Homotopy categories of unbounded complexes of projective modules

Yuji Yoshino

Department of Mathematics, Okayama University, Okayama, Japan

Correspondence
Yuji Yoshino, Department of Mathematics, Okayama University, Okayama 700-8530, Japan.
Email: yoshino@math.okayama-u.ac.jp

Funding information
JSPS, Grant/Award Number: 19K03448

Abstract
We develop in this paper the stable theory for projective complexes, by which we mean to consider a chain complex of finitely generated projective modules as an object of the factor category of the homotopy category modulo split complexes. As a result of the stable theory, we are able to prove that any complex of finitely generated projective modules over a generically Gorenstein ring is acyclic if and only if its dual complex is acyclic. This shows the dependence of total reflexivity conditions for modules over a generically Gorenstein ring.

MSC 2020
13D02 (primary), 18G35 (secondary)

Contents

1. INTRODUCTION ........................................... 101
2. PRELIMINARY OBSERVATION FOR COMPLEXES .................. 103
3. *TORSION-FREE AND *REFLEXIVE COMPLEXES ..................... 106
4. COMPLEXES OVER A GENERICALLY GORENSTEIN RING ............. 109
5. SPLIT COMPLEXES AND Add(𝑅) .................................. 111
6. THE STABLE CATEGORY OF 𝒦(𝑅) .................................. 116
7. Add(𝑅)-APPROXIMATIONS .................................... 119
8. CONTRACTIONS ............................................. 126
9. REMARKS ON PARTIAL Add(𝑅)-RESOLUTIONS .................... 132
10. COUNIT MORPHISM FOR THE ADJOINT PAIR (Σⁿ, Ωⁿ) ............ 139
11. THE MAIN THEOREM AND THE PROOF .............................. 143
12. APPLICATIONS ............................................. 148

© 2022 The Authors. The publishing rights in this article are licensed to the London Mathematical Society under an exclusive licence.
1 | INTRODUCTION

In this paper, we are mainly interested in unbounded cochain complexes consisting of finitely generated projective modules over a commutative Noetherian ring. Of most interest to us in the present paper are the properties of complexes that are independent of any additional split summands. For this purpose, we develop the stable theory for those complexes. For the module category, such an idea was first proposed and established by Auslander–Bridger [2] under the name of ‘stable module theory’. We apply their idea to the homotopy category of complexes of finitely generated projective modules.

The whole of our stable theory for complexes is devoted to prove the following single theorem.

**Theorem 1.1** (see Theorem 11.7). Let $R$ be a commutative Noetherian ring that is generically Gorenstein, and $X$ a complex of finitely generated projective $R$-modules. Then, $X$ is acyclic (that is, $H(X) = 0$) if and only if the $R$-dual $X^*$ is acyclic (that is, $H(X^*) = 0$).

Recall that a commutative Noetherian ring is called a generically Gorenstein ring if the total ring of quotients is a Gorenstein ring, or equivalently $R_p$ is a Gorenstein ring for every associated prime $p \in \text{Ass}(R)$. As a matter of fact, every Noetherian integral domain, a little more generally, every reduced Noetherian ring, is a generically Gorenstein ring. A similar theorem to Theorem 1.1, but in a more special setting, was considered in [20, Corollary 1.4]. Analogously to the stable module theory of Auslander and Bridger, we will introduce the parallel notion of torsion-freeness and reflexivity for complexes, which we call *torsion-free complexes and *reflexive complexes in this paper (Definition 3.1). We observe in Theorem 2.3 that there is an exact sequence similar to the Auslander–Bridger sequence. If the base ring $R$ is a generically Gorenstein ring, then as we shall show in Theorem 4.2, a complex $X$ is *torsion-free if and only if the cohomology modules $H^i(X^*) (i \in \mathbb{Z})$ are torsion-free as $R$-modules.

A crucial point for the proof of Theorem 1.1 is how one can relate a generic condition of the ring such as the generic Gorenstein condition with the *torsion-free or the *reflexive property for complexes. This will be accomplished by considering a factor category of the homotopy category. To be more precise, let $\mathcal{K}(R)$ be the homotopy category of all complexes of finitely generated projective modules over a commutative Noetherian ring $R$, and let $\text{Add}(R)$ be its additive full subcategory consisting of all split complexes. See Definition 5.2 and Theorem 5.8 for further details. We show in Lemmas 7.2 and 7.5 that $\text{Add}(R)$ is functorially finite in $\mathcal{K}(R)$ and hence every complex in $\mathcal{K}(R)$ can be resolved by complexes in $\text{Add}(R)$.

We define $\mathcal{K}(R)/\text{Add}(R)$ to be the factor category $\mathcal{K}(R)/\text{Add}(R)$ and call it the stable category. Then we are able to define the syzygy functor $\Omega$ and the cosyzygy functor $\Sigma$ on $\mathcal{K}(R)$, and as a result we have an adjoint pair $(\Sigma, \Omega)$ of functors (Theorem 7.11). Thus there is a natural counit morphism $\pi^n_X : \Sigma^n \Omega^n X \rightarrow X$ for any positive integer $n$ and for any complex $X$. In terms of syzygy functors, we can characterize the *torsion-free property for $X$ as the counit morphism $\pi^1_X$ is an isomorphism in $\mathcal{K}(R)$ (Theorem 7.14). We develop in Section 8 some new idea to construct complexes by successive use of mapping cone constructions, which we shall call the contraction. See Theorem and Definition 8.2.
Now taking the mapping cones of the counit morphisms in $\mathcal{K}(R)$, we have triangles of the form

$$\Delta^{(n,0)}(X) \longrightarrow \Sigma^n \Omega^n(X) \xrightarrow{\pi_X^n} X \longrightarrow \Delta^{(n,0)}(X)[1].$$

by which we define the complexes $\Delta^{(n,0)}(X)$ for $X$. Then we shall show that all such complexes $\Delta^{(n,0)}(X)$ have a finite $\text{Add}(R)$-resolution of length at most $n − 1$. See Theorem 10.2 for more precise statement, which is one of the key results in order to prove Theorem 1.1. After observing these facts, we prove in Lemma 11.1 that any syzygy complex $\Omega X$ is *torsion-free if $H(X^*) = 0$. This is the second key result to prove Theorem 1.1. As a consequence of this theorem, we are eventually able to prove the main Theorem 1.1 in Section 11.

The following are the corollaries that are proved straightforwardly from Theorem 1.1, and each one is proved in the last section of this paper.

**Corollary 1.2.** Assume that the ring $R$ is a generically Gorenstein ring. Let $f : X \to Y$ be a chain homomorphism between complexes of finitely generated projective modules over $R$. Then, $f$ is a quasi-isomorphism if and only if the $R$-dual $f^* : Y^* \to X^*$ is a quasi-isomorphism.

**Corollary 1.3.** Assume that the ring $R$ is a generically Gorenstein ring. Let $M$ be a finitely generated $R$-module. Then the following conditions are equivalent.

1. $M$ is a totally reflexive $R$-module.
2. $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$.
3. $M$ is an infinite syzygy, that is, there is an exact sequence of infinite length of the form $0 \to M \to P_0 \to P_1 \to P_2 \to \cdots$, where each $P_i$ is a finitely generated projective $R$-module.

**Corollary 1.4.** Under the assumption that $R$ is a generically Gorenstein ring, we have the equality of $G$-dimension;

$$G\text{-dim}_R M = \sup \{n \in \mathbb{Z} \mid \text{Ext}_R^n(M, R) \neq 0 \},$$

for a finitely generated $R$-module $M$.

Jorgensen and Şega [14] gave an example of a module over a non-Gorenstein Artinian ring for which the implication (2) $\Rightarrow$ (1) in Corollary 1.3 fails, hence the generic Gorensteinness assumption in the theorem is indispensable.

The following is a commutative version of Tachikawa conjecture that is also a consequence of Theorem 1.1. It should be noted that this has been proved by Avramov, Buchweitz and Şega [4].

**Corollary 1.5.** Let $R$ be a Cohen–Macaulay ring with canonical module $\omega$. Furthermore, assume that $R$ is a generically Gorenstein ring. If $\text{Ext}_R^i(\omega, R) = 0$ for all $i > 0$, then $R$ is Gorenstein.

**Corollary 1.6.** Assume that the ring $R$ is a generically Gorenstein ring. Let $X$ be a complex of finitely generated projective modules.

1. If $H(X)$ is bounded above, that is, $X \in D^-(R)$, then there is an isomorphism $X^* \cong \text{RHom}_R(X, R)$ in the derived category $D(R)$.
(2) If \( H(X) \) and \( H(X^*) \) are bounded above, that is, \( X, X^* \in D^-(R) \), then we have the isomorphism in the derived category:

\[
X \cong \text{RHom}_R(\text{RHom}_R(X, R), R).
\]

**Corollary 1.7.** Assume that the ring \( R \) is a generically Gorenstein ring. Let \( X \) be a complex of finitely generated projective modules. If all the cohomology modules \( H^i(X) \) (\( i \in \mathbb{Z} \)) have dimension at most \( \ell \) as \( R \)-modules, then so are the modules \( H^i(X^*) \) (\( i \in \mathbb{Z} \)). In particular, \( X \) has cohomology modules of finite length if and only if so does \( X^* \).

## 2 Preliminary Observation for Complexes

Throughout this paper, we assume that \( R \) is a commutative Noetherian ring. We denote by \( \text{mod}(R) \) the abelian category of finitely generated \( R \)-modules and \( R \)-module homomorphisms. Furthermore, we denote by \( \text{proj}(R) \) the additive subcategory of \( \text{mod}(R) \), which consists of all finitely generated projective \( R \)-modules.

We denote by \( \mathcal{C}(R) = \mathcal{C}(\text{proj}(R)) \) the additive category of complexes over \( \text{proj}(R) \) and chain homomorphisms. We also denote by \( \mathcal{K}(R) = \mathcal{K}(\text{proj}(R)) \) the homotopy category consisting of complexes over \( \text{proj}(R) \). Note by recalling the definition that objects of \( \mathcal{C}(R) \) and \( \mathcal{K}(R) \) are complexes consisting of finitely generated projective modules, which we denote cohomologically such as

\[
X = \left[ \cdots \to X^{i-1} \xrightarrow{d^{i-1}_X} X^i \xrightarrow{d^i_X} X^{i+1} \to \cdots \right],
\]

where each \( X^i \) belongs to \( \text{proj}(R) \). All cohomology modules \( H^i(X) \) (\( i \in \mathbb{Z} \)) are necessarily finitely generated \( R \)-modules for \( X \in \mathcal{C}(R) \). A morphism \( X \to Y \) in \( \mathcal{C}(R) \) is a cochain homomorphism, while a morphism \( X \to Y \) in \( \mathcal{K}(R) \) is a homotopy equivalence class of a chain homomorphism from \( X \) to \( Y \), that is,

\[
\text{Hom}_{\mathcal{K}(R)}(X, Y) = \text{Hom}_{\mathcal{C}(R)}(X, Y) / \text{chain homotopy}.
\]

Both of \( \text{Hom}_{\mathcal{C}(R)}(X, Y) \) and \( \text{Hom}_{\mathcal{K}(R)}(X, Y) \) have natural \( R \)-module structures. However, they are not necessarily finitely generated \( R \)-modules in general.

For example, consider the endomorphisms of the complex \( \left[ \cdots \to R \to R \to R \to \cdots \right] \).

Note that a complex \( X \in \mathcal{C}(R) \) is the zero object as an object of \( \mathcal{K}(R) \) if and only if it is a split exact sequence as a long exact sequence, which is called a null complex. (It is also known as a contractible complex.) Every complex \( X \in \mathcal{C}(R) \) has a direct sum decomposition in \( \mathcal{C}(R) \) such as \( X = X' \oplus N \), where \( N \) is a null complex and \( X' \) contains no null complex as a direct summand.\(^\dagger\)

We should note that such a decomposition is not unique in general.

---

\(^\dagger\)In general, let \( f : P \to Q \) be an \( R \)-homomorphism between finitely generated projective \( R \)-modules. Then there are direct sum decompositions \( P = P' \oplus P'' \), \( Q = Q' \oplus Q'' \) under which \( f \) is described as \( \begin{pmatrix} f' & 0 \\ 0 & f'' \end{pmatrix} \) where \( f'' : P'' \to Q'' \) is an isomorphism, and \( f' : P' \to Q' \) does not contain any direct summands that induces an isomorphism. This is possible since \( P \) and \( Q \) are finitely generated. If we are given a complex \( X = \left[ \cdots \to X^n \xrightarrow{d^n_X} X^{n+1} \to \cdots \right] \) in \( \mathcal{C}(R) \), applying this argument to each \( d^n_X \) we have desired decomposition.
It is clear and well known that a chain homomorphism $f$ in $\mathcal{C}(R)$ factors through a null complex if and only if $f$ is null homotopic. Therefore the category $\mathcal{K}(R)$ is a residue category of $\mathcal{C}(R)$ by the ideal generated by the object set consisting of all null complexes. It is easy to verify that $\mathcal{K}(R)$ is a Frobenius category with null complexes as relatively projective and injective objects. In such a sense, $\mathcal{K}(R)$ has a structure of triangulated category. Recall that the shift functor $X \mapsto X[1]$ is defined as $X[1]^n = X^{n+1}$ and $d^n_{X[1]} = -d^{n+1}_X$. Furthermore, there is a triangle $X \to Y \to Z \to X[1]$ in $\mathcal{K}(R)$ if and only if there is an exact sequence in $\mathcal{C}(R)$ of the form

$$0 \longrightarrow X \longrightarrow Y \oplus N \longrightarrow Z \longrightarrow 0,$$

where $N$ is a null complex. One can find such description of triangles in Happel [11, Chapter 1]. The more general references for complexes and triangulated categories are Weibel [18] and Gelfand–Manin [16].

A remarkable advantage of $\mathcal{K}(R)$ is that it possesses a duality. For $X \in \mathcal{K}(R)$, we are able to define the dual complex by

$$X^* = \text{Hom}_R(X, R), \quad d^n_{X^*} = (-1)^{n+1}\text{Hom}_R(d^{-n}_X, R).$$

Note that $X^*$ is again an object of $\mathcal{K}(R)$, since the dual of a finitely generated projective module is finitely generated projective. It is easy to see that the duality functor

$$(\cdot)^* : \mathcal{K}(R) \longrightarrow \mathcal{K}(R)^{\text{op}}, \quad X \mapsto X^*$$

is a triangle functor between triangulated categories. Since $X^{**}$ is naturally isomorphic to $X$, it actually yields the duality on $\mathcal{K}(R)$.

Notation 2.1. For a complex $X \in \mathcal{K}(R)$, we denote by $C(X)$ the cokernel of the differential mapping $d_X[-1] : X[-1] \to X$ as underlying graded $R$-module. So $C(X) = \bigoplus_{i \in \mathbb{Z}} C^i(X)$ where $C^i(X) = \text{Coker}(X^{i-1} \to X^i)$. Similarly, the cocycle $Z(X) = \bigoplus_{i \in \mathbb{Z}} Z^i(X)$ is the kernel of $d_X$ and the coboundary $B(X) = \bigoplus_{i \in \mathbb{Z}} B^i(X)$ is the image of $d_X$.

As $C(X) = X/B(X)$, there is a short exact sequence of graded $R$-modules such as

$$0 \longrightarrow H(X) \longrightarrow C(X) \longrightarrow B(X)[1] \longrightarrow 0. \quad (2.1)$$

Let $X$ be a complex in $\mathcal{K}(R)$ and let $M$ be an $R$-module. We denote by $K(\text{Mod}(R))$ the homotopy category of all complexes of any $R$-modules, and we regard $M$ as a complex concentrated in degree zero. Recall that $\text{Hom}_R(X, M)$ is the Hom complex and an element of the cohomology modules of this complex is nothing but the homotopy class of a chain map from $X$ to $M$, that is,

$$H^{-i}(\text{Hom}_R(X, M)) = \text{Hom}_{K(\text{Mod}(R))}(X[i], M).$$

Definition 2.2. Let $X \in \mathcal{K}(R)$, $M$ an $R$-module and $i \in \mathbb{Z}$. As noted above, each element $[f] \in H^{-i}(\text{Hom}_R(X, M))$ is a homotopy class of a chain map $f : X[i] \to M$, thus it induces a unique $R$-module homomorphism $H^0(f) : H^0(X[i]) = H^i(X) \to H^0(M) = M$, hence an element $H^0(f) \in$
Hom$_R(H^i(X), M)$. We define an $R$-module homomorphism

$$\rho_{X,M}^i : H^{-i}(\text{Hom}_R(X, M)) \rightarrow \text{Hom}_R(H^i(X), M)$$

by $\rho_{X,M}^i([f]) = H^0(f)$.†

**Theorem 2.3.** Under the circumstances in Definition 2.2, there is an exact sequence of $R$-modules;

$$0 \rightarrow \text{Ext}_R^1(C^{i+1}(X), M) \rightarrow H^{-i}(\text{Hom}_R(X, M)) \overset{\rho_{X,M}^i}{\rightarrow} \text{Hom}_R(H^i(X), M) \rightarrow \text{Ext}_R^2(C^{i+1}(X), M),$$

for each $i \in \mathbb{Z}$.

**Proof.** We see from the exact sequence (2.1) that there exists an exact sequence;

$$0 \rightarrow \text{Hom}_R(B^{i+1}(X), M) \rightarrow \text{Hom}_R(C^i(X), M) \rightarrow \text{Hom}_R(H^i(X), M) \rightarrow \text{Ext}_R^1(B^{i+1}(X), M),$$

where we should note that $\text{Ext}_R^1(B^{i+1}(X), M) \cong \text{Ext}_R^2(C^{i+1}(X), M)$, since $X^{i+1}/B^{i+1}(X) \cong C^{i+1}(X)$ and $X^{i+1}$ is projective.

Note that $\text{Hom}_R(X,M)^{-i} = \text{Hom}_R(X^i,M)$ for all $i \in \mathbb{Z}$. Thus, from the exact sequence $X^{i-1} \rightarrow X^i \rightarrow C^i(X) \rightarrow 0$, it follows that $0 \rightarrow \text{Hom}_R(C^i(X), M) \rightarrow \text{Hom}_R(X, M)^{-i} \rightarrow \text{Hom}_R(X, M)^{-i+1}$ is exact, hence we have an isomorphism

$$\lambda : Z^{-i}(\text{Hom}_R(X, M)) \rightarrow \text{Hom}_R(C^i(X), M),$$

where the left-hand side is the $(-i)$th cocycle module of the complex $\text{Hom}_R(X, M)$. On the other hand, the $(-i)$th coboundary $B^{-i}(\text{Hom}_R(X, M))$ is the image of the mapping $\text{Hom}_R(X^{i+1}, M) \rightarrow \text{Hom}_R(X^i, M)$. From the exact sequence $0 \rightarrow B^{i+1}(X) \rightarrow X^{i+1} \rightarrow C^{i+1}(X) \rightarrow 0$, we have an exact sequence

$$\text{Hom}_R(X^{i+1}, M) \overset{\nu}{\rightarrow} \text{Hom}_R(B^{i+1}(X), M) \overset{\text{Ext}_R^1(C^{i+1}(X), M)}{\rightarrow} 0.$$

Since there is a commutative diagram

$$X^i \overset{d_X^i}{\rightarrow} X^{i+1} \quad \text{and} \quad B^{i+1}(X),$$

we have a commutative diagram

$$\text{Hom}_R(X^{i+1}, M) \overset{(d_X^i)^*}{\rightarrow} \text{Hom}_R(X^i, M) \quad \text{with} \quad \nu \quad \text{and} \quad \text{Hom}_R(B^{i+1}(X), M).$$

†This is the standard Künneth map; see [6, IV.6].
from which we see that the mapping \( \nu \) has the image \( B^{-i}(\text{Hom}_R(X, M)) \). Hence, we have an exact sequence

\[
0 \longrightarrow B^{-i}(\text{Hom}_R(X, M)) \longrightarrow \text{Hom}_R(B^{i+1}(X), M) \longrightarrow \text{Ext}^1_R(C^{i+1}(X), M) \longrightarrow 0.
\]

Combine all the mappings together, we finally have a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B^{-i}(\text{Hom}_R(X, M)) & \longrightarrow & Z^{-i}(\text{Hom}_R(X, M)) & \longrightarrow & H^{-i}(\text{Hom}_R(X, M)) & \longrightarrow & 0 \\
& & \downarrow \lambda & & \downarrow \rho & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}_R(B^{i+1}(X), M) & \longrightarrow & \text{Hom}_R(C^{i}(X), M) & \longrightarrow & \text{Hom}_R(H^{i}(X), M) & \longrightarrow & \text{Ext}^2_R(C^{i+1}(X), M) \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \text{Ext}^1_R(C^{i+1}(X), M) & & & & & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & & & & & \\
\end{array}
\]

Since \( \lambda \) is an isomorphism, we have the desired exact sequence by the snake lemma.

Example 2.4. Let \( M \) be a finitely generated \( R \)-module and let

\[
P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0
\]

be a projective presentation of \( M \), where each \( P_i \) are finitely generated and projective. Recall that the transpose \( \text{Tr}(M) \) of \( M \) is defined to be the cokernel of the dual mapping \( f^* = \text{Hom}_R(f, R) \) of \( f \).

Now, set the complex \( X \) to be [0 \( \longrightarrow P_0 \xrightarrow{f} P_1 \xrightarrow{f} 0 \)]. Then we have that \( C^1(X) = \text{Tr}(M) \) and \( X^* = [0 \xrightarrow{f} P_1 \xrightarrow{f} P_0 \xrightarrow{f} 0] \). Therefore, \( H^0(X^*) = M \) and \( H^0(X)^* = M^{**} \) in this case. It is easily verified that the mapping \( \rho^0_{XR} \) is isomorphic to the canonical mapping \( M \rightarrow M^{**} \). Thus applying Theorem 2.3, we have an exact sequence

\[
0 \longrightarrow \text{Ext}^1(\text{Tr}(M), R) \longrightarrow M \longrightarrow M^{**} \longrightarrow \text{Ext}^2(\text{Tr}(M), R),
\]

as shown in [2, Chapter 2].

3 | *TORSION-FREE AND *REFLEXIVE COMPLEXES

Definition 3.1. Let \( X \in \mathcal{X}(R) \). We denote by \( X^* \) the \( R \)-dual complex \( \text{Hom}_R(X, R) \). As we remarked in Definition 2.2, we have a natural mapping

\[
\rho^i_{XR} : H^{-i}(X^*) \rightarrow H^i(X)^*
\]
for all \( i \in \mathbb{Z} \). We say that the complex \( X \) is \(*\)torsion-free if \( \rho^i_{XR} \) are injective mappings for all \( i \in \mathbb{Z} \). Likewise, we say that \( X \) is \(*\)reflexive if \( \rho^i_{XR} \) are isomorphisms for all \( i \in \mathbb{Z} \).

**Lemma 3.2.** Let \( X \) be a complex in \( \mathcal{K}(R) \).

1. \( X \) is \(*\)torsion-free if and only if it satisfies the following condition.
   \((\ast)\) For any chain map \( f : X \to R[i] \) with \( i \in \mathbb{Z} \), if \( H(f) = 0 \), then \( f = 0 \) as a morphism in \( \mathcal{K}(R) \).

2. Assume that \( X \) satisfies the condition \((\ast)\). Then \( X \) is \(*\)reflexive if and only if it satisfies the following condition.
   \((\ast\ast)\) If \( a : H^{-i}(X) \to R \) is an \( R \)-module homomorphism where \( i \in \mathbb{Z} \), then there is a chain map \( f : X \to R[i] \) such that \( H^{-i}(f) = a \).

**Proof.** This is just a restatement of the definition. \( \square \)

**Remark 3.3.**

1. Let \( X \) be \(*\)torsion-free (respectively, \(*\)reflexive). Then so are any shifted complexes \( X[i] \) for \( i \in \mathbb{Z} \). Any direct summands of \( X \) are also \(*\)torsion-free (respectively, \(*\)reflexive).

2. Any direct sums of \(*\)torsion-free complexes are \(*\)torsion-free. (As we will see in Section 5, the category \( \mathcal{K}(R) \) admits certain kind of infinite direct sums. This remark says that if \( \{X_i \mid i \in I\} \) is a set of \(*\)torsion-free complexes and if \( X = \coprod_{i \in I} X_i \) exists in \( \mathcal{K}(R) \), then \( X \) is also \(*\)torsion-free. The proof is clear from Lemma 3.2(1))

3. Any direct sums of finite number of \(*\)reflexive complexes are \(*\)reflexive.

The following is straightforward from Theorem 2.3.

**Theorem 3.4.** Let \( X \in \mathcal{K}(R) \).

1. \( X \) is \(*\)torsion-free if and only if \( \text{Ext}^1_R(C(X), R) = 0 \).

2. If \( \text{Ext}^1_R(C(X), R) = \text{Ext}^2_R(C(X), R) = 0 \), then \( X \) is \(*\)reflexive.

**Corollary 3.5.** If \( R \) is a Gorenstein ring of dimension zero, then every complex \( X \in \mathcal{K}(R) \) is \(*\)reflexive (and hence \(*\)torsion-free).

**Proof.** In this case, \( \text{Ext}^i_R(\cdot, R) = 0 \) for all \( i > 0 \). \( \square \)

**Example 3.6.** Let \( M \) be a finitely generated \( R \)-module and let

\[
\cdots \to P_2 \to P_1 \to P_0 \to M \to 0
\]

be a projective resolution of \( M \) with \( P_i \in \text{proj}(R) \) for all \( i > 0 \).

1. Setting

\[
X = \left[ \cdots \to P_2 \to P_1 \to P_0 \to 0 \right] \in \mathcal{K}(R),
\]

we can easily see that the following three conditions are equivalent.
(i) \( \Ext^i_R(M, R) = 0 \) for all \( i > 0 \).
(ii) \( X \) is *torsion-free.
(iii) \( X \) is *reflexive.

(2) Let \( n > 0 \) be an integer. Considering the truncation of \( X \), we set

\[
X(n) = \begin{array}{cccc}
0 & \rightarrow & P_n & \rightarrow & \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & 0
\end{array} \in \mathcal{K}(R).
\]

Then \( X(n) \) is *torsion-free if and only if \( \Ext^i_R(M, R) = 0 \) for \( 1 \leq i \leq n \), while \( X(n) \) is *reflexive if and only if \( \Ext^i_R(M, R) = 0 \) for \( 1 \leq i \leq n + 1 \).

**Proposition 3.7.** Let

\[
\begin{array}{c}
X \rightarrow Y \\
\downarrow^a \\
Z \rightarrow X[1]
\end{array}
\]

be a triangle in \( \mathcal{K}(R) \).

(1) Suppose that \( H(b)^* : H(Z)^* \rightarrow H(Y)^* \) is injective. If \( X \) and \( Z \) are *torsion-free, then so is \( Y \).

(2) Suppose that the sequence \( H(Z)^* \rightarrow H(Y)^* \rightarrow H(X)^* \) is exact. If \( Z \) is *reflexive and if \( Y \) is *torsion-free, then \( X \) is *torsion-free.

(3) Suppose that the sequence \( H(Z)^* \rightarrow H(Y)^* \rightarrow H(X)^* \) is exact. And assume that \( X \) and \( Z \) are *reflexive and that \( Y \) is *torsion-free. Then \( Y \) is *reflexive.

**Proof.**

(1) Let \( f : Y \rightarrow R[i] \) be a chain map with \( i \in \mathbb{Z} \). Assume \( H(f) = 0 \). Then \( H(fa) = H(f)H(a) = 0 \). Since \( X \) is *torsion-free, it follows that \( fa = 0 \) in \( \mathcal{K}(R) \). Then there is a morphism \( g : Z \rightarrow R[i] \) such that \( f = gb \). Thus we have \( 0 = H(f) = H(g)H(b) = H(b)^*(H(g)) \) and since \( H(b)^* \) is injective, it follows \( H(g) = 0 \). However, since \( Z \) is *torsion-free, we have \( g = 0 \). Therefore, \( f = gb = 0 \).

(2) Let \( f : X \rightarrow R[i] \) be a chain map for \( i \in \mathbb{Z} \) and we assume that \( H(f) = 0 \). Then, \( H(f \cdot c[-1]) = 0 \), and it follows that \( f \cdot c[-1] = 0 \), since \( Z \) is *torsion-free. Hence there is a morphism \( g : Y \rightarrow R[i] \) with \( f = ga \). Note that there is a commutative diagram of graded \( R \)-modules with an exact row:

\[
\begin{array}{cccc}
H(X) & \xrightarrow{H(a)} & H(Y) & \xrightarrow{H(b)} & H(Z) \\
\downarrow^{H(g)} & & & & \\
H(R[i]) & = & R[i].
\end{array}
\]

Since \( H(a)^*(H(g)) = H(g)H(a) = H(f) = 0 \), it follows from the assumption that \( H(g) \) induces a graded \( R \)-module homomorphism \( \epsilon : H(Z) \rightarrow H(R[i]) \) with \( H(g) = \epsilon H(b) \). Since \( Z \) is *reflexive, there is a chain map \( h : Z \rightarrow R[i] \) such that \( H(h) = \epsilon \). Then, we have \( H(g - hb) = H(g) - H(h)H(b) = 0 \). Since \( Y \) is *torsion-free, it follows that \( g = hb \). Thus \( f = ga = hba = 0 \) as desired.

(3) Let \( \alpha : H(Y) \rightarrow R \) be any element of \( H(Y)^* \). Since \( \alpha H(a) \in H(X)^* \) and since \( X \) is *reflexive, there is a morphism \( f : X \rightarrow R \) such that \( \alpha H(a) = H(f) \). Then we have \( H(f \cdot \alpha) = H(f) \) as desired.
\(c[-1]) = \alpha H(a)H(c[-1]) = 0\). Thus it follows from the *torsion-free property of \(Z\) that \(f \cdot c[-1] = 0\). Then there is a morphism \(g : Y \to R\) with \(f = ga\). Therefore we have \(\alpha H(a) = H(ga) = H(g)H(a)\), or equivalently \((\alpha - H(g))H(a) = 0\). By the exact sequence

\[
\begin{array}{c}
H(Z^*) \longrightarrow H(Y^*) \longrightarrow H(X)^*, \quad \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \approx \nu_{\text{hori}qonta} \\
\end{array}
\]

we find an element \(\beta \in H(Z)^*\) satisfying \(\alpha - H(g) = \beta H(b)\). Since \(Z\) is *reflexive, we have \(\beta = H(h)\) for some morphism \(h : Z \to R\). Thus we have \(\alpha = H(g) + \beta H(b) = H(g + hb)\).

\[\square\]

\section{Complexes over a Generically Gorenstein Ring}

Recall that a finitely generated \(R\)-module \(M\) is called \textit{torsionless} if it satisfies one of the following equivalent conditions: (See also Example 2.4.)

1. \(M\) is a submodule of a free \(R\)-module.
2. The natural mapping \(M \to M^{**}\) is injective.
3. \(\text{Ext}_R^1(\text{Tr}M, R) = 0\).

On the other hand, an \(R\)-module \(M\) is said to be \textit{torsion-free} if the natural mapping \(M \to S^{-1}M\) is injective, where \(S\) is the multiplicatively closed subset \(R \setminus \bigcup_{p \in \text{Ass}(R)} p\) consisting of all non-zero divisors of \(R\). Note that every torsionless module is torsion-free.

Recall that a Noetherian commutative ring \(R\) is said to be \textit{generically Gorenstein} if every localization \(R_p\) for \(p \in \text{Ass}(R)\) is a Gorenstein local ring, or equivalently the total quotient ring of \(R\) is a Gorenstein ring of dimension zero. The following lemma is well known.

**Lemma 4.1.** Let \(R\) be a generically Gorenstein ring. Then a finitely generated \(R\)-module \(M\) is torsionless if and only if \(M\) is torsion-free.

**Proof.** We have only to prove the 'if' part of the lemma. Let \(S = R \setminus \bigcup_{p \in \text{Ass}(R)} p\). There is a commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & \text{Hom}_R(\text{Hom}_R(M, R), R) \\
\downarrow{\beta} & & \downarrow \\
S^{-1}M & \xrightarrow{S^{-1}\alpha} & \text{Hom}_{S^{-1}R}(\text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}R), S^{-1}R),
\end{array}
\]

where the vertical arrows are the mapping induced by the localization at \(S\). If \(M\) is torsion-free, then \(\beta\) is injective. Since \(S^{-1}R\) is a Gorenstein ring of dimension zero, \(S^{-1}\alpha\) is an isomorphism. As a result, it follows that \(\alpha\) is injective, hence \(M\) is torsionless.

\[\square\]

**Theorem 4.2.** Let \(R\) be a generically Gorenstein ring. Then the following two conditions are equivalent for \(X \in \mathcal{K}(R)\).

1. \(X\) is *torsion-free.
2. Each cohomology module \(H^i(X^*)\) is a torsion-free \(R\)-module for \(i \in \mathbb{Z}\).

**Proof.** (1) \(\Rightarrow\) (2): Before the proof we recall that \(N^*\) is torsionless for any finitely generated \(R\)-module \(N\). (If \(R^m \to N\) is a surjective mapping of \(R\)-modules, then we have an injection \(N^*\) to a free module \((R^m)^*\).)
By definition, \( \rho^i_{X,R} : H^{-i}(X^*) \to H^i(X)^* \) is injective. Since \( H^i(X) \) is a finitely generated \( R \)-module, \( H^i(X)^* \) is torsionless. This forces \( H^{-i}(X^*) \) to also be torsionless, and hence torsion-free.

(2) \( \Rightarrow \) (1): Let \( S = R \setminus \bigcup_{\mathfrak{p} \in \text{Ass}(R)} \mathfrak{p} \) as in Lemma 4.1. Now let \( f : X \to R[i] \) be a chain map with \( i \in \mathbb{Z} \) and assume \( H(f) = 0 \). We want to show that \( f = 0 \) as an element of \( H^i(X^*) \).

Note that \( S^{-1}f : S^{-1}X \to S^{-1}R[i] \) is a chain map with \( H(S^{-1}f) = 0 \). Since \( S^{-1}R \) is a Gorenstein ring of dimension zero, we have from Corollary 3.5 that \( S^{-1}X \) is torsion-free as a complex over \( S^{-1}R \), hence \( S^{-1}f = 0 \) in \( K(S^{-1}R) \). This means that \( S^{-1}f = 0 \) as an element of \( H(\text{Hom}_{S^{-1}R}(S^{-1}X, S^{-1}R[i])) \).

This shows that there is an element \( s \in S \) with \( sf = 0 \) as an element of \( H^i(X^*) \). Since we assumed that \( H^i(X^*) \) is torsion-free \( R \)-module, we must have \( f = 0 \) as an element of \( H^i(X^*) \). \( \square \)

Remark 4.3. The implication (1) \( \Rightarrow \) (2) in the theorem is generally true without the assumption of generic Gorensteinness. But it is not the case for (2) \( \Rightarrow \) (1).

For example, let \((R, m, k)\) be a local ring with \( \dim R > 0 \) and \( \text{depth} R = 0 \). Note in this case that every \( k \)-vector space is torsionless, hence torsion-free, as an \( R \)-module, since \( k \) is isomorphic to a submodule in \( R \). Now let \( X \) be an \( R \)-free resolution of \( k \). Then it follows that \( H^i(X^*) \cong \text{Ext}_R^i(k, R) \) is a torsion-free \( R \)-module for each \( i \), hence \( X \) satisfies the condition (2). On the other hand, we note that \( H^i(X^*) \neq 0 \) only if \( i = 0 \). Hence the condition (1) forces \( \text{Ext}_R^i(k, R) = 0 \) for all \( i > 0 \), which is an equivalent condition for \( R \) to be a Gorenstein ring of dimension zero. Therefore, \( X \) does not satisfy the condition (1).

Note that a finitely generated module \( M \) over a commutative Noetherian ring \( R \) is said to be reflexive if the natural mapping \( M \to M^{**} \) is an isomorphism.

Recall that a commutative Noetherian ring \( R \) is said to be Gorenstein in depth one if each \( R_p \) is a Gorenstein ring for all the prime ideals \( p \) satisfying \( \text{depth} R_p \leq 1 \).

First we remark the following (perhaps well-known) lemma.

**Lemma 4.4.** Assume that \( R \) is Gorenstein in depth one.

1. If \( M \) is a finitely generated \( R \)-module, then \( M^* \) is a reflexive \( R \)-module.
2. Let \( M \subseteq N \) be a submodule of a finitely generated \( R \)-module which is equal in depth one, that is, \( M_p = N_p \) if \( \text{depth} R_p \leq 1 \). Furthermore assume that both \( M \) and \( N \) are reflexive. Then \( M = N \).

**Proof.**

(1) Since \( M^* \) is a torsionless module, the natural mapping \( \alpha : M^* \to M^{**} \) is injective. Set \( C \) to be the cokernel of this map, that is, \( C = \text{Coker}(\alpha) \). Then, by the assumption, we have \( C_p = 0 \) if \( \text{depth} R_p \leq 1 \). (Note that \( M^*_p \) are torsion-free, hence MCM’s over Gorenstein rings \( R_p \) for those \( p \), hence \( \alpha_p \) are isomorphisms.) To prove \( C = 0 \), let us assume that \( C \neq 0 \) and take a minimal prime ideal \( p \) in \( \text{Supp}(C) \). Then, by the above, we must have \( \text{depth} R_p \geq 2 \). Note that there is an exact sequence of \( R_p \)-modules \( 0 \to M^*_p \to M^{**}_p \to C_p \to 0 \), where \( C_p \) is a non-zero \( R_p \)-module of finite length. Remark here that both \( M^*_p \) and \( M^{**}_p \) are second syzygy modules over \( R_p \). Since \( \text{depth} R_p \geq 2 \), it follows that such second syzygy modules have depth at least 2.
Noticing that depth $C_p = 0$, we see that this contradicts the depth lemma (see [5, Proposition 1.2.9] or [3, 1.2.6]).

(2) Setting $C = N/M$, we want to show $C = 0$. By the assumption, if depth$R_p \leq 1$, then $C_p = 0$. Thus every prime $p$ in $\text{Supp}(C)$ satisfies depth$R_p \geq 2$. Assuming $C \neq 0$, we take a minimal prime ideal in $\text{Supp}(C)$. Then there is an exact sequence of $R_p$-modules $0 \to M_p \to N_p \to C_p \to 0$, where $C_p$ is of finite length. Since $M_p$ (respectively, $N_p$) is a reflexive $R_p$-module, the depth of $M_p$ (respectively, $N_p$) is at least 2. This contradicts the depth lemma again.

Theorem 4.5. Suppose that $R$ is Gorenstein in depth one and let $X$ be a complex in $\mathcal{X}(R)$. Then the following two conditions are equivalent.

(1) $X$ is *reflexive.

(2) Each cohomology module $H^i(X^*)$ is a reflexive $R$-module for $i \in \mathbb{Z}$.

Proof. (1) $\Rightarrow$ (2): Since $X$ is *reflexive, we have an isomorphism $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ for all $i \in \mathbb{Z}$. Note that each $H^i(X)$ is a finitely generated $R$-module. It thus follows from Lemma 4.4(1) that $H^i(X)^*$, hence $H^{-i}(X^*)$ as well, is a reflexive $R$-module.

(2) $\Rightarrow$ (1): We want to show that the natural mapping $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ is an isomorphism for each $i \in \mathbb{Z}$. We know, from Theorem 4.2, that $X$ is *torsion-free, hence all $\rho^i_{XR}$ are injective. Thus, applying Lemma 4.4(2), we have only to show that $(\rho^i_{XR})_p$ are isomorphisms for prime ideals $p$ with depth$R_p \leq 1$. Therefore, the proof is reduced to the case where the ring $R$ is a Gorenstein local ring of dimension at most one. Henceforth, we assume $R$ is such a ring. In this case, we have Ext$^2_R(C(X), R) = 0$, thus it results from Theorem 2.3 that $\rho^i_{XR} : H^{-i}(X^*) \to H^i(X)^*$ is surjective for each $i \in \mathbb{Z}$. Since we know already that this is injective, each $\rho^i_{XR}$ is an isomorphism.

5 | SPLIT COMPLEXES AND $\text{Add}(R)$

We note that $\mathcal{X}(R)$ admits finite direct sums, and moreover some kind of infinite direct sums can be possibly taken inside $\mathcal{X}(R)$. For example, let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathcal{X}(R)$ and assume that $X^i = \bigoplus_{j \in J} X^i_j$ is a finitely generated $R$-module for each $i \in \mathbb{Z}$. In such a case, the direct sum $X = \bigsqcup_{j \in J} X_j$ (or the coproduct in $\mathcal{X}(R)$) is well defined so that its $i$th component is $X^i$. Note in this case that the direct sum coincides with the direct product $\prod_{j \in J} X_j$, as we see in the next lemma.

The direct sum $\bigsqcup_{i \in \mathbb{Z}} R[i]$ is one of such typical examples of infinite direct sums, actually it is a complex of the form $[\cdots \to R \to R \to R \to \cdots]$ that belongs to $\mathcal{X}(R)$.

Lemma 5.1. Let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathcal{X}(R)$. Assume that, for each $i \in \mathbb{Z}$, there is a finite subset $I_i \subseteq J$ such that $X^i_j \neq 0$ only if $j \in I_i$. Then the coproduct $X = \bigsqcup_{j \in J} X_j$ exists in $\mathcal{X}(R)$. Moreover, in this case, the coproduct is a product in $\mathcal{X}(R)$, that is, $X = \prod_{j \in J} X_j$. Hence there is an isomorphism of $R$-modules

$$\text{Hom}_{\mathcal{X}(R)}(Y, \bigsqcup_{j \in J} X_j) \cong \prod_{j \in J} \text{Hom}_{\mathcal{X}(R)}(Y, X_j)$$

for all $Y \in \mathcal{X}(R)$. 
Proof. Let \( \text{Mod}(R) \) be the abelian category consisting of all (not necessarily finitely generated) \( R \)-modules and we denote by \( K(\text{Mod}(R)) \) the homotopy category of all complexes over \( \text{Mod}(R) \). Now regarding \( \{ X_j \mid j \in J \} \) as an object set in \( K(\text{Mod}(R)) \), we see that the coproduct \( X \) in \( K(\text{Mod}(R)) \) is given as \( X^i = \bigoplus_{j \in J} X_j^i \) with differentials defined diagonally by each \( d_i^j \). Similarly the product in \( K(\text{Mod}(R)) \) is given as \( \prod_{j \in J} X_j^i \). Now the assumption of the lemma assures that each \( X^i \) is finitely generated, hence the coproduct \( X \) in \( K(\text{Mod}(R)) \) lies in its full subcategory \( \mathcal{K}(R) \). This shows that \( X \) is in fact a coproduct in the category \( \mathcal{K}(R) \).

Moreover, under the assumption in the lemma, we have the equality \( \bigoplus_{j \in J} X_j^i = \prod_{j \in J} X_j^i \) as \( R \)-modules for all \( i \in \mathbb{Z} \). Hence the last half of the lemma follows.

Definition 5.2. Given an \( X \in \mathcal{K}(R) \), we define \( \text{Add}(X) \) as the smallest additive subcategory of \( \mathcal{K}(R) \) containing \( X \) that is closed under the shift functor and direct summands, and admits possibly infinite coproducts. Equivalently, \( \text{Add}(X) \) is the intersection of all the full subcategories \( U^\prime \) satisfying the following conditions.

(i) \( U^\prime \) is closed under isomorphism and \( X \in U^\prime \).
(ii) If \( Y \in U^\prime \), then \( Y[i] \in U^\prime \) for all \( i \in \mathbb{Z} \).
(iii) If \( Z \) is a direct summand of \( Y \in U^\prime \), then \( Z \in U^\prime \).
(iv) Let \( \{ Y_j \mid j \in J \} \) be a set of objects in \( U^\prime \) and assume that the coproduct \( \coprod_{j \in J} Y_j \) in \( \mathcal{K}(R) \) exists. Then \( \coprod_{j \in J} Y_j \in U^\prime \).

(Note that 0 is an object of \( U^\prime \) by (iii) and that all null complexes belong to \( U^\prime \) by (i).)

In the rest of the paper, we are particularly interested in \( \text{Add}(R) \), where \( R \) is regarded as a complex concentrated in degree 0.

If the complex

\[
X = \begin{array}{cccc}
\ldots & - & d_x^2 & - \\
& X & d_x^1 & X^0 \\
& d_x^0 & X^1 & d_x^1 & \ldots \\
& & &
\end{array}
\]

satisfies the equalities \( d_i^j = 0 \) for all \( i \in \mathbb{Z} \), then \( X \) belongs to \( \text{Add}(R) \), since \( X \) is a direct sum \( \coprod_{i \in \mathbb{Z}} X^i[i] \) with each \( X^i \) being a projective \( R \)-module. Such a complex \( X \) is characterized by the condition that \( X \cong H(X) \) in \( \mathcal{C}(R) \), where we regard the graded \( R \)-module \( H(X) \) as a complex with zero differentials.

Recall that a complex \( X \in \mathcal{C}(R) \) is called split if there is a graded \( R \)-module homomorphism \( s : X \to X[-1] \) satisfying \( d_x s d_x = d_x \). (Cf. [18, Definition (1.4.1)].) To state the following well-known lemma, we recall the notation \( C(X) = \text{Coker}(d_X) \) and \( B(X) = \text{Im}(d_X) \), for a complex \( X \in \mathcal{C}(R) \), as in Notation 2.1.

Lemma 5.3. The following conditions are equivalent for \( X \in \mathcal{C}(R) \).

(1) \( X \) is split.
(2) There is a direct sum decomposition \( X = X^i \oplus N \) in \( \mathcal{C}(R) \) where \( d_{X^i} = 0 \) and \( N \) is a null complex.
(3) \( C(X) = \bigoplus_{i \in \mathbb{Z}} C^i(X) \) is a projective \( R \)-module as underlying \( R \)-module.
(4) The natural inclusion map \( B(X) = \bigoplus_{i \in \mathbb{Z}} B^i(X) \hookrightarrow X = \bigoplus_{i \in \mathbb{Z}} X^i \) is a split monomorphism as graded \( R \)-modules.

Proof. The implications (2) \( \Rightarrow \) (1) \( \Rightarrow \) (4) \( \Rightarrow \) (3) are well known and easily proved. We have only to show (3) \( \Rightarrow \) (2).
If $C(X)$ is projective, then the natural exact sequences of graded $R$-modules
\[
0 \to B(X) \to X \to C(X) \to 0, \quad 0 \to H(X) \to C(X) \to B(X)[1] \to 0
\]
are splitting. Therefore, each $X^i$ decomposes to $X^i_0 \oplus X^i_1 \oplus X^i_2$ where $X^i_0 \cong H^i(X)$ and $X^i_1 \cong B^i(X)$, $X^i_2 \cong B^{i+1}(X)$ for $i \in \mathbb{Z}$, and the differential map $d^i_X$ yields an isomorphism $X^i_2 \to X^{i+1}_1$, while it is zero on $X^i_0 \oplus X^i_1$. Thus, part $X_1 \oplus X_2$ of $X$ defines a null subcomplex $N$. Therefore, setting $X' = X_0$ with zero differentials, we have a direct sum decomposition $X = X' \oplus N$. □

As a result of the equivalence $(1) \Leftrightarrow (2)$ in the lemma, we see that all the split complexes in $\mathcal{C}(R)$ belong to $\text{Add}(R)$. We can show that the uniqueness of the direct sum decomposition in the meaning of $(2)$ in the lemma holds for a split complex.

**Lemma 5.4.** Let $X$ be a split complex belonging to $\mathcal{C}(R)$. Assume there are decompositions $X = X_1 \oplus N_1 = X_2 \oplus N_2$ where $d_{X_i} = 0$ and $N_i$ is a null complex for $i = 1, 2$. Then we have isomorphisms $X_1 \cong X_2$, $N_1 \cong N_2$ in $\mathcal{C}(R)$.

**Proof.** Write the natural injection $X_1 \hookrightarrow X = X_2 \oplus N_2$ as \( \begin{pmatrix} a & b \end{pmatrix} \) where $a : X_1 \to X_2$ and $b : X_1 \to N_2$. Similarly, write the natural projection $X_2 \oplus N_2 = X \twoheadrightarrow X_1$ as $(c, d)$ with $c : X_2 \to X_1$ and $d : N_2 \to X_1$. Then we have $1_{X_1} = ca + db$. Since the morphism $db$ factors through a null complex, it is null homotopic. Hence it follows from the next remark that $db = 0$ as a morphism in $\mathcal{C}(R)$. Thus $ca = 1_{X_1}$. In the same way as this, one can show $ac = 1_{X_2}$. Hence $a : X_1 \to X_2$ is an isomorphism in $\mathcal{C}(R)$.

To show $N_1 \cong N_2$ in $\mathcal{C}(R)$ we remark that, for a null complex $N$, we have $Z(N) \cong N/Z(N)[-1]$ as graded $R$-modules and $N$ is isomorphic to the mapping cone of the identity mapping on $N/Z(N)$. Since $Z(X_i) = X_i$ for $i = 1, 2$, we have $X/Z(X) = (X_1 \oplus N_1)/(Z(X_1) \oplus Z(N_1)) \cong N_1/Z(N_1)$, therefore $N_1/Z(N_1) \cong N_2/Z(N_2)$ as graded $R$-modules. Since both $N_1$ and $N_2$ are null complexes, we have an isomorphism $N_1 \cong N_2$ in $\mathcal{C}(R)$, as remarked above. □

**Remark 5.5.** Let $f : X \to Y$ be a morphism in $\mathcal{C}(R)$, where we assume that $d_X = d_Y = 0$. If $f$ is null homotopic, then $f = 0$ in $\mathcal{C}(R)$. In fact, this follows from that $f = d_Y h - hd_X = 0$ for a homotopy $h$.

By a similar proof to the lemma above, we can also show the following lemma.

**Lemma 5.6.** Let $X$ and $Y$ be complexes in $\mathcal{C}(R)$ such that $d_X = 0$. If $X$ is a direct summand of $Y$ in $\mathcal{C}(R)$, then it is also a direct summand of $Y$ in $\mathcal{C}(R)$.

**Proof.** Assume there are morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}(R)$ such that $gf$ is chain homotopic to the identity morphism $1_X$ on $X$. Then it follows from the remark above that $1_X - gf = 0$ as a morphism in $\mathcal{C}(R)$. □

**Proposition 5.7.** Let \( \{X_j \mid j \in J \} \) be a set of complexes in $\mathcal{C}(R)$ such that $d_{X_j} = 0$ for all $j \in J$. Assume that the coproduct $\coprod_{j \in J} X_j$ in $\mathcal{C}(R)$ exists. Then, for any $i \in \mathbb{Z}$, there is a finite subset $J_i \subseteq J$ such that $X^i_j \neq 0$ only if $j \in J_i$. In this case, the coproduct is an ordinary direct sum of complexes. Hence $\coprod_{j \in J} X_j$ has zero differentials, and it is a split complex as well.
Proof. Set $P = \bigsqcup_{j \in J} X_j$. By definition, we have an isomorphism

$$\text{Hom}_\mathcal{K}(R)(P, -) \cong \prod_{j \in J} \text{Hom}_\mathcal{K}(R)(X_j, -) \cong \text{Hom}_{\text{Mod}(R)}(\bigoplus_{j \in J} X_j, -) \big|_{\mathcal{K}(R)}$$

as functors on $\mathcal{K}(R)$, where $\bigoplus_{j \in J} X_j$ denotes the coproduct in $K(\text{Mod}(R))$. Therefore, there is a morphism $\bigoplus_{j \in J} X_j \to P$ in $K(\text{Mod}(R))$, by which any finite direct sum $\bigoplus_{k=1}^r X_{j_k}$ is a direct summand of $P$ in the category $\mathcal{K}(R)$. In particular, for each $i \in \mathbb{Z}$, any finite direct sum $\bigoplus_{k=1}^r X_{j_k}^i$ of $R$-modules is a direct summand of $P^i$.

Under such a circumstance, for any $i \in \mathbb{Z}$, we claim that $X_{j_k}^i = 0$ for almost all $j \in J$ (that is, for all $j \in J$ except a finite number of them).

In fact, if $R$ is an integral domain, then this is true by rank argument, since $P^i$ is finitely generated and hence it has a finite rank. For general cases, set $\text{Min}(R) = \{p_1, \ldots, p_\ell\}$ and it follows from the domain case that $X_{j_k}^i \otimes_R R/p_i = 0$ for almost all $j \in J$. Then we have

$$X_{j_k}^i \subseteq p_1 X_{j_k}^i \subseteq p_1^2 X_{j_k}^i \subseteq \cdots \subseteq p_1^N X_{j_k}^i \subseteq p_1^N p_2 X_{j_k}^i \subseteq \cdots \subseteq (p_1 p_2 \cdots p_\ell)^N X_{j_k}^i$$

for almost all $j \in J$ and any $N > 0$. Taking $N$ enough large so that $(p_1 p_2 \cdots p_\ell)^N = 0$, we have $X_{j_k}^i = 0$ for all such $j \in J$.

Now we are able to state a main result of this section.

**Theorem 5.8.** The following conditions are equivalent for $X \in \mathcal{K}(R)$.

1. $X$ belongs to $\text{Add}(R)$.
2. $X$ is a split complex.
3. The natural mapping

$$H : \text{Hom}_\mathcal{K}(R)(X, Y) \longrightarrow \text{Hom}_{\text{graded}-\text{mod}}(H(X), H(Y))$$

which sends $f$ to $H(f)$ is injective for all $Y \in \mathcal{K}(R)$.
4. The natural mapping $H$ in the condition (3) is bijective for all $Y \in \mathcal{K}(R)$.

**Proof.** We have shown the implication $(2) \Rightarrow (1)$ in Lemma 5.3.

$(1) \Rightarrow (2)$: Let $\mathcal{U}'$ be the subcategory of $\mathcal{K}(R)$ consisting of all split complexes. Note that $R \in \mathcal{U}'$ and that $\mathcal{U}'$ is closed under shift functor, and taking direct summands. If we prove that $\mathcal{U}'$ is closed under taking coproducts in $\mathcal{K}(R)$, then $\text{Add}(R) \subseteq \mathcal{U}'$ by Definition 5.2 and the proof will be finished.

Let $\{X_j \mid j \in J\}$ be a set of complexes in $\mathcal{U}'$. By Lemma 5.4, each $X_j$ is uniquely decomposed into $X_j' \oplus N_j$ with $d_{X_j}' = 0$ and a null complex $N_j$. Since $X_j \cong X_j'$ in $\mathcal{K}(R)$, replacing $X_j$ with $X_j'$ we may assume $d_{X_j'} = 0$ for all $j \in J$. If the coproduct $\bigsqcup_{j \in J} X_j$ exists in $\mathcal{K}(R)$, then it follows from the previous proposition it is split again hence belongs to $\mathcal{U}'$.

$(2) \Rightarrow (4)$: As in the proof above, we may assume that $d_X = 0$, hence $X = H(X)$.

Let $f : X \to Y$ be a morphism in $\mathcal{K}(R)$ and assume that $H(f) = 0$. Then the image of $f^i$ is contained in the coboundary $B^i(Y)$ for $i \in \mathbb{Z}$. Since $X^i$ is a projective module, there is an $h^i : X^i \to Y^{i-1}$ with $f^i = d^{i-1} \cdot h^i$. Thus $\{h^i \mid i \in \mathbb{Z}\}$ gives a homotopy, and we have $f = 0$ as a morphism in $\mathcal{K}(R)$. 

\[\square\]
To show the surjectivity of $H$, let $a : H(X) \to H(Y)$ be a graded $R$-module homomorphism. Then each $a^i : H^i(X) = X^i \to H^i(Y)$ is lifted to an $R$-module mapping from $X^i$ to the cocycle module $Z^i(Y)$. These lifted maps define a chain map $f : X \to Y$ with $H(f) = a$.

(4) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (2): Let $M$ be a finitely generated $R$-module and let $P \in \mathcal{X}(R)$ be a projective resolution of $M$. Then note that the equality $\text{Hom}_{\mathcal{X}(R)}(X[i], P) = H^{-i}(\text{Hom}_R(X, M))$ holds. To show this, we introduce the notion of silly truncation

$$\sigma_{\leq 1}(X[i]) := [ \cdots \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^i \longrightarrow X^{i+1} \longrightarrow 0 ]$$

where $X^{i+1}$ sits in the $(+1)$st position. Note that this is $K$-projective. Then, since $P_i = 0$ for $i > 0$, we have

$$\text{Hom}_{\mathcal{X}(R)}(X[i], P) = \text{Hom}_{\mathcal{X}(R)}(\sigma_{\leq 1}(X[i]), P) = H^0(\text{Hom}_R(\sigma_{\leq 1}(X[i]), M)) = H^0(\text{Hom}_R(X[i], M)).$$

Therefore the mapping defined by taking cohomology modules $H : \text{Hom}_{\mathcal{X}(R)}(X[i], P) \to \text{Hom}_{\text{graded } R\text{-mod}}(H(X[i]), H(P)) = \text{Hom}_R(H^i(X), M)$ is just the same as $\rho^i_{XM}$ defined in Definition 2.2. Thus the condition (3) implies that $\rho^i_{XM}$ is injective for all $i \in \mathbb{Z}$ and for all $M \in \text{mod}(R)$. It then follows from Theorem 2.3 that $\text{Ext}^1_R(C(X), M) = 0$ for any $M \in \text{mod}(R)$, and therefore $C(X)$ is a projective $R$-module. Thus $X$ is split by Lemma 5.3. □

As a result of Theorem 5.8 and Lemma 5.3, we have the following corollary.

**Corollary 5.9.** Every complex $F$ in $\text{Add}(R)$ is decomposed as $F \cong \bigsqcup_{j \in \mathbb{Z}} H^j(F)[−j]$ where $H^j(F) \in \text{proj}(R)$ for all $j \in \mathbb{Z}$. (Note that $F^* = \bigsqcup_{j \in \mathbb{Z}} H^j(F)^*[j]$ in this case.) Moreover, every complex in $\text{Add}(R)$ is *reflexive.

**Proposition 5.10.** Let $X, F \in \mathcal{X}(R)$. Assume that $F$ belongs to $\text{Add}(R)$ and that $X$ is *torsion-free (respectively, *reflexive). Then the mapping

$$H : \text{Hom}_{\mathcal{X}(R)}(X, F) \longrightarrow \text{Hom}_{\text{graded } R\text{-mod}}(H(X), H(F)) ; f \mapsto H(f)$$

is injective (respectively, bijective).

**Proof.** We may take $F$ as it satisfies $d_F = 0$, hence $F = H(F)$. Then, as remarked above, $F = \bigsqcup_{j \in \mathbb{Z}} F^j[−j]$ with $F^j \in \text{proj}(R)$ and this coproduct is also a product. Therefore,

$$\text{Hom}_{\mathcal{X}(R)}(X, F) = \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{X}(R)}(X, F^j[−j]), \quad \text{and} \quad \text{Hom}_{\text{graded } R\text{-mod}}(H(X), H(F)) = \prod_{j \in \mathbb{Z}} (H^j(X), F^j[−j]).$$

According to $X$ is *torsion-free or *reflexive, we have that $H : \text{Hom}_{\mathcal{X}(R)}(X, F^j[−j]) \longrightarrow \text{Hom}_{\text{graded } R\text{-mod}}(H(X), F^j[−j])$ is injective or bijective for each $i, j \in \mathbb{Z}$. The proposition follows from this observation. □

The following theorem is one of the crucial results on *torsion-free complexes, on which the proof of the main Theorem 1.1 will deeply rely. See Sections 10 and 12.
Theorem 5.11. Assume that $X \in \mathcal{K}(R)$ is *torsion-free and that $F \in \text{Add}(R)$. Let $f \in \text{Hom}_{\mathcal{K}(R)}(X,F)$. Setting $S = R \setminus \bigcup_{p \in \text{Ass}(R)} p$, if $S^{-1}f = 0$ as a morphism $S^{-1}X \to S^{-1}F$ in $\mathcal{K}(S^{-1}R)$, then we have that $f = 0$ as a morphism in $\mathcal{K}(R)$.

Proof. If $S^{-1}f = 0$, then $H(S^{-1}f) = 0$ as an $S^{-1}R$-module homomorphism $H(S^{-1}X) \to H(S^{-1}F)$. Thus we see that $S^{-1}H(f) = 0$ as a mapping $S^{-1}H(X) \to S^{-1}H(F)$. Since $H(F)$ is a projective $R$-module, any elements of $S$ act on $H(F)$ as non-zero divisors. It thus follows that $H(f) = 0$ as a mapping $H(X) \to H(F)$. Then from Proposition 5.10, we have $f = 0$.

Corollary 5.12. If $X \in \mathcal{K}(R)$ is *torsion-free and $F \in \text{Add}(R)$, then $\text{Hom}_{\mathcal{K}(R)}(X,F)$ is a torsion-free $R$-module.

Proof. There is a commutative diagram of $R$-modules

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{K}(R)}(X,F) & \xrightarrow{\alpha} & S^{-1}\text{Hom}_{\mathcal{K}(R)}(X,F) \\
\downarrow{\gamma} & & \downarrow{\beta} \\
\text{Hom}_{\mathcal{K}(S^{-1}R)}(S^{-1}X,S^{-1}F) & & \\
\end{array}
$$

where $\alpha$ is a localization mapping by $S$ and $\gamma$ is a natural mapping that sends $f$ to $S^{-1}f$. Note that $\beta(f/s) = \gamma(f)/s$ for $f \in \text{Hom}_{\mathcal{K}(R)}(X,F)$ and $s \in S$. We have shown in Theorem 5.11 that $\gamma$ is injective. Thus $\alpha$ is also injective, and hence $\text{Hom}_{\mathcal{K}(R)}(X,F)$ is a torsion-free $R$-module.

Remark 5.13. In the proof of the corollary, we should note that the natural mapping

$$
\beta : S^{-1}\text{Hom}_{\mathcal{K}(R)}(X,F) \longrightarrow \text{Hom}_{\mathcal{K}(S^{-1}R)}(S^{-1}X,S^{-1}F)
$$

is not necessarily an isomorphism. For example, setting $X = F = \prod_{i \in \mathbb{Z}} R[-i]$, we have $\text{Hom}_{\mathcal{K}(R)}(X,F) = \prod_{i \in \mathbb{Z}} R$ and $\text{Hom}_{\mathcal{K}(S^{-1}R)}(S^{-1}X,S^{-1}F) = \prod_{i \in \mathbb{Z}} S^{-1}R$.

6 THE STABLE CATEGORY OF $\mathcal{K}(R)$

The main objective of this paper is to consider the nature of complexes in $\mathcal{K}(R)$ up to $\text{Add}(R)$-summands, which we call the stable theory after the paper [2].

Definition 6.1. We denote by $\overline{\mathcal{K}(R)}$ the factor category $\mathcal{K}(R)$ modulo the subcategory $\text{Add}(R)$:

$$\overline{\mathcal{K}(R)} = \mathcal{K}(R) / \text{Add}(R)$$

We call $\overline{\mathcal{K}(R)}$ the stable category of $\mathcal{K}(R)$.

The objects of $\overline{\mathcal{K}(R)}$ are the same as $\mathcal{K}(R)$, while the morphism set is given by

$$\text{Hom}_{\overline{\mathcal{K}(R)}}(X,Y) = \text{Hom}_{\mathcal{K}(R)}(X,Y) / \text{Add}(R)(X,Y),$$

for $X,Y \in \mathcal{K}(R)$, where $\text{Add}(R)(X,Y)$ is the $R$-submodule of $\text{Hom}_{\mathcal{K}(R)}(X,Y)$ consisting of all morphisms factoring through objects of $\text{Add}(R)$. The object sets of $\mathcal{K}(R)$ and $\overline{\mathcal{K}(R)}$ are identical, but for an object $X \in \mathcal{K}(R)$, to discriminate it with an object in $\overline{\mathcal{K}(R)}$, we often write $\overline{X}$ for the
corresponding object in $\mathcal{X}(R)$. Similarly, we denote by $f$ the corresponding morphism in $\mathcal{X}(R)$ for a given $f$ in $\mathcal{X}(R)$.

Since $\text{Add}(R)$ is stable under the action of shift functor in $\mathcal{X}(R)$, it should be noted that $\mathcal{X}(R)$ admits the shift functor so that $X[1] = X[1]$ for $X \in \mathcal{X}(R)$. However, $\mathcal{X}(R)$ is not a triangulated category, but merely an additive $R$-linear category with the shift functor that is an auto-functor on it. ($\mathcal{X}(R)$ is not triangulated, by which we mean that there is no triangle structure on $\mathcal{X}(R)$ that makes the natural functor $\mathcal{X}(R) \to \mathcal{X}(R)$ a triangle functor. This is true, since $\text{Add}(R)$ is not closed under triangles in $\mathcal{X}(R)$.)

First of all, we remark on the commutativity of a diagram in $\mathcal{X}(R)$.

**Lemma 6.2.** Let $f : X \to Z$, $g : X \to Y$, $h : Y \to Z$. Then $f = hg$ in $\mathcal{X}(R)$ if and only if there is a commutative diagram in $\mathcal{X}(R)$ of the following form:

![Diagram]

where $F \in \text{Add}(R)$.

**Proof.** If $f - hg$ factors through $F \in \text{Add}(R)$, then there are $a : F \to Z$ and $b : X \to F$ that satisfy the equality $f = hg + ab$. The converse is trivial since $g = b = 0$. □

Note from this lemma that $X = 0$ for $X \in \mathcal{X}(R)$ if and only if $X \in \text{Add}(R)$. In fact, if $1_X = 0$, then setting $X = Z$, $Y = 0$ and $f = 1_X$ in the lemma, we see that $X$ is a direct summand of $F \in \text{Add}(R)$ and hence $X \in \text{Add}(R)$. More generally, we should note the following corollary holds.

**Corollary 6.3.** Let $X, Y \in \mathcal{X}(R)$. Then $X \cong Y$ in $\mathcal{X}(R)$ if and only if $X \oplus F \cong Y \oplus F'$ in $\mathcal{X}(R)$ for some $F, F' \in \text{Add}(R)$.

**Proof.** If $g : X \to Y$ is an isomorphism whose inverse morphism is $h$, then it follows from Lemma 6.2 that $X$ is a direct summand of $Y \oplus F$ in $\mathcal{X}(R)$ for some $F \in \text{Add}(R)$. Therefore, there exists an isomorphism $Y \oplus F \to X \oplus F'$ in which the restricted map $Y \to X$ is given by $h$. We have to show that $F' \in \text{Add}(R)$. Since $F = 0$, we have an isomorphism $Y \to X \oplus F'$ in which $h : Y \to X$ is also an isomorphism. Taking the inverse of the isomorphism

$$\left( \begin{array}{c} h \\ a \end{array} \right) : Y \to X \oplus F',$$

we have

$$\left( \begin{array}{c} b \\ c \end{array} \right) : X \oplus F' \to Y$$

such that

$$\left( \begin{array}{cc} hb & hc \\ ab & ac \end{array} \right) = \left( \begin{array}{cc} 1_X & 0 \\ 0 & 1_{F'} \end{array} \right).$$
Since \( hb = 1_X \) where \( h \) is an isomorphism, \( b \) is also an isomorphism. Then, since \( ab = 0 \), we have \( a = 0 \). Therefore, \( 1_{F'} = ac = 0 \), and thus \( F' \cong 0 \), hence \( F' \in \text{Add}(R) \).

Remark 6.4. Recall from Theorem 5.8 that \( X \in \mathcal{K}(R) \) belongs to \( \text{Add}(R) \) if and only if \( X \) is a split complex. Hence, setting \( S \) to be the full subcategory of \( \mathcal{C}(R) \) that consists of all split complexes, we can also describe the stable category as \( \mathcal{K}(R) = \mathcal{C}(R)/S \). Therefore, one can also prove that \( X \cong Y \) in \( \mathcal{K}(R) \) if and only if \( X \oplus T \cong Y \oplus T' \) in \( \mathcal{C}(R) \) for some \( T, T' \in S \).

Definition 6.5. Let \( f : X \to Y \) be a morphism in \( \mathcal{K}(R) \). We say that \( f \) is cohomologically surjective if the cohomology mapping \( H(f) : H(X) \to H(Y) \) is surjective.

We also define the complex \( \text{Cone}(f) \in \mathcal{K}(R) \) by the triangle

\[
\begin{array}{ccc}
\text{Cone}(f)[-1] & \rightarrow & X \\
& & \downarrow f \\
& & Y \\
\end{array}
\]

in \( \mathcal{K}(R) \), which is actually the mapping cone of the chain map \( f \).

In general, for given morphisms \( f, g : X \to Y \) in \( \mathcal{K}(R) \), that \( f = g \) in \( \mathcal{K}(R) \) does not mean \( \text{Cone}(f) \cong \text{Cone}(g) \) in \( \mathcal{K}(R) \).

For an example of this, let \( R \) be a local ring and \( x \in R \) a non-zero divisor. Set \( X = Y = R \) and consider the morphisms \( f, g \in \text{Hom}_{\mathcal{K}(R)}(X, Y) \) defined by \( f(a) = xa, g(a) = 0 \) for \( a \in X = R \). Since \( R \) is a split complex as an object of \( \mathcal{K}(R) \), we have \( f = g = 0 \). In this case, \( \text{Cone}(f) = [0 \to R \to R \to 0] \) is the Koszul complex and \( \text{Cone}(f) \not\cong 0 \), while \( \text{Cone}(g) = [0 \to R \to R \to 0] \) is a split complex hence \( \text{Cone}(g) \cong 0 \). Note in this example that \( f \) and \( g \) are not cohomologically surjective.

Theorem 6.6. Let \( f : X \to Y \) and \( f' : X' \to Y \) be morphisms in \( \mathcal{K}(R) \). Assume that both \( f \) and \( f' \) are cohomologically surjective. Further assume that \( X \cong X' \) in \( \mathcal{K}(R) \) and that \( f \) corresponds to \( f' \) under the isomorphism \( \text{Hom}_{\mathcal{K}(R)}(X, Y) \cong \text{Hom}_{\mathcal{K}(R)}(X', Y) \). Then we have an isomorphism \( \text{Cone}(f) \cong \text{Cone}(f') \) in \( \mathcal{K}(R) \).

Proof. As the first step of the proof, we prove the following isomorphism:

\[
\text{Cone}(f) \cong \text{Cone}(fa) \quad \text{for any } F \in \text{Add}(R) \text{ and } (fa) : X \oplus F \to Y. \quad (6.1)
\]

In fact, there is a commutative diagram in \( \mathcal{K}(R) \) whose rows and columns are triangles:

\[
\begin{array}{ccccccc}
F[-1] & \rightarrow & Y[1] & \rightarrow & \text{Cone}(f)[-1] & \rightarrow & X \\
\downarrow \mu & & \downarrow \nu & & \downarrow \phi & & \downarrow (f) \\
F[-1] & \rightarrow & \text{Cone}(fa)[-1] & \rightarrow & X \oplus F & \rightarrow & Y \\
\downarrow \psi & & \downarrow (0, 1) & & \downarrow (0, 1) & & \downarrow 0 \\
F & \rightarrow & F.
\end{array}
\]
Since $H(f)$ is surjective, note in this diagram that $H(v)$ is injective. Then that $vu = 0$, and hence $H(v)H(u) = 0$, forces $H(u) = 0$. Thus by Theorem 5.8, we have $u = 0$, which shows an isomorphism $\text{Cone}(fa)[-1] \cong \text{Cone}(f)[-1] \oplus F$, and hence (6.1) is proved.

As the second step of the proof, we prove the theorem in the case of $X = X'$. In this case, we have $f' = f + ab$ for $a : F \to Y$ and $b : X \to F$ with $F \in \text{Add}(R)$, by virtue of Lemma 6.2. Then there is a commutative diagram in $\mathcal{K}(R)$

\[
\begin{array}{ccc}
X \oplus F & \xrightarrow{(f\ a)} & Y \\
\downarrow & & \downarrow \\
X \oplus F & \xrightarrow{(f\ a)} & Y.
\end{array}
\]

Since the left vertical arrow is an isomorphism, we have $\text{Cone}(fa) \cong \text{Cone}(f' a)$ in $\mathcal{K}(R)$, hence $\text{Cone}(f) \cong \text{Cone}(f')$ by using (6.1).

Now consider the general case of the theorem. Since $X \cong X'$, there is an isomorphism $g : X \oplus F \to X' \oplus F'$ for some $F, F' \in \text{Add}(R)$, and by the assumption we must have $f = f' \cdot g$. Consider the morphisms $(f\ 0) : X \oplus F \to Y$ and $(f'\ 0) : X' \oplus F' \to Y$, and we note that they are cohomologically surjective. On the other hand, since $F = F' = 0$, we have equalities

\[ (f\ 0) = f = f' \cdot g = (f'\ 0) \cdot g. \]

Thus, it follows from the second step of this proof that $\text{Cone}(f\ 0) \cong \text{Cone}(f'\ 0) \cdot g$. Note here that $\text{Cone}((f'\ 0) \cdot g) \cong \text{Cone}(f'\ 0)$ in $\mathcal{K}(R)$, since $g$ is an isomorphism in $\mathcal{K}(R)$. Hence the isomorphism $\text{Cone}(f) \cong \text{Cone}(f')$ follows from (6.1).

\[ \square \]

### 7 Add(R)-APPROXIMATIONS

We are able to show that the subcategory $\text{Add}(R)$ of $\mathcal{K}(R)$ is functorially finite in the sense of Auslander. (Cf. Auslander [1].) For this, we begin with recalling the definition of right approximations.

**Definition 7.1.** Let $X \in \mathcal{K}(R)$. A morphism $p : F \to X$ in $\mathcal{K}(R)$ is called a right $\text{Add}(R)$-approximation of $X$ if $F \in \text{Add}(R)$ and $\text{Hom}_{\mathcal{K}(R)}(G, p) : \text{Hom}_{\mathcal{K}(R)}(G, F) \to \text{Hom}_{\mathcal{K}(R)}(G, X)$ is surjective for any $G \in \text{Add}(R)$.

We should remark that the shift functor preserves the right $\text{Add}(R)$-approximation property, that is, $p : F \to X$ is a right $\text{Add}(R)$-approximation if and only if so is $p[n] : F[n] \to X[n]$ for any $n \in \mathbb{Z}$.

**Lemma 7.2.** Let $X \in \mathcal{K}(R)$ and $F \in \text{Add}(R)$. Then a morphism $p : F \to X$ in $\mathcal{K}(R)$ is a right $\text{Add}(R)$-approximation if and only if $p$ is cohomologically surjective. Moreover, there always exists a right $\text{Add}(R)$-approximation of $X$ for any $X \in \mathcal{K}(R)$.

**Proof.** If $p : F \to X$ is a right $\text{Add}(R)$-approximation, then $H^i(p) = \text{Hom}_{\mathcal{K}(R)}(R[-i], p)$ is surjective, since $R[-i] \in \text{Add}(R)$ for $i \in \mathbb{Z}$. 

\[ \square \]
Conversely assume that $H(p)$ is surjective, and let $g : G \to X$ be a morphism in $\mathcal{K}(R)$ with $G \in \text{Add}(R)$. Then $H(g) : H(G) \to H(X)$ factors through $H(p)$, since $H(G)$ is a graded projective $R$-module:

$$
\begin{array}{c}
H(G) \\
\downarrow \alpha \\
H(F) \\
\downarrow H(p) \\
H(X)
\end{array}
\xrightarrow{H(g)}
\begin{array}{c}
H(G) \\
\downarrow H(p) \\
H(X)
\end{array}
$$

Then, by Theorem 5.8, there is a morphism $\alpha : G \to F$ such that $H(\alpha) = \alpha$, and since $H(g) = H(p\alpha)$, we have $g = p\alpha$.

For the existence of right $\text{Add}(R)$-approximation of $X$, one has only to take a graded projective $R$-module $F$ which maps surjectively onto $H(X)$. Then it follows from Theorem 5.8 that this mapping is lifted to a chain homomorphism $F \to X$ which is in fact a right $\text{Add}(R)$-approximation of $X$.

If $p : F \to X$ is a right $\text{Add}(R)$-approximation, then as we have shown in Theorem 6.6, the mapping cone $\text{Cone}(p)$ is uniquely determined as an object of $\mathcal{K}(R)$.

**Definition 7.3.** Let $X \in \mathcal{K}(R)$ and $p : F \to X$ be a right $\text{Add}(R)$-approximation of $X$. We define $\Omega(X)$ (or simply denoted $\Omega X$) by the equality

$$
\Omega(X) = \text{Cone}(p)[-1],
$$

which is uniquely determined in the stable category $\mathcal{K}(R)$ by Theorem 6.6. Actually, $\Omega$ yields a functor $\mathcal{K}(R) \to \mathcal{K}(R)$ as follows:

Let $a : X \to Y$ be a morphism in $\mathcal{K}(R)$. If $p_X : F_X \to X$ and $p_Y : F_Y \to Y$ are right $\text{Add}(R)$-approximations, then, since $ap_X$ factors through $p_Y$, we have the following commutative diagram, and as a result the morphism $b : \Omega(X) \to \Omega(Y)$ is induced.

$$
\begin{array}{ccc}
\Omega(X) & \xrightarrow{q_X} & F_X \\
\downarrow b & & \downarrow p_X \\
\Omega(Y) & \xrightarrow{q_Y} & F_Y
\end{array}
\xrightarrow{a}
\begin{array}{ccc}
X & \xrightarrow{p_X} & X \\
\downarrow c & & \downarrow \alpha_X \\
Y & \xrightarrow{\omega_Y} & \Omega(Y)[1]
\end{array}
\xrightarrow{\omega_Y}
\begin{array}{c}
\Omega(X)[1] \\
\downarrow b[1]
\end{array}
$$

If $a$ factors through an object in $\text{Add}(R)$, then it factors through $p_Y$, that is, there is $c : X \to F_Y$ such that $a = p_Yc$. Then we have $b[1]\omega_X = \omega_Ya = \omega_Yp_Yc = 0$, hence there is a morphism $e : F_X \to \Omega(Y)$ with $b[1] = e[1]q_X[1]$ or $b = eq_X$. Thus $b$ factors through an object in $\text{Add}(R)$. In such a way, we see that the mapping

$$
\text{Hom}_{\mathcal{K}(R)}(X, Y) \to \text{Hom}_{\mathcal{K}(R)}(\Omega(X), \Omega(Y)) : a \mapsto b
$$

is well-defined, hence we can define $\Omega(a) = b$ for morphisms. We call $\Omega$ the syzygy functor on $\mathcal{K}(R)$.

**Definition 7.4.** Let $X \in \mathcal{K}(R)$. A morphism $q : X \to G$ in $\mathcal{K}(R)$ is called a left $\text{Add}(R)$-approximation of $X$ if $G \in \text{Add}(R)$ and $\text{Hom}_{\mathcal{K}(R)}(q, F) : \text{Hom}_{\mathcal{K}(R)}(G, F) \to \text{Hom}_{\mathcal{K}(R)}(X, F)$ are surjective mappings for all $F \in \text{Add}(R)$. 
Recall that the dual complex $X^\ast = \text{Hom}_R(X, R)$ is again a complex belonging to $\mathcal{K}(R)$ and $X^{**} \cong X$. Also note that $X^\ast \in \text{Add}(R)$ if and only if $X \in \text{Add}(R)$.

**Lemma 7.5.** Let $X \in \mathcal{K}(R)$ and $G \in \text{Add}(R)$. Then a morphism $q : X \to G$ in $\mathcal{K}(R)$ is a left $\text{Add}(R)$-approximation if and only if the dual $q^\ast : G^\ast \to X^\ast$ is a right $\text{Add}(R)$-approximation, and the latter is equivalent to that $q^\ast$ is cohomologically surjective. In particular, there exists a left $\text{Add}(R)$-approximation of $X$ for any $X \in \mathcal{K}(R)$.

**Proof.** Assume that $q^\ast : G^\ast \to X^\ast$ is a right $\text{Add}(R)$-approximation. Let $a : X \to F$ be a morphism in $\mathcal{K}(R)$ with $F \in \text{Add}(R)$. Then $a^\ast : F^\ast \to X^\ast$ is a morphism in $\mathcal{K}(R)$, hence it factors through $q^\ast$. As a result, $a$ factors through $q$, hence $q$ is a left $\text{Add}(R)$-approximation. The converse is proved similarly.

**Remark 7.6.** Suppose we have a commutative diagram in $\mathcal{K}(R)$ with $F, F' \in \text{Add}(R)$:

$$
\begin{array}{ccc}
F & \xrightarrow{p} & X \\
\downarrow{f} & & \downarrow{\rho'} \\
F' & \xrightarrow{\rho} & X
\end{array}
$$

In such a case, if $p$ is right $\text{Add}(R)$-approximation, then so is $p'$. In fact, if $H(p) = H(p')H(f)$ is surjective, then so is $H(p')$.

Similarly, if a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{q} & G \\
\downarrow{q'} & & \uparrow{G'} \\
G' & & \\
\uparrow{G'} & & \uparrow{G'}
\end{array}
$$

is commutative with $G, G' \in \text{Add}(R)$ and if $q$ is a left $\text{Add}(R)$-approximation, then $q'$ is a left $\text{Add}(R)$-approximation as well.

**Corollary 7.7.** Assume that $R$ is a Gorenstein ring of dimension zero, and let

$$
\begin{array}{ccc}
Y & \xrightarrow{q} & F \\
\downarrow{p} & & \downarrow{p} \\
F & \xrightarrow{p} & X & \longrightarrow & Y[1]
\end{array}
$$

be a triangle in $\mathcal{K}(R)$ where $F \in \text{Add}(R)$. Then, $p$ is a right $\text{Add}(R)$-approximation if and only if $q$ is a left $\text{Add}(R)$-approximation.

**Proof.** If $p$ is a right $\text{Add}(R)$-approximation, then $H(p)$ is a surjective $R$-module homomorphism. Then $H(p^\ast)$ is an injective homomorphism. Since $R$ itself is an injective $R$-module, noting that the equality $H(p^\ast) = H(p^\ast)$ holds, we see from the triangle $X^\ast \xrightarrow{p^\ast} F^\ast \xrightarrow{q^\ast} Y^\ast \to X^\ast[1]$ that $H(q^\ast)$ is surjective. Hence $q$ is a left $\text{Add}(R)$-approximation by Lemma 7.5. The converse is proved in a similar manner.

**Definition 7.8.** Let $X \in \mathcal{K}(R)$ and $q : X \to G$ be a left $\text{Add}(R)$-approximation of $X$. Embed $q$ into a triangle $X \xrightarrow{q} G \to Z \to X[1]$. We denote the resulted $Z$ by $\Sigma(X)$ (or simply $\Sigma X$). It follows
from Lemma 7.5 that

\[ \Sigma(X) = \Omega(X^*), \]

which is uniquely determined as an object in the stable category \( \mathcal{K}(R) \). Actually, \( \Sigma \) yields a well-defined functor \( \mathcal{K}(R) \to \mathcal{K}(R) \) in a similar manner to the case of \( \Omega \). We call \( \Sigma \) the \textit{cosyzygy functor} on \( \mathcal{K}(R) \).

**Remark 7.9.** If \( R \) is a Gorenstein ring of dimension zero, then Corollary 7.7 says that \( \Sigma \) is actually the inverse of \( \Omega \) as a functor on \( \mathcal{K}(R) \).

**Proposition 7.10.** Let \( S \) be a multiplicative closed subset of \( R \). Then the functor \( S^{-1} : \mathcal{K}(R) \to \mathcal{K}(S^{-1}R) \) is defined naturally by taking the localization by \( S \).

1. If \( p : F \to X \) is a right \( \text{Add}(R) \)-approximation in \( \mathcal{K}(R) \), then \( S^{-1}p : S^{-1}F \to S^{-1}X \) is a right \( \text{Add}(S^{-1}R) \)-approximation in \( \mathcal{K}(S^{-1}R) \).
2. If \( q : Y \to G \) is a left \( \text{Add}(R) \)-approximation in \( \mathcal{K}(R) \), then \( S^{-1}q : S^{-1}Y \to S^{-1}G \) is a left \( \text{Add}(S^{-1}R) \)-approximation in \( \mathcal{K}(S^{-1}R) \).
3. Let \( \Omega_{S^{-1}R} \) and \( \Sigma_{S^{-1}R} \) be the syzygy and cosyzygy functors on \( \mathcal{K}(S^{-1}R) \). Then the following squares are commutative:

\[
\begin{array}{ccc}
\mathcal{K}(R) & \xrightarrow{\Omega} & \mathcal{K}(R) \\
\downarrow S^{-1} & & \downarrow S^{-1} \\
\mathcal{K}(S^{-1}R) & \xrightarrow{\Omega_{S^{-1}R}} & \mathcal{K}(S^{-1}R) \\
\end{array}
\quad
\begin{array}{ccc}
\mathcal{K}(R) & \xrightarrow{\Sigma} & \mathcal{K}(R) \\
\downarrow S^{-1} & & \downarrow S^{-1} \\
\mathcal{K}(S^{-1}R) & \xrightarrow{\Sigma_{S^{-1}R}} & \mathcal{K}(S^{-1}R) \\
\end{array}
\]

**Proof.**

1. If \( p \) is cohomologically surjective, then so is \( S^{-1}p \).
2. is clear from the fact that \( S^{-1}\text{Hom}_R(q, R) \cong \text{Hom}_{S^{-1}R}(S^{-1}q, S^{-1}R) \) and Lemma 7.5.
3. follows from (1) and (2). \( \square \)

In general, \( \Sigma \) is not necessarily the quasi-inverse of \( \Omega \), but we see that \( \Sigma \) is a left adjoint to \( \Omega \).

**Theorem 7.11.** As functors from \( \mathcal{K}(R) \) to itself, \( (\Sigma, \Omega) \) is an adjoint pair, that is, there are functorial isomorphisms

\[ \text{Hom}_{\mathcal{K}(R)}(\Sigma X, Y) \cong \text{Hom}_{\mathcal{K}(R)}(X, \Omega Y), \]

for all \( X, Y \in \mathcal{K}(R) \).

**Proof.** To prove the theorem, let \( a : \Sigma X \to Y \) be a morphism in \( \mathcal{K}(R) \). Then it induces the following commutative diagram:

\[
\begin{array}{cccccc}
X & \xrightarrow{q} & G_X & \xrightarrow{\Sigma} & \Sigma X & \xrightarrow{a} & X[1] \\
\downarrow b & & \downarrow & & \downarrow & & \downarrow \\
\Omega Y & \xrightarrow{p} & F_Y & \xrightarrow{\Omega} & Y & \xrightarrow{\Omega Y[1]},
\end{array}
\]
where \( p \) is a right Add\((R)\)-approximation and \( q \) is a left Add\((R)\)-approximation. Then, by the same reason in Definition 7.3 above, we see that

\[
\text{Hom}_{\mathcal{K}(R)}(\Sigma X, Y) \to \text{Hom}_{\mathcal{K}(R)}(X, \Omega Y) \ ; \ a \mapsto b
\]

is well defined. Conversely, given a morphism \( b : X \to \Omega Y \), one can easily find an \( a : \Sigma X \to Y \) that makes the diagram commutative. It thus gives the inverse to the above mapping:

\[
\text{Hom}_{\mathcal{K}(R)}(X, \Omega Y) \to \text{Hom}_{\mathcal{K}(R)}(\Sigma X, Y) \ ; \ b \mapsto a
\]

We should note that the similar arguments to ours in this section and also the similar content to Theorem 5.8 can be found in those papers of Christensen [7, Proposition 8.1] and Krause–Kussin [15, Lemma 2.5].

**Example 7.12.** Let \( M \) be a finitely generated \( R \)-module and let

\[
\cdots \longrightarrow P_n \xrightarrow{u_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} M \longrightarrow 0
\]

be a projective resolution of \( M \) where each \( P_i \) are finitely generated. Now, set the complex \( X \) to be \([0 \to P_1 \xrightarrow{u_1} P_0 \to 0]\). Then the right Add\((R)\)-approximation of \( X \) is given as

\[
F_X = \begin{bmatrix} 0 & P_2 & 0 & P_0 & 0 \end{bmatrix},
\]

\[
X = \begin{bmatrix} 0 & P_1 & u_1 & P_0 & 0 \end{bmatrix}.
\]

Hence we have

\[
\Omega X = \begin{bmatrix} 0 & P_2 & u_2 & P_1 & 0 \end{bmatrix}.
\]

More generally, we have

\[
\Omega^n X (= \Omega(\Omega^{n-1} X)) = \begin{bmatrix} 0 & P_{n+1} & u_{n+1} & P_n & 0 \end{bmatrix},
\]

for \( n > 0 \). On the other hand, let

\[
\cdots \longrightarrow Q_n \xrightarrow{\nu_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \xrightarrow{\nu_1} Q_0 \xrightarrow{\nu_0} M^* \longrightarrow 0
\]

be a projective resolution of \( M^* \). Then one can see that

\[
\Sigma X = \begin{bmatrix} 0 & P_0 & u & Q_0^* & 0 \end{bmatrix},
\]
where \( w \) is the composition \( P_0 \xrightarrow{u_0} M \xrightarrow{\text{natural}} M^{**} \xrightarrow{u_0^*} Q_0^* \). For \( n > 1 \), it can be easily seen that

\[
\Sigma^n X = \begin{bmatrix}
0 & Q_0^* & \overbrace{Q_{n-2}^* \xrightarrow{v_{n-1}^*} Q_{n-1}^*} & 0
\end{bmatrix}.
\]

See also Example 9.10.

**Lemma 7.13.** Let \( X \in \mathcal{K}(R) \). Then \( \Sigma X \) is *torsion-free as an object in \( \mathcal{K}(R) \).

**Proof.** Suppose we are given \( f \in H^i((\Sigma X)^*) = \text{Hom}_{\mathcal{K}(R)}(\Sigma X, R[i]) \) for some \( i \in \mathbb{Z} \) such that \( H(f) = 0 \). We want to show \( f = 0 \).

There is a triangle

\[
\begin{array}{ccc}
G_X & \xrightarrow{a} & \Sigma X & \xrightarrow{b} & X[1] & \xrightarrow{q} & G_X[1] \\
 & f \downarrow & & \downarrow & & \downarrow & \\
 & & R[i], & & & &
\end{array}
\]

where \( q \) is a left \( \text{Add}(R) \)-approximation. Since \( H(f \alpha) = H(f)H(\alpha) = 0 \), we have \( f \alpha = 0 \) by Theorem 5.8. Therefore, there is a morphism \( c : X[1] \to R[i] \) such that \( f = cb \). Since \( q \) is a left \( \text{Add}(R) \)-approximation and since \( R[i] \in \text{Add}(R) \), we have \( c = eq \) for some \( e : G_X[1] \to R[i] \). Thus, \( f = cb = eqb = 0 \) as desired. \( \square \)

**Theorem 7.14.** The following conditions are equivalent for \( X \in \mathcal{K}(R) \).

1. \( X \) is *torsion-free.
2. There are complexes \( Y \in \mathcal{K}(R) \) and \( F \in \text{Add}(R) \) such that \( X \) is a direct summand of \( \Sigma Y \oplus F \) in \( \mathcal{K}(R) \).
3. There is an isomorphism \( X \cong \Sigma \Omega X \) in \( \mathcal{K}(R) \).

**Proof.** The implication \( (2) \Rightarrow (1) \) follows from Lemma 7.13, since direct sums (or direct summands) of *torsion-free complexes are *torsion-free as well.

The condition (3) means exactly that \( X \oplus F_1 \cong \Sigma \Omega X \oplus F_2 \) in \( \mathcal{K}(R) \) for some \( F_1, F_2 \in \text{Add}(R) \). Hence (3) implies (2).

It remains to prove \( (1) \Rightarrow (3) \). Let

\[
\begin{array}{ccc}
\Omega X & \xrightarrow{a} & F & \xrightarrow{p} & X & \xrightarrow{\pi} & \Omega X[1] \\
\Omega X & \xrightarrow{q} & G & \xrightarrow{f} & \Sigma \Omega X & \xrightarrow{\pi} & \Omega X[1]
\end{array}
\]

be a commutative diagram in \( \mathcal{K}(R) \) whose rows are triangles in \( \mathcal{K}(R) \), where \( p \) (respectively, \( q \)) is a right (respectively, left) \( \text{Add}(R) \)-approximation and the morphism \( f \) is induced by the definition of left \( \text{Add}(R) \)-approximations. Note in this diagram that we can take such diagram in such a way that \( H(f) \) is a surjective graded \( R \)-module homomorphism. In fact, if necessary, we may replace \( q : \Omega X \to G \) by \( \{q \} : \Omega X \to G \oplus F \). Thus we may assume that \( L := \text{Cone}(f)[-1] \) belongs to \( \text{Add}(R) \). Note also that \( \pi \) is cohomologically surjective, since both \( f \) and \( p \) are so.
Since there is a commutative diagram by the octahedron axiom:

```
\[ \begin{array}{c}
\Omega X \\ a \\
\downarrow f \\
\Omega X \\
\downarrow q \\
L \\
\end{array} \xrightarrow{p} \begin{array}{c}
F \\ \uparrow \\
X \\
\uparrow \pi \\
\Sigma \Omega X \\
\downarrow \Sigma \Omega X \\
L[1] \\
\end{array} \xrightarrow{b} \begin{array}{c}
L[1] \\
\end{array} \]
```

we have the following triangle in \( \mathcal{K}(R) \):

\[
L \longrightarrow \Sigma \Omega X \xrightarrow{\pi} X \longrightarrow L[1].
\]

Since \( H(\pi) \) is surjective, we see that \( H(b) = 0 \) by the cohomology long exact sequence. Then, since we are assuming that \( X \) is *torsion-free, it follows from Proposition 5.10 that \( b = 0 \) as a morphism in \( \mathcal{K}(R) \). Thus the triangle splits and we have the isomorphism \( \Sigma \Omega X \cong X \oplus L \) in \( \mathcal{K}(R) \) with \( L \in \text{Add}(R) \).

Remark 7.15. By the adjoint property proved in Theorem 7.11, there is a natural counit morphism \( \pi : \Sigma \Omega X \to X \) for any \( X \in \mathcal{K}(R) \). Theorem 7.14 says that this is actually a right *torsion-free approximation of \( X \) in the following sense: If \( f : Y \to X \) is a morphism in \( \mathcal{K}(R) \) where \( Y \) is *torsion-free, then \( f \) factors through \( \pi \).

In fact, \( \pi_Y \) in the following commutative diagram in \( \mathcal{K}(R) \) is an isomorphism:

```
\[ \begin{array}{c}
\Sigma \Omega Y \\
\downarrow \pi_Y \\
Y \\
\end{array} \xrightarrow{f} \begin{array}{c}
\Sigma \Omega X \\
\downarrow \pi \\
X \\
\end{array} \]
```

Lemma 7.16. Suppose that \( R \) is a generically Gorenstein ring. If \( X \in \mathcal{K}(R) \) is *torsion-free, then \( \Sigma X \) is *reflexive.

Proof. We have shown in Lemma 7.13 that \( \Sigma X \) is *torsion-free. To prove that it is *reflexive, let \( \alpha : H^n(\Sigma X) \to R \) be a homomorphism of \( R \)-modules, where \( n \in \mathbb{Z} \). We want to show that there is a morphism \( \alpha : \Sigma X \to R[-n] \) in \( \mathcal{K}(R) \) satisfying \( H(\alpha) = \alpha \). By definition, there is a triangle

\[
X \xrightarrow{q} G_X \xrightarrow{p} \Sigma X \xrightarrow{r} X[1],
\]

where \( q \) is a left \( \text{Add}(R) \)-approximation. Therefore, we have a long exact sequence of cohomology modules

\[
\begin{align*}
H^n(X) \xrightarrow{H(q)} & H^n(G_X) \xrightarrow{H(p)} H^n(\Sigma X) \xrightarrow{H(r)} H^{n+1}(X) \\
& \downarrow a \\
& R
\end{align*}
\]
Note that $G_X \in \text{Add}(R)$ is *reflexive, and thus there is a morphism $b : G_X \rightarrow R[−n]$ such that $H(b) = \alpha H(p)$. Then it is clear that $H(bq) = \alpha H(p)H(q) = 0$. Since $X$ is *torsion-free, it follows that $bq = 0$. (See Lemma 3.2.) Thus there is a morphism $a : \Sigma X \rightarrow R[−n]$ with $b = ap$. Note that $(\alpha - H(a))H(p) = 0$. Let $S$ be the set of all non-zero divisors of $R$. Since $S^{-1}R$ is a Gorenstein ring of dimension zero,

\[
S^{-1}X \xrightarrow{S^{-1}q} S^{-1}G_X \xrightarrow{S^{-1}p} S^{-1}\Sigma X \xrightarrow{S^{-1}r} S^{-1}X[1]
\]

is a triangle in which $S^{-1}q$ is a left $\text{Add}(S^{-1}R)$-approximation and $S^{-1}p$ is a right approximation in $\mathcal{X}(S^{-1}R)$ by Corollary 7.7. In particular, $S^{-1}H(p) = H(S^{-1}p)$ is a surjective mapping by Lemma 7.2. Since $(\alpha - H(a))H(p) = 0$, we see that $S^{-1}(\alpha - H(a)) = 0$ as an element of $S^{-1}(H(\Sigma X)^*)$. Noting that the $R$-dual of any finitely generated module is torsion-free, we see that $H(\Sigma X)^*$ is a torsion-free $R$-module. Consequently, we have that $\alpha = H(a)$ as an element of $H^n(\Sigma(X))^*$.

Combining this lemma with Theorem 7.14 or with Lemma 7.13, we obtain the following theorem.

**Theorem 7.17.** Under the assumption that $R$ is generically Gorenstein, $\Sigma^2X = \Sigma(\Sigma X)$ is always *reflexive for any $X \in \mathcal{K}(R)$.

Similarly to Theorem 7.14, one can characterize the *reflexivity property for complexes as follows:

**Corollary 7.18.** Assume that $R$ is a generically Gorenstein ring. Then the following two conditions for $X \in \mathcal{K}(R)$ are equivalent.

1. $X$ is *reflexive.
2. $\Sigma^2\Omega^2X \cong X$ in $\mathcal{K}(R)$.

**Proof.** Theorem 7.17 says that (2) $\Rightarrow$ (1) holds. Assume $X$ is *reflexive. Then take a right $\text{Add}(R)$-approximation sequence $\Omega X \rightarrow F_0 \rightarrow X \rightarrow \Omega X[1]$. Since right $\text{Add}(R)$-approximations are cohomologically surjective, we have an exact sequence of $R$-modules:

\[
0 \rightarrow H(X)^* \rightarrow H(F_0)^* \rightarrow H(\Omega X)^*
\]

Thus we can apply Proposition 3.7(2), and we see that $\Omega X$ is *torsion-free. Then it follows from Theorem 7.14 that $\Sigma^2\Omega X = \Sigma(\Omega X) \cong \Omega X$. Thus applying $\Sigma$ to the both sides, we have $\Sigma^2\Omega^2X \cong \Sigma\Omega X$ and the last equals $X$, since $X$ is *torsion-free.

**8 CONTRACTIONS**

**Definition 8.1.** We say that a finite sequence of morphisms in $\mathcal{K}(R)$;

\[
0 \rightarrow X_n \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{\cdots} F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \rightarrow 0
\]

(8.1)
is $\mathcal{K}(R)$-exact if there are triangles

$$X_{i+1} \xrightarrow{q_{i+1}} F_i \xrightarrow{p_i} X_i \xrightarrow{\omega_i} X_{i+1}[1]$$

and equalities

$$f_i = q_i p_i$$

for $0 \leq i \leq n - 1$. The $\mathcal{K}(R)$-exact sequence (8.1) can be described in a single diagram as

We also call the $\mathcal{K}(R)$-exact sequence (8.1) a partial $\text{Add}(R)$-resolution of $X_0$ if $F_i \in \text{Add}(R)$ for all $0 \leq i < n$. By definition, an $\text{Add}(R)$-resolution of $X_0$ of length $n - 1$ is a partial $\text{Add}(R)$-resolution with $X_n = 0$.

Note that, in the paper [13, Notation 3.2], we call a $\mathcal{K}(R)$-exact sequence an $(n + 1)$-angle in $\mathcal{K}(R)$. However, in the present paper we are interested only in the ‘exactness’ of the sequence, and not in the length $n$. For this reason, we use the term ‘$\mathcal{K}(R)$-exact sequence’ instead of $(n + 1)$-angle.

If we are given such a $\mathcal{K}(R)$-exact sequence (8.1), we have a natural morphism $\tilde{\omega}_n : X_0 \to X_n[n]$ that is defined by the composition $\omega_{n-1}[n-1]\omega_{n-2}[n-2] \cdots \omega_1[1]\omega_0$ of the morphisms in the relevant triangles. Note that the morphism $\tilde{\omega}_n : X_0 \to X_n[n]$ is determined by $\omega_i$ in (8.1), and hence it is uniquely determined by the collection of triangles in (8.1), which we call the connecting morphism of the $\mathcal{K}(R)$-exact sequence (8.1).

**Theorem and Definition 8.2.** Suppose we are given a partial $\text{Add}(R)$-resolution:

$$0 \longrightarrow X_n \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_1 \xrightarrow{f_0} X_0 \longrightarrow 0. \quad (8.2)$$

Furthermore, we assume that each $F_i \in \text{Add}(R)$ contains no null complex as a direct summand for $0 \leq i \leq n - 1$. Then there is a triangle of the form

$$X_n[n-1] \xrightarrow{\psi_n} \tilde{F} \xrightarrow{\phi_n} X_0 \xrightarrow{\tilde{\omega}_n} X_n[n]. \quad (8.3)$$

where $\tilde{\omega}_n$ is the connecting morphism of the sequence (8.2) and the following conditions are satisfied.

1. There is an equality as underlying graded $R$-modules

$$\tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0.$$
(2) Let \( i_n : F_i[i] \to \bar{F} \) and \( p_{ri} : \bar{F} \to F_i[i] \) be, respectively, a natural injection and a natural projection of graded \( R \)-modules according to the direct sum decomposition in (1). Denoting by \( d_{\bar{F}} \) the differential mapping of \( \bar{F} \), we have equalities of graded \( R \)-module homomorphisms;
\[
\begin{cases}
pr_j d_{\bar{F}} i_n = 0 & \text{for } 0 \leq i \leq j \leq n - 1, \\
p_{ri-1} d_{\bar{F}} i_n = f_i[i] & \text{for } 1 \leq i \leq n - 1.
\end{cases}
\]

(3) The natural inclusion \( i_{n0} : F_0 \to \bar{F} \) and the natural projection \( p_{r_{n1}} : \bar{F} \to F_{n-1}[n-1] \) are chain maps and they yield the morphisms in \( \mathcal{K}(R) \) which make the following diagrams in \( \mathcal{K}(R) \) commutative;
\[
\begin{array}{ccc}
X_n[n-1] & \xrightarrow{\psi_n} & \bar{F} \\
\downarrow{\eta_n[n-1]} & & \downarrow{p_{r_{n1}}} \\
F_{n-1}[n-1] & \xleftarrow{i_{n0}} & \bar{F} \\
\end{array}
\quad \begin{array}{ccc}
F_0 & \xrightarrow{i_{n0}} & X_0 \\
\downarrow{\phi_n} & & \downarrow{\omega_n} \\
\bar{F} & \xleftarrow{p_{0}} & X_0[n]
\end{array}
\]

As an object of \( \mathcal{K}(R) \), such a complex \( \bar{F} \) is unique up to isomorphism.

We call \( \bar{F} \) the contraction of the partial \( \text{Add}(R) \)-resolution (8.2). The triangle (8.3) is called the contracted triangle of (8.2).

**Proof.** We prove the theorem by the induction on \( n \). If \( n = 1 \), then there is a triangle
\[
X_1 \xrightarrow{q_1} F_0 \xrightarrow{p_0} X_0 \xrightarrow{\omega_0} X_1[1].
\]
Hence, setting \( \bar{F} = F_0, \psi_1 = q_1 \) and \( \varphi_1 = p_0 \) in this case, we see the conditions (1)–(3) are satisfied.

Now we assume \( n > 1 \). Setting \( F' \) as the contraction of the partial \( \text{Add}(R) \)-resolution
\[
0 \longrightarrow X_{n-1} \xrightarrow{q_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} F_{n-3} \xrightarrow{f_{n-3}} \cdots F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0,
\]
we assume that the theorem holds for this partial resolution and \( F' \). Then we have the following octahedron diagram:
(8.5)
where $\alpha_{n-1} = \psi_{n-1} p_{n-1} [n - 2]$. In fact, the third column and the third row of this diagram are triangles by the induction hypothesis and the $\mathcal{K}(R)$-exactness assumption. The second column is a triangle given by setting $\widetilde{F} = \text{Cone}(\alpha_{n-1})$. The second row gives the desired triangle for the case of $n$.

Since $\widetilde{F}$ is the mapping cone of the morphism $\alpha_{n-1}$, it equals $F_{n-1} [n - 1] \oplus \widetilde{F}'$ as an underlying graded $R$-module, and since $d_{\widetilde{F}_{n-1}} = 0$, the summand $F_{n-1} [n - 1]$ is mapped by $\alpha_{n-1} [1]$ into $\widetilde{F}' [1]$ under the differential $d_{\widetilde{F}}$. This proves $pr_{n-1} d_{\widetilde{F}} in_{n-1} = 0$ and the restriction of $d_{\widetilde{F}}$ to $\widetilde{F}'$ is its own differential $d_{\widetilde{F}'}$. It then follows from the induction hypothesis on $\widetilde{F}'$ that $pr_{j} d_{\widetilde{F}} in_{i} = 0$ holds for $0 \leq i \leq j \leq n - 1$. If $1 \leq i \leq n - 2$, then the equality $pr_{i-1} d_{\widetilde{F}} in_{i} = pr_{i-1} d_{\widetilde{F}'} in_{i} = f_{i}[i]$ holds by the induction hypothesis. To prove that $pr_{n-2} d_{\widetilde{F}} in_{n-1} = f_{n-1} [n - 1]$, we note that $d_{\widetilde{F}} in_{n-1} = \alpha_{n-1} [1]$ and that $pr'_{n-2} : \widetilde{F}' \rightarrow F_{n-2} [n - 2]$ is a chain map by the induction hypothesis. We consider the following diagram:

$$
\begin{array}{ccc}
F_{n-1} [n - 1] & \xrightarrow{\alpha_{n-1} [1]} & \widetilde{F}' [1] \\
\downarrow p_{n-1} [n - 1] & & \downarrow \phi_{n-1} [1] \\
X_{n-1} [n - 1] & \xrightarrow{\psi_{n-1} [1]} & F_{n-2} [n - 1]
\end{array}
$$

The upper left triangle is commutative by the definition of $\alpha_{n-1}$, and the lower right triangle is also commutative by the induction hypothesis for $\widetilde{F}'$. Therefore, the square above is a commutative diagram, and thus we have $pr'_{n-2} [1] \alpha_{n-1} [1] = q_{n-1} [n - 1] p_{n-1} [n - 1] = f_{n-1} [n - 1]$. This proves $pr_{n-2} d_{\widetilde{F}} in_{n-1} = f_{n-1} [n - 1]$, and the conditions (1) and (2) are proved.

We can see from (2) that $d_{\widetilde{F}} in_{0} = 0$ and $pr_{n-1} d_{\widetilde{F}} = 0$, hence $in_{0}$ and $pr_{n-1}$ are chain maps. The commutativity of the left triangle in (8.4) follows from the diagram (8.5). To prove that the right triangle in (8.4) is also commutative, note that $p_{0} = \varphi_{n-1} in'_{0}$ holds by the induction hypothesis, where $in'_{0} : F_{0} \rightarrow \widetilde{F}'$ is the natural injection. Note from the diagram (8.5) that $\varphi_{n} t_{n-1} = \varphi_{n-1}$ and $t_{n-1} in'_{0} = in_{0}$. Thus we obtain $\varphi_{n} in_{0} = \varphi_{n} t_{n-1} in'_{0} = \varphi_{n} in'_{0} = p_{0}$.

Since the connecting morphism $\alpha_{n}$ is uniquely determined by the $\mathcal{K}(R)$-exact sequence, noting that $\widetilde{F} \cong \text{Cone}(\alpha_{n} [-1])$ in $\mathcal{K}(R)$, we see that $\widetilde{F}$ is unique up to isomorphism in $\mathcal{K}(R)$.

Remark 8.3. Another interpretation of the conditions (1)–(3) in the theorem is the following:

Now returning to the setting of Theorem 8.2, the contraction of the $\mathcal{K}(R)$-exact sequence (8.2) is $F_{n-1} [n - 1] \oplus \cdots \oplus F_{1} [1] \oplus F_{0}$ as an underlying graded $R$-module and the differential $d_{\widetilde{F}}$ is given by a matrix of the form:

$$
d_{\widetilde{F}} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & f_{n-1} [n - 1] & 0 & \cdots & 0 \\
a_{n-1} n-3 & f_{n-2} [n - 2] & \cdots & 0 & 0 \\
a_{n-1} n-4 & a_{n-2} n-4 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} 0 & a_{n-2} 0 & \cdots & f_{1} [1] & 0
\end{pmatrix},
$$

(8.7)
where each \( a_{ij} : F_i[j] \to F_j[j + 1] \) is a graded \( R \)-homomorphism. On the other hand, the commutativity of the diagrams (8.4) says that as underlying graded \( R \)-module homomorphisms \( \psi_n \) and \( \varphi_n \) are represented, respectively, by the following matrices:

\[
\psi_n = \begin{pmatrix}
q_{n+1} \\
ar_{n-2} \\
a_{n0}
\end{pmatrix} : X_n[n-1] \to F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0
\]

\[
\varphi_n = (b_{n-1} \cdots b_1 p_0) : F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0 \to X_0
\]

for some graded \( R \)-homomorphisms \( a_{ni} : X_n[n-1] \to F_i[i] \) and \( b_{i0} : F_i[i] \to X_0 \).

**Definition 8.4.** Assume that we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & X_n & \xrightarrow{q_n^F} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0^F} & X_0 & \to & 0 \\
0 & \to & Y_n & \xrightarrow{q_n^G} & G_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_1} & G_0 & \xrightarrow{p_0^G} & Y_0 & \to & 0,
\end{array}
\]

(8.8)

where the rows are partial Add(\( R \))-resolutions. We say that the diagram (8.8) gives a morphism between the partial Add(\( R \))-resolutions if there are commutative diagrams

\[
\begin{array}{ccccccc}
X_{i+1} & \xrightarrow{q_{i+1}^F} & F_i & \xrightarrow{p_i^F} & X_i & \xrightarrow{r_i} & X_{i+1}[1] \\
Y_{i+1} & \xrightarrow{q_{i+1}^G} & G_i & \xrightarrow{p_i^G} & Y_i & \xrightarrow{r_i} & Y_{i+1}[1],
\end{array}
\]

where each row is a triangle in \( \mathcal{X}(R) \) for \( 0 \leq i < n \), and \( f_i = q_i^F p_i^F \), \( g_i = q_i^G p_i^G \) for \( 1 \leq i < n \).

In such a case, we have a morphism \( \tilde{s} \) between the contractions with the diagram;

\[
\begin{array}{ccccccc}
X_n[n-1] & \xrightarrow{\psi_n^F} & \widetilde{F} & \xrightarrow{\varphi_n^F} & X_0 & \xrightarrow{\omega_n^F} & X_n[n] \\
Y_n[n-1] & \xrightarrow{\psi_n^G} & \widetilde{G} & \xrightarrow{\varphi_n^G} & Y_0 & \xrightarrow{\omega_n^G} & Y_n[n],
\end{array}
\]

(8.9)

Since the morphisms \( t_0 \) and \( t_n \) are given beforehand, such a morphism \( \tilde{s} \) obviously exists so that the diagram (8.9) will be commutative. Unfortunately, it is not unique in general.

But we can prove the following theorem.

**Theorem 8.5.** Under the circumstances in Definition 8.4, we furthermore assume that all the \( F_i, G_i \) (\( 1 \leq i < n \)) do not contain any null complexes as direct summands. Then we can take a morphism \( \tilde{s} : \widetilde{G} \to \widetilde{F} \) so that it is represented by the following type of lower triangular matrix as an underlying graded \( R \)-module homomorphism according to the direct sum decompositions
\[ \tilde{G} = G_{n-1}[n-1] \oplus G_{n-2}[n-2] \oplus G_0 \quad \text{and} \quad \tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus F_0. \]

\[
\begin{pmatrix}
  s_{n-1}[n-1] & 0 & 0 & \cdots & 0 \\
  * & s_{n-2}[n-2] & 0 & \cdots & 0 \\
  * & * & s_{n-3}[n-3] & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  * & * & * & \cdots & s_0
\end{pmatrix}
\]

In this case, the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{pr_{n-1}^F} & F_{n-1}[n-1] \\
\uparrow & & \uparrow \\
\tilde{G} & \xrightarrow{pr_{n-1}^G} & G_{n-1}[n-1].
\end{array}
\]

**Proof.** The last part of the theorem follows from the first, since \( pr_{n-1}^F \) and \( pr_{n-1}^G \) are represented by matrices of the form \((1 \ 0 \ \cdots \ 0)\).

We prove the first part by induction on \( n \).

If \( n = 1 \), then \( \tilde{F} = F_0, \tilde{G} = G_0 \) and \( \tilde{s} = s_0 \). Hence the statement is true.

Assume \( n \geq 2 \). Under the same notation as in the proof of Theorem 8.2, \( \tilde{F} \) and \( \tilde{G} \) are the mapping cones of the morphisms \( \alpha_{n-1}^F = \psi_{n-1}^F p_{n-1}^F[n-2] \) and \( \alpha_{n-1}^G = \psi_{n-1}^G p_{n-1}^G[n-2] \), respectively. By the induction hypothesis for \( n-1 \), we may assume that we have such an \( \tilde{s}' \) that is represented by a lower triangular matrix and that makes the following diagram commutative.

\[
\begin{array}{ccc}
X_{n-1}[n-2] & \xrightarrow{\psi_{n-1}^F} & \tilde{F}' \\
\downarrow t_{n-1}[n-2] & & \uparrow \tilde{s} \\
Y_{n-1}[n-2] & \xrightarrow{\psi_{n-1}^G} & \tilde{G}'
\end{array}
\]

Since there is a triangle \( X_n \to F_{n-1} \xrightarrow{p_{n-1}^F} X_{n-1} \to X_n[1] \) in \( \mathcal{K}(R) \), we see that \( X_n \) is isomorphic to the mapping cone of \( p_{n-1}^F[-1] \), that is, \( F_{n-1} \oplus X_{n-1}[-1] \) is its underlying graded \( R \)-module and it has the differential \( d_{X_n} = \left( \begin{array}{cc} 0 & 0 \\ p_{n-1}^F & d_{X_{n-1}} \end{array} \right) \). Similarly, \( Y_n \cong G_{n-1} \oplus Y_{n-1}[-1] \) as an underlying graded \( R \)-module with \( d_{Y_n} = \left( \begin{array}{cc} 0 & 0 \\ p_{n-1}^G & d_{Y_{n-1}} \end{array} \right) \). Note that \( X_{n-1}[-1] \) is a subcomplex of this mapping cone, and \( F_{n-1} \) is a quotient of it. So the commutative diagram with rows being triangles;

\[
\begin{array}{cccc}
X_{n-1}[-1] & \xrightarrow{t_{n-1}[-1]} & F_{n-1} & \xrightarrow{t_{n}} & X_n \\
Y_{n-1}[-1] & \xrightarrow{t_{n-1}[-1]} & G_{n-1} & \xrightarrow{t_{n}} & Y_n
\end{array}
\]

is represented by a commutative diagram of exact sequences;

\[
\begin{array}{cccc}
0 & \xrightarrow{t_{n-1}[-1]} & X_{n-1}[-1] & \xrightarrow{t_{n}} & X_n \\
& & F_{n-1} & \xrightarrow{t_{n}} & 0 \\
0 & \xrightarrow{t_{n-1}[-1]} & Y_{n-1}[-1] & \xrightarrow{t_{n}} & Y_n \\
& & G_{n-1} & \xrightarrow{t_{n}} & 0
\end{array}
\]
Since $t_n$ maps the subcomplex $Y_{n-1}[-1]$ into the subcomplex $X_{n-1}[-1]$, we can see that $t_n$ is represented by a matrix

$$G_{n-1} \oplus Y_{n-1}[-1] \xrightarrow{\begin{pmatrix} s_{n-1} & 0 \\ u & t_{n-1}[-1] \end{pmatrix}} F_{n-1} \oplus X_{n-1}[-1],$$

where $u : G_{n-1} \longrightarrow X_{n-1}[-1]$ is a graded $R$-homomorphism. Identifying those relevant complexes under such isomorphisms, we also see from the inductive construction of $\tilde{F}$ in the proof of Theorem 8.2 that the morphism $\psi_n^F : X_n[n-1] \longrightarrow \tilde{F}$ is given by the chain map

$$F_{n-1}[n-1] \oplus X_{n-1}[n-2] \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \psi_{n-1}^F \end{pmatrix}} F_{n-1}[n-1] \oplus \tilde{F}'.$$

See the diagram (8.5). Similarly, $\varphi_n^F : \tilde{F} \longrightarrow X_0$ is represented by

$$F_{n-1}[n-1] \oplus \tilde{F}' \xrightarrow{\begin{pmatrix} 0 & \varphi_{n-1}^F \end{pmatrix}} X_0.$$

Finally, it is easy to see that the following diagram is commutative:

$$G_{n-1}[n-1] \oplus Y_{n-1}[n-2] \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \psi_{n-1}^F \end{pmatrix}} G_{n-1}[n-1] \oplus \tilde{G}' \xrightarrow{\begin{pmatrix} 0 & s_{n-1}^F \end{pmatrix}} Y_0.$$

Therefore, we can take the matrix $\begin{pmatrix} s_{n-1}[n-1] & 0 \\ \psi_{n-1}^F & u[n-1] \end{pmatrix}$ as $\tilde{s}$. Since $\tilde{s}'$ is taken to be a lower triangular matrix by the induction hypothesis, so is $\tilde{s}$.

\section{Remarks on Partial Add($R$)-Resolutions}

\textbf{Definition 9.1.}

(1) We say that a partial Add($R$)-resolution (8.2) is \textit{split} if each $q_i$ in Definition 8.1 has a left inverse, that is, $q_i$ is a split monomorphism, for all $1 \leq i \leq n$. This is equivalent to $\omega_i = 0$ for all $0 \leq i < n$, with the notation in Definition 8.1. See [11, Lemma 1.4].

(2) We say that a partial Add($R$)-resolution (8.2) is \textit{degenerate} if one can choose the differential $d_{\tilde{F}}$ such that $pr_j \circ d_{\tilde{F}} \circ m_i = 0$ for all $1 \leq i, j \leq n - 1$ with $j \neq i - 1$ under the notation of Theorem 8.2. This is equivalent to saying that one can take the differential of the
form
\[
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & f_{n-2}[n-2] & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & f_{n-3}[n-3] & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & 0 & \ldots & f_1[1] & 0
\end{pmatrix}
\] (9.1)

as a graded \( R \)-module mapping from \( \tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0 \) to \( \tilde{F}[1] \). Note in this case that we have an equality
\[
\tilde{F} = \bigoplus_{i \in \mathbb{Z}} \begin{array}{cccccc}
0 & F_{n-1}^i & F_{n-2}^i & \cdots & F_1^i & F_0^i & 0
\end{array} [-i].
\]

The following proposition will be necessary in a later argument of this paper.

**Proposition 9.2.** Let
\[
0 \longrightarrow F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \longrightarrow 0
\] (9.2)
be an \( \text{Add}(R) \)-resolution of length \( n-1 \) where each \( F_i \) has no null complex as a direct summand and let \( \tilde{F} \) be its contraction. Assume that \( n \geq 2 \) and \( f_{n-1} \) has a left inverse in \( \mathcal{K}(R) \). Then \( X_0 \cong \tilde{F} \) and the morphism \( pr_{n-1} : \tilde{F} \longrightarrow F_{n-1}[n-1] \) is zero in \( \mathcal{K}(R) \).

**Proof.** The isomorphism \( X_0 \cong \tilde{F} \) follows from the contraction sequence (8.3) in Theorem 8.2 by setting \( X_n = 0 \). Note that \( pr_{n-1} \) is represented by the matrix
\[
\begin{pmatrix}
1 & 0 & \cdots & 0
\end{pmatrix} : F_{n-1}[n-1] \oplus \cdots \oplus F_1[1] \oplus F_0 \longrightarrow F_{n-1}[n-1].
\]

Let \( v \) be a left inverse of \( f_{n-1} \), i.e. \( v : F_{n-2} \rightarrow F_{n-1} \) such that \( vf_{n-1} = 1_{F_{n-1}} \) and set \( \tilde{v} : \tilde{F}[1] \rightarrow F_{n-1}[n-1] \) as a graded \( R \)-homomorphism given by the matrix
\[
\begin{pmatrix}
0 & v[n-1] & 0 & \cdots & 0
\end{pmatrix} : F_{n-1}[n] \oplus F_{n-2}[n-1] \oplus \cdots \oplus F_0 \longrightarrow F_{n-1}[n-1].
\]

Then, since the differential \( d_{\tilde{F}} \) is represented by the matrix (8.7), it is easy to see that \( pr_{n-1} = \tilde{v}d_{\tilde{F}} \). Hence \( pr_{n-1} \) is null homotopic.

**Corollary 9.3.** Assume that the \( \text{Add}(R) \)-resolution (9.2) is split and \( n \geq 2 \). Then \( pr_{n-1} = 0 \) in \( \mathcal{K}(R) \).

**Lemma 9.4.** We assume that the following conditions are satisfied for the partial \( \text{Add}(R) \)-resolution (8.2).

1. \( X_0, X_n \) belong to \( \text{Add}(R) \) and they have no null complexes as direct summands.
(2) As a sequence of graded \( R \)-modules, the sequence
\[
0 \to X_n \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{p_0} X_0 \to 0
\]
is exact.

Then the partial \( \text{Add}(R) \)-resolution is degenerate. The contracted triangle (8.3) is realized by the morphisms represented by the following of underlying graded \( R \)-module homomorphisms:

\[
\psi_n = \begin{pmatrix} q_n[n-1] \\ 0 \\ \vdots \\ 0 \end{pmatrix} : X_n[n-1] \to \tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0
\]

\[
\varphi_n = (0 \cdots 0 p_0) : \tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_1[1] \oplus F_0 \to X_0.
\]

Proof. Set \( d_{\tilde{F}} \) as in (9.1) and we see by a straightforward computation that \( d_{\tilde{F}}^2 = 0, d_{\tilde{F}} \psi_n = 0 \) and \( \varphi_n d_{\tilde{F}} = 0 \) where \( \psi_n \) and \( \varphi_n \) are given as in the lemma. Therefore, the matrices given in the lemma define chain homomorphisms. It is then easy to see that the sequence
\[
X_n[n-1] \xrightarrow{\psi_n} \tilde{F} \xrightarrow{\varphi_n} X_0 \xrightarrow{0} X_n[n]
\]
is a triangle in \( \mathcal{K}(R) \). (By the condition (2), the mapping \( p_0 : F_0 \to X_0 \) is surjective as an underlying graded module homomorphism, hence it is a split epimorphism in \( \text{Add}(R) \). Thus, in the triangle
\[
X_1 \xrightarrow{q_1} F_0 \xrightarrow{p_0} X_0 \xrightarrow{\omega_0} X_1[1],
\]
we have \( \omega_0 = 0 \). Therefore \( \overline{\omega}_n = 0 \) in the contracted triangle above.)

\[\square\]

Corollary 9.5. If a partial \( \text{Add}(R) \)-resolution is split, then it is degenerate.

We should note that all partial \( \text{Add}(R) \)-resolutions of length \( n \leq 2 \) are degenerate. In fact, if \( n = 2 \), then the \( \mathcal{K}(R) \)-exact sequence is \( 0 \to X_2 \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 \to X \to 0 \) where one can take \( F_1 \) and \( F_2 \) have no null complexes as summands, and \( \tilde{F} \) is the mapping cone of \( f_1 \), therefore the sequence is degenerate.

Note also that, even if there is a \( \mathcal{K}(R) \)-exact sequence
\[
0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \to X \to 0
\]
for assigned \( F_i \in \text{Add}(R) \) and \( f_i \), the rightmost complex \( X \) is not uniquely determined.

In fact, \( X \) depends not only on \( f_i \) but also on \( p_i, q_i \) with \( f_i = p_i q_i \) as in Definition 8.2.

For example, consider the \( \text{Add}(R) \)-resolution
\[
0 \to F_2 = R \xrightarrow{(\iota)} F_1 = R \oplus R \xrightarrow{(\iota, \phi)} F_0 = R[1] \oplus R \to X_0 \to 0,
\]
where \(a, b \in R\). Then, under the notation of (8.2), \(q_1\) is described as

\[
\begin{array}{ccc}
X_1 & = & [0 \rightarrow R \rightarrow R^2 \rightarrow 0] \\
q_1 & & c \\
F_0 & = & [0 \rightarrow R \rightarrow R \rightarrow 0],
\end{array}
\]

where one can take any element of \(R\) as \(c\). If \(c = 0\), then

\[
X_0 = R[1] \oplus [0 \rightarrow R \rightarrow R^2 \rightarrow R \rightarrow 0].
\]

If \(c = 1\), then

\[
X_0 = [0 \rightarrow R^2 \rightarrow R \rightarrow 0]. \quad \square
\]

For \(X \in \mathcal{X}(R)\) and for an integer \(n > 0\), we define the \(n\)th syzygy and cosyzygy by induction on \(n\);

\[
\Omega^0 X = \Sigma^0 X = X, \quad \Omega^n X = \Omega(\Omega^{n-1} X), \quad \Sigma^n X = \Sigma(\Sigma^{n-1} X).
\]

Recall from Definitions 7.3 and 7.8 that \(\Omega^n X\) and \(\Sigma^n X\) are uniquely determined as objects in \(\mathcal{X}(R)\), or in other words they are unique up to Add(\(R\))-summands as objects in \(\mathcal{X}(R)\). Actually they define the functors \(\Omega^n, \Sigma^n : \mathcal{X}(R) \rightarrow \mathcal{X}(R)\), and Theorem 7.11 assures that \((\Sigma^n, \Omega^n)\) is an adjoint pair for each \(n > 0\).

**Definition 9.6.** Let \(X \in \mathcal{X}(R)\) and take a right Add(\(R\))-approximation \(p_0 : F_0 \rightarrow X\). We embed \(p_0\) into a triangle to get the first syzygy \(\Omega X\);

\[
\Omega X \xrightarrow{q_1} F_0 \xrightarrow{p_0} X \xrightarrow{\omega^X_1} \Omega X[1].
\]

Similarly but as for the dual version to this, we have a triangle for any \(Y \in \mathcal{X}(R)\);

\[
\Sigma Y[-1] \xrightarrow{\omega^Y_{-1}} Y \xrightarrow{q^0} G_0 \xrightarrow{\rho^0} \Sigma Y,
\]

where \(q^0\) is a left Add(\(R\))-approximation. In such a way, we have morphisms \(\omega^X_1\) and \(\omega^Y_{-1}\). Now let \(n\) be a positive integer. We define inductively

\[
\omega^X_n = \omega^X_{n-1} \omega^X_1 : X \rightarrow \Omega^n X[n], \quad \omega^Y_{-n} = \omega^Y_{-1} \omega^Y[-1] : \Sigma^n Y[-n] \rightarrow Y.
\]

Let \(X\) be an arbitrary object in \(\mathcal{X}(R)\). Note from the definition of \(\Omega^i\) that there are triangles

\[
\Omega^{i+1} X \xrightarrow{q_{i+1}} F_i \xrightarrow{p_i} \Omega^i X \xrightarrow{\omega^X_{i+1}} \Omega^{i+1} X[1],
\]

where \(\omega^X_{i+1}\) is a a morphism such that

\[
\omega^X_{i+1} = \omega^X_{i+1} \omega^X_i : \Omega^{i+1} X \rightarrow \Omega^{i+1} X[1].
\]
where \( F_i \in \text{Add}(R) \) and \( p_i \) is a right \( \text{Add}(R) \)-approximation of \( \Omega^i X \) for all \( i \geq 0 \). Hence, when \( n \) is a positive integer, we have a partial \( \text{Add}(R) \)-resolution of the form:

\[
0 \longrightarrow \Omega^n X \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{p_0} X \longrightarrow 0, \tag{9.3}
\]

where \( F_i \in \text{Add}(R) \) and \( f_i = q_i p_i \) for \( 0 \leq i < n \). We note here that we may assume that all the \( F_i \) (\( 0 \leq i < n \)) have zero differentials, because we can take them up to isomorphisms in \( \mathcal{K}(R) \). (Cf. Theorem 5.8.) Therefore, from now on we assume that the functions \( F_i \) have zero differentials. Hence \( H(F_i) = F_i \) for all \( 0 \leq i < n \). Note also that \( \omega^n_X \) defined in Definition 9.6 is the connecting morphism of the partial \( \text{Add}(R) \)-resolution (9.3).

The following theorem is a partial restatement of Theorem 8.2.

**Theorem 9.7.** Under the circumstances above, there is a triangle in \( \mathcal{K}(R) \):

\[
\Omega^n X[n - 1] \xrightarrow{\varphi_n} \widetilde{F} \xrightarrow{\varphi_n} X \xrightarrow{\omega^n_X} \Omega^n X[n],
\]

where the morphisms \( \varphi_n : \Omega^n X[n - 1] \rightarrow \widetilde{F} \) and \( \varphi_n : \widetilde{F} \rightarrow X \) make the following diagrams commutative:

\[
\begin{array}{ccc}
\Omega^n X[n - 1] & \xrightarrow{\varphi_n} & \widetilde{F} \\
\downarrow q_{n[n-1]} & & \downarrow p_{n[n-1]} \\
F_{n-1}[n - 1], & & X.
\end{array}
\tag{9.4}
\]

**Remark 9.8.** We shall make several remarks on the partial \( \text{Add}(R) \)-resolution (9.3). Firstly, we see from Lemma 7.2 that the following is an exact sequence of graded \( R \)-modules:

\[
0 \longrightarrow H(\Omega^n X) \xrightarrow{H(q_n)} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{H(p_0)} H(X) \longrightarrow 0,
\]

which means that there are exact sequences of \( R \)-modules

\[
0 \longrightarrow H^i(\Omega^n X) \xrightarrow{H(q_n)} F_{n-1}^i \xrightarrow{f_{n-1}^i} \cdots \xrightarrow{f_1^i} F_0^i \xrightarrow{H(p_0)} H^i(X) \longrightarrow 0,
\]

for all \( i \in \mathbb{Z} \). The diagram (9.4) induces the commutative diagram of cohomology modules:

\[
\begin{array}{ccc}
H(\Omega^n X)[n - 1] & \xrightarrow{H(\varphi_n)} & H(\widetilde{F}) \\
\downarrow H(q_{n[n-1]}) & & \downarrow H(p_{n[n-1]}) \\
F_{n-1}[n - 1], & & H(\widetilde{F}).
\end{array}
\]

Since \( H(q_n) \) is injective, so is \( H(\varphi_n) \). Similarly, \( H(\varphi_n) \) is surjective, as \( H(p_0) \) is surjective. As a consequence, it follows from the contracted triangle in Theorem 9.7 that there is an exact sequence of graded \( R \)-modules:

\[
0 \longrightarrow H(\Omega^n X)[n - 1] \xrightarrow{H(\varphi_n)} H(\widetilde{F}) \xrightarrow{H(\varphi_n)} H(X) \longrightarrow 0.
\]
Example 9.9. Let $M$ be a finitely generated $R$-module and

$$Y = \left[ \cdots \longrightarrow P_n \overset{u_n}{\longrightarrow} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \overset{u_1}{\longrightarrow} P_0 \longrightarrow 0 \right]$$ (9.5)

be an $R$-projective resolution of $M$, that is, $Y \in \mathcal{K}(R)$ and there is a quasi-isomorphism $Y \to M$. In this case, it is obvious that $\Omega^n Y$ is the truncated complex $[\cdots \to P_{n+1} \overset{u_n}{\to} P_n \to 0]$, and there is a $\mathcal{K}(R)$-exact sequence

$$0 \to \Omega^n Y \to P_{n-1} \overset{u_{n-1}}{\to} P_{n-2} \to \cdots \to P_1 \overset{u_1}{\to} P_0 \to Y \to 0.$$ In this case, the contraction of this partial Add($R$)-resolution is the complex

$$0 \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to 0,$$

hence it is degenerate.

Even for such natural constructions, we should remark that there are partial Add($R$)-resolutions of the form (9.3) that are not degenerate.

Example 9.10. Let $M$ be a finitely generated $R$-module and $Y$ a projective resolution of $M$ given as in (9.5). We consider a complex of length one;

$$X = \left[ 0 \to P_1 \overset{u_1}{\to} P_0 \to 0 \right].$$

As we remarked in Example 7.12, we see that

$$\Omega^n X = \left[ 0 \to P_{n+1} \overset{u_{n+1}}{\to} P_n \to 0 \right].$$

In fact, set

$$f_{i+1} = \begin{pmatrix} u_{i+1} & 0 \\ 0 & u_{i+3}[1] \end{pmatrix} : F_{i+1} := P_{i+1} \oplus P_{i+3}[1] \to F_i := P_i \oplus P_{i+2}[1]$$

for $i \geq 0$, where each $F_i$ is a complex with zero differential mappings. Furthermore, we set

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & u_2[1] \end{pmatrix} : F_0 = P_0 \oplus P_2[1] \to X = P_0 \oplus P_1[1],$$

and

$$q_n = \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix} : \Omega^n X = P_n \oplus P_{n+1}[1] \to F_{n-1} = P_{n-1} \oplus P_{n+1}[1].$$

Then we have a partial Add($R$)-resolution

$$0 \to \Omega^n X \overset{q_n}{\to} F_{n-1} \overset{f_{n-1}}{\to} \cdots \to F_1 \overset{f_1}{\to} F_0 \overset{p_0}{\to} X \to 0,$$

as in (9.3).
In this example, we can observe that if \( n \geq 3 \), then the partial \( \text{Add}(R) \)-resolution is never degenerate.

For example, in the case \( n = 3 \), setting a graded \( R \)-module homomorphism

\[
g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : F_2[2] = P_2[2] \oplus P_4[3] \longrightarrow F_0[1] = P_0[1] \oplus P_2[2],
\]

we can see that the differential of \( \tilde{F} \) is given by

\[
d_{\tilde{F}} = \begin{pmatrix} 0 & 0 & 0 \\ f_2[2] & 0 & 0 \\ g & f_1[1] & 0 \end{pmatrix} : \tilde{F} = F_2[2] \oplus F_1[1] \oplus F_0 \longrightarrow \tilde{F}[1] = F_2[3] \oplus F_1[2] \oplus F_0[1],
\]

which shows that the sequence is not degenerate. (The reason why this is not degenerate is clear from the following observation: As underlying graded modules, \( X \) and \( \Omega^3 X \) do not contain \( P_2 \) as a direct summand. Therefore, the \( P_2 \) component in \( \tilde{F} \) must split off.)

**Definition 9.11.** We say that a partial \( \text{Add}(R) \)-resolution

\[
\begin{array}{cccccccc}
0 & \longrightarrow & X_n & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & X_0 & \longrightarrow & 0.
\end{array}
\]

is **generically split** if the localized \( \mathcal{K}(R) \)-exact sequence

\[
\begin{array}{cccccccc}
0 & \longrightarrow & S^{-1}X_n & \longrightarrow & S^{-1}F_{n-1} & \longrightarrow & S^{-1}F_{n-2} & \longrightarrow & \cdots & \longrightarrow & S^{-1}F_1 & \longrightarrow & S^{-1}F_0 & \longrightarrow & S^{-1}X_0 & \longrightarrow & 0
\end{array}
\]

is split in \( \mathcal{K}(S^{-1}R) \) in the sense of Definition 9.1(1), where \( S = R \setminus \bigcup_{p \in \text{Ass}(R)} p \).

**Example 9.12.** Let \( a \) be an element of \( R \). Then for the complex \( X_1 := [0 \rightarrow R \overset{a}{\rightarrow} R \rightarrow 0] \) there are triangles:

\[
R \overset{a}{\longrightarrow} R \longrightarrow X_1 \longrightarrow R[1], \quad X_1 \longrightarrow R[1] \overset{a}{\longrightarrow} R[1] \longrightarrow X_1[1].
\]

Hence \( X_0 := R[1] \) has the following type of finite \( \text{Add}(R) \)-resolution:

\[
0 \longrightarrow R \overset{a}{\longrightarrow} R \overset{0}{\longrightarrow} R[1] \overset{a}{\longrightarrow} X_0 = R[1] \longrightarrow 0.
\]

If \( a \) is a non-zero divisor, then this resolution is generically split. However, whenever \( a \) is a non-unit, the sequence is not split and not degenerate.
Example 9.13. Let \( a, b \in R \) and assume that \( a, b \) is a regular sequence on \( R \). Now let

\[
X_0 = \begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R^2 \begin{array}{c}
(a, b)
\end{array} \longrightarrow R \longrightarrow 0, 
\quad X_1 = \begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R \begin{array}{c}
(\frac{b}{a})
\end{array} \longrightarrow R^2 \longrightarrow 0,
\]

and note that \( X_0 \) is *torsion-free but not *reflexive, while \( X_1 \) is not *torsion-free. One can easily see that there is an Add(\( R \))-resolution of \( X_0 \):

\[
\begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R \begin{array}{c}
(\frac{b}{a})
\end{array} \longrightarrow R^2 \begin{array}{c}
\begin{pmatrix}
0 & 0 \\
\ast & \ast
\end{pmatrix}
\end{array} \longrightarrow R[1] \oplus R \begin{array}{c}
p_0
\end{array} \longrightarrow X_0 \longrightarrow 0,
\]

where \( p_0, p_1 \) are chain maps defined, respectively, as

\[
\begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R \longrightarrow 0 
\quad \begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R \longrightarrow 0,
\]

\[
\begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R^2 \begin{array}{c}
(a, b)
\end{array} \longrightarrow 0 
\quad \begin{array}{c}
0 \\
\longrightarrow
\end{array} \longrightarrow R \begin{array}{c}
(\frac{b}{a})
\end{array} \longrightarrow 0.
\]

It is easy to see that \( p_0 \) and \( p_1 \) are right Add(\( R \))-approximations, hence \( \Omega X_0 = X_1 \) and \( \Omega^2 X_0 = 0 \) in \( \mathcal{H}(R) \). We should note that the Add(\( R \))-resolution above may not be a split sequence, but may generically split.

10 | COUNIT MORPHISM FOR THE ADJOINT PAIR (\( \Sigma^n, \Omega^n \))

It follows from Theorem 7.11 that there is an isomorphism

\[
\Hom_{\mathcal{K}(R)}(\Sigma^{n-i} \Omega^n X, \Omega^i X) \cong \Hom_{\mathcal{K}(R)}(\Omega^n X, \Omega^n X),
\]

(10.1)

for all \( X \in \mathcal{K}(R) \) and \( 0 \leq i \leq n \). Thus we can take a morphism in \( \mathcal{K}(R) \);

\[
\pi^{(n,i)}_X : \Sigma^{n-i} \Omega^n X \rightarrow \Omega^i X
\]

which yields a unique element of \( \Hom_{\mathcal{K}(R)}(\Sigma^{n-i} \Omega^n X, \Omega^i X) \) that corresponds to the identity on \( \Omega^n X \) in the right-hand side of (10.1).

If \( i = 0 \), then \( \pi^{(n,0)}_X \in \Hom_{\mathcal{K}(R)}(\Sigma^n \Omega^n X, X) \) is a counit morphism for the adjoint pair \( (\Sigma^n, \Omega^n) \).

If \( i = n \), then \( \pi^{(n,n)}_X \) is the identity on \( \Omega^n X \).

Adding an Add(\( R \))-summand to \( \Sigma^{n-i} \Omega^n X \) if necessary, we may take the morphism \( \pi^{(n,i)}_X \) as cohomologically surjective. Under such a circumstance, we make a triangle

\[
\Delta^{(n,i)}(X) \longrightarrow \Sigma^{n-i} \Omega^n X \begin{array}{c}
\pi^{(n,i)}_X
\end{array} \longrightarrow \Omega^i X \longrightarrow \Delta^{(n,i)}(X)[1]
\]

and define \( \Delta^{(n,i)}(X) \in \mathcal{K}(R) \) by this triangle.
Note that there is a short exact sequence of graded $R$-modules;

$$0 \to H(\Delta^{(n,i)}(X)) \to H(\Sigma^{n-1} \Omega^n X) \to H(\Omega^i X) \to 0,$$

for all $X \in \mathcal{K}(R)$ and $0 \leq i \leq n$.

Since $\pi^{(n,i)}_X$ is uniquely determined as a morphism in $\mathcal{K}(R)$, Theorem 6.6 implies the following lemma.

**Lemma 10.1.** For each $X \in \mathcal{K}(R)$ and positive integers $0 \leq i \leq n$, the complex $\Delta^{(n,i)}(X)$ defined above is uniquely determined as an object of $\mathcal{K}(R)$. □

As in Section 9, we have triangles of the form;

$$\Omega^{i+1} X \xrightarrow{\eta_{i+1}} F_i \xrightarrow{p_i} \Omega^i X \to \Omega^{i+1} X[1],$$

$$\Sigma' \Omega^n X \xrightarrow{q^{n-i}} G_{n-i-1} \xrightarrow{p_{n-i-1}} \Sigma^{i+1} \Omega^n X \to \Sigma^i \Omega^n X[1],$$

where $F_i$, $G_{n-i-1} \in \text{Add}(R)$ and $p_i$ (respectively, $q^{n-i}$) is a right (respectively, left) $\text{Add}(R)$-approximation for all $0 \leq i < n$.

Since each $q^{i+1}$ is a left $\text{Add}(R)$-approximation, setting as $v_n$ the identity morphism on $\Omega^n X$, by induction on $n-i$, we find morphisms $v_i : \Sigma^{n-i} \Omega^n X \to \Omega^i X$ and $a_i : G_i \to F_i$ that make the following diagrams commutative:

for $0 \leq i < n$. Here we can take such $a_i$ to be surjective graded $R$-module homomorphisms. Actually, if we add $F_i$ to $G_i$ and $\Sigma^{n-i} \Omega^n X$ as a direct summand, then the following diagram is also commutative.

Replacing $a_i$ by $(a_i, 1)$, we assume that all functions $a_i$ are surjective $R$-module homomorphisms. Then, since $p_i a_i$ is cohomologically surjective, we see that $v_i$ is also cohomologically surjective. Therefore, we may take all such $v_i$ to be equal to $\pi^{(n,i)}_X$ for $0 \leq i < n$.

Thus we have a commutative diagram in which the rows are $\mathcal{K}(R)$-exact sequences;

$$0 \to \Omega^n X \xrightarrow{q_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{p_0} X \xrightarrow{\pi^{(n,0)}_X} 0$$

and

$$0 \to \Omega^n X \xrightarrow{q^n} G_{n-1} \xrightarrow{\xi^{n-1}} \cdots \xrightarrow{\xi^1} G_0 \xrightarrow{\rho^0} \Sigma^n \Omega^n X \xrightarrow{\pi^{(n,0)}_X} 0.$$
where \( F_i, G_i \in \text{Add}(R) \), \( f_i = q_i p_i \) and \( q^j = q^i p^j \) for \( 1 \leq i < n \). This diagram is divided into two commutative diagrams whose rows are \( \mathcal{K}(R) \)-exact sequences as well:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^j X & \rightarrow & F_{j-1} & \rightarrow & \cdots & \rightarrow & F_0 & \rightarrow & P_0 & \rightarrow & X & \rightarrow & 0 \\
& & \downarrow \sigma_{X}^{(n,i)} & & \downarrow a_{j-1} & & \cdots & & \downarrow a_0 & & \downarrow \sigma_{X}^{(n,0)} & & \downarrow \sigma_{X}^{(n,0)} & & 0, \\
0 & \rightarrow & \Sigma^{j-1} \Omega^n X & \rightarrow & G_{j-1} & \rightarrow & \cdots & \rightarrow & G_0 & \rightarrow & P^0 & \rightarrow & \Sigma^n \Omega^n X & \rightarrow & 0.
\end{array}
\tag{10.3}
\]

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^n X & \rightarrow & F_{n-1} & \rightarrow & \cdots & \rightarrow & F_i & \rightarrow & P_i & \rightarrow & \Omega^i X & \rightarrow & 0 \\
& & \downarrow a_{n-1} & & \downarrow a_{i-1} & & \cdots & & \downarrow a_i & & \downarrow \sigma_{X}^{(n,i)} & & \downarrow \sigma_{X}^{(n,i)} & & 0, \\
0 & \rightarrow & \Omega^n X & \rightarrow & G_{n-1} & \rightarrow & \cdots & \rightarrow & G_i & \rightarrow & P^i & \rightarrow & \Sigma^{n-i} \Omega^n X & \rightarrow & 0.
\end{array}
\tag{10.4}
\]

Now set \( L_i = \text{Ker} \ a_i \) the kernel as a graded \( R \)-module homomorphism for \( 0 < i \leq n \). Since each \( a_i \) is surjective as a graded \( R \)-module homomorphism, we see that \( L_i \in \text{Add}(R) \). Then the successive use of octahedron axiom to the diagram (10.4) will show that there is a commutative diagram whose columns are triangles and rows are \( \mathcal{K}(R) \)-exact sequences:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^n X & \rightarrow & F_{n-1} & \rightarrow & \cdots & \rightarrow & F_i & \rightarrow & P_i & \rightarrow & \Omega^i X & \rightarrow & 0 \\
& & \downarrow a_{n-1} & & \downarrow a_{i-1} & & \cdots & & \downarrow a_i & & \downarrow \sigma_{X}^{(n,i)} & & \downarrow \sigma_{X}^{(n,i)} & & 0, \\
0 & \rightarrow & \Omega^n X & \rightarrow & G_{n-1} & \rightarrow & \cdots & \rightarrow & G_i & \rightarrow & P^i & \rightarrow & \Sigma^{n-i} \Omega^n X & \rightarrow & 0.
\end{array}
\tag{10.5}
\]

In fact, we prove by induction on \( n - i \) that the third row of the diagram (10.5) is a \( \mathcal{K}(R) \)-exact sequence. If \( n - i = 1 \), then the following octahedron diagram proves this.

\[
\begin{array}{c}
L_{n-1} \xrightarrow{\cong} \Delta^{(n,n-1)}(X) \\
\Omega^n X \xrightarrow{q_n} F_{n-1} \xrightarrow{p_{n-1}} \Omega^{n-1} X \rightarrow \Omega^n X[1] \\
\xrightarrow{a_{n-1}} \Omega^n X \xrightarrow{q^n} G_{n-1} \xrightarrow{p^{n-1}} \Sigma \Omega^n X \rightarrow \Omega^n X[1] \\
\xrightarrow{b^{n-1}} L_{n-1} \xrightarrow{\cong} \Delta^{(n,n-1)}(X)
\end{array}
\]
If \( n - i \geq 2 \), then applying the induction hypothesis, we see that in the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Omega^n X & \xrightarrow{\partial_n} & F_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_{i+2}} & F_{i+1} & \xrightarrow{p_{i+1}} & \Omega^{i+1} X & \rightarrow & 0 \\
0 & \rightarrow & \Omega^n X & \xrightarrow{\partial^n} & G_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_{i+2}} & G_{i+1} & \xrightarrow{p_{i+1}} & \Sigma^{n-i-1} \Omega^n X & \rightarrow & 0 \\
0 & \rightarrow & L_{n-1} & \xrightarrow{\epsilon_{n-1}} & \cdots & \xrightarrow{\epsilon_{i+2}} & L_{i+1} & \xrightarrow{\Delta^{n,i+1}(X)} & 0
\end{array}
\]

the third row is \( H(R) \)-exact. On the other hand, there is a commutative diagram where all rows and columns are triangles:

\[
\begin{array}{ccccccccc}
\Delta^{(n,i+1)}(X)[1] & \rightarrow & L_i[1] & \rightarrow & \Delta^{(n,i)}(X)[1] \\
\Omega^{i+1} X & \xrightarrow{q_{i+1}} & F_i & \xrightarrow{p_i} & \Omega^i X \\
\Sigma^{n-i-1} \Omega^n X & \xrightarrow{\varphi^{n(i+1)}} & G_i & \xrightarrow{p_i'} & \Sigma^{n-i} \Omega^n X \\
\Delta^{(n,i+1)}(X) & \rightarrow & L_i & \rightarrow & \Delta^{(n,i)}(X)
\end{array}
\]

(First we take the middle left square with \( a_i \) being surjective, then apply the nine lemma (cf. [18, Exercise 10.2.6, p. 378]) to get this commutative diagram.)

In particular, we have a \( H(R) \)-exact sequence

\[
0 \rightarrow \Delta^{(n,i+1)}(X) \rightarrow L_i \rightarrow \Delta^{(n,i)}(X) \rightarrow 0.
\]

Combining the sequences above, we finally obtain the \( H(R) \)-exact sequence:

\[
0 \rightarrow L_{n-1} \xrightarrow{\epsilon_{n-1}} L_{n-2} \rightarrow \cdots \xrightarrow{\epsilon_{i+1}} L_i \rightarrow \Delta^{(n,i)}(X) \rightarrow 0. \quad (10.6)
\]

This proves that all the rows in the diagram (10.5) are \( H(R) \)-exact sequences.

Letting \( L^{(n,i)} \) be the contraction of the Add\((R)\)-resolution (10.6), we have the isomorphism \( \Delta^{(n,i)}(X) \cong L^{(n,i)} \). We have thus proved the following theorem.

**Theorem 10.2.** Let \( X \in H(R) \) and \( 0 \leq i \leq n \). Then \( \Delta^{(n,i)}(X) \) has a finite Add\((R)\)-resolution of length \( n - i - 1 \). In this case, \( \Delta^{(n,i)}(X) \) is isomorphic in \( H(R) \) to the contraction of such a finite Add\((R)\)-resolution.

**Remark 10.3.**

(1) If \( R \) is a Gorenstein ring of dimension zero (that is, a self-injective algebra), then we can take all the \( a_i \) to be isomorphisms and hence one can take \( L_i = 0 \) for all \( 0 \leq i < n \). Thus we have \( \Delta^{(n,i)}(X) = 0 \) or \( \Delta^{(n,i)}(X) \in \text{Add}(R) \) for all \( 0 \leq i < n \). Moreover, for any choice of \( a_i \) the sequence (10.6) is a split sequence in this case. See Corollary 7.7 and Remark 7.9.
(2) Let $S$ be a multiplicatively closed subset of $R$. Then localizing by $S$ and going through the whole procedure again leads to the localization of the sequence (10.6). As a consequence, we observe an isomorphism

$$S^{-1} \Delta^{(n,i)}_{R}(X) \cong \Delta^{(n,i)}_{S^{-1}R}(S^{-1}X),$$

in the stable category $\mathcal{K}(S^{-1}R)$ for all $0 \leq i < n$, and the localized sequence of (10.6) by $S$ is an $\text{Add}(S^{-1}R)$-resolution of $\Delta^{(n,i)}_{S^{-1}R}(S^{-1}X)$.

By this remark, if $R$ is a generically Gorenstein ring, then the $\text{Add}(R)$-resolution (10.6) is generically split. Thus we have proved the following theorem.

**Theorem 10.4.** Let $R$ be a generically Gorenstein ring. For any $X \in \mathcal{K}(R)$ and $0 \leq i \leq n$, $\Delta^{(n,i)}(X)$ has a finite $\text{Add}(R)$-resolution of length $n - i - 1$ that is generically split.

11 | THE MAIN THEOREM AND THE PROOF

The following lemma is one of the most essential observations to prove the main theorem.

**Lemma 11.1.** Let $R$ be a generically Gorenstein ring, and let $X \in \mathcal{K}(R)$. If $H(X^*) = 0$, then $\Omega^r X$ is torsion-free for each non-negative integer $r$.

To prove this lemma, we prepare several preliminary lemmas.

**Lemma 11.2.** Let $X$ be a complex in $\mathcal{K}(R)$ and assume that $H(X^*) = 0$. Then we have $\text{Hom}_{\mathcal{K}(R)}(X,F) = 0$ for all $F \in \text{Add}(R)$.

**Proof.** Note that $H(X^*)$ is the cohomology module of the complex $\text{Hom}_{R}(X,R)$, hence we have the equality $H(X^*) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{K}(R)}(X,R[i])$. Thus if $H(X^*) = 0$, then we see that $\text{Hom}_{\mathcal{K}(R)}(X,P[i]) = 0$ for any finitely generated projective $R$-module $P$ and an integer $i$. Recall from Theorem 5.8 and Proposition 5.7 that any complex $F \in \text{Add}(R)$ is isomorphic to a direct sum $\bigoplus_{i \in \mathbb{Z}} F^i[-i]$ with $F^i$ being a finitely generated projective $R$-module for each $i \in \mathbb{Z}$. On the other hand, it follows from Lemma 5.1 the direct sum is a product in $\mathcal{K}(R)$. Hence $\text{Hom}_{\mathcal{K}(R)}(X,F) = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{K}(R)}(X,F^i[-i]) = 0$ as desired. \hfill \Box

**Lemma 11.3.** Let $X,Y \in \mathcal{K}(R)$. Assume the following conditions.

1. $Y$ has an $\text{Add}(R)$-resolution of finite length.
2. $H(X^*) = 0$.

Then we have $\text{Hom}_{\mathcal{K}(R)}(X,Y) = 0$.

**Proof.** This is obvious from the previous lemma and utilizing the induction on the length $\ell$ of the $\text{Add}(R)$-resolution of $Y$. In fact, if $\ell = 0$, then $Y \in \text{Add}(R)$ hence $\text{Hom}_{\mathcal{K}(R)}(X,Y) = 0$ by Lemma 11.2. If $\ell > 0$, then there is a triangle

$$Y' \longrightarrow F_0 \longrightarrow Y \longrightarrow Y'[1],$$

where \( F_0 \in \text{Add}(R) \) and \( Y' \) has an \( \text{Add}(R) \)-resolution of length \( \ell' - 1 \). Thus \( \text{Hom}_{\mathcal{H}(R)}(X, Y'[i]) = 0 \) for all \( i \in \mathbb{Z} \) by the induction hypothesis. Since there is an exact sequence of \( R \)-modules:

\[
\text{Hom}_{\mathcal{H}(R)}(X, F_0) \longrightarrow \text{Hom}_{\mathcal{H}(R)}(X, Y) \longrightarrow \text{Hom}_{\mathcal{H}(R)}(X, Y'[1]),
\]

which results that \( \text{Hom}_{\mathcal{H}(R)}(X, Y) = 0 \).

Now we proceed to the proof of Lemma 11.1.

In this proof, we assume that \( R \) is generically Gorenstein and \( H(X^*) = 0 \). It is clear that \( X \) is *torsion-free, since \( H(X^*) = 0 \rightarrow H(X)^* \) is injective. We shall prove that so is \( \Omega^r X \) for \( r \geq 1 \).

Let \( n \geq 2 \) be an integer. We have the following commutative diagram from (10.5):

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Omega^n X & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^n X & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & \Sigma^n \Omega^n X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{n-1} & \longrightarrow & \cdots & \longrightarrow & L_0 & \longrightarrow & \Delta^{(n,0)}(X) & \longrightarrow & 0,
\end{array}
\]

where the rows are \( \mathcal{H}(R) \)-exact sequences and the columns are triangles. Taking the contracted triangles of the rows, we obtain the following commutative diagram whose rows and columns are triangles:

\[
\begin{array}{cccccccc}
\tilde{L}[1] & \equiv & \Delta^{(n,0)}(X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^n X[n - 1] & \longrightarrow & \tilde{F} & \longrightarrow & X & \longrightarrow & \Omega^n X[n] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^n X[n - 1] & \longrightarrow & \tilde{G} & \longrightarrow & \Sigma^n \Omega^n X & \longrightarrow & \Omega^n X[n] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\tilde{L} & \equiv & \Delta^{(n,0)}(X),
\end{array}
\]

where \( \tilde{a} \) and \( \tilde{b} \) are the induced morphisms from \( \{a_i\} \) and \( \{b_i\} \), respectively. See Definition 8.4 and Theorem 8.5. (In fact, to see the second column is a triangle, note all the other columns and rows are triangles and apply the octahedron axiom.) We know from Theorem 10.2 that \( \Delta^{(n,0)}(X)[1] \) has a finite \( \text{Add}(R) \)-resolution. Hence it follows from Lemma 11.3 that \( \sigma \) in the diagram is zero. Thus \( \lambda \) is also zero in the diagram by the commutativity of the upper square. This means that the second and the third columns are split triangles, hence \( \tilde{a} \) and \( \pi_X^{(n,0)} \) have right inverses. Note from this that \( \tilde{L} \), and hence \( \Delta^{(n,0)}(X) \) as well, is *torsion-free, since it is a direct summand of \( \Sigma^n \Omega^n X \). See Lemma 7.13 and Theorem 7.17.
Taking all functions $F_i$ and $G_i$ to have no null complex as direct summands, we note that the following diagram is commutative (cf. Theorem 8.5).

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{pr_{n-1}^F} & F_{n-1}[n-1] \\
\tilde{a} & & a_{n-1}[n-1] \\
\tilde{G} & \xrightarrow{pr_{n-1}^G} & G_{n-1}[n-1] \\
\tilde{b} & & b_{n-1}[n-1] \\
\tilde{L} & \xrightarrow{pr_{n-1}^L} & L_{n-1}[n-1] \\
\end{array}
\] (11.3)

We shall now prove that $pr_{n-1}^L = 0$ in $\mathcal{K}(\mathcal{R})$.

To prove this, we note from Theorem 10.4 that $\tilde{L}$ has an Add($\mathcal{R}$)-resolution of the form (10.6) that is generically split. Therefore, we see from Corollary 9.3 that $S^{-1}(pr_{n-1}^L) = 0$ in $\mathcal{K}(S^{-1} \mathcal{R})$, where $S$ is the set of all non-zero divisors in $\mathcal{R}$ as before. Since $\tilde{L}$ is *torsion-free and $L_{n-1} \in$ Add($\mathcal{R}$), it follows from Theorem 5.11 that $pr_{n-1}^L = 0$ in $\mathcal{K}(\mathcal{R})$ as desired.

Then we have $pr_{n-1}^G \tilde{b} = 0$ by the commutativity of the diagram (11.3). Hence there is a morphism $e : \tilde{F} \rightarrow G_{n-1}[n-1]$ in $\mathcal{K}(\mathcal{R})$ such that $e \tilde{a} = pr_{n-1}^G$.

Let $\tilde{F}'$ and $\tilde{G}'$ be the contractions of the partial Add($\mathcal{R}$)-resolutions appearing in the following diagram, thus $\tilde{F}' = F_{n-2}[n-2] \oplus \cdots \oplus F_0 \subseteq \tilde{F} = F_{n-1}[n-1] \oplus F_{n-2}[n-2] \oplus \cdots \oplus F_0$ and the same for $\tilde{G}'$.

\[
\begin{array}{cccccccc}
0 & \rightarrow & \Omega^{n-1} X & \xrightarrow{q_{n-1}} & F_{n-2} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_1} & F_0 & \xrightarrow{p_0} & X & \rightarrow & 0 \\
& & \pi_X^{(n-1)} & & \sigma_{n-2} & & & & \alpha_0 & & \pi_X^{(n)} & & \\
0 & \rightarrow & \Sigma \Omega^n X & \xrightarrow{q^n} & G_{n-2} & \xrightarrow{g_{n-2}} & \cdots & \xrightarrow{g_1} & G_0 & \xrightarrow{p^0} & \Sigma \Omega^n X & \rightarrow & 0.
\end{array}
\] (11.4)

We note that $\tilde{F}'$ and $\tilde{G}'$ are subcomplexes of $\tilde{F}$ and $\tilde{G}$, respectively. Recall that $\tilde{a}$ is a splitting epimorphism in $\mathcal{K}(\mathcal{R})$ and it is represented by a lower triangular matrix whose diagonal entries are $a_{n-1}, \ldots, a_0$ which are all split epimorphisms of graded $\mathcal{R}$-modules. Thus we have that $\tilde{F}' = \tilde{a}(\tilde{G}')$.

There is a diagram whose rows are triangles and squares are commutative;

\[
\begin{array}{ccc}
\tilde{F}' & \xrightarrow{pr_{n-1}^G} & F_{n-1}[n-1] \\
\tilde{a}' & & a_{n-1}[n-1] \\
\tilde{G}' & \xrightarrow{pr_{n-1}^G} & G_{n-1}[n-1] \\
\tilde{a} & & a_{n-1}[n-1] \\
\end{array}
\] (11.5)

Since $pr_{n-1}^G(\tilde{G}') = 0$, we have $e(\tilde{F}') = e\tilde{a}(\tilde{G}') = pr_{n-1}^G(\tilde{G}') = 0$. Thus $e$ induces a morphism $f : F_{n-1}[n-1] \rightarrow G_{n-1}[n-1]$ such that $e = f pr_{n-1}^G$. Hence it holds that $f pr_{n-1}^G \tilde{a} = pr_{n-1}^G \tilde{a}$.

Now recalling in the diagram (11.1) that $q_n[n-1] = pr_{n-1}^F \psi_n$ and $q^n[n-1] = pr_{n-1}^G \psi_n$ by Theorem and Definition 8.2, we have equalities;

\[
q_n[n-1] = pr_{n-1}^G \psi_n = f pr_{n-1}^G \tilde{a} \psi_n = f pr_{n-1}^F \psi_n = f q_n[n-1],
\]

where we use the commutative diagram (11.2) for the third equality.
This shows the commutativity of the following diagram in which the rows are triangles:

\[
\begin{array}{cccccc}
\Omega^n X[n-1] & \xrightarrow{q[n-1]} & F_{n-1}[n-1] & \xrightarrow{f} & \Omega^{n-1} X[n-1] \\
\Omega^n X[n-1] & \xrightarrow{q[n-1]} & G_{n-1}[n-1] & \xrightarrow{} & \Sigma \Omega^n X[n-1]
\end{array}
\]

Recall that \(q[n-1]\) is a left Add(\(R\))-approximation. Then it is easy to see from Remark 7.6 that \(q[n-1]\) is also a left Add(\(R\))-approximation. As a consequence of this, we have \(\Omega^{n-1} X[n-1] \cong \Sigma \Omega^n X[n-1]\) in \(\mathcal{X}(R)\) (cf. Theorem 6.6). Thus \(\Omega^{n-1} X\) is *torsion-free by Theorem 7.14. Since \(n\) is any integer not less than 2, this completes the proof of Lemma 11.1.

Lemma 11.1 can be strengthened as in the following form.

**Theorem 11.4.** Let \(R\) be a generically Gorenstein ring. Assume \(H(X^*) = 0\) for \(X \in \mathcal{X}(R)\). Then \(\Omega^r X\) is *reflexive for any non-negative integer \(r\).

**Proof.** Let \(r\) be a non-negative integer. Note from the definition that there is a triangle

\[
\Omega^{r+1} X \xrightarrow{q} F_r \xrightarrow{p} \Omega' X \xrightarrow{} \Omega^{r+1} X[1],
\]

where \(p\) is a right Add(\(R\))-approximation. Hence the sequence of graded \(R\)-modules

\[
0 \rightarrow H(\Omega^{r+1} X) \xrightarrow{H(q)} H(F_r) \xrightarrow{H(p)} H(\Omega' X) \rightarrow 0
\]

is exact. Given a graded \(R\)-module homomorphism \(\alpha : H(\Omega' X) \rightarrow R[i]\) for some \(i \in \mathbb{Z}\), we find a morphism \(b : F_r \rightarrow R[i]\) in \(\mathcal{X}(R)\) with \(H(b) = \alpha H(p)\), since \(F_r\) is *reflexive. Then we have \(H(bq) = H(b)H(q) = \alpha H(pq) = 0\). As we have shown in Lemma 11.1, \(\Omega^{r+1} X\) is *torsion-free, we have that \(bq = 0\). (See Lemma 3.2 and also Theorem 5.8.) Then there is an \(a : \Omega' X \rightarrow R[i]\) that satisfies \(b = ap\). Since \(H(p)\) is a surjection, it thus follows that \(H(a) = \alpha\). Then one can apply Lemma 3.2(2) to conclude that \(\Omega' X\) is *reflexive. \(\square\)

**Proposition 11.5.** Let \(Y \in \mathcal{X}(R)\). Assume the following conditions are satisfied.

1. \(Y\) is *torsion-free.
2. \(\Omega Y\) is *reflexive.

Then we have \(\text{Ext}^1_R(H(Y), R) = 0\).

**Proof.** From the definition of \(\Omega Y\), there is a triangle in \(\mathcal{X}(R)\);

\[
\Omega Y \xrightarrow{q} F \xrightarrow{p} Y \xrightarrow{\omega} \Omega Y[1],
\]

where \(p\) is a right Add(\(R\))-approximation. Hence there is an exact sequence of graded \(R\)-modules

\[
0 \rightarrow H(\Omega Y) \xrightarrow{H(q)} H(F) \xrightarrow{H(p)} H(Y) \rightarrow 0,
\]
where $H(F)$ is a projective graded $R$-module. Thus we have an exact sequence

$$
0 \to H(Y)^* \xrightarrow{H(p)^*} H(F)^* \xrightarrow{H(q)^*} H(\Omega Y)^* \to \text{Ext}^1_R(H(Y), R) \to 0.
$$

On the other hand, we also have a triangle

$$
Y^* \xrightarrow{p^*} F^* \xrightarrow{q^*} (\Omega Y)^* \xrightarrow{\omega^*[1]} Y^*[1].
$$

Therefore, we have the following commutative diagram of $R$-modules whose rows are exact sequences of graded $R$-modules:

$$
\begin{array}{c}
H((\Omega Y)^*)[-1] \xrightarrow{H(\omega^*)} H(Y)^* \xrightarrow{H(p)^*} H(F)^* \xrightarrow{H(q)^*} H((\Omega Y)^*) \xrightarrow{H(\omega^*[1])} H(Y)^*[1] \\
0 \xrightarrow{0} H(Y)^* \xrightarrow{H(p)^*} H(F)^* \xrightarrow{H(q)^*} H((\Omega Y)^*) \xrightarrow{\text{Ext}^1_R(H(Y), R)} 0.
\end{array}
$$

Since $Y$ is $^*$torsion-free, $\rho_{Y^*}$ is injective and hence so is $H(p^*)$. It thus follows that $H(\omega^*) = 0$. Then we must have that $H(q^*)$ is surjective. Since $\Omega Y$ is $^*$reflexive, $\rho_{Y^*}$ is bijective. As a result, we have that $H(q^*)$ is surjective as well as $H(\omega^*)$. Thus it is concluded from the exactness of the second row that $\text{Ext}^1_R(H(Y), R) = 0$.

Combining Proposition 11.4 with Proposition 11.5, we obtain the following proposition that is a key for the proof of Theorem 11.7.

**Proposition 11.6.** Let $R$ be a generically Gorenstein ring, and assume that $H(X^*) = 0$ for $X \in \mathcal{X}(R)$. Then we have

$$
\text{Ext}^r_R(H(X), R) = 0,
$$

for all $r > 0$.

**Proof.** Recall from the argument after Theorem 9.7 that one can take a partial $\text{Add}(R)$-resolution of $X$

$$
0 \to \Omega^r X \xrightarrow{q_r} F_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{p_0} X \to 0,
$$

where each functions $F_i$ contains no null complex as a direct summand. Then it induces an exact sequence of graded $R$-modules

$$
0 \to H(\Omega^r X) \xrightarrow{H(q_r)} H(F_{r-1}) \xrightarrow{H(f_{r-1})} \cdots \xrightarrow{H(f_1)} H(F_0) \xrightarrow{H(p_0)} H(X) \to 0,
$$

with $H(F_i) = F_i$ for all $0 \leq i \leq r - 1$. Since $R$ is generically Gorenstein and $H(X^*) = 0$, it follows from Theorem 11.4 that $\Omega^r X$ is $^*$reflexive for each $r > 0$. Note also that $X$ is $^*$torsion-free by Lemma 11.1. Then, by Proposition 11.5, we have $\text{Ext}^1_R(H(\Omega^r X), R) = 0$ for all $r > 0$. Thus, it follows from the long exact sequence above of graded $R$-modules that $\text{Ext}^r_R(H(X), R) = 0$ for $r > 0$.

\[\square\]
The following is the main theorem of this paper, which we can now prove as a result of the previous theorems and propositions.

**Theorem 11.7.** Let $R$ be a generically Gorenstein ring, and let $X \in \mathcal{K}(R)$. Then, $H(X) = 0$ if and only if $H(X^*) = 0$.

**Proof.** We have only to prove that if $H(X^*) = 0$, then $H(X) = 0$ under the assumption that $R$ is generically Gorenstein. The other implication follows from this by the duality $X^{**} \cong X$. Thus in this proof we assume that $H(X^*) = 0$ and our aim is to show that $H(X) = 0$.

(1st step): We may assume that $(R, m, k)$ is a local ring, which is generically Gorenstein. Furthermore, we may assume that $\dim R > 0$.

Note that $H(X) = 0$ if and only if $H(X_m) = H(X)_m = 0$ for all maximal ideals $m$ of $R$, and that $\text{Hom}_R(X, R)_m = \text{Hom}_{R_m}(X_m, R_m)$. It is also obvious that if $R$ is generically Gorenstein, then so are all $R_m$. The first half is clear from these observations.

If $\dim R = 0$, then $R$ is a Gorenstein ring by the generic Gorenstein assumption and the theorem is trivial in this case, since $R$ is an injective $R$-module. Hence we may avoid this case.

(2nd step): We may assume that $mH(X) = 0$.

To show this, let $x \in m$ and consider the Koszul complex

$$K(x) = \begin{array}{c}
0 \rightarrow R \xrightarrow{x} R \rightarrow 0
\end{array}.$$

Set $X' = X \otimes_R K(x)$, and since there is a triangle $\xrightarrow{x} X \rightarrow X' \rightarrow X[1]$ in $\mathcal{K}(R)$, we have the equivalence $H(X) = 0 \iff H(X') = 0$ by Nakayama Lemma. Taking the dual of the triangle above, we have a triangle $(X')^* \rightarrow X^* \rightarrow (X')^*[1]$, thus there is an isomorphism $(X')^*[1] \cong X^* \otimes_R K(x)$. Therefore, we see that $H(X^*) = 0 \iff H(X''') = 0$ as well.

Now take a generating set $x_1, \ldots, x_m$ of the maximal ideal $m$, and consider the Koszul complex $X'' = X \otimes_R K(x_1, \ldots, x_m) = X \otimes_R K(x_1) \otimes_R \cdots \otimes_R K(x_m)$. Then we have the equivalences $H(X'') = 0 \iff H(X) = 0$, and also $H(X'''') = 0 \iff H(X^*) = 0$. Thus it is enough to show that $H(X'''') = 0$ implies $H(X'') = 0$. It is also clear that for any element $x \in m$, the multiplication map on $X''$ is trivial in $\mathcal{K}(R)$, hence $mH(X'') = 0$.

(3rd step): Now assume that $H(X) \neq 0$. Then there is an integer $i$ with $H^i(X) \neq 0$. By the second step of this proof, $H^i(X)$ is a non-trivial $k$-module, where $k = R/m$. On the other hand, we have shown in Proposition 11.6 that $\text{Ext}_R^r(H(X), k) = 0$ for all $r > 0$ under the condition that $H(X^*) = 0$. Therefore, we have

$$\text{Ext}_R^r(k, R) = 0 \quad \text{for all } r > 0.$$

This requires that $R$ is a Gorenstein ring of dimension zero, which is not the case by the first step. Hence, $H(X) = 0$ and the proof of the theorem is completed. \hfill \Box

## 12 APPLICATIONS

Recall that a chain homomorphism $f$ between complexes is called a quasi-isomorphism if the cohomology mapping $H(f)$ is an isomorphism of modules.
Theorem 12.1 (Corollary 1.2). Assume that the ring $R$ is a generically Gorenstein ring. Let $f : X \to Y$ be a morphism in $\mathcal{K}(R)$. Then, $f$ is a quasi-isomorphism if and only if the $R$-dual $f^* : Y^* \to X^*$ is a quasi-isomorphism.

Proof. In fact, let $X \xrightarrow{f} Y \to Z \to X[1]$ be a triangle in $\mathcal{K}(R)$. Then $f$ is a quasi-isomorphism if and only if $H(Z) = 0$. We have shown in Theorem 11.7 that $H(Z) = 0$ if and only if $H(Z^*) = 0$. Since $Z^*[-1] \to Y^* \xrightarrow{f^*} X^*$ is a triangle, Theorem 12.1 follows.

Now we recall the definition of totally reflexive modules. A finitely generated module $M$ over a commutative Noetherian ring $R$ is called a totally reflexive module or a module of G-dimension zero if $\text{Ext}^i_R(M, R) = 0$ for all $i > 0$. See [2, (3.8)] or [14]. This is equivalent to the following three conditions.

(i) $M$ is reflexive, that is, the natural mapping $M \to M^{**}$ is bijective.
(ii) $\text{Ext}^i_R(M, R) = 0$ for all $i > 0$.
(iii) $\text{Ext}^i_R(M^*, R) = 0$ for all $i > 0$.

See [2] for the detail on totally reflexive modules. The following theorem says that condition (ii) is sufficient for totally reflexivity if the ring $R$ is generically Gorenstein.

Theorem 12.2 (Corollary 1.3). Assume that the ring $R$ is a generically Gorenstein ring. Let $M$ be a finitely generated $R$-module. Then the following conditions are equivalent.

(1) $M$ is a totally reflexive $R$-module.
(2) $\text{Ext}^i_R(M, R) = 0$ for all $i > 0$.
(3) $M$ is an infinite syzygy, that is, there is an exact sequence of infinite length of the form $0 \to M \to P_0 \to P_1 \to P_2 \to \cdots$, where each $P_i$ is a finitely generated projective $R$-module.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are well known and easily proved. For example, see [2, Proposition (3.8)].

(2) $\Rightarrow$ (1): Take projective resolutions for $M$ and $M^*$, respectively, as

$$
\cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \xrightarrow{0} \cdots \quad \text{and} \quad \cdots \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M^* \xrightarrow{0} \cdots.
$$

Then we consider the complex

$$
X = \left[ \cdots \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} F_0^* \xrightarrow{f_0^*} F_1^* \xrightarrow{f_1^*} F_2^* \xrightarrow{f_2^*} \cdots \right],
$$

which belongs to $\mathcal{K}(R)$, and acyclic by the condition (2). Hence by Theorem 11.7, the dual $X^*$ is acyclic as well. Since

$$
X^* = \left[ \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{g_0^*} G_0^* \xrightarrow{g_1^*} G_1^* \xrightarrow{g_2^*} \cdots \right],
$$

is an exact sequence, it follows that $M \cong M^{**}$ and $\text{Ext}^i_R(M^*, R) = 0$ for $i > 0$. 
As in (3) we assume that there is an exact sequence
\[ 0 \rightarrow M \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots. \]

Then combining this with the projective resolution \[ \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \] of \( M \), we have an acyclic complex in \( \mathcal{X}(R) \)
\[ Y = \left[ \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \right]. \]

It follows from Theorem 11.7 that \( Y^* \) is acyclic again. In particular, the sequence \( F_0^* \rightarrow F_1^* \rightarrow F_2^* \rightarrow \cdots \) is exact, and hence \( \text{Ext}^i_R(M, R) = 0 \) for \( i > 0 \).

Recall that a finitely generated module \( M \) has the G-dimension at most \( n \), denoted by \( \text{G-dim}_R M \leq n \), if there is an exact sequence of the form
\[ 0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0, \]
where all \( G_i \) (\( 0 \leq i \leq n \)) are totally reflexive.

**Theorem 12.3.** Under the assumption that \( R \) is a generically Gorenstein ring, we have the equality
\[ \text{G-dim}_R M = \sup \{ i \in \mathbb{Z} \mid \text{Ext}^i_R(M, R) \neq 0 \}, \]
for a finitely generated \( R \)-module \( M \).

**Proof.** Setting \( n = \sup \{ i \in \mathbb{Z} \mid \text{Ext}^i_R(M, R) \neq 0 \} \), we have only to consider the case \( n < +\infty \). In this case, it is easy to see that \( n \leq \text{G-dim}_R M \). (This is just because \( \text{Ext}^i_R(M, R) = 0 \) for any \( i > \text{G-dim}_R M \).) Now we take part of projective resolution of \( M \) and get the \( n \)th syzygy module \( \Omega^n_R(M) \), that is,
\[ 0 \rightarrow \Omega^n_R(M) \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \]
is an exact sequence of \( R \)-modules with each \( P_j \) being projective. Then, since it holds that \( \text{Ext}^i_R(\Omega^n_R(M), R) = 0 \) for \( i > 0 \), \( \Omega^n_R(M) \) is totally reflexive by Theorem 12.2. Therefore, we have \( \text{G-dim}_R M \leq n \).

Jorgensen and Şega [14] gave an example of a module over a non-Gorenstein Artinian ring for which the implication \( 2 \Rightarrow 1 \) in Theorem 12.2 does not hold, hence the generic Gorensteinness assumption in the theorem is indispensable.

The following is a commutative version of Tachikawa conjecture, which we obtain as a corollary to Theorem 11.7. It should be noted that it has been proved by Avramov, Buchweitz and Şega [4].

**Theorem 12.4 (Corollary 1.5).** Let \( R \) be a Cohen–Macaulay ring with canonical module \( \omega \). Furthermore assume that \( R \) is a generically Gorenstein ring. If \( \text{Ext}^i_R(\omega, R) = 0 \) for all \( i > 0 \), then \( R \) is Gorenstein.
Proof. Assume \( \text{Ext}_R^i(\omega, R) = 0 \) for all \( i > 0 \). It is enough to show that \( \omega \) is a projective \( R \)-module. We see from Theorem 12.2 that \( \omega \) is a totally reflexive \( R \)-module, and hence there is an exact sequence of the form \( 0 \to \omega \to P_0 \to P_1 \to P_2 \to \cdots \), where each \( P_i \) is a finitely generated projective \( R \)-module. Setting \( M = \text{Ker}(P_1 \to P_2) \), we note that \( M \) is an maximal Cohen–Macaulay module, since there is an exact sequence \( 0 \to M \to P_1 \to P_2 \to \cdots \).

Therefore, we have \( \text{Ext}_R^i(M, \omega) = 0 \) by the local duality theorem. It, however, means that a short exact sequence \( 0 \to \omega \to P_0 \to M \to 0 \) splits, and \( \omega \) is a direct summand of the projective module \( P_0 \), and it is projective.

\[ \square \]

Remark 12.5. In Theorem 12.4, the canonical module has finite Gorenstein dimension, and existence of a finitely generated module of finite injective dimension and finite Gorenstein dimension implies that the ring is Gorenstein. This was announced in [10, 4.1]; it was proved in [12, 3.2] and can also be deduced from earlier results [8, 3.3.5] and [9, 8.3].

**Theorem 12.6** (Corollary 1.6). Assume that the ring \( R \) is a generically Gorenstein ring. Let \( X \) be a complex of finitely generated projective modules.

1. If \( H(X) \) is bounded above, that is, \( X \in D^{-}(R) \), then there is an isomorphism \( X^* \cong R\text{Hom}_R(X, R) \) in the derived category \( D(R) \).

2. If \( H(X) \) and \( H(X^*) \) are bounded above, that is, \( X, X^* \in D^{-}(R) \), then we have the isomorphism in the derived category:

\[
X \cong R\text{Hom}_R(R\text{Hom}_R(X, R), R).
\]

Proof. (1) There is an integer \( a \) such that \( H^i(X) = 0 \) for all \( i \geq a \). Let

\[
\cdots \longrightarrow W^{a-3} \xrightarrow{d^{a-3}_W} W^{a-2} \xrightarrow{d^{a-2}_W} W^{a-1} \xrightarrow{\alpha} Z^a(X) \longrightarrow 0
\]

be an \( R \)-projective resolution of the \( a \)th cocycle \( Z^a(X) = \ker(X^a \xrightarrow{d^a_X} X^{a+1}) \), where each \( W^i \) are finitely generated projective. Then we consider the complex

\[
Y = [ \cdots \longrightarrow W^{a-2} \xrightarrow{d^{a-2}_W} W^{a-1} \xrightarrow{\beta} X^a \xrightarrow{d^a_X} X^{a+1} \xrightarrow{d^{a+1}_X} \cdots ],
\]

where \( \beta \) is the composition of \( \alpha \) with the natural injection \( Z^a(X) \hookrightarrow X^a \). Then we see that \( Y \in \mathcal{K}(R) \) and \( H(Y) = 0 \) by the construction.

Then the identity mappings on \( X^i \) for \( i \geq a \) can be extended to a chain map \( \varphi : X \to Y \) as follows:

\[
\cdots \longrightarrow X^{a-3} \xrightarrow{\varphi^{a-3}_X} X^{a-2} \xrightarrow{\varphi^{a-2}_X} X^{a-1} \xrightarrow{\varphi^{a-1}_X} X^a \xrightarrow{\varphi^a_X} X^{a+1} \xrightarrow{\varphi^{a+1}_X} \cdots
\]

Now set \( U = \text{Cone}(\varphi) \). Then \( U \) is isomorphic in \( \mathcal{K}(R) \) to the complex

\[
\cdots \longrightarrow X^{a-2} \oplus W^{a-3} \longrightarrow X^{a-1} \oplus W^{a-2} \longrightarrow W^{a-1} \longrightarrow 0.
\]
Since there is a triangle $X \to Y \to U \to X[1]$ in $\mathcal{K}(R)$ and since $H(Y) = 0$, there is an isomorphism $U \cong X[1]$ or $U[-1] \cong X$ in $D(R)$. Since $U[-1]$ is a complex bounded above that consists of projective modules, we have an isomorphism in $D(R)$;

$$\text{RHom}_R(X, R) \cong \text{RHom}_R(U[-1], R) \cong (U[-1])^\ast.$$ 

On the other hand, taking the $R$-dual, we also have a triangle in $\mathcal{K}(R)$;

$$U^\ast \to Y^\ast \to X^\ast \to (U[-1])^\ast.$$ 

Since $H(Y^\ast) = 0$ by Theorem 11.7, we have $X^\ast \cong (U[-1])^\ast$ in $D(R)$. Combining this with the above, we consequently have $X^\ast \cong (U[-1])^\ast \cong \text{RHom}_R(X, R)$ as desired.

(2) is clear from (1), since the isomorphism $X^{**} \cong X$ holds in $D(R)$ as well as in $\mathcal{K}(R)$. □

As a miscellaneous result we obtain the following.

**Theorem 12.7** (Corollary 1.7). Assume that the ring $R$ is a generically Gorenstein ring. Let $X$ be a complex of finitely generated projective modules.

If all the cohomology modules $H^i(X)$ ($i \in \mathbb{Z}$) have dimension at most $\ell$ as $R$-modules, then so do the modules $H^i(X^\ast)$ ($i \in \mathbb{Z}$).

**Proof.** The assumption exactly means that $X_p$ is acyclic for a prime ideal $p$ with $\dim R/p > \ell$. Note that each localization $R_p$ is generically Gorenstein. Therefore, $(X^\ast)_p = \text{Hom}_{R_p}(X_p, R_p)$ is acyclic again for such $p$ with $\dim R/p > \ell$, by Theorem 11.7. □

Now we introduce the dimension of total cohomology module of a complex $X$ as

$$\dim_R H(X) = \sup \{\dim R^i(X) \mid i \in \mathbb{Z}\},$$

which is the dimension of the big support of $H(X)$. (Note that we use the convention that $\dim_R M = -1$ for the trivial $R$-module $M = \{0\}$.) Then the theorem above includes the following generalization of Theorem 11.7.

**Corollary 12.8.** Let $R$ be a generically Gorenstein ring. Then, for a complex $X \in \mathcal{K}(R)$, we have the equality $\dim_R H(X) = \dim_R H(X^\ast)$.

**ACKNOWLEDGEMENTS**

The author is grateful to Dr. Shinya Kumashiro for pointing out several errors in the initial version of the paper and giving him a lot of valuable comments. The author also thanks Osamu Iyama, Ryo Takahashi, Luchezar Avramov, Srikanth Iyengar, Henning Krause and Bernhard Keller for their useful comments. Finally he thanks the anonymous referee for his careful reading of the manuscript and his many insightful suggestions.

This research was supported by JSPS Grant-in-Aid for Scientific Research 19K03448.

**JOURNAL INFORMATION**

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and
mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. M. Auslander, Functors and morphisms determined by objects, Representation theory of algebras, Lecture Notes in Pure Appl. Math., vol. 37, Dekker, New York, NY, 1978, pp. 1–244.
2. M. Auslander and M. Bridger, Stable module theory, Memoirs of the American Mathematical Society, No. 94, Amer. Math. Soc., Providence, R.I., 1969.
3. L. L. Avramov, Infinite free resolutions, Six lectures on commutative algebra, Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.
4. L. L. Avramov, R.-O. Buchweitz, and L. M. Şega, Extensions of a dualizing complex by its ring: commutative versions of a conjecture of Tachikawa, J. Pure Appl. Algebra 201 (2005), no. 1–3, 218–239.
5. W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1993.
6. H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, Princeton, N.J., 1999.
7. J. D. Christensen, Ideals in triangulated categories: phantoms, ghosts and skeleta, Adv. Math. 136 (1998), no. 2, 284–339.
8. L. W. Christensen, Gorenstein dimensions, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000.
9. L. W. Christensen, Semi-dualizing complexes and their Auslander categories, Trans. Amer. Math. Soc. 353 (2001), 1839–1883.
10. H. B. Foxby, Gorenstein dimension over Cohen-Macaulay rings, Proceedings of International Conference on Commutative Algebra, Universität Osnabrück, 1994.
11. D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, Lond. Math. Soc. Lect. Note Series, vol. 119, Cambridge Univ. Press, Cambridge, 1988.
12. H. Holm, Gorenstein derived functors, Proc. Amer. Math. Soc. 132 (2004).
13. O. Iyama and Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, Invent. Math. 172 (2008), no. 1, 117–168.
14. D. Jorgensen and L. M. Şega, Independence of the total reflexivity conditions for modules, Algebr. Represent. Theory 9 (2006), no. 2, 217–226.
15. H. Krause and D. Kussin, Rouquier’s theorem on representation dimension, Trends in representation theory of algebras and related topics, Contemp. Math., vol. 406, Amer. Math. Soc., Providence, R.I., 2006, pp. 95–103.
16. Y. Manin and S. Gelfand, Methods of homological algebra, Springer-Verlag, Berlin, New York, 2003.
17. H. Masumura, Commutative algebra, 2nd ed., Math. Lecture Note Series, vol. 56. Benjamin/Cummings Publishing Co., Inc., Reading, MA, 1980, pp. xv+313.
18. C. A. Weibel, An introduction to homological algebra, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge, 1994.
19. Y. Yoshino, A functorial approach to modules of G-dimension zero, Illinois J. Math. 49 (2005), no. 2, 345–367.
20. Y. Yoshino, A remark on vanishing of chain complexes, Acta Math. Vietnam. 40 (2015), no. 1, 173–177.