Random geometric graphs in high dimension

Vittorio Erba\textsuperscript{1,2,a}, Sebastiano Ariosto\textsuperscript{1} Marco Gherardi\textsuperscript{1,2} and Pietro Rotondo\textsuperscript{2}

\textsuperscript{1} Dipartimento di Fisica dell’Università di Milano, 
\textsuperscript{2} INFN, sezione di Milano, 
Via Celoria 16, 20100 Milano, Italy.

\textsuperscript{a} Corresponding author. Email: vittorio.erba@unimi.it

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Abstract

Many machine learning algorithms used for dimensional reduction and manifold learning leverage on the computation of the nearest neighbours to each point of a dataset to perform their tasks. These proximity relations define a so-called geometric graph, where two nodes are linked if they are sufficiently close to each other. Random geometric graphs, where the positions of nodes are randomly generated in a subset of $\mathbb{R}^d$, offer a null model to study typical properties of datasets and of machine learning algorithms. Up to now, most of the literature focused on the characterization of low-dimensional random geometric graphs whereas typical datasets of interest in machine learning live in high-dimensional spaces ($d \gg 10^2$). In this work, we consider the infinite dimensions limit of hard and soft random geometric graphs and we show how to compute the average number of subgraphs of given finite size $k$, e.g. the average number of $k$-cliques. This analysis highlights that local observables display different behaviors depending on the chosen ensemble: soft random geometric graphs with continuous activation functions converge to the naive infinite dimensional limit provided by Erdös-Rényi graphs, whereas hard random geometric graphs can show systematic deviations from it. We present numerical evidence that our analytical insights, exact in infinite dimensions, provide a good approximation also for dimension $d \gtrsim 10$.

Random geometric graphs (RGGs) are networks whose nodes are $d$-dimensional randomly generated vectors from some probability distribution over $\mathbb{R}^d$, and edges link nodes only if their distance does not exceed a threshold distance $r$. As such, their connectivity structure encodes information about the spatial structure of the nodes, and on the space they are embedded in: for this reason they are widely used in modeling complex systems in which geometric constraints play a fundamental role, such as wireless and social networks. Most of the results on RGGs have been established in the low-dimensional regime $d \leq 3$. However, the high-dimensional limit $d \to \infty$ has recently gathered interest. Indeed in the era of big data and machine learning, typical datasets are made of vectors of hundreds of components (think for instance to the workhorse model in computer vision, the MNIST dataset of handwritten digits); understanding how high-dimensional geometry works, and how it affects the proximity structure of datasets is crucial for the correct usage of manifold learning algorithms, and for the creation of novel procedures tailored for the high-dimensional regime with benefits for dimensional reduction and clustering algorithms. With this idea in mind, high-dimensional RGGs become a perfect null model for unstructured data, to benchmark and compare against real world datasets.

On the more mathematical side, it is an open problem to understand whether high-dimensional RGGs converge (as a statistical ensemble) to Erdös-Rényi graphs; rigorous results for RGGs with nodes uniformly distributed on the sphere can be found in and suggest that high-$d$
RGGs are similar to Erdös-Rényi graphs. On the other hand, the clustering coefficient of RGGs with nodes uniformly distributed on the hypercube shows systematic deviations from the ERG prediction [17].

In this work, we present a general framework for the computation of the average value of local observables of high-dimensional hard and soft RGGs. To this end, we exploit a multivariate version of the central limit theorem (CLT) to show that the joint probability of rescaled distances between nodes is normal-distributed, and we compute and characterize its correlation matrix.

We evaluate the average number of $M$-cliques, i.e. of fully-connected subgraphs with $M$ vertices, in high-dimensional RGGs. We point out that these local observables show systematic deviations from the ERG prediction in hard-RGGs (whenever the hypothesis of the CLT are satisfied), whereas we observe convergence to ERG for soft-RGGs with continuous activation functions. This implies that the form of the activation function as well as the probability distribution on the nodes are crucial elements in studying the convergence of RGGs to ERGs.

Finally, we present numerical evidence that our analytical results do not hold only for $d \to \infty$, but provide a good approximation even in finite dimensions as low as $d \sim 10$. This suggests that the high-dimensional limit of RGGs could be seen as a 0-th order term of a series expansion in $d$, possibly giving perturbative access to analytical results for low-dimensional RGGs.

The manuscript is organized as follows. In Section 1 we introduce the notation and define the ensembles of RGGs that we will study. In Section 2 we use a multivariate version of the central limit theorem to derive an explicit expression for the joint probability distribution of the distances of $M$ randomly drawn vectors in the limit of high dimension. This will be the crucial tool to compute averages of observables in high-dimensional RGGs. Finally, in Section 3 we present our results on the average number of $M$-cliques for hard and soft RGGs alongside with numerical simulations.

1 **Hard and soft random geometric graphs**

**Note on terminology:** in the literature, random geometric graphs are those with hard activation function (see later in this Section). Here, when omitting the adjectives "hard" or "soft" we generically refer to both.

A **random geometric graph** is a graph whose nodes are random points in $\mathbb{R}^d$, and whose edges...
are randomly generated based on the mutual distances between the nodes (see Figure 1). Let us be more precise, starting by nodes. We consider a probability distribution \( \nu \) over \( \mathbb{R}^d \), and we draw \( N \) i.i.d. samples \( \{\vec{x}_i\}_{i=1}^N \) from \( \nu \); these will be the nodes of the random geometric graph. Among the possible choices of \( \nu \), a very common one is the uniform distribution on the \( d \)-dimensional hypercube \([0, 1]^d\), i.e.

\[
\nu_{\text{cube}}(\vec{x}) = \prod_{k=1}^d \theta(x^k)\theta(1-x^k)
\]

where \( \theta \) is the Heaviside theta, and superscripts denote coordinates. We will consider more in general probability distributions \( \nu \) that are factorized and equally distributed over the coordinates, i.e.

\[
\nu(\vec{x}) = \prod_{k=1}^d \tau(x^k)
\]

where \( \tau \) is a probability distribution on \( \mathbb{R} \) with finite first and second moments. In this case, the coordinates of all nodes \( \{x^k_i\} \), with \( 1 \leq i \leq N \) and \( 1 \leq k \leq d \) are i.i.d. random variables with law \( \tau \).

Now, for each pair of nodes \( x, y \) we compute the distance \( d(x, y) \) and we add the link \( e = (x, y) \) to the edge set of the random geometric graph with probability \( h(d(x, y)) \), where \( h : \mathbb{R}^+ \rightarrow [0, 1] \) is the so-called activation function of the random geometric graph. The activation function describes how likely it is for two nodes to be linked based on their distance, and will typically be a monotone decreasing function, with the idea that closer nodes will be linked with higher probability than further ones; we will consider monotone decreasing activation functions, with \( h(0) = 1 \) and \( h(\infty) = 0 \).

Usually, the activation function is labeled by a parameter \( r \in \mathbb{R}^+ \) that describes the typical distance at which a pair of nodes will be considered close enough to be linked with a non-trivial probability, for example \( h_r(r) = \frac{1}{2} \). In this case, the statistical properties of random geometric graphs can be investigated as functions of \( r \).

In this work, we will consider two types of activation functions. The first one is that of hard random geometric graphs, i.e.

\[
h_{r}^\text{hard}(x) = \theta(r-x).
\]

In this case, all pairs of nodes with distance smaller than \( r \) will be deterministically linked by an edge. The second one is that of soft random geometric graphs, i.e. those with \( h_r(x) \) at least continuous in \( x \). A common choice in the literature (see for example [2, 13]) is to employ the so-called Rayleigh fading activation functions, i.e.

\[
h_{r}^\text{rayleigh}(x) = \exp\left[-\xi \left(\frac{x}{r}\right)^\eta\right],
\]

where \( \xi = \log(2) \) guarantees that \( h_r(r) = \frac{1}{2} \).

The last ingredient to be discussed is the distance function \( d(x, y) \). We will consider the \( p \)-norms \( \mathbb{R}^d \)

\[
||\vec{x}||_p = \sqrt[p]{\sum_{i=1}^d |x^i|^p}.
\]

Notice that \( p \)-norms are norms only for \( p \geq 1 \), as for \( 0 < p < 1 \) the triangle inequality is not satisfied. It this case, one can show that \( ||\vec{x} - \vec{y}||_p^{\min(1,p)} \) defines nonetheless a distance. Thus, we will define and consider the distances

\[
d_p(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||_p^{\min(1,p)}.
\]
2 A central limit theorem for distances in high dimension

As a first step in our analysis, we are interested in computing the high-dimensional limit of the joint probability distribution of the distances between $M$ random points $\{\vec{x}_i\}_{i=1}^M \subset \mathbb{R}^d$, drawn independently from the factorized distribution $\nu$ in Eq. (2):

$$
\Pi(d_{(1,2)}, d_{(1,3)}, \ldots, d_{(M-1,M)}) = 
\int \prod_{i=1}^M \nu(\vec{x}_i) dx_i \prod_{1 \leq i < j \leq M} \delta \left( d_p(\vec{x}_i, \vec{x}_j) - d_{(i,j)} \right). 
$$

(7)

Since the distance $d_p(\vec{x}, \vec{y})$ between two vectors $\vec{x}, \vec{y}$ is a function of the sum of $d$ i.i.d. random variables, we expect that for $d \to \infty$ it converges to its average value $d \mu$ by the law of large numbers. Correspondingly, let us define the rescaled variables

$$
q_{(i,j)} = \frac{[d_p(\vec{x}, \vec{y})]^{\max(1,p)} - d \mu}{\sqrt{d}} 
$$

(8)

Notice that the random vectors $q_k = (q_{(1,2)}^k, q_{(1,3)}^k, \ldots, q_{(M-1,M)}^k) \in \mathbb{R}^{(M^2)}$, with $1 \leq k \leq d$, are statistically independent and identically distributed, and that by definition the expected value of $q_k$ is the null vector. Notice also that the components of the vectors $q_k$ are naturally indexed by lexicographically ordered multi-indices, as they are related to the distances between pairs of points along the $k$-th dimension; to distinguish such vectors from the Euclidean ones, we type them in boldface.

The vector $q = (q_{(1,2)}, q_{(1,3)}, \ldots, q_{(M-1,M)})$ is a sum of i.i.d. multivariate random variables, and satisfies the following central limit theorem:

**Theorem 1** (Multivariate central limit theorem). Let $q_1, q_2, \ldots, q_d$ be i.i.d. random vectors in $\mathbb{R}^{(M^2)}$ with null mean and covariance matrix $\Sigma_{(i,j),(k,l)} = E \left[ q_{(i,j)}^k q_{(k,l)}^l \right]$. Then

$$
q = \frac{1}{\sqrt{d}} \sum_{k=1}^d q_k
$$

(10)

is Gaussian-distributed with null mean and covariance $\Sigma$.

The general formal proof can be found in [19]. A "physicist" approach to the proof would be to compute the characteristic function of $q$ and to expand it to the leading order for large $d$. It is worth noticing that the first neglected term in the expansion is of relative order $1/\sqrt{d}$, and may depend on $M$. Thus, this $d \to \infty$ limit is to be intended at fixed $M$, and the result can be used either to treat generic observables for graphs where the total number of nodes is fixed, or to treat observables that depend only on a finite number of nodes at a time in graphs where the total number of nodes may scale with $d$.

The CLT presented above holds for the variable $q$, and not for the actual distances. However this is not an issue, as the joint distribution for distances can be derived by a simple coordinate change, factorized over each direction. Moreover, as we will see in the following, it is often easy to obtain the observables of interest in terms of the $q$ variable.
We now focus on the explicit form of the covariance matrix $\Sigma$ (notice that, as the vectors $q_k$, the covariance matrix is indexed by multi-indices). By definition, one has

$$
\Sigma_{(i,j),(k,l)} = \mathbb{E}[(|y_i - y_j|^p - \mu)(|y_k - y_l|^p - \mu)]
$$

(11)

where $y_i, y_j, y_k, y_l$ are all i.i.d. random variables with distribution $\tau$, and $1 \leq i < j \leq M, 1 \leq k < l \leq M$. By permutational symmetry, only three different cases are possible:

- **Diagonal correlations ($i = k$ and $j = l$)**
  $$
  \Sigma_{(i,j),(i,j)} := \alpha = \int dx dy \tau(x) \tau(y)|x - y|^{2p} - \mu^2;
  $$
  (12)

- **Triangular correlations ($i = k$ and $j \neq l$ or $i \neq k$ and $j = l$)**
  $$
  \Sigma_{(i,j),(i,k)} = \Sigma_{(i,j),(k,j)} := \beta = \int dx dy dz \tau(x) \tau(y) \tau(z)|x - y|^p|x - z|^p - \mu^2;
  $$
  (13)

- **Pair-pair correlations ($i, j, k, l$ are all distinct)**
  $$
  \Sigma_{(i,j),(k,l)} := \gamma = \left(\int dx dy \tau(x) \tau(y)|x - y|^p\right)^2 - \mu^2 = 0.
  $$
  (14)

In the case of the hypercube $\nu = \nu^\text{cube}$, $\tau(x) = \theta(x)\theta(1-x)$, the coefficients $\alpha$ and $\beta$ are given by:

$$
\alpha^{\text{cube}} = \frac{p^2(p+5)}{(p+1)^2(p+2)^2(2p+1)},
\beta^{\text{cube}} = \frac{2}{(p+1)^2} \times \left[\frac{p^2 - 2}{(2p+3)(p+2)^2} + \frac{\Gamma(p+2)^2}{\Gamma(2p+4)}\right],
$$

(15)

where $\Gamma(x)$ is the Euler gamma function. In general, $\alpha$ and $\beta$ depend only on the choice of $\tau$.

The general form of a matrix with the symmetries of $\Sigma$ is given by (see Figure 2):

$$
\Delta_{(i,j),(k,l)}(M,\alpha,\beta,\gamma) = (\alpha - 2\beta + \gamma)\delta_{i,k}\delta_{j,l} + (\beta - \gamma)(\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l}) + \gamma,
$$

(16)

where $\delta_{i,j}$ is the Kronecker delta, and $\binom{M}{2} \times \binom{M}{2}$ is the size of the matrix. We collect properties of such matrices in the following Proposition:

**Proposition 1.** Let $\Delta$ be a matrix of the form of Equation 16, then:

1. it can be written as
  $$
  \Delta(M,\alpha,\beta,\gamma) = (\alpha - \gamma)I + (\beta - \gamma)J + \gamma U
  $$
  (17)

  where $I$ is the identity matrix, $U$ is the matrix with all elements equal to one and $J$ is the adjacency matrix (with null diagonal) of the Johnson graph $J(M,2)$, which is the line graph of the complete graph over $M$ vertices;

2. the eigenvalues are:
\[ \Delta(N, \alpha, \beta, \gamma) \]

- \( N = 8 \)
- \( N = 5 \)

![Figure 2](image)

**Figure 2:** (Left) Example of a matrix \( \Delta(N, \alpha, \beta, \gamma) \) for \( N=8 \). The entries with value equal to \( \beta \) have the same structure of the adjacency matrix of the Johnson graph. (Right) Example of Johnson graph with \( N=5 \).

- \( \lambda_1 = \alpha + 2(N - 2)\beta + \frac{(N-2)(N-3)}{2}\gamma \) with multiplicity 1;
- \( \lambda_2 = \alpha + (N - 4)\beta - (N - 3)\gamma \) with multiplicity \( N - 1 \);
- \( \lambda_3 = \alpha - 2\beta + \gamma \) with multiplicity \( \frac{N(N-3)}{2} \).

3. The inverse matrix \( \Delta^{-1} \) is of the same form of \( \Delta \) with inverse eigenvalues, and its parameters \( \alpha', \beta' \) and \( \gamma' \) can be found by solving the linear system

\[
\lambda_i(\Delta(M, \alpha, \beta, \gamma)) \times \lambda_i(\Delta(M, \alpha', \beta', \gamma'))^{-1} = 1
\]

\( i = 1, 2, 3 \). \hspace{1cm} (18)

**Proof.**

1) Follows from the explicit expression of \( I, J \) and \( U \).

2) \( I, J \) and \( U \) commute between each other, and can be diagonalized simultaneously. The contribution of \( I \) is trivial. \( J \) and \( U \) share a non-degenerate eigenvector (that with all components equal to one), that accounts for \( \lambda_1 \). In the orthogonal subspace, \( U \) represents the null operator, and does not contribute. Thus, the remainder of the spectrum is determined by that of \( J \), which is known [20].

3) Follows from the fact that a matrix and its inverse share the same eigenvectors. \( \square \)

### 3 Number of cliques reveals non-trivial structure of hard geometric graphs

We are now ready to compute observables on random geometric graphs in the limit of infinite dimensions; in particular, we aim to characterize the average number of subgraph with a given structure. Recall that the adjacency matrix of a graph \( g \) with \( M \) nodes is the \( M \times M \) matrix with entry \( A_{ij}(g) = 1 \) if \( (i, j) \) is an edge of \( g \), and \( A(g)_{ij} = 0 \) otherwise.

In general, the average number of a certain subgraph \( g \) with \( M \) nodes of a random geometric graph can be factored in two terms. The first one is a combinatorial factor \( \binom{N}{M} \), that accounts for the number of ways in which one can extract \( M \) nodes from a set of \( N \) of them. The second one is the so-called density \( \rho_g(r) \) of the subgraph \( g \) at scale \( r \), that is the probability that \( M \) random points are close enough with respect to the cutoff radius \( r \) to form a subgraph with the same
adjacency matrix of $g$:

$$
\rho_g(r) = \int d\Pi(d) \prod_{1 \leq i < j \leq M} \left[ h_r(d_{(i,j)}) \right]^{A_{ij}(g)}
$$

$$
= \int dq \mathcal{N}(0, \Sigma)(q) \times \prod_{1 \leq i < j \leq M} \left[ h_r\left( \left[ d\mu + \sqrt{d}q_{(i,j)} \right]^{\min(1, \frac{1}{2})} \right) \right]^{A_{ij}(g)}
$$

(19)

where $\mathcal{N}(0, \Sigma)$ is the multivariate Gaussian with null mean and covariance $\Sigma$, i.e.

$$
\mathcal{N}(0, \Sigma)(q) = \frac{e^{-\frac{1}{2}q^T\Sigma q}}{(2\pi)^{(M/2)} \det \Sigma}.
$$

(20)

As a paradigmatic example, we consider the average density of $M$-cliques $\rho_M(r)$, i.e. fully connected subgraphs with $M$ vertices, on random geometric graphs with generic activation function $h_r(x)$; in this specific case, $A_{ij}$ has only unit entries, so that

$$
\rho_M(r) = \int dq \mathcal{N}(0, \Sigma)(q) \times \prod_{1 \leq i < j \leq M} h_r\left( \left[ d\mu + \sqrt{d}q_{(i,j)} \right]^{\min(1, \frac{1}{2})} \right).
$$

(21)

In the case of hard activation function $h_{\text{hard}}$, we observe that

$$
h_{\text{hard}}^\text{hard}(x) = h_{\text{hard}}(x^p)
$$

$$
h_{\text{hard}}^\text{hard}(x) = h_{\text{hard}}(cx), \quad \forall c \in \mathbb{R}^+
$$

(22)

so that the $p$-th root can be discarded along with a factor of $\sqrt{d}$, and the integral reduces to

$$
\rho_M^\text{hard}(r) = \rho_M^\text{hard}\left( \frac{x^{\max(1,p)} - d\mu}{\sqrt{d}} \right),
$$

(23)

with

$$
\frac{\rho_M^\text{hard}}{\rho_M^\text{hard}}(x) = \frac{\rho_M^\text{hard}}{\rho_M^\text{hard}}(x) = \prod_{1 \leq i < j \leq M} h_x^\text{hard}(q_{i,j}) = \prod_{1 \leq i < j \leq M} \theta(x - q_{i,j}),
$$

(24)

which is a multivariate Gaussian cumulative distribution function. Eq. (23) highlights the simple dependence of $\rho_M^\text{hard}$ on the parameters $p, d$ and $\mu$.

In the case $M = 2$, the integral in Eq. (24) can be explicitly solved as it reduces to the computation of an error function, giving

$$
\rho_2^\text{hard}(x) = \frac{1}{2} \left[ 1 + \text{Erf}\left( \frac{x}{\sqrt{2}d} \right) \right].
$$

(25)

The simple dependence of $\rho_M(r)$ on $p, d$ and $\mu$ suggests to study the quantities

$$
\omega_M^\text{hard}(x) = (\rho_M^\text{hard} \circ (\rho_2^\text{hard}^{-1}))(x) = (\rho_M^\text{hard} \circ (\rho_2^\text{hard}^{-1}))(x),
$$

(26)
as in this way all these dependences cancel out, and the curves at different values of the parameters all lie in the domain $x \in [0,1]$. By changing variable from $r$ to $(\rho_2(r))^{-1}$, we are expressing our observables as functions of the average probability that two nodes are linked, so that we are explicitly factoring out the typical scale of separation between the nodes from the observables.

In the case of continuous activation functions, one can expand $h_r(x)$ to the 0-th order in powers of $1/\sqrt{d}$, obtaining that in the limit of high dimension

$$
\begin{align*}
\rho_M^{\text{regular}}(r) &= \left[\rho_2^{\text{regular}}(r)\right]^{\left(\frac{M}{2}\right)} \\
\rho_2^{\text{regular}}(r) &= h_r\left((d\mu)_{\min}^{(1+\frac{1}{\eta})}\right)
\end{align*}
$$

(27)

Here, the relation between $\rho_M$ and $\rho_2$ reduces to that of Erdős-Rényi graphs with linking probability $\rho_2^{\text{regular}}(r)$, i.e.,

$$
\omega_M^{\text{soft}}(x) = \omega_M^{\text{ER}}(x) = x^{\left(\frac{M}{2}\right)}.
$$

(28)

In the special case of Rayleigh fading activation function $h^{\text{rayleigh}}$, one has

$$
\rho_2^{\text{rayleigh}}(r) = \exp\left[-\xi\left(\frac{d\mu}{r}\right)^{\eta_{\min}^{(1+\frac{1}{\eta})}}\right].
$$

(29)

Intuitively, the difference between hard and soft RGGs depends on the freedom in performing the rescaling of the cutoff radius in the former case [see Eq. (23)], which is lost in the latter.

We performed extensive numerical simulations to study Equation (24) and to check our analytical predictions in infinite dimension and finite dimensional simulations for both hard and soft random geometric graphs. The convergence to the limit is fast, and even at $d = 20$ our analytical prediction provides a good approximation of the simulated observables. More quantitatively, we observe relative deviations from the analytical predictions of the order of ~10% in $d = 20$ and ~2% in $d = 200$ in both the hard and soft case for $k = 3$. For $k = 4,5$ relative errors are slightly larger, mainly due to the fact the we are measuring $\rho_k(r)$ by a random sampling procedure (see Appendix A) that needs more and more samples as $k$ increases.
4 Discussion

In this work we exploited a multivariate version of the central limit theorem to compute average observables of random geometric graphs in the limit of infinite dimension. In particular, we obtained the average number of $M$-cliques in hard and soft RGGs for different distance functions induced by $p$-norms.

Our approach highlights that convergence to the ERG prediction for local observables depends on the choice of the ensemble: soft RGGs in particular seem to approach this naive limit for $d \to \infty$, whereas hard RGGs whose probability distribution of the nodes fulfills the CLT hypothesis deviate systematically from it. This result suggests that the latter provide a non-trivial null model to benchmark empirical data.

Since the CLT can be formulated in a much more general setting than the one reported in this manuscript, we expect that our findings hold (possibly with slight modifications) for several probability distributions of the nodes not included here, e.g. not factorized over coordinates, but with mild inter-coordinate correlations; factorized over coordinates, but not identically distributed; factorized over coordinates, but with infinite second moment.

As a second point, our numerical simulations show that the infinite dimensional limit is a good approximation even in finite dimensions of order $d \sim 10$. This hints at the possibility to improve our results by computing higher order corrections to the CLT, and using $d$ as a perturbative parameter to access the low dimensional regime of RGGs.

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To numerically evaluate the integrals of Equation (24), we implemented the algorithm described in [21], allowing very fast run times for the small values of $M$ ($M \lesssim 10$) we where interested in; notice that the dimension of the integral is already of order $10^2$ for $M = 10$. Higher values of $M$ would require finer techniques.

To compute the density of $M$-cliques in simulated hard RGGs, we implemented a simple random sampling procedure, as exhaustive enumeration scales poorly, i.e. as $O(N^M)$, with the total number of nodes. For each realization of the nodes (with $r_{\text{cube}}$ and $N = 10^4$), we extracted $\sim 5 \cdot 10^5$ $M$-uples of nodes, computing the minimum cutoff distance at which they formed a clique. The cumulative distribution of the minimal distances obtained, averaged over different realization of the nodes, reconstructs $\rho_{\text{hard}}(r)$. We noticed that as $N$ grows, the last average is well approximated by a single realization of the nodes, suggesting a self-averaging property for the density of $M$-cliques; in practice, not averaging does not affect the results of the simulations.

To compute the density of cliques in simulated soft RGGs with generic activation function, we implemented again a random sampling procedure. This time, for each realization of the nodes (as above) and for a fixed radius $r$, we counted how many of $\sim 10^4$ $M$-uples of nodes $\{y_i\}_{i=1}^M$ where $M$-cliques, considering each of them to be a $M$-clique with probability $h_r(d(\vec{y}_i, \vec{y}_j))$.

Normalizing the count over the total number of candidate cliques and averaging over different realizations of the nodes (order $10^4$) gives an empirical estimation for $\rho_M(r)$ in the soft case.