CENTRAL ELEMENTS OF THE ALGEBRAS $U'_q(\text{so}_m)$ AND $U_q(\text{iso}_m)$

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Abstract

The aim of this paper is to give a set of central elements of the algebras $U'_q(\text{so}_m)$ and $U_q(\text{iso}_m)$ when $q$ is a root of unity. They are surprisingly arise from a single polynomial Casimir element of the algebra $U'_q(\text{so}_3)$. It is conjectured that the Casimir elements of these algebras under any values of $q$ (not only for $q$ a root of unity) and the central elements for $q$ a root of unity derived in this paper generate the centers of $U'_q(\text{so}_m)$ and $U_q(\text{iso}_m)$ when $q$ is a root of unity.

1. The algebra $U'_q(\text{so}_m)$ is a nonstandard $q$-deformation of the universal enveloping algebra $U(\text{so}_m)$ of the Lie algebra $\text{so}_m$. It was defined in [1] as the associative algebra (with a unit) generated by the elements $I_{21}, I_{32}, \cdots, I_{m,m-1}$ satisfying the defining relations

\begin{align*}
I_{i+1,i}^2I_{i,i-1}^2 - (q + q^{-1})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i,i-1}^2I_{i+1,i} &= -I_{i+1,i}, \\
I_{i,i-1}^2I_{i+1,i} - (q + q^{-1})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i,i-1}I_{i+1,i}^2 &= -I_{i,i-1}, \quad (2)
\end{align*}

\begin{equation}
[I_{i,i-1}, I_{j,j-1}] = 0 \quad \text{for} \quad |i - j| > 1, \quad (3)
\end{equation}

where $[,]$ denotes the usual commutator. In the limit $q \to 1$ formulas (1)–(3) give the relations defining the universal enveloping algebra $U(\text{so}_m)$. Note also that the relations (1) and (2) differ from the $q$-deformed Serre relations in the approach of Drinfeld and Jimbo to quantum orthogonal algebras (see, for example, [2]) by presence of nonzero right hand sides in (1) and (2).

For the algebra $U'_q(\text{so}_3)$ the relations (1)–(3) are reduced to the following two relations:

\begin{align*}
I_{32}^2I_{21} - (q + q^{-1})I_{21}I_{32}I_{21} + I_{21}^2I_{32} &= -I_{32}, \quad (4)
I_{32}^2I_{21} - (q + q^{-1})I_{32}I_{21}I_{32} + I_{21}I_{32}^2 &= -I_{21}. \quad (5)
\end{align*}

Denoting $I_{21}$ and $I_{32}$ by $I_1$ and $I_2$, respectively, and introducing the element $I_3 := q^{1/2}I_1I_2 - q^{-1/2}I_2I_1$ we find that relations (4) and (5) are equivalent to three relations

\begin{align*}
q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 &= I_3, \\
q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 &= I_1, \quad (7)
q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 &= I_2. \quad (8)
\end{align*}
The Inonu–Wigner contraction applied to the algebra $U_q'(\mathfrak{so}_m)$ leads to the algebra $U_q'(\mathfrak{so}_{m-1})$ which was defined in [3]. The algebra $U_q'(\mathfrak{so}_m)$ is the associative algebra (with a unit) generated by the elements $I_{21}, I_{32}, \ldots, I_{m,m-1}, T_m$ such that the elements $I_{21}, I_{32}, \ldots, I_{m,m-1}$ satisfy the defining relations of the algebra $U_q'(\mathfrak{so}_m)$ and the additional defining relations

$$I_{m,m-1}^2 T_m - (q + q^{-1}) I_{m,m-1} T_m I_{m,m-1} + T_m I_{m,m-1}^2 = -T_m,$$

$$I_{m,m-1}^2 T_m - (q + q^{-1}) T_m I_{m,m-1} T_m + T_m^2 I_{m,m-1} = -I_{m,m-1},$$

$$[I_{k,k-1}, T_m] := I_{k,k-1} T_m - T_m I_{k,k-1} = 0 \quad \text{if} \quad k < m.$$

If $q = 1$, then these relations define the universal enveloping algebra $U(\mathfrak{so}_m)$ of the Lie algebra $\mathfrak{so}_m$ of the Lie group $ISO(m)$.

The algebra $U(\mathfrak{so}_2)$ is generated by two elements $I_{21}$ and $T_2$ satisfying the relations

$$I_{21}^2 T_2 - (q + q^{-1}) I_{21} T_2 I_{21} + T_2 I_{21}^2 = -T_2, \quad (9)$$

$$I_{21} T_2^2 - (q + q^{-1}) T_2 I_{21} T_2 + T_2^2 I_{21} = -I_{21}. \quad (10)$$

Denoting $I_{21}$ by $I$ and introducing the element $T_1 := [I, T_2]_q \equiv q^{1/2} I T_2 - q^{-1/2} T_2 I$, we find that the relations (9) and (10) are equivalent to the relations

$$[I, T_2]_q = T_1, \quad [T_1, I]_q = T_2, \quad [T_2, T_1]_q = 0. \quad (11)$$

2. In $U_q'(\mathfrak{so}_m)$ we can determine [4] elements analogous to the basis matrices $I_{ij}, \ i > j$, (defined, for example, in [5]) of $\mathfrak{so}_m$. In order to give them we use the notation $I_{k,k-1}^+ \equiv I_{k,k-1}^+, I_{k,k-1}^- \equiv I_{k,k-1}^-$. Then for $k > l + 1$ we define recursively

$$I_{kl}^\pm := [I_{l+1,l}, I_{k,l+1}]_{q^\pm} \equiv q^{\pm 1/2} I_{l+1,l} I_{k,l+1} - q^{-\pm 1/2} I_{k,l+1} I_{l+1,l}. \quad (12)$$

The elements $I_{kl}^+, k > l$, satisfy the commutation relations

$$[I_{in}, I_{kl}^+]_q = I_{kn}^+, \quad [I_{kl}^+, I_{kn}^+]_q = I_{in}^+, \quad [I_{kn}^+, I_{ln}^+]_q = I_{kl}^+ \quad \text{for} \quad k > l > n, \quad (13)$$

$$[I_{kl}^+, I_{kn}^+] = 0 \quad \text{for} \quad k > l > n > r \quad \text{and} \quad k > n > r > l, \quad (14)$$

$$[I_{kl}^+, I_{kn}^+] = (q - q^{-1})(I_{ln}^+ I_{kn}^+ - I_{kn}^+ I_{ln}^+) \quad \text{for} \quad k > n > l > r. \quad (15)$$

For $I_{kl}^-, k > l$, the commutation relations are obtained by replacing $I_{kl}^+$ by $I_{kl}^-$ and $q$ by $q^{-1}$.

Using the diamond lemma (see, for example, Chapter 4 in [2]), N. Iorgov proved the Poincaré–Birkhoff–Witt theorem for the algebra $U_q'(\mathfrak{so}_m)$ (proof of it will be published):

**Theorem 1.** The elements $I_{21}^{n_{21}} I_{32}^{n_{32}} I_{31}^{n_{31}} \cdots I_{m1}^{n_{m1}}, n_{ij} = 0, 1, 2, \ldots$, form a basis of the algebra $U_q'(\mathfrak{so}_m)$.

This theorem is true if the elements $I_{ij}^+$ are replaced by the corresponding elements $I_{ij}^-$. 

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Using the generating elements $I_{21}, I_{32}, \cdots, I_{m,m-1}$ of the algebra $U_q(\text{so}_3)$ we define by formula (12) the elements $I_{ij}^+, i > j$, in this algebra. Besides, in $U_q(\text{so}_3)$ we also define recursively the elements
\[ T_k^+ := [I_{k+1,k}, T_{k+1}]_{q^+}, \quad k = 1, 2, \cdots, m - 1. \]

It is shown in [6] that the elements $I_{ij}^+, i > j$, and $T_k^+, 1 \leq k \leq m$, satisfy the commutation relations (13)–(15) and the relations
\[
[I_n, I_m^+] = T_n^+ + [I_n^+, I_m] = T_l^+ \quad \text{for} \quad l > n,
\]
\[
[T_l^+, I_m] = 0 \quad \text{for} \quad l > n > p \quad \text{or} \quad n > p > l,
\]
\[
[T_l^+, I_p] = (q - q^{-1})(T_n^+ I_p - T_p^+ I_n) \quad \text{for} \quad n > l > p,
\]
\[
[T_l^+, I_n^+] = 0 \quad \text{for} \quad n < l.
\]

For $U_q(\text{so}_3)$ the Poincaré–Birkhoff–Witt theorem is formulated as

**Theorem 2.** The elements $I_{21}^{n_1} I_{32}^{n_2} \cdots I_{m1}^{n_m} T_1^{+n_1} T_2^{+n_2} \cdots T_m^{+n_m}$ with $n_{ij}, n_k = 0, 1, 2, \cdots$, form a basis of the algebra $U_q(\text{so}_3)$.

3. It is easy to check that for any value of $q$ the algebra $U'_q(\text{so}_3)$ has the Casimir element
\[ C_q = q^2 I_1^2 + I_2^2 + q^2 I_3^2 + q^{1/2}(1 - q^2) I_1 I_2 I_3. \]

As in the case of quantum algebras (see, for example, Chapter 6 in [2]), at $q$ a root of unity this algebra has additional central elements.

**Theorem 3.** Let $q^n = 1$ for $n \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < n$. Then the elements
\[
C^{(n)}(I_j) = \sum_{j=0}^{[n-1]/2} \binom{n-j}{j} \frac{1}{n-j} \left(\frac{i}{q-q^{-1}}\right)^{2j} I_j^{n-2j}, \quad j = 1, 2, 3,
\]

belong to the center of $U'_q(\text{so}_3)$, where $[x]$ for $x \in \mathbb{R}$ denotes the integral part of $x$.

The proof of this theorem is rather complicated (see [7]). First it is proved that $C^{(n)}(I_1)$ belongs to the center of $U'_q(\text{so}_3)$. This proof is based on the formula
\[
I_3 I_1^m = p_m(I_1) I_2 + q_m(I_1) I_3,
\]
where
\[
p_m(x) = q^{-\frac{1}{2}} \left(\frac{x(q+q^{-1})}{2}\right)^m \sum_{t=0}^{[m-1]/2} \binom{m}{2t+1} \left(\frac{q-q^{-1}}{x(q+q^{-1})}\right)^2 t,
\]
\[
q_m(x) = -q^\frac{1}{2} \left(\frac{x(q+q^{-1})}{2}\right)^m \sum_{t=0}^{[m]/2} \binom{m}{2t} \left(\frac{q-q^{-1}}{x(q+q^{-1})}\right)^2 t - (\frac{2}{x(q+q^{-1})})^2 t.
\]

The proof also needs deep combinatorial identities, such that
\[
\sum_{t=0}^{[N-1]/2} \binom{N}{2t+1} \binom{N-1-t}{t} \binom{t}{M} =
\]
\[
= 4^{C-M}(C_M)(N - 2C(1 - N')) \sum_{j=0}^{d} \left( \frac{n-j}{j} \right)^{2d-2j+1+n(2d-1)} \frac{q^{n-j}}{c-j} \sum_{n-j}^{(n-1-c-d)!} \frac{(2j-2)!}{(2j)!} \frac{1}{q^{n-j}} (n-j)!(n-1-c-d)!(n-2d+n'-1)!
\]

where \( N, n \in \mathbb{N} \), \( 0 \leq C, M \leq \left[ \frac{N-1}{2} \right] \), \( 0 \leq c, d \leq \left[ \frac{n-1}{2} \right] \) and \( N', n' = 0 \) or 1 such that \( N' = N \mod 2 \) and \( n' = n \mod 2 \). One also needs an extensive use of the fact that \( q \) is a root of unity.

If it is proved that \( C^{(n)}(I_1) \) belongs to the center of \( U'_q(so_3) \), then we have to use the automorphism \( \rho: U'_q(so_3) \rightarrow U'_q(so_3) \) defined by relations \( \rho(I_1) = I_2, \rho(I_2) = I_3, \rho(I_3) = I_1 \). This automorphism shows that \( C^{(n)}(I_2) \) and \( C^{(n)}(I_3) \) also belong to the center of \( U'_q(so_3) \).

**Conjecture 1.** If \( q \) is a root of unity as above, then the elements \( C_q \) and \( C^{(n)}(I_j) \), \( j = 1, 2, 3 \), generate the center of \( U'_q(so_3) \).

4. Central elements of the algebra \( U'_q(so_m) \) for any value of \( q \) are found in [8] and [9]. They are given in the form of homogeneous polynomials of elements of \( U'_q(so_m) \). If \( q \) is a root of unity, then (as in the case of quantum algebras) there are additional central elements of \( U'_q(so_m) \).

**Theorem 4.** Let \( q^n = 1 \) for \( n \in \mathbb{N} \) and \( q^j \neq 1 \) for \( 0 < j < n \). Then the elements

\[
C^{(n)}(I_{kl}^+) = \sum_{j=0}^{[n-j]} \left( \frac{n-j}{j} \right) \frac{1}{n-j} \left( \frac{i}{q-q^{-1}} \right)^{2j} I_{kl}^{n-2j}, \quad k > l,
\]

belong to the center of \( U'_q(so_m) \).

Let us prove this theorem for the algebra \( U'_q(so_4) \) (for the general case a proof is the same). This algebra is generated by the elements \( I_{43}, I_{32}, I_{21} \). We introduce the elements \( I_{31} \equiv I_{31}^+, I_{42} \equiv I_{42}^+, I_{41} \equiv I_{41}^+ \) defined as indicated above. Then the elements \( I_{ij}, i > j \), satisfy the relations

\[
[I_{43}, I_{21}] = 0, \quad [I_{32}, I_{31}] = I_{21}, \quad [I_{21}, I_{32}] = I_{31},
\]

\[
[I_{31}, I_{21}] = I_{32}, \quad [I_{43}, I_{42}] = I_{32}, \quad [I_{32}, I_{43}] = I_{42},
\]

\[
[I_{42}, I_{41}] = I_{43}, \quad [I_{31}, I_{43}] = I_{41}, \quad [I_{21}, I_{42}] = I_{41},
\]

\[
[I_{41}, I_{21}] = I_{42}, \quad [I_{41}, I_{31}] = I_{43}, \quad [I_{42}, I_{41}] = I_{21},
\]

\[
[I_{41}, I_{32}] = 0, \quad [I_{43}, I_{41}] = I_{31}, \quad [I_{42}, I_{31}] = (q-q^{-1})(I_{21}I_{43} - I_{41}I_{32}).
\]

If one wants to prove that an element \( X \) belongs to the center of \( U'_q(so_4) \), it is sufficient to prove that \( [X, I_{21}] = [X, I_{32}] = [X, I_{43}] = 0 \).
Let us consider the element $C^{(n)}(I_{21})$. It belongs to the subalgebra $U'_q(\mathfrak{so}_3)$ generated by $I_{21}, I_{32}$ and $I_{31}$:

\[
\begin{bmatrix}
I_{21} & I_{31} & I_{41} \\
I_{32} & I_{42} & I_{43}
\end{bmatrix}
\]

It follows from Theorem 3 that $C^{(n)}(I_{21})$ commutes with element $I_{32}$. Using the first relation in (18) we easily see that $C^{(n)}(I_{21})$ commutes with $I_{43}$ and therefore $C^{(n)}(I_{21})$ belongs to the center of $U'_q(\mathfrak{so}_4)$.

Let us consider the element $C^{(n)}(I_{32})$. In $U'_q(\mathfrak{so}_4)$ we separate two subalgebras $U'_q(\mathfrak{so}_3)$:

\[
\begin{bmatrix}
I_{21} & I_{31} & I_{41} \\
I_{32} & I_{42} & I_{43}
\end{bmatrix}
\]

From Theorem 3 we have $[C^{(n)}(I_{32}), I_{21}] = [C^{(n)}(I_{32}), I_{43}] = 0$ and $C^{(n)}(I_{32})$ belongs to the center of $U'_q(\mathfrak{so}_4)$. A proof that the element $C^{(n)}(I_{43})$ belongs to the center is the same as for $C^{(n)}(I_{21})$.

The elements $C^{(n)}(I_{31}), C^{(n)}(I_{42})$ and $C^{(n)}(I_{41})$ belong to the center of $U'_q(\mathfrak{so}_4)$ since they belong to the subalgebras $U'_q(\mathfrak{so}_3)$ generated by triplets

\[
I_{41}, I_{31}, I_{43} \quad \text{and} \quad I_{21}, I_{41}, I_{42}.
\]

(Note that $C^{(n)}(I_{31})$ and $C^{(n)}(I_{42})$ commute with $I_{42}$ and $I_{31}$, respectively, since $I_{42} = [I_{32}, I_{43}]_q$ and $I_{31} = [I_{21}, I_{32}]_q$.) Theorem is proved.

**Conjecture 2.** If $q$ is a root of unity as above, then the central elements of [9] and of Theorem 4 generate the center of $U'_q(\mathfrak{iso}_m)$.

5. Let us consider the associative algebra $U'_{q,\varepsilon}(\mathfrak{so}_3)$ (where $\varepsilon \geq 0$) generated by three generators $J_1, J_2, J_3$ satisfying the relations:

\[
[J_1, J_2]_q := q^{1/2}J_1J_2 - q^{-1/2}J_2J_1 = J_3, \quad [J_2, J_3]_q = J_1, \quad [J_3, J_1]_q = \varepsilon^2J_2.
\]

It is easily proved that this algebra is isomorphic to the algebra $U'_q(\mathfrak{so}_3)$ and the corresponding isomorphism is uniquely defined by $J_1 \to \varepsilon I_1, J_3 \to \varepsilon I_3, J_2 \to I_2$. Therefore, the elements

\[
\bar{C}^{(n)}(J_i, \varepsilon) := n\varepsilon^nC^{(n)}(J_i/\varepsilon), \quad i = 1, 3, \quad \bar{C}^{(n)}(J_2, \varepsilon) := C^{(n)}(J_2)
\]

belong to the center of $U'_{q,\varepsilon}(\mathfrak{so}_3)$ if $q^n = 1$. By means of the contraction $\varepsilon \to 0$ we transform the algebra $U'_{q,\varepsilon}(\mathfrak{so}_3)$ into the algebra $U'_q(\mathfrak{iso}_2)$. Under this contraction the commutation relations $[\bar{C}^{(n)}(J_1, \varepsilon), J_k] = 0$ transform into the relations $[\bar{C}^{(n)}(J_1, 0), J_k] = 0$. In other words, we have proved the following

**Theorem 5.** Let $q^n = 1$ for $n \in \mathbb{N}$ and $q^j \neq 1$ for $0 < j < n$. Then the elements $T_1^n, T_2^n$ and $C^{(n)}(I)$ belong to the center of the algebra $U'_q(\mathfrak{iso}_2)$. 

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It was shown in [10] that the element

\[ C_q = q^{-1}T_1^2 + qT_2^2 + q^{-3/2}(1 - q^2)T_1T_2I \]

is central in \( U_q'(\text{iso}_2) \).

**Conjecture 3.** If \( q \) is a root of unity as above, then the elements \( C_q, T_1^n, T_2^n \) and \( C^{(n)}(I) \) generate the center of \( U_q'(\text{iso}_2) \).

Using Theorem 5 and repeating the proof of Theorem 4 we prove the following theorem:

**Theorem 6.** Let \( q^n = 1 \) for \( n \in \mathbb{N} \) and \( q^j \neq 1 \) for \( 0 < j < n \). Then the elements

\[ C^{(n)}(I_{ij}), \quad i > j, \quad T_j^n, \quad j = 1, 2, \ldots, m, \]

belong to the center of the algebra \( U_q'(\text{iso}_m) \).

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