SHORT NOTE ON THE CONVOLUTION OF BINOMIAL COEFFICIENTS

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Abstract. We know \cite{1} that, for every non-negative integer numbers \( n, i, j \) and for every real number \( \ell \),

\[
\sum_{i+j=n} \binom{2i-\ell}{i} \binom{2j+\ell}{j} = \sum_{i+j=n} \binom{2i}{i} \binom{2j}{j},
\]

which is well-known to be \( 4^n \). We extend this result by proving that, indeed,

\[
\sum_{i+j=n} \binom{a+i+k-\ell}{i} \binom{a+j+\ell}{j} = \sum_{i+j=n} \binom{a+i+k}{i} \binom{a+j}{j}
\]

for every integer \( a \) and for every real \( k \), and present new expressions for this value.

We consider the sequence \( \{\binom{a}{n}\}_{n=0}^\infty \), where \( a \) is any integer number, negative, zero or positive, and take the convolution of this sequence with itself, defined by \( P_a(n) = \sum_{i+j=n} \binom{a}{i} \binom{a}{j} \).

When \( a = 2 \), the former is sequence A000984 of \cite{2}, the central binomial coefficients, and the latter is sequence A000302 of \cite{2}, the powers of 4. In fact (cf. \cite{1}), this can be proved directly using (1), and then the inclusion-exclusion principle. Note that

\[
2 \sum_{i=0}^{n+1} \binom{2n+1}{i} = 2^n (n+1).
\]

For another identity, define as usual \([n] = \{1,\ldots, n\}\) for any natural number \( n \), and consider the collection of the subsets of \([2n]\) with more than \( n \) elements with the same \((n+1)\)-th element, say \( p \). Note that \( p = n+1+i \) for some \( i = 0,\ldots, n-1 \) and that there are \( \binom{n+i}{n} \) \( 2^{n-i-1} \) subsets in the collection. It follows that the number of all subsets of \([2n]\) is

\[
P_2(n) = 2^{2n} = 2 \sum_{i=0}^{n-1} 2^{n-i-1} \binom{n+i}{i} + \binom{2n}{n} = \sum_{i=0}^{n} 2^{n-i} \binom{n+i}{i}.
\]

We generalize these identities, namely \cite{1}, \cite{3} and \cite{4}. When \( a = 3 \) and \( a = 4 \), we have sequences A006256 and A078995 of \cite{2}, and no such simple formulas for \( P_3(n) \) and \( P_4(n) \) are known as in case \( a = 2 \). For these sequences, we obtain, for every real \( \ell \),

\[
\sum_{i+j=n} \binom{3i}{i} \binom{3j}{j} = \sum_{i+j=n} 2^i \binom{3n+1}{j} = \sum_{i+j=n} 3^i \binom{2n+j}{j} = \sum_{i+j=n} \binom{3i-\ell}{i} \binom{3j+\ell}{j},
\]

\[
\sum_{i+j=n} \binom{4i}{i} \binom{4j}{j} = \sum_{i+j=n} 3^i \binom{4n+1}{j} = \sum_{i+j=n} 4^i \binom{3n+j}{j} = \sum_{i+j=n} \binom{4i-\ell}{i} \binom{4j+\ell}{j}.
\]

More generally we obtain the following theorem.
Theorem 1. For every non-negative integer numbers \(i, j\) and \(n\), and for every real numbers \(k\) and \(\ell\),
\[
\sum_{i+j=n} \binom{ai+k-\ell}{i} \binom{aj+\ell}{j} = \sum_{i+j=n} \binom{ai+k}{i} \binom{aj}{j}
\]
(5)
\[
= \sum_{i=0}^{n} (a-1)^{n-i} \binom{an+k+1}{i}
\]
(6)
\[
= \sum_{i=0}^{n} a^{n-i} \binom{(a-1)n+k+i}{i}
\]
where we take \(0^0 = 1\).

For the proof of this theorem we need some technical results.

Lemma 2. Let, for any real \(\ell\) and integers \(a\) and \(n\) such that \(n \geq 0\),
\[
S_{a,\ell}(n) = \sum_{i=0}^{n} (-1)^i \binom{\ell-(a-1)i}{i} \binom{\ell-a i}{n-i}
\]
Then
\[
\sum_{i=0}^{n} \binom{n}{p} S_{a,\ell}(p) = S_{a+1,\ell+n}(n).
\]

Proof. First note that we may assume that \(\ell\) is a natural number, since \(S_{a,\ell}(n)\) is a polynomial in \(\ell\), and thus is constant. Now, suppose that \(S_{a,\ell}(p) = x^p\) for some numbers \(a, \ell, p\) and \(x\). Then, from Lemma 2 it follows that \(S_{a+1,\ell+n}(n) = (1+x)^n\). Hence, all we must prove is that \(S_{a,\ell}(n) = 0\) when \(a = 1\) and \(\ell \in \mathbb{N}\).

For this purpose, define \(A = A_\emptyset\) as the set of \(n\)-subsets of the set \([\ell] = \{1, 2, \ldots, \ell\}\) and, for every non-empty subset \(T\) of \([\ell]\), \(A_T = \{A \in A \mid A \cap T = \emptyset\}\). Now, the result follows immediately from the inclusion-exclusion principle applied to this family. \(\square\)

Lemma 3. With the notation of the previous lemma,
\[
S_{a,\ell}(n) = (a-1)^n.
\]

Proof. First note that we may assume that \(\ell\) is a natural number, since \(S_{a,\ell}(n)\) is a polynomial in \(\ell\), and thus is constant. Now, suppose that \(S_{a,\ell}(p) = x^p\) for some numbers \(a, \ell, p\) and \(x\). Then, from Lemma 3 it follows that \(S_{a+1,\ell+n}(n) = (1+x)^n\). Hence, all we must prove is that \(S_{a,\ell}(n) = 0\) when \(a = 1\) and \(\ell \in \mathbb{N}\).

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Lemma 4. Let $s$ and $t$ be positive integers. Then

$$\binom{s+t+1}{j} = \sum_{i=0}^{j} \binom{s-i}{s-j} \binom{t+i}{i}.$$  

Proof. Given a subset $S$ of $[n]$ with $k$ elements and $p \in [n] \setminus S$, let $\text{Bef}_p(S) = S \cap [p-1]$ and $\text{Aft}_p(S) = \{t \in [n-p] \mid t+p \in S\}$.

Now, let $A$ be a subset of $[s+t+1]$ with $j$ elements and $p(A)$ be the $s-j+1$ smallest element of $[s+t+1]$ which is not in $A$. In other words, $\#\{x \in A \mid x < p(A)\} = j-i$ and $\#\{x \in A \mid x > p(A)\} = i$. One can easily see that the mapping

$$\varphi : \mathcal{P}_j([s + t + 1]) \rightarrow \bigcup_{0 \leq i \leq j} \mathcal{P}_{j-i}([s - i]) \times \mathcal{P}_i([t + i])$$

$$A \quad \mapsto \quad (\text{Bef}_p(A), \text{Aft}_p(A))$$

is a bijection, with inverse given by $\psi(B, C) = B \cup \{c + \#C \mid c \in C\}$, and the union is disjoint. □

Proof of Theorem 5. Let $\mathcal{G} = \sum_{i+j=n} \left(\begin{array}{c} a + k - i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} a + j + \ell \vspace{1pt} \\ j \end{array}\right) = \sum_{i+j=n} (-1)^i \left(\begin{array}{c} \ell - k' - (a - 1)i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} a + n + k' \vspace{1pt} \\ m \end{array}\right)$, with $k' = k + 1$. Then, by Vandermonde’s convolution,

$$\mathcal{G} = \sum_{i+j=n} \left(\begin{array}{c} a + k - i \vspace{1pt} \\ i \end{array}\right) \sum_{p+m=j} \left(\begin{array}{c} a + k' \vspace{1pt} \\ p \end{array}\right) \left(\begin{array}{c} \ell - k' - a i \vspace{1pt} \\ m \end{array}\right)$$

$$= \sum_{p=0}^{n} \left(\begin{array}{c} a + k' \vspace{1pt} \\ p \end{array}\right) \sum_{i+m=n-p} (-1)^i \left(\begin{array}{c} \ell - k' - (a - 1)i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} \ell - k' - a i \vspace{1pt} \\ m \end{array}\right).$$

Now, (5) follows immediately from Lemma 3 and (6) from Lemma 4. □

We end this article with a new result that, when we represent by $\binom{n}{k}$ the number of $k$-multisets of elements of an $n$-set, can be formulated in the following elegant terms.

Theorem 5. For every real $\ell$ and integers $a, n, i, j$ such that $n, i, j \geq 0$,

$$\sum_{i+j=n} (-1)^i \left(\begin{array}{c} \ell - a i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} \ell - a i \vspace{1pt} \\ j \end{array}\right) = a(a - 1)^{n-1}.$$  

Proof. By Pascal’s rule,

$$\sum_{i+j=n} (-1)^i \left(\begin{array}{c} \ell - 1 - (a - 1)i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} \ell - a i \vspace{1pt} \\ j \end{array}\right) = \sum_{i=0}^{n} (-1)^i \left(\begin{array}{c} \ell - (a - 1)i \vspace{1pt} \\ i \end{array}\right) \left(\begin{array}{c} \ell - a i \vspace{1pt} \\ n - i \end{array}\right)$$

$$- \sum_{i=1}^{n} (-1)^i \left(\begin{array}{c} \ell - (a - 1)i - 1 \vspace{1pt} \\ i - 1 \end{array}\right) \left(\begin{array}{c} \ell - a i \vspace{1pt} \\ n - i \end{array}\right)$$

$$= S_{a,\ell}(n) + S_{a,\ell-a}(n - 1).$$

Problem 6. Give a full combinatorial proof of Theorem 5. □

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