ASYMPTOTIC STABILITY OF N-SOLITONS OF THE FPU LATTICES

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Abstract. We study stability of \( N \)-soliton solutions of the FPU lattice equation. Solitary wave solutions of FPU cannot be characterized as a critical point of conservation laws due to the lack of infinitesimal invariance in the spatial variable. In place of standard variational arguments for Hamiltonian systems, we use an exponential stability property of the linearized FPU equation in a weighted space which is biased in the direction of motion.

The dispersion of the linearized FPU equation balances the potential term for low frequencies, whereas the dispersion is superior for high frequencies. We approximate the low frequency part of a solution of the linearized FPU equation by a solution to the linearized KdV equation around an \( N \)-soliton.

We prove an exponential stability property of the linearized KdV equation around \( N \)-solitons by using the linearized Bäcklund transformation and use the result to analyze the linearized FPU equation.

1. Introduction

In this paper, we study stability of multi-pulse solutions of lattice equations which describe motion of infinite particles connected by nonlinear springs:

\[
\ddot{q}(t, n) = V'(q(t, n) - q(t, n - 1)) - V'(q(t, n + 1) - q(t, n)) \quad \text{for } (t, n) \in \mathbb{R} \times \mathbb{Z},
\]

where \( q(t, n) \) denotes the displacement of the \( n \)-th particle at time \( t \), \( V(r) \) denotes a kinetic potential and \( \cdot \) denotes differentiation with respect to \( t \). Making use of the change of variables \( p(t, n) = \dot{q}(t, n), r(t, n) = q(t, n + 1) - q(t, n) \) and \( u(t, n) = (r(t, n), p(t, n)) \), we can translate (1.1) into a Hamiltonian system

\[
\frac{du}{dt} = J H'(u),
\]

where \( J = \begin{pmatrix} 0 & e^{\theta} - 1 \\ 1 - e^{-\theta} & 0 \end{pmatrix} \), \( e^{\pm \theta} \) are the shift operators defined by \((e^{\pm \theta})f(n) = f(n \pm 1)\) and

\[
H(u(t)) = \sum_{n \in \mathbb{Z}} \left( \frac{1}{2} p(t, n)^2 + V(r(t, n)) \right) \quad \text{(Hamiltonian)}.
\]

Typical examples of (1.1) are the \( \alpha \)-FPU equation \((V(r) = \frac{1}{2} r^2 + \frac{1}{6} r^3)\) and the Toda lattice equation \((V(r) = e^{-r} - 1 + r)\).

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Originally, Fermi-Pasta and Ulam [4] studied the FPU lattice numerically to observe the equipartition of the energy among all Fourier modes and found an almost recurrence phenomena contrary to their expectation. Zabusky and Kruskal [30] numerically found multi-solitons of KdV that was known to describe the long wave solutions of FPU and interpreted their result as an explanation of the FPU recurrent phenomena. For recent development of metastability results on solitary waves of the finite FPU lattice, see [1] and the references therein.

The FPU lattice equation has solitary wave solutions due to a balance of nonlinearity and dispersion induced by discreteness. This was indicated by [3] by numerics before being proved by Friesecke and Wattis [9] by using a concentration compactness theorem. See also [28] for the Toda lattice equation that is integrable and has explicit $N$-soliton solutions.

Eq. (1.2) has two parameter family of solitary wave solutions \( \{u_c(n - ct - \gamma) : c \in (-\infty, -1) \cup (1, \infty), \gamma \in \mathbb{R}\} \), where \( u_c = \begin{pmatrix} r_c \\ p_c \end{pmatrix} \) is a solution of

\[
\partial_x u_c + JH'(u_c) = 0.
\]

In the case where \( c \) is close to 1 or \(-1\), Friesecke and Pego [5] prove that solitary wave solutions are unique up to translation and their shape are similar to KdV 1-solitons. We remark that a solitary wave solution \( u_c(. - ct) \) is small if \( c \) is close to 1 or \(-1\) and \( \lim_{c \to \pm 1} H(u_c) = 0 \).

Friesecke and Pego also prove in [6, 7, 8] that small solitary waves of FPU are asymptotically stable in an exponentially weighted space. Their idea is to compare spectral property of the linearized FPU equation and the linearized KdV equation and to make use of the phenomena that the main solitary wave moves fastest to the right (or to the left) and it outruns from the rest of the solution as Pego and Weinstein [23] did for KdV. See also Mizumachi and Pego [22] that prove stability of Toda lattice 1-solitons of any size. More recently, Mizumachi [20] has proved stability of 1-soliton solutions of FPU in the energy space and Hoffman and Wayne proved stability of two solitary waves which propagate to the opposite directions.

Our goal is to prove stability of \( N \)-solitons in the energy space. In this paper, we assume

\[
(H1) \quad V \in C^\infty(\mathbb{R}; \mathbb{R}), \quad V(0) = V'(0) = 0, \quad V''(0) = 1, \quad V'''(0) = \frac{1}{6},
\]

and use the following properties of solitary wave solutions proved by [5].

(P1) Let \( c_* > 1 \) be a constant sufficiently close to 1 and let \( a \in [0, 2) \). For any \( c \in (1, c_*) \), there exists a unique single hump solution of (1.3) in \( l^2 \) up to translation in \( x \). Moreover, \( \sqrt{6(c - 1)} =: \varepsilon \mapsto e^{-2u_c(\varepsilon)} \in H^5(\mathbb{R}; e^{2a|x|}) \) is \( C^2 \).

(P2) There exists an open interval \( I \) such that \( V''(r) > 0 \) for every \( r \in I \) and that \( \{r_c(x) : x \in \mathbb{R}\} \subset I \) for every \( c \in (1, 1 + c_*) \).

(P3) The solitary wave energy \( H(u_c) \) satisfies \( dH(u_c)/dc \neq 0 \) for \( c \in (1, c_*) \).
(P4) As $c$ tends to 1, a shape of solitary wave solution becomes similar to that of a KdV 1-soliton. More precisely,
\[
\sum_{j=0}^{2} \varepsilon^j \left\| \frac{\partial^j}{\partial \varepsilon^j} \left( \varepsilon^{-2} r_c \left( \frac{x}{\varepsilon} \right) - \text{sech}^2 x \right) \right\|_{H^5(\mathbb{R}, e^{2\varepsilon|x|} dx)} = O(\varepsilon^2).
\]

Now we state our main result.

**Theorem 1.1.** Let $0 < k_1 < \cdots < k_N$ and $c_{i,0} = 1 + \frac{k_i^2 + 2}{6}$ $(1 \leq i \leq N)$. There exist positive numbers $\varepsilon_0$, $\gamma_0$, $A_0$ and $\delta_0$ satisfying the following: Suppose $\varepsilon \in (0, \varepsilon_0)$ and that $u(t)$ is a solution to (1.2) such that $\|v_0\|_{l^2} < \delta_0 \varepsilon^2$,
\[
u(\cdot,0) = \sum_{i=1}^{N} u_{c_{i,0}}(\cdot - x_{i,0}) + v_0,
\]
(1.4)
\[
L := \min_{2 \leq i \leq N} \varepsilon(x_{i,0} - x_{i-1,0}) \geq \frac{1}{k_1} | \log(\delta_0 \varepsilon) |.
\]
(1.5)

Then there exist $C^1$-functions $x_i(t)$ ($i = 1, \cdots, N$) such that
\[
\sup_{t \geq 0} \left\| u(\cdot, t) - \sum_{i=1}^{N} u_{c_{i,0}}(\cdot - x_i(t)) \right\|_{l^2} < A_0(\|v_0\|_{l^2} + \varepsilon^2 e^{-\gamma_0 L}).
\]
(1.6)

Furthermore, there exist $c_{N,+} > \cdots > c_{1,+} > 1$ and $c_\ast \in (1, (1 + c_{1,0})/2)$ such that
\[
\lim_{t \to \infty} \left\| u(\cdot, t) - \sum_{i=1}^{N} u_{c_{i,+}}(\cdot - x_i(t)) \right\|_{l^2(n \geq c_\ast t)} = 0,
\]
(1.7)
\[
\lim_{t \to \infty} \tilde{x}_i(t) = c_{i,+} \quad \text{and} \quad |c_{i,+} - c_{i,0}| < A_0(\varepsilon^{-1}\|v_0\|_{l^2} + \varepsilon^2 e^{-\gamma_0 L}) \quad \text{for } 1 \leq i \leq N.
\]
(1.8)

**Remark 1.1.** Eq. (1.6) implies orbital stability of FPU co-propagating $N$-solitons since by (P4),
\[
\|u_{c_{i,0}}\|_{l^2}^2 = 2 \int_{\mathbb{R}} r_{c_{i,0}}(x)^2 dx(1 + o(1)) = \frac{8k_i^3 \varepsilon^3}{3}(1 + o(1)).
\]

**Remark 1.2.** The solitary waves moving to the same direction interact more strongly than counter-propagating solitary waves because they interact each other through their tails for a longer period. Noting that the relative speeds between solitary waves are of $O(\varepsilon^2)$, we see that the total impulse caused by the interaction of solitary waves is of $O(\varepsilon^{\frac{3}{2}} e^{-k_1 L}) = O(\varepsilon^{\frac{3}{2}})$ in the setting of Theorem 1.1 whereas the total impulse caused by the interaction among counter-propagating solitary waves is of $O(\varepsilon^2)$ (1.3).

 Orbital stability of KdV multi-solitons was first studied by Maddocks and Sachs [16] (see Kapitula [14] for other integrable systems). In the nonintegrable case, Perelman [24, 25], Rodnanski-Schlag-Soffer [27] proved stability of multi-solitons of nonlinear Schrödinger equations that have super critical nonlinearities by using scattering theory.
Martel-Merle-Tsai [18, 19] studied stability of multi-soliton solutions of gKdV and NLS by combining a variational argument ([2, Chapter 8]) and some propagation estimates. Their approach seems more favorable because FPU has a subcritical nonlinearity. However, a solitary wave solution cannot be characterized as a local minimizer because FPU does not have a conservation law corresponding to momentum for KdV because the spatial variable is defined on $\mathbb{Z}$.

Instead of using the positivity of the second variation of a conservation law as is done in [18, 19], we will use exponential linear stability property of the multi-soliton. The idea of using exponential linear stability property was applied to FPU by Friesecke and Pego [5, 6, 7, 8] and lately used by Mizumachi [20] to prove orbital stability of 1-soliton solutions of FPU.

We remark that most of propagation estimates of linearized dispersive equations around multi-solitons are obtained in the case where relative speed between solitary waves are large (Perelman [24, 25], Rodgnanski-Schlag-Soffer [27], Hoffman-Wayne [13]) so that a dispersive wave mostly interacts with one solitary wave and virtually has no interaction with the others. In these cases, the problem can be reduced to that of 1-soliton solutions by using Fourier analysis or cut-off functions. The other extreme case is where the relative speed is small (Mizumachi [21]). In that case, 2-soliton solutions can be treated as a multi-bump bound state for a sufficiently long time.

In our problem, a dispersive wave effectively interacts with all the solitary waves which locate behind the dispersive wave at initial time because the group velocity of plane waves is $\pm \cos \frac{\xi}{2} \in [-1, 1]$ and velocity of solitary waves are larger than 1. Therefore, we need to consider exponential linear stability of $N$-solitons without using cut-off functions in the spatial variable.

To prove exponential linear stability of FPU $N$-solitons, we translate the linearized equation into a system of a high frequency part, a middle frequency part and a low frequency part. The high frequency part is governed by a linearized FPU equation around the null solution and the middle and low frequencies are in the KdV regime. The behavior of middle frequency modes is approximated by $u_t + u_{xxx} = 0$ because the potential term turns out to be negligible in this region. For low frequency modes, the dispersion and the potential term are of the same order and its behavior is governed by a linearized KdV equation around $N$-soliton solutions.

Haragus and Sattinger [11] proved exponential linear stability of linearized KdV equations in a class of analytic functions. In this paper, we show the exponential linear stability in weighted $L^2$ spaces.

Before we state our result, let us introduce several notations. Let $0 < k_1 < \cdots < k_N$, $\gamma_i \in \mathbb{R}$, $\theta_i = k_i(x - 4k_i^2t - \gamma_i)$ for $i = 1, \ldots, N$ and let $k = (k_1, \cdots, k_N)$, $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ and

$$ C_N = \left[ \frac{1}{k_i + k_j} e^{-(\theta_i + \theta_j)} \right]_{i = 1, \ldots, N}^{j = 1, \ldots, N}.$$
Then $\varphi_N(t, x; k, \gamma) := \partial_2^2 \log \det (I + C_N)$ is an N-soliton solution of KdV
\begin{equation}
\partial_t u + \partial_x (\partial_2^2 u + 6u^2) = 0 \quad \text{for } x \in \mathbb{R} \text{ and } t > 0.
\end{equation}

Especially $\varphi_1(t, x; k, \gamma) = k^2 \sech^2 k(x - 4k^2t - \gamma)$.

Let $a > 0$ and
\[ \mathcal{P}(t, k, \gamma) : L^2_0 \rightarrow \text{span}\{\partial_{\gamma_i} \varphi_N(t, y; k, \gamma), \partial_{k_i} \varphi_N(t, y; k, \gamma) : 1 \leq i \leq N\}, \quad Q(t, k, \gamma) = I - \mathcal{P}(t) \]
be projections associated with
\begin{equation}
\partial_t v + \partial_x (\partial_2^2 v + 12 \varphi_N(k, \gamma)v) = 0 \quad \text{for } x \in \mathbb{R} \text{ and } t > 0.
\end{equation}

such that for $v \in Q(t, k, \gamma)$ and $i = 1, \cdots, N$,
\begin{align}
\int_{\mathbb{R}} v(x) \int_{-\infty}^{x} \partial_{\gamma_i} \varphi_N(t, y; k, \gamma) dy dx &= 0, \\
\int_{\mathbb{R}} v(x) \int_{-\infty}^{x} \partial_{k_i} \varphi_N(t, y; k, \gamma) dy dx &= 0.
\end{align}

If $v$ is a solution of (1.10) and $v(s) \in Q(s)$, then $v(t) \in Q(t)$ for every $t \geq s$.

**Theorem 1.2.** Let $0 < k_1 < \cdots < k_N$, $0 < a < 2k_1$, $\theta \geq 0$, $\eta \in (0, 1)$ and let $v(t, x)$ be a solution of (1.10). Then there exists a positive constant $K$ such that for every $t > s$ and $c$, $x_0 \in \mathbb{R}$,
\begin{align*}
\|e^{a(-ct-x_0)} Q(t)v(t)\|_{L^2} &\leq Ke^{-a(c-a^2)(t-s)}\|e^{a(-cs-x_0)} Q(s)v(s)\|_{L^2}, \\
\|e^{a(-ct-x_0)} Q(t)v(t)\|_{L^2} &\leq K(t-s)^{-\frac{\theta}{2}}e^{-\eta a(c-a^2)(t-s)}\|e^{a(-cs-x_0)} Q(s)v(s)\|_{H^{-\theta}}.
\end{align*}

Our plan of the present paper is as follows. In Section 2 we decompose a solution that is close to a family of N-solitons into a sum of an N-soliton part and several remainder parts and derive modulation equations on parameters of speed and phase shift of the N-soliton part. In Section 3 we estimate the energy norm of the remainder parts and prove virial identities for each remainder part. In Section 4 we prove orbital and asymptotic stability of N-solitons assuming exponential linear stability of N-solitons of FPU. In Section 5 we will prove exponential linear stability of small N-soliton solutions of FPU assuming exponential stability property of KdV. In Section 6 we will use a linearized Bäcklund transformation to prove Theorem 1.2 following the idea of Mizumachi and Pego [22]. We will show that a linearized Bäcklund transformation determines an isomorphism that connects solutions of $u_t + u_{xxx} = 0$ and solutions of (1.10) satisfying (1.11) and (1.12) whose operator norm is uniformly bounded with respect to $t$.

Finally, let us introduce some notations. Let $\langle u, v \rangle := \sum_{n \in \mathbb{Z}} (u_1(n)u_2(n) + v_1(n)v_2(n))$ for $\mathbb{R}^2$-sequences $u = (u_1, u_2)$ and $v = (v_1, v_2)$ and let $\|u\|_{L^2} = (\langle u, u \rangle)^{\frac{1}{2}}$ and $\|u\|_{L^2} = \|e^{an}u(n)\|_{L^2}$. We use notations $\|u\|_{L^2(\mathbb{R})} = \|e^{ax}u(x)\|_{L^2(\mathbb{R})}$ and $\|u\|_{L^2(\mathbb{R})} = \|e^{ax}u(x)\|_{H^1(\mathbb{R})}$. 
For Banach spaces $X$ and $Y$, we denote by $B(X,Y)$ the space of all linear continuous operators from $X$ to $Y$ and abbreviate $B(X,X)$ as $B(X)$. We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \leq Cb$. For any $f \in l^2$, 
\[
(F_n f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} f(n)e^{-in\xi},
\]
and $(f_1 \ast_T f_2)(x) = \int_{\mathbb{T}} f_1(x-y)f_2(y)dy$ for $f_1, f_2 \in L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We denote by $\tau_h$ a translation operator defined by $(\tau_h f)(x) := f(x + h)$.

2. Decomposition of the solution

Let $u(t)$ be a solution to (1.2) which lies in a tubular neighborhood of
\[
\mathcal{M} = \left\{ \sum_{i=1}^{N} u_{c_i,0}(\cdot - y_i) : y_{i+1} - y_i > L \text{ for } i = 1, \cdots, N - 1 \right\},
\]
where $L$ is sufficiently large.

We decompose a solution around $\mathcal{M}$ as
\[
u(t) = \sum_{1 \leq i \leq N} u_{c_i(t)}(\cdot - x_i(t)) + v(t),
\]
where $u_{c_i(t)}(\cdot - x_i(t)) (i = 1, \cdots, N)$ denote solitary waves and $c_i(t)$ and $x_i(t)$ are modulation parameters of the speed and the phase shift of each solitary wave, respectively. Let $U_N(t) = \sum_{i=1}^{N} u_{c_i(t)}(\cdot - x_i(t))$. Substituting (2.1) into (1.2), we have
\[
\partial_t v = JH''(U_N)v + l + R,
\]
where $R = R_1 + R_2$ and
\[
R_1 = JH'(U_N + v) - JH'(U_N) - JH''(U_N)v,
\]
\[
R_2 = JH'(U_N) - \sum_{i=1}^{N} JH'(u_{c_i(t)}(\cdot - x_i(t)));
\]
\[
l = -\sum_{i=1}^{N} \{c_i \partial_x u_{c_i}(\cdot - x_i(t)) - (\dot{c}_i - c_i) \partial_x u_{c_i}(\cdot - x_i(t))\}.
\]

Now we decompose $v(t)$ into the sum of a small solution $v_1(t)$ to (1.2) and a remainder term which belongs to $l^2_a$ and is localized around solitary waves. Let $v_1(t)$ be a solution to
\[
\left\{ \begin{array}{l}
\partial_t v_1 = JH'(v_1), \\
v_1(0) = v_0,
\end{array} \right.
\]
and $v_2(t) = v(t) - v_1(t)$. By [20, Proposition 3], we see that $u(t) - v_1(t)$ remains in $l^2_a$ for every $0 \leq a < 2 \min_{1 \leq i \leq N} \kappa(c_{i,0})$ and $t \in \mathbb{R}$, where $\kappa(c)$ is a positive root of $c = \sinh \kappa/\kappa$. 

Suppose \( x_i(t) \) and \( c_i(t) \) are of class \( C^1 \). Then if \( u(t) = U_N(t) + v_1(t) + v_2(t) \) is a solution to (1.2),

\[
\begin{cases}
\partial_t v_2 = JH''(U_N(t))v_2 + l(t) + \tilde{R}(t), \\
v_2(0) = 0,
\end{cases}
\]

(2.4)

where \( \tilde{R}(t) = R(t) - JH'(v_1(t)) + JH''(U_N(t))v_1 \). Our strategy is to derive modulation equations on \( x_i(t) \) and \( c_i(t) \) and \( \text{a priori} \) estimates on \( v_2, x_i \) and \( c_i \) \((1 \leq i \leq N)\) to prove that \( u \) remains in a tubular neighborhood of \( \mathcal{M} \) in \( l^2 \). To prove convergence of speed parameters \( c_i(t) \) \((1 \leq i \leq N)\), we need to estimate \( v_2(t) \) in an exponential weighted space. Since \( e^{-k_1 x_1(t)}\|v_2(t)\|_{k_1^2} \) may grow as \( t \to \infty \) due to the interaction between \( v_1(t) \) and solitary waves \( u_{c_i}(\cdot - x_i(t)) \) \((i \geq 2)\), we will decompose \( v_2(t) \) into a sum of \( N \) functions \( v_{2k} \) \((1 \leq k \leq N)\) such that each \( v_{2k}(t) \) remains small in a weighted space

\[
X_k(t) = \left\{ v \in l^2_{k_1^2} : \|v\|_{X_k(t)} = \left( \sum_{n \in \mathbb{Z}} e^{k_1^2(n-x_{N+1-k}^{(1)})} \|v(n)\|^2 \right)^{\frac{1}{2}} < \infty \right\}.
\]

Let \( Q_k(t) : l^2_a \to l^2_a \) be an operator defined by

\[
Q_k(t)f = f - \sum_{N+1-k \leq i \leq N} (\alpha_i(f) \partial_x u_{c_i}(\cdot - x_i(t)) + \beta_i(f) \partial_c u_{c_i}(\cdot - x_i(t)))
\]

for \( a > 0 \), where \( \alpha_i(f) \) and \( \beta_i(f) \) \((i = 1, \ldots, N)\) are real numbers satisfying

\[
\langle Q_k(t)f, J^{-1} \partial_x u_{c_i}(\cdot - x_i(t)) \rangle = \langle Q_k(t)f, J^{-1} \partial_c u_{c_i}(\cdot - x_i(t)) \rangle = 0
\]

for \( N + 1 - k \leq i \leq N \) and let \( P_k(t) = I - Q_k(t) \). We remark that if \( a > 0 \),

\[
J^{-1} = \begin{pmatrix} 0 & \sum_{k=-\infty}^{0} e^{k\theta} \\ \sum_{k=-\infty}^{0} e^{-k\theta} & 0 \end{pmatrix}
\]

(2.5)

is a bounded operator on \( l^2_a \) because \( \|e^{-\theta u}\|_{l^2_a} = e^{-a\|u\|_{l^2_a}} \) and that \( J^{-1} \partial_c u_c \) and \( J^{-1} \partial_x u_c \) belong to \( l^2_a \) for any \( a \in (0, 2\kappa(c)) \).

Let \( v_{2k}(t) \) \((1 \leq k \leq N - 1)\) be a solution of

\[
\begin{cases}
\partial_t v_{2k} = JH''(U_k)v_{2k} + l_k + Q_k(t)JR_k, \\
v_{2k}(0) = 0,
\end{cases}
\]

(2.6)

where \( w_0 = v_1, \ w_k = v_1 + \sum_{1 \leq i \leq k} v_{2i} \) \((1 \leq k \leq N)\),

\[
R_k = H'(U_k + w_k) - H'(u_{c_{N+1-k}}) - H'(U_{k-1} + w_{k-1}) - H''(U_k)v_{2k},
\]

\[
l_k = \sum_{N+1-k \leq j \leq N} (\alpha_{j,k} \partial_c u_{c_j} + \beta_{j,k} \partial_x u_{c_j}),
\]

and \( \alpha_{j,k} \) and \( \beta_{j,k} \) \((N + 1 - k \leq j \leq N, 1 \leq k \leq N - 1)\) are continuous functions that will be defined later.
Let $v_{2N}(t) = v_2(t) - \sum_{1 \leq i \leq N-1} v_{2i}(t)$. To fix the decomposition (2.10), we will define $c_i(t)$ and $x_i(t) (1 \leq i \leq N)$ so that

\begin{align}
(2.7) & \quad \langle v_{2N}(t), J^{-1} \partial_x u_{c_i(t)}(\cdot - x_i(t)) \rangle = 0 \quad \text{for } i = 1, \ldots, N, \\
(2.8) & \quad \langle v_{2N}(t), J^{-1} \partial_x u_{c_i(t)}(\cdot - x_i(t)) \rangle = 0 \quad \text{for } i = 1, \ldots, N.
\end{align}

By (2.2), (2.3) and (2.6),

\[
\partial_t v_{2N} = J \left\{ H'(U_N + v) - \sum_{k=1}^N H'(u_{c_k}) - H'(v_1) \right\} + l
\]

\[
- \sum_{k=1}^{N-1} (JH''(U_k)v_{2k} + Q_k(t)JR_k + l_k)
\]

\[
= JH''(U_N)v_{2N} + JR_N + \sum_{k=1}^{N-1} (P_k(t)JR_k - l_k) + l.
\]

Let $A_k = (A_{i,j})_{i=N+1-k, \ldots, N\downarrow}$, $F_{j,k} = t(F_{j,k}^1, F_{j,k}^2)$ and

\[
A_{i,j} = \begin{pmatrix}
\varepsilon^{-1}(\partial_t u_{c_j}, J^{-1} \partial_x u_{c_i}) \\
\varepsilon^2(\partial_t u_{c_j}, J^{-1} \partial_x u_{c_i})
\end{pmatrix}, \\
F_{j,k}^1 = \varepsilon^{-4}(v_{2k}, (H''(U_k) - H''(u_{c_j}))\partial_x u_{c_j})
\]

\[
+ \varepsilon^{-4}\{(\dot{x}_j - c_j)(v_{2k}, J^{-1} \partial_x^2 u_{c_j}) - \dot{c}_j(v_{2k}, J^{-1} \partial_x u_{c_j})\},
\]

\[
F_{j,k}^2 = \varepsilon^{-1}\{(v_{2k}, (H''(U_k) - H''(u_{c_j}))\partial_x u_{c_j})
\]

\[
+ \varepsilon^{-1}\{(\dot{x}_j - c_j)(v_{2k}, J^{-1} \partial_x^2 u_{c_j}) - \dot{c}_j(v_{2k}, J^{-1} \partial_x^2 u_{c_j})\}.
\]

If $\alpha_{j,k}(t)$ and $\beta_{j,k}(t)$ are chosen to be a solution of

\[
(2.10) \quad A_k \begin{pmatrix}
\varepsilon^{-3}\alpha_{j,k} \\
\beta_{j,k}
\end{pmatrix}_{N+1-k \leq j \leq N\downarrow} = \begin{pmatrix}
F_{j,k}^1 \\
F_{j,k}^2
\end{pmatrix}_{N+1-k \leq j \leq N\downarrow},
\]

then $v_{2k} (1 \leq k \leq N - 1)$ satisfy secular term conditions.

**Lemma 2.1.** Suppose that $x_i(t)$ and $c_i(t)$ (1 \leq i \leq N) are of class $C^1$ on $[0, T]$ and that $v_{2k}$ (1 \leq k \leq N - 1) satisfy (2.6) and (2.10) for 1 \leq k \leq N - 1 and $t \in [0, T]$. Then

\[
(2.11) \quad \langle v_{2k}, J^{-1} \partial_x u_{c_i} \rangle = \langle v_{2k}, J^{-1} \partial_x u_{c_i} \rangle = 0
\]

for every $N + 1 - k \leq i \leq N$, 1 \leq k \leq N - 1 and $t \in [0, T]$.

**Proof.** First, we recall that $H(\cdot - ct)$ does not depend on $t$ and

\[
(2.12) \quad \langle \partial_x u_{c_i}, J^{-1} \partial_x u_{c_i} \rangle = -\frac{1}{c}(\partial_x u_{c_i}, H'(u_{c_i})) = \frac{1}{c} \frac{d}{dt}H(\cdot - ct) = 0,
\]

\[
(2.13) \quad \langle \partial_x u_{c_i}, J^{-1} \partial_x u_{c_i} \rangle = -\langle \partial_x u_{c_i}, J^{-1} \partial_x u_{c_i} \rangle = \frac{1}{c} \frac{d}{dc}H(u_{c_i}) > 0.
\]
Differentiating \(\text{(1.3)}\) with respect to \(x\) and \(c\), we have
\[
(2.14) \quad c\partial_x^2 u_c + JH''(u_c)\partial_x u_c = 0, \quad c\partial_c \partial_x u_c + JH''(u_c)\partial_c u_c = -\partial_x u_c.
\]

Using \((2.6), (2.14)\), \(J^* = -J\) and the fact that \(J^{-1}\partial_x u_c\) and \(J^{-1}\partial_c u_c\) \((N + 1 - k \leq j \leq N)\) are orthogonal to the range of the projection \(Q_k(t)\), we have for \(N + 1 - k \leq j \leq N\) and \(1 \leq k \leq N - 1\)
\[
\frac{d}{dt} \langle v_{2k}, J^{-1}\partial_x u_c \rangle (-x_j(t))
= \langle JH''(U_k)v_{2k} + l_k + Q_kJR_k, J^{-1}\partial_x u_c \rangle
- \dot{x}_j \langle v_{2k}, J^{-1}\partial_x^2 u_c \rangle + \dot{c}_j \langle v_{2k}, J^{-1}\partial_x u_c \rangle
= \langle l_k, J^{-1}\partial_x u_c \rangle + \langle v_{2k}, (H''(u_c) - H''(U_k))\partial_x u_c \rangle + \langle v_{2k}, J^{-1}\partial_x u_c \rangle
+ \dot{c}_j \langle v_{2k}, J^{-1}\partial_x^2 u_c \rangle - (\dot{x}_j - c_j) \langle v_{2k}, J^{-1}\partial_x^2 u_c \rangle
= \sum_{i=N+1-k}^{N} \left( \alpha_{i,k} \langle \partial_x u_c, J^{-1}\partial_x u_c \rangle + \beta_{i,k} \langle \partial_x u_c, J^{-1}\partial_x u_c \rangle \right)
- \langle v_{2k}, (H''(U_k) - H''(u_c))\partial_x u_c \rangle
- (\dot{x}_j - c_j) \langle v_{2k}, J^{-1}\partial_x^2 u_c \rangle + \dot{c}_j \langle v_{2k}, J^{-1}\partial_x^2 u_c \rangle + \langle v_{2k}, J^{-1}\partial_x u_c \rangle.
\]

In the course of calculations, we abbreviate \(u_{c_j(t)}(-x_j(t))\) as \(u_{c_j}\). Substituting \((2.10)\) into the above, we have for \(N + 1 - k \leq j \leq N\),
\[
\frac{d}{dt} \langle v_{2k}(t), J^{-1}\partial_x u_c \rangle = 0, \quad \frac{d}{dt} \langle v_{2k}(t), J^{-1}\partial_c u_c \rangle = \langle v_{2k}, J^{-1}\partial_x u_c \rangle.
\]
Since \(v_{2k}(0) = 0\), we have \((2.11)\) for every \(1 \leq j \leq N, N + 1 - k \leq k \leq N - 1\) and \(t \in [0, T]\).
Thus we complete the proof. \(\square\)

Next we will derive modulation equations of \(x_i\) and \(c_i\) so that \(v_{2N}\) satisfies \((2.7)\) and \((2.8)\).
Lemma 2.2. Let $u(t)$ be a solution of (1.2) and $v_1(t)$ be a solution of (2.3). There exist positive numbers $L$, $\varepsilon_0$ and $\delta$ satisfying the following: Suppose $\varepsilon \in (0, \varepsilon_0)$, that $c_i(t)$ and $x_i(t)$ $(i = 1, \cdots, N)$ are $C^1$-functions satisfying (2.7) and (2.8) on $[0, T]$ and that

$$\max_{1 \leq i \leq N} \sup_{t \in [0, T]} (|c_i(t) - c_{i,0}| + |\dot{x}_i(t) - c_i(t)|) \leq \delta \varepsilon^2,$$

$$\min_{1 \leq i \leq N-1} \inf_{t \in [0, T]} (x_{i+1}(t) - x_i(t)) \geq \varepsilon^{-1} L,$$

$$\sup_{t \in [0, T]} (\|v_1(t)\|_{W(t)} + \sum_{1 \leq k \leq N} \|v_{2k}(t)\|_{W(t) \cap X_k(t)}) \leq \delta \varepsilon^2.$$

Let $\sigma = \frac{1}{2} \varepsilon^{-2} \min_{2 \leq i \leq N} (c_{i,0} - c_{i-1,0})$. Then for $t \in [0, T]$,

$$\frac{d}{dt} \left\{ c_i(t) \left( 1 - \theta_1(c_i(t))^{-1} (v_1(t) + \sum_{k=1}^{N-i} v_{2k}(t), \rho_{c_i(t)}) \right) \right\}$$

(2.15)

$$= O \left( \varepsilon^2 \left( \|v_1(t)\|_{W(t)}^2 + \sum_{k=1}^{N} \|v_{2k}(t)\|_{W(t) \cap X_k(t)}^2 \right) + \varepsilon^5 e^{-2k_1(\sigma \varepsilon^3 t + L)} \right),$$

$$\dot{x}_i(t) - c_i(t)$$

(2.16)

$$= O \left( \varepsilon^\frac{1}{2} \left( \|v_1(t)\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}(t)\|_{W(t)} \right) + \varepsilon^2 e^{-k_1(\sigma \varepsilon^3 t + L)} \right),$$

where $\theta_1(c) = d\mathcal{H}(u_c)/dc$, $\rho_c = \partial_x(c\partial_x + J)^{-1}(H'(u_c) - u_c)$ and

$$\|u\|_{W(t)} = \sum_{1 \leq i \leq N} \|e^{-k_i \varepsilon |x_i(t)/2|} u\|_2, \quad \|u\|_{X_k(t) \cap W(t)} = \|u\|_{X_k(t)} + \|u\|_{W(t)}.$$

Remark 2.1. A solution of a system (2.3), (2.6), (2.9), (2.10), (2.15) and (2.16) (more precisely (2.21)) exists at least locally in time. If it satisfies an initial condition

$$v_1(0) = v_1, \quad v_{21}(0) = \cdots = v_{2N}(0) = 0, \quad x_i(0) = x_{i,0}, \quad c_i(0) = c_{i,0},$$

then $u(t) = \sum_{i=1}^{N} u_{c_i(t)}(\cdot - x_i(t)) + v_1(t) + \sum_{k=1}^{N} v_{2k}(t)$ becomes a solution to (1.2) and

$$u(0) = \sum_{i=1}^{N} u_{c_{i,0}}(\cdot - x_{i,0}) + v_0.$$

To prove Lemma 2.2 we need the following:

Lemma 2.3. Suppose that $c_i(t)$ and $x_i(t)$ be as in Lemma 2.2. Then there exists a positive constant $C$ depending only on $k_1, \cdots, k_N$, $\varepsilon_0$, $\delta$ and $L_0$ such that

$$\sup_{t \in [0, T]} (|A_{i,j}| + |A_k^{-1}|) \leq C \quad \text{for } 1 \leq i, j, k \leq N.$$
Lemma 2.4. Suppose that $c_i(t)$ and $x_i(t)$ be as in Lemma 2.2. Then there exists a positive constant $C$ depending only on $k_1, \cdots, k_N, \varepsilon_0, \delta$ and $L_0$ such that
\[
\sup_{t \geq 0}(\|P_k(t)\|_{B(0, 1)} + \varepsilon^{-1}\|P_k(t)J\|_{B(\frac{1}{\varepsilon})}) \leq C \quad \text{for } 1 \leq k \leq N.
\]

Proof of Lemma 2.3. Let $\theta_2(c) = \langle \partial_c p_c, 1 \rangle \langle \partial_c r_c, 1 \rangle$, $\theta_3(c_i, c_j) = \langle \partial_c p_{c_i, c_j}, 1 \rangle \langle \partial_c r_{c_i, c_j}, 1 \rangle + \langle \partial_c p_{c_j, c_i}, 1 \rangle \langle \partial_c r_{c_j, c_i}, 1 \rangle$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_1(c) = -(c c^{-1})\theta_1(c)\sigma_3 + \varepsilon^2\theta_2(c)$, $B_2(c_i, c_j) = \varepsilon^2\theta_3(c_i, c_j)$, $B_3(c_i, c_j) = -B_1(c_i)^{-1}B_2(c_i, c_j)B_1(c_j)^{-1}$.

By (2.12) and (2.13), we have $A_{ii} = B_1(c_i)$. Since
\[
x_i(t) - x_j(t) \geq x_i(0) - x_j(0) + \int_0^t (\dot{x}_i(s) - \dot{x}_j(s))ds \\
\geq \varepsilon^{-1}L + (c_i, 0 - c_j, 0 - 2\delta^2)\varepsilon \tau \\
\geq \sigma\varepsilon^2 t + \varepsilon^{-1}L \quad \text{for } i > j,
\]
it follows from Claims A.3 and A.7 that
\[
A_{i,j} = \begin{cases} 
B_2(c_i, c_j) + O(e^{-k_1(\sigma^3t+L)}) & \text{if } i < j, \\
O(e^{-k_1|x_i-x_j|}) & \text{if } i > j.
\end{cases}
\]

By a simple computation,
\[
A_N^{-1} = \begin{pmatrix} 
B_1(c_1)^{-1} & B_3(c_1, c_2) & \cdots & B_3(c_1, c_k) \\
B_1(c_2)^{-1} & B_3(c_2, c_3) & \cdots & \vdots \\
& \cdots & \cdots & \cdots \\
O & \cdots & B_1(c_{k-1})^{-1} & B_3(c_{k-1}, c_k) \\
& \cdots & O & B_1(c_k)^{-1}
\end{pmatrix}
+ O(e^{-k_1(\sigma^3t+L)}).
\]

Next we prove that $B_1(c_i)$, $B_1(c_i)\}^{-1}$ and $B_2(c_i, c_j)$ are uniformly bounded in $\varepsilon$ in the case where $V(r) = e^r - 1 - r$ (the Toda lattice). By [28],
\[
q_c(x) = -\log \frac{\cosh \{k(x - 1)\}}{\cosh kx},
\]
\[
p_c(x) = -\alpha \partial_x q_c(x), \quad r_c(x) = q_c(x + 1) - q_c(x),
\]
\[
H(u_c) = \sinh 2\kappa - 2\kappa.
\]

In view of the above, we have $\langle r_c, 1 \rangle = 2\kappa$, $\langle p_c, 1 \rangle = -2\kappa$ and
\[
(2.18) \quad \lim_{\varepsilon \downarrow 0} (c_i \varepsilon)^{-1}\theta_1(c_i) = 12k_i, \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2\theta_2(c_i) = \frac{36}{k_i^2}, \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2\theta_3(c_i, c_j) = \frac{72}{k_i k_j}.
\]
Since the Toda lattice equation satisfies (H1), its 1-soliton solution satisfies (P4) as well as solitary wave solutions of \((1.2)\). Thus we see that \((2.18)\) holds for \((1.2)\) with nonlinearity satisfying (H1) and that \(B_1(c_i), B_1(c_i)^{-1}\) and \(B_2(c_i, c_j)\) are uniformly bounded in \(\varepsilon \in (0, \varepsilon_0)\).

\[\square\]

**Proof of Lemma 2.4.** By the definition of \(P_k(t)\) and Cramer’s rule,

\[
P_k(t)f = \left(\varepsilon^3 \partial_x u_{c_j}, \partial_x u_{c_j}\right)_{j=N+1-k, \ldots, N-N\rightarrow \Lambda_k^{-1}} \left(\varepsilon^{-4}\left(f, J^{-1}\partial_x u_{c_i}\right)\right)
\]

\[
(2.19)
\]

\[
= \frac{1}{|A_k|} \sum_{j=1}^{N} \left| \begin{array}{cccc}
A_{11} & \ldots & \Delta_{1j} & \ldots & A_{1k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_{k1} & \ldots & \Delta_{kj} & \ldots & A_{kk}
\end{array} \right| = \left| \begin{array}{cccc}
A_{11} & \ldots & \Delta_{1j} & \ldots & A_{1k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_{k1} & \ldots & \Delta_{kj} & \ldots & A_{kk}
\end{array} \right|
\]

where

\[
\Delta_{ij} = \begin{pmatrix}
\varepsilon^{-1}\left(f, J^{-1}\partial_x u_{c_i}\right) & \varepsilon^{-4}\left(f, J^{-1}\partial_x u_{c_i}\right) \\
\varepsilon^2\left(f, J^{-1}\partial_x u_{c_i}\right) & \varepsilon^{-1}\left(f, J^{-1}\partial_x u_{c_i}\right)
\end{pmatrix}.
\]

We have

\[
\|\text{the first column of } \Delta_{ij}^1\|_{k_1\varepsilon} + \|\text{the second column of } \Delta_{ij}^2\|_{k_1\varepsilon} \
\leq e^{-4}(\|\partial_x u_{c_i}\|_{k_1\varepsilon} + \varepsilon^3\|\partial_x u_{c_i}\|_{k_1\varepsilon}) (\|J^{-1}\partial_x u_{c_i}\|_{k_1\varepsilon} + \varepsilon^3\|\partial_x u_{c_i}\|_{k_1\varepsilon}) \|\bar{f}\|_{k_1\varepsilon} \
\leq e^{k_1\varepsilon(x_j-x_i)} \|\bar{f}\|_{k_1\varepsilon}.
\]

On the other hand, for \(m = 2i - 1, 2i\), and \(n = 2j - 1, 2j\), the \((m, n)\) cofactor of \(\mathcal{A}_N\) decays as \(e^{-k_1\varepsilon(x_j-x_i)}\) if \(i \leq j\). Indeed, since the components of \(\mathcal{A}_{i',j'}\) decays as \(e^{-k_1\varepsilon|x_j-x_j'|}\) if \(i' \geq j'\), the \((m, n)\) cofactor of \(\mathcal{A}_N\) decays as

\[
\max_{\tau \in \Theta} \prod_{[r(k+1)/2]>[k+1/2]} \exp \left( -(k_1\varepsilon(x_{[r(k+1)/2]} - x_{[k+1/2]})\right) \leq e^{-k_1\varepsilon(x_j-x_i)},
\]

where \(\Theta\) is a set of all permutations from \(\{1, \cdots, m-1, m+1, \cdots, 2N\}\) to \(\{1, \cdots, n-1, n+1, \cdots, 2N\}\). Thus we conclude that \(P_k(t)\) is uniformly bounded in \(l_{k_1\varepsilon}^2\). We see that \(\|P_k\|_{B(F)} = O(\varepsilon)\) follows immediately from \((2.19)\) and Claim \([A.1]\).

\[\square\]

To prove Lemma 2.2, we start with the following:
Lemma 2.5. Let \( u(t), v_1(t), c_i(t) \) and \( x_i(t) \) \( (i = 1, \cdots , N) \) be as in Lemma 2.2. Then for \( t \in [0, T] \),

\[
\sum_{i=1}^{N} (\varepsilon^{-3} |\dot{c}_i| + |\dot{x}_i - c_i|) \leq \varepsilon^{2.5} \left( \|v_1\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}\|_{W(t)} \right) + \varepsilon^2 e^{-k_1(\sigma(t+L))},
\]

\[ \|l_k\|^2 + \|l_k\|_{X_k(t)} \leq \|v_{2k}(t)\|_{X_k(t)} \left\{ \varepsilon^{2.5} \left( \|v_1(t)\|_{W(t)} + \sum_{1 \leq k \leq N} \|v_{2k}(t)\|_{W(t)} \right) + \varepsilon^3 e^{-k_1(\sigma(t+L))} \right\}. \]

**Proof.** Differentiating (2.11) for \( k = N \) with respect to \( t \) and substituting (2.9) and (2.14) into the resulting equation, we have

\[
\frac{d}{dt} \langle v_{2N}, J^{-1} \partial_{x} u_{c_j} (\cdot - x_j(t)) \rangle = \langle \partial_{t} v_{2N}, J^{-1} \partial_{x} u_{c_j} \rangle - \dot{x}_j \langle v_{2N}, J^{-1} \partial_{x}^2 u_{c_j} \rangle + \dot{c}_j \langle v_{2N}, J^{-1} \partial_{x} u_{c_j} \rangle = (l - \sum_{1 \leq k \leq N} l_k, J^{-1} \partial_{x} u_{c_j} \rangle - \langle v_{2N}, (H''(U_N) - H''(u_{c_j})) \rangle \partial_{x} u_{c_j} \rangle
\]

\[
- (\dot{x}_j - c_j) \langle v_{2N}, J^{-1} \partial_{x}^2 u_{c_j} \rangle + \dot{c}_j \langle v_{2N}, J^{-1} \partial_{x} u_{c_j} \rangle + \sum_{k=1}^{N} \langle P_k J R_k, \partial_{c} u_{c_j} \rangle = 0,
\]

and

\[
\frac{d}{dt} \langle v_{2N}, J^{-1} \partial_{\dot{c}} u_{c_j} (\cdot - x_j(t)) \rangle = \langle \partial_{t} v_{2N}, J^{-1} \partial_{\dot{c}} u_{c_j} \rangle - \dot{x}_j \langle v_{2N}, J^{-1} \partial_{\dot{c}}^2 u_{c_j} \rangle + \dot{c}_j \langle v_{2N}, J^{-1} \partial_{\dot{c}} u_{c_j} \rangle = (l - \sum_{1 \leq k \leq N} l_k, J^{-1} \partial_{\dot{c}} u_{c_j} \rangle - \langle v_{2N}, (H''(U_N) - H''(u_{c_j})) \rangle \partial_{\dot{c}} u_{c_j} \rangle
\]

\[
- (\dot{x}_j - c_j) \langle v_{2N}, J^{-1} \partial_{\dot{c}}^2 u_{c_j} \rangle + \dot{c}_j \langle v_{2N}, J^{-1} \partial_{\dot{c}} u_{c_j} \rangle + \sum_{k=1}^{N} \langle P_k J R_k, \partial_{\dot{c}} u_{c_j} \rangle = 0.
\]

By (2.22), (2.23) and (2.19),

\[
(A_N - \delta A) \left( \varepsilon^{-3} \dot{c}_i \right)_{i = 1, \cdots , N} + \sum_{1 \leq k \leq N-1} A_k \left( \varepsilon^{-3} \alpha_{j,k} \right)_{j = N+1-k, \cdots , N} + R_1 + R_2 = 0,
\]
By Claims A.1 and A.3, it follows from Claim A.1 that 

\begin{align*}
\delta A &= \text{diag}(\delta A_i)_{1 \leq i \leq N}, \\
\delta A_i &= \left( \begin{array}{cc}
\varepsilon^{-1} & \varepsilon^{-1} \\
\varepsilon^{2} & \varepsilon^{2}
\end{array} \right) \\
\tilde{R}_1 &= \sum_{k=1}^{N} \tilde{A}_k A_k^{-1} \left( \begin{array}{c}
\varepsilon^{-4} R_k, \partial_x u_{c_i} \\
\varepsilon^{-1} R_k, \partial_x u_{c_i}
\end{array} \right)_{i=N+1-k, \ldots, N}, \\
\tilde{R}_2 &= \left( \begin{array}{c}
\varepsilon^{-4} v_{2N}, (H''(U_N) - H''(u_{c_i})) \partial_x u_{c_i} \\
\varepsilon^{-1} v_{2N}, (H''(U_N) - H''(u_{c_i})) \partial_x u_{c_i}
\end{array} \right)_{1 \leq i \leq N}. 
\end{align*}

Since \( \|J^{-1}\|_{l^2_{-k_1\epsilon}} = O(\varepsilon) \) and \( x_i(t) \geq x_1(t) \) for any \( i \geq 1 \),

\[ |\delta A_i| \lesssim \|v_{2N}(t)\|_{X_N(t)} \varepsilon^{-4} \|\partial_x^2 u_{c_i}\|_{l^2_{-k_1\epsilon}} + \varepsilon^{-1} \|\partial_x \partial_x u_{c_i}\|_{l^2_{-k_1\epsilon}} + \varepsilon^{2} \|\partial_x^2 u_{c_i}\|_{l^2_{-k_1\epsilon}} \lesssim \varepsilon^{-\frac{3}{2}} \|v_{2N}(t)\|_{X_N(t)} \]

follows from Claim A.1.

Let \( R_k = R_{k1} + R_{k2} + R_{k3} \) and

\begin{align*}
R_{k1} &= H'(U_k + w_k) - H'(U_k + w_{k-1}) - H''(U_k)v_{2k}, \\
R_{k2} &= H'(U_k) - H'(U_{k-1}) - H'(u_{c_{N+1-k}}), \\
R_{k3} &= H'(U_k + w_{k-1}) - H'(U_{k-1} + w_{k-1}) - H'(U_k) + H'(U_{k-1}).
\end{align*}

Then by the mean value theorem,

\begin{align}
|R_{k1}| &\lesssim (|w_{k-1}| + |v_{2k}|)|v_{2k}|, & |R_{k2}| &\lesssim |u_{c_{N+1-k}}||U_{k-1}|, \\
|R_{k3}| &\lesssim |u_{c_{N+1-k}}||w_{k-1}|.
\end{align}

(2.25)

It follows from Claim A.1 that

\begin{align}
|\langle R_{k1}, \partial_x u_{c_i} \rangle| &\lesssim \varepsilon^3 \left( \|v_1\|_{W(t)} + \sum_{i=1}^{k} \|v_{2i}\|_{W(t)} \right) \|v_{2k}\|_{W(t)}, \\
|\langle R_{k1}, \partial_x^2 u_{c_i} \rangle| &\lesssim \left( \|v_1\|_{W(t)} + \sum_{i=1}^{k} \|v_{2i}\|_{W(t)} \right) \|v_{2k}\|_{W(t)}.
\end{align}

(2.26)

By Claims A.1 and A.3

\begin{align}
|\langle R_{k2}, \partial_x u_{c_i} \rangle| + \varepsilon^3 |\langle R_{k2}, \partial_x^2 u_{c_i} \rangle| &\lesssim \varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)}, \\
|\langle R_{k3}, \partial_x u_{c_i} \rangle| + \varepsilon^3 |\langle R_{k3}, \partial_x^2 u_{c_i} \rangle| &\lesssim \varepsilon^2 \|w_{k-1}(t)\|_{W(t)}.
\end{align}

(2.27) (2.28)
Thus we have
\begin{equation}
|\tilde{R}_1| \lesssim \varepsilon \frac{1}{2} \|v_1(t)\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}\|_{W(t)} + \varepsilon^2 e^{-k_1(\sigma^2 t + L)}.
\end{equation}

By Claims \(A.1\), \(A.3\) and \(A.4\),
\begin{equation}
\tilde{R}_2 = O(\varepsilon \frac{1}{2} \|v_{2N}\|_{W(t)} e^{-k_1(\sigma^2 t + L)}).
\end{equation}

In view of the definition of \(F_{j,k}\),
\begin{equation}
|F_{j,k}| \lesssim \varepsilon^{-\frac{3}{2}} e^{k_1 \varepsilon (x_{N+1-k} - x_j)} \|v_{2k}\|_{X_k(t)} \left( \varepsilon^2 e^{-k_N - k_{N+1-k}} + \varepsilon^{-3} |\dot{c}_j| + |\dot{x}_j - c_j| \right),
\end{equation}

and it follows from (2.10), (2.31), Lemma 2.3 and its proof that
\begin{equation}
\sum_{j=N+1-k}^{N} e^{k_1 \varepsilon (x_j - x_{N+1-k})} (\varepsilon^{-3} |\alpha_{j,k}| + |\beta_{j,k}|)
\end{equation}
\begin{equation}
\lesssim \varepsilon^{-\frac{3}{2}} \|v_{2k}\|_{X_k(t)} \left( \varepsilon^2 e^{-k_N - k_{N+1-k}} + \sum_{j=N+1-k}^{N} (\varepsilon^{-3} |\dot{c}_j| + |\dot{x}_j - c_j|) \right).
\end{equation}

Combining (2.21), (2.29), (2.30) and (2.32), we obtain (2.20). Moreover, since
\[
\tilde{A}_k A_k^{-1} = E_k + O(e^{-k_1(\sigma^2 t + L)}) , 
E_k = (\delta_{i+k-N,j})_{j=1,\dots,N_\downarrow} ,
\]
we have
\begin{equation}
= \sum_{k=1}^{N} E_k \left( e^{-4} \langle R_{k3}, \partial_x u_{c_i} \rangle \right) + O \left( e^{-1} \left( \|v_1\|_{W(t)}^2 + \sum_{1 \leq k \leq N} \|v_{2k}\|_{X_k(t)}^2 + \varepsilon^2 e^{-k_1(\sigma^2 t + L)} \right) \right).
\end{equation}

Substituting (2.20) into (2.32), we have (2.21). Thus we complete the proof. \(\square\)

The right hand side of (2.33) is not necessarily integrable in time. We will use normal form method to retrieve bad parts from this term to prove convergence of speed parameters \(c_i(t)\) \((1 \leq i \leq N)\) as \(t \to \infty\).

Proof of Lemma 2.2 By Claim \(A.4\)

\[
R_{k3} = (H''(U_k) - H''(U_{k-1}))w_{k-1} + O(w_{k-1}^2)
\]
\begin{equation}
= (H''(u_{c_{N+1-k}}) - I)w_{k-1} + \sum_{N+1-k \leq i, j \leq N}^{N+1-k \leq i, j \leq N} O(|w_{k-1}|(|u_{c_i}| |u_{c_j}| + |w_{k-1}|)).
\end{equation}
Thus we have
\[
\langle R_{k3}, \partial_x u_{cN+1-k} \rangle = \langle w_{k-1}, (H''(u_{cN+1-k}) - I)\partial_x u_{cN+1-k} \rangle \\
+ O(\varepsilon^3 \|w_{k-1}\|_{H^s(t)}(\|w_{k-1}\|_{W(t)} + \varepsilon \frac{3}{4} e^{-k_1(\sigma^3 t + L)})).
\]

(2.35)

and for \( i \neq N + 1 - k \),
\[
\langle R_{k3}, \partial_x u_{c_i} \rangle = O(\varepsilon^3 \|w_{k-1}\|_{H^s(t)}(\|w_{k-1}\|_{W(t)} + \varepsilon \frac{3}{4} e^{-k_1(\sigma^3 t + L)})).
\]

(2.36)

By (2.33),
\[
\frac{d}{dt} \langle v_{2k}, \rho_{c_i(t)}(\cdot - x_i(t)) \rangle \\
= \langle JH'(v_1), \rho_{c_i} \rangle - \dot{x}_i \langle v_{2k}, \partial_x \rho_{c_i} \rangle + \dot{c}_i \langle v_{2k}, \rho_{c_i} \rangle \\
= - \langle v_{2k}, (c_i \partial_x + J)\rho_{c_i} \rangle + R_4,
\]

(2.37)

where
\[
R_4 = \langle J(H'(v_1) - v_1), \rho_{c_i} \rangle + \dot{c}_i \langle v_{2k}, \rho_{c_i} \rangle - (\dot{x}_i - c_i) \langle v_{2k}, \partial_x \rho_{c_i} \rangle.
\]

For \( i \leq N - k \), it follows from (2.30) that
\[
\frac{d}{dt} \langle v_{2k}, \rho_{c_i(t)}(\cdot - x_i(t)) \rangle \\
= \langle JH''(U_k)v_{2k} + l_k + Q_k J R_k, \rho_{c_i} \rangle - \dot{x}_i \langle v_{2k}, \partial_x \rho_{c_i} \rangle + \dot{c}_i \langle v_{2k}, \partial_x \rho_{c_i} \rangle \\
= - \langle v_{2k}, (c_i \partial_x + J)\rho_{c_i} \rangle + R_5,
\]

(2.38)

where
\[
R_5 = \langle l_k, \rho_{c_i} \rangle + \dot{c}_i \langle v_{2k}, \partial_x \rho_{c_i} \rangle - (\dot{x}_i - c_i) \langle v_{2k}, \partial_x \rho_{c_i} \rangle \\
- \langle v_{2k}, (H''(U_k) - I)J\rho_{c_i} \rangle + \langle Q_k J R_k, \rho_{c_i} \rangle.
\]

By Claim [A.5] we have \( \rho_{c_i} \in L^2_a \cap L^2_a \) for any \( a \in (0, 2k_1 \varepsilon) \) and
\[
|R_4| \lesssim \varepsilon^\frac{3}{2} (\|\dot{x}_i - c_i\| + \varepsilon^{-3}\|\dot{c}_i\|) \|v_1\|_{W(t)} + O(\varepsilon^3 \|v_1\|^2_{H^s(t)}) \\
\lesssim (\|v_1\|_{W(t)} + \sum_{1 \leq i \leq k-1} \|v_{2i}\|_{W(t)})^2 + \varepsilon^6 e^{-k_1(\sigma^3 t + L)}.
\]

(2.39)

Let \( \|u\|_{W(t)} = \min_{1 \leq i \leq N} \|e^{k_1 \varepsilon \cdot - x_i(t)} u\|_{L^2} \). By Claims [A.1] and [A.3]
\[
\|\langle v_{2k}, (H''(U_k) - I)J\rho_{c_i} \rangle \| \lesssim \|v_{2k}\|_{W(t)} \|H''(U_k) - I)J\rho_{c_i}(\cdot - x_i(t))\|_{W(t)} \\
\lesssim \varepsilon^\frac{3}{2} e^{-k_1(\sigma^3 t + L)} \|v_{2k}\|_{W(t)} \\
\lesssim \varepsilon^\frac{3}{2} e^{-k_1(\sigma^3 t + L)} \|v_{2k}\|_{W(t)}.
\]
By Claim [A.5] and (2.32),

\[
\|(l_k, \rho_{c_i})\| \leq \sum_{N+1-k \leq j \leq N} |\alpha_{j,k} \langle \partial_c u_{c_j}, \rho_{c_i}\rangle + \beta_{j,k} \langle \partial_{x} u_{c_j}, \rho_{c_i}\rangle| \\
\lesssim \sum_{N+1-k \leq j \leq N} (\varepsilon |\alpha_{j,k}| + \varepsilon^4 |\beta_{j,k}|) e^{-k_1(\sigma\varepsilon^3 t + L)} \\
\lesssim \sum_{N+1-k \leq j \leq N} \varepsilon^2 e^{-k_1(\sigma\varepsilon^3 t + L)} \|v_{2k}\|_{X_k(t)} \left(\varepsilon^2 |\dot{c}_j - c_j| + \varepsilon^{-3} |\dot{c}_j|\right).
\]

By (2.25) and Claim [A.5]

\[
\|(Q_k J R_k, \rho_{c_i})\| \lesssim \varepsilon^3 \|v_{2k}\|_{W(t)} \left(\|v_1\|_{W(t)} + \sum_{1 \leq i \leq k} \|v_{2i}\|_{W(t)}\right) + \varepsilon^6 e^{-k_1(\sigma\varepsilon^3 t + L)} \\
+ \varepsilon^2 e^{-k_1(\sigma\varepsilon^3 t + L)} \left(\|v_1\|_{W(t)} + \sum_{1 \leq i \leq k-1} \|v_{2i}\|_{W(t)}\right).
\]

Combining the above with Lemma 2.5 and Claim A.5 we have

\[
(2.40) \quad |\mathcal{R}_5| \lesssim \left(\|v_1\|_{W(t)} + \sum_{1 \leq i \leq k-1} \|v_{2i}\|_{W(t)}\right)^2 + \varepsilon^6 e^{-k_1(\sigma\varepsilon^3 t + L)}.
\]

In view of Lemma 2.5 and (2.35) - (2.40),

\[
(2.41) \quad \left|\langle w_{k-1}, (H''(u_{c_{N+1-k}}) - I)\partial_{x} u_{c_{N+1-k}}\rangle + \frac{d}{dt}(w_{k-1}, \rho_{c_{N+1-k}}(\cdot - x_{N+1-k}))\right| \\
\lesssim \varepsilon^3 \left(\|v_1\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}\|_{X_k(t) \cap W(t)}\right)^2 + \varepsilon^6 e^{-k_1(\sigma\varepsilon^3 t + L)}.
\]

Since $B_1(c_i)$ and $B_2(c_i, c_j)$ ($1 \leq i, j \leq N$) are lower triangular matrices, it follows from Lemma 2.5 (2.20) and (2.41) that

\[
(2.42) \quad \mathcal{B} \frac{dc}{dt} + \frac{d}{dt} \mathcal{R}_6 = \mathcal{R}_7,
\]

where $c(t) = (c_1(t), \ldots, c_N(t))$,

\[
\mathcal{B}(t) = \text{diag} \left(\frac{\theta_1(c_i(t))}{c_i(t)}\right)_{1 \leq i \leq N}, \quad \mathcal{R}_6 = \left(\langle w_{N-i}, \rho_{c_i}\rangle\right)_{i=1,\ldots,N},
\]

\[
\mathcal{R}_7 = O \left(\varepsilon^3 \left(\|v_1\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}\|_{X_k(t) \cap W(t)}\right)^2 + \varepsilon^6 e^{-k_1(\sigma\varepsilon^3 t + L)}\right).
\]

Thus we have

\[
(2.43) \quad \frac{d}{dt}(c + B^{-1} \mathcal{R}_6) = B^{-1} \mathcal{R}_7 + \left(\frac{d}{dt}(\mathcal{B})^{-1}\right) \mathcal{R}_6.
\]
By (2.13), (2.20) and the definition of $B$, we have $|B^{-1}| + |\partial_c B| = O(\varepsilon^{-1})$ and

$$|\dot{B}| \leq \sum_{1 \leq i \leq N} |\partial_{c_i} B| |\dot{c_i}|$$

$$\lesssim \varepsilon^{\frac{5}{2}} \left( \|v_1\|_{W(t)} + \sum_{1 \leq k \leq N} \|v_{2k}\|_{W(t)} \right) + \varepsilon^4 e^{-k_1(\sigma \varepsilon^3 t + L)}.$$ 

Since $|R_6| \lesssim \varepsilon^{\frac{5}{2}} (\|v_1\|_{W(t)} + \sum_{1 \leq k \leq N} \|v_{2k}\|_{W(t)})$ by Claim A.5,

$$\left( \frac{d}{dt} (B)^{-1} \right) R_6 \leq \varepsilon^{2} |\dot{B}| |R_6|$$

$$\lesssim \varepsilon^{2} \left( \|v_1\|_{W(t)} + \sum_{1 \leq k \leq N} \|v_{2k}\|_{W(t)} \right)^2 + \varepsilon^{5} e^{-2k_1(\sigma \varepsilon^3 t + L)}.$$ 

Combining the above with (2.43), we obtain (2.15). Thus we complete the proof. \hfill \Box

3. Energy identities and virial identities

First, we will estimate energy norm of $v(t)$ and $v_{2k}(t)$ by adopting an argument of [6] that uses the convexity of Hamiltonian and the orthogonality condition (2.7).

**Lemma 3.1.** Let $u(t)$ be a solution to (1.2) satisfying $u(0) = \sum_{1 \leq i \leq N} u_{c_i,0} (\cdot - x_{0,i}) + v_0$ and let $c_{i,0}$ and $x_{i,0}$ be as in Theorem 1.1. Then there exist positive numbers $\varepsilon_0$, $\delta$, $L_0$ and $C$ satisfying the following: Suppose $\varepsilon \in (0, \varepsilon_0)$, that $v_{2k}$ $(1 \leq k \leq N)$ satisfy (2.11) for $N + 1 - k \leq i \leq N$ and $t \in [0, T]$, and that

$$\sup_{t \in [0, T]} \left\{ \varepsilon^{-2} |c_i(t) - c_{i,0}| + \sum_{k=1}^{N} \varepsilon^{-\frac{3}{2}} \|v_{2k}(t)\|_{L^2} \right\} \leq \delta,$$

$$L = \inf_{t \in [0, T]} \min_{1 \leq i \leq N-1} \varepsilon \max \{ x_{i+1}(t) - x_i(t) \} \geq L_0.$$

Then for $t \in [0, T]$,

$$\|v_1(t)\|_{L^2} \leq C \|v_0\|_{L^2},$$

$$\|v(t)\|_{L^2}^2 \leq C \left( \varepsilon \sum_{i=1}^{N} |c_i(t) - c_0| + \varepsilon^2 \left( \|v_0\|_{L^2} + \sum_{k=1}^{N-1} \|v_{2k}\|_{W(t)} + \|v_0\|_{L^2}^2 + \varepsilon^3 e^{-k_1 L} \right) \right),$$

$$\|v_{2k}\|_{L^2}^2 \leq C \left( \varepsilon \sum_{i=N+1-k}^{N} |c_i(t) - c_0| + \varepsilon^2 \|v_0\|_{L^2} + \|v_0\|_{L^2}^2 + \varepsilon^3 e^{-k_1 L} \right)$$

$$+ C \left\{ \varepsilon^\frac{3}{2} \sum_{i=1}^{k-1} \|v_{2i}\|_{W(t)} + \varepsilon^3 \left( \|v_1\|_{L^2(0, T; W(t))} + \sum_{i=1}^{N} \|v_{2i}\|_{L^2(0, T; W(t) \setminus X_i(t))} \right)^2 \right\}.$$
\textbf{Proof.} Since $H(v_1(t)) = H(v_0)$ for $t \in \mathbb{R}$, there exists a nondecreasing function $C(r)$ such that $\|v_1(t)\|_2 \leq C(\|v_0\|_2)\|v_0\|_2 a$. Thus we have (3.1).

By (P2), there exists a positive constant $C'$ independent of $\varepsilon$ such that

$$\delta H := H(u(t)) - \sum_{1 \leq i \leq N} H(u_{ci,0}) = H(U_N(t) + v(t)) - \sum_{1 \leq i \leq N} H(u_{ci,0}) = I_1 + I_2 + \frac{1}{2} \langle H''(U_N)v, v \rangle + O(\|v\|_1^2) \geq \varepsilon' v(t) \|v(t)\|_2^2 + I_1 + I_2,$$

where $I_1 = \langle H'(U_N), v \rangle$ and $I_2 = H(U_N(t)) - \sum_{i=1}^{N} H(u_{ci,0})$. By (2.7) and (2.11),

$$\langle H'(u_{ci}(\cdot - x_i(t))), v(t) \rangle = - c_i \langle v(t), J^{-1} \partial_x u_{ci}(\cdot - x_i(t)) \rangle = - c_i \langle w_{N-i}(t), J^{-1} \partial_x u_{ci}(\cdot - x_i(t)) \rangle.$$

Hence it follows from Claims \textbf{A.3} and \textbf{A.4} that

$$|I_1| \leq \left| \left\langle H'(U_N(t)) - \sum_{1 \leq i \leq N} H'(u_{ci}(\cdot - x_i(t))), v \right\rangle \right| + \sum_{1 \leq i \leq N} |c_i(t)| \left| \left\langle w_{N-i}(t), J^{-1} \partial_x u_{ci}(\cdot - x_i(t)) \right\rangle \right|$$

$$\lesssim \|v(t)\|_2 \left\| H'(U_N(t)) - \sum_{1 \leq i \leq N} H'(u_{ci}(\cdot - x_i(t))) \right\|_2$$

$$+ \varepsilon^\frac{3}{2} \left( \|v_1(t)\|_{W(t)} + \sum_{i=1}^{N-1} \|v_{2k}(t)\|_{W(t)} \right)$$

$$\lesssim \varepsilon^\frac{3}{2} e^{-k_1(\sigma\varepsilon^3 t + L)} \|v(t)\|_2^2 + \varepsilon^\frac{3}{2} \left( \|v_1(t)\|_{W(t)} + \sum_{i=1}^{N-1} \|v_{2k}(t)\|_{W(t)} \right),$$

$$|I_2| \leq \sum_{1 \leq i \leq N} |H(u_{ci}(t)) - H(u_{ci,0})| + \left| H(U_N(t)) - \sum_{1 \leq i \leq N} H(u_{ci}(t)) \right|$$

$$\lesssim \sum_{1 \leq i \leq N} \theta_1(c_i,0) |c_i(t) - c_i,0| + \sum_{j \neq i} \|u_{ci}(\cdot - x_i(t))u_{cj}(\cdot - x_j(t))\|_1$$

$$\lesssim \varepsilon \sum_{1 \leq i \leq N} |c_i(t) - c_i,0| + \varepsilon^3 e^{-k_1(\sigma\varepsilon^3 t + L)}.$$
Since $H(u(t))$ does not depend on $t$, we have $|\delta H| \leq |I_3| + |I_4|$, where

$$
I_3 = H(U_N(0) + v_0) - H(U_N(0)),
$$
$$
I_4 = H(U_N(0)) - \sum_{1 \leq i \leq N} H(u_{c_i,0}(-x_i,0)).
$$

By the assumption and Claims A.1 and A.3,

$$
|I_3| \leq |\langle H'(U_N(0)), v_0 \rangle| + O(\|v_0\|_2^2) \lesssim \varepsilon^3 \|v_0\|_2 + \|v_0\|_2^2,
$$
$$
|I_4| \lesssim \varepsilon^3 e^{-k_1 L}.
$$

Combining the above, we conclude (3.2).

Finally we will prove (3.3). By (2.31) and the definitions of $U_k$,

$$
\begin{align*}
\partial_t(U_k + w_k) &= J H'(U_k + w_k) + \tilde{i}_k + \sum_{i=1}^k (l_i - P_i J R_i),
\end{align*}
$$

where $\tilde{i}_k = \sum_{i=N+1-k}^N (\dot{c}_i \partial_c u_{c_i} - (\dot{x}_i - c_i) \partial_x u_{c_i})$. Since $J$ is skew-adjoint, it follows from (3.4) that

$$
\begin{align*}
\frac{d}{dt} H(U_k + w_k) &= \left\langle H'(U_k + w_k), \tilde{i}_k + \sum_{i=1}^k (l_i - P_i J R_i) \right\rangle \\
&= \sum_{i=1}^k \left\langle H'(U_k + w_k), l_i \right\rangle,
\end{align*}
$$

where $U_{k, \text{int}} = H'(U_k) - \sum_{i=N+1-k}^N H'(u_{c_i})$ and

$$
\begin{align*}
II_1 &= \sum_{i=1}^k \langle H'(U_k + w_k), l_i \rangle, \\
II_2 &= -\sum_{i=1}^k \sum_{j=N+1-k}^N \langle H'(u_{c_j}), P_i J R_i \rangle, \\
II_3 &= -\sum_{i=1}^k \langle U_{k, \text{int}}, P_i J R_i \rangle, \\
II_4 &= -\sum_{i=1}^k \langle H'(U_k + w_k) - H'(U_k), P_i J R_i \rangle, \\
II_5 &= \sum_{j=N+1-k}^N \langle H'(u_{c_j}), \tilde{i}_k \rangle, \\
II_6 &= \langle H(U_k + w_k) - \sum_{j=N+1-k}^N H(u_{c_j}), \tilde{i}_k \rangle.
\end{align*}
$$

By (2.21) and the fact that $\|H'(U_k + w_k)\|_2 = O(\varepsilon^{\frac{3}{2}})$,

$$
\begin{align*}
|II_1| &\lesssim \varepsilon^3 \sum_{i=1}^k \|l_i\|_2 \\
&\lesssim \varepsilon^3 \left(\|v_1\|_{W(t)} + \sum_{k=1}^N \|v_{2k}\|_{W_{k}(t)} \right)^2 + \varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)}.
\end{align*}
$$
Next, we will estimate \( II_2 \). Using (2.25) and the fact that \((P_iJ)^*H'(u_{c_j}) = c_j \partial_x u_{c_j} \) for \( j \geq N + 1 - i \) and \((P_iJ)^*H'(u_{c_j}) = O(\varepsilon e^{-k_1\varepsilon x_{N+1-i-j}}) \) for \( j \leq N - i \), we have

\[
II_2 = - \sum_{i=1}^{k} \sum_{j=N+1-k}^{N} c_j \langle R_{i3}, \partial_x u_{c_j} \rangle \\
+ O \left( \varepsilon^3 \left( \|v_1\|_{W(t)}^2 + \sum_{1 \leq i \leq k} \|v_{2i}\|_{W(t)}^2 \right) + e^6 e^{k_1(\sigma e^3 t + L)} \right) \\
= - \sum_{i=1}^{k-1} \sum_{j=N+1-i}^{N} c_j \langle w_{i-1}, (H''(u_{c_{N+1-i-1}}) - I) \partial_x u_{c_j} \rangle \\
+ O \left( \varepsilon^3 \left( \|v_1\|_{W(t)}^2 + \sum_{1 \leq i \leq k} \|v_{2i}\|_{W(t)}^2 \right) + e^6 e^{k_1(\sigma e^3 t + L)} \right).
\]

(3.7)

Secondly, we will estimate \( II_3 \) and \( II_4 \). In view of (2.19), Claim A.1 and the proof of Lemma 2.4, we have \( \|PJ\|_{B(W(t), W(t)^*)} = O(\varepsilon) \), \( \|PJ u^2\|_{W(t)^*} \lesssim \varepsilon^2 \|u\|_{W(t)}^2 \). Hence it follows from (2.25) that

\[
|II_3| \leq \|U_{k, int}\| \sum_{i=1}^{k} \|P_i J R_i\|_{W(t)}^2 \\
\lesssim \varepsilon^4 e^{-k_1(\sigma e^3 t + L)} \left( \|v_1\|_{W(t)} + \sum_{i=1}^{k} \|v_{2i}\|_{W(t)} + \varepsilon^2 \right)^2,
\]

\[
|II_4| \leq \|w_k(t)\|_{W(t)} \sum_{i=1}^{k} \|P_i J R_i\|_{W(t)^*} \\
\lesssim \varepsilon^2 \|w_k\|_{W(t)} \left( \|v_1\|_{W(t)} + \|w_k\|_{W(t)} \right)^2 \\
+ \|w_k\|_{W(t)} \left\{ \varepsilon^3 \left( \|v_1\|_{W(t)} + \sum_{i=1}^{k} \|v_{2i}\|_{W(t)} \right) + e^6 e^{k_1(\sigma e^3 t + L)} \right\}.
\]

(3.9)
By (2.12), (2.13) and Claim A.3,

\[ II_5 = \sum_{1 \leq i \leq k} \left\{ \theta_1(c_i) \dot{c}_i + O(e^{-k_1(\sigma \varepsilon^3 t + L)}(\varepsilon |\dot{c}_i| + \varepsilon^4 |\dot{x}_i - c_i|) \right\} \]

(3.10)

\[ = \sum_{i = N+1-k}^N \theta_1(c_i) \dot{c}_i + O(\varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)} + O \left( \varepsilon^3 \left( \|v_1\|_{W(t)}^2 + \sum_{i=1}^N \|v_{2i}\|^2_{X_i(t) \cap W(t)} \right) \right) \]

By (2.20),

\[ |II_6| \leq (\|U_{k, int}\|_{t^2} + \|w_k\|_{W(t)}) \|k\|_{W(t)} \]

(3.11)

\[ \leq (\varepsilon k_1(\sigma \varepsilon^3 t + L) + \|w_k\|_{W(t)}) \sum_{i = N+1-k}^N (\varepsilon \sum_{k=1}^\infty \|v_{2k}\|^2_{X_i(t)}) + \varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)} \]

Using (2.37) and (2.38) and following the proof of Lemma 2.2, we have

(3.12)

\[ II_2 + II_5 = O \left( \varepsilon^3 (\|v_1\|_{W(t)} + \sum_{1 \leq i \leq N} \|v_{2i}\|_{W(t) \cap X_i(t)})^2 + \varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)} \right) \]

By (3.5), (3.3), (3.9), (3.11) and (3.12),

\[ \left| \frac{d}{dt} H(U_k + w_k) \right| \]

(3.13)

\[ \leq \varepsilon^3 \left( \|v_1\|^2_{W(t)} + \sum_{k=1}^N (\|v_{2k}\|^2_{W(t)} + \|v_{2k}\|^2_{X_i(t)}) \right) + \varepsilon^6 e^{-k_1(\sigma \varepsilon^3 t + L)} \]

Integrating (3.13) over \([0, t]\), we obtain

\[ H(U_k(t) + w_k(t)) - H(U_{k,0} + v_0) \]

(3.14)

\[ = O \left( \varepsilon^3 \left( \|v_1\|^2_{L^2(0,T;W(t))} + \sum_{i=1}^N \|v_{2i}\|^2_{L^2(0,T;X_i(t) \cap W(t))} + e^{-k_1 L} \right) \right) \]

Using the convexity of the Hamiltonian, we conclude

\[ \|w_k(t)\|^2 \leq \varepsilon \sum_{N+1-k \leq i \leq N} |c_i(t) - c_0| + \varepsilon \sum_{i=1}^N \|v_{2i}\|^2_{W(t)} + \varepsilon^3 e^{-k_1 L} \]

(3.15)

\[ + \varepsilon^{\frac{k-1}{2}} \sum_{i=1}^{k-1} \|v_{2i}\|_{W(t)} + \varepsilon^3 \left( \|v_1\|^2_{L^2(0,T;W(t))} + \sum_{i=1}^N \|v_{2i}\|^2_{L^2(0,T;W(t) \cap X_i(t))} \right)^2 \]

from (3.14) in exactly the same way as the proof of (3.2). Combining (3.15) with (3.1) and (3.2), we obtain (3.3). \qed
Since \( v_1(t) \) is small, it moves slowly and will be decoupled from the \( N \)-soliton part of the solution. The following is an analog of virial lemma for small solutions in Martel and Merle [17] and was used in [20] to prove orbital stability of 1-solitons of the FPU lattice equations. Here we confirm how coefficients of the virial identity depend on \( \varepsilon \).

**Lemma 3.2.** Let \( v_1(t) \) be a solution to (2.3). Let \( a > 0 \), \( \tilde{x}(t) \) be a \( C^1 \)-function and \( \psi_a(t, x) = 1 + \tanh a(x - \tilde{x}(t)) \). There exist positive numbers \( \varepsilon_0, \delta \) and \( C \) such that if \( \inf_{t \geq 0} \tilde{x}_t \geq 1 + k_1^2 \varepsilon^2 / 24 \) and \( a \varepsilon + \|v_0\|_2 \leq \delta \varepsilon^2 \) for an \( \varepsilon \in (0, \varepsilon_0) \), then

\[
\|\psi_a(t) \mathbf{v}(t)\|_{L^2}^2 + C a \varepsilon^2 \int_0^t \| \text{sech} (t + \tilde{x}(s)) \psi_a(t,s) \|_{L^2}^2 ds \leq \|\psi_a(0) \mathbf{v}_0\|_{L^2}^2.
\]

*Proof.* Let \( v_1(t, n) = \psi_1(t, n), p_1(t, n) = \frac{1}{2} p_1(t, n)^2 + V(r_1(t, n)) \) and \( \tilde{\psi}_a(t, x) = a \frac{\tilde{x}}{2} \text{sech} a(x - \tilde{x}(t)) \). By (3.1),

\[
\begin{align*}
V(r_1(t, n)) - \frac{1}{2} V'(r_1(t, n))^2 & \leq \|v_0\|_2 |r_1(t, n)|^2, \\
|V'(r_1(t, n)) - r_1(t, n)| & \leq \|v_0\|_2 |r_1(t, n)|.
\end{align*}
\]

Using (1.2) and the above, we have

\[
\begin{align*}
\frac{d}{dt} \sum_{n \in \mathbb{Z}} \psi_a(t, n) h_1(t, n) & = \sum_{n \in \mathbb{Z}} p_1(t, n) V'(r_1(t, n - 1))(\psi_a(t, n - 1) - \psi_a(t, n)) + \sum_{n \in \mathbb{Z}} \partial_t \psi_a(t, n) h_1(t, n) \\
(3.16) & \leq - \frac{\tilde{x}_1(t)}{2} \sum_{n \in \mathbb{Z}} \tilde{\psi}_a(t, n)^2 p_1(t, n)^2 \\
& + (1 + C \|v_0\|_2) \sum_{n \in \mathbb{Z}} (\psi_a(t, n) - \psi_a(t, n - 1)) |p_1(t, n) r_1(t, n - 1)| \\
& - \frac{\tilde{x}_1(t)}{2} (1 - C' \|v_0\|_2) \sum_{n \in \mathbb{Z}} \tilde{\psi}_a(t, n - 1)^2 r_1(t, n - 1)^2,
\end{align*}
\]

where \( C' \) is a positive constant.

Substituting

\[
\psi_a(t, n) - \psi_a(t, n - 1) = \sinh a \text{sech} a(n - \tilde{x}(t)) \text{sech} a(n - \tilde{x}(t) - 1)
\]

\[
= \tilde{\psi}_a(t, n) \tilde{\psi}_a(t, n - 1)(1 + O(a^2))
\]

into (3.16) and using Hölder inequality, we obtain

\[
\frac{d}{dt} \sum_{n \in \mathbb{Z}} \psi_a(t, n) h_1(t, n) \leq - \frac{\tilde{x}_1(t)}{2} (1 - C'' (\|v_0\|_2 + a^2)) \sum_{n \in \mathbb{Z}} \tilde{\psi}_a(t, n)^2 (p_1(t, n)^2 + r_1(t, n)^2)
\]

for a \( C'' > 0 \). Thus we have

\[
\frac{d}{dt} \sum_{n \in \mathbb{Z}} \psi_a(t, n) h_1(t, n) \leq - C \varepsilon^2 \sum_{n \in \mathbb{Z}} \tilde{\psi}_a(t, n)^2 (p_1(t, n)^2 + r_1(t, n)^2)
\]
for a $C > 0$ if $\delta > 0$ is sufficiently small. We have thus proved Lemma 3.2.

Finally, we will prove propagation estimates on $v_{2k}$.

**Lemma 3.3.** Let $u(t)$ be as in Theorem 1.1 and let $\psi_{a,i}(t, x) = 1 + \tanh a(x - x_i(t))$. Then there exist positive numbers $\varepsilon_0$, $\delta$, $L_0$ and $C$ satisfying the following: Suppose that

$$
(3.18) \quad a\varepsilon + \sup_{t \in [0,T]} \left\{ \|v_1(t)\|_2 + \sum_{k=1}^N \|v_{2k}(t)\|_2 \right\} \leq \delta\varepsilon^2,
$$

$$
\inf_{t \in [0,T]} \min_{1 \leq i \leq N} \varepsilon(x_{i+1}(t) - x_i(t)) \geq L,
$$

$$
\min_{1 \leq i \leq N} \inf_{t \in [0,T]} \dot{x}_i(t) \geq 1 + \frac{k_1^2\varepsilon^2}{24}
$$

for $\varepsilon \in (0, \varepsilon_0)$, $L \geq L_0$ and $T \geq 0$. Then for $t \in [0,T]$ and $1 \leq k \leq N$,

$$
\|\psi_{a,1}(t)\|_{L^2(0,T;W(t))} \leq C \left( \|v_0\|_2 + \varepsilon^\frac{3}{2} \sum_{i=1}^k \|v_{2i}(t)\|_{L^2(0,T;X_k(t))} + \varepsilon^\frac{3}{2}e^{-k_1L} \right).
$$

**Proof.** In order to prove the lemma, it suffices to show that

$$
(3.19) \quad \|\psi_{a,1}w_k\|_2 + \varepsilon^\frac{3}{2}\|w_k\|_{L^2(0,T;W(t))} \leq \|v_0\|_2 + \varepsilon^\frac{3}{2} \left( \|v_{2k}\|_{L^2(0,T;X_k(t))} + \sum_{i=1}^{k-1} \|v_{2i}\|_{L^2(0,T;W(t))} + e^{-k_1L} \right)
$$

for $1 \leq k \leq N$. Indeed, it follows from (3.19)

$$
\|\psi_{a,1}v_{2k}\|_2 + \varepsilon^\frac{3}{2}\|v_{2k}\|_{L^2(0,T;W(t))} \leq \|\psi_{a,1}w_k\|_2 + \|\psi_{a,1}w_{k-1}\|_2 + \varepsilon^\frac{3}{2} \left( \|w_k\|_{L^2(0,T;W(t))} + \|w_{k-1}\|_{L^2(0,T;W(t))} \right)
$$

$$
\lesssim \|v_0\|_2 + \varepsilon^\frac{3}{2} \left( \|v_{2k}\|_{L^2(0,T;X_k(t))} + \|v_{2k-1}\|_{L^2(0,T;X_{k-1}(t))} \right)
$$

$$
+ \varepsilon^\frac{3}{2} \sum_{i=1}^{k-1} \|v_{2i}\|_{L^2(0,T;W(t))} + \varepsilon^\frac{3}{2}e^{-k_1L}
$$

$$
\lesssim \|v_0\|_2 + \varepsilon^\frac{3}{2} \left( \sum_{i=1}^k \|v_{2i}\|_{L^2(0,T;X_i(t))} + e^{-k_1L} \right).
$$

Let $u = ^t(r,p)$, $h(u) = \frac{1}{2}p^2 + V(r)$ and $h'(u) = ^t(V'(r), p)$,

$$
H_{k,i} = \langle h(U_k + w_k) - h(U_k) - h'(U_k) \cdot w_k, \psi_{a,i} \rangle_{L^2},
$$
where \( \cdot \) denotes the inner product in \( \mathbb{R}^2 \). Then
\[
\frac{dH_{k,i}}{dt} = -\dot{x}_i(h(U_k + w_k) - h(U_k)) - h'(U_k) \cdot w_k, \psi'_{a,i}\]
\[
+ \langle H'(U_k + w_k) - H'(U_k), \psi_{a,i} \partial_t(U_k + w_k) \rangle - \langle H''(U_k) \partial_t U_k, \psi_{a,i} w_k \rangle
\]
\[= I + II.\]

By the mean value theorem, there exists a \( \theta = \theta(t, n) \in (0, 1) \) such that
\[
I = -\frac{\dot{x}_i}{2} (H''(U_k + \theta w_k) w_k, \psi'_{a,i} w_k).
\]

Since \( \|U_k w_k^2\|_{l^1} \lesssim \varepsilon^2 \|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}^2 \), we have
\[
I = -\frac{\dot{x}_i}{2} (1 + O(\|w_k\|_{l^\infty})) \|\tilde{\psi}_{a,i} w_k\|_{l^2}^2 + O(\varepsilon^3 (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}^2)),
\]
where \( \tilde{\psi}_{a,i} = a^\frac{3}{2} \text{sech} \ a(x - x_i(t)) \). By (3.4) and the definition of \( U_k(t) \), we have
\[
II = \left\langle H'(U_k + w_k) - H'(U_k), \psi_{a,i} JH'(U_k + w_k) + \sum_{i=1}^{k} \psi_{a,i} (l_i - P_i JH_l) \right\rangle
\]
\[
+ \langle \tilde{R}_3, \psi_{a,i} \tilde{l}_k \rangle - \sum_{i=N+1-k}^{N} \langle \psi_{a,i} \partial_t H''(U_k) w_k, JH'(u_{c_i}) \rangle
\]
\[= \sum_{i=1}^{6} II_i,
\]
where \( \tilde{R}_3 = H'(U_k + w_k) - H'(U_k) - H''(U_k) w_k \) and
\[
II_1 = \langle H'(U_k + w_k) - H'(U_k), \psi_{a,i} J(H'(U_k + w_k) - H'(U_k)) \rangle,
\]
\[
II_2 = \langle \tilde{R}_3, \psi_{a,i} JH'(U_k) \rangle, \quad II_3 = \langle \tilde{R}_3, \psi_{a,i} \tilde{l}_k \rangle,
\]
\[
II_4 = \sum_{i=1}^{k} \langle H'(U_k + w_k) - H'(U_k), \psi_{a,i} l_i \rangle,
\]
\[
II_5 = -\sum_{i=1}^{k} \langle H'(U_k + w_k) - H'(U_k), \psi_{a,i} P_i JH_l \rangle,
\]
\[
II_6 = \langle H''(U_k) w_k, \psi_{a,i} JU_{k,int} \rangle.
\]

Using the Schwarz inequality and (3.17), we have
\[
|II_1| \leq \frac{1}{2} \|\tilde{\psi}_{a,i} \| \left( \|H'(U_k + w_k) - H'(U_k)\|_{l^2}^2 (1 + O(a^2)) \right)
\]
as in the proof of Lemma 3.2. Since

\[ \| \tilde{\psi}_{a,i}(H'(U_k + w_k) - H'(U_k))\|_2 \]
\[ \leq \| \tilde{\psi}_{a,i}w_k \|_2 (1 + O(\|w_k\|_\infty)) + O(\|\tilde{\psi}_{a,i}\|_\infty \|U_k w_k\|_2) \]
\[ \leq \| \tilde{\psi}_{a,i}w_k \|_2 (1 + O(\|w_k\|_\infty)) + O(\varepsilon^3(\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})) \]

there exists a \( \delta' > 0 \) such that

\[ I + II_1 \leq - \frac{\dot{x}_i - 1 + O(\delta \varepsilon^2)}{2} \| \tilde{\psi}_{a,i}w_k \|_2^2 + O(\varepsilon^3(\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}^2)) \]
\[ \leq - \delta' \varepsilon^2 \| \tilde{\psi}_{a,i}w_k \|_2^2 + O(\varepsilon^3(\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}^2)). \]

Let

\[ \|u\|_{W_k(t)} = \sum_{i=N+1-k}^{N} \|e^{-k_1|\cdot-x_i(t)|} u\|_2, \quad \|u\|_{W_k(t)^*} = \min_{i=N+1-k} \|e^{k_1|\cdot-x_i(t)|} u\|_2, \]
\[ \|u\|_{\tilde{W}_k(t)} = \sum_{i=N+1-k}^{N} \|e^{-k_1|\cdot-x_i(t)|} u\|_1, \quad \|u\|_{\tilde{W}_k(t)^*} = \min_{i=N+1-k} \|e^{k_1|\cdot-x_i(t)|} u\|_1. \]

By Claim A.1

\[ |II_2| \lesssim \|u_k^2\|_{\tilde{W}_k(t)} \sum_{i=N+1-k}^{N} \|Ju_{c_i}\|_{\tilde{W}_k(t)^*} \lesssim \varepsilon^3(\|v_{2k}\|_{X_k(t)}^2 + \|w_{k-1}\|_{W(t)}^2). \]

By (2.20), (3.18) and Claim A.1

\[ |II_3| \lesssim \|u_k^2\|_{\tilde{W}_k(t)} \|\tilde{I}_k\|_{\tilde{W}_k(t)^*} \]
\[ \lesssim (\|v_{2k}\|_{X_k}^2 + \|w_{k-1}\|_{W(t)}^2) \sum_{i=N+1-k}^{N} (|\dot{c}_i| + |\dot{x}_i - c_i| \varepsilon^3) \]
\[ \lesssim \varepsilon^3(\|v_{2k}\|_{X_k}^2 + \|w_{k-1}\|_{W(t)}^2). \]

By (2.21),

\[ |II_4| \lesssim \|w_k\|_{W_k(t)} \sum_{i=1}^{k} \|I_i\|_{W_k(t)^*} \]
\[ \lesssim \varepsilon^3(\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}) \]
\[ \times \|v_{2k}\|_{X_k(t)} \{e^{-k_1L} + \varepsilon^{-\frac{1}{2}}(\|v_1\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k}\|_{W(t)}) \} \]
\[ \lesssim \varepsilon^3\|v_{2k}\|_{X_k(t)}(\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}). \]
In view of (2.19) and Claim A.1 we have \( \|PJ\|_{B(\overline{W}_k(t),W_k(t)^*)} = O(\varepsilon^{3}) \) for \( i \leq k \). Thus by (2.25),

\[
|II_5| \leq \|H'(U_k + w_k) - H'(U_k)\|_{W_k(t)} \sum_{i=1}^{k} \|P_iJR_i\|_{W_k(t)^*}
\leq \varepsilon^{\frac{3}{2}} \|w_k(t)\|_{W_k(t)} \sum_{i=1}^{k} (\|R_{i1}\|_{\overline{W}_k(t)} + \|R_{i2}\|_{\overline{W}_k(t)} + \|R_{i3}\|_{\overline{W}_k(t)})
\leq \varepsilon^{\frac{3}{2}} (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})^3
+ \varepsilon^{\frac{5}{2}} (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)}) e^{-k_1(\sigma\varepsilon^3t + L)}
+ \varepsilon^3 (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})^2
\leq \varepsilon^3 (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})^2 + \varepsilon^6 e^{-2k_1(\sigma\varepsilon^3t + L)},
\]

and

\[
|II_6| \leq \|w_k\|_{W_k(t)} \|JU_{k,\text{int}}\|_{W_k(t)^*}
\leq \varepsilon^{\frac{3}{2}} e^{-k_1(\sigma\varepsilon^3t + L)} (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})
\leq \varepsilon^3 (\|v_{2k}\|_{X_k(t)} + \|w_{k-1}\|_{W(t)})^2 + \varepsilon^6 e^{-2k_1(\sigma\varepsilon^3t + L)}
\]

as in the proof of Lemma 3.1. Combining the above, we obtain

\[
\frac{dH_{k,i}}{dt} + \delta \varepsilon^2 \|\tilde{\psi}_{a,i} w_k\|^2_{L^2}
\leq \varepsilon^3 \left( \|v_1\|_{W(t)} + \|v_{2k}\|_{X_k(t)} + \sum_{i=1}^{k-1} \|v_{2i}\|_{W(t)} \right)^2 + \varepsilon^6 e^{-2k_1(\sigma\varepsilon^3t + L)}.
\]

Integrating (3.20) over \([0,T]\) and summing up for \( 1 \leq i \leq k \), we have

\[
\sum_{i=1}^{N} \left\{ H_{k,i}(t) - H_{k,i}(0) + \varepsilon^2 \int_{0}^{T} \|\tilde{\psi}_{a,i}(t) w_k(t)\|^2_{L^2} dt \right\}
\leq \int_{0}^{T} \left\{ \varepsilon^3 (\|v_{2k}\|^2_{X_k(t)} + \sum_{i=1}^{k-1} \|v_{2i}\|^2_{W(t)}) + \varepsilon^6 e^{-2k_1(\sigma\varepsilon^3t + L)} \right\} dt.
\]

Since \( H_{k,i} = \|\tilde{\psi}_{a,i} w_k\|^2_{L^2} (1 + O(\|U_k\|_{l^\infty} + \|w_k\|_{l^\infty})) \), we have (3.19). Thus we prove Lemma 3.3. \( \square \)
4. Proof of Theorem 1.1

In this section, we will show \textit{a priori} estimates on $v_1, v_{2k}, x_i$ and $c_i$ to prove stability of $N$-soliton solutions. Let

\[ M_1(T) = \varepsilon^{-2} \sup_{t\in[0,T]} \sum_{1\leq i\leq N} \left( |c_i(t) - c_{i,0}| + |\dot{x}_i(t) - c_i(t)| \right), \]
\[ M_2(T) = \varepsilon^{-3} \sum_{k=1}^{N} \sup_{0\leq t\leq T} \|v_{2k}(t)\|_{l^2}^2, \]
\[ M_3(T) = \varepsilon^{-\frac{3}{2}} \sup_{0\leq t\leq T} \|v_1(t)\|_{l^2} + \|v_1\|_{L^2(0,T;W(t))}, \]
\[ M_4(T) = \sum_{1\leq k\leq N} \left( \varepsilon^{-\frac{3}{2}} \sup_{0\leq t\leq T} \|\psi_{k,1}v_{2k}(t)\|_{l^2} + \|v_{2k}\|_{L^2(0,T;W(t))} \right), \]
\[ M_5(T) = \sum_{1\leq k\leq N} \left( \varepsilon^{-\frac{3}{2}} \|v_{2k}\|_{L^\infty(0,T;X_k(t))} + \|v_{2k}\|_{L^2(0,T;X_k(t))} \right). \]

Lemmas 2.2, 3.1, 3.2 and 3.3 imply a priori bound on $M_i$ ($1 \leq i \leq 4$) by $\|v_0\|_{H^1}$ and $M_5$.

**Lemma 4.1.** There exists a positive constant $\delta$ such that if

\[ \|v_0\|_{l^2} + \varepsilon^{\frac{3}{2}} \sum_{i=1}^{5} M_i(T) \leq \delta \varepsilon^{\frac{3}{4}}, \]

(4.1) \hspace{1cm} M_1(T) \lesssim \varepsilon^{-\frac{3}{2}} \|v_0\|_{l^2} + M_5(T) + e^{-k_1L},
(4.2) \hspace{1cm} M_2(T) \lesssim \varepsilon^{-\frac{3}{2}} \|v_0\|_{l^2} + M_5(T) + e^{-k_1L},
(4.3) \hspace{1cm} M_3(T) \lesssim \varepsilon^{-\frac{3}{2}} \|v_0\|_{l^2},
(4.4) \hspace{1cm} M_4(T) \lesssim \varepsilon^{-\frac{3}{2}} \|v_0\|_{l^2} + M_5(T) + e^{-k_1L}.$
Lemma 4.2. There exists a positive constant $\delta$ such that for $t \in [0, T]$,
\[
\sum_{i=1}^{N} |c_i(t) - c_i,0| \\
\leq \sum_{i=1}^{N} \sum_{k=1}^{N} \theta_1(c_i(t))^{-1} |(w_k(t), \rho_{c_i}(t))| \\
+ C\varepsilon^2 \int_{0}^{t} \left\{ \left\| v_1 \right\|_{W(s)}^2 + \sum_{k=1}^{N} \left\| v_{2k} \right\|_{X_k(s) \cap W(s)}^2 + \varepsilon^3 e^{-k_1(s^3 s + L)} \right\} ds \\
\lesssim \varepsilon^{\frac{1}{2}} \sum_{i=1}^{N} \left\| \psi_{k_1,0}(t) \frac{1}{2} w_i(t) \right\|_{L^2} \\
+ \varepsilon^2 \left( \left\| v_1 \right\|_{L^2(0,T;W(t))}^2 + \sum_{k=1}^{N} \left\| v_{2k} \right\|_{L^2(0,T;X_k(t) \cap W(t))}^2 + e^{-k_1 L} \right) \\
\lesssim \varepsilon^{\frac{1}{2}} \left\{ M_4(T) + (M_3(T) + M_4(T) + M_5(T))^2 + e^{-k_1 L} \right\},
\]
and
\[
|x_i(t) - c_i(t)| \lesssim \varepsilon^{\frac{1}{2}} \left( \left\| v_1 \right\|_{W(t)} + \sum_{i=1}^{N} \left\| \psi_{k_1,0}(t) \frac{1}{2} v_{2i}(t) \right\|_{L^2} \right) + \varepsilon^2 e^{-k_1 L} \\
\lesssim \varepsilon^{\frac{1}{2}} (M_3(T) + M_4(T) + e^{-k_1 L}).
\]

Lemmas 3.1, 3.2 and 3.3 imply (4.3), (4.4) and
\[
M_5(T) \lesssim \frac{1}{2} + e^{-\frac{3}{2}} \left\| v_0 \right\|_{L^2}^2 + e^{-k_1 L} + M_3(T) + M_4(T) + M_5(T).
\]
Substituting (4.3) and (4.4) into (4.5)-(4.7), we obtain (4.11) and (4.12). Thus we prove Lemma 4.1.

Now we will estimate $M_5(T)$.

Lemma 4.2. There exists a positive constant $\delta$ such that if
\[
\left\| v_0 \right\|_{L^2} + \varepsilon^{\frac{3}{2}} \sum_{i=1}^{N} M_i(T) \leq \delta \varepsilon^\frac{5}{2},
\]
then $M_5(T) \lesssim \varepsilon^{-\frac{3}{2}} \left\| v_0 \right\|_{L^2} + e^{-k_1 L}$.

To prove Lemma 4.2, we need the following exponential stability result of $k$-soliton solutions ($1 \leq k \leq N$).

Lemma 4.3. Let $x_{i,0}(t) = c_{i,0}t + x_{i,0}$ and $\bar{U}_k(t) = \sum_{i=N, 1-k}^{N} u_{c_{i,0}}(i - x_{i,0}(t))$. Let $\zeta = t(\zeta_1, \zeta_2) \in C^1(\mathbb{R}^2)$, $\mathcal{F}_0 \zeta \in L^1(T)$, $F_1, F_2 \in C([0, \infty); l_{k_1 }^2)$ and let $w(t) \in C^1(\mathbb{R}; l_{k_1 }^2)$ be a solution of
\[
\partial_t w(t) = JH''(\bar{U}_k(t) + \zeta(t))w(t) + F_1(t) + JF_2(t).
\]
There exist positive numbers $\varepsilon_0$, $L_0$, $\delta_1$, $\delta_2$, $M$ and $b$ satisfying the following: Suppose $\varepsilon \in (0, \varepsilon_0)$, $0 \leq T_1 \leq T_2 \leq \infty$ and that

$$\inf_{t \in [T_1, T_2]} \min_{0 \leq j \leq N} \varepsilon (x_{j,0} - x_{j-1,0}) \geq L_0,$$

$$\sup_{t \in [T_1, T_2]} \sup_{x \in \mathbb{R}} (|\zeta_1(t, x)| + \varepsilon^{-1} |\partial_x \zeta_1(t, x)|) \leq \delta_1 \varepsilon^2,$$

and

$$\varepsilon^{-\frac{3}{2}} |\langle w(t), J^{-1} \partial_x u_{c_i,0} (\cdot - x_{i,0}(t)) \rangle| + \varepsilon^2 |\langle w(t), J^{-1} \partial_x u_{c_i} (\cdot - x_{i,0}(t)) \rangle|$$

$$\leq \delta_2 \|e^{\varepsilon k_1 \langle -x_{N+1-k,0}(t) \rangle} w(t)\|_2$$

for $N + 1 - k \leq i \leq N$ and $t \in [T_1, T_2]$. Then for every $t, t_1 \in [T_1, T_2]$ satisfying $t \geq t_1$,

$$\|e^{\varepsilon k_1 \langle -x_{N+1-k,0}(t) \rangle} w(t)\|_2$$

$$\leq M e^{-bp(t-t_1)} \|e^{\varepsilon k_1 \langle -x_{N+1-k,0}(t_1) \rangle} w(t_1)\|_2$$

$$+ M \int_{t_1}^t e^{-bp(t-s)} \|e^{\varepsilon k_1 \langle -x_{N+1-k,0}(s) \rangle} F_1(s)\|_2 \, ds$$

$$+ M \varepsilon^{-\frac{3}{2}} \int_{t_1}^t (t-s)^{-\frac{1}{2}} \|e^{\varepsilon k_1 \langle -x_{N+1-k,0}(s) \rangle} F_2(s)\|_2 \, ds.$$

Lemma 4.3 follows immediately from Lemma 5.1. See Appendix D.

**Proof of Lemma 4.2.** Let $\{t_j\}_{j \geq 0}$ be a monotone increasing sequence such that $t_0 = 0$ and $\sup_{j \geq 0} [t_j, t_{j+1}] = [0, T]$ that satisfies (4.10) and (4.13) below. We remark that $t_{j+1} - t_j \sim \varepsilon^{-3}$.

To begin with, we will show that Lemma 4.3 is applicable provided $\delta$ is small. Let $x_{ij}(t) := x_i(t_j) + c_i,0(t-t_j)$, $h_{ij}(t) = x_i(t) - x_{ij}(t)$ and $U_{kj}(t) = \sum_{i=N+1-k}^N u_{c_i,0}(\cdot - x_{ij}(t))$. Lemma 4.1 implies that for $t \in [t_j, t_{j+1}]

$$|h_{ij}(t)| \leq \int_{t_j}^t (|\dot{x}_i(s) - c_i(s)| + |c_i(s) - c_{i,0}|) \, ds$$

$$\leq \varepsilon^2 M_1(T) (t_{j+1} - t_j).$$

Thus there exists an $A_2 > 0$ such that for $t \in [t_j, t_{j+1}]

$$\sup_x |U_k(t) - U_{kj}(t)|$$

$$\leq \sum_{i=N+1-k}^N \left( \|\partial_x u_{c_{i,0}}\|_{L^\infty} |x_i(t) - x_{ij}(t)| + \sup_{|c_{i,0}| \leq \delta \varepsilon^2} \|\partial_x u\|_{L^\infty} |c_i(t) - c_{i,0}| \right)$$

$$\leq A_2 \varepsilon^2 M_1(T) \{ \varepsilon^3 (t_{j+1} - t_j) + 1 \},$$
and
\[ \sup_x |\partial_x U_k(t) - \partial_x U_{kj}(t)| \]
\[ \leq \sum_{i=N+1-k}^N \left( \| \partial_x^2 u_{c_i,0} \|_{L^\infty} |x_i(t) - x_{ij}(t)| + \sup_{|c-c_{i,0}| \leq \delta \varepsilon^2} \| \partial_x \partial_c u \|_{L^\infty} |c_i(t) - c_{i,0}| \right) \]
\[ \leq A_2 \varepsilon^3 M_1(T) \{ \varepsilon^3 (t_{j+1} - t_j) + 1 \}. \]

Suppose
\[ A_2 \delta \{ 1 + \varepsilon^3 \sup_{j \geq 0} (t_{j+1} - t_j) \} < \delta_1. \]

Since \( \sup_{t \in [t_j, t_{j+1}]} \varepsilon |x_i(t) - x_{ij}(t)| = O(\delta) \), there exist positive constants \( c_1 \) and \( c_2 \) such that
\[ c_1 \| e^{k_1 \varepsilon (-x_{k,j}(t))} u \|_{l^2} \leq \| u \|_{X_k(t)} \leq c_2 \| e^{k_1 \varepsilon (-x_{k,j}(t))} u \|_{l^2} \]
for every \( t \in [t_j, t_{j+1}] \), \( j \geq 0 \) and \( u \in l^2_{k_1 \varepsilon} \). Hence it follows from Lemma 2.5 that for \( t \in [t_j, t_{j+1}], \) \( j \geq 0 \) and \( 1 \leq k \leq N - 1, \)
\[ \| v_{2k}(t) \|_{X_k(t)} \lesssim e^{-bc^3(t-t_j)} \| v_{2k}(t_j) \|_{X_k(t_j)} \]
\[ + \int_{t_j}^t e^{-bc^3(t-s)} \left( \| l_k(s) \|_{X_k(s)} + \| [Q_k(s), J] R_k \|_{X_k(s)} \right) ds \]
\[ + \varepsilon^{-k_1 \varepsilon} \int_{t_j}^t e^{-bc^3(t-s)} (t - s)^{-\frac{2}{3}} \| Q_k(s) R_k \|_{X_k(s)} ds. \]

By Lemma 2.5
\[ \| l_k \|_{X_k(t)} \lesssim \varepsilon^3 \| v_{2k} \|_{X_k(t)} (\| v_1 \|_{l^2} + \sum_{i=1}^N \| v_{2i} \|_{l^2} + \varepsilon^{\frac{3}{2}} e^{-k_1 (\varepsilon^3 t + L)}) \]
\[ \lesssim \delta \varepsilon^3 \| v_{2k} \|_{X_k(t)}. \]

By (2.25)
\[ \| R_k \|_{X_k(t)} \lesssim \| R_{k1} \|_{X_k(t)} + \| R_{k2} \|_{X_k(t)} + \| R_{k3} \|_{X_k(t)} \]
\[ \lesssim \varepsilon^3 \| v_{2k} \|_{X_k(t)} (\| v_1 \|_{l^2} + \| w_{k-1} \|_{l^2}) + e^{\frac{3}{2}} e^{-k_{N+1-k} (\varepsilon^3 t + L)} + \varepsilon^2 \| w_{k-1} \|_{W(t)} \]
\[ \lesssim \delta \varepsilon^2 \| v_{2k} \|_{X_k(t)} + e^{\frac{3}{2}} e^{-k_{N+1-k} (\varepsilon^3 t + L)} + \varepsilon^2 \| w_{k-1} \|_{W(t)}. \]
Substituting the above inequalities and \( \|Q_k(s), J\|_{B(X_k(s))} = O(\varepsilon) \) into (4.11), we have
\[
\|v_{2k}(t)\|_{X_k(t)} \leq e^{-b_1\varepsilon^3(t-t_j)}\|v_{2k}(t_j)\|_{X_k(t_j)} + \varepsilon^2 \int_{t_j}^{t} e^{-b_1\varepsilon^3(t-s)} \left( 1 + e^{-\frac{3}{2}(t-s)^{-\frac{1}{2}}} \right) e^{-k_1(\sigma s^3 + L)} ds
\]
\[
+ \varepsilon^3 \int_{t_j}^{t} e^{-b_1\varepsilon^3(t-s)} \left( 1 + e^{-\frac{3}{2}(t-s)^{-\frac{1}{2}}} \right) \|v_{2k}(s)\|_{X_k(s)} ds
\]
\[
+ \varepsilon^3 \int_{t_j}^{t} (1 + e^{-\frac{3}{2}(t-s)^{-\frac{1}{2}}} e^{-b_1\varepsilon^3(t-s)} (\|v_1(s)\|_{W(s)} + \sum_{i=1}^{k-1} \|v_{2i}(s)\|_{W(s)}) ds
\]
where \( b_1 = \min\{\frac{L}{4}, \frac{L}{4}\} \). Applying Gronwall's inequality ([12, Lemma 7.1.1]) to the above, we see that for small \( \delta \), there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\|v_{2k}(t)\|_{X_k(t)} \leq C_1 \{ e^{-b_1\varepsilon^3(t-t_j)}\|v_{2k}(t_j)\|_{X_k(t_j)} + \varepsilon^2 e^{-b_1\varepsilon^3(t-t_j) + k_1 L} \}
\]
(4.12)
\[
+ C_2 \varepsilon^3 \int_{t_j}^{t} e^{-b_1\varepsilon^3(t-s)} (t-s)^{-\frac{1}{2}} \left( \|v_1(s)\|_{W(s)} + \sum_{i=1}^{k-1} \|v_{2i}(s)\|_{W(s)} \right) ds
\]
for every \( t \in [t_j, t_{j+1}], j \geq 0 \) and \( 1 \leq k \leq N - 1 \). Suppose that \( \{t_j\}_{j \geq 0} \) satisfies
\[
C_1 \sup_{j \geq 0} e^{-b_1\varepsilon^3(t_{j+1}-t_j)} \leq \frac{1}{2}.
\]
(4.13)
Lemma 3.3 implies
\[
\|v_{2i}\|_{W(t)} \leq \|v_0\|_{W} + \varepsilon \sum_{j=1}^{i} \|v_{2j}\|_{L^2(0,T;X_{j}(t))}
\]
(4.14)
By (4.12), (4.14) and Lemma 3.2 there exists a positive constant \( C_3 \) such that
\[
\|v_{2k}(t_{j+1})\|_{X_k(t_{j+1})}
\]
\[
\leq \frac{1}{2} \{ \|v_{2k}(t_j)\|_{X_k(t_j)} + \varepsilon^2 e^{-k_1 L} \}
\]
\[
+ C_2 \varepsilon^3 e^{-b_1\varepsilon^3(t-t_j) + k_1 L} \sup_{t \in [0,T]} \left( \|v_1(t)\|_{W(t)} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{W(t)} \right)
\]
\[
\leq \frac{1}{2} \|v_{2k}(t_j)\|_{X_k(t_j)} + C_3 \left\{ \|v_0\|_{W} + \varepsilon \left( e^{-k_1 L} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{L^2(0,T;X_{i}(t))} \right) \right\}
\]
for any \( j \geq 0 \). Thus we have

\[
\sup_{j \geq 0} \|v_{2k}(t_j)\|_{X_k(t_j)} \lesssim \|v_0\|_{l^2} + \varepsilon^{\frac{3}{2}} \left( e^{-k_1 L} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{L^2(0,T;X_i(t))} \right).
\]

Substituting the above into (4.12) and applying Young’s inequality to the resulting equation and using Lemmas 3.2 and 3.3 again, we have for \( 1 \leq k \leq N - 1 \),

\[
\|v_{2k}\|_{L^2(0,T;X_k(t))} \lesssim \varepsilon^{\frac{3}{2}} \|v_0\|_{l^2} + e^{-k_1 L} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{L^2(0,T;X_i(t))} \] + \varepsilon^{\frac{3}{2}} \|v_{2k}\|_{L^2(0,T;X_k(t))} \lesssim \varepsilon^{\frac{3}{2}} \|v_0\|_{l^2} + e^{-k_1 L} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{L^2(0,T;X_i(t))}.
\]

Similarly, we have

\[
\sup_{t \in [0,T]} \|v_{2k}(t)\|_{X_k(t)} \lesssim \|v_0\|_{l^2} + \varepsilon^{\frac{3}{2}} \left( e^{-k_1 L} + \sum_{i=1}^{k-1} \|v_{2i}(t)\|_{L^2(0,T;X_i(t))} \right)
\]

by using (4.12) and (4.14). Thus we conclude that for \( 1 \leq k \leq N - 1 \),

\[
\sup_{t \in [0,T]} \|v_{2k}(t)\|_{X_k(t)} + \varepsilon^{\frac{3}{2}} \|v_{2k}\|_{L^2(0,T;X_k(t))} \lesssim \|v_0\|_{l^2} + \varepsilon^{\frac{3}{2}} e^{-k_1 L}.
\]

Finally, we will estimate \( \|v_{2N}\|_{X_N(t)} \). Eq. (2.9) is transformed into

\[
\begin{cases}
\partial_t v_{2N} = JH''(U_N)v_{2N} + l_N + Q_NJR_N, \\
v_{2N}(0) = 0,
\end{cases}
\]

where \( l_N = P_N(t)(\partial_t - JH''(U_N(t)))v_{2N} = -\dot{P}_N(t)v_{2N} - P_N(t)JH''(U_N(t))v_{2N}. \)

Let

\[
f_N = \begin{pmatrix} f_{N,1}^1 \\ f_{N,1}^2 \\ \vdots \\ f_{N,1}^N \end{pmatrix} = \begin{pmatrix} \varepsilon^{-4}\langle v_{2N}, (H''(U_N) - H''(u_{c_1}))\partial_x u_{c_1} \rangle \\ \varepsilon^{-4}\langle v_{2N}, (H''(U_N) - H''(u_{c_1}))\partial_c u_{c_1} \rangle \\ \vdots \\ \varepsilon^{-4}\langle v_{2N}, J^{-1}\{((\dot{x}_i - c_i)\partial_x^2 u_{c_1} - \dot{c}_i\partial_x\partial_c u_{c_1}) \} \rangle \\ \varepsilon^{-4}\langle v_{2N}, J^{-1}\{((\dot{x}_i - c_i)\partial_x u_{c_1} - \dot{c}_i\partial_c^2 u_{c_1}) \} \rangle \end{pmatrix} \]

i = 1, \ldots, N_P.

By (2.19) and (2.14), we have

\[
l_N = (\varepsilon^3 \partial_x u_{c_j}, \partial_x u_{c_j})_{j=1, \ldots, N} = A_N^{-1} f_N
\]

\[
= \frac{1}{A_N} \sum_{j=1}^{N} \begin{pmatrix} A_{11} & \cdots & \tilde{A}_{1j} & \cdots & A_{1N} \\ \vdots & & \vdots & & \vdots \\ A_{N_1} & \cdots & \tilde{A}_{Nj} & \cdots & A_{NN} \end{pmatrix} + \begin{pmatrix} A_{11} & \cdots & \tilde{A}_{1j} & \cdots & A_{1N} \\ \vdots & & \vdots & & \vdots \\ A_{N_1} & \cdots & \tilde{A}_{Nj} & \cdots & A_{NN} \end{pmatrix}
\]
where

\[
\tilde{\Delta}_{ij}^1 = \left( \varepsilon^3 \partial_x u_c f_{N_i}^1, \varepsilon^{-4} \langle \partial_x u_c, J^{-1} \partial_x u_{c_i} \rangle \right),
\]

\[
\tilde{\Delta}_{ij}^2 = \left( \varepsilon^{-1} \langle \partial_x u_c, J^{-1} \partial_x u_{c_i} \rangle, \varepsilon^2 \langle \partial_c u_c f_{N_i}^1, \partial_c u_c f_{N_i}^2 \rangle \right).
\]

Noting that

\[
\|\text{the first column of } \tilde{\Delta}_{ij}^1 \|_{X_N(t)} + \|\text{the second column of } \tilde{\Delta}_{ij}^2 \|_{X_N(t)} \lesssim \varepsilon^3 \left\{ \varepsilon^{-2} \sum_{k=1}^{N} \|v_{2k}\|_{W(t)} + \sum_{k=1}^{N} \|v_{2k+1}\|_{W(t)} + e^{-k_1(\sigma \varepsilon^3 t + L)} \right\} e^{|x_j - x_i|} \|v_{2N}\|_{X_N(t)},
\]

and following the argument of the proof of Lemma 2.4, we have

\[
\|l_N(t)\|_{X_N(t)} \lesssim \varepsilon^3 (\delta + e^{-k_1 L}) \|v_{2N}(t)\|_{X_N(t)}.
\]

Thus we have

\[
\sup_{t \in [0, T]} \|v_{2N}\|_{X_N(t)} + \varepsilon^3 \sum_{k=1}^{N} \|v_{2k}\|_{L^2(0; X_N(t))} \lesssim \|v_0\|_{\ell^2} + \varepsilon^3 e^{-k_1 L} + \sum_{1 \leq k \leq N-1} \|v_{2k}\|_{L^2(0; X_k(t))}
\]

exactly in the same way as \((4.15)\).

Now we are in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \((v_1, v_{21}, \cdots, v_{2N}, x_1, c_1, \cdots, x_N, c_N)\) be a solution to the system \((2.3), (2.6), (2.9), (2.10), (2.24)\) satisfying the initial condition \((2.17)\). It exists as long as \(v_{2k} (1 \leq k \leq N)\) and \(c_i\) remain bounded. Let \(\delta\) be a positive number given in Lemmas 4.1 and 4.2. By \((2.16)\) and \((2.17)\),

\[
\|v_0\|_{\ell^2} + \varepsilon^3 \sum_{i=1}^{5} M_i(0) = 2\|v_0\|_{\ell^2} + \varepsilon^{-\frac{1}{2}} \sum_{i=1}^{5} |\dot{x}_i(0) - c_i(0)| \lesssim \delta_0 \varepsilon^2 + \varepsilon^3 e^{-k_1 L}.
\]

If \(\delta_0\) is sufficiently small and \(L\) is sufficiently large,

\[
\|v_0\|_{\ell^2} + \varepsilon^3 \sum_{i=1}^{5} M_i(0) \leq \frac{\delta}{2} \varepsilon^2.
\]
Let \( T_* = \sup\{ T_1 \geq 0 : \| v_0 \|_{L^2} + \varepsilon^2 \sum_{i=1}^{5} M_i(T) \leq \delta \varepsilon \hat{\gamma}^2 \text{ for } 0 \leq T \leq T_1 \} \). Lemmas 4.1 and 4.2 imply that there exists a \( C > 0 \) such that

\[
\| v_0 \|_{L^2} + \varepsilon^2 \sum_{i=1}^{5} M_i(T) \leq C(\| v_0 \|_{L^2} + \varepsilon^2 e^{-k_1 L}) < \delta \varepsilon \hat{\gamma}^2 \text{ for } 0 \leq T \leq T_*
\]

provided \( \varepsilon_0, \delta_0 \) are sufficiently small and \( L \) is sufficiently large. Thus we have \( T_* = \infty \) and (1.6). We can prove (1.7) and (1.8) in exactly the same way as [20, pp.140-143]. Thus we complete the proof of Theorem 1.1. \( \square \)

5. Linear estimate

In this section, we prove exponential linear stability of small \( N \)-soliton solutions of (1.2). Let \( T = t/24, X = x - t \) and

\[
r_{N, \varepsilon}(t, x; k, \gamma) = \varphi_N (T, X; \varepsilon k, \varepsilon^{-1} \gamma) = \varepsilon^2 \varphi_N (\varepsilon^3 T, \varepsilon X; k, \gamma),
\]

\[
u_{N, \varepsilon}(t, n; k, \gamma) = \ell (r_{N, \varepsilon}(t, n; k, \gamma), -r_{N, \varepsilon}(t, n; k, \gamma)).
\]

Gardner et al. [10] tells us that an \( N \)-soliton \( u_{N, \varepsilon} \) uniformly converges to a train of solitary waves \( u_{c_i, \varepsilon}(n - c_i t - \varepsilon^{-1} \gamma_i) \) \((1 \leq i \leq N)\) as \( t \to \infty \) (see also [11]). Since solitary waves of (1.2) are approximated by KdV 1-solitons in the continuous limit ([5]), \( u_{N, \varepsilon} \) is an approximate solution of (1.2).

The linearized equation of (1.2) around \( u_{N, \varepsilon} \) has a similar exponential stability property as the linearized KdV equation (1.10) if \( \varepsilon \) is close to 0.

**Lemma 5.1.** Let \( 0 < k_1 < \cdots < k_N, \ \zeta = (\zeta_1, \zeta_2) \in C^1(\mathbb{R}), \ F_n \zeta \in L^1(\mathbb{T}) \) and \( F_1, F_2 \in C([0, \infty); L^2_{k_{1, \varepsilon}}) \). Let \( w(t) \in C^1(\mathbb{R}; L^2_{k_{1, \varepsilon}}) \) be a solution of

\[
\partial_t w(t) = JH''(u_{N, \varepsilon}(t, \cdot; k, \gamma) + \zeta(t, \cdot))w(t) + F_1(t) + JF_2(t).
\]

There exist positive numbers \( \varepsilon_0, \delta_1, \delta_2, M \) and \( b \) satisfying the following: If \( \varepsilon \in (0, \varepsilon_0), \sup_{t,x}(|\zeta_1(t, x)| + \varepsilon^{-1} |\partial_x \zeta_1(t, x)|) \leq \delta_1 \varepsilon^2 \) and

\[
\sum_{1 \leq i \leq N} (\| w(t), J^{-1} \partial_{\gamma_i} u_{N, \varepsilon}(t) \| + \| w(t), J^{-1} \partial_{k_i} u_{N, \varepsilon}(t) \|)
\]

\[
\leq \delta_2 \varepsilon^2 \| e^{k_1 (-c_1 t - \varepsilon^{-1} \gamma_1)} w(t) \|_{L^2}
\]

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for $1 \leq i \leq N$ and $t \geq t_1$, then for every $t \geq t_1 \geq 0$,
\[
\left\| e^{\varepsilon k_1(-c_1, \varepsilon t)}w(t) \right\|_2 \leq M e^{-b_{\varepsilon_1}(t-s)} \left\| e^{\varepsilon k_1(-c_1, \varepsilon t_1)}w(t_1) \right\|_2 + M \int_{t_1}^t e^{-b_{\varepsilon_1}(t-s)} \left\| e^{\varepsilon k_1(-c_1, \varepsilon s)}F_1(s) \right\|_2 ds
\]
\[+ \frac{M}{\varepsilon} \int_{t_1}^t e^{-b_{\varepsilon_1}(t-s)}(t-s)^{-\frac{1}{2}} \left\| e^{\varepsilon k_1(-c_1, \varepsilon s)}F_2(s) \right\|_2 ds.
\]

Let
\[
\tilde{f} = \begin{pmatrix} 0 & e^{i\xi} - 1 \\ 1 - e^{-i\xi} & 0 \end{pmatrix}, \quad P(\xi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\xi/2} \\ -e^{-i\xi/2} & 1 \end{pmatrix},
\]
\[
f(t, \xi) = \begin{pmatrix} f_+(t, \xi) \\ f_-(t, \xi) \end{pmatrix} = e^{ic_1, \varepsilon t \xi} P(\xi)^* F_n w(t, \xi),
\]
\[
f_\#(t, \xi) = e^{-ic_1, \varepsilon t \xi} (f_+(t, \xi) + e^{i\xi} f_-(t, \xi)),
\]
\[
G_1(t, \xi) = \frac{e^{ic_1, \varepsilon t \xi}}{\sqrt{2\pi}} \left( \tilde{r}_{N, \varepsilon}(t, \xi; k, \gamma) + \tilde{\zeta}_1(t, \xi) \right) * f_\#(t, \xi),
\]
\[
G_2(t, \xi) = \begin{pmatrix} G_{2,+}(t, \xi) \\ G_{2,-}(t, \xi) \end{pmatrix} = ie^{ic_1, \varepsilon t \xi} P(\xi)^* \tilde{F}_1(t, \xi),
\]
\[
G_3(t, \xi) = \begin{pmatrix} G_{3,+}(t, \xi) \\ G_{3,-}(t, \xi) \end{pmatrix} = -2e^{ic_1, \varepsilon t \xi} \sigma_3 P(\xi)^* \tilde{F}_2(t, \xi).
\]

By the definition, $f_\#$ is $2\pi$-periodic in $\xi$. Using $P(\xi)^* \tilde{J} P(\xi) = -2i \sin \frac{\xi}{2} \sigma_3$, we see that (5.1) translates into
\[
\partial_t f = ic_1, \varepsilon f + e^{ic_1, \varepsilon t \xi} P(\xi)^* F_n (JH''(u_{N, \varepsilon} + \zeta) w) - i \left( G_2 + \sin \frac{\xi}{2} G_3 \right)
\]
\[= \Lambda_\varepsilon f + \frac{e^{ic_1, \varepsilon t \xi}}{\sqrt{2\pi}} P(\xi)^* \tilde{f} \left\{ \tilde{r}_{N, \varepsilon} + \tilde{\zeta}_1 \right\} * f_\#(t, \xi)
\]
\[= \Lambda_\varepsilon f - i \left( G_1(t, \xi) + G_{3,+}(t, \xi) \right) \sin \frac{\xi}{2} + G_{2,+}(t, \xi) - (G_1(t, \xi)e^{-i\xi/2} - G_{3,-}(t, \xi)) \sin \frac{\xi}{2} + G_{2,-}(t, \xi),
\]
where $\Lambda_\varepsilon = \text{diag}(i\lambda_{+}, i\lambda_{-})$ and $\lambda_{\pm}(\xi) = c_1, \varepsilon \xi \mp 2 \sin(\xi/2)$ for $\xi \in [-\pi, \pi]$. By Parseval's equality, we have
\[
\left\| e^{\varepsilon k_1(-c_1, \varepsilon t)}w(t) \right\|_2 = e^{-\varepsilon k_1c_1, \varepsilon t} \left\| \tau_{ik_1, \varepsilon} F_n w(t) \right\|_{L^2(T)}
\]
\[= \left\| e^{ic_1, \varepsilon t \xi} P(\cdot + i\varepsilon k_1)f(t, \cdot + i\varepsilon k_1) \right\|_{L^2(-\pi, \pi)} \lesssim \left\| \tau_{ik_1, \varepsilon} f(t) \right\|_{L^2(-\pi, \pi)}.
\]
Thus to prove Lemma 5.1, it suffices to estimate \( \| \tau_{t_k} f(t) \|_{L^2(\mathbb{T})} \).

To begin with, we will show the lower bound of \( \Im \lambda_\pm \).

**Lemma 5.2.** Let \( a \in (0, 2k_1) \) and \( \delta \in (0, \pi) \). Then there exist positive numbers \( K \) and \( \varepsilon_0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \),

\[
\lambda_{+,\varepsilon}(\eta + ia) = \frac{\varepsilon^3}{24} \left( (\eta + ia)^3 + 4k_1^2(\eta + ia) \right) + O(\varepsilon^5(\eta)^5) \quad \text{for } \eta \in [-2K, 2K],
\]

\[
\Im \lambda_{+,\varepsilon}(\eta + ia) \geq \frac{\varepsilon^3}{16} a^2 \quad \text{for } \eta \in [-2\delta\varepsilon^{-1}, -K] \cup [K, 2\delta\varepsilon^{-1}],
\]

\[
\Im \lambda_{+,\varepsilon}(\eta + ia) \geq \varepsilon a (1 - \cos \delta) \quad \text{for } \eta \in [-\pi\varepsilon^{-1}, -\delta\varepsilon^{-1}] \cup [\delta\varepsilon^{-1}, \pi\varepsilon^{-1}],
\]

\[
\Im \lambda_{-,\varepsilon}(\eta + ia) \geq \varepsilon a \quad \text{for } \eta \in [-\pi\varepsilon^{-1}, \pi\varepsilon^{-1}].
\]

**Proof.** Let \( \xi = \varepsilon(\eta + ia) \). For \( \eta \in [-2K, 2K] \), we have

\[
\lambda_{+,\varepsilon}(\xi) = \frac{\varepsilon^3}{24} \left( (\eta + ia)^3 + 4k_1^2(\eta + ia) \right) + O(\varepsilon^5(\eta)^5)
\]

Since

\[
\lambda_{-,\varepsilon}(\xi) = \varepsilon c_{1,\varepsilon}(\eta + ia) = 2 \left( \sin \frac{\varepsilon \eta}{2} \cosh \frac{\varepsilon a}{2} + i \cos \frac{\varepsilon \eta}{2} \sinh \frac{\varepsilon a}{2} \right),
\]

we have \( \Im \lambda_{-,\varepsilon}(\xi) \geq \varepsilon c_{1,\varepsilon} a \geq \varepsilon a \) for \( \eta \in [-\pi\varepsilon^{-1}, \pi\varepsilon^{-1}] \), and

\[
\Im \lambda_{+,\varepsilon}(\xi) = \varepsilon c_{1,\varepsilon} a - 2 \sinh \frac{\varepsilon a}{2} \cos \frac{\varepsilon \eta}{2}
\]

\[
= 2 \sinh \frac{\varepsilon a}{2} \left( 1 - \cos \frac{\varepsilon \eta}{2} \right) + \varepsilon c_{1,\varepsilon} a - 2 \sinh \frac{\varepsilon a}{2}
\]

\[
\geq \frac{\varepsilon^3}{8} a \left( 1 + O(\delta^2) \right) \eta^2 + O(\varepsilon^3) \quad \text{for } \eta \in [K, \delta\varepsilon^{-1}] \cup [-\delta\varepsilon^{-1}, -K],
\]

\[
\Im \lambda_{+,\varepsilon}(\xi) \geq 2 \sinh \frac{\varepsilon a}{2} \left( 1 - \cos \delta \right) + \varepsilon c_{1,\varepsilon} a - 2 \sinh \frac{\varepsilon a}{2}
\]

\[
\geq \varepsilon a (1 - \cos \delta) + O(\varepsilon^3) \quad \text{for } \eta \in [-\pi\varepsilon^{-1}, \delta\varepsilon^{-1}] \cup [\delta\varepsilon^{-1}, \pi\varepsilon^{-1}].
\]

\qed

We need the following lemma to estimate the potential term of \((5.3)\).

**Lemma 5.3.** (1) Suppose \( f \in L^\infty(\mathbb{R}) \), \( F_n f \in L^1(\mathbb{T}) \) and \( g \in L^2(\mathbb{T}) \). Then

\[
\left\| \int_T \tilde{f}(\xi_1) g(\xi - \xi_1) d\xi_1 \right\|_{L^2(\mathbb{T})} \leq \| f \|_{L^\infty(\mathbb{R})} \| g \|_{L^2(\mathbb{T})}.
\]

(2) Let \( 0 < \delta < \pi(4\sum_{n=1}^{N} k_i)^{-1} \). Then as \( \varepsilon \to 0 \),

\[
\sup_{t \geq 0, \xi \in [-\pi, \pi], \gamma \in \mathbb{R}^N} \left| \tilde{\tau}_{N,\varepsilon}(t, \xi_1, \gamma) - \tilde{\tau}_{N,\varepsilon}(t, \xi_1, \gamma) \right| = O(e^{-\pi\delta/\varepsilon}).
\]
See Appendix B for the proof. Now we start to prove Lemma 5.1

**Proof of Lemma 5.1 (the former part).** Since \( \inf_{\xi \in \mathbb{R}} t^{-1} \log |e^{t \Lambda_{c}(\xi + ik_{1}\varepsilon)}| \lesssim -\varepsilon^{3} \) and is of the same order as the size of the potential term of (5.3), Lemma 5.2 is not sufficient to prove exponential linear stability. We will decompose solutions into a high frequency part, a middle frequency part and a low frequency part.

Let \( \chi \) and \( \tilde{\chi} \) be nonnegative smooth functions such that \( \chi + \tilde{\chi} = 1 \) and \( \chi(\xi) = 0 \) if \( |\xi| \geq 2 \). Let \( \chi_{b}(\xi) = \chi(\xi/b) \) and \( \tilde{\chi}_{b}(\xi) = \tilde{\chi}(\xi/b) \). Let \( K \) be a large number satisfying \( K_{\varepsilon}^{2} \leq 1, \xi_{\varepsilon} = \xi + ik_{1}\varepsilon \) and

\[
\begin{align*}
    f_{1,+}(t,\xi) &= \chi K_{\varepsilon}(\xi)f_{+}(t,\xi), \\
    f_{2,+}(t,\xi) &= (\chi_{b}(\xi) - \chi K_{\varepsilon}(\xi))f_{+}(t,\xi), \\
    f_{3,+}(t,\xi) &= \tilde{\chi}_{b}f_{+}(t,\xi), \\
    f_{3,-}(t,\xi) &= (f_{3,+}(t,\xi), f_{-}(t,\xi)).
\end{align*}
\]

Then by (5.3),

\[
\begin{align*}
    \partial_{t}f_{1,+}(t,\xi) &= i\lambda_{+,\varepsilon}(\xi_{\varepsilon})f_{1,+}(t,\xi) \\
    &\quad - i\chi K_{\varepsilon}(\xi)\left( G_{2,+}(t,\xi_{\varepsilon}) + (G_{1}(t,\xi_{\varepsilon}) + G_{3,+}(t,\xi_{\varepsilon})) \sin \frac{\xi_{\varepsilon}}{2} \right), \\
    \partial_{t}f_{2,+}(t,\xi) &= i\lambda_{+,\varepsilon}(\xi_{\varepsilon})f_{2,+}(t,\xi) \\
    &\quad - i(\chi_{b}(\xi) - \chi K_{\varepsilon}(\xi))\left( G_{2,+}(t,\xi_{\varepsilon}) + (G_{1}(t,\xi_{\varepsilon}) + G_{3,+}(t,\xi_{\varepsilon})) \sin \frac{\xi_{\varepsilon}}{2} \right), \\
    \partial_{t}f_{3,+}(t,\xi) &= i\lambda_{+,\varepsilon}(\xi_{\varepsilon})f_{3,+}(t,\xi) \\
    &\quad - i\tilde{\chi}_{b}(\xi)\left( G_{2,+}(t,\xi_{\varepsilon}) + (G_{1}(t,\xi_{\varepsilon}) + G_{3,+}(t,\xi_{\varepsilon})) \sin \frac{\xi_{\varepsilon}}{2} \right), \\
    \partial_{t}f_{-}(t,\xi) &= i\lambda_{-,\varepsilon}(\xi_{\varepsilon})f_{-}(t,\xi) \\
    &\quad + i\left( (G_{1}(t,\xi_{\varepsilon})e^{\frac{i\xi_{\varepsilon}}{2}} - G_{3,-}(t,\xi_{\varepsilon})) \sin \frac{\xi_{\varepsilon}}{2} - G_{2,-}(t,\xi_{\varepsilon}) \right).
\end{align*}
\]

Except for the low frequency part \( f_{1,+} \), potential terms of the above equations are negligible. In the former part of the proof, we will estimate \( \|f_{2,+}\|_{L^{2}} \) and \( \|f_{3}\|_{L^{2}} \).

Lemma 5.2 implies that \( \exists \lambda_{-,\varepsilon}(\xi_{\varepsilon}) \geq k_{1}\varepsilon \) for \( \xi \in [-\pi, \pi] \) and that there exists \( \alpha \in (0, k_{1}) \) such that \( \exists \lambda_{+,\varepsilon}(\xi_{\varepsilon}) \geq \alpha \varepsilon \) for \( \xi \in \supp \tilde{\chi}_{b} \). Using the variation of constants formula and Minkowski’s inequality, we have

\[
\|f_{3,+}(t)\|_{L^{2}} \lesssim e^{-\alpha t}\|f_{3,+}(0)\|_{L^{2}} + \int_{0}^{t} e^{-\alpha(t-s)}(\|G_{1}(s,\xi_{\varepsilon})\|_{L^{2}} + \|G_{2}(s,\xi_{\varepsilon})\|_{L^{2}} + \|G_{3}(s,\xi_{\varepsilon})\|_{L^{2}})ds.
\]

Using Parseval’s identity, we have

\[
\|G_{2}(s,\xi_{\varepsilon})\|_{L^{2}} \lesssim e^{-k_{1}c_{1}\varepsilon s}\|F_{1}(s)\|_{L^{2}_{k_{1}\varepsilon}}, \quad \|G_{3}(s,\xi_{\varepsilon})\|_{L^{2}} \lesssim e^{-k_{1}c_{1}\varepsilon s}\|F_{2}(s)\|_{L^{2}_{k_{1}\varepsilon}}.
\]
Since $\|r_{N,\varepsilon}\|_{L^\infty} = O(\varepsilon^2)$, it follows from Lemma 5.3 that

$$\|G_1(s,\xi)\|_{L^2} \lesssim \|r_N + \zeta_1\|_{L^\infty} \left( \|f_+(s,\xi)\|_{L^2} + \|f_-(s,\xi)\|_{L^2} \right)$$

$$\lesssim \varepsilon^2 \left( \|f_{1,+}(s)\|_{L^2} + \|f_{2,+}(s)\|_{L^2} + \|f_3(s)\|_{L^2} \right).$$

Combining the above, we obtain

$$\|f_{3,+}(t)\|_{L^2} \lesssim e^{-\alpha\varepsilon t} \|f_{3,+}(0)\|_{L^2} + \int_0^t e^{-\alpha\varepsilon(t-s)} e^{-k_1\varepsilon s} \left( \|F_1(s)\|_{L^2} + \|F_2(s)\|_{L^2} \right) ds$$

$$+ \varepsilon^2 \int_0^t e^{-\alpha\varepsilon(t-s)} \left( \|f_{1,+}(s)\|_{L^2} + \|f_{2,+}(s)\|_{L^2} + \|f_3(s)\|_{L^2} \right) ds. \quad (5.4)$$

Next we will estimate $\|f_{2,+}(t)\|_{L^2}$. Noting that $\Im\lambda_{+,-\varepsilon} \geq k_1\varepsilon^2/16$ and $\left| \sin \frac{\varepsilon}{2} \right| \lesssim |\xi|$ on $\text{supp}(\chi_{\delta} - \chi_{K\varepsilon})$ and using the variation of constants formula, we have

$$\|f_{2,+}(t)\|_{L^2} \lesssim e^{-\frac{k_1\varepsilon^2(t-s)}{16}} \|f_{2,+}(0)\|_{L^2} + \int_0^t \|\xi e^{-\frac{k_1\varepsilon^2(t-s)}{16}} G_1(s,\xi)\|_{L^2} ds$$

$$+ \int_0^t \left( \|\xi e^{-\frac{k_1\varepsilon^2(t-s)}{16}} G_2(s,\xi)\|_{L^2} + \|\xi e^{-\frac{k_1\varepsilon^2(t-s)}{16}} G_3(s,\xi)\|_{L^2} \right) ds.$$}

Since $|\xi| e^{-\frac{k_1\varepsilon^2(t-s)}{16}} \lesssim (\varepsilon(t-s))^{\frac{1}{2}} e^{-\frac{k_1\varepsilon^2(t-s)}{32}}$ for $\xi \in \text{supp}(\chi_{\delta} - \chi_{K\varepsilon})$,

$$\|f_{2,+}(t)\|_{L^2} \lesssim e^{-\frac{k_1\varepsilon^2(t-s)}{16}} \|f_{2,+}(0)\|_{L^2} + \int_0^t e^{-\frac{k_1\varepsilon^2(t-s)}{16}} e^{-k_1\varepsilon s} \|F_1(s)\|_{L^2} ds$$

$$+ \varepsilon^2 \int_0^t (t-s)^{\frac{1}{2}} e^{-\frac{k_1\varepsilon^2(t-s)}{32}} e^{-k_1\varepsilon s} \|F_2(s)\|_{L^2} ds$$

$$+ \varepsilon^2 \int_0^t (t-s)^{\frac{1}{2}} e^{-\frac{k_1\varepsilon^2(t-s)}{32}} \left( \|f_{1,+}(s)\|_{L^2} + \|f_{2,+}(s)\|_{L^2} + \|f_3(s)\|_{L^2} \right) ds. \quad (5.5)$$

For the low frequency part, both the dispersion induced by discreteness of spatial variable and the potential produced by an $N$-soliton $r_{N,\varepsilon}$ are the same order. We will show that the balance between the dispersion and the potential is described by the linearized KdV equation around an $N$-soliton solution as was observed by [8] for a 1-soliton solution.

We need that $P(\tau)$ is uniformly bounded for $\tau \geq 0$.

**Lemma 5.4.** Let $0 < k_1 < \cdots < k_N$, $a \in (0, 2k_1)$ and $\tau_0 \in \mathbb{R}$. There exists a positive constant $C$ depending only on $k_1, \cdots, k_N$ and $a$ such that if $4k_1^2\tau_0 + \gamma_1 \leq \cdots \leq 4k_N^2 + \gamma_N$,

$$\sup_{\tau \geq \tau_0} \|P(\tau)\|_{B(L^2_\varepsilon)} \leq C.$$
To estimate $\|f_{1,+}\|_{L^2}$, we need to show that the low frequency part $f_{1,+}$ approximately satisfies the secular term condition for a linearized KdV equation \((1.11)\) and \((1.12)\). Let $P_1(\tau) = e^{k_1\tau}P(\tau)\tau_{-4k_1^2}e^{-k_1y}$ and let $h_i(\tau)$ be an $L^2(\mathbb{R})$-function such that

$$h_i(\tau, y) = \frac{1}{\sqrt{2\pi}} \int_{-\pi^{-1}}^{\pi^{-1}} f_{i,+}(t, \varepsilon \eta) e^{i\eta y} dy \quad \text{for } i = 1, 2.$$

**Lemma 5.5.** If $w(t)$ satisfies \((5.2)\), then

$$\varepsilon^\frac{1}{2} \|P_1(\tau) h_1(\tau)\|_{L^2} \lesssim (\varepsilon^2 + \delta_2 + K^{-2}) \|\tau_{ik_1} f(t)\|_{L^2(T)}.$$

The proof of Lemmas \(5.4\) and \(5.5\) will be given in Appendix \(C\).

**Proof of Lemma 5.4 (continued).** Finally, we will estimate $f_{1,+}$. Let $\tau = \varepsilon^3 t/24$, $\xi = \varepsilon \eta$ and

$$G_4(t, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ic_1 \xi \eta} \tilde{\tau}_{N,\xi}(t, \xi; k, \gamma) f_{\#}(t, \xi - \xi_1) d\xi_1,$$

$$G_5(t, \xi) = \frac{e^{ic_1 \xi \eta} \sin \frac{\xi}{2}}{\sqrt{2\pi \xi}} (\hat{\zeta} * T f_{\#})(t, \xi, \xi).$$

Lemma\(5.3\) implies that for any $N > 0$,

$$\left\| \chi_{K}(\eta) \left((G_1(t, \xi_1) - G_4(t, \xi_\varepsilon)) \sin \frac{\xi_\varepsilon}{2} - \xi_\varepsilon G_5(t, \xi_\varepsilon) \right) \right\|_{L^2_0(\mathbb{R})} \lesssim \varepsilon^{-\frac{1}{2}} e^{-k_1 c_1 \varepsilon t} \left\| \int_{-\pi}^{\pi} \left(\tilde{r}_{N,\xi}(t, \xi_1; k, \gamma) - \tilde{\tau}_{N,\xi}(t, \xi_1; k, \gamma)\right) f_{\#}(t, \xi - \xi_1) d\xi_1 \right\|_{L^2(-\pi, \pi)} \lesssim \varepsilon^N \left(\|h_1\|_{L^2(\mathbb{R})} + \|h_2\|_{L^2(\mathbb{R})}\right).$$

Using Parseval’s identity and the fact that $\sup_{t, n} |\zeta_\varepsilon| = O(\delta_1 \varepsilon^2)$, we have

$$\left\| \chi_{K} G_5 \right\|_{L^0_2(\mathbb{R})} \lesssim \varepsilon^{-\frac{1}{2}} \left\| \tilde{\zeta} \ast T f_{\#} \right\|_{L^2(T)} \lesssim \delta_1 \varepsilon^2 (\|h_1\|_{L^2(\mathbb{R})} + \|h_2\|_{L^2(\mathbb{R})}).$$

Since $\sin(\varepsilon(\eta + ia)) = \frac{\xi}{2}(\eta + ia) + O(\varepsilon^3 (\eta)^3)$ for $\eta \in [-K, K]$ and

$$e^{ic_1 \xi \eta} \tilde{r}_{N,\xi}(t, \xi; k, \gamma) = e^{4ik_1^2 \eta} \tilde{r}_{N}(\tau, \eta; k, \varepsilon \gamma)$$

by the definition of $r_{N,\xi}$, it follows that

$$\left\| \chi_{K}(\eta) \left(\sin \frac{\varepsilon(\eta + ik_1)}{2} - \frac{\varepsilon(\eta + ik_1)}{2}\right) G_4(t, \xi_\varepsilon) \right\|_{L^2_0(\mathbb{R})} \lesssim \varepsilon^3 \left\| G_4(t, \xi_\varepsilon) \right\|_{L^2(-2K, 2K)} \lesssim \varepsilon^5 \left\| \int_{-\pi}^{\pi} |\tilde{r}_{N}(\tau, \eta)| e^{ic_1 \xi \eta} f_{\#}(t, \varepsilon(\eta - \eta_1 + ik_1)) d\eta_1 \right\|_{L^2(-2K, 2K)} \lesssim \varepsilon^5 (\|h_1(\tau)\|_{L^2(\mathbb{R})} + \|h_2(\tau)\|_{L^2(\mathbb{R})}).$$
Let
\[ G_6(\tau, y) = -\frac{\varepsilon^3}{2}(\partial_y - k_1) \{ \varphi_N(\tau, y + 4k_1^2\tau; k, \varepsilon\gamma)(h_1(\tau, y) + h_2(\tau, y)) \} =: G_{6,1} + G_{6,2}. \]

By (5.6),
\[ \tilde{G}_6(\tau, \eta) = -\frac{i\varepsilon^3(\eta + ik_1)}{2\sqrt{2\pi}} \int_\mathbb{R} e^{i(k_1^2\tau(\eta - \eta_1))} \widetilde{\varphi_N}(\tau, \eta - \eta_1; k, \varepsilon\gamma)(\hat{h}_1(\tau, \eta_1) + \hat{h}_2(\tau, \eta_1)) d\eta_1. \]

Since supp\( h_i(t, \cdot) \subset [-\pi/\varepsilon, \pi/\varepsilon] \) for \( i = 1, 2 \),
\[ \tilde{G}_6(\tau, \eta) + \frac{i\varepsilon\eta}{2} G_4(t, \xi) = \frac{i\xi}{2\sqrt{2\pi}} \left( \int_{-\pi}^{\pi} - \int_{-\pi+\varepsilon}^{\pi+\varepsilon} \right) e^{i\varepsilon\xi_1 \gamma} \tilde{r}_N(t, \xi - \xi_1) f_\#(t, \xi_1) d\xi_1. \]

If \( \xi \in [-2K\varepsilon, 2K\varepsilon] \) and \( |\xi_1 \pm \eta| \leq |\xi| \), we have \( \tilde{r}_N(t, \xi - \xi_1) = O(\varepsilon^{-\pi^2/(8\sum_{i=1}^N k_i\varepsilon)}) \) and
\[ \left\| \chi_N(\eta) \left( \tilde{G}_6 + \frac{i\xi}{2} G_4 \right) \right\|_{L^2_\eta(\mathbb{R})} \lesssim K^{\frac{1}{4}} e^{-\pi^2/(8\sum_{i=1}^N k_i\varepsilon)} e^{-k_1\varepsilon\gamma}\| f_\# \|_{L^2(-\pi, \pi)} \lesssim \varepsilon^N \left( \| h_1 \|_{L^2(\mathbb{R})} + \| h_2 \|_{L^2(\mathbb{R})} \right) \]
for any \( N \geq 1 \).

Since
\[ \lambda_+(\xi) = \frac{\varepsilon^3}{24}(\eta + ik_1)((\eta + ik_1)^2 + 4k_1^2 + O(\varepsilon^2(\eta)^4)) \]
for \( \eta \in [-2K, 2K] \),
\[ \partial_t f_{1,+}(t, \xi) - i\lambda_+(\xi) f_{1,+}(t, \xi) = \frac{\varepsilon^3}{24} F_y \{ \partial_y h_1 - 4k_1^2(\partial_y - k_1)h_1 + (\partial_y - k_1)^3 h_1 + O(\varepsilon^2 h_1) \}. \]

Combining the above, we obtain
\[ \partial_t h_1 + \{ (\partial_y - k_1)^3 - 4k_1^2(\partial_y - k_1) \} h_1 + 12(\partial_y - k_1)\{ \varphi_N(\tau, y + 4k_1^2\tau)h_1 \} \]
\[ = 24\varepsilon^{-3} F^{-1}_y \{ \chi_N \tilde{G}_{6,2} - \chi_N \tilde{G}_{6,1} - i\chi_N(\xi_\varepsilon G_5 + G'_2 + G'_3 \sin \frac{\xi_\varepsilon}{2}) \} + O(\varepsilon^2(h_1 + h_2)), \]
where \( G'_2(\tau, \eta) := G_{2,+}(t, \xi_\varepsilon) \) and \( G'_3(\tau, \eta) := G_{3,+}(t, \xi_\varepsilon) \).
By [5.7] and Theorem 1.2,
\[
||Q(\tau)h_1(\tau)||_{L^2} \lesssim e^{-3k_1^2\tau}||Q(0)h_1(0)||_{L^2} + \varepsilon^2 \int_0^T e^{-3k_1^2(\tau-\tau_1)}(||h_1||_{L^2} + ||h_2||_{L^2})d\tau_1 \\
+ e^{-3} \int_0^T e^{-3k_1^2(\tau-\tau_1)}(\tau-\tau_1)^{-\frac{3}{2}}||\eta||^{-\frac{3}{2}}\chi_{K}G_{6,1}||_{L^2}d\tau_1 \\
(5.8) + e^{-3} \int_0^T e^{-3k_1^2(\tau-\tau_1)}(\tau-\tau_1)^{-\frac{3}{2}}\{e(||G_0'\|_{L^2} + ||G_5\|_{L^2}) + ||\eta||^{-\frac{1}{2}}G_{6,2}||_{L^2}\}d\tau_1 \\
+ e^{-3} \int_0^T e^{-3k_1^2(\tau-\tau_1)}||G_2'||_{L^2}d\tau_1 \\
\lesssim a(\tau) + \int_0^T e^{-3k_1^2(\tau-\tau_1)}\{\varepsilon^2 + \delta_1(\tau-\tau_1)^{-\frac{3}{2}} + K^{-\frac{1}{2}}(\tau-\tau_1)^{-\frac{3}{2}}\}||Q(\tau_1)h_1(\tau_1)||_{L^2}d\tau_1,
\]
where
\[
a(\tau) = e^{-3k_1^2\tau}||Q(0)h_1(0)||_{L^2} \\
+ \int_0^T e^{-3k_1^2(\tau-\tau_1)}(\tau-\tau_1)^{-\frac{3}{2}}||h_2(\tau_1)||_{L^2} + \varepsilon^{-2}||G_3'||_{L^2}d\tau_1 \\
+ \int_0^T e^{-3k_1^2(\tau-\tau_1)}(\varepsilon^2||h_2(\tau_1)||_{L^2} + \varepsilon^{-3}||G_2'||_{L^2})d\tau_1 \\
+ \int_0^T e^{-3k_1^2(\tau-\tau_1)}\{\varepsilon^2 + \delta_1(\tau-\tau_1)^{-\frac{3}{2}} + K^{-\frac{1}{2}}(\tau-\tau_1)^{-\frac{3}{2}}\}||P(\tau_1)h_1(\tau_1)||_{L^2}d\tau_1.
\]
Applying Gronwall’s inequality to (5.8), we have
\[
||Q(\tau)h_1(\tau)||_{L^2} \lesssim a(\tau) + \int_0^T e^{-2k_1^2(\tau-\tau_1)}(\tau-\tau_1)^{-\frac{3}{4}}a(\tau_1)d\tau_1
\]
if \(\varepsilon, \delta_1\) and \(K^{-\frac{1}{2}}\) are sufficiently small. Now we use the following computation result.

**Claim 5.1.** Let \(b \succ a \succ 0, 0 < \alpha, \beta < 1, t \geq 0\) and \(g(t)\) be a nonnegative measurable function. Then
\[
\int_0^t e^{-b(t-s)}(t-s)\beta \left(\int_0^s e^{-a(s-\tau)}(s-\tau)^{-\alpha}g(\tau)d\tau\right)ds \\
\lesssim \int_0^t e^{-a(t-s)}(t-s)^{1-(\alpha+\beta)}g(s)ds.
\]

By Lemma 5.5, the definition of \(h_2\) and Claim 5.1, we have
\[
||Q(\tau)h_1(\tau)||_{L^2} \lesssim e^{-2k_1^2\tau}||Q(0)h_1(0)||_{L^2} \\
+ \int_0^T e^{-2k_1^2(\tau-\tau_1)}\{e^{-3}||G_3'||_{L^2} + \varepsilon^{-2}||G_2'||_{L^2}\}d\tau_1 \\
+ \varepsilon^{-\frac{1}{2}} \int_0^T e^{-2k_1^2(\tau-\tau_1)}(\tau-\tau_1)^{-\frac{3}{4}}(\delta_3||f_1||_{L^2} + ||f_2||_{L^2} + ||f_3||_{L^2})d\tau_1,
\]
where \( \delta_3 = (\varepsilon^2 + \delta_2 + K^{-2})(\varepsilon^2 + \delta_1 + K^{-\frac{1}{2}}) \). Combining the above and Lemma 5.5 we obtain

\[
\|f_{1,+}(t)\|_{L^2} \lesssim e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} \|f_{1,+}(0)\|_{L^2} + \delta_4(\|f_{2,+}(t)\|_{L^2} + \|f_3(t)\|_{L^2})
\]

\[
+ \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} e^{-k_1 \varepsilon c_1, \varepsilon s} \{\|F_1(s)\|_{L^2} + \varepsilon^{\frac{-1}{2}}(t-s)^{-\frac{1}{2}}\|F_2(s)\|_{L^2}\}ds
\]

\[
+ \varepsilon^\frac{3}{4} \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} (t-s)^{-\frac{3}{4}} (\delta_3 \|f_{1,+}(s)\|_{L^2} + \|f_{2,+}(s)\|_{L^2} + \|f_3(s)\|_{L^2})ds,
\]

where \( \delta_4 = \varepsilon + \delta_2 + K^{-2} \).

By (5.5) and (5.4) and the fact that

\[
e^{-\alpha \varepsilon (t-s)} \lesssim \max\{e^{-\frac{3}{4}}(t-s)^{-\frac{3}{4}}, e^{-\frac{1}{4}}(t-s)^{-\frac{1}{4}}\},
\]

we have

\[
\|f_{2,+}(t)\|_{L^2} + \|f_3(t)\|_{L^2} \lesssim e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} (\|f_{1,+}(0)\|_{L^2} + \|f_{2,+}(0)\|_{L^2} + \|f_3(0)\|_{L^2})
\]

\[
+ \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} e^{-k_1 \varepsilon c_1, \varepsilon s} \{\|F_1(s)\|_{L^2} + \varepsilon^{\frac{-1}{2}}(t-s)^{-\frac{1}{2}}\|F_2(s)\|_{L^2}\}ds
\]

\[
+ \varepsilon^\frac{3}{4} (K^{-\frac{1}{2}} + \varepsilon^\frac{1}{4}) \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} (t-s)^{-\frac{3}{4}}
\]

\[
\times (\|f_{1,+}(s)\|_{L^2} + \|f_{2,+}(s)\|_{L^2} + \|f_3(s)\|_{L^2})ds.
\]

Let \( X(t) = \|f_{1,+}(t)\|_{L^2} + \delta_4^{\frac{1}{4}}(\|f_{2,+}(t)\|_{L^2} + \|f_3(t)\|_{L^2}) \). By (5.9) and (5.10),

\[
X(t) \lesssim a_1(t) + \delta_4^{\frac{1}{4}} \varepsilon^\frac{3}{4} \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} (t-s)^{-\frac{3}{4}} X(s)ds,
\]

where

\[
a_1(t) = e^{-\frac{k_1^2 \varepsilon^3 t}{12}} X(0)
\]

\[
+ \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{12}} e^{-k_1 \varepsilon c_1, \varepsilon s} \{\|F_1(s)\|_{L^2} + \varepsilon^{\frac{-1}{2}}(t-s)^{-\frac{1}{2}}\|F_2(s)\|_{L^2}\}ds.
\]

Applying [12] Lemma 7.1.1] to the above and using Claim 5.1 we obtain

\[
X(t) \lesssim a_1(t) + \varepsilon^\frac{3}{4} \int_0^t e^{-\frac{k_1^2 \varepsilon^3 t}{12} + O(\delta_4^{\frac{1}{4}})} (t-s)^{-\frac{3}{4}} a_1(s)ds
\]

\[
\lesssim e^{-\frac{k_1^2 \varepsilon^3 t}{24}} X(0) + \int_0^t e^{-\frac{k_1^2 \varepsilon^3 (t-s)}{24}} \{\|F_1(s)\|_{L^2} + \varepsilon^{\frac{-1}{2}}(t-s)^{-\frac{1}{2}}\|F_2(s)\|_{L^2}\}ds.
\]

Thus we prove Lemma 5.11

\[\square\]
6. Exponential stability property of KdV N-solitons

In this section, we will prove linear stability property of an \(N\)-soliton solution of KdV equation (1.9). We find that linear stability of an \(N\)-soliton in \(L^2_a(\mathbb{R})\) is equivalent to that of an \((N-1)\)-soliton connected by the Bäcklund transformation (6.2) and it turns out that exponential stability property of \(N\)-solitons in \(L^2_a(\mathbb{R})\) follows from that of the null solution.

First, we recall the Bäcklund transformation of KdV. If \(u\) is a solution of (1.9) and \(v(t,x) = -\int_x^\infty u(t,y)dy\),
\[
\partial_t v + \partial_x^3 v + 6(v_x)^2 = 0 \quad \text{for} \; x \in \mathbb{R} \; \text{and} \; t > 0.
\]
Eq. (6.1) admits a Bäcklund transformation determined by the equations
\[
\begin{aligned}
\partial_x (v' + v) &= k^2 - (v' - v)^2 \\
\partial_t (v' + v) &= 2(v' - v)\partial_x^2 (v' - v) - 4\{\partial_x v'\partial_x (v' + v) + (\partial_x v')^2\}.
\end{aligned}
\]
If \(v\) and \(v'\) satisfy (6.2) and \(v\) is a solution of (6.1), then \(v'\) is necessarily a solution of (6.1).

To begin with, we recall that the Bäcklund transformation (6.2) creates a 1-soliton solution from the null solution and an \(N\)-soliton solution from an \((N-1)\)-soliton solution (see [29]). Let \(0 < k_1 < \cdots < k_N\), \(k^m = (k_1, \cdots, k_m)\), \(\gamma^m = (\gamma_1^m, \cdots, \gamma_m^m)\), \(\theta_i^m = k_i(x - 4k_i^2t - \gamma_i^m)\),
\[
C_m = \left(\frac{e^{-(\theta_i^m + \theta_j^m)}}{k_i + k_j}\right)_{m \times m},
\]
\[
\Delta_m = \begin{cases} 
\exp(-\sum_{i=1}^N \theta_i^N) & \text{if} \; m = 0, \\
\exp(-\sum_{i=m+1}^N \theta_i^N) \det(I + C_m) & \text{if} \; 1 \leq m \leq N - 1, \\
\det(I + C_N) & \text{if} \; m = N.
\end{cases}
\]
Then \(v^m = \partial_x \log \Delta_m \; \text{(}0 \leq m \leq N\)\) is a solution of (6.1) and \(\varphi_m(t, x; k^m, \gamma^m) := \partial_x^2 \log \Delta_m\) is an \(m\)-soliton solution of (1.9) (see [10]).

An \(m\)-soliton solution is connected to an \((m-1)\)-soliton solution by (6.2).

**Lemma 6.1.** Suppose \(1 \leq m \leq N\) and that
\[
\gamma_i^{m-1} = \gamma_i^m + \frac{1}{2k_i} \log \left(\frac{k_m - k_i}{k_m + k_i}\right) \quad \text{for} \; 1 \leq i \leq m - 1.
\]
Then
\[
\partial_x (v^m + v^{m-1}) = k_m^2 - (v^m - v^{m-1})^2.
\]
**Proof.** By the definition,
\[
v^0 = -\sum_{i=1}^N k_i \quad \text{and} \quad v^1 = -\sum_{i=2}^N k_i - \frac{2k_1}{1 + e^{2\theta_1}}.
\]
and (6.4) is true for \(m = 1\).
Let \( m \geq 2 \) and let \( Q_{ij}^m \) be the \((i,j)\) cofactor of \( I + C_m \). Following the argument of \cite{10} p.121, we have
\[
\psi_m := \sum_{l=1}^m e^{-\theta_m^l} Q_{lm}^m = e^{-\theta_m^m} \frac{\det(I + C_{m-1})}{\det(I + C_m)} = e^{k_m (\gamma_m^m - \gamma_{N,m})} \Delta_{m-1}^{-1},
\]
whence
\[
(6.6) \quad v^{m-1} - v^m = \partial_x \log \psi_m.
\]
On the other hand, Theorem 3.2 in \cite{10} implies that
\[
\partial_x^2 \psi_m = (k_m^2 - 2 \partial_x v^m) \psi_m.
\]
Thus we have
\[
\partial_x (v^m + v^{m-1}) = \partial_x^2 \log \psi_m + 2 \partial_x \partial_x v^m
\]
\[
= \frac{\partial_x^2 \psi_m}{\psi_m} - \left( \frac{\partial_x \psi_m}{\psi_m} \right)^2 + 2 \partial_x v^m
\]
\[
= k_m^2 - (v^m - v^{m-1})^2.
\]
□

Now we linearize the Bäcklund transformation \((6.2)\) around \( v = v^m \) and \( v' = v^{m-1} \). Then we obtain a linearized Bäcklund transformation
\[
(6.7) \quad \partial_x (w^m + w^{m-1}) = -2 (v^m - v^{m-1}) (w^m - w^{m-1}).
\]
The semiflows generated by
\[
(6.8) \quad \partial_t w^m + \partial_x^3 w^m + 12 (\partial_x v^m)(\partial_x w^m) = 0 \quad \text{for} \quad x \in \mathbb{R},
\]
\[
(6.9) \quad \partial_t w^{m-1} + \partial_x^3 w^{m-1} + 12 (\partial_x v^{m-1})(\partial_x w^{m-1}) = 0 \quad \text{for} \quad x \in \mathbb{R},
\]
leave the linearized Bäcklund transformation \((6.7)\) invariant. Note that \((6.8)\) is a linearized equation of \((6.1)\) around \( v^m \) and the adjoint equation of \((1.9)\) if \( m = N \) and \( \partial_x v_m = \varphi_N \).

**Lemma 6.2.** Let \( a > 0 \), \( t_0 \in \mathbb{R} \) and let \( w^m \), \( w^{m-1} \in C((-\infty, t_0]; L_a^2(\mathbb{R})) \) be solutions of \((6.8)\) and \((6.9)\), respectively. If \((6.7)\) holds at \( t = t_0 \), it holds for every \( t \leq t_0 \).

Before we start to prove Lemma 6.2, we remark that linearized equation of \((6.1)\) around \( v^m \) is well posed in \( L_a^2 \) (see e.g. \cite{15}).

**Lemma 6.3.** Let \( a > 0 \), \( \varphi \in L_a^2(\mathbb{R}) \) and \( t_0 \) be a real number. There exists a unique solution of
\[
\begin{cases}
\partial_t w + \partial_x^3 w + (\partial_x v^m) \partial_x w = 0 & \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t < t_0, \\
w(t_0) = \varphi,
\end{cases}
\]
in the class \( C((-\infty, t_0]; L_a^2(\mathbb{R})) \).
Proof of Lemma 6.3. Let
\[ W = (w^m + w^{m-1})_x + 2(v^m - v^{m-1})(w^m - w^{m-1}). \]
By (6.8), (6.9) and the fact that \( v^m \) and \( v^{m-1} \) are solutions of (6.1), we have
\[ W_t + W_{xxx} + 6(v^m + v^{m-1})_x W_x = -6\{(v^m + v^{m-1})_x(w^m + w^{m-1})_x \}
\[ + 24(v_x^{m-1} w_x^{m-1} - v_x^m w_x^m) + 12(w^m - w^{m-1})(v_x^{m-1})^2 - (v_x^m)^2). \]
Using (6.7) twice and (6.4), we find
\[ \Phi_1 \]
by using [15, Lemma 9.1]. Applying Gronwall’s inequality to (6.10), we have
\[ W(t) = 0. \]
Let \( \tilde{W}(t, x) = (\partial_x^{-1} W)(t, x) = \int_{-\infty}^{x} W(t, y)dy \) and \( b = 6(v^m + v^{m-1})_{xx} \). Then
\[ \tilde{W}_t + \tilde{W}_{xxx} = b\tilde{W}_x - b_x \tilde{W} - \partial_x^{-1}(b_{xx} \tilde{W}), \]
\[ \tilde{W}(t_0) = 0. \]
Since \( \partial_x^{-1} \) is bounded on \( L^2_\alpha(\mathbb{R}) \) \((a > 0)\), we have \( \tilde{W} \in C((-\infty, t_0]; L^2_\alpha) \) and
\[ \|\tilde{W}(t)\|_{L^2_\alpha} \leq \int_{t_0}^{t} (1 + (s - t)^{-\frac{1}{2}})\|\tilde{W}(s)\|_{L^2_\alpha} ds \quad \text{for} \quad t \leq t_0. \]
by using [15] Lemma 9.1. Applying Gronwall’s inequality to (6.10), we have \( \tilde{W}(t) = 0 \) and \( W(t) = \partial_x \tilde{W}(t) = 0 \) for every \( t \geq 0 \).

The linearized Bäcklund transformation (6.7) defines an isomorphism between \( L^2_\alpha \) and its subspace
\[ X_\alpha(t, \gamma^m) = \left\{ w \in L^2_\alpha : \int_{\mathbb{R}} w \partial_{k_m} \partial_x v^m dx = \int_{\mathbb{R}} w \partial_{k_m} \partial_x v^m dx = 0 \right\}. \]
First, let us consider the case \( m = 1 \).

Lemma 6.4. Let \( a \in (-2k_1, 2k_1) \). Then for any \( w^0 \in L^2_\alpha(\mathbb{R}) \), there exists a unique \( w^1 \in X_1(t, \gamma^1) \) satisfying (6.11). Furthermore the map \( \Phi_1(t, \gamma^1) : L^2_\alpha \rightarrow X_1(t, \gamma^1) \) defined by (6.11) is isomorphic and
\[ \sup_{t, \gamma^1} (\|\Phi_1(t, \gamma^1)\|_{B(L^2_\alpha; X_1(t, \gamma^1))} + \|\Phi_1(t, \gamma^1)^{-1}\|_{B(X_1(t, \gamma^1); L^2_\alpha)}) < \infty. \]

Proof. Substituting (6.5) into (6.7) with \( m = 1 \), we have
\[ \partial_x(w^1 + w^0) = -2(\partial_x v^1)(w^1 - w^0). \]
Since \( \|\Phi_1(t, \gamma^1)\|_{B(L^2_\alpha; X_1(t, \gamma^1))} \) and \( \|\Phi_1(t, \gamma^1)^{-1}\|_{B(X_1(t, \gamma^1); L^2_\alpha)} \) do not depend on \( t \) and \( \gamma^1 \), we may assume \( t = 0 \) and \( \gamma^1 = (0) \).

Let \( c = 4k_1^2 \) and \( \phi_c(x) = k_1^2 \text{sech}^2 k_1 x \). Then (6.11) and be rewritten as
\[ \partial_x(w^1 + w^0) = \frac{\partial_x \phi_c}{\phi_c}(w^1 - w^0). \]
By (6.12), there exists a real constant $\alpha$ such that

\[(6.13) \quad w_1(x) = -w_0(x) - (I_1w_0)(x) + \alpha \phi_c(x),\]

where

\[(I_1w)(x)_0 = 2\phi_c(x) \int_0^x \frac{\partial \phi_c(y)}{\phi_c(y)^2} w_0(y) dy.\]

The constant $\alpha$ is uniquely determined by the orthogonality conditions. Hereafter, we use the notation $(f, g) := \int_\mathbb{R} f(x)g(x)dx$ in this section. Since $d\|\phi_c\|^2_{L^2(\mathbb{R})}/dc \neq 0$ and $\int_\mathbb{R} \partial_x \phi_c dx = 0$, there exists a unique $\alpha = \alpha(w_0)$ such that

\[(6.14) \quad (w_1, \partial_x \phi_c) = -(w_0 + I_1 w_0, \partial_x \phi_c) + \alpha(\phi_c, \partial_x \phi_c) = 0,
\]

and

\[(w_1, \partial_x \phi_c) = -(w_0 + I_1 w_0 + \alpha \phi_c, \partial_x \phi_c)
= -(w_0, \partial_x \phi_c) + (w_0, \partial_x \phi_c) = 0.\]

Next we prove that $\Phi_1 : w^0 \mapsto w^1$ is continuous linear operator from $L^2_a$ to $X_1$. Noting that

\[
\phi_c(x)|\partial_x \phi_c(y)|\phi_c(y)^{-2} \lesssim \cosh^2(k_1 y) \sech^2(k_1 x) 
\lesssim e^{-\sqrt{\epsilon}|x-y|} \quad \text{for any } y \in (-|x|, |x|),
\]

we see that $I_1$ is a bounded linear operator on $L^2_a$. Eq. (6.14) and the boundedness of $I_1$ imply that $\alpha(w_0)$ is continuous linear functional on $L^2_a$. Thus we prove that (6.12) defines $\Phi \in B(L^2_a, X_1)$.

Next, we will prove that $\Phi_1$ has a bounded inverse. By (6.12),

\[
\partial_x \{\phi_c(w^1 + w^0)\} = 2w_1 \partial_x \phi_c,
\]

and

\[
w_0(x) = -w^1(x) - (J_1w^1)(x),
\]

where

\[
(J_1f)(x) = 2\phi_c(x)^{-1} \int_x^\infty \partial_x \phi_c(y)f(y)dy = -2\phi_c(x)^{-1} \int_\infty^x \partial_x \phi_c(y)f(y)dy
\]

for any $f \in X_1$. Noting that

\[
\phi_c(x)^{-1}|\partial_x \phi_c(y)| \lesssim e^{-\sqrt{\epsilon}|x-y|} \quad \text{for } 0 \leq x \leq y \text{ or } y \leq x \leq 0,
\]

we have

\[
\|J_1f\|_{L^2_a} \lesssim \|e^{-\sqrt{\epsilon}|x-y|}\|_{L^1} \|f\|_{L^2_a} \lesssim \|f\|_{L^2_a}.
\]

Thus we see that (6.12) defines a bounded linear operator

\[
\Psi_1w^1 := w^0 = -w^1 - 2J_1w^1
\]

from $X_1$ to $L^2_a$. 
Lemma 6.6. Suppose that
Proof of Lemma 6.5. (6.16)
By (6.6), (6.15) and (6.16), we complete the proof of Lemma 6.4.
□

Next we will consider the case where 2 \leq m \leq N.

Lemma 6.5. Suppose \( a \in (-2k_m, 2k_m) \) and (6.3). Then for any \( w^{m-1} \in L^2_a(\mathbb{R}) \), there exists a unique \( w^m \in X_m \) satisfying (6.7). Furthermore the map \( \Phi_m(t, \gamma^m) : L^2_a \to X_m \) defined by (6.7) is isomorphic and

\[
\sup_{t, \gamma^m} \left( \| \Phi_m(t, \gamma^m) \|_{B(L^2_a; X_m)} + \| \Phi_m(t, \gamma^m)^{-1} \|_{B(X_m; L^2_a)} \right) < \infty.
\]

To prove Lemma 6.5 we need the following:

Lemma 6.6. Suppose (6.3). Then there exist positive constants \( C_1 \) and \( C_2 \) depending only on \( k^m (1 \leq i \leq N) \) such that

\[
C_1 \operatorname{sech} \theta^m_m \leq \psi_m \leq C_2 \operatorname{sech} \theta^m_m.
\]

Proof. Expanding \( \det(I + C_m) \), we obtain the sum of all the principal minors of \( C_m \) of every order:

\[
\det(I + C_m) = 1 + \sum_{l=1}^m \sum_{1 \leq i_1 \leq \cdots \leq i_l} C_{i_1, \ldots, i_l} e^{-\theta^m_{i_1} - \cdots - \theta^m_{i_l}},
\]

where \( C_{i_1, \ldots, i_l} \) are positive constants depending only on \( k_1, \ldots, k_N \) (see [10, p.110]). By (6.3) and the above, there exist positive constants \( C_1 \) and \( C_2 \) depending only of \( k_1, \ldots, k_N \) such that

\[
\frac{2C_1 e^{-\theta^m_m}}{1 + e^{-2\theta^m_m}} \leq e^{\theta^m_m} \psi_m = \frac{\det(I + C_{m-1})}{\det(I + C_m)} \leq \frac{2C_2 e^{-\theta^m_m}}{1 + e^{-2\theta^m_m}}.
\]

Now we are in position to prove Lemma 6.5.

Proof of Lemma 6.5. Without loss of generality, we may assume \( t = 0 \). Let \( A = \partial_x + 2(v^m - v^{m-1}) \) and \( B = -\partial_x + 2(v^m - v^{m-1}) \). Differentiating (6.4) with respect to \( k_m \) and \( \gamma^m_m \), we have

\[
A \partial_{\gamma^m_m} v^m = B^* \partial_{\gamma^m_m} v^m = 0, \quad A \partial_{k_m} v^m = B^* \partial_{k_m} v^m = 2k_m.
\]

First, we solve (6.7) for \( w^m \). Eq. (6.7) can be translated into

\[
A(w^m + w^{m-1}) = 4(v^m - v^{m-1})w^{m-1}.
\]

By (6.16), (6.15) and (6.16),

\[
w^m = -w^{m-1} + I_m(w^{m-1}) + \alpha \partial_{\gamma^m_m} v^m,
\]
where $\alpha$ is a real number and

$$I_m(f) := 4 \int_{\gamma_m^m}^{x_m} \left( v^m(y) - v^{m-1}(y) \right) \frac{\psi_m(t, x, k^m, \gamma^m)^2}{\psi_m(t, y, k^m, \gamma^m)^2} f(y) dy.$$ 

Lemma 6.6 implies that there exists a positive constant $C_3$ depending only on $k^m$ such that for every $x \geq y \geq \gamma_m^m$ or $x \leq y \leq \gamma_m^m$,

$$\frac{\psi_m(t, x, k^m, \gamma^m)^2}{\psi_m(t, y, k^m, \gamma^m)^2} \leq C_3 \frac{\sech \theta_m(t, x)^2}{\sech \theta_m(t, y)^2} \leq 4C_3 e^{-2k_m|x-y|}.$$ 

Thus we have $I_m \in B(L_2^m)$ for $a \in (0, 2k_m)$.

Next, we will show that $w^m \in X_m(t, \gamma^m)$. By (6.7) and the definitions of $A$ and $B$,

$$Aw^m = Bw^{m-1} \quad \text{and} \quad \partial_x = (B^* - A^*)/2.$$ 

Using (6.15) and the above, we have

$$2(w^m, \partial_x \partial_{\gamma_m} v^m) = (w^m, (B^* - A^*) \partial_{\gamma_m} v^m)$$

$$= -(Aw^m, \partial_{\gamma_m} v^m)$$

$$= -(Bw^{m-1}, \partial_{\gamma_m} v^m)$$

$$= -(w^{m-1}, B^* \partial_{\gamma_m} v^m) = 0,$$

and

$$2(\partial_{\gamma_m} v^m, \partial_x \partial_{k_m} v^m) = (\partial_{\gamma_m} v^m, (B^* - A^*) \partial_{k_m} v^m)$$

$$= (\partial_{\gamma_m} v^m, B^* \partial_{k_m} v^m)$$

$$= 2k_m(\partial_{\gamma_m} v^m, 1)$$

$$= -2k_m \left[ \partial_{\gamma_m} \log \Delta_m \right]_{x=-\infty}^x$$

$$= 2k_m \left[ \frac{\partial_{\gamma_m} \det(I + C_m)}{\det(I + C_m)} \right]_{x=-\infty}^x$$

$$= -2k_m \left. \frac{\partial_{\gamma_m} \det C_m}{\det C_m} \right|_{x=-\infty} = -4k_m^2 \neq 0.$$ 

Hence there exists a unique $\alpha = \alpha(w^{m-1})$ such that $(w^m, \partial_x \partial_{k_m} v^m) = 0$. Moreover, $\alpha(w^{m-1})$ is a continuous linear functional on $w^{m-1} \in L_2^\alpha$. Thus we prove $\Phi_m(t, \gamma^m) = -I + 4I_m + \alpha(\cdot) \partial_{\gamma_m} v^m$ satisfies $\sup_{t, \gamma^m} \|\Phi_m(t, \gamma^m)\|_{B(L_2^m, X_m(t, \gamma^m))} < \infty$.

Finally, we will prove $\sup_{t, \gamma^m} \|\Phi_m(t, \gamma^m)^{-1}\|_{B(X_m(t, \gamma^m), L_2^\alpha)} < \infty$. Let us solve (6.7) for $w^{m-1}$. Since $\ker(B) = \{0\}$ in $L_2^\alpha$ and

$$B(w^{m-1} + w^m) = -4(v^m - v^{m-1})w^m,$$
we have for any \( w^m \in C^1_0(\mathbb{R}) \cap X_m(t, \gamma_m) \),

\[
\begin{aligned}
    w^{m-1}(x) &= -w^m(x) + 4 \int_{x}^{\infty} \frac{\psi_m(t, y, k^m, \gamma_m)^2}{\psi_m(t, x, k^m, \gamma_m)^2} w^m(y) dy \\
    &= -w^m(x) - 4 \int_{-\infty}^{x} \frac{\psi_m(t, y, k^m, \gamma_m)^2}{\psi_m(t, x, k^m, \gamma_m)^2} w^m(y) dy \\
    &= -w^m(x) + J_m(w^m)(x).
\end{aligned}
\]

(6.18)

Lemma 6.6 implies that there exists a positive constant \( C \) depending only on \( k^m \) such that

\[
\frac{\psi_m(t, y, k^m, \gamma_m)^2}{\psi_m(t, x, k^m, \gamma_m)^2} \leq Ce^{-2k_m|x-y|}
\]

for \( \gamma_m^m \leq y \leq x \) or \( x \leq y \leq \gamma_m^m \). Hence \( J_m \) can be uniquely extended on \( X_m(t, \gamma_m) \) and \( \Psi_m := -I + J_m \in L^2(\mathbb{R}) \) satisfies \( \sup_{t, \gamma_m} \| \Psi_m \|_{L^2(\mathbb{R})} < \infty \). By the definitions of \( I \) and \( \Psi_m \), it is clear that \( \psi_m \Phi_m = I \) on \( X_m(t, \gamma_m) \). Thus we prove (6.7) defines an isomorphism between \( X_m(t, \gamma_m) \) and \( L^2_0 \) uniformly bounded with respect to \( t \) and \( \gamma_m \).

\[ \square \]

Let

\[
Y_m(t, \gamma_m) = \left\{ w \in L^2_0 : \int_{\mathbb{R}} w \partial_x \partial_{\gamma_i} v^m dx = \int_{\mathbb{R}} w \partial_x \partial_{k_i} v^m dx = 0 \right\}
\]

Note that \( \partial_x \partial_{\gamma_i} v^m \) and \( \partial_x \partial_{k_i} v^m \) (\( 1 \leq i \leq m \)) are secular mode solutions of the adjoint equation of (6.8). We will show that \( w^{m-1} \) satisfies the symplectical orthogonality condition for \( v^m \) if and only if \( w^m \) satisfy the symplectical orthogonality condition for \( v^m \).

**Lemma 6.7.** Let \( a \in (-2k_1, 2k_1) \) and let \( \Phi(t, \gamma_m) \) be as in Lemma 6.5. Suppose \( 2 \leq m \leq N \) and (6.3). Then \( \Phi_m(t, \gamma_m)(Y_m(t, \gamma_m)) = Y_{m-1}(t, \gamma_{m-1}) \).

**Proof.** We abbreviate \( \gamma_i^m \) as \( \gamma_i \) (\( 1 \leq i \leq m \)) if there is no confusion. Differentiating (6.4) with respect to \( \gamma_i \) and \( k_i \) (\( 1 \leq i \leq m-1 \)), we have

\[
B_i^* \partial_{\gamma_i} v^m = A_i^* \partial_{\gamma_i} v^{m-1}, \quad B_i^* \partial_{k_i} v^m = A_i^* \partial_{k_i} v^{m-1} + (\partial_{k_i} \gamma_{m-1}^i - \partial_{\gamma_i} v^{m-1})
\]

Using (6.19) and the fact that \( Aw^m = Bw^{m-1} \) and \( 2\partial_x = B^* - A^* \), we compute

\[
2(w^m, \partial_x \partial_{\gamma_i} v^m) = (w^m, (B^* - A^*) \partial_{\gamma_i} v^m) = (w^m, A^*(\partial_{\gamma_i} v^{m-1} - \partial_{\gamma_i} v^m)) = (Bw^{m-1}, \partial_{\gamma_i} v^{m-1} - \partial_{\gamma_i} v^m) = (w^{m-1}, (B^* - A^*) \partial_{\gamma_i} v^{m-1}) = 2(w^{m-1}, \partial_x \partial_{\gamma_i} v^{m-1}),
\]

and

\[
(w^m, \partial_x \partial_{k_i} v^m) = (w^{m-1}, \partial_x \partial_{k_i} v^{m-1}) + (\partial_{k_i} \gamma_{m-1}^i)(w^{m-1}, \partial_x \partial_{\gamma_i} v^{m-1}).
\]

Therefore \( w^m \in Y_m(t, \gamma_m) \) if and only if \( w^{m-1} \in Y_{m-1}(t, \gamma_{m-1}) \). This completes the proof of Lemma 6.7. \[ \square \]
Now we are in position to prove linear stability of $N$-soliton solutions. We first establish a decay estimate for (6.8).

**Proposition 6.8.** Let $0 < k_1 < \cdots < k_N$, $a \in (0, 2k_1)$ and let $t_0$ be a real number. Suppose that $w^N \in C((-\infty, t_0]; L^2_a)$ is a solution of
\begin{equation}
(6.20) \quad \begin{cases}
\partial_t w^N + \partial_x^3 w^N + 12(\partial_x v^N)(\partial_x w^N) = 0 \text{ for } x \in \mathbb{R}, \ t < t_0, \\
w^N(t_0) \in Y_N(t_0, \gamma^N).
\end{cases}
\end{equation}
Then $w^N(t) \in Y_N(t, \gamma^N)$ for $t \leq t_0$ and
\[ \|w^N(t)\|_{L^2_a} \leq M e^{-\alpha^3(t-s)} \|w^N(s)\|_{L^2_a} \text{ for every } t \leq s \leq t_0, \]
where $M$ is a positive constant depending only on $k_1, \cdots, k_N$. Furthermore, there exists a positive constant $M' = M'(k, l, b)$ for any $l \in \mathbb{N}$ and $b > a^3$ such that
\[ \|e^{-\alpha x}w^N(t)\|_{H^l} \leq M'(t - s)^{-\frac{l}{2}} e^{-b(t-s)} \|w^N(s)\|_{L^2_a} \text{ for every } t < s \leq t_0. \]

**Proof of Proposition 6.8.** First, we will prove that $w^N \in Y_N(t, \gamma^N)$ for every $t \leq s$. Since $v^N$ is a solution of (6.1) and $\partial_i v^N$ and $\partial_k v^N$ ($1 \leq i \leq N$) are solutions of (1.10) with $\varphi = \partial_x v_N$, we have for $1 \leq i \leq N$,
\[ \frac{d}{dt}(w^N, \partial_i v^N) = (\partial_t w^N, \partial_i v^N) + (w^N, \partial_t \partial_i v^N) = 0, \]
\[ \frac{d}{dt}(w^N, \partial_k v^N) = (\partial_t w^N, \partial_k v^N) + (w^N, \partial_t \partial_k v^N) = 0. \]
Combining the above with $w^N(t_0) \in Y_N(t_0, \gamma^N)$, we have $w^N(t) \in Y_m(t, \gamma^m)$ for every $t \leq t_0$.

Let $w^0(t) = \Phi_1(t, \gamma^1)^{-1}\cdots\Phi_N(t, \gamma^N)^{-1}w^N(t)$. Lemmas 6.4 and 6.5 imply that a map $\Phi_1(t, \gamma^1)^{-1}\cdots\Phi_N(t, \gamma^N)^{-1}$ is well defined on $Y_N(t, \gamma^N)$ and we have $w^0(t) \in C([0, \infty); L^2_a(\mathbb{R}))$ and
\begin{equation}
(6.21) \quad C^{-1}\|w^0(t)\|_{L^2_a} \leq \|w^N(t)\|_{L^2_a} \leq C\|w^0(t)\|_{L^2_a},
\end{equation}
where $C$ is positive constant depending only on $k^N$ and $a \in (0, 2k_1)$. Combining (6.21) with (6.7) for $m = 1, \cdots, N$, we see that there exists a $C_l > 0$ depending only on $k$ and $l \in \mathbb{N}$ such that
\begin{equation}
(6.22) \quad C_l^{-1}\|e^{-\alpha x}w^0(t)\|_{H^l} \leq \|e^{-\alpha x}w^N(t)\|_{H^l} \leq C_l\|e^{-\alpha x}w^0(t)\|_{H^l}.
\end{equation}

Lemma 6.2 implies that
\begin{equation}
(6.23) \quad \partial_t w^0 + \partial_x^3 w^0 = 0 \text{ for } t > s \text{ and } x \in \mathbb{R}.
\end{equation}
It follows from [13] Lemma 9.1 that for any $a > 0$ and $t \leq s$,
\begin{align*}
(6.24) \quad &\|w^0(t)\|_{L^2_a(\mathbb{R})} \leq e^{-a^3(t-s)}\|w^0(s)\|_{L^2_a(\mathbb{R})}, \\
(6.25) \quad &\|e^{-\alpha x}w^0(t)\|_{H^l(\mathbb{R})} \leq \{1 + (3a(t-s))^{-\frac{l}{2}}\} e^{-a^3(t-s)}\|w^0(s)\|_{L^2_a(\mathbb{R})}.
\end{align*}
Proposition 6.8 follows immediately from (6.21), (6.22), (6.24) and (6.25). Thus we complete the proof.

Proof of Theorem 1.2. Let $U(t, s)$ denote the evolution operator associated with

$$\tag{6.26}
\begin{align*}
\partial_tw + \partial_x^2 w^N + 12\partial_x((\partial_x v^N)(t))w &= 0 \quad \text{for } x \in \mathbb{R}, \ t > s, \\
w(s) &\in L_0^2.
\end{align*}
$$

Since (6.26) is the adjoint equation of (6.20), it follows from Proposition 6.8 that for every $t \geq s$ and $f \in L_{-a}^2$,

$$\|Q(s)^*U(t, s)^*Q(t)^*(t)f\|_{L_{-a}^2} \leq Me^{\alpha (t-s)}\|f\|_{L_{-a}^2},$$

$$\|e^{-ax}Q(s)^*U(t, s)^*Q(t)^*(t)f\|_{H^1} \leq M'(t-s)^{-\frac{1}{2}}e^{b(t-s)}\|f\|_{L_{-a}^2},$$

since $Q(t)^*$ is a projection to $Y_N(t, \gamma^N)$ associated with (6.20). By a standard duality argument,

$$\|U(t, s)Q(s)f\|_{L_2^2} \leq Me^{\alpha (t-s)}\|f\|_{L_2^2},$$

$$\|U(t, s)Q(s)f\|_{L_2^2} \leq M' e^{b(t-s)}(t-s)^{-\frac{1}{2}}\|e^{ax}f\|_{H^{-1}}.$$ 

Thus we prove Theorem 1.2. \hfill \Box

**Appendix A. Size of $u_c$ and $\rho_c$**

**Claim A.1.** Let $c = 1 + \frac{1}{6}\varepsilon^2$, $a \in (\frac{1}{4}, \frac{7}{4})$ and let $i$ and $j$ be nonnegative integers. Then

$$\|\partial_x^i \partial_t^j u_c\|_{L_0^2} = O(\varepsilon^{\frac{1}{2}+i-2j}), \quad \|J^{-1} \partial_x^i \partial_t^j u_c\|_{L_0^2} = O(\varepsilon^{\frac{1}{2}+i-2j}),$$

$$\|\partial_x^i \partial_t^j u_c\|_{\infty \cap L_0^\infty} = O(\varepsilon^{2+i-2j}), \quad \|J^{-1} \partial_x^i \partial_t^j u_c\|_{\infty \cap L_0^\infty} = O(\varepsilon^{1+i-2j}).$$

To estimate $L^2$-norm of $u_c$, we need the following.

**Claim A.2.** Let $f \in H^1(\mathbb{R})$. Then $\sum_{n \in \mathbb{Z}} f(n)^2 \leq 2\|f\|_{H^1}^2$.

*Proof.* Since $f(n)^2 \leq 2 \int_n^{n+1} (f(x)^2 + f'(x)^2)dx$ for any $n \in \mathbb{Z}$, we have

$$\sum_{n \in \mathbb{Z}} f(n)^2 \leq 2 \sum_{n \in \mathbb{Z}} \int_n^{n+1} (f(x)^2 + f'(x)^2)dx = 2\|f\|_{H^1(\mathbb{R})}^2.$$ \hfill \Box

*Proof of Claim A.1* Claim A.1 follows from (P4), Claim A.2 and the fact that $\|J^{-1}\|_{B(L_0^2)} = O(\varepsilon^{-1}).$ \hfill \Box
Claim A.3. Let \( 0 < k_1 < k_2 \) and \( a \in [0, \frac{7}{2} \varepsilon) \). Then there exists an \( \varepsilon_* > 0 \) such that if \( \varepsilon \in (0, \varepsilon_*) \) and \( c_i = 1 + \frac{k_i \varepsilon^2}{6} \) for \( i = 1, 2 \),

\[
\| \partial_x^\alpha \partial_c^\beta u_{c_i} (\cdot - x_1) \partial_x^\alpha \partial_c^\beta u_{c_1} (\cdot - x_2) \|_{L^\infty} = O(\varepsilon^{4+\alpha_1+\alpha_2-2(\beta_1+\beta_2)} e^{-k_1 a|x_2(t)-x_1(t)|}),
\]

\[
\| \partial_x^\alpha \partial_c^\beta u_{c_i} (\cdot - x_1) \partial_x^\alpha \partial_c^\beta u_{c_1} (\cdot - x_2) \|_{L^1} = O(\varepsilon^{3+\alpha_1+\alpha_2-2(\beta_1+\beta_2)} e^{-k_1 a|x_2(t)-x_1(t)|}).
\]

Proof. Claim [A.3] follows from Claim [A.1].

Claim A.4. Let \( a_1, \ldots, a_N \in \mathbb{R} \) and \( I = \{ \sum_{i=1}^N \theta_i a_i : 0 \leq \theta_i \leq 1 \text{ for } 1 \leq i \leq N \} \). Suppose \( f \in C^2(\mathbb{R}) \) and \( f(0) = 0 \). Then

\[
\left| f\left( \sum_{1 \leq i \leq N} a_i \right) - \sum_{1 \leq i \leq N} f(a_i) \right| \leq \sup_{x \in I} |f''(x)| \sum_{i \neq j} |a_i a_j|.
\]

Proof. Let \( b = \sum_{1 \leq i \leq N} a_i \). By the mean value theorem,

\[
\left| f(b) - \sum_{1 \leq i \leq N} f(a_i) \right| = \sum_{1 \leq i \leq N} \int_0^1 (f'(s_1 b) - f'(s_1 a_i)) ds_1 a_i
\]

\[
= \sum_{1 \leq i \leq N} \int_0^1 \int_0^1 f''(s_1 (s_2 b + (1 - s_2) a_i)) ds_1 ds_2 a_i (b - a_i)
\]

\[
\leq \sup_{x \in I} |f''(x)| \sum_{i=1}^N |a_i| |b - a_i|.
\]

Thus we prove Claim [A.4].

Now we estimate size of \( \rho_c \).

Claim A.5. Let \( a \in [0, 2k_1 \varepsilon) \). Then

\[
\| \partial_x^i \partial_c^j \rho_c \|_{L^2 a} + \| \partial_x^i \partial_c^j \rho_c \|_{L^2 a} = O(\varepsilon^{\frac{3}{2}+i-2j}),
\]

\[
\| \partial_x^i \partial_c^j \rho_c \|_{L^2 a} + \| \partial_x^i \partial_c^j \rho_c \|_{L^2 a} = O(\varepsilon^{2+i-2j}).
\]

Proof. Noting that \( (H''(u) - I) \partial_x u_c = O(r_c \partial_x r_c) \), we see that Claim [A.3] follows from Claim [A.1] and Claim [A.6] below.

Claim A.6. Let \( c = 1 + \frac{\varepsilon^2}{6} \) and \( a \in (0, 2) \). There exists a positive number \( \varepsilon_0 \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \varepsilon^2 \| \partial_x (c \partial_x + J)^{-1} \|_{B(L^2 a \cap L^2 a)} < \infty.
\]

Proof. Since

\[
\mathcal{F} \partial_x (c \partial_x + J)^{-1} = \frac{i \xi}{\varepsilon^2 \xi^2 - 4 \sin^2 \frac{\xi}{2} \left( -ci \xi - e^{i \xi} - 1 \right)},
\]

we have

\[
\| \partial_x (c \partial_x + J)^{-1} \|_{B(L^2 a)} \leq \sup_{\xi \in \mathbb{R}} |m(\xi + i \varepsilon)|,
\]

where

\[
m(\xi) = \frac{\xi}{\sqrt{\xi^2 - 4 \sin^2 \frac{\xi}{2}} - 2i \varepsilon \sin \frac{\xi}{2} - 4 \varepsilon^2 \sin^2 \frac{\xi}{2}}.
\]

Therefore,
where \( m(\xi) = \xi^2 (e^2 \xi^2 - 4 \sin^2 \frac{\xi}{2})^{-1} \).

Using

\[
e^2 - \frac{4 \sin^2 \frac{\xi}{2}}{\xi^2} = \frac{1}{12} (\xi^2 + 4\varepsilon^2) + O(\xi^4 + \varepsilon^4),
\]

we have

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{\xi \in (-\frac{\pi}{2}, \frac{\pi}{2})} |m(\xi + i\varepsilon)| < \infty.
\]

Suppose \( |\xi| \geq \varepsilon \frac{\pi}{2} \). Obviously,

\[
\inf_{\varepsilon \in (0, \varepsilon_0)} \inf_{|\xi| \geq \varepsilon \frac{\pi}{2}} \left| c + \frac{2 \sin \frac{\xi + i\varepsilon}{2}}{\xi + i\varepsilon} \right| > 0,
\]

and since \( 0 \leq \cosh \frac{a\varepsilon}{2} - 1 = O(\varepsilon^2) \) and \( 1 - a\frac{2 \sin \frac{\varepsilon}{2}}{\xi} \geq \varepsilon \frac{\pi}{2} \),

\[
\left| c(\xi + i\varepsilon) - 2 \sin \frac{\xi + i\varepsilon}{2} \right| \geq |\xi| \left( c - \cosh \frac{a\varepsilon}{2} \right)
\]

\[
\geq (c - 1)|\xi|
\]

\[
\geq \varepsilon^2 |\xi| + i\varepsilon|\xi|.
\]

Combining the above, we conclude Claim A.6.

To prove Lemma 5.3, we need the following:

Claim A.7. Let \( a \) be a positive number, \( u = (u_1, u_2) \in l^2_a \cap l^2_{-a} \) and \( v = (v_1, v_2) \in l^2_a \cap l^2_{-a} \). Then

\[
\langle u, J^{-1} v \rangle = \langle u_1, 0 \rangle \sum_{k=-\infty}^{0} e^{k\vartheta} v_2 + \langle v_1, \sum_{k=1}^{\infty} e^{k\vartheta} u_2 \rangle.
\]

Especially, \( \langle u, J^{-1} u \rangle = \langle u_1, 1 \rangle \langle u_2, 1 \rangle \), and as \( l \to \infty \),

\[
\langle u, J^{-1} e^{i\vartheta} v \rangle = O(a^{-1} e^{-la} \|u\|_{l^2_a} \|v\|_{l^2_{-a}}),
\]

\[
\langle u, J^{-1} e^{-i\vartheta} v \rangle = \langle u_1, 1 \rangle \langle v_2, 1 \rangle + \langle u_2, 1 \rangle \langle v_1, 1 \rangle + O(a^{-1} e^{la} \|u\|_{l^2_a} \|v\|_{l^2_{-a}}).
\]

Proof. Eq. (A.1) follows from (2.3) and the others follows immediately from (A.1).

Appendix B. Proof of Lemma 5.3

Proof of Lemma 5.3. Let \( a(n) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{T}} g(\xi) e^{in\xi} d\xi \). By Parseval’s identity,

\[
\left\| \int_{\mathbb{T}} \tilde{f}(\xi) g(\xi - \xi_1) d\xi \right\|_{L^2(\mathbb{T})} = \|f(n) a(n)\|_{l^2} \leq \|f\|_{L^\infty(\mathbb{R})} \|g\|_{L^2}.
\]

Next we prove (ii). By [10], there exist positive constants \( A_{i_1, \ldots, i_n} \) such that

\[
\det(1 + C_N) = 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq N} A_{i_1, \ldots, i_n} e^{-2(\theta_{i_1} + \cdots + \theta_{i_n})}.
\]
Hence $\varphi_N(t, z; k, \gamma)$ is analytic on $\{ z \in \mathbb{C} : |z| \leq \delta \}$ and $\sup_{|y| \leq \delta} \| \varphi_N(t, \cdot + iy; k, \gamma) \|_{L^1(\mathbb{R})} < \infty$. By the Paley-Wiener theorem [26, Theorem 9.14],

$$(B.1) \quad \widehat{r_{N, \varepsilon}}(t, \xi, k, \gamma) = \varepsilon \widehat{r_{N, 1}}(t, \varepsilon^{-1} \xi, k, \gamma) = O(e^{-|\xi|/\varepsilon}).$$

Making use of (B.1) and the Poisson summation formula, we have

$$\sum_{n \neq 0} \widehat{\tau_N}(t, \xi + 2n\pi, \gamma) \leq \sum_{n \geq 1} e^{-n\pi \delta/\varepsilon} \leq e^{-\pi \delta/\varepsilon} \quad \text{for } \xi \in [-\pi, \pi].$$

\[ \square \]

Appendix C. Relation between secular term conditions of FPU and KdV

A multi-soliton solution resolves into a train of 1-solitons as $t \to \infty$ ([10]). In fact, we have the following.

Lemma C.1. Let $0 < k_1 < \cdots < k_N$ and $\gamma_i \in \mathbb{R}$ for $1 \leq i \leq N$. Then

$$\varphi_N(t, x; k, \gamma) = \sum_{1 \leq j \leq N} k_j^2 \text{sech}^2 \theta_j + 2 \frac{d^2}{dx^2} \log(1 + R),$$

where $\theta_j = k_j(x - 4k_j^2 t - \tilde{\gamma}_j)$ and

$$\tilde{\gamma}_N = \gamma_N - \frac{1}{2k_N} \log(2k_N),$$

$$\tilde{\gamma}_i = \gamma_i - \frac{1}{2k_i} \log(2k_i) - \frac{1}{2k_i} \sum_{j=i+1}^N \log \left( \frac{k_j + k_i}{k_j - k_i} \right) \quad \text{for } 1 \leq i \leq N - 1,$

and there exist positive numbers $a$, $b$ and $\delta$ such that

$$(C.1) \quad \sup_{1 \leq i \leq N} \sup_{x \in \mathbb{R}} \left| \cosh(ax) \partial_x^{\alpha_1} \partial_{k_i}^{\alpha_2} \partial_{\gamma_i}^{\alpha_3} R(t, x) \right| \leq \delta e^{-bt} \quad \text{for } t \geq 0,$$

where $\delta$ is chosen as a function of $L := \inf_{1 \leq j \leq N-1} (\gamma_{j+1} - \gamma_j)$ satisfying $\delta(L) \to 0$ as $L \to \infty$. Moreover, for any $a \in [0, 2)$, there exists a positive number $b' > 0$ such that

$$\sum_{1 \leq i \leq N} \| e^{-ab_1} \partial_x^{\alpha_1} \partial_{k_i}^{\alpha_2} \partial_{\gamma_i}^{\alpha_3} R \|_{L^2} \leq \delta e^{-b't} \quad \text{for } t \geq 0.$$

Proof. The former part of Lemma C.1 is a slight modification of Theorem 2.1 in Haragus-Sattinger [11] and can be seen easily from their proof. The latter part also follows immediately from their proof. In fact, [11] tells us that

$$\left| \partial_x^{\alpha_1} \partial_{k_i}^{\alpha_2} \partial_{\gamma_i}^{\alpha_3} R \right| \leq \sum_{2 \leq m \leq N} \frac{1}{1 + e^{-2\theta_m}},$$
and \[
\frac{1}{1 + e^{-2\theta_1}} = \frac{1}{1 + \exp\left(-\frac{2\theta_1}{k^2}\right) \exp\{8k_m(k_m^2 - k_1^2)t + 4k_m(\gamma_m - \gamma)\}}.
\]
Thus we have Lemma C.1. \(\square\)

Now we are in position to prove Lemma 5.4.

Proof of Lemma 5.4. For \(i = 1, \cdots, N\), let
\[
\xi_i^1(\tau) = \partial_{\gamma_i} \varphi_N(\tau, x; k, \gamma), \quad \xi_i^2(\tau) = \partial_{\nu_i} \varphi_N(\tau, x; k, \gamma),
\]
\[
\eta_i^1(\tau) = \int_{-\infty}^{x} \partial_{\gamma_i} \varphi_N(\tau, y; k, \gamma) dy, \quad \eta_i^2(\tau) = \int_{-\infty}^{x} \partial_{\nu_i} \varphi_N(\tau, y; k, \gamma) dy,
\]
and let
\[
\mathcal{A}_{KdV} = \left(\mathcal{A}_{KdV}^{ij}\right)_{i=1, \cdots, N \downarrow j=1, \cdots, N \downarrow}, \quad \mathcal{A}_{KdV}^{ij} = \left(\langle \xi_i^1, \eta_j^1 \rangle \langle \xi_j^2, \eta_i^2 \rangle \right).
\]
Then we have
\[
\mathcal{P}(\tau) f = \sum_{i=1}^{N} \left( \alpha_i \xi_i^1(\tau) + \beta_i \xi_i^2(\tau) \right),
\]
where \(\alpha_i\) and \(\beta_i\) are given by
\[
\mathcal{A}_{KdV} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}_{i=1, \cdots, N \downarrow} = \left(\langle f, \eta_j^1(\tau) \rangle \langle f, \eta_j^2(\tau) \rangle \right)_{j=1, \cdots, N \downarrow}.
\]

Since \(\xi_i^k\) \((1 \leq i \leq N, k = 1, 2)\) are solutions of (1.10) and \(\eta_i^j\) are solutions of the adjoint equation of (1.10), \(\langle \xi_i^k, \eta_j^j \rangle\) are independent of \(t\). Let \(\phi_k(x) = k^2 \text{sech}^2 kx\). By Lemma C.1
\[
\eta_j^1 = -\phi_{kj}(x - 4k_j^2t - \bar{\gamma}_j) + R_{1,j}, \tag{C.2}
\]
\[
\eta_j^2 = \int_{-\infty}^{x} \partial_k \phi_{kj} (y - 4k_j^2t - \bar{\gamma}_j) dy - \sum_{m < j} \partial_{\nu_m} \phi_{km} (y - 4k_m^2t - \bar{\gamma}_i) + R_{2,j}, \tag{C.3}
\]
where \(R_{1,j} = 2\partial_{\nu_j} \phi_{kj} \log(1 + R)\) and \(R_{2,j} = 2\partial_{\nu_j} \phi_{kj} \log(1 + R)\). Observing limit as \(t \to \infty\), we have \(\langle \xi_i^k, \eta_j^j \rangle = 0\) if \(i \neq j\) and \((k, l) \neq (2, 2)\), and
\[
\langle \xi_i^1, \eta_i^1 \rangle = 0, \quad \langle \xi_i^1, \eta_i^2 \rangle = -\langle \xi_i^2, \eta_i^1 \rangle = \frac{1}{d} \left\| \phi_{ki} \right\|_{L^2}^2 \neq 0 \quad \text{for} \ i = 1, \cdots, N.
\]
If \(i < j\),
\[
\langle \xi_i^2, \eta_j^2 \rangle = \lim_{t \to \infty} \left( \partial_{\nu_j} \phi_{ki} - \sum_{l=1}^{i-1} \frac{\partial \bar{\gamma}_l}{\partial \nu_l} \partial_{\nu_l} \phi_{kl}, \int_{-\infty}^{x} \partial_{\nu_j} \phi_{kj} dy - \sum_{m=1}^{j-1} \frac{\partial \bar{\gamma}_m}{\partial \nu_m} \phi_{km} \right) = 0.
\]
It follows from above that $A_{KdV}^{ij} = O$ if $i < j$, that $A_{KdV}$ is invertible, and that
\[
\|P(\tau)f\|_{L^2} \lesssim \sum_{l,m} \sum_{i \leq j} \| \langle f, \eta_j^m \rangle \| \| \xi_i^l \|_{L^2}
\]
\[
\lesssim \sum_{l,m} \sum_{i \leq j} e^{-a((4(k_j^2-k_i^2)t+\gamma_j-\gamma_i)} \| e^{-a(-4k_j^2t-\gamma_j)} \eta_j^m \|_{L^2}
\]
\[
\times \| e^{a(-4k_j^2t-\gamma_j)} \xi_i^l \|_{L^2} \| f \|_{L^2}
\]
\[\leq \| f \|_{L^2}.\]

Thus we complete the proof of Lemma 5.4. \hfill\Box

Next we prove Lemma 5.5.

**Proof of Lemma 5.5.** By (5.2) and Parseval’s identity,
\[
\left| \langle w(t), J^{-1} \partial_{\eta} u_{N, \varepsilon} \rangle \right|
= \left| \left\langle f(t, \xi), e^{i c_1 t \xi} P(\xi)^* J^{-1} F_{n-1} \partial_{\eta} u_{N, \varepsilon}(t, \xi, \gamma) \right\rangle \right|
= \frac{1}{2} \left| \langle \tau_{ik_1} e f(t, \xi), \tau_{-ik_1} \{ e^{i c_1 t \xi} (\sin \frac{\xi}{\varepsilon})^{-1} \sigma_3 P(\xi)^* \} F_{n} \partial_{\eta} u_{N, \varepsilon}(t, \xi, \gamma) \rangle \right|
= \leq \varepsilon^{\frac{1}{2}} \delta_2 e^{-k_1 \gamma_1} \| \tau_{ik_1} f(t) \|_{L^2}.
\]

As in the proof of Lemma 5.3, we see that
\[
\| F_{n} \partial_{\eta} u_{N, \varepsilon}(t, \xi - ik_1 \varepsilon, \gamma) - F_{n} \partial_{\eta} u_{N, \varepsilon}(t, \xi - ik_1 \varepsilon, \gamma) \|_{L^2(-\pi, \pi)} = O(e^{-c/\varepsilon})
\]
for $a > 0$. Combining the above with $P(0)^* \partial_{\eta} u_{N, \varepsilon} = t (\sqrt{r_{N, \varepsilon}}, 0)$ and the facts that
\[
|P(\xi - ik_1 \varepsilon)^* - P^*(0)| + \left| \frac{1}{\sin \frac{\xi - ik_1 \varepsilon}{2}} - \frac{2}{\xi - ik_1 \varepsilon} \right| \lesssim |\xi - ik_1 \varepsilon| \quad \text{for } \xi \in [-\pi, \pi],
\]
and that $\| e^{-k_1 \varepsilon (-c_1 t - \varepsilon^{-1} \gamma_1)} \partial_{\eta} \partial_{\eta} u_{N, \varepsilon}(t, \cdot, k, \gamma) \|_{L^2} = O(\varepsilon^{\frac{1}{2}})$, we have
\[
\left\langle \tau_{ik_1} e f(t), \tau_{-ik_1} \{ e^{i c_1 t \xi} \xi^{-1} \partial_{\eta} r_{N, \varepsilon}(t, \xi; k, \gamma) \} \right\rangle
= O(\varepsilon^{\frac{1}{2}} (\delta_2 + \varepsilon^2) e^{-k_1 \gamma_1} \| \tau_{ik_1} f(t) \|_{L^2}).
\]

Let $h_2, h_3 \in L^2(\mathbb{R})$ such that
\[
h_1(\tau, y) + h_2(\tau, y) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i c_1 t (\eta + ik_1)} f_{\#}(t, \varepsilon(\eta + ik_1)) e^{i\eta y} dy,
\]
\[
h_3(\tau, y) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (f_{2,+}(t, \varepsilon\eta) + f_{3,+}(t, \varepsilon\eta)) e^{i\eta y} dy.
\]
Then
\[
\left\langle \tau_{ik_1} f_+ (t), \tau_{-ik_1} \left\{ e^{i \epsilon t \xi} \xi^{-1} \hat{\partial}_\gamma \varphi_{N, \xi} (t, \xi; k, \gamma) \right\} \right\rangle
\]
\[
= \epsilon \left\langle \hat{h}_1 + \hat{h}_3, \tau_{-ik_1} \left\{ \eta^{-1} e^{4ik_1^2 \gamma_1} \hat{\partial}_\gamma \varphi_{N} (\tau, \eta; k, \gamma) \right\} \right\rangle
\]
\[
= \epsilon \left\langle h_1 + h_3, e^{-k_1 y} \int_{-\infty}^{y} \partial_\gamma \varphi_{N} (\tau, y_1 + 4k_1^2 \tau; k, \gamma) dy_1 \right\rangle.
\]

Since \( \hat{h}_3 (\tau, \eta) = 0 \) for \( \eta \in [-K, K] \), it follows from Lemma C.1 that
\[
\left\| h_3, e^{-k_1 y} \int_{-\infty}^{y} \partial_\gamma \varphi_{N} (\tau, y_1 + 4k_1^2 \tau; k, \gamma) dy_1 \right\| \leq \epsilon^{-\frac{1}{2}} K^{-2} \| h_3 \|_{L^2} + (\epsilon^2 + \delta_2) \| \tau_{ik_1 \epsilon} f \|_{L^2}
\]
(C.5)
\[
\leq (K^{-2} + \epsilon^2 + \delta_2) \| \tau_{ik_1 \epsilon} f \|_{L^2}.
\]

Similarly,
\[
\epsilon^{\frac{1}{2}} \left\| \hat{h}_1 + \hat{h}_3, \tau_{-ik_1} \left\{ \eta^{-1} e^{4ik_1^2 \gamma_1} \hat{\partial}_\gamma \varphi_{N} (\tau, \eta; k, \gamma) \right\} \right\rangle
\]
(C.6)
\[
\leq (K^{-2} + \epsilon^2 + \delta_2) \| \tau_{ik_1 \epsilon} f \|_{L^2}.
\]

By (C.5), (C.6) and (C.4), we have
\[
\| L_2 \| \leq \sum_{l, m} \sum_{i \leq j} \left| \left\langle h_1 (\tau), e^{-k_1 y} \eta_j^m (\tau, \cdot + 4k_1^2 \tau) \right\rangle \right| e^{k_1 y} \xi^i (\tau, \cdot + 4k_1^2 \tau) \| L^2
\]
\[
\leq \sum_{l, m} \sum_{i \leq j} e^{-a \left( 4k_1^2 - k_1^2 \right) \tau + \gamma_j - \gamma_i} \left| \left\langle h_1 (\tau), e^{-k_1 \left( y - 4k_1^2 \tau \gamma_j - \gamma_i \right) \eta_j^m (\tau, \cdot + 4k_1^2 \tau) \right\rangle \right|
\]
\[
\leq \epsilon^{-\frac{1}{2}} (K^{-2} + \epsilon^2 + \delta_2) \| \tau_{ik_1 \epsilon} f \|_{L^2}.
\]

Thus we complete the proof of Lemma 5.5. 

APPENDIX D. PROOF OF LEMMA 4.3

To begin with, we compare spectral projection associated with a solitary wave solution of FPU and that associated with KdV 1-soliton.
Lemma D.1. Let $\varepsilon > 0$, $a \in (\varepsilon/8, 2\varepsilon)$ and $c = 1 + \varepsilon^2/6$. Then
\[
\left\| J^{-1} \partial_x u_c + \phi_c \left( \frac{1}{1} - 1 \right) \right\|_{L^2_a} = O(\varepsilon^{\frac{3}{2}}),
\]
\[
\left\| J^{-1} \partial_c u_c + \int_{-\infty}^{n} \partial_c \phi_c \left( \frac{1}{1} - 1 \right) \right\|_{L^2_a} = O(\varepsilon^{-\frac{1}{2}}).
\]

To prove Lemma D.1 we need the following:

Claim D.1. Suppose $a \in (0, 1)$ and $f \in C_0^\infty(\mathbb{R})$. Then
\[
\left\| (e^\partial - 1)^{-1} \partial_x f \right\|_{L^2_a} \lesssim \left\| f \right\|_{L^2_a} + a^{-1} \left\| \partial_x f \right\|_{L^2_a},
\]
\[
\left\| (e^\partial - 1)^{-1} \partial_x f - f \right\|_{L^2_a} \lesssim a \left\| f \right\|_{L^2_a} + a^{-1} \left\| \partial_x f \right\|_{L^2_a},
\]
\[
\left\| (e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 f \right\|_{L^2_a} \lesssim \left\| f \right\|_{L^2_a} + a^{-2} \left\| \partial_x^2 f \right\|_{L^2_a},
\]
\[
\left\| (e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 f - f \right\|_{L^2_a} \lesssim a^2 \left\| f \right\|_{L^2_a} + a^{-2} \left\| \partial_x^4 f \right\|_{L^2_a}.
\]

Proof. Let $g(x) = e^{ax} f(x)$. Using $|e^{i\xi-a} - 1| \geq 1 - e^{-\alpha} \geq a$ and
\[
|e^{i\xi-a} - i\xi + a - 1| \lesssim a^2 + |\xi|^2,
\]
we have
\[
\left\| (e^\partial - 1)^{-1} \partial_x f \right\|_{L^2_a} = \left\| \frac{i\xi - a}{e^{i\xi-a} - 1} \hat{g} \right\|_{L^2} \lesssim \left\| f \right\|_{L^2_a} + a^{-1} \left\| \partial_x f \right\|_{L^2_a},
\]
and
\[
\left\| (e^\partial - 1)^{-1} \partial_x f - f \right\|_{L^2_a} = \left\| \frac{e^{i\xi-a} - i\xi + a - 1}{e^{i\xi-a} - 1} \hat{g} \right\|_{L^2} \lesssim a \left\| \hat{g} \right\|_{L^2} + a^{-1} \left\| \partial_x \hat{g} \right\|_{L^2} \lesssim a \left\| f \right\|_{L^2_a} + a^{-1} \left\| \partial_x f \right\|_{L^2_a}.
\]
Similarly, by using $|e^{i\xi-a} + e^{-i\xi+a} - 2| \geq 4 \sinh^2(a/2)$ and
\[
|e^{i\xi-a} + e^{-i\xi+a} - 2 - (i\xi - a)^2| \lesssim \xi^4 + a^4,
\]
we have
\[
\left\| (e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 f \right\|_{L^2_a} = \left\| \frac{(i\xi - a)^2}{e^{i\xi-a} - 2 + e^{-i\xi+a}} \hat{g} \right\|_{L^2} \lesssim \left\| f \right\|_{L^2_a} + a^{-2} \left\| \partial_x^2 f \right\|_{L^2_a},
\]
and
\[
\left\| (e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 f - f \right\|_{L^2_a} = \left\| \frac{(i\xi - a)^2}{e^{i\xi-a} - 2 + e^{-i\xi+a}} \hat{g} - \hat{g} \right\|_{L^2} \lesssim a^2 \left\| \hat{g} \right\|_{L^2} + a^{-2} \left\| \partial_x^4 \hat{g} \right\|_{L^2} \lesssim a^2 \left\| f \right\|_{L^2_a} + a^{-2} \left\| \partial_x^4 f \right\|_{L^2_a}.
\]
Claim D.2. Let $a \in \mathbb{R}$ and $f \in H^1(\mathbb{R})$. Then

$$\left\| f(x) - \int_x^{x+1} f(y)dy \right\|_{L^2(\mathbb{R})} \leq \max(1, e^{-a}) \left\| f' \right\|_{L^2(\mathbb{R})}.$$

Proof. Since

$$\left| f(x) - \int_x^{x+1} f(y)dy \right| = \left| \int_x^{x+1} f'(t)dt \right| \leq \left( \int_x^{x+1} f'(t)^2 dt \right)^{\frac{1}{2}},$$

we have

$$\left\| f(x) - \int_x^{x+1} f(y)dy \right\|_{L^2(\mathbb{R})} \leq \int \left( e^{2ax} \int_x^{x+1} f'(t)^2 dt \right) dx \leq \max(1, e^{-2a}) \left\| f' \right\|_{L^2(\mathbb{R})}^2.$$

Proof of Lemma [D.7]. By the definition of $u_c$, we have

$$p_c = -c(e^\partial - 1)^{-1} \partial_x r_c, \quad J^{-1} \partial_x u_c = (-c(e^\partial - 2 + e^{-\partial})^{-1} \partial_x r_c, (e^\partial - 1)^{-1} \partial_x r_c).$$

Thus by Claims [A.2] and [D.1],

$$\left\| J^{-1} \partial_x u_c + \phi \varepsilon - (c(e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 (r_c - \phi \varepsilon) \right\|_{L^2(\mathbb{R})} \leq \left\| \left(c(e^\partial - 2 + e^{-\partial})^{-1} \partial_x^2 (r_c - \phi \varepsilon) \right) \right\| + \left\| \left(-(e^\partial - 1)^{-1} \partial_x + 1\right) \phi \varepsilon \right\| \leq \left\| r_c - \phi \varepsilon \right\|_{H^{-1}_a} + a^{-2} \left\| \partial_x^2 (r_c - \phi \varepsilon) \right\|_{H^{-1}_a} + \varepsilon^2 \left( \left\| \phi \varepsilon \right\|_{H^{-1}_a} + a^{-2} \left\| \partial_x^2 \phi \varepsilon \right\|_{H^{-1}_a} \right) + a^2 \left\| \phi \varepsilon \right\|_{H^{-1}_a} + a^2 \left\| \partial_x \phi \varepsilon \right\|_{H^{-1}_a} + a^{-1} \left\| \partial_x^2 \phi \varepsilon \right\|_{H^{-1}_a} \leq \left( \varepsilon^2 + a \varepsilon^2 \right) \left( 1 + \frac{\varepsilon}{a} \right)^2 + a^2 \left( \varepsilon^2 + \frac{\varepsilon^2}{a^2} \right)^2 = O(\varepsilon^2). \]
Since \( \|J^{-1}\|_{B(l^2_a \times l^2_a)} \lesssim a^{-1}, \)
\[
\left\| J^{-1} \partial_c u_c + \int_{-\infty}^n \partial_c \phi_\varepsilon \left( \frac{1}{1} \right) \right\|_{l^2_a} \\
\lesssim a^{-1} \left\| \partial_c u_c + \int_{-\infty}^n \partial_c \phi_\varepsilon \left( \frac{1}{1} \right) \right\|_{l^2_a} \\
\lesssim a^{-1} \left\| \partial_c r_c - \int_x^{x+1} \partial_c \phi_\varepsilon \right\|_{H^{1-a}} + a^{-1} \left\| \partial_c p_c + \int_{x-1}^x \partial_c \phi_\varepsilon \right\|_{H^{1-a}}.
\]

By (D.1) and Claim D.1,
\[
\| \partial_c p_c + \partial_c r_c \|_{l^2_a} \\
\lesssim \| (e^\theta - 1)^{-1} \partial_x r_c \|_{l^2_a} + \| (c(e^\theta - 1)^{-1} \partial_x - 1) \partial_c r_c \|_{l^2_a} \\
\lesssim \| r_c \|_{H^{1-a}} + a^{-1} \| \partial_x r_c \|_{H^{1-a}} + a \| \partial_c r_c \|_{H^{1-a}} + a^{-1} \| \partial_x^2 \partial_c r_c \|_{H^{1-a}} + \varepsilon^2 \| \partial_c r_c \|_{H^{1-a}} \\
\lesssim \varepsilon^{\frac{3}{2}} (1 + a^{-1}) \varepsilon + a^{-1} \varepsilon^{\frac{1}{2}} (1 + a^{-2} \varepsilon^2) = O(\varepsilon^{\frac{3}{2}}).
\]

Combining the above with (P4) and Claim D.2, we have
\[
\left\| J^{-1} \left( \partial_c r_c + \int_{-\infty}^n \partial_c \phi_\varepsilon \left( \frac{1}{1} \right) \right) \right\|_{l^2_a} \\
\lesssim a^{-1} (\| \partial_c r_c - \partial_c \phi_\varepsilon \|_{H^{1-a}} + \| \partial_c p_c + \partial_c \phi_\varepsilon \|_{H^{1-a}} + \| \partial_x \partial_c \phi_\varepsilon \|_{H^{1-a}}) \\
\lesssim a^{-1} \varepsilon^{\frac{1}{2}} = O(\varepsilon^{\frac{1}{2}}).
\]

Finally, we will prove Lemma 4.3.

**Proof of Lemma 4.3.** We assume that \( k = N \). The other cases can be shown in the same way.

By (P4) and Lemma C.1, we can choose \( k \) and \( \gamma \) so that
\[
\sum_{i=0,1} \sup_{t \geq 0, x \in \mathbb{R}} \left| \partial_x^i (U_N(t) - u_{N,\varepsilon}(t, x, \gamma)) \right| \leq \delta(L) \varepsilon^{2+i} + O(\varepsilon^4).
\]

Combining Lemmas C.1 and D.1 with (C.2) and (C.3), we obtain (5.2) from (4.9). Thus we prove Lemma 4.3.

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