Bi-Colored Expansions of Geometric Theories

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Abstract

This paper concerns the study of Bi-colored expansions of geometric theories in the light of the Fraïssé-Hrushovski construction method. Substructures of models of a geometric theory $T$ are expanded by a color predicate $p$, and the dimension function associated with the pre-geometry of the $T$-algebraic closure operator together with a real number $0 < \alpha \leq 1$ is used to define a pre-dimension function $\delta_\alpha$. The pair $(\mathcal{K}_\alpha^+, \leq_\alpha)$ consisting of all such expansions with a hereditary positive pre-dimension along with the notion of substructure $\leq_\alpha$ associated to $\delta_\alpha$ is then used as a natural setting for the study of generic bi-colored expansions in the style of Fraïssé-Hrushovski construction. Imposing certain natural conditions on $T$, enables us to introduce a complete axiomatization $T_\alpha$ for the class of rich structures in this class. We will show that if $T$ is a dependent theory (NIP) then so is $T_\alpha$. We further prove that whenever $\alpha$ is rational the strong dependence transfers to $T_\alpha$. We conclude by showing that if $T$ defines a linear order and $\alpha$ is irrational then $T_\alpha$ is not strongly dependent.

Keywords: Geometric Theories; Fraïssé-Hrushovski construction; Bi-Colored Expansions; Rich Structures; Dependence (NIP); Strong Dependence.

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1 Introduction

Extending methods of stability and applying them to a larger class of theories is a dominating theme in current research of pure model theory. In addition to showing the prevalence of these methods, this line of research also presents potential applications beyond model theory. To instantiate general concepts in stability-hierarchy and perhaps examine some related open questions/conjectures one would need to look for some new examples, conceivably through adapting known model-theoretic methods. Our main aim in this paper is to study the Fraïssé-Hrushovski method beyond the realm of stability/simplicity. We aim to take the first step of the more general plan of modifying the Fraïssé-Hrushovski construction to study Bi-colored expansions of geometric theories without the independence property (NIP).

There are two major (independent) research themes with which our proposed plan is naturally connected, as explained below.

Our first source of contribution is to the Bi-colored fields, that is the comprehensive studies concerning the expansion of algebraically closed fields with a unary predicate $p$—often called a color predicate—interpreted either by an arbitrary set (Black fields) [1, 2, 19], an additive subgroup (Red fields) [5, 20], or a multiplicative subgroup (Green fields) [6, 20]. All examples obtained by this construction are $\omega$-stable, either with Morley rank $\omega$ (non-collapsed constructions) [2, 19, 20] or finite Morley rank (collapsed constructions) [1, 6].

The other theme that our results are naturally connected to is the study of the generic expansions of models of geometric theories by a unary predicate which can be interpreted either by a submodel (Lovely pairs) [7, 8, 21, 26] or more generally by submodels of reducts [9, 10, 11, 13].

While our methodology is inspired by the first subject, in our study of the bi-colored expansions of theories, we do not restrict ourselves, a priori, to any particular class of theories in stability hierarchy. This would give a clearer understanding of the axiomatization of the resulting expansion in the style of the second subject. Our presentation somehow shows that both subjects could be put in a common framework where the Fraïssé-Hrushovski construction is seen as a sort of generalization of the theme of the generic expansions.

In more technical terms, we assume that $T$ is a complete geometric theory without finite models in a countable language $\mathcal{L}$. As a convenient routine, we also assume that $T$ admits elimination of quantifiers in $\mathcal{L}$. As $T$
is geometric, the algebraic closure operator gives rise to a pre-geometry and subsequently a notion of dimension function $\dim$, where $\dim(\bar{a})$ for any finite tuple $\bar{a}$ is the size of a basis of $\bar{a}$. We further require that $T$ satisfies the free-amalgamation property and its corresponding pre-geometry is indecomposable. Subsequently, we expand $\mathcal{L}$ to $\mathcal{L}_p = \mathcal{L} \cup \{p\}$ by adding a unary predicate $p$ called the coloring predicate. We consider the class of all $\mathcal{L}_p$-structures whose universe $M$ is a model of $T^{a'}$, fix a real number $0 < \alpha \leq 1$, and for $M \models T^{a'}$ and a finite subset $A$ of $M$, define the pre-dimension function

$$\delta_{\alpha}(A) = \dim(A) - \alpha|p(A)|.$$

Now, as $T$ is geometric, the dimension function enjoys certain definability conditions which makes $\mathcal{K}^+_\alpha$, the class of $\mathcal{L}_p$-structures $M \in \mathcal{K}^+_\alpha$ such that $\delta_{\alpha}(A) \geq 0$ for all $A \subseteq_{\text{fin}} M$, $\mathcal{L}_p$-axiomatizable.

There is a notion $\leq_{\alpha}$ of closed substructures in $\mathcal{K}^+_\alpha$ associated to the pre-dimension function $\delta_{\alpha}$. The free-amalgamation of $T$ implies that $(\mathcal{K}^+_\alpha, \leq_{\alpha})$ has the amalgamation property, and this guaranties that $(\mathcal{K}^+_\alpha, \leq_{\alpha})$ has $\lambda$-rich models, for each $\lambda \geq \aleph_0$.

Using ideas from [18], we give a complete axiomatization $T_{\alpha}$ for the class all $\omega_1$-rich structures. This axiomatization together with a description of types enables us to characterize coheirs in models of $T_{\alpha}$ and prove that if $T$ is dependent then so is $T_{\alpha}$.

Yet we will show that our construction separates between rational and irrational $\alpha$ via the notion of strong dependence. The strong dependence is transferred to $T_{\alpha}$ for rational $\alpha$, but this is not the case when $\alpha$ is irrational. Indeed, assuming that $T$ defines a linear order and $\alpha$ is irrational we show that $T_{\alpha}$ is not strongly dependent. This fact mimics the well-known result in ab-initio Fraïssé-Hrushovski construction that differentiate between rational/irrational $\alpha$ via $\omega$-stability/strict-stability.

The structure of the paper is as follows. In section 2 after fixing the setting for our underlying theory $T$, we introduce the class $(\mathcal{K}^+_\alpha, \leq_{\alpha})$. In Section 3 we prove certain properties of this class, present an axiomatization $T_{\alpha}$ for its rich structures, and prove it to be a complete theory. Finally in Section 4 we have given the mentioned results on dependence and strong dependence of $T_{\alpha}$.
2 Bi-Colored Structures

2.1 Preliminaries, Setting and Conventions

Let $T$ be as in the introduction. It is clear, by the quantifier-elimination assumption, that $\text{acl}_{M_1}(M_0) \cong \text{acl}_{M_2}(M_0)$ for all $M_0 \subseteq M_1, M_2 \models T$.

We use capital letters $M, N, P, K$ to denote the $\mathcal{L}$-structures and $A, B, C, D$ and $X, Y, Z$ respectively for finite and infinite sets. Instead of $X \cup Y$ we would write $XY$.

For tuples $\bar{a}, \bar{b}$ in a model of $M$ of $T$, by the quantifier-free type of $\bar{a}$ over $\bar{b}$, denoted by $\text{qftp}_\mathcal{L}(\bar{a}/\bar{b})$, we mean the set of all quantifier-free formulas $\varphi(\bar{x}, \bar{y})$ such that $M \models \varphi(\bar{a}, \bar{b})$.

The theory $T$ is assumed to be geometric, so it eliminates the quantifier $\exists^\infty$ and the algebraic closure, acl, satisfies the exchange property in its models. We denote the dimension obtained by acl in $T$ by dim. So for tuples $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_m)$ of $M$ and a set $Y \subseteq M$, we let $\dim(\bar{a}/Y)$ be the size of the acl-base of $\{a_1, \ldots, a_n\}$ over $Y$, and $\dim(\bar{a}/\bar{b}) = \dim(\bar{a}/\{b_1, \ldots, b_m\})$. If $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{b} \in M[\bar{y}]$, then $\dim(\varphi(M, \bar{b})) = \max\{\dim(\bar{a}/\bar{b}) : M \models \varphi(\bar{a}, \bar{b})\}$.

We say that $Y$ is dim-independent from $Z$ over $X$, and write $Y \downarrow^\dim_X Z$, if $\dim(\bar{a}/X) = \dim(\bar{a}/XZ)$ for each tuple $\bar{a} \in Y[\bar{a}]$. If in addition, $Y \cap Z = X$, we say that $Y$ and $Z$ are free over $X$. To emphasize the freeness of $Y$ and $Z$ over $X$ we would write the set $YZ$ as $Y \oplus_X Z$ (we also use the same notation for the free-amalgam of structures).

Some properties of the dim function and dim-independence is recalled in the following for later reference, assuming that all subsets and tuples are from (a sufficiently saturated) model $M$ of $T$.

**Fact 2.1.** The dim function has the following properties.

- **Finite character.** $\dim(\bar{a}/Y) = \min\{\dim(\bar{a}/B) : B \subseteq_{\text{fin}} Y\}$.
- **Additivity.** $\dim(\bar{a}\bar{b}/Z) = \dim(\bar{a}/Z) + \dim(\bar{b}/\bar{a}Z)$.
- **Monotonicity.** $\dim(\bar{a}/Y) \geq \dim(\bar{a}/Z)$ for $Y \subseteq Z$.
- **Definability.** For each formula $\varphi(\bar{x}, \bar{y})$ and $k \leq |\bar{x}|$ the set of tuples $\bar{b} \in M[\bar{y}]$ with $\dim(\varphi(M, \bar{b})) = k$ is definable by a formula $d_{\varphi,k}(\bar{y})$.
- **$\forall$-Definability.** If $\dim(\bar{a}/\bar{b}) \leq n$, then there is a formula $\psi(\bar{x}, \bar{y})$ such that $\psi(\bar{x}, \bar{b}) \in \text{qftp}_\mathcal{L}(\bar{a}/\bar{b})$ and if $M \models \psi(\bar{a}', \bar{b})$ then $\dim(\bar{a}'/\bar{b}') \leq n$, for any $\bar{a}', \bar{b}'$. 


We need a stronger version of Definability, as stated below. The first part of the following statement appears in [10], Lemma 2.3.

**Lemma 2.2.** Let $M$ be an $\omega$-saturated model of $T$, $\varphi(\bar{x}; \bar{y})$ an $L$-formula and $k$ a natural number. Then

1. There is a formula $H_\varphi(\bar{y})$ that defines the set of tuples $\bar{b} \in M^{[\bar{y}]}$ such that there exists $\bar{a}$ satisfying $\varphi(\bar{x}; \bar{b})$ with $\bar{a} \cap \text{acl}(\bar{b}) = \emptyset$.

2. There exists an $L$-formula $D_{\varphi,k}(\bar{y})$ which defines the set of tuples $\bar{b} \in M^{[\bar{y}]}$ for which there exists $\bar{a}$ satisfying $\varphi(\bar{x}; \bar{b})$ with $\dim(\bar{a}/\bar{b}) \geq k$ and $\bar{a} \cap \text{acl}(\bar{b}) = \emptyset$.

**Proof.** We use 1 to prove 2. Certainly, we may assume that $k \leq |\bar{x}| = n$. For $\sigma \in S_n$ and $\psi(x_{\sigma(1)}; x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \bar{y}) = \varphi(\bar{x}, \bar{y})$ the assertion

$$\exists x_{\sigma(1)} [\varphi(\bar{x}, \bar{y}) \land x_{\sigma(1)} \notin \text{acl}(x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \bar{y})]$$

is definable by a formula $H_\psi(x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \bar{y})$. Now by applying 1 to $H_\psi$ and iterating this procedure the following statement

$$\exists x_{\sigma(1)} \ldots \exists x_{\sigma(k)} [\varphi(\bar{x}, \bar{y}) \land \bigwedge_{i=1}^{k} x_{\sigma(i)} \notin \text{acl}(x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}, \bar{y})]$$

is definable by a $L$-formula $\rho(x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}, \bar{y})$. Informally $\rho$ states that there exists a set $\{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\} \subseteq \{x_1, \ldots, x_n\}$ which has dimension at least $k$ over $\bar{y}$. Now by applying 1 to $\rho$ we get that the statement

$$\exists x_{\sigma(k+1)} \ldots \exists x_{\sigma(n)} (\rho(x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}, \bar{y}) \land \bigwedge_{i=k+1}^{n} x_{\sigma(i)} \notin \text{acl}(\bar{y}))$$

is definable by an $L$-formula $D_{\varphi,k}(\bar{y})$. Let $D_{\varphi,k}(\bar{y}) = \bigvee_{\sigma \in S_n} D_{\varphi,k,\sigma}(\bar{y})$. Note that any tuple $\bar{b} \in M$ satisfies $D_{\varphi,k}(\bar{y})$ if and only if there exists a tuple $\bar{a}$ which satisfies $\varphi(\bar{x}, \bar{b})$, avoids $\text{acl}(\bar{b})$ and includes a subtuple $(a_{i_1}, \ldots, a_{i_k})$ with $\dim((a_1, \ldots, a_n)/\bar{b}) \geq \dim((a_{i_1}, \ldots, a_{i_k})/\bar{b}) = k$. Therefore $D_{\varphi,k}(\bar{y})$ is the desired formula. \qed

**Fact 2.3.**

1. *Symmetry.* $Y \downarrow_X^\dim Z$ if and only if $Z \downarrow_X^\dim Y$.  

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2. *Transitivity.* \( Y \dim_{X} Z_1 Z_2 \) if and only if \( Y \dim_{X} Z_1 \) and \( Y \dim_{X} Z_2 \).

3. *acl-Preservation.* \( Y \dim_{X} Z \) if and only if \( \text{acl}(Y) \dim_{\text{acl}(X)} \text{acl}(Z) \).

4. *dim-Morley Sequences.* Any order indiscernible sequence \( \{a_i : i \in I\} \) over \( X \) is a dim-Morley sequence, i.e. for any two disjoint subsequences \( J_1 \) and \( J_2 \) of \( I \) we have \( J_1 \dim_{X} J_2 \).

We will need formulas that are “strong” in the sense that they carry enough information about dim. These are characterized by the following Definition.

**Definition 2.4.** We say that an \( \mathcal{L} \)-formula \( \varphi(\bar{x}, \bar{y}) \) is strong with respect to the distinct tuples \( \bar{d}, \bar{a} \) if \( \varphi(\bar{x}, \bar{a}) \in \text{qftp}_\mathcal{L}(\bar{d}/\bar{a}) \) and for each \( \bar{d} \bar{a}' \) in a sufficiently saturated model \( M \models T \),

1. if \( M \models \varphi(\bar{d}, \bar{a}') \) then \( \bar{d} \bar{a}' \) is distinct, and
2. if \( M \models \varphi(\bar{d}, \bar{a}) \) and \( \dim(\bar{d}/\bar{a}) = \dim(\bar{d}/\bar{a}') \) then for each partition \( P = (\bar{x}_1, \bar{x}_2) \) of \( \bar{x} \), \( \dim(\bar{d}_2/\bar{d}_1 \bar{a}') \leq \dim(\bar{d}_2/\bar{d}_1 \bar{a}) \).

**Remark 2.5.** The second item above implies, in particular, that \( \dim(\bar{d}_1/\bar{a}') \geq \dim(\bar{d}_1/\bar{a}) \).

The \( \bigvee \)-Definability (Fact 2.1) implies that a strong formula for any pair of distinct sequence always exists. We shortly give a proof of this in the following, where for a sequence of distinct variables \( \bar{x} \) we denote by \( P \) the set of all partitions \( P = (\bar{x}_1, \bar{x}_2) \) of \( \bar{x} \).

**Lemma 2.6.**

1. Suppose that \( \bar{d}, \bar{a} \) is a pair of distinct sequences. Then there is a strong formula \( \varphi(\bar{x}, \bar{y}) \) with respect to \( \bar{d}, \bar{a} \).
2. If \( \varphi(\bar{x}, \bar{y}) \) is strong with respect to \( \bar{d}, \bar{a} \), \( T \models (\theta(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y})) \) and \( \theta(\bar{x}, \bar{a}) \in \text{qftp}_\mathcal{L}(\bar{d}/\bar{a}) \) then \( \theta(\bar{x}, \bar{y}) \) is also strong.

**Proof.** We prove only item 1, and 2 would be clear. For each partition \( P = (\bar{x}_1, \bar{x}_2) \) of \( \bar{x} \), by the \( \bigvee \)-Definability (Fact 2.1) there is a formula \( \varphi_P(\bar{x}_2, \bar{x}_1, \bar{y}) \) such that \( \varphi_P(\bar{x}_2, \bar{d}_1, \bar{a}) \in \text{qftp}_\mathcal{L}(\bar{d}_2/\bar{d}_1 \bar{a}) \) and if \( M \models \varphi_P(\bar{d}_2, \bar{d}_1, \bar{a}) \) then \( \dim(\bar{d}_2/\bar{d}_1 \bar{a}) \leq \dim(\bar{d}_2/\bar{d}_1 \bar{a}) \). Now we simply let \( \varphi(\bar{x}, \bar{y}) \) be the conjunction of all formulas \( \varphi_P \) with the formula that states \( \bar{x} \) and \( \bar{y} \) are distinct. \( \square \)
In the next sections, we deal with a class of bi-colored expansions of models of $T$. For this class to have certain desired features, we need $T$ to satisfy yet two other properties, that is the free-amalgamation, and the indecomposability, as introduced in the following. These features will later lead us to an axiomatizations for the “rich” structures.

Recall that an $L$-embedding $f : M \to N$ is algebraically closed if $\text{acl}(f(M)) = f(M)$.

**Definition 2.7.** We say that $T$ has the free-amalgamation property over algebraically closed substructures if for each $M_0, M_1, M_2 \models T$ and each pair of algebraically closed embeddings $f_1 : M_0 \to M_1$, $f_2 : M_0 \to M_2$, there are $M \models T$ and embeddings $g_1 : M_1 \to M$, $g_2 : M_2 \to M$ such that

1. $g_1 \circ f_1 = g_2 \circ f_2$, and
2. $g_1(M_1)$ and $g_2(M_2)$ are free over $g_1 \circ f_1(M_0)$ in $M$.

We call $M$ a free amalgam of $M_1$ and $M_2$ over $M_0$, and although $M$ is not unique up to isomorphism and by abuse of notation, denote it by $M_1 \oplus_{M_0} M_2$.

**Observation 2.8.** Let $T$ have the free-amalgamation property and $M$ be a $\lambda$-saturated model of $T$. Suppose that $\Sigma(\bar{x})$ is a partial type over $X$ which has a solution with no intersection with $\text{acl}(X)$. Then for each $Y \supseteq X$ with $|Y| < \lambda$ there is a solution $\bar{d}$ to $\Sigma$ in $M$ such that $\bar{d} \cap Y = \emptyset$. Furthermore $\bar{d}$ can be chosen in such that $X \bar{d}$ and $Y$ are free over $X$.

**Proof.** There is a solution $\bar{d}$ in $M \oplus_{\text{acl}(X)} M$ which satisfies the conditions required. By saturation we can find a solution $\bar{d}$ in $M$ such that $X \bar{d}$ and $Y$ are free over $X$. \qed

**Definition 2.9.** We call $T$ indecomposable if no finite-dimensional algebraically closed set $X$ can be written as a finite union $X = Y_1 \ldots Y_n$ with $\dim(Y_i) < \dim(X)$ for $i \leq n$.

The assumption of indecomposability is meant to provide the following desirable property for bases.

**Lemma 2.10.** Assume that $T$ is indecomposable and $M \models T$. Let $B = \{d_1, \ldots, d_m\}$ be an independent set over $A \subseteq M$. Then for each natural number $n$ there is a subset $D \subseteq \text{acl}(Ad_1, \ldots, d_m)$ with $|D| = n$ such that each $m$-element subset of $BD$ is a base for $\text{acl}(Ad_1, \ldots, d_m)$. 
Proof. By assumption \( \dim(d_1, \ldots, d_m/A) = m \). Let \( S_k \) be the group of permutations of \( \{1, \ldots, k\} \). Since \( T \) is indecomposable, the set

\[
D_{m+1} := \text{acl}(Ad_1, \ldots, d_m) \setminus \bigcup_{i=1}^{m} \text{acl}(Ad_1, \ldots, \hat{d}_i, \ldots, d_m)
\]

is non-empty. Let \( d_{m+1} \in D_{m+1} \) be arbitrary. By the exchange property, the set \( \{d_{\sigma(1)}, \ldots, d_{\sigma(m)}\} \) is a base for \( \{d_1, \ldots, d_m, d_{m+1}\} \) over \( A \) for any \( \sigma \in S_{m+1} \).

Again by indecomposability of \( T \), the set

\[
D_{m+2} := \text{acl}(Ad_1, \ldots, d_{m+1}) \setminus \bigcup_{1 \leq i < j \leq m+1} \text{acl}(Ad_1, \ldots, \hat{d}_i, \ldots, \hat{d}_j, \ldots, d_{m+1})
\]

is non-empty. Now let \( d_{m+2} \) be an arbitrary element of \( D_{m+2} \). Again by the exchange property the set \( \{d_{\sigma(1)}, \ldots, d_{\sigma(m)}\} \) is base for \( \{d_1, \ldots, d_{m+2}\} \) over \( A \) for any \( \sigma \in S_{m+2} \).

Iterating this process we obtain the set \( D = \{d_{m+1}, \ldots, d_{m+n}\} \) as desired. \( \Box \)

Examples of indecomposable geometric theories with free-amalgamation include strongly minimal expansions of divisible abelian groups, \( \omega \)-minimal expansions of ordered divisible abelian groups (ODAG), any completion of the theory of algebraically closed valued fields, and the theory of \( p \)-adic fields. The indecomposability of the mentioned theories indeed stems in the fact that no vector space over an infinite field can be written as a finite union of its proper subvector spaces.

2.2 Bi-colored Expansions

From here onward we assume that \( T \) is an indecomposable geometric theory satisfying the free-amalgamation property. Before proceeding, we need to fix some more notation and conventions.

Notation and Conventions. We expand \( \mathcal{L} \) to \( \mathcal{L}_p = \mathcal{L} \cup \{p\} \) by adding a unary predicate \( p \) called the coloring predicate. We denote by \( \mathcal{K} \) the class of all \( \mathcal{L}_p \)-structures whose universe \( M \) is a model of \( T^v \). For \( X \subseteq M \), by \( \langle X \rangle_M \), or \( \langle X \rangle \) we denote the \( \mathcal{L}_p \)-structure generated by \( X \) in \( M \). In this sense we call \( M \) finitely generated if there is a finite set \( A \subseteq M \) such that \( M = \langle A \rangle \). For technical convenience, we take \( \emptyset \) also as a finitely generated structure in
\( \mathcal{K} \). For the ease of reference, we say \( x \) is colored if \( M \models p(x) \). We would simply write \( p(x) \) instead of \( M \models p(x) \) when the meaning is clear from the context. We write \( p(\bar{x}) \) instead of \( \bigwedge_{i=1}^n p(x_i) \), when \( \bar{x} = (x_1, \ldots, x_n) \). Also by \( p(X/Y) \) we mean the set of colored elements of \( X \setminus Y \). Finally we fix a real number \( 0 < \alpha \leq 1 \) for the rest of the paper. We will occasionally add the assumption that \( \alpha \) is/is not rational. The \( \mathcal{L} \) and \( \mathcal{L}_p \)-algebraic closures are respectively denoted by \( \text{acl} \) and \( \text{Acl} \).

**Definition 2.11.** For \( M \in \mathcal{K} \) and a finite subset \( A \) of \( M \), define

\[
\delta_\alpha(A) = \dim(A) - \alpha |p(A)|.
\]

More generally for \( A \subseteq \text{fin} \ M \) and \( X \subseteq M \) we define \( \delta_\alpha(A/X) = \dim(A/X) - \alpha |p(A/X)| \). The function \( \delta_\alpha \) is called a pre-dimension map.

The assumption of quantifier elimination makes \( \delta_\alpha(A) \) independent from the ambient \( M \). It is also routine to check that

\[
\delta_\alpha(AB/C) = \delta_\alpha(A/BC) + \delta_\alpha(B/C),
\]

and that \( \delta_\alpha \) is submodular, that is

\[
\delta_\alpha(AB) + \delta_\alpha(A \cap B) \leq \delta_\alpha(A) + \delta_\alpha(B).
\]

**More on Notation.** Denote by \( \mathcal{K}_\alpha^+ \) the subclass of \( \mathcal{K} \) consisting of the \( \mathcal{L}_p \)-structures \( M \in \mathcal{K}_\alpha^+ \) such that \( \delta_\alpha(A) \geq 0 \) for all \( A \subseteq \text{fin} \ M \). When \( A \) is a subset of a structure \( M \) in the \( \mathcal{K}_\alpha^+ \), we simply say that \( A \) is in \( \mathcal{K}_\alpha^+ \). Furthermore in this situation for \( N \in \mathcal{K}_\alpha^+ \) we say that \( f : A \to N \) is an \( \mathcal{L}_p \)-embedding if \( \text{qftp}_\mathcal{L}(f(A)) = \text{qftp}_\mathcal{L}(A) \) and \( f(A) \) has the same color in \( N \) as \( A \) in \( M \). Finally, to avoid repeating the assumption that a structure is in \( \mathcal{K}_\alpha^+ \) in each statement, we simply assume that each structure and set in the following belongs to \( \mathcal{K}_\alpha^+ \) unless we emphasize otherwise.

It is clear by our previous conventions that \( \emptyset \in \mathcal{K}_\alpha^+ \). Also note that by \( \bigvee \)-Definability (Fact 2.1), \( \mathcal{K}_\alpha^+ \) is axiomatizable in \( \mathcal{L}_p \) by the theory consisting of the \( \mathcal{L}_p \)-sentences

\[
\neg \exists x_1, \ldots, x_n(\psi_1(x_1, \ldots, x_n) \land \bigvee_{Y \subseteq \{x_1, \ldots, x_n\}} \min \{p(Y) \mid |Y| \geq k\})
\]
where $\psi_l$ asserts that the dimension of $(x_1, \ldots, x_n)$ is bounded by $l$, and
$l < \alpha k$.

Notice that since $\alpha > 0$ all algebraic points over $\emptyset$ are non-colored.

We need to review some basic concepts concerning the properties of the
pre-dimension function $\delta_\alpha$.

**Definition 2.12.**

1. For $A \subseteq \text{fin } M$ and $X \subseteq M$ we say that $A$ is closed in $X$ and write $A \leq_{\alpha} X$ if $A \subseteq X$ and $\delta_\alpha(B/A) \geq 0$ for all $A \subseteq B \subseteq \text{fin } X$.

2. For $X, Y$ arbitrary subsets of $M$, we say $X$ is closed in $Y$ and write $X \leq_{\alpha} Y$ if $X \subseteq Y$ and $\delta_\alpha(A/X) \geq 0$ for all $A \subseteq \text{fin } Y$.

3. For structures $M, N$ we say that $M$ is closed in $N$ and write $M \leq_{\alpha} N$ if $M$ is a substructure of $N$ and $M \leq_{\alpha} N$ as sets in the sense above.

For simplicity, we omit the subscript $\alpha$ from $\leq_{\alpha}$ and $\delta_\alpha$. It is easy to see that if $X \leq M$, then $\neg \rho(x)$ for all $x \in \acl_M(X) \setminus X$. Also in this case, both $\langle X \rangle_M$ and $\acl_M(X)$ are closed in $M$.

**Remark 2.13.** $(\mathcal{K}_{\alpha}^+, \leq)$ is a so-called smooth class, that is for all $M, M_1, M_2, X$,

1. $\emptyset, M \leq M$.

2. If $M \leq M_1$ and $M_1 \leq M_2$, then $M \leq M_2$.

3. If $M \leq M_2$ then $M \leq M_1$ for all $M \subseteq M_1 \subseteq M_2$.

4. if $M \leq M_1$ then $M \cap X \leq X$ for all $X \subseteq M_1$.

Note also that items 2 and 4 above easily imply that if $M_1, M_2 \leq M$ then $M_1 \cap M_2 \leq M$.

**Definition 2.14.** Assuming that $X, A, B \subseteq M$,

1. The closure of $X$ in $M$, denoted by $\text{cl}_M(X)$, is the smallest subset $Y$ of $M$ containing $X$ such that $Y \leq M$.

2. We say that $B$ is an intrinsic extension of $A$ and write $A \leq i B$ if $A \subseteq B$ but there is no $A' \neq B$ with $A \subseteq A' \subseteq B$, equivalently, $\delta(B) \preceq \delta(A')$ for all $A \subseteq A' \subsetneq B$. 

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3. A pair \((A, B)\) is called *minimal* if \(A \subseteq B\), \(A \not\subseteq B\) but \(A \leq C\) for all \(A \subseteq C \subsetneq B\).

It is clear that if \((A, B)\) is a minimal pair, then \(A \leq_i B\). Moreover if \(B\) is an intrinsic extension of \(A\) then it is possible to find a tower \(B_0 = A \subseteq B_1 \subseteq \ldots \subseteq B_n = B\) where each \((B_i, B_{i+1})\) is minimal.

The following statements are well-known facts about basic properties of \(\text{cl}_M\).

**Fact 2.15.**

1. \(\text{cl}_M(X)\) is the intersection of all closed subsets of \(M\) that contain \(X\).
2. \(\text{cl}_M(X) = \bigcup_{A \subseteq X} \text{cl}_M(A)\).
3. \(\text{cl}_M(A) = \bigcup \{B \subseteq M : A \leq_i B\}\).

When \(M\) is clear from the context, we write \(\text{cl}(X)\) instead of \(\text{cl}_M(X)\).

**Definition 2.16.** For \(A \subseteq M\) and natural number \(n\), we define \(\text{cl}_M^n(A)\) as the union of all \(B \subseteq M\) such that \(A \leq_i B\) and \(|B - A| < n\).

**Remark 2.17.** Let \(B, M \in \mathcal{K}_\alpha^+\) and \(A \subseteq M\). Then

1. There is \(f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) such that for any embedding \(g : B \to M\) the size of \(\text{cl}_M^n(g(B))\) is bounded by \(f(|B|, n)\).
2. \(\text{cl}_M(A) \subseteq \text{Acl}_M(A)\).
3. When \(\alpha\) is rational, then \(\text{cl}_M(A)\) is finite; hence \(\text{cl}_M(A) = \text{cl}_M^n(A)\) for some \(n\).
4. By item 1, \(\text{cl}_M(A)\) is countable when \(\alpha\) is irrational.

The following definition singles out two different types of closed extensions. This distinction will be highlighted in the axiomatization of rich structures given in section 3.1.

**Lemma 2.18.** Suppose that \(A \subseteq M\) is finite. Then there is a finite set \(B\) containing \(A\) such that \(\langle A \rangle = \langle B \rangle\) and and \(B \leq \langle B \rangle = \langle A \rangle\).
Proof. If $c_1, \ldots, c_k \in \langle A \rangle \setminus A$ are colored elements then
\[ 0 \leq \delta(Ac_1, \ldots, c_k) = \delta(A) - k\alpha. \]
Hence if $C \subseteq \langle A \rangle$ and each elements of $C$ is colored then
\[ |C \setminus A| \leq \delta(A)/\alpha \leq |A|/\alpha. \]
So let $B = A \cup p(\langle A \rangle)$. \hfill \square

Remark and Convention. For $B$ as in the above lemma, each $x \in \langle B \rangle - B$ is non-colored. Hence we make the convention that whenever we write $\langle A \rangle$ we assume that $A \subseteq \langle A \rangle$. Notice that by this convention for $A \subseteq B$ we have that $\langle A \rangle \subseteq \langle B \rangle$ if and only if $A \subseteq B$.

Definition 2.19. Assume that $\langle A \rangle \subseteq \langle B \rangle$ and $A \subseteq B$. Call $B$

1. algebraic over $A$ if $\dim(b/A) = 0$ for all $b \in B \setminus A$, and
2. transcendental over $A$ if $\dim(b/A) = 1$ for all $b \in B \setminus A$.

Remark 2.20. It can be easily seen that if $A \subseteq B \subseteq \langle B \rangle$ then there exists $B_1$ such that $A \subseteq B_1 \subseteq B$ with $B_1$ algebraic over $A$ and $B$ transcendental over $B_1$. Furthermore, if $B$ is algebraic over $A$ and $f : \langle B \rangle \to M$ is an $L_p$-embedding in some $M \in K^+_\alpha$ then $\text{cl}_M(f(B)) = \text{cl}_M(f(A)) \oplus f(A)f(B)$. So, in particular, for each natural $n$ we have $\text{cl}_M^n(f(B)) = \text{cl}_M^n(f(A)) \oplus f(A)f(B)$.

3 Rich Structures and their Axiomatization

3.1 Rich Structures

We have already defined what we mean by $T$ having the free-amalgamation property (see Definition 2.7). Along the same lines, we say that $(K^+_\alpha, \leq)$ has the free-amalgamation property (defined more precisely after the following definition) if the conditions in Definition 2.7 hold with $M_i, M$ structures in the class $K^+_\alpha$, and $f_i, g_i$ strong embeddings in the sense of the following definition.

Definition 3.1. By a strong embedding $f : M \to N$ we mean an $L_p$-embedding $f$ such that $f(M) \leq N$.
So by the class \((K_\alpha^+, \leq)\) having the amalgamation property we mean that for each \(M_0, M_1, M_2\) and each pair of strong embeddings \(f_1 : M_0 \to M_1, f_2 : M_0 \to M_2\), there are \(M\) and strong embeddings \(g_1 : M_1 \to M, g_2 : M_2 \to M\) such that \(g_1 \circ f_1 = g_2 \circ f_2\).

**Lemma 3.2.** The class \((K_\alpha^+, \leq)\) has the amalgamation property. Moreover, if both \(f_1, f_2\) are algebraically-closed then we can choose \(M \in K_\alpha^+\) such that \(g_1(M_1)\) and \(g_2(M_2)\) are free over \(g_1 \circ f_1(M_0)\) in \(M\).

**Proof.** Assume that \(L\)-structures \(M_0, M_1, M_2\) in \(K_\alpha^+\) and the strong embeddings \(f_1, f_2\) are as above.

We can assume, without loss of generality that \(M_1, M_2\) are models of \(T\). This can be obtained by \(L\)-embedding of each \(M_1\) and \(M_2\) into a model of \(T\) and a suitable extension of the coloring.

We may also assume that \(f_1, f_2\) are algebraically-closed strong embeddings. This is because \(\text{acl}_{M_1}(M_0)\) and \(\text{acl}_{M_2}(M_0)\) are \(L\)-isomorphic over \(M_0\), as both \(f_1\) and \(f_2\) are strong (see the explanation after Definition 2.12).

Now since \(T\) has the free-amalgamation property, there are a model \(M\) and \(L\)-embeddings \(g_i : M_i \to M\), such that \(g_1 \circ f_1 = g_2 \circ f_2\). Defining \(p(M) = g_1(p(M_1)) \cup g_2(p(M_2))\), it is clear that \(M \in K_\alpha^+\).

It remains only to prove that \(g_1, g_2\) are strong embeddings. For notational simplicity we denote again by \(M_1, M_2\) and \(M_0\) their respective images in \(M\), that is \(g_1(M_1), g_2(M_2)\) and \(g_1 \circ f_1(M_0)\). In the following we show that \(M_1 \leq M\) (and \(M_2 \leq M\) follows similarly).

Let \(A \subseteq M \setminus M_1\) be arbitrary. Writing \(A = BC\) where \(B \subseteq M_2, C \subseteq M \setminus M_2\) we have

\[
\delta(A/M_1) = \delta(C/M_1) + \delta(B/M_1 C).
\]

Now \(\delta(C/M_1) \geq 0\) because all elements of \(C\) are non-colored. Also \(\delta(B/M_1 C) = \delta(B/M_0 C) \geq 0\), because \(M_1 \downarrow_{M_0} M_2\).

\[\Box\]

As we have let \(\emptyset\) in \(K_\alpha^+\), the amalgamation property implies the joint embedding property. It is also clear that if \(M\) is in \(K_\alpha^+\) then so are all its substructures. In this sense, we call \((K_\alpha^+, \leq)\) a *Fraïssé class*, and it is natural to look for its *Fraïssé limit*, or the *rich* structures in the sense below.

**Definition 3.3.** An \(L\)-structure \(M\) in \(K_\alpha^+\) is called \(\lambda\)-rich, for \(\lambda \geq \aleph_0\) a cardinal, if
1. $M \models T$,

2. If $M_1 \leq M_2$ are generated by a set of cardinality $< \lambda$, then each strong embedding $f : M_1 \to M$ extends to a strong embedding $g : M_2 \to M$.

The Amalgamation and Joint Embedding properties together with the fact that $\mathcal{K}^+_\alpha$ is closed under the union of $\leq$-chains, imply that the $\lambda$-rich structures exist. Indeed a union of $\leq$-chains of models of $T$, and the quantifier-elimination (for this union to be a model) give such a structure. We prefer to state this also as the following fact.

**Fact 3.4.** $(\mathcal{K}^+_\alpha, \leq)$ contains $\lambda$-rich structures, for all $\lambda \geq \aleph_0$.

**Lemma 3.5.** Each $\lambda$-rich structure is $\lambda$-saturated as a model of $T$.

**Proof.** Let $M$ be $\lambda$-rich. Assume that $\Sigma(x)$ is a partial 1-type over a set $X \subseteq M$, where $|X| < \lambda$ and without loss of generality assume that $X$ is closed in $M$. Let $d \notin M$ be a solution of $\Sigma(x)$ in an $\mathcal{L}$-elementary extension $N$ of $M$. Extend the coloring of $M$ to $N$ by letting $\neg p(x)$ for each $x \in N - M$, so that now $N \in \mathcal{K}^+_\alpha$. Observe that $\text{cl}_M(X) \subseteq M, N$ and hence we keep the notation $\text{cl}(X)$. It is clear that $\langle \text{cl}(X)d \rangle \in \mathcal{K}^+_\alpha$, $\langle \text{cl}(X) \rangle \leq \langle \text{cl}(X)d \rangle$ and $\langle \text{cl}(X) \rangle \leq M$. Now since $M$ is $\lambda$-rich, there is a strong embedding $f : \langle \text{cl}(X)d \rangle \to M$, and hence $f(d)$ is the solution of $\Sigma(x)$ in $M$. \qed

It is worth noting that in the literature, a $\lambda$-rich structure is sometimes called $\lambda$-ultra-homogeneous. Moreover, it is clear from the definition (letting $M_1 = \emptyset$) that for each $M_2$ as above, there is a strong embedding $g : M_2 \to M$. This property of $M$ is called $\lambda$-universality.

We are now set to provide an axiomatization for the rich structures.

### 3.2 Axiomatization

Below for the $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and a natural number $k$, we let $D_{\varphi,k}(\bar{y})$ and $d_{\varphi,k}(\bar{y})$ be the formulas introduced in Lemma 2.2 and Fact 2.1 respectively.

**Definition 3.6.** Let $T_\alpha$ be an $\mathcal{L}_\alpha$-theory whose models $\mathbb{M}$ satisfy the following.

1. $\mathbb{M} \models T$

2. $\mathbb{M} \in \mathcal{K}^+_\alpha$ (that is $\delta(A) \geq 0$ for all $A \subseteq \text{fin} \mathbb{M}$).
3. For each $\bar{a} \subseteq \bar{d}\bar{a}$, where $\bar{d} = (d_1, ..., d_n)$ is transcendental over $\bar{a}$, $\dim(\bar{d}/\bar{a}) = k$, and $\varphi(\bar{x}, \bar{y})$ is strong with respect to $\bar{d}, \bar{a}$,

$$\mathbb{M} \models \forall \bar{y} \left[ D_{\varphi,k}(\bar{y}) \land d_{\varphi,k}(\bar{y}) \rightarrow \exists \bar{z} \left( \varphi(\bar{z}, \bar{y}) \land \bigwedge_{p(d_i)} p(z_i) \land \bigwedge_{\neg p(d_i)} \neg p(z_i) \right) \right]$$

The items above can be expressed as first-order axiom-schemes (for the second item this is pointed out before Definition 2.12). In the rest of this section we aim to establish the following theorem.

**Theorem.** $T_\alpha$ is a complete theory that axiomatizes $\omega_1$-rich structures of $\mathcal{K}_\alpha^+$. 

Towards a proof of the theorem above, what we can prove in the rest of this subsection with the information so far, is that $\omega_1$-rich structures in general, and $\omega$-rich structures when $\alpha$ is rational, are a model of $T_\alpha$. Proving that models of $T_\alpha$ are rich and therefore giving the full proof of the theorem will be possible only in Section 3.3, after some more auxiliary lemmas in the next section.

**Lemma 3.7.** Any $\omega_1$-rich structure $\mathbb{M}$ is a model of $T_\alpha$ (and hence $T_\alpha$ is consistent).

**Proof.** By Lemma 3.3, $\mathbb{M}$ is $\omega_1$-saturated as a model of $T$. Now we only need to prove that item 3 in Definition 3.6 holds. Let $\bar{a}, \bar{d}$ and $\varphi$ be as in there, and $\bar{a}'$ in $\mathbb{M}$ be such that $\mathbb{M} \models D_{\varphi,k}(\bar{a}') \land d_{\varphi,k}(\bar{a}')$. So there is $\bar{d}' \in \mathbb{M}[\bar{d}]$ such that $\mathbb{M} \models \varphi(\bar{d}', \bar{a})$, $\bar{d}' \land \text{acl}(\bar{a}') = \emptyset$, and $\dim(\bar{d}/\bar{a}') = k$. Since $\langle \text{cl}(\bar{a}') \rangle$ is countable, by $\omega_1$-saturation of $\mathbb{M}$ and Observation 2.3 we may assume that $\langle \text{cl}(\bar{a}') \rangle \cap \bar{d}' = \emptyset$ and, $\bar{d}'$ and $\langle \text{cl}(\bar{a}') \rangle$ are free over $\bar{a}'$. Consider the $L$-structure generated by $\text{cl}(\bar{a}')\bar{d}'$ in $\mathbb{M}$ and make it into the $L_p$-structure $N$ by coloring $\text{cl}(\bar{a}')\bar{d}'$ with the same colors as $\text{cl}(\bar{a})\bar{d}$ and leaving the rest of the elements non-colored. It is clear that $\langle \text{cl}(\bar{a}') \rangle \subseteq \mathbb{M}$. We claim that also $\langle \text{cl}(\bar{a}') \rangle \subseteq N$. If this claim is proved then since $\mathbb{M}$ is $\omega_1$-rich, there is a strong embedding $f : N \rightarrow \mathbb{M}$ which fixes $\langle \text{cl}(\bar{a}') \rangle$ pointwise. If we put $f(\bar{d}') = \bar{e}$, then

$$\mathbb{M} \models \left( \varphi(\bar{e}, \bar{a}') \land \bigwedge_{p(d_i)} p(e_i) \land \bigwedge_{\neg p(d_i)} \neg p(e_i) \right).$$

Now to prove the claim, let $\bar{d}'_1 \subseteq \bar{d}'$ and $\bar{e} \subseteq N \setminus \text{cl}(\bar{a}')\bar{d}'$. Notice that $\dim(\bar{d}'_1/\langle \text{cl}(\bar{a}') \rangle) = \dim(\bar{d}'_1/\bar{a}')$ since $\bar{d}' \downarrow_{\text{cl}(\bar{a}')/\bar{a}'}$. Moreover as $\varphi$ is strong, by
Remark 2.5 we have \( \dim(\bar{d}_1/\bar{a}) \geq \dim(\bar{d}_1/\bar{a}) \). Therefore

\[
\delta(\bar{d}_1/\langle \text{cl}(\bar{a}) \rangle) = \dim(\bar{d}_1/\langle \text{cl}(\bar{a}) \rangle) - \alpha|p(\bar{d}_1)| \\
= \dim(\bar{d}_1/\langle \text{cl}(\bar{a}) \rangle) - \alpha|p(\bar{d}_1)| \\
\geq \dim(\bar{d}_1/\bar{a}) - \alpha|p(\bar{d}_1)| \geq 0.
\]

It follows from the proof above that in case \( \alpha \) is rational, \( \omega \)-rich structures are a model of \( T_\alpha \).

### 3.3 Auxiliary Lemmas

The following lemmas and their proofs are inspired by [18], Lemmas 3.1 and 4.1. When \( A \subseteq B \) and \( B \) is transcendental over \( A \), these lemmas provide us with a way to find a set \( D \) containing \( B \) (in both cases of rational and irrational \( \alpha \)) whose pre-dimension is as much close from below to \( \delta(B) \) as we wish, yet \( A \subseteq D \) and \( D \) is transcendental over \( A \).

We will in particular utilize Lemmas 3.9, 3.10, and 3.11 to show that any \( \omega_1 \)-saturated model of \( T_\alpha \), for an irrational \( \alpha \) is in fact semi-generic (Theorem 3.15). Similarly we will make use of Lemmas 3.12 and 3.13 to prove that any \( \omega \)-saturated model of \( T_\alpha \), for a rational \( \alpha \) is \( \omega \)-rich (Theorem 3.21).

**Lemma 3.8.** For each \( n \in \omega \) there exists an \( \epsilon_n > 0 \) such that for each \( A \subseteq B \), if \( |B \setminus A| < n \) and \( \delta(B/A) < 0 \) then \( \delta(B/A) < -\epsilon_n \).

**Proof.** The set \( V_n = \{ \delta(B/A) : A \subseteq B, |B \setminus A| < n \} \) is finite. Therefore there exists \( \epsilon_n > 0 \), depending on \( n \), which keeps every negative value in \( V_n \) bounded away from 0, as required by the lemma.

Recall that by the finite version of the \( \Delta \)-system lemma [17], if \( \{B_i\}_{i \in \omega} \) is a countably-infinite family of finite sets of bounded size, then there is an arbitrarily-large finite subfamily of \( B_i \)'s with the same mutual intersection. The following lemma shows that when the \( B_i \) are subsets of a structure in \( K_\alpha^* \), then it is possible that the intersection in question is closed in each \( B_i \).

**Lemma 3.9.** For \( k \geq 1 \), let \( \{B_i : i \in \omega\} \) be a family of \( k \)-element subsets of a structure \( M \). Then for each natural number \( n \) there are a subset \( \Omega \) of \( \omega \) with \( |\Omega| \geq n \) and \( A \subseteq_{\text{fin}} M \) such that
1. \( A \leq B_i \), for each \( i \in \Omega \) and
2. \( B_i \cap B_j = A \), for \( i \neq j \in \Omega \).

**Proof.** Let \( \epsilon_k \) be as in Lemma 3.8. By the finite version of \( \Delta \)-system lemma, for each natural number \( n \) there are a finite subset \( \Omega' \subset \omega \) with \( |\Omega'| \geq n + [k/\epsilon_k] \) and a finite subset \( A \) of \( M \) such that \( B_i \cap B_j = A \) for each \( i \neq j \in \Omega' \). But we need a further step to show that there are at most \([k/\epsilon_k]-\)many \( B_i \) in which \( A \) is not closed.

Put \( Z = \{ i \in \Omega' : A \not\subseteq B_i \} = \{ i_1, \ldots, i_m \} \). We show that \( m \leq [k/\epsilon_k] \). For \( i \in Z \) since \( A \not\subseteq B_i \), there is \( C_i \) with \( A \subseteq C_i \subseteq B_i \) and \( \delta(C_i/A) < 0 \). Put
\[
C = \bigcup_{i \in Z} C_i.
\]
Then
\[
\delta(C/A) = \delta(C_{i_m}/AC_{i_1}, \ldots, C_{i_{m-1}}) + \ldots + \delta(C_{i_2}/AC_{i_1}) + \delta(C_{i_1}/A) \leq -m\epsilon_k.
\]
So \( 0 \leq \delta(C) \leq -m\epsilon_k + \delta(A) \leq -m\epsilon_k + k \). Therefore \( m \leq [k/\epsilon_k] \). \( \square \)

**Lemma 3.10.** (\( \alpha \) irrational) Assume that \( A \leq B \) and \( B \) is transcendental over \( A \). Then for each \( \epsilon > 0 \), there is a finite set \( D \) containing \( B \) and transcendental over \( A \) such that
1. \( -\epsilon < \delta(D/B) \leq 0 \),
2. \( \delta(B) \leq \delta(D') \) for all \( B \subseteq D' \not\subsetneq D \),
3. \( A \leq D \).

**Proof.** Without loss of generality we may assume that \( 0 < \epsilon < \alpha \) and \( \delta(B/A) > \epsilon \). Note that the set \( \{ m - \alpha n : m, n \in \mathbb{N} \} \) is dense in \( \mathbb{R} \) (by Dirichlet’s rational approximation theorem). Hence there are natural numbers \( 1 \leq s < k \) such that \( s/k < \alpha < (s + \epsilon)/k \).

By Lemma 2.10 and richness there is a set \( D = Bd_1, \ldots, d_k \) such that \( p(d_i) \) for all \( i \leq k \) and each \( s \)-element subset of \( \{ d_1, \ldots, d_k \} \) together with \( B \) is a base for \( D \). We claim such \( D \) satisfies the conditions required by the lemma.

For the first condition note that \( -\epsilon < \delta(D/B) = s - k\alpha < 0 \). For the second condition, let \( B \subseteq D' \subseteq D \) and \( |D' - B| = l \). If \( l \leq s \) then \( \delta(D'/B) = l - l\alpha > 0 \) because \( \alpha < 1 \). Also if \( s < l < k \) then \( \delta(D'/B) = s - l\alpha > s - \alpha k > 0 \). For the third condition, note that \( A \leq E \) for any proper subset \( E \) of \( D \) containing \( A \). It only remains to show that \( \delta(D/A) > 0 \). But this is also easy to see, since \( \delta(D/A) = \delta(D/B) + \delta(B/A) \) where \( \epsilon < \delta(B/A) \) and \( \delta(D/B) > -\epsilon \). Finally each \( d_i \) is transcendental over \( B \), and hence over \( A \). \( \square \)
Lemma 3.11. (α irrational) Assume that $A \leq B$ and $B$ is transcendental over $A$. Then for each $\mu > 0$ and natural number $n$ there is $D^*$ containing $B$ such that

1. $A \leq D^*$ and $D^*$ is transcendental over $A$,
2. $\delta(D^*/A) < \mu$,
3. $B \subseteq C$, for each $C$ with $B \subseteq C \subseteq D^*$ and $|C \setminus B| < n$.

Proof. Consider a real number $0 < \lambda < \min\{\delta(B/A), \frac{1}{n} \epsilon_n\}$, where $\epsilon_n$ is as in Lemma 3.8. By Lemma 3.10 there exists $D$ such that $-\lambda < \delta(D'/B) < 0$, $A \leq D$ and $D$ is transcendental over $A$. Let $\gamma = -\delta(D'/B)$, so $0 < \gamma < \lambda$. Also let $k \geq 1$ be the natural number such that

$$k\gamma \leq \delta(B/A) < (k+1)\gamma.$$ 

Let $D^* = D_1 \oplus_B \ldots \oplus_B D_k$ be obtained as the union of $k$ copies of $D$ (with the same colors) free from one another over $B$. We first show that $A \leq D^*$ (hence $D^* \in K^+_\alpha$) and $\delta(D^*/A) < \mu$.

To show that $A \leq D^*$, we need to show that $\delta(C/A) \geq 0$ for all $C$ with $A \subseteq C \subseteq D^*$. So let $B_0 = C \cap B$ and $C_i = C \cap D_i$ for each $i < k$. There are two cases to consider:

Case 1. $B_0 = B$.

By Lemma 3.10, $\delta(D'/B) > -\gamma$ for all $D'$ that $B \subseteq D' \subseteq D$. So $\delta(C_i/B) > -\gamma$ for each $i < k$. Therefore

$$\delta(C) = \delta(B) + \sum_{i<k} \delta(C_i/B) > \delta(B) - k\gamma \geq \delta(A).$$

Case 2. $B_0 \neq B$.

Since $A \leq B$, it follows that $\delta(B_0/A) \geq 0$. Furthermore each $C_i$ is proper subset of $D_i$ which means $\delta(C_i/B_0) \geq 0$ for all $i < k$ by Lemma 3.10. Hence

$$\delta(C/A) = \sum_{i<k} \delta(C_i/B_0) + \delta(B_0/A) \geq 0.$$ 

So in general $A \leq D^*$ and hence $D^* \in K^+_\alpha$.

Now it is easy to see that $\delta(D^*/A) < \mu$, because

$$\delta(D^*/A) = \delta(D^*/B) + \delta(B/A) \leq \delta(B/A) - k\gamma < \gamma < \lambda < \mu.$$
Finally we prove that item 3 in the statement of the lemma is fulfilled. Let $C$ be subset of $D^*$ containing $B$ with $|C \setminus B| < n$. Let $C_i = C \cap D_i$ for each $i < k$. Since $|C \setminus B| < n$ it follows that $|\{i; C_i \neq B\}| < n$. Hence

$$\delta(C) = \delta(B) + \sum_{i<k} \delta(C_i/B) \geq \delta(B) - n\lambda.$$ 

So $\delta(C/B) \geq -n\lambda \geq -\epsilon_n$. But Lemma 3.8 implies that $\delta(C/B) \geq 0$. \hfill \qed

**Lemma 3.12.** (\(\alpha\) rational) Assume that $\alpha = \frac{m}{n}$ with $m, n$ co-prime and $0 < m < n$. Let $A \subseteq B$, $B$ be transcendental over $A$, $\delta(B/A) > 0$ and $t$ be a natural number. Then there is $D$ containing $B$ such that

1. $A \subseteq D$ and $D$ is transcendental over $A$.
2. $(B, D)$ is a minimal pair, $|D \setminus B| > t$, and $\delta(D/B) = -\frac{1}{n}$.

**Proof.** Similar to the proof of Lemma 3.10 we introduce suitable $s < k$ and $D = Bd_1, \ldots, d_k$ such that $p(d_i)$ for all $i \leq k$, and each of the $s$-element subsets of $\{d_1, \ldots, d_k\}$ together with $B$ forms a basis for $D$.

Let $1 \leq k', s' < nt^{i+1}$ such that $mt^{i+1}k' - 1 = s'n^{i+1}$ witness the fact that $(mt^{i+1}, n^{i+1}) = 1$. Put $s'n^i = s$ and $k'm^i = k$. Then $\delta(D/B) = s - \frac{m}{n}k = -\frac{1}{n}$.

The other requirements on $D$ can be verified in a similar way to the proof of Lemma 3.10. \hfill \qed

**Lemma 3.13.** (\(\alpha\) rational) Under the assumptions of the previous lemma, there is a transcendental extension $D^*$ of $A$ such that $\delta(D^*/A) = 0$ and for each $D'$ containing $B$ with $|D' \setminus B| < t$ we have $B \subseteq D'$.

**Proof.** Suppose that $\delta(B/A) = \frac{p}{n}$. Similar to the proof of Lemma 3.11 we consider $D^*$ to be a disjoint free union of $p$-copies of $D$ over $A$ and show that $D^*$ has the desired properties. \hfill \qed

### 3.4 Completeness

We now turn to the theory $\mathbb{T}_\alpha$ and showing that it is complete.
3.4.1 $\alpha$ irrational

In this situation to prove the completeness of $T_\alpha$ we adopt the notion of semi-genericity as appears in \[4\].

Recall the notation from subsection 2.2 that for a set $A \in K_\alpha^+$ an embedding $f : A \to M$ is called $L_p$-embedding if $\text{qftp}_L(f(A)) = \text{qftp}_L(A)$ and for each $x \in A$, $x$ and $f(x)$ have the same color.

**Definition 3.14.** An $L_p$-structure $M \in K_\alpha^+$ is called semi-generic, if

1. $M \models T$, and

2. whenever $A \subseteq B$, $B$ is transcendental over $A$ and $f : A \to M$ is an $L_p$-embedding then for each natural number $n$ there exists an $L_p$-embedding $\hat{f}_n : B \to M$ extending $f$ and such that

$$\text{cl}_M^n(\hat{f}_n(B)) = \hat{f}_n(B) \oplus f(A) \text{cl}_M^n(f(A)). \quad (*)$$

The proof of the following lemma is inspired by [18], Proposition 4.4.

**Theorem 3.15.** Each $\omega_1$-saturated model of $T_\alpha$ is semi-generic.

**Proof.** We find it beneficial to provide a sketch of the proof beforehand. Assuming that $A, B$ are as in item 2 in Definition 3.14 we use the axioms and saturation to find infinitely-many mutually-free copies over $A$ of a set $D^*$ as in Lemma 3.11, whose pre-dimension is close enough to that of $A$. We will then show that there are only finitely many of these copies in which the corresponding image of $B$ violates $(*)$ in Definition 3.14 and therefore there is a copy with the desired property.

Now, assume that $M$ is an $\omega_1$-saturated model of $T_\alpha$, $A \subseteq B$, $B$ is transcendental over $A$ and $f : A \to M$ is an $L_p$-embedding. Let $n$ be a given natural number, $\epsilon_n$ be as in Lemma 3.8 and $0 < \epsilon < \epsilon_n$. By Lemma 3.11 there exists $D^* \in K_\alpha^+$ extending $B$ such that $A \subseteq D^*$, $D^*$ is transcendental over $A$ and $\delta(D^*/A) < \epsilon$. Moreover as in the mentioned lemma, $B \leq_i D^*$ and for every $B \subseteq C \subseteq D^*$ with $|C \setminus B| < n$ we have $B \leq C$. Now by applying Observation 2.8 to $X = f(A)$, $Y = \text{cl}(f(A))$, and $\Sigma(\bar{x}) = \{\phi(\bar{x}, f(\bar{a})) : \phi(\bar{x}, \bar{a}) \in \text{qftp}_L(D^*/A)\}$ regarding the fact that $D^*$ is transcendental over $A$, there exists an $L$-embedding $h : D^* \to M$ such that $h(D^*) \cap \text{cl}(f(A)) = f(A)$ and $h(D^*)$ and $\text{cl}(f(A))$ are free over $f(A)$.
Now for a finite subset \( A' \) of \( \text{cl}(f(A)) \) including \( f(A) \) we let \( D' = A' \oplus_{f(A)} h(D^*) \). We color the \( L \)-structure \( \langle D', h \rangle \) by letting \( h(D^*) \) have the same color as \( D^* \), while keeping the color of \( A' \) and leaving the rest non-colored. Denote the resulting \( L' \)-structure by \( N \). Then \( D' \in \mathcal{K}_A^+ \), since \( N \in \mathcal{K}_A^+ \). Also \( A' \leq D' \) and \( D' \) is transcendental over \( A' \).

Let \( \{ D'_i \} \), indexed by the natural numbers, be disjoint copies of \( D \) mutually free over \( A \). For each natural number \( m \) let \( D'_m = D'_1 \oplus_A \ldots \oplus_A D'_m \) be the free union of \( m \) disjoint copies of \( D' \) over \( A' \) (so \( D'_i \equiv_{A'} D' \)). Axioms of \( \mathbb{T}_\alpha \) (Axiom 3) and \( \omega_1 \)-saturation of \( M \) ensure that there exists a copy of \( D_m \) over \( A' \), since \( D'_m \) is transcendental over \( A' \). Thus for each \( m \) there are \( m \) copies of \( D' \) disjoint over \( f(A) \) satisfying \( \text{qftp}_L(h(D')/A') \). Hence by \( \omega_1 \)-saturation there are infinity many copies of \( D^* \) disjoint over \( f(A) \) satisfying \( \text{qftp}_L(h(D^*)/f(A)) \).

Let \( g_i : D' \rightarrow M \) be the embedding that represents the \( i \)'th copy of \( D' \) over \( f(A) \), \( B_i = g_i(B) \) and \( D_i^* = g_i(D') \). By the construction, \( D_i^* \downarrow_{f(A)} \text{cl}(f(A)) \) and consequently \( B_i \downarrow_{f(A)} \text{cl}(f(A)) \).

Let \( Z = \{ i \in \omega : \text{cl}^n(B_i) \not\subseteq \text{cl}(f(A)) \oplus_{f(A)} B_i \} \). Under the assumption that \( Z \) is finite, the rest of the proof goes as follows. Pick some \( i \notin Z \). We claim that \( \text{cl}^n(B_i) = \text{cl}^n(f(A)) \oplus_{f(A)} B_i \), that is the condition of semi-genericity is satisfied by \( \hat{f}_n = g_i \). Suppose that \( E \subseteq M \) such that \( B_i \leq E \) and \( |E - B_i| < n \). Then \( E \subseteq \text{cl}(f(A)) \oplus_{f(A)} B_i \). If \( E_1 = E \setminus B_i \), then \( E_1 \subseteq \text{cl}(f(A)) \) and \( E_1 \downarrow_{f(A)} B_i \). Therefore \( f(A) \leq i, f(A) \cup E_1 \) and \( E_1 \subseteq \text{cl}^n(f(A)) \). Hence \( E \subseteq \text{cl}^n(f(A)) \oplus_{f(A)} B_i \) and \( \text{cl}^n(B_i) = \text{cl}^n(f(A)) \oplus_{f(A)} B_i \).

Now it remains to show that the assumption that \( Z \) is infinite leads to a contradiction. For \( i \in Z \), denote by \( C_i \subseteq M \) an intrinsic extension of \( B_i \) not included in \( \text{cl}(f(A)) \oplus_{f(A)} B_i \) with \( |C_i \setminus B_i| < n \). Put \( H_i = D_i^* \cup C_i \) and choose a natural number \( s \) such that \( |H_i| < s \) for all \( i \in Z \). As every subset of \( D_i^* \) that includes \( B_i \) and has less than \( n \) more elements than \( B_i \) is closed in \( D_i^* \), we have \( C_i \not\subseteq D_i^* \).

By Lemma 3.9 there are an arbitrary large finite set \( \Omega \subseteq Z \) and a set \( F \) such that \( H_i \cap H_j = F \) and \( F \leq H_i \) for all \( i \neq j \in \Omega \). If \( A' = F \cap \text{cl}(f(A)) \) then \( A' \leq F \) and by transitivity \( A' \leq H_i \) and \( \delta(H_i/A') > 0 \).

On the other hand we have that \( C_i \setminus D_i^* \neq \emptyset \). So

\[
\delta(H_i/A') = \delta(D_i^* \cup C_i/A') = \delta(D_i^*/A') + \delta(C_i/D_i^*).
\]

Notice that \( \delta(C_i/D_i^*) < 0 \). Hence as we have \( |C_i \setminus D_i^*| < n \), it follows that
\[ \delta(C_i/D_i^+) < -\epsilon_n < -\epsilon. \] Furthermore as \( D_i^+ \downarrow_{f(A)}^{\dim} \text{cl}(f(A)) \), we have that 
\[ \delta(D_i^+ / A') = \delta(D_i^+ / A) < \epsilon. \] Hence \( \delta(H_i / A') < 0 \), a contradiction. \( \square \)

**Definition 3.16.** By \( X \cong Y \), read as \( X \) is weakly isomorphic to \( Y \), we mean that there is an \( \mathcal{L} \)-isomorphism \( f : \langle X \rangle \to \langle Y \rangle \) mapping \( X \) to \( Y \) and such that \( p(x) \leftrightarrow p(f(x)) \) for all \( x \in X \).

**Remark 3.17.** Notice that the above notion of isomorphism is weaker than \( \mathcal{L} \)-isomorphism, since the colours are preserved only for elements of \( X \). Weak isomorphism of \( X \) and \( Y \), as in the definition above, also implies that for every \( A \subseteq X \) and natural number \( i \), if \( \text{cl}_i^X(A) \subseteq X \) and \( \text{cl}_i^X(f(A)) \subseteq Y \) then \( f(\text{cl}_i^X(A)) = \text{cl}_i^Y(f(A)) \) and therefore \( \text{cl}_i^X(A) \cong \text{cl}_i^Y(f(A)) \).

**Theorem 3.18.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two \( \omega \)-saturated models of \( T_\alpha \) and \( \phi(\bar{x}) \) an \( \mathcal{L}_\rho \)-formula. Then there exists \( n = n_\varphi \) such that for all \( \bar{a}_1 \in \mathcal{M}_1^{[\bar{n}_\varphi]} \) and \( \bar{a}_2 \in \mathcal{M}_2^{[\bar{n}_\varphi]} \),

\[ \text{cl}_{\mathcal{M}_1}^n(\bar{a}_1)^{\mathcal{M}_1} \cong \text{cl}_{\mathcal{M}_2}^n(\bar{a}_2) \Rightarrow \left( \mathcal{M}_1 \models \varphi(\bar{a}_1) \iff \mathcal{M}_2 \models \varphi(\bar{a}_2) \right). \]

Note that the idea of the following proof is borrowed from [4], Lemma 1.30, but there are essential alterations that make the rewrite necessary.

**Proof.** Assume that \( \mathcal{M}_1, \mathcal{M}_2, \bar{a}_1 \in \mathcal{M}_1^{[\bar{n}_\varphi]} \) and \( \bar{a}_2 \in \mathcal{M}_2^{[\bar{n}_\varphi]} \) are as in the statement of the theorem. We proceed by induction on the complexity of \( \mathcal{L}_\rho \)-formulas. As it appears, the induction step for the \( \mathcal{L} \)-atomic formulas as well as the boolean connectives are straightforward. For an \( \mathcal{L}_\rho \)-atomic formula of the form \( \varphi := p(t(x_1, \ldots, x_n)) \) where \( t(x_1, \ldots, x_n) \) is an \( \mathcal{L} \)-term, we may take \( n_\varphi = 1 \). Then notice that \( \text{acl}_{\mathcal{M}_i}(\bar{a}_i) \cap p(\mathcal{M}_i) \subseteq \text{cl}_{\mathcal{M}_i}^1(\bar{a}_i) \) for \( i = 1, 2 \). Hence \( \text{cl}_{\mathcal{M}_1}^1(\bar{a}_1)^{\mathcal{M}_1} \cong \text{cl}_{\mathcal{M}_2}^1(\bar{a}_2) \) implies that \( \text{acl}_{\mathcal{M}_1}(\bar{a}_1)^{\mathcal{M}_1} \cong \text{acl}_{\mathcal{M}_2}(\bar{a}_2) \). Therefore \( t^{\mathcal{M}_1}(\bar{a}_1) \in p(\mathcal{M}_1) \) if and only if \( t^{\mathcal{M}_2}(\bar{a}_2) \in p(\mathcal{M}_2) \).

Now assuming that the statement holds for \( \psi(\bar{y}, \bar{x}) \) we need to introduce a natural number \( n_\varphi \) so that it holds also for the formula \( \varphi(\bar{x}) = \exists \bar{y} \psi(\bar{y}, \bar{x}) \).

By Remark 2.17 let the natural number \( p_1 \) be sufficiently large so that \( |\text{cl}_N(\bar{a})| < p_1 \) for all \( N \in \mathcal{K}_\alpha^+ \) and \( \bar{a}, d \in N \) with \( |\bar{a}| = |\bar{d}| \). Put \( p = \max\{p_1, n_\varphi\} \). For \( N \in \mathcal{K}_\alpha^+ \) and \( \bar{a} \in N^{[\bar{n}_\varphi]} \) and \( i < p \), we define inductively \( A_i^N(\bar{a}) \) by setting \( A_0^N(\bar{a}) = \bar{a} \) and \( A_{i+1}^N(\bar{a}) = \text{cl}_N(A_i^N(\bar{a})) \). Now let \( n_\varphi > 1 \) be such that \( A_i^N(\bar{a}) \subseteq \text{cl}_N(\bar{a}) \) for all \( N \in \mathcal{K}_\alpha^+ \) and \( \bar{a} \in N^{[\bar{n}_\varphi]} \). We will show that \( n_\varphi \) satisfies the requirement of the theorem.
Let $\bar{a}_1 \in M_1[\bar{x}]$ and $\bar{a}_2 \in M_2[\bar{x}]$ be such that $cl^{n,\psi}_{M_2}(\bar{a}_1) \sim w cl^{n,\psi}_{M_2}(\bar{a}_2)$. For $b_1 \in M_1$ we show that there is $b_2 \in M_2$ with $cl^{n,\psi}_{M_1}(\bar{a}_1 b_1) \sim w cl^{n,\psi}_{M_2}(\bar{a}_2 b_2)$.

Denote by $H_0$ the set with elements $\bar{a}_1 b_1$ and let $H_1 = cl^{n,\psi}_{M_1}(H_0)$. Since $cl^{n,\psi}_{M_1}(\bar{a}_1) \sim w cl^{n,\psi}_{M_2}(\bar{a}_2)$, for each $i \leq p$, $A_i^{M_1} := A_i^{M_1}(\bar{a}_1) \sim w A_i^{M_2} := A_i^{M_2}(\bar{a}_2)$.

Since $|H_1 \setminus H_0| < p$, there is $j < p$ such that

$$(A_{j+1}^{M_1} \setminus A_j^{M_1}) \cap (H_1 \setminus H_0) = \emptyset.$$  

Moreover the fact that $p > |H_1 \setminus A_j^{M_1}|$ implies $A_j^{M_1} \subseteq A_j^{M_1} H_1$.

Now by Remark 2.20 we may find $G_1 \subseteq H_1$ such that $A_j^{M_1} G_1$ is algebraic over $A_j^{M_1}$ and $A_j^{M_1} H_1$ is transcendental over $A_j^{M_1} G_1$. Note that since $n,\psi \geq 1$, we have $acl_{M_1}(\bar{a}_1) \sim w acl_{M_1}(\bar{a}_2)$. Hence there is $G_2 \subseteq M_2$ such that $A_j^{M_1} G_1 \sim w A_j^{M_2} G_2$. Furthermore, again by Remark 2.20

$$cl^{P}_{M_2}(A_j^{M_2} G_2) = cl^{P}_{M_2}(A_j^{M_2}) \oplus_{A_j^{M_2}} A_j^{M_2} G_2. \quad (*)$$

Let $g : A_j^{M_1} G_1 \to M_2$ be the $L_p$-embedding witnessing $A_j^{M_1} G_1 \sim w A_j^{M_2} G_2$.

Now since $A_j^{M_1} H_1$ is transcendental over $A_j^{M_1} G_1$, by semi-genericity of $M_2$, we find an $L_p$-embedding $\hat{g} : A_j^{M_1} H_1 \to M_2$ extending $g$ such that

$$cl^{P}_{M_2}(A_j^{M_2} \hat{g}(H_1)) = cl^{P}_{M_2}(A_j^{M_2} G_2) \oplus_{A_j^{M_2} G_2} A_j^{M_2} \hat{g}(H_1).$$

Now by $(*)$ we have

$$cl^{P}_{M_2}(A_j^{M_2} \hat{g}(H_1)) = cl^{P}_{M_2}(A_j^{M_2}) \oplus_{A_j^{M_2}} A_j^{M_2} \hat{g}(H_1),$$

that is

$$cl^{P}_{M_2}(A_j^{M_2} \hat{g}(H_1)) = A_{j+1}^{M_2} \oplus_{A_j^{M_2}} A_j^{M_2} \hat{g}(H_1).$$

Now letting $b_2 = \hat{g}(b_1)$, we will show that $cl^{n,\psi}_{M_2}(\bar{a}_1 b_1) \sim w cl^{n,\psi}_{M_2}(\bar{a}_2 b_2)$.

By the definition of $\hat{g}$ we have $A_j^{M_2} \hat{g}(H_1) \sim w A_j^{M_1} H_1$ and $cl^{n,\psi}_{M_1}(\bar{a}_1, b_1) = H_1 \subseteq A_j^{M_1} H_1$. Note that

$$cl^{n,\psi}_{M_2}(\bar{a}_2, b_2) \subseteq cl^{P}_{M_2}(\bar{a}_2, b_2) \subseteq cl^{P}_{M_2}(A_j^{M_2} \hat{g}(H_1)) = A_{j+1}^{M_2} \hat{g}(H_1).$$
Moreover, since $A_{j+1}^{M_2}$ and $\hat{g}(H_1)$ are free over $A_j^{M_2}$, we have $cl^n_\psi^{M_2}(\bar{a}_2, b_2) \subseteq A_j^{M_2}\hat{g}(H_1)$. So it follows that $cl^n_\psi^{M_1}(\bar{a}_1, b_1) = cl^n_\psi^{(A_j^{M_1}H_1)}(\bar{a}_1, b_1)$ and $cl^n_\psi^{M_2}(\bar{a}_2, b_2) = cl^n_\psi^{(A_j^{M_2}\hat{g}(H_1))}(\bar{a}_2, b_2)$. On the other hand by Remark 3.17 we have that

$$cl^n_\psi^{(A_j^{M_1}H_1)}(\bar{a}_1, b_1) \overset{w}{=} cl^n_\psi^{(A_j^{M_2}\hat{g}(H_1))}(\bar{a}_2, b_2).$$

Therefore $cl^n_\psi^{M_1}(\bar{a}_1, b_1) \overset{w}{=} cl^n_\psi^{M_2}(\bar{a}_2, b_2).$ 

The completeness of $T_\alpha$ is finally established as follows.

**Theorem 3.19.** $T_\alpha$ is complete.

**Proof.** If $M_1$ and $M_2$ are two $\omega_1$-saturated models of $T_\alpha$ then by Theorem 3.18 $M_1 \equiv M_2$. Therefore $T_\alpha$ is complete. 

**Corollary 3.20.** Let $C$ be a monster model of $T_\alpha$.

1. Assume that $\bar{a}_1, \bar{a}_2$ in $C$ are small tuples and $X$ is a closed small subset of $C$. Then,

$$tp(\bar{a}_1/X) = tp(\bar{a}_2/X) \iff \langle cl(X\bar{a}_1) \rangle \cong \langle X \rangle \langle cl(X\bar{a}_2) \rangle$$

2. $C$ is $\lambda$-rich, for all $\lambda < |C|$.

### 3.4.2 $\alpha$ rational

When $\alpha$ is rational, we show the completeness of $T_\alpha$ by proving that any $\omega$-saturated model of $T_\alpha$ is $\omega$-rich and that any two $\omega$-rich structures of $K^+_\alpha$ are back and forth equivalent.

**Theorem 3.21.** Suppose that $M$ is an $\omega$-saturated model of $T_\alpha$ then $M$ is $\omega$-rich.

**Proof.** Let $M \subseteq N$ be two finitely generated structures in $K^+_\alpha$ and $f : M \rightarrow M$ be a strong $L_p$-embedding. We prove that there is a strong $L_p$-embedding $g : N \rightarrow M$ extending $f$.

Since $M, N$ are finitely generated, there is $A \subseteq \text{fin} M$ and $B \subseteq \text{fin} N$ such that $M = \langle A \rangle$, $N = \langle B \rangle$ and $A \leq B \leq N$. By Remark 2.20 we have two specific cases to consider.
Case 1. $B$ is algebraic over $A$.
Since $A \leq N \in K^+_\alpha$, any $x \in B \setminus A$ is non-colored. So by Remark 2.20 it is clear that there is a strong $L_p$-embedding $g : B \to M$ extending $f$.

Case 2. $B$ is transcendental over $A$.
There are two subcases to consider:

Case 2.1 $\delta(B/A) > 0$. By Lemma 3.13 there is $D \in K^+_\alpha$ extending $B$ with $A \leq D$, $D$ transcendental over $A$ and $\delta(D/A) = 0$. Using the same method as in the proof of Theorem 3.15 for each natural number $n$ we can find an $L_p$-embedding $g : D \to M$ extending $f$ such that $g(B)$ is $n$-strong, i.e. for each $C \subseteq M$ containing $g(B)$ with $|C \setminus g(B)| < n$ we have $B \leq C$.
Hence by $\omega$-saturation there is indeed a strong $L_p$-embedding $g : B \to M$ extending $f$.

Case 2.2. $\delta(B/A) = 0$. This can be dealt with as Case 2.1 by taking $D$ above in the place of $B$.

Now by Remark 2.20 there is $B_1$ with $A \leq B_1 \leq B$ such that $B_1$ is algebraic over $A$ and $B$ is transcendental over $B_1$. By Case 1 there is an $L_p$-embedding $g_1 : B_1 \to M$ extending $f$ and by Case 2, there is an $L_p$-embedding $g : B \to M$ extending $g_1$.

It is easy to see that $\omega$-rich structures are back and forth equivalent and hence so are any two $\omega$-saturated models of $T_\alpha$. This yields the following sought-for theorem.

Theorem 3.22. $T_\alpha$ is complete.

Remark 3.23. Corollary 3.20 also holds for rational $\alpha$.

Remark 3.24. If $T$ is an $o$-minimal expansion of a densely ordered group and $\alpha = 1$ then $T_\alpha$ happens to be the theory $T^\text{indep}$ as introduced in [12].

4 Dependence and Strong Dependence

In this section, we will prove that if $T$ is dependent, that is when it has Notethe-Independence Property, then so is $T_\alpha$. We will show that $T_\alpha$ inherits the strong dependence from $T$ as well, when $\alpha$ is rational. Moreover we will show that when $T$ defines a linear order, $T_\alpha$ is not strongly dependent.

Recall that an $L'$-theory $T'$ is called dependent if each $L'$-formulas $\varphi(\bar{x}, \bar{y})$ is dependent, that is there are no model $M' \models T'$, and sequences $\{\bar{a}_i : i \in \omega\}$
and \{\bar{b}_J : J \subseteq \omega\} such that \( M \models \varphi(\bar{a}_i, \bar{b}_J) \) if and only if \( i \in J \). It is known that in this definition suffices it to assume \(|x| = 1\).

From now on we add the further assumption that \( T \) is dependent, and let \( \mathfrak{C} \) be a monster model of \( T_\alpha \). In this section we use boldface letters \( \mathbb{M}, \mathbb{N} \) for models of \( T_\alpha \).

By standard facts on dependent theories, to show that \( T_\alpha \) is dependent, we verify that for a given model \( M \) of \( T_\alpha \), the number of coheir extensions of any 1-type \( p(x) \) over \( M \) is at most \( 2^{|M|} \). The following notion of \( D \)-independence would be of great help ([23]) for our counting of the coheirs.

**Definition 4.1.** Let \( \mathbb{M} \) be an arbitrary model of \( T_\alpha \), \( A, B \subseteq_{\text{fin}} \mathbb{M} \) and \( X \subseteq \mathbb{M} \). Define

1. \( D(A) = \inf \{\delta(C) : A \subseteq C \subseteq_{\text{fin}} \mathbb{M}\} \),
2. \( D(B/A) = D(BA) - D(A) \),
3. \( \text{CL}(A) = \{x \in \mathbb{M} : D(xA) = D(A)\} \),
4. \( D(A/X) = \inf \{D(A/X_0) : X_0 \subseteq_{\text{fin}} X\} \).

Note that the function \( D \) is automorphism invariant, i.e. for any automorphism \( f : \mathfrak{C} \to \mathfrak{C} \) we have \( D(A/X) = D(f(A)/f(X)) \) for any \( A, X \subseteq \mathfrak{C} \).

**Fact 4.2.** When \( \alpha \) is rational it can be easily seen that \( D(A) = \delta(\text{cl}(A)) \) and \( D(B/A) = \delta(\text{cl}(BA)) - \delta(\text{cl}(A)) \). Therefore the set containing all positive values \( D(B/A) \) is discrete, and it does not contain any infinite decreasing sequence.

**Definition 4.3.** Let \( \mathbb{M} \models T_\alpha \), \( A, B \subseteq_{\text{fin}} \mathbb{M} \) and \( Z \subseteq \mathbb{M} \). We say that \( A, B \) are \( D \)-independent over \( Z \) and write \( A \downarrow^D_Z B \) whenever \( D(A/Z) = D(A/ZB) \) and \( \text{cl}(AZ) \cap \text{cl}(BZ) = \text{cl}(Z) \). Moreover for arbitrary subsets \( X, Y \) of \( \mathbb{M} \) we say that \( X, Y \) are \( D \)-independent over \( Z \) and write \( X \downarrow^D_Z Y \) if \( C \downarrow^D_Z E \) for any \( C \subseteq_{\text{fin}} X \) and \( E \subseteq_{\text{fin}} Y \). In this case we also say that \( \text{tp}(X/YZ) \) is a \( D \)-independent extension of \( \text{tp}(X/Z) \).

The following facts, and the technique to prove the following lemma, are available in [3], Section 3.

**Fact 4.4.** The relation \( \downarrow^D \) has the following properties.
1. **D-symmetry.** If \( A \downarrow_B^D C \), then \( C \downarrow_B^D A \).

2. **D-transitivity.** \( A \downarrow_B^D C \) and \( A \downarrow_B^{DCE} \) if and only if \( A \downarrow_B^D CE \).

3. **D-local character.** For any \( A \) and \( X \) there exists a countable set \( X_0 \subseteq X \) such that \( A \downarrow^D_{X_0} X \). In the case of a rational \( \alpha \), \( X_0 \) is finite.

4. **D-existence.** For all sets \( X, Y, Z \) with \( Y \subseteq Z \) there exists \( X' \) such that \( \text{tp}(X/Y) = \text{tp}(X'/Y) \) and \( X' \downarrow_D Z \).

**Fact 4.5.** By D-existence, it is possible to find a D-Morley sequence for a given type \( p(\bar{x}) \in S_{\leq |M|}(M) \); that is a sequence \( (\bar{b}_i)_{i<\omega} \) of realizations of \( p(\bar{x}) \) such that \( \bar{b}_0, \ldots, \bar{b}_n \downarrow_D^{M} \bar{b}_{n+1} \) for each \( n < \omega \).

**Lemma 4.6.** Let \( Z \subseteq M \) and \( X, Y \subseteq M \). Then \( X \downarrow_D^Z Y \) if and only if

1. \( \text{cl}(XZ) \cap \text{cl}(YZ) = Z \),
2. \( \text{cl}(XY) = \text{cl}(XZ) \cup \text{cl}(YZ) \),
3. \( \text{cl}(XZ) \downarrow^\text{dim}_Z \text{cl}(YZ) \).

The above three conditions imply that \( \text{cl}(XY) = \text{cl}(XZ) \oplus_Z \text{cl}(YZ) \).

**Lemma 4.7.** Suppose that \( q(\bar{x}) \in S(C) \) is a coheir extension of \( p(\bar{x}) \in S(M) \). Then \( q(\bar{x}) \) is a D-independent extension of \( p(\bar{x}) \).

**Proof.** For \( \bar{a} \models q(\bar{x}) \), we need to show that \( \bar{a} \downarrow_M^D C \). Take a small set \( N \) with \( M \leq N \leq C \) and \( D(\bar{a}/N) = D(\bar{a}/C) \). By D-existence and D-transitivity (Fact 4.4) we can also find \( N' \) such that \( N' \downarrow_M^D N \) and \( N' \equiv_M N \).

**Claim.** \( D(\bar{a}/N') = D(\bar{a}/N) \).

**Proof of claim.** Otherwise, assume that \( D(\bar{a}/N') = \gamma > D(\bar{a}/N) \). So we can find \( \bar{b} \subseteq \text{fin} \text{ cl}(\bar{a}N) \) and \( \bar{e} \subseteq \text{fin} \text{ N} \) such that \( \delta(\bar{b}/\bar{e}) < \gamma \). By Fact 2.1 there is \( \varphi(\bar{z}, \bar{x}, \bar{e}) \in \text{tp}(\bar{a}N/\bar{e}) \) such that for any \( \bar{b} \) and \( \bar{e} \) if \( C \models \varphi(\bar{b}, \bar{a}, \bar{e}) \) then \( \bar{b} \subseteq \text{cl}(\bar{a}\bar{e}') \) and \( \delta(\bar{b}/\bar{a}\bar{e}') < \gamma \). So the formula \( \exists \bar{z} \varphi(\bar{z}, \bar{x}, \bar{e}) \) is in \( \text{tp}(\bar{a}/C) \). But since \( \text{tp}_C(\bar{a}/C) \) is a coheir extension of \( \text{tp}_C(\bar{a}/M) \), it is also \( M \)-invariant. So there is \( \bar{e}' \subseteq N' \) such that \( \exists \bar{z} \varphi(\bar{z}, \bar{x}, \bar{e}') \) is in \( \text{tp}_C(\bar{a}/C) \). Therefore \( \delta(\bar{b}/\bar{a}\bar{e}') < \gamma \) for some \( \bar{b} \subseteq \text{cl}(\bar{a}N) \). This implies that \( D(\bar{a}/N') < \gamma \), a contradiction. \( \square \)
Writing the above claim as $\bar{a} \downarrow^D_{\mathbb{N}} \mathbb{N}$ together with our assumption that $\mathbb{N}' \downarrow^D_{\mathbb{M}} \mathbb{N}$ and $D$-transitivity gives $\mathbb{N}' \bar{a} \downarrow^D_{\mathbb{M}} \mathbb{N}$. Therefore again by $D$-transitivity we have $\bar{a} \downarrow^D_{\mathbb{M}} \mathbb{N}$, and hence $\bar{a} \downarrow^D_{\mathbb{M}} \mathfrak{C}$ because $D(\bar{a}/\mathfrak{C}) = D(\bar{a}/\mathbb{N})$.

Finally we have to show that $\text{cl}(\bar{a}\mathbb{M}) \cap \mathfrak{C} = \mathbb{M}$. Let $e \in \text{cl}(\bar{a}\mathbb{M}) \cap \mathfrak{C}$. By Remark 2.17 there exists an algebraic $L_p$-formula $\varphi(\bar{x}, e, \bar{m}) \in q(\bar{x})$ witnessing $e \in \text{cl}(\bar{a}\mathbb{M}) \cap \mathfrak{C}$. If $tp(e/\mathbb{M})$ has infinitely many realizations $(e_i)_{i<\omega}$ in $\mathfrak{C}$ then as $q(\bar{x})$ is a coheir extension of $p(\bar{x})$, the formula $\varphi(\bar{a}, e_i, \bar{m})$ holds for each $i < \omega$, which contradicts the fact that $e \in \text{acl}(\bar{a}\mathbb{M})$. Therefore $tp(e/\mathbb{M})$ is algebraic and hence $e \in \mathbb{M}$. 

Recall that coheir extensions are in particular non-forking. In the following proposition using a similar idea as in [14] Lemma 4.3, we address the fact that when $\alpha$ is rational, the above lemma holds even for non-dividing extensions. However we would not require this in the rest.

**Proposition 4.8.** (\(\alpha\) rational) Assume that $\mathbb{M}, \mathbb{N}$ are models of $T_\alpha$ such that $\mathbb{M} \subseteq \mathbb{N} \subseteq \mathfrak{C}$. If $tp(\bar{a}/\mathbb{N})$ is a non-dividing extension of $tp(\bar{a}/\mathbb{M})$, then $\bar{a} \downarrow^D_{\mathbb{M}} \mathbb{N}$.

**Proof.** We first show that $D(\bar{a}/\mathbb{M}) = D(\bar{a}/\mathbb{N})$. Assume on the contrary that $D(\bar{a}/\mathbb{M}) > D(\bar{a}/\mathbb{N})$. So there is $\bar{b} \subseteq \mathbb{N} \setminus \mathbb{M}$ such that $D(\bar{a}/\mathbb{M}) > D(\bar{a}/\mathbb{M}\bar{b})$. Assume that $(\bar{b}_i)_{i<\omega}$ is a Morley sequence of $tp(\bar{b}/\mathbb{M})$ with respect to $\downarrow^D_{\mathbb{M}}$, as in Fact 4.5. Clearly $D(\bar{a}/\mathbb{M}) > D(\bar{a}/\mathbb{M}\bar{b}_i)$ for each $i < \omega$. Now by the $D$-symmetry $D(\bar{b}_i/\mathbb{M}) > D(\bar{b}_i/\mathbb{M}\bar{a})$. On the other hand,

$$D(\bar{b}_i/\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1}) \triangleright D(\bar{b}_i/\mathbb{M}\bar{a}\bar{b}_0, \ldots, \bar{b}_{i-1}). \quad (1)$$

Now if equality occurred in (1) then we would have

$$D(\bar{b}_i/\mathbb{M}) = D(\bar{b}_i/\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1})$$

$$= D(\bar{b}_i/\mathbb{M}\bar{a}\bar{b}_0, \ldots, \bar{b}_{i-1})$$

$$\leq D(\bar{b}_i/\mathbb{M}\bar{a})$$

which is a contradiction. So we have

$$D(\bar{b}_i/\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1}) \triangleright D(\bar{b}_i/\mathbb{M}\bar{a}\bar{b}_0, \ldots, \bar{b}_{i-1}).$$

Writing the above as $\bar{b}_i \triangleright^D_{\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1}} \bar{a}$, again by the $D$-symmetry we have $\bar{a} \triangleright^D_{\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1}} \bar{b}_i$. That is

$$D(\bar{a}/\mathbb{M}\bar{b}_0, \ldots, \bar{b}_{i-1}) \triangleright D(\bar{a}/\mathbb{M}\bar{b}_0, \ldots, \bar{b}_i),$$

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for each \( i < \omega \), which is a contradiction, because in the case of a rational \( \alpha \) we cannot have a decreasing sequence of \( D \)-dimensions (see Fact 4.2).

Now the fact that \( \text{cl}(\bar{a}M) \cap N = M \) follows from a similar argument to the proof of Lemma 4.7.

We now get to the point to prove that the dependence is transferred from \( T \) to \( T_\alpha \).

**Theorem 4.9.** If \( T \) is dependent, then so is \( T_\alpha \).

**Proof.** Let \( M \) and \( N \) be models of \( T_\alpha \) such that \( M \subseteq N \subseteq \mathcal{C} \). Assume that \( q_1(x), q_2(x) \in S_1(N) \) are two coheir extensions of a type \( p(x) \in S_1(M) \). If \( a_1 \models q_1(x) \) and \( a_2 \models q_2(x) \) then by Corollary 3.20 there is an \( \mathcal{L}_p \)-isomorphism between \( \text{cl}(a_1M) \) and \( \text{cl}(a_2M) \) mapping \( a_1 \) to \( a_2 \) and fixing \( M \) pointwise, and therefore \( tp(\text{cl}(a_1M)/M) = tp(\text{cl}(a_2M)/M) \). Since \( \text{cl}(a_iM) \) is contained in the \( \mathcal{L}_p \)-algebraic closure of \( a_iM \), it follows that \( tp(\text{cl}(a_iM)/N) \) is a coheir of \( \text{cl}(a_1M)/M) \). Thus by Lemma 4.6 and Lemma 4.7 we have \( \text{cl}(a_iN) = \text{cl}(a_iM) \oplus_M N \). This means that \( q_1(x) = q_2(x) \) if and only if there is an \( \mathcal{L} \)-isomorphism between \( \text{cl}(a_1M) \) and \( \text{cl}(a_2M) \) mapping \( a_1 \) to \( a_2 \) and fixing \( N \) pointwise. In other words, \( q_1(x) = q_2(x) \) if and only if \( \text{tp}_L(\text{cl}(a_1M)/N) = \text{tp}_L(\text{cl}(a_2M)/N) \).

Now since \( T \) is dependent, there are at most \( 2^{[M]} \) \( \mathcal{L} \)-coheirs of \( \text{tp}_L(\text{cl}(a_1M)/M) \) over \( N \). On the other hand by the above argument any coheir extension of \( p(x) \) is uniquely determined by \( \text{tp}_L(\text{cl}(dM)/N) \) for any realization \( a \) of \( p(x) \). Therefore there are at most \( 2^{[M]} \) coheirs of \( p(x) \). \( \square \)

By the proof of the theorem above, the number of coheirs is the same in \( T \) and \( T_\alpha \). This gives also the following corollary on the stability.

**Corollary 4.10.** If \( T \) is stable, then so is \( T_\alpha \).

**Proof.** If \( T \) is stable, then for any \( M \subseteq N \models T_\alpha \) every 1-type over \( M \) has a unique coheir extension to \( N \), and hence \( T_\alpha \) is also stable. \( \square \)

Our next result is that when \( \alpha \) is rational, the strong dependence also transfers from \( T \) to \( T_\alpha \). Before stating the result, let us recall some basic definitions concerning the strong dependence.

Let \( T' \) be an \( \mathcal{L} \)-theory and \( X \) an index set. Let \( I_t = \{ \bar{a}_{ti} : i < \omega \} \), for \( t \in X \), be sequences of tuples in a monster model of \( T' \) such that \( |\bar{a}_{ti}| = |\bar{a}_{sj}| \), for each \( i, j \in \omega \) and \( t, s \in X \). For a small set of parameters \( A \), we say that
sequences \( \{I_t : t \in X\} \) are \textit{mutually indiscernible} over \( A \) if each sequence \( I_t \) is indiscernible over \( A \cup \bigcup \{I_s : s \neq t\} \).

Let \( \Sigma(\bar{x}) \) be a partial type over \( A \) and \( \kappa \) be a (finite or infinite) cardinal. We say that \( \text{dp-rk}(\Sigma, A) < \kappa \) if for each family \( \{I_t : t < \kappa\} \) of mutually indiscernible sequences over \( A \) and for any realization \( \bar{b} \) of \( \Sigma \), there is \( t < \kappa \) such that \( I_t \) is indiscernible over \( A \bar{b} \).

A dependent theory \( T' \) is \textit{strongly dependent} if for \( \bar{x} \) of any length, \( \text{dp-rk}(\bar{x} = \bar{x}, \emptyset) < \aleph_0 \) (for an account on strong-dependence see [23]).

The following fact states some equivalent forms of strong dependence which are use for Theorem 4.13.

\textbf{Fact 4.11 (\cite{12,22})}. The following are equivalent for a complete theory \( T' \).

1. \( T' \) is strongly dependent.

2. For every indiscernible sequence \( I = \{\bar{b}_i : i < \kappa\} \) with tuples \( \bar{b}_i \) at most countable, and every finite set \( C \), there is a convex equivalence relation \( \sim \) on \( \kappa \) with finitely many classes and such that \( \text{tp}(\bar{b}_i/C) \) depends only on the \( \sim \)-class of \( i \). In other words, the set \( \{\text{tp}(\bar{b}_i/C) : i \in \kappa\} \) is finite and for each \( i \in \kappa \) the set \( \kappa(i) = \{j : \text{tp}(\bar{b}_j/C) = \text{tp}(\bar{b}_i/C)\} \) is a convex subset of \( \kappa \).

3. For every indiscernible sequence \( \{\bar{b}_i : i \in I\} \) with tuples \( \bar{b}_i \) at most countable, and every finite set \( C \), there is a convex equivalence relation \( \sim \) on \( I \) with finitely many classes and such that \( \{\bar{b}_i : i \in j/\sim\} \) is indiscernible over \( C \). In other words, in the above item the sequence \( I_i = \{\bar{b}_j : j \in \kappa(i)\} \) is indiscernible over \( C \).

\textbf{Lemma 4.12}. Let \( I = \{\bar{a}_i : i < \kappa\} \) be an \( L_p \)-indiscernible sequence over \( A \). Then

1. \( I \) is \( L_p \)-indiscernible also over \( \text{cl}(A) \).

2. There is a suitable enumeration \( \bar{b}_i = \bar{a}_i \bar{c}_i \) of each \( \langle \text{cl}(A\bar{a}_i) \rangle \) such that \( I' = \{\bar{b}_i : i < \kappa\} \) is an \( L_p \)-indiscernible sequence over \( A \).

\textit{Proof.} 1. Let \( \{\bar{c}_i : i < \kappa\} \) be an \( L_p \)-indiscernible sequence over \( \text{cl}(A) \) that realizes the \( EM \)-type of \( I \). This sequence has the same type over \( A \) as the type of \( I \), and therefore, there is an \( L_p \)-automorphism \( \sigma : \mathfrak{C} \to \mathfrak{C} \) fixing \( A \) pointwise such that \( \sigma(\bar{c}_i) = \bar{a}_i \), for each \( i < \kappa \). So, \( \{\bar{a}_i : i < \kappa\} \) is \( L_p \)-indiscernible over \( \sigma(\text{cl}(A)) = \text{cl}(A) \).
2. As \( \langle \text{cl}(A\bar{a}_i) \rangle \subseteq \text{dcl}(\text{cl}(A\bar{a}_i)) \), it is enough to prove 2 for the sequence \( I' = \{ \text{cl}(A\bar{a}_i) : i < \kappa \} \). Since \( \bar{a}_i \) and \( \bar{a}_0 \) have the same type over \( A \), enumerate each \( \text{cl}(A\bar{a}_i) \) in accordance with \( \text{cl}(A\bar{a}_0) \) using the \( L_p \)-automorphism \( \sigma_i \) that fixes \( A \) pointwise and sends \( \bar{a}_0 \) to \( \bar{a}_i \). Denote this enumeration by \( \bar{d}_i \) and put \( J = \{ \bar{d}_i : i < \kappa \} \). By standard application of Ramsey’s theorem, let \( I' = \{ \bar{b}_i : i < \kappa \} \) be the \( A \)-indiscernible sequence satisfying the \( EM \)-type of \( J \) over \( A \). Since each \( \bar{b}_i \) contains an automorphic image of \( \bar{a}_i \), by strong homogeneity of \( \mathfrak{C} \), we may assume that each \( \bar{b}_i \) includes \( \bar{a}_i \).

We now need to show that \( \bar{b}_i \) is indeed an enumeration of \( \text{cl}(A\bar{a}_i) \). Recall that \( \bar{d}_i \) is already an enumeration of \( \text{cl}(A\bar{a}_i) \). Let \( A_0 \) be a finite subset of \( A \) and \( (s_1, \ldots, s_l) \) be a finite subtuple of \( \bar{a}_0 \). Then \( \text{cl}^n(A_0 s_1, \ldots, s_l) = \{e_1, \ldots, e_k\} \) is a finite subtuple of \( \bar{d}_0 \). Thus, \( |\text{cl}^n(A_0, \sigma_1(s_1), \ldots, \sigma_l(s_l))| = k \), for each \( i < \kappa \).

Now the following two facts are covered by the \( EM \)-type of \( J \). First that for each \( i < \kappa \), \( \bar{b}_i \) includes \( \text{cl}(A\bar{a}_i) \). Second, that for any finite subset \( A_0 \) of \( A \) and finite subtuple \( (t_1, \ldots, t_l) \) of \( \bar{a}_i \), there is a finite subtuple \( (f_1, \ldots, f_k) \) of \( \bar{b}_i \) such that \( \text{cl}^n(A_0 t_1, \ldots, t_l) = \{f_1, \ldots, f_k\} \). Hence \( \bar{b}_i \) is an enumeration of \( \text{cl}(A\bar{a}_i) \), as required.

**Theorem 4.13.** (\( \alpha \) rational) If \( T \) is strongly dependent, then so is \( T_\alpha \).

**Proof.** Let \( I = \{ \bar{b}_i : i < \kappa \} \) be an indiscernible sequence and \( C \) a finite set, as in Fact 4.11. By \( D \)-local character and since \( \alpha \) is rational, there exists a finite set \( H \subseteq I \) such that \( C \downarrow_H I \), and in particular \( C \downarrow_H \bar{b}_i \) for each \( i \).

So by Lemma 4.10 for every \( i < \kappa \), \( \langle \text{cl}(\bar{b}_iCH) \rangle = \langle \text{cl}(\bar{b}_iH) \rangle \oplus_{\text{cl}(H)} \langle \text{cl}(CH) \rangle \).

Hence by Corollary 3.20 \( \text{tp}(\bar{b}_i/\text{cl}(CH)) \) depends only on \( \text{tp}(\bar{b}_i/\text{cl}(H)) \) and \( \text{tp}_L(\langle \text{cl}(\bar{b}_iH)/\text{cl}(HC) \rangle) \). We will find a convex partition of \( \kappa \) in each part of which these two are fixed, and the statement follows from Fact 4.11

Since \( H \) is a finite subset of \( I \) and \( I \) is \( L_p \)-indiscernible, there is a finite partition \( J_1, \ldots, J_k \) of \( \kappa \) into convex sets such that each \( I_j = \{ \bar{b}_i : i \in J_j \} \) is indiscernible over \( H \). So in particular \( \text{tp}(\bar{b}_i/H) \)-determined by \( \langle \text{cl}(\bar{b}_iH) \rangle \) is fixed for each \( i \in J_j \) and \( 1 \leq j \leq k \). We may assume that each \( J_j \) is infinite, as otherwise, the set \( I_j \) can be added to \( H \).

Since \( I_j \) is indiscernible over \( H \), by Lemma 4.12 it is also indiscernible over \( \text{cl}(H) \). Moreover again by Lemma 4.12 we may assume that \( I'_j = \{ \langle \text{cl}(\bar{b}_iH) \rangle : i \in J_j \} \) is indiscernible over \( H \).

Now since \( T \) is strongly dependent, by Fact 4.11 there exists a convex equivalence relation \( \sim_j \) over \( J_j \) with finitely many equivalence classes such that \( \{ \langle \text{cl}(\bar{b}_iH) \rangle : i \in \alpha / \sim_j \} \) is \( L \)-indiscernible over \( \text{cl}(CH) \). So, in particular, \( \text{tp}_L(\langle \text{cl}(\bar{b}_iH)/\text{cl}(CH) \rangle) \) is fixed on any equivalence class of \( \sim_j \). Hence
\( \sim = \bigcup_{j=1}^{k} \sim_j \) forms a convex equivalence relation on \( \kappa \) with finitely many equivalence classes such that \( \text{tp}(b_i / \text{cl}(HC)) \) (and hence \( \text{tp}(b_i / C) \)) depends only on the \( \sim \)-class of \( i \).

We will show in the following that in the particular case that \( T \) defines a linear order, \( T_\alpha \) is not strongly dependent for any irrational \( \alpha \).

**Theorem 4.14.** Assume that \( T \) defines a linear order and \( \alpha \) is irrational. Then \( T_\alpha \) is not strongly dependent.

**Proof.** Let \( <^\mathcal{E} \) be the interpretation of the order defined by \( T \). We construct \( \mathcal{L}_\mathcal{P} \)-mutually indiscernible sequences \( \{I_i : i < \omega\} \) and find \( b_0 \in \mathcal{E} \) such that \( I_i = \{a_{ij} : j < \omega\} \) is not \( \mathcal{L}_\mathcal{P} \)-indiscernible over \( b_0 \) for each \( i < \omega \).

Let \( J = \{c_{ij} : (i, j) \in \omega \times \omega\} \) be an \( \mathcal{L} \)-indiscernible sequence over \( \emptyset \), ordered by \( \omega \times \omega \) with horizontal lexicographic order. For convenience, we may assume the sequence \( J \) is increasing with respect to \( <^\mathcal{E} \). We now inductively define a sequence \((D_n, E_n, F_n)_{n<\omega} \) of subsets of \( \mathcal{E} \) and a sequence \((s_{n+1}, k_{n+1})_{n<\omega} \) of tuples of natural numbers which, in particular, have the following properties:

1. \( E_n \subseteq J_n = \{c_{nj} : j \in \omega\} \) and \( |E_{n+1}| = s_{n+1} \).
2. \( |F_{n+1}| = k_{n+1} - s_{n+1} \) and \( F_{n+1} \subseteq \text{acl}(D_n E_{n+1}) \). Furthermore each \( s_{n+1} \)-element subset of \( E_{n+1} F_{n+1} \) is a basis over \( D_n \).
3. \( D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \). \( D_n \subseteq \text{acl}(\bigcup_{i=0}^{n} E_i) \) and \( D_n \cap J = \bigcup_{i=0}^{n} E_i \).

For \( n = 0 \), let \( E_0 = D_0 = \{c_{01}\} \) and \( F_0 = \emptyset \). Suppose by induction that the sets \((D_n, E_n, F_n)\) are constructed and \( D_n \) satisfies condition 3 above. By Dirichlet’s rational approximation theorem there is a pair of natural numbers \( s_{n+1}, k_{n+1} \) with \( -(1-\alpha)2^{n+1} < s_{n+1} - \alpha k_{n+1} < 0 \). Let \( E_{n+1} \subseteq J_{n+1} \) be the set \( \{c_{(n+1)1}, c_{(n+1)2}, \ldots, c_{(n+1)s_n}\} \). Since \( J \) is an indiscernible sequence over \( \emptyset \), by Fact 2.3 \( E_{n+1} \) is dim-independent from \( D_n \) over \( \emptyset \). Hence \( \dim(E_{n+1} / D_n) = s_{n+1} \). By indecomposability of \( T \) let \( F_{n+1} \) be the set consisting of \( k_{n+1} - s_{n+1} \) elements in \( \text{acl}(E_{n+1} D_n) \), with the property that each \( s_{n+1} \)-element subsets of \( E_{n+1} F_{n+1} \) forms a basis over \( D_n \). Finally set \( D_{n+1} = D_n \cup E_{n+1} \cup F_{n+1} \). Note that \( \dim(D_{n+1} / D_n) = s_n \) and \( |D_{n+1} \setminus D_n| = k_n \). Furthermore, \( D_{n+1} \subseteq \text{acl}(\bigcup_{i=0}^{n+1} E_i) \).

Now set \( D = \bigcup_{i=0}^{\infty} D_i \) and \( I = J \setminus D \). Notice that \( I = J \setminus \bigcup_{i=0}^{\infty} E_i \). So \( I \) is also an \( \mathcal{L} \)-indiscernible sequence over \( \emptyset \). So we can re-enumerate \( I \) and
represent it by \( \{ d_{ij} : (i, j) \in \omega \times \omega \} \). By the construction, \( D \subseteq \text{acl}( \bigcup_{i=0}^{\infty} E_i ) \).

On the other hand since \( J \) is \( \mathcal{L} \)-indiscernible, it follows from Fact 2.3 that \( D \downarrow_{\emptyset} \dim I \) and \( \text{acl}(I) \cap \text{acl}(D) = \text{acl}(\emptyset) \).

Let \( H = \langle D \cup I \rangle \). Then \( H = \langle D \oplus_{\emptyset} I \rangle = \langle D \rangle \oplus_{\emptyset} \langle I \rangle \). Now we recolor \( H \) by letting only the elements in \( D \) colored and the rest non-colored. Hence the following conditions are fulfilled.

- For each \( n, D_n \in K_\alpha^+ \), and hence \( D \in K_\alpha^+ \).
- \( I, H \in K_\alpha^+ \).
- Every finite subset of \( I \) as well as \( I \) itself are closed in \( H \).
- \( (D_n, D_{n+1}) \) is a minimal pair, for each \( n \in \omega \). Hence \( \text{cl}_H(c_{01}) = D \).

Suppose that \( f : H \to C \) is a strong embedding and \( b_n = f(c_{n1}) \) and \( a_{ij} = f(d_{ij}) \) for each \( i, j, n \in \omega \). Then it is clear that \( I_i = \{ a_{ij} : j \in \omega \} \) are \( \mathcal{L}_p \)-mutually indiscernible sequences over \( \emptyset \). Furthermore \( a_{n0} <^C b_n <^C a_{n1} \) for each \( n \in \omega \). We claim that non of the \( I_i \) is \( \mathcal{L}_p \)-indiscernible over \( b_0 \) for each \( i \in \omega \).

Assume for a contradiction, that \( I_i \) is indiscernible over \( b_0 \), for some \( i \).

Note that since \( b_i \in \text{cl}(b_0) \subseteq \text{Acl}(b_0) \), there exists an algebraic \( \mathcal{L}_p \)-formula \( \psi(x, b_0) \) which is satisfied by \( b_i \). Let \( \varphi(x, y, b_0) \) be the formula

\[
\exists t \ (x <^C t <^C y \land \psi(t, b_0)).
\]

Then \( \varphi(a_{i0}, a_{i1}, b_0) \) holds. But since \( I_i \) is indiscernible over \( b_0 \), \( \varphi(a_{ij}, a_{ij+1}, b_0) \) holds for each \( j \in \omega \).

But this gives infinitely many elements satisfying \( \psi(x, b_0) \) which contradicts the algebraicity of \( \psi(x, b_0) \).

\[ \square \]

**Corollary 4.15.** If \( \alpha \) is irrational then \( \text{ODAG}_\alpha \) and \( \text{RCF}_\alpha \) are not strongly dependent.

Finally bring the paper to an end with a curious counterexample on the distality of \( T_\alpha \) assuming \( T \) itself is distal.

Recall from [24] that a dependent theory \( T' \) is distal if for any indiscernible sequence \( I \), every set \( A \), tuple \( \bar{b} \) and \( A \)-indiscernible sequence \( I' = I_1 + I_2 \) with \( I_1 \) and \( I_2 \) without endpoints and \( \text{EM-tp}(I') = \text{EM-tp}(I) \), if \( I_1 + \bar{b} + I_2 \) is indiscernible, then it is \( A \)-indiscernible.
Unlike the strong dependence we will see that distality does not transfer to $T_\alpha$, both for rational and irrational $\alpha$, even when we start with as distal a theory as the o-minimal theories.

**Example 4.16.** Let $T$ be any o-minimal expansion of ODAG. Then $T_\alpha$ is not distal for any $0 < \alpha \leq 1$.

Let $A = \{a\}$ be a one-element set, $I' = I_1 + I_2$ with both $I_1$ and $I_2$ sequences of elements in $C$ ordered by rational numbers and an element $b \in C$ such that $I_1 + b + I_2$ is $L$-indiscernible over $A$. Put $I = I_1$. It follows from the Fact 4.5 that $I_1 + b + I_2$ is a dim-Morley sequence over $A$. In particular $a + b \notin \langle I_A \rangle$. Now to find an $L_p$-indiscernible sequence which violates the distality we color the elements of $P = \langle I' \{b\} A \rangle$ by letting $a + b$ colored and the rest non-colored. Then it follows that $P \in K_\alpha^+$ and if $f : P \to C$ is a strong embedding then we have that $f(I')$ is an $L_p$-indiscernible over $f(A)$ and $f(I_1) + f(b) + f(I_2)$ is an $L_p$-indiscernible but it is not the case that $f(I_1) + f(b) + f(I_2)$ is an $L_p$-indiscernible over $f(A)$. Hence this shows that $T_\alpha$ is not distal.

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