Abstract

Spaces of differential forms over configuration spaces with Poisson measures are constructed. The corresponding Laplacians (of Bochner and de Rham type) on forms and associated semigroups are considered. Their probabilistic interpretation is given.

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1 Introduction

Stochastic differential geometry of infinite-dimensional manifolds has been a very active topic of research in recent times. One of the important and intriguing problems discussed concerns the construction of spaces of differential forms over such manifolds and the study of the corresponding Laplace operators and associated (stochastic) cohomologies. A central role in this framework is played by the concept of the Dirichlet operator of a differentiable measure, which is actually an infinite-dimensional generalization of the Laplace–Beltrami operator on functions, respectively the Laplace–Witten–de Rham operator on differential forms. The study of the latter operator and the associated semigroup on finite-dimensional manifolds was the subject of many works, and it leads to deep results on the interface of stochastic analysis, differential geometry and topology, and mathematical physics, see, e.g., [20], [21], [17], [35], [16]. Dirichlet forms and processes in connection with noncommutative $C^*$-algebras were considered in e.g. [23, 4, 18].

The interest in the infinite-dimensional case is motivated by relations with supersymmetric quantum field theory. De Rham type operators acting on differential forms over Hilbert spaces were considered in [11], [12], [13], [5]. In this relation, the mostly discussed example of an infinite-dimensional non-flat space is the loop space of a compact manifold, see [37], [25], [27]. Another important example given by the infinite product of compact manifolds was discussed in [1], [2], [14].

At the same time, there is a growing interest in geometry and analysis on Poisson spaces, i.e., on spaces of locally finite configurations in noncompact manifolds equipped with the Poisson measure. In [6], [7], [8], an approach to these spaces as to infinite-dimensional manifolds was initiated. This approach is motivated by the connection of such spaces with the theory of representations of diffeomorphism groups, see [22], [36], [24] (these references and [8], [10] also contain discussion of relations with quantum physics). In fact, the configuration
space, which does not possess the structure of a smooth manifold in the proper sense, can be equipped with some “Riemannian-like” structure generated by the action of the diffeomorphism group of the initial manifold. We refer the reader to [9], [10], [33], and references therein for further discussion of analysis on Poisson spaces and applications.

In the present work, we develop this point of view. We define spaces of differential forms over Poisson spaces and study Laplace operators acting in these spaces. We show, in particular, that the corresponding de Rham Laplacian can be expressed in terms of the Dirichlet operator on functions on the Poisson space and the Witten Laplacian on the initial manifold associated with the intensity of the corresponding Poisson measure. We give a probabilistic interpretation and investigate some properties of the associated semigroups.

The main general aim of our approach is to develop a framework which extends to Poisson spaces (as infinite-dimensional manifolds) the finite-dimensional Hodge–de Rham theory.

The results of the present paper in the special case of 1-forms were presented in [3].

A different approach to the construction of differential forms and related objects over Poisson spaces, based on the “transfer principle” from Wiener spaces, is proposed in [30], see also [28] and [29].

2 Differential forms over configuration spaces

The aim of this section is to define differential forms over configuration spaces (as infinite-dimensional manifold). First, we recall some known facts and definitions concerning “manifold-like” structures and functional calculus on these spaces.

2.1 Functional calculus on configuration spaces

Our presentation in this subsection is based upon [8], however for later use in the present paper we give a different description of some objects and results occurring in [8].

Let $X$ be a complete, connected, oriented, $C^\infty$ (noncompact) Riemannian manifold of dimension $d$. We denote by $\langle \bullet, \bullet \rangle_x$ the corresponding inner product in the tangent space $T_x X$ to $X$ at a point $x \in X$. The associated norm will be denoted by $| \bullet |_x$. Let also $\nabla^X$ stand for the gradient on $X$.

The configuration space $\Gamma_X$ over $X$ is defined as the set of all locally finite subsets (configurations) in $X$:

$$\Gamma_X := \{ \gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X \}.$$  

Here, $|A|$ denotes the cardinality of the set $A$.

We can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure $\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(X)$, where $\varepsilon_x$ is the Dirac measure with mass at $x$, $\sum_{x \in \emptyset} \varepsilon_x$ := zero measure, and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on the Borel $\sigma$-algebra $\mathcal{B}(X)$. The space $\Gamma_X$ is endowed
with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on $\Gamma_X$ such that all maps

$$\Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x) \gamma(dx) \equiv \sum_{x \in \gamma} f(x)$$

are continuous. Here, $f \in C_0(X)$ (the set of all continuous functions on $X$ with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel $\sigma$-algebra.

Following [8], we define the tangent space to $\Gamma_X$ at a point $\gamma$ as the Hilbert space $T_\gamma \Gamma_X := L^2(X \to TX; d\gamma)$, or equivalently

$$T_\gamma \Gamma_X = \bigoplus_{x \in \gamma} T_x X \quad (2.1)$$

(compare also with [36, Appendix 3]). The scalar product and the norm in $T_\gamma \Gamma_X$ will be denoted by $\langle \bullet, \bullet \rangle_\gamma$ and $\|\bullet\|_\gamma$, respectively. Thus, each $V(\gamma) \in T_\gamma \Gamma_X$ has the form $V(\gamma) = (V(\gamma)_x)_{x \in \gamma}$, where $V(\gamma)_x \in T_x X$, and

$$\|V(\gamma)\|_\gamma^2 = \sum_{x \in \gamma} |V(\gamma)_x|^2_x.$$

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $\mathcal{O}_{\gamma,x}$ we will denote an arbitrary open neighborhood of $x$ in $X$ such that the intersection of the closure of $\mathcal{O}_{\gamma,x}$ in $X$ with $\gamma \setminus \{x\}$ is the empty set. For any fixed finite subconfiguration $\{x_1, \ldots, x_k\} \subset \gamma$, we will always consider open neighborhoods $\mathcal{O}_{\gamma,x_1}, \ldots, \mathcal{O}_{\gamma,x_k}$ with disjoint closures.

Now, for a measurable function $F: \Gamma_X \to \mathbb{R}$, $\gamma \in \Gamma_X$, and $\{x_1, \ldots, x_k\} \subset \gamma$, we define a function $F_{x_1, \ldots, x_k}(\gamma, \bullet): \mathcal{O}_{\gamma,x_1} \times \cdots \times \mathcal{O}_{\gamma,x_k} \to \mathbb{R}$ by

$$\mathcal{O}_{\gamma,x_1} \times \cdots \times \mathcal{O}_{\gamma,x_k} \ni (y_1, \ldots, y_k) \mapsto F_{x_1, \ldots, x_k}(\gamma, y_1, \ldots, y_k) := F((\gamma \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}) \in \mathbb{R}.$$ 

Since we will be interested only in the local behavior of the function $F_{x_1, \ldots, x_k}(\gamma, \bullet)$ around the point $(x_1, \ldots, x_k)$, we will not write explicitly which neighborhoods $\mathcal{O}_{\gamma,x}$ we use.

**Definition 2.1** We say that a function $F: \Gamma_X \to \mathbb{R}$ is differentiable at $\gamma \in \Gamma_X$ if for each $x \in \gamma$ the function $F_x(\gamma, \bullet)$ is differentiable at $x$ and

$$\nabla^X F(\gamma) = (\nabla^X F(\gamma)_x)_{x \in \gamma} \in T_\gamma \Gamma_X,$$

where

$$\nabla^X F(\gamma)_x := \nabla^X F_x(\gamma, x).$$

We will call $\nabla^X F(\gamma)$ the gradient of $F$ at $\gamma$. 


For a function $F$ differentiable at $\gamma$ and a vector $V(\gamma) \in T_\gamma \Gamma_X$, the directional derivative of $F$ at the point $\gamma$ along $V(\gamma)$ is defined by

$$\nabla_{\gamma}^V F(\gamma) := \langle \nabla^V F(\gamma), V(\gamma) \rangle_{\gamma}.$$ 

In what follows, we will also use the shorthand notation

$$\nabla^X F(\gamma) := \nabla^X F_x(\gamma, x), \quad (2.2)$$

so that

$$\nabla^\Gamma F(\gamma) = (\nabla^X F(\gamma))_{x \in \gamma}.$$ 

It is easy to see that the operation $\nabla^\Gamma$ satisfies the usual properties of differentiation, including the Leibniz rule.

We define a class $\mathcal{F}C$ of smooth cylinder functions on $\Gamma_X$ as follows:

**Definition 2.2** A measurable bounded function $F : \Gamma_X \rightarrow \mathbb{R}$ belongs to $\mathcal{F}C$ iff:

(i) there exists a compact $\Lambda \subset X$ such that $F(\gamma) = F(\gamma_{\Lambda})$ for all $\gamma \in \Gamma_X$, where $\gamma_{\Lambda} := \gamma \cap \Lambda$;

(ii) for any $\gamma \in \Gamma_X$ and $\{x_1, \ldots, x_k\} \subset \gamma$, $k \in \mathbb{N}$, the function $F_{x_1,\ldots,x_k}(\gamma, \bullet)$ is infinitely differentiable with partial derivatives uniformly bounded in $\gamma$ and $x_1, \ldots, x_k$ (i.e., the majorizing constant depends only on the order of differentiation but not on the specific choice of $\gamma \in \Gamma_X$, $k \in \mathbb{N}$, and $\{x_1, \ldots, x_k\} \subset \gamma$).

Let us note that, for $F \in \mathcal{F}C$, only a finite number of coordinates of $\nabla^\Gamma F(\gamma)$ are not equal to zero, and so $\nabla^\Gamma F(\gamma) \in T_\gamma \Gamma_X$. Thus, each $F \in \mathcal{F}C$ is differentiable at any point $\gamma \in \Gamma_X$ in the sense of Definition 2.1.

**Remark 2.1** In [8], the authors introduced the class $\mathcal{F}C_b^\infty(D, \Gamma_X)$ of functions on $\Gamma_X$ of the form

$$F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle), \quad (2.3)$$

where $g_F \in C_b^\infty(\mathbb{R}^N)$ and $\varphi_1, \ldots, \varphi_N \in \mathcal{D} := C^\infty(\mathbb{R})(:= \text{the set of all } C^\infty\text{-functions on } X \text{ with compact support})$. Evidently, we have the inclusion

$$\mathcal{F}C_b^\infty(D, \Gamma_X) \subset \mathcal{F}C,$$

and moreover, the gradient of $F$ of the form (2.3) in the sense of Definition 2.1,

$$\nabla^\Gamma F(\gamma)_x = \sum_{i=1}^N \frac{\partial g_F}{\partial s_i}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \nabla^X \varphi_i(x),$$

coincides with the gradient of this function in the sense of [8].
2.2 Tensor bundles and cylinder forms over configuration spaces

Our next aim is to introduce differential forms on $\Gamma_X$.

Vector fields and first order differential forms on $\Gamma_X$ will be identified with sections of the bundle $TTX$. Higher order differential forms will be identified with sections of tensor bundles $\wedge^n(TTX)$ with fibers

$$\wedge^n(T_\gamma \Gamma_X) := \wedge^n \left( L^2(X \to TX; \gamma) \right),$$

where $\wedge^n(\mathcal{H})$ (or $\mathcal{H}^{\wedge n}$) stands for the $n$-th antisymmetric tensor power of a Hilbert space $\mathcal{H}$. In what follows, we will use different representations of this space. Because of (2.1), we have

$$\wedge^n(T_\gamma \Gamma_X) = \wedge^n \left( \bigoplus_{x \in \gamma} T_x X \right). \quad (2.4)$$

Let us introduce the factor space $X^n/S_n$, where $S_n$ is the permutation group of $\{1, \ldots, n\}$ which naturally acts on $X^n$:

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$ 

The space $X^n/S_n$ consists of equivalence classes $[x_1, \ldots, x_n]$ and we will denote by $[x_1, \ldots, x_n]_d$ an equivalence class $[x_1, \ldots, x_n]$ such that the equality $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ can hold only for $k \leq d$ points. (In other words, any equivalence class $[x_1, \ldots, x_n]$ is a multiple configuration in $X$, while $[x_1, \ldots, x_n]_d$ is a multiple configuration with multiplicity of points $\leq d$.) In what follows, instead of writing $[x_1, \ldots, x_n] : \{x_1, \ldots, x_n\} \subset \gamma$, we will use the shortened notation $[x_1, \ldots, x_n]_d \subset \gamma$, though $[x_1, \ldots, x_n]_d$ is not, of course, a set. We then have from (2.4):

$$\wedge^n(T_\gamma \Gamma_X) = \bigoplus_{[x_1, \ldots, x_n]_d \subset \gamma} T_{x_1} X \wedge T_{x_2} X \wedge \cdots \wedge T_{x_n} X. \quad (2.5)$$

Here, the space $T_{x_1} X \wedge T_{x_2} X \wedge \cdots \wedge T_{x_n} X$ is understood as a subspace of the Hilbert space $(T_{y_1} X \oplus T_{y_2} X \oplus \cdots \oplus T_{y_k} X)^{\otimes n}$, where $\{y_1, \ldots, y_k\} \subset \gamma$.

To see that (2.5) holds, notice that

$$(T_{y_1} X \oplus T_{y_2} X \oplus \cdots \oplus T_{y_k} X)^{\otimes n} \simeq \left( T_{y_{\nu(1)}} X \oplus T_{y_{\nu(2)}} X \oplus \cdots \oplus T_{y_{\nu(k)}} X \right)^{\otimes n}, \quad \nu \in S_k \quad (2.6)$$

(where $\simeq$ means isomorphism), and moreover $T_{x_1} X \wedge T_{x_2} X \wedge \cdots \wedge T_{x_n} X$ and $T_{x_{\nu(1)}} X \wedge T_{x_{\nu(2)}} X \wedge \cdots \wedge T_{x_{\nu(n)}} X$, $\sigma \in S_n$, coincide as subspaces of the space (2.6).

Thus, under a differential form $W$ of order $n$, $n \in \mathbb{N}$, over $\Gamma_X$, we will understand a mapping

$$\Gamma_X \ni \gamma \mapsto W(\gamma) \in \wedge^n(T_\gamma \Gamma_X). \quad (2.7)$$

We denote by $W(\gamma)[x_1, \ldots, x_n]_d$ the corresponding component of $W(\gamma)$ in the decomposition (2.5).

In particular, in the case $n = 1$, a 1-form $V$ over $\Gamma_X$ is given by the mapping

$$\Gamma_X \ni \gamma \mapsto V(\gamma) = (V(\gamma)_x)_{x \in \gamma} \in T_\gamma \Gamma_X.$$
For fixed $\gamma \in \Gamma_X$ and $x \in \gamma$, we consider the mapping

$$O_{\gamma,x} \ni y \mapsto W_x(\gamma, y) : = W(\gamma_y) \in \wedge^n(T_{\gamma_y} \Gamma_X),$$

where $\gamma_y := (\gamma \setminus \{x\}) \cup \{y\}$, which is a section of the Hilbert bundle

$$\wedge^n(T_{\gamma_y} \Gamma_X) \ni y \in O_{\gamma,x} \quad (2.8)$$

over $O_{\gamma,x}$. The Levi–Civita connection on $TX$ generates in a natural way a "product" connection on this bundle. We denote by $\nabla^X_x W(\gamma) := \nabla^X_{\gamma,x} W_x(\gamma, x) \in T_x X \otimes (\wedge^n(T_\gamma \Gamma_X))$

if the section $W_x(\gamma, \bullet)$ is differentiable at $x$. Analogously, we denote by $\Delta^X_\gamma$ the corresponding Bochner Laplacian associated with the volume measure $m$ on $O_{\gamma,x}$ (see subsec. 3.2 where the notion of Bochner Laplacian is recalled).

Similarly, for a fixed $\gamma \in \Gamma_X$ and $\{x_1, \ldots, x_k\} \subset \gamma$, we define a mapping

$$O_{\gamma,x_1} \times \cdots \times \bigcup_{\gamma,x_k} \ni (y_1, \ldots, y_k) \mapsto W_{x_1,\ldots,x_k}(\gamma, y_1, \ldots, y_k) :=
\wedge^n(T_{\gamma y_1,\ldots,y_k} \Gamma_X),$$

where $\gamma_{y_1,\ldots,y_k} := (\gamma \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}$, which is a section of the Hilbert bundle

$$\wedge^n(T_{\gamma y_1,\ldots,y_k} \Gamma_X) \ni (y_1, \ldots, y_k) \in O_{\gamma,x_1} \times \cdots \times O_{\gamma,x_k} \quad (2.9)$$

over $O_{\gamma,x_1} \times \cdots \times O_{\gamma,x_k}$.

Let us remark that, for any $\eta \subset \gamma$, the space $\wedge^n(T_\gamma \Gamma_X)$ can be identified in a natural way with a subspace of $\wedge^n(T_\eta \Gamma_X)$. In this sense, we will use expressions of the type $W(\gamma) = W(\eta)$ without additional explanations.

A set $\mathcal{F} \Omega^n$ of smooth cylinder $n$-forms over $\Gamma_X$ will be defined as follows.

**Definition 2.3** $\mathcal{F} \Omega^n$ is the set of $n$-forms $W$ over $\Gamma_X$ which satisfy the following conditions:

(i) there exists a compact $\Lambda = \Lambda(W) \subset X$ such that $W(\gamma) = W(\gamma_\Lambda)$;

(ii) for each $\gamma \in \Gamma_X$ and $\{x_1, \ldots, x_k\} \subset \gamma$, the section $W_{x_1,\ldots,x_k}(\gamma, \bullet)$ of the bundle (2.9) is infinitely differentiable at $(x_1, \ldots, x_k)$, and bounded together with partial derivatives component-wise in the sense of decomposition (2.5), uniformly in $\gamma, x_1, \ldots, x_k$, and the component.

**Remark 2.2** For each $W \in \mathcal{F} \Omega^n$, $\gamma \in \Gamma_X$, and any open bounded $\Lambda \supset \Lambda(W)$, we can define the form $W_{\Lambda,\gamma}$ on $O_{\gamma,x_1} \times \cdots \times O_{\gamma,x_k}$ by

$$W_{\Lambda,\gamma}(y_1, \ldots, y_k) = \text{Proj}_{\wedge^n(T_{\gamma y_1} X \oplus \cdots \oplus T_{\gamma y_k} X)} W((\gamma \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}), \quad (2.10)$$

where $\{x_1, \ldots, x_k\} = \gamma \cap \Lambda$. The item (ii) of Definition 2.3 is obviously equivalent to the assumption that $W_{\Lambda,\gamma}$ is smooth and bounded together with all partial derivatives component-wise uniformly in $\gamma$ (for some $\Lambda$ and consequently for any $\Lambda \supset \Lambda(W)$).
Definition 2.4  We define the covariant derivative $\nabla^F W$ of a form $W$ given by (2.7) as the mapping

$$\Gamma_X \ni \gamma \mapsto \nabla^F W(\gamma) := (\nabla^x_x W(\gamma))_{x \in \gamma} \in T_{\gamma} \Gamma_X \otimes (\wedge^n (T_{\gamma} \Gamma_X))$$

if for all $\gamma \in \Gamma_X$ and $x \in \gamma$ the form $W_x(\gamma, \bullet)$ is differentiable at $x$ and the $\nabla^F W(\gamma)$ just defined indeed belongs to $T_{\gamma} \Gamma_X \otimes (\wedge^n (T_{\gamma} \Gamma_X))$.

Remark 2.3 For each $W \in \mathcal{F} \otimes^n$, the covariant derivative $\nabla^F W$ exists, and moreover only a finite number of the coordinates $\nabla^F W(\gamma)_{x_1, \ldots, x_n|d}$ in the decomposition

$$T_{\gamma} \Gamma_X \otimes (\wedge^n (T_{\gamma} \Gamma_X)) = \bigoplus_{x \in \gamma, [x_1, \ldots, x_n|d] \subset \gamma} T_x \Gamma X \otimes (T_x X \wedge \cdots \wedge T_x X)$$

are not equal to zero.

Remark 2.4 For each $W \in \mathcal{F} \Omega^n$, $\gamma \in \Gamma_X$, $x \in \gamma$, and $[x_1, \ldots, x_n|d] \subset \gamma$, we define the mapping $W_x(\gamma, \bullet)_{[x_1, \ldots, x_n|d]}$ as follows: if $x \neq x_j$ for all $j = 1, \ldots, n$, then

$$\mathcal{O}_{\gamma,x} \ni y \mapsto W(x, y)_{[x_1, \ldots, x_n|d]} := W((\gamma \setminus \{x\}) \cup \{y\})_{[x_1, \ldots, x_n|d]} \in T_{x_1} X \wedge \cdots \wedge T_{x_n} X,$$

and if $x = x_i$ for some $x_i \in \{x_1, \ldots, x_n\}$, then

$$\mathcal{O}_{\gamma,x} \ni y \mapsto W(x, y)_{[x_1, \ldots, x_n|d]} := W((\gamma \setminus \{x\}) \cup \{y\})_{[y_1, \ldots, y_n|d]} \in T_{y_1} X \wedge \cdots \wedge T_{y_n} X,$$

where $y_j = x_j$ if $x \neq x_j$ and $y_j = y$ otherwise. Then, the condition (ii) of Definition 2.3 yields, in particular, that the mapping $W_x(\gamma, \bullet)_{[x_1, \ldots, x_n|d]}$ is $\mathcal{C}^\infty$ for all $x \in \gamma$ and $[x_1, \ldots, x_n|d] \subset \gamma$.

Now, we have

$$\nabla^F W(\gamma)_{x_1, \ldots, x_n, d} = \nabla^x_x W(\gamma)_{x_1, \ldots, x_n, d},$$

where

$$\nabla^x_x W(\gamma)_{x_1, \ldots, x_n, d} := \nabla^x_x W_x(\gamma, x)_{x_1, \ldots, x_n, d}.$$

Notice that, in the case where $x \neq x_j$ for all $j = 1, \ldots, n$, $\nabla^x_x W_x(\gamma, \bullet)_{[x_1, \ldots, x_n|d]}$ means, in fact, the usual derivative of a mapping defined on $\mathcal{O}_{\gamma,x}$ and taking values in the fixed vector space $T_{x_1} X \wedge \cdots \wedge T_{x_n} X$. On the other hand, if $x$ does coincide with some $x_i \in \{x_1, \ldots, x_n\}$, then the expression $\nabla^x_x W_x(\gamma, x)_{[x_1, \ldots, x_n|d]}$ can be understood as the $T_x \Gamma X \otimes (T_x X \wedge \cdots \wedge T_x X)$-coordinate of the covariant derivative of the $n$-form

$$\mathcal{O}_{\gamma,y_1 \times \cdots \times \mathcal{O}_{\gamma,y_k} \ni (z_1, \ldots, z_k) \mapsto \text{Proj}_{\wedge^n (T_{x_1} X \otimes \cdots \otimes T_{x_k} X)} W((\gamma \setminus \{y_1, \ldots, y_k\}) \cup \{z_1, \ldots, z_k\})}$$

(2.11)

at the point $(y_1, \ldots, y_k)$, where $(y_1, \ldots, y_k)$ is the set of all the different $x_j$’s, $j = 1, \ldots, n$. In fact, the last sentence was just an alternative description of the notion of covariant derivative $\nabla^x_x W_x(\gamma, x)_{[x_1, \ldots, x_n|d]}$ of the mapping $W_x(\gamma, \bullet)_{[x_1, \ldots, x_n|d]}$ in the case where $x$ coincides with some $x_i$. 

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Proposition 2.1 For arbitrary $W^{(1)}, W^{(2)} \in F \Omega^n$, we have

$$\nabla^T \langle W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)} = \langle \nabla^T W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)} + \langle W^{(1)}(\gamma), \nabla^T W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)}.$$ 

Proof. We have, for any fixed $\gamma \in \Gamma_X$,

$$\nabla^T \langle W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)} = \sum_{x \in \gamma} \nabla_X \langle W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)}$$

$$= \sum_{x \in \gamma} \sum_{[x_1, \ldots, x_n] \subseteq \gamma} \langle W^{(1)}(\gamma)[x_1, \ldots, x_n], W^{(2)}(\gamma)[x_1, \ldots, x_n] \rangle_{T \gamma \Gamma_X}$$

$$= \sum_{x \in \gamma} \sum_{[x_1, \ldots, x_n] \subseteq \gamma} [\nabla_X \langle W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)}]$$

$$= \langle \nabla^T W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)} + \langle W^{(1)}(\gamma), \nabla^T W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)}.$$ 

(All the sums above are actually finite because of the definition of $F \Omega^n$.) ■

2.3 Square-integrable $n$-forms

Our next goal is to give a description of the space of $n$-forms that are square-integrable with respect to the Poisson measure.

Let $m$ be the volume measure on $X$, let $\rho: X \to \mathbb{R}$ be a measurable function such that $\rho > 0$ m-a.e., and $\rho^{1/2} \in H^{1/2}_{\text{loc}}(X)$, and define the measure $\sigma(dx) := \rho(x) m(dx)$. Here, $H^{1/2}_{\text{loc}}(X)$ denotes the local Sobolev space of order 1 in $L^2_{\text{loc}}(X; m)$. Then, $\sigma$ is a nonatomic Radon measure on $X$.

Let $\pi_\sigma$ stand for the Poisson measure on $\Gamma_X$ with intensity $\sigma$. This measure is characterized by its Laplace transform

$$\int_{\Gamma_X} e^{\langle f, \gamma \rangle} \pi_\sigma(d\gamma) = \exp \int_X (e^{f(x)} - 1) \sigma(dx), \quad f \in \mathcal{D}.$$ 

Let $F \in L^1(\Gamma_X; \pi_\sigma)$ be cylindrical, that is, there exits a compact $\Lambda \subset X$ such that $F(\gamma) = F(\gamma \Lambda)$. Then, one has the following formula, which we will use many times:

$$\int_{\Gamma_X} F(\gamma) \pi_\sigma(d\gamma) = e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) \sigma(dx_1) \cdots \sigma(dx_n). \quad (2.12)$$

We define on the set $F \Omega^n$ the $L^2$-scalar product with respect to the Poisson measure:

$$(W^{(1)}, W^{(2)})_{L^2_{\pi_\sigma} \Omega^n} := \int_{\Gamma_X} \langle W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T \gamma \Gamma_X)} \pi_\sigma(d\gamma). \quad (2.13)$$
As easily seen, for each \( W \in \mathcal{F} \Omega \), there exists \( \varphi \in D \), \( \varphi \geq 0 \), such that
\[
|\langle W(\gamma), W(\gamma) \rangle_{\wedge^n T_\gamma \Gamma_X} | \leq \langle \varphi^{\otimes n}, \gamma^{\otimes n} \rangle.
\]
Hence, the function under the sign of integral in (2.13) indeed belongs to \( L^1(\Gamma_X; \pi_\sigma) \), since the Poisson measure has all moments finite. Moreover, \((W, W)^{L_2_\pi_\sigma \Omega} > 0\) if \( W \) is not identically zero. Hence, we can define the Hilbert space
\[
L_2^2(\pi_\sigma \Omega)^n := L^2(\Gamma_X \rightarrow \wedge^n (T T_X); \pi_\sigma)
\]
as the closure of \( \mathcal{F} \Omega \) in the norm generated by the scalar product (2.13).

We will give now an isomorphic description of the space \( L^2_\pi_\sigma \Omega^n \) via the space \( L^2_\pi_\sigma(\Gamma_X) := L^2(\Gamma_X \rightarrow \mathbb{R}; \pi_\sigma) \) and some special spaces of square-integrable forms on \( X_m \), \( m = 1, \ldots, n \).

We need first some preparations. Let \( X_m \) be the \( m \)-th Cartesian power of the manifold \( X \). We have
\[
\wedge^n (T_{(x_1, \ldots, x_m)} X^m) = \bigoplus_{0 \leq k_1, \ldots, k_m \leq d, k_1 + \cdots + k_m = n} (T_{x_1} X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}. \tag{2.14}
\]

For an \( n \)-form \( \omega \) on \( X^m \), we denote by \( \omega(x_1, \ldots, x_m)_{k_1, \ldots, k_m} \) the corresponding component of \( \omega(x_1, \ldots, x_m) \) in the decomposition (2.14).

Let
\[
\tilde{X}^m := \{ \tilde{x} = (x_1, \ldots, x_m) \in X^m | x_i \neq x_j \text{ if } i \neq j \}.
\]

We introduce a set \( \Psi^n_0(\tilde{X}^m) \) (resp. \( \Psi^n_0(X^m) \)) of bounded \( n \)-forms \( \omega \) over \( \tilde{X}^m \) (resp. on \( X^m \)), which have compact support, smooth on \( \tilde{X}^m \) (resp. on \( X^m \)), and satisfy the following assumptions:

(i) \( \omega(x_1, \ldots, x_m)_{k_1, \ldots, k_m} = 0 \) if \( k_j = 0 \) for some \( j \in \{1, \ldots, m\} \);

(ii) \( \omega \) is symmetric:
\[
\omega(x_1, \ldots, x_m) = \omega(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \quad \text{for each } \sigma \in S_m. \tag{2.15}
\]

(we identify the spaces \( \wedge^n (T_{(x_1, \ldots, x_m)} X^m) \) and \( \wedge^n (T_{x_{\sigma(1)}, \ldots, x_{\sigma(m)}} X^m) \), see (2.14) and the explanation just after formula (2.5)).

Evidently, \( \Psi^n_0(X^m) \subset \Psi^n_0(\tilde{X}^m) \).

Let \( \gamma^{\otimes m} : \) be the measure on \( X^m \) given by
\[
\gamma^{\otimes m} : (dx_1, \ldots, dx_m) := \sum_{\{y_1, \ldots, y_m\} \subset \gamma} e_{y_1} \otimes \cdots \otimes e_{y_m} (dx_1, \ldots, dx_m),
\]
where
\[
e_{y_1} \otimes \cdots \otimes e_{y_m} (dx_1, \ldots, dx_m) := \frac{1}{m!} \sum_{\sigma \in S_m} e_{y_{\sigma(1)}} \otimes \cdots \otimes e_{y_{\sigma(m)}} (dx_1, \ldots, dx_m).
\]
We will use the notation
\[ T_{\{x_1, \ldots, x_m\}}^{(n)} X^m := \bigoplus_{1 \leq k_1, \ldots, k_m \leq d} (T_{x_1} X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}. \] (2.16)

By virtue of (2.5), we have
\[ \wedge^n (T_\gamma \Gamma_X) = \bigoplus_{m=1, \ldots, n} \bigoplus_{\{x_1, \ldots, x_m\} \subset \gamma} T_{\{x_1, \ldots, x_m\}}^{(n)} X^m. \] (2.17)

For \( W \in \mathcal{F}\Omega^n \), we denote by \( W_m(\gamma) \in \bigoplus_{\{x_1, \ldots, x_m\} \subset \gamma} T_{\{x_1, \ldots, x_m\}}^{(n)} X^m \) the corresponding component of \( W(\gamma) \) in the decomposition (2.17). Thus, for \( \{x_1, \ldots, x_m\} \subset \gamma, W_m(\gamma)(x_1, \ldots, x_m) \) is equal to the projection of \( W(\gamma) \in \wedge^n (T_\gamma \Gamma_X) \) onto the subspace \( T_{\{x_1, \ldots, x_m\}}^{(n)} X^m \).

For \( \bar{x} = (x_1, \ldots, x_m) \in \bar{X}^m \) we set \( \{\bar{x}\} := \{x_1, \ldots, x_m\} \).

**Lemma 2.1** For \( W, V \in \mathcal{F}\Omega^n \), we have
\[ (W(\gamma), V(\gamma))_{\wedge^n (T_\gamma \Gamma_X)} = \sum_{m=1, \ldots, n} \int_{X^m} (W_m(\gamma)(\bar{x}), V_m(\gamma)(\bar{x}))_{T_{\{x_1, \ldots, x_m\}}^{(n)} X^m : \gamma^\otimes m : (d\bar{x}) \] (2.18)

The proof can be obtained by a direct calculation.

Let us remark that each \( \omega \in \Psi_0^n(\bar{X}^m) \) generates a cylinder form \( W \in \mathcal{F}\Omega^n \) by the formula
\[ W_k(\gamma)(x_1, \ldots, x_k) = \begin{cases} \omega(x_1, \ldots, x_m), & k = m, \\ 0, & k \neq m. \end{cases} \]

Let us denote by \( L^2_\sigma \Psi_0^n(X^m) \) the space obtained as the completion of \( \Psi_0^n(X^m) \) in the \( L^2 \)-scalar product w.r.t. the measure \( \sigma^\otimes m \). Evidently, \( \Psi_0^n(\bar{X}^m) \) is a dense subset of \( L^2_\sigma \Psi_0^n(X^m) \).

We have

**Proposition 2.2** The space \( L^2_\pi^n \Omega^n \) is unitarily isomorphic to the space
\[ L^2_\pi^n (\Gamma_X) \otimes \Bigg[ \bigoplus_{m=1}^n L^2_\sigma \Psi^n(X^m) \Bigg] = \bigoplus_{m=1}^n L^2_\pi^n (\Gamma_X) \otimes L^2_\sigma \Psi^n(X^m), \] (2.19)

where the corresponding isomorphism \( I^n \) is defined by the formula
\[ I^n_m W(\gamma, \bar{x}) := (m!)^{-1/2} W_m(\gamma \cup \{\bar{x}\})(\bar{x}), \quad m = 1, \ldots, n. \] (2.20)

Here, \( I^n_m W := (I^n W)_m \) is the \( m \)-th component of \( I^n V \) in the decomposition (2.19).

**Remark 2.5** Actually, the formula (2.20) makes sense only for \( \bar{x} \in \bar{X}^m \). However, since the set \( X^m \setminus \bar{X}^m \) is of zero \( \sigma^\otimes m \) measure, this does not lead to a contradiction.
Proof. First, we recall an extension of the Mecke identity (e.g. [26]) to the case of functions of several variables [31]:

\[ \int_{\Gamma_X} \left[ \int_{X^m} f(\gamma, \bar{x}) : \gamma^\otimes m : (d\bar{x}) \right] \pi_\sigma(d\gamma) \]

\[ = \frac{1}{m!} \int_{\Gamma_X} \left[ \int_{X^m} f(\gamma \cup \{ \bar{x} \}, \bar{x}) \sigma^\otimes m(d\bar{x}) \right] \pi_\sigma(d\gamma), \quad (2.21) \]

where \( f: \Gamma_X \times X^m \to \mathbb{R} \) is a measurable function for which at least one of the double-integrals in (2.21) exists (this formula can be easily proved by a direct calculation using (2.12) for \( f(\gamma, \bar{x}) = F(\gamma)g(\bar{x}) \), where \( F(\gamma) \) is bounded and cylindrical and \( g(\bar{x}) \) is bounded and has compact support).

Next, let us specify the scalar product of two cylinder \( n \)-forms \( W, V \in \mathcal{F}\Omega^n \). We have, according to (2.18),

\[ (W(\gamma), V(\gamma))_{\Lambda^n(T, \Gamma_X)} = \]

\[ = \sum_{m=1}^{n} \int_{X^m} (W_m(\gamma)(\bar{x}), V_m(\gamma)(\bar{x}))_{\bar{x}} : \gamma^\otimes m : (d\bar{x}) \]

\[ = \sum_{m=1}^{n} \int_{X^m} (W_m(\gamma \cup \{ \bar{x} \})(\bar{x}), V_m(\gamma \cup \{ \bar{x} \})(\bar{x}))_{\bar{x}} : \gamma^\otimes m : (d\bar{x}), \quad (2.22) \]

where \((\bullet, \bullet)_{\bar{x}} := (\bullet, \bullet)_{\Gamma(\bar{x})} \setminus \bar{x} \in X^m\) (we used the evident equality \( \gamma \cup \{ \bar{x} \} = \gamma \) for \( \{ \bar{x} \} \subset \gamma \)). The application of the Mecke identity (2.21) to the function

\[ f(\gamma, \bar{x}) = (W_m(\gamma \cup \{ \bar{x} \})(\bar{x}), V_m(\gamma \cup \{ \bar{x} \})(\bar{x}))_{\bar{x}} \]

shows that

\[ (W, V)_{L_{\sigma}^2 \Omega^n} = \sum_{m=1}^{n} \frac{1}{m!} \int_{\Gamma_X} \int_{X^m} (W_m(\gamma \cup \{ \bar{x} \})(\bar{x}), V_m(\gamma \cup \{ \bar{x} \})(\bar{x}))_{\bar{x}} \sigma^\otimes m(d\bar{x}) \pi_\sigma(d\gamma). \]

The space \( \mathcal{F}\Omega^n \) is dense in \( L_{\sigma}^2 \Omega^n \), and so it remains only to show that \( I^n(\mathcal{F}\Omega^n) \) is a dense subspace of \( \bigoplus_{m=1}^{n} L_{\sigma}^2(\Gamma_X) \otimes L_{\sigma}^2(\Psi^n(X^m)), \) i.e., \( I^n(\mathcal{F}\Omega^n) \) is a dense subspace of \( L_{\sigma}^2(\Gamma_X) \otimes L_{\sigma}^2(\Psi^n(X^m)), m = 1, \ldots, n. \)

For \( F \in \mathcal{F}C \) and \( \omega \in \Psi_0^n(X^m) \), we define a form \( W \) by setting

\[ W_k(\gamma) := 0 \quad \text{for} \ k \neq m, \]

\[ W_m(\gamma)(\bar{x}) := (m!)^{1/2} F(\gamma \setminus \{ \bar{x} \})\omega(\bar{x}). \quad \text{(2.23)} \]

Evidently, we have \( W \in \mathcal{F}\Omega^n \) and

\[ I_k^n W(\gamma, \bar{x}) = 0 \quad \text{for} \ k \neq m, \]

\[ I_m^n W(\gamma, \bar{x}) = F(\gamma)\omega(\bar{x}) \quad \text{(2.24)} \]
for each $\gamma \in \Gamma X$ and $\bar{x} \in \mathcal{X}^m$ such that $\{\bar{x}\} \cap \gamma = \emptyset$. Since $\gamma$ is a set of zero $\sigma$ measure, we conclude from (2.24) that

$$I^m_{\gamma} W = F \otimes \omega.$$  

Noting that the linear span of such $F \otimes \omega$ is dense in $L^2_{\sigma}(\Gamma X) \otimes L^2_{\sigma}(X^m)$, we obtain the result. 

In what follows, we will denote by $D^n_\Omega$ the linear span of forms $W$ defined by (2.23), $m = 1, \ldots, n$. As we already noticed in the proof of Proposition 2.2, $D^n_\Omega$ is a subset of $F^n_\Omega$ and is dense in $L^2_{\sigma}(\Gamma X) \otimes L^2_{\sigma}(X^m)$. 

**Corollary 2.1** We have the unitary isomorphism

$$\mathcal{I}^n: L^2_{\sigma}(\Omega^n) \rightarrow \text{Exp}(L^2(X; \sigma)) \otimes \bigoplus_{m=1}^n L^2_{\sigma}(\Psi^n(X^m))$$

given by

$$\mathcal{I}^n := (U \otimes 1) I^n,$$

where $U$ is the Wiener–Itô–Segal isomorphism between the Poisson space $L^2_{\sigma}(\Gamma X)$ and the symmetric Fock space $\text{Exp}(L^2(X; \sigma))$ over $L^2(X; \sigma)$, see e.g. [8].

### 3 Dirichlet operators on differential forms over configuration spaces

In this section, we introduce Dirichlet operators associated with the Poisson measure on $\Gamma X$ which act in the spaces of square-integrable forms. These operators generalize the notions of Bochner and de Rham–Witten Laplacians on finite-dimensional manifolds. But first, we recall some known facts and definitions concerning the usual Dirichlet operator of the Poisson measure and Laplace operators on differential forms over finite-dimensional manifolds.

#### 3.1 The intrinsic Dirichlet operator on functions

In this subsection, we recall some theorems from [8] which concern the intrinsic Dirichlet operator in the space $L^2_{\sigma}(\Gamma X)$, to be used later.

Let us recall that the logarithmic derivative of the measure $\sigma$ is given by the vector field

$$X \ni x \mapsto \beta_\sigma(x) := \frac{\nabla X \rho(x)}{\rho(x)} \in T_x X$$

(where as usually $\beta_\sigma := 0$ on $\{\rho = 0\}$). We wish now to define a logarithmic derivative of the Poisson measure, and for this we need a generalization of the notion of vector field.

For each $\gamma \in \Gamma X$, consider the triple

$$T_{\gamma, \infty} \Gamma X \supset T_{\gamma} \Gamma X \supset T_{\gamma, 0} \Gamma X.$$
Here, $T_{\gamma,0} \Gamma_X$ consists of all finite sequences from $T_{\gamma} \Gamma_X$, and $T_{\gamma,\infty} \Gamma_X := (T_{\gamma,0} \Gamma_X)'$ is the dual space, which consists of all sequences $V(\gamma) = (V(\gamma)_x)_{x \in \gamma}$, where $V(\gamma)_x \in T_{\gamma,x}$. The pairing between any $V(\gamma) \in T_{\gamma,\infty} \Gamma_X$ and $v(\gamma) \in T_{\gamma,0} \Gamma_X$ with respect to the zero space $T_{\gamma,0} \Gamma_X$ is given by

$$\langle V(\gamma), v(\gamma) \rangle_\gamma = \sum_{x \in \gamma} \langle V(\gamma)_x, v(\gamma)_x \rangle_x$$

(the series is, in fact, finite). From now on, under a vector field over $\Gamma_X$ we will understand mappings of the form $\Gamma_X \ni \gamma \mapsto V(\gamma) \in T_{\gamma,\infty} \Gamma_X$.

The logarithmic derivative of the Poisson measure $\pi_{\sigma}$ is defined as the vector field

$$\Gamma_X \ni \gamma \mapsto B_{\pi_{\sigma}}(\gamma) = (\beta_{\sigma}(x))_{x \in \gamma} \in T_{\gamma,\infty} \Gamma_X$$

(i.e., the logarithmic derivative of the Poisson measure is the lifting of the logarithmic derivative of the underlying measure).

The following theorem is a version of Theorem 3.1 in [8] (for more general classes of functions and vector fields).

**Theorem 3.1 (Integration by parts formula on the Poisson space)**

For arbitrary $F(1), F(2) \in FC$ and a smooth cylinder vector field $V \in FV := FC^1$, we have

$$\int_{\Gamma_X} \nabla_V^\Gamma F(1)(\gamma)F(2)(\gamma) \pi_{\sigma}(d\gamma) = -\int_{\Gamma_X} F(1)(\gamma)\nabla_V^\Gamma F(2)(\gamma) \pi_{\sigma}(d\gamma)$$

$$-\int_{\Gamma_X} F(1)(\gamma)F(2)(\gamma) \left[ \langle B_{\pi_{\sigma}}(\gamma), V(\gamma) \rangle_\gamma + \text{div}^\Gamma V(\gamma) \right] \pi_{\sigma}(d\gamma),$$

where the divergence $\text{div}^\Gamma V(\gamma)$ of the vector field $V$ is given by

$$\text{div} V(\gamma) = \sum_{x \in \gamma} \text{div}^X V(\gamma) = \langle \text{div}^X V(\gamma), \gamma \rangle,$$

$$\text{div}^X V(\gamma) := \text{div}^X V_x(\gamma, x), \quad x \in \gamma,$$

$\text{div}^X$ denoting the divergence on $X$ with respect to the volume measure $m$.

**Proof.** The theorem follows from formula (2.12) and the usual integration by parts formula on the space $L^2(\Lambda^n, \sigma^{\otimes n})$ (see also the proof of Theorem 3.3 below). ■

Following [8], we consider the intrinsic pre-Dirichlet form on the Poisson space

$$\mathcal{E}_{\pi_{\sigma}}(F(1), F(2)) = \int_{\Gamma_X} \langle \nabla_V^\Gamma F(1)(\gamma), \nabla_V^\Gamma F(2)(\gamma) \rangle_\gamma \pi_{\sigma}(d\gamma)$$

(3.2)

with domain $D(\mathcal{E}_{\pi_{\sigma}}) := FC$. By using the fact that the measure $\pi_{\sigma}$ has all moments finite and noting that there exists a function $\varphi \in D$, $\varphi \geq 0$, such that

$$|\langle \nabla_V^\Gamma F(1)(\gamma), \nabla_V^\Gamma F(2)(\gamma) \rangle_\gamma| \leq \langle \varphi, \gamma \rangle,$$

one concludes that the expression (3.2) is well-defined.
Let $H_\sigma$ denote the Dirichlet operator in the space $L^2(X;\sigma)$ associated to the pre-Dirichlet form

$$
\mathcal{E}_\sigma(\varphi,\psi) = \int_X \langle \nabla^X \varphi(x), \nabla^X \psi(x) \rangle \sigma(dx), \quad \varphi, \psi \in \mathcal{D}.
$$

This operator acts as follows:

$$
H_\sigma \varphi(x) = -\Delta^X \varphi(x) - \langle \beta_\sigma(x), \nabla^X \varphi(x) \rangle \sigma, \quad \varphi \in \mathcal{D},
$$

where $\Delta^X := \text{div}^X \nabla^X$ is the Laplace–Beltrami operator on $X$.

Then, by using Theorem 3.1, one gets

$$
\mathcal{E}_{\pi\sigma}(F^{(1)}, F^{(2)}) = \int_{\Gamma X} H_{\pi\sigma} F^{(1)}(\gamma) F^{(2)}(\gamma) \pi\sigma(d\gamma), \quad F^{(1)}, F^{(2)} \in \mathcal{FC}. \quad (3.3)
$$

Here, the intrinsic Dirichlet operator $H_{\pi\sigma}$ is given by

$$
H_{\pi\sigma} F(\gamma) := \sum_{x \in \gamma} H_{\sigma,x} F(\gamma) \equiv \langle H_{\sigma,\bullet} F(\gamma), \gamma \rangle,
$$

$$
H_{\sigma,x} F(\gamma) := H_{\sigma} F_x(\gamma, x), \quad x \in \gamma, \quad (3.4)
$$

so that the operator $H_{\pi\sigma}$ is the lifting to $L^2(\Gamma X)$ of the operator $H_\sigma$ in $L^2(X;\sigma)$.

Upon (3.3), the pre-Dirichlet form $\mathcal{E}_{\pi\sigma}$ is closable, and we preserve the notation for the closure of this form.

**Theorem 3.2** [8] Suppose that $(H_\sigma, \mathcal{D})$ is essentially self-adjoint on $L^2(X;\sigma)$. Then, the operator $H_{\pi\sigma}$ is essentially self-adjoint on $\mathcal{FC}$.

**Remark 3.1** This theorem was proved in [8], Theorem 5.3. (We have already mentioned in Remark 2.1 that the inclusion $\mathcal{FC}_b(\mathcal{D}, \Gamma X) \subset \mathcal{FC}$ holds.) We would like to stress that this result is based upon the theorem which says that the image of the operator $H_{\pi\sigma}$ under the isomorphism $U$ between the Poisson space and the Fock space $\text{Exp}(L^2(X;\sigma))$ is the differential second quantization $d\text{Exp} H_\sigma$ of the operator $H_\sigma$.

**Remark 3.2** The condition of Theorem 3.2 is satisfied if e.g.

$$
\|\beta_\sigma\|_{TX} \in L^p_{\text{loc}}(X;\sigma) \quad (3.5)
$$

for some $p > \text{dim } X$, see [8].

In what follows, we will suppose for simplicity that

the function $\rho$ is infinitely differentiable on $X$ and $\rho(x) > 0$ for all $x \in X. \quad (3.6)$

Evidently, estimate (3.5) is implied by (3.6).

Finally, we mention the important fact [8] that the diffusion process which is properly associated with the Dirichlet form $(\mathcal{E}_{\pi\sigma}, D(\mathcal{E}_{\pi\sigma}))$ is the usual independent infinite particle process (or distorted Brownian motion on $\Gamma X$), introduced by Doob [19].
3.2 Laplacians on differential forms over finite-dimensional manifolds

We recall now some facts on the Bochner and de Rham–Witten Laplacians on differential forms over a finite-dimensional manifold.

Let $M$ be a Riemannian manifold equipped with the measure $\mu(dx) = e^{\phi(x)}dx$, $dx$ being the volume measure and $\phi$ a $C^2$-function on $M$. We consider a Hilbert bundle

$$\mathcal{H}_x \mapsto x \in M$$

over $M$ equipped with a smooth connection, and denote by $\nabla$ the corresponding covariant derivative in the spaces of sections of this bundle. Let $L^2(M \to H; \mu)$ be the space of $\mu$-square integrable sections. The operator $H^B_\mu := \nabla^* \nabla$ in $L^2(M \to H; \mu)$, where $\nabla^* \mu$ is the adjoint of $\nabla$, will be called the Bochner Laplacian associated with the measure $\mu$. One can easily write the corresponding differential expression on the space of twice differentiable sections. In the case where $\phi \equiv 0$ and $\mathcal{H}_x = \wedge^n(T_xM)$, we obtain the classical Bochner Laplacian on differential forms (see e.g. [17]).

Now, let $d$ be the exterior differential in spaces of differential forms over $M$. The operator $H^R_\mu := d^* d + dd^*$ acting in the space of $\mu$-square integrable forms, where $d^*$ is the adjoint of $d$, will be called the de Rham Laplacian associated with the measure $\mu$ (or the Witten Laplacian associated with $\phi$, see e.g. [17]).

We will use sometimes more extended notations $H^B_{\mu,n}(M)$, $H^R_{\mu,n}(M)$ for the Bochner and de Rham–Witten Laplacians on the space of $\mu$-square integrable $n$-forms over $M$.

The relation of the Bochner and de Rham–Witten Laplacians on differential forms is given by the Weitzenböck formula (cf. [17], [20]), which will be recalled now.

Fix $x \in M$ and let $(e_j)_{j=1}^{\dim M}$ be an orthonormal basis in $T_xM$. Denote by

$$a_j : \wedge^{n+1}(T_xM) \to \wedge^n(T_xM),$$

$$a_j^* : \wedge^n(T_xM) \to \wedge^{n+1}(T_xM)$$

the annihilation and creation operators, respectively, defined by

$$a_j u^{(n+1)} = \sqrt{n+1} (e_j, u^{(n+1)})_x, \quad u^{(n+1)} \in \wedge^{n+1}(T_xM),$$

$$a_j^* u^{(n)} = \sqrt{n} e_j \wedge u^{(n)}, \quad u^{(n)} \in \wedge^n(T_xM)$$

(3.7)

(the pairing in the expression $(e_j, u^{(n+1)})_x$ is carried out in the first “variable,” so that $a_j^*$ becomes adjoint of $a_j$).

Let us introduce the operator $R_n(x)$ in $\wedge^n(T_xM)$ by

$$R_n(x) := \sum_{i,j,k,l=1}^{\dim M} R_{ijkl}(x) a_i^* a_j a_k^* a_l,$$
where $R_{ijkl}$ is the curvature tensor on $M$. It can be shown that the definition of this operator is independent of the specific choice of basis.

Next, let $(\nabla^M \beta_\mu(x))^n$ be the operator in $\wedge^n(T_xM)$ given by

$$(\nabla^M \beta_\mu(x))^n := \nabla^M \beta_\mu(x) \otimes 1 \cdots \otimes 1 + 1 \otimes \nabla^M \beta_\mu(x) \otimes 1 \cdots \otimes 1$$

$$+ \cdots + 1 \otimes \cdots \otimes 1 \otimes \nabla^M \beta_\mu(x), \quad (3.9)$$

$\nabla^M \beta_\mu(x)$ being understood as an operator in $T_xM$.

Then, the Weitzenböck formula writes as follows:

$$H^R_\mu \omega_n(x) = H^B_\mu \omega_n(x) + R_\mu(x)\omega_n(x), \quad (3.10)$$

where $\omega_n$ is an n-form on $X$, and $R_\mu(x)\omega_n(x) = R_{\mu,n}(x)\omega_n(x)$,

$$R_{\mu,n}(x) := R_n(x) - (\nabla^M \beta_\mu(x))^n. \quad (3.11)$$

**Remark 3.3** The classical Weitzenböck formula is related, in fact, to the case where $\phi \equiv 0$, see e.g. [17], [20]. Formula (3.10) can be obtained by a direct calculation using similar arguments, cf. [1].

### 3.3 Bochner Laplacian on forms over the configuration space

Let us consider the pre-Dirichlet form

$$\mathcal{E}^B_{\pi_\sigma}(W^{(1)}, W^{(2)}) = \int_{\Gamma_X} \langle \nabla^\Gamma W^{(1)}(\gamma), \nabla^\Gamma W^{(2)}(\gamma) \rangle_{T^\gamma \Gamma_X \otimes \wedge^n(T^\gamma \Gamma_X)} \pi_\sigma(d\gamma), \quad (3.12)$$

where $W^{(1)}, W^{(2)} \in \mathcal{F}\Omega^n$. As easily seen, there exists $\varphi \in \mathcal{D}$, $\varphi \geq 0$, such that

$$|\langle \nabla^\Gamma W^{(1)}(\gamma), \nabla^\Gamma W^{(2)}(\gamma) \rangle_{T^\gamma \Gamma_X \otimes \wedge^n(T^\gamma \Gamma_X)}| \leq \langle \varphi^{\otimes(n+1)}, \gamma^{\otimes(n+1)} \rangle,$$

so that the function under the sign of integral in (3.12) is integrable with respect to $\pi_\sigma$.

**Theorem 3.3** For any $W^{(1)}, W^{(2)} \in \mathcal{F}\Omega^n$, we have

$$\mathcal{E}^B_{\pi_\sigma}(W^{(1)}, W^{(2)}) = \int_{\Gamma_X} \langle H^B_{\pi_\sigma} W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\wedge^n(T^\gamma \Gamma_X)} \pi_\sigma(d\gamma),$$

where $H^B_{\pi_\sigma}$ is the operator in the space $L^2_{\pi_\sigma} \Omega^n$ with domain $\mathcal{F}\Omega^n$ given by

$$H^B_{\pi_\sigma} W(\gamma) = -\Delta^\Gamma W(\gamma) - \langle \nabla^\Gamma W(\gamma), B_{\pi_\sigma}(\gamma) \rangle_\gamma, \quad W \in \mathcal{F}\Omega^n. \quad (3.13)$$

Here,

$$\Delta^\Gamma W(\gamma) := \sum_{x \in \gamma} \Delta^X_x W(\gamma) \equiv \langle \Delta^X_x W(\gamma), \gamma \rangle, \quad (3.14)$$

where $\Delta^X_x$ is the Bochner Laplacian of the bundle $\wedge^n(T_{y^\gamma} \Gamma_X) \mapsto y \in \mathcal{O}_{\gamma,x}$ with the volume measure.
Proof. First, we note that, for $W \in \mathcal{F}\Omega^n$,
\[
\Delta^X_{\sigma} W(\gamma)|_{x_1,\ldots,x_n} := \Delta^X_{\sigma} W(\gamma)|_{x_1,\ldots,x_n}, \quad x \in \gamma, \ [x_1,\ldots,x_n]|_d \subset \gamma.
\]

Fix now $W^{(1)}, W^{(2)} \in \mathcal{F}\Omega^n$ and let $\Lambda_1, \Lambda_2$ be compact subsets of $X$ as in Definition 2.3 corresponding to $W^{(1)}, W^{(2)}$, respectively. Let $\Lambda$ be an open set in $X$ with compact closure such that both $\Lambda_1$ and $\Lambda_2$ are subsets of $\Lambda$. Then, by using (2.12),
\[
\int_{\Gamma X} \langle \nabla^T W^{(1)}(\gamma), \nabla^T W^{(2)}(\gamma) \rangle_{T_x\Gamma X \otimes \Lambda^n(T_x\Gamma X)} \pi_\sigma(d\gamma) =
\]
\[
eq e^{-\sigma(\Lambda)} \sum_{k=0}^\infty \frac{1}{k!} \sum_{i=1}^k \sum_{y_1,\ldots,y_n} \int_{\Lambda} \langle \nabla^X_{x_{1i}} W^{(1)}(\gamma)|_{y_1,\ldots,y_n}|_d \rangle_{\Lambda \cap \Lambda_1 \cup \Lambda \cap \Lambda_2} \sigma(dx_1) \cdots \sigma(dx_k)
\]
\[
\nabla^X_{x_{1i}} W^{(2)}(\gamma)|_{y_1,\ldots,y_n}|_d \sigma(dx_1) \cdots \sigma(dx_k)
\]
\[
eq e^{-\sigma(\Lambda)} \sum_{k=0}^\infty \frac{1}{k!} \sum_{i=1}^k \sum_{y_1,\ldots,y_n} \int_{\Lambda} \langle \Delta^X_{x_{1i}} W^{(1)}(\gamma)|_{y_1,\ldots,y_n}|_d \rangle_{\Lambda \cap \Lambda_1 \cup \Lambda \cap \Lambda_2} \sigma(dx_1) \cdots \sigma(dx_k)
\]
\[
+ \langle \nabla^X_{x_{1i}} W^{(2)}(\gamma)|_{y_1,\ldots,y_n}|_d \rangle_{\Lambda \cap \Lambda_1 \cup \Lambda \cap \Lambda_2} \sigma(dx_1) \cdots \sigma(dx_k)
\]
\[
= \int_{\Gamma X} \langle H^B_{\pi_\sigma} W^{(1)}(\gamma), W^{(2)}(\gamma) \rangle_{\Lambda^n(T_x\Gamma X)} \pi_\sigma(d\gamma). \quad \blacksquare
\]

Remark 3.4 We can rewrite the action of the operator $H^B_{\pi_\sigma}$ in the two following forms:

1) We have from (3.13) and (3.14) that
\[
H^B_{\sigma,x} W(\gamma) = \sum_{x \in \gamma} H^B_{\sigma,x} W(\gamma) \equiv \langle H^B_{\sigma,x} W(\gamma), \gamma \rangle, \quad W(\gamma) \in \mathcal{F}\Omega^n,
\]
where
\[
H^B_{\sigma,x} W(\gamma) := -\Delta^X_{\sigma} W(\gamma) - \langle \nabla^X_{\sigma} W(\gamma), \beta(\gamma) \rangle_x.
\]

Thus, the operator $H^B_{\sigma,x}$ is the lifting of the Bochner Laplacian on $X$ with the measure $\sigma$.

2) As easily seen, the operator $H^B_{\pi_\sigma}$ preserves the space $\mathcal{F}\Omega^n$, and we can always take $\Lambda(H^B_{\pi_\sigma} W) = \Lambda(W)$. Then, for any open bounded $\Lambda \supset \Lambda(W)$ (cf. Remark 2.2), we have
\[
(H^B_{\pi_\sigma} W)_{\Lambda,\gamma} = H^B_{\sigma,\Lambda(\Lambda \cap \gamma)} (X|_{\Lambda(\Lambda \cap \gamma)} |) W_{\Lambda,\gamma},
\]
where $H^B_{\sigma,\Lambda(\Lambda \cap \gamma)} (X|_{\Lambda(\Lambda \cap \gamma)} |)$ is the Bochner Laplacian of the manifold $X|_{\Lambda(\Lambda \cap \gamma)} |$ with the product measure $\sigma|_{\Lambda(\Lambda \cap \gamma)} |$ (cf. (2.10)). The equality (3.17) holds on $O_{\gamma,x_1} \times \cdots \times O_{\gamma,x_{|\Lambda \cap \gamma|}}$. 

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where \( \{x_1, \ldots, x_{|\Lambda \cap \gamma|}\} = \Lambda \cap \gamma \). Notice that, since the operator \( H^B_{\sigma|\Lambda \cap \gamma|}(X^{|\Lambda \cap \gamma|}) \) acts locally on (smooth) forms on \( X^{|\Lambda \cap \gamma|} \), the expression on the right hand side of (3.17) is well defined as a form on \( O_{\gamma,x_1} \times \cdots \times O_{\gamma,x_{|\Lambda \cap \gamma|}} \).

It follows from Theorem 3.3 that the pre-Dirichlet form \( \mathcal{E}^B_{\pi} \) is closable in the space \( L^2_{\pi} \Omega^n \). The generator of its closure (being actually the Friedrichs extension of the operator \( H^B_{\pi}, \) for which we will use the same notation) will be called the Bochner Laplacian on \( n \)-forms over \( \Gamma_X \) corresponding to the Poisson measure \( \pi \).

For linear operators \( A \) and \( B \) acting in Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively, we introduce the operator \( A \circ B \) in \( \mathcal{H} \otimes \mathcal{K} \) by

\[
A \circ B := A \otimes 1 + 1 \otimes B, \quad \text{Dom}(A \circ B) := \text{Dom}(A) \otimes_{a} \text{Dom}(B),
\]

where \( \otimes_{a} \) stands for the algebraic tensor product. Next, for operators \( A_1, \ldots, A_n \) acting in Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_n \), respectively, let \( \bigoplus_{i=1}^n A_i \) denote the operator in \( \bigoplus_{i=1}^n H_i \) given by

\[
\left( \bigoplus_{i=1}^n A_i \right) (f_1, \ldots, f_n) = (A_1 f_1, \ldots, A_n f_n), \quad f_i \in \text{Dom}(A_i).
\]

Theorem 3.4 1) On \( \mathcal{D} \Omega^n \) we have

\[
H^B_{\pi} = (I^n)^{-1} \left[ H_{\pi} \bigcirc \left( \bigoplus_{m=1}^n H^B_{\sigma,(n,m)} \right) \right] I^n, \quad (3.18)
\]

where \( H^B_{\sigma,(n,m)} \) denotes the restriction of the Bochner Laplacian \( H^B_{\sigma \otimes m,n}(X^m) \) acting in the space \( L^2(X^m \to \Lambda^n(TX^m); \sigma^{\otimes m}) \) to the subspace \( L^2_{\pi} \Psi^n(X^m) \).

2) Suppose that, for each \( m = 1, \ldots, n \), the Bochner Laplacian \( H^B_{\sigma,m}(X) \) is essentially self-adjoint on the set of smooth forms with compact support. Then, \( \mathcal{D} \Omega^n \) is a domain of essential self-adjointness of \( H^B_{\pi} \), and the equality (3.18) holds for the closed operators \( H^B_{\pi} \) and \( H = \bigoplus_{m=1}^n H^B_{\sigma,(n,m)} \) (where the latter operator is closed from its domain of essential self-adjointness \( I^n(\mathcal{D} \Omega^n) \)).

Remark 3.5 The essential self-adjointness of the Bochner Laplacian \( H^B_{\sigma} \) on the set of smooth forms with compact support is well-known in the case where \( \sigma \) is the volume measure, see e.g. [20]. More generally, it is sufficient to assume that \( \beta_\sigma \), together with its derivatives up to order 2, is bounded.

Proof of Theorem 3.4. 1) Let \( W \in \mathcal{D} \Omega^n \) be given by the formula (2.23). Then, using (3.15), (3.16), and (3.4), we get

\[
(H^B_{\pi} W)_k(\gamma) = 0 \quad \text{for } k \neq m,
\]

\[
(H^B_{\pi} W)_m(\gamma)(\bar{x}) = \left( \sum_{x \in \gamma} H^B_{\sigma,x} W \right)_m(\gamma)(\bar{x})
\]
On the set of smooth forms. Space and it is enough to show that the Bochner Laplacian \( H \) concludes from the theory of operators admitting separation of variables [15, Ch. 6] that the \( \Omega \) operator proves (3.18).

Hence, by virtue of (2.24), we get

\[
= (m!)^{1/2} \left[ \left( \sum_{x \in \gamma \setminus \{ \bar{x} \}} H_{\sigma,x}F \right)(\gamma \setminus \{ \bar{x} \})\omega(\bar{x}) + F(\gamma \setminus \{ \bar{x} \}) \left( \sum_{x \in \{ \bar{x} \}} H_{\sigma,x}\omega \right)(\bar{x}) \right]
\]

\[
= (m!)^{1/2} \left[ (H_{\sigma,F})(\gamma \setminus \{ \bar{x} \})\omega(\bar{x}) + F(\gamma \setminus \{ \bar{x} \}) (H_{\sigma,(n,m)}\omega)(\bar{x}) \right]. \tag{3.19}
\]

(Notice that the Bochner Laplacian in the space \( L^2(X^m \to \wedge^n(X^m); \sigma^{\otimes m}) \) leaves the set \( \Psi_0^n(X^m) \) invariant.) Therefore,

\[
(I_k^m H_{\sigma,F}^\lambda W)(\gamma, \bar{x}) = \begin{cases} 0, & \text{for } k \neq m, \\ (H_{\sigma,F})(\gamma \setminus \{ \bar{x} \})\omega(\bar{x}) + F(\gamma \setminus \{ \bar{x} \}) (H_{\sigma,(n,m)}\omega)(\bar{x}), & \text{for } k = m. \end{cases} \tag{3.20}
\]

Hence, by virtue of (2.24), we get

\[
\left[ H_{\sigma,F} \bigoplus \left( \bigoplus_{i=1}^n H_{\sigma,(n,i)}^{\lambda} \right) \right] I^\lambda W \bigg( \gamma, \bar{x} \bigg) = (I_k^m H_{\sigma,F}^\lambda W)(\gamma, \bar{x}), \quad k = 1, \ldots, n
\]

which proves (3.18).

2) Let \( \Omega_0^n(X^m) \) denote the set of all smooth forms \( \omega: X^m \to \wedge^n(X^m) \) with compact support. It is not hard to see that the essential self-adjointness of \( H_{\sigma,m}(X) \) for each \( m = 1, \ldots, n \) implies that

the Bochner Laplacian \( H^B := H_{\sigma^{\otimes m},n}^B(X^m) \) is essentially self-adjoint on \( \Omega_0^n(X^m) \). \( \tag{3.21} \)

Indeed, by using the decomposition (2.14), we have

\[
L^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m}) = L^2(X^m \ni (x_1, \ldots, x_m) \to \wedge^n(Tx_1 \ldots x_m X^m); \sigma^{\otimes m}) \\
= \bigoplus_{0 \leq k_1, \ldots, k_m \leq d} L^2(X^m \ni (x_1, \ldots, x_m) \to (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}; \sigma^{\otimes m})
\]

and it is enough to show that the Bochner Laplacian \( H^B \) is essentially self-adjoint in each space

\[
L^2(X^m \ni (x_1, \ldots, x_m) \to (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}; \sigma^{\otimes m}) \tag{3.22}
\]
on the set of smooth forms.

On the other hand, by using the essential self-adjointness of each operator \( H_{\sigma,m}^B(X) \) on \( \Omega_0^n(X) \) and that of the operator \( H_{\sigma} \) in the space \( L^2(X; \sigma) \) on the set \( D \) (Remark 3.2), we conclude from the theory of operators admitting separation of variables [15, Ch. 6] that the operator

\[
H_{\sigma,k_1}^B(X) \bigoplus \cdots \bigoplus H_{\sigma,k_m}^B(X), \quad H_{\sigma,0}^B(X) := H_{\sigma,0}, \tag{3.23}
\]
is essentially self-adjoint in the space
\[
L^2(X \to \wedge^k(TX); \sigma) \otimes \cdots \otimes L^2(X \to \wedge^m(TX); \sigma) = L^2(X^m \ni (x_1, \ldots, x_m) \to (T_x X)^{\wedge k_1} \otimes \cdots \otimes (T_{x_m} X)^{\wedge k_m}; \sigma^{\otimes m})
\]  
(3.24)
on the algebraic product of the domains of the operators \(H^{B_{\sigma,k_i}}(X)\).

Next, we note that, for each \((x_1, \ldots, x_m) \in \tilde{X}^m\), there exists an intrinsic unitary isomorphism \[\text{Iso}_{k_1,\ldots,k_m} : (T_{x_1} X)^{\wedge k_1} \otimes \cdots \otimes (T_{x_m} X)^{\wedge k_m} \to (T_{x_1} X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}\] that is given by the formula
\[
\text{Iso}_{k_1,\ldots,k_m}(u_1(1) \wedge \cdots \wedge u_{k_1}(1)) \otimes \cdots \otimes (u_1(m) \wedge \cdots \wedge u_{k_m}(m)) := \sqrt{\frac{(k_1+\cdots+k_m)!}{k_1! \cdots k_m!}} u_1(1) \wedge \cdots \wedge u_{k_1}(1) \wedge \cdots \wedge u_1(m) \wedge \cdots \wedge u_{k_m}(m), \quad u_j \in T_{x_j} X,
\] and then it is extended by linearity. As easily seen, this definition is independent of the representation of a vector from \((T_{x_1} X)^{\wedge k_1} \otimes \cdots \otimes (T_{x_m} X)^{\wedge k_m}\). Hence, for any \((k_1, \ldots, k_m)\), we can construct the unitary \(U_{k_1,\ldots,k_m}\) between the spaces (3.24) and (3.22) by setting
\[
(U_{k_1,\ldots,k_m} F)(x_1, \ldots, x_m) := \text{Iso}_{k_1,\ldots,k_m}(F(x_1, \ldots, x_m)).
\]
Under this unitary, the operator (3.23) goes over into the operator \(H^B\) in the space (3.22), while the image of its domain consists of linear combinations of the form \(U_{k_1,\ldots,k_m}(\omega(k_1) \otimes \cdots \otimes \omega(k_m)), \omega(k_i) \in \Omega^k_0(X)\). From here, the assertion (3.21) follows.

Let \(\tilde{L}^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m})\) denote the subspace of \(L^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m})\) consisting of all symmetric forms, i.e., the forms \(\omega \in L^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m})\) for which the equality (2.15) holds for \(\sigma^{\otimes m}\)-a.a. \((x_1, \ldots, x_m) \in X^m\). Evidently, the orthogonal projection \(P^m\) onto this subspace is given by the formula
\[
(P^m \omega)(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \omega(x_{\sigma(1)}, \ldots, x_{\sigma(m)})
\]  
(3.25)
and
\[
P^m \Omega^n_0(X^m) = \Omega^n_{0,\text{sym}}(X^m),
\]  
(3.26)
where \(\Omega^n_{0,\text{sym}}(X^m)\) denotes the set of symmetric smooth forms \(\omega : X^m \to \wedge^n(X^m)\) with compact support.

The assertion (3.21) and the nonnegative definiteness of \(H^B\) yield that the set \((H^B + 1)\Omega^n_0(X^m)\) is dense in \(\tilde{L}^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m})\), see e.g. [32], Section 10.1. Therefore, the set \(P^m(H^B + 1)\Omega^n_0(X^m)\) is dense in \(\tilde{L}^2(X^m \to \wedge^n(TX^m); \sigma^{\otimes m})\). But upon (3.25) and (3.26),
\[
P^m(H^B + 1) \Omega^n_0(X^m) = (H^B P^m + P^m) \Omega^n_0(X^m) = (H^B + 1) \Omega^n_{0,\text{sym}}(X^m),
\]
which implies that the Bochner Laplacian $H^B$ in the space $\tilde{L}^2(X^m \to \wedge^n(TX^m);\sigma^{\otimes m})$ is essentially self-adjoint on $\Omega^n_{0,\text{sym}}(X^m)$.

Because $H^B$ acts invariantly on the subspace $L^2_0\Psi^n(X^m)$ and also on its orthogonal complement in $\tilde{L}^2(X^m \to \wedge^n(TX^m);\sigma^{\otimes m})$, we conclude that $H^B_{\sigma,(n,m)}$ is essentially self-adjoint on $\Psi^n_0(X^m)$. Consequently, the operator $\bigoplus_{m=1}^n H^B_{\sigma,(n,m)}$ is essentially self-adjoint on the direct sum of the sets $\Psi^n_0(X^m)$, $m = 1, \ldots, n$.

Finally, taking to notice that the operator $H_{\pi,\sigma}$ is essentially self-adjoint on $\mathcal{F}C$ (Theorem 3.2), we conclude again from the theory of operators admitting separation of variables that $\Pi^n(D\Omega^n)$ is a domain of essential self-adjointness of the operator $H_{\pi,\sigma} \bigoplus (\bigoplus_{m=1}^n H^B_{\sigma,(n,m)})$ in the space $L^2_\sigma(\Gamma_X) \otimes [\bigoplus_{m=1}^n L^2_\sigma \Psi^n(X^m)]$. Thus, (3.18) yields the statement. ■

We give also a Fock space representation of the operator $H^B_{\pi,\sigma}$. Corollary 2.1 implies the following

**Corollary 3.1** Let the conditions of Theorem 3.4, 2) be satisfied. Then,

$$\mathcal{I}^n H^B_{\pi,\sigma}(\mathcal{I}^n)^{-1} = d\text{Exp} H_{\sigma} \bigoplus \left( \bigoplus_{m=1}^n H^B_{\sigma,(n,m)} \right),$$

cf. Remark 3.1.

### 3.4 De Rham Laplacian on forms over the configuration space

We define linear operators

$$d^\Gamma: \mathcal{F}\Omega^n \to \mathcal{F}\Omega^{n+1}, \quad n \in \mathbb{N}_0, \quad \mathcal{F}\Omega^0 := \mathcal{F}C,$$

by

$$(d^\Gamma W)(\gamma) := \sqrt{n+1} \text{AS}_{n+1}(\nabla^\Gamma W(\gamma)),$$ (3.28)

where $\text{AS}_{n+1}: (T_\gamma \Gamma_X)^{\otimes(n+1)} \to \wedge^{n+1}(T_\gamma \Gamma_X)$ is the antisymmetrization operator. It follows from this definition that

$$(d^\Gamma W)(\gamma) = \sum_{x \in \gamma} (d^X_x W)(\gamma),$$ (3.29)

where

$$(d^X_x W)(\gamma) := \sum_{[x_1, \ldots, x_n] \subset \gamma} d^X(W_x(\gamma, x)[x_1, \ldots, x_n])$$

$$= \sum_{[x_1, \ldots, x_n] \subset \gamma} \sqrt{n+1} \text{AS}_{n+1}(\nabla^X W_x(\gamma, x)[x_1, \ldots, x_n])$$ (3.30)

with $\text{AS}_{n+1}: T_x X \otimes (T_{x_1} X \wedge \cdots \wedge T_{x_n} X) \to T_x X \wedge T_{x_1} X \wedge \cdots \wedge T_{x_n} X$ being again the antisymmetrization. Therefore, we have indeed the inclusion $d^\Gamma \omega \in \mathcal{F}\Omega^{n+1}$ for each $\omega \in \mathcal{F}\Omega^n$.

Suppose that, in local coordinates on the manifold $X$, the form $W_x(\gamma, \bullet)[x_1, \ldots, x_n]$ has the representation

$$O_{\gamma, x} \ni y \to W_x(\gamma, y)[x_1, \ldots, x_n] = w(y) h_1 \wedge \cdots \wedge h_n,$$ (3.31)

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where \( w : \mathcal{O}_{\gamma,x} \to \mathbb{R} \) and \( h_1 \land \cdots \land h_n \in T_{x_1} \land \cdots \land T_{x_n} \). Then,

\[
\text{AS}_{n+1}(\nabla XW_x(\gamma,x)|_{x_1,\ldots,x_n}) = \nabla Xw(x) \land h_1 \land \cdots \land h_n, \tag{3.32}
\]

which, upon (3.30), describes the action of \( d^X \).

Let us consider now \( d^T \) as an operator acting from the space \( L^2_{\pi_\sigma} \Omega^n \) into \( L^2_{\pi_\sigma} \Omega^{n+1} \). Analogously to the proof of Theorem 3.3, we get the following formula for the adjoint operator \( d^{\Gamma}_\sigma \) restricted to \( \mathcal{F} \Omega^{n+1} \):

\[
(d^{\Gamma}_\sigma W)(\gamma) = \sum_{x \in \gamma} \langle d^{X,x}_\sigma W \rangle (\gamma), \tag{3.33}
\]

where

\[
(d^{X,x}_\sigma W)(\gamma) = \sum_{[x_1,\ldots,x_{n+1}]d \in \gamma: \{x_1,\ldots,x_{n+1}\} \in \gamma} d^{X,x}_\sigma(W_x(\gamma,x)|_{x_1,\ldots,x_{n+1}}). \tag{3.34}
\]

Suppose, analogously to the above, that in local coordinates on the manifold \( X \)

\[
\mathcal{O}_{\gamma,x} \ni y \mapsto W_x(\gamma,y)|_{x_1,\ldots,x_{n+1}} = w(y) h_1 \land \cdots \land h_{n+1}, \tag{3.35}
\]

where \( w : \mathcal{O}_{\gamma,x} \to \mathbb{R} \) and \( h_1 \land \cdots \land h_{n+1} \in T_{x_1}X \land \cdots \land T_{x_{n+1}}X \). Then,

\[
d^{X,x}_\sigma(W_x(\gamma,x)|_{x_1,\ldots,x_{n+1}}) = -\frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} \delta_{x,x_i} \left[ (\nabla Xw(x), h_i) \right]_{\gamma} \]

\[
+ w(x) \langle \beta_{g}(x), h_i \rangle_{\gamma} h_1 \land \cdots \land \hat{h}_i \land \cdots \land h_{n+1}. \tag{3.36}
\]

Here,

\[
\delta_{x,x_i} = \begin{cases} 1, & \text{if } x = x_i, \\ 0, & \text{otherwise}, \end{cases}
\]

and \( \hat{h}_i \) denotes the absence of \( h_i \).

Upon (3.33)–(3.36)

\[
d^{\Gamma}_\sigma : \mathcal{F} \Omega^{n+1} \to L^2_{\pi_\sigma} \Omega^n.
\]

For \( n \in \mathbb{N} \), we define the pre-Dirichlet form \( \mathcal{E}^R_{\pi_\sigma} \) by

\[
\mathcal{E}^R_{\pi_\sigma}(W(1),W(2)) := \int_{\Gamma_X} \left[ \langle d^T W(1)(\gamma), d^T W(2)(\gamma) \rangle_{\wedge^{n+1}(T_{\gamma}X)} \right. \\
+ \left. \langle d^{\Gamma}_\sigma W(1)(\gamma), d^{\Gamma}_\sigma W(2)(\gamma) \rangle_{\wedge^{n-1}(T_{\gamma}X)} \right] \pi_\sigma(d\gamma), \tag{3.37}
\]

where \( W(1), W(2) \in \mathcal{F} \Omega^n \). Analogously to the case of Bochner, we conclude that the function under the sign of integral in (3.37) is polynomially bounded, so that the integral exists.

The next theorem follows from (3.28)–(3.36).

**Theorem 3.5** For any \( W(1), W(2) \in \mathcal{F} \Omega^n \), we have

\[
\mathcal{E}^R_{\pi_\sigma}(W(1),W(2)) = \int_{\Gamma_X} \langle H^R_{\pi_\sigma} W(1)(\gamma), W(2) \rangle_{\wedge^n(T_{\Gamma_{\pi_\sigma}}X)} \pi_\sigma(d\gamma).
\]
Here, \( H^R_{\pi \sigma} = d^R d_{\pi \sigma}^* + d_{\pi \sigma}^* d \) is an operator in the space \( L^2_{\pi \sigma} \Omega^n \) with domain \( \mathcal{F} \Omega^n \). It can be represented as follows:

\[
H^R_{\pi \sigma} W(\gamma) = \sum_{x \in \gamma} H^R_{\sigma, x} W(\gamma) = \langle H^R_{\sigma, \cdot} W(\gamma), \gamma \rangle, \quad W \in \mathcal{F} \Omega^n ,
\]

where

\[
H^R_{\sigma, x} = d_{x}^X d_{\sigma, x}^X + d_{\sigma, x}^X d_{x}^X .
\]

From Theorem 3.5 we conclude that the pre-Dirichlet form \( \mathcal{E}^R_{\pi \sigma} \) is closable in the space \( L^2_{\pi \sigma} \Omega^n \). The generator of its closure (being actually the Friedrichs extension of the operator \( H^R_{\pi \sigma} \), for which we preserve the same notation) will be called the de Rham Laplacian on \( \Gamma \) corresponding to the Poisson measure \( \pi \sigma \). By (3.38) and (3.39), \( H^R_{\pi \sigma} \) is the lifting of the de Rham Laplacian on \( X \) with measure \( \sigma \).

**Remark 3.6** Similarly to (3.17), the operator \( H^R_{\pi \sigma} \) preserves the space \( \mathcal{F} \Omega^n \), and we can always take \( \Lambda(H^R_{\pi \sigma} W) = \Lambda(W) \). Then, for any open bounded \( \Lambda \supset \Lambda(W) \), we have

\[
(H^R_{\pi \sigma} W)_{\Lambda, \gamma} = H^R_{\sigma \otimes |\Lambda \cap \gamma|} (X^{[\Lambda \cap \gamma]}) W_{\Lambda, \gamma},
\]

where \( H^R_{\sigma \otimes |\Lambda \cap \gamma|} (X^{[\Lambda \cap \gamma]}) \) is the de Rham Laplacian of the manifold \( X^{[\Lambda \cap \gamma]} \) with the product measure \( \sigma \otimes |\Lambda \cap \gamma| \).

Analogously to Theorem 3.4, we get

**Theorem 3.6** 1) On \( \mathcal{D} \Omega^n \) we have

\[
H^R_{\pi \sigma} = (I^n)^{-1} \left[ H_{\pi \sigma} \bigoplus \left( \bigoplus_{m=1}^n H^R_{\sigma, (n,m)} \right) \right] I^n ,
\]

where \( H^R_{\sigma, (n,m)} \) denotes the restriction of the de Rham Laplacian acting in the space \( L^2(X^m \rightarrow \Lambda^n(TX^m); \sigma^{\otimes m}) \) to the subspace \( L^2_\sigma \Psi^n(X^m) \).

2) Suppose that, for each \( m = 1, \ldots, n \), the de Rham Laplacian \( H^R_{\sigma, m}(X) \) is essentially self-adjoint on the set of smooth forms with compact support. Then, \( \mathcal{D} \Omega^n \) is a domain of essential self-adjointness of \( H^R_{\pi \sigma} \), and the equality (3.41) holds for the closed operators \( H^R_{\pi \sigma} \) and \( H_{\pi \sigma} \bigoplus \left( \bigoplus_{m=1}^n H^R_{\sigma, (n,m)} \right) \) (where the latter operator is closed from its domain of essential self-adjointness \( I^n(\mathcal{D} \Omega^n) \)).

**Remark 3.7** The essential self-adjointness of the de Rham Laplacian \( H^R_{\sigma} \) on the set of smooth forms with compact support is well-known in the case where \( \sigma \) is the volume measure, see e.g. [20]. It is also sufficient to assume that \( \beta \sigma \), together with its derivatives up to order 3, as well as the the curvature tensor of \( X \), together with its derivatives up to order 2, are bounded, cf. Remark 3.5.

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Proof of Theorem 3.6. Upon (3.7), (3.8), (3.27)–(3.36), (3.38), and (3.39), we get, for any $W \in \mathcal{D}^n\Omega$ given by the formula (2.23),
\[
(H^R_{\pi_\sigma} W)_{k}(\bar{x}) = 0 \quad \text{for } k \neq m, \\
(H^R_{\sigma,x} W)_{m}(\bar{x}) = \begin{cases} 
\frac{1}{2} (m!)^{1/2} (H_{\sigma,x} F)(\gamma \setminus \{\bar{x}\}) \omega(\bar{x}), & x \in \gamma \setminus \{\bar{x}\}, \\
\frac{1}{2} (m!)^{1/2} F(\gamma \setminus \{\bar{x}\})(H^R_{\sigma,x} \omega)(\bar{x}), & x \in \{\bar{x}\}.
\end{cases}
\]

Hence, analogously to (3.19) and (3.20), we derive
\[
(I^n_k H^R_{\pi_\sigma} W)(\gamma, \bar{x}) = \begin{cases} 
0, & k \neq m, \\
(H_{\pi_\sigma} F)(\gamma) \omega(\bar{x}) + F(\gamma)(H^R_{\sigma,(n,m)} \omega)(\bar{x}), & k = m,
\end{cases}
\]
which easily yields (3.41).

2) The proof is similar to that of Theorem 3.4, 2). □

Again, analogously to Corollary 3.1, we get a Fock space representation of the operator $H^R_{\pi_\sigma}$.

Corollary 3.2 Let the conditions of Theorem 3.6, 2) be satisfied. Then,
\[
\mathcal{I}^n H^R_{\pi_\sigma}(\mathcal{I}^n)^{-1} = d \exp H_{\sigma} \boxplus \left( \bigoplus_{m=1}^{n} H^R_{\sigma,(n,m)} \right).
\]

3.5 Weitzenb"{o}ck formula on the configuration space

In this section, we will derive a generalization of the Weitzenb"{o}ck formula to the case of the Poisson measure on the configuration space. In other words, we will derive a formula which gives a relation between the Bochner and de Rham Laplacians.

Analogously to (3.7), (3.8), we define for each $V(\gamma) \in T_{\gamma} \Gamma_X$, $\gamma \in \Gamma_X$, the annihilation and creation operators
\[
a(V(\gamma)): \wedge^{n+1}(T_{\gamma} \Gamma_X) \to \wedge^n(T_{\gamma} \Gamma_X), \\
a^*(V(\gamma)): \wedge^n(T_{\gamma} \Gamma_X) \to \wedge^{n+1}(T_{\gamma} \Gamma_X)
\]
as follows:
\[
a(V(\gamma))W_{n+1}(\gamma) = \sqrt{n+1} \langle V(\gamma), W_{n+1}(\gamma) \rangle_{\gamma}, \quad W_{n+1}(\gamma) \in \wedge^{n+1}(T_{\gamma} \Gamma_X), \\
a^*(V(\gamma))W_n(\gamma) = \sqrt{n+1} V(\gamma) \wedge W_n(\gamma), \quad W_n(\gamma) \in \wedge^n(T_{\gamma} \Gamma_X).
\]

Now, for a fixed $\gamma \in \Gamma_X$ and $x \in \gamma$, we define the operator $R(\gamma)$ as follows:
\[
R(\gamma) = \sum_{x \in \gamma} R(\gamma, x), \quad D(R(\gamma)) := \wedge^n_0(T_{\gamma} \Gamma_X),
\]
\[
R(\gamma, x) := \sum_{i,j,k,l=1}^{d} R_{ijkl}(x) a_i^*(e_i) a(e_j) a^*(e_k) a(e_l).
\]
Here, $\{e_j\}_{j=1}^d$ is a fixed orthonormal basis in the space $T_x X$ considered as a subspace of $T_x \Gamma_X$, and $\wedge_0^n (T_\gamma \Gamma_X)$ consists of all $W(\gamma) \in \wedge^n (T_\gamma \Gamma_X)$ having only a finite number of nonzero coordinates in the direct sum expansion (2.5).

Next, we note that

$$\nabla^\Gamma B_{\pi_\sigma}(\gamma) = (\nabla_x^X B_{\pi_\sigma}(\gamma))_{x \in \gamma} = (\nabla_x^X (B_{\pi_\sigma}(\gamma)_y))_{x, y \in \gamma} = (\delta_{x, y} \nabla^X \beta_\sigma(y))_{x, y \in \gamma} \in (T_{\gamma, \infty \Gamma_X})^\otimes 2.$$  

Hence, for any $V(\gamma) \in T_{\gamma, 0 \Gamma_X}$,

$$\nabla^\Gamma B_{\pi_\sigma}(\gamma) := (\nabla^\Gamma B_{\pi_\sigma}(\gamma), V(\gamma))_{\gamma} = \left( \sum_{y \in \gamma} \delta_{x, y} \nabla^X \beta_\sigma(y), V(\gamma)_y \right)_{x \in \gamma} = ((\nabla^X \beta_\sigma(x), V(\gamma)_x)_{x \in \gamma} \in T_{\gamma, 0 \Gamma_X}.$$  

Thus, $\nabla^\Gamma B_{\pi_\sigma}(\gamma)$ determines the linear operator in $T_{\gamma, 0 \Gamma_X}$ given by

$$T_{\gamma, 0 \Gamma_X} \ni V(\gamma) \mapsto \nabla^\Gamma B_{\pi_\sigma}(\gamma)V(\gamma) := \nabla^\Gamma B_{\pi_\sigma}(\gamma) \in T_{\gamma, 0 \Gamma_X}.$$  

Analogously to (3.9), we define in $\wedge_0^n (T_\gamma \Gamma_X)$ the operator

$$(\nabla^\Gamma B_{\pi_\sigma}(\gamma))^\wedge n := \nabla^\Gamma B_{\pi_\sigma}(\gamma) \otimes 1 \cdots \otimes 1 + 1 \otimes \nabla^\Gamma B_{\pi_\sigma}(\gamma) \otimes 1 \otimes \cdots \otimes 1$$  

$$+ \cdots + 1 \otimes \cdots \otimes 1 \otimes \nabla^\Gamma B_{\pi_\sigma}(\gamma).$$  

**Theorem 3.7 (Weitzenböck formula on the Poisson space)** We have on $\mathcal{F} \Omega^n$

$$H_{\pi_\sigma}^R = H_{\pi_\sigma}^B + R_{\pi_\sigma}(\gamma),$$  

(3.42)

where

$$R_{\pi_\sigma}(\gamma) := R(\gamma) - (\nabla^\Gamma B_{\pi_\sigma}(\gamma))^\wedge n.$$  

(3.43)

**Proof.** Fix $W \in \mathcal{F} \Omega^n$ and $\gamma \in \Gamma_X$. Let $\Lambda(W) \subset X$ be a compactum as in Definition 2.3 corresponding to $W$, and let $\Lambda$ be an open set in $X$ with compact closure such that $\Lambda(W) \subset \Lambda$. Next, let $W_{\Lambda, \gamma}$ be the form on $\mathcal{O}_{\gamma, x_1} \times \cdots \times \mathcal{O}_{\gamma, x_k}$, $\{x_1, \ldots, x_k\} = \gamma \cap \Lambda$, defined by (2.10).

It follows from Remarks 3.4, 2) and 3.6 that

$$\text{Proj}_{\wedge^n (T_{x_1} \cdots \otimes T_{x_k})} (H_{\pi_\sigma}^B W(\gamma)) = H_{\sigma \oplus |\Lambda \cap \gamma|}^B (X^{|\Lambda \cap \gamma|}) W_{\Lambda, \gamma}(x_1, \ldots, x_k),$$

$$\text{Proj}_{\wedge^n (T_{x_1} \cdots \otimes T_{x_k})} (H_{\pi_\sigma}^R W(\gamma)) = H_{\sigma \oplus |\Lambda \cap \gamma|}^R (X^{|\Lambda \cap \gamma|}) W_{\Lambda, \gamma}(x_1, \ldots, x_k),$$

and $H_{\pi_\sigma}^B W(\gamma)_{[y_1, \ldots, y_n]}_d = H_{\pi_\sigma}^R W(\gamma)_{[y_1, \ldots, y_n]}_d = 0$, $[y_1, \ldots, y_n]_d \subset \gamma$, if at least one $y_i \in \{y_1, \ldots, y_n\}$ does not belong to $\Lambda$.

Now, the formulas (3.42), (3.43) follow from the usual Weitzenböck formula (3.10), (3.11) for the operators $H_{\sigma \oplus |\Lambda \cap \gamma|}^B (X^{|\Lambda \cap \gamma|})$ and $H_{\sigma \oplus |\Lambda \cap \gamma|}^R (X^{|\Lambda \cap \gamma|})$. \hfill \blacksquare
We will show now that the Weitzenböck correction term $R_{\sigma,\kappa}$ is a lifting of the Weitzenböck correction terms $R_{\sigma,k}$ of the manifold $X$.

Given operator fields
\[
X \ni x \mapsto J_k(x) \in \mathcal{L}(\wedge^k(T_xX)), \quad k = 1, \ldots, \min\{n, d\},
\]
which are supposed to be uniformly bounded, we define a “diagonal” operator field
\[
\tilde{X}^m \ni \tilde{x} \mapsto J_{n,m}(\tilde{x}) \in \mathcal{L}(\mathbb{T}^{(n)}_{\{\tilde{x}\}} X^m), \quad m = 1, \ldots, n,
\]
as follows. First, we define for each $\tilde{x} = (x_1, \ldots, x_m) \in \tilde{X}^m$ operators
\[
J_{n,m}^{k_1,\ldots,k_m}(x_1, \ldots, x_m) \in \mathcal{L}((T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m}X)^{k_m}),
\]
\[\begin{align*}
1 \leq k_1, \ldots, k_m &\leq d, \quad k_1 + \cdots + k_m = n,
\end{align*}\]
by setting
\[
J_{n,m}^{k_1,\ldots,k_m}(x_1, \ldots, x_m) u_1^{(k_1)} \wedge \cdots \wedge u_m^{(k_m)} := (J_{k_1}(x_1) u_1^{(k_1)}) \wedge (J_{k_2}(x_2) u_2^{(k_2)}) \wedge \cdots \wedge (J_{k_m}(x_m) u_m^{(k_m)}),
\]
\[\begin{align*}
+ u_1^{(k_1)} \wedge (J_{k_2}(x_2) u_2^{(k_2)}) \wedge \cdots \wedge u_m^{(k_m)} + \cdots + u_1^{(k_1)} \wedge \cdots \wedge u_{m-1}^{(k_{m-1})} \wedge (J_{k_m}(x_m) u_m^{(k_m)}),
\end{align*}\]
and extending the operator $J_{n,m}^{k_1,\ldots,k_m}(x_1, \ldots, x_m)$ by linearity and continuity to the whole space. Then, the operator $J_{n,m}(x_1, \ldots, x_m) \in \mathcal{L}(\mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m)$ is defined by setting its diagonal blocks in the decomposition (2.16) of the space $\mathbb{T}^{(n)}_{\{x_1, \ldots, x_m\}} X^m$ to be $J_{n,m}^{k_1,\ldots,k_m}(x_1, \ldots, x_m)$ and the other blocks to be equal to zero.

Notice that, for each $\nu \in S_m$, the operators $J_{n,m}(x_1, \ldots, x_m)$ and $J_{n,m}(x_{\nu(1)}, \ldots, x_{\nu(m)})$ coincide, so that (3.45) naturally determines the operator field
\[
\tilde{X}^m / S_m \ni \{\tilde{x}\} \mapsto J_{n,m}(\{\tilde{x}\}) \in \mathcal{L}(\mathbb{T}^{(n)}_{\{\tilde{x}\}} X^m), \quad m = 1, \ldots, n.
\]

Now, we define an operator field
\[
\Gamma_X \ni \gamma \mapsto J(\gamma) \in \mathcal{L}(\wedge^n(T\gamma X))
\]
setting $J(\gamma)$ to be again the block-diagonal operator in the decomposition (2.17) with the diagonal blocks $J_{n,m}(\{\tilde{x}\})$ and the other blocks equal to zero.

In what follows, we suppose, for simplicity, that

the curvature tensor $R_{ij\kappa}(x)$ and $\nabla^X \beta_\sigma(x)$ are uniformly bounded in $x \in X$.

As easily seen, for each $k \in \mathbb{N}$, the Weitzenböck correction term $R_{\sigma,k}(\bullet)$ on the manifold $X$ is now a uniformly bounded operator field taking values in $\wedge^k(TX)$ (cf. 3.11). Thus we can define an operator field $R_\sigma(\gamma)$ through the operator fields $R_{\sigma,k}(x)$.

**Proposition 3.1** Let (3.49) hold. Then,
\[
R_{\pi,\sigma} = R_\sigma.
\]

**Proof.** The result can be easily seen directly from the definition of $R_\sigma(\gamma)$, $R(\gamma)$ and $B_{\pi,\sigma}(\gamma)$.

\[\blacksquare\]
4 Probabilistic representation of the Laplacians

Let \( \xi_x(t) \) be the Brownian motion on \( X \) with the drift \( \beta \)—the logarithmic derivative of \( \sigma \)—which starts at a point \( x \in X \). We suppose the following:

- for each \( x \in X \), the process \( \xi_x(t) \) has infinite life-time;
- the semigroup
  \[
  T_0(t) f(x) := \mathbb{E} f(\xi_x(t))
  \]
  preserves the space \( C^2_b(X) \) and can be extended to a strongly continuous semigroup of contractions in \( L^2(X; \sigma) \), and its generator \( H_0 \) is essentially self-adjoint on the space \( D \) (in this case \( H_0 = H_\sigma \)).

**Remark 4.1** The above conditions are fulfilled if e.g. \( \beta_\sigma \), together with its derivatives up to order 3, is bounded.

We denote by \( \xi_\gamma(t) \) the corresponding independent infinite particle process which starts at a point \( \gamma \in \Gamma_X \),

\[
\xi_\gamma(t) = (\xi_x(t))_{x \in \gamma}.
\]

As we already mentioned in subsec. 3.1, this process is properly associated with the Dirichlet form \( \mathcal{E}_{\pi_\sigma} \), see [8].

**Remark 4.2** The process \( \xi_\gamma \) lives in general on the bigger state space \( \Gamma^\infty_X \) consisting of all \( \mathbb{R}^+ \)-valued Radon measures on \( X \) (the space \( \Gamma_X \) being Polish). Notice, however, that at each fixed moment of time \( t \in \mathbb{R}^+ \) the value \( \xi_\gamma(t) \) belongs to \( \Gamma_X \) a.s. Moreover, it was proven in [34] that, in the special case \( X = \mathbb{R}^d \) with \( d \geq 2 \), the process \( \xi_\gamma \) lives a.s. in \( \Gamma_X \).

Let \( T_0(t) F(\gamma) := \mathbb{E} F(\xi_\gamma(t)) \) be the corresponding semigroup. As shown in [8], it can be extended from \( FC^\infty_b(\Gamma_X) \) to a strongly continuous semigroup in \( L^2(\pi_\sigma; \Gamma_X) \) with the generator \( H_0 = H_{\Gamma_\pi_\sigma} \).

Given operator fields (3.44) which are now supposed to be uniformly bounded, continuous, and symmetric, we define again operator fields (3.45) in the same way as in subsec. 3.5. We have obviously \( J_{n,m}(\bar{x})^* = J_{n,m}(\bar{x}) \).

Let

\[
P_{\xi_\gamma}^{J_{n,m}}(t): \mathcal{T}^{(n)}_{\{\xi_\gamma(t)\}} X^m \rightarrow \mathcal{T}^{(n)}_{\{\xi_\gamma(t)\}} X^m, \quad m = 1, \ldots, n
\]

be the parallel translation along the path \( \xi_\gamma(t) := (\xi_{x_i}(t))_{i=1,\ldots,m} \) with the potential \( J_{n,m} \).

That is, \( \eta(t) = P_{\xi_\gamma}^{J_{n,m}}(t)h \) satisfies the SDE

\[
\frac{D}{dt} \eta(t) = J_{n,m}(\eta(t)), \quad \eta(0) = h,
\]

where \( \frac{D}{dt} \) denotes the covariant differentiation along the paths of the process \( \xi \) (see [20]). It is easy to see that the symmetry of the potential \( J_{n,m}(\bar{x}) \) w.r.t. a permutation of the components
of \( \bar{x} \) implies the same symmetry of \( P^{J,n,m}_{\xi} (t) \). Thus, analogously to (3.47), we get the operator field

\[
\hat{X}^m / S_m \ni \{ \bar{x} \} \mapsto P^{J,n,m}_{\xi} (t).
\] (4.2)

Now, for \( \pi_{\sigma} \)-a.e. \( \gamma \in \Gamma_X \), we define the operator

\[
P^J_{\xi \gamma} (t) : \wedge^n (T_{\xi} \Gamma_X) \to \wedge^n (T_{\xi \gamma} (t) \Gamma_X)
\]

by setting its diagonal blocks in the decomposition (2.17) to be \( P^{J,n,m}_{\xi} (t) \) and the other blocks to be equal to zero.

It is known that

\[
\| P^{J,n,m}_{\xi} (t) \| \leq e^{t C_m}, \quad m = 1, \ldots, n,
\] (4.3)

where \( C_m \) is the supremum of the spectrum of \( J_{n,m}(\bar{x}) \).

**Lemma 4.1** For \( \pi_{\sigma} \)-a.e. \( \gamma \in \Gamma_X \), we have

\[
\| P^J_{\xi \gamma} (t) \| \leq e^{t C}, \quad C = \max_{m=1, \ldots, n} C_m.
\] (4.4)

**Proof.** The result follows directly from the definition of \( P^J_{\xi \gamma} (t) \) and estimate (4.3). ■

Let us define a semigroup \( T^J_n (t) \) acting in the space of \( n \)-forms as follows:

\[
T^J_n (t) W(\gamma) := E (P^J_{\xi \gamma} (t))^* W(\xi \gamma (t)), \quad W \in F^\Omega^n.
\] (4.5)

Let \( T^J_{n,m} (t) \) be the semigroup acting in the space \( L^2_{\sigma} \Psi^n (X^n) \) as

\[
T^J_{n,m} (t) \omega(\bar{x}) := E (P^{J,n,m}_{\xi} (t))^* \omega(\xi \bar{x} (t)).
\] (4.6)

By virtue of (4.2) and estimate (4.3), we conclude the correctness of the definition of \( T^J_{n,m} (t) \) (in the sense that \( T^J_{n,m} (t) \) is uniquely defined) and its strong continuity. The following result describes the structure and properties of the semigroup \( T^J_n (t) \).

**Proposition 4.1** 1) \( T^J_n (t) \) satisfies the estimate

\[
\| T^J_n (t) V(\gamma) \|_{\wedge^n (T, \Gamma_X)} \leq e^{t C} T^J_{0} (t) \| V(\gamma) \|_{\wedge^n (T, \Gamma_X)}
\] (4.7)

for \( \pi_{\sigma} \)-a.e. \( \gamma \in \Gamma_X \).

2) Under the isomorphism \( I^n \), \( T^J_n (t) \) takes the following form:

\[
I^m_n T^J_n (t) = T^J_{0} (t) \otimes T^J_{n,m} (t) I^m_n, \quad m = 1, \ldots, n.
\] (4.8)

In particular, for 1-forms

\[
I^1 T^J_1 (t) = T^J_{0} (t) \otimes T^J_{1,1} (t) I^1.
\] (4.9)

3) \( T^J_n (t) \) extends to a strongly continuous semigroup in \( L^2_{\pi_{\sigma}} \Omega^n \).
Proof. 1) The result follows from formula (4.4).

2) For simplicity, we give the proof only in the case of 1-forms. Let $V \in \mathcal{D}\Omega^1$ be given by $I^1V = F \otimes v$. By the definition of $T^J_1(t)$ and the construction of the process $\xi_\gamma$, we have

$$T^J_1(t)V(\gamma)_x = \mathbb{E} F(\xi_\gamma(t) \setminus \{x_\gamma(t)\}) (P^{J_{1,1}}_{\xi_\gamma}(t))^* v(\xi_\gamma(t))$$

$$= \mathbb{E} F(\xi_\gamma(t) \setminus \{x_\gamma(t)\}) \mathbb{E}_{\xi_\gamma} (P^{J_{1,1}}_{\xi_\gamma}(t))^* v(\xi_\gamma(t))$$

$$= T_0(t) F(\gamma \setminus \{x\}) T^J_1(t) v(x),$$

$\mathbb{E}_{\xi_\gamma}$ meaning the expectation w.r.t. the process $\xi_\gamma(t)$, from where the result follows. The general case can be proved by similar arguments.

3) The result follows from the corresponding results for semigroups $T_0(t)$ and $T^J_{n,m}(t)$, which are well-known (see [6], resp. [20]).

Let $H^n_\sigma$ and $H^J_{n,m}$ be the generators of $T^n_\sigma(t)$ and $T^J_{n,m}(t)$, respectively.

Now, we will give probabilistic representations of the semigroups $T^B_\sigma(t)$ and $T^R_\sigma(t)$ associated with the operators $H^B_\sigma$ and $H^R_\sigma$, respectively. We set

$$J^{(1)}_m := 0, \quad J^{(2)}_m(x) := R_{\sigma,m}(x), \quad m = 1, \ldots, \min\{n, d\}$$

(cf. (3.11)). Let us remark that $P^{J^{(i)}_{\xi_\gamma,m}}_{\xi_\gamma}(t) \equiv P^{J_{\xi_\gamma}}_{\xi_\gamma}(t)$ is the parallel translation of the $n$-forms along the path $\xi_\gamma$, and we have

$$H^{J^{(1)}_{n,m}}_\sigma = -H^B_{\sigma,(m,m)}, \quad H^{J^{(2)}_{n,m}}_\sigma = -H^R_{\sigma,(m,m)} \quad \text{on } \Psi^n_\sigma(X^m).$$

Theorem 4.1 1) For $W \in \mathcal{D}\Omega^m$, we have

$$H^B_\sigma W = -H^{J^{(1)}_n} W, \quad H^R_\sigma W = -H^{J^{(2)}_n} W. \quad (4.10)$$

2) As $L^2$-semigroups,

$$T^B_\sigma(t) = T^{J^{(1)}_n}(t), \quad T^R_\sigma(t) = T^{J^{(2)}_n}(t). \quad (4.11)$$

3) The semigroups $T^B_\sigma(t)$ and $T^R_\sigma(t)$ satisfy the estimates

$$\|T^B_\sigma(t)V(\gamma)\|_\gamma \leq T_0(t)\|V(\gamma)\|_\gamma$$

and

$$\|T^R_\sigma(t)V(\gamma)\|_\gamma \leq e^{tC} T_0(t)\|V(\gamma)\|_\gamma$$

for $\pi_\sigma$-a.e. $\gamma \in \Gamma_X$.

Proof. 1) It follows directly from the decomposition (4.8) that, on $\mathcal{D}\Omega^n$, we have

$$I^n_m \mathbf{H}^J_n = (H_0 \boxplus H^J_{n,m}) I^n_m, \quad (4.12)$$

where $H_0$ is the generator of $T_0(t)$. Setting respectively $J_m := J^{(1)}_m$ and $J_m := J^{(2)}_m$ and comparing (3.18) with (4.12), we obtain the result.

2) The statement follows from (4.10) and the essential self-adjointness of $H^B_\sigma$ and $H^R_\sigma$ on $\mathcal{D}\Omega^n$ by applying Proposition 4.1, 3), with $J_m = J^{(1)}_m$ and $J_m = J^{(2)}_m$.

3) The result follows from (4.11) and (4.7).
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References

[1] S. Albeverio, A. Daletskii, and Yu. Kondratiev, Stochastic analysis on product manifolds: Dirichlet operators on differential forms, Preprint SFB 256 No. 598, Universität Bonn, 1999, to appear in J. Funct. Anal.

[2] S. Albeverio, A. Daletskii, and Yu. Kondratiev, De Rham complex over product manifolds: Dirichlet forms and stochastic dynamics, to appear in “Festschrift of L. Streit” (eds. S. Albeverio et al.), World Scientific, 2000.

[3] S. Albeverio, A. Daletskii, and E. Lytvynov, Laplace operators and diffusions in tangent bundles over Poisson spaces, Preprint SFB 256 No. 629, Universität Bonn, 1999, to appear in “Proceedings of KNAW.”

[4] S. Albeverio and R. Høegh-Krohn, Dirichlet forms and Markov semigroups on C*-algebras, Comm. Math. Phys. 56 (1977), 173–187.

[5] S. Albeverio and Yu. Kondratiev, Supersymmetric Dirichlet operators, Ukrainian Math. J. 47 (1995), 583–592.

[6] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Differential geometry of Poisson spaces, C. R. Acad. Sci. Paris 323 (1996), 1129–1134.

[7] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces, C. R. Acad. Sci. Paris 323 (1996), 1179–1184.

[8] S. Albeverio, Yu. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, J. Func. Anal. 154 (1998), 444–500.

[9] S. Albeverio, Yu. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces: The Gibbsian case, J. Func. Anal. 157 (1998), 242–291.

[10] S. Albeverio, Yu. Kondratiev, and M. Röckner, Diffeomorphism groups and current algebras: Configuration spaces analysis in quantum theory, Rev. Math. Phys. 11 (1999), 1–23.

[11] A. Arai, Supersymmetric extension of quantum scalar field theories, in “Quantum and Non-Commutative Analysis” (eds. H. Araki et al.), Kluwer Academic Publishers, Dordrecht, 1993, 73–90.
[12] A. Arai, Dirac operators in Boson–Fermion Fock spaces and supersymmetric quantum field theory, *J. Geometry and Physics* **11** (1993), 465–490.

[13] A. Arai and I. Mitoma, De Rham–Hodge–Kodaira decomposition in $\infty$-dimensions, *Math. Ann.* **291** (1991), 51–73.

[14] A. Bendikov and R. Léandre, Regularized Euler–Poincaré number of the infinite dimensional torus, *IDAQP* **2** (1999) 617–626.

[15] Yu. M. Beresansky, “Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables,” Amer. Math. Soc., Providence, R.I., 1986.

[16] A. Connes and H. Moscovici, The $L^2$-index theorem for homogeneous spaces of Lie groups, *Ann. of Math.* **115** (1982), 291–330.

[17] L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, “Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry,” Springer, Berlin, New York, 1987.

[18] E. B. Davies and J. M. Lindsay, Non-commutative symmetric Markov semigroups, *Math. Zeit.* **210** (1992), 379–411.

[19] J. L. Doob, “Stochastic Processes,” Wiley & Sons, New York, London, 1953.

[20] K. D. Elworthy, Geometric aspects of diffusions on manifolds, In “Lecture Notes in Math.,” Vol. 1362, pp. 276–425, Springer, Berlin, New York, 1988.

[21] K. D. Elworthy, Y. Le Jan, and X.-M. Li, “On the Geometry of Diffusion Operators and Stochastic Flow,” Lecture Notes in Mathematics, Vol. 1720, Springer, Berlin, 1999.

[22] G. A. Goldin, J. Grodnik, R. T. Powers, and D. H. Sharp, Nonrelativistic current algebra in the $N/V$ limit, *J. Math. Phys.* **15** (1974), 88–100.

[23] L. Gross, Hypercontractivity and logarithmic Sobolev inequalities for Clifford–Dirichlet forms, *Duke Math. J.* **43** (1975), 383–386.

[24] R. S. Ismagilov, “Representations of Infinite-Dimensional Groups,” Amer. Math. Soc., Providence, R. I., 1996.

[25] J. D. S. Jones and R. Léandre, A stochastic approach to the Dirac operator over the free loop space, in “Loop Spaces and the Group of Diffeomorphisms,” Proceedings of Steklov Institute, Vol. 217, 1997, 253–282.

[26] J. Kerstan, K. Matthes, and J. Mecke, “Infinite Divisible Point Processes,” Akademie-Verlag, Berlin, 1978.

[27] R. Léandre and S. S. Roan, A stochastic approach to the Euler–Poincaré number of the loop space of developable orbifold, *J. Geometry and Physics* **16** (1995), 71–98.
[28] J. J. Prat, and N. Privault, Explicit stochastic analysis of Brownian motion and point measures on Riemannian manifolds, *J. Funct. Anal.* **167** (1999), 201–242.

[29] N. Privault, Equivalence of gradients on configuration spaces, *Random Oper. Stoch. Eq.* **7** (1999), 241–262.

[30] N. Privault, Connections and curvature in the Riemannian geometry of configuration spaces, La Rochelle preprint, 1999.

[31] A. L. Rebenko and G. V. Shchepan’uk, The convergence of the cluster expansion for continuous systems with many-body interaction, *J. Stat. Physics* **88** (1997), 665–689.

[32] M. Reed and B. Simon, “Methods of Modern Mathematical Physics, Vol. 2. Fourier Analysis, Self-Adjointness,” Academic Press, New York, London, 1972.

[33] M. Röckner, Stochastic analysis on configuration spaces: Basic ideas and recent results, *in* “New Directions in Dirichlet Forms” (eds. J. Jost et al.), Studies in Advanced Mathematics, Vol. 8, American Math. Soc., 1998, 157–232.

[34] M. Röckner and B. Schmuland, A support property for infinite-dimensional interacting diffusion processes, *C. R. Acad. Sci. Paris* **326** (1998), Série I, 359–364.

[35] J. Roe, An index theorem on open manifolds, I, II, *J. Diff. Geometry* **27** (1988), 87–113, 115–136.

[36] A. M. Vershik, I. M. Gel’fand, and M. I. Graev, Representations of the group of diffeomorphisms, *Russian Math. Surv.* **30** (1975), 1–50.

[37] W. E. Witten, The index of the Dirac operator in loop space, *in* “Elliptic Curves and Modular Forms in Algebraic Topology,” Lecture Notes in Mathematics (ed. P. S. Landweber), pp. 161–181, Springer, Berlin, New York, 1988.