Spectral lower bounds for the orthogonal and projective ranks of a graph

Pawel Wocjan∗ Clive Elphick†

August 17, 2018

Abstract

The orthogonal rank of a graph $G = (V,E)$ is the smallest dimension $\xi$ such that there exist non-zero column vectors $x_v \in \mathbb{C}^\xi$ for $v \in V$ satisfying the orthogonality condition $x_v^\dagger x_w = 0$ for all $vw \in E$. We prove that many spectral lower bounds for the chromatic number, $\chi$, are also lower bounds for $\xi$. This result complements a previous result by the authors, in which they showed that spectral lower bounds for $\chi$ are also lower bounds for the quantum chromatic number $\chi_q$. It is known that the quantum chromatic number and the orthogonal rank are incomparable.

We conclude by proving an inertial lower bound for the projective rank $\xi_f$, and conjecture that the spectral lower bounds for $\xi$ are also lower bounds for $\xi_f$.

1 Introduction

For any graph $G$ let $V$ denote the set of vertices where $|V| = n$, $E$ denote the set of edges where $|E| = m$, $A$ denote the adjacency matrix, $\chi(G)$ denote the chromatic number, $\omega(G)$ denote the clique number, $\alpha(G)$ the independence number and $\overline{G}$ the complement of $G$. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ denote the eigenvalues of $A$ and then the inertia of $G$ is the ordered triple $(n^+, n^0, n^-)$, where $n^+$, $n^0$ and $n^-$ are the numbers of positive, zero and negative eigenvalues of $A$, including multiplicities. Note that $\text{rank}(A) = n^+ + n^-$ and $\text{null}(A) = n^0$. A graph is called non-singular if $n^0 = 0$.

Let $D$ be the diagonal matrix of vertex degrees, and let $L = D - A$ denote the Laplacian of $G$ and $Q = D + A$ denote the signless Laplacian of $G$. The eigenvalues of $L$ are $\theta_1 \geq \ldots \geq \theta_n = 0$ and the eigenvalues of $Q$ are $\delta_1 \geq \ldots \geq \delta_n$.

Let $\chi_v(G)$ denote the vector chromatic number as defined by Karger et al [13] and $\chi_{sv}(G)$ denote the strict vector chromatic number. Karger et al [13] proved that $\chi_{sv}(G) = \theta(\overline{G})$, where $\theta$ is the Lovasz theta function [15], and let $\theta^+$ denote Szegedy’s [20] variant of $\theta$. Let $\chi_f(G)$ and $\chi_c(G)$ denote the fractional and circular chromatic numbers and let $\chi_q(G)$ and

∗wocjan@cs.ucf.edu, Department of Computer Science, University of Central Florida, USA
†clive.elphick@gmail.com, School of Mathematics, University of Birmingham, Birmingham, UK
\(\chi_q^{(r)}(G)\) denote the quantum and rank-\(r\) quantum chromatic numbers, as defined by Cameron et al [2].

**Definition 1** (Orthogonal rank \(\xi(G)\)). The orthogonal rank of \(G\) is the smallest positive integer \(\xi(G)\) such that there exists an orthogonal representation, that is a collection of non-zero column vectors \(x_v \in \mathbb{C}^{\xi(G)}\) for \(v \in V\) satisfying the orthogonality condition
\[
x_v^\dagger x_w = 0
\]
for all \(vw \in E\).

The normalized orthogonal rank of \(G\) is the smallest positive integer \(\xi'(G)\) such that there exists an orthogonal representation, with the added restriction that the entries of each vector must all have the same modulus.

Let \(\xi_f(G)\) denote the projective rank which was defined by Mancinska and Roberson [16], who showed that \(\omega(G) \leq \xi_f(G) \leq \xi(G)\). We use the definition of the \(r\)-fold orthogonal rank \(\xi^{[r]}(G)\) due to Hogben et al. in [10] Section 2.1. and their results in [10] Section 2.2. to provide an equivalent and simpler definition of the projective rank.

**Definition 2** \((r\text{-fold orthogonal rank} \xi^{[r]}(G))\) and projective orthogonal rank \(\xi_f(G))\). A \(d/r\)-representation of \(G = (V, E)\) is a collection of rank-\(r\) orthogonal projectors \(P_v\) for \(v \in V\) such that \(P_v P_w = 0_d\) for all \(vw \in E\).

The \(r\)-fold orthogonal rank \(\xi^{[r]}(G)\) is defined as follows:
\[
\xi^{[r]}(G) = \min \left\{ d : G \text{ has a } d/r\text{-representation} \right\}.
\]
The projective rank, \(\xi_f(G)\), is defined as follows:
\[
\xi_f(G) = \lim_{r \to \infty} \frac{\xi^{[r]}(G)}{r}, \text{ and this limit exists.}
\]
The projective rank is also called the fractional orthogonal rank.

Clearly, the vectors \(x_v \in \mathbb{C}^{\xi(G)}\) of an orthogonal representation correspond to the rank-1 orthogonal projectors \(P_v = x_v x_v^\dagger \in \mathbb{C}^{\xi(G) \times \xi(G)}\) of a \(\xi(G)/1\)-representation. It is also clear that \(\xi^{[1]}(G) = \xi(G)\).

**Definition 3** (Vectorial chromatic number \(\chi_{\text{vect}}(G)\)). Paulsen and Todorov [19] defined the vectorial chromatic number, \(\chi_{\text{vect}}(G)\), as follows. Let \(G = (V, E)\) be a graph and \(c \in \mathbb{N}\). A vectorial \(c\)-coloring of \(G\) is a set of vectors \((x_{v,i} : v \in V, i \in [c]\) in a Hilbert space such that the following conditions are satisfied:
\[
\sum_{i=1}^c x_{v,i} = \sum_{i=1}^c x_{w,i}, \quad \| \sum_{i=1}^c x_{v,i} \| = 1, \quad \forall v, w \in V \tag{3}
\]
\[
\langle x_{v,i}, x_{v,j} \rangle = 0, \quad \forall v \in V, i \neq j \in [c] \tag{4}
\]
\[
\langle x_{v,i}, x_{w,i} \rangle = 0, \quad \forall vw \in E, i \in [c]. \tag{5}
\]
The least integer $c$ for which there exists a vectorial $c$-coloring will be denoted $\chi_{\text{vect}}(G)$ and called the vectorial chromatic number of $G$.

Note that $\chi_{\text{vect}}$ differs from $\chi$. Cubitt et al [4] (Corollary 16) proved the following (unexpected) equality between a chromatic number and a theta function:

$$\chi_{\text{vect}}(G) = \lceil \theta^+(\overline{G}) \rceil,$$

and provided an example of a graph with $\chi_{\text{vect}} < \chi_q$. Paulsen et al [18] (Theorem 7.3) proved that $\theta^+(\overline{G}) \leq \xi_f(G) \leq \xi(G)$, so $\chi_{\text{vect}}(G) = \lceil \theta^+(\overline{G}) \rceil \leq \xi(G)$.

2 Hierarchies of graph parameters

There are numerous graph parameters that lie between the clique number and the chromatic number. The following chains of inequalities summarise the relationships between many of them, and combines results in Cameron et al [2], Mancinska and Roberson ([17] and [16]), Paulsen et al [18] and Elphick and Wocjan [6]. The chains are broken into two parts so the rightmost ends of (6) and leftmost ends of (7) coincide.

\begin{align*}
\omega(G) & \longrightarrow \chi_v(G) & \longrightarrow \chi_{sv}(G) = \theta(\overline{G}) & \longrightarrow \theta^+(\overline{G}) & \longrightarrow \xi_f(G) \\
\chi_{\text{vect}}(G) & \longrightarrow \xi(G) & \xi_f(G) & \longrightarrow \chi_q(G) & \longrightarrow \chi_q^{(1)}(G) & \longrightarrow \xi'(G) & \longrightarrow \lceil \chi_c(G) \rceil = \chi(G) \\
\chi_f(G) & \longrightarrow \chi_c(G)
\end{align*}

(6)

(7)

As illustrated above, Mancinska and Roberson ([17] and [16]) demonstrated that $\xi$ and $\chi_q$ are incomparable, as are $\chi_f$ and $\chi_q$; and also $\chi_f$ and $\xi$. They also proved that $\xi_f$ is a lower bound for $\xi$, $\chi_q$ and $\chi_f$. Cubitt et al [4] demonstrated that $\chi_{\text{vect}}$ and $\xi_f$ are incomparable. We can also demonstrate that $\chi_{\text{vect}}$ and $\chi_f$ are incomparable as follows. It is straightforward that for $C_5$, $\chi_{\text{vect}} > \chi_f$. However if we consider the disjunctive product $C_5 \ast K_3$, then from [4] $\chi_{\text{vect}}(C_5 \ast K_3) \leq 7$ but $\chi_f(C_5 \ast K_3) = 7.5$, because $\chi_f$ is multiplicative for the disjunctive product. Note that $\xi, \xi', \chi_{\text{vect}}, \chi_q, \chi_q^{(1)}$ are integers, $\chi_f$ is rational but $\xi_f$ may be irrational.
These hierarchies of parameters resolve a question raised by Wocjan and Elphick (see Section 2.4 of [21]) of whether $\chi_v \leq \xi'$. Wocjan and Elphick [22] proved that many spectral lower bounds for $\chi(G)$ are also lower bounds for $\chi_q(G)$. In this paper we prove that many spectral lower bounds for $\chi(G)$ are also lower bounds for $\xi(G)$. In Theorem 1 we prove an inertial lower bound for $\xi(G)$ by strengthening a proof in [6]. In Theorem 2 we prove several eigenvalue lower bounds for $\xi(G)$ by proving lower bounds for $\chi_{\text{vect}}(G)$. We conjecture that all of these bounds are also lower bounds for $\xi_f(G)$, and make limited progress in this direction in Theorem 3.

**Theorem 1** (Inertial lower bound for orthogonal rank). Let $\xi(G)$ be the orthogonal rank of a graph $G$ with inertia $(n^+, n^0, n^-)$. Then

$$1 + \max \left( \frac{n^+}{n^-}, \frac{n^-}{n^+} \right) \leq \xi(G).$$

**Theorem 2** (Eigenvalue lower bounds for vectorial chromatic number). Let $\xi(G)$ be the orthogonal rank and $\chi_{\text{vect}}(G)$ be the vectorial chromatic number of a graph $G$. Then

$$1 + \max \left( \frac{\mu_1}{\mu_n}, \frac{2m-n\delta_n}{2m-n\delta_n}, \frac{\mu_1}{\mu_1 - \delta_1 + \theta_1} \right) \leq \chi_{\text{vect}}(G) \leq \xi(G). \quad (8)$$

These bounds, reading from left to right, have been proved to be lower bounds for $\chi(G)$ by Hoffman [11], Lima et al [14] and Kolotilina [12].

**Theorem 3** (Inertial lower bound for projective rank). Let $\xi_f(G)$ be the projective rank of a graph $G$ with inertia $(n^+, n^0, n^-)$. Then,

$$1 + \max \left( \frac{n^+}{n^- + n^0}, \frac{n^-}{n^+ + n^0} \right) \leq \xi_f(G).$$

In particular, when the graph $G$ is non-singular the lower bounds in Theorems 1 and 3 coincide.

**Remark 1.** All results also apply to weighted adjacency matrices $W \odot A$, where $W$ is an arbitrary Hermitian matrix and $\odot$ denotes the Hadamard product (also called Schur product).

### 3 Proof of the inertial lower bound on the orthogonal rank $\xi(G)$

Let $f_1, \ldots, f_n \in \mathbb{C}^n$ denote the eigenvectors of unit length corresponding to the eigenvalues $\mu_1 \geq \ldots \geq \mu_n$. Let $A = B - C$, where

$$B = \sum_{i=1}^{n^+} \mu_i f_i f_i^\dagger \quad \text{and} \quad C = \sum_{i=n-n^-+1}^{n} (-\mu_i) f_i f_i^\dagger. \quad (9)$$
Note that $B$ and $C$ are positive semidefinite and that $\text{rank}(B) = n^+$ and $\text{rank}(C) = n^-$. Let

$$P^+ = \sum_{i=1}^{n^+} f_i f_i^\dagger, \quad P^- = \sum_{i=n^-+1}^{n} f_i f_i^\dagger$$

denote the orthogonal projectors onto the subspaces spanned by the eigenvectors corresponding to the positive and negative eigenvalues respectively. Note that $B = P^+ A P^+$ and $C = -P^- A P^-$. 

**Lemma 1.** Let $X$ and $Y \in \mathbb{C}^{n \times n}$ be two positive semidefinite matrices satisfying $X \succeq Y$, that is, their difference $X - Y$ is positive semidefinite. Then,

$$\text{rank}(X) \geq \text{rank}(Y). \quad (10)$$

**Proof.** Assume to the contrary that $\text{rank}(X) < \text{rank}(Y)$. Then, there exists a non-trivial vector $v$ in the range of $Y$ that is orthogonal to the range of $X$. Consequently,

$$v^\dagger (X - Y) v = -v^\dagger Y v < 0$$

contradicting that $X - Y$ is positive semidefinite. □

**Remark 2.** Let $x_v = (x_v^1, \ldots, x_v^\xi)^T \in \mathbb{C}^\xi$ for $v \in V$ be an orthogonal representation. Note that we may assume that the first entries of these vectors are all equal to 1, that is,

$$x_v^1 = 1$$

for all $v \in V$ for the following reason. If we apply any unitary transformation $U \in \mathbb{C}^{\xi \times \xi}$ to $x_v$ we obtain an equivalent orthogonal representation $y_v = U x_v$. Clearly, there must exist a unitary matrix $U$ such that the resulting orthogonal representation $y_v = (y_v^1, \ldots, y_v^\xi)^T$ satisfies the condition $y_v^1 \neq 0$ for all $v \in V$ due to a simple parameter counting argument. We can now rescale each vector to additionally achieve $y_v^1 = 1$.

We now have all the tools to prove Theorem 1. □

**Proof.** Let $x_v = (x_v^1, \ldots, x_v^\xi)^T$ for $v \in V$ be an orthogonal representation satisfying the additional condition $x_v^1 = 1$ as in the remark above. We define $\xi$ diagonal matrices

$$D_i = \text{diag}(x_v^i : v \in V) \in \mathbb{C}^{n \times n}$$

for $i = 1, \ldots, \xi$. Due to this construction, we have

$$\sum_{i=1}^{\xi} D_i^\dagger A D_i = (s_{vw}) \text{, with } s_{vw} = a_{vw} \cdot x_v^i x_w^i \text{ for } v, w \in V.$$ 

We see that this sum is the zero matrix because all its entries $s_{vw}$ are zero either due to the orthogonality condition of the orthogonal representation $x_v^i x_w^i = 0$ for $vw \in E$ or due to $a_{vw} = 0$ for $vw \notin E$. Observe that $D_1 = I$ due to the above remark. We obtain

$$\sum_{i=2}^{\xi} D_i^\dagger A D_i = -A. \quad (11)$$
Equation (11) can be rewritten as

\[ \sum_{i=2}^{\xi} D_i^\dagger (B - C) D_i = C - B. \]

Multiplying both sides by \( P^- \) from left and right yields:

\[ P^- \left( \sum_{i=2}^{\xi} D_i^\dagger (B - C) D_i \right) P^- = C. \]

Using that

\[ P^- \left( \sum_{i=2}^{\xi} D_i^\dagger C D_i \right) P^- \]

is positive semidefinite, it follows that

\[ P^- \left( \sum_{i=2}^{\xi} D_i^\dagger B D_i \right) P^- \geq C. \]

Then using that the rank of a sum is less than or equal to the sum of the ranks of the summands, that the rank of a product is less than or equal to the minimum of the ranks of the factors, and Lemma 11, we have that \((\xi - 1)n^+ \geq n^-\). Similarly \((\xi - 1)n^- \geq n^+\) is obtained by multiplying equation (11) by -1 and repeating the arguments (but multiplying by \( P^+ \) instead of \( P^- \) from the left and right).

\[ \square \]

4 Proof of eigenvalue lower bounds on the orthogonal rank \( \xi(G) \)

Conditions (3) and (4) in Definition 3 imply that there exist orthogonal projectors \( P_{v,i} \in \mathbb{C}^{d \times d} \) and a unit (column) vector \( y \in \mathbb{C}^d \) such that the \( P_{v,i} \) form a resolution of the identity \( I_d \)

\[ \sum_{i \in [c]} P_{v,i} = I_d \quad (12) \]

for all \( v \in V \) and

\[ x_{v,i} = P_{v,i} y \]

for all \( v \in V \) and \( i \in [c] \).

Let \( e_v \) denote the standard basis (column) vectors of \( \mathbb{C}^n \) corresponding to the vertices \( v \in V \) so that \( A = \sum_{v,w \in V} a_{vw} e_v e_w^\dagger \). For \( i \in [c] \), define the block-diagonal projectors \( P_i \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{d \times d} \) by

\[ P_i = \sum_{v \in V} e_v e_i^\dagger \otimes P_{v,i}. \]
They form a resolution of the identity $I_n \otimes I_d$, which follows by applying condition (12) to each block of these projectors.

Moreover, condition (5) in Definition 3 implies that

$$\left(I_n \otimes yy^\dagger\right)P_i(A \otimes I_d)P_i\left(I_n \otimes yy^\dagger\right) = 0_n \otimes 0_d.$$ 

for all $i \in [c]$.

To abbreviate, set $P = I_n \otimes yy^\dagger$. Note that the multiplication of $P_i(A \otimes I_d)P_i$ by the projector $P$ from the left and right is a so-called compression.

We now make use of [22, Lemma 1] to construct unitary matrix $U$ from the pinching projectors $P_i$ such that

$$\sum_{\ell \in [c]} PU^\ell(A \otimes I_d)(U^\dagger)^\ell P = 0_n \otimes 0_d$$

for any matrix $X \in \mathbb{C}^{d \times d}$. We obtain:

$$\sum_{\ell \in [c]} PU^\ell(E \otimes I_d)(U^\dagger)^\ell P = c E \otimes yy^\dagger$$

for any diagonal matrix $E \in \mathbb{C}^{n \times n}$. We now have the tools to prove Theorem 2. Note that we did not make use of condition (2).

### 4.1 Proof of the Lima bound in Theorem 2

**Proof.** The proof is almost identical to the proof for the chromatic number. We use the identity $D - Q = -A$. We have:

$$A \otimes yy^\dagger = P(A \otimes I_d)P$$

$$= \sum_{\ell=1}^{c-1} PU^\ell(-A \otimes I_d)(U^\dagger)^\ell P$$

$$= \sum_{\ell=1}^{c-1} PU^\ell((D - Q) \otimes I_d)(U^\dagger)^\ell P$$

$$= (c - 1)(D \otimes yy^\dagger) - \sum_{\ell=1}^{c-1} PU^\ell(Q \otimes I_d)(U^\dagger)^\ell P$$

Define the column vector $v = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^\dagger \otimes y$. Multiply the left and right most sides of the above matrix equation by $v^\dagger$ from the left and by $v$ from the right to obtain

$$\frac{2m}{n} = v^\dagger(A \otimes yy^\dagger)v = (c - 1)\frac{2m}{n} - \sum_{\ell=1}^{c-1} v^\dagger PU^\ell(Q \otimes I_d)(U^\dagger)^\ell P v \leq (c - 1)\frac{2m}{n} - (c - 1)\delta_n.$$
This uses that $v^\dagger(A \otimes yy^\dagger)v = v^\dagger(D \otimes yy^\dagger)v = 2m/n$, which is equal to the sum of all entries of respectively $A$ and $D$ divided by $n$ due to the special form of the vector $v$, and that $v^\dagger P U^\ell(Q \otimes I_d)(U^\dagger)^\ell P v = v^\dagger U^\ell(Q \otimes I_d)(U^\dagger)^\ell v \geq \lambda_{\min}(Q) = \delta_n$.  

4.2 Proof of the Hoffman and Kolotilina bounds in Theorem \[2\]

Proof. Let $E \in \mathbb{C}^{n \times n}$ be an arbitrary diagonal matrix. Using (13) and (14), we obtain

$$\sum_{\ell=1}^{c-1} P U^\ell(E \otimes I_d - A \otimes I_d)(U^\dagger)^\ell P = (c - 1)E \otimes yy^\dagger + A \otimes yy^\dagger.$$  

Using that $\lambda_{\max}(X) \geq \lambda_{\max}(PXP)$ and $\lambda_{\max}(X) + \lambda_{\max}(Y) \geq \lambda_{\max}(X + Y)$ for arbitrary Hermitian matrices $X$ and $Y$, we obtain

$$\lambda_{\max}(E - A) = \lambda_{\max}(E \otimes I_d - A \otimes I_d)$$
$$\geq \lambda_{\max}\left(E \otimes yy^\dagger + \frac{1}{c - 1}A \otimes yy^\dagger\right)$$
$$= \lambda_{\max}\left(E + \frac{1}{c - 1}A\right).$$

[5, Corollary 5] shows that the above eigenvalue bound implies

$$\lambda_{\max}(E - A) \geq \lambda_{\max}(E + A) - \frac{c - 2}{c - 1}\lambda_{\max}(A),$$

or equivalently

$$c \geq 1 + \frac{\lambda_{\max}(A)}{\lambda_{\max}(A) - \lambda_{\max}(E + A) + \lambda_{\max}(E - A)},$$

from which the Hoffman and Kolotilina bounds are obtained by setting $E = 0$ and $E = D$, respectively.  

4.3 Inertial and generalized Hoffman and Kolotilina bounds

We do not know whether the inertial bound in Theorem \[1\] or the generalized (multi-eigenvalue) bounds in \[5\] are also lower bounds for the vectorial chromatic number. The difficulty seems to be in determining what happens to the entire spectrum of the various matrices when they are compressed by $P = I_n \otimes yy^\dagger$. The Kolotilina and Lima bounds only use the maximal and/or minimal eigenvalues.

Bilu \[1\] proved that the Hoffman bound is a lower bound for $\chi_v(G)$. The Kolotilina and Lima bounds equal the Hoffman bound for regular graphs, but we do not know if these bounds lower bound the vector chromatic number for all graphs.
5 Proof of the inertial lower bound on the projective rank $\xi_f(G)$

We conjecture that for all graphs $G$ the projective rank $\xi_f(G)$ is lower bounded by

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \leq \xi_f(G).$$

Unfortunately, we are not able to settle this question by either providing a counterexample or proving this bound for all graphs. However, we are able to prove the weaker lower bound in Theorem 3.

We derive two lemmas to better organize the proof of Theorem 3.

**Lemma 2.** Let $P$ be an orthogonal projector and $X$ a positive semidefinite matrix in $\mathbb{C}^{m \times m}$. Then, we have

$$\text{rank}(PX^P) \geq \text{rank}(P) - \text{null}(X).$$

**Proof.** There exist positive semidefinite matrices $Y$ and $\Delta$ such that $Y$ has full rank, $\Delta$ has rank null($X$), and $X + \Delta = Y$. Using that rank($M + N$) $\leq$ rank($M$) + rank($N$) for arbitrary matrices, we obtain

$$\text{rank}(PX^P) + \text{rank}(P\Delta P) \geq \text{rank}(PY^P).$$

Using that rank($MN$) $\leq$ rank($M$) for arbitrary matrices $M$ and $N$, we obtain

$$\text{rank}(PX^P) \geq \text{rank}(PY^P) - \text{rank}(\Delta).$$

We can write

$$PY^P = (Y^{1/2}P)^\dagger(Y^{1/2}P).$$

Using that rank($M^\dagger M$) = rank($M$) for arbitrary matrices, we obtain

$$\text{rank}(PY^P) = \text{rank}(Y^{1/2}P) = \text{rank}(P)$$

because $Y^{1/2}$ has full rank.

**Lemma 3.** Let $P_v$ be the projectors of a $(d/r)$-orthogonal representation. Define the block diagonal projector

$$P = \sum_{v \in V} e_v e_v^\dagger \otimes P_v.$$

Then, we have

$$P(A \otimes I_d)P = 0_n \otimes 0_d$$

and

$$\text{rank}(P) = nr.$$
Proof. This follows directly from the orthogonality condition $P_v P_w = 0_d$ for all $vw \in E$. We refer the reader to [22], where a similar result is proved for the quantum chromatic number. The projectors $P_v$ have rank $r$ for all $v \in V$ so $\text{rank}(P) = nr$. \qed

We are now ready to prove Theorem 3.

Proof. Let $A = B - C$, defined as in Section 3, so $\text{rank}(B) = n^+$ and $\text{rank}(C) = n^-$. Note that Lemma 3 implies

$$P(B \otimes I_d)P = P(C \otimes I_d)P,$$

so that

$$P(B \otimes I_d)P = \frac{1}{2} P((B + C) \otimes I_d)P. \quad (15)$$

Clearly, the rank of the left hand side of (15) is bounded from above by $n^+ d = \text{rank}(B \otimes I_d)$.

We now bound the rank of the right hand side of (15) from below. Observe that $B + C = |A|$, where $|A| = \sum_{i=1}^{n} |\mu_i| e_i e_i^\dagger$ and $\mu_i$ and $e_i$ are the eigenvalues and eigenvectors of $A$, respectively. Clearly, $|A|$ is positive semidefinite, its rank is equal to $\text{rank}(A) = n^+ + n^-$ and its nullity is equal to $\text{null}(A) = n_0$. Therefore, $|A| \otimes I_d$ is positive semidefinite, its rank is equal to $(n^+ + n^-) d$ and its nullity is equal to $n_0 d$. We can now apply Lemma 2 to obtain

$$\text{rank} \left( P(|A| \otimes I_d)P \right) \geq \text{rank}(P) - n_0 d = nr - n_0 d.$$ Combining the upper and lower bounds on the ranks, we obtain

$$n^+ d \geq nr - n_0 d \iff \frac{d}{r} \geq 1 + \frac{n^-}{n^+ + n^0}.$$ The result

$$\frac{d}{r} \geq 1 + \frac{n^+}{n^- + n^0}$$

is obtained by considering $P(C \otimes I_d)P$ on the left hand side of (15). \qed

6 On the equivalence of $\xi^{[r]}(G)$ and $\xi(G^{[r]})$

We now show that the $r$-fold orthogonal rank $\xi^{[r]}(G)$ of $G$ is equal to the orthogonal rank $\xi(G^{[r]})$, where $G^{[r]}$ arises from $G$ by a simple graph operation.

Remark 3. A $(d/r)$-orthogonal representation for $G$ in $\mathbb{C}^d$ can be equivalently characterized by a collection of suitable vectors $x_{v,k} \in \mathbb{C}^d$ for $v \in V$ and $k \in [r]$, where $[r] = \{1,\ldots,r\}$. The relationship between the orthogonal projectors $P_v$ and the vectors $x_{v,i}$ is given by

$$P_v = \sum_{k=1}^{r} x_{v,k} x_{v,k}^\dagger.$$ so that the following two properties hold:
\[ x_{v,k}^\dagger x_{v,\ell} = 0 \text{ for all } k, \ell \in [r] \text{ with } k \neq \ell \]
\[ x_{v,k}^\dagger x_{w,\ell} = 0 \text{ for all } vw \in E \text{ and all } k, \ell \in [r]. \]

For each \( v \in V \), the vectors \( x_{v,k} \) for \( k \in [r] \) form an orthonormal basis for the subspaces \( S_v \), which form a so-called \((d; r)\) orthogonal subspace representation [10].

**Definition 4.** The \( r \)-fold vertex-cloned graph \( G^{[r]} = (V^{[r]}, E^{[r]}) \) is defined to be the lexicographic product of \( G \) and \( K_r \), where \( K_r \) denotes the complete graph on \( r \) vertices. The adjacency matrix \( A^{[r]} \) of \( G^{[r]} \) is given by
\[
A \otimes J_r + I_n \otimes K_r,
\]
where \( J_r \) is the all-one-matrix of size \( r \times r \) and \( K_r = J_r - I_r \) is the adjacency matrix of the complete graph on \( r \) vertices.

Its vertex set \( V^{[r]} \) is equal to \( V \times [r] \), and its edge set \( E^{[r]} \) is defined as follows:

- \((v, k)(v, \ell) \in E^{[r]} \) for all \( k, \ell \in [r] \) with \( k \neq \ell \) and all \( v \in V \)
- \((v, k)(w, \ell) \in E^{[r]} \) for all \( vw \in E \) and all \( k, \ell \in [r] \).

Observing that the above two conditions precisely fit the two conditions in the remark above, the following lemma is readily proved:

**Lemma 4.** For any graph \( G \), the \( r \)-fold orthogonal rank \( \xi^{[r]}(G) \) of \( G \) is equal to the orthogonal rank \( \xi(G^{[r]}) \) of the \( r \)-fold vertex-cloned graph \( G^{[r]} \), that is
\[
\xi^{[r]}(G) = \xi(G^{[r]}).
\]

Consequently, we have:
\[
\xi_f(G) = \lim_{r \to \infty} \frac{\xi(G^{[r]})}{r}.
\]

Unfortunately, we do not know how to leverage this lemma to prove a useful spectral lower bound on the projective rank.

### 7 Implications for the projective rank

The following examples demonstrate that the inertial bound is exact for \( \xi_f \) for various classes of graphs. We also use Theorem 3 to derive the value of \( \xi_f \) for some graphs.

- For odd cycles, \( C_{2k+1} \) (see [6] and [16]):
  \[
  1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = \chi_f = \xi_f = 2 + \frac{1}{k}; \text{ but } \chi_{vect} = \chi_q = \xi = \chi = 3.
  \]
• For Kneser graphs, $K_{p,k}$ (see [6], [9] and [18]):

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = \chi_v = \chi_f = \xi_f = \frac{p}{k}; \chi_{\text{vect}} = \left\lceil \frac{p}{k} \right\rceil; \text{ but } \chi = p - 2k + 2.$$

• The orthogonality graph, $\Omega(n)$, has vertex set the set of $\pm 1$-vectors of length $n$, with two vertices adjacent if they are orthogonal. With $n$ a multiple of 4 (see [16, Lemma 4.2 and Theorem 6.4]):

$$\chi_{sv} = \chi_{\text{vect}} = \xi_f = \xi = \chi_q = n; \text{ but } \chi_f > n \text{ and } \chi \text{ is exponential in } n.$$

$\Omega(4)$ has spectrum $(6^2, 0^8, -2^6)$ so when $n = 4$:

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^+}{n^-} = 4 = \xi_f = \chi_q,$$

but for $n > 4$ this inertial bound is less than $\xi_f$.

• The Andrásfai graphs, And$(k)$, are $k$-regular with $(3k - 1)$ vertices. It is known ([8] and [7]) that

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^+}{n^-} = 1 + \frac{2k - 1}{k} = 3 - \frac{1}{k} = \chi_f$$

but

$$\chi = \chi_q = \xi = 3.$$

The Andrásfai graphs are non-singular, so using Theorem 3 and that $\xi_f \leq \chi_f$ it follows that $\xi_f = 3 - 1/k$.

• The Clebsch graph on 16 vertices has spectrum $(5^1, 1^{10}, -3^5)$ and $\chi_f = 3.2$ (see [7]). Therefore

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^+}{n^-} = 3.2 = \chi_f; \text{ but } \chi = \xi = 4.$$

The Clebsch graph is non-singular, so using Theorem 3 and that $\xi_f \leq \chi_f$, it follows that $\xi_f = 3.2$.

More generally, if the inertial bound is exact for the fractional chromatic number of a non-singular graph, then it is also exact for the projective rank. Vertex transitive graphs have $\xi_f \leq \chi_f = n/\alpha$, so if a non-singular vertex transitive graph has

$$\alpha = \min (n^+, n^-), \text{ then } \xi_f = \chi_f = \frac{n}{\alpha}.$$
8 Conclusion

We have proved that many lower bounds for \( \chi(G) \) are also lower bounds for \( \xi(G) \). We have also proved that for non-singular graphs

\[
1 + \max \left( \frac{n^+}{n^-}, \frac{n^-}{n^+} \right) \leq \xi_f(G).
\]

Elphick and Wocjan [6] proved this lower bound for \( \chi_f \) for non-singular graphs, using a simpler proof technique.

Costello et al [3] proved that almost all (random) graphs with no isolated vertices are non-singular. This provides limited support for our conjecture that all of the spectral lower bounds described in this paper are also lower bounds for \( \xi_f(G) \), and consequently for \( \chi_f(G) \).

Acknowledgements

This research has been supported in part by NSF Award 1525943.

References

[1] Y. Bilu, Tales of Hoffman: Three extensions of Hoffman’s bound on the chromatic number, J. Combin. Theory Ser. B, 96, (2006), 608 - 613.

[2] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini and A. Winter, On the quantum chromatic number of a graph, Elec. J. Combinatorics, 14, (2007), R81.

[3] K. P. Costello, T. Tao and V. Vu, Random symmetric matrices are almost surely non-singular, Duke Math. J. 135, (2006), 395 - 413.

[4] T. Cubitt, L. Mancinska, D. E. Roberson. S. Severini, D. Stahlke and A. Winter, Bounds on Entanglement-Assisted Source-Channel Coding via the Lovász \( \theta \) number and its variants, IEEE Transations on Inf. Theory, 60, 11 (2014), 7330 - 7344.

[5] C. Elphick and P. Wocjan, Unified spectral bounds on the chromatic number, Discussiones Mathematicae Graph Theory 35 4, (2015), 773-780

[6] C. Elphick and P. Wocjan, An inertial lower bound for the chromatic number of a graph, Elec. J. Combinatorics, 24(1), (2017), P1.58.

[7] C. Godsil, Interesting graphs and their Colourings, (2006).

[8] C. Godsil and G. F. Royle, Algebraic Graph Theory, Springer, Graduate Texts in Mathematics 207, (2013).
[9] C. Godsil, D. E. Roberson, B. Rooney, R. Samal and A. Varvitsiotis, *Graph homomorphisms via vector colorings*, https://arxiv.org/abs/1610.10002, (2016).

[10] L. Hogben, K. F. Palmowski, D. E. Roberson and S. Severini, *Orthogonal representations, Projective Rank and Fractional Minimum Positive Semidefinite Rank : Connections and New Directions*, Elec. J. Linear Algebra 32 (2017), 98 - 115.

[11] A. J. Hoffman, *On eigenvalues and colorings of graphs*, in: Graph Theory and its Applications (B. Harris ed.) Acad. Press, New York, (1970).

[12] L. Yu. Kolotilina, *Inequalities for the extreme eigenvalues of block-partitioned Hermitian matrices with applications to spectral graph theory*, J. Math. Sci 176 (2011), 44 - 56 (translation of the paper originally published in Russian in Zapiski Nauchnykh Seminarov POMI 382 (2010), 82 - 103.)

[13] D. Karger, R. Motwani and M. Sudan, *Approximate graph coloring by semidefinite programming*, J. ACM 45(2), (1998), 246 - 265.

[14] L. S. de Lima, C. S. Oliveira, N. M. M. de Abreu and V. Nikiforov, *The smallest eigenvalue of the signless Laplacian*, Linear Algebra Appl., 435, (2011), 2570 - 2584.

[15] L. Lovasz, *On the Shannon capacity of a graph*, IEEE Trans. Inf. Th., 25(1), (1979), 1 - 7.

[16] L. Mancinska and D.E. Roberson, *Graph homomorphisms for quantum players*, J. Combinatorial Theory, Ser. B, 118, (2016), 228 - 267.

[17] L. Mancinska and D. E. Roberson, *Oddities of Quantum Colorings*, Baltic J. Modern Computing, 4, (2016), 846 - 859.

[18] V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov and A. Winter, *Estimating quantum chromatic numbers*, J. Functional Analaysis, 270(6), (2016), 2180 - 2222.

[19] V. I. Paulsen and I. G. Todorov, *Quantum chromatic numbers via operator systems*, Quarterly J. Mathematics, 66(2), (2015), 677 - 692.

[20] M. Szegedy, *A note on the \( \theta \) number of Lovász and the generalized Delsarte bound*, in Proc. Annual Symposium on Foundations of Computer Science 1994, (1994), 36 - 39.

[21] P. Wocjan and C. Elphick, *New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix*, Elec. J. Combinatorics, 20(3), (2013), P39.

[22] P. Wocjan and C. Elphick, *Spectral lower bounds on the quantum chromatic number*, https://arxiv.org/abs/1805.08334, (2018).