NEGATIVELY CURVED LEFT-INARIANT METRICS
ON LIE GROUPS†

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ABSTRACT. We discuss negatively curved homogeneous spaces admitting a simply transitive group of isometries, or equivalently, negatively curved left-invariant metrics on Lie groups. Negatively curved spaces have a remarkably rich and diverse structure and are interesting from both a mathematical and a physical perspective. As well as giving general criteria for having left-invariant metrics with negative Ricci curvature scalar, we also consider special cases, like Einstein spaces and Ricci nilsolitons. We point out the relevance these spaces play in some higher-dimensional theories of gravity. In particular, we show that the Ricci nilsolitons are Riemannian solutions to certain higher-curvature gravity theories.

1. INTRODUCTION

Given a mathematical structure, some of the most interesting and basic mathematical objects are the symmetries of this structure. For a Riemannian space, $(M, g)$, we can consider the metric-preserving symmetries; i.e. the isometries defined by

$$\phi : M \rightarrow M, \quad \phi^* g = g.$$  

The isometries of a Riemannian space form a group, namely the isometry group

$$\text{Isom}(M) = \{ \phi : M \rightarrow M \mid \phi^* g = g \}.$$  

(1)

The isometry group is a topological group, and, in fact, a Lie group [1]. This property of the isometry group makes the study of symmetries of Riemannian spaces interlinked with the theory of Lie groups.

If two points, $p, q \in M$, are related via an isometry, i.e. $\exists \phi \in \text{Isom}(M)$ such that $\phi(p) = q$, the metric at $p$ and $q$ will be mathematically indistinguishable. In a physical context this means that there is no measurement which can distinguish the two points. Hence, the two points are equivalent and all physical properties (derived from the metric tensor) are identical.

A homogeneous Riemannian space $(M, g)$ is a space where the isometry group acts transitively; for any pair $p, q \in M$, there exist a $\phi \in \text{Isom}(M)$ such that $\phi(p) = q$. Clearly, homogeneous spaces have $\dim \text{Isom}(M) \geq \dim(M)$. In the case where $\dim \text{Isom}(M) = \dim(M)$ and the action is transitive, we call the homogeneous space simply transitive, and if there there is a strict inequality we say the space is multiply transitive. It is common to write a connected homogeneous space as the space of cosets, $M = G/H$, where $G$ the isometry group, and $H$ is a compact

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Let us assume that we have a homogeneous Riemannian space $M = G/H$ which is connected and simply connected and allows for a simply transitive subgroup $K \subset G$ acting on $M$ so that $\dim(K) = \dim(M) = n$. This implies that $M$ is a manifold which can be identified as the group $K$. Hence, the spaces we are considering are group manifolds, and, in fact, Lie groups. Thus an equivalent point of view is to consider left-invariant metrics on Lie groups. In the following we will use these two views interchangeably.

The connection between the Lie group structure of the manifold and the left-invariant Riemannian structure is most easily seen by using a left-invariant frame. The symmetries of the space can be expressed locally in terms of the Killing vectors. The simply transitive group of isometries can be taken to be the Lie group $K$ whose Lie algebra will be denoted $\mathfrak{k}$. Then, there exist a “Killing frame” $\{\xi_i\}$ whose span is isomorphic to the Lie algebra of $K$; i.e.

\[
[\xi_i, \xi_j] = \tilde{C}^k_{ij} \xi_k,
\]

where the $\tilde{C}^k_{ij}$'s are the structure constants of the Lie algebra $\mathfrak{k}$. A left-invariant frame $\{e_i\}$ can now be found by requiring

\[
\ell_{\xi_j} e_i = [\xi_j, e_i] = 0.
\]

This left-invariant frame will also form a Lie algebra:

\[
[e_i, e_j] = C^k_{ij} e_k.
\]

The structure constants $C^k_{ij}$ are related to $\tilde{C}^k_{ij}$ in the following way: If we choose

\[
\{\xi_i\} \text{ and } \{e_i\}
\]

to coincide at one point, then $C^k_{ij} = -\tilde{C}^k_{ij}$ which implies that these Lie algebras are isomorphic. The dual one-forms, $\omega^i$, of the vectors $e_i$ are also left-invariant (i.e. $\ell_{\xi_j} \omega^i = 0$) and obey

\[
d\omega^k = -\frac{1}{2} C^k_{ij} \omega^i \wedge \omega^j.
\]

These left-invariant one-forms are manifestly invariant under the group action so a particularly convenient choice is to use $\{\omega^i\}$ as our orthonormal Cartan co-frame. Henceforth we will assume that our frame is orthonormal so that our metric components are $g_{ij} = \delta_{ij}$.

Given a Lie algebra $\mathcal{A}$ with structure constants $C^k_{ij}$ we define the $GL(n)$-orbit,

\[
O_{GL(n)}(\mathcal{A}) \equiv \left\{ \tilde{C}^k_{ij} \mid \tilde{C}^k_{ij} = (M^{-1})^k_i C^l_{mn} M^m_i M^n_j, \ M \in GL(n) \right\},
\]

and the $O(n)$-orbit,

\[
O_{O(n)}(\mathcal{A}) \equiv \left\{ \tilde{C}^k_{ij} \mid \tilde{C}^k_{ij} = (M^{-1})^k_i C^l_{mn} M^m_i M^n_j, \ M \in O(n) \right\}.
\]

From the definition of a Lie algebra, two Lie algebras are isomorphic if and only if they lie in the same $GL(n)$-orbit $O_{GL(n)}(\mathcal{A})$. Regarding the Riemannian structure,

\[1\] Here and throughout the paper the Einstein summation convention has been used; i.e. summation over repeated indices.
a frame rotation corresponds to an \( O(n) \)-transformation; i.e. two left-invariant frames whose structure constants lie in the same orbit \( O_{O(n)}(A) \) are isometric as Riemannian spaces. Since, \( O_{O(n)}(A) \subset O_{GL(n)}(A) \), we can by varying the structure constants \( C^k_{ij} \) consider various left-invariant Riemannian structures on the Lie group. In fact, all left-invariant Riemannian metrics on a given Lie group can in this way be obtained by varying the structure constants, \( C^k_{ij} \in O_{GL(n)}(A) \), while keeping the metric fixed.

Explicitly, in terms of \( C^k_{ij} \) the Ricci curvature can be written

\[
R_{ij} = \mathcal{R}_{ij} - \frac{1}{2} \mathcal{K}_{ij} - Z_{ij},
\]

where \( \mathcal{K}_{ij} \) is the Killing form of \( \mathfrak{k} \),

\[
\mathcal{K}_{ij} = C^a_{bi} C^b_{aj},
\]

and \( \mathcal{R}_{ij} \) and \( Z_{ij} \) are defined by

\[
\mathcal{R}_{ij} = \frac{1}{4} C_{iab} C^a_{b} C^b_{ij} - \frac{1}{2} C^a_{ab} C^b_{ij},
\]

\[
Z_{ij} = C^a_{ab} C^b_{(ij)}.
\]

In this way we see how the Lie algebra structure of the group is directly involved in the Riemannian structure. For various Lie groups the constants \( C^k_{ij} \) can only be of certain forms. For example, \( Z_{ij} = 0 \) for unimodular groups, \( \mathfrak{k} \) and \( \mathcal{K}_{ij} = 0 \) for nilpotent groups. Hence, the group theoretical properties will to some extent be reflected in the Ricci curvature.

Let us first discuss some of the global aspects of Lie groups/Lie algebras. Any Lie group defines uniquely a Lie algebra; however, to any given Lie algebra there may be many different Lie groups. For example, the groups \( SU(2) \) and \( SO(3) \) are different but they have isomorphic Lie algebras. Notwithstanding this, groups having isomorphic Lie algebras are locally the same ( [8] page 109):

**Theorem 2.1.** Two connected Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

Hence, \( SU(2) \) and \( SO(3) \) can only differ at a global scale. Assuming simply connectedness removes this ambiguity completely ( [2] page 113):

**Theorem 2.2.** Let \( G \) and \( H \) be connected Lie groups with Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \), respectively. If \( G \) is simply connected, then for every homomorphism of \( \mathfrak{g} \) into \( \mathfrak{h} \) there exists a homomorphism of \( G \) into \( H \).

This implies that if two connected and simply connected Lie groups have isomorphic Lie algebras, then they must be isomorphic as Lie groups. These results reduce the problem of classifying simply connected groups \( K \) to classifying their Lie algebras \( \mathfrak{k} \). We will also assume that there is a Riemannian structure on \( M \) (which is not unique) given in terms of a \( K \)-invariant metric \( \mathfrak{g} \).

2A group is called unimodular if it admits a bi-invariant measure. In terms of the structure constants, \( C^k_{ij} = 0 \).

3A Lie algebra is nilpotent if the decending series \( \mathfrak{g}_0 = \mathfrak{g} \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}_{i-1}] \) terminates; i.e. there exists a \( k \) such that \( \mathfrak{g}_k = 0 \). A group is called nilpotent if and only if its Lie algebra is nilpotent.

4Regarding global properties of Lie groups we refer the reader to the book *Theory of Lie Groups* by Chevalley [2]. Here, many classic results on the global structure of groups are given.
3. THE RICCI CURVATURE SCALAR

A question one would like to answer is: Can we say anything about the Ricci curvature scalar, defined as $R = R_{ij}$, from the properties of the Lie group? In other words, to what extent does the curvature of left-invariant metric depend on the properties of the Lie group?

Let us consider the sign of the Ricci curvature scalar of $g$. First we ask: Which Lie groups admit a left-invariant metric with negative Ricci scalar, $R < 0$? The answer is provided by the following theorem [4]:

**Theorem 3.1.** If the Lie algebra of $K$ is non-commutative, then $K$ possesses a left-invariant metric of strictly negative scalar curvature.

**Proof.** The proof utilizes the following fact: For any non-Abelian Lie algebra $\mathfrak{k}$, we have either of the two (mutually exclusive) possibilities

1. There exists a Lie algebra contraction $\phi$ such that $\mathfrak{k} \rightarrow \mathfrak{n}_3 + \mathbb{R}^m$, where $\mathfrak{n}_3$ is the 3-dimensional Heisenberg Lie algebra.
2. Any left-invariant metric on $K$ has constant negative curvature.

The first implies that there exists a continuous curve $c(t)$ in the orbit $O_{GL(n)}(\mathfrak{k})$ which has $\lim_{t \to \infty} c(t) \in O_{GL(n)}(\mathfrak{n}_3 + \mathbb{R}^m)$, and hence, we can come arbitrary close to the Lie algebra $\mathfrak{n}_3 + \mathbb{R}^m$. Since the Heisenberg group has strictly negative Ricci scalar curvature there must exist, by continuity, a left-invariant metric on $\mathfrak{k}$ with negative Ricci scalar curvature as well. The second case has trivially negative Ricci scalar curvature. □

So if the group $K$ is not Abelian, $K$ admits a left-invariant metric such that $R < 0$. This may be surprising to many, but this means that even compact groups – e.g. $SU(2) \cong S^3$ – admits negatively curved left-invariant metrics.

A natural follow-up would be to consider Lie groups for which all left-invariant metrics have $R < 0$. To find these the following theorem by Bérard Bergery [3] comes to our aid:

**Theorem 3.2** (Bérard Bergery). Let $M = G/H$ be a homogeneous space on a connected Lie group $G$, admitting a $G$-invariant metric. Then the following 2 statements are equivalent:

1. The universal cover of $M$ is diffeomorphic to a Euclidean space.
2. Every $G$-invariant metric on $M$ is either flat or has strictly negative Ricci curvature scalar.

This theorem is very powerful and gives us the necessary criterion for non-positiveness of $R$. Firstly, the only homogeneous manifolds whose universal covers are diffeomorphic to $\mathbb{R}^n$ and has $R = 0$, are the flat ones. Secondly, it lifts the local statement about the Ricci scalar to a statement of the global structure of $M$. Hence, to check whether $M$ has a non-positive Ricci curvature scalar we have to check the global structure of the group $K$.

Another helpful result is the Iwasawa decomposition [5]:

**Theorem 3.3** (Iwasawa). If $K$ is a connected Lie group, then

1. Every compact subgroup is contained in a maximal compact subgroup $H$, which is necessary a connected Lie group.

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5This theorem is based on a conjecture by Milnor [4].
This maximal compact subgroup is unique up to conjugation.

As a topological space, $K$ is homeomorphic with the product of $H$ and some Euclidean space $\mathbb{R}^m$:

$$K \cong H \times \mathbb{R}^m$$

An immediate consequence is the following [4]:

**Corollary 3.4.** The universal covering of $K$ is homeomorphic to Euclidean space if and only if every compact subgroup is commutative.

This is exactly what we need to find out whether a Lie group has a universal cover diffeomorphic to a Euclidean space.

### 3.1. The classification of Real Lie algebras

To check the global structure of $M$ it is necessary to remind ourselves of some classification results regarding real Lie algebras. We shall use the symbol $s \oplus h$ for the semidirect sum, writing the ideal $s$ first, and the subalgebra $h$ second (an ordinary $+$ is written for the direct sum).

Using $[-,-]_s$ and $[-,-]_h$, we can endow the semidirect sum with a Lie algebra structure as follows:

$$[e_i,e_J] = D(e_i) \ast e_J, \quad e_i \in h, \quad e_J \in s,$$

where $D(e_i)$ is a linear mapping, $D(e_i) : s \mapsto s$. This leads to the following relations

$$[s,s] \subset s, \quad [h,h] \subset h, \quad [s,h] \subset s.$$  

Furthermore, the Jacobi identity implies that $D(e_i)$ is a derivation of $s$:

$$D(e_i) \ast [e_J,e_K] = [D(e_i) \ast e_J,e_K] + [e_J,D(e_i) \ast e_K].$$

The set $\{D(e_i)\}$ is itself a Lie algebra, and the homomorphism $e_i \mapsto D(e_i)$ must be a representation of the algebra $h$. The fundamental Levi-Malcev theorem says that, for an arbitrary Lie algebra $g$ with radical $s$, a semisimple subalgebra $h$ exists such that

$$g = s \oplus h.$$  

The semisimple subalgebra $h$ is called the **Levi factor**. This immediately implies that the Lie algebras fall in three categories:

1. The solvable algebras.
2. The semisimple algebras.
3. The semidirect sums of solvable and semisimple algebras.

**The solvable algebras.** Given a Lie algebra $g$. The algebra $g$ is called **solvable** if the derived series $g_i$, defined iteratively by $g_0 = g$ and $g_{i+1} = [g_i, g_i]$, terminates; i.e. there exists an $m$ such that $g_m = 0$.

In spite of the fact that the solvable ones are the most numerous (see Table [1]), they are easy to deal with due to a result by É. Cartan [6]: *An $n$-dimensional group is diffeomorphic to $\mathbb{R}^n$ if it is solvable, connected and simply connected.* In particular, this means that solvable algebras do not have any non-Abelian compact subgroups. More generally, if $s$ is solvable, then there does not exist a semisimple subalgebra $h$ such that $s \supset h \neq \{0\}$.

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[1]: See also [7]. This paper also considers the more general case when the solvable group is not necessarily simply connected.
Table 1. The number of equivalence classes of indecomposable real Lie algebras up to dimension 8. These numbers should be used with care since some equivalence classes are actually continuous families of non-isomorphic Lie algebras.

| dim | simple | semi-direct sum | solvable |
|-----|--------|-----------------|---------|
| 1   | 1      |                 |         |
| 2   | 1      |                 |         |
| 3   | 2      |                 | 7       |
| 4   |        |                 | 12      |
| 5   | 1      |                 | 40      |
| 6   | 1      |                 | 4       |
| 7   | 4      |                 | 164     |
| 8   | 3      |                 | 22      |

A complete classification of the solvable Lie algebras exists only up to dimension 6. In this regard, Turkowski found the remaining ones of dimension 6 [9]. (For the real and complex 7-dimensional nilpotent Lie algebras, see [10–12].)

The semisimple algebras. The study of semisimple groups reduces to studying the simple groups. The real simple algebras come in two classes [8]: (I), the simple Lie algebras over \( \mathbb{C} \), considered as real Lie algebras, and (II), the real forms of simple Lie algebras over \( \mathbb{C} \).

The semisimple Lie algebras over \( \mathbb{C} \) are completely classified so we can use these results. First, for the class I real algebras note that they all have the subalgebra \( \mathfrak{sl}(2, \mathbb{C}) \) (by inspecting, for example, the Dynkin diagrams). Due to the well-known diffeomorphisms

\[
\tilde{SL}(2, \mathbb{C}) \cong SU(2) \times \mathbb{R}^3 \cong S^3 \times \mathbb{R}^3,
\]

none of the class I real Lie groups will be diffeomorphic to a Euclidean space. The Lie algebras of class II are slightly more subtle. Using the Dynkin diagrams and the root systems, we see that all the Lie algebras over \( \mathbb{C} \), except \( A_1 \) and \( B_2 \), have an \( A_2 \)-subalgebra. \( A_2 \) has the following real forms:

\[
(14) \quad \tilde{SL}(3, \mathbb{R}) \cong SU(2) \times \mathbb{R}^5, \quad SU(3), \quad \tilde{SU}(2, 1) \cong SU(2) \times \mathbb{R}^5.
\]

The algebra \( B_2 \) has the following real forms [8]:

\[
(15) \quad \tilde{SO}(5), \quad \tilde{SO}(4, 1) \cong SU(2) \times SU(2) \times \mathbb{R}^4, \quad \tilde{SO}(3, 2) = SU(2) \times \mathbb{R}^7.
\]

Finally, the algebra \( A_1 \) has the following real forms:

\[
(16) \quad \tilde{SL}(2, \mathbb{R}) \cong \mathbb{R}^3, \quad SU(2) \cong S^3.
\]

Using Corollary 3.3, the above analysis implies that none of the simple Lie algebras, except for \( \mathfrak{sl}(2, \mathbb{R}) \), give rise to groups whose universal cover is diffeomorphic to a Euclidean space.

Lie algebras which are semidirect sums. For these algebras, there is one thing to note. Assume that \( l = s \oplus h \), where \( s \) is solvable and \( h \) is semisimple and let \( M \)
\( \tilde{M}_s \) and \( \tilde{M}_h \) be the universal cover of their corresponding groups. Then we have the diffeomorphism

\[
\tilde{M}_l \cong \tilde{M}_s \times \tilde{M}_h.
\]  

(17)

Hence, the topology of these groups is the direct product of the topology of their respective components. Since \( \tilde{M}_s \) is diffeomorphic to a Euclidean space, we can understand the groups in this category by understanding the semisimple groups.

All of these algebras are not known explicitly; however Turkowski has found all the real Lie algebras having a non-trivial Levi decomposition up to dimension 9 \([13, 14]\).

As an example, consider the only non-trivial 5-dimensional Lie algebra of this type, which gives rise to the group \( \mathbb{R}^2 \times SL(2, \mathbb{R}) \). A left-invariant frame on this group can be taken to be

\[
\begin{align*}
\omega^1 &= dx + \cosh 2\theta dy, \\
\omega^2 &= e^{-2x} (d\theta + \sinh 2\theta dy), \\
\omega^3 &= e^{2x} (d\theta - \sinh 2\theta dy) \\
\omega^4 &= e^{-x} (e^{-y} \cosh \theta du - e^y \sinh \theta dv), \\
\omega^5 &= e^{x} (-e^{-y} \sinh \theta du + e^y \cosh \theta dv).
\end{align*}
\]

These left-invariant one-forms obey eq. (15) where the \( C^k_{ij} \)'s represent the Lie algebra \( L_{5,1} \) in \([13]\).

### 3.2. Negatively curved left-invariant metrics.

We can summarise our investigation in the following theorem.

**Theorem 3.5.** Let \( M = G/H \) be a connected \( n \)-dimensional homogeneous Riemannian manifold admitting a \( G \)-invariant metric. Assume further that the universal cover \( \tilde{M} \) of \( M \) admits a group \( K \) acting simply transitive on \( \tilde{M} \). Then the following statements are equivalent:

1. Every \( G \)-invariant metric on \( M \) has non-positive Ricci curvature scalar.
2. The Lie algebra of \( K \) does not have a subalgebra isomorphic to the Lie algebra \( \mathfrak{su}(2) \).

**Proof.** Note that all of the simple real Lie algebras, except for \( \mathfrak{su}(2, \mathbb{R}) \), have a subalgebra isomorphic to \( \mathfrak{su}(2) \). Hence, this requirement excludes all semisimple algebras which cannot give rise to a group whose universal cover is not diffeomorphic to \( \mathbb{R}^n \) by virtue of the Iwasawa decomposition. Using Theorem 3.2 the theorem follows from the above analysis. \( \square \)

This theorem gives a very precise criterion for which Lie groups admit only negatively curved left-invariant metrics. The existence of an \( \mathfrak{su}(2) \) subgroup makes it possible to find a positively curved left-invariant metric on the Lie group.

### 4. Einstein metrics

A Lie group usually possess many non-isometric left-invariant metrics. However, for a given Lie group, is there a particularly nice or distinguished left-invariant metric? Such metrics are, for example, Einstein metrics for which

\[
R_{\mu\nu} = \lambda g_{\mu\nu},
\]

(18)

where \( \lambda \) is a constant. However, what Lie groups admit left-invariant Einstein metrics? Here we will discuss some known results regarding Einstein metrics on Lie groups.
Besse [15] has devoted a whole book to Riemannian manifolds with Einstein metrics. The complete answer for which Lie groups admit such metrics is not known. However, we have the following rough classification [15].

**Theorem 4.1.** Let \((M, g)\) be a homogeneous Einstein manifold, \(R_{\mu \nu} = \lambda g_{\mu \nu}\);

1. If \(\lambda > 0\), then \(M\) is compact with finite fundamental group.
2. If \(\lambda = 0\), then \(M\) is flat.
3. If \(\lambda < 0\), then \(M\) is non-compact.

A discussion of the positively curved ones is done in [15], and the above theorem implies that the \(\lambda = 0\) consists of only the flat ones (this was proven in [16]). Our concern here will be the negatively curved ones, \(\lambda < 0\).

From the analysis above we note that the best candidates for Lie groups with Einstein metrics are the solvable groups. However, not all solvable groups allows for an Einstein metric; e.g. Dotti Miattello showed that a solvable unimodular Lie group does not admit a left-invariant Einstein metric of strictly negative curvature [17]; the only Einstein metric allowed on a solvable unimodular Lie group is the flat one. (Note that a special class of the unimodular solvable groups are the nilpotent ones. This will be of significance a bit later.)

Consider a real Lie algebra, \(\mathfrak{s}\), with the following properties:

1. The Iwasawa decomposition has the following orthogonal decomposition:

\[
\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \quad [\mathfrak{s}, \mathfrak{s}] = \mathfrak{n},
\]

where \(\mathfrak{a}\) is Abelian, and \(\mathfrak{n}\) is nilpotent.
2. All operators \(\text{ad}_X, X \in \mathfrak{a}\) are symmetric.
3. For some \(X_0 \in \mathfrak{a}\), \(\text{ad}_{X_0}|_n\) has positive eigenvalues.

Since \(\mathfrak{s}\) is a solvable algebra, the corresponding group manifolds are so-called solv-manifolds. These geometries are strong candidates for Einstein spaces with negative curvature [18, 19]. The simplest possible \(\mathfrak{s}\) corresponds to real hyperbolic space, \(\mathbb{H}^n\). In fact, all known examples of homogeneous Einstein manifolds with negative curvature are of the above type [20].

Some examples of Einstein solvmanifolds can be found among the symmetric spaces:

\[
\begin{align*}
\mathbb{H}^n &= SO_0(n,1)/SO(n), \quad \text{(real hyperbolic space)} \\
\mathbb{H}^n_C &= SU(n,1)/S(U(n) \times U(1)), \quad \text{(complex hyperbolic space)} \\
\mathbb{H}^n_Q &= Sp(n,1)/Sp(n) \times Sp(1), \quad \text{(quaternionic hyperbolic space)} \\
\mathbb{H}^2_{Cay} &= F_4^{-20}/\text{Spin}(9), \quad \text{(Cayley hyperbolic plane)}
\end{align*}
\]

In all of the above examples, the nilpotent algebra \(\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]\) is of generalised Heisenberg type. By considering all the so-called generalised Heisenberg algebras (which are all two-step nilpotent) we can produce many more examples of such Einstein solvmanifolds [21]. For each of the generalised Heisenberg algebras, \(\mathfrak{n}\), there is a solvable extension \(\mathfrak{s}\) (i.e. \(\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]\)) whose Lie group admits an Einstein metric. These solvable extensions of generalised Heisenberg algebras are usually called Damek-Ricci spaces. These Damek-Ricci spaces thus provides us with even more examples of Einstein solvmanifolds.

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The simplest possible \(\mathfrak{s}\) is the Lie algebra defined by \([X_0, X_i] = X_i, \quad i = 1...n-1\), with all other commutators being zero. The corresponding Lie group acts simply transitively on \(n\)-dimensional real hyperbolic space, \(\mathbb{H}^n\).

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More recently, Jorge Lauret developed a new method of producing rank-one (i.e. dim(\(a\)) = 1) solvmanifolds \([22–24]\). The method relies on the observation that if \((n, g_n)\) is a so-called *Ricci nilsoliton*, then there is a metric solvable extension \((s, g_s)\) which is Einstein. This result, along with some other useful techniques Lauret developed, reduces the problem of finding rank-one Einstein solvmanifolds to finding Ricci soliton metrics on homogeneous nilmanifolds. Lauret explicitly used his methods to find all rank-one Einstein solvmanifolds up to dimension 6 \([24]\) while Cynthia Will found all of dimension 7 \([25]\).

4.1. Inhomogeneous Einstein metrics from Black Holes. The above Einstein solvmanifolds are homogeneous by construction. However, there is a simple way of constructing some inhomogeneous Einstein manifolds using these solvmanifolds as a starting point. The resulting inhomogeneous metrics have the property that they are asymptotically isometric to Einstein solvmanifolds and have an interesting Lorentzian interpretation.

The paper \([26]\) investigated Einstein solvmanifolds and explicitly found some Lorentzian black hole solutions. These black hole solutions are inhomogeneous and incomplete but their inhomogeneous Riemannian counterparts can be made complete and everywhere regular by an appropriate periodical identification.

By Wick-rotating the black hole solutions in \([26]\) – i.e. \(t = i\tau\) – the solutions can be written

\[
ds^2 = e^{-2p w} F(w) d\tau^2 + \frac{dw^2}{F(w)} + \sum_i e^{-2q_i w} (\omega^i)^2,
\]

\[(19) \quad F(w) = 1 - M e^{\sigma w},\]

where \(\sigma = p + \sum_i q_i\) and \(M > 0\).

The Lorentzian version of this metric describes a higher-dimensional black hole where the horizon geometry are products of solvgeometries and nilgeometries. For example, by considering the solvable extension of a generalised Heisenberg group we get black holes having a generalised Heisenberg group as a horizon. In this way we see that there is a great variety of black hole solutions for negatively curved spaces.

We do a change of coordinates

\[(20) \quad r^2 = 1 - M e^{\sigma w}, \quad \tau = \frac{2}{\sigma M \pi} \theta,\]

which transforms the metric into

\[
ds^2 = \frac{4}{\sigma^2} \left[ \frac{r^2 d\theta^2}{(1 - r^2)^2} + \frac{dr^2}{(1 - r^2)^2} \right] + \sum_i e^{-2q_i w(r)} (\omega^i)^2,
\]

where \(w(r)\) is implicitly given via eq. (20).

We note that by choosing \(\theta\) to be periodic with \(0 \leq \theta < 2\pi\) and \(0 \leq r < 1\), the metric becomes everywhere regular\(^8\). Near the horizon \((r = 0)\) the metric closes off regularly and hence, the metric describes a complete inhomogeneous Riemannian metric which is Einstein. The mass of the black hole, \(M > 0\), is arbitrary and

\(^8\) Writing \(\tau = \beta \theta\), the constant \(2\pi \beta\) can be interpreted as the inverse temperature of the black hole; i.e. \(2\pi \beta = 1/T\). This implies that the temperature is \(T \propto M^{1/\sigma}\), and hence, \(T\) is monotonically increasing in \(M\).
parametrises a one-parameter family of inhomogeneous spaces. The spaces are asymptotically Einstein solvmanifolds as \( r \to 1 \).

5. Ricci Nilsolitons

We have already pointed out that nilpotent groups do not allow for a left-invariant Einstein metric; the Ricci tensor, as given in eq. (8), will for nilpotent groups have both positive and negative eigenvalues. We thus might wonder whether there are any other “distinguished” metrics on nilpotent groups.

Lauret [22] noted that some nilpotent groups allow for metrics which obey

\[
R_{\mu\nu} = \lambda g_{\mu\nu} + D_{\mu\nu},
\]

where \( D^\mu_\nu \) as a linear map, \( D : n \to n \), is a derivation of \( n \); i.e.

\[
D ([X, Y]) = [D(X), Y] + [X, D(Y)].
\]

These metrics have a nice interpretation in terms of special solutions of the Ricci flow [27]. For a curve \( g(t) \) of Riemannian metrics on a manifold \( M \), the Ricci flow is defined by the equation

\[
\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}.
\]

If a solution to the Ricci flow moves by a diffeomorphism and is also scaled by a factor at the same time, we call the solution a homothetic Ricci soliton. In other words, if \( \phi_t \) is a one-parameter family of diffeomorphisms generated by some vector field, then if there is a solution of the form

\[
g(t) = c(t)\phi_t^* g,
\]

then \( g \) is a homothetic Ricci soliton.

Ricci nilsolitons are nilmanifolds with left-invariant metrics allowing for such solutions. Moreover, by inspection, all Einstein metrics will be so too. Hence, this is a way of generalising the Einstein requirement to nilpotent groups. These Ricci nilsolitons are unique up to isometry and scaling and can therefore be taken to be distinguished left-invariant metrics on nilmanifolds.

Lauret also showed the following property

**Theorem 5.1** (Lauret [22]). A homogeneous nilmanifold \((n, g_n)\) is a Ricci nilsoliton if and only if \((n, g_n)\) admits a metric solvable extension \((s, g_s)\) \((s = a \oplus n)\) with an Abelian corresponding solvmanifold is Einstein.

Hence, this result intertwines the problem of finding Einstein solvmanifolds and finding Ricci nilsolitons.

Furthermore, Lauret also noted that the Ricci nilsolitons are fixed points of the functional

\[
g \mapsto S[g_{\mu\nu}] = \int_M R_{\mu\nu} R^{\mu\nu} \sqrt{g} d^n x,
\]

restricted to a certain subset (basically the unit circle) of the space of all nilpotent brackets.

Examples of such Ricci nilsolitons are easy to find. For example, all metric groups of generalised Heisenberg type are Ricci nilsolitons. Moreover, all nilpotent

\[\text{Details of this variational procedure can be found in his papers [22, 24, 28].}\]
groups of dimension 6 or lower, admit a Ricci nilsoliton metric (see [25] where all the 6-dimensional ones are given).

5.1. **Higher-curvature gravitational nil-instantons.** Many fundamental theories of physics are defined via an action principle. For example, Einstein gravity can be defined through the Einstein-Hilbert action:

\[ g \mapsto S_{EH}[g_{\mu\nu}] = \int_M (R - 2\Lambda) \sqrt{|g|} d^n x. \]  

By requiring that the metric should be a fixed point of this map, implies the Einstein equation to be satisfied:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \]  

Compact Riemannian manifolds (with a totally geodesic boundary) which are solutions to this equation are usually called **gravitational instantons**.

It is easy to see that the Einstein equation (26) is equivalent to saying that the metric is Einstein; i.e. \( R_{\mu\nu} = \lambda g_{\mu\nu} \). We noted earlier that nilmanifolds do not admit an Einstein metric so there will be no non-trivial nilmanifolds being solutions to this equation. However, we will in the following rephrase the problem and ask: Are there actions for which the Ricci nilsolitons are solutions?

The result of Lauret gives us some hints of what such actions look like; they seem to contain quadratic terms like \( R_{\mu\nu} R^{\mu\nu} \). Let us therefore consider a class of higher-curvature gravity theories for which the action takes the form

\[ g \mapsto S_{(\alpha, \beta)}[g_{\mu\nu}] = \int_M (R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} - 2\Lambda) \sqrt{g} d^n x. \]  

The variation of the above action implies that the metric have to obey the generalised Einstein equation:

\[ \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \]  

where

\[ \Phi_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 2\alpha R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \\
+ (2\alpha + \beta) (g_{\mu\nu} \boxtimes - \nabla_{\mu} \nabla_{\nu}) R + \beta \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \\
+ 2\beta \left( R_{\mu\sigma\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho} \right) R^{\sigma\rho}, \]  

and \( \boxtimes \equiv \nabla^\mu \nabla_\mu \).

First note that all Einstein manifolds, \( R_{\mu\nu} = \lambda g_{\mu\nu} \), are solutions to eq.(28); hence, in some sense, this equation generalises the Einstein equation.

Regarding the nilsolitons, we also note that they are solutions for particular values of the parameter values \((\alpha, \beta, \Lambda)\). For example, consider the four-dimensional nilsoliton, \( \text{Nil}^4 \):

\[ ds^2 = dw^2 + (dx - ay dw)^2 + (dy - az dw)^2 + dz^2, \]

\( ^{10} \text{A nilmanifold can be compactified if and only if there exists a frame such that } [e_i, e_j] = C_{ij}^k e_k, \) where \( C_{ij}^k \) are all rational constants [29].
where $a$ is a constant. This is a solution to eq. (28) if and only if

$$
\text{Nil}^4 : \quad 2\alpha + 3\beta = \frac{1}{a^2}, \quad \Lambda = -\frac{a^2}{4}.
$$

Interestingly, the action for this solution is zero: $S_{(\alpha, \beta)}[\text{Nil}^4] = 0$.

Let us consider some other examples. We consider the generalised Heisenberg groups with the appropriate nilsoliton metric, $\mathcal{H}_{m,n}$. In Heber’s notation [20], the metric solvable extension of this group admits an Einstein metric of eigenvalue type $(1 < 2; n, m)$. This means that $\dim \mathcal{H}_{m,n} = m + n$. Note that equation (28) is invariant under a rescaling of the metric $ds^2 \mapsto \ell^2 ds^2$, along with a simultaneous rescaling of the parameters: $(\alpha, \beta, \Lambda) \mapsto (\ell^2 \alpha, \ell^2 \beta, \ell^{-2} \Lambda)$. After an appropriate rescaling, the spaces $\mathcal{H}_{m,n}$ are solutions to eq. (28) if

$$
\mathcal{H}_{m,n} : \quad \alpha + \frac{n + 4m \beta}{nm} = \ell^2, \quad \Lambda = -\frac{1}{8\ell^2}.
$$

Also for these solutions, the action is zero: $S_{(\alpha, \beta)}[\mathcal{H}_{m,n}] = 0$.

Thus it seems that the nilsolitons are solutions of certain higher-curvature theories of gravity. Admittedly, the approach we have followed here is in some sense the opposite of what is common. We have a set of spaces with particularly nice metrics, then we are trying to find the appropriate action which has these as solutions.

We should also point out what happens for other types of Lie algebras. There are also metrics on solvable Lie groups being solutions of eq. (28). Also in these cases eq. (28) seems to pick out a particularly nice metric. Regarding the semisimple groups, assuming $\beta > 0$, implies that the metric must be Einstein.

6. Outlook

In low-dimensional topology and geometry the homogeneous spaces seem to play an important role. According to a conjecture by Thurston [30], any compact 3-manifold can be decomposed into primes where each prime has to be one of 8 geometries. These 8 geometries are all homogeneous and are called “model geometries” [31]. Model geometries are special classes of homogeneous spaces and may be more interesting in a topological context. One of the biggest problems of proving Thurston’s conjecture for 3-manifolds is the enormous variety and richness of compact hyperbolic 3-manifolds [32]. In spite of the fact that they seem to be the most numerous, these negatively curved spaces seem to be the least understood of the 8 model geometries.

Another set of ”distinguished metrics” are bi-invariant metrics. For semi-simple groups, the bi-invariant metrics are related to Einstein metrics [4], and allowing for arbitrary signature, the relation is even more striking. For an investigation of bi-invariant metrics (not necessary Riemannian) on Lie groups, see, for example, [33].

Another thing it would be interesting to investigate is the relation between the symmetries of the negatively curved spaces and the symmetries of their conformal boundaries. For the simplest cases, namely the real hyperbolic spaces, we have the amazing correlation: $\text{Isom}(\mathbb{H}^n) = \text{Conf}(\partial \mathbb{H}^n)$ (see e.g. [32]). For $\mathbb{H}^n$ the isometries in the interior are uniquely determined by their action the conformal boundary. This interplay between the structure in the interior and on the conformal boundary ultimately gives rise to the celebrated AdS/CFT correspondence in theoretical physics. One might contemplate whether a similar correspondence holds for other negatively curved spaces.
This work has entirely been devoted to Riemannian manifolds; however, Lorentzian manifolds are equally interesting, particularly for theoretical physicists. Classifying the Lorentzian manifolds prove to be harder than the Riemannian case because some of the uniqueness results fail for Lorentzian manifolds. For example, homogeneous Riemannian manifolds are uniquely characterised by their curvature invariants [34]; however, for Lorentzian manifolds even after requiring that the metric to be homogeneous and Einstein, the curvature invariants do not uniquely determine the metric [35]. Notwithstanding, due to the role Lorentzian spaces play in fundamental theories of our physical universe, we believe that understanding the Lorentzian case would be extremely valuable.

Here we have only given a flavour of the enormous variety of geometrical structure these negatively curved homogeneous spaces have to offer. More study is clearly required to fully uncover and understand their geometric properties. Maybe in the future their apparently almost unlimited potential will be fully appreciated.

Note added:

Since these notes were written (June 2004), there have been further developments in the field. In particular, Lauret [36] has showed that all Einstein solvmanifolds are indeed standard, i.e., of the form described in section 5 (see also, [37]). Furthermore, the existence of the higher-curvature nil-instantons led me to realise that higher-curvature theories of gravity have a peculiar set of solutions which, from a cosmological standpoint, inflate anisotropically [38] (in fact, these exact anisotropically inflating solutions are solvmanifolds). This may have interesting consequences for early-universe cosmology and the cosmic microwave background [39, 40]. Moreover, the inhomogeneous Einstein metrics obtained from the black hole solutions were also discussed in [41].

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