Shared Purity of Multipartite Quantum States

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Outline of the talk

1. Motivation
2. Shared purity: definition & properties
3. Bipartite and multipartite mixed states: some examples
4. Monogamy properties
5. Application: detecting criticality in quantum spin models
6. Summary
Quantum Information protocols: some examples

- Quantum teleportation
- Superdense coding
- Entanglement based quantum crytography
- Quantum nonlocality without entanglement
- Deterministic quantum computation with one qubit
- Secure deterministic communication without entanglement

Quantum information protocols can achieve higher efficiencies than their classical counterparts, if they exist.
Why is the efficiency higher?

Quantum mechanical resource
- Quantum entanglement
- Some of them do not employ entanglement

What then is the resource?
- Quantum discord seems to be the answer
- However, intriguing questions remain

- Quantum entanglement $\rightarrow$ entanglement-separability
- Quantum discord $\rightarrow$ information-theoretic
Quantum Correlation measures

**Entanglement-separability paradigm**
1. Concurrence
2. Logarithmic negativity
3. Geometric measure, etc.

**Information-theoretic paradigm**
1. Quantum discord
2. Quantum work deficit
3. Symmetric discord, etc.

Shared purity $\rightarrow$ not a measure of quantum correlation
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2. **Shared purity: definition & properties**

3. Bipartite and multipartite mixed states: some examples

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Pure quantum states are privileged

- Maximum advantage $\rightarrow$ for pure shared states
- Pure quantum states $\rightarrow$ vanishing entropy $\rightarrow$ full information of the system is available

- Fidelity $\rightarrow$ distance of a state from another state

- Shared purity defined for an arbitrary quantum state of shared systems of an arbitrary number of parties in arbitrary dimensions
The “global fidelity” of an $N$-party arbitrary (pure or mixed) quantum state, $\rho_{1...N}$, on $\mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N}$,

$$F_G = \max_{\{|\phi\rangle_{1...N} \in \mathcal{H}\}} 1...N \langle \phi | \rho_{1...N} | \phi \rangle_{1...N},$$

where the maximization is performed over all elements (pure states) of $\mathcal{H}$.

- Measures the lack of disorder present in the system
- Unity for all pure states in arbitrary dimensions
Proposition

For an arbitrary mixed state \( \rho_{1...N} \), \( F_G \) is the largest eigenvalue in the spectrum of the state.

Proof.

- \( \rho_{1...N} = \sum_{i} \lambda_i |e_i\rangle\langle e_i|, \{ |e_i\rangle \} \rightarrow \) (orthonormal basis spanning \( \mathcal{H} \))

- \( |\phi\rangle_{1...N} = \sum_{i} a_i |e_i\rangle, \sum_{i} |a_i|^2 = 1 \)

- \( F_G = \max_{a_i} \sum_{i} |a_i|^2 \lambda_i \)

- \( \lambda_r \rightarrow \) largest eigenvalue

- \( a_r = 1 \) and \( a_i, i \neq r = 0 \) (assumption) \( \implies F_G \geq \lambda_r \)

- \( F_G \leq \max_{a_i} \sum_{i} |a_i|^2 \lambda_r = \lambda_r, \) since \( \lambda_i \leq \lambda_r \ \forall i \)

- \( F_G = \lambda_r \)
Local fidelity

Definition

The “local fidelity”, of the $N$-party quantum state $\rho_{1...N}$

$$F_L = \max_{\{|\phi\rangle_{1...N} \in S\}} \langle \phi | \rho_{1...N} | \phi \rangle_{1...N},$$

where the maximization is carried out over a certain set $S$, of pure product states.

Hierarchy of local fidelities depending on $S$

Two extreme cases:

- fully separable states
- genuinely multiparty entangled states
Proposition

For an arbitrary pure $N$-party state $|\psi\rangle_{1...N}$, $F_{L}^{n-gen}$ is the square of the maximal Schmidt coefficient among all bipartitions. (optimization over states that are not genuinely multiparty entangled)

Proof.

\begin{itemize}
  \item $F_{L} = \max_{\{|\phi\rangle_{1...N} \in S_{n-gen}\}} |\langle \phi | \psi \rangle |^2 = 1 - \mathcal{E}$, $\mathcal{E} \rightarrow$ generalized geometric measure
  \item $\mathcal{E}(|\psi\rangle) = 1 - \max\{\lambda_{A:B}^2 | A \cup B = \{1, \ldots, N\}, A \cap B = \emptyset\}$, where $\lambda_{A:B}$ is the maximal Schmidt coefficient in the $A : B$ bipartition
\end{itemize}
**Theorem**

For an arbitrary bipartite (pure or mixed) state, on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the minimum value attained by $F_L$ is $\lambda_r/d$, where $d = \min\{d_1, d_2\}$, and $\lambda_r$ is the largest eigenvalue in the spectrum of $\rho$.

**Proof.**

We have

$$F_L(\rho) = \max_{\{ |\phi\rangle \in S_L \}} \langle \phi | \rho | \phi \rangle = \max_{\{ |\phi\rangle \in S_L \}} \sum_i \lambda_i |\langle \phi | e_i \rangle|^2,$$

where $\sum_i p_i |e_i\rangle\langle e_i|$ is a spectral decomposition of the bipartite quantum state $\rho$. Therefore,

$$F_L(\rho) \geq \max_{\{ |\phi\rangle \in S_L \}} \lambda_r |\langle \phi | e_r \rangle|^2.$$

The property follows from the fact that $F_L \geq 1/d$ for any pure state in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$.

**Corollary 3.1.** For an arbitrary bipartite (pure or mixed) state, on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the maximum value attained by $S_P$ is $\lambda_r(1 - 1/d)$, where $d = \min\{d_1, d_2\}$, and $\lambda_r$ is the largest eigenvalue in spectrum of $\rho$. 

$\square$
Properties

Shared purity: \( S_P = F_G - F_L \)

- The shared purity vanishes for pure product states of the form \( |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \).
- For an arbitrary \( N \)-party pure state \( |\psi\rangle_{1\ldots N} \) in arbitrary dimensions, the shared purity is a geometric measure of entanglement.
- The shared purity is invariant under local unitary operations.
- For classically correlated states, the global and local fidelities are equal.
- For a state of the form \( \rho_1 \otimes \ldots \otimes \rho_N \) on \( \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N} \), shared purity vanishes.
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Admixtures of a Bell state with a pure product state

\[ \rho_{\text{ent}} = p|00\rangle\langle00| + (1 - p)|\psi^\text{-}\rangle\langle\psi^\text{-}| \]

\[ |\psi^\text{-}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \text{ and } 0 \leq p \leq 1 \]

The state is **entangled** for any value of \( p < 1 \).

\[ S_P = \frac{(1 - p)(1 - 2p)}{2 - 3p}, \quad 0 \leq p < \frac{1}{2}, \]

\[ = 0, \quad \frac{1}{2} \leq p \leq 1. \]

- Shared purity \( \rightarrow \) zero for this mixed **entangled** states
- Distillable entanglement \( \rightarrow \) zero for some **entangled** states
Figure: The most interesting region is $\frac{1}{2} \leq p < 1$, where the shared purity vanishes, although the state has a nonzero entanglement there.
Admixtures of pure states with noise

- \( \rho_{gen} = p|\psi\rangle\langle\psi| + \frac{(1-p)}{4}I \otimes I \),
- \( |\psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle \) with \( 0 \leq \theta \leq \pi/4, \ 0 \leq p \leq 1 \)
- \( S_P = p \sin^2 \theta \)
- \( \rho_{gen} \xrightarrow{} \) Werner state when \( \theta = \pi/4, \ S_P = \frac{p}{2} \)
- Werner state is entangled for \( p > 1/3 \)

- Quantum discord and quantum work-deficit are non-vanishing for \( p > 0 \)
- Shared purity can be positive for separable states
Figure: $S_P$ is always nonzero for $p > 0$ & $\theta > 0$. 
Bell mixtures

\[ \rho_{\text{Bell}} = p|\psi^-\rangle\langle\psi^-| + (1 - p)|\psi^+\rangle\langle\psi^+|, \]

\[ |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad \text{and} \quad 0 \leq p \leq 1. \]

Note that the state is entangled for all values of \( p \) except \( p = 1/2 \).

\[ S_P = p - \frac{1}{2} \quad \text{for} \quad p \geq \frac{1}{2}, \]
\[ = \frac{1}{2} - p \quad \text{for} \quad p < \frac{1}{2}. \]

Just like any quantum correlation measure, shared purity, in this case, is also a mirror reflection with respect to the \( p = 1/2 \) line. This is a result of the local unitary invariance of shared purity.
Figure: $S_P$ is vanishing only at $p = 1/2$. 
$N$-party Greenberger-Horne-Zeilinger state, mixed with white noise, in $(\mathbb{C}^d)^{\otimes N}$.

\[ \rho_{GHZ_N} = p|\psi\rangle\langle\psi| + (1 - p) \left( \frac{1}{d} I_d \otimes \cdots \otimes \frac{1}{d} I_d \right), \]

\[ |\psi\rangle = \frac{1}{\sqrt{d}} (|0_1 \ldots 0_N\rangle + \ldots + |(d - 1)_1 \ldots (d - 1)_N\rangle) \text{ and } 0 \leq p \leq 1. \]

$I_d$ denotes the identity operator on $\mathbb{C}^d$, and $\{|i_j\rangle\}_{i=0}^{d-1}$ for $j = 1, \ldots, N$ forms an orthonormal basis in the Hilbert space of the $j$th particle.

\[ S_P = p \left( 1 - \frac{1}{d} \right). \]

Note that the shared purity never vanishes except at $p = 0$. 

\[ \rho_{GHZ_N} = p|\psi\rangle\langle\psi| + (1 - p) \left( \frac{1}{d} I_d \otimes \cdots \otimes \frac{1}{d} I_d \right), \]
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Monogamy

- Multiparty quantum system in a state \( \rho_{1\ldots N} \)
- \( Q \) \( \rightarrow \) two-party physical quantity
- \( Q \) will be monogamous if a high amount of \( Q(\rho_{12}) \) implies that neither the party 1 nor the party 2 will be able to share a substantial amount of \( Q \) with any other party
- Sharing of classical correlations of a multiparty quantum system does not have any such restriction
Theorem

If two quantum systems, irrespective of their dimensions, have the maximal amount of shared purity, they cannot share any purity with any third quantum system.

Proof.

- Three-party system $\rho_{123}$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- Assumption: $S_P(\rho_{12}) = (1 - 1/d) \rightarrow \text{(maximal)}$, where $d = \min\{d_1, d_2\}$
- Now, $S_P(\rho_{12}) = 1 - 1/d$, only for a pure state $\rho_{12}$
- Therefore, $\rho_{123}$ must be of the form $\rho_{12} \otimes \rho_3$
- Therefore, $S_P(\rho_{13}) = S_P(\rho_{23}) = 0$
- Shared purity is qualitatively monogamous
Monogamy: quantitative investigation

The monogamy condition

\[ S_P(\rho_{12}) + S_P(\rho_{13}) \leq S_P^{1:23}(\rho_{123}) \]

- The generalized GHZ states always satisfy the monogamy condition
- The generalized W states always violate it
- However, the above statements are not true for the GHZ class and W class states
Generalized GHZ state

- $|\psi\rangle^G_{GHZ} = \cos \theta |000\rangle + e^{i\phi} \sin \theta |111\rangle$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$
- $\rho^G_{1j} = \cos^2 \theta |00\rangle \langle 00|_j + \sin^2 \theta |11\rangle \langle 11|_j$, $j \in \{2, 3\}$
- Since the state is classically correlated, $S_P(\rho^G_{1j}) = 0 \ \forall j$
- Since $|\psi\rangle^G_{GHZ}$ is a pure state, $F^1:23_G = 1$
- $F^1:23_L = \max\{\cos^2 \theta, \sin^2 \theta\}$ (calculated)
- $S^1:23_P(|\psi\rangle^G_{GHZ}) = 1 - \max\{\cos^2 \theta, \sin^2 \theta\}$
- Monogamy condition is satisfied
Generalized W state

\[ |\psi\rangle^G_W = \sin \theta_1 \cos \theta_2 |001\rangle + \sin \theta_1 \sin \theta_2 e^{i\phi_1} |010\rangle + \cos \theta_1 e^{i\phi_2} |100\rangle \]

\[ \delta_{SP} = S_P(\rho_{1:23}) - (S_P(\rho_{12}) + S_P(\rho_{13})) \rightarrow \text{shared purity monogamy score} \]
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Anisotropic quantum XY spin chain in one dimension

\[ H_{XY} = \frac{J}{2} \left( \sum_{i=1}^{N} (1 + \gamma)\sigma_i^x \sigma_{i+1}^x + (1 - \gamma)\sigma_i^y \sigma_{i+1}^y \right) + h \sum_{i=1}^{N} \sigma_i^z \]

- \( J \rightarrow \) coupling constant for the nearest neighbor interaction
- \( \gamma \in (0, 1] \rightarrow \) anisotropy parameter
- \( \sigma \)'s \rightarrow \) the Pauli spin matrices
- \( h \rightarrow \) the external transverse magnetic field
- Periodic boundary condition is assumed
\( H_{XY} \) can be diagonalized by applying Jordan-Wigner, Fourier, and Bogoliubov transformations successively

At zero temperature, the system undergoes a quantum phase transition driven by the external transverse magnetic field

Concurrence, geometric measures, quantum discord \( \rightarrow \) detects this transition
Figure: Concurrence detects quantum phase transition.

A. Osterloh, L. Amico, G. Falci and R. Fazio, Nature 416, 608, (2002)
We investigate the behavior of the shared purity of the nearest neighbor density matrix of the ground state near the known quantum critical point at $\lambda = \frac{\hbar}{J} = 1$

- $F_G \rightarrow$ maximum eigenvalue of the density matrix $\rho_{AB}$
- $F_L \rightarrow$ obtained by numerical maximization of the density matrix $\rho_{AB}$ with respect to the product states in $\mathbb{C}^2 \otimes \mathbb{C}^2$
Figure: Shared purity detects quantum phase transition.
Scaling analysis

- Scaling analysis helps us to understand the viability of detecting the critical point in finite-sized systems

- The point of divergence approaches \( \lambda = \lambda_c \) as \( N^{-1.40} \), i.e.,
  \[
  \lambda = \lambda_c + kN^{-1.40}
  \]

- For concurrence, \( \lambda \rightarrow \lambda_c \) as \( N^{-1.87} \)

- For quantum discord, \( \lambda \rightarrow \lambda_c \) as \( N^{-1.28} \)
Figure: Finite-size scaling analysis for shared purity. ($\gamma = 0.8$)
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Summary

- New property of shared quantum systems
- Different from quantum correlations
  - can be nonzero for unentangled states
  - can be zero for entangled states
- Quantum property (monogamous)
- Scaling in transverse Ising model different from both entanglement & discord

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A. B., A. Sen(De), U. Sen, arXiv:1306.0669
Thank you.