ON A DISTINGUISHED FAMILY OF RANDOM VARIABLES AND PAINLEVÉ EQUATIONS

Theodoros Assiotis, Benjamin Bedert, Mustafa Alper Gunes and Arun Soor
ON A DISTINGUISHED FAMILY OF RANDOM VARIABLES AND PAINLEVÉ EQUATIONS

THEODOROS ASSIOTIS, BENJAMIN BEDERT, MUSTAFA ALPER GUNES AND ARUN SOOR

A family of random variables $X(s)$, depending on a real parameter $s > -\frac{1}{2}$, appears in the asymptotics of the joint moments of characteristic polynomials of random unitary matrices and their derivatives (Assiotis et al. 2020), in the ergodic decomposition of the Hua–Pickrell measures (Borodin and Olshanski 2001 and Qiu 2017), and conjecturally in the asymptotics of the joint moments of Hardy’s function and its derivative (Hughes 2001 and Assiotis et al. 2020). Our first main result establishes a connection between the characteristic function of $X(s)$ and the $\sigma$-Painlevé III’ equation in the full range of parameter values $s > -\frac{1}{2}$. Our second main result gives the first explicit expression for the density and all the complex moments of the absolute value of $X(s)$ for integer values of $s$. Finally, we establish an analogous connection to another special case of the $\sigma$-Painlevé III’ equation for the Laplace transform of the sum of the inverse points of the Bessel point process.

1. Introduction

1A. Motivation. We first give the precise definition of the random variables $X(s)$, with $s > -\frac{1}{2}$, as principal value sums of certain determinantal point processes. We then elaborate on three distinct reasons why someone would be interested in them.

Definition 1.1. Let $s \in \mathbb{R}$ and $s > -\frac{1}{2}$. Let $C^{(s)}$ then be the determinantal point process on $\mathbb{R}^* = (-\infty, 0) \cup (0, \infty)$ with correlation kernel

$$K^{(s)}(x, y) = \frac{1}{2\pi} \frac{(\Gamma(s + 1))^2}{\Gamma(2s + 1)\Gamma(2s + 2)} \frac{T^{(s)}(x)R^{(s)}(y) - T^{(s)}(y)R^{(s)}(x)}{x - y},$$

where $\sum$ is over all $n$-tuples of pairwise distinct points of the point configuration $X \in \text{Conf}(E)$; see [Borodin 2011].
where $T^{(s)}(x), R^{(s)}(x)$ are given by the formulas

$$T^{(s)}(x) = 2^{2s-\frac{1}{4}} \Gamma(s + \frac{1}{2}) \cdot \frac{1}{|x|^{\frac{1}{2}}} J_{s-1/2}(\frac{1}{|x|}),$$

$$R^{(s)}(x) = 2^{2s+\frac{1}{4}} \Gamma(s + \frac{3}{2}) \cdot \frac{1}{|x|^{\frac{1}{2}}} J_{s+1/2}(\frac{1}{|x|}),$$

where $J_\nu$ denotes the Bessel function with parameter $\nu$. Then, $X(s)$ is defined to be the following principal value sum of the points of $C^{(s)}$, shown to be well defined by the results of Qiu [2017]:

$$X(s) = \lim_{m \to \infty} \left[ \sum_{x \in C^{(s)}} x \mathbb{1}_\mathbb{R}(|x| > \frac{1}{m^2}) \right].$$

Joint moments of characteristic polynomials of random unitary matrices. Let $U \in \mathbb{U}(N)$ be a Haar-distributed random matrix, where $\mathbb{U}(N)$ is the group of $N \times N$ unitary matrices, and let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ denote its eigenvalues.

Define the characteristic polynomial of $U$,

$$S_U(\theta) = \det(I - e^{-i\theta} U)$$

and consider

$$G_U(\theta) = e^{\frac{iN}{2}(\theta + \pi)} - i \sum_{k=1}^N \frac{i}{\pi} S_U(\theta),$$

so that $|S_U(\theta)| = |G_U(\theta)|$ and $G_U(\theta)$ is real-valued for $\theta \in [0, 2\pi)$.

Let $s \in \mathbb{R}$ and $s > -\frac{1}{2}$. Then we define, for $-\frac{1}{2} < h < s + \frac{1}{2}$, the following quantities, that we call the joint moments:

$$R_N(s, h) = \int_{\mathbb{U}(N)} |G_U(0)|^{2s-2h} \left| \frac{dG_U}{d\theta} \right|_{\theta=0}^{2h} d\mu_N(U),$$

where $d\mu_N(U)$ is the Haar probability measure on the group of unitary matrices $\mathbb{U}(N)$.

In his thesis, Hughes [2001], partly motivated by connections with number theory that we will say more about below, made the following conjecture about the asymptotics of the joint moments:

$$\frac{R_N(s, h)}{N^{n^2+2h}} \xrightarrow{N \to \infty} R(s, h),$$

for some unidentified (for generic real values of the exponents) quantity $R(s, h)$. The conjecture was proven for $s \in \mathbb{N}^2$ and $h \in \mathbb{N}$ or $h \in \mathbb{N} - \frac{1}{2}$ in a number of works, using a variety of methods, and different expressions for $R(s, h)$ (for integer or half-integer parameters\(^3\)) were obtained; see [Bailey et al. 2019; Basor et al. 2019; Conrey et al. 2006; Dehaye 2008; 2010; Hughes 2001; Winn 2012] for more details.

Recently, by employing a more probabilistic approach in [Assiotis et al. 2020] the conjecture was proven for general real values of the exponents $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$. For these parameter values

\[^3\text{Some of these expressions for } R(s, h) \text{ made sense for noninteger values of } s \text{ as well. However, all of these formulae required the parameter } h \text{ to be an integer or a half-integer in order to make sense.} \]
the main result of [Assiotis et al. 2020] reads as follows:

$$\lim_{N \to \infty} \frac{R_N(s, h)}{N^{s^2 + 2h}} = R(s, h) = \frac{G(s + 1)^2}{G(2s + 1)} 2^{-2h} E(|X(s)|^{2h}).$$

(6)

Here, $G$ denotes the Barnes G-function, given by

$$G(1 + z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2(1 + \gamma)}{2}\right) \prod_{j=0}^{\infty} \left(1 + \frac{z}{j}\right)^j \exp\left(\frac{z^2}{2j} - z\right)$$

(7)

where $\gamma$ is the Euler–Mascheroni constant. Our results below will lead to the first explicit evaluation of $R(s, h)$ for generic real values of $h$ when the parameter $s$ is an integer.

**Joint moments of Hardy’s function.** As mentioned above, part of Hughes’ motivation in studying the asymptotics of the joint moments $R_N(s, h)$ was to obtain a precise conjecture for the asymptotics of the joint moments of Hardy’s function $Z$, defined as

$$Z(y) = \pi^{-\frac{1}{2}iy} \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}iy\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}iy\right)} \zeta\left(\frac{1}{2} + iy\right).$$

(8)

where $\zeta$ denotes the Riemann zeta function.

Building on the seminal work of Keating and Snaith [2000a; 2000b], where the connection between moments of the Riemann zeta function and moments of characteristic polynomials was first understood, Hughes [2001] conjectured the following; see also [Hall 2002]:

$$\frac{1}{x} \int_0^x |Z(y)|^{2s - 2h} \left|\frac{dZ}{dy}\right|^{2h} dy \sim a(s) R(s, h)(\log(x))^{s^2 + 2h},$$

(9)

as $x \to \infty$ where the arithmetic factor $a(s)$ is given by

$$a(s) := \prod_{\text{primes } p} \left(1 - p^{-1}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} p^{-k} \left(\frac{\Gamma(k + s)}{\Gamma(k + 1)\Gamma(s)}\right)^2.$$  

(10)

The conjecture agrees with rigorous results of Hardy and Littlewood [1916] for $s = 1, h = 0$, Ingham [1927] for $s = 2, h = 0$ and $s = 1, h = 1$, Conrey [1988] for $s = 2, h = 1$ and $s = 2, h = 2$ and Conrey and Ghosh [1984] for $s = 1, h = \frac{1}{2}$. Hughes [2001, §6.3.] also stated an analogous conjecture for the joint moments of the Riemann zeta function itself and showed that for $s, h \in \mathbb{N}$ the two conjectures are equivalent; in particular it is possible to obtain one leading order coefficient from the other; see [Hughes 2001, 6.105].

The formula (6), expressing $R(s, h)$ in terms of the moments of $X(s)$, thus leads to the following refinement of (9), as given in [Assiotis et al. 2020]:

$$\frac{1}{x} \int_0^x |Z(y)|^{2s - 2h} \left|\frac{dZ}{dy}\right|^{2h} dy \sim a(s) \frac{G(s + 1)^2}{G(2s + 1)} 2^{-2h} E(|X(s)|^{2h})(\log(x))^{s^2 + 2h},$$

(11)

---

4Here we use the conventional notation $f(x) \sim g(x)$ to denote asymptotic equivalence, i.e., that $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$. 

---
valid for \( s > -\frac{1}{2} \) and \( h \in [0, s + \frac{1}{2}) \). The results of this paper lead to a further refinement of the conjecture above by explicitly evaluating the right-hand side of (11) when the parameter \( s \) is an integer.

**Ergodic decomposition of the Hua–Pickrell measures.** Lastly, the random variable \( X(s) \) arises naturally in the problem of ergodic decomposition of the Hua–Pickrell measures on the space of infinite Hermitian matrices \( \mathbb{H}(\infty) \).\(^5\) Indeed, this was the first setting [Borodin and Olshanski 2001; Qiu 2017] in which \( X(s) \) appeared.

The classification of the ergodic measures invariant under the action of \( \mathbb{U}(\infty) \) by conjugation was derived by Pickrell [1991] and Olshanski and Vershik [1996]. These measures can be parametrized by the following space of real parameters \( \Omega \subset \mathbb{R}^{2\infty+2} \):

\[
\Omega := \left\{ \omega = \left( \{ \alpha_n^+ \}_{n \in \mathbb{N}}, \{ \alpha_n^- \}_{n \in \mathbb{N}}, \gamma_1, \gamma_2 \right) \in \mathbb{R}^{2\infty+2} : \gamma_2 \geq 0, \ \alpha_n^+ \geq 0 \text{ and } \alpha_n^+ \geq \alpha_n^- \right\}.
\]

The parameters of a point \( \omega = (\{ \alpha_n^+ \}_{n \in \mathbb{N}}, \{ \alpha_n^- \}_{n \in \mathbb{N}}, \gamma_1, \gamma_2) \in \Omega \) do have a concrete meaning\(^7\) as explained in [Olshanski and Vershik 1996; Borodin and Olshanski 2001]. Moreover, explicit expressions for the characteristic functions for these ergodic measures \( M_\omega \), \( \omega \in \Omega \), are also known (which uniquely determine them); see, for example, [Olshanski and Vershik 1996] and [Borodin and Olshanski 2001].

A distinguished family of probability measures on \( \mathbb{H}(\infty) \) were constructed in [Borodin and Olshanski 2001], depending on a parameter \( s \in (-\frac{1}{2}, \infty) \), with the property that when projected onto the top-left \( N \times N \) submatrix, the classical generalised Cauchy, also known as Hua–Pickrell, ensemble is recovered\(^8\) and we will say more about this in Section 2A. Moreover, it was demonstrated in [Borodin and Olshanski 2001] that for this Hua–Pickrell measure, which we denote here by \( \mathcal{M}^{(s)} \) (and indeed any \( \mathbb{U}(\infty) \)-invariant probability measure on \( \mathbb{H}(\infty) \)), there is a unique probability measure \( \mu^{(s)} \) on \( \Omega \) describing its decomposition into the ergodic measures:

\[
\mathcal{M}^{(s)}(dH) = \int_{\Omega} \mu^{(s)}(d\omega)M_\omega(dH).
\]

---

\(^5\)This is defined as the projective limit of the spaces of finite Hermitian matrices \( \mathbb{H}(N) \) under the natural projections \( \mathbb{H}(N+1) \to \mathbb{H}(N) \) given by restriction to the top left \( N \times N \) submatrix.

\(^6\)This is defined as the inductive limit of the finite unitary groups \( \mathbb{U}(N) \) under the natural inclusions \( \mathbb{U}(N) \leftrightarrow \mathbb{U}(N+1) : U \mapsto \text{diag}(U, 1) \).

\(^7\)Informally, for \( \omega = (\{ \alpha_n^+ \}_{n \in \mathbb{N}}, \{ \alpha_n^- \}_{n \in \mathbb{N}}, \gamma_1, \gamma_2) \in \Omega \) the \( \alpha \) parameters are asymptotic (as \( N \to \infty \)) normalised eigenvalues, \( \gamma_1 \) is the asymptotic normalised trace and \( \gamma_2 \) is closely related to the asymptotic normalised sum of squares of eigenvalues of the \( N \times N \) top left submatrix of a random \( M_\omega \)-distributed matrix on \( \mathbb{H}(\infty) \) (where \( M_\omega \) is the ergodic measure parametrized by \( \omega \in \Omega \)). All these limits exist \( M_\omega \)-almost surely and are deterministic and coincide with the numerical values of the parameters of \( \omega \); see [Olshanski and Vershik 1996; Borodin and Olshanski 2001] for more details and proofs.

\(^8\)See [Bourgade et al. 2009; 2013; Bufetov and Qiu 2016; Neretin 2015; Qiu 2017; Borodin and Olshanski 2001; Hua 1979] for information on this ensemble and the related Hua–Pickrell measures; see [Pickrell 1987; Bufetov 2015; 2016a; 2016b; Assiotis 2019] for more information on similar measures; see [Witte and Forrester 2000] for a relation to Painlevé transcendents; see [Assiotis 2018; 2020] for relations to stochastic processes.
**Remark.** A more precise version of (13) is the following. Both $\mathbb{H}(\infty)$ and $\Omega$ are Borel spaces; see [Olshanski and Vershik 1996] and [Borodin and Olshanski 2001] for the details. Then, for any bounded Borel function $\mathfrak{F}$ on $\mathbb{H}(\infty)$ we have

$$M^{(s)}(\mathfrak{F}) = \int_\Omega \mu^{(s)}(d\omega) M_\omega(\mathfrak{F}),$$

where $M^{(s)}(\mathfrak{F})$ and $M_\omega(\mathfrak{F})$ are the integrals of $\mathfrak{F}$ with respect to $\mathfrak{M}^{(s)}$ and the ergodic measure $M_\omega$, respectively and for any such $\mathfrak{F}$ the function $\omega \mapsto M_\omega(\mathfrak{F})$ is a Borel function on $\Omega$; see in particular Sections 4 and 9 in [Borodin and Olshanski 2001] for further details and proofs.

Borodin and Olshanski [2001] were able to explicitly describe the distribution of the parameters $\{\alpha_\pm_n\}_{n \in \mathbb{N}}$ under $\mu^{(s)}$. However, the problem of determining the distribution of $\gamma_1$ and $\gamma_2$ under $\mu^{(s)}$ was unresolved for many years. In an important work, Qiu [2017] proved that almost surely $\gamma_2 = 0$ and that $\gamma_1$ is precisely $X^{(s)}$.

**1B. Main results.**

**Painlevé equation.** From the discussion above it is evident that understanding the family of random variables $\{X^{(s)}\}_{s \in (-\frac{1}{2}, \infty)}$ is important. A number of results, including explicit combinatorial formulae for the even moments of these random variables, were proven in [Assiotis et al. 2020]. Moreover, one of the results in [Assiotis et al. 2020] established a connection between the characteristic function of $X^{(s)}$ and a special case of the $\sigma$-Painlevé III' equation. It was however, for a reason that we explain below, restricted to integer parameters $s$ and the equation was only shown to hold in a small interval around the origin. Our first main result removes both of these restrictions and thus establishes the connection with Painlevé for the full range of parameter values $s > -\frac{1}{2}$.

**Theorem 1.2.** Let $s \in \mathbb{R}$ and $s > -\frac{1}{2}$. Define

$$\phi^{(s)}(t) = \mathbb{E}(e^{itX^{(s)}})$$

and the associated function

$$\tau^{(s)}(t) := t \frac{d}{dt} \log \phi^{(s)}(t).$$

Then $\tau^{(s)}(t)$ is $C^\omega$ on $\mathbb{R}^*$ and is a solution to a special case of the $\sigma$-Painlevé III' equation for $t \in \mathbb{R}^*$:

$$\left(t \frac{d^2 \tau^{(s)}}{dt^2}\right)^2 = -4t \left(\frac{d \tau^{(s)}}{dt}\right)^3 + (4s^2 + 4\tau^{(s)}) \left(\frac{d \tau^{(s)}}{dt}\right)^2 + t \frac{d \tau^{(s)}}{dt} - \tau^{(s)}.$$  \hfill (16)

Moreover, we have the boundary conditions

$$\begin{align*}
\tau^{(s)}(0) &= 0, \quad \text{for } s > 0, \\
\frac{d}{dt} \tau^{(s)}(t)|_{t=0} &= 0, \quad \text{for } s > \frac{1}{2}.
\end{align*}$$  \hfill (17)

\footnote{We use the notation $C^\omega$ to denote the space of real analytic functions.}
Remark 1.3. By using the well-known relation between derivatives of the characteristic function of a random variable and its positive integer moments, along with formula (6), Theorem 1.2 readily gives the following expression for $R(s, h)$, for $h \in \mathbb{N}$ and any $s > h - \frac{1}{2}$ (recall that this restriction is necessary for the joint moments to exist):

$$R(s, h) = (-1)^h \frac{G(s + 1)^2}{G(2s + 1)} \frac{d^{2h}}{dt^{2h}} \left[ \exp \left( \int_0^t \frac{\tau(s)(u)}{u} \, du \right) \right] \bigg|_{t=0},$$

where the function $\tau(s)$ solves the Painlevé equation in Theorem 1.2. This generalises to noninteger parameter $s$ the formula (1.16) of Theorem 2, in [Basor et al. 2019], which established this result for $s, h \in \mathbb{N}$ (with $s > h - \frac{1}{2}$).

For noninteger $h$ one could in principle use Fourier inversion to obtain the probability density\(^{10}\) of $X(s)$ (we note that $X(s)$ is a symmetric random variable and thus its density is even), then integrate to get the (fractional) moments $\mathbb{E}(|X(s)|^{2h})$ and thus obtain an integral expression for $R(s, h)$ in terms of this Painlevé transcendent. However, in order to obtain an explicit formula for $R(s, h)$ one would need as a starting point explicit exact\(^{11}\) expressions for $\phi(s)$. These exist for $s \in \mathbb{N} \cup \{0\}$ (the corresponding functions $\tau(s)$ are so-called classical solutions of the $\sigma$-Painlevé III equation; see [Umemura 1990; Umemura and Watanabe 1998; Murata 1995] for the precise definition of this terminology) and this is really the essence of Theorems 1.4 and 1.5 below.

We prove Theorem 1.2 by a limiting procedure, making use of the connection to the finite $N$ Hua–Pickrell measures that we will explain in Section 2A. This strategy was also employed in [Assiotis et al. 2020], however there a result from [Basor et al. 2019] was used as input that required $s$ to be an integer.\(^{12}\) The reason the result from [Basor et al. 2019] was restricted to integer parameters $s$ was because the proof was using as a starting point a certain formula of Winn [2012] involving an $s \times s$ determinant with entries given by Laguerre polynomials.

The starting point of our work is the observation that it is possible to use instead a different formula from [Winn 2012], that we recall in Proposition 2.2 below, valid for all $s > -\frac{1}{2}$, which gives rise to a Hankel determinant of a deformed Laguerre weight. This Hankel determinant formula is well-adapted for an application of the so-called ladder operator method; see [Chen and Its 2010; Basor and Chen 2015], which gives as output a Painlevé representation. We initially found that this computation was performed in the applied mathematics literature in [Chen and McKay 2012] in relation to applications to wireless communications (with some restrictions on the parameters which can be removed by a short analytic continuation argument). However, as pointed out to us by one of the referees, an equivalent result had in

\(^{10}\)This implicitly assumes that $\phi(s) \in L^1(\mathbb{R})$, which we expect to be true in general and in fact show for $s \in \left( -\frac{1}{2}, 0 \right) \cup \mathbb{N}$ in the course of proving our results, so that the probability density of $X(s)$ with respect to the Lebesgue measure exists. However, even without this assumption it is still possible to write down a more involved formula or alternatively one could use one of the numerous known expressions which give the fractional moments of the absolute value of a random variable in terms of its characteristic function, all of which involve some kind of integral operator.

\(^{11}\)Namely, not simply asymptotic expansions.

\(^{12}\)The second restriction, namely that the equation only holds on an interval is due to the fact that a priori it is not clear whether $\phi(s)$ is nonvanishing on the real line. This requires separate arguments that we present in Section 2B.
fact already been obtained a decade before [Chen and McKay 2012] by Forrester and Witte [2002] by a
different method that uses Okamoto’s $\tau$-function theory of the Painlevé equations [1987]. We say a bit
more about the history before the statement of Proposition 2.3.

Now, in order to take the $N \to \infty$ limit our arguments are completely different to the ones in [Assiotis
et al. 2020]. They are complex analytic in nature, while the ones in [Assiotis et al. 2020] are mainly
based on convergence of moments\(^{13}\) (and thus closely related to real analysis). Making use of the power
of complex analysis allows us to circumvent a number of technical issues that arise in the proof.

Although the statement of Theorem 1.2 is complete, as it covers the full parameter range, it would be
very desirable to have an alternative direct proof from the definition of $X(s)$ using Fredholm determinants.

Finally, using similar arguments we establish in Section 3 a connection to another special case of the
$\sigma$-Painlevé III’ equation for the Laplace transform of the sum of inverse points of the Bessel point process.

Explicit expressions for integer $s$. We now obtain explicit expressions for the density and all the complex
moments of the absolute value of the random variables $X(s)$, when the parameter $s$ is an integer. This
leads to the first explicit evaluation of $R(s, h)$ for general real values of $h$, when $s$ is an integer. When on
the other hand $h$ is an integer while $s$ is a general real number, then explicit expressions for $R(s, h)$ already
exist in the literature. The case $h = 0$ (in particular $X(s)$ does not appear) is due to the seminal work of
Keating and Snaith [2000b]. For integer $h \geq 1$ the reader is referred to [Assiotis et al. 2020] for more details.

We first consider the case $s = 0$ which is in some sense exceptional. We show that $X(0)$ is actually a
Cauchy random variable. Moreover, we extend the convergence of the joint moments of the characteristic
polynomial $R_N(0, h)$ to complex $h$ and cover the full range $\Re(h) \in (-\frac{1}{2}, \frac{1}{2})$. It is a remarkable fact that
there is no need to take a large $N$ limit but rather we simply have $N^{-2h}R_N(0, h) = R(0, h)$, for all $N \geq 1$.

**Theorem 1.4.** The random variable $X(0)$ is Cauchy distributed, namely, it has probability density with
respect to the Lebesgue measure given by

$$
\frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.
$$

(18)

Moreover, if $\Re(h) \in (-\frac{1}{2}, \frac{1}{2})$, then

$$
\frac{R_N(0, h)}{N^{2h}} = R(0, h) = 2^{-2h} \frac{1}{\cos(\pi h)}, \quad \text{for all } N \geq 1.
$$

(19)

We now turn our attention to the case $s \in \mathbb{N}$. First, we recall the standard definition of hypergeometric
functions:

$$
\binom{a_1, \ldots, a_p}{b_1, \ldots, b_q} ; z = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!},
$$

where $(a)_j$ denotes the Pochhammer symbol given by $(a)_j := \prod_{i=1}^{j} (a + i - 1)$ and $(a)_0 := 1$. In the
statement and proof of Theorem 1.5, we will slightly abuse notation by writing, for any $h$ such that the

\(^{13}\)It is important to note that we do not say anything new regarding convergence of the moments in this paper, except for the
special case $s = 0$ that we solve explicitly in Theorem 1.4.
right-hand side exists,
\[ \mathcal{R}(s, h) = \frac{G(s + 1)^2}{G(2s + 1)} 2^{-2h} \mathbb{E}(X(s)^{2h}). \] 
(20)

As explained earlier, from the main result of [Assiotis et al. 2020], this expression coincides with the original definition of \( \mathcal{R}(s, h) \) as the limit of the rescaled joint moments \( \lim_{N \to \infty} N^{-s^2-2h} \mathcal{R}_N(s, h) \) in the range \( h \in [0, s + \frac{1}{2}] \).

**Theorem 1.5.** The random variable \( X(1) \) has probability density with respect to the Lebesgue measure given by
\[ -1 + e^{\frac{2}{1+ix}} \cos \frac{2x}{1+ix}, \quad x \in \mathbb{R} \] 
(21)
and hence for \( \mathfrak{R}(h) \in (-\frac{1}{2}, \frac{3}{2}) \) we have that
\[ \mathcal{R}(1, h) = 2^{-2h} \frac{1}{\cos(\pi h)} \, {}_1F_1\left(-\frac{2h}{2}; 2\right). \] 
(22)

The random variable \( X(2) \) has probability density with respect to the Lebesgue measure given by
\[ \frac{1}{\pi} \times \mathfrak{R}\left(\frac{1}{1-ix} \, {}_2F_2\left[\frac{5}{2}, 1; 1\right], \frac{8}{1-ix}\right), \quad x \in \mathbb{R} \] 
(23)
and hence for \( \mathfrak{R}(h) \in (-\frac{1}{2}, \frac{5}{2}) \) we have that
\[ \mathcal{R}(2, h) = 2^{-2h} \frac{1}{12 \cos(\pi h)} \, {}_2F_2\left[\frac{5}{2}, -2h; 5, 4; 8\right]. \] 
(24)

Finally, for all \( s \in \mathbb{N} \), we have the following general expression for the density of \( X(s) \) with respect to the Lebesgue measure:
\[ \rho^{(s)}(x) = (-1)^{s(s-1)/2} \frac{G(2s + 1)}{G(s + 1)^2} 2\pi \]
\[ \times \mathfrak{R}\left(\sum_{k=0}^{\infty} \left[ \sum_{k_1+\ldots+k_s=k} \det \left[ \frac{1}{(k_i+i+j-1)!} \right]_{i,j=1,\ldots,s} \, \left( \frac{2}{1-ix} \right)^{k+1} \right], \quad x \in \mathbb{R}. \] 
(25)

Hence, for \( s \in \mathbb{N}, \, \mathfrak{R}(h) \in (-\frac{1}{2}, s + \frac{1}{2}) \) we have
\[ \mathcal{R}(s, h) = (-1)^{s(s-1)/2} 2^{-2h} \frac{1}{\cos(\pi h)} \]
\[ \times \left( \sum_{k=0}^{\infty} \left[ \sum_{k_1+\ldots+k_s=k} \det \left[ \frac{1}{(k_i+i+j-1)!} \right]_{i,j=1,\ldots,s} \, \left( \frac{2}{1-ix} \right)^{k+1} \right], \quad x \in \mathbb{R}. \] 
(26)

\[ ^{14} \text{In particular, in this paper we do not pursue further the problem of convergence of the rescaled joint moments beyond this range, although we do expect it to hold.} \]
Remark 1.6. We note that the general expressions (25) and (26) readily specialize to the ones for \( s = 1 \), namely, (21) and (22). To obtain expressions (23) and (24) for \( s = 2 \) from the general ones we need to take into account a simplification, due to Vandermonde’s identity, that we present in the proof of Theorem 1.5. We also note that the expression in (26) is indeterminate for \( h \in \mathbb{N} - \frac{1}{2} \) and should be understood as a limit of \( h \to m + \frac{1}{2} \) for some \( m \in \mathbb{N} \), which can be computed via L’Hôpital’s rule (see Remark 2.7).

An immediate corollary of Theorems 1.4 and 1.5 is a refinement of conjecture (11) when \( s \) is an integer. We write out the conjecture fully only for \( s = 0, 1, 2 \) since these are the simplest and most elegant cases. The general form of the conjecture can be obtained from (26).

Conjecture 1.7. For \( h \in \left[ 0, \frac{1}{2} \right) \), we have

\[
\frac{1}{x} \int_0^x \left| Z(y) \right|^{-2h} \left| \frac{dZ}{dy} \right|^{2h} dy \sim 2^{-2h} \frac{1}{\cos(\pi h)} (\log(x))^{2h}.
\]  

(27)

For \( h \in \left[ 0, \frac{3}{2} \right) \), we have

\[
\frac{1}{x} \int_0^x \left| Z(y) \right|^{2-2h} \left| \frac{dZ}{dy} \right|^{2h} dy \sim 2^{-2h} \frac{1}{\cos(\pi h)}^2 \, _1F_1 \left[ \frac{-2h}{2} \right] (\log(x))^{1+2h}.
\]  

(28)

For \( h \in \left[ 0, \frac{5}{2} \right) \), we have

\[
\frac{1}{x} \int_0^x \left| Z(y) \right|^{4-2h} \left| \frac{dZ}{dy} \right|^{2h} dy \sim \frac{1}{2\pi^2} 2^{-2h} \frac{1}{\cos(\pi h)^2} \, _2F_2 \left[ \frac{5}{2}, \frac{-2h}{5}, 4; \frac{8}{2} \right] (\log(x))^{4+2h}.
\]  

(29)

The sine process. The sine process, that we denote by \( S \), is the determinantal point process on \( \mathbb{R} \) with correlation kernel given by\(^{15}\) (see [Dyson 1962; Mehta 1991])

\[
J_{\sin}(x, y) = \frac{\sin(x - y)}{x - y}.
\]  

(30)

\( S \) is arguably the most fundamental object in random matrix theory: it is the universal scaling limit of eigenvalues of complex Hermitian random matrices in the bulk of the spectrum; see for example [Johansson 2001; Bleher and Its 1999; Deift et al. 1999] for precise statements. It also has close connections to the pair correlations between zeroes of the Riemann zeta function high up the critical line; see for example [Montgomery 1973] and [Bourgade and Keating 2013].

Here, as an immediate corollary of Theorem 1.4, we obtain an alternative proof of the fact that the principal value sum\(^{16}\) of the inverse points of the sine process is Cauchy distributed. This fact was already established, as a corollary of more general results, by Aizenman and Warzel [2015] using spectral theory methods.

\(^{15}\)Here we have rescaled the process by a factor of \(-\pi\), for aesthetic purposes.

\(^{16}\)The sine process is translation invariant and thus it can be shown that if one simply takes the sum of inverse points without the cutoff then this sum does not converge.
Corollary 1.8. Let $S$ denote the sine process on $\mathbb{R}$. Then, the random variable
\[
\lim_{m \to \infty} \left[ \sum_{y \in S} \frac{1}{y} \mathbb{1}(|y| < m^2) \right]
\]
(31)
is Cauchy distributed, namely with probability density given by (18).

2. Proofs of the main results

2A. Preliminaries. We begin with a number of preliminaries. Let $\mathbb{H}(N)$ denote the linear space of $N \times N$ complex Hermitian matrices. We define, for a parameter $s \in \mathbb{R}$, $s > -\frac{1}{2}$, the Hua–Pickrell measure $\mathcal{M}^{(s)}_N$ on $\mathbb{H}(N)$ as
\[
\mathcal{M}^{(s)}_N(dH) := \text{const} \cdot \det((1 + H^2)^{-s-N}) \times dH,
\]
(32)
where $dH$ is the Lebesgue measure on $\mathbb{H}(N)$ and the constant is chosen so that this is a probability measure on $\mathbb{H}(N)$.

The distribution of the eigenvalues of a random matrix from the ensemble in (32) is given by the following probability measure $m^{(s)}_N$ on $\mathbb{R}^N / S(N)$, where $S(N)$ denotes the $N$-th symmetric group; see [Borodin and Olshanski 2001]:
\[
m^{(s)}_N(dx) := \frac{1}{Z^{(s)}_N} \cdot \Delta(x)^2 \prod_{j=1}^{N} (1 + x_j^2)^{-s-N} dx_j,
\]
(33)
where $\Delta(x)$ is the Vandermonde determinant:
\[
\Delta(x) = \prod_{1 \leq l < k \leq N} (x_k - x_l),
\]
and the normalization constant is given by
\[
Z^{(s)}_N = \pi^N 2^{-N(N+2s-1)} \cdot \prod_{j=0}^{N-1} \frac{j!\Gamma(2s+N-j)}{\Gamma(s+N-j)^2}.
\]
Throughout this paper we denote expectations taken with respect to the measures $\mathcal{M}^{(s)}_N$ and $m^{(s)}_N$ by $\mathbb{E}^{(s)}_N$.

We now make concrete the connection\footnote{In fact this is how the abstract ergodic decomposition results are proven; see [Borodin and Olshanski 2001; Qiu 2017] for more details.} between the Hua–Pickrell measures and the random variable $X(s)$. If $H_N$ is a random matrix distributed according to the probability measure $\mathcal{M}^{(s)}_N$, it was proven by Borodin and Olshanski [2001] that the sequence of random variables $\{ \frac{1}{N} \text{Tr}(H_N) \}_{N \geq 1}$ is convergent in distribution, and by Qiu [2017] that the limiting distribution can be identified with that of $X(s)$, so that we have
\[
\frac{1}{N} \text{Tr}(H_N) \xrightarrow{d} X(s).
\]
(34)
Hence, if we consider the characteristic function of the scaled trace of $H_N$:

$$\phi_N^{(s)}(t) := \mathbb{E}_N^{(s)}(e^{\mu t \operatorname{Tr}(H_N)}) ,$$

we note that by (34) we have that

$$\mathbb{E}_N^{(s)}(e^{\mu t \operatorname{Tr}(H_N)}) \xrightarrow{N \to \infty} \mathbb{E}(e^{\frac{\mu t}{2} X(s)}),$$

uniformly on compact subsets of $\mathbb{R}$. Finally, we define

$$\tau_N^{(s)}(t) := t \frac{d}{dt} \log(\phi_N^{(s)}(t)).$$

2B. Proofs.

**Proposition 2.1.** Let $s \in \mathbb{R}$ and $s > -\frac{1}{2}$. Then, for $t \in \mathbb{R}^+$, $\tau_N^{(s)}(t)$ is a solution to a particular Painlevé V equation:

$$\left(\frac{d^2 \tau_N^{(s)}}{dt^2}\right)^2 = -4t \left(\frac{d \tau_N^{(s)}}{dt}\right)^3 + \left(4s^2 + 4 \tau_N^{(s)} + \frac{t^2}{N^2}\right) \left(\frac{d \tau_N^{(s)}}{dt}\right)^2$$

$$+ t \left(1 + \frac{2s}{N} - \frac{2 \tau_N^{(s)}}{N^2}\right) \frac{d \tau_N^{(s)}}{dt} - \left(1 + \frac{2s}{N} - \frac{\tau_N^{(s)}}{N^2}\right) \tau_N^{(s)}.$$  

Our starting point is the following remarkable integral identity due to [Winn 2012, Proposition 3]. For the convenience of the reader and completeness of the paper we outline Winn’s proof.

**Proposition 2.2 (Winn).** Let $s \in \mathbb{C}$ with $\Re(s) > -\frac{1}{2}$ and $t > 0$. Then,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{e^{itx_j}}{(1 + x_j^2)^{s+N}} \Delta(x)^2 \, dx$$

$$= \frac{\pi N}{2^{(N+2s-1)}N} \prod_{j=0}^{N-1} \frac{1}{\Gamma(s + 1 + j)^2} \cdot e^{-Nt} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{j=1}^{N} (y_j + 2t)^s y_j^s e^{-y_j} \Delta(y)^2 \, dy.$$  

(39)

**Proof.** It suffices to prove the equality up to a constant. The constant can then be obtained by taking the limit $t \to 0$, since both sides then have explicit evaluations using the Selberg integral; see [Forrester and Warnaar 2008].

Using the homogeneity of $\Delta(x)$, rewrite the integral on the left-hand side of (39) as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{e^{itx_j}}{(1 + x_j^2)^{s+1}} \Delta\left(\frac{1}{1 + i x}\right) \Delta\left(\frac{1}{1 - i x}\right) \, dx$$

$$= N! \det \left[ \int_{-\infty}^{\infty} \frac{e^{itx} \, dx}{(1 + i x)^{s+1+k}(1 - i x)^{s+1+j}} \right]_{j,k=0,...,N-1},$$

(40)

where in the last line we have used the Andréief identity. It was then shown in [Winn 2012, 4.15] that we

---

18 Here we use the notation $f(x) = (f(x_1), \ldots, f(x_n))$ to denote the evaluation of a scalar function $f : \mathbb{R} \to \mathbb{R}$ at a vector argument, e.g., $\frac{1}{x} = (\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2})$. 
have the following relation between one dimensional integrals:

\[
\int_{-\infty}^{\infty} \frac{e^{ix} \, dx}{(1 + i x)^{t+k} (1 - i x)^{t-k+1}} = Ce^{-t} \int_{0}^{\infty} y^{s+k} (y + 2t)^{s+k} e^{-y} \, dy,
\]

for a constant $C$ independent of $j$, $k$ and $t$. Therefore, combining the last line of (40) with the right-hand side of (41), and using the Andréief identity once again, noting that $\Delta(y + 2t) = \Delta(y)$, we obtain (39), up to a constant.

**Proof of Proposition 2.1.** By definition, we have that

\[
\phi^{(s)}_{N}(t) = \prod_{j=1}^{N} \Delta(j) \Gamma(2s+j).
\]

Now making use of the integral identity (39), we obtain for $t \geq 0$,

\[
\phi^{(s)}_{N}(t) = \frac{1}{C_{N}^{(s)}} e^{-t/2} \int_{0}^{\infty} \ldots \int_{0}^{\infty} \prod_{j=1}^{N} \Delta(j) \Gamma(2s+j). \quad (43)
\]

where $C_{N}^{(s)}$ is a normalization constant, given by

\[
C_{N}^{(s)} = N! \prod_{j=1}^{N} \Gamma(j) \Gamma(2s+j). \quad (44)
\]

An application of the Andréief identity to (43) yields

\[
\phi^{(s)}_{N}(t) = \frac{N!}{C_{N}^{(s)}} e^{-t/2} \det \left[ \int_{0}^{\infty} y^{j+k} \left( y + \frac{t}{N} \right)^{y} e^{-y} \, dy \right]_{j,k=0,\ldots,N-1},
\]

so that

\[
\tau^{(s)}_{N}(t) = -\frac{t}{2} + t \frac{d}{dt} \log \det \left[ \int_{0}^{\infty} y^{j+k} \left( y + \frac{t}{N} \right)^{y} e^{-y} \, dy \right]_{j,k=0,\ldots,N-1}. \quad (45)
\]

Note that $\phi^{(s)}_{N}(t)$ and hence $\tau^{(s)}_{N}(t)$ are even functions, which can be seen by the change of variables $x_j \mapsto -x_j$ in (42). Taking this into account, a simple calculation reveals that if $\tau^{(s)}_{N}(t)$ satisfies the Painlevé equation (38) for $t > 0$, then it is also a solution for $t < 0$. Thus, we may restrict to $t > 0$, and the result follows immediately as a corollary of Proposition 2.3.

Proposition 2.3, in an equivalent form, is originally due to Forrester and Witte [2002, Proposition 3.2], using Okamoto’s $\tau$-function theory of the Painlevé equations [Okamoto 1987]. A decade later this result was proven again\(^{19}\) using the ladder operator method (see, for example, [Basor and Chen 2015; Basor et al. 2012] for more on this technique) in [Chen and McKay 2012], but with the restriction $\alpha > 0$. The restriction to $\alpha > 0$ is due to the fact that the proofs of certain intermediate results in [Chen and McKay 2012] require a number of integrations by parts which are no longer valid in the range $\alpha < 0$. Here,\(^{19}\) In fact, in the intervening years the result was also studied in the physics literature, see [Osipov and Kanzieper 2010], by yet another method.
instead of simply citing the equivalent proposition from [Forrester and Witte 2002], we give a short analytic continuation argument that extends the result of [Chen and McKay 2012] to the full range of parameters mainly because analogous arguments can also be used to extend the intermediate results in [Chen and McKay 2012], on certain orthogonal polynomials, which are of independent interest.

**Proposition 2.3.** Let

\[ F_N(t; \alpha) = \det \left[ \int_0^\infty y^{j+k} w(y; t, \alpha) \, dy \right]_{j,k=0,\ldots,N-1} \]

and the associated function

\[ H_N(t; \alpha) = t \frac{d}{dt} \log F_N(t; \alpha) \]

where

\[ w(y; t, \alpha) = (y + t)^{\lambda} y^{\alpha} e^{-y}. \]

Then, we have that for \( \alpha > -1 \), \( t > 0 \) and \( \lambda \in \mathbb{R} \),

\[ \left( t \frac{d^2 H_N}{dt^2} \right)^2 = \left( t \frac{d H_N}{dt} - H_N + \frac{d H_N}{dt} (2N + \alpha + \lambda) + N\lambda \right)^2 - 4 \frac{d H_N}{dt} \left( t \frac{d H_N}{dt} - H_N + N(\alpha + \lambda) \right) \left( \frac{d H_N}{dt} + \lambda \right). \]

**Proof.** As mentioned above, in [Chen and McKay 2012] the result was proven for \( \alpha > 0 \) and here we simply extend the validity of (49) to \( \alpha > -1 \) by proving that \( H_N, \frac{d}{dt} H_N \) and \( \frac{d^2}{dt^2} H_N \) can be extended as functions of \( \alpha \), for fixed \( t > 0 \) and \( \lambda \in \mathbb{R} \), analytically to a neighbourhood \( V_N \subseteq \mathbb{C} \) of the part of the real line \((-1, \infty)\). For the remainder of this proof, let \( t > 0 \) and \( \lambda \in \mathbb{R} \) be fixed and arbitrary. Firstly, note that the function

\[ \beta \mapsto \int_0^\infty y^\beta (y + t)^{\lambda} e^{-y} \, dy \]

is holomorphic on \( \Re(\beta) > -1 \), which can be seen by combining Fubini’s and Morera’s theorems. Hence, by using rules for derivatives of determinants and differentiation under the integral sign we get that \( \alpha \mapsto \frac{d^p}{dt^p} F_N(t; \alpha) \) is holomorphic for \( p = 0, 1, 2, 3, \ldots \). Now, by rewriting \( F_N(t; \alpha) \) as in the integral in (43) via the Andréief identity, we see that

\[ F_N(t; \alpha) > 0, \quad \text{for all } \alpha \in (-1, \infty), \]

so that by continuity of \( \alpha \mapsto F_N(t; \alpha) \), there exists \( V_N \) such that \((-1, \infty) \subseteq V_N \subseteq \mathbb{C} \) such that \(|F_N(t; \alpha)| > 0\), for all \( \alpha \in V_N \). Hence, the left-hand side and right-hand side of (49) are two analytic functions on \( V_N \) that agree on \((0, \infty)\) so they must agree on the whole of \( V_N \). The proof for the case \( \alpha > -1 \) is now complete. \( \square \)

In order to prove Theorem 1.2 we will need the following proposition. It is important to note that the use of the obvious candidates to perform these analytic extensions in the proposition below, namely the
expressions as characteristic functions, will not work. These expressions do not exist off the real line due to the fact that the random variables involved only have a finite number of integer moments.

**Proposition 2.4** (analytic continuation). Let \( s \in \mathbb{R} \) and \( s > -\frac{1}{2} \). Then, there exist holomorphic functions \( f_N \) for \( N = 1, 2, \ldots \) and \( f \) on \( \{z \in \mathbb{C} : \Re z > 0\} \), with \( f_N(0) = f(0) = 1 \), such that

\[
\phi_N^{(s)} = f_N|_{\{0, \infty\}} \quad \text{and} \quad \phi^{(s)} = f|_{\{0, \infty\}}.
\]  

(52)

Moreover, for \( \Re z > 0 \), \( p = 1, 2, \ldots \) we have

\[
\lim_{N \to \infty} \frac{d^p}{dz^p} f_N(z) = \frac{d^p}{dz^p} f(z).
\]  

(53)

**Proof of Proposition 2.4.** We use an argument based on Montel’s theorem and Morera’s theorem; see [Stein and Shakarchi 2003]. For \( \{z \in \mathbb{C} : \Re z > 0\} \), we define as in (43):

\[
f_N(z) := \frac{1}{C_N^{(s)}} \cdot \int_0^\infty \cdots \int_0^\infty e^{-z/2} \prod_{j=1}^N \left(y_j + \frac{z}{N}\right)^s y_j^s e^{-y_j} \Delta(y)^2 \, dy.
\]  

(54)

Then, as in (43) we have that \( \phi_N(t) = f_N(t) \) for \( t \in [0, \infty) \). We claim that \( f_N(z) \) is holomorphic on \( \{z \in \mathbb{C} : \Re z > 0\} \). Towards this end, we note that the integrand in (54) is bounded by an integrable function on any compact subset of \( \{z \in \mathbb{C} : \Re z > 0\} \), and hence by Fubini’s theorem, for any closed path \( \gamma \) contained in \( \{z \in \mathbb{C} : \Re z > 0\} \) we have that

\[
\int_\gamma f_N(z) \, dz = 0.
\]  

(55)

where we have used Cauchy’s theorem. Hence, by Morera’s theorem \( f_N(z) \) is holomorphic on \( \{z \in \mathbb{C} : \Re z > 0\} \). We now claim further that the family \( \{f_N\}_{N \geq 1} \) is uniformly bounded on compact subsets of \( \{z \in \mathbb{C} : \Re z > 0\} \). For, if \( K \) is such a compact subset, there is a constant \( M_K > 0 \) such that

\[
|y + \frac{z}{N}| \leq \left| y + \frac{M_K}{N} \right|,
\]

uniformly for \( z \in K, \ y > 0 \) and \( N \geq 1 \). Therefore, in the case \( s > 0 \), it follows from (54) that

\[
|f_N(z)| \leq |e^{-\frac{z}{2}}| \phi_N^{(s)}(M_K) \leq e^{\frac{M_K}{2}},
\]

uniformly for \( z \in K \) and \( N \geq 1 \), where in the last inequality we have used the fact that \( \phi_N^{(s)} \) is a characteristic function (42). In the case \( s < -\frac{1}{2} \), we note that \( |f_N(z)| \leq 1 \) whenever \( \Re z > 0 \), for all \( N \geq 1 \). Hence, by Montel’s theorem the family \( \{f_N\}_{N \geq 1} \) is normal, and so every subsequence has a subsequence converging uniformly on compact sets of \( \{z \in \mathbb{C} : \Re z > 0\} \). As all these limits are holomorphic and agree on \( (0, \infty) \) by (36), they are equal. This property implies that the family \( \{f_N\}_{N \geq 1} \) is convergent uniformly on compacts to a holomorphic function which we denote by \( f(z) \). By properties of uniform limits of holomorphic functions, the sequences of derivatives \( \left\{ \frac{d^p}{dz^p} f_N(z) \right\}_{N \geq 1} \), \( p = 0, 1, 2, \ldots \), also converge (uniformly on compacts) to \( \frac{d^p}{dz^p} f(z) \). Thus \( f(z) \) is the required analytic continuation to \( \{z \in \mathbb{C} : \Re z > 0\} \), and we have the required convergence of derivatives. \( \Box \)
Proof of Theorem 1.2. We first show that the characteristic functions \( \phi^{(s)}_N(t) \) and \( \phi^{(s)}(t) \) are strictly positive for \( s > -\frac{1}{2} \).

For \( s > 0 \), using the integral representation of \( \phi^{(s)}_N(t) \) in (43) we can see that \( \phi^{(s)}_N(t)e^{\frac{t}{2}} \) is increasing in \( t \) for \( t \geq 0 \). Hence, we get that for all \( N \geq 1 \) and \( t > 0 \),

\[
\phi^{(s)}_N(t) \geq e^{-\frac{t}{2}}\phi^{(s)}_N(0) = e^{-\frac{t}{2}}
\]

and hence, as \( \phi^{(s)}(-t) = \phi^{(s)}(t) \), which can be seen by the change of variables \( x_j \mapsto -x_j \) in (42), we have that for all \( t \in \mathbb{R} \),

\[
\phi^{(s)}_N(t) \geq \exp\left(\frac{-|t|}{2}\right) > 0.
\]  

By (36), we see that (56) implies that \( \phi^{(s)}(t) \) is nonvanishing for \( s > 0, t \in \mathbb{R} \). For \( s \in (-\frac{1}{2}, 0] \) we need to argue more indirectly. By (43) we see that \( \phi^{(s)}_N(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( N \geq 1 \), and hence \( \phi^{(s)}(t) \geq 0 \) for all \( t \in \mathbb{R} \). Moreover, for \( s \in (-\frac{1}{2}, 0] \), by (43), for all \( N \geq 1 \), \( \phi^{(s)}_N(t) \), and thus also \( \phi^{(s)}(t) \), are nonincreasing on \((0, \infty)\). Hence, if these functions vanish at some \( r > 0 \) then they are identically 0 on \([r, \infty)\). However, since by Proposition 2.4 they are restrictions of holomorphic functions, this would imply that they are identically 0 for \( t > 0 \), which is a contradiction since a characteristic function is nonvanishing on a real neighbourhood around zero. Hence, \( \phi^{(s)}_N(t) \) and \( \phi^{(s)}(t) \) are strictly positive for \( t \in (0, \infty) \) and the result follows from noting that these functions are even.

This implies that \( \tau^{(s)}_N(t) \) and \( \tau^{(s)}(t) \) are well defined for all \( N \geq 1 \), \( s > -\frac{1}{2} \), and \( t \in \mathbb{R} \). Hence, by Proposition 2.4, and using that the functions \( \tau^{(s)}_N \) are even, we know that for all \( t \in \mathbb{R}^* \) and \( p = 0, 1, 2, \ldots \),

\[
\frac{d^p}{dt^p}\tau^{(s)}_N(t) \xrightarrow{N \to \infty} \frac{d^p}{dt^p}\tau^{(s)}(t).
\]

Hence the Painlevé equation (16) now follows immediately by taking the limit \( N \to \infty \) in (38).

We now prove \( \tau^{(s)} \) is real-analytic on \( \mathbb{R}^* \). Let \( f \) denote the analytic continuation of \( \phi^{(s)} \) to \( \{z : \Re z > 0\} \), as in Proposition 2.4. We know from the above that for \( t \in (0, \infty) \),

\[
\Re(f(t)) = \Re(\phi^{(s)}(t)) = \phi^{(s)}(t) > 0.
\]

By continuity this implies that \( \Re(f(t)) > 0 \) on an open set \( V \) with \((0, \infty) \subseteq V \subseteq \{z : \Re z > 0\} \). Therefore we can define a branch of \( \log(z) \) such that \( \log(f(z)) \) is holomorphic on \( V \). As \( \tau^{(s)} \) is even, this implies that it is real-analytic on \( \mathbb{R}^* \).

To establish the boundary conditions for \( s > \frac{1}{2} \) we have to employ a different method. Whenever \( r < 2s+1 \), we note that \( \mathbb{E}\left(\left|\sum_{i=1}^{N} x_i\right|^r\right) < \infty \) and that the sequence \( \left\{\left|\sum_{i=1}^{N} x_i\right|^r\right\}_{N \geq 1} \) is uniformly integrable, see [Assiotis et al. 2020, Proposition 2.10], and also Proposition 3.5 below, where we prove a similar statement using the same idea, and hence, also making use of (36),

\[
\frac{d^p}{dt^p}\phi^{(s)}_N(t) \xrightarrow{N \to \infty} \frac{d^p}{dt^p}\phi^{(s)}(t),
\]

for any \( t \in \mathbb{R} \) and \( p \in \{0, 1\} \) whenever \( s > 0 \), and \( p \in \{0, 1, 2\} \) whenever \( s > \frac{1}{2} \). Hence, we conclude,
again using that the $\phi^{(s)}_N$ and $\phi^{(s)}$ are nonvanishing, that
\[
\frac{d^p}{dt^p} \tau^{(s)}_N(t) \xrightarrow{N \to \infty} \frac{d^p}{dt^p} \tau^{(s)}(t),
\]
(57)
for $t \in \mathbb{R}$ and $p = 0$ whenever $s > 0$, and $p \in [0, 1]$ whenever $s > \frac{1}{2}$. The boundary conditions $\tau^{(s)}_N(0) = 0$ for $s > 0$ and $\frac{d}{dt} \tau^{(s)}_N(t)_{t=0} = 0$ for $s > \frac{1}{2}$ are computed in [Assiotis et al. 2020]. Hence by (57) we deduce the boundary conditions (17).

Remark 2.5. The simple observation made earlier that for $s \leq 0$, the integral representation of $\phi^{(s)}_N(t) \cdot e^{t/2}$ given in (43) is nonincreasing in $t$ gives the bound $\phi^{(s)}(t) \leq e^{-t/2}$. By Fourier inversion, since the characteristic function of $X(s)$ is in $L^1(\mathbb{R})$, this readily implies the nontrivial result that the law of $X(s)$ has a bounded and continuous density with respect to the Lebesgue measure. In fact, due to the exponential decay of the characteristic function, the density is $C^\infty$-smooth. We expect this result to be true for $s > 0$ as well.\(^{20}\) This is likely to require a more elaborate argument and we do not pursue it further in this paper.

We now prove Theorem 1.4. It is worth noting that from the definition of the characteristic function $\phi^{(s)}_N$ it is unclear whether any value of the parameter $s$ is special, while if one looks at formula (43) it becomes evident that $s = 0$ is exceptional. We believe this remarkable observation was missed in the literature due to the fact that formulae such as (39) were not thought of in probabilistic terms, but rather as simply some intermediate formulae required to prove the determinantal representation in terms of Laguerre polynomials\(^{21}\) mentioned in the introduction.

Proof of Theorem 1.4. When $s = 0$, the integral in (43) evaluates simply to $C_N^{(0)}$, as in (44). Therefore, it follows that $\phi^{(0)}_N(t) = e^{-|t|/2}$ for all $N = 1, 2, \ldots$. Thus, by (36) we get that $\mathbb{E}(e^{i t X(0)/2}) = e^{-|t|/2}$, i.e., $X(0)$ is Cauchy distributed.

For the second part, note that by [Assiotis et al. 2020, Proposition 2.7], for $\Re(h) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$,\(^{22}\)
\[
\frac{\mathcal{R}_N(0, h)}{N^{2h}} = 2^{-2h} \mathbb{E}^{(s)}_N \left( \sum_{i=1}^{N} \frac{x_i}{N} \right)^{2h}.
\]
(58)
Now, for $s = 0$, for all $N \geq 1$, $\sum_{i=1}^{N} \frac{x_i}{N}$ and $X(s)$ are identically distributed. Thus, for all $N \geq 1$, $\Re(h) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ we have
\[
\frac{\mathcal{R}_N(0, h)}{N^{2h}} = 2^{-2h} \mathbb{E}^{(0)}_N \left( \sum_{i=1}^{N} \frac{x_i}{N} \right)^{2h} = \mathcal{R}(0, h) = 2^{-2h} \mathbb{E}(|X(0)|^{2h}) = \int_0^\infty \frac{x^{2h}}{\pi(x^2 + 1)} \, dx.
\]
(59)
From [Gradshteyn and Ryzhik 2007, 3.241.2], we have that, for $\Re(h) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$,
\[
\int_0^\infty \frac{x^{2h}}{\pi(x^2 + 1)} \, dx = \frac{1}{2 \cos(\pi h)}.
\]
\(^{20}\)Clearly, for $s \in \mathbb{N}$ we already have explicit expressions for the density of $X(s)$ from Theorem 1.5.
\(^{21}\)In fact, we will also make use of this expression in the proof of Proposition 2.6 below.
\(^{22}\)The result in [Assiotis et al. 2020, Proposition 2.7] is stated for real $h$ but the argument goes through verbatim for complex $h$ as well. See also [Winn 2012, Proposition 2].
Thus we see, by expanding the determinant in (60) as a sum over
\[ \det[I_{j+k+1}(2\sqrt{|t|})]_{j,k=0,1,\ldots,s-1}, \]
where \( I_a \) denotes the modified Bessel function of the first kind and \( G \) denotes the Barnes \( G \)-function.

Proof. By [Winn 2012, Proposition 4.5.], for \( s \in \mathbb{N} \) we have that
\[ \phi_N^{(s)}(t) = (-1)^{s(s-1)/2} \frac{G(2s+1)}{G(s+1)^2} \frac{\det[I_{j+k+1}(2\sqrt{|t|})]_{j,k=0,1,\ldots,s-1}}{e^{|t|/2}|t|^{s/2}}, \]
where \( L_n^{(\alpha)}(x) \) denotes the Laguerre polynomial of order \( n \) and parameter \( \alpha \) (see [Winn 2012]).

The large \( N \) limit of the logarithmic derivative of the right-hand side of (61) was first established in [Forrester and Hughes 1994] and also using Riemann–Hilbert problem methods in [Basor et al. 2019]. Thus, using, for example, [Basor et al. 2019, eq. 5-79] we have, in our notation, that for \( s \in \mathbb{N} \),
\[ \frac{d}{dt} \log \phi_N^{(s)}(t) \xrightarrow{N \to \infty} \frac{d}{dt} \log \left( \frac{\det[I_{j+k+1}(2\sqrt{|t|})]_{j,k=0,1,\ldots,s-1}}{e^{|t|/2}|t|^{s/2}} \right). \]

Noting that \( \frac{d}{dt} \log \phi_N^{(s)}(t) \xrightarrow{N \to \infty} \frac{d}{dt} \log \phi^{(s)}(t) \) for \( s > 0 \) by using the results in the proof of Theorem 1.2, we obtain equality (60) up to a multiplicative constant. To recover this constant we observe that both sides of (60) must equal 1 at \( t = 0 \) and we note that the evaluation of the right-hand side of (60) at \( t = 0 \) can be obtained by taking \( h = 0 \) in [Assiotis et al. 2020, Corollary 1.5].

Proof of Theorem 1.5. We first seek to recover the density of \( X(s) \), where here and for the rest of the proof we assume \( s \in \mathbb{N} \). Observe that using the bound (see [Luke 1972])
\[ I_n(t) \leq \frac{\pi^2}{2^n n!} t^n e^{-t/2} \]
we see, by expanding the determinant in (60) as a sum over \( \mathcal{S}(s) \), that \( \phi^{(s)} \) is in \( L^1(\mathbb{R}) \). Therefore, by Fourier inversion, we can obtain an expression for the density function \( \rho^{(s)}(x) \) of \( X(s) \). Namely, we have that, for \( x \in \mathbb{R} \),
\[ \rho^{(s)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \phi^{(s)}(2t) \, dt = \frac{1}{2\pi} \text{Re} \left( \int_{0}^{\infty} e^{ixt} \phi^{(s)}(2t) \, dt \right), \]
using the fact that the characteristic function \( \phi^{(s)}(2t) \) is even. Now, using the expression (60), we get
\[ \rho^{(s)}(x) = \sqrt{s} \frac{1}{2\pi} \sqrt{s} \int_{0}^{\infty} e^{(ix-1)t} \times \frac{\det[I_{j+k+1}(2\sqrt{2t})]_{j,k=0,1,\ldots,s-1}}{(2t)^{s/2}} \, dt, \]
\[ \frac{1}{2} \text{ for } s^2/t, \] which has been corrected here.

---

23 There is a typo in [Basor et al. 2019, eq. 5-79], namely a missing factor of \( \frac{1}{2} \) for \( s^2/t \), which has been corrected here.
where for brevity here and for the rest of the proof we will write
\[ V^{(s)} = (-1)^{s(s-1)/2} \frac{G(2s + 1)}{G(s + 1)^2}. \]

Hence, expanding the determinant and using the substitution \( t \mapsto t^2 \) gives
\[ \rho^{(s)}(x) = V^{(s)} \frac{1}{\pi} \Im \left( \sum_{\sigma \in \mathcal{S}(s)} \sgn(\sigma) \int_0^\infty e^{(ix-1)t^2} \times \frac{\prod_{j=1}^s [I_{j+\sigma(j)-1}(2\sqrt{2t})]}{2^{s/2-1} t^{s/2-1}} \, dt \right). \] (64)

Noting that modified Bessel functions of the first kind have the expansion
\[ I_\alpha(x) = \sum_{k=0}^\infty \frac{x^{2k+\alpha}}{k! \Gamma(k + \alpha + 1) 2^{2k+\alpha}}, \] (65)
we can expand a finite product of modified Bessel functions of the first kind with integer parameters as follows:
\[ \prod_{j=1}^s I_{\nu_j}(2t) = \sum_{k_0=0}^{\infty} \sum_{k_1+\ldots+k_s=k} \left[ \prod_{j=1}^s \frac{1}{k_j!(k_j + \nu_j)!} \right] t^{2k+\nu_1+\ldots+\nu_s}. \]

Using this, we can simplify the expression for the density as
\[ \rho^{(s)}(x) = V^{(s)} \frac{1}{\pi} \sum_{\sigma \in \mathcal{S}(s)} \sgn(\sigma) \times \sum_{k=0}^{\infty} \left[ \sum_{k_1+\ldots+k_s=k} \left[ \prod_{j=1}^s \frac{1}{k_j!(k_j + \sigma(j) - 1)!} \right] \Im \left( \int_0^\infty e^{(ix-1)t^2} \times 2^{k+1} t^{2k+1} \, dt \right) \right], \]

because interchanging the sum and integral is justified as we explain next. Indeed, for each fixed permutation \( \sigma \in \mathcal{S}(s) \), we have, for all \( x \in \mathbb{R} \), that
\[ \int_0^\infty \sum_{k=0}^{\infty} \left[ \sum_{k_1+\ldots+k_s=k} \left[ \prod_{j=1}^s \frac{1}{k_j!(k_j + \sigma(j) - 1)!} \right] e^{(ix-1)t^2} \times 2^{k+1} t^{2k+1} \, dt < \infty, \] (66)
which can be seen by noting that \( |e^{(ix-1)t^2}| = e^{-t^2} \) so that the integral in (66) is equal to the integral in (64) for a fixed \( \sigma \) and \( x = 0 \), which is finite by the bound for modified Bessel functions in (63). Hence, by Fubini’s theorem, we get the desired interchange of summation and integration. The remaining integrals are standard and an explicit evaluation yields
\[ \rho^{(s)}(x) = V^{(s)} \frac{1}{2\pi} \sum_{\sigma \in \mathcal{S}(s)} \sgn(\sigma) \sum_{k=0}^{\infty} \Im \left[ \left[ \sum_{k_1+\ldots+k_s=k} \left[ \prod_{j=1}^s \frac{1}{k_j!(k_j + \sigma(j) - 1)!} \right] k! \left( \frac{2}{1 - ix} \right)^{k+1} \right] \right]. \] (67)

Finally, we can plug in the value of the constant and rewrite the sum over \( \mathcal{S}(s) \) as a determinant to obtain the expression in (25).
To compute the moments $\mathbb{E}(|X(s)|^{2h})$, and thus $\mathcal{R}(s, h)$ by the relation (20), we temporarily restrict to $h \in (-\frac{1}{2}, 0)$. Now, for $h \in (-\frac{1}{2}, 0)$ we have, for all $k \geq 1$, that

$$
\int_0^\infty \left| \mathfrak{H} \left( \frac{x^{2h}}{(1-ix)^k+1} \right) \right| dx \leq \int_0^\infty \frac{x^{2h}}{(1+x^2)^{k+1}} dx \leq \int_0^\infty \frac{x^{2h}}{\sqrt{1+x^2}} dx < \infty,
$$

and hence,

$$
\sum_{k=0}^\infty \left( \sum_{k_1+\ldots+k_s=k} \prod_{j=1}^s \frac{1}{k_j!} (k_j + j + \sigma(j) - 1)! \right) k! 2^{k+1} \int_0^\infty \left| \mathfrak{H} \left( \frac{x^{2h}}{(1-ix)^{k+1}} \right) \right| dx < \infty,
$$

where the finiteness of the sum is seen by using the inequalities in (68) and comparing to the infinite sum for $\rho^{(s)}(0)$. Now, we simply apply Fubini’s theorem to integrate the infinite series for $\rho^{(s)}(x)$ term-by-term, using the following evaluation from [Gradshteyn and Ryzhik 2007, 3.194.3]:

$$
\int_0^\infty \frac{x^{2h}}{(1-ix)^k} dx = -i\pi e^{i\pi h} \frac{(-2h)_{k-1}}{(k-1)! \sin(2\pi h)}.
$$

Thus, we obtain the equality

$$
\cos(\pi h) \mathbb{E}(|X(s)|^{2h}) = \nu^{(s)} \left( \sum_{k=0}^\infty \sum_{k_1+\ldots+k_s=k} \frac{1}{(k_i + i + j - 1)!} \prod_{j=1}^s \frac{1}{k_j!} (-2h)_k 2^k \right)
$$

for $h \in (-\frac{1}{2}, 0)$. Now, letting $M$ be a compact subset of the strip $D := \{ z \in \mathbb{C} : -\frac{1}{2} < \Re(z) < s + \frac{1}{2} \}$, and noting that there exists a positive constant $\alpha_M$ such that $|(-2h)_k| < \alpha_M k!$, for all $h \in M$ whenever $k$ is large enough (this can be seen immediately via Stirling’s approximation), we see that, by comparison with the sum in (25), which converges for $x = 0$, the infinite sum on the right-hand side of (69) converges uniformly on compact subsets of $D$, and thus it is analytic in $h$ on $D$. Note also that

$$
\mathbb{E}(|X(s)|^{2h}) = \int_{-\infty}^\infty |x|^{2h} \rho^{(s)}(x) dx
$$

is an analytic function of $h$ for $h \in D$, as can be seen by a combination of Fubini’s and Morera’s theorems. More precisely, let $\Gamma$ be a closed path contained in $D$. Setting $\alpha_1 := \inf_{h \in \Gamma} \Re(h) > -\frac{1}{2}$ and $\alpha_2 := \sup_{h \in \Gamma} \Re(h) < s + \frac{1}{2}$, we see that $||x|^{2h}| \leq |x|^{2\alpha_1}$ whenever $|x| \leq 1$ and $||x|^{2h}| \leq |x|^{2\alpha_2}$ whenever $|x| \geq 1$. Therefore, we have that $||x|^{2h}| \leq |x|^{2\alpha_1} + |x|^{2\alpha_2}$ for all $x \in \mathbb{R}$ and $h \in \Gamma$, so that

$$
\int_{\Gamma} \int_{-\infty}^\infty ||x|^{2h} \rho^{(s)}(x)| dxdh \leq \int_{\Gamma} \int_{-\infty}^\infty (|x|^{2\alpha_1} + |x|^{2\alpha_2}) \rho^{(s)}(x) dxdh < \infty,
$$

where the finiteness of the double integral is justified by the fact that $\Gamma$ is a finite length path, and that the inner integral is finite by the computation above when $\alpha_1, \alpha_2 \in (-\frac{1}{2}, 0)$, and by finiteness of $\mathbb{E}(|X(s)|^{2\alpha_i})$ for $\alpha_i \in [0, s + \frac{1}{2}]$ (see [Assiotis et al. 2020]). Hence, we have that

$$
\int_{\Gamma} \int_{-\infty}^\infty |x|^{2h} \rho^{(s)}(x) dxdh = \int_{-\infty}^\infty \int_{\Gamma} |x|^{2h} \rho^{(s)}(x) dhdx = 0,
$$

where we have used Cauchy’s theorem. Hence, by Morera’s theorem, we see that $\mathbb{E}(|X(s)|^{2h})$ is an
analytic function of \( h \) on \( D \). Thus, it is now clear that \( \cos(\pi h)\mathbb{E}(|X(s)|^{2h}) \) and the infinite sum on the right-hand side of (69) are two analytic functions of \( h \) for \( h \in D \), which agree for \( h \in (-\frac{1}{2}, 0) \), and so the equality (69) holds for all \( h \in D \).

We note that (25) and (26) simplify to the corresponding expressions given in Theorem 1.5 for \( s = 1, 2 \), which can be seen immediately for \( s = 1 \), and for \( s = 2 \) by using Vandermonde’s identity:\(^{24}\)

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{s}{t+k} = \binom{n+s}{n+t}.
\]

The proof of Theorem 1.5 is now complete. \( \square \)

**Remark 2.7.** Note that from (22) we may immediately recover the previously known (see [Hughes 2001, §6.2.]) value \( R(1, 1) = \frac{1}{12} \). Note also that by the equality (69), valid for \( \Re(h) \in (-\frac{1}{2}, s + \frac{1}{2}) \), and using that the moments \( \mathbb{E}(|X(s)|^{2h}) \) are finite for \( \Re(h) \in (-\frac{1}{2}, s + \frac{1}{2}) \) (as seen in the proof of Theorem 1.5), we deduce the seemingly nontrivial fact that the sum on the right-hand side of (69) vanishes for all \( h \in \mathbb{N} - \frac{1}{2} \) in this range. Therefore, calculations for the moments \( R(s, h) \) for \( h \in \mathbb{N} - \frac{1}{2} \) can be performed by applying L’Hôpital’s rule to the relevant formulae in Theorem 1.5. For instance, the value \( R(1, \frac{1}{2}) = \frac{e^{2-5}}{4\pi} \) (as calculated in [Winn 2012]) can be recovered:

\[
R(1, \frac{1}{2}) = -\frac{1}{2\pi} \lim_{h \to \frac{1}{2}} \frac{d}{dh} \left( \sum_{k=0}^{\infty} \frac{(-2h)^k}{k!(k+1)!} 2^k \right) = -\frac{1}{2\pi} \left( -2 + \sum_{k=2}^{\infty} \frac{1}{(k-1)k(k+1)!} 2^{k+1} \right) = \frac{e^{2-5}}{4\pi}, \tag{72}
\]

where in the penultimate equality we have used the following limit formula, obtained via the product rule:

\[
\lim_{h \to \frac{1}{2}} \frac{d}{dh} ((-2h)_k) = \begin{cases} 
2(k - 2)! & \text{if } k \geq 2, \\
-2 & \text{if } k = 1, \\
0 & \text{if } k = 0.
\end{cases} \tag{73}
\]

The interchange of limits in (72) can be justified by standard arguments using uniform convergence. We also remark that the specific values of (22) for \( h = \frac{-1}{4}, \frac{1}{4}, \frac{3}{4} \), and \( \frac{5}{4} \), which had not been computed before, can be expressed as a combination of \( I_0(1) \) and \( I_1(1) \):

\[
\begin{align*}
R(1, -\frac{1}{4}) &= 2e(I_0(1) - I_1(1)), \\
R(1, \frac{1}{4}) &= \frac{e}{5}(-I_0(1) + 3I_1(1)), \\
R(1, \frac{3}{4}) &= \frac{e}{30}(5I_0(1) - 9I_1(1)), \\
R(1, \frac{5}{4}) &= \frac{e}{140}(5I_0(1) - 3I_1(1)).
\end{align*} \tag{74}
\]

These values are taken from known special values of the confluent hypergeometric function. Finally, from (24) we can immediately recover the previously known (see [Hughes 2001, §6.2.]) values \( R(2, 1) = \frac{1}{720} \)

---

\(^{24}\)It is not clear whether such a combinatorial simplification exists for \( s \geq 3 \). We note that also Winn, when computing the half-integer moments in Section 7 of [Winn 2012], observed that some combinatorial structures seem to break down for \( s \geq 3 \).
and $\mathcal{R}(2, 2) = \frac{1}{6720}$. Again using L’Hôpital’s rule we can recover the half-integer values:

$$
\mathcal{R}(2, \frac{1}{2}) = \frac{7}{180\pi} \left( \frac{15}{7} - 3 F_3 \left[ \frac{9}{7}, 1, 1 \frac{3}{1}, 6, 7 ; 8 \right] \right),
$$

$$
\mathcal{R}(2, \frac{3}{2}) = \frac{11}{3360\pi} \left( -\frac{28}{33} + 3 F_3 \left[ \frac{13}{5}, 1, 1 \frac{3}{1}, 8, 9 ; 8 \right] \right).
$$

(75)

These were previously calculated using Maple by Winn [2012, §6.2.], who used combinatorial expressions valid only for half-integer parameters.

**Remark 2.8.** Using the fact that for fixed $s$, the series in (26) vanishes for half-integer $h$ in the range $(0, s + \frac{1}{2})$ and noting the known values for integer $h$ again in this range (see [Basor et al. 2019, eq. (4-46)]), we can write $\mathcal{R}(s, h)$ as

$$
\mathcal{R}(s, h) = \frac{G(s+1)^2}{G(2s+1)} 2^{-2h} \frac{1}{\cos(\pi h)} \sum_{k=0}^{\infty} a_k(s)(-2h)_k
$$

(76)

where $a_0(s), \ldots, a_{2m}(s)$ can be simplified to rational functions of $s$ so that this rational expression of the first $2m+1$ coefficients in (26) is valid for all $s \geq m$. For instance, for $s \geq 2$ there exists an expansion of $\mathcal{R}(s, h)$ in the form (76), where the first five terms of the series are explicitly given as follows:

$$
\mathcal{R}(s, h) = \frac{G(s+1)^2}{G(2s+1)} 2^{-2h} \frac{1}{\cos(\pi h)}
\times \left( 1 + (-2h)_1 + \frac{4s^2 - 2}{4s^2 - 1} (-2h)_2 + \frac{4s^2 - 4}{4s^2 - 1} (-2h)_3 + \ldots \right).
$$

(77)

Note also that the first three coefficients in the expansion above are correct for all $s \geq 1$, but the simplifications for $a_3(s), a_4(s)$ are only valid for $s \geq 2$. It is tempting to try to find an expansion in the form of (76) such that $a_0(s), \ldots, a_{2m+1}(s)$ are all rational functions of $s$, valid for all $s \geq m$ where $m$ is a positive integer. However, if one proves that $\lim_{h \to m + \frac{1}{2}} \mathcal{R}(m, h) = \infty$, which we expect to be true (we have verified this for $m = 1, 2, 3, 4$ by evaluating the right-hand side of (69) at $h = m + \frac{1}{2}$ for $m = 1, 2, 3, 4$ and finding that it did not vanish), but do not pursue further in this paper,\(^{25}\) then one can see that there cannot be an expansion in the form of (76) valid for all $s \geq m$ with $a_0(s), \ldots, a_{2m+1}(s)$ rational functions of $s$. Indeed if such an expansion existed, then we would have, for all integer $s > m$, that

$$
0 = \frac{G(2s+1)}{G(s+1)} 2^{2m+1} \cos((m + \frac{1}{2})\pi) \mathcal{R}(s, m + \frac{1}{2})
= a_0(s)(-2m - 1)_0 + \ldots + a_{2m+1}(s)(-2m - 1)_{2m+1},
$$

so that since $a_0(s) + \ldots + a_{2m+1}(s)(-2m - 1)_{2m+1}$ is a rational function of $s$ with infinitely many zeros, it is zero identically. Using the fact that the sum on the right-hand side of (26) is analytic on $\Re(h) > -\frac{1}{2}$, this would imply that $\mathcal{R}(m, m + \frac{1}{2}) < \infty$, which contradicts the assumption that $\lim_{h \to m + \frac{1}{2}} \mathcal{R}(m, h) = \infty$.

\(^{25}\)We note that for finite $N$, one can show by expanding the Haar measure in (4) that $\lim_{h \to m + \frac{1}{2}} \mathcal{R}_N(m, h) = \infty$; see [Dehaye 2008].
(which still remains to be proven for \( m \geq 5 \)); hence, we have a contradiction and such an expansion cannot exist.

**Proof of Corollary 1.8.** The sine process \( S \) with kernel given by (30) is obtained from the process \( C^{(0)} \) under the mapping \( x \mapsto \frac{1}{x} \); see [Borodin and Olshanski 2001, Theorem I]. The result now follows immediately from the definition (1) of \( X(s) \) and Theorem 1.4.

\( \square \)

Alternative proof of (60) in the case \( s = 1 \). We use the following version of Aomoto’s integral formula [Forrester and Warnaar 2008]:

\[
a^{(\alpha)}_{N,k} := \int_{[0,\infty)^N} \prod_{j=1}^k y_j \prod_{j=1}^N y_j^{\alpha-1} e^{-y_j} \Delta(y)^2 \, dy = C_N^{(\alpha)} \times \prod_{j=1}^k (\alpha + N - j),
\]

where \( C_N^{(\alpha)} \) is defined as in (44). Thus, expanding the factor \( \prod_{j=1}^N (y_j + \frac{t}{N}) \) in (43) yields, for \( t \geq 0 \),

\[
\phi_N^{(1)}(t) = e^{\frac{t}{N}} \sum_{r=0}^N \binom{N}{r} \binom{r}{N} \times a^{(2)}_{N,N-r} \times \frac{1}{C_N^{(1)}} = e^{\frac{t}{N}} \sum_{r=0}^N \frac{(N-r+1)r}{N^r} \frac{1}{r!(r+1)!} t^r.
\]

Hence, taking the limit \( N \to \infty \) in the last equality of (79), and using the fact that the functions \( \phi_N^{(1)}(t) \) are even, yields

\[
\phi^{(1)}(t) = e^{-\frac{|t|}{2}} \sum_{r=0}^\infty \frac{1}{r!(r+1)!} |t|^r = e^{-\frac{|t|}{2}} \frac{I_1(2\sqrt{|t|})}{\sqrt{|t|}},
\]

for all \( t \in \mathbb{R} \).

\( \square \)

3. The sum of inverse points of the Bessel process

In this section we establish a connection between the Laplace transform of the sum of inverse points of the Bessel point process and the \( \sigma \)-Painlevé III' equation, in analogy to Theorem 1.2. Let \( v > -1 \) and recall that the Bessel point process (with parameter \( v \)), that we denote by \( P^{(v)} \), is the determinantal point process on \((0, \infty)\) with infinitely many points, whose correlation kernel is given, see [Forrester 1993; Tracy and Widom 1994], by

\[
K^{(v)}(x, y) = \frac{\sqrt{x} J_{v+1}(\sqrt{x}) J_v(\sqrt{y}) - \sqrt{y} J_{v+1}(\sqrt{y}) J_v(\sqrt{x})}{2(x-y)},
\]

where \( J_v \) denotes the Bessel function with parameter \( v \). The Bessel point process is a fundamental object which appears as the universal scaling limit of the eigenvalues of random matrices at the hard edge; see, for example, [Kuijlaars and Vanlessen 2002; Lubinsky 2008; Rider and Waters 2019].

Then, we define\(^{26}\)

\[
Y(v) = \sum_{x \in P^{(v)}} \frac{1}{x}.
\]

\(^{26}\)The fact that \( Y(v) \) is almost surely finite follows from the results of [Assiotis 2019].
The random variable $Y(\nu)$ plays a similar role to $X(s)$ in the ergodic decomposition of another distinguished family of unitarily invariant probability measures on $\mathbb{H}(\infty)$, the inverse Laguerre measures; more precisely it is equal in distribution with the parameter $\gamma_1$; see [Assiotis 2019]. The main result of this section is the following:

**Theorem 3.1.** Let $\nu > -1$. Define

$$h^{(\nu)}(t) := \frac{\nu^2}{4} + \xi^{(\nu)}(t),$$

where

$$\xi^{(\nu)}(t) := t \frac{d}{dt} \log \psi^{(\nu)}(t) \quad \text{and} \quad \psi^{(\nu)}(t) := \mathbb{E}(e^{-4tY^{(\nu)}}).$$

Then, $h^{(\nu)}(t)$ is $C^\omega$ on $(0, \infty)$ and is a solution to a special case of the $\sigma$-Painlevé III equation with one parameter for $t \in (0, \infty)$:

$$\left( t \frac{d^2 h^{(\nu)}(t)}{dt^2} \right)^2 = 4 \left( \frac{dh^{(\nu)}(t)}{dt} \right)^2 \left( h^{(\nu)}(t) - t \frac{dh^{(\nu)}(t)}{dt} \right) + 2\nu \frac{dh^{(\nu)}(t)}{dt} + 1. \quad (83)$$

Furthermore, we have the boundary conditions

$$\begin{align*}
h^{(\nu)}(0) &= \frac{\nu^2}{4}, \quad \text{for } \nu > 0, \\
\left. \frac{d}{dt} h^{(\nu)}(t) \right|_{t=0} &= -\frac{1}{\nu}, \quad \text{for } \nu > 1.
\end{align*} \quad (84)$$

Before discussing some of the relevant literature we give a brief outline of the strategy of proof which is similar to the one of Theorem 1.2. Namely, we will consider the Laplace transforms of a sequence of random variables that converge, in distribution, to $Y(\nu)$. We begin with some preliminaries.

Let $M_N$ be an $N \times N$ random matrix taken from the Laguerre unitary ensemble (LUE) with parameter $\nu > -1$, having the law

$$\text{const} \cdot \det(H)^\nu \exp(-\text{Tr}(H)) \mathbb{1}_{\{H \in \mathbb{H}_+(N)\}} dH,$$

where $dH$ is the Lebesgue measure on $\mathbb{H}(N)$ and $\mathbb{H}_+(N)$ denotes the space of $N \times N$ positive-definite Hermitian matrices, and the constant is chosen so that this is a probability measure on $\mathbb{H}_+(N)$. Then, the eigenvalues of $M_N$ are distributed according to the following probability measure on $\mathbb{R}_+/S(N)$:

$$\frac{1}{\tilde{C}_N^{(\nu)}} \cdot \Delta(x)^2 \prod_{j=1}^N x_j^{-\nu} e^{-x_j} dx_j, \quad (85)$$

where

$$\tilde{C}_N^{(\nu)} = \prod_{j=1}^N \Gamma(j) \Gamma(\nu + j). \quad (86)$$

Via the transformation $M_N \leftrightarrow \frac{2}{M_N}$, the LUE is transformed to the inverse Laguerre ensemble, whose eigenvalue distribution is given by the following probability measure on $\mathbb{R}_+/S(N)$:

$$\frac{1}{E_N^{(\nu)}} \cdot \Delta(y)^2 \prod_{j=1}^N y_j^{-\nu-2N} e^{-y_j^2} dy_j,$$

where

$$E_N^{(\nu)} = \prod_{j=1}^N \Gamma(j) \Gamma(\nu + j).$$  

(87)
with
\[ E_N^{(\nu)} = \prod_{j=1}^{N} \frac{((j - \nu - 2N)_{j-1})^2(2j - \nu - 2N - 1)}{2^{2j - \nu - 2N - 1}\Gamma(-j + \nu + 2N + 1)(j - 1)!}, \]  
(88)

where \((a)_j\) denotes the Pochhammer symbol given by \((a)_j := \prod_{i=1}^{j}(a + i - 1), \ (a)_0 := 1\).

Then, similarly to the Hua–Pickrell case, by a combination of the results from [Borodin and Olshanski 2001] (the existence of the limit) and [Assiotis 2019] (the identification of the limit with \(Y(\nu)\)) we have that
\[ \sum_{i=1}^{N} \frac{2}{N x_i} \frac{d}{N} \frac{d}{\rightarrow \infty} 8Y(\nu), \]  
(89)

where \((x_1, x_2, \ldots, x_N)\) are distributed according to the probability measure in (85), whereas \((y_1, y_2, \ldots, y_N)\) are distributed according to the probability measure in (87). Then, if we let
\[ \psi_N^{(\nu)}(t) = \mathbb{E}(e^{-t \sum_{j=1}^{N} \frac{1}{x_j}}), \]
where the expectation is taken with respect to probability measure in (85), we have that
\[ \psi_N^{(\nu)}(t) \xrightarrow{N \rightarrow \infty} \psi^{(\nu)}(t) \]  
(90)

for all \(t \in [0, \infty)\), with the convergence being uniform on compacts. Finally, we define
\[ \xi_N^{(\nu)}(t) = t \frac{d}{dt} \log \psi_N^{(\nu)}(t). \]

Now, regarding the relevant literature, as far as we are aware, the first time a connection between \(\psi_N^{(\nu)}\) and Painlevé equations was established was in the physics literature in [Osipov and Kanzieper 2007]. Then, Chen and Its [2010] established this result rigorously, using the ladder operator method, and an alternative proof from the viewpoint of integrable systems theory was given in [Mezzadri and Simm 2013]. Later, a detailed study of the large \(N\) limit of \(\psi_N^{(\nu)}\) for \(\nu > 0\) (without the identification of the limit with the Laplace transform \(\psi^{(\nu)}\) of \(Y(\nu)\) proven in [Assiotis 2019]) was performed using Riemann–Hilbert problem methods in [Xu et al. 2014; 2015]; see also Section 3 in [Dai et al. 2019]. It would be possible, for \(\nu > 0\), to extract the required ingredients we need from [Xu et al. 2014; 2015; Dai et al. 2019] so that along with the discussion above we establish Theorem 3.1 (for \(\nu > 0\)). However, since we want to cover the entire range of parameter values, \(\nu > -1\), we instead adapt the relatively short and elementary arguments given in Section 2 to the present setting.

Our starting point is the following proposition, the analogue of Proposition 2.1, which as mentioned above is essentially due to [Chen and Its 2010; Mezzadri and Simm 2013; Osipov and Kanzieper 2007]. Namely, by making the change of variables \(t \mapsto tN\) and some algebraic manipulations it reduces to Theorem 3 of [Chen and Its 2010]. This theorem in [Chen and Its 2010] is stated only for \(\nu > 0\)\(^{27}\) but it can be extended to \(\nu > -1\) via an analytic continuation argument identical to the one presented in the

\(^{27}\)For the same reasons explained in the discussion before the statement of Proposition 2.3.
proof of Proposition 2.3. For an alternative approach to this result, see [Mezzadri and Simm 2013] and also the physics paper [Osipov and Kanzieper 2007].

**Proposition 3.2.** Let \( v > -1 \). Then, for \( t \in (0, \infty) \), \( \xi_N^{(v)}(t) \) is a solution to a particular Painlevé equation:

\[
\left( t \frac{d^2 \xi_N^{(v)}}{dt^2} \right)^2 \begin{aligned}
&= -4t \left( \frac{d \xi_N^{(v)}}{dt} \right)^3 + \left( v^2 + 4\xi_N^{(v)} + \frac{4t}{N} \right) \left( \frac{d \xi_N^{(v)}}{dt} \right)^2 + \left( 2v - \frac{4}{N} \xi_N^{(v)}(t) \right) \frac{d \xi_N^{(v)}}{dt} + 1.
\end{aligned}
\tag{91}
\]

Now, we prove a series of propositions that will allow us to deduce Theorem 3.1.

**Proposition 3.3.** Let \( v > -1 \). Then there exist holomorphic functions \( g_N \) for \( N = 1, 2, 3, \ldots \) and \( g \) on \( \{ z \in \mathbb{C} : \Re z > 0 \} \), with \( g_N(0) = g(0) = 1 \), such that

\[
\psi_N^{(v)} = g_N|_{(0, \infty)} \quad \text{and} \quad \psi^{(v)} = g|_{(0, \infty)}.
\tag{92}
\]

Moreover, for \( \Re z > 0 \), \( p = 1, 2, \ldots \), we have

\[
\lim_{N \to \infty} \frac{d^p}{dz^p} g_N(z) = \frac{d^p}{dz^p} g(z).
\tag{93}
\]

**Proof.** We use the same sequence of arguments we used in the proof of Proposition 2.4 by noting that \( |e^{-\frac{x}{x_N}}| < 1 \) uniformly on \( \{ z \in \mathbb{C} : \Re z > 0 \} \), \( x > 0 \) and \( N \geq 1 \). \( \square \)

**Proposition 3.4.** Let \( v > -1 \). Then, there exists an exchangeable sequence of random variables \( \{ e_i \}_{i=1}^{\infty} \) having the following inverse-Gamma distribution on \( (0, \infty) \):

\[
\frac{2^{v+1}}{\Gamma(v+1)} x^{-v-2} e^{-\frac{x}{2}} dx,
\tag{94}
\]

such that

\[
\sum_{i=1}^{N} \frac{2}{x_i} = \sum_{i=1}^{N} y_i = \sum_{i=1}^{N} e_i, \quad \text{for all } N \geq 1,
\tag{95}
\]

where \( (x_1, x_2, \ldots, x_N) \) are distributed according to the probability measure (85) while \( (y_1, y_2, \ldots, y_N) \) are distributed according to the probability measure (87).

**Proof.** We effectively follow the proof in [Assiotis et al. 2020, Proposition 2.11] verbatim, using the analogous results on the inverse Laguerre ensemble established in [Assiotis 2019], and noting that (94) is the law of \( y_1 \) in the case \( N = 1 \). \( \square \)

**Proposition 3.5.** Let \( v > r - 1 \geq 0 \). Then, the sequence of random variables

\[
\left\{ \sum_{j=1}^{N} \frac{1}{N x_j} \right\}_{N \geq 1}^r,
\]

where \( (x_1, x_2, \ldots, x_N) \) are distributed according to the measure in (85), is uniformly integrable.

\[\text{28}\text{The sequence } \{ e_i \}_{i=1}^{\infty} \text{ is simply given by the diagonal elements of an infinite inverse Laguerre distributed random matrix with parameter } v \text{ on } \mathbb{H}(\infty). \text{ This probability measure on } \mathbb{H}(\infty) \text{ was constructed in [Assiotis 2019].} \]
Proof. To show uniform integrability of \( \{ |\sum_{j=1}^{N} \frac{1}{N x_j}|^r \}_{N \geq 1} \) we simply show uniform boundedness of a higher moment: for all \( r \in [1, \nu + 1) \), there exists \( k \in (r, \nu + 1) \) so that, by Jensen’s inequality,

\[
\sup_{N \geq 1} E \left( \left| \sum_{j=1}^{N} \frac{1}{N x_j} \right|^k \right) \leq 2^{-k} E[|e_1|^k] = \frac{\Gamma(\nu + 1)}{2^{2k} \Gamma(\nu - k + 1)} < \infty,
\]

(97)

where we have used Proposition 3.4 for both the bound and the equality. This implies that the sequence \( \{ |\sum_{j=1}^{N} \frac{1}{N x_j}|^r \}_{N \geq 1} \) is uniformly integrable for all \( r \in [1, \nu + 1) \).

Now, we are finally in a position to prove the main result of this section:

Proof of Theorem 3.1. Using an argument similar to the proof of Proposition 2.4 we get the convergence of derivatives for \( p = 0, 1, 2, \ldots \):

\[
\frac{d^p \psi_N^{(\nu)}}{dt^p} \xrightarrow{N \to \infty} \frac{d^p \psi^{(\nu)}}{dt^p}
\]

(98)

for all \( t \in (0, \infty) \). Now, for \( \nu > -1 \), it was proven in [Assiotis 2019, Proposition 7.2] that \( Y(\nu) \) is finite almost surely. Moreover, it is clear from the definition of the eigenvalue density (85) that \( \mathbb{P}(\sum_{j=1}^{N} \frac{1}{N x_j} < 1) \) is strictly positive for all \( N \). Therefore, an application of Markov’s inequality yields that there are constants \( m_N^{(\nu)} > 0 \), \( m^{(\nu)} > 0 \) and \( M^{(\nu)} > 0 \) such that

\[
\psi_N^{(\nu)}(t) \geq m_N^{(\nu)} e^{-t} > 0 \quad \text{and} \quad \psi^{(\nu)}(t) \geq m^{(\nu)} e^{-M^{(\nu)} t} > 0, \quad \text{for all} \quad t \in (0, \infty).
\]

(99)

Therefore, \( \xi_N^{(\nu)} \) and its derivatives are well defined and we can take the limits for \( t \in (0, \infty), \ p = 0, 1, 2, \ldots \):

\[
\frac{d^p \xi_N^{(\nu)}(t)}{dt^p} \xrightarrow{N \to \infty} \frac{d^p \xi^{(\nu)}(t)}{dt^p}.
\]

(100)

Then, we simply take the limit as \( N \to \infty \) of (91) and substitute \( h^{(\nu)}(t) = \frac{\nu^2}{4} + \xi^{(\nu)}(t) \) to get the desired Painlevé equation (83). To show that \( \xi^{(\nu)}(t) \) is \( C^\omega \) on \( (0, \infty) \), we simply use the fact that \( \psi^{(\nu)}(t) > 0 \) on \( (0, \infty) \) and apply the same sequence of arguments as in the proof of Theorem 1.2.

For the boundary conditions, letting \( (x_1, x_2, \ldots, x_N) \) be distributed according to the probability measure in (85), we simply compute, using Proposition 3.4,

\[
E \left( \sum_{j=1}^{N} \frac{1}{N x_j} \right) = E \left[ \frac{e_1}{2} \right] = \frac{1}{\nu}
\]

(101)

for \( \nu > 0 \) and note that by Proposition 3.5 we have that

\[
E \left( \left( \sum_{j=1}^{N} \frac{1}{N x_j} \right)^2 \right) < \infty,
\]

(102)

for \( \nu > 1 \). Hence, we get the boundary conditions

\[
\begin{cases}
\xi_N^{(\nu)}(0) = 0, & \text{for} \ \nu > 0, \\
\frac{d}{dt} \xi_N^{(\nu)}(t)|_{t=0} = -\frac{1}{\nu}, & \text{for} \ \nu > 1,
\end{cases}
\]

(103)
for all \( N \geq 1 \). Now, since by Proposition 3.5 we have that \( \nu > r - 1 \geq 0 \) implies that the sequence \( \{\frac{1}{N} \sum_{j=1}^{N} e^{-\frac{1}{N} x_j} \}_{N \geq 1} \) is uniformly integrable, arguing as in the proof of Theorem 1.2 and using (99) we establish

\[
\frac{d^p}{dt^p} \xi_{N}^{(\nu)}(t) \xrightarrow{N \to \infty} \frac{d^p}{dt^p} \xi^{(\nu)}(t)
\]

for \( t \in [0, \infty) \), \( p = 0 \) when \( \nu > 0 \) and \( t \in [0, \infty) \), \( p \in \{0, 1\} \) when \( \nu > 1 \). Thus, the desired boundary conditions are obtained by taking the limit \( N \to \infty \) in (103).

\[\square\]

Acknowledgements

Bedert and Soor gratefully acknowledge the financial support from the Mathematical Institute, University of Oxford. Bedert also gratefully acknowledges the financial support from the EPSRC. Alper Gunes gratefully acknowledges the financial support from Prof. J.P. Keating’s start-up grant. Assiotis is grateful for financial support at the early stages of this work from ERC Advanced Grant 740900 (LogCorRM). We would like to thank the anonymous referees for a number of very useful comments and suggestions which have improved the presentation and we are also particularly grateful to them for pointing out a number of references.

References

[Aizenman and Warzel 2015] M. Aizenman and S. Warzel, “On the ubiquity of the Cauchy distribution in spectral problems”, Probab. Theory Related Fields 163:1-2 (2015), 61–87. MR Zbl

[Assiotis 2018] T. Assiotis, “A matrix Bougerol identity and the Hua–Pickrell measures”, Electron. Commun. Probab. 23 (2018), art. id. 7. MR Zbl

[Assiotis 2019] T. Assiotis, “Ergodic decomposition for inverse Wishart measures on infinite positive-definite matrices”, SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), art. id. 067. MR Zbl

[Assiotis 2020] T. Assiotis, “Hua–Pickrell diffusions and Feller processes on the boundary of the graph of spectra”, Ann. Inst. Henri Poincaré Probab. Stat. 56:2 (2020), 1251–1283. MR Zbl

[Assiotis et al. 2020] T. Assiotis, J. P. Keating, and J. Warren, “On the joint moments of the characteristic polynomials of random unitary matrices”, 2020. arXiv

[Bailey et al. 2019] E. C. Bailey, S. Bettin, G. Blower, J. B. Conrey, A. Prokhorov, M. O. Rubinstein, and N. C. Snaith, “Mixed moments of characteristic polynomials of random unitary matrices”, J. Math. Phys. 60:8 (2019), art. id. 083509. MR Zbl

[Basor and Chen 2015] E. Basor and Y. Chen, “Perturbed Laguerre unitary ensembles, Hankel determinants, and information theory”, Math. Methods Appl. Sci. 38:18 (2015), 4840–4851. MR Zbl

[Basor et al. 2012] E. Basor, Y. Chen, and L. Zhang, “PDEs satisfied by extreme eigenvalues distributions of GUE and LUE”, Random Matrices Theory Appl. 1:1 (2012), art. id. 1150003. MR Zbl

[Basor et al. 2019] E. Basor, P. Bleher, R. Buckingham, T. Grava, A. Its, E. Its, and J. P. Keating, “A representation of joint moments of CUE characteristic polynomials in terms of Painlevé functions”, Nonlinearity 32:10 (2019), 4033–4078. MR Zbl

[Bleher and Its 1999] P. Bleher and A. Its, “Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model”, Ann. of Math. (2) 150:1 (1999), 185–266. MR Zbl

[Borodin 2011] A. Borodin, “Determinantal point processes”, pp. 231–249 in The Oxford handbook of random matrix theory, edited by J. B. Gernot Akemann and P. D. Francesco, Oxford Univ. Press, 2011. MR Zbl

[Borodin and Olshanski 2001] A. Borodin and G. Olshanski, “Infinite random matrices and ergodic measures”, Comm. Math. Phys. 223:1 (2001), 87–123. MR Zbl
[Bourgade and Keating 2013] P. Bourgade and J. P. Keating, “Quantum chaos, random matrix theory, and the Riemann $\zeta$-function”, pp. 125–168 in Chaos, edited by S. N. Bertrand Duplantier and V. Rivasseau, Prog. Math. Phys. 66, Springer, 2013. MR Zbl

[Bourgade et al. 2009] P. Bourgade, A. Nikeghbali, and A. Rouault, “Circular Jacobi ensembles and deformed Verblunsky coefficients”, Int. Math. Res. Not. 2009:23 (2009), 4357–4394. MR Zbl

[Bourgade et al. 2013] P. Bourgade, J. Najnudel, and A. Nikeghbali, “A unitary extension of virtual permutations”, Int. Math. Res. Not. 2013:18 (2013), 4101–4134. MR Zbl

[Bufetov 2015] A. I. Bufetov, “Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures, I: Construction of infinite determinantal measures”, Izv. Ross. Akad. Nauk Ser. Mat. 79:6 (2015), 18–64. In Russian; translated in Izvestiya: Mathematics 79:6 (2015), 1111–1156. MR Zbl

[Bufetov 2016a] A. I. Bufetov, “Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures, II: Convergence of determinantal measures”, Izv. Ross. Akad. Nauk Ser. Mat. 80:2 (2016), 16–32. In Russian; translated in Izvestiya: Mathematics 80:2 (2016), 299–315. MR Zbl

[Bufetov 2016b] A. I. Bufetov, “Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures, III: The infinite Bessel process as the limit of radial parts of finite-dimensional projections of infinite Pickrell measures”, Izv. Ross. Akad. Nauk Ser. Mat. 80:6 (2016), 43–64. In Russian; translated in Izvestiya: Mathematics 80:6 (2016), 1035-1056. MR Zbl

[Bufetov and Qiu 2016] A. I. Bufetov and Y. Qiu, “The explicit formulae for scaling limits in the ergodic decomposition of infinite Pickrell measures”, Ark. Mat. 54:2 (2016), 403–435. MR Zbl

[Chen and Its 2010] Y. Chen and A. Its, “Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I”, J. Approx. Theory 162:2 (2010), 270–297. MR

[Chen and McKay 2012] Y. Chen and M. R. McKay, “Coulumb fluid, Painlevé transcendents, and the information theory of MIMO systems”, IEEE Trans. Inform. Theory 58:7 (2012), 4594–4634. MR Zbl

[Conrey 1988] J. B. Conrey, “The fourth moment of derivatives of the Riemann zeta-function”, Quart. J. Math. Oxford Ser. (2) 39:153 (1988), 21–36. MR Zbl

[Conrey and Ghosh 1984] J. B. Conrey and A. Ghosh, “On mean values of the zeta-function”, Mathematika 31:1 (1984), 159–161. MR Zbl

[Conrey et al. 2006] J. B. Conrey, M. O. Rubinstein, and N. C. Snaith, “Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function”, Comm. Math. Phys. 267:3 (2006), 611–629. MR Zbl

[Dai et al. 2019] D. Dai, P. J. Forrester, and S.-X. Xu, “Applications in random matrix theory of a PIII’ r-function sequence from Okamoto’s Hamiltonian formulation”, 2019. arXiv

[Dehaye 2008] P.-O. Dehaye, “Joint moments of derivatives of characteristic polynomials”, Algebra Number Theory 2:1 (2008), 31–68. MR Zbl

[Dehaye 2010] P.-O. Dehaye, “A note on moments of derivatives of characteristic polynomials”, pp. 681–692 in 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010) (San Francisco), Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010. 270. MR Zbl

[Deift et al. 1999] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, “Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory”, Comm. Pure Appl. Math. 52:11 (1999), 1335–1425. MR Zbl

[Dyson 1962] F. J. Dyson, “Statistical theory of the energy levels of complex systems, III”, J. Mathematical Phys. 3 (1962), 166–175. MR Zbl

[Forrester 1993] P. J. Forrester, “The spectrum edge of random matrix ensembles”, Nuclear Phys. B 402:3 (1993), 709–728. MR Zbl

[Forrester and Hughes 1994] P. J. Forrester and T. D. Hughes, “Complex Wishart matrices and conductance in mesoscopic systems: exact results”, J. Math. Phys. 35:12 (1994), 6736–6747. MR Zbl

[Forrester and Warnaar 2008] P. J. Forrester and S. O. Warnaar, “The importance of the Selberg integral”, Bull. Amer. Math. Soc. (N.S.) 45:4 (2008), 489–534. MR Zbl
FORRESTER AND WITTE 2002 P. J. Forrester and N. S. Witte, “Application of the r-function theory of Painlevé equations to random matrices: Pv, PIII, the LUE, JUE, and CUE”, Comm. Pure Appl. Math. 55:6 (2002), 679–727. MR Zbl

[Gradshteyn and Ryzhik 2007] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, 7th ed., Elsevier, Amsterdam, 2007. MR Zbl

[Hall 2002] R. R. Hall, “A Wirtinger type inequality and the spacing of the zeros of the Riemann zeta-function”, J. Number Theory 93:2 (2002), 235–245. MR Zbl

[Har 1979] L. K. Hua, Harmonic analysis of functions of several complex variables in the classical domains, Translations of Mathematical Monographs 6, American Mathematical Society, Providence, R.I., 1979. MR

[Hughes 2001] C. Hughes, On the characteristic polynomial of a random unitary matrix and the Riemann zeta function, Ph.D. thesis, 2001.

[Ingham 1927] A. E. Ingham, “Mean-value theorems in the theory of the Riemann Zeta-function”, Proc. London Math. Soc. (2) 27:4 (1927), 273–300. MR Zbl

[Johansson 2001] K. Johansson, “Universality for eigenvalue correlations from the modified Jacobi unitary ensemble”, Comm. Math. Phys. 215:3 (2001), 683–705. MR Zbl

[Kenning and Snaith 2000a] J. P. Keating and N. C. Snaith, “Random matrix theory and L-functions at s = 1/2”, Comm. Math. Phys. 214:1 (2000), 91–110. MR

[Kenning and Snaith 2000b] J. P. Keating and N. C. Snaith, “Random matrix theory and ζ(1/2 + it)”, Comm. Math. Phys. 214:1 (2000), 57–89. MR

[Kuijlaars and Vanlessen 2002] A. B. J. Kuijlaars and M. Vanlessen, “Universality for eigenvalue correlations from the modified Jacobi unitary ensemble”, Int. Math. Res. Not. 2002:30 (2002), 1575–1600. MR Zbl

[Lubinsky 2008] D. S. Lubinsky, “Universality limits at the hard edge of the spectrum for measures with compact support”, Int. Math. Res. Not. 2008 (2008), art. id. rnm 099. MR Zbl

[Luke 1972] Y. L. Luke, “Inequalities for generalized hypergeometric functions”, J. Approximation Theory 5 (1972), 41–65. MR Zbl

[Mehta 1991] M. L. Mehta, Random matrices, 2nd ed., Academic Press, Boston, 1991. MR Zbl

[Mezzadri and Simm 2013] F. Mezzadri and N. J. Simm, “Tau-function theory of chaotic quantum transport with β = 1, 2, 4”, Comm. Math. Phys. 324:2 (2013), 465–513. MR Zbl

[Montgomery 1973] H. L. Montgomery, “The pair correlation of zeros of the zeta function”, pp. 181–193 in Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV) (St. Louis Univ., St. Louis, Mo. (1972)), edited by H. G. Diamond, 1973. MR Zbl

[Murata 1995] Y. Murata, “Classical solutions of the third Painlevé equation”, Nagoya Math. J. 139 (1995), 37–65. MR Zbl

[Neretin 2015] Y. A. Neretin, “Matrix Beta-integrals: An Overview”, pp. 257–272 in Geometric Methods in Physics, edited by P. Kielanowski et al., Springer, Cham, 2015. Zbl

[Okamoto 1987] K. Okamoto, “Studies on the Painlevé equations, II: Fifth Painlevé equation Py”, Japan. J. Math. (N.S.) 13:1 (1987), 47–76. MR Zbl

[Olshanski and Vershik 1996] G. Olshanski and A. Vershik, “Ergodic unitarily invariant measures on the space of infinite Hermitian matrices”, pp. 137–175 in Contemporary mathematical physics, edited by M. A. S. R. L. Dobrushin, R. A. Minlos and A. M. Vershik, Amer. Math. Soc. Transl. Ser. 2 175, Amer. Math. Soc., Providence, RI, 1996. MR

[Osipov and Kanzieper 2007] V. A. Osipov and E. Kanzieper, “Are Bosonic Replicas Faulty?”, Phys. Rev. Lett. 99 (2007), art. id. 050602.

[Osipov and Kanzieper 2010] V. A. Osipov and E. Kanzieper, “Correlations of RMT characteristic polynomials and integrability: Hermitean matrices”, Ann. Physics 325:10 (2010), 2251–2306. MR Zbl

[Pickrell 1987] D. Pickrell, “Measures on infinite-dimensional Grassmann manifolds”, J. Funct. Anal. 70:2 (1987), 323–356. MR Zbl
[Pickrell 1991] D. Pickrell, “Mackey analysis of infinite classical motion groups”, Pacific J. Math. 150:1 (1991), 139–166. MR Zbl

[Qiu 2017] Y. Qiu, “Infinite random matrices & ergodic decomposition of finite and infinite Hua–Pickrell measures”, Adv. Math. 308 (2017), 1209–1268. MR Zbl

[Rider and Waters 2019] B. Rider and P. Waters, “Universality of the stochastic Bessel operator”, Probab. Theory Related Fields 175:1-2 (2019), 97–140. MR Zbl

[Stein and Shakarchi 2003] E. M. Stein and R. Shakarchi, Complex analysis, Princeton Lectures in Analysis 2, Princeton University Press, Princeton, NJ, 2003. MR Zbl

[Tracy and Widom 1994] C. A. Tracy and H. Widom, “Level spacing distributions and the Bessel kernel”, Comm. Math. Phys. 161:2 (1994), 289–309. MR Zbl

[Umemura 1990] H. Umemura, “Second proof of the irreducibility of the first differential equation of Painlevé”, Nagoya Math. J. 117 (1990), 125–171. MR Zbl

[Umemura and Watanabe 1998] H. Umemura and H. Watanabe, “Solutions of the third Painlevé equation, I”, Nagoya Math. J. 151 (1998), 1–24. MR Zbl

[Winn 2012] B. Winn, “Derivative moments for characteristic polynomials from the CUE”, Comm. Math. Phys. 315:2 (2012), 531–562. MR Zbl

[Witte and Forrester 2000] N. S. Witte and P. J. Forrester, “Gap probabilities in the finite and scaled Cauchy random matrix ensembles”, Nonlinearity 13:6 (2000), 1965–1986. MR Zbl

[Xu et al. 2014] S.-X. Xu, D. Dai, and Y.-Q. Zhao, “Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble”, Comm. Math. Phys. 332:3 (2014), 1257–1296. MR Zbl

[Xu et al. 2015] S.-X. Xu, D. Dai, and Y.-Q. Zhao, “Painlevé III asymptotics of Hankel determinants for a singularly perturbed Laguerre weight”, J. Approx. Theory 192 (2015), 1–18. MR Zbl

Received 14 Sep 2020. Revised 9 Jan 2021. Accepted 23 Feb 2021.

THEODOROS ASSIOTIS: theo.assiotis@ed.ac.uk
School of Mathematics, University of Edinburgh, Edinburgh, United Kingdom

BENJAMIN BEDERT: benjamin.bedert@sjc.ox.ac.uk
Mathematical Institute, University of Oxford, Oxford, United Kingdom

MUSTAFA ALPER GUNES: mustafa.gunes@st-hildas.ox.ac.uk
Mathematical Institute, University of Oxford, Oxford, United Kingdom

ARUN SOOR: arun.soor@sjc.ox.ac.uk
Mathematical Institute, University of Oxford, Oxford, United Kingdom
| Title                                                                 | Pages |
|-----------------------------------------------------------------------|-------|
| Covariant Symanzik identities                                          | 419   |
| Adrien Kassel and Thierry Lévy                                        |       |
| Quantitative spectral gaps for hypoelliptic stochastic differential   | 477   |
| equations with small noise                                             |       |
| Jacob Bedrossian and Kyle Liss                                         |       |
| A family of probability distributions consistent with the DOZZ formula | 533   |
| conjecture for the law of 2D GMC                                       |       |
| Dmitry Ostrovsky                                                       |       |
| Excess deviations for points disconnected by random interlacements    | 563   |
| Alain-Sol Sznitman                                                     |       |
| On a distinguished family of random variables and Painlevé equations  | 613   |
| Theodoros Assiotis, Benjamin Bedert, Mustafa Alper Gunes and Arun Soor|       |
| Geodesic networks in Liouville quantum gravity surfaces               | 643   |
| Ewain Gwynne                                                          |       |