THE CAUCHY PROBLEM FOR THE HOMOGENEOUS MONGE–AMPÈRE EQUATION, III. LIFESPAN

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ABSTRACT. We prove several results on the lifespan, regularity, and uniqueness of solutions of the Cauchy problem for the homogeneous complex and real Monge–Ampère equations (HCMA/HRMA) under various a priori regularity conditions. We use methods of characteristics in both the real and complex settings to bound the lifespan of solutions with prescribed regularity. In the complex domain, we characterize the $C^3$ lifespan of the HCMA in terms of analytic continuation of Hamiltonian mechanics and intersection of complex time characteristics. We use a conservation law type argument to prove uniqueness of solutions of the Cauchy problem for the HCMA. We then prove that the Cauchy problem is ill-posed in $C^3$, in the sense that there exists a dense set of $C^3$ Cauchy data for which there exists no $C^3$ solution even for a short time. In the real domain we show that the HRMA is equivalent to a Hamilton–Jacobi equation, and use the equivalence to prove that any differentiable weak solution is smooth, so that the differentiable lifespan equals the convex lifespan determined in our previous articles. We further show that the only obstruction to $C^3$ solvability is the invertibility of the associated Moser maps. Thus, a smooth solution of the Cauchy problem for HRMA exists for a positive but generally finite time and cannot be continued even as a weak $C^1$ solution afterwards. Finally, we introduce the notion of a “leafwise subsolution” for the HCMA that generalizes that of a solution, and many of our aforementioned results are proved for this more general object.

1. INTRODUCTION

This article is the third in a series [21][22] whose aim is to study existence, uniqueness, and regularity of solutions of the initial value problem (IVP) for geodesics in the space

$$
\mathcal{H}_\omega = \{ \varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}
$$

(1)

of Kähler metrics on a compact Kähler manifold $(M,\omega)$ in the class of $\omega$, where $\mathcal{H}_\omega$ is equipped with the metric [16][25][4]

$$
g^\omega(\zeta, \eta)_\varphi := \int_M \zeta \eta \omega^m_\varphi, \quad \varphi \in \mathcal{H}_\omega, \quad \zeta, \eta \in T_\varphi \mathcal{H}_\omega \cong C^\infty(M).
$$

This initial value problem is a special case of the Cauchy problem for the homogeneous complex/real Monge–Ampère equation (HCMA/HRMA). The IVP is long believed to be ill-posed, and a motivating problem is to prove that this is indeed the case, to determine which initial data give rise to solutions, especially those of
relevance in geometry (‘geodesic rays’), to construct the solutions, and to determine
the lifespan $T_{\text{span}}$ of solutions for general initial data.

In this article, we prove a number of results on the lifespan, regularity, and uniqueness
of solutions of the Cauchy problem for the HCMA and the HRMA equations
under various a priori regularity conditions. The results are based on a study of the
‘characteristics’ of the HCMA/HRMA equations, or more precisely on the relations
between solutions of these equations and Hamiltonian mechanics, and to solutions
of related Hamilton–Jacobi equations.

First, we characterize the $C^3$ lifespan of the HCMA and prove uniqueness of classical solutions. We then introduce the notion of a leafwise subsolution of the HCMA
that generalizes the notion of a solution, and derive obstructions to its existence.
This can be considered as a method of ‘complex characteristics’. Combining these
results we establish that the IVP for the HCMA is locally ill-posed in $C^3$. This puts
a restriction on Cauchy data, and addresses questions about the Cauchy problem
raised by the work of Mabuchi, Semmes, and Donaldson [16, p. 238], [25], [4, p. 27].
We then study the notion of a leafwise subsolution for the HRMA, and prove its
uniqueness. This allows us to characterize the Legendre transform subsolution of
the prequels [21, 22], and determine the $C^1$ lifespan of the HRMA. A key ingre-
dient here is an apparently new connection between HRMA and Hamilton–Jacobi

equations.

1.1. Obstructions to solvability, uniqueness, and the smooth lifespan of
the HCMA. We begin in the complex domain, where Semmes and Donaldson [25]
[4] gave a formal solution of the IVP in terms of holomorphic characteristics. Namely,
the Cauchy data $(\omega_{\varphi_0}, \dot{\varphi}_0)$ of the IVP determines a Hamiltonian flow $\exp t X_{\dot{\varphi}_0}^{\omega_{\varphi_0}}$.
If the orbits $\exp t X_{\dot{\varphi}_0}^{\omega_{\varphi_0}} z$ of the flow admit analytic continuations in time up to
imaginary time $T$, one obtains a family of maps

$$f_{\tau}(z) = \exp -\sqrt{-1} \tau X_{\dot{\varphi}_0}^{\omega_{\varphi_0}} z : S_T \times M \to M,$$

where $S_T = [0, T] \times \mathbb{R}$

with $\tau = s + \sqrt{-1} t \in S_T$, $s \in [0, T]$ and $t \in \mathbb{R}$. The formal solution $\varphi_s$ is then given
by the formula,

$$(f_s^{-1})^{*} \omega_{\varphi_0} - \omega_{\varphi_0} = \sqrt{-1} \partial \bar{\partial} \varphi_s, \quad s \in [0, T].$$

There are several obstructions to solving the IVP in this manner, which must
vanish if there exists a $C^3$ solution. The most obvious one is that the Hamilton orbits
need to possess analytic continuations to a strip $S_T$. This analytic extension problem
for orbits should already be an ill-posed problem, and we say that the Cauchy data
is “$T$-good” if the extension exists and $f_s$ is smooth (see Definitions 2.3–2.3). This
is a Cauchy problem for a holomorphic map into a nonlinear space, and we do not
study it directly here; but in §1.2 we describe some results on obstructions to closely
related linear Cauchy problems.

In several settings, such as torus-invariant Cauchy data on toric varieties, the
Hamilton orbits for smooth Cauchy data do possess analytic continuations (see
Proposition 1.4 below). As the following theorem shows, the only additional obstruction to solving the HCMA smoothly is that the space-time complex Hamilton orbits may intersect. To state the result precisely, let \((M, J, \omega)\) be a compact closed connected Kähler manifold of complex dimension \(n\). The IVP for geodesics is equivalent to the following Cauchy problem for the HCMA

\[
(\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^{n+1} = 0, \quad (\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n \neq 0, \text{ on } S_T \times M, \\
\varphi(0, t, \cdot) = \varphi_0(\cdot), \quad \partial_s \varphi(0, t, \cdot) = \dot{\varphi}_0(\cdot), \text{ on } \{0\} \times \mathbb{R} \times M.
\]  

(4)

where \(\pi_2 : S_T \times M \to M\) is the projection, and where \(\varphi\) is required to be \(\pi_2^*\omega\)-plurisubharmonic (psh) on \(S_T \times M\). The rest of the notions in the following theorem are defined in [2].

**Theorem 1.1.** (Smooth lifespan and uniqueness) Let \((M, \omega_{\varphi_0})\) be a compact Kähler manifold. The Cauchy problem (4) with \(\omega_{\varphi_0} \in C^1\) and \(\dot{\varphi}_0 \in C^3(M)\) has a solution in \(C^3(S_T \times M) \cap PSH(S_T \times M, \pi_2^*\omega)\) if and only if the Cauchy data is \(T\)-good and the maps \(f_s\) defined by (2) are \(C^1\) and admit a \(C^1\) inverse for each \(s \in [0, T]\). The solution is unique in \(C^3(S_T \times M) \cap PSH(S_T \times M, \pi_2^*\omega)\).

This result is important in clarifying the nature of the obstructions to solving the HCMA. The existence proof follows by a rather straightforward combination of the Semmes–Donaldson arguments [25, 4]. The uniqueness proof, somewhat surprisingly, does not readily adapt from the \(C^\infty\) setting studied by Bedford–Burns [2]. Unlike in their setting, the proof is not local in nature, and requires a global conservation law type argument. The key difference is that the stripwise equations vary from leaf to leaf, and one has to prove an a priori estimate that ensures that the stripwise elliptic problems are not degenerating. The uniqueness proof is also completely different from the corresponding proof for the Dirichlet problem, where the maximum principle is available.

Henceforth, we describe breakdown in terms of lifespan.

**Definition 1.2.** Let the \(C^{k,\alpha}\) lifespan \(T_{\text{span}}^{k,\alpha}\) (respectively, lifespan \(T_{\text{span}}\)) of the Cauchy problem (4) be the supremum over all \(T \geq 0\) such that (4) admits a solution in \(C^{k,\alpha}(S_T \times M) \cap PSH(S_T \times M, \pi_2^*\omega)\) (respectively, in \(PSH(S_T \times M, \pi_2^*\omega)\)).

We thus have the following characterization of the smooth lifespan of the HCMA. The same result holds also for the \(C^3\) lifespan \(T_{\text{span}}^3\).

**Corollary 1.3.** The smooth lifespan \(T_{\text{span}}^\infty\) of the Cauchy problem (4) with smooth initial data is the supremum over \(T \geq 0\) such that the Cauchy problem is \(T\)-good and the maps \(f_s\) defined by (7) are smoothly invertible for each \(s \in [0, T]\).

**1.2. Leafwise subsolutions for the HCMA and ill-posedness.** In the apparent absence of weak solutions beyond the convex lifespan, motivated by the detailed results of the prequel [22] on the Legendre subsolution in the special case of the HRMA, we are led to introduce a notion of a leafwise subsolution, that should be an “optimal” subsolution in some situations.

**Definition 1.4.** Assume that the Cauchy problem (4) is \(T\)-good. We call a \(\pi_2^*\omega\)-psh function \(\varphi\) on \(S_T \times M\) a \(T\)-leafwise subsolution of the HCMA (4) if it satisfies the...
initial conditions of the Cauchy problem \((4)\), if
\[
(\pi^*_2 \omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n \neq 0,
\]
and if for each \(z \in M\), we have
\[
\gamma_z^*(\pi^*_2 \omega + \sqrt{-1} \partial \bar{\partial} \varphi) = 0, \tag{5}
\]
where \(\gamma_z(\tau) = (\tau, f_\tau(z))\), and \(f_\tau(z) := \exp -\sqrt{-1} \tau X_{\varphi_0} \varphi\).

The proof of Theorem 1.1 shows that a \(C^3\) solution on \([0, T] \times M\) is a \(T\)-leafwise subsolution, but “leafwise subsolutions” are more general: a subsolution of HCMA may solve \((5)\) along leaves without solving the HCMA globally since the invertibility condition on \(f_s\) in Theorem 1.1 may fail, e.g., when the leaves intersect.

One of our main results is that the problem of existence of a leafwise subsolution for the Cauchy problem for the HCMA is already locally ill-posed in time. In particular, this implies the ill-posedness in \(C^3\) of the Cauchy problem for the HCMA itself.

**Theorem 1.5.** (Local ill-posedness) For each \(\varphi_0 \in \mathcal{H}_\omega\) there exists a dense set of \(\dot{\varphi}_0 \in C^3(M)\) for which \(T_{\text{span}}^\infty = T_{\text{span}}^3 = 0\), i.e., the IVP \((4)\) admits no \(C^3\) solution for any \(T > 0\).

The proof is given in Section 3. The Cauchy problem for the HCMA is a multidimensional, nonlinear generalization of the Cauchy problem for the Laplace equation \(\Delta u = 0\) on each strip \(S_T\), which is one of the classic ill-posed problems of Hadamard \([10, 15, 17]\). One might think that Theorem 1.5 could be obtained directly from the well-known ill-posedness of the Cauchy problem for the Laplace equation on a strip.

However, this is not the case: the leafwise equations \((5)\) are inhomogeneous and depend on the solution of the HCMA itself. Second, and perhaps more basic, is that the strip on which the problem is posed depends on the solution of the HCMA. The standard argument of Hadamard (for the classical Laplace equation) of perturbing the Cauchy data so as to lie outside the range of the Dirichlet-to-Neumann operator therefore cannot be applied directly as it would also perturb the leaves themselves!

The actual proof does employ the Dirichlet-to-Neumann operators along each leaf but also uses a geometric perturbation argument. First, we analyze the obstructions for a leafwise subsolution in detail, and show that, for each \(z \in M\), the pull-back of a leafwise subsolution under \(\gamma_z\) (the map of restriction to the leaf through \(z\) defined in Definition 1.4) satisfies a certain real-analyticity condition on the initial boundary \(\{0\} \times \mathbb{R}\) of \(S_T\); more precisely, a certain function of the Cauchy data is real analytic on \(\mathbb{R}\) and possesses an analytic continuation to a two-sided strip \([-T, T] \times \mathbb{R}\) of width precisely \(T\). We refer to Proposition 3.3 for the precise statement. Second, we combine Theorem 1.1 and Proposition 3.3 with a geometric perturbation argument and basic properties of the Hilbert transform to derive a real-analyticity condition, independent of \(T\).

Theorem 1.5 is thus based on the analytic continuation obstruction of Theorem 1.1. In the remainder of the paper we concentrate on the second obstruction, i.e., the invertibility of the Moser maps \(f_s\) appearing in Theorem 1.1. It is present even
in the simplest case of toric Kähler manifolds. As we will see, even when the strip-wise Cauchy problems can all be solved, there does not generally exist a global in time solution of the HCMA.

1.3. Complementary results for the HRMA. The local ill-posedness result, Theorem 1.5, does not apply to the study of the HRMA. The Cauchy problem for the HRMA arises precisely when the Cauchy data is torus-invariant, which is, of course, non-generic in the space of all possible Cauchy data. And in fact, the Cauchy problem for the HRMA has a positive smooth lifespan [22]. Moreover, as we observe in Proposition 1.9 there is no obstruction to analytically continuing orbits. In the remainder of the article our goal is thus to derive results for the HRMA that are somewhat of a complementary nature to those for the HCMA described above. First, we would like to understand how our characterization of the smooth (or \(C^3\)) lifespan specializes to the setting of the HRMA. Second, we would like to understand lifespan of solutions with less regularity than that described in Theorem 1.1, that is less than \(C^3\).

Theorem 1.1 clarifies the breakdown of classical solutions of the Cauchy problem for the HCMA already in the toric case. When the Cauchy data is \((S^1)^n\)-invariant, the equation reduces to the HRMA

\[
\text{MA} \psi = 0, \quad \text{on} \quad [0, T] \times \mathbb{R}^n, \quad \psi(0, \cdot) = \psi_0(\cdot), \quad \partial_s \psi(0, \cdot) = \dot{\psi}_0(\cdot), \quad \text{on} \quad \mathbb{R}^n, \quad (6)
\]

that describes geodesics in the space of toric Kähler metrics, where \(\text{MA}\) denotes the real Monge–Ampère operator that associates a Borel measure to a convex function and equals \(\det \nabla^2 f \, dx^1 \wedge \cdots \wedge dx^{n+1}\) on \(C^2\) functions (see [22 §2.2] and [19]). Here \(\psi_0\) is a smooth strictly convex function (moreover, with strictly positive Hessian) on \(\mathbb{R}^n\) that corresponds to a torus-invariant Kähler metric, i.e., \(\omega_{\psi_0} = \sqrt{-1} \partial \bar{\partial} \psi_0\) over the open orbit and \(\dot{\psi}_0\) is a smooth torus-invariant function on \(M\), considered as a smooth bounded function \(\dot{\psi}_0\) on \(\mathbb{R}^n\). Thus, we view \(\mathbb{R}^n\) as the real slice of \(M\) (minus its divisor at infinity). We also refer to the real slices of the leaves of the Monge–Ampère foliation as leaves. Also, \(\text{Im} \nabla \psi_0 = P\) is a compact convex polytope in \(\mathbb{R}^n\). Translating the definition of \(\pi_0^* \omega_0\)-psh solutions to the HCMA [4] to the real setting yields a corresponding class for the HRMA.

Before defining the class we recall the definition of the Monge–Ampère operator. Let \(M(\mathbb{R}^{n+1})\) denote the space of differential forms of degree \(n + 1\) on \(\mathbb{R}^{n+1}\) whose coefficients are Borel measures (i.e., currents of degree \(n + 1\) and order 0).

**Proposition 1.6.** (See [19 Proposition 3.1]) **Define by**

\[
\text{MA} f := d \frac{\partial f}{\partial x^1} \wedge \cdots \wedge d \frac{\partial f}{\partial x^{n+1}},
\]

**an operator** \(\text{MA} : C^2(\mathbb{R}^{n+1}) \to M(\mathbb{R}^{n+1})\). **Then** \(\text{MA}\) **has a unique extension to a continuous operator on the cone of convex functions.**
A result of Alexandrov shows that for any convex function \( f \), the measure \( MAf \), defined by \( (MAf)(E) := \text{Lebesgue measure of } \partial f(E) \), where \( \partial f \) denotes the sub-differential mapping of \( f \) (see [22, §2.1]), is a Borel measure ([19, Section 2]). Furthermore, according to Rauch–Taylor \( MAf = MAf \) for every convex function \( f \) on \( \mathbb{R}^{n+1} \) [19, Proposition 3.4].

**Definition 1.7.** A convex function \( \rho \) on \([0,T] \times \mathbb{R}^n\) is an Alexandrov weak solution of the HRMA \( MA\rho = 0 \), if the image of the subdifferential mapping \( \partial \rho : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n+1} \) is a set of Lebesgue measure zero.

We now define our class of “admissible solutions” to the HRMA to be Alexandrov weak solutions with the property that the image under the spatial sub-differential of the solution is a fixed polytope for all times. The assumption means that the solutions are potentials of non-degenerate Kähler metrics for each \( s \) that stay in the same Kähler class. It seems that only these solutions are relevant to toric Kähler geometry.

**Definition 1.8.** An admissible subsolution to \((6)\) is a convex function on \([0,T] \times \mathbb{R}^n\) satisfying (i) \( \psi(0, \cdot) = \psi_0(\cdot) \) and \( \partial_s \psi(0, \cdot) = \dot{\psi}_0(\cdot) \) on \( \mathbb{R}^n \), and (ii) \( \psi(s) : \mathbb{R}^n \to \mathbb{R} \) is strictly convex, and \( \text{Im} \partial \psi(s) = P \) for each \( s \in [0,T] \). An admissible solution in addition is a weak solution of \( MA \psi = 0 \) in the sense of Alexandrov.

Note that this definition assumes \( \psi_0 \) and \( \dot{\psi}_0 \) to satisfy the regularity, growth, and convexity assumptions of the previous paragraphs.

In the previous article, we showed that the Legendre transform method for solving the HRMA breaks down at the convex lifespan
\[
T_{\text{span}}^{\text{cvx}} := \sup \{ s > 0 : \psi_0^* - s \dot{\psi}_0 \circ (\nabla \psi_0)^{-1} \text{ is convex} \},
\] (7)
where \( \psi_0^* \) denotes the Legendre transform of \( \psi_0 \) [22, Theorem 1]. The next proposition shows that there is no obstruction for the Hamilton orbits to admit analytic extensions to strips nor for the maps \((2)\) to be smooth, and that the only obstruction to smooth solvability is the invertibility of these maps, that we refer to as Moser maps (see Definition 2.2).

**Proposition 1.9.** Let \((M,J,\omega_{\varphi_0})\) be a toric Kähler manifold, and let \( \dot{\varphi}_0 \in C^3(M) \) be torus-invariant. Then,

(i) The Cauchy problem \((4)\) for \((\omega_{\varphi_0},\dot{\varphi}_0)\) is \( T \)-good for every \( T > 0 \).

(ii) The maps \( f_s(z) = \exp -\sqrt{-1} s X_{\omega_{\varphi_0}}.z \) \((2)\) are invertible if and only if \( s \in [0,T_{\text{span}}^{\text{cvx}}) \).

This result, together with Theorem 1.1, determines the smooth lifespan for toric geodesics, as well as characterizes all smooth toric geodesic rays.

**Corollary 1.10.** (Characterization of smooth toric geodesics) (i) The smooth lifespan of the Cauchy problem \((5)\) with smooth Cauchy data coincides with the convex lifespan \((7)\), \( T_{\text{span}}^\infty = T_{\text{span}}^{\text{cvx}} \).

(ii) Smooth geodesic rays in the space of toric metrics are in one-to-one correspondence with admissible solutions of the Cauchy problem \((6)\) with \( \psi_0 \in C^\infty(\mathbb{R}^n) \),
\[ \nabla^2 \psi_0 > 0, \, \text{Im} \nabla \psi_0 = P, \, \dot{\psi}_0 \in C^\infty \cap L^\infty(\mathbb{R}^n), \text{ and } \dot{\psi}_0 \circ (\nabla \psi_0)^{-1} \text{ a concave function on } P. \]

Next, we show that in the case of the HRMA the leafwise obstruction vanishes and characterizes the Legendre transform subsolution among all subsolutions of the Cauchy problem.

**Proposition 1.11.** (i) The Legendre transform potential, given by
\[
\psi_L(s, x) : = (\psi_0 - s \dot{\varphi}_0 \circ (\nabla \psi_0)^{-1})^*(z), \quad x \in \mathbb{R}^n, \ s \in \mathbb{R}_+,
\]
(8)
is the unique admissible leafwise subsolution to the HRMA (6) for all \( T > 0 \).

(ii) The corresponding unique admissible leafwise subsolution to the HCMA (4) is given by
\[
\varphi_L(s + \sqrt{-1}t, e^{x+\sqrt{-1}t}) := \psi_L(s, x) - \psi_0(x). \quad (9)
\]

Observe that the uniqueness result in (i) holds under much weaker regularity than that needed in Theorem 1.1.

However, the possibility remains that a solution could persist beyond \( T_{\text{span}}^{\text{cvx}} \), but not be given by the Legendre transform method. But by following the lead of Theorem 1.1 in the case of the HRMA, we show that there cannot exist any \( C^1 \) weak solution in the Alexandrov sense beyond \( T_{\text{span}}^{\text{cvx}} \). The result is a regularity statement.

**Theorem 1.12.** (\( C^1 \) lifespan of HRMA) Any admissible \( C^1 \) weak solution to the Cauchy problem (6) with Cauchy data \( \psi_0 \in C^\infty(\mathbb{R}^n), \nabla^2 \psi_0 > 0, \, \text{Im} \nabla \psi_0 = P, \, \dot{\psi}_0 \in C^\infty \cap L^\infty(\mathbb{R}^n) \) is smooth. Thus, \( T^1_{\text{span}} = T_{\text{span}}^{\text{cvx}} \).

This generalizes a classical theorem of Pogorelov on the developability of flat (in a suitable sense) \( C^1 \) surfaces in \( \mathbb{R}^3 \). In the language of geodesics in the infinite dimensional symmetric space \( \mathcal{H}_\omega \) [16, 25, 4], it shows that the exponential map fails to be globally defined even when \( C^1 \) weak solutions are allowed. It is interesting to observe that Pogorelov’s result for \( n = 1 \) involves a quite intricate proof [18, 24]. In higher dimensions, this result has been known previously under the rather stronger assumption of \( C^2 \) regularity or more, i.e., for classical solutions [11, 7, 8, 28].

The proof of Theorem 1.12 uses the following characterization of the HRMA in terms of a Hamilton–Jacobi equation:

**Theorem 1.13.** (HRMA and Hamilton–Jacobi) \( \eta \in C^1([0, T \times \mathbb{R}^n) \) is an admissible weak solution of the HRMA (6) if and only if it is a classical solution of the Hamilton–Jacobi equation
\[
F(\nabla \eta) = 0, \quad \eta(0, \cdot) = \psi_0, \quad (10)
\]
where \( F(\sigma, \xi) = \sigma - \dot{\psi}_0 \circ (\nabla \psi_0)^{-1}(\xi) \), where \( \sigma \in \mathbb{R}, \xi \in \mathbb{R}^n \).

Theorem 1.13 reduces the HRMA to a first-order equation for which a well-known theory for solutions exists—based on the method of characteristics. The Hamilton–Jacobi equation is a ‘conservation law’ for the HRMA. It may be viewed as combining the conservation law \( \dot{\varphi}_s \circ f_s = \dot{\varphi}_0 \) of Proposition 6.3 (see (2) for notation) with the explicit formula for \( f_s^{-1} \) in (74); see also (60). This makes rigorous as well as
generalizes to weak solutions the folklore idea \cite{11,7,8,28} that classical solutions
of HRMA—despite being of second-order—can be obtained by integrating along
‘characteristics’ just like a first-order equation, indeed they are affine along lines
determined by the Cauchy data.

Theorem 1.12 follows from Proposition 1.11 and Theorem 1.13. Except for one
step (Proposition 1.14), the proof is short and we give it here:

Proof of Theorem 1.12. Given the results of \cite{22}, the main new step of the proof
of Theorem 1.12 is the following generalization to weak $C^1$ admissible solutions
of HRMA (6) of the fact (see Section 2) that every $C^3$ $\pi^*_2\omega$-psh solution of the
HCMA (4) is a leafwise subsolution.

Proposition 1.14. Let $(M,J,\omega,\varphi_0)$ be a toric Kähler manifold, and let $\varphi_0 \in C^\infty(M)$
be torus-invariant. Assume that the corresponding Cauchy problem for the HCMA
(4) is $T$-good. Then any $C^1$ $\pi^*_2\omega$-psh solution of the HCMA (4) up to time $T$
is the unique $T$-leafwise subsolution.

The proof of Proposition 1.14 is based on Theorem 1.13 and uniqueness of $C^1$
solutions of the Hamilton–Jacobi equation.

We now complete the proof Theorem 1.12 assuming Proposition 1.14. This is
possible since the $T$-good assumption is satisfied in the toric setting. The proof is
simple and is given in Lemma 4.1. By Proposition 1.11 there exists a unique leafwise
subsolution $\varphi_L$ of the toric HCMA (see \cite{9}), induced by the Legendre transform
potential \cite{8}. By Proposition 1.14 any $\pi^*_2\omega$-psh $C^1$ solution of (4) on a toric variety
must coincide with $\varphi_L$. However, $\varphi_L \not\in C^1$ for $T > T_{\text{span}}^{\infty}$ \cite{22} Proposition 1]. Hence,
there exists no admissible $C^1$ weak solution of the IVP for $T > T_{\text{span}}^{\infty}$, concluding
the proof of Theorem 1.12.

For sufficiently regular $\eta$, Theorem 1.13 can be proved in a symplectic geometric
way by observing that the Lagrangian submanifold $\Lambda_\eta := \text{graph}(d\eta)$ of $T^*(\mathbb{R} \times \mathbb{R}^n)$
lies in a level set of the Hamiltonian $F$. When $\Lambda_\eta$ is sufficiently smooth, it must
then be invariant under the Hamilton flow of $F$. When $\Lambda_\eta$ is Lipschitz, for instance,
we can use flat forms and chains to prove the latter statement, and obtain:

Proposition 1.15. Let $\eta \in C^{1,1}([0,T] \times \mathbb{R}^n)$ be an admissible weak solution of the
HRMA. Then the Lipschitz Lagrangian submanifold $\Lambda_\eta := \text{graph}(d\eta) \subset T^*\mathbb{R}^{n+1}$ is
foliated by straight line segments along each of which $\nabla \eta$ is constant.

We postpone the details of this symplectic approach to the HRMA and the proof
of this proposition to a sequel \cite{23}, where we also pursue a complex analogue for the
HCMA.

1.4. Organization. The characterization of the smooth lifespan and uniqueness of
classical solutions (Theorem 1.1) is proved in Section 2. The ill-posedness of the
leafwise problem, Theorem 1.5, is proved in Section 3. Proposition 1.9 concerning
the obstructions to solvability and the characterization of the smooth lifespan in the
toric setting is proved in Section 4. The characterization of the Legendre potential
as the unique leafwise subsolution is proved in Section 5. The characterization of the
C^1 lifespan for the HRMA is given in Section 6, where we also prove the equivalence between HRMA and a Hamilton–Jacobi equation.

2. Smooth lifespan of the HCMA: Proof of Theorem 1.1

Before proving Theorem 1.1 we need to introduce some terminology and background related to the ill-posedness of the Cauchy problem.

**Definition 2.1.** We say that the Cauchy problem \((4)\) with smooth initial data \((M, \omega_{\varphi_0}, \dot{\varphi}_0)\) is \(T\)-Hamiltonian analytic if for every \(z \in M\) the orbit of \(z\) under the Hamiltonian flow of \(\dot{\varphi}_0\) with respect to \(\omega_{\varphi_0}\) admits a holomorphic extension to the strip \(S_T\).

Here, by a holomorphic extension of a map \(\gamma : \mathbb{R} \to M\) to \(S_T\) we mean a holomorphic map \(\tilde{\gamma} : S_T \to M\) such that \(\tilde{\gamma}(0, t) = \gamma(t)\). Such an extension is unique when it exists (this can be seen either by the Cauchy-Riemann equations or by the Monodromy Theorem). When it exists for \(z \in M\) we denote it by \(\exp_{\sqrt{-1}} t X_{\dot{\varphi}_0} z, \tau \in S_T\) the holomorphic strip extending the Hamiltonian orbit \(\exp t X_{\dot{\varphi}_0} z\).

**Definition 2.2.** The Moser maps are defined by

\[
\{ (\tau, \Gamma_z(\tau)) : \tau \in S_T \} \subset S_T \times M, \quad \text{with } \Gamma_z : S_T \to M. \tag{12}
\]

For each \(\tau \in S_T\) define the map \(f_\tau : M \to M\) by

\[
f_\tau(z) := \Gamma_z(\tau). \tag{13}
\]

**Definition 2.3.** We say that the Cauchy problem \((4)\) is \(T\)-good if it is \(T\)-Hamiltonian analytic and if the Moser map \(f_\tau\) is a differentiable map of \(M\) for each \(\tau \in S_T\).
By the transversality condition and the fact that the leaves do not intersect each other (follows from uniqueness for ODEs with $C^1$ coefficients—here we used the $C^3$ assumption for the second time) it follows that $f_\tau$ is a $C^1$ diffeomorphism.

It remains to prove that the maps $f_\tau$ are Moser maps in the sense of Definition 2.2. Since the strips are constructed by integrating the vector field $\frac{\partial}{\partial \tau} + \frac{df}{dt}$ in $S_T \times M$, this vector field lies in the kernel of $\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi$. Therefore,

$$f_\tau^* \omega_{\varphi_\tau} = \omega_{\varphi_0}.$$  \hspace{1cm} (14)

Now, since $\varphi_0, \dot{\varphi}_0$ are invariant under the $\mathbb{R}$-action $(\tau, z) \mapsto (\tau + \sqrt{-1} c, z), c \in \mathbb{R}$, uniqueness of smooth solutions implies that so is $\varphi_\tau$. The proof of uniqueness is postponed to Lemma 2.6 below, however its proof does not rely on the rest of this subsection. By abuse of notation we write $\varphi_s = \varphi_\tau$ when no confusion arises, where $\tau = s + \sqrt{-1} t$.

Next,

$$\frac{df_\tau}{dt} = X^{\omega_{\varphi_s}} \circ f_\tau = -J \nabla g_{\varphi_s} \dot{\varphi}_s \circ f_\tau, \quad \frac{df_\tau}{ds} = -\nabla g_{\varphi_s} \dot{\varphi}_s \circ f_\tau, \quad f_0 = \text{id},$$  \hspace{1cm} (15)

since $\frac{\partial}{\partial \tau} - \nabla^{1,0}_{g_{\varphi_s}} \dot{\varphi}_s \in \ker(\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi)|_{(\tau, \Gamma_s(\tau))}$, indeed

$$\frac{\partial}{\partial \tau} (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi) = -\sqrt{-1} \partial \frac{\partial \varphi}{\partial \tau} = -\sqrt{-1} \partial \dot{\varphi}_s,$$

and

$$\nabla^{1,0}_{g_{\varphi_s}} \dot{\varphi}_s (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi) = \nabla^{1,0}_{g_{\varphi_s}} \dot{\varphi}_s \omega_{\varphi_s} = \mathcal{L}_{\dot{\varphi}_s} \varphi_s = \sqrt{-1} (\partial - \bar{\partial}) \dot{\varphi}_s,$$

and we use the convention $\frac{\partial}{\partial \tau} = \frac{1}{2} \frac{\partial}{\partial s} - \frac{\sqrt{-1}}{2} \frac{\partial}{\partial t}$ and $Y^{1,0} = \frac{1}{2} Y - \frac{\sqrt{-1}}{2} J Y$.

It then follows from (14) that

$$f_{s+\sqrt{-1} t} = h_{s+\sqrt{-1} t} \circ f_s,$$  \hspace{1cm} (16)

with $h_{s+\sqrt{-1} t}$ a $C^1$ symplectomorphism of $(M, \omega_{\varphi_s})$. Also, from (16) and (15)

$$h_{s+\sqrt{-1} t} = \exp t X^{\omega_{\varphi_s}}.$$  \hspace{1cm} (17)

We conclude therefore from (16) and (15) that the maps $f_\tau$ defined by (13) satisfy (11), i.e., for each $z \in M$, induce analytic continuation to the strip of the Hamiltonian orbit $\exp t X^{\omega_{\varphi_0}} z$. Hence we have shown both that the Cauchy data is $T$-good and that the Moser maps of Definition 2.2 are $C^1$ and admit $C^1$ inverses for each $s \in [0, T]$. This completes the proof of the first half of Theorem 1.1.

We conclude this subsection with some further properties of the Moser maps. In view of (3), the Moser maps which are relevant to the solution of HCMA are the ones with $t = 0$, and their definition only requires analytic continuation of the Hamiltonian flow of $X^{\omega_{\varphi_0}}$ to a rectangle $[0, T] \times (-\epsilon, \epsilon)$. However, such an analytic continuation necessarily induces one to the strip $S_T$.

**Corollary 2.4.** Suppose that $h_{\sqrt{-1} t} = \exp t X^{\omega_{\varphi_0}} z$ admits an analytic continuation to $[0, T] \times (-\epsilon, \epsilon)$. Then $h_{\sqrt{-1} t}$ admits an analytic continuation to $S_T$. 


Indeed, by (16), \( h_{s+\sqrt{-1}t}(z) \) is the orbit of a Hamiltonian flow for fixed \( s \) and varying \( t \in \mathbb{R} \). Hence it may be holomorphically extended by the group law
\[
\exp(t_1 + t_2)X_{\omega_s}^*(z) = \exp t_1 X_{\omega_s}^* \exp t_2 X_{\omega_s}^* z.
\]
Therefore, by (16), one may define \( f_{s+\sqrt{-1}t} \) for all \( s + \sqrt{-1}t \in S_T \).

**Remark 2.5.** In comparison to this group law for fixed \( s \), \( f_{s+\sqrt{-1}t} \) does not satisfy a group law in the complex parameter \( s + \sqrt{-1}t \) and thus we cannot conclude that the flow has an analytic continuation to a half-plane by the same argument. This may be seen from the fact that \( X_{\hat{\varphi}_s}^* \) does not Lie-commute with its image under \( J \). Indeed, commutativity fails even for generic Cauchy data in the case of toric varieties—see Remark [4.2](#).

### 2.2. Existence of classical solutions for the HCMA.

In this subsection we continue the proof of Theorem [1.1](#) and establish the existence of a classical solution to the HCMA under our assumptions.

We now assume that the Cauchy problem for \((\omega_{\varphi_0}, \varphi_0)\) is \( T \)-good and solve the HCMA under the additional assumption of invertibility.

**Lemma 2.6.** Let \( \omega_{\varphi_0} \in C^1 \) and \( \varphi_0 \in C^3 \). Assume that the Cauchy problem for \((\omega_{\varphi_0}, \varphi_0)\) is \( T \)-good, and that for each \( \tau \in S_T \) the map \( f_\tau \) given by (17) is smoothly invertible. Then the HCMA (4) admits a \( C^3 \pi^\tau \omega \)-psh solution.

**Proof.** Define a \( C^3 \) function on \( S_T \times M \) by
\[
\varphi(s + \sqrt{-1}t, z) := -\sqrt{-1}\partial^*_{\omega_{\varphi_0}} \bar{\partial}^*_{\omega_{\varphi_0}} G^2_{\omega_{\varphi_0}} ((f_\tau^{-1})^* \omega_{\varphi_0} - \omega_{\varphi_0})(z) + \varphi_0(z) + \frac{s}{V} \int_M \varphi_0 \omega_{\varphi_0}^n,
\]
where \( G_{\omega_{\varphi_0}} \) denotes Green’s function for the Laplacian \( \Delta_{\omega_{\varphi_0}} = -\partial^* \partial - \bar{\partial}^* \bar{\partial} \) acting on forms. The operator \( \sqrt{-1}\partial^*_{\omega_{\varphi_0}} \bar{\partial}^*_{\omega_{\varphi_0}} G^2_{\omega_{\varphi_0}} \) is a pseudo-differential operator of order \(-2\) with smooth coefficients. By our assumptions it then follows that \( \varphi \) is \( C^3 \).

We claim that \( \varphi \) solves the HCMA (4). First, observe that since \( f_{\sqrt{-1}t}(z) = \exp t X_{\omega_0}^* z \) is a symplectomorphism the formula (18) implies that \( \varphi((\sqrt{-1}t, z) = \varphi(0, z) = \varphi_0(z) \).

Next,
\[
\frac{\partial \varphi(\sqrt{-1}t, z)}{\partial s} = -\sqrt{-1}\partial^*_{\omega_{\varphi_0}} \bar{\partial}^*_{\omega_{\varphi_0}} G^2_{\omega_{\varphi_0}} \left( \mathcal{L}_{\frac{df_{\sqrt{-1}t}}{ds}} \omega_{\varphi_0} \right)(f_{\sqrt{-1}t}(z)) + \frac{1}{V} \int_M \varphi_0 \omega_{\varphi_0}^n,
\]
Since
\[
\frac{df_{\sqrt{-1}t}}{ds} = J X_{\omega_0}^* (f_{\sqrt{-1}t}(z)) = -\nabla_{\omega_0} \varphi_0 (f_{\sqrt{-1}t}(z)),
\]
we have
\[
\mathcal{L}_{\frac{df_{\sqrt{-1}t}}{ds}} \omega_{\varphi_0} = \sqrt{-1}\partial \bar{\partial} \varphi_0,
\]
and the $\partial\bar{\partial}$-lemma \cite{9} p. 149 implies that $\frac{\partial}{\partial s}\left(\sqrt{-1}t,z\right)$.

Finally, applying the $\partial\bar{\partial}$-lemma again implies that (14) holds where $\varphi_{\tau} := \varphi(\tau, \cdot)$, for all $\tau \in S_T$. Since $f_{\tau}$ is a diffeomorphism and moreover a smooth homotopy to the identity map it follows that $\omega_{\varphi_{\tau}}$ is a Kähler metric for each $\tau \in S_T$. In particular, $(\pi^{\star}_2\omega_{\varphi_0} + \sqrt{-1}\partial\bar{\partial}\varphi)^n \neq 0$. Differentiating (14) we find that $\frac{\partial}{\partial \tau} + df_{\tau}\frac{d}{d\tau}$ is a holomorphic vector field in the kernel of $\pi^{\star}_2\omega_{\varphi_0} + \sqrt{-1}\partial\bar{\partial}\varphi$. It follows that $(\pi^{\star}_2\omega_{\varphi_0} + \sqrt{-1}\partial\bar{\partial}\varphi)^n + 1 = 0$ on $S_T \times M$, as required. This concludes the proof of existence.

\section{Uniqueness of classical solutions for the HCMA.} In this subsection we complete the proof of Theorem 1.1 and establish the uniqueness of classical solutions to the HCMA under our assumptions.

Before giving the proof let us emphasize some of the subtleties involved.

First, the uniqueness we establish is essentially equivalent to showing that any solution must be $\mathbb{R}$-invariant when the Cauchy data is $\mathbb{R}$-invariant. A subtle point is that the HCMA is only equivalent to the geodesic equation under the assumption of $\mathbb{R}$-invariance, which is implicit in the arguments of Semmes and Donaldson. In general, the HCMA is equivalent to the more complicated WZW equation. Thus, the uniqueness proof cannot a priori use the identities we established in \S 2.1 for $C^3$ $\mathbb{R}$-invariant solutions. We need to derive these identities in the proof, and we do so by first establishing short-time uniqueness and then extending this to a global statement.

Thus, if we only wanted to prove uniqueness of $\mathbb{R}$-invariant solutions, the proof would simplify considerably. Alternatively, one could have defined the class of admissible subsolutions to be $\mathbb{R}$-invariant $\pi^{\star}_2\omega$-psh functions. It follows from Lemma 2.7 below that such a restriction would be redundant.

Second, the proof does not follow directly from the arguments of Bedford–Kalka \cite{3} and Bedford–Burns \cite{2} Proposition 1.1, where uniqueness is proved for a simpler situation, namely for the equation $(\sqrt{-1}\partial\bar{\partial}u)^m = 0$ on $\mathbb{C}^m$. Parts of the proof are local in nature, essentially the Cauchy–Kowalevskaya theorem on each strip, and thus adapt to our setting. However, the relative Kähler potential $\pi^{\star}_2\omega$ makes the situation more complicated since the leafwise equations are now not the fixed Laplace equation on $S_T$ but rather an inhomogeneous Poisson equation that varies from strip to strip, and one has to make sure that this equation does not degenerate. Thus, we need to invoke a global conservation law type argument that is special for our HCMA (4).

\begin{lemma}
Let $\omega_{\varphi_0} \in C^1$ and $\varphi_0 \in C^3$. Assume that the Cauchy problem for $(\omega_{\varphi_0}, \varphi_0)$ is $T$-good, and that for each $\tau \in S_T$ the map $f_{\tau}$ given by (17) is smoothly invertible. Then any $C^3$ $\pi^{\star}_2\omega$-psh solution of the HCMA (4) is unique, and in particular $\mathbb{R}$-invariant.
\end{lemma}

\begin{proof}
Assume that $\varphi, \rho \in C^3$ are both $\pi^{\star}_2\omega$-psh solutions of (4). Then the equation (4) and the equality of the Cauchy data implies that all the second derivatives of $\varphi$ and $\rho$, possibly with the exception of the second $s$ derivative, agree on the hypersurface $\Sigma := \{0\} \times \mathbb{R} \times M$. Now the form $\pi^{\star}_2\omega + \sqrt{-1}\partial\bar{\partial}\Phi$ restricts to a
positive form on Σ ensuring that $g_φ$ is non-degenerate (i.e., Σ is non-characteristic). Also, the Monge–Ampère equation on the initial hypersurface can be rewritten as

$$\bar{\phi}|_{s=0} = \frac{1}{2} |\nabla \bar{\phi}|_{g_\phi}^2 |_{s=0};$$

(19)

this was shown by Semmes [25] for all $s$, assuming $\phi$ is an R-invariant solution, but holds by his argument at $\{s = 0\}$ without that assumption since $\partial_t \phi, \partial_t^2 \phi, \partial_t \phi_0,$ and $\partial_t \partial_\phi \phi$ vanish on $\{0\} \times \mathbb{R} \times M$ as the initial data is R-invariant. Note that (19) expresses the second $s$ derivative of a solution in terms of the other second derivatives, all restricted to Σ. Since we know $\phi_0$ is a Kähler potential, it follows that $\phi$ and $\rho$ agree to second order on Σ. Thus, $\ker(\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi)|_\Sigma = \ker(\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \rho)|_\Sigma$ along the hypersurface. Thus, by the uniqueness of solutions of first order ODEs with C^1 coefficients, the leaves of the foliation by strips defined by each of the solutions $\phi, \rho$ must coincide. Thus the maps defined by (12) and (13) for $\phi$ and $\rho$ are identical, and we denote them simply by $\Gamma_z(\tau) = f_\tau(z)$. By the construction of the Monge–Ampère foliation, on each leaf the Kähler form $\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi$ satisfies (5). We claim that (15) always holds for $s = 0$. Recall, that we proved (15) for all $s \in [0, T]$, but only under the assumption of R-invariance of the solution. To prove this claim, note first

$$\imath_{\frac{\partial}{\partial \tau}} (\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi)|_{s=0} = \sqrt{-1} \partial \bar{\partial} \phi|_{s=0} = \sqrt{-1} \partial \bar{\partial} \phi_0, \quad \text{when } s = 0,$$

since $\phi_0$ is R-invariant. Similarly, since $\phi_0$ is R-invariant,

$$\imath_{\nabla_{g_{\phi,0}}} \phi_0 (\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi)|_{s=0} = \imath_{\nabla_{g_{\phi,0}}} \phi_0 \omega_{\phi,0} = \imath_{\phi_0} \omega_{\phi,0} = (\imath_{\phi_0} \omega_{\phi,0}) = \sqrt{-1} (\bar{\partial} - \partial) \phi_0.$$

Thus, $\frac{\partial}{\partial \tau} - \nabla_{g_{\phi,0}} \phi_0 \in \ker(\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi)|_{\{\sqrt{-1} \tau, \Gamma_z(\sqrt{-1} \tau)\}}$. Therefore, since also $\frac{\partial}{\partial \tau} + \frac{d f_\tau}{d s}|_{s=0} \in \ker(\pi^*_z \omega + \sqrt{-1} \partial \bar{\partial} \phi)|_{\{\sqrt{-1} \tau, \Gamma_z(\sqrt{-1} \tau)\}}$, we conclude that

$$\frac{d f_\tau}{d s}|_{s=0} = X_{\phi_0} \circ f_{\sqrt{-1} \tau} = -J_{g_{\phi,0}} \phi_0 \circ f_{\sqrt{-1} \tau}, \quad \frac{d f_\tau}{d s}|_{s=0} = -\nabla_{g_{\phi,0}} \phi_0 \circ f_{\sqrt{-1} \tau}, \quad (20)$$

as claimed.

Let $\Gamma_z$ be as in (12) and (11) and suppose that $\Gamma_z(S_T) \neq \{z\}$, i.e., that the leaf passing through $z$ is not trivial. For each $z \in M$, put $e_z := \gamma_1 \phi, \bar{e}_z := \gamma^*_1 \rho,$ and let $\omega_z := \gamma^*_1 \pi^*_z \omega = \Gamma_z^* \omega$. First, note that $\omega_z$ is strictly positive (1,1)-form on $S_T$. Indeed, write $\omega_z = \sqrt{-1} a_z \omega \wedge \omega \wedge dt$. Then by (20),

$$a_z(\sqrt{-1} \tau) = -\sqrt{-1} \omega \left( d\Gamma_z|_{\tau=\sqrt{-1} \tau} \left( \frac{\partial}{\partial \tau} \right), d\Gamma_z|_{\tau=\sqrt{-1} \tau} \left( \frac{\partial}{\partial \bar{\tau}} \right) \right) \bigg|_{\Gamma_z(\sqrt{-1} \tau)}$$

$$= -\sqrt{-1} \omega \left( \frac{\partial f_\tau(z)}{\partial \tau} \bigg|_{\tau=\sqrt{-1} \tau}, \frac{\partial f_\tau(z)}{\partial \bar{\tau}} \bigg|_{\tau=\sqrt{-1} \tau} \right) \bigg|_{f_{\sqrt{-1} \tau}(z)}$$

$$= -\frac{1}{4} \omega(-\nabla_{g_{\phi,0}} \phi_0 + \sqrt{-1} J_{g_{\phi,0}} \phi_0, -\nabla_{g_{\phi,0}} \phi_0 - \sqrt{-1} J_{g_{\phi,0}} \phi_0)|_{f_{\sqrt{-1} \tau}(z)}$$

$$= \frac{1}{2} \omega(\nabla_{g_{\phi,0}} \phi_0, J_{g_{\phi,0}} \phi_0)|_{f_{\sqrt{-1} \tau}(z)} = \frac{1}{2} |\nabla_{g_{\phi,0}} \phi_0|^2 \bigg|_{f_{\sqrt{-1} \tau}(z)} \geq 0.$$
Since \( g_{\varphi_0} \) and \( g \) are (strictly positive) metrics, \( a_z \) vanishes at some \( \sqrt{-1}t \in \{0\} \times \mathbb{R} \subset S_T \) if and only if \( d\varphi_0(f_{\sqrt{-1}t}(z)) = d\varphi_0(z) = 0 \) (by (20)). \( f_{\sqrt{-1}t} = \exp tX_{\varphi_0} \) so in particular \( f_{\sqrt{-1}t}^* \varphi_0 = \varphi_0 \). Thus, if \( a_z(\sqrt{-1}t) = 0 \) for some \( t \), then \( a_z(\sqrt{-1}t) \) for all \( t \in \mathbb{R} \). Now, for fixed \( z \in M \) and \( t \in \mathbb{R} \), equation (21) is an ODE in \( s \) for \( f_{s+\sqrt{-1}t}(z) \). If \( a_z(\sqrt{-1}t) = 0 \), then its initial condition is \( f_{\sqrt{-1}t}(z) = z \) and the initial derivative is zero. Thus, in this case \( f_{\tau}(z) = z \) for all \( \tau \in S_T \), and the leaf through \( z \) is trivial, i.e., \( \Gamma_z(S_T) = \{z\} \). Since we assumed at the beginning of this paragraph that the leaf through \( z \) was non-trivial, we thus conclude that \( a_z|_{s=0} > 0 \), and by continuity also \( C > a_z|_{s\in[0,\pi]} > 0 \), for some \( C, \epsilon > 0 \).

Denote the Laplacian associated to \( \omega_z \) by \( \Delta_z \). Then for each \( z \) with a non-trivial leaf, the leafwise problem (5) restricted to \( S_{2\epsilon} \times M \) is equivalent to the Cauchy problem,

\[
1 + \Delta_z a_z = 0, \quad \text{on} \quad S_{2\epsilon},
\]

\[
\alpha_z(\sqrt{-1}t) = \varphi_0(\Gamma_z(\sqrt{-1}t)) \quad \text{on} \quad \{0\} \times \mathbb{R},
\]

\[
\frac{\partial \alpha_z}{\partial s}(\sqrt{-1}t) = \varphi_0(\Gamma_z(\sqrt{-1}t)) - d\varphi_0(\nabla g_{\varphi_0} \varphi_0)(\Gamma_z(\sqrt{-1}t)), \quad \text{on} \quad \{0\} \times \mathbb{R}.
\]

The last equation follows from (20). Thus, \( e_z \) and \( \tilde{e}_z \) solve (22). Hence, since \( \Delta_z = a_z^{-1} \Delta_0 \), \( \zeta_z := e_z - \tilde{e}_z \) solves the Cauchy problem for \( \Delta_0 \zeta_z = 0 \) on \( S_{2\epsilon} \) with zero initial data, where \( \Delta_0 \) denotes the Euclidean Laplacian on \( S_{2\epsilon} \). It is well-known that bounded solutions to the Cauchy problem on bounded domains for this classical Euclidean equation are unique (cf., e.g., [15], p. 19)). However, we could not find a reference that treats our particular situation, namely the non-compact strip as in the following Lemma.

**Lemma 2.8.** Let \( u \in C^2 \cap L^\infty(S_T) \) be a solution of \( \Delta_0 u = 0 \) on \( S_T \), with \( u|_{s=0} = a \in C^2(\mathbb{R}) \), and \( \partial u/\partial s|_{s=0} = b \in C^2(\mathbb{R}) \). Then \( u \) is unique.

**Proof.** Since the equation is linear it suffices to consider the case of zero Cauchy data \( a = b = 0 \), and prove any solution must then vanish. Also, it suffices to consider the case \( T = \pi \), since if \( u \) is a non-trivial solution of \( \Delta_0 u = 0 \) on \( S_T \) with \( a = b = 0 \) then \( v(s, t) := u(T_s, T t) \) solves the same equation on \( S_\pi \).

Let \( P \) denote the Dirichlet Poisson kernel of the strip \( S_\pi \).

\[
P(s, t) = \frac{\sin s}{\cosh t - \cos s}.
\]

According to a theorem of Widder [29, Theorem 4], any harmonic function bounded below on the strip \( S_\pi \) can be expressed as

\[
u(s, t) - \inf u = [Ae^t + Be^{-t}] \sin s + \frac{1}{2\pi} \int_\mathbb{R} P(s, a-t)da(a) + \frac{1}{2\pi} \int_\mathbb{R} P(\pi-s, a-t)da(a),
\]

for some constants \( A, B \geq 0 \), and some (measurable) nondecreasing functions \( a, \beta : \mathbb{R} \to \mathbb{R} \). Moreover, the integrals converge in the interior of \( S_\pi \). Evaluating at \( s = 0 \) gives, by the continuity of \( u \)

\[
- \inf u = \frac{1}{2\pi} \lim_{s \to 0^+} \int_\mathbb{R} P(s, a-t)da(a).
\]
Thus $-\inf u\, dt = d\alpha(t)$. Plugging this back into (24), thus
\[
u(s, t) = [Ae^t + Be^{-t}]\sin s + \frac{1}{2\pi} \int_\mathbb{R} P(\pi - s, a - t)d\beta(a).
\] (25)

Therefore,
\[
0 = \frac{\partial \nu}{\partial s}(0, t) = Ae^t + Be^{-t} + \frac{1}{2\pi} \int_\mathbb{R} \frac{d\beta(a)}{\cosh(a - t) + 1}
\]
Since each of the terms is nonnegative they all vanish. Hence, $A = B = 0$, and
\[
\frac{1}{2\pi} \int_\mathbb{R} \frac{d\beta(a)}{\cosh(a - t) + 1} = 0,
\]
and therefore $\beta = 0$. Plugging back into (24), we conclude that $\nu = 0$, as desired.

It follows that $e_z = \tilde{e}_z$, whenever $\Gamma_z(S_T) \neq \{z\}$. On the other hand, if $\Gamma_z(S_T) = \{z\}$ then $\varphi(\tau, z) = \rho(\tau, z)$ by using (14), (19) and that $\varphi(\sqrt{-1}\tau, z) = \rho(\sqrt{-1}\tau, z)$. Since the foliation foliates all of $S_{2\pi} \times M$, it follows that $\varphi = \rho$ on that set. Thus, we have short-time uniqueness for $C^3$ solutions of the HCMA (4).

In particular, it follows that both $\varphi$ and $\rho$ are $\mathbb{R}$-invariant for $s \in [0, \epsilon]$. Also, (16) - (17) hold since again they were derived assuming only $\mathbb{R}$-invariance. Thus, (19) extends to a strip:
\[
\tilde{\varphi} = \frac{1}{2} |\nabla \tilde{\varphi}|^2_{g_z},
\] (26)
on $S_z \times M$. Consequently [25, 4],
\[
\tilde{\varphi}_s \circ f_s = \tilde{\varphi}_0.
\] (27)
Indeed, this holds when $s = 0$, and differentiating in $s$ and using (15), (16), (17), and (26) we obtain it must holds for all $s \in [0, \epsilon]$, where we used that $\tilde{\varphi}_s$ is constant along its Hamilton orbits (the factor of $1/2$ in (26) can be traced to our normalizations and corresponds to switching between the Hermitian and the Riemannian metrics associated to $\omega_\varphi$, cf. [20, §2.1.4.1, §2.2.3]). Finally, we can now also apply (15) which was valid for any $\mathbb{R}$-invariant solution, and compute
\[
a_z(s + \sqrt{-1}t) = -\sqrt{-1}\omega \left( d\Gamma_z \left( \frac{\partial}{\partial \tau}, d\Gamma_z \left( \frac{\partial}{\partial \tau} \right) \right) \right) |_{\Gamma_z(\tau)}
\]
\[
= -\sqrt{-1}\omega \left( \frac{\partial f_r(z)}{\partial \tau}, \frac{\partial f_r(z)}{\partial \tau} \right) |_{f_r(z)}
\]
\[
= -\frac{1}{4} \omega(-\nabla g_{g_z} \dot{\varphi}_s + \sqrt{-1}J\nabla g_{g_z} \dot{\varphi}_s, -\nabla g_{g_z} \dot{\varphi}_s - \sqrt{-1}J\nabla g_{g_z} \dot{\varphi}_s) |_{f_r(z)}
\]
\[
= \frac{1}{2} \omega(\nabla g_{g_z} \dot{\varphi}_s, J\nabla g_{g_z} \dot{\varphi}_s) |_{f_r(z)} = \frac{1}{2} |\nabla g_{g_z} \dot{\varphi}_s|^2_{g_0}(f_r(z)).
\] (28)
Therefore, by (14) - (27), and compactness it follows that if $0 < a_{s=0} < \text{then there exists constants } c, C > 0 \text{ determined by } z \text{ and the Cauchy data such that the a priori estimate } c < a_z(\tau) < C \text{ holds for each } \tau \in S_T \text{ for which a solution exists. Thus, we can now repeat the argument for the Cauchy problem with $\mathbb{R}$-invariant initial data given by } \varphi|_{\{t\} \times \mathbb{R}} = \rho|_{\{t\} \times \mathbb{R}} \text{ and } \dot{\varphi}|_{\{t\} \times \mathbb{R}} = \dot{\rho}|_{\{t\} \times \mathbb{R}}, \text{ and conclude that in fact (22) must hold on } S_T. \text{ Thus } \rho = \varphi. \text{ This concludes the proof of Lemma 2.7.} \quad \Box
Theorem 1.1 now follows by combining §2.1, and Lemmas 2.6 and 2.7.

As can be seen from the proof, \( \varphi \) is a smooth solution of the IVP (4) if and only if the Moser maps \( f_s \) are smoothly invertible and the ‘conservation law’ (27) holds. Of course, this is a weaker statement than Lemma 2.6. Nevertheless we record it here.

**Corollary 2.9.** Let \( \omega_{\varphi_0} \in C^1 \) and \( \dot{\varphi}_0 \in C^3 \). Assume that the Cauchy problem for \( (\omega_{\varphi_0}, \dot{\varphi}_0) \) is \( T \)-good, and that for each \( \tau \in S_T \) the map \( f_\tau \) given by (11) is smoothly invertible. Then (26) and (27) are equivalent.

We already saw that (26) implies (27). For the converse, note that under the assumptions, it follows from Lemma 2.6 that there exists a solution, and that the Moser maps determined by the Cauchy data satisfy (15)–(17); thus differentiating (27) immediately gives (26).

In the setting of the HRMA, we will interpret (27) in terms of a Hamilton–Jacobi equation (Theorem 1.13) and show that this ‘conservation law’ persists also for certain weak solutions (Proposition 6.3).

### 3. Ill-posedness of leafwise Cauchy problems

The goal of this section is to prove Theorem 1.5 showing that the Cauchy problem for the HCMA is not even locally well-posed. As the proof of Theorem 1.1 shows, the leaves of the Monge–Ampère foliation are obtained as the analytic continuation of the Hamiltonian flow of \( (\omega_{\varphi_0}, \dot{\varphi}_0) \). The Monge–Ampère distribution picks out as the \( M \)-component the Hamiltonian vector field associated to \( (\omega_{\varphi_0}, \dot{\varphi}_0) \) and not an arbitrary multiple of it precisely because the \( S_T \)-component of the distribution is \( \partial/\partial \tau \). In other words, the leaves (strips) of the foliation are graphs (of maps \( S_T \to S_T \times M \)) of (complex) time-parametrized Hamiltonian flow of \( (\omega_{\varphi_0}, \dot{\varphi}_0) \). As we will show, this puts a serious restriction on the Cauchy data.

So far, we have operated under the assumption that we have \( T \)-good Cauchy data (Definitions 2.1 and 2.3). Yet the analytic continuation of each Hamiltonian orbit should be an ill-posed problem. The closely related problem of solving the leafwise Cauchy problem for the equation (5) should also ill-posed, and the goal of this section is to give a proof of this latter ill-posedness. The latter problem seems simpler than the former since it is a linear problem for a function on a strip rather than a Cauchy problem for a holomorphic map into a nonlinear space. Hence we concentrate on the leafwise problem here. However, it is natural also to linearize the nonlinear problem (cf. [5]) and prove ill-posedness for the existence of \( T \)-Hamiltonian analytic data. We pursue this approach in a sequel.

As above, we suppose that we are given \( (\omega_{\varphi_0}, \dot{\varphi}_0) \) for which the orbit \( \exp tX_{\omega_{\varphi_0}} \) admits an analytic continuation to the strip \( S_T \). Let \( \gamma_z \) be as in Definition 1.4. Then \( \alpha_z = \gamma_z^* \varphi \) satisfies (22). Since \( \Gamma_z^* \omega \) has a global potential \( \Phi_z \) on \( S_T \), we may also write the equation in terms of the Euclidean Laplacian \( \Delta_0 \) as

\[
\Delta_0 \chi = 0, \quad \text{where} \quad \chi = (\Phi_z + \alpha_z).
\]

However, \( \Phi_z \) is not unique since the addition of any harmonic function on the strip gives another potential. In the case where the image of the complex Hamiltonian orbit \( \Gamma_z \) lies in an open set \( U \subset M \) in which \( \omega \) has a potential \( \Phi_0 \), we have...
\[ \Phi_z = \Gamma_z^* \Phi_0. \] In general, the closure of the image of \( \Gamma_z(\sqrt{-1}t) \) lies in the level set \( \{ \varphi_0 = \varphi_0(z) \} \). We will see that toric varieties always satisfy these conditions. However, simple examples (e.g., elliptic curves) show that there need not exist a potential for \( \omega \) defined in a neighborhood of the orbit. The following lemma shows that one may find a reasonable replacement for that, on each leaf separately. The growth estimate we derive here is not optimal, but suffices for our purposes.

**Lemma 3.1.** Let \( \varphi \) be a smooth solution to the HCMA [4]. There exists a global Kähler potential \( \Phi_z \) for \( \Gamma_z^* \omega \) on \( S_T \) with polynomial growth at infinity.

**Proof.** The claim would be obvious if there exists a potential for \( \omega \) on a neighborhood of the image of \( \Gamma_z(S_T) \), but as mentioned earlier such a potential need not exist. Instead, we will find suitable Kähler potentials along each leaf.

As before, denote \( \omega_z = \gamma_z^* \pi_z^* \omega = 2a_z ds \wedge dt \). As shown in the proof of Lemma 2.6, \( a_z > 0 \) if and only if the leaf through \( z \) is non-trivial, i.e., \( \Gamma_z(S_T) \neq \{ z \} \), which we assume throughout this section. Thus, by compactness, there exist some constants \( c, C > 0 \) (depending on \( z \)) such that \( 0 < c < a_z(s, t) < C \) on \( S_T \). For convenience, in this section we omit the subscript and denote \( a \equiv a_z \).

We wish to find \( \Phi_z \in C^\infty(S_T) \) of polynomial growth so that \( \Delta_0 \Phi_z = a \), i.e., \( \sqrt{-1} \partial \bar{\partial} \Phi_z = \gamma_z^* \pi_z^* \omega \). Throughout this proof \( \partial = \partial_t \).

We rewrite the Poisson equation above as

\[ \bar{\partial}(\sqrt{-1} \partial \Phi_z) = -2ads \wedge dt \] (30)

and use existence theorems for the inhomogeneous \( \bar{\partial} \)-equation on the strip. Introduce the subharmonic weight \( \psi = \log(1 + |\tau|^2) \) and observe that

\[ ads \wedge dt \in L^2_{(1,1)}(S_T, \psi) \]

where \( L^2_{(1,1)}(S_T, \psi) \) is the space of \( (1, 1) \) forms \( a(s, t) ds \wedge dt \) so that

\[ \int_{S_T} e^{-\psi} |a|^2 ds \wedge dt < \infty. \]

By Hörmander’s weighted \( L^2 \) existence theorem for the \( \bar{\partial} \)-equation [13, Theorem 4.4.2], there exists \( u \in L^2_{(0,1)}(S_T) \) such that \( \bar{\partial} u = -2ads \wedge dt \) and

\[ \int_{S_T} |u|^2 (1 + |\tau|^2)^{-3} ds \wedge dt \leq 4 \int_{S_T} |a|^2 e^{-\psi} ds \wedge dt. \]

Applying the same theorem to \( \bar{\partial} \Phi_z = \bar{u} \) with \( \psi = 3 \log(1 + |\tau|^2) \), we then obtain a solution \( \Phi_z \) of \( \sqrt{-1} \partial \bar{\partial} \Phi = 2ads \wedge dt \) satisfying

\[ \int_{S_T} |\Phi_z|^2 (1 + |\tau|^2)^{-5} ds dt < \infty. \] (31)

We now show that this \( L^2 \) estimate implies the polynomial growth of \( \Phi_z \). Note that \( \partial_s \Phi_z \) and \( \partial \Phi_z \) satisfy a Poisson equation on \( S_T \) satisfying the same estimates. Indeed, by [15] (and the assumption of existence of a smooth solution) under \( \Gamma_z \) these vector fields push-forward to the Hamilton vector fields for \( \varphi_z \), respectively \( J \) of these fields. Hence the Lie derivative with respect to these fields of \( \omega \) are bounded.
and we can use them as the right hand side in place of $ads\wedge dt$ above and repeat
the argument to get the estimate (31) for these derivatives and for repeated mixed
derivatives.

By the Sobolev inequality
\[ \sup_{S_T} f^2 \leq C \int_{S_T} |(1 - \Delta_0)f|^2 ds dt \]
for a strip, we have
\[ \sup_{S_T} \Phi_z^2 (1 + |\tau|^2)^{-5} \leq C \int_{S_T} |(1 - \Delta_0)(\Phi_z (1 + |\tau|^2)^{-5/2})|^2 ds \wedge dt. \]
It is straightforward to check that the integral is finite: this follows from the weighted
$L^2$ estimates for $\Phi_z$ and $\Delta_0 \Phi_z$, and the fact that derivatives of $(1 + |\tau|^2)^{-r}$ for $r > 0$
decay more rapidly with each derivative. It follows that
\[ |\Phi_z| \leq C (1 + |\tau|^2)^{5/2} \text{ on } S_T. \] (32)
□

\textbf{Remark 3.2.} In the proof of Theorem 1.5 we will be able to specialize to a situa-
tion where $\Phi_z$ is actually of the form $\Gamma_0 \Phi_0$. However, Lemma 3.1 is needed to derive the
general obstruction in Proposition 3.3 below that holds for all $z \in M$.

The obstruction to solvability of (22), and hence to the exis-
tence of a leafwise
subsolution (and in particular to the existence of a $C^3$ solution of the HCMA), is
summarized in the following propositon.

\textbf{Proposition 3.3.} Let $\varphi$ be a $C^3$ solution to the HCMA (4), and $z \in M$. Let
\[ D = \frac{d}{\sqrt{-1} dt} \text{ on } \mathbb{R}, \]
and set
\begin{align*}
q_z(t) &:= \frac{\partial \alpha_z}{\partial s}(\sqrt{-1} t) = \varphi_0(\Gamma_z(\sqrt{-1} t) - d\varphi_0(\nabla_{g_0} \varphi_0)(\Gamma_z(\sqrt{-1} t)), \\
p_z(t) &:= -\partial_s \Phi_z(0, t) - D \coth TD(\Phi_z + \gamma^*_z \varphi_0)(0, t),
\end{align*}
where $\Phi_z$ is given by Lemma 3.1. Then,
\[ (\overline{q_z - p_z})(\xi) = o(e^{-T|\xi|}). \]
Thus, $q_z - p_z$ admits an analytic continuation to the interior of $S_T \cup \overline{S_T} = [-T, T] \times \mathbb{R}$.

Henceforth we denote by $PW_T(\mathbb{R})$ the Paley–Wiener space
\[ PW_T(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : |\hat{f}(\xi)| = o(e^{-T|\xi|}) \}. \] (33)

Our convention for the Fourier transform is
\[ \mathcal{F}(f)(\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}} e^{-\sqrt{-1} t \xi} f(t) dt. \]
It is well-known that if $f \in PW_T(\mathbb{R})$, then $f$ is the restriction to $\mathbb{R}$ of a holomorphic
function on any two-sided strip $S_b \cup S_b = [-b, b] \times \mathbb{R}$ with $b < T$ [27, p. 121].

\textbf{Proof.} Despite the lack of uniqueness of $\Phi_z$ it seems simpler to work with the equa-
tion (29) rather than $\Delta_0 \alpha = -1$ since the Euclidean equation is simpler and it too has real analytic coefficients. We then wish to represent the solution $\chi_z$ as a Poisson integral in terms of its boundary values on $\partial S_T$. We first assume $T = \pi$. 
We recall the following theorem of Widder [29, Theorem 3]: If \(u(s,t)\) is (i) continuous on \(S_\pi\) and harmonic on its interior; (ii) satisfies the bounds \(u(0,t)e^{-|t|} \in L^1(\mathbb{R})\), \(u(\pi,t)e^{-|t|} \in L^1(\mathbb{R})\) and \(\int_0^\pi |u(s,t)|ds = o(e^{|t|})\), then
\[
u(s,t) = \frac{1}{2\pi} \int_\mathbb{R} P(s,a-t)u(0,a)da + \frac{1}{2\pi} \int_\mathbb{R} P(\pi-s,a-t)u(\pi,a)da, \tag{34}
\]
where \(P\) is defined by (23).

The assumptions for Widder’s theorem are satisfied when \(u = \chi = \alpha z + \Phi z\), with \(\Phi z\) the potential constructed in Lemma 3.1. Indeed, then \(\chi\) has polynomial growth at infinity on \(S_\pi\) itself is a bounded continuous function. Consequently, (34) is valid when \(u = \chi\).

We next consider the implications of this equation for \(q_z\). As in [30], it simplifies the notation to put
\[
Q(s,t) = \frac{1}{4} \cos \frac{\pi s}{2} \cosh \frac{\pi t}{2} + \sin \frac{\pi s}{2} = \frac{1}{4} P\left(\frac{\pi s}{2}, \frac{\pi t}{2}\right)
\]
on the strip \(s \in (-1,1), t \in \mathbb{R}\). One has [30] (5)
\[
Q(s,t) = \frac{1}{2\pi} \int_\mathbb{R} e^{-\sqrt{-1}\pi a} \sinh(1-s)a \sinh 2a \tilde{u}(0,a)da, \quad s \in (-1,1).
\]
Then,
\[
u(s,t) = \int_\mathbb{R} Q(s,a-t)u(-1,a)da + \int_\mathbb{R} Q(-s,a-t)u(1,a)da
= \int_\mathbb{R} e^{-\sqrt{-1}\pi a} \sinh(1-a) \sinh 2a \tilde{u}(-1,a)da + \int_\mathbb{R} e^{\sqrt{-1}\pi a} \sinh(1+s) \sinh 2a \tilde{u}(1,a)da.
\]

Note that this formula holds even when \(u(\pm 1, \cdot)\) is of polynomial growth. Then \(\tilde{u}(\pm 1, \cdot)\) is a temperate distribution while \(\frac{\sinh(1+s)a}{\sinh 2a}\) is a Schwartz function for \(s \in (-1,1)\), and so the second equality holds by the definition of the Fourier transform of a temperate distribution [14, Definition 7.1.9].

By a change of variable, for the strip \((s,t) \in [0,T] \times \mathbb{R}\) and for \(\Delta_0 \chi = 0\) with boundary values \(\chi(0,\cdot)\) and \(\chi(T,\cdot)\) we obtain,
\[
\chi(s,t) = \int_\mathbb{R} e^{-\sqrt{-1}\pi a} \sinh(T-s)a \sinh Ta \tilde{\chi}(0,a)da + \int_\mathbb{R} e^{-\sqrt{-1}\pi a} \sinh sa \sinh Ta \tilde{\chi}(T,a)da.
\]

Thus,
\[
\partial_s \chi(0,t) = -\int_\mathbb{R} e^{-\sqrt{-1}\pi a} \coth Ta \tilde{\chi}(0,a)da + \int_\mathbb{R} e^{-\sqrt{-1}\pi a} \frac{a}{\sinh Ta} \tilde{\chi}(T,a)da.
\]

Note that differentiation at the boundary is allowed since we can consider (35) as a distributional equation in \(t\) with parameter \(s\), and so one can pair (35) with any
Schwartz function of $t$ and then differentiate in $s$. Thus,

$$q_z(t) = -\partial_s \Phi_z(0, t) - D \coth TD \chi(0, \cdot) + \frac{D}{\sinh TD} \chi(T, \cdot)$$

$$= p_z(t) + \frac{D}{\sinh TD} \chi(T, \cdot),$$

where $D = \frac{1}{\sqrt{-1}} \frac{d}{dt}$ on $\mathbb{R}$. Inverting, $q_z$ must lie in the domain of the operator $A_{T,z} : \mathcal{S}' \to \mathcal{S}'$ given by

$$A_{T,z} : u \mapsto \frac{\sinh TD}{D} (u - p_z) = \int_{\mathbb{R}} e^{ita} \frac{\sinh Ta}{a}(u - p_z)(a) da.$$ 

Here $\mathcal{S}'$ denotes temperate distributions on $\mathbb{R}$. By our earlier estimates $\chi(T, \cdot)$ is continuous and of at most polynomial growth, hence belongs to $\mathcal{S}'$. Thus it follows that one has $(q_z - p_z)(\xi) = o(e^{-T|\xi|})$, since these are the Fourier coefficients of $\chi(T, \cdot)$.

In particular $q_z - p_z \in L^2(\mathbb{R})$, so $q_z - p_z \in PW_T(\mathbb{R})$. By a Paley–Wiener type theorem [27, p. 121] it follows that $q_z - p_z$ admits an analytic continuation

$$(q_z - p_z)(s + \sqrt{-1}t) := \int_{\mathbb{R}} e^{(s+\sqrt{-1}t)a} \frac{a}{\sinh Ta} \hat{\chi}(T, a) da$$

(36) to the interior of a two-sided strip of width $2T$.

As mentioned in the Introduction, the Paley–Wiener condition on $q_z$ may be viewed as characterizing the range of a Dirichlet-to-Neumann map. These leafwise Dirichlet-to-Neumann maps are induced by the global Dirichlet-to-Neumann map for the HCMA, defined by

$$\mathcal{N}^T(\varphi_0, \varphi_T) = \hat{\varphi}_0$$

(37) from the endpoint $\varphi_T$ at time $T$ of the geodesic arc from $\varphi_0$ to $\varphi_T$ to the initial velocity $\dot{\varphi}_0$ of the geodesic.

### 3.1. Lifespan of generic Cauchy data

We now complete the proof of Theorem 1.5.

Assume that the Cauchy problem for (44) with Cauchy data $(\varphi_0, \dot{\varphi}_0)$ admits a $C^3$ solution $\varphi$. For simplicity, we take the reference Kähler metric to be $\omega_{\varphi_0}$ and then the initial relative Kähler potential becomes zero. Then (44) reduces to

$$1 + \Delta_z \alpha_z = 0, \quad \text{on } S_T,$$

$$\alpha_z(\sqrt{-1}t) = 0, \quad \text{on } \{0\} \times \mathbb{R},$$

$$\frac{\partial \alpha_z}{\partial s}(\sqrt{-1}t) = \dot{\varphi}_0(z), \quad \text{on } \{0\} \times \mathbb{R}. $$

(38)

The last line follows since $\dot{\varphi}_0$ is constant along its Hamiltonian flow orbits. It follows that $q_z$ is a constant, and therefore Proposition 3.3 implies that

$$p_z = (\partial_s - A_T) \Phi_z s=0 \in PW_T(\mathbb{R})$$

(39) where

$$A_T := D \coth TD.$$
Note that $A_T$ is an approximation (with respect to $T$) to the Dirichlet-to-Neumann operator for the half-plane, which is the operator

$$|D|f(t) = \int_{\mathbb{R}} e^{\sqrt{-1}t} |a|^f(a) da.$$ 

We further observe that, at least for some $z$, $\Phi_z$ is the pullback of a Kähler potential defined in a neighborhood of $\Gamma_z(S_T)$. Indeed, let $z_0$ be a non-degenerate maximum point of $\hat{\varphi}_0$ (one always exists for a generic $\hat{\varphi}_0$, which as far as proving Theorem 1.5 we may assume is the case), so that the orbit of $z_0$ is $\{z_0\}$ and find a potential in a neighborhood of $z_0$. If $z$ is sufficiently close to $z_0$ then the orbit of $z$ under the Hamilton flow of $\hat{\varphi}_0$ is non-trivial and is contained in the level set $\{\hat{\varphi}_0 = \hat{\varphi}_0(z)\} \subset M$, which is close to $\{z_0\}$ (by the Morse theorem). Moreover, this Hamilton orbit is contractible in $M$ to $z_0$. By (15)–(17) the slices $\Gamma_z(\{s\} \times \mathbb{R})$ are all homotopic to the this initial Hamilton orbit. Hence, $\Gamma_z(S_T)$ itself is contractible to $z_0$. Since $\omega$ has a local Kähler potential on any contractible neighborhood of $z_0$ in $M$, the conclusion follows.

Assume from now on that

$$\Phi_z = \Gamma_z^* \Phi_0,$$  \hspace{1cm} (40)

where $\Phi_0$ is a smooth function defined on some neighborhood of $\Gamma_z(S_T)$ in $M$. For simplicity of notation, put

$$H := \hat{\varphi}_0, \quad X_H := X_{\hat{\varphi}_0}^\omega.$$ 

Let $\nabla$ denote the gradient with respect to the associated metric $g_{\hat{\varphi}_0}$. Then

$$\partial_s \Phi_z|_{s=0} = \Gamma_z^* J X_H \Phi_0|_{s=0}.$$ 

By (15) and (40), the conclusion of Proposition 3.3 can be rewritten as

$$\Gamma_z^* J X_H \Phi_0 + A_T \Gamma_z^* \Phi_0 = \Gamma_z^* d \Phi_0(\nabla H) + A_T \Gamma_z^* \Phi_0 \in PW_T(\mathbb{R}).$$  \hspace{1cm} (41)

We now study this equation under particular deformations of the Cauchy data. We denote by

$$C_{T,z} = \{(\varphi_0, \varphi_0, z) \in C^3(M) \times C^3(M) \times M : (\varphi_0, \varphi_0) \text{ is } T\text{-good and } (39) \text{ holds}\}.$$ 

We claim that for $z$ near a maximum point (as above), the complement of $C_{T,z}$ in $C^3(M) \times C^3(M) \times M$ is dense. By assumption $(0, \hat{\varphi}_0, z) \in C_{T,z}$. We fix such a $z$ for the rest of the argument. We may, and do, choose $z$ so that in addition it is a regular point for $H$. Our first goal is to find a perturbation of $\hat{\varphi}_0$ with the property that the orbit $\Gamma_z(\sqrt{-1}R)$ is unchanged.

Let $h$ be a $C^2$ function on $M$, and set for each $\epsilon \geq 0$,

$$H_\epsilon := H + \epsilon(H - H(z)) h,$$

$$V_\epsilon := \{w \in M : H_\epsilon(w) = H_\epsilon(z)\}.$$ 

Note that $V_\epsilon$ indeed is independent of $\epsilon \geq 0$; by assumption $z \in M$ is a regular point for $H$ so that $V_\epsilon$ is a (real) hypersurface. Also,

$$X_{H_\epsilon} := X_{H_\epsilon}^\omega = (1 + \epsilon h) X_H \quad \text{along } V_\epsilon.$$
Denote \( \hat{\Gamma}_z(t) := \Gamma_z(\sqrt{-1}t) \). Then define

\[
\hat{\Gamma}^e_z(t) := \exp t X_{H, \epsilon}(z) = \hat{\Gamma}_z(g_\epsilon(t)),
\]

since \( \Gamma_z(\sqrt{-1}R) \subset V_z \), where \( g_\epsilon : \mathbb{R} \to \mathbb{R} \) is a diffeomorphism defined by

\[
g'_\epsilon(t) := \frac{d}{dt} g_\epsilon(t) = 1 + \epsilon h \circ \hat{\Gamma}_z(t), \quad g_\epsilon(0) = 0. \tag{42}
\]

Thus,

\[
g_\epsilon(t) = t + \epsilon \int_0^t h \circ \hat{\Gamma}_z(a) da. \tag{43}
\]

To derive a contradiction, we assume that there exists some \( \epsilon_0 > 0 \) for which \( \{ (0, H_\epsilon, z) \}_{\epsilon \in [0, \epsilon_0]} \subset C_{T, z} \). In particular, by (41), for sufficiently small \( \epsilon \geq 0 \),

\[
(\hat{\Gamma}^e_z)^* d\Phi_0(\nabla H_\epsilon) + A_T(\hat{\Gamma}^e_z)^* \Phi_0 \in PW_T(\mathbb{R}). \tag{44}
\]

Thus,

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} \left( (\hat{\Gamma}^e_z)^* d\Phi_0(\nabla H_\epsilon) + A_T(\hat{\Gamma}^e_z)^* \Phi_0 \right) \in PW_T(\mathbb{R}). \tag{45}
\]

Note that

\[
d\Phi_0(\nabla H_\epsilon)|_{V_z} = (1 + \epsilon h) d\Phi_0(\nabla H)|_{V_z}.
\]

Also

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} (\hat{\Gamma}^e_z)^* \Phi_0 = h_z \left. \frac{d}{dt} \hat{\Gamma}^*_z \Phi_0 \right|_{t}, \tag{46}
\]

where by (43)

\[
h_z(t) = \int_0^t h \circ \hat{\Gamma}_z(a) da. \tag{47}
\]

Set also,

\[
\tilde{h}_z = \hat{\Gamma}^*_z h.
\]

Then

\[
\tilde{h}_z' = \tilde{h}_z. \tag{48}
\]

First,

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} (\hat{\Gamma}^e_z)^* d\Phi_0(\nabla H_\epsilon) = h_z \left( \hat{\Gamma}^*_z d\Phi_0(\nabla H) \right)' + \tilde{h}_z \hat{\Gamma}^*_z d\Phi_0(\nabla H). \tag{49}
\]

Second,

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon = 0} A_T(\hat{\Gamma}^e_z)^* \Phi_0 = A_T(h_z \left. \frac{d}{dt} \hat{\Gamma}^*_z \Phi_0 \right|_{t}). \tag{50}
\]

Combining (45), (49) and (50), we get that

\[
h_z \left( \hat{\Gamma}^*_z d\Phi_0(\nabla H) \right)' + \tilde{h}_z \hat{\Gamma}^*_z d\Phi_0(\nabla H) + A_T(h_z \left. \frac{d}{dt} \hat{\Gamma}^*_z \Phi_0 \right|_{t}) \in PW_T(\mathbb{R}), \tag{51}
\]

for all nonnegative functions \( h \in C^\infty(M) \) for which \( (0, H_\epsilon, z) \in C_{T, z} \).

In this formula,

\[
a_z := \hat{\Gamma}^*_z \Phi_0, \quad b_z := \hat{\Gamma}^*_z d\Phi_0(\nabla H) \tag{52}
\]

are two fixed functions on \( \mathbb{R} \) satisfying (by (41))

\[
b_z + A_T a_z \in PW_T(\mathbb{R}). \tag{53}
\]
By (48), rewrite equation (51) as
\[ h_z b'_z + h_z b_z + A_T(h_z a'_z) = (h_z b_z)' + A_T(h_z a'_z) \in PW_T(\mathbb{R}). \tag{54} \]

Our goal is now to show that (54) is impossible for a dense set of \( h \). We denote by \( \text{Hilb} : S' \rightarrow S' \) the Hilbert transform, defined by
\[ (\text{Hilb} f)(t) = -\int_{\mathbb{R}} \sqrt{-1} \text{sign}(\xi) e^{\sqrt{-1} t \xi} \hat{f}(\xi) d\xi. \]

**Lemma 3.4.** Let \( h_z \) be given by (47). Then \( \sqrt{-1} h_z b_z + h_z a'_z = 2h_z \Gamma_z^* \tilde{\Phi}_0(X_H) \) admits a holomorphic extension to \( S_T \).

**Proof.** By (54),
\[ \sqrt{-1} \xi \hat{h}_z b_z + \xi \coth T \xi \hat{h}_z a'_z = o(e^{-T|\xi|}). \]

Now, \( \coth T \xi - \text{sign}(\xi) = O(e^{-2T|\xi|}) \). Since \( a'_z \in L^\infty(\mathbb{R}) \) and \( h_z/\xi \in L^\infty(\mathbb{R}) \), this implies
\[ \sqrt{-1} \xi \hat{h}_z b_z + \xi \text{sign}(\xi) \hat{h}_z a'_z = o(e^{-T|\xi|}), \]
or \( \hat{h}_z b_z - \sqrt{-1} \text{sign}(\xi) \hat{h}_z a'_z = o(e^{-T|\xi|}) \), i.e., \( \hat{h}_z b_z + \text{Hilb}(h_z a'_z) \in PW_T \). Since by definition \( \text{Hilb} \) maps \( PW_T \) to itself, we also have \( \text{Hilb}(h_z b_z) - h_z a'_z \in PW_T \). Multiply the former equation by \( \sqrt{-1} \) and add it to the latter to obtain
\[ (I - \sqrt{-1} \text{Hilb})(\sqrt{-1} h_z b_z - h_z a'_z) \in PW_T, \]
and by conjugation \( (I + \sqrt{-1} \text{Hilb})(\sqrt{-1} h_z b_z + h_z a'_z) \in PW_T \). Since \( I + \sqrt{-1} \text{Hilb} \) is twice the orthogonal projection operator onto the positive frequency space, it follows that
\[ \mathcal{F}(\sqrt{-1} h_z b_z + h_z a'_z)(\xi) = o(e^{-T\xi}), \text{ for all } \xi > 0. \]

By the proof of [27] Theorem 3.1 it follows that \( \sqrt{-1} h_z b_z + h_z a'_z \) admits a holomorphic extension to the one-sided strip \( S_T \). The lemma now follows from (15) and (52), and \( d\Phi_0(X_H + \sqrt{-1} JX_H) = 2d\Phi_0(X_H^{0,1}) = 2\tilde{\Phi}_0(X_H). \]

**Lemma 3.5.** There exist \( \alpha < \beta \in \mathbb{R} \) and \( \epsilon \in (0, T] \) all independent of \( T \) (but depending on \( z \)) such that \( h_z \), given by (17), admits a holomorphic extension to the two-sided rectangle \( [-\epsilon, \epsilon] \times [\alpha, \beta] \subset \overline{S_T \cup \overline{S_T}} \). In particular, \( \Gamma_z^* h \) is real-analytic on \( [\alpha, \beta] \).

**Proof.** First, observe that (54) is true for the constant function \( h \equiv 1 \) on \( M \): then \( 0, (1 + \epsilon)\varphi_0, z \in C_{T', \epsilon} \) for all \( \epsilon \in [0, \epsilon_0] \), for some \( T' < T \). In fact, there exists then a solution to the HCMA (4) for some \( T' < T \) by reparametrizing \( \varphi \) in the \( s \) variable, and \( T' \rightarrow T \) as \( \epsilon_0 \rightarrow 0 \). This proves the claim.

Lemma 3.4 implies that \( h_z(\sqrt{-1} b_z + a'_z) \), respectively \( h_z(-\sqrt{-1} b_z + a'_z) \), admits a holomorphic extension to \( S_T \), respectively, \( S_T \). When \( h \equiv 1 \), then \( \hat{h}_z \equiv 1 \) and \( h_z(t) = t \) on \( \mathbb{R} \). Thus, by the previous paragraph, these estimates hold for \( h_z = t \). Hence, \( \sqrt{-1} b_z + a'_z \), respectively \( -\sqrt{-1} b_z + a'_z \), admits a holomorphic extension to \( S_T \), respectively, \( S_T \). By dividing, and since \( h_z \) is real, it follows by the Schwarz reflection principle that \( h_z \) admits a holomorphic extension to some rectangle \( [-\epsilon, \epsilon] \times [\alpha, \beta] \subset \overline{S_T} \) whenever \( \sqrt{-1} b_z + a'_z \) does not vanish on \( [\alpha, \beta] \) (here we also used the fact that
zeros of holomorphic functions cannot have an accumulation point). In particular, \( h_z \) is real analytic on \( \mathbb{R} \setminus W_z \), where
\[
W_z := \{ t \in \mathbb{R} : a'_z = \Gamma_z^* X_H \Phi_0(\sqrt{-1}t) = d\Phi_0(X_H) \circ \Gamma_z(\sqrt{-1}t) = 0 \}.
\]

The proof is complete if \( \mathbb{R} \setminus W_z \) contains an open interval. Since \( W_z \) is closed it thus suffices to rule out the case where \( W_z = \mathbb{R} \), i.e., \( a'_z = 0 \), and hence \( a_z = 0 \), on \( \mathbb{R} \). If this holds for every point in a neighborhood of \( z \) then \( \Phi_0 \) must be a function of \( H \) on some neighborhood of \( z \) in \( M \). Clearly, this is a non-generic property and perturbing either \( H \) or adding to \( \Phi_0 \) the real part of a generic local holomorphic function (this does not require changing \( \omega_{\Phi_0} \)) will destroy this property. Thus, \( h_z \) must be real-analytic at least on some open interval on \( \mathbb{R} \), and by (47) and differentiation so is \( \hat{\Gamma}^*_z h \). Now, let \([\alpha, \beta] \subset \mathbb{R} \setminus W_z \) be any nonempty interval, and note that \( W_z \) is independent of \( T \). \( \square \)

We now complete the proof of Theorem 4.5

By taking \( \beta - \alpha \) sufficiently small we may assume that \( \hat{\Gamma}_z : [\alpha, \beta] \to M \) is an embedded curve. Consider the map \( R_z : C^3(M) \to C^3([\alpha, \beta]) \) defined by \( R_z f := \Gamma_z^* f |_{[\alpha, \beta]} \). Observe that \( R_z \) is a bounded surjective linear operator. Hence, it defines an open map. Let \( B \) be any open ball in \( C^3(M) \) containing the zero function 0. If for some \( T > 0 \), \( \{(\varphi_0, \varphi_0 + f, z) : f \in B \} \subset C_{T, z} \) then Lemma 3.5 implies that \( R_z(B) \) is contained in the subset of real-analytic functions in \( C^3([\alpha, \beta]) \), with \([\alpha, \beta] \) independent of \( T > 0 \). However, the latter is not an open subset in \( C^3([\alpha, \beta]) \). This concludes the proof of Theorem 4.5.

4. The smooth lifespan of the HRMA

In this section we restrict to toric manifolds and prove Proposition 1.9 concerning the analytic continuation of orbits of Hamiltonian orbits and the invertibility of the associated Moser maps. The first part, concerning the infinite analytic continuation of the Hamiltonian flow defined by the Cauchy data, is proved in Lemma 4.1. The second part, concerning the invertibility of the Moser maps, is proved in Lemma 4.3.

4.1. Some background on toric Kähler manifolds. We briefly recall some background facts on toric Kähler manifolds. For more detailed background we refer to \([20, 21]\) and references therein.

A symplectic toric manifold is a compact closed Kähler manifold \((M, \omega)\) whose automorphism group contains a complex torus \((\mathbb{C}^*)^n\) whose action on a generic point is isomorphic to \((\mathbb{C}^*)^n\), and for which the real torus \((S^1)^n \subset (\mathbb{C}^*)^n\) acts in a Hamiltonian fashion by isometries.

We will work with coordinates on the open dense orbit of the complex torus given by \( z_j = e^{x_j/2 + \sqrt{-1} \theta_j}, j = 1, \ldots, n \), with \( (x, \theta) = (x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \in \mathbb{R}^n \times (S^1)^n \). Let \( M_0 \cong (\mathbb{C}^*)^n \) be the open orbit of the complex torus in \( M \) and write
\[
\omega |_{M_0} = \sqrt{-1} \partial \bar{\partial} \psi_\omega.
\] (55)

We call \( \psi_\omega \) the open-orbit Kähler potential of \( \omega \). The real torus \((S^1)^n \subset (\mathbb{C}^*)^n\) acts in a Hamiltonian fashion with respect to \( \omega \). The image of the moment map \( \nabla \psi_\omega \)
is a convex Delzant polytope $P \subset \mathbb{R}^n$ and depends only on $[\omega]$ (note that $\omega$ only determines $P$ up to translation; we fix a strictly convex $\psi_0$ satisfying (55) to fix $P$).

We further assume that this is a lattice polytope. Being a lattice Delzant polytope means that: (i) at each vertex meet exactly $n$ edges, (ii) each edge is contained in the set of points $\{ p + t u_{p,j} : t \geq 0 \}$ with $p \in \mathbb{Z}^n$ a vertex, $u_{p,j} \in \mathbb{Z}^n$ and 

$$\text{span}\{u_{p,1}, \ldots, u_{p,n}\} = \mathbb{Z}^n.$$ 

Equivalently, there exist outward pointing normal vectors $\{v_j\}_{j=1}^d \subset \mathbb{Z}^n$ that are primitive (i.e., their components have no common factor) to the $d$ facets in $\partial P$ and $P$ may be written as

$$P = \{ y \in \mathbb{R}^n : l_j(y) := \langle y, v_j \rangle - \lambda_j \leq 0, \quad j = 1, \ldots, d \},$$ 

with $\lambda_j = \langle p, v_j \rangle \in \mathbb{Z}$ with $p$ any vertex on the $j$-th facet, and $y$ the coordinate on $\mathbb{R}^n$.

Given a toric metric $\omega_\varphi$, its corresponding open-orbit Kähler potential $\psi$ is a strictly convex function on $\mathbb{R}^n$ in logarithmic coordinates. Therefore its gradient $\nabla \psi$ is one-to-one onto $P = \text{Im} \nabla \psi$. Its Legendre dual $u := \psi^\star$, called the symplectic potential, is a strictly convex function on $P$. Recall the following formulas that will be used throughout

$$(\nabla \psi)^{-1}(y) = \nabla u(y),$$ 

$$(\nabla^2 \psi)^{-1}_{|\nabla \psi^{-1}}(y) = \nabla^2 u|_y,$$

and if $\eta(s)$ is a one-parameter family of Kähler potentials and $u(s) := \eta(s)^\star$ the corresponding symplectic potentials then

$$\dot{\eta}(s) = -\dot{u}(s) \circ \nabla \eta(s).$$

The proofs of these identities, assuming at least $C^2$ regularity, can be found in [20, pp. 84–87].

### 4.2. Complexifying Hamiltonian flows on toric manifolds.

First we establish the following result regarding the existence of analytic continuations for the Hamiltonian orbits. It shows that on a toric manifold any smooth Cauchy data is good, and moreover gives an explicit expression for the associated Moser maps.

**Lemma 4.1.** Let $(M, J, \omega)$ be a toric Kähler manifold. Given a toric Kähler potential $\varphi_0$ let $\psi_0$ be a smooth strictly convex function on $\mathbb{R}^n$ such that over the open orbit $\omega_{\varphi_0} = \sqrt{-1} \partial \bar{\partial} \psi_0$, and let $\varphi_0$ be a smooth torus-invariant function on $M$. For every $z \in M_0$, the orbit of the Hamiltonian vector field $X^{\omega_{\varphi_0}}$ admits an analytic continuation to the strip $S_\infty$. Moreover, it is given explicitly by

$$f_\tau(z) = \exp -\sqrt{-1} \tau X^{\omega_{\varphi_0}} : z \mapsto z - \tau (\nabla^2 \psi_0)^{-1}|_{\nabla_x \varphi_0}, \quad \tau \in S_\infty.$$ 

This expression remains valid on the divisor at infinity if we restrict to the orbit coordinates $\tilde{x}$ on a slice containing $z$.

Here (and in similar expressions below) by $(\nabla^2 \psi_0)^{-1}|_{\nabla_x \varphi_0}$ we mean the usual matrix multiplication of the matrix $(\nabla^2 \psi_0)^{-1}(x)$ and the vector $\nabla_x \varphi_0(x)$. 
The moment coordinates \( y \) on the polytope \( P \) and the angular coordinates on the regular orbits are action-angle coordinates for the \( (S^1)^n \) Hamiltonian action on \( (M,\omega_{\psi_0}) \), in other words
\[
(\nabla \psi_0)_{*} \omega_{\psi_0} = \sum_{j=1}^{n} dy_j \wedge d\theta_j, \quad \text{over} \quad (P \setminus \partial P) \times (S^1)^n.
\]
The Hamiltonian vector field of \( \dot{\psi}_0 \) is not valid), and this essentially amounts to some toric bookkeeping. Let the complex coordinates \( z \) therefore the Hamiltonian flow of \( X_{\dot{\psi}_0} \) is given in these coordinates by
\[
(\nabla \psi_0)_{*} X_{\dot{\psi}_0} = \sum_{j=1}^{n} \frac{\partial \dot{\psi}_0}{\partial y_j} ((\nabla \psi_0)^{-1}(y)) \frac{\partial}{\partial \theta_j}, \quad y \in P \setminus \partial P.
\]

Therefore the Hamiltonian flow of \( X_{\dot{\psi}_0} \) is given, in terms of the moment coordinates, by
\[
\nabla \psi_0 \exp t X_{\dot{\psi}_0} \circ (\nabla \psi_0)^{-1}.(y,\theta) = (y,\theta - t \nabla y \dot{\psi}_0 \circ (\nabla \psi_0)^{-1}), \quad \text{over} \quad (P \setminus \partial P) \times (S^1)^n,
\]
and in terms of the coordinates on \( M_o \) by
\[
\exp t X_{\dot{\psi}_0}.(x,\theta) = (x,\theta - t(\nabla^2 \psi_0)^{-1}\nabla_{\tau} \dot{\psi}_0).
\]

It therefore admits a holomorphic extension to a map \( \exp \sqrt{-1} t X_{\dot{\psi}_0} \), \( \tau = s + \sqrt{-1} t \), given in these coordinates by (using \( \ref{58}\)–\( \ref{59} \))
\[
\exp -\sqrt{-1} t X_{\dot{\psi}_0}.(x,\theta) = (x - s(\nabla^2 \psi_0)^{-1}\nabla_{\tau} \dot{\psi}_0, \theta - t(\nabla^2 \psi_0)^{-1}\nabla_{\tau} \dot{\psi}_0), \quad s \in \mathbb{R}_+, t \in \mathbb{R}.
\]

For each \( z \in M_o \), this is a holomorphic map of \( S_\infty \) into \( M_o \subset M \) since in terms of the complex coordinates \( z_j := x_j + \sqrt{-1} \theta_j \) it is given by an affine map
\[
\tau \mapsto z - \tau(\nabla^2 \psi_0)^{-1}\nabla_{\tau} \dot{\psi}_0, \quad \tau \in S_\infty.
\]

It remains to consider orbits of points \( z \in M \setminus M_o \) (for these points Equation \( \ref{65} \) is not valid), and this essentially amounts to some toric bookkeeping. Let \( F \subset \partial P \) be a codimension \( k \) face of \( P \) cut out by the equations (see \( \ref{57} \))
\[
F := \{ y \in \partial P : \langle y, v_{j_i} \rangle = \lambda_{j_i}, \quad i = 1, \ldots, k \},
\]
and assume that \( z \) corresponds to a point in the interior of \( F \). More precisely, assume that for a sequence of points \( \{ z_j \} \subset M_o \) converging to \( z \) the points \( \nabla \psi_0(z_j) \) converge to a point in the interior of \( F \). On points in \( M \) that correspond to points in \( F \setminus \partial F \) the stabilizer of the \( (S^1)^n \)-action action is \( k \)-dimensional. In other words, when restricted to \( F \setminus \partial F \), the vector fields \( \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n} \) span an \( (n-k) \)-dimensional distribution. Without loss of generality we may assume that in \( \ref{66} \) we have \( \{ j_1, \ldots, j_k \} = \{ 1, \ldots, k \} \) (otherwise rename the labels). Let \( p \in \partial F \) be a vertex and let \( u_{p,1}, \ldots, u_{p,n} \) be the vector defining the edges emanating from \( p \), as in \( \ref{56} \). Without loss of generality assume the vectors \( u_{p,1}, \ldots, u_{p,n-k} \) span \( F \). On \( F \setminus \partial F \) the vectors \( v_1, \ldots, v_k, u_{p,1}, \ldots, u_{p,n-k} \) span \( \mathbb{R}^n \) and one may find \( n-k \) unit vectors \( \tilde{u}_{p,1}, \ldots, \tilde{u}_{p,n-k} \) such that \( \{ v_1, \ldots, v_k, \tilde{u}_{p,1}, \ldots, \tilde{u}_{p,n-k} \} \) form an orthonormal
basis. Let $U$ denote the orthogonal matrix obtained from these $n$ column vectors. Let $\tilde{y} := yU$ and $\tilde{\theta} := \theta U$. Then in these coordinates \textbf{(62)} becomes

$$\left(\nabla \psi_0\right)_\ast \omega_{\phi_0} = \sum_{j=1}^{n} d\tilde{y}_j \wedge d\tilde{\theta}_j, \quad \text{over } (P \setminus \partial P) \times (S^1)^n. \quad (67)$$

The advantage of this formula is that it specializes to the following formula when restricted to $F \setminus \partial F$:

$$\omega_{\phi_0}|_{(F \setminus \partial F) \times (S^1)^{n-k}} = \sum_{j=k+1}^{n} d\tilde{y}_j \wedge d\tilde{\theta}_j. \quad (68)$$

Hence, in these coordinates the Hamiltonian flow of $\phi_0$ is given by

$$(\tilde{y}, \tilde{\theta}) \mapsto (\tilde{y}, \tilde{\theta}_1, \ldots, \tilde{\theta}_k, \tilde{\theta}_{k+1} + t \nabla\tilde{y}_{k+1} \tilde{u}_0, \ldots, \tilde{\theta}_n + t \nabla\tilde{y}_n \tilde{u}_0).$$

In order to describe the complexification of this map in $M$, we use local holomorphic slice-orbit coordinates $(z', z'') \in \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ (see, e.g., [26]) that can be described as follows. The stabilizer of $(\mathbb{C}^*)^n$ at $z$ is $(\mathbb{C}^*)^k$. The tangent space $T_zM$ decomposes to the tangent space to the orbit of $z$, $T_z((\mathbb{C}^*)^{n-k}, z)$, and its normal $(T_z((\mathbb{C}^*)^{n-k}, z))\perp$. Intersecting each of these spaces with the unit ball in $T_zM$ we therefore obtain local holomorphic coordinates $(z', z'') \in \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$. The coordinates $z'$ are called the slice coordinates, while the $z''$ are called the orbit coordinates. We may write $z''_j = \tilde{x}_j/2 + \sqrt{-1}\tilde{\theta}_j, j = k+1, \ldots, n$, with $\tilde{x}_j = \nabla u_0(\tilde{y}_j)$. On $F \setminus \partial F$ the matrix $\nabla\tilde{y}u$ is of rank $n-k$ with the bottom $(n-k) \times (n-k)$ block an invertible matrix. The same reasoning as before now shows that we have a formula analogous to \textbf{(63)} where we replace $(\nabla^2 z \psi)^{-1}$ by that block of $\nabla^2\tilde{y}u$, and $\nabla x\phi_0$ by $(\nabla\tilde{x}_{k+1} \phi_0, \ldots, \nabla\tilde{x}_n \phi_0)$. Once again we see that the resulting maps extend to the strip $S_{\infty}$, and this concludes the proof of the Lemma.

\textbf{Remark 4.2.} As the Lemma shows, the Hamiltonian orbits admit an analytic continuation to the whole upper half plane. In relation to Remark \textbf{2.5}, we point out that nevertheless the Moser maps do not generically obey a group law in the holomorphic variable $\tau$. To see this, change variables to the action-angle variables $(y, \theta)$. Since $y = y(x)$, $X_H^{\omega_{\phi_0}} = -\sum_j \frac{1}{2} \frac{\partial H}{\partial y_j} \frac{\partial}{\partial y_j}$ and $JX_H^{\omega_{\phi_0}} = -\sum_j \frac{\partial H}{\partial y_j} \frac{\partial}{\partial y_j}$ (with a slight abuse of notation as compared to \textbf{(63)}). Then

$$[X_H^{\omega_{\phi_0}}, JX_H^{\omega_{\phi_0}}] = -\sum_{j,k} \frac{\partial H}{\partial y_k} \frac{\partial^2 H}{\partial x_k \partial x_j} \frac{\partial}{\partial \theta_j},$$

vanishing only if the matrix $\left(\frac{\partial^2 H}{\partial x_k \partial y_j}\right)$ has a kernel, which is generically false.

4.3. Moser flows on toric manifolds. Having derived an explicit expression for the analytic continuations of the Hamiltonian orbits for all imaginary time, we now turn to investigate the invertibility of the resulting Moser maps.

\textbf{Lemma 4.3.} Let $(M, J, \omega)$ be a toric Kähler manifold. Given a toric Kähler potential $\phi_0$ let $\psi_0$ be a smooth strictly convex function on $\mathbb{R}^n$ such that over the open orbit $\omega_{\phi_0} = \sqrt{-1} \partial \bar{\partial} \psi_0$, and let $\phi_0$ be a smooth torus-invariant function on $M$. The Moser
maps \( f_s(z) = \exp -\sqrt{-1} s X_{\psi_0} \) are smoothly invertible if and only if

\[
s < T_{\text{span}}^{\text{cvx}} := \sup \{ a > 0 : \psi_0^* - a \phi_0 \circ (\nabla \psi_0)^{-1} \text{ is convex} \}.
\]  

Note that the formula for \( T_{\text{span}}^{\text{cvx}} \) is well-defined independently of the choice of the open-orbit Kähler potential \( \psi_0 \) for \( \omega_{\psi_0} \).

**Proof.** From the proof of Lemma 4.1 (cf. (65)) we have the following formula for the Moser maps, restricted to the open orbit,

\[
f_s(z) = z - s (\nabla^2 \psi_0)^{-1} \nabla_x \phi_0, \quad \tau \in S_\infty, z \in M_0, \tag{70}
\]

or in terms of the moment coordinates

\[
f_s(\nabla u_0(y)) = \nabla_y u_0(y) + s \nabla_y \dot{u}_0, \quad s \in \mathbb{R}_+, y \in P \setminus \partial P. \tag{71}
\]

Since \( \nabla_y = \nabla_y^2 u_0. \nabla_z \), applying the gradient with respect to \( y \) to this equation we obtain

\[
\nabla^2 u_0(y) \nabla_x f_s(\nabla u_0(y)) = \nabla_y^2 (u_0 + s \dot{u}_0) \tag{72}
\]

Since \( \nabla^2 u_0 \) is invertible for \( y \in P \setminus \partial P \), it follows that the gradient of \( f_s \) is invertible at \( z \in M_0 \) if and only if \( u_0 + s \dot{u}_0 \) is strictly convex on \( P \setminus \partial P \). The analysis for \( z \in M \setminus M_0 \) is similar, following the technicalities outlined in the proof of Lemma 4.1. Since by definition \( u_0 = \psi_0^* \) and using (58) we obtain (69).

This concludes the proof of Proposition 1.9.

5. LEAFWISE SUBSOLUTIONS FOR HRMA

The toric setting is special in that first the Moser maps exist for all \( s \geq 0 \), and second that \( \omega \) admits a Kähler potential on the whole open orbit \( M_0 \). As in the discussion below (22), the Cauchy problem takes the following form:

\[
\Delta \chi_z = 0, \quad \text{on } S_\infty, \quad \chi_z(\sqrt{-1}t) = \psi_0 \circ f_{\sqrt{-1}t}(z), \quad \text{on } \partial S_\infty, \tag{73}
\]

\[
\frac{\partial \chi_z}{\partial s}(\sqrt{-1}t) = \phi_0 \circ f_{\sqrt{-1}t}(z) - \nabla_{g_{\psi_0}} \phi_0(\psi_0) \circ f_{\sqrt{-1}t}(z), \quad \text{on } \partial S_\infty.
\]

We now turn to proving that the HRMA (6) admits a unique leafwise subsolution.

**Proof of Proposition 1.11.** First, we record some useful formulas for the Moser maps on a toric manifold. They follow from the proof of Lemma 4.1 by substituting \( y = \nabla \psi_0 \) in (71) and using (58).

**Lemma 5.1.** Let \( \psi_s \) be a smooth solution of the HRMA (6), and let \( f_s \) denote the associated Moser diffeomorphisms given by Lemma 4.1. Then on the open-orbit,

\[
f_s^{-1} = (\nabla \psi_0)^{-1} \circ \nabla \psi_s = \nabla u_0 \circ (\nabla u_s)^{-1}, \quad s \in [0, T_{\text{span}}^{\text{cvx}}], \tag{73}
\]

and if we let \( u_s(y) = u_0(y) + s \dot{u}_0(y) \), then

\[
f_s = \nabla u_s \circ (\nabla u_0)^{-1}, \quad \text{all } s \geq 0. \tag{74}
\]

These expressions remain valid globally on \( M \) if we use the Euclidean gradient in the orbit coordinates \( \tilde{x} \) along each slice.
Observe that (73) and (74) are in agreement with Proposition 1.29 (i),(ii), respectively.

Next, we show that each of the Cauchy problems (72) admits a unique global smooth solution. In the toric setting the harmonic extension to the generalized leaves of the foliation is especially simple since the initial conditions are constant on the boundary of the strip. In particular, the harmonic functions must be linear along the leaves of the foliation.

**Lemma 5.2.** For every $z \in M_o$ the Cauchy problem for the Laplace equation (72) admits a unique smooth solution, given by

$$\chi_x(\tau) := \langle \nabla \psi_0(z), \nabla (u_0 + su_0) \circ \nabla \psi_0(z) \rangle - (u_0 + su_0) \circ \nabla \psi_0(z).$$

**(Proof.** Note first that uniqueness holds for the Cauchy problem for the Laplace equation on a half-plane (this can be obtained from a suitable generalization of Lemma 2.8 to the case $T = \infty$). We claim that a solution to (72) is given by (75).

First, $\chi_x$ is linear in $s$ and independent of $t$, hence harmonic. Moreover, by (58),

$$\chi_x(\sqrt{-1}t) = \langle \nabla \psi_0(z), \nabla u_0 \circ \nabla \psi_0(z) \rangle - u_0 \circ \nabla \psi_0(z) = u_0^*(z) = \psi_0(z),$$

and by (59) and (60),

$$\frac{\partial \chi_x}{\partial s}(\sqrt{-1}t) = \langle \nabla \psi_0(z), \nabla u_0 \circ \nabla \psi_0(z) \rangle - \dot{u}_0 \circ \nabla \psi_0(z)
= \langle \nabla \psi_0(z), -\nabla^2 \psi_0 \cdot \nabla \phi_0 \rangle + \dot{\phi}_0(z)
= -g_{\phi_0}(\nabla \psi_0(z), \nabla \phi_0) + \dot{\phi}_0(z)
= -\nabla g_{\phi_0} \dot{\phi}_0(\psi_0(z)) + \dot{\phi}_0(z).$$

Finally, observe that in (72) one may eliminate $f_{\sqrt{-1}t}$ since the data $(\psi_0, \dot{\phi}_0)$ is $(S^1)^n$-invariant. Thus, $\chi_x$ satisfies the initial conditions. □

**Remark 5.3.** To see how this Lemma fits in with Proposition 5.3 note that $\Phi_x = \gamma_x^*(\psi_0 - \phi_0)$. Thus, $p_x(t) = -\partial_t \Phi_x = d(\psi_0 + \phi_0)(\nabla g_{\phi_0} \phi_0)(\Gamma_x(\sqrt{-1}t))$, and $q_x - p_x = \phi_0(\Gamma_x(\sqrt{-1}t)) - d\psi_0(\nabla g_{\phi_0} \phi_0)(\Gamma_x(\sqrt{-1}t))$ is a constant (depending on $z$), and so naturally admits an analytic continuation to a whole half-plane.

We now turn to proving that the Cauchy problem admits a unique leafwise subsolution, equal precisely to the Legendre transform potential $\psi L$ given by (5). Note that we use interchangeably $z$ and $x = \log |z|^2$, as $\psi L$ is independent of $\theta$.

To show that (5) defines a leafwise subsolution on the open orbit it suffices to show that for every $z \in M_o$ the function $F^*_x \psi L$ solves the Cauchy problem (72). Now,

$$\psi L(s, z) = u^*_s(z) = \sup_{y \in P} [\langle y, x \rangle - u_s(y)],$$

with the supremum achieved in at least one point $y$ that is contained in the set $(\nabla u_s)^{-1}(z)$. It then follows from (71) and (75) that $F^*_x \psi L = u^*_s \circ f_x(z) = \chi_x(s + \sqrt{-1}t)$, and thus by Lemma 5.2 $\psi L$ defines a leafwise subsolution. This proves the existence part of Proposition 1.11 (i).
To prove the existence part of Proposition 1.11 (ii), it suffices to note that every leafwise subsolution for the HRMA (6) on the open orbit gives rise to a global leafwise subsolution for the HCMA (4) by letting
\[ \varphi(s + \sqrt{-1}t, z) = \psi_L(s, z) - \psi_0(z). \]
This can be seen as follows. Note first that according to Lemma 4.1 the maps \( F_z \) are smooth. Second, note that according to our description of the Moser maps in orbit coordinates, it follows that \( f_\tau \) preserves the interior of each codimension \( k \) toric subvariety of the divisor at infinity \( D \). And so, given \( z \in D = M \setminus M_0 \), the condition \( F_z^*(\pi^*_z\omega + \sqrt{-1}\partial\partial\varphi) \) is equivalent to a Cauchy problem for the Laplace equation, where we now let \( N \) be the open toric variety obtained as the interior of the codimension \( k \) toric subvariety containing \( z \). This Cauchy problem then admits a unique smooth global solution, by working in orbit coordinates, as in Lemma 4.1.

And since, as already noted, \( F_z \) preserves \( N \), the harmonicity of \( F_z^*(\psi_N + \varphi) \) implies that \( F_z^*(\pi^*_z\omega + \sqrt{-1}\partial\partial\varphi) = 0 \) on \( N \), where here \( \psi_N \) is a local Kähler potential for \( \omega \) on \( N \).

Finally, we prove the uniqueness of the leafwise subsolution just constructed. Let \( \eta(\tau, z) \) be another leafwise subsolution. It will suffice to prove that \( \eta = \psi_L \) on the product of \( S_\infty \) and the open-orbit \( M_0 \). Observe that by (74), (75), and the fact that \( \nabla \psi_0 : \mathbb{R}^n \to P \setminus \partial P \) is an isomorphism we have
\[ \eta(\tau, \nabla u_s(y)) = \langle y, \nabla u_s(y) \rangle - u_s(y). \]
for every \( y \in P \setminus \partial P \) and \( \tau \in S_\infty \). Since \( \psi_L \) satisfies the same equation and by [22, Lemma 7.1] \( \operatorname{Im}\nabla u_s|_{P \setminus \partial P} = \operatorname{Im}\nabla u_0|_{P \setminus \partial P} = \mathbb{R}^n \) it follows that \( \eta = \psi_L \). \( \square \)

**Remark 5.4.** As a by-product, Propositions 1.9 and 1.11 give an alternative and conceptual proof that the Legendre transform solves the homogeneous real Monge–Ampère equation. Of course, these results show considerably more since they give information for all time, where the Legendre duality breaks down to some extent. As studied in detail in [22], the leafwise subsolution \( \psi \) measures precisely the extent to which the Legendre duality breaks down.

6. HRMA and the Hamilton–Jacobi equation

We now turn to showing that there exists no admissible \( C^1 \) weak solution of the IVP for \( T > T_\text{span}^\infty \) and establishing the relation between the HRMA and the Hamilton–Jacobi equation. By a weak solution we mean a solution in the sense of Alexandrov.

The first step is the observation that any \( C^1 \) weak solution of HRMA is a classical solution of a Hamilton–Jacobi equation. Some steps resemble the arguments of Proposition 13.1 in [22, §13].

Recall that the initial Neumann data \( \psi_0 \) of the HRMA (6) is a bounded function on \( \mathbb{R}^n \) obtained by restricting the global Neumann data \( \dot{\varphi}_0 \) on the toric manifold to the open-orbit.

**Proof of Theorem 1.13.** Given the Cauchy data \( (\psi_0, \dot{\psi}_0) \) of (6), we set
\[ \dot{u}_0 := -\dot{\psi}_0 \circ (\nabla \psi_0)^{-1}. \]
Lemma 6.1. Let \( \eta \) be a \( C^1 \) admissible solution for the HRMA (70). Define the set-valued map,
\[
G : s \in \mathbb{R}_+ \mapsto \text{Im} \nabla \eta (\{s\} \times \mathbb{R}^n) \subset \mathbb{R}^{n+1}.
\]
Then \( G(s) = G(0) = \{(-\dot{u}_0(y), y) : y \in P \setminus \partial P \} \), for each \( s \in [0, T) \).

Proof. Since \( \eta \) is admissible, \( \nabla_u \eta (\mathbb{R}^n) = P \setminus \partial P \). Thus, \( \nabla \eta (\{s\} \times \mathbb{R}^n) \subset \mathbb{R} \times (P \setminus \partial P) \).

Note that \( G(0) \) is the graph of \(-\dot{u}_0 \) over \( P \setminus \partial P \). We now prove that \( G(s) \subset G(0) \).

The idea is that \( s \to G(s) \) is a continuous set-valued map. If \( G(s) \) is not contained in \( G(0) \), it would sweep out a set of positive Lebesgue measure in \( \mathbb{R} \times (P \setminus \partial P) \subset \mathbb{R}^{n+1} \) as \( s \) varies, contrary to the assumption that \( \eta \) is a weak solution. Since \( \eta(s) \) is \( C^1 \) and strictly convex, its gradient map \( \nabla_u \eta(s) : \mathbb{R}^n \to P \setminus \partial P \) is a homeomorphism to its image, i.e. has a \( C^0 \) single valued inverse, and for each \( x_0 \in P \setminus \partial P \), \( (\nabla_u \eta(s))^{-1} : P \setminus \partial P \to \mathbb{R}^n \) maps an open neighborhood of \( \nabla_u \eta(s,x_0) \in P \) to an open neighborhood \( U \) of \( x_0 \) in \( \mathbb{R}^n \).

To clarify the picture, consider the diagram:
\[
\begin{array}{ccc}
\{s\} \times \mathbb{R}^n & \xrightarrow{\nabla \eta} & G(s) = \{\eta(s,x), \nabla_u \eta(s,x)\} \subset \mathbb{R} \times (P \setminus \partial P) \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{R}^n & \xleftarrow{(\nabla_u \eta(s))^{-1}} & P \setminus \partial P
\end{array}
\]

(77)

Here, \( \pi \) is the natural projection. Since \( \nabla_u \eta(s,x) : \mathbb{R}^n \to P \setminus \partial P \) is a homeomorphism, also \( \pi : G(s) \to P \setminus \partial P \) is a homeomorphism. Thus, \( G(s) \) is a graph over \( P \setminus \partial P \).

Now suppose that there exists \( z = (\eta(s,x_0), \nabla_u \eta(s,x_0)) \in G(s) \setminus G(0) \), i.e. \( \eta(s,x_0) \neq -\dot{u}_0 \circ \nabla_u \eta(s,x_0) \). Then \( \pi^{-1}(U) \subset G(s) \) is a graph passing through \( z \notin G(0) \). Since \( G(s) \) is a continuous set-valued mapping and \( \pi : G(0) \to U \) is a different graph than \( \pi : G(s) \to U \), the intermediate graphs \( \pi : G(\sigma) \to U \) for \( \sigma \in [0,s] \) must fill out the region in between the graphs and create a set of positive Lebesgue measure. To be more precise, put \( S := \sup \{\sigma : G(\sigma) \text{ contains } (-\dot{u}_0 \circ \nabla_u \eta(s,x), \nabla_u \eta(s,x))\} \leq s \). Again by continuity, \( \mathbb{R} \times U \cap \left( \bigcup_{\sigma \in [0,s]} G(\sigma) \right) \) must contain a set of positive Lebesgue measure in \( \mathbb{R}^{n+1} \). This is impossible, though, by Definition [17]. Thus, we have shown that \( G(s) \subset G(0) \)

But since the projection of \( G(s) \) onto the \( \mathbb{R}^n \) factor equals \( P \setminus \partial P \) for each \( s \), and \( G(0) \) is a graph over \( P \setminus \partial P \), the containment just proved implies the equality \( G(s) = G(0) \).

Thus, by Lemma 6.1 and the differentiability assumption, for each \( (s, x) \in [0, T] \times \mathbb{R}^n \) there exists a unique \( y \in P \setminus \partial P \) such that
\[
\left( \frac{\partial \eta}{\partial s}(s, x), \nabla_u \eta(s, x) \right) = (-\dot{u}_0(y), y),
\]
or, in other words,
\[
\frac{\partial \eta}{\partial s}(s, x) = -\dot{u}_0 \circ \nabla_u \eta(s, x), \quad (78)
\]
which concludes the proof of one direction of Theorem 1.13.

For the converse, suppose that \( \eta \in C^1([0, T] \times \mathbb{R}^n) \) is a solution of the Hamilton–Jacobi equation (10). Then \( \text{Im} \nabla \eta \subset G(0) \), and since \( G(0) \) has zero Lebesgue measure in \( \mathbb{R}^{n+1} \), \( \eta \) is a weak solution of the HRMA. \( \square \)

**Remark 6.2.** The proof can be generalized to handle admissible solutions that are only partially \( C^1 \) regular in the sense of [22, §10].

**Proof of Proposition 1.14.** Let \( \psi_L \) denote the leafwise subsolution of the HRMA (10) given by Proposition 1.11 and let \( \eta \) be a \( C^1 \) admissible solution of (6) (see Definition 1.8). Both \( \psi \) and \( \eta \) are convex functions on \( [0, T] \times \mathbb{R}^n \). By Theorem 1.13 both \( \psi \) and \( \eta \) are solutions of the Hamilton–Jacobi equation (10). The method of characteristics implies that \( C^1 \) solutions of (10) are unique as long as the characteristics of the equation do not intersect each other. The equation for the projected characteristic curves \( x(s) \) is (see, e.g., [6, Chapter 3])

\[
\dot{x}(s) = (1, \nabla_x \hat{u}_0(p_\xi(s))), \quad x(0) = (0, x_0),
\]

while \( z(s) \), the solution at \( x(s) \), satisfies

\[
\dot{z}(s) = (1, \nabla_x \hat{u}_0(p_\xi(s))) \cdot (p_\sigma(s), p_\xi(s)), \quad z(0) = \psi_0(x_0),
\]

and \( p(s) = (p_\sigma(s), p_\xi(s)) \), the gradient of the solution at \( x(s) \), satisfies

\[
\dot{p}(s) = 0, \quad p(0) = (\psi_0(x_0), \nabla \psi_0(x_0)).
\]

Therefore, \( x(s) = \left(s, x_0 + s \nabla \hat{u}_0(\nabla \psi_0(x_0))\right) \). Thus, the projected characteristic do not intersect as long as the map \((s, x) \mapsto (s, x + s \nabla \hat{u}_0(\nabla \psi_0(x)))\) is invertible, or equivalently as long as

\[
x \mapsto \nabla u_0 \circ \nabla \psi_0(x) + s \nabla \hat{u}_0 \circ \nabla \psi_0(x)
\]

is invertible on \( \mathbb{R}^n \); this is precisely as long as \( \nabla u_0 + s \nabla \hat{u}_0 \) is invertible on \( P \setminus \partial P \), or as long as \( u_0 + s\hat{u}_0 \) is strictly convex, i.e., precisely for \( s < T_{\text{span}} \). Thus \( \eta = \psi_L \) for \( s \leq T_{\text{span}} \). In fact, the equation for \( x(s) \) shows that the characteristics for the Hamilton–Jacobi equation precisely coincide with the leaves of the HRMA foliation. Moreover, the equation for \( z(s) \) shows that

\[
z(x(s)) = \psi_0(x_0) + s \hat{\psi}_0(x_0) + s \langle \nabla \hat{u}_0 \circ \nabla \psi_0(x_0), \nabla \psi_0(x_0) \rangle
\]

which concludes the proof of one direction of Theorem 1.13.

 Altogether, letting \( u_s := u_0 + s\hat{u}_0 \), we have

\[
z(\nabla u_s(y)) = -u_s(y) + \langle \nabla u_s(y), y \rangle,
\]
or in other words, \( z(s,x) = u^*_s(x) = \psi_L(s,x) \).

Note that in the proof above we show in essence that any \( C^1 \) solution of the HRMA is given by the Hopf–Lax formula \[12, 6\].

Finally, we relate the orbits of the Moser map, the Hamiltonian orbits, and the characteristics in \( \mathbb{R}^{n+1} \) of the HRMA. The following generalizes to weak \( C^1 \) solutions of the HRMA the well-known ‘conservation law’ \[27\] of smooth solutions of the HCM.

**Proposition 6.3.** Let \( \eta \) be a \( C^1 \) weak solution of the HRMA \[6\], and let \( \varphi = \eta - \psi_0 \), considered as a function \( M \). Also, let \( f_s \) be the Moser maps \( f_s(z) = \exp -\sqrt{-1} s X^\omega \varphi_0 \cdot z \) defined in \[2\] and Proposition \[1.9\]. Then

\[
\dot{\varphi}_s \circ f_s = \dot{\varphi}_0.
\]

Further, the \( f_s \)-orbits \((s,f_s(x))\) are the leaves of the real Monge–Ampère foliation, namely the projected characteristics of the Hamilton–Jacobi equation \[10\].

**Proof.** By combining \[78\], \[74\] and Propositions \[1.9\] and \[1.14\], one sees that this equation is equivalent to the Hamilton–Jacobi equation in Theorem \[1.13\].

To prove the last statement we note that the leaves of the Monge–Ampère foliation are orbits of the complexified Hamiltonian action \( \exp t X^\omega \varphi_0 \cdot \). The real orbits lie on the orbits of the Hamiltonian \((S^1)^n\)-action and the real slice of this torus orbit is a point. Hence the real slice is the imaginary time orbit, i.e., the orbit of \( f_s \).

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34

Y.A. RUBINSTEIN AND S. ZELDITCH

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