ON THE BOGOMOLOV-GIESEKER INEQUALITY IN
POSITIVE CHARACTERISTIC

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ABSTRACT. We prove a version of the Bogomolov-Gieseker inequality on smooth projective surfaces of general type in positive characteristic, which is stronger than the result by Langer when the ranks of vector bundles are sufficiently large. Our inequality enables us to construct Bridgeland stability conditions with full support property on all smooth projective surfaces in positive characteristic. We also prove the BG type inequality for higher dimensional varieties.

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1. INTRODUCTION

In the theory of algebraic surfaces in characteristic zero, the Bogomolov-Gieseker (BG) inequality \cite{Bog78, Gie79} is a powerful tool, which is the inequality for the Chern characters of slope semistable vector bundles:

\[ \Delta(E) = \chi_1(E)^2 - 2 \chi_0(E) \chi_2(E) \geq 0. \tag{1.1} \]

Among others, one of the important consequences of the BG inequality is the existence of Bridgeland stability conditions on surfaces \cite{ABIC13, Bri07, Bri08}. The theory of Bridgeland stability conditions on surfaces has been applied to various classical problems in algebraic geometry such as birational geometry \cite{ABCH13, BMW14, CH14, CH15, CH16, CHW17, LZ16, LZ18}, Brill-Noether problem \cite{Bay18, BL17}, higher rank Clifford indices \cite{FL18, Li19, Kos20a}, and so on.

In positive characteristic, it is known that the BG inequality (1.1) does not hold in general (cf. \cite{Muk13, Ray78}). When a surface \( S \) has Kodaira dimension \( \kappa(S) \leq 1 \) and it is not quasi-elliptic, Langer \cite{Lan16} proves that the inequality (1.1) holds. For surfaces of general type, although Langer \cite{Lan04, Lan15} proves that the inequality with a modified term, it is still a mysterious problem what the best possible form of the inequality is. In particular,
the result in [Lan04, Lan15] is not enough to construct Bridgeland stability conditions on surfaces in positive characteristic.

In this paper, we investigate the improvement of Langer’s results [Lan04, Lan15] for general type surfaces. The following is our main result in this paper:

**Theorem 1.1** (Theorem 3.5). Let $S$ be a smooth projective surface defined over an algebraically closed field of positive characteristic. Then there exists a constant $C_S \geq 0$, depending only on a birational equivalence class of $S$, satisfying the following condition: For every numerically non-trivial nef divisor $H$ on $S$ and $\mu_H$-semistable torsion free sheaf $E \in \text{Coh}(S)$ on $S$, the inequality

$$\Delta(E) + C_S \text{ch}_0(E)^2 \geq 0.$$

holds. Explicitly, we can take the constant $C_S$ as follows:

1. When $S$ is a minimal surface of general type, then $C_S = 2 + 5K_S^2 - \chi(\mathcal{O}_S)$.
2. When $\kappa(S) = 1$ and $S$ is quasi-elliptic, then $C_S = 2 - \chi(\mathcal{O}_S)$.
3. Otherwise, $C_S = 0$.

The above theorem is strong enough to construct Bridgeland stability conditions on an arbitrary surface in positive characteristic:

**Theorem 1.2** (Theorem 6.6). Let $S$ be a smooth projective surface defined over an algebraically closed field of positive characteristic. Then there exist Bridgeland stability conditions on $D^b(S)$ satisfying a full support property.

We also show a higher dimensional version of Theorem 1.1:

**Theorem 1.3** (Theorem 4.3). Let $X$ be a smooth projective variety of dimension $n \geq 2$, defined over an algebraically closed field of positive characteristic, and $H$ a very ample divisor on $X$. Let $H_1, \cdots, H_{n-2} \in |H|$ be general hyperplanes and put $S := H_1 \cap \cdots \cap H_{n-2}$. Define a constant $C_{X,H} := C_S$ as in Theorem 1.1. Then for every $\mu_H$-semistable torsion free sheaf $E \in \text{Coh}(X)$, we have

$$H^{n-2} \Delta(E) + C_{X,H} \text{ch}_0(E)^2 \geq 0.$$

Specializing to hypersurfaces in the projective spaces, we have the following better result:

**Theorem 1.4** (Theorem 5.3). Let $k$ be an algebraically closed field of positive characteristic. Let $S^d_d \subset \mathbb{P}^{n+1}_k$ be a smooth hypersurface of degree $d \geq 1$, dimension $n \geq 2$. Denote by $H$ the restriction of the hyperplane class on $\mathbb{P}^{n+1}$. Then for every $\mu_H$-semistable torsion free sheaf $E \in \text{Coh}(S^d_d)$, we have

$$\overline{\Delta}_H(E) := (H^{n-1} \text{ch}_1(E))^2 - 2H^n \text{ch}_0(E)H^{n-2} \text{ch}_2(E) \geq 0.$$

In particular, if $\text{Pic}(S^d_d) = \mathbb{Z}[H]$, then we have the usual BG inequality $H^{n-2} \Delta(E) \geq 0$. 
1.1. Idea of proof. One big part of the proof is the argument similar to the one in [Lan04], where he proves the BG inequality in positive characteristic, with the modified term depending on $\text{ch}_0(E)^4$. In addition to his arguments, the key fact we use is the “invariance” of the BG inequality under blow-ups and change of polarizations, discussed in [BLMS17, Lan16]. According to these invariance, we are able to proceed by induction on $\text{ch}_0(E)$ in the birational equivalence class of a surface, rather than the whole category of surfaces. As a result, we obtain a quadratic inequality as in Theorem 1.2 which is necessary for the construction of Bridgeland stability conditions in Theorem 1.2. It would be worth mentioning that our proof simplifies Langer’s original proof so that we do not need the careful analysis of slope semistability under the Frobenius pull-backs.

For higher dimension, to ensure the well-definedness of the constant $C_{X,H}$ in Theorem 1.3, we use a result from the minimal model program for three-folds in positive characteristic.

For Theorem 1.4, besides Langer’s induction argument, the key input is the BG inequality for surfaces $S^d_2$, which is proved in [Kos20b] using the theory of tilt-stability conditions on the derived categories.

1.2. Relation with the existing works. Theorem 1.1 was previously known when the Kodaira dimension $\kappa \leq 1$, except quasi-elliptic surfaces ([Lan16]). Our main contribution is the case of $\kappa = 2$. See [Sun19] for the case of product type varieties. Note also that Shepherd-Barron [SB91] obtains a similar result for rank two bundles on surfaces.

In [BLMS17], the authors consider a variant of Langer’s induction argument in the category of codimension two blow-ups of a given variety. The present paper is inspired by them.

The construction of Bridgeland stability conditions on surfaces in positive characteristic has been an open problem since its appearance.

1.3. Plan of the paper. This paper is organized as follows. In Section 2 we recall the notion of slope stability, and the result from the minimal model program. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.3. In Section 5 we discuss the BG inequality for hypersurfaces in the projective spaces, and prove Theorem 1.4. In Section 6 we recall the notion of Bridgeland stability conditions and prove Theorem 1.2.

Acknowledgement. The author would like to thank Professor Arend Bayer for useful discussions and comments, and Professor Adrian Langer for insightful comments. The author would also like to thank Masaru Nagaoka, Kenta Sato, and Professor Hiromu Tanaka for discussions related to MMP. In particular, the author learned the proof of Theorem 2.4 from Professor Hiromu Tanaka. The author was supported by ERC Consolidator grant WallCrossAG, no. 819864.

Notation and Convention. Throughout the paper, we work over an algebraically closed field of characteristic $p > 0$. We use the following notations:

- $\text{Coh}(X)$: the category of coherent sheaves on a variety $X$.
- $D^b(X) := D^b(\text{Coh}(X))$: the bounded derived category of coherent sheaves.
\[
\chi^B = (\chi^B_0, \chi^B_1, \cdots, \chi^B_n) := e^{-B}. \chi: \text{the } B\text{-twisted Chern character for an } \mathbb{R}\text{-divisor } B.
\]

2. Preliminaries

2.1. Notations on slope stability. Let \(X\) be a smooth projective variety of dimension \(n \geq 1\), defined over an algebraically closed field \(k\) of characteristic \(\text{char}(k) = p > 0\). We introduce the basic notions used in this paper:

**Definition 2.1.** Let \(D_1, \cdots, D_{n-1}\) be a collection of nef divisors such that the 1-cycle \(D_1 \cdots D_{n-1}\) is numerically non-trivial. For a torsion free sheaf \(E \in \text{Coh}(X)\), we define the \((D_1 \cdots D_{n-1})\)-slope of \(E\) as

\[
\mu_{D_1 \cdots D_{n-1}}(E) := \frac{D_1 \cdots D_{n-1} \chi_1(E)}{\chi_0(E)}.
\]

We then define the notion of \(\mu_{D_1 \cdots D_{n-1}}\)-(semi)stability (or \((D_1 \cdots D_{n-1})\)-(semi)stability) in the usual way. When we have \(H = D_1 = \cdots = D_{n-1}\), we also call it as \(\mu_H\)-stability.

**Definition 2.2.** For a coherent sheaf \(E \in \text{Coh}(X)\), we define the discriminant of \(E\) as follows:

\[
\Delta(E) := \chi_1(E)^2 - 2 \chi_0(E) \chi_2(E).
\]

For an ample divisor \(H\) on \(X\), we also define the \(H\)-discriminant as

\[
\Delta_H(E) := (H^{n-1} \chi_1(E))^2 - 2H^n \chi_0(E)H^{n-2} \chi_2(E).
\]

2.2. A result from minimal model program.

**Definition 2.3.** We say a morphism \(f: X \to Z\) is a projective contraction if \(X\) and \(Z\) are quasi-projective varieties and \(f\) is a projective morphism satisfying \(f_*\mathcal{O}_X = \mathcal{O}_Z\).

We use the following result from minimal model program (MMP) for threefolds in positive characteristic, built on the works \[Bir16, BW17, CT19, GNT19, HW19a, HW19b, HX15, Tan20\]:

**Theorem 2.4.** Let \(C\) be a smooth quasi-projective curve defined over an algebraically closed field of characteristic \(p > 0\). Let \(f: X \to C\) be a projective contraction. Suppose that the following conditions hold:

1. The variety \(X\) is a \(\mathbb{Q}\)-factorial terminal threefold,
2. For every closed point \(c \in C\), the fiber \(X_c := f^{-1}(c)\) is an irreducible surface, and the pair \((X, X_c)\) is plt.

Then we can run a \(K_X\)-MMP over \(C\). The MMP terminates with a minimal model over \(C\), or a Mori fiber space.

In particular, we can apply the above theorem when the morphism \(f: X \to C\) is smooth of relative dimension two.

**Proof.** When \(p \geq 5\), the assertion directly follows from \[HW19a, Theorem 1.2\]. We give a proof based on \[HW19b\], which works in an arbitrary characteristic.
Step 1. (Cone theorem and Contraction theorem) By the same argument as in [HW19b, Proof of Theorem 1.6], Cone theorem and Contraction theorem hold for the morphism $X \to C$. Hence if $X$ is not a minimal model over $C$, then there exists the contraction $\phi: X \to \overline{X}$ (over $C$) of a $K_X$-negative extremal ray. Note that if it is a divisorial contraction, then the variety $X_1 := \overline{X}$ satisfies the conditions (1) and (2).

Step 2. (Existence of a flip) If $\phi$ is a flippling contraction, let $R$ be the curve contracted by the morphism $\phi$. Then $R$ is contained in an $f$-fiber $X_o := f^{-1}(o)$ for some closed point $o \in C$. By assumption (2), the pair $(X, X_o)$ is plt and we have $X_oR = 0$. Hence by [HW19b, Proposition 4.1], there exists a flip $f_1: X_1 \to C$ of $\phi$. Furthermore, the pair $(X_1, f_1^{-1}(o))$ is again plt. Hence $X_1$ satisfies the conditions (1) and (2).

Step 3. (Termination) It remains to show the finiteness of terminal flips. By [Kol13, Corollary 3.10], the singular locus of a terminal threefold is zero dimensional. Hence we can check that the proof of [KM98, Theorem 6.17] works also in positive characteristic. We conclude that the program ends with finite steps. □

3. BG inequality on surfaces

The goal of this section is to prove Theorem 1.1.

3.1. Statement of theorems. We use the following terminology:

Definition 3.1. Let $S$ be a smooth projective surface. We say that a divisor $D$ on $S$ is a birational pull-back of a very ample divisor if there exists a birational morphism $\phi: S \to \overline{S}$ to a normal projective surface $\overline{S}$ and a very ample divisor $A$ on $\overline{S}$ such that $D = \phi^*A$. We define $\text{BD}(S)$ to be a set of all birational pull-backs of very ample divisors.

We define a modified version of the discriminant as follows:

Definition 3.2. Let $T$ be a smooth projective surface. We define a constant $\mathcal{C}[T]$, which only depends on the birational equivalence class of $T$, as follows:

1. If the Kodaira dimension $\kappa(T) = 2$, let $S$ be the minimal model of $T$. We set

$$d[T] := \min \{ K_S.H : H \in \text{BD}(S), H^2 \geq K_SH \},$$

$$C[T] := d[T] - \chi(O_S) + 2.$$

2. If $\kappa(T) = 1$ and $T$ is quasi-elliptic, we set $C[T] := 2 - \chi(O_T)$.
3. Otherwise, we set $C[T] := 0$.

We then define

$$\widetilde{\Delta}[T] := \Delta + C[T] \text{ch}_0^2.$$

Remark 3.3. Let $S$ be a minimal surface of general type. Then the divisor $5K_S$ is very ample considered as a divisor on the canonical model of $S$. Hence we have $5K_S \in \text{BD}(S)$ and the inequality $C[S] \leq 5K_S^2 - \chi(O_S) + 2$ holds.
Example 3.4. Let $S := S^2_d \subset \mathbb{P}^3$ be a hypersurface of degree $d \geq 5$. Denote by $H$ the restriction of a hyperplane class on $\mathbb{P}^3$. Then we have $K_S = (d - 4)H$ and it is easy to compute that
\[
C_{[S_d^2]} = \frac{5}{6}d^3 - 7d^2 + \frac{85}{6}d + 2,
\]
which is positive for $d \geq 5$.

We prove the following results:

**Theorem 3.5.** Let $T$ be a smooth projective surface. Then for every numerically non-trivial nef divisor $L$ on $T$ and $\mu_L$-semistable torsion free sheaf $E \in \text{Coh}(T)$, we have
\[
\tilde{\Delta}_{\mathcal{T}}(E) \geq 0.
\]

**Theorem 3.6.** Let $T$ be a smooth projective surface. $L$ a birational pullback of a very ample divisor. Let $E \in \text{Coh}(T)$ be a $\mu_L$-semistable torsion free sheaf on $T$. Assume that for any general member $C \in |L|$, the restriction $E|_D$ is not slope semistable. Then we have
\[
\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq L^2 \cdot \tilde{\Delta}_{\mathcal{T}}(E),
\]
where we denote by $r_i$ the ranks and the slopes of the Hardar-Narasimhan factors of $E|_C$.

When $\kappa(T) \leq 1$, the results essentially follows from [Lan16, Theorem 7.1]. For surfaces of general type, we prove the above theorems at the same time by induction on $\text{ch}_0(E)$, following the idea of Langer [Lan04] (see also [BLMS17]). Note that the assertions are trivial when $\text{ch}_0(E) = 1$. Let $T^1(r)$ (resp. $T^2(r)$) be the statement that Theorem 3.5 (resp. 3.6) holds when $\text{ch}_0(E) \leq r$. We prove that $T^2(r)$ implies $T^1(r)$, and that $T^1(r - 1)$ implies $T^2(r)$.

### 3.2. Invariance of the BG inequality.

In this subsection, we recall some results concerning the invariance of the BG inequality under the blow-ups and the change of polarizations, which are essentially proved in [Lan16].

**Lemma 3.7.** Let $T$ be a smooth projective surface, $L$ a numerically non-trivial nef divisor. Suppose that we have an exact sequence
\[
0 \to E_1 \to E \to E_2 \to 0
\]
of torsion free sheaves on $T$ with $\mu_L(E_1) = \mu_L(E) = \mu_L(E_2)$. Then we have
\[
\frac{\tilde{\Delta}_{\mathcal{T}}(E)}{\text{ch}_0(E)} \geq \frac{\tilde{\Delta}_{\mathcal{T}}(E_1)}{\text{ch}_0(E_1)} + \frac{\tilde{\Delta}_{\mathcal{T}}(E_2)}{\text{ch}_0(E_2)}.
\]

**Proof.** By the assumption, we have $\left(\frac{\text{ch}_1(E_1)}{\text{ch}_0(E_1)} - \frac{\text{ch}_1(E_2)}{\text{ch}_0(E_2)}\right) L = 0$. Hence by the Hodge index theorem, we have
\[
\frac{\tilde{\Delta}_{\mathcal{T}}(E)}{\text{ch}_0(E)} = \frac{\tilde{\Delta}_{\mathcal{T}}(E_1)}{\text{ch}_0(E_1)} + \frac{\tilde{\Delta}_{\mathcal{T}}(E_2)}{\text{ch}_0(E_2)} - \frac{\text{ch}_0(E_1) \text{ch}_0(E_2)}{\text{ch}_0(E)} \left(\frac{\text{ch}_1(E_1)}{\text{ch}_0(E_1)} - \frac{\text{ch}_1(E_2)}{\text{ch}_0(E_2)}\right)^2
\]
\[
\geq \frac{\tilde{\Delta}_{\mathcal{T}}(E_1)}{\text{ch}_0(E_1)} + \frac{\tilde{\Delta}_{\mathcal{T}}(E_2)}{\text{ch}_0(E_2)}.
\]
Proposition 3.8 ([Lan16, Proposition 6.2]). Let $T$ be a smooth projective surface, $r \geq 2$ an integer. Suppose that $T^1(r)$ holds for some numerically non-trivial nef divisor $L$. Then it also holds for every choice of numerically non-trivial nef divisors $M$.

Proof. Let $E$ be a $\mu_M$-semistable torsion free sheaf with $\text{ch}_0(E) \leq r$. We prove the assertion by induction on $\text{ch}_0(E)$. For $\text{ch}_0(E) = 1$, the assertion is trivial, so assume that $2 \leq \text{ch}_0(E) \leq r$. Let us put $M_t := tM + (1 - t)L$ for $t \in [0, 1]$. There are two possible cases. If $E$ is $\mu_L$-semistable, then the assertion follows from $T^1(r)$ for $L$.

If $E$ is not $\mu_L$-semistable, then there exists a real number $t \in (0, 1)$ such that $E$ is strictly $\mu_{M_t}$-semistable. Hence there exists an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

of torsion free $\mu_{M_t}$-semistable sheaves with $\mu_{M_t}(E_1) = \mu_{M_t}(E) = \mu_{M_t}(E_2)$.

Now by the induction hypothesis and Lemma 3.7, we get the result. □

Proposition 3.9 ([Lan16, Proposition 6.5]). Take an integer $r \geq 2$. Let $S$ be a smooth projective surface, $L$ a numerically non-trivial nef divisor on $S$. Let $\psi: T \to S$ be the blow-up at points. Suppose that $T^1(r)$ holds for $(S, L)$. Then it also holds for $(T, \psi^*L)$.

Proof. Let $E$ be a $\mu_{\psi^*L}$-semistable torsion free sheaf. By taking the double dual, we may assume that $E$ is locally free. Let $F := (\psi^*E)^{\ast\ast}$. Then $F$ is $\mu_L$-semistable. Moreover, by [Lan16, Lemma 6.4], we have $\Delta(F) \geq \Delta(E)$. Since we have $\text{ch}_0(F) = \text{ch}_0(E)$, the inequality $\tilde{\Delta}_{[T]}(F) \geq \tilde{\Delta}_{[T]}(E)$ also holds, and hence the assertion holds. □

3.3. Proof of Theorems 3.5 and 3.6 For surfaces with $\kappa(S) \leq 1$, we have the following result of Langer:

Theorem 3.10 ([Lan16, Theorem 7.1]). Let $T$ be a smooth projective surface with $\kappa(S) \leq 1$, $L$ be a numerically non-trivial nef divisor on $T$. Then for every $\mu_L$-semistable torsion free sheaf $E$ on $T$, we have $\tilde{\Delta}_{[T]}(E) \geq 0$.

Proof. We prove the assertion when $\kappa(S) = 1$ and $S$ is quasi-elliptic. In this case, we have $K_S^2 = 0$ and $K_S$ is numerically non-trivial. Hence by Proposition 3.8 it is enough to prove the assertion for $\mu_{K_S}$-semistable sheaves $E$. If $\text{ch}_0(E) = 1$, then the result trivially holds. Hence we may assume that $\text{ch}_0(E) \geq 2$. Now by Serre duality and the Riemann-Roch theorem, we have

$$\chi(O_S) \text{ch}_0(E)^2 - \Delta(E) = \chi(E, E) \leq \text{hom}(E, E) + \text{hom}(E, E(K_S)) \leq 2,$$

and the desired inequality holds.

For other cases, the result is proved in [Lan16, Theorem 7.1]. □

For surfaces of general type, the idea of the proof is similar as above. However, we need some more works to bound $\text{hom}(E, E(K_S))$.

We use the following easy lemma:
Lemma 3.11. Let \( C \) be a smooth projective curve, \( E \) a slope semistable vector bundle of rank \( r \). For every integer \( d \geq 0 \), we have

\[
\text{hom}(E, E(d)) \leq (d + 1)r^2.
\]

Proof. We prove it by induction on \( d \geq 0 \). For \( d = 0 \), the assertion follows from semistability of \( E \). Let us consider the case when \( d \geq 1 \). Pick a point \( p \in C \). By applying the functor \( \text{Hom}(E, -) \) to the exact sequence

\[
0 \to E(d - 1) \to E(d) \to E(d)|_p \to 0,
\]

we have

\[
\text{hom}(E, E(d)) \leq \text{hom}(E, E(d - 1)) + \text{hom}(E, E(d)|_p)
= \text{hom}(E, E(d - 1)) + r^2,
\]

and hence the assertion holds. \( \square \)

Now we are ready to prove Theorems 3.5 and 3.6:

\( T^1(r - 1) \) implies \( T^2(r) \). This direction follows from the arguments as in [Lan04, Subsection 3.9]. For the completeness, we include the proof here.

Let \( \Lambda \subset |L| \) be a general pencil, let \( \tilde{T} := \{(t, C) \in T \times \Lambda : t \in C \} \) be the incidence variety. Let us put \( d := L^2 \). Since \( \Lambda \) is general, the base locus \( \text{Bs}(\Lambda) \) consists of \( d \)-distinct points, and the projection \( q: \tilde{T} \to T \) is the blow-up at \( \text{Bs}(\Lambda) \). Denote by \( p: \tilde{T} \to \Lambda \) the projection, and by \( f \) a class of \( p \)-fiber.

Let \( E \in \text{Coh}(T) \) be a \( \mu_L \)-semistable torsion free sheaf with \( \text{ch}_0(E) = r \), suppose that the restriction \( E|_C \) is not slope semistable for general \( C \in \Lambda \). Let \( E_i \subset q^*E \) be the \( p \)-relative Hardar-Narasimhan (HN) filtration, which is same as the HN filtration with respect to \( \mu_f \)-stability. As \( f \) is a numerically non-trivial nef divisor, we can apply \( T^1(r - 1) \) to the factors \( F_i := E_i/E_{i-1} \) to get the inequality

\[
(3.1) \quad \Delta_{[\tilde{T}]}(F_i) \geq 0.
\]

Note also that as \( \tilde{T} \to T \) is the blow-up, we have \( \Delta_{[\tilde{T}]} = \Delta_{[T]} \).

Let \( N_1, \cdots, N_d \) be the \( q \)-exceptional divisors. Then there exist divisors \( M_i \) on \( T \) and integers \( b_{ij} \) such that \( \text{ch}_1(F_i) = q^*M_i + \sum_j b_{ij}N_j \). Put \( b_i := \sum_j b_{ij}/d \). Then we have

\[
(3.2) \quad \mu_i = \frac{f \text{ch}_1(F_i)}{r_i} = \frac{M_iL + b_id}{r_i}.
\]

On the other hand, as \( E \) is \( \mu_L \)-semistable and \( q_*E_i \subset E \), we have

\[
\frac{\sum_{j \leq i} M_jL}{\sum_{j \leq i} r_j} \leq \mu_L(E)
\]

and hence

\[
(3.3) \quad \sum_{j \leq i} b_jd \geq \sum_{j \leq i} r_j(\mu_j - \mu_L(E)).
\]
Now using the inequality (3.1), we have
\[ \frac{d\tilde{\Delta}_{[T]}(E)}{r} = \sum_i d\tilde{\Delta}_{[T]}(F_i) - \frac{d}{r} \sum_{i<j} r_ir_j \left( \frac{\text{ch}_1(F_i)}{r_i} - \frac{\text{ch}_1(F_j)}{r_j} \right)^2 \]
\[ \geq \frac{d}{r} \sum_{i<j} r_ir_j \left( d \left( \frac{b_i}{r_i} - \frac{b_j}{r_j} \right)^2 - \left( \frac{M_i - M_j}{r_i} \right)^2 \right) \]
\[ \geq \frac{1}{r} \sum_{i<j} r_ir_j \left( d^2 \left( \frac{b_i}{r_i} - \frac{b_j}{r_j} \right)^2 - \left( \frac{M_iL - M_jL}{r_i} \right)^2 \right), \]
where the last inequality follows from the Hodge index theorem. By using the equation (3.2), the bottom line of the above inequalities becomes
\[ 2 \sum_i db_i\mu_i - \frac{1}{r} \sum_{i<j} r_ir_j(\mu_i - \mu_j)^2. \]
By the inequality (3.3), we get
\[ \sum_i db_i\mu_i = \sum_i \left( \sum_{j \leq i} db_j \right) (\mu_i - \mu_{i+1}) \]
\[ \geq \sum_i \left( \sum_{j \leq i} r_j(\mu_j - \mu_L(E)) \right) (\mu_i - \mu_{i+1}) \]
\[ = \sum_i r_i\mu_i^2 - r\mu_L(E)^2 = \sum_{i<j} r_ir_j(\mu_i - \mu_j)^2, \]
and hence \( T^2(r) \) holds. \( \square \)

\( T^2(r) \) implies \( T^1(r) \). By Theorem 3.10 we may assume that \( \kappa(T) = 2 \). Moreover, by Propositions 3.8 and 3.9 it is enough to prove the assertion for a minimal surface \( S \), and a divisor \( L \in \text{BD}(S) \) with \( KS-L = d|S| \).

Assume for a contradiction that there exists a \( \mu_L \)-semistable torsion free sheaf \( E \in \text{Coh}(S) \) with \( \text{ch}_0(E) = r \) such that \( \tilde{\Delta}_{|S|}(E) < 0 \). Then, by \( T^2(r) \), the restriction \( E|_C \) is slope semistable for a general member \( C \in |L| \).

Note that we may assume that \( E \) is \( \mu_L \)-stable. By Serre duality and the Riemann-Roch theorem, we have
\[ \chi(\mathcal{O}_S)r^2 - \Delta(E) = \chi(E, E) \]
\[ \leq \text{hom}(E, E) + \text{hom}(E, E(K_S)) \]
\[ = 1 + \text{hom}(E, E(K_S)). \]

On the other hand, by the exact sequence
\[ 0 \to E(K_S - L) \to E(K_S) \to E(K_S)|_C \to 0, \]
Lemma 3.11 and the equation \( KS-L = d|S| \), we have
\[ \text{hom}(E, E(K_S)) \leq \text{hom}(E, E(K_S - L)) + \text{hom}(E, E(K_S)|_C) \]
\[ \leq 1 + \text{hom}(E|_C, E(K_S)|_C) \]
\[ \leq 1 + (d|S| + 1)r^2. \]
Combining with the inequality (3.5), we obtain
\[ \Delta(E) + (d_{\mathcal{S}} - \chi(\mathcal{O}_{\mathcal{S}}) + 1) r^2 + 2 \geq 0, \]
which is a contradiction. \[\square\]

**Remark 3.12.** Let \( S \) be a minimal surface of general type. For rank two slope semistable bundles \( E \) on \( S \), a similar bound is obtained in [SB91, Theorem 12]. In fact, we have
\[
\begin{cases}
\Delta(E) + K_S^2 \geq 0 & (\text{char}(k) = p \geq 3),
\Delta(E) + \max\{K_S^2, K_S^2 - 3\chi(\mathcal{O}_{\mathcal{S}}) + 2\} \geq 0 & (p = 2).
\end{cases}
\]

**Example 3.13 (Counter-examples to Kodaira vanishing).** It is known that there exist a surface \( S \) of general type and an ample divisor \( L \) on \( S \) such that \( H^1(S, L^{-1}) \neq 0 \) (cf. [Muk13, Ray78]). Our Theorem 3.5 gives an upper bound on the intersection number \( L^2 \) for such a divisor \( L \). Indeed, consider the non-trivial extension
\[
0 \to \mathcal{O}_S \to E \to L \to 0,
\]
which exist as we assume \( H^1(S, L^{-1}) \neq 0 \). Then the bundle \( E \) is \( \mu_L \)-stable, and we have \( \Delta(E) = -L^2 \). Hence by Theorem 3.5 we have
\[
4 \left( 5K_S^2 - \chi(\mathcal{O}_{\mathcal{S}}) + 2 \right) \geq 4C_{\mathcal{S}} \geq L^2.
\]

See [SB91] (and Remark 3.12 above) for the detailed study on the vanishing theorem in positive characteristic.

## 4. BG Inequality in higher dimension

Let \( X \) be a smooth projective variety of dimension \( n \geq 3 \), and \( H \) a very ample divisor. Let \( \Pi := |H| \) be the complete linear system, and define the incidence variety \( S \) as
\[
S := \left\{ (x, (H_i)) \in X \times \mathcal{I}^{(n-2)} : x \in H_i \text{ for all } i \right\}.
\]
We have the following diagram:
\[
\begin{array}{ccc}
S & \xrightarrow{p} & \mathcal{I}^{(n-2)} \\
\downarrow{q} & & \\
X.
\end{array}
\]

For a point \( t \in \mathcal{I}^{n-2} \), we denote by \( S_t := p^{-1}(t) \) the scheme-theoretic fiber of \( t \). First we need to define a well-defined discriminant with a modification term.

**Definition-Proposition 4.1.** There exists a non-empty open subset \( U \subset \mathcal{I}^{(n-2)} \) satisfying the following properties:

1. For every point \( t \in U \), the fiber \( S_t \) is smooth,
2. The Kodaira dimension \( \kappa(S_t) \) is constant for all \( t \in U \),
3. When \( \kappa(S_t) \geq 0 \), let \( R_t \) be the minimal model of \( S_t \). Then the invariants \( K_{R_t}^2, \chi(\mathcal{O}_{R_t}) \) are constant for all \( t \in U \).
In particular, the number
\[ C_{X,H} := \begin{cases} 0 & (\kappa(S_t) \leq 0), \\ 5K_R^2 - \chi(O_{R_t}) + 2 & (\kappa(S_t) \geq 1) \end{cases} \]
is independent on the points \( t \in U \). Moreover, we have \( C_{X,T} \geq C_{[S_t]} \) for all \( t \in U \), where the constant \( C_{[S_t]} \) is defined as in Definition 3.2.

**Proof.** First take an open subset \( U' \subset \Pi^{(n-2)} \) so that all the fibers \( S_t \) are smooth, and put \( S_U := p^{-1}(U') \). It is enough to show the following claim: Let \( C \rightarrow U' \) be a non-constant morphism from a smooth quasi-projective curve \( C \), and set \( S_C := C \times_U S_U \). Then for a general point \( c \in C \), the Kodaira dimension \( \kappa(S_c) \) and the invariants \( K_S^2, \chi(O_S) \) are constant.

Note that the morphism \( S_C \rightarrow C \) is a smooth projective morphism of relative dimension two. By Theorem 2.4, we can run an MMP for \( S_C \) over \( C \):
\[ S_C = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_t. \]

Observe the following facts:
- If \( R_i \rightarrow R_{i+1} \) contracts a divisor to a point or is a flip, then it does not change the general fiber of \( R_i \rightarrow C \).
- If \( R_i \rightarrow R_{i+1} \) contracts a divisor to a curve, then it contracts \((-1)\)-curves of the general fiber of \( R_i \rightarrow C \).

Hence for general \( t \in C \), the morphism \( S_t \rightarrow (R_t)_t \) contracts the same number of \((-1)\)-curves. Moreover, the invariants \( K_S^2, \chi(O_S) \) are constant for all \( t \in C \). Hence the same holds for the minimal models \( (R_t)_t \) with \( t \in C \) general.

So we have proven the existence of an open subset \( U \subset \Pi^{(n-2)} \) with the required properties. For the inequality \( C_{X,H} \geq C_{[S_t]} \), see Remark 3.3. □

Following [BLMS17], we use the following notion:

**Definition 4.2.**
1. For general hyperplanes \( H_1, H_2 \in |H| \), we call the intersection \( H_1 \cap H_2 \) as a codimension two linear subspace of \( X \).
2. Let \( V = (V_1, \cdots, V_m) \) be an ordered configuration of codimension two linear subspaces in \( X \). We say that \( \psi: Y \rightarrow X \) is a blow-up along \( V \) if \( \psi \) is the iterated blow-up along the strict transforms of the \( V_i \)’s.

For the blow-up \( \psi: Y \rightarrow X \) along an ordered configuration of codimension two linear subspaces, we define a discriminant \( \Delta_{Y,\psi^*H} \) as
\[ \Delta_{Y,\psi^*H} := \psi^*H^{n-2}\Delta + C_{X,H} \text{ch}_0^2. \]

**Theorem 4.3.** Let \( \psi: Y \rightarrow X \) be the blow-up of an ordered configuration of codimension two linear subspaces. Let \( E \in \text{Coh}(Y) \) be a \( \psi^*H \)-slope semistable torsion free sheaf. Then we have
\[ \Delta_{Y,\psi^*H}(E) \geq 0. \]

**Theorem 4.4.** Let \( \psi: Y \rightarrow X \) be the blow-up of an ordered configuration of codimension two linear subspaces. Let \( E \in \text{Coh}(Y) \) be a \( \psi^*H \)-slope semistable torsion free sheaf. Assume that for any general member \( D \in |\psi^*H| \), the restriction \( E|_D \in \text{Coh}(D) \) is not \( \psi^*H|_D \)-semistable. Let
Let \( r_i, \mu_i \) be the ranks and slopes of its Harder-Narasimhan factors. Then we have

\[
\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq H^n \cdot \tilde{\Delta}_{Y, \psi^* H}(E).
\]

Let \( T^1(r) \) (resp. \( T^2(r) \)) be the statement that Theorem \( 4.3 \) (resp. \( 4.4 \)) holds when \( \text{ch}_0(E) \leq r \). As in the surface case, we prove that \( T^2(r) \) implies \( T^1(r) \), and that \( T^1(r-1) \) implies \( T^2(r) \).

\( T^1(r - 1) \) implies \( T^2(r) \). The proof is almost same as the surface case. We take a general pencil \( \Lambda \subset |\psi^* H| \) and consider the incidence variety

\[
\tilde{Y} := \{(y, D) \in Y \times \Lambda : y \in D\}.
\]

Denote by \( q : \tilde{Y} \to Y \), \( p : \tilde{Y} \to \Lambda \) the projections, and by \( f \) the class of a \( p \)-fiber. Note that the composition \( \tilde{\psi} : \tilde{Y} \to Y \to X \) is again the blow-up of an ordered configuration of codimension two linear subspaces. Let \( E_1 \subset q^* E \) be the \( p \)-relative HN filtration, which is same as the HN filtration with respect to \( f. \psi^* H^{n-2} \)-slope stability. By Proposition \( 4.5 \) below, each HN factor \( F_i := E_i / E_{i-1} \) satisfies the inequality \( \tilde{\Delta}_{Y, \tilde{\psi}^* H}(F_i) \geq 0 \). Now the remaining computations are same as in the surface case.

\[ \Box \]

**Proposition 4.5 ([BLMS17, Proposition 8.9]).** Suppose \( T^1(r - 1) \) holds. Let \( F \in \text{Coh}(\tilde{Y}) \) be a \( \psi^* H^{n-2} \)-slope semistable torsion free sheaf with \( \text{ch}_0(F) \leq r - 1 \). Then the inequality

\[
\tilde{\Delta}_{Y, \tilde{\psi}^* H}(F) \geq 0
\]

holds.

**Proof.** The proof of [BLMS17, Proposition 8.9] works, as our modified discriminant differs from the usual one only by a term \( \text{ch}_2 \).

\[ \Box \]

\( T^2(r) \) implies \( T^1(r) \). Let \( E \in \text{Coh}(Y) \) be a \( \psi^* H \)-semistable torsion free sheaf with \( \text{ch}_0(E) = r \). Assume for a contradiction that we have \( \Delta_{Y, \psi^* H}(E) < 0 \). As we assume \( T^2(r) \), the restriction \( E|_D \) to a general member \( D \in |\psi^* H| \) is \( \psi^* H|_D \)-slope semistable. Note that by taking it general, we may assume that \( D \) is again the blow-up of an ordered configuration codimension two linear subspaces of a general member of \( |H| \). Hence by induction on \( n = \dim X \), we may assume that \( X \) is a surface. Since we have \( C_{X,H} \geq C_{[X]} \) (see Definition-Proposition \( 3.1 \)), we get a contradiction by Theorem \( 3.5 \).

\[ \Box \]

5. **BG inequality on hypersurfaces**

The goal is to prove a variant of the BG inequality for the \( H \)-discriminant on hypersurfaces in the projective spaces. Let \( S^n_d \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of dimension \( n \geq 2 \), degree \( d \geq 1 \). Denote by \( H \) the restriction of the hyperplane class of \( \mathbb{P}^{n+1} \). We prepare several notations:

**Definition 5.1.** Let \( \psi : Y \to S^n_d \) be the blow-up of an ordered configuration of codimension two linear subspaces. Denote by \( E_1, \ldots, E_l \) the \( \psi \)-exceptional prime divisors.
(1) We denote by $\Lambda_Y := \mathbb{Z} \oplus \bigoplus_i \mathbb{Z}[E_i]$. We define a quadratic form $q_Y$ on $\Lambda_Y$ as follows:

$$q_Y \left( b, \sum_i a_i E_i \right) := b^2 + H^n \psi^* H^{n-2} \left( \sum_i a_i E_i \right)^2.$$ 

(2) Identify $\text{NS}(Y)$ with $\psi^* \text{NS}(S^n_d) \oplus \bigoplus_i \mathbb{Z}[E_i]$. Then we define a group homomorphism $\text{pr}_H : \text{NS}(Y) \to \Lambda_Y$ as follows:

$$\text{pr}_H \left( \psi^* M, \sum_i a_i E_i \right) := \left( H^{n-1} M, \sum_i a_i E_i \right).$$

We denote by $\text{ch}_1^H := \text{pr}_H \circ \text{ch}_1 : K(Y) \to \text{NS}(Y) \to \Lambda_Y$ the composition.

(3) For a coherent sheaf $E \in \text{Coh}(Y)$, we define

$$Q_{Y, \psi^*}^H(E) := q_Y \left( \text{ch}_1^H(E) \right) - 2H^n \text{ch}_0(E) \psi^* H^{n-2} \text{ch}_2(E).$$

Remark 5.2. The quadratic form $Q_{Y, \psi^*}^H$ is a natural generalization of the usual discriminant. Indeed, we have the following:

(1) We have $Q_{S^n_d, H} = \Delta_H$.

(2) If $\text{Pic}(S^n_d) = \mathbb{Z}[H]$, then we have $Q_{Y, \psi^*}^H = H^n \cdot H^{n-2} \Delta$.

As similar to the previous sections, we prove the following results at the same time:

**Theorem 5.3.** Let $\psi : Y \to S^n_d$ be the blow-up of an ordered configuration of codimension two linear subspaces. For every $\mu_{\psi^*}^H$-semistable torsion free sheaf $E \in \text{Coh}(Y)$, we have

$$Q_{Y, \psi^*}^H(E) \geq 0.$$ 

**Theorem 5.4.** Let $\psi : Y \to S^n_d$ be the blow-up of an ordered configuration of codimension two linear subspaces. Let $E \in \text{Coh}(Y)$ be a $\psi^* H$-slope semistable torsion free sheaf. Assume that for any general member $D \in |\psi^* H|$, the restriction $E|_D \in \text{Coh}(D)$ is not $\psi^* H|_D$-semistable. Let $r_i, \mu_i$ be the ranks and slopes of its Hardar-Narasimhan factors. Then we have

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq Q_{Y, \psi^*}^H(E).$$

5.1. **The case of $n = 2$**. The key input is the following result:

**Theorem 5.5** ([Kos205 Corollary 6.4]). Take an integer $d \geq 1$. For every $\mu_H$-semistable sheaf $E \in \text{Coh}(S^n_d)$, we have

$$Q_{S^n_d, H}^H(E) = \Delta_H(E) \geq 0.$$ 

For blow-ups, we use the following easy lemma:

**Lemma 5.6.** Let $\psi : Y \to S^n_d$ be the blow-up at points. Let $E \in \text{Coh}(Y)$ be a torsion free sheaf. The following statements hold:

(1) The sheaf $\psi_* E \in \text{Coh}(S^n_d)$ is torsion free, and the sheaf $R^1 \psi_* (E)$ is supported on a zero dimensional subscheme.

(2) We have $\text{ch}_0(\psi_* E) = \text{ch}_0(E)$. 


(3) We have $\psi^*H \cdot ch_1(E) = H \cdot ch_1(\psi_*, E)$.
(4) If $E$ is $\mu_{\psi^*H}$-semistable, then $\psi_*, E$ is $\mu_H$-semistable.

**Proposition 5.7.** Let $\psi: Y \to S^2_d$ be the blow-up at $l$-distinct points. For every $\mu_{\psi^*H}$-semistable torsion free sheaf, we have

$$Q_{Y, \psi^*H}(E) \geq 0.$$ 

*Proof.* Let $E_1, \cdots, E_l$ be the $\psi$-exceptional divisors. There exists a line bundle $M$ on $S^2_d$ and integers $a_i \in \mathbb{Z}$ such that $ch_1(E) = \psi^*M + \sum a_i E_i$. Observe that the number $Q_{Y, \psi^*H}(E)$ is invariant under tensoring line bundles $\mathcal{O}(E_i)$, hence we may assume that $0 \leq a_i < ch_0(E)$. Now we obtain

$$0 \leq Q_{S^2_d, H}(\psi_*, E) \leq Q_{S^2_d, H}(R \psi_*, E) = \left((HM)^2 - 2H^2 \cdot ch_0(E) \left(ch_2(E) + \frac{\sum_i a_i}{2}\right) \right) \leq \left((HM)^2 - H^2 \sum_i a_i^2 - 2H^2 \cdot ch_0(E) \cdot ch_2(E) = Q_{Y, \psi^*H}(E). \right.$$ 

For the first inequality, we use Theorem 5.5 and Lemma 5.6 (4); for the second inequality, we use Lemma 5.6 (1); the third equality follows from the Grothendieck-Riemann-Roch theorem for the morphism $p$, together with Lemma 5.6 (2), (3); the fourth inequality follows from the assumption that $0 \leq a_i < ch_0(E)$. $\square$

5.2. The case of $n \geq 3$. For higher dimension, we need a variant of Proposition 4.5. Let $\psi: Y \to S^n_d$ be the blow-up of an ordered configuration of codimension two linear subspaces. Let $\Lambda \subset |\psi^*H|$ be a general pencil and consider the incidence variety $\bar{Y}$ as in (4.3). Denote by $q: \bar{Y} \to Y$, $p: \bar{Y} \to \Lambda$ the projections. Let $e$ be the $q$-exceptional divisor and $f$ the $p$-fiber. Note that, by taking the pencil $\Lambda$ general, the composition $\psi: \bar{Y} \to Y \to S^n_d$ is again the blow-up along an ordered configuration of codimension two linear subspaces. Put $h := \psi^*H$.

For a real number $t \geq 0$, we put $h_t := th + f$. Define a group homomorphism $Z_t: H^0(\bar{Y}, \mathbb{Q}) \oplus \Lambda_{\bar{Y}} \to \mathbb{C}$ as follows:

$$Z_t \left(r, b, \sum a_i E_i\right) := -(t + 1)b + h^{n-2} f \left(\sum a_i E_i\right) + \sqrt{-1} r.$$ 

We denote as $\mu_{Z_t} := -3Z_t/3Z_t$. For a coherent sheaf $E \in \text{Coh}(\bar{Y})$, we have

$$\mu_{h^{n-2} h_t}(E) = \mu_{Z_t} \left(ch_0(E), ch_1^H(E)\right).$$ 

**Lemma 5.8.** Let the notations as above. Then $Q_{\bar{Y}, h_t}$, considered as a quadratic form on $(\Lambda_{\bar{Y}})_\mathbb{R}$, is negative semi-definite on the kernel $\text{Ker}(Z_t) \subset H^0(\bar{Y}, \mathbb{Q}) \oplus \Lambda_{\bar{Y}}$.

*Proof.* Take an element $(r, b, \sum a_i E_i) \in \text{Ker}(Z_t)$. Then we have $r = 0$.

On the other hand, let us put

$$\widehat{h}_t := pr_H(h_t) = (t + 1)h^n - e,$$ 

where $pr_H$ is the projection from $H^0(\bar{Y}, \mathbb{Q}) \oplus \Lambda_{\bar{Y}}$ to $H^0(\bar{Y}, \mathbb{Q})$, and $\Lambda_{\bar{Y}}$ is considered as a subspace of $H^0(\bar{Y}, \mathbb{Q})$. Since $\Lambda_{\bar{Y}}$ is a subspace of $H^0(\bar{Y}, \mathbb{Q})$, the projection $pr_H$ is well-defined.

We have $\mu_{\widehat{h}_t}(E) = \mu_{Z_t} \left(ch_0(E), ch_1^H(E)\right)$. Since $\mu_{Z_t}$ is negative semi-definite on $\text{Ker}(Z_t)$, we have $\mu_{\widehat{h}_t}(E) < 0$ for $E \in \text{Ker}(Z_t)$. Therefore, $Q_{\bar{Y}, h_t}$ is negative semi-definite on $\text{Ker}(Z_t)$. $\square$
where \( e \) denotes the \( q \)-exceptional divisor. Then we have
\[
q_Y(\overline{h}_t) = (t + 1)^2(H^n)^2 - (H^n)^2 \geq 0,
\]
\[
q_Y\left(\prod_{i}(b_i, \sum a_i E_i)\right) = H^n\left(-\Re Z_t\left(b_i, \sum a_i E_i\right)\right) = 0.
\]

Since \( q \tilde{Y} \) has signature \((1, l)\) for some \( l \geq 1 \), we get
\[
Q_{Y,h}(\prod(a_i E_i)) = q_{\tilde{Y}}(b_i, \sum a_i E_i) \leq 0
\]
for \( t > 0 \). When \( t = 0 \), as \( \overline{h}_0 \) is a limit of \( \overline{h}_t \), the assertion also holds. □

**Proposition 5.9.** Fix a positive integer \( r \geq 2 \). Suppose that Theorem 5.3 holds for \( \text{ch}_0(E) \leq r - 1 \). Let \( F \in \text{Coh}(\tilde{Y}) \) be a torsion free \( f.\tilde{\psi}^*H^{n-2} \)-slope semistable sheaf with \( \text{ch}_0(F) \leq r - 1 \). Then we have
\[
Q_{Y,\tilde{\psi}^*H}(F) \geq 0.
\]

**Proof.** The proof of [BLMS17, Proposition 8.9] works by Lemma 5.8 above. □

**Proof of Theorems 5.3 and 5.4.** Now we can check that the same proofs as in the previous sections work, using Propositions 5.7 and 5.9. □

6. **Construction of Bridgeland stability conditions**

In this section, we construct Bridgeland stability conditions on surfaces, defined over an algebraically closed field of positive characteristic. We refer to papers [BMS16, BMT14, Bri07, Bri08] and a lecture note [MS17] for the basics of the theory of stability conditions.

Let us first recall the definition of Bridgeland stability conditions [Bri07]:

**Definition 6.1.** Let \( \mathcal{A} \) be an abelian category.

1. A group homomorphism \( Z: K(\mathcal{A}) \to \mathbb{C} \) is called a stability function if we have \( Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H} \cup \mathbb{R}_{<0} \), where we denote by \( \mathcal{H} \) the upper half plane.
2. For a stability function \( Z: K(\mathcal{A}) \to \mathbb{C} \) and an object \( E \in \mathcal{A} \), we define the \( Z \)-slope of \( E \) as
   \[
   \mu_Z(E) := -\frac{\Re Z(E)}{\Im Z(E)}.
   \]
3. We say that an object \( E \in \mathcal{A} \) is \( Z \)-(semi)stable if for every non-zero proper subobject \( 0 \neq F \subsetneq E \), we have an inequality \( \mu_Z(F) \leq \mu_Z(E) \).

**Definition 6.2.** Let \( \mathcal{D} \) be a triangulated category. A stability condition on \( \mathcal{D} \) is a pair \((Z, \mathcal{A})\) consisting of the heart \( \mathcal{A} \subset \mathcal{D} \) of a bounded t-structure and a group homomorphism \( Z: K(\mathcal{A}) \to \mathbb{C} \) satisfying the following axioms:

1. The group homomorphism \( Z: K(\mathcal{A}) \to \mathbb{C} \) is a stability function.
2. It satisfies the Hardar-Narasimhan (HN) property, i.e., for every object \( E \in \mathcal{A} \setminus \{0\} \), there exists a Hardar-Narasimhan filtration of \( E \) with respect to \( Z \)-stability.

We also call a homomorphism \( Z \) as a central charge.
Definition 6.3. Let $D$ be a triangulated category. Fix a finitely generated free abelian group $\Lambda$ and a group homomorphism $\chi: \mathbb{K}(D) \to \Lambda$. We say that a stability condition $(Z, A)$ on $D$ satisfies the support property with respect to $(\Lambda, \chi)$ if the central charge $Z$ factors through $\chi$, i.e., $Z: \mathbb{K}(D) \to \Lambda \to \mathbb{C}$, and there exists a quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ such that

1. The kernel $\ker Z \subset \Lambda$ is $Q$-negative definite,
2. For every $Z$-semistable object $0 \neq E \in \mathcal{A}$, we have $Q(\chi(E)) \geq 0$.

In the following, we explain the construction of Bridgeland stability conditions on the derived category $D^b(S)$ of a smooth projective surface $S$. We fix a lattice $H^*_{alg}(S) := \text{Im}(\chi: \mathbb{K}(X) \to H^{2*}(X, \mathbb{Q}_{\ell}))$.

We use the following central charge function:

Definition 6.4. Let $S$ be a smooth projective surface, $H$ an ample $\mathbb{R}$-divisor and $B$ an arbitrary $\mathbb{R}$-divisor. Let $C[\mathcal{S}]$ be a constant defined in Definition 3.2. We define a group homomorphism $Z^{[\mathcal{S}]}_H: \mathbb{K}(X) \to \mathbb{C}$ as follows:

$$Z^{[\mathcal{S}]}_H := -\chi^B + \left( \frac{C[\mathcal{S}]}{2H^2} + \frac{1}{2} \right) H^2 \chi^B + \sqrt{-1} H \chi^1.$$

For the heart, we need the theory of torsion pairs and tilting (see [HRS96]). Let us fix an ample $\mathbb{R}$-divisor $H$ on $S$, and an arbitrary $\mathbb{R}$-divisor $B$. We define full subcategories $T_{H, B}, \mathcal{F}_{H, B} \subset \text{Coh}(S)$ as follows:

$$T_{H, B} := \langle T \in \text{Coh}(S): T \text{ is } \mu_H\text{-semistable with } \mu_H(T) > HB \rangle,$$

$$\mathcal{F}_{H, B} := \langle F \in \text{Coh}(S): F \text{ is } \mu_H\text{-semistable with } \mu_H(F) \leq HB \rangle,$$

where, for a set $S \subset \text{Coh}(S)$, we denote by $\langle S \rangle \subset \text{Coh}(S)$ the extension closure. We then define the tilted-heart as the extension closure in the derived category:

$$\text{Coh}^{H, B}(S) := \langle \mathcal{F}_{H, B}[1], T_{H, B} \rangle \subset D^b(S).$$

We need the following lemma:

Lemma 6.5 ([BMT14, Corollary 7.3.3]). Let $H$ be an ample divisor on $S$. There exists a constant $C_H \geq 0$ such that for every effective divisor $D$ on $S$, we have

$$C_H(HD)^2 + D^2 \geq 0.$$

Theorem 6.6. Let $S$ be a smooth projective surface defined over an algebraically closed field of positive characteristic. Let $H$ be an ample $\mathbb{R}$-divisor and $B$ an arbitrary $\mathbb{R}$-divisor. Let $C[\mathcal{S}]$ be the constant defined as in Definition 3.2. Then the pair $(Z^{[\mathcal{S}]}_H, \text{Coh}^{H, B}(S))$ defines a Bridgeland stability condition on $D^b(S)$ and satisfies the support property with respect to $(\chi, H^*_{alg}(S))$.

Proof. We check that the following condition holds: Let $E \in \text{Coh}(S)$ be a $\mu_H$-semistable torsion free sheaf with $\mu_H(E) = HB$. We then have $\Re Z^{[\mathcal{S}]}_H(E) > 0$. Indeed, by Theorem 3.3 we have

$$0 \leq \Delta(E) + C[\mathcal{S}] \chi_0(E)^2 \leq \Delta_{H, B}(E) + C[\mathcal{S}] \chi_0(E)^2 = \chi_0^B(E) \left( -2H^2 \chi_2^B(E) + C[\mathcal{S}] \chi_0^B(E) \right),$$
where we define $\Delta_{H,B} := (H \cdot \text{ch}_1 B)^2 - 2H^2 \cdot \text{ch}_2 B$, and the second inequality follows from the Hodge index theorem. Hence the desired inequality $\Re \mathbb{Z}^{[S]}_{H,B}(E) > 0$ holds. Now we can show that the pair $(\mathbb{Z}^{[S]}_{H,B}, \text{Coh}^{H,B}(S))$ defines a stability condition by the standard arguments as in [AB13, Bri08].

For the support property, let $C_H \geq 0$ be as in Lemma 6.5. We can show that the quadratic form $\tilde{\Delta}_{[S]} + C_H(H \cdot \text{ch}_1 B)^2$ satisfies the required property, see [BMS16, Theorem 3.5] for the details. □

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