Simple description of generalized electromagnetic and gravitational hopfions

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Received 26 July 2018, revised 10 October 2018
Accepted for publication 25 October 2018
Published 20 November 2018

Abstract
The generalization of electromagnetic and gravitational hopfions is performed in terms of a complex scalar field. New definition of topological charge for linearized gravity is given. Quasi-local (super-)energy densities are compared for gravitational hopfion.

Keywords: linearized gravity, hopfions, electromagnetism, topological conserved quantities

1. Introduction

A hopfion or Hopf soliton is a ‘solitony’ solution of spin-N field which has rich topological structure related to Hopf fibration. The Hopf fibration is the simplest non-trivial fibration of 3D sphere. We will study electromagnetic and gravitational solutions based on the Hopf projection which is a surjective map sending great circles on $S^3$ to points on $S^2$. These circles weave nested toroidal surfaces and each is linked with every other circle exactly once, creating the characteristic Hopf fibration. The characteristic structure of hopfion can be easily seen on the integration curves of the vector field (see [11]). The structure of closed, linked field lines of hopfions propagates without intersections along the light cone.

In 1977 Trautman [28] proposed the first electromagnetic solutions which were derived from the Hopf fibration. Rañada developed them to propagating solutions in [22, 23]. Last years, these little known solutions become more interesting because hopfions have successful applications in many areas of physics including electromagnetism [11, 24], magnetohydrodynamics [19], hadronic physics [25] and Bose-Einstein condensate [20]. The definition of hopfion was extended in [27]. It includes a class of spin N-fields and uses this to classify the electromagnetic and gravitational hopfions by algebraic type.

In [13] one of us proposed a kind of reduced data for weak gravitational field. Such reduced data which are represented as complex scalar field $\Psi$ enables one to obtain quasi-locally a full gravito-electric (magnetic) tensor for linearized gravity. Similar construction can be obtained...
also for electromagnetic field. We would like to highlight two advantages of the presented approach:

- Considerations (recovery procedure of full gravito-electromagnetic field, investigation of integral quantities, gauge-invariance, etc) in our framework simplify significantly when the reduced data $\Psi$ for linearized gravity have one multipole structure. That happens in the case of hopfions. It holds also in electromagnetic case. See the comments nearby the equations (2.6) and (2.12) for electromagnetic case. Analogically, (3.3) for linearized gravity.
- Presented approach is consistent with non-local nature of gravitational field. Non local physical quantities, like energy (see section 3.2) or topological charge (see section 3.5), can be easily represented in terms of our reduced data and its derivatives.

Our framework can be easily generalized to curved spacetimes which possess spherical symmetry. With the help of conformal Yano–Killing tensors the approach can be generalized for type D spacetimes. The construction presented in [18] (see also appendix B.1) shows how to define the scalar for type D spacetimes.

The paper has two main parts: the first is related to electromagnetic generalization of hopfion, the second one presents the linearized gravity case.

In the first part, we briefly present a description of electromagnetic hopfion-like solution with the help of a complex scalar field $\Phi$. The description is an original approach developed by one of the authors. Its particular application to hopfions drastically simplifies the description and enables one to generalize this notion easily. We demonstrate a simple parametrization of such class of generalized hopfions by scalar wave function $\Phi$.

The constructed scalar represents true degrees of freedom of the field which carries a gauge independent information of the field. The description of E-M field in terms of $\Phi$ is presented in the appendix B. The reconstruction of electromagnetic field from such function is presented. Next, we show the condition for conservation of topological charge—electric (magnetic) helicity in time in terms of $\Phi$.

In the second part, the description of linearized gravity hopfions in terms of the complex scalar field $\Psi$ is presented. This approach is also an original idea introduced by one of us. $\Psi$ plays analogical role to $\Phi$ in electromagnetism. The constructed scalar represents true degrees of freedom of the weak gravitational field which carries a gauge independent information of the field. Gravitational hopfions in terms of $\Psi$ have a simple form and they can be easily generalized to a new class of solutions (3.3). The reconstruction of a gravitational hopfion from such complex, scalar function is performed. We propose a new definition of a topological charge for spin-2 field in analogy to the electromagnetic case. Hamiltonian energy for linearized gravity is discussed. To indicate the difference between the super-energy of spin-2 field and the Hamiltonian energy which is a physical energy of gravitational field we present a few quasi-local (super-)energy densities in terms of the complex scalar $\Psi$. We compare such quasi-local (super-)energy densities for gravitational hopfion.

To clarify the exposition, a full explanation of a complex, scalar framework for electromagnetism/linearized gravity has been placed in the appendix.

### 1.1. Notation

For convenience we use index notation with Einstein summation convention. Minkowski spacetime is the background with the metric $g = -dt^2 + \delta_{ab}dx^a dx^b$. The 3D spatial metric is denoted by $\delta_{ab}$. Small latin indices, except $t$ and $r$, run spatial coordinates on $\Sigma = \{ t = \text{const.} \}$

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1. The E-M field can be equivalently represented by the complex scalar field $\Phi = (E + iB) \cdot r$, where $E$—electric vector field, $B$—magnetic vector field, $r$—position vector field and $i^2 = -1$. See equation (2.1) and the appendix B.
slice. The distinguished \( t \) and \( r \) are respectively time and radius; \( \mathbf{r} \) is a 3D position vector. Bold letters mean 3D spatial vectors and ‘·’ is a 3D scalar product. For example, \( \mathbf{E} \cdot \mathbf{C} = E^iC_i\partial_i \). Capital letters represent the axial coordinates on the unit sphere, \( ' \), \( ' \) denotes the partial derivative \( \partial \). \( ' | \) is a 3D spatial covariant derivative on \( \{ t = \text{const.} \} \) surface and \( ' || ' \) denotes a 2D covariant derivative on the sphere of radius \( r \). The 2D trace is denoted by \( \mathbf{X} = g^{CD}X_{CD} \) and the 2D traceless part \( \mathbf{X}_{\alpha\beta} = X_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}X^{\alpha\beta} \).

\[ T_{\alpha\beta\cdots} = \frac{1}{2}g_{\alpha\beta}T_{\gamma\delta\cdots} \]

\[ T_{[\alpha\beta]} = \frac{1}{2}g_{\alpha\beta}T_{\gamma\delta\cdots} \]

\[ Y_l = \text{spherical harmonic with a specified degree} \]

\[ \frac{\Delta Y_l}{-l(l+1)}Y_l \]

\[ \text{is a linear combination of spherical harmonics} \]

\( Y_{lm} \) with any order \( m \), where \( m \in \{-l, -l+1, \ldots, l-1, l\} \). See (2.7) and the comments below for precise formulation.

2. Generalized hopfions in electrodynamics

2.1. Class of generalized hopfions

Consider a class of complex functions on the Minkowski background which are harmonic:

\[ \square \Phi = 0 \quad (2.1) \]

where \( \square \) is the d’Alembert operator in Minkowski spacetime.

There exists a bijection between electromagnetic solutions and such complex scalar fields. For a given Riemann–Silberstein vector \( \mathbf{Z} := \mathbf{E} + \mathbf{iB} \), complex combination of electric vector field \( \mathbf{E} \) and magnetic vector field \( \mathbf{B} \), we simply define \( \Phi := \mathbf{Z} \cdot \mathbf{r} \), i.e. \( \Phi \) is the scalar product of Riemann–Silberstein vector and position vector (see equation (B.7) and comments nearby). To check the inverse mapping we need to show the reconstruction of the full EM data \( \mathbf{Z} \) from a wave function \( \Phi \).

From now, we restrict ourselves to use \( (t, \theta, \phi, r) \) coordinates with metric

\[ \delta_{ab}dx^adx^b = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

The procedure presented below describes how to recover Riemann–Silberstein vector field \( \mathbf{Z} \) from \( \Phi \). We would like to stress that the presented procedure can be used for any smooth solution of (2.1).

The definition of \( \Phi \) and some of the vacuum Maxwell equations (B.9) and (B.10) in terms of scalar \( \Phi \) (in index notation) take the form

\[ \partial_r (r\Phi) = -r^2Z^A|_A \quad (2.2) \]

\[ \partial_\alpha \Phi = \varepsilon^{AB}Z_A|_B \quad (2.3) \]

\[ \Phi = rZ^r \quad (2.4) \]

where \( \varepsilon^{AB} \) is a Levi-Civita tensor on a sphere \( t = \text{const.}, r = \text{const} \). Hence, quasi-locally the above formulae enable one to reconstruct \( \mathbf{Z} \). More precisely, according to Hodge–Kodaira

\[ i \in \mathbb{C} \quad r \in \mathbb{R} \quad \theta, \phi \text{ parametrize the two-sphere.} \]

\[ \varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } \{t, \theta, \phi, r\} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } \{t, \theta, \phi, r\} \\ 0 & \text{in any other case.} \end{cases} \]

For lower dimensional case, we have \( \varepsilon_{abc} = \varepsilon_{tabc} \) and \( \varepsilon_{AB} = \varepsilon_{tAB} \).
theory applied to differential forms on a sphere\textsuperscript{4}. $Z_\alpha \, dx^A$ can be decomposed into a gradient and co-gradient of some functions $\alpha$ and $\beta$

\[ Z_\alpha = \alpha_A + \varepsilon_A^B \beta_B. \quad (2.5) \]

The equations (2.2)–(2.5) allow to obtain $\Delta \alpha$ and $\Delta \beta$. The 2D Laplace operator $\Delta$ on the unit sphere can be quasi-locally inverted with the help of methods which are presented in appendix A. From now, we restrict ourselves to the function $\Phi$ which is a $\ell$-pole like in the formula (2.12). For convenience, we define the time-radius part $\phi$ of $\Phi$:

\[ \Phi = \phi(t, r) Y_l(\theta, \varphi) \quad (2.6) \]

where $Y_l$ is a spherical harmonic of $l$th degree. The construction presented in the paper, except section 3.4, requires only two properties of spherical harmonics:

- Spherical harmonics are eigenfunctions of the 2D Laplace operator:
  \[ \Delta Y_l = -l(l+1)Y_l. \quad (2.7) \]

- There are two distinguished cases with specified order $m$ of spherical harmonic: axially symmetrical harmonic (order of multipole $m = 0$) and a harmonic with a maximal order $m = \pm l$.

For convenience of the reader, we choose the following representation of spherical harmonics:

\[ Y_{lm} = P_{lm}(\cos \theta) e^{im\varphi}. \quad (2.8) \]

Often the order $m$ of a spherical harmonic does not have to be specified. In that case, $Y_l$ denotes a linear combination of spherical harmonics with distinguished degree $l$ and any order $m$.

We can split $\alpha$ and $\beta$ into multipoles. We highlight that the multipole decomposition is convenient to use in the examined case but the reconstruction procedure does not require multipole splitting in general. Equation (2.6) suggests that only one $\ell$-pole will be non-vanishing in the expansion

\[ Z_\alpha = a(t, r)(Y_\ell)^A + b(t, r)e^{A B}(Y_\ell)_B. \quad (2.9) \]

Combining (2.2) with (2.9), the direct formula for complex scalar function $a(t, r)$ is obtained:

\[ a(t, r) = \frac{\partial}{\partial r} \left( r \phi(t, r) \right) \frac{l}{l(l+1)}. \quad (2.10) \]

Analogically, using (2.3) and (2.9), we obtain the function $b(t, r)$:

\[ b(t, r) = -i \frac{r}{l(l+1)} \partial_t \phi(t, r). \quad (2.11) \]

We reconstruct the 2D part of $Z$. The radial component of $Z$ is algebraically related with $\Phi = Z' r$. We recover the full form of $Z$ in that way.

In the context of hopfions, the interesting set of solutions of (2.1) is

\[ \Phi_H = \frac{r^l Y_l}{\left[ r^2 - (t - i)^2 \right]^{l+1}}. \quad (2.12) \]

\textsuperscript{4}See appendix A.3. There are no harmonic one-forms on a two-sphere.
The dipole solution from (2.12) is related to Hopfion solution from [27], so we call (2.12) generalized hopfions. The properties of solutions (2.12) are discussed in the sequel at the end of section 2.

2.2. Chandrasekhar–Kendall vector potential

A vector potential is defined up to a gradient of some function by the formula $Z = \text{curl} \, V$—see appendix B, equation (B.11). The field $Z$ for presented class of generalized hopfions (2.12) has simple multipole structure. It leads to a similar form of $V$. We propose for $V$ following ansatz:

\begin{align*}
V^r &= s(t, r)Y_l \\
V^A &= p(t, r)(Y_l)^A + q(t, r)\epsilon^{AB}(Y_l)_B.
\end{align*}

The above formulae and the Maxwell equation (B.11) imply

\begin{align*}
\partial_r \phi(t, r) &= l(l + 1)q(t, r) \\
a(t, r) &= \partial_r[q(t, r)] \\
b(t, r) &= s(t, r) - \partial_r[p(t, r)].
\end{align*}

For solutions (2.12), freedom of choice of $V$ enables one to construct vector potential in Chandrasekhar–Kendall (C–K) form. C–K potential is an eigenvector of the curl operator $Z = \lambda(t, r)V$ where $\lambda(t, r)$ is a complex, scalar function. It leads to an overdetermined system of equations

\begin{align*}
\phi(t, r) &= \lambda s(t, r) \\
a(t, r) &= \lambda p(t, r) \\
b(t, r) &= \lambda q(t, r).
\end{align*}

It turns out that the equations (2.15)–(2.17) and (2.19)–(2.21) for solutions (2.12) are self-consistent. For (2.12), we introduce the time-radius part (2.6) denoted by $\phi_H(t, r)$. The solutions are the following functions

\begin{align*}
s(t, r) &= \frac{r\phi_H(t, r)^2}{\partial_t \phi_H(t, r)} \\
p(t, r) &= \frac{r\phi_H(t, r)\partial_t[r\phi_H(t, r)]}{l(l + 1)\partial_r \phi_H(t, r)} \\
q(t, r) &= \frac{r\phi_H(t, r)}{l(l + 1)}
\end{align*}

which represent eigenvector of (2.18) with the following eigenvalue:

\begin{align*}
\lambda(t, r) &= -i\partial_t \ln(\phi_H(t, r)).
\end{align*}

5 Chandrasekhar–Kendall potential is part of a family of fields known as force-free fields and is of broad importance in plasma physics and fluid dynamics. See [11] and the citations within.
2.3. Conservation of topological charge in time

For electric and magnetic field fulfilling constraints one can introduce vector potentials:

\[ E = \text{curl } C, \quad B = \text{curl } A, \quad V := C + \imath A, \quad Z = \text{curl } V. \]

See appendix B for details. Helicity integrals measure topological properties of field lines. For electromagnetic field, electric helicity

\[ h_E = \int_\Sigma C \cdot E \]  

and magnetic helicity

\[ h_M = \int_\Sigma A \cdot B \]

are quantities which are related to a number of linkedness and knotness of the integral curves of the electric \( E \) (magnetic \( B \)) vector field. \( C \) and \( A \) are vector potentials for \( E \) and \( B \) respectively (see appendix B for details). \( \Sigma \) means the whole spatial space on a slice \( \{ t = \text{const.} \} \)

Helicities quantifies various aspects of field structure. Examples of fields which poses non-vanishing helicity include twisted, linked, knotted or kinked flux tubes, sheared layers of flux, and force-free fields. The origins of helicity integrals are related with Gauss linking integral. See [2] for detailed review.

It is convenient to present the helicities in terms of Riemann–Silberstein vector field \( Z \) and its vector potential \( V \):

\[ h_E + h_M = \int \Re \left( Z \cdot \nabla \bar{V} \right) = \Re \int_\Sigma Z^a \bar{V}^b \delta_{ab} d^3x \]  

\[ h_E - h_M = \int \Re \left( Z \cdot V \right) = \Re \int_\Sigma Z^a V^b \delta_{ab} d^3x \]

where \( \bar{V} \) is the complex conjugate of \( V \) and \( \Re \) denotes the real part. Using the scalar description of E-M fields (appendix B) we can express total helicity (2.28) in terms of \( \Phi \):

\[ h_E + h_M = \int \Re \left[ \imath \left( \Phi \Delta^{-1} \partial_r \bar{\Phi} - \bar{\Phi} \Delta^{-1} \partial_r \Phi \right) \right] \]  

where \( r^2 = -1 \) and \( \Delta^{-1} \) is an inverse operator to the 2D Laplace operator on the unit sphere (see appendix A).

The equations (2.30) and (2.1) imply conservation law for total helicity:

\[ \partial_t (h_E + h_M) = \lim_{R \to \infty} \int_{B(0,R)} \Re \left[ \imath \left( \Phi \Delta^{-1} \partial_r \bar{\Phi} - \bar{\Phi} \Delta^{-1} \partial_r \Phi \right) \right] \]  

\[ = \lim_{R \to \infty} \int_{\partial B(0,R)} \Re \left[ \imath \left( \Phi \Delta^{-1} r^2 \partial_r \bar{\Phi} - r^2 \partial_r \Phi \Delta^{-1} \bar{\Phi} \right) \right] \]

where \( B(0,R) = \{ x \in \Sigma : |x| \leq R \} \). We assume the E-M fields are localized\(^6\), hence the boundary terms at infinity can be neglected. In general, for the quantity \( h_E - h_M \) (2.29) we have no time dependence. However, in terms of \( \Phi \) we have

\(^6\) By localized we mean compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms.
and
\[ \partial_t (h_E - h_M) = -2 \int_\Sigma \Re \partial_t \left( \imath \Phi \Delta^{-1} \partial_t \Phi \right) \] (2.33)

which lead to the following

**Proposition 2.1.** For localized fields, the helicities (2.26) and (2.27) are preserved in time if and only if
\[ \int_\Sigma \Re \partial_t \left( \imath \Phi \Delta^{-1} \partial_t \Phi \right) = 0 \] (2.34)

The following quasi-local equality
\[ \int_{\partial B(0,R)} Z \cdot Z = \int_{\partial B(0,R)} \partial_t \left( \Phi \Delta^{-1} \partial_t \Phi \right) \] (2.35)
gives equivalence to the Ráñada result in [22]. We highlight that \( \cdot \) denotes scalar product without complex conjugate.

### 2.4. Discussion

The conservation of topological charge imposes an additional condition (2.34) for solutions (2.12). The integral in (2.34) for solutions (2.12) contains an integral of a square of a single multipole \( Y_l \) over a 2D sphere. \( \int_0^\pi d\theta \int_0^{2\pi} d\varphi (Y_l)^2 \) is equal to zero for non-zero order \( m \) of multipole. We denote \( Y_l = Y_{lm} \) where \( l \) and \( m \) are respectively a degree and an order \(^7\) of multipole. Hence these values of \( m \) for each \( l \) lead to an E-M field which preserves the topological charge. Such E-M solution is a generalization of the null hopfion. For \( l = 1 \) our solutions with the maximal order are equal (up to a constant) to the null hopfion described in [27]. The case \( l = 1, m = 0 \) corresponds to non-null hopfion from [27].

### 3. Spin-2 field and generalized gravitational hopfions

Consider a weak gravitational field on the Minkowski background. The used complex scalar framework is related to the linearized Weyl tensor splitted into a tidal (gravito-electric) part \( E_{kl} \) and frame-dragging (gravito-magnetic) part \( B_{kl} \) (see appendix C.2). Both \( E_{kl} \) and \( B_{kl} \) are symmetric and traceless. With the help of the constraint equations,
\[ E_{kl} \big|_\nu = 0 \] (3.1)
\[ B_{kl} \big|_\nu = 0 \] (3.2)
we can quasi-locally describe the field in the terms of complex scalar field \( \Psi \). See the appendix C.3 for precise formulation and details. The used notation and denotings are presented in appendix C.

\(^7\)Physicists usually use the naming convention which is associated with quantum mechanics. The degree of multipole is related to spin number and the order of multipole corresponds to magnetic spin number.
3.1. Reconstruction for linearized gravity field

The reconstruction for linearized gravity field is a generalization of the procedure for electromagnetic field described in the section 2.1. Constraint equations and the Hodge–Kodaira decomposition for 2D tensors on a sphere (see appendices A.4 and C.3) enable one to encode quasi-locally a spin-2 field into a complex scalar field. For a given \( l \)-pole field the reconstruction is simplified because the inverse operator to the 2D Laplacian has a simple form. In the context of hopfions, we consider a class of complex scalar fields in the following form

\[
\Psi_H = \frac{r^l Y_l}{[r^2 - (t - i)^2]^{l+1}}
\]

for \( l \geq 2 \). For convenience, we define

\[
\psi_H = \frac{r^l}{[r^2 - (t - i)^2]^{l+1}}.
\]

\( \Psi_H = \psi_H Y_l \) is the same function as \( \Phi_H \) (2.12) for the set of generalized electromagnetic hopfions. For \( l = 2 \), the solution (3.3) is related to gravitational hopfion\(^8\), so we call the set of solutions (2.12) generalized gravitational hopfions. \( \Psi_H \) fulfills wave equation \( \Box \Psi_H = 0 \) and represents gauge-invariant reduced data for linearized vacuum Einstein equation. For given \( l \)-pole field (3.3) the structure of reconstructed gravo-electromagnetic tensor \( Z_{\alpha \beta} \) is as follows:

\[
Z^\alpha \equiv a_\alpha(t, r) Y_l
\]

\[
Z^{\alpha A} = b_\alpha(t, r)(Y_l)^{|A} + c_\alpha(t, r)\varepsilon^{AB}(Y_l)^{|B}
\]

\[
Z^{(2)} = -a_\alpha(t, r) Y_l
\]

\[
\overset{\circ}{Z}_{\alpha \beta} = d_\alpha(t, r) \left( (Y_l)^{|AB} - \frac{1}{2} \frac{g_{AB}}{r^2} \right) + e_\alpha(t, r)(Y_l)^{|C} \varepsilon_{AB}^C
\]

where

\[
a_\alpha(t, r) = \frac{\psi_H}{r^2}
\]

\[
b_\alpha(t, r) = \frac{\partial_r (r \psi_H)}{l(l+1)}
\]

\[
c_\alpha(t, r) = \frac{\partial_t \psi_H}{l(l+1)}
\]

\[
d_\alpha(t, r) = \frac{\partial_r (r \partial_t (r \psi_H))}{l(l+1)} - \frac{l(l+1)}{l(l+1) - 2} \frac{\partial_r \psi_H}{l(l+1) - 2}
\]

\[
e_\alpha(t, r) = \frac{\partial_r (r^2 \partial_t \psi_H)}{l(l+1)}
\]

which are similar to (2.10) and (2.11) for electromagnetic case.

\(^8\)The type \( N \) hopfion from [27] covers (up to a constant) with the solution from class (3.3) for \( l = 2 \) and for the spherical harmonic with maximum spin number \( m = l = 2 \).
3.2. Hamiltonian energy for linearized gravity

Energy of a gravitational field is an issue which is problematic in various contexts. The ambiguity of energy of gravitational system can be observed even in Newtonian theory. Consider Newtonian gravitational potential \( \phi \) on the flat 3D space with a gravitational force given by one-form \( d\phi = \phi_k dx^k \). According to Galileo–Eötvös experiment, i.e. the principle of equivalence, there is an ambiguity in the gravitational force: It is determined only up to an additive constant covector field \( \omega_k \), and hence by an appropriate transformation \( \phi_k \rightarrow \phi_k + \omega_k \) the gravitational force \( \phi_k \) at a given point \( p \in \Sigma \) can be made zero. Thus, at this point both the gravitational energy density and the spatial stress have been made vanishing. On the other hand, they can be made vanishing on an open subset \( \mathcal{O} \subset \Sigma \) only if the tidal force, \( \phi_{ik} \), is vanishing on \( \mathcal{O} \). Therefore, the gravitational energy and the spatial stress cannot be localized to a point, i.e. they suffer from the ambiguity in the gravitational force above. For a more detailed discussion of the energy in the (relativistically corrected) Newtonian theory, see [9].

In the case of general relativity theory the issue is more complicated and well-known. Brill and Deser have published a series of classical papers [4–6] in which the issue of ambiguity and positivity of energy is discussed.

Sections 3.2–3.4 have been written to point out that the real (Hamiltonian) energy of weak gravitational field can not be localized. We compare such quasi-local Hamiltonian energy density with chosen well-known (super-) energy densities using hopfions as an example.

In [17] (see also [12] and [13]) one of us proposed energy functional \( \mathcal{H} \) which takes the following form in Minkowski spacetime:

\[
\mathcal{H} = \frac{1}{32\pi} \int_{\Sigma} \left[ (r\dot{x}) \Delta^{-1}(\Delta + 2)^{-1}(r\dot{x}) + (r\dot{y}) \Delta^{-1}(\Delta + 2)^{-1}(r\dot{y}) \\
+ (rx)\Delta^{-1}(\Delta + 2)^{-1}(rx) - x(\Delta + 2)^{-1}x \\
+ (ry)\Delta^{-1}(\Delta + 2)^{-1}(ry) - y(\Delta + 2)^{-1}y \right] dr \sin \theta d\theta d\phi \tag{3.14}
\]

where \( x \) and \( y \) are defined by relation \( \Psi = x + iy \). \( \mathcal{H} \) have simpler form in terms of \( \Psi \):

\[
\mathcal{H} = \frac{1}{32\pi} \int_{\Sigma} \left[ (r\dot{\psi}) \Delta^{-1}(\Delta + 2)^{-1}(r\dot{\psi}) \Psi \\
+ (r\psi)\Delta^{-1}(\Delta + 2)^{-1}(r\psi) \right] dr \sin \theta d\theta d\phi. \tag{3.15}
\]

The formula has its origins in the canonical (Hamiltonian) formulation of the linearized theory of gravity. In this sense it describes a true energy of linearized gravitational field. In section 3.4 we remind another two super-energy functionals\(^8\) \( \Theta_0 \) (3.23) (see also [14]) and super-energy (3.30) which arises for spin-2 field in a natural way. In particular, the integrals (3.23) and (3.15) differ by the operator \( (\Delta + 2)^{-1} \), hence for each spherical mode (i.e. after spherical harmonics decomposition) they are proportional to each other. Hamiltonians for whom functions in multipole expansions differ by a constant multiplicative factor lead to the same dynamics. We will discuss in a separate paper [16] how the functional \( \mathcal{H} \) is related to the following expression:

\(^8\) We remind a quasi-local densities of such energy functionals. The difference is only to integrate over the radial coordinate i.e. \( \Theta_0 = \int_0^\infty U_{\Theta_0} dr \).


\[ 16\pi \mathcal{H}_V = \int_V \left( E_{ab} (-\Delta)^{-1} E_{ab} + B_{ab} (-\Delta)^{-1} B_{ab} \right) \]

\[ = \int \int_{\Sigma \times \Sigma} \left[ \frac{E_{ab}(r') E_{ab}(r'')} {4\pi \|r' - r''\|} + \frac{B_{ab}(r') \cdot B_{ab}(r'')} {4\pi \|r' - r''\|} \right] dr' dr'' \quad \text{(3.16)} \]

\[ = \int \int_{\Sigma \times \Sigma} \left[ Z_{ab}(r') Z_{ab}(r'') \right] \frac{dr' dr''} {4\pi \|r' - r''\|} \quad \text{(3.17)} \]

which is proposed by Bialynicki-Birula [3] and has a nice property—it is manifestly covariant with respect to the Euclidean group. In the future we also plan to incorporate boundary terms because we want to generalize the above formulae to finite region with boundary.

Let us consider localized initial data on \( \Sigma \), i.e. compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms. The following theorem (to be presented in detail in [16])

**Theorem 3.1.** For localized data \( \mathcal{H} = \mathcal{H}_V \).

can be checked as follows:

**Proof.** Let us observe that \( x = 2 \dot{x} x E_{\dot{t}t} \), \( y = 2 \dot{y} x B_{\dot{t}t} \). If we introduce transverse–traceless potentials\(^{10}\) \( e \) and \( h \):

\[-\Delta e_{\dot{t}t} = E_{\dot{t}t}, \quad -\Delta h_{\dot{t}t} = B_{\dot{t}t} \]

where \( \Delta \) is the 3D Laplacian\(^{11}\), then for \( a := 2 \dot{x} x e_{\dot{t}t} \), \( b := 2 \dot{y} x h_{\dot{t}t} \) we get

\[-\Delta a = x, \quad -\Delta b = y. \]

Moreover, for finite region \( V \subset \Sigma \)

\[ 16\pi \mathcal{H}_V := \int_V \left( e_{\dot{t}t} E_{\dot{t}t} + h_{\dot{t}t} B_{\dot{t}t} \right) d^3x \quad \text{(3.18)} \]

\[ \begin{align*}
\quad = \frac{1}{2} \int \int_{\Sigma} \frac{1}{\sqrt{r^2}} \left( \frac{\partial}{\partial r} (-\Delta)^{-1} (r\dot{x}) + \partial_r (r\dot{x}) (-\Delta)^{-1} \partial_r (r\dot{x}) + \frac{1}{2} ax \right. \\
\quad \quad \left. + \frac{1}{2} \Delta a \right) \Delta^{-1} (\Delta + 2) \partial_r (r^2 \dot{x}) + \frac{1}{4} ax \\
\quad \quad + \left( \partial_r [r \partial_r (ra)] + \frac{1}{2} \Delta a \right) \Delta^{-1} (\Delta + 2) \partial_r (r^2 \dot{x}) + \frac{1}{4} ax \\
\quad \quad + \left( \partial_r [r \partial_r (ra)] + \frac{1}{2} \Delta a \right) \Delta^{-1} (\Delta + 2) \partial_r (r^2 \dot{x}) + \frac{1}{4} ax \\
\quad \quad + \left( \partial_r [r \partial_r (ra)] + \frac{1}{2} \Delta a \right) \Delta^{-1} (\Delta + 2) \partial_r (r^2 \dot{x}) + \frac{1}{4} ax \\
\quad \quad + \left( \partial_r [r \partial_r (ra)] + \frac{1}{2} \Delta a \right) \Delta^{-1} (\Delta + 2) \partial_r (r^2 \dot{x}) + \frac{1}{4} ax \quad \text{dr sin} \theta d\theta d\phi. \quad \text{(3.19)}
\end{align*} \]

\(^{10}\) Transverse–traceless symmetric tensor-field \( h_{\dot{t}t} \) means \( h_{\dot{t}t} \delta^{\dot{t}t} = 0 \) and \( h^t_t = 0 \).

\(^{11}\) In Cartesian coordinates it is simply \( \Delta = \sum_{i=1}^3 \left( \frac{\partial}{\partial x^i} \right)^2 \).
Now, we have to integrate by parts many times and finally we obtain energy (3.15) up to boundary terms
\[
16\pi \mathcal{H}_V = \frac{1}{2} \int_V \left[ (-r^2 \Delta a) \Delta^{-1}(\Delta + 2)^{-1} x + \partial_t (-r \Delta a) \Delta^{-1}(\Delta + 2)^{-1} \partial_t (\Delta + 2)^{-1} x \\
+ \Delta a (\Delta + 2)^{-1} x + \partial_t (-r \Delta b) \Delta^{-1}(\Delta + 2)^{-1} \partial_t (\Delta + 2)^{-1} y \\
+ (-r^2 \Delta b) \Delta^{-1}(\Delta + 2)^{-1} y + \Delta b (\Delta + 2)^{-1} y \right] dr \sin \theta d\theta d\varphi \\
+ \frac{1}{2} \int_{\partial V} \left[ \partial_t (r^2 a) \Delta^{-1}(\Delta + 2)^{-1} x - \frac{1}{2r^2} \partial_t (r^2 a) (\Delta + 2)^{-1} x \\
+ \partial_t (r^2 b) \Delta^{-1}(\Delta + 2)^{-1} y - \frac{1}{2r^2} \partial_t (r^2 b) (\Delta + 2)^{-1} y \\
+ \left( r \Delta a + \frac{1}{r} \partial_t (ra) \right) \Delta^{-1}(\Delta + 2)^{-1} \partial_t (\Delta + 2)^{-1} x \\
+ \partial_t (r^2 b) \Delta^{-1}(\Delta + 2)^{-1} y - \frac{1}{2r^2} \partial_t (r^2 b) (\Delta + 2)^{-1} y \right] \sin \theta d\theta d\varphi. \tag{3.20}
\]

More precisely, the volume term in the above formula equals $16\pi \mathcal{H}$ given by (3.15).

3.3. Quasi-local (super-)energy density for spin-2 field

We present quasi-local (q-l) energy and super-energy densities for spin-2 field and linearized gravity. By q-l density we mean a functional which is an integral over a 2D topological boundary. The compared q-l densities listed below are presented in general form—they are valid for every localized weak gravitational field represented as a complex harmonic function $\Psi$. The q-l (super-)energy densities can be organized as follows:

1. Related to the canonical (Hamiltonian) theory:
   
   (a) The q-l energy density of hamiltonian energy $\mathcal{H}$ (3.15) derived from the canonical formulation of linearized theory of gravity.

   \[
   U_\mathcal{H} = \frac{1}{32\pi} \int_{S(t,r)} \sin \theta \left[ (r \partial_t \Psi) \Delta^{-1}(\Delta + 2)^{-1} (r \partial_t \Psi) \\
   + (r \Psi) \Delta^{-1}(\Delta + 2)^{-1} (r \partial_t \Psi) \right] 
   \tag{3.21}
   \]

   where $S(t, r)$ denotes $\{ t = \text{const.}, r = \text{const.} \}$ surface.

   (b) $\Theta_0$ functional is obtained with the help of conformal Yano–Killing (CYK) tensors. The contraction of CYK tensor $Q^{\mu\nu}$ with Weyl tensor $W_{\mu\nu\alpha\beta}$ is a two-form $F_{\alpha\beta} = Q^{\mu\nu} W_{\mu\nu\alpha\beta}$, where $Q^{\mu\nu} \partial_{\nu} \wedge \partial_{\nu} = \mathcal{D} \wedge \partial_t$ is a CYK tensor for Minkowski spacetime and $\mathcal{D} = x^\nu \partial_{\nu}$ is a generator of dilatations in Minkowski spacetime. $F_{\alpha\beta}$ fulfills vacuum Maxwell equations. $\Theta_0$ is an electromagnetic energy calculated for $F_{\alpha\beta}$ from stress–energy tensor.
\[ T^\text{EM}_{\mu\nu}(F) := \frac{1}{2}(F_{\mu\sigma}F^{\sigma}_{\nu} + F^{*}_{\mu\rho}F^{*}_{\nu}^{\rho}) \]  

(3.22)

where \( F^{*\mu\lambda} = \frac{1}{2}\varepsilon^{\mu\lambda\rho\sigma}F_{\rho\sigma} \). See [14] for details. The q-l density of \( \Theta_0 \) is

\[ 4\pi U_{\Theta_0} = \int_{S(t,r)} T^\text{EM}(\partial_t, \partial_r, F(W, D \wedge T)) r^2 \sin \theta d\theta d\varphi \]

\[ = \frac{1}{2} \int_{S(t,r)} r^2 (E_k E^k + B_k B^k) r^2 \sin \theta d\theta d\varphi \]

\[ = \frac{1}{4} \int_{S(t,r)} \left[ \partial_t (r\Psi)(-\Delta)^{-1} \partial_t (r\overline{\Psi}) + \partial_r (r\Psi)(-\Delta)^{-1} \partial_r (r\overline{\Psi}) + \Psi \overline{\Psi} \right] \sin \theta d\theta d\varphi. \]  

(3.23)

(c) We compare q-l energy densities for linearized gravity with q-l electromagnetic energy densities for the corresponding electromagnetic solution (compare (3.36) and (3.37)). Let us define

\[ F_1(\Phi) := \left[ (r\partial_t \Phi)(-\Delta^{-1})(r\partial_t \overline{\Phi}) + (r\Phi)_t(-\Delta^{-1})(r\overline{\Phi}) + \Phi \overline{\Phi} \right] \sin \theta \]  

(3.24)

\[ F_2(\Phi) := \left[ \partial_r (r\Phi)\Delta^{-1}r\partial_r \overline{\Phi} + \partial_r (r\overline{\Phi})\Delta^{-1}r\partial_r \Phi \right] \sin \theta. \]  

(3.25)

The electromagnetic q-l energy density in terms of electromagnetic scalar \( \Phi \) is equal to

\[ 4\pi U_{\text{EM}} = \int_{S(t,r)} T^\text{EM}(\partial_t, \partial_r, \Phi) r^2 \sin \theta d\theta d\varphi \]

\[ = \frac{1}{4} \int_{S(t,r)} F_1(\Phi) d\theta d\varphi. \]  

(3.26)

The electromagnetic q-l energy density for the conformal field

\[ \mathcal{K} = 2rt\partial_t + (t^2 + r^2) \partial_r \]  

(3.27)

is the following

\[ 4\pi U_{\text{CEM}} = \int_{S(t,r)} T^\text{EM}(\mathcal{K}, \partial_t, \Phi) r^2 \sin \theta d\theta d\varphi \]

\[ = \frac{1}{4} \int_{S(t,r)} \left[ (r^2 + t^2) F_1(\Phi) + 2rtF_2(\Phi) \right] d\theta d\varphi \]  

(3.28)

2. Associated to Bel–Robinson tensor. The Bel–Robinson tensor has the structure

\[ T_{\mu\nu\kappa\lambda}^\text{KR} := W_{\mu\rho\kappa} W^{\rho\lambda}_{\nu} + W^{*}_{\mu\rho\kappa} W^{*\rho\lambda}_{\nu} \]  

(3.29)
where \((W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}^{\mu\nu} \varepsilon_{\mu\nu\gamma\delta}\). The spin-2 field equations (C.16) and (C.17) remain invariant under the global \(U(1)\) transformation \(Z^{kl} \rightarrow e^{i\alpha} Z^{kl}\). The duality invariance\(^{12}\) is a property of Bel–Robinson tensor. The q-l density of super-energy fulfills

\[
4\pi u_S = \int_{S(t,r)} \frac{1}{2} T_{BR}^{\mu\nu\kappa\lambda} (\partial_\mu, \partial_\nu, \partial_\kappa, \partial_\lambda, \Psi) r^2 \sin \theta d\theta d\varphi
= \int_{S(t,r)} u_S r^2 \sin \theta d\theta d\varphi
= \frac{1}{4} \int_{S(t,r)} F_3 (\Psi) (3.30)
\]

where \(F_3 (\Psi)\) is given by (3.31). Let us introduce

\[
F_3 (\Psi) := \left( \frac{1}{2} \left[ \frac{1}{2} \left( \frac{(\partial_\mu \partial_\nu \partial_\kappa \partial_\lambda) (r^2 \partial_\mu \Psi) \Delta^{-1} (\Delta + 2)^{-1} \partial_\lambda (r^2 \partial_\lambda \Psi) \right] \right) \right) \sin \theta
\]

\[
F_4 (\Psi) := \left( \frac{1}{2} \left[ \frac{1}{2} \left( \frac{(\partial_\mu \partial_\nu \partial_\kappa \partial_\lambda) (r^2 \partial_\mu \Psi) \Delta^{-1} (\Delta + 2)^{-1} \partial_\lambda (r^2 \partial_\lambda \Psi) \right] \right) \right) \sin \theta
\]

The Bel–Robinson charge for a conformal field is as follows

\[
4\pi U_{CS} = \int_{S(t,r)} \frac{1}{2} T_{BR}^{\mu\nu\kappa\lambda} (K, \partial_\mu, \partial_\nu, \partial_\kappa, \partial_\lambda, \Psi) r^2 \sin \theta d\theta d\varphi
= \frac{1}{4} \int_{S(t,r)} \left[ (r^2 + r^2) F_3 (\Psi) + 2rF_4 (\Psi) \right] d\theta d\varphi (3.34)
\]

3.4. Comparison of the energies for hopfions

In [27], the following super-energy density

\[
u S = \frac{E_{ab} B^{ab} + B_{ab} B^{ab}}{2 (3.35)}
\]

\(^{12}\) Introducing \(W_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + i W_{\alpha\beta\gamma\delta}\), Bel–Robinson tensor (3.29) has the form \(T_{BR}^{\mu\nu\kappa\lambda} := W_{\mu\nu\kappa} \overline{W}_{\rho\sigma} \varepsilon_{\lambda}^{\mu \nu \rho \sigma}\), All components of \(W_{\mu\nu\kappa}\) depends of \(Z_{kl}\) without complex conjugate \(Z_{kl}\). It means that the components of Bel–Robinson tensor are proportional to ‘\(ZZ\)’ which are invariant under \(Z^a \rightarrow e^{i\alpha} Z^a\) transformation.
has been calculated for gravitational type N hopfion. We highlight that type N hopfion overlap (up to a constant) with the solution from our class (3.3) for \( l = 2 \) and for the spherical harmonic with maximal order \( (m = l = 2) \). For such quadrupole solution

\[
\Psi_q := \frac{r^2 Y_{22}}{[r^2 - (t - i)^2]^3}
\]

we analyze q-l (super-)energy densities for linearized gravity (3.21), (3.23), (3.30) and (3.34) which are presented in the previous section. We compare them with the electromagnetic q-l energy densities (3.26) and (3.28) for the corresponding to \( \Psi_q \) (3.36) electromagnetic quadrupole solution

\[
\Phi_q := \frac{r^2 Y_{22}}{[r^2 - (t - i)^2]^3}.
\]

Let us define

\[
\xi(t,r) := \frac{r^4}{((r + t)^2 + 1)^5 ((r - t)^2 + 1)^3} \left[ t^4 + \left( \frac{14}{5} r^2 + 2 \right) r^2 + (r^2 + 1)^2 \right]
\]

(3.38)

\[
\kappa(t,r) := \frac{r^5 t}{((r + t)^2 + 1)^5 ((r - t)^2 + 1)^3} \left[ r^2 + t^2 + 1 \right]
\]

(3.39)

\[
\eta(t,r) := \frac{r^2}{((r + t)^2 + 1)^5 ((r - t)^2 + 1)^3} \left[ r^8 + (12 r^6 + 4) r^6 \right.
\]

\[+ \left( \frac{126}{5} r^4 + 28 r^2 + 6 \right) r^4 + (12 r^6 + 28 r^4 + 20 r^2 + 4) r^2 + (r^2 + 1)^4 \left. \right] \right]
\]

(3.40)

\[
\tau(t,r) := \frac{r^3 (r^2 + t^2 + 1)}{((r + t)^2 + 1)^5 ((r - t)^2 + 1)^3} \left[ t^4 + \left( \frac{22}{5} r^2 + 2 \right) r^2 + (r^2 + 1)^2 \right].
\]

(3.41)

The results for quadrupole hopfion are the following:

\[
U_H(\Psi_q) = \frac{1}{24} \xi(t,r)
\]

(3.42)

\[
U_{eh}(\Psi_q) = \frac{1}{3} \xi(t,r)
\]

(3.43)

\[
U_{EM}(\Phi_q) = \frac{1}{3} \xi(t,r)
\]

(3.44)

\[
U_S(\Psi_q) = \frac{1}{2} \eta(t,r)
\]

(3.45)

\[
U_{CEM}(\Phi_q) = \frac{4}{15} \left[ \frac{5}{4} (t^2 + r^2) \xi(t,r) - 6 r t \kappa(t,r) \right]
\]

(3.46)

\[
U_{CS}(\Psi_q) = \frac{1}{2} \left[ (t^2 + r^2) \eta(t,r) + 2 r t \tau(t,r) \right].
\]

(3.47)
One can observe the following:

1. The q-l energy densities can be divided into two sets:

   \[ X_1 = \{ U_H, U_{\Theta}, U_{EM}; U_S \} \]  
   \[ X_2 = \{ U_{CEM}, U_{CS} \}. \]

   Functions in each set have similar properties. It means:

   (a) In the set \( X_1 \) we can distinguish a subset \( \{ U_H, U_{\Theta}, U_{EM} \} \). Q-l (super-) energy densities in the subset differ by a multiplicative constant. Simple, single-multipole structure of the solutions (3.36) and (3.37) is responsible for proportionality of q-l (super-)energy densities in the subset. For solutions with the richer multipole structure relations between the densities will be more complicated.

   (b) The set \( X_2 \) contains q-l energy densities for the conformal field \( K \) (3.27). For \( t = 0 \), the conformal q-l densities are proportional to theirs counterparts for \( \partial_t \) field. \( r^2 \) is the proportional factor

   \[ U_{CEM}(\Phi_q, t = 0) = r^2 U_{EM}(\Phi_q, t = 0) \]
   \[ U_{CS}(\Psi_q, t = 0) = r^2 U_{EM}(\Psi_q, t = 0). \]

2. All the above presented q-l (super-)energy densities are localized on light cones for large \( t \) and \( r \).

### 3.5. Topological charge

We were not able to find a definition of a topological charge for weak gravitational field in the literature. We propose a quantity which can be a good candidate for a topological charge and investigate its properties. Consider the following non-local objects:

\[ h_{GE} = \int \Sigma E^{ab} (-\Delta)^{-1} S_{ab} = \int \int_{\Sigma \times \Sigma} \frac{E^{ab}(r') S_{ab}(r'')}{4\pi ||r' - r''||} dr' dr'' \]

\[ h_{GB} = \int \Sigma B^{ab} (-\Delta)^{-1} P_{ab} = \int \int_{\Sigma \times \Sigma} \frac{B^{ab}(r') P_{ab}(r'')}{4\pi ||r' - r''||} dr' dr'' \]

where \( \Delta^{-1} \) is an inverse operator to the 3D Laplacian \( \Delta \) (details in appendix A). \( P_{ab} \) and \( S_{ab} \) are respectively ADM momentum and its dual counterpart discussed nearby (C.10) and (C.11). For convenience we work with complex objects\(^\text{13}\)

\[ h_G = h_{GE} - h_{GB} = \Re \int \Sigma Z^{ab} (-\Delta)^{-1} V_{ab} \]

\[ \tilde{h}_G = h_{GE} + h_{GB} = \Re \int \Sigma Z^{ab} (-\Delta)^{-1} \tilde{V}_{ab}. \]

We list a few properties which support our hypothesis that (3.52) and (3.53) play a role of a ‘topological charge’:

\(^\text{13}\) We use \( Z^{ab} = E^{ab} + i B^{ab} \) and \( V^{ab} = S^{ab} + i P^{ab} \). See appendices C.2 and C.3 for more details.
• (3.50) and (3.51) are well-defined and gauge invariant\textsuperscript{14}.

• Similarities with the electromagnetic case:
  – Analogy to the electromagnetic helicity—the quantity (3.52) in terms of complex scalar field (3.54) is similar to (2.32).
  – Analogy to the conservation law—(3.52) is conserved in time if (3.58) is fulfilled. It is analogous to (2.34).

• (3.52) is conserved in time for an example of gravitational type N hopfion described in \cite{27}.

• Structure comparable to other quantities defined for linearized gravity field, for example the energy (3.17).

To highlight the analogy with electromagnetic field we express equation (3.52) in terms of complex scalar field. Using the reduction presented in appendices A.4 and C.3 the result is as follows:

\begin{align*}
  h_G &= -\Re \int_\Sigma i \Psi \Delta^{-1} (\Delta + 2)^{-1} \partial_t \Psi \\
  \tilde{h}_G &= \frac{1}{2} \int_\Sigma \Re \left[ i \left( \Psi \Delta^{-1} (\Delta + 2)^{-1} \partial_t \Psi - \bar{\Psi} \Delta^{-1} (\Delta + 2)^{-1} \partial_t \Psi \right) \right].
\end{align*}

(3.54)

(3.55)

If we compare $\partial_t h_G$ and the real part of $\int_\Sigma Z^{\mu \ell} \Delta^{-1} Z_{\mu \ell}$ in terms of the scalar $\Psi$ then turns out that they are equal up to the factor 2

\[ \partial_t h_G = -2 \Re \int_\Sigma i Z^{\mu \ell} (-\Delta^{-1}) Z_{\mu \ell} = -\Re \int_\Sigma i \partial_t \left( \Psi \Delta^{-1} (\Delta + 2)^{-1} \partial_t \Psi \right). \]

(3.56)

The gravitational helicity $\tilde{h}_G$ is preserved in time for all $\Psi$ which fulfill wave equation
\[ \partial_t \tilde{h}_G = \frac{1}{2} \lim_{\mathcal{R} \to \infty} \int_{\mathcal{B}(0, \mathcal{R})} \Re \left( \Psi \Delta^{-1} (\Delta + 2)^{-1} r^2 \partial_r \nabla - r^2 \partial_r \Psi \Delta^{-1} (\Delta + 2)^{-1} \nabla \right) \]  

(3.57)

where \( \mathcal{B}(0, \mathcal{R}) = \{ x \in \Sigma : ||x|| \leq \mathcal{R} \} \). We assume the linearized gravity fields are localized\(^{15}\). It implies that (3.57) vanishes. Only (3.56) leads to a condition for conservancy of topological charges. The results (3.56) and (3.57) give

Proposition 3.1. For localized fields, the objects \( h_{GE} \) (3.50) and \( h_{GB} \) (3.51) are preserved in time if and only if

\[ 2 \Re \int_\Sigma Z^{kl} (-\Delta)^{-1} Z_{kl} = \Re \int_\Sigma \partial_t \left( \Psi \Delta^{-1} (\Delta + 2)^{-1} \partial_r \Psi \right) = 0 \]  

(3.58)

The above theorem corresponds to proposition 2.1 in electrodynamics.

4. Conclusions

In the paper, the electromagnetic hopfions are described in terms of the complex scalar \( \Phi \) which contains the full information about Maxwell field—two unconstrained degrees of freedom. The scalar \( \Phi \) formalism for electrodynamics is presented in appendix \( B \). We generalize the electromagnetic hopfions by the natural generalization\(^{16}\) of \( \Phi \) to the higher multipole solution (2.12). The physical quantities, like energy or helicity, are expressed in terms of the scalar.

The electromagnetic case can be treated as a ‘toy-model’ for the linearized gravity. Next, the scalar \( \Psi \) description of gravito-electromagnetic formulation of linearized gravity is presented (appendix C.3). In analogy to electromagnetism, we generalize the gravitational hopfion to higher multipole solution (3.3). We propose the notion of helicity for linearized gravity \( h_{GE} \) (3.50) and \( h_{GB} \) (3.50). The properties of gravitational helicities in terms of the scalar \( \Psi \) are similar to electromagnetic ones. We compare gravitational quasi-local densities for quadrupole solution (3.36). The results are presented and discussed in section 3.4.

The structure of the theory for electromagnetism and linearized gravity can be illustrated on the figure 1:

We would like to point out the following:

1. Spin-2 field theory (see appendix \( C.2 \)) starts with linearized Weyl tensor as a primary object and Bianchi identities play a role of evolution equations. Theory of linearized gravity has a richer structure. It contains ‘potentials’ for curvature tensors: metric, momenta and their dual counterparts. See rhs of figure 1. That simple observation has consequences for (non-)locality of densities of energy and helicity.

2. The energy functional for Maxwell theory is constructed from electromagnetic vector fields \( \mathbf{E} \) and \( \mathbf{B} \). The energy density is local at a point in terms of \( \mathbf{E} \) and \( \mathbf{B} \). In the case of linearized gravity the Hamiltonian energy density (see (3.17)) becomes local as a combination of the metric and curvature. However, in terms of spin-2 field, the energy functional (3.17) contains non-local \( Z^{kl} (-\Delta)^{-1} Z_{kl} \) term. More precisely, the object \( (-\Delta)^{-1} Z_{kl} = (\kappa^{-1})^2 Z_{kl} \) is locally related to a combination of metrics \( h_{ab} \) and \( k_{ab} \) (see rhs of figure 1). Another form of the energy functional \( ((\kappa^{-1} Z_{kl})^2 \) —square of momenta)

\(^{15}\)By localized we mean compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms.

\(^{16}\)The scalar \( \Phi \) for hopfions is equal to \( \frac{\partial \Phi}{\partial t} \). The type of hopfion, namely null or non-null (see [27]) is related to the order of the dipole.
contains non-local integral operator $\kappa^{-1}$ which is responsible for the non-locality of energy density described by (3.17).  

3. For helicity of linearized gravity the similar problems occur like for energy. The natural objects for helicity functional to be local are metric and momenta.

The precise description of $\kappa$ operator and the structure of linearized gravity will be presented in a separate paper [16].

Acknowledgments

This work was supported in part by Narodowe Centrum Nauki (Poland) under Grant No. 2016/21/B/ST1/00940.

Appendix A. Mathematical supplement

A.1. Three-dimensional Laplace operator and its inverse

Consider Laplace equation

$$\Delta G(r, r') = -\delta^{(3)}(r - r')$$ (A.1)

with a solution on an open set without boundary. $\delta^{(3)}(r - r')$ is a 3D Dirac delta. $G(r, r')$ is the following Green function of (A.1)

$$G(r, r') = \frac{1}{4\pi ||r - r'||}.$$ (A.2)

The solution of Poisson equation

$$\Delta u(r) = -f(r)$$ (A.3)

is the convolution of $f(r)$ and Green function

$$u(r) = \int_{\Sigma} f(r') G(r, r') dr' = \int_{\Sigma} \frac{f(r')}{4\pi ||r - r'||} dr'.$$ (A.4)

A.2. Two-dimensional Laplace operator and its inverse

Consider 2D unit sphere in $\mathbb{R}^3$, parameterized by a unit position vector $n$. One of the main differences is that the domain of the solutions is the compact surface without boundary. The conclusions of the Stokes theorem ($\int_{S^2} \nabla u(n) = 0$) require a modified problem to be examined than in the 3D case. Consider the following 2D Laplace equation with an additional condition

$$\Delta G(n, n') = 1 - \delta^{(2)}(n - n')$$ (A.5)

$$\int_{S^2} \sigma G(n, n') dn' = 0$$ (A.6)

where $\sigma$ is area element on $S^2$. We have the solution

$$G(n, n') = -\frac{1}{4\pi} \left( \ln \left( \frac{1 - n \cdot n'}{2} \right) + 1 \right)$$ (A.7)
where\(^\cdot\) is a scalar product of the position vectors\(^\text{17}\). The solution of the Poisson equation
\[
\Delta s(n) = -f(n) \quad (A.8)
\]
\[
\int_{S^2} \sigma f(n) d\mathbf{n} = 0 \quad (A.9)
\]
is the convolution of \(f(n')\) and the Green function
\[
u(n) = \int_{S^2} f(n') G(n, n') d\mathbf{n}' = -\int_{S^2} \frac{1}{4\pi} \left( \ln \left( \frac{1 - \mathbf{n} \cdot \mathbf{n}'}{2} \right) + 1 \right) f(n') d\mathbf{n}'. \quad (A.10)
\]
See [8] and [15] for detailed view [26], is a specialized literature on the subject.

A.3. Operations on the sphere

Let us denote by \(\Delta\) the Laplace–Beltrami operator associated with the standard metric \(h_{AB}\) on \(S^2\). Let \(SH^l\) denote the space of spherical harmonics of degree \(l\)
\[(g \in SH^l) \iff \Delta g = -l(l + 1)g.\]
Consider the following sequence
\[V^0 \oplus V^0 \overset{i_{01}}{\rightarrow} V^1 \overset{i_{12}}{\rightarrow} V^2 \overset{i_{21}}{\rightarrow} V^1 \overset{i_{10}}{\rightarrow} V^0 \oplus V^0.\]
Here \(V^0\) is the space of, say, smooth functions on \(S^2\), \(V^1\)—that of smooth covectors on \(S^2\), and \(V^2\)—that of symmetric traceless tensors on \(S^2\). The various mappings above are defined as follows:
\[
i_{01}(f, g) = f|_a + \varepsilon^a_b g|_b
\]
\[
i_{12}(v) = v|_a + v|_b - h_{ab}v|_c
\]
\[
i_{21}(\chi) = \chi^a_b
\]
\[
i_{10}(v) = (\varepsilon^a_b|_a, \varepsilon^a_b v|_b)
\]
where \(|\) is used to denote the covariant derivative with respect to the Levi-Civita connection of the standard metric \(h_{AB}\) on \(S^2\). For more details see appendix E in [15].

A.4. Identities on the sphere

We have used the following identities on a sphere
\[
-\int_{S(r)} \pi^A v_A = \int_{S(r)} (r\pi^A|_A) \Delta^{-1} (r\pi^A|_A) + \int_{S(r)} (r\pi^A|_B \varepsilon_{AB}) \Delta^{-1} (r\varepsilon_{AB}|_B)
\]
and similarly for the traceless tensors we have
\[
\int_{S(r)} \tilde{\pi}^{AB} v_{AB} = 2 \int_{S(r)} (r^2 \varepsilon^{AC} \tilde{\pi}^B_A|_B) \Delta^{-1} (\Delta + 2)^{-1} (r^2 \varepsilon^{AC} \tilde{v}_A^B|_B) + 2 \int_{S(r)} (r^2 \tilde{\pi}^{AB}|_B) \Delta^{-1} (\Delta + 2)^{-1} (r^2 \tilde{v}_{AB}|_B). \quad (A.12)
\]
\(^\text{17}\) For a given point \((\theta, \varphi)\) in spherical coordinates on the unit sphere, the 3D position vector in the Cartesian embedding is \(n = \sin \theta \cos \varphi \partial_\theta + \sin \theta \sin \varphi \partial_\varphi + \cos \theta \partial_z\). Then we use scalar product with Euclidean metric.
Appendix B. Scalar representation of electromagnetic field

Let us consider an electromagnetic field on Minkowski background. We present how to describe electromagnetism in terms of complex scalar function $\Phi$. The section is organized as follows: we start with a description of standard electric $E$ and magnetic $B$ fields with help of complex Riemann–Silberstein vector $Z = E + iB$. Then, we decompose $Z$, in the spherical coordinate system, into radial and angular part. We show that the radial part is sufficient to recover quasi-locally the whole $Z$ vector.

The vacuum Maxwell equations for electric field vector $E$, magnetic field $B$, and vector potential $A$ are

\begin{align}
\text{div} \, E &= 0 \\
\text{div} \, B &= 0 \\
\text{curl} \, E &= -\frac{\partial B}{\partial t} \\
\text{curl} \, B &= \frac{\partial E}{\partial t} \\
B &= \text{curl} \, A.
\end{align}

If electric field $E$ is sourceless then additional vector potential can be introduced\textsuperscript{18}. It is defined up to a gradient of a function in the following way

$$E = \text{curl} \, C.$$  \hspace{1cm} (B.6)

It is convenient to use one complex electromagnetic vector field $Z$, called Riemann–Silberstein vector, instead of $E$ and $B$. $Z$ is defined as follows

$$Z = E + iB.$$  \hspace{1cm} (B.7)

where $i^2 = -1$. For sake of simplicity, we will use complex vector potential $V$ instead of $C$ and $A$:

$$V = C + iA.$$  \hspace{1cm} (B.8)

The vacuum Maxwell equations (B.1)–(B.5) with vector potential $C$ (B.6) can be written in the form of three complex, differential equations for vector fields

\begin{align}
\text{div} \, Z &= 0 \\
\text{curl} \, Z &= i\frac{\partial Z}{\partial t} \\
Z &= \text{curl} \, V.
\end{align}

In the next part of the section, we will use spherical coordinate system. Each vector $w = (w^r, w^A)$ can be decomposed into its radial part $w^r$ and 3D angular part $w^A$. The capital letter index runs angular coordinates.

We split 2D vector into its longitudinal $w^A|_A$ and transversal part $\varepsilon_{A|B}w^A|_B$. The Maxwell equations in terms of the decomposition have the form:

\textsuperscript{18} We remark that the description of electromagnetism with the help of complex scalar function holds also without the additional potential.
\( rZ' = \Phi \) \hspace{1cm} (B.12)
\( r^2Z_{|A} = -\partial_r(r\Phi) \) \hspace{1cm} (B.13)
\( r_{|AB}Z^{[A|B} = -i\partial_t\Phi \) \hspace{1cm} (B.14)
\( \Delta V_r - V_{C,r}^{[C} = -i\partial_t\Phi \) \hspace{1cm} (B.15)
\( r_{|AB}V_{A|B} = -\Phi \) \hspace{1cm} (B.16)

### B.1. Scalar representation of electromagnetic field in curved spacetimes

The description of electromagnetism in terms of complex scalar field can be generalized for type D spacetimes (Petrov classification). For example, in [18] it has been done for Kerr spacetime. The generalization of \( \Phi \) for Kerr, we denote it by \( \Phi_K \), is constructed from conformal Yano–Killing tensor \( Q_{\mu\nu} \), its dual companion \( *Q_{\mu\nu} \) and Maxwell field \( F_{\mu\nu} \). \(*\) denotes the Hodge duality. The contraction
\[
\Phi_K := \frac{1}{2} F^{\mu\nu} [Q_{\mu\nu} - i(*Q_{\mu\nu})]
\]
fulfills extended wave equation
\[
\hat{\Box} \Phi_K + \frac{2m}{(r - \frac{ia}{cos \theta})^3} \Phi_K = 0 \hspace{1cm} (B.17)
\]
which is called Fackerell–Ipser equation. \( \hat{\Box} \) is d’Alembert operator for Kerr metric. \( m \) and \( ma \) are respectively mass and angular momentum of Kerr black hole. \( r \) and \( \theta \) belong to Boyer–Lindquist coordinates. For detailed results and discussion see [18].

### B.2. Simple observation

Let us consider 4D Laplace equation in the 4D Euclidean space
\[
^{(4)} \Delta f(x) = -\delta(x) \hspace{1cm} (B.18)
\]
where \( \delta(r) \) is the Dirac delta. We focus on the following solution of (B.18) given in the Cartesian coordinates:
\[
f(x) = \frac{1}{(x^0)^2 + ||r||^2} \hspace{1cm} (B.19)
\]
where \( ||r|| = r = \sqrt{\sum_{i=1}^{3} x_i^2} \). The solution (B.19) can be extended analytically on Minkowski spacetime by the transformation
\[
x^0 = u + 1. \hspace{1cm} (B.20)
\]
We receive
\[
\tilde{f}(t, r) = \frac{1}{r^2 - (t - i)^2} \hspace{1cm} (B.21)
\]
which fulfills wave equation on Minkowski background.

The imaginary shift has been successfully used many times. For example, it is useful for Bateman construction [1]. New knotted solution can also be obtained in that way [10]. Now, consider $l$th order differential operator $A_l$ which

1. generates a function $F_l(t, r)$ from $\tilde{f}(t, r)$ which is proportional to $l$th spherical mode$^{19}$, 2. commutes with d’Alembert operator.

For $l = 1$ we have the following example

$$A_1 = \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2} \quad (B.22)$$

and

$$0 = A_1 \Box \tilde{f}(t, r) = \Box A_1 \tilde{f}(t, r) = \Box F_1(t, r) \quad (B.23)$$

hence $F_1(t, x)$ fulfills wave equation. The above simple observation enables one to generate a solution similar (up to a constant) to (2.12).

Appendix C. Scalar description of linearized gravity

C.1. Equivalent definitions of spin-2 field

Let us start with the standard formulation of a spin-2 field $W_{\mu\nu\alpha\beta}$ in the Minkowski spacetime equipped with a flat metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. We consider vacuum case. The field $W$ can be also interpreted as a Weyl tensor for linearized gravity (see [7, 12, 15]).

The following algebraic properties:

$$W_{\mu\nu\alpha\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha}[\nu\beta]} , \quad W_{\mu[\alpha\beta]} = 0 , \quad g^{\mu\nu}W_{\mu\nu\alpha\beta} = 0 \quad (C.1)$$

and Bianchi identities which play a role of field equations

$$\nabla^{(4)} W_{\mu\nu\alpha\beta} = 0 \quad (C.2)$$

can be used as a definition of spin-2 field $W$. The $*$-operation defined as

$$(^*W)_{\alpha\beta\gamma\delta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\gamma\delta} , \quad (W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}^{\mu\nu} \varepsilon_{\mu\nu\gamma\delta}$$

has the following properties:

$$(^*W^*)_{\alpha\beta\gamma\delta} = \frac{1}{4} \varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\rho\sigma} \varepsilon_{\rho\sigma\gamma\delta} , \quad *W = W^* , \quad (*W) = *W = -W$$

where $\varepsilon_{\mu\nu\gamma\delta}$ is a Levi-Civita skew-symmetric tensor$^{20}$ and $*W$ is called dual spin-2 field. The above formulae are also valid for general Lorentzian metrics.

$^{19}$ $l$th mode from spherical harmonics decomposition which fulfills $\triangle F_l = -l(l + 1)F_l$.

$^{20}$ Defined in footnote 3.
C.2. Gravito-electric and gravito-magnetic formulation

Following Maartens [21], spin-2 field can be equivalently described in terms of gravito-electric and gravito-magnetic tensors. We perform a $(3 + 1)$-decomposition of the Weyl tensor. The ten independent components of $W$ split into two 3D symmetric, traceless tensors: the electric part

$$E(X, Y) := W(X, \partial_t, \partial_t, Y)$$

and the magnetic part

$$B(X, Y) := {}^*W(X, \partial_t, \partial_t, Y).$$

The following relations between $W$ and the 3D tensors hold:

$$W_{0k0} = E_{kl}, \quad W_{0kij} = B_{kl} \varepsilon^{ij}_{lp} E_{lp}, \quad W_{klmn} = \varepsilon^{i}_{kl} \varepsilon^{j}_{mn} E_{ij}.$$  

The classical formulation of gravito-electromagnetism uses the constraint equations

$$E_{kl} |_l = 0$$

and the dynamical equations

$$\partial_t E_{kl} = \varepsilon^{pq(l} B_{k|q}^{|p}$$

where $[\text{curl } X]_{ab} := \varepsilon_{cd(a} X_{b)} d^{de}$ is the symmetric curl operator for tensors.

C.2.1 ADM momentum $P$ and the dual counterpart $S$ as "potentials" for Weyl tensor. Analogically to electromagnetic case we introduce potentials for Weyl tensor in gravito-electromagnetic formulation. The potential for gravito-magnetic part is the ADM momentum $P$. It fulfills

$$B_{ab} = \varepsilon_{cd(a} P_{b)} d^{de}.$$  

The second potential can be introduced for gravito-electrical part

$$E_{ab} = \varepsilon_{cd(a} S_{b)} d^{de}.$$  

The potentials fulfill constraint equations

$$P_{kl} |_l = 0$$

and its potentials $P_{kl}$ and $S_{kl}$

$$V_{kl} = S_{kl} + i P_{kl}.$$  

The equations (C.6)–(C.13) in terms of complex objects are

$$Z_{kl} |_l = 0.$$  

$$(C.16)$$
\[ \dot{Z}^{kl} = -\epsilon^{pq(k}Z_{q|l} \]  
(C.17)

\[ Z_{ab} = \varepsilon_{cd(a}V_{b)}d^{c} \]  
(C.18)

\[ V^{kl}_{|l} = 0. \]  
(C.19)

C.3. Scalar representation of spin-2 field

Spin-2 field can be represented as a complex, scalar function defined analogically to the electromagnetic case\(^{21}\).

In the spherical coordinates it has the form

\[ \Psi = 2Z_{kl}x^{k}x^{l} = 2Z_{rr}r^{2}. \]  
(C.20)

A counterpart of gravito-electromagnetic equations (C.6)–(C.9) for \( \Psi \) is

\[ \Box \Psi = 0 \]  
(C.21)

where \( \Box \) is a d’Alembert operator for Minkowski background. The recovery procedure of the \( Z_{kl} \) field from \( \Psi \) uses the constraint equations for linearized Weyl tensor and the dynamical equations. The \((2 + 1)\)-splitting of the constraint (C.16):

\[ \partial_{r}(r^{3}Z^{rr}) + r^{3}Z^{rA}_{|A} = 0 \]  
(C.22)

\[ \partial_{r}(r^{4}Z^{rA}_{|A}) + r^{4}Z^{AB}_{|AB} - \frac{1}{2}r^{2}\Delta Z^{rr} = 0 \]  
(C.23)

\[ \partial_{r}(r^{4}Z^{rA}_{|A}\varepsilon^{AB}) + r^{4}Z^{AB}_{|AB}\varepsilon^{AC} = 0 \]  
(C.24)

and the \((2 + 1)\)-decomposition of the dynamical equation (C.17):

\[ \partial_{r}Z^{rr} = -r^{2}\varepsilon^{AB}Z_{rA||B} \]  
(C.25)

\[ \partial_{r}(r^{2}\partial_{t}Z^{rr}) = -ir^{4}Z^{AB}_{|AB}\varepsilon^{AC} \]  
(C.26)

enables one to express explicitly all electromagnetic components of the Weyl tensor in terms of \( \Psi \) and \( \partial_{t}\Psi \):

\[ r^{2}Z^{rr} = \frac{1}{2}\Psi \]  
(C.27)

\[ r^{2}Z_{rA||B}\varepsilon^{AB} = -\frac{1}{2}i\partial_{t}\Psi \]  
(C.28)

\[ r^{3}Z^{rA}_{|A} = -\frac{1}{2}\partial_{t}(r\Psi) \]  
(C.29)

\[ r^{3}(c)Z = -\frac{1}{2}\Psi \]  
(C.30)

\(^{21}\) See the equation (2.1) and the comments below.
\[ r^4 \overset{\circ}{Z}^{AB} \bigg|_{AB} = \frac{1}{2} \partial_r (r \partial_r (r \Psi)) + \frac{1}{4} \Delta \Psi \tag{C.31} \]

\[ r^3 \overset{\circ}{Z}^B_{\ B|C} \overset{\circ}{Z}^{AC} = \frac{1}{2} \partial_\gamma (r^2 \partial_\gamma \Psi) \tag{C.32} \]

where \( Z = g_{AB} Z^{AB} \) and \( \overset{\circ}{Z}^{AB} = Z^{AB} - \frac{1}{2} g_{AB} \overset{\circ}{Z} \). The scalar is related to a gauge-independent part of the potential \( V_{ab} \). The \( (2 + 1) \)—splitting of (C.18), (C.19) and use of (C.27)–(C.32) gives

\[ \Delta (\Delta + 2) V' = - \left( 2i \partial_r \Psi + 2i (r \Pi)^r + 2 (\Delta + 2) \Pi \right) \tag{C.33} \]

\[ (\Delta + 2) V'^{A|A} = \frac{i \partial_\gamma \Psi + (r \Pi)^r}{r} \tag{C.34} \]

\[ 2r^2 V'^{A|B} \varepsilon_{rAB} = - \Psi \tag{C.35} \]

\[ r^3 (\overset{\circ}{V}^{(2)} V) = \Psi \tag{C.36} \]

\[ 2r^2 \overset{\circ}{V}^{AB} \bigg|_{AB} = -i (\partial_r \Psi - \Pi) \tag{C.37} \]

\[ 2r^4 \overset{\circ}{V}^{C}_{|AB} \varepsilon_{CB} = (r^2 \Psi)^r \tag{C.38} \]

where \( \Pi = 2r V'^{A|A} + \Delta V' \) is a gauge dependent part.

The presented formulation of linearized gravity also holds in the case of sources. Described decomposition of linearized Einstein equation can be repeated with non-vanishing stress–energy tensor. Monopole and dipole part of reduced data \( \Psi \) are related to stationary fields in that case. See [13] for details.

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