Particle Trapping in Axisymmetric Electron Holes

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Abstract

Electron orbits are calculated in solitary two-dimensional axisymmetric electrostatic positive potential structures, typical of plasma electron holes, in order to establish the conditions for the particles to remain trapped. Analytic calculations of the evolution of the parallel energy caused by the perturbing radial electric field (breaking magnetic moment invariance) are shown to agree well with full numerical orbit integration Poincaré plots. The predominant mechanism of detrapping is resonance between the gyrofrequency in the parallel magnetic field and harmonics of the parallel bounce frequency. A region of phase space adjacent to the trapped-passing boundary in parallel energy is generally stochastic because of island overlap of different harmonics, but except for very strong radial electric field perturbation, more deeply trapped orbits have well-defined islands and are permanently confined. A simple universal quantitative algorithm is given, and its results plotted as a function of magnetic field strength and hole radial scale length, determining the phase space volume available to sustain the electron hole by depression of the permanently trapped distribution function.

1. Background

Electron holes are steady solitary electrostatic positive potential structures that sustain themselves by an electron density deficit arising from depressed phase space density on trapped orbits (Eliasson & Shukla, 2006; Hutchinson, 2017; Schamel, 1986; Turikov, 1984). They are frequently observed in one-dimensional nonlinear simulations of plasma kinetic instabilities (Berk et al., 1970; Hutchinson, 2017; Miyake et al., 1998; Morse & Nielson, 1969; Omura et al., 1996), and in observations of space plasmas (Andersson et al., 2009; Bale et al., 1998; Ergun et al., 1998; Hutchinson & Malaspina, 2018; Malaspina et al., 2013, 2014; Mangeney et al., 1999; Matsumoto et al., 1994; Mozer et al., 2016, 2018; Pickett et al., 2008; Vasko et al., 2015; Wilson et al., 2010). The one-dimensional theory of these hole equilibria is well established, being a type of BGK mode (Bernstein et al., 1957). However, in multiple dimensions, both the equilibrium and stability of these self-sustaining structures is far less well understood. Satellite observations show that electron holes are generally three-dimensional (Franz et al., 2000; Holmes et al., 2018; Tong et al., 2018; Vasko et al., 2017), oblate structures, more extended in the direction perpendicular to the ambient magnetic field, than parallel, but by an amount that varies with plasma and hole parameters. Also, analysis and simulation have shown that initially one-dimensional holes are subject to instabilities (Goldman et al., 1999; Hutchinson, 2018a, 2018b, 2019a, 2019b; Lu et al., 2008, 2008; Miyake et al., 1998; Mottez et al., 1997; Muschietti et al., 2000; Oppenheim et al., 1999, 2001; Singh et al., 2001) that break them up in the transverse dimension, forming multidimensional remnants.

A significant magnetic field is known theoretically to be necessary for the existence of multidimensional electron hole equilibria in nonpathological background electron distributions (Krasovsky et al., 2004; Ng & Bhattacharjee, 2005; Ng et al., 2006). When the field is strong enough that the gyroradius (ρ) is very small, the equilibrium becomes locally one-dimensional (Chen & Parks, 2002; Jovanović et al., 2002), with minor corrections to Poisson's equation to account for any transverse electric field divergence, but eventually negligible influence on the particle orbits. At the other extreme, the magnetic field cannot be so weak as to make the gyroradius large compared with the hole's transverse dimensions, otherwise it provides little transverse confinement. But there is a big parameter range between these two limits, in which virtually no theory beyond order of magnitude heuristics has been completed. A high proportion of observed electron holes have equilibrium parameters lying in this unexplained region (see, e.g., Franz et al., 2000; Holmes et al., 2018; Vasko et al., 2017).

This article presents a first step to carry out rigorous analysis of multidimensional electron hole equilibria. It adopts a model potential that is axisymmetric (independent of the angle θ in a cylindrical coordinate system),
which is a representative subset of three-dimensional holes. The electron orbits in this equilibrium are analyzed and calculated numerically, to discover which regions of phase space are permanently trapped, and, in contrast, the regions that initially possess small enough parallel kinetic energy to be trapped by the parallel electric field but evolve soon to become untrapped, by the transfer of energy from perpendicular gyration. By time reversal symmetry, equivalent orbits (in equal numbers) experience evolution of parallel energy from passing to become trapped. There thus arises a large effective parallel energy diffusion across the trapped/passing boundary. These detrapping/trapping orbits cannot sustain depressed electron phase space density and so cannot contribute to maintaining the hole’s positive potential, because the important detrapping occurs usually on a short timescale of a moderate number of bounces, that is, approximately of plasma periods.

The present work does not solve the (still unsolved) full problem of finding a self-consistent equilibrium in which only the velocity distribution function on the permanently trapped orbits is allowed to differ from the background distribution. But it does give limits on what fully self-consistent solutions can exist and indicates what their distribution functions might look like.

2. Orbits in Axisymmetric Electron Holes

We consider the orbits of electrons in a potential \( \phi(r, z) \) that is axisymmetric about the coordinate \( z \). This is a 2-D problem, meaning there is just one ignorable coordinate \( \theta \). A 2-D cartesian geometry in which one cartesian coordinate is ignorable would give essentially the same result and can be considered to be the limit in which the radius \( r \) is large.

In the 1-D case where \( \phi \) depends only on \( z \), there are two exact constants of the motion, which are the total energy \( W \) and the magnetic moment, which in the present uniform magnetic field can be taken as the perpendicular kinetic energy \( W_\perp \). The perpendicular motion is then entirely decoupled from the parallel and can be ignored. However, when \( \phi \) varies with radius \( r \), and a transverse electric field \( E_r \) exists, the \( W_\perp \) (magnetic moment) invariance is broken, and the only strict invariant in addition to \( W \) is the canonical angular momentum about the \( z \) axis: \( p_\theta = r(v_\perp - \Omega r/2) \), where \( \Omega \) is the gyrofrequency. The effect of \( p_\theta \) conservation is mostly to restrict the range of variation of the orbit’s radius to what in the probe literature are called “magnetic bottles” (e.g., Laframboise & Sonmor, 1993). At radii greater than approximately the gyroradius \( \rho = v_\perp / \Omega \), conservation of \( p_\theta \) contributes little to parallel particle dynamics, serving mostly to localize the orbit in radial position, within approximately one gyroradius. Figure 1 illustrates the kind of orbit that results.

For an electron hole to sustain itself requires a substantial fraction of the particle orbits to be trapped. These orbits can then permanently possess a phase space density \( f \) less than those of untrapped orbits. Because in
a collisionless plasma \( f \) is constant along orbits, the *untrapped* orbits have phase space density corresponding to their distribution function at infinity; whereas the permanently *trapped* orbits have \( f \) determined by initial conditions: the hole formation processes, etc. The key question concerning the existence of a steady solitary electron hole equilibrium is whether there are enough permanently trapped orbits to provide a negative electron density perturbation that can sustain the potential structure self-consistently.

Isotropic multidimensional electron hole equilibria do not exist because the trapped phase space is then only orbits, which have \( W < 0 \), and in \( d \) dimensions this volume \( \alpha \phi^{d/2} \) is insufficient when \( d > 1 \) (Krasovsky et al., 2004; Ng & Bhattacharjee, 2005; Ng et al., 2006). Particle in cell simulation and drift-orbit analysis show that there exist axisymmetric 2-D equilibria, with anisotropic \( f(\mathbf{r}) \), when the magnetic field is strong enough that the gyroradius \( \rho \) is negligibly small. Essentially, this existence arises because of the adiabatic invariance of \( W_\perp \) and hence \( W_\parallel = W - W_\perp \) in the limit of small \( \rho \). The trapped phase space volume is then larger \( (\alpha \phi^{3/2}) \), requiring only \( W_\parallel < 0 \) and extending to large positive \( W_\perp \). For the intermediate case, where \( \rho \) is finite, yet transverse \( \phi \) variation exists, the challenge is this. Given that, for finite magnetic field strength, energy can be exchanged during the orbit between \( W_\parallel \) and \( W_\perp \), one can quantify whether and to what extent the amount exchanged is limited, and an orbit remains trapped in the \( z \) direction \( (W_\parallel < 0) \) even if it has so large a \( W_\perp \) that \( W > 0 \)? An earlier attempt on this problem used a more complicated treatment based on an “approximate invariant” (Krasovsky et al., 2006) but was not carried through to a full result. The present paper overcomes the challenge.

Although the calculation will remain as general as possible, we shall have in mind equipotentials that are oblate: varying faster in the parallel (\( \rho \)) direction than in the transverse (\( \theta \)) direction. For convenience we assume that the \( z \) dependence of \( E_\parallel = -\nabla_\perp \phi \) is the same as that of \( \phi \), as would be the case if \( \phi \) is of separable form \( \phi_\rho (\rho)\phi_\theta (\theta) \).

We work in units where time is measured in inverse plasma frequencies \( (\omega_\theta = \sqrt{n_\theta e^2 / m_\theta e_0}) \), length in Debye lengths \( (\lambda_\text{De} = \sqrt{e_0 T_e / e n_e}) \), and energies (and potential) in electron temperatures \( (T_e) \). Thus, if primes denote dimensioned parameters, and unprimed the normalized quantities, \( t = \omega_\theta^{-1} \; \mathbf{x} = \mathbf{x}' / \lambda_\text{De} \), and energy \( W = W'/T_e \). The parameters \( T_e \) and \( n_e \) are the temperature and density of the unperturbed electron distribution far from the hole. In these units the electron mass is eliminated from the equations, and the electron charge is \( q_e = -1 \), so the total energy of an electron can be written \( W = \frac{1}{2} v^2 - \phi \). We shall refer to the parallel energy as \( W_\parallel = \frac{1}{2} v_{\parallel}^2 - \phi \) and perpendicular as \( W_\perp = \frac{1}{2} v_{\perp}^2 = \frac{1}{2} (v_{\parallel}^2 + v_{\theta}^2) \). The magnetic field strength is represented by the (normalized) cyclotron frequency \( \Omega = \Omega' / \omega_\theta \). The equation of an electron orbit is then

\[
\frac{dv}{dt} = \nabla \phi - v \times \Omega \mathbf{\hat{z}}. \tag{1}
\]

We treat changes in magnetic moment as slow. This is justified if the transverse electric field (arising from transverse non-uniformity of \( \phi \)) is small in the sense that \( E_\perp / \phi \ll 1 / \rho \), which may also be written \( L_\perp \gg \rho \), where \( L_\perp \equiv \phi / E_\perp = \phi / (d\phi / d\mathbf{r}) \) is the transverse length scale of potential variation. Starting from the drift limit (which is essentially \( \rho / L_\perp \to 0 \)), we recognize that the orbit’s gyrocenter moves freely along \( z \) under the influence of the parallel electric field and simultaneously rotates azimuthally in \( \theta \) under the (time-varying) influence of \( \mathbf{E}_\parallel \times \mathbf{B}/B^2 \). Trapped orbits (our main focus) bounce in \( z \) and experience an effectively periodic \( E_\parallel \), as a consequence. The mean value of \( E_\parallel \) over a period determines the average azimuthal rotation. The varying component of \( E_\parallel \) is the perturbation responsible for the transfer between perpendicular and parallel energy. To first order, the fractional transfer of energy in a bounce period is small. Then during a single period, \( z(t) \) can be approximated as being given by parallel motion with fixed \( W_\parallel \), which is simply the 1D orbit problem. Moreover, \( E_\parallel (t) = E_\parallel (r, z(t)) = -d\phi / dr \), \( r = r(t) \), can be approximated as being at fixed radius \( r \) (again provided \( \rho \) is small enough relative to \( L_\perp \)). The instantaneous energy transfer rate (recalling that \( W = W_\parallel + W_\perp \) is exactly conserved) is simply the rate of doing work on the electron by \( E_\parallel \), namely,

\[
\frac{dW_\parallel}{dt} = -\frac{dW_\perp}{dt} = -E_\parallel (t) v_\parallel (t). \tag{2}
\]
The important velocity component in this equation arises from the gyromotion of the electron, \( v_r = v_\perp \cos(\Omega t) \). When this equation is integrated over many bounces and gyroperiods, large excursions in \( W_\parallel \) will occur if there is a resonance between the gyrofrequency and a harmonic of the bounce frequency \( \omega_b \). These are the orbits that are liable to lead to detrapping, because the energy transfer is consistently unidirectional (between \( W_\perp \) and \( W_\parallel \)) over many bounce periods.

Figure 2 illustrates an orbit (red) that quickly becomes detrapped and a permanently trapped orbit (blue), all as a function of parallel position. The top frame shows the track \( (r(t), z(t)) \) in the \( r-z \) plane combining gyromotion and parallel reflections (the blue orbit is omitted in this frame for clarity but has the same radial excursion). The middle and bottom frames show instead the parallel energy \( W_\parallel(z, t) \) and the bottom shows \( -\phi(z) \) and (read on the right scale) \( E_r(z) \). The detrapped orbit has two \( z \) reflections before acquiring \( W_\parallel > 0 \) at the right-hand hole extremity and leaving the hole, becoming untrapped. The trapped orbit has many more bounces with excursions in \( W_\parallel \) never reaching zero because it lies somewhat deeper in the potential well. If near-resonant orbits do not lead directly to detrapping, by raising \( W_\parallel \) above 0, like the red orbit here, then they generally take the form of “islands” in the coordinate space \( W_\parallel \) versus relative phase angle (to be explained more fully in a moment). The result then is that the orbits remain trapped. The blue orbit and essentially all in this hole with even more negative initial \( W_\parallel \) are of this type.

Since there are multiple resonances arising from the harmonics of \( \omega_b \), the orbits can become stochastic and the islands broken up. Very generally, stochasticity begins in Hamiltonian systems approximately when there is overlap between the separatrices of adjacent islands (Chirikov, 1979; Meiss, 1992). Indeed, this principle is called the Chirikov criterion in recognition of its discoverer who studied resonances between gyromotion and bounces along the magnetic field in magnetic traps (Chirikov, 1960): a close analog of our current concern. If an orbit is stochastic, it is generally not permanently trapped and in principle cannot contribute to hole sustainment. Our analytic determination of the orbit trajectories in \( W_\parallel \) disregards the radial variation of \( \phi \) and \( E_r \); an appropriate approximation for small gyroradius (but the full orbit integration, also presented, does not).

3. Islands in Energy

When discussing resonant perturbation islands in a Hamiltonian system, one generally requires an angle-like coordinate that amounts to the phase difference between the Hamiltonian orbit and the
perturbation. In the magnetized electron hole with (presumed) uniform $\Omega$, the phase difference we require is between the gyromotion (of $v_\perp$ and hence phase of $v_r$) and a perturbing electric field, which we will take as the Fourier component $E_n$ at some harmonic $n$ of the slowly varying bounce frequency: $\omega_n = n\omega_b$. The Fourier component has a fixed phase with respect to the $z$ motion, which we will take as 0 when $z = 0$. But because $\omega_b$ varies with $W_\parallel$, the bounce phase has a variable rate of change with respect to the gyrophase, whose phase we have taken as zero when $v_r = v_\perp$. We shall write the phase difference between bounce and gyromotion as $\xi$, so that

$$\frac{d\xi}{dt} = \omega_n - \Omega,$$

(3)

and seek the locus of orbit motion in the plane $\xi W_\parallel$. Orbits will then have

$$\frac{dW_\parallel}{dt} = -E_n v_\perp \cos(\omega_n t) \cos(\Omega t)$$

$$= -E_n v_\perp \frac{1}{2} [\cos(\omega_n t + \Omega t) + \cos(\omega_n - \Omega)]$$

$$\approx -\frac{1}{2} E_n v_\perp \cos \xi,$$

(4)

and we have dropped the term $\cos(\omega_n + \Omega)t$, because it is a fast oscillation, compared with the presumed slow evolution of $\xi = (\omega - \Omega)t$. We shall mention it later.

Let us suppose for initial illustrative purposes that the variation of $\omega_n$ with $W_\parallel$ can be approximated linearly as

$$\frac{d\xi}{dt} = \omega_n - \Omega = \frac{d\omega_n}{dW_\parallel} (W_\parallel - W_{1R}) = \frac{d\omega_n}{dW_\parallel} \Delta W_\parallel,$$

(5)

where $W_{1R}$ is the value of $W_\parallel$ at which exact resonance occurs ($\omega_n = \Omega$) and that we can take $E_n$ and $\frac{d\omega_n}{dW_\parallel}$ to be independent of $W_\parallel$. Then Equation 2 becomes

$$\frac{d\omega_n}{dW_\parallel} \Delta W_\parallel \frac{dW_\parallel}{d\xi} = \frac{d\omega_n}{dW_\parallel} \frac{d\Delta W_1^2}{2d\xi} = -E_n v_\perp \frac{1}{2} \cos \xi,$$

(6)

This expression can be integrated as

$$\frac{d\omega_n}{dW_\parallel} \Delta W_1^2 = -E_n v_\perp \sin \xi + C.$$  

(7)

This is the island locus, and different values of the integration constant, $C$, give rise to different trajectories, effectively different starting $W_\parallel$s. The island’s separatrix corresponds to $C = E_n v_\perp$. The $x$ point (if $\frac{dW_\parallel}{d\omega_n}$ is negative) is at $\xi = -\pi/2$ and the (maximum) half-width of the separatrix (at $\xi = \pi/2$) is then

$$|\Delta W_1| = \sqrt{2E_n v_\perp} \left| \frac{dW_\parallel}{d\omega_n} \right|.$$  

(8)

The island center is at $\xi = \pi/2$, $C = -E_n v_\perp \frac{dW_\parallel}{d\omega_n}$.

The result is most usefully plotted as contours of the constant $C$, in the $\xi - W_\parallel$ plane, which trace the trajectories of the orbits. Figure 3 shows examples from a more elaborate calculation (and will be explained more fully in section 5.1), but the blue contours in it, centered on $W_\parallel/\psi = -0.5$, have approximately the shape obtained with the simple approximations used in this introductory section.
\[ \omega_b / \sqrt{\psi} = \left[ \left( -W_i / \psi \right)^{-1/4} - 1 + 2^{1/4} \right]^{-2} / \sqrt{2} \]  

represents the dependence over the entire trapped energy range extremely well, within approximately the thickness of the line. The inverse of this expression is

\[ -W_i / \psi = \left( 2\omega_b^2 / \psi \right)^{-1/4} + 1 - 2^{1/4} \]  

The shallow \( W_i \to 0 \) limit line \( \omega_b = \sqrt{-W_i / 2} \) is indicated by the dotted line. For approximate analytic purposes (to avoid the eventual necessity to evaluate hypergeometric functions) it is adequate to adopt a more approximate form

\[ \omega_b / \sqrt{\psi} = \left( -W_i / \psi \right)^{1/2} / 2, \]  

the dash-dotted line with constant slope of 1/2 in Figure 4.

Now we must relate the bounce motion to the time harmonics of \( E_r \). First, observe that for a mirror symmetric potential such as Equation 9 the period of the variation of \( E_r = -\partial \phi / \partial r \) with \( z \) at constant \( r \) is actually \( \pi / \omega_b \), and so only even harmonics \( n \omega_b \) are nonzero. The harmonics \( n > 2 \) arise from the anharmonic motion and the resulting deviations of \( E_r(t) \) from a pure sinusoid.

Let us introduce convenient energy parameter notation involving positive values normalized to \( \psi \), and (for future use) cyclotron frequency to \( \sqrt{\psi} \) as

\[ w_i \equiv -W_i / \psi, \quad w \equiv W / \psi, \quad b \equiv \Omega / \sqrt{\psi}; \]  

so trapped orbits have \( w_i \) running from 0 to 1, and the orbits that can become untrapped have \( w > 0 \). For a
given magnetic field value $b$, and harmonic number $n$, the resonance condition is $n\omega_b/\sqrt{\psi} = b$, which gives a resonant parallel energy

$$w_{0R} = \left[ (2b/\pi)^{1/4} + 1 - 2^{1/4} \right]^{-1/4}$$

(14)

corresponding to Equation 11.

4.2. Shallow Trapped Orbits

For shallow-trapped orbits, the $E_r(t)$ has the form of a train of relatively narrow impulses of width $\sim\tau$, and period $\pi/\omega_b$, which peak briefly as the orbit passes rapidly through $z = 0$. The orbit spends most of its time near the extrema of the $z$, where the parallel electric field is very small, and this dwell duration determines the period (Hutchinson, 2019a). The total impulse in a single passage can be written $A = \int E_r dt$, and its duration is approximately the potential width divided by the peak speed $\tau_s \approx 8/\sqrt{\psi + w_{0R}}$. When $w_{0R}/\psi \rightarrow 0$, the integral $\int E_r(z) dt = \int E_{ro} \operatorname{sech}^{4}(z/4) dz/v_0(z)$ can be performed exactly and yields $A = 8E_{ro}/\sqrt{2\psi}$. The Fourier decomposition of $E_r(t)$ then gives the following Fourier mode amplitudes $E_n$, for even $n$ when $\tau_s \ll \pi/\omega_n$ (i.e., $\sqrt{-W_4}/\psi = \sqrt{w_{0R}} \lesssim \pi/4n$):

$$E_n = A \frac{2\omega_b}{\pi} \approx E_{ro} \frac{8}{\pi} \sqrt{-W_4}/\psi = E_{ro} \frac{8}{\pi} \sqrt{w_{0R}}.$$

(15)

We will refer to this as the impulse limit.

4.3. High Bounce Harmonics

An alternative perspective of the impulse limit is to note that each impulse gives an energy change $\delta w_b = -A\nu_0 = -A\nu_0 \cos \xi$, every $\delta t = \pi/\omega_b$. If $\delta \xi$ and $\delta w_b$ during a single passage through $z = 0$ are small, we may approximate the effect as an average energy rate of change

$$\frac{d\nu_0}{dt} = \frac{\delta w_b}{\delta t} = -\frac{A\nu_0 \omega_b}{\pi} \cos \xi,$$

(16)

in agreement with Equations 15 and 4.

However, if $\Omega \gg \omega_b$, so that only high harmonics of $\omega_b$ are resonant, the continuum limit is inappropriate. Moreover, it will always be the case that $\Omega \gg \omega_n$ near the trapping boundary, $w_b \rightarrow 0$, because $\omega_b \rightarrow 0$ there.

When there are many cyclotron periods during one bounce period, but the cyclotron period is still long compared with the impulse duration, $\tau_s$ (which does not itself become significantly longer as $w_b \rightarrow 0$), the cyclotron phase ($\xi$) at which each succeeding impulse occurs becomes effectively random relative to the previous impulse. So, rather than a systematic continuous flight in the $(\xi, w_0)$ space, the evolution consists of steps of virtually random amplitude $\delta w_b \cos \xi$ cosine-distributed between $\pm A\nu_0$. This represents an effective diffusion in $w_b$ with a diffusion coefficient $\sim (A\nu_0)^2 \omega_b/\pi$. Moreover, for passing particles, which are addressed in a recent paper (Vasko et al., 2018) concerning the scattering of passing particles by successive encounters with different electron holes, one similarly arrives at velocity space diffusion. The diffusion connects the trapped orbit region $w_b < 0$ with the untrapped region $w_b > 0$ across the nominal phase space separatrix $w_b = 0$, with the result that the distribution function in this region has only limited gradient $|df/dw_b|$, and a value approximately equal to the external distribution $f_{ex}$ at $v_b = 0$ (in the frame of reference of the hole). This is one crucial constraint on possible electron hole equilibria.

4.4. Deeply Trapped Orbits

Orbits that are deeply trapped, having $-W_4/\psi$ a significant fraction of unity, are not accurately described by the impulse approximation of the previous section. Instead of being strongly anharmonic, the $\phi(z)$ is approximately parabolic for them, and their orbit’s $z$ position varies approximately sinusoidally in time. In the limit $w_b \rightarrow 1$, only the lowest Fourier mode, $n = 2$ is important and the higher harmonics become negligible. Moreover, $E_r$ variation depends on the orbit’s $z$ excursion size; so even for the lowest relevant harmonic $\omega_n = 2\omega_b$, the electric field Fourier amplitude $E_n = E_2$ can become small.
The Taylor expansion of the potential \( \phi(z) = \psi \text{sech}^4(z/4) \) about \( z = 0 \) is
\( \phi(z) \approx \psi [1 - z^2/8 + 7z^4/768 + O(z^6)] \),
which leads to the sinusoidal bounce frequency \( \omega_b = \sqrt{\psi}/2 \) when \( z^2 \) and higher terms are dropped.
The presumed similar radial electric field likewise has \( E_r(z) \approx E_{\psi0}(1 - z^2/8) \), of which the time-varying part is only the second term. For given parallel energy, \( w_b \) the amplitude \( z_{\text{max}} \) of the \( z \) oscillation satisfies \( z_{\text{max}}^2/8 = (1 - w_b) \), and
\[ E_r(t) = E_{\psi0}[1 - z_{\text{max}}^2 \sin^2(\omega_b t)/8] = E_{\psi0}[1 - z_{\text{max}}^2/16] + z_{\text{max}}^2 \cos(2\omega_b t) \]
then yields
\[ E_2 = z_{\text{max}}^2 E_{\psi0}/16 = E_{\psi0}(1 - w_b)/2. \] (17)
This dependence on \( 1 - w_b \) replaces the \( \sqrt{-w_b} \) dependence of Equation 15.

### 4.5. Interpolated \( E_n \) Expression

It is helpful to have an approximate analytic interpolation for the Fourier harmonics \( E_n \) that spans the entire range \( 0 < w_b < 1 \). Observe that
\[ 1 - w_b = (1 - \sqrt{w_b})/(1 + \sqrt{w_b}) \]
so an alternative expression to Equation 17, which is equally valid in the limit \( w_b \to 1 \), is
\[ E_2 = E_{\psi0}(1 - \sqrt{w_b}) \]
Realize also that the higher harmonics, \( n = 4, 6, \ldots, \) arise from correspondingly higher-order terms in the Taylor expansion of \( \phi(z) \) and that, therefore, as \( w_b \to 1 \), \( E_n \) will become proportional to correspondingly higher powers: \((1 - \sqrt{w_b})^n/2^n \). Consider then the following proposed interpolation between the two limits of \( w_b \):
\[ E_n = E_{\psi0}\left[ n/2 \left(1 - \sqrt{w_b}\right)^m + \frac{\pi}{8} \frac{1}{\sqrt{w_b}} \right]^{-1}, \]
(18)
where for most purposes \( m = n/2 \). The first term predominates as \( w_b \to 1 \), and the second as \( w_b \to 0 \). In their respective limits, these two terms give the correct values for \( E_2 \), in agreement with Equations 15 and 17. In the \( w_b \to 1 \) limit, the higher harmonics have appropriate scaling with \( 1 - \sqrt{w_b} \).
Their numerator \( n/2 \) has not been derived; and, more crucially, neither has the inverse form of the interpolation. Nevertheless, a comparison between this expression and numerical calculation of the Fourier harmonics shows quite good agreement, as can be seen in Figure 5. This agreement is sufficient for many purposes, but some moderate discrepancies remain especially at high \( n \). They are significantly reduced if an ad hoc adjustment is made by substituting
\[ m = \text{n int}(n/2 + \max(n/2 - 3.3, 0) + 0.75) \]
(19)
(instead of \( m = n/2 \)) into Equation 18. The adjustment benefits from retaining convenient integrability.

### 5. Solving the \( w_{||} \) Trajectories

#### 5.1. Analytic Calculation

Using the approximate expression (12) for \( \omega_b \) giving \( \omega_n = (n/2)\sqrt{\psi w_{||}} \), the energy trajectory Equation 4 ignoring the fast \( \omega_n + \Omega \) term becomes
\[
\frac{dw_{||}}{dt} = (\omega_n - \Omega) \frac{dw_{\perp}}{d\xi} = \frac{n\sqrt{\psi}}{2} (\sqrt{w_{||}} - \sqrt{w_{||R}}) \frac{dw_{\perp}}{d\xi}
= \frac{1}{2}(E_n/\psi)v_{\perp} \cos \xi,
\]
(20)
where \( w_{||R} \) is the resonant parallel energy at which \( \omega_n = \Omega \). Substituting the interpolation for \( E_n \) from Equation 18, and \( v_{\perp} = \sqrt{2(\psi + w_b)/\psi} \), it can be written

![Figure 5. Comparison between numerically integrated Fourier coefficients for a sech\(^4\) potential variation (solid lines) and the interpolation Equation 18 with \( m = n/2 \) (dotted) or using Equation 19 (dashed lines), for different harmonic number (line labels).](image-url)
This equation can be integrated analytically in terms of elementary functions to obtain

$$F_n(w_l, w, w_{lr}) = (E_{r0}/\psi) \sin \xi = \text{const.},$$

where for each $n = 2, 4, 6, \ldots$, $F_n$ is a fairly complicated algebraic expression detailed in the appendix. For chosen total energy, magnetic field strength, and perturbing field (i.e., $w, w_{lr}$ and $E_{r0}/\psi$) the trajectories can most easily be plotted as contours of the left-hand side expression, $F_n - (E_{r0}/\psi) \sin \xi$, in the plane ($\xi, w_l$). In these calculations it improves accuracy to use the more accurate Equation 14 for $w_{lr}$ in terms of $b$, in $F_n$; and we adopt this practice forthwith, ignoring the minor inconsistency.

Figure 3 are shown examples of the energy trajectories for $w = 1, E_r/\psi = 0.01 \psi = 1$, and three values of the magnetic field strength, and hence of the resonance energy $w_{lr}$ for the lowest harmonic $n = 2$. The perturbing field is quite strong, and we can see that the trajectories near the top or bottom of the potential energy well (i.e., near to $-w_l = 0$ or $-1$) are compressed asymmetrically at those limits because of the form of $F_n$. For an energy away from those limits, the contours are approximately symmetric about the resonant energy.

In Figure 6a instead shows trajectories for fixed magnetic field, and hence fixed $n = 2$ resonance frequency, $b = \Omega/\sqrt{\psi} = \sqrt{0.8}, E_{r0}/\psi = 0.005$, but for harmonic numbers $n = 2, 4, 6, 8, \ldots$. The resonance energy is $w_{lr} = [(2b^2/n^2)^{-1/4} + 1 - 2^{1/4}]/4$. The higher harmonics bunch together near the top of the potential energy well, corresponding to low bounce frequency. And in fact the $n = 6$ and $n = 8$, islands overlap, indicating that this region of energy has stochastic orbits, and so the orbits there are not permanently trapped. Lower in the well, no overlap occurs with the $n = 2$ island, so orbits there are permanently trapped.

### 5.2. Numerical Orbits: Poincaré Plots

In order to verify the analytic calculation and to show what happens when its applicable parameter limits are exceeded, it is helpful to perform a numerical integration of the trapped orbits. The full (nonrelativistic) equations of motion for the model potential have been implemented in cylindrical coordinates using a fourth-order Runge-Kutta numerical scheme with time step chosen short enough that the (known)
conservation of $W_\parallel$ and $p_\theta$ are reproduced for long orbits to no worse than 10 times machine precision. This is observed to require $\Omega dt \lesssim 0.05$. Figures 1 and 2 are examples of orbits so calculated.

Poincaré plots of the energy trajectories for such orbits are obtained by collecting values of $W_\parallel$ and the phase of $v_r$ (i.e., $\text{atan2}(v_\theta, v_r)$) at successive instants when the orbit passes through $z = 0$ (at which the orbit bounce phase is 0 or $\pi$ and the phase of $E_n$ is 0 for all even $n$). The phase difference, $\xi$, thus equals the phase of $v_r$. We place a point at each of the corresponding positions in $\xi, W_\parallel$ space. We also, for convenience, start all orbits at $z = 0$ and with $v_\theta = 0$, $v_r$ positive: $\xi = 0$. We abandon as escaped any orbits that acquire positive $W_\parallel$ or pass beyond $|z| = 20$. A technical subtlety is that it is most appropriate to use for $W_\parallel = v_r^2/2 - \phi$, not the value of $\phi$ at the orbit, but rather the value of $\phi$ at the gyrocenter, which gives significantly smaller oscillatory excursions of $W_\parallel$. It therefore more effectively suppresses the $\omega_n + \Omega$ term and expresses the approximate magnetic moment conservation.

Figure 6b shows an example of a Poincaré plot, alongside its analytic energy trajectories (Figure 6a). The agreement is excellent. Orbits are initialized at equally spaced $W_\parallel$ values. Of course, they cannot trace island contours well inside their separatrices where the island does not extend past $\xi = 0$. The position and $W_\parallel$ width of the $n = 2$ island agree quantitatively very well between (a) and (b). And the $n = 4$ and $n = 6$ islands are also readily seen at their expected positions. Between the islands, the Poincaré points trace the open contours. Above the position of the $n = 6$ island ($W_\parallel/\psi \geq -0.05$) and near its $x$ point the plot shows rather incoherent scatter of the points. Orbits above this energy are stochastic and terminate after some tens of bounces by leaving the domain. Again, this agrees well with the analytic observation of overlap between $n = 6$ and 8, but not between $n = 4$ and 6 islands.

Figure 7, by comparison, shows what happens if the amplitude of the perturbing transverse field is increased by a factor of 8, other parameters unchanged. The $n = 4$ and 6 islands now overlap strongly, and the entire region $W_\parallel/\psi \geq -0.3$ becomes stochastic. Below it, the Poincaré plots show orbits to be permanently trapped. Small island chains with higher mode numbers in phase $\xi$ become visible. For example, the chain of three islands at $W_\parallel/\psi \simeq -0.45$, or of two islands at $W_\parallel/\psi \simeq -0.34$. These additional chains arise from nonlinearity and are not represented in the analytic linearized approximation. Still, the overall extent of the $n = 2$ island is quite well captured by the analytics, which predict that it should remain intact, as it does. If the perturbing $E_r/\psi$ is increased to 0.1, then overlap and stochasticization of even the $n = 2$ island begins, as illustrated by Figure 8. Soon beyond it, by $E_r/\psi = 0.13$, essentially the whole of the phase space becomes stochastic.

Figure 9 shows what happens for a lower magnetic field, $\Omega/\sqrt{\psi} = 0.6$. In this case, a field $E_r/\psi = 0.04$ is sufficient to make the $n = 2$ island stochastic, but when that happens, there still remain some permanently trapped orbits at energies sufficiently below the resonance value ($\sim 0.6^2$).
In contrast, as shown in Figure 10, increasing the magnetic field to $\Omega/\sqrt{\psi} = 1.8$, removes the $n = 2$ resonance, and because the higher resonances are weaker, the orbits can sustain higher $E_r/\psi$ before becoming stochastic. This stabilizing effect is enhanced by the resulting reduction in gyroradius $\rho$.

6. Island Widths, Overlap, and Trapped Phase Space

In the previous section we have shown that the island overlap criterion successfully predicts which trajectories are stochastic (and hence become untrapped) and which are permanently trapped. We therefore rely on this success and formulate an analytic condition for particles at different locations in phase space to be permanently trapped. We will take those parallel energies $W_\parallel$ to be trapped which lie below the bottom of the lowest overlapped island and all others to be subject to detrapping. This criterion describes within typically 10% in $W_\parallel$ what has been observed in the example cases we have shown.

The function $F_n$, for fixed $w$ and $w_{\parallel R}$, is stationary at resonance ($\sqrt{w_\parallel} = \sqrt{w_{\parallel R}}$), and its derivative in the vicinity of the resonance can be taken from Equation 21 as

$$\frac{\partial F_n}{\partial \sqrt{w_\parallel}} = 2\sqrt{w_\parallel} \frac{\partial F_n}{\partial w_\parallel}$$

$$= \frac{n(\sqrt{w_\parallel} - \sqrt{w_{\parallel R}})}{2\sqrt{w + w_\parallel}} \left[ \frac{\sqrt{w_\parallel} \, n/2}{(1 - \sqrt{w_\parallel})^m} + \frac{\pi}{8} \right].$$

Consequently, the width of the island separatrix, which occurs at $\xi = \pi/2$, is determined by the $\sqrt{w_\parallel}$ value for which $F_n(w_\parallel) - F_n(w_\parallel = w_{\parallel R}) \approx \frac{1}{2} (\sqrt{w_\parallel} - \sqrt{w_{\parallel R}})^2 \frac{\partial^2 F_n}{\partial \sqrt{w_\parallel}^2}$ is equal to $E_{\parallel R}/\psi$. Therefore, regarding the second derivative as constant (adopting just the second-order term in a Taylor expansion of $F_n$), we can express the island (half-)width as

$$\delta_n \equiv \sqrt{w_\parallel} - \sqrt{w_{\parallel R}}$$

$$\approx \left[ \frac{E_{\parallel R}2\sqrt{2}}{\psi \, n} \left[ \frac{\sqrt{w_{\parallel R}} \, n/2}{(1 - \sqrt{w_{\parallel R}})^m} + \frac{\pi}{8} \right] \right]^{1/2}.$$  \hspace{1cm} (24)

We write $w + w_{\parallel R} = w_\perp (\sqrt{2} w_\perp = v_\perp/\sqrt{\psi})$ and recognize that together the parameters $n, E_{\parallel R}v_\perp/\psi^{3/2}$, and $w_{\parallel R}$ determine $\delta_n$ as follows.
Analytic Algorithm

Equation 14 \( w^\parallel_{Rn} = \left[ \frac{n^2/2b^2}{b^2} + 1 - 2^{1/4} \right]^{-4} \) enables us to find the energy of the upper and lower island limits of island \( n \) as

\[
\sqrt{w_{Rn}^\parallel} \pm \delta_n = \sqrt{w_{Rn}^\parallel} \pm \left[ \frac{E_{\psi n/2}}{\sqrt{\psi_n}} \right]^{1/2} \left[ \frac{\sqrt{w_{Rn}^\parallel n/2}}{(1 - \sqrt{w_{Rn}^\parallel})^m + \pi} \right]^{-1/2}.
\]  

(25)

It is in this equation that one must use the adjustment of \( m \) of Equation 19 for high harmonics. Overlap occurs between the \( n \) and \( n + 2 \) harmonic islands when \( \sqrt{w_{Rn}^\parallel} < \sqrt{w_{Rn+2}^\parallel + \delta_{n+2}} \). Beginning at the lowest value of \( n \) for which a resonance exists (requiring \( w_{Rn}^\parallel < 1 \)) determine from evaluation of \( \sqrt{w_{Rn}^\parallel} - \delta_n \) and \( \sqrt{w_{Rn+2}^\parallel + \delta_{n+2}} \) whether it overlaps with the \( n + 2 \) island. If so, then it is the lowest energy overlapped island; if not, increment \( n \) by 2 and repeat until overlap is found. The resulting \( n \) is the harmonic whose island’s lower energy limit is sought, which is

\[
W_{lt}/\psi = -w_{lt} = -\left( \sqrt{w_{Rn}^\parallel + \delta_n} \right)^2.
\]

(26)
Energies below this approximate bound are predicted trapped, energies above have stochastic orbits and are detrapped.

Figure 11a shows the universal contours that result. Where \( W_{\parallel t}/\psi \) is close to 0 (light regions), very few orbits are detrapped; while where \( W_{\parallel t}/\psi \) is close to \(-1\) (dark regions) almost all orbits are detrapped. Discontinuities in \( W_{\parallel t} \) occur where \( n \) changes: It starts at 2 at the bottom (right, below \( \Omega = \sqrt[4]{\psi} \approx 1 \)) and increments through 4, 6, ..., as one moves to larger \( \Omega = \sqrt[4]{\psi} \). To avoid almost complete detrapping for \( \Omega \approx 1 \), extremely weak perturbation is required. In contrast, for \( \Omega \approx 2 \) there is a substantial region of permanently trapped orbits even up to the largest perturbation strength shown.

In Figure 11b are shown vertical profiles through the contours at four values of the perturbation strength, giving \( W_{\parallel t} \) as a function of \( b \). These lines are each accompanied by points, each of which comes from full numerical orbit integration. A point gives the lowest starting energy that escapes during the first 200 bounces (which might take as many as a million time steps). We observe that there is very good agreement (even in respect of the discontinuities) between the points and the lines.

A more approximate form of the island widths can be obtained by using \( m = n/2 \), substituting the more approximate frequency fit \( \omega_{Rt} = 4\Omega^2/n^2\psi = 4b^2/n^2 \) so that \( \sqrt{W_{Rt} - \sqrt{W_{Rt} + 2}} \approx 4b/n^2 \), and approximating
\[
(1 - \sqrt{\frac{\psi}{\Omega}})^{n/2} = (1 - 2b/n)^{n/2} \approx e^{-b}. \]

Then we find \(\delta_n\) is approximately proportional to \(1/\sqrt{n}\) and can be written

\[
\delta_n \approx \left[ \frac{E_0}{\psi} \frac{2\sqrt{\psi}}{n} \right]^{1/2} \left[ be^b + \frac{\pi}{8} \right]^{-1/2}. \tag{27}
\]

With reference to this approximation, the behavior can readily be understood as follows. Island overlap \(2\delta_n \gtrsim \sqrt{\epsilon} - \sqrt{\epsilon + 2}\) leading to stochastic trajectories occurs if \(\delta_n\) is too large, that is, if \(E_0/\psi\) is too large provided \(b (= \Omega/\sqrt{\psi})\) is not large; or else if \(n\) is too large, making \(\sqrt{\epsilon} - \sqrt{\epsilon + 2}\) too small. The last of these cases (high \(n\) at modest \(E_0\) and \(b\)) predicts that there is in principle always a stochastic region at very small \(\sqrt{\psi}\), where the bounce frequency is correspondingly small and the resonant bounce harmonic number large, regardless of the exact \(E_0\) and \(b\) values. Consequently, a steady electron hole of

**Figure 11.** (a) Contours of the energy boundary \(W_{\|}\) between trapped and detrapped orbits as a function of perturbation strength \(E_0\sqrt{\psi}/\sqrt{\psi}\) and magnetic field \(\Omega/\sqrt{\psi}\). (b) The energy boundary \(W_{\|}\) between trapped and detrapped orbits from Equation 26 compared with the lowest detrapped orbits found from numerical orbit integration.

**Figure 12.** The boundary in velocity space measured at \(z = 0\) between trapped and untrapped orbits at different \(\psi\) values: (a) for low magnetic field \(\Omega = 0.5\); (b) for higher \(\Omega = 1.2\).
limited transverse extent will always have a stochastic transition between trapped and passing orbits that in practice smooths out any steep $f$ gradients at the separatrix. Our numerical orbit integration confirms this prediction.

When $b$ ($= \Omega/\sqrt{\psi}$) is large, the term $e^{b}$ makes $\delta_n$ small, regardless of $E_{\ell n}$, and suppresses overlap. This effect can be considered to arise because when the gyroperiod is small compared with the central transit time ($t_t \propto 1/\sqrt{\psi}$, the duration of the impulse), the Fourier transform of a single impulse has become exponentially small at the cyclotron frequency. The suppression applies at essentially all $w_0$ up to 1, because the impulse width is a rather weak (slowly increasing) function of $w_0$. Only the exponentially large $n$ orbits at exponentially small $w_0$ will then be stochastic. And the region of stochasticity is limited to very small $w_0$. High enough magnetic field thus justifies the drift orbit treatment and eventually imposes no minimum $L_{\perp}$ requirement for a long-lived hole to exist.

The opposite case $b \ll 1$ (weak magnetic field) preserves the assumed localization in $r$ only if the transverse length scale remains greater than the gyroradius $E_{\ell 0}/\psi = 1/L_{\perp} \leq 1/\rho = \Omega/v_\perp = b\sqrt{\psi}/v_\perp$ so $E_{\ell 0}v_\perp/\psi^{3/2} \leq \Omega/\sqrt{\psi}$. The valid region of Figure 11a is therefore above the diagonal straight line $E_{\ell 0}v_\perp/\psi^{3/2} = \Omega/\sqrt{\psi}$ drawn in purple. And in Figure 11b, the lines are drawn only in the valid region. In the invalid region one can expect the permanent trapping to be poor, and this is confirmed by the points.

A perhaps more intuitive way to portray typical results is as in Figure 12, where are shown examples of boundaries between trapped and untrapped orbits in velocity space (based on the island overlap calculation). The important regions of this domain extend to thermal velocities ($v_\perp \geq 1$, not just $v_\perp \geq \sqrt{\psi}$). We need orbits to be permanently trapped for most of the range of possible $v_\perp$, to allow the depression of $f(v_\parallel, v_\perp)$ to contribute sufficient positive charge to sustain the hole. For smaller $\psi$ the effective perturbation strength $\propto v_\perp/\sqrt{\psi}L_{\perp}$ becomes stronger for given $L_{\perp}$, which makes orbits more easily detrapped. However, the effects of varying resonance condition as $\psi$ changes are very strong, so the boundaries do not behave monotonically with $\psi$. When the $n = 2$ resonance is avoided, as in Figure 12b, the boundary lies at fairly high velocity near $W_0 = 0$. That leads us to expect qualitatively that a distribution $f(v_\parallel, v_\perp)$ that is approximately flat above $v_{\text{tan}}$ in the stochastic region, can still sustain an electron hole with these parameters.

In all cases, increasing $L_{\perp}$ and making the hole more oblate, that is, closer to one-dimensional reduces the detrapped phase space area. But unless $\Omega/\sqrt{\psi} \gtrsim 2$, holes of large transverse dimension are unstable to transverse perturbations that grow in a few hundred plasma periods and break up the holes into shorter transverse lengths, causing them to collapse. So there is a competition between the requirements of equilibrium and stability.

### 7. Summary

It has been shown that parallel energies of deeply trapped orbits in axisymmetric electron holes have limited excursions in parallel energy, provided the transverse electric field perturbation is weak enough. There is a parallel energy threshold which is a function of perturbation strength and magnetic field, above which the parallel energy trajectory becomes stochastic, and is no longer limited in extent, instead becoming detrapped. Such orbits cannot therefore contribute to the electron deficit needed to sustain the hole. The stochasticity arises when trajectory islands overlap, as has been confirmed by numerical orbit integration. The parallel energy threshold for detrapping has been quantitatively evaluated using the Analytic Algorithm as a universal function of the hole parameters. Magnetic fields strong enough that $\Omega/\sqrt{\psi} \gtrsim 2$ allow a large fraction of the orbits with negative parallel energy to be permanently trapped, even for quite short transverse scale lengths. However, lower magnetic field strengths $\Omega/\sqrt{\psi} \leq 1$ have most of their orbits detrapped unless the transverse scale length is rather large. Although fully self-consistent hole equilibria have not yet been calculated, the present results appear to give an explanation based upon equilibrium trapping constraints for the observation that holes with lower magnetic field and lower peak potential generally must have greater transverse extent than those with greater field or greater potential. Future work will aim to use the quantitative results of this trapped-phase-space calculation, illustrated in Figures 11 and 12, to explore when fully self-consistent 2-D holes can exist and what their forms are likely to be.
Appendix A: Mathematical Function Details

The integrated expressions for \( F_n \) are as follows

\[
F_n = \frac{n^2}{2\sqrt{2}} g_n + \frac{n\pi}{16\sqrt{2}} g_0 \tag{A1}
\]

with

\[
g_0 = \int \frac{(\sqrt{w_1} - \sqrt{w_{ij}})dw_i}{\sqrt{w + w_1 \sqrt{w_1}}} \tag{A2}
\]

\[
g_m = \int \frac{(\sqrt{w_1} - \sqrt{w_{ij}})dw_i}{\sqrt{w + w_1 (1 - \sqrt{w_j})}} \tag{A3}
\]

The first function is easy: \( g_0 = 2[\sqrt{w + w_1} - w\sqrt{w_{ij}} \ln(\sqrt{w + w_1} + \sqrt{w_1})] \). To evaluate \( g_m \), define the integrals

\[
I_m(a, x) = \int \frac{dx}{\sqrt{a + x^2(1 - x)^m}} \tag{A4}
\]

\[
J_m(a, x) = \int \frac{x dx}{\sqrt{a + x^2(1 - x)^m}} \tag{A5}
\]

then, since \( x^2 = (x-1)x + x \), it is easy to show that

\[
g_m = 2[(1 - \sqrt{w_{ij}})J_m(w, \sqrt{w_{ij}}) - J_m(w, \sqrt{w_1})]. \tag{A6}
\]

The \( J_m \) and \( I_m \) are related by

\[
J_m(a, x) = \int \frac{(x-1) + 1}{\sqrt{a + x^2(1 - x)^m}} dx = I_m - I_{m-1}. \tag{A7}
\]

Also, \( J_m \) can be integrated by parts as

\[
J_m = \frac{\sqrt{a + x^2}}{(1 - x)^m} - m\left[\frac{(1 - x)^2 - 2(1 - x) + 1 + a}{\sqrt{a + x^2(1 - x)^m + 1}} \right] dx \tag{A8}
\]

Eliminating \( J_m \) between A7 and A8 and gathering terms we obtain the following recursion relation:

\[
I_{m+1} = \left[\frac{\sqrt{a + x^2}}{(1 - x)^m} + (2m - 1)I_m - (m - 1)I_{m-1}\right] \frac{1}{m(a + 1)}. \tag{A9}
\]

Given \( J_0 = \sqrt{a + x^2} \), and the initial values of the recursion: \( I_0 = \ln(\sqrt{a + x^2} + x) \), and \( I_1 = \left[\ln(\sqrt{a + 1}\sqrt{a + x^2 + a + x}) - \ln(1 - x)\right]/\sqrt{a + x^2} \), we can efficiently obtain by iteration \( I_m \) and \( J_m \) for \( m \) as high as required. This iterative scheme has been implemented and verified and is used to give the island plots in this paper, which use simply \( m = n/2 \).

Data Availability Statement

The codes that were used to do the calculations and create the figures in this article are publically available as doi:10.5281/zenodo.3746740 at https://zenodo.org/record/3746740 or at https://github.com/ihutch/AxisymOrbits website.
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