ON THE BRANCHING GEOMETRY OF ALGEBRAIC FUNCTIONS

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Abstract. This paper describes an algorithm for determining the branching geometry of algebraic functions. The Newton polygon algorithm can be used to compute power expansions centered at the origin for the branches but these series often represent only a small portion of the function. In order to obtain power expansions for the remainder of the function, a method is described using a variation of Laurent’s Theorem applied to algebraic functions. After the branching and power expansions are computed, a geometric method is used to determine the region of convergence of these series to arbitrary precision. The software used to implement the algorithm is Mathematica ver. 11.1.

1. Introduction

The objects studied in this paper are algebraic functions \( w(z) \) expressed implicitly by the equation

\[
f(z, w) = a_0(z) + a_1(z)w + a_2(z)w^2 + \cdots + a_n(z)w^n = 0,
\]

with \( z \) and \( w \) complex variables and the coefficients, \( a_i(z) \), polynomials in \( z \) with rational coefficients. The degree of the function is the highest power of \( w \). By the Implicit Function Theorem, this equation defines locally, an analytic function \( w(z) \) when \( \frac{\partial f}{\partial w} \neq 0 \). And it is known from the general theory of algebraic functions that the solution, \( w(z) \), has \( n \) fractional power series expansions around the origin. These fractional power series can be computed by the Newton polygon method and often have finite radii of convergences determined by the nearest impinging singular point so do not in general represent the entire function. The remainder of the function can be represented by annular power series. These series are also fractional and can be computed by a version of Laurent’s Theorem applied to algebraic functions.

The purpose of this paper is four-fold:

1. Describe an algorithm to determine the branching geometry of an algebraic function,
2. Describe a method of applying Laurent’s Theorem to compute annular power series expansions of an algebraic function,
3. Determine the region of convergence of a fractional power series representation of \( w(z) \),
4. Present a software tool for visually investigating the branching of algebraic functions.

2. Some properties of algebraic functions used in this paper

Fractional power expansions of algebraic functions are called Puiseux series and usually have finite radii of convergences. As stated earlier, series centered at the origin can be computed by the Newton Polygon method but usually represent a small portion of (1). Power series for the remainder of the function can in principle, be computed using a variation of Laurent’s Theorem for multivalued functions. The Laurent expansions computed this way are again fractional power expansions. In this way, the branching geometry of \( w(z) \) can be represented in the form of singular points segregating the \( z \)-plane into annular regions where the function is analytic and ramifies into multivalued branches. This branching, both around the singular points and annular regions, is represented by the notation \( \{ s_1, s_2, \ldots, s_n \} \) where each \( s_i \) represents an analytic and single-valued sheet of the branch. For example the notation \( \{ 1, 2 \} \) represents a 2-cycle branch, like \( \sqrt{z} \), and the numbers 1 and 2 represent the sort order of the function value at predetermined points in the \( z \)-plane called the annular or singular reference points. Figure 1 shows these points as the blue and red dots labeled ‘arp’ and ‘srp’.

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The resultant of \( f(z, w) \) with its derivative \( f_w \) is denoted by \( R(f, f_w) \). A point \( z \) where \( R(f, f_w) \) is zero is a singular point of \( f \). However this does not tell us which branch sheet is singular. A point where \( a_\alpha(z) = 0 \) is also a pole, possibly ramified, of the function. Singular points are sorted first by the real component and then the imaginary component and labeled \( s_n \) with \( n \) ranging from one to the total number of singular points including the point at infinity. Even though the function may not ramify and thus not be singular at infinity, it is still included in the list of singular points.

A singular point may not affect all coverings of an algebraic function unless the function fully-ramifies at the singular point. For example, a 10-degree function may only ramify into a single 2-cycle branch at a singular point with the remaining coverings single-cycled and unramified. In this case, the 2-cycle covering is singular. The eight single-cycle coverings are not analytically affected at this singular point unless affected by a pole of the function. However, if the function fully-ramifies into a 10-cycle branch at this singular point, all coverings would be affected. It is for this reason a power expansion of an algebraic function often has a region of convergence extending beyond the nearest singular point: the branch coverings may simply not be affected by the singular point. Only when the covering becomes singular does the convergence region of its power expansion become established. The main objective of this paper is to identify which singular points are interrupting the analyticity of branch cycles thereby establishing the analytic domain of the branch and subsequently, the region of convergence of their power expansions.

The following conventions are used in this paper:

1. The Puiseux expansions of (1) consist of a set of \( d \)-valued branches. A branch is sometimes called a \( d \)-cycle where \( d \) is a positive integer. A power series in \( z^{1/3} \) would be a 3-cycle branch. It has three coverings over the complex \( z \)-plane just like \( \sqrt[3]{z} \). The sum of the cycles is always equal to the degree of the function in \( w \).

2. The concept of “branch” is used throughout this paper and refers to a multi-valued \( d \)-cycle of \( w(z) \).

3. The following discussion makes use of the term, “extending a branch over a singular point”. This is in reference to the discussion above about singularities and the coverings they affect.
(4) $r_i$ is a positive integer representing the radius of a circular ring with center at the origin. These rings are created by arranging the finite singular points in order of increasing absolute value. The smallest non-zero singular point is therefore on ring one with radius $r_1$. Singular points with the next largest absolute value are on ring two with radius $r_2$ and so forth.

(5) $a_i$ is an annular region between successive rings. $a_1$ is the disk $D(r_0, r_1)$, punctured if the origin is a singular point. $a_2$ is the next annular region between $r_1$ and $r_2$ and so forth.

(6) Power series for algebraic branches are fractional power series called Puiseux series. Two methods are used to compute these power series: Newton Polygon and Laurent’s Theorem applied to algebraic functions. For brevity, only a few terms of a series are listed. Actual computations for this paper used 150 terms with 300 digits of precision.

3. Determining Monodromies

The finite singular points of the function are computed by setting the resultant, $R(f, f_n)$, equal to zero. This gives a list of $z$ values where the singular points are located. The list is then sorted in order of absolute value and used to create rings around the origin. In the annular regions between rings, the function is analytic and splits into multi-valued branches.

The next objective is to determine the branching geometry around each annular region, and then determine the annular domain of analyticity for each branch. The branching is called monodromy or ‘ramified covering’ and these terms are used interchangeably. The monodromy is computed by numerical integration: If an analytically-continuous $2\pi$ circular path over an annular domain is followed, then at the end of the path, we may or may not return to the same starting point on the branch. If we return to the same value, we have identified a single-cycle branch. If we need to traverse a $4\pi$ circuit to return to the same starting point, then we have traversed a 2-cycle branch and so forth. In practice however, a somewhat different algorithm is used but the principle is similar. This gives us the local monodromy of each annular region. The monodromy around each singular point is similarly computed. Naturally, this requires a comparison of floating-point numbers and some type of numerical accuracy must be established to identify terms considered identical. To accomplish this, a certain number of digits to the right of the decimal place is determined without rounding. For example, we could require five digits of accuracy between $w(0)$ and $w(2\pi)$ to be considered equal. However care must be taken to avoid rounding or cases like $0.99999999$. The later case is converted to 1.

Consider first the function

$$f(z, w) = w^2 - (z - (1 + i))^2(z - (1 - i))^2(z - 2) = 0.$$  

(2)

Setting $R(f, f_w) = 0$ we obtain as the singular points, $\{1 - i, 1 + i, 2, \infty\}$ and the rings in Figure 1. The black circles demarcate the annular regions separated by one or more singular points shown in the figure as black dots. During the analysis, we integrate around annular regions and singular points beginning at reference points labeled ‘arp’ in the figure for annular regions and ’srp’ for singular points. The annular integration paths are shown as the red dashed circles. The blue dashed circles are the singular point integration paths.

Numerical integration over $2\pi$ routes around each annular region are used to determine the monodromy. And in order to minimize errors, a route midway in the region is taken to maximize the distance to the nearest singular point. So for example if a region was given by $1 \leq r \leq 2$, we would integrate at 1.5 using for example $z(t) = 1.5e^{it}$. But how do we decide if values are identical? We can do this by extracting the actual digits of the results without rounding and chop off the rest. For example, consider the value $1.23901950708 - 0.278993512i$. Taking an accuracy of 5 digits, we construct the following approximate integer representation of this number:

$$((1, (1, 2, 3, 9, 0), 1), (-1, (2, 7, 8, 9, 9), 0)),$$

with the format $\{s, \{d_1, d_2, \ldots, d_n\}, e\}$ with $s$ being the sign of the number, $(d_1, d_2, \ldots, d_n)$ the digits of the number and $e$ being the number of digits to the left of the decimal place. Notice that we did not round the number. So we basically convert a floating point number to an integer sequence so that we can make exact comparisons of floating point numbers up to a desired accuracy. In practice, we are comparing a
number against a set of numbers and we check that the smallest difference between the set of numbers is larger than the accuracy of the comparison to minimize errors.

Now consider a 10-degree algebraic function. We wish to integrate over one of the annular regions and then compare the starting and ending values of the function over each $2\pi$ route of each branch sheet. Table 1 gives actual results for a 10-degree function. Looking carefully at the starting and ending values we see the $w_1$ root goes back to the $w_1$ root so this is a single cycle branch $\{1\}$. Next, the $w_2$ root goes to $w_7$ then $w_6$ back to $w_2$. This is a 4-cycle branch, $\{2, 7, 6, 4\}$. We then have two single cycle branches $w_3$ and $w_5$, then a 3-cycle branch $w_8, w_9, w_{10}$. Therefore the annular monodromy is $\{\{1\}, \{2, 7, 6, 4\}, \{3\}, \{5\}, \{8, 9, 10\}\}$.

| Root | Starting Value | Ending Value |
|------|----------------|--------------|
| 1    | (-1,1,0,5,0,8,1),(-1,4,0,8,8,9,9,-1) | (-1,1,0,5,0,8,1),(-1,4,0,8,8,9,9,-1) |
| 2    | (-1,4,7,6,5,7,0),(-1,1,9,8,2,8,-1) | (1,5,8,2,9,7,0),(-1,9,7,6,1,3,0) |
| 3    | (1,4,0,9,0,9,0),(-1,1,7,5,3,2,4,0) | (-1,4,0,9,0,9,0),(-1,1,7,5,3,2,4,0) |
| 4    | (-1,2,5,8,4,2,0),(-1,3,8,2,8,1) | (-1,4,7,6,5,7,0),(-1,1,9,8,2,8,-1) |
| 5    | (-1,2,3,9,4,1,0),(-1,7,6,4,2,7,0) | (-1/2,3,9,4,1,0),(-1,7,6,4,2,7,0) |
| 6    | (-1,3,8,0,2,3,1),(-1,1,7,0,9,5,3,1) | (-1,2,5,8,4,2,0),(-1,1,3,8,2,8,1) |
| 7    | (1,5,8,2,9,7,0),(-1,9,7,6,1,3,0) | (-1,3,8,0,2,3,1),(-1,1,7,0,9,5,3,1) |
| 8    | (1,9,5,0,3,7,0),(-1,4,4,9,0,4,0) | (1,1,1,0,7,5,1),(-1,5,2,0,1,7,1) |
| 9    | (1,1,1,0,7,5,1),(-1,5,2,0,1,7,1) | (1,1,1,4,4,9,1),(-1,6,2,4,2,6,0) |
| 10   | (1,1,1,4,4,9,1),(-1,6,2,4,2,6,0) | (1,9,5,0,3,7,0),(-1,4,4,9,0,4,0) |

4. Numerically Solving the Monodromy Differential Equations

In order to determine monodromies, the function is integrated around each annular region and around each singular point. This is done by solving the monodromy differential equation: Given $f(z, w) = a_0(z) + a_1(z)w + \cdots + a_n(z)w^n = 0$, then $\frac{dw}{dt} = -\frac{f_z}{f_w} \frac{dz}{dt}$. In order to compute the monodromy, this differential equation is solved for $w(t)$ over a circular path around annular regions and singular points. For example, the following $n$ initial value problems for a function of degree $n$ would be solved:

$$\frac{dw}{dt} = -\frac{f_z}{f_w} \frac{dz}{dt}, \quad (z_0, w_i), \quad i = 1, 2, \cdots, n$$

where $w_i$ are the roots to the expression $f(z_0, w) = 0$. We then compile the values of $w(t)$ at the beginning and ending of each $2\pi$ route. However, in order to obtain accurate results, the working precision and step size of the numerical integration is adjusted as needed to achieve the desired accuracy. To economize this, a loop of decreasing step size with increasing precision ranging from $(1/1000, 20)$ to $(1/50000, 65)$ is set up. However, even at the high range of precision, we may not be able to resolve the branching if for example the region is very small or the branch sheets are very close to each other. For random functions, if the annular size, $|r_i - r_{i+1}|$, is larger than $1/5000$, and the difference between branch sheet is greater than the desired accuracy, then a 10-degree function can usually be processed successfully. Therefore, this paper deals only with functions of degree 10 or less in $w$ and $z$ with $|r_i - r_{i+1}| \geq 1/5000$.

5. Determining the Branching Geometry

We divide this section into the following sub-sections:

1. Computation of singular points,
2. Construction of annular regions,
3. Compute singular point monodromies,
(4) Compute annular monodromies, 
(5) Determine branch-continuation support, 
(6) Compute branch continuations, 
(7) Determine continuations over poles, 
and use the following function to illustrate the concepts:
\[ f(z, w) = (9) + (-7)w + (7 - 2z - 4z^2 - z^3)w^2 + (7z - 2z^3)w^3. \] 

\[ \text{(3)} \]

\[ \text{Table 2. Singular points} \]

| Singularity | Value                                      |
|-------------|--------------------------------------------|
| \(s_1\)    | 0.                                         |
| \(s_2\)    | -0.21713 - 0.255535 i                      |
| \(s_3\)    | -0.21713 + 0.255535 i                      |
| \(s_4\)    | 1.36571                                    |
| \(s_5\)    | -1.83037 - 0.0204249 i                     |
| \(s_6\)    | -1.83037 + 0.0204249 i                     |
| \(s_7\)    | -1.87083                                   |
| \(s_8\)    | 1.87083                                    |
| \(s_9\)    | 2.21932 - 1.22168 i                        |
| \(s_{10}\) | 2.21932 + 1.22168 i                        |
| \(s_{11}\) | -6.85468 - 6.05671 i                       |
| \(s_{12}\) | -6.85468 + 6.05671 i                       |
| \(s_{13}\) | \(\infty\)                                |

5.1. **Computation of singular points.** Using the built-in Mathematica function `NSolve`, the finite singular points can be determined to practically arbitrary precision given rational exponents of (1). However, in practice these are computed to 75 digits of precision, and this precision is approximately maintained throughout the calculations. Table 2 lists the singular points in blue where we have identified poles in red.

\[ \text{Table 3. Rings and Singular points} \]

| Ring | Radius | Singularity | Value                                      |
|------|--------|-------------|--------------------------------------------|
| \(r_1\) | 0.335327 | \(s_1\) | -0.21713 - 0.255535 i                      |
|       |        | \(s_2\) | -0.21713 + 0.255535 i                      |
| \(r_2\) | 1.36571 | \(s_3\) | 1.36571                                    |
| \(r_3\) | 1.83048 | \(s_4\) | -1.83037 - 0.0204249 i                     |
|       |        | \(s_5\) | -1.83037 + 0.0204249 i                     |
| \(r_4\) | 1.87083 | \(s_6\) | -1.87083                                   |
|       |        | \(s_7\) | 1.87083                                    |
| \(r_5\) | 2.53336 | \(s_8\) | 2.21932 - 1.22168 i                        |
|       |        | \(s_9\) | 2.21932 + 1.22168 i                        |
| \(r_6\) | 9.14715 | \(s_{10}\) | -6.85468 - 6.05671 i                      |
|       |        | \(s_{11}\) | -6.85468 + 6.05671 i                      |

5.2. **Construction of rings and annular regions.** Once the singular points are computed, they are arranged in order of increasing absolute values which then segregate the z-plane into annular regions separated by rings. On each ring lies one or more singular points. Table 3 lists the rings, \(r_i\), which will be used to define the region of convergence of power expansions of branches. For example, a branch may have a region of convergence given by \((r_1, r_3)\).
Between each ring is an annular region devoid of singular points. Table 4 lists the annular regions separated by singular points. The last region is simply determined by an arbitrary distance from the most distant finite singular point. In this case, this distance is 4.

### Table 4. Annular Regions

| Annulus | Annulus/singular point |
|---------|------------------------|
| $a_1$   | $\{0.00005,0.335277\}$ |
|         | $-0.21713-0.255535\,i$ |
|         | $-0.21713+0.255535\,i$ |
| $a_2$   | $\{0.335377,1.36566\}$ |
|         | 1.36571                 |
| $a_3$   | $\{1.36576,1.83043\}$  |
|         | $-1.83037-0.0204249\,i$ |
|         | $-1.83037+0.0204249\,i$ |
| $a_4$   | $\{1.83053,1.87078\}$  |
|         | -1.87083                |
|         | 1.87083                 |
| $a_5$   | $\{1.87088,2.53331\}$  |
|         | 2.21932 $-1.22168\,i$  |
|         | 2.21932 $+1.22168\,i$  |
| $a_6$   | $\{2.53341,9.1471\}$   |
|         | $-6.85468-6.05671\,i$  |
|         | $-6.85468+6.05671\,i$  |
| $a_7$   | $\{9.1472,13.1471\}$   |
|         | $\infty$               |

5.3. **Computation of annular monodromies.** Once the annular regions are defined, the annular monodromies can be computed. One way to compute this is to simply integrate around annuli over $2\pi$ routes, one route for each branch sheet for a total of $n$ sheets and determine how many routes to return to a starting point as was shown above in Table 1 until all the branch sheets have been processed. This necessarily involves comparing floating-point numbers, but if we are willing to accept a tolerance say of five decimal digits to the right of the decimal point or other numerical accuracy, experience has shown we can obtain reliable results.

As stated earlier, to effect this integration, we integrate the monodromy differential equation. In the case of the annular regions, a circular path midway in the region with starting value $z(\theta_0)$ is chosen with $\theta_0$ somewhat arbitrary to best effect the integration. For example, $\theta$ could be chosen to maximize the distance from most singular points. We then form a table of starting and ending values as described in Section 3 above for each path and then determine the monodromies through integer comparisons. This gives the monodromy results in Table 5.

5.4. **Computing singular point monodromies.** The procedure for computing singular point monodromies is identical to that for the annular regions: We solve the monodromy DE for a circular path around each singular point containing no other singular points. In the case of the monodromy around infinity, we integrate over a closed contour enclosing all finite singular points. These results are shown in Table 6.

5.5. **Determining support.** Once the monodromies for both the annular regions and singular points are computed, the possible support of each annular branch across intervening singular points is determined. A necessary condition for branch continuation across a singular point is that that singular monodromy between successive annuli must support a sufficient number of single-cycle branches to continue the branch.
### Table 5. Annular Monodromies

| Annulus | Annulus/singular point | Monodromies |
|---------|------------------------|-------------|
| $a_1$   | $\{0.00005,0.335277\}$ | $((1),(2),(3))$ |
|         | $-0.21713-0.255535\ i$ |             |
|         | $-0.21713+0.255535\ i$ |             |
| $a_2$   | $\{0.335377,1.36566\}$ | $((1,3,2))$ |
|         | 1.36571                 |             |
| $a_3$   | $\{1.36576,1.83043\}$  | $((1),(3),(2))$ |
|         | $-1.83037-0.0204249\ i$ |             |
|         | $-1.83037+0.0204249\ i$ |             |
| $a_4$   | $\{1.83053,1.87078\}$  | $((1,2),(3))$ |
|         | $-1.87083$              |             |
|         | 1.87083                 |             |
| $a_5$   | $\{1.87088,2.53331\}$  | $((1,2),(3))$ |
|         | 2.21932                |             |
|         | $-1.22168\ i$          |             |
|         | 2.21932                |             |
|         | $+1.22168\ i$          |             |
| $a_6$   | $\{2.53341,9.1471\}$   | $((1,2),(3))$ |
|         | $-6.85468-6.05671\ i$  |             |
|         | $-6.85468+6.05671\ i$  |             |
| $a_7$   | $\{9.1472,13.1471\}$   | $((1),(2,3))$ |
|         | $\infty$               |             |

### Table 6. Annular and Singular Point Monodromies

| Annulus | Annulus/singular point | Monodromies |
|---------|------------------------|-------------|
| $a_1$   | $\{0.00005,0.335277\}$ | $((1),(2),(3))$ |
|         | $-0.21713-0.255535\ i$ |             |
|         | $-0.21713+0.255535\ i$ |             |
| $a_2$   | $\{0.335377,1.36566\}$ | $((1,3,2))$ |
|         | 1.36571                 |             |
| $a_3$   | $\{1.36576,1.83043\}$  | $((1),(3),(2))$ |
|         | $-1.83037-0.0204249\ i$ |             |
|         | $-1.83037+0.0204249\ i$ |             |
| $a_4$   | $\{1.83053,1.87078\}$  | $((1,2),(3))$ |
|         | $-1.87083$              |             |
|         | 1.87083                 |             |
| $a_5$   | $\{1.87088,2.53331\}$  | $((1,2),(3))$ |
|         | 2.21932                |             |
|         | $-1.22168\ i$          |             |
|         | 2.21932                |             |
|         | $+1.22168\ i$          |             |
| $a_6$   | $\{2.53341,9.1471\}$   | $((1,2),(3))$ |
|         | $-6.85468-6.05671\ i$  |             |
|         | $-6.85468+6.05671\ i$  |             |
| $a_7$   | $\{9.1472,13.1471\}$   | $((1),(2,3))$ |
|         | $\infty$               |             |

into the next annular region. For example, if a 3-cycle branch is to be continued across a singular point, then the monodromy around the singular point must have three single-cycle branches. And likewise for other cycles.
A second necessary condition is that the next or post-annular monodromy must have the same branch cycle type we are attempting to continue. Branch support is shown in Table 7.

| Annulus | Annulus/singular point | Monodromies | Support |
|---------|-----------------------|-------------|---------|
| 0       | {0.00005, 0.335277}  | ((1), (2), (3)) |         |
|         | {-0.21713 - 0.255535 i} | ((1), (2), (3)) |         |
|         | {-0.21713 + 0.255535 i} | ((1), (2), (3)) |         |
| a1      | {0.335377, 1.36566}  | ((1, 3), 2) |         |
|         | 1.36571               | ((1), (2), (3)) |         |
| a2      | {1.36576, 1.83043}   | ((1, 3), (2)) | (2)     |
|         | -1.83037 - 0.0204249 i | ((1), (2), (3)) |         |
|         | -1.83037 + 0.0204249 i | ((1), (2), (3)) |         |
| a3      | {1.83053, 1.87078}   | ((1, 2), (3)) | (1, 2), (3) |
|         | -1.87083              | ((1), (2), (3)) |         |
|         | 1.87083               |             |         |
| a4      | {1.87088, 2.53331}   | ((1, 2), (3)) | (3)     |
|         | 2.21932 - 1.22168 i  | ((1, 2), (3)) |         |
|         | 2.21932 + 1.22168 i  | ((1, 2), (3)) |         |
| a5      | {2.53341, 9.1471}    | ((1, 2), (3)) | (3)     |
|         | -6.85468 - 6.05671 i | ((1, 2), (3)) |         |
|         | -6.85468 + 6.05671 i | ((1, 2), (3)) |         |
| a6      | {9.1472, 13.1471}    | ((1), (2), (3)) |         |
| a7      | ∞                     | ((1), (2), (3)) |         |

For example, consider the first and second annuli in Table 7 and the intervening singular points: The first annulus has three single-cycle branches. In order to continue one or more of these branches into annulus two, the singular points between these regions must support single-cycle branches. In this case, the singular points do have single-cycle branches. However, annulus 2 is fully-ramified into a 3-cycle branch so does not support continuing any of the single-cycle branches. And likewise for annulus 2: In order to continue this branch into annulus 3, the intervening singular points would need three single-cycle branches. In the case of annulus 3 with monodromy \{\{1, 3\}, \{2\}\}, the 2-cycle branch cannot be continued into annulus 4 since the intervening singular points do not support two single-cycle branches. However there is support to continue branch \{2\}: the singular point has a single-cycle branch and annulus 4 does as well. Now consider annuli 4 and 5 and the two poles between them: the poles do not ramify but rather consists of three single-cycle branches and so supports continuing both the \{1, 2\} branch and the \{3\} branch however not holomorphically: one or more of the branches will contain a pole.

### 5.6. Determining branch continuations.

In the previous section, a necessary condition for branch continuation was defined: The intervening singular points must support continuation. A sufficient condition is that each branch sheet be analytically continuous over all singular points on the bordering ring and continuing into the next annular region onto a branch with the same monodromy sequence. For example, to continue a 3-cycle branch \{2, 1, 3\} across a singular point from annulus \(k\) to annulus \(k + 1\), then the singular monodromy must support three single-cycle branches, and annulus \(k + 1\) must have a 3-cycle branch such as \{1, 2, 3\} such that sheet 2 in annulus \(k\) is continued onto sheet 1 of annulus \(k + 1\), sheet 1 is continued onto sheet 2, and sheet 3 is continued onto sheet 3 or:

\[
\begin{align*}
2 & \quad 1 \quad 3 \\
\downarrow & \quad \downarrow \quad \downarrow \\
1 & \quad 2 \quad 3
\end{align*}
\]
After the possible continuations are determined, numerical integration is used to determine if each branch covering is analytically continuous around the singular points. In order to show how this integration is effected, Equation (2) is solved along the contour shown in Figure 2:

To check if a branch sheet in region 1 is continued onto a branch sheet in region 2 across the singular point at $1 + i$, the branch sheet is analytically continued to the branch sheet of region 2 by integrating from point $a$ to $b$ in Figure 2 along the red, green, blue, and orange contours. The dotted black line is the argument where all annular monodromies are taken at their associated reference point, arp. At point $a$ in the diagram, the annular monodromy of the annulus 1 is determined. At point $b$, the monodromy of annulus 2 is determined. In this way we can compare the monodromies between annulus 1 and 2 to determine if a branch in region 1 analytically continues to a branch in region 2. However we cannot simply integrate from point $a$ to $b$ as we may be integrating too close to another singular point. Rather, we integrate over the third blue leg, around the singular point to point $d$ where we choose the radius of the blue leg such that it minimizes the distance to the nearest singular point while remaining in the bordering rings. We now have integrated over an analytically-continuous route from region 1 to region 2 to point $d$. And our objective is to get to point $b$ where we can compare annular monodromies. And although we could in principle integrate directly from point $d$ to point $b$, we take a more symmetrical route first along the short purple leg and then the orange leg to $b$.

At the point $b$ in the figure, the monodromies of region 2 are compared to those in region 1. The resulting continuations are shown in the fourth column of Figure 3 with arrows between branch regions signifying continuations.

5.7. **Continuations over poles.** The arrows between branches in annuli 4 and 5 in Figure 3 shows the \{1, 2\} and \{3\} cycles in ring 4 continuing into ring 5 across two singular points. However, since the intervening singular points are poles, one or more of the branch continuations may be meromorphic, that is, contain a pole.
In cases involving poles, we can determine which branch sheet is affected by the pole by computing the Puiseux expansion around each singular point and then computing the value of each series at the singularity reference point \( c \) in Figure 2. First, the function at point \( c \) is computed and the values sorted. Then each branch sheet is continued to point \( c \) and the values likewise sorted. The Puiseux series at point \( c \) is computed. Note however, the point at \( c \) for the Puiseux expansions will be offset by the singular point. For each singular point, the annulus, branch, sheet, singular point, sort index, the value of \( z \) at the singular point reference point, and it’s offset, \( zP \) when we translate axes to the singular point are tabulated. These results are shown in Table 8.

### Table 8. Singular Witness Data

| Line Num | Annulus | Next Cycle | Sheet | sing pt | Sort index | zstart  | zP       |
|----------|---------|------------|-------|---------|------------|---------|----------|
| 1        | 4       | \( \{1,2\} \) | 1     | -1.87083 | 2          | -1.85741 | 0.0134149 |
| 2        | 4       | \( \{1,2\} \) | 2     | -1.87083 | 1          | -1.85741 | 0.0134149 |
| 3        | 4       | \( \{1,2\} \) | 1     | 1.87083  | 1          | 1.85741  | -0.0134149 |
| 4        | 4       | \( \{1,2\} \) | 2     | 1.87083  | 3          | 1.85741  | -0.0134149 |
| 5        | 4       | \( \{3\} \)  | 3     | -1.87083 | 3          | -1.85741 | 0.0134149 |
| 6        | 4       | \( \{3\} \)  | 3     | 1.87083  | 2          | 1.85741  | -0.0134149 |

Now look at the first line in the table: We are checking the first sheet of the 2-cycle branch \( \{1,2\} \) in annulus 4 across singular point \(-1.87083\) and when it continues to the singular point reference point \( c \) in Diagram 2, it does so onto the second root of the function at this point. The value of \( z \) at point \( c \) is \(-1.85741\) and it’s offset value around the singular point is 0.0134149. Next, the Puiseux expansions at this
singular point is computed:

\[
w_1(z) = (1.06397 - 1.26645i) + (3.01595 - 6.43854i)z + (0.4263 - 73.5684i)z^2
- (426.834 + 907.601i)z^3 - (12030.5 + 9669.1i)z^4 - (249021. + 55181.5i)z^5 + \cdots
\]

\[
w_2(z) = (1.06397 + 1.26645i) + (3.01595 + 6.43854i)z + (0.4263 + 73.5684i)z^2
- (426.834 - 907.601i)z^3 - (12030.5 - 9669.1i)z^4 - (249021. - 55181.5i)z^5 + \cdots
\]

\[
w_3(z) = -1.76336 + \frac{0.234968}{z} - 5.65797z - 0.676369z^2 + 853.755z^3 + 24061.1z^4 + \cdots
\]

And then the value of each series at the offset point \( z_P = 0.0134149 \) is determined:

\[
w_1(z_P) = 1.10295 - 1.36858i
\]
\[
w_2(z_P) = 1.10295 + 1.36858i
\]
\[
w_3(z_P) = 15.6793
\]

Now, when we compute the sorted list of the function at the point \( c \) we obtain:

\[
1.10295 - 1.36858i
\]
\[
1.10295 + 1.36858i
\]
\[
15.6793
\]

and note \( w_3(z) \) is the pole and it’s value at point \( c \) is approx 15.6793 and that is the third index into the sorted list of function values at point \( c \). Since we determined that the first sheet of branch \( \{1,2\} \) was continued onto the second index, then we know this sheet is not affected by the pole. And in the second line of the table, we see the second sheet of this branch is continued onto index 1. Therefore, since branch \( \{1,2\} \) is not continued onto the pole sheet, we know this 2-cycle branch is not affected by this singular point.

Now look at lines 3 and 4 in Table 8 as we attempt to continue the \( \{1,2\} \) cycle across 1.87083. At point \( c \), the branches continue onto indexes 1 and 3. At the singular reference point for this pole (the equivalent of point \( c \) for the pole), we again compute the Puiseux series at 1.87083:

\[
w_1(z) = -0.951784 + 0.494400z - 0.146001z^2 + 0.0672361z^3 - 0.0564447z^4 + 0.0251513z^5 + \cdots
\]

\[
w_2(z) = 0.546915 - 0.405452z + 0.375163z^2 - 0.435910z^3 + 0.584467z^4 - 0.853222z^5 + \cdots
\]

\[
w_3(z) = -0.566856 - \frac{1.23497}{z} + 0.179982z - 0.377397z^2 + 0.449107z^3 - 0.571336z^4 + \cdots
\]

and note \( w_3(z) \) is the pole. When we compute the values of the series at the singular reference point we obtain:

\[
w_1(z_P) = -0.958443
\]
\[
w_2(z_P) = 0.552423
\]
\[
w_3(z_P) = 91.4904
\]

And the sorted list of function values at the witness mark is:

\[
-0.958443
\]
\[
0.552423
\]
\[
91.4904
\]

so that index 3 is the pole. Thus the second sheet of this branch continues across this singular point but in a meromorphic fashion. When we analyze the single cycle branch in the same manner, it continues meromorphically across the negative pole. We identify these continuations across singular points in Table 8 as the small red monodromies along the continuation arrows. In the first case, the second sheet of \( \{1,2\} \) was continuous onto the third sheet of the negative pole to the second sheet of the post annulus. This is the \( \{2,3,2\} \) on the side of the first arrow. And since the single cycle branch \( \{3\} \) was meromorphically continued onto the positive pole to \( \{3\} \) on the post annulus, we label this as \( \{3,3,3\} \) next to the continuation arrow between these branches. So that all branches in this annulus are affected by poles. Therefore the annular
region of convergence of the branches in $c_4$ is contained in the annulus. This is shown as $(r_4, r_5)$ below the branch monodromies in the table. The only function branch which extends beyond one annular region is branch $\{3\}$ in $c_5$ where it has a region of convergence $(r_5, r_7)$. To give a concrete example of this, the annular Puiseux series for this branch is

$$\{3\}(z) = \cdots + \frac{1.8811}{z^3} - \frac{0.9475}{z^2} + \frac{1.3012}{z} + 0.0072z - 0.00053z^2 + 0.000031z^3 + \cdots$$

and has a region of convergence of $(r_5, r_7)$ or approximately $(1.87088, 9.1471)$.

6. Computation of annular Puiseux series

The Newton Polygon algorithm computes power series for the function centered at the origin. These power series have radii of convergences equal to the absolute value of the branch singular point which often extends only into the first few rings. In order to compute power series for the function in the remaining annular rings, we use a version of Laurent’s Theorem applied to algebraic functions:

$$w_n(z) = \sum_{k=0}^{\infty} c_k (z^{1/n})^k + \sum_{k=1}^{\infty} \frac{b_k}{(z^{1/n})^k}$$

$$c_k = \frac{1}{2n\pi i} \int \frac{w_n(z)}{(z^{1/n})^{k+n}} dz$$

$$b_k = \frac{1}{2n\pi i} \int w_n(z) (z^{1/n})^{k-n} dz.$$ 

(4)

Or in symmetrical form:

$$w_n(z) = \sum_{p=-\infty}^{\infty} c_p (z^{1/n})^p,$$

(5)

where the integral symbol $\mathcal{I}$ indicates the integration is over a closed analytically continuous route along the integrand branch surfaces. For example, if the integrand contained a 4-cycle branch, the integration would proceed over the entire branch surface along an $8\pi$ route of winding number 4.

In order to demonstrate these formulas, we use

$$f(z, w) = (-z^2 + z^3)$$

$$+ (-4z + 3z^2)w$$

$$+ (-z^3 - 9z^4)w$$

$$+ (-2 + 8z + 4z^2 - 4z^3)w^3$$

$$+ (6 - 8z^2 + 7z^3 + 8z^4)w^4.$$ 

(6)

The ring and singular points are shown in Table 9 and the continuations are in Table 4. Consider now the first annular region in Figure 4 having a 2-cycle and two single cycle branches. Using the Newton-Polygon algorithm, we compute (numerically), the power series representation for these branches:

$$\{1, 3\}(z) = -1.4142i^{1/2} - 2.8750z + 13.3300i z^{3/2} + 83.9648z^2 - 611.2916i z^{5/2} + \cdots$$

$$\{2\}(z) = -0.2500z + 0.0703z^2 + 0.02075z^3 + 0.00399z^4 - 0.1061z^5 + \cdots$$

$$\{4\}(z) = 0.3333 + 4.6667z - 168.222z^2 + 9523.5z^3 - 65812z^4 + 5.056 \cdot 10^7 z^5 + \cdots.$$ 

(7)

From the branch table, we know the $\{1, 3\}$ cycle in $c_1$ has a radius of convergences of $r_1$ while the $\{2\}$ branch extends to $r_4$. These are labeled below the branches in the Branch columns. A good demonstration of this is to compute the power series representation of the $\{4\}$ branch in annulus 4 using Equation (4).
Table 9. Rings and Singular points

| Ring | Radius       | Singularity | Value                      |
|------|--------------|-------------|----------------------------|
| $r_1$ | 0.00919971  | $s_1$       | -0.00919971                |
| $r_2$ | 0.597463    | $s_2$       | -0.597463                  |
| $r_3$ | 0.632598    | $s_3$       | 0.632598                   |
| $r_4$ | 0.692915    | $s_4$       | 0.692915                   |
| $r_5$ | 0.81757     | $s_5$       | 0.644655 - 0.502832 i      |
|       |              | $s_6$       | 0.644655 + 0.502832 i      |
| $r_6$ | 0.85077     | $s_7$       | 0.296412 - 0.797464 i      |
|       |              | $s_8$       | 0.296412 + 0.797464 i      |
| $r_7$ | 0.855943    | $s_9$       | -0.0728759 - 0.852835 i    |
|       |              | $s_{10}$    | -0.0728759 + 0.852835 i    |
| $r_8$ | 0.859144    | $s_{11}$    | -0.859144                  |
| $r_9$ | 0.86077     | $s_{12}$    | -0.86077                   |
| $r_{10}$ | 0.87273  | $s_{13}$    | 0.72046 - 0.492539 i       |
|       |              | $s_{14}$    | 0.72046 + 0.492539 i       |
| $r_{11}$ | 0.901619 | $s_{15}$    | -0.901619                  |
| $r_{12}$ | 0.960847 | $s_{16}$    | 0.859329 - 0.429862 i      |
|       |              | $s_{17}$    | 0.859329 + 0.429862 i      |
| $r_{13}$ | 0.966603 | $s_{18}$    | 0.966603                   |
| $r_{14}$ | 1.19237   | $s_{19}$    | -1.16276 - 0.264081 i      |
|       |              | $s_{20}$    | -1.16276 + 0.264081 i      |
| $r_{15}$ | 1.29612    | $s_{21}$    | -1.29612                   |
| $r_{16}$ | 1.30354    | $s_{22}$    | -1.30354                   |
| $r_{17}$ | 1.4026    | $s_{23}$    | 0.280488 - 1.37427 i       |
|       |              | $s_{24}$    | 0.280488 + 1.37427 i       |

above. In order to integrate (4), we first compute the analytic continuation of the branch along the annular integration path by solving the monodromy differential equation for a $2\pi$ route. This gives us an analytically continuous version of the branch in annulus 4. And we likewise do the same for the base function $b(z) = z$ over a similar route. If we were dealing with a 2-cycle branch, the base function would be $\sqrt{z}$ and we would create an analytically-continuous version of the square-root function in annulus 4 via the monodromy DE for the function $f(z, w) = w^2 - z$. Likewise for other cycles. Once we have the analytically-continuous version of the branch sheet and base function, we can compute the coefficients $c_n$ and $b_n$ of (4) and then substitute these values into the summation formula for this branch. Doing this we obtain:

$$w(z) = -0.25z + 0.07031z^2 + 0.02075z^3 + 0.00399z^4 - (0.10608 - 0.0000242586i)z^5 + \cdots$$

which is numerically similar to the power series computed by Newton Polygon for the \{4\} branch in annulus 1. That is, it is the same branch.

Now consider the four single-cycle branches in annulus 4. Computing the annular Puiseux series using Equation (4) we obtain:

$$w_1(z) = \cdots + \frac{0.0182}{z^3} + \frac{0.0011}{z^2} + \frac{0.1437}{z} - 1.52381z - 0.144502z^2 - 2.23321z^3 + \cdots$$

$$w_2(z) = \cdots - \frac{0.0094}{z^3} - \frac{0.0236}{z^2} - \frac{0.0743}{z} + 0.462614z - 0.0102267z^2 + 0.515712z^3 - \cdots$$

$$w_3(z) = -0.25z + 0.0703z^2 + 0.0207z^3 + 0.0034z^4 - 0.1061z^5 + \cdots$$

$$w_4(z) = \cdots - \frac{0.0089}{z^3} + \frac{0.0225}{z^2} - \frac{0.0694}{z} - 0.0221411z - 0.137806z^2 + 0.196744z^3 + \cdots$$
According to Figure 4, this branch analytically continues into annulus 7 with a region of convergence does not. But looking at Figure 4, we see the branch in annulus 1 so would not be expected to have a singular series. In fact the two series are the same.

Consider now annulus 5 where we have two single cycle branches and one double-cycle. Cycle \( \{2, 3\} \) extends to annulus 7 and we wish to use 4 to compute the power series for this ring:

\[
\{2, 3\}(z) = \cdots - \frac{0.0162i}{z^{3/2}} - \frac{0.0371}{z} - \frac{0.0443i}{\sqrt{z}} + 0.4046i\sqrt{z} + 0.1063z - 0.1521iz^{3/2} + \cdots .
\]

And according to Figure 4, this branch analytically continues into annulus 7 with a region of convergence \((r_4, r_7)\).
Once we have the continuations, we can update Table 4 with the branch regions in column five. These lists the branch and the annular region of convergence of the associated power expansion. For example, in the monodromy column, we see \{2\} continues from the first annulus to the fourth. Therefore, the region column does not list the intervening continuations of this branch in the second third and fourth annular regions as these are part of the same \{2\} branch with a region of convergence given by \{r_0, r_4\} or the punctured disk \(D(0, r_4)\). And from Figure 4, we see the annular domain of convergence is approximately \((0, 0.692915)\). Likewise, the \{4\} branch in annulus 3 has a region of convergence of \((r_2, r_6)\) as it continues to the sixth annulus. This function therefore has 29 branches where we do not treat a meromorphic continuation across a pole as a single branch but rather two distinct branches.

In order to show graphically how the power series for a branch computed in one annular domain could be used to compute the value of the branch in another domain in which the branch was analytically-continuous, we first compute the Puiseux series for branch \{2, 3\} in \(a_5\), and use the series to compute the value of the branch in \(a_7\) since it extends into \(a_7\).
7. Convergence of power series

Figure 5 shows four plots. Each has the real part of branch \{2, 3\} in \(a_7\) along the branch integration path in blue. The red plots are generated by using the Puiseux series computed for \(a_7\) along the same path. The series are clearly converging in the Fourier sense as more terms are added.

8. Plotting the results

For readers interested in investigating equation 6 and its branches further, the author has a website [7] with an interactive 3D viewer. The viewer can be used to illustrate each branch. See Algebraic Functions.

9. Conclusions

The software used in this paper was able to compute successfully, branch cycles of algebraic functions and convergence domains of their power expansions. This was shown in Figure 5 by the success of the power expansions to converge to the function in annular domains outside bordering singular points. The success of the algorithm to adjust precision and step size automatically until a desired accuracy was reached was very high over many trial runs although the results of which were not included here to avoid clutter. However, as these are numerical calculations, the possibility for error is present. Errors can be minimized. For example, monodromy results are confirmed to be permutations, integration results are checked against function results. And Figure 5 graphically demonstrates how the series computed in one annular domain converges to the function in an analytically-continuous domain. This clearly would not have been the case if an intervening singular point impinged upon the branch sheet.

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