Phase Separation Induced by Curved Spacetime:
Mobility Edge, Nonergodic Critical Phase and Sub-Extended Phase

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We construct a quasiperiodic lattice model in a curved spacetime to explore the crossover concerning both condensed matter and curved spacetime physics. We study the related Anderson localization and find that the model has a clear boundary of localized-extended phase separation, which leads to the mobility edge, i.e., the coexistence of nonergodic localized, critical and extended phases. A novel self-consistent segmentation method is developed to calculate the analytical expression of the critical position of phase separation, and the rich phase diagram is determined by calculating the fractal dimension and scaling index in multifractal analysis.

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Introduction.—The past few decades have witnessed extensive studies on the Anderson localization [1–6], with many important results being achieved both theoretically [7–19] and experimentally [20–35]. To explain the disappearance of spin diffusion for low doping density [1, 2], P. W. Anderson proposed the famous theory of Anderson localization [3], which states that the ergodic property of electrons in a system without interaction will be destroyed and the system will transform into the localized phase when the intensity of disorder exceeds the critical value, thus making the system turn from a metallic to an insulating phase. N. F. Mott further proposed the concept of mobility edge [4], which indicates that the localized and extended phases of the system can coexist under certain circumstances. Previous studies have suggested that one-dimensional (1D) and 2D systems will exhibit localized behavior when uncorrelated disorder potential is introduced. In the 3D case, however, the introduction of disorder will cause mobility edge, i.e., there appear both localized and extended states in the system [6, 36, 37].

In addition to disordered systems, the 1D Aubry-André-Harper (AAH) model is a quasiperiodic system and has been eye-catching as one of the simplest systems to demonstrate the localized-extended phase transition [38, 39]. The quasiperiodic potential is incommensurate with the lattice space, which exhibits disorder in a small scale while order in a large scale. In other words, a quasiperiodic AAH model can be regarded as a limbo system between disorder and order. Due to the self-duality of AAH model, the system is characterized by the extended phase (localized phase) when the quasiperiodic potential is less than (greater than) the critical value, and the corresponding eigenstates are all in extended (localized) states [39]. The AAH model is also valuable in studying topological phases in quasicrystals for the reason that it can be mapped to the 2D integer quantum Hall effect by a continuous U(1) gauge transformation [40–52]. Apart from the standard AAH model, the extended one has also become a hot topic in recent studies, where the mobility edge can be obtained by introducing a long-range correlation [53, 54] or reconstructing the quasiperiodic potential [55–60], and many-body localization can be studied by exerting many-body interaction [61–65]. So far, the quasiperiodic system has been realized experimentally in various platforms [66–69].

On the other hand, in 1981, G.E. Unruh proposed the sonic horizon, which was the first attempt to simulate a black hole horizon and the relevant CST physics in the laboratory [70, 71]. The seminal work provides an effective way in exploring black holes, CST physics and general relativity. After 40 years of intensive efforts, long-lived black hole horizon and CST have been successfully simulated in various tabletop experiments, such as water flume [70, 72–75], Bose-Einstein condensates [76–80], exciton-polaritons [81], and nonlinear optics [82–87]. These milestone achievements provide a solid platform for studying the CST physics, which has deepened our understanding of the nature of gravity, e.g., helping us to reveal the relation between (1+1)D Jackiw-Teitelboim gravity and Sachdev-Ye-Kitaev model [88–92]. So far, despite the successful simulation of Hawking radiation and lots of effort on the verification of Unruh effect [70–95], little attention has been focused on the condensed matter properties in CST [96–101].

Motivated by these works on artificial black holes horizon and CST, we construct a generalized 1D AAH model in CST to explore the crossover concerning both condensed matter and CST physics. We find that condensed matter systems with Anderson phase transition can exhibit phase separation in CST. A novel segmentation method is developed to calculate the analytical expression of the critical phase separation position. Furthermore, mobility edge, nonergodic critical phase and sub-extended phase can coexist and separated by a clear boundary. Our work provides a CST-version of Anderson localization and mobility edge theory.

Model.—Based on the lattice model in CST [96, 98], we
Figure 1: (Color online). (a) Schematic diagram of the CST-AAH model in the vicinity of a black hole, where the nearest neighboring hopping is power-law position-dependent. (b) AAH chain in CST shows phase separation, while the whole chain is divided into the localized (shows phase separation, while the whole chain is divided into the localized) and extended (shows phase separation, while the whole chain is divided into the extended) regions. (c)-(f) The evolution of a test wavepacket in flat space ($\sigma = 0$) and in CST ($\sigma = 1$). The first (second) row corresponds to the case where the quasiperiodic potential is absent (present). The centers of the initial states are at $j_0 = 1000$ and $2300$, respectively. The system size $N = 2584$ and the corresponding phase separation boundary $j_c = 1937$.

construct a quasiperiodic lattice model under the influence of a supermassive compact object [see Fig. 1(a)(b)]. The corresponding Hamiltonian reads [see Supplemental Materials (SM) [103] for details]

$$H = \sum_{j=1}^{N-1} J_j (\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) + \sum_{j=1}^{N} V_j \hat{c}_j^\dagger \hat{c}_j, \quad (1)$$

where $\hat{c}_j^\dagger (\hat{c}_j)$ is the fermionic creation (annihilation) operator at the $j$th site. The on-site potential $V_j = \lambda \cos(2\pi \phi j + \theta)$, where $\lambda$ denotes the strength of the incommensurate potential, $\phi$ is an irrational number, and $\theta \in [0,2\pi]$ is a phase angle [104]. For CST, the nearest neighboring hopping strength $J_j = J (\frac{1}{2\sigma})^\lambda$, which depends on the site index $j$ and the parameter $\sigma$ indicating warping degree of spacetime [98]. Without loss of generality, hereafter we take $J = 1$ as the unit of energy, and select $\phi = (\sqrt{5} - 1)/2$ as a typical irrational number.

The standard AAH model (without spacetime warping for the condition of $\sigma = 0$ and thus $J_j = J$) exhibits a phase transition at $\lambda_c = 2J$ [39]: The system is of extended (localized) phase for $\lambda < \lambda_c$ ($\lambda_c$). Properties induced by CST emerge when $\sigma \geq 1$, where the corresponding hoppings gradually increase from 0 to 1 with the lattice index growing from small to large [97, 98, 100]. As is known, nothing, even as minuscule as photons, can be spared from being pulled in the vicinity of a super gravitational source. The closer the little thing is to the gravitational monster, the greater the influence it will feel. Therefore, it is reasonable to assume that for the near-end of AAH chain to the event horizon, hopping becomes difficult due to the extreme attraction, while the rear-end of the chain, farther away from the gravitational pull, exhibits normal flat-spacetime hopping.

Therefore, the CST-AAH chain [Eq. (1)] can well reflect the lattice-gravity correspondence and the relevant dynamical properties in the vicinity of an event horizon. The model proves an ideal simulator to reveal the event horizon dynamics of free particles in (1+1)D anti-de Sitter space, where the particles slow down exponentially as they move towards the event horizon, and vice versa [71, 103].

Phase Separation.—An analytical expression of mobility edge for the AAH model with constant hopping strength ($J_j = J$) can be obtained by Avila’s global theorem [102], but this method does not work for a general $J_j$. We here propose a method of segmentation to explore the localization properties of CST-AAH model. Cases of two, four and N-1 segments are considered in a stepwise way, so as to gradually approach the real CST-AAH model (see SM [103] for details). Based on the known critical point of the phase transition in the flat spacetime AAH model, we solve approximately the CST-AAH model. The results show that the phase separation will occur, which divides the whole AAH chain into two parts (the localized and the extended) with a clear boundary in between. As shown in SM [103], the analytical expression of phase separation’s critical site $j_c$ can be obtained as

$$j_c = \left[ \frac{\lambda}{2J} \right]^\frac{1}{\sigma} (N - 1). \quad (2)$$

Here $[\ast]$ denotes floor function, which is defined to round down the number inside the function to an integer. When $\sigma = 0$, the system reduces to the case of flat spacetime, and the expression Eq. (2) becomes $j_c = \left[ \left( \frac{1}{2J} \right)^{\frac{1}{\sigma}} (N - 1) \right]$. One can see that as $\lambda$ increases, the system will exhibit extended ($0 < \frac{\lambda}{2J} < 1$) and localized ($\frac{\lambda}{2J} > 1$) phase. Before (after) $\lambda$ exceeds the threshold value $\lambda = 2J$, $j_c = 0$ ($j_c = \infty$), which indicates no coexistent localized and extended phases. That is to say, the system can only be a pure extended or a pure localized state, which is consistent with the previous studies [103]. Meanwhile, when spacetime is infinitely curved ($\sigma = \infty$), we learn from the expression that the boundary of phase separation is always at the rightmost end of the chain, and then the whole chain of the system exhibits the localized phase, which
is quite in line with our knowledge: infinitely curved spacetime means completely frozen hopping in the whole chain.

The corresponding wavepacket dynamics can effectively reflect properties of spacetime. One can consider a general Gaussian function as the initial state, i.e.,

$$\psi(j, t = 0) = \frac{1}{\sqrt{\pi w}} e^{-\frac{1}{2} \left(\frac{j-j_0}{w}\right)^2} \rho_{j_0},$$

where the width of the wavepacket $w = 50$, and the initial momentum $p_0 = -\pi/2$ in our numerical calculation. The results are plotted in Fig. 1(c)-(f), and calculation details are contained in [103]. As a comparison, we show in Fig. 1(c) the case of $\lambda = 0$. The results show that the wavepacket in the flat spacetime ($\sigma = 0$) is ergodic over the entire chain, while that in CST behaves more like an object on the verge of the black hole horizon, featuring continuous deceleration and localization [Fig. 1(d)]. The presence of quasiperiodic potential in the system can give rise to a novel phenomenon of phase separation, with a clear boundary $j_c$, existing between the localized and the extended regions [dashed line in Fig. 1(e)(f)].

The wave function exhibits the localized (extended) characteristics if it is initially placed in the localized (extended) region.

Mobility Edge and Nonergodic Phases.—In order to explore the localization properties of AAH model in CST, we calculate the fractal dimension and scaling indices, both of which are core observables in the investigation of the localization and mobility edge.

Firstly, we calculate the fractal dimension defined as

$$\Gamma(\beta) = -\lim_{N \to \infty} \frac{\ln \xi(\beta)}{\ln N},$$

where $\xi(\beta) = \sum_{j=1}^{N} |\psi_j(\beta)|^4$ denotes the inverse participation rate, and $\beta$ is the energy level index of the particle eigenstate. The fractal dimension $\Gamma \to 0$ ($\to 1$) corresponds to the localized (extended) state, while $\Gamma \in (0, 1)$ to critical state [58–60, 105]. Previous studies suggested two possible ways to induce mobility edge in the AAH model, i.e., by introducing a long-range hopping or energy dependent quasiperiodic potential [53–60]. Here we show that warping the spacetime is another way to induce mobility edge [see Fig. 2(a)].

To better understand the generation mechanism of mobility edge shown in Fig. 2(a), we compute the eigenvalues versus level index $\beta$ of localized and extended subchains, respectively. As shown in Fig. 2(b), the results reveal that there are three different phases: pure localized phase (region $I_1$ and $I_2$), mixed critical phase (region $II_1$, $II_2$ and $II_3$), and pure extended phase (region $III$). Above two pure extended or localized eigenvalues ($E_{ext}$ and $E_{loc}$), while the overlapping of the extended and localized eigenvalues leads to a mixed critical phase.

Furthermore, one can distinguish different phases by the behavior of the fractal dimension versus system size. To this end, we perform the scaling analysis and plot the results of different $N$ in Fig. 2(c). One can find that the fractal dimension $\Gamma$ in region $III$ (region $I_1$ and $I_2$) increases (decreases) with the increasing system size $N$, which exhibits the properties of the extended (localized) phase. However, $\Gamma$ is independent of the system size in regions $II_1$, $II_2$ and $II_3$, which is an evidence of the critical phase. Both the scaling behavior of fractal dimension and the overlap of subchain eigenvalues agree well with each other, and thus corroborate the emergence of the critical phases [105–107].

Secondly, we calculate the scaling index to further explore different phases [108–111]. The probability of a particle occupied in site $j$ is represented by the modulus square of the wave function $P_j = |\psi_j|^2$, which satisfies the normalization condition $\sum_j |\psi_j|^2 = 1$. The scaling index of multifractal analysis $\alpha_j$ is defined by the probability measure $P_j$ as

$$P_j = N^{-\alpha_j}.$$
Since the occupation probability on all sites is $P_j = 1/N$ for a completely extended wave function, the corresponding scaling index $\alpha_j = 1$. For a localized wave function, the occupation probability is non-zero at just a few sites, therefore $\alpha \to 0$ for such occupied sites and $\alpha \to \infty$ for the other sites. For a critical wave function, the scaling index $\alpha$ is distributed in a finite interval $[\alpha_{\min}, \alpha_{\max}]$. Thus, by considering the thermodynamic limit $N \to \infty$, one can characterize the localization properties of a wave function by $\alpha_{\min}$. To be specific, for $N \to \infty$, $\alpha_{\min} = 1$ (0) indicates the extended (localized) states, whereas $0 < \alpha_{\min} < 1$ corresponds to the critical state.

To better demonstrate the properties of the wave functions in different regions, a routine approach is to calculate the mean values of $\Gamma$ and $\alpha_{\min}$ in different regions, which are defined as

\begin{equation}
\Gamma = \frac{1}{\eta_R} \sum_R \Gamma, \quad \alpha_{\min} = \frac{1}{\eta_R} \sum_R \alpha_{\min},
\end{equation}

where $\eta_R$ denotes the total number of eigenstates in the region denoted as $R = I_1, I_2, \Pi_1, \Pi_2, \Pi_3, \Pi_3$. By taking a finite number of system size, we extrapolate $\Gamma$ and $\alpha_{\min}$ to the case of thermodynamic limit ($N \to \infty$) by linear fitting as shown in Fig. 2(d)(e). Under this condition, the values of $\Gamma$ and $\alpha_{\min}$ in the localized region decrease with the growing lattice size and finally approach zero. However, the values of $\Gamma$ and $\alpha_{\min}$ in the extended regions increase with the increasing lattice size until they approach a fixed value. Note that, for the AAH model in flat spacetime, both $\Gamma$ and $\alpha_{\min}$ in the extended region will eventually be close to 1. The mechanism behind the above interesting phenomenon lies in the phase separation featured by AAH chain in CST. Though the wave function of the extended state can experience all states of the extended subchain, it is all the way prohibited from entering the localized subchain. In other words, although a subchain in the extended region follows the eigenstate thermalization hypothesis, a complete AAH chain in CST does not exhibit the same behavior as the AAH model in flat spacetime.

The inserts of Fig. 2(d)(e) exhibit the $\Gamma$ and $\alpha_{\min}$ of all eigenstates versus the CST parameter $\sigma$ at $\lambda = 0$, and the results support that the larger the degree of spacetime warping, the smaller the value of $\Gamma(N \to \infty)$ and $\alpha_{\min}(N \to \infty)$.

Furthermore, the distribution of the eigenstate wave functions can soundly reveal the effects of the localized-extended phase separations. Fig. 3(a)-(c) exhibit the distribution of eigenstate wave functions of three typical phases, and we randomly select four phase angle $\theta$ in the calculation. The results reveal that the wave functions of localized phase [Fig. 3(a)] remain localized regardless of the value of $\theta$, and the wave functions are all localized in the subchain of $j < j_c$. On the other hand, while the wave functions of the sub-extended phase [Fig. 3(c)] still display extended behaviors for different $\theta$, the extended state is confined to the region of extended subchains ($j > j_c$). For wave functions of the critical phase [Fig. 3(b)], localization characteristics of the system depend on the value of parameter $\theta$. In other words, different values of $\theta$ may produce either extended or localized states, which represents a whole new style of critical phase that has not been reported.

Finally, the full phase diagram of CST-AAH model is obtained as shown in Fig. 4. One can see that with the increase of the quasiperiodic potential parameter $\lambda$, the system experiences four intermediate phases from the extended to the final localized state [Fig. 4(b)-(d)]. Stepwise analyses of Fig. 4 are given as follows. When $\lambda = 0$, the whole system resides in the sub-extended phase [Fig. 4(a)], while the localized properties become increasingly salient as $\lambda$ grows larger. First, there appears a multilayered structure composed of the sub-extended and critical phases [Fig. 4(b)]. Then $\lambda$ continues to grow, leading to a much richer phase diagram that contains the sub-extended, critical and localized phases [Fig. 4(c)]. With the ever-increasing $\lambda$, the localized properties gradually gain the upper hand [Fig. 4(d)], occupying an overwhelming majority of regions in the multilayered structure [Fig. 4(e)]. Eventually, when $\lambda$ exceeds the critical value, the entire AAH chain will become localized [Fig. 4(f)].

**Conclusion.**—In summary, we have proposed a CST-AAH model to explore the properties of condensed matter in CST. We found a phase separation of CST-AAH model with a clear boundary, where the entire AAH chain can be regarded as a combination of the localized and the extended subchains. By applying a novel method of segmentation, the analytical expression of phase separation of the critical position is obtained. Furthermore, we find that the phase separation gives rise to mobility edge and nonergodic phases, i.e., the localized, critical and sub-extended phases. In CST, the broken ergodicity of AAH chain makes it impossible to be fully expanded even for the wave function originally in the extended phase, hence the “sub-extended state”. Our work is devoted
to constructing the CST-version of Anderson localization and mobility edge theory, helping foster a crossover research concerning both condensed matter and CST physics. Nowadays, the ever advancing experimental techniques have enabled black hole horizons and CST to be simulated in various artificial systems [104]. Thus, it is promising that the phenomenon predicted here will be realized in experiments in the near future.

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S1. The construction of a quasiperiodic model in curved spacetime

As suggested by refs. [1, 2], to establish a connection between continuum field theory and the condensed matter, one can start from 1D Jackiw-Teitelboim gravitation gauge with dilaton scalar field to obtain a 1D CST lattice model with position-dependent hopping strength at last, where the corresponding Hamiltonian reads

$$H = \sum_{j=1}^{N-1} J_j (\hat{c}^\dagger_j \hat{c}_{j+1} + \text{H.c.}),$$  \hspace{1cm} (S1)$$

where $\hat{c}^\dagger_j$ and $\hat{c}_j$ correspond to the creation and annihilation operators, respectively, and $J_j = J(\frac{j}{N-1})^\sigma$ denotes the hopping strength between site $j$ and site $j+1$. Under the thermodynamic limit ($N \to \infty$), the hopping strength of the two nearest neighboring sites can be regarded as a constant, thus one can obtain an approximate localized band structure, i.e., $\epsilon(j,k) \approx -2(\frac{j}{N})^\sigma \cos k$. Therefore, the corresponding dispersion relation of the Hamiltonian Eq. (S1) at $k = \pm \pi/2$ has a dirac cone shape, and its quasiparticle shows the Dirac fermionic property. The position-dependent group velocity of the quasiparticle is similar to that of the Dirac field in a 1D Jackiw-Teitelboim gravitational background. When $\sigma > 0$, the quasiparticle’s group velocity vanishes at the sites of $j \to 0$, and the quasiparticle shows in its behavior the group velocity of the light cone approaching the event horizon of a black hole. One can capture this interesting phenomenon through the wavepacket evolution. We consider a general Gaussian initial state as follows,

$$\psi(j,t=0) = \frac{1}{\sqrt{\pi w}} e^{-\frac{1}{2} \left(\frac{j-j_0}{w}\right)^2} e^{ip_0 j},$$  \hspace{1cm} (S2)$$

where $w$ is the width of the wavepacket, $j_0$ is the position of the center of the wavepacket, and $p_0 = -\pi/2$ is the initial momentum.

We can see from Fig. S1(a) that, when $\sigma = 0.5$, since the wavepacket bounces back after touching the boundary, it can not simulate the deceleration process of the wavepacket approaching a black hole horizon.

However, as shown in Fig. S1(b)-(d), the evolving wavepacket slows down at the sites $j \to 0$ and resides near $j = 0$. When $\sigma \geq 1$, to be specific, the system can effectively simulate the dynamical properties of the wavepacket in the vicinity of the black hole horizon, i.e., the wavepacket becomes slower and more localized as it approaches the black hole horizon.

Based on the above analysis, we develop a quasiperiodic model in CST by applying quasiperiodic potential energy at each site, which can be used to explore the CST-version of Anderson localization theory. The corresponding Hamiltonian takes the form

$$H = \sum_{j=1}^{N-1} J_j (\hat{c}^\dagger_j \hat{c}_{j+1} + \text{H.c.}) + \sum_{j=1}^{N} V_j \hat{c}^\dagger_j \hat{c}_j,$$  \hspace{1cm} (S3)$$

where $\hat{c}^\dagger_j$ ($\hat{c}_j$) is the fermionic creation (annihilation) operator at the $j$th site. The on-site potential $V_j = \lambda \cos(2\pi \phi j + \theta)$, where $\lambda$ denotes the strength of the incommensurate potential, $\phi$ is an irrational number, and $\theta \in [0, 2\pi]$ is the phase angle. Similarly,
we can study the evolution behavior of wavepackets in the CST-AAH model. Without loss of generality, here we discuss the case of $\sigma = 1$ to study the dynamical evolution of the wavepacket by adjusting the strength of the quasiperiodic potential. Fig. S2 shows the cases of $\lambda = 0.1, 0.5, 1, 1.5$, respectively. The initial position of the wavepacket is set at $j_0 = 2300$, $w = 50$ and the initial momentum $p_0 = -\pi/2$. The results show that wavepackets can get close to the black hole horizon for small $\lambda$. However, by increasing $\lambda$, the site where the wavepacket can reach will be further and further away from the black hole horizon ($j = 0$), i.e., as $\lambda$ grows, the left region becomes even more prohibitive. To shed more light on this novel phenomenon in the CST-AAH model, we propose a “segmentation” method to expand our knowledge from flat to curved spacetime.

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**Figure S3**: (color online). Schematic diagram of the segmentation method: approximating the CST-AAH model by the standard AAH model in flat spacetime. Throughout, for convenience, we choose $J = 1$ as the unit of energy.
S2. THE SEGMENTATION METHOD

In this section, we briefly demonstrate how the segmentation method works as shown in Fig. S3. First, we consider a standard AAH chain with the nearest neighboring hopping strength $J$ and system size $N \to \infty$. Then we cut the chain in the middle to get two new AAH chains, with the hopping strength of the left chain being set at $0.5J$ and the right at $J$. In a system of size $N \to \infty$, the two new chains acquired by segmentation can be regarded as two individual flat-spacetime AAH models featuring different hopping strengths. Simply put, we have an AAH chain which consists of two standard flat-spacetime AAH subchains that are step-different in hopping coefficient.

In the same way as above, we cut the two subchains in the middle, respectively. That means the original AAH chain has been cut three times to become four segments, and then we set the hopping strengths of the four newly acquired subchains at $0.25J$, $0.5J$, $0.75J$, and $J$ from left to right. By doing so, we obtain a coupled AAH chain with three step-changes in hopping strength, which is composed of four individual flat-spacetime AAH models featuring different hopping strengths. Repeat the above segmentation $N - 2$ times and we will obtain a flat-spacetime AAH model comprising $N - 1$ segments with a series of hopping strengths. This reinvented AAH model can be used to theoretically analyze the CST-AAH model with $N$ sites, because the CST-AAH model with system size $N$ also exhibits hopping strength that changes with $N - 1$ times. Note that, the above segmentation model is equivalent to the CST-AAH model when both the system size and the number of segmentation approach infinity. In the next section, we will take a closer look at an example of the segmentation method.

S3. PHASE SEPARATION

I. The coupled AAH chain of two segments

The first important discovery of the segmentation method is phase separation, where a complete AAH chain in CST is divided into two parts, with the more warped end (near a black hole) featuring localized properties while the other end (away from a black hole) maintaining extended properties. Between the localized and the extended regions exists a clear boundary, whose analytical expression, based on the segmentation method, can be obtained through simple logical deduction. Next, we will focus on the phenomenon of phase separation during the transition from flat to curved spacetime by exploring the dynamical evolution of particles in the segmentation model.

For a standard AAH model in flat spacetime, the Hamiltonian takes the form

$$H = \sum_{j=1}^{N-1} J(\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) + \sum_{j=1}^{N} V_j \hat{c}_j^\dagger \hat{c}_j,$$  \hspace{1cm} (S4)

where $J$ is the hopping strength, which is equal at all sites. $\lambda$ is the strength of the quasiperiodic potential, and the rest of parameters are of the same property as in Eq. (1) (see the main text). The critical point of localized-extended phase transition for the standard AAH model is $\lambda_c = 2J$, i.e., when $\lambda \leq 2J$ ($\lambda \geq 2J$), all sites exhibit extended (localized) properties, simultaneously. We split it once and set the hopping strength of the left half of the chain at $0.5J$. The corresponding Hamiltonian reads

$$H_{s=2} = H_1 + H_2,$$  \hspace{1cm} (S5)

with

$$H_1 = \sum_{j=1}^{N} J(\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) + \sum_{j=1}^{N} \lambda \cos(2\pi \alpha j + \theta) \hat{c}_j^\dagger \hat{c}_j,$$ \hspace{1cm} (S6)

$$H_2 = \sum_{j=\frac{N}{2}+1}^{N-1} J(\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) + \sum_{j=\frac{N}{2}+1}^{N} \lambda \cos(2\pi \alpha j + \theta) \hat{c}_j^\dagger \hat{c}_j,$$ \hspace{1cm} (S7)

where the subscript $s = 2$ indicates that the Hamiltonian consists of two parts. One can find from the Hamiltonian that the phase transition point is at $\lambda_{c1} = 2J_1 = 2(0.5J) = J$ for a subchain satisfying $H_1$, while for $H_2$ subchain, the critical point is at $\lambda_{c2} = 2J_2 = 2J$. Therefore, as the quasiperiodic potential $\lambda$ increases, $H_1$ subchain will first enter the localized phase ($J < \lambda < 2J$), and then $H_2$ subchain will follow suit ($\lambda > 2J$).

In Fig. S4, we show the evolution behavior of wavepackets whose initial state positions are placed in different subchains. The wavepacket whose initial state is placed in the extended subchain will never enter the localized subchain. The evolution results
of wave functions again corroborate that the left half of the chain \( J_1 = 0.5J \) enters the localized phase first, while the right half enters the localized phase later. Remarkably, we find that when the parameters are taken between the critical points of the phase transition of the two chains \( J < \lambda < 2J \), there will appear a clear boundary between the localized and extended phases [see Fig. S4 column (b)]. This is like water and oil mixed together, which will, after being put still for a while, become spatially explicitly separated from each other. Numerical and theoretical results are consistent that when \( \lambda = 1.5 \), the critical site of phase separation lies in the middle of the whole chain, i.e., \( j_c = N/2 \) [see Fig. S4(b)].

II. The coupled AAH chain of four segments

Now we discuss the four subchains case, where the corresponding Hamiltonian reads

\[
H_{\text{c=4}} = H_1 + H_2 + H_3 + H_4, \tag{S8}
\]

with

\[
H_1 = \sum_{j=1}^{N} J \left( c_j^\dagger c_{j+1} + H.c. \right) + \sum_{j=1}^{N} \lambda \cos(2\pi \alpha j + \theta) c_j^\dagger c_j, \tag{S9}
\]

\[
H_2 = \sum_{j=\frac{N}{2}+1}^{N} J \left( c_j^\dagger c_{j+1} + H.c. \right) + \sum_{j=\frac{N}{2}+1}^{N} \lambda \cos(2\pi \alpha j + \theta) c_j^\dagger c_j, \tag{S10}
\]

\[
H_3 = \sum_{j=\frac{N}{2}+1}^{N} J \left( c_j^\dagger c_{j+1} + H.c. \right) + \sum_{j=\frac{N}{2}+1}^{N} \lambda \cos(2\pi \alpha j + \theta) c_j^\dagger c_j, \tag{S11}
\]

\[
H_4 = \sum_{j=\frac{N}{2}+1}^{N} J \left( c_j^\dagger c_{j+1} + H.c. \right) + \sum_{j=\frac{N}{2}+1}^{N} \lambda \cos(2\pi \alpha j + \theta) c_j^\dagger c_j, \tag{S12}
\]

Just as the two subchains case, the phase transition points of the four new AAH subchains are \( \lambda_{c1}, \lambda_{c2}, \lambda_{c3}, \lambda_{c4} = 0.5J, J, 1.5J, 2J \), respectively. Reusing the above analysis, we find that the system enters the localized phase in the following order: \( H_1 \) subchain, \( H_2 \) subchain, \( H_3 \) subchain, \( H_4 \) subchain. We show in Fig. S5, through the evolution behavior of the wavepacket over time, the phase separation phenomenon with different values of parameter \( \lambda \). Other calculation parameters, like the system size et al., are marked in the figure.

Fig. S5 shows the wavepacket dynamics of a coupled chain with four segments. The results show that when \( \lambda < \lambda_{c1} \), all the subchains of the system are in the extended phase, so the initial state placed in any position can be ergodic throughout the entire
Figure S5: (color online). The wavepacket evolution for (a) \( \lambda = 0.2 \), (b) \( \lambda = 0.7 \), (c) \( \lambda = 1.2 \), (d) \( \lambda = 1.7 \) and (e) \( \lambda = 3 \). The initial wavepacket are placed at \( j_0 = 1000 \), 2400 in column (a), \( j_0 = 300 \), 2400 in column (b), \( j_0 = 800 \), 2400 in column (c), \( j_0 = 1000 \), 2400 in column (d), and \( j_0 = 1000 \), 2400 in column (e). The system size \( N = 2584 \). The corresponding \( j_c = 646 \), 1292, 1938 in (b), (c) and (d), respectively.

chain [see Fig. S5 column (a)]. As \( \lambda \) increases, when \( \lambda_{c4} < \lambda < \lambda_{c4} \), a part of the entire chain will enter the localized phase and the rest will be in the extended phase. Therefore, there is a clear phase separation boundary \( j_c \) in the entire chain, and the wave function evolutions on both sides of the boundary exhibit localized and extended properties, respectively [see Fig. S5 column (b)(c)(d)]. Furthermore, as \( \lambda \) continues to increase, the entire chain becomes localized. Therefore, the wavepacket evolution exhibits localized characteristics [see Fig. S5 column (e)].

III. From the coupled AAH chain of N-1 segments to CST-AAH model

Finally, let’s turn to the case of \( N-1 \) segments. As shown above, under the limit of thermodynamics, subchains formed by segmentation can still be regarded as the standard AAH model in flat spacetime. Therefore, all the newly produced AAH subchains possess self-duality, which means that we can still deduce the localized-extended critical points of the subchains of the segmented AAH model from the conclusions of the standard AAH model. Since the hopping strengths difference \( \Delta J = J_1 - J_2 \), the critical point satisfies the expression

\[
\lambda = 2J_{j_c} = 2J\left(\frac{j_c}{N-1}\right).
\]

(S13)

As \( \lambda \) increases, subchains satisfying \( H_{j=1,2,\ldots,N-1} \) will successively turn from the extended phase to the localized phase from left to right. If we consider the more general case where hopping difference \( \Delta J \) between subchains is not a fixed value, then we need to modify the above expression as

\[
\lambda = 2J_{j_c} = 2J\left(\frac{j_c}{N-1}\right)^\sigma
\]

where \( \sigma \) is the correction coefficient which controls the changing rate of hopping strength.

In thermodynamic limit, the system is divided into infinite segments. Under such circumstances, the segmentation model is equivalent to the AAH model in CST. Therefore, through similar analysis, the analytical expression of the critical site \( j_c \) of the localized-extended phase separation of the CST-AAH model with system size \( N \) can be obtained as

\[
j_c = \left\lfloor \left(\frac{\lambda}{\Delta J}\right)^\frac{1}{\sigma} (N-1) \right\rfloor.
\]

(S14)

where \( \lfloor \cdot \rfloor \) is the symbol of floor function in mathematics, which rounds down the numbers to the nearest smaller integer. At the critical site \( j_c \), a clear phase boundary appears. For the fixed system size \( N \), we can see from the analytical expression that \( j_c \) is directly proportional to the strength of the quasiperiodic potential \( \lambda \) and inversely proportional to the hopping strength \( J \). Meanwhile, the CST parameter \( \sigma \) determines the changing rate of the phase separation critical point \( j_c \) versus \( \Delta J \).

Then, let’s discuss two extreme cases. We first consider the flat-spacetime scenario, where the CST parameter \( \sigma = 0 \) and the expression becomes

\[
j_c = \left\lfloor \left(\frac{\lambda}{\Delta J}\right)^{\infty} (N-1) \right\rfloor.
\]

As \( \lambda \) grows, the system will change from an extended to a critical phase. Specifically, when \( \lambda < 2J \), we get \( 0 < \frac{\lambda}{\Delta J} < 1 \), and the corresponding \( j_c = 0 \), thus the whole AAH chain is of the extended state. On the other hand, however, when \( \lambda > 2J \), one can get \( \frac{\lambda}{\Delta J} > 1 \), and then the corresponding critical site \( j_c = \infty \), which leads to the localized phase in the entire AAH chain. This is in accordance with our knowledge of the standard AAH model, where no phase separation can occur.

Now let’s turn to the other extreme, where the spacetime is severely warped. Without loss of generality, we take \( \sigma = \infty \). From Eq. (S14), we have the corresponding phase separation critical site at \( j_c = N-1 \), which is independent of the parameters \( \lambda \) and
Therefore, the extreme CST will completely freeze the hopping inside lattice systems of condensed matter to make the whole region localized.

Finally, we analyze the general case through Eq. (S14). Phase separation occurs when the spacetime curvature value is finite. Taking the parameters discussed in the main text as an example, when $\sigma = 1$, one can obtain the corresponding expression as $j_c = \left\lfloor \left( \frac{1}{2} \right) (N - 1) \right\rfloor$. From the above expression, one can obtain the exact position of the site where phase separation occurs, and here the $j < j_c$ part of the whole chain exhibits localized (extended) properties.

Similarly, we can depict this characteristic of the system through the wavepacket evolution. Fig. S6 shows the influence of different $\lambda$ on the position of the critical point of the phase separation. The wavepacket used for diagnosis are placed on both sides of the phase separation point, i.e., one in the extended chain, and the other one in the critical chain. In the above calculation, we fix the system size as $N = 2584$. The results well confirm our theoretical prediction that it is CST that induces the localized-extended phase separation.

![Figure S6: (color online). The wavepacket evolution of $\sigma = 1$ CST-AAH model with $\lambda = 0.1$ (a), 0.5 (b), 1 (c) and 1.5 (d). The initial positions of the first wave packet are at $j_0 = 50$ in (a), 300 in (b), 800 in (c) and 1000 in (d). The initial positions of wavepacket in extended region are all at $j_0 = 2300$. The width $w = 20$ for the bottom figure of column (a) and $w = 50$ for the rest of the figure. The system size $N = 2584$ and the corresponding $j_c = 129, 645, 1291, 1937$, respectively.](image)

**S4. MOBILITY EDGE AND NONERGODIC PHASES**

The occurrence of phase separation can affect the localization property of the system. In this section, we dive deeper into the nonergodic phases generated in the system based on the segmentation method.

**I. Two segments**

First, we discuss the two-segment case, where fractal dimensions are calculated to characterize the localization property of the system. As shown in Fig. S7(a), when the parameters $J < \lambda < 2J$, the fractal dimension indicates the existence of mobility edge in the system. Previous studies have shown that there are two ways to induce mobility edge in the standard AAH model, i.e., one can either introduce long-range hopping or provide an energy-dependent quasiperiodic potential. We take in this paper, however, a quite different approach of CST to induce mobility edge and get satisfactory results. Next, we will analyze the mobility edge through the Eigenenergy distribution and scaling behavior of the system. Without loss of generality, we fix the parameter $\lambda = 1.5$. In this case, the $H_1$ subchain becomes localized, while the $H_2$ subchain still remains extended. We show in Fig. S7(b) the Eigenenergy distribution of $E_{\text{ext}}$ and $E_{\text{loc}}$ of $H_1$ and $H_2$ subchains with respect to the level index $\beta$. The results show that there are three different phases in the system, namely, the localized phase (Region $I_1$ and $I_2$), the extended phase (Region $III_1$ and $III_2$) and the critical phase (Region $II_1$, $II_2$ and $II_3$). Their corresponding wave function eigenenergies show three different distributions, namely, the pure local subchain eigenvalues, the pure extended subchain eigenvalues, and the superposition of eigenvalues of the localized and extended subchains. To better understand the nonergodic phases, we scale the fractal dimensions of the eigenstate of the system. As shown in Fig. S7(c), the fractal dimension of the localized (extended phase) decreases (increases) with the ever-growing system size, while the gray region is independent of the system size, which is the evidence of the critical phase.
II. Four segments

Let’s move on to the four-segment case. As has been analyzed before, since hopping strengths of the four subchains are $J_1 = 0.25J$, $J_2 = 0.5J$, $J_3 = 0.75J$, $J_4 = J$, respectively, they will enter the localized phase successively. Fig. S8(a) shows the fractal dimension of the eigenstates, and we find that the mobility edge appears in the system when the parameter ranges $0.5J < \lambda < 2J$. Since the critical points of the four subchains entering the localized phase are different, the mobility edge in the Fig. S8(a) is actually composed of mobility edge of $H_{1,2,3,4}$ subchains. Then we fix $\lambda = 0.7J$ and take it as an example. In this case, only $H_1$ subchain is in the localized phase while the other three subchains are all in the extended phase.

Fig. S8(b) shows the eigenvalues’ distribution of the localized and the extended subchain versus level indices $\beta$, and the gray area indicates where the eigenvalues overlap. The distribution of eigenvalues can clearly depict the existence of multiple energy regions in the system, which belong to three different phases, namely, the localized, the extended and the critical phases. Through the analytical expression, we calculate the critical site of phase separation at $j_c = 646$. To prove the occurrence of three different phases in the system, we conduct scaling analysis again in different energy regions. The scaling behavior well reflects the differences between the regions as well as the three phases occurring in the system. This is consistent with the conclusion in the two-segment case, reaffirming that phase separation produces rich localized behaviors and mobility edge.

III. N-1 segments

Based on the above analysis, now we discuss the case of $N - 1$ segments, so as to generalize the theory to CST-AAH model. One can see that in the two-segment and four-segment cases, mobility edge displayed by the fractal dimension is zigzagged.
because the whole chain was segmented far less than enough. However, for the $N - 1$ segment case, the result of segmentation method is the same as that obtained by directly calculating the CST-AAH model, i.e., a very smooth mobility edge appears in the system. In Fig. S9, we illustrate how the fractal dimension of the CST-AAH model varies with the parameter $\lambda$. The results show that the mobility edge appears as the spacetime curves, and to what degree the spacetime curves has impact on the structure of the mobility edge. It is worth mentioning that since a very large CST parameter $\sigma$ will lead to a dwindling sub-extended region, the expansion of the system is inhibited. Therefore, the fractal dimension of the extended state demonstrated in the figure cannot turn red, i.e., no matter what the value of $\lambda$ is, the fractal dimension $\Gamma$ is always less than 1[see Fig. S9(d)].

Figure S9: (color online). Fractal dimensions with different CST parameters $\sigma = 1.5$ (a), 2 (b), 3 (c) and 100 (d). The system size $N = 2584$.

**S5. THE SPATIAL DISTRIBUTION OF EIGENSTATES FOR DIFFERENT PHASES**

To reveal phase separation, we also calculate the eigenstates of different phases versus quasiperiodic potential $\lambda$ in $\sigma = 1$. As shown in Fig. S10, the localized (extended) region becomes wider (narrower) with increasing $\lambda$. The eigenstates of localized and sub-extended phases are independent of $\theta$, while that of the critical phase depend on $\theta$, which agrees well with the previous analysis.

Figure S10: (color online). Density distribution of eigenstates for localized (top row), nonergodic critical (middle row) and sub-extended phase (bottom row). The system size $N = 2584$ and the corresponding $j_c = 904, 1291, 1937$ for column (a), (b) and (c), respectively.

**S6. THE COMPLETE PHASE DIAGRAM**

In the end, we provide a complete phase diagram of AAH model in CST by fractal dimensions under different parameters. In Fig. S11, by showing scaling behaviors of the fractal dimension at different energy $E$, we list and summarize all possible phases of the CST-AAH model in Fig. 4 in the main text. Different dashed boxes in Fig. S11 correspond to the results of $\sigma = 0$.
(flat spacetime), $\sigma = 1$ (CST) and $\sigma = \infty$ (CST to the extreme), respectively. One can see that although the CST-AAH model still experiences a change from the extended to the localized phases as the quasiperiodic potential $\lambda$ increases, the process of phase transition differs from AAH model in flat spacetime. To be specific, during phase transition of the CST-AAH model there appears four transitory stages, i.e., the extended phase steps into the critical phase first, and then gradually enters the localized phase.

Another way to distinguish among the three different phases and confirm the mobility edge is to examine the wave function itself. This method is called multifractal analysis and is often used to study the localized behavior of AAH models [6]. Through the scaling index in the multifractal analysis, we again test the correctness of the above results. The concrete calculation results are shown in Fig. S12. The fractal dimension and scaling index consistently prove the existence of mobility edge and nonergodic phases in CST-AAH model.

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Figure S12: (color online). The minimum of the scaling index $\alpha_{\text{min}}$ for $\sigma = 0$ (traditional AAH), $\sigma = 1$ (CST-AAH), $\sigma = \infty$ (the most curved case). The system size $N = 4181$ (green), $N = 10946$ (blue) and $N = 17711$ (red). The quasiperiodic potential strength $\lambda = 0$ (a), $\lambda = 1$ (b), $\lambda = 0$ (c), $\lambda = 0.5$ (d), $\lambda = 0.7$ (e), $\lambda = 1$ (f), $\lambda = 1.5$ (g), $\lambda = 3$ (h), $\lambda = 1$ (i), and $\lambda = 3$ (j). In computation, 100 times quasiperiodic averages have been performed on $\theta$.

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