A Decomposition Approach to Counterfactual Analysis in Game-Theoretic Models

Nathan Canen and Kyungchul Song
University of Warwick and CEPR, and University of British Columbia

Abstract. Decomposition methods are often used for producing counterfactual predictions in non-strategic settings. When the outcome of interest arises from a game-theoretic setting where agents are better off by deviating from their strategies after a new policy, such predictions, despite their practical simplicity, are hard to justify. We present conditions in Bayesian games under which the decomposition-based predictions coincide with the equilibrium-based ones. In many games, such coincidence follows from an invariance condition for equilibrium selection rules. To illustrate our message, we revisit an empirical analysis in Ciliberto and Tamer (2009) on firms’ entry decisions in the airline industry.

Key words: Counterfactual Analysis, Game-Theoretic Models, Bayes Correlated Equilibria, Decomposition Method

JEL Classification: C30, C57

1. Introduction

One of the central goals of empirical research in economics is to quantify the effects of new policies that are yet to be implemented. Examples include analyzing the effects of increasing minimum wages on labor outcomes, the effects of different government legislation in healthcare and the effects of different market characteristics on firm entry, to name only a few.

To obtain appropriate counterfactual predictions in strategic environments, researchers typically specify a game-theoretic model and estimate structural parameters of the model, the
results of which are used for generating post-policy predictions. The main virtue of this approach is that its predictions are incentive compatible, that is, agents have no incentive to deviate from the strategies yielding the post-policy predictions. However, it requires specification of the game to fine details, often made to address computational challenges in implementation. These challenges frequently arise from the presence of multiple equilibria in the game, which may result in identified sets that are too large for meaningful policy analysis. In fact, it is not uncommon that they produce conflicting conclusions (e.g., see Aguirregabiria and Nevo (2013), p.111).

An alternative approach for counterfactual analysis is to use decomposition methods. Such methods extrapolate the observed relationship between an outcome and an explanatory variable to a counterfactual environment. They are widely used in labor economics. For instance, a researcher studying the effects of minimum wages on wage inequality (e.g., DiNardo, Fortin, and Lemieux (1996)) would first estimate this relationship in the observed data, then use those estimates with different (counterfactual) minimum wages to evaluate the distribution of wages under the new policy. The decomposition-based approach has several practical merits: it is computationally simple, yields point-identified predictions, and does not rely on detailed specifications (e.g., of payoffs or unobserved heterogeneity). Furthermore, statistical inference is often straightforward: we can just use bootstrap. However, to the best of our knowledge, decomposition-methods have been used mostly in non-strategic settings.

What prevents us from using decomposition methods for counterfactual analysis in a strategic setting? Decomposition methods assume that the causal relationship between the outcome variable and the policy variable remains the same in the counterfactual environment. This assumption can be violated if an agent has an incentive to deviate from the equilibrium strategies in the original game after a new policy. In such cases, we cannot put forth a decomposition-based prediction as a sound counterfactual prediction.

In this paper, we present a set of sufficient conditions under which the predictions obtained from the decomposition method coincide with the equilibrium-based predictions (i.e., the predictions defined in terms of the equilibria in the game in combination with the unknown equilibrium selection rule in the data generating process). These results are derived in a general strategic environment, where the policy of interest affects a component of the model with sampling variation (e.g., observed variables in the payoff function).\footnote{Throughout the paper, we focus on counterfactual analysis where a policy of interest changes an observed random vector constituting the payoff state and the target of prediction is the action of the agents (or a known function of such actions). While this restriction covers a large set of policy analysis settings, it does exclude many important situations of counterfactual analysis such as those that focus on a change in the welfare. We clarify this restriction on the scope later in the paper.} For a formal analysis, we consider a generic game, which includes games with various information structures, with
the solution concept of Bayes Correlated Equilibria of Bergemann and Morris (2016) (which includes Nash equilibria and other concepts as special cases).

It turns out that the coincidence between equilibrium-based predictions and those from decomposition methods applies to both complete information and incomplete information games. It does not depend on whether we use Nash equilibria as a solution concept or not. Rather, the coincidence depends on the class of counterfactual policies that are considered.

The sufficient conditions for the validity of decomposition methods can be summarized as follows: (a) the policy alters only a publicly observed component of the payoff state (observed by players and the researcher), (b) the policy keeps the payoff state within its support in the pre-policy game, and (c) the equilibrium selection rule conditional on a payoff state remains the same if the set of equilibrium actions conditional on the payoff state remains the same after the policy. When these conditions are met, researchers can use a decomposition-based prediction as a counterfactual prediction because this prediction is the same as the equilibrium-based one.

Condition (a) is satisfied by many policies, including those that affect taxes, tariffs or laws, which are often observed by all agents. Most of all, complete information games satisfy this condition immediately.

Condition (b) is also satisfied in many empirical applications. Even when the condition is not met, we show below that we can obtain bounds for the equilibrium-based prediction using decomposition-based predictions. These bounds are generally informative and easy to estimate using data.

The invariance condition (c) in this paper requires that, for each value of the payoff state, if the pre-policy set of equilibrium actions and the post-policy set of equilibrium actions are the same, so are their selection probabilities. While the invariance condition is not innocuous, it seems intuitive because, if the equilibrium action profiles remain the same at a payoff state after the policy, this means that the policy has not altered the prediction of the game at the payoff state. If the equilibrium selection probability at the payoff state nevertheless changed after the policy, our counterfactual predictions would depend on variations which have nothing to do with the prediction of the game-theoretic model.

Furthermore, the invariance is already widely used in empirical work in various disguises. For example, this condition is implicitly used when we focus on a specific equilibrium played (e.g., a Pareto superior equilibrium, or the most profitable for a specific firm as in Jia (2008)), assumes that the same equilibrium is played after the policy (as discussed in Aguirregabiria and Mira (2010)), or parametrizes the equilibrium selection rule and uses the estimated selection function for counterfactual analysis (e.g., Bajari, Hong, and Ryan (2010)). In their analysis of counterfactual predictions on games, Aguirregabiria and Mira (2013) explicitly considered an invariance condition for the equilibrium selection rule though differently from ours. On the other
hand, there are alternative approaches that are fully agnostic about the equilibrium selection rule or even about part of the structure of the game, such as the partial identification approach of Ciliberto and Tamer (2009) and Haile and Tamer (2003).

Our results can be applied to many empirical settings, including counterfactual exercises in entry games where a policy changes part of the market characteristics (e.g., Jia (2008)) or regulatory policy (e.g., Ciliberto and Tamer (2009)), in auction markets where the focus is on the impact of a change in reserve prices on auction outcomes, in various policy settings in labor economics such as increases in taxes or minimum wages on employment, among others. (We provide details on the validity and its implementation below.) In all of these examples, under the invariance condition, it suffices to run a decomposition-based prediction to recover the equilibrium-based prediction without further assumptions on payoffs or distributions of unobserved heterogeneity.

Overall, our results show the tradeoffs that researchers face when they perform counterfactual analysis using a game-theoretic model. They can use the decomposition approach which is more computationally tractable and does not require specifying the fine details of the model, at the expense of focusing on a rather narrower class of counterfactual policies under the invariance of the equilibrium selection rules. Hence, although the decomposition approach requires weaker assumptions (e.g., on parametrizations and information structure), it cannot generally be used for other counterfactuals, including those that involve changes in parts of the model that do not have sample variation (e.g., to structural parameter values) or those that are players' private information. It further requires the invariance condition on the equilibrium selection rules described above. The right balance in the tradeoff will depend on the details of the empirical settings and the policy questions of interest. The primary contribution of our paper is to provide formal results that clarify these tradeoffs for researchers.

Finally, we emphasize that even if one were to use decomposition-based predictions to point- or interval-identify the equilibrium-based predictions, it does not eliminate entirely the need to use a game-theoretic model for counterfactual analysis. On the contrary, to check the validity of decomposition-based predictions for counterfactual analysis, we need to clarify the strategic environment, the agents' information structure, and then consider what components of the strategic environment are invariant to the policy of interest. We believe that our results are useful for this step, as they show which specifications are needed (and which ones are not) for the use of decomposition-based predictions in such settings. Once the validity of decomposition-based predictions is confirmed, we do not need to specify further details of the game for counterfactual analyses.

As an illustration of our results, we revisit the empirical application of Ciliberto and Tamer (2009). Using a model of a complete information entry game, they studied the U.S. airline
market and assessed the effects on entry from a repeal of the Wright amendment, a law that restricted entry in routes using Dallas Love Field Airport. Due to the multiplicity of equilibria, they pursued a set identification approach and reported maximum entry probabilities as counterfactual predictions. In our application, we produce a decomposition-based prediction and compare it with the prediction from Ciliberto and Tamer (2009). We also compare these predictions with the actual results following the repeal of the Wright amendment in 2014. We find that the decomposition-based predictions using the pre-policy data perform well out-of-sample.

As a second empirical application, we follow Goolsbee and Syverson (2008) and revisit the effects of a shock which decreases Southwest’s costs of operating a given market on its competitors’ decisions. Our focus, though, is on its effects on competitors’ entry, rather than prices. We find that other airlines’ entry behavior is consistent, on average, with deterrence. However, this effect is heterogeneous across markets, and driven primarily by small markets. In large markets, they appear to “accommodate” Southwest (i.e., the competitors do not change their own entry behavior when Southwest has a lower cost of operating that same market). This is consistent with evidence in other industries (e.g., Tenn and Wendling (2014)). We then extend this exercise to settings where multiple competitors potentially have lower costs to operate that same route. While airlines are generally more likely to enter when the number of potential competitors decreases from low benchmarks (i.e., when they are likely to become monopolists), this is not the case when very competitive markets become slightly less competitive.
Related Literature

The idea that we may not need to identify a full structural model to identify policy effects of interest goes back at least to Marschak (1953), as pointed out by Heckman (2010). See Wolpin (2013) for further examples and discussion. Recent approaches exploring similar insights include the sufficient statistics approach (most notably in public finance, e.g., Chetty (2009); see Kleven (2020) for a recent overview) and policy relevant treatment effects proposed by Ichimura and Taber (2000) and Heckman and Vytlacil (2001). See Heckman (2010) for a review of this literature. We convey a similar message in this paper by studying conditions for the validity of decomposition-based predictions in strategic settings.

The decomposition approach is widely used in economics, most notably in labor economics. In the decomposition approach, the researcher estimates their model and then decomposes the variation in the outcome into the effects from different covariates. One example is the study of the effects of minimum wages on wage inequality (e.g., DiNardo, Fortin, and Lemieux (1996)): while there might be other mechanisms that affect wage inequality (e.g., unionization), once the model has been estimated, we can check how the distribution of wages would have differed if minimum wages had increased. Salient examples of the decomposition approach include the early work of Oaxaca (1973) and Blinder (1973), Juhn, Murphy, and Pierce (1993), the nonparametric/semiparametric approach of DiNardo, Fortin, and Lemieux (1996), and other extensions which has been very widely applied, see Fortin, Lemieux, and Firpo (2011) for a survey. The connection between the decomposition methods and methods of program evaluations have been pointed out by Fortin, Lemieux, and Firpo (2011). Kline (2011) shows that the estimated decomposition-based prediction can be interpreted as a reweighting estimator in the program evaluation setting.

More relevant to our paper is the decomposition approach used for counterfactual policy predictions. Rothe (2010) and Chernozhukov, Fernández-Val, and Melly (2013) provided inference on the full counterfactual distributions. Hsu, Lai, and Lieli (2022) introduced the quantile counterfactual treatment effects on a different population, and showed how we can identify and perform inference on the treatment effects using an invariance condition. While the main emphasis of this literature is on the general method of inference on various counterfactual distributions, our focus is on presenting a generic set of conditions in game-theoretic models under which such decomposition-based predictions can be accepted as valid predictions.

A common way to generate counterfactual predictions in a strategic environment is to specify and estimate a game-theoretic model, and then use the predictions from its equilibria. In many cases, point-identification of these models is not possible due to multiple equilibria. Researchers often either choose one equilibrium from the game (e.g, Jia (2008)) or conduct inference on the identified set allowing for all equilibria (e.g., Ciliberto and Tamer (2009)).
In light of these challenges, a recent literature studies point-identification of counterfactual predictions without identifying the full details of the model. These developments have been centered around dynamic discrete choice models – e.g., Aguirregabiria and Suzuki (2014), Norets and Tang (2014), Arcidiacono and Miller (2020), Kalouptsidi, Scott, and Souza-Rodrigues (2020). While many structural models within this class are unidentified - see the discussion in Aguirregabiria and Suzuki (2014), some counterfactuals are point-identified such as those characterized by linear changes in payoffs (the so called “additive transfers counterfactuals” in Kalouptsidi, Scott, and Souza-Rodrigues (2017)). See also Kalouptsidi, Kitamura, Lima, and Souza-Rodrigues (2020) for partial identification of counterfactuals in a similar context. Jun and Pinkse (2020) explored various approaches to produce a point-decision on partially identified counterfactual predictions from a game. Recently, Gu, Russell, and Stringham (2021) found a useful characterization of an identified set for counterfactual predictions in a general class of discrete-outcome models.

The message in this paper is related to that of Kocherlakota (2019). Using a model of a dynamic game between the private sector and the government, he showed that to obtain an optimal policy, we can just use predictions by regressing policy payoffs on policies using historical data, without relying on a structural macroeconomic model. There are major differences between his framework and ours. First, his model is a dynamic model where the policy-maker is a player of the game, whereas ours is a static one in which the policy-maker is outside the model. Second, his model is a macroeconomic model where the analysis is based on one equilibrium which generated the data. In our setting, many independent copies of a static game are observed so we need to deal with the problem of multiple equilibria.

We use the solution concept of BCE due to its generality, which makes it easy to demonstrate that the decomposition method is not limited to a specific equilibrium concept. Furthermore, through the use of BCE, we show that our result holds even if the counterfactual policy changes players’ information structures in specific ways – see Section 2.5.2. BCE has recently gained interest as a solution concept in an empirical setting due to its robustness properties and computational tractability. Magnolfi and Roncoroni (2022) adopt it in their study of entry decisions in the supermarket sector. They focus on characterizing the identified set obtained under a weak assumption on the information structure of the game, and use the set to study a policy that changes a covariate (presence of large malls). Other empirical examples using BCE and obtaining partial identification include Gualdani and Sinha (2020) on discrete choice models and Syrgkanis, Tamer, and Ziani (2018) on auctions. Finally, Bergemann, Brooks, and Morris (2022) present a general framework to produce a set of counterfactual predictions in games exploiting the requirement that observations be consistent with some unknown information structure of the games. Their counterfactual policies are concerned with a change of payoff
functions or the set of action profiles, whereas in our setting, the policies are mainly a change in the distribution of the payoff state vector. More importantly, our emphasis is on the use of decomposition-based predictions without requiring the researcher to specify details on the payoff functions and the distribution of the payoff states.

This paper is organized as follows. In Section 2, we begin with a brief overview of our main results using a simple example. Then, we present the formal results of the validity of decomposition-based predictions for generic Bayesian games and provide discussions, extensions and examples. In Section 3, we provide empirical applications. Section 4 concludes. The mathematical proofs, details on generalizations of our results, details on the data and the implementation of decomposition-based predictions are found in the Online Appendix to this paper.

2. Counterfactual Analysis Using Game-Theoretic Models

2.1. An Overview with a Simple Example

2.1.1. Environment. To fix ideas, we revisit a complete information entry game that has been widely used in the empirical literature. There are two firms, \( i = 1, 2 \), that choose a binary action \( Y_i \in \{0, 1\} \), \( Y_i = 1 \) representing entry in the market, and \( Y_i = 0 \) staying out of the market. As in many empirical models used in the literature, we may consider specifying the payoff function parametrically. While our main results do not require such specifications, we consider one possible parametrization of the payoff function (e.g., Bresnahan and Reiss (1991), Jia (2008), Ciliberto and Tamer (2009)):

\[
u_i(Y, W_i) = Y_i(-\delta Y_{-i} + X_i' \beta_i + \epsilon_i),
\]

where \( W_i = (X_i, \epsilon_i) \) is the payoff state with \( X_i \) and \( \epsilon_i \) denoting the characteristics of firm \( i \) observable and unobservable by the researcher, \( \delta > 0 \) is a parameter measuring the effect of competition and \( \beta_i \) coefficient vector for the observed covariates. We denote the set of equilibria as \( E \). For illustration purposes, we assume that \( E \) is a finite set.

2.1.2. Counterfactual Predictions: Our focus is on predicting the entry probability for the firms when a policy changes the payoff state \((X, \epsilon)\) into \((f(X), \epsilon)\), for some map \( f \). In this setting, for example, Ciliberto and Tamer (2009) studied a change in an observable policy (modeled as a dummy variable in the payoff state), Jia (2008) a change in the market size (i.e., increasing the population size by 10%), Grieco (2014) a change in the presence of a supercenter, and Magnolfi and Roncoroni (2022) a decrease in the presence of large malls. All such policies change an observable payoff state \( X \) to \( f(X) \), without affecting the unobserved \( \epsilon \).
Given policy $f$, our target parameter is the average entry probabilities of the two firms after the (observable) payoff state is changed from $X$ into $f(X)$, as predicted by the entry game model. We call this **Average Equilibrium-based Prediction (AEP)**. It can be written as:

$$AEP \equiv \mathbb{E} \left[ \sum_{g \in E_f} g(f(X), \epsilon) e_f(g \mid f(X), \epsilon) \right],$$

where the expectation $\mathbb{E}$ is over the joint distribution of $(X, \epsilon)$, $E_f$ is the set of post-policy equilibria, and $e_f$ denotes the post-policy equilibrium selection rule, i.e., the probability of choosing equilibrium $g \in E_f$ *after* the policy $f$, given state $W = (f(X), \epsilon)$. It is usually difficult to estimate this prediction because we need to recover both the post-policy equilibria, $E_f$ and the equilibrium selection rule, $e_f$, from pre-policy data.

**2.1.3. The Decomposition-Based Prediction.** An alternative approach to prediction in this environment is to extrapolate the pre-policy relationship between $X$ and $Y$. One example, which we call the **Average Decomposition-based Prediction (ADP)** due to its link to decomposition methods in labour economics, is:

$$ADP = \mathbb{E}[m(f(X))1\{f(X) \in \mathbb{S}_X\}],$$

where $m(x) = \mathbb{E}[Y \mid X = x]$ and $\mathbb{S}_X$ denotes the support of $X$. By extrapolating pre-policy relationships, the ADP implicitly uses the pre-policy equilibrium selection mechanism ($e(\cdot)$) and set of equilibria ($\mathcal{E}$) for predictions, even if both may change following the policy.

The practical advantage of using the ADP is clear: we do not need to recover the set of equilibria or the equilibrium selection rule from data for the counterfactual prediction. The quantity is point-identified, and statistical inference on it is simple. However, its predictions may fail to be incentive compatible in the post-policy game.

**2.1.4. Main Results.** Our main contribution is to provide sufficient conditions for ADP to serve as useful bounds for the target parameter AEP. The sufficient conditions are that the policy $f$ affects only the publicly observable component of $X$ and the pre and post-policy equilibrium selection rules satisfy a type of invariance condition (described below), which is applied in many empirical settings. Under these conditions, we show that

$$ADP \leq AEP \leq ADP + \Delta,$$

where $\Delta$ is the 2-dim vector with the same entry $P\{f(X) \notin \mathbb{S}_X\}$, ADP and AEP are 2-dim vectors and the inequalities are point-wise. The entry of $\Delta$ represents the probability that the policy sends the payoff state outside of the support of the payoff state in the pre-policy game. It is identified and can be estimated because $X$ is observable. It tends to increase as the policy $f$
transforms the payoff state further away from the support of the payoff state in the pre-policy game $G$.

Hence, we can identify bounds on the counterfactual predictions of the game relying only on the nonparametric causal structure of the game above without invoking more detailed specifications of the game, such as the parametric specification of payoffs as in (1). Furthermore, there is no need to compute the set of equilibria under a new policy, and this is no small convenience in practice. Finally, if $f(X) \in S_X$, this result strengthens to point-identification as

$$\text{AEP} = \text{ADP}.$$  

In this case, we can benefit from the computational simplicity of the ADP, while preserving the equilibrium-based nature of the AEP.

2.1.5. Invariance Condition on Equilibrium Selection Rules. As mentioned before, the sufficient conditions for the bounds in (4) are that (i) the counterfactual policy $f$ affects a publicly observable part of the payoff state $X$, and (ii) an invariance condition on the equilibrium selection rules is satisfied. Condition (i) accommodates many applications in the empirical literature, although not all. Section 2.7 below describes its limitations.

As for (ii), the condition says that if the set of equilibrium action profiles remains the same after the policy for each fixed $w = (x, \epsilon)$, their selection probability given $w$ remains the same as well. For example, suppose that we model the equilibrium selection rule as a function:

$$e(g \mid x, \epsilon) = h_g(x, \epsilon), \quad g \in \mathcal{E},$$

for some function $h_g$. Suppose further that under the counterfactual policy $f$, the equilibrium selection rule becomes

$$e(g \mid f(x), \epsilon) = h_g(f(x), \epsilon), \quad g \in \mathcal{E}.$$  

Then, the invariance condition holds.  

As mentioned in the introduction, some empirical applications have already used some versions of the invariance conditions as we discuss below.

Example 1 (Games with a Unique Equilibrium): The literature of empirical auctions often assumes the existence of a unique equilibrium bidding strategy (Guerre, Perrigne, and Vuong (2009)). In this case, the invariance condition for the equilibrium selection rule is vacuously satisfied. If the counterfactual prediction is with regards to a change of a payoff component that belongs to public information, our result shows that we can apply the decomposition

\footnote{Equation (5) in Bajari, Hong, and Ryan (2010), for instance, sets $h_g(x, \epsilon)$ as a logistic function. Here, we do not specify the functional form of the equilibrium selection rule or require it to be identified or partially identified from data. Instead, we clarify the precise form of the invariance condition that is required for the decomposition-based prediction in this paper.}
method and recover the counterfactual prediction without making assumptions required for the identification of the value distribution of the bidders. Details are found in Section 2.6.2 below.

**Example 2 (The Same Equilibrium in the Data and in the Counterfactual):** A common assumption in the empirical literature is that the same equilibrium is played in the data and in the counterfactual (see Aguirregabiria and Mira (2010), for example). In our notation, this means that \( e(g \mid x, \varepsilon) = e_f(g \mid x, \varepsilon) = 1 \) for some \( g \in \mathcal{E} \cap \mathcal{E}_f \), whenever \((x, \varepsilon)\) realizes in the common support of \(X\) and \(f(X)\). In this case, the invariance condition for the equilibrium selection rule is satisfied.

**Example 3 (Focusing on Specific Equilibria):** Some researchers prefer to focus on specific equilibria for their counterfactual analysis. This includes the highest profit equilibrium action for a certain firm (Jia (2008)) or the Pareto efficient equilibrium (see de Paula (2013)), among many others. For example, the highest profit equilibrium action does not change once the payoff state is fixed. If we let the highest profit equilibrium action profile at the payoff state \((X, \varepsilon)\) be denoted by \(g(x, \varepsilon)\), the policy does not change \(g(x, \varepsilon)\) at the same payoff state \((x, \varepsilon)\). Our focusing on this equilibrium action profile means that \( e(g \mid x, \varepsilon) = e_f(g \mid x, \varepsilon) = 1 \). Hence, the invariance condition for the equilibrium selection rule is satisfied.

### 2.2. A Finite-Player Bayesian Game

We turn to presenting our main results in a general set-up of strategic interactions. For this, we first introduce a finite player Bayesian game following Bergemann and Morris (2016) (BM, hereafter).\(^3\) In our model, the Bayesian game is populated by a finite set of players indexed by \(N = \{1, \ldots, n\}\). Let \(\mathcal{W}\) denote the set from which the payoff state \(W\) takes value, and \(\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_n\) the set from which the action profile \(Y = (Y_1, \ldots, Y_n)\) takes values. Each player’s action space \(\mathcal{Y}_i\) can be a continuum or a countable set.

Each player \(i\) is endowed with a payoff function \(u_i : \mathcal{Y} \times \mathcal{W} \rightarrow \mathbb{R}\), and chooses an action from \(\mathcal{Y}_i\). Let the payoff state \(W\) be drawn from a distribution \(\mu_W\) on \(\mathcal{W}\).\(^4\) Following BM, we call \(B = (\mathcal{Y}, \mathcal{W}, u, \mu_W)\) a basic game, where \(u = (u_i)_{i \in N}\) denotes the payoff profile. Each player \(i\) observes a signal vector \(T_i\) taking values in the space \(\mathcal{T}_i \subset \mathbb{R}^{d_T}, d_T \geq 1\). Define the signal profile \(T = (T_1, \ldots, T_n) \in \mathcal{T}\), where \(\mathcal{T} = \mathcal{T}_1 \times \ldots \times \mathcal{T}_n\). Once the payoff state is realized to be

---

\(^3\)Bergemann and Morris (2016) considered the case where the state space and the signal space are finite sets for simplicity. In this paper, we consider more general spaces for the state and signal spaces because the researcher’s models often involve both discrete and continuous variables.

\(^4\)We assume that \(\mathcal{Y} \times \mathcal{W}\) is endowed with a topology and a Borel \(\sigma\)-field. Throughout the section, we suppress measure-theoretic qualifiers, such as Borel sets, measurable functions, or a statement holding almost everywhere. Details of the mathematical set-up are found in the Online Appendix.
among others. Above, they include widely used models, including those applied in entry games and auctions, of distributions. For each \( \sigma \) \( \in \mathcal{S} \), let \( \sigma_i(\cdot \mid w, t) \) be a conditional distribution on \( \mathcal{Y}_i \), when the payoff state and the signal profile are realized to be \((w, t) \in \mathcal{W} \times \mathcal{T}\). Let \( \Sigma \) be a collection of such conditional distributions. For each \( i = 1, \ldots, n \), let \( \sigma_i(\cdot \mid w, t) \) be the \( i \)-th coordinate marginal conditional distribution of \( \sigma(\cdot \mid w, t) \). Following BM, we call each \( \sigma \in \Sigma \) a decision rule.

For each \( i = 1, \ldots, n \), \( t_i \in \mathcal{T}_i \) and \( \sigma \in \Sigma \) and a transform \( \tau_i : \mathcal{Y}_i \to \mathcal{Y}_i \), we write the expected payoff of player \( i \) as

\[
U_i(\tau_i, t_i; \sigma) = \int \int u_i(\tau_i(y_i), y_{-i}, w) d\sigma(y_i, y_{-i} \mid w, t_i, t_{-i}) d\mu_{W,T\mid \mathcal{T}_i}(w, t_{-i} \mid t_i),
\]

where \( \mu_{W,T\mid \mathcal{T}_i}(\cdot \mid t_i) \) denotes the conditional distribution of \((W, T_{-i})\) given \( T_i = t_i \) under the joint distribution \( \mu_{W,T} \) of \((W, T)\). The quantity \( U_i(\tau_i, t_i; \sigma) \) denotes the conditional expected payoff of player \( i \) given her signal \( T_i = t_i \) when the player \( i \) deviates from the action \( y_i \) recommended to her according to the decision rule \( \sigma \) and chooses \( \tau_i(y_i) \) instead.

We say that a decision rule \( \sigma \in \Sigma \) is a Bayesian Correlated Equilibrium (BCE) for \( G \) if for each \( i = 1, \ldots, n \), and each \( \tau_i : \mathcal{Y}_i \to \mathcal{Y}_i \) and \( t_i \) in the support of \( T_i \),

\[
U_i(\text{Id}, t_i; \sigma) \geq U_i(\tau_i, t_i; \sigma),
\]

where \( \text{Id} \) is the identity map on \( \mathcal{Y}_i \). Denote by \( \Sigma_{\text{BCE}}(G) \) the set of BCE’s of game \( G \). BCE generalizes other solution concepts, such as Bayes Nash equilibria. Later we show how our result carries over to these other solution concepts.

Concrete examples of the set-up and our results are provided in Section 2.6. As anticipated above, they include widely used models, including those applied in entry games and auctions, among others.
2.3. Counterfactual Predictions from a Game

2.3.1. Predictions from a Bayesian Game. Predictions from a game $G$ are generated from the distribution of the observed action profile $Y$ conditional on the payoff state $W$, when such an action profile is generated by an equilibrium of the game $G$.

Given $\sigma \in \Sigma$ and the information structure $I = (\mathbb{W}, \mathbb{T}, S)$, we define a probability measure $\rho_\sigma(A \mid w)$ on $\mathbb{Y}$ for each $w \in \mathbb{W}$ as follows:\(^5\)

\[\rho_\sigma(A \mid w) \equiv \int \sigma(A \mid w, t) d\alpha(t \mid w).\] (8)

Hence, $\rho_\sigma(A \mid w)$ indicates the probability of the action profile realizing in the set $A$ when the payoff state is $W = w$, and the actions are drawn by the decision rule $\sigma$. For example, in an entry game where firms $i = 1, \ldots, n$ are deciding whether to enter ($Y_i = 1$) or not ($Y_i = 0$), and this depends on market and firm-level characteristics $W = w$, then $\rho_\sigma(A \mid w)$ is the probability of the entry profile $(Y_1, \ldots, Y_n)$ being realized in $A$ given market conditions $w$.

The researcher observes only one action profile $Y$ from the game $G$, even when there are multiple equilibria in this game. In order to complete the description of how $Y$ is generated, we introduce a generic form of the equilibrium selection rule. For a game $G$, we define the equilibrium selection rule (denoted by $e(\cdot ; G)$) to be a distribution on $\Sigma_{BCE}(G)$. Thus, the generation of $Y$ is described as follows:

Step 1: An equilibrium $\sigma \in \Sigma_{BCE}$ is drawn from the distribution $e(\cdot ; G)$.

Step 2: The value of the payoff state $W = w$ is drawn from the distribution $\mu_W$.

Step 3: The action profile $Y$ is drawn from the distribution $\rho_\sigma(\cdot \mid w)$.

The three steps summarize the causal structure of the model for observed actions $Y$ and the payoff state $W$. Given a game $G$ and $w \in \mathbb{W}$, we define a (randomized) reduced form of game $G$ as\(^6\), for $A \subset \mathbb{Y}$,

\[\rho_G(A \mid w) \equiv \int_{\Sigma_{BCE}(G)} \rho_\sigma(A \mid w) d\sigma(G).\] (9)

The term randomized reduced form is due to the generation of $Y$ being completely described by the couple $(\rho_G, \mu_W)$. It can be represented by first drawing $W = w$ from $\mu_W$, then drawing $Y$ from the distribution $\rho_G(\cdot \mid w)$. Thus, $\rho_G(A \mid w)$ gives the probability of $Y$ taking a value in a set $A$ when $W$ is fixed to be $w$. Hence the reduced form $\rho_G$ gives the prediction rule of the game in counterfactual analysis.

\(^5\)The conditional probability $\rho_\sigma$ corresponds to what Bergemann and Morris (2016) called an “outcome induced by the decision rule $\sigma$”.

\(^6\)The integral with respect to the equilibrium selection rule is an integral of a real function on the space of conditional distributions which we topologize appropriately. Details can be found in the Online Appendix.
From here on, any probability statements (including expectation and conditional expectation) involving \((Y, W)\) are with respect to the joint distribution defined as follows:

\[
P\{Y \in A, W \in S\} = \int_S \rho_G(A \mid w) d\mu_W(w), \quad A \subset \mathcal{Y}, \quad S \subset \mathcal{W}.
\]

2.3.2. **Counterfactual Predictions.** Our counterfactual experiment involves a new policy that changes the payoff state \(W\) into \(f(W)\) for some map \(f : \mathcal{W} \to \mathcal{W}\). The policy changes the basic game \(B\) into

\[
B_f = (\mathcal{Y}, \mathcal{W}, u, \mu_W \circ f^{-1}),
\]

where \(\mu_W \circ f^{-1}\) is the distribution of \(f(W)\) when \(W\) is drawn from \(\mu_W\). Thus our counterfactual analysis involves a policy that transforms the pre-policy game \(G = (B, I)\) into a post-policy game\(^7\)

\[
G_f = (B_f, I).
\]

As we describe in Section 2.6, this can include changes to market characteristics (population size, or changes to existing market policies) or changes to observable reserve prices in auctions. It further extends to many other settings where the policy affects a random variable with sampling variation, such as individual income, taxes, etc.

A counterfactual prediction at \(f(w)\) in game \(G_f\) can be made from the reduced form \(\rho_\sigma\) induced by \(\sigma \in \Sigma_{\text{BCE}}(G_f)\), i.e.,

\[
\rho_\sigma(A \mid f(w)) = \int \sigma(A \mid f(w), t) d\alpha(t \mid f(w)).
\]

Let \(e(\cdot ; G_f)\) be the equilibrium selection rule of the game \(G_f\). Then the **equilibrium-based prediction of the post-policy game** \(G_f\) is given by

\[
\rho_{G_f}(A \mid f(w)) \equiv \int_{\Sigma_{\text{BCE}}(G_f)} \rho_\sigma(A \mid f(w)) d\epsilon(\sigma; G_f), \quad A \subset \mathcal{Y}.
\]

The quantity \(\rho_{G_f}(A \mid f(w))\) represents the probability of the action profile from the post-policy game \(G_f\) realizing in \(A\) when the payoff state is \(f(w)\).

An alternative way of generating a prediction is to use the following:

\[
\rho_G(A \mid f(w)) \equiv \int_{\Sigma_{\text{BCE}}(G)} \rho_\sigma(A \mid f(w)) d\epsilon(\sigma; G).
\]

\(^7\)It is important to note that here we regard the information structure \(I\) as invariant to the policy. This does not mean that the signals remain the same after the policy. Note that the post-policy signals are generated from the distribution \(\alpha(\cdot \mid f(w))\). Hence if the payoff states are drawn differently, this affects realized signals.
We call $\rho_G(A \mid f(w))$ the **decomposition-based prediction from game $G$**. When $f(w)$ is in the support of $W$, the last integral is equal to

$$P\{Y \in A \mid W = f(w)\}.$$

Hence, this prediction extrapolates the relation between $Y$ and $W$ in the pre-policy game to the post-policy game.

In general, decomposition-based predictions do not coincide with equilibrium-based predictions in (12), when $\rho_G$ is not policy-invariant. Our main result below presents a set of sufficient conditions under which the equilibrium-based predictions have upper and lower bounds that can be identified using only decomposition-based predictions. The bounds coincide when $f(W)$ is in the support of $W$.

### 2.4. Interval-Identification of Equilibrium-Based Predictions

#### 2.4.1. The Main Result

The main result of this paper shows that the equilibrium-based prediction is interval-identified by the decomposition-based prediction under two sufficient conditions. The first condition is concerned with the information structure and requires that the policy affects only part of the payoff states that are commonly observed by all the players. The second condition is an invariance condition on the equilibrium selection rules.

**Assumption 2.1** *(Information Structure Condition)*

(i) $W = W_C \times W_I$, for some sets $W_C$ and $W_I$, and

$$f(w) = (\tilde{f}(w_C), w_I), \quad w = (w_C, w_I) \in W,$$

for some map $\tilde{f} : W_C \to W_C$.

(ii) For each $i = 1, ..., n$, $T_i = T_{C,i} \times T_{I,i}$, where $T_{C,i} = W_C$, and

$$\alpha(A \mid w) = 1\{s(w) \in A\},$$

for $A \subset T$, for a map $s = (s_1, ..., s_n) : W \to T$, where for each $i = 1, ..., n$,

$$s_i(w) = (w_C, \varphi_i(w_I)), \quad w = (w_C, w_I) \in W,$$

for some map $\varphi_i : W_I \to T_{I,i}$.

Assumption 2.1(i) requires that the change of the payoff state by the policy $f$ is restricted to the first component $W_C$ of the payoff state $W = (W_C, W_I)$. This assumption guarantees that the counterfactual payoff state is changing a publicly observed payoff state, rather than one that is private information. This is used to ensure that players’ posterior beliefs remain invariant after

---

8The subscript $C$ is mnemonic for “common” and $I$ for “idiosyncratic”. These subscripts are used in place of 1 and 2 to avoid a conflict with subscripts $i$ for individual players we use later.
the policy. The Online Appendix provides a simple example of a private information entry game which shows that when the policy changes the private information component of the payoff state, decomposition-based predictions fail. The failure is because a change in a private component of players’ payoff state leads to changes to posterior beliefs and creates incentives to deviate from the strategies used in the pre-policy game.

Assumption 2.1 (ii) says that, in terms of the relation between signals and payoff state vector, we have

\[ T = s(W) = (s_1(W), \ldots, s_n(W)), \quad s_i(W) = (W_C, \varphi_i(W_i)), \]

for some maps, \( s \) and \( \varphi_i, i = 1, ..., n \). Hence, every player observes \( W_C \) that is subject to a change by the policy in (i). Assumption 2.1 allows for a setting where the signals include payoff irrelevant signals, by payoff functions \( u_i \) specified to depend on only part of \( W \).

Assumption 2.1 admits a broad class of information structures, as we see through examples below. In these examples, we specify the payoff state vector as \( W = (W_1, \ldots, W_n) \), where

\[ W_i = (X_i, \epsilon_i), \quad (14) \]

so that each individual player \( i \)'s payoff depends on \( W \) only through \( W_i \). Assumption 2.1 in this setting does not put any restrictions on the joint dependence of \( W_i \) across players. Hence we can accommodate the information structures considered by Magnolfi and Roncoroni (2022) in the entry game setting.

**Example 1: Complete Information:** In the case of a complete information game, we simply specify \( T_i = W \), so that each map \( s_i \) is an identity map. In this case, any policy \( f \) that changes \( W \) satisfies Condition (i) of Assumption 2.1 because we can simply take \( W_C \) to be \( W \) here.

**Example 2: Public-Private Dichotomy of Signals:** In the case of a public-private information structure where \( X_i \) is publicly observed while \( \epsilon_i \) belongs to private information, we specify \( W = (W_1, \ldots, W_n) \), \( W_i = (X_i, \epsilon_i) \), with \( T_i = s_i(W) \) such that \( s_i(W) = (X, \epsilon_i) \), where \( X = (X_1, \ldots, X_n) \). In this case, Condition (i) of Assumption 2.1 is satisfied, whenever the policy \( f \) is one that alters \( X \), not \( \epsilon_i \).

**Example 3: Observable Private Signals:** The researcher may observe signals that are part of the private information. To reflect this setting, we specify \( W = (W_1, \ldots, W_n) \), \( W_i = (X_i, \epsilon_i) \), with \( X_i = (X_{1,i}, X_{2,i}) \), and take \( T_i = s_i(W) \), with \( s_i(W) = (X_{1,i}, X_{2,i}, \epsilon_i) \), \( X_1 = (X_{1,i})_{i=1}^n \), so that \( X_{2,i} \) belongs to private information and is observed by the researcher. In this case, as long as \( f \) alters \( X_1 \), not \( (X_{2,i}, \epsilon_i) \), Condition (i) of Assumption 2.1 is satisfied.
Let us introduce the invariance condition on equilibrium selection rules. For each Bayesian game $G$ and $w \in \mathcal{W}$, let

$$
\Sigma_{BCE,w}(G) = \left\{ \sigma(\cdot \mid w, s(w)) : \sigma \in \Sigma_{BCE}(G) \right\}.
$$

This is the $w$-section of the set of BCE for game $G$, i.e., each member of the set $\Sigma_{BCE,w}(G)$ is an equilibrium probability over the action profiles given the payoff state $w$ and the signal $s(w)$. Similarly, we denote the $w$-section of the equilibrium selection rule $e(\cdot; G)$ as $e_w(\cdot; G)$: for each $w \in \mathcal{W}$ and for each $A \subset \Sigma_{BCE,w}(G)$,

$$
e_w(A; G) = e\left(\{ \sigma \in \Sigma_{BCE}(G) : \sigma(\cdot \mid w, s(w)) \in A \} ; G\right).
$$

We also define $e_w(\cdot; G_f)$ to be the $w$-section of $e(\cdot; G_f)$. Now, the invariance condition is stated as follows.

**Assumption 2.2** (Invariance of Equilibrium Selection Rules). For each $w \in \mathcal{S}_W \cap \mathcal{S}_{f(W)}$, if $\Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f)$, then, we have

$$
e_w(\cdot; G) = e_w(\cdot; G_f),
$$

where $\mathcal{S}_W$ and $\mathcal{S}_{f(W)}$ are the supports of $W$ and $f(W)$ under $\mu_W$ respectively.

The invariance condition requires that, if the equilibrium action profiles remain the same after the policy at the same payoff state $w$, their selection probability remains the same at the payoff state as well. In other words, once the payoff state is realized, and the set of action profiles in equilibrium in the post-policy game is the same as that in the pre-policy game, the action profile in the post-policy game is selected with the same probability as in the pre-policy game.\(^9\)

The invariance condition naturally arises from a consistency condition among equilibrium selection rules across different games. The consistency condition requires that, for each Bayesian game $G$, $e_w(\cdot; G)$ is derived from a primitive probability measure on $\Sigma$ as a conditional probability concentrated on $\Sigma_{BCE,w}(G)$. More specifically, let $\lambda$ be a probability distribution over $\Sigma$, and we define $\lambda_w$ from $\lambda$ in the same way as we defined $e_w$ from $e$. Then, for any set $A$ of probability measures on $\mathcal{Y}$, the consistency condition states that, for each $w \in \mathcal{W}$, we have

$$
e_w(A; G) = \frac{\lambda_w(A \cap \Sigma_{BCE,w}(G))}{\lambda_w(\Sigma_{BCE,w}(G))}.
$$

\(^9\)This assumption speaks only to the case where $\Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f)$. Recall that the payoff state $w$ includes both observable and unobservable states.
Then, Assumption 2.2 is satisfied, because $e_w(\cdot; G)$ depends on $G$ only through $\Sigma_{BCE,w}(G)$.\(^\text{10}\)

The invariance condition is already satisfied by many equilibrium selection rules used in the literature of empirical research. This includes the common assumption that the same equilibrium is played in the data and in the counterfactual (see Aguirregabiria and Mira (2010)). Relatedly, many papers choose a specific equilibrium which is analyzed in both the data and the counterfactual. This includes the highest profit equilibrium for a certain firm (Jia (2008)), the Pareto superior equilibrium (see de Paula (2013)), or the equilibrium studied by Milgrom and Weber (1982) in common value auctions. Furthermore, this invariance condition already holds when an equilibrium selection rule is kept the same for counterfactual predictions (e.g., in Bajari, Hong, and Ryan (2010), or as assumed in Berry and Haile (2014)). It is also satisfied if it is deemed constant across the equilibrium in the data and counterfactual (e.g., Aguirregabiria (2012), applied in Aguirregabiria and Ho (2012)). Our invariance condition is closely related to the one used in Aguirregabiria and Mira (2013). Their invariance condition requires that the equilibrium selection rule depend on the payoff state $w$ and the structural parameters $\theta$ only through the payoff function evaluated at $(w, \theta)$. On the other hand, our invariance condition is satisfied when the equilibrium selection rule is determined by the payoff state $w$ only.

Sometimes the invariance condition may be reasonable in a setting with small changes to the payoff state (e.g., small increases in minimum wages), because small changes can be well approximated by keeping the same game environment. In fact, many of the policy experiments in entry games in empirical research are “small policies” which change payoff states (or the set of affected players) by only a small amount, such as 10-15% changes in observed policy variables (as opposed to, say, doubling). For example, Jia (2008) considered the increase in market size by a small amount, 10%, and Magnolfi and Roncoroni (2022) considered a change in a market characteristic affecting only 13% of markets. This local argument is also used in Aguirregabiria (2012) and Aguirregabiria and Ho (2012), since they explore local approximations of the counterfactual values around the data.

In order to use the invariance condition, we need to ensure that, for any $w \in S_w \cap S_{f(w)}$, we have $\Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f)$. The essential role of Assumption 2.1 is to ensure this by maintaining each player’s posterior after the policy.

\(^{10}\)We call this condition consistency of equilibrium selection rules, as it is analogous to the consistency of beliefs in a Bayesian game where individual players’ beliefs are derived from a common prior on the true state of the game.
For a real map $h$ on $\mathcal{Y} \times \mathcal{W}$, let us define

$$
\operatorname{AEP}(h) \equiv \int \int_{\mathcal{Y} \times \mathcal{W}} h(y, w) d\rho_{\mathcal{G}}(y | w) d(\mu_w \circ f^{-1})(w),
$$

and

$$
\operatorname{ADP}(h) \equiv \int \mathbb{E}[h(Y, W) | W = f(w)]1\{f(w) \in \mathcal{S}_W\} d\mu_w(w).
$$

The quantity $\operatorname{AEP}(h)$ represents our target parameter which is the equilibrium-based expectation of $h(Y, f(W))$ after the policy\textsuperscript{11}, whereas $\operatorname{ADP}(h)$ represents its decomposition-based counterpart. Depending on the choice of $h$, which can also be vector-valued, we can express various objects of interest as listed below.

**Example 1: Expected Actions** We simply take $h(y, w) = y$. Then $\operatorname{AEP}(h)$ represents the expected action profile after the policy.

**Example 2: Distribution of the Action Profile** We take $h(y, w) = 1\{y \leq t\}$, $t \in \mathbb{R}^n$. Then $\operatorname{AEP}(h)$ is the CDF of the action profile at $t$, after the policy.

**Example 3: Distribution of Maximum Actions** We take $h(y, w) = 1\{\max_{1 \leq i \leq n, w_i \in S} y_i \leq t\}$, $t \in \mathbb{R}$, $w = (w_1, \ldots, w_n)$, for some set $S \subset \mathcal{W}$. Then $\operatorname{AEP}(h)$ represents the CDF at $t$ of the maximum action among those players $i$ with $W_i \in S$, after the policy.

We are prepared to present our main result.

**Theorem 2.1.** Suppose that Assumptions 2.1 and 2.2 hold for the pre-policy game $G = (B, I)$ and the post-policy game $G_f = (B_f, I)$. Let a map $h : \mathcal{Y} \times \mathcal{W} \to \mathbb{R}$ be such that, for all $w \in \mathcal{W}$ such that $f(w) \notin \mathcal{S}_W$,

$$
\underline{h}(w) \leq \inf_{y \in \mathcal{Y}} h(y, w) \leq \sup_{y \in \mathcal{Y}} h(y, w) \leq \overline{h}(w),
$$

for some maps $\underline{h}, \overline{h} : \mathcal{W} \to \mathbb{R}$.

Then,

$$
\operatorname{ADP}(h) + \Delta(\underline{h}) \leq \operatorname{AEP}(h) \leq \operatorname{ADP}(h) + \Delta(\overline{h}),
$$

where

$$
\Delta(\overline{h}) \equiv \mathbb{E}\left[\overline{h}(f(W))1\{f(W) \notin \mathcal{S}_W\}\right], \text{ and }
$$

$$
\Delta(\underline{h}) \equiv \mathbb{E}\left[\underline{h}(f(W))1\{f(W) \notin \mathcal{S}_W\}\right].
$$

When $Y_i \in [h_L, h_U]$, $h_L < h_U$, the theorem yields bounds for the conditional average predicted outcome of player $i$. More specifically, for $i = 1, \ldots, n$, define $h_i : \mathcal{Y} \times \mathcal{W} \to \mathbb{R}$ as $h_i(Y, W) = Y_i$.\textsuperscript{11}Note the change of variables within the integral, so that we are integrating $f(w)$.
Then the theorem above yields that, for each \( i = 1, \ldots, n \),
\[
\mu_i(f) + h_L \cdot P \{ f(W) \notin S_W \} \leq \text{AEP}(h_i) \leq \mu_i(f) + h_U \cdot P \{ f(W) \notin S_W \},
\]
where
\[
\mu_i(f) \equiv \int E[Y_i | W = f(w)] 1 \{ f(w) \in S_W \} d\mu_W(w).
\]

In an entry game where \( Y_i \in \{0, 1\} \), the bounds above with \( h_U = 1 \) and \( h_L = 0 \) are bounds for the predicted entry probability of firm \( i \) after the policy.

We can use the results to obtain bounds for the average effect of the policy \( f \) on the player \( i \)'s outcome. The average treatment effect (denoted by \( \text{ATE}_i \)) is defined as
\[
\text{ATE}_i \equiv \text{AEP}(h_i) - E[Y_i].
\]

Then we obtain the following bounds for the average treatment effect:
\[
\mu_i(f) - E[Y_i] + h_L \cdot P \{ f(W) \notin S_W \} \leq \text{ATE}_i \leq \mu_i(f) - E[Y_i] + h_U \cdot P \{ f(W) \notin S_W \}.
\]

One might ask whether the bounds in Theorem 2.1 are sharp. The following proposition gives a sense in which the answer is affirmative.

**Proposition 2.1.** Suppose that a policy \( f : \mathbb{W} \to \mathbb{W} \) and \( \mu_w \) are given such that the support of \( W \) overlaps that of \( f(W) \). Suppose further that maps \( h : \mathbb{Y} \times \mathbb{W} \to \mathbb{R}, h^L, h^R : \mathbb{W} \to \mathbb{R} \), are given as in Theorem 2.1, where \( \mathbb{Y} \) is a countable set and for all \( w \in \mathbb{W} \), \( \overline{h}(w) = \sup_{y \in \mathbb{Y}} h(y, w) \) and \( \underline{h}(w) = \inf_{y \in \mathbb{Y}} h(y, w) \).

Then, there exists a Bayesian game \( G \) such that Assumptions 2.1 and 2.2 hold and either of the two inequalities in (18) holds with equality.

Note that simple shape constraints such as \( \rho_G(\cdot | w) \) being monotone or concave in \( w \) in all the BCEs do not help improve the bounds because a constant map also satisfies such constraints trivially. However, shape constraints may help estimate \( \text{ADP}(h) \) more accurately. The next section overviews the heuristics of the proof of Theorem 2.1, while the subsequent section overviews identification of the bounds in the theorem. Concrete examples applying those results to entry games and auctions are then provided in Section 2.6.

**2.4.2. Heuristics behind Theorem 2.1.** To understand how Theorem 2.1 follows from Assumptions 2.1 and 2.2, let us assume for simplicity that \( h(Y, W) = Y \) and \( Y \in \{0, 1\} \). First, note that
\[
E[Y | W = f(w)] = \int y d\rho_G(y | f(w)).
\]

\(^{12}\)The condition of countability of \( \mathbb{Y} \) can be removed, for example, if \( h \) is a continuous map and \( \mathbb{Y} \) is compact.
Suppose that we have proved that, for \( w \in S_w \cap S_{f(w)} \),
\[
\rho_G(\cdot \mid f(w)) = \rho_{G_f}(\cdot \mid f(w)).
\] (19)

Then,
\[
E[Y \mid W = f(w)]1\{f(w) \in S_w \cap S_{f(w)}\} = \int yd\rho_G(y \mid f(w))1\{f(w) \in S_w \cap S_{f(w)}\}
= \int yd\rho_{G_f}(y \mid f(w))1\{f(w) \in S_w \cap S_{f(w)}\}
\leq \int yd\rho_{G_f}(y \mid f(w)).
\]

The last integral is bounded by
\[
\int yd\rho_{G_f}(y \mid f(w))1\{f(w) \in S_w \cap S_{f(w)}\} + 1\{f(w) \notin S_w \cap S_{f(w)}\}
= \int yd\rho_G(y \mid f(w))1\{f(w) \in S_w \cap S_{f(w)}\} + 1\{f(w) \notin S_w \cap S_{f(w)}\}.
\]

Integrating out \( w \) in the terms using \( \mu_W \), we obtain the desired bounds in Theorem 2.1. Thus, the crucial step is to show (19).

By the invariance condition in Assumption 2.2, it suffices for (19) to show that, for all \( w \in S_w \cap S_{f(w)} \), we have
\[
\Sigma_{\text{BCE},w}(G) = \Sigma_{\text{BCE},w}(G_f).
\]

This can be shown under the conditions for the information structure in Assumption 2.1. In fact, since the policy changes only the publicly observable payoff component, it can be shown that the sets \( \Sigma_{\text{BCE}}(G) \) and \( \Sigma_{\text{BCE}}(G_f) \) coincide when we restrict the payoff state to \( S_w \cap S_{f(w)} \).

We refer the reader to the Online Appendix for details.

2.4.3. Identification of the Bounds. If we observe the actions \( Y \) and the payoff state \( W \) of the pre-policy game, then we can recover the bounds in Theorem 2.1 from data without specifying the details of the game. However, in practice, we often do not observe the whole vector \( W \).

Suppose that \( W_i = (X_i, \epsilon_i) \), where \( X_i \) is observed and \( \epsilon_i \) unobserved by the researcher. Let the policy \( f \) be of the form \( f(W) = (f_1(W_1), \ldots, f_n(W_n)) \), where
\[
f_i(W_i) = (f_i^*(X_i), \epsilon_i), \quad i = 1, \ldots, n,
\] (20)

for some map \( f_i^* \). As for \( h \), we consider \( h(y, w) = h^*(y, x), \bar{h}(w) = \bar{h}^*(x) \), and \( \bar{h}(w) = \bar{h}^*(x) \), for some maps \( h^*, \bar{h}^* \), and \( \bar{h}^* \), which depend only on observable states. The identification of the bounds in Theorem 2.1 can be derived in this setting as follows.
First, by the choice of \( f \) in (20), we can identify

\[
\Delta(h) = \mathbb{E}\left[h^*(f^*(X)) \mathbb{1}\{f^*(X) \notin \mathcal{S}_X\}\right] \quad \text{and} \quad \Delta(\widetilde{h}) = \mathbb{E}\left[\widetilde{h}^*(f^*(X)) \mathbb{1}\{f^*(X) \notin \mathcal{S}_X\}\right],
\]

where \( \mathcal{S}_X \) denotes the support of \( X \), and \( f^*(X) = (f^*_1(X_1), \ldots, f^*_n(X_n)) \). Thus, for the interval identification of \( \text{ADP}(h) \), it suffices to identify \( \text{ADP}(h) \). However, the identification strategy of \( \text{ADP}(h) \) depends on whether \( X \) and \( \epsilon \) are independent or not. The following proposition considers a setting where \( X \) and \( \epsilon \) are independent.

**Proposition 2.2.** Suppose that \( X \) and \( \epsilon \) are independent. Then,

\[
\text{ADP}(h) = \int \mathbb{E}[h^*(Y, X) \mid X = f^*(x)] \mathbb{1}\{f^*(x) \in \mathcal{S}_X\} d\mu_X(x),
\]

where \( \mu_X \) denotes the distribution of \( X \).

Note that the identification result allows for \( \epsilon_1, \ldots, \epsilon_n \) to be correlated; this correlation can come from some unobserved characteristics of the game.

Suppose that \( X \) and \( \epsilon \) are potentially correlated. In this case, the decomposition-based approach may still be implemented using a control function approach (Blundell and Powell (2003) and Imbens and Newey (2009)). More specifically, suppose that \( X_i = (X_{i,a}, X_{i,b}) \) and

\[
f^*_i(X_i) = (g_i(X_{i,a}), X_{i,b}),
\]

for some map \( g_i \) so that the policy alters only \( X_{i,a} \). Define

\[
X_a = (X_{1,a}, \ldots, X_{n,a}) \quad \text{and} \quad X_b = (X_{1,b}, \ldots, X_{n,b}),
\]

and let \( g(x_a) = (g_1(x_{1,a}), \ldots, g_n(x_{n,a})), x_a = (x_{1,a}, \ldots, x_{n,a}) \). Then, we obtain the following identification result.

**Proposition 2.3.** Suppose that \( X_a \) and \( \epsilon \) are conditionally independent given \( X_b \). Then,

\[
\text{ADP}(h) = \int \mathbb{E}[h^*(Y, X) \mid (X_a, X_b) = (g(x_a), x_b)] \mathbb{1}\{g(x_a) \in \mathcal{S}_{X_a}\} d\mu_{X_a X_b}(x_a, x_b),
\]

where \( \mu_{X_a X_b} \) is the distribution of \( (X_a, X_b) \), and \( \mathcal{S}_{X_a} \) denotes the support of \( X_a \).

---

\(^{13}\) Game-theoretic models often involve a simultaneous system of equations. Note that we exclude the setting where the policy variable is part of the endogenous outcomes in such equations. For example, in a two-player game, with outcomes, \( Y = (Y_1, Y_2) \), we focus on a policy that changes the payoff state \( X \) which is not one of the two outcomes. When one of the endogenous outcomes is a policy variable, the structural equations need to be transformed into a triangular system of equations to apply a control function approach. Blundell and Matzkin (2014) present precise conditions for such a transform to exist. These conditions may be implausible in some empirical applications.
In general, when $X_a$ and $\epsilon$ are dependent due to some unobserved game-specific characteristics, we may consider $X_b$ as an observed vector including game characteristics such that conditioning on $X_b$ removes the stochastic dependence between $X_a$ and $\epsilon$.

In many examples, the payoff state $W_i$ of each player $i$ enters the payoff as a partial index form: $W_i = (X_i, V_i, \epsilon_i)$, where $V_i = X'_{i,b} \theta_i$, with coefficient $\theta_i$. If $X_a$ and $\epsilon$ are conditionally independent given $V = (V_1, ..., V_n)$, which is an assumption weaker than the previous assumption that $X$ and $\epsilon$ are independent, we can rewrite (23) as:

$$\text{ADP}(h) = \int E[h^*(Y, X) | (X_a, V) = (g(x_a), v)] 1 \{g(x_a) \in S_{X_a}\} d\mu_{X_a, V}(x_a, v),$$

where $\mu_{X_a, V}$ is the distribution of $(X_a, V)$, and $V = (V_1, ..., V_n)$. We can identify and estimate $\theta_1, ..., \theta_n$ (up to a scale) following the literature of multi-index models (see Ichimura and Lee (1991), Lee (1995), Donkers and Schafghans (2008), Xia (2008) and Ahn, Ichimura, Powell, and Ruud (2018) and references therein.) The main difference here is that we do not introduce the multi-index structure as a semiparametric restriction on a nonparametric function; it naturally follows from the index structure of the payoff function in the game. The multi-index models are useful for dimension reduction when the game involves only a few players, and the dimension of $X_i$ is large. We provide some implementation details in the Online Appendix.

2.5. Extensions

2.5.1. Extension to Other Solution Concepts. As mentioned in the main text, our results extend to other solution concepts beyond BCE. In particular, we can accommodate any refinement to BCE, which includes Bayes-Nash Equilibria, among others.

To see this, we let $\Sigma' \subset \Sigma$ be a given subcollection of decision rules $\sigma$ and consider the restricted BCE:

$$\Sigma'_{\text{BCE}}(G) = \Sigma_{\text{BCE}}(G) \cap \Sigma'.$$

We call this set the set of Bayes Correlated Equilibria (BCE) restricted to $\Sigma'$.

For example, suppose that $\Sigma'$ is the collection of decision rules $\sigma$ of the following form: for any $A = A_1 \times ... \times A_n$,

$$\sigma(A \mid w, t) = \prod_{i=1}^n \beta_i(A_i \mid w, t_i),$$

where $\beta_i(\cdot \mid w, t_i)$ is a conditional distribution on $Y_i$ given $(W, T_i) = (w, t_i)$. Then a BCE restricted to $\Sigma'$ is the set of Bayes Nash Equilibria (BNE). We can add further restrictions such as symmetry or differentiability depending on the application.
Let us turn to the interval-identification of equilibrium-based predictions in terms of a BCE restricted to $\Sigma'$. First, similarly as before, define the equilibrium selection rules $e'(\cdot; G)$ and $e'(\cdot; G_f)$ as a distribution on $\Sigma'_{\text{BCE}}(G)$ and $\Sigma'_{\text{BCE}}(G_f)$ respectively. Similarly, we define $\rho'_G$ and $\rho'_G$, using $\Sigma'_{\text{BCE}}(G)$, $\Sigma'_{\text{BCE}}(G_f)$, $e'(\cdot; G)$, and $e'(\cdot; G_f)$. Let $ADP'(h)$ and $AEP'(h)$ be the same as $ADP(h)$ and $AEP(h)$ except that we substitute $\rho'_G$ and $\rho'_G_f$ for $\rho_G$ and $\rho_{G_f}$ in the definition in (16). Similarly as in the case of BCE, we assume that the invariance condition on $e'(\cdot; G)$ and $e'(\cdot; G_f)$ holds in terms of the BCEs restricted to $\Sigma'$. Then, we obtain an analogue of Theorem 2.1 as follows.

**Corollary 2.1.** Suppose that Assumptions 2.1 and 2.2 (in terms of the BCEs restricted to $\Sigma'$) hold for the pre-policy game $G = (B, I)$ and the post-policy game $G_f = (B_f, I)$. Suppose further that maps $h : \mathbb{Y} \times \mathbb{W} \to \mathbb{R}$, $\underline{h} : \mathbb{W} \to \mathbb{R}$, are given as in Theorem 2.1. Then,

\[ (27) \quad ADP'(h) + \Delta(h) \leq AEP'(h) \leq ADP'(h) + \Delta(h). \]

Hence, decomposition-based predictions can still be used for counterfactual analysis in a setting with various other solution concepts such as BNE or its further restricted versions. They coincide with the equilibrium-based predictions when $S_{f(w)} \subset S_W$.

### 2.5.2. When the Policy Also Changes the Information Structure.

When a government policy is announced in advance, additional signals are often created through various reports of analysis on the policy and may be used by agents. Thus, the policy may change the information structure of the players as well. In the Online Appendix, we extend our main results to such cases.

In particular, we use the connection between information structures and the set of equilibrium reduced-forms in BM and show that the bounds in Theorem 2.1 accommodate scenarios where the policy also introduces new signals, as long as the latter do not reveal other players’ pre-policy signals and the payoff state beyond what has been known to the player. This includes the cases where the policy may be used as a coordination device by players (e.g., sunspots, as in Shell (1989); Peck and Shell (1991)), or when this signal is about future policy implications (e.g., in one empirical application below, government reports discuss market structure following the policy repeal, but are unlikely to reveal pre-policy signals of individual agents.)

### 2.6. Examples

#### 2.6.1. Entry Games.

Consider an entry game which received a great deal of attention in the literature (e.g., Ciliberto and Tamer (2009), Jia (2008), Grieco (2014) and Magnolfi and Roncoroni (2022)). Suppose that there are $n$ firms, $i = 1, \ldots, n$, who choose a binary action $Y_i \in \{0, 1\}$,
\[
Y = (Y_1, ..., Y_n), \text{ where } Y_i = 1 \text{ represents entry in the market and } Y_i = 0 \text{ staying out of the market. We specify the payoff generically as } u_i(y, W), \quad W = (X, \varepsilon), \text{ where } X = (X_1, ..., X_n) \in \mathbb{R}^{nd_x} \text{ is observed and } \varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^{nd_x} \text{ unobserved by the researcher.}
\]

As for the information structure, we focus on two information structures that are often used in the literature of empirical research.

**Assumption 2.3.** Either of the following two conditions is satisfied:

(i) The game is of complete information, i.e., \( T_i = (X, \varepsilon) \), for \( i = 1, ..., n \).

(ii) The game has a public-private dichotomy of signals, i.e., \( T_i = (X, \varepsilon_i) \), for \( i = 1, ..., n \).

For example, Assumption 2.3(i) is satisfied in the complete information environments of Bresnahan and Reiss (1991), Jia (2008) and Ciliberto and Tamer (2009). The public-private dichotomy case is studied in depth by Grieco (2014).

As for the policy, we assume that \( X \) is subject to a change by a policy whereas \( \varepsilon \) is not.

**Assumption 2.4.** \( f(X, \varepsilon) = (f^*(X), \varepsilon) \), for some map \( f^* \).

A counterfactual policy \( f \) in the above assumption was considered by all the papers cited at the beginning of this section. For example, Ciliberto and Tamer (2009) and Grieco (2014) set the values of a dummy variable in \( X_i \) to 0. Meanwhile, Jia (2008) changes market size, a variable in \( X_i \).

Let us describe the invariance condition for the equilibrium selection rules in this setting. Let \( G \) denote the pre-policy game and \( G_f \) the post-policy game. We define \( \Sigma_{BCE,w}(G) \) and \( \Sigma_{BCE,w}(G_f) \) as in Assumption 2.2 with \( w = (x, \varepsilon) \). Then, the invariance condition we require takes the following form:

**Assumption 2.5.** For each \( w = (x, \varepsilon) \) in the intersection of the supports of \( (X, \varepsilon) \) and \( (f^*(X), \varepsilon) \) such that \( \Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f) \), we have \( e_w(\cdot; G) = e_w(\cdot; G_f) \).

Our target parameter is the conditional probability of \( Y = a \), \( a \in \{0, 1\}^n \), after the policy, given \( f^*(X) \in C \) for some set \( C \), which is defined as follows:

\[
p_f(Y = a | C) = \frac{1}{P \{ f^*(X) \in C \}} \int_{C \times \mathbb{R}^{nd_x}} \int_{\Sigma_{BCE(G_f)}} \sigma(a | w, s(w)) de(\sigma; G_f) d(\mu_W \circ f^{-1})(w),
\]

where \( w = (x, \varepsilon) \). For example, \( p_f(Y = (1, ..., 1) | C) \) denotes the conditional probability of all firms entering the market given \( f^*(X) \in C \), after the policy. The result below shows how this probability is bounded by decomposition-based predictions.
Corollary 2.2. Suppose that Assumptions 2.3-2.5 hold. Then for each \( a \in \{0, 1\}^n \) and \( C \subset \mathbb{R}^{nd_X} \),
\[
\mathbb{E} [ \pi(a \mid f^*(X)) \mid f^*(X) \in C] \leq p_f(Y = a \mid C) \\
\leq \mathbb{E} [\pi(a \mid f^*(X)) \mid f^*(X) \in C] + P\{f^*(X) \notin \mathbb{S}_X \mid f^*(X) \in C\},
\]
where \( \pi(a \mid x) = P\{Y = a \mid X = x\}1\{x \in \mathbb{S}_X\} \) and \( \mathbb{S}_X \) denotes the support of \( X \).

The results do not rely on any further specification of the payoff function, or a parametric assumption for the distribution of \( \varepsilon_i \). The conditional expectation \( \mathbb{E} [\pi(a \mid f^*(X)) \mid f^*(X) \in C] \) is identified using the data from the pre-policy game. We provide a step-by-step empirical implementation of this result in Section 3 and the Online Appendix.

For example, suppose that the researcher would like to obtain the predicted joint entry probability of the firms under a policy that changes \( X \) into \( f^*(X) \) such that \( f^*(X) \) lies in the support of \( X \). Then, Corollary 2.2 says that the counterfactual prediction is point-identified as
\[
\int P\{Y = (1, \ldots, 1) \mid X = f^*(x)\} dP_X(x),
\]
where \( P_X \) denotes the distribution of \( X \). It is quite simple to obtain this prediction: we use the nonparametric regression of \( 1\{Y = (1, \ldots, 1)\} \) on \( X \), and integrate the regression function after transforming the regressors by the map \( f^* \). More importantly, this prediction is obtained without specifying further details of the game such as a functional form restrictions or the parametric specification of the payoff function or a parametric distribution of unobserved heterogeneity.

2.6.2. Auctions. A common approach for counterfactual analysis in the empirical auction literature is to first nonparametrically identify and estimate the distribution of valuations from the distribution of bids, and use those estimates for counterfactual analysis (see Athey and Haile (2007) for a survey.) One may wonder if we can use the decomposition approach to generate counterfactual predictions without identifying the valuation distribution from data. While we can in a more general setting, for the sake of concreteness, we focus on the setting of the first-price sealed bid, independent private value auction of Guerre, Perrigne, and Vuong (2009), and consider a counterfactual policy that alters the reserve price (see Paarsch, 1997; Haile and Tamer, 2003 for two examples of such a policy). The reserve price is often a publicly observed state variable.

---

14Reserve prices are set by the seller before the auction takes place. They are the minimum value for which the seller is willing to sell the good: if no bid is higher than the reserve price during the auction, then the good remains unsold. As a result, in empirical work, they are usually considered as an auction characteristic (primitive). The reserve price is often observed by the bidders and by the researcher. If the focus is on auctions with unobserved reserve prices (as in Elyakime, Laffont, Loisel, and Vuong (1997)) our results do not apply. If the researchers are interested in predicting bids after setting the reserve price beyond its support in the data (which is suggested in Table 4 of Haile and Tamer (2003)), they may use the bounds approach outlined above.
In this model, there are potential bidders $i = 1, \ldots, n$, who observe both private valuation $V_i$ drawn from a distribution with common support, $S_V$, and commonly observe the vector of auction specific characteristics $(X, \eta)$, where $X = (X_1, R)$, with $R$ denoting the reserve price, which is observed by the researcher, while $\eta$ represents unobserved auction heterogeneity. Each potential bidder $i$ chooses to enter or not depending on the value of $(V_i, X, \eta)$. The entry rule is modeled as a reduced form $I : S_V \times S_X \times S_\eta \rightarrow \{0, 1\}$, where $S_X$ and $S_\eta$ denote the support of $X$ and $\eta$. Each participating bidder $i$ uses a bidding strategy (bid) $s_i : S_V \times S_X \times S_\eta \rightarrow \mathbb{R}_+$, and wins if they submit the highest bid which is higher than the reserve price. The policy of interest is a change in the reserve price: a change of $R$ and wins if they submit the highest bid which is higher than the reserve price. The policy of interest is a change in the reserve price: a change of $R$ into $f(R)$ for some map $f$. We denote $S_R$ and $S_{f(R)}$ the supports of $R$ and $f(R)$. For each auction, let $\tilde{N} = \{ i : I(V_i, X, \eta) = 1 \}$ be the set of participants, and $s^* = (s^*_i)_{i \in \tilde{N}}$ a symmetric pure strategy BNE in the post-entry auction game. The researcher observes $(X, B)$, where $B = (B_i)_{i \in \tilde{N}}$ and $B_i = s^*_i(V_i, X, \eta), i \in \tilde{N}$.

The following assumption summarizes the features of this game relevant to the decomposition approach.

**Assumption 2.6.** (i) $(X, \eta)$ is publicly known, but valuations, $V_i$, are private information.

(ii) The policy changes the reserve price $R$ into $f^*(R)$ for some map $f^*$.

(iii) The auction has a unique symmetric pure strategy BNE.

The uniqueness of the pure strategy symmetric BNE in this auction is well studied in the literature. (See Guerre, Perrigne, and Vuong (2009).) Since the equilibrium is unique, the invariance condition for equilibrium selection rules is trivially satisfied. As we saw before, the decomposition-based approach applies to a setting where the researcher focuses on a subset of BCE satisfying restrictions such as symmetry or differentiability.

We introduce an additional assumption that is used to identify the bounds in the decomposition approach.

**Assumption 2.7.** $(V, \eta)$ is conditionally independent of $R$ given $X_1$.

This assumption requires selection on observables, i.e., the source of dependence between $(V, \eta)$ and the reserve price $R$ is fully captured by observed auction characteristics $X_1$.

Our object of interest is the conditional distribution of the auction revenue under the counterfactual reserve price $f^*(R)$ given $X_1 \in C$ for some set $C$ in the support of $X_1$:

$$p_f(A \mid C) \equiv P \left\{ \max_{i \in \tilde{N}^f} B^f_i \in A \mid X_1 \in C \right\}, \quad A \subset \mathbb{R}_+,$$

where $B^f_i = s^*_i(V_i, X_1, f^*(R), \eta)$ and $\tilde{N}^f$ denotes the set of participants after the policy. We define

$$p(A \mid x_1, r) \equiv P \left\{ \max_{i \in \tilde{N}} B_i \in A \mid (X_1, R) = (x_1, r) \right\}.$$
The conditional probability $p(A \mid x_1, r)$ is identified for all $(x_1, r)$ in the support of $(X_1, R)$, and can be estimated from the pre-policy auction data without recovering the value distribution from the data. The following result is a corollary to Theorem 2.1 and Proposition 2.3.

**Corollary 2.3.** Suppose that Assumptions 2.6-2.7 hold, and let $C$ be a subset of the support of $X_1$. Then, for each $A \subset \mathbb{R}$,

$$E[p(A \mid X_1, f^*(R))1\{f^*(R) \in S_R\} \mid X_1 \in C] \leq p_f(A \mid C) \leq E[p(A \mid X_1, f^*(R))1\{f^*(R) \in S_R\} \mid X_1 \in C] + P\{f^*(R) \notin S_R \mid X_1 \in C\}.$$

Furthermore, if the support of the reserve price after the policy is within the support of the reserve price before the policy (i.e., $S_{f^*(R)} \subset S_R$), for each $A \subset \mathbb{R}$,

$$p_f(A \mid C) = E[p(A \mid X_1, f^*(R))1\{f^*(R) \in S_R\} \mid X_1 \in C].$$

The corollary says that when $S_{f^*(R)} \subset S_R$, the counterfactual quantity $p_f(A \mid C)$ can be directly recovered from data, without first recovering the valuation distribution. Hence, we can obtain point-identification of the counterfactual prediction without relying on the conditions invoked in the literature to ensure the nonparametric identification of the valuation functions. This also simplifies the estimation procedure, as there is no need to estimate the latter from data.

For example, **Haile and Tamer (2003)** are interested in the reserve price $r$ that maximizes expected auction revenue. We can apply the decomposition method in finding the optimal reserve price. First, from (28), the expected auction revenue when the reserve price $R$ is counterfactually fixed at $r \in S_R$ is given by

$$\int E\left[\max_{i \in N} B_i \mid X_1 = x_1, R = r\right] dP_{X_1}(x_1).$$

Hence, the optimal reserve price is identified as one that maximizes this expected revenue, as long as the optimal reserve price is within the support of the reserve price in the data. In this case, we do not need to recover the valuation function from data.

### 2.7. The Scope of Decomposition-Based Predictions

While our result is widely applicable to many settings of counterfactual predictions from game-theoretic models, there are important examples that its scope does not cover. First and foremost, our main result restricts the counterfactual policies to those that affect the observed payoff state. In doing so, we keep other aspects of the environment unchanged after the policy. For instance, the result does not generally apply when the counterfactual policy alters the functional form of the payoff, the action space, or the set of (possible) players (e.g., mergers). The latter cases include **Hortaçsu and McAdams (2010)**, who study the effects of different
auction formats for selling treasuries on bidder expected surplus, and Roberts and Sweeting (2013) who study the effect of changing the mechanism by which a good is sold (from an auction set-up to a sequential bidding design) on expected revenues and payoffs in an incomplete information game. A change in the mechanism alters (unobserved) payoffs and expected revenues.

In other cases, researchers are interested in counterfactuals involving a change in the agents’ choice sets. For example, Keane and Wolpin (2010) present and estimate a model where women choose labor supply, fertility and welfare participation, among other outcomes. One of their counterfactuals eliminates a welfare program (and hence, the agents’ possibility to choose to participate in it) to study the program’s effects on labor supply across racial groups. The results on decomposition methods in this paper do not apply in this context either, as the probability of playing such (deleted) actions in the data cannot be extrapolated to the counterfactual environment. See Kalouptsidi, Scott, and Souza-Rodrigues (2020) for further examples and some identification results when the policy changes agents’ choice sets.

Our paper’s framework takes a policy variable among the payoff states $W$, not among the endogenous outcomes $Y$ in the game. However, in some applications, we may be interested in the counterfactual analysis which involves a policy that changes an endogenous outcome variable, such as a policy that changes the price in a simultaneous system of equations for price and quantity. Our framework excludes such counterfactual analysis.

Another key conceptual feature of this restriction is that our policy variable only changes (exogenous) variables or their index (such as $X'/\beta$) with sampling variation, thereby excluding policies that affect structural parameters or those that cannot be mapped to observable random variables. Thus our framework excludes counterfactual analysis of a policy’s effect on the welfare or the profits of the agents in the game, where the identification of the welfare or the profits require identification of structural parameters in the first place.

3. Empirical Applications

3.1. Ciliberto and Tamer (2009) Revisited

We revisit the counterfactual analysis in Ciliberto and Tamer (2009) using our results from Section 2.6.1. They investigated the effect of the repeal of the Wright amendment on airline entry in markets out of Dallas Love Field Airport. The Wright amendment had been implemented in 1979 to stimulate the use of the newer (and not as central) Dallas Fort Worth (DFW) Airport. As of the early 2000’s, it restricted the flights out of the central Dallas Love Field to other cities in Texas or those from some neighboring states. A full repeal of the amendment was agreed by
the major airlines and DFW Airport in 2008\textsuperscript{15} and was to be fully implemented in 2014. This repeal could have led to significant changes in market entry and, hence, on consumer welfare.

Ciliberto and Tamer (2009) produced a counterfactual prediction of the outcomes after the repeal of the Wright amendment, after estimating the identified set from a complete information entry game permitting multiple equilibria. This is the game presented in Example 2.6.1.\textsuperscript{16} A market was defined by a route between two airports, irrespective of directions or layovers. Thus, this included connecting flights through a third airport. They modeled the Wright amendment as a dummy variable covariate, \( X_{i,m}^{\text{Wright}} \), which equaled 1 if market \( m \) was affected by the Wright amendment (affecting all the firms in the market) and 0 otherwise. For a counterfactual analysis, they considered the counterfactual experiment of repealing the Wright amendment, setting \( X_{i,m}^{\text{Wright}} \) to 0, and studied its effects on market entry. The support of \( X_{i,m}^{\text{Wright}} \) in the data is \( \{0, 1\} \), and hence contains its post-policy support that is \( \{0\} \). Hence, Corollary 2.2 says that the decomposition-based prediction coincides with the equilibrium-based prediction, where the latter prediction can be obtained from estimating all structural parameters under the equilibrium invariance condition. As discussed in Section 2.5.2, this result holds even if the policy introduces new signals to agents which may be used as coordination devices. (In this complete information setting, such new signals cannot reveal private information unbeknownst to the firms.) For example, the results still hold if the congressional hearings associated with the repeal of the Wright amendment led firms to coordinate towards equilibria more desirable to specific airlines (e.g., Southwest), or to equilibria more desirable to the regulator (e.g., with more entry). Further discussion is provided in the Supplemental Note.

3.1.1. Our Set-up. For this application, we follow their work and focus on the decisions of the four main airlines in their analysis (American Airlines, Delta Airlines, Southwest Airlines and United Airlines) and use their same dataset. We perform a dimension reduction to resolve near multicollinearity between covariates. This reduction is useful because our decomposition-based prediction must include all firm-level covariates in \( X_m \) (the covariates \( X_{j,m} \) for every \( j \neq i \) impact \( i \)'s entry decision in equilibrium through affecting \( j \)'s decision to enter).\textsuperscript{17} We drop three out of eight market level variables (market size, per capita income growth and Dallas market)

\textsuperscript{15}The agreement involved, most notably, decreasing the number of gates in Dallas Love Field to restrict its impact on Dallas Fort Worth.

\textsuperscript{16}We provide extensive details on the empirical application, including the description of the covariates, estimators and inference procedures in the Online Appendix. Following Ciliberto and Tamer (2009), we assume that unobservable payoff components \( \epsilon_i \) are i.i.d. and independent of all covariates, so we do not need to use a control function approach.

\textsuperscript{17}In this context there are 8 market-level covariates and 2 variables at the firm-market level. They are described in detail in the Online Appendix. This generates a total of 16 covariates to be included in the analysis. While the parametric estimator is robust to including all 16 covariates due to its additional structure, the performance of the nonparametric estimator is improved with a smaller subset. In general, although our identification results are nonparametric, nonparametric estimation can be challenging when the dimensionality of \( X \) is large. Discretization
that lack variation for nonparametric estimation. This is motivated both by data considerations, as well as by the theoretical construction of the dropped variables.\footnote{Both market size and per capita income growth appear well predicted by income per capita and market presence (variables that already capture economic performance at the market level and included in the analysis). Meanwhile the binary Dallas market variable is highly correlated with the Wright amendment variable - by construction, any market in Dallas that does not use Dallas Love Field Airport must be using Dallas Fort Worth Airport instead. However, Dallas Fort Worth is the hub for American Airlines - and American Airlines’ market presence is already included as a covariate. Details are provided in the Online Appendix.} We drop one additional firm-market level covariate (a proxy for the firm’s cost) using the causal structure of the game, because this variable is a function of other covariates in the analysis, such as route distance, by construction.

3.1.2. **Results.** We compare the results from the decomposition approach to the original results in \textit{Ciliberto and Tamer} (2009). The results of this exercise are shown in Columns 1 and 2 of Table 1 using two different estimators (a linear/parametric and a nonparametric estimator), while their original results are shown in Column 3.

In the first column, we assume that the expected entry of \(i\) in market \(m\) is given by the linear form \(\mathbb{E}[Y_{i,m} | X_m = x_m] = x_m^\prime \gamma_i\) and we estimate it using Ordinary Least Squares in a linear regression framework, where \(Y_{i,m}\) denotes the indicator of entry by firm \(i\) in market \(m\). We present this specification as a simple benchmark. We then present the counterfactual estimate which is the estimated change in entry in the post-policy game relative to the data for the markets previously affected by the policy. This estimated change can be written as

\[
\frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} f^*(X_m) \hat{\gamma}_i - \overline{Y}_i,
\]

where \(\hat{\gamma}_i\) is the estimator of \(\gamma_i\), \(\mathcal{M}\) represents the set of markets previously affected by the Wright amendment, \(f^*(X_m)\) represents the values of the covariates for market \(m\) after the policy, and \(\overline{Y}_i\) is the average outcome in the data for firm \(i\) in those markets in \(\mathcal{M}\). In the second column, we estimate the conditional expectation \(g_i(x) = \mathbb{E}[Y_{i,m} | X_m = x]\) nonparametrically. We use a leave one out kernel estimator with a bandwidth chosen by cross-validation. (See the Online Appendix for details.)

While Column 1’s results for Southwest Airlines and United Airlines are very similar to those in Column 2, we prefer the latter as our main specification. This is because a linear reduced form (Column 1) cannot be induced from equilibria in the entry game specification. Meanwhile, Column 2 is consistent with equilibria of the entry game specification and, by virtue of being nonparametric, illustrates further benefits of the decomposition approach (i.e., not requiring parametrizations of the utility function, unobserved heterogeneity, etc.). In the third
column, we restate the results of counterfactual predictions from the main specification in Ciliberto and Tamer (2009) (Table VII, Column 1). These are the maximum predicted increase in the share of Dallas Love Field markets that are served by each airline following the 2014 repeal of the Wright amendment according to their estimates. We find that our results across specifications are broadly consistent with theirs, as they are below their estimated maximum entry.

Now we take this exercise one step further. The Wright amendment was actually repealed in 2014. This means that we can observe how airlines entered the markets after the repeal of the Wright Amendment and after any new information arose during its implementation. We compile 2015 data from the DB1B Market and Ticket Origin and Destination Dataset of the U.S. Department of Transportation (the same source as the original dataset), and treat it the same way as the original authors - see the Online Appendix for details. We focus on the same 81 markets from the original paper. The actual change in entry in 2015 in the data relative to the original data is shown in Column 4 of Table 1. We then compare the counterfactual estimates from Ciliberto and Tamer (2009) and our decomposition approach to the actual changes.

The results show that the decomposition method (Columns 1-2) using pre-repeal data performs well relative to the empirical outcomes in Column 4. Both the parametric and non-parametric estimates of the decomposition approach capture the large increase in entry by Southwest Airlines, and the negligible post-repeal entry by American Airlines and United Airlines. This lack of entry by American and United post-repeal is broadly consistent with the multiple equilibria in an entry model: Southwest and Delta entered frequently after the repeal, but American and United stayed out of those markets. The empirical values are also within the maximum bounds reported in Ciliberto and Tamer (2009). However, as the authors only reported the maximum predicted entry, their results appear further apart from the realized values for American and United. While the lack of entry results for these firms is consistent with Ciliberto and Tamer (2009)’s results of maximum predicted entry, this would imply that their counterfactual analysis predicted a range of 0 to 50% of markets entered by those airlines, a large range for policy analysis.

While the decomposition-based results perform well for American, United and Southwest, the method performs worse in predicting entry by Delta Airlines. This could simply be a feature of out-of-sample prediction, possibly due to changes to Delta between 2008-2014 (including the acquisition of Northwest Airlines, which was completed in 2010), and/or due to the definition of markets in this dataset.19 Nevertheless, we consider that the decomposition-based prediction performed well overall in this out-of-sample exercise, particularly as it used data

---

19Delta Airlines only operates from Dallas Love Field to Atlanta, but there are multiple connecting flights from Atlanta. Routes that include layovers are considered as separate markets by Ciliberto and Tamer (2009).
TABLE 1. Ciliberto and Tamer (2009) Revisited: Model Predicted and Empirical Counterfactuals of the Repeal of the Wright Amendment

| Decomposition Method | Decomposition Method | Ciliberto & Tamer (2009) | Empirical |
|----------------------|----------------------|--------------------------|-----------|
| Linear Model         | Nonparametric Model  | Maximum Predicted Entry  |           |
| American Airlines    | -0.030               | 0.128                    | 0.463     | -0.04 |
|                      | (0.037)              | (0.036)                  |           |       |
| Delta Airlines       | -0.023               | 0.174                    | 0.499     | 0.46  |
|                      | (0.043)              | (0.039)                  |           |       |
| Southwest Airlines   | 0.508                | 0.451                    | 0.474     | 0.471 |
|                      | (0.037)              | (0.056)                  |           |       |
| United Airlines      | -0.009               | 0.043                    | 0.147     | 0     |
|                      | (0.031)              | (0.016)                  |           |       |

Notes: We report the estimated counterfactual changes to the entry of major airlines into Dallas Love Field markets following the repeal of the Wright Amendment. In Columns 1-2, we use our decomposition approach to provide point estimates of this counterfactual effect, using the same pre-2014 dataset of Ciliberto and Tamer (2009). Column 1 uses a linear model, while Column 2 reports a nonparametric estimate. Standard errors for these columns are computed by the bootstrap, following the approach in the Online Appendix with $B = 999$ replications. In the third column, we restate the results in Table VII, Column 1 of Ciliberto and Tamer (2009), who presented the maximum change in entry of those airlines. Finally, the Wright Amendment was fully repealed in 2014, allowing all airlines to enter those markets. The final column shows the realized values of the change in entry for those airlines in affected markets in 2015, after the repeal.

from years before the policy was implemented and matched well with multiple observed outcomes.20

3.2. The Threat of Southwest Airlines on Competitors’ Entry Behavior

We can pursue further counterfactual exercises beyond those in Ciliberto and Tamer (2009). One salient example is to study competition effects in the airline industry. This includes the role of Southwest Airlines’ rise on its competitors’ behavior, which has received attention due to the latter’s status as a new, lower-cost entry with new organizational strategies.

20In the Online Appendix, we provide results for the other component in the typical aggregate decomposition (e.g., Oaxaca-Blinder): the average difference in entry across markets that are not subject/subject to the Wright amendment due to observable characteristics alone: after all, airlines would be less likely to enter smaller markets, even if the Wright amendment was not present/repealed. We show that a naive comparison of entry across different markets (ignoring the difference in characteristics) would overstate the effects of the policy.
(see Knorr and Arndt (2002), for example). Goolsbee and Syverson (2008) studied one angle to this question, focusing on whether Southwest Airlines' threat of entry affected established airlines’ (e.g., American, Delta, United) pricing behavior. In this section, we use the decomposition-based prediction within the same entry game above to extend their analysis beyond pricing.

We use the same dataset and the same model specification from the last section. This includes the entry game environment with complete information. It also includes the definition of market entry/operation (i.e., a route between two airports, irrespective of directions or layovers). Our only change is regarding the policy of interest: we replace ‘Wright Amendment’ by a binary variable equaling 1 if Southwest operates in both endpoints of a market \( m \), and 0 otherwise. This means that the variable is 1 if Southwest has entered markets that include airports in both endpoints of \( m \). This is the same variable as in Goolsbee and Syverson (2008), which they interpret as a “discontinuous” threat of Southwest entry. However, despite its name as a “threat of entry”, it is actually interpretable as a cost shock.\(^21\) By operating in markets that share both endpoints as market \( m \), Southwest has a lower cost of operating market \( m \) (since it has established facilities, labor, etc. on both endpoints of a route). However, a smaller cost for Southwest to operate in market \( m \) does not restrict Southwest - or any other airline - from entering market \( m \) or any other market.

Our outcomes of interest are whether the threat of Southwest entry changes established airlines’ actual entry behavior (as opposed to pricing, as in Goolsbee and Syverson (2008)) and whether such effects vary across smaller or larger markets (as found in Ellison and Ellison (2011) and Tenn and Wendling (2014) in the pharmaceutical industry).\(^22\)

3.2.1. Results.

Our counterfactual policy sets the Southwest Threat of Entry variable to 0 for all markets. Therefore, we identify whether American, Delta and United are more/less likely to enter markets when Southwest is no longer a threat to entry (i.e., when it no longer has a lower cost of operating in such markets). The arguments for the validity of our decomposition-based prediction are analogous to those in the previous exercise: (i) the policy is observable to players, since Southwest’s routes are observable and (ii) the policy is within the support of the data, as there are many markets where Southwest does not operate in both endpoints. Hence, under the invariance condition on the equilibrium selection rules, Corollary 2.2 can be

\(^{21}\)Note that entry decisions in other markets are independent from those in market \( m \) due to the maintained assumption of independence of \( \epsilon_{i,m} \). Hence, it can be used as a policy variable in this exercise. However, we cannot condition on actual entry in a certain market, whether by Southwest or another airline, since that is a simultaneous decision in this environment.

\(^{22}\)Since our framework is static, the interpretation of our results also differs from those cited above. In a static environment, there is no dynamic choice of capacity and there are no incumbents, so the latter cannot “accommodate” entry over time. Nevertheless, we think the present exercise is still informative about whether a state in which Southwest is likely to enter induces differential (strategic) behavior by established airlines.
TABLE 2. *Goolsbee and Syverson (2008)* Revisited: Effects of Removing the Threat of Southwest Airlines Entry on Other Airlines’ Probability of Entry

| Outcome: Change in the Entry Prob. after Removing Southwest Threat of Entry |
|---------------------------------------------------------------|
| **All Markets** | **Small Markets** | **Large Markets** |
| Linear | Nonparametric | Linear | Nonparametric | Linear | Nonparametric |
| American Airlines | -0.080 | -0.074 | -0.153 | -0.189 | -0.006 | 0.052 |
| (0.023) | (0.021) | (0.034) | (0.031) | (0.033) | (0.027) |
| Delta Airlines | -0.079 | -0.078 | -0.109 | -0.134 | -0.037 | -0.015 |
| (0.024) | (0.021) | (0.037) | (0.028) | (0.034) | (0.028) |
| United Airlines | -0.056 | -0.047 | -0.073 | -0.079 | -0.056 | 0.006 |
| (0.023) | (0.020) | (0.029) | (0.025) | (0.032) | (0.026) |

Notes: We report the estimated counterfactual changes to the entry of major airlines (American, Delta, United) after the removal of the threat of Southwest Airlines entry. The effect is estimated on the markets originally subject to that threat, as defined in *Goolsbee and Syverson (2008)*. The first two columns show the effects for all such markets, for both linear and nonparametric estimates. Columns 3-4 show the results for markets affected by such a threat, but below the median market size, while the last shows the effects for markets larger than the median. Standard errors for these columns are computed by the bootstrap, following the approach in the Online Appendix with \( B = 999 \) replications. A negative coefficient represents a decrease in entry if there were no threat of Southwest entry, relative to there being such a threat.

As we can see from Table 2, a threat of Southwest entry increases the average entry probability of American Airlines (7.4-8%), Delta Airlines (7.8-7.9%) and United Airlines (4.7-5.6%) in such markets (i.e., removing a threat of Southwest entry, thereby increasing Southwest’s cost, decreases competitors’ average entry probabilities). The results suggest that in markets where a Southwest entry is likely, the established firms (with larger market presence) will be more likely to enter. While our results cannot be strictly interpreted as endogenous deterrence or accommodation, they are consistent with the multiple equilibria in the game, together with an equilibrium selection mechanism whereby larger and established firms are more likely to enter when many firms are willing to do so. This is an additional outcome affected by competition, beyond pricing (*Goolsbee and Syverson (2008)* and *Tenn and Wendling (2014)*).

To further understand our results, we follow *Ellison and Ellison (2011)* and *Tenn and Wendling (2014)* and check whether such effects depend on market size. To do so, we re-estimate the
model for markets below and above the median market size. Consistent with those papers, we find nonmonotonic effects of the Southwest threat of entry on its competitors’ decisions. In particular, we find that the increased entry due to Southwest’s threat is driven by small markets. In small markets (i.e., below the median market size), where profits are more limited, the threat of Southwest entry induces other firms to do so. This is consistent with such airlines coordinating on entry as an equilibrium “deterrence” to Southwest (beyond pricing). In contrast, the threat of Southwest does not induce entry in larger markets. This is consistent with (a static interpretation of) “accommodation” in large markets: when profit is large enough, all firms enter for that state even if others are likely to enter, as suggested in Tenn and Wendling (2014).

3.2.2. Heterogeneous Effects depending on Number of Airlines Threatening Entry. We can extend the previous exercise beyond Southwest Airlines. For instance, researchers may be interested whether the competition effects differ across the number of potential entrants (i.e., number of airlines with lower costs). This can also be answered within our framework.

Indeed, we can redefine our policy variable as the number of airline $i$’s competitors that threaten entry in market $m$. We keep the same definition of threat of entry as in the last section: i.e., a competitor operates flights out of each endpoint of a route – thereby lowering its costs, but not the route itself. Since our emphasis is on the same four airlines (American, Delta, Southwest, United), each firm $i$ may face $\{0, 1, 2, 3\}$ competitors threatening entry in each market. We conduct three separate counterfactuals exercises to study how a decrease in the number of airlines threatening entry affect $i$’s choice to enter. Each exercise decreases the number of potential entrants by one (i.e., making markets with three entrants have only two entrants, etc.). The results are shown in Table 5 in the Online Appendix.

As we can see, airlines would generally increase entry if they had no potential competitors rather than one. After all, the airline is more likely to benefit from the market’s profits and less likely to compete in such a market. However, there is a net decrease in entry for American, Delta and United in markets facing higher threat of entry. Indeed, decreasing the number of $i$’s competitors from three to two decreases average entry in such markets - consistent with the previous section’s results.

---

23Table 4 in the Online Appendix shows the results for the second term in the average aggregate decomposition (i.e., the role of market characteristics, keeping the same threat of Southwest Airlines entry).

24We note that the different exercises are not generally comparable, because the effects are calculated over different markets (i.e., those markets with two competitors threatening entry for American are likely to be very different than those with only one threat).
4. Conclusion

Decomposition methods are appealing in counterfactual analysis for their computational tractability and simplicity. However, in strategic settings, predictions generated by those methods may fail to be incentive compatible after the policy or to account for additional coordination possibilities induced by the policy. In this paper, we have presented a set of core conditions that validate the use of the methods in strategic settings. Most importantly, we have provided a precise formulation of the invariance condition on the equilibrium selection rules that is required for the approach. Essentially, under the invariance condition, the agents can be viewed as playing “the same game” after the policy. As demonstrated in this paper, this condition already encompasses many existing assumptions on equilibrium selection in empirical research. Our result opens up a new approach of counterfactual analysis in a strategic setting, where we do not need to recover the structural parameters and the set of equilibria for the analysis. The result’s primary contribution is to clarify conditions for such an approach to work.

There are several extensions from this paper’s proposal that look promising to us. A most prominent extension is to explore a set of conditions for the decomposition-based approach in a dynamic game setting. A policy in these games generally induces a change of the agents’ posteriors through a change of a future path of the payoff states, and it seems nontrivial to maintain the invariance of the posterior after the policy. It appears interesting in this regard to note the approach of Kocherlakota (2019) who introduced independent shocks to the policy so that the posterior of the private sector for future policies remains invariant. Future work can expand this insight and explore the validity of decomposition-based predictions in a dynamic setting.

Our proposal can generate a wide range of intermediate approaches, where the target of the prediction is of the form $h(Y, W; \theta)$, and $\theta$ is part of the structural parameters in the game. Then, the message of our paper is that under the conditions stated in this paper, we can perform a counterfactual analysis using the decomposition method, after identifying $\theta$. The main point here is that we do not need to recover the full set of structural parameters of the game or the set of equilibria from data. For example, such an intermediate approach can be used to extend this paper’s framework to counterfactual analysis where the target of the prediction is welfare or profits.

5. Acknowledgements

We thank Aimee Chin, Sukjin Han, Chinhui Juhn, Arvind Magesan, Daniela Scur, Eduardo Souza-Rodrigues, Ko Sugiura, Andrea Szabo, Xun Tang, and participants at many seminars and conferences for their helpful comments. All errors are ours. Song acknowledges that
this research was supported by Social Sciences and Humanities Research Council of Canada. Corresponding address: Kyungchul Song, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada. Email address: kysong@mail.ubc.ca.

References

AGUIRREGABIRIA, V. (2012): “A Method for Implementing Counterfactual Experiments in Models with Multiple Equilibria,” *Economics Letters*, 114(2), 190–194.

AGUIRREGABIRIA, V., AND C.-Y. HO (2012): “A Dynamic Oligopoly Game of the US Airline Industry: Estimation and Policy Experiments,” *Journal of Econometrics*, 168, 156–173.

AGUIRREGABIRIA, V., AND P. MIRA (2010): “Dynamic Discrete Choice Structural Models: A Survey,” *Journal of Econometrics*, 156(1), 38–67.

—— (2013): “Identification of Games of Incomplete Information with Multiple Equilibria and Common Unobserved Heterogeneity,” *University of Toronto Department of Economics Working Paper*, 474.

AGUIRREGABIRIA, V., AND A. NEVO (2013): “Recent Developments in Empirical IO: Dynamic Demand and Dynamic Games,” *Advances in Economics and Econometrics*, 3, 53–122.

AGUIRREGABIRIA, V., AND J. SUZUKI (2014): “Identification and Counterfactuals in Dynamic Models of Market Entry and Exit,” *Quantitative Marketing and Economics*, 12(3), 267–304.

AHN, H., H. CHIMURA, J. L. POWELL, AND P. A. RUUD (2018): “Simple Estimators for Invertible Index Models,” *Journal of Business and Economic Statistics*, 36, 1–10.

ARCIDIACONO, P., AND R. A. MILLER (2020): “Identifying Dynamic Discrete Choice Models of Short Panels,” *Journal of Econometrics*, 215(2), 473–485.

ATHY, S., AND P. A. HAILE (2007): “Nonparametric Approaches to Auctions,” *Handbook of econometrics*, 6, 3847–3965.

BAJARI, P., H. HONG, AND S. P. RYAN (2010): “Identification and Estimation of a Discrete Game of Complete Information,” *Econometrica*, 78, 1529–1568.

BERGEMANN, D., B. A. BROOKS, AND S. MORRIS (2022): “Counterfactuals with Latent Information,” *American Economic Review*, 112, 343–368.

BERGEMANN, D., AND S. MORRIS (2016): “Bayes Correlated Equilibrium and the Comparison of Information Structures in Games,” *Theoretical Economics*, 11, 487–522.

BERRY, S. T., AND P. A. HAILE (2014): “Identification in Differentiated Products Markets Using Market Level Data,” *Econometrica*, 82(5), 1749–1797.

BLINDER, A. S. (1973): “Wage Discrimination: Reduced Form and Structural Estimates,” *Journal of Human Resources*, pp. 436–455.

BLUNDELL, R., AND R. L. MATZKIN (2014): “Control Functions in Nonseparable Simultaneous Equations Models,” *Quantitative Economics*, 5(2), 271–295.
BLUNDELL, R., AND J. L. POWELL (2003): “Endogeneity in Nonparametric and Semiparametric Regression Models,” in Advances in Economics and Econometrics, ed. by L. Dewatripont, L. Hansen, and S. Turnovskv, vol. 2, pp. 312–357. Cambridge University Press, Cambridge.
BRESNAHAN, T. F., AND P. C. REISS (1991): “Empirical Models of Discrete Games,” Journal of Econometrics, 48, 57–81.
CHERNOZHUKOV, V., I. FERNÁNDEZ-VAL, AND B. MELLY (2013): “Inference on Counterfactual Distributions,” Econometrica, 81(6), 2205–2268.
CHETTY, R. (2009): “Sufficient Statistics for Welfare Analysis: A Bridge Between Structural and Reduced-Form Methods,” Annual Review of Economics, 1, 451–487.
CILIBERTO, F., AND E. TAMER (2009): “Market Structure and Multiple Equilibria in Airline Markets,” Econometrica, 77, 1791–1828.
DiNARDO, J., N. M. FORTIN, AND T. LEMIEUX (1996): “Labor Market Institutions and the Distribution of Wages, 1973-1992: A Semiparametric Approach,” Econometrica, 64, 1001–1044.
DONKERS, B., AND M. SCHAFGANS (2008): “Specification and Estimation of Semiparametric Multiple-Index Models,” Econometric Theory, 24, 1584–1606.
ELLISON, G., AND S. F. ELLISON (2011): “Strategic Entry Deterrence and the Behavior of Pharmaceutical Incumbents Prior to Patent Expiration,” American Economic Journal: Microeconomics, 3(1), 1–36.
ELYAKIME, B., J.-J. LAFFONT, P. LOISEL, AND Q. VUONG (1997): “Auctioning and Bargaining: An Econometric Study of Timber Auctions with Secret Reservation Prices,” Journal of Business & Economic Statistics, 15(2), 209–220.
FORTIN, N., T. LEMIEUX, AND S. FIRPO (2011): “ Decomposition Methods in Economics,” in Handbook of Labor Economics, vol. 4, pp. 1–102. Elsevier.
GOOLSBEE, A., AND C. SYVerson (2008): “How do Incumbents Respond to the Threat of Entry? Evidence from the Major Airlines,” The Quarterly Journal of Economics, 123(4), 1611–1633.
GRIECO, P. L. E. (2014): “Discrete Games with Flexible Information Structures: An Application to Local Grocery Markets,” RAND Journal of Economics, 45, 303–340.
GU, J., T. RUSSELL, AND T. STRINGHAM (2021): “Counterfactual Identification and Latent Space Enumeration in Discrete Outcome Models,” Working Paper.
GUARDANI, C., AND S. SInHA (2020): “Identification and Inference in Discrete Choice Models with Imperfect Information,” Working Paper.
GUERRE, E., I. PERRIGNE, AND Q. VUONG (2009): “Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions,” Econometrica, 77(4), 1193–1227.
HAILE, P. A., AND E. TAMER (2003): “Inference with an Incomplete Model of English Auctions,” Journal of Political Economy, 111, 1–51.
HECKMAN, J. J. (2010): “Building Bridges Between Structural and Program Evaluation Approaches to Evaluating Policy,” *Journal of Economic Literature*, 48, 356–398.

HECKMAN, J. J., AND E. VYTLACIL (2001): “Policy-Relevant Treatment Effects,” *American Economic Review*, 91(2), 107–111.

HORTAÇSU, A., AND D. MCADEMS (2010): “Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market,” *Journal of Political Economy*, 118(5), 833–865.

HSU, Y.-C., T.-C. LAI, AND R. P LIELI (2022): “Counterfactual Treatment Effects: Estimation and Inference,” *Journal of Business and Economic Statistics*, 40(1), 240–255.

ICHIMURA, H., AND L. F. LEE (1991): “Semiparametric Least Squares Estimation of Multi-Index Models: Single Equation Estimation,” in *Nonparametric and Semiparametric Estimation Methods in Econometrics and Statistics*, ed. by W. A. Barnett, J. Powell, and G. E. Tauchen, pp. 3–49. Cambridge University Press.

ICHIMURA, H., AND C. TABER (2000): “Direct Estimation of Policy Impacts,” *NBER Working Paper*, (t0254).

IMBENS, G. W., AND W. K. NEWEY (2009): “Identification and Estimation of Triangular Simultaneous Equations Models Without Additivity,” *Econometrica*, 77, 1481–1512.

JIA, P (2008): “What Happens When Wal-Mart Comes to Town: an Empirical Analysis of the Discount Retailing Industry,” *Econometrica*, 76, 1263–1316.

JUHN, C., K. M. MURPHY, AND B. PIERCE (1993): “Wage Inequality and the Rise in Returns to Skill,” *Journal of Political Economy*, 101(3), 410–442.

JUN, S. J., AND J. PINKSE (2020): “Counterfactual Prediction in Complete Information Games: Point Prediction Under Partial Identification,” *Journal of Econometrics*, 216(2), 394–429.

KALOUPTSIDI, M., Y. KITAMURA, L. LIMA, AND E. A. SOUZA-RODRIGUES (2020): “Partial Identification and Inference for Dynamic Models and Counterfactuals,” Discussion paper, National Bureau of Economic Research.

KALOUPTSIDI, M., P T. SCOTT, AND E. SOUZA-RODRIGUES (2017): “On the Non-Identification of Counterfactuals in Dynamic Discrete Games,” *International Journal of Industrial Organization*, 50, 362–371.

——— (2020): “Identification of Counterfactuals in Dynamic Discrete Choice Models,” *Quantitative Economics*.

KEANE, M., AND K. WOLPIN (2010): “The Role of Labor and Marriage Markets, Preference Heterogeneity, and the Welfare System in the Life Cycle Decisions of Black, Hispanic, and White Women,” *International Economic Review*, 51, 851–891.

KLEVEN, H. (2020): “Sufficient Statistics Revisited,” *National Bureau of Economic Research Working Paper Series*, (w27242).
KLINE, P. (2011): “Oaxaca-Blinder as a Reweighting Estimator,” American Economic Review: Papers and Proceedings, 101(4), 532–537.

KNORR, A., AND A. ARNDT (2002): Successful Entry Strategies on the Deregulated US Domestic Market—the Case of Southwest Airlines. Citeseer.

KOCHERLAKOTA, N. R. (2019): “Practical Policy Evaluation,” Journal of Monetary Economics, Forthcoming.

LEE, L. F. (1995): “Semiparametric Maximum Likelihood Estimation of Polychotomous and Sequential Choice Models,” Journal of Econometrics, 65, 381–428.

MAGNOLFI, L., AND C. RONCORONI (2022): “Estimation of Discrete Games with Weak Assumptions on Information,” forthcoming in Review of Economic Studies.

MARSCHAK, J. (1953): “Economic Measurements for Policy and Prediction,” in Studies of Econometric Method, ed. by T. K. WC Hood, vol. Cowles Comm. Res. Econ., Monogr. No. 14, pp. 1–26. New York: Wiley.

MILGROM, P. R., AND R. J. WEBER (1982): “A Theory of Auctions and Competitive Bidding,” Econometrica: Journal of the Econometric Society, pp. 1089–1122.

MORETS, A., AND X. TANG (2014): “Semiparametric Inference in Dynamic Binary Choice Models,” Review of Economic Studies, 81(3), 1229–1262.

OAXACA, R. (1973): “Male-Female Wage Differentials in Urban Labor Markets,” International Economic Review, pp. 693–709.

PAARSCH, H. J. (1997): “Deriving an Estimate of the Optimal Reserve Price: an Application to British Columbian Timber Sales,” Journal of Econometrics, 78(1), 333–357.

DE PAULA, Â. (2013): “Econometric Analysis of Games with Multiple Equilibria,” Annual Review of Economics, 5, 107–131.

PECK, J., AND K. SHELL (1991): “Market Uncertainty: Correlated and Sunspot Equilibria in Imperfectly Competitive Economies,” Review of Economic Studies, 58(5), 1011–1029.

ROBERTS, J. W., AND A. SWEETING (2013): “When Should Sellers Use Auctions?,” American Economic Review, 103, 1830–1861.

ROTHE, C. (2010): “Nonparametric Estimation of Distributional Policy Effects,” Journal of Econometrics, 155, 56–70.

SHELL, K. (1989): “Sunspot Equilibrium,” in General Equilibrium, pp. 274–280. Springer.

SYRGKANIS, V., E. TAMER, AND J. ZIANI (2018): “Inference on Auctions with Weak Assumptions on Information,” ArXiv Preprint, arXiv:1710.03830.

TENN, S., AND B. W. WENDLING (2014): “Entry Threats and Pricing in the Generic Drug Industry,” Review of Economics and Statistics, 96(2), 214–228.

WOLPIN, K. I. (2013): The Limits of Inference Without Theory. MIT Press.
XIA, S. (2008): “A Multiple-Index Model and Dimension Reduction,” *Journal of the American Statistical Association*, 103(484), 1631–1640.
Online Appendix to “A Decomposition Approach to Counterfactual Analysis in Game-Theoretic Models”

Nathan Canen and Kyungchul Song
University of Warwick and NBER, and University of British Columbia

This Online Appendix consists of five appendices. Appendix A formally introduces Bayesian games and counterfactual predictions. This appendix gives a general result on decomposition-based methods on counterfactual predictions in the Bayesian games. In particular, the appendix considers a setting where a policy changes the information structure, and provides a sufficient condition for the information structure for the decomposition-based predictions to work. For this, we use the ordering between information structures developed in Bergemann and Morris (2016) (BM, hereafter). Appendix B focuses on a setting under Assumption 2.1 in the main text and presents a result on the decomposition-based predictions. Appendix C presents the proofs of the results in the main text. Appendix D provides details on the implementation of the decomposition method. This includes a step-by-step overview of the nonparametric estimation method used in Section 3 in the main text, as well as details about the empirical application (including data collection, empirical specifications and the implementation of the parametric and nonparametric estimators). It also provides additional results for the empirical applications. Finally, Appendix E provides the example referenced in Section 2, where decomposition-based predictions do not coincide with equilibrium-based predictions, due to the failure of incentive compatibility in the counterfactual game after the policy.
## Contents

Appendix A. Counterfactual Predictions in Bayesian Games ........................................ 45  
  A.1. Bayesian Games and Counterfactual Policies ................................................. 45  
  A.2. Randomized Reduced Forms and Counterfactual Predictions ....................... 47  
  A.3. A General Result for the Decomposition Method .......................................... 48  
  A.4. Low Level Conditions for Assumption A.1 .................................................. 50  

Appendix B. Counterfactual Predictions from Bayesian Games under Assumption 2.1 ........ 54  
  B.1. Preliminary Results .......................................................................................... 54  
  B.2. The Main Result ............................................................................................. 58  

Appendix C. Proofs of the Results in the Main Text .................................................... 60  

Appendix D. Implementation of Decomposition-Based Predictions ............................. 63  
  D.1. Implementation Using a Nonparametric Decomposition Method .................. 64  
  D.2. Implementation Using a Multi-Index Structure ............................................. 66  
  D.3. Details on the Empirical Application ............................................................. 67  
  D.4. Additional Empirical Results ......................................................................... 69  

Appendix E. An Example of a Game that Fails the Decomposition-Based Approach. .......... 73  
  E.1. Characterization of the Threshold $\overline{w}$ ......................................................... 73  
  E.2. Failure of Incentive Compatibility in the Counterfactual Game ..................... 74  

References .................................................................................................................. 75
Appendix A. Counterfactual Predictions in Bayesian Games

A.1. Bayesian Games and Counterfactual Policies

A.1.1. Bayesian Games. Let us give a mathematical set-up of the general Bayesian game with \( n \) players where each player is indexed by \( i = 1, \ldots, n \). We assume that each of the spaces \( \mathbb{U} \) (the action space), \( \mathbb{V} \) (the space for payoff state) and \( \mathbb{T} \) (the signal space) is a complete separable metric space, and their Borel \( \sigma \)-fields are denoted by \( \mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}), \) and \( \mathcal{B}(\mathbb{T}) \). Here \( \mathbb{U} = \mathbb{U}_1 \times \cdots \times \mathbb{U}_n \) and \( \mathbb{T} = \mathbb{T}_1 \times \cdots \times \mathbb{T}_n \), so that \( \mathbb{U} \) represents the space of action profiles and \( \mathbb{T} \) that of the signal profiles. Consider the measurable space \( (\mathbb{V} \times \mathbb{T}, \mathcal{B}(\mathbb{V} \times \mathbb{T})) \), where \( \mathcal{B}(\mathbb{V} \times \mathbb{T}) \) denotes the Borel \( \sigma \)-field of \( \mathbb{V} \times \mathbb{T} \) with respect to the product topology.

Let \( \mu_\mathbb{V} \) be a probability measure on \( (\mathbb{V}, \mathcal{B}(\mathbb{V})) \). The measure \( \mu_\mathbb{V} \) represents the prior distribution from which the payoff state \( \mathbb{V} \) for the players is drawn. For each \( i = 1, \ldots, n \), let \( u_i \) be a measurable function: \( \mathbb{U} \times \mathbb{V} \to \mathbb{R} \), which represents the payoff function of player \( i \). Then, we follow BM and take \( \mathcal{B} = (\mathbb{U}, \mathbb{V}, u, \mu_\mathbb{V}) \) to be the basic game, where \( u = (u_i)_{i=1}^n : \mathbb{U} \times \mathbb{V} \to \mathbb{R} \) denotes the profile of payoff functions. We let \( \mathcal{S} = \{\alpha(\cdot \mid w) : w \in \mathbb{V}\} \), where \( \alpha : \mathcal{B}(\mathbb{T}) \times \mathbb{V} \to [0,1] \) is a Markov kernel, and take \( \mathcal{I} = (\mathbb{V}, \mathbb{T}, \mathcal{S}) \) to be the information structure denoted by \( \mathcal{I} \). (The existence of such a Markov kernel is ensured here, see, e.g., Theorem 5.3 of Kallenberg (1997).) Note that the Markov kernel \( \alpha \) is introduced separately from \( \mu_\mathbb{V} \), so that \( \alpha(\cdot \mid w) \) is well defined for \( w \in \mathbb{V} \) outside of the support of \( \mu_\mathbb{V} \).

Following BM, we call the pair \( \mathcal{G} = (\mathcal{B}, \mathcal{I}) \) a Bayesian game.

The Markov kernel \( \alpha \), together with \( \mu_\mathbb{V} \), induces the joint distribution \( \mu_{\mathbb{W},\mathbb{T}} \) on \( (\mathbb{V} \times \mathbb{T}, \mathcal{B}(\mathbb{V} \times \mathbb{T})) \) as follows (see e.g., Section 4.3 of Pollard (2011)):

\[
\mu_{\mathbb{W},\mathbb{T}} = \mu_\mathbb{V} \otimes \alpha,
\]

where for \( A \in \mathcal{B}(\mathbb{V}) \) and \( B \in \mathcal{B}(\mathbb{T}) \),

\[
(\mu_\mathbb{V} \otimes \alpha)(A \times B) = \int_A \alpha(B \mid w) d\mu_\mathbb{V}(w).
\]

Hence, the \( \mathbb{T} \)-marginal \( \mu_\mathbb{T} \) of \( \mu_{\mathbb{W},\mathbb{T}} \) depends on both \( \alpha \) and \( \mu_\mathbb{V} \) in the Bayesian game \( \mathcal{G} \). From here on, we let \( (\mathbb{W}, \mathbb{T}) \), with \( \mathbb{T} = (\mathbb{T}_1, \ldots, \mathbb{T}_n) \), be a \( (\mathbb{V} \times \mathbb{T}) \)-valued random element which follows \( \mu_{\mathbb{W},\mathbb{T}} \).

Now we introduce the space of decision rules which plays the role of the strategy space for the Bayesian game \( \mathcal{G} \). Let the set \( \Sigma \) be the class of Markov kernels \( \sigma(\cdot \mid \cdot, \cdot) : \mathcal{B}(\mathbb{U}) \times \mathbb{V} \times \mathbb{T} \to [0,1] \) so that, for each \( A \in \mathcal{B}(\mathbb{U}) \), \( \sigma(A \mid \cdot, \cdot) \) is Borel measurable, and for each \( (w, t) \in \mathbb{V} \times \mathbb{T} \), \( \sigma(\cdot \mid w, t) \) is a probability measure on \( \mathcal{B}(\mathbb{U}) \). We fix a \( \sigma \)-finite measure \( \nu \) on \( \mathcal{B}(\mathbb{V} \times \mathbb{T}) \) and endow \( \Sigma \) with weak topology which is the weakest topology that makes the maps \( \sigma \mapsto \int (f \otimes h)d(\nu \otimes \sigma) \) continuous for all \( f \in \mathcal{C}_1(\mathbb{V}) \) and all \( h \in \mathcal{C}_b(\mathbb{U}) \), where \( \mathcal{C}_b(\mathbb{U}) \) denotes the set of continuous and bounded real maps on \( \mathbb{U} \),

\[
\int (f \otimes h)d(\nu \otimes \sigma) = \int \int f(w, t)h(y)d\sigma(y \mid w, t)d\nu(w, t),
\]
and $L^1(\nu)$ denotes the class of real integrable maps on $\mathcal{W} \times \mathcal{T}$ with respect to $\nu$. (See Definition 2.2 of Háusler and Luschgy (2015).) Then $\Sigma$ is a complete separable metric space with respect to the weak topology (e.g., Proposition A.2.5.III of Daley and Vere-Jones (2003), p.400). Let us define $\mathcal{B}(\Sigma)$ to be the Borel $\sigma$-field of $\Sigma$. Following BM, we call each element $\sigma \in \Sigma$ a decision rule.

For each $i = 1, \ldots, n$, $t_i \in \mathcal{T}_i$ and $\sigma \in \Sigma$ and a measurable function $\tau_i : \mathcal{Y}_i \rightarrow \mathcal{Y}_i$, we write the expected payoff of player $i$ as

\begin{equation}
U_i(\tau_i, t_i; \sigma) = \int \int u_i(\tau_i(y_i), y_{-i}, w) d\sigma(y_i, y_{-i} | w, t_i, t_{-i}) d\mu_{W,T_i|T}(w, t_{-i} | t_i),
\end{equation}

where $\mu_{W,T_i|T}(\cdot, \cdot | t_i)$ denotes the conditional distribution of $(W, T_{-i})$ given $T_i = t_i$ under the joint distribution $\mu_{W,T}$ of $(W, T)$ constructed as in (29). Following BM, we say that a decision rule $\sigma \in \Sigma$ is a Bayes Correlated Equilibrium (BCE) for $G$, if for each $i = 1, \ldots, n$, and each measurable $\tau_i : \mathcal{Y}_i \rightarrow \mathcal{Y}_i$,

\[U_i(\text{Id}, t_i; \sigma) \geq U_i(\tau_i, t_i; \sigma),\]

for each $t_i$ in the support of $T_i$,

where $\text{Id}$ is the identity map $\mathcal{Y}_i \rightarrow \mathcal{Y}_i$. We let $\Sigma_{\text{BCE}}(G)$ be the set of BCEs of the Bayesian game $G$. We consider only those Bayesian games $G$ such that $\Sigma_{\text{BCE}}(G) \in \mathcal{B}(\Sigma)$.

A.1.2. Counterfactual Policies. The counterfactual policy of interest alters the payoff state $W$ into $f(W)$ for some measurable map $f : \mathcal{W} \rightarrow \mathcal{W}$. The nature of counterfactual predictions depends on what we take to be invariant from the original game. This is made clear once we specify the counterfactual Bayesian game after the policy.

Given the original Bayesian game $G = (B, I)$, we consider a setting where the policy changes the basic game $B = (\mathcal{Y}, \mathcal{W}, u, \mu_W)$ into $B_f = (\mathcal{Y}, \mathcal{W}, u, \mu_W \circ f^{-1})$. Thus, $(\mathcal{Y}, \mathcal{W}, u)$ from the basic game remains invariant. The policy changes the distribution of the payoff state from $\mu_W$ into $\mu_W \circ f^{-1}$, i.e., the distribution of $W$ under $\mu_W$ into that of $f(W)$ under $\mu_W$. In the transformed game $G_f = (B_f, I)$, the joint distribution of $(f(W), T)$ is then given as follows:

\begin{equation}
\mu_{f(W),T} = (\mu_W \circ f^{-1}) \otimes \alpha,
\end{equation}

that is, for $A \in \mathcal{B}(\mathcal{W})$ and $B \in \mathcal{B}(\mathcal{T})$,

\[\mu_{f(W),T}(A \times B) = \int_A \alpha(B | w) d(\mu_W \circ f^{-1})(w).\]

Compared with (29), it is clear that the distribution of $(f(W), T)$ in game $G_f$ is induced while keeping the information structure $I$ invariant. For future reference, we let $\mathcal{S}_{f(W)}$ denote the support of $\mu_W$, i.e., the smallest closed set $C \subset \mathcal{W}$ such that $\mu_W(\mathcal{W} \setminus C) = 0$. Similarly, we let $\mathcal{S}_{f(W)}$ denote the support of $\mu_W \circ f^{-1}$.

Let us consider a setting where the policy also alters the information structure from $I = (\mathcal{W}, \mathcal{T}, \mathcal{S})$ into $I^* = (\mathcal{W}, \mathcal{T}^*, \mathcal{S}^*)$, where $\mathcal{S}^* = \{\alpha^* \cdot \cdot | w : w \in \mathcal{W}\}$, with $\alpha^*$ denoting a Markov kernel on $\mathcal{B}(\mathcal{T}^*) \times \mathcal{W}$ and $\mathcal{T}^* = \mathcal{T}_1^* \times \ldots \times \mathcal{T}_n^*$. Similarly as in $\mathcal{T}$, we assume that $\mathcal{T}^*$ is a complete separable metric space.
Thus the counterfactual game after the policy is a Bayesian game $G^*_f = (B_f, I^*)$. The problem of counterfactual predictions can be summarized as that of predicting the outcome of the counterfactual game $G^*_f$ using data generated from the original game $G$.

A.2. Randomized Reduced Forms and Counterfactual Predictions

We consider a policy which changes a Bayesian game $G = (B, I)$ into $G^*_f = (B_f, I^*)$ so that a policy changes the payoff state $W$ into $f(W)$ for some map $f$ and the information structure $I$ into $I^*$. For the decomposition method to work, the causal relation between the policy variable and the outcome of the game needs to remain invariant before and after the policy. To make precise the meaning of the “causal relation”, we introduce the notion of randomized reduced forms and reduced-form selection rules.

First, let $R$ be the set of Markov kernels $\rho : B(Y) \times \mathcal{W} \to [0, 1]$. Given the $\mathcal{W}$-marginal $\nu_W$ of $\nu$, we endow $R$ with the usual weak topology that makes the map $\rho \mapsto \int (f \otimes h) d(\nu_W \otimes \rho)$ continuous for all $f \in L^1(\nu_W)$ and $h \in C_b(\mathcal{Y})$. For each decision rule $\sigma \in \Sigma$ and an information structure $\alpha$, we define $\rho_\sigma$ to be the Markov kernel on $B(Y) \times \mathcal{W}$ as follows:

(33) $\rho_\sigma = \alpha \otimes \sigma$,

that is, for $A \in B(\mathcal{Y})$,

$$\rho_\sigma(A \mid w) = \int \sigma(A \mid w, t) d\alpha(t \mid w), \text{ for } w \in \mathcal{W}.$$ 

We call $\rho_\sigma$ a (randomized) reduced form. It is induced by a decision rule $\sigma$ through the given information structure $\alpha$. The randomized reduced form $\rho_\sigma$ summarizes the causal relationship between the outcome and the payoff state according to the decision rule $\sigma$ of the players.

For a given Bayesian game $G = (B, I)$, we define

$$\mathcal{R}_{BCE}(G) = \{\rho_\sigma \in R : \sigma \in \Sigma_{BCE}(G)\}.$$ 

The set $\mathcal{R}_{BCE}(G)$ is the collection of randomized reduced forms induced by the BCEs of the game $G$.

For counterfactual predictions, we face ambiguity about the causal relationship between the outcome and the payoff state due to the multiplicity of BCEs. We let $\gamma(\cdot ; G)$ be the reduced-form selection rule for the game $G$, which is defined as a probability measure on $(R, B(\mathcal{R}))$ such that

$$\gamma(\mathcal{R}_{BCE}(G) ; G) = 1,$$

where $B(\mathcal{R})$ denotes the Borel $\sigma$-field of $\mathcal{R}$. Thus, through the reduced-form selection rule $\gamma$, we obtain the (randomized) reduced form of the game $G$ as follows:

$$\rho_G(A \mid w) = \int \rho(A \mid w) d\gamma(\rho ; G).$$ 

The reduced form of the game $\rho_G$ summarizes the causal relationship between the policy variable and the outcome of the game. Indeed, we can describe the generation of the outcome $Y$ from the game $G$ as follows:
Step 1: A reduced form $\rho \in \mathcal{R}_{\text{BCE}}(G)$ is drawn from the distribution $\gamma(\cdot; G)$.  
Step 2: The value of the payoff state $W = w$ is drawn from the distribution $\mu_W$.  
Step 3: The action profile $Y$ is drawn from the distribution $\rho(\cdot \mid w)$. 

This data generating process captures the essential causal structure of the game as relevant to the policy prediction problem of interest. Using this structure, we can clarify the necessary invariance conditions for the decomposition method to work.

Suppose that the pre-policy game is given by $G = (B, I)$ and the post-policy game is $G^*_f = (B_f, I^*)$, where $B_f = (Y, W, u, \mu_W \circ f^{-1})$ and $I^* = (W, T^*, S^*)$. Let us define our target parameter in this setting: for a real measurable map on $h$ on $\mathcal{Y} \times \mathcal{W}$,

$$
\text{AEP}^*(h) \equiv \int_{\mathcal{W}} \int_{\mathcal{Y}} h(y, w) d\rho_{G^*_f}(y \mid w) d(\mu_W \circ f^{-1})(w), 
$$

where $\rho_{G^*_f}$ represents the reduced form of the counterfactual game $G^*_f$. The target parameter $\text{AEP}^*(h)$ gives the expected value of $h(Y, f(W))$ when $Y$ is drawn from the conditional distribution $\rho_{G^*_f}(\cdot \mid W)$ given the payoff state $W$ and $W$ is drawn from the distribution $\mu_W$. This parameter is a modified version of $\text{AEP}(h)$ in the main text which involves the game $G^*_f$ instead of the game $G_f$. (The AEP stands for the “Average Equilibrium-based Prediction”.) Of course, when $I = I^*$, we have $\text{AEP}^*(h) = \text{AEP}(h)$. We refer the reader to the main text for examples of the map $h$.

A.3. A General Result for the Decomposition Method

We focus on the validity of a decomposition method for counterfactual predictions when a policy changes a Bayesian game $G$ into $G^*_f$. We define the decomposition-based prediction as

$$
\text{ADP}(h) \equiv \int_{\mathcal{S}_W} \int_{\mathcal{Y}} h(y, w) d\rho_G(y \mid w) d(\mu_W \circ f^{-1})(w).
$$

If we compare (34) and (35), we can see that when $\mathcal{S}_{f(W)} \subset \mathcal{S}_W$, the decomposition-based prediction $\text{ADP}(h)$ coincides with $\text{AEP}^*(h)$ as long as

$$
\rho_G = \rho_{G^*_f},
$$

i.e., the causal relationship between the outcome of the game and the payoff state remains the same after the policy. The rest of our focus is on exploring lower level conditions for this equality.

We begin by introducing two main conditions: (1) the invariance of the set of $w$-sections of randomized reduced forms induced by BCEs and (2) the invariance of the reduced-form selection rules. Let us introduce notation. For each $w \in \mathcal{W}$, we define

$$
\mathcal{R}_{\text{BCE}, w}(G) = \{\rho_{\sigma}(\cdot \mid w) : \sigma \in \Sigma_{\text{BCE}}(G)\}.
$$

Hence, $\mathcal{R}_{\text{BCE}, w}(G)$ denotes the set of $w$-sections of the randomized reduced forms induced by the BCEs of the Bayesian game $G$. The first condition we introduce is that the set of the $w$-sections of reduced forms induced by BCEs remains invariant after the policy.
Assumption A.1 (Invariance of the $w$-Sections of the BCE Reduced Forms). For each $(\mu_w \circ f^{-1})$-a.e. $w \in S_W \cap S_f(W)$,

$$\mathcal{R}_{BCE, w}(G) = \mathcal{R}_{BCE, w}(G_f^*) .$$

Recall that the randomized reduced form of a Bayesian game is determined by the BCEs of the game and the information structure. We will provide lower level conditions for Assumption A.1 in terms of the BCEs and the information structure later.

The second condition imposes an invariance condition on the selection rule from the reduced forms induced by BCEs. Let $P_{\mathbb{Y}}$ be the set of probability measures on $\mathbb{Y}$ endowed with the usual weak topology, and let $B(P_{\mathbb{Y}})$ be the Borel $\sigma$-field of $P_{\mathbb{Y}}$. For each $w \in \mathbb{W}$ and $D \in B(P_{\mathbb{Y}})$, we define

$$(37) \quad \gamma_w(D; G) = \gamma(\{\rho \in \mathcal{R}_{BCE}(G) : \rho(\cdot \mid w) \in D\}; G).$$

In other words, $\gamma_w(\cdot; G)$ is the push-forward of $\gamma(\cdot; G)$ under the map $\rho \mapsto \rho(\cdot \mid w)$. We introduce the following invariance condition for the reduced-form selection rule $\gamma$.

Assumption A.2 (Invariance of Reduced-Form Selection Rules). For each $w \in S_W \cap S_f(W)$ such that $\mathcal{R}_{BCE, w}(G) = \mathcal{R}_{BCE, w}(G_f^*)$, we have

$$\forall w \in S_W \cap S_f(W) \rightarrow \mathbb{R}$$

$$\gamma_w(D; G) = \gamma_w(D; G_f^*) ,$$

for each $D \in B(P_{\mathbb{Y}})$.

This assumption requires that, for any $w \in S_W \cap S_f(W)$ such that the $w$-section of BCE reduced forms remain the same, the $w$-section of the reduced-form selection rule remains the same as well after the policy. We now state the main theorem for the counterfactual predictions using the decomposition method.

Theorem A.1. Suppose that $G = (B, I)$ and $G_f^* = (B_f, I_f^*)$ are given such that Assumptions A.1 and A.2 hold. Let $h : \mathbb{Y} \times \mathbb{W} \rightarrow \mathbb{R}$ be a measurable map such that, for all $w \in \mathbb{W}$ such that $f(w) \notin S_W$,

$$(38) \quad \underline{h}(w) \leq \inf_{y \in \mathbb{Y}} h(y, w) \leq \sup_{y \in \mathbb{Y}} h(y, w) \leq \overline{h}(w),$$

for some measurable maps $\underline{h}, \overline{h} : \mathbb{W} \rightarrow \mathbb{R}$.

Then,

$$\text{ADP}(h) + \Delta(h) \leq \text{AEP}^*(h) \leq \text{ADP}(h) + \Delta(\overline{h}),$$

where

$$\Delta(\overline{h}) = \int_{\mathbb{W} \setminus S_W} \overline{h}(w) d(\mu_w \circ f^{-1})(w),$$

and $\Delta(h)$ is defined similarly.

Proof: First, we let

$$m_G(h; w) = \int_{\mathbb{Y}} h(y, w) d\rho_G(y \mid w).$$
Then, we obtain the following inequalities:

\[
\begin{align*}
(39) \quad & \int_{S_w} m_{G^*_j}(h;w)\mu_{f^{-1}}(w) + \int_{W\setminus S_w} h(w)\mu_{f^{-1}}(w) \\
& \leq \int_{S_w} m_{G_j}(h;w)\mu_{f^{-1}}(w) \\
& \leq \int_{S_w} m_{G_j}(h;w)\mu_{f^{-1}}(w) + \int_{S_w \setminus S_w} h(w)\mu_{f^{-1}}(w),
\end{align*}
\]

where \( m_{G_j}(h;w) \) is \( m_G(h;w) \) with \( G \) replaced by \( G^*_j \). It suffices to show that, for \((\mu_{f^{-1}})\text{-a.e.} \ w \in S_w \cap S_{f(W)}\),

\[(40) \quad m_G(h;w) = m_{G^*_j}(h;w).\]

By Assumptions A.1 and A.2, for \((\mu_{f^{-1}})\text{-a.e.} \ w \in S_w \cap S_{f(W)}\) and all Borel \( A \subset Y \),

\[
\rho_G(A \mid w) = \int_{\mathcal{R}_{BCE}(G)} \rho(A \mid w)\gamma(\rho;G) = \int_{\mathcal{R}_{BCE}(G)} \rho(A)\gamma_w(\rho;G) = \int_{\mathcal{R}_{BCE}(G^*_j)} \rho(A)\gamma_w(\rho;G^*_j) = \rho_{G^*_j}(A \mid w).
\]

Hence we obtain (40). Thus the desired result follows. ■

Now, the rest of our development focuses on studying low level conditions for Assumption A.1 and A.2 and explore their meaning in specific contexts of Bayesian games.

A.4. Low Level Conditions for Assumption A.1

We explore low level conditions for Assumption A.1. We reduce the assumption to two low level conditions, where the first condition is concerned with the relation between the pre-policy and post-policy information structures \( I \) and \( I^* \), and the second condition is about the invariance of the set of \( w \)-sections of the BCEs.

To clarify the conditions for the information structure \( I^* \), the ordering between information structures proposed by Bergemann and Morris (2016) (BM, hereafter) is crucially used here. Following BM, we say that \( I \) is \textit{individually sufficient} for \( I^* \) if there exist a distribution \( \mu_{W,T,T^*} \) on \( W \times T \times T^* \) such that any random element \((W,T,T^*)\) having the distribution \( \mu_{W,T,T^*} \) satisfies the following two conditions:

(a) The distributions of \((W,T)\) and \((W,T^*)\) are equal to \( \mu_{W,T} = \mu_W \otimes \alpha \) of \( G \) and \( \mu_{W,T^*} = \mu_W \otimes \alpha^* \) of \( G^* \) respectively.

(b) For all \( i = 1, ..., n \),

\[T^*_i \perp (T_{-i}, W) \mid T_i,\]

under \( \mu_{W,T,T^*} \), that is, \( T^*_i \) is conditionally independent of \((T_{-i}, W)\) given \( T_i \) under \( \mu_{W,T,T^*} \).
When $I$ is individually sufficient for $I^*$, the signal $T^*_i$ of agent $i$ does not reveal anything new about the payoff state $W$ or other players’ signals $T_{-i}$ beyond the signal $T_i$. We interpret the information structure $I$ as “richer” than $I^*$ if $I$ is individually sufficient for $I^*$.

If $I$ is individually sufficient for $I^*$ and $I^*$ is individually sufficient for $I$, we say that $I$ and $I^*$ are **mutually individually sufficient**. Note that mutual individual sufficiency of $I$ and $I^*$ does not imply that the information structures are the same. We give two examples.

**Example A.1.** Suppose that $T_i = (V_i, S_i)$ and $T^*_i = (V_i, S^*_i)$, and that $(S_i, S^*_i)$s are independent across $i$’s and independent of $(W, V)$, $V = (V_1, \ldots, V_n)$, under $\mu_{W,T,T^*}$. Then we have

$$T^*_i \perp \perp (T_{-i}, W) | T_i,$$

under $\mu_{W,T,T^*}$. Hence $I$ is individually sufficient for $I^*$. It follows by symmetry that $I$ and $I^*$ are mutually individually sufficient.

**Example A.2.** We assume that $T^*_i = (T_i, S^*_i)$, where $S^*_i$ denotes the additional signal that player $i$ receives after a policy. Define $T^* = (T^*_i)_{i=1}^n$ and $S^* = (S^*_i)_{i=1}^n$. For each player $i \in N$, we assume that

$$S^*_i \perp \perp (T_{-i}, W) | T_i,$$

under $\mu_{W,T,T^*}$. The assumption requires that the additional individual signal of player $i$ does not reveal other players’ individual signals and the payoff state once conditioned on the player’s previous signal.\(^{25}\) Hence $I$ is individually sufficient for $I^*$. On the other hand, since $T^*_i$ contains $T_i$, we have

$$T_i \perp \perp (T^*_i, W) | T^*_i.$$ 

Hence, $I^*$ is individually sufficient for $I$. We find that $I$ and $I^*$ are mutually individually sufficient.

Theorem 2 of BM implies that when $I$ is individually sufficient for $I^*$, the set of randomized reduced forms induced by the BCE of $G = (B, I)$ is contained in the set induced by the BCE of $G^* = (B, I^*)$. This establishes a useful relation between the ordering among the information structures and that among the sets of randomized reduced forms induced by the BCEs.

Recall that $\mathcal{R}$ denotes the set of Markov kernels $\rho: B(\mathcal{Y}) \times \mathcal{W} \rightarrow [0, 1]$. The following lemma is an extension of a part of Theorem 2 of BM from the setting with the action space and the type space as finite sets to general, potentially infinite sets. In proving the lemma, we follow the arguments in the proof of Theorem 2 of BM.

\(^{25}\)In the context of the empirical application in Section 3, this requirement is fulfilled if new signals were introduced by congressional hearings, bureaucratic reports and extensive media coverage of the repeal of the Wright Amendment and were used as correlation devices. After all, the decision to repeal the Wright Amendment came after congressional hearings and court procedures with statements by the Department of Justice, and its implementation was preceded by statements from the Federal Aviation Authority (FAA), among other forms of communication including extensive media coverage. For example, see the hearing on the Wright Amendment on November 10, 2005 before the Senate Subcommittee on Aviation from the Committee on Transportation and Infrastructure in 2006 (S. Hrg. 109-1098) and the statement by Michael Cirillo (FAA) on “Reforming the Wright Amendment” before the same subcommittee.
**Lemma A.1.** Suppose that two Bayesian games \( G = (B, I) \) and \( G^* = (B, I^*) \) are given such that \( I \) is individually sufficient for \( I^* \). Then,

\[
\mathcal{R}_{BCE}(G) \subset \mathcal{R}_{BCE}(G^*).
\]

**Proof:** First, we show necessity. Suppose that \( I \) is individually sufficient for \( I^* \). Fix any Bayesian games \( G = (B, I) \) and \( G^* = (B, I^*) \). Let \( \sigma \in \Sigma_{BCE}(G) \) and \( \rho_\sigma \in \mathcal{R}_{BCE}(G) \). Then, for (42), it suffices to find \( \sigma^* \in \Sigma_{BCE}(G^*) \) such that

\[
\rho_\sigma = \rho_{\sigma^*},
\]

where

\[
\rho_{\sigma^*} = \rho^* \otimes \sigma^*.
\]

Let \( \mu_{T|\Gamma^*, W} \) be the conditional distribution of \( T \) given \( (T^*, W) \), when \( (W, T, T^*) \) follows the distribution \( \mu_{W,T,T^*} \). Recall that \( \rho^* \) is the Markov kernel belonging to the information structure \( I^* = (W, T^*, S^*) \). Then, we have

\[
\alpha^* \otimes \mu_{T|\Gamma^*, W} = \alpha,
\]

by the construction of \( \mu_{T|\Gamma^*, W} \).

For \( (w, t^*) \in \mathbb{W} \times T^* \), define \( \sigma^*(\cdot \mid w, t^*) \) as: for any Borel \( A \subset \mathbb{Y} \),

\[
\sigma^*(A \mid w, t^*) = \int T \sigma(A \mid W, T) d \mu_{T|\Gamma^*, W}(t^* \mid w, t).
\]

The expected payoff \( U_i(\tau_i, t^*_i, \sigma^*) \) in \( G^* \) is written as

\[
\int \int Y u_i(\tau_i(y_i), y_{-i}, w) \sigma^*(y_i, y_{-i} \mid w, t^*) d \mu_{W,T^*_i,t_i}^*(w, t^*_i, t_i)
\]

\[
= \int \int Y u_i(\tau_i(y_i), y_{-i}, w) \sigma(y_i, y_{-i} \mid w, t) d \mu_{W,T^*_i,t_i}^*(w, t^*_i, t_i),
\]

using (46). By integrating out \( T^*_i \) in the integral, we rewrite the last double integral as

\[
\int \int Y u_i(\tau_i(y_i), y_{-i}, w) \sigma(y_i, y_{-i} \mid w, t) d \mu_{W,T^*_i,t_i}^*(w, t^*_i, t_i)
\]

\[
= \int \int Y u_i(\tau_i(y_i), y_{-i}, w) \sigma(y_i, y_{-i} \mid w, t) d \mu_{W,T^*_i,t_i}^*(w, t^*_i, t_i) d \mu_{T|\Gamma_i^*}(t_i \mid t_i^*).
\]

By the individual sufficiency of \( I \) for \( I^* \), the last integral becomes

\[
\int \int Y u_i(\tau_i(y_i), y_{-i}, w) \sigma(y_i, y_{-i} \mid w, t) d \mu_{W,T^*_i,t_i}^*(w, t_{-i} \mid t_i) d \mu_{T|\Gamma_i^*}(t_i \mid t_i^*)
\]

\[
= \int_{T_i} U_i(\tau_i, t_i; \sigma) d \mu_{T|\Gamma_i^*}(t_i \mid t_i^*).
\]
Hence,
\begin{equation}
U_i(\tau_i, t^*_i, \sigma^*) = \int_{T_i} U_i(\tau_i, t_i; \sigma) d\mu_{T_i|T'_i}(t_i | t^*_i).
\end{equation}

Since \( \sigma \in \Sigma_{\text{BCE}}(G) \),
\begin{align*}
U_i(\tau_i, t^*_i, \sigma^*) &= \int_{T_i} U_i(\tau_i, t_i; \sigma) d\mu_{T_i|T'_i}(t_i | t^*_i) \\
&\leq \int_{T_i} U_i(\text{Id}, t_i; \sigma) d\mu_{T_i|T'_i}(t_i | t^*_i) = U_i(\text{Id}, t^*_i; \sigma^*),
\end{align*}

where the last equality follows from (47) with \( \tau_i \) replaced by \( \text{Id} \). Hence we have shown that \( \sigma^* \in \Sigma_{\text{BCE}}(G^*) \). Note that, for \( w \in \mathbb{W} \), by (44), (45), and (46),
\begin{align*}
\rho^*_{\sigma}(A | w) &= \int \sigma^*(A | w, t^*) d\alpha^*(t^* | w) \\
&= \int \int_{T} \sigma(A | w, t) d\mu_{T|T',W}(t | t^*, w) d\alpha^*(t^* | w) \\
&= \int_{T} \sigma(A | w, t) d\alpha(t | w) = \rho_{\sigma}(A | w).
\end{align*}

We have shown (43) and hence (42). 

Therefore, by symmetry, when \( G \) and \( G^* \) are such that \( I \) and \( I^* \) are mutually individually sufficient, we have the following result:
\begin{equation*}
\mathcal{R}_{\text{BCE}}(G) = \mathcal{R}_{\text{BCE}}(G^*).
\end{equation*}

In other words, changing the information structure maintaining mutual individual sufficiency does not change the set of the randomized reduced forms induced by the BCEs. This means that a counterfactual prediction from \( G^* \) after a policy changes \( G \) into \( G^* \) remains the same in so far as the prediction is based on the randomized reduced forms induced by the BCEs of the game \( G^* \).

We are ready to provide two low level conditions for Assumption A.1. The first condition is about the information structures \( I \) and \( I^* \).

**Assumption A.3** (Mutual Individual Sufficiency of Information Structures). The information structures \( I \) and \( I^* \) are mutually individually sufficient.

The examples of mutually individually sufficient information structures have been discussed previously. The second condition is about the \( w \)-sections of the BCEs. For each \( w \in \mathbb{W} \), we define
\begin{equation}
\Sigma_{\text{BCE},w}(G) = \{ \sigma(\cdot | w, \cdot) : \sigma \in \Sigma_{\text{BCE}}(G) \}.
\end{equation}

**Assumption A.4** (Invariance of the \( w \)-Sections of the BCEs). For \((\mu_w \circ f^{-1})\)-a.e. \( w \in \mathbb{S}_w \cap \mathbb{S}_f(W) \),
\[ \Sigma_{\text{BCE},w}(G) = \Sigma_{\text{BCE},w}(G_f), \]
where $G_f = (B_f, I)$.

This assumption says that, for $(\mu_w \circ f^{-1})$-a.e. $w \in S_w \cap S_f(w)$, the set of $w$-sections of the BCEs of $G$ and $G_f$ remain the same. In the next section, we show that this assumption is fulfilled under Assumption 2.1 in the main text.

Now, we present the result that Assumptions A.3 and A.4 yield Assumption A.1.

**Corollary A.1.** Suppose that Bayesian games $G = (B, I)$ and $G_f^* = (B_f, I^*)$ satisfy Assumptions A.3 and A.4. Then, Assumption A.1 holds.

**Proof:** By Lemma A.1, for $w \in S_w \cap S_f(w)$, we have

\[ R_{BCE,w}(G_f) = R_{BCE,w}(G_f^*), \]

because the information structures $I$ and $I^*$ in games $G_f = (B_f, I)$ and $G_f^* = (B_f, I^*)$ are mutually individually sufficient. Let $\mathcal{W}_1 \subset S_w \cap S_f(w)$ be the set of $w$ such that (50) holds and

\[ \Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f). \]

We can choose $\mathcal{W}_1$ such that $(\mu_w \circ f^{-1})(\mathcal{W} \setminus \mathcal{W}_1) = 0$ by Assumption A.4. We fix $w \in \mathcal{W}_1$. Note that $G = (B, I)$ and $G_f = (B_f, I)$ have the same information structure. Hence, if $\tilde{\rho} \in R_{BCE,w}(G)$, then $\tilde{\rho}(\cdot) = \rho_{\sigma}(\cdot \mid w)$ for some $\sigma \in \Sigma_{BCE}(G)$. However, recall the definition of $\rho_{\sigma}$ in (33). Since $\Sigma_{BCE,w}(G) \subset \Sigma_{BCE,w}(G_f)$ from (51), we have $\rho_{\sigma}(\cdot \mid w) \in R_{BCE,w}(G_f)$. Therefore, $R_{BCE,w}(G) \subset R_{BCE,w}(G_f)$. Following the same arguments now using $\Sigma_{BCE,w}(G_f) \subset \Sigma_{BCE,w}(G)$ from (51), we find that $R_{BCE,w}(G_f) \subset R_{BCE,w}(G)$. Hence, we conclude that $R_{BCE,w}(G) = R_{BCE,w}(G_f)$ for all $w \in \mathcal{W}_1$. ■

**Appendix B.** Counterfactual Predictions from Bayesian Games under Assumption 2.1

**B.1. Preliminary Results**

In this appendix, we focus on a setting where the policy changes the payoff state $W$ but does not change the information structure. Hence, the game involved in the data generating process is $G = (B, I)$, whereas the counterfactual game is $G_f = (B_f, I)$. The main goal is to derive Theorem 2.1 under Assumptions 2.1 and 2.2. For this, we show how Assumption A.4 follows from Assumption 2.1. The result is Lemma B.4 below. To obtain this, we need to introduce some results.

We begin with restating Assumption 2.1 with measure-theoretic details made explicit.

**Assumption B.1** (Information Structure Condition). (i) $\mathcal{W} = \mathcal{W}_C \times \mathcal{W}_I$, for some complete separable metric spaces $\mathcal{W}_C$ and $\mathcal{W}_I$, and

\[ f(w) = (\tilde{f}(w_C), w_I), \quad w = (w_C, w_I) \in \mathcal{W}, \]

for some measurable map $\tilde{f} : \mathcal{W}_C \to \mathcal{W}_C$.  

(ii) For each \(i = 1, \ldots, n\), \(T_i = T_{C,i} \times T_{I,i}\), where \(T_{C,i} = W_C\), and \(T_{I,i}\) is a complete separable metric space, and \(\alpha\) in game \(G\) is such that, for each \(A \in B(T)\),
\[
\alpha(A \mid w) = 1 \{ s(w) \in A \},
\]
for a map \(s = (s_1, \ldots, s_n) : W \to T\), where for each \(i = 1, \ldots, n\),
\[
s_i(w) = (w_C, \varphi_i(w_I)), \quad w = (w_C, w_I) \in W,
\]
for some measurable map \(\varphi_i : W_I \to T_{I,i}\).

If we restrict the payoff state space and the signal space appropriately, we can compare the pre-policy and post-policy games by connecting the BCEs from a restricted game with the restricted BCEs from the original game.

For simplicity, we write \(W_1 = S_W \cap S_{f(W)}\) and \(W_2 = W \setminus W_1\).

For \(\ell = 1, 2\), we let \(\mu_{W,\ell}\) be the conditional distribution of \(W\) given \(W \in W_\ell\) under \(\mu_W\). Let \(T_{i,1}\) be the support of \(\mu_{W,1} \circ s_i^{-1}\), and let \(T_{i,2} = T_i \setminus T_{i,1}\). We define
\[
T'_\ell = \prod_{i=1}^n T_{i,\ell}, \quad \ell = 1, 2.
\]

For \(\ell = 1, 2\), we construct a Bayesian game whose signal space is restricted to \(T'_\ell\), \(\ell = 1, 2\). For this, define a Markov kernel \(\alpha_\ell : B(T'_\ell) \times W_\ell \to [0, 1]\) as follows: for any Borel \(A \subset T'_\ell\), and \(w \in W_\ell\),
\[
\alpha_\ell(A \mid w) = 1 \{ w \in s^{-1}(A) \cap W_\ell \}.
\]

This yields the following information structure: \(\ell = 1, 2\),
\[
I_\ell = (W_\ell, T'_\ell, S_\ell),
\]
where \(S_\ell = \{ \alpha_\ell(\cdot \mid w) : w \in W_\ell \}\). We define a basic game \(B_\ell = (Y, W_\ell, u, \mu_{W,\ell})\). Similarly, we let \(\mu_{W,\ell}'\) be the conditional distribution of \(f(W)\) given \(W \in W_\ell\) under \(\mu_W\), and define a post-policy basic game \(B_\ell' = (Y, W_\ell, u, \mu_{W,\ell}')\). We construct the pre-policy and post-policy games restricted to the signal space \(T'_\ell\) as
\[
G_\ell = (B_\ell, I_\ell), \quad \text{and} \quad G'_\ell = (B_\ell', I_\ell).
\]

The following lemma shows that \(G_1\) and \(G'_1\) have the same set of BCEs.

**Lemma B.1.** Suppose that a game-policy pair \((G, f)\) satisfies Assumption B.1. Then,
\[
\Sigma_{\text{BCE}}(G_1) = \Sigma_{\text{BCE}}(G'_1).
\]
Proof: Recall that, for \( t_i = (w_c, t_{i,i}) \in T_i, \)

\[
U_i(\tau_i, t_i, \sigma) = \int \int u_i(\tau_i(y), y_{-i}, w) d\sigma(y | w, s(w)) d\mu_{W|T_i}(w | w_c, t_{i,i}).
\]  

Note that we have \( \sigma \in \Sigma_{BCE}(G_1) \) if and only if for each \( i = 1, \ldots, n, \) for any measurable transform \( \tau_i : \mathcal{Y}_i \rightarrow \mathcal{Y}_i, \) and for each \( t_i \) in the support of \( \mu_{W,1} \circ s_i^{-1}, \)

\[
U_i(\tau_i, t_i, \sigma) \leq U_i(\text{Id}, t_i, \sigma).
\]

Also, we have \( \sigma \in \Sigma_{BCE}(G'_i) \) if and only if the above inequality holds for each \( i = 1, \ldots, n, \) for any measurable transform \( \tau_i : \mathcal{Y}_i \rightarrow \mathcal{Y}_i, \) and for each \( t_i \) in the support of \( \mu_{W,1} \circ s_i^{-1}. \) However, by the construction of \( \mathcal{W}_1, \) the supports of \( \mu_{W,1} \circ s_i^{-1} \) and \( \mu_{W,1} \circ s_i^{-1} \) coincide. Hence, the desired result follows. 

Lemma B.3 below establishes that the set of BCE's of \( G_1 \) consists precisely of \( \sigma|_{\mathcal{W}_i} \) with \( \sigma \in \Sigma_{BCE}(G), \) where \( \sigma|_{\mathcal{W}_i} \) is a Markov kernel on \( \mathcal{B}(\mathcal{Y}) \times \mathcal{W}_1 \times T'_1, \) such that

\[
\sigma|_{\mathcal{W}_i}(A | w, t) = \sigma(A | w, t),
\]

for all \( A \in \mathcal{B}(\mathcal{Y}) \) and for all \( (w, t) \in \mathcal{W}_1 \times T'_1. \) That is, \( \sigma|_{\mathcal{W}_i} \) is a restriction of \( \sigma \) to \( \mathcal{B}(\mathcal{Y}) \times \mathcal{W}_1 \times T'_1. \) For a Bayesian game \( G \) and for \( \mathcal{W}' \subset \mathcal{W}, \) we define

\[
\Sigma_{BCE}(G)|_{\mathcal{W}} = \{ \sigma|_{\mathcal{W}} : \sigma \in \Sigma_{BCE}(G) \},
\]

where \( \sigma|_{\mathcal{W}} \) is a Markov kernel on \( \mathcal{B}(\mathcal{Y}) \times \mathcal{W}' \times T \) as a restriction of \( \sigma \) to \( \mathcal{B}(\mathcal{Y}) \times \mathcal{W}' \times T. \)

We introduce the following technical lemma.

Lemma B.2. Suppose that Assumption B.1 holds. Then, for each \( i = 1, \ldots, n, \)

\[
T_{i,2} \subset \text{supp}(\mu_{W,2} \circ s_i^{-1}),
\]

where \( \text{supp}(\mu_{W,2} \circ s_i^{-1}) \) denotes the support of \( \mu_{W,2} \circ s_i^{-1}. \)

Proof: First, since \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) partition \( \mathcal{W}, \) we have

\[
\overline{s_i(\mathcal{W})} \subset \text{supp}(\mu_{W,1} \circ s_i^{-1}) \cup \text{supp}(\mu_{W,2} \circ s_i^{-1}),
\]

where \( \overline{s_i(\mathcal{W})} \) denotes the closure of \( s_i(\mathcal{W}). \) Therefore,

\[
\overline{s_i(\mathcal{W})} \setminus \text{supp}(\mu_{W,1} \circ s_i^{-1}) \subset \text{supp}(\mu_{W,2} \circ s_i^{-1}).
\]

Since \( T_{i,2} \subset \overline{s_i(\mathcal{W})} \setminus T_{i,1} \) and \( T_{i,1} = \text{supp}(\mu_{W,1} \circ s_i^{-1}), \) we obtain the desired result. 

Lemma B.3. Suppose that a game-policy pair \((G, f)\) satisfies Assumption B.1. Then,

\[
\Sigma_{BCE}(G)|_{\mathcal{W}_i} = \Sigma_{BCE}(G_1) \quad \text{and} \quad \Sigma_{BCE}(G_f)|_{\mathcal{W}_i} = \Sigma_{BCE}(G'_1).
\]

Proof: Whenever \( \sigma \in \Sigma_{BCE}(G), \) for each \( t_i \) in the support of \( \mu_W \circ s_i^{-1}, \) we have

\[
U_i(\tau_i, t_i, \sigma) \leq U_i(\text{Id}, t_i, \sigma).
\]
The inequality continues to hold if we change the information structure \( \alpha \) into \( \alpha_1 \), because this change is tantamount to restricting \( t_i \) to the support of \( \mu_{W,1} \circ s_i^{-1} \). Therefore, we find that

\[
(56) \quad \Sigma_{\text{BCE}}(G) \mid_{W_1} \subset \Sigma_{\text{BCE}}(G_1).
\]

Now, let us show that

\[
(57) \quad \Sigma_{\text{BCE}}(G_1) \subset \Sigma_{\text{BCE}}(G) \mid_{W_1}.
\]

First, we choose \( \sigma_1 \in \Sigma_{\text{BCE}}(G_1) \) and \( \sigma_2 \in \Sigma_{\text{BCE}}(G_2) \), and for each \( w \in W \), let

\[
(58) \quad c(\sigma_1, \sigma_2)(\cdot \mid w) = \sum_{t=1}^2 \sigma_t(\cdot \mid w, s(w))1\{s(w) \in T'_t\}.
\]

Then \( c(\sigma_1, \sigma_2) \in \Sigma \). We construct a set of decision rules in \( \Sigma \) as follows:

\[
\bigotimes_{t=1}^2 \Sigma_{\text{BCE}}(G_t) = \{c(\sigma_1, \sigma_2) : \sigma_1 \in \Sigma_{\text{BCE}}(G_1) \text{ and } \sigma_2 \in \Sigma_{\text{BCE}}(G_2)\}.
\]

If \( w \in W_1 \), then \( s(w) \in T'_1 \). Hence, \( c(\sigma_1, \sigma_2)(\cdot \mid w) \) with \( w \) restricted to \( W_1 \) is equal to \( \sigma_1(\cdot \mid w, s(w)) \).

Therefore,

\[
\Sigma_{\text{BCE}}(G_1) = \bigotimes_{t=1}^2 \Sigma_{\text{BCE}}(G_t) \bigg|_{W_1}.
\]

Hence, it suffices for (57) to show that

\[
(59) \quad \bigotimes_{t=1}^2 \Sigma_{\text{BCE}}(G_t) \bigg|_{W_1} \subset \Sigma_{\text{BCE}}(G) \mid_{W_1}.
\]

Take \( c(\sigma_1, \sigma_2) \in \bigotimes_{t=1}^2 \Sigma_{\text{BCE}}(G_t) \) for some \( \sigma_1 \in \Sigma_{\text{BCE}}(G_1) \) and \( \sigma_2 \in \Sigma_{\text{BCE}}(G_2) \). Choose any \( i = 1, \ldots, n \) and any measurable map \( \tau_i : Y_i \to Y_i \). For \( t_i = (w_c, \varphi_i(w_i)) \) in the support of \( (W_c, \varphi_i(W_i)) \) under \( \mu_W \),

\[
U_i(\tau_i, t_i, c(\sigma_1, \sigma_2)) = U_i(\tau_i, t_i, \sigma_1)1\{t_i \in T_{i,1}\} + U_i(\tau_i, t_i, \sigma_2)1\{t_i \in T_{i,2}\}
\]

\[
\leq U_i(Id, t_i, \sigma_1)1\{t_i \in T_{i,1}\} + U_i(Id, t_i, \sigma_2)1\{t_i \in T_{i,2}\},
\]

\[
= U_i(Id, t_i, c(\sigma_1, \sigma_2)).
\]

To see that the first equality holds, note that we have \( s(w) \in T'_i \), with \( w = (w_c, w_i) \), if and only if \( w_c \) belongs to the intersection of the supports of \( W_c \) and \( f(W_c) \) under \( \mu_W \). However, the latter statement holds if and only if \( s_i(w) \in T_{i,1} \). Thus, the first equality follows from the definition of \( c(\sigma_1, \sigma_2) \) in (58). The inequality above follows because we have chosen \( \sigma_\ell \in \Sigma_{\text{BCE}}(G_\ell) \), \( \ell = 1, 2 \), and \( T_{i,1} \) is equal to the support of \( \mu_{W,1} \circ s_i^{-1} \), and by Lemma B.2, \( T_{i,2} \) is contained in the support of \( \mu_{W,2} \circ s_i^{-1} \). Hence \( c(\sigma_1, \sigma_2) \in \Sigma_{\text{BCE}}(G) \). We find that

\[
\bigotimes_{t=1}^2 \Sigma_{\text{BCE}}(G_t) \subset \Sigma_{\text{BCE}}(G).
\]
By restricting both sides to $\mathcal{W}_1$, we obtain the inclusion (59), and hence, (57), which, in combination with (56), yields the first statement of the lemma.

As for the second statement, similarly as before, whenever $\sigma \in \Sigma_{BCE}(G_f)$, for $t_i$ in the support of $\mu_w \circ f^{-1} \circ s_i^{-1}$, we have

\begin{equation}
U_i^f(\tau_i, t_i, \sigma) \leq U_i^f(Id, t_i, \sigma),
\end{equation}

where $U_i^f$ is the same as $U_i$ except that $\mu_{W,T}$ is replaced by $(\mu_w \circ f^{-1}) \otimes \alpha$. The inequality continues to hold if we restrict $w$ to $\mathcal{W}_1$, or if we change the information structure $\alpha$ into $\alpha_1$. Therefore,

$$\Sigma_{BCE}(G_f)|_{\mathcal{W}_1} \subset \Sigma_{BCE}(G^f_1).$$

The proof of the reverse inclusion can be done in the same way as before, and hence omitted. ■

The following lemma shows that under Assumption B.1, the set of equilibrium actions remain the same between $G$ and $G_f$ once we restrict the payoff state $w$ to $\mathcal{W}_W \cap \mathcal{S}_f(W)$.

\textbf{Lemma B.4.} Suppose that a game-policy pair $(G, f)$ satisfies Assumption B.1. Then,

$$\Sigma_{BCE}(G)|_{\mathcal{W}_W \cap \mathcal{S}_f(W)} = \Sigma_{BCE}(G_f)|_{\mathcal{W}_W \cap \mathcal{S}_f(W)}.$$

\textbf{Proof:} Consider the restricted, pre-policy and post-policy games, $G_1 = (B_1, I_1)$ and $G^f_1 = (B^f_1, I_1)$, as in (53). By Lemma B.1, $\Sigma_{BCE}(G_1) = \Sigma_{BCE}(G^f_1)$. In combination with Lemma B.3, this yields the following:

\begin{equation}
\Sigma_{BCE}(G)|_{\mathcal{W}_1} = \Sigma_{BCE}(G_1) = \Sigma_{BCE}(G^f_1) = \Sigma_{BCE}(G_f)|_{\mathcal{W}_1}.
\end{equation}

■

\section*{B.2. The Main Result}

\subsection*{B.2.1. Invariance of Equilibrium Selection Rules}

We introduce equilibrium selection rules and an invariance condition under a policy of changing the payoff state $W$ of a Bayesian game into $f(W)$. Consider the space $(\Sigma, \mathcal{B}(\Sigma))$, where $\Sigma$ is the complete separable metric space of Markov kernels introduced before. We denote $e(\cdot; G)$ to be the \textit{equilibrium selection rule} for the game $G$, which is defined as a probability measure on $(\Sigma, \mathcal{B}(\Sigma))$ such that

$$e(\Sigma_{BCE}(G); G) = 1.$$

We connect the equilibrium selection rule $e$ to the reduced-form selection rule $\gamma$ as follows: for any $A \in \mathcal{B}(\mathcal{R})$, we define

\begin{equation}
\gamma(A; G) = e(\{\sigma \in \Sigma : \rho_{\sigma} \in A\}; G).
\end{equation}

In other words, $\gamma(\cdot; G)$ is defined as the push-forward of $e$ by the map $\sigma \mapsto \rho_{\sigma}$. Then, it is not hard to see that $\gamma(R_{BCE}(G); G) = 1$. Thus, for each $w \in \mathcal{W}$, and $D \in \mathcal{B}(\mathcal{P}_{\psi})$, we have

\begin{equation}
\gamma_w(D; G) = e(\{\sigma \in \Sigma : \rho_{\sigma}(\cdot \mid w) \in D\}; G),
\end{equation}
by the definition of $\gamma_w$ in (37).

Recall the definition of $\Sigma_{BCE,w}(G)$ in (49). Under Assumption B.1, we can rewrite

(64) $$\Sigma_{BCE,w}(G) = \{\sigma(\cdot \mid w, s(w)) : \sigma \in \Sigma_{BCE}(G)\},$$

where $s$ is the map in Assumption B.1. Recall that $\mathcal{P}_\mathcal{Y}$ be the set of probability measures over $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$. We endow $\mathcal{P}_\mathcal{Y}$ with the usual weak topology, and let $\mathcal{B}(\mathcal{P}_\mathcal{Y})$ be the Borel $\sigma$-field of $\mathcal{P}_\mathcal{Y}$. For each $w \in \mathcal{W}$, we define $e_w(\cdot; G)$ as follows: for each $A \in \mathcal{B}(\mathcal{P}_\mathcal{Y})$,

$$e_w(A; G) = e(\{\sigma \in \Sigma : \sigma(\cdot \mid w, s(w)) \in A\}; G).$$

We are ready to introduce the invariance condition for the equilibrium selection rules.

**Assumption B.2** (Invariance of Equilibrium Selection Rules). For $w \in \mathcal{S}_W \cap \mathcal{S}_{f(W)}$ such that $\Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f)$, we have

$$e_w(A; G) = e_w(A; G_f),$$

for each $A \in \mathcal{B}(\mathcal{P}_\mathcal{Y})$.

The invariance condition in Assumption B.2 is used for the decomposition approach when a policy does not alter the information structure, whereas the invariance condition Assumption A.2 is used when the policy alters the information structure as well.

When the equilibrium selection rules are consistent in an appropriate sense, these invariance conditions are satisfied. More specifically, suppose that there exists a probability measure $\lambda$ on $(\Sigma, \mathcal{B}(\Sigma))$ such that, for each $w \in \mathcal{W}$ and for each $A \in \mathcal{B}(\mathcal{P}_\mathcal{Y})$,

(65) $$e_w(A; G) = \frac{\lambda_w(A \cap \Sigma_{BCE,w}(G))}{\lambda_w(\Sigma_{BCE,w}(G))},$$

where for each $B \in \mathcal{B}(\Sigma_\mathcal{Y})$,

$$\lambda_w(B) = \lambda(\{\sigma \in \Sigma : \sigma(\cdot \mid w, s(w)) \in B\}).$$

Then it is immediately seen that the invariance condition (Assumptions B.2) holds.

### B.2.2. The Main Result

The following theorem is a restatement of Theorem 2.1 in the main text.

**Theorem B.1.** Suppose that Assumptions B.1 and B.2 hold for the pre-policy game $G = (B, I)$ and the post-policy game $G_f = (B_f, I)$. Suppose further that maps $h : \mathcal{Y} \times \mathcal{W} \rightarrow \mathbb{R}$, $\underline{h}, \overline{h} : \mathcal{W} \rightarrow \mathbb{R}$ are given as in Theorem A.1.

Then,

$$\text{ADP}(h) + \Delta(\underline{h}) \leq \text{AEP}(h) \leq \text{ADP}(h) + \Delta(\overline{h}).$$

**Proof:** For the proof, we apply Theorem A.1. For this, we verify Assumptions A.1 and A.2. By Lemma B.4, for $w \in \mathcal{S}_W \cap \mathcal{S}_{f(W)}$, $\Sigma_{BCE,w}(G) = \Sigma_{BCE,w}(G_f)$. Since the information structures of games $G$ and $G_f$
are the same, they are trivially mutually individually sufficient. Therefore, by Corollary A.1, we find that Assumption A.1 is satisfied.

Let us turn to Assumption A.2. By Assumption B.1, we have
\[ \rho_\sigma(\cdot \mid w) = \sigma(\cdot \mid w, s(w)) \]
for all \( \sigma \in \Sigma \). Therefore, there is a one-to-one correspondence between \( \Sigma_{\text{BCE}}(G) \) and \( R_{\text{BCE}}(G) \). From (62) and (63), we find that Assumption B.2 implies Assumption A.2. The desired result follows from Theorem A.1, because \( \text{AEP}(h) = \text{AEP}^*(h) \), as \( G \) and \( G_f \) have the same information structure. ■

**Appendix C. Proofs of the Results in the Main Text**

In this section, we provide the proofs of the results in the main text.

**Proof of Theorem 2.1:** The theorem immediately follows from Theorem B.1. ■

**Proof of Proposition 2.1:** Let \( f : \mathbb{W} \to \mathbb{W}, \mu_W \) and \( h : \mathbb{Y} \times \mathbb{W} \to \mathbb{R} \) be given such that \( \overline{h}(w) = \sup_{y \in \mathbb{Y}} h(y, w) \) and \( \underline{h}(w) = \inf_{y \in \mathbb{Y}} h(y, w) \) exist in \( \mathbb{R} \). For each \( w \in \mathbb{W} \), we fix \( \varepsilon(w) > 0 \) and we take \( \overline{a}(w), \underline{a}(w) \in \mathbb{Y} \) such that
\[ \overline{h}(w) - \varepsilon(w) \leq h(\overline{a}(w), w) \leq \overline{h}(w), \quad \text{and} \]
\[ \underline{h}(w) \leq h(\underline{a}(w), w) \leq \underline{h}(w) + \varepsilon(w). \]

First, we show the sharpness of the upper bound. For each player \( i = 1, ..., n, a = (a_1, ..., a_n) \in \mathbb{Y} \), and \( w \in \mathbb{W} \), we set
\[ u_i(a, w) = \begin{cases} 1 \{a_i = \overline{a}_i(w)\}, & \text{if } w \in S_f(w), \text{ and} \\ 1 \{a_i = \underline{a}_i(w)\}, & \text{if } w \notin S_f(w), \end{cases} \]
where \( \overline{a}(w) = (\overline{a}_1(w), ..., \overline{a}_n(w)) \) and \( \underline{a}(w) = (\underline{a}_1(w), ..., \underline{a}_n(w)). \) Then, for all player \( i \), playing \( \overline{a}_i(w) \) is a strictly dominant strategy when \( w \in S_f(w) \), while \( \underline{a}_i(w) \) is strictly dominant when \( w \notin S_f(w) \). We take \( I \) to be the complete information structure by taking \( s_i \) to be the identity map for each \( i = 1, ..., n \). Let \( G = (B, I) \) and \( G_f = (B_f, I) \), where \( B = (\mathbb{Y}, \mathbb{W}, u, \mu_W) \) and \( B_f = (\mathbb{Y}, \mathbb{W}, u, \mu_W \circ f^{-1}) \). The game \( G_f \) has a unique BCE that is a unique pure-strategy Nash equilibrium such that
\[ \rho_{\sigma}(\{\overline{a}(w)\} \mid w) = 1 \{w \in S_f(w)\} \quad \text{and} \quad \rho_{\sigma}(\{\underline{a}(w)\} \mid w) = 1 \{w \notin S_f(w)\}. \]
The equilibrium described above is the same before and after the policy conditional on \( w \), and the equilibrium selection rule puts the probability mass of one to this equilibrium for all \( w \in S_f(w) \), as it is unique. Thus, we have
\[ \int_{\mathbb{W}} \int_{\mathbb{Y}} h(y, w) d \rho_{G_f}(y \mid w) d(\mu_W \circ f^{-1})(w) \]
\[ = \int_{S_f(w)} \int_{\mathbb{Y}} h(y, w) d \rho_{G_f}(y \mid w) d(\mu_W \circ f^{-1})(w) + \int_{\mathbb{W} \setminus S_f(w)} h(\overline{a}(w), w) d(\mu_W \circ f^{-1})(w), \]
because the integral domain is \( S_f(w) \times \mathbb{Y} \). Since \( G \) is a complete information game, and \( \Sigma_{\text{BCE},w}(G) = \Sigma_{\text{BCE},w}(G_f) \) which is a singleton for any \( w \in S_W \cap S_f(w) \), we have \( \rho_G(\cdot | w) = \rho_{G_w}(\cdot | w) \) for all \( w \in S_W \cap S_f(w) \). Hence,
\[
\int_{\mathbb{Y}} \int_{S_W} h(y,w) d\rho_G(y | w) d(\mu_w \circ f^{-1})(w) = \text{ADP}(h).
\]
Furthermore,
\[
\int_{\mathbb{Y}} \int_{W \setminus S_W} h(\overline{a}(w),w) d(\mu_w \circ f^{-1})(w) \leq \mathbb{E} \left[ \mathbb{E}(f(W))1\{f(W) \notin S_W\} \right] - \mathbb{E}[\epsilon(W)].
\]
Since the choice of \( \epsilon(\cdot) \) was arbitrary, we find that the upper bound is achieved.

As for the sharpness of the lower bound, we consider again a complete information game. We set
\[
u_i(a,w) = \begin{cases} 1\{a_i = \overline{a_i}(w)\}, & \text{if } w \in S_W \cap S_f(w), \text{ and} \\ 1\{a_i = a_i(w)\}, & \text{if } w \notin S_W \cap S_f(w).
\end{cases}
\]
With this \( u = (u_1, \ldots, u_n) \), we construct \( G \) and \( G_f \) as before. Then, as above, the game \( G_f \) has a unique BCE that is a unique pure-strategy Nash equilibrium, such that each player \( i \) playing \( \overline{a}_i(w) \) is a strictly dominant strategy when \( w \in S_W \cap S_f(w) \), while \( a_i(w) \) is strictly dominant when \( w \notin S_W \cap S_f(w) \). Hence,
\[
\rho_G(\{\overline{a}(w)\} | w) = 1\{w \in S_W \cap S_f(w)\}, \text{ and } \rho_G(\{a(w)\} | w) = 1\{w \notin S_W \cap S_f(w)\}.
\]

Note that
\[
\int_{\mathbb{Y}} \int_{W \setminus S_W} h(\overline{a}(w),w) d(\mu_w \circ f^{-1})(w) = \int_{S_W} \int_{\mathbb{Y}} h(y,w) d\rho_{G_i}(y | w) d(\mu_w \circ f^{-1})(w) + \int_{W \setminus S_W} h(\overline{a}(w),w) d(\mu_w \circ f^{-1})(w).
\]
Again, we have
\[
\int_{S_W} \int_{\mathbb{Y}} h(y,w) d\rho_{G_i}(y | w) d(\mu_w \circ f^{-1})(w) = \text{ADP}(h),
\]
and
\[
\int_{W \setminus S_W} h(a(w),w) d(\mu_w \circ f^{-1})(w) \leq \mathbb{E} \left[ \mathbb{E}(h(W))1\{f(W) \notin S_W\} \right] + \mathbb{E}[\epsilon(W)].
\]
Since the choice of \( \epsilon(\cdot) \) was arbitrary, the lower bound is achieved. \( \blacksquare \)

**Proof of Proposition 2.3:** Let \( S_{X_a} \) and \( S_{\epsilon} \) be the support of \( X_b \) and \( \epsilon \). For \((x_a, x_b, \epsilon) \in S_{X_a} \times S_{X_a} \times S_{\epsilon}\), we define
\[
m_h(x_a, x_b, \epsilon) = \mathbb{E}[h^*(Y, X) | (X_a, X_b, \epsilon) = (x_a, x_b, \epsilon)].
\]
Let \( \mu_{\epsilon | X_a}(\cdot | x_a) \) denote the conditional distribution of \( \epsilon \) given \( X_b = x_b \). We define similarly \( \mu_{\epsilon | X_a, X_b}(\cdot | x_a, x_b) \). Since \( X_a \) and \( \epsilon \) are conditionally independent given \( X_b \), for each \((x_a, x_b, \epsilon) \in S_{X_a} \times S_{X_b} \times S_{\epsilon}\), we have
\[
\mu_{\epsilon | X_a, X_b}(\cdot | x_a, x_b) = \mu_{\epsilon | X_a}(\cdot | x_a).
\]
such that \( g(x_a) \in S_{X_a} \), we use Proposition 4.36 of Breiman (1992), and note that

\[
\mathbb{E}_{}[h^*(Y,X) \mid (X_a,X_b) = (g(x_a),x_b)] = \int m_h(g(x_a), x_b, \varepsilon) d\mu_{\varepsilon \mid X_a,X_b} (\varepsilon \mid g(x_a), x_b)
\]

\[
= \int m_h(g(x_a), x_b, \varepsilon) d\mu_{\varepsilon \mid X_b} (\varepsilon \mid x_b).
\]

Now, we write

\[
\text{ADP}(h) = \int m_h(g(x_a), x_b, \varepsilon) 1 \{ g(x_a) \in S_{X_a} \} d\mu_{X_a,X_b,\varepsilon}(x_a, x_b, \varepsilon)
\]

\[
= \int \int m_h(g(x_a), x_b, \varepsilon) d\mu_{\varepsilon \mid X_b} (\varepsilon \mid x_b) 1 \{ g(x_a) \in S_{X_a} \} d\mu_{X_a,X_b}(x_a, x_b),
\]

where \( \mu_{\varepsilon \mid X_b} \) denotes the conditional distribution of \( \varepsilon \) given \( X_b \), and \( \mu_{X_a,X_b,\varepsilon} \) is defined similarly. By (66), the last double integral is equal to

\[
\int \mathbb{E}_{}[h^*(Y,X) \mid (X_a,X_b) = (g(x_a),x_b)] 1 \{ g(x_a) \in S_{X_a} \} d\mu_{X_a,X_b}(x_a, x_b),
\]

delivering the desired identification result. ■

**Proof of Corollary 2.2:** If we take \( h(y,w) = h^*(y,x) \), where \( h^*(y,x) = 1 \{ y = a \} 1 \{ x \in C \} \). Then, we can write

\[
\text{AEP}(h) = (\mu_w \circ f^{-1})(C) \times p_f(Y = a \mid C).
\]

Also, note that by (22) (using independence between \( X \) and \( \varepsilon \)),

\[
\text{ADP}(h) = \int \mathbb{E}_{}[h^*(Y,X) \mid X = f^*(x)] 1 \{ f^*(x) \in S_X \} d\mu_X(x)
\]

\[
= \int P \{ Y = a \mid X = f^*(x) \} 1 \{ f^*(x) \in C \cap S_X \} d\mu_X(x),
\]

where the last integral is equal to

\[
\mathbb{E}_{}[\pi(a \mid f^*(X)) \mid f^*(X) \in C] P \{ f^*(X) \in C \}.
\]

(Here \( \mu_w \) denotes the distribution of \( X \) under \( \mu_w \).) Also observe that \( \overline{h^*}(x) = 1 \{ x \in C \} \) and \( \underline{h^*}(x) = 0 \).

Hence, we have

\[
\Delta(\overline{h}) = P \{ f^*(X) \notin S_X, f^*(X) \in C \}, \text{ and } \Delta(\underline{h}) = 0.
\]

It is not hard to see that the conditions of Theorem 2.1 are met by the assumptions of this corollary, and hence by the theorem, we obtain that

\[
\text{ADP}(h) \leq \text{AEP}(h) \leq \text{ADP}(h) + \Delta(\overline{h}).
\]

By dividing both sides of the equation by \( P \{ f^*(X) \in C \} \), we obtain the desired result. ■
Proof of Corollary 2.3: The proof comes from applying Theorem 2.1. To map this setting to the general setting of the theorem, we set \( Y = (B_1, ..., B_n) \) and \( W = (X_1, R, V, \eta) \), identify policy \( f(W) \) as \( f^*(R) \), and take

\[
h(Y, W) = 1 \left\{ \max_{i: I(V, X, \eta) = 1} B_i \in A, X_1 \in C \right\} \\
= 1 \left\{ \max_{i: I(V, X_1, R, \eta) = 1} s_i^*(V, X_1, R, \eta) \in A, X_1 \in C \right\}.
\]

Note that \( h(Y, W) \) is measurable with respect to the \( \sigma \)-field generated by \( W \). Then, we have

\[
\mathbb{E}[h(Y, W) \mid W = f(w)] = 1 \left\{ \max_{1 \leq i \leq n: I(v, x_1, f^*(r), \eta) = 1} s_i^*(v, x_1, f^*(r), \eta) \in A, X_1 \in C \right\},
\]

whenever \( f(w) \in S_W \). Hence we can write ADP(h) as

\[
\int C \int \int \int 1 \left\{ i: I(v, x_1, f^*(r), \eta) = 1 \right\} s_i^*(v, x_1, f^*(r), \eta) \in A, x_1 \in C, f^*(r) \in S_R \} dF(v, \eta, x_1, r)
\]

\[
= \int C \int \int \int 1 \left\{ i: I(v, x_1, f^*(r), \eta) = 1 \right\} s_i^*(v, x_1, f^*(r), \eta) \in A, f^*(r) \in S_R \} dF(v, \eta \mid x_1) dF(r \mid x_1) dF(x_1)
\]

\[
= \int C \int \int \int 1 \left\{ i: I(v, x_1, r, \eta) = 1 \right\} s_i^*(v, x_1, r, \eta) \in A, r \in S_R \} dF(v, \eta \mid x_1) d(F \circ f^{*-1})(r \mid x_1) dF(x_1),
\]

where we use the same notation \( F \) to denote the distribution of \( W \) and conditional distributions under it for simplicity. The first equality follows from the conditional independence of \( (V, \eta) \) from \( R \) given \( X_1 \), and the second equality from change of variables. The last term is written as

\[
\int C \int_{S_R} P \left\{ \max_{i \in N} B_i \in A \mid (X_1, R) = (x_1, r) \right\} d(F \circ f^{*-1})(r \mid x_1) dF(x_1)
\]

\[
= \int C \int_{S_R} P(A \mid x_1) d(F \circ f^{*-1})(r \mid x_1) dF(x_1) = \mathbb{E}[p(A \mid f^*(R), X_1) 1\{f^*(R) \in S_R, X_1 \in C}] .
\]

Observe that \( \mathbf{h}^*(x) = 1 \{ x_1 \in C \} \) and \( h^*(x) = 0 \), and that

\[
\Delta(\mathbf{h}) = P \{ f^*(R) \notin S_R, X_1 \in C \} \text{ and } \Delta(h) = 0.
\]

Thus, we obtain the desired result by applying Theorem 2.1 and dividing the terms in the inequalities by \( P \{ X_1 \in C \} \). □

Appendix D. Implementation of Decomposition-Based Predictions

In this section, we explain the implementation details on estimating the ADP defined in (2.24). Throughout this section, we assume that \( X \) and \( \varepsilon \) are independent. As in Section 2.4.3, we consider the policy of the following form: \( f(W) = (f^*(X), \varepsilon) \) for some map \( f^* \). If we define

\[
\mu(x) = \mathbb{E}[Y \mid X = x],
\]

(68)
the ADP is identified as follows:

\[(69) \quad \text{ADP} = \mathbb{E}[\mu(f^*(X))1\{f^*(X) \in S_X\}],\]

where we recall that \(S_X\) denotes the support of \(X\) (in the data from the pre-policy game). Our explanation divides into two settings: the setting with a fully nonparametric method and the setting that makes use of a multi-index structure.

We introduce a sample unit notation, \(m\), (mnemonic for ‘market’), so we assume we observe i.i.d. draws \((Y_m,X_m), m = 1,\ldots,M\), from the joint distribution of \((Y,X)\) in an equilibrium of the Bayesian game. Hence the sample size is \(M\) which we assume to go to infinity in asymptotic inference while the number of players \(n\) is fixed. For each market \(m = 1,\ldots,M\), we have \(Y_m = (Y_{1,m},\ldots,Y_{n,m})\), the profile of actions in the \(m\)-th market, where \(Y_{i,m}\) represents the observed action of player \(i\) in market \(m\). Similarly we have \(X_m = (X_{1,m},\ldots,X_{n,m})\), where \(X_{i,m}\) denotes the \(d\)-dimensional vector of observed payoff characteristics for player \(i\) in market \(m\).

### D.1. Implementation Using a Nonparametric Decomposition Method

#### D.1.1. Implementation. **Decomposition-based predictions** can be generated using various nonparametric estimation methods. For illustration, we explain a way to construct such predictions using a kernel estimation method. Suppose that we observe \(M\) markets which we view as i.i.d. draws from an \(n\)-player representative game. For each player \(i\), let \(X_{i,m} = [X_{i,m,1},X_{i,m,2}]'\), where \(X_{i,m,1}\) takes values from a finite subset of \(\mathbb{R}^{d_1}\) and \(X_{i,m,2} \in \mathbb{R}^{d_2}\), \(d = d_1 + d_2\), is a vector of continuous variables in market \(m = 1,\ldots,M\). We also assume that \(Y_i \geq 0\) for all \(i = 1,\ldots,n\). In this environment, let us focus on estimating ADP defined in (69). We first construct a leave-one-out kernel estimator of the \(i\)-th component \(\mu_i(x) = \mathbb{E}[Y_i | X = x]\) of \(\mu(x) = \mathbb{E}[Y | X = x]\) for each \(m = 1,\ldots,M\) as

\[(70) \quad \hat{\mu}_{i,m,b}(x) = \frac{\sum_{k=1,k\neq m}^M Y_{i,k}1\{X_{k,1} = x_1\}K_b(X_{k,2} - x_2)}{\sum_{k=1,k\neq m}^M 1\{X_{k,1} = x_1\}K_b(X_{k,2} - x_2)}, \quad x = (x_1, x_2),\]

where \(Y_{i,k}\) is the \(i\)-th component of \(Y_k\), \(X_{k,1} = [X_{1,k,1}',\ldots,X_{n,k,1}']', X_{k,2} = [X_{1,k,2}',\ldots,X_{n,k,2}']'\) and \(K_b(x_2) = K(x_{2,1}/b,\ldots,x_{2,d}/b)/b^d, x_2 = [x_{2,1}',\ldots,x_{2,d}'],\) and \(K\) is a univariate kernel and \(b\) is a bandwidth. For an estimator of ADP, we consider the following:

\[(71) \quad \hat{\text{ADP}} = \frac{1}{M} \sum_{m=1}^M \hat{\mu}_{m,b}(f^*(X_m))1\{f^*(X_m) \in S_X\},\]

where \(\hat{\mu}_{m,b}(x) = [\hat{\mu}_{1,m,b}(x),\ldots,\hat{\mu}_{n,m,b}(x)]'\). The estimated interval of the average policy effect on player \(i\)’s action is given by

\[\left[\hat{\text{ADP}}_i - \bar{Y}_i, \quad \hat{\text{ADP}}_i - \bar{Y}_i + \sup Y_i \cdot \Delta\right],\]
where \( \widehat{\text{ADP}}_i \) is the \( i \)-th component of \( \widehat{\text{ADP}} \),
\[
\Delta = \frac{1}{M} \sum_{m=1}^{M} 1\{f^*(X_m) \notin S_X\}, \quad \text{and} \quad \overline{Y}_i = \frac{1}{M} \sum_{m=1}^{M} Y_{i,m}.
\]
For bandwidths, we use cross-validation for each player \( i = 1, \ldots, n \). Define
\[
(72) \quad CV_i(b) = \frac{1}{M} \sum_{m=1}^{M} (Y_{i,m} - \hat{\mu}_{i,m,b}(X_m))^2,
\]
where \( \hat{\mu}_{i,m,b}(x) \) denotes the \( i \)-th component of \( \hat{\mu}_{m,b}(x) \). We choose a minimizer of \( CV_i(b) \) over a range of \( b \)'s as our bandwidth.

When the support of \( X_m \) contains that of \( f^*(X_m) \), we can simply estimate the average treatment effect as
\[
(73) \quad \widehat{\text{ATE}}_i \equiv \widehat{\text{ADP}}_i - \overline{Y}_i.
\]

For the bootstrap standard errors for (73), we first draw \( (Y_{m}^*, X_m^*)_{m=1}^{M} \), i.i.d. from the empirical distribution of \( (Y_m, X_m)_{m=1}^{M} \). Using the bootstrap sample, we obtain \( \hat{\mu}_{m,b}(x) \) (where \( b \) is the bandwidth chosen from the cross-validation). Define
\[
(74) \quad \overline{\text{ATE}}_i^* = \frac{1}{M} \sum_{m=1}^{M} \hat{\mu}_{i,m,b}^*(f^*(X_m^*)) - \overline{Y}_i^*\text{,}
\]
where \( \overline{Y}_i^* \) is the sample mean of \( \{Y_{i,m}^*\}_{m=1}^{M} \), and \( Y_{i,m}^* \) denotes the \( i \)-th component of \( Y_m^* \). In order to construct a bootstrap standard error, we can adopt the bootstrap interquartile range method used in Chernozhukov, Fernández-Val, and Melly (2013). First, let \( q_{0.75} \) and \( q_{0.25} \) be the 0.75 and 0.25 quantiles of the bootstrap distribution of \( \sqrt{M}(\overline{\text{ATE}}_i - \overline{\text{ATE}}_i^*) \), and let \( z_{0.75} \) and \( z_{0.25} \) be the 0.75 and 0.25 quantiles of the standard normal distribution. Then, the bootstrap standard error is taken to be \( M^{-1/2}(q_{0.75} - q_{0.25})/(z_{0.75} - z_{0.25}) \).

D.1.2. (Near) Multicollinearity for Non-Parametric Counterfactual Predictions. For our empirical application, we reduced the dimension of \( X_m \) due to near multicollinearity in the counterfactual environment. We now expand on this point, showing why multicollinearity in counterfactuals may be a concern even if it is absent in-sample. For simplicity of exposition, we omit the subscript \( m \).

Let us split \( X = (X_1, X_2) \), so we write:
\[
\mathbb{E}[Y] = \int \mu(x_1, x_2) dF_{X_1,X_2}(x_1, x_2).
\]
This quantity is certainly identified even if \( X_1 \) and \( X_2 \) are perfectly correlated. This is because, if that were the case, we would be integrating over the values of \( X_1 \) and \( X_2 \) such that \( X_1 = X_2 \).

However, this is different in a counterfactual setting. If the counterfactual setting is such that \( \mu(x_1, x_2) \) have \( x_1 \) and \( x_2 \) varying independently, we cannot identify \( \mu(x_1, x_2) \) for independent variations of \( x_1 \) and \( x_2 \). As such, the decomposition method may not work. Ultimately this consideration
can be subsumed into the setting where the support of the exogenous variables go outside the support after the policy. Below, we outline our choices for our empirical applications.

D.2. Implementation Using a Multi-Index Structure

Let us explain how we can obtain an estimator of $ADP(h)$ using a multi-index structure in Section 2.4.3. Here we assume a setting that leads to the identification result in (24), and make explicit the dependence of random variables on the market index $m$. More specifically, we write $Y_{i,m}, X_{i,m,a}, X_{i,m,b}$ and $V_{i,m}$ in place of $Y_i, X_{i,a}, X_{i,b}$, and $V_i$. We also write $Y_m = (Y_{1,m}, ..., Y_{n,m}), X_{m,a} = (X_{1,m,a}, ..., X_{n,m,a}), X_{m,b} = (X_{1,m,b}, ..., X_{n,m,b})$ and $V_m = (V_{1,m}, ..., V_{n,m})$, in place of $Y = (Y_1, ..., Y_n), X_a = (X_{1,a}, ..., X_{n,a}), X_b = (X_{1,b}, ..., X_{n,b})$ and $V = (V_1, ..., V_n)$.

For each player $i = 1, ..., n$, and market $m = 1, ..., M$, define

$$m_i(h; v, x_a) = E[h(Y_{i,m}, X_{i,m}) \mid X_{m,a} = x_a, V_m = v].$$

Then, we can write

$$ADP_i(h) = E[m_i(h; V_m, g(X_{m,a})) \mathbb{1}\{g(X_{m,a}) \in S_{X_a}\}].$$

Now, let us discuss estimation of $m_i$ and $ADP_i(h)$. For this, we first estimate coefficients $\theta_i$ in

$$V_{i,m} = X'_{i,m,b} \theta_i,$$

using methods in the literature of multi-index models. Here we assume that $V_m$ is a continuous random vector, and adopt the proposal by Ahn, Ichimura, Powell, and Ruud (2018) which is based on the pairwise differencing approach of Ahn and Powell (1993). First, we define

$$\gamma_{i,m} = m_i(h; X'_{1,m,b} \theta_1, ..., X'_{n,m,b} \theta_n, X_{m,a}), \quad i = 1, ..., n.$$

The approach of Ahn, Ichimura, Powell, and Ruud (2018) invokes the invertibility of this system of equations so that it is assumed that there exist functions $\varphi_i, i = 1, ..., n$, such that

$$X'_{i,m,b} \theta_i = \varphi_i(\gamma_{1,m}, ..., \gamma_{n,m}; X_{m,a}), \quad i = 1, ..., n.$$

Furthermore, we assume that $X_{i,m,b} = [X_{i,m,b,A}, X'_{i,m,b,b}]'$, and the coefficient of the first random variable $X_{i,m,b,A}$ is not zero, and normalized to be one. By reparametrizing $\theta_i = [1, -\beta_i]'$, we can rewrite the previous equation as

$$X_{i,m,b} = X'_{i,m,b,b} \beta_i + \varphi_i(\gamma_{m}; X_{m,a}),$$

where $\gamma_m = (\gamma_{1,m}, ..., \gamma_{n,m})$. Hence when $\gamma_m \approx \gamma_k$, we have

$$X_{i,m,b,A} - X_{i,k,b,A} \approx (X_{i,m,b,B} - X_{i,k,b,B})' \beta_i.$$
This observation suggests the following estimator of $\beta_i$:\footnote{We could use an optimal weighting matrix here to improve efficiency as explained in \textit{Ahn, Ichimura, Powell, and Ruud (2018)}. Here we choose a simple form for the estimator for the sake of expositional simplicity.}

\[
\hat{\beta}_i = \left( \sum_{m=1}^{M} \sum_{k=1,k\neq m}^{M} (X_{i,m,b,B} - X_{i,k,b,B})(X_{i,m,b,A} - X_{i,k,b,A}) \hat{\Gamma}_{m,k} \right)^{-1} \\
\times \left( \sum_{m=1}^{M} \sum_{k=1,k\neq m}^{M} (X_{i,m,b,B} - X_{i,k,b,B})(X_{i,m,b,A} - X_{i,k,b,A}) \hat{\Gamma}_{m,k} \right),
\]

where $\hat{\Gamma}_m = [\hat{\gamma}_{1,m}, ..., \hat{\gamma}_{n,m}]'$,

\[
\hat{\Gamma}_{m,k} = K \left( \frac{\hat{\gamma}_m - \hat{\gamma}_k}{b} \right), \quad \text{and} \quad \hat{\gamma}_{i,m} = \sum_{k=1,k\neq m}^{M} L \left( \frac{X_{m,b,B} - X_{k,b,B}}{b}, \frac{X_{m,a,A} - X_{k,a,A}}{b} \right) h(Y_{i,k}, X_{k,a})
\]

and $K$ is an $n$-variate kernel and $b$ is a bandwidth, $X_{m,b,B} = [\tilde{X}_{1,m,b,B}', ..., \tilde{X}_{n,m,b,B}]'$ is the $nd_2$-dimensional vector and $L$ is an $nd$-variate kernel, where $d = d_1 + d_2$ and $d_1$ denotes the dimension of $X_{m,a}$ and $d_2$ denotes the dimension of $X_{m,b}$.ootnote{In constructing $\hat{\gamma}_{i,m}$ using a kernel regression estimator, we are assuming $X_m$ is a continuous random vector. When $X_m$ contains a discrete subvector, we can obtain $\hat{\gamma}_{i,m}$ similarly as in (70).} Then, for each $i = 1, ..., n$, we set $\hat{\theta}_i = [1, -\hat{\beta}_i]'$ and construct

\[
\hat{m}_i(h; v, x_1) = \frac{\sum_{k=1,k\neq m}^{M} H \left( \frac{\hat{v}_k - v}{b}, \frac{X_{k,a} - x_1}{b} \right) h(Y_{i,k}, X_{k,a})}{\sum_{k=1,k\neq m}^{M} H \left( \frac{\hat{v}_k - v}{b}, \frac{X_{k,a} - x_1}{b} \right)}, \quad r = [r_1, ..., r_n]',
\]

where $H$ is an $(n + 1)$-variate kernel, and $\hat{v}_k = [X_{1,k,b}', \hat{\theta}_1, ..., X_{n,k,b}', \hat{\theta}_n]'$. Using $\hat{\theta}_i$ and $\hat{m}_i$, we can obtain $\hat{\text{ADP}}_i(h)$ as

\[
\hat{\text{ADP}}_i(h) = \frac{1}{M} \sum_{m=1}^{M} \hat{m}_i(h; \hat{v}_m; g(X_{m,a})) \mathbb{1} \{ g(X_{m,a}) \in S_{x_1} \}.
\]

When the support of $X_{m,a}$ contains that of $g(X_{m,a})$, we can estimate the average policy effect as in (73). The bootstrap standard error for this effect can be constructed similarly as before, constructing $\hat{\text{ADP}}_i(h)$ using the bootstrap sample.

### D.3. Details on the Empirical Application

In this empirical application, each game $m$ corresponds to a market $m$ defined by a route (airport 'A' to airport 'B'). Regarding the additional data collection for the out-of-sample comparison, we used the same data sources to collect 2015 data as the authors did for their original (pre-2009) dataset.
particular, we collected all 2015 trimester data from the DB1B Market and Ticket Origin and Destination Dataset of the U.S. Department of Transportation and then aggregated them to the yearly level. We also followed the same data cleaning choices: most notably, we set entry equal to 0 for markets with less than 20 passengers, and dropped any markets with 6 or more market coupons - see Section S.1 in the Supplemental Note of Ciliberto and Tamer (2009). Then, we merged the 2015 data to the same 2,742 markets considered in their work. We note that this data is only used for the out-of-sample comparison, as our policy estimates use the same pre-repeal data as the authors.

The choice of covariates in our application follows Ciliberto and Tamer (2009) pages 1808-1811. Recall that, for firm $i$, their covariates $X_{i,m}$ are given by (1) $i$’s market presence in $m$ (the share of markets served by airline $i$ among all of those starting from $m$’s endpoints), (2) $m$’s “cost” (approximated by the difference between the origin/destination locations to the closest hub of firm $i$ relative to the nonstop distance of this route), (3) whether market $m$ is affected by the Wright Amendment, (4) Dallas market (whether this route includes an airport in Dallas), (5) market size (the geometric mean of population size at the route’s endpoints), (6) average per capita income (average income of the cities at market endpoints) (7) income growth rate (similarly defined), (8) market distance (nonstop distance between the airports defining this market), (9) distance to the center of the U.S. (sum of the distances from each endpoint of the route to the geographical center of the U.S.), and (10) close airport (minimum of the distances from each airport to the closest alternative a passenger could use). Covariates (1)-(2) vary at the firm-market level, while covariates (3)-(10) vary only at the market level.

Our interest is in changing the distribution of the Wright Amendment variable, keeping the others constant. The role of the remaining variables is to control for a variety of confounders (costs, returns from serving those markets, outside options to consumers, geographical characteristics).

Since the conditional expectation in (68) must include all $X_{j,m}$ for every $j$, we are left with possibly 8 (market specific) plus $2 \times 4$ (firm level) covariates to be included in the analysis. This is a large number for nonparametric estimation, although our linear results are robust to using the full set. As described in the main text, we perform a dimension reduction to facilitate nonparametric estimation. First, we note that 3 of those 16 variables (the binary Dallas market, market size, and income growth rate - numbered (4), (5), (7) above) were highly collinear or did not provide enough variation for nonparametric estimation (in the sense explained in Appendix D.1.2 above). We dropped such variables from our analysis. To see this, first note that every market affected by the Wright Amendment was already in Dallas. There are only 110 observations (out of 2,742) with different values for the Dallas market variable and the Wright amendment one. However, these 110 observations refer to markets using Dallas Fort Worth Airport, the hub for American Airlines. As a result, this variation is already captured by measures of market income and market presence. Regarding market size and income growth rate, they appear to be nonlinearly predicted from market income and market presence.\footnote{Including one of these three variables generated numerical instability in nonparametric estimation. See Section D.1.2 for further discussion.} Intuitively, the latter variables already (nonlinearly) reflect economic prosperity of those markets. Finally, we make use of a causal structure argument: we do not include the proxy for a firm’s cost (covariate (2) above) because it is a
function of market distance and market presence by construction. As a result, we use the remaining 9 covariates in the specifications.

Let us now describe the implementation of our estimators. Denote the set of markets affected by the Wright amendment in the original dataset by $\mathcal{M}$, which has cardinality $|\mathcal{M}| = 81$. Our counterfactual covariates are denoted by $f^*(X_m)$. They are equal to $X_m$ for all firms and markets, except for the Wright amendment variable, which is set to 0 for all markets $m \in \mathcal{M}$ and for all firms.

For the linear estimator, we simply parametrize $E[Y_{i,m}|X_m = x] = x'\gamma_i$ and estimate this model on the full dataset using Ordinary Least Squares. We then use the recovered estimate $\hat{\gamma}_i$ with the counterfactual covariates, $f^*(X_m)$. Our linear estimator of ATE$_i$ for firm $i$ is then given by:

$$ (76) \quad \overline{\text{ATE}}_i = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} f^*(X_m)\hat{\gamma}_i - \overline{Y}_i, $$

where $\overline{Y}_i$ is the average of $Y_{i,m}$’s over the markets in $\mathcal{M}$.

For the nonparametric case, we follow the previous section in estimating $E[Y_{i,m} | X_m = x] = g(x)$ and use a quartic kernel for continuous covariates. Standard errors for both estimators are computed by the bootstrap approach outlined in the previous section.

### D.4. Additional Empirical Results

#### D.4.1. The “Other Term” in Aggregate Decomposition: The Role of Market (Observable) Characteristics

Table 1 presents the results for the (average) effect of the repeal of the Wright amendment on the set of markets affected by the policy. However, this is only one of two components in the typical aggregate decomposition (e.g., Oaxaca-Blinder - see Fortin, Lemieux, and Firpo (2011) for an overview). The other term in such exercises is the “observable effect”: i.e., how the characteristics of markets affected and not affected by the repeal differ.

To see this, let us write the payoff state as $X_m = (X_m^{\text{Wright}}, \tilde{X}_m)$, which makes explicit which variables are related to the policy ($X_m^{\text{Wright}}$) and which other states are firm-market specific ($\tilde{X}_m$). Let $\tilde{X}_m^1$ and $\tilde{X}_m^0$ denote the (observable) payoff states for firm-markets that have $X_m^{\text{Wright}} = 1$ or 0, respectively. Then, the change in the (average) probability of entry by firms not subject to the Wright amendment, to that by those subject to the policy (i.e., the effect of a repeal of the amendment) can be written as:\(^\text{29}\)

$$ \begin{align*}
\mathbb{E}[Y_{i,m}|\tilde{X}_m^0, X_m^{\text{Wright}} = 0] - \mathbb{E}[Y_{i,m}|\tilde{X}_m^1, X_m^{\text{Wright}} = 1] \\
= \mathbb{E}[Y_{i,m}|\tilde{X}_m^0, X_m^{\text{Wright}} = 0] - \mathbb{E}[Y_{i,m}|\tilde{X}_m^1, X_m^{\text{Wright}} = 0] + \mathbb{E}[Y_{i,m}|\tilde{X}_m^1, X_m^{\text{Wright}} = 0] - \mathbb{E}[Y_{i,m}|\tilde{X}_m^1, X_m^{\text{Wright}} = 1].
\end{align*} $$

The second term on the right-hand side is the effect of the policy (repeal of the amendment), keeping fixed the distribution of observable characteristics for firms/markets subject to the Wright amendment. Its estimates are shown in Table 1. However, the other term shows the average difference in entry

\(^{29}\)We note that this has a flipped sign relative to standard aggregate decompositions because we study the repeal of the Wright amendment, rather than imposing such a policy. Hence, the comparison is between markets not affected to those affected, rather than the other way round.
TABLE 3. Ciliberto and Tamer (2009) Revisited: The Effects from Differences in Market Characteristics

| Outcome: Change in Probability of Entry in Dallas-Love Markets |
|--------------------------------------------------------------|
|                  | Linear Model | Nonparametric Model |
| American Airlines | 0.431        | 0.272               |
|                   | (0.037)      | (0.027)             |
| Delta Airlines    | 0.565        | 0.369               |
|                   | (0.039)      | (0.022)             |
| Southwest Airlines| -0.406       | -0.350              |
|                   | (0.029)      | (0.051)             |
| United Airlines   | 0.292        | 0.241               |
|                   | (0.033)      | (0.009)             |

Notes: We report the estimated counterfactual changes to the entry of major airlines into Dallas Love Field markets following the repeal of the Wright Amendment, only due to observable (non-policy) characteristics. The specification follows Table 1, while estimation/inference is described above. Column 1 uses a linear model, while Column 2 reports a nonparametric estimate. Standard errors for these columns are computed by the bootstrap, following the approach in the Online Appendix with $B = 999$ replications.

across markets that are not affected/affected by the amendment, which are due solely to differences in observable characteristics (as the policy variable does not change).

It is straightforward to estimate the new term. First, we note that it is, itself, composed by two separate terms. The first is the mean entry for firms in markets that are not affected by the Wright amendment. It can be estimated using the empirical mean for such markets. The second is the counterfactual entry for markets subject to the Wright amendment, if the policy were to be repealed. This has already been estimated for Table 1, as it is a component of the policy effect. Hence, differences between the “linear” and “nonparametric” estimates in the observable effect are solely due to differences in the latter, already present in Table 1. The results for this second term of an aggregate decomposition for the Wright Amendment application is shown in Table 3, while the results for the application of Southwest Airlines’ Threat of Entry is shown in Table 4 below.

The effects in Table 3 are much larger for American, Delta and United relative to Table 1. For Southwest, the effect is 20% lower than those in Table 1. This can be explained by the characteristics of markets subject to the Wright amendment. Such markets, including Dallas Love Airport, are strictly in the South, often smaller than elsewhere, and with lower market presence of the major airlines. Hence, it would be natural that airlines are less likely to enter such markets, even if the Wright amendment was not present/repealed. Hence, a naive comparison of entry across different markets would overstate the effects of the policy, with effects of over 50% for most airlines (found by aggregating the estimates in
| Outcome: Change in Probability of Entry After Removal of Southwest Threat |
|---------------------------------------------------------------|
| **Linear Model** | **Decomposition Method** | **Nonparametric Model** |
| American Airlines | -0.043 | -0.048 |
|                    | (0.024) | (0.024) |
| Delta Airlines     | 0.115  | 0.115  |
|                    | (0.024) | (0.024) |
| United Airlines    | -0.000 | -0.009 |
|                    | (0.022) | (0.020) |

Notes: We report the estimated counterfactual changes to the entry of major airlines when Southwest Airlines no longer threatens entry into such markets. The effect presented here is due only to observable (non-policy) characteristics, complementing the policy results from Table 2. The specification follows the one for Table 2, while estimation/inference is described above. Column 1 uses a linear model, while Column 2 reports a nonparametric estimate. Standard errors for these columns are computed by the bootstrap, following the approach in the Online Appendix with $B = 999$ replications.

Tables 1 and 3). However, we estimate that the repeal itself would generate entry into affected markets as seen in Table 1.

D.4.2. **Additional Results Referenced in the Main Text.** Table 5 provides an additional set of results referenced in Section 3.2 in the main text. It reports the effects from decreasing the number of competitors threatening entry on each airline’s entry decision. There are three such exercises, reflecting the heterogeneous effects depending on the number of airlines currently threatening entry. This complements the main results presented in Table 1.
TABLE 5. How Decreasing the Number of Competitors Threatening Entry Affects Airlines’ Behavior

Policy: Decreasing Number of Competitors Threatening Entry by 1

| Policy: Decreasing Number of Competitors Threatening Entry by 1 | Decrease from 1 to 0 | Decrease from 2 to 1 | Decrease from 3 to 2 |
|---------------------------------------------------------------|----------------------|----------------------|----------------------|
|                                                               | Linear   | Nonparametric       | Linear   | Nonparametric       | Linear   | Nonparametric       |
| American Airlines                                            | -0.026   | 0.071               | -0.025   | -0.248              | -0.063   | -0.094              |
|                                                               | (0.016)  | (0.042)             | (0.015)  | (0.028)             | (0.015)  | (0.022)             |
| Delta Airlines                                               | -0.021   | 0.153               | -0.020   | 0.088               | -0.041   | -0.129              |
|                                                               | (0.015)  | (0.026)             | (0.012)  | (0.028)             | (0.013)  | (0.023)             |
| Southwest Airlines                                           | 0.002    | -0.029              | 0.041    | 0.020               | 0.005    | -0.039              |
|                                                               | (0.008)  | (0.038)             | (0.009)  | (0.024)             | (0.006)  | (0.013)             |
| United Airlines                                              | 0.012    | -0.024              | 0.007    | -0.068              | -0.016   | 0.010               |
|                                                               | (0.017)  | (0.019)             | (0.012)  | (0.035)             | (0.014)  | (0.018)             |

Notes: We report the estimated counterfactual changes to the entry of major airlines after the number of competitors threatening entry is decreased by one. There are three such exercises: reducing the number of potential entrants in markets that had three such competitors to two; those that had two to one; and those that had one potential entrant to having none. In doing so, we note that the results across exercises are not directly comparable, as they are based on different markets. Following Table 1, we present results for linear and nonparametric specifications. Standard errors for these columns are computed by the bootstrap, following the approach in the Online Appendix with $B = 999$ replications.
Appendix E. An Example of a Game that Fails the Decomposition-Based Approach

Our main result that shows the validity of the decomposition-based prediction for counterfactual predictions requires that the policy changes only the publicly observable part of the payoff component. Let us give an example where the decomposition-based prediction fails when this requirement is not met. Our example is based on the entry game of Section 2.6.1 with two players, \( i = 1,2 \), but with \( W = (W_1, \ldots, W_n) \), where \( T_i = W_i \), i.e., each payoff type \( W_i \) is private information. We assume that the payoffs are given by: with \( i = 1, 2 \),

\[
 u_i(y, W_i) = y_i(\delta y_{-i} + W_i),
\]

where \( W_i \) is standard normally distributed, \( W_1 \) and \( W_2 \) are independent, and \( \delta < 0 \). The counterfactual policy changes the payoff state \( W_i \) to \( f(W_i) = \alpha + W_i \) for \( \alpha > 0 \). Thus, this policy violates Assumption 2.1 in the main text, because the policy changes the private information.

We consider a symmetric threshold Bayes-Nash equilibrium of the game, characterized by \( i \)'s strategy of entering the market if \( W_i > w \), where the threshold \( w \) is such that firm \( i \) is indifferent between entering and not entering the market. For simplicity, we will consider an equilibrium selection rule that always chooses this BNE in the pre-policy game or its analogue in the counterfactual. This means that the lack of validity of the decomposition-based prediction in this example is not driven by the equilibrium selection rule.

E.1. Characterization of the Threshold \( \bar{w} \)

The equilibrium described above is defined by firm \( i \) playing “Enter” if \( W_i > \bar{w} \) and “Not Enter” if and only if \( W_i \leq \bar{w} \), where the (symmetric) threshold \( \bar{w} \) was such that firm \( i \) is indifferent between entering and not entering the market. This is the BNE counterpart to the BCE in Magnolfi and Roncoroni (2022) (i.e. where a mediator suggests this strategy). However, we cannot use the same threshold \( \bar{w} \) from their example because we assume that \( W_i \) follows a standard normal distribution instead of a uniform distribution.\(^{30}\) To make sure this equilibrium is well defined, we now show that \( \bar{w} \) in our setting exists and that it is unique.

The thresholds \( \{\bar{w}_i\}_{i=1,2} \) that equalize the expected utility from entering the market to that from not entering for firm \( i \) are the solutions to:

\[
\bar{w}_1 + \delta(1 - \Phi(\bar{w}_1)) = 0, \quad \text{and} \quad \bar{w}_2 + \delta(1 - \Phi(\bar{w}_2)) = 0,
\]

where \( \Phi \) is the CDF of \( N(0, 1) \). As we seek a symmetric equilibrium, let us guess for a symmetric threshold \( \bar{w} = \bar{w}_1 = \bar{w}_2 \) such that \( \bar{w} + \delta(1 - \Phi(\bar{w})) = 0 \). The function \( h(w) = w + \delta(1 - \Phi(w)) \) is continuous in \( w \), negative for \( w < 0 \) (since \( \delta < 0 \)) and positive for \( w > -\delta \). A solution \( \bar{w} \) such that \( h(\bar{w}) = 0 \) exists by the Intermediate Value Theorem. This threshold is unique because \( h(\cdot) \) is strictly

\(^{30}\)We use a normal distribution assumption because it allows for our counterfactual \( f(W_i) = \alpha + W_i \) to have the same support as \( W_i \), and to also follow a normal distribution.
increasing, since \( h'(w) = 1 - \delta \phi(w) > 0 \), as \( \delta < 0 \) and \( \phi(\cdot) > 0 \), where \( \phi \) is the density function of \( N(0, 1) \).

E.2. Failure of Incentive Compatibility in the Counterfactual Game

The decomposition-based prediction of the entry decisions after the policy \( W_i \mapsto W_i + \alpha \) extrapolates from the pre-policy game equilibrium strategy so that it predicts that firm \( i \) enters if and only if \( W_i + \alpha > \bar{w} \). We show that this prediction cannot be a valid prediction in the counterfactual game.

The expected utility for \( i \) in the post-policy game from following this strategy, conditional on the competitor also doing so, is given by:

\[
1 \{ W_i + \alpha > \bar{w} \} (\delta (1 - \Phi(w - \alpha)) + W_i + \alpha).
\]

We now show that there exists a profitable deviation from this strategy for \( i \) in the post-policy game. Consider the following possible deviation from the recommendation for firm \( i \): “Enter” if \( W_i + \alpha > \bar{w} + \epsilon \), where \( \epsilon \) is a constant such that \( 0 < \epsilon < -\delta (1 - \Phi(w - \alpha)) - \bar{w} \). We note that such an \( \epsilon \) exists because:

\[
0 = \bar{w} + \delta (1 - \Phi(w)) > \bar{w} + \delta (1 - \Phi(w - \alpha)),
\]

where the inequality comes from that the map \((w, a) \mapsto w + \delta (1 - \Phi(w - a))\) is strictly decreasing in \( a \), and \( \alpha > 0 \).

The difference in expected utilities from deviating relative to following the recommendation is given by:

\[
(1 \{ W_i + \alpha > \bar{w} + \epsilon \} - 1 \{ W_i + \alpha > \bar{w} \}) (\delta (1 - \Phi(w - \alpha)) + W_i + \alpha).
\]

This difference is equal to 0 except for the event \( \{ \bar{w} < W_i + \alpha < \bar{w} + \epsilon \} \), in which case the first term of (78) is equal to \(-1\). Let us focus on this subset. Then, deviating to the alternative strategy is a profitable deviation as long as \( \delta (1 - \Phi(w - \alpha)) + W_i + \alpha < 0 \). This is true if and only if

\[
W_i + \alpha < -\delta (1 - \Phi(w - \alpha)).
\]

However, by the assumption on \( \epsilon \) above, we have that

\[
\bar{w} + \epsilon < -\delta (1 - \Phi(w - \alpha)).
\]

But if \( \bar{w} < W_i + \alpha < \bar{w} + \epsilon \), using equation (79), we find that

\[
W_i + \alpha < \bar{w} + \epsilon < -\delta (1 - \Phi(w - \alpha)).
\]

This is exactly the expression required for there to be a profitable deviation for player \( i \). It follows that the decomposition-based prediction based on this strategy (i.e. \( W_i + \alpha > \bar{w} \)) cannot be a BNE in the post-policy game.
References

Ahn, H., H. Ichimura, J. L. Powell, and P. A. Ruud (2018): “Simple Estimators for Invertible Index Models,” *Journal of Business and Economic Statistics*, 36, 1–10.

Ahn, H., and J. L. Powell (1993): “Semiparametric Estimation of Censored Selection Models With a Nonparametric Selection Mechanism,” *Journal of Econometrics*, 58, 3–29.

Bergemann, D., and S. Morris (2016): “Bayes Correlated Equilibrium and the Comparison of Information Structures in Games,” *Theoretical Economics*, 11, 487–522.

Breiman, L. (1992): *Probability*. Society for Industrial and Applied Mathematics, Reading.

Chernozhukov, V., I. Fernández-Val, and B. Melly (2013): “Inference on Counterfactual Distributions,” *Econometrica*, 81(6), 2205–2268.

Ciliberto, F., and E. Tamer (2009): “Market Structure and Multiple Equilibria in Airline Markets,” *Econometrica*, 77, 1791–1828.

Daley, D. J., and D. Vere-Jones (2003): *An Introduction to the Theory of Point Processes*. Springer, New York.

Fortin, N., T. Lemieux, and S. Firpo (2011): “Decomposition Methods in Economics,” in *Handbook of Labor Economics*, vol. 4, pp. 1–102. Elsevier.

Goolsbee, A., and C. Syverson (2008): “How do Incumbents Respond to the Threat of Entry? Evidence from the Major Airlines,” *The Quarterly Journal of Economics*, 123(4), 1611–1633.

Häusler, E., and H. Luschgy (2015): *Stable Convergence and Stable Limit Theorems*. Springer, New York.

Kalkeenberg, O. (1997): *Foundations of Modern Probability*. Springer-Verlag, Berlin.

Magnolfi, L., and C. Roncoroni (2022): “Estimation of Discrete Games with Weak Assumptions on Information,” *Forthcoming in Review of Economic Studies*.

Pollard, D. (2011): *A User’s Guide to Measure Theoretic Probability*. Cambridge University Press, New York.