A THEORETICAL ANALYSIS OF LOGISTIC REGRESSION AND BAYESIAN CLASSIFIERS

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ABSTRACT

This study aims to show the fundamental difference between logistic regression and Bayesian classifiers in the case of exponential and unexponential families of distributions, yielding the following findings. First, the logistic regression is a less general representation of a Bayesian classifier. Second, one should suppose distributions of classes for the correct specification of logistic regression equations. Third, in specific cases, there is no difference between predicted probabilities from correctly specified generative Bayesian classifier and discriminative logistic regression.

JEL Classification: C13, C18, C25, C35.

Keywords: Bayes’ theorem, distribution families, logistic regression, method of moments.

1 Introduction

The comparison of generative and discriminative models has been a perennial topic of discussion. An example is a comparison of naive Bayes classifier and logistic regression in terms of a classification quality or an error (Ng and Jordan, 2002; L. Mary Gladence and Anu, 2015; Tsangaratos and Ilia, 2016; Prabhat and Khullar, 2017; Pranckevicius and Marcinkevičius, 2017; Aborisade and Anwar, 2018; Helmi Setyawan et al., 2018; Hasanli and Rustamov, 2019; Seka et al., 2019; Itoo et al., 2020). However, what has been unreasonably ignored in this debate is the fundamental relationship between the models, when both classes have a multivariate normal distribution with similar covariance matrices (Efron, 1975; Rubinstein and Hastie, 1997).

Efron (1975) stated that the conditional likelihood for logistic regression ‘… is valid under general exponential family assumptions …’ However, this study shows that it is correct only under the linearity by the features assumption of the discriminant function.

The novelty of this study is that it offers:

• a generalization for the multiclass task;
• a solution without a likelihood maximization;
• a covariance dissimilarity for classes with multivariate normal distribution;
• an analysis in terms of families of distributions, not prediction accuracy.

2 Problem statement

2.1 General case

Suppose we have a sample \( \{y_i, x_i\}_{i=1}^N \) where \( y_i \in \{1, 2, \ldots, M\} \) denotes the correct class label, and \( x_i \) denotes the feature vector of size \( d \) for observation \( i \). There are two well-known approaches to estimate parameters with
According to the given specification, there are \( M \) normally distributed classes, with only one feature \((x_i)\) is 1-dimensional). Then, for each class \( s \) a number of equations (5) are valid \((\text{Nielsen and Garcia}, 2011)\). But there is no guarantee that the discriminant function \( z_{m,s}(x_i) = \ln \rho_{m,s}^{\text{cond}}(x_i) + \ln \rho_{m,s}^{\text{uncond}} \) can be represented as \( \beta_0 + \beta^T x_i \).

### 3 Particular solutions

#### 3.1 Examples for exponential families

##### 3.1.1 Univariate normal distribution

According to the given specification, there are \( M \) normally distributed classes, with only one feature \((x_i)\) is 1-dimensional). Then, for each class \( s \) a number of equations (5) are valid \((\text{Nielsen and Garcia}, 2011)\).
\[ \theta_s = (\theta_{s,1}, \theta_{s,2}) = \left( \frac{\mu_s}{\sigma_s^2}, -\frac{1}{2\sigma_s^2} \right) \]

\[ t_s(x_i) = (x_i, x_i^2) \]

\[ F_s(\theta_s) = -\frac{\theta_{s,1}^2}{4\theta_{s,2}} + \frac{1}{2} \ln \left( -\frac{\pi}{\theta_{s,2}} \right) = \]

\[ = \frac{1}{2} \left( \frac{\mu_s^2}{\sigma_s^2} \right) + \frac{1}{2} \ln (2\pi) + \frac{1}{2} \ln \left( \sigma_s^2 \right) \]

\[ k_s(x_i) = 0 \]

The discriminant function for a couple and will take this form (6).

\[ z_{m,s}(x_i) = \alpha_{m,s} + \beta_{m,s} \times x_i + \gamma_{m,s} \times x_i^2 \]

\[ \alpha_{m,s} = \ln \left( \frac{n_m}{n_s} \right) - \frac{1}{2} \ln \left( \frac{\sigma_m^2}{\sigma_s^2} \right) + \left( \frac{\mu_m^2}{\sigma_m^2} - \frac{\mu_s^2}{\sigma_s^2} \right) \]

\[ \beta_{m,s,j} = \frac{1}{2} \sum_{h=1}^{d} \left( \omega_{m,j,h} \mu_m - \omega_{s,j,h} \mu_s \right) \]

\[ \gamma_{m,s,j,h} = -\frac{1}{2} \left( \omega_{m,j,h} - \omega_{s,j,h} \right) \]

The solution for the specific case leads to the following findings. First, the discriminant function is not linear by feature, in general. Second, if feature variances are the same for both classes, then the logistic regression equations are linear.

### 3.1.2 Multivariate normal distribution

This section presents the solution for a set of multivariate normally distributed classes. It describes the generalization of the previous subsection when \( x_i \) is \( d \)-dimensional. According to the current task, equations (7) are valid.

\[ \Theta_s = (\theta_s, \Theta_s) = \left( \frac{1}{2} \Sigma_s^{-1}, \mu_s, \sigma_s^{-1} \right) \]

\[ t_s(x_i) = (x_i, -x_i^T) \]

\[ F_s(\Theta_s) = \frac{1}{4} tr \left( \Theta_s^{-1} \theta_s \Theta_s^{-1} \theta_s^T \right) - \frac{1}{2} \ln (\det \Theta_s) + \frac{d}{2} \ln (\pi) \]

\[ = \frac{1}{2} \mu_m^T \Sigma_m^{-1} \mu_m + \frac{1}{2} \ln (\det \Sigma_m) + \frac{d}{2} \ln (2\pi) \]

\[ k_s(x_i) = 0 \]

For simplicity, let \( \Omega_s = \Sigma_s^{-1} \) for any class \( s \) and \( \omega_{s,j,h} \) be the corresponding element of the matrix \( \Omega_s \). Then the discriminant function can be represented as (8).

\[ z_{m,s}(x_i) = \alpha_{m,s} + \sum_{j=1}^{d} \beta_{m,s,j} x_{i,j} + \sum_{j=1}^{d} \sum_{h=1}^{d} \gamma_{m,s,j,h} x_{i,j} x_{i,h} \]

\[ \alpha_{m,s} = \ln \left( \frac{n_m}{n_s} \right) - \frac{1}{2} \ln \left( \frac{\det \Sigma_m}{\det \Sigma_s} \right) + \left( \mu_m^T \Sigma_m^{-1} \mu_m - \mu_s^T \Sigma_s^{-1} \mu_s \right) \]

\[ \beta_{m,s,j} = \sum_{h=1}^{d} \left( \omega_{m,j,h} \mu_m - \omega_{s,j,h} \mu_s \right) \]

\[ \gamma_{m,s,j,h} = -\frac{1}{2} \left( \omega_{m,j,h} - \omega_{s,j,h} \right) \]
The analytical solution in this section yields two additional findings. First, if features are not independent (at least for one class), then feature interactions (including squares) should be added to the regression equations. Second, if the features are independent and their variance is the same (for all classes), then the regression equations are linear.

3.2 Example for unexponential families

3.2.1 Univariate uniform distribution

Consider two classes: \( x_i \mid y_i = 1 \sim U[a, c] \) and \( x_i \mid y_i = 2 \sim U[b, d] \), where \( a < b < c < d \). Then according to the Bayesian classifier, the estimated probability is that the observation i drawn from the class m equals (9).

\[
P(y_i = 1 \mid x_i) = \begin{cases} 
0, & \text{if } x_i < a \\
1, & \text{if } a \leq x_i < b \\
\frac{1}{1+\frac{P(x_i | y_i = 2) \times P(y_i = 2)}{P(x_i | y_i = 1) \times P(y_i = 1)}} & \text{if } b \leq x_i \leq c \\
0, & \text{if } x_i > c 
\end{cases}
\] (9)

Obviously, the task cannot be solved by logistic regression with a simple linear equation, especially for probability 1. As a result, standard logistic regression does not work on an unexponential family, unlike a Bayesian classifier.

4 Conclusion

As models, Bayesian classifiers are more general than simple logistic regression. They are appropriate for both exponential and unexponential distribution families (in the cases considered). Under Bayesian classifiers, there always exists a distribution assumption and there can be an additional restriction on parameters, but they should not be naive by default. The main assumptions of classical logistic regression are: a) exponential distribution family of classes, and b) linearity by features of the discriminant function. Both assumptions do not always hold, but if the first is appropriate and the regression equations are correctly specified, then logistic regression is a useful representation (especially in econometrics) and leads to a correct solution. Moreover, learning Bayesian classifier is more computationally efficient because moment’s method can be used. For logistic regression, only variations of gradient ascent are available.

Conflict of Interest

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