Exactness of the replica method in perturbation

Hisamitsu Mukaida
Department of Physics, Saitama Medical University, 38 Moro hongo, Moroyama-cho, Iruma-gun, Saitama, 350-0495, Japan

Yoshinori Sakamoto
Laboratory of Physics, College of Science and Technology, Nihon University, 7-24-1, Narashino-dai, Funabashi-city, Chiba, 274-8501, Japan

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The replica method for a quenched disordered system is considered in a perturbative field theory. Since correction in a finite-order perturbation is given in a polynomial of the replica number $n$, the zero-replica limit $n → 0$ is regarded as extracting the constant term from the polynomial, which mathematically makes sense. The meaning of the extraction is clarified comparing with a direct calculation.

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I. INTRODUCTION

The replica method is widely used for studying quenched disordered systems 1. It was developed for getting around difficulty involved in taking average $\mathbb{E} (\cdot)$ over quenched disorder. For example, in the case of a free energy, we compute $\mathbb{E} Z^n$ instead of $\mathbb{E} \log Z$, where $Z$ is the partition function of the system computed under fixed quenched disorder, and use the formula

$$\mathbb{E} \log Z = \lim_{n \to 0} \frac{1}{n} (\mathbb{E} Z^n - 1).$$

Here, $Z^n$ is realized by introducing $n$ “replicants” identical with the original system, so that $n$ is a positive integer, and thus the limit $n \to 0$ is ill-defined. Although there are specific models in which the replica method is proven to be mathematically rigorous 2, 3, 4, justification for the zero-replica limit is left beyond the scope of investigation in most literatures. Nevertheless, since this method gives reliable results to various quenched disordered systems, it has survived for more than three decades. In this paper, we wish to point out that the replica method can be justified within the framework of perturbation.

The main idea for the justification is that a physical quantity in the replicated system calculated in a finite-order perturbation is a (finite-degree) polynomial in $n$. The zero-replica limit can be regarded as the extraction of the constant term from the polynomial. A similar idea is used by Brunet and Derrida in Ref. 5.

Generally, if a function $f(n)$ defined on positive integers is a polynomial with the degree $m$, the constant term is obtained solving a linear equation for all the coefficients of $f(n)$ generated by $f(1), ..., f(m+1)$. We thus find that the constant term is extracted by

$$\sum_{n=1}^{m+1} (-1)^{n-1} \binom{m+1}{n} f(n) \equiv P_0 f(n).$$

(1)

It should be noted that $P_0$ in (1) becomes ill-defined when $m = \infty$, which implies that the constant term of a power series in $n$ becomes ambiguous. For instance, consider the series $\sum_{k=0}^{\infty} (-1)^k (n\pi)^{2k} / (2k + 1)!$, which apparently indicates that its constant term is one. On the other hand, the summation results in $\sin n\pi / (n\pi)$, which vanishes for all $n = 1, 2, ...$. Such ambiguity causes probability densities where the replica method does not work 6, 7.

In this paper, we show that the constant term in $n$ of a physical quantity calculated in the replicated system equals the disordered average of the corresponding physical quantity in the original system within finite-order perturbation. It is also found that the $n$-dependent terms in the replicated system originate from the disorder correlation between...
the corresponding physical quantity and the free energy in the original system, which are nothing to do with the disorder average we want. The limit \( n \to 0 \) can be interpreted as realization of extracting the constant term.

Our argument is also applicable to perturbative renormalization group, which includes functional renormalization group in perturbation [8, 9]. We will see that a beta function calculated in perturbation becomes a polynomial in \( n \) in the replicated system. The constant term in it is precisely equal to the beta function for the correlators characterizing probability density of quenched disorder.

Now we evaluate the right-hand side in finite-order perturbation. Letting

\[
\langle A[\phi] \rangle_{H[\phi;v]} = \frac{\int D\phi A[\phi] e^{-H[\phi;v]}}{\int D\phi e^{-H[\phi;v]}},
\]

where \( H[\phi;v] \) is the Hamiltonian (times \( 1/kT \)) of this model. We can put \( H[0;v] = 0 \) for an arbitrary \( v \) without loss of generality since the quotient \( \frac{H[0]}{H[0]} \) is independent of \( H[0;v] \).

The probability density for \( v \) is usually characterized by a given set of correlators among itself, which is denoted by \( u \). The average and the cumulants for the quenched disorder are respectively described as \( E \) and \( \kappa \). E.g., \( \kappa (v) = E v \), \( \kappa (v_1, v_2) = E (v_1 v_2) - E v_1 E v_2 \).

Our interest is to compute expectation values such as \( E \langle A[\phi] \rangle_{H[\phi;v]} \) by means of the replica method. Introducing \( n \) identical systems, we define the replica partition function

\[
Z \equiv E Z^n = \int D\phi e^{-\sum_{\alpha=1}^n H[\phi_\alpha;v]}. \tag{3}
\]

The replica Hamiltonian is defined as

\[
-\mathcal{H}[\phi;u] = \log E e^{-\sum_{\alpha=1}^n H[\phi_\alpha;v]}, \tag{4}
\]

where we have used the notation \( \phi \equiv (\phi_1, ..., \phi_n) \). We restrict ourselves to the case where the cumulant expansion for the right-hand side terminates at some finite number:

\[
-\mathcal{H}[\phi;u] = \sum_{l=1}^M \frac{(-1)^l}{l!} \kappa \left( \sum_{\alpha_1=1}^n H[\phi_{\alpha_1};v], ..., \sum_{\alpha_l=1}^n H[\phi_{\alpha_l};v] \right), \tag{5}
\]

which excludes introducing infinite number of replica indices. The physical quantity in the replicated system corresponding to \( E \langle A[\phi] \rangle_{H[\phi;v]} \) is \( \langle A[\phi_1] \rangle_{\mathcal{H}[\phi;u]} \), where the subscript 1 is the replica index. It can be written as

\[
\langle A[\phi_1] \rangle_{\mathcal{H}[\phi;u]} = \frac{E \left( \langle A[\phi] \rangle_{H[\phi;v]} Z^n \right)}{E Z^n}. \tag{6}
\]

Now we evaluate the right-hand side in finite-order perturbation. Letting \( W = \log Z \), the cumulant expansion has the form of

\[
\sum_{k=0}^\infty \frac{1}{k!} \kappa \left( \langle A[\phi] \rangle_{H[\phi;v]} , \underbrace{nW, ..., nW}_{k} \right). \tag{7}
\]
We write the unperturbed Hamiltonian as $H_0[\phi; u]$. Here, the expression $H_0[\phi; u]$ instead of $H_0[\phi; v]$ implies that the disorder average for $H_0$ is already taken. Perturbation is defined as $V \equiv H[\phi; v] - H_0[\phi; u]$. The entries in the cumulant $\mathcal{K}$ have the following perturbative series:

$$
\langle A[\phi]\rangle_{H[\phi; v]} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left\langle A[\phi]; V; \ldots; V \right\rangle_{H_0[\phi; u]}^{m},
$$

$$
W = \log Z_0 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \left\langle V; \ldots; V \right\rangle_{H_0[\phi; u]}^{m},
$$

which are obviously independent of $n$. Here the semicolons in the angle brackets mean to take the connected part, e.g., $\langle A; B \rangle = \langle A \rangle \langle B \rangle$, and the partition function for $H_0$ is denoted by $Z_0$. The summations in $\mathcal{K}$ and $\mathcal{P}_0$ are truncated at a finite number under a finite-order perturbation, so that $\langle A[\phi]\rangle_{H[\phi; u]}$ is expressed in a polynomial in $n$. Let $[Y]_m$ be the perturbative expansion of an arbitrary quantity $Y$ up to and including the $m$-th order in $V$. Since the constant term of $\mathcal{K}$ is given by the term with $k = 0$, we have

$$
\mathcal{P}_0 \left[ \langle A[\phi]\rangle_{H[\phi; u]} \right]_m = k \left( \left[ \langle A[\phi]\rangle_{H[\phi; v]} \right]_m \right)
\quad = \mathbb{E} \left[ \langle A[\phi]\rangle_{H[\phi; v]} \right]_m.
$$

It shows that the perturbative expansion for $\mathbb{E} \left[ \langle A[\phi]\rangle_{H[\phi; v]} \right]_m$ is given by the constant term in $n$ of the perturbative expansion for $\langle A[\phi]\rangle_{H[\phi; u]}$. We also find from $\mathcal{K}$ that the higher-order terms in $n$ give disorder correlation between $\langle A[\phi]\rangle_{H[\phi; v]}$ and the free energy $W$, which should be generally removed. We comment that substituting $W$ for $\langle A[\phi]\rangle_{H[\phi; v]}$ yields the moments of the free energy from the higher-order terms.

### III. Renormalization Group

The perturbation becomes more powerful combined with the renormalization group (RG), which consists of a coarse graining and a rescaling. The coarse graining means integrating over high-momentum components. Let the momentum space with a cutoff $\Lambda$ be $K \equiv \{ k : 0 \leq |k| \leq \Lambda \}$. Introducing $L > 1$, we divide $K$ into low- and high-momentum spaces defined as $K_\Lambda \equiv \{ k : 0 \leq |k| \leq L^{-1}\Lambda \}$ and $K_\Lambda' \equiv \{ k : L^{-1}\Lambda < |k| \leq \Lambda \}$ respectively. The Hamiltonian $H[\phi; v]$ is also decomposed into $H_\Lambda[\phi; v]$ and $H_\Lambda'[\phi; v]$, where $H_\Lambda[\phi; v]$ contains only the low-momentum component $\phi(p)$ ($p \in K_\Lambda$), and $H_\Lambda'[\phi; v]$ denotes the remainders. Integrating over $\phi(q)$ ($q \in K_\Lambda'$) in $Z$, we have the correction term $\delta H[\phi; v]$ of the Hamiltonian generated as

$$
\int \prod_{q \in K_\Lambda'} d\phi(q) e^{-H_\Lambda'[\phi; v]} = e^{-\delta H[\phi; v]}.
$$

The rescaling procedure is carried out introducing the renormalized field $\phi'(k) \equiv L^{-\theta} \phi(L^{-1} k)$ and the disorder $v'$. Here, a number $\theta$ and a renormalized disorder $v'$ is determined in such a way that a main part of the Hamiltonian remains the same form:

$$
- H_\Lambda'[\phi; v] - \delta H[\phi; v] = -H[\phi'; v'] - \delta v_0 + \cdots,
$$

where $\delta v_0$ means $\phi$-independent but $v$-dependent term, which should be added to the Hamiltonian because we put $H[0; v] = 0$. The dots in the right-hand side represents irrelevant terms other than $\delta v_0$, which can be neglected in low-energy physics. The same procedure leads to the RG transformation (RGT) for $\langle A[\phi]\rangle_{H[\phi; v]}$.

$$
\langle A[\phi]\rangle_{H[\phi; v]} = \langle A[L^\theta \phi']\rangle_{H[\phi'; v']},
$$

Taking the average over quenched disorder, we get

$$
\mathbb{E} \langle A[\phi]\rangle_{H[\phi; v]} = \mathbb{E} \langle A[L^\theta \phi']\rangle_{H[\phi'; v']},
$$
Next, we calculate the counterpart in the replicated system. We apply to the RGT to $Z$ in (3) before taking the random average:

$$Z = \mathbb{E} \int \prod_{\alpha=1}^{n} D\phi_{\alpha}^0 \ e^{-\sum_{\alpha=1}^{n} H[\phi_{\alpha}^0, u']} - n\delta v_0,$$

(14)

which indicates that the renormalized replica Hamiltonian satisfies

$$-H[\phi'; u] = \log \mathbb{E} \ e^{-\sum_{\alpha=1}^{n} H[\phi_{\alpha}^0, u'] - n\delta v_0}$$

(15)

up to the irrelevant terms. Thus, the RGT for the replicated system is written as

$$\langle A[\phi_1] \rangle_{H[\phi; u]} = \langle A[L^{\theta} \phi_{1}'] \rangle_{H[\phi'; u']}.$$

(16)

Since there are no free replica sums in the right-hand side of (15), $u'$ explicitly depends on $n$ through $n\delta v_0$. Furthermore, employing the cumulant expansion (5), the renormalized coupling constants are expressed as polynomials in $n$. Combining (9), (10) and (13), we get

$$\mathcal{P}_0 \left[ \langle A[L^{\theta} \phi_{1}'] \rangle_{H[\phi'; u']} \right]_m = \mathbb{E} \left[ \langle A[L^{\theta} \phi_{1}'] \rangle_{H[\phi'; u']} \right]_m.$$

(17)

It clearly shows that the right-hand side of (13) can be computed from the RGT in the replicated system with the constant-term extraction. Note that $\mathcal{P}_0$ removes $n$ dependence of $u'$. It implies that the beta function, which is defined by the linear response of $u'$ under the infinitesimal change $L \rightarrow L + \delta L$, is independent of $n$.

### IV. EXAMPLE

Now we exhibit a couple of examples. One of the simplest example is the random-field Gaussian model in $d$ dimensions given by the following Hamiltonian

$$H_1[\phi; v] = \int_k \phi(k) v_1(k) + \frac{1}{2} \int_{k_1, k_2} v_2(k_1, k_2) \phi(k_1) \phi(k_2),$$

(18)

where $v_1$ and $v_2$ are quenched disorder. For simplicity, we ignore fluctuation of $v_2$ and fix

$$v_2(k_1, k_2) = (k_1^2 + t) f(K),$$

(19)

where $f(K) \equiv (2\pi)^d \delta(K)$ with $K \equiv \sum_i k_i$, while $\phi_1$ obeys

$$\kappa (v_1(k_1), v_1(k_2)) = \Delta f(K).$$

(20)

The subscript $k$ for the integral in (18) means the measure $d^d k/(2\pi)^d$ on $K$. We regard the second term in (18) as the unperturbed Hamiltonian $H_0$ and the first term as perturbation $V$. Since cumulants of $v_1$ higher than (20) vanish, the higher order terms with $m \geq 3$ in the first line of (3) vanish when we take the disorder average. Hence we can readily derive the exact result

$$\mathbb{E} \langle \phi(k_1) \phi(k_2) \rangle_{H_1[\phi; v]} = (G_0(k_1) + \Delta G_0(k_1)^2) f(K),$$

(21)

where $G_0(k) = 1/(k^2 + t)$. The same quantity is computed by the replica method as shown below. Employing (5) and (20), the replica Hamiltonian to (18) is

$$\mathcal{H}_1[\phi; u] = \sum_{\alpha, \beta=1}^{n} \frac{1}{2} \int_k \phi_{\alpha} (k) \left( (k^2 + t) \delta_{\alpha \beta} - \Delta \right) \phi_{\beta}(-k).$$

(22)

Perturbative expansion with respect to $\Delta$ shows that

$$\left[ \langle \phi_1(k_1) \phi_1(k_2) \rangle_{\mathcal{H}_1[\phi; u]} \right]_{2m} = \left( G_0(k_1) + \sum_{j=1}^{m} n^{j-1} \Delta^j G_0^{j+1}(k_1) \right) f(K).$$

(23)
FIG. 1: Vertices appeared in the Hamiltonian (27), which are generated by $\kappa(v_4)$ and $\kappa(v_3, v_1)$ respectively. The open circle denotes $v_1$.

FIG. 2: The one-loop correction discussed in the main text. A low momentum component is shown in a broken line, while the solid line means $G_0(q)$. A cross on a solid line depicts $\Delta$ carrying $\Delta G_0(q)^2$, which is generated by $\kappa(v_1, v_1)$.

We see that the constant term in $n$ is identical with (21), as expected. On the other hand, if we take $m \to \infty$ in (23), we obtain the exact two-point function in the replicated system:

$$\langle \phi_1(k_1) \phi_1(k_2) \rangle_{H_1[\phi; u]} = G_0(k_1) \left( 1 + \frac{\Delta}{(k_1^2 + t - n\Delta)} \right) f(K).$$

(24)

It is argued in Ref.[10] that, when $\phi^4$ coupling constants are taken into account, the limiting procedure putting $t = n \Delta$ and then $n \to 0$ in (24) may generate a correction singular in $n$ to the coupling constants due to $n\Delta$ in the denominator. However, it is easily checked that the higher-order terms with $j \geq 1$ in (23) correspond to disorder correlation between $\langle \phi(k_1) \phi(k_2) \rangle_{H[\phi; v]}$ and the free energy $W$, which should be removed.

Next, in order to include $\phi^4$ interactions discussed above, consider the following Hamiltonian inspired by [10, 12, 18]:

$$H[\phi; u] = H_1[\phi; u] + \sum_{\alpha, \beta=1}^{n} \sigma_{\alpha\beta} \left( \frac{u_1}{4!} \delta_{\alpha\beta} + \frac{u_2}{3!} \right),$$

(27)

where $H_1$ is given in (22) and

$$\sigma_{\alpha\beta} = \int_{k_1, \ldots, k_4} f(K) \phi_{\alpha}(k_1) \phi_{\alpha}(k_2) \phi_{\alpha}(k_3) \phi_{\beta}(k_4).$$

(28)

Here we focus on the one-loop corrections to $u_1$ having one $\Delta$. We treat $H_1[\phi; u]$ as the unperturbed Hamiltonian, while the remaining terms in (27) the perturbation graphically represented as Fig. 1. As we discussed in the above example, the free propagator is (24) with $n\Delta$ removed. The $n$-dependent, one-loop diagrams having one $\Delta$ for $u'_1$ are presented in Fig. 2 which are calculated as [10]

$$\frac{\Delta}{4!} \left( 3u_1^2 + 6nu_1u_2 + 3n^2u_2^2 \right) \int_{q \in K>} G_0(q)^3.$$

(29)
The novel limiting procedure proposed in [10] is $n \to 0$ with $g_2 \equiv n u_2$ fixed, (29) yields the following beta function $\beta_1$ for $g_1 \equiv \Delta u_1$:

$$
\beta_1 = (6 - d)g_1 - (3g_1^2 + 6\Delta g_1 g_2 + 3\Delta^2 g_2^2).
$$

(30)

It means that $g_2$ can affect flow of $g_1$. In Ref. [10], a similar mechanism is argued in the context of failure of dimensional reduction.

Contrary to this procedure, our general argument indicates that the second and the third terms in (29) should be removed because they are generated by the disordered average in (15) containing $n\delta v_0$. In fact, the diagrams in Fig. 3 which corresponds to those in Fig. 2 before taking the random average, have $nw(q)G_0(q)v_1(-q)$ contained in perturbative expansion in $n\delta v_0$. Thus the beta function for $g_1$ according to our argument becomes

$$
\bar{\beta}_1 = (6 - d)g_1 - 3g_1^2.
$$

(31)

Here, $g_2$ does not affect flow of $g_1$ within the one-loop correction, which leads to dimensional reduction. Although we have demonstrated the simpler model, a tedious but straightforward computation in the original model [10] shows that all the terms depending on $n$ and remaining in the novel limiting procedure are caused by disorder correlations with $n\delta v_0$. These terms drastically change stability of fixed points and lead to breakdown of dimensional reduction [10]. However, in the case when the higher order terms in $n$ are dropped in the beta function, this scenario does not happen and dimensional reduction does occur [12, 19]. Although it is still debated whether dimensional reduction occurs or not near the upper critical dimensions [8, 10, 11, 12, 18, 19, 20, 21, 22, 23], the seeming ambiguity of the limiting procedure $n \to 0$ in perturbation cannot explain its breakdown.

V. SUMMARY AND DISCUSSION

We have shown that a physical quantity in a quenched disordered system calculated in a finite-order perturbation can be exactly derived by the replica method where the limit $n \to 0$ means to extract the constant term in $n$. The $n$-dependent terms give disorder correlations between the physical quantity we want to compute and the free energy, which should be removed. In this sense, the limiting procedure is uniquely determined as long as the physical quantity is represented in a polynomial in $n$. Our claim can provide a part of mathematical basis for various results by perturbative RG with the replica method where the limit is properly understood.

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