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Nicholas Proudfoot

Abstract
We establish a formalism for working with incidence algebras of posets with symmetries, and we develop equivariant Kazhdan–Lusztig–Stanley theory within this formalism. This gives a new way of thinking about the equivariant Kazhdan–Lusztig polynomial and equivariant $Z$-polynomial of a matroid.

1. Introduction
The incidence algebra of a locally finite poset was first introduced by Rota, and has proved to be a natural formalism for studying such notions as Möbius inversion [11], generating functions [4], and Kazhdan–Lusztig–Stanley polynomials [12, Section 6].

A special class of Kazhdan–Lusztig–Stanley polynomials that have received a lot of attention recently is that of Kazhdan–Lusztig polynomials of matroids, where the relevant poset is the lattice of flats [5, 9]. If a finite group $W$ acts on a matroid $M$ (and therefore on the lattice of flats), one can define the $W$-equivariant Kazhdan–Lusztig polynomial of $M$ [7]. This is a polynomial whose coefficients are virtual representations of $W$, and has the property that taking dimensions recovers the ordinary Kazhdan–Lusztig polynomial of $M$. In the case of the uniform matroid of rank $d$ on $n$ elements, it is actually much easier to describe the $S_n$-equivariant Kazhdan–Lusztig polynomial, which admits a nice description in terms of partitions of $n$, than it is to describe the non-equivariant Kazhdan–Lusztig polynomial [7, Theorem 3.1].

While the definition of Kazhdan–Lusztig–Stanley polynomials is greatly clarified by the language of incidence algebras, the definition of the equivariant Kazhdan–Lusztig polynomial of a matroid is completely ad hoc and not nearly as elegant. The purpose of this note is to define the equivariant incidence algebra of a poset with a finite group of symmetries, and to show that the basic constructions of Kazhdan–Lusztig–Stanley theory make sense in this more general setting. In the case of a matroid, we show that this approach recovers the same equivariant Kazhdan–Lusztig polynomials that were defined in [7].

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2. THE EQUIVARIANT INCIDENCE ALGEBRA

Fix once and for all a field $k$. Let $P$ be a locally finite poset equipped with the action of a finite group $W$. We consider the category $\mathcal{C}^W(P)$ whose objects consist of

- a $k$-vector space $V$
- a direct product decomposition $V = \prod_{x \leq y \in P} V_{xy}$, each $V_{xy}$ finite dimensional
- an action of $W$ on $V$ compatible with the decomposition.

More concretely, for any $\sigma \in W$ and any $x \leq y \in P$, we have a linear map

$$\varphi^\sigma_{xy} : V_{xy} \to V_{\sigma(x)\sigma(y)},$$

and we require that $\varphi^\sigma_{xy} = \text{id}_{V_{xy}}$ and that $\varphi^\sigma_{\sigma(x)\sigma(y)} \circ \varphi^\sigma_{xy} = \varphi^\sigma_{xz} \circ \varphi^\sigma_{yz}$. Morphisms in $\mathcal{C}^W(P)$ are defined to be linear maps that are compatible with both the decomposition and the action. This category admits a monoidal structure, with tensor product given by

$$(U \otimes V)_{xz} := \bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz}.$$

Let $I^W(P)$ be the Grothendieck ring of $\mathcal{C}^W(P)$; we call $I^W(P)$ the equivariant incidence algebra of $P$ with respect to the action of $W$.

**Example 2.1.** If $W$ is the trivial group, then $I^W(P)$ is isomorphic to the usual incidence algebra of $P$ with coefficients in $\mathbb{Z}$. That is, it is isomorphic as an abelian group to a direct product of copies of $\mathbb{Z}$, one for each interval in $P$, and multiplication is given by convolution.

**Remark 2.2.** If $W$ acts on $P$ and $\psi : W' \to W$ is a group homomorphism, then $\psi$ induces a functor $F_\psi : \mathcal{C}^W(P) \to \mathcal{C}^{W'}(P)$ and a homomorphism $R_\psi : I^W(P) \to I^{W'}(P)$.

We now give a second, more down to earth description of $I^W(P)$. Let $\text{VRep}(W)$ denote the ring of finite dimensional virtual representations of $W$ over the field $k$. A group homomorphism $\psi : W' \to W$ induces a ring homomorphism

$$\Lambda_\psi : \text{VRep}(W) \to \text{VRep}(W').$$

For any $x \in P$, let $W_x \subset W$ be the stabilizer of $x$. We also define $W_{xy} := W_x \cap W_y$ and $W_{xyz} := W_x \cap W_y \cap W_z$. Note that, for any $x, y \in P$ and $\sigma \in W$, conjugation by $\sigma$ gives a group isomorphism

$$\psi^\sigma_{xy} : W_{xy} \to W_{\sigma(x)\sigma(y)},$$

which induces a ring isomorphism

$$\Lambda_{\psi^\sigma_{xy}} : \text{VRep}(W_{\sigma(x)\sigma(y)}) \to \text{VRep}(W_{xy}).$$

An element $f \in I^W(P)$ is uniquely determined by a collection

$$\{ f_{xy} \mid x \leq y \in P \},$$

where $f_{xy} \in \text{VRep}(W_{xy})$ and for any $\sigma \in W$ and $x \leq y \in P$, $f_{xy} = \Lambda_{\psi^\sigma_{xy}}(f_{\sigma(x)\sigma(y)})$.

The unit $\delta \in I^W(P)$ is characterized by the property that $\delta_{xx}$ is the 1-dimensional trivial representation of $W_x$ for all $x \in P$ and $\delta_{xy} = 0$ for all $x < y \in P$. The following proposition describes the product structure on $I^W(P)$ in this representation.

**Proposition 2.3.** For any $f, g \in I^W(P)$,

$$(fg)_{xz} = \sum_{x \leq y \leq z} \frac{|W_{xyz}|}{|W_{xy}|} \text{Ind}_{W_{xy}}^{W_{xyz}}\left( (\text{Res}_{W_{xyz}}^{W_{xy}} f_{xy}) \otimes (\text{Res}_{W_{xyz}}^{W_{xz}} g_{yz}) \right).$$
Proof. By Proposition 2.3, an element inverses are unique and they coincide. We could in fact replace the sum over $[x, z]$ with a sum over one representative of each $W_{xz}$-orbit in $[x, z]$ and then eliminate the factor of $\frac{\vert W_{xz} \vert}{\vert W_{xy} \vert}$. Including the fraction in the equation allows us to avoid choosing such representatives.

Remark 2.4. It may be surprising to see the fraction $\frac{\vert W_{xz} \vert}{\vert W_{xy} \vert}$ in the statement of Proposition 2.3, since $\text{VRep}(W_{xy})$ is not a vector space over the rational numbers. If we could in fact replace the sum over $[x, z]$ with a sum over one representative of each $W_{xz}$-orbit in $[x, z]$ and then eliminate the factor of $\frac{\vert W_{xz} \vert}{\vert W_{xy} \vert}$. Including the fraction in the equation allows us to avoid choosing such representatives.

Remark 2.5. Proposition 2.3 could be taken as the definition of $I^W(P)$. It is not so easy to prove associativity directly from this definition, though it can be done with the help of Mackey’s restriction formula (see for example [3, Corollary 32.2]).

Remark 2.6. Suppose that $\psi : W^r \to W$ is a group homomorphism, and for any $x, y, z \in P$, consider the induced group homomorphism $\psi_{xy} : W_{xy} \to W_{xy}$. For any $f \in I^W(P)$, we have, $R_{\psi}(f) = \Lambda_{\psi_{xy}}(f_{xy})$. In particular, if $W^r$ is the trivial group, then $R_{\psi}(f)_{xy}$ is equal to the dimension of the virtual representation $f_{xy} \in \text{VRep}(W_{xy})$.

Before proving Proposition 2.3, we state the following standard lemma in representation theory.

Lemma 2.7. Suppose that $E = \bigoplus_{x \in S} E_x$ is a vector space that decomposes as a direct sum of pieces indexed by a finite set $S$. Suppose that $G$ acts linearly on $E$ and acts by permutations on $S$ such that, for all $s \in S$ and $\gamma \in G$, $\gamma \cdot E_s = E_{\gamma \cdot s}$. For each $x \in S$, let $G_x \subset G$ denote the stabilizer of $s$. Then there exists an isomorphism

\[ E \cong \bigoplus_{x \in S} \frac{\vert G_s \vert}{\vert G \vert} \text{Ind}_{G_x}^G (E_s) \]

of representations of $G$.[1]

Proof of Proposition 2.3. By linearity, it is sufficient to prove the proposition in the case where we have objects $U$ and $V$ of $c^W(P)$ with $f = [U]$ and $g = [V]$. This means that, for all $x \leq y \leq z \in P$, $f_{xy} = [U_{xy}] \in \text{VRep}(W_{xy})$, $g_{yz} = [V_{yz}] \in \text{VRep}(W_{yz})$, and

\[(fg)_{xz} = [(U \otimes V)_{xz}] = \left[ \bigoplus_{x \leq y \leq z} U_{xy} \otimes V_{yz} \right] \in \text{VRep}(W_{xz}).\]

The proposition then follows from Lemma 2.7 by taking $E = (U \otimes V)_{xz}$, $S = [x, z]$, and $G = W_{xz}$.

Let $R$ be a commutative ring. Given an element $f \in I^W(P) \otimes R$ and a pair of elements $x \leq y \leq z \in P$, we will write $f_{xy}$ to denote the corresponding element of $\text{VRep}(W_{xy}) \otimes R$.

Proposition 2.8. An element $f \in I^W(P) \otimes R$ is (left or right) invertible if and only if $f_{xz} \in \text{VRep}(W_x) \otimes R$ is invertible for all $x \in P$. In this case, the left and right inverses are unique and they coincide.

Proof. By Proposition 2.3, an element $g$ is a right inverse to $f$ if and only if $g_{xz} = f_{xz}^{-1}$ for all $x \in P$ and

\[
\sum_{x \leq y \leq z} \frac{\vert W_{xy} \vert}{\vert W_{xz} \vert} \text{Ind}_{W_{xz}}^{W_{xy}} \left( (\text{Res}_{W_{xy}}^{W_{yz}} f_{xy}) \otimes (\text{Res}_{W_{xy}}^{W_{yz}} g_{yz}) \right) = 0
\]

[1] As in Remark 2.4, we may eliminate the fraction at the cost of choosing one representative of each $W$-orbit in $S$.
Following the notation of [9, Section 2.1], we define \( f \) as a polynomial whose coefficients are virtual representations of \( f \) along with \( W \) as a graded virtual representation of \( f \). If \( g \) is left inverse to \( f \), then \( g \) is also left inverse to some function, which we will denote \( h \). We then have

\[
f = f \delta = f(gh) = (fg)h = \delta h = h,
\]

so \( g \) is left inverse to \( f \), as well. \( \square \)

### 3. Equivariant Kazhdan–Lusztig–Stanley theory

In this section we take \( R \) to be the ring \( \mathbb{Z}[t] \) and for each \( f \in I^W(P) \otimes \mathbb{Z}[t] \) and \( x \leq y \in P \), we write \( f_{xy}(t) \) for the corresponding component of \( f \). One can regard \( f_{xy}(t) \) as a polynomial whose coefficients are virtual representations of \( W_{xy} \), or equivalently as a graded virtual representation of \( W_{xy} \). We assume that \( P \) is equipped with a \( W \)-invariant weak rank function in the sense of [2, Section 2]. This is a collection of natural numbers \( \{r_{xy} \in \mathbb{N} \mid x \leq y \in P \} \) with the following properties:

1. \( r_{xy} > 0 \) if \( x < y \)
2. \( r_{xy} + r_{yz} = r_{xz} \) if \( x \leq y \leq z \)
3. \( r_{xy} = r_{\sigma(x) \sigma(y)} \) if \( x \leq y \) and \( \sigma \in W \).

Following the notation of [9, Section 2.1], we define

\[
\mathcal{F}^W(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \deg f_{xy}(t) \leq r_{xy} \text{ for all } x \leq y \right\}
\]

along with

\[
\mathcal{F}_{W/2}^W(P) := \left\{ f \in I^W(P) \otimes \mathbb{Z}[t] \mid \deg f_{xy}(t) < r_{xy}/2 \text{ and } f_{xx}(t) = \delta_{xx}(t) \right\}
\]

Note that \( \mathcal{F}^W(P) \) is a subalgebra of \( I^W(P) \), and we define an involution \( f \mapsto \overline{f} \) of \( \mathcal{F}^W(P) \) by putting \( \overline{f}_{xy}(t) := t^{r_{xy}} f_{xy}(t^{-1}) \). An element \( \kappa \in \mathcal{F}^W(P) \) is called a \( P \)-kernel if \( \kappa_{xx}(t) = \delta_{xx}(t) \) for all \( x \in P \) and \( \overline{\kappa} = \kappa^{-1} \).

**Theorem 3.1.** If \( \kappa \in \mathcal{F}^W(P) \) is a \( P \)-kernel, there exists a unique pair of functions \( f, g \in \mathcal{F}_{W/2}^W(P) \) such that \( \overline{f} = \kappa f \) and \( \overline{g} = g \kappa \).

**Proof.** We follow the proof in [9, Theorem 2.2]. We will prove existence and uniqueness of \( f \); the proof for \( g \) is identical. Fix elements \( x \leq w \in P \). Suppose that \( f_{yw}(t) \) has been defined for all \( x < y \leq w \) and that the equation \( \overline{f} = \kappa f \) holds where defined. Let

\[
Q_{xw}(t) := \sum_{x \leq y < w} \left| W_{xyw} \right| \mathrm{Ind}_{W_{xyw}}^{W_{xyw}} \left( \left( \operatorname{Res}_{W_{xyw}} \kappa_{xy}(t) \right) \otimes \left( \operatorname{Res}_{W_{xyw}} f_{yw}(t) \right) \right),
\]

which is an element of \( \operatorname{VRep}(W_{xw}) \otimes \mathbb{Z}[t] \). The equation \( \overline{f} = \kappa f \) for the interval \([x, w]\) translates to

\[
\overline{f}_{xw}(t) = f_{xw}(t) = Q_{xw}(t).
\]

\(^{(2)}\)If the ring \( R \) has integer torsion, then we rewrite this equation without the fractions as described in Remark 2.4.
It is clear that there is at most one polynomial $f_{xw}(t)$ of degree strictly less than $r_{xw}/2$ satisfying this equation. The existence of such a polynomial is equivalent to the statement

$$t^{r_{xw}}Q_{xw}(t^{-1}) = -Q_{xw}(t).$$

To prove this, we observe that

$$t^{r_{xw}}Q_{xw}(t^{-1}) = t^{r_{xw}} \sum_{x<y \leq w} \frac{|W_{xyz}|}{|W_{xw}|} \text{Ind}_{W_{xyz}}^{W_{xw}} \left( (\text{Res}_{W_{xyz}} W_{xy} \kappa_{xy}(t^{-1})) \otimes (\text{Res}_{W_{xyz}} W_{wy} f_{wy}(t^{-1})) \right)$$

$$= \sum_{x<y \leq w} \frac{|W_{xyz}|}{|W_{xw}|} \text{Ind}_{W_{xyz}}^{W_{xw}} \left( (\text{Res}_{W_{xyz}} W_{xy} t^{r_{xy}} \kappa_{xy}(t^{-1})) \otimes (\text{Res}_{W_{xyz}} W_{wy} t^{r_{wy}} f_{wy}(t^{-1})) \right)$$

$$= \sum_{x<y \leq w} \frac{|W_{xyz}|}{|W_{xw}|} \text{Ind}_{W_{xyz}}^{W_{xw}} \left( (\text{Res}_{W_{xyz}} W_{xy} \pi_{xy}(t)) \otimes (\text{Res}_{W_{xyz}} W_{wy} \pi_{wy}(t)) \right)$$

$$= \sum_{x<y \leq w} \frac{|W_{xyz}|}{|W_{xw}|} \text{Ind}_{W_{xyz}}^{W_{xw}} \left( (\text{Res}_{W_{xyz}} W_{xy} \pi_{xy}(t)) \otimes (\text{Res}_{W_{xyz}} W_{wy} (\kappa f)(t)) \right).$$

This is formally equal to the expression for $(\pi(\kappa f))_{xw} - (\kappa f)_{xw}$, which by associativity is equal to the expression for

$$(\pi \kappa f)_{xw} - (\kappa f)_{xw} = f_{xw} - (\kappa f)_{xw}.$$ 

Thus we have

$$t^{r_{xw}}Q_{xw}(t^{-1}) = - \sum_{x<y \leq w} \frac{|W_{xyz}|}{|W_{xw}|} \text{Ind}_{W_{xyz}}^{W_{xw}} \left( (\text{Res}_{W_{xyz}} W_{xy} \pi_{xy}(t)) \otimes (\text{Res}_{W_{xyz}} W_{wy} f_{wy}(t)) \right)$$

$$= -Q_{xw}(t).$$

Thus there is a unique choice of polynomial $f_{xw}(t)$ consistent with the equation $\mathcal{F} = \kappa f$ on the interval $[x, w]$.

We will refer to the element $f \in \mathcal{F}^W(P)$ from Theorem 3.1 is the right equivariant KLS-function associated with $\kappa$, and to $g$ as the left equivariant KLS-function associated with $\kappa$. For any $x \leq y$, we will refer to the graded virtual representations $f_{xy}(t)$ and $g_{xy}(t)$ as (right or left) equivariant KLS-polynomials. When $W$ is the trivial group, these definitions specialize to the ones in [9, Section 2].

**Example 3.2.** Let $\zeta \in \mathcal{F}^{W}(P)$ be the element defined by letting $\zeta_{xy}(t)$ be the trivial representation of $W_{xy}$ in degree zero for all $x \leq y$, and let $\chi := \zeta^{-1} \zeta$. The function $\chi$ is called the equivariant characteristic function of $P$ with respect to the action of $W$. We have $\chi^{-1} = \zeta^{-1} \zeta = \chi$, so $\chi$ is a $P$-kernel. Since $\zeta = \zeta \chi$, $\zeta$ is equal to the left KLS-function associated with $\chi$. However, the right KLS-function $f$ associated with $\chi$ is much more interesting! See Propositions 4.1 and 4.3 for a special case of this construction.

We next introduce the equivariant analogue of the material in [9, Section 2.3]. If $\kappa$ is a $P$-kernel with right and left KLS-functions $f$ and $g$, we define $Z := g \kappa f \in \mathcal{F}^W(P)$, which we call the equivariant $Z$-function associated with $\kappa$. For any $x \leq y$, we will refer to the graded virtual representation $Z_{xy}(t)$ as an equivariant $Z$-polynomial.

**Proposition 3.3.** We have $Z = Z$.

**Proof.** Since $g = g \kappa$, we have $Z = g \kappa f = g f$. Since $\mathcal{F} = \kappa f$, we have $Z = g \kappa f = g \mathcal{F}$. Thus $Z = g \mathcal{F} = g \mathcal{F} = g \mathcal{F} = Z$. \qed
Suppose that \( \kappa \in I^W(P) \) is a \( P \)-kernel and \( f, g, Z \in I^W(P) \) are the associated equivariant KLS-functions and equivariant Z-function. It is immediate from the definitions that, if \( \psi : W' \to W \) is a group homomorphism, then \( R_\psi(f), R_\psi(g), R_\psi(Z) \in I^W(P) \) are the equivariant KLS-functions and equivariant Z-function associated with the \( P \)-kernel \( R_\psi(\kappa) \in I^W(P) \). In particular, if we take \( W' \) to be the trivial group, then Remark 2.6 tells us that the ordinary KLS-polynomials and \( Z \)-polynomials are recovered from the equivariant KLS-polynomials and \( Z \)-polynomials by sending virtual representations to their dimensions.

4. Matroids

Let \( M \) be a matroid, let \( L \) be the lattice of flats of \( M \) equipped with the usual weak rank function, and let \( W \) be a finite group acting on \( L \). Let \( OS^W_M(t) \) be the Orlik–Solomon algebra of \( M \) \cite{Nicholas Proudfoot 2005}, regarded as a graded representation of \( W \). Following \cite[Section 2]{Nicholas Proudfoot 2005}, we define

\[
H^W_M(t) := t^{rk_M} OS^W_M(-t^{-1}) \in \text{VRep}(W) \otimes \mathbb{Z}[t].
\]

If \( W \) is trivial, then \( H^W_M(t) \in \mathbb{Z}[t] \) is equal to the characteristic polynomial of \( M \). For any \( F \leq G \in L \), let \( M_{FG} \) be the minor of \( M \) with lattice of flats \( \{F, G\} \) obtained by deleting the complement of \( G \) and contracting \( F \); this matroid inherits an action of the stabilizer group \( W_{FG} \subset W \). Define \( H \in \mathcal{A}^W(L) \) by putting \( H_{FG}(t) = H^W_{M_{FG}}(t) \) for all \( F \leq G \).

Proposition 4.1. The function \( H \) is the equivariant characteristic function of \( L \).

Proof. It is proved in \cite[Lemma 2.5]{Nicholas Proudfoot 2005} that \( \zeta H = \overline{\zeta} \). Multiplying on the left by \( \zeta^{-1} \), we have \( H = \zeta^{-1} \iota \), which is the definition of the equivariant characteristic function.

Remark 4.2. The proof of \cite[Lemma 2.5]{Nicholas Proudfoot 2005} is surprisingly difficult. Consequently, Proposition 4.1 is a deep fact about Orlik–Solomon algebras, not just a formal consequence of the definitions.

The equivariant Kazhdan–Lusztig polynomial \( P^W_M(t) \in \text{VRep}(W) \otimes \mathbb{Z}[t] \) was introduced in \cite[Section 2.2]{Nicholas Proudfoot 2005}. Define \( P \in \mathcal{A}^W(L) \) by putting \( P_{FG}(t) = P^W_{M_{FG}}(t) \) for all \( F \leq G \). The defining recursion for \( P^W_M(t) \) in \cite[Theorem 2.8]{Nicholas Proudfoot 2005} translates to the formula \( P = HP \), which immediately implies the following proposition.

Proposition 4.3. The function \( P \) is the right equivariant KLS-function associated with \( H \).

The equivariant \( Z \)-polynomial \( Z^W_M(t) \in \text{VRep}(W) \otimes \mathbb{Z}[t] \) was introduced in \cite[Section 6]{Nicholas Proudfoot 2005}. Define \( Z \in \mathcal{A}^W(L) \) by putting \( Z_{FG}(t) = Z^W_{M_{FG}}(t) \) for all \( F \leq G \). The defining recursion for \( Z^W_M(t) \) in \cite[Section 6]{Nicholas Proudfoot 2005} translates to the formula \( Z = \overline{\zeta} P \).

Proposition 4.4. The function \( Z \) is the \( Z \)-function associated with \( H \).

Proof. Example 3.2 tells us that the right KLS-function associated with \( H \) is \( \zeta \) and Proposition 4.3 tells us that the left KLS-function associated with \( H \) is \( P \), thus the \( Z \)-function is equal \( \zeta HP = \overline{\zeta} P = Z \).

The following corollary was asserted without proof in \cite[Section 6]{Nicholas Proudfoot 2005}, and follows immediately from Propositions 3.3 and 4.4.

Corollary 4.5. The polynomial \( Z^W_M(t) \) is palindromic. That is,

\[
t^{rk_M} Z^W_M(t^{-1}) = Z^W_M(t).
\]

\(^{(3)}\)The difficult part appears in the proof of Lemma 2.4, which is then used to prove Lemma 2.5.
When $W$ is the trivial group, Gao and Xie define polynomials $Q_M(t)$ and $\hat{Q}_M(t) = (-1)^{r_M} Q_M(t)$ with the property that $(P^{-1})_{FG}(t) = \hat{Q}_{M_{FG}}(t)$ [6]. If $\hat{0}$ and $\hat{1}$ are the minimal and maximal flats of $M$, this is equivalent to the statement that $Q_M(t) = (-1)^{r_M} (P^{-1})_{\hat{0}\hat{1}}(t)$. The polynomial $Q_M(t)$ is called the inverse Kazhdan–Lusztig polynomial of $M$.

Using the machinery of this paper, we may extend their definition to the equivariant setting by defining the equivariant inverse Kazhdan–Lusztig polynomial

$$Q_W^M(t) := (-1)^{r_M} (P^{-1})_{\hat{0}\hat{1}}(t).$$

If we then define $\hat{Q} \in \mathcal{I}_{W/\hat{2}}(L)$ by putting $\hat{Q}_{FG}(t) = (-1)^{r_{FG}} Q_{M_{FG}}(t)$ for all $F \leq G$, we immediately obtain the following proposition.

**Proposition 4.6.** The functions $P$ and $\hat{Q}$ are mutual inverses in $\mathcal{I}_{W}(L)$.

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(4) The reason for bestowing this name on $Q_M(t)$ rather than $\hat{Q}_M(t)$ is that $Q_M(t)$ has non-negative coefficients; this was conjectured in [6, Conjecture 4.1] and proved in [1, Theorem 1.4].

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