On Universality of the S Combinator

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Abstract

In combinatory logic it is known that the set of two combinators \( S \) and \( K \) are universal; in the sense that any other combinator can be expressed in terms of these two. We show that the \( K \) combinator cannot be expressed only in terms of the \( S \) combinator. This will answer a question raised by Stephen Wolfram [15] as “Is the \( S \) combinator on its own computation universal?”

1 Introduction

Combinatory logic introduced by Schönfinkel [10] and developed by Curry [2]. Wolfram’s book, [15], provides an extensive historical background of its development. Here we consider combinatory logic as a rewiring (or substitution) system.

Here are the rewriting rules of some combinators, with the names given by Smullyan [13]:

\[
\begin{align*}
K_{xy} & \triangleright x & \quad & \text{(Kestrel)}, \\
S_{xyz} & \triangleright xz(yz) & & \text{(Starling)}, \\
B_{xyz} & \triangleright x(yz) & & \text{(Bluebird)}, \\
I_x & \triangleright x & & \text{(Identity)}, \\
J_{xyzw} & \triangleright xy(xwz) & & \text{(Jay)}, \\
L_{xy} & \triangleright x(yy) & & \text{(Lark)},
\end{align*}
\]

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We denote the reflexive, transitive closure of $\triangleright$ by $\triangleright^*$; i.e., $X \triangleright^* Y$ if and only if there is a sequence $X_1, \ldots, X_n$, $n \geq 1$, such that $X_1 = X$, $X_n = Y$, and $X_i \triangleright X_{i+1}$, for $1 \leq i \leq n - 1$.

**Definition 1.1 (Terms of Combinatory Logic)** The language of combinatory logic consists of an infinite set of variables $x_0, x_1, \ldots$ and two atomic constants $K$ and $S$, called basic combinators. The set of expressions called combinatory logic terms, or simply terms, is defined inductively as follows:

1. all variables and atomic constants are terms;
2. if $X$ and $Y$ are terms, then so is $(X \cdot Y)$.

A combinator is a term having no occurrence of any variable. ■

In the following, for simplicity, we use “$x$”, “$y$”, “$z$”, etc., to represent variables (distinct, unless otherwise stated). Also sometimes parentheses will be omitted following the convention of association to the left, so that $(((Sx)y)z)$ will be abbreviated to $Sxyz$, and $((SK)(KS))$ will be abbreviated to $SK(KS)$. Also, we write $(X \cdot Y)$ simply as $(XY)$ or $XY$.

According to the above definition, $K$ and $S$ are the only primitive combinator and the other combinators defined by (3)-(7) can be defined in terms of the two primitive ones; for example:

$$B := S(KS)K,$$
$$I := SKK,$$
$$L := ((S((S(KS))K))(K((S((SK)K))((SK)K))))) ,$$
$$M := S(SKK)(SKK).$$

**Definition 1.2** For a set $\{C_1, \ldots, C_k\}$ of combinators, $\mathcal{CL}(C_1, \ldots, C_k)$ is the set of the combinators built only from $C_1, \ldots, C_k$ by means of application. ■

Thus $\mathcal{CL}(K, S)$ is the set of all combinators. There has been studies of some subsets of $\mathcal{CL}(K, S)$. Giraudo [5] investigated $\mathcal{CL}(M)$ as an ordered set. Sprenger and Wymann-Böni [12] showed that $\mathcal{CL}(L)$ is decidable.
Probst and Studer [9] studied CL(J) to provide an elementary proof of the strong normalization property of J. Waldmann [14] studied CL(S) to show that this term rewriting system admits no ground loops. This extends the known result of the absence of cycles. Also, the paper provides a procedure that decides whether an S-term has a normal form. In [1], Barendregt et al. surveyed different problems regarding CL(S).

In this paper we investigate the universality of the combinator S. This is a natural question, as \{K, S\} is a universal basis for combinators; in the sense that every rewriting rule can be represented as a combinator in CL(K, S). This is a question that Wolfram [15] has raised as “Is the S combinator on its own computation universal?” We provide a negative answer to this question: every combinator \Sigma \in CL(K, S) that satisfies the rewriting rule \Sigma x \triangleright^* x does not belong to CL(S).

Our proof is based on a model of combinatory logic. We are not using the elegant Scott’s \(D_\infty\) model (see, e.g., [11, 7]), but a simpler set-theoretic model introduced by Engeler [3, 4, 6, 7], also mentioned by Plotkin [8]. We show that in this model every combinator in CL(S) corresponds with a set that is “closed under a substitution rule.” Then we show that interpretation of every combinator \Sigma that satisfies the rewriting rule \Sigma x \triangleright^* x does not has this property. This proves that K is not in CL(S), in the sense that there is no combinator \Sigma_0 \in CL(S) such that \Sigma_0 xy \triangleright^* x.

2 A Model for Combinatory Logic

Throughout this paper the notation “\(a \rightsquigarrow b\)” means the ordered pair “\(a, b\)”, following the suggestion of [3], “to make notation mnemonic.”

**Definition 2.1 (The set \(\mathcal{G}\))** We define the sets \(G_n\) recursively:

\[
G_0 = \{0, 1, 2, \ldots\};
\]

\[
G_{n+1} = G_n \cup \{(\alpha \rightsquigarrow b) : \alpha \subseteq G_n, \alpha \text{ is finite, and } b \in G_n\}.
\]

Then

\[
\mathcal{G} = \bigcup_{n \geq 0} G_n.
\]

The members of \(\mathcal{G}\) can be presented as trees. In tree representation of \(\alpha \rightsquigarrow b\), the left branch is labeled by the subset \(\alpha \subseteq \mathcal{G}\) and the right branch by the element \(b \in \mathcal{G}\) (see Figure [1]).
We adopt the following definition of a model for the combinatory logic originally introduced by Engeler [3], also Plotkin [8] proposed a similar definition.

**Definition 2.2 (The Model \( \mathbb{D} \), [3, 4, 6, 7])** The model \( \mathbb{D} \) is consists of the background set

\[
\mathcal{P} = \text{the set of all subsets of } \mathcal{G},
\]

and the binary operation \( \bullet \) on \( \mathcal{P} \):

\[
M \bullet N = \{ s : \text{there exists a finite } \alpha \subseteq N \text{ such that } (\alpha \mapsto s) \in M \}.
\]

The interpretations of the basic combinators are defined as follows:

\[
\begin{align*}
\llbracket K \rrbracket &= \{(\{t\} \mapsto (\emptyset \mapsto t)) : t \in \mathcal{G} \}, \quad (10) \\
\llbracket S \rrbracket &= \left\{ \left. \tau \mapsto \{r_1, \ldots, r_n \} \mapsto s \right| \mapsto \left( \sigma_1 \mapsto r_1, \ldots, \sigma_n \mapsto r_n \right) \mapsto (\sigma \mapsto s) \right\} : \\
&\quad n \geq 0, r_1, \ldots, r_n \in \mathcal{G}, \sigma = \tau \cup (\cup_i \sigma_i) \in \mathcal{P}, \sigma \text{ finite}, \quad (11) \\
\llbracket (X \cdot Y) \rrbracket &= \llbracket X \rrbracket \bullet \llbracket Y \rrbracket. \quad \blacksquare (12)
\end{align*}
\]

In the original definition [3] of the model \( \mathbb{D} \), the interpretation of \( \mathcal{K} \) was defined as

\[
\{(\alpha \mapsto (\beta \mapsto t)) : \alpha, \beta \subseteq \mathcal{G}, t \in \alpha, \alpha \text{ and } \beta \text{ are finite} \}.
\]
Figure 2: Tree representation of a generic member (11) of $[S]$; here $\tau, \sigma_i$ are finite subsets of $G$ and $s, r_i$ are members of $G$.

Here we use the simpler definition of [4].

Figure 1 (b) shows a tree representation of a member of $[K]$ and Figure 2 shows a tree representation of a generic member of $[S]$.

**Theorem 2.1** ([3, 6, 7]) For subsets $M$, $N$, and $L$ of $G$, we have

$$[K] \cdot M \cdot N = M,$$

(13)

$$[S] \cdot M \cdot N \cdot L = M \cdot L \cdot (N \cdot L).$$

(14)

**Example 1.** The combinator $I$, defined by rewriting rule (4). In $CL(K, S)$, the combinator $I$ is define as $I := SKK$, because

$$Ix = SKKx \triangleright Kx(Kx) \triangleright x.$$

In fact, $I$ also can be defined as $SKC$, where $C \in CL(K, S)$ is an arbitrary combinator. Then

$$[I] = [SKK] = [S] \cdot [K] \cdot [K]$$

$$= \{ s : \exists \alpha_1, \alpha_2 \subseteq [K] \text{ such that } (\alpha_1 \rightarrow (\alpha_2 \rightarrow s)) \in [S] \}$$

$$= \{ s : \exists t \in G, \exists \alpha_2 \subseteq [K] \text{ such that } (\{ t \} \rightarrow (\emptyset \rightarrow t)) \rightarrow (\alpha_2 \rightarrow s)) \in [S] \}.$$  

Comparing the last condition with (11), it follows that here $n = 0$, $\alpha_2 = \emptyset$, and $s = (\{ t \} \rightarrow t)$. Therefore,

$$[I] = \{ (\{ t \} \rightarrow t) : t \in G \}.$$  

(15)
Note that if we used the definition $I = SKC$, for some other combinator $C$, then we would get the same result.

**Example 2.** Using (10) and (15), the interpretation of the combinator $KI$ is

$$[KI] = [K] \cdot [I]$$

$$= \{ X : \exists \alpha \subseteq [I] \text{ such that } (\alpha \rightarrow X) \in [K] \}$$

$$= \{ (\emptyset \mapsto (\{t\} \mapsto t)) : t \in \mathcal{G} \}.$$

Let

$$K^{**} := K(KI).$$

Then

$$K^{**}xyz = K(KI)xyz \triangleright KIyz \triangleright lz \triangleright z.$$

The interpretation of the combinator $K^{**}$ is

$$[K^{**}] = [K(KI)]$$

$$= [K] \cdot [KI]$$

$$= \{ X : \exists \alpha \subseteq [KI] \text{ such that } (\alpha \rightarrow X) \in [K] \}$$

$$= \{ (\emptyset \mapsto (\emptyset \mapsto (\{t\} \mapsto t)) : t \in \mathcal{G} \}. \quad \blacksquare$$

### 3 Substitution

#### 3.1 Templates for the generic members

The equations (10), (11), and (15) define templates for the generic member of $[K]$, $[S]$, and $[I]$, respectively. Each template consists of variables, like $\tau$ and $s$ in (11), which represent an arbitrarily finite subset or a member of $\mathcal{G}$.

The same is true for any combinator $\Sigma \in \mathcal{CL}(K, S)$, in the sense that there is a templates for the generic member of $[\Sigma]$, consists of variables denoting either arbitrarily finite subsets or members of $\mathcal{G}$. To obtain this template, suppose that $\Sigma = \Sigma_1 \cdot \Sigma_2$, where $\Sigma_1, \Sigma_2 \in \mathcal{CL}(K, S)$. There are the templates $\mathcal{T}_1 = Y \rightarrow X$ and $\mathcal{T}_2$ for $\Sigma_1$ and $\Sigma_2$, respectively. Consider $\mathcal{T}_2 \rightarrow X$ and modify $X$ to $X'$ such that $\mathcal{T}_2 \mapsto X'$ follows the template $\mathcal{T}_1$. Then $X'$ is the template for the generic member of $[\Sigma]$. 

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Example 1. Consider the combinator $\Sigma_1 = (S \cdot K)$. From (11), the template for the generic member of $[S]$ is
\[ T_1 = \left\{ \tau \mapsto (\{r_1, \ldots, r_n\} \mapsto s) \right\} \mapsto \left\{ \sigma_1 \mapsto r_1, \ldots, \sigma_n \mapsto r_n \right\} \mapsto \left( (\tau \cup \cup_i \sigma_i) \mapsto s \right), \] (16)
for $n \geq 0$ and $\sigma = \tau \cup (\cup_i \sigma_i)$. From (10), the template of $[K]$ is
\[ T_2 = (\{t\} \mapsto (\emptyset \mapsto t)). \]

Now $T_2 \mapsto T'$ follows the template $T_1$ if and only if
\[ (\{t\} \mapsto (\emptyset \mapsto t)) = (\tau \mapsto (\{r_1, \ldots, r_n\} \mapsto s)), \]
\[ T' = (\{\sigma_1 \mapsto r_1, \ldots, \sigma_n \mapsto r_n\} \mapsto ((\tau \cup \cup_i \sigma_i) \mapsto s)). \]

Thus, $\tau = \{t\}$, $n = 0$, and $s = t$. Therefore, the template for the generic member of $[S \cdot K]$ is
\[ T' = (\emptyset \mapsto (\{t\} \mapsto t)). \]

In another words,
\[ [S \cdot K] = \{(\emptyset \mapsto (\{t\} \mapsto t)) : t \in \mathcal{G}\}. \]

Example 2. Consider the combinator $\Sigma_2 = (S \cdot S)$, and the template $T_1$ of equation (16) for the generic member of $[S]$. Now, $T_1 \mapsto T''$ follows the template of $[S]$ of the following form
\[ \left\{ \tau' \mapsto (\{r'_1, \ldots, r'_n\} \mapsto s') \right\} \mapsto \left\{ \sigma'_1 \mapsto r'_1, \ldots, \sigma'_n \mapsto r'_n \right\} \mapsto ((\tau' \cup \cup_i \sigma'_i) \mapsto s') \]
if and only if
\[ T_1 = (\tau' \mapsto (\{r'_1, \ldots, r'_n\} \mapsto s')), \]
\[ T'' = (\{\sigma'_1 \mapsto r'_1, \ldots, \sigma'_n \mapsto r'_n\} \mapsto ((\tau' \cup \cup_i \sigma'_i) \mapsto s')). \]

Thus,
\[ \tau' = (\tau \mapsto (\{r_1, \ldots, r_n\} \mapsto s)), \]
\[ r'_i = (\sigma_i \mapsto r_i), \quad 1 \leq i \leq n, \]
\[ s' = ((\tau \cup \cup_i \sigma_i) \mapsto s). \]
Therefore, $\mathcal{T}''$, the template for the generic members of $\llbracket S \cdot S \rrbracket$, has the following form

$$
\mathcal{T}'' = \left\{ \sigma'_1 \mapsto (\sigma_1 \mapsto r_1), \ldots, \sigma'_n \mapsto (\sigma_n \mapsto r_n) \right\} \mapsto \left( (\tau' \cup \cup_i \sigma'_i) \mapsto ((\tau \cup \cup_i \sigma_i) \mapsto s) \right).
$$

Thus $\mathcal{T}''$ is the template for the generic members of $\llbracket S \cdot S \rrbracket$; i.e.,

$$
\llbracket S \cdot S \rrbracket = \left\{ \left\{ \sigma'_1 \mapsto (\sigma_1 \mapsto r_1), \ldots, \sigma'_n \mapsto (\sigma_n \mapsto r_n) \right\} \mapsto \left( (\tau' \cup \cup_i \sigma'_i) \mapsto ((\tau \cup \cup_i \sigma_i) \mapsto s) \right) : n \geq 0, \tau' = (\tau \mapsto \{r_1, \ldots, r_n\} \mapsto s) \right\}, s \in \mathcal{G}, r_i \in \mathcal{G}, \tau, \sigma_i, \text{ and } \sigma'_i \text{ finite subsets of } \mathcal{G}.
$$

**Theorem 3.1** If $\mathcal{T}$ is the template for the generic member of $\llbracket \Sigma \rrbracket$, where $\Sigma \in \text{CL}(S)$, then $\mathcal{T}$ does not contain $\{t\}$, as a variable denoting a subset of $\mathcal{G}$.

**Proof.** Note that variables of (16), the template for the generic member of $\llbracket S \rrbracket$, denote either members of $\mathcal{G}$ or finite subsets of it; i.e., no variable of the form $\{t\}$ as a variable representing a subset. We prove the theorem by induction on the number of occurrences of $S$ in $\Sigma$. Then the base case, where $\Sigma = S$, is obvious. For the induction step, suppose that $\Sigma = \Sigma_1 \cdot \Sigma_2$, where $\Sigma_1, \Sigma_2 \in \text{CL}(S)$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be templates for the generic members of $\llbracket \Sigma_1 \rrbracket$ and $\llbracket \Sigma_2 \rrbracket$, respectively. By induction hypothesis, $\mathcal{T}_1$ and $\mathcal{T}_2$ do not contain any $\{t\}$, as a variable denoting a subset of $\mathcal{G}$. The template $\mathcal{T}$ for the generic member of $\llbracket \Sigma \rrbracket$ is obtained from $\mathcal{T}_1 \rightarrow \mathcal{T}$ by forcing it to follow the template $\mathcal{T}_1$. This process does not introduce any subset variable of the form $\{t\}$. ■

### 3.2 Companion

First, some useful definitions and notations.

**Definition 3.1** $(B_0 \text{ and } B_\mu)$ Let

$$
B_0 = (\{0\} \mapsto 0),
$$

$$
B_\mu = (\{0\} \mapsto \mu).
$$

The value of the integer $\mu \geq 1$ will be determined later. ■

**Definition 3.2** $(B_0\text{-Base})$ A member $X \in \mathcal{G}$ has $B_0$-base if and only if

$$
X = (\alpha_1 \mapsto (\cdots (\alpha_n \mapsto B_0) \cdots)),
$$

where $n \geq 0$ and $\alpha_i \subseteq \mathcal{G}$ is finite. In the special case of $n = 0$, $B_0$ has $B_0$-base. ■
**Definition 3.3** $(B_\mu\text{-Substitution})$ Suppose $X \in \mathcal{G}$ has $B_0$-base and is of the general form (17). The $B_\mu$-substitution of $X$, denoted as $\text{sub}_\mu(X)$, is

$$\text{sub}_\mu(X) = \left( \alpha_1 \mapsto ( \cdots ( \alpha_n \mapsto B_\mu) \cdots ) \right).$$

**Definition 3.4** $(B_\mu\text{-Companion})$ Suppose that $X \in \mathcal{G}$, of the form (17), for $\Sigma \in \text{CL}(S)$, has $B_0$-base. Let $T$ be the template for the generic member of $\mathcal{G}$. Therefore, $X$ is obtained from $T$ by substituting variables of $T$ by members or finite subsets of $\mathcal{G}$. Then there are two possible cases. (i) There is variable $s$ which is substituted by a $B_0$-base $Y \in \mathcal{G}$ to obtain $X$. (ii) There are variables $t$ and $\sigma$ which are substituted by $0$ and $\{0\}$, respectively, to obtain $X$. Then the $B_\mu$-companion of $B_0$, denoted by $\text{comp}_\mu(X)$, is obtained as follows: in case (i) by replacing the variable $s$ by $\text{sub}_\mu(Y)$; in case (ii) by replacing the variable $t$ by $\mu$. Here we assume that the integer $\mu$ is bigger than any number appearing in $X$.

**Theorem 3.2** If $\Sigma \in \text{CL}(S)$, $X \in \mathcal{G}$, $X$ has $B_0$-base, and $\mu$ is bigger than any number appearing in $X$, then $\text{comp}_\mu(X) \in \mathcal{G}$.

**Proof.** If $\text{comp}_\mu(X)$ is obtained using rule (i) of Definition 3.4 then obviously $\text{comp}_\mu(X) \in \mathcal{G}$. If the rule (ii) is used, then the theorem follows from Theorem 3.1.

### 4 Combinators generated by $S$

There are combinators in $\text{CL}(K, S)$ which define the same rewriting rule as $I := SKK$. For example, $SK(SKSK) \triangleright^* SKK$. Also, for the combinator

$$\Sigma_0 = S(S(S(SK)(S(KK)S(KK)I))(KI)))(KI)K,$$

we have $\Sigma_0 x \triangleright^* x$, for all $x$; while it is not the case that $\Sigma_0 \triangleright^* SKK$. The following theorem shows that interpretation of such combinators in $\mathcal{D}$ is a super set of $[I]$.

**Theorem 4.1** Let $\Sigma \in \text{CL}(K, S)$ such that $\Sigma x \triangleright^* x$, for all $x$. Then $[I] \subseteq \mathcal{G}$. 

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**Proof.** From (15), it is enough to show that \( \{ t \rightarrow t \} \in [\Sigma] \), for every \( t \in \mathcal{S} \). From the relations (1), (2), and (12)-(14) it follows that for every \( M \subseteq \mathcal{S} \),

\[
[\Sigma] \cdot M = M.
\]

Let \( t \in \mathcal{S} \). Then

\[
\{ t \} = [\Sigma] \cdot \{ t \} = \{ s : \exists \alpha \subseteq \{ t \} \text{ such that } (\alpha \rightarrow s) \in [\Sigma] \}.
\]

If \( \alpha = \emptyset \), then \((\emptyset \rightarrow t) \in [\Sigma]\), which implies \( t \in \emptyset = [\Sigma] \cdot \emptyset \). Therefore, \( \alpha = \{ t \} \); and \((\{ t \} \rightarrow t) \in [\Sigma]\).

\[\blacksquare\]

**Theorem 4.2** Let \( \Sigma \in \text{CL}(\mathcal{S}) \). Then it is not the case that \( \Sigma x \triangleright^* x \), for all \( x \).

**Proof.** Suppose, by contradiction, that \( \Sigma x \triangleright^* x \), for all \( x \). From (15) and Theorem 4.1, it follows that \( B_0 = (\{ 0 \} \rightarrow 0) \in [\mathbb{I}] \subseteq [\Sigma] \). Note that from proof of Theorem 4.1, \( [\Sigma] \cdot M = M \), for every \( M \subseteq \mathcal{S} \). Then from the Theorem 3.2 it follows that \( \text{comp}_\mu(B_0) = (\{ 0 \} \rightarrow \mu) \in [\Sigma] \); which implies \( \mu \in \{ 0 \} = [\Sigma] \cdot \{ 0 \} \). This contradicts the assumption \( \mu > 0 \). \[\blacksquare\]

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