Schrödinger uncertainty relation with Wigner-Yanase skew information

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Abstract. We shall give a new Schrödinger type uncertainty relation for a quantity representing a quantum uncertainty, introduced by S.Luo in [8]. Our result improves the Heisenberg uncertainty relation shown in [8] for a mixed state.

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1 Introduction

In quantum mechanical system, the expectation value of an observable (self-adjoint operator) $H$ in a quantum state (density operator) $\rho$ is expressed by $\text{Tr}[\rho H]$. Also, the variance for a quantum state $\rho$ and an observable $H$ is defined by $V_\rho(H) \equiv \text{Tr}[\rho (H - \text{Tr}[\rho H]I)^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$. It is famous that we have the Heisenberg uncertainty relation [1]:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2 \quad (1)$$

for a quantum state $\rho$ and two observables $A$ and $B$. The further strong result was given by Schrödinger [2]:

$$V_\rho(A)V_\rho(B) - |\text{Re}\{\text{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2, \quad (2)$$

where the covariance is defined by $\text{Cov}_\rho(A, B) \equiv \text{Tr}[\rho (A - \text{Tr}[\rho A]I) (B - \text{Tr}[\rho B]I)]$.

On the other hand, as a degree for non-commutativity between a quantum state $\rho$ and an observable $H$, the Wigner-Yanase skew information $I_\rho(H)$ was defined in [3] (See Definition 2.1 in Section 2). It is famous that the convexity of the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr} [(i[\rho^\alpha, H]) (i[\rho^{1-\alpha}, H])]$, $\alpha \in [0, 1]$, which is a one-parameter extension of the Wigner-Yanase skew information $I_\rho(H)$, with respect to $\rho$ was successfully proven by E.H.Lieb in [4]. We have the relation between $I_\rho(H)$ and $V_\rho(H)$ such that $0 \leq I_\rho(H) \leq V_\rho(H)$ so that it is quite natural to consider that we have the further sharpened uncertainty relation for the Wigner-Yanase skew information:

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho [A, B]]|^2.$$

However, the above relation failed. (See [5] [6] [7].) Then, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (3)$$

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then he succeeded to show a new Heisenberg uncertainty relation on $U_\rho(H)$ in [8]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2. \quad (4)$$

As stated in [8], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state $\rho$, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [9, 10]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state, by the use of the quantity $U_\rho(H)$.

Recently, K.Yanagi gave a one-parameter extension of the inequality (4) in [11], using the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$. Note that we have the following ordering among three quantities:

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (5)$$

The inequality (5) is a refinement of the original Heisenberg’s uncertainty relation (1) in the sense of the above ordering (5).

In this brief report, we show the further strong inequality (Schrödinger type uncertainty relation) for the quantity $U_\rho(H)$ representing a quantum uncertainty.

## 2 Main results

To show our main theorem, we prepare the definition for a few quantities and a lemma representing properties on their quantities.

**Definition 2.1** For a quantum state $\rho$ and an observable $H$, we define the following quantities.

(i) The Wigner-Yanase skew information:

$$I_\rho(H) \equiv \frac{1}{2} \text{Tr} \left[ \left(i\rho^{1/2}, H_0\right)^2 \right] = \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^{1/2}H_0\rho^{1/2}H_0],$$

where $H_0 \equiv H - \text{Tr}[\rho H]I$ and $[X, Y] \equiv XY - YX$ is a commutator.

(ii) The quantity associated to the Wigner-Yanase skew information:

$$J_\rho(H) \equiv \frac{1}{2} \text{Tr} \left[ \left(\rho^{1/2}, H_0\right) \right]^2 = \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^{1/2}H_0\rho^{1/2}H_0] - 2(\text{Tr}[\rho H])^2 = 2V_\rho(H) - I_\rho(H), \quad (6)$$

which implies $I_\rho(H) \leq J_\rho(H)$. In addition, we have the following relations.
\textbf{Lemma 2.2} \hspace{5mm} (i) For a quantum state $\rho$ and an observable $H$, we have the following relation among $I_\rho(H)$, $J_\rho(H)$ and $U_\rho(H)$:

\[ U_\rho(H) = \sqrt{I_\rho(H)J_\rho(H)}. \]

(ii) For a spectral decomposition of $\rho = \sum_{j=1}^\infty \lambda_j |\phi_j\rangle \langle \phi_j|$, putting $h_{ij} \equiv \langle \phi_i|H_0|\phi_j\rangle$, we have

\[ I_\rho(H) = \sum_{i<j} \left( \sqrt{\lambda_i} - \sqrt{\lambda_j} \right)^2 |h_{ij}|^2, \]

(iii) For a spectral decomposition of $\rho = \sum_{j=1}^\infty \lambda_j |\phi_j\rangle \langle \phi_j|$, putting $h_{ij} \equiv \langle \phi_i|H_0|\phi_j\rangle$, we have

\[ J_\rho(H) \geq \sum_{i<j} \left( \sqrt{\lambda_i} + \sqrt{\lambda_j} \right)^2 |h_{ij}|^2. \]

(i) immediately follows from Eq.(6). See [11] for the proofs of (ii) and (iii).

\textbf{Theorem 2.3} For a quantum state (density operator) $\rho$ and two observables (self-adjoint operators) $A$ and $B$, we have

\[ U_\rho(A)U_\rho(B) - |\text{Re} \left\{ \text{Corr}_\rho(A, B) \right\}|^2 \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2, \tag{7} \]

where the correlation measure is defined by

\[ \text{Corr}_\rho(X, Y) \equiv \text{Tr}[\rho X^* Y] - \text{Tr}[\rho^{1/2} X^* \rho^{1/2} Y] \]

for any operators $X$ and $Y$.

\textbf{Proof:} We take a spectral decomposition $\rho = \sum_{j=1}^\infty \lambda_j |\phi_j\rangle \langle \phi_j|$. If we put $a_{ij} = \langle \phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle \phi_j|B_0|\phi_i\rangle$, where $A_0 = A - \text{Tr}[\rho A]I$ and $B_0 = B - \text{Tr}[\rho B]I$, then we have

\[ \text{Corr}_\rho(A, B) = \text{Tr}[\rho AB] - \text{Tr}[\rho^{1/2} A \rho^{1/2} B] \]
\[ = \text{Tr}[\rho A_0 B_0] - \text{Tr}[\rho^{1/2} A_0 \rho^{1/2} B_0] \]
\[ = \sum_{i,j=1}^\infty (\lambda_i - \lambda_i^{1/2} \lambda_j^{1/2})a_{ij}b_{ji} \]
\[ = \sum_{i \neq j} (\lambda_i - \lambda_i^{1/2} \lambda_j^{1/2})a_{ij}b_{ji} \]
\[ = \sum_{i<j} \left\{ (\lambda_i - \lambda_i^{1/2} \lambda_j^{1/2})a_{ij}b_{ji} + (\lambda_j - \lambda_j^{1/2} \lambda_i^{1/2})a_{ji}b_{ij} \right\}. \]

Thus we have

\[ |\text{Corr}_\rho(A, B)| \leq \sum_{i<j} \left\{ |\lambda_i - \lambda_i^{1/2} \lambda_j^{1/2}| |a_{ij}| |b_{ji}| + |\lambda_j - \lambda_j^{1/2} \lambda_i^{1/2}| |a_{ji}| |b_{ij}| \right\}. \]

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, taking a square of both sides and then using Schwarz inequality and Lemma 2.2, we have

\[ |\text{Corr}_\rho(A, B)|^2 \leq \left\{ \sum_{i<j} \left\{ |\lambda_i - \lambda_i^{1/2} \lambda_j^{1/2}| + |\lambda_j - \lambda_j^{1/2} \lambda_i^{1/2}| \right\} |a_{ij}| |b_{ji}| \right\}^2. \]
By the similar way, we also have
\[ |\text{Corr}_{\rho}(A, B)|^2 \leq I_{\rho}(B)J_{\rho}(A) \]
Thus we have
\[ |\text{Corr}_{\rho}(A, B)|^2 \leq U_{\rho}(A)U_{\rho}(B), \]
which is equivalent to the inequality:
\[ U_{\rho}(A)U_{\rho}(B) - |\text{Re} \{ \text{Cov}_{\rho}(A, B) \}|^2 \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2, \]
since we have
\[ |\text{Im} \{ \text{Corr}_{\rho}(A, B) \}|^2 = \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2. \]

Theorem 2.3 improves the uncertainty relation (4) shown in [8], in the sense that the upper bound of the right hand side of our inequality (7) is tighter than that of S.Luo’s one (4).

Remark 2.4 For a pure state \( \rho = |\varphi\rangle\langle\varphi| \), we have \( I_{\rho}(H) = V_{\rho}(H) \) which implies \( U_{\rho}(H) = V_{\rho}(H) \) for an observable \( H \) and \( \text{Corr}_{\rho}(A, B) = \text{Cov}_{\rho}(A, B) \) for two observables \( A \) and \( B \). Therefore our Theorem 2.3 coincides with the Schrödinger uncertainty relation (2) for a particular case that a given quantum state is a pure state, \( \rho = |\varphi\rangle\langle\varphi| \).

Remark 2.5 As a similar problem, we may consider the following uncertainty relation:
\[ U_{\rho}(A)U_{\rho}(B) - |\text{Re} \{ \text{Cov}_{\rho}(A, B) \}|^2 \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2. \]
However, the above inequality does not hold in general, since we have a counter-example as follows. We take
\[ \rho = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
then we have
\[ U_{\rho}(A)U_{\rho}(B) - |\text{Re} \{ \text{Cov}_{\rho}(A, B) \}|^2 - \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 = -\frac{3}{4}. \]

Remark 2.6 From Theorem 2.3 and Remark 2.5, we may expect that the following inequality holds:
\[ |\text{Re} \{ \text{Cov}_{\rho}(A, B) \}|^2 \geq |\text{Re} \{ \text{Corr}_{\rho}(A, B) \}|^2. \]
However, the above inequality does not hold in general, since we have a counter-example as follows. We take
\[ \rho = \frac{1}{10} \begin{pmatrix} 5 & 4 & 4 \\ 4 & 5 & 4 \\ 4 & 4 & 1 \end{pmatrix}, A = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}. \]
then we have
\[ |Re \{ Cov_\rho(A, B) \}|^2 - |Re \{ Corr_\rho(A, B) \}|^2 \simeq -0.1539. \]

Actually, from Theorem 2.3, the example in Remark 2.5 and the above example, we find that there is no ordering between \( |Re \{ Cov_\rho(A, B) \}|^2 \) and \( |Re \{ Corr_\rho(A, B) \}|^2 \).

**Remark 2.7** The example given in Remark 2.5 shows
\[ V_\rho(A)V_\rho(B) - |Re \{ Cov_\rho(A, B) \}|^2 - (U_\rho(A)U_\rho(B) - |Re \{ Corr_\rho(A, B) \}|^2) \simeq -0.232051. \]
The example given in Remark 2.6 also shows
\[ V_\rho(A)V_\rho(B) - |Re \{ Cov_\rho(A, B) \}|^2 - (U_\rho(A)U_\rho(B) - |Re \{ Corr_\rho(A, B) \}|^2) \simeq 13.7862. \]
Therefore there is no ordering between \( V_\rho(A)V_\rho(B) - |Re \{ Cov_\rho(A, B) \}|^2 \) and \( U_\rho(A)U_\rho(B) - |Re \{ Corr_\rho(A, B) \}|^2 \) so that we can conclude that neither the inequality (2) nor the inequality (7) is uniformly better than the other.

3 **Conclusion**

As we have seen, we proved a new Schrödinger type uncertainty relation for a quantum state (generally a mixed state). Our result coincides with the original Schrödinger uncertainty relation for a particular case that a quantum state is a pure state. In addition, our result improves the uncertainty relation shown in [3] and obviously does the original Heisenberg uncertainty relation. Moreover, it is impossible to conclude that our result is better than the original Schrödinger uncertainty relation for a mixed state, in the sense of finding a tighter upper bound for \( \frac{1}{2}|Tr[\rho[A, B]]|^2 \), where \( Tr[\rho[A, B]] \) can be regarded as an average of the commutator \([A, B]\) for two observables \(A\) and \(B\) in a quantum state \(\rho\). However, in other words, it is also impossible to conclude that our result is a trivial one, since there is no ordering between the left hand side of the inequality (2) and that of one (7).

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