A Central Limit Theorem for biased random walks on Galton-Watson trees

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Abstract

Let $\mathcal{T}$ be a rooted Galton-Watson tree with offspring distribution $\{p_k\}$ that has $p_0 = 0$, mean $m = \sum k p_k > 1$ and exponential tails. Consider the $\lambda$-biased random walk $\{X_n\}_{n \geq 0}$ on $\mathcal{T}$; this is the nearest neighbor random walk which, when at a vertex $v$ with $d_v$ offspring, moves closer to the root with probability $\lambda/(\lambda + d_v)$, and moves to each of the offspring with probability $1/(\lambda + d_v)$. It is known that this walk has an a.s. constant speed $v = \lim_n |X_n|/n$ (where $|X_n|$ is the distance of $X_n$ from the root), with $v > 0$ for $0 < \lambda < m$ and $v = 0$ for $\lambda \geq m$. For all $\lambda \leq m$, we prove a quenched CLT for $|X_n| - nv$. (For $\lambda > m$ the walk is positive recurrent, and there is no CLT.) The most interesting case by far is $\lambda = m$, where the CLT has the following form: for almost every $\mathcal{T}$, the ratio $|X_{nt}|/\sqrt{n}$ converges in law as $n \to \infty$ to a deterministic multiple of the absolute value of a Brownian motion. Our approach to this case is based on an explicit description of an invariant measure for the walk from the point of view of the particle (previously, such a measure was explicitly known only for $\lambda = 1$) and the construction of appropriate harmonic coordinates.

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1 Introduction and statement of results

Let $\mathcal{T}$ be a rooted Galton-Watson tree with offspring distribution $\{p_k\}$. That is, the numbers of offspring $d_v$ of vertices $v \in \mathcal{T}$ are i.i.d. random variables, with $P(d_v = k) = p_k$. Throughout this paper, we assume that $p_0 = 0$, and that $m := \sum k p_k > 1$. In particular, $\mathcal{T}$ is almost surely an infinite tree. For technical reasons, we also assume the existence of exponential moments, that is the existence of some $\beta > 1$ such that $\sum \beta^k p_k < \infty$. We let $|v|$ stand for the distance of a vertex $v$ from the root of $\mathcal{T}$, and let $o$ denote the root of $\mathcal{T}$.

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We are interested in \( \lambda \)-biased random walks on the tree \( T \). These are Markov chains \( \{X_n\}_{n \geq 0} \) with \( X_0 = o \) and transition probabilities

\[
P_T(X_{n+1} = w|X_n = v) = \begin{cases} 
\frac{\lambda}{\lambda + d(v)}, & \text{if } v \text{ is an offspring of } w, \\
1/(\lambda + d(v)), & \text{if } w \text{ is an offspring of } v.
\end{cases}
\]

Let \( GW \) denote the law of Galton-Watson trees. Lyons \cite{Lyons13} showed that

- If \( \lambda > m \), then for \( GW \)-almost every \( T \), the random walk \( \{X_n\} \) is positive recurrent.
- If \( \lambda = m \), then for \( GW \)-almost every \( T \), the random walk \( \{X_n\} \) is null recurrent.
- If \( \lambda < m \), then for \( GW \)-almost every \( T \), the random walk \( \{X_n\} \) is transient.

In the latter case, \( \lambda < m \), it was later shown in \cite{Lyons16} and \cite{Lyons17} that \( |X_n|/n \to v > 0 \) almost surely, with a deterministic \( v = v(\lambda) \) (an explicit expression for \( v \) is known only for \( \lambda = 1 \)).

Our interest in this paper is mainly in the critical case \( \lambda = m \). Then, \( |X_n|/n \) converges to 0 almost surely. Our main result is the following.

**Theorem 1** Assume \( \lambda = m \). Then, there exists a deterministic constant \( \sigma^2 > 0 \) such that for \( GW \)-almost every \( T \), the processes \( \{|X_{\lfloor nt\rfloor}|/\sqrt{\sigma^2 n}\}_{t \geq 0} \) converges in law to the absolute value of a standard Brownian motion.

Theorem 1 is proved in Section 6 by coupling \( \lambda \)-biased walks on \( GW \) trees to \( \lambda \)-biased walks on auxiliary trees, which have a marked ray emanating from the root. The ergodic theory of walks on such trees turns out (in the special case of \( \lambda = m \)) to be particularly nice. We develop this model and state the Central Limit Theorem (CLT) for it, Theorem 2, in Section 2. The proof of Theorem 2, which is based on constructing appropriate martingales and controlling the associated corrector, is developed in Sections 3, 4 and 5.

We conclude by noting that when \( \lambda > m \), the biased random walk is positive recurrent, and no CLT limit is possible. On the other hand, \cite{Lyons17} proved that when \( \lambda < m \) and the walk is transient, there exists a sequence of stationary regeneration times. Analyzing these regeneration times, one deduces a quenched invariance principle with a proper deterministic centering, see Theorem 3 in Section 7 for the statement. We note in passing that this improves the annealed invariance principle derived in \cite{Lyons20} for \( \lambda = 1 \).

## 2 A CLT for trees with a marked ray

We consider infinite trees \( T \) with one (semi)-infinite directed path, denoted \text{Ray}, starting from a distinguished vertex, called the \text{root} and denoted \( o \). For vertices \( v, w \in T \), we let \( d(v, w) \) denote the length of the (unique) geodesic connecting \( v \) and \( w \) (we consider the geodesic as containing both \( v \) and \( w \), and its length as the number of vertices in it minus one). A vertex \( w \) is an offspring of a vertex \( v \) if
Figure 1: Tree, Ray and horocycle distance

$d(v, w) = 1$ and either $d(w, \text{Ray}) > d(v, \text{Ray})$ or $v, w \in \text{Ray}$ and $d(v, o) > d(w, o)$. In particular, the root is an offspring of its unique neighbor on Ray. For any vertex $v \in T$, we let $d_v$ denote the number of offspring of $v$.

For $v$ a vertex in $T$, let $R_v \in \text{Ray}$ denote the intersection of the geodesic connecting $v$ to Ray with Ray, that is $d(v, R_v) = d(v, \text{Ray})$. For $v_1, v_2 \in T$, let $h(v_1, v_2)$ denote the horocycle distance between $v_1$ and $v_2$ (possibly negative), which is defined as the unique function $h(v_1, v_2)$ which equals to $h(x, v_2) − h(x, v_1)$ for all vertices $x$ such that both $v_1$ and $v_2$ are descendants of $x$. (A vertex $w \in T$ is a descendant of $v$ if the geodesic connecting $w$ to $v$ contains an offspring of $v$.) We also write $h(v) = h(o, v)$; The quantity $h(v)$, which may be either positive or negative, is the level to which $v$ belongs, see Figure 1. Let $D_n(v)$ denote the descendants of $v$ in $T$ at distance $n$ from $v$. Explicitly,

\[
D_n(v) = \{ w \in T : d(w, v) = h(w) − h(v) = n \}.
\]  

We let $Z_n(v) = |D_n(v)|$ be the number of descendants of $v$ at level $h(v) + n$. Then $\{Z_n(v)/m^n\}_{n \geq 1}$ forms a martingale and converges a.s., as $n \to \infty$, to a random variable denoted $W_v$. Moreover, $W_v$ has exponential tails, and there are good bounds on the rate of convergence, see [1].

Motivated by [15], we next describe a measure on the collection of trees with marked rays, which we denote by $\text{IGW}$. Fix a vertex $o$ (the root) and a semi-infinite ray, denoted Ray, emanating from it. Each vertex $v \in \text{Ray}$ with $v \neq o$ is assigned independently a size-biased number of offspring, that is $P_{\text{IGW}}(d_v = k) = kp_k/m$, one of which is identified with the descendant of $v$ on Ray. To each offspring of $v \neq o$ not on Ray, and to $o$, one attaches an independent Galton-Watson tree of offspring distribution $\{p_k\}_{k \geq 1}$. The resulting random tree $T$ is distributed according to $\text{IGW}$. An alternative characterization of $\text{IGW}$
is obtained as follows, see [15] for a similar construction.

**Lemma 1** Consider the measure $Q_n$ on rooted trees with root $r$, obtained from $GW$ by size-biasing with respect to $|D_n(r)|$ (that is, $dQ_n/dGW = |D_n(r)|/m^n$). Choose a vertex $o \in D_n(r)$ uniformly, creating a (finite) ray from $o$ to the root of the original tree, and extend the ray from $r$ to obtain an infinite ray, creating thus a random rooted tree with marked ray emanating from the new root $o$. Call $IGW_n$, the distribution thus obtained. Then, $IGW$ is the weak limit of $IGW_n$.

Sometimes, we also need to consider trees where the root has no ancestors. Often, these will be distributed according to the Galton-Watson measure $GW$. There is however another important measure that we will use, described in [15], namely the size-biased measure $\hat{GW}$ corresponding to $GW$. It is defined formally by $d\hat{GW}/dGW = W_o$. An alternative construction of $\hat{GW}$ is by sampling, size-biased, a particular trunk.

We let $\{X_n\}$ denote the $\lambda$-biased random walk on the tree $T$, where $\lambda = m$. Explicitly, given a tree $T$, $X_n$ is a Markov process with $X_0 = o$ and transition probabilities

$$P_T(X_{n+1} = u|X_n = v) = \begin{cases} \frac{\lambda}{\lambda + d_v}, & \text{if } 1 = d(u, v) = h(u, v) \\ \frac{1}{\lambda + d_v}, & \text{if } 1 = d(u, v) = h(v, u) \\ 0, & \text{else}. \end{cases}$$

That is, the walker moves with probability $\lambda/(\lambda + d_v)$ toward the ancestor of $v$ and with probability $1/(\lambda + d_v)$ toward any of the offspring of $v$. We recall that the model of $\lambda$-biased random walk on a rooted tree is reversible, and possesses an electric network interpretation, where the conductance between $v \in D_n(o)$ and an offspring $w \in D_{n+1}(o)$ of $v$ is $\lambda^{-n}$ (see e.g. [14] for this representation, and [9] for general background on reversible random walks interpreted in electric networks terms). With a slight abuse of notation, we let $P_T^v$ denote the law, conditional on the given tree $T$ and $X_0 = v$, on the path $\{X_n\}$. We refer to this law as the quenched law. Our main result for the $IGW$ trees is the following.

**Theorem 2** Under $IGW$, the horocycle distance satisfies a quenched invariance principle. That is, for some deterministic $\sigma^2 > 0$ (see [16] below for the value of $\sigma$), for $IGW$-a.e. $T$, the processes $\{h(X_{\lfloor nt\rfloor})/\sqrt{n} \sigma \}_{t \geq 0}$ converge in distribution to a standard Brownian motion.

### 3 Martingales, stationary measures, and proof of Theorem 2

The proof of Theorem 2 takes the bulk of this paper. We describe here the main steps.

- In a first step, we construct in this section a martingale $M_t$, whose increments consist of the normalized population size $W_{X_{t+1}}$ when $h(X_{t+1}) - h(X_t) = 1$ and $-W_{X_t}$ otherwise. (Thus, the increments of the martingale
depend on the “environment as seen from the particle”). This martingale provides “harmonic coordinates” for the random walk, in the spirit of [12] and, more recently, [21] and [3].

In the next step, we prove an invariance principle for the martingale \( M_t \). This involves proving a law of large numbers for the associated quadratic variation. It is at this step that it turns out that \( IGW \) is not so convenient to work with, since the environment viewed from the point of view of the particle is not stationary under \( IGW \). We thus construct a small modification of \( IGW \), called \( IGWR \), which is a reversing measure for the environment viewed from the point of view of the particle, and is absolutely continuous with respect to \( IGW \) (see Lemma 2). This step uses crucially that \( \lambda = m \).

In the final step, we introduce the corrector \( Z_t \), which is the difference between a constant multiple \( 1/\eta \) of the harmonic coordinates \( M_t \) and the position of the random walk, \( X_t \). As in [3], we seek to show that the corrector is small, see Proposition 1. The proof of Proposition 1 is postponed to Section 4, and is based on estimating the time spent by the random walk at any given level.

In the sequel (except in Section 6), we often use the letters \( s, t \) to denote time, reserving the letter \( n \) to denote distances on the tree \( T \). Set \( M_0 = 0 \) and, if \( X_t = v \) for a vertex \( v \) with parent \( u \) and offspring \( Y_1, \ldots, Y_d_v \), set

\[
M_{t+1} - M_t = \begin{cases} 
-W_v, & X_{t+1} = u \\
W_{Y_j}, & X_{t+1} = Y_j.
\end{cases}
\]

Quenched (i.e., given the realization of the tree), \( M_t \) is a martingale with respect to the natural filtration \( \mathcal{F}_t = \sigma(X_1, \ldots, X_t) \), as can be seen by using the relation \( W_v = \sum_{j=1}^{d_v} W_{Y_j}/m \). Also, for \( v \in T \), let \( g_v \) denote the geodesic connecting \( v \) with \( \text{Ray} \) (which by definition contains both \( v \) and \( R_v \)), and set

\[
S_v = \begin{cases} 
\sum_{u \in g_v, u \neq o} W_u, & \text{if } R_v = o, \\
\sum_{u \in g_v, u \neq R_v} W_u - \sum_{u \in \text{Ray}, 0 \geq h(u) > h(R_v)} W_u, & \text{if } R_v \neq o.
\end{cases}
\]

Then, \( M_t = S_{X_t} \).

Set \( \eta = E_{GW} W_o^2 (= E_{GW} W_o) \) and \( Z_t = M_t/\eta - h(X_t) \). Fix

\[
\alpha = 1/3, \epsilon_0 < 1/100, \delta \in (1/2 + \alpha + 4\epsilon_0, 1 - 4\epsilon_0).
\]

(The reason for the particular choice of constants here will become clearer in the course of the proof.) For any integer \( t \), let \( \tau_t \) denote an integer valued random variable, independent of \( T \) and \( \{X_s\}_{s \geq 0} \), uniformly chosen in \([t, t + [t^\delta]]\). We prove in Section 4 the following estimate, which shows that \( M_t/\eta \) is close to \( h(X_t) \). The variable \( \tau_t \) is introduced here for technical reasons as a smoothing device, that allows us to consider occupation measures instead of pointwise in time estimates on probabilities.
Proposition 1  With the above notation, for any \( \epsilon < \epsilon_0 \),
\[
\lim_{t \to \infty} P^o_T(|Z_{\tau t}| \geq \epsilon \sqrt{t}) = 0, \quad \text{IGW-a.s.} \tag{3}
\]
Further,
\[
\lim_{t \to \infty} P^o_T \left( \sup_{r,s \leq t, |r-s| < t^\delta} |h(X_r) - h(X_s)| > t^{1/2-\epsilon} \right) = 0, \quad \text{IGW-a.s.} \tag{4}
\]

The interest in the martingale \( M_t \) is that we can prove for it a full invariance principle. Toward this end, one needs to verify that the normalized quadratic variation process
\[
V_t = \frac{1}{t} \sum_{i=1}^{t} E^o_T((M_{i+1} - M_i)^2|\mathcal{F}_i) \tag{5}
\]
converges IGW-a.s. Note that if \( X_i = v \) with offspring \( Y_1, \ldots, Y_{d_v} \), then
\[
E^o_T((M_{i+1} - M_i)^2|\mathcal{F}_i) = \frac{m}{m + d_v} W_v^2 + \frac{1}{m + d_v} \sum_{j=1}^{d_v} W_{Y_j}^2 \tag{6}
\]
\[
= \frac{1}{m + d_v} \sum_{j=1}^{d_v} W_{Y_j}^2 + \frac{1}{m(m + d_v)} \left( \sum_{j=1}^{d_v} W_{Y_j} \right)^2 =: \mu_v^2.
\]

It turns out that to ensure the convergence of \( V_t \), it is useful to introduce a new measure on trees, denoted \( \text{IGWR} \), which is absolutely continuous with respect to the measure \( \text{IGW} \), and such that the “environment viewed from the point of view of the particle” becomes stationary under that measure, see Lemma 2 below. The measure \( \text{IGWR} \) is similar to \( \text{IGW} \), except at the root. The root \( o \) has an infinite path \( v_j \) of ancestors, which all possess an independent number of offspring which is size-biased, that is
\[
P(d_{v_j} = k) = kp_k/m, \quad \text{for all } j, k > 0.
\]
The number of offspring at the root itself is independent of the variables just mentioned, and possesses a distribution which is the average of the original and the size-biased laws, that is:
\[
P(d_o = k) = (m + k)p_k/(2m), \quad \text{for all } k > 0.
\]

All other vertices have the original offspring law. All these offspring variables are independent. In other words, \( d\text{IGWR} / d\text{IGW} = (m + d_o)/2d_o \). Consequently, we can use the statements “\( \text{IGW-a.s.} \)” and “\( \text{IGWR-a.s.} \)” interchangeably.

For \( v \) a neighbor of \( o \), let \( \theta^v T \) denote the tree which is obtained by shifting the location of the root to \( v \) and adding or erasing one edge from \( \text{Ray} \) in the only way that leaves an infinite ray emanating from the new root. We also write, for an arbitrary vertex \( w \in \mathcal{T} \) with geodesic \( g_w = (v_1, v_2, \ldots, v_{|w|-1}, w) \) connecting \( o \) to \( w \), the shift \( \theta^w \mathcal{T} = \theta^w \circ \theta^{|w|-1} \circ \ldots \theta^1 \mathcal{T} \). Finally, we set \( \mathcal{T}_t = \theta^{X_t} \mathcal{T} \). It
is evident that $\mathcal{T}_t$ is a Markov process, with the location of the random walk being frozen at the root, and we write $P_{\mathcal{T}}(\cdot)$ for its transition density, that is $P_{\mathcal{T}}(A) = P_{\mathcal{T}}(\mathcal{T}_t \in A)$. What is maybe surprising at first is that $\text{IGWR}$ is reversing for this Markov process. That is, we have.

**Lemma 2** The Markov process $\mathcal{T}_t$ with initial measure $\text{IGWR}$ is stationary and reversible.

**Proof of Lemma** Suppose that $\mathcal{T}_0$ is picked from $\text{IGWR}$, and $\mathcal{T}_1$ is obtained from it by doing one step (starting with $X_0 = o$) of the critically biased walk on $\mathcal{T}_0$, then moving the root to $X_1$ and adjusting Ray accordingly. We must show that the ordered pair $(\mathcal{T}_0, \mathcal{T}_1)$ has the same law as $(\mathcal{T}_1, \mathcal{T}_0)$.

Let $\mathcal{T}_F$ be finite tree of depth $\ell$ rooted at $\rho$, and let $u, v$ be adjacent internal nodes of $\mathcal{T}_F$, at distance $k$ and $k+1$, respectively, from $\rho$ (see figure 2).

Let $A(\mathcal{T}_F, u)$ be the cylinder set of infinite labeled rooted trees $\mathcal{T}$ in the support of $\text{IGWR}$ which locally truncate to $\mathcal{T}_F$ rooted at $u$, that is, the connected component of the root of $\mathcal{T}$ among levels between $-k$ and $\ell-k$ in $\mathcal{T}$ is identical to $\mathcal{T}_F$ once the root of $\mathcal{T}$ is identified with $u$, and Ray in $\mathcal{T}$ goes through the vertex identified with $\rho$ in $\mathcal{T}$. Let $\{w : \rho \leq w < u\}$ denote the set of vertices on the path from $\rho$ (inclusive) to $u$ (exclusive) in $\mathcal{T}_F$. Then

$$P_{\text{IGWR}}[A(\mathcal{T}_F, u)] = P_{\text{GW}}(\mathcal{T}_F) \prod_{\{w : \rho \leq w < u\}} \left[ \frac{d_w}{m} \cdot \frac{1}{d_w} \right] \frac{m + d_u}{2m},$$

(7)

where the factors $d_w/m$ and $(m+d_u)/(2m)$ come from the density of the $\text{IGWR}$ offspring distributions with respect to the $\text{GW}$ offspring distribution, and the factors $1/d_w$ comes from the uniformity in the choice of Ray. Thus

$$P_{\text{IGWR}}[A(\mathcal{T}_F, u)] = P_{\text{GW}}(\mathcal{T}_F) m^{-k-1} (m + d_u)/2,$$

(8)
and similarly
\[ P_{\text{IGWR}}[A(T_F, v)] = P_{GW}(T_F)m^{-k-2}(m + d_v)/2. \] (9)

Since the transition probabilities for the critically biased random walk are \( p(u, v) = 1/(m + d_v) \) and \( p(v, u) = m/(m + d_v) \), we infer from (8) and (9) that
\[ P_{\text{IGWR}}[A(T_F, u)]p(u, v) = P_{\text{IGWR}}[A(T_F, v)]p(v, u) \]
as required.

With \( V_t \) as in (5), the following corollary is of crucial importance.

**Corollary 1**
\[ V_t \rightarrow E_{\text{IGWR}}\hat{\mu}_0^2 =: \sigma^2\eta^2, \text{ IGWR} - \text{a.s.} \] (10)

**Proof of Corollary** That IGWR is absolutely continuous with respect to IGW is obvious from the construction. By Lemma 2, IGWR is invariant and reversible under the Markov dynamics induced by the process \( T \). Thus, (10) holds as soon as one checks that \( \mu_0 \in L^2(\text{IGWR}) \), which is equivalent to checking that with \( v_i \) denoting the offspring of \( o \), it holds that \( (\sum_{i=1}^{d_o} W_{v_i})^2 \in L^1(\text{IGWR}) \). This in turn is implied by \( E_{GW}(W_o^2) < \infty \), which holds due to [1].

**Proof of Theorem 2** In what follows, we consider a fixed \( T \), with the understanding that the statements hold true for IGW almost every such tree. Due to (10) and the invariance principle for the Martingale \( M_t \), see [4, Theorem 14.1], it holds that for IGWR almost every \( T \), \( \{M_{\lfloor nt\rfloor}/\sqrt{\eta^2\sigma^2n}\}_{t \geq 0} \) converges in distribution, as \( n \to \infty \), to a standard Brownian motion. Further, by [4, Theorem 14.4], so does \( \{M_{\tau nt}/\sqrt{\eta^2\sigma^2n}\}_{t \geq 0} \). By (3), it then follows that the finite dimensional distributions of the process \( \{h(X_{\lfloor nt\rfloor})/\sqrt{\sigma^2n}\}_{t \geq 0} \) converge, as \( n \to \infty \), to those of a standard Brownian motion. On the other hand, due to [4], the sequence of processes \( \{Y^n_{\lfloor nt\rfloor}\}_{t \geq 0} \) is tight, and hence converges in distribution to standard Brownian motion. Applying again [4, Theorem 14.4], we conclude that the sequence of processes \( \{h(X_{\lfloor nt\rfloor})/\sqrt{\sigma^2n}\}_{t \geq 0} \) converges in distribution to a standard Brownian motion, as claimed.

**4 Proof of Proposition 1**

**Proof of Proposition** For any tree with root \( o \), we write \( D_n \) for \( D_n(o) \), c.f. (1). Recall that \( E_{GW}W_o = \eta \). For \( \epsilon > 0 \), let \( A'_n = A_n(T) = \{v \in D_n : |n^{-1}S_v - \eta| > \epsilon\} \), noting that for GW or \( \hat{\text{GW}} \) trees, \( S_v = \sum_{u \in g_v, u \neq o} W_u \). We postpone for a moment the proof of the following.

**Lemma 3** For any \( \epsilon > 0 \) there exists a deterministic \( \nu = \nu(\epsilon) > 0 \) such that
\[ \limsup_{n \to \infty} \frac{1}{n} \log P_{GW}(\frac{1}{n} \log |A'_n|/D_n) > -\nu) \leq -\nu/2, \] (11)
and
\[
\limsup_{n \to \infty} \frac{1}{n} \log P_{GW} \left( \frac{1}{n} \log \frac{|A_n^\epsilon|}{D_n} > -\nu \right) \leq -\nu/2. \tag{12}
\]

Turning our attention to trees governed by the measure IGW, for any vertex \( w \in T \) we set
\[
S_{Ray}^w = \sum_{v \in T \setminus Ray(w)} W_v.
\]
Let \( B_\epsilon^n(T) = \{ w \in T : d(w, Ray) = n, |n^{-1}S_{Ray}^w - \eta| > \epsilon \} \), and set
\[
Q_\epsilon(T) = \{ w \in T : d(w, Ray) \leq \epsilon \}. \tag{13}
\]

The following proposition will be proved in Section 5.

**Proposition 2**

\[
\limsup_{t \to \infty} P_{T}^o (X_{\tau_t} \in Q_\epsilon(T)) = 0, \text{ IGW - a.s.}. \tag{14}
\]

We can now prove the following.

**Lemma 4** With the preceding notation, it holds that for any \( \epsilon > 0 \),
\[
\lim_{t \to \infty} P_T^o (X_{\tau_t} \in \bigcup_m B_\epsilon^m(T)) = 0, \text{ IGW - a.s.}. \tag{15}
\]

**Proof of Lemma 4** By (14),
\[
a_t := P_T^o (X_{\tau_t} \in Q_\epsilon(T)) \to_{t \to \infty} 0, \text{ IGW - a.s.}. \tag{15}
\]

Letting \( \gamma_m = \min \{ t : X_t \in B_\epsilon^m(T) \} \), we have (using \( t + \lceil t^\delta \rceil \leq 2t \)),
\[
P_T^o (X_{\tau_t} \in \bigcup_m B_\epsilon^m(T)) \leq a_t + \sum_{\ell = t^\delta}^{2t} P_T^o (\gamma_\ell \leq 2t). \tag{16}
\]

Consider the excursions of \( \{X_i\} \) down the GW trees whose starting points are offspring of a vertex in Ray, where an excursion is counted between visits to such a starting point. The event \( \{ \gamma_\ell^\epsilon \leq 2t \} \) implies that of the first \( 2t \) such excursions, there is at least one excursion that reaches level \( \ell - 1 \) below the corresponding starting point, at a vertex \( v \) with \( |(\ell - 1)^{-1}v - \eta| > \epsilon \). Therefore, with \( \tau_o = \min \{ t > 0 : X_t = o \} \), for \( \ell \) large so that \( \{ x > 0 : |(\ell - 1)^{-1}x - \eta| > \epsilon/2 \} \subset \{ x > 0 : |(\ell - 1)^{-1}x - \eta| > \epsilon/2 \} \),
\[
P_{IGW}^o (\gamma_\ell^\epsilon \leq 2t) \leq 2tP_{GW}^o (\gamma_{\ell-1}^{\epsilon/2} \leq 2t \wedge \tau_o), \tag{17}
\]
where we set for a GW rooted tree, \( \gamma_{\ell-1}^{\epsilon/2} = \min \{ i > 0 : X_i \in A_{\ell-1}^{\epsilon/2} \} \). But, for a GW rooted tree, the conductance \( C(o \leftrightarrow A_{\ell-1}^{\epsilon/2}) \) from the root to the vertices in \( A_{\ell-1}^{\epsilon/2} \).
is at most $\lambda^{-\ell}|A^{\ell/2}_n|$. Note that with $Z_n := |D_n|m^{-n}$ it holds that $E_{GW}(Z_n) = 1$ and

$$E_{GW}(Z_{n+1}^2) = E_{GW}(Z_n^2) + \frac{E_{GW}(d_n^2 - d_o)}{\lambda^2}(E_{GW}(Z_n))^2$$

and hence $E_{GW}(Z_n^2) \leq c\ell$ for some deterministic constant $c$. Therefore,

$$P_0^{GW}(\gamma^{\ell/2}_n \leq \tau_o) \leq E_{GW}(\mathbb{C}(o \leftrightarrow A^{\ell/2}_{\ell-1})) \leq E_{GW}(\lambda^{-\ell+1}|A^{\ell/2}_{\ell-1}|) = E_{GW}(Z_{\ell-1} | D_{\ell-1})$$

$$\leq \left[E_{GW}(Z_{\ell-1}^2)\right]^{1/2} \left[E_{GW}(\frac{|A^{\ell/2}_{\ell-1}|}{|D_{\ell-1}|})\right]^{1/2} \leq e^{-\nu(\ell/2)\ell/4}.$$ 

for $\ell$ large, where Lemma 3 was used in the last inequality. Combined with (17), we conclude that

$$\sum_{\ell=0}^{2t} P_0^{IGW}(\gamma^{\ell}_{\ell} \leq 2t) \leq e^{-\nu(\ell/2)\ell/8}.$$ 

By Markov’s inequality and the Borel-Cantelli lemma, this implies that

$$\lim_{t \to \infty} \frac{1}{t^{\nu/16}} \sum_{\ell=0}^{2t} P_0^{IGW}(\gamma^{\ell}_{\ell} \leq 2t) = 0, \quad IGW - a.s.$$ 

Substituting in (16) and using (15), one concludes the proof of Lemma 4.

**Proof of Lemma 3** Recall the construction of the measures $\hat{GW}$ and $\hat{GW}_*$, see [15, Pg 1128]. Note that $\hat{GW}_*$ is a measure on rooted trees with a marked ray emanating from the root. We let $v^*_n$ denote the marked vertex at distance $n$ from the root.

By [15, (2.1),(2.2)], and denoting by $T_n$ the first $n$ generations of the tree $T$, it holds that

$$\hat{GW}_*(v^*_n \in A^*_{\ell} = E_{\hat{GW}} \left( \frac{1}{|D_n|} \sum_{v \in D_n} P_{\hat{GW}}(v \in A^*_{\ell} | T_n) \right).$$

We show below that there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$\hat{GW}_*(v^*_n \in A^*_{\ell}) \leq e^{-2\delta_1 n}. \quad (18)$$

We assume that (18) has been proved, and complete the proof of the lemma. By Markov’s inequality, (18) implies that

$$P_{\hat{GW}} \left( E_{\hat{GW}} \left( \frac{A^*_{\ell}}{|D_n|} | T_n \right) \geq e^{-\delta_1 n} \right) \geq e^{-2\delta_1 n} \epsilon_{-\delta_1 n}. \quad (19)$$

$$= P_{\hat{GW}} \left( \frac{1}{|D_n|} \sum_{v \in D_n} P_{\hat{GW}}(v \in A^*_{\ell} | T_n) \geq e^{-\delta_1 n} \right) \leq e^{-2\delta_1 n} e^{-\delta_1 n} = e^{-\delta_1 n}. \quad (19)$$
We thus get
\[
P^{\hat{G}_W}
\left(\frac{|A^r_n|}{|D_n|} > e^{-\delta_1 n/2}\right) = E^{\hat{G}_W}
\left(P^{\hat{G}_W}
\left(\frac{|A^r_n|}{|D_n|} > e^{-\delta_1 n/2} \mid T_n\right)\right) \\
\leq E^{\hat{G}_W}
\left(E^{\hat{G}_W}
\left(\frac{|A^r_n|}{|D_n|} \mid T_n\right) e^{|\delta_1 n/2|}\right) \\
\leq e^{-\delta_1 n/2} + e^{\delta_1 n/2} E^{\hat{G}_W}
\left(E^{\hat{G}_W}
\left(\frac{|A^r_n|}{|D_n|} \mid T_n\right) \geq e^{-\delta_1 n}\right) \leq 2e^{-\delta_1 n/2},
\]
where Markov’s inequality was used in the first inequality and (19) in the last. This proves (11). While (12) could be proved directly, one notes that, with \(r > 1\) such that \(p_1/m^{r-1} < 1\),
\[
P^{\hat{G}_W}
\left(\frac{1}{n} \log \frac{|A^r_n|}{|D_n|} > -\nu\right) = E^{\hat{G}_W}
\left(W^{-1} e^{\frac{1}{r} \log \frac{|A^r_n|}{|D_n|} - \nu}\right) \\
\leq (E^{\hat{G}_W} e^{\nu W_o})^{1/r} \left(P^{\hat{G}_W}
\left(\frac{1}{n} \log \frac{|A^r_n|}{|D_n|} > -\nu\right)\right)^{1-1/r},
\]
where H"older’s inequality with exponent \(r > 1\) was used. Since 
\[
E^{\hat{G}_W} e^{\nu W_o} < \infty
\]
by [18, Theorem 1], (12) follows from (11).

It remains to prove (18). We use the following: Since 
\[
(E^{\hat{G}_W} e^{\xi W_o})^2 \leq E^{\hat{G}_W} (W_o^2) E^{\hat{G}_W} e^{2\xi W_o} < \infty
\]
for some \(\xi > 0\), where the last inequality is due to [14], it follows that there exists a \(\xi > 0\) such that 
\[
E^{\hat{G}_W} e^{\xi W_o} = E^{\hat{G}_W} e^{\xi W_o} < \infty. \tag{20}
\]

For a marked vertex \(v^*_k\), we let \(\tilde{Z}^n_{v^*_k}\) denote the size of the subset of vertices in 
\(D_n(v^*_k)\) whose ancestral line does not contain \(v^*_{k+1}\), and we define \(\tilde{W}_k\) as the a.s. limit (as \(n \to \infty\)) of \(\tilde{Z}^n_{v^*_k}/m^n\), which exists by the standard martingale argument. Note that by construction, for \(k < n\), with \(W_k = W_{v^*_k}\),
\[
W_k = \tilde{W}_k + \frac{\tilde{W}_{k+1}}{m} + \ldots + \frac{\tilde{W}_{n-1}}{m^{n-k-1}} + \frac{W_n}{m^{n-k}}. \tag{21}
\]
Therefore,
\[
S_{v^*_n} = \sum_{k=0}^{n-1} \tilde{W}_k C_k + W_n C_n,
\]
where \(C_k = 1+1/m+(1/m)^2+\ldots+(1/m)^k\). Due to (20), we have the existence of a \(\delta_2 > 0\) such that
\[
P^{\hat{G}_W}
\left(|W_n C_n| > cn/4\right) \leq P^{\hat{G}_W}
\left(|W_o| > (1-1/m)cn/4\right) \leq e^{-\delta_2 n}. \tag{22}
\]
Also,
\[
\begin{align*}
P_{\tilde{GW}} \left( \sum_{k=0}^{n-1} \tilde{W}_k [C_\infty - C_k] > \epsilon n/4 \right) &= P_{\tilde{GW}} \left( \sum_{k=0}^{n-1} \frac{\tilde{W}_k}{m^{k+1}(1-1/m)} > \epsilon n/4 \right) \\
&\leq n P_{\tilde{GW}}(\tilde{W}_0 > \epsilon n/4) \\
&\leq n P_{\tilde{GW}}(\tilde{W}_0 > c_{\epsilon,m} n) \\
&\leq e^{-\delta n}.
\end{align*}
\]
for some constant $c_{\epsilon,m}$, where (20) was used in the second inequality. On the other hand,
\[
\eta = E_{\tilde{GW}}[W_k] = E_{\tilde{GW}}[C_\infty \tilde{W}_0],
\]
where the first equality follows from the construction of $\tilde{GW}$ and the definition of $\eta$, and the second from (21). The random variables $\tilde{W}_k$ are i.i.d. under $\tilde{GW}$. Therefore, using (22) and (23),
\[
P_{\tilde{GW}} \left( \left| S_{\tilde{V}_n}/n - \eta \right| > \epsilon \right) \leq 2 e^{-\delta n} + P_{\tilde{GW}} \left( \sum_{k=0}^{n-1} \left| C_\infty \tilde{W}_k - \eta \right| > \epsilon/n \right).
\]
Standard large deviations (applied to the sum of i.i.d. random variables $\tilde{W}_k$ that possess exponential moments) together with (20) now yield (18) and complete the proof of Lemma 3.

Continuing with the proof of Proposition 1, let $v_n$ denote the vertex on Ray with $h(v_n) = -n$. By the same construction as in the course of the proof of Lemma 3, it holds that
\[
S_{v_n}/n \rightarrow n \rightarrow \infty -\eta, \text{ IGWR-a.s.}
\]
(24)
Let $R_t = R_{X_{\tau_t}}$. Note that $S_{X_{\tau_t}} = -S_{R_t} + S_{X_{\tau_t}}^{\text{Ray}}$. Thus,
\[
\left| Z_{\tau_t} \right| \leq \left| S_{R_t}/\eta + h(R_t) \right| + \left| S_{X_{\tau_t}}^{\text{Ray}}/\eta - h(R_t, X_{\tau_t}) \right|.
\]
Note that since the random walk restricted to Ray is transient, $h(R_t) \rightarrow t \rightarrow \infty -\infty$, and hence by (21), $S_{R_t}/\eta h(R_t) \rightarrow 1$. Therefore, for any positive $\epsilon_1$, for all large $t$, using that $\left| S_{R_t}/\eta + h(R_t) \right| \leq \epsilon_1 \sup_{s \leq 2t} |M_s|$. Similarly, for any $\epsilon_1 < \epsilon$, on the event $X_{\tau_t} \notin \cup_{m} B^{\text{Ray}}_{m}(T)$, it holds that for large $t$, $|S_{X_{\tau_t}}/\eta - h(R_t, X_{\tau_t})| \leq \epsilon_1 \sup_{s \leq 2t} |M_s|$ for all $t$ large. Thus, for such $\epsilon_1$, $\left| Z_{\tau_t} \right| \leq 2\epsilon_1 \sup_{s \leq 2t} |M_s|$ for all $t$ large. From Lemma 4
\[
\limsup_{t \rightarrow \infty} P_T^\theta(X_{\tau_t} \in \cup_{m} B^{\text{Ray}}_{m}(T)) = 0.
\]
But, since the normalized increasing process $V_t$ is IGWR-a.s. bounded, standard Martingale inequalities imply that
\[
\lim_{\epsilon_1 \rightarrow 0} \limsup_{t \rightarrow \infty} P_T^\theta (\sup_{s \leq t} |M_t| > \epsilon\sqrt{t}/2\epsilon_1) = 0.
\]
It follows that
\[ \lim_{t \to \infty} P_{t}^{\tau}(|Z_{\tau_{t}}| \geq \varepsilon \sqrt{t}) = 0, \]
as claimed.

The proof of (4) is provided in Section 5, see (35). This completes the proof of Proposition 1.  \( \square \)

5 Auxiliary computations and proof of (4)

We begin by an a-priori annealed estimate on the displacement of the random walk in a GW tree.

**Lemma 5** For any \( u, t \geq 1 \), it holds that
\[ P_{GW}^{\tau}(|X_{i}| \geq u \text{ for some } i \leq t) \leq 4te^{-u^{2}/2t}. \] (26)

**Proof of Lemma 5** Throughout, we write \( |v| = d(v, o) \). Let \( T_{u} \) denote the truncation of the tree \( T \) at level \( u \), and let \( T^{\ast} \) denote the graph obtained from \( T_{u} \) by adding an extra vertex (denoted \( o^{\ast} \)) and connecting it to all vertices in \( D_{u} \). Let \( X_{i}^{\ast} \) denote the random walk on \( T^{\ast} \), with
\[ P_{T}(X_{i+1}^{\ast} = w|X_{i}^{\ast} = v) = \begin{cases} P_{T}(X_{i+1} = w|X_{i} = v), & \text{if } v \notin D_{u}, \\ 1/2, & \text{if } v \in D_{u} \text{ and } d(v, w) = 1, \\ 1/|D_{u}|, & \text{if } v = o^{\ast} \text{ and } d(v, w) = 1. \end{cases} \]

Then,
\[ P_{GW}^{\tau}(|X_{i}| \geq u \text{ for some } i \leq t) \leq \sum_{i=1}^{t} P_{GW}^{\tau}(|X_{i}^{\ast}| = u \text{ for some } i \leq t) \leq 2 \sum_{i=1}^{t+1} P_{GW}^{\tau}(|X_{i}^{\ast}| = o^{\ast}). \] (27)

By the Carne-Varopoulos bound, see [7, 23, 14, Theorem 12.1],
\[ P_{T}^{\tau}(|X_{i}^{\ast}| = o^{\ast}) \leq 2\sqrt{\lambda^{-u}|D_{u}|/d_{o}e^{-u^{2}/2t}}. \]

Hence, since \( E_{GW}|D_{u}| = \lambda^{u} \),
\[ 2 \sum_{i=1}^{t+1} P_{GW}^{\tau}(|X_{i}^{\ast}| = o^{\ast}) \leq 4te^{-u^{2}/2t}. \]

Combining the last estimate with (27), we get (26).

We get the following.

**Corollary 2** It holds that
\[ P_{IGWR}^{\tau}(|h(X_{i})| \geq u \text{ for some } i \leq t) \leq 8te^{-(u-1)^{2}/2t}. \] (28)

and
\[ P_{IGW}^{\tau}(|h(X_{i})| \geq u \text{ for some } i \leq t) \leq 16te^{-(u-1)^{2}/2t}. \] (29)
Therefore, by (28), for any $\epsilon_0$, the increments

$$P_{\IGWR}^o(h(X_i) \geq u) \leq P_{\IGWR}^o(3j < i : h(X_i) - h(X_j) \geq u, h(X_i) - h(X_j) > 0 \forall t \in \{j + 1, \ldots, i\})$$

$$\leq \sum_{j=0}^{i-1} P_{\IGWR}^o(h(X_i) - h(X_j) \geq u, h(X_i) - h(X_j) > 0 \forall t \in \{j + 1, \ldots, i\}) .$$

Using the stationarity of IGWR, we thus get

$$P_{\IGWR}^o(h(X_i) \geq u)$$

$$\leq \sum_{j=0}^{i-1} P_{\IGWR}^o(h(X_{i-j}) \geq u, h(X_s) > 0 \forall s \in \{1, \ldots, i-j\}),$$

$$\leq i \max_{r \leq i} P_{\IGWR}^o(h(X_r) \geq u, h(X_s) > 0 \forall s \in \{1, \ldots, r\}).$$

On the other hand, for $r, u > 1$,

$$P_{\IGWR}^o(h(X_r) \geq u, h(X_s) > 0 \forall s \in \{1, \ldots, r\}) \leq P_{\GW}^o(h(X_r) \geq u - 1),$$

because reaching level $u$ before time $r$ and before returning to the root or visiting Ray requires reaching level $u$ from one of the offspring of the root before returning to the root. Substituting in (30) we get

$$P_{\IGWR}^o(h(X_i) \geq u) \leq i \max_{r \leq i} P_{\GW}^o(h(X_r) \geq u - 1) \leq 4i^2e^{-(u-1)^2/2i},$$

where (29) was used in the last inequality. It follows from the above that

$$P_{\IGWR}^o(h(X_i) \geq u) \leq 4i^2e^{-(u-1)^2/2i} .$$

Recall the process $T_s = \theta^{X_t} \mathcal{T}$, which is reversible under $P_{\IGWR}^o$, and note that $h(X_i) - h(X_0)$ is a measurable function, say $H$, of $\{\mathcal{T}_j\}_{0 \leq j \leq i}$ (we use here that for IGWR-almost every $\mathcal{T}$, and vertices $u, w \in \mathcal{T}$, one has $\theta^u \mathcal{T} \neq \theta^w \mathcal{T}$). Further, with $T_j \equiv T_{-j}$, it holds that $H(\{T_j\}_{0 \leq j \leq i}) = -H(\{T_j\}_{0 \leq j \leq i})$. Therefore,

$$P_{\IGWR}^o(h(X_i) \leq -u) = P_{\IGWR}^o(h(X_i) \geq u) .$$

Applying (32), one concludes that

$$P_{\IGWR}^o(h(X_i) \leq -u) \leq 4i^2e^{-(u-1)^2/2i} .$$

Together with (33), the proof of (28) is complete. To see (29), note that IGW is absolutely continuous with respect to IGWR, with Radon-Nikodym derivative uniformly bounded by 2.

We can now give the

**Proof of (4)** The increments $h(X_{i+1}) - h(X_i)$ are stationary under $P_{\IGWR}^o$. Therefore, by (25), for any $\epsilon$ and $r, s \leq t$ with $|r - s| \leq t^\delta$,

$$P_{\IGWR}^o(|h(X_r) - h(X_s)| > t^{1/2-\epsilon}) = P_{\IGWR}^o(|h(X_{r-s})| > t^{1/2-\epsilon}) \leq 8t^3e^{-t^{-\delta-2\epsilon}} .$$
Therefore, by Markov’s inequality, for all $t$ large,

$$P_{\text{IGWR}} \left( P_T \left( |h(X_{r-s})| > t^{1/2-\epsilon} \right) \right) \geq t^{-2} e^{-t^{1-\delta-\epsilon}} \leq e^{-t^{1-\delta-\epsilon}}.$$  

Consequently,

$$P_{\text{IGWR}} \left( P_T \left( \sup_{r,s \leq t, |r-s| < t^\delta} |h(X_r) - h(X_s)| > t^{1/2-\epsilon} \right) \right) \leq e^{-t^{1-\delta-\epsilon}}.$$  

It follows that

$$\limsup_{t \to \infty} P_T \left( \sup_{r,s \leq t, |r-s| < t^\delta} |h(X_r) - h(X_s)| > t^{1/2-\epsilon} \right) \leq e^{-t^{1-\delta-\epsilon}}$$

completing the proof of (35) since the measures IGWR and IGW are mutually absolutely continuous.

We next control the expected number of visits to $D_n$ during one excursion from the root of a GW tree. We recall that $T_o = \min\{n \geq 1 : X_n = o\}$.

**Lemma 6** Let $N_o(n) = \sum_{i=1}^{T_o} 1_{X_i \in D_n}$. There exists a constant $C$ independent of $n$ such that

$$E_{\text{GW}}(N_o(n)|d_o) \leq C d_o \quad \text{and} \quad E_{\text{GW}}^o(N_o(n)|d_o) \leq C d_o.$$  

Further,

$$\limsup_{n \to \infty} E_{\text{GW}}(N_o(n)) < \infty, \quad \text{GW} - a.s. \quad (37)$$

**Proof of Lemma 6** We begin by conditioning on the tree $T$, and fix a vertex $v \in D_n$. Let $\Gamma_v$ denote the number of visits to $v$ before $T_o$. Then,

$$E_T^o(\Gamma_v) = P_T^o(T_v < T_o) E_T^\nu(\Gamma_v).$$

Note that the walker performs, on the ray connecting $o$ and $v$, a biased random walk with holding times. Therefore, by standard computations,

$$P_T^o(T_v < T_o) = \frac{1}{d_o[1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1}]},$$

and, when starting at $v$, $\Gamma_v$ is a Geometric random variable with parameter $\lambda^n/[(\lambda + d_v)(1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1})]$. Therefore, for some deterministic constant $C$,

$$E_T^\nu(\Gamma_v) \leq C \lambda^{-n} d_v.$$  

Thus,

$$E_T^o(N_o(n)) \leq C \sum_{v \in D_n} \lambda^{-n} d_v.$$  

(38)
Since the random variables $d_v$ are i.i.d., independent of $D_n$, and possess exponential moments, and since $|D_n|\lambda^{-n} \to_{n \to \infty} W_o < \infty$, it holds that
\[
\limsup_{n \to \infty} \sum_{v \in D_n} \lambda^{-n}d_v < \infty.
\]
Together with \((38)\), this proves \((37)\). Further, it follows from \((38)\) that
\[
E_{GW}^o \left( |N_o(n)| d_o \right) \leq C\lambda^{-n}E_{GW} \left( |D_n| d_o \right) = Cd_o.
\]
The proof for $\hat{GW}$ is similar. 

We return to $GW$ trees. Recall that $Q_t(T) = \{ w \in T : d(w, \text{Ray}) \leq t^\alpha \}$, and set $N_t(\alpha) = \sum_{i=1}^t 1_{X_i \in Q_t(T)}$.

**Lemma 7** For each $\epsilon > 0$ it holds that for all $t$ large enough,
\[
E_{IGW}^o(N_t(\alpha)) \leq t^{1/2+\alpha+\epsilon}. \tag{39}
\]

**Proof of Lemma 7** Let $U_t = \min\{ h(X_i) : i \leq t \}$ and $t_\epsilon = [t^{1/2+\epsilon}/4]$. By \((29)\), for $t$ large,
\[
P_{GW}^o(U_t \leq -t_\epsilon) \leq 16\epsilon^2 e^{-t_\epsilon/2} \tag{40}
\]
Let $\xi_t = \min\{ s : h(X_s) = -i \}$. It follows from \((40)\) that for all $t$ large,
\[
E_{GW}^o(N_t(\alpha)) \leq 1 + E_{GW}^o(N_t(\alpha); U_t \leq -t_\epsilon) \leq 1 + E_{GW}^o(N_t(\alpha); \xi_t \geq t_\epsilon) \tag{41}
\]
For all $k \geq 0$, let $v_k$ be the unique vertex on $\text{Ray}$ satisfying $h(v_k) = -k$, and set $d_k = d_{v_k}$. We next claim that there exists a constant $C_1 = C_1(\epsilon)$ independent of $t$ such that, with
\[
\Upsilon_{t, \epsilon} := \{ \max_{k \leq [t^\alpha]} d_k \leq C_1(\log t_\epsilon) \},
\]
it holds that
\[
P_{GW}^o(\Upsilon_{t, \epsilon}) \leq \frac{1}{t} \tag{42}
\]
Indeed, with $\beta' = 1 + (\beta - 1)/2 > 1$,
\[
P_{GW}^o(\Upsilon_{t, \epsilon}) \leq t_\epsilon P_{GW}^o(d_0 > C_1 \log t_\epsilon) \leq t_\epsilon \sum_{j=1}^{C_1 \log t_\epsilon} \frac{t_\epsilon^{\beta'} - C_1 \log t_\epsilon}{\log t_\epsilon} \sum_{j=1}^{\infty} j \beta^j \leq \sum_{j=1}^{\infty} j \beta^j, \tag{43}
\]
from which \((42)\) follows if $C_1$ is large enough since $\sum \beta^j p_j < \infty$ by assumption. Combined with the fact that $N_t(\alpha) \leq t$ and \((41)\), we conclude that for such $C_1$,
\[
E_{GW}^o(N_t(\alpha)) \leq 2 + E_{GW}^o(N_t(\alpha); \xi_t \geq t; \Upsilon_{t, \epsilon}). \tag{44}
\]
For the next step, let \( \theta_0 = 0 \) and, for \( \ell \geq 1 \), let \( \theta_\ell \) denote the \( \ell \)-th visit to \( \text{Ray} \), that is \( \theta_\ell = \min \{ t > \theta_{\ell-1} : X_t \in \text{Ray} \} \). Let \( H_\ell = X_{\theta_\ell} \) denote the skeleton of \( X_t \) on \( \text{Ray} \). Note that \( h_\ell = h(H_\ell) \) is a (biased) random walk in random environment with holding times; that is,

\[
P(h_{\ell+1} = j|h_\ell = k) = \begin{cases} 
\lambda/(\lambda + d_k), & j = k - 1, \\
1/(\lambda + d_k), & j = k + 1, \\
(d_k - 1)/(\lambda + d_k), & j = k.
\end{cases}
\]

(45)

Let \( h^*_\ell \) denote the homogeneous Markov chain on \( \mathbb{Z} \) with \( h^*_0 = 0 \) and transitions as in (44) corresponding to a homogeneous environment with \( d_k = C_1 \log t_\ell \), and set \( \eta_i = \min \{ \ell : h_\ell = -i \} \) and \( \eta^*_i = \min \{ \ell : h^*_\ell = -i \} \). The chain \( h^*_\ell \) possesses the same drift as the chain \( h_\ell \), and on the event \( \Upsilon_{t, \epsilon} \), its holding times dominate those of the latter chain. Therefore,

\[
1_{\Upsilon_{t, \epsilon}} P_{\tilde{\eta}_t}^* (\eta_t > m) \leq P(\eta^*_t > m).
\]

Further, setting \( \tilde{\theta}_0 = 0 \) and, for \( j \geq 1 \), using \( \tilde{\theta}_j = \min \{ i > \tilde{\theta}_{j-1} : h^*_i \neq h^*_{\tilde{\theta}_{j-1}} \} \) to denote the successive jump time of the walk \( h^*_i \), one can write

\[
\eta^*_i = \sum_{j : \tilde{\theta}_j < \eta^*_i} G_j
\]

where the \( G_j \) are independent geometric random variables with parameter \((\lambda + 1)/(\lambda + C_1 \log t_\ell)\) that represent the holding times. Therefore, for any constants \( C_2, C_3 \) independent of \( \epsilon \) and \( t \),

\[
P(\eta^*_t > C_2 t_\epsilon (\log t_\epsilon)^2) \leq P(\tilde{\theta}_{C_3 t_\epsilon} < \eta^*_t) + P(\sum_{j=1}^{C_3 t_\epsilon} G_j > C_2 t_\epsilon (\log t_\epsilon)^2).
\]

The event \( \{ \tilde{\theta}_{C_3 t_\epsilon} < \eta^*_t \} \) has the same probability as the event that a biased nearest neighbor random walk on \( \mathbb{Z} \) started at 0, with probability \( \lambda/(\lambda + 1) \) to increase at each step, does not hit \( t_\epsilon \) by time \( C_3 t_\epsilon \). Because \( \lambda > 1 \), choosing \( C_3 = C_3(\epsilon) \) large, this probability can be made exponentially small in \( t_\epsilon \), and in particular bounded above by \( 1/t \) for \( t \) large. Fix such a \( C_3 \). Now,

\[
P(\sum_{j=1}^{C_3 t_\epsilon} G_j > C_2 t_\epsilon (\log t_\epsilon)^2) \leq C_3 t_\epsilon P(G_1 > C_2 (\log t_\epsilon)^2 / C_3).
\]

By choosing \( C_2 = C_2(\epsilon) \) large, one can make this last term smaller than \( 1/t \). Therefore, with such a choice of \( C_2 \) and \( C_3 \), and writing \( \tilde{\Upsilon}_{t, \epsilon} = \Upsilon_{t, \epsilon} \cap \{ \eta_t < C_2 t_\epsilon (\log t_\epsilon)^2 \} \), we obtain from (43) that for all \( t \) large,

\[
E_{\tilde{\Upsilon}_{t, \epsilon}}^\omega (N_t(\alpha)) \leq 4 + E_{\tilde{\Upsilon}_{t, \epsilon}}^\omega (N_t(\alpha); \xi_t \geq t; \tilde{\Upsilon}_{t, \epsilon}).
\]

(46)

On the event \( \Upsilon_{t, \epsilon} \), all excursions \( \{ X_{\ell t} : \ell = \eta_{t-1}, \ldots, \eta_t - 1 \} \) away from \( \text{Ray} \) that start at \( v \in \text{Ray} \) with \( h(v) > -t_\epsilon \) are excursions into GW-trees where the degree
of the root is bounded by $C_1(\log t_\epsilon) - 1$. Therefore,

$$E^\omega_{\hat{\text{IGW}}} \left( \sum_{\ell = \eta_{n-1}}^{n} 1_{X_{\ell} \in Q_t(T); \hat{Y}_{t,\epsilon, h(X_{\eta_{n-1}})} > -t_\epsilon} \right) \quad (47)$$

$$\leq \max_{d \leq C_1(\log t_\epsilon) - 1} E^\omega_{\hat{\text{GW}}} \left( \sum_{\ell = 0}^{T_{\epsilon}} 1_{h(X_{\ell}) \leq \epsilon^n} |d_\epsilon = d \right)$$

$$= \max_{d \leq C_1(\log t_\epsilon) - 1} \left( \sum_{j = 0}^{\epsilon^n} E^\omega_{\hat{\text{GW}}} (\mathcal{N}_\epsilon(j) | d_\epsilon = d \right) .$$

Therefore, for all $t$ large,

$$E^\omega_{\hat{\text{IGW}}} (\mathcal{N}_\epsilon(\alpha; \xi_{t_\epsilon} \geq t; \hat{Y}_{t,\epsilon})) \leq E^\omega_{\hat{\text{IGW}}} \left( C_2 t_\epsilon (\log t_\epsilon)^2 \sum_{i = 1}^{\mathcal{N}_\epsilon(\alpha; \xi_{t_\epsilon} \geq t_\epsilon)} \sum_{\ell = \eta_{n-1}}^{n} 1_{X_{\ell} \in Q_t(T); \hat{Y}_{t,\epsilon}} \right)$$

$$\leq C_2 t_\epsilon (\log t_\epsilon)^2 \max_{d \leq C_1(\log t_\epsilon) - 1} \left( \sum_{j = 0}^{\epsilon^n} E^\omega_{\hat{\text{GW}}} (\mathcal{N}_\epsilon(j) | d_\epsilon = d \right)$$

$$\leq t^{1/2 + \alpha + \epsilon/2} , \quad (48)$$

where the second inequality uses (47), and (49) was used in the last inequality. Combined with (49), this completes the proof of Lemma 7. \hfill \Box

**Corollary 3** For each $\epsilon > 0$ there exists a $t_1 = t_1(T, \epsilon) < \infty$ such that for all $t \geq t_1$,

$$E^\omega_{\hat{\text{IGW}}} (\mathcal{N}_\epsilon(\alpha)) \leq t^{1/2 + \alpha + 2\epsilon} \hat{\text{IGW}} - a.s.$$ \quad (49)

**Proof of Corollary** From Lemma 7 and Markov’s inequality we have

$$P_{\hat{\text{IGW}}} (E^\omega_{\hat{\text{IGW}}} (\mathcal{N}_\epsilon(\alpha)) > c_\epsilon t^{1/2 + \alpha + 3\epsilon/2}) \leq t^{-\epsilon/2} .$$

Therefore, with $t_k = 2^k$, it follows from Borel-Cantelli that there exists an $k_1 = k_1(T, \epsilon)$ such that for $k > k_1$,

$$E^\omega_{\hat{\text{IGW}}} (\mathcal{N}_\epsilon(\alpha)) \leq c_\epsilon t_k^{1/2 + \alpha + 3\epsilon/2} \hat{\text{IGW}} - a.s.$$ But for $t_k < t < t_{k+1}$ one has that $\mathcal{N}_\epsilon(\alpha) \leq \mathcal{N}_{t_{k+1}}(\alpha)$. The claim follows. \hfill \Box

**Proof of Proposition** Note that the number of visits of $X_i$ to $Q_t(T)$ between time $i = t$ and $i = t + \lceil t^3 \rceil$ is bounded by $\mathcal{N}_{t + \lceil t^3 \rceil}(\alpha)$. Therefore,

$$P^\omega_{\hat{\text{IGW}}} (X_{t^3} \in Q_t(T)) = \frac{1}{t^3} \sum_{i = t}^{t + \lceil t^3 \rceil} P^\omega_{\hat{\text{IGW}}} (X_i \in Q_t(T)) \leq \frac{1}{t^3} E^\omega_{\hat{\text{IGW}}} (\mathcal{N}_{t + \lceil t^3 \rceil}(\alpha)) .$$

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Applying Corollary 3 with our choice of \( \epsilon_0 \), see (2), it follows that for all \( t > t_1(\mathcal{T}, \epsilon_0) \), for \( \text{IGW} \)-almost every \( \mathcal{T} \),

\[
P^0_\mathcal{T}(X_{\tau_t} \in Q_t(\mathcal{T})) \leq \frac{(t + \lfloor t^\delta \rfloor)^{1/2 + \alpha + 3\epsilon_0}}{t^\delta} \leq \frac{1}{t^\epsilon_0}.
\]

\[\square\]

6 From \( \text{IGW} \) to \( \text{GW} \): Proof of Theorem 1

Our proof of Theorem 1 is based on constructing a shifted coupling between the random walk \( \{X_n\} \) on a \( \text{GW} \) tree and a random walk \( \{Y_n\} \) on an \( \text{IGW} \) tree. We begin by introducing notation. For a tree (finite or infinite, rooted or not) \( \mathcal{T} \), we let \( \mathcal{L}\mathcal{T} \) denote the collection of leaves of \( \mathcal{T} \), that is of vertices of degree 1 in \( \mathcal{T} \) other than the root. We set \( \mathcal{T}^0 = \mathcal{T} \setminus \mathcal{L}\mathcal{T} \). For two trees \( \mathcal{T}_1, \mathcal{T}_2 \) with roots (finite or infinite) and a vertex \( v \in \mathcal{L}\mathcal{T}_1 \), we let \( \mathcal{T}_1 \circ v \mathcal{T}_2 \) denote the tree obtained by gluing the root of \( \mathcal{T}_2 \) at the vertex \( v \). Note that if \( \mathcal{T}_1 \) has an infinite ray emanating from the root, and \( \mathcal{T}_2 \) is a finite rooted tree, then \( \mathcal{T}_1 \circ v \mathcal{T}_2 \) is a rooted tree with a marked infinite ray emanating from the root.

Given a \( \text{GW} \) tree \( \mathcal{T} \) and a path \( \{X_n\} \) on the tree, we construct a family of finite trees \( \mathcal{T}_i \) and of finite paths \( \{u^i_n\} \) on \( \mathcal{T}_i \) as follows. Set \( \tau_0 = 0, \eta_0 = 0 \), and let \( \mathcal{U}_0 \) denote the rooted tree consisting of the root \( o \) and its offspring. For \( i \geq 1 \), let

\[
\tau_i = \min\{n > \eta_{i-1} : X_n \in \mathcal{L}\mathcal{U}_{i-1}\}, \quad \text{(Excursion start)}
\]

\[
\eta_i = \min\{n > \tau_i : X_n \in \mathcal{U}_{i-1}\}, \quad \text{(Excursion end)}
\]

\[
v_i = X_{\tau_i}, \quad \text{(Excursion start location)}.
\]

We then set

\[
\mathcal{V}_i = \{v \in \mathcal{T} : X_n = v \text{ for some } n \in [\tau_i, \eta_i]\},
\]

define \( \overline{\mathcal{V}}_i = \mathcal{V}_i \cup \{v \in \mathcal{T} : v \text{ is an offspring of some } w \in \mathcal{V}_i\} \), and let \( \mathcal{T}_i \) denote the rooted subtree of \( \mathcal{T} \) with vertices in \( \overline{\mathcal{V}}_i \) and root \( v_i \). We also define the path \( \{u^i_n\}_{n=0}^{\eta_i-\tau_i} \) by \( u^i_n = X_{n+\tau_i} \), noting that \( u^i_n \) is a path in \( \mathcal{T}_i \). Finally, we set

\[
\mathcal{U}_i = \mathcal{U}_{i-1} \circ v_i \mathcal{T}_i.
\]

Note that \( \mathcal{U}_i \) is a tree rooted at \( o \) since \( v_i \in \mathcal{L}\mathcal{U}_{i-1} \). Further, by the \( \text{GW} \)-almost sure recurrence of the biased random walk on \( \mathcal{T} \), it holds that \( \mathcal{T} = \lim_{i} \mathcal{U}_i \).

Next, we construct an \( \text{IGW} \) tree \( \tilde{\mathcal{T}} \) with root \( o \) and an infinite ray, denoted \( \text{Ray} \), emanating from the root, and a (\( \lambda \)-biased) random walk \( \{Y_n\} \) on \( \tilde{\mathcal{T}} \), as follows. First, we choose a vertex denoted \( o \) and a semi-infinite directed path \( \text{Ray} \) emanating from it. Next, we let each vertex \( v \in \text{Ray} \) have \( d_v \) offspring, where \( P(d_v = k) = kp_k/m \), and the \( \{d_v\}_{v \in \text{Ray}} \) are independent. For each vertex \( v \in \text{Ray}, v \neq o \), we identify one of its offspring with the vertex \( w \in \text{Ray} \) that satisfies \( d(w, o) = d(v, o) - 1 \), and write \( \hat{\mathcal{U}}_0 \) for the resulting tree with root \( o \) and marked ray \( \text{Ray} \).

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Set next $\hat{\tau}_0 = \hat{\eta}_0 = 0$. We start a $\lambda$-biased random walk $Y_n$ on $\hat{\mathcal{U}}_0$ with $Y_0 = o$, and define

$$\hat{\tau}_1 = \min\{n > 0 : Y_n \in \mathcal{L}\hat{\mathcal{U}}_0\}.$$ 

Let $\hat{v}_1 = Y_{\hat{\tau}_1}$. We now set $\hat{U}_1 = \hat{U}_0 \circ \hat{T}_1$ and $\hat{\eta}_1 = \hat{\tau}_1 + \hat{\eta}_1 - \hat{\tau}_1$, and for $\hat{\tau}_1 \leq n \leq \hat{\eta}_1 - 1$, set $Y_n = w^n_{\hat{\nu}_n - \hat{\tau}_1}$. Finally, with $\hat{w}_1$ the ancestor of $\hat{v}_1$, we set $Y_{\hat{\eta}_1} = Y_{\hat{w}_1}$.

The rest of the construction proceeds similarly. For $i > 1$, start a $\lambda$-biased random walk $\{Y_n\}_{n \geq \hat{\eta}_{i-1}}$ on $\hat{U}_{i-1}$ with $Y_{\hat{\eta}_{i-1}} = \hat{w}_{i-1}$ and define

$$\begin{align*}
\hat{\tau}_i &= \min\{n > \hat{\eta}_{i-1} : Y_n \in \mathcal{L}\hat{U}_{i-1}\}, \quad \text{(Excursion start)}, \\
\hat{v}_i &= Y_{\hat{\tau}_i}, \quad \text{(Excursion start location),} \\
\hat{\eta}_i &= \hat{\tau}_i + \hat{\eta}_i - \hat{\tau}_i, \quad \text{(Excursion end),} \\
\hat{U}_i &= \hat{U}_{i-1} \circ \hat{T}_i, \quad \text{(Extended tree),} \\
Y_n &= X_n - \hat{\tau}_i, n \in [\hat{\tau}_i, \hat{\eta}_i) \quad \text{(Random walk path during excursion),} \\
Y_{\hat{\eta}_i} &= \hat{w}_i = \text{ancestor of } \hat{v}_i.
\end{align*}$$

Finally, with $\hat{U} = \lim_i \hat{U}_i$, define the tree $\hat{T}$ by attaching to each vertex of $\mathcal{L}\hat{U}$ an independent Galton-Watson tree, thus obtaining an infinite tree with root $o$ and infinite ray emanating from it. The construction leads immediately to the following.

**Lemma 8** a) The tree $\hat{T}$ with root $o$ and marked ray Ray is distributed according to IGW.

b) Conditioned on $\hat{T}$, the law of $\{Y_n\}$ is the law of a $\lambda$-biased random walk on $\hat{T}$.

Let $\mathcal{R}_n = h(Y_n) - \min_{1 \leq i \leq n} h(Y_i) \geq 0$. Due to Theorem 2 for IGW-almost all $\hat{T}$, the process $\mathcal{R}_n(\cdot)/\sqrt{n}$ converges to a Brownian motion reflected at its running minimum, which possesses the same law as the absolute value of a Brownian motion, see e.g. [11, Theorem 6.17]. Our efforts are therefore directed toward estimating the relation between the processes $\{X_n\}$ and $\{\mathcal{R}_n\}$. Toward this end, let $I_n = \max\{i : \tau_i \leq n\}$ and $\hat{I}_n = \max\{i : \hat{\tau}_i \leq n\}$ measure the number of excursions started by the walks $\{X_n\}$ and $\{Y_n\}$ before time $n$, and set $\Delta_n = \sum_{i=1}^{I_n}(\tau_i - \eta_{i-1})$, and $\hat{\Delta}_n = \sum_{i=1}^{\hat{I}_n}(\hat{\tau}_i - \hat{\eta}_{i-1})$. Set also $B_n = \max_{s \leq t \leq n : Y_s \in \text{Ray}, Y_t \in \text{Ray}}(h(Y_t) - h(Y_s))$ ($B_n$ measures the maximal amount the random walk $\{Y_n\}$ backtracks, that is moves against the drift, along Ray before time $n$). Next set, recalling (13),

$$\begin{align*}
\Delta_n^\alpha &= \sum_{i=1}^{I_n} \sum_{t \in [\eta_{i-1}, \tau_i]} 1_{|X_t| \leq n^\alpha}, \\
\hat{\Delta}_n^\alpha &= \sum_{i=1}^{\hat{I}_n} \sum_{t \in [\hat{\eta}_{i-1}, \hat{\tau}_i]} 1_{Y_t \in Q_n^\alpha(\hat{T})}.
\end{align*}$$

Clearly, $\Delta_n^\alpha \leq \Delta_n$ and $\hat{\Delta}_n^\alpha \leq \hat{\Delta}_n$. We however can say more.
Figure 3: The coupling between the GW and IGW walks. X marks the location of the walker.
Lemma 9 Let \( A_n = \{ \Delta_n^a = \Delta_n \} \) and \( \hat{A}_n = \{ \hat{\Delta}_n^a = \hat{\Delta}_n \} \). Then,

\[
\lim_{n \to \infty} P_T^0(A_n^c) = 0, \quad \text{GW} - a.s., \tag{54}
\]

and

\[
\lim_{n \to \infty} P_T^0(\hat{A}_n^c) = 0, \quad \text{IGW} - a.s. \tag{55}
\]

Further,

\[
\limsup \frac{\Delta_n}{n} = 0, \quad \text{GW} - a.s., \tag{56}
\]

and

\[
\limsup \frac{\hat{\Delta}_n}{n} = 0, \quad \text{IGW} - a.s. \tag{57}
\]

Finally,

\[
\limsup \frac{B_n}{\sqrt{n}} = 0, \quad \text{IGW} - a.s., \tag{58}
\]

We postpone for the moment the proof of Lemma 9. Note that on the event \( A_n \cap \hat{A}_n \), one has

\[
\min_{s: |s-n| \leq \Delta_n + \hat{\Delta}_n} |X_s - R_s| \leq 2n^a + B_n. \tag{59}
\]

(To see that, note that the position \( |X_n| \) consists of sums of excursions \( \{u_t^i\} \), up to an error coming from the parts of the path not contained in these excursions, all contained in a distance at most \( n^a \) from the root. Similarly, for some \( s \) with \( |s-n| \leq \Delta_n + \hat{\Delta}_n \), \( R_s \) consists of the sum of the same excursions, up to an error coming from the parts of the path not contained in these excursions, which sum up to a total distance of at most \( n^a \) from \( \text{Ray} \) in addition to the amount \( B_n \) of backtracking along \( \text{Ray} \).

In view of Lemma 9, the convergence in distribution (for IGW-almost every \( \hat{T} \)) of \( R_{|s|}/\sqrt{n} \) to reflected Brownian motion, together with (60), complete the proof of Theorem 1.

Proof of Lemma 9 Consider a rooted tree \( T \) distributed according to GW, and a random walk path \( \{X_t\}_{t \geq 0} \) with \( X_0 = 0 \) on it. We introduce some notation. For \( k \geq 1 \), let \( \alpha_k = \sum_{j=1}^k \tau_j \), \( \beta_k = \sum_{j=1}^{k-1} \eta_j \), and \( J_k = [a_k - b_k + k, a_{k+1} - b_{k+1} + k] \) (the length of \( J_k \) is the time spent by the walk between the \( k \)-th and the \( k+1 \)-th excursions). For \( s \in J_k \), we define \( t(s) = \eta_k + s - (a_k - b_k + k) \). Finally, we set \( \tilde{X}_0 = 0, \tilde{X}_1 = X_\tau_1 = X_1, \) and \( \tilde{X}_s = X_{t(s)} \) (note that the process \( \tilde{X}_s \) travels on vertices “off the coupled excursions”). Note that even conditioned on \( T \), the nearest neighbor process \( \{\tilde{X}_s\}_{s \geq 0} \) on \( T \) is neither Markovian nor progressively measurable with respect to its natural filtration. To somewhat address this issue, we define the filtration \( G_n = \sigma(X_i, i \leq t(s)) \), and note that conditioned on \( \mathcal{T}, \{\tilde{X}_s\}_{s \geq 0} \) is progressively measurable with respect to the filtration \( G_n \).

The statement (61) will follow as soon as we prove the statement

\[
\lim_{n \to \infty} P_T^0(\max_{s \in \bigcup_{k=1}^n J_k} |\tilde{X}_s| \geq n^a) = 0, \quad \text{GW} - a.s., \tag{60}
\]
The proof of (60) will be carried out in several steps. The first step allows us to control the event that the time spent by the process $X_t$ inside excursions is short. The proof is routine and postponed.

**Lemma 10** For all $\epsilon > 0$,

$$
\lim_{n \to \infty} P^n_T(\sum_{i=1}^{n^{1/2+\epsilon}} (\eta_i - \tau_i) < n) = 0, \quad \text{GW} - a.s.
$$

(61)

Further, with

$$\bar{T}_n = \min\{t : W_{X_t} > (\log n)^2\},$$

it holds that

$$
\lim_{n \to \infty} nP^n_T(\bar{T}_n \leq n) = 0, \quad \text{GW} - a.s.
$$

(62)

Our next step involves “coarsening” the process $\{\tilde{X}_s\}$ by stopping it at random times $\{\Theta_i\}$ in such a way that if the stopped process has increased its distance from the root between two consecutive stopping times, with high probability one of the intervals $J_k$ has been covered. More precisely, define $\Theta_0 = 0$, and for $i \geq 1$,

$$\Theta_i = \min\{s > \Theta_{i-1} : |\tilde{X}_s| - |\tilde{X}_{\Theta_{i-1}}| = \gamma \}.$$

We emphasize that the $\Theta_i$ depend on $n$, although this dependence is suppressed in the notation. The following lemma, whose proof is again routine and postponed, explains why this coarsening is useful.

**Lemma 11** With the notation above,

$$
\lim_{n \to \infty} P^n_T(\text{for some } k \leq I_n, \Theta_{i-1}, \Theta_i \in J_k, |\tilde{X}_{\Theta_i}| - |\tilde{X}_{\Theta_{i-1}}|) = 0, \quad \text{GW} - a.s.
$$

(63)

We have now prepared all needed preliminary steps. Fix $\epsilon > 0$. Note first that due to (11) and the Borel-Cantelli lemma, for all large $n$, $|A^\alpha_{n\alpha}| \leq |D_{n\alpha}| e^{-\nu(\epsilon) n^{\alpha}/2}$, GW-a.s. On the other hand, since $E_{GW}[D_{n\alpha}] = m^{n\alpha}$, Markov’s inequality and the Borel-Cantelli lemma imply that for all large $n$, $|D_{n\alpha}| \leq m^{n\alpha} e^{\nu(\epsilon) n^{\alpha}/2}$, GW-a.s. Combining these facts, it holds that for all large

$$|A^\alpha_{n\alpha}| \leq m^{n\alpha} e^{-\nu(\epsilon) n^{\alpha}/2}, \quad \text{GW} - a.s.
$$

(64)

For any vertex $v \in D_{n\alpha}$, by considering the trace of the random walk on the path connecting $o$ and $v$ it follows that

$$P^n_T(X_t = v \text{ for some } t \leq n) \leq 1 - (1 - \lambda^{-n\alpha})^n \leq n\lambda^{-n\alpha}, \quad \text{GW} - a.s.
$$

Using this and (64) in the first inequality, and (62) in the second, we get

$$
\limsup_{n \to \infty} P^n_T(\max_{s \in \cup_{k=1}^{n} J_k} |\tilde{X}_s| \geq n^{\alpha})
\leq \limsup_{n \to \infty} P^n_T(\exists s \in \cup_{k=1}^{n} J_k : |\tilde{X}_s| = n^{\alpha}, S_{\tilde{X}_s} \geq \eta m^{\alpha}/2)
\leq \limsup_{n \to \infty} P^n_T(\exists s \in \cup_{k=1}^{n} J_k : |\tilde{X}_s| = n^{\alpha}, S_{\tilde{X}_s} \geq \eta m^{\alpha}/2, t(s) \leq \bar{T}_n).
$$

(65)
We next note that by construction,

$$|\{i \in \{1, \ldots, t\} : |\bar{X}_{\Theta_i}| > |\bar{X}_{\Theta_{i-1}}|\}| \geq \ell/2.$$ 

Hence, with $P_T$ probability approaching 1 as $n$ goes to infinity, $t(\Theta_{2n^{1/2+\epsilon}}) > n$ because of (61) and Lemma 11. From this and (65), we conclude that

$$\limsup_{n \to \infty} P_T^o \left( \max_{s \in \bigcup_{i=1}^{n} J_k} |\bar{X}_s| \geq n^\alpha \right)$$ 

$$\leq \limsup_{n \to \infty} \sum_{i=1}^{2n^{1/2+\epsilon}} P_T^o \left( |\bar{X}_{\Theta_i}| \geq n^\alpha - (\log n)^2, S_{\bar{X}_{\Theta_i}} \geq \eta n^\alpha/2 - (\log n)^4, T_n > t(\Theta_i) \right).$$

On the event $T_n > t(\Theta_i)$ it holds that $|S_{\bar{X}_{\Theta_i}} - S_{\bar{X}_{\Theta_{i-1}}}| \leq (\log n)^4$. Therefore, decomposing according to return times of $\bar{X}_{\Theta_i}$ to the root,

$$\limsup_{n \to \infty} P_T^o \left( \max_{s \in \bigcup_{i=1}^{n} J_k} |\bar{X}_s| \geq n^\alpha \right) \tag{66}$$

$$\leq \limsup_{n \to \infty} \sum_{i=0}^{2n^{1/2+\epsilon}} \sum_{j=i+1}^{2n^{1/2+\epsilon}} P_T^o \left( |\bar{X}_{\Theta_j}| \geq n^\alpha - (\log n)^2, \bar{X}_{\Theta_i} = o, S_{\bar{X}_{\Theta_j}} \geq \eta n^\alpha/2 - (\log n)^4, \right.$$

$$\left. |\bar{X}_{\Theta_k}| > 0 \text{ and } |S_{\bar{X}_{\Theta_k}} - S_{\bar{X}_{\Theta_{k-1}}}| \leq (\log n)^4 \text{ for } i < k \leq j \right)$$

$$=: \limsup_{n \to \infty} \sum_{i=0}^{2n^{1/2+\epsilon}} \sum_{j=i+1}^{2n^{1/2+\epsilon}} P_{i,j,n}.$$

Fixing $i$, set for $t \geq 1$, $\tilde{M}_t = S_{\bar{X}_{\Theta_{i+1}}}$. Introduce the random time

$$K_n = \min \{t > 1 : X_n = o \text{ for some } s \in [t(\Theta_{i+1}), t(\Theta_{i+t})] \}
\text{ or } |\tilde{M}_t - \tilde{M}_{t-1}| \geq (\log n)^4,$$

and the filtration $\tilde{G}_t = G_{\Theta_{i+t}}$. The crucial observation is that $\{\tilde{M}_{t \land K_n} - \tilde{M}_1\}$ is a supermartingale for the filtration $\tilde{G}_t$, with increments bounded in absolute value by $(\log n)^4$ for all $t < K_n$, and bounded below by $-(\log n)^4$ even for $t = K_n$ (it fails to be a martingale due to the “defects” at the boundary of each of the intervals $J_k$, at which times $r$ the conditional expectation of the increment $S_{\bar{X}_{r+1}} - S_{\bar{X}_r}$ is negative). Let $\tilde{M}'_t = \tilde{M}_t$ if $t < K_n$ or $t = K_n$ but $\tilde{M}_t < \tilde{M}_{t-1} + (\log n)^4$, and $\tilde{M}'_t = \tilde{M}_{K_n-1}$ otherwise. That is, $\tilde{M}'_t - \tilde{M}_1$ is a truncated version of the supermartingale $\tilde{M}_{t \land K_n} - \tilde{M}_1$. It follows that for some non-negative process $a_t$, $\{\tilde{M}'_t - \tilde{M}_1 + a_t\}$ is a martingale with increments
bounded for all $t \leq K_n$ by $2(\log n)^4$. Therefore, by Azuma’s inequality [2], for $j \leq n^{1/2+\epsilon}$, and all $n$ large,

$$P_{i,j,n} \leq P_1^n \left( \max_{1 \leq k \leq 2n^{1/2+\epsilon}} \left[ \tilde{M}_k' - \tilde{M}_1 \right] \geq \eta n^{\alpha/3} \right) \leq e^{-n^{2\alpha}/n^{1+3\epsilon}}.$$  

Since this estimate did not depend on $i$ or $j$, together with (66), this completes the proof of (60), and hence of (54). The proofs of (55) and (58) are similar and omitted.

We next turn to the proof of (57). Recall that from Lemma 7, for any $\epsilon > 0$, and all $n > n_0(\epsilon)$,

$$P_0^\text{IGW}(N_n(\alpha) \geq n^{1/2+\alpha+2\epsilon}) \leq n^{-\epsilon}.$$  

Therefore, noting the monotonicity of $N_n(\alpha)$ in $n$, an application of the Borel-Cantelli lemma (to the sequence $n_k = 2^k$) shows that

$$\frac{N_n(\alpha)}{n^{1/2+\alpha+3\epsilon}} \to_{n \to \infty} 0, \quad \text{IGW - a.s.}$$  

Since $\epsilon$ can be chosen such that $1/2 + \alpha + 3\epsilon < 1$, c.f. (2), and $\tilde{\Delta}_n \leq N_n(\alpha)$, (67) follows.

We finally turn to the proof of (66). In what follows, we let $C_i = C_i(T)$ denote constants that may depend on $T$ (but not on $n$). Let $T_\epsilon(n) = \min\{t : |X_t| = n^{1/2+\epsilon}\}$. By Lemma 5

$$P_0^\text{GW}(T_\epsilon(n) \leq n) \leq 4ne^{-n^{2\epsilon}/2}.$$  

In particular, by the Borel-Cantelli lemma, for $\text{GW}$-almost every $T$,

$$P_1^n(T_\epsilon(n) \leq n) \leq C_4(T)e^{-n^{\epsilon}}.$$  

Let $C_{\alpha,\ell}$ denote the conductance between the root and $D_\ell$. That is, define a unit flow $f$ on $T$ as a collection of non-negative numbers $f_{v,w}$, with $v \in T$ and $w \in T$ an offspring of $v$, such that Kirchoff’s current law hold: $1 = \sum_{w \in D_1} f_{o,w}$ and $f_{v,w} = \sum_{w' : w' \text{ is an offspring of } w} f_{w,w'}$. Then,

$$C_{\alpha,\ell}^{-1} = \inf_{f,f \text{ is a unit flow}} \sum_{i=0}^{\ell-1} \sum_{v \in D_1} \sum_{w : w \text{ is an offspring of } v} f_{v,w}^2 \lambda^i.$$  

By [19, Theorem 2.2], for $\text{GW}$-almost every $T$ there exists a constant $C_5(T)$ and a unit flow $f$ such that

$$\sum_{v \in D_1} \sum_{w : w \text{ is an offspring of } v} f_{v,w}^2 \leq C_5(T) \lambda^{-i}.$$  

It follows that

$$C_{\alpha,\ell}^{-1} \leq C_5(T) \ell.$$  

(68)
On the other hand, by standard theory, see [14] Exercise 2.47, for a given tree \( \mathcal{T} \), with \( L_o(j) \) denoting the number of visits to the root before time \( j \),

\[
E_T^o L_o(T_o(n)) = d_o \zeta_{o,1/2+\epsilon}^{-1}.
\]

Hence, \( E_T^o L_o(T_o(n)) \leq d_o C_\delta(T)n^{1/2+\epsilon} \). By Lemma 6, we also have that \( E_T^o (N_o(\ell)) \leq C_\delta(T) \), for any \( \ell \). Thus, using \( \bar{N}_n(\alpha) = \sum_{t=1}^n 1_{\{|X_t| \leq n^\alpha\}} \),

\[
E_T^o (\bar{N}_n(\alpha) ; T_o(n) \geq n) \leq E_T^o L_o(T_o(n)) E_T^o \left( \sum_{\ell=0}^{n^\alpha} N_o(\ell) \right) \leq d_o C_\delta(T) C_\delta(T)n^{1/2+\epsilon+\alpha}.
\]

It follows from this that

\[
E_T^o (\bar{N}_n(\alpha)) \leq n P_T^o (T_o(n) \leq n) + d_o C_\delta(T) C_\delta(T)n^{1/2+\epsilon+\alpha}.
\]

Using (72), and the fact that \( \bar{N}_n(\alpha) \geq \Delta_n^\alpha \), together with (53), completes the proof of (66), and hence of Lemma 6. \( \square \)

**Proof of Lemma 10** We note first that under the annealed measure \( \mathsf{GW} \), the random times \( (\eta_i - \tau_i) \), which denote the length of the excursions, are i.i.d., and for all \( x \),

\[
P_{\mathsf{GW}}^o(\eta_i - \tau_i \geq x) \geq \frac{1}{\lambda + 1} P_{\mathsf{GW}}^o(T_o \geq x),
\]

where \( T_o = \min\{t \geq 1 : X_t = o\} \) denotes the first return time of \( X_t \) to \( o \).

Throughout, the constants \( C_i(T) \), that depend only on the tree \( T \), are as in the proof above. Let \( x_t = t^{1/2 + \epsilon/2} \) and set \( T_x = \min\{t : |X_t| = z\} \). Then,

\[
P_T^o (T_0 \geq t) \geq P_T^o (T_x, t < T_o) P_T^o (T_x, t \geq t T_x, T_o) .
\]  

(69)

Note however that \( P_T^o (T_x < T_o) \) is bounded by the effective conductance between the root and \( D_{x_t} \), which by (68) is bounded below by \( C_\delta(T)x_t^{-1} \). In particular,

\[
P_T^o (T_x < T_o) \geq C_\delta(T) x_t^{-1}.
\]  

(70)

On the other hand, using (70) and the Carne-Varopoulos bound (see [14] Theorem 12.1, [7] 23) in the second inequality,

\[
P_T^o (T_x < t | T_x < T_o) \leq \frac{P_T^o (T_x < t)}{P_T^o (T_x < T_o)} \leq C_\gamma(T) x_t e^{-t x_t}.
\]  

(71)

It follows that for all \( t \) large,

\[
P_T^o (T_x \geq t | T_x < T_o) > 1/2 ,
\]

implying with (69) and (70) that for all \( t \) large,

\[
P_T^o (T_o \geq t) \geq \frac{C_\delta(T)}{2 t^{1/2+\epsilon/2} }.
\]  

(72)
It follows that for some deterministic constant $C$ and all $t$ large,

$$P^o_{\text{GW}}(T_o \geq t) \geq \frac{C}{t^{1/2+\epsilon/2}}.$$  \hfill (73)

Hence,

$$P^o_{\text{GW}}(\sum_{i=1}^{n^{1/2+\epsilon}} (\eta_i - \tau_i) < n) \leq \left(1 - \frac{P^o_{\text{GW}}(T_o \geq n)}{\lambda + 1}\right) n^{1/2+\epsilon} \leq \left(1 - \frac{C}{n^{1/2+\epsilon/2}}\right) n^{1/2+\epsilon} \leq e^{-Cn^{\epsilon/2}}.$$

An application of the Borel-Cantelli lemma yields (61).

To see (62), note that by time $n$ the walker explored at most $n$ distinct sites. We say that $t$ is a fresh time if $X_s \neq X_t$ for all $s < t$. Then,

$$P^o_{\text{GW}}(W_{X_t} \geq (\log n)^2, t \text{ is a fresh time}) \leq P^o_{\text{GW}}(W_o \geq (\log n)^2) \leq e^{-c(\log n)^2},$$

by the tail estimates on $W_o$, see [1]. Therefore,

$$P^o_{\text{GW}}(W_{X_t} \geq (\log n)^2, \text{ for some } t \leq n) \leq \sum_{t=0}^{n} P^o_{\text{GW}}(W_{X_t} \geq (\log n)^2, t \text{ is a fresh time}) \leq (n + 1)e^{-c(\log n)^2},$$

from which (62) follows by an application of the Borel-Cantelli lemma. \qed

**Proof of Lemma 11** Let $G_n$ denote the event inside the probability in the left hand side of (63). The event $G_n$ implies the existence of times $t_0 < t_1 < t_2 \leq n$ and vertices $u, v$ such that $X_{t_0} = u = X_{t_2}, X_{t_1} = v,$ and $|v| = |u| - \lfloor (\log n)^{3/2} \rfloor$. Thus, using the Markov property,

$$P^o_T(G_n) \leq |\{(t_0, t_1) : t_0 < t_1 \leq n\}| \max_{\substack{u, v \in \mathbb{T}, |v| = |u| - \lfloor (\log n)^{3/2} \rfloor}} P^v(X_t = u \text{ for some } t \leq n).$$

Noting that for each fixed $u, v$ as above, the last probability is dominated by the probability of a $\lambda$-biased (toward 0) random walk on $\mathbb{Z}_+$ reflected at 0 to hit location $\lfloor (\log n)^{3/2} \rfloor$ before time $n$, we get

$$P^o_T(G_n) \leq n^2e^{-c(\log n)^{3/2}},$$

for some $c > 0$, which implies (63). \qed

### 7 The transient case

Recall that when $\lambda < m$, it holds that $|X_n|/n \to_{n \to \infty} v > 0$, \text{GW-\text{a.s.}}$, for some non-random $v = v(\lambda)$ (see [17]). Our goal in this section is to prove the following:
Theorem 3 Assume \( \lambda < m \) and \( p_0 = 0, \sum_k \beta^k p_k < \infty \) for some \( \beta > 1 \). Then, there exists a deterministic constant \( \sigma^2 > 0 \) such that for \( GW \)-almost every \( T \), the processes \( \{(|X_{nt}| - nt)/\sqrt{\sigma^2 n}\}_{t \geq 0} \) converges in law to standard Brownian motion.

Before bringing the proof of Theorem 3 we need to derived an annealed invariance principle, see Corollary 4 below. The proof of the latter proceeds via the study of regeneration times, which are defined as follows: we set

\[
\tau_1 := \inf\{t : |X_i| > |X_s| \text{ for all } s < t, \text{ and } |X_u| \geq |X_i| \text{ for all } u \geq t\},
\]

and, for \( i \geq 1 \),

\[
\tau_{i+1} := \inf\{t > \tau_i : |X_i| > |X_s| \text{ for all } s < t, \text{ and } |X_u| \geq |X_i| \text{ for all } u \geq t\}.
\]

We recall (see [17]) that under the assumptions of the theorem, there exists \( GW \)-a.s. an infinite sequence of regeneration times \( \{\tau_i\}_{i \geq 1} \), and the sequence \( \{(|X_{\tau_{i+1}}| - |X_{\tau_i}|), (\tau_{i+1} - \tau_i)\}_{i \geq 1} \) is i.i.d. under the \( GW \) measure, and the variables \( |X_{\tau_2} - |X_{\tau_1}| \) and \( |X_{\tau_1}| \) possess exponential moments (see [8] Lemma 4.2 for the last fact). A key to the proof of an annealed invariance principle is the following

Proposition 3 When \( \lambda < m \), it holds that \( E_{GW}((\tau_2 - \tau_1)^k) < \infty \) for all integer \( k \).

Proof of Proposition 3: By coupling with a biased (away from 0) simple random walk on \( Z_+ \), the claim is trivial if \( \lambda < 1 \). The case \( \lambda = 1 \) is covered in [20, Theorem 2]. We thus consider in the sequel only \( \lambda \in (1, m) \). Let \( T_o = \inf\{t > 0 : X_t = o\} \) denote the first return time to the root and \( T_n = \min\{t > 0 : |X_t| = n\} \) denote the hitting time of level \( n \). Let \( o' \in D_1 \) be an arbitrary offspring of the root. By [3] (4.25), the law of \( \tau_2 - \tau_1 \) under \( GW \) is identical to the law of \( \tau_1 \) for the walk started at \( v \), under the measure \( GW' \cdot|T_o = \infty) \). Therefore,

\[
E_{GW}^o((\tau_2 - \tau_1)^k) = E_{GW}^{o'}(\tau_1^k | T_o = \infty) = \frac{E_{GW}^{o'}((\tau_1^k ; T_o = \infty)}{P_{GW}^{o'}(T_o = \infty)}
\]

where in the last equality we used that \( P_{GW}^{o'}(T_o = \infty) = P_{GW}^{o'}(T_o = \infty) \). Thus, with \( c \) denoting a deterministic constant whose value may change from line to line,

\[
E_{GW}^o((\tau_2 - \tau_1)^k) \leq c \sum_{n=1}^{\infty} E_{GW}^{o'}(\tau_1^k ; |X_{\tau_1}| = n, T_o = \infty)
\]

\[
= c \sum_{n=1}^{\infty} E_{GW}^{o'}(T_n ; |X_{\tau_1}| = n, T_o = \infty)
\]

\[
\leq c \sum_{n=1}^{\infty} E_{GW}^{o'}(T_n^{2k} ; T_o = \infty)^{1/2} P_{GW}^{o'}(|X_{\tau_1}| = n)^{1/2}
\]

\[
\leq c \sum_{n=1}^{\infty} e^{-n/c} E_{GW}^{o'}(T_n^{2k} ; T_o = \infty)^{1/2},
\]

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where the last inequality is due to the above mentioned exponential moments on \(|X_{r_1}|\). Therefore,

\[
E_{GW}^{\omega}((\tau_2 - \tau_1)^k) \leq c \sum_{n=1}^{\infty} e^{-n/c} n^{10k} \left( \sum_{j=0}^{\infty} (j + 1)^{2k} P_{GW}^{\omega}(T_n > jn^{10} \mid T_o = \infty) \right)^{1/2}.
\]

We proceed by estimating the latter probability. For \(j \geq 1\), let

\[
A_{1,j,n} = \{ \text{there exists a } t \leq jn^{10} \text{ such that } d_{X_t} \geq (\log jn^{10})^2 \}.
\]

Note that by the assumption \(\sum \beta^k p_k < \infty\) for some \(\beta > 1\), there exists a constant \(c\) such that for all \(j\) and all \(n\) large,

\[
P_{GW}^{\omega}(A_{1,j,n}) \leq e^{-c(\log(jn^{10}))^2} \leq e^{-c(\log n)^2 - c(jn^{10})^2},
\]

We next recall that \(t\) is a fresh time for the random walk if \(X_s \neq X_t\) for all \(s < t\). Let \(N_{j,n} := |\{t \leq jn^{10} : t \text{ is a fresh time}\}|\) (i.e., \(N_{j,n}\) is the number of distinct vertices visited by the walk up to time \(jn^{10}\)). Set

\[
A_{2,j,n} = \{ N_{j,n} < \sqrt{jn^{10}} \cap \{ T_o = \infty \} \}.
\]

Note that on the event \(A_{2,j,n} \cap A_{1,j,n}\), there is a time \(t \leq jn^{10}\) and a vertex \(v\) with \(d_v \leq (\log(jn^{10}))^2\) such that \(X_t = v\) and \(v\) is subsequently visited \(\sqrt{jn^{10}}\) times with no visit at the root. Considering the trace of the walk on the ray connecting \(v\) and \(o\), and conditioning on \(X_t = v\), the last event has a probability bounded uniformly (in \(t, v\)) by \((1 - c/(\log(jn^{10}))^2)\sqrt{jn^{10}}\), since \(\lambda > 1\). Hence, for all \(n\) large, using (75),

\[
P_{GW}^{\omega}(A_{2,j,n}) \leq e^{-c(\log(jn^{10}))^2} + jn^{10} \left( 1 - \frac{c}{(\log(jn^{10}))^2} \right)^{\sqrt{jn^{10}}}
\leq e^{-c(\log n)^2 - c(jn^{10})^2} + jn^{10} e^{-(jn^{10})^{1/4}}.
\]

The event \(A_{2,j,n} \cap \{ T_o = \infty \} \) entails the existence of at least \(j^{1/2}n^3\) fresh times which are at distance at least \(n^2\) from each other. Letting \(t_1 = \min\{ t > 0 : t \text{ is a fresh time} \}\) and

\[
t_i = \min\{ t > t_{i-1} + n^2 : t \text{ is a fresh time} \},
\]

we observe that if \(|X_{t_i}| < n\) then \(P_{GW}^{X_{t_i}}(T_n < n^2 | F_{t_i}) > c > 0\) (since from each fresh time, the walk has under the \(GW\) measure a strictly positive probability to escape with positive speed without backtracking to the fresh point). Thus,

\[
P_{GW}^{\omega}(T_n > jn^{10}, T_o = \infty, A_{2,j,n}) \leq (1 - c)^{1/2}n^3.
\]

Combining (76) and (77), we conclude that

\[
\sum_{j=0}^{\infty} (j + 1)^{2k} P_{GW}^{\omega}(T_n > jn^{10}, T_o = \infty) \leq c.
\]
Substituting in (74), the lemma follows.

A standard consequence of Proposition 3 and the regeneration structure (see e.g. [22, Theorem 4.1], [24, Theorem 3.5.24]) is the following:

**Corollary 4** There exists a constant $c^2$ such that, under the annealed measure $\mathbb{G}_W$, the process $\{(|X_{[nt]}| - nt\sqrt{\lambda})/\sqrt{n}\}_{t \geq 0}$ converges in distribution to a Brownian motion.

**Proof of Theorem 3** Our argument is based on the technique introduced by Bolthausen and Sznitman in [5], as developed in [6]. Let $B^n_i = B^n([X_i]) = \frac{(|X_{[nt]}| - nt\sqrt{\lambda})/\sqrt{n}}{\sqrt{n}}$, and let $\mathbb{B}_p^n([X_i])$ denote the polygonal interpolation of $(k/n) \rightarrow B^n_{k/n}$. Consider the space $C_T$ of continuous functions on $[0, T]$, endowed with the distance $d_T(u, u') = \sup_{t \leq T} |u(t) - u'(t)| \wedge 1$. By [22, Lemma 4.1], Theorem 3 will follow from Corollary 4 once we show that for all bounded by 1 Lipschitz function $F$ on $C_T$ with Lipschitz constant 1, and $b \in (1, 2]$,

$$\sum_k \text{var}_{\mathbb{G}_W} \left( E_T^F[F(B^{|b|})] \right) < \infty. \quad (78)$$

In the sequel, fix $b$ and $F$ as above. For the same tree $T$, let $X^1$ and $X^2$ be independent $\lambda$-biased random walks on $T$, and set $\mathbb{B}[i, k] = \mathbb{B}^{|b|}([X_i])$ and $\mathbb{B}[i, k, s] = \mathbb{B}^{|b|}([X_i^s]_{-s} - [X_i^s], i = 1, 2$. Set

$$\tau_i, k = \min\{t > \lfloor b^{k/4} \rfloor : t \text{ is a regeneration time for } X^i \},$$

$$A^1_k := \{(X^1_s, s \leq \tau_1^i, k) \cap X^2_{k, s} = \emptyset \}, \quad A^2_k := \{(X^2_s, s \leq \tau_2^i, k) \cap X^1_{s, k} = \emptyset \},$$

$$A_{k} = A^1_k \cup A^2_k,$$

$$B^i_k := \{\tau_i^k \leq \lfloor b^{k/3} \rfloor\}.$$  

Note that on the event $A^1_k$, the paths $\{X^1_s, s \geq \tau_1^i, k\}$ and $\{X^2_s, s \geq \tau_2^i, k\}$ can intersect only if $X^2_{s, k}$ is a descendant of $X^1_{s, k}$. Applying the same reasoning for the symmetric event $A^2_k$, we conclude that on the event $A_k$, these two paths do not intersect.

By construction, for any path $X$ on $T$, the path $\mathbb{B}^{|b^i|}([X_i])$ is Lipschitz with Lipschitz constant bounded by $b^{i/2}$. Hence, since

$$\max_i |\mathbb{B}[i, k]|_t - \mathbb{B}[i, k, \tau_i^k, k]|_t \leq \frac{\tau_i^k}{b^{k/2}}$$

and using the fact that $F$ is a Lipschitz function with Lipschitz constant 1, we have that on the event $B^i_k$, $|F(\mathbb{B}[i, k]) - F(\mathbb{B}[i, k, \tau_i^k, k])| \leq b^{k/3}/b^{k/2}$, and thus, since $|F| \leq 1$,

$$\begin{align*}
\text{var}_{\mathbb{G}_W} \left( E_T^F[F(B^{|b|})] \right) \\
= E_{\mathbb{G}_W}[F(\mathbb{B}[1, k])F(\mathbb{B}[2, k])] - E_{\mathbb{G}_W}[F(\mathbb{B}[1, k])]E_{\mathbb{G}_W}[F(\mathbb{B}[2, k])] \\
\leq 4P_{\mathbb{G}_W}(B^1_k) + 4b^{k/3-k/2} + E_{\mathbb{G}_W}[F(\mathbb{B}[1, k, \tau_1^k, k])F(\mathbb{B}[2, k, \tau_2^k, k])] \\
- E_{\mathbb{G}_W}[F(\mathbb{B}[1, k, \tau_1^k, k])]E_{\mathbb{G}_W}[F(\mathbb{B}[2, k, \tau_2^k, k])].
\end{align*}$$

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Conditioning on the event \( A_k \) and using again that \(|F| \leq 1\), we get
\[
\text{var}_{GW}\left(E_F^\varphi [F(B[k])]\right) \leq 4P_{GW}(\{B_k\}^c) + 4P_{GW}(A_k^c) + 4b^{k/3 - k/2} + E_{GW}[F(B[1, k, \tau^{1,k}])F(B[2, k, \tau^{2,k}])|A_k] - E_{GW}[F(B[1, k, \tau^{1,k}])|A_k]E_{GW}[F(B[2, k, \tau^{2,k}])|A_k].
\]
Conditioned on the event \( A_k \), the paths \( B[1, k, \tau^{1,k}] \) and \( B[2, k, \tau^{2,k}] \) are independent under the \( GW \) measure. Therefore, we conclude that with Markov’s inequality was used in the third step. Let \( \tau_i^1 \) denote the successive regeneration times for \( X_j \), \( j = 1, 2 \). The event \( \{\{B_k\}^c]\cap\{\tau_i^1 \leq b^{k/4}\} \) implies that at least one of the first \( b^{k/4} \) inter-regeneration times \( \tau_{i+1}^1 - \tau_i^1 \) is larger than \( b^{k/3} \). Therefore,
\[
P_{GW}(\{B_k\}^c) \leq P_{GW}(\tau_i^1 > b^{k/4}) + P_{GW}(\text{one of } (\tau_{i+1}^1 - \tau_i^1)_{i=1}^{b^{k/4}} \text{ is larger than } b^{k/3})
\leq P_{GW}(\tau_1^1 > b^{k/4}) + b^{k/4}P_{GW}(\tau_2^1 - \tau_1^1 > b^{k/3})
\leq P_{GW}(\tau_1^1 > b^{k/4}) + b^{k/4 - k/3}E_{GW}(\tau_2^1 - \tau_1^1)
\leq P_{GW}(\tau_1^1 > b^{k/4}) + cb^{k/4 - k/3},
\]
where Markov’s inequality was used in the third step. Let \( T_\ell = \min\{t > 0 : |X_t| = \ell\} \). Let \( Y_\ell \) be a nearest neighbor random walk on \( \mathbb{Z}_+ \) with \( P(Y_{t+1} = Y_t - 1|Y_t) = \lambda/(\lambda + 1) \) whenever \( Y_t \neq 0 \). \( Y_\ell \) and \( X_\ell \) can be constructed on the same probability space, such that \( T_\ell \leq \min\{t > 0 : Y_t = \ell\} =: T^{Y}_\ell \) for all \( \ell \). On the other hand, using the Markov property, for any constant \( c \) and all \( \ell \) large,
\[
P(T^{Y}_\ell > c^{\ell}) \leq \left(1 - \left(\frac{1}{1 + \lambda}\right)^\ell\right)^{c^{\ell}/\ell}
\]
In particular, there exists a \( c_1 = c_1(\lambda) > 0 \) such that \( P_{GW}(T_\ell > c^{\ell}) \leq e^{-\ell/c_1} \) (better bounds are available but not needed). Thus, for some deterministic constants \( c_i = c_i(\ell, b) > 0, i \geq 2, \) and all \( k \) large,
\[
P_{GW}(\tau_i > b^{k/4}/2) \leq P_{GW}(|X_{\tau_i}| > c_2 k) + P_{GW}(\tau_i > b^{k/4}/2, |X_{\tau_i}| \leq c_2 k)
\leq P_{GW}(|X_{\tau_i}| > c_2 k) + P_{GW}(T_{ce^k} > b^{k/4}/2) \leq e^{-c_3 k},
\]
where we have used the above mentioned fact that \( |X_{\tau_i}| \) possesses exponential moments. We conclude that with \( c_4 \leq c_3 \),
\[
P_{GW}(\{B_k\}^c) \leq b^{-c_4 k}.
\]
It remains to estimate \( P_{GW}(A_k^c) \leq 2P_{GW}(\{A_k\}^c) \). Let
\[
C_{k,i} := \{\tau_i^1 < b^{k/4}/2\}, C_{k,i}'' := \{\tau_i^1_{b/k} < b^{k/4}\}, C_k := C_{k,1} \cap C_{k,2} \cap C_{k,1}'' \cap C_{k,2}'',
\]
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Using (80), it follows that $P_{GW}(C_{k,1}^c) \leq b^{-c_3k}$. On the other hand, the event $C'_{k,1} \cap (C''_{k,1})^c$ implies that the sum of the difference $\tau_{i+1}^1 - \tau_i^1$, $i = 1, \ldots, \lfloor b^{k/8} \rfloor$, is larger than $b^{k/4}/2$, and hence, by Markov’s inequality,

$$P_{GW}(C_{k,1}^c \cap (C''_{k,1})^c) \leq 2b^{k/8} E_{GW}(\tau_2^1 - \tau_1^1) \leq b^{-c_3k},$$

for some deterministic constant $c_3 < c_4$. Since the same estimates are valid also for $C'_{k,2}$ and $C''_{k,2}$ replacing $C'_{k,1}$ and $C''_{k,1}$, it follows that

$$P_{GW}(C_k^c) \leq 4b^{-c_3k}. \quad (81)$$

On the other hand, let $Z_i$ denote the collection of vertices in $D_{\lfloor b^{k/8} \rfloor}$ hit by $X_i$. On $C_k$ there are at most $b^{k/4}$ vertices in $Z_1$. The event $(A_k^1)^c \cap C_k$ implies that the path $X^2$ intersected the path $X^1$ at a distance at least $\lfloor b^{k/8} \rfloor$ from the root, and this has to happen before time $\tau_{\lfloor b^{k/8} \rfloor}^2$, i.e. before time $b^{k/4}$, for otherwise $Z_1 \cap Z_2 = \emptyset$. Therefore,

$$P_{GW}((A_k^1)^c \cap C_k) \leq E_{GW} P_T^v(X^2 \text{ visits } Z^1 \text{ before time } b^{k/4}) \leq b^{k/4} E_{GW} \max_{v \in D_{\lfloor b^{k/8} \rfloor}} P_T^v(X^2 \text{ visits } v \text{ before time } b^{k/4}). \quad (82)$$

When $\lambda > 1$, there exists a constant $c_6 < c_5$ such that uniformly in $v \in D_{\lfloor b^{k/8} \rfloor}$,

$$P_T^v(X^2 \text{ visits } v \text{ before time } b^{k/4}) \leq b^{k/4} e^{-c_6 b^{k/8}}.$$

On the other hand, even when $p_1 > 0$, Lemma 2.2 of [S] shows that there exists a $\beta > 0$ such that with $M_v = |\{w \text{ is an ancestor of } v : d_w \geq 2\}|$, it holds that

$$\limsup_{\ell \to \infty} P_{GW}(\min_{v \in D_{\ell}} M_v/\ell < \beta) < 0.$$

It immediately follows, reducing $c_6$ if necessary, that when $\lambda \leq 1$, for all $k$ large,

$$E_{GW} \max_{v \in D_{\lfloor b^{k/8} \rfloor}} P_T^v(X^2 \text{ visits } v \text{ ever}) \leq e^{-c_6 b^{k/8}}.$$

Substituting in (82), we conclude that whenever $\lambda < m$,

$$P_{GW}(A_k^c \cap C_k) \leq 2P_{GW}((A_k^1)^c \cap C_k) \leq e^{-c_3 b^{k/8}}.$$

Together with (81), (80), and (79), we conclude that (78) holds and thus conclude the proof of Theorem 3.

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