THE ∂-COMPLEX ON WEIGHTED BERGMAN SPACES ON HERMITIAN MANIFOLDS

FRIEDRICH HASLINGER AND DUONG NGOC SON

Abstract. In this paper, we generalize several results about the ∂-complex on the Segal-Bargmann space of $\mathbb{C}^n$ to weighted Bergman spaces on Hermitian manifolds. We also study in detail the ∂-complex on the unit ball with the complex hyperbolic metric and a non-Kähler metric. The former case turns out to have duality properties similar to the Segal-Bargmann space while the latter exhibits a different behavior. We apply these results to solve the ∂-equation on the Bergman spaces in the unit ball of $\mathbb{C}^n$ with the “exponential” and “standard” weights.

1. Introduction

In a recent paper [9], the author studies the ∂-complex on the Segal-Bargmann spaces of $(p,0)$-forms

$$A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) := \left\{ u = \sum_{|J|=p} u_J dz^J : \int_{\mathbb{C}^n} |u|^2 e^{-|z|^2} d\lambda < \infty, \text{ } u_J \text{ are holomorphic} \right\}.$$ 

It is well-known that the forms with polynomial coefficients are dense in the Segal-Bargmann space and hence ∂ is a densely defined operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. Furthermore, it is proved in [9] that the associated complex Laplacian $\tilde{\Box}$ is an unbounded self-adjoint operator acting on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ which has a bounded and compact inverse $\tilde{N}_p$. This exposes a difference between the ∂-complex and the well-known $\bar{\partial}$-complex on the weighted $L^2$ space with the same weight function.

The inspiration for [9] comes from quantum mechanics, where the annihilation operator $a_j$ can be represented by the differentiation with respect to $z_j$ on $A^2(\mathbb{C}^n, e^{-|z|^2})$ and its adjoint, the creation operator $a_j^*$, by the multiplication by $z_j$, both operators being unbounded densely defined (see [4]). One can show that $A^2(\mathbb{C}^n, e^{-|z|^2})$ with this action of the $a_j$ and $a_j^*$ is an irreducible representation $M$ of the Heisenberg group, by the Stone-von Neumann theorem it is the only one up to unitary equivalence. Physically $M$ can be thought of as the Hilbert space of a harmonic oscillator with $n$ degrees of freedom and Hamiltonian operator

$$H = \sum_{j=1}^n \frac{1}{2}(P^2_j + Q^2_j) = \sum_{j=1}^n \frac{1}{2}(a_j^* a_j + a_j a_j^*).$$

The main purpose of this paper is to generalize several results of [9] to the ∂-complex on weighted Bergman spaces on Hermitian manifolds (or the generalized Segal-Bargmann space), i.e., the restriction of the ∂-complex of weighted $L^2$ spaces to the subspaces of forms with holomorphic coefficients. We give several conditions under which the complex...
Laplacian $\tilde{\Box}$ on generalized Segal-Bargmann spaces is a densely defined self-adjoint operator. Under these conditions, we study the coercivity of $\tilde{\Box}$ and its inverse, and the Neumann operators $\tilde{N}$. We also study two models on the unit ball $B := \{ z \in \mathbb{C}^n : |z|^2 < 1 \}$ with the complex hyperbolic metric and a (conformally Kähler) non-Kähler (when $n \geq 2$) metric. These models have close relations with the so-called Bergman spaces with “exponential” and “standard” weights.

2. The $\partial$-operators on weighted Bergman spaces

In this section, we study some general properties of the $\partial$-complex on the Bergman spaces on Hermitian manifolds. For the reader’s convenience, we recall here some basic facts and fix some notations. For general references regarding Hermitian manifolds and the $\partial$-complex, we refer to [11, 1] and [8], respectively.

Let $(M, h)$ be a Hermitian manifold. In holomorphic coordinates $z^1, \ldots, z^n$, the metric $h$ has the form

$$h_{j\bar{k}} dz^j \otimes dz^\bar{k},$$

(2.1)

where $[h_{j\bar{k}}]$ is a positive definite Hermitian matrix with smooth coefficients. This metric induces a volume element which we denoted by $d\text{vol}_h$. If $\psi$ is a weight function on $M$, then the Hilbert space of $L^2$ integrable functions with respect to the measure $d\mu := e^{-\psi} d\text{vol}_h$ is defined by

$$L^2(M, e^{-\psi} d\text{vol}_h) = \left\{ f : M \to \mathbb{C} \text{ measurable} : \int_M |f|^2 e^{-\psi} d\text{vol}_h < +\infty \right\}. \quad (2.2)$$

The weighted Bergman space with weight $\psi$ is defined to be

$$A^2(M, e^{-\psi} d\text{vol}_h) = L^2(M, e^{-\psi} d\text{vol}_h) \cap O(M). \quad (2.3)$$

Here, $O(M)$ denotes the space of holomorphic functions on $M$. Under a suitable condition on $\psi$, the Bergman space $A^2(M, e^{-\psi} d\text{vol}_h)$ is a closed subspace of $L^2(M, e^{-\psi} d\text{vol}_h)$ and thus it is a Hilbert space (although it can be trivial, finite, or infinite dimensional.)

The Hermitian metric $h$ induces a metric on tensors of every degree. For example, if in local coordinates $u = u_j dz^j$ and $v = v_j dz^j$ are $(1, 0)$-forms, then

$$\langle u, v \rangle_h = h^{j\bar{k}} u_j v_{\bar{k}}, \quad \|u\|^2_h = \langle u, u \rangle_h \quad (2.4)$$

where $[h^{j\bar{k}}]$ is the transpose of the inverse matrix of $[h_{j\bar{k}}]$. We define the weighted spaces of $(p, 0)$-forms

$$L^2_{(p,0)}(M, h, e^{-\psi}) = \left\{ u \text{ is a } (p, 0)\text{-form} : \int_M |u|^2_h e^{-\psi} d\text{vol}_h < \infty \right\}, \quad 0 \leq p \leq n, \quad (2.5)$$

with inner product

$$\langle u, v \rangle_{h, \psi} = \int_M \langle u, v \rangle_h e^{-\psi} d\text{vol}_h. \quad (2.6)$$

We say that a $(p, 0)$-form $u$ is holomorphic if in local holomorphic coordinates, we can write

$$u = \sum_{|J|=p} \langle u_J dz^J \rangle \quad (2.7)$$
with holomorphic coefficients \( u_J \) and with summation over increasing multiindices. Observe that this notion does not depend on the chosen coordinates (cf. \[9\]) and hence is well-defined on complex manifolds. We define the Bergman space of \((p,0)\)-forms to be

\[
A^2_{(p,0)}(M, h, e^{-\psi}) = \left\{ u \text{ is a holomorphic } (p,0)\text{-form} : \int_M |u|^2_h e^{-\psi} dv_{\lambda} < \infty \right\}. \tag{2.8}
\]

For smooth forms, the \(\bar{\partial}\)-operator is defined in local coordinates by

\[
\frac{\partial u}{\partial z^j} := \sum_{|J|=p} \sum_{j=1}^n \frac{\partial u_J}{\partial z^j} dz^j . \tag{2.9}
\]

Thus, if \( u \) is holomorphic, then so is \( \partial u \).

For a \((p,0)\)-form \( u \) in \( A^2_{(p,0)}(M, h, e^{-\psi}) \), it is not necessary that the \((p+1,0)\)-form \( \partial u \) is in \( A^2_{(p+1,0)}(M, h, e^{-\psi}) \). Therefore, we introduce the subspace

\[
\text{dom}(\partial_p) = \left\{ u \in A^2_{(p,0)}(M, h, e^{-\psi}) : \partial u \in A^2_{(p+1,0)}(M, h, e^{-\psi}) \right\}. \tag{2.10}
\]

Clearly, \(\text{dom}(\partial_p)\) also depends on both the metric \( h \) and the weight function \( \psi \).

The interesting situation is when \(\text{dom}(\partial_p)\) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \), for each \( p \). In this case, \(\bar{\partial}\) is a densely defined (bounded or unbounded) operator:

\[
\partial_p : A^2_{(p,0)}(M, h, e^{-\psi}) \to A^2_{(p+1,0)}(M, h, e^{-\psi}),
\]

and the powerful theory of unbounded operators applies.

Although for general Hermitian manifolds, it is difficult to determine when \(\text{dom}(\partial_p)\) restricted to the weighted Bergman space is dense, this is the case in many interesting situations.

**Example 2.1.** Let \( M = \mathbb{C}^n \) and suppose that \( h \) is the standard Euclidean metric and \( \psi : \mathbb{C}^n \to \mathbb{R} \) is convex as a function of \( 2n \) real variables (e.g., \( \psi(z) = |z|^2 \) satisfies this convexity assumption). By a result of B.A. Taylor \[15\], the polynomials are dense in \( A^2(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \), provided that \( A^2(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \) contains the polynomials. More generally, \((p,0)\)-forms with polynomial coefficients are dense in \( A^2_{(p,0)}(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \). In this case, since the \(\bar{\partial}\)-operator sends \((p,0)\)-forms with polynomial coefficients to \((p+1,0)\)-forms with polynomial coefficients, \(\bar{\partial}\) is densely defined on \( A^2_{(p,0)}(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \). The case \( \psi(z) = |z|^2 \) corresponds to the Segal–Bargmann space and has been treated thoroughly in \[9\]. Similarly, if for some \( \psi \) all the exponentials are dense in \( A^2(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \), then \(\text{dom}(\bar{\partial})\) is also dense in \( A^2(\mathbb{C}^n, e^{-\psi} dv_{\lambda}) \).

In the next two propositions, we establish the relation between the \(\bar{\partial}\)-operators on the weighted Bergman spaces and on the weighted \(L^2\) spaces. We denote by \( D_p \) the maximal extension (in the sense of distributions) of the \(\bar{\partial}\)-operator acting on \( L^2_{(p,0)}(M, h, e^{-\psi} dv_{\lambda}) \).

**Proposition 2.2.** Let \((M, h, \psi)\) be as above. Then for each \( p \geq 0 \), it holds that

\[
\text{dom}(\partial_p) = \text{dom}(D_p) \cap A^2_{(p,0)}(M, h, e^{-\psi}). \tag{2.11}
\]

**Proof.** If \( u \in \text{dom}(\partial) \subset A^2_{(p,0)}(M, h, e^{-\psi}) \), then \( u \) has holomorphic coefficients and \( \partial u \in A^2_{(p+1,0)}(M, h, e^{-\psi}) \subset L^2_{(p+1,0)}(M, h, e^{-\psi}) \). Hence \( u \in \text{dom}(D) \cap A^2_{(p,0)}(M, h, e^{-\psi}) \), as desired. Conversely, if \( u \) belongs to the right hand side of \( (2.11) \), then \( u \) has holomorphic coefficients and \( |\partial u|_h \) is \(L^2\)-integrable with respect to \( d\mu \). This clearly implies \( u \in \text{dom}(\partial) \). The proof is complete. □
Suppose that \( \text{dom}(\partial) \) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \). Then the Hilbert space adjoint of \( \partial \) in \( A^2_{(p,0)}(M, h, e^{-\psi}) \) is well-defined and denoted by \( \partial^* \) (see \[8\] Definition 4.1). Clearly,\
\[
\text{dom}(\partial^*) = \left\{ g \in A^2_{(p+1,0)}(M, h, e^{-\psi}) : f \mapsto (\partial f, g)_{h,\psi} \text{ is continuous on } \text{dom}(\partial) \right\}.
\]
Since \( (M, h) \) is complete, \( D \) is densely defined in \( L^2_{(p,0)}(M, h, e^{-\psi}) \) and hence the Hilbert space adjoint \( D^* \) of \( D \) is well-defined. Assume that \( g \in \text{dom}(D^*) \), then \( f \mapsto (\partial f, g)_{h,\psi} \) is continuous on \( \text{dom}(D) \) and hence on \( \text{dom}(\partial) \) since \( \text{dom}(\partial) \subset \text{dom}(D) \). Thus, we obtain

**Proposition 2.3.** Suppose that \( \text{dom}(\partial) \) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \). Then
\[
\text{dom}(D^*) \cap A^2_{(p+1,0)}(M, h, e^{-\psi}) \subset \text{dom}(\partial^*). \tag{2.12}
\]

It is natural to ask when is the left hand side of \( (2.12) \) dense in the right hand side? Suppose that \( v \) is a \( (p + 1, 0) \)-form in \( \text{dom}(\partial^*) \); in particular, \( v \) has holomorphic coefficients. Then \( v \in \text{dom}(D^*) \) if and only if \( u \mapsto (\partial u, v)_{h,\psi} = (P_{h,\psi}(\partial u), v)_{h,\psi} \) is continuous on \( \text{dom}(D) \), where \( P_{h,\psi} \) is the Bergman orthogonal projection
\[
P_{h,\psi}: L^2_{(p,0)}(M, h, e^{-\psi}) \longrightarrow A^2_{(p,0)}(M, h, e^{-\psi}), \tag{2.13}
\]
which is well-defined under the admissibility condition in the sense of \[14\] (the map is \textit{a priori} continuous on the subspace \( \text{dom}(D) \cap A^2(M, h, e^{-\psi})) \).

**Proposition 2.4.** Suppose that \( \text{dom}(\partial) \) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \). Then the operators \( \partial \) and \( \partial^* \) are closed operators.

**Proof.** By the general theory for closed unbounded operators (see, e.g., \[8\] Lemma 4.5), we only need to prove the statement for \( \partial \). Suppose that \( \{u_j\} \) is a sequence in \( A^2_{(p,0)}(M, h, e^{-\psi}) \) that converges to \( u \) in \( L^2 \)-topology and suppose that \( \partial u_j \) converges to \( v \) also in \( L^2 \)-topology. Since \( d\mu \) is a positive measure with smooth positive density with respect to the Lebesgue measure in any local coordinate patch \( U \), it is “admissible” in the sense of \[14\]. Thus, the coefficients of \( u_j \) converge to those of \( u \) uniformly on compact sets of \( U \) and hence \( \partial u = v \). Since \( v \in A^2_{(p+1,0)}(M, h, e^{-\psi}) \) by assumption, we obtain that \( u \in \text{dom}(\partial) \). The proof is complete. \( \square \)

Thus, if \( \text{dom}(\partial) \) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \), then \( \text{dom}(\partial^*) \) is dense in \( A^2_{(p+1,0)}(M, h, e^{-\psi}) \), see, e.g., \[8\] Lemma 4.6.

**Proposition 2.5.** Suppose that \( \text{dom}(\partial) \) is dense in \( A^2_{(p,0)}(M, h, e^{-\psi}) \) and \( v \in \text{dom}(\partial^*) \). If \( w \in L^2_{(p,0)}(M, h, e^{-\psi}) \) such that
\[
(\partial u, v)_{h,\psi} = (u, w)_{h,\psi}, \quad \forall u \in \text{dom}(\partial), \tag{2.14}
\]
then
\[
\partial^* v = P_{h,\psi}(w). \tag{2.15}
\]
In particular, if \( v \in \text{dom}(D^*) \cap A^2_{(p+1,0)}(M, h, e^{-\psi}) \) then
\[
\partial^* v = P_{h,\psi}(D^* v). \tag{2.16}
\]

**Proof.** Since \( v \in \text{dom}(\partial^*) \), then for all \( u \in \text{dom}(\partial) \), one has \( (u, \partial^* v)_{h,\psi} = (\partial u, v)_{h,\psi} \). Thus
\[
(u, w)_{h,\psi} = (u, \partial^* v)_{h,\psi}, \quad \forall u \in \text{dom}(\partial). \tag{2.17}
\]
Thus, by the density of \( \text{dom}(\partial) \) in \( A^2_{(p,0)}(M, h, e^{-\psi}) \) and the definition of the Bergman projection, (2.15) follows. If, in addition, \( v \in \text{dom}(D^*) \), then (2.15) holds with \( w = D^*v \) and hence (2.16) follows.

We point out that (2.16) is an extension of equation (3.3) in [9] in which (2.14) was verified using the Green-Gauß theorem. We shall improve this result in Proposition 2.6.

If \((M, h)\) is a Hermitian manifold, then there exists a canonical linear connection on \( M \), the Chern connection of \( h \), which parallelizes both the metric \( h \) and the complex structure of the underlying manifold (see, e.g., [11, 6]). In local coordinates \( z^1, \ldots, z^n \), the nonvanishing Christoffel symbols for the Chern connection are

\[
\Gamma^i_{jk} = h^i_{\bar{l}} \partial_j h_{k \bar{l}}, \quad \overline{\Gamma^i_{jk}} = \overline{\Gamma^i_{kj}}. \tag{2.18}
\]

Since \( \partial_j (h^i_{\bar{l}} h_{k \bar{l}}) = \partial_j (\delta_{ik}) = 0 \), we also have \( \Gamma^i_{jk} = -h_{k \bar{l}} \partial_j h^i_{\bar{l}} \). The covariant derivatives can be explicitly expressed in local coordinates. For examples, if in local coordinate \( u = u_k dz^k \) is a \((1,0)-\)form, then

\[
\nabla_j u_k = \partial_j u_k - \Gamma^l_{jk} u_l, \quad \nabla_{\bar{j}} u_k = \partial_{\bar{j}} u_k. \tag{2.19}
\]

Note that in the second equation, the Christoffel symbols of “mixed type” vanish and hence the covariant derivative of \((1,0)-\)forms along \((0,1)-\)direction reduces essentially to the partial derivatives of its components.

For a general Hermitian metric, the torsion tensor may be nontrivial; we define the torsion \( T^i_{jk} \) by

\[
T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad \overline{T^i_{jk}} = \overline{T^i_{jk}}. \tag{2.20}
\]

The torsion \((1,0)-\)form is then obtained by taking the trace:

\[
\tau = T^i_{j i} dz^j. \tag{2.21}
\]

In local coordinates, the volume element is given by \( d\text{vol}_h = \det(h_{j \bar{l}}) d\lambda \), where \( d\lambda \) is the Lebesgue measure in that coordinate patch. Thus, if \( \psi \) is an weight function, then we can write (locally) \( d\mu = e^{-\psi} d\text{vol} = e^{-\varphi} d\lambda \), with

\[
\varphi = \psi - \log \det(h_{j \bar{l}}). \tag{2.22}
\]

Therefore, since \( \Gamma^i_{k i} = \partial_k \log \det(h_{j \bar{l}}) \) (see, e.g., [11, p. 111], but mind that the Kählerian condition is not assumed here), we obtain

\[
\varphi^k = \psi^k - \varphi^k \log \det(h_{j \bar{l}}) = \psi^k - \Gamma^i_{k i}. \tag{2.23}
\]

Suppose that \( u = u_j dz^j \) is a smooth \((1,0)-\)form and \( v \) is compactly supported function. We assume that \( v \) has support contained in a coordinate patch \((U, z)\). Then

\[
(u, \partial v)_{h, \psi} = \int_U u_j \overline{\partial_k h^j_k} e^{-\varphi} d\lambda
= -\int_U \partial_k \left(u_j h^j_k e^{-\varphi}\right) \varphi d\lambda
= -\int_M \left( \partial_k u_j h^j_k + u_j \partial_k h^j_k - u_j \psi^k h^j_k\right) \varphi d\mu. \tag{2.24}
\]
where \( v_k := \frac{\partial u}{\partial z^k} \). Observe that
\[
\partial_k h^{jk} = -h^{jk} \Gamma^i_{lk} = h^{jk} \left( \tau^i_k - \Gamma^i_{kl} \right)
\]
\[
= h^{jk} (\tau^i_k - \partial_k \log \det(h_k^i))
\]
\[
= h^{jk} (\tau^i_k + \varphi^i_k - \psi_k). \tag{2.25}
\]
Plugging this into (2.24), we obtain the integration-by-part formula:
\[
(u, \partial v)_{h,\psi} = -\int_M h^{jk} [\partial_k u_j + u_j (\tau^i_k - \psi_k)] \, d\mu. \tag{2.26}
\]
For the case of general compactly supported \( v \), we can use the partition of unity to reduce to the case above; we omit the details.

Thus, if additionally \((M, h)\) is a complete manifold (so that the Andreotti–Vesentini density lemma applies) and \( u \in \text{dom}(D^*) \), then we have a local expression for \( D^* u \) as follows (see, e.g., [6, (6.20)] or [2]):
\[
D^* u = -\nabla_k u^k + (\psi_k - \tau_k) u^k, \quad u^k := h^{jk} u_j. \tag{2.27}
\]
We shall derive a similar formula for the adjoint \( \partial^* \) on \( \text{dom}(\partial^*) \). It turns out that, similar to the Segal–Bargmann case [9], the adjoint \( \partial^* \) is closely related to the Bergman projection [2,13], which is nonlocal.

**Proposition 2.6.** Let \((M, h)\) be a complete Hermitian manifold and \( e^{-\psi} \) a smooth weight on \( M \). Suppose that \( \text{dom}(\partial) \) is dense in \( A^2_{(1,0)}(M, h, e^{-\psi}) \) and let \( u = u_j dz^j \in A^2_{(1,0)}(M, h, e^{-\psi}) \).

Let \( \partial^* \) be the Hilbert space adjoint of \( \partial \). If \( \langle u, \partial \psi - \tau \rangle_h \in L^2(M, h, e^{-\psi}) \), then \( u \) belongs to \( \text{dom}(D^*) \) and hence \( u \in \text{dom}(\partial^*) \). Moreover,
\[
\partial^* u = P_{h,\psi}(\langle u, \partial \psi - \tau \rangle_h). \tag{2.28}
\]
where \( \tau = \tau_j dz^j \) is the torsion \((1,0)\)-form.

**Remark 1.** This proposition implies that if \( |\partial \psi - \tau|_h \) is bounded, then \( \partial^* \) is a bounded operator. Example [14] exhibits a situation with finite dimensional generalized Bergman spaces so that \( \partial \) and \( \partial^* \) are bounded operators and \( |\partial \psi - \tau|_h \) is bounded. For the Segal-Bargmann model (cf. [9]), \( |\partial \psi - \tau|_h = |z_j dz^j| \) is unbounded and so are \( \partial \) and \( \partial^* \). See also Section 5 where we exhibit two unbounded examples.

Compared to the local expression (2.27), the formula for \( \partial^* \) in (2.28) is global as it involves the Bergman projection.

**Proof of Proposition 2.6.** We shall use the usual cut-off procedure on complete Riemannian manifolds (see, e.g., [13, pp. 48]). For a fixed point \( p_0 \in M \), the distance function \( d(\cdot, p_0) \) is Lipschitz on \( M \). Let \( \rho(x) \) be a smoothing of \( d(x, x_0) \) and choose a function \( \chi : \mathbb{R} \to \mathbb{R} \) such that \( \chi|_{(-\infty, 1]} = 1 \) and \( \supp \chi \subset (-\infty, 2] \). Put
\[
\chi_R(x) = \chi(\rho(x)/R), \quad R > 1. \tag{2.29}
\]
Then \( \chi_R \) has compact support and \( |\partial \chi_R| \leq c/R \) for some \( c > 0 \). Suppose that \( u = u_j dz^j \in A^2_{(1,0)}(M, h, e^{-\psi}) \) and \( v \in \text{dom}(D) \). In local coordinates, \( \partial v = v_k dz^k \) where \( v_k = \partial v / \partial z^k \).
Using integration by parts (2.26)

\[ (\chi_R u, \partial v)_{h,\psi} = -\int_M h^{jk} [\partial_k (\chi_R u_j) + \chi_R u_j (\tau_k - \psi_k)] \bar{v} \, d\mu \]

\[ = -\int_M h^{jk} u_j (\partial_k \chi_R) \bar{v} \, d\mu + \int_M h^{jk} \chi_R u_j (\tau_k - \psi_k) \bar{v} \, d\mu \]  \hspace{1cm} (2.30)

Here we use \( \partial_k u_j = 0 \) since \( u \) is holomorphic.

Observe that the first integral tends to 0 as \( R \to \infty \). Indeed,

\[ \left| \int_M h^{jk} u_j (\partial_k \chi_R) \bar{v} \, d\mu \right| \leq \int_M |v| |u|_h |\partial \chi_R| \, d\mu \to 0 \quad \text{as} \quad R \to \infty \]  \hspace{1cm} (2.31)

since \( |\partial \chi_R| < c/R \). Letting \( R \to \infty \), we obtain that

\[ (u, \partial v)_{h,\psi} = \lim_{R \to \infty} \int_M \chi_R \bar{v} (u, \partial \psi - \tau) \, d\mu. \]  \hspace{1cm} (2.32)

Thus, if \( \langle u, \partial \psi - \tau \rangle_h \in L^2(M, h, e^{-\psi}) \), then the limit on the right hand side is

\[ \lim_{R \to \infty} \int_M \chi_R \bar{v} (u, \partial \psi - \tau) \, d\mu = \int_M \bar{v} (u, \partial \psi - \tau) \, d\mu \]

\[ = \int_M \bar{v} P_{h,\psi} (\langle u, \partial \psi - \tau \rangle_h) \, d\mu, \]  \hspace{1cm} (2.33)

since \( v \) is holomorphic. Therefore, \( u \in \text{dom}(\partial^*) \) and

\[ \partial^* u = P_{h,\psi} \left( \langle u, \partial \psi - \tau \rangle_h \right). \]  \hspace{1cm} (2.34)

The proof is complete. \( \square \)

In view of this result, we call \( \delta u := P_{h,\psi} \left( \langle u, \partial \psi - \tau \rangle_h \right) \) the formal adjoint of \( \partial \), whenever the right hand side is defined.

**Example 2.7** (cf. [9]). Let \( M = \mathbb{C}^n \) with the standard Euclidean metric and \( \psi(z) = |z|^2 \).

Let \( u_j dz^j \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \). If \( \sum_{j=1}^n z^j u_j \in L^2(\mathbb{C}^n, e^{-|z|^2}) \) then \( u \in \text{dom}(\partial^*) \) and \( \partial^* u = \sum_{j=1}^n z^j u_j \).

Next, we give a condition under which \( \partial^* \) agrees with \( D^* \) for \( u \in \text{dom}(D^*) \) having holomorphic coefficients. To this end, the following notion is crucial for us:

**Definition 2.8.** Suppose that \( \xi = \xi^j \partial_j \) is a \((1,0)\)-vector field expressed in a local coordinate patch \( U \). We say that \( \xi \) is holomorphic if each coefficient \( \xi^j \) is holomorphic in \( U \).

The notion of a holomorphic \((1,0)\)-vector field does not depend on the choice of coordinate. For a \((0,1)\) form \( w = w_k \, dz^k \), the “musical operator” \( \sharp \) acts on \( w \) and produces an \((1,0)\) vector field \( w^\sharp := h^{kj} w_j \partial_k \). If \( u \) and \( v \) are \((1,0)\) forms, then \( \langle u, v \rangle_h = \langle u, \bar{v}^\sharp \rangle \) where the right hand side is the dual pairing between vectors and covectors. Thus, if \( u \in A^2_{(1,0)}(M, h, e^{-\psi}) \) and \( \bar{\psi}^\sharp \) is a holomorphic vector field, then \( \langle u, v \rangle_h \) is a holomorphic function. Thus, we obtain the following

**Corollary 2.9.** Suppose that \( \partial \) is densely defined in the weighted Bergman space and \((\partial \psi - \tau)^\sharp \) is a holomorphic vector field, then for each \( u \in \text{dom}(D^*) \cap A^2_{(1,0)}(M, h, e^{-\psi}) \), one has

\[ \partial^* u = D^* u. \]  \hspace{1cm} (2.35)
illustrate the calculations, we start with $p$-Kähler manifolds to preserve the type of forms (i.e., to send $(\cdot)$, see, e.g., [12]). It is necessary and sufficient for the weighted Dirichlet forms $Z_f$ corresponding to the same weight to be of the form $\mathcal{Z}$. Necessary and sufficient for the weighted Hodge Laplacian with weight $\mathcal{Z}$ problems in the literature, especially for the Kähler case. For examples, the condition is as in Corollary 2.9 also plays an important role in several (similar, but not directly related)

Moreover, since $v_{pq} = -v_{qp}$, we find that

$$\langle \partial u, v \rangle_h = \sum_{j,k,p,q} v_{pj} h^{jk} \partial_{\bar{z}^q} \frac{\partial u_k}{\partial z^j}.$$  \hfill (2.40)

Assuming $(M, h)$ is complete, we use an usual cut-off function technique as in the proof of Proposition 2.6 and apply the integration by parts without boundary terms. The calculations can be done in a local coordinate patch as follows:

$$\langle \partial u, v \rangle_{h, \psi} = \int_M \sum_{j,k,p,q} \bar{v}_{pj} \left( \frac{\partial u_k}{\partial z^j} \right) \left( h^{jk} h^{\bar{q}\bar{q}} \right) e^{-\varphi} d\lambda = -\int_M \sum_{j,k,p,q} \bar{v}_{pj} u_k \partial_j \left( e^{-\varphi} h^{jk} h^{\bar{q}\bar{q}} \right) d\lambda.$$  \hfill (2.41)

Expanding the right hand side in local coordinates using (2.29) and (2.30), we obtain

$$\partial^* v = P_{h, \psi} \left( -\langle \psi \rangle_j \bar{v}_{pj} h^{\bar{q}\bar{q}} d\bar{z}^p + \frac{1}{2} T_{jk}^\tau h^{jk} h^{\bar{q}\bar{q}} v_{pq} d\bar{z}^p \right).$$  \hfill (2.42)

Here, $P_{h, \psi}$ is the orthogonal projection from $L^2_{(2,0)}(M, h, e^{-\varphi})$ onto $A^2_{(2,0)}(M, h, e^{-\varphi})$.

Proof. If $u \in \text{dom}(D^*) \cap A^2_{(1,0)}(M, h, e^{-\varphi})$, then clearly

$$D^* u = \langle u, \partial \psi - \tau \rangle_h \in L^2(M, h, e^{-\varphi})$$  \hfill (2.36)

and thus $u \in \text{dom}(\partial^*)$.

On the other hand, $(\bar{\partial} \psi - \bar{\tau})^j$ is the $(1,0)$-vector field expressed in local coordinates by

$$(\bar{\partial} \psi - \bar{\tau})^j = h^{jk} \left( \psi_k - \tau_k \right) \frac{\partial}{\partial z^j}. \hfill (2.37)$$

The holomorphicity to $(\bar{\partial} \psi - \bar{\tau})^j$ means that for each $j$, $h^{jk} \left( \psi_k - \tau_k \right)$ is holomorphic. Thus, by (2.28), $\partial^* u = P_{h, \psi}(D^* u) = D^* u$. The proof is complete. \hfill \Box

Remark 2. It is worth pointing out that the condition on the holomorphicity of $(\bar{\partial} \psi - \bar{\tau})^j$ as in Corollary 2.9 also plays an important role in several (similar, but not directly related) problems in the literature, especially for the Kähler case. For examples, the condition is necessary and sufficient for the weighted Hodge Laplacian with weight $d\mu := e^{-\varphi} d\nu \text{vol}$ on Kähler manifolds to preserve the type of forms (i.e., to send $(p, q)$-forms into $(p, q)$-forms, see, e.g., [12]). It is also necessary and sufficient for the weighted Dirichlet forms $d^* \psi$ corresponding to the same weight to be of the form $Z_f$ for some holomorphic $(1, 0)$-vector field $Z$ (see, e.g., [7] for more details).

In the rest of this section, we describe the formula for $\partial^*$ on $(p, 0)$-forms with $p \geq 2$. To illustrate the calculations, we start with $p = 2$. For $v \in A^2_{(2,0)}(\mathcal{Z}, h, d\mu)$ and write

$$v = \frac{1}{2} \sum_{j,k} v_{jk} \bar{z}^j \wedge d\bar{z}^k = \sum_{j<k} v_{jk} \bar{z}^j \wedge d\bar{z}^k,$$  \hfill (2.38)

where $v_{jk} = -v_{kj}$ are holomorphic. If $u = u_j \bar{z}^j$, we have

$$\partial u = \frac{1}{2} \sum_{j,k} \left( \frac{\partial u_k}{\partial \bar{z}^j} - \frac{\partial u_j}{\partial z^k} \right) \bar{z}^j \wedge d\bar{z}^k.$$  \hfill (2.39)

Moreover, since $v_{pq} = -v_{qp}$, we find that

$$\langle \partial u, v \rangle_h = \sum_{j,k,p,q} \bar{v}_{pq} h^{jk} \bar{h}^{\bar{q}\bar{q}} \frac{\partial u_k}{\partial z^j}.$$  \hfill (2.40)

Assuming $(M, h)$ is complete, we use an usual cut-off function technique as in the proof of Proposition 2.6 and apply the integration by parts without boundary terms. The calculations can be done in a local coordinate patch as follows:

$$\langle \partial u, v \rangle_{h, \psi} = \int_M \sum_{j,k,p,q} \bar{v}_{pj} \left( \frac{\partial u_k}{\partial z^j} \right) \left( h^{jk} h^{\bar{q}\bar{q}} \right) e^{-\varphi} d\lambda = -\int_M \sum_{j,k,p,q} \bar{v}_{pj} u_k \partial_j \left( e^{-\varphi} h^{jk} h^{\bar{q}\bar{q}} \right) d\lambda.$$  \hfill (2.41)

Expanding the right hand side in local coordinates using (2.29) and (2.30), we obtain

$$\partial^* v = P_{h, \psi} \left( -\langle \psi \rangle_j \bar{v}_{pj} h^{\bar{q}\bar{q}} d\bar{z}^p + \frac{1}{2} T_{jk}^\tau h^{jk} h^{\bar{q}\bar{q}} v_{pq} d\bar{z}^p \right).$$  \hfill (2.42)

Here, $P_{h, \psi}$ is the orthogonal projection from $L^2_{(2,0)}(M, h, e^{-\varphi})$ onto $A^2_{(2,0)}(M, h, e^{-\varphi})$. 
Observe that the participation of the torsion is rather involved in the case $p = 2$. For general $p \geq 2$, we follow [4], which is somewhat implicit. Precisely, if $\eta \in A^2_{(p,0)}(M, h, e^{-\psi})$ is written as $\eta = \frac{1}{p!} \sum_{|\alpha|=p} \eta_\alpha dz^{\alpha}$, we define, for $p \geq 1$,

$$T: A^2_{(p,0)}(M, h, e^{-\psi}) \to A^2_{(p+1,0)}(M, h, e^{-\psi})$$

by [6] (6.9)]

$$T(\eta) = \frac{1}{(p-1)!} \sum T_{j_1 k_1}^{i_1} \eta_{i_1 \cdots i_p} dz^{j_1} \wedge dz^{k_1} \wedge \cdots \wedge dz^{j_p}$$

(2.43)

and, for $p \geq 2$, the “adjoint” $T^\dagger: A^2_{(p,0)}(M, h, e^{-\psi}) \to A^2_{(p-1,0)}(M, h, e^{-\psi})$ by

$$\langle T^\dagger \eta, \xi \rangle_h = \langle \eta, T \xi \rangle_h.$$ 

(2.44)

If $\eta \in \text{dom}(D^*) \cap A^2_{(p,0)}(M, h, e^{-\psi})$, then by [6], and the fact that $\eta$ is holomorphic,

$$D^* \eta = \frac{(-1)^{p-1}}{(p-1)!} h^{j_k} \eta_{i_1 \cdots i_p-1} dz^{i_1} \wedge \cdots \wedge dz^{i_p-1} - T^2(\eta).$$ 

(2.45)

More generally, if $\eta \in A^2_{(p,0)}$ such that the right hand side of (2.44) belongs to $L^2$, then $\eta$ belongs to $\text{dom}(D^*)$ and hence in $\text{dom}(\partial^* \partial)$. Moreover,

$$\partial^* \eta = P_{h,\psi} \left( \frac{(-1)^{p-1}}{(p-1)!} h^{j_k} \eta_{i_1 \cdots i_p-1} dz^{i_1} \wedge \cdots \wedge dz^{i_p-1} - T^2(\eta) \right).$$ 

(2.46)

This generalizes the formula in Proposition 2.6 (cf. [9] Eq. (3.3)]. The proof uses a calculation as in [6] and a density lemma by Andreotti–Vesentini [1] Lemma 4. We omit the details.

**Corollary 2.10.** Let $(M, h)$ be a complete Kähler manifold and $e^{-\psi}$ is a weight on $M$. Assume that $\partial_p$ is densely defined in the Bergman space $A^2_{(p,0)}(M, h, e^{-\psi})$. If $(\partial^* \partial)^2$ is holomorphic, then

$$D^* \eta = \partial^* \partial \eta$$ 

(2.47)

for all $\eta \in \text{dom}(D^*) \cap A^2_{(p+1,0)}(M, h, e^{-\psi})$.

One can state a version of this corollary for general Hermitian non-Kähler manifold. However, when $p > 1$, the hypothesis is more technical due to the presence of the term $T^2$ in (2.46) above.

### 3. The complex Laplacian and the basic estimate

In this section, we study the complex Laplacian associated to the $\partial$-operator restricted to the Bergman spaces. Let $(M, h)$ be a Hermitian manifold and $e^{-\psi}$ is a weight on $M$. Suppose that $\text{dom}(\partial)$ is dense in $A^2_{(p,0)}(M, h, e^{-\psi})$ and $A^2_{(p-1,0)}(M, h, e^{-\psi})$. Then the Laplacian

$$\square_p = \partial \partial^* + \partial^* \partial$$

(3.1)

is well-defined (for $p = 0$, we define $\square_0 = \partial^* \partial$). This operator was studied earlier in [9] for the case $M = \mathbb{C}^n$, $h$ is the Euclidean metric, and $\psi = |z|^2$. Under the density assumption, $\square$ acts as an (bounded or unbounded) self-adjoint operator on the Bergman space $A^2_{(p,0)}(M, h, e^{-\psi})$. We point out that since the compactly supported forms can not have holomorphic coefficients, they are not useful for several problems considered here such as the density of $\text{dom}(\partial)$. In particular, it is a nontrivial question whether $\partial$ is densely defined on the weighted Bergman spaces. Fortunately, in several interesting cases when $M$
is an open subset of $\mathbb{C}^n$, we can use $(p,0)$-forms with polynomial coefficients as a substitute to prove that $\bar{\partial}$ is densely defined.

**Definition 3.1** (Basic estimate). Let $(M, h, \psi)$ be a Hermitian manifold with smooth weight $e^{-\psi}$ such that the $\bar{\partial}$-operator is densely defined in $A^2_{(p,0)}(M, h, e^{-\psi})$. We say that the $\bar{\partial}$-complex satisfy the basic estimate on holomorphic $(p,0)$-forms if for each $u \in \text{dom}(\bar{\partial}_p) \cap \text{dom}(\bar{\partial}_h)$, we have

$$\|\partial u\|_{h,\psi}^2 + \|\partial^* u\|_{h,\psi}^2 \geq c\|u\|_{h,\psi}^2$$

for some constant $c > 0$.

Similarly to the $L^2$-theory for the $\bar{\partial}$-complex, the basic estimate (3.2) implies various useful properties for the complex Laplacian $\Box$ (cf. Chapter 8 of [8]).

In the following, we describe a simple situation in which the basic estimate for $\bar{\partial}$-complex holds. For this purpose, we first let $\Theta$ be the Chern-Ricci form of $(M, h)$ and therefore, (3.4) follows immediately from the well-known identity for the $\bar{\partial}$-complex.

**Corollary 3.3.** Let $(M, h)$ be a complete Hermitian manifold. Suppose that $\text{dom}(\bar{\partial})$ is dense in $A^2_{(1,0)}(M, h, e^{-\psi})$. If $u$ is a $(1,0)$-form in $\text{dom}(\bar{\partial}^*)$ such that $(u, \bar{\partial}^* u - \tau)_h \in L^2(M, h, e^{-\psi})$ and $\partial u - Tu = L^2_{(1,0)}(M, h, e^{-\psi})$, then

$$\|\partial u - Tu\|_h^2 + \|\partial^* u\|_h^2 = 2\|\nabla u\|_h^2 + (i\partial \bar{\partial} \psi + \Theta, u \wedge \bar{u})_{h,\psi} - ||(I - P_{h,\psi})(\langle u, \partial \psi - \tau \rangle_h)||^2.$$  

**(3.4)**

**Proof.** By Proposition 2.6, $u \in \text{dom}(\partial^*)$ and $D^* u = \langle u, \partial \psi - \tau \rangle_h$. Thus,

$$\partial^* u = P_{h,\psi}(D^* u) = P_{h,\psi}(\langle u, \partial \psi - \tau \rangle_h).$$

(3.5)

Consequently,

$$\|\partial u - Tu\|_h^2 + \|\partial^* u\|_h^2 = \|\partial u - Tu\|_h^2 + \|D^* u\|_h^2 - ||(I - P_{h,\psi}(D^* u))||,$$

and therefore, (3.4) follows immediately from the well-known identity for the $\bar{\partial}$-complex (see, e.g., [2] or [6]).

**(3.6)**

If $(M, h)$ is not a complete manifold but a relatively compact domain in a complex manifold with smooth boundary, we can still formulate a similar basic identity with boundary term. We shall not use such an identity and hence omit the details.

Next, we define the torsion $(1,1)$ form as follows (cf. [2])

$$T \circ \overline{T} := h^{ab} h^{mk} h_{ij}^{p} T_{ab}^{pq} \overline{T}_{pq}^{lm} dz^j \wedge dz^k.$$  

(3.7)

Observe that $(i T \circ \overline{T}, \overline{u} \wedge u)_h = |Tu|^2$.

**Corollary 3.3.** Let $(M, h)$ be a complete Hermitian manifold and $e^{-\psi}$ a smooth weight on $M$. Suppose that the following conditions hold.

(i) $\text{dom}(\Box)$ is dense in $A^2_{(1,0)}(M, h, e^{-\psi})$,

(ii) $\partial \psi - \tau \in L^2_{(1,0)}(M, h, e^{-\psi})$ and $(\partial \psi - \tau)^2$ is holomorphic,
(iii) on $M$,  
\[ i\partial\bar{\partial}\psi + \Theta - i\left(\frac{\sigma}{\sigma - 1}\right) T \circ \bar{T} \geq \sigma b^2 \omega_h, \quad b > 0, \quad \sigma > 1. \]  
(3.8)

If $Tu \in L^2_{1,0}(M, h, e^{-\psi})$, then  
\[ \|\partial u\|^2 + \|\partial^* u\|^2 \geq b^2 \|u\|^2. \]  
(3.9)

Proof. If $(\partial\psi - \tau)^{\sharp}$ is holomorphic, then $\langle u, \partial\psi - \tau \rangle$ is holomorphic. Thus,  
\[ (I - P_{h,\psi})(\langle u, \partial\psi - \tau \rangle) = 0. \]  
(3.10)

The basic identity reduces to  
\[ \|\partial u - Tu\|^2 + \|\partial^* u\|^2 = 2\|\nabla u\|^2 + \langle i\partial\bar{\partial}\psi + \Theta, u \land \bar{u} \rangle_{h,\psi}. \]  
(3.11)

From this, we can argue similarly to [2] to obtain (3.9). We omit the details. □

We conclude this section by pointing out that although the basic identity is very useful to establish the basic estimate for the Laplacian associated to a $\bar{\partial}$-complex, the condition (3.8) in Corollary 3.3 seems to be rather strong (compared to the situation for $L^2$-complex) for the $\bar{\partial}$-complex in Bergman spaces since we only require (3.2) to hold for holomorphic $(p,0)$-forms. In Section 5, we shall meet two situations in which (3.8) either fails or holds with a non-optimal constant.

4. The $\partial$-Neumann operator on weighted Bergman spaces

In this section, we assume that for our $\partial$-complex, the basic estimate (3.2) holds. Corollary 3.3 in the last section provides a concrete condition for this to be true, however, there are several situations in which the basic estimate can be proved directly (see Section 5). In these situations, it is natural to study the bounded inverse $\tilde{N} := \bar{\partial}^{-1}$. More precisely,

**Proposition 4.1.** Suppose that $\text{dom}(\partial_p)$ is dense in $A_{(p,0)}(M, h, e^{-\psi})$ for $p = 0, 1$ and suppose that the basic estimate (3.2) holds. Then $\partial$ and $\partial^*$ have closed ranges. If we endow $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ with the graph norm  
\[ f \mapsto (\|\partial f\|^2 + \|\partial^* f\|^2)^{\frac{1}{2}} \]  
(4.1)

then the subspace $\text{dom}(\partial) \cap \text{dom}(\partial^*)$ becomes a Hilbert space.

Proof. As usual $\ker \partial = (\text{im } \partial^*)^\perp$. Therefore,  
\[ (\ker \partial)^\perp = \text{im } \partial^* \subseteq \ker \partial^*. \]  
(4.2)

If $u \in \ker \partial \cap \ker \partial^*$, we have by (3.2) that $u = 0$. Hence  
\[ (\ker \partial)^\perp = \ker \partial^*. \]  
(4.3)

If $u \in \text{dom}(\partial) \cap (\ker \partial)^\perp$, then $u \in \ker \partial^*$, and (3.2) also implies  
\[ \|u\| \leq \frac{1}{b} \|\partial u\|. \]  
(4.4)

To conclude the proof, we can use general results of unbounded operators on Hilbert spaces (see for instance [8, Chapter 4]) to show that $\text{im } \partial$ and $\text{im } \partial^*$ are closed. The last assertion follows again by (3.2). □
Theorem 4.2. Suppose that $\text{dom}(\partial_p)$ is dense in $A_{(p,0)}(M,h,e^{-\psi})$ for $p = 0,1$ and suppose that the basic estimate (3.2) holds. Then the operator $\tilde{N}: \text{dom}(\tilde{\Box}) \rightarrow A_{(1,0)}^{2}(M,h,e^{-\psi})$ is bijective and has a bounded inverse

\[ \tilde{N}: A_{(1,0)}^{2}(M,h,e^{-\psi}) \rightarrow \text{dom}(\tilde{\Box}). \]  

In addition

\[ \|\tilde{N}u\| \leq \frac{1}{b}\|u\|, \]  

for each $u \in A_{(1,0)}^{2}(M,h,e^{-\psi})$.

Theorem 4.3. Suppose that $\text{dom}(\partial_p)$ is dense in $A_{(p,0)}^{2}(M,h,e^{-\psi})$ for $p = 0,1$ and suppose that the basic estimate (3.2) holds. Let $\eta \in A_{(1,0)}^{2}(M,h,e^{-\psi})$ with $\partial \eta = 0$. Then $u_{0} := \partial^* \tilde{N}_{1}\eta$ is the canonical solution of $\partial u = \eta$, this means $\partial u_{0} = \eta$ and $u_{0} \in (\ker \partial)^{\perp}$. Moreover,

\[ \left\|\partial^* \tilde{N}_{\eta}\right\| \leq b^{-1/2}\|\eta\|. \]  

Example 4.4. Consider the complex plane $\mathbb{C}$ and a radial weight function $\psi(|z|^2)$, where $\psi(t)$ is a real-valued function of one real variable. Suppose that $\psi(t) > 0$ and put $h = \psi(|z|^2)dz \otimes d\bar{z}$. Then $d\mu = e^{-\psi(|z|^2)}d\text{vol}_{h} = e^{-\psi(|z|^2)}\psi(|z|^2)d\lambda$. Clearly, $(\partial \psi)^{2} = z\partial_{z}$ is a holomorphic vector field. The special case $\psi(t) = t$ leads to the case of Segal–Bargmann space and was studied thoroughly in [9].

For a constant $\alpha \geq 2$, we put $\psi(z) = \alpha \log(1 + |z|^2)$ and consider the complete metric

\[ h = (1 + |z|^2)^{-1}dz \otimes d\bar{z}. \]  

(This metric is often referred to as Hamilton’s cigar soliton in the literature). Put

\[ e^{-\psi}d\text{vol}_{h} = (1 + |z|^2)^{-(\alpha+1)}d\lambda, \]  

where $d\lambda$ is the standard Lebesgue measure. The Bergman space $A^{2}(\mathbb{C},e^{-\psi}d\text{vol}_{h})$ is the finite dimensional space of polynomials of degree $\leq \alpha - 1$.

If $u(z) \in A^{2}(\mathbb{C},e^{-\psi}d\text{vol}_{h})$, i.e., $u(z)$ is a polynomial of degree $k \leq \alpha - 1$, then

\[ |\partial u|^2_h = |u'(z)dz|^2_h = |u'(z)|^2(1 + |z|^2). \]  

Since $u'(z)$ is of degree $k - 1$, we can easily see that then $\partial u \in A_{(1,0)}^{2}(\mathbb{C},h,e^{-\psi})$. That is, $\partial$ maps $A^{2}(\mathbb{C},h,e^{-\psi})$ into $A_{(1,0)}^{2}(\mathbb{C},h,e^{-\psi})$. This is an operator between finite dimensional Hilbert spaces and hence bounded. Notice that $|\partial \psi|_{h} = \alpha |z|/\sqrt{1 + |z|^2}$ is also bounded (cf. Proposition 2.6).

Let $D$ be the $\partial$-operator on the weighted $L^2$ spaces, then

\[ D^*(udz) = -h^{-1}\partial_zu + h^{-1}(\partial_z\psi)u = -(1 + |z|^2)\partial_zu + \alpha zu. \]  

Therefore, the weighted Bergman space adjoint is a “multiplication” operator:

\[ \partial^*(udz) = \alpha zu, \]  

where $u$ is a holomorphic polynomial of degree $\leq \alpha - 2$.

The formula for $\tilde{\Box}$ is rather simple. Indeed, for a polynomial $f \in A^{2}(\mathbb{C},h,e^{-\psi})$, one has

\[ \tilde{\Box}_{\alpha}f = \alpha f'(z). \]  

Clearly, $\ker \tilde{\Box}_{\alpha}$ is the constants. Moreover, $\alpha, 2\alpha, \ldots, ([\alpha] - 1)\alpha$ are the eigenvalues with corresponding eigenvectors $z, z^2, \ldots, z^{[\alpha]} - 1$, respectively.
For a holomorphic $(1,0)$-form $udz$, one has
\[ \tilde{\Box}_1(udz) = \alpha(u + zu')dz. \] (4.14)
Thus, $\ker \tilde{\Box}_1 = \{0\}$. Moreover, the eigenvalues are $\alpha k$ for $k = 1, \ldots, |\alpha| - 1$. The basic identity gives $\tilde{\Box}_1 \geq (\alpha + 1)/2$, as quadratic forms, but in fact we have a stronger inequality $\tilde{\Box}_1 \geq \alpha$. Observe that $\tilde{\Box}_1$, expected to be of second order, is actually of first order when restricted to holomorphic forms.

5. Weighted Bergman spaces on the unit ball of $\mathbb{C}^n$

In this section, we shall compute the spectra of $\tilde{\Box}$ for two (Kählerian and non-Kählerian) models on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$. As in [11], we shall use the following result.

Lemma 5.1 (see Lemma 1.2.2 of [3]). Let $A$ be a symmetric operator on a Hilbert space $H$ with domain $\text{dom}(A)$ and suppose that $\{x_k\}_k$ is a complete orthonormal system in $H$. If each $x_k$ lies in $\text{dom}(A)$ and there exists $\lambda_k \in \mathbb{R}$ such that $Ax_k = \lambda_k x_k$ for every $k \in \mathbb{N}$, then $A$ is essentially self-adjoint and the spectrum of $\lambda$ is the closure in $\mathbb{R}$ of the set of all $\lambda_k$.

5.1. The complex hyperbolic metric. We study in detail the $\partial$-complex on the weighted Bergman space on the complex hyperbolic space with an appropriate weight. The weight $\psi$ is chosen such that the vector field $(\partial \psi - \tau)^t$ is holomorphic (in fact, $\tau = 0$ in this case).

Precisely, consider the unit ball $\mathbb{B} \subset \mathbb{C}^n$ endowed with the Bergman–Kähler metric:
\[ h_{jk} = -\partial_j \partial_k \log(1 - |z|^2) = (1 - |z|^2)^{-1} \delta_{jk} + (1 - |z|^2)^{-2} z^j \bar{z}^k. \] (5.1)
Here, $|z|^2 := \sum_{j=1}^n |z_j|^2$. Let the weight function be
\[ \psi(z) = \frac{\alpha}{1 - |z|^2}, \quad \alpha > 0, \] (5.2)
so that
\[ d\mu = e^{-\psi} d\text{vol}_k = (1 - |z|^2)^{-n-1} \exp \left( -\frac{\alpha}{1 - |z|^2} \right) d\lambda. \] (5.3)
It turns out that this Bergman space with the so-called “exponential weight” has duality properties similar to the Segal-Bargmann space, so it can be seen as a version of the Segal-Bargmann space on a bounded domain. We will show that the adjoint of the densely defined unbounded operator $\partial$ is the operator multiplication by $\alpha z$. But in this case we have to take care of the Hermitian metric on $\mathbb{B}$ and of the fact that $\partial$ maps $A_{(p,0)}^2(\mathbb{B}, h, d\mu)$ into $A_{(p+1,0)}^2(\mathbb{B}, h, d\mu)$ and $\partial^*$ maps $A_{(p+1,0)}^2(\mathbb{B}, h, d\mu)$ into $A_{(p,0)}^2(\mathbb{B}, h, d\mu)$.

Since the weight is radial, the polynomials are dense in Bergman space $A_{(p,0)}^2(\mathbb{B}, h, \psi)$ (see [10] or the proof of Proposition 2.6 in [16] for the case $p = 0$ with “standard” weight; the case of general radial weights and $p \geq 1$ follows easily). Moreover, the monomials $z^J$’s (each $J$ is a multi-index) are orthogonal in $A^2(\mathbb{B}, d\mu)$. For each $k$, put
\[ a_k = \int_0^1 (1 - t)^{-n-1} \exp \left( -\frac{\alpha}{1 - t} \right) t^{n+k-1} dt = \frac{1}{\alpha^n} \int_0^\infty s^{n+k-1} e^{-s-\alpha} ds. \] (5.4)
Then by the density of the polynomials, an orthonormal basis for $A^2(\mathbb{B}, d\mu)$ can be taken as
\[ e_J := \frac{(|J| + n - 1)!}{\sqrt{n!|J|!}} z^J, \] (5.5)
where $J$ is a multi-index.
Observe that
\[ h^{j \bar{k}} = (1 - |z|^2) (\delta_{jk} - z^j \bar{z}^k). \] (5.6)

Therefore, if \( u = u_j dz^j \), then
\[ |u|^2_h = u_j u_k h^{j \bar{k}} = (1 - |z|^2) \left( \sum_{j=1}^{n} |u_j|^2 - \sum_{j,k} z^j \bar{z}^k u_j u_k \right). \] (5.7)

Hence, the holomorphic \((1,0)\)-forms with polynomial coefficients are in \( L^2_{(1,0)}(\mathbb{B}, h, e^{-\psi}) \).

We also compute,
\[ \psi^k = \frac{\alpha z^k}{(1 - |z|^2)^2}, \] (5.8)
and find that (sum over \( k \))
\[ h^{j \bar{k}} \psi^k = \alpha z^j \] (5.9)
are holomorphic. Consequently, for \( u = u_j dz^j \)
\[ \langle u, \partial \psi \rangle_h = \alpha z^j u_j. \] (5.10)

Thus, if \( u_j \)'s are holomorphic polynomials, then \( \langle u, \partial \psi \rangle_h \) is a holomorphic polynomial and hence \( u \in \text{dom}(\partial^*) \). Moreover, by Proposition 2.6
\[ \partial^* (u_j dz^j) = P_h,\psi (u, \partial \psi)_h = \alpha z^j u_j. \] (5.11)

Since the restrictions of polynomials are dense in \( L^2(\mathbb{B}, h, e^{-\psi}) \), formula (5.11) for \( \partial^* \) holds for every \( u \in \text{dom}(\partial^*) \).

Using Taylor series expansion (in sake of simplicity we take \( n = 1 \)) we can directly verify that for \( f \in \text{dom}(\partial) \) and \( g dz \in \text{dom}(\partial^*) \). We have
\[ \langle \partial f, g dz \rangle_h,\psi = \langle f' dz, g dz \rangle_h,\psi = \langle f, \alpha z g \rangle_h,\psi. \] (5.12)

Each \( f \in A^2(\mathbb{B}, h, d\mu) \) can be represented in the form \( f = \sum_{k=0}^{\infty} f_k e_k \), where \( e_k = \frac{z^k}{k!} \) and \( (f_k)_k \in l^2 \) and each \( F \in A^2_{(1,0)}(\mathbb{B}, h, d\mu) \) can be represented in the form \( F = \sum_{k=0}^{\infty} F_k E_k \), where \( E_k = \frac{z^k}{k!} \) and \( (F_k)_k \in l^2 \). We write \( f = \sum_{k=0}^{\infty} f_k e_k \), where \( (f_k)_k \in l^2 \), and \( g = \sum_{k=0}^{\infty} g_k e_k \), where \( (g_k)_k \in l^2 \) and have
\[ c_k^2 = 2\pi \int_0^1 r^{2k+1}(1 - r^2)^{-2} \exp\left(\frac{-\alpha}{1 - r^2}\right) dr = \frac{\pi}{\alpha} \int_0^{\infty} \left(\frac{s}{s + \alpha}\right)^k e^{-s - \alpha} ds, \] (5.13)
and
\[ d_k^2 = 2\pi \int_0^1 r^{2k+1} \exp\left(\frac{-\alpha}{1 - r^2}\right) dr = \pi\alpha \int_0^{\infty} \frac{s^k}{(s + \alpha)^{k+2}} e^{-s - \alpha} ds. \] (5.14)

Now we obtain
\[ \langle f' dz, g dz \rangle_h,\psi = \int_{\mathbb{B}} f'(z) g(z) \exp\left(\frac{-\alpha}{1 - |z|^2}\right) d\lambda \]
\[ = \sum_{k=0}^{\infty} f_{k+1} (k + 1) \frac{1}{c_k c_{k+1}} \overline{g}_k d_k^2. \] (5.15)

On the other hand,
\[ \langle f, \alpha z g \rangle_h,\psi = \alpha \sum_{k=0}^{\infty} f_{k+1} \frac{c_{k+1}}{c_k} \overline{g}_k. \] (5.16)
In order to prove (5.12) we have to show that
\[ \alpha c_{k+1}^2 = (k+1)d_k^2. \] (5.17)

Observe that
\[ \frac{d}{ds} \left[ \left( \frac{s}{s + \alpha} \right)^{k+1} \right] = (k+1) \left( \frac{s}{s + \alpha} \right)^k \frac{\alpha}{(s + \alpha)^2}. \] (5.18)

Using partial integration we get
\[ \alpha c_{k+1}^2 = \pi \int_0^\infty \left( \frac{s}{s + \alpha} \right)^{k+1} e^{-s - \alpha} ds \]
\[ = \pi \alpha (k + 1) \int_0^\infty \frac{s^k}{(s + \alpha)^{k+2}} e^{-s - \alpha} ds = (k + 1) d_k^2. \] (5.19)

Thus, (5.17) and (5.12) follow.

Remark 3. Since \( \partial \) is unbounded, \( |\partial \psi|_h \) must be unbounded by Proposition 2.6. One can also see this by direct calculation: \( |\partial \psi|^2 = \alpha |z|^{2(1 - |z|^2)^{-2}} \) is unbounded in \( \mathbb{B} \).

Next, we compute \( \widehat{\partial}_1 \) for general \( n \). For this, we let \( v \in A^2_{2,0}(\mathbb{B}, h, d\mu) \) and write
\[ v = \frac{1}{2} \sum_{j,k} v_{jk} dz^j \wedge dz^k = \sum_{j < k} v_{jk} dz^j \wedge dz^k, \] (5.22)
where \( v_{jk} = -v_{kj} \) are holomorphic. By (2.42) and (5.9),
\[ \partial^* v = -\sum_{j,k,l} v_{kj} \bar{\psi} h^j d^k = \alpha \sum_{j,k} v_{jk} z^j d^k. \] (5.23)

For \( u = u_j dz^j \), we have
\[ \partial u = \frac{1}{2} \sum_{j,k} \left( \frac{\partial u_k}{\partial z^j} - \frac{\partial u_j}{\partial z^k} \right) d^j \wedge dz^k. \] (5.24)

Thus,
\[ \partial^* \partial u = \alpha \sum_{j,k} \left( \frac{\partial u_k}{\partial z^j} - \frac{\partial u_j}{\partial z^k} \right) z^j dz^k. \] (5.25)
On the other hand, since
\[ \partial^* u = \alpha \sum_j u_j z^j, \] (5.26)
we have
\[ \partial \partial^* u = \alpha \sum_k \left( u_k + \sum_j z^j \frac{\partial u_j}{\partial z^k} \right) dz^k. \] (5.27)
Consequently,
\[ \tilde{\Box}_1 u = \alpha \left[ u + \sum_{j,k} z^j \frac{\partial u_k}{\partial z^j} dz^k \right]. \] (5.28)
This is similar to the formula for \( \tilde{\Box}_1 \) on the Segal-Bargmann space given in [9, Eq. (2.5)].
Thus, we have

**Proposition 5.2.** Let \( \alpha > 0 \). Then \( \tilde{\Box}_1 \) has a bounded inverse \( \tilde{N}_1 \), which is a compact operator on \( A^2_{(1,0)}(\mathbb{B}, h, e^{-\psi}) \) with spectrum \( \{ \alpha k : k \in \mathbb{N} \} \), where each eigenvalue \( \alpha k \) has multiplicity \( n(n+k-2-n^{-1}) \).

In addition we have
\[ \| \tilde{N}_1 u \| \leq \frac{1}{\alpha} \| u \|, \] (5.29)
for each \( u \in A^2_{(1,0)}(\mathbb{B}, h, e^{-\psi}) \).

Consequently, if \( \eta = \eta_j dz^j \in A^2_{(1,0)}(\mathbb{B}, h, e^{-\psi}) \) with \( \partial \eta = 0 \), then \( f := \partial^* \tilde{N}_1 \eta \) is the canonical solution of \( \partial f = \eta \), this means \( \partial f = \eta \) and \( f \in (\text{ker } \partial)^\perp \). Moreover,
\[ \int_{\mathbb{B}} |f|^2 (1-|z|^2)^{-n-\alpha} \exp \left( -\frac{\alpha}{1-|z|^2} \right) d\lambda \leq \frac{1}{\alpha} \int_{\mathbb{B}} \left[ \sum_{j=1}^n |\eta_j|^2 - \sum_{j,k=1}^n \eta_j \eta_k z^j \bar{z}^k \right] (1-|z|^2)^{-n} \exp \left( -\frac{\alpha}{1-|z|^2} \right) d\lambda. \] (5.30)

**Proof.** The proof is similar to that of [9, Theorem 4.8] and uses Theorem 4.3. Indeed, the coercivity of \( \tilde{\Box}_1 \) follows directly from the fact that its spectrum consists of the point eigenvalues \( \alpha k \), \( k = 1, 2, \ldots \), each with finite multiplicity. Thus, \( \tilde{\Box}_1 \) has a bounded inverse \( \tilde{N}_1 \).

Since \( \eta \in \ker(\partial_0) \subset A^2_{(1,0)}(\mathbb{B}, h, \psi) \), we can define \( f = \partial^* \tilde{N}_1 \eta \). Standard arguments implies that \( f \) is orthogonal to \( A^2(\mathbb{B}, (1-|z|^2)^{-n-\alpha}) \) and \( \partial f = \eta \). Moreover,
\[ \| f \|^2 = \left( \partial^* \tilde{N}_1 \eta, f \right)_{h,\psi} = \left( \tilde{N}_1 \eta, \partial f \right)_{h,\psi} = \left( \tilde{N}_1 \eta, \eta \right)_{h,\psi} \leq \frac{1}{\alpha} \| \eta \|^2, \]
and hence (5.30) follows. The proof is complete. \( \square \)

**Remark 4.** If \( n = 1 \) and \( u = u dz \in A^2_{(1,0)}(\mathbb{B}, h, e^{-\psi}) \), then
\[ \tilde{\Box}_1 u = \partial \partial^* u = \alpha \left( u + z \frac{\partial u}{\partial z} \right) dz. \] (5.31)
By Proposition 5.2, we have for \( \alpha > 0 \) that
\[ (\tilde{\Box}_1 u, u)_{h,\psi} \geq \alpha \| u \|^2. \] (5.32)
For any \( v \in A^2_{(1,0)}(\mathbb{B}, h, e^{-\phi}) \), there exists \( w \in A^2_{(0,0)}(\mathbb{B}, h, e^{-\phi}) \) such that
\[
\partial w = v,
\]
and
\[
\|w\|^2 \leq \frac{1}{\alpha} \|v\|^2.
\]
The operator \( \tilde{\Box}_1 \) is has an bounded inverse. We point out that Theorem 4.3 only gives a weaker estimate under a stronger assumption \( \alpha > 2 \). This is also the case in higher dimension. To see this, we compute the Ricci form \( \Theta = -\bar{i} \partial \bar{\partial} \log \det(h_{j\bar{k}}) = -(n+1)\omega_h \), here \( \omega_h = \bar{i} h_{j\bar{k}} dz^j \wedge d\bar{z}^k \) is the Kähler form. Therefore, for \( \epsilon > 0 \) and \( \gamma := n + 1 + \epsilon \),
\[
i \partial \bar{\partial} \psi + \Theta - \epsilon \omega_h = i \left[ \frac{\alpha - \gamma(1 - |z|^2)}{(1 - |z|^2)^2} \delta_{jk} + \frac{2 \alpha - \gamma(1 - |z|^2)}{(1 - |z|^2)^3} \bar{j} \bar{k} \right] dz^j \wedge d\bar{z}^k.
\]
Thus, \( i \partial \bar{\partial} \psi + \Theta \geq \epsilon \omega_h \) on \( \mathbb{B} \) if and only if \( \alpha \geq B \) or \( \epsilon \leq \alpha - n - 1 \). However, we can only deduce from Corollary 3.3 the basic estimate \( 8.2 \) with constant \( c = \alpha - n - 1 \) which is much smaller than the lowest eigenvalue \( \lambda_1 = \alpha \) of \( \tilde{\Box}_1 \).

5.2. A non-Kählerian metric. We give an example of a space of holomorphic functions on the unit ball in \( \mathbb{C}^n \) endowed with a non-Kählerian metric (for \( n \geq 2 \)) such that the \((1,0)\)-vector field \( \overline{(\partial \psi - \bar{\tau})} \) is holomorphic and the formal adjoint of \( D \) sends \((2,0)\)-forms with holomorphic coefficients to \((1,0)\)-forms with holomorphic coefficients. These facts allow us to obtain an explicit formula for the complex Laplacian \( \tilde{\Box}_1 \).

To construct this example, we first compute in the case \( n = 2 \) the \((1,0)\)-vector field \( \xi := \overline{(\partial \psi - \bar{\tau})} \) in terms of the metric: the components of \( \xi \) are
\[
\xi^1 = \frac{1}{\det(h_{\bar{\tau}})} (h_{\bar{\tau}}(\partial \psi) - h_{\bar{\tau}}(\partial \psi) - \partial h_{\bar{\tau}} \partial_{\bar{\tau}} h_{\bar{\tau}}),
\]
\[
\xi^2 = \frac{1}{\det(h_{\bar{\tau}})} (-h_{\bar{\tau}}(\partial \psi) + h_{\bar{\tau}}(\partial \psi) - \partial_{\bar{\tau}} h_{\bar{\tau}} \partial_{\bar{\tau}} h_{\bar{\tau}}).
\]
This calculation suggests that, for the unit ball \( \mathbb{B}^2 \subset \mathbb{C}^2 \), we choose \( h_{\bar{\tau}} = \delta_{jk}(1 - |z|^2)^{-1} \) for \( z \in \mathbb{B}^2 \) and \( \psi(z) = \alpha \log(1 - |z|^2), \alpha \in \mathbb{R} \). An easy computation shows that
\[
(\partial \psi - \bar{\tau})^2 = -\alpha + 1 \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right),
\]
which means that this \((1,0)\)-vector field is holomorphic. For general dimension \( n \geq 2 \), the same choices of \( h_{\bar{\tau}} \) and \( \psi \) also work. Indeed, we can check that
\[
T^{i}_{jk} = (1 - |z|^2)^{-1} (\bar{z}_j \delta_{ik} - \bar{z}_k \delta_{ij}), \quad \tau_j = T^{i}_{ji} = (n - 1)(1 - |z|^2)^{-1} \bar{z}_j,
\]
and
\[
\psi_j = -\alpha(1 - |z|^2)^{-1} \bar{z}_j.
\]
Therefore,
\[
(\partial \psi - \bar{\tau})^2 = (1 - n - \alpha) \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}
\]
is also holomorphic. In this case we have, for \( p \geq 0 \),
\[
L^2_{(p,0)}(\mathbb{B}^n, h, e^{-\phi}) = \left\{ \sum_{|J|=p} \cdot u_J dz^J : \sum_{|J|=p} \int_{\mathbb{B}^n} |u_J(z)|^2 (1 - |z|^2)^{p-n-\alpha} d\lambda(z) < \infty \right\}.
\]
For \( p = 0 \), the Bergman space \( A^2_{(0,0)}(\mathbb{B}, h, e^{-\psi}) \) coincides with the usual Bergman space \( A^2_{-1}(\mathbb{B}) \), with \( \gamma = 1 - n - \alpha \) (see, e.g., [14]). As usual, we assume \( \gamma > 0 \), or equivalently, \( \alpha < 1 - n \). Then for each \( p \) the weighted Bergman space \( A^2_{(p,0)}(\mathbb{B}^n, h, e^{-\psi}) \) is closed in the weighted Lebesgue space \( L^2_{(p,0)}(\mathbb{B}^n, h, e^{-\psi}) \) by Corollary 2.5 in [16]. Moreover, the monomials

\[
\left\{ \frac{z^J}{c_J} : |J| \geq 0 \right\}
\]

form an orthonormal basis in \( A^2_{(0,0)}(\mathbb{B}^n, h, e^{-\psi}) \cong A^2(\mathbb{B}, (1 - |z|^2)^{-\alpha}d\lambda) \) and the \((1,0)\)-forms with monomial coefficients

\[
\left\{ \frac{z^J}{d_J} dz^k : |J| \geq 0, k = 1, 2, \ldots, n \right\}
\]

form an orthonormal basis in \( A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi}) \), where \( J = (j_1, \ldots, j_n) \) is a multi-index,

\[
c^2_J = \int_{\mathbb{B}} |z^J|^2 (1 - |z|^2)^{-\alpha} d\lambda = \frac{\omega_n n! J! \Gamma(1 - n - \alpha)}{\Gamma(|J| + 1 - \alpha)}, \tag{5.40}
\]

and similarly

\[
d^2_J = \int_{\mathbb{B}} |z^J dz_k|^2 h (1 - |z|^2)^{-\alpha} d\lambda = \frac{\omega_n n! J! \Gamma(2 - n - \alpha)}{\Gamma(|J| + 2 - \alpha)}. \tag{5.41}
\]

Here \( \omega_n \) is the volume of the unit ball. Note that the metric \( h \) is not complete and hence Andreotti-Vesentini density lemma does not apply. However, in view of the calculations for the hyperbolic metric on \( \mathbb{B} \), we get for a \((1,0)\)-form \( u = \sum_{j=1}^n u_j dz_j \in \text{dom}(\partial^\ast) \) that

\[
\partial^\ast(u) = (1 - n - \alpha) \sum_{j=1}^n z^j u_j \tag{5.42}
\]

if we can show that

\[
(1 - n - \alpha)c^2_{j+k+1} = (j_k + 1)d^2_J, \tag{5.43}
\]

here \( J + k \) denotes the multi-index \( (j_1, \ldots, j_{k-1}, j_k + 1, j_{k+1}, \ldots, j_n) \). Equation (5.43) corresponds to (15.17) and follows easily from (5.40) and (5.41).

To compute \( \partial^\ast \) for \((2,0)\)-forms, we write \( v = \frac{1}{2} v_{jk} dz^j \wedge dz^k \), \( v_{jk} = -v_{kj} \) and use (5.37) to obtain

\[
\frac{1}{2} \sum_{j,k=1}^n T_{jk}^z h^{\tilde{h}j} h^{\tilde{s}k} h^{\tilde{p}l} v_{rs} dz^p = z^\ast v_{rs} dz^\ast. \tag{5.44}
\]

This turns out to be holomorphic for holomorphic \((2,0)\)-forms \( v \). Plugging this and (5.39) into (2.42) (which is valid since the boundary terms in the integration-by-parts argument vanish due to the factor \( 1 - |z|^2 \)), we find that

\[
\partial^\ast v = (2 - n - \alpha) z^\ast v_{rs} dz^\ast. \tag{5.45}
\]

Here the orthogonal projection \( P_{h,\psi} \) has no effect since the coefficients \( v_{rs} \)'s are holomorphic. We get, for \( \gamma := 1 - n - \alpha > 0 \),

\[
\square_1 u = \gamma u + \left[ (1 + \gamma) \sum_{j,k=1}^n z^j \frac{\partial u_k}{\partial z_j} - \sum_{j,k=1}^n z^j \frac{\partial u_j}{\partial z^k} \right] dz^k, \tag{5.46}
\]

Unlike the case of complex hyperbolic metric, \( \square_1 \) is not diagonal in the basis \( \{ z^l dz^k : |J| \geq 0, l = 1, 2, \ldots, n \} \).
The subspaces
\[ A^2_{1(1,0)}(m) := \text{span} \left\{ c_{j,l} z^j d^l : , |J| = m, l = 1, 2, \ldots, n \right\} \]
are invariant under the action of \( \boxdot_i \). Using Lemma \[5.1\] we can study the spectrum of \( \boxdot_1 \) by studying the spectra of its restrictions onto finite dimensional subspaces \( A^2_{1(1,0)}(m) \). When \( m = 0 \), \( A^2_{1(1,0)}(0) \) is spanned by \( dz^1, dz^2, \ldots, dz^n \) and \( \boxdot_1(dx^k) = \gamma dx^k \) and hence \( \gamma \) is an eigenvalue for \( \boxdot_1 \). When \( m = 1 \), \( A^2_{1(1,0)}(1) \) has dimension \( n^2 \) and is spanned by \( z^j dz^k \), \( j, k = 1, \ldots, n \); For example, if \( n = 2 \) then the matrix representation of \( \boxdot_1 \) in the basis \( e_1 := z_1dz_1, e_2 := z_1dz_2, e_3 := z_2dz_1 \), and \( e_4 := z_2dz_2 \) is
\[
\begin{pmatrix}
2\gamma & 0 & 0 & 0 \\
0 & 2\gamma + 1 & -1 & 0 \\
0 & -1 & 2\gamma + 1 & 0 \\
0 & 0 & 0 & 2\gamma
\end{pmatrix}
\]
and the eigenvalues are \( 2(\gamma + 1) \) and \( 2\gamma \); the later has multiplicity 3, and the matrix is diagonalizable. In the general case, by straightforward calculations, we obtain that the smallest eigenvalue of \( \boxdot_1 \) is coercive and has bounded inverse \( \tilde{N}_1 \), which is a compact operator on \( A^2_{1(1,0)}(m) \). When \( m = 1 \), \( A^2_{1(1,0)}(1) \) has dimension \( n^2 \) and is spanned by \( z^j dz^k \), \( j, k = 1, \ldots, n \); For example, if \( n = 2 \) then the matrix representation of \( \boxdot_1 \) in the basis \( e_1 := z_1dz_1, e_2 := z_1dz_2, e_3 := z_2dz_1 \), and \( e_4 := z_2dz_2 \) is
\[
\begin{pmatrix}
2\gamma & 0 & 0 & 0 \\
0 & 2\gamma + 1 & -1 & 0 \\
0 & -1 & 2\gamma + 1 & 0 \\
0 & 0 & 0 & 2\gamma
\end{pmatrix}
\]
and the eigenvalues are \( 2(\gamma + 1) \) and \( 2\gamma \), the later has multiplicity 3, and the matrix is diagonalizable. In the general case, by straightforward calculations, we obtain that the smallest eigenvalue of \( \boxdot_1 \) on \( A^2_{1(1,0)}(m) \) is \( (m + 1)\gamma \) while the largest one is smaller than \( \gamma + m(2 + \gamma) \), as simple consequences of a theorem of Geršgorin [4]. Moreover, the corresponding matrix is diagonalizable by the self-adjointness of \( \boxdot_1 \). Thus, by Lemma \[5.1\] the spectrum of \( \boxdot_1 \) consists of point eigenvalues which are those of the finite dimensional restrictions and each has finite multiplicity.

**Proposition 5.3.** If \( \gamma := 1 - n - \alpha > 0 \), then \( \boxdot_1 \) is coercive and has bounded inverse \( \tilde{N}_1 \), which is a compact operator on \( A^2_{1(0)}(\mathbb{B}^n, h, \psi) \) with discrete spectrum.

Consequently, for every \( \eta_1, \eta_2, \ldots, \eta_n \in A^2_{1}(\mathbb{B}) \) such that \( \partial \eta_j/\partial z^k = \partial \eta_k/\partial z^j \) for every pair \( j, k = 1, 2, \ldots, n \), there exists \( f \in A^2_{1}(\mathbb{B}) \) such that \( \partial f/\partial z^k = \eta_k \) for every \( k = 1, 2, \ldots, n \), and
\[
\int_\mathbb{B} |f|^2 (1 - |z|^2)^{\gamma - 1} d\lambda \leq \frac{1}{\gamma} \int_\mathbb{B} \sum_{k=1}^n |\eta_k|^2 (1 - |z|^2)^{\gamma} d\lambda. \tag{5.47}
\]

**Proof.** The coercivity of \( \boxdot_1 \) and the existence and compactness of \( \tilde{N}_1 \) follow directly from the fact that its spectrum consists of the point eigenvalues with finite multiplicity.

Define \( \eta := \sum_{k=1}^n \eta_k dz^k \), with \( \eta_k \)'s are holomorphic, \( \partial \eta = 0 \), and
\[
\| \eta \|^2 = \int_\mathbb{B} |\eta|^2 d\mu = \int_\mathbb{B} \sum_{k=1}^n |\eta_k|^2 (1 - |z|^2)^\gamma d\lambda < \infty. \tag{5.48}
\]

Then \( \eta \in \ker(\partial h) \subset A^2_{1(1,0)}(\mathbb{B}, h, \psi) \). Define \( f = \partial^* \tilde{N}_1 \eta \). Standard arguments implies that \( \tilde{f} \) is orthogonal to \( A^2(\mathbb{B}, (1 - |z|^2)^{\gamma - 1}) \) and \( \partial f = \eta \). Moreover,
\[
\| f \|^2 = (\partial^* \tilde{N}_1 \eta, f)_{h, \psi} = (\tilde{N}_1 \eta, \partial f)_{h, \psi} \leq \frac{1}{\gamma} \| \eta \|^2. \tag{5.49}
\]

The last inequality follows from the fact that the lowest eigenvalue of \( \boxdot_1 \) is \( \lambda_1 = \gamma \). The proof is complete. \( \square \)

We point out again that the usual basic identity as in Corollary \[5.3\] is not useful for the metric \( h_{\bar{j}k} = (1 - |z|^2)^{-1} \delta_{\bar{j}k} \) as above for \( n \geq 2 \). To see this, we compute,
\[
i \partial \bar{\partial} \psi + \Theta = i(n + \alpha) \partial \bar{\partial} \log(1 - |z|^2) \tag{5.50}
\]
and
\[ i T \circ \overline{T} = \frac{2(|z|^2 - z^j z^k)}{(1 - |z|^2)^2} dz^j \wedge dz^k. \tag{5.51} \]

Consequently, for any \( \mu > 1 \),
\[
i \partial \bar{\partial} \psi + \Theta - \mu i T \circ \overline{T} - \epsilon \omega_h = i \left[ \frac{(n + \alpha + \epsilon - \mu)|z|^2 - n - \alpha - \epsilon}{(1 - |z|^2)^2} \delta_{jk} + \frac{(2\mu - n - \alpha) \bar{z}^j z^k}{(1 - |z|^2)^2} \right] dz^j \wedge dz^k.
\]

For this to be nonnegative at the origin, \( n + \alpha + \epsilon < 0 \). But near the boundary, the hermitian matrix in the bracket on the right-hand side is a rank-one perturbation of the negative constant multiple of the identity matrix and hence can not be nonnegative.

REFERENCES

[1] A. Andreotti and E. Vesentini. Carleman estimates for the Laplace-Beltrami equation on complex manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (25):81–130; Erratum: 27 (1965), 153–155, 1965.
[2] F. Berger, G. M. Dall’Ara, and D. N. Son. Exponential decay of Bergman kernels on complete Hermitian manifolds with Ricci curvature bounded from below. *arXiv preprint arXiv:1804.07540*, 2018.
[3] E. B. Davies. *Spectral theory and differential operators*, volume 42. Cambridge University Press, 1995.
[4] G. B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, 1989.
[5] S. Geršgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Izv. Akad. Nauk SSSR Ser. Mat*, 1(7):749–755, 1931.
[6] P. A. Griffiths. The extension problem in complex analysis. II. Embeddings with positive normal bundle. *Amer. J. Math.*, 88:366–446, 1966.
[7] L. Gross. Hypercontractivity over complex manifolds. *Acta Math.*, 182(2):159–206, 1999.
[8] F. Haslinger. The \( \partial \)-Neumann Problem and Schrödinger Operators. de Gruyter Expositions in Mathematics, vol. 59. Walter de Gruyter GmbH & Co KG, 2014.
[9] F. Haslinger. The \( \partial \)-complex on the Segal-Bargmann space. *Ann. Polon. Math.*, Online first, (doi: 10.4064/ap180715-2-11), 2019.
[10] S. N. Mergelyan. On completeness of systems of analytic functions. *Uspekhi Matematicheskikh Nauk*, 8(4):3–63, 1953.
[11] J. A. Morrow and K. Kodaira. *Complex manifolds*, volume 355. Amer. Math. Soc., 1971.
[12] O. Munteanu and J. Wang. Kähler manifolds with real holomorphic vector fields. *Math. Ann.*, 363(3-4):893–911, 2015.
[13] T. Ohsawa. \( L^2 \) Approaches in Several Complex Variables. Springer Monographs in Mathematics, 2015.
[14] Z. Pasternak-Winiarski. On the dependence of the reproducing kernel on the weight of integration. *J. Funct. Anal.*, 94(1):110–134, 1990.
[15] B. A. Taylor. On weighted polynomial approximation of entire functions. *Pacific J. of Math.*, 36(2):523–539, 1971.
[16] K. Zhu. *Spaces of holomorphic functions in the unit ball*, volume 226. Springer Science & Business Media, 2005.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

E-mail address: friedrich.haslinger@univie.ac.at

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

E-mail address: son.duong@univie.ac.at