\partial_x^{2k+1} K$, for $k = 1, 2, \ldots$, are continuous functions, and $\partial_x^{2k} K(x, z) = \partial_x^{2k} K(x, z)$ for $k = 1, 2, \ldots$. Form (50) can be easily verified by the representation of $K$ given in (47). Notice that

$$\partial_{xxxx} K(x, z) = \partial_{zzzz} K(x, z) = \gamma_1 \delta^{-1} \partial_{xx} K(x, z) - \delta^{-1} K(x, z),$$

it implies that for $k \geq 2$,

$$\partial_x^{2k} K(x, z) = \gamma_1 \delta^{-1} \partial_x^{2(k-1)} K(x, z) - \delta^{-1} \partial_x^{2(k-2)} K(x, z),$$

and inductively,

$$\partial_x^{2k} K(x, z) = P_{k-1}(\delta^{-1}) \partial_{xx} K(x, z) + P_{k-2}(\delta^{-1}) K(x, z)$$

where $P_{k-1}, P_{k-2}$ are polynomial functions of degree $k - 1, k - 2$ respectively. This together with (50) completes (51). \(\Box\)

**Proposition 5.** The operator $K$ maps $C^1_b(\mathbb{R}^+)$ to $C^{k+3}_b(\mathbb{R}^+)$, and the mapping is continuous. Moreover, if $v \in C_b(\mathbb{R}^+)$ and $\lim_{x \to \infty} v(x) = 0$, then

$$\lim_{x \to \infty} K v(x) = \lim_{x \to \infty} \frac{d}{dx} K v(x) = \lim_{x \to \infty} \frac{d^2}{dx^2} K v(x) = \lim_{x \to \infty} \frac{d^3}{dx^3} K v(x) = 0.$$

**Proof.** As the integral kernel $K$ has properties that

$$K(x, z), \quad K_x(x, z) \in C_b(\mathbb{R}^+ \times \mathbb{R}^+),$$

and

$$\sup_{x \in \mathbb{R}^+} \int_0^\infty |K(x, z)| \, dz < \infty, \quad \sup_{x \in \mathbb{R}^+} \int_0^\infty |K_x(x, z)| \, dz < \infty,$$

so for any $v \in C_b(\mathbb{R}^+)$,

$$K v, \quad \frac{d}{dx} K v(x) = \int_0^\infty K_x(x, z) v(z) \, dz, \quad \frac{d^2}{dx^2} K v(x) = \int_0^\infty K_{xx}(x, z) v(z) \, dz$$

(52)
all lie in \( C_b(\mathbb{R}^+) \) with
\[
\| K v \|_{C_b(\mathbb{R}^+)} \leq \sup_{x \in \mathbb{R}^+} \int_0^\infty \| K(x, z) \| v \|_{C_b(\mathbb{R}^+)} \ , \tag{53}
\]
\[
\| \frac{d}{dx} K v(x) \|_{C_b(\mathbb{R}^+)} \leq \sup_{x \in \mathbb{R}^+} \int_0^\infty \| K_x(x, z) \| \| v \|_{C_b(\mathbb{R}^+)} \ , \tag{54}
\]
and
\[
\| \frac{d^2}{dx^2} K v(x) \|_{C_b(\mathbb{R}^+)} \leq \sup_{x \in \mathbb{R}^+} \int_0^\infty \| K_{xx}(x, z) \| \| v \|_{C_b(\mathbb{R}^+)} . \tag{55}
\]

On the other hand, by property (50),
\[
\frac{d^3}{dx^3} K v(x) = (K_{xx}(x, x) - K_{xx}(x, x+)) v(x)
+ \int_0^x K_{xxx}(x, z) v(z) \, dz + \int_x^\infty K_{xxx}(x, z) v(z) \, dz
= -\frac{1}{\delta} v(x) + \int_0^\infty K_{xxx}(x, z) v(z) \, dz . \tag{56}
\]
It still lies in \( C_b(\mathbb{R}^+) \) and
\[
\| \frac{d^3}{dx^3} K v \|_{C_b(\mathbb{R}^+)} \leq \left( \frac{1}{\delta} + \sup_{x \in \mathbb{R}^+} \int_0^\infty \| K_{xxx}(x, z) \| \right) \| v \|_{C_b(\mathbb{R}^+)} . \tag{57}
\]
It is seen that the operator \( K \) maps \( C_b^m(\mathbb{R}^+) \) to \( C_b^m(\mathbb{R}^+) \) continuously and \( K v \) has regularity exactly three orders higher than that of \( v \). Suppose that \( K \) maps \( C_b^m(\mathbb{R}^+) \) to \( C_b^{m+3}(\mathbb{R}^+) \) continuously for \( 0 \leq m \leq k - 1 \), that is, there is a constant \( c_m > 0 \) such that
\[
\| K v \|_{C_b^{m+3}(\mathbb{R}^+)} \leq c_m \| v \|_{C_b^m(\mathbb{R}^+)}
\]
for \( v \in C_b^m(\mathbb{R}^+) \), then for \( v \in C_b^k(\mathbb{R}^+) \),
\[
\frac{d^4}{dx^4} K v(x) = -\frac{1}{\delta} \frac{d}{dx} v(x) + (K_{xxx}(x, x) - K_{xxx}(x, x+)) v(x)
+ \int_0^x K_{xxxx}(x, z) v(z) \, dz + \int_x^\infty K_{xxxx}(x, z) v(z) \, dz
= -\frac{1}{\delta} \frac{d}{dx} v(x) + \int_0^x K_{xxxx}(x, z) v(z) \, dz + \int_x^\infty K_{xxxx}(x, z) v(z) \, dz
= -\frac{1}{\delta} \frac{d}{dx} v(x) + \frac{1}{\delta} \int_0^x (\gamma_1 K_{xx}(x, z) - K(x, z)) v(z) \, dz
+ \frac{1}{\delta} \int_x^\infty (\gamma_1 K_{xx}(x, z) - K(x, z)) v(z) \, dz
\]
\[
= -\frac{1}{\delta} \frac{d}{dx} v(x) + \frac{\gamma_1}{\delta} \frac{d^2}{dx^2} K v(x) - \frac{1}{\delta} K v(x) \tag{58}
\]
implies that
\[
\frac{d^{k+3}}{dx^{k+3}} K v(x) = \frac{d^{k-1}}{dx^{k-1}} \frac{d^4}{dx^4} K v(x)
= -\frac{1}{\delta} \frac{d}{dx} v(x) + \frac{\gamma_1}{\delta} \frac{d^{k+1}}{dx^{k+1}} K v(x) - \frac{1}{\delta} \frac{d^{k-1}}{dx^{k-1}} K v(x) \tag{59}
\]
lies in \( C_b(\mathbb{R}^+) \) with
\[
\| K v(x) \|_{C_b^{k+3}(\mathbb{R}^+)} \leq c_k \| v \|_{C_b^k(\mathbb{R}^+)}
\]
for some constant $c_k$. By induction, $\mathcal{K}$ maps $C^k_b(\mathbb{R}^+)$ to $C^{k+3}_b(\mathbb{R}^+)$ continuously for all $k = 0, 1, 2, \cdots$.

If $v \in C_b(\mathbb{R}^+)$ and $\lim_{x \to \infty} v(x) = 0$, then for any $\epsilon > 0$, there exists $M = M_\epsilon > 0$ such that $|v(x)| \leq \epsilon$ for $|x| > M$. Whence

$$|\int_0^\infty K(x, z)v(z) \, dz| \leq \int_0^M |K(x, z)v(z)| \, dz + \int_M^\infty |K(x, z)v(z)| \, dz \leq \int_0^M |K(x, z)| \, dz \|v\|_{C_b(\mathbb{R}^+)} + \epsilon \int_M^\infty |K(x, z)| \, dz.$$ 

As $\lim_{x \to \infty} K(x, z) = 0$ uniformly for $z \in [0, M]$ and $\sup_z \int_0^\infty |K(x, z)| \, dz = c < \infty$, it transpires that

$$\lim_{x \to \infty} \int_0^\infty K(x, z)v(z) \, dz \leq 0 + c\epsilon = c\epsilon.$$

That $\epsilon > 0$ is arbitrary indicates

$$\lim_{x \to \infty} Kv(x) = \lim_{x \to \infty} \int_0^\infty K(x, z)v(z) \, dz = 0.$$ 

Similar argument shows that

$$\lim_{x \to \infty} \frac{d}{dx} Kv(x) = \lim_{x \to \infty} \int_0^\infty \partial_x K(x, z)v(z) \, dz = 0,$$

$$\lim_{x \to \infty} \frac{d^2}{dx^2} Kv(x) = \lim_{x \to \infty} \int_0^\infty \partial_{xx} K(x, z)v(z) \, dz = 0$$

and

$$\lim_{x \to \infty} \frac{d^3}{dx^3} Kv(x) = \frac{1}{\delta} \lim_{x \to \infty} v(x) + \lim_{x \to \infty} \int_0^\infty \partial_{xxx} K(x, z)v(z) \, dz = 0.$$ 

The proposition is established. \qed

**Proposition 6.** The operator $\mathcal{K}$ defined in (46) maps $H^k = H^k(\mathbb{R}^+)$ to $H^{k+3}$ continuously for $k = 0, 1, 2, \cdots$. However, it maps $L_1 = L_1(\mathbb{R}^+)$ only to $H^2$ continuously.

**Proof.** Introduce a function $\tilde{K} : \mathbb{R} \to \mathbb{R}$ as follows,

$$\tilde{K}(x) = \begin{cases} 
\frac{1}{2\sqrt{4\delta^2 - 4\delta}} \left( -e^{-\rho_2|x|} + e^{-\rho_1|x|} \right) & \text{if } \delta \in (0, \frac{\gamma_1^2}{4}), \\
\frac{1}{4\delta^\frac{3}{2}} |x| e^{-\rho|x|} & \text{if } \delta = \frac{\gamma_1^2}{4}, \\
\frac{1}{\sqrt{4\delta - \gamma_1}} e^{-\rho|x| \cos \theta} \sin (\rho|x| \sin \theta) & \text{if } \delta > \frac{\gamma_1^2}{4},
\end{cases}$$

(60)

in which $\rho = \delta^{-\frac{3}{4}}$, and $\rho_1, \rho_2, \theta$ depend on $\delta$ and $\gamma_1$ are given in the corresponding context in (27) and (37), respectively. Then when $x$ and $z$ are restricted to $\mathbb{R}^+$,

$$K(x, z) = \tilde{K}(x + z) + \text{sgn}(x - z) \tilde{K}(x - z).$$

(61)

If $v$ is a function from $\mathbb{R}^+ \to \mathbb{R}$, then for $x \in \mathbb{R}^+$,

$$\mathcal{K}v(x) = \int_0^\infty \text{sgn}(x - z) \tilde{K}(x - z)v(z) \, dz + \int_{-\infty}^0 \text{sgn}(x - z) \tilde{K}(x - z)v(-z) \, dz$$

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\[ \int_{-\infty}^{\infty} \text{sgn}(x-z) \tilde{K}(|x-z|) v(|z|) \, dz. \quad (62) \]

Since the integral kernel and its first derivatives are continuous functions,

\[ \frac{d}{dx} \mathcal{K}v(x) = \int_{-\infty}^{\infty} \tilde{K}'(|x-z|) v(|z|) \, dz, \quad (63) \]

\[ \frac{d^2}{dx^2} \mathcal{K}v(x) = \int_{-\infty}^{\infty} \text{sgn}(x-z) \tilde{K}''(|x-z|) v(|z|) \, dz. \quad (64) \]

However, the second order derivative of the integral kernel has a jump discontinuity at \( x = z \) as follows,

\[ \frac{d^3}{dx^3} \mathcal{K}v(x) = -\frac{1}{\delta} v(x) + \int_{-\infty}^{\infty} \tilde{K}'''(|x-z|) v(|z|) \, dz. \quad (65) \]

It is seen that for \( v \in L_2(\mathbb{R}^+) \),

\[ \frac{d}{dx} \mathcal{K}v, \frac{d^2}{dx^2} \mathcal{K}v, \frac{d^3}{dx^3} \mathcal{K}v \in L_2(\mathbb{R}^+) \]

satisfy

\[ \| \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \| \tilde{K} \|_{L_1(\mathbb{R})} \| v \|_{L_2(\mathbb{R}^+)}. \quad (66) \]

Similarly,

\[ \| \frac{d}{dx} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \| \tilde{K}' \|_{L_1(\mathbb{R})} \| v \|_{L_2(\mathbb{R}^+)}; \]

\[ \| \frac{d^2}{dx^2} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \| \tilde{K}'' \|_{L_1(\mathbb{R})} \| v \|_{L_2(\mathbb{R}^+)} \]

and

\[ \| \frac{d^3}{dx^3} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \frac{1}{\delta} \| v \|_{L_2(\mathbb{R}^+)} + \| \tilde{K}''' \|_{L_1(\mathbb{R})} \| v \|_{L_2(\mathbb{R}^+)}. \]

Inductively, if \( v \in H^k(\mathbb{R}^+) \), then \( \frac{d^k}{dx^k} \mathcal{K}v \in L_2(\mathbb{R}^+) \) satisfies

\[ \| \frac{d^{k+3}}{dx^{k+3}} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \frac{1}{\delta} \| v^{(k)} \|_{L_2(\mathbb{R}^+)} + \frac{\gamma_1}{\delta} \| \frac{d^{k+1}}{dx^{k+1}} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} + \frac{1}{\delta} \| \frac{d^{k-1}}{dx^{k-1}} \mathcal{K}v \|_{L_2(\mathbb{R}^+)}. \]

for some constant \( c > 0 \). The first part of the proposition is established.

Since \( \tilde{K}, \tilde{K}', \tilde{K}'' \in L_p(\mathbb{R}^+) \) for all \( p \geq 1 \), if \( v \in L_1(\mathbb{R}^+) \), then

\[ \mathcal{K}v, \frac{d}{dx} \mathcal{K}v, \frac{d^2}{dx^2} \mathcal{K}v \in H^2(\mathbb{R}^+) \]

with

\[ \| \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \sqrt{2} \| \tilde{K} \|_{L_2(\mathbb{R})} \| v \|_{L_1(\mathbb{R}^+)}. \quad (67) \]

Similarly,

\[ \| \frac{d}{dx} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \sqrt{2} \| \tilde{K}' \|_{L_2(\mathbb{R})} \| v \|_{L_1(\mathbb{R}^+)}, \]

\[ \| \frac{d^2}{dx^2} \mathcal{K}v \|_{L_2(\mathbb{R}^+)} \leq \sqrt{2} \| \tilde{K}'' \|_{L_2(\mathbb{R})} \| v \|_{L_1(\mathbb{R}^+)}. \]

Which is equivalent to say that \( \mathcal{K}v \in H^2(\mathbb{R}^+) \) with

\[ \| \mathcal{K}v \|_{H^2(\mathbb{R}^+)} \leq c \| v \|_{L_1(\mathbb{R}^+)} \]

for some constant \( c > 0 \). It means that \( \mathcal{K} \) maps \( L_1(\mathbb{R}^+) \) to \( H^2(\mathbb{R}^+) \) continuously. The proposition is complete. \( \square \)
Lemma 3.1. If boundary data $g,j \in C(\mathbb{R}^+)$, the initial data $u_0 \in C_b^2(\mathbb{R}^+)$ and the triple $(u_0,g,j)$ is admissible, then the integral equation (42) is locally well posed in time. Precisely, there is a small value $T$ that depends on $\|u_0\|_{C_b^2(\mathbb{R}^+)}$, $\|\phi_1\|_{C_b^2(\mathbb{R}^+)}$, $\|\phi_2\|_{C_b^2(\mathbb{R}^+)}$, $g$ and $j$ such that (42) has a unique solution $u$ which lies in space $C(0,T;C_b^2(\mathbb{R}^+))$. Moreover, the correspondence between initial-boundary data $(u_0,g,j)$ and the associated solution $u$ of (42) is a Lipschitz continuous mapping from any bounded set of $((u_0,g,j) \in C_b^2(\mathbb{R}^+) \times C(\mathbb{R}^+) \times C(\mathbb{R}^+)) \mapsto (\psi,g,j)$ is admissible} into $C(0,T;C_b^2(\mathbb{R}^+))$, and $u$ satisfies $u(0,0) = g(0)$, $u_{xx}(0,0) = j(0)$. 

Proof. Let $T > 0$ be arbitrary and $u \in C(0,T;C_b^2(\mathbb{R}^+))$, then it is clear that

$$f(x,t) = f(u_x, u_u, u_{xx}) = u + \gamma_2 u_x + \frac{3}{4} u^2 + \gamma u_{xx} - \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \in C(0,T;C_b(\mathbb{R}^+)).$$

Whence, $Kf(u_x, u_u, u_{xx}) \in C(0,T;C_b(\mathbb{R}^+))$ as $K$ maps $C_b(\mathbb{R}^+)$ to $C_b^2(\mathbb{R}^+)$ continuously, and the operator $A$ defined in (42) maps $C(0,T;C_b^2(\mathbb{R}^+))$ to itself. Choose $t_0 > 0$ such that

$$\max_{t \in [0,t_0]} |g(t) - g(0)| \leq 1, \quad \max_{t \in [0,t_0]} |j(t) - j(0)| \leq 1. \quad (68)$$

Let

$$R = 2\|u_0\|_{C_b^2(\mathbb{R}^+)} + 2\|\phi_1\|_{C_b^2(\mathbb{R}^+)} + 2\|\phi_2\|_{C_b^2(\mathbb{R}^+)}$$

and fix $T$ as

$$T = \min \left\{ t_0, \frac{1}{2(\kappa(1 + \gamma_2) + \frac{3}{4} R + 12\gamma R + \frac{7}{48} R^2)} \right\}, \quad (70)$$

where

$$\kappa = \inf \left\{\|Ku\|_{C_b^2(\mathbb{R}^+)} : u \in C_b(\mathbb{R}^+), \|u\|_{C_b(\mathbb{R}^+)} = 1\right\}.$$ It is straightforward to check that $A$ is a contraction mapping on the ball $B_R(0) = \{u \in C(0,T;C_b^2(\mathbb{R}^+)) : \|u\|_{C(0,T;C_b^2(\mathbb{R}^+))} \leq R\}.$

Since $B_R(0)$ is a complete metric space under the metric $\| \cdot \|_{C(0,T;C_b^2(\mathbb{R}^+))}$, there is a unique $u \in C(0,T;C_b^2(\mathbb{R}^+))$ such that

$$Au = u.$$

Because the solution obtained is the fixed point of a contractive mapping, naturally the mapping from the initial-boundary data to the solution

$$(u_0, g, j) \rightarrow u$$

is Lipschitz continuous. To see the point, let $u$ and $\check{u}$ be the solutions of (42) corresponding to the auxiliary data $(u_0, g, j)$ and $(\check{u}_0, \check{g}, \check{j})$ respectively. Then on the time interval $[0,T]$, we have

$$u = A_{(u_0, g, j)} u, \quad \check{u} = A_{(\check{u}_0, \check{g}, \check{j})} \check{u},$$

where $A_{(u_0, g, j)}$ is contractive. It provides that

$$\|u - \check{u}\|_{C(0,T;C_b^2(\mathbb{R}^+))} = \|A_{(u_0, g, j)} u - A_{(\check{u}_0, \check{g}, \check{j})} \check{u}\|_{C(0,T;C_b^2(\mathbb{R}^+))} \leq \|A_{(u_0, g, j)} u - A_{(\check{u}_0, \check{g}, \check{j})} \check{u}\|_{C(0,T;C_b^2(\mathbb{R}^+))} + \|A_{(u_0, g, j)} \check{u} - A_{(\check{u}_0, \check{g}, \check{j})} \check{u}\|_{C(0,T;C_b^2(\mathbb{R}^+))} \leq \|u - \check{u}\|_{C(0,T;C_b^2(\mathbb{R}^+))} + \|u_0 - \check{u}_0\|_{C_b^2(\mathbb{R}^+)} + \|\phi_1\|_{C_b^2(\mathbb{R}^+)} \|g - \check{g}\|_{C(0,T)} + \|\phi_2\|_{C_b^2(\mathbb{R}^+)} \|j - \check{j}\|_{C(0,T)}.$$
where $\theta < 1$ is the contractive constant for the operator $K_{(u_0, g, j)}$. It follows readily that
\[
\|u - \tilde{u}\|_{C(0,T;C^2_b(\mathbb{R}^+))} \leq \frac{1}{1 - \theta} \left( \|u_0 - \tilde{u}_0\|_{C^2_b(\mathbb{R}^+)} + \|\phi_1\|_{C^2_b(\mathbb{R}^+)}\|g - \tilde{g}\|_{C(0,T)} + \|\phi_2\|_{C^2_b(\mathbb{R}^+)}\|\tilde{j} - j\|_{C(0,T)} \right).
\]
The Lipschitz continuity from $(u_0, g, j)$ to $u$ has been established. It remains to check the initial and boundary conditions. By (42), we have
\[
|u(x, t) - u_0(x)| \leq \sup_{x \in \mathbb{R}^+} |\phi_1(x)| |g(t) - g(0)| + \sup_{x \in \mathbb{R}^+} |\phi_2(x)| |j(t) - j(0)| + t \int_0^t |Kf(x, s)| \, ds.
\]
The property that $Kf(x, t) \in C(0,T;C^2_b(\mathbb{R}^+))$ together with $\phi_1, \phi_2 \in C^\infty_b(\mathbb{R}^+)$ and $g, j \in C(\mathbb{R}^+)$ implies that
\[
\lim_{t \to 0} u(x, t) = u_0(x)
\]
in $C(0,T;C^2_b(\mathbb{R}^+))$. Similarly, we have
\[
|u(x, t) - g(t)| \leq |u_0(x) - \phi_1(x)g(0)| + |(\phi_1(x) - 1)g(t)| + |\phi_2(x)(j(t) - j(0))|
\]
\[
+ \int_0^t \int_0^\infty |K(x, z)f(z, s)| \, dz \, ds,
\]
\[
\leq |u_0(x) - \phi_1(x)u_0(0)| + |(\phi_1(x) - 1)g(t)| + |\phi_2(x)(j(t) - j(0))|
\]
\[
+ t\|f\|_{C(0,T;C^2_b(\mathbb{R}^+))} \int_0^\infty |K(x, z)| \, dz.
\]
It together with $\lim_{x \to 0} \int_0^\infty |K(x, z)| \, dz = 0$, $\phi_1(0) = 1$ and $\phi_2(0) = 0$ indicates readily that
\[
\lim_{x \to 0} u(x, t) = g(t)
\]
uniformly for $t \in [0, T]$. In a same fashion,
\[
|ux(x, t) - j(t)| \leq |u''_0(x) - \phi''_2(x)j(0)| + |\phi''_1(x)(g(t) - g(0))|
\]
\[
+ |(\phi''_2(x) - 1)j(t)| + t\|f\|_{C(0,T;C^2_b(\mathbb{R}^+))} \int_0^\infty |K_{xx}(x, z)| \, dz,
\]
plus that $u''_0(0) = j(0)$, $\lim_{x \to 0} \int_0^\infty |K_{xx}(x, z)| \, dz = 0$, $\phi''_1(0) = 0$ and $\phi''_2(0) = 1$, it concludes
\[
\lim_{x \to 0} u_{xx}(x, t) = j(t)
\]
uniformly for $t \in [0, T]$.

The lemma is established.

\textbf{Corollary 1.} If the triple $(u_0, g, j)$ is admissible, and moreover, the initial value $u_0 \in C^m_b(\mathbb{R}^+) \cap C^{k,s}(\mathbb{R}^+)$ and $g, j \in C^{m,s}(\mathbb{R}^+)$ for some non-negative integers $m \geq 0, k \geq 2$ and $0 < s, \sigma \leq 1$, then the solution $u$ of (42) is a member of $C^{m,\sigma}(0,T;C^2_b(\mathbb{R}^+) \cap C^{k,s}(\mathbb{R}^+))$, where $T$ only depends on the value of $\|u_0\|_{C^2_b(\mathbb{R}^+)}$, and properties of $g$ and $j$ near $t = 0$, not their derivatives.

\textbf{Proof.} The assumptions on the auxiliary data mean that $u_0 \in C^2_b(\mathbb{R}^+)$ and $g, j \in C(\mathbb{R}^+)$. By the last theorem, equation (42) has a unique solution $u$ which lies in space
\[
C(0,T;C^2_b(\mathbb{R}^+))
\]
where $T$ is provided in (70) which only depends on $\|u_0\|_{C^2_b(\mathbb{R}^+)}$, and properties of $g$ and $j$ near $t = 0$, see (68)-(69)-(70). Repeat (42) in an alternative form
\[
  u(x, t) - u_0(x) - \mu(x, t) = \int_0^t Kf(u, u_x, u_{xx})(x, s)\, ds,
\]
and notice that $\mu$ defined in (43) lies in $C^{m,\sigma}(0, \infty; H^\infty(\mathbb{R}^+))$. The smoothing property of $K$ implies that the right-hand side is a member of $C^1(0, T; C^3_b(\mathbb{R}^+))$. That says $u \in C^{0,\sigma}(0, T; C^2_b(\mathbb{R}^+) \cap C^{2,\sigma}(\mathbb{R}^+))$ is true if $m = 0, k = 2$. Inductively, one can show that
\[
  u(x, t) - u_0(x) - \mu(x, t) \in C^{m+1}(0, T; C^{k+1}_b(\mathbb{R}^+)) \subset C^{m,\sigma}(0, T; C^1_b(\mathbb{R}^+) \cap C^{k,\sigma}(\mathbb{R}^+)),
\]
and therefore, the solution $u$ of (42) is a member of $C(0, T; C^1_b(\mathbb{R}^+) \cap C^{k,\sigma}(\mathbb{R}^+))$.

The corollary is established.

**Theorem 3.2.** Suppose that boundary data $g, j \in C^1(\mathbb{R}^+)$, the initial data $u_0 \in C^4_b(\mathbb{R}^+)$ and the triple $(u_0, g, j)$ is admissible. Then the solution $u$ of the integral equation (42) which lies in space $C(0, T; C^4_b(\mathbb{R}^+))$ for some $T > 0$, (see detail in Lemma 3.1), is, in fact, a classical solution of (25). More precisely, $\partial^k_x u, \partial^k_t u \in C(0, T; C^4_b(\mathbb{R}^+))$ for $k = 0, 1, 2, 3, 4$, with $\lim_{t \to 0} u(x, t) = u_0(x)$ in $C^4_b(\mathbb{R}^+)$, $\lim_{x \to 0} u(x, t) = g(t)$ and $\lim_{x \to 0} u_{xx}(x, t) = j(t)$ uniformly for $t \in [0, T]$.

The continuous function
\[
  u_t + u_x - \gamma_1 u_{xtt} + \gamma_2 u_{xxx} + \phi u_{xxxx} + \phi_3 u_{xxxxx} + \frac{3}{4}\phi_4 + \phi_5 u_{xxxxxx} + \frac{7}{48}\phi_6 u_{xxxxxx} - \frac{1}{8}\phi_7 u_{xxxxxx}
\]
is identically equal to zero for all $(x, t) \in \mathbb{R}^+ \times [0, T]$.

**Proof.** Since $u \in C(0, T; C^4_b(\mathbb{R}^+))$ is the solution of the integral equation (42), $\phi_1, \phi_2 \in C^\infty(\mathbb{R}^+)$ and $g, j \in C^1(\mathbb{R}^+)$,
\[
f = f(x, t) = u(x, t) + \gamma_2 u_{xx}(x, t) + \frac{3}{4}u^2(x, t) + \gamma_2 u_{xxx}(x, t) - \frac{7}{48}u^2(x, t) - \frac{1}{8}u^3(x, t)
\]
lies in space $C(0, T; C^4_b(\mathbb{R}^+))$. Hence, for any $\Delta t \neq 0$,
\[
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{\phi_1(x) g(t) + \frac{\phi_2(x) j(t) + j(t)}{\Delta t}}{\Delta t} + \frac{1}{\Delta t} \int_t^{t+\Delta t} \mathcal{K}f(x, s)\, ds.
\]
The limit of the right-hand side as $\Delta t \to 0$ exists, that is
\[
u_t(x, t) = \phi_1(x) g' (t) + \phi_2(x) h'(t) + \mathcal{K}f(x, t),
\]
and it lies in $C(0, T; C^1_b(\mathbb{R}^+))$. From Proposition 5, the operator $\mathcal{K}$ is an order three smoothing operator, so
\[
u_x(x, t) = u_0'(x) + \phi_1'(x)(g(t) - g(0)) + \phi_2'(x)(j(t) - j(0)) + \int_0^t \partial_x \mathcal{K}f(x, s)\, ds,
\]
\[
u_{xx}(x, t) = u_0''(x) + \phi_1''(x)(g(t) - g(0)) + \phi_2''(x)(j(t) - j(0)) + \int_0^t \partial_{xx} \mathcal{K}f(x, s)\, ds,
\]
\[
u_{xxx}(x, t) = \phi_1'''(x)(g(t) - g(0)) + \phi_2'''(x)(j(t) - j(0)) + \int_0^t \partial_{xxx} \mathcal{K}f(x, s)\, ds,
\]
and
\[
u_{xxxx}(x, t) = \phi_1''''(x)(g(t) - g(0)) + \phi_2''''(x)(j(t) - j(0)) + \int_0^t \partial_{xxxx} \mathcal{K}f(x, s)\, ds
\]
all lie in $C(0, T; C_b(\mathbb{R}^+))$. It transpires that $f_x \in C(0, T; C_b(\mathbb{R}^+))$ as well. By (58),
\[
\partial_x^4 u = u_0^{(4)}(x) + \phi_1^{(4)}(x)(g(t) - g(0)) + \phi_2^{(4)}(x)(j(t) - j(0)) + \int_0^t \partial_x^4 K_f(x,s) \, ds
\]
\[
= u_0^{(4)}(x) + \phi_1^{(4)}(x)(g(t) - g(0)) + \phi_2^{(4)}(x)(j(t) - j(0)) + \int_0^t \left( -\frac{1}{\delta} f_x(x, s) + \frac{\gamma_1}{\delta} \partial_x x K_f(x, s) - \frac{1}{\delta} K_f(x, s) \right) \, ds.
\]
Take derivative with respect to $t$, it follows that
\[
\partial_x^4 \partial_t u = \phi_1^{(4)}(x) g'(t) + \phi_2^{(4)}(x) j'(t) - \frac{1}{\delta} f_x(x, t) + \frac{\gamma_1}{\delta} \partial_x x K_f(x, t) - \frac{1}{\delta} K_f(x, t)
\]
\[
= \phi_1^{(4)}(x) g'(t) + \phi_2^{(4)}(x) j'(t) - \frac{1}{\delta} f_x(x, t) + \frac{\gamma_1}{\delta} \left( u_{xtt} - \phi_1''(x) g'(t) - \phi_2''(x) j'(t) \right)
\]
\[
- \frac{1}{\delta} \left( u_t(x, t) - \phi_1(x) g'(t) - \phi_2(x) j'(t) \right).
\]
Both $\partial_x^4 u$ and $\partial_x^4 \partial_t u$ belong to $C(0, T; C_b(\mathbb{R}^+))$. Because $\phi_1$ and $\phi_2$ satisfy (44) and (45), respectively,
\[
\partial_x^4 \partial_t u = -\frac{1}{\delta} f_x(x, t) + \frac{\gamma_1}{\delta} u_{xxt} - \frac{1}{\delta} u_t(x, t). \tag{71}
\]
It is seen that the continuous function
\[
u_t - \gamma_1 u_{xxt} + \delta u_{xxtt} + f_x(x, t)
\]
is identically equal to zero for all $(x, t) \in \mathbb{R}^+ \times [0, T]$. Whence, $u$ is the classical solution of the equation
\[
u_t + u_x - \gamma_1 u_{xxt} + \gamma_2 u_{xxx} + \delta u_{xxtt} + \frac{3}{4} (u^2)_x + \gamma (u^2)_{xxx} - \frac{7}{48} (u_x^3)_x - \frac{1}{8} (u^3)_x = 0.
\]
The proof of
\[
\lim_{t \to 0} u(x, t) = u_0(x), \quad \lim_{x \to 0} u(x, t) = g(t) \quad \text{and} \quad \lim_{x \to 0} u_{xx}(x, t) = j(t)
\]
is as same as that in proof of Lemma 3.1. \hfill \Box

Remark. The solution $u$ cannot acquire more spatial and time regularity than that of the initial data $u_0(x)$ and boundary data $g(t)$ and $j(t)$, respectively. Under the weak regularity assumptions for $g$ and $u_0$, it is no longer expected that there is a classical solution to equation (25). Instead, we search for solutions of (42) in the weak regularity assumptions for $g, j$. Indeed, it is clear that a solution of (42) solve the initial-boundary-value problem (25) in the sense of distributions. Indeed, it is clear that a solution of (42) satisfies the initial condition since $g(t) \to g(0), j(t) \to j(0)$ and the double integral term vanishes as $t \to 0$. Because $\lim_{x \to 0} K(x, z) = \lim_{x \to 0} K_{xx}(x, z) = 0$ for all $z > 0$, the dominated convergence theorem implies that the double integral and its second
order derivative with respect to \( x \) again tend to zero in the limit \( x \to 0 \). Since \( u_0(x) \to g(0) \), \( u_0''(x) \to j(0) \), \( \phi_1(x) \to 1, \phi_2(x) \to 0 \), \( \phi_1''(x) \to 0 \), \( \phi_2''(x) \to 1 \) as \( x \to 0 \), it thus follows that the boundary condition is also satisfied. The fact that a solution of (42) is a distributional solution of (25) follows from the fact that it thus follows that the boundary condition is also satisfied. The fact that a solution of (42) is a distributional solution of (25) follows from the fact that a solution of (42) is a distributional solution of (25) follows from the fact that

\[
u_t(x,t) = g'(t)\phi_1(x) + h'(t)\phi_2(x) + \int_0^\infty K(x,z)f(u(z,t),u_z(z,t),u_{zz}(z,t))\,dz,
\]

and by the fundamental theorem of calculus

\[
(I - \gamma_1 \partial_x^2 + \delta \partial_x^4) \int_0^\infty K(x,z)f(u(z,t),u_z(z,t),u_{zz}(z,t))\,dz = -\partial_x f(u, u_x, u_{xx}),
\]

see (71) together with identities (48) and (49). Of course, \( g', j' \) and \( f_x \) are taken in the sense of distributions.

Conversely, a distributional solution \( u \) of the initial-boundary-value problem (25) with \((u_0, g, j)\) being admissible is a solution of the integral equation (42). This latter fact is seen by following the steps outlined earlier for the derivation of the integral equation. Thus the two problems are equivalent as far as admissible solutions are concerned and we will not distinguish between them further.

The following theorem is one of the principal results of the current paper.

**Theorem 3.3.** If boundary data \( g, j \in C(\mathbb{R}^+) \), the initial data \( u_0 \in H^2 = H^2(\mathbb{R}^+) \) and \((u_0, g, j)\) is admissible, then the integral equation (42) is locally well posed in time. Precisely, there is a small value \( T \) depending on \( \|u_0\|_2 \) and local property of \( g \) and \( j \) at \( t = 0 \) such that (42) has unique solution \( u \) which lies in space \( C(0,T; H^2) \). Moreover, the correspondence between initial and boundary data \((u_0, g, j)\) and the associated solution \( u \) of (42) is a uniformly Lipschitz continuous mapping from any bounded subset of \( H^2 \times C(0,T) \times C(0,T) \) into \( C(0,T; H^2) \), \( \lim_{t \to 0} \|u(\cdot, t) - u_0\|_2 = 0 \), and \( u \) is twice differentiable near \( x = 0 \) satisfying \( u(0, t) = g(t) \) and \( u_{xx}(0, t) = j(t) \).

**Proof.** Like the proof of Lemma 3.1, we show that the operator \( \mathcal{A} \) defined in (42) has a fixed point in \( C(0,T; H^2) \) for some small \( T > 0 \).

First, let \( T > 0 \) be any number and \( u \in C(0,T; H^2) \), then the nonlinear function \( f = f(u, u_x, u_{xx}) \in C(0,T; L_2) \), so \( \mathcal{K} f \in C(0,T; H^2) \), and therefore, \( \mathcal{A} \) maps \( C(0,T; H^2) \) to itself because \( u_0 \in H^2 \).

Introduce \( t_0 > 0 \), \( \kappa \) and \( R \) as follows:

\[
g(t) - g(0) \leq 1, \quad j(t) - j(0) \leq 1 \quad \text{for all} \quad t \in [0, t_0]
\]

\[
\kappa = \min \left\{ \|Kv\|_2 : v \in L_2, \|v\| = 1 \right\}
\]

and

\[
R = 2\left( \|u_0\|_2 + \|\phi_1\|_2 + \|\phi_2\|_2 \right).
\]

Fix \( T \) as

\[
T = \min \left\{ t_0, \frac{1}{2\kappa \left( 1 + |\gamma_2| + \frac{3}{2} R + 12 |\gamma| R + \frac{7}{2} R^2 + \frac{3}{2} R^2 \right)} \right\},
\]

and denote

\[
\mathcal{B}_R(0) = \left\{ v \in C(0,T; H^2) : \|v\|_{C(0,T; H^2)} \leq R \right\}.
\]

It is straightforward to verify that for any \( u \in \mathcal{B}_R(0) \),

\[
\|\mathcal{A} u\|_{C(0,T; H^2)} \leq \frac{1}{2} R + T \|\mathcal{K} f(u, u_x, u_{xx})\|_{C(0,T; H^2)}
\]
\[
\leq \frac{1}{2} R + \kappa T \left( R + |\gamma_2| R + \frac{3}{4} R^2 + 4|\gamma| R^2 + \frac{7}{48} R^2 + \frac{1}{8} R^3 \right)
\]

and for any \( u, v \in B_R(0) \),

\[
\|Au - Av\|_{C(0,T;H^2)} \leq \kappa T \left( 1 + |\gamma_2| + \frac{3}{2} R + 12|\gamma| R + \frac{7}{24} R^2 + \frac{3}{8} R^2 \right) \|u - v\|_{C(0,T;H^2)}
\]

\[
\leq \frac{1}{2} \|u - v\|_{C(0,T;H^2)}.
\]

It is seen that the operator \( A \) is a contraction mapping on \( B_R(0) \), and whence there is a unique point \( u \in B_R(0) \) such that

\[
u = Au.
\]

The argument of showing that the mapping from initial and boundary data \((u_0, g, j)\) to the associated solution \( u \) of (42) is a uniformly Lipschitz continuous mapping from any bounded subset of \( H^2 \times C(0,T) \times C(0,T) \) into \( C(0,T;H^2) \) is as same as that in the proof of Lemma 3.1, so it is not repeated here. What remains is to check the initial and boundary conditions.

Since \( f(\cdot, t) = f(u(\cdot, t), u_x(\cdot, t), u_{xx}(\cdot, t)) \) is a member of \( L_2 \) for any \( t \in [0,T] \) and \( K \) maps \( L_2 \) to \( H^3 \) continuously, \( Kf(\cdot, t) \) is a member of \( H^3 \) for any \( t \in [0,T] \). It means that

\[
u(x, t) - u_0(x) - \phi_1(x)(g(t) - g(0)) - \phi_2(x)(j(t) - j(0))
\]

\[
= \int_0^t Kf(x, s) \, ds
\]

\[
= \int_0^t \int_0^\infty K(x, z)f(z, s) \, dz \, ds
\]

lies in \( C(0,T;H^3) \). (In fact, it lies in \( C^1([0,T];H^3) \)). It transpires that

\[
\|u(\cdot, t) - u_0\|_3 \leq \|\phi_1\|_3|g(t) - g(0)| + \|\phi_2\|_3|j(t) - j(0)|
\]

\[
\quad + \kappa t \|f(u, u_x, u_{xx})\|_{C(0,T;L_2)}
\]

(72)

the initial condition

\[
\lim_{t \to 0} \|u(\cdot, t) - u_0\|_{C^2(R^3)} \leq \lim_{t \to 0} \|u(\cdot, t) - u_0\|_3 = 0
\]

holds true. Now we turn to verify the boundary condition. Elementary calculations show that

\[
|u(x, t) - g(t)| \leq |u_0(x) - \phi_1(x)g(0)| + |\phi_1(x) - 1||g(t)| + |\phi_2(x)||j(t) - j(0)|
\]

\[
\quad + \int_0^t \int_0^\infty |K(x, z)f(z, s)| \, dz \, ds
\]

and

\[
|u_{xx}(x, t) - j(t)| \leq |u_0''(x) - \phi_2''(x)j(0)| + |\phi_2''(x)(g(t) - g(0))|
\]

\[
\quad + \left(|\phi_2''(x) - 1||j(t)| + \int_0^t \int_0^\infty |K_{xx}(x, z)f(z, s)| \, dz \right).
\]

Because \( u_0(x) \) is twice differentiable at \( x = 0 \) satisfying \( u_0(0) = g(0) \) and \( u_0''(0) = j(0) \), and \( \phi_1, \phi_2 \) satisfy \( \phi_1''(0) = 0, \phi_2''(0) = 1 \),

\[
\lim_{x \to 0} \int_0^\infty |K(x, z)f(z, s)| \, dz \leq \lim_{x \to 0} \left( \int_0^\infty |K(x, z)|^2 \, dz \right)^{\frac{1}{2}} \|f\|_{C(0,T;L_2)} = 0
\]
and
\[
\lim_{x \to 0} \int_0^\infty |K_{xx}(x,z)f(z,s)| \,dz \leq \lim_{x \to 0} \left( \int_0^\infty |K_{xx}(x,z)|^2 \,dz \right)^{\frac{1}{2}} \|f\|_{C(0,T;L_2)} = 0.
\]
It immediately follows that
\[
\lim_{x \to 0} u(x,t) = g(t), \quad \lim_{x \to 0} u_{xx}(x,t) = j(t)
\]
uniformly for \( t \in [0, T] \). The theorem is established. \( \square \)

**Theorem 3.4.** In Theorem 3.3, if the initial data \( u_0 \in H^s \) for some \( s > 2 \), and the boundary data \( g, j \in C^{m,\sigma}(0, \infty) \) for some \( m \in \mathbb{N} \) and \( 0 < \sigma \leq 1 \), then the solution \( u \) lies in space \( C^{m,\sigma}(0,T;H^s) \).

**Proof.** Since it is already established that the solution \( u \in C(0,T;H^2) \) in last theorem, and the fact that \( f \in C(0,T;L_2) \) and \( Kf \in C(0,T;H^3) \) indicates that
\[
u(x,t) - u_0(x) - \phi_1(x)(g(t) - g(0)) - \phi_2(x)(j(t) - j(0)) = \int_0^t Kf(x,s) \,ds
\]
is a member of \( C^2(0,T;H^3) \). By bootstrapping or mathematical induction,
\[
u(x,t) - u_0(x) - \phi_1(x)(g(t) - g(0)) - \phi_2(x)(j(t) - j(0))
\in C^{m+1}(0,T;H^{[\sigma]})
\subset C^{m,\sigma}(0,T;H^s).
\]
Therefore, \( u_0 \in H^s \) and \( g, j \in C^{m,\sigma}(0, \infty) \) imply
\[
u \in C^{m,\sigma}(0,T;H^s).
\]
The theorem is established. \( \square \)

**Remark.** The time length \( T \) depends on \( \|u_0\|_2 \) (instead of \( \|u_0\|_s \)) and local property of \( g, j \) at \( t = 0 \) as described in Theorem 3.3. The proof is basically applying smoothing property of the operator \( K \).

Now we turn our attention to the case where the initial data \( u_0 \in H^1(\mathbb{R}^+) \), the operator \( A \) does not map \( C(0,T;H^1(\mathbb{R}^+)) \) to itself due to the terms \( K(u^2)_{xx} \) and \( K_{xx} \). To see this point, using integrations by parts and noticing that \( K(x, x-) - K(x, x+) = K_z(x, x-) - K_z(x, x+) = K_z(x, 0) = 0 \), we have
\[
K(u^2)_{xx}(x,t) = \int_0^\infty K(x,z)(u^2(z,t))_{zz} \,dz
= -2K(x,0)u(0,t)u_x(0,t) + \int_0^\infty K_{zz}(x,z)u^2(z,t) \,dz
= -2K(x,0)u(0,t)u_x(0,t) + \partial_{xx}K(u^2)(x,t),
\]
and similarly,
\[
K_{xx}(x,t) = -K(x,0)u_x(x,0,t) + \partial_{xx}Ku(x,t),
\]
where
\[
K(x,0) = \begin{cases} 
\frac{1}{\sqrt{\gamma_1^2 - 4\delta}} (e^{-\rho x} - e^{-\rho z}) & \text{for } \delta \in (0, \frac{\gamma_1^2}{4}), \\
\frac{1}{2\delta^2} xe^{-\rho x} & \text{for } \delta = \frac{\gamma_1^2}{4}, \\
\frac{2}{\sqrt{4\delta - \gamma_1^2}} e^{-\rho x \cos \theta} \sin(\rho x \sin \theta) & \text{for } \delta > \frac{\gamma_1^2}{4},
\end{cases}
\]
in which \( \rho = \delta^{-\frac{1}{2}}, \rho_1, \rho_2 \) and \( \theta \in (0, \frac{\pi}{2}) \) are constants given in (27) and (37). Apparently, \( K(x, 0) \in H^\infty, \partial_x \mathcal{K}(u^2) \in C(0, T; H^2) \), therefore, \( \mathcal{K}u_{xx}, \mathcal{K}(u^3)_{xx} \in C(0, T; H^1) \) if and only if \( u_x \) has trace at \( x = 0 \) which is not warranted by \( u \) being a member of \( C(0, T; H^1) \). On the other hand, if \( u_x \) is known to have trace at \( x = 0 \), then \( \mathcal{K}u_{xx}, \mathcal{K}(u^3)_{xx} \in C(0, T; H^2) \) with \( \|u(\cdot, t)\|_2 \) dependent on the trace \( u_x(0, t) \) as well as \( \|u(\cdot, t)\|_1 \). It follows immediately the following result.

**Proposition 7.** If \( \Sigma = \Sigma(x, t) \in C(0, \infty; H^1) \) and \( \Sigma_x \) has trace at \( x = 0 \) for all \( t \geq 0 \), then

\[
\mathcal{K}\{f(\Sigma, \Sigma_x, \Sigma_{xx})\} = \mathcal{K}\left(\Sigma + \gamma_2 \Sigma_{xx} + \frac{3}{4} \Sigma^2 + \gamma(\Sigma^2)_{xx} - \frac{7}{48} \Sigma^2 - \frac{1}{8} \Sigma^3\right)
= \mathcal{K}\left(\Sigma + \frac{3}{4} \Sigma^2 - \frac{7}{48} \Sigma^2 - \frac{1}{8} \Sigma^3\right)
- \gamma_2 K(x, 0) \Sigma_x(0, t) + \frac{7}{48} K(x, 0) \Sigma_x(0, t) + \gamma \partial_x \mathcal{K}(\Sigma^2)
\]

lies in \( C(0, \infty; H^2) \).

**Proof.** The condition \( \Sigma \in C(0, \infty; H^1) \) implies that \( \Sigma_x^2 \in C(0, \infty; L^1) \) and \( \frac{3}{4} \Sigma^2, \frac{1}{8} \Sigma^3 \in C(0, \infty; H^1) \). The first part of Proposition 6, that is \( \mathcal{K} \) maps \( H^1 \) to \( H^1 \) continuously, implies

\[
\mathcal{K}\left(\Sigma + \frac{3}{4} \Sigma^2 - \frac{1}{8} \Sigma^3\right) - \gamma_2 K(x, 0) \Sigma_x(0, t) + \frac{7}{48} K(x, 0) \Sigma_x(0, t) + \gamma \partial_x \mathcal{K}(\Sigma^2)
\]

lies in \( C(0, \infty; H^2) \), and the second part, i.e \( \mathcal{K} \) maps \( L^1 \) to \( H^2 \) continuously, indicates

\[
\mathcal{K}\left(\Sigma_x^2\right)
\]
also lies in the same space \( C(0, \infty; H^2) \). Hence, the proposition follows. 

**Theorem 3.5.** If the boundary data \( g, j \in C(\mathbb{R}^+) \), the initial data \( u_0 \in H^1 \) and \( (u_0, g, j) \) is admissible, then the integral equation (42) is locally well posed in time. Precisely, there is a small value \( T \) which depends on \( \|u_0\|_1, u_0', u_0(0), \) local property of \( g \) and \( j \) at the origin such that (42) has a unique solution \( u \) which lies in space \( C(0, T; H^1) \). Moreover, the correspondence between the initial-boundary data \( (u_0, g, j) \) and the associated solution \( u \) of (42) is a Lipschitz continuous mapping from any bounded subset of \( \{(u_0, g, j) \in H^1 \times C(\mathbb{R}^+) \times C(\mathbb{R}^+) : (u_0, g, j) \text{ is admissible}\} \) into \( C(0, T; H^1) \), \( \lim_{t \to 0} u(x, t) = u_0(x) \) uniformly for \( x \in \mathbb{R}^+ \), \( u_{xx}(x, t) \) is continuous at \( x = 0 \) and satisfies \( u(0, t) = g(t), u_{xx}(0, t) = j(t) \).

**Remark.** In fact, the difference \( u(x, t) - u_0(x) \) lies in \( C(0, T; H^2) \) and \( \lim_{t \to 0} \|u(\cdot, t)\|_2 - u_0\|_2 = 0 \). It will appear in the proof more clearly.

**Proof.** Introduce a new dependent variable \( w = w(x, t) \) as follows,

\[
w = u(x, t) - \Sigma(x, t)
\]
where

\[
\Sigma(x, t) = u_0(x) + \phi_1(x)(g(t) - g(0)) + \phi_2(x)(j(t) - j(0))
\]

(76)

\( \phi_1 \) and \( \phi_2 \) are defined in (44) and (45), respectively. So \( \Sigma \in C(0, \infty; H^1) \) and \( \Sigma_x \) has trace at \( x = 0 \) for all \( t \geq 0 \). In this new regime, (42) reads as follows...
\[ w(x, t) = \int_0^t K\left( f(\Sigma, \Sigma_x, \Sigma_{xx}) \right)(x, s) \, ds + \int_0^t K\left( f(w, w_x, w_{xx}) \right)(x, s) \, ds \\
+ \int_0^t K\left( \frac{3}{2} \Sigma w - \frac{7}{24} \Sigma_x w_x - \frac{3}{8} \Sigma^2 w - \frac{3}{8} \Sigma w^2 \right)(x, s) \, ds \\
- 2\gamma \int_0^t \left( K(x, 0)(\Sigma w)_x(0, s) - \partial_x K(\Sigma w)(x, s) \right) \, ds \\
=: A\Sigma w(x, t). \]

It turns to show that the operator \( A\Sigma \) has a unique fixed point in \( C(0, T; H^2) \) for sufficiently small value of \( T \). We now attempt to find a proper space in which the operator \( A\Sigma \) is contractive mapping.

By Proposition 7,

\[ Kf(\Sigma, \Sigma_x, \Sigma_{xx}) = K\left( \Sigma + \gamma_2 \Sigma_x + \frac{3}{4} \Sigma^2 + \gamma(\Sigma^2)_{xx} - \frac{7}{48} \Sigma^2 x - \frac{1}{8} \Sigma^3 \right) \]

is a member of \( C(0, \infty; H^2) \). It is seen that \( A\Sigma \) maps \( C(0, T; H^2) \) to itself for any \( T > 0 \). Let \( t_0 > 0 \) so that \( |g(t) - g(0)| \leq 1, |j(t) - j(0)| \leq 1 \) for all \( t \in [0, t_0] \). Let

\[ \kappa_{2,0} = \min \left\{ \| Ku \|_2 : u \in L_2, \| u \| = 1 \right\}, \quad \kappa_{4,1} = \min \left\{ \| Ku \|_4 : u \in H^1, \| u \|_1 = 1 \right\}, \]

\[ \sigma = \| u_0 \|_1 + \| \phi_1 \|_1 + \| \phi_2 \|_1, \quad R = 4t_0 \| Kf(\Sigma, \Sigma_x, \Sigma_{xx}) \|_{C(0, t_0; H^2)}, \]

and fix \( T \) as

\[ T = \min \left\{ t_0, \frac{1}{4\kappa_{2,0} \left( 1 + |\gamma_2| + \frac{3}{8} R + 4|\gamma| R + \frac{3}{8} R^2 \right)}, \frac{1}{8|\gamma| \left( \| K(\cdot, 0) \|_2 \max_{t \leq t_0} \{ |\Sigma(0, t)| + |\Sigma(0, t)| \} + \kappa_{4,1} \sigma \right)} \right\}. \]

Denote,

\[ B_R(0) = \{ v \in C(0, T; H^2) : \| v \|_{C(0, T; H^2)} \leq R \}. \]

It is straightforward to verify that \( B_R(0) \) is a complete metric space. What remains is to show that \( A\Sigma \) maps \( B_R(0) \) to itself and is contractive. To do so, for any \( w \in B_R(0) \), we need to check that \( A\Sigma w \) lies in \( C(0, T; H^2) \). Let \( t \leq T \), then,

\[ \| A\Sigma w(\cdot, t) \|_2 \leq \int_0^t \| Kf(\Sigma, \Sigma_x, \Sigma_{xx})(\cdot, s) \|_2 \, ds + \int_0^t \| Kf(w, w_x, w_{xx})(\cdot, s) \|_2 \, ds \\
+ \int_0^t \| K\left( \frac{3}{2} \Sigma w - \frac{7}{24} \Sigma_x w_x - \frac{3}{8} \Sigma^2 w - \frac{3}{8} \Sigma w^2 \right)(\cdot, s) \|_2 \, ds \\
+ 2|\gamma| \int_0^t \| \left( K(\cdot, 0)(\Sigma w)_x(0, s) - \partial_x K(\Sigma w)(\cdot, s) \right) \|_2 \, ds \\
\leq \frac{T}{4t_0} R + \int_0^t \kappa_{2,0} \| f(w, w_x, w_{xx})(\cdot, s) \|_2 \, ds \\
+ \int_0^t \kappa_{2,0} \| \left( \frac{3}{2} \Sigma w - \frac{7}{24} \Sigma_x w_x - \frac{3}{8} \Sigma^2 w - \frac{3}{8} \Sigma w^2 \right)(\cdot, s) \|_2 \, ds \\
+ 2|\gamma| \int_0^t \left( \| K(\cdot, 0) \|_2 \| \Sigma w(0, s) \| + \kappa_{4,1} \| \Sigma w(\cdot, s) \|_1 \right) \, ds. \]
Indeed, $A$ maps $B_R(0)$ to itself. For any $w_1, w_2 \in B_R(0)$,

$$
A\Sigma w_1(x, t) - A\Sigma w_2(x, t)
= \int_0^t K\left(f(w_1, w_{1x}, w_{1xx}) - f(w_2, w_{2x}, w_{2xx})\right)(x, s) \, ds
+ \int_0^t K\left(\frac{3}{2} \Sigma(x, t) - w_2 - \frac{7}{24} \Sigma(x, t) - w_2 - \frac{3}{8} \Sigma(x, t) - w_2\right)(x, s) \, ds
- 2\gamma \left(\int K(x, t) (\Sigma(x, t))_{s} \, ds - \partial_t \Sigma(x, t) \right)(x, s) \, ds,
$$
take the $C(0, T; H^2)$-norm,

$$
\|A\Sigma w_1 - A\Sigma w_2\|_{C(0, T; H^2)}
\leq \kappa_{2,0} T \left(1 + |\gamma_2| + \frac{3}{2} R + 4 |\gamma| R + \frac{7}{24} R + \frac{3}{8} R^2\right) \|w_1 - w_2\|_{C(0, T; H^2)}
+ \kappa_{2,0} T \left(\frac{3}{2} \sigma + \frac{7}{24} \sigma + \frac{3}{8} \sigma^2 + \frac{3}{4} \sigma R\right) \|w_1 - w_2\|_{C(0, T; H^2)}
+ 2T|\gamma| \left(\|K(\cdot, 0)\|_{H^2} \max_{t \leq T} \|\Sigma(0, t)\| + \|\Sigma(0, t)\|\right) + \kappa_{4,1} \|w_1 - w_2\|_{C(0, T; H^2)}
\leq \frac{3}{4} \|w_1 - w_2\|_{C(0, T; H^2)}.
$$

In consequence, the operator $A\Sigma$ is a contraction mapping on $B_R(0)$, and hence there is a unique point $w \in B_R(0)$ such that

$$
w = A\Sigma w.
$$

Therefore, $u = w + \Sigma \in C(0, T; H^1)$ is the unique solution of (42). The Lipschitz continuity follows from the contraction mapping principle. The proof is as same as that in the proof of Lemma 3.1. We now check the initial and boundary conditions.

Notice that the fixed point $w$ satisfies

$$
\|w(\cdot, t)\|_2 \leq tP(R)
$$

for a cubic polynomial function $P(R)$, it follows

$$
\lim_{t \to 0} \|u(\cdot, t) - u_0\|_2 \leq \lim_{t \to 0} \left(\|w(\cdot, t)\|_2 + \|\phi_1\|_2 \|g(t) - g(0)\| + \|\phi_2\|_2 \|j(t) - j(0)\|\right) = 0.
$$

It implies the initial condition

$$
\lim_{t \to 0} u(x, t) = u_0(x)
$$

uniformly for $x \in \mathbb{R}^+$. To estimate the boundary condition, reorganize (77) by singling $\gamma_2, \gamma$-terms out as follows,
Applying the results in (58),

\[
K(x,0)\Sigma_x(0,s) \quad \text{and} \quad -2\gamma \int_0^t K(x,0)\Sigma(0,s) \Sigma_x(0,s) \, ds + \gamma \int_0^t \int_0^\infty K_{xx}(x,z) \Sigma^2(z,s) \, dz \, ds
\]

(78)

\[
\]
holds uniformly for $t$.

In consequence,

$$
\lim_{x \to 0} w_{xx}(x, t) = 0.
$$

It is seen that $w_{xx}$ is in fact continuous pointwise, the dominated convergence theorem together with properties that $\lim_{x \to 0} K_{xx}(x, 0) = -\frac{1}{8}$ and $\lim_{x \to 0} K(x, z) = \lim_{x \to 0} K_{xx}(x, z) = 0$ for any $z > 0$ yield

$$
\lim_{x \to 0} w_{xx}(x, t) = 0.
$$

In consequence,

$$
\lim_{x \to 0} u_{xx}(x, t) = j(t)
$$

holds uniformly for $t \in [0, T]$. The theorem is proved.

As we see from the proof of the last theorem,

$$
w(x, t) = u(x, t) - u_0(x) - \phi_1(x)(g(t) - g(0)) - \phi_2(x)(j(t) - j(0))
$$

is, in fact, a member of $C^1(0, T; H^2 \cap C^2(\mathbb{R}^+))$, the following corollary follows immediately.

**Corollary 2.** In the last theorem, if $u_0 \in H^s$ where $1 \leq s < 2$, then the solution $u \in C(0, T; H^s)$.

4. **Global well-posedness.** Local well-posedness study in last section is irrelevant to size of $\alpha$ and $\beta$. However, it matters in global well-posedness. So we turn our attention back to the initial-boundary value problem (9) where the parameters $\alpha$ and $\beta$ as of order $o(1)$ constants are in explicit display. Here is our theorem on global well-posedness.

**Theorem 4.1.** Assume that conditions $\gamma = \frac{7}{12}$, $(\eta, G, J) \in H^2 \times C(\mathbb{R}^+) \times C(\mathbb{R}^+)$ and $G, J \in L_1 \cap L_3$ with $\|\eta\|_2 + |G|_1 + |G|_3 + |J|_1 + |J|_3 = O(1)$ hold true, then the initial-boundary value problem (9) with the compatibility condition (7) is globally well-posed. Furthermore, the solution $u \in C(0, \infty; H^2)$ has the following growth bound in time

$$
\|u(\cdot, t)\|_2 \leq c \left(1 + \alpha |G(\beta^2 t)| + \alpha \beta |J(\beta^2 t)|\right)
$$

where $c = O(1)$ is a constant only dependent on $\|\eta\|_2, |G|_1, |G|_3, |J|_1$ and $|J|_3$.

**Proof.** Theorem 3.3 provides that the problem (9) is locally well-posed in $H^2$, the solution $u$ lies in space $C(0, T; H^2)$ for some $T > 0$. To show the global well-posedness, it is sufficient to show (79) to be true.

For simplicity, the lower case letter $c = O(1)$ represents a constant independent of time $t$. It may depend on norms of the auxiliary data $\|\eta\|_2, |G|_1, |G|_3, |J|_1$ and $|J|_3$. 

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+ $\gamma \int_0^t \int_0^\infty \{\frac{\gamma_1}{\delta} K_{xx}(x, z)\Sigma^2(z, s) - \frac{1}{\delta} K(x, z)\Sigma^2(z, s)\} \, dz \, ds$

+ $\int_0^t \int_0^\infty K_{xx}(x, z) f(w, w_x, w_{xx})(z, s) \, dz \, ds$

+ $\int_0^t \int_0^\infty K_{xx}(x, z) \left(\frac{3}{2}\Sigma w - \frac{7}{24}\Sigma w_x - \frac{3}{8}\Sigma^2 w_x - \frac{3}{8}\Sigma^2 w\right)(z, s) \, dz \, ds$

$- 2\gamma \int_0^t K_{xx}(x, 0)(\Sigma w)_x(0, s) \, ds + 2\gamma \int_0^t -\frac{1}{\delta}(\Sigma w)_x(x, s) \, ds$

$+ 2\gamma \int_0^t \int_0^\infty \left(\frac{\gamma_1}{\delta} K_{xx}(x, z)\Sigma w(z, s) - \frac{1}{\delta} K(x, z)\Sigma w(z, s)\right) \, dz \, ds.$
Introduce a new dependent variable

\[ v = v(x, t) = u(x, t) - \alpha \phi_1(x)G(\beta^2 t) - \alpha \beta \phi_2(x)J(\beta^2 t), \]  

(80)

where \( \phi_1 \) and \( \phi_2 \) are given in (44) and (45), respectively. Denote

\[ \mu = \mu(x, t) = \alpha \phi_1(x)G(\beta^2 t) + \alpha \beta \phi_2(x)J(\beta^2 t), \]

then that \( u \in C(0, T; H^2) \) is a solution of (9) guarantees that \( v \in C(0, T; H^2) \) satisfies the following initial-boundary value problem,

\[
\begin{aligned}
\partial_t(I - \gamma_1 \partial_{xx} + \delta \partial_{xxxx})v &+ \partial_x f(v, v_x, v_{xx}) + \partial_x f(\mu, \mu_x, \mu_{xx}) \\
&+ \partial_x \left( \frac{3}{2} \mu v + \frac{7}{24}(\mu v)_{xx} - \frac{7}{24} \mu_x v_x - \frac{3}{8} \mu^2 v - \frac{3}{8} \mu v^2 \right) = 0,
\end{aligned}
\]

(81)

\[ v(x, 0) = \alpha \eta_0(\beta^2 x) - \alpha \eta_0(0) \phi_1(x) - \alpha \beta J(0) \phi_2(x), \]

\[ v(0, t) = 0, \quad v_{xx}(0, t) = 0, \]

where \( f \) is

\[ f(v, v_x, v_{xx}) = v + \gamma_2 v_{xx} + \frac{3}{4} v^2 + \frac{7}{48} (v^2)_{xx} - \frac{7}{48} v_x^2 - \frac{1}{8} v^3. \]

Multiply the equation by \( 2v \) and integrate over \( \mathbb{R}^+ \), and notice that the boundary data is homogeneous, so after integrations by parts, \( \int_0^\infty 2v \partial_x f(v, v_x, v_{xx}) \, dx = 0 \). It transpires that

\[
\frac{d}{dt} \int_0^\infty (v^2(x, t) + \gamma_1 v_x^2(x, t) + \delta v_{xx}^2(x, t)) \, dx \\
= - \int_0^\infty 2v \partial_x f(\mu, \mu_x, \mu_{xx}) \, dx \tag{82}
\]

We now estimate the last two integrals:

\[
- \int_0^\infty 2v \partial_x f(\mu, \mu_x, \mu_{xx}) \, dx \\
\leq 2\|v(\cdot, t)\|\|\partial_x f(\mu, \mu_x, \mu_{xx})\| \\
\leq c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^2 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^2 \\
+ \alpha^3 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^3 \right) \|v(\cdot, t)\| \tag{83}
\]

and

\[
- \int_0^\infty 2v \partial_x \left( \frac{3}{2} \mu v + \frac{7}{24}(\mu v)_{xx} - \frac{7}{24} \mu_x v_x - \frac{3}{8} \mu^2 v - \frac{3}{8} \mu v^2 \right) \, dx \\
= \int_0^\infty 2v \left( \frac{3}{2} \mu v + \frac{7}{24} \mu_x v_x + \frac{7}{24} \mu_{xx} v_x + \frac{7}{24} \mu v_{xx} - \frac{3}{8} \mu^2 v - \frac{3}{8} \mu v^2 \right) \, dx \\
= \int_0^\infty \left( 3 \mu v v_x + \frac{7}{12} \mu_{xx} v_{xx} + \frac{7}{12} \mu_x v_x^2 + \frac{7}{12} \mu v_{xx} v_x - \frac{3}{4} \mu^2 v v_x - \frac{3}{4} \mu v^2 v_x \right) \, dx \\
= \int_0^\infty \left( - \frac{3}{2} \mu_x v^2 - \frac{7}{24} \mu_{xx} v^2 + \frac{7}{24} \mu v_{xx}^2 + \frac{3}{4} \mu_x v^2 + \frac{1}{4} \mu_x v^3 \right) \, dx
\]
\[ \leq \frac{3}{2} \mu_x(\cdot, t) + \frac{7}{24} \mu_{xx}(\cdot, t) + \frac{3}{4} \mu_x(\cdot, t) \| v(\cdot, t) \|^2 + \frac{7}{24} |\mu_x(\cdot, t)|_\infty \| v_x(\cdot, t) \|^2 \\
+ \frac{1}{4} |\mu_x(\cdot, t)|_\infty \| v(\cdot, t) \|^2 \]

Notice that

\[ \sigma = \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^2 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^2 \| v(\cdot, t) \|^2 \\
+ c \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) \| v(\cdot, t) \|^2. \]

Denote

\[ \sigma(t) = \left( \int_0^\infty \left( v^2(x, t) + \gamma_1 v_x^2(x, t) + \delta v_{xx}^2(x, t) \right) dx \right)^{\frac{1}{2}}. \]

Notice that \( \sigma(t) \) is equivalent to \( H^2(\mathbb{R}^+) \)-norm \( \| v(\cdot, t) \|_2 \), the last three forms together indicate the following.

\[ \frac{d\sigma^2(t)}{dt} \leq c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^2 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^2 \right) \sigma(t) \]

\[ + c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^2 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^2 \right) \sigma^2(t) \]

\[ + c \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) \sigma^3(t). \]

This is equivalent to

\[ \frac{d\sigma(t)}{dt} \leq c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^3 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^3 \right) (1 + \sigma(t))^2. \]

Or, what is the same,

\[ (1 + \sigma(t))^{-2} \frac{d\sigma(t)}{dt} \leq c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^3 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^3 \right). \]

Integrate with respect to \( t \) over \([0, t]\), we have

\[ \left( 1 + \sigma(t) \right) - \frac{1}{1 + \sigma(t)} + \frac{1}{1 + \sigma(0)} \]

\[ \leq \int_0^t c \left( \alpha \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right) + \alpha^3 \left( |G(\beta^2 t)| + \beta |J(\beta^2 t)| \right)^3 \right) dt \]

\[ \leq c \alpha \beta^{-\frac{1}{2}} \int_0^\infty \left( |G(t)| + \beta |J(t)| \right) \right) dt + c \alpha \beta^{-\frac{1}{2}} \int_0^\infty \alpha^2 \left( |G(t)| + \beta |J(t)| \right)^3 dt. \]

Denote

\[ \kappa = \int_0^\infty \left( |G(t)| + \beta |J(t)| + \alpha^2 \left( |G(t)| + \beta |J(t)| \right)^3 \right) dt, \]

notice that \( G, J \in L_1 \cap L_3 \), so that \( \kappa = |G|_1 + O(\beta) \). Solve the last inequality, it yields

\[ \sigma(t) \leq \frac{\sigma(0) + c \kappa \left( 1 + \sigma(0) \right) \alpha \beta^{-\frac{1}{2}}}{1 - c \kappa \left( 1 + \sigma(0) \right) \alpha \beta^{-\frac{1}{2}}} = \sigma(0) + O(\alpha \beta^{-\frac{1}{2}}). \]

Because \( \sigma(t) \) is equivalent to \( H^2 \)-norm of \( v(\cdot, t) \) and \( \alpha \sim \beta = o(1) \), there is number \( c^*_2 > 0 \) such that

\[ \| v(\cdot, t) \|_2 \leq c^*_2 \quad \text{for} \quad t \in [0, \infty). \]

That is to say the problem (9) is well-posed globally in time and its solution

\[ u(x, t) = v(x, t) + \alpha \phi_1(x) G(\beta^2 t) + \alpha \beta \phi_2(x) J(\beta^2 t) \in C(0, \infty; H^2). \]
Therefore,
\[ \|u(\cdot, t)\|_2 \leq c_2^* + \alpha \|\phi_1\|_2 |G(\beta^{\frac{1}{2}} t)| + \alpha \beta \|\phi_2\|_2 |J(\beta^{\frac{1}{2}} t)|. \]

The bound form (79) follows. The theorem is established. \(\Box\)

**Theorem 4.2.** In addition to the assumptions on \(\gamma, \eta_0, G, J\) described in Theorem 4.1, suppose \(\eta_0 \in H^s\) for some \(s > 2\) which is not necessarily integer, then the solution \(u\) of problem (9) is a member of \(C(0, \infty; H^s)\). Furthermore, for any integer \(k \in [2, s]\), there are polynomial functions \(P_{k-2}(t)\) of degree \(k-2\) such that
\[ \|u(\cdot, t)\|_k \leq c_0 \left( |G(\beta^{\frac{1}{2}} t)| + |J(\beta^{\frac{1}{2}} t)| \right) + P_{k-2}(t); \]
if \(s\) is not an integer, then
\[ \|u(\cdot, t)\|_s \leq c_0 \left( |G(\beta^{\frac{1}{2}} t)| + |J(\beta^{\frac{1}{2}} t)| \right) + P_{[s]-2}(t), \]
where \(c = O(1)\) is a constant independent of \(t\), \([s] = \inf\{n \in \mathbb{N} : n \geq s\}\).

**Proof.** The strategy for the proof is bootstrapping. Based on Theorem 4.1, problem (9) has a unique solution \(u \in C(0, \infty; H^2)\) which has property (79), i.e. (86) is true for \(k = 2\). Because \(u\) satisfies the integral equation (42) and the operator \(K\) maps \(H^k\) to \(H^{k+3}\) continuously, it is guaranteed that
\[ u(x, t) - \left( \alpha \eta_0(\beta^{\frac{1}{2}} x) + \phi_1(x) \left( \alpha G(\beta^{\frac{1}{2}} t) - \alpha G(0) \right) + \phi_2(x) \left( \alpha \beta J(\beta^{\frac{1}{2}} t) - \alpha \beta J(0) \right) \right) \]
\[ = \int_0^t K \left\{ u + \gamma_2 u_{xx} + \frac{3}{4} u^2 + \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \right\} (x, s) \, ds \]
is a member of \(C(0, \infty; H^3)\), and
\[ \|u(\cdot, t) - \left( \alpha \eta_0(\beta^{\frac{1}{2}} \cdot) + \phi_1(\cdot) \left( \alpha G(\beta^{\frac{1}{2}} t) - \alpha G(0) \right) + \phi_2(\cdot) \left( \alpha \beta J(\beta^{\frac{1}{2}} t) - \alpha \beta J(0) \right) \right)\|_3 \]
\[ \leq \kappa_{3,0} \int_0^t \| \left( u + \gamma_2 u_{xx} + \frac{3}{4} u^2 + \frac{7}{24} u_x^2 + \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \right)(\cdot, s) \| ds \]
\[ \leq c \int_0^t \left( \|u(\cdot, s)\|_2 + \|u(\cdot, s)\|_2^2 + \|u(\cdot, s)\|_2^3 \right) ds, \]
where \(\kappa_{3,0} = \sup\{\|K\|_3 : u \in L_2, \|u\| = 1\}\). Inequality (79) and the property that \(G, J \in L_1\) implies that
\[ \int_0^t \|u(\cdot, s)\|_2^2 ds \leq c \int_0^t \left( 1 + |\alpha G(\beta^{\frac{1}{2}} s)| + |\alpha \beta J(\beta^{\frac{1}{2}} s)| \right) \, ds \]
\[ \leq c(t + \alpha \beta^{-\frac{1}{2}} |G|_1 + \alpha \beta^{\frac{1}{2}} |J|_1) \]
\[ = P_1(t). \]

Similarly, due to \(G, J \in L_1 \cap L_3\), (hence \(G, J \in L_2\) as well,) we have
\[ \int_0^t \|u(\cdot, s)\|_2^3 ds = c \int_0^t \left( 1 + |\alpha G(\beta^{\frac{1}{2}} s)| + \alpha \beta |J(\beta^{\frac{1}{2}} s)| \right)^2 \, ds \]
\[ \leq c(t + \alpha^2 \beta^{-\frac{1}{2}} |G|_2^2 + \alpha^2 \beta^{\frac{1}{2}} |J|_2^2) \]
\[ = P_1(t). \]
and in a same fashion,
\[
\int_0^t \|u(\cdot,s)\|_2^2 ds \leq c \int_0^t \left( 1 + \alpha |G(\beta^{\frac{3}{2}}s)| + \alpha \beta |J(\beta^{\frac{3}{2}}s)| \right)^3 ds
\]
\[
\leq c(t + \alpha^3 \beta^{-\frac{3}{2}} |G|_3^3 + \alpha^3 \beta^2 |J|_3^3)
= P_1(t).
\]

Therefore,
\[
\|u(\cdot,t) - (\alpha \eta_0(\beta^{\frac{3}{2}} \cdot) + \phi_1(\cdot)(\alpha G(\beta^{\frac{3}{2}} t) - \alpha G(0)) + \phi_2(\cdot)(\alpha \beta J(\beta^{\frac{3}{2}} t) - \alpha \beta J(0)))\|_3
\leq P_1(t). \tag{89}
\]

This indicates two folds. First if \( s = 3 \), i.e. \( \eta_0 \in H^3 \), then
\[
\|u(\cdot,t)\|_3 \leq \alpha \beta^{-\frac{3}{2}} \|\eta_0\|_3 + \alpha \|\phi_1\|_3 |G(\beta^{\frac{3}{2}} t) - G(0)| + \alpha \beta \|\phi_2\|_3 |J(\beta^{\frac{3}{2}} t) - J(0)| + P_1(t).
\]

Estimate (86) is true. Second, if \( 2 < s < 3 \), then \([s] = 3\), (89) implies that the solution \( u \in C(0, \infty; H^s) \) and
\[
\|u(\cdot,t)\|_s \leq \|u(\cdot,t) - (\alpha \eta_0(\beta^{\frac{3}{2}} \cdot) + \phi_1(\cdot)(\alpha G(\beta^{\frac{3}{2}} t) - \alpha G(0)) + \phi_2(\cdot)(\alpha \beta J(\beta^{\frac{3}{2}} t) - \alpha \beta J(0)))\|_3
\]
\[
+ \alpha \beta^{-\frac{3}{2}} \|\eta_0\|_s + \alpha \|\phi_1\|_s |G(\beta^{\frac{3}{2}} t) - G(0)| + \alpha \beta \|\phi_2\|_s |J(\beta^{\frac{3}{2}} t) - J(0)|
\leq P_1(t) + c \|G(\beta^{\frac{3}{2}} t) + J(\beta^{\frac{3}{2}} t)|.
\]

This is form (87). The theorem is proved for \( 2 < s < 3 \).

Consideration now is for \( 3 < s \leq 4 \), identity (88) indicates \( u - \alpha \eta_0(\beta^{\frac{3}{2}} \cdot) \in C(0, \infty; H^4) \), hence \( u = \alpha \eta_0(\beta^{\frac{3}{2}} \cdot) + (u - \alpha \eta_0(\beta^{\frac{3}{2}} \cdot)) \in C(0, \infty; H^s) \). Attention now is turned back to differential equation (9). Integrating with respect to \( t \), it follows
\[
\delta \partial_t^2 x(u(x,t) - \alpha \eta_0(\beta^{\frac{3}{2}} x))
= \alpha \eta_0(\beta^{\frac{3}{2}} x) - u(x,t) + \gamma_1 \partial_{xx}(u(x,t) - \alpha \eta_0(\beta^{\frac{3}{2}} x))
\]
\[
+ \int_0^t \left( u_x + \gamma_2 u_{xxx} + \frac{3}{4} (u^2)_x + \frac{7}{48} (u^2)_{xxx} - \frac{1}{8} (u^3)_x \right)(x,t) dt
\]

Taking \( L_2 \)-norm, it follows that
\[
\delta \|\partial_t^2 x(u(\cdot,t) - \alpha \eta_0(\beta^{\frac{3}{2}} \cdot))\|
\leq \|\alpha \eta_0(\beta^{\frac{3}{2}} \cdot) - u(\cdot,t)\| + \gamma_1 \|\partial_{xx}(u(\cdot,t) - \alpha \eta_0(\beta^{\frac{3}{2}} \cdot))\|
\]
\[
+ \int_0^t \left( \|u_x(\cdot,t)\| + \|u_{xxx}(\cdot,t)\| + \frac{3}{2} \|u^2(\cdot,t)\| \|u_x(\cdot,t)\| + \frac{7}{24} \|u^3(\cdot,t)\| \|u_x(\cdot,t)\| \right) dt.
\]

Notice that
\[
\|\alpha \eta_0(\beta^{\frac{3}{2}} \cdot) - u(\cdot,t)\| + \gamma_1 \|\partial_{xx}(u(\cdot,t) - \alpha \eta_0(\beta^{\frac{3}{2}} \cdot))\| \leq c \left( 1 + \alpha (|G(\beta^{\frac{3}{2}} t) + \beta |J(\beta^{\frac{3}{2}} t)|) \right),
\]
and the integrant is bounded by
\[
c \left( 1 + \alpha (|G(\beta^{\frac{3}{2}} t) + \beta |J(\beta^{\frac{3}{2}} t)|) \right) \left( 1 + \alpha (|G(\beta^{\frac{3}{2}} t) + \beta |J(\beta^{\frac{3}{2}} t)|) + P_1(t) \right)
\]
\[
\int \right) \left( 1 + \alpha (|G(\beta^{\frac{3}{2}} t) + \beta |J(\beta^{\frac{3}{2}} t)|) + P_1(t) \right)
\]
+ c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right) P_1(t).

Whence,
\delta \| \partial_x^{m+1} \left( u(\cdot, t) - \alpha \eta_0(\beta \frac{\partial}{\partial x}) \right) \|
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3 dt
\leq c (\alpha (|G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})|) + P_2(t)).

Similar argument as in case 2 < s \leq 3 shows (86) and (87) for 3 < s \leq 4.

Suppose the results stated in the theorem is true for m - 1 < s \leq m, i.e. the solution u \in C(0, \infty; H^s) and (86) for k = 2, 3, \ldots, \lfloor s \rfloor and (87) hold true. Then for m < s \leq m + 1, identity (88) indicates u - \alpha \eta_0(\beta \frac{\partial}{\partial x}) \in C(0, \infty; H^{m+1})$, hence u = \alpha \eta_0(\beta \frac{\partial}{\partial x}) + (u - \alpha \eta_0(\beta \frac{\partial}{\partial x})) \in C(0, \infty; H^s)$. In (90), taking derivative with respect to x m - 3 times, it yields
\delta \| \partial_x^{m+1} \left( u(\cdot, t) - \alpha \eta_0(\beta \frac{\partial}{\partial x}) \right) \|
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3 dt
\leq c (\alpha (|G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})|) + P_2(t))

By the inductive assumption and the property that G, J \in L_1,
\delta \| \partial_x^{m+1} \left( u(\cdot, t) - \alpha \eta_0(\beta \frac{\partial}{\partial x}) \right) \|
\leq c (\alpha (|G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})|) + P_{m-3}(t)
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3 dt
\leq c (\alpha (|G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})|) + P_{m-3}(t) + \frac{3}{4} \| \partial_x^{m-2} u(\cdot, t) \|
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3 dt
\leq c (\alpha (|G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})|) + P_{m-3}(t) + \frac{7}{48} \| \partial_x^{m-2} (2u(\cdot, t) u_{xx}(\cdot, t) + u_x^2(\cdot, t)) \|
\leq c \left( 1 + \alpha \left( |G(\beta \frac{\partial}{\partial x})| + |J(\beta \frac{\partial}{\partial t})| \right) \right)^3 dt.

It is clear that the growth bound in time t is determined by \| \partial_x^{m-2} (2u_{xx} + u_x^2) \|
which involves the highest order of derivatives. Since
\partial_x^{m-2} (2u_{xx} + u_x^2) = \sum_{n=0}^{m-2} \binom{m-2}{n} \left( 2 \partial_x^n u \partial_x^{m-n} u + \partial_x^{n+1} u \partial_x^{m-n-1} u \right).
we have,
\[
\|\partial_x^{m-2} (2u(\cdot, t) u_{xx}(\cdot, t) + u_x^2(\cdot, t))\| \\
\leq c \sum_{n=0}^{m} \|\partial_x^n u(\cdot, t) \partial_x^{m-n} u(\cdot, t)\|
\]
\[
\leq c \left( \|u(\cdot, t)\|_\infty \|\partial_x^n u(\cdot, t)\| + |u_x(\cdot, t)| |\partial_x^{m-1} u(\cdot, t)\| + \sum_{n=2}^{m-2} \|\partial_x^n u(\cdot, t)\|_\infty \|\partial_x^{m-n} u(\cdot, t)\| \right)
\]
\[
\leq c \left( \|u(\cdot, t)\|_\infty \|\partial_x^n u(\cdot, t)\| + |u_x(\cdot, t)| |\partial_x^{m-1} u(\cdot, t)\| \\
+ \sum_{n=2}^{m-2} \|\partial_x^n u(\cdot, t)\| \frac{1}{2} \|\partial_x^{n+1} u(\cdot, t)\| \frac{1}{2} \|\partial_x^{m-n} u(\cdot, t)\| \right).
\]

By the inductive assumption,
\[
\|\partial_x^n u(\cdot, t)\| \leq c \alpha \left( |G(\beta \frac{1}{2} t)| + \beta |J(\beta \frac{1}{2} t)| + P_{n-2}(t) \right) \quad \text{for all} \quad n = 2, 3, \ldots, m - 1,
\]
it obtains,
\[
\|\partial_x^n u(\cdot, t)\| \leq c \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{m-2}(t) \right)
\]
and
\[
\|\partial_x^n u(\cdot, t)\| \frac{1}{2} \|\partial_x^{n+1} u(\cdot, t)\| \frac{1}{2} \|\partial_x^{m-n} u(\cdot, t)\|
\]
\[
\leq c \left( \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{n-2}(t) \right) \right) \frac{1}{2}
\]
\[
\times \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{n-1}(t) \right) \frac{1}{2}
\]
\[
\times \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{m-2}(t) \right)
\]
\[
\leq c \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{m-2}(t) \right).
\]

It transpires that
\[
\delta \|\partial_x^{n+1} u(\cdot, t) - \beta \eta_0 (\beta t)\| \\
\leq c \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_m(t) \right)
\]
\[
+ c \int_0^t \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| \right) \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_m(t) \right) \right) dt
\]
\[
\leq c \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| + P_{m-3}(t) \right) + c \int_0^t \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| \right) \frac{1}{2} dt
\]
\[
+ c \left( \int_0^t \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| \right) \frac{1}{2} dt \right) \frac{1}{2} \left( \int_0^t P_{m-2}(t) \right) \frac{1}{2}
\]
\[
\leq c \left( 1 + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| \right) + P_{m-1}(t).
\]

If \( s = m + 1 \) then
\[
\|u(\cdot, t)\|_{m+1} \leq \|\alpha \eta_0 (\beta t)\|_{m+1} + \|u(\cdot, t) - \alpha \eta_0 (\beta t)\|_{m+1}
\]
\[
\leq c \left( \alpha \beta \frac{1}{2} \|\eta_0\|_{m+1} + \alpha |G(\beta \frac{1}{2} t)| + \alpha \beta |J(\beta \frac{1}{2} t)| \right) + P_{m-1}(t),
\]
form (86) is true; and if \( s \) is not an integer, then \( \lceil s \rceil = m + 1 \),

\[
\|u(\cdot, t)\|_s \leq \|\alpha \eta_0(\beta^{\frac{1}{2}} \cdot)\|_s + \|u(\cdot, t) - \alpha (\beta^{\frac{1}{2}} \cdot)\|_{m+1} \\
\leq c(\alpha \beta^{-\frac{1}{2}} \|\eta_0\|_s + \alpha \|G(\beta^{\frac{1}{2}} t)\| + \alpha \beta |J(\beta^{\frac{3}{2}} t)|) + P_{m-1}(t),
\]

form (87) is true. Mathematical induction finishes off the theorem.

\[\square\]

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