Flow damping in stellarators close to quasisymmetry

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Abstract

Quasisymmetric stellarators are a type of optimized stellarators for which flows are undamped to lowest order in an expansion in the normalized Larmor radius. However, perfect quasisymmetry is impossible. Since large flows may be desirable as a means to reduce turbulent transport, it is important to know when a stellarator can be considered to be sufficiently close to quasisymmetry. The answer to this question depends strongly on the size of the spatial gradients of the deviation from quasisymmetry and on the collisionality regime. Recently, criteria for closeness to quasisymmetry have been derived in a variety of situations. In particular, the case of deviations with large gradients was solved in the $1/\nu$ regime. Denoting by $\alpha$ a parameter that gives the size of the deviation from quasisymmetry, it was proven that particle fluxes do not scale with $\alpha^{3/2}$, as typically claimed, but with $\alpha$. It was also shown that ripple wells are not the main cause of transport. This paper reviews those works and presents a new result in another collisionality regime, in which particles trapped in ripple wells are collisional and the rest are collisionless.

Keywords: stellarator, quasisymmetry, flow damping

(Some figures may appear in colour only in the online journal)
second order, but it is necessarily violated to third order in the expansion parameter. Then, it only makes sense to talk about stellarators close to quasisymmetry. It is actually possible to be close to quasisymmetry, as the existence of the HSX stellarator [8] shows. An experimental study of flow damping in HSX can be found in [9]. Let \( B = B_0 + \alpha B_1 \) be the magnetic field, where \( B_0 \) is quasisymmetric, \( \alpha B_1 \) is the deviation from quasisymmetry and \( 0 < \alpha \ll 1 \) is a small parameter. The theoretical question is to understand when \( \alpha \) is sufficiently small for the stellarator to be considered quasisymmetric in practice.

The systematic approach to this question was started in references [10 and 11] by exploiting the equivalence of quasisymmetry and intrinsic ambipolarity [4], i.e. the fact that

\[
\langle J_n \nabla \psi \rangle_\psi = 0
\]

(1)

for any density, temperature and radial electric field profiles is equivalent to having no flow damping. Here, \( J_n \) is the neoclassical electric current density and \( \psi \) is a flux-label coordinate. In the previous equation \( \langle \cdot \rangle_\psi \) denotes the flux surface average, defined as

\[
\langle f \rangle_\psi = \frac{1}{V} \int_0^{2\pi} \int_0^\alpha \sqrt{g} f(\psi, \Theta, \zeta) \, d\Theta d\zeta.
\]

(2)

where \( \Theta \) is a poloidal angle, \( \zeta \) a toroidal angle, \( \sqrt{g} \) is the square root of the metric determinant, \( V(\psi) \) is the plasma volume enclosed by the surface labeled by \( \psi \), and its derivative is given by

\[
V'(\psi) = \int_0^{2\pi} \int_0^\alpha \sqrt{g} \, d\Theta d\zeta.
\]

(3)

In [4] it is shown that in a stellarator far from quasisymmetry flow damping is given precisely by \( \langle J_n \nabla \psi \rangle_\psi \). Thus, this quantity vanishes identically for a quasisymmetric magnetic field, i.e. for \( \alpha = 0 \). Therefore, it is the scaling of \( \langle J_n \nabla \psi \rangle_\psi \) with \( \alpha \) that needs to be determined in order to derive criteria for closeness to quasisymmetry. How the determination of the scaling allows to obtain the criterion is explained in section 2.

The calculation of the scalings, and the scalings themselves, have revealed to be quite different depending on the size of the gradients of the perturbation, and so will be the criteria derived from them. Let us give a summary of some of the results contained in [10 and 11]. From now on, we assume that \( (\psi, \Theta, \zeta) \) are Boozer coordinates [12]. They exist as long as \( J \nabla \psi = 0 \), where \( J \) is the total electric current density, and are defined by requiring that the magnetic field can be simultaneously written as

\[
B = -\nabla \psi + \frac{I_\psi(\psi)}{2\pi} \nabla \Theta + \frac{I_{\psi \zeta}(\psi)}{2\pi} \nabla \zeta
\]

(4)

and as

\[
B = \frac{\psi_{\psi}(\psi)}{2\pi} \nabla \zeta \times \nabla \psi + \frac{\psi_{\psi \zeta}(\psi)}{2\pi} \nabla \psi \times \nabla \Theta.
\]

(5)

Here, \( I_\psi \) and \( I_{\psi \zeta} \) are flux functions, \( \psi_{\psi} \) is the toroidal flux, \( \psi_{\psi \zeta} \) is the poloidal flux, \( \nabla \psi(\psi, \Theta, \zeta) \) is a singly-valued function and primes denote differentiation with respect to \( \psi \). Two properties of Boozer coordinates will be relevant for us. The first one is that \( \sqrt{g} \) takes the form

\[
\sqrt{g} = \frac{\psi'(B^2)_\psi}{4 \pi^2 B^2}.
\]

(6)

The second one is that if \( B \) is quasisymmetric, then its magnitude \( B \) depends on a single linear combination, or helicity, of the Boozer angles.

With the above notation, \( B_0 \) depends only on one helicity, \( B_0 \equiv B_0(\psi, \Theta \Theta N \zeta) \), and we can take \( B_1(\psi, \Theta, \zeta) \) such that it does not contain the helicity \( \Theta \Theta N \zeta \). Then, we look at \( \partial_\alpha B_1 \) \( \partial_\alpha B_0 \) and \( \partial_\alpha B_1 \partial_\alpha B_0 \). As shown in reference [10], if

\[
\frac{\partial_\alpha B_1}{\partial_\alpha B_0} \sim \alpha,
\]

(7)

then

\[
\langle J_n \nabla \psi \rangle_\psi \sim \alpha^2 k
\]

(8)

for any value of the collisionality (the quadratic scaling was obtained in [13] for high collisionality). The scaling of the function \( k \) with the collision frequency depends on the collisionality regime. If, on the contrary,

\[
\frac{\alpha \partial_\alpha B_1}{\partial_\alpha B_0} \sim 1,
\]

(9)

then

\[
\langle J_n \nabla \psi \rangle_\psi = O(\alpha^0)
\]

(10)

and the quasisymmetric properties of \( B_0 \) have been completely destroyed by the perturbation. However, as explained in [11], if perturbations with large gradients are present and the extra assumption

\[
\psi_{\Theta \Theta N \zeta} = \psi_{\Theta \Theta N \zeta}^{(0)} \ll \psi_{\Theta \Theta N \zeta}^{(0)}
\]

(12)

is made, where \( \psi_{\Theta \Theta N \zeta}^{(0)} \) is the radial magnetic drift corresponding to \( B_0 \), then a scaling less favorable than \( \alpha^3 \) but more favorable than \( \alpha^5 \) is obtained. Hence, when designing a stellarator close to quasisymmetry large helicity perturbations should be avoided; but if this is not possible, and it probably is not, condition (12) should, in principle\(^4\), be a design goal.

Denote by \( L_0 \) the typical variation length of \( B_0 \), \( L_0^{-1} \sim |V \ln B_0| \). In particular, the wells of \( B_0 \) along a magnetic field line have size \( L_0 \). It is easy to realize that when the perturbation \( \alpha B_1 \) has strong gradients, \( B = B_0 + \alpha B_1 \) can have, in addition, ripple wells of size \( L_1 \sim |V \ln B_0| \) (see figure 1). Sometimes [15], it has been mistakenly thought that the original results of [16] apply to stellarators close to quasisymmetry. This has led to the assumption that ripple wells are the main cause of transport, and that in the so-called 1/\( v \) regime particle and energy fluxes scale with \( \alpha^{3/2} \). In [11], it was shown that this is not

\(^4\) In reference [14] it is proven that the extra assumption (12) can actually be relaxed in the 1/\( v \) regime. However, we do not have a similar proof for other collisionality regimes.
correct. A rigorous calculation was carried out assuming (9) and (12) and the result was found to be, in the 1/\(\nu\) regime,
\[
\langle J_x \nabla \psi \rangle \sim \frac{a c_i^2 \nu_i}{\nu_i} m_i n_i \nabla \psi|_0. \tag{13}
\]

Furthermore, the calculation shows that ripple wells do not dominate transport. Even if they contribute to the fluxes with the scaling given on the right side of (13), the same scaling is produced by orbits trapped in the wells of size \(L_0\); more specifically, by the part of the orbit near the bounce points. Here, \(v_i\) is the ion thermal speed, \(n_i\) is the ion equilibrium density, \(\nu_i\) is the ion-ion collision frequency, \(|\nabla \psi|_0\) is a characteristic value of \(|\nabla \psi|\) on the flux surface, and \(\epsilon_i = \rho_i L_0\), where \(\rho_i\) is the ion Larmor radius. Throughout the paper the ion and electron temperatures are assumed to be comparable, and an expansion in the square root of the ratio of the electron and ion masses, \(\sqrt{m_i/m_e}\), is employed.

The rest of the paper gives a brief summary of the calculations leading to the above results and presents the criteria for closeness to quasisymmetry implied by them. We also derive the scaling of the fluxes with \(\alpha\), and the corresponding criterion, in a new collisionality regime. In [11] passing particles, particles trapped in wells of size \(L_0\), and particles trapped in wells of size \(L_1\) were assumed collisionless (this is what is understood by \(1/\nu\) regime). Here, we extend the computation to the case when particles in ripple wells are collisional and the rest are collisionless. Finally, we provide a treatment of the collisional boundary layers more detailed than in our previous works. The results of [10] and [11], and those presented for the first time here, also apply to tokamaks with ripple.

Before finishing this Introduction, it is pertinent to comment on the size of the plasma flows that we are dealing with. In this paper we always assume that the ion flow \(V_i\) is subsonic, i.e. \(V_i \ll v_i\). The presence of sonic flows, \(V_i \sim v_i\), immediately implies quasisymmetry [17]. Hence, the absence of damping of subsonic flows is a necessary condition to achieve sonic flows. However, it is not sufficient. It has recently been proven [18] that even in quasisymmetric stellarators there are global obstructions that forbid strictly sonic flows, but they do not exclude regimes with \(\epsilon_i v_i \ll V_i \ll v_i\). The actual flow size that can be reached in a stellarator that is not perfectly quasisymmetric will be the subject of further research.

2. Formulation of the problem

We use the subindex \(\sigma\) to denote different species and define \(\epsilon_\sigma = \rho_\sigma L_0\), the Larmor radius over the macroscopic scale. We also need to define the collisionality \(\nu_{\sigma\sigma'} = v_{\sigma\sigma'}/L_0\), where \(v_{\sigma\sigma'}\) is the collision frequency of species \(\sigma\) with \(\sigma'\). Here, \(v_{\sigma\sigma'} = \sqrt{T_\sigma/m_\sigma T_{\sigma'}}\), \(T_\sigma\), and \(m_\sigma\) are the thermal speed, temperature, and mass of species \(\sigma\). Denote by \(\nu_{\sigma}\) the largest of all collisionalities \(\nu_{\sigma\sigma'}\) when \(\sigma'\) runs over species. If \(\epsilon_\sigma \ll \nu_{\sigma}\) for all species, one can eliminate the degree of motion corresponding to the gyration of particles around the magnetic field by expanding the fields and the kinetic equations in \(\epsilon_\sigma\) \(\ll 1\) and averaging order by order in the gyrophase.

We employ phase-space coordinates \((\rho, \Theta, \xi, v, \lambda, s)\), where \(v\) is the magnitude of the velocity, \(\lambda = B^{-1}v_\perp^2/\nu\) is the pitch-angle coordinate, \(v_\perp\) is the magnitude of the velocity component perpendicular to the magnetic field, and \(s = \pm 1\) is the sign of the parallel velocity \(v_\parallel\), that can be expressed as
\[
v_\parallel = s v \sqrt{1 - \lambda^2 B}. \tag{14}
\]

Let us use the notation \(F_\sigma = F_{\sigma 0} + F_{\sigma 1} + \ldots \) \(\varphi = \varphi_0 + \varphi_1 + \ldots\) for the expansions of the distribution function and the electrostatic potential, where \(F_{\sigma 0}/F_{\sigma 1} = O(\epsilon_\sigma)\), \(\varphi_{\sigma 0}/\varphi_{\sigma 1} = O(\epsilon_\sigma)\), \(\varphi_{\sigma 0}/T_\sigma = O(1)\) and \(\epsilon\) is the proton charge. To lowest order, one obtains that the distribution function is Maxwellian
\[
F_{\sigma 0}(R, u, \mu) = n_e \left( \frac{m_\sigma}{2\pi T_\sigma} \right)^{3/2} \exp \left( -\frac{m_\sigma (u^2 / 2 + \mu B)}{T_\sigma} \right) \tag{15}
\]
and that the density \(n_e\), the temperature \(T_\sigma\) and \(\varphi_0\) are flux functions. The next order pieces \(F_{\sigma 1}\) and \(\varphi_1\) have both neoclassical and turbulent components, but only the former, that varies in macroscopic length scales, matters for our calculation. Thus, \(F_{\sigma 1}\) and \(\varphi_1\) stand for the neoclassical components of the corrections to \(F_{\sigma 0}\) and \(\varphi_0\). Denote by \(G_{\sigma 1} = F_{\sigma 1} + (Z_{\sigma e}/\varphi_{\sigma}/T_{\sigma}) F_{\sigma 0}\) the non-adiabatic component of the correction to the distribution function. The equation that determines \(G_{\sigma 1}\) is called the drift-kinetic equation [19] and reads
\[
\nu_{\sigma} \hat{\mathbf{b}} \cdot \nabla G_{\sigma 1} + \nabla \varphi_{\sigma, F_{\sigma 0}} = C_{\sigma}^e [G_{\sigma 1}]. \tag{16}
\]

Here,
\[
\nu_{\sigma} := \frac{Z_{\sigma e}}{T_{\sigma}} \partial_\phi \varphi_0 + \frac{1}{n_\sigma} \partial_\phi n_\sigma + \left( \frac{m_\sigma v_\perp^2}{2 T_\sigma} - \frac{3}{2} \right) \frac{1}{T_\sigma} \partial_\phi T_\sigma, \tag{17}
\]

\(\nu_{\sigma, F_{\sigma 0}} = \nabla_M \cdot \nabla \varphi\) is the radial magnetic drift,
\[
\nabla_M \cdot \nabla \varphi = \frac{\nu_\sigma^2 (1 - \lambda B)}{\Omega_\sigma} (\hat{\mathbf{b}} \cdot \hat{\mathbf{b}}) + \frac{\nu_\sigma^2}{\Omega_\sigma^2} (\hat{\mathbf{b}} \times \nabla \hat{\mathbf{b}}) \times \nabla B. \tag{18}
\]
\(\Omega_\sigma = Z_{\sigma e} B (m_\sigma c)\) is the gyrofrequency of species \(\sigma\), \(Z_{\sigma e}\) is the electric charge, \(c\) is the speed of light and \(C_{\sigma}^e [G_{\sigma 1}]\) is the linearized collision operator, whose explicit expression is not needed (see, for example, [20]).

The well-known neoclassical expression for the flux-surface averaged radial electric current is
\[\langle \mathbf{J}_0 \cdot \nabla \psi \rangle_{\psi} = \sum_{\alpha} \int_{m_{\alpha}}^{m_{\alpha}} Z_{\alpha} \int_{0}^{B_{\alpha}} d \psi \int_{0}^{B_{\alpha}} d \ell \frac{e^2 B_{\alpha} v_{\alpha,0} G_{\alpha}}{1 - \lambda B_{\alpha}} \psi_{\alpha} \cdot \mathbf{G}_{\alpha} \frac{d \psi}{d \ell} \]  

(19)

From here on, and for the sake of simplicity, we only deal with the ion-drift kinetic equation. We note that if the ion and electron temperatures are comparable and just lowest-order terms in an expansion in \(m_i/m_e\) are kept, then only the ion particle flux needs to be taken into account in (19) and ion-electron collisions can be neglected.

Consider an expansion in \(\psi\) of the total flux-surface averaged radial electric current, \(\langle \mathbf{J} \cdot \nabla \psi \rangle_{\psi}\) (which, of course, vanishes due to quasineutrality). In a generic stellarator the right-hand side of (19) dominates because it scales with \(\psi_i \), whereas turbulent and higher-order neoclassical contributions to the total radial current are \(O(\psi_i)\), as explained in detail in reference [10]. These \(O(\psi_i)\) contributions include higher-order flow damping terms and the polarization current. Schematically,

\[\langle \mathbf{J} \cdot \nabla \psi \rangle_{\psi} = (e_i^2 A + e_i C) e_{\psi,0} \beta \nabla \psi_{\psi} + \ldots\]  

(20)

For perfectly quasisymmetric stellarators, \(A\) is identically zero. Then, for stellarators close to quasisymmetry, it is expected that \(A \sim \Delta \alpha t \nu_{\psi, i} E\), with \(q > 0\) and \(\Delta = O(1)\). Then,

\[\langle \mathbf{J} \cdot \nabla \psi \rangle_{\psi} = (e_i^2 \Delta A_{\psi, i} \nu_{\psi, i} E + e_i C) e_{\psi,0} \beta \nabla \psi_{\psi} \]  

(21)

Note that we have replaced one instance of \(\psi_i\) by \(\nu_{\psi, i}\) in the first term of (20). We have done this just to point out that, in general, this is the actual scaling of that term; this comes from the fact that \(G_{\nu, i}\) depends linearly on \(\nu_{\psi, i}\), and the latter sets the magnitude of the flow. The two terms in equation (21) are comparable when \(V_i \sim e_i^2 \Delta \alpha t \nu_{\psi, i} E\). The stellarator exhibits a quasisymmetric behavior if flows of size \(V_i \sim \psi_i E_i\) are reached, which gives the criterion

\[\alpha < (e_i \psi_i E_i)^{1/q} \]  

(22)

Without further information, we have assumed \(C = O(1)\) above. Our task consists of finding the powers \(q\) and \(r\), that depend on the geometry of the deviation from quasisymmetry and on the collisionality regime.

3. Criteria for closeness to quasisymmetry

In this section we compute the scaling with \(\alpha\) (and with \(\nu_{\psi, i}\) when the result depends on the collisionality regime) of the right side of (19). As advanced in the Introduction, this depends strongly on the size of the gradient of \(B_t\), the magnitude of the perturbation to the quasisymmetric configuration \(B_0\).

3.1. Perturbations with small gradients

The calculation when (7) is satisfied was carried out in [10] and reviewed in [11]. Hence, we do not repeat it here. We simply recall that it consists of Taylor expanding the right side of (19) written in Boozer coordinates. The result is given in equation (8) and therefore the stellarator can be considered quasisymmetric if

\[\alpha < \psi_i^{1/2}\]  

(23)

for a generic value \(\nu_{\psi, i} \sim 1\). In the regime \(1/\nu_e\) the function \(k\) appearing in (8) scales with \(\nu_{\psi, i}^{-1}\) and the criterion (23) can be more precisely written as

\[\alpha < \sqrt{\nu_{\psi, i} \psi_i} \]  

(24)

In the next subsection we show that when \(B_t\) has large gradients the calculation is more complicated and that the scaling with \(\alpha\) is not quadratic, but more unfavorable.

3.2. Perturbations with large gradients

The straightforward approach of Taylor expanding the drift-kinetic equation (16) fails when the perturbation has large gradients in the sense of (9). For example, it is clear that the parallel streaming operator \(\mathbf{v}_i \sqrt{1 - \lambda B} \mathbf{b} \cdot \nabla\) cannot be expanded at points where \(\mathbf{V}_i \sim \alpha^{-1} L_{g, i} B_0\).

As mentioned in the Introduction, if (9) is satisfied and no additional condition is imposed, the perturbation to the source term of the drift-kinetic equation (16), \(v_{M, i} = \nu_{M, i}^0\), is \(O(1)\) and the stellarator cannot be viewed as a perturbation of a quasisymmetric one. Hence, we assume \(v_{M, i} = \nu_{M, i}^0 \ll v_{M, i}^0\).

In [11] a detailed calculation of the scaling has been given in the 1/\(\nu_e\) regime, i.e. when passing particles, particles trapped in larges wells, and particles trapped in ripple wells are collisionless. Here, we will extend those results to the case when passing particles and particles trapped in larges wells are collisionless, but particles trapped in ripple wells are collisional.

It will be useful to employ \(\psi, \chi, \Theta\) as spatial coordinates, with \(\chi = \Theta - \psi_i E\). The coordinate \(\psi\) acts as a magnetic field line label and \(\Theta\) as the coordinate that gives the position along the magnetic field line. We make \(\epsilon_i \ll \nu_{\psi, i} \ll 1\), that defines what is usually understood by low collisionality. The drift-kinetic equation and its solutions can then be expanded in \(\nu_{\psi, i}\). Taking

\[G_i = G_i^{[1]} + G_i^{[0]} + O(\nu_{\psi, i}^2), \]  

(25)

with \(G_i^{[n]} \sim \nu_{\psi, i}^n e_i F_{i0}\), one finds that

\[\mathbf{b}_i \cdot \nabla G_i^{[1]} = 0.\]  

(26)

For passing particles this means that \(G_i^{[1]}\) is a flux function, whereas for trapped particles it implies that \(G_i^{[1]}\) does not depend on \(\Theta\) and on the sign of the parallel velocity, \(s\). Hence, one can write

\[G_i^{[1]}(\psi, \chi, v, \lambda, s) = g_i(\psi, v, \lambda, s) + \partial_{\lambda} h_i( \psi, \chi, v, \lambda),\]  

(27)

with \(h_i \equiv 0\) in the passing region. The function \(G_i^{[1]}\) is determined by going to next order in the \(\nu_{\psi, i}\) expansion,

\[v_{\psi, i} \mathbf{b}_i \cdot \nabla G_i^{[0]} + v_{\psi, i} G_i^{[0]} = C_i G_i^{[1]}\]  

(28)

For trapped trajectories, we multiply (28) by \(v_{\psi, i}^2 \mathbf{b}_i \cdot \nabla \Theta\) and integrate over the orbit, finding the constraint

\[\oint v_{\psi, i} \frac{v_{\psi, i} \mathbf{b}_i \cdot \nabla F_{i0} d\Theta}{v_{\psi, i} \mathbf{b}_i \cdot \nabla \Theta} = \oint v_{\psi, i} C_i G_i^{[1]} d\Theta.\]  

(29)
Figure 2. The different regions into which phase space is divided to solve the drift-kinetic equation.

An entropy production argument shows (see reference [11]) that $g_{\alpha} \equiv 0$ up to terms $O(\alpha)$. We will see below that terms $O(\alpha)$ are negligible, so that for our purposes $G^\alpha_{L_1}$ is zero in the passing region of velocity space and

$$\int_0^{2\pi} G^\alpha_{L_1} d\chi = 0 \quad (30)$$

in the trapped region. Then, we focus on (29) to find the scaling of $G^\alpha_{L_1}$ for trapped particles. As discussed in the Introduction, when $B_1$ has large gradients, small wells of size $L_1 \sim L_0$ are typically created. Wells of size $L_0$ and ripple wells of size $L_1$ have to be treated separately. More specifically, the calculation is arranged by dividing the trapped part of phase-space into the regions depicted in figure 2. Region I corresponds to a ripple well of size $L_1$, and Regions II and III correspond to a well of size $L_0$ already present in $B_0$. We will give the pertinent form of the drift-kinetic equation in each region, derive the scaling of the distribution function, and then will investigate the matching conditions. We will see that Regions IV and V are collisional boundary layers that develop to heal discontinuities at the interfaces among the first three regions.

We start with particles trapped in wells of size $L_0$. The key is the computation of the scaling of the radial magnetic drift integrated over the trajectory. The rigorous calculation is rather complicated and is presented in detail in [11]. Here, we only state the result,  

$$\int_{\Theta_{\alpha}} \frac{v_{\alpha j}}{v_{\parallel} B} \frac{d\Theta}{V} \sim a^{1/2} \psi. \quad (31)$$

Employing (31), and the fact that in Regions II and III $\partial_{\parallel} \sim B_0$, it is easy to see that (29) implies

$$G^\alpha_{II} \sim a^{1/2} v_{\alpha j} e_i F_0 \quad (32)$$

$$G^\alpha_{III} \sim a^{1/2} v_{\alpha j} e_i F_0 \quad (33)$$

and also

$$\partial_{\parallel} G^\alpha_{II} \sim a^{1/2} v_{\alpha j} e_i B_0 F_0 \quad (34)$$

$$\partial_{\parallel} G^\alpha_{III} \sim a^{1/2} v_{\alpha j} e_i B_0 F_0 \quad (35)$$

Observe that the size of these pieces of the distribution function justifies neglecting terms $O(\alpha)$ of the distribution function in the passing region, as advanced above.

Regions II and III give contributions to the neoclassical radial electric current

$$\langle j_\parallel \nabla \psi \rangle_{\alpha j} \sim \frac{\alpha}{\nu_{\alpha j}} e_i c_i \nabla_{\parallel} | \nabla \psi | \quad (36)$$

and

$$\langle j_\parallel \nabla \psi \rangle_{\alpha j} \sim \frac{\alpha}{\nu_{\alpha j}} e_i c_i \nabla_{\parallel} | \nabla \psi | \quad (37)$$

We turn to find the distribution function in the ripple well, Region I. We want to solve the regimes

$$\alpha^{1/2} \nu_{\alpha j} \ll 1, \quad (38)$$

corresponding to the case in which particles trapped in ripple wells are collisional (the case solved in [11]), and

$$\alpha^{1/2} \nu_{\alpha j} \gg 1, \quad (39)$$

to lowest order

$$B_0^{-1} \partial_\parallel G^\alpha_{I} \gg \frac{v_{\parallel} \partial_\parallel G^\alpha_{I}}{v_{\parallel} B_0} \quad (40)$$

where $\nu_{\alpha j}(\psi)$ is the pitch-angle scattering frequency. Here, we have approximated $B = B_{\text{max}}$ and $\lambda \approx \lambda_{\alpha} \approx B_{\text{max}}$, with $\lambda_{\alpha}$ defined in figure 1 along with other quantities that will be employed below. In order to get the above simplified equation, one needs the relations

$$B_0^{-1} \partial_\parallel G^\alpha_{I} \gg \frac{v_{\parallel} \partial_\parallel G^\alpha_{II}}{v_{\parallel} B_0} \quad (41)$$

obtained by using $v_{\parallel} \sim a^{1/2} v_{\alpha j} B_0 \nabla \sim (\alpha L_0)^{-1}$ and $\partial_\parallel \sim B_0 / \alpha$. If $\alpha^{1/2} \nu_{\alpha j} \ll 1$, then to lowest order $G^\alpha_{I}$ does not depend on $\Theta$ and it is determined by integrating (40) over the orbit,  

$$\int_{\Theta_{\alpha i}}^{\Theta_{\alpha f}} \frac{d\Theta}{v_{\parallel} B_0} \frac{v_{\parallel} \partial_\parallel G^\alpha_{I}}{v_{\parallel} B_0} = \int_{\Theta_{\alpha i}}^{\Theta_{\alpha f}} \frac{d\Theta}{v_{\parallel} B_0} T_{\nu \alpha j} F_0. \quad (42)$$

Imposing a regularity condition at the bottom of the well $\lambda_0$, we can find an explicit expression for $\partial_\parallel G^\alpha_{I}$ in the well. Using again that the size of the well in $\lambda$ is $O(\alpha)$, we get

$$\partial_\parallel G^\alpha_{I} \sim v_{\alpha j} e_i F_0. \quad (43)$$

If $\alpha^{1/2} \nu_{\alpha j} \gg 1$, one can drop the parallel streaming term in (40) to obtain

$$\int_{\Theta_{\alpha i}}^{\Theta_{\alpha f}} \frac{d\Theta}{v_{\parallel} B_0} \frac{v_{\parallel} \partial_\parallel G^\alpha_{I}}{v_{\parallel} B_0} = \int_{\Theta_{\alpha i}}^{\Theta_{\alpha f}} \frac{d\Theta}{v_{\parallel} B_0} T_{\nu \alpha j} F_0. \quad (44)$$

This also yields $\partial_\parallel G^\alpha_{I} \sim v_{\alpha j} e_i F_0$. In this case $\partial_\parallel G^\alpha_{I}$ can depend on $\Theta$.

Hence, the size of $\partial_\parallel G^\alpha_{I}$ is the same for both regimes, $\alpha^{1/2} \nu_{\alpha j} \ll 1$ and $\alpha^{1/2} \nu_{\alpha j} \gg 1$. Whereas we have been able to determine the size of $\partial_\parallel G^\alpha_{I}$ by using only the drift-kinetic equation in the ripple well, the size of $G^\alpha_{I}$ is still unknown. It is determined by matching with $G^\alpha_{II}$ and $G^\alpha_{III}$, and found to be, therefore,
Assuming a number of ripple wells per magnetic field line $O(\alpha^{-1})$, and a number of magnetic field lines with ripple wells $O(\alpha^{-1})$, we find the contribution of Region I to the flux-surface averaged radial current, 
\[
\langle J_v \nabla \psi \rangle_\psi \sim \frac{\alpha}{L_0} c_\ell^2 e n_{v_\ell} \nabla \psi |_{\psi = 0}.
\]  
(46)

In deriving the scalings (36), (37) and (46), we have skipped the proof that the matching among the three regions can be done consistently. Actually, it cannot be done without Regions IV and V. One can show that the values of $\partial_{\lambda} G_I$ and $\partial_{\lambda} G_{II}$ do not match exactly, and the same happens with $\partial_{\lambda} G_{II}$ and $\partial_{\lambda} G_{III}$. The correct matching is provided by the emergence of collisional layers at the interfaces between I and II (denoted by Region IV) and II and III (denoted by Region V). A careful treatment of the collisional layers, Regions IV and V, gives 
\[
\langle J_v \nabla \psi \rangle_\psi \sim \frac{\alpha}{L_0} c_\ell^2 e n_{v_\ell} \nabla \psi |_{\psi = 0},
\]  
(47) and 
\[
\langle J_v \nabla \psi \rangle_\psi \sim \frac{\alpha}{L_0} c_\ell^2 e n_{v_\ell} \nabla \psi |_{\psi = 0},
\]  
(48) where the assumption that the number of ripple wells is $O(\alpha^{-2})$ has been employed again. We postpone the details on the boundary layers to section 4. In particular, the estimations of section 4 show that the discontinuities in the absence of the collisional boundary layers are small enough that the estimates (36), (37) and (46) are correct.

It is clear that the contribution of Region IV is always negligible. When $\alpha^{-1/2} \lambda_{\ell_i} \ll 1$, Regions I, II and III contribute the same and the contribution of Region V is negligible, giving 
\[
\langle J_v \nabla \psi \rangle_\psi \sim \frac{\alpha}{L_0} c_\ell^2 e n_{v_\ell} \nabla \psi |_{\psi = 0}.
\]  
(49) However, when $\alpha^{-1/2} \lambda_{\ell_i} \gg 1$, the contribution of Region V dominates and one obtains 
\[
\langle J_v \nabla \psi \rangle_\psi \sim \frac{\alpha}{L_0} c_\ell^2 e n_{v_\ell} \nabla \psi |_{\psi = 0}.
\]  
(50)

Finally, we write the criteria to assess closeness to quasisymmetry inferred from the above results. Namely, if the perturbation has large gradients and $\alpha^{-1/2} \lambda_{\ell_i} \ll 1$, then the criterion is 
\[
\alpha < \nu_{qi} c_\ell.
\]  
(51)

If the perturbation has large helicities and $\alpha^{-1/2} \lambda_{\ell_i} \gg 1$, then the criterion is 
\[
\alpha < c_\ell^2.
\]  
(52)

One can compare these criteria with (24) and confirm that, for the same value of $\alpha$, large helicity perturbations degrade more efficiently the quasisymmetric properties of the stellarator than small helicity ones, as expected.

4. The collisional boundary layers

Recall that the different regions into which we split phase space to compute the distribution function are shown in figure 2. We have explained how to determine the distribution function in Regions I, II, and III, and here we focus on the collisional boundary layers, Regions IV and V. In fact, as already pointed out, the correct treatment of the collisional boundary layers is also needed to ensure that the results on Regions I, II, and III are consistent. Below we show how to find two functions $\delta G_{IV}^\lambda$ and $\delta G_{V}^\lambda$ such that $G_{II}^\lambda + \delta G_{IV}^\lambda$ smoothly matches $G_{I}^\lambda$, and $G_{III}^\lambda + \delta G_{V}^\lambda$ smoothly matches $G_{II}^\lambda + G_{IV}^\lambda$. This section is based on the rigorous discussion given in [14]. In this reference, the boundary layers are calculated for a stellarator close to omnigeniety, proving that their effect on transport is negligible. As advanced in subsection 3.2, the boundary layers are more relevant in stellarators close to quasisymmetry because they can dominate flow damping in the regime in which particles trapped in ripple wells are collisional and the rest are collisionless.

Let us start by Region IV and recall the notation used in figure 1. We define a function $\delta G_{IV}^\lambda$ supported in $\lambda \in [\lambda_{\ell_i}, -K \delta \lambda_{IV}, \lambda_{\ell_i}]$ and $\Theta \in \Theta_\psi(\lambda), \Theta_{\psi}(\lambda)$. Here, $\delta \lambda_{IV}$ is the characteristic width of the layer and $K \gg 1$. The equation for $\delta G_{IV}^\lambda$ is 
\[
\nu_{qi} \mathbf{b} \nabla \Theta \partial_{\psi} \delta G_{IV}^\lambda - \frac{\nu_{qi}}{v_{B_{\text{max}}}^\psi} \delta_{\psi}(\nu_{qi} \partial_{\psi} \delta G_{IV}^\lambda) = 0,
\]  
(53) which has to be solved with the boundary conditions $\partial_{\lambda} \delta G_{IV}^\lambda = \partial_{\lambda} G_{II}^\lambda - \partial_{\lambda} G_{II}^\lambda = 0$ at $\lambda = \lambda_{\ell_i}$, $G_{IV}^\lambda = 0$ at $\lambda = \lambda_{\ell_i} - K \delta \lambda_{IV}$, $\delta G_{IV}^\lambda = \delta G_{V}^\lambda$ at $\Theta = \Theta_\psi(\lambda)$, the interface between the two collisional boundary layers, and $G_{IV}^\lambda(s = 1) = \delta G_{IV}^\lambda(s = -1)$ at $\Theta_\psi(\lambda_{\ell_i})$. If both terms in (53) have to be of the same order, and in the layer $\nu_{qi} \sim \alpha^{1/2} \lambda_{\ell_i}^2$ and $\mathbf{b} \nabla \sim \alpha^{-1/2} L_0^{-1}$, one infers 
\[
\delta \lambda_{IV} \sim \alpha^{3/4} \nu_{qi}^{-1/2} L_0^{-1}.
\]  
(54)

The boundary condition on $\partial_{\psi} \delta G_{IV}^\lambda$ at $\lambda = \lambda_{\ell_i}$ gives 
\[
\partial_{\psi} \delta G_{IV}^\lambda = B_0 \psi_{\psi} \langle e_i \psi \rangle_\psi,\n\]  
(55) and therefore 
\[
\delta G_{IV}^\lambda \sim \alpha^{3/4} \nu_{qi}^{-1/2} e_i F_0.\n\]  
(56)

We proceed to deal with the collisional layer denoted by Region V. In this case, $\delta G_{V}^\lambda$ satisfies 
\[
\nu_{qi} \mathbf{b} \nabla \Theta \partial_{\psi} \delta G_{V}^\lambda - \frac{\nu_{qi}}{v_{B_{\text{max}}}^\psi} \delta_{\psi}(\nu_{qi} \partial_{\psi} \delta G_{V}^\lambda) = 0
\]  
(57) and is defined on $\lambda \in [\lambda_{\ell_i} - K \delta \lambda_{IV}, \lambda_{\ell_i} + K \delta \lambda_{IV}]$. $\Theta \in \Theta_\psi(\lambda), \Theta_{\psi}(\lambda)$ for $\lambda < \lambda_{\ell_i}$, and $\Theta \in \Theta_{\psi}(\lambda), \Theta_\psi(\lambda)$ for $\lambda > \lambda_{\ell_i}$. The boundary conditions are $\partial_{\lambda} \delta G_{V}^\lambda(\lambda_{\ell_i}^+) = \partial_{\lambda} \delta G_{V}^\lambda(\lambda_{\ell_i}^-) = - (\partial_{\lambda} G_{III}^\lambda(\lambda_{\ell_i}^-) - \partial_{\lambda} G_{II}^\lambda(\lambda_{\ell_i}^-))$ at $\lambda = \lambda_{\ell_i}$, $G_{V}^\lambda = 0$ at $\lambda_{\ell_i} - K \delta \lambda_{IV}$ and $\lambda_{\ell_i} + K \delta \lambda_{IV}$, $\delta G_{V}^\lambda = \delta G_{IV}^\lambda$, for $\lambda < \lambda_{\ell_i}$ at $\Theta = \Theta_\psi(\lambda)$, $\delta G_{V}^\lambda(s = 1) = \delta G_{V}^\lambda(s = -1)$ for $\lambda > \lambda_{\ell_i}$ at $\Theta_\psi(\lambda)$, and $\delta G_{V}^\lambda(s = 1) = \delta G_{V}^\lambda(s = -1)$, for any $\lambda$, at $\Theta_\psi(\lambda)$.

Balancing the two terms in (57), and employing $\nu_{qi} \sim \nu_{qi}$ and $\mathbf{b} \nabla \sim L_0^{-1}$, one gets 
\[
\delta \lambda_{IV} \sim \nu_{qi}^{1/2} B_0^{-1}.
\]  
(58)

Estimating the size of $\delta G_{V}^\lambda$ is slightly more involved. It can be shown [14] that phase space continuity between Regions IV and V implies
\[ \delta G_i^V \sim a^{3/2} \nu_{\alpha_i}^{1/2} e_i f_i. \] (59)

Finally, we emphasize that the size of both \( \delta G_i^V \) and \( \delta G_i^{\text{HI}} \) is very small compared to \( G_i^\text{II} \) and \( G_i^{\text{III}} \), given in (32). This implies, in particular, that the estimation (45) is correct.

5. Conclusions

In this paper we have first reviewed the results of references [10 and 11]. In those references, we started a systematic approach to understand how much one can deviate from perfect quasisymmetry without spoiling some of the properties that make quasisymmetric stellarators interesting; in particular, the absence of flow damping in the symmetry direction. We show that the answer depends on the size of the helicity of the deviations from quasisymmetry and, in general, on the collisionality regime. Formal criteria are derived in a variety of situations, to assess whether a stellarator can be considered quasisymmetric in practice. The survey of [10 and 11] is completed by extending those results to a new collisionality regime. All these results apply to tokamaks with ripple as well.

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