A Kähler Structure of Triplectic Geometry

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We study the geometry of the triplectic quantization of gauge theories. We show that underlying the triplectic geometry is a Kähler manifold $N$ with a pair of transversal polarizations. The antibrackets can be brought to the canonical form if and only if $N$ admits a flat symmetric connection that is compatible with the complex structure and the polarizations.

1 Introduction

The $Sp(2)$-symmetric Lagrangian quantization [1, 2] of general gauge theories generalizes the standard BV-formalism [3] so that ghosts and antighosts enter it in a symmetric way. The triplectic quantization [4, 5, 6] has been formulated as the corresponding analogue of the covariant formulation of the BV scheme [7, 8, 9, 10, 11], where “covariant” refers to the space of fields. An essential point in such a formulation is to ensure that the antibracket(s) can be locally brought to the canonical (“Darboux”) form, since only then the equivalence with the Hamiltonian quantization has been established.

In this paper, we investigate the geometry underlying the triplectic quantization procedure. There are considerable differences from the usual BV formalism. By construction, the covariant version of the BV scheme does not differentiate between fields and antifields, which simply become non-invariant notions. In the triplectic formalism, on the other hand, the antibrackets are degenerate, therefore one can single out the marked functions (Casimir functions, or “zero modes”) of the antibrackets; the marked functions then span the space of antifields. In this sense, the antifields are already encoded in the triplectic data.

As we will see, the triplectic geometry is essentially concentrated on the “manifold of antifields.” This turns out to be a complex manifold $N$, the complex structure originating from, and giving the geometric interpretation of, the $e$-structure entering the weakly canonical antibrackets from [12]. Further, the existence of two antibrackets induces a polarization on $N$, and the symmetrized Jacobi identities [1] imply then that the associated Nijenhuis tensor vanishes. Finally, the one-form $F$ that enters the triplectic data [4, 12] (the “potential” for the odd vector fields) induces a symplectic structure on $N$, which together with the complex structure makes it into a Kähler manifold.

The properties of the “antifield” manifold $N$ are, in particular, responsible for the possibility of bringing the triplectic antibrackets to the canonical form. The condition for the general triplectic antibrackets to allow the transformation to the canonical form reformulates as the requirement that $N$ admit a flat symmetric connection that is compatible with the complex structure and the polarization. This solves the problem posed in [12] and addressed recently in [12].

In Sec. 2, we briefly recall the triplectic formulation and reformulate the structures known from [12]. In Sec. 3, we show how these translate into the language of Kähler geometry. An important fact proved in Sec. 3 is that these geometric structures distinguish different triplectic structures up to local equivalence. The geometric reformulation, further, allows us to derive the conditions for the existence of canonical coordinates for the antibrackets (Sec. 4). In Sec. 5, we briefly discuss the $Sp(2)$ action, and in Sec. 6, we describe the geometric restrictions arising on the manifold of fields of the theory.
2 Geometry of triplectic manifolds

2.1 Basic definitions

The geometric background of triplectic quantization is a \((2N+2k|4N-2k)\)-dimensional supermanifold \(\mathcal{M}\) endowed with a pair of compatible antibrackets and an even 1-form \(\mathcal{F}\), which we briefly recall. Let \(\mathcal{C}_\mathcal{M}\) be the algebra of smooth functions on \(\mathcal{M}\). An antibracket \((\cdot, \cdot)\) is an odd skew-symmetric bilinear map \(\mathcal{C}_\mathcal{M} \times \mathcal{C}_\mathcal{M} \to \mathcal{C}_\mathcal{M}\) satisfying the Leibniz rule and Jacobi identity. The triplectic antibrackets \((\cdot, \cdot)\) and \((\cdot, \cdot)^1\) are compatible in the following way:

\[
(\Psi_{(F,G)}(\phi) - (\Psi_{(F,G)}(\psi) \cdot \mathcal{F}^1)) + \text{cycle}(F,G,H) = 0, \quad F, G, H \in \mathcal{C}_\mathcal{M},
\]

where the curly brackets stand for symmetrization of indices. This condition is often referred to as the symmetrized Jacobi identity \([\overline{1}]\). The antibrackets can be specified in terms of two bivector fields \(E^a : \Omega_\mathcal{M} \times \Omega_\mathcal{M} \to \mathcal{C}_\mathcal{M}\) determined by \(E^a(dF,dG) = (F, G)\), with \(d\) being the De Rham differential of \(\mathcal{M}\) and \(\Omega_\mathcal{M}\) being the space of 1-forms on \(\mathcal{M}\) (while \(E^a(\phi_1, \phi_2)\) denotes the bivector \(E^a\) evaluated on the 1-forms \(\phi_1\) and \(\phi_2\)). In the local coordinates \(z^A, A = 1, \ldots, 6N\) on \(\mathcal{M}\), we have \(E^a_{AB} = E^a(dz^A, dz^B) = (z^A, z^B)^a\).

The bivector \(E^a\) determines a mapping from 1-forms into vector fields that sends every 1-form \(\phi\) to the vector field \(X^a = E^a\phi\) such that \((E^a\phi)G = E^a(\phi, dG)\), \(G \in \mathcal{C}_\mathcal{M}\). In particular, the even 1-form \(\mathcal{F}\) gives rise to a pair of odd vector fields \(V^a = E^a\mathcal{F}\). The triplectic quantization prescription requires \(V^a = E^a\mathcal{F}\) to be compatible with the antibrackets,

\[
V^{(a}(F,G)b) - (V^{(a}F,G)b) - (1)^{\epsilon(F)+1}(F, V^{(a}G)b) = 0, \quad F, G \in \mathcal{C}_\mathcal{M},
\]

which can be rewritten as

\[
\Psi(\epsilon(E^{(a}\phi_1, E^{b}\phi_2)) = 0, \quad \Psi = d\mathcal{F},
\]

for any 1-forms \(\phi_1\) and \(\phi_2\). (Here \(E^a\phi_1\) and \(E^a\phi_2\) stand for the vector fields on which the 2-form \(\Psi\) is evaluated; see the Appendix for precise definitions of differential-geometric objects).

In local coordinates \(z^A\), we write \(\mathcal{F}_A = \mathcal{F}(\frac{\partial}{\partial z^A}), \Psi_{AB} = \Psi(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B})\). Then Eq. (2.3) takes the form \([\overline{1}]\)

\[
E^{aAC}\Psi_{CD}E_b^{dB} = 0, \quad \Psi_{AB} = \frac{1}{2}(\partial_A\mathcal{F}_B - (1)^{\epsilon(A)}\epsilon(B)\partial_B\mathcal{F}_A)(-1)^{\epsilon(B)+1}.
\]

The additional constraints imposed on the triplectic data \([\overline{12}]\) are formulated in terms of the marked functions of the antibrackets. A function \(\varphi \in \mathcal{C}_\mathcal{M}\) is called a marked function of the antibracket \((\cdot, \cdot)\) if \((F, \varphi) = 0\) for any \(F \in \mathcal{C}_\mathcal{M}\). Two compatible antibrackets \((\cdot, \cdot)^a, a = 1, 2\) are called mutually commutative if any marked functions \(\phi\) and \(\psi\) of the first antibracket \((\cdot, \cdot)^1\) satisfy \((\phi, \psi)^2 = 0\) and conversely, the first antibracket vanishes when evaluated on marked functions of the second antibracket. A pair of antibrackets is called jointly nondegenerate if the antibrackets do not have common marked functions (i.e., bivectors \(E^1\) and \(E^2\) do not have common zero modes).

We now introduce the notion of triplectic manifolds (see \([\overline{2}]\) for the details).
Definition 2.1 A $(2N+2k|4N-2k)$-dimensional supermanifold $\mathcal{M}$ endowed with a pair of compatible antibrackets and even 1-form $F$ is called triplectic if

1. the antibrackets are jointly nondegenerate and mutually commutative,
2. each of the antibrackets is of rank $4N$,
3. the 2-form $\Psi = dF$ is compatible with the antibrackets (i.e. satisfies (2.4)) and is of rank $4N$.

By the ranks of an antibracket and a 2-form $\Psi$, we mean the ranks of the respective supermatrices $E^{aAB}$, $a = 1, 2$, and $\Psi_{AB}$.

2.2 Geometric objects on the triplectic manifold

Let $\mathcal{M}$ be a triplectic manifold. In some neighbourhood $U$ of any point of $\mathcal{M}$ we can choose functions $\xi_{i1}, \xi_{2\alpha}, i, \alpha = 1, \ldots, 2N$ in such a way that $\xi_{i1}$ ($\xi_{2\alpha}$) is a minimal set that generates the algebra of marked functions of the second (respectively, the first) antibracket. We have shown in [12] that there exist functions $x^i$ such that $(\xi_{i1}, \xi_{2\alpha}, x^i)$ is a local coordinate system in $U$ in which the antibrackets take the form

$$(F,G)^1 = F \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_{i1}} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G),$$

$$(F,G)^2 = F \frac{\partial}{\partial x^i} \epsilon_\alpha^i \frac{\partial}{\partial \xi_{2\alpha}} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G),$$

where $\epsilon_\alpha^i$ depend only on the marked functions $\xi_{i1}$ and $\xi_{2\alpha}$. This form is called weakly canonical. Now the symmetrized Jacobi identity (2.1) rewrites as

$$\frac{\partial}{\partial \xi_{i1}} \epsilon_\alpha^i - (-1)^{(\epsilon(i)+1)(\epsilon(j)+1)} \frac{\partial}{\partial \xi_{ij}} \epsilon_\alpha^j = 0,$$

$$\epsilon_\alpha^i \frac{\partial}{\partial \xi_{2\alpha}} \epsilon_\beta^j - (-1)^{(\epsilon(i)+1)(\epsilon(j)+1)} \epsilon_\alpha^j \frac{\partial}{\partial \xi_{2\alpha}} \epsilon_\beta^j = 0,$$

where we use the following Grassmann parity assignments: $\epsilon(x^i) = \epsilon(i)$, $\epsilon(\xi_{i1}) = \epsilon(i) + 1$, $\epsilon(\xi_{2\alpha}) = \epsilon(\alpha) + 1$. It also follows from the above rank condition that $\epsilon_\alpha^i$ is an invertible matrix.

Each antibracket determines foliations $\mathcal{M}_a \rightarrow \mathcal{M}$, where $\mathcal{M}_1$ ($\mathcal{M}_2$) is the symplectic leaf of the first (respectively, the second) antibracket. In the local coordinates $(\xi_{i1}, \xi_{2\alpha}, x^i)$, every submanifold $\mathcal{M}_1$ ($\mathcal{M}_2$) is singled out by the equations $\xi_{2\alpha} = \text{const}_\alpha$ (respectively $\xi_{i1} = \text{const}_i$). We also consider the foliation $i: \mathcal{L} \rightarrow \mathcal{M}$ with the fibres $\mathcal{L} = \mathcal{M}_1 \cap \mathcal{M}_2$.

Using the weakly canonical coordinate system also allows us to simplify the 2-form $\Psi = dF$. First of all we note that compatibility condition (2.3) implies that the 2-form $\Psi$ vanishes on a pair of vectors that are tangent to $\mathcal{M}_1$ or $\mathcal{M}_2$. Condition (2.3) also implies that the vectors tangent to $\mathcal{L}$ are zero modes of $\Psi$. Thus the only nonvanishing coefficients of $\Psi$ are $\Psi^{i\alpha} = (-1)^{(\epsilon(i)+1)} \Psi^{i\alpha} = \Psi(\xi_{i1}, \xi_{2\alpha})$.

$$\Psi = 2d\xi_{i1} \wedge d\xi_{2\alpha} \Psi^{i\alpha}.$$  

(2.7)

Since $\Psi$ is exact, the coefficients $\Psi^{i\alpha}$ are independent of $x^i$. In addition, the rank condition requires $\Psi^{i\alpha}$ to be an invertible matrix. Finally, inserting (2.7) into (2.3), we obtain the condition

$$\epsilon_\alpha^i(\xi_1, \xi_2) \Psi^{i\alpha} + (-1)^{(\epsilon(i)+1)} \epsilon_\alpha^j(\xi_1, \xi_2) \Psi^{j\alpha} = 0.$$  

(2.8)
An interesting feature of triplectic geometry is that the triplectic data determine a Poisson bracket on the entire manifold $\mathcal{M}$. This originates from the bivector field $\omega(\phi, \phi) = -\frac{1}{2} \epsilon_{ab} \Psi(E^a \phi, E^b \phi)$, which gives rise to (see [12] for the details)

$$\{F, G\} = \omega(\partial F, \partial G) = F \cdot \frac{\partial}{\partial z^A} \omega^{AB} \cdot \frac{\partial}{\partial z^B} G,$$

with $\omega^{AB} = \omega(dz^A, dz^B)$. In the weakly-canonical coordinates, the only nonvanishing coefficients of $\omega$ are $\omega^{ij} = \omega(dx^i, dx^j)$, and therefore, the bracket (2.9) rewrites as

$$\{F, G\} = F \cdot \frac{\partial}{\partial x^i} \omega^{ij} \cdot \frac{\partial}{\partial x^j} G, \quad \omega^{ij} = e^i_\alpha \Psi^{(ij)},$$

where, moreover, $\omega^{ij}$ is an $x^i$-independent nondegenerate matrix. Thus the foliation into symplectic leaves of Poisson bracket (2.10) coincides with the foliation $i: \mathcal{L} \to \mathcal{M}$ mentioned above. In particular, every leaf $\mathcal{L}$ is a symplectic submanifold.

### 2.3 The I structure

The above structures defined on $\mathcal{M}$ give rise to another structure on the manifold.

**Proposition 2.2** On a triplectic manifold $\mathcal{M}$, there exists a tensor field $I: \text{Vect}_\mathcal{M} \to \text{Vect}_\mathcal{M}$ (and the transposed mapping $I^T: \Omega_\mathcal{M} \to \Omega_\mathcal{M}$) satisfying

$$I^2 = 1,$$

$$E^1(I^T \phi_1, I^T \phi_2) = E^2(\phi_1, \phi_2)$$

for arbitrary 1-forms $\phi_1, \phi_2$, and

$$I|_{\mathcal{T}} = \mathcal{I}.$$

$I$ acts on $d\xi_{1i}, d\xi_{2a}$ as

$$I^T d\xi_{2a} = d\xi_{1i} e^i_\alpha, \quad I^T d\xi_{1i} = d\xi_{2a} \bar{e}^a_i,$$

where $\bar{e}^a_i$ is the inverse matrix to $e^i_\alpha$ (i.e., $(e^i_\alpha \bar{e}^a_j) = \delta_i^j$).

For a tensor field $I: \text{Vect}_\mathcal{M} \to \text{Vect}_\mathcal{M}$, the transposed mapping $I^T: \Omega_\mathcal{M} \to \Omega_\mathcal{M}$ is defined by

$$\langle X, I^T \phi \rangle = \langle IX, \phi \rangle,$$

where $\langle X, \phi \rangle = I_X \phi$ is the contraction of the vector field $X$ with the 1-form $\phi$. In the local coordinates $z^A$, we write $I^2 \frac{\partial}{\partial z^A} = I_A^B \frac{\partial}{\partial z^B}$ and $I d z^A = d z^B I_A^B$; then conditions (2.11) and (2.13) become

$$I_B^C I^A_C = \delta^A_B, \quad (-1)^{(A) + \epsilon(C)} I^A_C E^{1CD} I^B_D = E^{2AB},$$

The existence of a linear mapping satisfying (2.11) and (2.12) can be easily checked using the explicit form (2.25) of the antibrackets in the weakly canonical coordinates. Such a mapping is not unique. However, every $I$ satisfying (2.14) and (2.15) can be restricted to the vector fields tangent to $\mathcal{L}$. Indeed, we can represent $X \in \text{Vect}_\mathcal{M}$ that is tangent to $\mathcal{L}$ as $X = E^1 \phi = E^2 \psi$ for some $\phi, \psi \in \Omega_\mathcal{M}$, because $T \mathcal{L} = T \mathcal{M}_1 \cap T \mathcal{M}_2$. Then we have $I X = I(E^1 \phi) = E^2(I^T \phi)$, which is thus tangent to $\mathcal{M}_1$; on the other hand $I X = I(E^2 \psi) = E^1(I^T \psi)$ is tangent to $\mathcal{M}_2$, which shows that $I X$ is tangent to $\mathcal{L}$. 

4
This allows us to impose condition (2.13). Even this does not completely fix the arbitrariness of \( I \). However, the action of \( I \) (in fact, of \( I^T \)) on the 1-forms with vanishing restrictions to \( \mathcal{L} \) is now unambiguous. In particular, \( I^T \) acts in a well-defined way on the differentials \( d\xi_{1i} \) and \( d\xi_{2\alpha} \). In order to find the explicit form of this action we consider the vector fields

\[
X_i^1 = (\xi_{1i}, \cdot)^1 = E^1 d\xi_{1i}, \quad X_\alpha^2 = (\xi_{2\alpha}, \cdot)^2 = E^2 d\xi_{2\alpha}.
\]

(2.17)

By definition, \( X_i^1 \) and \( X_\alpha^2 \) are tangent to \( \mathcal{L} \); in addition, we have \( I^2 \) \( X_\alpha^2 = (-1)^{e(i)+1} e^\alpha X_i^1 \), which we now rewrite as

\[
IE^1(I^T d\xi_{2\alpha}) = (-1)^{(e(i)+1)e(\alpha)} e^\alpha I^1 d\xi_{1i}.
\]

(2.18)

Observing that \( IE^2 I^T d\xi_{2\alpha} = 0 \Rightarrow E^2 I^T d\xi_{2\alpha} = 0 \Rightarrow I^T d\xi_{2\alpha} = d\xi_{1i} A^i_\alpha \) for some \( A^i_\alpha \), we conclude that \( E^1(I^T d\xi_{2\alpha}) \) is tangent to \( \mathcal{L} \), which in turn implies that \( E^1(I^T d\xi_{2\alpha}) = E^1(d\xi_{1i} A^i_\alpha) \). Thus, \( x \equiv I^T d\xi_{2\alpha} - d\xi_{1i} A^i_\alpha \) is a zero mode of \( E^1 \). Now, it is easy to see that \( E^2 x = 0 \) as well, which means that \( x = 0 \) in view of the conditions imposed on the antibrackets. This shows (2.14).

3 From triplexic to Kähler geometry

As we have seen, a given triplexic structure determines a foliation \( i : \mathcal{L} \rightarrow \mathcal{M} \) of the triplexic manifold \( \mathcal{M} \) (the leaves being at the same time the symplectic leaves of Poisson bracket (2.10)). For a sufficiently small neighbourhood \( U \) in \( \mathcal{M} \), this foliation is a fibration with base \( U_N \) and the projection \( \pi : U \rightarrow U_N \). When the entire \( \mathcal{M} \) is a fibration, we will write \( \pi : \mathcal{M} \rightarrow \mathcal{N} \), then \( U_N \) will be a neighbourhood in \( \mathcal{N} \); however, it is not necessarily assumed that \( \mathcal{N} \) exists globally, since we mainly work with local statements. We identify the algebra \( \mathcal{C}_{U_N} \) of smooth functions on \( U_N \) with the functions on \( U \) that are constant along the fibres; this gives precisely the algebra generated by the marked functions of the antibrackets in the neighbourhood. Further, the weakly canonical coordinates provide us with a diffeomorphism \( \psi : U \rightarrow U_N \times U_L \) that identifies \( U \) with a neighbourhood in the product of linear (super)spaces \( U_N \times U_L \) (such that the first component of \( \psi \) is \( \pi \), i.e., \( \psi = (\pi, \rho) \)).

In the present section, we assume for simplicity that the base \( \mathcal{N} \) and the projection \( \pi : \mathcal{M} \rightarrow \mathcal{N} \) exist globally. Thus, smooth functions on \( \mathcal{N} \) can be identified with functions on \( \mathcal{M} \) that are constant along the fibres, i.e., with the algebra generated by the marked functions of the antibrackets on \( \mathcal{M} \).

In particular, we can choose a coordinate system \( \tilde{\xi}_{1i}, \tilde{\xi}_{2\alpha} \) on \( \mathcal{N} \) such that the functions \( \xi_{1i} = \pi^* \tilde{\xi}_{1i} \) (respectively, \( \xi_{2\alpha} = \pi^* \tilde{\xi}_{2\alpha} \)), where \( \pi^* \) is the pullback associated with the projection \( \pi \), generate the algebra of marked functions of the second (respectively, the first) antibracket. In what follows, we will not write the tilde over the coordinates on \( \mathcal{N} \) and thus identify functions on \( \mathcal{N} \) with their pullbacks to \( \mathcal{M} \), in accordance with the one-to-one correspondence between functions from \( \mathcal{C}_N \) and the functions that are constant along \( \mathcal{L} \).

Further, since \( e^\alpha_\alpha \) are constant along \( \mathcal{L} \), the 1-forms \( d\xi_{1i} e^i_\alpha \) and \( d\xi_{2\alpha} e^\alpha_j \) are the pullbacks of some 1-forms on \( \mathcal{N} \) (as, obviously, are the 1-forms \( d\xi_{1i} \) and \( d\xi_{2\alpha} \)). Then, according to proposition 2.2, we conclude that \( \tilde{I}^T : \Omega_M \rightarrow \Omega_M \) determines a mapping \( \tilde{I}^T : \Omega_N \rightarrow \Omega_N \), and thus \( \tilde{I} \) is well-defined on \( \mathcal{N} \).

In the local coordinates \( \xi_{1i}, \xi_{2\alpha} \) on \( \mathcal{N} \), we have

\[
\tilde{I} \frac{\partial}{\partial \xi_{1i}} = e^i_\alpha \frac{\partial}{\partial \xi_{2\alpha}}, \quad \tilde{I} \frac{\partial}{\partial \xi_{2\alpha}} = e^\alpha_i \frac{\partial}{\partial \xi_{1i}}.
\]

(3.1)
Further, it follows from (2.7) that there exists a 2-form \( \hat{\Psi} \) on \( \mathcal{N} \) whose pullback coincides with \( \Psi = d\mathcal{F} \) from (2.3). The rank assumption implies that \( \hat{\Psi} \) is nondegenerate and, thus, \( \mathcal{N} \) is a symplectic manifold\(^2\). As can be seen from (2.7) and (2.8), the structures identified on \( \mathcal{N} \) are related by

\[
\hat{\Psi}(\hat{Y}_1, \hat{Y}_2) = -\hat{\Psi}(Y_1, Y_2)
\]  

for arbitrary vector fields \( Y_1, Y_2 \) on \( \mathcal{N} \).

As regards vector fields, we have, obviously,

**Proposition 3.1**  
Every vector field \( X : \mathcal{C}_M \rightarrow \mathcal{C}_M \) preserving the space of functions that are constant along \( \mathcal{L} \), determines a unique vector field \( \tilde{X} \) on \( \mathcal{N} \).

We now show that the symplectic manifold \( \mathcal{N} \) is endowed with a pair of transversal polarizations. We first recall that an integrable distribution \( P : \mathcal{N} \rightarrow T\mathcal{N} \) is called a polarization of the symplectic manifold \( \mathcal{N} \) if the image \( P_x \subset T_x\mathcal{N} \) at any point \( x \in \mathcal{N} \) is a Lagrangian subspace of \( T_x\mathcal{N} \). Two polarizations \( P^1 \) and \( P^2 \) are called transversal if \( T_x\mathcal{N} = P^1_x \oplus P^2_x \).

In the case at hand, we observe that the vector fields on \( \mathcal{N} \) annihilating the marked functions of the first antibracket (of the second antibracket) considered as functions on \( \mathcal{N} \) determine a foliation of \( \mathcal{N} \) and, thus, an integrable distribution \( P^1 : \mathcal{N} \rightarrow T\mathcal{N} \) (respectively, \( P^2 : \mathcal{N} \rightarrow T\mathcal{N} \)). In the \( \xi^{1i}, \xi^{2\alpha} \) coordinate system on \( \mathcal{N} \), we see that \( P^1 \) (respectively, \( P^2 \)) is generated by the vector fields

\[
\frac{\partial}{\partial \xi^{1i}} \quad \text{(respectively, } \frac{\partial}{\partial \xi^{2\alpha}} \text{). }
\]

The explicit form of \( P^1 \) and \( P^2 \) shows that \( T_x\mathcal{N} = P^1_x \oplus P^2_x \) at any point \( x \in \mathcal{N} \). It is easy to see that the symplectic form \( \hat{\Psi} \) vanishes on \( P^1_x \) as well as on \( P^2_x \), and, thus, \( P^1 \) and \( P^2 \) are a pair of transversal polarizations.

Now, one can represent any vector field \( X \) on \( \mathcal{N} \) as a sum \( X = X^1 + X^2 \), where \( X^1 \in P^1 \) and \( X^2 \in P^2 \). This allows us to introduce the mapping \( K : \text{Vect}_\mathcal{N} \rightarrow \text{Vect}_\mathcal{N} \) as

\[
KX = X^1 - X^2.
\]  

(3.3)

It is easy to see that \( K \) satisfies

\[
K^2 = KK = 1, \quad K\hat{I} + \hat{I}K = 0, \quad \hat{\Psi}(KY_1, KY_2) = -\hat{\Psi}(Y_1, Y_2), \quad Y_1, Y_2 \in \text{Vect}_\mathcal{N}.
\]  

(3.4)

Given the mappings \( \hat{I} \) and \( K \) on \( \mathcal{N} \), we can consider the product \( J = \hat{I}K \). It follows from (3.2) and (3.4) that \( J \) satisfies

\[
J^2 = JJ = -1, \quad \hat{\Psi}(JY_1, JY_2) = \hat{\Psi}(Y_1, Y_2).
\]  

(3.5)

Thus \( J \) is an almost complex structure which is compatible with the symplectic form. In the local coordinates \( \xi^{1i}, \xi^{2\alpha} \), we have

\[
J \frac{\partial}{\partial \xi^{1i}} = e_i^\alpha \frac{\partial}{\partial \xi^{2\alpha}}, \quad J \frac{\partial}{\partial \xi^{2\alpha}} = -e_\alpha^i \frac{\partial}{\partial \xi^{1i}}.
\]  

(3.6)

Next, we show that this almost complex structure is integrable. For \( J \) to be integrable it is sufficient that the Nijenhuis tensor \( N_{J, J} \) vanish. The Nijenhuis tensor of \( J \) is the mapping \( N_{J, J} : \text{Vect}_\mathcal{N} \times \text{Vect}_\mathcal{N} \rightarrow \text{Vect}_\mathcal{N} \) given by

\[
N_{J, J}(X, Y) = [JJ, Y] + [JX, JY] - J[X, JY] - J[JX, Y], \quad X, Y \in \text{Vect}_\mathcal{N}.
\]  

(3.7)

\(^2\)Note that the 2-form \( \Psi \) is in general closed but not exact, whereas \( \Psi = d\mathcal{F} \) is evidently exact.
Using the explicit form of $J$ given in (3.6) we conclude that $N_{J,J} = 0$ in view of Eqs. (2.6). Therefore, $N$ is a complex manifold.

Putting everything together, we have

**Theorem 3.2** The manifold $N$ is Kähler. The corresponding fundamental 2-form is $\hat{\Psi}$.

Since $N$ is in general a supermanifold, we actually have the super analogue of a Kähler manifold. Also, we have not required $h$ to be positive definite, which means that $N$ is in fact a pseudo-Kähler manifold.

Explicitly, the Kähler metric is $h(X,Y) = \hat{\Psi}(JX,Y)(-1)^{\varepsilon(Y)}$, $X,Y \in \text{Vect}_N$. It follows from the above that $h$ is nondegenerate and satisfies

$$ h(X,Y) = (-1)^{\varepsilon(X)\varepsilon(Y)} h(Y,X), \quad h(JX, JY) = h(X,Y), \quad X,Y \in \text{Vect}_N. \tag{3.8} $$

### 4 Local equivalence of triplectic manifolds

We show in Sec 4.1 that the geometric structures induced on $U_N$ (see the beginning of Sec. 3) distinguish different triplectic structures up to local equivalence. In particular, the condition for the triplectic antibrackets to admit the canonical form also reformulates in terms of some objects on $U_N$, as we show in Sec. 4.2.

#### 4.1 The equivalence theorem

For a sufficiently small neighbourhood $U \subset M$, the triplectic data give rise to the projection $\pi : U \to U_N$ along the leaves $L$ of the foliation $i : L \to M$. The triplectic antibrackets, further, induce a complex structure and a pair of transversal polarizations on $U_N$. Similarly, the 2-form $\Psi = d\mathcal{F}$ determines the fundamental form of $U_N$.

We will say that two pairs of triplectic antibrackets\footnote{For brevity, we consider only the antibrackets, disregarding the odd nilpotent vector fields (i.e., the 1-form $\mathcal{F}$); the equivalence statement given below can easily be generalized to include $\mathcal{F}$.} $(\ _{\alpha} , \ )$ and $(\ _{\beta} , \ )$ are locally equivalent if for any sufficiently small neighbourhood $U \subset M$ there exists a diffeomorphism $\phi : U \to U$ such that

$$ (F,G)_{\alpha} = (\phi^{-1})^* (\phi^* F, \phi^* G)^{\beta}, \tag{4.1} $$

Let now $(\ _{\alpha} , \ )$ and $(\ _{\beta} , \ )$ be two triplectic structures on $M$. We show that different triplectic structures are locally distinguished by geometries on $U_N$.

**Theorem 4.1** The following conditions are equivalent:

1. The triplectic structures $(\ _{\alpha} , \ )$ and $(\ _{\beta} , \ )$ are locally equivalent.
2. For every sufficiently small neighbourhood $U \subset M$, there exists a diffeomorphism $\phi_0 : U_N \to U_N$ such that

$$ \phi_0^* (K\psi) = K(\phi_0^* \psi), \quad \phi_0^* (J\psi) = J(\phi_0^* \psi), \tag{4.2} $$

for arbitrary 1-form $\psi$ on $U_N$, where $\pi : U \to U_N$ and $\overline{\pi} : U \to \overline{U}_N$ are the projections associated with the respective triplectic structures.
To show this, let $\phi : U \to U$ be a diffeomorphism satisfying (4.1). Let also $\xi_{1i}$ ($\theta_{1i}$) be the marked functions of the $(\cdot, \cdot)^2$ antibracket (respectively, of $(\cdot, \cdot)^2$) and $\xi_{2a}$ ($\theta_{2a}$) be the marked functions of $(\cdot, \cdot)^1$ (respectively, of $(\cdot, \cdot)^1$). It follows from (4.1) that $\phi^*\theta_{1i}$ and $\phi^*\theta_{2a}$ are marked functions of the brackets $(\cdot, \cdot)^2$ and $(\cdot, \cdot)^1$, respectively, and therefore, $\phi^*\theta_{1i}$ is a function of $\xi_{1i}$, which we write as $\phi^*\theta_{1i} = \xi_{1i}(\xi_1)$ and similarly, $\phi^*\theta_{2a} = \xi_{2a}(\xi_2)$. Thus $\phi$ induces a mapping from the marked functions of the $(\cdot, \cdot)^2$ antibrackets to the marked functions of the $(\cdot, \cdot)^1$ antibrackets. We now consider the vector fields generated by the marked functions

$$(\xi_{2a}, \cdot)^2 = -(1)^{e(i)+1}e(\alpha)e^i_\alpha(\xi_{1i}, \cdot)^1, \quad (\theta_{2a}, \cdot)^2 = -(1)^{e(i)+1}e(\alpha)\tau^i_\alpha(\theta_{1i}, \cdot)^1,$$

where $e$ and $\tau$ are the corresponding e-structures. According to (4.2), we have

$$(\phi^*\theta_{2a}, \cdot)^2 = -(\cdot)^{e(i)+1}e(\alpha)\phi^*\tau^i_\alpha(\phi^*\theta_{1i}, \cdot)^1.$$  

Since, as we have seen, $\phi^*\theta_{1i} = \xi_{1i}(\xi_1)$ and $\phi^*\theta_{2a} = \xi_{2a}(\xi_2)$, we have

$$e^i_\alpha = \frac{\partial \xi_{1i}}{\partial \xi_1}(\phi^*\tau^j_\beta) \frac{\partial \xi_{2a}}{\partial \xi_{2\beta}}.$$  

(4.5)

Taking the marked functions $\xi_1$, $\xi_2$ as the coordinates on $U_N$ and, similarly, $\theta_1$, $\theta_2$ as the coordinates on $\overline{U}_N$, we see that $\phi$ restricts to a diffeomorphism $\phi_0 : U_N \to \overline{U}_N$. Recalling that $\phi^*$ maps marked functions into the corresponding marked functions and also using Eq. (4.5), we conclude that $\phi_0$ is as required in the theorem.

Conversely, let $U_N$ and $\overline{U}_N$ be related by a diffeomorphism $\phi_0$ satisfying (4.2). We then choose a coordinate system $\xi_{1i}, \xi_{2a}$ (a coordinate system $\theta_{1i}, \theta_{2a}$) on $U_N$ (respectively, on $U_N'$) such that $K$ and $J$ (respectively, $\overline{K}$ and $\overline{J}$) act on the basis 1-forms as $K^T d\xi_{1i} = d\xi_{1i}$, $K^T d\xi_{2a} = -d\xi_{2a}$ and $J^T d\xi_{1i} = -\xi_{2a} d\xi_{1i}$, $J^T d\xi_{2a} = \xi_{1i} d\xi_{2a}$. Then Eqs. (4.2) imply that $\phi_0^* d\theta_{1i} = A^j_i d\xi_{1j}$. This, in turn, shows that $\phi_0^* \theta_{1i}$ are functions of only $\xi_1$. Similarly, $\phi_0^* \theta_{2a}$ is a function of only $\xi_2$. This allows us to choose coordinates $\overline{\theta}_{1i}, \overline{\theta}_{2a}$ in $U_n$ such that $\phi_0^* \overline{\theta}_{1i} = \xi_{1i}$ and $\phi_0^* \overline{\theta}_{2a} = \xi_{2a}$. In coordinates, we write $\overline{J}^T d\overline{\theta}_{2a} = (\overline{\tau})^i_\alpha d\overline{\theta}_{1i}$. Then the second equation in (4.2) implies

$$e^i_\alpha = \phi_0^*(\overline{\tau})^i_\alpha,$$  

where we view $(\overline{\tau})^i_\alpha$ for each $i$ and $\alpha$, as functions on $\overline{U}_N$. Further, we consider $\xi_{1i}, \xi_{2a}$ and $\overline{\theta}_{1i}, \overline{\theta}_{2a}$ as functions on $U \subset M$, where they are the marked functions of corresponding antibrackets. Choosing the functions $x^i$ and $y^j$ on $U \subset M$ in such a way that $x^i, \xi_{1i}, \xi_{2a}$ (respectively, $y^j, \overline{\theta}_{1i}, \overline{\theta}_{2a}$) be the weakly canonical coordinates for the antibrackets $(\cdot, \cdot)^2$ (respectively, $(\cdot, \cdot)^1$), we consider the diffeomorphism $\phi : U \to U$ determined by

$$\phi^* y^j = x^i, \quad \phi^* \overline{\theta}_{1i} = \xi_{1i}, \quad \phi^* \overline{\theta}_{2a} = \xi_{2a}.$$  

(4.7)

It is easy to check that $\phi$ satisfies (4.1) and, thus, two triplectic structures are locally equivalent. This completes the proof.

### 4.2 Finding the canonical coordinates

As we are going to see, the question of whether the antibrackets can be (locally) brought to the canonical form is solved in terms of geometric structures on $U_N$. Recall that having chosen the bases
of marked functions of the antibrackets, one arrives at the structure $e^i_\alpha(\xi_1, \xi_2)$, which is in general a local obstruction to finding the canonical coordinates for the triplectic antibrackets. It follows from (2.18) that if we choose new bases of the marked functions as $\xi'_{1i} = \xi'_{1i}(\xi_1)$ and $\xi'_{2\alpha} = \xi'_{2\alpha}(\xi_2)$, the matrix $e$ transforms as follows:

$$e'^i_\alpha = \frac{\partial}{\partial \xi'_{1i}} e^j_\beta \frac{\partial}{\partial \xi'_{2\beta}}, \quad (\xi'_{2\alpha}, \cdots)^2 = (-1)^{(\epsilon(i)+1)\epsilon(\alpha)} e'^i_\alpha (\xi'_{1i}, \cdots)^1. \quad (4.8)$$

The structure $e$ is called reducible if there exist bases of the marked functions $\xi'_{1i} = \xi'_{1i}(\xi_1)$ and $\xi'_{2\alpha} = \xi'_{2\alpha}(\xi_2)$ such that $e'^i_\alpha = \delta^i_\alpha$. Once $e$ is reducible, there exists a coordinate system where both antibrackets take the canonical (“Darboux”) form.

We now reformulate the problem of reducibility in terms of differential geometry on $U_N$ from the previous section (the proof of the following proposition is immediate from the explicit form of $J$, $K$ and $I = JK$, where we remove the hat over $I$).

**Proposition 4.2** Let $U$ be a (sufficiently small) neighbourhood in $\mathcal{M}$ and $\pi: U \to U_N$ be the projection associated with the triplectic structure. Then the following conditions are equivalent:

1. The $e$-structure associated with the antibrackets is reducible.
2. There exists a coordinate system in $U_N$, where the components of the tensor fields $I$, $K$, and $J = IK$ are constants.
3. There exists a flat symmetric linear connection $\nabla$ on $U_N$ such that the tensor fields $I$, $K$, and $J$ are parallel with respect to $\nabla$,

$$\nabla K = 0, \quad \nabla I = 0, \quad \nabla J = 0. \quad (4.9)$$

In items 2 and 3, it suffices to have the conditions satisfied for any two structures of $I$, $K$, and $J$. The covariant derivative $\nabla$ is viewed as a mapping $\nabla: \text{Vect}_{U_N} \times \text{Vect}_{U_N} \to \text{Vect}_{U_N}$ satisfying

$$\nabla_{FX}Y = F\nabla_X Y, \quad \nabla_X (FY) = (XF)(Y) + (-1)^{\epsilon(X)\epsilon(F)} F\nabla_X Y, \quad (4.10)$$

where $F \in \mathcal{C}_{U_N}$ and $X, Y \in \text{Vect}_{U_N}$. The action of $\nabla$ on a tensor field $I : \text{Vect}_{U_N} \to \text{Vect}_{U_N}$ is defined by

$$(\nabla_X I)Y = \nabla_X (IY) - I(\nabla_X Y), \quad X, Y \in \text{Vect}_{U_N}. \quad (4.11)$$

Taking $\nabla$ symmetric means the vanishing of torsion $T(X,Y) = \nabla_X Y - (-1)^{\epsilon(X)\epsilon(Y)} \nabla_Y X - [X,Y]$. With the Christoffel symbols defined in local coordinates $z^A$ on $U_N$ as $\Gamma^C_{AB} = (\nabla_A \frac{\partial}{\partial z^B})z^C$ and $I_{AB} = \frac{\partial}{\partial z^A} \frac{\partial}{\partial z^B}$, we have

$$(\nabla_A I)_{BC} = \partial_A I_{BC} - \Gamma^D_{AB} I_{DC} + (-1)^{(\epsilon(D)+\epsilon(B))\epsilon(A)} I_{DC} \Gamma^C_{AD}. \quad (4.12)$$

We further observe that the flat connection from item 3 is unique. Indeed, it follows from the first equation in (4.9) and definition (1.3) of $K$ that the only nonvanishing connection coefficients in the coordinates $\xi_{1i}, \xi_{2\alpha}$ are $\Gamma^i_{jk}$ and $\Gamma^\alpha_{\beta\gamma}$. Then, we use the equation $\nabla I = 0$ (and the explicit form (1.11) of $I$) to obtain

$$\Gamma^i_{jk} = \left(\frac{\partial}{\partial \xi_{1i}} e^j_\alpha\right) e^\alpha_k, \quad \Gamma^\alpha_{\beta\gamma} = \left(\frac{\partial}{\partial \xi_{2\alpha}} e^\beta_j\right) e^\gamma_j, \quad (4.13)$$

which shows that the symmetric connection $\nabla$ satisfying $\nabla I = \nabla J = \nabla K = 0$ is unique.
Looking at the zero-curvature conditions, we see that the only nonvanishing curvature components are $[\nabla^1, \nabla^{2\alpha}]$, therefore the flatness condition becomes
\[ \frac{\partial}{\partial \xi_{1i}} \left( \frac{\partial}{\partial \xi_{2\alpha}} e_\gamma^\beta \right) e_i^\gamma = 0, \quad \frac{\partial}{\partial \xi_{2\alpha}} \left( \frac{\partial}{\partial \xi_{1i}} e_\gamma^\beta \right) e_k^\beta = 0. \] (4.14)

We now recall that this vanishing curvature condition on $U_N$ can be traced back to the reducibility of $e_\alpha^i$ on $M$. This gives the following theorem on the transformation of the triplectic antibrackets to the canonical form.

**Theorem 4.3** The structure $e_\alpha^i$ corresponding to the triplectic antibrackets is reducible if and only if it satisfies (4.14). Thus, the triplectic antibrackets (i.e., a pair of rank-$4N$ compatible antibrackets that are jointly nondegenerate and mutually commutative) admit canonical coordinates if and only if the corresponding $e$-structure satisfies (4.14).

5 The $Sp(2)$ action on $N$

In this section, we return, for simplicity, to the situation described in Sec. 3, where the base $N$ is assumed to exist globally.

An essential ingredient of the ghost-antighost symmetric quantization is the $Sp(2)$ action [1]. In the covariant formulation, this takes the form of the requirement that $M$ should carry an action of $Sp(2)$ [12], i.e., for every $G \in Sp(2)$ there is a mapping $\phi_G : M \to M$ such that $\phi_G \circ \phi_G = \phi_G$.

A pair of antibrackets and a 1-form $F$ are called $Sp(2)$-covariant if there exists an action $\phi$ of $Sp(2)$ on $M$ such that
\[ \phi_\ast_G((f,g)^a) = G^a_b (\phi_\ast_G(f), \phi_\ast_G(g))^b, \quad \phi_\ast_G F = F, \quad G \in Sp(2), \quad f, g \in C_M, \] (5.1)
where $G^a_b$ is the $2 \times 2$ matrix representation of $Sp(2)$. Infinitesimally, this reformulates as a homomorphism from the Lie algebra $sp(2)$ to Vect$_M$ such that
\[ L_Y E^a = g^a_b E^b, \quad L_Y F = 0, \] (5.2)
where $Y$ is the vector field corresponding to $g \in sp(2)$ (and $L$ is the Lie derivative). This has an important consequence that Poisson bracket (2.10) is $sp(2)$-invariant:
\[ L_Y \omega = 0. \] (5.3)

Next, we observe that the $Sp(2)$ action maps the marked functions $(\xi_{1i}, \xi_{2\alpha})$ into marked functions (but does not, obviously, preserve the separation of the marked functions into those of the first and the second antibracket). A convenient way to see this is to note that the collection $(\xi_a) = (\xi_{1i}, \xi_{2\alpha})$ of marked functions can be characterized by the fact that these are marked functions of the Poisson bracket, $\{F, \xi_a\} = 0$ for any $F$. Applying now an $Sp(2)$ transformation, we have
\[ 0 = \phi_\ast_G(\{F, \xi\}) = \{\phi_\ast_G(F), \phi_\ast_G(\xi)\}, \] (5.4)
which means that $\phi_\ast_G(\xi)$ is again a marked function of the Poisson bracket. This implies, further, that the vector fields representing the $sp(2)$ action on $M$ project onto $N$ [4]. We will denote the projection of $Y$ again by $Y$. Since $L_Y F = 0$ and hence $L_Y \Psi = 0$, we arrive at

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4 As noted above, we now assume that $N$ exists globally.

5 Although the shortest way to this statement is to recall the Poisson bracket, it can also be shown without resorting to the Poisson structure.
Proposition 5.1  The manifold $\mathcal{N}$ carries an $Sp(2)$ action that preserves the symplectic structure on $\mathcal{N}$.

Thus, the vector fields $Y$ are locally Hamiltonian on $\mathcal{N}$.

Choosing now $Y^\pm$ and $Y^0$ to correspond to the basis in $sp(2)$ where

\[
\begin{align*}
[J^+, J^-] &= -2J^0, & [J^+, J^0] &= -J^+, & [J^-, J^0] &= J^-,
\end{align*}
\]

we see that the structures $I$, $J$, and $K$ furnish the three-dimensional representation of $sp(2)$:

\[
\begin{align*}
L_Y I &= K, & L_Y - I &= K, & L_Y 0 &= J,
\end{align*}
\]

\[
\begin{align*}
L_Y J &= -K, & L_Y - J &= -K, & L_Y 0 &= J, & L_Y K &= I - J, & L_Y 0 &= K. 
\end{align*}
\]

Apart from the global properties of the group action on a manifold, the issue of $Sp(2)$ covariance of triplectic antibrackets is solved for the entire class of equivalent triplectic structures and, therefore, can be solved in terms of geometry on $\mathcal{N}$—it amounts to the existence of an $Sp(2)$ action on $\mathcal{N}$ satisfying (5.6).

6 Geometry of $\mathcal{L}$

In this section, we will show that a pair of antibrackets induce an additional structure on every submanifold $\mathcal{L}$ (every leaf of the foliation $i : \mathcal{L} \to \mathcal{M}$). Besides the known symplectic structure on $\mathcal{L}$, the conditions imposed on the triplectic objects (see Definition 2.1) imply the existence of a flat connection on $\mathcal{L}$:

Theorem 6.1  A pair of triplectic antibrackets induce a flat symmetric connection on each leaf $\mathcal{L}$ of the foliation $i : \mathcal{L} \to \mathcal{M}$.

To prove the theorem, we choose a fixed leaf $\mathcal{L} \subset \mathcal{M}$; let $\{U_n\}$ be an atlas of $\mathcal{M}$ such that in each neighbourhood $U_n$ there exist weakly canonical coordinates $x^i, \xi_{1i}, \xi_{2a}$. Let $U_1$ and $U_2$ be coordinate neighbourhoods on $\mathcal{M}$ such that $U_1 = U_1 \cap \mathcal{L}$ and $U_2 = U_2 \cap \mathcal{L}$, and also $U_1 \cap U_2$, are non-empty; let also $x^i, \xi_{1i}, \xi_{2a}$ and $y^j, \theta_{1j}, \theta_{2a}$ be weakly canonical coordinates on $U_1$ and $U_2$, respectively. Then the functions $\varphi^i = x^i|\mathcal{L}$ (respectively, $\varphi^j = y^j|\mathcal{L}$) are local coordinates on $U_1$ (respectively, $U_2$). We have seen that the vector fields

\[
X_i = (\xi_{1i}, \cdot)^1 = \frac{\partial}{\partial x^i}, \quad Y_j = (\theta_{1j}, \cdot)^1 = \frac{\partial}{\partial y^j}, \quad (6.1)
\]

satisfy $[X_i, X_j] = 0$ in $U_1$, $[Y_i, Y_j] = 0$ in $U_2$, and $[X_i, Y_j] = 0$ in $U_1 \cap U_2$. These vector fields are tangent to $\mathcal{L}$ and, thus, determine commuting vector fields $\overline{X}_i = \frac{\partial}{\partial \varphi^i}$ and $\overline{Y}_i = \frac{\partial}{\partial \varphi^j}$ on $U_1$ and $U_2$, respectively. In $U_1 \cap U_2$, we have

\[
0 = [\overline{X}_i, \overline{Y}_j] = \frac{\partial}{\partial \varphi^k} \overline{Y}_j \frac{\partial}{\partial \varphi^k} - \frac{\partial}{\partial \varphi^k} \overline{X}_i \frac{\partial}{\partial \varphi^k}, \quad (6.2)
\]

where $\overline{Y}_j = Y_j$, $\overline{X}_i = \frac{\partial}{\partial \varphi^i}$, $\overline{Y}_j = \frac{\partial}{\partial \varphi^j}$ are the coefficients of the vector field $\overline{Y}_j$ in the coordinates $\varphi^i$. This means that $\overline{Y}_j$, which is the Jacobi matrix associated with the change of coordinates $\varphi \to \overline{\varphi}$, is a constant
matrix. Thus, there exists an atlas on $\mathcal{L}$ such that the Jacobi matrices are constant. This is equivalent to the statement of the theorem.

This raises the question as to the geometric structures on $\mathcal{L}$ that give rise to a pair of compatible antibrackets on a vector bundle over $\mathcal{L}$. Recall that in the $Sp(2)$-symmetric quantization, the manifold $\mathcal{L}$ is the space that includes the original fields of the theory to be quantized, the ghosts, antighosts, and the auxiliary fields in the $Sp(2)$-symmetric quantization $1$, which in the triplectic case include also the 'symplectic' partners $2, 3, 4$. One then adds antifields, thereby constructing the triplectic manifold $\mathcal{M}$. In the case where $\mathcal{L}$ is a linear (super)space, the known construction $1, 4, 5$ works by assigning each field $\phi^A$ (a coordinate on $\mathcal{L}$) a pair of antifields $\phi^{*aA}$. Then the nonvanishing antibrackets read as $$ (\phi^A, \phi^{*Bb})^a = \delta^a_0 \delta^A_B, $$ these antibrackets are evidently covariant under the linear transformation of $\mathcal{L}$ combined with the induced transformations of $\phi^{*aA}$. When $\mathcal{L}$ is not a linear (super)space, we see that it cannot be arbitrary: it has to admit a flat connection. Once the connection is given, we can consider the 'duplicated' cotangent bundle $\mathcal{M} = \Pi T^* \mathcal{L} \oplus \Pi T^* \mathcal{L}$ over $\mathcal{L}$ with the reversed parity of the fibers. In some coordinate neighbourhood on $\mathcal{L}$, the coordinates on the fibers of $\Pi T^* \mathcal{L} \oplus \Pi T^* \mathcal{L}$ read as $\xi_{ai}$, $a = 1, 2$, $\epsilon(\xi_{ai}) = \epsilon(x^i) + 1$. Under coordinate changes on $\mathcal{L}$, the coordinates $\xi_{ai}$ transform as $\frac{\partial}{\partial x^i}$. We now construct the antibrackets as $$ (x^i, \xi_{bj})^a = \delta_0^a \delta^i_j, \quad (\xi_{ai}, \xi_{bj})^c = \delta^c_0 \Gamma^m_{ij} \xi_{am} - \delta^c_a \Gamma^m_{ij} \xi_{bm}, \quad (6.3) $$ where $\Gamma^m_{ij}$ are the Christoffel coefficients of $\nabla$ in the coordinate system $x$ on $\mathcal{L}$. Introducing $\nabla = \frac{\partial}{\partial x^i} + \Gamma^m_{ij} \xi_{bm} \frac{\partial}{\partial \xi_{bj}}$, we can rewrite (6.3) as $$ (F, G)^a = -G \frac{\partial}{\partial \xi_{ai}} \nabla_i F + (-1)^{\epsilon(F)+1} \epsilon(G)+1 \frac{\partial}{\partial \xi_{ai}} \nabla_i G \cdot (6.4) $$ The symmetrized Jacobi identities for this pair of antibrackets are satisfied because the curvature and torsion of $\nabla$ vanish.

Thus, we have seen that in triplectic quantization, the manifold of fields $\phi^A$ is required to admit a flat symmetric connection. This is in contrast with the standard BV-scheme, where no additional requirements are imposed on the manifold of fields. This may be viewed as a restriction on the applicability of the covariant $Sp(2)$ quantization.

7 Conclusions

We have uncovered the geometric structures underlying the triplectic quantization of gauge theories. The most essential of these is the Kähler manifold with additional polarizations (a certain analogue of a hyper-Kähler manifold, however with a “wrong” signature of two complex structures).

As we have seen, however, the requirements on the marked functions from $12$ that lead eventually to a Darboux-like theorem restrict the spaces involved in the quantization by a number of flatness conditions. This may be viewed as a limitation of the entire $Sp(2)$-symmetric quantization approach; alternatively, it can be attributed to the properties of the axioms imposed in $12$. Thus, one may speculate that if the mutual commutativity condition imposed on the antibrackets is relaxed, one may still be able to identify some interesting geometries; the key question would then be about the meaning of the quantization procedure (e.g., in the construction of path integral). We thank I. Batalin for a discussion on this point.

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6 We thank I. Batalin for a discussion on this point.
As a final remark, note that the geometric structures that we have identified in the triplectic quantization (the symplectic and complex structures and the transversal polarizations) are those entering the Geometric Quantization (see, e.g., [13]) of symplectic manifolds. One may also note some formal similarities with the structures discussed in [14] in the context of BV geometry.

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Appendix

In this paper, we make use of some conventions of differential geometric objects on the supermanifold \( \mathcal{M} \). The main object is the graded associative algebra \( \mathcal{C}_M \) of (globally defined) smooth functions on \( \mathcal{M} \). A vector field \( X \) on \( \mathcal{M} \) is the differential of \( \mathcal{C}_M \), which means that

\[
X(FG) = (XF)G - (-1)^{\epsilon(F)}F(XG), \tag{A.1}
\]

where \( \epsilon(F) \) is the Grassmann parity of a function \( F \). The Grassmann parity \( \epsilon(X) \) of a vector field \( X \) is defined as \( \epsilon(X) = \epsilon(XF) + \epsilon(F), F \in \mathcal{C}_M \). Vector fields on \( \mathcal{M} \) constitute a left module over \( \mathcal{C}_M \); for any \( F \in \mathcal{C}_M \) and any vector field \( X \), we define \( (FX) \) by its action on arbitrary \( G \in \mathcal{C}_M \) as \( (FX)G = F(XG) \).

An \( N \)-form \( \Phi \) is defined as a multilinear mapping \( \Phi : \text{Vect}_M \times \ldots \text{Vect}_M \rightarrow \mathcal{C}_M \) satisfying

\[
\Phi(X_1, \ldots, X_N) = (-1)^{\epsilon(X_j)+1}\epsilon(X_{j+1})\Phi(X_1, \ldots, X_{j-1}, X_{j+1}, X_j, X_{j+2}, \ldots, X_N), \tag{A.2}
\]

\[
\Phi(FX_1, \ldots, X_N) = F\Phi(X_1, \ldots, X_N), \quad X_1, \ldots, X_N \in \text{Vect}_M, \quad F \in \mathcal{C}_M.
\]

The Grassmann parity \( \epsilon(\Phi) \) of the \( N \)-form \( \Phi \) is defined as

\[
\epsilon(\Phi) = \epsilon(\Phi(X_1, \ldots, X_N)) + \epsilon(X_1) + \ldots + \epsilon(X_N). \tag{A.3}
\]

The differential forms are a right module over \( \mathcal{C}_M \). For any \( F \in \mathcal{C}_M \) and \( N \)-form \( \Phi \) we have

\[
(\Phi F)(X_1, \ldots, X_N) = \Phi(X_1, \ldots, X_N)F. \tag{A.4}
\]

Contraction of the vector field \( X \) and an \( N \)-form \( \Phi \) reads as

\[
(i_X \Phi)(X_1, \ldots, X_{N-1}) = p(\Phi)\Phi(X_1, \ldots, X_{N-1}, X). \tag{A.5}
\]

where \( p(\Phi) \) is the degree of the form \( \Phi \). It is useful to define the outer product \( \wedge \) of forms such that \( i_X \) differentiate the outer product,

\[
i_X(\Phi \wedge \Psi) = (i_X \Phi) \wedge \Psi + (-1)^{\epsilon(X)+1}p(\Phi) \Phi \wedge (i_X \Psi), \tag{A.6}
\]

Now the De Rham differential \( d \) is by definition the nilpotent linear operator satisfying

\[
i_X dF = (X, dF) = XF, \quad X \in \text{Vect}_M, \quad F \in \mathcal{C}_M, \tag{A.7}
\]

\[
d(\Phi \wedge \Psi) = (d\Phi) \wedge \Psi + (-1)^{\epsilon(\Phi)+1}\Phi \wedge (d\Psi),
\]

In particular, for the 1-form \( \Phi \) we have

\[
(d\Phi)(X_1, X_2) = \frac{1}{2}(X_1 \Phi(X_2) - (-1)^{\epsilon(X_1)+\epsilon(X_2)}X_2\Phi(X_1) - \Phi([X_1, X_2]))(-1)^{\epsilon(X_2)+1}, \tag{A.8}
\]

where \( X_1, X_2 \in \text{Vect}_M \).
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