EVALUATION OF SOME SUMS INVOLVING POWERS OF HARMONIC NUMBERS

CE XU AND XIXI ZHANG
School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, P.R.C
JIANQIANG ZHAO
Department of Mathematics, The Bishop’s School, La Jolla, CA 92037, USA

Abstract. In this note, we extend the definition of multiple harmonic sums and apply their stuffle relations to obtain explicit evaluations of the sums \( R_n(p, t) = \sum_{m=0}^{n} m^p H_m^t \), where \( H_m \) are harmonic numbers. When \( t \leq 4 \) these sums were first studied by Spieß around 1990 and, more recently, by Jin and Sun. Our key step first is to find an explicit formula of a special type of the extended multiple harmonic sums. This also enables us to provide a general structural result of the sums \( R_n(p, t) \) for all \( t \geq 0 \).

Keywords: Bernoulli number; harmonic number; multiple harmonic sum; extended multiple harmonic sum.

AMS Subject Classifications (2020): 05A19; 11B73; 11M32; 68R05.

1. Introduction

We begin with some basic notations. Let \( \mathbb{N} \) (resp. \( \mathbb{Z} \)) be the set of positive integers (resp. integers) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). A finite sequence \( \mathbf{k} := (k_1, \ldots, k_r) \in \mathbb{N}^r \) is called a composition. We define its weight and the depth of \( \mathbf{k} \), respectively, by

\[ |\mathbf{k}| := k_1 + \cdots + k_r \quad \text{and} \quad \text{dep}(\mathbf{k}) := r. \]

For any \( n \in \mathbb{N} \) and \((k_1, \ldots, k_r) \in \mathbb{Z}^r \), we define the extended multiple harmonic sums by

\[ H_n(k_1, \ldots, k_r) := \sum_{n \geq n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{1.1} \]

We set \( H_n(\emptyset) := 1 \) and \( H_n(k_1, \ldots, k_r) := 0 \) if \( n < r \). When \( r = 1 \) and \( k > 0 \), \( H_n(k) = \sum_{j=1}^{n} 1/j^k \) is the \( n \)-th generalized harmonic number of order \( k \), and furthermore, if \( k = 1 \) then \( H_n := H_n(1) \) is the classical \( n \)-th harmonic number. Identities involving harmonic numbers and multiple harmonic sums (i.e., when all \( k_j > 0 \)) have been extensively studied in the literature (see, e.g., [6] and the references therein).

Email: cexu2020@ahnu.edu.cn (C. Xu), XixiZhang2019@163.com (X. Zhang), zhaoj@ihes.fr (J. Zhao: Corresponding author).
Let \((k_1, \ldots, k_r) \in \mathbb{N}^r\). In this note, we are particularly interested in the sums of the form

\begin{equation}
H_n(-k_1,k_2,\ldots,k_r) = \sum_{n_1 > \cdots > n_r > 0} n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r} = \sum_{m=1}^n m^{k_1} H_{m-1}(k_2, \cdots, k_r).
\end{equation}

Recall that there are two versions of Bernoulli numbers defined by the generating functions

\begin{equation}
\frac{te^t}{e^t - 1} = \sum_{n=0}^\infty B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{t}{e^t - 1} = \sum_{n=0}^\infty \tilde{B}_n \frac{t^n}{n!}.
\end{equation}

It is well-known that \(B_j = \tilde{B}_j\) if \(j \neq 1\), \(B_1 = 1/2\) and \(\tilde{B}_1 = -1/2\). By Faulhaber’s formula,

\begin{equation}
H_n(-k) = \sum_{m=1}^n m^k = \frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j n^{k+1-j} = n^k + \frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} \tilde{B}_j n^{k+1-j}.
\end{equation}

In [3], Spieß studied the summation

\begin{equation}
R_n(d,t) := \sum_{m=0}^n m^d H_m^t \quad (n \in \mathbb{N}, d, t \in \mathbb{N}_0)
\end{equation}

and proved the following structure theorem (see [3 Thm. 30]).

**Theorem 1.1.** Let \(p(m)\) be a polynomial in \(m\) of degree \(d\). Then for \(t = 1, 2\) or \(3\), there exist polynomials \(q_0(n), \ldots, q_t(n)\) and \(C(n)\) of degree at most \(d + 1\) such that

\begin{equation}
\sum_{m=0}^n p(m) H_m^t = \sum_{i=0}^t q_i(n) H_n^i + C(n) H_n(2),
\end{equation}

for all nonnegative integers \(n\). Moreover, \(C(n) = 0\) when \(t = 1, 2\).

Spieß also conjectured that Theorem 1.1 holds for any positive integer \(t \geq 4\). However, Jin and Sun [4] showed that the sum \(R_n(0, 4) = \sum_{m=0}^n H_n^4\) cannot be represented by the form as conjectured by Spieß. Moreover, Jin and Sun [4 Thms. 1.2 and 1.3] proved that

\begin{equation}
R_n(d, 3) = \sum_{m=1}^n m^d H_n^3 = H_n(-d)H_n^3 + q_1(n) H_n^2 + q_2(n) H_n + q_3(n) + \frac{\tilde{B}_d}{2} H_n(2),
\end{equation}

where \(q_1(n), q_2(n)\) and \(q_3(n)\) are polynomial in \(n\) of degree at most \(d + 1\). They also provided similar formulas for the sums

\begin{equation}
\sum_{m=0}^n p(m) H_m^t \quad \text{and} \quad \sum_{m=0}^n m^d H_m H_m(2).
\end{equation}

Nevertheless, the explicit formula of the polynomials \(q_1(n), q_2(n), q_3(n)\) were not found.

The primary goal of this note is to establish explicit formulas of (1.7) and (1.8). Our main idea is to express these sums using (1.2) for which we are able to derive an explicit formula in general (see Theorem 2.1). At the end of the note, we present a result generalizing Theorem 1.1 to all \(t \geq 0\) (see Theorem 2.15).
Theorem 2.1. Let $k - k' \leq k$ where $k' = (k_1, \ldots, k_r)$.

Some Explicit Formulas of General MHSs.

2.1. **Explicit Formulas of General MHSs.** To derive the general formula for $H_n(-p, k)$ where $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we set $k_i = (k_1, \ldots, k_i)$ for all $1 \leq i \leq r$ and $|k_i| = k_0 + \cdots + k_i$ for all $0 \leq i \leq r$.

**Theorem 2.1.** Let $p \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $k_1, \ldots, k_r \in \mathbb{N}$. Put $k_0 = j_0 = 0$, $k'_r = k_{r+1} = 1$ and $k'_l = k_l$ for all $l < r$. Then

$$H_n(-p, k_1, \ldots, k_r) =$$

$$-\sum_{l=1}^{r}(-1)^l \left( \sum_{0 \leq j_i \leq p+i-|k'_i|-|j_i|} \prod_{h=0}^{l-1} \frac{(p+h+1-|k_h|-|j_h|)B_{j_h+1}}{p+h+1-|k_h|-|j_h|} \right) n^{p+l-|k'_l|-|j_l|} \cdot H_n(k_l, \ldots, k_r)$$

$$+\sum_{l=1}^{r}(-1)^l \left( \sum_{p+l+1-|k_i| \leq |j_i| \leq 0 \leq p+l-1-|k_{i-1}|} \prod_{h=0}^{l-1} \frac{(p+h+1-|k_h|-|j_h|)B_{j_h+1}}{p+h+1-|k_h|-|j_h|} \right) H_n(|k_i|+|j_i|-l-p, k_{l+1}, \ldots, k_r)$$

$$+(-1)^r \left( \sum_{0 \leq j_i \leq p+l-|k_i|-|j_i|} \prod_{h=0}^{r} \frac{(p+h+1-|k_h|-|j_h|)B_{j_h+1}}{p+h+1-|k_h|-|j_h|} \right) n^{p+r+1-|k_r|-|j_{r+1}|},$$

where we set the binomial coefficients \( \binom{n}{b} = 0 \) if \( b < 0 \).

**Proof.** According to definition (1.2), we have the following recurrence relation

$$H_n(-p, k_1, \ldots, k_r)$$

$$= \sum_{m=1}^{n} m^p H_{m-1}(k_1, \ldots, k_r) = \sum_{m=1}^{n} m^p \sum_{m>n_1>\cdots>n_r} \frac{1}{n_1 \cdots n_r}$$

$$= \sum_{m=1}^{n} m^p \left( H_m(k_1, \ldots, k_r) - H_{m-1}(k_2, \ldots, k_r) \right)$$

$$= \sum_{m=1}^{n} m^p H_m(k_1, \ldots, k_r) - \sum_{m=1}^{n} m^{p-k_1} H_{m-1}(k_2, \ldots, k_r)$$
Corollary 2.2. For a composition $\mathbf{k} = (k_1, \ldots, k_r)$ and $p \in \mathbb{N}_0$, the sum $H_n(-p, \mathbf{k})$ can be expressed in terms of a combination of products of polynomial in $n$ of degree $\leq p + 1$ and
multiple harmonic sums with depth \( \leq r \). In particular, if \( k = \{1\}_r \) then we have

\[
(2.7) \quad H_n(-p, \{1\}_r) = H_n(-p)H_n(\{1\}_r) + \sum_{i=1}^{r} (-1)^i C_{0,0}^{(p)}(n) H_n(\{1\}_{r-i}),
\]

where \( \{1\}_r \) means the string obtained by repeating 1 exactly \( r \) times.

**Proof.** Observe that the \( l \)-th term in the second sum of (2.3) is vacuous if \( k_l = 1 \). The rest of the proof is straight-forward. \( \square \)

Setting \( r = 2 \) in (2.7) and noting the fact that

\[
(2.8) \quad H_n(1,1) = \frac{H_n^2 - H_n(2)}{2},
\]

we get the following corollary.

**Corollary 2.3.** For \( p \in \mathbb{N}_0 \),

\[
H_n(-p, 1, 1) = H_n(-p)H_n(\{1\}_2) - C_{0,0}^{(p)}(n)H_n + C_{0,0}^{(p)}(n)
\]

\[
(2.9) \quad = \frac{1}{2} H_n(-p)(H_n^2 - H_n(2)) - C_{0,0}^{(p)}(n)H_n + C_{0,0}^{(p)}(n).
\]

Taking \( r = 2 \) in (2.3) we can get the following corollary.

**Corollary 2.4.** For positive integers \( k_1, k_2 \) and nonnegative \( p \), we have

\[
H_n(-p, 1, 2) = H_n(-p)H_n(k_1, k_2) - \frac{1}{p+1} \sum_{j=p+2-k_1}^p \binom{p+1}{j} B_j H_n(k_1 + j - p - 1, k_2)
\]

\[
+ \frac{1}{p+1} \sum_{j_1=0}^{p+1-k_1} \sum_{j_2=p+3-k_1-k_2-j_1}^{p+1-k_1-j_1} \binom{p+1}{j_1} \binom{p+2-k_1-j_1}{j_2} \frac{B_{j_1}B_{j_2}H_n(k_1 + k_2 + j_1 + j_2 - p - 2)}{(p+2-k_1-j_1)(p+3-k_1-k_2-j_1-j_2)}
\]

\[
+ \frac{1}{p+1} \sum_{j_1=0}^{p+1-k_1} \sum_{j_2=0}^{p+2-k_1-k_2-j_1} \sum_{j_3=0}^{p+2-k_1-k_2-j_1-j_2} \binom{p+1}{j_1} \binom{p+2-k_1-j_1}{j_2} \binom{p+3-k_1-k_2-j_1-j_2}{j_3}
\]

\[
\times B_{j_1}B_{j_2}B_{j_3} H_n^3(p+3-k_1-k_2-j_1-j_2-j_3).
\]

\[
(2.10)
\]

For \( p \in \mathbb{N}_0 \), put \( D_{a_2,\ldots,a_r}(B) := C_{a_2,\ldots,a_r}(B)/B \) with \( B^i \) understood as \( B_i \), which is the convention in umbral calculus. Setting \( (k_1, k_2) = (1, 2) \) and (2, 1) in (2.10), respectively, we get the following two formulas:

\[
(2.11) \quad H_n(-p, 1, 2) = H_n(-p)H_n(1, 2) + D^{(p)}(B)H_n - C_{0,0}^{(p)}(n)H_n(2) + C_{0,1}^{(p)}(n),
\]

\[
(2.12) \quad H_n(-p, 2, 1) = H_n(-p)H_n(2, 1) - B_pH_n(1,1) - C_{1,1}^{(p)}(n)H_n + C_{1,1}^{(p)}(n).
\]

For \( p = 0, 1 \) in (2.11) and (2.12), we get

\[
H_n(0, 1, 2) = nH_n(1, 2) - nH_n(2) + H_n,
\]

\[
H_n(0, 2, 1) = nH_n(2, 1) - \frac{H_n^2 - H_n(2)}{2}.
\]
Letting $k = 2$ in (2.10) we see that
\[ H_n(-p, 2) = H_n(-p)H_n(2) - B_pH_n - C_1^{(p)}(n). \]
Combining the above with (2.9), we arrive at (2.13) immediately.

\[ \sum_{m=1}^{n} m^p H_{m-1}^2 = C^{(p)}(n) H_n^2 - (2C_0^{(p)}(n) + B_p) H_n + 2C_0^{(p)}(n) - C_1^{(p)}(n). \]

**Proof.** By the stuffle relation (2.8) we have
\[ \sum_{m=1}^{n} m^p H_{m-1}^2 = 2H_n(-p, 1, 1) + H_n(-p, 2). \]

**Example 2.6.** Letting $p = 0, 1$ in (2.13) we get
\[ \sum_{m=1}^{n} H_{m-1}^2 = nH_n^2 - (2n + 1)H_n + 2n, \]
\[ \sum_{m=1}^{n} mH_{m-1}^2 = \frac{n(n + 1)}{2} H_n^2 - \frac{n^2 + 3n + 1}{4} H_n + \frac{n(n + 5)}{4}. \]

**Theorem 2.7.** For $p \in \mathbb{N}_0$, we have
\[ \sum_{m=1}^{n} m^p H_{m-1}^2 H_{m-1}(2) = H_n(-p)H_n H_n(2) - \frac{B_p}{2} H_n^2 + \left( D^{(p)}(B) - C_1^{(p)}(n) - \frac{p}{2} B_{p-1} \right) H_n \]
\[ - \left( C_0^{(p)}(n) + \frac{B_p}{2} \right) H_n(2) + C_0^{(p)}(n) + C_1^{(p)}(n) - C_2^{(p)}(n). \]

**Proof.** Applying identities (2.2), (2.6), (2.11) and (2.12), we obtain the explicit evaluation of sum $\sum_{m=1}^{n} m^p H_{m-1}^2 H_{m-1}(2)$ by a direct calculation.

**Example 2.8.** Letting $p = 0, 1$ in (2.14), we get
\[ \sum_{m=1}^{n} H_{m-1}^2 H_{m-1}(2) = nH_n H_n(2) - \frac{1}{2} H_n^2 - \frac{2n + 1}{2} H_n(2) + H_n, \]
\[ \sum_{m=1}^{n} mH_{m-1}^2 H_{m-1}(2) = \frac{n(n + 1)}{2} H_n H_n(2) - \frac{1}{4} H_n^2 - \frac{n^2 + 3n + 1}{4} H_n(2) + \frac{1 - 2n}{4} H_n + \frac{3}{4} n. \]

**Theorem 2.9.** For $p \in \mathbb{N}_0$, we have
\[ \sum_{m=1}^{n} m^p H_{m-1}^3 H_{m-1}(2) = H_n(-p)H_n^3 - 3 \left( C_0^{(p)}(n) + \frac{B_p}{2} \right) H_n^2 + \frac{B_p}{2} H_n(2) \]
Taking $\sum_{m=1}^{n} m^p H_m^3 = 6H_n(-p, \{1\}_3) + 3H_n(-p, 1, 2) + 3H_n(-p, 2, 1) + H_n(-p, 3)$. Taking $k = 3$ in (2.6) we see that

$$H_n(-p, 3) = H_n(-p)H_n(3) - B_p H_n(2) - \frac{p}{2} B_{p-1} H_n - \sum_{j_1+j_2 \leq p-2, j_1, j_2 \geq 0} \frac{(p+1) (p-1-j_1) B_{j_1} B_{j_2}}{(p+1)(p-1-j_1)} n^{p-1-j_1-j_2} = H_n(-p)H_n(3) - B_p H_n(2) - \frac{p}{2} B_{p-1} H_n - C_2^{(p)}(n).$$

Combining the above with (2.7) (taking $r = 3$), (2.11) and (2.12), we can complete the proof of the theorem by a straightforward simplification.

**Example 2.10.** Taking $p = 0, 1, 2$ in Theorem 2.9 and using the relation

$$\sum_{m=0}^{n} m^p H_m^3 = n^p H_n^3 + \sum_{m=1}^{n} (m-1)^p H_{m-1}^3$$

we can confirm the first three examples in the last section of [4].

**Theorem 2.11.** Let $d \in \mathbb{N}_0$. Then for any polynomial $F(x) = \sum_{p=0}^{d} a_p x^p \in \mathbb{Q}[x]$ we have

$$\sum_{m=1}^{n} F(m) H_{m-1}^4 = H_n^4 \sum_{m=0}^{n} F(m) + \sum_{j=0}^{3} Q_j(n) H_n^j + P(n) H_n(2) + 2F(B)H_n(2, 1) + F(B)H_n(3)$$

where we should replace $B^i$ by $B_i$ for all $i$ in $F(B)$, and $P(x)$ and $Q_j(x)$, $0 \leq j \leq 3$, are all polynomials of $n$ of degree at most $d+1$ defined by

$$P(n) = \sum_{p=0}^{d} a_p \left( -6D^{(p)}(B) + 4D^{(p)}(n) + \frac{p}{2} B_{p-1} \right),$$

$$Q_0(n) = \sum_{p=0}^{d} a_p \left( 24C_{0,0,0,0}^{(p)}(n) - 12C_{0,0,1,1}^{(p)}(n) - 12C_{0,1,1,1}^{(p)}(n) - 12C_{1,1,1,1}^{(p)}(n) + 4C_{0,2}^{(p)}(n) + 6C_{1,2}^{(p)}(n) + 4C_{2,2}^{(p)}(n) - C_3^{(p)}(n) \right),$$

$$Q_1(n) = \sum_{p=0}^{d} a_p \left( -24C_{0,0,0,0}^{(p)}(n) - 12D_0^{(p)}(B) + 12C_{0,1}^{(p)}(n) + 12C_{1,1}^{(p)}(n) + 2 \frac{d}{dx} D^{(p)}(x) \bigg|_{x=B} \frac{6(D^{(p)}(x) - B_p)}{x} \bigg|_{x=B} - 4C_2^{(p)}(n) - \frac{p(p-1)}{6} B_{p-2} \right),$$

$$Q_2(n) = \sum_{p=0}^{d} a_p \left( 12C_{0,0}^{(p)}(n) + 6D^{(p)}(B) - 6C_1^{(p)}(n) - pB_{p-1} \right),$$

$$Q_3(n) = \sum_{p=0}^{d} a_p \left( -12C_{0,0,0,0}^{(p)}(n) - 12D_0^{(p)}(B) + 12C_{0,1}^{(p)}(n) + 12C_{1,1}^{(p)}(n) + 2 \frac{d}{dx} D^{(p)}(x) \bigg|_{x=B} \frac{6(D^{(p)}(x) - B_p)}{x} \bigg|_{x=B} - 4C_2^{(p)}(n) - \frac{p(p-1)}{6} B_{p-2} \right).$$
Proof. We first consider \( \sum_{m=1}^{n} m^p H_{m-1}^4 \) for any \( p \in \mathbb{N}_0 \). Then by Theorem 2.1

\[
H_n(-p, \{1\}_4) = H_n(-p) H_n(\{1\}_3) - C_0^{(p)}(n) H_n(\{1\}_2) + \]
\[
C_1^{(p)}(n) H_n(\{1\}_2) + D_0^{(p)}(B) H_n(\{1\}_2) - C_0^{(p)}(n) H_n(\{1\}_2) - C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2),
\]
\[
H_n(-p, 1, 2, 1) = H_n(-p) H_n(1, 2, 1) - C_0^{(p)}(n) H_n(2, 1) + D_0^{(p)}(B) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2) + C_0^{(p)}(n) H_n(\{1\}_2),
\]
\[
H_n(-p, 1, 3) = H_n(-p) H_n(1, 3) - C_0^{(p)}(n) H_n(3) + D_0^{(p)}(n) H_n(2) + \frac{1}{2} \frac{d}{dx} D_0^{(p)}(x) \bigg|_{x=B} H_n(1, 3),
\]
\[
H_n(-p, 2, 2) = H_n(-p) H_n(2, 2) - B_p H_n(1, 2) - C_1^{(p)}(n) H_n(2) + \frac{D_0^{(p)}(x) - B_p}{x} \bigg|_{x=B} H_n(2, 2),
\]
\[
H_n(-p, 3, 1) = H_n(-p) H_n(3, 1) - B_p H_n(2, 1) - \frac{p}{2} B_{p-1} H_n(\{1\}_2) - C_2^{(p)}(n) H_n(3) + C_2^{(p)}(n) H_n(3),
\]
\[
H_n(-p, 4) = H_n(-p) H_n(4) - B_p H_n(3) - \frac{p}{2} B_{p-1} H_n(2) - \frac{p(p-1)}{6} B_{p-2} H_n - C_3^{(p)}(n).
\]

The theorem now follows directly from the threeuffle relations: (2.5) and

\[
H_n^3 = 6H_n(\{1\}_3) + 3H_n(1, 2) + 3H_n(2, 1) + H_n(3),
\]
\[
H_n^2 = 24H_n(\{1\}_4) + 12 \left( H_n(1, 1, 2) + H_n(1, 2, 1) + H_n(2, 1, 1) \right)
\]
\[
+ 6H_n(2, 2) + 4H_n(1, 3) + 4H_n(3, 1) + H_n(4).
\]

This concludes the proof of the theorem. \( \Box \)

In order to see the consistence of our result with that of Jin and Sun in [4], we may use the convention of umbral calculus (see, e.g., [2]) to rewrite the two versions of Bernoulli numbers defined by (1.3) as

\[
\frac{te^t}{e^t - 1} = e^{Bt}, \quad \frac{t}{e^t - 1} = e^{\tilde{B}t}
\]

where in the expansion of the right-hand side we understand \( B^i \) (resp. \( \tilde{B}^i \)) as \( B_i \) (resp. \( \tilde{B}_i \)) for all \( i \in \mathbb{N}_0 \). This notation scheme helps us to prove the next lemma in a straight-forward manner.

**Lemma 2.12.** Let \( F(x) \in \mathbb{R}[x] \). Then \( F(\tilde{B}) \equiv 0 \) if and only if \( F(B - 1) \equiv 0 \).

**Proof.** With umbral calculus notation we have

\[
\sum_{i=0}^{\infty} (B - 1)^i \frac{t^i}{i!} = e^{(B-1)t} = e^{Bt} e^{-t} = \frac{te^t}{e^t - 1} \cdot e^{-t} = \frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \tilde{B}_i^i \frac{t^i}{i!}.
\]

Thus \( (B - 1)^i = \tilde{B}_i^i \) for all \( i \in \mathbb{N}_0 \). The lemma follows immediately. \( \Box \)

**Remark 2.13.** If \( F(B) \equiv 0 \) holds in Theorem 2.11 then we can recover an essentially equivalent form of [4] Theorem 1.3 by Lemma 2.12 since

\[
\sum_{n=0}^{\infty} F(m) H_m^4 = F(n) H_n^4 + \sum_{m=1}^{n} F(m - 1) H_m^4.
\]
We point out that the Bernoulli numbers in [4] are denoted by $B_i$ in the current note.

**Example 2.14.** By the examples at the end of [4]

$$\sum_{k=0}^{n} (2k+1)H_k^4 \quad \text{nor} \quad \sum_{k=0}^{n} (3k^2+k)H_k^4$$

involves $H_n(2,1)$ or $H_n(3)$. This can also follow easily from $2\tilde{B}_1 + \tilde{B}_0 = 0$ and $3\tilde{B}_2 + B_1 = 0$ by [4, Theorem 1.3]. This is consistent with our result since

$$\sum_{k=0}^{n} (2k+1)H_k^4 = (2n+1)H_n^4 + \sum_{m=1}^{n} (2m-1)H_m^4,$$

and

$$\sum_{k=0}^{n} (3k^2+k)H_k^4 = (3n^2+n)H_n^4 + \sum_{m=1}^{n} (3m^2-5m+2)H_m^4,$$

Thus by Theorem 2.11 we see that neither (2.15) nor (2.16) involves $H_n(2,1)$ or $H_n(3)$.

By looking back at Theorem 2.11 and Remark 2.13, we find that not only Spieß’s conjecture fails in general as shown by Jin and Sun in [4] but also linear combinations of multiple harmonic sums with coefficients given by rational polynomials of $n$ should suffice to express $R_n(d,t)$ for all $d, t \in \mathbb{N}$. We now conclude our note by the following general theorem.

**Theorem 2.15.** Suppose $n \in \mathbb{N}$, $t \in \mathbb{N}_0$, and the polynomial $F(x) \in \mathbb{Q}[x]$ has degree $d$. Put $S_n(F) = \sum_{m=1}^{n} F(m)$ and denote by $V_n(d,t)$ the $\mathbb{Q}$-vector space generated by $P(n)H_n(1)$ for all $P(x) \in \mathbb{Q}[x]$ of degree less than $d+2$ and compositions $1$ of depth less than $t$. Then we have

$$\sum_{m=1}^{n} F(m)H_m^t \subseteq \sum_{m=0}^{n} F(m)H_m^t \in S_n(F)H_n^t + V_n(d,t).$$

**Proof.** Observe that

$$\sum_{m=0}^{n} F(m)H_m^t = F(n)H_n^t + \sum_{m=1}^{n} F(m-1)H_m^{t-1}.$$ 

It suffices to prove the theorem for $\sum_{m=1}^{n} F(m)H_m^{t-1}$, which can be further reduced to the case when $F(m) = m^p$ for some $p \leq d$. By the stuffle relation have

$$\sum_{m=1}^{n} m^pH_m^{t-1} - t!H_n(-p, \{1\}_t) = \sum_{1 \text{-dep}(l) < t, |l| = t} c_lH_n(-p,l)$$

for some suitable integer coefficients $c_l$. We see by Theorem 2.11 that each term on the right-hand side of the above lies inside $V_n(d,t)$. This concludes the proof of the theorem.

**References**

[1] M. E. Hoffman, Quasi-shuffle products, J. Algebraic Combin. 11(2000), 49–68.
[2] S. Roman, The Umbral Calculus, Dover Publications, Reprint edition (April 17, 2019).
[3] J. Spieß, Some identities involving harmonic numbers, Math. Comp. 192(1990), 839–863.
[4] H. Jin, L.H. Sun, On Spieß’s conjecture on harmonic numbers, Discrete Appl. Math. 161(13-14)(2013), 2038–2041.
[5] C. Xu, W. Wang, Explicit formulas of Euler sums via multiple zeta values, J. Symb. Comput. 101(2020), 109–127.

[6] J. Zhao, Multiple Zeta Functions, Multiple Polylogarithms and Their Special Values, Series on Number Theory and Its Applications: Volume 12, World Scientific Publishing, 2016.