ON VOLTERRA INTEGRAL OPERATORS WITH HIGHLY OSCILLATORY KERNELS

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Dedicated to Professor Arieh Iserles on the occasion of his 65th birthday

Abstract. We study the high-oscillation properties of solutions to integral equations associated with two classes of Volterra integral operators: compact operators with highly oscillatory kernels that are either smooth or weakly singular, and noncompact cordial Volterra integral operators with highly oscillatory kernels. In the latter case the focus is on the dependence of the (uncountable) spectrum on the oscillation parameter. It is shown that the results derived in this paper merely open a window to a general theory of solutions of highly oscillatory Volterra integral equations, and many questions remain to be answered.

1. Introduction. We consider linear Volterra integral equations (of the first and second kind) corresponding to the following two classes of Volterra integral operators on $C[0,T]$ with highly oscillatory kernels that are either smooth or weakly singular.

I. Compact Volterra integral operators:
If the kernel $K = K(t,s)$ and the oscillator $g = g(t,s) \geq 0$ are continuous on $D := \{(t,s): 0 \leq s \leq t \leq T\}$, the Volterra integral operators

$$ (V_\omega u)(t) := \int_0^t K_\omega(t,s)u(s)\,ds $$

and

$$ (V_{\omega,\alpha} u)(t) := \int_0^t K_{\omega,\alpha}(t,s)u(s)\,ds \quad (0 < \alpha < 1), $$

with kernels

$$ K_\omega(t,s) := K(t,s)e^{i\omega g(t,s)} $$

and

$$ K_{\omega,\alpha}(t,s) := (t-s)^{\alpha-1}K_\omega(t,s) \quad (0 < \alpha < 1), $$

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are bounded and compact (as operators on $C[0,T]$) for all $\omega \geq 0$ and all $\alpha \in (0,1]$. One of the fundamental properties of these Volterra integral operators is that their spectrum is $\sigma(V_{\varphi,\alpha}) = \{0\}$ for all $\alpha \in (0,1]$ and all $\omega$; there are no eigenvalues.

II. Cordial (non-compact) Volterra integral operators:

Volterra integral operators with certain types of weakly singular kernels (other than the one in (1.4)) are not compact, as the following example reveals (cf. [1, p.20]). Consider the operator $V_\varphi : C[0,1] \to C[0,1]$ given by

$$ (V_\varphi u)(t) := \int_0^t (t^2 - s^2)^{-1/2} u(s) \, ds = \int_0^t t^{-1} [1 - (s/t)^2]^{-1/2} u(s) \, ds, \quad t \in [0,1], $$

(1.5)

where $\varphi(s/t) := [1 - (s/t)^2]^{-1/2}$ denotes the ‘core’ of this operator. A simple calculation shows that for any $r \in \mathbb{R}^+$ the function $\phi_r(t) := t^r$ is an eigenfunction corresponding to the eigenvalue

$$ \lambda_r := \int_0^{\pi/2} \sin^r(\theta) \, d\theta = \frac{1}{2} B\left(\frac{1+r}{2}, \frac{1}{2}\right) \in (0, \pi/2], $$

(1.6)

where $B(\cdot, \cdot)$ denotes the Euler beta function. This implies that the spectrum $\sigma(V_\varphi)$ is uncountable, and hence $V_\varphi$ is not compact (compare also Section 3.1).

This is a simple example of what Vainikko [22] has called a cordial Volterra integral operator. The general form of such an operator is

$$ (V_\varphi u)(t) := \int_0^t t^{-1} \varphi(s/t) K(t,s) u(s) \, ds, $$

(1.7)

where the so-called core $\varphi$ is assumed to be in $L^1(0,1)$ and the (continuous) kernel $K$ satisfies $K(0,0) \neq 0$ (if $K(0,0) = 0$, the Volterra operator $V_\varphi$ is compact; cf. [23]). Linear cordial Volterra integral operators (1.7) were studied by Vainikko in the papers [23, 24].

The following three representative examples of cores reveal the nature of the underlying kernel singularities in the cordial Volterra integral operator (1.7) (see also [22]); we shall return to them in Section 3.

**Example 1.1.** $\varphi(z) = (1 - z^\gamma)^{\alpha-1}$, with $\gamma > 1$, $0 < \alpha < 1$, $\gamma(\alpha - 1) = -1$.

This core $\varphi$ is associated with the kernel singularity

$$ (t^\gamma - s^\gamma)^{\alpha-1} = t^{\gamma(\alpha-1)}[1 - (s/t)^\gamma]^{\alpha-1} = t^{-1}[1 - (s/t)^\gamma]^{-1/\gamma}. $$

The Volterra operator of (1.5) corresponds to $\gamma = 2$, $\alpha = 1/2$, while the operator in Lighthill’s (nonlinear) integral equation arising in certain heat transfer problems ([17]; see also [6] and [9]) corresponds to the values $\gamma = 3/2$, $\alpha = 1/3$.

**Example 1.2.** $\varphi(z) = z^{\beta-1}$, with $\beta > 0$.

Second-kind Volterra integral equations with the kernel singularity

$$ t^{-\beta} s^{\beta-1} = t^{-1} (s/t)^{\beta-1} = t^{-1} \varphi(s/t) $$

were studied in [11] and [7] (see also for additional references).

**Example 1.3.** $\varphi(z) = (1 - z)^{-\nu}$, with $0 < \nu < 1$.

If a kernel has boundary and diagonal singularities (cf. [18]) such as

$$ t^{\nu-1} (t-s)^{-\nu} = t^{\nu-1} (1 - (s/t))^{-\nu} = t^{-1} (1 - (s/t))^{-\nu}, $$

it corresponds to the core $\varphi(z) = (1 - z)^{-\nu}$. 

The highly oscillatory version of the cordial Volterra integral operator \( \mathcal{V}_\varphi \) in (1.7) is
\[
(\mathcal{V}_\omega \varphi u)(t) := \int_0^t t^{-1} \varphi(s/t) K(t, s) e^{i \omega g(t, s)} u(s) \, ds \quad (\omega \gg 1),
\] (1.8)
where \( K \in C(D) \), with \( K(0, 0) \neq 0 \), the oscillator \( g \) is smooth and nonnegative on \( D \), and \( K \) and \( g \) do not depend on \( \omega \). We will assume throughout this paper that the given functions \( K \) and \( g \) are real-valued.

This paper is concerned with the following questions:

- Let \( u \in C(I) \) be a solution of the second-kind VIE
  \[
  u(t) = f(t) + \int_0^t K_{\omega, \alpha}(t, s) u(s) \, ds, \quad t \in I \quad (\omega \gg 1),
  \] (1.9)
or of the analogous first-kind VIE
  \[
  \int_0^t K_{\omega, \alpha}(t, s) u(s) \, ds = f(t), \quad t \in I \quad (\omega \gg 1),
  \] (1.10)
where the highly oscillatory kernel \( K_{\omega, \alpha} \) \((0 < \alpha \leq 1)\) is given by (1.3) or (1.4). Is \( u \) necessarily highly oscillatory?

- Does the spectrum of the cordial Volterra operator (1.8) with highly oscillatory kernel (1.3) always (i.e. for all oscillators \( g(t, s) \)) depend on \( \omega \)?

- If \( \mu \in \mathbb{R} \) is not in the spectrum of the cordial Volterra integral operator \( \mathcal{V}_\varphi \) defined in (1.7), does the highly oscillatory cordial VIE
  \[
  \mu u(t) = f(t) + \int_0^t t^{-1} \varphi(s/t) K_{\omega}(t, s) u(s) \, ds, \quad t \in I,
  \] (1.11)
with \( K_{\omega}(t, s) \) as in (1.3), have a unique solution \( u \in C(I) \) for all \( \omega > 0 \), and is it highly oscillatory? An analogous question can be asked about the solution of the cordial VIE of the first kind,
  \[
  \int_0^t t^{-1} \varphi(s/t) K_{\omega}(t, s) u(s) \, ds = f(t), \quad t \in I.
  \] (1.12)
The functions \( f \) in (1.11) and (1.12) are assumed to be independent of \( \omega \).

The content of the paper is as follows. In Section 2 we shall look at the representation of solutions to highly oscillatory VIEs with smooth or weakly singular kernels. The results will yield some first insight into the answer of the first of the above questions; they will also reveal that the oscillatory behaviour of solutions to a first-kind VIE can be rather different from the one of the analogous second-kind VIE. Section 3 is concerned with the spectra of cordial VIEs with highly oscillatory kernels. It turns out that for certain oscillators the spectra do not depend on \( \omega \). Section 4 is devoted to a brief description of some open problems: it will be seen that a complete understanding of the oscillatory behaviour of solutions to VIEs corresponding to general oscillators is still lacking.

2. Highly oscillatory VIEs with compact operators.

2.1. VIEs of the second kind. In order to obtain some first insight into the nature of the solution of a second-kind VIE with highly oscillatory kernels (1.3) or (1.4) we consider the ‘separable’ oscillator
\[
\varphi(t, s) = g_0(t) - g_0(s), \quad (t, s) \in D,
\] (2.1)
with \( g_0 \) smooth, strictly increasing and \( g_0(0) \geq 0 \). The linear oscillator
\[
g(t, s) = t - s
\]
(2.2)
corresponds to \( g_0(t) = t \) and represents the most obvious special case.

The following theorem reveals that the resolvent kernel \( R_{\omega, \alpha}(t, s) \) inherits the highly oscillatory term of the kernel \( K_{\omega, \alpha}(t, s) \) for all \( 0 < \alpha \leq 1 \) in (1.4).

**Theorem 2.1.** Assume that the given functions \( K \) and \( g \) in (1.3) and (1.4) have the properties

(i) \( K \) is in \( C(D) \) and does not depend on \( \omega \);

(ii) the oscillator \( g \) is separable, i.e. \( g(t, s) = g_0(t) - g_0(s) \), satisfies the conditions stated in (2.1) and does not depend on \( \omega \).

Then the following statements are true:

(a) For any \( \alpha \in (0, 1] \), the resolvent kernel associated with the kernel \( K_{\omega, \alpha}(t, s) \) in (1.4) is given by
\[
R_{\omega, \alpha}(t, s) = R_{\alpha}(t, s)e^{i\omega g(t, s)},
\]
where the resolvent kernel \( R_{\alpha}(t, s) \) of the kernel \( (t - s)^{\alpha-1}K(t, s) \) does not depend on \( \omega \).

If \( 0 < \alpha < 1 \), the resolvent kernel inherits the weak singularity: we have
\[
R_{\omega, \alpha}(t, s) = (t - s)^{\alpha-1}Q_{\omega, \alpha}(t, s), \quad 0 \leq s < t \leq T,
\]
where
\[
Q_{\omega, \alpha}(t, s) := Q_{\alpha}(t, s)e^{i\omega g(t, s)},
\]
for some \( Q_{\alpha} \in C(D) \) that does not depend on \( \omega \).

(b) For any \( f \in C(I) \) the highly oscillatory VIE (1.9) possesses a unique solution \( u \in C(I) \) that has the form
\[
u(t) = f(t) + \int_0^t R_{\omega, \alpha}(t, s)f(s)\,ds, \quad t \in I.
\]
(2.5)

**Proof.** Multiply both sides of (1.9) by \( e^{-i\omega g_0(t)} \) and set
\[
u_\omega(t) := e^{-i\omega g_0(t)} \quad \text{and} \quad f_\omega(t) := e^{-i\omega g_0(t)}.
\]
(2.6)

This yields a second-kind VIE for \( u_\omega \), namely
\[
u_\omega(t) = f_\omega(t) + \int_0^t (t - s)^{\alpha-1}K(t, s)u_\omega(s)\,ds \quad (0 < \alpha \leq 1).
\]
(2.7)

It follows from the classical Volterra theory (see for example [3, Section 6.1.2]) that its unique solution \( u_\omega \in C(I) \) has the form
\[
u_\omega(t) = f_\omega(t) + \int_0^t R_\alpha(t, s)f_\omega(s)\,ds,
\]
where \( R_\alpha(t, s) \) denotes the resolvent kernel of the given kernel \( (t - s)^{\alpha-1}K(t, s) \). If \( 0 < \alpha < 1 \) the resolvent kernel corresponding to \( (t - s)^{\alpha-1}K(t, s) \) inherits the weak singularity and is given by
\[
R_\alpha(t, s) = (t - s)^{\alpha-1}Q_\alpha(t, s),
\]
for some \( Q_\alpha \in C(D) \). The assertions (a) and (b) then follow immediately from the definitions (2.6). \( \Box \)
We use two examples to illustrate the statements (a) and (b) of the above theorem for the case of the linear oscillator (2.2) (corresponding to \(g_0(t) = t\) in (2.1)), and to obtain insight into the damping of highly oscillatory solutions.

**Example 2.1.** If \(f(t) = 1\), \(K(t, s) = \lambda \neq 0\) in (1.3) \((\alpha = 1)\), the resolvent kernel of \(K(t, s)\) is \(R(t, s) = \lambda e^{\lambda(t-s)}\). Thus, by (2.5) the solution of the corresponding highly oscillatory VIE (1.9) with \(\alpha = 1\) (continuous kernel) is readily seen to be

\[
u(t) = 1 - \frac{\lambda}{\lambda + i\omega} + \frac{\lambda e^{\lambda t}}{\lambda + i\omega} e^{i\omega t}, \quad t \in I.
\]

It is highly oscillatory for large \(\omega\) but tends to the value 1 as \(\omega\) tends to \(\infty\). (For \(\omega = 0\) we have of course \(u(t) = e^{\lambda t}\).) We observe that the highly oscillatory term in the expression for \(u(t)\) contains the damping factor \((\lambda + i\omega)^{-1}\).

This raises the question as to whether stronger damping of the highly oscillatory terms in the corresponding solutions of (1.9) is possible for certain classes of smooth (nonconstant) functions \(f\). The following theorem provides the answer.

**Theorem 2.2.** Assume that \(f \in C^{q+1}(I)\) for some \(q \geq 0\). Then the solution of the second-kind VIE

\[
u(t) = f(t) + \lambda \int_0^t e^{i\omega(t-s)} u(s) \, ds, \quad t \in I,
\]

has the form

\[
u(t) = f(t) - \lambda \sum_{j=0}^q \frac{f^{(j)}(t)}{(\lambda + i\omega)^{j+1}} + \lambda \sum_{j=0}^q \frac{f^{(j)}(0)}{(\lambda + i\omega)^{j+1}} e^{\lambda t} e^{i\omega t}
\]

\[+ \frac{\lambda}{(\lambda + i\omega)^{q+1}} \int_0^t e^{(t-s)(\lambda + i\omega)} f^{(q+1)}(s) \, ds.
\]

**Proof.** The resolvent kernel of the constant kernel \(K(t, s) = \lambda\) is \(R_1(t, s) = \lambda e^{\lambda(t-s)}\). Hence, according to Theorem 2.1 \((\alpha = 1)\) the solution of the VIE is given by

\[
u(t) = f(t) + \lambda \int_0^t e^{\lambda(t-s)} e^{i\omega(t-s)} f(s) \, ds, \quad t \in I.
\]

The integral in the above expression is highly oscillatory. In order to derive an asymptotic expansion in terms of powers of \((\lambda + i\omega)^{-1}\) we adapt the well-known technique used for highly oscillatory integrals of the form \(\int_0^1 e^{i\omega s} f(s) \, ds\) (see for example [14, 8]), with the difference that now the upper limit of integration is variable. Thus, integration by parts leads to the desired result. \(\square\)

**Remark 2.1.** If \(f \in C^{q+1}(I)\) \((q \geq 1)\) is such that \(f^{(j)}(0) = 0\) for \(j = 0, 1, \ldots, q-1\), then the damping factor in the highly oscillatory term of the solution \(u(t)\) is \((\lambda + i\omega)^{-(q+1)}\). In Example 2.1 we have \(f(0) = 1 \neq 0\) and hence \(q = 1\), which leads to the damping factor \((\lambda + i\omega)^{-1}\).

**Example 2.2.** Consider the VIE (1.9) with \(f \in C(I)\) and \(K(t, s) = \lambda/\Gamma(\alpha)\) \((\lambda \neq 0, 0 < \alpha < 1)\). The unique solution \(u \in C(I)\) of the corresponding highly oscillatory VIE (1.9),

\[
u(t) = f(t) + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{i\omega(t-s)} u(s) \, ds,
\]

\[(2.8)\]
is given by
\[ u(t) = f(t) + \int_0^t R_\alpha(t,s)e^{i\omega(t-s)}f(s) \, ds, \quad t \in I \]
(recall Theorem 2.1(b)). Here,
\[ R_\alpha(t,s) = \frac{d}{dt}E_\alpha(\lambda(t-s)^\alpha) = (t-s)^\alpha - 1 \sum_{n=1}^\infty \frac{(\lambda \Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{(n-1)\alpha}, \quad (t,s) \in D. \]
We see that the solution of the weakly singular VIE (2.8) is highly oscillatory, similarly to Example 2.1.

2.2. VIEs of the first kind. We shall see that the oscillation properties of solutions to highly oscillatory second-kind VIEs of the form (1.9) may be rather different from the ones of solutions to analogous first-kind VIEs (1.10). Before presenting some general results, we look at the solution of a simple singularly perturbed VIE (compare also the review paper [15]) linking the two types of VIEs.

Example 2.3. Consider the highly oscillatory, singularly perturbed VIE
\[ \varepsilon u(t) = f(t) + \int_0^t K(t,s)e^{i\omega g(t,s)}u(s) \, ds, \quad t \in I \quad (\omega \gg 1, \ 0 < \varepsilon \ll 1). \quad (2.9) \]
The reduced VIE (corresponding to \( \varepsilon = 0 \)),
\[ 0 = f(t) + \int_0^t K(t,s)e^{i\omega g(t,s)}u(s) \, ds, \quad t \in I, \quad (2.10) \]
is a first-kind VIE with highly oscillatory kernel. Suppose now that in (2.9) we choose \( f(t) = \sin(t) \), \( K(t,s) \equiv -1 \), and the linear oscillator \( g(t,s) = t-s \). It is readily verified that for \( \varepsilon > 0 \) the unique solution \( u \in C(I) \) of this VIE is
\[ u(t) = \gamma_\varepsilon[\cos(t) + \varepsilon(1 - i\omega/\varepsilon - \omega^2)\sin(t) - e^{-t/\varepsilon}e^{i\omega t}], \]
where \( \gamma_\varepsilon := [1 + \varepsilon^2(1 - \omega^2) - 2i\omega\varepsilon]^{-1} \) and \( u(0) = 0 \). This reveals that when \( 0 < \varepsilon \ll 1 \), the solution contains the highly oscillatory boundary-layer term (‘inner solution’) \(-\gamma_\varepsilon e^{-t/\varepsilon}e^{i\omega t}\). On the other hand, the solution
\[ u_0(t) = \cos(t) - i\omega \sin(t), \quad \text{with} \quad u_0(0) = 1 - i\omega \neq 0, \]
of the reduced first-kind VIE (2.10) is non-oscillatory.

We now show that for a certain class of first-kind VIEs with smooth highly oscillatory kernels, the solutions are non-oscillatory.

Theorem 2.3. The solution of the first-kind VIE
\[ \int_0^t K_\omega(t,s)u(s) \, ds = f(t), \quad t \in I, \quad (2.11) \]
with highly oscillatory kernel
\[ K_\omega(t,s) := \frac{(t-s)^r}{r!}e^{i\omega(t-s)} \quad (\omega \gg 1, \ r \in \mathbb{N}), \quad (2.12) \]
with \( f \in C^r(I) \) not depending on \( \omega \) and satisfying \( f^{(j)}(0) = 0 \ (j = 0, \ldots, r-1) \), is non-oscillatory:
\[ u(t) = f^{(r+1)}(t) + \sum_{j=1}^{r+1} \binom{r+1}{j}(-i\omega)^j f^{(r+1-j)}(t). \quad (2.13) \]
For \( r = 0 \) we obtain
\[
    u(t) = f'(t) - i\omega f(t).
\]

**Proof.** The result is trivial when \( r = 0 \). If \( r \geq 1 \), we differentiate the VIE (2.11) \( r + 1 \) times to obtain the desired result (2.13). (We note that the VIE (2.11) with kernel (2.12) does not have a unique continuous solution on the closed interval \([0, T]\) when \( f^{(j)}(0) = 0 \) for \( j = 0, \ldots, q \) \((q < r - 1)\) but \( f^{(r-1)}(0) \neq 0\). \)

**Remark 2.2.** We observe that the solution (2.13) reduces to the well-known solution \( u(t) = f(r+1)(t) \) of (2.11) when \( \omega = 0 \) in (2.12). When \( \omega > 0 \) this result is modified by a non-oscillatory \( \omega \)-perturbation’ involving lower-order derivatives of \( f(t) \). This raises the obvious question as to whether this remains true (i) for first-kind VIEs with more general (continuous) kernels, and (ii) for first-kind VIEs with weakly singular kernels?

As a first step towards answering this question, we consider a first-kind VIE with linear oscillator,
\[
    \int_0^t k(t-s)e^{i\omega(t-s)}u(s)\,ds = f(t), \quad t \in I,
\]
where \( k(t-s) \) is a smooth, non-constant convolution kernel satisfying \( k(0) \neq 0 \) and \( f \in C^1(I) \), with \( f(0) = 0 \). If we resort again to the substitutions (2.6) we see that the differentiated form of (2.15) may be written as
\[
    u_\omega(t) + \int_0^t k'(t-s)u_\omega(s)\,ds = f_\omega'(t), \quad t \in I
\]
(where we have assumed without loss of generality that \( k(0) = 1 \).

**Theorem 2.4.** Assume that the resolvent kernel associated with the kernel \(-k'(t-s)\) in (2.16) is \( r_1(t-s) \). Then the (unique) solution of the first-kind VIE (2.15) is given by
\[
    u(t) = f'(t) + \int_0^t r_1(t-s)e^{i\omega(t-s)}f'(s)\,ds \,
    -i\omega \left( f(t) + \int_0^t r_1(t-s)e^{i\omega(t-s)}f(s)\,ds \right), \quad t \in I.
\]

**Proof.** The proof is straightforward, since by using again the notation of (2.6) and the fact that the solution of (2.16) is
\[
    u_\omega(t) = f_\omega'(t) + \int_0^t r_1(t-s)f_\omega'(s)\,ds,
\]
where
\[
    f_\omega'(t) = [-i\omega f(t) + f'(t)]e^{-i\omega t},
\]
we obtain the desired result (2.17). \)

**Example 2.4.** Let \( k(t-s) = 1 + t - s \) in (2.15). Since the resolvent kernel \( r_1(t-s) \) corresponding to \(-k'(t-s)\) is \( r_1(t-s) = -e^{-(t-s)} \) (cf. Example 2.1 with \( \lambda = -1 \)), it is readily verified that the unique solution of
\[
    \int_0^t (1 + t-s)e^{i\omega(t-s)}u(s)\,ds = f(t), \quad t \in I,
\]

\( \blacksquare \)
corresponding to \( f(t) = t \) is

\[
u(t) = 1 - \frac{1}{(1 - i\omega)^2} + \frac{(i\omega)^2}{1 - i\omega}t + \frac{e^{-t}}{(1 - i\omega)^2}i\omega t.
\]

Hence the solution of the first-kind VIE (2.15) is highly oscillatory, in contrast to (2.13) and (2.14). Moreover, the highly oscillatory term contains the damping factor \((1 - i\omega)^{-2}\). This is similar to the damping observed in Example 2.1 for second-kind VIEs.

It turns out that stronger damping of the highly oscillatory component of the solution is possible for certain classes of smooth functions \( f \) in the first-kind VIE (2.15), in analogy to the result of Theorem 2.2. The following theorem makes this precise.

**Theorem 2.5.** Assume that \( f \in C^{q+1}(I) \) \((q \geq 1)\), with \( f(0) = 0 \). Then the unique solution \( u \in C(I) \) of the first-kind VIE (2.18) can be written in the form

\[
u(t) = f'(t) - i\omega f(t) + \sum_{j=1}^{q} \frac{(-1)^j f^{(j)}(t)}{(1 - i\omega)^j} + i\omega \sum_{j=0}^{q} \frac{(-1)^j f^{(j)}(0)}{(1 - i\omega)^j}
\]

\[+ \frac{(-1)^{q+1}}{(1 - i\omega)^q} \int_0^t e^{-(t-s)(1-i\omega)} f^{(q+1)}(s) ds + i\omega \frac{(-1)^{q+1}}{(1 - i\omega)^q+1} \int_0^t e^{-(t-s)(1-i\omega)} f^{(q+1)}(s) ds.
\]

**Proof.** We proceed as in the proof of Theorem 2.2, using integration by parts for the two highly oscillatory integrals in the solution representation (2.17), where the resolvent kernel is \( r_1(t) = e^{-(t-s)} \). The asymptotic expansion of the second integral is the one we used in that proof; this then yields the expansion of the first integral, by replacing \( f(s) \) by \( f'(s) \). We leave the details to the reader.

We conclude this section by stating the analogue of Theorem 2.3 for the weakly singular first-kind VIE

\[
\int_0^t (t-s)^{-\alpha-1} K(t, s)e^{i\omega g(t,s)} u(s) ds = f(t), \quad t \in I \quad (0 < \alpha < 1),
\]

with linear oscillator \( g(t,s) = t-s \). Here, \( K, g \) and \( f \) are assumed to be smooth and independent of \( \omega \), with \( K(t, t) \neq 0 \) for \( t \in I \).

**Theorem 2.6.** Assume that

\[
K(t, s) = \frac{(t-s)^{r+\alpha-1}}{\Gamma(r+\alpha)} \quad (r \in \mathbb{N}, \ 0 < \alpha < 1).
\]

Then the solution of the first-kind VIE (2.19), with oscillator \( g(t,s) = t-s \) and with \( f \in C^{r+1}(I) \), \( f^{(j)}(0) = 0 \) \((j = 0, \ldots, r)\), has the form

\[
u(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} e^{i\omega(t-s)} f^{(r+1)}(s) ds \quad (2.20)
\]

\[+ i\omega \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} e^{i\omega(t-s)} \sum_{j=1}^{r+1} \binom{r+1}{j} (-i\omega)^j f^{(r+1-j)}(s) ds, \quad t \in I.
\]
For $r = 0$ the solution reduces to

$$u(t) = \int_0^t (t-s)^{-\alpha} \frac{e^{i\omega(t-s)}}{\Gamma(1-\alpha)} f'(s) ds$$

$$-i\omega \int_0^t (t-s)^{-\alpha} \frac{e^{i\omega(t-s)}}{\Gamma(1-\alpha)} f(s) ds, \quad t \in I.$$  \hfill (2.21)

Proof. If we multiply both sides of (2.19) by $e^{-i\omega t}$ and resort once more to the substitutions (2.6), we obtain the first-kind VIE

$$\int_0^t (t-s)^{r+\alpha-1} \frac{u_\omega(s)}{\Gamma(r+\alpha)} ds = f_{\omega}(t)$$

for $u_\omega(t) = e^{-i\omega t}u(t)$. It then follows from the classical inversion formula for Abel integral equations (cf. [3, Section 6.1.4]) and our assumptions on $f^{(j)}(0)$ ($j = 0, \ldots, r$) that its solution is given by

$$u_\omega(t) = \int_0^t (t-s)^{-\alpha} \frac{f^{(r+1)}(s)}{\Gamma(1-\alpha)} ds,$$  

where $f_\omega(t) = e^{-i\omega t}f(t)$. The result (2.20) then follows. \hfill □

Remark 2.3. We observe that, as for first-kind VIEs with smooth, highly oscillatory kernel (cf. Remark 2.1), the solution (2.20) of the weakly singular VIE (2.19) may again be viewed as an ‘$\omega$-perturbation’ of the solution corresponding to $\omega = 0$ (which is the fractional derivative of order $r + 1$ of $f$ in (2.19)).

2.3. Other types of oscillators. We show briefly, by means of two examples, that the solutions of first-kind VIEs with other types of highly oscillator kernels behave in a way similar to what we have already seen in Example 2.4 and in Theorem 2.6.

Example 2.5. If $f \in C^1(I)$ and $f(0) = 0$, the (unique) solution $u \in C(I)$ of the VIE

$$\int_0^t \cos(\omega(t-s))u(s) ds = f(t) \quad (\omega \gg 1)$$

is

$$u(t) = f'(t) + \omega^2 \int_0^t f(s) ds, \quad t \in I.$$  

It is clearly not highly oscillatory.

Example 2.6. Consider the weakly singular first-kind VIE

$$\int_0^t (t-s)^{-1/2} \cos(\omega(t-s))u(s) ds = f(t), \quad t \in I.$$  \hfill (2.22)

Using again Laplace transform techniques it is seen that, under the assumptions $f \in C^2(I)$ and $f(0) = f'(0) = 0$, its unique solution in $C(I)$,

$$u(t) = \frac{2}{\pi \omega} \int_0^t (t-s)^{-1/2} \sin(\omega(t-s))[f''(s) + \omega^2 f(s)] ds$$

(see also [19, p.81]) is highly oscillatory. For $\omega = 0$ it reduces to the well-known inversion formula for the non-oscillatory version of (2.23): we have

$$u(t) = \frac{1}{\pi} \int_0^t (t-s)^{-1/2} f'(s) ds,$$

for any $f \in C^1(I)$ with $f(0) = f'(0) = 0$. 
2.4. Fredholm integral equations with highly oscillatory kernels. Ursell [21] studied the asymptotic behaviour (as $\omega \to \infty$) of the solution of the Fredholm integral equation
\begin{equation}
  u(x) = f(x) + \nu(\mathcal{F}_\omega u)(x), \quad x \in [-1,1],
\end{equation}
with $\nu \neq 0$,
\begin{equation}
  (\mathcal{F}_\omega u)(t) := \int_{-1}^{1} K(x,y)e^{i\omega g(x,y)}u(y)\,dy, \quad (\omega \gg 1),
\end{equation}
and oscillator $g(x,y) = |x-y|$. He showed that when $\nu^{-1}$ is not in the spectrum of $\mathcal{F}_\omega$, and if $f$ and $K$ are continuous and independent of $\omega$, the solution $u = u(x;\omega)$ of (2.24) behaves like
\begin{equation}
  u(x;\omega) - f(x) = o(1) \quad \text{as} \quad \omega \to \infty.
\end{equation}

Ursell’s results for the above oscillator $g(x,y)$ were refined in [4]. The papers [5] and [2] address the analogous problem for the highly oscillatory Fredholm integral operator (2.25) with the so-called ‘Fox-Li’ oscillator $g(x,y) = (x-y)^2$ which arises in laser and maser engineering ([16], [20, Section 60], [2], and their references). Since these bounded linear Fredholm integral operators are compact, they have at most a countable number of (complex) eigenvalues that accumulate at the origin. However, the derivation of rigorous results on their existence and asymptotic behaviour is rather difficult since these Fredholm integral operators are complex-symmetric, but not Hermitian. As Landau says in [16], “this presents a major obstacle to a theoretical understanding of the equation [(2.24)] – indeed, even the existence of eigenvalues is difficult to prove”.

We note in passing that, to the author’s knowledge, the analysis of the asymptotic behaviour of the eigenvalues of (2.25) with weakly singular kernel $|x-y|^{-\alpha}$ ($0 < \alpha < 1$) and oscillators $|x-y|$ or $(x-y)^2$ remains open.

As we shall show in Section 3.2, the analogous problem for highly oscillatory cordial (non-compact) Volterra integral operators (and corresponding second-kind VIEs) is much more simple: here, the eigenvalues, as well as their asymptotic properties as $\omega \to \infty$, are now completely known (see Theorem 3.2 and Corollary 3.3).

3. Cordial VIEs with highly oscillatory kernels.

3.1. Basic properties of cordial Volterra operators. The following definition is due to Vainikko (cf. [22, 23]).

**Definition 3.1.** The Volterra integral operator
\begin{equation}
  (\mathcal{V}_\varphi u)(t) := \int_0^t t^{-1}\varphi(s/t)K(t,s)u(s)\,ds = \int_0^1 \varphi(x)K(t,tx)u(tx)\,dx, \quad t \in [0,T],
\end{equation}
is called a cordial Volterra integral operator if its core $\varphi$ is in $L^1(0,1)$, and $K \in C^m(D)$ for some $m \geq 0$, with $K(0,0) \neq 0$. It will be seen in Lemma 3.1 that under these assumptions $\mathcal{V}_\varphi$ is a non-compact operator. (We note that if $K(0,0) = 0$ then $\mathcal{V}_\varphi$ is compact.)

We shall occasionally state results for the basic cordial Volterra integral operator
\begin{equation}
  (\mathcal{V}_\varphi^0 u)(t) := \int_0^t t^{-1}\varphi(s/t)u(s)\,ds = \int_0^1 \varphi(x)u(tx)\,dx.
\end{equation}
It corresponds to the operator $\mathcal{V}_\varphi$ in (3.1) with $K(t,s) \equiv 1$. 
We first review some relevant basic properties of the cordial Volterra integral operators \((3.1)\) and \((3.2)\). The results are due to Vainikko ([22, 23]).

**Lemma 3.1.** Under the assumptions stated above, the following statements are true:

(a) \(\mathcal{V}_\varphi\) is a bounded linear operator that maps \(C(I)\) into itself, and we have
\[
\|\mathcal{V}_\varphi\|_\infty \leq \|\varphi\|_1\|K\|_\infty.
\]
(b) \(\mathcal{V}_\varphi\) is not compact.
(c) The spectrum of \(\mathcal{V}_\varphi\) is uncountable; it is related to the spectrum of the basic cordial Volterra operator \(\mathcal{V}_\varphi^0\) by
\[
\sigma(\mathcal{V}_\varphi) = K(0,0)\sigma(\mathcal{V}_\varphi^0) = \{0\} \cup \{K(0,0)\hat{\varphi}(\lambda) : \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0\},
\]
where
\[
\hat{\varphi}(\lambda) := \int_0^1 \varphi(z)z^\lambda dz \quad (\text{Re}(\lambda) \geq 0)
\]
is an eigenvalue of \(\mathcal{V}_\varphi^0\).
(d) In \((3.3)\), \(\{0\}\) belongs to the closure of the set \(\{K(0,0)\hat{\varphi}(\lambda) : \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0\}\), and
\[
\lim_{|\lambda| \to \infty} |\hat{\varphi}(\lambda)| = 0.
\]
(e) The basic cordial Volterra operator \(\mathcal{V}_\varphi^0\) maps polynomials of degree \(r\) into polynomials of the same degree; moreover, the function \(\hat{\varphi}(\lambda)(t) := t^\lambda\) is an eigenfunction for the eigenvalue \((3.4)\) of \(\mathcal{V}_\varphi^0\).
(f) If \(K \in C^m(I)\), then \(\mathcal{V}_\varphi\) maps \(C^m(I)\) into itself for any \(m \geq 1\).

**Example 3.1.** For the cores \(\varphi(z)\) introduced in Examples 1.1, 1.2 and 1.3,
\[
\varphi(z) = (1 - z)^{\gamma-1} / \gamma, \quad \varphi(z) = z^{\beta-1}, \quad \varphi(z) = (1 - z)^{-\nu},
\]
the eigenvalues of the corresponding basic cordial Volterra operators \(\mathcal{V}_\varphi^0\) of \((3.2)\) (i.e. \((3.1)\) with \(K(t, s) \equiv 1\)) are given by
\[
\hat{\varphi}(\lambda) = \gamma^{-1}\Gamma(1-\gamma^{-1})\Gamma(\gamma^{-1}(\lambda+1))/\Gamma(\gamma^{-1}\lambda+1) \quad (\gamma > 1, \text{Re}(\lambda) \geq 0),
\]
\[
\hat{\varphi}(\lambda) = 1/((\beta+\lambda)) \quad (\beta > 0, \text{Re}(\lambda) \geq 0),
\]
\[
\hat{\varphi}(\lambda) = \Gamma(1-\nu)\Gamma(1+\lambda)/\Gamma(\lambda+2-\nu) \quad (0 < \nu < 1, \text{Re}(\lambda) \geq 0),
\]
respectively; see [22, Section 6]. (This section also contains a good geometrical description of the sets \(\{\hat{\varphi}(\lambda) : \text{Re}(\lambda) \geq 0\}\), with \(\hat{\varphi}(\lambda)\) defined in \((3.4)\), for the above cores \(\varphi(z)\).)

3.2. The spectra of highly oscillatory cordial Volterra operators. Suppose that the cordial Volterra integral operator \(\mathcal{V}_\varphi\) defined by \((3.1)\) possesses the spectrum \(\sigma(\mathcal{V}_\varphi)\) given by \((3.3)\). How is the spectrum \(\sigma(\mathcal{V}_{\omega,\varphi})\) of the corresponding highly oscillatory cordial Volterra operator
\[
(\mathcal{V}_{\omega,\varphi}u)(t) = \int_0^t t^{-1}\varphi(s/t)K_\omega(t, s)u(s)ds \quad (\omega \gg 1),
\]
with
\[
K_\omega(t, s) := K(t, s)e^{i\omega g(t, s)},
\]
related to \(\sigma(\mathcal{V}_\varphi)\)?
**Theorem 3.2.** Assume that the oscillator \(g(t, s)\) defining the highly oscillatory kernel \(K_{\omega}(t, s)\) of the cordial Volterra operator \(V_{\omega, \varphi}\) in (3.9) is smooth, nonnegative, and independent of \(\omega\), and let \(K \in C(D)\) be independent of \(\omega\) and satisfy \(K(0, 0) \neq 0\). Then the spectrum \(\sigma(V_{\omega, \varphi})\) depends on \(\omega\) if, and only if, \(g(0, 0) \neq 0\): we have

\[
\sigma(V_{\omega, \varphi}) = e^{i\omega g(0, 0)} \sigma(V_{\varphi}).
\]

The eigenvalues \(\hat{\varphi}_{\omega}(\lambda)\) of \(V_{\omega, \varphi}\) satisfy

\[
|\hat{\varphi}_{\omega}(\lambda)| = |\hat{\varphi}(\lambda)|.
\]

**Proof.** Since the kernel \(K_{\omega}(t, s)\) of \(V_{\omega, \varphi}\) satisfies

\[
K_{\omega}(0, 0) = K(0, 0)e^{i\omega g(0, 0)},
\]

with \(K(0, 0) \neq 0\), Lemma 3.1(c) implies that the spectrum of \(V_{\omega, \varphi}\) is given by

\[
\sigma(V_{\omega, \varphi}) = e^{i\omega g(0, 0)} \sigma(V_{\varphi}).
\]

Hence the result of Theorem 3.2 follows. \(\square\)

As we have indicated in Section 2.4, the analysis for the oscillator \(g(t, s) = (t - s)^2\) or \(g(t, s) = ts\) is quite difficult (in fact, it appears that the analysis for the oscillator \(g(t, s) = xy\) has not yet been carried out). The situation is completely different for the highly oscillatory cordial Volterra integral operators (3.9) with the analogous oscillators, as the following corollary shows.

**Corollary 3.3.** For any \(\omega > 0\) the spectra of the highly oscillatory cordial Volterra integral operators \(V_{\omega, \varphi}\) in (3.9), with \(K(t, s) \equiv 1\) and oscillators

\[
g(t, s) = (t - s)^2 \quad \text{or} \quad g(t, s) = ts,
\]

are given by

\[
\sigma(V_{\omega, \varphi}) = \sigma(V_{\varphi}) \quad (= \sigma(V_{\varphi}^{0})).
\]

**Proof.** If the oscillator \(g(t, s)\) has the property that \(g(0, 0) = 0\), the statement (3.10) of Theorem 3.2 becomes our assertion. \(\square\)

**Remark 3.1.** As a simple example of an oscillator \(g(t, s)\) that does not vanish at \((0, 0)\) we consider, similarly to Iserles [12, p.29], the ‘irregular oscillator’

\[
g(t, s) = \frac{1}{2}(a + t - s)^2 \quad (a > 0),
\]

for which we have \(g(0, 0) = a^2/2\). By Theorem 3.2, the spectrum of \(V_{\omega, \varphi}\) with kernel \(K(t, s) \equiv 1\) and the above oscillator is given by

\[
\sigma(V_{\omega, \varphi}) = e^{i\omega a^2/2} \sigma(V_{\varphi}).
\]

Thus, if \(V_{\varphi} = V_{0, \varphi}\) is as in Example 1.1 with \(\gamma = 2\) (and core \(\varphi(z) = (1 - z^2)^{-1/2}\)), its real eigenvalues \(\hat{\varphi}(r) = \frac{1}{2}B(\frac{1}{2}, \frac{1}{2}) \in (0, \pi/2] \quad (r \geq 0); \text{ see also (1.5)) become complex when } \omega > 0\): they are

\[
\hat{\varphi}_{\omega}(r) = \hat{\varphi}(r)e^{i\omega r^2 / 2} \quad (r \geq 0),
\]

with \(\hat{\varphi}(0) = \pi/2\) and, by Lemma 3.1(d), \(\hat{\varphi}(r) \to 0\) as \(r \to \infty\).

The above analysis shows, as we have already mentioned in Section 2.4, that the derivation of the asymptotic behaviour of the spectra of highly oscillatory cordial Volterra integral operators is in sharp contrast to the one of the behaviour of the spectra of analogous highly oscillatory Fredholm integral operators with oscillators like \(g(x, y) = (x - y)^2\).
3.3. Cordial VIEs of the second kind. Consider the cordial VIE
\[\mu u(t) = f(t) + (V_{\omega, \varphi} u)(t), \quad t \in [0, T],\] (3.12)
where the highly oscillatory cordial Volterra integral operator \(V_{\omega, \varphi}\) is defined in (3.9) and \(\mu \neq 0\). We assume as always that \(K(0, 0) \neq 0\), and that \(f, K\) and the oscillator \(g\) do not depend on \(\omega\).

**Theorem 3.4.** Assume that the given functions in the highly oscillatory cordial VIE (3.12) do not depend on \(\omega\) and are subject to the conditions \(f \in C(I), K \in C(D), K(0, 0) \neq 0\), with \(g\) smooth and nonnegative on \(D\). If \(\mu \notin \sigma(V_{\varphi})\) (that is, the non-oscillatory cordial VIE
\[\mu u(t) = f(t) + (V_{\varphi} u)(t), \quad t \in I,\] has a unique solution \(u \in C(I)\), then the highly oscillatory cordial VIE (3.12) has a unique solution \(u \in C(I)\) for all \(\omega > 0\), regardless of whether \(g(0, 0) = 0\) or \(g(0, 0) \neq 0\).

**Proof.** According to (3.10) in Theorem 3.2, any eigenvalue \(\hat{\varphi}(\lambda)\) of \(V_{\varphi, \omega}\) is of the form
\[\hat{\varphi}(\lambda) = e^{i\omega g(0,0)} \hat{\varphi}(\lambda) \quad (\text{Re}(\lambda) \geq 0),\]
where \(\hat{\varphi}(\lambda)\) is an eigenvalue of \(V_{\varphi}\). Thus, \(\mu \notin \sigma(V_{\varphi})\) implies \(\mu \notin \sigma(V_{\varphi, \omega})\). For \(g(0, 0) = 0\) the assertion is trivial.

**Example 3.2.** The cordial VIE
\[\mu u(t) = f(t) + \int_0^t (t^2 - s^2)^{-1/2} u(s) \, ds \quad (\mu \neq 0)\] (3.13)
corresponds to the cordial Volterra integral operator \(V_{\varphi}^0\) in (3.2) with core \(\varphi(z) = (1 - z^2)^{-1/2}\) (recall (1.5)). The spectrum of this operator is
\[\sigma(V_{\varphi}^0) = \{0\} \cup \{\hat{\varphi}(\lambda) = \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{1}{2}\right) (1 + \lambda) / \Gamma(1 + \frac{1}{2} \lambda) : \text{Re}(\lambda) \geq 0\}\]
(recall (3.6) with \(\gamma = 2\)). Since the spectrum of the corresponding highly oscillatory cordial Volterra operator \(V_{\omega, \varphi}\) is \(\sigma(V_{\omega, \varphi}) = \sigma(V_{\varphi})\) for all \(\omega > 0\) (Theorem 3.2), the highly oscillatory cordial VIE
\[\mu u(t) = f(t) + \int_0^t (t^2 - s^2)^{-1/2} e^{i\omega g(t,s)} u(s) \, ds\]
possesses a unique solution \(u \in C(I)\) for any \(\omega > 0\), whenever \(\mu \notin \sigma(V_{\varphi})\).

3.4. Cordial VIEs of the first kind. In [24] Vainikko extended his results [22] on the theory of second-kind cordial VIEs to analogous first-kind cordial VIEs of the form
\[\int_0^t t^{-1} \varphi(s/t) K(t,s) u(s) \, ds = f(t), \quad t \in I,\] (3.14)
with \(K(t,s) \equiv 1\); here, \(\varphi \in L^1(0,1)\) and \(f\) is at least in \(C^1(I)\) (or in an analogous weighted space). Under certain boundedness conditions on the moments of \(\varphi\) and \(\varphi'\) and the assumption that \(\varphi(1) \neq 0\) ([24, Section 4]), (3.14) can be transformed into an equivalent cordial VIE of the second kind.

The highly oscillatory version of (3.14) is
\[(V_{\omega, \varphi} u)(t) = f(t), \quad t \in I,\] (3.15)
where \(V_{\omega, \varphi}\) is as in (3.9).
Example 3.3. Using the well-known inversion formula

\[ u(t) = \frac{2}{\pi} \left( f(0) + t \int_0^t (t^2 - s^2)^{-1/2} f'(s) \, ds \right) , \quad 0 < t \leq T \]

(3.16)

for the first-kind Abel-type VIE

\[ \int_0^t (t^2 - s^2)^{-1/2} u(s) \, ds = f(t), \quad t \in I, \]

with \( f \in C^1(I) \) (see for example [10, p.24]), together with an obvious modification of the substitutions (2.6), we see that the solution of the highly oscillatory first-kind VIE

\[ \int_0^t (t^2 - s^2)^{-1/2} e^{i\omega g(t,s)} u(s) \, ds = f(t), \quad t \in I, \]

with \( f \in C^1(I), \ f(0) = 0, \) and separable oscillator \( g(t,s) = g_0(t) - g_0(s) \) as in (2.1), has the form

\[ u(t) = \frac{2t}{\pi} \int_0^t (t^2 - s^2)^{-1/2} e^{i\omega[g_0(t)-g_0(s)]} f'(s) \, ds \]

\[ -\frac{2t\omega}{\pi} \int_0^t (t^2 - s^2)^{-1/2} e^{i\omega[g_0(t)-g_0(s)]} f(s) g_0'(s) \, ds, \quad t \in I. \]

We observe that this solution structure is analogous to the one shown in (2.20): for \( \omega > 0 \) we have again an \('\omega\)-perturbation' of the classical solution (3.16) of (3.14) corresponding to \( \omega = 0 \).

4. Future work. While the results of Sections 2 and 3 have yielded considerable insight into the oscillatory or non-oscillatory behaviour of solutions to highly oscillatory VIEs with smooth or weakly singular kernels whose oscillators \( g(t,s) \) are either linear (cf. (2.2)) or separable ((2.3)), much more work needs to be done before we have a complete understanding of the oscillation properties of solutions corresponding to \textit{general} nonlinear oscillators.

These observations raise also a challenging problem for the \textit{numerical analyst}. Suppose we do not know a priori whether or not the solution of first- or second-kind VIE with a highly oscillatory kernel is highly oscillatory. Can we detect a highly oscillatory solution numerically, by means of some suitably designed computational scheme (e.g. collocation or discontinuous Galerkin) which employs highly accurate approximations of the underlying highly oscillatory moment integrals (cf. [12], [13], [14], and [8] and its references)?

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REFERENCES

[1] K. E. Atkinson, “The Numerical Solution of Integral Equations of the Second Kind,” Cambridge University Press, Cambridge, 1997.
[2] A. Böttcher, H. Brunner, A. Iserles and S. P. Norsett, On the singular values and eigenvalues of the Fox-Li and related operators, New York J. Math., 16 (2010), 539–561.
[3] H. Brunner, “Collocation Methods for Volterra Integral and Related Functional Equations,” Cambridge University Press, Cambridge, 2004.
[4] H. Brunner, A. Iserles and S. P. Norsett, “The Spectral Problem for a Class of Highly Oscillatory Fredholm Integral Operators,” IMA J. Numer. Anal., 30 (2010), 108–130.
[5] H. Brunner, A. Iserles and S. P. Norsett, The computation of the spectra of highly oscillatory Fredholm integral operators, J. Integral Equations Appl., 23 (2010), 467–519.
[6] S. N. Curle, Solution of an integral equation of Lighthill, Proc. Roy. Soc. London Ser A, 364 (1978), 435–441.
[7] T. Diogo and P. Lima, Superconvergence of collocation methods for a class of weakly singular Volterra integral equations, J. Comput. Appl. Math., 218 (2008), 307–316.
[8] B. Engquist, A. Fokas, E. Hairer and A. Iserles, “Highly Oscillatory Problems,” London Math. Soc. Lecture Note Ser., 366, Cambridge University Press, Cambridge, 2009.
[9] N. B. Franco, S. McKee and J. Dixon, A numerical solution of Lighthill’s integral equation for the surface temperature distribution of a projectile, Mat. Apl. Comput., 2 (1983), 257–271.
[10] R. Gorenflo and S. Vessella, “Abel Integral Equations: Analysis and Applications,” Lecture Notes in Math., 1461, Springer-Verlag, Berlin-Heidelberg, 1991.
[11] W. Han, Existence, uniqueness and smoothness results for second-kind Volterra equations with weakly singular kernels, J. Integral Equations Appl., 6 (1994), 365–384.
[12] A. Iserles, On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators, IMA J. Numer. Anal., 25 (2005), 25–44.
[13] A. Iserles and S. P. Norsett, On quadrature methods for highly oscillatory integrals and their implementation, BIT, 44 (2004), 755–772.
[14] A. Iserles and S. P. Norsett, Efficient quadrature of highly oscillating integrals using derivatives, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), 1383–1399.
[15] J.-P. Kauthen, A survey of singularly perturbed Volterra equations, Appl. Numer. Math., 24 (1997), 95–114.
[16] H. Landau, The notion of approximate eigenvalues applied to an integral equation of laser theory, Quart. Appl. Math., 35 (1977/78), 165–172.
[17] M. J. Lighthill, Contributions to the theory of heat transfer through a laminar boundary layer, Proc. Roy. Soc. London Ser. A, 202 (1950), 359–377.
[18] A. Pedas and G. Vainikko, Integral equations with diagonal and boundary singularities of the kernel, Z. Anal. Anwend., 25 (2006), 487–516.
[19] H. M. Srivastava and R. G. Buschmann, “Convolution Integral Equations,” Wiley, New York, 1977.
[20] Li. N. Trefethen and M. Embree, “Spectra and Pseudospectra. The Behavior of Nonnormal Matrices and Operators,” Princeton University Press, Princeton, NJ, 2005.
[21] F. Ursell, Integral equations with a rapidly oscillating kernel, J. London Math. Soc., 44 (1969), 449–459.
[22] G. Vainikko, Cordial Volterra integral equations 1, Numer. Funct. Anal. Optim., 30 (2009), 1145–1172.
[23] G. Vainikko, Cordial Volterra integral equations 2, Numer. Funct. Anal. Optim., 31 (2010), 191–219.
[24] G. Vainikko, First-kind cordial Volterra integral equations 1, Numer. Funct. Anal. Optim., 33 (2012), 680–704.

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