CONDITIONED FUNCTIONAL LIMITS AND APPLICATIONS TO QUEUES

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Abstract

We consider a renewal process that is conditioned on the number of events in a fixed time horizon. We prove that a centered and scaled version of this process converges to a Brownian bridge, as the number of events grows large, which relies on first establishing a functional strong law of large numbers result to determine the centering. These results are consistent with the asymptotic behavior of a conditioned Poisson process. We prove the limit theorems over triangular arrays of exchangeable random variables, obtained by conditionning a sequence of independent and identically distributed renewal processes. We construct martingale difference sequences with respect to these triangular arrays, and use martingale convergence results in our proofs. To illustrate how these results apply to performance analysis in queueing, we prove that the workload process of a single server queue with conditioned renewal arrival process can be approximated by a reflected diffusion having the sum of a Brownian Bridge and Brownian motion as input to its regulator mapping.

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1. Introduction

The objective of this paper is to prove limit theorems for renewal processes conditioned on hitting a fixed integer level \( n \) in a fixed time horizon; we denote the corresponding event as \( A_n \). To be precise, we establish functional strong law (FSLLN) and functional central limit theorems (FCLT) as \( n \) tends to infinity. A well known result in stochastic process theory is the ordered statistics (OS) property of Poisson processes; viz., the arrival epochs of a homogeneous Poisson process conditioned on \( A_n \) are equal in distribution to the ordered statistics of \( n \) independent and identically distributed (i.i.d.) uniform random variables. It follows, for each \( n \), that an appropriately defined ‘conditioned’ Poisson process is equal in distribution to the empirical distribution process constructed from the ordered statistics. A further corollary to this property is that, appropriately scaled, the conditioned Poisson processes satisfy a version of Donsker’s FCLT for empirical processes so that a sequence of diffusion-scaled conditioned Poisson processes converges to a Brownian bridge process, as \( n \to \infty \). We provide a proof of this result in Theorem 3.1.

As [12, 20] show, the only renewal process that satisfies the OS property is Poisson, due to the independent increments property. Nonetheless, a natural conjecture is that a conditioned renewal process (appropriately scaled) satisfies a counterpart to the Poisson FCLT in Theorem 3.1. The primary result of this paper, Theorem 4.1, establishes precisely this result in a triangular array setting. However, the proof is more subtle and we construct a sequence of probability sample spaces by conditioning on the sequence of events \( \{ A_n, n \geq 1 \} \); note that there are no measurability issues owing to the fact that by definition \( A_n \) has positive measure. Now, renewal processes display weak dependence, specifically exchangeability of the inter-arrival times, when conditioned on the event \( A_n \). We use the sequence of probability sample spaces to construct a triangular array of exchangeable random variables representing the inter-arrival times of a sequence of conditioned renewal processes.

Consequently, Theorem 4.1 shows that the diffusion-scaled conditioned renewal process converges weakly to a Brownian bridge process, akin to Theorem 3.1. The proof follows from Proposition 4.1 and Proposition 4.2 which prove FSLLN and FCLT’s (respectively) for the partial sums constructed from the triangular array. The proof of
Proposition 4.1 follows by the construction of a triangular array martingale sequence, and then using the martingale convergence theorem. Similarly, for Proposition 4.2, we construct a triangular martingale difference array from standardized inter-arrival times, and show that this martingale difference array satisfies a martingale FCLT.

As an application of the limit results, in Section 5 we briefly consider the performance analysis of a queueing system that sees a fixed finite number of jobs applying for service over a fixed time horizon. Examples of such systems include clinics, certain call centers, airline check-in queues, and even certain cloud-based computing systems where client systems contact a centralized server for updates. All of these receive a fixed finite number of jobs over a finite time period. Further, no jobs are carried over from one ‘on period’ to the next. One approach to modeling such systems is to use a single server queue with a conditioned renewal arrival process. This leads to a reflected diffusion approximation that depends on a Brownian bridge process, which is in contrast with the conventional heavy-traffic diffusion approximation that is a reflected Brownian motion. We contrast these two approximations in Section 5.1. An important distinction between the approximations is the fact that we do not assume an explicit heavy-traffic condition in the conditioned renewal case.

**Existing Literature** There is a substantial literature related to weak convergence of conditioned random walks and partial sum processes. In particular, we note [21] where conditioned limit theorems are proved in some generality for multivariate processes with i.i.d. increments. [9, 16, 17, 19] (and many others) study the question of the limit behavior of random walks conditioned to stay positive. There is a much less extensive literature on conditioned limit theorems for sums of weakly dependent sequences, which would be relevant to this paper; see [8] for instance. Conditioned limit theorems have also been used in the context of performance analysis of queues. For instance [15] develops functional limits for the workload process conditioned on the event that the number of customers in a busy period exceeds or equals some pre-specified level, as this level tends to infinity. It is shown that the workload process converges to the Brownian excursion process. The limit results in [1] come closest to the current paper. There, limit theorems for random walks conditioned on exceeding a certain level in finite time are derived under the assumption that the random walk has
negative drift. It is shown that the ‘polygonized’ random walk sample path converges to a Brownian bridge process. This is then used to study the $GI/G/1$ waiting time process in a busy period.

The diffusion approximation derived for the workload process in Section 5 is similar to that of the $\Delta(i)/G/1$ queue, derived in [14]. In the latter model, the arrival epochs of a large but finite number of arrivals are modeled as i.i.d. random variables, and the arrival process is defined as the empirical distribution defined with respect to these random samples. The workload diffusion approximation is shown to be a function of a Brownian bridge process, but the resulting limit is more general than that derived in Section 5. In particular, it is shown that the reflection is through the so-called ‘directional derivative’ reflection map [26, Chapter 9]. The primary reason for the difference in the approximations is the fact that in [14] the fluid limit is non-linear (and time-varying) whereas in the current paper, the fluid limit is trivial. More recently, [2] studied the $\Delta(i)/G/1$ queue under a heavy-traffic condition on the initial work and exponentially distributed arrival epochs, and showed that the diffusion approximation to the queue length is a reflected Brownian motion process with parabolic drift. The diffusion approximation in Theorem 3.1 can also be contrasted with the diffusion approximation for the $Mt/M/1$ queue implied by the results in [23]. In particular, a martingale strong approximation argument from [11, Chapter 7] is used to show that the compensated Poisson process (which is a martingale) converges to a Brownian motion process. In contrast, the conditioned Poisson process is not a compensated martingale process, since it is defined with respect to the conditioned measure on the set $A_n$, and requires a different treatment.

2. Notation

Let $(\Omega, \mathcal{F}, P)$ be the sample space with respect to which we define the random elements of interest. The topology of convergence is $(\mathcal{D}, U)$ where $\mathcal{D}$ is the space of functions that are right continuous with left limits (cadlag) and $U$ is the uniform metric topology on compact sets of $[0, \infty)$. Weak convergence is represented as $\Rightarrow$; if necessary, we also note the probability measure as well: for instance, $\Rightarrow_P$. Stochastic dominance and equivalence in distribution are represented as $\leq_{st}$ and $\overset{d}{=}$. respectively.
3. The Conditioned Poisson Model

Let \((P(t), t \geq 0)\) be a unit rate Poisson process defined with respect to \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\gamma : [0, \infty) \to [0, \infty)\) be a non-negative function such that \(\Gamma(t) := \int_0^t \gamma(s) ds\) is finite for each \(t \in [0, \infty)\). It follows that \(P \circ \Gamma\) is a nonhomogeneous Poisson process with time-varying rate function \(\gamma\). Fix \(T \in (0, \infty)\), then

\[
F(t) := \frac{\Gamma(t)}{\Gamma(T)} \quad \forall t \in [0, T], \tag{3.1}
\]

is a continuous probability distribution function.

**Definition 1.** (OS Property.) Let \(n \geq 1\). Conditioned on the event \(\{P(T) = n\}\), the arrival epochs \((T_1, \ldots, T_n)\), are distributed as the ordered statistics of \(n\) independent and identically distributed (i.i.d.) random variables sampled from the distribution \(F(t)\) for \(t \in [0, T]\).

Let \(0 \leq t_1 < \cdots < t_d \leq T\) represent an arbitrary partition of \([0, T]\). Then, the independent increments property of the Poisson process implies that the following sequence of measures is well-defined:

\[
\mu_n(z_1, z_2, \ldots, z_d) := \mathbb{P} \left( P(t_1) = l_1, \ P(t_2) - P(t_1) = l_2, \right.
\]

\[
\left. \ldots, P(t_d) - P(t_{d-1}) = l_d \bigg| P(T) = n \right), \tag{3.2}
\]

where \(l_i = z_i - z_{i-1}\) (with \(z_0 = 0\)) and \(\sum_{i=1}^d l_i \leq n\). From Kolmogorov’s Extension Theorem [10 Section A.7] it follows that there exists a stochastic process \(\hat{P}_n\) such that for any \((x_1, \ldots, x_d) \in \mathbb{R}^d\),

\[
\mathbb{P} \left( \hat{P}_n(t_1) \leq x_1, \ldots, \hat{P}_n(t_d) \leq x_d \right) = \mathbb{P} \left( \frac{1}{\sqrt{n}} \left( P(t_1) - nF(t_1) \right) \leq x_1, \right.
\]

\[
\left. \ldots, \frac{1}{\sqrt{n}} \left( P(t_d) - nF(t_d) \right) \leq x_d \bigg| P(T) = n \right) \nonumber
\]

\[
= \int_{B_n(x_1, \ldots, x_d)} \mu_n(dz_1, \ldots, dz_d),
\]

where \(B_n(x_1, \ldots, x_d) := \{(z_1, \ldots, z_d) \in \mathbb{Z}^d : z_1 \leq nF(t_1) + x_1\sqrt{n}, \ldots, z_d \leq nF(t_d) + x_d\sqrt{n}\}\). By exploiting the OS property we can easily obtain a FCLT satisfied by the process \(\hat{P}_n\). Let \(W^0\) be a Brownian bridge process defined with respect to \((\Omega, \mathcal{F}, \mathbb{P})\).
Theorem 3.1. We have, \( \hat{P}_n \Rightarrow W^0 \circ F \) in \((D, U)\), as \( n \to \infty \).

Proof. For a fixed \( n \geq 1, x \in \mathbb{R} \) and \( t \in [0, T] \) we have
\[
\mathbb{P}(\hat{P}_n(t) \leq x) = \mathbb{P}(P(t) \leq x\sqrt{n} + nF(t)|P(T) = n).
\]

Let \( T := \{T_1, \ldots, T_n\} \) be a collection of i.i.d. random variables, with distribution function \( F \) (defined in (3.1)). Let \( A_n(t) := \sum_{i=1}^{n} 1_{\{T_i \leq t\}} \) and \( \hat{A}_n(t) := \sqrt{n} \left( \frac{A_n(t)}{n} - F(t) \right) \) be the empirical process associated with \( T \). The OS property implies that \( \mathbb{P}(P(t) = l|P(T) = n) = \mathbb{P}(A_n(t) = l), \) using the fact that \( \{T_l \leq t < T_{l+1}\} = \{P(t) = l\}. \)

Therefore, we have
\[
\mathbb{P}(\hat{P}_n(t) \leq x) = \mathbb{P}(A_n(t) \leq x\sqrt{n} + nF(t))
\]
\[
= \mathbb{P}(\hat{A}_n(t) \leq x)
\]
\[
\Rightarrow (W^0 \circ F)(t) \text{ as } n \to \infty
\]

the convergence following from Donsker’s theorem for empirical distributions [3, Chapter 13], proving the pointwise convergence of the process \( \hat{P}_n \).

Next, consider the partition \( 0 < t_1 < \cdots < t_d < T \) and observe that
\[
\mathbb{P}\left(\hat{P}_n(t_1) \leq x_1, \ldots, \hat{P}_n(t_d) \leq x_d\right) = \mathbb{P}\left(\hat{A}_n(t_1) \leq x_1, \ldots, \hat{A}_n(t_d) \leq x_d\right).
\]

From the proof of Donsker’s theorem it follows that the increments of the diffusion-scaled empirical process satisfies \( (\hat{A}_n(t_1) - \hat{A}_n(t_2)), \ldots, (\hat{A}_n(t_d) - \hat{A}_n(t_{d-1}) \Rightarrow ((W^0 \circ F)(t_1), (W^0 \circ F)(t_2) - (W^0 \circ F)(t_1) \cdots, (W^0 \circ F)(t_d) - (W^0 \circ F)(t_{d-1}), \) implying that the finite dimensional distribution of \( \hat{P}_n \) too converges to the same limit. Next, the tightness of the sequence \( \hat{A}_n \) implies that \( \hat{P}_n \) is tight. Therefore, by [3, Theorem 8.1], \( \hat{P}_n \Rightarrow W^0 \circ F \) as \( n \to \infty \). \( \square \)

4. Conditioned Renewal Model

The proof of Theorem 3.1 is a consequence of the OS property. However, [20, Theorem 1] shows that a renewal process satisfies the OS property if and only if it is Poisson (see [12] as well). Consequently, it is not \textit{a priori} obvious that the conditioned renewal process satisfies an analogous result to Theorem 3.1. Furthermore, while we could argue the existence of the ‘conditioned’ process, \( \hat{P}_n \), by appealing to the extension
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dear and the independent increments property of the Poisson process, we can no
longer do that in the case of a renewal process. Instead, we prove the conditioned
functional limit theorems in this section by working with the properties of the inter-
arrival times, when conditioned on the event \( A_n \).

Let \( F : [0, \infty) \to [0, 1] \) now represent the distribution of a non-negative random
variable, with well defined density function \( f(t) := \frac{dF(t)}{dt} \). Without loss of generality,
we also assume that \( \int_0^\infty (1 - F(t))dt < \infty \). Recall the definition of a finitely exchangeable
sequence [18, Chapter 9].

**Definition 2. (Finitely Exchangeable.)** Let \( \{X_1, \ldots, X_n\} \) be a collection of random
variables defined with respect to the sample space \((\Omega, F, \mathbb{P})\). Then, this collection
is said to be finitely exchangeable if \( \{X_1, \ldots, X_n\} \overset{d}{=} \{X_{\pi(1)}, \ldots, X_{\pi(n)}\} \), where \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \) is a permutation function on the index of the collection.

Recall that an infinitely exchangeable sequence of random variables satisfies the per-
mutation condition in Definition 2 for every finite subset of random variables in the
sequence. Thus, finitely exchangeable random variables differ from infinitely exchange-
able sequences. Consequently, important results such as de Finetti’s Theorem, which
could have been used to represent the weakly dependent ensemble as a mixture of
independent random variables, are unavailable; see [18, Chapter 9]. In the ensuing
discussion, we will refer to finitely exchangeable collections of random variables as
merely ‘exchangeable’ for brevity.

Renewal processes satisfy an exchangeable (or E) property as summarized in the
following lemma.

**Lemma 4.1. (E Property.)** Let \( \{\xi_i, i \geq 1\} \) be a sequence of i.i.d. positive random
variables and define \( A(t) := \sup\{k > 0| \sum_{i=1}^k \xi_i \leq t\} \), for all \( t > 0 \) to be the associated
renewal counting process. Fix \( T \in (0, \infty) \). Then, the collection \( \Xi_n := (\xi_1, \ldots, \xi_n) \) is
finitely exchangeable when conditioned by the event \( A_n = \{A(T) = n\} \).

Before proceeding to the proof, note that the event \( \mathbb{P}(A_n) > 0 \), under the conditions
of the theorem.

**Proof.** Let \( (x_1, \ldots, x_n) \in [0, \infty)^n \) and consider the measure of the event \( \{\xi_i \in

Recall that \( \{A(T) = n\} = \left\{ \sum_{l=1}^{n} \xi_l \leq T < \sum_{l=1}^{n+1} \xi_l \right\} \). Now, using the fact that under the measure \( P \), \( \xi_i \) are i.i.d. random variables, it follows that the measure of the joint event is invariant under any permutation of the first \( n \) random variables. That is, if \( \pi(\cdot) \) is a permutation of \( \{1, \ldots, n\} \), then we have

\[
P\left( (\xi_1 \in dx_1, \ldots, \xi_n \in dx_n), A(T) = n \right) = P\left( \xi_{\pi(1)} \in dx_1, \ldots, \xi_{\pi(n)} \in dx_n, \sum_{l=1}^{n} \xi_{\pi(l)} \leq T, \sum_{l=1}^{n} \xi_{\pi(l)} + \xi_{n+1} > T \right),
\]

implying that

\[
P(\xi_1 \in dx_1, \ldots, \xi_n \in dx_n | A_n) = P(\xi_{\pi(1)} \in dx_1, \ldots, \xi_{\pi(n)} \in dx_n | A_n).
\]

Next, suppose that \( \tilde{\pi}(\cdot) \) is a permutation of \( \{1, \ldots, n+1\} \). Then, it is possible that \( \sum_{i=1}^{n} \xi_{\tilde{\pi}(i)} > T \), since \( \xi_{n+1} > T - \sum_{i=1}^{n} \xi_i > 0 \) conditioned on \( \{A(T) = n\} \). Thus, \( \Xi_n \) cannot be extended to a larger collection of exchangeable random variables, implying that it is finitely exchangeable.

Intuitively, the collection \( \Xi_n \) is finitely exchangeable owing to the fact that \( \sum_{i=1}^{n} \xi_i \leq T \). This hard bound forces the random variables to not only take values in a finite interval but to also be weakly dependent on one another, when conditioned on \( A_n \).

Consider \( \{(\xi_{n,i}, \ i = 1, \ldots, n+1), \ n \geq 1\} \), a row-wise independent triangular array of i.i.d. random variables. Define the counting process

\[
A_n(t) := \sup \left\{ 0 \leq m \leq n \bigg| S_n \left( \frac{m}{n} \right) := \sum_{l=1}^{m} \xi_{n,l} \leq t \right\}.
\]

By Lemma 4.1, we know that \( \Xi_n := \{\xi_{n,1}, \ldots, \xi_{n,n}\} \) is an exchangeable collection when conditioned on the event \( A_n := \{ \sum_{i=1}^{n} \xi_{n,i} \leq T < \sum_{i=1}^{n+1} \xi_{n,i} \} = \{A_n(T) = n\} \). Then, the collection \( \{\Xi_n, \ n \geq 1\} \) forms a triangular array of exchangeable random variables with independent rows.

Our analysis will proceed down the triangular array as \( n \to \infty \). Note that the conditioned probability measure changes for each row of the array. Classical triangular
array results assume that the array is defined with respect to the same probability space. In order to facilitate the proofs, we first construct a product probability space that covers the entire array \( \{ \Xi_n, n \geq 1 \} \).

### 4.1. A Product Sample Space

For a fixed \( n \geq 1 \) and \( T > 0 \), we define the restricted sample space, \((\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\), where \( \Omega_n = \Omega \cap \{ A_n(T) = n \} \), \( \mathcal{F}_n := \sigma \{ A \cap \{ A_n(T) = n \} : A \in \mathcal{F} \} \) and \( \mathbb{P}_n(B) := \frac{\mathbb{P}(B \cap \{ A_n(T) = n \})}{\mathbb{P}(A_n(T) = n)} \) for any \( B \in \mathcal{F}_n \). Clearly \( \{ \Omega_n, n \geq 1 \} \) forms a partition of \( \Omega \). Next, we construct a new product space from the restricted sample spaces \((\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\) as follows.

Let, \( \bar{\Omega} := \Omega_1 \times \Omega_2 \times \cdots \), so that \( A \subset \bar{\Omega} = A_1 \times A_2 \times \cdots \) for sets \( A_n \subset \Omega_n \). The product \( \sigma \)-algebra, \( \bar{\mathcal{F}} := \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \) is the \( \sigma \)-algebra generated from cylinder sets of the type \( R = \{ (\omega_1, \omega_2, \cdots ) \in \bar{\Omega} \mid \omega_{i_1} \in A_{i_1}, \cdots , \omega_{i_k} \in A_{i_k} \} \), where \((i_1, \ldots , i_l)\) is an arbitrary subset of \( \mathbb{N} \) of size \( k \geq 1 \) and \( A_{i_n} \in \mathcal{F}_n \). The existence of such a product \( \sigma \)-algebra is well-justified by [13, Proposition 1.3]. Finally, we define \( \bar{\mathbb{P}}(R) = \Pi_{i=1}^{k} \mathbb{P}_i(A_i) \), for the cylinder sets. This extends to \( \bar{\mathbb{P}} = \mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \), which is the natural product measure on the measure space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})\), by standard arguments showing that the measure is countably additive on \( \bar{\mathcal{F}} \). The definition of the Lebesgue integral on the space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})\) now follows from standard definitions of integration on product spaces. However, we introduce some notation to help the following discussion. In particular, consider a function defined in the following manner: \( \bar{X} := X \times \Pi_{i \neq n} I_{\{ \Omega_i \}} \), where \( X \) is measurable and integrable with respect to \((\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\), and \( I_{\{ \} } \) is the indicator function. Then

\[
E_{\bar{\mathbb{P}}}[\bar{X}] = \int_{\Omega_n} X d\mathbb{P}_n \int_{\Omega_{i \neq n} \Omega_i} I_{\{ \Omega_i \}} d\mathbb{P}_i
\]

is well-defined, and we write this as \( E_{\bar{\mathbb{P}}}[X] \), where it is to be understood that the integration is actually of \( \bar{X} \).

### 4.2. Asymptotics of Conditioned Renewal Processes

Let \( \mu_n := \mathbb{E}[\xi_n, i | A_n] = \mathbb{E}[\xi_n, i] \) be the conditional mean of the inter-arrival times; the exchangeable property implies that these random variables are identically distributed. Observe that, for a fixed \( n \), the conditioning is with respect to a fixed event \( A_n \). Therefore, \( \mu_n \) is not a random variable.

**Lemma 4.2.** The conditional mean \( \mu_n \) satisfies
\((i)\) \(\mu_n \to 0\) as \(n \to \infty\),
\((ii)\) \(\operatorname{Var}(\xi_{n,1}) := E[(\xi_{n,1} - \mu_n)^2|A_n(T) = n] \to 0\) as \(n \to \infty\), and
\((iii)\) \(n\mu_n \to T\) as \(n \to \infty\).

Now, consider the sequence of partial sum processes \(\{S_n, n \geq 1\}\) defined as \(S_n(t) := \sum_{i=1}^{[nt]} \xi_{n,i} \forall t \in [0,T]\).

**Proposition 4.1.** (Partial Sum FSLN.) The partial sum sequence satisfies
\[
S_n := \frac{S_n}{n} \xrightarrow{\mathbb{P}} \frac{e}{T} \text{ in } (\mathcal{D}, \mathcal{U}) \bar{\mathbb{P}} - \text{a.s.}
\]
as \(n \to \infty\).

Next, we prove an FCLT for the partial sum sequence \(\{S_n, n \geq 1\}\). Specifically, consider \(\{\phi_{n,i}, l = 1, \ldots, n\}\) defined with respect to \(\Xi_n\) as
\[
\phi_{n,i} := \frac{\xi_{n,i} - \mu_n}{\sqrt{n}}.
\]

Following [24] and [3, Theorem 24.2], the following theorem characterizes the sequence \(\phi_{n,i}\) and shows that the partial sums of these random variables converge weakly to a Brownian bridge process. We assume that the Brownian bridge process \(W^0\) is well-defined with respect to the product sample space \((\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})\).

**Proposition 4.2.** Let \(\{(\phi_{n,1}, \ldots, \phi_{n,n}), n \geq 1\}\) be the triangular array of random variables defined above and define \(\tilde{S}_n(t) := \sum_{i=1}^{[nt]} \phi_{n,i}, t \in [0,T]\). Then,
\((i)\) \(\sum_{i=1}^{n} \phi_{n,i} \Rightarrow \mathbb{P} 0\),
\((ii)\) \(\max_{1 \leq i \leq n} |\phi_{n,i}| \Rightarrow \mathbb{P} 0\),
\((iii)\) \(\sum_{i=1}^{n} \phi_{n,i}^2 \Rightarrow \mathbb{P} 1\), and
\((iv)\) \(\tilde{S}_n \Rightarrow \mathbb{P} W^0\) in \((\mathcal{D}, \mathcal{U})\), as \(n \to \infty\).

The conditions in Proposition 4.2 are natural in the context of the conditioned limit result we seek. Note that the conditioned limit result is akin to proving a diffusion limit for a tied-down random walk (see [22, 25]). The first condition here enforces a type of “asymptotic tied down” property. The second condition is a necessary and sufficient condition for the limit process to be infinitely divisible (see [6] for more on this). The third condition is necessary to ensure that the Gaussian limit, when \(t = 1\), has variance 1.
Now, define and the ‘inverse’ process, corresponding to $S_n$, as

$$S_n^{-1}(t) := \inf \left\{ p \in [0, 1] | A_n(p) > t \right\},$$

and the scaled counting process $\tilde{A}_n := n^{-1}A_n$.

**Lemma 4.3.** We have,

(i) $\sup_{0 \leq t \leq T} |\tilde{A}_n(t) - S_n^{-1}(t)| \to 0$ $\tilde{P}$-a.s. as $n \to \infty$, and

(ii) $\sqrt{n} \sup_{0 \leq t \leq T} (\tilde{A}_n(t) - S_n^{-1}(t)) \to 0$ $\tilde{P}$-a.s. as $n \to \infty$.

We can now state and prove the main result of this section, proving the FSLLN and FCLT for the counting process $A_n$ in (4.1).

**Theorem 4.1.** The counting process $A_n$ satisfies

(i) $\tilde{A}_n \to \frac{eT}{\tilde{P}}$ in $(D, U)$ $\tilde{P}$-a.s. as $n \to \infty$, where $e : [0, \infty) \to [0, \infty)$ is the identity map, and

(ii) $\sqrt{n}(\tilde{A}_n - \frac{eT}{\tilde{P}}) \Rightarrow -W^0$ in $(D, U)$ as $n \to \infty$, where $W^0$ is the Brownian bridge limit process observed in Proposition 4.2.

**Proof.** Applying [26, Theorem 7.8.1], the FSLLN in Proposition 4.1 implies the convergence of the corresponding inverse function, $S_n^{-1}$ to $e$. Part (i) of Lemma 4.3, in turn, implies the convergence of the counting process $\tilde{A}_n$. This proves part (i).

Next, [26, Theorem 7.8.2] and the FCLT in Proposition 4.2 (iv) implies that $\sqrt{n}(S_n^{-1} - eT^{-1}) \Rightarrow -W^0$. Part (ii) of the theorem follows from Lemma 4.3 (ii). □

We now proceed to the proofs of the lemmas and propositions.

**Proof of Lemma 4.3.** (i) The conditional intensity function (CIF) of $A_n(t)$ is $l^*(t)dt := E[A_n(dt)|\mathcal{H}_t] = \frac{l(t)dt}{1 - F(t)} \geq 0$, where $\mathcal{H}_t$ is the filtration generated by $A_n$. Let $L^*(t) = \int_0^t l^*(s)ds$ be the integrated CIF, so that $A_n(t) - L^*(t)$ is a compensated Martingale process. [7, Theorem 7.4.1] shows that $\tilde{A}_n(t) = A_n(L^*(t))$ is a unit rate Poisson process. That is, if $\{T_1, T_2, \ldots \}$ is a realization of the event epochs of process $A_n$, then $\{\tilde{T}_i = L^*(T_i)\}$ is equal (in distribution) to a realization from a unit rate Poisson process. As $L^*$ is non-decreasing, it follows that $\{\tilde{T}_{n+1} > L^*(T) \geq \tilde{T}_n\}$ if and only if $\{T_{n+1} > T \geq T_n\}$. [7, Theorem 7.4.1] implies that $\{A_n(T) = n\} = \{\tilde{A}_n(L^*(T)) = n\}$.

Now, we have

$$P(\xi_{n,1} > u | A_n(T) = n) = P(\phi_1 \geq L^*(u) | \tilde{A}_n(L^*(T)) = n),$$
where \( \phi_1 := L^*(\xi_{n,1}) \). Recall that a Poisson process satisfies the OS property in Definition 1. It follows that

\[
\mathbb{P}(\phi_1 \geq L^*(u) | \hat{A}_n(L^*(T))) = n) = \left(1 - \frac{L^*(u)}{L^*(T)}\right)^n.
\]

Now, by definition

\[
\mu_n = \int_0^T \mathbb{P}(\xi_1 > u | A_n(T) = n) du
= \int_0^T \left(1 - \frac{\Lambda^*(u)}{\Lambda^*(T)}\right)^n du.
\] (4.2)

Since \( \Lambda^*(t) \) is a non-decreasing function of \( t \), it follows that the integrand in (4.2) is bounded above by 1 for all \( n \geq 1 \). Furthermore, for every \( t \in (0, T] \), \( \lim_{n \to \infty} \left(1 - \frac{\Lambda^*(t)}{\Lambda^*(T)}\right)^n = 0 \). Therefore, by the bounded convergence theorem it follows that \( \lim_{n \to \infty} \mu_n = 0 \).

(ii) Observe that, by definition, \( \int \xi_{n,1}^2 dP_n \leq T^2 \forall n \geq 1 \), implying \( \Xi_n \) is a uniformly integrable (U.I.) family of random variables. Then, fixing \( \epsilon > 0 \), part (i) of the lemma implies that \( \mathbb{P}(\xi_{n,1} > \epsilon) \to 0 \) as \( n \to \infty \). We now have

\[
\int \xi_{n,1}^2 dP_n = \int_{(\xi_{n,1} > \epsilon)} \xi_{n,1}^2 dP_n + \int_{(\xi_{n,1} \leq \epsilon)} \xi_{n,1}^2 dP_n
\leq T^2 \mathbb{P}(\xi_{n,1} > \epsilon) + \epsilon \int \xi_{n,1} dP_n,
\]

implying that \( \int \xi_{n,1}^2 dP_n \to 0 \) as \( n \to \infty \). Thus, \( \text{Var}(\xi_{n,1}) \to 0 \) as \( n \to \infty \).

(iii) Let \( S_n := \sum_{i=1}^n \xi_{n,i} \) and \( S_{n+1} = S_n + \xi_{n,n+1} \). Since the random variables \( \xi_{n,i} \) are identical in distribution,

\[
\mu_n = \mathbb{E}[\xi_{n,1} | A_n(T) = n] = \frac{1}{n} \mathbb{E}[S_n | A_n(T) = n].
\]

By definition \( \{A_n(T) = n\} = \{S_n \leq T < S_{n+1}\} \), implying that \( \mathbb{E}[S_n | A_n(T) = n] \leq T \) for all \( n \geq 1 \). Next, fix \( \epsilon > 0 \), and note that

\[
\mathbb{E}[S_n | A_n(T) = n] = \frac{\mathbb{E}[S_n \mathbf{1}_{\{S_n \leq T < S_{n+1}\}}]}{\mathbb{P}(A_n(T) = n)}
\geq (T - \epsilon) \frac{\mathbb{P}(T - \epsilon < S_n \leq T < S_{n+1})}{\mathbb{P}(S_n \leq T < S_{n+1})}.
\]

Now, consider the partition of \( \{S_n \leq T < S_{n+1}\} = \{T - \epsilon < S_n \leq T < S_{n+1}\} \cup \{S_n < T - \epsilon, S_{n+1} > T\} \). We have that, under \( \mathbb{P} \),

\[
\mathbb{P}(S_n < T - \epsilon, S_{n+1} > T) \leq \mathbb{P}(S_n < T).
\]
By the strong law of large numbers, it follows that $S_n \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$. Therefore, $\mathbb{P}(S_n < T) \to 0$ as $n \to \infty$. Thus, it follows that

$$
\lim_{n \to \infty} \frac{\mathbb{P}(T - \epsilon < S_n \leq T < S_{n+1})}{\mathbb{P}(S_n \leq T < S_{n+1})} = 1.
$$

Therefore,

$$
\lim_{n \to \infty} \mathbb{P}(S_n < T) \to 0 \text{ as } n \to \infty.
$$

Thus, it follows that

$$
\lim_{n \to \infty} \mathbb{P}(T - \epsilon < S_n \leq T < S_{n+1}) = 1.
$$

Therefore,

$$
\lim_{n \to \infty} \mathbb{P}(S_n \leq T < S_{n+1}) = \frac{1}{n},
$$

(4.3)

Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{n \to \infty} n\mu_n = \lim_{n \to \infty} \mathbb{E}[S_n | A_n(T) = n] = T$.

**Proof of Proposition 4.1** Without loss of generality let $T = 1$. Consider, for $t \in [0, 1]$,

$$
\left| \sum_{l=1}^{\lfloor nt \rfloor} (\xi_{n,l} - \mu_n) \right| \leq \left| \sum_{l=1}^{\lfloor nt \rfloor} (\xi_{n,l} - \mu_n) \right| + |\lfloor nt \rfloor \mu_n - t|.
$$

The second term on the RHS tends to 0, as a consequence of Lemma 4.2 (iii). Define the martingale sequence, $z_{n,l} := (\xi_{n,l} - \mu_n) - \mathbb{E}_{\tilde{\mathbb{P}}}(\xi_{n,l} - \mu_n | \mathcal{F}_{n,l-1})$, where $\mathcal{F}_{n,l} := \sigma\{(\xi_{n,1} - \mu_n), \ldots, (\xi_{n,l} - \mu_n), \sum_{i=1}^{n} \xi_{n,i} - \mu_n\}$. Note that expectation is taken with respect to the measure $\tilde{\mathbb{P}}$, implying that there is (implicitly) a conditioning with respect to the event $A_n$ as well.

It follows that

$$
\sum_{i=j}^{n} (\xi_{n,i} - \mu_n) = \sum_{i=j}^{n} \mathbb{E}_{\tilde{\mathbb{P}}}[\xi_{n,i} - \mu_n | \mathcal{F}_{n,j-1}]
$$

$$
= (n - j + 1) \mathbb{E}_{\tilde{\mathbb{P}}}[\xi_{n,j} - \mu_n | \mathcal{F}_{n,j-1}],
$$

where the last equality follows from the fact that the random variables are exchangeable (and hence identically distributed) under the measure $\tilde{\mathbb{P}}$. This implies that

$$
z_{n,l} = (\xi_{n,l} - \mu_n) - \frac{1}{n-l+1} \sum_{i=l}^{n} (\xi_{n,i} - \mu_n)
$$

Using the fact that $\xi_{n,l} \in [0, 1]$, under the measure $\tilde{\mathbb{P}}$, it follows that

$$
\sum_{l=1}^{n} \frac{1}{n-l+1} \sum_{i=l}^{n} \xi_{n,i} = \sum_{l=0}^{n-1} \frac{1}{n-l} \sum_{j=l+1}^{n} \xi_{n,j}
$$

$$
\leq \left( \sum_{l=0}^{n-1} \frac{1}{n-l} (n-l-1) \right)
$$

$$
\leq n.
$$
and consequently,
\[ \sum_{l=1}^{n} (\xi_{n,l} - \mu_n) \leq \sum_{l=1}^{n} z_{n,l} + n. \]

On the other hand, observe that
\[
\sum_{l=1}^{n} (\xi_{n,l} - \mu_n) = \sum_{l=1}^{n} z_{n,l} + \sum_{l=1}^{n} \frac{1}{n-l+1} \sum_{i=l}^{n} \xi_{n,i} - n\mu_n \\
\geq \sum_{l=1}^{n} z_{n,l} - n,
\]

where we have used the fact that \( \mu_n \leq 1 \) in the final inequality. Now, fix \( \epsilon > 0 \) and use the inequalities above to obtain
\[
\{ \omega \in \bar{\Omega} : |\sum_{l=1}^{n} (\xi_{n,l} - \mu_n)| > \epsilon \}
\subset \{ \omega \in \bar{\Omega} : \sum_{l=1}^{n} z_{n,l} + n > \epsilon \} \\
\cup \{ \omega \in \bar{\Omega} : \sum_{l=1}^{n} z_{n,l} < n - \epsilon \}.
\]

Since \( |z_{n,l}| \leq 2 \), the Azuma-Hoeffding inequality implies that
\[
\bar{\mathbb{P}} \left( \sum_{l=1}^{n} z_{n,l} > \epsilon - n \right) \leq \exp \left( \frac{-(\epsilon - n)^2}{8n} \right),
\]

and
\[
\bar{\mathbb{P}} \left( \sum_{l=1}^{n} z_{n,l} < -\epsilon \right) \leq \exp \left( \frac{-(n - \epsilon)^2}{8n} \right).
\]

Therefore, using the union bound
\[
\bar{\mathbb{P}} \left( \sum_{l=1}^{n} |\xi_{n,l} - \mu_n| > \epsilon \right) \leq 2e^{-\frac{(n-\epsilon)^2}{8n}}.
\]

Now, by Cauchy’s ratio test, it can be readily verified that for any \( \epsilon > 0 \)
\[
\sum_{n=1}^{\infty} \bar{\mathbb{P}} \left( \sum_{l=1}^{n} |\xi_{n,l} - \mu_n| > \epsilon \right) < \infty.
\]

Thus, by the First Borel-Cantelli Lemma, \( \bar{\mathbb{P}}(\sum_{l=1}^{n} |\xi_{n,l} - \mu_n| > \epsilon \text{ i.o.}) = 0 \). Therefore, \( \sum_{l=1}^{n} (\xi_{n,l} - \mu_n) \to 0 \) \( \bar{\mathbb{P}} \)-a.s. as \( n \to \infty \). Clearly, this holds for any \( t \in [0, 1] \), so that \( \sum_{l=1}^{[nt]} (\xi_{n,l} - \mu_n) \to 0 \) \( \bar{\mathbb{P}} \)-a.s. as \( n \to \infty \). The proof of uniform convergence on \([0, 1]\) follows from standard arguments (see [5, Chapter 5] for instance). \( \square \)
Proof of Proposition 4.2. First, note that the exchangeability of \( \{\phi_{n,i}\} \) follows directly from that of \( \{\xi_{n,i}\} \).

(i) The proof follows by using the definition of \( \tilde{P} \). Fix \( \epsilon > 0 \), and consider

\[
\tilde{P} \left( \left| \sum_{l=1}^{n} \phi_{n,l} \right| > \epsilon \right) = \tilde{P} \left( \left| \sum_{l=1}^{n} \phi_{n,l} \right| > \epsilon, A_n(T) = n \right) \\
= \tilde{P} \left( \left| \sum_{l=1}^{n} \xi_{n,l} - n\mu_n \right| > \epsilon \sqrt{n}, A_n(T) = n \right) \\
= \tilde{P} \left( \sum_{l=1}^{n} \xi_{n,l} > \epsilon \sqrt{n} + n\mu_n, A_n(T) = n \right) \\
\quad + \tilde{P} \left( \sum_{l=1}^{n} \xi_{n,l} < -\epsilon \sqrt{n} + n\mu_n, A_n(T) = n \right).
\]

Recall that \( \{A_n(T) = n\} = \{\sum_{l=1}^{n} \xi_{n,l} \leq T < \sum_{l=1}^{n} \xi_{n,l} + \xi_{n,n+1}\} \). It follows that for any \( \omega \in B_n := \{\sum_{l=1}^{n} \xi_{n,l} > \epsilon \sqrt{n} + n\mu_n, A_n(T) = n\} \) we have \( T \geq \sum_{l=1}^{n} \xi_{n,l} > \epsilon \sqrt{n} + n\mu_n \) and that \( n\mu_n = E_{\tilde{P}}[\sum_{l=1}^{n} \xi_{n,l}] \leq T \). Therefore, \( n\mu_n \) is uniformly bounded (for every \( n \geq 1 \)). Then, for a given \( T \), there exists a \( n_T \) such that for every \( n > n_T \), \( \sqrt{n}\epsilon + n\mu \geq T \). As \( \epsilon \) is arbitrary, asymptotically, \( B_n \) is an impossible event.

Next, consider the event \( C_n := \{\sum_{l=1}^{n} \xi_{n,l} < -\epsilon \sqrt{n} + n\mu_n, A_n(T) = n\} \). Using the facts that \( \xi_{n,l} \geq 0 \) and \( n\mu_n \leq T \) for all \( n \), we have \( -\epsilon \sqrt{n} + T \geq -\epsilon \sqrt{n} + n\mu_n > \sum_{l=1}^{n} \xi_{n,l} \geq 0 \). Clearly, as \( n \to \infty \), \( -\epsilon \sqrt{n} + T \to -\infty \) implying that \( -\epsilon \sqrt{n} + n\mu_n \to -\infty \). Since \( \epsilon > 0 \) is arbitrary, for large enough \( n \) \( C_n \) too is an impossible event. It follows that \( \phi_{n,l} \to 0 \) as \( n \to \infty \).

(ii) First, for a fixed \( \epsilon > 0 \) the union bound implies that

\[
\tilde{P} \left( \max_{1 \leq l \leq n} |\phi_{n,l}| > \epsilon \right) \leq \sum_{l=1}^{n} \tilde{P} (|\phi_{n,l}| > \epsilon) \\
\leq n \tilde{P} (|\phi_{n,1}| > \epsilon) \\
\leq n \frac{E_{\tilde{P}} |\xi_{n,1} - \mu_n|^2}{n\epsilon^2} = \frac{\text{Var}(\xi_{n,1})}{\epsilon^2},
\]

where the latter expression follows by an application of Chebyshev’s inequality under the \( \tilde{P} \) measure. Lemma 4.2 implies that \( \text{Var}(\xi_{n,1}) \to 0 \) as \( n \to \infty \). As \( \epsilon > 0 \) is arbitrary, (ii) is proved.

(iii) Define the martingale difference sequence \( Z_{n,l} := \phi_{n,l}^2 - E_{\tilde{P}}[\phi_{n,l}^2|\mathcal{F}_{n,l-1}] \), where \( \{\mathcal{F}_{n,l}\} \) is a filtration defined with respect to \( \phi_{n,l}^2 \) as \( \mathcal{F}_{n,l} = \sigma \left( \phi_{n,1}^2, \ldots, \phi_{n,l-1}^2, \sum_{i=1}^{l} \phi_{n,i}^2 \right) \).
Now, consider the conditional expectation in the definition of $Z_{n,l}$. Notice that we have,

$$\sum_{i=j}^{n} \phi_{n,i}^2 = E_{\bar{P}} \left[ \sum_{i=j}^{n} \phi_{n,i}^2 \mid \mathcal{F}_{n,j-1} \right] = E_{\bar{P}} \left[ \sum_{i=j}^{n} \phi_{n,i}^2 \mid \mathcal{F}_{n,j-1} \right] = (n - j + 1) E \left[ \phi_{n,j}^2 \mid \mathcal{F}_{n,j-1} \right].$$

The penultimate equation follows from the fact that $\phi_{n,l}^2$ are exchangeable, and the last by the fact that they are also identically distributed. It follows that

$$E_{\bar{P}} \left[ \phi_{n,j}^2 \mid \mathcal{F}_{n,j-1} \right] = \frac{1}{n - j + 1} \sum_{i=j}^{n} \phi_{n,i}^2,$$

and

$$Z_{n,l} = \phi_{n,l}^2 - \frac{1}{n - l + 1} \sum_{i=l}^{n} \phi_{n,i}^2.$$

Now, by definition $\phi_{n,l} \leq 2T/\sqrt{n}$ under the measure $\bar{P}$, so that

$$\sum_{l=1}^{n} \frac{1}{n - l + 1} \sum_{i=l}^{n} \phi_{n,i}^2 = \frac{1}{n - l + 1} \sum_{l=0}^{n-1} \frac{1}{n - l} \sum_{j=l+1}^{n} \phi_{n,j}^2 \leq \frac{4T^2}{n} \sum_{l=0}^{n-1} \left(1 - \frac{1}{n - l}\right) \leq 4T^2.$$

Thus, we have $\sum_{l=1}^{n} Z_{n,l} \geq \sum_{l=1}^{n} \phi_{n,l}^2 - 4T^2$. Fix $\epsilon > 0$, and use the Azuma-Hoeffding inequality to obtain

$$\bar{P} \left( \sum_{l=1}^{n} \phi_{n,l}^2 - 1 \geq \epsilon \right) \leq \bar{P} \left( \sum_{l=1}^{n} Z_{n,l} \geq \epsilon + 1 - 4T^2 \right) \leq \exp \left( - \frac{(\epsilon + 1 - 4T^2)^2}{n \times \frac{64T^4}{n^2}} \right),$$

where the bound in the numerator on the R.H.S. follows by the facts that

$$|Z_{n,l}| \leq |\phi_{n,l}^2| + \frac{1}{n - l + 1} \sum_{j=l}^{n} |\phi_{n,j}^2| \leq 2|\phi_{n,l}^2|,$$

and

$$\sum_{i=j}^{n} \phi_{n,i}^2 = E_{\bar{P}} \left[ \sum_{i=j}^{n} \phi_{n,i}^2 \mid \mathcal{F}_{n,j-1} \right] = E_{\bar{P}} \left[ \sum_{i=j}^{n} \phi_{n,i}^2 \mid \mathcal{F}_{n,j-1} \right] = (n - j + 1) E \left[ \phi_{n,j}^2 \mid \mathcal{F}_{n,j-1} \right].$$
and \( \phi^2_{n,l} \leq 2T^2/n \). It follows that
\[
\bar{P} \left( \sum_{l=1}^{n} \phi^2_{n,l} - 1 \geq \epsilon \right) \to 0 \text{ as } n \to \infty.
\]

Next, since \( \phi^2_{n,l} \geq 0 \) for all \( l \leq n \), it follows that \( \sum_{l=1}^{n} Z_{n,l} \leq \sum_{l=1}^{n} \phi^2_{n,l} \). Clearly,
\[
\bar{P} \left( \sum_{l=1}^{n} \phi^2_{n,l} < 1 - \epsilon \right) \leq \bar{P} \left( \sum_{l=1}^{n} Z_{n,l} < 1 - \epsilon \right).
\]

Using the Azuma-Hoeffding inequality again, we have
\[
\bar{P} \left( \sum_{l=1}^{n} Z_{n,l} < 1 - \epsilon \right) \leq \exp \left( -\frac{(1 - \epsilon)^2}{n \times 64T^4/n^2} \right),
\]

implying that
\[
\bar{P} \left( \sum_{l=1}^{n} \phi^2_{n,l} < 1 - \epsilon \right) \to 0 \text{ as } n \to \infty.
\]

Finally, it follows that \( \sum_{l=1}^{n} \phi^2_{n,l} \Rightarrow 1 \) as \( n \to \infty \).

(iv) Parts (i), (ii), (iii) verify [3, Theorem 24.2], implying that \( \hat{A}_n \Rightarrow W^0 \) in \((D, U)\) as \( n \to \infty \).

\[\square\]

Proof of Lemma 4.3 (i) Fix \( t \in [0, 1] \). By definition it follows that \( S_n(\hat{A}_n(t)) \leq t \) and \( S_n(S_n^{-1}(t)) > t \) (and \( S_n(S_n^{-1}(t) - ) \leq t \)). Thus, for any \( \epsilon > 0 \), \( S_n(\hat{A}_n(t) + \epsilon) > t \). In particular, \( \hat{A}_n(t) + \frac{1}{n} \geq S_n^{-1}(t) \). Since \( S_n \) is non-decreasing (since the increments \( \xi_{n,t} \geq 0 \)), it follows that
\[
\frac{1}{n} \geq S_n^{-1}(t) - \hat{A}_n(t) \geq 0,
\]
where the last inequality follows by definition.

(ii) The result is an obvious corollary of the argument for part (i).

\[\square\]

5. An Application to Transient Workload Analysis

We now demonstrate how the conditioned limit theorems developed in the previous section can be used to conduct a transient performance analysis of a single server queue. We will focus on the workload process, though the analysis can be extended to other performance metrics as well.
The ‘data’ of the queueing model are as follows: Let \( T = 1 \) (with out loss of generality) and consider a triangular array of tuples, \( \{(\xi_{n,1}, \nu_{n,1}), \ldots, (\xi_{n,n+1}, \nu_{n,n+1})\}, \ n \geq 1 \). For simplicity we will assume that \( \xi_{n,i} \) and \( \nu_{n,i} \) are independent for all \( i = 1, \ldots, n \). We will also assume that \( \xi_{n,i} \) are identically distributed, and that the unconditional mean satisfies \( \mathbb{E}[\xi_{n,i}] < \infty \), for all \( n \geq 1 \) and \( i \leq n \). Similarly, for \( \nu_{n,i} \) we assume i.i.d. random variables with \( \mathbb{E}[\nu_{n,i}] = 1 \) and variance \( \sigma^2 \) for all \( n \geq 1 \).

As in Section 4, we will focus on the sub-array, \( \{\Xi_n, n \geq 1\} = \{((\xi_{n,1}, \nu_{n,1}), \ldots, (\xi_{n,n}, \nu_{n,n})), n \geq 1\} \), and define the corresponding conditioned measures \( \{\mathbb{P}_n, n \geq 1\} \) and the joint distribution \( \bar{\mathbb{P}} \) as in Section 4.1. Observe that the independence of \( \{\xi_{n,l}, l \leq n\} \) and \( \{\nu_{n,l}, l \leq n\} \) implies that \( \mathbb{P}_n((\xi_{n,1}, \ldots, \xi_{n,n}) \in d\mathbf{x}, (\nu_{n,1}, \ldots, \nu_{n,n}) \in d\mathbf{z})\) is \( \mathbb{P}_n((\xi_{n,1}, \ldots, \xi_{n,n}) \in d\mathbf{x}) \mathbb{P}_n((\nu_{n,1}, \ldots, \nu_{n,n}) \in d\mathbf{z}) \).

Now, for the \( n \)th row \( \Xi_n \), we define the process

\[
\Gamma_n(t) := \frac{1}{n} \sum_{i=1}^{[nt]} \nu_{n,i} - \sum_{i=1}^{[nt]} \xi_{n,i} \forall t \in [0, 1].
\]

Then, the workload process is defined as

\[
\Phi(\Gamma_n) := \Gamma_n + \Psi(\Gamma_n),
\]

where \( \Psi(\Gamma_n)(\cdot) := \sup_{0 \leq s \leq (\cdot)} (-\Gamma_n(s))_+ \) is the Skorokhod regulator function.

**Proposition 5.1.** Conditional on the sequence of events \( \mathcal{A}_n := \{\sum_{i=1}^{n} \xi_{n,i} \leq 1 < \sum_{i=1}^{n+1} \xi_{n,i}\} \ n \geq 1 \), we have

(i) \( \Gamma_n \to 0 \) in \( (\mathcal{D}, \mathcal{U}) \) \( \bar{\mathbb{P}} \)-a.s. as \( n \to \infty \) and \( \Phi(\Gamma_n) \to 0 \) in \( (\mathcal{D}, \mathcal{U}) \) \( \bar{\mathbb{P}} \)-a.s. as \( n \to \infty \).
(ii) $\sqrt{n}\Gamma_n \Rightarrow \bar{P} W - W^0$ in $(\mathcal{D}, U)$ as $n \to \infty$ and $\sqrt{n}\Phi(\Gamma_n) \Rightarrow \bar{P} \Phi(W - W^0)$ in $(\mathcal{D}, U)$ as $n \to \infty$, where $W$ is a Brownian motion with zero drift and diffusion coefficient equal to $\sigma$, and $W^0$ is the Brownian bridge process defined in Proposition 4.2.

Proof. First, observe that if $\Gamma_n \to 0$ in $(\mathcal{D}, U)$ $\bar{P}$-a.s. and $\sqrt{n}\Gamma_n \Rightarrow \bar{P} W - W^0$ in $(\mathcal{D}, U)$ as $n \to \infty$, then the convergence for the workload process follows automatically from the continuity of the Skorokhod regulator map, $\Phi(\cdot)$.

From the FSLLN \[5\], Chapter 5] we have,

$$\frac{1}{n} \sum_{i=0}^{\lfloor nt \rfloor} \nu_{n,i} \to e \text{ in } (\mathcal{D}, U) \bar{P} \text{- a.s. as } n \to \infty,$$

and from the FCLT \[5\], Chapter 5],

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=0}^{\lfloor nt \rfloor} \nu_{n,i} - e \right) \Rightarrow \bar{P} W \text{ as } n \to \infty.$$

On the other hand, Proposition \[4.1\] and Proposition \[4.2\] imply that

$$\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \xi_{n,i} \to e \text{ in } (\mathcal{D}, U) \bar{P} \text{- a.s. as } n \to \infty$$

and

$$\sqrt{n} \left( \frac{1}{n} \sum_{l=1}^{\lfloor nt \rfloor} \xi_{n,l} - e \right) \Rightarrow \bar{P} W^0 \text{ in } (\mathcal{D}, U) \text{ as } n \to \infty.$$

Note that, for the latter result we also use the fact that $\mu_n \leq 1$ implies that

$$\sqrt{n} \left( \lfloor n \cdot \rfloor \mu_n - e \right) \leq \frac{1}{\sqrt{n}} \left( \lfloor n \cdot \rfloor - ne \right) \to 0 \text{ in } (\mathcal{D}, U) \text{ as } n \to \infty.$$

The continuity of the difference operator in the metric space $(\mathcal{D}, U)$ implies that

$$\Gamma_n \to 0 \text{ in } (\mathcal{D}, U) \bar{P} \text{- a.s. as } n \to \infty,$$

and

$$\sqrt{n}\Gamma_n = \sqrt{n} \left( \frac{1}{n} \sum_{l=0}^{\lfloor nt \rfloor} \left( \nu_{n,l} - \xi_{n,l} \right) \right) \Rightarrow \bar{P} \left( W - W^0 \right) \text{ in } (\mathcal{D}, U) \text{ as } n \to \infty.$$

□
5.1. Comparison with conventional heavy-traffic approximation

We begin by observing that the Brownian bridge limit does not assume a so-called “heavy-traffic condition” as is the case in standard heavy-traffic approximations. The standard heavy-traffic condition for a sequence of queueing models indexed by \( n \) having arrival \( \lambda_n \) and service rate \( \mu_n \) assumes that

\[
\sqrt{n} \left( \frac{\lambda_n}{n} - \frac{\mu_n}{n} \right) \to \theta \in \mathbb{R} \text{ as } n \to \infty.
\]

If \( \theta < 0 \), then the load factor \( \rho_n := \frac{\lambda_n}{\mu_n} < 1 \), implying that the sequence of models are ‘underloaded’; that is, in the long-term the workload process remains bounded. On the other hand, if \( \theta > 0 \), then \( \rho_n > 1 \) and the sequence of models are ‘overloaded’. In either case, however, \( \lim_{n \to \infty} \rho_n = 1 \). In other words, for large enough \( n \) there are many arrivals, but approximately a similar order of service completions as well. Note that we are not assuming the limit diffusion approximation has a steady state, hence considering \( \theta \geq 0 \) is acceptable in the current analysis.

The workload approximation for a GI/GI/1 queue can be developed by assuming \( l_n = n \) and \( \mu_n = n - \theta \sqrt{n} \). Let the sequence of i.i.d. random variables \( \{\nu_i, i \geq 1\} \) and \( \{\xi_i, i \geq 1\} \) represent the service times and inter-arrival times (respectively). Assume that \( \mathbb{E}[\xi_1] = 1 \) and \( \mathbb{E}[\nu_1] = n/\mu_n = n/(n - \theta \sqrt{n}) \) in the \( n \)th system. We define the workload process \( \Phi(\Gamma_n) \) following (5.2), with

\[
\Gamma_n(t) = \left( \frac{1}{n} \sum_{i=0}^{nt} \nu_i - \frac{nt}{\mu_n} \right) - \left( \frac{1}{n} \sum_{i=0}^{nt} \xi_i - \frac{nt}{l_n} \right) + nt \left( \frac{1}{\mu_n} - \frac{1}{l_n} \right),
\]

where we assume that \( \nu_0 = \xi_0 = 0 \) a.s. Note that we do not define a triangular array anymore, since \( \{\Gamma_n\} \) can be considered as a single sequence of stochastic processes. However, there are versions of the heavy-traffic approximation where triangular arrays can be used \[4, 27\].

Observe that \( n^{3/2} \left( \frac{1}{\mu_n} - \frac{1}{l_n} \right) \to \theta \) as \( n \to \infty \). This together with the FCLT \[5\] Chapter 5] implies that \( \sqrt{n}\Gamma_n \Rightarrow \theta e + W - W' \) as \( n \to \infty \), where \( W \) and \( W' \) are Brownian motion processes corresponding to FCLTs for the service times and inter-arrival time sequences (respectively) and \( e : \mathbb{R} \to \mathbb{R} \) is the identity map, as before. It follows that

\[
\sqrt{n}\Phi(\Gamma_n) \Rightarrow \Phi(\theta e + W - W') \text{ as } n \to \infty.
\]
Thus, the diffusion approximation in this case is a reflected Brownian motion. In contrast, the limit process in Proposition [5.1] (ii) is a reflected Brownian bridge process explicitly capturing a ‘depleting points effect,’ in the sense that as the day progresses, there are fewer and fewer remaining jobs to arrive; see [2] for a rigorous definition. This effect is a consequence of conditioning on the number of arrivals in the horizon.

From an operational analysis perspective, the ‘depletion of points’ effect also implies that the increments of the workload process display long-range correlations (if $\Theta(n)$ arrivals occur in $[0, t)$, then there is necessarily few arrivals in the remaining time). These effects are not present in the standard $GI/GI/1$ heavy-traffic analysis. Furthermore, these effects can affect operational decisions, and the choice of model is crucial and application dependent. On the other hand, from a simulation perspective, the main results in this paper suggest approximations that can be easily simulated in the conduct of ‘what if’ type simulation analyses of $GI/GI/1$ queueing models. For instance, it is of interest to ask what the distribution of the workload is going to be at time $T (= 1)$ conditioned on $A_n$. Simulating the workload process in Proposition [5.1] to compute this is quite straightforward.

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