REDUCTION OF INTEGRATION DOMAIN IN TRIEBEL–LIZORKIN SPACES

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Abstract. We investigate the comparability of generalized Triebel–Lizorkin and Sobolev seminorms on uniform and non-uniform sets when the integration domain is truncated according to the distance from the boundary. We provide numerous examples of kernels and domains in which the comparability does and does not hold.

1. Introduction

Let \( \Omega \) be a domain in \( \mathbb{R}^d \), \( d \geq 1 \), and let \( p, q \in (1, \infty) \). Let \( K : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty) \) be a homogeneous, radial kernel, i.e. \( K(x, y) = k(|x - y|) \), satisfying \( \int_{\mathbb{R}^d} (1 \wedge |y|^q) K(0, y) \, dy < \infty \) for \( x \in \mathbb{R}^d \). We define the (generalized) Triebel–Lizorkin space on \( \Omega \) as

\[
F_{p,q}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega} \left( \int_{\Omega} |u(x) - u(y)|^q K(x, y) \, dx \right)^{\frac{p}{q}} \, dy < \infty \right\}.
\]

The space obviously depends on \( K \), however we skip it in the notation for clarity.

\( F_{p,q}(\Omega) \) is endowed with the norm

\[
\|u\|_{F_{p,q}(\Omega)} = \|u\|_{L^p(\Omega)} + \left( \int_{\Omega} \left( \int_{\Omega} |u(x) - u(y)|^q K(x, y) \, dx \right)^{\frac{p}{q}} \, dy \right)^{\frac{1}{p}}.
\]

We are interested in the Gagliardo-type seminorm

\[
\left( \int_{\Omega} \left( \int_{\Omega} |u(x) - u(y)|^q K(x, y) \, dx \right)^{\frac{p}{q}} \, dy \right)^{\frac{1}{p}},
\]

which will be called the full seminorm. Let \( \theta > 0 \) and let \( \delta(y) = d(y, \partial \Omega) \). Our main goal is to establish the comparability of the full seminorm and the truncated seminorm

\[
\left( \int_{\Omega} \left( \int_{B(y, \theta \delta(y))} |u(x) - u(y)|^q K(x, y) \, dx \right)^{\frac{p}{q}} \, dy \right)^{\frac{1}{p}}
\]

for sufficiently regular \( K \) and \( \Omega \). Later on, such occurrence will be called a comparability result.

Here is our first comparability result.

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Theorem 1.1. Assume that $\Omega$ is a uniform domain, and that $K$ satisfies $A_1$, $A_2$ and $A_3$. Assume that $1 < q \leq p < \infty$. Then for every $0 < \theta \leq 1$

$$
\left( \int_{\Omega} \left( \int_{\Omega} |u(x) - u(y)|^q K(x,y) \, dx \right)^{\frac{p}{q}} \, dy \right)^{\frac{1}{p}} \approx \left( \int_{\Omega} \left( \int_{B(y,\theta \delta(y))} |u(x) - u(y)|^q K(x,y) \, dx \right)^{\frac{p}{q}} \, dy \right)^{\frac{1}{p}}.
$$

The comparability constant depends on $p,q,\theta$, and the constants in assumptions $A_2, A_3$.

This is a generalization of the result of Prats and Saksman [16] Theorem 1.6 who prove it in the fractional case: $K(x,y) = |x - y|^{-d-qs}$ for $s \in (0,1)$. We adapt their method of proof for a wide class of kernels of the form $K(x,y) = |x - y|^{-d}\phi(|x - y|)^{-q}$. The most technical assumption $A_2$ is tailored for the key Lemma 2.1, however in Subsection 4.2 we argue that it amounts to $O$-regular variation of $\phi$. For the formulation of the assumptions, see Subsection 2.1.

A result of this flavor was established earlier by Dyda [8] and was used to obtain Hardy inequalities for nonlocal operators. More recently, Bux, Kassmann and Schulze [6] studied comparability of full Sobolev seminorms with assumptions, see Subsection 2.1. However, their overall aim and scope are different than ours.

Here we mention that, independently of our work, M. Kassmann and V. Wagner have also proved comparability results which extend the ones from [16], allowing for kernels with scaling conditions, see [13]. However, their overall aim and scope are different than ours.

Notably, we go beyond the uniform domains, where the methods used by Prats and Saksman are no longer available. Namely, we prove that the comparability may hold for the fractional Sobolev spaces in strip domains.

Theorem 1.2. Assume that $p = q = 2$. Let $\Omega = \mathbb{R}^k \times (0,1)^l \subseteq \mathbb{R}^{k+l}$ with $k, l > 0$. For $d = k + l$ let $K(x,y) = |x - y|^{-d-\alpha}$. If $l = 1$ and $\alpha > 1$, or $l > 1$, then the seminorms (1.2) and (1.3) are comparable.

We also construct a counterexample for $\alpha < 1$ and $k = l = 1$. This shows an intriguing interplay between the kernel and the width of the domain. Informally speaking, the comparability is more likely to hold if the stochastic process corresponding to the jump kernel $K \cdot 1_{\Omega \times \Omega}$ traverses large distances in $\Omega$ by many small jumps rather than few large jumps.

Another object of our studies is the 0-order kernel $K(x,y) \approx |x - y|^{-d}$. We provide examples showing that the comparability does not hold in this case. In an attempt to repeat the proof of Theorem 1.1 we obtain an estimate of (1.2) by a truncated seminorm with a slightly more singular kernel.

The classical Triebel–Lizorkin spaces were introduced independently by Lizorkin [15] and Triebel [22]. The original definition is formulated using Paley–Littlewood theory and is widely used in analysis and applications, see e.g. [1] [5] [11]. For various cases of $p,q,d$ and $\Omega$ the classical definition was proved to be equivalent to (1.1) with $K(x,y) = |x - y|^{-d-sq}$, where $s \in (0,1)$, see [16] [20] [23].

The definition (1.1) seems more natural if the starting point is $p = q = 2$, e.g., fractional Sobolev spaces in nonlocal PDEs [7] [9] [18], or Dirichlet forms for Hunt processes [4] [10]. It is also a suitable definition for considering kernels $K$ more general than $|x - y|^{-d-sq}$ which is also of interest in the field of nonlocal operators and stochastic processes. In this paper we will not attempt to characterize the definition (1.1) in the spirit of
classical definitions by Triebel and Lizorkin in the full generality. However, we appreciate the Fourier methods in Section 5, where we compare spaces with kernels which are only slightly different from each other.

As we argue further in the article, the comparability results can be used to study a class of stochastic processes, whose jumps from \( y \) are restricted to the ball \( B(y, \theta \delta(y)) \). The truncated seminorms may also prove useful in peridynamics, as \( B(y, \theta \delta(y)) \) may be understood as the variable horizon, see e.g. [3] [14], and in particular [21] where horizons depending on the distance from the boundary are studied.

The paper is organized as follows. Section 2 contains notions, assumptions, and basic facts used further in our work. Section 3 is devoted to proving Theorem 1.1. In Section 4 we present positive and negative examples of kernels concerning the comparability results. Section 5 contains the analysis of 0-order kernels. In Section 6 we consider strip domains, in particular we prove Theorem 1.2. Section 7 presents the connection of our development with the theory of Hunt processes.

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2. Preliminaries and assumptions

2.1. Assumptions on the kernel. For the main result, we fix \( q > 1 \) and assume that the kernel \( K \) is of the form \( K(x, y) = |x - y|^{-d} \phi(|x - y|)^{-q} \), where \( \phi : (0, \infty) \to (0, \infty) \) satisfies

\[ A1 \quad (1 \wedge |y|^q)|y|^{-d} \phi(|y|)^{-q} \in L^1(\mathbb{R}^d), \]
\[ A2 \quad \phi \text{ is increasing and there exists } M = M(\phi, \Omega) > 0 \text{ such that for every } 0 < r < \text{diam}(\Omega) \sum_{k=1}^{\infty} \frac{\phi(r)^q}{r^{q(2^k + 1)}} \leq M. \]
\[ A3 \quad \text{If } C > 1, \text{ then there exists } C' = C'(C), \text{ such that the following implication holds for every } 0 < r, s < 3 \text{diam}(\Omega): \ r \leq Cs \implies (C')^{-1} \phi(s) < \phi(r) < C' \phi(s). \]

In particular, we allow unbounded domains in which the scaling conditions A2, A3 become global. Note that A1 is a Lévy-type condition. If \( q = 2 \) and \( \phi(t) = t^s, s \in (0, 1) \), then \( K \) corresponds to the fractional Laplacian of order \( s \) and all the assumptions are satisfied. The conditions A2 and A3 imply certain lower scaling for \( K \), see Subsection 4.2 for the details.

2.2. Whitney decomposition and uniform domains. For cubes \( Q, R \) in \( \mathbb{R}^d \) we consider \( l(R) \) – the length of the side of \( R \), and the long distance between \( Q \) and \( R \): \( D(Q, R) = l(Q) + d(Q, R) + l(R) \), where \( d \) is the Euclidean distance. The scaling of the cube is done from its center \( -x_Q \).

We say that a family of (closed) dyadic cubes \( \mathcal{W} \) is a Whitney decomposition of \( \Omega \) if for every \( Q, S \in \mathcal{W} \)

- if \( Q \neq S \), then \( \text{int}(Q) \cap \text{int}(S) = \emptyset \);
- if \( Q \cap S = \emptyset \), then \( l(Q) \leq 2l(S) \);
- if \( Q \subseteq 5S \), then \( l(S) \leq 2l(Q) \);
- there exists a constant \( C_\mathcal{W} \) such that \( C_\mathcal{W}l(Q) \leq d(Q, \partial \Omega) \leq 4C_\mathcal{W}l(Q) \).
The last two conditions yield $5Q \subseteq \Omega$ for every $Q \in \mathcal{W}$. A sequence of cubes $(Q, R_1, \ldots, R_n, S)$ is a chain connecting $Q$ and $S$, if every cube has nonempty intersection with its successor and predecessor (if it has one) – we call such cubes neighboring. We will denote the chain as $[Q, S]$ and the sum of the lengths of its cubes as $l([Q, S])$.

The Whitney decomposition is admissible, if there exists $\varepsilon > 0$ such that for every pair of cubes $Q, S$, there exists an $\varepsilon$-admissible chain $[Q, S] = (Q_1, Q_2, \ldots Q_n)$, i.e.

- $l([Q, S]) \leq \frac{1}{\varepsilon} D(Q, S)$,
- there exists $j_0 \in \{1, \ldots, n\}$ for which $l(Q_j) \geq \varepsilon D(Q, Q_j)$ for every $1 \leq j \leq j_0$, and $l(Q_j) \geq \varepsilon D(Q_j, S)$ for every $j_0 \leq j \leq n$. $Q_{j_0}$ will be denoted as $Q_S$ – it is the central cube of the chain $[Q, S]$.

A domain which has an admissible Whitney decomposition is called a uniform domain. Unless we state otherwise, $[Q, S]$ is an arbitrary $(\varepsilon)$-admissible chain connecting $Q$ and $S$.

Let us define the shadow of a cube: $\text{Sh}_\rho(Q) = \{ S \in \mathcal{W} : S \subseteq B(x_Q, \rho l(Q)) \}$. We also denote $\text{SH}_\rho(Q) = \bigcup \text{Sh}_\rho(Q)$. Note that we can take a sufficiently large $\rho_\varepsilon$ so that

- for every $\varepsilon$-admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$, we have $Q \in \text{Sh}_{\rho_\varepsilon}(P)$,
- if $[Q, S]$ is $\varepsilon$-admissible, then every cube from it belongs to $\text{Sh}_{\rho_\varepsilon}(Q_S)$,
- for every $Q \in \mathcal{W}$, $5Q \subseteq \text{SH}_{\rho_\varepsilon}(Q)$.

From now on we fix $\rho_\varepsilon$ and write $\text{Sh}(Q) = \text{Sh}_{\rho_\varepsilon}(Q)$ and $\text{SH}(Q) = \text{SH}_{\rho_\varepsilon}(Q)$.

The next result provides some inequalities for the Hardy–Littlewood maximal operator (denoted by $M$) with connection to the kernel $K$.

**Lemma 2.1.** Let $\Omega$ be a domain with Whitney covering $\mathcal{W}$, and let $\phi$ satisfy A1, A2 and A3. Assume that $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $0 < r < 3 \text{ diam}(\Omega)$. For every $\eta > q$, $Q \in \mathcal{W}$ and $x \in \mathbb{R}^d$, we have

$$
\int_{|y-x|>r} \frac{g(y) \, dy}{|y-x|^d \phi(|y-x|)^\eta} \lesssim \frac{Mg(x)}{\phi(r)^\eta}, \quad \sum_{S:D(Q,S)>r} \int_{S} g(y) \, dy D(Q,S)^d \phi(D(Q,S))^\eta \lesssim \frac{\inf_{x \in Q} Mg(x)}{\phi(r)^\eta}.
$$

and

$$
\sum_{S \in \mathcal{W}} \frac{l(S)^d}{D(Q,S)^d \phi(D(Q,S))^\eta} \lesssim \frac{1}{\phi(l(Q))^\eta}.
$$

**Proof.** Let us look at the first claim of (2.1). For clarity, assume that $x = 0$. Since $1/\phi$ is decreasing, we get

$$
\int_{|y|>r} \frac{\phi(r)^\eta g(y) \, dy}{|y|^d \phi(|y|)^\eta} = \sum_{k=1}^\infty \int_{2^{k-1}r < |y| < 2kr} \frac{g(y) \phi(r)^\eta \, dy}{|y|^d \phi(|y|)^\eta} \lesssim \sum_{k=1}^\infty \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta} \frac{1}{|B_{2^k r}|} \int_{2^{k-1}r < |y| < 2kr} g(y) \, dy \lesssim \sum_{k=1}^\infty \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta} Mg(0).
$$

The sum is bounded with respect to $r$ thanks to A2 and the fact that $\eta > q$. For the right hand side part of (2.1) note that if $D(Q,S) > r$, then for every $x \in Q$, $y \in S$, we have $|x-y| + r \lesssim D(Q,S)$. Therefore, by A3...
for every \( x \in Q \) we have
\[
\sum_{S:D(Q,S) > r} \frac{\phi(r)^n}{D(Q,S)^d \phi(D(Q,S))} \int_S g(y) \, dy \lesssim \int_{\mathbb{R}^d} \frac{g(y)\phi(r)^n}{(|x-y| + r)^d \phi(|x-y| + r)^n} dy.
\]

\[
\leq \int_{|x-y| > r} \frac{\phi(r)^n g(y)}{|x-y|^{d} \phi(|x-y|)^{n}} \, dy + \int_{|x-y| < r} \frac{\phi(r)^n g(y)}{(|x-y| + r)^d \phi(|x-y| + r)^n} \, dy.
\]

The claim follows from the previous estimate. Since the constants in the last inequality does not depend on \( x \), the same holds for the infimum.

\[(2.2)\] can be obtained by taking \( g \equiv 1 \) and \( r = l(Q) \) in \[(2.1)\]. In that case \( D(Q,S) > r \) for every \( S \), including \( Q \).

Under A3 the following fact can be proved identically as the analogue \([16] (2.7),(2.8)\).

**Lemma 2.2.** Let \( s > 0 \).
\[
\sum_{R:P \in Sh_p(R)} \phi(l(R))^{-s} \lesssim \phi(l(P))^{-s}.
\]

Furthermore, if \( S \in Sh(R) \), then
\[
\sum_{P \in [S,R]} \phi(l(P))^s \lesssim \phi(l(R))^s.
\]

For a further discussion of these results see Section 3 of the paper by Prats and Tolsa \([17]\).

### 3. Proof of Theorem 1.1

**Remark 3.1.** The proof involves a lot of ‘\( \lesssim \)’, and ‘\( \gtrsim \)’ signs. We would like to stress that any comparability for \( \phi \) stems from A2 and A3 with \( R = 3 \text{diam}(\Omega) \) and bounded \( C \). In particular, for fixed \( p, q \), the constants depend only on the constants in A2 and A3, and the geometry of \( \Omega \) (including the dimension).

**Proof of Theorem 1.1.** Obviously it suffices to show that the truncated seminorm dominates the full one up to a multiplicative constant.

We will work with dual norms, namely
\[
(3.1) \quad \sup_{g \geq 0} \int_{\Omega} \int_{\Omega} |f(x) - f(y)|^{q} \phi(|x-y|)^{-\frac{d}{q}} \, dy \, dx.
\]

From now on, \( g \) will be like in formula \[(3.1)\].

First let us take care of the case when \( x \) and \( y \) are close to each other. By the Hölder’s inequality, we get
\[
\sum_{Q \in W} \int_{Q} \int_{2Q} \frac{|f(x) - f(y)|g(x,y)}{|x-y|^{\frac{d}{q}} \phi(|x-y|)} \, dy \, dx \leq \sum_{Q \in W} \int_{Q} \left( \int_{2Q} \frac{|f(x) - f(y)|^{q}}{|x-y|^{d} \phi(|x-y|)^{q}} \, dy \right)^{\frac{1}{q}} \left( \int_{2Q} g(x,y)^{q} \, dy \right)^{\frac{1}{q}} \, dx,
\]

\[
\leq \left( \sum_{Q \in W} \int_{Q} \left( \int_{2Q} \frac{|f(x) - f(y)|^{q}}{|x-y|^{d} \phi(|x-y|)^{q}} \, dy \right)^{\frac{q}{q}} \, dx \right)^{\frac{1}{q}}.
\]
And since $2Q \subseteq \text{Sh}(Q)$, this part is finished.

What remains is $(\Omega \times \Omega) \setminus \bigcup_{Q \in W} Q \times 2Q = \bigcup_{Q \in W} Q \times (\Omega \setminus 2Q) = \bigcup_{Q,S \in W} Q \times (S \setminus 2Q)$. We claim that in this case $|x - y| \approx D(Q,S)$. Indeed, since $y \notin 2Q$, we immediately get $l(Q) \lesssim |x - y|$. Furthermore, if $l(S) \geq l(Q)$, and $|x - y| \leq 2l(S)$, then $Q \subseteq 5S$, and by the definition of the Whitney decomposition $l(Q) \geq \frac{1}{4}l(S)$ which proves the claim. Therefore, by A3 we get

$$
\sum_{Q,S} \int_Q \int_{S \setminus 2Q} |f(x) - f(y)| g(x,y) \frac{|x-y|}{\phi(|x-y|)} dy dx \lesssim \sum_{Q,S} \int_Q \int_S |f(x) - f(y)| g(x,y) D(Q,S)^{\frac{d}{q}} \phi(D(Q,S)) dy dx.
$$

Let $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$. By the triangle inequality, we can split the right hand side of (3.2) as follows:

(A) $$
\sum_{Q,S} \int_Q \int_S |f(x) - f(y)| g(x,y) D(Q,S)^{\frac{d}{q}} \phi(D(Q,S)) dy dx \leq \sum_{Q,S} \int_Q \int_S |f(x) - f_Q| g(x,y) D(Q,S)^{\frac{d}{q}} \phi(D(Q,S)) dy dx
$$

(B) $$
+ \sum_{Q,S} \int_Q \int_S |f_Q - f_Q_s| g(x,y) D(Q,S)^{\frac{d}{q}} \phi(D(Q,S)) dy dx.
$$

(C) $$
+ \sum_{Q,S} \int_Q \int_S |f_Q - f(x)| g(x,y) D(Q,S)^{\frac{d}{q}} \phi(D(Q,S)) dy dx.
$$

Using Hölder’s inequality and (2.2) we can estimate (A):

$$
(A) \leq \sum_Q \int_Q |f(x) - f_Q| \left( \int_{Q} g(x,y)^{q'} dy \right)^{\frac{1}{q'}} \left( \sum_S l(S)^{d} \frac{1}{D(Q,S)^{d}} \frac{\phi(D(Q,S))}{\phi(l(Q))} \right)^{\frac{1}{q}} dx
$$

$$
\lesssim \sum_Q \int_Q |f(x) - f_Q| \left( \int_{Q} g(x,y)^{q'} dy \right)^{\frac{1}{q'}} \frac{1}{\phi(l(Q))} dx
$$

$$
\lesssim \left( \sum_Q \int_Q \left( \frac{|f(x) - f_Q|}{\phi(l(Q))} \right)^{q} dx \right)^{\frac{1}{q'}}.
$$

Now, by the definition of $f_Q$, Jensen’s inequality, and the unimodality of $\frac{1}{\phi}$ we get

$$
(A) \leq \left( \sum_Q \int_Q \left( \int_Q \frac{|f(x) - f(y)|}{\phi(l(Q))} dy \right)^{q} dx \right)^{\frac{1}{q'}} \leq \left( \sum_Q \int_Q \left( \int_Q \frac{|f(x) - f(y)|}{|x-y|^{d}} \frac{1}{\phi(|x-y|)} dy \right)^{\frac{2}{q}} dx \right)^{\frac{1}{q'}}.
$$

Let us consider (B). If we denote the successor of $Q$ in a chain $[Q,S]$ as $\mathcal{N}(Q)$, then by the triangle inequality

$$
(B) \leq 2 \sum_{Q,S} \left( \int_Q \int_S \frac{g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} dy dx \sum_{P \in [Q,S]} |f_P - f_{\mathcal{N}(P)}| \right).
$$
Recall that $\mathcal{N}(P) \subseteq 5P$, and for every $P \in [Q,Q_S]$, $Q \in \text{Sh}(P)$. For such $P$ it is also true that $D(P,S) \approx D(Q,S)$, see [16] (2.6)). Therefore, by A3

$$(B) \lesssim \sum_P \left( \left| f_P - f_{\mathcal{N}(P)} \right| \sum_{Q \in \text{Sh}(P)} \int_Q \int_S \frac{g(x,y)}{D(P,S)^{\frac{d}{q}} d\phi(D(P,S))} \ dy \ dx \right)$$

$$\leq \sum_P \left( \frac{1}{|P||5P|} \int_P \int_{5P} |f(\xi) - f(\zeta)| d\xi d\zeta \sum_{Q \in \text{Sh}(P)} \int_Q \int_S \frac{g(x,y)}{D(P,S)^{\frac{d}{q}} d\phi(D(P,S))} \ dy \ dx \right).$$

By the Hölder’s inequality and (2.2),

$$(3.5) \quad (B) \lesssim \sum_P \left( \frac{1}{|P||5P|} \int_P \int_{5P} |f(\xi) - f(\zeta)| d\xi d\zeta \left( \int_{\text{Sh}(P)} \left( \int_Q g(x,y)^q \ dy \right)^{\frac{1}{q}} \frac{1}{l(P)} \ dx \right) \right).$$

Let $G(x) = (\int_{\Omega} g(x,y)^q \ dy)^{\frac{1}{q}}$. By [16] Lemma 2.7 we have $\int_{\text{Sh}(P)} G(x) \ dx \lesssim \inf_{y \in P} MG(y) l(P)^d$. Using this, the Jensen’s inequality, the Hölder’s inequality, and the fact that the maximal operator is continuous in $L^{p'}$, $p' > 1$, we obtain

$$(3.6) \quad (B) \lesssim \sum_P \left( \int_P \frac{l(P)^d}{l(P)^d \phi(l(P))} \int_{5P} |f(\xi) - f(\zeta)| d\xi d\zeta \right)^{\frac{1}{q}} \frac{1}{l(P)} \ dx.$$

Since $|x - y| \leq 5l(P)$, B is estimated.

Now let us deal with (C). Recall that for every admissible chain $[Q,S]$, we have $Q,S \in \text{Sh}(Q_S)$. Therefore the following manipulation is possible.

$$(C) \lesssim \sum_{R \in \mathcal{W}} \sum_{Q \in \text{Sh}(R)} \sum_{S \in \text{Sh}(R)} \int_Q \int_S \frac{|f_R - f_S(y)| g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \ dy \ dx.$$

Furthermore, since $D(Q,S) \approx l(Q_S)$, and in the above sums $l(R) \approx l(Q_S)$, A3 gives us

$$(3.7) \quad (C) \lesssim \sum_{R \in \mathcal{W}} \sum_{Q \in \text{Sh}(R)} \sum_{S \in \text{Sh}(R)} \int_Q \int_S \frac{|f_R - f_S(y)| g(x,y)^q}{l(R)^q \phi(l(R))} \ dy \ dx.$$

By Hölder’s inequality we get

$$(C) \lesssim \sum_{R \in \mathcal{W}} \frac{1}{l(R)^{\frac{d}{q}} \phi(l(R))} \left( \int_{\text{Sh}(R)} |f_R - f_S(y)|^q \ dy \right)^{\frac{1}{q}} \int_{\text{Sh}(R)} \left( \int_{\text{Sh}(R)} g(x,y)^q \ dy \right)^{\frac{1}{q}} \ dx$$

$$\leq \sum_{R \in \mathcal{W}} \frac{1}{l(R)^{\frac{d}{q}} \phi(l(R))} \left( \int_{\text{Sh}(R)} |f_R - f_S(y)|^q \ dy \right)^{\frac{1}{q}} \int_{\text{Sh}(R)} G(x) \ dx.$$
By \[16\] (2.13), the fact that \( \inf_R MG \leq \frac{1}{l(R)^q} \int_R MG \), and the Hölder’s inequality, we get that

\[
(C) \lesssim \sum_{R \in W} \frac{1}{l(R)^{\frac{d}{q}} \phi(l(R))} \left( \int_{S(R)} |f_R - f(y)|^q dy \right)^{\frac{1}{q}} \int_R MG(\xi) \, d\xi
\]

\[
\leq \left( \sum_{R \in W} \int_R \frac{1}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \int_{S(R)} |f_R - f(y)|^q dy \right)^{\frac{p}{q}} \, d\xi \right)^{\frac{1}{p}} \|MG\|_{L^p(\Omega)}
\]

\[
\leq \left( \sum_{R \in W} \frac{l(R)^d}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \sum_{S \in S(R)} \int_{S} |f_R - f(y)|^q dy \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}
\]

Let \([S, R]\) be an admissible chain between \( S \) and \( R \), and let \([S, R] = [S, R] \setminus \{R\}\). Then, after using \(|f_R - f(y)|^q \lesssim \|f_R - f_S\|^q + \|f_S - f(y)\|^q\), we get

\[
(C)^p \lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \sum_{S \in S(R)} \sum_{P \in [S, R]} |f_P - f_{N(P)}|^q l(S)^d \phi(l(P))^\frac{q}{p} \right)^{\frac{p}{q}}
\]

If we write \(f_P - f_{N(P)} = (f_P - f_{N(P)}) \phi(l(P))^\frac{q}{p},\) then by Hölder’s inequality we get

\[
(3.8) \quad (C1) \lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \sum_{S \in S(R)} \sum_{P \in [S, R]} \frac{|f_P - f_{N(P)}|^q l(S)^d}{\phi(l(P))^\frac{q}{p}} \phi(l(P))^\frac{q}{p} \right)^{\frac{p}{q}}
\]

By Lemma 2.2

\[
(C1) \lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \sum_{S \in S(R)} \sum_{P \in [S, R]} \frac{|f_P - f_{N(P)}|^q l(S)^d}{\phi(l(P))^\frac{q}{p}} \phi(l(P))^\frac{q}{p} \right)^{\frac{p}{q}}
\]

Let us take \( \rho_2 \) large enough for \( S \in \text{Sh}^2(R) := \text{Sh}_{\rho_2}(R), \) and \( P \in \text{Sh}^2(R) \) to occur. Then
\( \sum_{S \in S(R)} \sum_{P \in [S, R]} \lesssim \sum_{P \in \text{Sh}^2(R)} \sum_{S \in S(R)} \). We denote the sum of the neighbors of \( P \) as \( U_P \).
Since \( \sum_{S \in \text{Sh}^2(P)} l(S)^d \lesssim l(P)^d \), we get that

\[
(C1) \lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^{\frac{d}{q}} \phi(l(R))^p} \left( \sum_{P \in \text{Sh}^2(P)} \frac{|f_P - f_{U_P}|^q l(P)^d}{\phi(l(P))^\frac{q}{p}} \phi(l(P))^\frac{q}{p} \right)^{\frac{p}{q}}
\]
Since \( p \geq q \), we can use the Hölder’s inequality with exponent \( \frac{p}{q} \) to get

\[
(C1) \lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^q} \phi(l(R))^\frac{p}{q} \left( \sum_{\xi \in Sh(R)} |f'_{\phi} - f_{U_{\xi}}|^p l(l(R))^{\frac{d}{q}} \right) \left( \sum_{\xi \in Sh(R)} |f_{\phi} - f_{U_{\xi}}|^q l(l(R))^{\frac{d}{q}} \right)^{\frac{1}{2}} \]

\[
\lesssim \sum_{R \in W} \sum_{\xi \in Sh(R)} \phi(l(R))^\frac{p}{q} \frac{|f'_{\phi} - f_{U_{\xi}}|^p l(l(R))^{\frac{d}{q}}}{\phi(l(R))^\frac{p}{q}} \]

\[
\lesssim \sum_{R \in W} \sum_{\xi \in Sh(R)} \phi(l(R))^\frac{p}{q} \frac{|f'_{\phi} - f_{U_{\xi}}|^p l(l(R))^{\frac{d}{q}}}{\phi(l(R))^\frac{p}{q}} \sum_{R \in Sh(R)} \phi(l(R))^\frac{p}{q} - p.
\]

Since \( \frac{p}{q} - p < 0 \), Lemma 2.2 and Jensen’s inequality give

\[
(C1) \lesssim \sum_{R \in W} \int_{U_{\xi}} \frac{|f'_{\phi} - f_{\phi}|^p}{\phi(l(R))^p} \, d\xi \lesssim \sum_{R \in W} \int_{U_{\xi}} \left( \int_{U_{\xi}} \frac{|f(\zeta) - f(\xi)|^q}{l(l(R))^q} \, d\zeta \right)^\frac{p}{q} \, d\xi.
\]

Since \( U_{\xi} \subseteq 5P \) we have finished estimating \((C1)\).

The procedure for \((C2)\) is pretty similar. By Hölder’s inequality

\[
(C2) = \sum_{R \in W} \frac{l(R)^d}{l(R)^q} \phi(l(R))^p \left( \sum_{\xi \in Sh(R)} \int_S |f_{\xi} - f(y)|^q dy \frac{l(S)^{d(\frac{p}{q} - 1)}}{l(S)^{d(\frac{p}{q})}} \right)^\frac{p}{q}
\]

\[
\lesssim \sum_{R \in W} \frac{l(R)^d}{l(R)^q} \phi(l(R))^p \sum_{\xi \in Sh(R)} \frac{(\int_S |f_{\xi} - f(y)|^q dy)^\frac{p}{q}}{l(S)^{d(\frac{p}{q} - 1)}} \left( \sum_{\xi \in Sh(R)} l(S)^d \right)^\frac{p}{q} - 1
\]

\[
\lesssim \sum_{R \in W} \sum_{\xi \in Sh(R)} \frac{(\int_S |f_{\xi} - f(y)|^q dy)^\frac{p}{q}}{l(S)^{d(\frac{p}{q} - 1)}} \phi(l(R))^p.
\]

By rearranging and using Lemma 2.2 we obtain

\[
(C2) \lesssim \sum_{S \in W} \frac{(\int_S |f_{\xi} - f(y)|^q dy)^\frac{p}{q}}{l(S)^{d(\frac{p}{q} - 1)}} \sum_{R \in Sh(S)} \phi(l(R))^p \lesssim \sum_{S \in W} \left( \int_S \frac{|f_{\xi} - f(y)|^q}{l(S)^d} \, dy \right)^\frac{p}{q} \frac{l(S)^d}{\phi(l(S))^p}.
\]

Hence, by Jensen’s inequality,

\[
(C2) \lesssim \sum_{S \in W} \frac{l(S)^d}{\phi(l(S))^p} \int_S \frac{|f_{\xi} - f(y)|^p}{l(S)^d} \, dy = \sum_{S \in W} \int_S \frac{|f_{\xi} - f(y)|^p}{\phi(l(S))^p} \, dy.
\]

Thus we have arrived at the same situation as in \((3.9)\) and the proof is finished (we may need to enlarge the constant \( C_{\gamma W} \) which can be done by diminishing the cubes in the Whitney decomposition). \(\square\)
4. Examples of $\phi$

4.1. Positive examples. We will present some examples of kernels which satisfy $\textbf{A2}$ and $\textbf{A3}$.

**Example 4.1.** Stable-like scaling is more than enough for $\textbf{A2}$ to hold. Indeed, if we assume that for $0 < r < R$, $\lambda \geq 1$, there exist $s, t \in (0, 1)$ for which we have $\phi(\lambda r) \gtrsim \lambda^s \phi(r)$, and $\phi(\lambda^{-1} r) \lesssim \lambda^{-t} \phi(r)$, then the series in $\textbf{A2}$ is geometric and independent of $r$. Obviously, we also have $\textbf{A3}$.

Let us examine the constant $M$ in $\textbf{A2}$ for $p = q = 2$ and the kernels of the form $K(x, y) = (2 - \alpha) |x - y|^{-d - \alpha}$, i.e. $\phi(t) = (2 - \alpha)t^{\alpha/2}$. For every $r > 0$ we have

$$\sum_{k=1}^{\infty} \frac{\phi(r)}{\phi(2^k r)} = \sum_{k=1}^{\infty} \frac{1}{(2^{a/2})^k} = \frac{1}{2^{a/2} - 1}.$$

This quantity is bounded as $\alpha \to 2$. Since the constant in $\textbf{A3}$ is also bounded in this case, we get that the comparability in Theorem 1.1 is uniform for $\alpha \in (\varepsilon, 2)$ for every $\varepsilon > 0$.

**Example 4.2.** Assume that $\Omega$ is bounded. Let $s \in (0, 1)$, $\phi(x) = |\log(1 + |x|)|^s$ and let $R = \text{diam}(\Omega)$. We claim that $\frac{\phi(r)}{\phi(2^k r)} \lesssim C k^{-s}$ with some $C \geq 1$ independent of $k$, and $r \in (0, \text{diam}(\Omega))$. It suffices to verify that

$$(1 + r)^k \leq (1 + 2^k r)^C.$$

For $r < 1$ we have $(1 + r)^k = 1 + \sum_{l=1}^{k} \binom{k}{l} r^l \leq 1 + \sum_{l=1}^{k} \binom{k}{l} r \leq 1 + 2^k r$. If $r \geq 1$, then $(1 + 2^k r)^C \geq (2^k r)^C = 2^{kC} r^C = 2^{(C-1)k^2 r^C}.2^{(C-1)k^2 r^C}$. It is enough to take $C$ such that $2^{C-1} \geq (1 + R)$. Thus the claim is proved, and therefore, the summation condition is satisfied when $sq > 1$.

$\textbf{A3}$ is granted by Bernoulli’s inequality. Fix $C > 1$. For every $t > 0$ we have

$$\log(1 + Ct) \leq \log(1 + t)^C = C \log(1 + t).$$

4.2. O-regularly varying functions.

**Definition 4.3.** We say that $\phi$ is O-regularly varying at infinity if there exist $a, b \in \mathbb{R}$ and $A, B, R > 0$ such that

$$A \left( \frac{r_2}{r_1} \right)^a \frac{\phi(r_2)}{\phi(r_1)} \leq B \left( \frac{r_2}{r_1} \right)^b$$

holds whenever $R < r_1 < r_2$. Analogously, $\phi$ is O-regularly varying at zero if (4.1) holds for $0 < r_1 < r_2 < R$.

The supremum of $a$ and the infimum of $b$ for which (4.1) is satisfied are called lower, respectively upper, Matuszewska indices (or lower/upper indices).

Assume $\textbf{A2}$ and $\textbf{A3}$. Note that the first inequality in (4.1) is trivially satisfied with $a = 0$ and $A = 1$.

Let $0 < R < \text{diam}(\Omega)$ be fixed. If $\text{diam}(\Omega) = \infty$, then our assumptions are global and for every $r > R$ we have

$$M \phi(r)^{-q} \gtrsim \sum_{k=0}^{\infty} \frac{1}{\phi(2^k r)^q} \approx \sum_{k=0}^{\infty} \frac{1}{2^k r} \int_{2^k r}^{2^{k+1} r} ds = \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{ds}{s \phi(s)^q} = \int_{r}^{\infty} \frac{ds}{s \phi(s)^q}.$$
and
\[ \int_r^\infty \frac{ds}{s\phi(s)^q} \gtrsim \frac{1}{\phi(r)^q} \int_r^{2r} \frac{ds}{s} = \frac{\log 2}{\phi(r)^q}. \]

Therefore \( \phi(r)^{-q} \approx \int_r^\infty \frac{\phi(s)^{-q}}{s} ds \), which, by Proposition A.1 a) in [12], means that \( \phi^{-q} \) (and \( \phi^{-1} \)) is an O-regularly varying function with upper index \( \beta < 0 \).

Without assuming the unboundedness of \( \Omega \) we can show that \( \phi(r)^{-q} \approx \int_r^R \frac{\phi(s)^{-q}}{s} ds \) for \( 0 < r < R/2 \). Hence, by Proposition A.1 d) in [12], \( \phi^{-1} \) is O-regularly varying at 0 with upper index \( \alpha < 0 \).

Heuristically speaking, the conditions above tell us that \( K \) needs to have a strong singularity at the origin for bounded and unbounded \( \Omega \) and a relatively fast decay at infinity for unbounded \( \Omega \). In terms of comparability results this is expected, as we need to compensate for the part of the form corresponding to the points distant from each other. Here, in fact, we get \( K(x, y) \gtrsim |x - y|^{-d-\gamma} \) for some \( \gamma \in (0, q) \) and \( x \) close to \( y \), cf. A1. It remains unclear whether such scalings are necessary for the comparability to hold.

For further reading on O-regularly varying functions we refer to the book by Bingham et al. [2].

4.3. Negative examples. We will show some examples for which the seminorms (1.2) and (1.3) are not comparable. Assume for clarity that \( p = q = 2 \).

**Example 4.4.** Let \( \Omega = (0, 1) \subset \mathbb{R} \), and let \( K(x, y) \equiv 1 \). Consider the function \( f(x) = x^{-\gamma} \) with \( \gamma \in (0, \frac{1}{2}) \). A direct calculation shows that
\[ (4.2) \int_0^1 \int_0^1 (f(x) - f(y))^2 dy dx = 2 \left( \frac{1}{1 - 2\gamma} - \frac{1}{(1 - \gamma)^2} \right). \]

In particular, \( f \) belongs to the corresponding Sobolev space (actually the "Sobolev space" is \( L^2(\Omega) \) in this case).

Let \( \varepsilon \in (0, 1) \), and let \( A_\varepsilon \) be the integration domain of the truncated form. Then we have
\[
\int_0^1 \int_{x-\delta_x}^{x+\delta_x} (f(x) - f(y))^2 \, dy \, dx \leq \int_0^1 \int_{x-\varepsilon}^{x+\varepsilon} (f(x) - f(y))^2 \, dy \, dx
\]
(4.3)
\[
= \frac{2\varepsilon}{1-\gamma} + \frac{1}{(1-2\gamma)(1-\gamma)} \left[ (1+\varepsilon)^{1-2\gamma} - (1-\varepsilon)^{1-2\gamma} \right].
\]

As \( \gamma \to \frac{1}{2} \), the ratio of the right hand sides of (4.2) and (4.3) goes to infinity which shows that in this case the result from Theorem 1.1 does not hold.

Example 4.5. The preceding example gives an idea on how to show an analogous fact for any nonzero \( K \) such that \( K(0, \cdot) \in L^1([0,1]) \). In \( A_\varepsilon \) we have \( x \approx y \). Therefore \( \frac{1}{x^\gamma} - \frac{1}{y^\gamma} \leq \frac{1}{x^\gamma} \), hence

\[
\int_{A_\varepsilon} \left( \frac{1}{x^\gamma} - \frac{1}{y^\gamma} \right)^2 K(x,y) \, dy \, dx \leq \int_{A_\varepsilon} \frac{1}{x^{2\gamma}} K(x,y) \, dy \, dx = \int_0^1 \frac{1}{x^{2\gamma}} \int_{B(x,\delta_x)} K(x,y) \, dy \, dx.
\]
(4.4)

On the other hand, since \( K \) is nontrivial, there exists \( \eta > 0 \) such that for every \( x \in (0,\eta) \) we have \( \int_0^1 K(x,y) \, dy \geq C > 0 \). Therefore

\[
\int_0^1 \int_0^1 \left( \frac{1}{x^\gamma} - \frac{1}{y^\gamma} \right)^2 K(x,y) \, dy \, dx \geq \int_0^{\eta/2} \int_0^1 \left( \frac{1}{x^\gamma} - \frac{1}{y^\gamma} \right)^2 K(x,y) \, dy \, dx \geq \int_0^{\eta/2} \frac{1}{x^{2\gamma}} \int_0^1 K(x,y) \, dy \, dx
\]
(4.5)

The right hand side of (4.4) is of the form \( \int_0^1 \frac{f(x)}{x^{2\gamma}} \, dx \) with \( f(x) \) bounded and \( \lim_{x \to 0^+} f(x) = 0 \). Let us fix an arbitrarily small \( \xi > 0 \), and let \( \rho \) be sufficiently small so that \( f(x) \leq \xi \) for \( x \in (0,\rho) \). If we separate \( \int_0^1 = \int_0^\rho + \int_\rho^1 \), then we see that the ratio of the right hand sides of (4.4) and (4.5) tends to 0 as \( \gamma \to \frac{1}{2} \).

Remark 4.6. In previous examples the kernel was integrable. This means that

\[
\int_{\Omega} \int_{\Omega} (f(x) - f(y))^2 K(x,y) \, dx \, dy \leq 2 \int_{\Omega} \int_{\Omega} f(x)^2 K(x,y) \, dx \, dy \leq \|f\|_{L^2(\Omega)} \|K(0, \cdot)\|_{L^1(\mathbb{R}^d)}.
\]

Therefore, even though the quadratic forms (1.2) and (1.3) are incomparable, the Triebel-Lizorkin norm \( \|\cdot\|_{F_{p,q}(\Omega)} \) is comparable when we replace the full seminorm with the truncated one.

Example 4.7. For \( K(x,y) = |x-y|^{-1} \) on \( D = (0,1) \) the seminorms also fail to be comparable. Consider the functions \( f_n(x) = n \wedge \frac{1}{2} \). Since

\[
\int_0^1 \int_0^y (f(x) - f(y))^2 K(x,y) \, dx \, dy = \frac{1}{2} \int_0^1 \int_0^1 (f(x) - f(y))^2 K(x,y) \, dx \, dy
\]
(4.6)

we will assume that \( y > x \), and work only with the integral on the left hand side. Note that for \( f_n \), the integral over \( (0, \frac{1}{n})^2 \) vanishes. We split as follows

\[
\int_0^1 \int_0^y (f_n(x) - f_n(y))^2 K(x,y) \, dx \, dy = \int_{1/n}^1 \int_{1/n}^1 \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x,y) \, dx \, dy + \int_{1/n}^1 \int_0^{1/n} \left( \frac{1}{y} - \frac{1}{1} \right)^2 K(x,y) \, dx \, dy.
\]
(4.7)
We first compute (4.6). Note that the integrand is equal to \( \frac{(y-x)^2}{xy^2} \cdot \frac{1}{y-x} = \frac{y-x}{xy^2} \).

\[
\int_{1/n}^{1} \int_{1/n}^{y} \frac{y-x}{x} \, dx \, dy = \int_{1/n}^{1} \int_{1/n}^{y} \frac{1}{x} \, dx \, dy - \int_{1/n}^{1} \int_{1/n}^{y} \frac{1}{xy^2} \, dx \, dy = n \log n - 2n + \log n + 2.
\]

For (4.7) we only show the asymptotics.

\[
\int_{1/n}^{1} \int_{0}^{1/n} \left( n - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy = \int_{1/n}^{1} \left( n - \frac{1}{y} \right)^2 \left( \log y - \log \left( y - \frac{1}{n} \right) \right) \, dy
\]

\[
= -n^2 \int_{1/n}^{1} \left( 1 - \frac{1}{ny} \right)^2 \log \left( 1 - \frac{1}{ny} \right) \, dy = -n \int_{0}^{1} \frac{t^2}{(1-t)^2} \log t \, dt.
\]

For \( n > 2 \) we split the latter integral: \( \int_{0}^{1/n} = \int_{1/2}^{1/2} + \int_{1/2}^{1/n} \). The first one converges, i.e. it is a (negative) constant. In the second one \( t^2 \approx 1 \), and \( \frac{\log t}{1-t} \approx -1 \), therefore

\[
- n \int_{0}^{1/n} \frac{t^2}{(1-t)^2} \log t \, dt \approx n \left( 1 + \int_{1/2}^{1/n} \frac{dt}{1-t} \right) = n(1 + \log n - \log 2).
\]

Thus we get the asymptotics

\[
\int_{0}^{1} \int_{0}^{1/(n-1)} (f_n(x) - f_n(y))^2 K(x, y) \, dx \, dy \approx n \log n.
\]

Now consider the truncated case. For clarity, assume that \( \epsilon = \frac{1}{2} \).

\[
\int_{0}^{1} \int_{y/2}^{y} (f_n(x) - f_n(y))^2 K(x, y) \, dx \, dy = \int_{2/n}^{1} \int_{y/2}^{y} \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy
\]

\[
+ \int_{1/n}^{2/n} \int_{1/n}^{y} \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy
\]

\[
+ \int_{1/n}^{2/n} \int_{y/2}^{1/n} \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy.
\]

For (4.10) and (4.11) we note that

\[
\int_{2/n}^{1} \int_{y/2}^{y} \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy \leq \int_{2/n}^{1} \int_{y/2}^{y} \frac{1}{x^2 y} \, dx \, dy = \frac{n}{2} - 1,
\]

and

\[
\int_{1/n}^{2/n} \int_{1/n}^{y} \left( \frac{1}{x} - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy \leq \int_{1/n}^{2/n} \int_{1/n}^{y} \frac{1}{x^2 y} \, dx \, dy = n \log 2 - \frac{n}{2}.
\]

The last integral (4.12) is estimated as follows

\[
\int_{1/n}^{2/n} \int_{y/2}^{1/n} \left( n - \frac{1}{y} \right)^2 K(x, y) \, dx \, dy = \int_{1/n}^{2/n} \left( n - \frac{1}{y} \right)^2 \left( \log y - \log \left( y - \frac{1}{n} \right) \right) \, dy
\]

\[
= -n^2 \int_{1/n}^{2/n} \left( 1 - \frac{1}{ny} \right)^2 \left( \log \left( 1 - \frac{1}{ny} \right) + \log 2 \right) \, dy \leq -n \int_{0}^{1} \frac{t^2}{(1-t)^2} \log t \, dt \approx n.
\]
To conclude, we get
\[ (4.13) \int_0^1 \int_{B(y, \delta_y/2)} (f_n(x) - f_n(y))^2 K(x, y) \, dx \, dy \lesssim n. \]
Since the ratio of the left hand sides of (4.9) and (4.13) diverges as \( n \to \infty \), our claim is proven.

5. The 0-order kernel

**Theorem 5.1.** Let \( \Omega \) be a bounded uniform domain. Then, if \( 1 < q \leq p < \infty \), then for every \( 0 < \rho < 1 \)
\[ (5.1) \left( \int_{\Omega} \left( \int_{\Omega} \frac{|f(x) - f(y)|^q}{|x - y|^d} \, dy \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}} \lesssim \left( \int_{\Omega} \left( \int_{B(x, \rho \delta_x)} \frac{|f(x) - f(y)|^q}{|x - y|^d} \left( |\log |x - y|| \vee 1 \right) \, dy \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}. \]

In order to obtain this result we will prove an analogue of Lemma [2.1] for \( K(x, y) = |x - y|^{-d} \), i.e. \( \phi \equiv 1 \). For now every integral is restricted to \( \Omega \) by default.

**Lemma 5.2.** Let \( \Omega \) be a bounded domain with Whitney covering \( \mathcal{W} \) such that the largest cube has length at most \( 1/2 \). Assume that \( g \in L_{\text{loc}}^1(\mathbb{R}^d) \), and \( 0 < r < \left( \frac{1}{2} \wedge \text{diam}(\Omega) \right) \). Then for every \( Q \in \mathcal{W} \) and \( x \in \Omega \) we have
\[ (5.2) \int_{|y - x| > r} \frac{g(y) \, dy}{|y - x|^d} \lesssim Mg(x) |\log(r)|, \quad \sum_{S: D(Q, S) > r} \frac{\int_{S} g(y) \, dy}{D(Q, S)^d} \lesssim \inf_{x \in Q} Mg(x) |\log(r)|. \]
and
\[ (5.3) \sum_{S \in \mathcal{W}} \frac{l(S)^d}{D(Q, S)^d} \lesssim |\log(r)|. \]

**Proof.** Let \( x \in \Omega \). If we take \( R = \text{diam}(\Omega) \), then proceeding as in Lemma [2.1] we get
\[ \int_{|y - x| < r} \frac{g(y) \, dy}{|y - x|^d} \leq \sum_{k=1}^{[\log_2(R/r)]} \int_{2^{k-1}r \leq |y - x| < 2^k r} \frac{g(y) \, dy}{|x - y|^d} \lesssim Mg(x) |\log_2(R/r)| \lesssim Mg(x) |\log(r)|. \]
As in the proof of Lemma [2.1] in the second part of (5.2) we use the first part, and we are left with
\[ \int_{|x - y| < r} \frac{g(y) \, dy}{(|x - y| + r)^d} \lesssim \frac{1}{B(x, r)} \int_{B(x, r)} g(y) \, dy \leq Mg(x) \lesssim Mg(x) |\log(r)|, \]
since \( r < \frac{1}{2} \). (5.3) is obtained by taking \( r = l(Q) \) and \( g \equiv 1 \).

**Proof of Theorem 5.1.** Most importantly, note that the function \( (|\log(r)| \vee 1)^{-1} \) satisfies \( A3 \) hence Lemma [2.2] holds with \( \phi(r) = (|\log(r)| \vee 1)^{-1} \). Therefore we can use the same approach as in the proof of Theorem [1.1] only instead of using Lemma [2.1] we use Lemma [5.2] and we start with \( \phi \equiv 1 \). Let us discuss the crucial changes in the argument.

- The integrals over \( Q \times 2Q \) are trivial because the kernel on the right hand side of (5.1) is larger than the one on the right hand side.
- In (A)/ (B) Lemma [2.1] is used in (3.3) / (3.5). Using Lemma [5.2] instead, we get \( |\log(l(Q))|^{\frac{1}{q}} / |\log(l(P))|^{\frac{1}{q}} \). The exponents are compensated in (3.4) / (3.6).
Indeed, we have $Ω = (0, 1)$ and it is plausible that the converse inequality is not true. We will show the existence of a counterexample when $Ω = (0, 1)$, $p = q = 2$. For an open interval $I ⊆ ℝ$ we let

$$F_0(I) = \{ f ∈ L^2(I) : \int_I \int_I (f(x) - f(y))^2 |x - y|^{-1} dy dx < ∞ \},$$

$$F_{log}(I) = \{ f ∈ L^2(I) : \int_I \int_I (f(x) - f(y))^2 |x - y|^{-1}(|log |x - y|| ∨ 1) dy dx < ∞ \}.$$

**Theorem 5.3.** For every $θ > 0$, there exists $f ∈ F_0(0, 1) ∩ L^∞(0, 1)$ such that

$$\int_0^1 \int_{B(x, θδ_x)} (f(x) - f(y))^2 |x - y|^{-1}(|log |x - y|| ∨ 1) dy dx = ∞.$$

**Proof.**

**Step 1.** First, note that the finiteness of the left hand side of (5.4) implies that $f ∈ F_{log}(\frac{n}{2n+1}, \frac{n+1}{2n+1})$ for a sufficiently large $n ∈ ℕ$. Indeed, if $θ ≥ \frac{1}{n}$ for some natural number $n ≥ 2$, then

$$\int_0^1 \int_{B(x, θδ_x)} (\cdots) ≥ \int_0^1 \int_{B(x, δ_x/n)} (\cdots) ≥ \int_0^{n+1} \int_{B(x, \frac{1}{2n+1})} (\cdots) ≥ \int_0^{n+1} \int_{B(x, \frac{1}{2n+1})} (\cdots).$$

We fix a number $n$ for which (5.5) is satisfied.

**Step 2.** In order to construct the counterexample we will use the asymptotics of the Fourier expansions of functions in $F_0(I)$ and $F_{log}(I)$. Let $f$ be Borel measurable and satisfy $f(x + 1) = f(x)$ for $x ∈ ℝ$. Let $K(x, y)$ be equal to $|x - y|^{-1}$ (resp. $|x - y|^{-1}(|log |x - y|| ∨ 1)$). We claim that $f ∈ L^∞(0, 1)$ belongs to $F_0(0, 1)$ (resp. $F_{log}(0, 1)$) if and only if

$$\int_0^1 \int_0^1 (f(x) - f(x - h))^2 K(0, h) dh dx < ∞.$$

Indeed, we have

$$\int_0^1 \int_0^1 (f(x) - f(y))^2 K(x, y) dy dx = 2 \int_0^1 \int_0^x (f(x) - f(y))^2 K(x, y) dx dy$$

$$= \int_0^1 \int_0^x (f(x) - f(x - h))^2 K(0, h) dh dx.$$
Therefore, it suffices to verify that \( \int_0^1 \int_x^1 (f(x) - f(x-h))^2 K(0,h) \, dh \, dx < \infty \) for bounded \( f \). Clearly we can assume that \( K(x,y) = |x - y|^{-1} (|\log |x - y|| + 1) \).

\[
\int_0^1 \int_x^1 (f(x) - f(x-h))^2 K(0,h) \, dh \, dx \lesssim \int_0^1 \int_x^1 \frac{(-\log h) \vee 1}{h} \, dh \, dx = \int_0^{1/e} \int_x^1 \frac{\log h}{h} \, dh \, dx + \int_{1/e}^1 \int_x^1 \frac{1}{h} \, dh \, dx.
\]

Both integrals are finite, therefore the claim is proved.

By Parseval's identity and Tonelli's theorem we get

\[
\int_0^1 K(0,h) \int_0^1 (f(x) - f(x-h))^2 \, dh \, dx = \int_0^1 K(0,h) \sum_{m \in \mathbb{Z}} |\hat{f}(m)|^2 |1 - e^{2\pi ih}|^2 \, dh
\]

\[
\sum_{m \in \mathbb{Z}} |\hat{f}(m)|^2 \int_0^1 |1 - e^{2\pi ih}|^2 K(0,h) \, dh = 2 \sum_{m \in \mathbb{Z}} |\hat{f}(m)|^2 \int_0^1 (1 - \cos(2\pi mh)) K(0,h) \, dh.
\]

Now let us inspect the remaining integrals for both cases of \( K \). For \( m \neq 0 \) we have

\[
\int_0^1 \frac{1 - \cos(2\pi mh)}{h} \, dh = \int_0^{|m|} \frac{1 - \cos(2\pi h)}{h} \, dh \approx \log |m|.
\]

In the logarithmic case

\[
\int_0^1 \frac{1 - \cos(2\pi mh)}{h} (-\log h \vee 1) \, dh = \int_0^{|m|} \frac{1 - \cos(2\pi h)}{h} (-\log \frac{h}{|m|} \vee 1) \, dh \approx \log^2 |m|.
\]

To summarize, for bounded functions we can characterize \( F_0(0,1) \) by

\[
(5.6) \quad \sum_{m \in \mathbb{Z}, m \neq 0} |\hat{f}(m)|^2 \log |m| < \infty
\]

and \( F_{\log}(0,1) \) by

\[
(5.7) \quad \sum_{m \in \mathbb{Z}, m \neq 0} |\hat{f}(m)|^2 \log^2 |m| < \infty.
\]

The same characterizations hold for \( I = \left(\frac{n}{2n+1}, \frac{n+1}{2n+1}\right) \) and the respective Fourier expansion.

**Step 3.** We give an example of \( f \in F_0(0,1) \cap L^\infty(0,1) \) for which \((5.6)\) is finite and \((5.7)\) is infinite. For \( m = (2n+1)^l \), \( l = 0, 1, \ldots \), we put \( \hat{f}(m) = \frac{1}{m^2} \). For other \( m \) we let \( \hat{f}(m) = 0 \). Note that \( f \) is \( \frac{1}{2n+1} \)-periodic. Therefore the \( j \)-th Fourier coefficient of \( f \) on \( \left(\frac{n}{2n+1}, \frac{n+1}{2n+1}\right) \) is equal to \( \frac{1}{2n+1} \) of its \((2n+1)\cdot j\)-th Fourier coefficient on \((0,1)\). Since \( (\hat{f}(m))_{m \in \mathbb{Z}} \) is summable, \( f \) is bounded. Furthermore \( l^{-3} \log[(2n+1)^{2l}] = l^{-2} \log 2 + l^{-3} \log(2n+1) \) and \( l^{-3} \log^2[(2n+1)^{2l}] \approx l^{-1} \). Therefore \((5.6)\) is satisfied and \((5.7)\) is not. By \((5.5)\), the proof is finished.

\( \square \)

6. Uniformity is not a sharp condition

In this section we examine the strip \( \mathbb{R} \times (0,1) \) which is a non-uniform domain. We will show that the comparability fails for fractional Sobolev spaces with \( \alpha < 1 \). Then we show that for \( \alpha > 1 \) and slightly more general kernels the comparability holds. Later, we present a higher-dimensional case which shows that the comparability may also hold for \( \alpha < 1 \) in non-uniform domains. For clarity of presentation, we assume that \( p = q = 2 \).
Example 6.1. Let $\Omega = \mathbb{R} \times (0, 1)$ and let $K(x, y) = |x - y|^{-2-\alpha}$. Note that $\Omega$ is not uniform – if we take two cubes far from each other we will fail to find a sufficiently large central cube in any chain connecting them.

We will show for $\alpha \in (0, 1)$ the comparability does not hold. Consider a sequence of functions $(f_n)$ given by the formula $f_n(x_1, x_2) = (1 - \frac{|x_1|}{n}) \vee 0$. Since $f_n$ are constant on the second variable, for every $\xi \in (0, 1)$ we have

$$\int_0^1 \int_0^1 \frac{(f_n(x_1) - f_n(y_1))^2}{|x - y|^{2+\alpha}} \, dx \, dy \leq \int_0^1 \int_{B(y_1, 1)} (f_n(x_1, \xi) - f_n(y_1, \xi))^2 \kappa(x_1, y_1) \, dx \, dy.$$  

(6.1)

Let the integral over $(0, 1) \times (0, 1)$ be called $\kappa(x_1, y_1)$. We claim that $\kappa(x_1, y_1)$ is comparable with $|x_1 - y_1|^{-2-\alpha}$ if $|x_1 - y_1| \geq 1$ and with $|x_1 - y_1|^{-1-\alpha}$ otherwise. Indeed, we have $|x - y| \approx |x_1 - y_1| + |x_2 - y_2|$. If $|x_1 - y_1| \geq 1$, then

$$\int_0^1 \int_0^1 |x - y|^{-2-\alpha} \, dx \, dy \approx |x_1 - y_1|^{-2-\alpha} \int_0^1 \int_0^1 \, dx \, dy = |x_1 - y_1|^{-2-\alpha}.$$

For $|x_1 - y_1| < 1$ note that for fixed $a > 0$

$$a^{1+\alpha} \int_0^1 \int_0^1 (a + |x_2 - y_2|)^{-2-\alpha} \, dx \, dy \approx a^{1+\alpha} \int_0^1 \int_0^{y_2} (a + y_2 - x_2)^{-2-\alpha} \, dx \, dy = a^{1+\alpha} \int_0^1 (a^{1-\alpha} - (a + y_2)^{-1-\alpha}) \, dy_2 = 1 - \int_0^1 (1 + \frac{y_2}{a})^{-1-\alpha} \, dy_2.$$

Clearly, the latter integral does not exceed 1. Furthermore, for $a < 1$ we have $y_2/a \geq y_2$, so the integral is clearly smaller than 1. Thus the whole expression is approximately constant which proves our claim.

The shape of $\Omega$ grants that for every $\rho \in (0, 1)$ we have

$$\int_0^1 \int_{B(y, \rho \delta_y)} \frac{(f_n(x) - f_n(y))^2}{|x - y|^{2+\alpha}} \, dx \, dy \leq \int_0^1 \int_{B(y, 1)} (f_n(x_1, \xi) - f_n(y_1, \xi))^2 \kappa(x_1, y_1) \, dx \, dy.$$  

(6.2)

To simplify the notation we will write $f_n(x_1) = f_n(x_1, \xi)$ for some fixed $\xi \in (0, 1), x \in \mathbb{R}$. Since $f_n$ is Lipschitz, we have

$$\int_{B(y_1, 1)} (f_n(x_1) - f_n(y_1))^2 \kappa(x_1, y_1) \, dx \, dy = \int_{-n-1}^{n+1} \int_{B(y_1, 1)} (f_n(x_1) - f_n(y_1))^2 |x_1 - y_1|^{-1-\alpha} \, dx \, dy \approx \frac{1}{n} \int_{-n-1}^{n+1} \int_{B(y_1, 1)} |x_1 - y_1|^{-1-\alpha} \, dx \, dy \approx \frac{1}{n}.$$  

Thanks to the fact that $\alpha < 1$, the full seminorm is significantly greater as $n \to \infty$.

$$\int \int (f_n(x_1) - f_n(y_1))^2 \kappa(x_1, y_1) \, dx \, dy \geq \int_{-n/2}^{n/2} \int_{-\infty}^{\infty} |x_1 - y_1|^{-2-\alpha} \, dx \, dy \geq \frac{1}{1 + \alpha} \frac{n/2}{(n/2)^{1+\alpha}} \approx \frac{1}{n^\alpha}.$$  

Lemma 6.2. Let $\Omega = \mathbb{R} \times (0, 1)$. If $f : \mathbb{R}^2 \to [0, \infty)$ is radial, then $\int_\Omega |x| f(x) \, dx < \infty$ if and only if $\int_{\mathbb{R}^2} f(x) \, dx < \infty$. 


Then the seminorms (1.2) let

\[ B \]

We split the domain \( \Omega \) into open unit cubes. If we let

\[ L = \text{Int}[Q_{n-1} \cup Q_n \cup Q_{n+1}], \]

then \( L_n \) is a uniform domain, hence by Theorem 1.1,

\[ \int_{L_n} \int_{L_n} (f(x) - f(y))^2 K(x, y) \, dx \, dy \approx \int_{L_n} \int_{B(x, \rho \delta(x))} (f(x) - f(y))^2 K(x, y) \, dx \, dy \]

with constant independent of \( n \). Therefore for every \( 0 < \rho < 1 \)

\[ \int_{\Omega} \int_{B(x, \rho \delta(x))} (f(x) - f(y))^2 K(x, y) \, dx \, dy \approx \sum_{n \in \mathbb{Z}} \int_{L_n} \int_{L_n} (f(x) - f(y))^2 K(x, y) \, dx \, dy \]

(6.3)

\[ \approx \sum_{n \in \mathbb{Z}} \int_{Q_n} \int_{L_n} (f(x) - f(y))^2 K(x, y) \, dx \, dy, \]

so it suffices to show that the latter expression is comparable with the integral over \( \Omega \times \Omega \). We have

\[ \int_{\Omega} \int_{\Omega} (f(x) - f(y))^2 K(x, y) \, dx \, dy \approx \sum_{i, j \in \mathbb{Z}} \int_{Q_i} \int_{Q_j} (f(x) - f(y))^2 K(x, y) \, dx \, dy \]

\[ \approx \sum_{i \in \mathbb{Z}} \sum_{j+1 < i} \int_{Q_i} \int_{Q_j} (f(x) - f(y))^2 K(x, y) \, dx \, dy + \sum_{i \in \mathbb{Z}} \sum_{j+1 < i} \int_{Q_i} \int_{L_i} (f(x) - f(y))^2 K(x, y) \, dx \, dy. \]

Clearly it suffices to estimate the first summand. Since the cubes are far apart, we have \( |x - y| \approx |i - j| \) for \( x \in Q_i, y \in Q_j \). Hence

\[ \sum_{i \in \mathbb{Z}} \sum_{j+1 < i} \int_{Q_i} \int_{Q_j} (f(x) - f(y))^2 K(x, y) \, dx \, dy \lesssim \sum_{i \in \mathbb{Z}} \sum_{j+1 < i} \int_{Q_i} \int_{Q_j} (f(y) - f_{Q_i})^2 K(x, y) \, dx \, dy \]

(6.4)

\[ + \sum_{i \in \mathbb{Z}} \sum_{j+1 < i} \sum_{p \in \mathbb{Z}} \int_{Q_i} \int_{Q_j} (f_{Q_{n+1}} - f_{Q_n})^2 |x - y| K(x, y) \, dx \, dy. \]

In this inequality we have used \( (x_1 + \ldots + x_m)^2 \lesssim m(x_1^2 + \ldots + x_m^2) \) and \( |Q_i| = |Q_j| = 1 \). For the first term we use Jensen’s inequality and the fact that the sum over \( j \) is finite and does not depend on \( i \). Therefore

\[ \sum_{i \in \mathbb{Z}} \int_{Q_i} (f(y) - f_{Q_i})^2 \sum_{j+1 < i} \int_{Q_j} K(x, y) \, dx \, dy \lesssim \sum_{i \in \mathbb{Z}} \int_{Q_i} (f(y) - f(x))^2 \, dx \, dy, \]

which is smaller than (6.3).

**Proof.** Note that for \( n \in \mathbb{N} \) the area of \( \Omega \cap (B_n \setminus B_{n-1}) \) is comparable to the \( 1/n \)-th of the area of the annulus \( B_n \setminus B_{n-1} \). Therefore by the rotational symmetry of \( f \) we get

\[ \int_{\Omega} |x| f(x) \, dx \approx \sum_{n \in \mathbb{N}} \int_{\Omega \cap (B_n \setminus B_{n-1})} n f(x) \, dx \approx \sum_{n \in \mathbb{N}} \int_{B_n \setminus B_{n-1}} f(x) \, dx = \int_{\mathbb{R}^2} f(x) \, dx. \]
By Lemma 6.2 the additional integrability assumption on $K$ is equivalent to $\sum_{n \geq 1} \int_{B(0,n) \cap \Omega} |x| K(0,x) \, dx < \infty$. We change the order of summation and use that fact to get

$$\sum_{i \in \mathbb{Z}} \sum_{j+1 < i \leq n < i} (f_{n+1} - f_n)^2 \int_{Q_i} \int_{Q_j} |x - y| K(x,y) \, dx \, dy$$

$$= \sum_{n \in \mathbb{Z}} (f_{n+1} - f_n)^2 \sum_{i > n} \sum_{j+1 < i < n} \int_{Q_i} \int_{Q_j} |x - y| K(x,y) \, dx \, dy$$

$$\lesssim \sum_{n \in \mathbb{Z}} (f_{n+1} - f_n)^2 \sum_{n \in \mathbb{Z}} \int_{Q_n} \int_{Q_{n+1}} (f(x) - f(y))^2 |x - y|^{-2-\alpha} \, dx \, dy$$

Thus (6.4) is smaller than (6.3). \qed

**Proof of Theorem 1.1.** The idea is similar as above. We split $\Omega$ into a family of unit cubes $(Q_i)_{i \in \mathbb{Z}^k}$ and we let $L_i = \text{Int}[(Q_j): B(x_Q, \sqrt{d}) \cap Q_j \neq \emptyset]$. By Theorem 1.1 for $0 < \rho < 1$ we have

$$\int_{\Omega} \int_{B(x, \rho d(x))} (f(x) - f(y))^2 |x - y|^{-d-\alpha} \, dx \, dy \approx \sum_{i \in \mathbb{Z}^k} \int_{Q_i} \int_{L_i} (f(x) - f(y))^2 |x - y|^{-d-\alpha} \, dx \, dy.$$
To finish the proof we note that the double sum over \(i, j\) does not depend on \(n\), hence we take \(n = 0\) and get that
\[
\sum_{j \geq 0} \sum_{i < j + 1} \int_{Q_j} \int_{Q_i} |x - y|^{-d - \alpha + 1} \, dx \, dy \approx \sum_{j \geq 0} \int_{Q_j} \int_{B(y, |j|) \cap \Omega} |x - y|^{-d - \alpha + 1} \, dx \, dy
\]
\[
= \sum_{j \geq 0} \int_{Q_j} \sum_{m=0}^{\infty} \int_{(B(0,2^{m+1}|j|) \setminus B(0,2^m|j|)) \cap \Omega} |x|^{-d - \alpha + 1} \, dx \, dy \approx \sum_{j \geq 0} \int_{Q_j} \sum_{m=0}^{\infty} (2^m|j|)^k (2^m|j|)^{-d - \alpha + 1} \, dy
\]
\[
\approx \sum_{j \geq 0} |j|^{-d - \alpha + 1} = \sum_{j \geq 0} |j|^{-k - l - \alpha + 1},
\]
which is finite provided that \(l = 1\) and \(\alpha > 1\), or \(l > 1\). \(\square\)

7. Application: a new class of Markov processes

In this section we present how our comparability results (in particular, Theorem 1.1) can be applied to prove the existence of a Markov stochastic processes corresponding to the truncated seminorms (1.3). Hereafter we work with Sobolev spaces, i.e. \(p = q = 2\).

Due to demonstrative character of this section we refrain from formulating precise assumptions on \(K\) and \(\Omega\). Here we only mention that the censored and reflected processes have been investigated for domains more general than Lipschitz in case of the fractional Sobolev spaces [4] and for a class of subordinate Brownian motions [24].

The starting point is the pure-jump Lévy process \(X_t\) on \(\mathbb{R}^d\) with intensity of jumps given by the Lévy measure \(K(0, y) \, dy\). It is well-known [19] that for \(u \in C^2_0(\mathbb{R}^d)\), the generator of \(X_t\) is the singular integral operator
\[
Lu(x) = \lim_{\varepsilon \to 0^+} \int_{B(x, \varepsilon)^c} (u(y) - u(x))K(x, y) \, dy.
\]
The Dirichlet form associated with this operator is given by the formula (see [10, Example 1.4.1])
\[
\mathcal{C}(u, u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 K(x, y) \, dx \, dy
\]
for \(u \in L^2(\mathbb{R}^d)\) for which this quantity is finite. Note that \(\mathcal{C}\) is different from (1.2) where the integration domain is \(\Omega \times \Omega\).

The form (1.2), which we are interested in, corresponds to a generalization of the censored, or reflected stable process introduced by Bogdan, Burdzy and Chen in [4]. We define
\[
\mathcal{E}(u, u) = \int_{\Omega} \int_{\Omega} (u(x) - u(y))^2 K(x, y) \, dx \, dy
\]
and \(\mathcal{E}_1(u, u) = \mathcal{E}(u, u) + \|u\|_{L^2(\Omega)}\). We consider two domains for \(\mathcal{E}\):
\[
\mathcal{F} = \text{Completion of } C^\infty_c(\Omega) \text{ with respect to } \mathcal{E}_1',
\]
\[
\mathcal{F}^{\text{ref}} = \{ u \in L^2(\Omega) : \mathcal{E}(u, u) < \infty \}.
\]
The Dirichlet spaces \((\mathcal{E}, \mathcal{F}), (\mathcal{E}, \mathcal{F}^{\text{ref}})\) correspond to the censored and reflected process respectively. Note that in our notation \(\mathcal{F}^{\text{ref}} = F_{2,2}(\Omega)\).
For sufficiently regular $K$ and $\Omega$ the spaces $\mathcal{F}$ and $\mathcal{F}^{\text{ref}}$ (hence, the corresponding processes) were shown to be the same [4, Corollary 2.6], [24, Corollary 2.9].

For a fixed $\theta > 0$ let

$$\mathcal{E}^{\text{tr}}(u, u) = \int_{\Omega} \int_{B(x, \theta \delta(x))} (u(x) - u(y))^2 K(x, y) \, dy \, dx.$$  

Knowing that $\mathcal{E}$ and $\mathcal{E}^{\text{tr}}$ are comparable and using [10, Theorem 7.2.1], we get the existence of the process of the reflected type.

**Corollary 7.1.** $(\mathcal{E}^{\text{tr}}, F^{2,2}(\Omega))$ is a regular Dirichlet form. Consequently, there exists a symmetric Hunt process with the Dirichlet form $(\mathcal{E}^{\text{tr}}, F^{2,2}(\Omega))$.

Note that the existence of the censored-type process for $\mathcal{E}^{\text{tr}}$ does not require comparability results. It comes from the fact that $\mathcal{E}^{\text{tr}}$ is smaller than $\mathcal{E}$ and has a similar structure. Indeed, $(\mathcal{E}^{\text{tr}}, C_c^\infty(\Omega))$ is closable, and normal contractions [10, Section 1.1] operate on it.

Concerning these classes of processes, it would be interesting to estimate the transition probabilities, investigate other potential theoretic objects and verify whether the process hits the boundary in finite time.

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