Open sets avoiding integral distances

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Abstract

We study open point sets in Euclidean spaces $\mathbb{R}^d$ without a pair of points an integral distance apart. By a result of Furstenberg, Katznelson, and Weiss such sets must be of Lebesgue upper density zero. We are interested in how large such sets can be in $d$-dimensional volume. We determine the exact values for the maximum volumes of the sets in terms of the number of their connected components and dimension. Here techniques from diophantine approximation, algebra and the theory of convex bodies come into play. Our problem can be viewed as a counterpart to known problems on sets with pairwise rational or integral distances. This reveals interesting links between discrete geometry, topology, and measure theory.

1 Introduction

Is there a dense set $S$ in the plane so that all pairwise Euclidean distances between the points are rational? This famous open problem was posed by Ulam in 1945, see e.g. [17, 18, 40]. Unlike this, a construction of a countable dense set in the plane avoiding rational distances is not hard to find, see e.g. [29, Problem 13.4, 13.9]. If all pairwise distances between the points in $S$ are integral and $S$ is non-collinear, i.e. not all points are located on a line, then $S$ is finite [2, 16]. Having heard of this result, Ulam guessed that the answer to his question would be in the negative. Of course the rational numbers form a dense subset of a coordinate line with pairwise rational distances; also, on a circle there are dense sets with pairwise rational distances, see e.g. [1, 2]. It was proved by Solymosi and De Zeeuw [38] that the line and the circle are the only two irreducible algebraic curves containing infinite subsets of points with pairwise rational distances. Point sets with rational coordinates on spheres have been considered in [35]. There is interest in a general construction of a planar point set $S(n, k)$ of size $n$ with pairwise integral distances such that $S(n, k) = A \cup B$ where $A$ is collinear, $|A| = n - k$, $|B| = k$, and $B$ has no three collinear points. The current record is $k = 4$ [10]. And indeed, it is very hard to construct a planar point set, no three points on a line, no four points on a circle, with pairwise integral distances. Kreisel and Kurz [30] found such a set of size 7, but it is unknown if there exists one of size 8.

The present paper is concerned with a problem that may be considered as a counterpart to those just described, namely with large point sets in $\mathbb{R}^d$ without a pair of points an integral distance apart. We write $f_d(n)$ for the supremum of the volumes $\lambda_d(P)$ of open point sets $P \subset \mathbb{R}^d$ with $n$ connected components.
without a pair of points whose distance apart is a positive integer. We determine the exact values of the function \( f_d(n) \) for all \( d \) and \( n \).

This problem is related to the famous Hadwiger–Nelson open problem of determining the (measurable) chromatic number of \( \mathbb{R}^d \), see e.g. [12] Problem G10. Here one can also ask for the highest density of one color class in such a coloring, that is, we may ask for the densest set without a pair of points a distance 1 apart. In [32] such a construction in \( \mathbb{R}^3 \) has been given. In the plane the best known example, due to Croft [11], consists of the intersections of hexagons with circles and attains a density of 0.2294. The upper bounds are computed in [5][13]. Point sets avoiding a finite number \( k \) of prescribed distances are considered e.g. in [9] and [12, Problem G4], so the point sets avoiding all distances that are positive integers correspond to the case with an infinite number \( \kappa \) of excluded distances. It is known [21] that for each subset \( \mathcal{U} \) of the plane with positive density, there is a constant \( d(\mathcal{U}) \) such that all distances greater than \( d(\mathcal{U}) \) occur between the points of \( \mathcal{U} \). The same result is true in higher dimensions [34]. It follows that in every dimension \( d \geq 2 \), the Lebesgue measurable sets avoiding integral distances, which are of interest here, must be of upper density zero, so we consider the supremum of their volumes instead.

The paper is organized as follows: in Section 2 we introduce the basic notation and provide characterizations of arbitrary open point sets without pairs of points an integral distance apart. After stating first relationships between the upper bounds for the maximum volumes of those sets with different numbers of connected components we continue in Section 3 by considering a relaxed problem. We evaluate the maximum volumes of sets avoiding integral distances in the special case where the connected components we continue in Section 3 by considering a relaxed problem. We evaluate the maximum volumes of those sets with different numbers of connected components. Thus [22]. In Subsection 4.2 we provide some upper bounds for \( l_d(n) \). The main problem of evaluating \( f_d(n) \) generally is finally settled in Subsection 4.3. In Section 5 we give a summary of the results obtained and draw the appropriate conclusions.

### 2 General observations and basic notation

Denote by dist(\( x, y \)) the Euclidean distance between two points \( x, y \in \mathbb{R}^d \) and by dist(\( V, W \)) := \( \inf\{dist(x, y) \mid x \in V, y \in W\} \) the distance between two subsets \( V \) and \( W \) of \( \mathbb{R}^d \). The minimum width of \( V \), i.e. the minimum distance between parallel support hyperplanes of the closed convex hull of \( V \), will be denoted by width(\( V \)), and \( \lambda_d \) will stand for the Lebesgue measure in \( \mathbb{R}^d \).

At first we observe that the diameter of any connected component of an open set avoiding integral distances, i.e. having no points an integral distance apart, is at most 1.

**Lemma 1** Let \( \mathcal{P} \subseteq \mathbb{R}^d \) be an open set avoiding integral distances. Then for every connected component \( C \) of \( \mathcal{P} \) we have \( \text{diam}(C) \leq 1 \).

**Proof.** Suppose there is a connected component \( C \) with \( \text{diam}(C) > 1 \), then there exist \( x_1, x_2 \in C \) such that \( \text{dist}(x_1, x_2) > 1 \). Since \( \mathbb{R}^d \) is locally connected, \( C \) is open, so it is path connected. Hence there is a point \( x \) on the image curve of a continuous path in \( C \) joining \( x_1 \) and \( x_2 \) such that \( \text{dist}(x_1, x) = 1 \). \( \square \)

By the isodiametric inequality the open ball \( B_d \subseteq \mathbb{R}^d \) centered at the origin with unit diameter has the largest volume among measurable sets in \( \mathbb{R}^d \) of diameter at most 1, see e.g. [19], [6, chap. 2]. Thus we have

\[
 f_d(1) = \lambda_d(B_d) = \frac{\pi^{d/2}}{2^d \cdot \Gamma(\frac{d}{2} + 1)} = \begin{cases} 
 \frac{\pi^{d/2}}{2^d \cdot \Gamma(\frac{d}{2} + 1)} & \text{for } d \text{ even}, \\
 \left(\frac{d-1}{2d}\right) \pi^{(d-1)/2} & \text{for } d \text{ odd}.
\end{cases}
\]
The first few values are given by $\lambda_1(B_1) = 1$, $\lambda_2(B_2) = \frac{7}{3}$, $\lambda_3(B_3) = \frac{x}{3}$, and $\lambda_4(B_4) = \frac{x^2}{52}$. Note that the volume of the scaled ball $B$ with diameter $m$ in $\mathbb{R}^d$ is $\lambda_d(B) = m^d \lambda_d(B_d)$.

Next we characterize 1-dimensional open sets containing a pair of points an integral distance apart.

**Lemma 2** A non-empty open set $\mathcal{P} \subseteq \mathbb{R}$ contains a pair of points $x, y \in \mathcal{P}$ with $\text{dist}(x, y) \in \mathbb{N}$ if and only if either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of connected components (i.e., disjoint open intervals) $C_1, C_2$ of $\mathcal{P}$ such that $\text{dist}(C_1, C_2) \notin \mathbb{N}$ and $\lambda_1(C_1 \cup C_2) > \lceil \text{dist}(C_1, C_2) \rceil - \text{dist}(C_1, C_2)$. If $\lambda_1(\mathcal{P}) \leq 1$, then there exists a shift $f : x \mapsto x + a$ of $\mathbb{R}$ such that $f(\mathcal{P}) \cap \mathbb{Z} = \emptyset$.

**Proof.** The restriction of the canonical epimorphism $\phi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \mathbb{Z} = (x - \lfloor x \rfloor) + \mathbb{Z}$, to the interval $[0, 1)$ is a continuous bijection of $[0, 1)$ onto the 1-dimensional torus $T = \mathbb{R}/\mathbb{Z}$, the inverse map $\phi_{\lfloor 0,1 \rfloor}^{-1}$ being continuous at all points except $\phi(0) = 0 + \mathbb{Z} = \mathbb{Z} \in T$. We consider the retraction $\phi_{\lfloor 0,1 \rfloor} : \mathbb{R} \to [0, 1)$, that is, $\phi_{\lfloor 0,1 \rfloor}(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$ (i.e., $\phi_{\lfloor 0,1 \rfloor}(x) = x \mod 1$ is the fractional part of $x$). We observe that the image under $\phi_{\lfloor 0,1 \rfloor}$ of any open interval $(x, y)$ of length $y - x < 1$ is either the open interval $(\phi_{\lfloor 0,1 \rfloor}(x), \phi_{\lfloor 0,1 \rfloor}(y)) = (x - n, y - n)$ of the same length $\phi_{\lfloor 0,1 \rfloor}(y) - \phi_{\lfloor 0,1 \rfloor}(x) = (y - n) - (y - n) = y - x$, whenever both $x$ and $y$ are in $(n, n + 1)$, for some $n \in \mathbb{Z}$, or the union of two disjoint connected components $[0, \phi_{\lfloor 0,1 \rfloor}(y)) \cup (\phi_{\lfloor 0,1 \rfloor}(x), 1) = [0, y - n) \cup (1 - (n - x), 1)$ of the same total length $(y - n) + (n - x) = y - x$, whenever $x < n < y$, for some $n \in \mathbb{Z}$. If $y - x = 1$, then similarly either $\phi_{\lfloor 0,1 \rfloor}((n, n + 1)) = (0, 1)$ or $\phi_{\lfloor 0,1 \rfloor}([x, y)) = [0, y - n) \cup (1 - (n - x), 1) = (0, 1) \setminus \{y - n\}$ whenever $x < n < y$ for some $n \in \mathbb{N}$. Hence, in general, the total length of the connected components of $\phi_{\lfloor 0,1 \rfloor}((x, y))$ is $y - x$, whenever $y - x \leq 1$.

Let $\mathcal{P}$ be the disjoint union of open intervals $C_i$, say, with total length $\lambda_1(\mathcal{P}) = \sum_i \lambda_1(C_i) > 1$. Then by Lemma [1], $\lambda_1(C_i) \leq 1$ for all $i$. We thus have from above that the total length of the connected components of all the images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ equals $\sum_i \lambda_1(C_i) > 1$. Hence at least two images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ and $\phi_{\lfloor 0,1 \rfloor}(C_j)$ must overlap, so there exists $z \in \phi_{\lfloor 0,1 \rfloor}(C_i) \cap \phi_{\lfloor 0,1 \rfloor}(C_j)$, that is, $x_0 - x_0 = y_0 - y_0$ for some $x_0 \in C_i$ and $y_0 \in C_j$. Thus $x_0 - y_0 = (x_0 - y_0) \in \{0\} \setminus \{0\}$, hence dist$(x_0, y_0) \in \mathbb{N}$.

If $\lambda_1(C_1 \cup C_2) > \alpha$ for some connected components $C_1 = (a, b)$ and $C_2 = (c, d)$ of $\mathcal{P}$ with dist$(C_1, C_2) = c - b = m - \alpha$, where $m \in \mathbb{N}$, $0 < \alpha < 1$, so that $\lceil \text{dist}(C_1, C_2) \rceil - \text{dist}(C_1, C_2) = \alpha$, we can take a point $x$ in the leftmost interval, say $x \in C_1$ and a point $y \in C_2$ so that the length of $(x, b) \cup (c, y)$ is $\alpha < \lambda_1(C_1 \cup C_2) = (b - a) + (d - c)$. Then

$$\text{dist}(x, y) = (b - x) + \alpha + (y - c) = \alpha + m - \alpha = m \in \mathbb{N}.$$

Conversely, suppose there are $x, y \in \mathcal{P}$ with dist$(x, y) = k \in \mathbb{N}$. If $x$ and $y$ lie in the same connected component $C_i$ of $\mathcal{P}$, then $\lambda_1(C_i) > k \geq 1$ because $C_i$ is open, hence $\lambda_1(\mathcal{P}) > 1$. Suppose $x$ and $y$ lie in distinct connected components of $\mathcal{P}$, say $x < y$ and $x \in C_1 = (a, b)$, $y \in C_2 = (c, d)$, and let $\lambda_1(\mathcal{P}) \leq 1$. Then $(b - a) + (d - c) \leq 1$ as well whence the distance between the components is \text{dist}(C_1, C_2) = c - b \notin \mathbb{N}$, because $c - b < \text{dist}(x, y) < c - b + [(b - a) + (d - c)] \leq c - b + 1$. Let $c - b = m - \alpha$ where $m \in \mathbb{N}$, $0 < \alpha < 1$. Then

$$\alpha = m + b - c < m + 1 + b - c < (d - a) + (b - c) = (b - a) + (d - c) = \lambda_1(C_1 \cup C_2),$$

since $m + 1 < d - a$ because $m + 1 \leq k < d - a$. Thus either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of required connected components of $\mathcal{P}$.

If $\lambda_1(\mathcal{P}) \leq 1$, then $\lambda_1(C_i) \leq 1$ for all $i$, so the total length of the connected components of all the images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ equals $\sum \lambda_1(C_i) = \lambda_1(\mathcal{P})$, as shown previously. If $\lambda_1(\mathcal{P}) < 1$, then clearly, $\phi_{\lfloor 0,1 \rfloor}(\mathcal{P}) \neq [0, 1)$. If $\lambda_1(\mathcal{P}) = 1$, then again $\phi_{\lfloor 0,1 \rfloor}(\mathcal{P}) \neq [0, 1)$, whenever the images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ are not pairwise disjoint. Suppose all the images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ are pairwise disjoint and $\mathcal{P} \cap \mathbb{Z} \neq \emptyset$. Then there is exactly one $C_j = (a, b)$ that meets $\mathbb{Z}$. Hence the complement $[0, 1) \setminus \phi_{\lfloor 0,1 \rfloor}(C_j) = [\phi_{\lfloor 0,1 \rfloor}(b), \phi_{\lfloor 0,1 \rfloor}(a))$ is a non-open set in $\mathbb{R}$ that can not be covered by the images $\phi_{\lfloor 0,1 \rfloor}(C_i)$ of the other connected components of $\mathcal{P}$, since they are all open intervals, so $\phi_{\lfloor 0,1 \rfloor}(\mathcal{P}) \neq [0, 1)$ as well. Thus in all the cases we have $\phi_{\lfloor 0,1 \rfloor}(\mathcal{P}) \neq [0, 1)$. Take $\phi_{\lfloor 0,1 \rfloor}(a) \in [0, 1) \setminus \phi_{\lfloor 0,1 \rfloor}(\mathcal{P})$, $a \in \mathbb{R}$, that is, $\phi_{\lfloor 0,1 \rfloor}(a) \notin \phi_{\lfloor 0,1 \rfloor}(\mathcal{P}) = \emptyset$. Then $\phi(a) \cap \phi(\mathcal{P}) = \emptyset$, i.e. $(a + \mathbb{Z}) \cap (\mathcal{P} + \mathbb{Z}) = \emptyset$, so $(\mathcal{P} - a) \cap \mathbb{Z} = \emptyset$ and the required shift is $f : x \mapsto x + (-a)$.  

\qed
Lemma 3 Let \( \mathcal{P} \) be a \( d \)-dimensional disconnected open set all of whose connected components are of diameter at most 1. Then \( \mathcal{P} \) contains a pair of points with integral distance if and only if

\[
\left( \text{dist}(\mathcal{C}_1, \mathcal{C}_2), \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) \right) \cap \mathbb{N} \neq \emptyset
\]

for some of its connected components \( \mathcal{C}_1, \mathcal{C}_2 \).

**Proof.** Since all the connected components of \( \mathcal{P} \) are open with diameter at most 1, any two distinct points of \( \mathcal{P} \) with integral distance must be in two different components, say \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \). Let \( x \in \mathcal{C}_1 \), \( y \in \mathcal{C}_2 \) with \( \text{dist}(x, y) = n \in \mathbb{N} \). We then select two small closed balls \( \overline{B}(x, \varepsilon_1) \subseteq \mathcal{C}_1 \) and \( \overline{B}(y, \varepsilon_2) \subseteq \mathcal{C}_2 \) centered at \( x \) and \( y \) respectively with radii \( \varepsilon_1, \varepsilon_2 > 0 \). The line \( \mathcal{L} \) through \( x \) and \( y \) meets the two balls in the intervals, say \([x_1, x_2] \subseteq \overline{B}(x, \varepsilon_1)\) and \([y_1, y_2] \subseteq \overline{B}(y, \varepsilon_2)\), where \( x_1, x_2 \in \mathcal{C}_1 \) and \( y_1, y_2 \in \mathcal{C}_2 \). With this notation we have

\[
\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < \min_{1 \leq i,j \leq 2} \text{dist}(x_i, y_j) < \text{dist}(x, y) = n < \max_{1 \leq i,j \leq 2} \text{dist}(x_i, y_j) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2).
\]

Conversely, if \( \text{dist}(\mathcal{C}_1, \mathcal{C}_2) < n < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) \) for an integer \( n \), then there exist \( x_1, x_2 \in \mathcal{C}_1 \) and \( y_1, y_2 \in \mathcal{C}_2 \) such that

\[
\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < \text{dist}(x_1, y_1) < n < \text{dist}(x_2, y_2) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2).
\]

Joining \( x_1 \) with \( x_2 \) in \( \mathcal{C}_1 \) and \( y_1 \) with \( y_2 \) in \( \mathcal{C}_2 \) by continuous paths, we can find \( x \in \mathcal{C}_1 \) and \( y \in \mathcal{C}_2 \) on the image curves of these paths with \( \text{dist}(x, y) = n \). \( \square \)

Sometimes it is helpful, if we can assume that the connected components of the point sets in question are not too close to each other. Specifically, we will be using the fact that in such cases the connected components of the sets have disjoint closures.

**Lemma 4** Let \( \mathcal{C}_1, \mathcal{C}_2 \) be distinct connected components of a \( d \)-dimensional open point set \( \mathcal{P} \) without a pair of points an integral distance apart. If \( \lambda_d(\mathcal{C}_1 \cup \mathcal{C}_2) > \lambda_d(\mathcal{B}_d) \), then \( \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1 \).

**Proof.** Making use of the isodiametric inequality we deduce from \( \lambda_d(\mathcal{C}_1 \cup \mathcal{C}_2) > \lambda_d(\mathcal{B}_d) \) that \( \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) > 1 \). By Lemma \( \square \) we have \( \text{diam}(\mathcal{C}_1) \leq 1 \) and \( \text{diam}(\mathcal{C}_2) \leq 1 \). So we can choose \( x_1 \in \mathcal{C}_1 \), \( x_2 \in \mathcal{C}_2 \) with \( \text{dist}(x_1, x_2) > 1 \). If \( \text{dist}(\mathcal{C}_1, \mathcal{C}_2) < 1 \), then there exist \( \bar{x}_1 \in \mathcal{C}_1 \) and \( \bar{x}_2 \in \mathcal{C}_2 \) such that

\[
\text{dist}(\bar{x}_1, \bar{x}_2) < 1.
\]

Since \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are open, they are path connected, hence we can join \( x_1 \) and \( \bar{x}_1 \) by a continuous path in \( \mathcal{C}_1 \) and similarly \( x_2 \) and \( \bar{x}_2 \) in \( \mathcal{C}_2 \) and on the image curves of these paths we then find \( x'_1 \in \mathcal{C}_1 \) and \( x'_2 \in \mathcal{C}_2 \) such that \( \text{dist}(x'_1, x'_2) = 1 \), but \( \mathcal{P} \) avoids integral distances, a contradiction. Thus we have \( \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1 \). \( \square \)

As Lemma \( \square \) and Theorem \( \square \)(i) will be our main tools in estimating upper bounds for \( f_d(n) \), we denote by \( l_d(n) \) the supremum of the volumes \( \lambda_d(\mathcal{P}) \) of open point sets \( \mathcal{P} \subseteq \mathbb{R}^d \) with \( n \) connected components of diameter at most 1 each (condition (a)), and with total length of the intersection with every line at most 1 (condition (b)). Clearly \( l_d(1) = f_d(1) = \lambda_d(\mathcal{B}_d) \) and \( f_d(n) \leq l_d(n) \) for all \( d \) and \( n \). Note that omitting condition (b) trivializes the problem of estimating the extreme volumes, the extreme configurations obviously consist of \( n \) disjoint open \( d \)-dimensional balls of diameter 1. Dropping condition (a) makes the problem more challenging. It turns out that there are open connected \( d \)-dimensional point sets \( \mathcal{P} \) with infinite volume \( \lambda_d(\mathcal{P}) \) and diameter \( \text{diam}(\mathcal{P}) \) even though the length of the intersection of \( \mathcal{P} \) with every line \( \mathcal{L} \) is at most 1, i.e. \( \lambda_1(\mathcal{P} \cap \mathcal{L}) \leq 1 \) for all \( \mathcal{L} \).
Example 1 For integers \( n \geq 1 \) and \( d \geq 2 \), denote by \( A^d_n \) the \( d \)-dimensional open spherical shell, or annulus, centered at the origin with inner radius \( n \) and outer radius \( n + \frac{1}{dn^d} \), i.e. \( A^d_n \) are bounded by concentric \((d - 1)\)-dimensional spheres centered at the origin. These shells will guarantee that the volume of their union is unbounded as \( n \) increases. So far the constructed point set is disconnected. To obtain a connected point set, we denote by \( B^d_n \) the \( d \)-dimensional open spherical shell centered on the \( y \)-axis at \( n + \frac{3}{4} \) with inner radius \( 1 \) and outer radius \( 1 + \frac{1}{n^d} \). With this \( \mathcal{P} = \bigcup_{n \geq 30} (A^d_n \cup B^d_n) \) is open and connected with infinite volume and diameter even though the length of its intersection with every line is smaller than 1. In Figure 1 we depicted such a configuration in dimension \( d = 2 \) with first few annuli getting thinner and thinner \( A^2_n \) and \( B^2_n \) being blue and green respectively. The detailed computations demonstrating the assertions claimed are provided in the Appendix, see Subsection A.1.

![Figure 1: Concentric annuli with infinite area but small lengths of line intersections.](image)

In order to make the problem of evaluating the functions \( f_d(n) \) and \( l_d(n) \) more tractable, we consider both problems in the special case, where the connected components are restricted to \( d \)-dimensional open balls. We denote the corresponding maximum volumes by \( f^o_d(n) \) and \( l^o_d(n) \) respectively. Clearly we have \( f^o_d(n) \leq f_d(n) \) and \( f^o_d(n) \leq l^o_d(n) \leq l_d(n) \). In Section 3 we determine the exact values of both functions \( l^o_d(n) \) and \( f^o_d(n) \) for all \( d \) and \( n \).

Based on a simple averaging argument, any given upper bound on one of the four introduced maximum volumes for \( n \) connected components yields an upper bound for \( k \geq n \) connected components in the same dimension.

Lemma 5 For each \( k \geq n \) we have \( l_d(k) \leq \frac{k}{n} \cdot \Lambda_1 \) whenever \( l_d(n) \leq \Lambda_1 \) and \( f_d(k) \leq \frac{k}{n} \cdot \Lambda_2 \) whenever \( f_d(n) \leq \Lambda_2 \).

Proof. Let \( \mathcal{P} \) be a \( d \)-dimensional open set with corresponding property in either case and \( k \geq n \) connected components. The volume of each of the \( \binom{k}{n} \) different unions of \( n \) connected components inheriting these properties is at most \( \Lambda_1 \). Since each connected component occurs exactly \( \binom{k-1}{n-1} \) times in those unions, the stated inequalities hold.

3 Unions of open \( d \)-dimensional balls

Here we consider open point sets \( \mathcal{P} \) that are unions of \( n \) disjoint \( d \)-dimensional open balls of diameter at most 1 each such that they either do not contain a pair of points with integral distance or intersect each line in the intervals with total length at most 1. As introduced in the previous section, we denote the supremum of possible volumes of such \( \mathcal{P} \) by \( f^o_d(n) \) and \( l^o_d(n) \) respectively.
In dimension 1 we can consider one open interval of length \(1 - \varepsilon\) and \(n - 1\) open intervals of length \(\varepsilon / n\), where \(1 > \varepsilon > 0\), arranged in a unit interval so that they are pairwise disjoint. Clearly the set does not have a pair of points an integral distance apart and the total length of the \(n\) intervals tends to 1, as \(\varepsilon\) approaches 0. It follows from Theorem \(\text{I}\) that \(f_1^\circ(n) = l_1^\circ(n) = 1\) for all \(n\). For \(n = 1\), by the isodiametric inequality, only the volumes of \(d\)-dimensional open balls of diameter 1 attain the maximum value \(f_1^\circ(1) = l_1^\circ(1) = \lambda_1(B_d)\).

**Lemma 6** \(l_d^\circ(n) \leq \max(1, \frac{n}{2\pi}) \cdot \lambda_d(B_d)\) for all \(d, n \in \mathbb{N}\).

**Proof.** Consider \(n\) disjoint open \(d\)-dimensional balls with diameters \(X_1 \leq 1, \cdots, X_n \leq 1\), where we can assume w.l.o.g. that \(X_1 \leq \cdots \leq X_n \leq 1\). Clearly in dimension 1 we have \(l_1^\circ(n) = l_1(n) = 1 = \lambda_1(B_1)\) for all \(n \in \mathbb{N}\), and for all dimensions \(d\), we have \(l_d^\circ(1) = l_d(1) = \lambda_d(B_d)\), so in the cases where either \(d\) or \(n\) is 1 the stated inequality holds. Hence we can assume that \(d \geq 2\) and \(n \geq 2\). Then by Theorem \(\text{I (i)}\) we have \(X_i + X_j \leq 1\) for all \(1 \leq i < j \leq n\). If \(X_n \leq \frac{1}{2}\), then \(\sum X_i^d \leq \frac{n}{2\pi}\), so the required inequality holds. Otherwise we have \(X_i \leq 1 - X_n\) and it remains to maximize the function \(g_d(x) := x^d + (n-1)(1-x)^d\) with domain \([\frac{1}{2}, 1]\). Since \(g_d(x)^n = d(d-1)x^{d-2} + d(d-1)(n-1)(1-x)^{d-2} > 0\), every inner local extremum of \(g_d\) is a minimum, so the global maximum of \(g_d\) is attained at the boundary of the domain. Finally, we compute \(g_d(1) = 1, g_d\left(\frac{1}{2}\right) = \frac{n}{2\pi}\), so the lemma follows.

**Remark.** The special case of balls of diameter \(\frac{1}{2}\) is directly related to point sets with pairwise integral distances. Let \(P\) be the union of \(n\) \(d\)-dimensional open balls of diameter \(\frac{1}{2}\) each without a pair of points an integral distance apart. Then the distances between the centers of the balls \(i\) and \(j\) in some enumeration must be of the form \(d_{ij} + \frac{1}{2}\) for some integers \(d_{ij}\). By dilation with a factor of two we obtain the set of size \(n\) of the centers of the balls with pairwise odd integral distances. However, it has been shown in [23] that for such sets \(n \leq d + 2\), where equality holds if and only if \(d + 2 \equiv 0 \mod 16\). The exact maximum number of odd integral distances between points in the plane has been determined in [33].

**Theorem 2** \(l_d^\circ(n) = \max(1, \frac{n}{2\pi}) \cdot \lambda_d(B_d)\) for all \(n\) and \(d \geq 2\).

**Proof.** By Lemma \(\text{I}\) it suffices to provide configurations whose volumes (asymptotically) attain the upper bound.

For \(1 > \varepsilon > 0\), we consider the union of one \(d\)-dimensional open ball of diameter \(1 - \varepsilon\) and \(n - 1\) disjoint open balls of diameter \(\frac{\varepsilon}{n-1}\) arranged in the interior of an open ball of diameter 1. As \(\varepsilon\) approaches 0, the volume of the union tends to \(\lambda_d(B_d)\).

For the remaining part we consider the union of \(n\) open \(d\)-dimensional balls with diameter \(\frac{1}{2}\) centered at the vertices of a regular \(n\)-gon with circumradius \(k\). Clearly, for \(k\) large enough, every line hits at most two balls.

An alternative construction would be the union of \(n\) open \(d\)-dimensional balls of diameter \(\frac{1}{2}\) centered at \((i \cdot k, i^2 \cdot 0, \ldots, 0)\) for \(1 \leq i \leq n\). If \(k\) is large enough, then again there is no line intersecting three or more balls.

**Corollary 1** \(f_d^\circ(n) = l_d^\circ(n) = \lambda_d(B_d)\) for all \(d \geq 2\) and \(n \leq 2^d\).

It turns out that, in fact, the equalities \(f_2^\circ(n) = l_2^\circ(n) = \max(1, \frac{n}{2\pi}) \cdot \lambda_2(B_d)\) hold in all dimensions \(d \geq 2\). To explain the underlying idea, we first consider the special case where \(d = 2\) and \(n = 5\), i.e. the first case that is not covered by Corollary \(\text{I}\).

**Lemma 7**

\[f_2^\circ(5) = \frac{5\pi}{16} \approx 0.9817477.\]
PROOF. For each integer $k \geq 2$ and $\frac{1}{2} > \varepsilon > 0$, we consider a regular pentagon $P$ with side length $\frac{1}{2} - 2\varepsilon + k$ and the union $U$ of five open round discs of diameter $\frac{1}{2} - 2\varepsilon$ centered at the vertices of $P$, see Figure 2. Since each connected component of $U$ has diameter less than 1, there is no pair of points an integral distance apart inside a connected component. For every two points $a$ and $b$ from different components, we either have

\[ k < \text{dist}(a,b) < k + 1, \]

whenever the discs are adjacent with their centers located on an edge of $P$, or

\[ \left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5} - 1}{4} - 2\varepsilon < \text{dist}(a,b) < \left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{3 + \sqrt{5}}{4} - 5\varepsilon \]

otherwise.

Let $[\alpha]$ stand for the positive fractional part of a real number $\alpha$, i.e. $[\alpha] := \alpha - \lfloor \alpha \rfloor$. If, given $\varepsilon > 0$, one can find an integer $k$ such that $\left[\left(\frac{1 + \sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5} - 1}{4} - 2\varepsilon\right] < 3\varepsilon$, then the set $U$ with parameters $k$ and $\varepsilon$ does not contain a pair of points with integral distance.

Since $\frac{1 + \sqrt{5}}{2}$ is irrational, we can apply the equidistribution theorem, see e.g. [39, 41], to ensure that $\left(\frac{1 + \sqrt{5}}{2}\right) \cdot N$ is dense (even uniformly distributed) in $[0, 1)$. The same holds true if we shift the set by the fixed real number $\frac{\sqrt{5} - 1}{4} - 2\varepsilon > 0$. Thus we can find a suitable integer $k$ for each $\varepsilon > 0$. As $\varepsilon$ approaches 0, the total area of $U$ tends to $\frac{5\pi}{16}$, which is best possible by Lemma 6. \[ \square \]

We illustrate this by a short list of suitable values of $k$:

\[
\begin{align*}
[\frac{\sqrt{5} - 1}{4} + \left(\frac{1 + \sqrt{5}}{2}\right) \cdot 6] & \approx 0.01722, \\
[\frac{\sqrt{5} - 1}{4} + \left(\frac{1 + \sqrt{5}}{2}\right) \cdot 61] & \approx 0.00909, \\
[\frac{\sqrt{5} - 1}{4} + \left(\frac{1 + \sqrt{5}}{2}\right) \cdot 116] & \approx 0.00996, \\
[\frac{\sqrt{5} - 1}{4} + \left(\frac{1 + \sqrt{5}}{2}\right) \cdot 1103] & \approx 0.00051, \text{ and } [\frac{\sqrt{5} - 1}{4} + \left(\frac{1 + \sqrt{5}}{2}\right) \cdot 2090] & \approx 0.00005.
\end{align*}
\]

Figure 2: Integral distance avoiding open set for $d = 2$ and $n = 5$.

We shall generalize Lemma 7 to an arbitrary dimension $d \geq 2$ and arbitrary number $n$ of connected components. The idea is to locate the centers of $n$ small $d$-dimensional open balls of diameter slightly less than $\frac{1}{2}$ at some points $C_i$ in a two-dimensional sub-plane so that the set of different pairwise distances
$\alpha_i$ between their centers are linearly independent over the rational numbers, that is, the distances are either confluent or rationally independent. The appropriate candidates for the center points $C_i$ would be the vertices of a regular $p$-gon, where $p$ is an odd prime. We use a theorem of Mann, see [31], to prove the desired property of the set of distances. The condition that the point set in question avoids integral distances can be translated into a system of inequalities of the form $[\alpha_1 \cdot k] < \varepsilon, \ldots, [\alpha_l \cdot k] < \varepsilon$, where $k \in \mathbb{N}$, and we are looking for an integer $k$ such that the above fractional parts of the scaled pairwise distances are arbitrarily small. By a theorem of Weyl, see e.g. [41 Satz 3] or a textbook on Diophantine Approximation like e.g. [26], such systems have solutions whenever the $\alpha_i$ are irrational and linearly independent over $\mathbb{Q}$. (Weyl actually proves equidistribution while we only need denseness, a weaker result that Weyl himself attributes to Kronecker.)

Note that the same construction, using the vertices of a regular hexagon, does not work. Indeed, there would be only three distinct values for the lengths $l_i$ of the diagonals, namely $1$, $\sqrt{3}$, and $2$. The required inequalities

$$\left( k + \frac{1}{2} - 2\varepsilon \right) \cdot l_i - \left( \frac{1}{2} - 2\varepsilon \right) = k \cdot l_i + \left( \frac{1}{2} l_i - \frac{1}{2} \right) + (2 - 2l_i) \cdot \varepsilon < 3\varepsilon,$$

would trivially hold for $l_i = 1$, but fail for $l_i = 2$ and $\varepsilon$ small enough. We note in passing that quite recently Mann’s theorem was used in another problem from Discrete Geometry see [14, 36].

**Theorem 3** (Mann, 1965, [31]) Suppose we have

$$\sum_{i=1}^{k} a_i \zeta_i = 0,$$

with $a_i \in \mathbb{Q}$, $\zeta_i$ roots of unity, and no sub-relations $\sum_{i \in I} a_i \zeta_i = 0$, where $\emptyset \neq I \subseteq \{1, \ldots, k\}$. Then

$$(\zeta_i / \zeta_j)^m = 1$$

for all $i, j$, where $m = \prod_{p \text{ prime} \leq k} p$.

The vertices of a regular $p$-gon with a circumcircle of radius $1$ centered at the origin are given by

$$\left( \cos \left( \frac{j \cdot 2\pi}{p} \right), \sin \left( \frac{j \cdot 2\pi}{p} \right) \right)$$

for $0 \leq j \leq p - 1$. In standard complex number notation with $i := \sqrt{-1}$ they coincide with the $p$th roots of unity $\zeta_j' = \cos \left( \frac{j \cdot 2\pi}{p} \right) + i \cdot \sin \left( \frac{j \cdot 2\pi}{p} \right)$. The distance between the vertices $0$ and $j$ is equal to $2 \sin \left( \frac{2\pi}{2p} \right)$. Since $\sin(\pi - \alpha) = \sin(\pi)$, there are only $(p - 1)/2$ distinct distances in a regular $p$-gon, attained for $1 \leq j \leq (p - 1)/2$. We note in passing that this number is not far away from the minimum number of distinct distances in the plane, which is bounded below by $c \cdot \frac{p}{\log p}$ for a suitable constant $c$, see [24]. We can express these distances in terms of $2p$th roots of unity $\zeta_j = \cos \left( \frac{j \cdot 2\pi}{2p} \right) + i \cdot \sin \left( \frac{j \cdot 2\pi}{2p} \right)$ via

$$2 \sin \left( \frac{j \cdot 2\pi}{2p} \right) = \frac{\zeta_j - \zeta_{2p-j}}{i}$$

for all $1 \leq j \leq \frac{p-1}{2}$.

**Lemma 8** Given an odd prime $p$, let $\alpha_j = \frac{\zeta_j - \zeta_{2p-j}}{i}$ for $1 \leq j \leq \frac{p-1}{2}$, where the $\zeta_j$ are $2p$th roots of unity. Then the $\alpha_j$ are irrational and linearly independent over $\mathbb{Q}$. 

8
PROOF. A folklore result, see e.g. [27], states that $\sin(\pi q)$, where $q \in \mathbb{Q}$, is a rational number if and only if $\sin(\pi q) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. Since $p$ is odd, this cannot occur in our context. It remains to show that the irrational numbers $\alpha_j$ are linearly independent over $\mathbb{Q}$. Suppose to the contrary that there are rational numbers $b_j$ for $1 \leq j \leq l \leq \frac{n-1}{2}$ such that $\sum_{j=1}^{l} b_j \alpha_j = 0$. We then have

$$\sum_{j=1}^{l} (b_j \zeta_j - b_j \zeta_{2p-j}) = 0.$$ 

Now let $J$ be a subset of those indices $j$, $2p-j$ such that $\sum_{j \in J} a_j \zeta_j = 0$, where $a_j \in \{\pm b_j\}$, and no vanishing sub-combination. We have $|J| \leq p-1$. Hence by Mann’s Theorem $(\zeta_{j1}/\zeta_{j2})^2 = 1$ for all $j_1, j_2 \in J$, since

$$\gcd(2p, \prod_{t \text{ prime} \leq p-1} t) = 2.$$ 

This yields $j_2 = j_1 + p$ for $j_1 < j_2$. Since $J$ is a subset of

$$\left\{1, \ldots, \frac{p-1}{2}\right\} \cup \left\{2p - \frac{p-1}{2}, \ldots, 2p-1\right\},$$

this is impossible, so the numbers $\alpha_j$ have to be linearly independent over $\mathbb{Q}$. \hfill $\Box$

**Theorem 4** $f_d^2(n) = \max\{1, \frac{n}{2^d}\} \cdot \lambda_d(B_d)$ for all $n$ and $d \geq 2$.

**Proof.** Since $f_d^2(n) \leq k_d^2(n)$, by Theorem 2 $f_d^2(n) \leq \max\{1, \frac{n}{2^d}\} \cdot \lambda_d(B_d)$. By Corollary 1 we can assume that $n > 2^d$. For the construction we fix an odd prime $p$ with $p \geq n$. For each integer $k \geq 2$ and each $\frac{1}{p} > \varepsilon > 0$ we consider a regular $p$-gon $P$ with side lengths $2k \cdot \sin\left(\frac{\pi}{p}\right)$, i.e. with circumradius $k$. At $n$ arbitrarily chosen vertices of the $p$-gon $P$ we place the centers of $d$-dimensional open balls with diameter $\frac{1}{2} - 2\varepsilon$ and consider their union. Since each of the $n$ connected components of the union has a diameter less than $1$, there is no pair of points with integral distance inside the components. Now consider arbitrary points $a$ and $b$ from two different connected components (open balls). Let $\alpha$ stand for the distance between their centers. The triangle inequality yields

$$\alpha - \left(\frac{1 - 4\varepsilon}{2}\right) < \text{dist}(a, b) < \alpha + \left(\frac{1 - 4\varepsilon}{2}\right).$$

Since all possible distances $\alpha$ are given by $2k \cdot \sin\left(\frac{j\pi}{p}\right)$ for $1 \leq j \leq \frac{p-1}{2}$, we look for a solution of the system of inequalities

$$\left[2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + 2\varepsilon\right] \leq 4\varepsilon$$

where $k \in \mathbb{N}$. By Lemma 8, the factors $2 \sin\left(\frac{j\pi}{p}\right)$ are irrational and linearly independent over $\mathbb{Q}$, so by Weyl’s Theorem [41] such systems admit solutions for all $\varepsilon$.

Therefore, for every $0 < \varepsilon < \frac{1}{4}$ we can choose a suitable value of $k$ and construct an open $n$-component set without pairs of points an integral distance apart with volume $\frac{n}{2^d} \cdot \lambda_d(B_d)$. As $\varepsilon$ approaches $0$, this volume tends to $\frac{n}{2^d} \cdot \lambda_d(B_d)$, completing the proof. \hfill $\Box$

Thus, in the case of round connected components the values of $l_d^2(n)$ and $f_d^2(n)$ are completely determined. In the general case of arbitrary connected components the problem is more challenging for $n \geq 2$ and will be addressed in the following section.
4 Bounds for $l_d(n)$ and the exact value of $f_d(n)$

In dimension 1 we can consider the disjoint union $U$ of $n$ open intervals of length $\frac{1}{n}$ inside an open interval of length 1. Obviously $U$ avoids integral distances and the line intersection property holds trivially for $U$ the total length of $U$ being 1 which is the largest possible by Theorem 1. Thus $f_1(n) = l_1(n) = 1$ for all $n$. For $n = 1$, we have $f_d(1) = l_d(1) = \lambda_d(B_d)$ and only $d$-dimensional open balls of diameter 1 can have that large volume. For $n, d \neq 1$ the evaluation of $f_d(n)$ and $l_d(n)$ gets more involved. In Subsection 4.1 we treat the 2-component case $n = 2$. As to the general case, we only could find some bounds for $l_d(n)$ in Subsection 4.2 and succeeded in determining the exact values of the function $f_d(n)$ in Subsection 4.3.

4.1 Two components

At first we find an upper bound for $f_d(2)$. To this end, note that the condition in Lemma 3 can be restated as follows: $\text{diam}(C_1 \cup C_2) \leq |\text{dist}(C_1, C_2)| + 1$. We further use lemmas 3 and 4 to provide a structural property of the pairs of connected components $C_1, C_2$ of a $d$-dimensional open set $P$ avoiding integral distances. By Lemma 4 there exist parallel hyperplanes $H_2$ and $H_3$ such that, possibly after relabeling the components, $C_1$ is on the left hand side of $H_2$, $C_2$ is on the right hand side of $H_3$, and $H_2$ is on the left hand side of $H_3$. W.l.o.g. we can assume that $\text{dist}(H_2, H_3) \leq \text{dist}(C_1, C_2)$. By Lemma 3 there exist another pair of hyperplanes $H_1, H_4$ parallel to $H_2$ and $H_3$ such that $C_1$ is on the right hand side of $H_1$ and $C_2$ is on the left hand side of $H_4$, that is, $C_1$ lies between $H_1$ and $H_2$, $C_2$ lies between $H_3$ and $H_4$. W.l.o.g. we can assume that $\text{dist}(H_1, H_4) \leq \text{diam}(C_1 \cup C_2)$. Thus for $d_1 := \text{dist}(H_1, H_2)$ and $d_2 := \text{dist}(H_3, H_4)$ we have $d_1 + d_2 \leq 1$ by Theorem (i). Clearly $d_1$ and $d_2$ are upper bounds for the widths of $C_1$ and $C_2$ respectively.

For a convex body $K$ in $\mathbb{R}^d$ with diameter $D$ and minimum width $\omega$ an upper bound for its $d$-dimensional volume $V$ has been found in [25], Theorem 1, namely:

$$V \leq \lambda_{d-1}(B_{d-1}) \cdot D^d \int_0^{\arcsin \frac{\omega}{D}} \cos^d \theta \, d\theta. \tag{1}$$

Equality holds if and only if $K$ is the $d$-dimensional spherical symmetric slice with diameter $D$ and minimum width $\omega$. In the planar case some more inequalities relating several descriptive parameters of a convex set can be found in [37]. Since we will extensively use $d$-dimensional spherical symmetric slices with diameter 1 and width $\frac{1}{2}$, we denote them by $S_d$. Viewing $S_d$ as a truncated $d$-dimensional open ball of unit diameter we denote the two congruent cut-off bodies by $C_d$ and call them caps. Thus $\lambda_d(B_d) = \lambda_d(S_d) + 2 \cdot \lambda_d(C_d)$ where

$$\lambda_d(S_d) = \lambda_{d-1}(B_{d-1}) \int_0^{\frac{\pi}{2}} \cos^d \theta \, d\theta, \tag{2}$$

$$\lambda_d(C_d) = \frac{1}{2} \cdot (\lambda_d(B_d) - \lambda_d(S_d)). \tag{3}$$

In Table 1 we tabulated the first few exact values of the volumes of $S_d$ and $C_d$ and refer to the Appendix, Subsection X.2, for more information on these volumes as functions of $d$.

| $d$ | $\lambda_d(S_d)$ | $\lambda_d(C_d)$ |
|-----|------------------|------------------|
| 2   | $\frac{\sqrt{3}}{8} + \frac{\pi}{12}$ | $\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$ |
| 3   | $\frac{\pi}{8} + \frac{\pi}{12}$ | $\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$ |
| 4   | $\frac{\pi}{6} \approx 0.5236$ | $\frac{\pi}{6} - \frac{\sqrt{3}}{8} \approx 0.0818$ |
| 5   | $\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$ | $\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$ |

Table 1: Values of $\lambda_d(S_d)$ and $\lambda_d(C_d)$ for small dimensions.
Lemma 9 \( f_d(2) \leq 2\lambda_d(S_d) \) for all \( d \geq 2 \).

PROOF. With notation above we estimate making use of Inequality (1) the total volume of the closed convex hulls of the two connected components \( \text{conv}(C_1), \text{conv}(C_2) \), i.e.

\[
\lambda_d(\text{conv}(C_1)) + \lambda_d(\text{conv}(C_2))
\]

where both connected components are of diameter at most 1, \( C_1 \) is of width at most \( d_1 \), and \( C_2 \) is of width at most \( d_2 \). Thus we have

\[
\lambda_d(\text{conv}(C_1)) \leq \lambda(d(B_{d-1}) \int_{0}^{\arcsin d_1} \cos^d \theta \, d\theta
\]

and

\[
\lambda_d(\text{conv}(C_2)) \leq \lambda(d(B_{d-1}) \int_{0}^{\arcsin d_2} \cos^d \theta \, d\theta.
\]

Since both right hand sides are strictly monotone in \( d_1, d_2 \) respectively, we can assume w.l.o.g. that \( d_1 + d_2 = 1 \), so it suffices to maximize the following function of \( x \)

\[
\int_{0}^{\arcsin x} \cos^d \theta \, d\theta + \int_{0}^{\arcsin(1-x)} \cos^d \theta \, d\theta
\]

with domain \([0, 1]\). A straightforward calculation shows that the function attains its unique maximum value at \( x = \frac{1}{2} \).

□

Lemma 10 \( f_d(2) \geq 2 \cdot \lambda_d(S_d) \) for all \( d \)

PROOF. For an arbitrary integer \( k \geq 5 \) we place the center of a \( d \)-dimensional open ball with diameter \( 1 - \frac{2}{k} \) at the origin and cut off the spherical caps with the hyperplanes determined by the values \( \pm \left( \frac{1}{4} - \frac{1}{k} \right) \) of the first coordinate. We denote by \( S_1 \) the resulting truncated ball. We consider the copy \( S_2 \) of \( S_1 \) by shifting the center of \( S_1 \) \( dk + \frac{1}{2} - \frac{2}{k} \) units along the first coordinate axis (Figure 3 below illustrates the 2-dimensional case). Since both \( S_1 \) and \( S_2 \) have diameter less than 1 for all \( k \in \mathbb{N} \), they contain no pair of points with integral distance. For arbitrary points \( a \in S_1 \) and \( b \in S_2 \), we have

\[
dk < \text{dist}(a, b) < \sqrt{(d - 1) \left( 1 - \frac{2}{k} \right)^2 + \left( dk + 1 - \frac{4}{k} \right)^2} \leq dk + 1,
\]

so \( S_1 \cup S_2 \) has no pairs of points with integral distance.

It is easily seen that the volume of \( S_1 \cup S_2 \) approaches \( 2 \cdot \lambda_d(S_d) \), as \( k \) increases.

\[
\begin{align*}
\text{diameter } 1 - \frac{2}{k} & \quad \text{diameter } 1 - \frac{2}{k} \\
\frac{1}{2} - \frac{2}{k} & \quad \frac{1}{2} - \frac{2}{k}
\end{align*}
\]

Figure 3: Truncated circles – a construction of two components without integral distances.

Combining Lemmas 9 and 4 yields the following
Corollary 2 \( f_d(2) = 2\lambda_d(S_d) \) for all \( d \geq 2 \).

One might conjecture that the upper bound from Lemma 9 is also valid for \( l_d(2) \), see Conjecture 1. Technically, we have used Lemmas 3 and 4 but it is conceivable that there is an alternative approach not relying on these assertions.

Note that related problems can be quite complicated, e.g. it is hard to determine the equilateral \( n \)-gon with diameter 1 and maximum area \([3, 4]\).

Conjecture 1 \( l_d(n) = n \cdot \lambda_d(S_d) \) for all \( n \geq 2 \) and \( d \geq 2 \).

4.2 Bounds for \( l_d(n) \)

Using exhaustion over lines, we can find two first upper bounds for \( l_d(n) \).

Lemma 11 \( l_d(2) \leq \lambda_{d-1}(B_{d-1}) \cdot \left( \sqrt{\frac{2d}{d+1}} \right)^{d-1} \) for all \( d \geq 2 \).

**Proof.** By Lemma 1 both connected components, denoted by \( C_1 \) and \( C_2 \), are of diameter at most 1, so Jung’s theorem \([15, 28]\) yields the enclosing balls \( B_1, B_2 \) for these connected components of diameter \( \sqrt{\frac{2d}{d+1}} \). So there is an enclosing cylinder, having a \((d-1)\)-dimensional ball of diameter \( \sqrt{\frac{2d}{d+1}} \) as its base, containing the closed convex hull \( \text{conv}(B_1 \cup B_2) \). The diagram is depicted in Figure 4, note that in general the two enclosing balls \( B_1 \) and \( B_2 \) are not necessarily disjoint. By exhausting the cylinder with the lines parallel to the line through the centers of \( B_1 \) and \( B_2 \) and applying Theorem 1(i) we conclude, using a suitable Riemann integral or Fubini’s theorem, that the volume of \( C_1 \cup C_2 \) is at most \( \lambda_{d-1}(B_{d-1}) \cdot \left( \sqrt{\frac{2d}{d+1}} \right)^{d-1} \). \( \square \)

The estimates for the first few upper bounds of \( l_d(2) \) in Lemma 11 are: \( l_2(2) \leq \frac{2}{\sqrt{3}} \approx 1.1547 \), \( l_3(2) \leq \frac{3\pi}{8} \approx 1.1781 \), \( l_4(2) \leq \frac{8\sqrt{2\pi}}{15\sqrt{3}} \approx 1.0597 \), \( l_5(2) \leq \frac{25\pi^2}{288} \approx 0.8567 \) and \( l_d(2) \) tends to 0 as the dimension \( d \) increases.

Note that we used a bit wastefully the Jung enclosing balls. The universal cover problem, first stated in a personal communication of Lebesgue in 1914, asks for the minimum area \( A \) of a convex set \( U \) containing a congruent copy of any planar set of diameter 1, see \([8]\). For the currently best known bounds \( 0.832 \leq A \leq 0.844 \) and generalizations to higher dimensions we refer the interested reader to \([7, \text{Section 11.4}]\). In this paper we do not pursue the aim of finding more precise bounds for the maximum volumes using this idea. The restriction of the shape of connected components to \( d \)-dimensional open balls has already been treated in Section 3.

In dimension \( d = 2 \) the upper bound from Lemma 11 can easily be improved.

Lemma 12 \( l_2(2) \leq 1 \).
Figure 5: A connected component contained in the convex hull of another one.

PROOF. Let \( \mathcal{P} \) be a planar open point set with two connected components \( C_1 \) and \( C_2 \) of diameter at most 1 each. If one of them is contained in the closed convex hull of the other, see Figure 5 for an example, then we have \( \lambda_2(\mathcal{P}) \leq \lambda_2(B_2) = \frac{\pi}{4} < 1 \). Otherwise, we select any support line \( L \) through the boundary points of \( C_1 \) and \( C_2 \) so that both regions are in the same half-plane determined by \( L \). We then consider the strip parallel to this line with smallest possible width \( w \) containing both regions, see Figure 6. Since both \( C_1 \) and \( C_2 \) have diameter at most 1, we have \( w \leq 1 \). By exhausting the strip with the lines parallel to \( L \) and applying Theorem 1(i) we conclude, using Riemann integral or Steiner symmetrization with respect to a line orthogonal to \( L \), that the area of \( C_1 \cup C_2 \) is at most 1. \( \square \)

Figure 6: Two components between two parallel lines.

4.3 The exact value of \( f_d(n) \)

Combining Lemmas 9 and 5 yields the upper bound \( f_d(n) \leq n\lambda_d(S_d) \). In the remaining part of this subsection we will describe configurations whose volumes asymptotically attain this upper bound.

As a first step, we improve slightly the construction from Theorem 4. For \( d \geq 2 \), we choose an odd prime \( p \geq n \) and locate the centers of \( n \) open balls of diameter \( 1 - 2\varepsilon \), where \( \varepsilon \) is suitably chosen, at \( n \) consecutive vertices of a regular \( p \)-gon. For each two balls, we cut off spherical caps in the directions of the lines through their centers the resulting sets being of width \( \frac{1}{2} - 2\varepsilon \). We can assume that \( \varepsilon \) approaches 0, as the circumradius of the \( p \)-gon increases. For our purpose it suffices to consider a regular \( p \)-gon \( P \) of fixed circumradius \( > \frac{1}{2} \), locate the centers of \( n \) open balls at the consecutive vertices of \( P \), and cut off spherical caps so that the connected components of the resulting union are of width \( \frac{1}{2} \) in the direction of each line through the corresponding vertices, i.e. the centers of the \( n \) balls. For future reference we call this construction a \( p \)-gon construction. An example of such a construction for \( p = n = 5 \) in dimension \( d = 2 \) is depicted in Figure 7.

**Theorem 5** \( f_d(n) = n\lambda_d(S_d) \) for all \( d \geq 2 \) and \( n \geq 2 \).

PROOF. It follows from Lemmas 9 and 5 that \( f_d(n) \leq n\lambda_d(S_d) \). By Lemma 10 we can assume that \( n \geq 3 \). For arbitrary \( \varepsilon \) we denote by \( S_{d,\varepsilon} \) a \( d \)-dimensional spherical symmetrical slice with diameter \( 1 - 2\varepsilon \) and minimum width \( \frac{1}{2} - 2\varepsilon \). As \( \varepsilon \) approaches 0, the volume of \( S_{d,\varepsilon} \) tends to \( \lambda_d(S_d) \). Below we
Figure 7: $p$-gon construction: open set avoiding integral distances for $d = 2$ and $p = n = 5$.

provide a construction of an open $n$-component point set $P'$ avoiding integral distances each of whose connected components contains a congruent copy of $S_{d,\varepsilon}$.

Consider a regular $p$-gon $P$ with circumradius $k$, the parameters $p$ and $k$ are to be specified. We enumerate clockwise the vertices of $P$ from 1 to $p$ and assume w.l.o.g. that the line through the vertices 1 and 2 is the $x$-axis. At each vertex $1 \leq i \leq n \leq p$ we place the center of an open $d$-dimensional ball of diameter $1 - \varepsilon$. For each pair of the $n$ balls we cut off spherical caps in the direction of the lines through their centers resulting in a set of width $\frac{1}{2} - \varepsilon$. We denote the union of the resulting $n$ open sets by $P$.

Consider further all $2 \cdot \binom{n}{2}$ cutting hyperplanes that cut off the spherical caps from the initial open balls. As the number $p$ of vertices of the $p$-gon $P$ increases, with $n$ fixed, all those hyperplanes tend to be orthogonal to the $x$-axis. Now choose a prime $p$ large enough so that each connected component of $P$ contains a $d$-dimensional spherical symmetrical slice with diameter $1 - 2\varepsilon$ and minimal width $\frac{1}{2} - 2\varepsilon$ whose cutting hyperplanes are orthogonal to the $x$-axis. By $P'$ we denote the subset of $P$ which is the union of those $S_{d,\varepsilon}$'s.

There exists a number $k_1$ such that for $k \geq k_1$ each line hits at most two connected components of $P'$. Since the diameter of each of its connected components is at most $1 - 2\varepsilon$, the pairwise distances between the points within the same component are non-integral. Let $a$ and $b$ be two points in different connected components. By the construction the distance between the corresponding centers is given by

$$\text{dist}(a, b) \geq 2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + \varepsilon.$$  

There exists a number $k_2$ such that for $k \geq k_2$, we have

$$\text{dist}(a, b) \leq 2k \cdot \sin\left(\frac{j\pi}{p}\right) + \frac{1}{2} - \varepsilon,$$

since all the lines joining the centers of the connected components of $P'$ tend to be parallel to the $x$-axis, as $k$ increases. (cf. the proof of Lemma [10])

Thus, provided that for $k \geq \max\{k_1, k_2\}$, the system of inequalities

$$2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + \varepsilon \leq 2\varepsilon$$



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has a solution, the distance \( \text{dist}(a, b) \) can not be integral, so \( P' \) does not contain a pair of points an integral distance apart. By Lemma 8 and the Weyl theorem the above system indeed admits a solution for all \( k \). This completes the proof. \( \square \)

5 Conclusion

Problems related to point sets with pairwise rational or integral distances were one of Erdős’ favorite subjects in combinatorial geometry. In the present paper we study a counterpart to this type of problems by asking for the largest open \( d \)-dimensional set \( P \) of points without a pair of points an integral distance apart, i.e. that with the largest possible volume \( f_d(n) \), where \( n \) stands for the number of connected components of \( P \). As a relaxation we have also considered \( d \)-dimensional open point sets with \( n \) connected components of diameter at most 1 each whose intersection with every line has a total length of at most 1. The corresponding maximum volume has been denoted by \( l_d(n) \). While the assumption on the diameters of the connected components seems to be a bit technical, geometrical objects with specified intersections with lines or higher-dimensional subspaces are interesting in their own right. In this context we just mention the famous Kakeya problem of whether a Kakeya set in \( \mathbb{R}^d \), i.e. a compact set containing a unit line segment in every direction, has Hausdorff dimension \( d \), see e.g. the review [42] or [12, Problem G6].

By restricting the shapes of the connected components to \( d \)-dimensional open balls, we were able to determine the exact values of the corresponding maximum volumes \( f_d(n) \) and \( l_d(n) \) respectively. Also the values of \( f_d(n) \) have been determined exactly, while for \( l_d(n) \) we only have the lower bound \( l_d(n) \geq f_d(n) \), which we conjecture to be tight.

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In order to keep the main part of the paper more accessible we have moved some side remarks and necessary technical computations to this Appendix.

A.1 Details of the annuli construction

We shall show that Example 1 satisfies the properties as stated. First note that both $A_n^d$ and $A_{n+1}^d$ meet $B_n^d$ for $n \geq 1$. Thus $\mathcal{P}$ is a connected open set in $\mathbb{R}^d$. The volume $\lambda_d(A_n^d)$ is given by

$$\lambda_d(B_d) \cdot \left( \frac{2}{d \pi^2} \cdot \left( 2n^2 + \frac{2}{d \pi^2} \right)^d - (2n)^d \right) = \lambda_d(B_d) \cdot 2^d \cdot \left( n + \frac{1}{d \pi^2} \right)^d - n^d \geq \lambda_d(B_d) \cdot 2^d \cdot \frac{1}{n}.$$ 

Since the harmonic series diverges to infinity, the $d$-dimensional volume of $\mathcal{P}$ is unbounded.

Now we consider the intersection of a line $\mathcal{L}$ with a $d$-dimensional annulus $C_d(r_1, r_2)$ of inner radius $r_1$ and of outer radius $r_2$ centered at the origin. By symmetry we can assume that $\mathcal{L}$ is parallel to the $x$-axis, i.e. $\mathcal{L} = \{(1 \ 0 \ \ldots \ 0)^T \cdot \lambda + (0 \ a_2 \ \ldots \ a_d)^T \mid \lambda \in \mathbb{R} \}$. Furthermore, we can also assume by symmetry that $a_i \geq 0$ for all $2 \leq i \leq d$. To simplify notation we set $l := \sqrt{\sum_{i=2}^{d} a_i^2}$. Note that $C_d(r_1, r_2) \cap \mathcal{L} = \emptyset$ for $l^2 > r_2^2$. The $x$-coordinates of the intersections of $\mathcal{L}$ with the $d$-dimensional sphere

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Since the harmonic series diverges to infinity, the $d$-dimensional volume of $\mathcal{P}$ is unbounded.
of radius \( r_1 \) are given by \( \pm \sqrt{r_1^2 - l^2} \), as long as \( l^2 \leq r_1^2 \). Similarly the \( x \)-coordinates of the intersections of \( L \) and the \( d \)-dimensional sphere of radius \( r_2 \) are given by \( \pm \sqrt{r_2^2 - l^2} \), as long as \( l^2 \leq r_2^2 \). For \( l^2 \leq r_1^2 \), we have

\[
\lambda_1 (C_d(r_1, r_2) \cap L) = 2 \cdot \left( \sqrt{r_2^2 - \sum_{i=2}^{d} a_i^2} - \sqrt{r_1^2 - \sum_{i=2}^{d} a_i^2} \right).
\]

Since

\[
\frac{\partial h_1}{\partial a_i}(a_2, \ldots, a_d) = a_i \cdot \left( \frac{1}{\sqrt{r_2^2 - \sum_{i=2}^{d} a_i^2}} - \frac{1}{\sqrt{r_1^2 - \sum_{i=2}^{d} a_i^2}} \right) \geq 0
\]

we can assume \( l^2 \geq r_1^2 \) for the maximum length of the line intersection. If the \( a_i \) are restricted by an inequality \( l^2 \leq k^2 \leq r_1^2 \), the maximum length of the intersection is bounded above by \( 2\sqrt{r_1^2 - k^2} - 2\sqrt{r_1^2 - r_2^2} \).

For \( r_1^2 \leq \sum_{i=2}^{d} a_i^2 \leq r_2^2 \), we have

\[
\lambda_1 (C_d(r_1, r_2) \cap L) = 2 \cdot \sqrt{r_2^2 - \sum_{i=2}^{d} a_i^2}
\]

and

\[
\frac{\partial h_1}{\partial a_2}(a_2, \ldots, a_d) = -a_i \cdot \frac{1}{\sqrt{r_2^2 - \sum_{i=2}^{d} a_i^2}} \leq 0,
\]

so the extreme values are attained at \( \sum_{i=2}^{d} a_i^2 = r_1^2 \) where we have \( \lambda_1 (C_d(r_1, r_2) \cap L) \leq 2\sqrt{r_1^2 - r_2^2} \).

Thus for an arbitrary line \( L \), we have

\[
\lambda_1 \left( \bigcup_{n \geq 30} B_n^d \cap L \right) \leq \sum_{n=30}^{\infty} 2 \sqrt{\left( 1 + \frac{1}{n^2} \right)^2 - 1^2} \leq \sum_{n=30}^{\infty} \frac{2\sqrt{3}}{n^2} < 0.12.
\]

For the remaining part we restrict ourselves with lines parallel to the \( x \)-axis. If \( l < 30 \), then

\[
\lambda_1 \left( \bigcup_{n \geq 30} A_n^d \cap L \right) \leq 2 \sqrt{\left( 30 + \frac{1}{d \cdot 30^d} \right)^2 - 30^2} + \sum_{n=31}^{\infty} 2 \sqrt{\left( n + \frac{1}{dn^d} \right)^2 - l^2 - 2\sqrt{n^2 - l^2}}
\]

\[
\leq 0.366 + 2 \sum_{n=31}^{\infty} \frac{2n}{2\sqrt{n^2 - 30^2}} < 0.47.
\]

For \( l \geq 30 \), we have

\[
\lambda_1 \left( \bigcup_{n \geq 30} A_n^d \cap L \right) \leq 4 \sqrt{\left( \frac{l}{d \cdot \sqrt{l^2}} \right)^2 - \left| \frac{l}{d \cdot \sqrt{l^2}} \right|^2} + \sum_{n=|l+2|}^{\infty} 2 \sqrt{\left( n + \frac{1}{dn^d} \right)^2 - l^2 - 2\sqrt{n^2 - l^2}}
\]

\[
\leq 0.732 + 2 \int_{|l+1|}^{\infty} \frac{1}{x\sqrt{x^2 - l^2}} \, dx = 0.732 + \frac{2}{l} \cdot \arcsin \left( \frac{l}{|l+1|} \right)
\]

\[
\leq 0.732 + \frac{2}{l} \cdot \frac{\pi}{2} < 0.84.
\]

Since \( 0.12 + \max\{0.47, 0.84\} < 1 \), we have \( \lambda_1 (P \cap L) < 1 \) for all lines \( L \).
A.2 Volumes of truncated balls and caps

In Table 1 we presented the volumes of truncated \( d \)-dimensional open balls of unit diameter \( S_d \) and the cut-off bodies, i.e. caps \( C_d \), in small dimensions \( d \). Equations (2) and (3) enable us to compute the values 

\[
v(d) := \int_0^\pi \cos^d(x) \, dx.
\]

First few values are given by \( v(1) = \frac{1}{2}, \, v(2) = \frac{1}{8} \cdot \sqrt{3} + \frac{1}{12} \cdot \pi, \, v(3) = \frac{11}{24}, \) and \( v(4) = \frac{9}{64} \cdot \sqrt{3} + \frac{1}{16} \cdot \pi \). Integrating by parts we find

\[
v(d) = \begin{cases}
\frac{(2m-1)!!}{(2m)!} \cdot \left( \frac{1}{2} \cdot \sum_{k=0}^{m-1} \frac{(2k)!! \cdot \sqrt{3}}{2} \cdot \left( \frac{3}{2} \right)^k + \frac{3}{2} \right) & \text{for } d = 2m, \\
\frac{(2m)!!}{(2m+1)!} \cdot \left( \frac{1}{2} \cdot \sum_{k=0}^{m} \frac{(2k-1)!! \cdot (\frac{3}{4})^k}{2} \right) & \text{for } d = 2m + 1.
\end{cases}
\]

Given the integer sequence A091814 from the “On-line encyclopedia of integer sequences”, \( v(d) \) can be written as

\[
\frac{A091814(d) \cdot (d-\frac{1}{2})!!}{d \cdot 2^{d+1}}
\]

for all odd \( d \). Benoit Cloitre contributed the following second order recursion formula in this case:

\[
v(2n - 1) = \frac{1}{8n - 4} \cdot \left( (14n - 17) \cdot v(2n - 3) - 6(n - 2) \cdot v(2n - 5) \right)
\]

for \( n \geq 3 \). A similar recursion formula can be obtained for all even \( d \), where \( v(d) \) can be written in the form \( q(d) \cdot \sqrt{3} + \frac{(d-\frac{1}{2})!!}{2^d \cdot 3} \cdot \pi \) for some rational number \( q(d) \).

To determine the asymptotic behavior of \( v(d) \) as \( n \rightarrow \infty \) one can compute the corresponding ordinary generating function:

\[
F(z) := \sum_{k=0}^\infty v(k) z^k = \sum_{k=0}^\infty \int_0^\pi \left( \cos t \right)^k \, dt = \int_0^\pi \frac{dt}{1 - z \cos t} = \frac{2}{\sqrt{1-z}^2} \arctan \left( \frac{1+z}{1-z} \cdot \tan \frac{\pi}{12} \right).
\]

We apply the singularity analysis to determine the asymptotic behavior of \( a_n := F_\alpha(z)[z^n] \), where slightly more generally, \( F_\alpha(z) := \frac{2}{\sqrt{1-z}} \arctan \left( \sqrt{\frac{1+z}{1-z}} \cdot \alpha \right) \), see e.g. [20] chapter VI. The main singularity is at \( z = 1 \), since there is a compensation for \( z = -1 \). It follows from

\[
\arctan \left( \sqrt{\frac{1+z}{1-z}} \cdot \alpha \right) = \frac{\pi}{2} + O((1-z)^{\frac{1}{2}}),
\]

\[
\frac{2}{\sqrt{1+z}} = \sqrt{2} + O(1-z), \text{ and }
\]

\[
\left[ z^n \right] \frac{1}{\sqrt{1-z}} = \frac{1}{\sqrt{\pi n}} + O\left( \frac{1}{n^\frac{3}{2}} \right)
\]

that

\[
a_n = \sqrt{\frac{\pi}{2n}} + O\left( \frac{1}{n^\frac{3}{2}} \right).
\]

Thus \( v(d) \sim \sqrt{\pi} \).