A RIGOROUS REAL TIME FEYNMAN PATH INTEGRAL AND PROPAGATOR

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Abstract. We will derive a rigorous real time propagator for the Non-relativistic Quantum Mechanics $L^2$ transition probability amplitude and for the Non-relativistic wave function. The derivation will be explicitly given in terms of the time evolution operator. The derivation will be for all self-adjoint non-vector potential Hamiltonians. For systems with potential that carries at most a finite number of singularities and discontinuities, we will show that our propagator can be written in the form of a rigorous real time, time sliced Feynman path integral via improper Riemann integrals. We will also derive the Feynman path integral in Nonstandard Analysis formulation. Finally, we will compute the propagator for the harmonic oscillator using the Nonstandard Analysis Feynman path integral formulation; we will compute the propagator without using any knowledge of classical properties of the harmonic oscillator.

I. Introduction. Since Feynman’s invention of the path integral, much research have been done to make the real time Feynman path integral mathematically rigorous (see [6], [9], [10], [13], [18], [19], and [20]). In physics, the real time, time sliced Feynman path integral is formally given by (see [3], [4], and [5])

\begin{equation}
\bar{K}_t (\vec{x}, \vec{x}_0) = \lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^{(k-1)n}} \exp \left[ \frac{it}{\hbar} S_k (\vec{x} = \vec{x}_0, ..., \vec{x}_k) \right] d\vec{x}_1 ... d\vec{x}_{k-1},
\end{equation}

where the integral of the first equation in (1.1) is an improper Riemann integral, and the last line in (1.1) is the evolution operator operating on the delta function’s $\vec{y}$ variable. It is well known that mathematical rigor of (1.1) is lacking,

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and we know that an integral over path space in real time can not be well defined with measure theory (see [6]).

The problems with the objects in equation 1.1 are that we do not know if the improper Riemann integrals exist, we do not know if the $\lim_{k \to \infty}$ limit exists, and we do not know if the Feynman path integral in 1.1 produces the propagator. In his paper (see [11] footnote 13), Feynman observed that by using wave functions, ill-defined oscillatory integrals can be given rigorous meaning. With this observation, we will reformulate equation 1.1 into rigorous mathematical objects that represent the propagator.

From mathematics, we know that for some values of $t$, some propagators must be treated as distributions; the harmonic oscillator is one example (see [6] and [7]). Also, $\tilde{K}_t(\vec{x}, \vec{x}_0)$ given in (1.1) is a function of $\vec{x}$ and $\vec{x}_0$. Thus, it is natural to consider $\tilde{K}_t$ as a tempered distribution on the class of Schwartz test functions $S(\mathbb{R}^n \times \mathbb{R}^n)$. The space of square integrable functions is a subset of the space of tempered distributions. If we consider the wave function as a distribution and take its inner product with a test function, we can formally use (1.1) and get

\begin{equation}
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-i t \tilde{H}}{\hbar} \right) \right] \psi(\vec{x}) \, d\vec{x} =
\end{equation}

\begin{equation}
\int_{\mathbb{R}^n} \phi(\vec{x}) \int \tilde{K}_t(\vec{x}, \vec{x}_0) \psi(\vec{x}_0) \, d\vec{x}_0 \, d\vec{x} =
\end{equation}

\begin{equation}
\int \tilde{K}_t(\vec{x}, \vec{x}_0) \phi(\vec{x}) \psi(\vec{x}_0) \, d\vec{x}_0 \, d\vec{x}.
\end{equation}

Equation 1.2 will be the fifth theorem in section II.

As for the formal evolution of the delta function in (1.1), let us formally consider the following equation.

\begin{equation}
\lim_{\eta, \gamma \to 0} K(\vec{x}, \vec{x}_0, \eta, \gamma, t) =
\end{equation}

\begin{equation}
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} G_x(\vec{y}, \eta) \left[ \exp \left( \frac{-i t \tilde{H}}{\hbar} \right) \right] F_{\vec{x}_0}(\vec{z}, \gamma) \, (\vec{y}) \, d\vec{y} =
\end{equation}

\begin{equation}
\int \delta(\vec{x} - \vec{y}) \left[ \exp \left( \frac{-i t \tilde{H}}{\hbar} \right) \delta(\vec{z} - \vec{x}_0) \right] (\vec{y}) \, d\vec{y} =
\end{equation}

\begin{equation}
\left[ \exp \left( \frac{-i t \tilde{H}}{\hbar} \right) \right] \delta(\vec{z} - \vec{x}_0), \quad (\vec{x}),
\end{equation}

where the functions $K, F$ and $G$ are given by equation 2.1. If we are going to take the propagator as a distribution in the sense of (1.2), we might consider the limit in (1.3) as a distribution limit. Doing so produces the second theorem and in some sense the fourth theorem in section II (equations 2.2, 2.3, 2.6a, and 2.6b).

In mathematics, there exists a rigorous formulation for a real time, time sliced Feynman path integral (see [7] and [8]), it reads

\begin{equation}
\lim_{k \to \infty} \left[ \exp \left( \frac{-i t \tilde{H}}{\hbar} \right) \right] \psi(\vec{x}) =
\end{equation}

\begin{equation}
\lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^n} \exp \left[ \frac{ic}{\hbar} S_k(\vec{x}_k = \vec{x}, \ldots, \vec{x}_0) \right] \psi(\vec{x}_0) \, d\vec{x}_0 \, d\vec{x}_{k-1},
\end{equation}
where $\psi \in L^2$, the integral in (1.4) is an improper Lebesgue integral with convergence taken in the $L^2$ topology, and the $k$ limit in (1.4) is taken in the $L^2$ topology. Comparing (1.4) and (1.1), we see that rigorously we have a Feynman path integral that has all convergence taken in $L^2$ topology while in physics, formal improper Riemann integral and pointwise convergence is favored.

What we will do is convert all convergences in $L^2$ topology into pointwise convergences in $t$. The idea is to use a wave function as a convergence factor as observed by Feynman. For simplicity, suppose $f(x) \in L^2(\mathbb{R})$, $g(x,y) \in L^2(\mathbb{R} \times \mathbb{R})$ are such that they are bounded and continuous. Further, suppose that both

\begin{align}
  h(x) &= \int_{-a}^{b} g(x,y) \, dy, \\
  p(x) &= \lim_{a,b \to \infty} \int_{-a}^{b} g(x,y) \, dy,
\end{align}

are in $L^2(\mathbb{R})$ as a function of $x$. In (1.5), we take the integrals to be Lebesgue integrals and the limits are taken independent of each other in the $L^2$ norm. Notice that for $p(x)$, we can interpret the integral as an improper Lebesgue integral with convergence in the $L^2$ topology. Let us denote $\chi_{[-c, d]}$ to be the characteristic function on $[-c, d]$. Schwarz’s inequality then implies

\begin{align}
  \left| \int_{\mathbb{R}} f(x) p(x) \, dx - \int_{-c}^{d} \int_{-a}^{b} f(x) g(x,y) \, dxdy \right| &\leq ||f||_2 ||p - h||_2 + ||f - \chi_{[-c, d]}f||_2 ||h||_2 \to 0.
\end{align}

Thus, we can write

\begin{align}
  \int_{\mathbb{R}} f(x) p(x) \, dx &= \lim_{a,b,c,d \to \infty} \int_{-c}^{d} \int_{-a}^{b} f(x) g(x,y) \, dxdy,
\end{align}

where the limits are all taken independent of each other. Since $f$ and $g$ are bounded and continuous, the Lebesgue integral over $[-a,b] \times [-c,d]$ in (1.7) can be replaced by a Riemann integral. Since the limits are taken independent of each other, we can then interpret the right hand-side of (1.7) as an improper Riemann integral. If $f$ and $g$ carry singularities and discontinuities, care must be taken in the region of integration so that the replacement of Lebesgue integral with Riemann integrals can be done.

The technique of converting $L^2$ limits into pointwise limits as illustrated above is what we will use to prove all the theorems in the next section. It is the foundation of this work.

II. Results. In his paper (see [11] footnote 13), Feynman observed that by using wave functions, ill-defined oscillatory integrals can be given rigorous meaning. With this observation, we will reformulate equation 1.1 into a rigorous mathematical object that represents the propagator.

The goal of this paper is the following. First, we will elaborate on Feynman’s observation and use wave functions to provide a convergence factor in the derivation of a real time propagator that takes the form of an $L^2$ transition probability
amplitude. We will use wave functions to derive a real time, time sliced Feynman path integral. We will derive two Nonstandard Analysis formulations of the time sliced Feynman path integral. Finally, we will compute the propagator of the harmonic oscillator using our Nonstandard Feynman path integral representation. We will assume that the reader is familiar with Nonstandard Analysis (see [13]-[17] and references within).

The usual idea in using Nonstandard Analysis is to replace the time slice limit by a standard part (see [9],[13],[18],[19] and references within). We will derive a Nonstandard formulation that transfers the time slice limit into the nonstandard world and standard part is taken on infinitesimal parameters in wave functions. It was shown in [19] that for the harmonic oscillator, equation 1.1 can be cast into the language of Nonstandard Analysis where the time slice limit is replaced by a standard part. Further, using Nonstandard Analysis methods, one can rigorously compute the harmonic oscillator propagator without having prior knowledge of the classical path. We will follow the approach of [19] in the computations of this paper. In [19], we do not know if equation 1.1 is the propagator apriori; we are satisfied because the computation produced the correct results. In this paper, we will have a Feynman path integral representation that is known to produce the propagator and we will use it to compute the harmonic oscillator propagator.

What we will show is the following. Let $H = -\frac{\hbar^2}{2m} \Delta + V(\vec{x}) = H_0 + V(\vec{x})$ be essentially self-adjoint and the domain of $H$ contains the Schwartz space of rapidly decreasing test functions. Denote the closure of $H$ by $\overline{H}$. Let $t > 0$ and let

$$K(\vec{x},\vec{x}_0,\eta,\gamma,t) = \int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y},\eta) \left[ \exp \left( -\frac{i t \overline{H}}{\hbar} \right) F_{\vec{x}_0}(\vec{z},\gamma) \right] (\vec{y}) d\vec{y}.$$  

The notation for $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is that the evolution operator operates on the $\vec{z}$ variable while leaving $\vec{x}_0$ fixed and the result is a function of $\vec{x}_0$ and $\vec{y}$; finally, the $\vec{y}$ variable is integrated against $G$. We point out that $K$ in (2.1) is in the form of an $L^2$ transition probability amplitude, but neither $F$ nor $G$ are wave functions since they are not normalized to 1 in the $L^2$ norm. Also, the form of $K$ is similar to the form of the propagator given in [12] where Prugovecki provides a theory of stochastic Quantum Mechanics. It will be interesting to see the relationship between (2.1) and the stochastic propagator derived by Prugovecki. One immediate difference between (2.1) and Prugovecki’s formulation is that (2.1) stays within the popular representation of Quantum Mechanics where as Prugovecki uses a different representation (see [12]).

The existence of $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is immediate since both functions in the integrand are in $L^2$. We will show that

**First Theorem.** $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is continuous as a function of $(\vec{x},\vec{x}_0) \in \mathbb{R}^{2n}$ and it is uniformly bounded as a function of $(\vec{x},\vec{x}_0) \in \mathbb{R}^{2n}$.
The kernel in the first theorem will play the role of an integral kernel in the following sense.

**Second Theorem Part a.** Let \( \phi, \psi \in L^2(\mathbb{R}^n) \) Let \( H \) be essentially self-adjoint, then

\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-i t H}{\hbar} \right) \psi(\vec{x}) \right] d\vec{x} =
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^{2n}} \phi(\vec{x}) \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) d\vec{x}_0 d\vec{x},
\]

where the notation for the integral in the right hand side of the equality in (2.2) means an improper Lebesgue integral with the convergence at infinity taken pointwise in \( t \) (see equation 5.10 for more details), and the limits are taken independent of each other and pointwise in \( t \).

Notice that in the above theorem, the convergence of the improper Lebesgue integral is pointwise in \( t \) as opposed to convergence in the \( L^2 \) topology in equation 1.4. A pointwise convergence might provide computational advantages.

We will not attempt to pass the limits in (2.2) inside the integral since some real time propagators does not exist as a function for all time and must be treated as distributions (see [1]). We will make a connection between \( K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \) and the theory of distributions (see remark 6.2 and equation 6.4). On the other hand, we would like to be able to pass the limits inside the improper Lebesgue integral when the kernel in the first theorem and the propagator for the evolution are well behaved. The next theorem provides us with that opportunity.

**Second Theorem Part b.** Let \( \phi, \psi \in L^2 \cap L^1 \). Let \( H \) be essentially self-adjoint, then

\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-i t H}{\hbar} \right) \psi(\vec{x}) \right] d\vec{x} =
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^{2n}} \phi(\vec{x}) \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) d\vec{x}_0 d\vec{x},
\]

where the integral in the right hand side of the equality is a Lebesgue integral and all limits are taken independent of each other and pointwise in \( t \).

In the above theorem, the first theorem and the fact that the wave functions are in \( L^1 \) provide us the opportunity to pass the limits inside the integral and produce the propagator for the evolution. We will do this for the harmonic oscillator Hamiltonian. Further, for the purpose of passing the limits, we will not attempt to generalize the wave functions to all of \( L^2 \) since integrating the propagator against two arbitrary \( L^2 \) wave functions in the sense of equation 1.2 is not always well defined; the free evolution propagator is one such example.

The kernel in (2.1) can be explicitly represented by a time sliced Feynman path integral.

**Third Theorem.** Let \( H \) be essentially self-adjoint, and the potential \( V \) be such that it has at most a finite number of discontinuities and singularities. Let

\[
w_{n,k} = \left( \frac{m}{2\pi i \hbar} \right)^{\frac{n}{2}}, \quad \epsilon = \frac{t}{k},
\]

\[
S_k (\vec{x}_{k+1}, \ldots, \vec{x}_1) = \sum_{j=2}^{k+1} \left[ \frac{m}{2} \left( \frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon} \right)^2 - V(\vec{x}_j) \right],
\]
then

\begin{equation}
K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = \\
\lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^{(k+1)n}} F_{\vec{x}_0}(\vec{x}_1, \gamma) \exp \left[ \frac{i\epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1) \right] G_{\vec{x}}(\vec{x}_{k+1}, \eta) \, d\vec{x}_1 \ldots d\vec{x}_{k+1}.
\end{equation}

In (2.5), the integral is an improper Riemann integral and the \( k \) limit is taken pointwise in \( t \).

In the third theorem, there is no restriction on the type of discontinuities and singularities on the potential as long as the Hamiltonian is essentially self-adjoint and by improper Riemann integral, we mean a Riemann integral with convergence at infinity taken pointwise in \( t \). Further, it is not necessary to put the restriction on potential in the above theorem and work with improper Riemann integrals if one is willing to work with improper Lebesgue integral as in part a of the second theorem (see remark 3.2), but for our purpose of computing the harmonic oscillator propagator (see section VIII), we will formulate the third theorem as above.

The problem with formulating a real time, time sliced Feynman path integrals is that Fubini’s theorem can not be applied due to the oscillatory nature of the integrand. We will see that the application of Fubini’s theorem can be justified in the derivation of (2.5) because the functions \( F \) and \( G \) play the role of convergence factors as Feynman pointed out.

The propagator is usually formulated for the wave function. If we wish to work with the wave function, we have the following.

**Fourth Theorem Part a.** Let \( \psi \in L^2(\mathbb{R}^n) \), \( H \) be essentially self-adjoint, then the following is true

\begin{equation}
\left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right](\vec{x}) = \lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \psi(\vec{x}_0) \, K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \, d\vec{x}_0,
\end{equation}

where the integral in (2.6a) is a Lebesgue integral and the limits are taken independent of each other in the \( L^2 \) topology.

Part a of the fourth theorem above provides us with another way to deal with arbitrary \( L^2 \) wave functions when equation 2.1 is considered as a distribution in \( \mathbb{R}^{2n} \).

**Fourth Theorem Part b.** Let \( \psi, \phi \in L^2(\mathbb{R}^n) \), \( H \) be essentially self adjoint, then the following is true

\begin{equation}
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right](\vec{x}) = \\
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left( \int_{\mathbb{R}^n} \psi(\vec{x}_0) \, K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \, d\vec{x}_0 \right) d\vec{x}
\end{equation}

where the integrals in (2.6b) are iterated Lebesgue integrals and the limits are taken independent of each other and pointwise in \( t \).

We will connect the second theorem to the theory of distributions.
Fifth Theorem. Let $S$ be the space of rapidly decreasing test functions. Suppose $H = H_0 + V$ is essentially self-adjoint, then there exists a tempered distribution $K_t(\bar{x}, \bar{x}_0)$ on $S(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\forall \phi(\bar{x}), \psi(\bar{x}_0) \in S(\mathbb{R}^n)$,

$$
(2.7) \quad \int_{\mathbb{R}^n} \phi(\bar{x}) \left[ \exp \left( -\frac{it\bar{H}}{\hbar} \right) \right] \psi(\bar{x}) d\bar{x} = \int K_t(\bar{x}, \bar{x}_0) \phi(\bar{x}) \psi(\bar{x}_0) d\bar{x}d\bar{x}_0,
$$

where the integral in the right hand side of equation (2.7) is a distribution inner product.

The second theorem above is linked to the theory of tempered distributions via the fifth theorem. We will prove some properties of the distribution $K_t(\bar{x}, \bar{x}_0)$ given in (2.7). We have in fact gone beyond the theory of distributions in the sense that the second, and fourth theorem are not just true for rapidly decreasing test functions, they are true for a much bigger class of functions.

Finally, the above theorems can be cast into the language of Nonstandard Analysis. The idea of using Nonstandard Analysis to formulate the Feynman path integral is not new (see [9],[13],[18],[19] and references within). The usual formulation is to replace the time slice limit with a standard part. Following that idea, the third theorem can be easily reformulated in the language of Nonstandard Analysis as follows.

Sixth Theorem. With the notations and conditions in the third theorem, we can write

$$
(2.8) \quad K(\bar{x}, \bar{x}_0, \eta, \gamma, t) = \lim_{k \to \infty} K_k(\bar{x}, \bar{x}_0, \eta, \gamma, t),
$$

where $K_k$ is given in 2.5. Let $\omega \in {^*}\mathbb{N} - \mathbb{N}$, then

$$
(2.9) \quad K(\bar{x}, \bar{x}_0, \eta, \gamma, t) = st({^*}K_\omega(\bar{x}, \bar{x}_0, \eta, \gamma, t)),
$$

where $st$ is the standard part.

Notice that $*K_\omega$ consists of an infinite(namely $\omega$) copies of $*$-improper Riemann integrals. Equation 2.9 is in fact a special case of the work done in reference [9].

We now come to a new formulation of the Feynman path integral in the language of Nonstandard Analysis.

Seventh Theorem. Under the conditions of the second theorem(part a or part b), let $\eta$ and $\gamma$ be positive infinitesimals in the language of Nonstandard Analysis, then

$$
(2.10a) \quad \int_{\mathbb{R}^n} \phi(\bar{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \right] \psi(\bar{x}) d\bar{x} =
$$

$$
st \left( {^*} \int_{\mathbb{R}^n} \phi(\bar{x}) \psi(\bar{x}_0) K(\bar{x}, \bar{x}_0, \eta, \gamma, t) d\bar{x}_0 d\bar{x} \right),
$$

$$
(2.10b) \quad \int_{\mathbb{R}^n} \phi(\bar{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \right] \psi(\bar{x}) d\bar{x} =
$$

$$
st \left( {^*} \int_{\mathbb{R}^n} \phi(\bar{x}) \psi(\bar{x}_0) K(\bar{x}, \bar{x}_0, \eta, \gamma, t) d\bar{x}_0 d\bar{x} \right),
$$
where equations 2.10a and 2.10b corresponds to theorem two part a and part b respectively, \( st \) means the standard part of the \( ^*\)-transformed improper Lebesgue integral(2.10a, in the sense in second theorem part a) and Lebesgue integral(2.10b, second theorem part b).

The implication of the seventh theorem on Feynman path integrals is the following. Suppose the Hamiltonian has a finite number of singularities and discontinuities, then the third theorem holds. \( ^*\)-transforming the third theorem, equation 2.5 reads: for all \( \eta, \gamma \in ^*\mathbb{R}^+ \),

\[
^* K (\vec{x}, \vec{x}_0, \eta, \gamma, t) = \lim_{k \to \infty} ^* K_k (\vec{x}, \vec{x}_0, \eta, \gamma, t),
\]

where the \( ^*\)-limit is a limit taken in the nonstandard world, and \( ^*K_k \) is \( k \) copies of \( ^*\)-improper Riemann integrals. In particular, we can let \( \eta \) and \( \gamma \) be positive infinitesimals and use equation 2.11 in the seventh theorem. The time sliced Feynman path integral now has time sliced limits taken in the nonstandard world and standard parts taken on \( \eta \) and \( \gamma \) in the sense of the seventh theorem. As mentioned earlier, this formulation differs from the popular usage of Nonstandard analysis on path integrals in that the time slice limit is not replaced by a standard part. This new formulation uses the fact that the functions \( F \) and \( G \) behave like delta functions when \( \eta \) and \( \gamma \) are positive infinitesimals.

Lastly, we will use equation 2.9 and theorem 2 part b to compute the harmonic oscillator propagator. We will compute the propagator in such a way that no prior knowledge of the classical path is needed. In fact, the classical part of the propagator naturally falls out from quantum considerations. The usual method to compute the harmonic oscillator with the Feynman path integral is to use the classical path and separate out the classical and quantum fluctuation parts(see [4] and [5]). From our computational point of view, the classical mechanics part comes purely from quantum considerations and that goes against the grain of Feynman’s original idea that quantum mechanics come from classical mechanics via the action integral in the integrand of integration over path space.

The Hamiltonian for the harmonic oscillator is \( H = \frac{-\hbar^2}{2m} \Delta + \frac{m\lambda^2}{2} \vec{x}^2 \). It is well known that for \( 0 < t < \frac{\pi}{\lambda} \), the \( n \) dimensional harmonic oscillator propagator is given by

\[
K (\vec{x}, \vec{x}_0, t) = (\frac{m}{2\pi i\hbar})^{\frac{n}{2}} (\frac{\lambda}{\sin \lambda t})^{\frac{n}{2}} \exp \left\{ \frac{i m}{\hbar} \frac{\lambda}{\sin \lambda t} [ (\vec{x}_0^2 + \vec{x}^2) \cos \lambda t - 2 \vec{x} \vec{x}_0] \right\} = \hbar \left( \frac{n}{2} \right) g (\vec{x}, \vec{x}_0, t),
\]

\[
\hbar \left( \frac{n}{2} \right) = \left( \frac{m}{2\pi i\hbar} \right)^{\frac{n}{2}} (\frac{\lambda}{\sin \lambda t})^{\frac{n}{2}},
\]

\[
g (\vec{x}, \vec{x}_0, t) = \exp \left\{ \frac{i m}{\hbar} \frac{\lambda}{\sin \lambda t} [ (\vec{x}_0^2 + \vec{x}^2) \cos \lambda t - 2 \vec{x} \vec{x}_0] \right\}
\]

where the \( g (\vec{x}, \vec{x}_0, t) \) is the classical part and the \( \hbar \left( \frac{n}{2} \right) \) is the quantum fluctuation.
Given 2.12, we would expect that

\[
K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = h \left( \frac{n}{2}, t \right) \int_{R^{2n}} g(\vec{y}, \vec{y}_0, t) F(\vec{y}_0, \vec{x}_0, \gamma) G(\vec{y}, \vec{x}, \eta) \, d\vec{y} d\vec{y}_0.
\]

Notice that in 2.13, the disturbance of the functions \( F \) and \( G \) affects only the classical part.

We conclude this section with a comment and a summary. The reader should compare the similarities and differences between the formulations above and that of the notion of weak integral kernels (see [2] and references within). In particular, one difference is that the above kernel exists for all essentially self-adjoint Hamiltonians. The main purpose of this paper is to derive a rigorous theory of real time propagators and real time Feynman path integrals. The propagator exists for all essentially self-adjoint Hamiltonians and is closely related to distributions. The Feynman path integral exists for potentials that carry at most a finite number of singularities and discontinuities, it is formulated via improper Riemann integrals, and it can be formulated with classical analysis and Nonstandard Analysis. Lastly, we use Nonstandard Analysis and compute the propagator for the harmonic oscillator without prior knowledge of the classical path.

**III. Proof of Third Theorem.** We start by giving a quick proof of the third theorem. The third theorem is a specific case of the work work done in reference [9]. For the full details of the proof, we refer the reader to [9].

We first set some notations. Suppose \( V \) is such that it has at most a finite number of singularities and discontinuities. Let \( k \in \mathbb{N} \) and \( 1 \leq l \leq k + 1 \). We will denote the interior of the \( l \)th box by

\[
A^l = (-a^l_1, b^l_1) \times \cdots \times (-a^l_n, b^l_n),
\]

for positive and large \( a \)'s and \( b \)'s. Let \( K = \{ \vec{y}_1, \ldots, \vec{y}_p \} \) be the set of discontinuous and singular points of \( V \). For each \( \vec{y}_q = (y^q_1, \ldots, y^q_n) \in K \), denote the \( l \)th box centered at \( \vec{y}_q \) by

\[
B^l_q = (y^q_1 - \frac{1}{c^l_1}, y^q_1 + \frac{1}{c^l_1}) \times \cdots \times (y^q_n - \frac{1}{d^l_n}, y^q_n + \frac{1}{d^l_n}),
\]

for positive and large \( c \)'s and \( d \)'s. Let

\[
C^l = A^l - \left\{ \bigcup_{q=1}^p B^l_q \right\}.
\]

For arbitrary large \( a \)'s, \( b \)'s, \( c \)'s, and \( d \)'s, \( C^l \) is a box which encloses the set \( K \) and at each point of \( K \), a small box centered at that point is taken out. Associated with \( C^l \) is a set of indices

\[
\{ j_l \} = \{ a^l_1, \ldots, a^l_n, b^l_1, \ldots, b^l_n, c^l_1, \ldots, c^l_n, d^l_1, \ldots, d^l_n \}
\]

We will denote by \( \{ j_l \} \to \infty \) to mean

\[
d^l_1, \ldots, d^l_n, b^l_1, \ldots, b^l_n, c^l_1, \ldots, c^l_n, d^l_1, \ldots, d^l_n \to \infty.
\]
where all indices go to infinity independent of each other. Notice that as \( \{j_l\} \to \infty \), we recover \( \mathbb{R}^n \) a.e. from \( C^l \). We will denote by \( \chi_{\{j_l\}} \) the characteristic function on \( C^l \). Notice that for \( f \in L^2(\mathbb{R}^n) \),

\[
\lim_{\{j_l\} \to \infty} \chi_{\{j_l\}} f = f \quad \text{a.e.,}
\]

where the limit in (3.6) is taken in the \( L^2 \) topology. Let us write

\[
D_{\{j_l\}} = C^l \times \cdots \times C^h, \quad l \leq h.
\]

Associated with \( D_{\{j_l\}} \) is a set of indices

\[
\{J_l^h\} = \bigcup_{\alpha = l}^h \{j_\alpha\},
\]

and as before, we will use the notation \( \{J_l^h\} \to \infty \) to mean

\[
\{j_l\} \to \infty, \ldots, \{j_h\} \to \infty,
\]

where the indices are taken to infinity independent of each other. Finally, we will denote by \( \int_O \) to be Riemann or improper Riemann integration over the region \( O \) and \( \int_O \) to be Lebesgue integration over the region \( O \).

**Theorem 3.1.** The third theorem in section II is true.

*Proof.* Trotter’s product formula (see [7],[8], and [10]) and Schwarz’s inequality implies that

\[
\int_{\mathbb{R}^n} G_{\bar{x}}(\bar{y}, \eta) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) F_{z_0}(\bar{z}, \gamma) \right] (\bar{y}) \, d\bar{y} = \]

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} G_{\bar{x}}(\bar{y}, \eta) \left[ \exp \left( \frac{-itV}{kh} \right) \exp \left( \frac{-itH_0}{kh} \right) \right]^{k} F_{z_0}(\bar{z}, \gamma) \right] (\bar{y}) \, d\bar{y},
\]

where the limit in (3.10) is taken pointwise as a function of \( t \). To the right of each of the operator \( \exp \left( \frac{-itH_0}{kh} \right) \) in (3.10), we put in the identity operator \( \lim_{\{j_l\} \to \infty} \chi_{\{j_l\}} \) for \( 1 \leq l \leq k \) in increasing order from right to left and the limit is taken in the \( L^2 \) topology. Since \( \exp \left( \frac{-itV}{kh} \right) \), \( \exp \left( \frac{-itH_0}{kh} \right) \), and multiplication by a characteristic function are all continuous operators, we can take all the limits.
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outside of the operators and get

\[
\left\{ \exp\left( -\frac{i t V}{\hbar} \right) \exp\left( -\frac{i t H_0}{\hbar} \right) \right\}^k F_{\vec{x}_0} = \\
\exp\left( -\frac{i t V}{\hbar} \right) \exp\left( -\frac{i t H_0}{\hbar} \right) \lim_{\{j_k\} \to \infty} \chi_{\{j_k\}} \ldots \\
\exp\left( -\frac{i t V}{\hbar} \right) \exp\left( -\frac{i t H_0}{\hbar} \right) \lim_{\{j_1\} \to \infty} \chi_{\{j_1\}} F_{\vec{x}_0} = \\
\lim_{\{J^h\} \to \infty} \exp\left( -\frac{i t V}{\hbar} \right) \exp\left( -\frac{i t H_0}{\hbar} \right) \chi_{\{j_1\}} F_{\vec{x}_0} = \\
\lim_{\{J^h\} \to \infty} w_{n,k} \int_{D_{\{J^h\}}} \exp \left[ \frac{i \epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1) \right] F_{\vec{x}_0}(\vec{x}_1, \gamma) \, d\vec{x}_1 \ldots d\vec{x}_k.
\]

In the last equality in (3.11), we used the integral representation of the free evolution operator; we emphasize that all limits in (3.11) are taken in the $L^2$ topology and are taken independent of each other. Equations (3.6), (3.10), (3.11) and Schwarz’s inequality imply that

\[
\int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y}, \eta) \left[ \exp\left( -\frac{i t \tilde{H}}{\hbar} \right) F_{\vec{x}_0}(\vec{z}, \gamma) \right](\vec{y}) \, d\vec{y} = \\
\lim_{k \to \infty} w_{n,k} \int_{\mathbb{R}^n} \lim_{\{j_{k+1}\} \to \infty} \chi_{\{j_{k+1}\}} G_{\vec{x}}(\vec{x}_{k+1}, \eta) \times \\
\lim_{\{J^h\} \to \infty} \lim_{\{J^h\} \to \infty} w_{n,k} \int_{D_{\{J^h\}}} \exp \left[ \frac{i \epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1) \right] F_{\vec{x}_0}(\vec{x}_1, \gamma) \, d\vec{x}_1 \ldots d\vec{x}_k = \\
\lim_{k \to \infty} w_{n,k} \lim_{\{J^h\} \to \infty} \int_{D_{\{J^h\}}} G_{\vec{x}}(\vec{x}_{k+1}, \eta) \times \\
\exp \left[ \frac{i \epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1) \right] F_{\vec{x}_0}(\vec{x}_1, \gamma) \, d\vec{x}_1 \ldots d\vec{x}_{k+1}.
\]

In (3.12), all limits inside the integrals are taken independent of each other in the $L^2$ topology and all limits taken outside of the integral are taken pointwise in $t$. By construction, the integrand

\[
G_{\vec{x}}(\vec{x}_{k+1}, \eta) \exp \left[ \frac{i \epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1) \right] F_{\vec{x}_0}(\vec{x}_1, \gamma)
\]

is bounded and continuous on $D_{\{J^{k+1}\}}$. Hence, the Lebesgue integral over $D_{\{J^{k+1}\}}$ in the last equality of (3.12) can be replaced by a Riemann integral over $D_{\{J^{k+1}\}}$. Since the $\{J^{k+1}\}$ limits in (3.12) are all taken independent of each other, we can interpret those limits and the integral as an improper Riemann integral. □
Remark 3.2. It is not necessary to use improper Riemann integrals or put the discontinuities and singularities restriction on the potential. We forget about the holes centered at each elements of $K$ as given in (3.2) and take $C^l = A^l$ as defined in (3.1). Proceeding in the same manner as the above proof, we get to (3.12). At this point, we do not replace the Lebesgue integral with Riemann integral since the integrand is not necessarily bounded and continuous over the region of integration. We are then left with an improper Lebesgue integral in which the convergence of the integral is taken pointwise in $t$.

IV. Proof of First Theorem. In this section, we prove the first theorem in section II. Let us denote

$$T^k = \left\{ \exp \left( \frac{-itV}{\hbar} \right) \exp \left( \frac{-itH_0}{\hbar} \right) \right\}^k,$$

$$\tilde{T}^k = \left\{ \exp \left( \frac{-itH_0}{\hbar} \right) \exp \left( \frac{-itV}{\hbar} \right) \right\}^k.$$

Theorem 4.1. With our previously defined notations, we have

$$|K(\vec{x}, \vec{x}_0, \eta, \gamma, t)| \leq C_{t, \eta, \gamma}$$

where $C_{t, \eta, \gamma}$ is a constant depending only on $t, \eta$, and $\gamma$.

Proof. Since the evolution operator has norm 1, using Schwarz’s inequality on the kernel in 2.1 gives

$$|K(\vec{x}, \vec{x}_0, \eta, \gamma, t)| \leq \left\| G_{\vec{x}}(\vec{y}, \eta) \right\|_2 \left\| F_{\vec{x}_0}(\vec{y}, \gamma) \right\|_2 \equiv C_{t, \eta, \gamma}. \quad \square$$

We will now show that $K(\vec{x}, \vec{x}_0, \eta, \gamma, t, \cdot)$ is continuous as functions of $(\vec{x}, \vec{x}_0) \in \mathbb{R}^n$.

Lemma 4.2. Let $f, g \in L^2(\mathbb{R}^n)$, then the following is true

$$\int_{\mathbb{R}^n} g(\vec{x}) \left[ \exp \left( \frac{-itH_0}{\hbar} \right) f \right](\vec{x}) d\vec{x} = \int_{\mathbb{R}^n} \left[ \exp \left( \frac{-itH_0}{\hbar} \right) g \right](\vec{x}) f(\vec{x}) d\vec{x}.$$

Proof. Let $\chi_\alpha$ be the characteristic function of the cube centered at the origin with sides of length $\alpha$, then

$$\left[ \exp \left( \frac{-itH_0}{\hbar} \right) f \right](\vec{x}) = \lim_{\alpha \to \infty} w_{n,1} \int_{\mathbb{R}^n} \chi_\alpha(\vec{y}) \exp \left[ \frac{imc}{2\hbar} \left( \frac{\vec{x} - \vec{y}}{\epsilon} \right)^2 \right] f(\vec{y}) d\vec{y},$$
where the limit in (4.5) is taken in the $L^2$ norm. Using Schwarz’s inequality on $\alpha$ and Lebesgue’s dominating convergence theorem on $\beta$, we have

\[
(4.6) \int_{\mathbb{R}^n} g(\vec{x}) \left[ \exp \left( -\frac{it\overline{H}_0}{\hbar} \right) f \right](\vec{x}) \, d\vec{x} =
\]

\[
\lim_{\beta,\alpha \to \infty} w_{n,1} \int_{\mathbb{R}^n} \chi_{\beta}(\vec{x}) g(\vec{x}) \left\{ \int_{\mathbb{R}^n} \chi_{\alpha}(\vec{y}) \exp \left[ \frac{im\epsilon}{2\hbar} \left( \frac{\vec{x} - \vec{y}}{\epsilon} \right)^2 \right] f(\vec{y}) \, d\vec{y} \right\} d\vec{x} =
\]

\[
\lim_{\beta,\alpha \to \infty} w_{n,1} \int_{\mathbb{R}^n} \chi_{\alpha}(\vec{y}) f(\vec{y}) \left\{ \int_{\mathbb{R}^n} \chi_{\beta}(\vec{x}) \exp \left[ \frac{im\epsilon}{2\hbar} \left( \frac{\vec{x} - \vec{y}}{\epsilon} \right)^2 \right] g(\vec{x}) \, d\vec{x} \right\} d\vec{y},
\]

where the limits are taken pointwise in $t$. Using Schwarz’s inequality on $\beta$ and Lebesgue’s dominating convergence theorem on $\alpha$ in the last expression in (4.6) gives (4.4). □

**Lemma 4.3.** Let $f, g \in L^2(\mathbb{R}^n)$, then the following is true

\[
(4.7) \int_{\mathbb{R}^n} g(\vec{x}) \left[ \exp \left( -\frac{it\overline{H}}{\hbar} \right) f \right](\vec{x}) \, d\vec{x} = \int_{\mathbb{R}^n} \left[ \exp \left( -\frac{it\overline{H}}{\hbar} \right) g \right](\vec{x}) \, f(\vec{x}) \, d\vec{x}.
\]

**Proof.** Intuitively, if we think of the evolution as an exponential of the Hamiltonian $\overline{H}$, we can expand the exponential in powers of $\overline{H}$ and put all powers of $\overline{H}$ from the function $f$ onto the function $g$ since $\overline{H}$ is self-adjoint. This is also true for lemma 4.2.

Schwarz’s inequality and Trotter’s formula implies

\[
(4.8) \int_{\mathbb{R}^n} g(\vec{x}) \left[ \exp \left( -\frac{it\overline{H}}{\hbar} \right) f \right](\vec{x}) \, d\vec{x} = \lim_{k \to \infty} \int_{\mathbb{R}^n} g(\vec{x}) \left[ T^k f \right](\vec{x}) \, d\vec{x},
\]

where the limit is taken pointwise in $t$ and $T^k$ is given in (4.1). Using Lemma 4.2, and $T^k$ as defined in (4.1), we obtain

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} g(\vec{x}) \left[ T^k f \right](\vec{x}) \, d\vec{x} = \lim_{k \to \infty} \int_{\mathbb{R}^n} \left[ T^k g \right](\vec{x}) \, f(\vec{x}) \, d\vec{x} =
\]

\[
\int_{\mathbb{R}^n} \left[ \exp \left( -\frac{it\overline{H}}{\hbar} \right) g \right](\vec{x}) \, f(\vec{x}) \, d\vec{x}. \quad \square
\]

**Theorem 4.4.** With our previously defined notations, the expression

\[
(4.10) \left| K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t) \right|
\]

goes to zero as $\vec{x}_0$ goes to $\vec{y}_0$. 
Proof. We first show that $K(\vec{x}, \vec{x}_0, \eta, \gamma, t)$ is separately continuous in $\vec{x}$ and $\vec{x}_0$, then jointly continuous. Schwarz’s inequality implies that

$$||K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{x}_0, \eta, \gamma, t)||^2 \leq$$

$$||G_{\vec{x}}(\vec{z}, \eta) - G_{\vec{y}}(\vec{z}, \eta)||^2 \times ||F_{\vec{x}_0}(\vec{z}, \gamma)||^2 =$$

$$||F_{\vec{x}_0}(\vec{z}, \gamma)||^2 \times \left(\frac{m}{2\pi \hbar \eta}\right)^2 \int_{\mathbb{R}^n} \exp\left[\frac{-m\eta}{2\hbar} (\vec{z} - \vec{\eta})^2\right] -$$

$$\exp\left[\frac{-m\eta}{2\hbar} (\vec{y} - \vec{\eta})^2\right]\right)^2 d\vec{z} = C_{\gamma,t} g(\vec{y}, \vec{x})$$

where $C_{\gamma,t}$ is a constant independent of $\vec{x}_0$ and $g(\vec{y}, \vec{x})$ is independent of $\vec{x}_0$. Using Lebesgue’s dominating convergence theorem on $\vec{x} \to \vec{y}$ in (4.11), we get $C_{\gamma,t} g(\vec{y}, \vec{x}) \to 0$. Using lemma 4.3 in equation 2.1, we can put the evolution operator on $G_{\vec{x}}(\vec{y}, \eta)$. With the same reasoning as 4.11, we get

$$|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)|^2 \leq$$

$$D_{\eta,t} f(\vec{y}_0, \vec{x}_0) \to 0$$

as $\vec{x}_0 \to \vec{y}_0$.

Finally,

$$|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)| \leq$$

$$|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{x}_0, \eta, \gamma, t)| +$$

$$|K(\vec{y}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)| \leq$$

$$\sqrt{C_{\gamma,t} g(\vec{y}, \vec{x})} + \sqrt{D_{\eta,t} f(\vec{y}_0, \vec{x}_0)} \to 0$$

as $(\vec{x}, \vec{x}_0) \to (\vec{y}, \vec{y}_0)$. □

V. Proof of Second, and Fourth theorem. We will now prove the second and fourth theorem in section II.

Proposition 5.1. Let $f, g \in L^2$, then

$$\int_{\mathbb{R}^n} f(\vec{z}) \left[\exp\left(\frac{-\eta H_0}{\hbar}\right) g\right](\vec{z}) d\vec{z} =$$

$$\int_{\mathbb{R}^n} g(\vec{z}) \left[\exp\left(\frac{-\eta H_0}{\hbar}\right) f\right](\vec{z}) d\vec{z} =$$

$$\int_{\mathbb{R}^{2n}} g(\vec{x}) G(\vec{x}, \vec{y}, \eta) f(\vec{y}) d\vec{x} d\vec{y},$$
where $G(\vec{x}, \vec{y}, \eta)$ is the Gaussian kernel given in equation 2.1.

Proof. Since $|f|$ and $|g|$ are in $L^2$, we have

$$
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |g(\vec{x})| |G(\vec{x}, \vec{y}, \eta)| |f(\vec{y})| \, d\vec{y} \right) \, d\vec{x} =
$$

$$
\int_{\mathbb{R}^n} |g(\vec{x})| \left( \int_{\mathbb{R}^n} G(\vec{x}, \vec{y}, \eta) |f(\vec{y})| \, d\vec{y} \right) \, d\vec{x} =
$$

$$
\int_{\mathbb{R}^n} |g(\vec{x})| \left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) |f| \right](\vec{x}) \, d\vec{x} < \infty.
$$

Equation 5.1 then follows from Fubini’s theorem. □

**Theorem 5.2.** Part b of the second theorem in section II is true.

Proof. First recall that in part b of the second theorem, the wave functions $\phi, \psi \in L^2 \cap L^1$. Using lemma 4.3, the kernel in (2.1) can be written as

$$
K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = \left[ \exp \left( \frac{-\gamma H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) G\vec{x}(\vec{z}, \eta) \right](\vec{x}_0).
$$

We have that

$$
\lim_{\eta, \gamma \to 0} \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right](\vec{x}) =
$$

$$
\int_{\mathbb{R}^n} K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \psi(\vec{x}_0) \, d\vec{x}_0.
$$
Thus,
\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-itH}{\hbar} \right) \psi \right] (\vec{x}) d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left( \int_{\mathbb{R}^n} K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \psi(\vec{x}_0) d\vec{x}_0 \right) d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^2} \phi(\vec{x}) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \psi(\vec{x}_0) d\vec{x}_0 d\vec{x},
\]
where the last equality in (5.7) is obtained from the first theorem in section II (theorems 4.1 and 4.2) and the fact that the wave functions \( \phi \) and \( \psi \) are in \( L^1 \).

**Theorem 5.3.** Part a of the second theorem in section II is true.

**Proof.** Let \( C^1 = A^1, C^2 = A^2 \) be as described in equation 3.1 and remark 3.2. Let \( \chi_{\{j_1\}}, \chi_{\{j_2\}} \) be the characteristic function on the region \( C^1, C^2 \) respectively (\( \{j_1\} \), and \( \{j_2\} \) are as described in equation 3.4 for \( C^1 \) and \( C^2 \)). Let \( \phi, \psi \in L^2 \), then
\[
\lim_{\{j_1\} \to \infty} \left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) \exp \left( \frac{-\gamma H_0}{\hbar} \right) \psi \right] =
\]
\[
\lim_{\{j_2\} \to \infty} \left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) \exp \left( \frac{-\gamma H_0}{\hbar} \right) \chi_{\{j_1\}} \psi \right],
\]
where all limits in (5.8) are taken in the \( L^2 \) topology. Using 5.4, 5.5 and 5.8, we have
\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right] (\vec{x}) d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) \exp \left( \frac{-\gamma H_0}{\hbar} \right) \psi \right] (\vec{x}) d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \lim_{\{j_2\} \to \infty} \chi_{\{j_1\}} \phi(\vec{x}) \times
\]
\[
\lim_{\{j_2\} \to \infty} \left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) \exp \left( \frac{-\gamma H_0}{\hbar} \right) \chi_{\{j_2\}} \psi \right] (\vec{x}) d\vec{x}.
\]
Finally, using Schwarz’s inequality and theorem 5.2 on the last expression in (5.9) gives
\[
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right] d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \lim_{\{j_1\}, \{j_2\} \to \infty} \int_{\mathbb{R}^n} \chi_{\{j_1\}} \phi(\vec{x}) \times
\]
\[
\left[ \exp \left( \frac{-\eta H_0}{\hbar} \right) \exp \left( \frac{-it\bar{H}}{\hbar} \right) \exp \left( \frac{-\gamma H_0}{\hbar} \right) \chi_{\{j_2\}} \psi \right] (\vec{x}) d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \lim_{\{j_1\}, \{j_2\} \to \infty} \int_{C^1 \times C^2} \phi(\vec{x}) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \psi(\vec{x}_0) d\vec{x}_0 d\vec{x} =
\]
\[
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) \psi(\vec{x}_0) d\vec{x}_0 d\vec{x}.
\]
where all limits outside of integrals are taken pointwise in $t$ and $\bar{\int}$ by definition is an improper Lebesgue integral with convergence at infinity taken pointwise in $t$. □

**Remark 5.1.** If we restrict $\phi, \psi \in L^2$ to have a finite number of singularities and discontinuities as in the proof of the third theorem (theorem 3.1), we can obtain an improper Riemann integral as opposed to an improper Lebesgue integral for theorem 5.3.

**Theorem 5.4.** Part a of the fourth theorem in section II is true.

*Proof.* Equation 5.6 is true for all $\psi \in L^2$. Taken $\lim_{\eta, \gamma \to \infty}$ on both sides of 5.6 in the $L^2$ topology gives part a of theorem four in section I equation 2.6a. □

**Theorem 5.6.** Part b of the fourth theorem in section II is true.

*Proof.* Let $\phi, \psi \in L^2$, then Schwarz's inequality and part a of the fourth theorem (theorem 5.4) implies

\[
\left( \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi(\vec{x}) \right] d\vec{x} \right) = \\
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left( \int_{\mathbb{R}^n} \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) d\vec{x}_0 \right) d\vec{x} = \\
\lim_{\eta, \gamma \to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left( \int_{\mathbb{R}^n} \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t) d\vec{x}_0 \right) d\vec{x},
\]

where limits inside the integral are taken in $L^2$ and limits taken outside the integrals are taken pointwise in $t$. □

**VI. Proof Fifth Theorem.** In this section, we prove the fifth theorem in section II and derive some properties of the tempered distribution in the fifth theorem. Since we will be working with tempered distributions, we will let $\phi$ and $\psi$ be in the class of rapidly decreasing test functions which we will denote by $S(\mathbb{R}^n)$. If $\phi$ and $\psi$ are elements of $L^2(\mathbb{R}^n)$, we can choose a sequence of test functions $\{\phi_l\}$ and $\{\psi_j\}$ such that $\phi_l \to \phi$ and $\psi_j \to \psi$ in $L^2$. Applying Schwarz's inequality and using the fact that the evolution operator has operator norm equal to 1, we can write

\[
\left( \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right] d\vec{x} \right) = \\
\lim_{j,l \to \infty} \int_{\mathbb{R}^n} \phi_l(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi_j \right] d\vec{x}.
\]

Thus, by taking limits in the sense of (6.1), we can always recover all of the $L^2$ wave functions in the theory.

**Theorem 6.1.** The fifth theorem in section I is true.

*Proof.* Suppose $\{\phi_k(\vec{x})\} \subset S(\mathbb{R}^n)$ with $\phi_k(\vec{x}) \to \phi(\vec{x})$ in $S(\mathbb{R}^n)$, then

\[
\left| \int_{\mathbb{R}^n} [\phi(\vec{x}) - \phi_k(\vec{x})] \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \right] d\vec{x} \right| \leq \\
\| \phi - \phi_k \|_2 \times \| \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi \|_2 \to 0.
\]
Suppose \( \{ \psi_k (x) \} \subset S(\mathbb{R}^n) \) with \( \psi_k (x) \to \psi (x) \) in \( S(\mathbb{R}^n) \), then

\[
\int_{\mathbb{R}^n} \phi (x) \left[ \exp \left( \frac{-iH}{\hbar} \right) (\psi - \psi_k) \right] (x) d\bar{x} \leq \left\| \phi \right\|_2 \times \left\| \exp \left( \frac{-iH}{\hbar} \right) (\psi - \psi_k) \right\|_2 = \left\| \phi \right\|_2 \times \left\| (\psi - \psi_k) \right\|_2 \to 0.
\]

Hence the theorem follows from Schwartz’s Kernel Theorem. \( \square \)

Remark 6.2. Notice that the second and the fifth theorem imply that

\[
\lim_{\eta, \gamma \to \infty} \int_{\mathbb{R}^n} \phi (x) \psi (x_0) K (x, x_0, \eta, \gamma, t) d\bar{x}_0 d\bar{x} = \int K_t (x, x_0) \phi (x) \psi (x_0) d\bar{x}_0 d\bar{x}_0,
\]

At \( t = 0 \), the evolution operator becomes the identity operator. In distributions language, We have the following

Theorem 6.3. At \( t = 0 \), \( K_t (x, x_0) \) satisfies \( K_0 (x, x_0) = \delta (x - x_0) \)

Proof. Since \( \left[ \exp \left( \frac{-iH}{\hbar} \right) \right] (x) = \psi (x) \) when \( t = 0 \), we have

\[
\int K_0 (x, x_0) \phi (x) \psi (x_0) d\bar{x}_0 d\bar{x} = \int_{\mathbb{R}^n} \phi (x) \psi (x) d\bar{x} = \int \phi (x) \delta (x - x_0) \psi (x_0) d\bar{x} d\bar{x}_0.
\]

We extend (6.5) to all of \( S(\mathbb{R}^n \times \mathbb{R}^n) \). Let \( \eta (x, x_0) \in S(\mathbb{R}^n \times \mathbb{R}^n) \). Choose a sequence of functions \( \eta_k (x, x_0) \) in \( S(\mathbb{R}^n \times \mathbb{R}^n) \) such that \( \eta_k \to \eta \) in \( S(\mathbb{R}^n \times \mathbb{R}^n) \) and \( \eta_k = \sum_{i=0}^{\infty} u_{i,k} (x) v_{i,k} (x_0) \) where \( u_{i,k} \) and \( v_{i,k} \in D(\mathbb{R}^n) \), the \( C^\infty \) compactly supported test functions, then

\[
\int_{\mathbb{R}^n} \sum_{i=0}^{\infty} u_{i,k} (x) v_{i,k} (x_0) d\bar{x} d\bar{x}_0 = \int_{\mathbb{R}^n} \eta (x, x_0) d\bar{x} d\bar{x}_0 = \int \delta (x - x_0) \eta (x, x_0) d\bar{x} d\bar{x}_0. \quad \square
\]

It is well known that the free propagator satisfies \( K_t^{\text{free}} (x, x_0) = K_t^{\text{free}} (x_0, x) \). We will show a similar property for the tempered distribution in theorem 6.1. Intuitively, it is reasonable to believe that the from lemma 4.5 we can conclude the following.

Theorem 6.4. \( K_t (x, x_0) = K_t (x_0, x) \) where equality is in the sense of distributions.
Proof. \( \forall \phi(\vec{x}), \psi(\vec{x}_0) \in S(\mathbb{R}^n) \), we have that

\[
\int K_t(\vec{x}, \vec{x}_0) \phi(\vec{x}) \psi(\vec{x}_0) \, d\vec{x} \, d\vec{x}_0 = \int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \psi(\vec{x}) \right] \, d\vec{x} = \\
\int_{\mathbb{R}^n} \psi(\vec{x}_0) \left[ \exp \left( \frac{-it\bar{H}}{\hbar} \right) \phi(\vec{x}) \right] \, d\vec{x}_0 = \int_{\mathbb{R}^n} \psi(\vec{x}_0) \phi(\vec{x}) \, d\vec{x}_0 = \\
\int K_t(\vec{x}, \vec{x}_0) \psi(\vec{x}_0) \phi(\vec{x}_0) \, d\vec{x} \, d\vec{x}_0 = \int K_t(\vec{x}_0, \vec{x}) \phi(\vec{x}_0) \psi(\vec{x}) \, d\vec{x} \, d\vec{x}_0.
\]

We extend (6.7) to all of \( S(\mathbb{R}^n \times \mathbb{R}^n) \). Let \( \eta(\vec{x}, \vec{x}_0) \in S(\mathbb{R}^n \times \mathbb{R}^n) \). Choose a sequence of functions \( \eta_k(\vec{x}, \vec{x}_0) \) in \( D(\mathbb{R}^{2n}) \) such that \( \eta_k \to \eta \) in \( S(\mathbb{R}^{2n}) \) and \( \eta_k = \sum_{i=0}^{n} u_{i,k}(\vec{x}) v_{i,k}(\vec{x}_0) \) where \( u_{i,k} \) and \( v_{i,k} \in D(\mathbb{R}^n) \), the \( C^\infty \) compactly supported test functions. We then have

\[
\int K_t(\vec{x}, \vec{x}_0) \eta(\vec{x}, \vec{x}_0) \, d\vec{x} \, d\vec{x}_0 = \lim_{k \to \infty} \int K_t(\vec{x}, \vec{x}_0) \eta_k(\vec{x}, \vec{x}_0) \, d\vec{x} \, d\vec{x}_0 = \\
\lim_{k \to \infty} \int K_t(\vec{x}_0, \vec{x}) \eta_k(\vec{x}_0, \vec{x}) \, d\vec{x} \, d\vec{x}_0 = \int K_t(\vec{x}_0, \vec{x}) \eta(\vec{x}_0, \vec{x}) \, d\vec{x} \, d\vec{x}_0.
\]
VIII The Harmonic Oscillator. We now compute the harmonic oscillator propagator for $0 < t < \frac{\pi}{\lambda}$ using the formulas above. Some of the techniques that we will use was previously worked out in [19], for full details, we will occasionally refer the reader to [19]. For the harmonic oscillator, equation 1.5 reads (with a shift in the indices)

$K(\vec{q}, \vec{q}_0, \eta, \gamma, t) =$

$$\lim_{k \to \infty} w_{n,k+1} \int_{\mathbb{R}^{(k+2)n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{x}_{k+1}, \eta) \times$$

$$\exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=1}^{k+1} \left[ \frac{m}{2} \left( \frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (\vec{x}_j)^2 \right] \right\} d\vec{x}_0 \ldots d\vec{x}_{k+1}.$$ 

Let us write $\vec{x}_j = (x_j^1, \ldots x_j^n)$, and

$$\exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=1}^{k+1} \left[ \frac{m}{2} \left( \frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (\vec{x}_j)^2 \right] \right\} =$$

$$\prod_{\alpha=1}^{n} \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=1}^{k+1} \left[ \frac{m}{2} \left( \frac{x_{\alpha j} - x_{\alpha j-1}}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (x_{\alpha j})^2 \right] \right\}.$$ 

The popular method to compute the time sliced harmonic oscillator path integral is to use (8.2) to decouple the integrals in 1.0 and reduce the problem to produces of one dimensional harmonic oscillators. Due to the extra $\vec{x}_0, \vec{x}_{k+1}$ integrals in 8.1, it is not immediately clear that we can use 8.2 to decouple the improper Riemann integrals.

For the moment, let us consider just one of the entries in the product of 8.2. To shorten notation, let us write

$$\frac{i\epsilon}{\hbar} \sum_{j=1}^{k+1} \left[ \frac{m}{2} \left( \frac{x_{\alpha j} - x_{\alpha j-1}}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (x_{\alpha j})^2 \right] =$$

$$\left( \frac{im}{2\hbar\epsilon} \right) \left[ (x_0^\alpha)^2 - 2x_0^\alpha x_1^\alpha + (x_{k+1}^\alpha)^2 - 2x_k^\alpha x_{k+1}^\alpha +$$

$$\sum_{j=1}^{k} 2 (x_j^\alpha)^2 - \sum_{j=1}^{k} 2x_j^\alpha x_{j-1}^\alpha - \epsilon^2 \lambda^2 \sum_{j=1}^{k+1} (x_j^\alpha)^2 \right] =$$
\[
\left( \frac{im}{2\hbar} \right) (x^\alpha)^t \left\{ \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & -1 \end{pmatrix} \right\} = \\
\lambda^2 \left( \begin{pmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \right) + \\
\left( \begin{pmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 2 & -1 & 0 & \ldots & \ldots \\ \vdots & -1 & 2 & -1 & 0 & \ldots \\ \vdots & 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & -1 & 2 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{pmatrix} \right) \equiv \left( \frac{im}{2\hbar} \right) (x^\alpha)^t T_k x^\alpha,
\]

where \( T_k \) is the \((k + 2)\) by \((k + 2)\) symmetric matrix,

\[
T_k = \left( \begin{array}{cccccc} 1 & -1 & 0 & \ldots & 0 \\ -1 & 0 & S_k & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & -1 & 1 - \epsilon^2 \lambda^2 \end{array} \right),
\]

with \( S_k \) being the \(k\) by \(k\) symmetric matrix \( S_k = A_k - \epsilon^2 \lambda^2 B_k \), where

\[
A_k = \left( \begin{array}{cccccc} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & \ldots \\ 0 & -1 & 2 & -1 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & -1 & 2 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{array} \right).
\]
Let \( w_\alpha \). Here, we allow \( \vec{x} \) and \( \vec{y} \) to be any finite path that starts at \( x_0^\alpha \) and ends at \( x_{k+1}^\alpha \). We do not assume prior knowledge of classical mechanics. We make the substitution \( x_j^\alpha = w^\alpha \left( \frac{j}{k+1} \right) \) and writing \( y_j^\alpha = y_j^\alpha \) (notice that \( y_0^\alpha = 0 = y_{k+1}^\alpha \) since \( w^\alpha (0) = x_0^\alpha \) and \( w^\alpha (t) = x_{k+1}^\alpha \). Using the fact that \( T_k \) is symmetric, we have

\[
(\vec{x}^\alpha)^T T_k \vec{x}^\alpha = (\vec{y}^\alpha + \vec{w}^\alpha)^T T_k (\vec{y}^\alpha + \vec{w}^\alpha) = \\
(\vec{w}^\alpha)^T T_k \vec{w}^\alpha + (\vec{y}^\alpha)^T T_k \vec{y}^\alpha + (\vec{w}^\alpha)^T T_k \vec{y}^\alpha + (\vec{y}^\alpha)^T T_k \vec{w}^\alpha = \\
(\vec{w}^\alpha)^T T_k \vec{w}^\alpha + (\vec{y}^\alpha)^T T_k \vec{y}^\alpha + (T_k \vec{w}^\alpha)^T \vec{y}^\alpha + (\vec{w}^\alpha)^T T_k \vec{y}^\alpha = \\
(\vec{w}^\alpha)^T T_k \vec{w}^\alpha + (\vec{y}^\alpha)^T T_k \vec{y}^\alpha + 2(T_k \vec{w}^\alpha)^T \vec{y}^\alpha,
\]

where

\[
\vec{y}^\alpha = \begin{pmatrix}
0 \\ y_1^\alpha \\ \vdots \\ y_k^\alpha \\ 0
\end{pmatrix}, \quad \vec{w}^\alpha = \begin{pmatrix}
w_0^\alpha = x_0^\alpha \\
w_1^\alpha \\ \vdots \\
w_k^\alpha \\
w_{k+1}^\alpha = x_{k+1}^\alpha
\end{pmatrix}.
\]

By using \( y_0^\alpha = 0 = y_{k+1}^\alpha \) and writing \( T_k \) as

\[
T_k = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & S_k & 0 & \vdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1 & -1 & 0 & \ldots & 0 \\
-1 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & -1 & \ddots & \vdots \\
0 & \ldots & 0 & -1 & 1 - e^2 \lambda^2
\end{pmatrix},
\]
we obtain

\[(8.10) \quad (\vec{x}^\alpha)^t T_k \vec{x}^\alpha = (\vec{w}^\alpha)^t T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t T_k \vec{y}^\alpha + 2 (T_k \vec{w}^\alpha)^t \vec{y}^\alpha = (\vec{w}^\alpha)^t T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t S_k \vec{y}^\alpha + 2 (\vec{w}^\alpha)^t \vec{y}^\alpha\]

where

\[(8.11) \quad \hat{y}^\alpha = \begin{pmatrix} y_0^1 \\ \vdots \\ y_k^\alpha \\ y_{k+1}^\alpha \end{pmatrix}, \quad \hat{\rho}^\alpha = S_k \begin{pmatrix} w_0^\alpha \\ \vdots \\ w_{k-1}^\alpha \\ w_k^\alpha \\ 0 \end{pmatrix} - \begin{pmatrix} w_{k+1}^\alpha = x_{k+1}^\alpha \\ \vdots \\ 0 \end{pmatrix} = S_k \hat{w}^\alpha - \hat{\omega}^\alpha.\]

**Lemma 8.1.** Let \( t \in \mathbb{R} \) and \( 0 < t < \frac{\pi}{\lambda} \). For any \( \omega \in *\mathbb{N} - \mathbb{N}, *S_\omega \) is positive definite in the \(*\)-transformed sense. Here, \(*S_\omega \) is the \(*\)-transform of the matrix \( S_k \) defined after equation 8.4.

**Proof.** See [19].

We now go back to equations 8.1 and 8.2 in Nonstandard Analysis form. With an abuse of notation, in Nonstandard Analysis 8.1 reads

\[(8.12) \quad K (\vec{q}, \vec{q}_0, \eta, \gamma, t) = \]

\[\text{st} \left\{ \int_{r^N(\omega+2)^n} \left[ \prod_{j=1}^{\omega+1} \left[ \frac{m}{2} \left( \frac{x_j - x_{j-1}}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (x_j^\alpha)^2 \right] \right] d\vec{x}_0 \ldots d\vec{x}_{\omega+1} \right\} = \]

\[\text{st} \left\{ \int_{r^N(\omega+2)^n} \left[ \prod_{j=1}^{\omega+1} \left[ \frac{m}{2} \left( \frac{x_j^\alpha - x_{j-1}^\alpha}{\epsilon} \right)^2 - \frac{m}{2} \lambda^2 (x_j^\alpha)^2 \right] \right] d\vec{x}_0 \ldots d\vec{x}_{\omega+1} \right\} = \]

\[\prod_{n=1}^{\omega+1} \left( \frac{m}{2 \hbar c} \right) \left( (x^\alpha)^t T_\omega x^\alpha \right) d\vec{x}_0 \ldots d\vec{x}_{\omega+1} \].

We perform a \(*\)-transform of the change of variable described in equation 8.7 on
\[ K(\vec{q}_1, \vec{q}_0, \eta, \gamma, t) = \left\{ \begin{array}{l}
w_\omega \int_{\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0} (\vec{x}_0, \gamma) G_{\vec{q}} (\vec{x}_{\omega+1}, \eta) \times \\
\prod_{\alpha=1}^{n} \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{w}_\alpha \right)^t T_\omega \vec{w}_\alpha \right\} \times \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{z}_\alpha \right)^t S_\omega \vec{z}_\alpha \right\} d\vec{x}_0 d\vec{x}_{\omega+1} d\vec{y}_1 \ldots d\vec{y}_\omega \end{array} \right\}. \]

Since \( S_\omega \) is positive definite, it is invertable. Since \( S_\omega \) is symmetric, the following is true

\[ (\vec{y}_\alpha)^t S_\omega \vec{y}_\alpha + 2 (\vec{\rho}_\alpha)^t \vec{y}_\alpha = (\vec{y}_\alpha + S_\omega^{-1} \vec{\rho}_\alpha)^t S_\omega (\vec{y}_\alpha + S_\omega^{-1} \vec{\rho}_\alpha) - (\vec{\rho}_\alpha)^t S_\omega^{-1} \vec{\rho}_\alpha. \]

Using 8.14 in 8.13 and performing the transformation \( z_j^\alpha = y_j^\alpha + (S_\omega^{-1} \vec{\rho}_\alpha)_j \), we obtain

\[ K(\vec{q}_1, \vec{q}_0, \eta, \gamma, t) = \left\{ \begin{array}{l}
w_\omega \int_{\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0} (\vec{x}_0, \gamma) G_{\vec{q}} (\vec{x}_{\omega+1}, \eta) \times \\
\prod_{\alpha=1}^{n} \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{w}_\alpha \right)^t T_\omega \vec{w}_\alpha \right\} \times \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{z}_\alpha \right)^t S_\omega \vec{z}_\alpha \right\} d\vec{x}_0 d\vec{x}_{\omega+1} d\vec{z}_1 \ldots d\vec{z}_\omega \end{array} \right\}. \]

Notice that before the improper limits are taken on the integrals, the limits of integration on the variables \( z_j^\alpha \) are dependent on \( \vec{x}_0 \) and \( \vec{x}_{\omega+1} \) due to the fact that \( \vec{\rho}_\alpha \) is dependent on them. Thus, we still can not decouple the improper Riemann integrals.

Let us take a look at the limits of integrations more closely. Before the improper limits on the integrals are taken in equation 8.15, we have

\[ w_\omega \int_{\mathcal{O}} F_{\vec{q}_0} (\vec{x}_0, \gamma) G_{\vec{q}} (\vec{x}_{\omega+1}, \eta) \times \\
\prod_{\alpha=1}^{n} \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{w}_\alpha \right)^t T_\omega \vec{w}_\alpha \right\} \times \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{i m}{2 \hbar} \right) \left( \vec{z}_\alpha \right)^t S_\omega \vec{z}_\alpha \right\} d\vec{x}_0 d\vec{x}_{\omega+1} d\vec{z}_1 \ldots d\vec{z}_\omega \].

where both \( \mathcal{O} \) and \( \mathcal{O} \) are \( * \)-compact and the boundry of \( \mathcal{O} \) depends on \( \vec{x}_0 \) , \( \vec{x}_{\omega+1} \) and a set of indices \( \{ J \} \) such that as \( \{ J \} \to \infty, \mathcal{O} \to *\mathbb{R}^{\omega n} \) in the \( * \)-transformed
sense. The reason for $\tilde{O}$'s dependent on $\vec{x}_0, \vec{x}_{\omega+1}$ is due to the fact that $\tilde{\rho}^\alpha$ is dependent on them (equation 8.11) and we performed the change of variables from equation 8.13 to equation 8.15. Further, the boundary of $\mathcal{O}$ is also indexed by a set similar to that of $\{J\}$. What we would like to do is pass the $\{J\}$ limits inside the $\mathcal{O}$ integral and decouple the improper Riemann integrals in equation 8.15 into improper Riemann integrals in $d\vec{z}_1 \ldots d\vec{z}_\omega$ then $d\vec{x}_0 d\vec{x}_{\omega+1}$.

It is well known from the one dimensional harmonic oscillator that

\begin{equation}
(8.17)
\int_{\mathbb{R}^k} \exp\left(\frac{im}{2\hbar}(z^\alpha)^t S_k z^\alpha\right) dz_1^\alpha \ldots dz_k^\alpha = \left(\frac{m}{2\pi i\hbar}\right)^{\frac{k}{2}} \sqrt{\frac{1}{\det S_k}}.
\end{equation}

Let us fix an $\mathcal{O}$. Since $\mathcal{O}$ is compact (see the construction in equation 3.12). We can $\ast$-transform and conclude from 8.16 and 8.17 that for any $\beta \in \ast \mathbb{R}^+$, there exists a fixed $M \in \ast \mathbb{R}^+$ that depends only on $\mathcal{O}$ such that

\begin{equation}
(8.18)
\left| w_{n,\omega+1} \int_{\mathcal{O}} \prod_{\alpha=1}^{\omega} \exp \left\{ \left(\frac{im}{2\hbar}\right) (z^\alpha)^t S_\omega z^\alpha \right\} d\vec{z}_1 \ldots d\vec{z}_\omega - \left(\frac{m}{2\pi i\hbar}\right)^{\frac{k}{2}} \sqrt{\frac{1}{\det S_\omega}} \right|^n < \beta
\end{equation}

whenever all entries of $\{J\}$ are bigger than $M$. In other words, because $\mathcal{O}$ is compact, for all $(\vec{x}_0, \vec{x}_{\omega+1}) \in \mathcal{O}$, equation 8.18 is true whenever all entries of $\{J\}$ are bigger than a fixed $M$; further, this $M$ depends on $\mathcal{O}$.

Equation 8.18 allows us to use $\ast$-Lebesgue dominating convergence theorem and pass the $\{J\}$ limits inside the $\mathcal{O}$ integral and decouples the improper Riemann integrals. Thus, we have proved the following
Theorem 8.2. For the harmonic oscillator,

(8.19)
\[ K(\vec{q}, \vec{q}_0, \eta, \gamma, t) = \]
\[ \text{st} \left\{ w_{n,\omega + 1} \int_{R^{(\omega + 2)n}} F_{\vec{q}_0} (\vec{x}_0, \gamma) G_{\vec{q}} (\vec{x}_{\omega + 1}, \eta) \times \prod_{\alpha = 1}^{n} \exp \left\{ \left( \frac{im}{2\hbar} \right) \left( (\vec{w}^\alpha)^t T_{\omega} \vec{w}^\alpha - (\vec{\rho}^\alpha)^t S_{\omega}^{-1} \vec{\rho}^\alpha \right) \right\} \times \prod_{\alpha = 1}^{n} \exp \left\{ \left( \frac{im}{2\hbar} \right) (\vec{z}^\alpha)^t S_{\omega} \vec{z}^\alpha \right\} d\vec{x}_0 d\vec{x}_{\omega + 1} d\vec{z}_1 \ldots d\vec{z}_{\omega} \right\} = \]
\[ \text{st} \left\{ w_{n,\omega + 1} \int_{R^{2n}} \prod_{\alpha = 1}^{n} \exp \left\{ \left( \frac{im}{2\hbar} \right) \left( (\vec{w}^\alpha)^t (\vec{w}^\alpha)^t S_{\omega}^{-1} \vec{\rho}^\alpha \right) \right\} d\vec{x}_0 d\vec{x}_{\omega + 1} \right\} = \]
\[ \text{st} \left\{ \left( \frac{m^{\frac{3}{2}}}{2\pi i\hbar} \right)^{\frac{n}{2}} \sqrt{\frac{1}{\det S_{\omega}}} \int_{R^{2n}} F_{\vec{q}_0} (\vec{x}_0, \gamma) G_{\vec{q}} (\vec{x}_{\omega + 1}, \eta) \times \prod_{\alpha = 1}^{n} \exp \left\{ \left( \frac{im}{2\hbar} \right) (\vec{z}^\alpha)^t S_{\omega} \vec{z}^\alpha \right\} d\vec{x}_0 d\vec{x}_{\omega + 1} \right\}. \]

Proof. See above. □

It remains now to compute the last equality in 8.19.

Proposition 8.3. With the previously defined notations,

(8.20)
\[ \text{st} \left\{ \left( \frac{m^{\frac{3}{2}}}{2\pi i\hbar} \right)^{\frac{n}{2}} \sqrt{\frac{1}{\det S_{\omega}}} \right\} = \left( \frac{m^{\frac{3}{2}}}{2\pi i\hbar} \right)^{\frac{n}{2}} \left( \frac{\lambda}{\sin \lambda t} \right)^{\frac{n}{2}}. \]

Proof. See [19]. □

Proposition 8.4. Let \( \vec{x}_0, \vec{x}_n = \vec{y} \in R^n \) be fixed. With the previously defined notations,

(8.21)
\[ \lim_{k \to \infty} \left\{ \prod_{\alpha = 1}^{n} \exp \left\{ \left( \frac{im}{2\hbar} \right) \left( (\vec{w}^\alpha)^t T_k \vec{w}^\alpha - (\vec{\rho}^\alpha)^t S_k^{-1} \vec{\rho}^\alpha \right) \right\} = \right. \]
\[ \left. \exp \left\{ \frac{im}{\hbar} \frac{\lambda}{\sin \lambda t} \left[ (\vec{x}_0^2 + \vec{y}^2) \cos \lambda t - 2\vec{y}\vec{x}_0 \right] \right\}. \]

Remark 8.5. Notice that proposition 8.4 does not take place in the nonstandard world.

Proof. This is just the classical version of the nonstandard results obtained from [19]. See [19] for more details.
Proposition 8.5. With the previously defined notations,

\[
\left\{ \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{x}_{\omega+1}, \eta) \times \right. \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{im}{2\hbar \epsilon} \right) (\vec{w}^{\alpha})^t T_{\omega} \vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^t S_{\omega}^{-1} \vec{\rho}^{\alpha} \right\} \left. \right\} d\vec{x}_0 d\vec{x}_{\omega+1} = \\
\int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{y}, \eta) \times \\
exp \left\{ \frac{im}{\hbar} \frac{\lambda}{\sin \lambda t} \left[ (\vec{x}_{0}^2 + \vec{y}^2) \cos \lambda t - 2\vec{y}\vec{x}_0 \right] \right\} d\vec{x}_0 d\vec{y}.
\]

Proof. Using proposition 8.4 and Lebesgue’s dominating convergence theorem, we obtain

\[
\left\{ \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{x}_{\omega+1}, \eta) \times \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{im}{2\hbar \epsilon} \right) (\vec{w}^{\alpha})^t T_{\omega} \vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^t S_{\omega}^{-1} \vec{\rho}^{\alpha} \right\} \left. \right\} d\vec{x}_0 d\vec{x}_{\omega+1} = \\
\lim_{k \to \infty} \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{y}, \eta) \times \\
\prod_{\alpha=1}^{n} \exp \left\{ \left( \frac{im}{2\hbar \epsilon} \right) (\vec{w}^{\alpha})^t T_{k} \vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^t S_{k}^{-1} \vec{\rho}^{\alpha} \right\} \left. \right\} d\vec{x}_0 d\vec{y} = \\
\int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{y}, \eta) \exp \left\{ \frac{im}{\hbar} \frac{\lambda}{\sin \lambda t} \left[ (\vec{x}_{0}^2 + \vec{y}^2) \cos \lambda t - 2\vec{y}\vec{x}_0 \right] \right\} d\vec{x}_0 d\vec{y}. \quad \Box
\]

Theorem 8.6. For the harmonic oscillator,

\[
K(\vec{q}, \vec{q}_0, \eta, \gamma, t) = \\
\left( \frac{m}{2\pi \hbar} \right)^{\frac{p}{2}} \left( \frac{\lambda}{\sin \lambda t} \right)^{\frac{p}{2}} \times \\
\int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0, \gamma) G_{\vec{q}}(\vec{y}, \eta) \exp \left\{ \frac{im}{\hbar} \frac{\lambda}{\sin \lambda t} \left[ (\vec{x}_{0}^2 + \vec{y}^2) \cos \lambda t - 2\vec{y}\vec{x}_0 \right] \right\} d\vec{x}_0 d\vec{y}.
\]

Proof. Equation 8.24 follows from theorem 8.2, proposition 8.3, and proposition 8.5. \( \Box \)
Theorem 8.7. Let $\phi, \psi \in L^1 \cap L^2$, then for the harmonic oscillator Hamiltonian and for $0 < t < \pi/\lambda$,

$$
\int_{\mathbb{R}^n} \phi(\vec{x}) \left[ \exp\left( \frac{-i t \hat{H}}{\hbar} \right) \psi \right](\vec{x}) \, d\vec{x} = \\
\left( \frac{m}{2\pi i\hbar} \right)^{\frac{n}{2}} \left( \frac{\lambda}{\sin \lambda t} \right)^{\frac{n}{2}} \times \\
\int_{\mathbb{R}^{2n}} \phi(\vec{q}_0) \psi(\vec{q}) \exp\left\{ \frac{im}{\hbar} \frac{\lambda}{\sin \lambda t} \left[ (\vec{q}_0^2 + \vec{q}^2) \cos \lambda t - 2\vec{q}_0 \vec{q} \right] \right\} \, d\vec{q}_0 \, d\vec{q}.
$$

Proof. Notice that $\left| K(\vec{q},\vec{q}_0,\eta,\gamma,t) \right| \leq C_{n,\lambda,t}$. Substituting equation 8.24 in equation 1.3 and using Lebesgue dominating theorem on the $\eta,\gamma$ limits give 8.25. □

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