On Rings of Weak Global Dimension at Most One

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Abstract: A ring \( R \) is of weak global dimension at most one if all submodules of flat \( R \)-modules are flat. A ring \( R \) is said to be arithmetical (resp., right distributive or left distributive) if the lattice of two-sided ideals (resp., right ideals or left ideals) of \( R \) is distributive. Jensen has proved earlier that a commutative ring \( R \) is a ring of weak global dimension at most one if and only if \( R \) is an arithmetical semiprime ring. A ring \( R \) is said to be centrally essential if either \( R \) is commutative or, for every noncentral element \( x \in R \), there exist two nonzero central elements \( y, z \in R \) with \( xy = z \). In Theorem 2 of our paper, we prove that a centrally essential ring \( R \) is of weak global dimension at most one if and only if \( R \) is a right or left distributive semiprime ring. We give examples that Theorem 2 is not true for arbitrary rings.

Keywords: ring of weak global dimension at most one; centrally essential ring; arithmetical ring; right distributive ring; left distributive ring

1. Introduction

We consider only nonzero associative unital rings. For a ring \( R \), we write w.gl.dim. \( R \leq 1 \) if \( R \) is a ring of weak global dimension at most one, i.e., \( R \) satisfies the following equivalent (The equivalence of the conditions is well known; e.g., see the conditions in [1] (Theorem 6.12)).

- For every finitely generated right ideal \( X \) of \( R \) and each finitely generated left ideal \( Y \) of \( R \), the natural group homomorphism \( X \otimes_R Y \to XY \) is an isomorphism.
- Every finitely generated right (resp., left) ideal of \( R \) is a flat right (resp., left) \( R \)-module.
- Every right (resp., left) ideal of \( R \) is a flat right (resp., left) \( R \)-module.
- Every submodule of any flat right (resp., left) \( R \)-module is flat.
- \( Tor^R_1(A, B) = 0 \) for all right (resp., left) \( R \)-modules \( A \) and \( B \).

Since every projective module is flat, any right or left (semi)hereditary ring is of weak global dimension at most one. (A module \( M \) is said to be hereditary (resp., semihereditary) if all submodules (resp., finitely generated submodules) of \( M \) are projective.) We also recall that a ring \( R \) is of weak global dimension zero if and only if \( R \) is a Von Neumann regular ring, i.e., \( r \in rRr \) for every element \( r \) of \( R \). Von Neumann regular rings are widely used in mathematics; see [2,3].

A ring \( R \) is said to be arithmetical if the lattice of two-sided ideals of \( R \) is distributive, i.e., \( X \cap (Y + Z) = X \cap Y + X \cap Z \) for any three ideals \( X, Y, Z \) of \( R \). A ring \( R \) is said to be semi prime (resp., prime) if \( R \) does not have nilpotent nonzero ideals (resp., the product of any two nonzero ideals of \( R \) are nonzero).

Theorem 1 (C.U. Jensen ([4], Theorem)). A commutative ring \( R \) is a ring of weak global dimension at most one if and only if \( R \) is an arithmetical semiprime ring.

A ring \( R \) with center \( C \) is said to be centrally essential if \( R_C \) is an essential extension of the module \( C_C \), i.e., for every nonzero element \( r \in R \), there exist two nonzero central elements \( x, y \in R \) with \( rx = y \). Centrally essential rings are studied in many papers; e.g., see [5].
There are many noncommutative centrally essential rings. For example, if $F$ is the field $\mathbb{Z}/2\mathbb{Z}$ and $Q_8$ is the quaternion group of order 8, then the group algebra $FQ_8$ is a finite noncommutative centrally essential ring; see [5].

Let $F$ be the field $\mathbb{Z}/3\mathbb{Z}$, and let $V$ be a vector $F$-space with basis $e_1, e_2, e_3$. It is known that the exterior algebra of the space $V$ is a finite centrally essential noncommutative ring. It is known that there exists a centrally essential ring $R$ such that the factor ring $R/J(R)$ with respect to the Jacobson radical is not a PI ring (in particular, the ring $R/J(R)$ is not commutative).

A module $M$ is said to be distributive (resp., uniserial) if the submodule lattice of $M$ is distributive (resp., is a chain). It is clear that a commutative ring is right (resp., left) distributive if and only if the ring is arithmetical.

The main result of this work is Theorem 2.

**Theorem 2.** For a centrally essential ring $R$, the following conditions are equivalent.
1. $R$ is a ring of weak global dimension at most one.
2. $R$ is a right (resp., left) distributive semiprime ring.
3. $R$ is an arithmetical semiprime ring

**2. Remarks and Proof of Theorem 2**

**Example 1.** The implication (1) $\Rightarrow$ (2) of Theorem 2 is not true for arbitrary rings. There exists a right hereditary ring $R$ of weak global dimension at most one that is neither right distributive nor semiprime; in particular, the right hereditary ring $R$ is of weak global dimension at most one. Let $F$ be a field, and let $R$ be the 5-dimensional $F$-algebra consisting of all $3 \times 3$ matrices of the following form: $\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$, where $f_{ij} \in F$. The ring $R$ is not semiprime, since the following set is a nonzero nilpotent ideal of $R$: $\left\{ \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$. Let $e_{11}$, $e_{22}$, and $e_{33}$ be ordinary matrix units. The ring $R$ is not right or left distributive, since every idempotent of a right or left distributive ring is central (see [6]), but the matrix unit $e_{11}$ of $R$ is not central. To prove that the ring $R$ is right hereditary, it is sufficient to prove that $R_R$ is a direct sum of hereditary right ideals. We have that $R_R = e_{11}R \oplus e_{22}R \oplus e_{33}R$, where $e_{22}R$ and $e_{33}R$ are projective simple $R$-modules; in particular, $e_{22}R$ and $e_{33}R$ are hereditary $R$-modules. Any direct sum of hereditary modules is hereditary; see ([7], 39.7, p. 332). Therefore, it remains to show that the $R$-module $e_{11}R = e_{11}F + e_{12}F + e_{13}F$ is hereditary, which is directly verified.

The following lemma is well known; e.g., see ([1], Assertion 6.13).

**Lemma 1.** Let $R$ be a ring in which the principal right ideals are flat. If $r$ and $s$ are two elements of $R$ with $rs = 0$, then there exist two elements $a, b \in R$ such that $a + b = 1$, $ra = 0$, and $bs = 0$.

**Lemma 2.** Let $R$ be a centrally essential ring in which the principal right ideals are flat. Then, the ring $R$ does not have nonzero nilpotent elements.

**Proof.** Indeed, let us assume that there exists a nonzero element $r \in R$ with $r^2 = 0$. Since the ring $R$ is centrally essential, there exist two nonzero central elements $x, y \in R$ with $rx = y$. Since $r^2 = 0$, we have that $y^2 = (rx)^2 = r^2x^2 = 0$. Since $y^2 = 0$, it follows from Lemma 1 that there exist two elements $a, b \in R$ such that $a + b = 1$, $ry = 0$, and $by = yb = 0$. Then, $y = y(a + b) = ya + yb = 0$. This is a contradiction. \(\square\)

**Lemma 3.** There exists right and left uniserial prime rings $R$ that have a non-flat principal right ideal.
Proof. There exists right and left uniserial prime rings \( R \) with two nonzero elements \( r, s \in R \) such that \( rs = 0 \); see ([8], p. 234, Corollary). The uniserial ring \( R \) is local; therefore, the invertible elements of \( R \) form the Jacobson radical \( J(R) \) of \( R \). The ring \( R \) is not a ring in which the principal right ideals are flat. Indeed, let us assume the contrary. By Lemma 1, there exist two elements \( a, b \in R \) such that \( a + b = 1, ra = 0 \), and \( bs = 0 \). We have that either \( aR \subseteq bR \) or \( bR \subseteq aR \); in addition, \( aR + bR = Ra + Rb \). Therefore, at least one of the elements \( a, b \) of the local ring \( R \) is invertible; in particular, this invertible element is not a right or left zero-divisor. This contradicts to the relations \( ra = 0 \) and \( bs = 0 \). \( \Box \)

Remark 1. It follows from Lemma 3 that the implication \((2) \Rightarrow (1)\) of Theorem 2 is not true for arbitrary rings.

Lemma 4. Every centrally essential semiprime ring \( R \) is commutative.

Proof. Assume the contrary. Then, the ring \( R \) does not coincide with its center \( C \) and \( xy - yx \neq 0 \) for some \( x, y \in R \). We note that \( A = \{ c \in C : xc \in C \} \) is an ideal of the ring \( C \). The set \( d \in C \mid dA = 0 \) is not empty, since we can take \( d = 0 \). We take any element \( d \in C \) with \( dA = 0 \). If \( xd \neq 0 \), then \( dxz \in C \setminus \{ 0 \} \) for some \( z \in C \). Hence \( dxz \in A \), and therefore, \( d(dxz) = 0 \) and \( (dz)^2 = 0 \). Thus, \( dz = 0 \) and \( dxz = 0 \); this is a contradiction. Therefore, \( xd = 0 \), and thus, \( d \in A \). Therefore, \( d^2 = 0 \) and \( d = 0 \). This implies that \( \text{Ann}_C(A) = 0 \). For any \( a \in A \), we have that \( xa = ax \in C \). Thus,

\[(xy - yx)a = x(ya) - y(xa) = xay - yax = 0\]

and \((xy - yx)A = 0\). However, \( c_1(xy - yx) = c_2 \) for some nonzero elements \( c_1, c_2 \in C \), so \( c_2A = 0 \) and, hence, \( \text{Ann}_C(A) \neq 0 \); this is a contradiction. Thus, \( R \) is commutative. \( \Box \)

The Completion of the Proof of Theorem 2

Proof. \((1) \Rightarrow (2)\). Since \( R \) is a centrally essential ring of weak global dimension at most one, it follows from Lemma 2 that the ring \( R \) does not have nonzero nilpotent elements.

By Lemma 4, the centrally essential semiprime ring \( R \) is commutative. By Theorem 1, \( R \) is an arithmetical semiprime ring. Any commutative arithmetical ring is right and left distributive.

The implication \((2) \Rightarrow (3)\) follows from the property that every right or left distributive ring is arithmetical.

\((3) \Rightarrow (1)\). Since \( R \) is a centrally essential semiprime ring, it follows from Lemma 4 that the ring \( R \) is commutative; in particular, \( R \) is centrally essential. In addition, \( R \) is arithmetical. By Theorem 1, the ring \( R \) is of weak global dimension at most one. \( \Box \)

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