Nonstationary quantum systems and entanglement in the tomographic-probability representation

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Abstract. Time-dependent integrals of motion for systems with both time-independent and time-dependent Hamiltonians are studied and expressed in terms of the evolution operator. The probability representation in which the system quantum states are described by tomographic probabilities (tomograms), is reviewed. Examples of systems of parametric oscillators and the entanglement phenomena induced by time-dependent coupling of the oscillators are discussed. Transition probabilities between the energy levels of parametric oscillators expressed in terms of special functions like Laguerre and Legendre polynomials are given in terms of tomographic probabilities.

1. Introduction

The systems with time-dependent Hamiltonian like classical or quantum oscillator with time-dependent frequency studied in [1] have specific time-dependent integrals of motion, e.g. quadratic in position and momentum [2] or linear in position and momentum [3–5].

Also the systems have the universal invariants which are analogs of universal Poincare-Cartan invariants. The theory of quantum universal invariants was given in [6, 7]. The application of universal invariants to the light propagation in waveguids was presented in [8]. The oscillator with time-dependent parameters and damping and its application to study the nonstationary Casimir effect [9] was considered in [10] by V. V. Dodonov and A. V. Dodonov in complete form. The Green functions are connected with the integrals of motion [11, 12]. The Green function of a charge in electromagnetic fields was used in [13].

There exists the probability representation of quantum mechanics [14–16], in which the tomographic probability is used instead of the density matrix or wave function for describing the quantum state. This representation is equivalent to all other available representations like the path-integral representation or Weyl representation, in which the quantum states are described by the Wigner function [17], the review of the representations is available in [18].

The aim of this work is to consider nonstationary quantum systems in the probability representation of quantum mechanics. We study the evolution of tomographic probabilities for a system of coupled parametric oscillators and time-dependent integrals of motion for this system. The quantum phenomenon of entanglement appearing in the process of evolution of the parametric oscillators is considered using the probability description of quantum states. It is worthy noting that entanglement in two system of parametric oscillators was studied in [19].
The paper is organized as follows: in Section 2 we consider linear integrals of motion for parametric oscillators. In Section 3 we review the probability representation of oscillator quantum states. The expression for probabilities between the energy levels is derived in Section 4. The entanglement of Gaussian states for bipartite oscillator system is discussed in Section 5. Conclusions and perspectives are given in Section 6.

2. Linear integrals of motion of quadratic systems

The linear in position and momentum integrals of motion exist for systems (both stationary and nonstationary ones) with Hamiltonians quadratic in the position and momentum operators. One can easily see this on the example of free classical motion of the particle with Hamiltonian (mass of the particle is taken to be equal to unity)

\[ H = \frac{p^2}{2}. \]  

The particle trajectory in the phase space reads

\[ p = p_0, \quad q = q_0 + p_0 t, \]  

where \( p = \dot{q} \) is the initial momentum and \( q_0 \) is the initial position, which are obviously constants of the motion. In fact, solving (2) for \( q_0 \) and \( p_0 \) as functions on the phase space, we obtain

\[ p_0 = p, \quad q_0 = q - pt. \]  

These two functions satisfy the equation

\[ \frac{df(q,p,t)}{dt} = \frac{\partial f(q,p,t)}{\partial t} + \frac{\partial f(q,p,t)}{\partial q} p + \frac{\partial f(q,p,t)}{\partial p} \dot{p} = 0, \]  

since \( \dot{p} = 0 \). Equation (4) is the equation for the classical integral of motion. Formula (3) can be presented in the vector form

\[ \vec{Q}_0 = \Lambda(t) \vec{Q}, \]  

where

\[ \vec{Q}_0 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}, \quad \vec{Q} = \begin{pmatrix} p \\ q \end{pmatrix} \]  

and the \( 2 \times 2 \) real matrix \( \Lambda(t) \) reads

\[ \Lambda(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}. \]  

Relations (3) or (5) can be interpreted as linear canonical transforms of the position and momentum preserving the Poisson brackets, since

\[ \{ p, q \} = \{ p_0, q_0 \}. \]  

The set of matrices \( \Lambda(t) \), preserving the Poisson brackets, is the real symplectic group \( \text{Sp}(2,R) \). The matrices (7) form nilpotent subgroup of the symplectic group. For a quantum free particle with the Hamiltonian

\[ \hat{H} = \frac{\hat{p}^2}{2}, \]  

the particle trajectory in the phase space reads

\[ \hat{p} = \hat{p}_0, \quad \hat{q} = \hat{q}_0 + \hat{p}_0 \hat{t}, \]  

where \( \hat{p} = \hat{\dot{q}} \) is the initial momentum and \( \hat{q}_0 \) is the initial position, which are obviously constants of the motion. In fact, solving (2) for \( \hat{q}_0 \) and \( \hat{p}_0 \) as functions on the phase space, we obtain

\[ \hat{p}_0 = \hat{p}, \quad \hat{q}_0 = \hat{q} - \hat{p} \hat{t}. \]  

These two functions satisfy the equation

\[ \frac{d\hat{f}(\hat{q},\hat{p},t)}{dt} = \frac{\partial \hat{f}(\hat{q},\hat{p},t)}{\partial t} + \frac{\partial \hat{f}(\hat{q},\hat{p},t)}{\partial \hat{q}} \hat{p} + \frac{\partial \hat{f}(\hat{q},\hat{p},t)}{\partial \hat{p}} \hat{\dot{p}} = 0, \]  

since \( \hat{\dot{p}} = 0 \). Equation (4) is the equation for the classical integral of motion. Formula (3) can be presented in the vector form

\[ \hat{\vec{Q}}_0 = \Lambda(t) \hat{\vec{Q}}, \]  

where

\[ \hat{\vec{Q}}_0 = \begin{pmatrix} \hat{p}_0 \\ \hat{q}_0 \end{pmatrix}, \quad \hat{\vec{Q}} = \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} \]  

and the \( 2 \times 2 \) real matrix \( \Lambda(t) \) reads

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The set of matrices \( \Lambda(t) \), preserving the Poisson brackets, is the real symplectic group \( \text{Sp}(2,R) \). The matrices (7) form nilpotent subgroup of the symplectic group. For a quantum free particle with the Hamiltonian
where the momentum operator in the position representation reads
\[ \hat{p} = -i \frac{\partial}{\partial x} \] (10)

(we take Planck’s constant \( \hbar = 1 \)). The quantum operator in the form (3), or (5) is preserved, i.e.
\[ \hat{p}_0 = \hat{p}, \quad \hat{q}_0 = \hat{q} - \hat{p}t \] (11)
satisfy the condition
\[ \frac{d\hat{f}(\hat{q}, \hat{p}, t)}{dt} = \frac{\partial \hat{f}(\hat{q}, \hat{p}, t)}{\partial t} + i[\hat{H}, \hat{f}(\hat{q}, \hat{p}, t)] = 0. \] (12)

The condition (12) is a quantum analog of the classical condition (4) for quantum integral of motion. Analogous invariants exist for n-dimensional free particle. The example of free motion shows that, for the system with quadratic Hamiltonian, there exist 2n integrals of motion \( \vec{p}_0 \) and \( \vec{q}_0 \) in the classical case and \( \hat{\vec{p}}_0 \) and \( \hat{\vec{q}}_0 \) operators in the quantum case, which have the physical meaning of the initial coordinates of the particle trajectory in the phase space of the system. The quantum integrals of motion can be described by a 2n-vector
\[ \hat{\vec{Q}}_0 = \begin{pmatrix} \hat{p}_0 \\ \hat{q}_0 \end{pmatrix} = \Lambda(t) \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \Lambda(t) \hat{\vec{Q}}, \] (13)

where the matrix \( \Lambda(t) \) belongs to symplectic group Sp(2n,R). This means that the commutation relations of the integrals of motion read
\[ [\hat{Q}_{0k}, \hat{Q}_{0j}] = [\hat{Q}_k, \hat{Q}_j] = \Sigma_{kj}, \quad (k, j = 1, 2, \ldots 2n). \] (14)

Here the antisymmetric \( 2n \times 2n \) matrix \( \Sigma \) reads
\[ \Sigma = i \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \] (15)

The above construction of the linear in positions and momenta integrals of motion can be extended for the system of coupled parametric oscillators with the Hamiltonian
\[ \hat{H} = \frac{1}{2} \sum_{j,k=1}^{2n} \hat{Q}_j B_{jk}(t) \hat{Q}_k. \] (16)

This system has the linear integrals of motion of the form (13) provided the symplectic matrix \( \Lambda(t) \) obeys to the differential equation
\[ \dot{\Lambda}(t) = -i\Lambda(t)\Sigma B(t), \] (17)

with the initial condition
\[ \Lambda(0) = 1_{2n}. \] (18)

For the stationary system \( B(t) = B \) and the system of equations (17) with the initial condition (18) has the solution
\[ \Lambda(t) = \exp(-it\Sigma B). \] (19)
For example, the one-dimensional harmonic oscillator has the unit matrix $B$ determining its Hamiltonian and the linear integrals of motion of the form

$$\hat{p}_0 = \hat{p} \cos t + \hat{q} \sin t, \quad \hat{q}_0 = -\hat{p} \sin t + \hat{q} \cos t. \quad (20)$$

The symplectic matrix $\Lambda(t)$ for the harmonic oscillator given by (19) reads

$$\Lambda(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \quad (21)$$

The integrals of motion give the system of equations for the Green function of the system evolution equation. For example, in the position representation, the Green function $G(\vec{x}, \vec{x}', t)$ satisfies the equations (see, e.g. [5])

$$-i \frac{\partial G(\vec{x}, \vec{x}', t)}{\partial x'} = i \frac{\partial G(\vec{x}, \vec{x}', t)}{\partial x}, \quad (22)$$

and

$$xG(\vec{x}, \vec{x}', t) + it \frac{\partial G(\vec{x}, \vec{x}', t)}{\partial x} = x'G(\vec{x}, \vec{x}', t). \quad (24)$$

Equation (23) yields

$$G(\vec{x}, \vec{x}', t) = F(x, -x', t) \quad (25)$$

and (24) has the solution

$$F(y, t) = e^{i\phi(t)} \frac{1}{\sqrt{2\pi it}} \exp \left\{ -\frac{i}{2} \left[ y^2 + 2y \lambda_3^{-1} \lambda_4 \bar{x} - 2 \bar{x} \lambda_3^{-1} \bar{x}' + \bar{x}' \lambda_3^{1} \lambda_4^{-1} \bar{x}' \right] \right\}. \quad (26)$$

The function $F(y, t)$ gives the Dirac delta-function for $t = 0$ provided the phase function $\phi(0) = 2\pi k$, $k = 0, \pm 1, \pm 2 \ldots$. To obtain the function $\phi(t)$, one needs to use the equation

$$i \frac{\partial G(\vec{x}, \vec{x}', t)}{\partial t} = \hat{H}G(\vec{x}, \vec{x}', t), \quad (27)$$

which gives the constant function $\phi(t) = 2\pi k$.

For the system of parametric oscillators with the Hamiltonian (16), the Green function reads [5]

$$G(\vec{x}, \vec{x}', t) = [\det(-2\pi i \lambda_3)]^{-1/2} \exp \left\{ -\frac{i}{2} \left[ \bar{x} \lambda_3^{-1} \lambda_4 \bar{x} - 2 \bar{x} \lambda_3^{-1} \bar{x}' + \bar{x}' \lambda_3^{1} \lambda_4^{-1} \bar{x}' \right] \right\}. \quad (28)$$

The matrix parameters determining the Gaussian expression (depending on time $t$) are the blocks of the symplectic matrix $\Lambda$ presented in the form

$$\Lambda(t) = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}. \quad (29)$$

We assume that $\det \lambda_3 \neq 0$. Using the explicit form of the matrix $\Lambda(t)$ for the one-dimensional free particle (7), one can insert into (28) the values $\lambda_3 = -t, \lambda_1 = \lambda_4 = 1$, and get the Green function

$$G(x, x', t) = \frac{1}{\sqrt{2\pi it}} \exp \left( i(x - x')^2 / 2t \right), \quad (30)$$
which coincides with the expression given by (26). Also using the matrix Λ(t) for the harmonic oscillator (21) and inserting into (28) the values \(\lambda_1 = \lambda_4 = \cos t, \lambda_2 = -\lambda_3 = \sin t\), we obtain the Green function of harmonic oscillator

\[
G(x, x', t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp \left[\frac{i}{2} \cot t (x^2 + x'^2) - \frac{i xx'}{\sin t}\right].
\]

(31)

In fact, formula (28) provides the expression of the Green function of the system of parametric oscillators in terms of its classical action

\[
S(\vec{x}, \vec{x}', t) = -\frac{1}{2} \left[\vec{x}^T \Lambda_3^{-1} \Lambda_4 \vec{x} - 2\vec{x}^T \Lambda_3^{-1} \vec{x}' + \vec{x}'^T \Lambda_1 \Lambda_3^{-1} \vec{x}'\right].
\]

(32)

Expression (28) yields the Green function of a charged parametric oscillator moving in nonstationary homogeneous magnetic field.

3. Probability representation of quantum states

In [14] it was shown that the quantum states can be described by the standard probabilities called symplectic tomograms \(w(X, \mu, \nu)\) which are related to the wave functions by means of the fractional Fourier transform (see [20])

\[
w(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left|\int \psi(y) \exp \left[\frac{i \mu y^2}{2\nu} - \frac{i X y}{\nu}\right] dy\right|^2.
\]

(33)

Here \(X, \mu\) and \(\nu\) are real variables. The argument \(X\) is a random position of a particle measured in a scaled and rotated reference frame in its phase space determined by the parameters \(\mu = s \cos \theta\) and \(\nu = s^{-1} \sin \theta\). The angle \(\theta\) is the rotation angle of the reference frame and \(s\) is a scaling parameter. For normalized wave function, the probability distribution \(w(X, \mu, \nu)\) is normalized, i.e.,

\[
\int w(X, \mu, \nu) dX = 1.
\]

(34)

One can get the density matrix of the pure state in terms of its tomogram

\[
\psi(x)\psi^*(x') = \frac{1}{2\pi} \int w(Y, \mu, x - x') \exp \left[i \left(Y - \mu \frac{x + x'}{2}\right)\right] dY d\mu.
\]

(35)

The symplectic tomogram can be expressed in terms of the Wigner function [17] of the pure state

\[
W(q, p) = \int \psi(q + u/2)\psi^*(q - u/2) e^{-ipu} du,
\]

(36)

using the integral Radon transform [21]

\[
w(X, \mu, \nu) = \int W(q, p) \delta(X - \mu q - \nu p) \frac{dq dp}{2\pi}.
\]

(37)

The inverse Radon transform yields the Wigner function in terms of the symplectic tomogram

\[
W(q, p) = \int w(X, \mu, \nu) e^{i(X - \mu q - \nu p)} \frac{dX d\mu d\nu}{2\pi}.
\]

(38)

For the n-dimensional system, the symplectic tomogram is expressed in terms of the Wigner function as follows:

\[
w(\vec{X}, \vec{\mu}, \vec{\nu}) = \int W(\vec{q}, \vec{p}) \prod_{k=1}^{n} \delta(X_k - \mu_k q_k - \nu_k p_k) \frac{dq_k dp_k}{2\pi}.
\]

(39)
The invariant connection of the density operator $\hat{\rho}$ of the system with its symplectic tomogram reads

$$\hat{\rho} = \int w(\vec{X}, \vec{\mu}, \vec{\nu}) \left( \prod_{k=1}^{n} \exp \left[ i (X_k - \mu_k \hat{q}_k - \nu_k \hat{p}_k) \right] \right) \frac{dX_k d\mu_k d\nu_k}{2\pi}. \quad (40)$$

Formulae (39) and (40) are valid for the both pure and mixed quantum states. Since the tomograms are the standard probability distributions, which determine the density operators of quantum states, the formulation of quantum mechanics, based on the using of the tomograms for description of quantum states is called the probability representation of quantum mechanics.

The evolution equation and equation for the energy spectra of quantum systems follow from the von Neumann equations for density matrix by applying the fractional Fourier (or Radon) transforms.

4. Separability and entanglement

The quantum states of composite systems have specific quantum correlations related to the entanglement phenomenon. Let us discuss this phenomenon in terms of the probability representation of quantum mechanics. For example, for a two-mode system, the symplectic tomogram is the joint probability distribution function of two random positions $X_1$ and $X_2$. The state is called separable, if its tomogram can be expressed as a convex sum of the joint probabilities, each of the probabilities being uncorrelated, e.g.,

$$w(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \sum_k p_k w^{(k)}_1(X_1, \mu_1, \nu_1) w^{(k)}_2(X_2, \mu_2, \nu_2). \quad (41)$$

Here $p_k \geq 0$ and $\sum_k p_k = 1$. The indices $k$ can belong to an arbitrary domain; also they can be continuous ones.

The property (41) corresponds to the property of density operator $\hat{\rho}(1, 2)$ of a bipartite system with components 1 and 2, determining the separable state

$$\hat{\rho}(1, 2) = \sum_k p_k \hat{\rho}^{(k)}(1) \otimes \hat{\rho}^{(k)}(2). \quad (42)$$

Here $\hat{\rho}^{(k)}(1)$ and $\hat{\rho}^{(k)}(2)$ are the density operators of subsystems states. If the density operator $\rho(1, 2)$ of a quantum state cannot be expressed as convex sum (42), the state is called entangled. In the probability representation this means that for entangled state the tomogram of the system state cannot be presented in the form of convex sum of the products of the subsystems tomograms of the form (41).

An example of symplectic tomogram of the two-mode oscillator ground state reads

$$w_0(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{\sqrt{\pi}} \exp \left(- \frac{X_1^2}{\mu_1^2 + \nu_1^2} - \frac{X_2^2}{\mu_2^2 + \nu_2^2} \right). \quad (43)$$

This Gaussian probability distribution corresponds to a separable state, since the tomogram is factorized

$$w_0(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{\sqrt{\pi}} \exp \left(- \frac{X_1^2}{\mu_1^2 + \nu_1^2} \right) \frac{1}{\sqrt{\pi}} \exp \left(- \frac{X_2^2}{\mu_2^2 + \nu_2^2} \right). \quad (44)$$
The two-mode squeezed vacuum state with wave function \( \alpha < 1 \)

\[
\psi_s(x, y) = \left( 1 - \alpha^2 \right)^{1/4} \exp \left( -x^2/2 - y^2/2 - \alpha xy \right)
\]  

(45)
is described by the Gaussian tomogram

\[
w_s(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{\sqrt{\det(2\pi \Sigma)}} \exp \left\{ -\frac{1}{2} (X_1, X_2) \Sigma^{-1} \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) \right\},
\]  

(46)

where the matrix \( \Sigma \) reads

\[
\Sigma = \left( \begin{array}{cc} \sigma_{X_1X_1} & \sigma_{X_1X_2} \\ \sigma_{X_2X_1} & \sigma_{X_2X_2} \end{array} \right)
\]  

(47)

and variances and covariances are

\[
\sigma_{X_1X_1} = \frac{1}{2}(\mu_1^2 + \nu_1^2), \quad \sigma_{X_2X_2} = \frac{1}{2}(\mu_2^2 + \nu_2^2), \quad \sigma_{X_1X_2} = \sigma_{X_2X_1} = \alpha(\mu_1\mu_2 + \nu_1\nu_2).
\]  

(48)
The squeezed vacuum state with tomogram (46) is entangled state, since this tomogram cannot be presented in the form (41) for any value of parameter \( \alpha \). It can be also checked using the partial scaling criterion of separability [22] or the partial transpose criterion [23–26]. For a \( n \)th energy level of the harmonic oscillator, the tomogram reads

\[
w_n(X, \mu, \nu) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{(\mu^2 + \nu^2)^n}} H_n^{2} \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right),
\]  

(49)

where \( H_n(y) \) is the Hermite polynomial.

If in formula (33) the wave function \( \psi(y) \) is replaced by shifted and scaled wave function

\[
\psi(y) \rightarrow \frac{1}{\sqrt{l}} \psi \left( \frac{y - a}{l} \right) = \tilde{\psi}(y)
\]  

(50)
the tomogram \( w_{l,a}(X, \mu, \nu) \), corresponding to the wave function \( \tilde{\psi}(y) \), is expressed in terms of the tomogram \( w(X, \mu, \nu) \), corresponding to the wave function \( \psi(y) \), as follows:

\[
w_{l,a}(X, \mu, \nu) = w(X - \mu a, l\mu, \nu/l).
\]  

(51)

Thus for the harmonic oscillator with scaled frequency and shifted rest position, the tomogram of the \( n \)th excited state reads

\[
w_{n,l,a}(X, \mu, \nu) = \frac{1}{\sqrt{\pi((l\mu)^2 + (\nu/l)^2)^n}} \exp \left( -\frac{(X - \mu a)^2}{(l\mu)^2 + (\nu/l)^2} \right) \frac{1}{2^n n!} H_n^{2} \left( \frac{X - \mu a}{\sqrt{(l\mu)^2 + (\nu/l)^2}} \right).
\]  

(52)
The transition probability between two quantum states \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) (or fidelity) is expressed in terms of the density operator as follows:

\[
P_{12} = \text{Tr} \hat{\rho}_1 \hat{\rho}_2.
\]  

(53)
For pure states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), this formula yields

\[
P_{12} = |\langle \psi_1 | \psi_2 \rangle|^2.
\]  

(54)
The transition probability can be expressed in terms of the symplectic tomograms

\[ P_{12} = \frac{1}{2\pi} \int w_1(X, \mu, \nu) w_2(Y, \mu, \nu) \cos(X - Y) dX dY d\mu d\nu. \]  

(55)

In terms of quantum tomographic probabilities the Frank-Condon factor for the transition between \( n \)th and \( m \)th energy levels of the harmonic oscillator, due to the shift of its equilibrium position and the frequency change reads

\[ F_{nm}(a, l) = \frac{1}{2^{n+m+1} n! m! \pi^2} \int \exp \left[ -\frac{X^2}{\mu^2 + \nu^2} - \frac{(Y-\mu a)^2}{(\mu l)^2 + (\nu/l)^2} \right] \]

\[ \times H_n^2 \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right) H_m^2 \left( \frac{(Y-\mu a)}{\sqrt{(\mu l)^2 + (\nu/l)^2}} \right) \cos(X - Y) dX dY d\mu d\nu. \]  

(56)

For \( a = 0, l = 1 \), the transition probability reads

\[ F_{nm}(0, 1) = \delta_{nm}. \]  

(57)

The fidelity

\[ F_{00}(a, 1) = \exp(-a^2/2) \]  

(58)

determines the fidelities given by (56)

\[ F_{nm}(a, 1) = \exp(-a^2/2) \left[ L_n(a^2/2) \right]^2, \]  

(59)

where \( L_n \) is the Laguerre polynomial.

The integral (56) provides generic transition probabilities due to the shift of the equilibrium position of the oscillator

\[ F_{nm}(a, 1) = \exp(-a^2/2)(a^2/2)^{2(m-n)} \left[ P_n(F_{00}(0, l)) \right]^2, \]  

(60)

The integral (56) provides also the fidelity

\[ F_{00}(0, l) = \frac{2l}{l^2 + 1} \]  

(61)

and the fidelity

\[ F_{nn}(0, l) = F_{00}(0, l) \left| P_n(F_{00}(0, l)) \right|^2, \]  

(62)

where \( P_n(x) \) is Legandre polynomial.

The transition probabilities of the oscillator with a varying frequency for \( n \geq m \) are given by (56) as follows:

\[ F_{nm}(0, l) = F_{00}(0, l) \frac{m!}{n!} \left[ F_{(n-m)/2}(F_{00}(0, l)) \right]^2. \]  

(63)

In a generic case of transition probabilities, the integral (56) is expressed in terms of multivariable Hermite polynomials [5].
5. Entanglement and symplectic matrix

For coupled oscillators, the initially separable state can become entangled state. Let us consider the initial coherent state of a two-mode harmonic oscillator with the wave function

\[ \Psi_{\alpha_1, \alpha_2}(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{|\alpha_1|^2}{2} - \frac{|\alpha_2|^2}{2} - \frac{x_1^2}{2} - \frac{x_2^2}{2} + \sqrt{2}(\alpha_1 x_1 + \alpha_2 x_2) - \frac{\alpha_1^2}{2} - \frac{\alpha_2^2}{2} \right). \]

The tomogram of this state is the Gaussian joint probability distribution without correlations

\[ w_{\alpha_1, \alpha_2}(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{\sqrt{2\pi \sigma_1^2}} \exp \left( -\frac{(X_1 - \bar{X}_1)^2}{2\sigma_1^2} \right) \frac{1}{\sqrt{2\pi \sigma_2^2}} \exp \left( -\frac{(X_2 - \bar{X}_2)^2}{2\sigma_2^2} \right), \]

where the dispersions and covariance are

\[ \sigma_1^2 = \frac{\mu_1 + \nu_1}{2}, \quad \sigma_2^2 = \frac{\mu_2 + \nu_2}{2}, \quad \sigma_{X_1 X_2} = 0 \]

and the mean values read

\[ \bar{X}_1 = \mu_1 \sqrt{2} \text{Re} \alpha_1 + \nu_1 \sqrt{2} \text{Im} \alpha_1, \quad \bar{X}_2 = \mu_2 \sqrt{2} \text{Re} \alpha_2 + \nu_2 \sqrt{2} \text{Im} \alpha_2. \]

The coherent state of the coupled oscillators with the Hamiltonian (16) evolves by means of the Green function (28). For Gaussian states, this evolution provides the Gaussian tomogram, which is the joint probability distribution with time dependent parameters determined by symplectic matrix \( \Lambda(t) \). Let us use the matrix \( \lambda(t) = \Lambda^{-1}(t) \). In terms of this matrix, the initial mean values of positions \( \langle \hat{q}_1 \rangle, \langle \hat{q}_2 \rangle \) and momenta \( \langle \hat{p}_1 \rangle, \langle \hat{p}_2 \rangle \) become the positions and momenta given by the vector

\[ \begin{pmatrix} \langle \hat{p}_1(t) \rangle \\
\langle \hat{p}_2(t) \rangle \\
\langle \hat{q}_1(t) \rangle \\
\langle \hat{q}_2(t) \rangle \end{pmatrix} = \lambda(t) \begin{pmatrix} \langle \hat{p}_1 \rangle \\
\langle \hat{p}_2 \rangle \\
\langle \hat{q}_1 \rangle \\
\langle \hat{q}_2 \rangle \end{pmatrix}. \]

For the coherent state \( \Psi_{\alpha_1, \alpha_2}(x_1, x_2) \),

\[ \langle \hat{p}_1 \rangle = \sqrt{2} \text{Im} \alpha_1, \quad \langle \hat{p}_2 \rangle = \sqrt{2} \text{Im} \alpha_2, \quad \langle \hat{q}_1 \rangle = \sqrt{2} \text{Re} \alpha_1, \quad \langle \hat{q}_2 \rangle = \sqrt{2} \text{Re} \alpha_2. \]

Thus formula (67) yields means

\[ X_1(t) = \mu_1 \langle \hat{q}_1(t) \rangle + \nu_1 \langle \hat{p}_1(t) \rangle, \quad X_2(t) = \mu_2 \langle \hat{q}_2(t) \rangle + \nu_2 \langle \hat{p}_2(t) \rangle. \]

The variances and covariances at time \( t \) are

\[ \sigma_{X_1 X_1}(t) = \mu_1^2 \sigma_{q_1 q_1}(t) + \nu_1^2 \sigma_{p_1 p_1}(t) + 2\mu_1 \nu_1 \sigma_{q_1 p_1}(t), \]
\[ \sigma_{X_2 X_2}(t) = \mu_2^2 \sigma_{q_2 q_2}(t) + \nu_2^2 \sigma_{p_2 p_2}(t) + 2\mu_2 \nu_2 \sigma_{q_2 p_2}(t), \]
\[ \sigma_{X_1 X_2}(t) = \sigma_{x_1 x_2}(t) = \mu_1 \mu_2 \sigma_{q_1 q_2}(t) + \mu_1 \nu_2 \sigma_{q_1 p_2}(t) + \nu_1 \mu_2 \sigma_{p_1 q_2}(t) + \nu_1 \nu_2 \sigma_{p_1 p_2}(t). \]

The time dependent parameters in (71) are given in terms of the matrix \( \lambda(t) \) as follows:

\[ \sigma(t) = \lambda(t) \sigma(0) \lambda^\dagger(t), \]

where \( \sigma(0) = \sigma_{0} \).

\[ \sigma_{0} \]

is the initial covariance matrix.
where the ordering of column and rows is chosen as \( p_1, p_2, q_1, q_2 \) and the initial dispersion matrix is a diagonal four-dimensional matrix

\[
\sigma(0) = \frac{1}{2} 1_{4}.
\]  

(73)

Thus one has

\[
\sigma(t) = \frac{1}{2} \lambda(t) \lambda^\text{T}(t).
\]

(74)

As it is clear from the derivation, the analogous results (68) and (74) follow for an arbitrary dimensional system of coupled oscillators. Thus the product of symplectic matrix \( \lambda(t) \) and its transpose determines the separability and entanglement properties of the Gaussian state of the system of coupled oscillators with initial coherent state. The matrix (74) is nonnegative (its eigenvalues are nonnegative). The partial scaling criterion of separability states that, if in the matrix \( \sigma(t) \) all the elements in first row and the first column are replaced as

\[
\sigma_{11}(t) \rightarrow \lambda^2 \sigma_{11}(t), \quad \sigma_{12}(t) \rightarrow \lambda \sigma_{12}(t), \quad \sigma_{13}(t) \rightarrow \lambda \sigma_{13}(t), \quad \sigma_{14}(t) \rightarrow \lambda \sigma_{14}(t),
\]

(75)

the obtained symmetric matrix \( \sigma(\lambda, t) \) for the separable state must satisfy the inequality

\[
\det(\sigma(\lambda, t) + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \geq 0.
\]

(76)

If \( \lambda = -1 \), this criterion coincides with the positive partial transpose criterion. The described method was used to study entanglement phenomenon in Raman scattering process in [27].

6. Conclusions

We have shown that the system of coupled parametric oscillators has 2n linear in positions and moments integrals of motion. The integrals of motion are obtained by means of a symplectic matrix \( \Lambda(t) \) determined by the Hamiltonian of the system of oscillators. The evolution of the system given by its Green function or by the tomographic probability which determines the quantum states in the probability representation of quantum mechanics, depends on the matrix elements of the symplectic matrix. We have shown that the separability and entanglement of the quantum state of the system of the oscillators depends on the properties of the symplectic matrix, which yields the linear integrals of motion. One can obtain entangled state of nonstationary oscillator system from the initial separable state, choosing the interaction matrix \( B \) in the Hamiltonian to get the matrix \( \lambda(t) \lambda^\text{T}(t) \), violating the condition of separability. The transition probabilities between the oscillator energy levels were discussed within the framework of the probability representation of quantum mechanics.

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