On \( b \)-open sets via infra soft topological spaces

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Abstract. This work aims to present the concept of infra soft \( b \)-open sets (IS-\( b \)-open sets) as a generalized new class of infra open sets (IS-open sets). We first investigate their basic properties and study their behaviours under infra soft homeomorphism maps. Then, we establish some soft operators such as interior, closure, limit and boundary using IS-\( b \)-open sets and IS-\( b \)-closed sets. The relationships between them are illustrated and discussed. Finally, we display some soft maps (S-map) defined using IS-\( b \)-open and IS-\( b \)-closed sets and scrutinize their master properties.

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1. Introduction

Molodtsov [62] proposed the idea of soft set (S-set) as a new mathematical tool to deal with vagueness. He presented some of its applications to some areas. Since the advent of S-set, they have been applied to address some problems and phenomena in different disciplines such as information system [9], economy [14], linear equations [27], computer science [50] and decision-making problems [53].

The main operations and operators via S-set theory such as the difference, intersection, and union between two S-sets, and a complement of an S-set were introduced by Maji et

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al. [61]. Then, new operations and operators between S-sets were presented in [27, 44]. Some extensions of S-sets were proposed with the goal of expanding the applications of S-sets such as bipolar S-sets [8] and double-framed S-sets [38].

Recently, topology has been applied to model some real-life issues as showed in [1, 11, 15, 16, 32, 46, 56, 64]. To study topology via S-set theory, Çağman et al. [51] and Shabir and Naz [65], in 2011, introduced the concept of soft topology(ST). They followed different techniques for studying ST. This article follows Shabir and Naz’ technique which is defined an ST over a fixed set of universe and a fixed set of parameters. The basic concepts and notions of classical topology have been studied in ST such as caliber and chain conditions [43], compactness [2, 26, 28, 29, 40, 49], local compactness [47] separation axioms [18, 21, 22, 42, 45, 52], fixed point theorem [7, 19], connectedness [54, 57, 60], mappings [20, 25, 30, 58], bioperators [48], covering properties [34, 35, 59], sum of topologies [31, 36] and generalized open sets [3]. Additionally, STs and supra STs were discussed in ordered settings as given in [24]. Al-shami and Kočinac [33] elucidated the conditions under which the soft operators and classical operators of interior and closure are interchangeable. It should be noted that some classical topological properties were generalized to STs without consideration for the divergences between STs and classical topologies, which causes some incorrect forms of some results; so some articles were conducted to put forward the correct frame of these results via soft structures; see [4–6].

In 2021, Al-shami [13] familiarized the structure of infra soft topologies(ISTs) [13] and showed the motivations for studying this structure. He with his coauthors continued investigating several topological concepts and properties via this structure such as compactness [12], homeomorphisms [10], connectedness [17], separation axioms [37, 39], infra soft semi-open (IS-semi-open) [23] and infra soft pre-open (IS-pre-open) sets [41]. In this article, we display the notion of soft b-open sets (S-b-open sets) and applied to initiate new operators and mappings via infra soft structures.

The structure of this article is designed as follows. In Sect. 2, we recall the main ideas and findings that make this work self-contained. In Sect. 3, we introduce the notion of infra soft b-open sets(IS-b-open sets) and establish its master characterizations. In Sect. 4, we define new operators and discuss their main properties. In Sect. 5, we explore novel kinds of mappings and demonstrated their features. Ultimately, we give the main contributions of the article and propose some future works.

2. Preliminaries

2.1. Soft set theory

Definition 1. [62] A mapping $\mathcal{H}$ from a set of parameters $\mathcal{O}$ into $2^X$, where $2^X$ is the power set of $X$, is called an S-set denoted by $\langle \mathcal{H}, \mathcal{O} \rangle$, and it can written as follows $\langle \mathcal{H}, \mathcal{O} \rangle = \{(o, \mathcal{H}(o)) : o \in \mathcal{O} \text{ and } \mathcal{H}(o) \in 2^X\}$. $C(\mathcal{X}_\mathcal{O})$ refers to the class of all S-sets over $X$ with the set of parameters $\mathcal{O}$. 
Definition 2. [44] A complement of an S-set $\mathcal{H}(\mathcal{O})$, denoted by $(\mathcal{H}, \mathcal{O}^c)$, is given as follows: for each $o \in \mathcal{O}$, $(\mathcal{H}, \mathcal{O}^c)$ is given as follows: $\mathcal{H}(o) = X \setminus \mathcal{H}(o)$ for each $o \in \mathcal{O}$.

Definition 3. [61] If $\mathcal{H}(o) = \emptyset$ (resp., $\mathcal{H}(o) = X$) for all $o \in \mathcal{O}$, then $(\mathcal{H}, \mathcal{O})$ is called a null S-set (resp., an absolute S-set) over $X$.

Definition 4. [63] $(\mathcal{H}, \mathcal{O})$ is called an S-set on $X$ if there is $o \in \mathcal{O}$ such that $\mathcal{H}(o) = x \in X$ and $\mathcal{H}(o') = \emptyset$ for each $o' \not= o$. The symbol of an S-point will be $\delta_o$.

Definition 5. [44] The intersection of S-sets $(\mathcal{H}, \mathcal{O})$ and $(\mathcal{F}, \Delta)$ on $X$, symbolized by $(\mathcal{H}, \mathcal{O}){\cap}(\mathcal{F}, \Delta)$, is an S-set $(\mathcal{G}, T)$, where $T = \mathcal{O} \cap \Delta \neq \emptyset$, and a map $\mathcal{G} : T \rightarrow 2^X$ is given by $\mathcal{G}(o) = \mathcal{H}(o) \cap \mathcal{F}(o)$ for each $o \in T$.

Definition 6. [61] The union of S-sets $(\mathcal{H}, \mathcal{O})$ and $(\mathcal{F}, \Delta)$ on $X$, symbolized by $(\mathcal{H}, \mathcal{O}){\cup}(\mathcal{F}, \Delta)$, is an S-set $(\mathcal{G}, T)$, where $T = \mathcal{O} \cup \Delta$ and a map $T : \mathcal{O} \rightarrow 2^X$ is given as follows:

$$
\mathcal{G}(o) = \begin{cases} 
\mathcal{H}(o) : o \in \mathcal{O} \setminus \Delta \\
\mathcal{F}(o) : o \in \Delta \setminus \mathcal{O} \\
\mathcal{H}(o) \cup \mathcal{F}(o) : o \in \mathcal{O} \cap \Delta
\end{cases}
$$

Definition 7. [55] A S-set $(\mathcal{H}, \mathcal{O})$ is a subset of an S-set $(\mathcal{F}, \Delta)$, symbolized by $(\mathcal{H}, \mathcal{O}){\subset}(\mathcal{F}, \Delta)$, if $\mathcal{O} \subseteq \Delta$ and $\mathcal{H}(o) \subseteq \mathcal{F}(o)$ for all $o \in \mathcal{O}$. If $(\mathcal{H}, \mathcal{O}){\subset}(\mathcal{F}, \Delta)$ and $(\mathcal{F}, \Delta){\subset}(\mathcal{H}, \mathcal{O})$, then $(\mathcal{H}, \mathcal{O})$ and $(\mathcal{F}, \Delta)$ are said to be soft equal.

The definition of soft maps (S-map) in [58] was adjusted as follows.

Definition 8. [10] Let $f : X \rightarrow S$ and $\psi : \mathcal{O} \rightarrow \Delta$ be two maps. A S-map $f_\psi$ of $C(\mathcal{O})$ into $C(\mathcal{S}_\Delta)$ is a relation such that any S-point in $C(\mathcal{O})$ is related to one and only one S-point in $C(\mathcal{S}_\Delta)$ such that $f_\psi(\delta_o^x) = \delta_{f(x)}^{\psi(o)}$ for any $\delta_o^x \in C(\mathcal{O})$.

In addition, $f_\psi^{-1}(\delta_{y}^x) = \bigsqcup_{\lambda \in \psi^{-1}(y)} \delta_{\lambda}^x$ for any $\delta_{y}^x \in C(\mathcal{S}_\Delta)$.

Definition 9. [63] For an S-map $f_\psi : C(\mathcal{O}) \rightarrow C(\mathcal{S}_\Delta)$, if $f$ and $\psi$ are injective (resp., surjective, bijective), then $f_\psi$ is called injective (resp., surjective, bijective).

2.2. Infra soft topological spaces

Definition 10. [13] A subfamily $\mu$ of $C(\mathcal{O})$ is called an infra soft topology (IST) on $X$ if it contains $\Phi$ and it is closed under finite intersection.

The triple $(X, \mu, \mathcal{O})$ is called ISTS. The elements of $\mu$ are called IS-open sets and their complements are called IS-closed sets.

Definition 11. [13] Let $(\mathcal{H}, \mathcal{O})$ be a subset of $(X, \mu, \mathcal{O})$. 

Proposition 1. [13] Let \((\mathcal{H}, \mathcal{O})\) and \((\mathcal{F}, \mathcal{O})\) subsets of an ISTS \((X, \mu, \nu, \Delta)\). Then

(i) the IS-closure points of \((\mathcal{H}, \mathcal{O})\), denoted by \(cl(\mathcal{H}, \mathcal{O})\), is the intersection of all IS-closed subsets of \((X, \mu, \nu, \Delta)\) containing \((\mathcal{H}, \mathcal{O})\).

(ii) the IS-interior points of \((\mathcal{H}, \mathcal{O})\), denoted by \(int(\mathcal{H}, \mathcal{O})\) is the union of all IS-open subsets of \((X, \mu, \nu, \Delta)\) which are contained in \((\mathcal{H}, \mathcal{O})\).

Proposition 2. [13] Let \((\mathcal{H}, \mathcal{O})\) be an IS-open set. Then

\[ (\mathcal{H}, \mathcal{O}) \cap cl(\mathcal{F}, \mathcal{O}) \subseteq cl[(\mathcal{H}, \mathcal{O}) \cap (\mathcal{F}, \mathcal{O})] \]
for any \((\mathcal{F}, \mathcal{O})\) in \((X, \mu, \nu, \Delta)\).

Proposition 3. [13] Let \((\mathcal{H}, \mathcal{O})\) be an IS-closed set. Then

\[ int[(\mathcal{H}, \mathcal{O}) \cap (\mathcal{F}, \mathcal{O})] \subseteq (\mathcal{H}, \mathcal{O}) \cap int(\mathcal{F}, \mathcal{O}) \]
for any \((\mathcal{H}, \mathcal{O})\) in \((X, \mu, \nu, \Delta)\).

Definition 12. [10] A bijective S-map \(f_\psi : (X, \mu, \nu, \Delta) \to (\mathcal{S}, \nu, \Delta)\) is said to be an IS-homeomorphism if it is IS-open (i.e., the image of any IS-open set is IS-open), and IS-continuous (i.e., the pre-image of any IS-open set is IS-open).

We call a property which is kept by any IS-homeomorphism an IS-topological property.

Definition 13. [10] Let \(f_\psi : (X, \mu, \nu, \Delta) \to (\mathcal{S}, \nu, \Delta)\) be an S-map and \(\mathcal{M} \neq \emptyset\) be a subset of \(X\). A S-map \(f_{\psi|\mathcal{M}} : (\mathcal{M}, \mu|\mathcal{M}, \nu) \to (\mathcal{S}, \nu, \Delta)\) which given by \(f_{\psi|\mathcal{M}}(\delta^m_o) = f_\psi(\delta^m_o)\) for every \(\delta^m_o \in \mathcal{M}\) is called a restriction S-map of \(f_\psi\) on \(\mathcal{M}\).

Lemma 1. [23, 41] Let \(f_\psi : (X_1, \mu_1, \nu_1) \to (X_2, \mu_2, \nu_2)\) be an IS-homeomorphism map. Then for any \((\mathcal{H}, \mathcal{O}_1)\) we have:

(i) \(f_\psi(int(\mathcal{H}, \mathcal{O}_1)) = int(f_\psi(\mathcal{H}, \mathcal{O}_1))\).

(ii) \(f_\psi(cl(\mathcal{H}, \mathcal{O}_1)) = cl(f_\psi(\mathcal{H}, \mathcal{O}_1))\).

3. Main properties of infra soft \(b\)-open sets

Definition 14. A S-set \((\mathcal{H}, \mathcal{O})\) in an ISTS \((X, \mu, \nu, \Delta)\) is said to be IS\(b\)-open if \((\mathcal{H}, \mathcal{O}) \subseteq int(cl(\mathcal{H}, \mathcal{O})) \cup cl(int(\mathcal{H}, \mathcal{O}))\). Its complement is said to be an IS\(b\)-closed set.

Proposition 4. Every IS-semi-open (IS-pre-open) set is IS\(b\)-open.

Proof. Let \((\mathcal{H}, \mathcal{O})\) be an IS-semi-open (resp. IS-pre-open) set. Then, \((\mathcal{H}, \mathcal{O}) \subseteq cl(int(\mathcal{H}, \mathcal{O}))\) (resp. \((\mathcal{H}, \mathcal{O}) \subseteq int(cl(\mathcal{H}, \mathcal{O}))\)). Automatically, we obtain \((\mathcal{H}, \mathcal{O}) \subseteq int(cl(\mathcal{H}, \mathcal{O})) \cup cl(int(\mathcal{H}, \mathcal{O}))\), which means that \((\mathcal{H}, \mathcal{O})\) is IS\(b\)-open.

The converse of the above proposition fails as the next example shows.
Example 1. Let $X = \{x_1, x_2, x_3\}$ and $\mathcal{O} = \{o_1, o_2\}$. Then $\mu = \{\Phi, \widetilde{X}, \mathcal{H}_1, \mathcal{O}, \mathcal{H}_2, \mathcal{O}\}$ is an IST on $X$, where

$$\mathcal{H}_1, \mathcal{O} = \{(o_1, \{x_1\}), (o_2, \{x_2, x_3\}\} \text{ and } \mathcal{H}_2, \mathcal{O} = \{(o_1, \{x_3\}), (o_2, \{x_1\})\}.$$ 

Let $(\mathcal{H}_5, \mathcal{O}) = \{(o_1, \{x_3\}), (o_2, \{x_2, x_3\})\}$ and $\mathcal{H}_6, \mathcal{O} = \{(o_1, \{x_1, x_2\}), (o_2, \{x_2, x_3\})\}$. Then $(\mathcal{H}_5, \mathcal{O})$ and $(\mathcal{H}_6, \mathcal{O})$ are IS-b-open sets because $\text{cl}(\mathcal{H}_5, \mathcal{O}) = \widetilde{X}$ and $\text{cl}(\text{int}(\mathcal{H}_6, \mathcal{O})) = (\mathcal{H}_6, \mathcal{O})$. But $(\mathcal{H}_5, \mathcal{O})$ is not IS-semi-open because $\text{int}(\mathcal{H}_5, \mathcal{O}) = \Phi$, and $(\mathcal{H}_6, \mathcal{O})$ is not IS-pre-open because $\text{int}(\text{cl}(\mathcal{H}_5, \mathcal{O})) = \{(o_1, \{x_1\}), (o_2, \{x_2, x_3\})\}$.

Proposition 5. The unions of IS-b-open sets is IS-b-open.

Proof. Consider $\{(\mathcal{H}_j, \mathcal{O}) : j \in J\}$ as a family of IS-b-open sets. Suppose $J \neq \emptyset$. Then $\bigcup_{j \in J}(\mathcal{H}_j, \mathcal{O}) \subseteq \text{int}(\text{cl}(\mathcal{H}_j, \mathcal{O})) \cup \text{cl}(\text{int}(\mathcal{H}_j, \mathcal{O}))$ for each $j \in J$. Thus, $\bigcup_{j \in J}(\mathcal{H}_j, \mathcal{O}) \subseteq \bigcup_{j \in J}[\text{int}(\text{cl}(\mathcal{H}_j, \mathcal{O})) \cup \text{cl}(\text{int}(\mathcal{H}_j, \mathcal{O})))$. Hence, $\bigcup_{j \in J}(\mathcal{H}_j, \mathcal{O})$ is IS-b-open.

Corollary 1. The intersections of IS-b-closed sets is IS-b-closed.

Proposition 6. If $(\mathcal{H}_1, \mathcal{O})$ is IS-open and $(\mathcal{H}_2, \mathcal{O})$ is IS-b-open, then $(\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O})$ is IS-b-open.

Proof. Let $(\mathcal{H}_1, \mathcal{O})$ and $(\mathcal{H}_2, \mathcal{O})$ be as given in the proposition. Then $(\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O}) \subseteq (\mathcal{H}_1, \mathcal{O}) \cap \text{int}(\text{cl}(\mathcal{H}_2, \mathcal{O})) \cup \text{cl}(\text{int}(\mathcal{H}_2, \mathcal{O}))$. It follows from Proposition 2 that $(\mathcal{H}_1, \mathcal{O}) \cap \text{int}(\text{cl}(\mathcal{H}_2, \mathcal{O})) \subseteq \text{int}(\text{cl}((\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O})))$ and $(\mathcal{H}_1, \mathcal{O}) \cap \text{cl}(\text{int}(\mathcal{H}_2, \mathcal{O})) \subseteq \text{cl}(\text{int}((\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O})))$. Hence, $(\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O})$ is an IS-b-open set.

Corollary 2. If $(\mathcal{H}_1, \mathcal{O})$ is IS-closed and $(\mathcal{H}_2, \mathcal{O})$ is IS-b-closed, then $(\mathcal{H}_1, \mathcal{O}) \cap (\mathcal{H}_2, \mathcal{O})$ is IS-b-closed.

Proposition 7. The image of an IS-b-open set under an IS-homeomorphism is IS-b-open.

Proof. Consider $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \rightarrow (X_2, \mu_2, \mathcal{O}_2)$ as an IS-continuous map and let $(\mathcal{H}, \mathcal{O})$ be an IS-b-open subset of $(X_1, \mu_1, \mathcal{O}_1)$. Then $f_\psi(\mathcal{H}, \mathcal{O}) \subseteq \text{int}(\text{cl}(\mathcal{H}, \mathcal{O})) \cup \text{int}(\text{cl}(\mathcal{H}, \mathcal{O})))$. It follows from Lemma 1 that $f_\psi(\mathcal{H}, \mathcal{O}) \subseteq \text{cl}(\text{int}(f_\psi(\mathcal{H}, \mathcal{O}))) \cup \text{int}(\text{cl}(f_\psi(\mathcal{H}, \mathcal{O})))$. Hence, $f_\psi(\mathcal{H}, \mathcal{O})$ is an IS-b-open subset of $(X_2, \mu_2, \mathcal{O}_2)$, as required.

4. Infra $b$-interior, infra $b$-closure, infra $b$-limit and infra $b$-boundary soft points of a soft set.

Definition 15. Let $(\mathcal{H}, \mathcal{O})$ be an $S$-set in $(X, \mu, \mathcal{O})$. Then:

(i) the IS-$b$-interior of $(\mathcal{H}, \mathcal{O})$, denoted by $\text{bint}(\mathcal{H}, \mathcal{O})$, is the union of all IS-$b$-open sets that are contained in $(\mathcal{H}, \mathcal{O})$. 
(ii) the IS-$b$-closure of $(\mathcal{H}, \mathcal{O})$, denoted by $bcl(\mathcal{H}, \mathcal{O})$, is the intersection of all IS-$b$-closed sets containing $(\mathcal{H}, \mathcal{O})$.

**Proposition 8.** We have the following properties.

(i) $(\mathcal{H}, \mathcal{O})$ is an IS-$b$-open subset of $(X, \mu, \mathcal{O})$ iff $bint(\mathcal{H}, \mathcal{O}) = (\mathcal{H}, \mathcal{O})$.

(ii) $(\mathcal{H}, \mathcal{O})$ is an IS-$b$-closed subset of $(X, \mu, \mathcal{O})$ iff $bcl(\mathcal{H}, \mathcal{O}) = (\mathcal{H}, \mathcal{O})$.

**Proof.** It comes from Proposition 5 and Corollary 1.

The two characterizations given in the the above proposition are generally false for IS-open and IS-closed sets.

**Proposition 9.** Let $(\mathcal{H}, \mathcal{O})$ be a subset of $(X, \mu, \mathcal{O})$.

(i) $\delta_o^x \in bint(\mathcal{H}, \mathcal{O})$ iff there exists an IS-$b$-open set $(\mathcal{F}, \mathcal{O})$ such that $\delta_o^x \in (\mathcal{F}, \mathcal{O}) \subseteq (\mathcal{H}, \mathcal{O})$.

(ii) $\delta_o^x \in bcl(\mathcal{H}, \mathcal{O})$ iff the intersection of any IS-$b$-open set $(\mathcal{F}, \mathcal{O})$ containing $\delta_o^x$ and $(\mathcal{H}, \mathcal{O})$ is non-null.

**Proof.** The proof of (i) is obvious, so we prove (ii).

Let $\delta_o^x \in bcl(\mathcal{H}, \mathcal{O})$. Then every IS-$b$-closed set contains $(\mathcal{H}, \mathcal{O})$ contains $\delta_o^x$ as well. Suppose that there exists an IS-$b$-open set $(\mathcal{F}, \mathcal{O})$ containing $\delta_o^x$ such that $(\mathcal{H}, \mathcal{O}) \cap (\mathcal{F}, \mathcal{O}) = \Phi$. Therefore, $(\mathcal{H}, \mathcal{O}) \subseteq (\mathcal{F}, \mathcal{O})$ which means that $\delta_o^x \notin bcl(\mathcal{H}, \mathcal{O})$. This is a contradiction. Conversely, suppose that there exists an IS-$b$-open set $(\mathcal{F}, \mathcal{O})$ containing $\delta_o^x$ such that $(\mathcal{H}, \mathcal{O}) \cap (\mathcal{F}, \mathcal{O}) = \Phi$. Therefore, $bcl(\mathcal{H}, \mathcal{O}) \subseteq (\mathcal{F}, \mathcal{O})$ which means that $\delta_o^x \notin bcl(\mathcal{H}, \mathcal{O})$. Hence, the result holds.

**Proposition 10.** Let $(\mathcal{H}, \mathcal{O})$ be a subset of $(X, \mu, \mathcal{O})$. Then:

(i) $(bint(\mathcal{H}, \mathcal{O}))^c = bcl(\mathcal{H}^c, \mathcal{O})$.

(ii) $(bcl(\mathcal{H}, \mathcal{O}))^c = bint(\mathcal{H}^c, \mathcal{O})$.

**Proof.** (i): $(bint(\mathcal{H}, \mathcal{O}))^c = \{ \bigcup_{j \in J} (\mathcal{F}_j, \mathcal{O}) : (\mathcal{F}_j, \mathcal{O}) \text{ is an IS-$b$-open set contained in } (\mathcal{H}, \mathcal{O}) \}^c = \bigcap_{j \in J} \{ (\mathcal{F}_j, \mathcal{O}) : (\mathcal{F}_j, \mathcal{O}) \text{ is an IS-$b$-closed set containing } (\mathcal{H}, \mathcal{O}) \} = bcl(\mathcal{H}^c, \mathcal{O})$.

The proof of (ii) is similar to (i).

**Proposition 11.** Let $(\mathcal{F}, \mathcal{O})$ be an IS-open set and $(\Lambda, \mathcal{O})$ be an IS-closed set in $(X, \mu, \mathcal{O})$. Then:

(i) $(\mathcal{F}, \mathcal{O}) \cap bcl(\mathcal{H}, \mathcal{O}) \subseteq bcl((\mathcal{F}, \mathcal{O}) \cap (\mathcal{H}, \mathcal{O}))$.

(ii) $bint((\Lambda, \mathcal{O}) \cup (\mathcal{H}, \mathcal{O})) \subseteq (\Lambda, \mathcal{O}) \cup bint(\mathcal{H}, \mathcal{O})$. 
Proof. (i): Let $\delta_o^x \in (F, O) \cap bcl(H, O)$. Then $\delta_o^x \in (F, O)$ and $\delta_o^x \in bcl(H, O)$. This implies $(U, O) \cap (F, O) \neq \Phi$ for every IS-b-open set $(U, O)$ containing $\delta_o^x$. It follows from Proposition 6 that $(F, O) \cap (H, O) \neq \Phi$ for every IS-b-open set $(H, O)$ containing $\delta_o^x$. Therefore, $[(F, O) \cap (U, O)] \cap (H, O) \neq \Phi$. Now, $(F, O) \cap (F, O) \cap (H, O) \neq \Phi$ which means that $\delta_o^x \in bcl((F, O) \cap (H, O))$. Hence, $(F, O) \cap bcl(H, O) \subseteq bcl((F, O) \cap (H, O))$.

One can prove (ii) following similar arguments.

Theorem 1. Let $(H, O)$ and $(F, O)$ are in $(X, \mu, O)$. Then we have:

(i) $bint(\bar{X}) = \bar{X}$.

(ii) $bint(H, O) \subseteq (H, O)$.

(iii) If $(F, O) \subseteq (H, O)$, then $bint(F, O) \subseteq bint(H, O)$.

(iv) $bint(bint(H, O)) = bint(H, O)$.

(v) $bint(F, O) \cap bint(H, O) \subseteq bint((F, O) \cap (H, O))$.

Proof. (i): Since $\bar{X}$ is IS-b-open, $bint(\bar{X}) = \bar{X}$.

(ii) and (iii) are obvious.

(iv): Clearly $bint(bint(H, O))$ is the largest IS-b-open set contained in $bint(H, O)$; however, $bint(H, O)$ is an IS-b-open set; hence, $bint(bint(H, O)) = bint(H, O)$.

(v): It comes from (iii).

Theorem 2. Let $(H, O)$ and $(F, O)$ be subsets of $(X, \mu, O)$. Then we have:

(i) $bcl(\Phi) = \Phi$.

(ii) $(H, O) \subseteq bcl(H, O)$.

(iii) If $(F, O) \subseteq (H, O)$, then $bcl(F, O) \subseteq bcl(H, O)$.

(iv) $bcl(bcl(H, O)) \subseteq bcl(H, O)$.

(v) $bcl((F, O) \cap (H, O)) = bcl(F, O) \cap bcl(H, O)$.

Proof. It can be proved following similar arguments given in the proof of Theorem 1.

Definition 16. A S-point $\delta_o^x$ is called an IS-b-limit point of a subset $(H, O)$ of $(X, \mu, O)$ provided that $[(F, O) \setminus \delta_o^x] \cap (H, O) \neq \Phi$ for any IS-b-open set $(F, O)$ containing $\delta_o^x$.

The S-set of all IS-b-limit points of $(H, O)$ is called an infra b-derived S-set. It is denoted by $(H, O)_{bs}$.

Proposition 12. Consider $(F, O)$ and $(H, O)$ as S-sets in $(X, \mu, O)$. Then

(i) $\Phi_{bs} = \Phi$ and $\bar{X}_{bs} \subseteq \bar{X}$. 
(ii) If \((F, O) \subseteq (H, O)\), then \((F, O)^{b\text{cl}} \subseteq (H, O)^{b\text{cl}}\).

(iii) If \(\delta_o^x \in (H, O)^{b\text{cl}}\), then \(\delta_o^x \in ((H, O) \setminus \delta_o^{b\text{cl}})\).

(iv) \((F, O)^{b\text{cl}} \cap (H, O)^{b\text{cl}} \subseteq ((F, O) \cap (H, O))^{b\text{cl}}\).

**Proof.** Straightforward.

**Theorem 3.** Let \((H, O)\) be an \(S\)-set in \((X, \mu, O)\). Then

(i) If \((H, O)\) is an IS-\(b\)-closed set, then \((H, O)^{b\text{cl}} \subseteq (H, O)\).

(ii) \(((H, O)^{b\text{cl}} \cap (H, O)^{b\text{cl}}) = (H, O)^{b\text{cl}}\).

(iii) \(bcl(H, O) = (H, O) \cap (H, O)^{b\text{cl}}\).

**Proof.**

(i) Consider \((H, O)\) as an IS-\(b\)-closed set such that \(\delta_o^x \not\in (H, O)\). Then \(\delta_o^x \in (H^c, O)\). Now, \((H^c, O)\) is an IS-\(b\)-open set such that \((H^c, O)^{\cap} (H, O) = \emptyset\) which means that \(\delta_o^x \not\in (H, O)^{b\text{cl}}\). Thus, \((H, O)^{b\text{cl}} \subseteq (H, O)\).

(ii) Consider \(\delta_o^x \not\in (H, O)^{\cap} (H, O)^{b\text{cl}}\). Then \(\delta_o^x \not\in (H, O)\) and \(\delta_o^x \not\in (H, O)^{b\text{cl}}\). Therefore, there exists an IS-\(b\)-open set \((F, O)\) such that

\[(F, O)^{\cap} (H, O) = \emptyset \tag{1}\]

This implies that

\[(F, O)^{\cap} (H, O)^{b\text{cl}} = \emptyset \tag{2}\]

It follows from (1) and (2) that \((F, O)^{\cap} ((H, O) \cap (H, O)^{b\text{cl}}) = \emptyset\). Thus, \(\delta_o^x \not\in ((H, O)^{\cap} (H, O)^{b\text{cl}})^{b\text{cl}}\). Hence, \(((H, O)^{\cap} (H, O)^{b\text{cl}})^{b\text{cl}} \subseteq ((H, O) \cap (H, O)^{b\text{cl}})^{b\text{cl}}\), as required.

(iii) It is clear that \((H, O)^{\cap} (H, O)^{b\text{cl}} \subseteq bcl(H, O)\). Conversely, let \(\delta_o^x \in bcl(H, O)\). Then for every IS-\(b\)-open set containing \(\delta_o^x\) we have \((H, O)^{\cap} (F, O) \neq \emptyset\). Without loss of generality, let \(\delta_o^x \not\in (H, O)\). Then 

\[|(H, O) \setminus \delta_o^x|^{\cap} (F, O) \neq \emptyset\]

Consequently, \(\delta_o^x \in (H, O)^{b\text{cl}}\). Hence, the proof is complete.

**Definition 17.** The IS-\(b\)-boundary points of a subset \((H, O)\) of \((X, \mu, O)\), denoted by \(bB(H, O)\), are all the S-points which belong to the complement of \(bint(H, O) \cap bint(H^c, O)\).

**Proposition 13.** Let \((H, O)\) be an \(S\)-set in \((X, \mu, O)\). Then:

(i) \(bB(H, O) = bcl(H, O) \cap bcl((H^c, O))\).

(ii) \(bB(H, O) = bcl(H, O) \setminus bint(H, O)\).

**Proof.**
Corollary 3. Let \((\mathcal{H}, \mathcal{O})\) be a subset of \((X, \mu, \mathcal{O})\). Then

(i) \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) = b\mathcal{B}(\mathcal{H}^c, \mathcal{O})\)

(ii) \(b\mathcal{L}(\mathcal{H}, \mathcal{O}) = b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap b\mathcal{C}(\mathcal{H}, \mathcal{O})\)

Proposition 14. Let \((\mathcal{H}, \mathcal{O})\) be a subset of \((X, \mu, \mathcal{O})\). Then

(i) \((\mathcal{H}, \mathcal{O})\) is IS-b-open iff \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap (\mathcal{H}, \mathcal{O}) = \Phi\).

(ii) \((\mathcal{H}, \mathcal{O})\) is IS-b-closed iff \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) \subseteq (\mathcal{H}, \mathcal{O})\).

Proof.

(i) \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap (\mathcal{H}, \mathcal{O}) = b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap b\mathcal{I}(\mathcal{H}, \mathcal{O}) = \Phi\). Conversely, let \(\delta_o^x \in (\mathcal{H}, \mathcal{O})\). Then \(\delta_o^{x'} \in b\mathcal{I}(\mathcal{H}, \mathcal{O})\) or \(\delta_o^{x'} \in b\mathcal{B}(\mathcal{H}, \mathcal{O})\). Since \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap (\mathcal{H}, \mathcal{O}) = \Phi\), \(\delta_o^{x'} \in b\mathcal{I}(\mathcal{H}, \mathcal{O})\). Thus, \((\mathcal{H}, \mathcal{O}) \subseteq b\mathcal{I}(\mathcal{H}, \mathcal{O})\) which means that \((\mathcal{H}, \mathcal{O}) = b\mathcal{I}(\mathcal{H}, \mathcal{O})\). Hence, \((\mathcal{H}, \mathcal{O})\) is IS-b-open.

(ii) \((\mathcal{H}, \mathcal{O})\) is IS-b-closed \(\iff (\mathcal{H}^c, \mathcal{O})\) is IS-b-open \(\iff b\mathcal{B}(\mathcal{H}^c, \mathcal{O}) \cap (\mathcal{H}, \mathcal{O}) = \Phi \iff b\mathcal{B}(\mathcal{H}, \mathcal{O}) \cap (\mathcal{H}^c, \mathcal{O}) = \Phi \iff b\mathcal{B}(\mathcal{H}, \mathcal{O}) \subseteq (\mathcal{H}, \mathcal{O})\).

Corollary 4. A subset \((\mathcal{H}, \mathcal{O})\) of \((X, \mu, \mathcal{O})\) is IS-b-open and IS-b-closed iff \(b\mathcal{B}(\mathcal{H}, \mathcal{O}) = \Phi\).

5. Infra soft \(b\)-homeomorphism maps

Definition 18. \(f_\psi : (X, \mu, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta)\) is said to be IS-b-continuous at \(\delta_o^x \in \tilde{X}\) if for any IS-b-open set \((\mathcal{F}, \Delta)\) containing \(f_\psi(\delta_o^x)\), there is an IS-b-open set \((\mathcal{H}, \mathcal{O})\) containing \(\delta_o^x\) such that \(f_\psi(\mathcal{H}, \mathcal{O}) \subseteq (\mathcal{F}, \Delta)\).

\(f_\psi\) is called IS-b-continuous if it is IS-b-continuous at all \(\delta_o^x \in \tilde{X}\).

Theorem 4. If \(f_\psi : (X, \mu, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta)\) is IS-b-continuous, then the next properties are equivalent.

(i) \(f_\psi\) is an IS-b-continuous map;
Theorem 5. If \( f \) is IS-continuous, then

(iii) \( \overline{f^{-1}(\mathcal{H}, \Delta)} \subseteq f^{-1}(\overline{\mathcal{H}, \Delta}) \) for each \((\mathcal{H}, \Delta) \subseteq \mathcal{S}\);

(iv) \( f_{\psi}(bcl(\mathcal{F}, \mathcal{O})) \subseteq bcl(f_{\psi}(\mathcal{F}, \mathcal{O})) \) for each \((\mathcal{F}, \mathcal{O}) \subseteq X\);

(v) \( f_{\psi}^{-1}(bint(\mathcal{H}, \Delta)) \subseteq bint(f_{\psi}^{-1}(\mathcal{H}, \Delta)) \) for each \((\mathcal{H}, \Delta) \subseteq \mathcal{S}\).

Proof. (i) \( \Rightarrow \) (ii): Let \((\mathcal{H}, \Delta) \) be an IS-b-open subset in \((\mathcal{S}, \nu, \Delta)\). Then \( f_{\psi}^{-1}(\mathcal{H}, \Delta) \) is an IS-b-open subset of \( X \). Obviously, \( f_{\psi}^{-1}(\mathcal{H}, \Delta) = X - f_{\psi}^{-1}(\mathcal{H}, \Delta) \); hence, \( f_{\psi}^{-1}(\mathcal{H}, \Delta) \) is an IS-b-closed subset of \( X \).

(ii) \( \Rightarrow \) (iii): According to (ii), \( f_{\psi}^{-1}(\overline{\mathcal{H}, \Delta}) \) is an IS-b-closed subset of \( X \). Then \( bcl(f_{\psi}^{-1}(\mathcal{H}, \Delta)) \subseteq bcl(f_{\psi}^{-1}(\overline{\mathcal{H}, \Delta})) = f_{\psi}^{-1}(\overline{\mathcal{H}, \Delta}) \).

(iii) \( \Rightarrow \) (vi): According to (iii), \( bcl(f_{\psi}^{-1}(\mathcal{F}, \mathcal{O})) \subseteq bcl(f_{\psi}^{-1}(\overline{\mathcal{F}, \mathcal{O}})) \).

(iv) \( \Rightarrow \) (v): According to (iv), \( f_{\psi}(bcl(\mathcal{F}, \mathcal{O})) \subseteq bcl(f_{\psi}(\overline{\mathcal{F}, \mathcal{O}})) \). Therefore, \( f_{\psi}(\mathcal{F}, \mathcal{O}) \subseteq bcl(f_{\psi}(\overline{\mathcal{F}, \mathcal{O}})) \).

(v) \( \Rightarrow \) (i): Let \((\mathcal{H}, \Delta) \) be an IS-b-open subset of \( \mathcal{S} \). According to (v), \( f_{\psi}^{-1}(\mathcal{H}, \Delta) \subseteq bint(f_{\psi}^{-1}(\mathcal{H}, \Delta)) \). This implies \( f_{\psi}^{-1}(\mathcal{H}, \Delta) = bint(f_{\psi}^{-1}(\mathcal{H}, \Delta)) \). Hence, \( f_{\psi} \) is IS-b-continuous.

Theorem 5. If \( f_{\psi} : (X, \mu, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta) \) is IS-b-continuous, then the restriction S-map \( f_{\psi \mid M} : (M, \mu_{M}, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta) \) is IS-b-continuous provided that \( M \) is an IS-open set.

Proof. Consider \((\mathcal{H}, \Delta) \) an IS-b-open set in \((\mathcal{S}, \nu, \Delta)\). By hypothesis, \( f_{\psi}^{-1}(\mathcal{H}, \Delta) \) is IS-b-open. Now, \( f_{\psi}^{-1}(\mathcal{H}, \Delta) = f_{\psi}^{-1}(\mathcal{H}, \Delta) \subseteq \overline{M} \). Since \( \overline{M} \) is an IS-open set, it follows from Proposition 6 that \( f_{\psi}^{-1}(\mathcal{H}, \Delta) \) is IS-b-open. Hence, \( f_{\psi \mid M} \) is an IS-b-continuous map.

Definition 19. If the image of each IS-b-open (resp., IS-b-closed) set under an S-map \( f_{\psi} : (X, \mu, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta) \) is IS-b-open (resp., IS-b-closed), then \( f_{\psi} \) is called IS-b-open (resp., IS-b-closed).

Proposition 15. \( f_{\psi} : (X, \mu, \mathcal{O}) \rightarrow (\mathcal{S}, \nu, \Delta) \) is an IS-b-open map iff \( f_{\psi}(\text{bint}(\mathcal{H}, \mathcal{O})) \subseteq \text{bint}(f_{\psi}(\mathcal{H}, \mathcal{O})) \) for each subset of \((\mathcal{H}, \mathcal{O})\) of \( X \).

Proof. \( \Rightarrow \): Let \((\mathcal{H}, \mathcal{O})\) be a subset of \( X \). Now, \( f_{\psi}(\text{bint}(\mathcal{H}, \mathcal{O})) \subseteq f_{\psi}(\mathcal{H}, \mathcal{O}) \) and \( \text{bint}(\mathcal{H}, \mathcal{O}) \) is an IS-b-open set. By hypothesis, \( f_{\psi}(\text{bint}(\mathcal{H}, \mathcal{O})) \) is IS-b-open. Therefore, \( f_{\psi}(\text{bint}(\mathcal{H}, \mathcal{O})) \subseteq \text{bint}(f_{\psi}(\mathcal{H}, \mathcal{O})) \).

\( \Leftarrow \): Let \((\Lambda, \mathcal{O})\) be an IS-open subset of \( X \). Then \( f_{\psi}(\mathcal{H}, \mathcal{O}) \subseteq \text{bint}(f_{\psi}(\mathcal{H}, \mathcal{O})) \). Therefore, \( f_{\psi}(\mathcal{H}, \mathcal{O}) = \text{bint}(f_{\psi}(\mathcal{H}, \mathcal{O})) \) which means that \( f_{\psi} \) is a IS-b-open map.
Proposition 16. Let \( f : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta) \) be an IS-\( b \)-open map iff \( \overline{f(bcl(\mathcal{H}, \mathcal{O}))} \subseteq f(bcl(\mathcal{H}, \mathcal{O})) \) for each subset \((\mathcal{H}, \mathcal{O})\) of \( X \).

\[ \subseteq \]

Proof. \( \Rightarrow \): Let \( f \) be an IS-\( b \)-open map and \((\mathcal{H}, \mathcal{O})\) be an \( S \)-set of \( X \). By hypothesis, \( f(bcl(\mathcal{H}, \mathcal{O})) \) is IS-\( b \)-open. Since \( f(\mathcal{H}, \mathcal{O}) \subseteq f(bcl(\mathcal{H}, \mathcal{O})) \), \( bcl(f(\mathcal{H}, \mathcal{O})) \subseteq f(bcl(\mathcal{H}, \mathcal{O})) \).

\( \Leftarrow \): Suppose that \((\mathcal{H}, \mathcal{O})\) is an IS-\( b \)-closed subset of \( X \). By hypothesis, \( f(\mathcal{H}, \mathcal{O}) \subseteq bcl(f(\mathcal{H}, \mathcal{O})) \). Therefore, \( f(\mathcal{H}, \mathcal{O}) \) is IS-\( b \)-closed. Hence, \( f \) is an IS-\( b \)-closed map.

Proposition 17. The concepts of IS-\( b \)-open and IS-\( b \)-closed maps are equivalent under bijectiveness.

Proof. It comes from the fact that a bijective soft map \( f : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta) \) implies \( f(\mathcal{H}^c, \mathcal{O}) = (f(\mathcal{H}, \mathcal{O}))^c \).

Proposition 18. Let \( f : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta) \) and \( F : (\mathcal{S}, \nu, \Delta) \to (\mathcal{V}, \sigma, \mathcal{U}) \) be two \( S \)-maps. Then:

(i) If \( f \) and \( F \) are IS-\( b \)-open maps, then \( F \circ f \) is an IS-\( b \)-open map.

(ii) If \( F \circ f \) is an IS-\( b \)-open map and \( f \) is a surjective IS-\( b \)-continuous map, then \( F \) is an IS-\( b \)-open map.

(iii) If \( F \circ f \) is an IS-\( b \)-open map and \( F \) is an injective IS-\( b \)-continuous map, then \( f \) is an IS-\( b \)-open map.

Proof.

(i) Straightforward.

(ii) Consider \((\mathcal{H}, \Delta)\) as an IS-\( b \)-open set in \((\mathcal{S}, \nu, \Delta)\). By hypothesis, \( f^{-1}(\mathcal{H}, \Delta) \) is an IS-\( b \)-open subset of \((X, \mu, \mathcal{O})\). Again, by hypothesis, \((F \circ f)(f^{-1}(\mathcal{H}, \Delta)) \) is an IS-\( b \)-open subset of \((\mathcal{V}, \sigma, \mathcal{U})\). Since \( F \circ f \) is surjective, then \((F \circ f)(f^{-1}(\mathcal{H}, \Delta)) = F(V(f^{-1}(\mathcal{H}, \Delta))) = F(\mathcal{H}, \Delta) \). Hence, \( F \) is an IS-\( b \)-open map.

(iii) Consider \((\mathcal{H}, \mathcal{O})\) as an IS-\( b \)-open subset of \((X, \mu, \mathcal{O})\). By hypothesis, \((F \circ f)(\mathcal{H}, \mathcal{O}) \) is an IS-\( b \)-open subset of \((\mathcal{V}, \sigma, \mathcal{U})\). Again, by hypothesis, \((F \circ f)(\mathcal{H}, \mathcal{O}) \) is an IS-\( b \)-open subset of \((\mathcal{S}, \nu, \Delta)\). Since \( F \circ f \) is injective, then \((F \circ f)(\mathcal{H}, \mathcal{O}) = (F^{-1}(F \circ f)(\mathcal{H}, \mathcal{O})) = f(\mathcal{H}, \mathcal{O}) \). Hence, \( f \) is an IS-\( b \)-open map.

Definition 20. A bijective \( S \)-map \( f : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta) \) is said to be an IS-\( b \)-homeomorphism if it is IS-\( b \)-continuous and IS-\( b \)-open.

The proofs of the following two results are easy and so is omitted.
**Proposition 19.** Let $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ and $F_\psi : (\mathcal{S}, \nu, \Delta) \to (\mathcal{V}, \sigma, \mathcal{U})$ be IS-$b$-homeomorphism maps. Then $F_\psi \circ f_\psi$ is an IS-$b$-homeomorphism map.

**Proposition 20.** If $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is a bijective $S$-map, then the following items are equivalent.

(i) $f_\psi$ is an IS-$b$-homeomorphism.

(ii) $f_\psi$ and $f_\psi^{-1}$ is IS-$b$-continuous.

(iii) $f_\psi$ is IS-$b$-closed and IS-$b$-continuous.

**Proposition 21.** If $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ is an IS-$b$-homeomorphism map, then the following items hold for each $(\mathcal{H}, \mathcal{O}) \in S(X)_A$.

(i) $f_\psi(bint(\mathcal{H}, \mathcal{O})) = bint(f_\psi(\mathcal{H}, \mathcal{O}))$.

(ii) $f_\psi(bcl(\mathcal{H}, \mathcal{O})) = bcl(f_\psi(\mathcal{H}, \mathcal{O}))$.

**Proof.** (i): According to Proposition 15 (i), we obtain $f_\psi(bint(\mathcal{H}, \mathcal{O})) \subseteq bint(f_\psi(\mathcal{H}, \mathcal{O}))$. Conversely, let $\delta^*_b \in bint(f_\psi(\mathcal{H}, \mathcal{O}))$. Then there is an IS-$b$-open set $(\mathcal{F}, \Delta)$ such that $\delta^*_b \in (\mathcal{F}, \Delta) \subseteq f_\psi(\mathcal{H}, \mathcal{O})$. By hypothesis, $\delta^*_b = f_\psi^{-1}(\delta^*_b) \in f_\psi^{-1}(\mathcal{F}, \Delta) \subseteq (\mathcal{H}, \mathcal{O})$ such that $f_\psi^{-1}(\mathcal{F}, \Delta)$ is an infra soft $b$-open set. So that, $\delta^*_b \in bint(\mathcal{H}, \mathcal{O})$ which means that $\delta^*_b \in f_\psi(bint(\mathcal{H}, \mathcal{O}))$. One can achieve item (ii) following similar arguments.

**Theorem 6.** The property of an IS-$b$-dense set is an IS-topological invariant.

**Proof.** Let $f_\psi : (X, \mu, \mathcal{O}) \to (\mathcal{S}, \nu, \Delta)$ be an IS-$b$-homeomorphism map and consider $(\mathcal{H}, \mathcal{O})$ as an IS-$b$-dense subset of $(X, \mu, \mathcal{O})$, i.e. $bcl(\mathcal{H}, \mathcal{O}) = \bar{X}$. It comes from Proposition 21 (ii) that $bcl(f_\psi(\mathcal{H}, \mathcal{O})) = f_\psi(bcl(\mathcal{H}, \mathcal{O})) = f_\psi(\bar{X}) = bcl(\bar{S}) = \bar{S}$. Thus, $f_\psi(\mathcal{H}, \mathcal{O})$ is an IS-$b$-dense set in $(\mathcal{S}, \nu, \Delta)$, as required.

We complete this section by studying the concept of fixed soft points with respect to IS-$b$-open sets.

**Definition 21.** We say that $(X, \mu, \mathcal{O})$ has a b-fixed $S$-point property provided that for every IS-$b$-continuous map $f_\psi : (X, \mu, \mathcal{O}) \to (X, \mu, \mathcal{O})$ there exists $\delta^*_b \in X$ such that $f_\psi(\delta^*_b) = \delta^*_b$.

**Proposition 22.** The property of being a b-fixed $S$-point is preserved under an IS-$b$-homeomorphism.

**Proof.** Consider $(X_1, \mu_1, \mathcal{O}_1)$ and $(X_2, \mu_2, \mathcal{O}_2)$ as two IS-$b$-homeomorphism. This means that there exists a bijective $S$-map $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ such that $f_\psi$ and $f_\psi^{-1}$ are IS-$b$-continuous. Suppose that $(X_1, \mu_1, \mathcal{O}_1)$ has the property of b-fixed soft point. That is any IS-$b$-continuous map $f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_1, \mu_1, \mathcal{O}_1)$ has a b-fixed S-point. Now, consider $C_\psi : (X_2, \mu_2, \mathcal{O}_2) \to (X_2, \mu_2, \mathcal{O}_2)$ is IS-$b$-continuous. It is clear that $C_\psi \circ f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_2, \mu_2, \mathcal{O}_2)$ is IS-$b$-continuous. Therefore, $f_\psi^{-1} \circ C_\psi \circ f_\psi : (X_1, \mu_1, \mathcal{O}_1) \to (X_1, \mu_1, \mathcal{O}_1)$ is IS-$b$-continuous. Since $(X_1, \mu_1, \mathcal{O}_1)$ has a b-fixed $S$-point property, $f_\psi^{-1}(h_\psi(f_\psi(\delta^*_b))) = \delta^*_b$ for some $\delta^*_b \in \bar{X}$. Thus, $f_\psi(f_\psi^{-1}(h_\psi(f_\psi(\delta^*_b)))) = f_\psi(\delta^*_b)$. This implies that $h_\psi(f_\psi(\delta^*_b)) = f_\psi(\delta^*_b)$. Hence, $f_\psi(\delta^*_b)$ is a b-fixed soft point of $C_\psi$ which means that $(X_2, \mu_2, \mathcal{O}_2)$ has a b-fixed $S$-point property.
6. Concluding remark and further work

In this paper, we have formulated the concept of IS-\(b\)-open sets and discussed its main properties. Then, we have defined novel operators and mappings between ISTSs depending on the classes of IS-\(b\)-open and IS-\(b\)-closed sets. We have revealed the relationships between these operators and mappings and investigated their basic features. As we have noted that several topological characterizations still have been valid via the structures of infra topologies, which confirms the importance of infra ST-structures.

Our future works will focus on studying further topological concepts and notions via infra ST-structures. Also, we will research the hybridizations structures obtained from ISTs and other structures such as rough soft and FS-structures.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] R Abu-Gdairi, M El-Gayar, TM Al-shami, AS Nawar, and MK El-Bably. Some topological approaches for generalized rough sets and their decision-making applications. *Symmetry*, 14(1), 2022.

[2] HH Al-jarrah, A Rawshdeh, and TM Al-shami. On soft compact and soft lindelöf spaces via soft regular closed sets. *Afrika Matematika*, 33(23), 2022.

[3] TM Al-shami. Soft somewhere dense sets on soft topological spaces. *Communications of the Korean Mathematical Society*, 33(4):1341–1356, 2018.

[4] TM Al-shami. Comment on soft mappings space. *The Scientific World Journal*, 2019, 2019.

[5] TM Al-shami. Investigation and corrigendum to some results related to g-soft equality and gf-soft equality relations. *Filomat*, 33(11):3375–3383, 2019.

[6] TM Al-shami. Comments on some results related to soft separation axioms. *Afrika Matematika*, 31(7):1105–1119, 2020.

[7] TM Al-shami. Soft separation axioms and fixed soft points using soft semiopen sets. *Journal of Applied Mathematics*, 2020, 2020.

[8] TM Al-shami. Bipolar soft sets: relations between them and ordinary points and their applications. *Complexity*, 2021, 2021.
REFERENCES

[9] TM Al-shami. Compactness on soft topological ordered spaces and its application on the information system. *Journal of Mathematics*, 2021, 2021.

[10] TM Al-shami. Homeomorphism and quotient mappings in infrasoft topological spaces. *Journal of Mathematics*, 2021, 2021.

[11] TM Al-shami. Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets. *Soft Computing*, 25(23):14449–14460, 2021.

[12] TM Al-shami. Infra soft compact spaces and application to fixed point theorem. *Journal of Function Spaces*, 2021, 2021.

[13] TM Al-shami. New soft structure: infra soft topological spaces. *Mathematical Problems in Engineering*, 2021, 2021.

[14] TM Al-shami. On soft separation axioms and their applications on decision-making problem. *Mathematical Problems in Engineering*, 2021, 2021.

[15] TM Al-shami. Soft somewhat open sets: Soft separation axioms and medical application to nutrition. *Computational and Applied Mathematics*, 41, 2022.

[16] TM Al-shami. Topological approach to generate new rough set models. *Complex & Intelligent Systems*, 2022.

[17] TM Al-shami and EA Abo-Tabl. Connectedness and local connectedness on infra soft topological spaces. *Mathematics*, 9(15), 2021.

[18] TM Al-shami and EA Abo-Tabl. Soft α-separation axioms and α-fixed soft points. *AIMS Mathematics*, 6(6):5675–5694, 2021.

[19] TM Al-shami, EA Abo-Tabl, and BA Asaad. Weak forms of soft separation axioms and fixed soft points. *Fuzzy Information and Engineering*, 12(4):509–528, 2020.

[20] TM Al-shami, I Alshammari, and BA Asaad. Soft maps via soft somewhere dense sets. *Filomat*, 34(10):3429–3440, 2020.

[21] TM Al-shami, ZA Ameen, AA Azzam, and ME El-Shafei. Soft separation axioms via soft topological operators. *AIMS Mathematics*, 7(8):15107–15119, 2022.

[22] TM Al-shami, BA Asaad, and EA Abo-Tabl. Separation axioms and fixed points using total belong and total non-belong relations with respect to soft β-open sets. *Journal of Interdisciplinary Mathematics*, 24(4):1053–1077, 2021.

[23] TM Al-shami and AA Azzam. Infra soft semiopen sets and infra soft semicontinuity. *Journal of Function Spaces*, 2021, 2021.

[24] TM Al-shami and ME El-Shafei. On supra soft topological ordered spaces. *Arab Journal of Basic and Applied Sciences*, 26(1):433–445, 2019.
[25] TM Al-shami and ME El-Shafei. Some types of soft ordered maps via soft pre open sets. *Applied Mathematics & Information Sciences*, 13(5):707–715, 2019.

[26] TM Al-shami and ME El-Shafei. Two types of separation axioms on supra soft topological spaces. *Demonstratio Mathematica*, 52(1):147–165, 2019.

[27] TM Al-shami and ME El-Shafei. T-soft equality relation. *Turkish Journal of Mathematics*, 44(4):1427–1441, 2020.

[28] TM Al-shami, ME El-Shafei, and M Abo-Elhamayel. Almost soft compact and approximately soft lindelöf spaces. *Journal of Taibah University for Science*, 12(5):620–630, 2018.

[29] TM Al-shami, ME El-Shafei, and M Abo-Elhamayel. Seven generalized types of soft semi-compact spaces. *Korean Journal of Mathematics*, 27(3):661–690, 2019.

[30] TM Al-shami, ME El-Shafei, and BA Asaad. Other kinds of soft β maps via soft topological ordered spaces. *European Journal of Pure and Applied Mathematics*, 12(1):176–193, 2019.

[31] TM Al-shami, ME El-Shafei, and BA Asaad. Sum of soft topological ordered spaces. *Advances in Mathematics: Scientific Journal*, 9(7):4695–4710, 2020.

[32] TM Al-shami, H Işık, AS Nawar, and RA Hosny. Some topological approaches for generalized rough sets via ideals. *Mathematical Problems in Engineering*, 2021, 2021.

[33] TM Al-shami and LDR Kočinac. The equivalence between the enriched and extended soft topologies. *Applied and Computational Mathematics*, 18(2):149–162, 2019.

[34] TM Al-shami and LDR Kočinac. Nearly soft menger spaces. *Journal of Mathematics*, 2020, 2020.

[35] TM Al-shami and LDR Kočinac. Almost soft menger and weakly soft menger spaces. *Applied and Computational Mathematics*, 21(1):35–51, 2022.

[36] TM Al-shami, LDR Kočinac, and BA Asaad. Sum of soft topological spaces. *Mathematics*, 8(6):990, 2020.

[37] TM Al-shami and J-B Liu. Two classes of infrasoft separation axioms. *Journal of Mathematics*, 2021, 2021.

[38] TM Al-shami and A Mhemdi. Belong and nonbelong relations on double-framed soft sets and their applications. *Journal of Mathematics*, 2021, 2021.

[39] TM Al-shami and A Mhemdi. Two families of separation axioms on infra soft topological spaces. *Filomat*, 36(4):1143–1157, 2022.
[40] TM Al-shami, A Mhemdi, A Rawshdeh, and HH Al-jarrah. Soft version of compact and lindelöf spaces using soft somewhere dense sets. *AIMS Math*, 6(8):8064–8077, 2021.

[41] TM Al-shami and HA Othman. Infra pre-open sets and their applications to generate new types of operators and maps. *European Journal of Pure and Applied Mathematics*, 15(1):261–280, 2022.

[42] TM Al-shami, A. Tercan, and A Mhemdi. New soft separation axioms and fixed soft points with respect to total belong and total non-belong relations. *Demonstratio Mathematica*, 54(1):196–211, 2021.

[43] JCR Alcantud, TM Al-shami, and AA Azzam. Caliber and chain conditions in soft topologies. *Mathematics*, 9(19), 2021.

[44] MI Ali, F Feng, X Liu, WK Min, and M Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[45] ZA Ameen, TM Al-shami, M Abdelwaheb, and ME El-Shafei. The role of soft θ-topological operators in characterizing various soft separation axioms. *Journal of Mathematics*, 2022:7 pages, 2022.

[46] ZA Ameen, AA Azzam, TM Al-shami, and ME El-Shafei. Generating soft topologies via soft set operators. *Symmetry*, 14(5), 2022.

[47] CG. Aras, TM Al-shami, A Mhemdi, and S Bayramov. Local compactness and paracompactness on bipolar soft topological spaces. *Journal of Intelligent and Fuzzy Systems*, 43(5):6755–6763.

[48] BA Asaad, TM Al-shami, and A Mhemdi. Bioperators on soft topological spaces. *AIMS Mathematics*, 6(11):12471–12490, 2021.

[49] A. Aygınoglu and H. Aygın. Some notes on soft topological spaces. *Neural computing and Applications*, 21(1):113–119, 2012.

[50] N. Çağman and S. Enginoğlu. Soft matrix theory and its decision making. *Computers & Mathematics with Applications*, 59(10):3308–3314, 2010.

[51] N Çağman, S Karataş, and S Enginoğlu. On soft topology. *Computers & Mathematics with Applications*, 62:351–358, 2011.

[52] ME El-Shafei, M Abo-Elhamayel, and TM Al-shami. Partial soft separation axioms and soft compact spaces. *Filomat*, 32(13):4755–4771, 2018.

[53] ME El-Shafei and TM Al-shami. Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem. *Computational and Applied Mathematics*, 39(3):1–17, 2020.
[54] ME El-Shafei and TM Al-shami. Some operators of a soft set and soft connected spaces using soft somewhere dense sets. *Journal of Interdisciplinary Mathematics*, 24(6):1471–1495, 2021.

[55] F Feng, C Li, B Davvaz, and MI Ali. Soft sets combined with fuzzy sets and rough sets: a tentative approach. *Soft computing*, 14(9):899–911, 2010.

[56] RA Hosny, BA Asaad, AA Azzam, and TM Al-shami. Various topologies generated from-neighbourhoods via ideals. *Complexity*, 2021, 2021.

[57] S Hussain. Binary soft connected spaces and an application of binary soft sets in decision making problem. *Fuzzy Information and Engineering*, 11(4):506–521, 2019.

[58] A Kharal and B Ahmad. Mappings on soft classes. *New Mathematics and Natural Computation*, 7(03):471–481, 2011.

[59] LDR Kočinac, TM Al-shami, and V Çetkin. Selection principles in the context of soft sets: Menger spaces. *Soft computing*, 25(20):12693–12702, 2021.

[60] F Lin. Soft connected spaces and soft paracompact spaces. *International Journal of Mathematical and Computational Sciences*, 7(2):277–283, 2013.

[61] PK Maji, R Biswas, and AR Roy. Soft set theory. *Computers & Mathematics with Applications*, 45(4-5):555–562, 2003.

[62] D Molodtsov. Soft set theory first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[63] Sk Nazmul and SK Samanta. Neighbourhood properties of soft topological spaces. *Annals of Fuzzy Mathematics and Informatics*, 6(1):1–15, 2013.

[64] AS Salama, A Mhemdi, OG Elbarbary, and TM Al-shami. Topological approaches for rough continuous functions with applications. *Complexity*, 2021, 2021.

[65] M Shabir and M Naz. On soft topological spaces. *Computers & Mathematics with Applications*, 61(7):1786–1799, 2011.