Gradient estimates and Liouville type theorems for a nonlinear elliptic equation

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Abstract. Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold. We consider gradient estimates and Liouville type theorems for positive solutions to the following nonlinear elliptic equation:

$$\Delta u + au \log u = 0,$$

where $a$ is a nonzero constant. In particular, for $a < 0$, we prove that any bounded positive solution of the above equation with a suitable condition for $a$ with respect to the lower bound of Ricci curvature must be $u \equiv 1$. This generalizes a classical result of Yau.

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1. Introduction. In this paper, we study positive solutions of the equation

$$\Delta u + au \log u = 0$$

(1.1)
on an $n$-dimensional complete Riemannian manifold $(M^n, g)$, where $a$ is a nonzero constant. The first study of this and related nonlinear equations can be traced back to Li [6], and later by Ma [7] and Yang [11], who derived various gradient estimates and Harnack estimates and noted the relation to gradient Ricci solitons, which are self-similar solutions to Ricci flow and arise in the blow-up procedure of the long time existence or convergence of the flow. Moreover, the Eq. (1.1) is closely related to the famous Gross Logarithmic...
Sobolev inequality, see [4]. For the recent research of (1.1), one can refer to [2,3,5,8,10] and the references therein.

It is well-known that for complete noncompact Riemannian manifolds with \( \text{Ric} \geq 0 \), Yau [12] showed that every positive or bounded solution to the equation

\[
\Delta u = 0
\]

is constant. We note that the Eq. (1.2) can be seen as a special case of (1.1) when \( a = 0 \). Therefore, a natural idea is to achieve similar Liouville type theorems for positive solutions to the nonlinear elliptic equation (1.1).

Inspired by the method used by Brighton in [1], this paper is concerned with gradient estimates and Liouville type theorems for positive solutions to the nonlinear elliptic equation (1.1) with \( a \neq 0 \) and obtain the following results:

**Theorem 1.1.** Let \((M^n, g)\) be an \( n \)-dimensional complete Riemannian manifold with \( \text{Ric}(B_p(2R)) \geq -K \), where \( K \geq 0 \) is a constant. Suppose that \( u \) is a positive solution to (1.1) on \( B_p(2R) \). Then on \( B_p(R) \), the following inequalities hold:

1. If \( a > 0 \), then
   \[
   |\nabla u| \leq M \sqrt{\frac{(n+3)^2}{2n}} (a + K) + \frac{C}{R^2} (1 + \sqrt{KR} \coth(\sqrt{KR}));
   \]
2. If \( a < 0 \), then
   \[
   |\nabla u| \leq M \sqrt{\frac{(5n+6)^2}{36n}} \max\{0, a + K\} + \frac{C}{R^2} (1 + \sqrt{KR} \coth(\sqrt{KR})),
   \]

where \( M = \sup_{x \in B_p(2R)} u(x) \) and the constant \( C \) depends only on \( n \).

Letting \( R \to \infty \), we obtain the following gradient estimates on complete noncompact Riemannian manifolds:

**Corollary 1.2.** Let \((M^n, g)\) be an \( n \)-dimensional complete noncompact Riemannian manifold with \( \text{Ric} \geq -K \), where \( K \geq 0 \) is a constant. Suppose that \( u \) is a positive solution to (1.1). Then the following inequalities hold:

1. If \( a > 0 \), then
   \[
   |\nabla u| \leq \frac{(n+3)M}{\sqrt{2n}} \sqrt{a + K};
   \]
2. If \( a < 0 \), then
   \[
   |\nabla u| \leq \frac{(5n+6)M}{6\sqrt{n}} \sqrt{\max\{0, a + K\}},
   \]

where \( M = \sup_{x \in M^n} u(x) \).

In particular, for \( a < 0 \), if \( a \leq -K \), then (1.6) shows that \( \max\{0, a+K\} = 0 \) and \(|\nabla u| \leq 0\) whenever \( u \) is a bounded positive solution to (1.1). This implies that \( u \equiv 1 \) is a constant. Therefore, the following Liouville-type result follows:
Corollary 1.3. Let \((M^n, g)\) be an \(n\)-dimensional complete noncompact Riemannian manifold with \(\text{Ric} \geq -K\), where \(K \geq 0\) is a constant. Suppose that \(u\) is a bounded positive solution defined on \(M^n\) to (1.1) with \(a < 0\). If \(a \leq -K\), then \(u \equiv 1\) is a constant.

In particular, we have the following result:

Corollary 1.4. Let \((M^n, g)\) be an \(n\)-dimensional complete noncompact Riemannian manifold with \(\text{Ric} \geq 0\). Suppose that \(u\) is a bounded positive solution defined on \(M^n\) to (1.1) with \(a < 0\), then \(u \equiv 1\) is a constant.

Remark 1.1. Clearly, our Corollary 1.4 generalizes a uniqueness result of Yau with respect to the heat equation to the nonlinear elliptic equation (1.1) with \(a < 0\).

Remark 1.2. For an \(n\)-dimensional complete noncompact Riemannian manifold with \(\text{Ric} \geq 0\), it has been shown in [5, Corollary 1.1] that for positive solutions of (1.1) with \(a < 0\),

\[
\frac{|\nabla u|^2}{u^2} + a\alpha \log u \leq -\frac{n\alpha a^2}{8} \tag{1.7}
\]

for any \(\alpha > 1\). By virtue of (1.7), one has

\[
a\alpha \log u \leq -\frac{n\alpha a^2}{8}. \tag{1.8}
\]

Since \(a < 0\), by letting \(\alpha \to 1\), we obtain from (1.8)

\[
u(x) \geq e^{-\frac{n}{8}} \tag{1.9}
\]

for all \(x \in M^n\). Hence, the estimate (1.9) is sharper than \(u(x) \geq e^{-\frac{n}{8}}\) of Yang in [11, Corollary 1.2].

On the other hand, from (1.7), we can not obtain the uniqueness theorem for any bounded positive solution of (1.1). Therefore, our Corollary 1.4 generalizes (2) in [5, Corollary 1.1] in this sense.

2. Proof of results. The proof of the main results will follow from applying the Bochner formula to an appropriate function \(h\) of a given positive solution \(u\).

Let \(h = u^\epsilon\), where \(\epsilon \neq 0\) is a constant to be determined. Then we have

\[
\log h = \epsilon \log u. \tag{2.1}
\]

A straightforward computation gives

\[
\Delta h = \Delta (u^\epsilon) = \epsilon (\epsilon - 1) u^{\epsilon - 2} |\nabla u|^2 + \epsilon u^{\epsilon - 1} \Delta u
\]

\[
= \epsilon (\epsilon - 1) u^{\epsilon - 2} |\nabla u|^2 - a\epsilon u^\epsilon \log u
\]

\[
= \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ah \log h. \tag{2.2}
\]
Hence, we have

\[ \nabla h \nabla \Delta h = \nabla h \left( \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ah \log h \right) \]

\[ = \frac{\epsilon - 1}{\epsilon} \nabla h \nabla \left( \frac{|\nabla h|^2}{h} \right) - a \nabla h \nabla (h \log h) \]

\[ = \frac{\epsilon - 1}{\epsilon h} \nabla h \nabla (|\nabla h|^2) - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} - ah \log h \frac{|\nabla h|^2}{h} - a|\nabla h|^2. \]

(2.3)

Applying (2.2) and (2.3) into the well-known Bochner formula to \( h \), we have

\[ \frac{1}{2} \Delta |\nabla h|^2 = |\nabla^2 h|^2 + \nabla h \nabla \Delta h + \text{Ric}(\nabla h, \nabla h) \]

\[ \geq \frac{1}{n} (\Delta h)^2 + \nabla h \nabla \Delta h - K|\nabla h|^2 \]

\[ = \frac{1}{n} \left( \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^2}{h} - ah \log h \right)^2 + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) \]

\[ - \frac{\epsilon - 1}{\epsilon} \frac{|\nabla h|^4}{h^2} - ah \log h \frac{|\nabla h|^2}{h} - (a + K)|\nabla h|^2 \]

\[ = \left( \frac{(\epsilon - 1)^2}{ne^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{|\nabla h|^4}{h^2} - a \left( \frac{2(\epsilon - 1)}{ne} + 1 \right) h \log h \frac{|\nabla h|^2}{h} \]

\[ + \frac{a^2}{n} (h \log h)^2 + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) - (a + K)|\nabla h|^2. \]

(2.4)

Now we let

\[ a \left( \frac{2(\epsilon - 1)}{ne} + 1 \right) \geq 0. \]

(2.5)

Then for a fixed point \( p \), if there exist a positive constant \( \delta \) such that \( h \log h \leq \delta \frac{|\nabla h|^2}{h} \), (2.4) becomes

\[ \frac{1}{2} \Delta |\nabla h|^2 \geq \left[ \frac{(\epsilon - 1)^2}{ne^2} - \frac{\epsilon - 1}{\epsilon} - a\delta \left( \frac{2(\epsilon - 1)}{ne} + 1 \right) \right] \frac{|\nabla h|^4}{h^2} \]

\[ + \frac{a^2}{n} (h \log h)^2 + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) - (a + K)|\nabla h|^2 \]

\[ \geq \left[ \frac{(\epsilon - 1)^2}{ne^2} - \frac{\epsilon - 1}{\epsilon} - a\delta \left( \frac{2(\epsilon - 1)}{ne} + 1 \right) \right] \frac{|\nabla h|^4}{h^2} \]

\[ + \frac{\epsilon - 1}{\epsilon} \frac{\nabla h}{h} \nabla (|\nabla h|^2) - (a + K)|\nabla h|^2. \]

(2.6)
On the contrary, at the point $p$, if $h \log h \geq \delta \frac{\|\nabla h\|^2}{h}$, then (2.4) becomes

$$
\frac{1}{2} \Delta |\nabla h|^2 \geq \left( \frac{(\epsilon - 1)^2}{n \epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) \frac{\|\nabla h\|^4}{h^2} + \left[ \frac{a^2}{n} - \frac{a}{\delta} \left( \frac{2(\epsilon - 1)}{n \epsilon} + 1 \right) \right] (h \log h)^2
$$

$$
+ \frac{\epsilon - 1}{\epsilon} \frac{\nabla(|\nabla h|^2)}{h} - (a + K)|\nabla h|^2
$$

$$
\geq \left\{ \left( \frac{(\epsilon - 1)^2}{n \epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) + \delta^2 \left[ \frac{a^2}{n} - \frac{a}{\delta} \left( \frac{2(\epsilon - 1)}{n \epsilon} + 1 \right) \right] \right\} \frac{\|\nabla h\|^4}{h^2}
$$

$$
+ \frac{\epsilon - 1}{\epsilon} \frac{\nabla(|\nabla h|^2)}{h} - (a + K)|\nabla h|^2
$$

$$
\geq \left\{ \left( \frac{(\epsilon - 1)^2}{n \epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) - a \delta \left( \frac{2(\epsilon - 1)}{n \epsilon} + 1 \right) \right\} \frac{\|\nabla h\|^4}{h^2}
$$

$$
+ \frac{\epsilon - 1}{\epsilon} \frac{\nabla(|\nabla h|^2)}{h} - (a + K)|\nabla h|^2
$$

(2.7)

as long as

$$
\frac{a^2}{n} - \frac{a}{\delta} \left( \frac{2(\epsilon - 1)}{n \epsilon} + 1 \right) > 0.
$$

(2.8)

In order to obtain the bound of $|\nabla h|$ by using the maximum principle for (2.7), it is sufficient to choose the coefficient of $\frac{\|\nabla h\|^4}{h^2}$ in (2.7) is positive, that is,

$$
\left( \frac{(\epsilon - 1)^2}{n \epsilon^2} - \frac{\epsilon - 1}{\epsilon} \right) - a \delta \left( \frac{2(\epsilon - 1)}{n \epsilon} + 1 \right) > 0.
$$

(2.9)

We next consider two cases:

**Case one:** $a > 0$.

In this case, provided $\epsilon \in \left( \frac{2}{n+2}, \frac{6}{5-\sqrt{13}}n+6 \right)$, there will exist an $\delta$ satisfying (2.5), (2.8), and (2.9). In particular, we choose

$$
\epsilon = \frac{3}{n + 3}
$$

(2.10)

and

$$
\delta = \frac{n}{2a},
$$

(2.11)

then (2.7) becomes

$$
\frac{1}{2} \Delta |\nabla h|^2 \geq \frac{5n}{18} \frac{\|\nabla h\|^4}{h^2} - \frac{n}{3} \frac{\nabla h}{h} \nabla(|\nabla h|^2) - (a + K)|\nabla h|^2.
$$

(2.12)

**Case two:** $a < 0$.

In this case, provided $\epsilon \in \left( \frac{6}{5+\sqrt{13}}n+6, \frac{2}{n+2} \right)$, there will exist an $\delta$ satisfying (2.5), (2.8), and (2.9). In particular, we choose

$$
\epsilon = \frac{6}{5n + 6}
$$

(2.13)

and

$$
\delta = \frac{3n}{4a}.
$$

(2.14)
In this case, (2.7) becomes
\[ \frac{1}{2} \Delta |\nabla h|^2 \geq \frac{37n}{36} \frac{|\nabla h|^4}{h^2} - \frac{5n}{6} \frac{\nabla h}{h} \nabla(|\nabla h|^2) - (a + K)|\nabla h|^2. \] (2.15)

**Proof of Theorem 1.1.** First, we prove the case of \( a > 0 \).

Denote by \( B_p(R) \) the geodesic ball centered at \( p \) with radius \( R \). Take a cut-off function \( \phi \) (see [9]) satisfying \( \text{supp}(\phi) \subset B_p(2R) \), \( \phi|_{B_p(R)} = 1 \), and
\[ \frac{|\nabla \phi|^2}{\phi} \leq \frac{C}{R^2}, \] (2.16)
\[ -\Delta \phi \leq \frac{C}{R^2}(1 + \sqrt{KR} \coth(\sqrt{KR})), \] (2.17)
where \( C \) is a constant depending only on \( n \). Define \( G = \phi |\nabla h|^2 \). Next we will apply the maximum principle to \( G \) on \( B_p(2R) \). Assume \( G \) achieves its maximum at the point \( x_0 \in B_p(2R) \) and assume \( G(x_0) > 0 \) (otherwise the proof is trivial). Then at the point \( x_0 \), it holds that
\[ \Delta G \leq 0, \quad \nabla(|\nabla h|^2) = -\frac{|\nabla h|^2}{\phi} \nabla \phi \]
and
\begin{align*}
0 &\geq \Delta G \\
&= \phi \Delta(|\nabla h|^2) + |\nabla h|^2 \Delta \phi + 2 \nabla \phi \nabla|\nabla h|^2 \\
&= \phi \Delta(|\nabla h|^2) + \frac{\Delta \phi}{\phi} G - 2 \frac{|\nabla \phi|^2}{\phi^2} G \\
&\geq 2\phi \left[ \frac{5n}{18} \frac{|\nabla h|^4}{h^2} - \frac{n}{3} h \nabla(|\nabla h|^2) - (a + K)|\nabla h|^2 \right] \\
&\quad + \frac{\Delta \phi}{\phi} G - 2 \frac{|\nabla \phi|^2}{\phi^2} G \\
&= \frac{5n}{9} \frac{G^2}{\phi h^2} + \frac{2nG}{3\phi} \nabla \phi \frac{\nabla h}{h} - 2(a + K)G \\
&\quad + \frac{\Delta \phi}{\phi} G - 2 \frac{|\nabla \phi|^2}{\phi^2} G, \quad (2.18)
\end{align*}
where the second inequality used (2.12). Multiplying both sides of (2.18) by \( \frac{\phi}{G} \) yields
\[ \frac{5n}{9} \frac{G}{h^2} \leq -\frac{2n}{3} \nabla \phi \frac{\nabla h}{h} + 2\phi(a + K) - \Delta \phi + 2 \frac{|\nabla \phi|^2}{\phi}. \] (2.19)

Applying the Cauchy inequality
\[ -\frac{2n}{3} \nabla \phi \frac{\nabla h}{h} \leq \frac{2n}{3} \frac{|\nabla \phi|}{\phi} \frac{|\nabla h|}{h} \]
\[ \leq \frac{n}{3\varepsilon} \frac{|\nabla \phi|^2}{\phi} + \frac{n\varepsilon}{3h^2} \phi |\nabla h|^2 \]
\[ = \frac{n}{3\varepsilon} \frac{|\nabla \phi|^2}{\phi} + \frac{n\varepsilon}{3h^2} G, \]
where \( \varepsilon \in (0, \frac{5}{3}) \) is a positive constant, into (2.19) gives

\[
\frac{(5 - 3\varepsilon)n}{9} G \frac{G}{h^2} \leq 2\phi(a + K) - \Delta \phi + \left(2 + \frac{n}{3\varepsilon}\right) \frac{|\nabla \phi|^2}{\phi} \\
\leq 2(a + K) - \Delta \phi + \left(2 + \frac{n}{3\varepsilon}\right) \frac{|\nabla \phi|^2}{\phi}.
\]  
(2.20)

In particular, choosing \( \varepsilon = \frac{1}{3} \) in (2.20) and applying (2.16) and (2.17), we obtain

\[
\frac{4nG}{9h^2} \leq 2(a + K) - \Delta \phi + (n + 2) \frac{|\nabla \phi|^2}{\phi} \\
\leq 2(a + K) + \frac{C}{R^2} (1 + \sqrt{K} \coth(\sqrt{K}R)).
\]  
(2.21)

It follows that for \( x \in B_p(R) \),

\[
\frac{4n}{9} G(x) \leq \frac{4n}{9} G(x_0) \\
\leq h^2(x_0) \left[ \frac{(n + 3)^2}{2n} (a + K) + \frac{C}{R^2} (1 + \sqrt{K}R \coth(\sqrt{K}R)) \right] .
\]  
(2.22)

This shows

\[
|\nabla u|^2(x) \leq M^2 \left[ \frac{(n + 3)^2}{2n} (a + K) + \frac{C}{R^2} (1 + \sqrt{K}R \coth(\sqrt{K}R)) \right] 
\]  
(2.23)

and

\[
|\nabla u| \leq M \sqrt{\frac{(n + 3)^2}{2n} (a + K) + \frac{C}{R^2} (1 + \sqrt{K}R \coth(\sqrt{K}R))},
\]  
(2.24)

where \( M = \sup_{x \in B_p(2R)} u(x) \). This yields the desired inequality (1.3) of Theorem 1.1.

Next, we prove the case of \( a < 0 \).

Define \( \tilde{G} = \phi |\nabla h|^2 \). We apply the maximum principle to \( \tilde{G} \) on \( B_p(2R) \). Assume \( \tilde{G} \) achieves its maximum at the point \( \tilde{x}_0 \in B_p(2R) \) and assume \( \tilde{G}(\tilde{x}_0) > 0 \) (otherwise the proof is trivial). Then at the point \( \tilde{x}_0 \), it holds that

\[
\Delta \tilde{G} \leq 0, \quad \nabla (|\nabla h|^2) = -\frac{|\nabla h|^2}{\phi} \nabla \phi
\]
and
\[
0 \geq \Delta \tilde{G} = \phi \Delta (|\nabla h|^2) + \frac{\Delta \phi}{\phi} \tilde{G} - 2 \frac{|
abla \phi|^2}{\phi^2} \tilde{G} \\
\geq 2\phi \left[ \frac{37n}{36} \frac{|\nabla h|^4}{h^2} - \frac{5n}{6} \nabla h \nabla (|\nabla h|^2) - (a + K)|\nabla h|^2 \right] \\
+ \frac{\Delta \phi}{\phi} \tilde{G} - 2 \frac{|
abla \phi|^2}{\phi^2} \tilde{G} \\
= \frac{37n}{18} \frac{\tilde{G}^2}{\phi h^2} + \frac{5n\tilde{G}}{3\phi} \nabla \phi \nabla h - 2(a + K)\tilde{G} \\
+ \frac{\Delta \phi}{\phi} \tilde{G} - 2 \frac{|
abla \phi|^2}{\phi^2} \tilde{G},
\]
(2.25)
where the second inequality used (2.15). Multiplying both sides of (2.25) by \( \frac{\phi}{\tilde{G}} \) yields
\[
\frac{37n}{18} \frac{\tilde{G}}{h^2} \leq - \frac{5n}{3} \frac{\nabla \phi}{h} \frac{\nabla h}{h} + 2\phi(a + K) - \Delta \phi + 2 \frac{|
abla \phi|^2}{\phi}.
\]
(2.26)
Inserting the Cauchy inequality
\[
- \frac{5n}{3} \frac{\nabla \phi}{h} \frac{\nabla h}{h} \leq \frac{5n}{3} \frac{|
abla \phi|}{h} \frac{|\nabla h|}{h} \\
\leq \frac{5n}{6\varepsilon} \frac{|
abla \phi|^2}{\phi} + \frac{5n\varepsilon}{6h^2} \tilde{G},
\]
where \( \varepsilon \in (0, \frac{37}{15}) \) is a positive constant, into (2.26) gives
\[
\frac{(37 - 15\varepsilon)n}{18} \frac{\tilde{G}}{h^2} \leq 2\phi(a + K) - \Delta \phi + \left( 2 + \frac{5n}{6\varepsilon} \right) \frac{|
abla \phi|^2}{\phi} \\
\leq 2 \max\{0, a + K\} - \Delta \phi + \left( 2 + \frac{5n}{6\varepsilon} \right) \frac{|
abla \phi|^2}{\phi}.
\]
(2.27)
Hence, choosing \( \varepsilon = \frac{1}{15} \) in (2.20) and applying (2.16) and (2.17), we have
\[
\frac{2n\tilde{G}}{h^2} \leq 2 \max\{0, a + K\} + \frac{C}{R^2} (1 + \sqrt{K} \coth(\sqrt{KR})).
\]
(2.28)
Finally, it holds on \( B_p(R) \)
\[
|\nabla u|^2 \leq M^2 \left[ \frac{(5n + 6)^2}{36n} \max\{0, a + K\} + \frac{C}{R^2} (1 + \sqrt{K} R \coth(\sqrt{KR})) \right].
\]
(2.29)
This concludes the proof of inequality (1.4) of Theorem 1.1.

\[\square\]

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