Features of radiation heat transfer in a medium with variable refractive index

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Abstract. In the literature one can find at least three different radiative transfer equations and various radiation diffusion equations for media with spatially varying refractive indices. We argue that the classical version of the transport equation seems to be valid. A generalized radiation diffusion equation, which coincides with the results of several authors, is derived. The equation allows one to take into account the reflection from the internal interfaces by means of the delta function in the radiation diffusion coefficient. This equation makes it possible to calculate heat exchange by radiation in a medium with a variable refractive index without additional computational costs. The physical meaning of the coefficients of the generalized radiation diffusion equation is discussed. The diffusion coefficient determines the momentum exchange between radiation and matter, and the absorption coefficient determines the exchange of energy.

1. Introduction
The radiative transfer equation (RTE) in absorbing-emitting and scattering media has applications in a wide variety of subjects, including optics, astrophysics, atmospheric science, neutron transport, biomedical optics, semitransparent crystal growth, fuel combustion, rocket propulsion, hypersonic shock layers, and heat transfer in porous media. The RTE for light refractive media was introduced and studied in astrophysics and physics of planetary atmospheres [1-3], physics of plasmas [4], and only recently attracted attention in biomedical optics and engineering applications [5-16].

Attempts to solve transport equation in a medium with a coordinate-dependent refractive index lead to considerable difficulties. Complexity arises as a result of the curvature of the rays due to contiguous change of the refractive index, and due to the occurrence of reflection from the surfaces of discontinuity of the refractive index. This occurs due to melting, pore collapse in insulations, gas ionization, and so on. Additional difficulties arise when the optical properties of the medium or the location of the boundaries are not known beforehand and are defined in the solution process, for example, when solving the Stefan problem. Refractive index alterations are especially significant in living biological tissues.

Because of the wide range of applications of the transport equation terminology varies between different fields. The RTE can be written out for intensity, specific intensity, brightness, radiance, and so on, with different definitions of those quantities. In a medium with a constant refractive index, this does not cause serious problems. However, discrepancies in notation cause problems in interpreting results when refractive index varies within the medium.
Interactions of light with the matter due to absorption, emission, and scattering are well understood, but the interaction of refraction is not. There are currently three [5] or even four [13] different versions of radiative transfer equation involving refraction for nonhomogeneous media. These equations are reduced to the same equation for homogeneous media when refractive index is assumed constant.

In classical monographs [1–4] and in most articles, the radiation intensity is defined either phenomenologically or is considered proportional to the photon’s distribution function $f$. The (specific) intensity of radiation is defined as $I = hv(c/n)f$ $I = hv(c/n)f$, where $hv$ is the energy of a single photon, $c$ is the speed of light in vacuum and $n$ is the refractive index [3]. This leads to well-known radiative transfer equation [1–4]

$$\frac{n\partial I}{c\partial t} + n^2 \frac{d}{ds} \frac{I}{n^2} = -(k + \beta)I + kn^2 I_p(r, t) + \beta \int_{4\pi} \sigma(\Omega, \Omega') I(\Omega') d\Omega' ,$$

(1)

where $\frac{d}{ds}$ is the derivative along the ray, $k$ is the absorption coefficient, $\beta$ is the scattering coefficient, $\sigma(\Omega, \Omega')$ is the scattering phase function, describing the probability of scattering from direction $\Omega'$ to $\Omega$, $T$ is the temperature, $I_p$ is the equilibrium (Planck) radiation intensity.

The equation (1) shows that the fundamental quantity $\frac{I}{n^2}$, the radiation intensity divided by the square of the refractive index, remains unchanged between collisions along the ray. The same quantity enters into the boundary conditions for the transport equation [1–4].

Serious difficulties arise when trying to determine the intensity in the geometric optics limit through the time averaged Poynting vector [17, 18]. From general considerations it is clear that the path from the time-reversible (at least in a vacuum) Maxwell equations to the irreversible kinetic equation of radiation transfer (1) cannot be simple and short. Having no place to discuss this issue here, we note that different definitions of radiation intensity lead to different conservation laws [12, 17], and apparently these problems are closely related to the Abraham-Minkowski controversy [22, 23].

Below we show that the reflection from the internal boundary can be accounted for by introducing delta-functions in the radiation diffusion coefficient. In this case, the boundary thermal radiation flux is continuous, and the radiation energy density has a discontinuity. It is shown that the generalized diffusion coefficient of radiation determines the momentum exchange between radiation and matter, and the generalized absorption coefficient determines the exchange of energy.

2. New versions of radiative transfer equations

Recently, several new versions of the radiative transfer equation were proposed and compared in reviews [5, 13]. The radiation transfer equation can be rewritten for the specific intensity $L(r, \Omega, t)$ in the form [5]

$$\frac{n\partial L}{c\partial t} + \nabla \cdot \nabla L + \Sigma + (k + \beta)L = \kappa n^2 L_p + \beta \int_{4\pi} \sigma(\Omega, \Omega') L(r, \Omega', t) d\Omega' ,$$

(2)

where $\nabla$ is the gradient with respect to position $r$. The specific intensity $L(r, \Omega', t)$ is the radiant power flux density at position $r$, direction $\Omega$ and time $L(r, \Omega', t)$. The intensity $I(s)$ and the specific intensity $t$, satisfy relation $I(s) = L(s) d\Omega(s)$ [5]. The term $\Sigma$ on the left-hand side is the streaming term, different among the three formulations of RTE from [5–16], which describes the changes of the specific intensity because of refraction. The term $\Sigma$ has three different forms:

- RTE1 ref. [3, 4, 6–8, 12, 14]:
  $$\Sigma^1 = \frac{1}{n} \left[ \nabla n - (\nabla n \cdot \Omega) \Omega \right] \cdot \nabla \Omega L - \frac{2}{n} \left( \nabla n \cdot \Omega \right) L ,$$

(3)

where $\nabla \Omega$ is the gradient with respect to direction $\Omega$

- RTE2 ref. [9, 10, 12, 16]:
  $$\Sigma^{21} = \frac{1}{n} \left[ \nabla n - (\nabla n \cdot \Omega) \Omega \right] \cdot \nabla \Omega L + \frac{1}{n} \left( \nabla^2 S \cdot \Omega \cdot \nabla n \right) L ,$$

(4)

- RTE3 ref. [5, 11]:
  $$\Sigma^3 = \frac{1}{n} \left[ \nabla n - (\nabla n \cdot \Omega) \Omega \right] \cdot \nabla \Omega L - \frac{2}{n} (\nabla n \cdot \Omega) L + \frac{1}{n} \left( \nabla^2 S \cdot \Omega \cdot \nabla n \right) L ,$$

(5)
where $S$ is the eikonal which is to be found by solving the eikonal equation [17]

$$|\nabla S|^2 = n^2 \ . \tag{6}$$

Three terms are used to describe the change of the specific intensity because of refraction. The first term $\frac{1}{n}[\nabla n - (\nabla \cdot \Omega)\Omega] \cdot \nabla L$, is due to the differential change of the specific intensity along a ray path, which is the same between the three formulations. The second term $-\frac{2}{n}(\nabla \cdot \Omega)L$, is due to the change of the directional differential along the ray path, which is in RTE1 and RTE3 but not in RTE2.

The third term $\frac{1}{n}(\nabla^2 S \cdot \Omega \cdot \nabla n)L$, is the ray divergence term due to the refractive variation of the transverse section of a ray tube along the ray path [5].

Ref. [5] argues that only the radiative transfer equation RTE3 is consistent with the intensity law of geometric optics. Ref. [13] prefers RTE2 and Tualle [14] showed that RTE1 do not contradict to RTE2. Both ref. [5, 12] believe that the difference of those results is because of the difference of the (specific) intensity definition.

In our opinion, the situation remains unclear and requires further research. Moreover, compliance with the laws of geometric optics is useful for diagnostic tasks. For radiative energy transfer problems, the integrated over solid angle characteristics of the radiation are more important. It is essential to fulfill the laws of conservation of energy and momentum in the whole heat and mass transfer problems [19, 20]. Therefore, we will proceed from the classical transport equation (1), RTE1 and investigate the radiation diffusion approximation.

3. Radiation diffusion equation

Radiation diffusion equation is one of the most popular methods of approximate solutions of the transport equation. This equation is always, at least qualitatively, correctly describes the characteristics of energy transfer by thermal radiation. To simplify the calculations, we consider the stationary one-dimensional problem and obtain the radiation diffusion equation in a medium with variable refractive index.

Strictly speaking, in the equation (1) $n(x)$ is a ray index of refraction [4]. However, in the case of an isotropic medium, it coincides with the usual refractive index. In an isotropic medium the Snell law $n(x)\sin \theta(x) = \text{const}$ is valid along the ray, where $\theta(x)$ is the angle between the ray and the positive direction of the coordinate axis. The Snell law is actually a consequence of the eikonal equation (6) and replaces it in the one-dimensional case.

From the Snell law follows

$$\frac{d}{ds} = \mu \frac{d}{dx} + \frac{1}{n} \frac{dn}{dx} (1 - \mu^2) \frac{d}{d\mu} \ , \tag{7}$$

where $\mu = \cos \theta$.

Substituting (7) into (1) and integrating over the angle, both directly and after multiplication on $\mu$ we obtain

$$\frac{d}{dx} K = -U + k n^2 U_p (T(x)) \ , \tag{8}$$

$$\frac{d}{dx} K = -\frac{n}{n'} (K - U) + \left( k + \beta(1 - \bar{\mu}) \right) q \ . \tag{9}$$

where $U = \int_{4\pi} l d\Omega$ is the energy density of the radiation (without a factor equal to the inverse group velocity of light in the medium), $q = \int_{4\pi} l \mu d\Omega$ is the energy flux density, $K = \int_{4\pi} l \mu^2 d\Omega$, $U_p = 4\pi l_p$ is the energy density of the equilibrium radiation in a vacuum, and $\bar{\mu}$ is the average cosine of the scattering.

Using asymptotic equality $K \approx U/3$ that is valid in the diffusion limit, we exclude $K$ from (9):

$$\frac{d}{dx} U = \frac{3(k + \beta(1 - \bar{\mu}))}{n^2} q \ , \tag{10}$$

which in a more conventional form can be rewritten as Fick's law for a medium with variable refractive index.
\[ q = -Dn^2 \frac{d}{dx} \frac{u}{n^2} = -\frac{n^2}{3(k+\beta(1-\mu))} \frac{d}{dx} \frac{u}{n^2}, \]  
\[ \text{where } D = 1/3(k + \beta(1 - \mu)) \text{ is the usual radiation diffusion coefficient.} \]

With ordinary law of energy conservation \((8)\), this expression gives the equation of radiation diffusion in a medium with variable refractive index.

\[ -\frac{d}{dx} Dn^2 \frac{d}{dx} \frac{u}{n^2} + kU = kn^2 U_p(T(x)) . \]  
\[ \text{The diffusion equation (12) should be supplemented by the usual boundary conditions at the outer boundaries, which can be written out in the form} \]
\[ -Dn^2 \frac{d}{dx} \frac{u}{n^2} \pm \frac{1}{2} \frac{1-R_{t\text{le}(ri)}}{2+R_{t\text{le}(ri)}} \left(U - n^2 U_p(T_{t\text{le}(ri)})\right) = 0 \text{, at } x = x_{t\text{le}(ri)} , \]
\[ \text{where } R_{t\text{le}(ri)} \text{ is the hemispherical reflection coefficient of the left (right) boundary and } T_{t\text{le}(ri)} \text{ is the temperature of the external medium.} \]

This equation is obviously generalized to the multidimensional case:

\[ -\nabla Dn^2 \frac{u}{n^2} + kU = kn^2 U_p(T(r)) . \]  
\[ \text{Equation (14) coincides with the diffusion equations obtained in Refs } 6-8. \text{ All these works use the classical radiative transfer equation (1) i.e. RTE1. However, in Refs } 12, 15, 16 \text{ three different equations for diffusion of radiation in a medium with a variable refractive index was obtained. But none of them can withstand criticism } 7. \]

4. Accounting for reflection from internal boundaries

So far, we have assumed that refractive index depends smoothly on the coordinate. Now suppose that the refractive index changes abruptly, for example, during a phase transition, or assume that there is an internal interface between media with known reflection and transmission coefficients. For the radiation intensity at the interface, in the general case, one can write

\[ I_{t\text{le}(ri)}^{-(+)}(\Omega') = \int_{0}^{\pi} r_{t\text{le}(ri)}(\Omega', \Omega) I_{t\text{le}(ri)}^{-(+)}(\Omega) \mu d\Omega + \int_{0}^{\pi} p_{t\text{le}(ri)}(\Omega', \Omega) I_{t\text{le}(ri)}^{-(+)}(\Omega) \mu d\Omega , \]

where \( r(\Omega', \Omega), p(\Omega', \Omega) \) is the bi-directional coefficients of reflection and transmission of the boundary.

Integrating these relations with respect to the angle, and using the reciprocity relations for the transmittances, we obtain

\[ \frac{u_r}{n_r^2} = \frac{u_l}{n_l^2} = \frac{2(2-p_{t\text{le}(ri)}-p_r)}{pn^2} q , \]

where \( pn^2 = p_r n^2(\xi + 0) = p_l n^2(\xi - 0) \) and \( p_{t\text{le}(ri)} \) is the transmission coefficient of the boundary when radiation propagates from the right (left) side of the boundary see Fig. 1.

\[ \text{Fig. 1 A boundary inside the medium} \]

Comparing expressions (10) and (16), it is easy to see that with the help of the delta function they can be written out in a uniform way

\[ \frac{d}{dx} \frac{u}{n^2} = -\left[ \frac{3(k+\beta(1-\mu))}{n^2} + \delta(x - \xi) \frac{2(2-p_{t\text{le}(ri)}-p_r)}{pn^2} \right] q , \]  
\[ \text{where } D = 1/3(k + \beta(1 - \mu)) \text{ is the usual radiation diffusion coefficient.} \]
where $\xi$ is the interface coordinate. Obviously, if there are several boundaries in the layer, delta functions should be added at each interface coordinate.

The equation (17) with the energy equation (10) and ordinary boundary conditions (13) form a closed system of Caratheodory ordinary differential equations with generalized functions in the coefficients [24]. The existence and uniqueness theorems and the continuous dependence of solutions on the coefficients are proved for the Caratheodory equations. This means that if we approximate the delta function $\delta(x)$ by a "cap" function $\delta_c(x)$ and solve the problem numerically with the help of ordinary homogeneous conservative difference schemes, we can expect good convergence to the discontinuous solution.

The radiation diffusion equations (10), (17) may be rewritten in the form (12) with new radiation diffusion coefficient

$$D_\varepsilon = \frac{1}{\{(k+\beta(1-\varepsilon))\}} + Pn^2\delta_\varepsilon(x - \xi),$$

where $P = 2(2 - p_l - p_r)/pm^2$. Such a diffusion equation uniformly describes not only a smooth change in the refractive index, but also possible reflection and scattering of radiation at jumps of the refractive index.

The relation at the boundary with the jump of the refractive index inside a medium can be approximated by discontinuous solutions. This convergence is to be verified by computational experiments.

If we introduce a new quantity $N = \frac{u}{n^2} = \int_{\Omega} I/n^2 d\Omega$, the record of the diffusion equation (14) can be future simplified

$$-\nabla M_\varepsilon \nabla N + EN = EU_p(T(x)),$$

where new coefficients $M_\varepsilon$ and $E$ of the new diffusion equation are related to old coefficients by relations

$$M_\varepsilon = D_\varepsilon n^2,$$

$$E = kn^2.$$
where 

\[
w(x) = \nabla q = E(U_p(T(x)) - N).
\]

Moreover, as measured in experiments, the coefficient of diffuse reflection or the emissivity of a plane layer [25] also depends only on the coefficients \(M_e\) and \(E\):

\[
1 - R = \varepsilon = 4n^2\sqrt{kDth(H/k/D)} = 4\sqrt{EMth(E/M)}.
\]

where \(H\) is the layer thickness. Thus, if optical properties of a material are measured in an experiment, then to calculate the energy transfer by radiation in the diffusion limit [25], one can completely eliminate the refractive index.

The same is obviously true in the multidimensional case. This means that in the diffusion limit for calculating energy transfer by radiation, known software packages developed for a constant refractive index can be used simply by redefining the coefficients of the radiation diffusion equation.

In one-dimensional case, the proposed approach has been successfully used in numerical modeling of laser ablation of porous heat shielding [21]. The work took into account significant changes in the refractive index due to softening and shrinkage of the material.

5. Discussion

From the equations (22) and (25), it is easy to see that the radiative heat transfer operator acting on the Planck radiation energy density and calculating the divergence of the radiant flux is self-adjoint operator. This is equivalent to conservativeness of the corresponding equations, that is, the validity of the energy conservation law. This is an important argument in favor of the validity of the obtained radiation diffusion equation in a medium with a variable refractive index.

When describing radiation transfer using equation (1), we actually divide a system into subsystems: radiation and matter, which, interacting, exchange energy and momentum with each other. For simplicity, we first consider modeling highly rarefied medium consisting of macroscopic particles (e.g., micron glass beads) in vacuum. When a photon is absorbed (or emitted) by a ball, the photon transfers (takes away) its energy \(hv\) and momentum \(hv/c\) from the radiation to the matter. The intensity of this process is obviously proportional to the absorption cross section \(\sigma_{av}\) of the particles or to the absorption coefficient \(k_v\), \(k_v = \sigma_{av}C\) where \(C\) is the particle concentration.

Since the scattering proceeds without changing the frequency (as is usually assumed), there is no exchange of energy between radiation and matter, but only momentum exchange occurs. It is easy to calculate that during scattering the photon transfers the momentum to the ball in proportion to the transport scattering cross section \(\sigma_{trv} = \sigma_{sv}(1-\mu_v)\) or \(\beta_v = \sigma_{sv}C\). In the expression for the radiation diffusion coefficient (17) absorption and scattering enter in such a way that the diffusion coefficient can be naturally interpreted as a coefficient describing the total (due to absorption and scattering) momentum exchange between radiation and matter. Similarly, the absorption coefficient can naturally be interpreted as a coefficient describing the energy exchange between radiation and matter.

In a dense refractive medium, the situation is not so obvious, since the definition of the momentum of a photon (or a wave packet) in a medium is the subject of the old Abraham-Minkowski controversy [22, 23]. Using symmetry considerations, it can be shown that in a dense medium the coefficients \(M_e\) and \(E\), as before, can be interpreted as the coefficients describing the exchange of momentum and energy between radiation and matter.

In conclusion, it is useful to try to derive the diffusion equation (22) directly from the fluctuation electrodynamics [26, 27], bypassing the transport equation. This theory is free from the limitations of geometric optics and, perhaps, may circumvent the above problems.

6. Conclusions

It is well known that light in a medium with a variable refractive index can propagate in a very bizarre manner, generating caustics and other sophisticated objects [12]. Even more surprising that in the radiation diffusion limit a very simple equation (22) arises, for the solution of which the refractive index does not appear to be needed at all. The coefficients of this equation describe the exchange of momentum and energy between radiation and matter.
The equation (22) allows one to take into account the reflection and scattering at internal boundaries uniformly by introducing a delta function in the radiation diffusion coefficient. Radiation diffusion equation (22) makes it possible to calculate heat exchange by radiation in a medium with a variable refractive index by known software packages without additional computational costs.

We started from the definition of intensity through the photon distribution function as \( I = \frac{\nu}{n^2} = \int_{4\pi} \frac{I}{n^2} d\Omega \) that can naturally be interpreted as the photon diffusion equation. The quantity \( N = U = \frac{\nu}{n^2} = \int_{4\pi} \frac{I}{n^2} d\Omega \) is proportional to the number of photons.

On the other hand, it is natural to require that in a medium is conserved a radiation energy density, which is proportional to \( n^2 \partial n(\nu) \nu/\partial \nu \) [28], where \( \nu \) is the radiation frequency. In our case, the radiation energy density obviously is not conserved. Apparently, this does not lead to significant problems. However, in our opinion, the choice of radiation transport equation and diffusion equation in a medium with a variable refractive index remains open question and requires further research.

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