Tightening the tripartite quantum memory assisted entropic uncertainty relation

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The uncertainty principle determines the distinction between the classical and quantum worlds. This principle states that it is not possible to measure two incompatible observables with the desired accuracy simultaneously. In quantum information theory, Shannon entropy has been used as an appropriate measure to express the uncertainty relation. According to the applications of entropic uncertainty relation, studying and trying to improve the bound of this relation is of great importance. Uncertainty bound can be altered by considering an extra quantum system as the quantum memory $B$ which is correlated with the measured quantum system $A$. One can extend the bipartite quantum memory assisted entropic uncertainty relation to tripartite quantum memory assisted entropic uncertainty relation in which the memory is split into two parts. In this work, we obtain a lower bound for the tripartite quantum memory assisted entropic uncertainty relation by adding two additional terms depending on the conditional von Neumann entropy, the Holevo quantity and mutual information. It is shown that the bound obtained in this work is more tighter than other bound. In addition, using our lower bound, a lower bound for the quantum secret key rate has been obtained.

I. INTRODUCTION

Uncertainty principle is undoubtedly one of the fundamental concepts in quantum theory. This principle defines the distinction between the classical and quantum world. It. This principle sets a bound on our ability to predict the measurement outcomes of two incompatible observables which simultaneously are measured on a quantum system. Actually, for two arbitrary observables, $X$ and $Z$, Robertson showed that [2, 3]

$$\Delta X \Delta Z \geq \frac{1}{2} |\langle [X, Z] \rangle|, \quad (1)$$

where $\Delta \kappa = \sqrt{\langle \kappa^2 \rangle - \langle \kappa \rangle^2}$ with $\kappa \in \{X, Z\}$ shows the standard deviation, $\langle \kappa \rangle$ represents the expectation value of operator $\kappa$, and $[X, Z] = XZ - ZX$. This form of uncertainty relation is still one of the most well-known uncertainty relations. It is important to note that the lower bound of the above uncertainty relation depends on the state of the system. Obviously, this dependence leads to trivial bound if $|\langle [X, Z] \rangle| = 0$. In quantum information theory, the uncertainty principle can be formulated in terms of the Shannon entropy. The most famous form of entropic uncertainty relation (EUR) was introduced by Deutsch [4] and then improved by Massen and Uffink [5]. They have shown that for two incompatible observables $X$ and $Z$, the following EUR holds

$$H(X) + H(Z) \geq \log_2 \frac{1}{c} = q_{MU}, \quad (2)$$

where $H(P) = -\sum_k p_k \log_2 p_k$ is the Shannon entropy of the measured observable $P \in \{X, Z\}$, $p_k$ is the probability of the outcome $k$, the quantity $c$ is defined as $c = \max_{X, Z} \langle |\langle x_i | z_j \rangle|^2 \rangle$, where $X = \{|x_i\}$ and $Z = \{|z_j\}$ are eigenstates of observables $X$ and $Z$, respectively and $q_{MU}$ is called incompatibility measure.

The EUR has a wide range of different applications in the field of quantum information, including quantum key distribution [6, 8], quantum cryptography [12, 13], quantum randomness [14, 15], entanglement witness [16], EPR steering [17, 18], and quantum metrology [19].

So far, many efforts have been made to expand and modify this relation [8, 20-42]. Berta et al. studied bipartite quantum memory assisted entropic uncertainty relation (QMA-EUR) [8] which can be explained by means of an interesting game between two players, Alice and Bob. At the beginning of the game, Alice and Bob share a quantum state $\rho_{AB}$. In the next step, Alice carries out a measurement on her quantum system $A$ by choosing one of the observables $X$ and $Z$, then she announces her choice of the measurement to Bob which keeps the quantum memory $B$. Bob’s task is to predict the outcome of Alice’s measurement. It is shown that the bipartite QMA-EUR can be written as [8]

$$S(X|B) + S(Z|B) \geq q_{MU} + S(A|B), \quad (3)$$

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where $S(P|B) = S(\rho_{PB}) - S(\rho_B)$ ($P \in \{X, Z\}$) are the conditional von Neumann entropies of the post-measurement states after measuring $X$ or $Z$ on the part $A$, 

$$\rho_{XB} = \sum_i |x_i\rangle\langle x_i|_A \otimes I_B \rho_{AB}|x_i\rangle\langle x_i|_A \otimes I_B,$$

and $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy. Note that when the conditional entropy $S(A|B)$ is negative which means that $A$ and $B$ are entangled, Bob can predict Alice’s measurement outcomes with better accuracy. Moreover, when the measured particle $A$ and the memory particle $B$ are maximally entangled, Bob can perfectly predict Alice’s measurement outcomes. Also, in the absence of a quantum memory, Eq. (3) reduces to

$$H(X) + H(Z) \geq q_{MU} + S(A),$$

which is tighter than the Maassen and Uffink EUR due to $S(A) \geq 0$.

Much effort has been made to improve the lower bound of the bipartite QMA-EUR\cite{22,30,31}. Pati et al. improved the Berta’s bound by adding a term to the lower bound in Eq. (3). The term depends on the classical correlation and quantum discord\cite{22}.

Adabi et al. provided a lower bound for the uncertainties $S(X|B)$ and $S(Z|B)$ by considering an additional term on the right-hand side of Eq. (6).\cite{6}

$$S(X|B) + S(Z|B) \geq q_{MU} + S(B) + \max\{0, \delta\},$$

where

$$\delta = I(A : B) - |I(X : B) + I(Z : B)|,$$

in which

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

is mutual information and

$$I(P : B) = S(\rho_B) - \sum_i p_i S(\rho_{Bi})$$

is the Holevo quantity. It is equal to the upper bound of the accessible information to Bob about Alice’s measurement outcomes. Note that when Alice measures the observable $P$ on the part $A$, the $i$-th outcome with probability $p_i = TR_{AB}(\Pi_i^A \rho_{AB} \Pi_i^A)$ is obtained and the part $B$ is left in the corresponding state $\rho_{Bi} = \frac{TR_{AB}(\Pi_i^A \rho_{AB} \Pi_i^A)}{p_i}$.

Adabi et al. showed that this lower bound is tighter than both the Berta and Pati lower bounds.

It is possible to extend the bipartite QMA-EUR to tripartite one. In tripartite scenario, two additional particle $B$ and $C$ are considered as the quantum memories. In fact, parts $A$, $B$, and $C$ are available to Alice, Bob, and Charlie, respectively. In this case, Alice, Bob, and Charlie share a quantum state $\rho_{ABC}$ and Alice carries out one of two measurements, $X$ and $Z$, on her quantum system. If she measures $X$, then Bob’s task is to minimize his uncertainty about $X$. If she measures $Z$, then Charlie’s task is to minimize his uncertainty about $Z$. It is shown that The tripartite QMA-EUR can be expressed as

$$S(X|B) + S(Z|C) \geq q_{MU},$$

where $q_{MU}$ is the same as that in Eq. (3). This equation was conjectured by Renes and Boileau\cite{7} and then proved by Berta et al.\cite{8}. The proof was simplified by Tomamichel and Renner\cite{9} and Coles et al.\cite{10}.

As can be seen the lower bound of Eq. (6) depends only on the complementarity of the observables. Although the bound should be depends on the state of system. According to our knowledge so far, there have been few improvement of the tripartite QMA-EURs. However, recently Ming et al.\cite{11} improved the lower bound of the tripartite QMA-EUR by adding a term to the lower bound in Eq. (6).

$$S(X|B) + S(Z|C) \geq q_{MU} + \max\{0, \Delta\},$$

where

$$\Delta = q_{MU} + 2S(A) - [I(A : B) + I(A : C)]$$

$$+ [I(Z : B) + I(X : C)] - H(X) - H(Z).$$

They showed that this lower bound is tighter than that of Eq. (6).

In this work, we introduce a lower bound for the tripartite QMA-EUR by adding two additional terms to the lower bound in Eq. (6) which one depending on the conditional von Neumann entropies and the other one depending on the mutual information and the Holevo quantity. we examine our lower bound for three examples and compare our lower bound with the other lower bound. we show that our lower bound for generalized GHZ state and Werner-type state states coincides with Ming et al. lower bound\cite{11} and for generalized W states our lower bound is tighter than that of Ming et al. Also, as an application, here we obtain a lower bound for the quantum secret key rate based on our results.

II. IMPROVED TRIPARTITE QMA-EUR

In this section, a lower bound for the tripartite QMA-EUR is obtained.
Theorem 1. Let $X$ and $Z$ be two incompatible observables with bases $X$ and $Z$, respectively. The following tripartite uncertainty relation holds for any state $\rho_{ABC}$,

$$S(X|B) + S(Z|C) \geq q_{MU} + \frac{S(A|B) + S(A|C)}{2} + \max\{0, \delta\},$$

where

$$\delta = \frac{I(A : B) + I(A : C)}{2} - [I(X : B) + I(Z : C)],$$

Proof. Regarding $S(X|B) = H(X) - I(X : B)$ and $S(Z|C) = H(Z) - I(Z : C)$, the left-hand side of Eq. 9 can be rewritten as

$$S(X|B) + S(Z|C) = H(X) + H(Z) - I(X : B) - I(Z : C) \geq q_{MU} + S(A) - I(X : B) - I(Z : C) = q_{MU} + S(A|B) + I(A : B) - [I(X : B) + I(Z : C)],$$

where the inequality follows from the Eq. 11 and last equality comes from the identity $S(A) = S(A|B) + I(A : B)$. Using $S(A) = S(A|C) + I(A : C)$ in the last line of the above equation, one can obtain

$$S(X|B) + S(Z|C) \geq q_{MU} + S(A|C) + I(A : C) - [I(X : B) + I(Z : C)],$$

from Eqs. 10 and 11, one arrives at

$$S(X|B) + S(Z|C) \geq q_{MU} + \frac{S(A|B) + S(A|C)}{2} + \frac{I(A : B) + I(A : C)}{2} - [I(X : B) + I(Z : C)].$$

It should be mentioned that the inequality $S(A|B) + S(A|C) \geq 0$ is always true due to strong subadditivity. Thus, our lower bound is stronger than Eq. 6.

III. EXAMPLES

A. Generalized GHZ state

First, let us consider the generalized Greenberger-Horne-Zeilinger (GHZ) state defined as

$$|GHZ\rangle = \cos\beta|000\rangle + \sin\beta|111\rangle,$$

where $\beta \in [0, 2\pi]$. Two complementary observables measured on the part $A$ of this state are assumed to be the Pauli matrices, $X = \sigma_1$ and $Z = \sigma_3$. In Fig. 1 different lower bounds of the the tripartite QMA-EUR for this state are plotted versus the parameter $\beta$. As can be seen, our lower bound coincides with those in Eqs. 10 and 7.

B. Werner-type state

As a second example, let us consider a Werner-type state defined as

$$\rho_{GHZ} = p|\Psi\rangle\langle\Psi| + \frac{1-p}{8}I_{ABC},$$

where $|\Psi\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$ is the GHZ state and $0 \leq p \leq 1$. For this state we find that our lower bound and Ming et al.’s lower bound completely coincide with each other, as shown in Fig. 2.

C. Generalized W state

As the last example, we consider the generalized $W$ state defined as

$$|W\rangle = \sin\theta \cos\phi|100\rangle + \sin\theta \sin\phi|010\rangle + \cos\theta|001\rangle,$$
where \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \). In Fig. 3, the lower bounds of the tripartite QMA-EUR for this state are plotted versus the parameter \( \theta \). As can be seen, our lower bound is tighter than that of Ming et al.

![Graph showing different lower bounds of the tripartite QMA-EUR for two complementary observables.](image)

**FIG. 3:** (Color online) Different lower bounds of the tripartite QMA-EUR for two complementary observables \( X = \sigma_1 \) and \( Z = \sigma_3 \) measured on the part \( A \) of the state in Eq. (15), versus the parameter \( \theta \), where \( \phi = \pi/4 \)

### IV. APPLICATION

The main purpose of the key distribution protocol is the agreement on a shared key between two honest parts (Alice and Bob) by communicating over a public channel in a way that the key is secret from any eavesdropping by the third part (Eve). It has been shown that the amount of the key \( K \) that can be extracted by Alice and Bob is lower bounded by \( K \geq S(Z|E) - S(Z|B) \), where the eavesdropper (Eve) prepares a quantum state \( \rho_{ABE} \) and sends the parts \( A \) and \( B \) to Alice and Bob, respectively, and keeps \( E \).

Note that the lower bound of the tripartite QMA-EUR is closely connected with the quantum secret key (QSK) rate. Eq. (6), leads us to

\[
S(Z|E) \geq q_{MU} - S(X|B). \tag{17}
\]

Regarding Eqs. (16) and (17), Berta et al. obtained the following relation for the bound of the QSK rate \( S \)

\[
K \geq q_{MU} - S(X|B) - S(Z|B). \tag{18}
\]

Using Eq. (9), one can obtain a new lower bound on the QSK rate which is

\[
K' \geq q_{MU} + \frac{S(A|B) + S(A|C)}{2} + \max\{0, \delta\} - S(X|B) - S(Z|B). \tag{19}
\]

Compared with Eq. (18), the QSK rate has lower bounded by two additional terms in Eq. (19). Since these terms are greater than or equal to zero, one comes to the result that \( K' \) is tighter than \( K \).

### V. CONCLUSION

In this work, we have obtained a lower bound for the tripartite QMA-EUR. We have compared our lower bound with the other lower bounds for some examples: especially, for the generalized W states, the comparison of the lower bounds is depicted in Fig. 3 where it is clear that our lower bound is tighter than that of Ming et al. Also, Regarding the tripartite QMA-EUR, we could derive a lower bound for the quantum secret key rate.

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