Convex Influences

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Abstract

We introduce a new notion of influence for symmetric convex sets over Gaussian space, which we term “convex influence”. We show that this new notion of influence shares many of the familiar properties of influences of variables for monotone Boolean functions $f : \{\pm 1\}^n \to \{\pm 1\}$.

Our main results for convex influences give Gaussian space analogues of many important results on influences for monotone Boolean functions. These include (robust) characterizations of extremal functions, the Poincaré inequality, the Kahn-Kalai-Linial theorem [KKL88], a sharp threshold theorem of Kalai [Kal04], a stability version of the Kruskal-Katona theorem due to O’Donnell and Wimmer [OW09], and some partial results towards a Gaussian space analogue of Friedgut’s junta theorem [Fri98]. The proofs of our results for convex influences use very different techniques than the analogous proofs for Boolean influences over $\{\pm 1\}^n$. Taken as a whole, our results extend the emerging analogy between symmetric convex sets in Gaussian space and monotone Boolean functions from $\{\pm 1\}^n$ to $\{\pm 1\}$. 
1 Introduction

Background: An intriguing analogy. This paper is motivated by an intriguing, but at this point only partially understood, analogy between monotone Boolean functions over the hypercube and symmetric convex sets in Gaussian space. Perhaps the simplest manifestation of this analogy is the following pair of easy observations: since a Boolean function \( f : \{\pm1\}^n \to \{\pm1\} \) is monotone if \( f(x) \leq f(y) \) whenever \( x_i \leq y_i \) for all \( i \), it is clear that “moving an input up towards \( 1^n \)” by flipping bits from \(-1\) to \(1\) can never decrease the value of \( f \). Similarly, we may view a symmetric\(^1\) convex set \( K \subseteq \mathbb{R}^n \) as a \(0/1\) valued function, and it is clear from symmetry and convexity that “moving an input in towards the origin” can never decrease the value of the function.

The analogy extends far beyond these easy observations to involve many analytic and algorithmic aspects of monotone Boolean functions over \( \{\pm1\}^n \) under the uniform distribution and symmetric convex subsets of \( \mathbb{R}^n \) under the Gaussian measure. Below we survey some known points of correspondence (several of which were only recently established) between the two settings:

1. Density increments. The well-known Kruskal-Katona theorem \([\text{Kru}63, \text{Kat}10]\) gives quantitative information about how rapidly a monotone \( f : \{\pm1\}^n \to \{\pm1\} \) increases on average as the input to \( f \) is “moved up towards \( 1^n \).” Let \( f : \{\pm1\}^n \to \{0,1\} \) be a monotone function and let \( \mu_f(j) \) be the fraction of the \( \binom{n}{j} \) many weight-\( j \) inputs for which \( f \) outputs 1; the Kruskal-Katona theorem implies (see e.g. \([\text{Lov}81]\)) that if \( k = cn \) for some \( c \) bounded away from 0 and 1 and \( \mu_f(k) \in (0.1,0.9) \), then \( \mu_f(k+1) \geq \mu_f(k) + \Theta(1/n) \). Analogous “density increment” results for symmetric convex sets are known to hold in various forms, where the analogue of moving an input in \( \{\pm1\}^n \) up towards \( 1^n \) is now moving an input in \( \mathbb{R}^n \) in towards the origin, and the analogue of \( \mu_f(j) \) is now \( \alpha_r(K) \), which is defined to be the fraction of the origin-centered radius-\( r \) sphere \( rS^{n-1} \) that lies in \( K \). For example, Theorem 2 of the recent work \([\text{DS}21]\) shows that if \( K \subseteq \mathbb{R}^n \) is a symmetric convex set (which we view as a function \( K : \mathbb{R}^n \to \{0,1\} \)) and \( r = \Theta(\sqrt{n}) \) satisfies \( \alpha_r(K) \in (0.1,0.9) \), then \( \alpha_K(r(1-1/n)) \geq \alpha_K(r) + \Theta(1/n) \).

2. Weak learning from random examples. Building on the above-described density increment for symmetric convex sets, \([\text{DS}21]\) showed that any symmetric convex set can be learned to accuracy \( 1/2 + O(1)/\sqrt{n} \) in \( \text{poly}(n) \) time given \( \text{poly}(n) \) many random examples drawn from \( \mathcal{N}(0,1)^n \). \([\text{DS}21]\) also shows that any \( \text{poly}(n) \)-time weak learning algorithm (even if allowed to make membership queries) can achieve accuracy no better than \( 1/2 + O(\log(n)/\sqrt{n}) \). These results are closely analogous to the known (matching) upper and lower bounds for \( \text{poly}(n) \)-time weak learning of monotone functions with respect to the uniform distribution over \( \{\pm1\}^n \): Blum et al. \([\text{BBL}98]\) showed that \( 1/2 + \Theta(\log(n)/\sqrt{n}) \) is the best possible accuracy for a \( \text{poly}(n) \)-time weak learner (even if membership queries are allowed), and O’Donnell and Wimmer \([\text{OW}09]\) gave a \( \text{poly}(n) \) time weak learner that achieves this accuracy using random examples only.

3. Analytic structure and strong learning from random examples. \([\text{BT}96]\) showed that the Fourier spectrum of any \( n \)-variable monotone Boolean function over \( \{\pm1\}^n \) is concentrated in the first \( O(\sqrt{n}) \) levels. Analogously, \([\text{KOS}08]\) showed that the same concentration holds for the first \( O(\sqrt{n}) \) levels of the Hermite spectrum\(^2\) of the indicator function of any convex set. In both cases this concentration gives rise to a learning algorithm, using random examples only,

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\(^1\)A set \( K \subseteq \mathbb{R}^n \) is symmetric if \(-x \in K \) whenever \( x \in K \).

\(^2\)The Hermite polynomials form an orthonormal basis for the space of square-integrable real-valued functions over Gaussian space; the Hermite spectrum of a function over Gaussian space is analogous to the familiar Fourier spectrum of a function over the Boolean hypercube. See Section 2 for details.
running in $n^{O(\sqrt{n})}$ time and learning the relevant class (either monotone Boolean functions over the $n$-dimensional hypercube or convex sets under Gaussian space) to any constant accuracy.

4. Qualitative correlation inequalities. The well-known Harris-Kleitman theorem [Har60, Kle66] states that monotone Boolean functions are non-negatively correlated: any monotone $f, g : \{\pm 1\}^n \to \{0, 1\}$ must satisfy $\mathbb{E}[fg] - \mathbb{E}[f] \mathbb{E}[g] \geq 0$. The Gaussian Correlation Inequality [Roy14] gives an exactly analogous statement for symmetric convex sets in Gaussian space: if $K, L \subseteq \mathbb{R}^n$ are any two symmetric convex sets, then $\mathbb{E}[KL] - \mathbb{E}[K] \mathbb{E}[L] \geq 0$, where now expectations are with respect to $\mathcal{N}(0, 1)^n$.

5. Quantitative correlation inequalities. Talagrand [Tal96] proved the following quantitative version of the Harris–Kleitman inequality: for monotone $f, g : \{\pm 1\}^n \to \{0, 1\}$,

$$\mathbb{E}[fg] - \mathbb{E}[f] \mathbb{E}[g] \geq \frac{1}{C} \cdot \Psi \left( \sum_{i=1}^{n} \text{Inf}_i[f] \text{Inf}_i(g) \right).$$

Here $\Psi(x) := x / \log(e/x)$, $C > 0$ is an absolute constant, $\text{Inf}_i[f]$ is the influence of coordinate $i$ on $f$ (see Section 2), and the expectations are with respect to the uniform distribution over $\{\pm 1\}^n$. In a recent work [DNS21] proved a closely analogous quantitative version of the Gaussian Correlation Inequality: for $K, L$ symmetric convex subsets of $\mathbb{R}^n$,

$$\mathbb{E}[KL] - \mathbb{E}[K] \mathbb{E}[L] \geq \frac{1}{C} \cdot \Upsilon \left( \sum_{i=1}^{n} \tilde{K}(2e_i) \tilde{L}(2e_i) \right),$$

where $\Upsilon : [0, 1] \to [0, 1]$ is $\Upsilon(x) = \min \left\{ x, \frac{x}{\log^2(1/x)} \right\}$, $C > 0$ is a universal constant, $\tilde{K}(2e_i)$ denotes the degree-2 Hermite coefficient in direction $e_i$ (see Section 2), and expectations are with respect to $\mathcal{N}(0, 1)^n$.

We remark that in many of the above cases the proofs of the two analogous results (Boolean versus Gaussian) are very different from each other even though the statements are quite similar. For example, the Harris-Kleitman theorem has a simple one-paragraph proof by induction on $n$, whereas the Gaussian Correlation Inequality was a famous conjecture for four decades before Thomas Royen proved it in 2014.

**Motivation.** We feel that the examples presented above motivate a deeper understanding of this “Boolean/Gaussian analogy.” This analogy may be useful in a number of ways; in particular, via this connection known results in one setting may suggest new questions and results for the other setting.\footnote{Indeed, the recent Gaussian density increment and weak learning results of [DS21] were inspired by the Kruskal-Katona theorem and the weak learning algorithms and lower bounds of [BBL98] for monotone Boolean functions. Similarly, the recent quantitative version of the Gaussian Correlation Theorem established in [DNS21] was motivated by the existence of Talagrand’s quantitative correlation inequality for monotone Boolean functions.}

Thus the overarching goal of this paper is to strengthen the analogy between monotone Boolean functions over $\{\pm 1\}^n$ and symmetric convex sets in Gaussian space. We do this through the study of a new notion of influence for symmetric convex sets in Gaussian space.

### 1.1 This Work: A New Notion of Influence for Symmetric Convex Sets

Before presenting our new notion of influence for symmetric convex sets in Gaussian space, we first briefly recall the usual notion for Boolean functions. For $f : \{\pm 1\}^n \to \{\pm 1\}^n$, the influence of
coordinate $i$ on $f$, denoted $\text{Inf}_i[f]$, is $\Pr[f(x) \neq f(x^{\oplus i})]$, where $x$ is uniform random over $\{\pm 1\}^n$ and $x^{\oplus i}$ denotes $x$ with its $i$-th coordinate flipped. It is a well-known fact (see e.g. Proposition 2.21 of [O’D14]) that for monotone Boolean functions $f$, we have $\text{Inf}_i[f] = \hat{f}(i)$, the degree-1 Fourier coefficient corresponding to coordinate $i$.

Inspired by the relation $\text{Inf}_i[f] = \hat{f}(i)$ for influence of monotone Boolean functions, and by the close resemblance between Equation (1) and Equation (2), [DNS21] proposed to define the influence of $K$ along direction $v$, for $K : \mathbb{R}^n \to \{0, 1\}$ a symmetric convex set and $v \in S^{n-1}$, to be

$$\text{Inf}_v[K] := -\tilde{K}(2v),$$

the (negated) degree-2 Hermite coefficient$^4$ of $K$ in direction $v$ (see Definition 8 for a detailed definition). [DNS21] proved that this quantity is non-negative for any direction $v$ and any symmetric convex $K$ (see Proposition 9). They also defined the total influence of $K$ to be

$$I[f] := \sum_{i=1}^n \text{Inf}_{e_i}[f] \quad (3)$$

and observed that this definition is invariant under different choices of orthonormal basis other than $e_1, \ldots, e_n$, but did not explore these definitions further.

The main contribution of the present work is to carry out an in-depth study of this new notion of influence for symmetric convex sets. For conciseness, and to differentiate it from other influence notions (which we discuss later), we will sometimes refer to this new notion as “convex influence.”

Inspired by well known results about influence of monotone Boolean functions, we establish a number of different results about convex influence which show that this notion shares many properties with the familiar Boolean influence notion. Intriguingly, and similar to the Boolean/Gaussian analogy elements discussed earlier, while the statements we prove about convex influence are quite closely analogous to known results about Boolean influences, the proofs and tools that we use (Gaussian isoperimetry, Brascamp-Lieb type inequalities, theorems from the geometry of Gaussian space such as the $S$-inequality [LO99], etc.) are very different from the ingredients that underlie the corresponding results about Boolean influence.

1.2 Results and Organization

We give an overview of our main results below.

Basics, examples, Margulis-Russo, and extremal functions. We begin in Section 3.1 by working through some basic properties of our new influence notion. After analyzing some simple examples in Section 3.2, we next show in Section 3.3 that the total convex influence for a symmetric convex set is equal to (a scaled version of) the rate of change of the Gaussian volume of the set as the variance of the underlying Gaussian is changed. This gives an alternate characterization of total convex influence, and may be viewed as an analogue of the Margulis-Russo formula for our new influence notion. We continue in Section 3.4 by giving some straightforward characterizations of extremal symmetric convex sets vis-a-vis our influence notion, namely the ones that have the largest individual influence in a single direction and the largest total influence. As one would expect, these extremal functions are the Gaussian space analogues of the Boolean dictator and majority functions respectively. Next, we compare our new influence notion with some other previously studied notions.

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$^4$We observe that if $K$ is a symmetric set then since its indicator function is even, the degree-1 Hermite coefficient $\tilde{K}(v)$ must be 0 for any direction $v$. 

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3
of influence over Gaussian space (Section 3.5). These include the “geometric influences” that were studied by [KMS12] as well as the standard notion (from the analysis of functions over product probability domains, see e.g. Chapter 8 of [O’D14]) of the expected variance of the function along one coordinate when all other coordinates are held fixed.

**Total influence lower bounds.** In Section 4 we give two lower bounds on the total convex influence (Equation (3)) for symmetric convex sets, which are closely analogous to the classical Poincaré and KKL Theorems. Our KKL analogue is quadratically weaker than the KKL theorem for Boolean functions; we conjecture that a stronger bound in fact holds, which would quantitatively align with the Boolean variant (see Item 1 of Section 1.4). Our proofs, which are based on the “S-inequality” of Latała and Oleskiewicz [LO99] and on the Gaussian isoperimetric inequality, are quite different from the proofs of the analogous statements for Boolean functions.

**A consequence of** Friedgut’s junta theorem. In Section 5 we establish a convex influences analogue of a consequence of Friedgut’s junta theorem. Friedgut’s junta theorem states that any Boolean function \( f : \{\pm 1\}^n \to \{0, 1\} \) with small total influence must be close to a junta. This implies that for any monotone function \( f : \{\pm 1\}^n \to \{0, 1\} \) with small total influence, “averaging out” over a small well-chosen set of input variables (the variables on which the approximating junta depends) results in a low-variance function. We prove a closely analogous statement for symmetric convex sets with small total convex influence, thus capturing a convex influence analogue of this consequence of Friedgut’s junta theorem. (We conjecture that a convex influence analogue holds for Friedgut’s original junta theorem; see Item 2 of Section 1.4.)

**Sharp thresholds for functions with all small influences.** In Section 6 we establish a “sharp threshold” result for symmetric convex sets in Gaussian space, which is analogous to a sharp threshold result for monotone Boolean functions due to Kalai [Kal04]. Building on earlier work of Friedgut and Kalai [FK96], Kalai [Kal04] showed that if \( f : \{\pm 1\}^n \to \{0, 1\} \) is a monotone Boolean function and \( p \in (0, 1) \) is such that (i) all the \( p \)-biased influences of \( f \) are \( o_n(1) \) and (ii) the expectation of \( f \) under the \( p \)-biased measure is \( \Theta(1) \), then \( f \) must have a “sharp threshold” in the following sense: the expectation of \( f \) under the \( p_1 \)-biased measure (\( p_2 \)-biased measure, respectively) is \( o_n(1) \) \((1 - o_n(1))\), respectively) for some \( p_1 < p < p_2 \) with \( p_2 - p_1 = o_n(1) \). For our sharp threshold result, we prove an analogous statement for symmetric convex sets, where now \( \mathcal{N}(0, \sigma^2) \) takes the place of the \( p \)-biased distribution over \( \{\pm 1\}^n \) and the \( \sigma \)-biased convex influences (see Definition 15) take the place of the \( p \)-biased influences. Interestingly, the sharpness of our threshold is quantitatively better than the known analogous result [Kal04] for monotone Boolean functions; see Section 6 for an elaboration of this point.

**A stable density increment result.** Finally, in Section 7, we use our new influence notion to give a Gaussian space analogue of a “stability” version of the Kruskal-Katona theorem due to O’Donnell and Wimmer [OW09]. In [OW09] it is shown that the \( \Omega(1/n) \) density increment of the Kruskal-Katona theorem (see Item 1 at the beginning of this introduction) can be strengthened to \( \Omega(\log(n)/n) \) as long as a “low individual influences”-type condition holds. We analogously show that a similar strengthening of the Gaussian space density increment result mentioned in Item 1 earlier can be achieved under the condition that the convex influence in every direction is low.
1.3 Techniques

We give a high-level overview here of the techniques for just one of our results, namely our analogue of the KKL theorem, Theorem 25. Several of our other results either employ similar tools (for example, our robust density increment result, Theorem 36, and our main sharp threshold result, Theorem 31) or else build off of Theorem 25 (for example, our analogue of a consequence of Friedgut’s junta theorem, Theorem 29).

The KKL theorem states that if \( f : \{\pm 1\}^n \to \{\pm 1\} \) has every coordinate influence small, specifically \( \max_{i \in [n]} \text{Inf}_i[f] \leq \delta \), then the total influence of \( f \) must be large compared to \( f \)'s variance, specifically it must hold that \( I[f] = \Omega(\text{Var}[f] \log(1/\delta)) \). This is a dramatic strengthening of the Poincaré inequality (which only states that \( I[f] \geq \text{Var}[f] \)) and is a signature result in the analysis of Boolean functions with many applications. The classical proof of the KKL theorem is based on hypercontractivity [Bon70, Bec75], and only recently [EG20, KKK’21] have proofs been given which avoid the use of hypercontractivity.

Our convex influences analogue of the KKL theorem states that if \( K \) is a symmetric convex set and the convex influence \( \text{Inf}_v[K] \) in every direction \( v \in S^{n-1} \) is at most \( \delta \) and \( \delta \leq \text{Var}[K]/10 \), then the total convex influence \( I[K] \) must be at least \( \Omega\left(\text{Var}[K]^{1/2} \log\left(\frac{\text{Var}[K]}{\delta}\right)\right) \). Our proof does not employ hypercontractivity but instead uses tools from convex geometry. It proceeds in two main conceptual steps:

1. First, we use a Brascamp–Lieb-type inequality due to Vempala [Vem10] to argue that the maximum convex influence of \( K \) in any coordinate can be lower bounded in terms of the Gaussian volume of \( K \) and its “width” (equivalently, the radius of the largest origin-centered ball contained in \( K \), which is called the in-radius of \( K \) and is denoted \( r_{\text{in}}(K) \)). This lets us show that \( r_{\text{in}}(K) \geq \Omega\left(\sqrt{\ln(\text{Var}[K]/\delta)}\right) \) (see Equation (19)).

2. Next, we argue that \( I[K] \geq \frac{1}{\sqrt{\pi}} \text{Var}[K] \cdot r_{\text{in}}(K) \) (see Equation (18)), which together with the lower bound on \( r_{\text{in}}(K) \) gives the result. This is shown using our Margulis-Russo analogue, the Gaussian Isoperimetric Theorem, and concavity of the Gaussian isoperimetric function.

1.4 Discussion and Future Work

We believe that much more remains to be discovered about this new notion of influences for symmetric convex sets. We list some natural concrete (and not so concrete) questions for future work:

1. **A stronger KKL-type theorem for convex influences?** We conjecture that the factor of \( \sqrt{\ln(\text{Var}[K]/\delta)} \) in our KKL analogue, Theorem 25, can be strengthened to \( \log(1/\delta) \). As witnessed by Example 14, this would be essentially the strongest possible quantitative result, and would align closely with the original KKL theorem [KKL88].

2. **An analogue of Friedgut’s theorem for convex influences?** As described earlier, our Theorem 29 establishes a Gaussian space analogue of a consequence of Friedgut’s Junta Theorem [Fri98] for Boolean functions over \( \{\pm 1\}^n \). The following would give a full-fledged Gaussian space analogue of Friedgut’s Junta Theorem:

   **Conjecture 1** (Friedgut’s Junta Theorem for convex influences). Let \( K \subset \mathbb{R}^n \) be a convex symmetric set with \( \text{Inf}[K] \leq I \). Then there are \( J \leq 2^{O(I/\varepsilon)} \) orthonormal directions \( v^1, \ldots, v^J \in S^{n-1} \) and a symmetric convex set \( L \subset \mathbb{R}^n \), such that
2 Preliminaries

In this section we give preliminaries setting notation and recalling useful background on convex geometry, log-concave functions, and Hermite analysis over $\mathcal{N}(0, \sigma^2)^n$.

2.1 Convex Geometry and Log-Concavity

Below we briefly recall some notation, terminology and background from convex geometry and log-concavity. Some of our main results employ relatively sophisticated results from these areas; we will recall these as necessary in the relevant sections and here record only basic facts. For a general and extensive resource we refer the interested reader to [AAGM15].

We identify sets $K \subseteq \mathbb{R}^n$ with their indicator functions $K : \mathbb{R}^n \to \{0, 1\}$, and we say that $K \subseteq \mathbb{R}^n$ is symmetric if $K(x) = K(-x)$. We write $B_r$ to denote the origin-centered ball of radius $r$ in $\mathbb{R}^n$. If $K \subseteq \mathbb{R}^n$ is a nonempty symmetric convex set then we let $r_{\min}(K)$ denote $\sup_{r \geq 0} \{r : B_r \subseteq K\}$ and we refer to this as the in-radius of $K$.

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is log-concave if its domain is a convex set and it satisfies $f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta}$ for all $x, y \in \text{domain}(f)$ and $\theta \in [0, 1)$. In particular, the 0/1-indicator functions of convex sets are log-concave.
Recall that the marginal of \( f : \mathbb{R}^n \to \mathbb{R} \) on the set of variables \( \{i_1, \ldots, i_k\} \) is obtained by integrating out the other variables, i.e. it is the function
\[
g(x_{i_1}, \ldots, x_{i_k}) = \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_n) dx_{j_1} \ldots dx_{j_{n-k}},
\]
where \( \{j_1, \ldots, j_{n-k}\} = [n] \setminus \{i_1, \ldots, i_k\} \). We recall the following fact:

**Fact 1** ([Din57, Lei72, Pre73a, Pre73b] (see Theorem 5.1, [LV07])). All marginals of a log-concave function are log-concave.

The next fact follows easily from the definition of log-concavity:

**Fact 2** ([Ibr56], see e.g. [An95]). A one-dimensional log-concave function is unimodal.

### 2.2 Gaussian Random Variables

We write \( z \sim \mathcal{N}(0,1) \) to mean that \( z \) is a standard Gaussian random variable, and will use the notation
\[
\varphi(z) := \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{and} \quad \Phi(z) := \int_{-\infty}^{z} \varphi(t) dt
\]
to denote the pdf and the cdf of this random variable.

Recall that a non-negative random variable \( r^2 \) is distributed according to the chi-squared distribution \( \chi^2(n) \) if \( r^2 = g_1^2 + \cdots + g_n^2 \) where \( g_i \sim \mathcal{N}(0,1) \), and that a draw from the chi distribution \( \chi(n) \) is obtained by making a draw from \( \chi^2(n) \) and then taking the square root.

We define the shell-density function for \( K, \alpha_K : [0, \infty) \to [0, 1] \), to be
\[
\alpha_K(r) := \Pr_{x \in \mathbb{S}^{n-1}}[x \in K],
\]
where the probability is with respect to the normalized Haar measure over \( \mathbb{S}^{n-1} \); so \( \alpha_K(r) \) equals the fraction of the origin-centered radius-\( r \) sphere which lies in \( K \). We observe that if \( K \) is convex and symmetric then \( \alpha_K(\cdot) \) is a nonincreasing function. A view which will be sometimes useful later is that \( \alpha_K(r) \) is the probability that a random Gaussian-distributed point \( g \sim \mathcal{N}(0,1)^n \) lies in \( K \), conditioned on \( \|g\| = r \).

### 2.3 Hermite Analysis over \( \mathcal{N}(0, \sigma^2)^n \)

Our notation and terminology here follow Chapter 11 of [O’D14]. We say that an \( n \)-dimensional **multi-index** is a tuple \( \alpha \in \mathbb{N}^n \), and we define
\[
|\alpha| := \sum_{i=1}^{n} \alpha_i.
\]

We write \( \mathcal{N}(0, \sigma^2)^n \) to denote the \( n \)-dimensional Gaussian distribution with mean 0 and variance \( \sigma^2 \), and denote the corresponding measure by \( \gamma_{n, \sigma}(\cdot) \). When the dimension \( n \) is clear from context we simply write \( \gamma_{\sigma}(\cdot) \) instead, and sometimes when \( \sigma = 1 \) we simply write \( \gamma \) for \( \gamma_1 \). For \( n \in \mathbb{N}_{>0} \) and \( \sigma > 0 \), we write \( L^2(\mathbb{R}^n, \gamma_{\sigma}) \) to denote the space of functions \( f : \mathbb{R}^n \to \mathbb{R} \) that have finite second moment \( \|f\|_2^2 \) under the Gaussian measure \( \gamma_{\sigma} \), that is:
\[
\|f\|_2^2 := \mathbb{E}_{z \sim \mathcal{N}(0, \sigma^2)^n} \left[ f(z)^2 \right]^{1/2} < \infty.
\]
We view \( L^2(\mathbb{R}^n, \gamma_{\sigma}) \) as an inner product space with \( \langle f, g \rangle := \mathbb{E}_{z \sim \mathcal{N}(0, \sigma^2)^n} [f(z)g(z)] \) for \( f, g \in L^2(\mathbb{R}^n, \gamma_{\sigma}) \). We define “biased Hermite polynomials,” which yield an orthonormal basis for \( L^2(\mathbb{R}^n, \gamma_{\sigma}) \):
In this section, we first introduce our new notion of influence for symmetric convex sets over Gaussian space and establish some basic properties. In Section 3.2 we analyze the influences of several natural symmetric convex sets, and in Section 3.3 we give an analogue of the Margulis-Russo formula (characterizing the influences of monotone Boolean functions) which provides an alternative equivalent view of our new notion of influence for symmetric convex sets in terms of the behavior of the sets under dilations. We characterize the symmetric convex sets which have extremal max influence and total influence in Section 3.4. Finally, in Section 3.5, we compare our new notion of influence with some previously studied influence notions over Gaussian space.
3.1 Definitions and Basic Properties

**Definition 8** (Influence for symmetric log-concave functions). Let \( f \in L^2(\mathbb{R}^n, \gamma) \) be a symmetric (i.e. \( f(x) = f(-x) \)) log-concave function. Given a unit vector \( v \in \mathbb{S}^{n-1} \), we define the **influence of direction** \( v \) **on** \( f \) as being

\[
\text{Inf}_v[f] := -\tilde{f}(2v) = \mathbb{E}_{x \sim N(0,1)^n} \left[ -f(x) h_2(v \cdot x) \right] = \mathbb{E}_{x \sim N(0,1)^n} \left[ f(x) \cdot \left( \frac{1 - (v \cdot x)^2}{\sqrt{2}} \right) \right],
\]

the negated “degree-2 Hermite coefficient in the direction \( v \).” Furthermore, we define the **total influence** of \( f \) as

\[
I[f] := \sum_{i=1}^{n} \text{Inf}_{e_i}[f].
\]

Note that the indicator of a symmetric convex set is a symmetric log-concave function, and this is the setting that we will be chiefly interested in. The following proposition (which first appeared in [DNS21], and a proof of which can be found in Appendix A) shows that these new influences are indeed “influence-like.” An arguably simpler argument for the non-negativity of influences is presented in Section 3.3.

**Proposition 9** (Influences are non-negative). If \( K \) is a centrally symmetric, convex set, then \( \text{Inf}_v[K] \geq 0 \) for all \( v \in \mathbb{S}^{n-1} \). Furthermore, equality holds if and only if \( K(x) = K(y) \) whenever \( x \perp v = y \perp v \) (i.e. the projection of \( x \) orthogonal to \( v \) coincides with that of \( y \)) almost surely.

We note that the total influence of a symmetric, convex set \( K \) is independent of the choice of basis; indeed, we have

\[
I[K] = \mathbb{E}_{x \sim N(0,1)^n} \left[ f(x) \left( \frac{n - \|x\|^2}{\sqrt{2}} \right) \right]
\]

which is invariant under orthogonal transformations. Hence any orthonormal basis \( \{v_1, \ldots, v_n\} \) could have been used in place of \( \{e_1, \ldots, e_n\} \) in defining \( I[K] \).

We note that (as is shown in the proof of Proposition 9), the influence of a fixed coordinate is not changed by averaging over some set of other coordinates:

**Fact 10.** Let \( K \subseteq \mathbb{R}^n \) be a symmetric, convex set, and define the log-concave function \( K_{e_i} : \mathbb{R} \to [0,1] \) as

\[
K_{e_i}(x) := \mathbb{E}_{x \sim N(0,1)^{n-1}} \left[ K(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \right].
\]

Then we have

\[
\text{Inf}_{e_i}[K] = \text{Inf}_{e_i}[K_{e_i}] = I[K_{e_i}].
\]

We conclude with the following useful relationship between the in-radius of a symmetric convex set \( K \) and its max influence along any direction. **Proposition 11** is proved in Appendix A.

**Proposition 11.** Let \( K \subseteq \mathbb{R}^n \) be a centrally symmetric convex set with \( \gamma(K) \geq \Delta \), and let \( r_{in} = r_{in}(K) \) be the in-radius of \( K \). Then there is some direction \( v \in \mathbb{S}^{n-1} \) such that

\[
\text{Inf}_v[K] \geq \frac{\Delta e^{-r_{in}^2}}{2^{3/2} \pi}.
\]
3.2 Influences of Specific Symmetric Convex Sets

In this subsection we consider some concrete examples by analyzing the influences of a few specific symmetric convex sets, namely “slabs”, balls, and cubes. As we will see, these are closely analogous to well-studied monotone Boolean functions (dictator, Majority, and Tribes, respectively).

Example 12 (Analogue of Boolean dictator: a “slab”). Given a vector $w \in \mathbb{R}^n$, define $\text{Dict}_w := \left\{ x \in \mathbb{R}^n : |\langle x, w \rangle| \leq 1 \right\}$. As suggested by the notation, this is the analogue of a single Boolean variable $f(x) = x_i$, i.e. a “dictatorship.” For simplicity, suppose $w := \frac{1}{c} \cdot e_1$ for some $c > 0$, i.e. $\text{Dict}_w = \left\{ x \in \mathbb{R}^n : |x_1| \leq c \right\}$. We then have

$$\text{Inf}^{\epsilon_i}_{\text{Dict}_w} = \begin{cases} \Theta \left( c \cdot \exp \left( -\frac{c^2}{2} \right) \right) & i = 1 \\ 0 & i \neq 1 \end{cases}.$$

Note that while in the setting of the Boolean hypercube there is only one “dictatorship” for each coordinate, in our setting given a particular direction we can have “dictatorships” of varying widths and volumes.

Example 13 (Analogue of Boolean Majority: a ball). Let $B_r := \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq r \right\}$ denote the ball of radius $r$. Analogous to the Boolean majority function, we argue that for $B = B_{\sqrt{n}}$ we have that $\text{Inf}^{\epsilon_i}_{B} = \Theta(1/\sqrt{n})$ for all $i \in [n]$.

Recall from Equation (6) that

$$I[B] = \frac{1}{\sqrt{2}} \mathbb{E}_{x \sim \mathcal{N}(0,1)^n} \left[ B(x) \left( n - \|x\|_2^2 \right) \right].$$

By the Berry-Esseen Central Limit Theorem (see [Ber41, Ess42] or, for example, Section 11.5 of [O’D14]), we have that for $t \in \mathbb{R}$,

$$\left| \mathbb{P}_{x \sim \mathcal{N}(0,1)^n} \left[ \frac{\|x\|_2^2 - n}{\sqrt{n}} \leq t \right] - \mathbb{P}_{y \sim \mathcal{N}(0,1)} [y \leq t] \right| \leq \frac{c}{\sqrt{n}}$$

for some absolute constant $c$. In particular, this implies that

$$\mathbb{P}_{x \sim \mathcal{N}(0,1)^n} \left[ \|x\|_2^2 \leq n - \sqrt{n} \right] \geq \mathbb{P}_{y \sim \mathcal{N}(0,1)} [y \leq -1] - \frac{c}{\sqrt{n}} \geq 0.15.$$

Since $\mathbb{P}_{x \sim \mathcal{N}(0,1)^n}[B(x) = 1] = \frac{1}{2} \pm o_n(1)$, and $B(x)(n - \|x\|_2^2)$ is never negative, it follows that

$$\mathbb{E}_{x \sim \mathcal{N}(0,1)^n} \left[ B(x) \left( n - \|x\|_2^2 \right) \right] \geq \Theta(\sqrt{n})$$

from which symmetry implies that $\text{Inf}^{\epsilon_i}_{B} \geq \Theta \left( \frac{1}{\sqrt{n}} \right)$ for all $i \in [n]$. The upper bound $\text{Inf}^{\epsilon_i}_{B} \leq \Theta \left( \frac{1}{\sqrt{n}} \right)$ follows from Parseval’s identity.

Our last example is analogous to the “Tribes CNF” function introduced by Ben-Or and Linial [BOL85] (alternatively, see Definition 2.7 of [O’D14]):
Example 14 (Analogue of Boolean Tribes: a cube). Let \( C_r := \{ x \in \mathbb{R}^n : |x_i| \leq r \text{ for all } i \in [n] \} \) denote the axis-aligned cube of side-length \( 2r \) and \( \gamma(C_r) = \frac{1}{2} \), i.e. let \( r > 0 \) be the unique value such that
\[
\Pr_{\mathbf{g} \sim N(0,1)}[|\mathbf{g}| \leq r] = \left( \frac{1}{2} \right)^{1/n} = 1 - \Theta\left( \frac{1}{n} \right).
\] (9)
By standard tail bounds on the Gaussian distribution, we have that \( r = \Theta(\sqrt{\log n}) \). Because of the symmetry of \( C_r \), we have \( \text{Inf}_{e_i}[C_r] = \text{Inf}_{e_j}[C_r] \) for all \( i, j \in [n] \). Note, however, that we can write
\[
C_r(x) = \prod_{i=1}^{n} \text{Dict}_{1/r}(x_i)
\]
where \( \text{Dict}_{1/r} : \mathbb{R} \to \{0,1\} \) is as defined in Example 12. By considering the Hermite representation of \( C_r(x) \), it is easy to see that
\[
\text{Inf}_{e_i}[C_r] = \mathbb{E}_{\mathbf{g} \sim N(0,1)} \left[ \text{Dict}_{1/r}(\mathbf{g}) \right]^{n-1} \mathbb{I}\left[ \text{Dict}_{1/r} \right].
\]
By our choice of \( r \) above, we have \( \mathbb{E}\left[ \text{Dict}_{1/r} \right] = \sqrt{1/2} \) and so
\[
\mathbb{E}_{\mathbf{g} \sim N(0,1)} \left[ \text{Dict}_{1/r}(\mathbf{g}) \right]^{n-1} = \Theta(1).
\]
From Example 12, we know \( \mathbb{I}\left[ \text{Dict}_{1/r} \right] = \Theta\left( r e^{-r^2/2} \right) \), and so we have
\[
\text{Inf}_{e_i}[C_r] = \Theta\left( r e^{-r^2/2} \right). \tag{10}
\]
We now recall the following tail bound on the normal distribution (see Theorem 1.2.6 of [Dur19] or Equation 2.58 of [Wai]):
\[
\varphi(r) \left( \frac{1}{r} - \frac{1}{r^3} \right) \leq \Pr_{\mathbf{g} \sim N(0,1)}[\mathbf{g} \geq r] \leq \varphi(r) \left( \frac{1}{r} - \frac{1}{r^3} + \frac{3}{r^5} \right), \tag{11}
\]
where \( \varphi(r) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2} \) is the density function of \( N(0,1) \). Combining Equation (9), Equation (10) and Equation (11) we get that \( \text{Inf}_{e_i}[C_r] = \Theta(r^2) \cdot \Pr_{\mathbf{g} \sim N(0,1)}[\mathbf{g} \geq r] = \Theta(\log(n)) \cdot \Theta(1/n) \), which corresponds to the influence of each individual variable on the Boolean “tribes” function.

3.3 Margulis-Russo for Convex Influences: An Alternative Characterization of Influences via Dilations

In this subsection we give an alternative view of the notion of influence defined above, in terms of the behavior of the Gaussian measure of the set as the variance of the underlying Gaussian is changed.\(^5\) This is closely analogous to the Margulis-Russo formula for monotone Boolean functions on \( \{-1,1\}^n \) (see [Rus81, Mar74] or Equation (8.9) in [O'D14]), which relates the derivative (with respect to \( p \)) of the \( p \)-biased measure of a monotone function \( f \) to the \( p \)-biased total influence of \( f \).

We start by defining \( \sigma \)-biased convex influences, which are analogous to \( p \)-biased influences from Boolean function analysis (see Section 8.4 of [O'D14]).

\(^5\)Since \( \gamma_y(K) = \gamma(K/\sigma) \), decreasing (respectively increasing) the variance of the underlying Gaussian measure is equivalent to dilating (respectively shrinking) the set.
Definition 15 ($\sigma$-biased influence). Given a centrally symmetric convex set $K \subseteq \mathbb{R}^n$, we define the $\sigma$-biased influence of direction $v$ on $K$ as being

$$\text{Inf}_{v}[K] := -\tilde{f}_{\sigma}(2v) = \mathbb{E}_{x \sim \mathcal{N}(0,1)^n} [-f(x)h_{2,\sigma}(v \cdot x)],$$

the negated degree-2 $\sigma$-biased Hermite coefficient in the direction $v$. We further define the $\sigma$-biased total influence of $K$ as

$$I^{(\sigma)}[K] := \sum_{i=1}^{n} \text{Inf}_{e_i}[K].$$

The proof of the following proposition, which asserts that the rate of the change of the Gaussian measure of a symmetric convex set $K$ with respect to $\sigma^2$ is (up to scaling) equal to the $\sigma$-biased total influence of $K$, is deferred to Appendix A. We note that this relation was essentially known to experts (see e.g. [LO99]), though we are not aware of a specific place where it appears explicitly in the literature.

Proposition 16 (Margulis-Russo for symmetric convex sets). Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex set. Then for any $\sigma > 0$ we have

$$\frac{d}{d\sigma^2} \mathbb{E}_{x \sim \mathcal{N}(0,\sigma^2)^n} [K(x)] = \frac{I^{(\sigma)}[K]}{\sigma^2 \sqrt{2}} = \frac{-1}{\sigma^2 \sqrt{2}} \sum_{i=1}^{n} \text{Inf}_{e_i}^{(\sigma)}[K].$$

Note that decreasing (respectively increasing) the variance of the background Gaussian measure is equivalent to dilating (respectively shrinking) the symmetric convex set while keeping the background measure fixed; this lets us write

$$I[K] = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n((1-\delta)K)}{\delta}$$

for a symmetric convex $K \subseteq \mathbb{R}^n$. We also note that Proposition 16 easily extends to the following coordinate-by-coordinate version (which also admits a similar description in terms of dilations):

Proposition 17 (Coordinate-wise Margulis-Russo). Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex set. Then for any $\sigma > 0$, we have

$$\frac{d}{d\sigma_i^2} \mathbb{E}_{x_{j \neq i} \sim \mathcal{N}(0,\sigma^2), x_i \sim \mathcal{N}(0,\sigma_i^2)} [K(x)] \bigg|_{\sigma_i^2 = \sigma^2} = \frac{-1}{\sigma^2 \sqrt{2}} \text{Inf}_{e_i}^{(\sigma)}[K].$$

In particular, we have

$$\text{Inf}_{e_i}[K] = -\sqrt{2} \frac{d}{d\sigma_i^2} \mathbb{E}_{x_{j \neq i} \sim \mathcal{N}(0,\sigma^2), x_i \sim \mathcal{N}(0,1)} [K(x)] \bigg|_{\sigma_i^2 = 1}. $$

Note that decreasing the variance of the underlying Gaussian measure along a coordinate direction cannot cause the volume of the set to decrease. It follows then that $\text{Inf}_{e_i}[K] \geq 0$ for all $e_i$. 

12
3.4 Extremal Symmetric Convex Sets

The unique maximizer of $\inf[f]$ across all monotone Boolean functions $f: \{\pm 1\}^n \rightarrow \{\pm 1\}$ is the dictator function $f(x) = x_1$. The next proposition gives an analogous statement for the “dictatorship” function $\text{Dict}_v$ from Example 12, for every possible Gaussian volume:

**Proposition 18.** Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set and let $v \in \mathbb{S}^{n-1}$. Let $c \geq 0$ be chosen so that the $\mathcal{N}(0, 1)^n$ Gaussian volume of $\text{Dict}_{cv}$ equals that of $K$, i.e. $\gamma(\text{Dict}_{cv}) = \gamma(K)$. Then $\inf_v[K] \leq \inf_v[\text{Dict}_{cv}]$.

**Proof.** Without loss of generality (for ease of notation) we take $v = e_1$. Let $g_K: \mathbb{R} \rightarrow [0, 1]$ be the function obtained by marginalizing out variables $x_2, \ldots, x_n$, so

$$g_K(x_1) = \mathbb{E}_{(x_2, \ldots, x_n) \sim \mathcal{N}(0, 1)^{n-1}}[K(x_1, x_2, \ldots, x_n)].$$

As noted following Proposition 9, we have that $\inf_{e_1}[K] = \inf_{e_1}[g_K]$. We observe that by definition we have

$$\inf_{e_1}[g_K] = \frac{1}{\sqrt{2}} \mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)}[(1 - x_1^2)g_K(x_1)].$$

Since $1 - t^2$ is a decreasing function of $t$ for all $t \geq 0$, it is easy to see that the symmetric $[0, 1]$-valued function $g_K$ that maximizes $\mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)}[(1 - x_1^2)g_K(x_1)]$ subject to having $\mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)}[g_K(x_1)] = \gamma(\text{Dict}_{e_1})$ is the function $g$ for which $g(t) = 1$ for $|t| \leq c$ and $g(t) = 0$ for $|t| > c$. This corresponds precisely to having $K = \text{Dict}_{e_1}$; so in fact taking $K = \text{Dict}_{e_1}$ maximizes $\inf_{e_1}[K]$ over all measurable subsets of $\mathbb{R}^n$ of Gaussian volume $\gamma(\text{Dict}_{e_1})$ (not just over all symmetric convex sets of that volume).

We note that a slight extension of this argument can be used to give a robust version of Proposition 18, showing that for any $c > 0$, any symmetric convex set $K$ (in fact any measurable set $K$) of Gaussian volume $\gamma(\text{Dict}_{cv})$ that has $\inf_v[K]$ close to $\inf_v[\text{Dict}_{cv}]$ must in fact be close to $\text{Dict}_{cv}$. This is analogous to the easy fact that any monotone Boolean function with $\inf_1[f]$ close to 1 must be close to the function $f(x) = x_1$.

Next we give a similar result but for total convex influence rather than influence in a single direction, analogous to the well known fact that the Majority function maximizes total influence across all $n$-variable monotone Boolean functions $f: \{\pm 1\}^n \rightarrow \{\pm 1\}$.

**Proposition 19.** Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set, and let $r \geq 0$ be chosen so that the $\mathcal{N}(0, 1)^n$ Gaussian volume of $B_r$ equals that of $K$, i.e. $\gamma(B_r) = \gamma(K)$. Then $I[K] \leq I[B_r]$.

**Proof.** The argument is similar to that of Proposition 18. We have

$$I[K] = \mathbb{E}_{x \sim \mathcal{N}(0, 1)^n} \left[ K(x) \left( \frac{n - \|x\|^2}{\sqrt{2}} \right) \right] = \frac{1}{\sqrt{2}} \mathbb{E}_{r \sim \chi(n)} \left[ (n - r^2)\alpha_K(r) \right]$$

(recall Equation (4)), where $\chi(n)$ is the $\chi$-distribution with $n$ degrees of freedom. We observe that taking $K = B_r$ results in $\alpha_K(t) = 1$ for $t \leq r$ and $\alpha_K(t) = 0$ for $t > r$, that the range of $\alpha_K(\cdot)$ is contained in $[0, 1]$ for any $K$, and that $n - t^2$ is a decreasing function of $t$ for all $t \geq 0$. Combining these observations, it is easily seen that taking $K = B_r$ in fact maximizes the expression on the RHS over all measurable subsets of $\mathbb{R}^n$ of volume $\gamma(B_r)$ (not just over all symmetric convex sets of that volume).

As before, the argument above can be used to establish a robust version of Proposition 19, showing that any symmetric convex set $K$ (in fact any measurable set $K$) of Gaussian volume $\gamma(B_r)$ that has $I[K]$ close to $I[B_r]$ must in fact be close to $B_r$. 

13
3.5 Other Notions of Influence

Here, we compare the notion of influence for symmetric convex sets proposed in Definition 8 with two previous notions of influence, namely i) the geometric influence introduced in [KMS12]; and ii) the expected variance along a fiber which coincides with the usual notion of influence for Boolean functions on the hypercube.

3.5.1 Geometric Influences

In [KMS12], Keller, Mossel, and Sen introduced the notion of geometric influence for functions over Gaussian space, and proved analogues of seminal results from the analysis of Boolean functions—including the KKL theorem, the Margulis–Russo lemma, and an analogue of Talagrand’s correlation inequality—for this notion of influence. Informally, the geometric influence captures the expected lower Minkowski content along each one-dimensional fiber of a set.

Definition 20 (Geometric influences). Given a Borel measurable set $K ⊆ \mathbb{R}$, its lower Minkowski content (with respect to the standard Gaussian measure), denoted $γ^+$, is defined as

$$γ^+(K) := \lim\inf_{r \to 0^+} \frac{γ(K + [-r, r]) - γ(K)}{r}.$$  

(Note that for $K = [a, b] ⊆ \mathbb{R}$, we have $γ^+(K) = ϕ(a) + ϕ(b)$.) For any Borel-measurable $K ⊆ \mathbb{R}^n$, for each $i ∈ [n]$ and $x ∈ \mathbb{R}^n$, define the fiber

$$K^x_i := \{y ∈ \mathbb{R} : (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) ∈ K\}.$$  

The geometric influence of coordinate $i$ on $K$ is

$$\text{Inf}_G^i[K] := E_{x \sim N(0,1)^n} \left[ γ^+(K^x_i) \right].$$  

For convex sets the total geometric influence admits a geometric interpretation as the change in the boundary of the set under uniform enlargement:

Proposition 21 (Remark 2.2 of [KMS12]). Let $K ⊆ \mathbb{R}^n$ be a convex set. Then we have

$$\lim_{r \to 0^+} \frac{γ_n(K + [-r, r]^n) - γ_n(K)}{r} = \sum_{i=1}^n \text{Inf}_G^i[K]$$  

where the right-hand side above is the total geometric influence of $K$.

Note that unlike $I[K]$ (see Definition 8), the total geometric influence is not invariant under rotations as the boundary of a convex set under enlargement is not rotationally invariant. It is possible for our convex influence notion to be much smaller than the geometric influence. For example, a routine computation shows that the $\sqrt{n}$-radius Euclidean ball $B := B_{\sqrt{n}}$ has $\text{Inf}_G^i[B] = Ω(1)$ for each $i ∈ [n]$, whereas as seen from Example 13, for convex influence we have $\text{Inf}_e^i[B] = O\left(\frac{1}{\sqrt{n}}\right)$. 

14
3.5.2 Variance Along a Fiber

In the setting of Boolean functions over the hypercube, the usual notion of influence of a coordinate on a function \( f : \{\pm 1\}^n \to \{\pm 1\} \) coincides with the expected variance of the function along a random fiber in the direction of that coordinate. This is also a standard notion of influence for product probability measures more generally, see e.g. Proposition 8.24 of [O’D14]. More formally, we have the following definition.

**Definition 22** (Expected variance along a fiber). Given a function \( f \in L^2(\mathbb{R}^n, \gamma) \), we define

\[
\text{VarInf}_i[f] := \mathbb{E}_{x \setminus \{x_i\} \sim \mathcal{N}(0,1)^{n-1}}[\text{Var}_{x_i}[f(x)]]
\]

to be the expected variance of \( f \) along the \( i \)th fiber.

We can express the expected variance along the \( i \)th fiber in terms of the Hermite expansion as \( \text{VarInf}_i[f] = \sum_{\alpha_i > 0} f(\alpha)^2 \) (see Proposition 8.23 of [O’D14]). In our setting it is possible for the convex influence of a symmetric convex set to be much smaller than the expected variance along a fiber. This is witnessed by the symmetric convex set \( \text{Dict}_v \subseteq \mathbb{R}^n \) given by

\[
\text{Dict}_v := \{x \in \mathbb{R}^n : |x \cdot v| \leq 1\}
\]

where \( v = \frac{1}{\sqrt{n}}(1, \ldots, 1) \).

A routine computation shows that \( \text{VarInf}_i[\text{Dict}_v] = \Theta\left(\frac{1}{\sqrt{n}}\right) \) for each \( i \in [n] \). On the other hand, since \( I[\text{Dict}_v] \) is rotationally invariant, it follows from Example 12 that \( I[\text{Dict}_v] = I[\text{Dict}_{e_1}] = \Theta(1) \), and consequently by symmetry, it follows that \( \text{Inf}_{e_i}[K] = \Theta\left(\frac{1}{n}\right) \).

4 Lower Bounds on Total Convex Influence

Two fundamental results on the influence of variables for Boolean functions \( f : \{\pm 1\}^n \to \{\pm 1\} \) are the Poincaré inequality and the celebrated “KKL Theorem” of Kahn, Kalai, and Linial [KKL88], both of which give lower bounds on total influence. The former states that the total influence of any \( f : \{\pm 1\}^n \to \{\pm 1\} \) is at least its variance, and has a very elementary proof (indeed it can be proved in a single line by comparing the Fourier expressions for the two quantities). The KKL theorem gives a more refined bound, showing (roughly speaking) that if all influences are small then the total influence must be somewhat large. Several proofs of the KKL Theorem are now known, using a range of different techniques such as the famous hypercontractive inequality [Bon70, Bec75] (the original approach), methods of stochastic calculus [EG20], and the Log-Sobolev inequality [KKK+21].

In this section we prove convex influence analogues of the Poincaré inequality and the KKL theorem. We use the “S-inequality” of Latala and Oleskiewicz [LO99] to give a relatively quick proof our Poincaré analogue, and prove our analogue of the KKL theorem using the Gaussian Isoperimetric Theorem [Bor75].

4.1 A Poincaré Inequality for Symmetric Convex Sets

Recall from Section 3.5 that our convex influence notion can be much smaller than the geometric influence defined in [KMS12] or the ordinary “variance along a fiber” influence notion of Section 3.5.2. Given this, it is of interest to give lower bounds on the total convex influence. Our first result along these lines is an analogue of the standard Poincaré inequality for Boolean functions (or more generally for functions over product domains):
Proposition 23 (Poincaré for symmetric convex sets). Let \( K \subseteq \mathbb{R}^n \) be a symmetric convex set. Then \( I[K] \geq \Omega(\text{Var}[K]) \).

The main tool we use for our proof of Proposition 23 is the following celebrated result of Latała and Oleskiewicz concerning the rate of growth of symmetric convex sets under dilations:

**Proposition 24** (S-inequality, Theorem 1 of [LO99]). Let \( K \subseteq \mathbb{R}^n \) be a symmetric convex set, and let \( \text{Dict}_w \subseteq \mathbb{R}^n \) be a symmetric strip (i.e. \( \text{Dict}_w = \{ x \in \mathbb{R}^n : |x \cdot w| \leq 1 \} \) for some fixed \( w \in \mathbb{R}^n \)) such that \( \gamma_n(A) = \gamma_n(\text{Dict}_w) \). Then

\[
\gamma_n(tK) \geq \gamma_n\left(\text{Dict}_{w/t}\right) \quad \text{for} \ t \geq 1,
\]

and

\[
\gamma_n(tK) \leq \gamma_n\left(\text{Dict}_{w/t}\right) \quad \text{for} \ 0 \leq t \leq 1.
\]

Intuitively, the above result says that among all convex symmetric sets of a given Gaussian volume, the Gaussian volume of dictatorships (see Example 12) grows the slowest under enlargement by dilations.

The proof of Proposition 23 combines Proposition 24 with our Margulis-Russo analogue (the characterization of influence in terms of dilations given in Section 3.3):

**Proof of Proposition 23.** Write \( \gamma_n(K) = \alpha \), and let \( \text{Dict}_{e_1/a} = \{ x \in \mathbb{R}^n : |x_1| \leq a \} \) be such that \( \gamma_1(\text{Dict}_{e_1/a}) = \alpha \), i.e. \( \gamma_n([-a,a]) = \alpha \). Recall from Section 3.3 that

\[
I[K] = \frac{1}{\sqrt{2\pi}} \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n((1-\delta)K)}{\delta}.
\]  

By the S-inequality (Proposition 24 above), for any fixed \( 0 < \delta \leq 1 \), we have

\[
\gamma_n((1-\delta)K) \leq \gamma_n((1-\delta)\text{Dict}_{e_1/a}) = \gamma_1\left([- (1-\delta) a, (1-\delta) a]\right),
\]

which implies that

\[
\gamma_n(K) - \gamma_n((1-\delta)K) \geq \gamma_1([-a,a]) - \gamma_1\left([- (1-\delta) a, (1-\delta) a]\right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \int_{-a}^{a} e^{-x^2/2} dx - \int_{-(1-\delta) a}^{(1-\delta) a} e^{-x^2/2} dx \right)
\]

\[
= \sqrt{\frac{2}{\pi}} \int_{(1-\delta) a}^{a} e^{-x^2/2} dx.
\]  

(14)

When \( \alpha \leq 1/2 \), we have \( \alpha \leq a \leq 1 \), and clearly

\[
\sqrt{\frac{2}{\pi}} \int_{(1-\delta) a}^{a} e^{-x^2/2} dx \geq \Omega(\delta a) \geq \Omega(\delta \alpha).
\]

(15)

Combining Equations (13) to (15) implies the desired result in this case. On the other hand, when \( \alpha \geq 1/2 \), we have

\[
\sqrt{\frac{2}{\pi}} \int_{(1-\delta) a}^{a} e^{-x^2/2} dx \geq \Omega\left(\delta a e^{-a^2/2}\right).
\]
Standard tail bounds on the Gaussian distribution give $ae^{-a^2/2} \geq \Omega(1 - \Phi(a))$ when $a \geq 1$ (see, for example, Theorem 1.2.6 of [Dur19]). It follows that if $\gamma_1([-a,a]) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-x^2} \, dx = \alpha$, then $ae^{-a^2/2} \geq \Omega(1 - \alpha)$. In particular, we have
\[
\sqrt{\frac{2}{\pi}} \int_{(1-\delta)a}^{a} e^{-x^2/2} \, dx \geq \Omega(\delta(1 - \alpha)) \tag{16}
\]
and combining Equations (13), (14) and (16) implies the desired result. \hfill \Box

4.2 A KKL Analogue for Symmetric Convex Sets

For the symmetric convex set $\text{Dict}_{e_1}$, both the total convex influence and the variance are $\Theta(1)$, so Proposition 23 is best possible (up to a constant factor) for arbitrary symmetric convex sets. But of course $\text{Dict}_{e_1}$ has very large (constant) influence in a single direction, analogous to a Boolean function with an individual coordinate of constant influence. The famous KKL theorem for Boolean functions over $\{-1,1\}^n$ states that if no coordinate influence is allowed to be large (each is at most $\delta$), then the total influence must be large (at least $\Omega(\text{Var}[f] \cdot \log(1/\delta))$). We now prove an analogous result for convex influences, though we only achieve a quadratically weaker bound in terms of the max influence:

**Theorem 25** (KKL for symmetric convex sets). Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set with $\text{Inf}_v[K] \leq \delta \leq \text{Var}[K]/10$ for all $v \in S^{n-1}$. Then
\[
\text{I}[K] \geq \Omega \left( \text{Var}[K] \sqrt{\log \left( \frac{\text{Var}[K]}{\delta} \right)} \right). \tag{17}
\]

Our proof of Theorem 25 is inspired by the approach of [LO05]. The main technical ingredient we use is the Gaussian isoperimetric inequality:

**Proposition 26** (Gaussian isoperimetric inequality, [Bor75]). Given any Borel set $A \subseteq \mathbb{R}^n$, we have
\[
\Phi^{-1}(\gamma_n(A_t)) \geq \Phi^{-1}(\gamma_n(A)) + t
\]
where $A_t := A + B_t$ is the $t$-enlargement of $A$.

We remark that it is easy to obtain Proposition 26 from the Ehrhard-Borell inequality [Ehr83, Bor03, Bor08], which we recall as Proposition 38 in Appendix A. We will also require the following easy estimate on the Gaussian isoperimetric function $\varphi \circ \Phi^{-1}(\cdot)$.

**Proposition 27.** Let $\Phi : \mathbb{R} \to [0,1]$ denote the cumulative distribution function of the standard one-dimensional Gaussian distribution, and let $\varphi := \Phi'$ denote its density. Then for all $\alpha \in (0,1)$, we have
\[
\varphi \circ \Phi^{-1}(\alpha) \geq \sqrt{\frac{2}{\pi}} \min(\alpha, 1 - \alpha).
\]

**Proof.** By symmetry, it suffices to show that $\varphi \circ \Phi^{-1}(\alpha) \geq \sqrt{\frac{2}{\pi}} \alpha$ for $\alpha \in \left[0, \frac{1}{2}\right]$. This is immediate from the fact that
\[
\varphi \circ \Phi^{-1}(0) = 0 \quad \text{and} \quad \varphi \circ \Phi^{-1} \left( \frac{1}{2} \right) = \frac{1}{\sqrt{2\pi}},
\]
and the concavity of $\varphi \circ \Phi^{-1}$ (see, for example, Exercise 5.43 of [O’D14]). \hfill \Box
Proof of Theorem 25. Let \( r_{in} \) denote the in-radius of \( K \). We will show that
\[
I[K] \geq \frac{1}{\sqrt{\pi}} \text{Var}[K] \cdot r_{in}
\] (18)
and that
\[
r_{in} \geq \Omega(\sqrt{\ln(\text{Var}[K]/\delta)})
\] (19)
from which the desired result follows.

For Equation (19), by Proposition 11 we have that for some direction \( v \in S^{n-1} \),
\[
\inf_{\hat{v}}[K] \geq \frac{\gamma(K)e^{-r_{in}^2/2\pi}}{2^3/2\pi} \geq \frac{\text{Var}[K]e^{-r_{in}^2}}{2^3/2\pi}.
\]
Combining this with \( \inf_{\hat{v}}[K] \leq \delta \) and recalling that \( \delta \leq \text{Var}[K]/10 \), we get Equation (19).

We turn now to establishing Equation (18). Recall from Equation (12) of Section 3.3 (our Margulis-Russo formula) that
\[
I[K] = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \frac{\gamma(K) - \gamma((1-\delta)K)}{\delta}.
\] (20)
We proceed to upper-bound \( \gamma((1-\delta)K) \) in terms of \( \gamma(K) \). Since \( r_{in} \) is the in-radius of \( K \), for all \( 0 < \delta \leq 1 \), we have that
\[
(1-\delta)K + \delta r_{in}B_1 = (1-\delta)K + B_{\delta r_{in}} \subseteq K.
\] (21)
Let \( K^c := \mathbb{R} \setminus K \), and let \( (K^c)_{\delta r_{in}} := K^c + B_{\delta r_{in}} \) be the \( \delta r_{in} \)-enlargement of \( K^c \). It follows from Equation (21) that \( (1-\delta)K \cap (K^c)_{\delta r_{in}} = \emptyset \), which in turn implies that
\[
\gamma((1-\delta)K) + \gamma((K^c)_{\delta r_{in}}) \leq 1, \quad \text{and so} \quad \gamma((1-\delta)K) \leq 1 - \gamma((K^c)_{\delta r_{in}}).
\] (22)
However, from the Gaussian isoperimetric inequality (Proposition 26), we know that
\[
\gamma((K^c)_{\delta r_{in}}) \geq \Phi\left(\Phi^{-1}(\gamma(K^c)) + \delta r_{in}\right).
\] (23)
Let \( \alpha = \gamma(K^c) \), so \( \gamma(K) = 1 - \alpha \). Putting Equations (20), (22) and (23) together, we get
\[
I[K] \geq \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \frac{\Phi(\Phi^{-1}(\alpha) + \delta r_{in}) - \alpha}{\delta}
\]
\[
= \frac{1}{\sqrt{2}} r_{in} \left( \lim_{\varepsilon \to 0} \frac{\Phi(\Phi^{-1}(\alpha) + \varepsilon) - \Phi(\Phi^{-1}(\alpha))}{\varepsilon} \right)
\]
\[
= \frac{1}{\sqrt{2}} r_{in} \cdot \Phi'(\Phi^{-1}(\alpha))
\]
\[
= \frac{1}{\sqrt{2}} r_{in} \cdot \varphi \circ \Phi^{-1}(\alpha)
\]
by making the change of variables \( \varepsilon := \delta r_{in} \) and using the fact that \( \varphi = \Phi' \). It follows then from Proposition 27 that
\[
I[K] \geq \frac{1}{\sqrt{2}} r_{in} \cdot \left( \sqrt{\frac{2}{\pi}} \min(\alpha, 1-\alpha) \right) \geq \frac{1}{\sqrt{\pi}} \text{Var}[K] \cdot r_{in}
\]
which completes the proof.

As discussed in Item 1 of Section 1.4, we conjecture that the RHS of Equation (17) can be strengthened to \( \Omega\left(\text{Var}[K] \log\left(\frac{1}{\delta}\right)\right) \), which would be the best possible bound by Example 14.
5 Towards a Junta Theorem for Convex Sets

Friedgut’s junta theorem [Fri98] is an important result in the analysis of Boolean functions. It says that Boolean functions with low total influence must be close to juntas; more precisely, if \( f : \{\pm 1\}^n \to \{0, 1\} \) has \( I[f] \leq I \), then \( f \) is \( \varepsilon \)-close to a junta on some set \( J \) of at most \( 2^{O(I/\varepsilon)} \) variables. Like the KKL theorem, the standard proof of Friedgut’s theorem uses the hypercontractive inequality (and is in fact quite similar to the proof of the KKL theorem; see Section 9.6 of [O’D14]).

An easy consequence of Friedgut’s junta theorem is that for any low-influence function, averaging out a well-chosen small set of coordinates makes the function have low variance:

**Corollary 28** (Corollary of Friedgut’s junta theorem). Let \( f : \{\pm 1\}^n \to \{0, 1\} \) be a function that has \( I[f] \leq I \). Let \( \varepsilon > 0 \) and let \( f_{-J} : \{\pm 1\}^n \to \{0, 1\} \) denote the function obtained by “averaging out” the coordinates in \( J \), i.e. \( f_{-J} \) is defined as

\[
f_{-J}(x^{[n]\setminus J}) := \mathbb{E}_{x \sim \{\pm 1\}^J}[f(x', x^{[n]\setminus J})],
\]

where \( J \) is the set of \( 2^{O(I/\varepsilon)} \) variables whose existence is given by Friedgut’s junta theorem (so \( f_{-J} \) depends only on the coordinates in \([n] \setminus J\)). Then \( \text{Var}[f_{-J}] \leq 4\varepsilon \).

**Proof.** Let \( \mu_f = \mathbb{E}[f] = \mathbb{E}[f_{-J}] \). Let \( g \) denote the \( J \)-junta which \( \varepsilon \)-approximates \( f \), and let \( \mu_g = \mathbb{E}[g_J] \); note that since \( f \) and \( g \) are both 0/1-valued, we have \( \mathbb{E}[(f-g)^2] = \varepsilon \). We have

\[
\text{Var}[f_{-J}] = \mathbb{E}[(f_{-J} - \mu_f)^2] \\
\leq 2 \left( \mathbb{E}[(f_{-J} - \mu_g)^2] + \mathbb{E}[(\mu_f - \mu_g)^2] \right) \\
\leq 2 \left( \mathbb{E}[(f-g)^2] + \mathbb{E}[(g-f)^2] \right) = 4\varepsilon. \tag*{\blacksquare}
\]

In this section we prove a Gaussian space analogue of **Corollary 28** for our convex influence notion:

**Theorem 29** (Analogue of Corollary 28 for convex influence). Let \( K \subseteq \mathbb{R}^n \) be a symmetric convex set with \( I[K] \leq I \). For any \( \varepsilon > 0 \), there exists a set of orthogonal directions \( S = \{v_1, \ldots, v_{\ell}\} \) with \( \ell = \exp \left( O \left( (I^2/\varepsilon^4) \right) \right) \) such that the following holds: For convenience, rename coordinates so that \( v_1 = e_1, \ldots, v_{\ell} = e_\ell \), and define \( K_{-S} \) to be the symmetric log-concave function

\[
K_{-S}(x) := \mathbb{E}_{(x_1, \ldots, x_\ell) \sim N(0, 1)^\ell}[K(x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_n)]
\]

(so \( K_{-S} \) depends only on the variables \( x_{\ell+1}, \ldots, x_n \)). Then \( \text{Var}[K_{-S}] \leq \varepsilon \).

5.1 The Main Technical Ingredient

The main technical ingredient in our proof of **Theorem 29** is the following generalization of **Theorem 25** (our convex influence analogue of the KKL theorem) to symmetric log-concave functions:

**Proposition 30** (KKL for symmetric logconcave functions). Let \( f : \mathbb{R}^n \to [0, 1] \) be a symmetric log-concave function with \( \text{Var}[f] \geq \sigma^2 \) and \( I[f] \leq I \). Then there exists some direction \( v^* \in \mathbb{S}^{n-1} \) such that

\[
\text{Inf}_{v^*}[f] \geq \Omega \left( \sigma^2 e^{-4\pi I^2/\sigma^4} \right).
\]

19
Proof. For \( t \in [0, 1] \), define the level set
\[
A_t := \{ x \in \mathbb{R}^n : f(x) \geq t \}.
\]
It is immediate from the log-concavity of \( f \) that \( A_t \) is a symmetric convex set for all \( t \in [0, 1] \), and that
\[
f(x) = \int_0^1 A_t(x) \, dt = \mathbb{E}_{t \in [0,1]}[A_t(x)],
\]
where we identified \( A_t \) with its indicator function. Next, note that for any \( x \in \mathbb{R}^n \), by Jensen’s inequality we have that
\[
\mathbb{E}_t \left[ \left( A_t(x) - \mathbb{E}_y[A_t(y)] \right)^2 \right] \geq \mathbb{E}_t \left[ \left( A_t(x) - \mathbb{E}_y[A_t(y)] \right)^2 \right] \geq \mathbb{E}_t \left[ (A_t(x) - \mathbb{E}[A_t])^2 \right].
\]
Averaging Equation (24) over \( x \sim \mathcal{N}(0,1)^n \), the LHS becomes \( \mathbb{E}_t [\text{Var}[A_t]] \) and the RHS becomes \( \text{Var}[f] \), so we get that
\[
\mathbb{E}_t [\text{Var}[A_t]] \geq \text{Var}[f] \geq \sigma^2.
\] Let \( r_{in}(A_t) \) denote the in-radius of \( A_t \). From the proof of Theorem 25 (in particular, see Equation (18)), we have that
\[
I[A_t] \geq \frac{1}{\sqrt{\pi}} r_{in}(A_t) \cdot \text{Var}[A_t]
\]
and as \( I[f] = \int_0^1 I[A_t] \, dt \), we have that
\[
I \geq I[f] \geq \frac{1}{\sqrt{\pi}} \int_0^1 r_{in}(A_t) \cdot \text{Var}[A_t] \, dt \quad \text{i.e.} \quad \sqrt{\pi} I \geq \mathbb{E}_{t \in [0,1]}[r_{in}(A_t) \text{Var}[A_t]].
\] Let \( \mathcal{D} \) be the distribution over \([0, 1] \) which samples each outcome of \( t \in [0, 1] \) with probability proportional to \( \text{Var}[A_t] \) (so the density function of \( \mathcal{D} \) at \( t \) is \( \text{Var}[A_t]/\int_0^1 \text{Var}[A_s] \, ds \)). Armed with \( \mathcal{D} \), we may infer from Equations (25) and (26) that \( \mathbb{E}_{t \sim \mathcal{D}}[r_{in}(A_t)] \leq \sqrt{\pi} / \sigma^2 \), so by Markov’s inequality we have that
\[
\text{Pr}_{t \sim \mathcal{D}}[r_{in}(A_t) \leq 2 \sqrt{\pi} I / \sigma^2] \geq 1/2.
\]
Recall that \( r_{in}(A_t) \) is non-increasing, and let \( t^* = \inf\{ t \in [0, 1] : r_{in}(A_t) \leq 2 \sqrt{\pi} I / \sigma^2 \} \). By definition of \( \mathcal{D} \) and Equation (27), we have that
\[
\int_{t=t^*}^1 \text{Var}[A_t] \, dt \geq \sigma^2 / 2.
\] Applying Proposition 11 to \( A_{t^*} \), we get that there exists some direction \( v^* \in S^{n-1} \) such that
\[
\inf_{v^*} [A_{t^*}] \geq \frac{\gamma(A_{t^*}) e^{-r_{in}(A_{t^*})^2}}{2^{3/2}\pi} \geq \frac{e^{-r_{in}(A_{t^*})^2}}{2^{3/2}\pi} \cdot \text{Var}[A_{t^*}].
\] Since \( A_t \subseteq A_{t^*} \) for \( t \geq t^* \), it follows from the proof of Proposition 11 (and the fact that \( r_{in}(A_t) \) is non-increasing in \( t \)) that the direction \( v^* \) has \( \inf_{v^*} [A_t] \geq \frac{e^{-r_{in}(A_{t^*})^2}}{2^{3/2}\pi} \cdot \text{Var}[A_t] \) for all \( t \geq t^* \). Hence
\[
\inf_{v^*}[f] = \int_{t=0}^1 \inf_{v^*}[A_t] \, dt \geq \int_{t=t^*}^1 \inf_{v^*}[A_t] \, dt \geq \frac{e^{-r_{in}(A_{t^*})^2}}{2^{3/2}\pi} \cdot \int_{t=t^*}^1 \text{Var}[A_t] \, dt \geq \frac{\sigma^2 e^{-r_{in}(A_{t^*})^2}}{2^{5/2}\pi} \geq \frac{\sigma^2 e^{-4\pi t^2/\sigma^4}}{2^{5/2}\pi},
\]
where Equation (30) is by Equation (28). \( \square \)
5.2 Proof of Theorem 29

If \( \text{Var}[K] \leq \varepsilon \), then the result clearly holds. If \( \text{Var}[K] > \varepsilon \), then by Proposition 30, there exists some direction \( v \in S^{n-1} \)—without loss of generality, say \( v = e_1 \)—such that

\[
\inf_{e_1}[K] = \inf_{e_1}[K] \geq c\varepsilon \cdot e^{-4\pi I^2/\varepsilon^2}
\]

for some absolute constant \( c > 0 \). Let \( K_{-\{e_1\}} : \mathbb{R}^{n-1} \to [0, 1] \) be the symmetric log-concave function obtained from \( K \) by averaging out the coordinate \( e_1 \), i.e. we define

\[
K_{-\{e_1\}}(x) := \mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)}[K(x_1, x_2, \ldots, x_n)].
\]

It follows from Fact 10 that for all \( i \neq 1 \), we have \( \inf_{e_1}[K_{-e_1}] = \inf_{e_1}[K] \), and so we have

\[
\mathbf{I}[K_{-e_1}] = \mathbf{I}[K] - \inf_{e_1}[K] \leq I - c\varepsilon \cdot e^{-4\pi I^2/\varepsilon^2}.
\]

If \( \text{Var}[K_{-e_1}] \leq \varepsilon \), then the claimed result holds; if not, then once again by Proposition 30, there exists some direction—without loss of generality, say \( e_2 \)—such that

\[
\inf_{e_2}[K_{-e_1}] \geq c\varepsilon \cdot e^{-4\pi I^2/\varepsilon^2},
\]

and we can average out \( e_2 \) to obtain \( K_{-\{e_1, e_2\}} \),

\[
K_{-\{e_1, e_2\}}(x) := \mathbb{E}_{x_1, x_2 \sim \mathcal{N}(0, 1)}[K(x_1, x_2, x_3, \ldots, x_n)],
\]

with

\[
\mathbf{I}[K_{-\{e_1, e_2\}}] \leq I - 2c\varepsilon \cdot e^{-4\pi I^2/\varepsilon^2}.
\]

If \( \text{Var}[K_{-\{e_1, e_2\}}] \leq \varepsilon \), then the desired result holds; if not, then we repeat as above. Note, however, that the maximum possible number of repetitions (before \( \mathbf{I}[K_{-S}] \) would become negative, which is impossible) is at most

\[
\frac{I}{c\varepsilon \cdot e^{-4\pi I^2/\varepsilon^2}} = \exp \left( O(I^2/\varepsilon^2) \right);
\]

so after at most this many repetitions it must be the case that \( \text{Var}[K_{-S}] \leq \varepsilon \). This concludes the proof of Theorem 29. \( \square \)

6 Sharp Threshold Results for Symmetric Convex Sets

For any symmetric convex set \( K \subseteq \mathbb{R}^n \), we have that \( \gamma_\sigma(K) = \gamma(K/\sigma) \), and hence the map \( \Psi_K : \sigma \mapsto \gamma_\sigma(K) \) is a non-increasing function of \( \sigma \) (since \( K/\sigma_1 \subseteq K/\sigma_2 \) whenever \( \sigma_1 \geq \sigma_2 \)). Given this, it is natural to study the rate of decay of \( \Psi_K \) for different symmetric convex sets \( K \subseteq \mathbb{R}^n \).

The \( S \)-inequality (Proposition 24) can be interpreted as saying that the slowest rate of decay across all symmetric convex sets of a given volume is achieved by a symmetric strip. Let \( K_* \) be such a strip, i.e. we may take \( K_* = \{ x \in \mathbb{R}^n : |x_1| \leq c_* \} \) where \( c_* = \Theta(\sqrt{\ln(1/\varepsilon)}) \) is chosen so that \( \Psi_{K_*}(1) = 1 - \varepsilon \) (and hence \( \gamma(K_*) = 1 - \varepsilon \)). With this choice of \( c_* \), it follows that \( \Psi_{K_*}(\sigma) = \varepsilon \) for \( \sigma = \Theta(1/\varepsilon) \). Hence, for the volume of \( K_* \) to shrink from \( 1 - \varepsilon \) to \( \varepsilon \), the variance of the underlying Gaussian has to increase very dramatically, by a factor of \( \tilde{O}(1/\varepsilon^2) \). Taking, for example, \( \varepsilon = 0.01 \), we see that in order for the symmetric strip \( K_* \) to have its Gaussian volume change from \( \gamma_1(K_*) = 0.99 \)
to $\gamma_\sigma(K_*) = 0.01$, the parameter $\sigma$ must vary over an interval of size $\Theta(1)$, so the strip $K_*$ does not exhibit a "sharp threshold."

Of course, as we have seen before, the symmetric strip $K_*$ has an extremely large (constant) convex influence in the direction $e_1$. We now show that large individual influences are the only obstacle to sharp thresholds, i.e. any symmetric convex set in which no direction has large convex influence must exhibit a sharp threshold:

**Theorem 31** (Sharp thresholds for symmetric convex sets with small max influence). Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex set. Suppose $\varepsilon, \delta > 0$ where $\delta \leq \varepsilon^{-10 \log(1/\varepsilon)}$ and $\varepsilon > 0$ is sufficiently small (at most some fixed absolute constant). Suppose that $\gamma(K) \leq 1 - \varepsilon$ and $\max_{v \in S^{n-1}} [\text{Inf}_v(K)] \leq \delta$. Then, for $\sigma = 1 + \Theta\left(\frac{\ln(1/\varepsilon)}{\sqrt{\ln(1/\delta)}}\right)$, we have $\gamma_\sigma(K) \leq \varepsilon$.

Setting $\varepsilon = 0.01$ and $\delta = o(1)$, the above theorem implies that for $K$ a symmetric convex set $K$ with $\max_{v \in S^{n-1}} [\text{Inf}_v(K)] = o(1)$, it must be the case that $\gamma_\sigma(K)$ changes from 0.99 to 0.01 as the underlying $\sigma$ changes from 1 to $1 + o(1)$.

**Discussion.** Theorem 31 can be seen as a convex influence analogue of a “sharp threshold” result due to Kalai [Kal04]. Building on [FK96], Kalai [Kal04] shows that if $f : \{\pm 1\}^n \to \{0, 1\}$ is monotone and its max influence is $o(1)$, then $\mu_p(f)$ must have a sharp threshold (where $\mu_p(f)$ is the expectation of $f$ under the $p$-biased measure) (see also Theorem 3.8 of [KS05]). This is closely analogous to Theorem 31, which establishes a sharp threshold for $\gamma_\sigma(K)$ under the assumption that the max convex influence of $K$ is $o(1)$. We note an interesting quantitative distinction between Theorem 31 and the result of [Kal04]: if the max influence of a monotone $f : \{\pm 1\}^n \to \{0, 1\}$ function is $\delta$, then [Kal04] shows that $\mu_p(f)$ goes from 0.01 to 0.99 in an interval of width $\approx 1/poly(\log \log(1/\delta))$ (see the discussion following Theorem 3.8 of [KS05]). In contrast, Theorem 31 shows that $\gamma_\sigma(K)$ goes from 0.01 to 0.99 in an interval of width $\approx 1/\sqrt{\log(1/\delta)}$, thus establishing an exponentially “sharper threshold” in the convex setting.\(^6\)

**Proof of Theorem 31.** We may assume that $\gamma(K) \geq \varepsilon$, since otherwise, there is nothing to prove. Let $r_{in} = r_{in}(K)$ be the in-radius of $K$. By Proposition 11 we get that

$$r_{in} \geq \sqrt{\ln \left(\frac{\gamma(K)}{2^{3/2} \pi \delta}\right)} \geq \sqrt{\ln \left(\frac{\varepsilon}{2^{3/2} \pi \delta}\right)}$$  \hspace{1cm} (31)

(note that our assumptions on $\delta$ and $\varepsilon$ imply that the right hand side of (31) is well-defined). Next, we observe that a *mutatis mutandis* modification of the proof of Equation (18) gives that

$$I^{(\sigma)}[K] \geq \frac{1}{\sqrt{\pi}} \cdot r_{in} \cdot \Var_\sigma[K].$$  \hspace{1cm} (32)

We further recall that by our Marguilis-Russo formula for symmetric convex sets (Proposition 16), we have

$$\frac{d\gamma_\sigma(K)}{d\sigma^2} = -\frac{1}{\sigma^2} I^{(\sigma)}[K].$$  \hspace{1cm} (33)

Combining (31), (32) and (33), we get that

$$\frac{d\gamma_\sigma(K)}{d\sigma^2} \leq -\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \Var_\sigma[K] \cdot \sqrt{\ln \left(\frac{\varepsilon}{2^{3/2} \pi \delta}\right)}.$$  

---

\(^6\)Roughly speaking, the extra exponential factor in [Kal04] arises because of Friedgut’s junta theorem; our proof takes a different path and does not incur this quantitative factor.
Expressing $\Var\sigma[K]$ as $\gamma_\sigma(K) \cdot (1 - \gamma_\sigma(K))$ and “solving” the above differential equation for $\gamma_\sigma(K)$, we get that
\[
\ln \left( \frac{\gamma_\sigma(K)}{1 - \gamma_\sigma(K)} \right) - \ln \left( \frac{\gamma(K)}{1 - \gamma(K)} \right) \leq \frac{-1}{\sqrt{2\pi}} \cdot \sqrt{\ln \left( \frac{\varepsilon}{2^{3/2} \pi \delta} \right)} \cdot 2 \ln \sigma. \tag{34}
\]

Using the assumption that $\gamma(K) \leq 1 - \varepsilon$, it follows that for $\sigma \geq 1$, we have
\[
\ln \left( \frac{\gamma_\sigma(K)}{1 - \gamma_\sigma(K)} \right) \leq \ln(1/\varepsilon) + \frac{-1}{\sqrt{2\pi}} \cdot \sqrt{\ln \left( \frac{\varepsilon}{2^{3/2} \pi \delta} \right)} \cdot 2 \ln \sigma.
\]

Recalling the assumption that $\delta \leq \varepsilon^{-10\log(1/\varepsilon)}$, we see that choosing
\[
\sigma = 1 + \Theta \left( \frac{\ln(1/\varepsilon)}{\sqrt{\ln(\varepsilon/\delta)}} \right),
\]
we get $\gamma_\sigma(K) \leq \varepsilon$ as claimed. \hfill \Box

**Remark 32.** We close this section by noting that the type of threshold phenomenon studied here has previously been considered in geometric functional analysis. In particular, the seminal work of Milman [Mil71], using concentration of measure to establish Dvoretzky’s theorem [Dvo61] on almost Euclidean sections of symmetric convex sets, implies a type of threshold phenomenon for symmetric convex sets. Milman’s result can be shown to imply that if the “Dvoretzky number” of a symmetric convex set is $\omega_n(1)$, then the set must exhibit a type of sharp threshold behavior. Indeed, Milman’s theorem can be used to give an alternate proof of a result that is similar to Theorem 31.

## 7 A Robust Kruskal-Katona Analogue for Symmetric Convex Sets

Recall from Equation (4) that for a symmetric convex set $K \subseteq \mathbb{R}^n$, the shell density function $\alpha_K : [0, \infty) \to [0, 1]$ is defined to be $\alpha_K(r) := \Pr_{x \in \mathbb{S}^{n-1}}[x \in K]$, and that $\alpha_K(\cdot)$ is non-increasing. In [DS21], De and Servedio established the following quantitative lower bound on the rate of decay of $\alpha_K(\cdot)$:

**Theorem 33** (Theorem 12 of [DS21]). Let $K \subseteq \mathbb{R}^n$ be a convex body that has in-radius $r_in > 0$. Then for $r > r_in$ such that $\min\{\alpha_K(r), (1 - \alpha_K(r))\} \geq e^{-n/4}$, as $\Delta r \to 0^+$ we have that
\[
\alpha_K(r - \Delta r) - \alpha_K(r) \geq \Omega \left( \frac{r_in \cdot \sqrt{n} \cdot r}{r^2} \right) \alpha_K(r) (1 - \alpha_K(r)).
\]

As alluded to in Item 1 of Section 1, the above result can be used to obtain a Kruskal-Katona type theorem for centrally symmetric convex sets. In particular, we have the following corollary:

**Corollary 34.** Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set and $r = \Theta(\sqrt{n})$ be such that $\alpha_K(r) \in [1/10, 9/10]$. Then, as $\varepsilon \to 0^+$, we have that
\[
\alpha_K(r(1 - \varepsilon)) - \alpha_K(r) = \Omega(\varepsilon).
\]

**Proof.** Let $r_in$ denote the in-radius of $K$, so for any $\zeta > 0$, there is a point $z_*$ such that $z_* \notin K$ and $\|z_*\|_2 = r_in + \zeta$. By the separating hyperplane theorem, it follows that there is a unit vector $\hat{v} \in \mathbb{R}^n$ such that
\[
K \subseteq K_* := \{x \in \mathbb{R}^n : |\hat{v} \cdot x| \leq r_in + \zeta\}. \tag{35}
\]
We next upper bound \( \alpha_{K_v}(r) \). For this, without loss of generality, we may assume that \( \hat{v} = e_1 \). We have

\[
\alpha_{K_v}(r) = \text{Pr}_{y \in S^{n-1}}[|y_1| \leq r_{in} + \epsilon] \leq O\left(\frac{(r_{in} + \epsilon)^{\frac{1}{2}}}{r}\right),
\]

where the upper bound is an easy consequence of well-known concentration of measure results for the \( n \)-dimensional unit sphere. Now, using (35) and letting \( \epsilon \to 0 \), we have

\[
\alpha_K(r) \leq \alpha_{K_v}(r) \leq O\left(\frac{r_{in} \cdot \sqrt{n}}{r}\right).
\]

Since \( \alpha_K(r) \geq 0.1 \) by assumption, it follows that \( r_{in} = \Omega(1) \). Corollary 34 now follows from Theorem 33.

**A Robust Analogue of Kruskal-Katona.** The lower bound given by Corollary 34 cannot be improved in general; for example, the convex set \( K = \text{Dict}_{e_1} := \{x : |x_1| \leq 1\} \) satisfies the conditions of Corollary 34 and has

\[
\alpha_{\text{Dict}_{e_1}}(r(1 - \epsilon)) - \alpha_{\text{Dict}_{e_1}}(r) = \Theta(\epsilon)
\]

for \( r = \Theta(\sqrt{n}) \). This is closely analogous to how the \( \Omega(1/n) \) density increment of the original Kruskal-Katona theorem for monotone Boolean functions (recall Item 1 of Section 1) cannot be improved in general because of functions like the Boolean dictator function \( f(x) = x_1 \). However, if “large single-coordinate influences” are disallowed then stronger forms of the Kruskal-Katona theorem, with larger density increments, hold for monotone Boolean functions. In particular, O’Donnell and Wimmer proved the following “robust” version of the Kruskal-Katona theorem:

**Theorem 35** (Theorem 1.3 of \([OW09]\)). Let \( f : \{\pm 1\}^n \to \{0, 1\} \) be a monotone function and let \( n/10 \leq j \leq 9n/10 \). If \( 1/10 \leq \mu_j(f) \leq 9/10 \) and it holds for all \( i \in [n] \) that

\[
\left| \text{Pr}_{x \sim \{\pm 1\}^n}[f(x) = 1|x_i = 1] - \text{Pr}_{x \sim \{\pm 1\}^n}[f(x) = 1|x_i = -1] \right| \leq \frac{1}{n^{1/10}},
\]

then

\[
\mu_{j+1}(f) \geq \mu_j(f) + \Omega\left(\frac{\log n}{n}\right).
\]

In words, under condition Equation (36) (which is akin to saying that each variable \( x_i \) has “low influence on \( f \)”), the much larger density increment \( \Omega(\log(n)/n) \) must hold.

Using our notion of convex influences, we now establish a robust version of Corollary 34 which is similar in spirit to the Boolean “robust Kruskal-Katona” result given by Theorem 35. Intuitively, our result says that if all convex influences are small, then we get a stronger density increment than Corollary 34:

**Theorem 36.** Let \( K \subseteq \mathbb{R}^n \) be a centrally symmetric convex set and \( \sqrt{n} \leq r = \Theta(\sqrt{n}) \) be such that \( \alpha_K(r) \in [1/10, 9/10] \). If \( \text{Inf}_v[K] \leq \delta \) for all \( v \in S^{n-1} \) then as \( \epsilon \to 0^+ \) we have that

\[
\alpha_K(r(1 - \epsilon)) - \alpha_K(r) = \Omega(\epsilon \sqrt{\ln(1/\delta)}).
\]

**Proof of Theorem 36.** We begin by proving that \( \gamma(K) = \Theta(1) \). Note that

\[
\gamma(K) = \int_{r=0}^{\infty} \alpha_K(r) \cdot \chi_n(r) dr \geq \int_{r=0}^{\sqrt{n}} \alpha_K(r) \cdot \chi_n(r) dr.
\]
where $\chi_n(\cdot)$ is the pdf of the $\chi$-distribution with $n$-degrees of freedom. Now, since $\alpha_K(\cdot)$ is non-increasing and $\int_{r=0}^{\sqrt{n}} \chi_n(r) = \Omega(1)$, it must be the case that

$$\gamma(K) \geq \alpha_K(\sqrt{n}) \cdot \int_{r=0}^{\sqrt{n}} \chi_n(r) dr = \Theta(1), \tag{37}$$

where the last equality uses the fact that $r \geq \sqrt{n}$ and $\alpha_K(r) \geq 1/10$.

Let $r_{in}$ denote the in-radius of $K$. Exactly as reasoned in the proof of Corollary 34, there exists a unit vector $v \in S^{n-1}$ such that $K \subseteq \{ x \in \mathbb{R}^n : |v \cdot x| \leq r_{in} + \zeta \}$ for any $\zeta > 0$. Since $\gamma(K) = \Omega(1)$, it now follows from Proposition 41 that there is a direction $v$ such that

$$\operatorname{Inf}_v[K] = \Omega(e^{-(r_{in}+\zeta)^2}).$$

By our hypothesis, we have that $\operatorname{Inf}_v[K] \leq \delta$, so taking $\zeta \to 0$ we get that $r_{in} = \Omega(\sqrt{\ln(1/\delta)})$ (note that we may assume $\delta$ is at most some sufficiently small constant, since otherwise the claimed result is given by Corollary 34). We now apply Theorem 33 to obtain that

$$\alpha_K(r(1-\varepsilon)) - \alpha_K(r) = \Omega(\varepsilon \sqrt{\ln(1/\delta)}),$$

thus proving Theorem 36. \hfill \square

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A Omitted Proofs from Section 3

A.1 Proof of Proposition 9

We will require the following fact about log-concave functions:
Fact 37 (Lemma 4.7 of [Vem10]). Let $g : \mathbb{R} \to \mathbb{R}^+$ be a log-concave function such that

$$\mathbb{E}_{x \sim \mathcal{N}(0, 1)} [xg(x)] = 0.$$ 

Then $\mathbb{E}[x^2 g(x)] \leq \mathbb{E}[g(x)]$, with equality if and only if $g$ is a constant function.

We will also require the following Brunn-Minkowski-type inequality over Gaussian space, as well as a recent characterization of the equality case:

Proposition 38 (Ehrhard-Borell inequality, [Ehr83, Bor03, Bor08]). Let $A, B \subseteq \mathbb{R}^n$ be Borel sets, identified with their indicator functions. Then

$$\Phi^{-1} \left( \gamma_n (\lambda A + (1 - \lambda)B) \right) \geq \lambda \Phi^{-1} \left( \gamma_n (A) \right) + (1 - \lambda) \Phi^{-1} \left( \gamma_n (B) \right) \tag{38}$$

where $\lambda A + (1 - \lambda)B := \{ \lambda x + (1 - \lambda)y : x \in A, y \in B \}$ is the Minkowski sum of $\lambda A$ and $(1 - \lambda)B$.

Proposition 39 (Theorem 1.2 of [SvH18]). Equality holds in the Ehrhard-Borell inequality (Equation (38)) if and only if either

- $A$ and $B$ are parallel halfspaces, i.e. we have
  $$A = \{ x : \langle a, x \rangle + b_1 \geq 0 \} \quad \text{and} \quad B = \{ x : \langle a, x \rangle + b_2 \geq 0 \}$$
  for some $a \in \mathbb{R}^n$, and $b_1, b_2 \in \mathbb{R}$; or
- $A$ and $B$ are convex sets with $A = B$.

Proof of Proposition 9. Without loss of generality, let $v = e_1$. We have

$$\inf_{e_1} [K] = -\overline{K}(2e_1) = \mathbb{E}_{x \sim \mathcal{N}(0, 1)^n} \left[ -K(x)h_2(x_1) \right]$$

where $g$ is a univariate function. From Fact 1, it follows that $K_{e_1}$ is log-concave, and $K_{e_1}$ is symmetric since $K$ is symmetric, so $\mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)} [h_2(x_1)g(x_1)] = 0$. Hence, using the fact that $h_2(x_1) = (x_1^2 - 1)/\sqrt{2}$, we get that

$$\inf_{e_1} [K] = -\mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)} [h_2(x_1)K_{e_1}(x_1)] = \frac{1}{\sqrt{2}} \mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)} \left[ K_{e_1}(x_1)(1 - x_1^2) \right] \geq 0,$$

where the inequality is by Fact 37.

Next, we move to the characterization of $\inf_{e_1} [K] = 0$. Note that if $K(x) = K(y)$ whenever $x_{e_1}^\perp = y_{e_1}^\perp$ (i.e. $K(x) = K(y)$ whenever $(x_2, \ldots, x_n) = (y_2, \ldots, y_2)$), then the function $K$ does not depend on the variable $x_1$. This lets us re-express Equation (39) as

$$\mathbb{E}_{x_1 \sim \mathcal{N}(0, 1)} \left[ -h_2(x_1) \right] \mathbb{E}_{(x_2, \ldots, x_n) \sim \mathcal{N}(0, 1)^{n-1}} \left[ K(\cdot, x_2, \ldots, x_n) \right].$$
As the first term in the above product is zero, we conclude that $\text{Inf}_{e_1}[K] = 0$.

To see the reverse direction, suppose $\text{Inf}_{e_1}[K] = 0$. From Equation (39) and Fact 37, it follows that $K_{e_1}(\cdot)$ is a constant function. Now, for any $\alpha \in \mathbb{R}$, define $K_\alpha \subseteq \mathbb{R}^{n-1}$ as follows:

$$K_\alpha := \{(x_2, \ldots, x_n) : (\alpha, x_2, \ldots, x_n) \in K\}.$$ 

Thus, $K_\alpha$ is the convex set obtained by intersecting $K$ with the affine plane $\{(x \in \mathbb{R}^n : x_1 = \alpha)\}$. Observe that $K_{e_1}(\alpha)$ is the $(n-1)$-dimensional Gaussian measure of $K_\alpha$. Let $K^*_\alpha := \frac{1}{2}(K_\alpha + K_{-\alpha})$. Note that $K^*_\alpha \subseteq \mathbb{R}^{n-1}$ is a centrally symmetric, convex set, and that $K^*_\alpha \subseteq K_0$ because of convexity. By the Ehrhard-Borell inequality, we have

$$\Phi^{-1}\left(\gamma_{n-1}\left(\frac{K_\alpha + K_{-\alpha}}{2}\right)\right) \geq \frac{1}{2}\Phi^{-1}\left(\gamma_{n-1}(K_\alpha)\right) + \frac{1}{2}\Phi^{-1}\left(\gamma_{n-1}(K_{-\alpha})\right).$$

However, $\gamma_{n-1}(K_\alpha) = \gamma_{n-1}(K_{-\alpha})$ because $K$ is centrally symmetric, so it follows that $\gamma_{n-1}(K^*_\alpha) \geq \gamma_{n-1}(K_\alpha)$. From our earlier observation that $K^*_\alpha \subseteq K_0$ and that $K_{e_1}(\cdot)$ is constant (which implies $\gamma_{n-1}(K_\alpha) = \gamma_{n-1}(K_0)$), it follows that $K_0 = K^*_\alpha$ up to a set of measure zero. In other words, we have equality in the application of the Ehrhard-Borell inequality above, and so by Proposition 39, we must have $K_\alpha = K_{-\alpha}$ (since $K_\alpha$ and $K_{-\alpha}$ cannot be parallel halfspaces). Consequently, up to a set of measure zero, we have that

$$K_0 = K^*_\alpha = \frac{K_\alpha + K_{-\alpha}}{2} = \frac{K_\alpha + K_{-\alpha}}{2} = K_\alpha.$$

As this is true for all $\alpha \in \mathbb{R}$, it follows that up to a set of measure zero, $K(x) = K(y)$ if $(x_2, \ldots, x_n) = (y_2, \ldots, y_n)$ (where we used the fact that $K(x) = K_x(x_2, \ldots, x_n)$).

### A.2 Proof of Proposition 11

We will use the following Brascamp–Lieb-type inequality.

**Lemma 40** (Final assertion of Lemma 4.7 of [Vem10]). If $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is log-concave and symmetric and supported in $[-c, c]$, then

$$\frac{\int_{-c}^{c} x^2 e^{-x^2/2} g(x) dx}{\int_{-c}^{c} e^{-x^2/2} g(x) dx} \leq 1 - \frac{1}{2\pi} e^{-c^2}.$$

We use this in the proof of the following claim, which will easily yield Proposition 11:

**Proposition 41.** Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex set with $\gamma(K) \geq \Delta$, and let $v \in S^{n-1}$ be a unit vector such that $K \subseteq \{x \in \mathbb{R}^n : |v \cdot x| \leq c\}$. Then we have

$$\text{Inf}_v[K] \geq \frac{\Delta e^{-c^2}}{2^{3/2}\pi}.$$

**Proof of Proposition 41.** For ease of notation, we take $v = e_1$ and so $K \subseteq \{x \in \mathbb{R}^n : |x_1| \leq c\}$. From Equations (7) and (8), we have that

$$\text{Inf}_v[K] = \text{Inf}_{e_1}[K] = \text{I}[K_{e_1}] = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} K_{e_1}(x)(1 - x^2)e^{-x^2/2} dx \quad (40)$$
where $K_{e_1} : \mathbb{R} \to [0, 1]$ is the symmetric log-concave function given by

$$K_{e_1}(x) := \mathbf{E}_{x \sim \mathcal{N}(0,1)^{n-1}} \left[ K(x, x_2, \ldots, x_n) \right].$$

As $K(x) = 0$ when $|x_1| > c$ we have that $\text{supp}(g) \subseteq [-c, c]$ and so it follows from Equation (40) that

$$\inf_v[K] = \frac{1}{2\sqrt{\pi}} \int_{-c}^{c} K_{e_1}(x)(1 - x^2)e^{-x^2/2} dx. \quad (41)$$

It follows then from Lemma 40 that

$$\inf_v[K] \geq \frac{1}{2^{3/2}\pi} \left( \frac{e^{-c^2}}{2\pi} \int_{-c}^{c} K_{e_1}(x)e^{-x^2/2} dx \right) = \frac{\Delta e^{-c^2}}{2^{3/2}\pi}$$

which completes the proof of Proposition 41.

**Proof of Proposition 11.** By definition of the in-radius and the supporting hyperplane theorem, there must exist some unit vector $\hat{v} \in \mathbb{R}^n$ such that

$$K \subseteq K_+ := \{ x \in \mathbb{R}^n : |\hat{v} \cdot x| \leq r_{in} \},$$

and hence by Proposition 41 we get that

$$\inf_{\hat{v}}[K] \geq \frac{\gamma(K)e^{-r_{in}^2}}{2^{3/2}\pi} \geq \frac{\text{Var}[K]e^{-r_{in}^2}}{2^{3/2}\pi},$$

giving Proposition 11 as claimed.

**A.3 Proof of Proposition 16**

This is an elementary calculation using the chain rule and the Leibniz rule.

**Proof of Proposition 16.** By the chain rule, we have

$$\frac{d}{d\sigma^2} \mathbf{E}_{x \sim \mathcal{N}(0,\sigma^2)^n} [K(x)] = \frac{d}{d\sigma^2} \left( \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \int_K \exp \left( -\frac{\|x\|^2}{2\sigma^2} \right) dx \right) =: A(\sigma^2)$$

$$= A(\sigma^2) \frac{d}{d\sigma^2} \left( \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \right) + \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \left( \frac{d}{d\sigma^2} A(\sigma^2) \right)$$

$$= -nA(\sigma^2) + \frac{1}{2\sigma^2 \sqrt{(2\pi\sigma^2)^n}} \left( \frac{d}{d\sigma^2} A(\sigma^2) \right)$$

$$= -\frac{n\mathbf{E}_{x \sim \mathcal{N}(0,\sigma^2)^n} [K(x)]}{2\sigma^2} + \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \left( \frac{d}{d\sigma^2} A(\sigma^2) \right).$$

31
Now, using the dominated convergence theorem to commute integration and differentiation, we get

\[
\frac{1}{\sqrt{(2\pi\sigma^2)^n}} \left( \frac{d}{d\sigma^2} A(\sigma^2) \right) = \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \left( \frac{d}{d\sigma^2} \int_K \exp \left( -\frac{\|x\|^2}{2\sigma^2} \right) dx \right)
\]

\[
= \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \int_K \frac{d}{d\sigma^2} \exp \left( -\frac{\|x\|^2}{2\sigma^2} \right) dx
\]

\[
= \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \int_K \exp \left( -\frac{\|x\|^2}{2\sigma^2} \right) \frac{\|x\|^2}{2\sigma^4} dx
\]

\[
= \frac{1}{2\sigma^2} \sqrt{\frac{2}{\pi}} \mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \left( \frac{\|x\|^2}{\sigma^4} \right) \right]
\]

This in turn implies that

\[
\frac{d}{d\sigma^2} \mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \right] = -\frac{n}{2\sigma^2} \mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \right] + \frac{\mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \left( \frac{\|x\|^2}{\sigma^4} \right) \right]}{2\sigma^2}
\]

\[
= \frac{1}{\sigma^2 \sqrt{2}} \mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \left( \frac{\|x\|^2}{\sigma^4} - \frac{n}{\sqrt{2}} \right) \right]
\]

\[
= \frac{1}{\sigma^2 \sqrt{2}} \sum_{i=1}^{n} \mathbb{E}_{x \sim \mathcal{N}(0, \sigma^2)^n} \left[ K(x) \left( \frac{(x_i)^2}{\sigma^4} - \frac{1}{\sqrt{2}} \right) \right]
\]

\[
= \frac{1}{\sigma^2 \sqrt{2}} \sum_{i=1}^{n} \tilde{K}_{\sigma}(2\sigma_i)
\]

where we used the fact that \( h_2(x) \frac{\sigma^2}{\sqrt{2}}. \)