Cohomology and Coquasi-bialgebras in the category of Yetter-Drinfeld modules

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COHOMOLOGY AND COQUASI-BIALGEBRAS IN THE CATEGORY OF YETTER-DRINFELD MODULES

IVÁN ANGIONO, ALESSANDRO ARDIZZONI, AND CLAUDIA MENINI

Abstract. We prove that a finite-dimensional Hopf algebra with the dual Chevalley Property over a field of characteristic zero is quasi-isomorphic to a Radford-Majid bosonization whenever the third Hochschild cohomology group in the category of Yetter-Drinfeld modules of its diagram with coefficients in the base field vanishes. Moreover we show that this vanishing occurs in meaningful examples where the diagram is a Nichols algebra.

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Introduction

Let $A$ be a finite-dimensional Hopf algebra over a field $k$ of characteristic zero such that the coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). Denote by $\mathcal{D}(A)$ the diagram of $A$. The main aim of this paper (see Theorem 5.6) is to prove that, if the third Hochschild cohomology group in $H^3_{\mathcal{YD}}$ of the algebra $\mathcal{D}(A)$ with coefficients in $k$ vanishes, in symbols $H^3_{\mathcal{YD}}(\mathcal{D}(A), k) = 0$, then $A$ is quasi-isomorphic to the Radford-Majid bosonization $E \# H$ of some connected bialgebra $E$ in $H^0_{\mathcal{YD}}$ with $gr(E) \cong \mathcal{D}(A)$ as bialgebras in $H^0_{\mathcal{YD}}$.

The paper is organized as follows. Let $H$ be a Hopf algebra over a field $k$. In Section 1 we investigate the properties of coalgebras with multiplication and unit in the category $H^1_{\mathcal{YD}}$ (in particular of coquasi-bialgebras) and their associated graded coalgebra. The main result of this section, Theorem 1.5, establishes that the associated graded coalgebra $grQ$ of a connected coquasi-bialgebra in $H^1_{\mathcal{YD}}$ is a connected bialgebra in $H^0_{\mathcal{YD}}$. 

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In Section 3 we study the deformation of coquasi-bialgebras in $H \mathcal{YD}$ by means of gauge transformations. In Proposition 2.5 we investigate its behaviour with respect to bosonization while in Proposition 2.6 with respect to the associated graded coalgebra.

In Section 4 we consider the associated graded coalgebra in case the Hopf algebra $H$ is semisimple and cosemisimple (e.g. $H$ is finite-dimensional cosemisimple over a field of characteristic zero). In particular, in Theorem 3.2, we prove that a f.d. connected coquasi-bialgebra $C$ equivalent to a connected bialgebra in $H \mathcal{YD}$ whenever $H \mathcal{YD} (\text{gr} Q, k) = 0$. This result is inspired by Proposition 2.3.

In Section 4 we focus on the link between $H \mathcal{YD} (B, k)$ and the invariants of $H^n (B, k)$, where $B$ is a bialgebra in $H \mathcal{YD} (B, k)$. In particular, in Proposition 4.7 we show that $H \mathcal{YD} (B, k)$ is isomorphic to $H^n (B, k)_{D(H)}$, which is a subspace of $H^n (B, k)^H \cong H^n (B \# H, k)$, see Corollary 4.3.

Section 5 is devoted to the proof of the main result of the paper, the aforementioned Theorem 5.4.

In Section 6 we provide examples where $H \mathcal{YD} (B, k) = 0$ in case $B$ is the Nichols algebra $B(V)$ of a Yetter-Drinfeld module $V$. In particular we show that that $H \mathcal{YD} (B(V), k)$ can be zero although $H^3 (B(V) \# H, k)$ is non-trivial.

**Preliminaries**

Given a category $C$ and objects $M, N \in C$, the notation $C (M, N)$ stands for the set of morphisms in $C$. This notation will be mainly applied to the case $C$ is the category of vector space $\text{Vec}_k$ over a field $k$ or $C$ is the category of Yetter-Drinfeld modules $H \mathcal{YD}$ over a Hopf algebra $H$. The set of natural numbers including 0 is denoted by $\mathbb{N}_0$ while $\mathbb{N}$ denotes the same set without 0.

1. **Yetter-Drinfeld**

**Definition 1.1.** Let $C$ be a coalgebra. Denote by $C_n$ the $n$-th term of the coradical filtration of $C$ and set $C_{-1} := 0$. For every $x \in C$, we set

$$|x| := \min \{i \in \mathbb{N}_0 : x \in C_i\} \quad \text{and} \quad \overline{x} := x + C_{|x|-1}.$$ 

Note that, for $x = 0$, we have $|x| = 0$. One can define the associated graded coalgebra

$$\text{gr} C := \oplus_{i \in \mathbb{N}_0} \frac{C_i}{C_{i-1}}$$

with structure given, for every $x \in C$, by

\begin{align*}
\Delta_{\text{gr} C} (\overline{x}) &= \sum_{0 \leq i \leq |x|} (x_1 + C_{i-1}) \otimes (x_2 + C_{|x|-i-1}), \\
\varepsilon_{\text{gr} C} (\overline{x}) &= \delta_{|x|,0} C (x).
\end{align*}

1.2. For every $i \in \mathbb{N}_0$, take a basis $\{\overline{x^{i,j}} \mid j \in B_i\}$ of the k-module $C_i/C_{i-1}$ with $\overline{x^{i,j}} \neq \overline{x^{i,l}}$ for $j \neq l$ and $|x^{i,j}| = i$.

Then $\{x^{i,j} \mid 0 \leq i \leq n, j \in B_i\}$ is a basis of $C_n$ and $\{x^{i,j} \mid i \in \mathbb{N}_0, j \in B_i\}$ is a basis of $C$. Assume that $C$ has a distinguished grouplike element $1 = 1_C \neq 0$ and take $i > 0$. If $\varepsilon (x^{i,j}) \neq 0$ then we have that

$$\overline{x^{i,j} - \varepsilon (x^{i,j})} 1 = \overline{x^{i,j}}$$

so that we can take $x^{i,j} - \varepsilon (x^{i,j}) 1$ in place of $x^{i,j}$. In other words we can assume

$$\varepsilon (x^{i,j}) = 0, \quad \text{for every } i > 0, j \in B_i.$$

It is well-known there is a $k$-linear isomorphism $\varphi : C \to \text{gr} C$ defined on the basis by $\varphi (x^{i,j}) := \overline{x^{i,j}}$.

We compute

$$\varepsilon_{\text{gr} C} \varphi (x^{i,j}) = \varepsilon_{\text{gr} C} (\overline{x^{i,j}}) \delta_{i,0} \varepsilon (x^{0,j}) \varepsilon (x^{i,j}).$$
Hence we obtain
\[(4) \quad \varepsilon_{\text{gr} C} \circ \varphi = \varepsilon.\]

Let \(H\) be a Hopf algebra. A **coalgebra with multiplication and unit** in \(H^H \mathcal{YD}\) is a datum \((Q, m, u, \Delta, \varepsilon)\) where \((Q, \Delta, \varepsilon)\) is a coalgebra in \(H^H \mathcal{YD}\), \(m : Q \otimes Q \to Q\) is a coalgebra morphism in \(H^H \mathcal{YD}\) called multiplication (which may fail to be associative) and \(u : k \to Q\) is a coalgebra morphism in \(H^H \mathcal{YD}\) called called unit. In this case we set \(1_Q := u(1_k)\).

Note that, for every \(h \in H, k \in k,\) we have
\[(5) \quad h_1 Q = hu(1_k) = u(h(1_k)) = u(\varepsilon_H(h) 1_k) = \varepsilon_H(h) u(1_k) = \varepsilon_H(h) 1_Q,\]
\[(6) \quad (1_Q)_{-1} \otimes (1_Q)_0 = (u(1_k))_{-1} \otimes (u(1_k))_0 = (1_k)_{-1} \otimes u((1_k)_0) = 1_H \otimes u(1_k) = 1_H \otimes 1_Q.\]

**Proposition 1.3.** Let \(H\) be a Hopf algebra and let \((Q, m, u, \Delta, \varepsilon)\) be a coalgebra with multiplication and unit in \(H^H \mathcal{YD}\). If \(Q_0\) is a subcoalgebra of \(Q\) in \(H^H \mathcal{YD}\) such that \(Q_0 \cap Q_0,\) then \(Q_n\) is a subcoalgebra of \(Q\) in \(H^H \mathcal{YD}\) for every \(n \in \mathbb{N}_0\). Moreover \(Q_a \cdot Q_b \subseteq Q_{a+b}\) for every \(a, b \in \mathbb{N}_0\) and the graded coalgebra \(grQ\), associated with the coradical filtration of \(Q\), is a coalgebra with multiplication and unit in \(H^H \mathcal{YD}\) with respect to the usual coalgebra structure and with multiplication and unit defined by
\[(7) \quad m_{grQ} ((x + Q_{a-1}) \otimes (y + Q_{b-1})) := xy + Q_{a+b-1},\]
\[u_{grQ} (k) := k1_Q + Q_{-1}.

**Proof.** The coalgebra structure of \(Q\) induces a coalgebra structure on \(grQ\). Since \(Q_0\) is a subcoalgebra of \(Q\) in \(H^H \mathcal{YD}\) and, for \(n \geq 1\), one has \(Q_n = Q_{n-1} \wedge Q_0\), then inductively one proves that \(Q_n\) is a subcoalgebra of \(Q\) in \(H^H \mathcal{YD}\). As a consequence one gets that \(grQ\) is a coalgebra in \(H^H \mathcal{YD}\) (this construction can be performed in the setting of monoidal categories under suitable assumptions, see e.g. [AM, Theorem 2.10]). Let us prove that \(grQ\) inherits also a multiplication and unit. Let us check that \(Q_a \cdot Q_b \subseteq Q_{a+b}\) for every \(a, b \in \mathbb{N}_0\). We proceed by induction on \(n = a + b\). If \(n = 0\) there is nothing to prove. Let \(n \geq 1\) and assume that \(Q_i \cdot Q_j \subseteq Q_{i+j}\) for every \(i, j \in \mathbb{N}_0\) such that \(0 \leq i + j \leq n - 1\). Let \(a, b \in \mathbb{N}_0\) be such that \(n = a + b\). Since \(\Delta(Q_a) \subseteq \sum_{i=0}^{a} Q_i \otimes Q_{a-i}\) and \(c_{Q,Q}(Q_a \otimes Q_b) \subseteq Q_{a+b} \otimes Q_a\), where \(c_{Q,Q}\) denotes the braiding in \(H^H \mathcal{YD}\), using the compatibility condition between \(\Delta\) and \(m\), one easily gets that \(\Delta(Q_a \cdot Q_b) \subseteq Q_{a+b-1} \otimes Q + Q \otimes Q_a\).

Therefore \(Q_a \cdot Q_b \subseteq Q_{a+b}\). This property implies we have a well-defined map in \(H^H \mathcal{YD}\)
\[m^{a,b}_{grQ} : Q_a \otimes Q_b \to Q_{a+b}\]
defined, for \(x \in Q_a\) and \(y \in Q_b\), by (8). This can be seen as the graded component of a morphism in \(H^H \mathcal{YD}\) that we denote by \(m_{grQ} \otimes grQ \to grQ\). Let us check that \(m_{grQ}\) is a coalgebra morphism in \(H^H \mathcal{YD}\). Consider a basis of \(Q\) with terms of the form \(x^{i,j}\) as in (4). Hence we can write the comultiplication in the form
\[\Delta(x^{a,u}) = \sum_{s+t \leq a} \sum_{l,m} q^{s,t,l,m} x^{a,u} \otimes x^{l,m}.

Now, using (1), one gets that
\[(8) \quad \Delta_{grQ}(x^{a,u}) = \sum_{0 \leq i \leq a} \sum_{l,m} q^{a-i+l,m} x^{a-i,u} \otimes x^{l,m}.

Using that \(\Delta_{grQ} \otimes grQ = (grQ \otimes c_{grQ,grQ} \otimes grQ)(\Delta_{grQ} \otimes \Delta_{grQ})\) and (8), it is straightforward to check that \((m_{grQ} \otimes m_{grQ}) \Delta_{grQ} \otimes grQ (x^{a,u} \otimes x^{b,v}) = \Delta_{grQ} m_{grQ} (x^{a,u} \otimes x^{b,v})\).

Moreover, since \(\varepsilon_{grQ} \otimes grQ = \varepsilon_{grQ} \otimes grQ\), we get that \(\varepsilon_{grQ} m_{grQ} (x^{a,u} \otimes x^{b,v}) = \varepsilon_{grQ} \otimes grQ (x^{a,u} \otimes x^{b,v})\).

This proves that \(m_{grQ}\) is a coalgebra morphism in \(H^H \mathcal{YD}\).

The fact that \(u_{grQ} : k \to grQ\), defined by \(u_{grQ}(k) := k1_Q + Q_{-1}\) is a coalgebra morphism in \(H^H \mathcal{YD}\) easily follows by means of (4) and (8).
Definition 1.4 ([ABM, Definition 5.2]). Let $H$ be a Hopf algebra. Recall that a coquasi-bialgebra $(Q, m, u, \Delta, \varepsilon, \alpha)$ in the pre-braided monoidal category $\mathcal{H}_H\mathcal{YD}$ is a coalgebra $(Q, \Delta, \varepsilon)$ in $\mathcal{H}_H\mathcal{YD}$ together with coalgebra homomorphisms $m : Q \otimes Q \to Q$ and $u : k \to Q$ in $\mathcal{H}_H\mathcal{YD}$ and a convolution invertible element $\alpha \in \mathcal{H}_H\mathcal{YD}(Q^\otimes 3, k)$ (braided reassociator) such that

\begin{align*}
(9) \quad & \alpha(Q \otimes Q \otimes m) \ast \alpha(m \otimes Q \otimes Q) = (\varepsilon \otimes \alpha) \ast \alpha(Q \otimes m \otimes Q) \ast (\alpha \otimes \varepsilon), \\
(10) \quad & \alpha(Q \otimes u \otimes Q) = \alpha(u \otimes Q \otimes Q) = \alpha(Q \otimes Q \otimes u) = \varepsilon_{Q \otimes Q}, \\
(11) \quad & m(Q \otimes m) \ast \alpha = \alpha \ast m(m \otimes Q), \\
(12) \quad & m(u \otimes Q) = \text{Id}_Q = m(Q \otimes u).
\end{align*}

Here $\ast$ denotes the convolution product, where $Q^\otimes 3$ is the tensor product of coalgebras in $\mathcal{H}_H\mathcal{YD}$ whence it depends on the braiding of this category. Note that in (10) any of the three equalities such as $\alpha(u \otimes Q \otimes Q) = \varepsilon_{Q \otimes Q}$ implies that $\alpha$ is unital.

**Theorem 1.5.** Let $H$ be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in $\mathcal{H}_H\mathcal{YD}$. Then $\text{gr}Q$ is a connected bialgebra in $\mathcal{H}_H\mathcal{YD}$.

**Proof.** By Proposition 1.3, we know that $\text{gr}Q$ is a coalgebra with multiplication and unit in $\mathcal{H}_H\mathcal{YD}$. We have to check that the multiplication is associative and unitary.

Given two coalgebras $D, E$ in $\mathcal{H}_H\mathcal{YD}$ endowed with coalgebras filtration $(D_n)_{n \in \mathbb{N}_0}$ and $(E_n)_{n \in \mathbb{N}_0}$ in $\mathcal{H}_H\mathcal{YD}$ such that $D_0$ and $E_0$ are one-dimensional, let us check that $C_n := \sum_{0 \leq i \leq n} D_i \otimes E_{n-i}$ gives a coalgebra filtration on $C := D \otimes E$ in $\mathcal{H}_H\mathcal{YD}$. First note that the coalgebra structure of $C$ depends on the braiding. Thus, we have

\[
\Delta_C(C_n) = (D \otimes c_{D,E} \otimes E)(\Delta_D \otimes \Delta_E)\left(\sum_{i=0}^{n} D_i \otimes E_{n-i}\right)
\]

\[
\subseteq (D \otimes c_{D,E} \otimes E)\left(\sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_a \otimes D_{i-a} \otimes E_b \otimes E_{n-i-b}\right)
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_a \otimes c_{D,E} D_{i-a} \otimes E_b \otimes E_{n-i-b}
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_a \otimes c_{D_{i-a},E} D_{i-a} \otimes E_b \otimes E_{n-i-b}
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{a=0}^{i} \sum_{b=0}^{n-i} D_a \otimes E_b \otimes D_{i-a} \otimes E_{n-i-b}
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{w=0}^{n} \sum_{0 \leq a \leq i} \sum_{0 \leq b \leq n-i} D_a \otimes E_b \otimes D_{i-a} \otimes E_{n-i-b}
\]

\[
\subseteq \sum_{i=0}^{n} \sum_{w=0}^{n} C(w) \otimes C_{n-w}.
\]

Moreover, by [SW, Proposition 11.1.1], we have that the coradical of $C$ is contained in $D_0 \otimes E_0$ and hence it is one-dimensional.

This argument can be used to produce a coalgebra filtration on $C := Q \otimes Q \otimes Q$ using as a filtration on $Q$ the coradical filtration. Let $n > 0$ and let $w \in C_n = \sum_{i+j+k \leq n} Q_i \otimes Q_j \otimes Q_k$. By [AMS], Lemma 3.69, we have that

\[
\Delta_C(w) - w \otimes (1Q)^\otimes 3 - (1Q)^\otimes 3 \otimes w \in C_{n-1} \otimes C_{n-1}.
\]

Thus we get

\[
w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1Q)^\otimes 3 - \Delta_C((1Q)^\otimes 3) \otimes w \in \Delta_C(C_{n-1}) \otimes C_{n-1}
\]

and hence, tensoring the first relation by $(1Q)^\otimes 3$ on the right and adding it to the second one, we get

\[
w_1 \otimes w_2 \otimes w_3 - w \otimes (1Q)^\otimes 3 \otimes (1Q)^\otimes 3 - (1Q)^\otimes 3 \otimes w \otimes (1Q)^\otimes 3 - (1Q)^\otimes 6 \otimes w \in C_{n-1} \otimes C_{n-1} \otimes C_{n-1}.
\]

For shortness, we set $\nu_n(z) := m(Q \otimes m)(z) + Q_{n-1}$ for every $z \in C$. Thus, by applying to the last displayed relation $C_{n-1} \otimes m(Q \otimes m) \otimes C_{n-1}$ and factoring out the middle term by $Q_{n-1}$,
we get
\[
\left[ w_1 \otimes \nu_n (w_2) \otimes w_3 - w \otimes \nu_n \left( (1_Q)^{\otimes 3} \right) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n (w) \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes \nu_n \left( (1_Q)^{\otimes 3} \right) \otimes w \right] \\
\in C_{(n-1)} \otimes \left( \nu_n \left( \frac{C_{(n-1)}}{Q_{n-1}} \right) \right) \subseteq C_{(n-1)} \otimes \frac{Q_{n-1}}{Q_{n-1}} \otimes C_{(n-1)} = 0.
\]
Thus we can express the first term with respect to the remaining ones as follows
\[
w_1 \otimes \nu_n (w_2) \otimes w_3 = w \otimes \nu_n \left( (1_Q)^{\otimes 3} \right) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n (w) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n \left( (1_Q)^{\otimes 3} \right) \otimes w.
\]

We have so proved that for \( n > 0 \) and \( w \in C_{(n)} \),
\[
w_1 \otimes \nu_n (w_2) \otimes w_3 = (1_Q)^{\otimes 3} \otimes \nu_n (w) \otimes (1_Q)^{\otimes 3}.
\]
The same equation trivially holds also in the case \( n = 0 \) as \( C_{(n)} \) is one-dimensional.

Let \( x, y, z \in Q \). Then \( x \otimes y \otimes z \in C_{|x|+|y|+|z|} \) so that
\[
(x \cdot y) \cdot z = ((x + Q_{|x|-1}) \cdot (y + Q_{|y|-1})) \cdot (z + Q_{|z|-1})
\]
\[
= ((xy) + Q_{|x|+|y|-1}) \cdot (z + Q_{|z|-1})
\]
\[
= (xy) \cdot z + Q_{|x|+|y|+|z|-1}
\]
\[
= \omega^{-1} ((x \otimes y \otimes z_1) \nu_{|x|+|y|+|z|} ((x \otimes y \otimes z)_2) \omega ((x \otimes y \otimes z)_3))
\]
\[
= \omega^{-1} (1_Q \otimes 1_Q \otimes 1_Q) \nu_{|x|+|y|+|z|} (x \otimes y \otimes z) \omega (1_Q \otimes 1_Q \otimes 1_Q)
\]
\[
= \nu_{|x|+|y|+|z|} (x \otimes y \otimes z)
\]
\[
= x \cdot y \cdot z + Q_{|x|+|y|+|z|-1} = x \cdot y \cdot z.
\]

Therefore the multiplication is associative. It is also unitary as
\[
\overline{x} \cdot \overline{1_Q} = (x + Q_{|x|-1}) \cdot (1_Q + Q_{-1}) = x \cdot 1_Q + Q_{|x|-1} = x + Q_{|x|-1} = \overline{x}
\]
and similarly \( \overline{1_Q} \cdot \overline{x} = \overline{x} \) for every \( x \in Q \).

2. Gauge deformation

**Definition 2.1.** Let \( H \) be a Hopf algebra and let \( (Q, m, u, \Delta, \varepsilon, \omega) \) be a coquasi-bialgebra in \( _H^H \mathcal{YD} \).

A **gauge transformation** for \( Q \) is a morphism \( \gamma : Q \otimes Q \to k \) in \( _H^H \mathcal{YD} \) which is convolution invertible in \( _H^H \mathcal{YD} \) and which is also unitary on both entries.

**Remark 2.2.** For \( \gamma \) as above, let us check that \( \gamma^{-1} \) is unitary whence a gauge transformation too.

First note that for all \( x \in Q \), by means of \([5]\) and \([6]\), one gets
\[
(1_Q \otimes x)_{1} \otimes (1_Q \otimes x)_{2} = 1_Q \otimes x_1 \otimes 1_Q \otimes x_2
\]
\[
(x \otimes 1_Q)_{1} \otimes (x \otimes 1_Q)_{2} = x_1 \otimes 1_Q \otimes x_2 \otimes 1_Q
\]
Thus
\[
\gamma^{-1} (1_Q \otimes x) = \gamma^{-1} (1_Q \otimes x_1) \varepsilon (x_2) = \gamma^{-1} (1_Q \otimes x_1) \gamma (1_Q \otimes x_2) = (\gamma^{-1} \ast \gamma) (1_Q \otimes x) = \varepsilon (x)
\]
and similarly \( \gamma^{-1} (x \otimes 1_Q) = \varepsilon (x) \).

**Lemma 2.3.** Let \( H \) be a Hopf algebra and let \( C \) be a coalgebra in \( _H^H \mathcal{YD} \). Given a map \( \gamma \in _H^H \mathcal{YD} (C, k) \), we have that \( \gamma \) is convolution invertible in \( _H^H \mathcal{YD} (C, k) \) if and only if it is convolution invertible in \( \text{Vec}_k (C, k) \). Moreover the inverse is the same.
Proof. Assume there is a $k$-linear map $\gamma^{-1} : C \to k$ which is a convolution inverse of $\gamma$ in $\mathbf{Vec}_\gamma(C,k)$. By \cite[Remark 2.4(ii)]{ABM}, $\gamma^{-1}$ is left $H$-linear. Let us check that $\gamma^{-1}$ is left $H$-colinear:

$$c_{-1} \otimes \gamma^{-1}(c_0) = (c_1)_{-1} 1_H \otimes \gamma^{-1}((c_1)_0)(c_2)\gamma^{-1}(c_3)$$

$$= (c_1)_{-1} (c_2)_{-1} \otimes \gamma^{-1}((c_1)_0)(c_2)\gamma^{-1}(c_3)$$

$$= (1_H \otimes \varepsilon_C)((c_1)_0) \gamma^{-1}(c_2)$$

$$= (1_H \otimes \varepsilon_C)\gamma^{-1}(c_2) = 1_H \otimes \gamma^{-1}(c)$$

where in (*) we used that the comultiplication or the counit of $C$ is left $H$-colinear. Thus $\gamma$ is convolution invertible in $H_\mathcal{YD}(C,k)$. The other implication is obvious. \hfill \Box

**Proposition 2.4.** Let $H$ be a Hopf algebra and let $(Q,m,u,\Delta,\varepsilon,\omega)$ be a coquasi-bialgebra in $H_\mathcal{YD}$. Let $\gamma : Q \otimes Q \to k$ be a gauge transformation in $H_\mathcal{YD}$. Then

$$Q^\gamma := (Q,m^\gamma,u,\Delta,\varepsilon,\omega^\gamma)$$

is a coquasi-bialgebra in $H_\mathcal{YD}$, where

$$m^\gamma := \gamma \ast m \ast \gamma^{-1}$$

$$\omega^\gamma := (\varepsilon \otimes \gamma) \ast \gamma (Q \otimes m) \ast \omega \ast \gamma^{-1} (m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon).$$

**Proof.** The proof is analogue to \cite[Proposition XV.3.2]{K} in its dual version. We include some details for the reader’s sake. Note that $Q^\gamma$ has the same underlying coalgebra of $Q$ which is a coalgebra in $H_\mathcal{YD}$. The unit is also the same and hence it is a coalgebra map in $H_\mathcal{YD}$. Since $m^\gamma$ is the convolution product of morphisms in $H_\mathcal{YD}$, it results that $m^\gamma$ is in $H_\mathcal{YD}$ as well.

Since $m$ is a coalgebra map in $H_\mathcal{YD}$ and $\gamma$ is convolution invertible with convolution inverse $\gamma^{-1}$, it follows that $m^\gamma$ is a coalgebra map in $H_\mathcal{YD}$.

By means of (13) and (15), one gets that $m^\gamma (1_Q \otimes x) = x = m^\gamma (x \otimes 1_Q)$.

Let us consider now $\omega^\gamma$. Since it is the convolution product of morphisms in $H_\mathcal{YD}$, it results that $\omega^\gamma$ is in $H_\mathcal{YD}$ as well.

Let us check that $\omega^\gamma$ is unitary. Consider the map $\alpha_2 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_2(x \otimes y) = x \otimes 1_Q \otimes y$. The equalities (13) and (15) yield

$$(\alpha_2(x \otimes y))_1 \otimes (\alpha_2(x \otimes y))_2 = \alpha_2(x_1 \otimes (x_2)_1 \varepsilon) \otimes \alpha_2((x_2)_0 \otimes y_2)$$

so that $\alpha_2$ is comultiplicative.

Thus

$$\omega^\gamma \alpha_2 := (\varepsilon \otimes \gamma) \alpha_2 \ast \gamma (Q \otimes m) \alpha_2 \ast \omega \alpha_2 \ast \gamma^{-1} (m \otimes Q) \alpha_2 \ast (\gamma^{-1} \otimes \varepsilon) \alpha_2$$

and computing the factors of this convolution products one gets

$$(\varepsilon \otimes \gamma) \alpha_2 = \varepsilon \otimes \varepsilon, \quad (Q \otimes m) \alpha_2 = \gamma, \quad \omega \alpha_2 = \varepsilon \otimes \varepsilon,$$

$$\gamma^{-1} (m \otimes Q) \alpha_2 = \gamma^{-1}, \quad (\gamma^{-1} \otimes \varepsilon) \alpha_2 = \varepsilon \otimes \varepsilon$$

and hence $\omega^\gamma \alpha_2 = \gamma \ast \gamma^{-1} = \varepsilon \otimes \varepsilon$, which means that $\omega^\gamma (x \otimes 1_Q \otimes y) = \varepsilon(x) \varepsilon(y)$ for every $x,y \in Q$.

Similarly, considering $\alpha_1 : Q \otimes Q \to Q \otimes Q \otimes Q$ defined by $\alpha_1(x \otimes y) = 1_Q \otimes x \otimes y$, one proves that $\omega^\gamma (1_Q \otimes x \otimes y) = \varepsilon(x) \varepsilon(y)$. A symmetric argument shows that $\omega^\gamma (x \otimes y \otimes 1_Q) = \varepsilon(x) \varepsilon(y)$.

Note that, by Lemma 2.3, $\omega^\gamma$ is convolution invertible in $H_\mathcal{YD}(D,k)$ as it is convolution invertible in $\mathbf{Vec}_k(D,k)$.

Let us check that the multiplication is quasi-associative. By \cite[Lemma 2.10 formula (2.7)]{ABM}, we have

$$m^\gamma (Q \otimes \gamma \ast m \ast \gamma^{-1}) = (\varepsilon \otimes \gamma) \ast m^\gamma (Q \otimes m) \ast (\varepsilon \otimes \gamma^{-1}),$$

where in (*) we used that the comultiplication or the counit of $C$ is left $H$-colinear. Thus $\gamma$ is convolution invertible in $H_\mathcal{YD}(C,k)$. The other implication is obvious. \hfill \Box
and the right regular action over $H$ we have that $m$ and $\omega$ are so that $\omega$ is $E$-diagonal action $H$-balanced, it is easy to check that $\Gamma$ is convolution invertible $H$-bilinear and $H$-balanced. Moreover $\Gamma^{-1}((x\#h)\otimes(x'\#h')) = \gamma^{-1}(x\otimes h'x') \in H(h')$. If $\alpha : (E\#H) \otimes (E\#H) \rightarrow E\#H$ is $H$-bilinear and $H$-balanced, it is easy to check that $\Gamma \ast \alpha \ast \Gamma^{-1}$ is $H$-bilinear and $H$-balanced too. In particular, since

$$m_{E\#H}((x\#h)\otimes(x'\#h')) = m(x \otimes h_1x') \otimes h_2h'$$

we have that $m_{E\#H}$ is $H$-bilinear and $H$-balanced where $E\#H$ carries the left $H$-diagonal action and the right regular action over $H$.

In analogy to the case of Hopf algebras, one can define the bosonization $E\# H$ of a coquasi-bialgebra in $H^*YD$ by a Hopf algebra $H$, see [ABM, Definition 5.4] for further details on the structure. The following result was originally stated for $E$ a Hopf algebra. Yorck Sommerhäuser suggested the present more general form which investigates the behaviour of the bosonization under a suitable gauge transformation.

**Proposition 2.5.** Let $H$ be a Hopf algebra and let $(E, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in $H^*YD$. Let $\gamma : E \otimes E \rightarrow k$ be a gauge transformation in $H^*YD$. Set

$$\Gamma : (E\#H) \otimes (E\#H) \rightarrow k : (x\#h) \otimes (x'\#h') \mapsto \gamma(x \otimes h'x') \varepsilon_{H}(h').$$

Then $\Gamma$ is a gauge transformation and $(E\#H)^{\Gamma} = E^{\gamma}\# H$ as ordinary coquasi-bialgebras.

**Proof.** By [ABM, Lemma 2.15 and what follows], we have that $\Gamma$ is convolution invertible $H$-bilinear and $H$-balanced. Moreover $\Gamma^{-1}((x\#h)\otimes(x'\#h')) = \gamma^{-1}(x\otimes h'x') \in H(h')$. If $\alpha : (E\#H) \otimes (E\#H) \rightarrow E\#H$ is $H$-bilinear and $H$-balanced, it is easy to check that $\Gamma \ast \alpha \ast \Gamma^{-1}$ is $H$-bilinear and $H$-balanced too. In particular, since

$$m_{E\#H}((x\#h)\otimes(x'\#h')) = m(x \otimes h_1x') \otimes h_2h'$$

we have that $m_{E\#H}$ is $H$-bilinear and $H$-balanced where $E\#H$ carries the left $H$-diagonal action and the right regular action over $H$. 

\[
(\varepsilon \otimes \gamma^{-1}) \ast (\varepsilon \otimes \gamma) = \varepsilon \otimes (\gamma^{-1} \ast \gamma) = \varepsilon \otimes \varepsilon \otimes \varepsilon,
\]

\[
m^{\gamma}(m^{\gamma} \otimes Q) = m^{\gamma}(\gamma \ast m \ast \gamma^{-1} \otimes Q) = (\gamma \otimes \varepsilon) \ast m^{\gamma}(m \ast \gamma^{-1} \otimes Q) = (\gamma \otimes \varepsilon) \ast m^{\gamma}(m \otimes Q) \ast (\gamma^{-1} \otimes \varepsilon),
\]

and $\omega^{\gamma}$ is $E$-diagonal action $H$-balanced, it is easy to check that $\Gamma$ is convolution invertible $H$-bilinear and $H$-balanced. Moreover $\Gamma^{-1}((x\#h)\otimes(x'\#h')) = \gamma^{-1}(x\otimes h'x') \in H(h')$. If $\alpha : (E\#H) \otimes (E\#H) \rightarrow E\#H$ is $H$-bilinear and $H$-balanced, it is easy to check that $\Gamma \ast \alpha \ast \Gamma^{-1}$ is $H$-bilinear and $H$-balanced too. In particular, since

$$m_{E\#H}((x\#h)\otimes(x'\#h')) = m(x \otimes h_1x') \otimes h_2h'$$

we have that $m_{E\#H}$ is $H$-bilinear and $H$-balanced where $E\#H$ carries the left $H$-diagonal action and the right regular action over $H$. 

\[
(\gamma^{-1} \otimes \varepsilon) \ast (\varepsilon \otimes \gamma) = ((\gamma^{-1} \ast \gamma) \otimes \varepsilon) = \varepsilon \otimes \varepsilon \otimes \varepsilon.
\]
Thus $m_{(E\#H)^c} = \Gamma \ast m_{E\#H} \ast \Gamma^{-1}$ is $H$-bilinear and $H$-balanced. Moreover, since $E^\gamma$ is also a coquasi-bialgebra in $H YD$ we have that $m_{E^\gamma \# H} : (E^\gamma \# H) \otimes (E^\gamma \# H) \to E^\gamma \# H$ is $H$-bilinear and $H$-balanced too.

Therefore, in order to check that $m_{(E\#H)^c} = m_{E^\gamma \# H}$, it suffices to prove that they coincide on elements of the form $(x \#_1 H) \otimes (x' \#_1 H)$.

Let us consider the multiplication

$$m_{(E\#H)^c} ((x \#_1 H) \otimes (x' \#_1 H))$$

$$= \Gamma ((x \#_1 H) \otimes (x' \#_1 H))$$

$$= \Gamma ((x \#_1 H)_1 \otimes (x' \#_1 H)_1) \cdot m_{E\#H} ((x \#_1 H)_2 \otimes (x' \#_1 H)_2) \cdot \Gamma^{-1} ((x \#_1 H)_3 \otimes (x' \#_1 H)_3).$$

Now, from

$$\Delta_{E\#H} (x \# h) = \sum \left( x^{(1)} \# x^{(2)}_{(-1)} h_1 \right) \otimes \left( x^{(2)}_{(0)} \# h_2 \right)$$

we get

$$(x \#_1 H)_1 \otimes (x \#_1 H)_2 \otimes (x \#_1 H)_3$$

$$= \sum \left( x^{(1)} \# x^{(2)}_{(-1)} x^{(3)}_{(-2)} \right) \otimes \left( x^{(2)}_{(0)} \# x^{(3)}_{(-1)} \right) \otimes \left( x^{(3)}_{(0)} \#_1 H \right)$$

so that

$$m_{(E\#H)^c} ((x \#_1 H) \otimes (x' \#_1 H))$$

$$= \Gamma ((x \#_1 H)_1 \otimes (x' \#_1 H)_1) \cdot m_{E\#H} ((x \#_1 H)_2 \otimes (x' \#_1 H)_2) \cdot \Gamma^{-1} ((x \#_1 H)_3 \otimes (x' \#_1 H)_3).$$
Thus we obtain

\[
\begin{align*}
\omega_{(E\#H)^r} &= \omega_{E\#H} = 1_{E\#H} = 1_{E\#H} = u_{E\#H}.
\end{align*}
\]

As a coalgebra \((E\#H)^r\) coincides with \(E\#H\) and hence with \(E\#H\).

Finally let us check that \(\omega_{E\#H}^r\) and \(\omega_{(E\#H)^r}\) coincide. To this aim, let us use the maps \(\mathcal{O}_{H,-}\) of [ABM, Lemma 2.15]. First note that \(\omega_{E\#H}^r = \mathcal{O}_{H,-}^r (\omega_{E\#H})\) by [ABM, Proposition 5.3]. Now

\[
\omega_{(E\#H)^r} = (\varepsilon_{E\#H} \otimes \Gamma) * (E\#H \otimes m_{E\#H}) * \omega_{E\#H} * \Gamma^{-1} (m_{E\#H} \otimes E\#H) \ast (\Gamma^{-1} \otimes \varepsilon_{E\#H})
\]

\[
= (\mathcal{O}_{H,E}^1 (\varepsilon) \otimes \mathcal{O}_{H,E}^2 (\gamma)) * \mathcal{O}_{H,E}^3 (\gamma) (E\#H \otimes m_{E\#H}) \ast \mathcal{O}_{H,E}^3 (\omega) \ast \mathcal{O}_{H,E}^2 (\gamma^{-1}) (m_{E\#H} \otimes E\#H) \ast (\mathcal{O}_{H,E}^2 (\gamma^{-1}) \otimes \mathcal{O}_{H,E}^1 (\varepsilon))
\]

One easily checks that

\[
\begin{align*}
\mathcal{O}_{H,E}^1 (\varepsilon) \otimes \mathcal{O}_{H,E}^2 (\gamma) &= \mathcal{O}_{H,\varepsilon}^2 (\varepsilon \otimes \gamma),
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_{H,E}^2 (\gamma) (E\#H \otimes m_{E\#H}) &= \mathcal{O}_{H,\varepsilon}^3 (\gamma (E \otimes m)),
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_{H,E}^2 (\gamma^{-1}) (m_{E\#H} \otimes E\#H) &= \mathcal{O}_{H,\varepsilon}^3 (\gamma^{-1}(m \otimes E)),
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_{H,E}^2 (\gamma^{-1}) \otimes \mathcal{O}_{H,E}^1 (\varepsilon) &= \mathcal{O}_{H,\varepsilon}^3 (\gamma^{-1} \otimes \varepsilon).
\end{align*}
\]

Thus we obtain

\[
\begin{align*}
\omega_{(E\#H)^r} &= \mathcal{O}_{H,\varepsilon}^3 (\varepsilon \otimes \gamma) * \mathcal{O}_{H,\varepsilon}^3 (\gamma (E \otimes m)) * \mathcal{O}_{H,\varepsilon}^3 (\omega) * \mathcal{O}_{H,\varepsilon}^3 (\gamma^{-1}(m \otimes E)) * \mathcal{O}_{H,\varepsilon}^3 (\gamma^{-1} \otimes \varepsilon)
\end{align*}
\]

\[
= \mathcal{O}_{H,\varepsilon}^3 [(\varepsilon \otimes \gamma) * \gamma (E \otimes m) * \omega * \gamma^{-1}(m \otimes E) * (\gamma^{-1} \otimes \varepsilon)]
\]

\[
= \mathcal{O}_{H,\varepsilon}^3 (\omega_{E\#H}) = \omega_{E\#H}^r.
\]

**Proposition 2.6.** Let \(H\) be a Hopf algebra and let \((Q,m,u,\Delta,\varepsilon,\omega)\) be a connected coquasi-bialgebra in \(H_{\varepsilon}^r\) and \(\text{gr}(Q)\) coincide as bialgebras in \(H_{\varepsilon}^r\).

**Proof.** By Proposition 2.3, \(Q^r\) is a coquasi-bialgebra in \(H_{\varepsilon}^r\). It is obviously connected as it coincides with \(Q\) as a coalgebra. By Theorem 1.3, both \(\text{gr}Q\) and \(\text{gr}(Q^r)\) are connected bialgebras in \(H_{\varepsilon}^r\). Let us check they coincide.

Note that, by Remark 2.3, we have that \(\gamma^{-1}\) is a gauge transformation, hence it is trivial on \(Q_1 \otimes Q\). Let \(C := Q \otimes Q\). Let \(n > 0\) and let \(w \in C^{(n)} = \sum_{i+j \leq n} Q_i \otimes Q_j\). By [AMSI, Lemma 3.69], we have that \(\Delta_C (w) = (1_Q)^{\otimes 2} \otimes (1_Q)^{\otimes 2} \otimes w \in C^{(n-1)} \otimes C^{(n-1)}\). Thus we get

\[
w_1 \otimes w_2 \otimes w_3 - \Delta_C (w) \otimes (1_Q)^{\otimes 2} - \Delta_C (1_Q)^{\otimes 2} \otimes w \in \Delta_C (C^{(n-1)}) \otimes C^{(n-1)}
\]

and hence

\[
w_1 \otimes w_2 \otimes w_3 - w \otimes (1_Q)^{\otimes 2} \otimes (1_Q)^{\otimes 2} \otimes w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 4} \otimes w \in C^{(n-1)} \otimes C^{(n-1)} \otimes C^{(n-1)}.
\]

Since \(C^{(n-1)} \subseteq Q_{n-1} \) we get

\[
w_1 \otimes m (w_2) \otimes w_3 - w \otimes 1_Q \otimes (1_Q)^{\otimes 2} \otimes w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes m (w) \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 3} \otimes w \in C^{(n-1)} \otimes Q_{n-1} \otimes C^{(n-1)}
\]

and hence

\[
(16) \quad w_1 \otimes (m (w_2) + Q_{n-1}) \otimes w_3 = (1_Q)^{\otimes 2} \otimes (m (w) + Q_{n-1}) \otimes (1_Q)^{\otimes 2}.
\]
Let $x, y \in Q$. We compute
\[
\pi \cdot y = (x + Q_{|x| - 1}) \cdot y + Q_{|y| - 1} = (x \cdot y) + Q_{|x| + |y| - 1} = \gamma((x \otimes y)_1) m((x \otimes y)_2) \gamma^{-1}((x \otimes y)_3) + Q_{|x| + |y| - 1}.
\]

Note that $Q^n$ and $Q$ have the same unit so that $\text{gr}Q$ and $\text{gr}Q^n$ have.

\[\square\]

3. (Co)semisimple case

Assume $H$ is a semisimple and cosemisimple Hopf algebra (e.g. $H$ is finite-dimensional cosemisimple over a field of characteristic zero). Note that $H$ is then separable (see e.g. [SI, Corollary 3.7] or [AMS], Theorem 2.34) whence finite-dimensional. Let $(Q, m, u, \Delta, \varepsilon)$ be a f.d. coalgebra with multiplication and unit in $H^1 YD$. Assume that the coradical $Q_0$ is a subcoalgebra of $Q$ in $H^1 YD$ such that $Q_0 \cdot Q_0 \subseteq Q_0$. Let $y^{n,i}$ with $1 \leq i \leq \dim (Q_n/Q_{n-1})$ be a basis for $Q_n/Q_{n-1}$. Consider, for every $n > 0$, the exact sequence in $H^1 YD$ given by
\[0 \rightarrow Q_{n-1} \xrightarrow{s_n} Q_n \xrightarrow{\pi_n} Q_n/Q_{n-1} \rightarrow 0\]

Now, since $H$ is semisimple and cosemisimple, by [Rn2 Proposition 7] the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [Ma, Proposition 10.6.16]) and by [Rl, Proposition 6], we get that the category $H^1 YD$ is a semisimple category. Therefore $\pi_n$ is a morphism $Q_n/Q_{n-1} \rightarrow Q_n$ in $H^1 YD$ such that $\pi_n \sigma_n = \text{Id}$. Let $u_n : k \rightarrow Q_n$ be the costructure of the unit $u : k \rightarrow Q$ and let $\varepsilon_n = \varepsilon_{Q_n} : Q_n \rightarrow k$ be the counit of the subcoalgebra $Q_n$. Set
\[\sigma'_n := \sigma_n - u_n \circ \varepsilon_n \circ \sigma_n\]

This is a morphism in $H^1 YD$. Moreover
\[\pi_n \circ \sigma'_n = \pi_n \circ \sigma_n - \pi_n \circ u_n \circ \varepsilon_n \circ \sigma_n = \pi_n \circ \text{Id}_{Q_n/Q_{n-1}} - \pi_n \circ \text{Id}_{Q_n/Q_{n-1}} = 0,\]
\[\varepsilon_n \circ \sigma'_n = \varepsilon_n \circ \sigma_n - \varepsilon_n \circ u_n \circ \varepsilon_n = \varepsilon_n \circ \sigma_n - \varepsilon_n \circ \sigma_n = 0.\]

Therefore, without loss of generality we can assume that $\varepsilon_n \circ \sigma_n = 0$. A standard argument on split short exact sequences shows that there exists a morphism $p_n : Q_n \rightarrow Q_{n-1}$ in $H^1 YD$ such that $s_n p_n + \sigma_n \pi_n = \text{Id}_{Q_n}$, $p_n s_n = \text{Id}_{Q_{n-1}}$, and $p_n \sigma_n = 0$. We set
\[x^{n,i} := \sigma_n(y^{n,i}).\]

Therefore
\[y^{n,i} = \pi_n \sigma_n(y^{n,i}) = \pi_n(x^{n,i}) = x^{n,i} + Q_{n-1} = x^{n,i}.\]

These terms $x^{n,i}$ define a $k$-basis for $Q$. As $Q$ is finite-dimensional, there exists $d \in \mathbb{N}_0$ such that $Q = Q_d$; fix $d$ minimal. For all $0 \leq a \leq b$, define the maps
\[p_{a,b} : Q_b \rightarrow Q_a, \quad p_{a,b} := p_{a+1} \circ p_{a+2} \circ \cdots \circ p_b \circ p_{b-1} \circ p_b,\]
\[s_{b,a} : Q_a \rightarrow Q_b, \quad s_{b,a} := s_b \circ s_{b-1} \circ \cdots \circ s_{a+2} \circ s_{a+1}.\]

Clearly one has
\[p_{a,b} \circ s_{b,a} = \text{Id}_{Q_a}.
\]

Thus, for $0 \leq i, a \leq b$ we have
\[p_{i,b} \circ s_{b,a} = \begin{cases} p_{i,b} \circ s_{b,i} \circ s_{i,a} & i > a \\ p_{i,a} \circ p_{a,b} \circ s_{i,a} & i \leq a \end{cases} = \begin{cases} s_{i,a} & i > a \\ p_{i,a} & i \leq a \end{cases}.
\]

Thus we get an isomorphism $\varphi : Q \rightarrow \text{gr}Q$ of objects in $H^1 YD$ given by
\[\varphi(x) := \{p_{0,d}(x) + \pi_1 p_{1,d}(x) + \pi_2 p_{2,d}(x) + \cdots + \pi_{d-2} p_{d-2,d}(x) + \pi_{d-1} p_{d-1,d}(x) + \pi_d(x)\}
\]
where we set
\[ \pi_0 = \text{Id}_{Q_0}, \quad p_{d,d} = \text{Id}_{Q_d}. \]
For \(0 \leq n \leq d\), we have
\[
\varphi (x^{n,i}) = \varphi (s_{d,n} (x^{n,i})) = \varphi (s_{d,n} \sigma_n (y^{n,i})) = \sum_{0 \leq t \leq d} \pi_t p_{t,d} s_{d,n} (\sigma_n (y^{n,i}))
\]
\[= \sum_{n < t \leq d} \pi_t p_{t,d} s_{d,n} (\sigma_n (y^{n,i})) + \sum_{0 \leq t \leq n} \pi_t p_{t,d} s_{d,n} (\sigma_n (y^{n,i})) \]
\[= \sum_{n < t \leq d} \pi_t s_{t,n} (\sigma_n (y^{n,i})) + \sum_{0 \leq t < n} \pi_t p_{t,n} \sigma_n (y^{n,i}) + \pi_n p_{n,d} s_{d,n} (\sigma_n (y^{n,i}))
\]
\[= \sum_{n < t \leq d} \pi_t s_{t,n} (\sigma_n (y^{n,i})) + \sum_{0 \leq t < n} \pi_t p_{t,n-1} \sigma_{n-1} (\sigma_n (y^{n,i}) + \pi_n \sigma_n (y^{n,i})
\]
\[= 0 + 0 + y^{n,i} = y^{n,i}. \]
Hence \(\varphi (x^{n,i}) = y^{n,i}\). Since \(y^{n,i}\) with \(1 \leq i \leq \dim (Q_n/Q_{n-1}) =: d_n\) form a basis for \(Q_n/Q_{n-1}\) we have that
\[hy^{n,i} \in \frac{Q_n}{Q_{n-1}}, \quad (y^{n,i})_0 \in H \otimes \frac{Q_n}{Q_{n-1}}. \]
Therefore there are \(\chi_{t,i}^n \in H^*\) and \(h_{t,i}^n \in H\) such that
\[hy^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (h) y^{n,t}, \quad (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} h_{t,i}^n \otimes y^{n,t}. \]
We have
\[h (h'y^{n,i}) = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n (h') hy^{n,s} = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n (h') \sum_{1 \leq t \leq d_n} \chi_{t,s}^n (h) y^{n,t}
\]
\[(hh') y^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (hh') y^{n,t}
\]
and hence
\[\chi_{t,i}^n (hh') = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n (h) \chi_{s,i}^n (h'). \]
Moreover
\[y^{n,i} = 1_H y^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n (1_H) y^{n,t}
\]
and hence
\[\chi_{t,i}^n (1_H) = \delta_{t,i}. \]
We also have
\[(y^{n,i})_0 - ((y^{n,i})_0)_0 = \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes \chi_{s,i}^n (h) y^{n,t}
\]
\[= \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes \sum_{1 \leq t \leq d_n} h_{s,t}^n \otimes y^{n,t}
\]
\[= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_{t,i}^n \otimes h_{s,t}^n \otimes y^{n,t}, \]
so that
\[\Delta_H (h_{t,i}^n) = \sum_{1 \leq s \leq d_n} h_{t,i}^n \otimes h_{s,t}^n. \]
Moreover
\[y^{n,i} = \varepsilon_H \left( (y^{n,i})_0 \right) (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} \varepsilon_H (h_{t,i}^n) y^{n,t}
\]
and hence
\[\varepsilon_H (h_{t,i}^n) = \delta_{t,i}. \]
Finally
\[ (h_1 y^{n,i})_{-1} h_2 \otimes (h_1 y^{n,i})_0 = \sum_{1 \leq s \leq d_n} \chi_{s,i} (h_1) (y^{n,s})_{-1} h_2 \otimes (y^{n,s})_0 \]
\[ = \sum_{1 \leq s \leq d_n} \chi_{s,i} (h_1) \sum_{1 \leq t \leq d_n} h_{s,t} h_2 \otimes y^{n,t} \]
\[ = \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_{s,t} \chi_{s,i} (h_1) h_2 \otimes y^{n,t}, \]
\[ h_1 (y^{n,i})_{-1} \otimes h_2 (y^{n,i})_0 = \sum_{1 \leq s \leq d_n} h_1 h_{n,s} \otimes h_2 y^{n,s} = \sum_{1 \leq s \leq d_n} h_1 h_{n,s} \otimes \sum_{1 \leq t \leq d_n} \chi_{t,s} (h_2) y^{n,t} \]
\[ = \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_1 \chi_{t,s} (h_2) h_{n,s} \otimes y^{n,t}. \]
Therefore, we get
\[ \sum_{1 \leq s \leq d_n} h_{s,t} \chi_{s,i} (h_1) h_2 = \sum_{1 \leq s \leq d_n} h_1 \chi_{t,s} (h_2) h_{n,s}. \]
We have
\[ h x^{n,i} = h \sigma_n (y^{n,i}) = \sigma_n (h y^{n,i}) = \sigma_n \left( \sum_{1 \leq t \leq d_n} \chi_{t,s} (h) y^{n,t} \right) = \sum_{1 \leq t \leq d_n} \chi_{t,s} (h) x^{n,t}, \]
\[ (x^{n,i})_{-1} \otimes (x^{n,i})_0 = (\sigma_n (y^{n,i}))_{-1} \otimes (\sigma_n (y^{n,i}))_0 = (y^{n,i})_{-1} \otimes \sigma_n ((y^{n,i})_0) = \sum_{1 \leq t \leq d_n} h_{n,t} \otimes x^{n,t}, \]
\[ \varepsilon_Q (x^{n,i}) = \varepsilon_n (x^{n,i}) = \varepsilon_n \sigma_n (y^{n,i}) = 0 \text{ for } n > 0. \]
If \( Q \) is connected, then \( d_0 = 1 \) so we may assume \( y^{0,0} := 1_Q + Q_{-1} \). Since \( \pi_0 = \text{Id}_{Q_0} \) we get
\[ \sigma_0 = \text{Id}_{Q_0} \circ \sigma_0 = \pi_0 \circ \sigma_0 = \text{Id}_{Q_0} \]
and hence
\[ x^{0,0} = \sigma_0 (y^{0,0}) = \sigma_0 (1_Q + Q_{-1}) = 1_Q. \]
Since, by Proposition 13, \( Q_a \cdot Q_{a'} \subseteq Q_{a+a'} \) for every \( a, a' \in \mathbb{N}_0 \), we can write the product of two elements of the basis in the form
\[ x^{a,l} x^{a',l'} = \sum_{u \leq a + a'} \sum_v \mu_{u,v}^{a,l,a',l'} x^{u,v}. \]
We compute
\[ \overline{x^{a,l} \cdot x^{a',l'}} = (x^{a,l} + Q_{a-1}) \left( x^{a',l'} + Q_{a'-1} \right) \]
\[ = x^{a,l} x^{a',l'} + Q_{a+a'-1} \]
\[ = \left( \sum_{u \leq a + a'} \sum_v \mu_{u,v}^{a,l,a',l'} x^{u,v} \right) + Q_{a+a'-1} \]
\[ = \sum_u \mu_{u+a'+v}^{a,l,a',l'} x^{a+a',v} + Q_{a+a'-1} \]
\[ = \sum_u \mu_{u+a'+v}^{a,l,a',l'} x^{a+a',v} + Q_{a+a'-1} \]
which gives
\[ \overline{x^{a,l} \cdot x^{a',l'}} = \sum_v \mu_{u+a'+v}^{a,l,a',l'} x^{a+a',v}. \]

Remark 3.1. Let \( H \) be a Hopf algebra and let \((A, m_A, u_A)\) be an algebra in \( H \text{-}\text{YD} \). Let \( \varepsilon_A : A \to \mathbb{k} \) be an algebra map in \( H \text{-}\text{YD} \). The Hochschild cohomology in a monoidal category is known, see e.g. [AMS2]. Consider \( \mathbb{k} \) as an \( A \)-bimodule in \( H \text{-}\text{YD} \) through \( \varepsilon_A \). Then, following [AMS2, 1.24], we can consider an analogue of the standard complex
\[ H \text{-}\text{YD} (\mathbb{k}, \mathbb{k}) \xrightarrow{\partial^0} H \text{-}\text{YD} (A, \mathbb{k}) \xrightarrow{\partial^1} H \text{-}\text{YD} (A \otimes^2 \mathbb{k}) \xrightarrow{\partial^2} H \text{-}\text{YD} (A \otimes^3 \mathbb{k}) \xrightarrow{\partial^3} \ldots \]
Explicitly, given \( f \) in the corresponding domain of \( \partial^n \), for \( n = 0, 1, 2, 3 \), we have
\[ \partial^n (f) = f (1) \varepsilon_A - \varepsilon_A f (1) = 0, \]
\[ \partial^1(f) = f \otimes \varepsilon_A - f m_A + \varepsilon_A \otimes f, \]
\[ \partial^2(f) = f \otimes \varepsilon_A - f (A \otimes m_A) + f (m_A \otimes A) - \varepsilon_A \otimes f, \]
\[ \partial^3(f) = f \otimes \varepsilon_A - f (A \otimes A \otimes m_A) + f (A \otimes m_A \otimes A) - f (m_A \otimes A \otimes A) + \varepsilon_A \otimes f. \]

For every \( n \geq 1 \) denote by

\[ Z^n_{YD}(A, k) := \ker (\partial^n), \quad B^n_{YD}(A, k) := \text{Im} (\partial^{n-1}), \quad \text{and} \quad H^n_{YD}(A, k) := \frac{Z^n_{YD}(A, k)}{B^n_{YD}(A, k)} \]

the abelian groups of \( n \)-cocycles, of \( n \)-coboundaries and the \( n \)-th Hochschild cohomology group in \( \text{H}^n_{YD} \) of the algebra \( A \) with coefficients in \( k \). We point out that the construction above works for an arbitrary \( A \)-bimodule \( M \) in \( \text{H}^n_{YD} \) instead of \( k \).

Next result is inspired by [EG, Proposition 2.3]. Two coquasi-bialgebras \( Q \) and \( Q' \) in \( \text{H}^n_{YD} \) will be called \textit{gauge equivalent} whenever there is some gauge transformation \( \gamma : Q \otimes Q \to k \) in \( \text{H}^n_{YD} \) such that \( Q' \cong Q'\gamma \) as coquasi-bialgebras in \( \text{H}^n_{YD} \), see Proposition 2.4 for the structure of \( Q' \).

**Theorem 3.2.** Let \( H \) be a semisimple and cosemisimple Hopf algebra and let \( (Q, m, u, \Delta, \varepsilon, \omega) \) be a f.d. connected coquasi-bialgebra in \( \text{H}^n_{YD} \). If \( H^3_{YD}(\text{gr} Q, k) = 0 \) then \( Q \) is gauge equivalent to a connected bialgebra in \( \text{H}^n_{YD} \).

**Proof.** For \( t \in \mathbb{N}_0 \), and \( x, y, z \) in the basis of \( Q \), we set

\[ \omega_t(x \otimes y \otimes z) := \delta_{|x|+|y|+|z|,t} \omega(x \otimes y \otimes z). \]

Let us check it defines a morphism \( \omega_t : Q \otimes Q \otimes Q \to k \) in \( \text{H}^n_{YD} \). It is left \( H \)-linear as, by means of (8), the definition of \( \omega_t \) and the \( H \)-linearity of \( \omega \), we can prove that \( \omega_t \left( h \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right) \right) = \varepsilon_H(h) \omega_t \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right). \)

Moreover it is left \( H \)-colinear as, by means of (8), the definition of \( \omega_t \) and the \( H \)-colinearity of \( \omega \), we can prove that

\[ \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right) \left( -1 \right)^{\omega_t} \omega_t \left( \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right) \left( 0 \right) \right) = 1_H \otimes \omega_t \left( x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right). \]

Clearly, for \( x, y, z \in Q \) in the basis, one has

\[ \sum_{t \in \mathbb{N}_0} \omega_t(x \otimes y \otimes z) = \sum_{t \in \mathbb{N}_0} \delta_{|x|+|y|+|z|,t} \omega(x \otimes y \otimes z) = \omega(x \otimes y \otimes z) \]

so that we can formally write

\[ \omega = \sum_{t \in \mathbb{N}_0} \omega_t. \]

Since \( \varepsilon \) is trivial on elements in the basis of strictly positive degree, one gets

\[ \omega_0 = \varepsilon \otimes \varepsilon \otimes \varepsilon. \]

If \( \omega = \omega_0 \) then \( Q \) is a (connected) bialgebra in \( \text{H}_{YD}^n \) and the proof is finished. Thus we can assume \( \omega \neq \omega_0 \) and set

\[ s := \min \left\{ i \in \mathbb{N} : \omega_i \neq 0 \right\}, \quad \widetilde{\omega}_s := \omega_s \left( \varphi^{-1} \otimes \varphi^{-1} \otimes \varphi^{-1} \right), \quad \overline{Q} := \text{gr} Q. \]

Note that \( \overline{Q} \) is a morphism in \( \text{H}_{YD}^n \) as a composition of morphisms in \( \text{H}_{YD}^n \).

Let \( n \in \mathbb{N}_0 \), let \( C^4 = Q \otimes Q \otimes Q \otimes Q \) and let \( u \in C^4_{(n)} = \sum_{i+k+l+\ell = n} Q_i \otimes Q_j \otimes Q_k \otimes Q_l. \)

A direct computation rewriting the cocycle condition using (21) proves that, for every \( n \in \mathbb{N}_0 \), and \( u \in C^4_{(n)} \),

\[ \sum_{0 \leq i+j \leq n} \left[ \omega_i (Q \otimes Q \otimes m) \ast \omega_j (m \otimes Q \otimes Q) \right] (u) \]
By means of these equalities one gets
\[ (\omega_s (Q \otimes m \otimes Q)) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \) so that there exists a morphism \( \overline{\omega} : \overline{Q} \otimes \overline{Q} \to k \) in \( H^1_{\mathcal{YD}} \) such that
\[ \overline{\omega}_s = \partial^2 \overline{\omega} = \overline{\varepsilon} \otimes \overline{\omega} - \overline{\varepsilon} \left( Q \otimes m \right) + \overline{m} \left( \overline{Q} \otimes \overline{Q} \right) - \varepsilon \overline{\omega} \overline{\varepsilon}. \]

Explicitly, on elements in the basis we get
\[ \omega_s \left( m_{Q \otimes Q} \otimes \overline{Q} \right) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \), one has that
\[ \omega_s (m_{Q \otimes Q} \otimes \overline{Q}) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \), so that there exists a morphism \( \overline{\omega} : \overline{Q} \otimes \overline{Q} \to k \) in \( H^1_{\mathcal{YD}} \) such that
\[ \overline{\omega}_s = \partial^2 \overline{\omega} = \overline{\varepsilon} \otimes \overline{\omega} - \overline{\varepsilon} \left( Q \otimes m \right) + \overline{m} \left( \overline{Q} \otimes \overline{Q} \right) - \varepsilon \overline{\omega} \overline{\varepsilon}. \]

Explicitly, on elements in the basis we get
\[ \omega_s \left( m_{Q \otimes Q} \otimes \overline{Q} \right) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \), so that there exists a morphism \( \overline{\omega} : \overline{Q} \otimes \overline{Q} \to k \) in \( H^1_{\mathcal{YD}} \) such that
\[ \overline{\omega}_s = \partial^2 \overline{\omega} = \overline{\varepsilon} \otimes \overline{\omega} - \overline{\varepsilon} \left( Q \otimes m \right) + \overline{m} \left( \overline{Q} \otimes \overline{Q} \right) - \varepsilon \overline{\omega} \overline{\varepsilon}. \]

Explicitly, on elements in the basis we get
\[ \omega_s \left( m_{Q \otimes Q} \otimes \overline{Q} \right) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \), so that there exists a morphism \( \overline{\omega} : \overline{Q} \otimes \overline{Q} \to k \) in \( H^1_{\mathcal{YD}} \) such that
\[ \overline{\omega}_s = \partial^2 \overline{\omega} = \overline{\varepsilon} \otimes \overline{\omega} - \overline{\varepsilon} \left( Q \otimes m \right) + \overline{m} \left( \overline{Q} \otimes \overline{Q} \right) - \varepsilon \overline{\omega} \overline{\varepsilon}. \]

Explicitly, on elements in the basis we get
\[ \omega_s \left( m_{Q \otimes Q} \otimes \overline{Q} \right) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]

This equality follows by using (23) and the definition of \( s \).

By assumption \( H^2_{\mathcal{YD}} (\text{gr} Q, k) = 0 \), so that there exists a morphism \( \overline{\omega} : \overline{Q} \otimes \overline{Q} \to k \) in \( H^1_{\mathcal{YD}} \) such that
\[ \overline{\omega}_s = \partial^2 \overline{\omega} = \overline{\varepsilon} \otimes \overline{\omega} - \overline{\varepsilon} \left( Q \otimes m \right) + \overline{m} \left( \overline{Q} \otimes \overline{Q} \right) - \varepsilon \overline{\omega} \overline{\varepsilon}. \]

Explicitly, on elements in the basis we get
\[ \omega_s \left( m_{Q \otimes Q} \otimes \overline{Q} \right) (u) = \omega_s (m_{Q \otimes Q} \otimes Q) (u) \]
and similarly for the other pieces so that one has to check that
\[ \omega_s (m \otimes Q \otimes Q) (u) + \omega_s (Q \otimes Q \otimes m) (u) = (\varepsilon \otimes \omega_s) (u) + \omega_s (Q \otimes m \otimes Q) (u) + (\omega_s \otimes \varepsilon) (u). \]
so that $v(x \otimes 1_Q) = 0$ and hence $\gamma(x \otimes 1_Q) = \varepsilon(x) + v(x \otimes 1_Q) = \varepsilon(x)$. Similarly one proves $\gamma(1_Q \otimes x) = \varepsilon(x)$. Hence $\gamma$ is unital. Note that the coalgebra $C = Q \otimes Q$ is connected as $Q$ is.

Thus, in order to prove that $\gamma : Q \otimes Q \to k$ is convolution invertible it suffices to check (see Lemma 5.2.10) that $\gamma_{|k1_Q \otimes k1_Q}$ is convolution invertible. But for $k, k' \in k$ we have

$$\gamma(k1_Q \otimes k'1_Q) = kk'\gamma(1_Q \otimes 1_Q) = kk'(1_Q) = k(k' \cdot (1_Q \otimes 1_Q))$$

Hence $\gamma_{|k1_Q \otimes k1_Q} = (\varepsilon \otimes \varepsilon)_{|k1_Q \otimes k1_Q}$ which is convolution invertible. Thus there is a $k$-linear map $\gamma^{-1} : Q \otimes Q \to k$ and such that

$$\gamma \ast \gamma^{-1} = \varepsilon \otimes \varepsilon = \gamma^{-1} \ast \gamma.$$ 

Note that, by Lemma 2.3, $\gamma \in H^l(YD)$ implies $\gamma^{-1} \in H^l(YD)$.

Therefore $\gamma$ is a gauge transformation in $H^l(YD)$. By Proposition 2.3, $Q^\gamma$ is a co-quasi-bialgebra in $H^l(YD)$. By Proposition 2.4, we have that $grQ^\gamma$ and $grQ$ coincide as bialgebras in $H^l(YD)$. Hence $H^3_{YD}(grQ^\gamma, k) = H^3_{YD}(grQ, k) = 0$. Therefore $Q^\gamma$ fulfills the same requirement of $Q$ as in the statement. Let us check that $\omega^\gamma = 0$ for $1 \leq t \leq s$ (this will complete the proof by an induction process as $Q$ is finite-dimensional).

Note that the definition of $\gamma$ and $25$ imply

$$(26) \quad \gamma(x \otimes y) = \delta_{|x|+|y|} \gamma(x \otimes y) + \delta_{|x|+|y|+1} \gamma(x \otimes y)$$

for $x, y$ in the basis.

Let $C^2 = Q \otimes Q$ and let $C^2_{(u)} = \sum_{i+j \leq n} Q_i \otimes Q_j$. For $u \in C^2_{(2s-1)}$ we have

$$[\gamma \ast ((\varepsilon \otimes \varepsilon) - v)](u) = (\varepsilon \otimes \varepsilon)(u) - v(u) + v(u) - v(u_1) + v(u_2) \quad (26) \quad (\varepsilon \otimes \varepsilon)(u).$$

Therefore $[\gamma \ast ((\varepsilon \otimes \varepsilon) - v)]C^2_{(2s-1)} = (\varepsilon \otimes \varepsilon)C^2_{(2s-1)}$. By uniqueness of the convolution inverse, we deduce

$$\gamma^{-1}(u) = (\varepsilon \otimes \varepsilon)(u) - v(u), \quad \text{for } u \in C^2_{(2s-1)}.$$

Let $x, y, z$ be in the basis. Set $\overline{u} := \overline{x} \otimes \overline{y} \otimes \overline{z}$ and $u := x \otimes y \otimes z$. We compute

$$(\omega^\gamma)_s(u) = \delta_{|x|+|y|+z} \omega^\gamma(u) = \delta_{|x|+|y|+z} \left[ (\varepsilon \otimes \varepsilon)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) + (\varepsilon \otimes \varepsilon)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) \right].$$

Now, all terms appearing in the last two lines, excepted $\omega_s$, vanish out of degrees $0$ and $s$ and coincide with $\varepsilon \otimes \varepsilon \otimes \varepsilon$ on degree $0$. On the other hand $\omega_s$ vanishes out of $s$. Since $\gamma := (\varepsilon \otimes \varepsilon) + v$ and in view of (27), the term $\delta_{|x|+|y|+z} \omega^\gamma$ forces the following simplification

$$(\omega^\gamma)_s(u) = \delta_{|x|+|y|+z} \left[ (\varepsilon \otimes \varepsilon)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) - (\varepsilon \otimes \varepsilon)(u) + \delta_{|x|+|y|+z} \omega_s(u) \right].$$

Now $\omega_s(u) = \overline{w}_s(\overline{u})$ while one proves that

$$\varepsilon \otimes v)(u) = \left( \overline{Q} \otimes \overline{m} \right)(\overline{u}), \delta_{|x|+|y|+z} v(Q \otimes m)(u) = \delta_{|x|+|y|+z} \left( m_{Q \otimes Q}^\gamma \right)(\overline{u})$$

and similarly for the other pieces of the equality.

Thus one gets

$$(\omega^\gamma)_s(u) = \delta_{|x|+|y|+z} \left[ (\varepsilon \otimes \varepsilon)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) - (\varepsilon \otimes \varepsilon)(u) + \delta_{|x|+|y|+z} \omega_s(u) \right].$$
\[
\frac{\partial}{\partial x} \mathbb{P} + \delta_{\abs{x+y+z} = \abs{\omega}} \mathbb{P} = 0.
\]

For \(0 \leq t \leq s - 1\), analogously to the above, we compute
\[
(\omega^\gamma)_t (u) = \delta_{\abs{x+y+z} = \abs{\omega}} (u)
\]
\[
= \delta_{\abs{x+y+z} = \abs{\omega}} \left[ (\varepsilon \otimes \gamma) * (Q \otimes m) * \omega * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
= \delta_{\abs{x+y+z} = \abs{\omega}} \left[ (\varepsilon \otimes \gamma) * (Q \otimes m) * \omega_0 * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
\partial_{\omega} \delta_{\abs{x+y+z} = \abs{\omega}} \left[ (\varepsilon \otimes \gamma) * (Q \otimes m) * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon) \right] (u)
\]
\[
= \delta_{\abs{x+y+z} = \abs{\omega}} \left( \varepsilon \otimes \varepsilon \otimes \varepsilon \right) (u) = \delta_{\omega, t} (\varepsilon \otimes \varepsilon \otimes \varepsilon) (u).
\]

Therefore, we can now repeat the argument on \(\omega^\gamma\) instead of \(\omega\). Deforming several times we will get a reassociator, say \(\omega'\), whose first non-trivial component \(\omega'_{t}\), with \(t \neq 0\), exceeds the dimension of \(Q\). In other words \(\omega' = \omega'_{0}\) which is trivial. Hence \(Q\) is gauge equivalent to a connected bialgebra in \(\mathcal{H}^{2}\).

\[\square\]

4. INVARINTS

Given a \(k\)-algebra \(A\), we denote by \(H^n (A, -)\) the \(n\)-th right derived functor of \(\text{Hom}_{A,A} (A, -)\) in the category of \(A\)-bimodules. In other words, for every \(A\)-bimodule \(M\), \(H^n (A, M)\) is the Hochschild cohomology group of \(A\) with coefficients in \(M\). Denote by \(Z^n (A, M)\) and \(B^n (A, M)\) the abelian groups of \(n\)-cocycles and of \(n\)-coboundaries respectively.

Let \(H\) be a Hopf algebra, let \(B\) be a left \(H\)-module algebra and let \(M\) be a \(B \# H\)-bimodule, where \(B \# H\) denotes the smash product algebra, see e.g. [Mo, Definition 4.1.3]. Then \(H^n (B, M)\) becomes an \(H\)-bimodule as follows. Its structure of left \(H\)-module is given via \(\varepsilon_H\) and its structure of right \(H\)-module is defined, for every \(f \in Z^n (B, M)\) and \(h \in H\), by setting
\[
[f] h := [\chi^n_H (M) (f)]
\]
where, for every \(k \in k, b_1, \ldots, b_n \in B\), we set
\[
\chi^0_H (M) (f) (k) := (1_B \# S (h_1)) f (k) (1_B \# h_2) \quad \text{for } n = 0 \quad \text{and for } n \geq 1
\]
\[
\chi^n_H (M) (f) (b_1 \otimes b_2 \otimes \cdots \otimes b_n) := (1_B \# S (h_1)) f (h_2 b_1 \otimes h_3 b_2 \otimes \cdots \otimes h_n b_n) (1_B \# h_{n+2}).
\]

Moreover
\[
\partial^n \circ \chi^h_H (M) = \chi^{h+1}_h (M) \circ \partial^n, \quad \text{for every } n \geq -1,
\]
where \(\partial^n : \text{Hom}_k (B^{\otimes n}, M) \to \text{Hom}_k (B^{\otimes (n+1)}, M)\) denotes the differential of the usual Hochschild cohomology.

Denote by \(H^n (B, M)^H\) the space of \(H\)-invariant elements of \(H^n (B, M)\).

**Proposition 4.1.** Let \(H\) be a semisimple Hopf algebra and let \(B\) be a left \(H\)-module algebra. Denote by \(A := B \# H\). Then, for each \(n \in \mathbb{N}_0\) and for every \(A\)-bimodule \(M\)
\[
H^n (B \# H, M) \cong H^n (B, M)^H.
\]

**Proof.** We will apply [St1, Equation (3.6.1)]. To this aim we have to prove first that \(A/B\) is an \(H\)-Galois extension such that \(A\) is flat as left and right \(B\)-module. Now, \(A = B \# H\) for \(\xi : H \otimes H \to B\) defined by \(\xi (x, y) = \varepsilon_H (x) \varepsilon_H (y) 1_A\), cf. [Mo, Definition 7.1.1]. Moreover a direct computation shows that \(\iota : B \to A : b \mapsto b \# 1_H\) is a right \(H\)-extension where \(A\) is regarded as a right \(H\)-comodule algebra via \(\rho : A \to A \otimes H : b \# h \mapsto (b \# h_1) \otimes h_2\). Thus, by [Mo, Proposition 7.2.7], we know that \(\iota : B \to A\) is \(H\)-cleft and hence, by [Mo, Theorem 8.2.4], it is \(H\)-Galois. The \(B\)-bimodule structure of \(A\) is induced by \(\iota\) so that, explicitly, we have
\[
b' (b \# h) = (b' \# 1_h) (b \# h) = b' b \# h,
\]
\[
(b \# h) b' = (b \# h) (b' \# 1_h) = b (h_1 b') \# h_2.
\]
Note that $A = B\#H$ is flat as a left $B$-module as $H$ is a free $k$-module ($k$ is a field). Now consider the map $\alpha : H \otimes B \to A$ defined by setting $\alpha(h \otimes b) := h_1 b \otimes h_2$ (note that it is defined as the braiding in $H_H^D$). We have

$$\alpha(h \otimes bb') = h_1(b') \otimes h_2 = (h_1 b)(h_2 b') = \alpha(h \otimes b) b'$$

so that $\alpha$ is right $B$-linear where $H \otimes B$ is regarded as a right module via $(h \# b)b' := h \# bb'$. Now $H$ is semisimple and hence separable (see [St1, Corollary 3.7]). Thus $H$ is finite-dimensional and hence it has bijective antipode $S_H$. Thus $\alpha$ is invertible with inverse given by $\alpha^{-1}(b \# h) := h_2 \otimes S_H^{-1}(h_1) b$. Therefore $\alpha$ is an isomorphism of right $B$-modules and hence $A$ is flat as a right $B$-module as $H \otimes B$ is.

We have now the hypotheses necessary to apply [St1, Equation (3.6.1)] and obtain

$$H^n(A, M) \cong \text{Hom}_{-H}(k, H^n(B, M)) = \text{Hom}_k(k, H^n(B, M))^H \cong H^n(B, M)^H.$$

□

Remark 4.2. Proposition 4.1 in the particular case when $M = k$ and $B$ is finite-dimensional is [SV, Theorem 2.17]. Note that in the notations therein, one has $E(B) = \oplus_{n \in \mathbb{N}} E_n(B, k)$ where $E_n(B, k) = \text{Ext}^n_B(k, k) \cong H^n(B, k)$. The latter isomorphism is [CE, Corollary 4.4, page 170].

Let $H$ be a Hopf algebra and let $B$ be a bialgebra in the braided category $H_H^D$. Denote by $A := B \# H$ the Radford-Majid bosonization of $B$ by $H$, see e.g. [Rm, Theorem 1]. Note that $A$ is endowed with an algebra map $\varepsilon_A : A \to k$ defined by $\varepsilon_A(b \# h) = \varepsilon_B(b) \varepsilon_H(h)$ so that we can regard $k$ as an $A$-bimodule via $\varepsilon_A$. Then we can consider $H^n(B, k)$ as an $H$-bimodule as follows. Its structure of left $H$-module is given via $\varepsilon_H$ and its structure of right $H$-module is defined, for every $f \in \mathbb{Z}^n(B, k)$ and $h \in H$, by setting

$$[f]h := [fh],$$

where $(fh)(z) = f(hz)$, for every $z \in B^{\otimes n}$. The latter is the usual right $H$-module structure of $\text{Hom}_k(B^{\otimes n}, k)$. Indeed, for every $n \geq -1$, the vector space $\text{Hom}_k(B^{\otimes n}, k)$ is an $H$-bimodule with respect to this right $H$-module structure and the left one induced by $\varepsilon_H$.

Corollary 4.3. Let $H$ be a semisimple Hopf algebra and let $B$ be a bialgebra in the braided category $H_H^D$. Set $A := B \# H$. Then, for each $n \in \mathbb{N}_0$

$$H^n(B \# H, k) \cong H^n(B, k)^H$$

and the differential $\partial^n : \text{Hom}_k(B^{\otimes n}, k) \to \text{Hom}_k(B^{\otimes (n+1)}, k)$ of the usual Hochschild cohomology is $H$-bilinear.

Proof. In the particular case when $M = k$, the right module $H$-structure used in Proposition 4.1 simplifies as follows. It is defined, for every $f \in \mathbb{Z}^n(B, k)$ and $h \in H$, by setting

$$[f]h := [\chi^h_n(k)(f)]$$

where, for every $k \in k, b_1, \ldots, b_n \in B$, we set

$$\chi^h_n(k)(f)(k) := \varepsilon_H(h)(f)(k) \text{ for } n = 0 \text{ while and for } n \geq 1$$

$$\chi^h_n(k)(f)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) := f(h_1 b_1 \otimes h_2 b_2 \otimes \cdots \otimes h_n b_n).$$

More concisely $\chi^h_n(k)(f)(z) = f(hz)$ for every $n \in \mathbb{N}_0$ and $z \in B^{\otimes n}$ i.e. $[f]h := [fh]$ where $fh := \chi^h_n(k)(f)$.

Now consider the differential $\partial^n : \text{Hom}_k(B^{\otimes n}, k) \to \text{Hom}_k(B^{\otimes (n+1)}, k)$ of the usual Hochschild cohomology. Note that for each $n \in \mathbb{N}_0$, $\text{Hom}_k(B^{\otimes n}, k)$ is regarded as a bimodule over $H$ using the left $H$-module structures of its arguments. By [Rm], we have

$$\partial^n \chi^h_n(k)(f) = \chi^h_{n+1}(k) \partial^n(f)$$

Since $\chi^h_n(k)(f) = fh$, the last displayed equality becomes $\partial^n(fh) = \partial^n(f)h$ for every $n \in \mathbb{N}_0$. Thus $\partial^n$ is right $H$-linear. Since $hf = \varepsilon_H(h)f$ for every $f \in \text{Hom}_k(B^{\otimes n}, k)$, $h \in H$, we get that $\partial^n$ is also left $H$-linear whence $H$-bilinear. □
Remark 4.4. Note that, in the context of the proof of [EC, Proposition 5.1], one has
\[ H^3(B(V) \# C[Z_p], C) \cong H^3(B(V), C)^Z_p. \]
This is a particular case of Corollary 4.3 where \( H = C[Z_p], V \in \mathcal{Y}D \) and \( B = B(V) \).

Proposition 4.5. Let \( \mathcal{C} \) and \( \mathcal{D} \) be abelian categories. Let \( r, \omega : \mathcal{C} \to \mathcal{D} \) be exact functors such that \( r \) is a subfunctor of \( \omega \) i.e. there is a natural transformation \( \eta : r \to \omega \) which is a monomorphism when evaluated on objects. If \( X \) is a subobject of \( Y \) then \( r(X) = \omega(X) \cap r(Y) \). Moreover, for every morphism \( f : X \to Y \) in \( \mathcal{C} \) one has
\[
\ker(r(f)) = r(\ker(f)) = \omega(\ker(f)) \cap r(X) = \ker(\omega(f)) \cap r(X),
\]
\[
\text{Im}(r(f)) = \text{Im}(\omega(f)) \cap r(Y) = r(\text{Im}(f)).
\]

Proof. The proof is similar to [Sta, Proposition 1.7, page 138].

Remark 4.6. From Corollary 4.3, we have
\[
H^n(B, k)^H = \{ [f] \mid f \in Z^n(B, k), \varepsilon_H(h)[f] = [f]h, \text{ for every } h \in H \} = \{ [f] \mid f \in Z^n(B, k), [\varepsilon_H(h)f] = [fh], \text{ for every } h \in H \}
\]
where, for every \( z \in B^{\otimes n} \), we have
\[
(fh)(z) = f(hz).
\]

Note that, for any \( H \)-bimodule \( M \) one has
\[
\text{Hom}_{H, H}(H, M) \cong M^H = \{ m \in M \mid hm = mh, \text{ for every } h \in H \}.
\]

Note also that \( H \) is a separable \( k \)-algebra whence it is projective in the category of \( H \)-bimodules. As a consequence \( \text{Hom}_{H, H}(H, -) \cong (-)^H : H_{\mathcal{M}} \to \mathcal{M} \) is an exact functor (here \( H_{\mathcal{M}} \) is the category of \( H \)-bimodules and \( \mathcal{M} \) the category of \( k \)-vector spaces). By Proposition 4.3 applied to the case when \( r := (-)^H : H_{\mathcal{M}} \to \mathcal{M} \) and \( \omega \) is the forgetful functor, for every morphism \( f : X \to Y \) of \( H \)-bimodules one has
\[
\ker(f^H) = \ker(f) \cap X^H = (\ker(f))^H \quad \text{and} \quad \text{Im}(f^H) = \text{Im}(f) \cap Y^H = (\text{Im}(f))^H.
\]

Still by Corollary 4.3, we know that the differential \( \partial^n : \text{Hom}_k(B^{\otimes n}, k) \to \text{Hom}_k(B^{\otimes (n+1)}, k) \) of the usual Hochschild cohomology is \( H \)-bilinear. Thus we can apply the argument above to get
\[
\ker(\partial^n)^H = \ker(\partial^n) \cap \text{Hom}_k(B^{\otimes n}, k)^H = (\ker(\partial^n))^H, \quad \text{and}
\]
\[
\text{Im}(\partial^{n-1})^H = \text{Im}(\partial^{n-1}) \cap \text{Hom}_k(B^{\otimes n}, k)^H = (\text{Im}(\partial^{n-1}))^H.
\]

Now \( \text{Hom}_k(B^{\otimes n}, k)^H = \text{Hom}_{H, -}(B^{\otimes n}, k) \) so that we get
\[
Z^n_{H, \text{-Mod}}(B, k) = Z^n(B, k) \cap \text{Hom}_{H, -}(B^{\otimes n}, k) = Z^n(B, k)^H \quad \text{and}
\]
\[
B^n_{H, \text{-Mod}}(B, k) = B^n(B, k) \cap \text{Hom}_{H, -}(B^{\otimes n}, k) = B^n(B, k)^H.
\]

where \( Z^n_{H, \text{-Mod}}(B, k) \) and \( B^n_{H, \text{-Mod}}(B, k) \) denotes the the abelian groups of \( n \)-cocycles, of \( n \)-coboundaries for the cohomology of the algebra \( B \) with coefficients in \( k \) computed in the monoidal category \( H \)-Mod of left \( H \)-modules. The corresponding \( n \)-th Hochschild cohomology group is
\[
H^n_{H, \text{-Mod}}(B, k) := \frac{Z^n_{H, \text{-Mod}}(B, k)}{B^n_{H, \text{-Mod}}(B, k)} = \frac{Z^n(B, k)^H}{B^n(B, k)^H} \cong \left( \frac{Z^n(B, k)}{B^n(B, k)} \right)^H = H^n(B, k)^H.
\]

Denote by \( D(H) \) the Drinfeld double, see e.g. the first structure of [Ma, Theorem 7.1.1].

Proposition 4.7. In the setting of Corollary 4.3, assume that \( H \) is also cosemisimple. Then, for \( n \in \mathbb{N}_0 \)
\[
Z^n_{\mathcal{Y}D}(B, k) = Z^n(B, k)^{D(H)}, \quad B^n_{\mathcal{Y}D}(B, k) = B^n(B, k)^{D(H)} \quad \text{and} \quad H^n_{\mathcal{Y}D}(B, k) \cong H^n(B, k)^{D(H)}.
\]

where \( Z^n(B, k) \) and \( B^n(B, k) \) are regarded as \( D(H) \)-subbimodules of \( \text{Hom}_k(B^{\otimes n}, k) \) whose structure is induced by the left \( D(H) \)-module structures of its arguments.
Moreover $H^n(B,k)^{D(H)}$ is a subspace of $H^n(B,k)^H$.

**Proof.** For shortness, in this proof, we denote $D(H)$ by $D$. Consider the analogue of the standard complex as in Remark 3.3:

\[
\begin{array}{ccccccc}
H^n \mathcal{YD}(k,k) & \xrightarrow{\emptyset^n} & H^n \mathcal{YD}(B,k) & \xrightarrow{\emptyset^1} & H^n \mathcal{YD}(B^\otimes 2,k) & \xrightarrow{\emptyset^2} & \cdots
\end{array}
\]

where $\emptyset^n$ is induced by the differential $\emptyset^n : \text{Hom}_k(B^\otimes n,k) \rightarrow \text{Hom}_k(B^\otimes (n+1),k)$ of the ordinary Hochschild cohomology. Now, since $H$ is semisimple, it is finite-dimensional (whence it has bijective antipode) so that, by a result essentially due to Majid (see \[Ma, Proposition 10.6.16\]) and by \[RT, Proposition 6\], we get a category isomorphism $H\mathcal{YD} \cong D\mathcal{YD}$. Thus the complex above can be rewritten as follows

\[
\begin{array}{ccccccc}
\text{Hom}_{D,-}(k,k) & \xrightarrow{\emptyset^0} & \text{Hom}_{D,-}(B,k) & \xrightarrow{\emptyset^1} & \text{Hom}_{D,-}(B^\otimes 2,k) & \xrightarrow{\emptyset^2} & \cdots
\end{array}
\]

Now, since, for each $n \in \mathbb{N}_0$, we have $\text{Hom}_{D,-}(B^\otimes n,k) = \text{Hom}_k(B^\otimes n,k)^D$, we obtain the complex

\[
\begin{array}{ccccccc}
\text{Hom}_k(k,k)^D & \xrightarrow{\emptyset^0} & \text{Hom}_k(B,k)^D & \xrightarrow{\emptyset^1} & \text{Hom}_k(B^\otimes 2,k)^D & \xrightarrow{\emptyset^2} & \cdots
\end{array}
\]

We will write $(\emptyset^n)^D$ instead of $\emptyset^n$ when we would like to stress that the map considered is the one induced on invariants. Thus we will write equivalently

\[
\begin{array}{ccccccc}
\text{Hom}_k(k,k)^D & \xrightarrow{(\emptyset^n)^D} & \text{Hom}_k(B,k)^D & \xrightarrow{(\emptyset^n)^D} & \text{Hom}_k(B^\otimes 2,k)^D & \xrightarrow{(\emptyset^n)^D} & \cdots
\end{array}
\]

Now, assume $H$ is also cosemisimple. Since $H$ is both semisimple and cosemisimple, by \[Ra2, Proposition 7\] the Hopf algebra $D$ is semisimple as an algebra. Thus, as in Remark 3.3 in case of $H$, the functor $(-)^D : D\mathcal{YD} \rightarrow \mathcal{YD}$ is exact (here $D\mathcal{YD}$ is the category of $D$-bimodules and $\mathcal{YD}$ the category of $k$-vector spaces). By Proposition 3.3 applied to the case when $r := (-)^D : D\mathcal{YD} \rightarrow \mathcal{YD}$ and $\omega$ is the forgetful functor, for every morphism $f : X \rightarrow Y$ of $D$-bimodules one has

\[
\ker(f^D) = \ker(f) \cap X^D = (\ker(f))^D \quad \text{and} \quad \text{Im}(f^D) = \text{Im}(f) \cap Y^D = (\text{Im}(f))^D.
\]

In particular we get

\[
\ker((\emptyset^n)^D) = \ker(\emptyset^n) \cap \text{Hom}_k(B^\otimes n,k)^D = \ker((\emptyset^n)^D) \quad \text{and} \quad \text{Im}((\emptyset^n-1)^D) = \text{Im}(\emptyset^n-1) \cap \text{Hom}_k(B^\otimes n,k)^D = \text{Im}((\emptyset^n-1)^D)
\]

and hence

\[
\begin{array}{ccc}
\mathcal{Z}^n_B(B,k) & = & \mathcal{Z}^n(B,k) \cap \text{Hom}_{D,-}(B^\otimes n,k)^D = \mathcal{Z}^n(B,k)^D \quad \text{and} \quad \\
\mathcal{B}^n_B(B,k) & = & \mathcal{B}^n(B,k) \cap \text{Hom}_{D,-}(B^\otimes n,k)^D = \mathcal{B}^n(B,k)^D
\end{array}
\]

Then we obtain

\[
\begin{array}{ccc}
\mathcal{H}^n_B(B,k) & = & \frac{\mathcal{Z}^n_B(B,k)}{\mathcal{B}^n_B(B,k)} = \frac{\mathcal{Z}^n(B,k)^D}{\mathcal{B}^n(B,k)^D} \cong H^n(B,k)^D.
\end{array}
\]

Let us prove the last part of the statement. The correspondence between the left $D$-module structure and the structure of Yetter-Drinfeld module over $H$ is written explicitly in \[Ma, Proposition 7.1.6\]. In particular $D = H^* \otimes H$ and given $v \in H^0 \mathcal{YD}$, the two structures are related by the following equality $(f \otimes h) \triangleright v = f((h \triangleright v)_{-1})(h \triangleright v)_0$ for every $f \in H^*$, $h \in H$, $v \in V$. Thus $(\varepsilon_H \otimes h) \triangleright v = h \triangleright v$. Moreover $H$ is a Hopf subalgebra of $D$ via $h \mapsto \varepsilon_H \otimes h$, where $D$ is considered with the first structure of \[Ma, Theorem 7.1.1\]. Since the $D$-bimodule structure of $H^n(B,k)$ is induced by the one of $\text{Hom}_k(B^\otimes n,k)$ which comes from the left $D$-module structures of its arguments and similarly for the $H$-bimodule structure of $H^n(B,k)$, we deduce that $H^n(B,k)^D$ is a subspace of $H^n(B,k)^H$. \qed
Example 4.8. In the setting of the proof of [AM, Theorem 4.1.3], a Nichols algebra \( B(V) \) such that \( H^3(B(V), k) = 0 \) is considered where \( k \) is a field of characteristic zero. By Proposition 1.5 applied in the case \( H = k \mathbb{Z}_m \) and \( B = B(V) \), we have that \( H^3_{YD}(B(V), k) \cong H^3(B(V), k)^{D(H)} \) is a subspace of \( H^3(B(V), k)^H = H^3(B(V), k)^{\mathbb{Z}_m} = 0 \). Thus we get \( H^3_{YD}(B(V), k) = 0 \). Therefore, in view of Theorem 2.5 if \( (Q, m, u, \Delta, \varepsilon, \omega) \) is a d.f. connected coquasi-bialgebra in \( \mathcal{H} \) such that \( \text{gr} Q \cong B(V) \) (as above) as augmented algebras in \( \mathcal{H} \) (the counit must be the same in order to have the same Yetter-Drinfeld module structure on \( k \)), then we can conclude that \( Q \) is gauge equivalent to a connected bialgebra in \( \mathcal{H} \).

Remark 4.9. Let \( A \) be a finite-dimensional coquasi-bialgebra with the dual Chevalley property i.e. the coradical \( H \) of \( A \) is a coquasi-subbialgebra of \( A \) (in particular \( H \) is cosemisimple). Assume the coquasi-bialgebra structure of \( H \) has trivial reassociator (i.e. it is an ordinary bialgebra) and also assume it has an antipode (i.e. it is a Hopf algebra). Then, by [AM, Corollary 4.6], \( \text{gr} A \) is isomorphic to \( R \# H \) as a coquasi-bialgebra, where \( R \) is a suitable connected bialgebra in \( \mathcal{H} \). Note that \( R \# H \) is the usual Radford-Majid bosonization as \( H \) has trivial reassociator, see [AM, Definition 5.4]. Hence we can compute

\[
H^3(\text{gr} A, k) = H^3(R \# H, k).
\]

Assume further that \( H \) is semisimple. Then, by Corollary 2.5, we have

\[
H^n(R \# H, k) \cong H^n(R, k)^H
\]

so that \( H^3(\text{gr} A, k) \cong H^3(R, k)^H \). Thus, if \( H^3(R, k)^H = 0 \), one gets \( H^3(\text{gr} A, k) = 0 \) which is the analogue of the condition [EG, Proposition 2.3] (note that our \( A \) is the dual of the one considered therein) which guarantees that \( A \) is gauge equivalent to an ordinary Hopf algebra, if \( A \) has an a quasi-antipode and \( k = \mathbb{C} \). Next we will give another approach to arrive at the same conclusion but just requiring \( H^3_{YD}(R, k) = 0 \). Note that a priori \( H^3_{YD}(R, k) \cong H^3(R, k)^{D(H)} \) is smaller than \( H^3(R, k)^H \).

5. Dual Chevalley

The main aim of this section is to prove Theorem 5.4. Let \( A \) be a Hopf algebra over a field \( k \) of characteristic zero such that the coradical \( H \) of \( A \) is a sub-Hopf algebra (i.e. \( A \) has the dual Chevalley Property). Assume \( H \) is finite-dimensional so that \( H \) is semisimple. By [ABM, Theorem 1], there is a gauge transformation \( \zeta : A \otimes A \to k \) such that \( A^\zeta \) is isomorphic, as a coquasi-bialgebra, to the bosonization \( Q\# H \) of a connected coquasi-bialgebra \( Q \) in \( \mathcal{H} \) by \( H \). By construction \( \zeta \) is \( H \)-bilinear and \( H \)-balanced: this follows from [ABM, Proposition 5.7] (note that gauge transformation \( v_B : B \otimes B \to k \), used therein for \( B := R^\#_\xi H \), is \( H \)-bilinear and \( H \)-balanced, as observed in the proof) and the fact that there is an \( H \)-bilinear Hopf algebra isomorphism \( \psi : B \to A \) (see [ABM, Proof of Theorem 1, page 36 and Theorem 6.1] which is a consequence of [AMS1, Theorem 3.64]) where \( (R, \xi) \) is a suitable connected pre-bialgebra with cocycle in \( \mathcal{H} \) (note that \( \lambda = v_B \circ (\psi^{-1} \otimes \psi^{-1}) \)): here by connected pre-bialgebra we mean that the coradical \( R_0 \) of \( R \) is \( k \mathbb{1}_R \) (by the properties of \( 1_R \) this implies that \( R_0 \) is a subcoalgebra in \( \mathcal{H} \) of \( R \)). Assume that \( A \) is finite-dimensional. Then \( Q \# H \) and hence \( Q \) is finite dimensional.

Thus, by Theorem 5.2, if \( H^3_{YD}(\text{gr} Q, k) = 0 \), then \( Q \) is gauge equivalent to a connected bialgebra in \( \mathcal{H} \).

First let us check which condition on \( Q \) guarantee that \( H^3_{YD}(\text{gr} Q, k) = 0 \). Note that by construction \( Q = R^\omega \) (see [ABM, Proposition 5.7]) where \( v := (\lambda \xi)^{-1} \), the convolution inverse of \( \lambda \xi \) and \( \lambda : H \to k \) denotes the total integral on \( H \). Thus we can rewrite \( \text{gr}(Q) \) as \( \text{gr}(R^\omega) \).

Moreover \( v_B \) is given by \( v_B ((r \# h) \otimes (r' \# h')) = v (r \otimes h r') \varepsilon_H (h') \) for every \( r, r' \in R, h, h' \in H \). By [AMS1, Proposition 2.5], \( \text{gr}(R) \) inherits the pre-bialgebra structure in \( \mathcal{H} \) of \( R \). This is proved by checking that \( R_i \cap R_j \subseteq R_{i+j} \) for every \( i, j \in \mathbb{N}_0 \), where \( R_i \) denotes the \( i \)-th term of the coradical filtration of \( R \). Moreover \( R_i \) is a subcoalgebra of \( R \) in \( \mathcal{H} \).
**Lemma 5.1.** Keep the above hypotheses and notations. Then $\text{gr}(R^n)$ and $\text{gr}(R)$ coincide as bialgebras in $H^*_Y \mathcal{D}$ where the structures of $\text{gr}(R)$ are induced by the ones of $(R, \xi)$.

**Proof.** By Theorem [1.5], $\text{gr}(R^n) = \text{gr}(Q)$ is a connected bialgebra in $H^*_Y \mathcal{D}$.

Note that $R^n$ and $R$ coincide as coalgebras in $H^*_Y \mathcal{D}$ so that $\text{gr}(R^n)$ and $\text{gr}(R)$ coincide as coalgebras in $H^*_Y \mathcal{D}$. They also have the same unit. It remains to check that their two multiplications coincide too.

Since $\xi$ is unital, by [AMS1, Proposition 4.8], we have that $v$ is unital and this is equivalent to $v^{-1}$ unital (see the proof therein).

Let $C := R \otimes R$. Let $n > 0$ and let $w \in C(n) = \sum_{i+j \leq n} R_i \otimes R_j$. By [AMS1] Lemma 3.69], we have that

$$\Delta_C (w) - w \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes w \in C(n-1) \otimes C(n-1).$$

Thus we get

$$w_1 \otimes w_2 \otimes w_3 - \Delta_C (w) \otimes (1_R)^{\otimes 2} - \Delta_C \left((1_R)^{\otimes 2}\right) \otimes w \in \Delta_C \left(C(n-1)\right) \otimes C(n-1)$$

and hence

$$w_1 \otimes w_2 \otimes w_3 - w \otimes (1_R)^{\otimes 2} \otimes (1_R)^{\otimes 2} - w \otimes (1_R)^{\otimes 2} \otimes (1_R)^{\otimes 2} - \otimes w \in C(n-1) \otimes C(n-1).$$

Since $m \left(C(n-1)\right) \subseteq \sum_{i+j \leq n} m \left(R_i \otimes R_j\right) \subseteq R_{n-1}$ we get

$$w_1 \otimes m (w_2) \otimes w_3 - w \otimes 1_R \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes m (w) \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 3} \otimes w \in C(n-1) \otimes R_{n-1} \otimes C(n-1)$$

and hence

$$(29) \quad w_1 \otimes (m (w_2) + R_{n-1}) \otimes w_3 = (1_R)^{\otimes 2} \otimes (m (w) + R_{n-1}) \otimes (1_R)^{\otimes 2}.$$ 

Let $x, y \in R$. We compute

$$\mathcal{P} (x \otimes y) = (x + R_{|x|-1}) \cdot v (y + R_{|y|-1}) = (x \cdot y) + R_{|x|+|y|-1} = M_x (x \otimes y) + R_{|x|+|y|-1}$$

and hence

$$v (x \otimes y) \cdot v (y \otimes x) \cdot v^{-1} (x \otimes y) \cdot v^{-1} (y \otimes x) = v (x \otimes y) \cdot v (y \otimes x) \cdot v^{-1} (x \otimes y) \cdot v^{-1} (y \otimes x).$$

The following result is inspired by [AMS1, Theorem 3.71].

**Lemma 5.2.** Let $H$ be a cosemisimple Hopf algebra. Let $C$ be a left $H$-comodule coalgebra such that $C_0$ is a one-dimensional left $H$-comodule subcoalgebra of $C$. Let $B = C \# H$ be the smash coproduct of $C$ by $H$ i.e. the coalgebra defined by

$$(30)\quad \Delta_B (c \# h) = \sum (c_1 \# (c_2)_{(-1)} h_1) \otimes (c_2)_{(0)} \# h_2.$$ 

Then, for every $n \in \mathbb{N}_0$, we have $B_n = C_n \# H$.

**Proof.** Since $C_0$ is a subcoalgebra of $C$ in $H \mathfrak{M}$, and, for $n \geq 1$, one has $C_n = C_{n-1} \wedge C_0$, then inductively one proves that $C_n$ is a subcoalgebra of $C$ in $H \mathfrak{M}$. Set $B(n) := C_n \# H$ for every $n \in \mathbb{N}_0$.

Let us check that $B(n) = B_n$ by induction on $n \in \mathbb{N}_0$.

Let $n = 0$. First note $B = \cup_{m \in \mathbb{N}_0} B(m)$ and, since $\Delta_C (C_m) \subseteq \sum_{0 \leq i \leq m} C_i \otimes C_{m-i}$, we also have

$$\Delta_B (B(m)) = \Delta_B (C_m \# H) \subseteq \sum_{0 \leq i \leq m} C_i \# (C_{m-i})_{(-1)} (H)_1 \otimes ((C_{m-i})_{(0)} \# (H)_2) \subseteq \sum_{0 \leq i \leq m} (C_i \# H) \otimes (C_{m-i} \# (H)) = \sum_{0 \leq i \leq m} B(i) \otimes B(m-i).$$
Therefore $(B^{(m)})_{m \in \mathbb{N}_0}$ is a coalgebra filtration for $B$ and hence, by [32, Proposition 11.1.1], we get that $B^{(0)} \supseteq B_0$. Since $C_0$ is one-dimensional, there is a grouplike element $1_C \in C_0$ such that $C_0 = k1_C$. Moreover one checks that $C_0$ is a coideal subalgebra of $C$ in $H \otimes \mathfrak{M}$ implies $\sum (1_C \# (1_C)^{(-1)} \otimes (1_C)^{(0)}) = 1_H \otimes 1_C$.

Let $\sigma : H \to C \otimes H : h \mapsto 1_C \otimes h$ be the canonical injection. We have

$$\Delta_B \sigma (h) = \Delta_B (1_C \otimes h) = \sum \left( 1_C \# (1_C)^{(-1)} h_1 \right) \otimes \left( (1_C)^{(0)} \# h_2 \right)$$

$$= \sum (1_C \# 1_H h_1) \otimes (1_C \# h_2) = \sum \sigma (h_1) \otimes \sigma (h_2) = (\sigma \otimes \sigma) \Delta_H (h),$$

$$\varepsilon_B \sigma (h) = \varepsilon_B (1_C \otimes h) = \varepsilon_C (1_C) \varepsilon_H (h) = \varepsilon_H (h)$$

so that $\sigma$ is a coalgebra map. Since $H$ is cosemisimple and $\sigma$ an injective coalgebra map we deduce that also $\sigma (H) = C_0 \otimes H = B^{(0)}$ is a cosemisimple subcoalgebra of $B$ whence $B^{(0)} \subseteq B_0$.

Let $n > 0$ and assume that $B_i = B^{(i)}$ for $0 \leq i \leq n - 1$. Let $\sum c_i h_i \in B_n$. Then

$$\Delta_B \left( \sum_{i \in I} c_i \# h_i \right) \in B_{n-1} \otimes B + B \otimes B_0 = C_{n-1} \otimes H \otimes C \otimes H + C \otimes H \otimes C_0 \otimes H.$$ Let $p_n : C \to \frac{C_n}{C_n}$ be the canonical projection. If we apply $(p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)$ we get

$$0 = (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \Delta_B \left( \sum_{i \in I} c_i \# h_i \right)$$

$$= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \left( \sum_{i \in I} \left( (c_i)_1 \# ((c_i)_2)^{(-1)} (h_i)_1 \right) \otimes \left( ((c_i)_2)^{(0)} \# (h_i)_2 \right) \right)$$

$$= (p_{n-1} \otimes p_0 \otimes H) \left( \sum_{i \in I} (c_i)_1 \otimes (c_i)_2 \otimes h_i \right) = ((p_{n-1} \otimes p_0) \Delta_C \otimes H) \left( \sum_{i \in I} c_i \# h_i \right).$$

Thus $\sum c_i \# h_i \in \ker ((p_{n-1} \otimes p_0) \Delta_C \otimes H) = [\ker ((p_{n-1} \otimes p_0) \Delta_C)] \otimes H = C_n \otimes H = B^{(n)}$. Thus $B_n \subseteq B^{(n)}$. On the other hand, form $\Delta_C (C_n) \subseteq C_{n-1} \otimes C + C \otimes C_0$ we deduce

$$\Delta_B (B^{(n)}) = \Delta_B (C_n \otimes H)$$

$$\subseteq \sum \left( (C_n)_1 \# ((C_n)_2)^{(-1)} (H)_1 \right) \otimes \left( ((C_n)_2)^{(0)} \# (H)_2 \right)$$

$$\subseteq \sum \left( C_{n-1} \# (C)^{(-1)} H \right) \otimes \left( (C)^{(0)} \# H \right) + \sum \left( C \# (C_0)^{(-1)} H \right) \otimes \left( (C_0)^{(0)} \# H \right)$$

$$\subseteq \sum \left( C_{n-1} \# H \otimes C \# H \right) + \left( C \# H \otimes C_0 \# H \right)$$

$$= B^{(n-1)} \otimes B + B \otimes B^{(0)} = B_{n-1} \otimes B + B \otimes B_0$$

and hence $B_n \subseteq B^{(n)}$. \hfill $\square$

**Definition 5.3.** Let $A$ be a Hopf algebra over a field $k$ such that the coradical $H$ of $A$ is a sub-Hopf algebra (i.e. $A$ has the dual Chevalley Property). Set $G := \text{gr} (A)$. There are two canonical Hopf algebra maps

$$\sigma_G : H \to \text{gr} (A) : h \mapsto h + A_{-1},$$

$$\pi_G : \text{gr} (A) \to H : a + A_{n-1} \mapsto a \delta_{n,0}, \quad n \in \mathbb{N}_0.$$ The diagram of $A$ (see [AS1, page 659]) is the vector space

$$D(A) := \{ d \in \text{gr} (A) | \sum d_1 \otimes \pi_G (d_2) = d \otimes 1_H \}.$$ It is a bialgebra in $H^H \otimes \mathcal{YD}$ as follows. $D(A)$ is a subalgebra of $G$. The left $H$-action, the left $H$-coaction of $D(A)$, the comultiplication and counit are given respectively by

$$h \triangleright d := \sum \sigma_G (h_1) d \sigma GS (h_2), \quad \rho (d) = \sum \pi_G (d_1) \otimes d_2,$$
\[ \Delta_{D(A)}(d) := \sum d_1 \sigma_G S_H \pi_G(d_2) \otimes d_3, \quad \varepsilon_{D(A)}(d) = \varepsilon_G(d). \]

Although the following result seems to be folklore, we include here its statement for future references.

**Proposition 5.4.** Let \( A \) be a Hopf algebra over a field \( \mathbb{k} \) such that the coradical \( H \) of \( A \) is a sub-Hopf algebra. Let \( A' \) be a Hopf algebra over a field \( \mathbb{k} \). Let \( f : A' \to A \) be an isomorphism of Hopf algebras. Then \( H' := f^{-1}(H) \cong H \) is the coradical of \( A' \) and it is a sub-Hopf algebra of \( A' \). Thus we can identify \( H' \) with \( H \). Moreover \( f \) induces an isomorphism \( D(f) : D(A') \to D(A) \) of bialgebras in \( H^H \mathcal{YD} \).

**Proposition 5.5.** Keep the hypotheses and notations of the beginning of the section. Then \( D(A) \cong D(R \# \xi H) \cong \text{gr}(R) \) as bialgebras in \( H^H \mathcal{YD} \).

**Proof.** Apply Proposition 5.4 to the canonical isomorphism \( \psi : B := R \# \xi H \to A \) that we recalled at the beginning of the section to get that \( D(R \# \xi H) \cong D(A) \). Note that, by \( H \)-linearity we have
\[ \psi(1_R \# h) = \psi((1_R \# 1_H)(1_R \# h)) = \psi((1_R \# 1_H) h) = \psi(1_R \# 1_H) h = h \]
so that \( \psi(k1_R \otimes H) = H \) and hence \( H' = \psi^{-1}(H) = k1_R \otimes H \) with the notation of Proposition 5.4.

Thus \( B_0 = k1_R \otimes H = R_0 \otimes H \) so that we can identify \( B_0 \) with \( H \) via the canonical isomorphism \( H \to R_0 \otimes H : h \mapsto 1_R \otimes h \). Its inverse is \( R_0 \otimes H \to H : r \otimes h \mapsto \varepsilon_R(r) h \). With this identification and by setting \( G := \text{gr}(B) \), we can consider the canonical bialgebra maps
\[ \sigma_G : H \to \text{gr}(B) : h \mapsto 1_R \# h + (R \# \xi H)_{-1}, \]
\[ \pi_G : \text{gr}(B) \to H : r \# h + (R \# \xi H)_{-1} \to \varepsilon_R(r) h \delta_{n,0}, \]
where \( r \# h \in (R \# \xi H)_n \), \( n \in \mathbb{N}_0 \).

Since the underlying coalgebra of \( B \) is exactly the smash coproduct of \( R \) by \( H \) and \( (R, \xi) \) is a \( \mathbb{N}_0 \)-connected pre-bialgebra with cocycle in \( H \mathcal{YD} \), by Lemma 5.3 we have that \( B_n = R_n \otimes H \). Let us compute \( D := D(B) \). As a vector space it is
\[ D := \left\{ d \in G \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}. \]
By \([AS1] \), Lemma 2.1, we have that \( D = \bigoplus_{n \in \mathbb{N}_0} D^n \) where \( D^n = D \cap G^n = D \cap B_{n-1} \). Let \( d := \sum_i r_i \# h_i \in D^n \) where we can assume \( \sum_i r_i \# h_i \in B_n \setminus B_{n-1} \) and, for every \( i \in I \), \( r_i \# h_i \in B_n \setminus B_{n-1} \).

Then \( \sum_i r_i \# h_i = \sum_i r_i \# h_i \) and hence the fact that \( d \) is coinvariant rewrites as
\[ \sum_{i \in I} (r_i \# h_i) \otimes \pi_G((r_i \# h_i)_2) = \sum_{i \in I} (r_i \# h_i) \otimes 1_H. \]
By definition of \( \pi_G \) and \([11]\), the left-hand side becomes
\[ \sum_{i \in I} (r_i \# h_i)_1 \otimes \pi_G((r_i \# h_i)_2) = \sum_{i \in I} ((r_i \# (h_i)_1) + B_{n-1}) \otimes (h_i)_2 \]
so that \([31]\) becomes
\[ \sum_{i \in I} ((r_i \# (h_i)_1) + B_{n-1}) \otimes (h_i)_2 = \sum_{i \in I} r_i \# h_i \otimes 1_H = \sum_{i \in I} (r_i \# h_i + B_{n-1}) \otimes 1_H \]
i.e.
\[ \sum_{i \in I} (r_i \# (h_i)_1) \otimes (h_i)_2 - \sum_{i \in I} r_i \# h_i \otimes 1_H \in B_{n-1} \otimes H = R_{n-1} \otimes H \otimes H. \]
If we apply \( R \otimes \varepsilon_H \otimes H \), we get
\[ \sum_{i \in I} r_i \otimes h_i - \sum_{i \in I} r_i \varepsilon_H (h_i) \otimes 1_H \in R_{n-1} \otimes H = B_{n-1}. \]
Thus \( \sum_{i \in I} r_i \# h_i = \sum_{i \in I} r_i \# h_i = \sum_{i \in I} (r_i \# h_i + B_{n-1}) = \sum_{i \in I} (r_i \varepsilon_H (h_i) \otimes 1_H) + B_{n-1}. \)
Since $\sum_{i \in I} r_i \# h_i \in B_n \setminus B_{n-1}$ we get that $\left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right) \otimes 1_H \notin B_{n-1}$ and hence $\sum_{i \in I} r_i \varepsilon_H (h_i) \notin R_{n-1}$ and we can write

$$\sum_{i \in I} r_i \# h_i = \left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right) \otimes 1_H.$$ 

Therefore we have proved that the map

$$\varphi_n : \frac{R_n}{R_{n-1}} \to D^n : r \mapsto r \otimes 1_H,$$

which is well-defined as $D^n = D \cap G^n = D \cap \frac{B_n}{R_{n-1}} = D \cap \frac{B_{n-1} \otimes H}{R_{n-1} \otimes H}$, is also surjective.

It is also injective as $\varphi_n (\bar{r}) = \varphi_n (\bar{s})$ implies $\varphi_n (r) = \varphi_n (s)$ which is equivalent to $\varphi (r) = \varphi (s)$ by applying (1), (30), the definition of $\rho_{\bar{r}}$ and we get $r - s \in R_{n-1}$ i.e. $\bar{r} = \bar{s}$. Therefore $\varphi_n$ is an isomorphism such that $\sum_{i \in I} r_i \# h_i = \varphi_n \left( \sum_{i \in I} r_i \varepsilon_H (h_i) \right)$ and hence

$$\varphi_n^{-1} \left( \sum_{i \in I} r_i \# h_i \right) = \sum_{i \in I} r_i \varepsilon_H (h_i).$$

Clearly this extends to a graded $k$-linear isomorphism

$$\varphi : \text{gr} \, (R) \to D.$$ 

Let us check that $\varphi$ is a morphism in $H \text{YD}$. First note that, for every $r \in R_n$, we have

$$\varphi (r + R_{n-1}) = \delta_{|r|, n} \varphi (r + R_{n-1}) = \delta_{|r|, n} \varphi_n (r + R_{n-1}) = \delta_{|r|, n} \varphi_n (\bar{r})$$

$$= \delta_{|r|, n} r \otimes 1_H = \delta_{|r|, n} \left( r \otimes 1_H + (R \# \xi H)_{n-1} \right) = r \otimes 1_H + (R \# \xi H)_{n-1}.$$ 

Thus

$$(32) \quad \varphi (r + R_{n-1}) = r \otimes 1_H + (R \# \xi H)_{n-1}, \quad \text{for every } r \in R_n.$$ 

For every $r \in R_n \setminus R_{n-1}$, by using (32), it is straightforward to prove that $h \triangleright \varphi (\bar{r}) = \varphi (h \bar{r})$.

Moreover, by applying (1), (30), the definition of $\pi_G$ and (32), we get that $\rho_{\bar{r}} \varphi (\bar{r}) = (H \otimes \varphi) \rho (\bar{r})$.

Let us check that $\varphi$ is a morphism of bialgebras in $H \text{YD}$. Fix $r \in R_n \setminus R_{n-1}$.

Using the definition of $\Delta_D$, (1), (31), the definition of $\pi_G$, the definition of $\sigma_G$, (32) and (1), again, we obtain $\Delta_D \varphi (\bar{r}) = (\varphi \otimes \varphi) \Delta_{\text{gr}(R)} (\bar{r})$.

Let us check $\varphi$ is comultiplicative. Let $s \in R_m \setminus R_{m-1}$. Then, by definition of $\varphi$, of $m_D$ and of the multiplication of $R \# \xi H$, we have that

$$m_D (\varphi \otimes \varphi) (\bar{s} \otimes \bar{r}) = \sum \left( s^{(1)} \left( \left( s^{(2)} \right)_{(-1)} \right) (r^{(1)}) \right) \# \xi \left( \left( s^{(2)} \right)_{(0)} \otimes r^{(2)} \right) + (R \# \xi H)_{m+n-1}.$$ 

Now write $\sum s^{(1)} \otimes s^{(2)} = \sum_{0 \leq i \leq m} s_i \otimes s_{i-1}^{(1)} \otimes s_{i-1}^{(2)}$ for some $s_i, s_{i-1}^{(1)}, s_{i-1}^{(2)} \in R_i$ and similarly $\sum r^{(1)} \otimes r^{(2)} = \sum_{0 \leq j \leq n} r_j \otimes r_{j-1}^{(-1)}$ for some $r_j, r_{j-1}^{(-1)} \in R_j$. Then

$$m_D (\varphi \otimes \varphi) (\bar{s} \otimes \bar{r}) = \sum_{0 \leq i \leq m} \delta_{i, 0} \varepsilon_B (s_i) \left( \left( s_{i-1}^{(1)} \right)_{(-1)} \right) r_j \# \xi \left( \left( s_{i-1}^{(1)} \right)_{(0)} \otimes r_{j-1}^{(-1)} \right) + (R \# \xi H)_{m+n-1}$$

$$= \sum_{0 \leq j \leq m} \delta_{i, m} \varepsilon_B (s_i) \left( \left( s_{i-1}^{(1)} \right)_{(-1)} \right) r_j \# \xi \left( \left( s_{i-1}^{(1)} \right)_{(0)} \otimes r_{j-1}^{(-1)} \right) + (R \# \xi H)_{m+n-1}$$

$$= \sum_{0 \leq i \leq m} \left( s_{i-1} \right) r_n \# \xi \left( \left( s_{i-1}^{(1)} \right)_{(-1)} \right) + (R \# \xi H)_{m+n-1}$$

$$= \sum_{R_n = \{ 0 \}}^{k \cdot n} s_m \left( \left( s_m^{(1)} \right)_{(-1)} \right) r_n \# \xi \left( \left( s_m^{(1)} \right)_{(0)} \otimes r_n \right) + (R \# \xi H)_{m+n-1}$$

$$= \sum_{R_n = \{ 0 \}}^{k \cdot n} s_m \left( \left( s_m^{(1)} \right)_{(-1)} \right) r_n \# \xi \left( \left( s_m^{(1)} \right)_{(0)} \otimes r_n \right) + (R \# \xi H)_{m+n-1}$$
Let us check \( \varphi \) is unitary. We have
\[
\varphi (1_{\text{gr}(R)}) = \varphi (1_R + R-1) = \varphi (1_R) = 1_R \otimes 1_H = (1_R \otimes 1_H) = (R \# \xi H)_{-1} = 1_B = B - 1 = 1_G.
\]

Summing up we have proved that
\[
\text{gr}(Q) = \text{gr}(R) \overset{\text{Lem. \ref{lem:gr}(B), Prop. \ref{prop:gr}(D)}}{=} \text{gr}(R) \overset{\text{Prop. \ref{prop:gr}(D)}}{=} D(\# \xi H) \overset{\text{Prop. \ref{prop:gr}(D)}}{=} D(A)
\]
as bialgebras in \( H^2 \). Therefore \( H^2 \) of the algebra \( D(A) \) with values in \( k \) if, and only if, \( H^2 \) is quasi-isomorphic to the Radford-Majid bosonization \( E \# H \) as coquasi-bialgebras. By Proposition \ref{prop:gr}(B), we have that \( (E \# H)^{\gamma} = E^{-\gamma} \# H \) as an ordinary coquasi-bialgebras. Recall that two coquasi-bialgebras \( A \) and \( A' \) are called gauge equivalent or quasi-isomorphic whenever there is some gauge transformation \( \gamma : Q \otimes Q \rightarrow k \) in \( \text{Vec}_k \) such that \( A \overset{\gamma}{\cong} A' \) as coquasi-bialgebras. We point out that, if \( A \) and \( A' \) are ordinary bialgebras and \( A \cong A' \), then \( \gamma \) comes out to be a unitary cocycle. This is encoded in the triviality of the reassociators of \( A \) and \( A' \).

**Theorem 5.6.** Let \( A \) be a finite-dimensional Hopf algebra over a field \( k \) of characteristic zero such that the coradical \( H \) of \( A \) is a sub-Hopf algebra (i.e. \( A \) has the dual Chevalley Property). If \( H_2 D(A), k = 0 \), then \( A \) is quasi-isomorphic to the Radford-Majid bosonization \( E \# H \) of some connected bialgebra \( E \) in \( H^2 \) by \( H \). Moreover \( \text{gr}(E) \cong D(A) \) as bialgebras in \( H^2 \).

**Proof.** By the foregoing \( A \cong Q \# H \cong E^{\gamma} \# H = (E \# H)^{\gamma} \) as coquasi-bialgebras. Now \( A \) is quasi-isomorphic to \( A \) which is quasi-isomorphic to \( E \# H \) so that \( A \) is quasi-isomorphic to \( E \# H \). Moreover
\[
\text{gr}(E) = \text{gr}(E^{\gamma}) = \text{gr}(Q) \cong D(A).
\]
where the first equality holds by Proposition \ref{prop:gr}(B).

More generally, given \( A \) a (finite-dimensional) Hopf algebra whose coradical \( H \) is a sub-Hopf algebra, then if \( H \) is also semisimple, we expect that \( A \) is quasi-isomorphic to the Radford-Majid bosonization \( E \# H \) of some connected bialgebra \( E \) in \( H^2 \) by \( H \). See e.g. [AGM, Corollary 3.4 and the proof therein] and [AGMV, AG] for a further clue in this direction.

6. **Examples**

We notice that the Hochschild cohomology of a finite-dimensional Nichols algebras has been computed in few examples. We consider here those Nichols algebras to compute \( H^2 D(B(V), k) \).
6.1. Braiding of Cartan type. Let $A = (a_{ij})_{1 \leq i, j \leq \theta}$ be a finite Cartan matrix, $\Delta$ the corresponding root system, $(\alpha_i)_{1 \leq i \leq \theta}$ a set of simple roots and $W$ its Weyl group. Let $w_0 = s_{i_1} \cdots s_{i_M}$ be a reduced expression of the element $w_0 \in W$ of maximal length as a product of simple reflections, $\beta = s_{i_1} \cdots s_{i_{M-1}}(\alpha_{i_M}), 1 \leq j \leq M$. Then $\beta_j \neq \beta_k$ if $j \neq k$ and $\Delta^+ = \{ \beta_j | 1 \leq j \leq M \}$, see [AS, page 25 and Proposition 3.6].

Let $\Gamma$ be a finite abelian group, $\hat{\Gamma}$ its group of characters. $D = (\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ is a datum of finite Cartan type [AS2] associated to $\Gamma$ and $A$ if $g_i \in \Gamma$, $\chi_i \in \hat{\Gamma}$, $1 \leq i, j \leq \theta$, satisfy $\chi_i(g_i) \neq 1$, $\chi_j(g_i) \chi_i(g_i) = \chi_i(g_i)^{a_{ij}}$ for all $i, j$. Set $q = (q_{ij})_{1 \leq i, j \leq \theta}$, where $q_{ij} = \chi_j(g_i)$.

In what follows $V$ denotes the Yetter-Drinfeld module over $k\Gamma$, $\dim V = \theta$, with a fixed basis $x_1, \ldots, x_{\theta}$, where the action and the coaction over each $x_i$ is given by $\chi_i$ and $g_i$, respectively. Then the associated braiding is $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j$. Let $B_q = B(V)$. The tensor algebra $T(V)$ is $\mathbb{N}_0^\theta$-graded with grading $\alpha_i$ for each $x_i$. For $\beta = \sum_{i=1}^{\theta} \alpha_i \alpha_i \in \Delta^+$, set

$$g_\beta = g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_\beta = \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \quad q_\beta = \chi_\beta(g_\beta).$$

Given $\alpha, \beta \in \Delta^+$, we denote $q_{\alpha \beta} = \chi_{\beta}(g_\alpha)$.

We assume as in [AS2, MPSW] that the order of $q_{ii}$ is odd for all $i$, and not divisible by 3 for each connected component of the Dynkin diagram of $A$ of type $G_2$. Therefore the order of $q_{ii}$ is the same for all the $i$ in the same connected component $J$. Given $\beta \in J$, we denote by $N_\beta$ the order of the corresponding $q_{ii}$ in $J$, which is also the order of $q_\beta$.

By [AS] there exist homogeneous elements $x_\beta$ of degree $\beta$, $\beta \in \Delta^+$, such that the Nichols algebra $B_q$ of $V$ is presented by generators $x_1, \ldots, x_\theta$ and relations

$$(ad_x x_i)^{1-a_{ii}-1} x_j = 0, \quad 1 \leq i \neq j \leq \theta;$$
$$x_\beta^{N_\beta} = 0, \quad \beta \in \Delta^+. $$

Moreover $\{x_\beta^{n_1} \cdots x_\beta^{n_M} | 0 \leq n_i < N_\beta \}$ is a basis of $B_q$.

We shall prove that $H^2_{DF}(B_q, k) = 0$. We need first some technical results.

**Lemma 6.1.** Let $\alpha, \beta \in \Delta^+$. Then either $g_\alpha g_\beta^{N_\beta} \neq e$, or else $\chi_{\alpha}^{N_\beta} \neq e$.

**Proof.** Suppose on the contrary that $g_\alpha g_\beta^{N_\beta} = e$, $\chi_{\alpha}^{N_\beta} = e$. Then

$$q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta^{N_\beta}(g_\beta^{N_\beta}) = q_\beta^{N_\beta} = 1,$$

since $q_\beta$ is a root of unity of order $N_\beta$. But this is a contradiction, since $q_\alpha \neq 1$. \hfill $\square$

**Lemma 6.2.** Let $\alpha, \beta, \gamma \in \Delta^+$ be pairwise different. Then either $g_\alpha g_\beta g_\gamma \neq e$, or else $\chi_{\alpha} \chi_{\beta} \chi_{\gamma} \neq e$.

**Proof.** Suppose on the contrary that $g_\alpha g_\beta g_\gamma = e$ and $\chi_{\alpha} \chi_{\beta} \chi_{\gamma} = e$. Then

$$q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta \chi_\gamma (g_\beta g_\gamma) = \chi_\beta g_\gamma q_\beta g_\gamma, \quad q_\beta = q_\alpha q_\gamma q_\beta q_\gamma q_\alpha, \quad q_\gamma = q_\alpha q_\beta q_\alpha q_\beta q_\alpha.$$ 

Notice that $\alpha, \beta, \gamma$ belong to the same connected component. Indeed, if $\gamma$ belongs to a different connected component, then $q_\beta, q_\gamma q_\beta = q_\alpha q_\gamma q_\alpha = 1$. Thus $q_\beta = q_\alpha q_\gamma = q_\beta q_\gamma$, so $q_\gamma = 1$, which is a contradiction. Therefore we may assume that the Dynkin diagram is connected.

One can prove that $q_{s_\alpha}(\alpha) = q_\alpha$ for every $\alpha \in \Delta$. As we observed that $\Delta^+ = \{ \beta_j | 1 \leq j \leq M \}$, we deduce that for every $\beta \in \Delta^+$ there is some $j$ such that $q_{\beta_j} = q_j$. One can prove that there is some $q \in k$ such that $q_{\alpha} = q^{(\alpha, \alpha)/2}$ and $q_{\alpha \gamma} q_{\alpha} = q^{(\alpha, \gamma)}$, where $(\cdot, \cdot)$ is the invariant bilinear form on the simple Lie algebra $\mathfrak{g}$ associated with the finite Cartan matrix [AS, Ch. VI, §1, Proposition 3 and Definition 3] and the basis of the root systems given in [AS, Ch. VI, §4] should be normalized in such a way that $q = q_\beta$, $(\delta, \delta) = 2$ for each short root $\delta \in \Delta$. Note that $q_\alpha = q^{(\alpha, \alpha)/2} \neq 1$ for all $\alpha$ as $(\alpha, \alpha) \neq 0$. Thus

- $q_\alpha = q_\beta = q_\gamma = q$ if the Dynkin diagram is simply laced,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^2\}$ if the Dynkin diagram has a double arrow,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^3\}$ if the Dynkin diagram is of type $G_2$. 

If the Dynkin diagram is simply laced, then, by [33], we have $q_{\alpha+\beta}q_{\gamma}\beta = q_{\alpha+\beta}q_{\gamma}\alpha = q_{\alpha+\beta}q_{\gamma}\alpha = q^{-1}$. Then $q^{(\alpha,\gamma)} = q^{-1}$. Now set $n(\alpha,\beta) := 2(\alpha,\beta) = (\alpha,\beta)$. Then $n(\alpha,\beta)$ is symmetric whence, by [DP], Ch. VI, §1, page 148] we have $(\alpha, \gamma) = -1$ as the order of $q$ is odd, so $\alpha + \gamma \leq \Delta^+$. By [DP, Ch. VI, §1, Corollary, page 149] the same argument we used above shows that also $1(\alpha + \gamma, \beta) = -2$, so $\alpha + \beta + \gamma \leq \Delta^+$, since $\alpha + \gamma \neq -\beta$ (as $\alpha + \gamma$ and $\beta$ are both in $\Delta^+$). But $q_{\alpha+\beta+\gamma} = q_{\alpha+\beta}q_{\alpha+\gamma}q_{\alpha}q_{\alpha+\beta}q_{\beta} = q^{-1}$, which is a contradiction.

If the Dynkin diagram has a double arrow, then $q_{\alpha}, q_{\beta}, q_{\gamma} \in \{q, q^2\}$. If $q_{\alpha} = q_{\beta} = q_{\gamma}$, then the proof follows as for the simply-laced case because $n(u, v) = n(v, u)$ for $u, v \in \{\alpha, \beta, \gamma\}$. If $q_{\alpha} = q_{\beta} = q$ and $q_{\gamma} = q^2$, then $q_{\beta}q_{\gamma}q_{\alpha} = q^{-2}$, and $q_{\beta}q_{\alpha}q_{\beta} = 1$, by [33]. Then a simple calculation yields $(\beta, \gamma) = -2$ so that $\beta + \gamma \in \Delta^+$. One also gets $(\alpha, \beta) = 0$ and $(\alpha, \gamma) = -2$ so that $(\alpha, \beta + \gamma) = (\alpha, \gamma) = -2 < 0$ by the conditions on the order of $q$, so again $\alpha + \beta + \gamma \leq \Delta^+$; but again we obtain $q_{\alpha+\beta+\gamma} = 1$, which is a contradiction. The proof for $q_{\alpha} = q_{\beta} = q^2$ and $q_{\gamma} = q$ follows analogously.

Finally, if the Dynkin diagram is of type $G_2$, then a similar analysis gives a contradiction. □

For each $1 \leq k \leq M$, set $B_{k}(k)$ as the subspace of $B_k$ spanned by $\{x_{\beta}^n | 0 \leq n_i < N_\beta\}$. By [DP] this gives an algebra filtration, and the graded algebra $Gr B_{k}$ associated to this filtration is presented by generators $x_{\beta}, \beta \in \Delta^+$, and relations

$$x_{\beta}x_{\gamma} = q_{\beta}\gamma x_{\beta}, \quad x_{\beta}^N = 0, \quad \beta < \gamma \in \Delta_+.$$

In [MPSW] $Gr B_{k}$ is viewed as an algebra in $\mathcal{kD}$, which (as an algebra) is the Nichols algebra of Cartan type $A_1 \times \cdots \times A_1$, $M$ copies, with action and coaction on $x_{\beta}$ given by $\chi_{\beta}, g_{\beta}$, respectively. By [MPSW, Theorem 4.1], $H^*(Gr B_{k}, \mathcal{k})$ is the algebra generated by $\xi_{\beta}, \eta_{\beta}, \beta \in \Delta^+$, where $\deg \xi_{\beta} = 2$, $\deg \eta_{\beta} = 1$, and relations

$$\xi_{\beta}\xi_{\gamma} = q_{\beta}\gamma N_{\gamma,\beta,\xi_{\beta}}\xi_{\gamma}, \quad \eta_{\beta}\xi_{\gamma} = q_{\beta}\gamma N_{\gamma,\beta,\eta_{\beta}}\xi_{\gamma} - q_{\beta}\gamma N_{\gamma,\beta,\eta_{\beta}}\eta_{\gamma}, \quad \beta, \gamma \in \Delta^+.$$

As we assume that all the $q_{\beta}$ have odd order, we deduce in particular from the last equality that $q_{\beta}^2 = 0$ for all $\beta \in \Delta^+$. As an algebra in $\mathcal{kD}$, the action and coaction on $\xi_{\beta}$ is given by $\chi_{\beta}^{-N_{\beta}}, g_{\beta}^{-N_{\beta}}$, while the action and coaction on $\eta_{\beta}$ is given by $\chi_{\beta}^{-1}, g_{\beta}^{-1}$.

**Theorem 6.3.** $H^3_{\mathcal{D}}(B_{k}, \mathcal{k}) = 0$.

**Proof.** First we will prove that $H^3(Gr B_{d}, \mathcal{k}) = 0$ for $D := D(\mathcal{K})$. Now, the invariants are with respect to the $D$-bimodule structure that $H^3(Gr B_{d}, \mathcal{k})$ inherits from $\text{Hom}(Gr B_{d}, \mathcal{k})$ (this is a $D$-bimodule as its arguments are left $D$-modules). Since the left $D$-module structure is induced by the one of $\mathcal{k}$, it is trivial. Thus the invariants of $H^3(Gr B_{d}, \mathcal{k})$ as a $D$-bimodule reduce to the its invariants as a right $D$-module. Since right $D$-modules are equivalent to left $D$-modules, via the antipode of $D$ which is invertible as $D$ is finite-dimensional, the right $D$-module structure of $H^3(Gr B_{d}, \mathcal{k})$ becomes the structure of object in $\mathcal{kD}$ described above. Thus, in order to prove that $H^3(Gr B_{d}, \mathcal{k}) = 0$ we just have to check that the invariants of $H^3(Gr B_{d}, \mathcal{k})$ as a left-left Yetter-Drinfeld modules are zero.

Now, by the defining relations of $H^3(Gr B_{d}, \mathcal{k})$, a basis $B$ of $H^3(Gr B_{d}, \mathcal{k})$ is given by $\{\xi_{\alpha}\eta_{\beta}\} \cup \{\eta_{\alpha}\eta_{\beta}\eta_{\gamma}|\alpha < \beta < \gamma\}$. If $v \in H^3(Gr B_{d}, \mathcal{k})$ is invariant, then $v$ is written as a linear combination of elements in the trivial component. Indeed, write $v = \sum_{b \in B} c_b b$ for some $c_b \in \mathcal{k}$, and let $g_{\beta}, \chi_{\beta}$ be the elements describing the component of $b$. Then

$$v = g \cdot v = \sum_{b \in B} c_b g \cdot b = \sum_{b \in B} c_b \chi_{\beta}(g) b, \quad 1 \otimes v = \rho(v) = \sum_{b \in B} c_b \rho \cdot b = \sum_{b \in B} c_b g_{\beta} \otimes b.$$

If $c_b \neq 0$, then $\chi_{\beta}(g) = 1$ for all $g \in \Gamma$ so $\chi_{\beta} = \epsilon$, and $g_{\beta} = 1$. Thus $b$ is invariant. We have so proved that the existence of $v \neq 0$ invariant implies the existence of $b \in B$ invariant. Hence, if $B$ has no invariant element then there is no invariant element at all. Note that, for all $h \in H$, we have $h \cdot (\xi_{\alpha}\eta_{\beta}) = (\chi_{\alpha}^{-N_{\alpha}}\chi_{\beta}^{-1})(h)\xi_{\alpha}\eta_{\beta}$ and $h \cdot (\xi_{\alpha}\eta_{\beta}) = g_{\alpha}^{-N_{\alpha}}g_{\beta}^{-1} \otimes \xi_{\alpha}\eta_{\beta}$ so that, by Lemma [6.1], the element
\(\xi, \eta, \gamma\) is not \(D\)-invariant. A similar argument, using Lemma 6.1, shows that also \(n, \eta, \gamma\eta\) is not \(D\)-invariant. Thus the elements in \(B\) are not \(D\)-invariant, so \(H^3(\text{Gr} B_3, k)_D = 0\). Since the elements in \([\eta_{\beta_1}, \ldots, \eta_{\beta_i}]\) are eigenvectors for \(D\), we can mimic the argument in \(\text{[MPSW], Section 5}\) by taking into account the spectral sequence associated to the filtration of algebras therein; see for example \(\text{[MPSW], Corollary 5.5}\) for a similar argument. Thus \(H^3(\text{Gr} B_3, k)_D = 0\). 

Remark 6.4. Notice that \(H^3_{2D}(B_3, k) \cong H^3(B_3, k)^{(3)} = 0\) although \(H^3(B_3 \# k\Gamma, k) \cong H^3(B_3, k)\) can be non-trivial, see for example \(\text{[MPSW], Example 5.8}\).

6.2. **Braidings of non-diagonal type.** For \(n \geq 3\), \(\mathcal{FK}_n\) denotes the quadratic algebra \(\mathcal{FK}\) with a presentation by generators \(x_{ij}\), \(1 \leq i < j \leq n\), and relations

\[
\begin{align*}
  x^2_{ij} & = 0, \\
  x_{ij}x_{jk} & = x_{jk}x_{ik} + x_{ik}x_{ij}, \\
  x_{jk}x_{ij} & = x_{ik}x_{jk} + x_{ij}x_{ik}, \\
  x_{ij}x_{ki} & = x_{ikh}x_{ij}, \quad \{i, j, k, l\} = 4.
\end{align*}
\]

According to \(\text{[MS]}\) each \(\mathcal{FK}_n\) is a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group \(S_n\), generated as an algebra by the vector space \(V_n\), with basis \(\{x_{ij} \mid 1 \leq i < j \leq n\}\). The action is described by identifying \((ij)\) with the corresponding transposition in \(S_n\) and then consider the conjugation twisted by the sign, while the coaction is given by declaring \(x_{ij}\) a homogeneous element of degree \(\sigma\). Then the braiding on \(V_n\) becomes

\[
c(\sigma \otimes \tau) = \chi(\sigma, \tau)x_{\sigma \tau \sigma^{-1} \otimes x_{\sigma}}, \quad \chi(\sigma, \tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij), i < j, \\ -1 & \text{otherwise}, \end{cases}
\]

where \(\sigma\) and \(\tau\) are transpositions. Moreover \(\mathcal{FK}_n\) projects onto the Nichols algebra \(B(V_n)\). For \(n = 3, 4, 5\), it is known that \(\mathcal{FK}_n = B(V_n)\) and has dimension, respectively, 12, 576 and 8294400.

The Hochschild cohomology of \(\mathcal{FK}_3\) is a consequence of the results in \(\text{[SV]}\) as follows.

**Theorem 6.5.** \(H^*_{\text{grS}_3, Mod}(\mathcal{FK}_3, k)\) is isomorphic to the graded algebra

\(k[X, U, V]/(U^2 - V - UV)^2\), where \(\deg U = \deg V = 2\), \(\deg X = 4\).

**Proof.** By \(\text{[SV], Theorem 4.19}\), we have that \(E(B \# kS_3)\) is isomorphic to the algebra in the claim, where \(B = \mathcal{FK}_3\). By \(\text{[SV], Theorem 2.17}\), we know that \(E(B \# kS_3) \cong E(B)^{kS_3}\) as graded algebras. As observed in Remark 4.2, we have that \(E(B) \cong H^* (B, k)\). By Remark 4.6 we have \(H^* (B, k)^{kS_3} \cong H^*_{\text{grS}_3, Mod}(\mathcal{FK}_3, k)\).

From this result we get \(H^3_{\text{grS}_3, Mod}(\mathcal{FK}_3, k) = 0\) so that, by Proposition 4.7 we conclude that

**Corollary 6.6.** \(H^3_{2D}(\mathcal{FK}_3, k) = 0\).

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