Quantum superposition of multiple clones and the novel cloning machine

Arun Kumar Pati
SEECS, Dean Street, University of Wales, Bangor LL 57 1UT, UK

(Received: January 31, 2018)

We envisage a novel quantum cloning machine, which takes an input state and produces an output state whose success branch can exist in a linear superposition of multiple copies of the input state and the failure branch exist in a superposition of composite state independent of the input state. We prove that unknown non-orthogonal states chosen from a set S can evolve into a linear superposition of multiple clones by a unitary process if and only if the states are linearly independent. We derive a bound on the success probability of the novel cloning machine. We argue that the deterministic and probabilistic clonings are special cases of our novel cloning machine.

PACS NO: 03.67.-a, 03.65.Bz, 89.70.+c
e-mail:akpati@sees.bangor.ac.uk

The information theoretical approach to the foundations of quantum theory is to regard the debatable issues as the facts of the quantum world and to exploit these facts in a constructive way so as to achieve classically forbidden technological applications in quantum information processing. In recent years the quantum mechanical principles such as linearity, unitarity and inseparability have been utilised to realise quantum computers [1], quantum teleportation [2], quantum cryptography [2] and so on. In one hand these principles enhance the possibility of information processing and on the other they put some limitations, too. That an unknown quantum state cannot be perfectly copied is a consequence of linearity of quantum theory [3]. Later it was shown that the unitarity of the quantum theory does not allow to clone two non-orthogonal states and thus it is impossible to measure the wavefunction of a single quantum system [3]. For mixed states a generalisation of "no-cloning" theorem [3] says that there is no physical means for broadcasting an unknown quantum state onto two separate parties. Though perfect copies cannot be produced, there exist the possibility of producing approximate copies of an unknown quantum state [3]. Optimal and universal quantum cloning machines have been constructed and it was shown [3] that if the device can make infinite number of copies then it is as good as a classical copying machine. The universal and state dependent quantum cloning machines have been studied and applied to quantum cryptography [2]. Remarkably, if one allows unitary and measurement processes then a set of linearly independent non-orthogonal states can be cloned perfectly with a non-zero probability [3]. Recently, we have proposed a protocol for producing copies and complement copies of an unknown qubit using minimal communication from a state preparer. The problem of quantum cloning can be regarded as a special case of state separation and unambiguous discrimination of non-orthogonal quantum states [3]. As some applications one finds the possibility of decompressing quantum entanglement using local copying [7]. Interestingly, the quantum "no-cloning" theorem has also been invoked to explain the information loss in side a black hole [3].

In the past various authors have asked the question: If we have an unknown state $|\psi\rangle$ is there a device which will produce either $|\psi\rangle \rightarrow |\psi\rangle \otimes^2$, $|\psi\rangle \rightarrow |\psi\rangle \otimes^3$, $|\psi\rangle \rightarrow |\psi\rangle \otimes^M$ or $|\psi\rangle \otimes^N \rightarrow |\psi\rangle \otimes^M$ copies of an unknown state in a deterministic or probabilistic fashion. This is a "classicalised" way of thinking about a quantum cloning machine. If we pause for a second, and think the working style of a classical Xerox machine, then we know that it does exactly the same thing. If we feed a paper with some amount of information into a Xerox machine containing $M$ blank papers, we can either get 1 $\rightarrow$ 2, or 1 $\rightarrow$ 3, or 1 $\rightarrow$ M copies by just pressing the number of copies we want. However, the quantum world is different where one can have linear superposition of all possibilities with appropriate probabilities. If a real quantum cloning machine would exist it should exploit this basic feature of the quantum world and it should produce simultaneously $|\psi\rangle \rightarrow |\psi\rangle \otimes^2$, $|\psi\rangle \rightarrow |\psi\rangle \otimes^3$, and $|\psi\rangle \rightarrow |\psi\rangle \otimes^M$ copies. We ask if it is possible by some physical process to produce an output state of an unknown quantum state which will be in a linear superposition of all possible multiple copies each in the same original state? A device that can perform this task we call "novel quantum cloning machine" (NQCM).

In this letter we show that the non-orthogonal states secretly chosen from a set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ can evolve into a linear superposition of multiple copy states together with a failure term described by a composite state (independent of the input state) by a unitary process if and only if the states are linearly independent. Further, we show that the von Neumann measurement of a "Xerox number" operator can yield a distribution of
perfect copies according to the standard rules of quantum theory. We also prove a bound on the success probability of the novel cloning machine for non-orthogonal states. We point out that the recently proposed probabilistic cloning machine of Duan-Guo ([34] can be thought of as a special case of a more general “novel cloning machine”. We hope that the existence of such a machine would greatly facilitate the quantum information processing in a quantum computer.

Consider an unknown input state $|\psi_i\rangle$ from a set $S$ which belongs to a Hilbert space $\mathcal{H}_A = C^{N_A}$. Let $|\Sigma\rangle_B$ be the state of the ancillary system $B$ (analogous to bunch of blank papers) which belongs to a Hilbert space $\mathcal{H}_B$ of dimension $N_B = N_A^M$, where $M$ is the total number of blank states each having dimension $N_A$. In fact we can take $|\Sigma\rangle_B = |0\rangle^\otimes M$. Let there be a probe state of the cloning device which can measure the number of copies that have been produced and $|P\rangle$ be the initial state of the probing device. Let $|P_1\rangle, |P_2\rangle, ..., |P_M\rangle, ..., |P_{N_C}\rangle$ are orthonormal basis states of the probing device. The set $\{|P_n\rangle\} \in \mathcal{H}_C = C^{N_C}$ such that $N_C > M$. If a novel cloning machine exists, then it should be represented by a linear, unitary operator that acts on the combined states of the composite system. The question is: Is it possible to have a quantum superposition of the multiple clones of the input state given by

$$|\psi_i(\Sigma)|P\rangle \rightarrow U(|\psi_i(\Sigma)|P\rangle) = \sqrt{p_1^{(i)}}|\psi_i\rangle|0\rangle^{(n)}|P_1\rangle + \sqrt{p_2^{(i)}}|\psi_i\rangle|\psi_i\rangle|0\rangle^{(n)}|P_2\rangle + .... + \sqrt{p_M^{(i)}}|\psi_i\rangle|\psi_i\rangle...|\psi_i\rangle|P_M\rangle,$$

where $p_n^{(i)} (n = 1, 2, ..., M)$ is the probability with which $n$-copies of the original input quantum state can be produced. However, we [19] have recently shown that such an ideal novel cloning machine based on unitarity of quantum theory cannot exist. The cloning machine should fail some time and the failure branch should be described by a state independent of the input state. Nevertheless, novel cloning machines which can create linear superposition of multicoopies with non-unit total success probability do exists. The existence of such a machine is proved by the following theorem.

Theorem: There exists a unitary operator $U$ such that for any unknown state chosen from a set $S = \{|\psi_i\rangle\} (i = 1, 2, ..., k)$ the machine can create a linear superposition of multiple clones together with failure copies given by

$$U(|\psi_i(\Sigma)|P\rangle) = \sum_{n=1}^{M} \sqrt{p_n^{(i)}}|\psi_i\rangle^{(n+1)}|0\rangle^{(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C} \sqrt{f_l^{(i)}}|\Psi_l\rangle_{AB}|P_l\rangle,$$

(2)

if and only if the states $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_k\rangle$ are linearly independent. In the above equation $p_n^{(i)}$ and $f_l^{(i)}$ are success and failure probabilities for the $i$th input state to produce $n$ exact copies and to remain in the $l$th failure component, respectively. The states $|\Psi_l\rangle_{AB}$’s are normalised states of the composite system $AB$ and they are not necessarily orthogonal.

We prove the existence of such a unitary operator in two stages. First, we prove that if an unknown quantum state chosen from a set $S$ exists in a linear superposition of multiple copy states then the set $S$ is linearly independent. Second, (which is the converse of the above statement) we prove that if the set $S = \{|\psi_i\rangle\}$ is linearly independent then any state chosen from a set $S$ can evolve into a linear superposition of multiple clones.

Consider an arbitrary state $|\psi\rangle = \sum_{i} c_i |\psi_i\rangle$. If we feed this state, then, the unitary evolution yields

$$U(|\psi(\Sigma)|P\rangle) = \sum_{n=1}^{M} \sqrt{p_n^{(i)}}|\psi_i\rangle^{(n+1)}|0\rangle^{(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C} \sqrt{f_l^{(i)}}|\Psi_l\rangle_{AB}|P_l\rangle.$$

(3)

However, by linearity of quantum theory each of $|\psi_i\rangle$ would undergo transformation under (2) and we have

$$U(\sum_{i} c_i |\psi_i(\Sigma)|P\rangle) = \sum_{i} c_i \sum_{n=1}^{M} \sqrt{p_n^{(i)}}|\psi_i\rangle^{(n+1)}|0\rangle^{(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C} \sum_{i} c_i \sqrt{f_l^{(i)}}|\Psi_l\rangle_{AB}|P_l\rangle.$$

(4)

Since the final states in (3) and (4) are different a quantum state represented by $|\psi\rangle$ cannot exist in a linear superposition of all possible copy states. We know that if a set contains distinct vectors $\{|\psi\rangle, |\psi_1\rangle, |\psi_2\rangle, ..., |\psi_k\rangle\}$ such that $|\psi\rangle$ is a linear combination of other $|\psi_i\rangle$’s then the set is linearly dependent. Thus linearity prohibits us creating linear superposition of multiple copy states chosen from a linearly dependent set. Therefore, the unitary (linear) evolution (2) exists for any state secretly chosen from $S$ only if its elements are linearly independent. This proves the first part of the theorem.

Now we prove the converse of the statement, i.e., we show that if the set $S$ is linearly independent then there exists a unitary evolution, which can create linear superposition of multiple copy states with some success and failure. If the unitary evolution (2) holds, then the overlap of two distinct output states $|\psi_i\rangle$ and $|\psi_j\rangle$ secretly chosen from $S$ after they have passed through the device would be given by

$$\langle \psi_i|\psi_j\rangle = \sum_{n=1}^{M} p_n^{(i)} \langle \psi_i|\psi_j\rangle^{n+1} + \sum_{l=M+1}^{N_C} f_l^{(i)} f_l^{(j)}.$$

(5)
Conversely, if (5) holds, there exists a unitary operator to satisfy (2). In the sequel we will prove that if the set $\mathcal{S}$ is linearly independent then (5) holds. The above equation can be generically expressed as a $k \times k$ matrix equation

$$G(1) = \sum_{n=1}^{M} A_n G(n+1) A_n^\dagger + \sum_{l} F_l,$$

where the matrices $G(1) = \left[|\psi_i\rangle\langle\psi_j|\right]$ is the Gram matrix, $G(n+1) = \left[|\psi_i\rangle\langle\psi_j|^{(n+1)}\right]$, $A_n = A_n^\dagger = diag(\sqrt{p_n^{(1)}}, \sqrt{p_n^{(2)}}, \ldots, \sqrt{p_n^{(k)}})$ and $F_l = \left[\sqrt{f_l^{(1)}} f_l^{(2)} \ldots f_l^{(k)}\right]$. Now proving the existence of a unitary evolution given in (2) is equivalent to showing that (6) holds for a positive definite matrix $A_n$. It can be shown that if the states $\{|\psi_i\rangle\}$ are linearly independent, then the Gram matrix $G(1)$ is a positive definite and its rank is equal to the dimension of the space spanned by the vectors $\{|\psi_i\rangle\}$. Similarly, we can show that the matrix $G(n+1)$ is also positive definite. Because for an arbitrary vector $\alpha = col(c_1, c_2, \ldots, c_k)$, we can write $\alpha^\dagger G(n+1) \alpha = \sum_{i,j=1}^{k} c_i^* c_j G_{ij}^{(n+1)} = \langle \beta | \beta \rangle$, where $|\beta\rangle = \sum_i c_i |\psi_i\rangle^{\otimes (n+1)}$. Since square of the length of a vector is positive and cannot go to zero (if the set is linearly independent), this shows that $G(n+1)$ is a positive definite matrix.

Also, the matrix $A_n$ is positive definite which suggests $A_n G(n+1) A_n^\dagger$ is also a positive definite matrix. Further, we know that sum of positive definite matrices is also a positive definite one. From the continuity argument for a small enough $A_n$, the matrix $G(1) - \sum_{n=1}^{M} A_n G(n+1) A_n^\dagger$ is also a positive definite matrix. Therefore, we can diagonalise the Hermitian matrix by a suitable unitary operator $V$. Thus we have $V^\dagger G(1) V = diag(a_1, a_2, \ldots, a_k)$, where the eigenvalues $\{a_i\}$ are positive real numbers. Now we can choose the matrix $F_l$ to be $F_l = V dia(g_{l(1)} g_{l(2)} \ldots g_{l(k)}) V^\dagger$ such that $\sum_l g_{l(i)} = a_i, (i = 1, 2, \ldots k)$. Thus the matrix equation (5) is satisfied with a positive definite matrix $A_n$ if the states are linearly independent. Once (2) holds, we see that the success and failure probabilities are summed to unity, i.e., $\sum_n p_n^{(i)} + \sum_l f_l^{(i)} = 1$ as expected. This completes the proof of our main result.

Here we discuss the generality of our novel cloning machine. For example, if $\{|0\rangle\}$ and $\{|1\rangle\}$ are the computational basis, then a qubit secretly chosen from a set $\{|0\rangle, \alpha|0\rangle + \beta|1\rangle\}$ or from a set $\{|1\rangle, \alpha|0\rangle + \beta|1\rangle\}$ can exist in a linear superposition of multiple clones. But a state chosen from a set $\{|0\rangle, |1\rangle, \alpha|0\rangle + \beta|1\rangle\}$ cannot exist in such a superposition of multiple clones as the set is not linearly independent. It may be remarked that “no-cloning” theorem is a special case of our result. Because, the linear superposition of multiple clones fails if the machine does not fail with some probability. When all the failure probabilities are zero, we have “no-supersetion of multi-clones” theorem [19]. Then if we take one of the success probability as one, then we get Wootters-Zurek-Dick’s “no-cloning” theorem [20].

Our result is consistent with the known results on cloning. In the unitary evolution if one of the positive number in success branch is one (i.e., $p_n^{(i)} = 1$ for some $n$ and all $i$) and all others (including failure branches) are zero, then we have $U(|\psi_i\rangle |\mathcal{S}_{i}\rangle |P\rangle) = |\psi_i\rangle^{\otimes (n+1)} |0\rangle^{\otimes (M-n)} |P\rangle$. This tells us that the matrix equation would be $G(1) = G(n+1)$ since $A_n = I$. This will be possible only when the states chosen from a set are orthogonal to each other. Thus a single quanta in an orthogonal state can be perfectly cloned [3]. Here we discuss the condition under which all $f^{(i)}$’s are zero. The orthogonality relation $\langle \psi_i | \psi_j \rangle = \delta_{ij}$ is a necessary and sufficient condition on the set $\mathcal{S}$ for all $f^{(i)}$’s to be zero. The converse can be also proved, i.e., if all $f^{(i)}$’s are zero then the states are orthogonal. When all $f^{(i)}$’s are zero, from (5) we can obtain $|\langle \psi_i | \psi_j \rangle| \leq \sum_{n=1}^{M} \sqrt{p_n^{(i)} p_n^{(j)}} |\langle \psi_i | \psi_j \rangle|^{n+1}$. Using two inequalities $(p_n^{(i)} + p_n^{(j)}) \leq \sqrt{2 (p_n^{(i)} + p_n^{(j)})}$ and $|\langle \psi_i | \psi_j \rangle|^n \leq |\langle \psi_i | \psi_j \rangle| (1 - |\langle \psi_i | \psi_j \rangle|) \leq 0$. Since the quantity in the bracket is positive and $|\langle \psi_i | \psi_j \rangle|$ cannot be negative it must be zero. Therefore, the states have to be orthogonal when all $f^{(i)}$’s are zero. Note that another interesting result follows from our proposed cloning machine. If the states are orthogonal and all $p_n^{(i)}$’s are non-zero, then unitarity allows us to have a linear superposition of multiple copies of orthogonal states as the matrix equation is always satisfied. We mention that it would be interesting to investigate the extension of $U$ beyond the elements of $\mathcal{S}$ in future.

After the input state chosen from the set $\mathcal{S}$ undergo unitary evolution in order to know how many copies are produced by the novel cloning machine, one needs to do a von Neumann measurement onto the probe basis. This can be thought of as a measurement of a Hermitian operator. We introduce such an operator, which is called “Xerox number” operator $N_X$, defined as

$$N_X = \sum_{n=1}^{M} n|P_n\rangle \langle P_n|.$$

The probe states $|P_n\rangle$ are eigenstates of the Xerox number operator with eigenvalue $n$ where $n$ is the number of clones produced with a probability distribution $p_n^{(i)}$. The measurement of Xerox number operator will give us information about how many copies have been produced by the cloning machine. For example, the novel cloning machine would produce $1 \rightarrow 2$ copies with probability $p_1$, $1 \rightarrow 3$ copies with probability $p_2$, ... and $1 \rightarrow M + 1$ copies with probability $p_M$ in accordance with the usual rules of quantum mechanics.

Here, we derive a bound on the success probability of producing multiple clones through a unitary machine (2). Taking the overlap of two distinct states we find
\[
|\langle \psi_i | \psi_j \rangle| \leq \sum_{n=1}^{M} \sqrt{p_n^{(i)} p_n^{(j)}} |\langle \psi_i | \psi_j \rangle|^{n+1} + \sum_{l=M+1}^{N_M} \sqrt{f_l^{(i)} f_l^{(j)}}.
\]

(8)

On simplifying (8) we get the tight bound on the individual success probability for cloning of two distinct non-orthogonal states as

\[
\frac{1}{2} \sum_n (p_n^{(i)} + p_n^{(j)})(1 - |\langle \psi_i | \psi_j \rangle|^{n+1}) \leq (1 - |\langle \psi_i | \psi_j \rangle|).
\]

(9)

The above bound is related to the distinguishable metric of the quantum state space. Since the “minimum-normed-distance” \([20]\) between two non-orthogonal states \(|\psi_i\rangle\) and \(|\psi_j\rangle\) is

\[
D^2(|\psi_i\rangle, |\psi_j\rangle) = 2(1 - |\langle \psi_i | \psi_j \rangle|)
\]

and the “minimum-normed-distance” between \(n + 1\) clones is

\[
D^2(|\psi_i\rangle^\otimes n+1, |\psi_j\rangle^\otimes n+1) = 2(1 - |\langle \psi_i | \psi_j \rangle|^{n+1}),
\]

the tight bound can be expressed as

\[
\sum_n p_n D^2(|\psi_i\rangle^\otimes n+1, |\psi_j\rangle^\otimes n+1) \leq D^2(|\psi_i\rangle, |\psi_j\rangle),
\]

(10)

where we have defined total success probability \(p_n\) for \(n^\text{th}\) clones as \(p_n = \frac{1}{2}(p_n^{(i)} + p_n^{(j)})\). The “minimum-normed-distance” function is a measure of distinguishability of two non-orthogonal quantum states. Therefore, the above bound can be interpreted physically as the sum of the weighted distance between two distinct states of \(n + 1\) clones is always bounded by the the original distance between two non-orthogonal states. Since cloning transformation is a physical procedure for making two states more distinguishable, any two state which pass through our machine has to satisfy this strict inequality. Also, our bound is consistent with the known results on cloning. For example, if we have 1 \(\to\) 2 cloning, then in the evolution we have \(p_1^{(i)}\) and \(p_1^{(j)}\) are non-zero and all others are zero. In this case our bound reduces to

\[
\frac{1}{2} (p_1^{(i)} + p_1^{(j)}) \leq \frac{1}{2} |\langle \psi_1 | \psi_1 \rangle|,
\]

which is nothing but the Duan-Guo bound \([14]\) for producing two clones in a probabilistic fashion. Similarly if we have 1 \(\to\) \(M\) cloning, then in the evolution we have \(p_M^{(i)}\) and \(p_M^{(j)}\) are non-zero and all others are zero. In this case our bound reduces to

\[
\frac{1}{2} (p_M^{(i)} + p_M^{(j)}) \leq \frac{1}{2} |\langle \psi_M | \psi_M \rangle|,
\]

which is nothing but Cheffes-Barnett \([16]\) bound, obtained using quantum state separation method.

We can imagine a more general novel cloning machine and then show that the probabilistic cloning machine discussed by Duan and Guo \([4]\) can be considered as a special case of the general novel cloning machine. Instead of the unitary evolution (2) one could describe a general unitary evolution of the composite system \(ABC\) as

\[
U(|\psi_i\rangle |\Sigma\rangle |P\rangle) = \sum_{n=1}^{M} \sqrt{p_n^{(i)}} |\psi_i\rangle^\otimes (n+1) |0\rangle^\otimes (M-n) |P_n\rangle + \sum_l c_d |\Psi_l\rangle_{ABC}.
\]

(11)

Here, the first term has the usual meaning and the second term represents the failure term. The states \(|\Psi_l\rangle\) \(\in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C\) are normalised states of the composite system. For simplicity we assume that they are orthonormal. Further, since the measurement of Xerox number operator should yield perfect copies (say \(n\)) of the input state with probability \(p_n^{(i)}\), this entails that \(|P_n\rangle|\langle P_n|\Sigma\rangle = 0\) for any \(n\) and \(l\). Imposing this physical condition, we find from eq.(11) that the inner product of two distinct states gives

\[
\langle \psi_i | \psi_j \rangle = \sum_{n=1}^{M} \sqrt{p_n^{(i)}} |\psi_i\rangle^\otimes (n+1) \sqrt{p_n^{(j)}} + \sum_l c_d^j c_d l.
\]

This can be expressed as a matrix equation \(G^{(1)}(n) = \sum_{n=1}^{M} A_n G^{(n+1)}(\eta) A_n^\dagger + C^\dagger C\), where \(C = [c_{ij}]\). From our earlier theorem we can now prove that with a positive definite matrix \(A_n\) we can diagonalise \(G^{(1)} - \sum_{n=1}^{M} A_n G^{(n+1)} A_n^\dagger\) and with a particular choice of the matrix \(C\) the unitary evolution exists.

To see that from our machine Duan-Guo machine follows as a special case, let us take one of the \(p_n^{(i)}\) is non-zero and all others are zero in the above unitary transformation. Then we have the following evolution for the non-orthogonal states

\[
U(|\psi_i\rangle |\Sigma\rangle |P\rangle) = \sqrt{p_n^{(i)}} |\psi_i\rangle^\otimes (n+1) |P_n\rangle + \sum_l c_d |\Psi_l\rangle_{ABC}.
\]

(12)

where we have assumed that there are \(n\) blank states. This is nothing but Duan-Guo type of probabilistic machine for producing 1 \(\to\) \(n\) copies. If one does a measurement of the probe with a postselection of the measurement results, then this will yield \(n(n = 1, 2, \ldots M)\) exact copies of the unknown quantum states. Since all other deterministic cloning machines are special cases of Duan-Guo machine, we can say, in fact, that all deterministic and probabilistic cloning machines are special cases of our novel cloning machines.

To conclude this letter, we discovered yet another surprising feature of cloning transformation, which says that unitarity allows us to have linear superposition of multiple clones of non-orthogonal states along with a failure term if and only if the states are linearly independent. We derived a tight bound on the success probability of passing two non-orthogonal states through a novel cloning machine. We proved that the probabilistic and deterministic clonings are special cases of our novel cloning machine. We hope that the existence of linear superposition of multiple clones will be very much useful in quantum state engineering, easy preservation of important quantum information, quantum error correction and parallel storage of information in a quantum computer.
Since multiple clones remains in various branches of the output state in a quantum computer one might think of manipulating different clones in a desired way using controlled operations.

I thank S. L. Braunstein for useful discussions. I thank L. M. Duan for useful discussions and suggestions. I gratefully acknowledge financial support by EPSRC.

[1] D. Deutsch, Proc. R. Soc. A 400, 97 (1985).
[2] C. H. Bennett, G. Brassard, Proc. IEEE International Conference on Computers, Systems and Signal processing (New York) pp. 175, IEEE, 1984, Bangalore, India, December 1984.
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[4] W. K. Wootters and W. H. Zurek, Nature 299, 802 (1982).
[5] D. Dieks, Phys. Lett. A 92, 271 (1982).
[6] H. P. Yuen, Phys. Lett. A 113, 405 (1986).
[7] G. M. D’Ariano and H. P. Yuen, Phys. Rev. Lett. 76, 2832 (1996).
[8] H. Barnum, C. M. Caves, C. A. Fuchs, R. Josza, and B. Schumacher, Phys. Rev. Lett. 76, 2818 (1996).
[9] V. Bužek and M. H. Hillery, Phys. Rev. A 54, 1844 (1996).
[10] V. Bužek, S. L. Braunstein, M. H. Hillery, D. Bruß, Phy. Rev. A. 56, 3446 (1997).
[11] N. Gisin and S. Massar, Phys. Rev. Lett. 79, 2153 (1997).
[12] D. Bruß, et al, Phys. Rev. A 57, 2368 (1998).
[13] L. M. Duan and G. C. Guo, Phys. Lett. A 243, 261 (1998).
[14] L. M. Duan and G. C. Guo, Phys. Rev. Lett. 80, 4999 (1998).
[15] A. K. Pati, Phys. Rev. A (in press) (1999).
[16] A. Chefles and S. M. Barnett, J. Phys. A 31, 10097 (1998).
[17] V. Bužek, V. Vedral, M. B. Plenio, P. L. Knight, and M. H. Hillery, Phys. Rev. A 55, 3327 (1997).
[18] L. Susskind and J. Uglun, String Physics and Black holes, Nuclear Phys. (Proc. Suppl.) 45 B, C, 115 (1996).
[19] A. K. Pati, Multiple quantum clones cannot be superposed, (Preprint, 1999).
[20] A. K. Pati, Phys. Lett. A 159, 105 (1991).