The emergence of synchronization in systems of coupled agents is a pivotal phenomenon in physics, biology, computer science, and neuroscience. Traditionally, interaction systems have been described as networks, where links encode information only on the pairwise influences among the nodes. Yet, in many systems, interactions among the units often take places in larger groups. Recent work has shown that the presence of higher-order interactions between oscillators can significantly affect the emerging dynamics. However, these early studies have mostly limited their investigations to interactions up to 4 oscillators at time, or are limited to well-mixed homogeneous scenarios. Here, we propose a general framework that allows us to effectively study populations of oscillators where higher-order interactions of all possible orders are considered. For this, we introduce a multi-order Laplacian whose spectrum determines the stability of the synchronized solution. Our framework is validated on three structures of interactions of increasing complexity. First, we study a population with all-to-all interactions at all orders, for which we can derive in a full analytical manner the Lyapunov exponents of the system, and for which we investigate the effect of including attractive and repulsive interactions. Second, we apply the multi-order Laplacian framework to synchronization on a synthetic model with heterogeneous higher order interactions. Finally, we compare the dynamics of coupled oscillators with higher-order and pairwise couplings only, for a real dataset describing the macaque brain connectome. Taken together, in this work we highlight the importance of faithfully representing the complexity of interactions in real-world systems, proposing an analytical framework to characterise the emergent behaviour in coupled nonlinear systems with higher-order interactions.

I. INTRODUCTION

The emergence of order in populations of interacting oscillators – a phenomenon known as synchronization – is ubiquitous in natural and man-made systems [1, 2]. Typical examples of synchronization include the flashing of fireflies, or the clapping of an audience. In the last decades synchronization has been the subject of intense research, and has been applied to a wide range of areas, including neuroscience [3], circadian rhythms [4], or the cardio-vascular system [5, 6]. In particular, much attention has been devoted to unveil the relationship between the structure of the network of interactions and the emerging collective behavior [7, 8]. As outcome of these investigations, noticeable examples include the discovery of abrupt synchronization induced by degree-frequency correlation [9], and cluster synchronization induced by structural symmetries [10].

Most interacting systems have so far been represented as networks, a collection of nodes and links describing relationship and influences between them at the level of pairs. However, many real-world systems are better modeled by including higher-order interactions, i.e. interactions between more than two nodes at a time. A typical example is that of human collaborations, which often occur at the level of groups. In this cases, a traditional network representation is misleading, as it would associate the same structure – a triangle – both to the case of a triplet of people collaborating on a single task, and to the case of three individuals collaborating as three distinct pairs in different projects. This indistinguishability can be solved by making use of higher-order network representations, such as hypergraphs [11] and simplicial complexes [12]. A stream of research has recently been focused on correctly characterizing the structure of systems with higher-order interactions [13–19]. Interestingly, considering this additional level of complexity sometimes leads to changes in the emerging dynamics of a complex system, including social contagion [20, 21], activity driven models [22], diffusion [23, 24], random walk [25, 26] and evolutionary games [27].

Higher-order interactions can influence the nature of the dynamics also for systems of coupled oscillators. A few studies have recently investigated their effect experimentally [28, 29], and from a network inference point of view [30–32]. From a theoretical point of view, higher-order interactions were considered in the context of global nonlinear coupling [30], were shown to arise from phase reduction beyond the first approximation [33–36], and can facilitate chaotic behavior [33]. The Kuramoto model, where phase oscillators interact in pairs, is often invoked as the most simple way to describe the emergence of synchronization in a population of interacting
nodes with local dynamics. Yet, only a few studies have so far considered higher-order generalizations of the Kuramoto model, where they were shown to favor cluster synchronization [37, 38] and explosive transitions [38–40]. In these studies, analytical descriptions of the emerging phenomena are limited to all-to-all settings. Orders no higher than 3 were considered, and calculations were valid in limit of large networks. Interestingly, a different approach was introduced in [40], where a Kuramoto phase oscillator is associated to each simplex.

In this work we study a population of coupled oscillators where higher-order interactions of all orders are possible, by introducing a generalization of the Kuramoto model of identical phase oscillators to simplicial complexes. For this model, we show that the stability of the fully synchronized state is determined by the eigenvalues of a newly defined multi-order Laplacian, which can be applied to any complex topology. We validate this Laplacian framework on several toy-models describing simplicial complexes of increasing complexity. First, we investigate all-to-all interactions at all orders, for which the eigenvalues can be derived fully analytically. We further characterize this system by considering three sub-cases: (i) attractive coupling only, (ii) interplay between attractive and repulsive orders, (iii) and decaying coupling strength (that we link to higher-order phase reduction studies). We confirm our analytical findings with numerical simulations. Second, we consider the star-clique model, a toy model specifically generated to highlight some simple spectral properties of the multi-order Laplacian. Finally, we investigate the effect of higher-order interactions on synchronization on a real macaque brain dataset. Results indicate that higher-order interactions modify the Laplacian spectrum, enhancing stability. Taken together, our work sheds new light on the effect of higher-order interactions in a population of coupled oscillators, and unveils how new emergent phenomena can be captured analytically through the introduction of a suitable Laplacian framework, which naturally generalizes the traditional approach to networks beyond pairwise interactions.

II. GENERALIZED HIGHER-ORDER INTERACTIONS

We study the effect of generalized higher-order interactions in a population of $N$ identical phase oscillators. Specifically, we consider the following simplicial complex describing many-body interactions among oscillators of any order $d = 1, \ldots, d_{\text{max}}$

\[
\dot{\theta}_i = \omega + \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) + \frac{\gamma_2}{2!(K^{(2)})} \sum_{j,k=1}^{N} A_{ij} A_{jk} A_{ki} \sin(\theta_j + \theta_k - 2\theta_i) + \frac{\gamma_3}{3!(K^{(3)})} \sum_{j,k,l=1}^{N} [(A_{ij} A_{jk} A_{kl}) (A_{il} A_{lj}) A_{kl}] \sin(\theta_j + \theta_k + \theta_l - 3\theta_i) + \ldots
\]

where each oscillator has natural frequency $\omega$ and participates in interactions of order up to $d_{\text{max}}$. The largest value that $d_{\text{max}}$ can take is $N - 1$, which corresponds to $N$-oscillator interactions, the highest order possible. The dynamics of system (1) is a generalization of the canonical Kuramoto model [41] to higher-order interactions of any order. Indeed, the first two terms on the right side correspond to the traditional Kuramoto, where pairs of oscillators are coupled, with strength $\gamma_1$, via a sinusoidal and diffusive function, and the (undirected) adjacency matrix $A$ determines the structure of those 2-oscillator interactions: $A_{ij} = 1$ if there is a 2-oscillator interaction $(i, j)$, i.e. a link, between oscillators $i$ and $j$ and 0 otherwise. The following terms refer instead to the higher-order interactions taking place in the system, and for which we adopt the terminology of simplicial complexes. A link describes the lowest simplicial interaction of order 1. The general interaction scheme of (1) is illustrated with an example in Fig. 1. There a 3-oscillator interaction is represented by a 2-simplex, and any $(d + 1)$-oscillator interaction is represented by a $d$-simplex, (also called a simplex of order $d$), as illustrated in Fig. 1(b). Following those definitions, $\gamma_2$ is the coupling strength of 2-simplex interactions, i.e. interactions of 3 oscillators. The structure of those 2-simplex interactions is determined by a product of the same adjacency...
matrix: $A_{ij}A_{jk}A_{ki} = 1$ only if there is a 3-oscillator interaction $(i, j, k)$ between oscillators $i$, $j$, and $k$, and 0 otherwise. This means that there can be a 2-simplex (3-oscillator) interaction $(i, j, k)$ only if the three 1-simplex (2-oscillator) interactions $(i, j)$, $(j, k)$, and $(k, i)$ also occur: that is a requirement of simplicial complexes (at variance with hypergraphs). The strength of those 2-simplex interactions is divided by $2!$ to ensure each 2-simplex is counted only once. To normalize the interactions we scale them by a factor $\langle K^{(d)} \rangle$, the average 2-simplex degree $K_i^{(2)}$ of nodes in the simplicial complex, i.e. the average number of distinct 2-simplex interactions nodes are part of. Similarly, $\gamma_3$ is the coupling strength of 3-simplex interactions, i.e. interactions of 4 oscillators. The structure at order 3 is also determined by the same adjacency matrix: $A_{ij}A_{jk}A_{ki}A_{il}A_{lj}A_{jk} = 1$ if there is a 3-simplex interaction $(i, j, k, l)$ and 0 otherwise. The factor $3!$ ensures each 3-simplex interaction is counted only once, and we normalize by $\langle K^{(3)} \rangle$, the average 3-simplex degree $K_i^{(3)}$.

Finally, the description at any order $d$ requires us to introduce an index of indices: we define a set of $N$ indices $\{j_1, j_2, \ldots, j_{d_{\text{max}}+1}\}$, with $d_{\text{max}} = N - 1$, and define $j_1 \equiv i$, and $j_2 \equiv j$ for convenience. In general, at order $d$ with $d = 1, \ldots, N - 1$, the coupling strength is given by $\gamma_d$ and the structure of the $d$-simplex interactions is determined by the following product: $\prod_{m,n=1}^{d+1} A_{j_m j_n} = 1$ if there is a $d$-simplex interaction $(j_1, j_2, \ldots, j_{d+1})$ and 0 otherwise. Similarly to the cases above, we divide by $d!$ and normalize by $\langle K^{(d)} \rangle$, the average $d$-simplex degree $K_i^{(d)}$.

Let us emphasize two features of model (1). First, the coupling function at each order $d$ is symmetric with respect to $i$, i.e. it is invariant for any permutation of the remaining indices. This is the most natural generalization of the pairwise Kuramoto coupling function. Other choices are possible, as we will discuss later, e.g. that of reference [39], where a similar generalization is proposed up to order 4 with distributed natural frequencies. Second, the normalization factors $\langle K^{(d)} \rangle$ at each order $d$ ensure that the total contribution of each order $d$ can be compared across orders. In other words, even if a given simplicial complex contains many more 2-simplices than 1-simplices, the contribution of orders 1 and 2 can be compared and will be of equal relative strength if $\gamma_1 = \gamma_2$.

Finally, it is sometimes convenient to look at the interactions at each order $d$ separately, rather than at the full simplicial complex (1). This can be achieved by turning on one order of interaction while all others are turned off. For example, a classical network with pairwise interaction corresponds to the pure 1-simplex version of the full simplicial complex, i.e. $\gamma_d = 0$ for $d \neq 1$. Similarly, the pure 2-simplex case is that with only 3-oscillator interactions, i.e. $\gamma_2 = 0$ but $\gamma_d = 0$ for $d \neq 2$. In general, we have a pure $d^*$-simplex when $\gamma_{d^*} \neq 0$ but $\gamma_d = 0$ for $d \neq d^*$. All these cases are graphically represented in Fig. (1)(c). We will use such a decomposition in the rest of the paper.

For the introduced model (1), existence of the fully synchronized state, $\theta_i = \theta_j$ for all $i$ and $j$, is trivially guaranteed and implies the solution $\psi_i(t) = \theta_0 t$. Here, we are interested in the stability of that fully synchronized state. For convenience, we start by going to the rotating reference frame $\psi_i = \theta_0 - \omega t$. This is equivalent to applying the transformation $\theta_i \mapsto \psi_i$ and $\omega \mapsto 0$ to the original system (1). In this new reference frame, the synchronized solution is given by $\psi_i(t) = 0$ for all $i = 1, \ldots, N$.

The linear stability of the synchronized state is determined by the dynamics of heterogeneous perturbations $\delta \psi_i(t)$, which satisfy the linearized dynamics

$$
\delta \dot{\psi}_i = + \frac{\gamma_1}{\langle K^{(1)} \rangle} \sum_{j=1}^{N} A_{ij} (\delta \psi_j - \delta \psi_i) + \frac{\gamma_2}{2! \langle K^{(2)} \rangle} \sum_{j,k=1}^{N} A_{ij} A_{jk} A_{ki} (\delta \psi_j + \delta \psi_k - 2 \delta \psi_i) + \frac{\gamma_3}{3! \langle K^{(3)} \rangle} \sum_{j,k,l=1}^{N} [(A_{ij} A_{jk} A_{ki}) (A_{il} A_{lj} A_{lk})] (\delta \psi_j + \delta \psi_k + \delta \psi_l - 3 \delta \psi_i) + \ldots
$$

For networks with pairwise interactions only, the dynamics of those perturbations – and hence the stability of the system – is typically assessed by using the so-called Laplacian formalism. In the next section, we see how we can characterize the stability of system (1) with higher-order interactions up to any order $d$ by extending the tra-
III. MULTI-ORDER LAPLACIAN FRAMEWORK

Different generalizations of the Laplacian operator have been proposed so far in the literature to include higher orders of interactions: from the simplest versions for uniform hypergraphs [42, 43], to those more complicated associated to simplicial complexes [44–46] and Hodge Laplacians [24, 47], to mention a few. Let us notice that these Laplacians describe the hierarchy among building blocks of the topology, and different orders are associated to Laplacian matrices of different sizes, where the order zero is the traditional node point of view; the first order represents the edge perspective where the Laplacian size is equal to the number of pairwise connections and each entry is associated to edge adjacency; the second order Laplacian has a different size again, being based on the existing triangles, and so on.

Here, we propose a multi-order Laplacian: a new operator which generalizes the typical pairwise Laplacian framework and allows us to analytically describe the effect of higher-order couplings on node oscillatory dynamics for any simplicial interactions. This is different from the previously defined Laplacians, where interactions between, e.g. triangles, are seen from the point of view of triangles and not from the point of view of the nodes in those triangles.

First, we show that each $d$-simplex interaction term in Eq. (2) can be written in terms of a generalized Laplacian of order $d$. Second, we display how the full system (2) can be written in terms of a multi-order Laplacian.

A. Laplacian

Pure 1-simplex interactions. We start by recalling the classical Laplacian formalism for pairwise interactions, i.e. 1-simplex interactions. We do so by switching
off all higher-order interactions in system (1): \( \gamma_d = 0 \) for \( d \geq 2 \). The linearized system (2) is thus

\[
\delta \dot{\psi}_i = + \frac{\gamma_1}{\langle K(1) \rangle} \sum_{j=1}^{N} A_{ij} (\delta \psi_j - \delta \psi_i),
\]

(3)

The degree \( K^{(1)}_i \) (of order 1) of oscillator \( i \), i.e. the number of pairwise (1-simplex) interactions it has with other oscillators, and the Laplacian matrix \( L^{(1)} \) (of order 1) are typically defined as

\[
K^{(1)}_i = \sum_{j=1}^{N} A_{ij},
\]

(4)

\[
L^{(1)}_{ij} = K^{(1)}_i \delta_{ij} - A^{(1)}_{ij},
\]

(5)

where \( \delta_{ij} \) denotes the Kronecker delta, and where we write \( A^{(1)}_{ij} \) and \( A_{ij} \) interchangeably to refer to the (first order) adjacency matrix to avoid unnecessarily heavy notation. For example, in Fig. 1(a), the red node has a degree of order 1 equal to 7. With those definitions in hand, one can rewrite Eq. (3) as

\[
\delta \dot{\psi}_i = - \frac{\gamma_1}{\langle K(1) \rangle} \sum_{j=1}^{N} L^{(1)}_{ij} \delta \psi_j,
\]

(6)

of which the dynamics is determined by the eigenvalues of \( L^{(1)} \), as we shall see in the next section. By definition the Laplacian is a positive semidefinite matrix, thus characterized by a non-negative spectrum. For undirected networks the eigenvalues are also real, given the symmetry of \( L^{(1)} \). Moreover, its rows sum up to zero, implying that zero is always an eigenvalue, corresponding to an eigenvector with all identical entries.

In the following we show how the higher-order interactions can all be cast into a similar Laplacian framework.

**Pure 2-simplex interactions.** We now re-write 3-oscillator interactions, i.e. those interactions represented by filled orange triangles in Fig. 1, to fit our Laplacian framework. We switch off all interactions of other orders, \( \gamma_d = 0 \) for all \( d \neq 2 \) in Eq. (2), and re-write

\[
\delta \dot{\psi}_i = \frac{\gamma_2}{2\langle K(2) \rangle} \sum_{j,k=1}^{N} A_{ik} A_{jk} (\delta \psi_j + \delta \psi_k - 2\delta \psi_i),
\]

(7)

\[
= \frac{\gamma_2}{2\langle K(2) \rangle} \sum_{j,k=1}^{N} A_{ik} A_{jk} \delta \psi_j + \delta \psi_k - \delta \psi_i
\]

\[
+ \frac{\gamma_2}{2\langle K(2) \rangle} \sum_{j,k=1}^{N} A_{ij} A_{jk} \delta \psi_j - \delta \psi_i,
\]

(8)

\[
= \frac{\gamma_2}{\langle K(2) \rangle} \sum_{j,k=1}^{N} A_{ij} A_{jk} \delta \psi_j - \delta \psi_i,
\]

(9)

by using the symmetry of (7) with respect to indices \( j \) and \( k \). Note that this symmetry is needed both in the adjacency matrix \( A_{ij} \) and the coupling function. Notably, this symmetry allows us to write the 2-simplex interaction of 3 phases as 2 identical terms of the difference of only 2 phases. The phase difference in expression (9) is identical to that in (3), and this allows us to naturally generalize the Laplacian formalism to 2-simplex interactions.

Indeed, we start by generalizing two important 1-simplex quantities to 2-simplex interactions. First, we write the 2-degree \( K^{(2)}_i \) of node \( i \), i.e. the number of distinct 2-simplices node \( i \) is part of, as

\[
K^{(2)}_i = \frac{1}{2!} \sum_{j,k=1}^{N} A_{ij} A_{jk} A_{ki},
\]

(10)

where the factor \( 2! \) ensures each 2-simplex is counted only once. For example, in Fig. 1(a), the red node has a degree of order 2 of \( K^{(2)}_i = 4 \).

Finally, we define the adjacency matrix of order 2, whose entries \( A^{(2)}_{ij} \) represent the number of 2-simplices shared by the pair \((i,j)\), as

\[
A^{(2)}_{ij} = A_{ij} \sum_{k=1}^{N} A_{jk} A_{ki},
\]

(11)

which is a natural generalization of the usual pairwise adjacency matrix \( A \). Indeed, just as \( A_{ij} = 1 \) if \( i \) and \( j \) are part of a common 1-simplex interaction but 0 otherwise, \( A^{(2)}_{ij} = n \) if \( i \) and \( j \) are part of \( n \) common (but distinct) 2-simplex interactions, but 0 otherwise.

With these definitions in hand, we can re-write the 2-simplex interaction term (9) as follows

\[
\delta \dot{\psi}_i = \frac{\gamma_2}{\langle K(2) \rangle} \sum_{j,k=1}^{N} A_{ij} A_{jk} A_{ki} (\delta \psi_j - \delta \psi_i)
\]

(12)

\[
= \gamma_2 \frac{\langle K(2) \rangle}{\langle K(2) \rangle} \left[ \sum_{j=1}^{N} A^{(2)}_{ij} \delta \psi_j - \delta \psi_i 2! K^{(2)}_i \right],
\]

(13)

\[
= \gamma_2 \frac{\langle K(2) \rangle}{\langle K(2) \rangle} \sum_{j=1}^{N} \left[ A^{(2)}_{ij} - 2 K^{(2)}_i \delta_{ij} \right] \delta \psi_j,
\]

(14)

\[
= - \gamma_2 \frac{\langle K(2) \rangle}{\langle K(2) \rangle} \sum_{j=1}^{N} L^{(2)}_{ij} \delta \psi_j.
\]

(15)

where we obtained the last line by defining the Laplacian of order 2

\[
L^{(2)}_{ij} = 2 K^{(2)}_i \delta_{ij} - A^{(2)}_{ij},
\]

(16)

as a natural generalization of the usual Laplacian of order 1 (5). This newly defined Laplacian of order 2 can be shown to have the expected properties of a Laplacian matrix: it is symmetric, and its rows sum to zero. Moreover, its eigenvalues are all non-negative, as we shall see in the next section. With Eq. (15), we have explicitly shown the link between the structure of the 2-simplex interactions
and the dynamics of the oscillators, by casting it into a Laplacian form.

**Pure 3-simplex interactions.** Now, we briefly show how to rewrite the 3-simplex interactions in a similar way, i.e. those interactions represented by filled green tetrahedra in Fig. 1. Although it is very similar to the 2-simplex case, the case of 3-simplices better prepares us for the next step: the general \(d\)-simplex case. We switch off all interactions of other orders, \(\gamma_{d'} = 0\) for all \(d' \neq 3\) in Eq. (2), and rewrite the 3-simplex interactions the differences of two phases

\[
\delta \psi_i = \frac{\gamma_3}{3! \langle K^{(3)} \rangle} \left( \sum_{j,k,l=1}^{N} (A_{ij} A_{jk} A_{kl}) (A_{il} A_{lj} A_{kl}) (\delta \psi_j + \delta \psi_k + \delta \psi_l - 3 \delta \psi_i) \right),
\]

(17)

where the factor \(2! = 3! / 3\) comes from the intermediate step where the expression is written as the sum of 3 identical terms.

Similarly to the 2-simplex case, we now define the \(K_i^{(3)}\) of order 3, adjacency matrix \(A_{ij}^{(3)}\) of order 3, and the Laplacian \(L_{ij}^{(3)}\) of order 3

\[
K_i^{(3)} = \frac{1}{3!} \sum_{j,k,l=1}^{N} (A_{ij} A_{jk} A_{kl}) (A_{il} A_{lj} A_{kl}) A_{ik},
\]

(19)

\[
A_{ij}^{(3)} = \frac{1}{2!} \sum_{k,l=1}^{N} (A_{ij} A_{kl}) (A_{ij} A_{kl}) A_{ik},
\]

(20)

\[
L_{ij}^{(3)} = 3K_i^{(3)} \delta_{ij} - A_{ij}^{(3)}.
\]

(21)

The degree of order 3, \(K_i^{(3)}\), of node \(i\) is the number of distinct 3-simplex interactions it is part of, and \(A_{ij}^{(3)}\) is the number of shared 3-simplices including nodes \(i\) and \(j\). With these definitions, we can now rewrite the 3-simplex term (18) as

\[
\delta \psi_i = \frac{\gamma_3}{2! \langle K^{(3)} \rangle} \sum_{j,k,l=1}^{N} (A_{ij} A_{jk} A_{kl}) (A_{il} A_{lj} A_{kl}) (\delta \psi_j - \delta \psi_i)
\]

(22)

\[
= \frac{\gamma_3}{2! \langle K^{(3)} \rangle} \sum_{j=1}^{N} 2! A_{ij}^{(3)} \delta \psi_j - \delta \psi_i 3! K_i^{(3)}
\]

(23)

\[
= \frac{\gamma_3}{\langle K^{(3)} \rangle} \sum_{j=1}^{N} \left[ A_{ij}^{(3)} - 3K_i^{(3)} \delta_{ij} \right] \delta \psi_j,
\]

(24)

\[
= -\frac{\gamma_3}{\langle K^{(3)} \rangle} \sum_{j=1}^{N} L_{ij}^{(3)} \delta \psi_j.
\]

(25)

This shows that the dynamics of the 3-simplex interactions, i.e. of 4 oscillators, can be rewritten in terms of a Laplacian of order 3. Indeed, once again such an operator fulfills the requirements to be a Laplacian, being symmetric, positive semi-definite and zero row-sum.

**Pure \(d\)-simplex interactions.** We are now ready to show that our generalized Laplacian formalism can be used to describe higher-order interactions of any order \(d\). We switch off all interactions of other orders, \(\gamma_{d'} = 0\) for all \(d' \neq d\) in Eq. (2), and rewrite the \(d\)-simplex interactions as the differences of two phases

\[
\delta \psi_i = \frac{\gamma_d}{d! \langle K^{(d)} \rangle} \sum_{\{j_2, \ldots, j_{d+1}\}=1}^{N} \prod_{m,n=1}^{d+1} A_{k_m k_n} \left( \sum_{j_m=2}^{d+1} \delta \psi_{j_m} - d \delta \psi_i \right),
\]

(26)

\[
\delta \psi_i = \frac{\gamma_d}{d! \langle K^{(d)} \rangle} \sum_{\{j_2, \ldots, j_{d+1}\}=1}^{N} \prod_{m,n=1}^{d+1} A_{j_m j_n} (\delta \psi_j - \delta \psi_i).
\]

(27)

Similarly to the previous lower orders, we now define at order \(d\) the degree, \(K_i^{(d)}\), the adjacency matrix \(A_{ij}^{(d)}\), and the Laplacian \(L_{ij}^{(d)}\)

\[
K_i^{(d)} = \frac{1}{d!} \sum_{\{j_2, \ldots, j_{d+1}\}=1}^{N} \prod_{m,n=1}^{d+1} A_{j_m j_n},
\]

(28)

\[
A_{ij}^{(d)} = \frac{1}{(d-1)!} \sum_{\{j_2, \ldots, j_{d+1}\}=1}^{N} \prod_{m,n=1}^{d+1} A_{j_m j_n},
\]

(29)

\[
L_{ij}^{(d)} = dK_i^{(d)} \delta_{ij} - A_{ij}^{(d)}.
\]

(30)

With these definitions, similarly as before, we can rewrite
the $d$-simplex interaction term
\[
\delta \psi_i = \frac{\gamma_d d}{d!(K(d))} \sum_{\{j_2, \ldots, j_{d+1}\}=1}^N \prod_{m,n=1}^{d+1} A_{j_mj_n} (\delta \psi_j - \delta \psi_i)
\]
\[= \frac{\gamma_d}{(d-1)! (K(d))} \sum_{j=1}^N 2d A_{ij} \delta \psi_j - \delta \psi_i d! K_i^d(1)
\]
\[= \frac{\gamma_d}{(K(d))} \sum_{j=1}^N [A_{ij}^d - d K_i^d \delta \psi_j] \delta \psi_j,
\]
\[= - \frac{\gamma_d}{(K(d))} \sum_{j=1}^N L_{ij}^{(d)} \delta \psi_j.
\]
This shows that, at any order $d$, the dynamics caused by interactions of order $d$ ($d + 1$ oscillators) is determined by the matrix $L^{(d)}$. Such a matrix fulfills the properties necessary to be a Laplacian so that its eigenvalues are non-negative and include at least one zero. The definitions of the degree, adjacency matrix, and Laplacian matrix of order $d$ are natural generalizations of their usual pairwise counterparts. Indeed, $K_i^{(d)}$ is the number of $d$-simplex interactions node $i$ is part of, the adjacency matrix $A_{ij}^{(d)} = n$ where $n$ is the number of distinct $d$-simplex interactions shared by nodes $i$ and $j$. Finally, $L_{ij}^{(d)}$ is defined in terms of $A_{ij}^{(d)}$ and $K_i^{(d)}$ identically to how $L_{ij}^{(1)}$ is defined in terms of $A_{ij}^{(1)}$ and $K_i^{(1)}$, up to factor $d$. Note that the usual pairwise definitions are recovered for $d = 1$.

Now that we can cast each order $d$ into this Laplacian framework, we show that the full system (1) can also be rewritten with a multi-order Laplacian matrix.

**Multi-order interactions.** We go back to our original system (1), with all interactions at all orders switched on, $\gamma_d \neq 0$ for all $d$. We know that the stability of the synchronized solution of system (1) is determined by system (2), which we now can write
\[
\dot{\delta \psi_i} = - \sum_{j=1}^N \gamma_d L_{ij}^{(M)} \delta \psi_j,
\]
where we have defined
\[
L_{ij}^{(M)} = \sum_{d=1}^{N-1} \frac{\gamma_d}{(K(d))} L_{ij}^{(d)}
\]
the multi-order Laplacian $L_{ij}^{(M)}$ as a weighted sum of the Laplacian matrices of order $d$. Hence, by definition, the multi-order Laplacian gives an equal weight to each order, even if the networks contains more, say, 2-simplices than 5-simplices. This is different from the Laplacians of order $d$ which are not normalised in this way.

It is sometimes convenient to consider interactions of oscillators up to a given order $d_{max}$, i.e. $\gamma_d = 0$ for all $d > d_{max}$. For this purpose, we introduce the notation
\[
L_{ij}^{(-d_{max})} = \sum_{d=1}^{d_{max}} \frac{\gamma_d}{(K(d))} L_{ij}^{(d)}
\]
where the maximum value of $d_{max}$ is $N - 1$, which corresponds to an $N$-oscillator interaction. Note that we consider "maximal" simplicial complexes following the scheme studied in [39] up to order 3.

In this section, our generalized framework showed us two things. First, how to rewrite interactions at each order with a Laplacian matrix $L_{ij}^{(d)}$ of order $d$. And second, how to rewrite the full system, including all higher-order interactions, with a multi-order Laplacian matrix $L_{ij}^{(M)}$. Additionally we showed how the latter matrix is just a weighted sum of the former matrices.

So far, we have an analytical expression for the Laplacian matrix of a given simplicial complex. It is the eigenvalues of this Laplacian that quantify the stability or instability of the synchronized state of the system. However, even though the multi-order Laplacian is the weighted sum of each $d$-Laplacian, its eigenvalues cannot be obtained in general by summing the eigenvalues of the Laplacians of order $d$. In general, we need to numerically compute the eigenvalues of the Laplacian. While this is generally true, special cases exist where the eigenvalues can nonetheless be summed and the system characterised in a fully analytical manner. We present such case in Sec. IV.

**B. Stability and Lyapunov exponents**

We are finally able to study the stability of our system of oscillators. The synchronized state is stable if the perturbation $\delta \psi_i$ on each node $i$ converges to zero.

Let us first consider pure $d$-simplex interactions. To this scope, we need to solve Eq. (34) to obtain the temporal evolution of the perturbation. To do so we can make use of the Laplacian eigenvalues $\Lambda^{(d)}(\alpha)$ and eigenvectors $\phi^{(d)}(\alpha)$ defined by $\sum_{j=1}^{N} \Lambda^{(1)}(\alpha) L_{ij}^{(1)} \phi^{(d)}(\alpha) = \Lambda^{(d)}(\alpha) \phi^{(d)}(\alpha)$, with $\alpha = 1, \ldots, N$. Indeed, this eigendecomposition can be used to project the perturbation vector $\delta \psi_i(t) = \sum_{\alpha=1}^{N} c_\alpha(t) \phi^{(d)}(\alpha)$, where the $c_\alpha$ are real constants. By plugging this solution into system (34), we can decouple our system of $N$ equations and obtain the $N$ Lyapunov exponents of the synchronized state
\[
\lambda^{(d)}(\alpha) = -\frac{\gamma_d}{(K(d))} \Lambda^{(d)}(\alpha).
\]

The Lyapunov exponents are a measure a stability: the system with pure $d$-simplex interactions is stable if all values $\lambda^{(d)}(\alpha)$ are negative, so that the perturbations $\delta \psi_i$ tend to zero over time. By convention, the Lyapunov
exponents are ordered: $\lambda_1^{(d)} \geq \lambda_2^{(d)} \geq \ldots \lambda_N^{(d)}$. The Laplacian eigenvalues are non-negative by definition so that if all $\gamma_d > 0$ (attractive coupling), the synchronized state is always stable up to a global phase shift: $0 = \lambda_1^{(d)} > \lambda_2^{(d)} \geq \ldots \lambda_N^{(d)}$.

In the multi-order system (35) the stability of the synchronized equilibrium is determined by the interplay of all different orders, as encoded in the multi-order Laplacian. We can then analogously use the spectrum and eigenbasis of $L^{(M)}$, hence obtaining the $N$ Lyapunov exponents

$$\lambda_{\alpha}^{(M)} = -\Lambda_{\alpha}^{(M)}.$$ (39)

The Laplacian eigenvalues are non-negative by definition so that if all $\gamma_d > 0$ (attractive coupling), the synchronized state is always stable. The Lyapunov exponent that determines the long-term behaviour the second Lyapunov exponent, $\lambda_2^{(M)}$, i.e. the smallest non-zero one. Its value determines the resilience of the system to perturbations, i.e. how fast the system comes back to the stable state after a perturbation. In particular, the more negative the $\lambda_2^{(M)}$, the more stable the synchronized state.

In the spirit of Eq. (37), we also define the Lyapunov exponents of the system with interactions up to order $d_{\text{max}}$ in terms of the eigenvalues $\Lambda_{\alpha}^{-d_{\text{max}}}^{(-d_{\text{max}})}$ of the Laplacian $L_{ij}^{(-d_{\text{max}})}$

$$\lambda_{\alpha}^{(-d_{\text{max}})} = -\Lambda_{\alpha}^{(-d_{\text{max}})}.$$ (40)

### IV. STABILITY IN HOMOGENEOUS HIGHER-ORDER NETWORKS

In this section we analyse a simple case – dubbed *higher-order all-to-all* – which can be solved analytically. This setting is a generalization of the usual “all-to-all” (or global) coupling scheme in traditional networks. Indeed, in networks with pairwise interactions, all-to-all coupling indicates that every possible pairwise interaction takes place. Similarly, in a network with higher-order interactions, higher-order all-to-all indicates that every possible $(d + 1)$-oscillator interaction occurs, for all orders $d$.

In the higher-order all-to-all setting, system (1) reads

$$\dot{\psi}_i = + \frac{\gamma_1}{\langle K^{(1)} \rangle} \sum_{j=1}^{N} \sin(\psi_j - \psi_i)$$

$$+ \frac{\gamma_2}{2! \langle K^{(2)} \rangle} \sum_{j,k=1}^{N} \sin(\psi_j + \psi_k - 2\psi_i)$$

$$+ \frac{\gamma_3}{3! \langle K^{(3)} \rangle} \sum_{j,k,l=1}^{N} \sin(\psi_j + \psi_k + \psi_l - 3\psi_i)$$

$$+ \ldots$$

$$+ \frac{\gamma_d}{d! \langle K^{(d)} \rangle} \sum_{\{j_2, \ldots, j_{d+1}\} = 1}^{N} \sin \left( \sum_{m=2}^{d+1} \psi_{j_m} - d \psi_i \right),$$ (41)

and the linearised system reads

$$\dot{\delta \psi}_i = + \frac{\gamma_1}{\langle K^{(1)} \rangle} \sum_{j=1}^{N} (\delta \psi_j - \delta \psi_i)$$

$$+ \frac{\gamma_2}{2! \langle K^{(2)} \rangle} \sum_{j,k=1}^{N} (\delta \psi_j + \delta \psi_k - 2\delta \psi_i)$$

$$+ \frac{\gamma_3}{3! \langle K^{(3)} \rangle} \sum_{j,k,l=1}^{N} (\delta \psi_j + \delta \psi_k + \delta \psi_l - 3\delta \psi_i)$$

$$+ \ldots$$

$$+ \frac{\gamma_d}{d! \langle K^{(d)} \rangle} \sum_{\{j_2, \ldots, j_{d+1}\} = 1}^{N} \left( \sum_{m=2}^{d+1} \delta \psi_{j_m} - d \delta \psi_i \right).$$ (42)

Now that we have defined the coupling scheme, we can explicitly write the multi-order Laplacian.

#### A. Higher-order degree and adjacency matrices

In this section, we explicitly write the higher-order adjacency matrix and connectivity defined in the above section, for each order $d$, in the higher-order all-to-all setting.

First, we start with the adjacency matrices of order $d$. In a traditional all-to-all setting, the pairwise adjacency matrix has all entries equal to one, but 0 on the diagonal, which can be written $A_{ij}^{(1)} = 1 - \delta_{ij}$. At order $d$, the generic entry $(i,j)$ of the adjacency matrix $A_{ij}^{(d)}$ is equal to the number of distinct $d + 1$-oscillator interactions including both oscillators $i$ and $j$, as described above. Hence, the matrix is simply given by the number of ways to pick $d - 1$ oscillators among the $N - 1$ oscillators left. This number is given by the combinatorics formula $A_{ij}^{(d)} = \binom{N-2}{d-1} (1 - \delta_{ij})$, where $(1 - \delta_{ij})$ ensures that entries with $i = j$ are equal to zero.

Second, we proceed similarly to write explicitly the degree of order $d$. In a usual all-to-all setting with only
pairwise interactions, every oscillators has a pairwise interaction with all \(N-1\) oscillators left, i.e. \(K^{(1)}_{ij} = N-1\). The degree of order \(d\) is equal to the number of \(d+1\) oscillator interactions that oscillator \(i\) is part of. Hence, it is given by the number of ways to pick \(d\) oscillators out of the \(N-1\) left, which can be written \(K^{(d)}_{ij} = \binom{N-1}{d}\) in combinatorics notation.

For the sake of clarity, here are the expanded expressions for the degree of the higher-order all-to-all setting at orders 1, 2, 3, and \(d\)

\[
K^{(1)}_i = N - 1, \quad K^{(2)}_i = (N - 1)(N - 2)/2!, \quad K^{(3)}_i = (N - 1)(N - 2)(N - 3)/3!, \quad \vdots \quad K^{(d)}_i = (N - 1) \cdots (N - d)/d!,
\]

and similarly for the adjacency matrix at orders 1, 2, 3, and \(d\)

\[
A^{(1)}_{ij} = 1 - \delta_{ij}, \quad A^{(2)}_{ij} = (1 - \delta_{ij})(N - 2), \quad A^{(3)}_{ij} = (1 - \delta_{ij})(N - 2)(N - 3)/2!, \quad \vdots \quad A^{(d)}_{ij} = (1 - \delta_{ij})(N - 2) \cdots (N - d)/(d - 1)!
\]

Additionally, the following identities will prove useful in the next section, to relate the quantities at order \(d\) to their traditional counterparts (at order 1)

\[
\begin{align*}
A^{(d)}_{ij} &= [(N - 2) \cdots (N - d)/(d - 1)!] A^{(1)}_{ij}, \\
K^{(d)}_i &= [(N - 2) \cdots (N - d)/d!] K^{(1)}_i.
\end{align*}
\]

Now that we have these expressions, we have everything in hand to write an explicit formula for the multi-order Laplacian \((30)\) in the next section. In the higher-order all-to-all setting, all nodes have the same degree of order \(d\). Hence, in this section, we write \(K^{(d)}_{ij}\) and \(K^{(d)}\) interchangeably.

**B. Higher-order Laplacians are proportional to the traditional pairwise Laplacian**

We now show that at each order \(d\), the higher-order all-to-all Laplacian \(L^{(d)}\) is proportional to the usual pairwise all-to-all Laplacian \(L^{(1)}\). Consequently, since the analytical spectrum of the latter is known, the analytical spectrum of \(L^{(d)}\) can also be obtained, and in turn, that of \(L^{(M)}\).

**Pure 1-simplex interactions.** We start with the usual Laplacian describing pairwise oscillator interactions. Injecting expression \(48\) into its definition \(5\) yields

\[
L^{(1)}_{ij} = K^{(1)}_{ij} \delta_{ij} - A^{(1)}_{ij},
\]

\[
= N \delta_{ij} - 1.
\]

We now go straight to the general case of order \(d\).

**Pure \(d\)-simplex interactions.** For the general case of order \(d\), analogously injecting expressions \(53\) into definition \(30\) yields

\[
L^{(d)}_{ij} = d K^{(d)}_{ij} \delta_{ij} - A^{(d)}_{ij},
\]

\[
= [(N - 2) \cdots (N - d)/(d - 1)!] [K^{(1)}_{ij} \delta_{ij} - A^{(1)}_{ij}],
\]

\[
= [(N - 2) \cdots (N - d)/(d - 1)!] L^{(1)}_{ij},
\]

which can be rewritten

\[
L^{(d)}_{ij} = d \frac{K^{(d)}}{N - 1} L^{(1)}_{ij}.
\]

Hence, we have shown that the Laplacian of order \(d\) is proportional to the usual pairwise Laplacian, for any order \(d\). In addition, Eq. \(60\) indicates that \(L^{(d)}_{ij}\) is linearly growing with the order of the interactions \(d\), and with \(K^{(d)}\), the number of \(d\)-simplex interactions each oscillator has.

We will discuss the implications that these dependencies have on the stability in more detail in Sec. IV C. For now, we only stress that the analytical formula \(60\) serves as a limit case to understand the behaviour of \(L^{(d)}\) in more complex coupling schemes than the higher-order all-to-all scheme. Examples of those will be investigated in Sec. V, in which a full analytical derivation is not always possible.

**Multi-order interactions.** Oscillators in the higher-order network have multi-oscillator interactions with the other oscillators at all orders \(d\). In general, the stability of the synchronized solution is determined by the eigenvalues of the multi-order Laplacian \((36)\) \(L^{(M)}\), as we have shown in Sec. III. In the higher-order all-to-all setting \((41)\), by injecting Eq. \(60\), this Laplacian reduces to

\[
L^{(M)}_{ij} = \left( \sum_{d=1}^{N-1} \frac{\gamma_d d}{N - 1} \right) L^{(1)}_{ij},
\]

which, for convenience, we write as follows when only considering orders up to \(d_{\text{max}}\)

\[
L^{(-d_{\text{max}})}_{ij} = \left( \sum_{d=1}^{d_{\text{max}}} \frac{\gamma_d d}{N - 1} \right) L^{(1)}_{ij}.
\]

We remind that the maximum value \(d_{\text{max}}\) can take is \(N - 1\), corresponding to all \(N\) oscillators having an interaction.

Hence, the stability of the synchronized solution is only determined by the usual pairwise Laplacian, as well as
order of those interactions. We give the full analytical expressions for the Lyapunov exponents of each interaction of order \( d \), for different interaction orders. We study the case where attractive and repulsive interactions are associated to different interaction orders.

In this section, we give an analytical formula for the eigenvalues of the higher-order all-to-all Laplacian matrix, and consequently for the Lyapunov exponents which determine the stability of the synchronized solution. We remind that the attractiveness or repulsiveness of interaction at a given order \( d \) is determined by the sign of the coupling strength \( \gamma_d \): positive and negative coupling strengths correspond to attractive and repulsive interactions, respectively. We consider different scenarios, where we tune at will the sign and intensity of the coupling strengths \( \gamma_d \).

**Attracting couplings at all orders.** We have shown in the previous section that the \( d \)-order Laplacians are intrinsically connected to the usual pairwise Laplacian \( L^{(1)} \). Therefore, this last shapes the Laplacian spectrum at each order \( d \), and consequently also at the multi-order, determining the stability of \( (41) \). For higher-order all-to-all networks the spectrum of \( L^{(1)} \) is simply given by

\[
\Lambda_1^{(1)} = 0 \quad \Lambda_2^{(1),\ldots,N} = N
\]

from which we derive the Lyapunov exponents of each order \( d \):

\[
\lambda_1^{(d)} = 0 \quad \lambda_2^{(d),\ldots,N} = -\gamma_d d \frac{N}{N-1}.
\]

The second Lyapunov exponent is reported in Fig. 2(c) for different values of \( d \) as a function of the number of nodes \( N \). It appears clear that interactions of higher orders \( d \) stabilise the synchronized state more (more negative \( \lambda_2^{(d)} \) as \( d \) increases). Interestingly, this is true despite each order being given an equal weight through the normalisation in system \( (41) \). This is illustrated with numerical simulations for the pure orders \( d = 1 \) and \( d = 2 \), respectively shown in Figs. 2(a) and (b). Indeed, trajectories converge faster in with pure 2-simplex interactions than with pure pairwise ones.

Then, combining all orders, we obtain the multi-order Lyapunov exponents

\[
\lambda_1^{(M)} = 0 \quad \lambda_2^{(M),\ldots,N} = -\frac{N}{N-1} \sum_{d=1}^{d_{\max}} \gamma_d d.
\]

and its counterpart for orders up to \( d_{\max} \)

\[
\lambda_1^{(-d_{\max})} = 0 \quad \lambda_2^{(-d_{\max}),\ldots,N} = -\frac{N}{N-1} \sum_{d=1}^{d_{\max}} \gamma_d d.
\]

From this formula, we see that the more orders are taken into account, i.e. \( d_{\max} \) is increased, the more negative \( \lambda_2^{(-d_{\max})} \) is, as shown in Fig. 2(d). Physically, additional attractive higher-order interactions tend to stabilise the synchronized state.

**Attractive even orders and repulsive odd orders.** Here, we apply our analytical framework to investigate the interplay between attractive and repulsive interactions. For simplicity, we study the case where attractive and repulsive relationship are associated to different interaction orders.
First, we restrict ourselves to only 1-simplex (2-oscillator) and repulsive 2-simplex (3-oscillator). Depending on the sign of $\gamma_1$, 1-simplex interactions are attractive ($\gamma_1 > 0$) or repulsive ($\gamma_1 < 0$). The attractiveness or repulsiveness of 2-simplex interactions depends identically on the sign of $\gamma_2$. Physically, attractive interactions favour synchronization by stabilising the synchronized state. By contrast, repulsive interactions will favour incoherence by destabilising the synchronized state. The result of this interplay between the interactions at various orders depends on the respective coupling strengths $\gamma_d$: $\lambda_2^{(2)} = -\frac{N}{N-1}(\gamma_1 + 2\gamma_2)$ which is negative if $\gamma_1 + 2\gamma_2 > 0$. The value of $\lambda_2^{(2)}$ is shown in Fig. 3(c) for a range of positive and negative values of $\gamma_1$ and $\gamma_2$. We see that synchronization can be stable even if the traditional pairwise (1-simplex) interactions are repulsive, as long as 3-oscillator (2-simplex) interactions are attractive enough to counterbalance them, as illustrated in Fig. 3(a). This highlights a potential benefit of considering higher-order interactions: indeed they might stabilise the systems, in cases when the purely pairwise system is unstable. This confirms a similar result obtained in [38] for phases oscillators with distributed frequencies. In addition, attractive pairwise interactions might be outplayed by repulsive 3-oscillators ones. This can result in an unstable synchronized state, as illustrated in Fig. 3(b).

Second, we generalize to all orders up to $d_{\text{max}}$ with alternating signs. Specifically, we consider all interactions of an even and odd order $d$ to be attractive and repulsive, respectively, by setting

$$\gamma_{2n} = -1 \quad \quad \gamma_{2n+1} = +1.$$  

Now, the Lyapunov exponent is given by the alternating series

$$\lambda_2^{(-d_{\text{max}})} = -\frac{N}{N-1} \sum_{d=1}^{d_{\text{max}}} d (-1)^{d+1},$$

which diverges as $d_{\text{max}} \to \infty$ (which requires $N \to \infty$). For increasing but finite values of $d_{\text{max}}$, however, the Lyapunov exponent alternates between positive and negative values, as illustrated in Fig. 3(d). In other words, if the highest order of interaction considered is odd, the synchronized state is stable. However, if it is even, the synchronized state is unstable. This is due to the fact that the contribution to $\lambda_2^{(-d_{\text{max}})}$ of each order $d$ is proportional to $d$: adding one repulsive or attractive order outplays all lower-order interactions. Finally, we note that the factor $N/(N-1) \to 1$ in the limit of large networks $N \to \infty$, so that it can be seen as a finite size correction factor.

**Weaker higher orders: link to phase reduction.**

Here, we make a brief link between our formalism and the higher-order phase reductions approaches developed for example in [32–34, 36]. In these phase reductions studies, the authors obtain a phase model from an original network of nonlinear oscillators, by performing a sophisticated perturbative expansion in a small parameter. This small parameter is usually linked to the original pairwise coupling strength $\gamma_1$. The authors find that, at higher orders in the expansion, i.e. at higher powers of $\gamma_1$, there appear terms including higher-order interactions, i.e. interactions between more than 2 phases.

Motivated by these studies, we consider a scenario of decaying coupling strengths. Specifically, we set $0 \leq \gamma_1 < \infty$.
1, and couplings at higher orders as powers of the pairwise coupling strength

$$\gamma_d = \gamma_1^d,$$

which means that interactions between many oscillators are weak. In this case, the stability is given by the series

$$\lambda_2(\to{d_{\max}}) = - \frac{N}{N-1} \sum_{d=1}^{d_{\max}} d \gamma_1^d < 0. \quad (70)$$

which is always negative, and hence stability of the synchronized state is ensured. In addition, this is a geometric series which converges to $-\gamma_1/(\gamma_1 - 1)^2$ when $d_{\max}$ (and hence $N$) tend to infinity. This convergence is illustrated in Fig. 3(e). Even though the qualitative result, i.e. stability, was expected since interactions at all orders are attractive, such a quantitative result has more predictive power. In particular, we note that $-\gamma_1/(\gamma_1 - 1)^2 \to -\infty$ as we go away from the domain of validity of the perturbative regime $\gamma_1 \to 1$.

V. STABILITY IN HETEROGENEOUS HIGHER-ORDER NETWORKS

In this section, we apply our multi-order Laplacian framework to simplicial complexes with complex heterogeneous structures. We consider two cases: first, a toy model—the star-clique model—and second, a real network, i.e. a macaque brain dataset.

A. The star-clique model

As first example of complex topology with higher-order interactions, we consider a toy model that we call star-clique [26]. We make use of this model in order to observe in a simple case the multi-order Laplacian properties and thus the stability conditions for synchronization. The star-clique, as its name indicates, is composed of two subnetworks: a star of size $N_s$ and a clique of size $N_c$, which we treat analogously to a higher-order all-to-all network of size $N_c$ from the previous section. The subnetworks are connected only by one link from the center of the star to a single node of the clique, as illustrated in Fig. 4(a). By construction of this simplicial complex, the two subnetworks are almost disconnected, and work almost as two independent systems. We will see that, however, that at each pure order $d \geq 2$ the subnetworks are actually disconnected. This allows us to use our analytical understanding from the previous section to comprehend the present case with complex topology.

We start by analyzing the structure of the 1-simplex interactions. The Lyapunov spectrum of the pure 1-simplex interactions is shown in blue in Fig. 4(b). The spectrum reflects the quasi-independence of the subnetworks. Indeed, since the star and the clique are almost independent, the adjacency matrix $A_{ij}$ is a slightly perturbed matrix with two independent blocks, one for each subnetwork. The same holds for the Laplacian matrix, and hence its eigenvalues and the Lyapunov exponents. We know that the $N_s$ Laplacian eigenvalues $\Lambda_0^{(1)}$ of a star network with pairwise interactions are given by $\{0, 1, \ldots, 1, N_s + 1\}$. We also know that the $N_c$ Laplacian eigenvalues $\Lambda_0^{(1)}$ of a pairwise all-to-all network are given by $\{0, N_c, \ldots, N_c\}$. Hence, the spectrum of $N$ eigenvalues $\Lambda_0^{(1)}$ of the 1-simplex Laplacian $L^{(1)}$ is the union of both spectra, only slightly perturbed. This can be seen on Fig. 4(b) for the Lyapunov exponents, i.e. scaled Laplacian eigenvalues (blue symbols).

Moving to higher-order interactions, $d \geq 2$, the star and the clique actually are disconnected. In fact, in these pure $d$-simplex cases, there is no star, since it has no higher-order interactions. Hence the higher-order Laplacians only “see” the clique. This implies that spectrum of $L_{ij}^{(2)}$ is the union of $N_s$ zeros and the spectrum of a higher-order all-to-all that we derived analytically in the previous section: one element zero and the others with
FIG. 5. Real macaque brain dataset: structure and dynamics. Panels (a), (b) and (c) depict the simplicial complex as it appears at orders 1, 4 and 9 respectively, also reporting the histogram indicating how much each order is represented in the dataset. (d) Number of d-simplices in the simplicial complex, for each d. The highest order of interaction in the network is \( d = 10 \). (e) Lyapunov spectra with only 1-simplex interactions, and with higher-order interactions (with \( d \) up to 4). (f) Comparison of the Lyapunov spectrum at order \( d = 1 \) with the multi-order.

value \(-\gamma dN_c/(N_c-1)\). Analogously at orders \( d \) higher than 2, the spectrum is given by \( N_c+1 \) zeros and \( N_c-1 \) elements \(-\gamma dN_c/(N_c-1)\). See Fig. 4(b) for the spectrum at different orders \( d \).

For what concerns the spectrum of the total Laplacian, it is not exactly the sum of the spectrum of order \( d \), as in the all-to-all case, but almost. Indeed, the only thing that distinguishes it from the previous case, is that the pairwise Laplacian of the star-clique network is not composed of two disconnected blocks, but almost. Let us observe that, if the pure orders \( d \geq 2 \) can be simply described by the all-to-all higher order spectrum, the order \( d = 1 \) (traditional pairwise) and the multi-order, reflect the structure of the two almost disconnected subgraphs, the star and the clique. Their spectrum is compared in Fig. 4(c). It is important to notice that \( \lambda_2^{(1)} \) and \( \lambda_2^{(M)} \) are identical. The two spectra are however globally different and in particular \( \lambda_N^{(1)} \) and \( \lambda_N^{(M)} \) are very very different values, suggesting that in the repulsive case (all \( \gamma_d < 0 \)) the two Laplacian would yields dynamics with very different timescales.

B. Real network: macaque brain

In this section, we demonstrate the use of our multi-order Laplacian framework on a real dataset. We use a macaque brain dataset publicly available at [48], consisting of 91 nodes and 628 edges.

We assume that each fully connected \((d+1)\)-node clique captures multi-order interactions that can be described as a d-simplex. The obtained simplicial complex as it appears at orders 1, 4 and 9 is depicted in Fig. 5, respectively (a), (b) and (c).

The full distribution of the number of d-simplices present in the simplicial complex is shown in Fig. 5(a). The highest order of interaction is 10. As expected, the distribution follows a bell curve: the simplicial complex contains only less than 1000 1-simplices, i.e. 2-oscillator interactions, and 10-simplices, i.e. 11-oscillator interactions, but has over 6000 5-simplices.

In addition to the distribution of d-simplices across orders d, the dynamics is affected by the structure of the d-simplex interactions at each order d. Specifically, we wonder how 1-, 2-, ..., and d-simplices are distributed among nodes. It turns out that at each order d, the
interactions are very centralized: only few nodes have many $d$-simplex interactions whereas the majority of the nodes have very few or no $d$-simplex interaction at all. This is shown for order 4 and 9 in Figs. 5(b) and 5(c), respectively. The higher the order $d$ is, the more nodes have zero $d$-simplex interactions.

How is the stability of the synchronized state affected by this inter- and intra-order structure? To assess this, we compute the Laplacian (30) at each order $d$ as well as the multi-order Laplacian (36), and obtain the Lyapunov exponents from the eigenvalue spectra. First, we assess the stability of each pure case, at orders $d$ from 1 to 4, by computing their respective Lyapunov spectra shown in Fig. 5(e). Orders higher than 4 are not shown for visualization purposes. These spectra reflect the structure we described above: the higher the order $d$, the more nodes that have no $d$-simplex interactions, i.e. that are disconnected of the main component in the pure $d$-simplex network. These disconnected nodes translate into eigenvalues of value 0 in the spectra. We observe also two other stages of the spectra: a first step where higher orders correspond to lower Lyapunov exponents, which is similar to what happens for the star-clique model because of the clique subnetwork and may reflect, also in this example, the aggregated nature of the network at each order $d$. Then, we observe a strong descending behavior for the last eigenvalues, which confirms the stronger stability of the higher orders.

Finally, we compare the stability of the full system with combined higher-order interactions against what is obtained on the corresponding projecting graph, where nodes coupled at any order are linked with a pairwise edge. This is illustrated in Fig. 5(f).

VI. SUMMARY AND CONCLUSIONS

In summary, in this work we have studied the effects of higher-order interactions, i.e. multi-oscillator interactions, on the synchronization of identical phase oscillators on simplicial complexes. We considered interactions up to any order in an extension of the Kuramoto model, by introducing a multi-order Laplacian framework, which makes the system amenable to analytical treatment, and determines the stability of the fully synchronized state. In particular, we obtained a quantitative measure of that stability by computing the Lyapunov exponents of the state, that are proportional to the Laplacian spectrum. We show the application of the multi-order Laplacian in settings of increasing complexity.

As a first example, we considered a case that is fully tractable analytically: the higher-order all-to-all setting. In this setting, all nodes interact in all possible $d$-simplex interactions for any order $d$. In this case, we showed that the Laplacian of each order $d$, $L^{(d)}$, is proportional to the traditional pairwise Laplacian $L^{(1)}$, and to the value of $d$ itself, see Eq. (60). As a consequence, the stability of the multi-order system can be linearly decomposed into the stability of each of the pure $d$-simplex systems. Indeed, the multi-order Lyapunov exponents are merely a weighted sum of the Lyapunov exponents of order $d$.

We confirm our findings with numerical simulations. Finally, this fully tractable higher-order all-to-all setting also serves as a limit case to help us understand what happens in more complex topologies.

In this setting, we considered two additional scenarios analytically. First, we investigated the interplay between orders with attractive interactions ($\gamma_d > 0$) and orders with repulsive interactions ($\gamma_d < 0$). For example, we showed that their opposite effect on synchronization means that repulsive pairwise couplings can be countered effectively by higher-order attractive couplings. In general, we showed that when odd orders are attractive and even orders are repulsive, taking more higher-orders into account can stabilize or destabilize the system, depending on the highest order considered. Second, to link our work to higher-order phase reduction techniques, we considered attractive coupling strength $\gamma_d$ that decay as the order $d$ increases. We derived the convergence of the Lyapunov exponents as higher orders are considered.

Then, we considered two cases with more complex topologies.

First, we applied the multi-order Laplacian framework to a toy model: the star-clique network. With this model, we illustrated spectral properties of the multi-order Laplacian, and of the Laplacian at each order. In that specific topology, the star subnetwork does not contain higher-order interactions and do not influence the value of the first non-zero Lyapunov exponent. They do drastically change the other values of the spectrum, however, as we showed.

Second, we considered a real-world topology: a macaque brain network. In this setting, we show how the complex shape of the multi-order Laplacian spectrum can be understood from the structure of the simplicial complex. More importantly, our analysis confirms changes in all Lyapunov exponents due to the inclusion of higher-order interactions, as compared to only pairwise. Specifically, higher-order interactions tend to stabilize synchronization, with more drastic change on the more negative part of the spectrum, due to the hub-like structure of the dataset.

Many open questions remain about the effect of higher-order interactions on the synchronization of oscillators, even for identical phase oscillators. We mention here two possible directions for future investigations. First, the choice of the coupling functions for the higher-order interactions is of interest. In this work, we chose the coupling functions to be symmetric with respect to $i$. However, other choices are possible, see e.g. [39]. It would be interesting to have a systematic study extending what we know about pairwise coupling functions [49] to higher-order ones. Besides, it is worth to note that the multi-order Laplacian framework presented here can be readily extended to other higher-order interactions settings such as hypergraphs, which include all simplicial complexes.
Computationally, even if our model is applicable to each order of interaction, the numerical setup becomes slower for each additional order and this could represent a computational problem when aiming to inspect the stability of very large higher-order networks.

In conclusion, in this paper, we introduced a multi-order Laplacian framework to assess the stability of synchronization in populations of oscillators with higher-order interactions. Our framework has two main strengths: (i) it is a natural generalization of traditional and well known pairwise Laplacian framework and (ii) it can be applied to simplicial complexes with arbitrary complex topology and interactions up to any order $d$. We applied this multi-order Laplacian framework to three settings of increasing complexity, including a fully tractable all-to-all setting and a real brain dataset. Finally, we confirmed our analytical results with numerical simulations.

The multi-order Laplacian is a very general framework that could be used widely to understand the properties of synchronization when oscillators have higher-order interactions: an empirical feature of many real world networks, which is responsible for modifying the emergent behavior of many complex systems.

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