A note on the Hawking radiation calculated by the quasi-classical tunneling method

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Since Parikh and Wilczek’s tunneling method was proposed, there have been many generalizations, such as its application to massive charged particles’ tunneling and other spacetimes. Moreover, an invariant tunneling method was also recently proposed by Angheben et al that it was independent of coordinates. However, there are some subtleties in the calculation of Hawking radiation, and particularly is the so-called factor of 2 problem during calculating the Hawking temperature. The most popular opinion on this problem is that it is just a problem of the choice of coordinates. However, following other treatments we show that we can also consider this problem to be that we do not consider the contribution from P(absorption). Moreover, we also clarify some subtleties in the balance method and give some comparisons with other treatments. In addition, as Parikh and Wilczek’s original works have showed that if one takes the tunneling particles’ back-reaction into account, the Hawking radiation would be modified, and this modification is underlying consistent with the unitary theory, we further find that this modification is also underlying correlated with the laws of black hole thermodynamics. Furthermore, we show that this tunneling method may be valid just when the tunneling process is reversible.

Keywords: tunneling method, Hawking radiation, black hole thermodynamics

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I. INTRODUCTION

In 2000, Parikh and Wilczek proposed a new method to reconsider the Hawking radiation. Besides treating the Hawking radiation as a tunneling process, they also took the tunneling particles’ back-reaction into account. And after that, they obtained the corresponding modified spectrum. The most interesting result was that they found this modified spectrum was implicitly consistent with the unitary theory and could support the conservation of information \[1–4\]. Following this tunneling method, there have been many generalizations, such as its application to other spacetimes and the massive charged particles’ tunneling cases \[5–49\]. In addition, it can also be used to calculate the black hole temperature \[50\] (Note that the semi-classical tunneling method for Hawking radiation has also been separately proposed by G.E. Volovik in \[51, 52\]). However, Parikh and Wilczek’s tunneling method is dependent on coordinates, which means that it should find a Painleve-like coordinates. Recently, Angheben et al found an invariant tunneling method which was independent of coordinates and called the Hamilton-Jacobi tunneling method to calculate the Hawking temperature \[53\]. This variant tunneling method could also be considered as an extension of the method used by Padmanabhan et al \[54–58\].

In the above two tunneling methods, they both involve calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon. And according to the WKB approximation, the tunneling probability usually related to the imaginary part is

\[
\Gamma \propto \exp(-2\text{Im}S_{\text{out}}) = \exp(-2\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr).
\]

(1.1)

where \(S\) is the action of the trajectory. Thus if we ignore the tunneling particles’ back-reaction effect, we can give the Hawking temperature from

\[
\Gamma \propto \exp(-2\text{Im}S_{\text{out}}) = \exp(-\beta \omega).
\]

(1.2)

where \(\beta\) is the inverse temperature of the horizon and \(\omega\) is the energy of the tunneling particle. In other words, the Hawking temperature can be recovered at linear order when

\[2\text{Im}S_{\text{out}} = \beta \omega + o(\omega^2),\]

and the higher order terms are a self-interaction effect resulting from energy conservation \[50, 59, 61\]. However, recently some authors proposed that the formalism \(\text{Im}S_{\text{out}} = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr\) in (1.1) was not invariant under canonical transformations, thus the tunneling probability was not a proper observable \[62, 65\]. Moreover, if we calculate
the imaginary part of the action in different coordinate systems by using the Hamilton-Jacobi tunneling method and (1.1), we can obtain different results which means obtaining different temperatures. For example, if we calculate the Schwarzschild black hole’s temperature in Schwarzschild static coordinates by using the Hamilton-Jacobi tunneling method, we will obtain the temperature \( T = \frac{1}{4\pi M} \), which is twice the temperature originally calculated by Hawking or other methods \([54–58, 66–70]\). It is the so-called factor of 2 problem \([63–65]\). Note that, however, if we also use (1.1) and the Hamilton-Jacobi tunneling method to calculate the Hawking temperature in the Painleve coordinates, we will obtain the correct Hawking temperature. Thus it can be naturally simply considered the factor of 2 problem just as a problem of the choice of coordinates. However, there are also other treatments. And basically expect to generalize the tunneling method to be independent of coordinates, Angheben et al in the original paper of the Hamilton-Jacobi tunneling method have proposed a treatment to solve this problem. And they argued that one had to make a change of some spatial variable to be the corresponding proper spatial variable defined by the spatial metric \([53]\). However, as Akhmedov et al found that it couldn’t in fact solve the factor of 2 problem, which could be simply understood from the fact that the simple change of spatial variable shouldn’t change the value of the integral (more details of argument could be seen in the references \([53, 63–65]\), and the key point is that the contour associated with the divergent part of the integral is also changed after making the radial variable to be the proper spatial variable). Thus, following the treatment of generalizing the tunneling method to be independent of coordinates, some authors argued that perhaps there was a temporal imaginary contribution from the time part of action which was usually neglected in previous works \([63–65]\). And others proposed that an integration constant should be inserted into the action like \( S = \int_{r_{in}}^{r_{out}} p_r dr + C \), in which the constant \( C \) may also give the contribution to the imaginary part of the action \([77, 78]\). In this note, we show that we can also consider this problem not as a problem of the choice of coordinates but the problem that we do not consider the contribution from \( \Pi \) (absorption). And we show again that if we consider the thermal balance and take the tunneling probability such that \( \Gamma \propto \frac{P(\text{emission})}{P(\text{absorption})} \) (the so-called balance method), there is no the factor of 2 problem, as have been implicated by some previous works which recover the Hawking temperature \([54–58]\). In addition, this tunneling probability is invariant under canonical transformations. Furthermore, we also clarify some subtleties in the balance method and give some comments and comparisons
with other treatments. And here the meaning of showing again is that some authors have also considered the thermal balance before but they gave the different result that the balance method could not solve the factor of 2 problem \[77\].

In addition, note that the back-reaction of the tunneling particles is neglected when we recover the usual Hawking temperature. And if one takes the back-reaction of the tunneling particles into account \[59–61\], the Hawking radiation would be modified. In Parikh and Wilczek’s original works they have showed that if one takes the tunneling particles’ back-reaction which comes from the conservation of energy into account, the modified spectrum could also be calculated by the tunneling method. And the modified spectrum is underlying consistent with the unitary theory \[1–4\]. In this note, by taking the general Reissner-Nordstrom black hole (R-N black hole) and kerr black hole as examples \[5, 6\], we could further find that this modification is also underlying correlated with the laws of black hole thermodynamics. Furthermore, we show that this tunneling method may be valid only when the tunneling process is reversible \[79\].

The outline of our paper is as follows. In Sec. II, by taking also the simple Schwarzschild black hole as an example, we emphasize that the balance method could indeed solve the factor of 2 problem well. And we also clarify some subtleties in this method and give comparisons with other treatments. In Sec. III, we take the R-N black hole and kerr black hole as examples to show the underlying correlation between the modification spectrum and the laws of black hole thermodynamics. Section. IV. is devoted to conclusion and discussion.

II. THERMAL BALANCE’S TUNNELING PROBABILITY AND THE TEMPERATURE

The Schwarzschild black hole in the static coordinates is

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2.
\]

(2.1)

In the Hamilton-Jacobi tunneling method, the classical action \(S\) of the tunneling particle satisfies the relativistic Hamilton-Jacobi equation \[50, 53, 63, 65\]

\[
g^{\mu\nu}\partial_\mu S \partial_\nu S + m^2 = 0.
\]

(2.2)

For this metric and radial trajectories which are independent of \(\theta, \varphi\), the Hamilton-Jacobi
equation becomes

\[-(1 - \frac{2M}{r})^{-1}(\partial_t S)^2 + (1 - \frac{2M}{r})(\partial_r S)^2 + m^2 = 0\] (2.3)

As usual, due to the symmetries of the metric, there exists a solution of the form

\[S = -\omega t + W(r)\] (2.4)

where

\[\partial_t S = -\omega, \quad \partial_r S = W'(r)\]

Solving \(W(r)\) yields

\[W(r) = \pm \int \frac{dr}{(1 - \frac{2M}{r})} \sqrt{\omega^2 - m^2(1 - \frac{2M}{r})}\] (2.5)

where the \(+(-)\) sign in front of this integral expresses the ingoing(outgoing) particles. Thus, it can be easily found that if we simply consider the tunneling probability

\[\Gamma \propto \exp(-2\text{Im}S_{\text{out}}) = \exp(2\text{Im} \int_{2M-\epsilon}^{2M+\epsilon} \frac{dr}{(1 - \frac{2M}{r})} \sqrt{\omega^2 - m^2(1 - \frac{2M}{r})}) = \exp(-4\pi M \omega)\] (2.6)

where the contour lies in the upper complex plane and the minus dropped in front of the imaginary part of the action corresponds to the initial condition that \(\partial_r S > 0\) at \(r = 2M - \epsilon < 2M\) [54–58], we will obtain the Hawking temperature \(T = 1/4\pi M\). It is twice the temperature originally calculated by Hawking and other methods [50, 53, 63–65]. That is the so-called factor of 2 problem. However, if we also use (1.1) and the Hamilton-Jacobi tunneling method to calculate the Hawking temperature in the Painleve coordinates, we will obtain the correct Hawking temperature. And the Painleve coordinates could be obtained by the following transformation from the Schwarzschild coordinates

\[t \rightarrow t_p = \int \frac{\sqrt{2M/r}}{1 - 2M/r} dr\] (2.7)

and the metric in the Painleve coordinates is

\[ds^2 = -(1 - \frac{2M}{r})dt_p^2 + 2\sqrt{\frac{2M}{r}}dt_p dr + dr^2 + r^2d\Omega^2\] (2.8)

where one of the advantages in this coordinates is that the metric is regular at the horizon [1–4] (note that here the contour of the divergent part of the integral in the Painleve coordinates
is the same as that in the static Schwarzschild coordinates, which is different from the contour
in the proper spatial variable coordinates referred in [53]). In addition, another well-known
point is that the original tunneling method first proposed by Parikh and Wilczek should
be calculated in the Painleve coordinates. Thus it can be naturally simply considered the
factor of 2 problem just as a problem of the choice of coordinates. However, there are also
other treatments which expect to generalize the tunneling method to be independent of
coordinates. And some authors argued that when calculating the Hawking temperature in
the Schwarzschild static coordinates perhaps there was a temporal imaginary contribution
from the time [63–65], while some authors thought that one should consider the integration
constant $C$ when yields $W(r)$ in (2.5) [74, 78]. Here, what we will show is mainly that we can
also consider it not to be a problem of the choice of coordinates but to be the problem that we
do not consider the contribution from $P$ (absorption). And we show again that if we consider
the thermal balance and take the tunneling probability as $\Gamma \propto P(\text{emission})/P(\text{absorption})$
(the so-called balance method), there is no the factor of 2 problem as some works have
implied [54–58, 80], which could be easily checked in the static Schwarzschild coordinates as
a simple example that

$$\Gamma \propto \frac{P(\text{emission})}{P(\text{absorption})} = \frac{\exp(-2\text{Im}S_{\text{out}})}{\exp(-2\text{Im}S_{\text{in}})} = \frac{\exp(-4\pi M\omega)}{\exp(4\pi M\omega)} = \exp(-8\pi M\omega) \tag{2.9}$$

which can recover the Hawking temperature $T = 1/8\pi M$. It should be noted that although
the above treatment has already been proposed by T.Padmanabhan et al, there are some
subtleties in this method. And these subtleties may be the direct motivation of the other two
treatments. The first subtlety is that $P(\text{absorption})$ in (2.9) is greater than unity. Thus, if
it is considered as the absorption probability, it would be unphysical. However, considering
$P(\text{absorption})$ as the absorption probability is not suitable here because we have neglected
the unitary factor in front of the wave function. And it can be easily checked out that consi-
dering $P(\text{emission})/P(\text{absorption})$ as the tunneling probability is always suitable. The
second subtlety is that the tunneling probability $\Gamma \propto P(\text{emission})/P(\text{absorption})$ is indeed
invariant under canonical transformations, which means that the tunneling probability has
the same result in any coordinates. The third subtlety is that whether the time in (2.4)
can give the contribution to the imaginary part of the action. In the present status, we
think that it is not necessary to consider its contribution. The main reason is just be-
because the temporal imaginary contribution from the time can be canceled automatically
in $P(\text{emission})/P(\text{absorption})$. In fact, after some simple calculations, we could find that during the calculation in the Schwarzschild static coordinates the contribution from the $\exp(-2\text{Im}S_{in})$ in (2.9) is just equal to the contribution from the time in references [63–65] or the integration constant in references [77, 78]. Moreover, it should also be easily found that the contribution from the $\exp(-2\text{Im}S_{in})$ is just equal to unit in the Painleve coordinates (which is correlated with the order that the metric in Painleve-like coordinates should be regular at the horizon, then $S_{in}$ is real), thus the tunneling probability is $\Gamma \propto \exp(-2\text{Im}S_{out})$ which is just the familiar formalism (1.1) in the original tunneling method’s treatment [1–4]. Thus it can also be considered as another underlying reason why it should find a Painleve-like coordinates and the tunneling probability in (1.1) can work well in the original tunneling method.

**III. THE MODIFIED HAWKING RADIATION AND ITS RELATIONSHIP WITH LAWS OF BLACK HOLE THERMODYNAMICS**

In the above section, we calculate the Hawking temperature by dropping effect of the back-reaction of the tunneling particle. In this section, we take it into account [1–4, 59–61]. In Parikh and Wilczek’s original works they have shown that the back-reaction effect will give a correction to the Hawking radiation (i.e, modified spectrum), and this modification is underlying consistent with the unitary theory and can support the conservation of information during the evolution of black hole [1–4]. In addition, it has also been found that this modification is underlying correlated with the laws of thermodynamics. Because although the behavior of massive particles’ tunneling is different from that of massless particles’ tunneling, two integrations are the same after we first integrate the radial part during the calculation of the imaginary part of the action, and the integration is direct proportional to the inverse of the temperature. Moreover, this integration can be expressed as the formalism of laws of thermodynamics in some cases [32, 36, 79]. In the following, taking the more general R-N black hole and Kerr black hole as examples, we further show that the underlying relationship between the modification and laws of thermodynamics is universal. Furthermore, we show that this tunneling method may be valid only when the tunneling process is reversible.
A. The R-N black hole

Recently, Parikh and Wilczek’s original works have been extended to the R-N black hole. As a general solution with a simple charge in Einstein equation, we first take it as an example. According to Ref. [6], after taking the back-reaction of the tunneling massive charged particle into account, the imaginary part of the action for the classically forbidden trajectory is

\[
\text{Im}S = \text{Im}\left\{ \int_{r_i}^{r_f} \left[ p_r - \frac{p_{At}}{r} \right] dr \right\}
\]

\[
= -\text{Im}\left\{ \int_{r_i}^{r_f} \int_{(M,Q)}^{(M-\omega;Q-q)} \frac{2r\sqrt{2Mr-Q^2}}{r^2-2Mr+Q^2} dM - \frac{2\sqrt{2Mr-Q^2}Q}{r^2-2Mr+Q^2} dQ \right\}
\]

\[
= -\pi \int_{(M,Q)}^{(M-\omega;Q-q)} \frac{(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}} dM - \frac{(M+\sqrt{M^2-Q^2})Q}{\sqrt{M^2-Q^2}} dQ
\]

\[
= -\frac{1}{2} \Delta S_{BH}.
\]

(3.1)

where \( A_t = \frac{Q}{r} \) is the first component of the 4-Dimensional electromagnetic potential, and \( p_{At} \) is the corresponding canonical momentum conjugate. As the conclusion discussed in Ref. [6], from (3.1) we can easily obtain the modified spectrum and find that the original temperature of R-N black hole is just the case of neglecting the higher-order terms of \( \omega \) and \( q \). Moreover, it can also be found that this modified spectrum is consistent with the underlying unitary theory, which supports the conservation of information. Here, we further show that an interesting result could also be obtained in the same equation (3.1) if we start from the viewpoint of laws of black hole thermodynamics, which is expressed in the following. And the underlying relationship between the tunneling process and the laws of black hole thermodynamics may give a new insight into the tunneling process.

As we know, for the R-N black hole, when a charged massive particle tunnels across the event horizon, the mass and the charge of black hole will be changed as a consequence. According to the first law of black hole thermodynamics, the differential Bekenstein-Smarr equation of the R-N black hole is [81, 82]

\[
dM = \frac{\kappa}{8\pi} dA + V dQ \ (J = 0),
\]

(3.2)

Furthermore, if the tunneling process is considered as a reversible process, according to the
The second law of black hole thermodynamics, \( (3.2) \) can be rewritten as
\[
dM = TdS + VdQ. \tag{3.3}
\]
Equally, it can be rewritten as
\[
dS = \frac{dM}{T} - \frac{VdQ}{T}. \tag{3.4}
\]
The temperature and the potential are respectively \( (3.5) \)
\[
T = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2}, \quad V = \frac{Q}{M + \sqrt{M^2 - Q^2}}. \tag{3.5}
\]
Substituting (3.5) into (3.4), we can obtain
\[
dS = \frac{2\pi(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}}dM - \frac{2\pi(M + \sqrt{M^2 - Q^2})Q}{\sqrt{M^2 - Q^2}}dQ. \tag{3.6}
\]
Thus, using (3.6) we can rewrite (3.1) as
\[
\text{Im}S = -\pi \int_{(M,Q)}^{(M-\omega,Q-q)} \left[ \frac{(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}}dM - \frac{(M + \sqrt{M^2 - Q^2})Q}{\sqrt{M^2 - Q^2}}dQ \right] = -\frac{1}{2} \int_{S_i}^{S_f} dS = -\frac{1}{2} \Delta S_{BH}. \tag{3.7}
\]
Which is also the same result as (3.1). The difference is that the result in (3.7) implicates that Hawking radiation via tunneling is correlated with the laws of black hole thermodynamics.

### B. The Kerr black hole

As another general solution, the Kerr black hole spacetime with a simple angular momentum, we also take it as an example to show the relationship between the modified spectrum and the laws of black hole thermodynamics. For the sake of simplicity, we only investigate the case of a massless particle with angular momentum tunneling across the outer event horizon.

According to Ref [5], after taking the tunneling massless particles’ back-reaction into
account, the imaginary part of the action for the classically forbidden trajectory is

\[
\text{Im} S = \text{Im} \left[ \int_{r_i}^{r_f} p_r dr - \int_{\phi_i}^{\phi_f} p_{\phi} d\phi \right]
\]

\[
= \text{Im} \left[ \int_{r_i}^{r_f} \int_{M}^{M-\omega} \frac{\sqrt{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}}{\rho^2 - \sqrt{\rho^2 (\rho^2 - \Delta)}} dr dM \right.

\]

\[
\left. - \int_{r_i}^{r_f} \int_{M}^{M-\omega} \frac{\sqrt{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}}{\rho^2 - \sqrt{\rho^2 (\rho^2 - \Delta)}} a\Omega dr dM \right]
\]

\[
= \int_{M}^{M-\omega} \frac{-2\pi (M^2 + M\sqrt{M^2 - a^2})}{\sqrt{M^2 - a^2}} dM + \int_{M}^{M-\omega} \frac{\pi a^2}{\sqrt{M^2 - a^2}} dM
\]

\[
= \pi [M^2 - (M - \omega)^2 + M\sqrt{M^2 - a^2} - (M - \omega)\sqrt{(M - \omega)^2 - a^2}]
\]

\[
= -\frac{1}{2} \Delta S_{BH}. \tag{3.8}
\]

As the same discussion as that of R-N black hole, we next rewrite (3.8) from the viewpoint of laws of black hole thermodynamics. For the Kerr black hole, the first law of black hole thermodynamics is \(^{[81, 82]}\)

\[
dM = \frac{\kappa}{8\pi} dA + \Omega dJ \ (Q = 0), \tag{3.9}
\]

And if the tunneling process is a reversible process, we can obtain

\[
dS = \frac{dM}{T} - \frac{\Omega dJ}{T}. \tag{3.10}
\]

The temperature, the angle velocity and the angular momentum of Kerr black hole are respectively \(^{[5]}\)

\[
T = \frac{\sqrt{M^2 - a^2}}{4\pi (M^2 + M\sqrt{M^2 - a^2})}, \quad \Omega = \frac{a}{r_+^2 + a^2} = \frac{a}{2(M^2 + M\sqrt{M^2 - a^2})}, \quad J = aM. \tag{3.11}
\]

Thus, substituting (3.11) into (3.10), we get

\[
dS = \frac{4\pi (M^2 + M\sqrt{M^2 - a^2})}{\sqrt{M^2 - a^2}} dM - \frac{2\pi a^2}{\sqrt{M^2 - a^2}} dM. \tag{3.12}
\]

After comparing (3.12) with (3.8), it is easy to find that we can also rewrite the imaginary part of the action as follows

\[
\text{Im} S = \int_{M}^{M-\omega} \frac{-2\pi (M^2 + M\sqrt{M^2 - a^2})}{\sqrt{M^2 - a^2}} dM + \int_{M}^{M-\omega} \frac{\pi a^2}{\sqrt{M^2 - a^2}} dM
\]

\[
= -\frac{1}{2} \int_{M}^{M-\omega} \left[ 4\pi (M^2 + M\sqrt{M^2 - a^2}) \right. \frac{\sqrt{M^2 - a^2}}{\sqrt{M^2 - a^2}} dM \left. - \frac{2\pi a^2}{\sqrt{M^2 - a^2}} dM \right]
\]

\[
= -\frac{1}{2} \int_{S_i}^{S_f} dS = -\frac{1}{2} \Delta S_{BH}. \tag{3.13}
\]

which shows again that the Hawking radiation via tunneling is also correlated with the laws of black hole thermodynamics for the stationary axial symmetry case.
IV. CONCLUSION AND DISCUSSION

In this paper, we mainly give a note and make a discussion on the Hawking radiation calculated by the quasi-local tunneling method. And we show that the original Hawking temperature will be recovered if we neglect the tunneling particles’ back-reaction effect. And if we take the effect into account, the original Hawking radiation would be modified.

During the calculation of original Hawking temperature, following the treatments which do not simply consider the factor of 2 problem just as a problem of choice of coordinates, we show again that the so-called factor of 2 problem could indeed be solved well by considering the thermal balance and taking the tunneling probability as \( \Gamma \propto P(\text{emission})/P(\text{absorption}) \). Moreover, we also clarify some subtleties in this balance method and compare this treatment with other treatments. In addition, we also find out why the familiar formalism (1.1) works well in the original tunneling method proposed by Parikh and Wilczek. Because the underlying advantage of Painleve-like coordinates can make the \( P(\text{absorption}) \) to be unit. In spite of that, here we would also give some comments on other treatments. In the constant C treatment [77, 78], it may be a little unphysical since one essentially just picks C to be whatever is needed to get the original Hawking result. Thus there is no physical content in this approach. More mathematically notice that S is a definite integral (i.e. you have limits \( r_{in} \) and \( r_{out} \)) thus there could be no integration constant. In addition, note that a direct motivation of this treatment is that because \( P(\text{absorption}) \) may be greater than unity. However, as we have discussed in Sec II, this is a subtlety in the balance method. And it is just because we neglect the unitary factor in front of the wave function. While in the treatment considering the temporal imaginary contribution from the time, although there are some works based on it [17-25, 63-65], in fact, after we consider the thermal balance, the temporal imaginary contribution from the time could be considered as a redundancy because it can be canceled automatically in \( P(\text{emission})/P(\text{absorption}) \). In other words, if we do not consider the temporal imaginary contribution from the time, we can also have the same results such as the modified temperature in some previous works [17-25]. In addition, note also that the transformation in (2.7) can not be considered as the evidence that the Schwarzschild time has the temporal contribution. And it just shows how to get the non-singular coordinates defined in the whole region from two unconnected patches which have the same coordinate system. Therefore, in the present status, it is not neces-
sary to consider the temporal contribution. And in fact the contributions from the time (or the constant) is just equal to the contribution from \( \exp(-2\text{Im}S_{in}) \) in the balance method’s treatment. Note that, another interesting result may be also obtained after we considering \( P(\text{absorption}) \) in the tunneling probability. It is that the ratio between \( P(\text{emission}) \) and \( P(\text{absorption}) \) can be just the relative scattering amplitude viewed from the Damour-Ruffini method \[68\]. Thus the balance treatment may be underlying consistent with the Damour-Ruffini method. In addition, the Damour-Ruffini method is underlying related with two different vacua \[109\]. Therefore, it would be interesting to give a further light on the balance treatment viewed from the Damour-Ruffini method and two non-equivalent vacua. In addition, note also that the thermodynamics on dynamical spacetime is still an open question until now. As we know, some methods calculating the Hawking temperature may be invalid in the dynamical spacetimes such as the Euclidean method or anomaly method \[67, 69, 76\], thus the tunneling method may be a good tool to investigate the Hawking temperature of the dynamical spacetime \[37, 38\]. And more details about the research on the thermodynamics of dynamical spacetimes could be seen in \[83, 108\].

On the other hand, during the calculation of the modified Hawking radiation, we mainly focus on researching the underlying contents. As Parikh and Wilczek’s original works have showed that this modification is underlying consistent with the unitary theory and can support the conservation of information, we further show that this modification is also underlying correlated with the laws of black hole thermodynamics by taking the general R-N black hole and Kerr black hole for examples. And this new relationship may give a new insight into the tunneling process. As a simple consequence, it may imply that the tunneling method is valid only when the tunneling process is reversible. And if the tunneling process is irreversible, the conservation of information may be violated \[79\]. However, how to consider or add the non-equilibrium effects is also an open question and deserves further investigations.

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[1] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000) [arXiv:hep-th/9907001];
[2] M. K. Parikh, Phys. Lett. B 546, 189 (2002) [arXiv:hep-th/0204107];
[3] M. K. Parikh, arXiv:hep-th/0402166;
[4] M. K. Parikh, Int. J. Mod. Phys. D 13, 2351 (2004) [Gen. Rel. Grav. 36, 2419 (2004)] [arXiv:hep-th/0405160].
[5] J. Y. Zhang and Z. Zhao, Mod. Phys. Lett. A 20, 1673 (2005).
[6] J. Y. Zhang and Z. Zhao, JHEP 0510, 055 (2005).
[7] E. C. Vagenas, Phys. Lett. B 503, 399 (2001) [arXiv:hep-th/0012134];
[8] E. C. Vagenas, Mod. Phys. Lett. A 17, 609 (2002) [arXiv:hep-th/0108147];
[9] E. C. Vagenas, Phys. Lett. B 533, 302 (2002) [arXiv:hep-th/0109108];
[10] E. C. Vagenas, Phys. Lett. B 559, 65 (2003) [arXiv:hep-th/0209185];
[11] A. J. M. Medved and E. C. Vagenas, Mod. Phys. Lett. A 20, 2449 (2005) [arXiv:gr-qc/0504113];
[12] A. J. M. Medved and E. C. Vagenas, Mod. Phys. Lett. A 20, 1723 (2005) [arXiv:gr-qc/0505015];
[13] M. Arzano, A. J. M. Medved and E. C. Vagenas, JHEP 0509, 037 (2005) [arXiv:hep-th/0505266].
[14] M. R. Setare and E. C. Vagenas, Phys. Lett. B 584, 127 (2004) [arXiv:hep-th/0309092];
[15] M. R. Setare and E. C. Vagenas, Int. J. Mod. Phys. A 20, 7219 (2005) [arXiv:hep-th/0405186];
[16] M. R. Setare, Int. J. Mod. Phys. A 23, 2047 (2008) [arXiv:0807.0273 [hep-th]].
[17] R. Banerjee and B. R. Majhi, Phys. Lett. B 662, 62 (2008) [arXiv:0801.0200 [hep-th]];
[18] R. Banerjee, B. R. Majhi and S. Samanta, Phys. Rev. D 77, 124035 (2008) [arXiv:0801.3583 [hep-th]].
[19] R. Banerjee and B. R. Majhi, JHEP 0806, 095 (2008) [arXiv:0805.2220 [hep-th]];
[20] B. R. Majhi and S. Samanta, arXiv:0901.2258 [hep-th];
[21] R. Banerjee and B. R. Majhi, arXiv:0808.3688 [hep-th];
[22] B. R. Majhi, arXiv:0809.1508 [hep-th];
[23] R. Banerjee and B. R. Majhi, arXiv:0812.0497 [hep-th];
[24] R. Banerjee, B. R. Majhi and D. Roy, arXiv:0901.0466 [hep-th];
[25] S. K. Modak, Phys. Lett. B 671, 167 (2009) [arXiv:0807.0959 [hep-th]].
[26] Q. Q. Jiang and S. Q. Wu, Phys. Lett. B 635, 151 (2006) [Erratum-ibid. 639, 684 (2006)]
[arXiv:hep-th/0511123];
[27] Q. Q. Jiang, S. Q. Wu and X. Cai, Phys. Rev. D 73, 064003 (2006) [Erratum-ibid. D 73,
069902 (2006)] [arXiv:hep-th/0512351];
[28] S. Q. Wu and Q. Q. Jiang, JHEP 0603, 079 (2006) [arXiv:hep-th/0602033];
[29] Q. Q. Jiang, S. Q. Wu and S. Z. Yang, Int. J. Mod. Phys. A 22 (2007) 777;
[30] D. Y. Chen, Q. Q. Jiang, S. Z. Yang and X. T. Zu, Class. Quant. Grav. 25, 205022 (2008)
[arXiv:0803.3248 [hep-th]];
[31] Q. Q. Jiang, Phys. Rev. D 78, 044009 (2008) [arXiv:0807.1358 [hep-th]].
[32] Y. P. Hu, J. Y. Zhang and Z. Zhao, Mod. Phys. Lett. A 21, 2143 (2006) [arXiv:gr-qc/0611026];
[33] Y. P. Hu, J. Y. Zhang and Z. Zhao, Int. J. Mod. Phys. D 16, 847 (2007) [arXiv:gr-qc/0611055];
[34] Y. P. Hu, L. Gao, and Z. Zhao, Int. J. Theor. Phys. 45, 2001 (2006).
[35] J. Y. Zhang and Z. Zhao, Phys. Lett. B 618, 14 (2005);
[36] J. Y. Zhang and Z. Zhao, Nucl. Phys. B 725, 173 (2005);
[37] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini and G. Zoccatelli, Phys. Lett. B 657,
107 (2007) [arXiv:0707.4425 [hep-th]];
[38] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini and S. Zerbini, arXiv:0806.0014
[gr-qc];
[39] R. G. Cai, L. M. Cao and Y. P. Hu, Class. Quant. Grav. 26, 155018 (2009) [arXiv:0809.1554
[hep-th]].
[40] S. Hemming and E. Keski-Vakkuri, Phys. Rev. D 64, 044006 (2001) [arXiv:gr-qc/0005115];
[41] K. Nozari and S. H. Mehdipour, Class. Quant. Grav. 25, 175015 (2008) [arXiv:0801.4074
[gr-qc]];
[42] K. Nozari and S. Hamid Mehdipour, Europhys. Lett. 84, 20008 (2008) [arXiv:0804.4221]
[gr-qc]);

[43] S. Zhou and W. Liu, Phys. Rev. D 77, 104021 (2008);
[44] T. Clifton, Class. Quant. Grav. 25, 175022 (2008) arXiv:0804.2635 [gr-qc];
[45] C. Z. Liu, J. Y. Zhang and Z. Zhao, Phys. Lett. B 639, 670 (2006);
[46] C. Z. Liu and Z. Zhao, Mod. Phys. Lett. A 23, 539 (2008);
[47] H. Z. Fang, J. Ren and Z. Zhao, Int. J. Mod. Phys. D 14, 1699 (2005);
[48] W. Liu, Phys. Lett. B 634, 541 (2006) arXiv:gr-qc/0512099;
[49] S. P. Kim, JHEP 0711, 048 (2007) [arXiv:0710.0915 [hep-th]].
[50] R. Kerner and R. B. Mann, Phys. Rev. D 73, 104010 (2006) [arXiv:hep-th/0503019].
[51] G. E. Volovik, Pisma Zh. Eksp. Teor. Fiz. 69, 662 (1999) [JETP Lett. 69, 705 (1999)]
  arXiv:gr-qc/9901077;
[52] G. E. Volovik, arXiv:cond-mat/9902171.
[53] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, JHEP 0505, 014 (2005)
  arXiv:hep-th/0503081.
[54] K. Srinivasan and T. Padmanabhan, Phys. Rev. D 60, 024007 (1999) arXiv:gr-qc/9812028;
[55] S. Shankaranarayanan, K. Srinivasan and T. Padmanabhan, Mod. Phys. Lett. A 16, 571
  (2001) arXiv:gr-qc/0007022;
[56] S. Shankaranarayanan, T. Padmanabhan and K. Srinivasan, Class. Quant. Grav. 19, 2671
  (2002) arXiv:gr-qc/0010042;
[57] S. Shankaranarayanan, Phys. Rev. D 67, 084026 (2003) arXiv:gr-qc/0301090;
[58] E. C. Vagenas, Nuovo Cim. B 117, 899 (2002) arXiv:hep-th/0111047.
[59] P. Kraus and F. Wilczek, Nucl. Phys. B 433, 403 (1995) arXiv:gr-qc/9408003;
[60] E. Keski-Vakkuri and P. Kraus, Nucl. Phys. B 491, 249 (1997) arXiv:hep-th/9610045;
[61] P. Kraus and F. Wilczek, Nucl. Phys. B 437, 231 (1995) arXiv:hep-th/9411219.
[62] B. D. Chowdhury, Pramana 70, 593 (2008) [Pramana 70, 3 (2008)] arXiv:hep-th/0605197;
[63] T. Pilling, Phys. Lett. B 660, 402 (2008) arXiv:0709.1624 [gr-qc].
[64] V. Akhmedova, T. Pilling, A. De Gill and D. Singleton, Phys. Lett. B 666, 269 (2008)
  arXiv:0804.2289 [hep-th];
[65] E. T. Akhmedov, V. Akhmedova and D. Singleton, Phys. Lett. B 642 (2006) 124
  arXiv:hep-th/0608098.
[66] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].
16

[67] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2752 (1977).

[68] T. Damour and R. Ruffini, Phys. Rev. D 14, 332 (1976).

[69] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. 95, 011303 (2005) [arXiv:gr-qc/0502074];

[70] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. Lett. 96, 151302 (2006) [arXiv:hep-th/0602146];

[71] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. D 74, 044017 (2006) [arXiv:hep-th/0606018];

[72] E. C. Vagenas and S. Das, JHEP 0610, 025 (2006) [arXiv:hep-th/0606077];

[73] S. Das, S. P. Robinson and E. C. Vagenas, Int. J. Mod. Phys. D 17, 533 (2008) [arXiv:0705.2233 [hep-th]]; [74] R. Banerjee and S. Kulkarni, arXiv:0810.5683 [hep-th];

[75] R. Banerjee and S. Kulkarni, Phys. Lett. B 659, 827 (2008) [arXiv:0709.3916 [hep-th]]; [76] R. Banerjee and S. Kulkarni, Phys. Rev. D 77, 024018 (2008) [arXiv:0707.2449 [hep-th]].

[77] P. Mitra, Phys. Lett. B 648, 240 (2007) [arXiv:hep-th/0611265];

[78] S. Stotyn, K. Schleich and D. Witt, arXiv:0809.5093 [gr-qc].

[79] J. Y. Zhang, Y. P. Hu and Z. Zhao, Mod. Phys. Lett. A 21, 1865 (2006) [arXiv:hep-th/0512121];

[80] B. Zhang, Q. Y. Cai and M. S. Zhan, Phys. Lett. B 671, 310 (2009) [arXiv:0901.0591 [hep-th]].

[81] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973);

[82] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).

[83] S. A. Hayward, Phys. Rev. D 49, 6467 (1994);

[84] S. A. Hayward, Phys. Rev. D 53, 1938 (1996) [arXiv:gr-qc/9408002];

[85] S. A. Hayward, Class. Quant. Grav. 15, 3147 (1998) [arXiv:gr-qc/9710089];

[86] S. A. Hayward, S. Mukohyama and M. C. Ashworth, Phys. Lett. A 256, 347 (1999) [arXiv:gr-qc/9810006];

[87] R. G. Cai and L. M. Cao, Phys. Rev. D 75, 064008 (2007) [arXiv:gr-qc/0611071];

[88] M. Akbar, arXiv:0808.3308 [gr-qc];

[89] M. Akbar, arXiv:0808.0169 [gr-qc];

[90] R. G. Cai, L. M. Cao and Y. P. Hu, JHEP 0808, 090 (2008) [arXiv:0807.1232 [hep-th]]; [91] S. F. Wu, G. H. Yang and P. M. Zhang, arXiv:0805.4041 [hep-th];

[92] T. Zhu, J. R. Ren and S. F. Mo, arXiv:0805.1162 [gr-qc];

[93] S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, arXiv:0801.2688 [hep-th];
[94] R. G. Cai, Prog. Theor. Phys. Suppl. 172, 100 (2008) [arXiv:0712.2142 [hep-th]];

[95] S. F. Wu, B. Wang and G. H. Yang, Nucl. Phys. B 799, 330 (2008) [arXiv:0711.1209 [hep-th]];

[96] S. F. Wu, G. H. Yang and P. M. Zhang, [arXiv:0710.5394 [hep-th]];

[97] J. Zhou, B. Wang, Y. Gong and E. Abdalla, Phys. Lett. B 652, 86 (2007) [arXiv:0705.1264 [gr-qc]];

[98] J. R. Ren and R. Li, [arXiv:0705.4339 [gr-qc]];

[99] Y. Gong and A. Wang, Phys. Rev. Lett. 99, 211301 (2007) [arXiv:0704.0793 [hep-th]];

[100] A. Sheykhi, B. Wang and R. G. Cai, Phys. Rev. D 76, 023515 (2007) [arXiv:hep-th/0701261];

[101] A. Sheykhi, B. Wang and R. G. Cai, Nucl. Phys. B 779, 1 (2007) [arXiv:hep-th/0701198];

[102] T. Padmanabhan and A. Paranjape, Phys. Rev. D 75, 064004 (2007) [arXiv:gr-qc/0701003];

[103] R. G. Cai and L. M. Cao, Nucl. Phys. B 785, 135 (2007) [arXiv:hep-th/0612144];

[104] M. Akbar and R. G. Cai, Phys. Lett. B 648, 243 (2007) [arXiv:gr-qc/0612089];

[105] M. Akbar and R. G. Cai, Phys. Rev. D 75, 084003 (2007) [arXiv:hep-th/0609128];

[106] M. Akbar and R. G. Cai, Phys. Lett. B 635, 7 (2006) [arXiv:hep-th/0602156];

[107] R. G. Cai and S. P. Kim, JHEP 0502, 050 (2005) [arXiv:hep-th/0501055];

[108] R. G. Cai, L. M. Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D 78, 124012 (2008) [arXiv:0810.2610 [hep-th]].

[109] S. Sannan, Gen. Rel. Grav. 20, 239 (1988).