SMOOTHING METHODS TO ESTIMATE THE HAZARD RATE UNDER DOUBLE TRUNCATION

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ABSTRACT

In Survival Analysis, the observed lifetimes often correspond to individuals for which the event occurs within a specific calendar time interval. With such interval sampling, the lifetimes are doubly truncated at values determined by the birth dates and the sampling interval. This double truncation may induce a systematic bias in estimation, so specific corrections are needed. A relevant target in Survival Analysis is the hazard rate function, which represents the instantaneous probability for the event of interest. In this work we introduce a flexible estimation approach for the hazard rate under double truncation. Specifically, a kernel smoother is considered, in both a fully nonparametric setting and a semiparametric setting in which the incidence process fits a given parametric model. Properties of the kernel smoothers are investigated both theoretically and through simulations. In particular, an asymptotic expression of the mean integrated squared error is derived, leading to a data-driven bandwidth for the estimators. The relevance of the semiparametric approach is emphasized, in that it is generally more accurate and, importantly, it avoids the potential issues of nonexistence or nonuniqueness of the fully nonparametric estimator. Applications to the age of diagnosis of Acute Coronary Syndrome (ACS) and AIDS incubation times are included.

Keywords Bandwidth selection, Doubly truncated data, Interval sampling, Kernel smoothing, Survival Analysis
1 Introduction

Acute coronary syndrome (ACS) is still one of the main causes of death in Europe and worldwide. Coronary heart disease mortality has decreased in the last decades in high-income countries because of primary prevention and improvement in treatment of patients with ACS [1]. EPIHeart [2] is a prospective cohort study which includes patients with confirmed diagnosis of type 1 (primary spontaneous) ACS who were consecutively admitted to the Cardiology Department of two tertiary hospitals in Portugal between August 2013 and December 2014. EPIHeart cohort comprises 939 cases with ages at diagnosis ranging from 30 to 94 years. Due to the aforementioned sampling scheme, the age at diagnosis in EPIHeart cohort is observed conditionally on being larger than the time from birth to the study onset (August 2013) and smaller than the age at the end of the study (December 2014); i.e., the event time is doubly truncated by the sampling interval. This restriction in the observation of the ages at infarction may entail biases in estimation, unless appropriate corrections are used [3], [4].

In general, interval sampling occurs when only those individuals whose event lies within a certain calendar time window are observed [5]. Individuals with event out of such sampling interval are not observed, and no information about them is available to the investigator. As mentioned, interval sampling induces double truncation on the event times; this phenomenon is less known, and much more complicated, than one-sided truncation, where only left or right observational limits are present. In particular, the nonparametric maximum-likelihood estimator (NPMLE) for doubly truncated data has no explicit form, and iterative methods are needed [6]. Moreira et al, [7] implemented three different iterative algorithms to compute the NPMLE. However, convergence of these algorithms for a particular dataset does not imply that the NPMLE actually exists. Indeed, the NPMLE may not exist, and may not be unique [8]. Therefore, in practice a preliminary inspection of these potential issues is required. Complications under double truncation arise in theoretical developments too; see for instance [9] for recent updates and a discussion. Summarising, double truncation requires attention and is far from simple.

One of the goals of EpiHeart study is to evaluate the risk of infarction along time from such cohort and, therefore, the estimation of the hazard rate is of interest. The hazard rate function reports the instantaneous probability of death or failure along time, and it plays an essential role in Epidemiology and particularly in Survival Analysis. The shape of the hazard rate function helps in apprehending the mechanism that affects survival and hence, in absence of shape constraints, nonparametric methods turn out to be particularly useful for the estimation of the true curve. In this context, kernel based estimation of the hazard rate function, following the spirit of kernel density estimation [10], has received particular attention in the literature.

Nonparametric estimation of density and hazard rate functions under right censoring was investigated by [11] and [12], while [13] considered the issue of dependent censoring. [14] studied kernel density and hazard rate estimation based on the product-limit estimator from left truncated data [15]. The mean squared error for the kernel estimator of the hazard rate from left truncated and right censored data was derived in [16]. [17] introduced kernel-type density estimation for doubly truncated data. These authors considered a purely nonparametric density estimator, as well as a semiparametric estimator which exploits information on the truncation distribution [18]. The two estimators were shown to be asymptotically equivalent but, as expected, the semiparametric estimator exhibited a better finite-sample performance. [19] explored several bandwidth selection procedures for kernel density estimation under double truncation, which are appropriate modifications of the normal reference rule, the least squares cross-validation procedure, plug-in procedures, and a bootstrap based method. However, smoothing methods for the hazard rate under double truncation have not been investigated yet.

In this paper we propose and investigate nonparametric and semiparametric kernel smoothers for the hazard rate when the data are subject to double truncation. We also illustrate how the application of the semiparametric approach may mitigate the aforementioned issues for the NPMLE. The rest of the paper is organized as follows. In Section 2 we introduce the needed notation and we review the NPMLE and the semiparametric maximum likelihood estimator (SPMLE) for doubly truncated data; the smooth hazard rate estimators are introduced too. In Section 3 the main theoretical properties of the proposed estimators for the hazard rate are given. The finite-sample performance of the estimators is investigated through simulations in Section 4. Section 5 is devoted to the analysis of two real datasets: the EPIHeart cohort data, and the well-known CDC AIDS blood transfusion data [20]. The main conclusions of our research are given in the Section 6, whereas the technical proofs and the details on bandwidth selection and the smoothed bootstrap are deferred to the Appendix.

2 Notation and estimators

Let $X^*$ be the random variable of ultimate interest, with continuous df $F$, and assume that it is doubly truncated by the random pair $(U^*, V^*)$ with joint df $T$, where $U^*$ and $V^*$ ($U^* \leq V^*$) are the left and right truncation variables.
respectively. This means that the triplet \((U^*, X^*, V^*)\) is observed if and only if \(U^* \leq X^* \leq V^*\), while no information is available when \(X^* < U^*\) or \(X^* > V^*\). In the special case of interval sampling we have \(V^* = U^* + \tau\), for some constant \(\tau > 0\) (the width of the sampling interval). It is assumed that \((U^*, V^*)\) is independent of \(X^*\); we also assume that the supports of \(X^*, U^*\) and \(V^*\) are such that the identifiability conditions for \(F\) are satisfied [21].

2.1 The hazard function estimator

The ordinary kernel estimator for the hazard function is given by

\[
\lambda_h(x) = \int K_h(x - t) A_n(dt) = \int K_h(x - t) \frac{F_n(dt)}{1 - F_n(t^-)}
\]

where \(K_h(t) = K(t/h)/h\) is the re-scaled kernel function, \(h = h_n\) is a deterministic bandwidth sequence with \(h_n \to 0\) as \(n \to \infty\). \(A_n\) is the empirical cumulative hazard, and \(F_n\) is the empirical cumulative distribution function [10]. For doubly truncated data \(F_n\) is the Efron and Petrosian estimator, which is the NPMLE in such a setting [6]. We revisit this estimator in the following lines.

Let \((U_i, X_i, V_i)\), \(1 \leq i \leq n\), be the observed data, which are independent copies with the conditional distribution of \((U^*, X^*, V^*)\) given \(U^* \leq X^* \leq V^*\). Here, without loss of generality we assume that the NPMLE is a discrete distribution supported by the set of observed data. Let \(\varphi = (\varphi_1, \ldots, \varphi_n)\) be a distribution putting probability \(\varphi_i\) on \(X_i\), \(i = 1, \ldots, n\). Similarly, let \(\psi = (\psi_1, \ldots, \psi_n)\) be a distribution putting joint probability \(\psi_i\) on \((U_i, V_i)\), \(i = 1, \ldots, n\). Under the assumption of independence between \(X^*\) and \((U^*, V^*)\) the full likelihood, \(\mathcal{L}(\varphi, \psi)\), can be decomposed as a product of the conditional likelihood of the \(X_i\)’s given the \((U_i, V_i)\)’s, say \(\mathcal{L}_1(\varphi)\), and the marginal likelihood of the \((U_i, V_i)\)’s, say \(\mathcal{L}_2(\varphi, \psi)\):

\[
\mathcal{L}(\varphi, \psi) = \prod_{j=1}^{n} \frac{\varphi_j}{\Phi_j} \times \prod_{j=1}^{n} \frac{\Phi_j \psi_j}{\sum_{i=1}^{n} \Phi_i \psi_i} = \mathcal{L}_1(\varphi) \times \mathcal{L}_2(\varphi, \psi)
\]

where \(\Phi_i\) is defined through \(\Phi_i = \sum_{m=1}^{n} \varphi_m J_{im}, i = 1, \ldots, n\) with \(J_{im} = I(U_i \leq X_m \leq V_i)\) the indicator of the event \(U_i \leq X_m \leq V_i\).

The conditional NPMLE of \(F\) [6] is defined as the maximizer of \(\mathcal{L}_1(\varphi)\) in equation (2): \(\hat{\varphi} = \arg\max_{\varphi} \mathcal{L}_1(\varphi)\). The conditional NPMLE \(F_n(x) = \sum_{i=1}^{n} \hat{\varphi}_i I(X_i \leq x)\) maximizes indeed the full likelihood, which can be also written as the product

\[
\mathcal{L}(\varphi, \psi) = \prod_{j=1}^{n} \psi_j / \Psi_j \times \prod_{j=1}^{n} \frac{\psi_j \varphi_j}{\sum_{i=1}^{n} \psi_i \varphi_i} = \mathcal{L}_1^*(\psi) \times \mathcal{L}_2^*(\psi, \varphi)
\]

where \(\Psi_i = \sum_{m=1}^{n} \psi_m J_{mi}, i = 1, \ldots, n\). Here, \(\mathcal{L}_1^*(\psi)\) denotes the conditional likelihood of the \((U_i, V_i)\)’s given the \(X_i\)’s and \(\mathcal{L}_2^*(\psi, \varphi)\) refers to the marginal likelihood of the \(X_i\)’s. Introduce \(\hat{\psi} = (\hat{\psi}_1, \ldots, \hat{\psi}_n)\) as the maximizer of \(\mathcal{L}_1^*(\psi)\); then, \(T_n(u, v) = \sum_{i=1}^{n} \hat{\psi}_i I(U_i \leq u, V_i \leq v)\) is the NPMLE of \(T\) [22].

The NPMLE of \(F\) also admits the representation

\[
F_n(x) = \alpha_n \int_{a_F}^{x} \frac{F_n^*(dt)}{G_n(t)}
\]

where \(a_F\) is the lower limit of the support of \(F\), \(F_n^*\) is the ordinary empirical df of the \(X_i\)’s.
A simple necessary condition for the existence and uniqueness of the NPMLE can be derived from Proposition 1. Let \(\theta\) be the parameter

\[
\theta = \left\{ \left( X_i, U_i, V_i \right) \mid i = 1, \ldots, n \right\}
\]

There exists a unique NPMLE if and only if the graph \(G\) is strongly connected.\(^\text{(8)}\)

Proposition 1 (Xiao and Hudgens, 2019) There exists a unique NPMLE if and only if the graph \(G\), is strongly connected.

In practice the NPMLE may have some limitations. When analysing a particular doubly truncated dataset, the existence or uniqueness of the NPMLE may be compromised. For instance, the convergence of the iterative algorithms proposed by \(^\text{[23]}\) or \(^\text{[6]}\) does not imply that a NPMLE actually exists. In fact, such an estimate may not exist, in which case the estimates provided by the iterative algorithm may be misleading. Xiao and Hudgens \(^\text{[8]}\) presented a necessary and sufficient graphical condition, based on \(^\text{[24]}\), to determine the existence and uniqueness of the NPMLE. The graphical condition is based on graphs theory applied to the observed triplets \((X_i, U_i, V_i)\), \(i = 1, \ldots, n\). Considering that each of these triplets represents one of the \(n\) vertices of the graph \(G\), a directed edge from vertex \(i\) to vertex \(j\) exists if and only if \(X_j \in [U_i, V_i]\). A graph \(G\) is strongly connected if, for any two vertices \(i\) and \(j\), there exists a directed path from \(i\) to \(j\) and a directed path from \(j\) to \(i\), see \(^\text{[8]}\) for more details. Specifically, the result is as follows.

A simple necessary condition for the existence and uniqueness of the NPMLE can be derived from Proposition 1. Let

\[
S_j = \sum_{i=1}^{n} J_{ij} \quad \text{and} \quad \hat{S}_j = \sum_{i=1}^{n} I(U_j \leq X_i \leq V_j).
\]

If \(S_j = 1\) or \(\hat{S}_j = 1\) for some \(j\) then the NPMLE does not exist or is not unique.

Even when existing, the NPMLE is of little use when its variance is extremely large, something with may occur for special truncation patterns. In Section 5 we provide illustrations of the referred limitations. An alternative to the NPMLE is the semiparametric approach, which is discussed in the following subsection.

2.3 The SPMLE of the cumulative df

\(^\text{[13]}\) derived the asymptotic results for the SPMLE of the cumulative df under double truncation, and concluded that it may be more efficient than the Efron–Petrosian NPMLE. On other hand, the use of the semiparametric approach circumvents the eventual limitation of the non-existence or non-uniqueness of the NPMLE. In the nonparametric approach, the NPMLE of \(F\) was represented as an inverse probability weighted estimator (IPWE), where the \(X_i\)'s are upweighted according to their estimated sampling probabilities \(G_n(X_i)\); see equation (5). In the semiparametric setting \(T\) is assumed to belong to a parametric family of df's \(\{T_\theta\}_{\theta \in \Theta}\), where \(\theta\) is a vector of parameters and \(\Theta\) stands for the parametric space. As a consequence, \(G(t)\) becomes

\[
G_\theta(t) = \int_{\{u \leq t \leq v\}} T_\theta(du, dv).
\]

The parameter \(\theta\) is estimated by the maximizer \(\hat{\theta}\) of the conditional likelihood of the \((U_i, V_i)\)'s given the \(X_i\)'s, that is,

\[
\mathcal{L}_1^*(\psi) = \mathcal{L}_1^*(\hat{\theta}) = \prod_{i=1}^{n} \frac{g_\theta(U_i, V_i)}{G_\theta(X_i)}.
\]
where \( g_\theta(u, v) = \frac{\partial^2}{\partial u \partial v} P(U^* \leq u, V^* \leq v) = T_\theta(du, dv) \) stands for the joint density of \((U^*, V^*)\). In the case of interval sampling one has \( V^* = U^* + \tau \) for some constant \( \tau > 0 \) (the sampling interval width) so the joint density of \((U^*, V^*)\) does not exist; in this case one should rather use \( g_\theta(u, v) = \frac{\partial}{\partial v} P(U^* \leq u) = L_\theta(du) \) and \( G_\theta(x) = L_\theta(x) - L_\theta(x - \tau) \) in \( L_\theta^*(\theta) \), where \( L_\theta(.) \) stands for the df of the parametric model assumed for \( U^* \).

Once \( \theta \) is estimated, the SPMLE of \( F \) is introduced through

\[
F_\hat{\theta}(x) = \alpha_\hat{\theta} \int_a^x \frac{F_n^*(dt)}{G_\hat{\theta}(t)},
\]

where \( \alpha_\hat{\theta} = \left( \int_{\alpha_F} G_\hat{\theta}^{-1}(t) F_n^*(dt) \right)^{-1} \). [18] established the asymptotic normality of both \( \hat{\theta} \) and \( F_\hat{\theta} \). As a drawback, the semiparametric estimator requires preliminary specification of a parametric family, which may eventually introduce a bias component when it is far away from reality [18].

Following (1), we introduce the semiparametric kernel estimator for the hazard function as

\[
\lambda_{\hat{\theta},h}(x) = \int K_h(x-t) \frac{F_\hat{\theta}(dt)}{1 - F_\hat{\theta}(t^-)} = \alpha_\hat{\theta} \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i) \frac{G_\hat{\theta}(X_i)}{1 - F_\hat{\theta}(X_i)},
\]

(4)

3 Hazard rate estimators: main properties

Both \( G_n \) and \( G_\hat{\theta} \) defined in the previous section are \( \sqrt{n} \)-consistent estimators of \( G \). For \( G_\hat{\theta} \) this follows from the \( \sqrt{n} \)-consistency of \( \hat{\theta} \), provided that \( G_\theta \) is a smooth function of \( \theta \) [18]. For \( G_n \), the result may be obtained by noting that

\[
G_n(x) = \alpha_n \int \int_{\{u \leq x \leq v\}} \frac{T_n^*(du, dv)}{\int_{\{u \leq t \leq v\}} F_n(dt)},
\]

and

\[
\alpha_n = \int \int \frac{T_n^*(du, dv)}{\int_{\{u \leq t \leq v\}} F_n(dt)},
\]

where \( T^*_n \) is the ordinary empirical df of the truncation times. Hence, \( \sqrt{n} \)-consistency of \( G_n \) is a consequence of that of \( F_n \) and \( T^*_n \); see [9] for formal derivations. Since both \( G_n \) and \( G_\hat{\theta} \) approach to \( G \) at a \( \sqrt{n} \)-rate, which is faster than the nonparametric rate \( \sqrt{nh} \), the asymptotic properties of \( \lambda_h \) and \( \lambda_{\hat{\theta},h} \) are expected to be the same, and will coincide with those of the artificial estimator based on the true \( G \). The same heuristic argument suggests that \( F_n \), \( F_\hat{\theta} \), \( \alpha_n \) and \( \alpha_\hat{\theta} \) can be replaced by their limits \( F \) and \( \alpha \) for asymptotic analysis. This is in parallel with the approach in [17] for density estimation.

Introduce the asymptotically equivalent version of \( \lambda_h \) and \( \lambda_{\hat{\theta},h} \) through

\[
\bar{\lambda}_h(x) = \alpha \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i) \frac{G(X_i)}{1 - F(X_i)},
\]

(5)

As discussed, under regularity one has \( (nh)^{1/2}(\lambda_h(x) - \bar{\lambda}_h(x)) = o_P(1) \) and \( (nh)^{1/2}(\lambda_{\hat{\theta},h}(x) - \bar{\lambda}_h(x)) = o_P(1) \). The function \( G(.) \) may be constant; for example, this happens when \( V^* - U^* \) is degenerated (that is, with interval sampling) provided that the left-truncation time \( U^* \) is uniformly distributed in a suitable interval. In such a case, the correction for truncation vanishes and the usual kernel hazard estimators for complete data is obtained. This is not surprising, since a constant \( G \) indicates that there is no sampling bias. In general, however, the function \( G \) will not be flat and the correction for double truncation becomes relevant. In the next result we establish the strong consistency and the asymptotic normality of \( \bar{\lambda}_h(x) \). Throughout this Section we implicitly assume \( G(x) > 0 \) for each \( x \) in the support of \( X^* \). Note that this condition is needed to ensure the identifiability of \( F \) along its whole support.
Theorem 1 (i) If $K$ is bounded on a compact support, $h$ is such that $\sum_{n=1}^{\infty} \exp(-\eta hn) < \infty$ for each $\eta > 0$, $G$ is continuous at $x$, and $x$ is a Lebesgue point of $\lambda$, then $\overline{\lambda}_h(x) \to \lambda(x)$ with probability 1.

(ii) If, in addition to the conditions in (i), $R$ is obtained without truncation (constant $G$), the asymptotic bias is that of the complete data case (no truncation). The variance can be smaller or larger than the one obtained without truncation (constant $G$) depending on the particular $x$ value. This is intuitive, since the sampling bias due to the double truncation may result in an oversampling of certain lifetime values, while other may be undersampled.

Theorem 2 Under (A1)-(A3) we have, as $n \to \infty$,

$$E \left[ \overline{\lambda}_h(x) \right] = \lambda(x) + \frac{1}{2} h^2 \lambda''(x) \mu_2(K) + o(h^2),$$

$$Var \left[ \overline{\lambda}_h(x) \right] = (nh)^{-1} \alpha \frac{G(x)^{-1}}{1 - F(x)} \lambda(x) R(K) + o((nh)^{-1}).$$

Proof The proof follows standard steps, see e.g. [10]. A second-order Taylor expansion of $\lambda$, respectively of $G^{-1} \lambda / (1 - F)$, around $x$ is used, and the assumptions on the kernel and the bandwidth are enough to conclude. Details are omitted.

From Theorem 2 it can be seen that the asymptotic variance of $\overline{\lambda}_h(x)$ is affected by the double truncation issue, while the asymptotic bias is that of the complete data case (no truncation). The variance can be smaller or larger than the one obtained without truncation (constant $G$) depending on the particular $x$ value. This is intuitive, since the sampling bias due to the double truncation may result in an oversampling of certain lifetime values, while other may be undersampled.

The global error of $\overline{\lambda}_h$ can be measured through the mean integrated squared error (MISE), namely

$$MISE(\overline{\lambda}_h) = \int MSE(\overline{\lambda}_h(x))dx,$$

where

$$MSE(\overline{\lambda}_h(x)) = \left[ E[\overline{\lambda}_h(x)] - \lambda(x) \right]^2 + Var(\overline{\lambda}_h(x)).$$

Under regularity, the following asymptotic expression for the $MISE(\overline{\lambda}_h)$ is immediately derived from the previous results:

$$AMISE(\overline{\lambda}_h) = \frac{1}{4} h^4 R(\lambda'') \mu_2(K)^2 + (nh)^{-1} \alpha R(K) \int \frac{G^{-1}}{1 - F} \lambda$$

where $R(\lambda'') = \int (\lambda'')^2$. Minimization of $AMISE(\overline{\lambda}_h)$ w.r.t. $h$ leads to the asymptotically optimal bandwidth

$$h_{AMISE} = \left[ \frac{\alpha R(K) \int \frac{G^{-1}}{1 - F} \lambda}{R(\lambda'') \mu_2(K)^2} \right]^{1/5} n^{-1/5}.$$

Of course, this expression depends on unknown quantities that must be estimated in practice. There exist several criteria to select the bandwidth from the data at hand. In the Appendix, a least-squares cross-validation (LSCV) bandwidth selector is derived. The cross-validation bandwidth is used in the real data analyses of Section 5.
4 Simulations

In this section we illustrate the finite sample behavior of the purely nonparametric estimator \( \lambda_h(x) \) and the semiparametric estimator \( \lambda_{\hat{\theta},h}(x) \) through simulations. We analyze the influence of the bandwidth in the MISE of the estimators, and we measure the amount of efficiency which is gained by using the semiparametric information.

We simulate \( U^* \) independently of \( X^* \) and then we take \( V^* = U^* + \tau \) for some constant \( \tau > 0 \). Such scenario represents interval sampling, and follows the spirit of the two real data examples presented in Section 5, with terminating events (acute coronary syndrome or AIDS diagnosis) falling between two specific dates. Different models are simulated. We take \( \tau = 0.25 \) and \( U^* \sim U(0,1), X^* \sim 0.75\text{Beta}(3/4,1) + 0.25 \) for Model 1 and \( U^* \sim U(0,1), X^* \sim 0.75N(0.5,0.15) + 0.25 \) for Model 2. In Model 3 we take \( U^* \sim U(0.25,1), X^* \sim 0.75\text{Beta}(3/4,1) + 0.25 \) and \( V^* = U^* + \tau \), considering a decreasing sequence of widths for the sampling interval: \( \tau = 0.25, 0.15 \) and 0.10 (Model 3.1, 3.2, 3.3 respectively). As parametric information on the truncation distribution we take a \( \text{Beta}(\theta_1,1) \) model for \( U^* \). For each model, we simulate 1000 trials with final sample size \( n = 100, 250, \) or 500.

The functions \( G \) corresponding to the aforementioned models, based on a Monte Carlo approximation from a single sample of size \( n = 20,000 \), are shown in Figure 1. The depicted functions indicate that small values of the variable of interest (\( X^* \)) are observed with a relatively small probability in the last three models, while there is no observational bias in Models 1 and 2 (\( G \) remains constant).

In Table 1 we report the optimal bandwidths (in the sense of the MISE) and the corresponding minimum MISEs for the nonparametric estimator and the semiparametric estimator. The theoretical MISE function is approximated by the average of the integrated squared error (ISE) along the \( M = 1000 \) trials, namely

\[
\text{TSE}(\lambda_h) = \frac{1}{M} \sum_{m=1}^{M} \int (\lambda_{h}^{m} - \lambda)^2 , \\
\text{TSE}(\lambda_{\hat{\theta},h}) = \frac{1}{M} \sum_{m=1}^{M} \int (\lambda_{\hat{\theta},h}^{m} - \lambda)^2 ,
\]

where \( \lambda_{h}^{m} \) and \( \lambda_{\hat{\theta},h}^{m} \) are respectively the nonparametric and the semiparametric estimators when based on the \( m \)-th Monte Carlo trial.
From Table 1 it is seen that the optimal bandwidths and the MISEs decrease when increasing the sample size; besides, the semiparametric estimator has an error which is smaller than that of the nonparametric estimator. It is also seen that the optimal bandwidths for the semiparametric estimator are generally smaller than those of the nonparametric estimator, according to the extra amount of information. As the sample size grows, the relative efficiency of the nonparametric estimator approaches to one; this is in agreement to the asymptotic equivalence of the semiparametric and the nonparametric hazard estimators discussed in Section 2. Results for Models 3.1 to 3.3 reveal that the MISE increases as the width of the sampling interval decreases, due to the large variance attached to a small observational window. The semiparametric estimator however behaves much better than the nonparametric estimator, offering moderate MISE values even in the extreme situation with $\tau = 0.1$.

In Table 2 we report the biases and the variances of the nonparametric and semiparametric hazard estimators at some selected time points, corresponding to the quartiles of $\lambda$, for sample sizes $n=100, 500$, along the 1,000 Monte Carlo trials. It is seen that the squared bias is always of a smaller order when compared to the variance, so the resulting mean squared errors (MSEs) are mainly determined by the dispersion of the estimates. For all the cases, the local MSEs of the semiparametric estimator are smaller than those pertaining to the nonparametric estimator, as expected.

Table 1: Optimal bandwidths (h$_{opt}$) and minimum MISEs of the hazard estimators: nonparametric estimator (NP) and semiparametric estimator (SP). Averages along 1000 trials of a sample size $n$.

| Model | $n$  | $h_{opt}$ | NP  | SP  | $MISE(h_{opt})$ |
|-------|------|-----------|-----|-----|-----------------|
|       | 100  | 0.100     | 0.090 | 7.723 | 4.649         |
| 1     | 250  | 0.020     | 0.020 | 3.432 | 2.381         |
| 2     | 500  | 0.020     | 0.020 | 1.532 | 1.031         |
| 1     | 100  | 0.091     | 0.078 | 9.392 | 4.713         |
| 2     | 250  | 0.090     | 0.076 | 5.473 | 2.763         |
| 3     | 500  | 0.070     | 0.070 | 2.483 | 1.754         |
| 1     | 100  | 0.073     | 0.066 | 8.726 | 6.026         |
| 3.1   | 250  | 0.014     | 0.014 | 7.788 | 5.489         |
| 500   | 0.011 | 0.011  | 4.290 | 3.823 |
| 3.2   | 100  | 0.093     | 0.068 | 15.968 | 10.759       |
| 2     | 250  | 0.014     | 0.014 | 11.595 | 9.925        |
| 3.3   | 500  | 0.011     | 0.011 | 5.869 | 4.150         |
| 1     | 100  | 0.18      | 0.066 | 28.651 | 10.808        |
| 3.3   | 250  | 0.115     | 0.014 | 17.0180 | 10.043       |
| 500   | 0.011 | 0.011  | 8.517 | 4.162 |

In Figures 2 to 6 we report for each simulated model: (i) the target hazard function together with its semiparametric and nonparametric estimators averaged along the 1000 Monte Carlo trials (bottom row); and (ii) the ratio between the MISEs of the semiparametric and the nonparametric estimators along a grid of bandwidths (top row). From these Figures 2 to 6 several interesting features can be seen. First, for each given smoothing degree, the MISE of the semiparametric estimator is less than that of the nonparametric estimator; the relative benefits of using the semiparametric information are more clearly seen when working with relatively smaller bandwidths, when the variance component of the MISE is larger. This illustrates how the semiparametric estimator achieves a variance reduction w.r.t. the NPMLE. Also importantly, we see that the ratios of the MISEs approach to one as the sample size increases. This was expected, since (as discussed in Section 3) both estimators are asymptotically equivalent.

Simulations above are informative about the relative performance of the two proposed estimators when the parametric information on the truncation distribution is correctly specified. However, in practice, some level of misspecification in the parametric model may occur. To investigate the sensitivity of the semiparametric estimator to the misspecification degree, we have repeated the simulation of Model 1 but changing the $U(0, 1)$ distribution of $U^*$ for a Beta($\theta_1, 1$) distribution, with $\theta_1 \neq 1$, so the parametric information Beta($\theta_1, 1$) on $U^*$ is misspecified. Note that the misspecification degree increases as $\theta_1$ departs from 1. Results on the bandwidth, the MISE, and the local MSE of both the semiparametric and nonparametric hazard estimators are reported in Tables 3 and 4 for the case $n = 500$ (results based on 1,000 trials).

From Table 3 it is seen that the semiparametric estimator may be still equivalent or even preferred to the nonparametric estimator in all cases of misspecification. Table 4 indicates that, when the parametric information is misspecified, the variance of the semiparametric estimator remains smaller than that of the nonparametric estimator for almost all the cases (exceptions for $\theta_1 = 5$ are found in the first two quartiles).
The EPIHeart cohort is a prospective study assembled between August 2013 and December 2014 to the Cardiology

Table 2: Biases ($\times 10^3$) and variances ($\times 10$) of the nonparametric estimator ($NP$) and semiparametric estimator ($SP$) at the quartiles of $F$, for sample sizes n=100, 500, along 1000 Monte Carlo trials.

| Model | n  | $x$  | NP Bias | NP Var | SP Bias | SP Var |
|-------|----|------|---------|--------|---------|--------|
| 1     | 100| q.25 | 37.20   | 2.44   | 149.80  | 2.01   |
|       |    | q.50 | 72.20   | 9.90   | 468.10  | 8.31   |
|       |    | q.75 | 110.30  | 37.95  | 155.60  | 32.75  |
|       | 500| q.25 | 20.34   | 1.66   | 8.39    | 0.44   |
|       |    | q.50 | 28.69   | 3.04   | 16.24   | 0.75   |
|       |    | q.75 | 72.09   | 6.58   | 63.28   | 3.06   |
| 2     | 100| q.25 | 62.29   | 6.93   | 69.50   | 4.86   |
|       |    | q.50 | 66.05   | 11.98  | 13.30   | 2.86   |
|       |    | q.75 | 726.60  | 44.92  | 325.00  | 16.95  |
| 3.1   | 100| q.25 | 12.66   | 7.37   | 5.88    | 2.39   |
|       |    | q.50 | 130.62  | 10.86  | 34.18   | 2.88   |
|       |    | q.75 | 299.28  | 24.22  | 159.63  | 11.04  |
| 3.2   | 500| q.25 | 2.37    | 1.40   | 2.47    | 0.56   |
|       |    | q.50 | 17.93   | 2.07   | 9.66    | 0.68   |
|       |    | q.75 | 67.40   | 5.15   | 67.18   | 2.94   |
| 3.3   | 100| q.25 | 157.70  | 32.72  | 1792.10 | 1.47   |
|       |    | q.50 | 435.79  | 39.31  | 959.42  | 3.53   |
|       |    | q.75 | 882.44  | 64.47  | 1070.40 | 15.54  |
|       | 500| q.25 | 16.050  | 38.89  | 115.50  | 0.17   |
|       |    | q.50 | 82.48   | 0.68   | 82.48   | 0.68   |
|       |    | q.75 | 188.52  | 3.26   | 188.52  | 3.26   |
|       | 3   | 500  | 2731.90 | 34.94  | 1705.30 | 0.88   |
|       |    | q.50 | 6355.11 | 41.19  | 947.58  | 3.43   |
|       |    | q.75 | 2796.60 | 37.41  | 2553.54 | 0.06   |

Table 3: Optimal bandwidths ($h_{opt}$) and minimum MISEs of the hazard estimators: nonparametric estimator ($NP$) and semiparametric estimator ($SP$). Averages along 1000 trials of a sample size $n = 500$. Similar as Model 1, but $U^*$ is simulated as a $Beta(1, a)$ random variable.

$$h_{opt} \quad MISE(h_{opt})$$

| a     | NP  | SP  | NP  | SP  |
|-------|-----|-----|-----|-----|
| 1/5   | 0.027 | 0.037 | 1.632 | 1.492 |
| 1/2   | 0.028 | 0.030 | 1.586 | 1.443 |
| 1     | 0.031 | 0.031 | 1.309 | 1.116 |
| 3/2   | 0.025 | 0.025 | 2.794 | 1.850 |
| 5     | 0.024 | 0.026 | 6.024 | 3.386 |

5 Real data illustration

5.1 Acute Coronary Syndrome data

For illustration purposes, in this section we consider the aforementioned data on the age at diagnosis of ACS. In Portugal, with a population of 10.3 million inhabitants, there are 38 public hospitals with resources for structured care of patients with ACS, out of which 16 have catheterisation laboratory facilities. Public hospitals provide treatment for the majority of the acute coronary events and the number of patients submitted to primary Percutaneous coronary intervention (PCI) increased by 37.0% from 2009 to 2013, although at the regional level access to this procedure varied. The EPIHeart cohort is a prospective study assembled between August 2013 and December 2014 to the Cardiology Department of two tertiary hospitals in two regions in Northern Portugal (Hospital de São João, Porto, covering the
metropolitan area of Porto in the coast; and Hospital de São Pedro, Vila Real, covering the interior, northeastern region). The inclusion criteria to the cohort were admission with a diagnosis of ACS type I, aged 18 years or older, living in the catchment area of the referred hospitals (Porto, Vila Real, Bragança or Viseu) with confirmed diagnosis of type I (primary spontaneous) ACS. Data was collected through structured interviews within the first 48 hours after admission. Of the 1297 patients initially considered, 939 were included in the cohort due the inclusion criteria. The age at diagnosis (ranging from 30 to 94 years old) was doubly truncated by \((U^*, V^*)\), where \(V^*\) stands for the elapsed time (in years) between birth and end of the study (December 2014), and \(U^* = V^* - 1\).

For this dataset it happens \(\sum_{j=1}^{n} I(U_j \leq X_i \leq V_j) = 1\) for the three largest values of \(X_i\), corresponding to ages \(X_i = 90.70, 91.99\) and \(93.80\). Then, the NPMLE \(F_n\) does not exist or is not unique for the ACS data [8]. We redefined our sample restricting to the largest dataset in which the Proposition 1 is satisfied, i.e, the conditional NPMLE exists and is unique. Thus, our final sample is composed by 917 patients, 680 male and 237 female, with ages at ACS diagnosis between 39 and 90 years.

The nonparametric and semiparametric kernel estimators for the hazard function of \(X^*\) computed from the \(n = 917\) patients together with the pointwise confidence bands at 95% level are given in Figure 7, top row. For the semiparametric estimator a \(Beta(\theta_1, \theta_2)\) model for \(U^*\) was assumed, and the parameters were estimated by maximizing the conditional likelihood of the truncation times. The 95% pointwise confidence bands were computed from the smoothed bootstrap; the parametric information was included in the bootstrap when dealing with \(\lambda_{\hat{\theta},h}(x)\) (see the Appendix for more details).

The reason to use a smoothed bootstrap procedure (as opposed to a non-smoothed one) is the same as in [25], namely without smoothing the bootstrap would be inconsistent.

Generally speaking it is seen that the hazard of ACS increases with age. A local mode, located approximately at 78 years old, is suggested by the nonparametric estimator. This mode can be medically ignored since it is a result of the large variability of the estimator; the bootstrap confidence intervals around this mode are very wide indeed. This is in well agreement with our findings in Section 3, Models 3.1 to 3.3, in which large MISE values were found when the

![Figure 2: (i) the target hazard (solid line) together with its semiparametric (dashed line), nonparametric (dotted line) and naive (dashed-dotted line for n=500) estimators averaged along the 1000 Monte Carlo trials for Model 1 (top row); (ii) The ratio between the MISE’s of the semiparametric and the nonparametric estimators along a grid of bandwidths (bottom row).](image-url)
Figure 3: (i) the target hazard (solid line) together with its semiparametric (dashed line), nonparametric (dotted line) and naive (dashed-dotted line for n=500) estimators averaged along the 1000 Monte Carlo trials for Model 2 (top row); (ii) The ratio between the MISE’s of the semiparametric and the nonparametric estimators along a grid of bandwidths (bottom row).

sampling interval was too narrow. The situation here is even worse in that the width of the sampling interval (1.42 years) is only 3% the width of the support of the target variable \(X^*\). In situations like this the NPMLE may be expected to be irrelevant, and the SPMLE becomes a useful alternative.

The optimal bandwidths, derived from the LSCV method in the Appendix, are \(h = 0.028\) and \(h = 0.031\) for the nonparametric and semiparametric estimators, respectively. For comparison purposes the naive estimator is also depicted in Figure 7 \((h = 0.042)\). It is seen that the three estimators are close to each other, suggesting no impact of the double truncation issue in the hazard function; this can be further investigated through the estimation of the biasing function \(G(\cdot)\).

The biasing function, together with the 95% pointwise confidence bands based on the bootstrap are displayed in Figure 7 bottom panel. It can be seen that both biasing functions \(G_n\) and \(\hat{G}_\theta\) are roughly flat, and this explains why the semiparametric hazard estimator mimics the naive one which does not correct for double truncation. Note that, although \(G_n\) exhibits some bumps, these are not significant according to the confidence limits.

5.2 AIDS Blood Transfusion data

In this subsection we use epidemiological data on transfusion-related Acquired Immune Deficiency Syndrome (AIDS). The AIDS Blood Transfusion data were collected from the Centers for Disease Control (CDC), which is from a registry database, a common source of medical data; see [26] and [20]. The variable of interest \((X^*)\) is the induction or incubation time, which is defined as the time elapsed from Human Immunodeficiency virus (HIV) infection to the clinical manifestation of AIDS. The CDC AIDS Blood Transfusion Data can be viewed as being doubly truncated. The data were retrospectively ascertained for all transfusion-associated AIDS cases in which the diagnosis of AIDS occurred prior to the end of the study, thus leading to right-truncation. Besides, because HIV was unknown prior to
1982, any cases of transfusion-related AIDS before this time would not have been properly classified and thus would have been missed. Thus, in addition to right-truncation, the observed data were also truncated from the left. See [26] Section 5.2, for further discussions.

Data included 494 cases reported to the CDC prior to January 1, 1987, and diagnosed prior to July 1, 1986. Of the 494 cases, 295 had consistent data, and the infection could be attributed to a single transfusion or short series of transfusions. Our analyses are restricted to this subset, which is entirely reported in [20], Table 1. Values of $U^*$ were obtained by measuring the time from HIV infection to January 1, 1982; while $V^*$ was defined as time from HIV infection to the end of study (July 1, 1986). Note that the difference between $V^*$ and its respective $U^*$ is always 4.5 years. The times were considered in months.

After checking the existence and uniqueness of the NPMLE, the semiparametric and the nonparametric kernel estimators for the hazard rate function of $X^*$ were computed from the $n = 295$ cases, together with 95% bootstrap pointwise confidence bands. The results are displayed in Figure 8 top row. The transformation $(t + 49)/95$ has been used for the ages at diagnosis and the truncation variables. With this transformation, the $U^*$ is supported on the $(0, 1)$ interval. For better analysis of the figures, the ages in the horizontal axis are reported in their original scale (months). As in our first real data illustration for the semiparametric estimator, we assume a $Beta(a_1, a_2)$ model for $U^*$, and the parameters are estimated by maximizing the conditional likelihood of the truncation times. This parametrization has been used since it permits a range of different curves to describe the data.

The optimal (LSCV) bandwidths for the nonparametric and semiparametric estimators were $h = 0.61$ and $h = 0.53$ respectively. For comparison purposes, the naive kernel hazard estimator which does not correct the double truncation issue is also reported. Figure 8 top row, reveals that the hazard increases with the induction times, which is in accordance with the literature. It is also seen that the proposed estimators are close to each other along their whole support, while the naive estimator clearly overestimates the hazard function.

In Figure 8 bottom row, we display the parametric and nonparametric biasing functions together with the 95% pointwise
confidence bands based on the bootstrap. The two estimators are roughly equivalent, and they both suggest a sampling probability which decreases as the induction time increases. This decreasing shape of the function $G$ is responsible for the positive bias of the naive hazard estimator. Indeed, it can be proved in general that, when $G$ is non-increasing, the hazard rate corresponding to the observed $X$ is greater than the target. Both the nonparametric and the semiparametric estimators declare a mode around 65-68 months for the hazard, although with no epidemiological interpretation.

6 Conclusions

In this paper we have introduced kernel hazard estimators for a variable which is observed under random double truncation. Two estimators have been proposed. The first one is purely nonparametric, and it is defined as a convolution of a kernel function with the NPMLE of the cumulative hazard. The second estimator is based on a parametric specification for the df of the truncation variables, thus being semiparametric. Asymptotic properties of the two estimators have been discussed, including a formula for the asymptotic MISE. A LSCV bandwidth selection criterion for the automatic application of the proposed smoothers has been derived.

The nonparametric and semiparametric estimators are asymptotically equivalent in the sense of having the same asymptotic MISE. However, simulations have shown that, for moderate sample sizes, the semiparametric estimator may outperform the nonparametric estimator. Importantly, the nonparametric estimator may be misleading due to its relatively large variance, providing spurious bumps in particular applications. Furthermore, the NPMLE may not exist and, therefore, the semiparametric estimator may be the only way out in estimation. The referred issues are particularly present with interval sampling when the sampling interval is very narrow. This is in agreement with the intuition that short sampling intervals may result in little, or too deteriorated, information on the target.

Two real data illustrations were provided. For the ACS data, some features concerning the NPMLE, such as nonexistence and non-uniqueness, were encountered. In order to make the application of the nonparametric estimator
Figure 6: (i) the target hazard (solid line) together with its semiparametric (dashed line), nonparametric (dotted line) and naive (dashed-dotted line for n=500) estimators averaged along the 1000 Monte Carlo trials for Model 3.3 (top row); (ii) The ratio between the MISE’s of the semiparametric and the nonparametric estimators along a grid of bandwidths (bottom row).

possible, the sample was reduced to the largest dataset for which the NPMLE exists and is unique. In this reduced dataset it was seen that the effect of double truncation was almost negligible, the proposed estimators being close to the ordinary kernel smoother. In contrast, for the Blood Transfusion data the effect of double truncation was found critical and the standard kernel estimator exhibited a gross positive bias. Thus, in practice, taking the double truncation issue into account is very important.

In the two real data applications the semiparametric estimator provided confidence intervals much narrower than the nonparametric estimator, indicating the importance of modeling the truncation distribution. This was much more evident for the ACS data. In practice, the semiparametric estimator may be recommended when a suitable parametric family for the truncation distribution is available. A two-parameter beta model worked well in our applications. Besides, in general the semiparametric estimator is well defined, so no sample reduction is needed. Application of the semiparametric estimator to the full ACS dataset \( n = 939 \) provided an estimate similar to that in Figure 7, top-right plot (results not shown). The analysis of the full dataset through the nonparametric hazard estimate was not possible due to the aforementioned nonexistence of the NPMLE.

Appendix

Proof of Theorem 1

Proof 2 For (i) introduce \( \tilde{\lambda}_h(x) = A \frac{G(x)}{1 - F(x)} f_h^*(x) \) where

\[
f_h^*(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i)
\]
Table 4: Bias ($\times 10^3$) and variances ($\times 10$) of the nonparametric estimator (NP) and semiparametric estimator (SP) at the quartiles of $F$, for sample sizes $n=500$, along 1000 Monte Carlo trials. Misspecified parametric model.

| a   | $x$ | NP Bias | NP Var | SP Bias | SP Var |
|-----|-----|---------|--------|---------|--------|
| 1/5 | $q_{0.25}$ | 0.44 | 0.51 | 11.80 | 0.44 |
|     | $q_{0.50}$ | 7.02 | 1.26 | 6.45 | 0.99 |
|     | $q_{0.75}$ | 1825.00 | 11.64 | 4687.00 | 6.86 |
| 1/2 | $q_{0.25}$ | 0.60 | 0.44 | 0.92 | 0.35 |
|     | $q_{0.50}$ | 7.18 | 1.08 | 1.83 | 0.73 |
|     | $q_{0.75}$ | 36.84 | 4.18 | 2.89 | 2.93 |
| 1   | $q_{0.25}$ | 0.80 | 0.39 | 0.99 | 0.30 |
|     | $q_{0.50}$ | 7.70 | 1.06 | 8.80 | 0.87 |
|     | $q_{0.75}$ | 38.28 | 4.47 | 37.98 | 3.79 |
| 3/2 | $q_{0.25}$ | 0.58 | 40.42 | 24.23 | 0.34 |
|     | $q_{0.50}$ | 0.02 | 1.13 | 55.54 | 1.03 |
|     | $q_{0.75}$ | 39.09 | 4.88 | 171.48 | 4.59 |
| 5   | $q_{0.25}$ | 1.57 | 48.41 | 1.98 | 51.30 |
|     | $q_{0.50}$ | 14.75 | 1.36 | 293.40 | 1.53 |
|     | $q_{0.75}$ | 3176.00 | 8.90 | 2037.00 | 8.58 |

is the ordinary kernel density estimator computed from the observed data. By [28] we have $f_h^*(x) \to f^*(x)$ almost surely, where $f^*$ is the density of $F^*(x) = P(X_1 \leq x)$. Now, if the support of $K$ is contained in $[-a, a]$, $\bar{\lambda}_h(x) - \lambda(x) = O(h^2)$.

$$
\int \bar{\lambda}_h(x) - \lambda(x) \leq \alpha f^*_h(x) \sup_{x-ah \leq y \leq x+ah} \frac{G(y) - G(x)}{1-F(y) - 1-F(x)},
$$

and the supremum goes to zero as $h \to 0$ by the continuity of $G$ at $x$. This ends with the proof to (i). Statement (ii) is proved similarly to Section 2 of [29], by following such lines we obtain

$$(nh)^{1/2} (\bar{\lambda}_h(x) - E\bar{\lambda}_h(x)) \to N(0, \alpha \frac{G(x)}{1-F(x)}(x)R(K))$$

in distribution. Now, a two-term Taylor expansion (and the fact that $K$ is even) gives $E\bar{\lambda}_h(x) = \lambda(x) + O(h^2)$. Since $nh^3 \to 0$, this implies the claimed result.

LSCV bandwidth selection

We give the details of the LSCV bandwidth selector for the semiparametric estimator $\lambda_{\check{\theta},h}$. The LSCV selector for the nonparametric estimator is analogous.

LSCV aims to estimate the $MISE$ and then to minimize the $MISE$ with respect to $h$. Given the estimator $\lambda_{\check{\theta},h}$ of the hazard $\lambda$, the mean integrated squared error can be written as

$$MISE(\lambda_{\check{\theta},h}) = E[ISE(\lambda_{\check{\theta},h})] = E \left[ \int (\lambda_{\check{\theta},h}(x) - \lambda(x))^2 dx \right] = E \left[ \int \lambda^2_{\check{\theta},h}(x) dx - 2 \int \lambda_{\check{\theta},h}(x) \lambda(x) dx + \int \lambda^2(x) dx \right].$$

The term $\int \lambda^2(x) dx$ does not depend on $h$, and so minimizing $MISE(\lambda_{\check{\theta},h})$ is equivalent to minimizing

$$S(\lambda_{\check{\theta},h}) = MISE(\lambda_{\check{\theta},h}) - \int \lambda^2(x) dx = E \left[ \int \lambda^2_{\check{\theta},h}(x) dx - 2 \alpha \int \lambda_{\check{\theta},h}(x) \frac{G^{-1}(x)}{1-F(x)} F^*(dx) \right].$$

Now, in order to construct an estimator of $S(\lambda_{\check{\theta},h})$, let $\lambda_{\check{\theta},h,-i}$ be the hazard estimator constructed from all data points except $X_i$, i.e.

$$\lambda_{\check{\theta},h,-i}(x) = \frac{1}{n-1} \sum_{j \neq i} K_h(x - X_j) \frac{G^{-1}_{\check{\theta},-i}(X_j)}{1 - F_{\check{\theta},-i}(X_j)},$$

where $G_{\check{\theta},-i}(\cdot)$ and $F_{\check{\theta},-i}(\cdot)$ are the estimators of $G$ and $F(\cdot)$, defined in Section 2, except that the $i$-th data point is not used for estimating $\check{\theta}$. Introduce

$$LSCV(h) = \int \lambda^2_{\check{\theta},h}(x) dx - 2 \alpha \int \lambda_{\check{\theta},h,-i}(x) \frac{G^{-1}_{\check{\theta},-i}(X_i)}{1 - F_{\check{\theta},-i}(X_i)}.$$
Figure 7: Top row: Nonparametric hazard estimator (left panel) and the semiparametric hazard estimator (right panel) for the age at diagnosis (solid line), ACS data (n=917), with pointwise confidence bands at level 95% (dotted lines) and the naive estimator (dashed line). Bottom row: Nonparametric estimator $G_n$ (left panel, solid line) with pointwise confidence bands at level 95% (dotted lines) and $\hat{G}_\theta$ based on a beta model (right panel, dotted lines), ACS data (n=917).

and estimate the optimal $h$ by minimizing $LSCV(h)$ over $h$:

$$\tilde{h}_{LSCV} = \arg\min_{h} LSCV(h).$$

Note that

$$E\left[ n^{-1} \sum_{i=1}^{n} \alpha_{\theta,\cdot} G^{-1}_{\hat{\theta},\cdot} (X_i) \frac{G^{-1}_{\hat{\theta},\cdot} (X_i)}{1 - F_{\hat{\theta},\cdot} (X_i)} \right]$$

$$= E\left[ \alpha_{\theta,\cdot} G^{-1}_{\hat{\theta},\cdot} (X_1) \frac{G^{-1}_{\hat{\theta},\cdot} (X_1)}{1 - F_{\hat{\theta},\cdot} (X_1)} \right]$$
Figure 8: Top row: Nonparametric hazard estimator (left panel) and the semiparametric hazard estimator (right panel) for the age at diagnosis (solid line), AIDS data (n=295), with pointwise confidence bands at level 95% (dotted lines) and the naive estimator (dashed line). Bottom row: Nonparametric estimator $G_n$ (left panel, solid line) with pointwise confidence bands at level 95% (dotted lines) and $G_{\hat{\theta}}$ based on a beta model (right panel, dotted lines), AIDS data (n=295).

\[
E\left[\alpha \int \lambda_{\hat{\theta},h}^{-1}(x) \frac{G_{\hat{\theta},h}^{-1}(x)}{1 - F_{\hat{\theta},h}^{-1}(x)} F^*(dx)\right],
\]

and this is asymptotically equivalent to $E[\int \lambda_{\hat{\theta},h}^{-1}(x) \frac{G_{\hat{\theta},h}^{-1}(x)}{1 - F_{\hat{\theta},h}^{-1}(x)} F^*(dx)] = E[\int \Lambda(x) \Lambda(dx)] = E[\int \lambda_{\hat{\theta},h}^{-1}(x) \lambda(x) dx]$, where $\Lambda$ stands for the cumulative hazard of $F$. Last equality follows from the fact that $E[\lambda_{\hat{\theta},h}(x)]$ depends only on the kernel and the bandwidth, and not on the sample size. Hence, $E[LSCV(h)]$ is asymptotically equivalent to $S(\lambda_{\hat{\theta},h})$, which suggests that we can expect $h_{LSCV}$ to be close to the minimizer of $S(\lambda_{\hat{\theta},h})$, that is, the minimizer of $MISE(\lambda_{\hat{\theta},h})$. 

17
Smoothed bootstrap

The smoothed bootstrap procedure can be described as follows. In order to simplify the presentation, we restrict our attention to the semiparametric estimator $\hat{\lambda}_{\hat{\theta}, h}$. The proposed method can be adapted to the nonparametric estimator in an obvious way; see below. For fixed $B$ and for $b = 1, \ldots, B$:

1. Let $X_{b,i}^0, i = 1, \ldots, n$, be an i.i.d. sample from $f_{\hat{\theta}, h_0}(x) = \alpha_n n^{-1} \sum_{i=1}^n K_{h_0}(x - X_i) G_n(X_i)^{-1}$, where the pilot bandwidth $h_0$ is chosen to be $\hat{h}_{LSCV}$ (other choices for $h^0$ are possible as well). Let $(U_{b,i}^0, V_{b,i}^0)$, $i = 1, \ldots, n$, be an i.i.d. sample from $T_{\hat{\theta}, h}^0$. Next, for each $i = 1, \ldots, n$, we keep the triplet $(U_{b,i}^0, X_{b,i}^0, V_{b,i}^0)$ in the resample only if the condition $t_{b,i}^0 \leq X_{b,i}^0 \leq v_{b,i}^0$ is fulfilled. If not, the same resampling procedure is repeated until a triplet satisfying the inequality is found.

2. Let $\hat{\theta}_{b}^{\text{boot}}$ and $\lambda_{\hat{\theta}_{b}^{\text{boot}}, b, h}^{\text{boot}}$ be the estimator of $\theta$ (that is, $\hat{\theta}$) and of the hazard $\lambda(\hat{\theta}_{b, h})$ respectively, obtained from the bootstrap sample $(U_{b,i}^{\text{boot}}, X_{b,i}^{\text{boot}}, V_{b,i}^{\text{boot}}), i = 1, \ldots, n$.

Variability of $\hat{\theta}$ and $\lambda_{\hat{\theta}, h}$ is then estimated by that of the $B$ bootstrap evaluations $\hat{\theta}_{b}^{\text{boot}}$ and $\lambda_{\hat{\theta}_{b}^{\text{boot}}, b, h}^{\text{boot}}$, $1 \leq b \leq B$. Note that the resampling plan above is an obvious bootstrap, as opposed to the simple bootstrap which directly resamples with replacement from the triplets $(U_{i}, X_{i}, V_{i})$ [27]. This allows for an easy introduction of the semiparametric information, since the truncation couple is resampled from the estimated parametric model $T_{\hat{\theta}}$. When the focus is the nonparametric estimator $\lambda_{b}$, the parametric distribution $T_{b}$ is replaced by the NPMLE $T_{b}^{\text{NP}}$ in Step 1 above; and, obviously, the semiparametric density estimator $f_{\hat{\theta}, h_0}(x)$ is replaced by its nonparametric counterpart $f_{b}(x) = \alpha_n n^{-1} \sum_{i=1}^n K_{h_0}(x - X_i) G_n(X_i)^{-1}$.

References

[1] Nichols, M., N. Townsend, P. Scarborough, and M. Rayner, Cardiovascular disease in Europe 2014: epidemiological update, *European Heart Journal* 35(42), 2950–2959 (2014)

[2] Araújo, C., O. Laszczyńska, M. Viana, F. Melão, A. Henriques, A. Borges, M. Severo, M. J. Maciel, I. Moreira, and A. Azevedo, Sex differences in presenting symptoms of acute coronary syndrome: the epiheart cohort study, *European Heart Journal* 35, 2950–2959 (2014)

[3] Mandel, M., de Uña-Álvarez I., Simon D. K., and Betensky R. A., Inverse probability weighted cox regression for doubly truncated data, *Biometrika* 101(3), 519–533. (2014)

[4] de Uña-Álvarez, J. and Van Keilegom, I., An asymptotic representation of Efron-Petrosian integrals with covariates: an asymptotic analysis, *Bernoulli* 27, 249–273 (2021)

[5] Moreira, C., J. de Uña-Álvarez, and R. Crujeiras, Dtda: an r package to analyze randomly truncated data, *Journal of Multivariate Analysis* 167, 1–20 (2018)

[6] Xiao, J. and M. G. Hudgens, On nonparametric maximum likelihood estimation with double truncation, *Biometrika* 106(3), 519–533 (2019)

[7] Efron, B. and V. Petrosian, Nonparametric methods for doubly truncated data, *Journal of the American Statistical Association* 94, 824–834 (1999)

[8] Rennert, L. and S. X. Xie, Bias induced by ignoring double truncation inherent in autopsy-confirmed survival studies of neurodegenerative diseases, *Statistics in Medicine* 38, 3599–3613 (2019)

[9] Zhu, H. and M.-C. Wang, Nonparametric inference on bivariate survival data with interval sampling: association estimation and testing, *Biometrika* 101(3), 519–533 (2014)

[10] Wand, M. P. and M. C. Jones, *Kernel Smoothing*, Volume 60 of *Monographs on Statistics and Applied Probability*, London: Chapman and Hall Ltd. (1985)

[11] Diehl, S. and W. Stute, Kernel density and hazard function estimation in the presence of censoring, *Journal of Multivariate Analysis* 25(2), 299 – 310 (1988)

[12] Müller, H. G. and J. L. Wang, Hazard rate estimation under random censoring with varying kernels and bandwidths, *Biometrika* 50(1), 61–76 (1994)

[13] Cai, Z., Kernel density and hazard rate estimation for censored dependent data, *Journal of Multivariate Analysis* 67(1), 23 – 34 (2018)
[14] Zhou, Y. Asymptotic representations for kernel density and hazard function estimators with left truncation, *Statistica Sinica* 9(2), 521–533 (1999)

[15] Lynden-Bell, D., A method for allowing for known observational selection in small samples applied to 3cr quasars, *Monthly Notices of the Royal Astronomical Society* 155, 95–118 (1971)

[16] Lemdani, M. and E. Ould-Saïd, Asymptotic behavior of the hazard rate kernel estimator under truncated and censored data, *Communications in Statistics - Theory and Methods* 36(1), 155–173 (2007)

[17] Moreira, C. and J. de Uña-Álvarez, Kernel density estimation with doubly truncated data, *Electron. J. Statist.* 6, 501–521 (2012)

[18] Moreira, C. and J. de Uña-Álvarez (2010a), A semiparametric estimator of survival for doubly truncated data, *Statistics in Medicine* 29, 3147–3159 (2010)

[19] Moreira, C. and I. V. Keilegom Bandwidth selection for kernel density estimation with doubly truncated data, *Computational Statistics & Data Analysis* 61, 107 – 123 (2013)

[20] Kalbfleisch, J. D. and J. F. Lawless, Inference based on retrospective ascertainment: An analysis of the data on transfusion-related aids, *American Statistical Association* 84, 360–372 (1989)

[21] Woodroofe, M., Estimating a distribution function with truncated data, *The Annals of Statistics* 13, 163–177 (1985)

[22] Shen, P., Nonparametric analysis of doubly truncated data, *Annals of the Institute of Statistical Mathematics* 62, 835–853 (2010)

[23] Turnbull, Bruce W., The Empirical Distribution Function with Arbitrarily Grouped, Censored and Truncated Data, *Journal of the Royal Statistical Society. Series B* 38, 290–295 (1976)

[24] Vardi, Y., Empirical distributions in selection bias models, *Ann. Statist* 13, 178—203 (1985)

[25] Silverman, B.W., Density Estimation for Statistics and Data Analysis. *Monographs on Statistics and Applied Probability 26 Chapman and Hall* (1986)

[26] Bilker, W.B. and M.-C. Wang, A semiparametric extension of the mann-whitney test for randomly truncated data, *Biometrics*, 52, 10–20 (1996)

[27] Moreira, C. and J. de Uña-Álvarez, Bootstraping the npmle for doubly truncated data, *Journal of Nonparametric Statistics* 22, 567–583 (2010)

[28] Devroye, L.P. and Wagner, T.J., The $L^1$ convergence of kernel density estimates, *The Annals of Statistics* 7, 1136–1139 (1979)

[29] Parzen, E., On estimation of a probability density function and mode, *Annals of Mathematical Statistics 33*, 1065–1076 (1962)