Abstract: Let \( A \) be any unital \(*\)-algebra over the real or complex field \( F \), and let \( \xi \in F \) with \( \xi \neq 1 \). Assume that \( \Phi: A \to A \) is a map. It is shown that, \( \Phi \) satisfies \( \Phi(A)\Phi(B) - \xi \Phi(B)\Phi(A) = AB - \xi BA \) for all \( A, B \in A \) if and only if \( \Phi(I) \in Z(A) \), the center of \( A \), \( \Phi(I)^2 = I \) and \( \Phi(A) = \Phi(I)A \) for all \( A \in A \); if \( \Phi(I) = \Phi(I)^* \), then \( \Phi \) satisfies \( \Phi(A)\Phi(B) - \xi \Phi(B)\Phi(A)^* = AB - \xi BA^* \) for all \( A, B \in A \) if and only if \( \Phi(I) \in Z(A) \), \( \Phi(I)^2 = I \) and \( \Phi(A) = \Phi(I)A \) for all \( A \in A \); if \( |\xi| = 1 \) and \( \Phi \) is surjective, then \( \Phi \) satisfies \( \Phi(A)\Phi(B) - \xi \Phi(B)\Phi(A)^* = AB - \xi BA^* \) for all \( A, B \in A \) if and only if \( \Phi(I) = \Phi(I)^* \in Z(A) \), \( \Phi(I)^2 = I \), and \( \Phi(A) = \Phi(I)A \) for all \( A \in A \).

1. Introduction

Let \( R \) be a ring. Then \( R \) is a Lie ring under the Lie product \( [A, B] = AB - BA \). Recall that a map \( \Phi: R \to R \) preserves commutativity if \( \Phi([A, B]) = 0 \) whenever \( [A, B] = 0 \) for all \( A, B \in R \). The problem of characterizing linear (additive) maps preserving commutativity had been studied intensively on various rings and algebras (see [Brešar, 1993; Brešar & Šemrl, 2005; Choi, Jafarian, & Radjavi, 1987]) and the references therein).

In Bell and Daif (1994), the authors gave the conception of strong commutativity preserving maps. Let \( S \subseteq R \). A map \( \Phi: S \to R \) is called strong commutativity preserving if \( \Phi([T, S]) = [T, \Phi(S)] \) for all \( T, S \in S \). Note that a strong commutativity preserving map must be commutativity preserving, but the inverse is not true generally. Bell and Daif (1994) proved that \( R \) must be commutative, if \( R \) is a prime ring and \( R \) admits a derivation or a non-identity endomorphism which is strong commutativity preserving on a right ideal of \( R \). Brešar and Miers (1994) proved that every strong commutativity preserving additive map \( \Phi \) on a semiprime ring \( R \) is of the form \( \Phi(A) = \lambda A + \mu(A) \), where \( \lambda \in C \), the extended centroid of \( R \), \( \lambda^2 = 1 \), and \( \mu: R \to C \) is an additive map. Recently, Lin and Liu (2008) obtained the similar result on a noncentral Lie ideal of a prime ring. Qi and Hou (2010; 2012) gave a complete characterization of strong commutativity preserving surjective maps (without the assumption of additivity) on prime rings and triangular algebras, respectively.

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PUBLIC INTEREST STATEMENT

Preserver problem had attracted many mathematicians’ attentions for many years. In this paper, the authors discuss the general maps preserving strong (skew) \( \xi \)-Lie commutativity on any algebras and give a complete characterization for such maps.
Let $R$ be a $*$-ring. For any $A, B \in R, [A, B] = AB - BA^*$ denotes the skew Lie product of $A$ and $B$.

This kind of product is found playing a more and more important role in some research topics such as representing quadratic functionals with sesquilinear functionals, and its study has attracted many authors’ attention (see [Brešar & Fosňar, 2000; Chebotar, Fong, & Lee, 2005; Cui & Hou, 2006] and the reference therein). Molnár (1996) initiated the systematic study of this skew Lie product, and studied the relation between subspaces and ideals of $B(H)$, the algebra of all bounded linear operators acting on a Hilbert space $H$.

Recall that a map $\Phi : R \to R$ is called zero skew Lie product preserving, if $\Phi(A)\Phi(B) - \Phi(B)\Phi(A)^* = 0$ whenever $AB - BA^* = 0$ for any $A, B \in R$. Additive or linear maps preserving zero skew Lie products on various rings and algebras had been studied by many authors (see, Bell & Daif, 1994 and the references therein). More specially, $\Phi$ is strong skew commutativity preserving, if $[\Phi(A), \Phi(B)]_c = [A, B]$, for all $A, B \in R$. It is obvious that strong skew commutativity preserving maps must be zero skew Lie product preserving. However, the inverse is not true generally. In Cui and Park (2012), they proved that, if $R$ is a factor von Neumann algebra, then every strong skew commutativity preserving map $\Phi$ on $R$ has the form $\Phi(A) = \Psi(A) + h(A)I$ for all $A \in R$, where $\Psi : R \to R$ is a linear bijective map satisfying $\Psi(A)\Psi(B) - \Psi(B)\Psi(A)^* = AB - BA^*$ for all $A, B \in R$ and is a real linear functional of $R$ with $h(0) = 0$; particularly, if $R$ is of type I, then $\Phi(A) = CA + h(A)I$ for each $A \in R$, where $c \in \{-1, 1\}$. Recently, Qi and Hou (2013) generalized the above result to von Neumann algebras without central summand of type $I_1$.

Recall that $A$ commutes with $B$ up to a factor $\xi \in F$ if $AB = \xi BA$. Note that the concept of commutativity up to a factor for pairs of operators is important and has been studied in the context of operator algebras and quantum groups (see Brooke, Busch, & Pearson, 2002 and Kassel, 1995). Motivated by this, a binary operation $[A, B] = AB - \xi BA$, called $\xi$-Lie product of $A$ and $B$, was introduced in Qi and Hou (2009). Thus, we also can define the skew $\xi$-Lie product of $A$ and $B$. Let $A$ be a $*$-algebra over $F$, where $F$ is a field with an involution $\xi$. For $A, B \in A$ and $\xi \in F$, we call $AB - \xi BA^*$ the skew $\xi$-Lie product of $A$ and $B$. It is obvious that the skew $\xi$-Lie product is the skew Lie product if $\xi = 1$. Now, based on these concepts, we say that a map $\Phi : A \to A$ is preserving strong $\xi$-Lie commutativity if $[\Phi(T), \Phi(S)]_c = [T, S]$, for all $T, S \in A$; is preserving strong skew $\xi$-Lie commutativity if $\Phi(T)\Phi(S) - \xi\Phi(S)\Phi(T)^* = TS - \xi ST^*$ for all $T, S \in A$.

The purpose of this paper is to consider nonlinear strong (skew) $\xi$-Lie commutativity preserving maps on general algebras with $\xi \neq 1$. Let $A$ be any unital algebra over any field $F$ and $\xi \in F$ with $\xi \neq 1$. Denote by $\mathcal{Z}(A)$ the center of $A$. Assume that $\Phi : A \to A$ is a map. In Section 2, we prove that $\Phi$ preserves strong $\xi$-Lie commutativity if and only if $\Phi(I) \in \mathcal{Z}(A)$, $\Phi(I)^2 = I$, and $\Phi(A) = \Phi(IA)$ for all $A \in A$ (Theorem 2.1). In Section 3, we furthermore assume that $A$ is a $*$-algebra. It is shown that, if $\Phi(I) = \Phi(I)^*$, then $\Phi$ preserves strong skew $\xi$-Lie commutativity if and only if $\Phi(I) \in \mathcal{Z}(A)$, $\Phi(I)^2 = I$ and $\Phi(A) = \Phi(IA)$ for all $A \in A$ (Theorem 3.1); if $|\xi| = 1$ and $\Phi$ is surjective, then $\Phi$ preserves strong skew $\xi$-Lie commutativity if and only if $\Phi(I) = \Phi(I)^* \in \mathcal{Z}(A)$, $\Phi(I)^2 = I$ and $\Phi(A) = \Phi(IA)$ for all $A \in A$ (Theorem 3.2).

2. Maps preserving strong $\xi$-Lie commutativity

In this section, we will give a characterization of nonlinear strong $\xi$-Lie commutativity preserving maps on general algebras. The following is our main result.

\textbf{THEOREM 2.1} Let $A$ be any algebra with unit $I$ over a field $F$, and let $\xi \in F$ with $\xi \neq 1$. Assume that $\Phi : A \to A$ is a map. Then $\Phi$ preserves strong $\xi$-Lie commutativity, that is, $\Phi$ satisfies $[\Phi(A), \Phi(B)]_c = [A, B]_c$ for all $A, B \in A$, if and only if $\Phi(I) \in \mathcal{Z}(A)$, $\Phi(I)^2 = I$ and $\Phi(A) = \Phi(IA)$ for all $A \in A$.

\textbf{Proof} The “if” part is obvious. For the “only if” part, since $[\Phi(I), \Phi(I)]_c = [I, I]_c$, we have $(1 - \xi)\Phi(I)^2 = (1 - \xi)I$. It follows that $\Phi(I)^2 = I$ as $\xi \neq 1$. 

In the sequel, we will complete the proof by considering two cases.

Case 1: \( \xi = -1 \).

Take any \( A \in \mathcal{A} \). Then

\[
2A = AI + IA = \Phi(A)\Phi(I) + \Phi(I)\Phi(A)
\]

Multiplying \( \Phi(I) \) from the left- and the right-hand side in Equation 2.1, respectively, one gets

\[
2\Phi(I)A = \Phi(I)\Phi(A)\Phi(I) + \Phi(I)\Phi(A) = \Phi(I)\Phi(A)\Phi(I) + \Phi(A)
\]

and

\[
2\Phi(I) = \Phi(A)\Phi(I)\Phi(A) + \Phi(I)\Phi(A) = \Phi(A) + \Phi(I)\Phi(A)\Phi(I)
\]

Comparing the above two equations, we obtain \( \Phi(I)A = A\Phi(I) \) for each \( A \in \mathcal{A} \). It follows from the arbitrariness of \( A \in \mathcal{A} \) that \( \Phi(I) \in \mathcal{Z}(\mathcal{A}) \). This and Equation 2.1 imply \( \Phi(I)\Phi(A) = A \). Note that \( \Phi(I)^2 = I \). So \( \Phi(A) = \Phi(I)A \) holds for all \( A \in \mathcal{A} \), completing the proof of the theorem.

Case 2: \( \xi \neq -1 \).

Take any \( A, B \in \mathcal{A} \) and note that \( (1 - \xi)[A, B]_{\xi} = [A, B]_{\xi} + [B, A]_{\xi} \). Since \( \Phi \) preserves strong \( \xi \)-Lie commutativity, we have

\[
(1 - \xi)[A, B]_{\xi} = [\Phi(A), \Phi(B)]_{\xi} + [\Phi(B), \Phi(A)]_{\xi} = (1 - \xi)(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)).
\]

That is,

\[
AB + BA = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)
\]

holds for all \( A, B \in \mathcal{A} \). Now, by Case 1, the theorem is true.

Combining Case 1 and Case 2, the proof of the theorem is complete.

3. Maps preserving strong skew \( \xi \)-Lie commutativity

In this section, we will discuss the maps preserving strong skew \( \xi \)-Lie commutativity on general algebras.

**Theorem 3.1** Let \( \mathcal{A} \) be any \( ^* \)-algebra with unit \( I \) over the real or complex field \( \mathbb{F} \) and let \( \xi \in \mathbb{F} \) with \( \xi \neq 1 \). Assume that \( \Phi: \mathcal{A} \to \mathcal{A} \) is a map. If \( \Phi(I) = \Phi(I)^* \), then \( \Phi \) preserves strong skew \( \xi \)-Lie commutativity, that is, \( \Phi \) satisfies \( \Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)^* = AB - \xi BA^* \) for all \( A, B \in \mathcal{A} \), if and only if \( \Phi(I) \in \mathcal{Z}(\mathcal{A}) \), \( \Phi(I)^2 = I \) and \( \Phi(A) = \Phi(I)A \) for all \( A \in \mathcal{A} \).

**Proof** Still, one only needs to prove the “only if” part.

By the assumption, for any \( A, B \in \mathcal{A} \), we have

\[
\Phi(A)\Phi(B) - \xi\Phi(B)\Phi(A)^* = AB - \xi BA^*
\]

Taking \( A = B = I \) in Equation (3.1), one gets \( \Phi(I)^2 - \xi\Phi(I)\Phi(I)^* = (1 - \xi)I \). Note that \( \Phi(I) = \Phi(I)^* \) and \( \xi \neq 1 \). We obtain
Taking $A = I$ in Equation 3.1, one gets $\Phi(I)\Phi(\xi) - \xi \Phi(I)\Phi(I)^* = (1 - \xi)B$, that is,

$$\Phi(I)\Phi(A) - \xi \Phi(A)\Phi(I) = (1 - \xi)A \text{ for all } A \in \mathcal{A} \tag{3.3}$$

Taking $B = I$ in Equation 3.1, one has

$$\Phi(A)\Phi(I) - \xi \Phi(I)\Phi(A)^* = A - \xi A^* \tag{3.4}$$

This implies

$$\Phi(I)^*\Phi(A)^* - \xi \Phi(A)\Phi(I)^* = A^* - \xi A \text{ for all } A \in \mathcal{A}$$

Multiplying $\xi$ from both sides in the above equation, we get

$$\xi \Phi(I)^*\Phi(A)^* - |\xi|^2\Phi(A)\Phi(I)^* = \xi A^* - |\xi|^2A \text{ for all } A \in \mathcal{A} \tag{3.5}$$

Combining Equations 3.4 and 3.5, we have

$$\Phi(A)\Phi(I) - |\xi|^2\Phi(I)^* - \xi (\Phi(I) - \Phi(I)^*)\Phi(A)^* = (1 - |\xi|^2)A \text{ for all } A \in \mathcal{A} \tag{3.6}$$

Note that $\Phi(I)^* = \Phi(I)^\dagger$. Equation 3.6 implies

$$(1 - |\xi|^2)\Phi(A)\Phi(I) = (1 - |\xi|^2)A \text{ for all } A \in \mathcal{A} \tag{3.7}$$

In the following, we will prove the theorem by two cases.

Case 1 $|\xi| \neq 1$.

In this case, Equation 3.7 implies

$$\Phi(A)\Phi(I) = A \text{ for all } A \in \mathcal{A} \tag{3.8}$$

Multiplying $\Phi(I)$ from the right-hand side in Equation 3.8, by Equation (3.2), one gets

$$\Phi(A) = A\Phi(I) \text{ for all } A \in \mathcal{A} \tag{3.9}$$

On the other hand, combining Equations 3.3 and 3.8, one has $\Phi(I)\Phi(A) = A$. Multiplying $\Phi(I)$ from the left-hand side in this equation and noting that Equation (3.2), one gets

$$\Phi(A) = \Phi(I)A \text{ for all } A \in \mathcal{A} \tag{3.10}$$

It follows from Equations 3.9 to 3.10 that $\Phi(I) \in \mathcal{Z}$. The proof is finished.

Case 2 $|\xi| = 1$.

Multiplying $\Phi(I)$ from the left- and the right-hand side in Equation 3.3, respectively, by Equation 3.2 again, one can obtain

$$\Phi(A) - \xi \Phi(I)\Phi(A)\Phi(I) = (1 - \xi)\Phi(I)A$$

and
\(\Phi(I)\Phi(A)\Phi(I) - \xi \Phi(A) = (1 - \xi)A\Phi(I)\)

Comparing the above two equations gets \((1 - \xi^2)\Phi(A) - \xi(1 - \xi)A\Phi(I) = (1 - \xi)\Phi(I)A\), that is,

\[
(1 + \xi)\Phi(A) = \xi A\Phi(I) + \Phi(I)A \quad \text{holds for all } A \in \mathcal{A} \tag{3.11}
\]

We claim \(A\Phi(I) = \Phi(I)A\), and so \(\Phi(I) \in \mathcal{Z}(\mathcal{A})\). In fact, if \(\xi = -1\), Equation (3.11) implies \(A\Phi(I) = \Phi(I)A\); if \(\xi \neq -1\), multiplying \(\xi\) from both sides in Equation (3.11), one has

\[
(1 + \xi)\Phi(A) = \xi A\Phi(I) + \Phi(I)A \quad \text{for all } A \in \mathcal{A} \tag{3.12}
\]

as \(|\xi| = 1\). On the other hand, Equation 3.4 implies \((1 + \xi)\Phi(A)\Phi(I) = \xi(1 + \xi)\Phi(I)\Phi(A)^* + (1 + \xi)A - \xi(1 + \xi)A^*\). This and Equations 3.11–3.12 yield

\[
(\xi A\Phi(I) + \Phi(I)A)\Phi(I) = \xi \Phi(I)(A\Phi(I) + \xi \Phi(I)A)^* + (1 + \xi)A - \xi(1 + \xi)A^*
\]

Note that \(\Phi(I) = \Phi(I)^*\) and Equation 3.2. The above equation can be reduced to

\[
\Phi(I)\Phi(A) = \xi^2 \Phi(I)A^* \Phi(I) + A - \xi^2 A^*
\]

Multiplying \(\Phi(I)\) from the right side in the above equation yields

\[
\Phi(I)A - A\Phi(I) = \xi^2(\Phi(I)A^* - A^* \Phi(I)) \quad \text{for all } A \in \mathcal{A} \tag{3.13}
\]

Replacing \(A\) by \(iA\) in Equation (3.13), one can get \(\Phi(I)(iA) - (iA)\Phi(I) = \xi^2(\Phi(I)(iA)^* - (iA)^* \Phi(I))\), that is,

\[
\Phi(I)A - A\Phi(I) = \xi^2(-\Phi(I)A^* + A^* \Phi(I)) \quad \text{for all } A \in \mathcal{A} \tag{3.14}
\]

Combining Equations 3.13 and 3.14, one achieves \(\Phi(I)A = A\Phi(I)\) for all \(A \in \mathcal{A}\). The claim holds.

Now, it follows from Equation 3.3 that \(\Phi(I)\Phi(A) = \Phi(A)\Phi(I) = A\), and so \(\Phi(A) = \Phi(I)A = A\Phi(I)\) holds for all \(A \in \mathcal{A}\). The theorem holds.

We complete the proof of the theorem.

If \(\Phi\) is surjective and \(|\xi| = 1\), then the condition \(\Phi(I) = \Phi(I)^*\) in Theorem 3.1 can be deleted.

**THEOREM 3.2** Let \(\mathcal{A}\) be any *-algebra with unit \(I\) over the real or complex field \(\mathbb{F}\), and let \(\xi \in \mathbb{F}\) with \(\xi \neq 1\) and \(|\xi| = 1\). Assume that \(\Phi:\mathcal{A} \rightarrow \mathcal{A}\) is a surjective map. Then \(\Phi\) preserves strong skew \(\xi\)-Lie commutativity if and only if \(\Phi(I) = \Phi(I)^* \in \mathcal{Z}(\mathcal{A})\), \(\Phi(I)^2 = I\) and \(\Phi(A) = \Phi(I)A\) for all \(A \in \mathcal{A}\).

**Proof** By Theorem 3.1, to complete the proof of the theorem, one only needs to prove \(\Phi(I) = \Phi(I)^*\).

Indeed, by checking the proof of Theorem 3.1, Equation 3.6 still holds, that is,

\[
\Phi(A)(\Phi(I) - |\xi|^2 \Phi(I)^*) - \xi(\Phi(I) - \Phi(I)^*)\Phi(A)^* = (1 - |\xi|^2)A \quad \text{holds for all } A \in \mathcal{A}
\]

Since \(|\xi| = 1\), the above equation reduces to \(\Phi(A)(\Phi(I) - \Phi(I)^*) - \xi(\Phi(I) - \Phi(I)^*)\Phi(A)^* = 0\) for all \(A \in \mathcal{A}\). As \(\Phi\) is surjective, there exists \(A \in \mathcal{A}\) such that \(\Phi(A) = I\). So \(\Phi(I) - \Phi(I)^* = \xi(\Phi(I) - \Phi(I)^*)\). It follows from the fact \(\xi \neq 1\) that \(\Phi(I) = \Phi(I)^*\).
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