FLIP CYCLES IN PLABIC GRAPHS

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Abstract. Planar bicolored (plabic) graphs are combinatorial objects introduced by Postnikov to give parameterizations of the positroid cells of the totally nonnegative Grassmannian $\text{Gr}^{\geq 0}(n,k)$. Any two plabic graphs for the same positroid cell can be related by a sequence of certain moves. The flip graph has plabic graphs as vertices and has edges connecting the plabic graphs which are related by a single move. A recent result of Galashin shows that plabic graphs can be seen as cross-sections of zonotopal tilings for the cyclic zonotope $Z(n,3)$. Taking this perspective, we show that the fundamental group of the flip graph is generated by cycles of length 4, 5, and 10, and use this result to prove a related conjecture of Dylan Thurston about triple crossing diagrams. We also apply our result to make progress on an instance of the generalized Baues problem.

1. Introduction

A flip graph for our purposes is the graph whose vertices form the set of all diagrams of a particular class, and whose edges correspond to flips in these diagrams, which are mutations which transform one diagram into a similar diagram with one small thing changed. A common question to ask is if the flip graph is connected, that is, can any two objects in the set be related by a sequence of flips? For Postnikov’s moves on plabic graphs [11], the answer is affirmative. In this paper we investigate the next natural topological question for some flips graphs that are known to be connected: is there a nice set of “simple cycles” which generate the fundamental group of the flip graph?

Possibly the most famous example of a flip graph is that of triangulations of an $n$-gon, whose flip graph forms the 1-skeleton of the Stasheff associahedron. Although not novel (known at least since Stasheff’s work [15]), a nice corollary of our result is that the fundamental group of the 1-skeleton of the associahedron is generated by cycles of length four and five. Another famous flip graph has domino tilings of a planar region as its vertices; in [18] it is proved that the flip graph is connected (provided that the region is simply connected) through a height function on tilings, which also gives formula for computing the distance between tilings. Dylan Thurston [17] introduced triple crossing diagrams, which are a generalization of domino tilings, proved that the flip graph is connected, and made a conjecture about the fundamental group of the flip graph. One of the results of this paper is a proof of that conjecture.

We will study several different flip graphs, whose objects and flips are as follows:

- Fine zonotopal tilings of the cyclic zonotope $Z(n,d)$, with flips corresponding to switching between the two tilings of $Z(d+1,d)$.
- Reduced trivalent plabic graphs for a given strand connectivity, with flips corresponding to the moves (M1)–(M3) in Figure 3.

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Reduced bipartite plabic graphs for a given connectivity, considered modulo the contraction/uncontraction moves (see Figure 3). The flips are only given by the square move (M2).

Triple crossing diagrams for a given connectivity, with flips being $2 \leftrightarrow 2$ moves (see Figure 5).

For definitions of these objects and flips, see Section 2 for zonotopal tilings, Section 3 for plabic graphs, and Section 5 for triple crossing diagrams.

Our main theorem regards generating sets for the fundamental group of the flip graphs when considered as a 1-complex. Though we later phrase our theorems as proving that a 2-complex made out of the flip graph with certain 2-cells glued is simply connected, here we will simply state the sizes of the cycles which generate the fundamental group.

**Theorem 1.1.** The fundamental groups for our flips graphs on the following objects are generated by cycles with sizes as follows

1. Fine zonotopal tilings of $Z(n,d)$, by cycles of sizes 4 and $2d + 4$.
2. Reduced trivalent plabic graphs, by cycles of sizes 4, 5 (two types), and 10 (two types).
3. Reduced bipartite plabic graphs, by cycles of sizes 4 and 5.
4. Triple crossing diagrams, by cycles of sizes 4, 5, and 10.

The first part is the subject of Section 2 and is a result of using Ziegler’s results on the higher Bruhat order poset [19] to generalize the proof for $d = 2$ given by Henriques and Speyer [6] (for the flips in the rhombus tilings of the regular $2n$-gon). The second result is completely new to our knowledge, and uses Galashin’s [5] interpretation of plabic graphs as cross-sections of fine zonotopal tilings. An alternate proof for the case where the plabic graphs have a cyclic strand connectivity (corresponding to the totally positive Grassmannian $Gr^+0(n,k)$, [11]) is given in the appendix, and uses the first result. The relationship between zonotopal tilings and plabic graphs is elaborated on further in Section 3, and the result is proven in Section 4. The last two results are corollaries of the second, with the fourth result proving a conjecture of Dylan Thurston [17, Conjecture 21], see Section 5.

In the case $d = 3$, the fine zonotopal tiling flip graph is generated by squares and decagons. The vertices of the decagons correspond to fine zonotopal tilings of $Z(5,3)$, one of which is shown in Figure 1. The plabic graph cycles of length 5 and 10 occur in the cross-sections of the tilings as shown in Figure 4.

The questions we discuss regard the relations between flips in our flip graphs. One can further ask about relations between relations, and so on. A natural way to pose this question formally involves the topology of the poset, whose minimal elements are our configurations, rank 1 elements are the flips, rank 2 elements are relations between flips, etc. A formal way to do it is known as the generalized Baues problem, which we discuss in Section 6. We prove there that the Baues poset of the Grassmannian graphs, conjectured [12] to have the homotopy type of sphere, is simply connected.

## 2. Cycles for Zonotopal Tilings

**Definition 2.1.** Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be any collection of distinct vectors on the dimension $d$ moment curve parameterized by $(1, t, t^2, \ldots, t^{d-1})$ (the indexing of the $v_i$ is increasing in $t$).

The *cyclic zonotope* $Z(n,d)$ consists of all points which can be written as $\sum_{i=1}^n c_i v_i$ for some $(c_i)_{i=1}^n \in [0, 1]^n$, that is, $Z(n,d)$ is the Minkowski sum of the intervals $[0, v_i]$. 
Figure 1. A fine zonotopal tiling of $D = Z(5, 3)$ with the tile $\tau_{\{\{5\},\{3\}\}}$ highlighted in green. The four nontrivial cross-sections shown are planar duals to plabic graphs, and exhibit four of the five types of 2-cells in $X_{\pi(n,k)}$ (see Theorem 4.6). Figure from Galashin [5].

Remark 2.2. Sometimes cyclic zonotopes (and cyclic polytopes) are defined more generally, cf. [12, Definition 9.1]. Everything we prove could be related to those slightly more general objects as well.

Following Galashin [5], we define tilings of the cyclic zonotope as collections of signed subsets. A pair $X = (X^+, X^-)$ of disjoint subsets of $[n]$ is a signed subset of $[n]$, and we also define $X^0 := [n] \setminus (X^+ \cup X^-)$. Then the signed subsets are exactly the strings in $\{+,-,0\}^n$. For $X$ a signed subset, the tile $\tau_X$ consists of all points which can be written as $\sum_{i \in X^+} v_i + \sum_{j \in X^0} c_j v_j$ for some $(c_j)_{j \in X^0} \in [0,1]^{\vert X^0 \vert}$.

Definition 2.3. A collection $\Delta$ of signed subsets of $[n]$ is called a fine zonotopal tiling of $Z(n,d)$ provided that

1. $Z(n,d) = \bigcup_{X \in \Delta} \tau_X$,
2. Whenever $\tau_X \cap \tau_Y \neq \emptyset$ for $X, Y \in \Delta$, there exists $Z \in \Delta$ such that $\tau_X \cap \tau_Y = \tau_Z$ is a face of both $\tau_X$ and $\tau_Y$, and
3. For all $X \in \Delta$, we have $\vert X^0 \vert \leq d$.

When the third condition fails, $\Delta$ is a zonotopal tiling but is not fine. Fine zonotopal tilings $\Delta$ of $Z(n,d)$ can be related to each other through a series of mutations. Geometrically, these mutations consist of finding a tiling copy of $Z(d + 1, d)$ inside $\Delta$, which has only two fine tilings, and flipping the way it is tiled. We will use the combinatorial definition in terms of
signed subsets. Suppose that \( S \in \binom{[n]}{d+1} \) has elements \( i_1 < \cdots < i_{d+1} \). For \( i \in S \), let \( X_i \) be the unique signed subset in \( \Delta \) for which \( X_i^+ = S \setminus \{i\} \). Let \( S_i^+ := X_i^+ \setminus S \) and \( s_i := 1_{X_i^+}(i) \).

**Definition 2.4.** A *flip* of the set \( S \) is available in a fine zonotopal tiling \( \Delta \) if \( S_i^+ = S_j^+ \) for all \( i, j \in S \) and \( s_i \neq s_{i+1} \) for all \( \ell \in [n = d] \). Performing the flip of the set \( S \) results in a new fine zonotopal tiling \( \Delta \) for which all the signed subsets are identical except that all the values of \( s_i \) have changed.

The *flip graph* is the graph which has fine zonotopal tilings as vertices and edges connecting those tilings which are related by a single flip. It is a fact that any two fine zonotopal tilings of \( Z(n, d) \) can be related by a series of flips, so the flip graph is connected, as we will see.

Henriques and Speyer ([6, Proposition 3.14]) prove that the fundamental group of the flip graph of \( Z(n, 2) \) as a 1-complex is generated by 4-cycles and 8-cycles, where the 4-cycles correspond to pairs of commuting flips and the 8-cycles correspond to copies of \( Z(4, 2) \). In this section we generalize this result to any dimension using Ziegler’s [19] results on the higher Bruhat order. Ziegler [19] shows (with different language) that the flip graph for \( Z(n, d) \) is isomorphic to the Hasse diagram for the higher Bruhat order graded poset \( B(n, k) \) for \( k = d \). We will not bother to define the higher Bruhat order, rather, we will state the relevant results about it in the language of fine zonotopal tilings. Flips in zonotopal tilings correspond to covering relations in \( B(n, d) \), and the functional \( \phi \) used in [6, Proposition 3.14] on tilings can be related to the rank function on \( B(n, 2) \).

**Theorem 2.5 ([19, Theorem 4.1]).** The edges of the flip graph for \( Z(n, d) \) form the Hasse diagram for a graded poset with unique minimal and maximal elements \( \Delta_{\min} \) and \( \Delta_{\max} \) at ranks 0 and \( \binom{n}{d+1} \). The set of minimal-length paths of flips between \( \Delta_{\min} \) and \( \Delta_{\max} \) modulo commutation of unrelated flips is in natural bijection with the elements of \( Z(n, d + 1) \), such that flips in tilings of \( Z(n, d + 1) \) swap the order in which \( d + 2 \) flips occur in the corresponding path.

It follows from the above that \( Z(d + 2, d + 1) \) has only two fine zonotopal tilings, so \( Z(d + 2, d) \) has only two paths from \( \Delta_{\min} \) to \( \Delta_{\max} \) up to commutation. There are also no pairs of commuting flips in tilings of \( Z(d + 2, d) \), so its flip graph must be a single \((2d + 4)\)-cycle. We are now ready to characterize the cycles in the flip graph for zonotopal tilings.

**Theorem 2.6.** Let \( Z_{n,d} \) be the 2-complex formed by the flip graph for \( Z(n, d) \) with the following 2-cells glued:
- quadrilaterals, wherever there is a cycle of length four corresponding to commuting pairs of flips;
- \((2d + 4)\)-gons, wherever there is a cycle of length \((2d + 4)\) whose vertices are all refinements of a particular zonotopal tiling which is fine except for a single signed subset which creates a tile isomorphic to \( Z(d + 2, d) \).

Then \( Z_{n,d} \) is simply connected.

**Proof.** We will use a technique similar to the proof in [6, Proposition 3.14], and use results about the higher Bruhat order as a black box to generalize to higher dimensions.

Let \( \gamma = S_1S_2 \cdots S_m \), where each \( S_i \) is a flip which turns tiling \( \Delta_i \) into \( \Delta_{i+1} \) and \( \Delta_1 = \Delta_{m+1} \), be a loop in the flip graph for \( Z(n, d) \) which connects the tilings \( \Delta_1, \Delta_2, \ldots, \Delta_{m+1} = \Delta_1 \). It suffices to show that \( \gamma \) can be continuously deformed to a point in \( Z_{n,d} \). All we know is that the squares and the cycles corresponding to the \((2d + 4)\)-gon from copies of \( Z(d + 2, d) \) are
nullhomotopic, so our only tool is to replace paths in $\gamma$ with their complement in a square or $(2d+4)$-gon.

First suppose that $\gamma$ is a loop of length $2(\begin{pmatrix} n \\ d+1 \end{pmatrix})$ that includes $\Delta_{\min}$ and $\Delta_{\max}$. Since $\gamma$ connects the minimal and maximal elements twice in the shortest possible time, it can be divided into two parts, $\alpha$ and $\beta$, each of which is a series of monotonic in terms of rank flips in $Z(n,d)$. Then by Theorem 2.5, $\alpha$ and $\beta$ are each representative elements of some equivalence classes of paths between $\Delta_{\min}$ and $\Delta_{\max}$ given by fine zonotopal tilings $A$ and $B$ of $Z(n,d+1)$, respectively. The flip graph for $Z(n,d+1)$ is connected, so there exists a sequence of flips to transform $A$ into $B$. Along the way, commutation of flips in $\alpha$ is required to get the right representative element of $A$, to allow the flips in $Z(n,d+1)$ to be realized as $(2d+4)$-gons in $Z_{n,d}$. The flips in $Z(n,d+1)$ involve $d+2$ tiles in a copy of $Z(d+2,d+1)$, which appear as $d+2$ flips in $\alpha$, all inside a copy of $Z(d+2,d)$. Therefore commutation of flips moves $\alpha$ over a quadrilateral, while flips of $A$ involve moves $\alpha$ over a $(2d+4)$-gon. At each step, a continuous deformation of $\alpha$ occurs, eventually transforming it to $\beta$, at which point $\gamma$ is trivial because it is $\beta \beta^{-1}$.

Now suppose $\gamma$ is any arbitrary loop as before. Then for each vertex $\Delta_i$ in $\gamma$, draw a path $\delta_i$ of length $\binom{n}{d+1}$ between $\Delta_{\min}$ and $\Delta_{\max}$ which goes through $\Delta_i$, using Theorem 2.5. Let’s say that $\delta_i = \delta_i^+ \delta_i^-$, where $\delta_i^+$ connects $\Delta_{\min}$ to $\Delta_i$, and then $\delta_i^-$ connects $\Delta_i$ to $\Delta_{\max}$, both in the shortest possible time. Suppose that the loops $S_i \delta_i^+ (\delta_i^-)^{-1}$ are all deformable to point. Then after a continuous deformation we could compute to conclude the result (the brackets denote the homotopy class of a curve with fixed endpoints):

$$[\gamma] = \prod_{i=1}^{m} [\delta_i^+][\delta_i^+]^{-1} = [\delta_1^+ \left( \prod_{i=2}^{m} [\delta_i^+][\delta_i^-]^{-1}[\delta_i^+] \right)] [\delta_{m+1}^+][\delta_{m+1}^-]^{-1} = 1.$$  

Each flip $S_i$ is either an upward flip or a downward flip, depending on whether $\Delta_{i+1}$ has a higher or lower rank than $\Delta_i$ when seen in the higher Bruhat order. If it is an upward flip, then $\delta_i^- (\delta_i^-)^{-1} S_i \delta_{i+1}^+ (\delta_i^-)^{-1}$ is a cycle of minimal length which includes $\Delta_{\min}$ and $\Delta_{\max}$, and so is trivial in $Z_{n,d}$ as we showed in the previous paragraph. If it is a downward flip, then $S_i (\delta_i^-)^{-1} \delta_{i+1}^- (\delta_i^-)^{-1}$ is similarly trivial. In either case, the new loop is certainly homotopic to $S_i \delta_{i+1}^+ (\delta_i^-)^{-1}$, completing the proof.  

This result will be crucial for an alternate proof of our result for plabic graphs, which is described in the appendix.

3. Plabic Graphs in Zonotopal Tilings

Let $G$ be an embedding of a planar graph in a disk with each vertex colored white or black (adjacent vertices need not be different colors). Also add $n$ black boundary vertices $b_1, \ldots, b_n$ in clockwise order outside of the disk, each with a single edge to one of the vertices of $G$. This configuration is called a plabic graph and we refer to it by $G$ (see Figure 2).

**Definition 3.1.** A strand $s_i$ in a plabic graph $G$ is a path which starts at $b_i$, and proceeds along the edges of $G$ until it reaches some boundary vertex $b_j$, according to the rules of the road; when $s_i$ reaches a white (resp. black) vertex $v$ through edge $e$, if makes a sharp left (resp. right) turn. That is, if the edges of $v$ are shown in a circle, then $s_i$ should traverse the next edge clockwise (resp. counterclockwise) of $e$. The strand permutation of $G$ is the permutation $\pi_G \in S_n$ such that if $s_i$ ends at $b_j$ then $\pi_G(i) = j$.  

We will only deal with a special class of plabic graph. A *bad double crossing* is when two distinct strands both traverse edge $e_1$ and later traverse another edge $e_2$.

**Definition 3.2** (cf. [11, Theorem 13.2]). A plabic graph $G$ is *reduced* if and only if

- For any edge $e$ between non-boundary vertices, exactly two distinct strands $s_i$ and $s_j$ traverse $e$.
- $G$ does not contain any bad double crossings.
- When $\pi_G(i) = i$, the vertex $b_i$ is connected to a single isolated vertex of $G$.

We will only be considering reduced plabic graphs, and so will often omit the word “reduced”.

The *decorated strand permutation* $\pi_G$, for $G$ a reduced plabic graph, is identical to $\pi_G$ except that the fixed points of $\pi_G$ are *decorated* (black) if the single isolated vertex they are connected to is black, otherwise they are *undecorated* (white).

Postnikov described how the *boundary measurements* for reduced plabic graphs parameterize certain “positroid cells” in the totally non-negative Grassmannian $\text{Gr}^{\geq 0}(n,k)$ [11, Thm. 12.7]. The positroid cell parameterized by a plabic graph depends only on its decorated strand permutation. The cyclic permutation which sends $i$ to $i + k$ (modulo $n$) corresponds to the top cell of $\text{Gr}^{\geq 0}(n,k)$, which is the totally positive Grassmannian $\text{Gr}^{> 0}(n,k)$ and so is of special interest; we refer to this cyclic permutation by $\pi(n,k)$.

When Postnikov [11] introduced plabic graphs, he gave some moves to relate them (they correspond to certain simple reparametrizations of the positroid cell). One can check that the moves in Figure 3 preserve the strand connectivity and whether the plabic graph is reduced. Plabic graphs where all vertices have degree three are called *trivalent*, and $(M1), (M3)$ are
called trivalent moves. Through uncontraction moves, any plabic graph can be made trivalent. If all possible contraction moves are performed, the resulting graph will be bipartite.

**Theorem 3.3** (Postnikov [11]). Any two reduced trivalent plabic graphs with the same connectivity can be related by a sequence of the moves (M1), (M2), and (M3) in Figure 3.

We will primarily deal with trivalent plabic graphs and so only use (M1)–(M3), but the contraction/uncontraction moves will be relevant for triple crossing diagrams.

![Move Diagram](image)

**Figure 3.** Moves in plabic graphs, from Galashin [5]

Pavel Galashin [5] shows that the $k$-th cross-section, $1 \leq k \leq n - 1$, of fine tilings of the three dimensional cyclic zonotope correspond to trivalent reduced plabic graphs with connectivity $\pi(n,k)$. Let $\Delta$ be a fine zonotopal tiling of $Z(n,3)$. The cross-section $\Sigma_k$ of $\Delta$ with the plane $\{(k,x_2,x_3) \in \mathbb{R}^3\}$ is a triangulation of an $n$-gon, possibly with some interior vertices. We call it a *plabic triangulation*. The vertices of $\Sigma_k$ are labeled by strings in $\{+,-\}^n$ with exactly $k$ ‘+’ symbols, or equivalently, by elements of $\binom{[n]}{k}$. For any triangle in $\Sigma_k$, either the union of the labels of the vertices has $k + 1$ elements, or the intersection of the labels of the vertices has $k - 1$ elements, depending on the location of the triangle has a cross-section of a single parallelepiped tile. In the first case consider the triangle to be *black*, in the second case consider it *white*. Let $G_k$ be the planar dual to the triangulation $\Sigma_k$, and color the vertices of $G_k$ according to the color of the triangle to which it belonged.

**Theorem 3.4** (Galashin [5]). $G_k$ is a trivalent reduced plabic graph with strand connectivity $\pi(n,k)$. Further, for any trivalent reduced plabic graph $G$ with strand connectivity $\pi(n,k)$, there exists a fine zonotopal tiling of $Z(n,3)$ for which $G_k = G$.

If we erase all the edges in $\Sigma_k$ between the regions of the same color, we get what is called a *plabic tiling*. The planar dual of a plabic tiling is a bipartite plabic graph. The vertices of $\Sigma_k$ appear in the faces of $G_k$, and so we will refer to their labels as the *face labels* of $G_k$.

We would like to see how these trivalent plabic moves relate to the three-dimensional cyclic zonotopal flips. Galashin [5] observed that a zonotopal flip at height $k$ performs a square move in $G_k$, a white trivalent move in $G_{k-1}$, and a black trivalent move in $G_{k+1}$.
Lemma 3.5. For any zonotopal tiling $\Delta$, the available flips are in bijective correspondence with the available square moves in the plabic graphs $\{G_k\}_{k=1}^n$.

Proof. A proof does not completely appear in [5], so we will include one here. Take any available flip $S \in \binom{[n]}{4}$ for $\Delta$, say $S = \{a, b, c, d\}$ with $a < b < c < d$. Let $k := |S^+| + 2$. Then the intersection $\tau_{X_a} \cap \tau_{X_b} \cap \tau_{X_c} \cap \tau_{X_d} =: v$ is a vertex in $\Delta$ which is in the cross-section $\Sigma_k$. Further, the cross-sections of the tiles $\tau_{X_i}$ at level $k$ are triangles in $\Sigma_k$ which include $v$ as a vertex. The color of the triangle corresponding to $X_i$ is determined by the whether the plane $x = k$ cuts the tile at a lower or higher part, and so depends only on the value of $s_i$. Then the triangles from $X_a, X_c$ are of one color and $X_b, X_d$ have the other color, by Definition 2.4. Finally, the intersections $\tau_{X_a} \cap \tau_{X_b}, \tau_{X_b} \cap \tau_{X_c}, \tau_{X_c} \cap \tau_{X_d}, \tau_{X_d} \cap \tau_{X_a}$ all appear as edges connected to $v$ in $\Sigma_k$, because each of these tiles is a quadrilateral with two vertices at height $k$, one of which is $v$. Therefore $G_k$ has a square move pattern formed by the vertices in the four vertices from the four tiles. Performing this flip performs this square move and no other square moves in any other layer.

It now suffices to invert this map. That is, take any available square move in any layer $G_k$, and recover the unique flip which performs that square move. Well, the square move is formed by four triangles in $\Sigma_k$, whose five vertices, when considered as strings in $\{+,-\}^n$, agree in all but four coordinates. This can be seen by noting that all five strings have exactly $k$ ‘+’ symbols, and that when two vertices are adjacent they can only differ in two coordinates. These four coordinates $a < b < c < d$ form our set $S$, and if the flip corresponding to $S$ is available, then it must correspond to this square move in the map described in the previous paragraph. It then suffices to check that $S$ satisfies the conditions in Definition 2.4. Indeed, $S_a^+ = S_b^+ = S_c^+ = S_d^+$, because the vertices agree on all coordinates outside of $S$. Now, of the outer four vertices, two are white and two are black, so two of $a, b, c, d$ will have $s_i = 1$ and two will have $s_i = 0$. Moreover, these colors are oriented in a cyclically alternating fashion, and they also correspond to the signed subsets $X_a, X_b, X_c, X_d$ in a cyclic fashion. Therefore we must have $X_a = X_c \neq X_b = X_d$, so we can conclude that $S$ is an available flip in $\Delta$. \hfill $\square$

We would like to know when the other two plabic moves can be performed as well. A white or black trivalent move depends on the existence of a square move in a neighboring layer, so the following result about the relationship between the graphs $G_k$ is helpful.

Lemma 3.6 (Galashin [5]). Let $\Sigma_k$ be a colored and labeled triangulation for some tiling $\Delta$. Then $\Sigma_{k+1}$ is fixed up to the triangulation of the white regions and $\Sigma_{k-1}$ is fixed up to the triangulation of the black regions.

Proof. By Galashin’s [5] Corollary 4.4], the vertex labels of $\Sigma_{k+1}$ and $\Sigma_{k-1}$ are completely determined by $\Sigma_k$. The white triangles in $S_k$ cut a tile of $\Delta$ which is cut by a black triangle in $\Sigma_{k+1}$, and all black triangles in $\Sigma_{k+1}$ correspond to a white triangle in $\Sigma_k$. Similarly, the black triangles in $\Sigma_k$ give the white triangles in $\Sigma_{k-1}$. Therefore all the white and black regions are determined in both $\Sigma_{k+1}$ and $\Sigma_{k-1}$, and indeed all that is left is the triangulation of the white regions in $\Sigma_{k+1}$ and the black regions in $\Sigma_{k-1}$. \hfill $\square$

This allows us to define UP and DOWN operations on plabic graphs with cyclic strand connectivity.

Definition 3.7. Let $G$ be a trivalent plabic graph with strand connectivity $\pi(n,k)$, and let $\Delta$ be any fine zonotopal tiling of $Z(n,3)$ such that $\Sigma_k$ is dual to $G$. Then $\text{UP}(G,\Delta)$ is the plabic graph dual to $\Sigma_{k+1}$, and $\text{DOWN}(G,\Delta)$ is the plabic graph dual to $\Sigma_{k-1}$. If $\Delta$
is not specified, UP($G$) and DOWN($G$) can be computed with an arbitrary valid $\Delta$, and the corresponding plabic triangulations are determined up to white and black triangulation, respectively.

By Lemma 3.6, we can determine UP($G$) up to white trivalent moves, and DOWN($G$) up to black trivalent moves. In particular, the face labels are determined exactly. In the next section, we will extend the definition of UP and DOWN to plabic graphs with any strand connectivity, and use these operations to prove our main result. The same operations can be applied to plabic triangulations, or even plabic tilings (and non-trivalent plabic graphs) if an arbitrary triangulation is chosen.

4. Cycles in the Plabic Flip Graph

For a given decorated permutation $\pi$, any two trivalent reduced plabic graphs with connectivity $\pi$ can be related by a sequence of the moves (M1)–(M3). The flip graph $F_{\pi}$ is the graph whose vertices are trivalent reduced plabic graphs with connectivity $\pi$ and whose edges connect plabic graphs related by a move. Cycles in the flip graph correspond to sequences of moves which leave the plabic graph unchanged. We’re going to use the UP and DOWN operations to study these cycles; since we want our proof to apply to plabic graphs of any connectivity, we want to be able to include any plabic graph inside one with the cyclic connectivity. To do this we need to introduce some more terminology.

Definition 4.1 ([11, Definition 16.1]). A Grassmann necklace $I = (I_1, I_2, \ldots, I_n)$ is a sequence of subsets of $[n]$ of the same size such that for all $i$ there exists $j$ such that $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, where as an index $i$ is considered modulo $n$.

Grassmann necklaces are in bijection with decorated permutations via juggling patterns (see [7, Section 3]) of period $n$ and throws of height at most $n$, and all of these are in correspondence with positroids inside $\binom{[n]}{k}$ (originally shown directly in [11]). Let $\phi$ be the bijection sending necklaces to permutations, and call the size of any element of $\phi^{-1}(\pi)$ the helicity of $\pi$.

Definition 4.2. Suppose $I = (I_1, \ldots, I_n)$ is a Grassmann necklace such that $\phi(I)$ is not an identity decorated permutation. Define UP($I$) to be the necklace $(I_1 \cup I_{i(1)}, I_2 \cup I_{i(2)}, \ldots, I_n \cup I_{i(1)})$, where $i(j)$ is the next after $j$ index (in the cyclic order) such that $I_j \neq I_{i(j)}$. (One can check that this is indeed a Grassmann necklace.) Similarly, DOWN($I$) is the necklace $(I_1 \cap I_{i(1)}, I_2 \cap I_{i(2)}, \ldots, I_n \cap I_{i(1)})$. For decorated permutations $\pi$, define DOWN($\pi$) := $\phi$(DOWN($\phi^{-1}(\pi)$)) and UP($\pi$) := $\phi$(UP($\phi^{-1}(\pi)$)).

Now we state a useful result of Oh, Postnikov, and Speyer [9] (for our purposes it does not matter what positroids are, or what it means for a collection to be weakly separated).

Theorem 4.3 ([9, Theorems 1.3 and 1.5]). Suppose $\mathcal{M} \subset \binom{[n]}{k}$ is a positroid, $I$ is the corresponding Grassmann necklace, and $\pi := f(I)$ the corresponding decorated permutation. Then the maximal by inclusion weakly separated collections inside $\mathcal{M}$ all contain $I$ and are exactly the collections of face labels for the plabic graphs with connectivity $\pi$.

The positroid corresponding to the top cell $Gr^{>0}(n, k)$ is all of $\binom{[n]}{k}$ and the corresponding decorated permutation is $\pi(n, k)$. A plabic graph is completely determined by its face labels. Therefore plabic graphs for the cyclic permutations are the maximal by inclusion weakly
separated collections in \( \binom{n}{k} \). Then if \( G \) is any plabic graph, then its collection of face labels is a subset of those for some plabic graph \( G' \) for the cyclic permutation.

Since Grassmann necklaces and decorated permutations are in bijection, we can talk about the class of plabic graphs for a certain Grassmann necklace instead. If a plabic graph \( G \) has strand permutation \( \pi \), and the corresponding necklace is \( \mathcal{I} = \phi^{-1}(\pi) \), then the face labels of the perimeter regions of the graph form \( \mathcal{I} \) as well. To summarize: Theorem 4.3 implies that any plabic graph \( G \) with the strand permutation \( \pi \) of helicity \( k \) can be embedded as a subgraph into another plabic graph \( G' \) corresponding to the cyclic permutation \( \pi(n, k) \).

Let \( \Sigma_k \) be a plabic triangulation dual to \( G' \), constructed as in Section 3. The Grassmann necklace \( \mathcal{I} \), considered as a polygonal line in \( \Sigma_k \), encloses the triangulation \( \sigma \), which we call the plabic triangulation for \( G \). If \( G \) was trivalent, \( \sigma \) is defined unambiguously; otherwise, \( \sigma \) is defined up to triangulating some of its convex monochromatic regions, corresponding to the vertices of \( G \) of degree greater than three. We regard \( \sigma \) as a geometric realization of \( G \) in the plane \( \{(k,x_2,x_3) \in \mathbb{R}^3\} \). An equivalent realization of plabic graphs appeared in [10], but without its relation to zonotopal tilings.

Note that plabic triangulations as defined above might have degeneracies. For example, the plabic triangulation corresponding to the necklace \( (134, 234, 134, 145, 135) \) consists of a black triangle together with a hanging edge. If the corresponding decorated permutation is an identity then the triangulation consists of a single vertex; if the corresponding decorated permutation is a transposition then the triangulation is just an edge.

We can use the above remarks to generalize Definition 3.7.

**Definition 4.4.** Let \( G \) be a plabic graph with non-identity strand permutation \( \pi \) of helicity \( k \), and let \( \Delta \) be a fine zonotopal tiling of \( Z(n, 3) \) for which the dual \( G' \) of \( \Sigma_k \) contains \( G \) as a subgraph. Then define \( \text{UP}(G, \Delta) \) to be the subgraph of \( \text{UP}(G', \Delta) \) surrounded by \( \text{UP}(\phi^{-1}(\pi)) \), and \( \text{DOWN}(G, \Delta) \) to be the subgraph of \( \text{DOWN}(G', \Delta) \) surrounded by \( \text{DOWN}(\phi^{-1}(\pi)) \). If \( \Delta \) is not specified, \( \text{UP}(G) \) and \( \text{DOWN}(G) \) are determined up to white and black trivalent moves, respectively.

**Lemma 4.5.** Let \( \mathcal{I} = \phi(\pi) \) be a necklace such that \( \pi \) is not an identity and \( \text{DOWN}(\pi) \) is not an identity. Then the necklace \( \mathcal{J} := \text{UP}(\text{DOWN}(\mathcal{I})) \) is nested in \( \mathcal{I} \), when embedded geometrically in the plane.

**Proof.** Let \( \mathcal{I} = (I_1, \ldots, I_n) \) be such a necklace, let \( \mathcal{L} = (L_1, \ldots, L_n) \) be \( \text{DOWN}(\mathcal{I}) \), and let \( \mathcal{J} = (J_1, \ldots, J_n) \) be \( \text{UP}(\mathcal{L}) \). Say that \( \iota(j) \) denotes the next index cyclically after \( j \) such that \( I_j \neq I_{\iota(j)} \), and \( \lambda(j) \) denotes the next index cyclically after \( j \) such that \( L_j \neq I_{\lambda(j)} \). Note that \( \iota \) and \( \lambda \) are well-defined because \( \phi^{-1}(\mathcal{I}) \) and \( \phi^{-1}(\mathcal{L}) \) are both not an identity, so not all of the \( I_j \) and \( L_j \) are the same. Then by Definition 4.5, we have \( J_j = (I_j \cap I_{\iota(j)}) \cup (I_{\lambda(j)} \cap I_{\iota(\lambda(j))}) \).

Now, what is \( I_{\lambda(j)} \)? Well, \( \lambda(j) \) must come at least as late as \( \iota(j) \) cyclically, and moreover, \( \lambda(j) = \iota^m(j) \) for some \( m \geq 1 \). If \( \lambda(j) = \iota^m(j) \), then we claim that \( J_j = I_{\iota^m(j)} = I_{\lambda(j)} \). Indeed, \( J_j = (I_j \cap I_{\iota(j)}) \cup (I_{\iota^m(j)} \cap I_{\iota^{m+1}(j)}) \subseteq I_{\lambda(j)} \), since \( I_j \cap I_{\iota(j)} = L_j = I_{\iota^{m-1}(j)} \cap I_{\iota^m(j)} \subseteq I_{\lambda(j)} \) and \( I_{\lambda(j)} \cap I_{\iota(\lambda(j))} \subseteq I_{\lambda(j)} \). On the other hand, \( |J_j| = |I_{\lambda(j)}| \), so it follows that \( J_j = I_{\lambda(j)} \). We have now shown that \( J_j \in \{I_j, I_{\iota(j)}, I_{\iota^2(j)}, \ldots\} \) for all \( j \), which certainly implies that \( \mathcal{J} \) is nested in \( \mathcal{I} \).

We are now ready to prove our main result.
Theorem 4.6. Let $X_{\pi^*}$ be the 2-complex given by the flip graph of trivalent reduced plabic graphs with connectivity $\pi^*$, with the following 2-cells glued to it (cf. Figure 4):

- A quadrilateral, wherever there is a 4-cycle generated by two moves occurring in separate parts of a plabic graph;
- A pentagon, wherever there is a 5-cycle generated by five white or five black trivalent moves, such that all flips take place in a subgraph which forms a plabic graph with connectivity $\pi(5, 1)$ or $\pi(5, 4)$;
- A decagon, wherever there is a 10-cycle consisting of 5 plabic moves alternating with 5 white or 5 black trivalent moves, such that all flips in the cycle take place in a subgraph which forms a plabic graph with connectivity $\pi(5, 2)$ or $\pi(5, 3)$.

Then $X_{\pi^*}$ is simply connected.
Proof. We induct on the helicity of \( \pi' \), which we denote \( k \). The case \( k = 0 \) is trivial, since the only decorated permutation of helicity 0 is the identity with all elements white, there is just one plabic graph of this strand connectivity. The case \( k = 1 \) is not trivial but well-known. The vertices of \( X_\pi' \) are just triangulations of a convex polygon, the flips correspond to the switch of the diagonal in a quadrilateral, and the only non-trivial relation between flips is the 5-cycle of flips in a pentagon. This result follows from the original work of Stasheff [15], but in Appendix we include an independent proof of that, involving Theorem 2.6.

Now assume \( k \geq 2 \). Let \( \mathcal{I} \) the Grassmann necklace corresponding to \( \pi' \). In the rest of this proof, we use words “(trivalent) plabic graph” and “plabic triangulation” interchangeably, assuming the bijection between them.

Start with a loop \( L \) of plabic graphs with necklace \( \mathcal{I} = \phi(\pi') \). If all the moves along \( L \) are just white trivalent moves, then the result follows from the base case \( k = 1 \): All the moves are triangulation flips in a few convex white regions, and the contraction of \( L \) can be constructed as a sequence of independent contractions for each white region.

If \( \pi' \) is an identity decorated permutation, there is just one plabic graph of that strand connectivity, so the theorem follows trivially. If \( \pi' \) is not an identity, \( \text{DOWN}(\pi') \) is well-defined. If \( \text{DOWN}(\pi') \) happens to be an identity, it follows that \( \mathcal{I} \) encloses a white region, and this case was already discussed above. So we assume that \( \text{DOWN}(\pi') \) is not an identity, so that \( \mathcal{J} := \text{UP}(\text{DOWN}(\mathcal{I})) \) is well-defined.

First we define a loop \( \text{DOWN}(L) \) of plabic graphs within the necklace \( \text{DOWN}(\mathcal{I}) \), with strand permutation \( \text{DOWN}(\pi') \). For each graph \( G \) in \( L \), the graph \( \text{DOWN}(G) \) is determined up to the triangulation of black regions. Choose those triangulations in an arbitrary way (we keep the notation \( \text{DOWN}(G) \) for those plabic triangulations). For every edge \( G - G' \) in \( L \), the graphs \( \text{DOWN}(G) \) and \( \text{DOWN}(G') \) differ by (at most) one move corresponding to the move between \( G \) and \( G' \), plus maybe some black trivalent moves. Connect \( \text{DOWN}(G) \) and \( \text{DOWN}(G') \) by a chain of plabic graphs through those moves (the chain might be of length zero if the move \( G - G' \) was white trivalent and \( \text{DOWN}(G) = \text{DOWN}(G') \)). All the chains, over all edges \( G - G' \) of \( L \) can be incorporated into a loop of plabic graphs with necklace \( \text{DOWN}(\mathcal{J}) \). We call this loop \( \text{DOWN}(L) \). Note that this loop consists of more than one vertex, because \( L \) contained moves apart from white trivalent ones.

All those graphs in \( \text{DOWN}(L) \) have helicity \( k - 1 \), so we can apply the inductive hypothesis to contract it in the complex \( X_{\text{DOWN}(\pi')} \). In other words, there is a map from the two-dimensional disk to \( X_{\text{DOWN}(\pi')} \), whose restriction onto the boundary follows \( \text{DOWN}(L) \). The map can be chosen to be cellular, so with some abuse of language we treat it as a polyhedral surface \( D \), glued out of 2-cells of \( X_{\text{DOWN}(\pi')} \) (maybe with repetitions of vertices), and whose boundary is the loop \( \text{DOWN}(L) \). We would like to “lift” \( D \) back to the \( k \)-th level, that is, to construct a polyhedral disk \( \text{UP}(D) \) in \( X_\pi \), whose boundary would be \( L \).

The surface \( D \) consists of polygons \( P_i \). Each of them can be lifted separately in the following manner, which can be easily guessed if one looks at the cross-sections of the 10-cycle of the zonotopal tilings of \( Z(5,3) \), see Figure 4.

- If \( P_i \) is a pentagon consisting of five black trivalent moves, \( \text{UP}(P_i) \) is a degenerate polygon, consisting of a single graph, lying above all those five graphs. The triangulation of white regions can chosen in an arbitrary way.
- If \( P_i \) is a decagon of five black trivalent moves alternating with square moves, \( \text{UP}(P_i) \) is a pentagon of five black trivalent moves, corresponding to the square moves of \( P_i \).
The triangulation of white regions should be chosen to be the same for the five graphs of $UP(P_i)$, but it can be done in an arbitrary way.

- If $P_i$ is a decagon of five white trivalent moves alternating with square moves, then $UP(P_i)$ is a decagon of five square moves alternating with five black trivalent moves, corresponding to the moves of $P_i$. The triangulation of white regions should be chosen so that all the moves happen in a copy of $\pi(5,3)$-necklace, and outside of it the white regions are triangulated in the same (but arbitrary) manner.
- If $P_i$ is a pentagon of five white trivalent moves, then $UP(P_i)$ is a decagon of five white trivalent moves alternating with square moves, corresponding to the moves of $P_i$. The triangulation of white regions should be chosen so that all the moves happen in a copy of $\pi(5,2)$-necklace, and outside of it the white regions are triangulated in the same (but arbitrary) manner.
- If $P_i$ is a quadrilateral built of two commuting moves, then $UP(P_i)$ is either a quadrilateral, or an edge, or a vertex, depending on how many of those moves were white trivalent. The triangulation of white regions is again arbitrary but consistent.

Now we make an important adjustment to this lifting. All the graphs in the lifted polygons correspond to the necklace $J = UP(DOWN(I))$, which might not coincide with $I$. If it does not, use Lemma 4.5 to extend those graphs (in the same manner) so that they all correspond to the necklace $I$.

So far we lifted every $P_i$ separately. Now we glue them to one another and to $L$, using the gluing pattern of $D$. The following steps should be done over all edges and all vertices of $D$.

1. Let $e$ be an edge of $D$ not from its boundary, and let $P$, $Q$ be the two adjacent polygons.
   - If $e$ is a black trivalent move (so that it gets contracted to a vertex after lifting), then we connect the vertices corresponding to $e$ in $UP(P)$ and $UP(Q)$ by a chain of white trivalent moves.
   - If $e$ is a square move or a white trivalent move, there are edges corresponding to $e$ in both $UP(P)$ and $UP(Q)$, call them $p_1 - p_2$ and $q_1 - q_2$, correspondingly. Note that $p_1$ and $q_1$ differ by some white trivalent moves, and $p_2$ and $q_2$ differ by the same exact white trivalent moves. Let $p_1 - a_1 - \ldots - z_1 - q_1$ and $p_2 - a_2 - \ldots - z_2 - q_2$ be chains of white trivalent moves such that every pair $a_1 - a_2, \ldots, z_1 - z_2$ is related by the same move as the pairs $p_1 - p_2$ and $q_1 - q_2$ (this move arises from lifting $e$). This way we connect the edges $p_1 - p_2$ and $q_1 - q_2$ by a sequence of quadrilaterals.

2. Let $e$ be a boundary edge of $D$, which belongs to a polygon $P$. We glue $UP(P)$ to $L$ following the same scheme as in the previous item. If $e$ is a black trivalent move, we connect the vertices corresponding to $e$ in $UP(P)$ and $L$ by a chain of white trivalent moves. If $e$ is a square move or a white trivalent move, there are edges corresponding to $e$ in both $UP(P)$ and $L$, and we connect them by a sequence of quadrilaterals in the same way as above.

3. Let $e$ be a boundary edge of $D$ not adjacent to any of the polygons. Then $e$ is bypassed by $DOWN(L)$ an even number of times, and the occurrences of $UP(e)$ in $L$ could be broken into pairs. For each pair, we connect the two corresponding vertices in $L$ by a chain of white trivalent moves (is $e$ was black trivalent), or the two corresponding edges in $L$ by a strip of quadrilaterals (is $e$ was black trivalent), just as above.
(4) Let \( v \) be an internal (not lying on the boundary) vertex of \( D \). Let \( Q_1, \ldots, Q_m \) be the polygons in \( D \) sharing a common vertex \( v \), indexed in the order they follow around \( v \), so that \( Q_i \) and \( Q_{i+1} \) (cyclic indexing) share an edge. We already connected the polygons \( \text{UP}(Q_i) \) cyclically by strips of quadrilaterals and/or chains of white trivalent moves. Those strips/chains bound a loop whose all vertices match with \( \text{UP}(v) \), and differ only by white trivalent moves. Such a loop can be contracted as it follows from the base case \( k = 1 \). That is, we can glue a disk consisting of pentagons and quadrilaterals to this loop.

(5) Let \( v \) be a boundary vertex of \( D \). Write down the sequence \( Q_1, \ldots, Q_m \) of the adjacent to \( v \) polygons in the order they follow around \( v \), including \( L \) in this sequence every time the adjacent region is the outside of \( D \). Then proceed the same way as if \( v \) were internal, by filling in a loop of white trivalent moves.

The construction above forms a polyhedral disk \( \text{UP}(D) \) whose boundary is \( L \), which finishes the proof.

We can consider any two reduced trivalent plabic graphs to be equivalent if they can be related by only white and black trivalent moves. Then there is a square flip graph, whose vertices are equivalence classes of plabic graphs for each connectivity \( \pi \), and whose edges connect equivalence classes of graphs which have a pair of representative elements related by a square move. By Theorem 3.3, the square flip graph is connected for every decorated permutation \( \pi' \). Our result can be restricted to the square flip graph as follows.

**Corollary 4.7.** Let \( Y_{\pi'} \) be the 2-complex given by the square flip graph for plabic graphs with connectivity \( \pi' \), with the following 2-cells glued to it:

- A quadrilateral, wherever there is a 4-cycle generated by two square moves occurring in separate parts of a plabic graph;
- A pentagon, wherever there is a 5-cycle generated by five square moves which take place in a subgraph which forms a plabic graph with connectivity \( \pi(5,2) \) or \( \pi(5,3) \).

Then \( Y_{\pi'} \) is simply connected.

**Proof.** Any loop \( \gamma \) in \( Y_{\pi'} \) can be extended to a loop \( \gamma' \) in \( X_{\pi'} \) by adding the necessary extra white and black trivalent moves. Contract \( \gamma' \) to a point step-by-step by moving it across the 2-cells in \( X_{\pi'} \). Each 2-cell in \( X_{\pi'} \) corresponds to either a point or a 2-cell in \( Y_{\pi'} \), so \( \gamma \) may also be continuously deformed while maintaining the correspondence between \( \gamma \) and \( \gamma' \). Then once \( \gamma' \) has been deformed to a point, so has \( \gamma \). Therefore \( Y_{\pi'} \) is simply connected.

**5. Triple Crossing Diagrams**

Dylan Thurston [17] introduced triple crossing diagrams as a generalization of the domino tilings and their flip operation. Just as the space of domino tilings is flip-connected [18], so is the space of (minimal) triple crossing diagrams with a given connectivity [17]. We will consider only what Thurston [17] calls minimal triple crossing diagrams, defined as when introduced by Postnikov to study perfect orientations of plabic graphs [11].

**Definition 5.1.** Consider a disk with boundary vertices labeled \( b_1, b'_1, \ldots, b_n, b'_n \) in clockwise order. A triple crossing diagram with connectivity (strand permutation) \( \pi \in S_n \) consists of \( n \) oriented strands drawn inside the disk which start at \( b_i \) and end at \( b'_{\pi(i)} \) for each \( i \in [n] \), satisfying the following properties
(1) Wherever two strands intersect, exactly three distinct strands meet in a *triple crossing*.

(2) When considered in cyclic order, the orientation of the six rays from any triple crossing alternates.

(3) The diagram contains no *bad double crossings*, defined as when two distinct strands both arrive at triple crossing $c_1$ followed by triple crossing $c_2$.

It follows from [17, Theorem 7] that this definition is equivalent to Thurston’s definition for minimal triple crossing diagrams.

Similar to plabic graphs, triple crossing diagrams on $n$ strands have a connectivity $\pi \in S_n$ given by the final positions of the strands. There is also a notion of flip in a triple crossing diagram, the $2 \leftrightarrow 2$ move, shown in Figure 5. Dylan Thurston [17, Theorem 5] proved that all minimal triple crossing diagrams with the same connectivity can be related by a series of $2 \leftrightarrow 2$ moves.

**Figure 5.** A $2 \leftrightarrow 2$ move (shaded) in a triple crossing diagram

Postnikov gave the following correspondence between triple crossing diagrams and plabic graphs. For any triple crossing diagram $D$, the plabic graph $\phi(D)$ has white vertices corresponding to triple crossings in $D$, black vertices corresponding to regions bounded by counterclockwise-oriented strands that aren’t in the middle of a possible $2 \leftrightarrow 2$ move, and edges corresponding to counterclockwise regions bordered by triple crossings and to triple crossing which could be involved in a $2 \leftrightarrow 2$ move together.

**Lemma 5.2 ([11, Lemma 14.4]).** The map $\phi$ described above gives a bijection between triple crossing diagrams with strand connectivity $\pi$ and reduced plabic graphs for the connectivity $\pi$ (fixed points undecorated) with all white vertices trivalent and no edges with both endpoints black.

Such plabic graphs can be considered to be trivalent plabic graphs where the configuration of the edges between black vertices is arbitrary (choose any sequence of uncontraction moves
on the black vertices with degree more than three). We observe that the flips in the two contexts correspond nicely

**Lemma 5.3.** Let $D$ and $D'$ be triple crossing diagrams related by a single $2 \leftrightarrow 2$ move in $D$. Then $\phi(D)$ and $\phi(D')$ are related by a square move and several black contraction/uncontraction moves if the interior region of the $2 \leftrightarrow 2$ move was oriented clockwise, otherwise they are related by a single white trivalent move. Conversely, if $G$ and $G'$ are reduced plabic graphs with all white vertices trivalent, and no edges with both endpoints black which are related by a single white trivalent move or a square move and several black contraction/uncontraction moves, then $\phi^{-1}(G)$ and $\phi^{-1}(G')$ are related by a single $2 \leftrightarrow 2$ move.

**Proof.** Examine how $\phi$ transforms $2 \leftrightarrow 2$ moves and $\phi^{-1}$ transforms plabic moves locally. \[\square\]

Dylan Thurston [17] conjectured the following, which we now prove as a theorem

**Theorem 5.4.** Let $T_\pi$ be the 2-complex given by the flip graph of triple crossing diagrams with connectivity $\pi$, with the following 2-cells glued to it (cf. [17, Figure 4])

- A quadrilateral, wherever two flips are commuting in different parts of the diagram;
- A pentagon, wherever there is a 5-cycle taking place in a subset of the diagram which is a triple crossing diagram with connectivity $\pi(5, 1)$ or $\pi(5, 3)$;
- A decagon, wherever there is a 10-cycle taking place in a subset of the diagram which is a triple crossing diagram with connectivity $\pi(5, 2)$.

Then $T_\pi$ is simply connected for all permutations $\pi$.

**Proof.** Let $\gamma$ be a cycle $D_1, D_2, \ldots, D_{m+1} = D_1$ of triple diagrams with connectivity $\pi$ related by $2 \leftrightarrow 2$ moves. Then let $\phi(\gamma)$ be the cycle $G_1, G_2, \ldots, G_{M+1} = G_1$ of plabic graphs in $X_\pi$ constructed by Lemma 5.3 which contains $\phi(D_1), \phi(D_2), \ldots, \phi(D_{m+1}) = \phi(D_1)$ in that order. By Theorem 4.6 $\phi(\gamma)$ can be continuously deformed to a point in $X_\pi$ by moving it across the cells in $X_\pi$. By construction of $T_\pi$ and Lemma 5.3 the cells in $X_\pi$ correspond to either points or cells in $T_\pi$. In particular, if $\phi(\gamma)'$ is a deformation of $\phi(\gamma)$ from moving across a cell in $X_\pi$, then there is a (possibly trivial) cell in $T_\pi$ which $\gamma$ can be moved across to create $\gamma'$ such that $\phi(\gamma') = \phi(\gamma)'$. Then after deforming $\phi(\gamma)$ to a point in $X_\pi$ while doing the corresponding deformations to $\gamma$, the cycle $\gamma$ must also have been deformed to a point. Therefore $T_\pi$ is simply connected. \[\square\]

6. **Generalized Baues Problem**

Recall the model example of a flip graph, whose vertices are the triangulations of a convex polygon, and whose edges are the flips between triangulations, defined by flipping a diagonal in any quadrilateral. The relations between flips are understood well in this case (see Appendix below). One can also consider the relations between relations, and so on, and all those higher relations lead to the classical construction of Stasheff’s associahedron, which captures in a sense the “topology of triangulations”. A different way to study this topology comes from considering the map $p : C(n, d) \to C(n, 2)$ projecting the $d$-dimensional cyclic polytope onto the convex $n$-gon. The triangulations of $C(n, 2)$ arise as the homeomorphic images of certain two-dimensional subcomplexes of $C(n, d)$, and the associahedron can be recovered in this interpretation as the Minkowski average of the fibers of the map $p$, as introduced in [3]. We can go further and consider coarser subdivisions of $C(n, 2)$ as well, in a sense induced by the map $p$, then form a poset out of them, and ask questions about its topology. The generalized Baues problem is such a question. It was posed in [2], for arbitrary linear maps
of polytopes, as a generalization of original Baues’s question from [1], where the target was one-dimensional. Even though in general the GBP-conjecture is false [14], there are positive results in some special cases. The GBP for \( p : C(n, d) \rightarrow C(n, 2) \) was solved affirmatively in [13]; the solution for the GBP for the zonotopal tilings, arising from the projection of a hypercube, follows from the results of [16].

In this section, we interpret Theorem 4.6 in terms of the GBP for the cyclic projection of a hypersimplex. We pose the relevant special case of the GBP, following [12]. The hypersimplex \( \Delta_{kn} \) is the \( k \)-th cross-section of the \( n \)-dimensional hypercube:

\[
\Delta_{kn} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \sum x_i = k \right\}.
\]

If we affinely project the \( n \)-hypercube onto \( \mathbb{R}^3 \) so that the image is the cyclic zonotope \( Z(n, 3) \) (this is achieved by sending the \( i \)-th basis vector to \( v_i \in \mathbb{R}^3 \), where the \( v_i \) are as in Definition 2.1) then the hypersimplex \( \Delta_{kn} \) goes to the \( k \)-th horizontal layer of \( Z(n, 3) \), which we call \( Q_{kn} \):

\[
Q_{kn} = Z(n, 3) \cap \{(k, x_2, x_3) \in \mathbb{R}^3\}.
\]

The affine projection \( \pi : \Delta_{kn} \rightarrow Q_{kn} \) gives rise to a family of \( \pi \)-induced subdivisions of \( Q_{kn} \). By definition, a subdivision \( \Sigma \) of \( Q_{kn} \) into convex polygons is \( \pi \)-induced if there is a polyhedral subcomplex \( \mathcal{S} \) of \( \Delta_{kn} \) such that \( \pi \) establishes an isomorphism between \( \mathcal{S} \) and \( \Sigma \), as polyhedral complexes. The \( \pi \)-induced subdivisions form a poset, called the Baues poset and denoted by \( \omega(k, n, 2) \) in [12], whose order relation comes from the inclusion relation on the subcomplexes \( \mathcal{S} \) of \( \Delta_{kn} \). The minimal elements of \( \omega(k, n, 2) \) correspond to the finest \( \pi \)-induced subdivisions, arising from the subcomplexes of the 2-skeleton of \( \Delta_{kn} \) that get projected down onto \( Q_{kn} \) homeomorphically. We readily recognize those subdivisions as the tilings dual to the trivalent plabic graphs of connectivity \( \pi(n, k) \) [5]. The maximal element \( \hat{1} \) of \( \omega(k, n, 2) \) is unique and corresponds to the trivial subdivision of \( Q_{kn} \), consisting of a single 2-face. The GBP in this case asks the following.

**Question 6.1** (Special case of [12] Problem 10.3). *Does the poset \( \omega(k, n, 2) - \hat{1} \) (the Baues poset with the maximal element removed) have the homotopy type of the \((n - 4)\)-dimensional sphere?*

Another part of [12] Problem 10.3 asks whether the poset \( \omega(k, n, 2) \) coincides with another poset of interest, arising from the zonotopal tilings of \( Z(n, 3) \). Every zonotopal tiling \( \Delta \) (not necessarily a fine one) induces a tiling of \( \Delta_{kn} \) in its \( k \)-th horizontal section. Those tilings of \( \Delta_{kn} \), ordered by refinement, form the poset of lifting subdivisions of \( \Delta_{kn} \), denoted as \( \omega_{lift}(k, n, 2) \). After Theorem 11.7 in [12], Postnikov asks if \( \omega_{lift}(k, n, 2) \) coincides with \( \omega(k, n, 2) \). We note that this is indeed so.

**Lemma 6.2.**  \( \omega(k, n, 2) = \omega_{lift}(k, n, 2) \).

**Proof.** Fix a Grassmannian graph \( G \) with helicity \( k \) on \( n \) boundary vertices. It suffices to show that it is realized as the \( k \)-th cross-section of some (not fine) tiling of \( Z(n, 3) \). Galashin [4] proved this for \( G \) plabic, and we proved it for \( G \) almost-plabic previously [reference]. Say \( v_i \) are the vertices of \( G \), each with some helicity \( k_i \) and degree \( d_i \). Let \( G_i \) be a plabic graph with connectivity \( \pi(d_i, k_i) \) for each \( i \). Then we can form a plabic graph \( G' \) by replacing each vertex \( v_i \) in \( G \) with \( G_i \). We will form a fine tiling \( \Delta \) of \( Z(n, 3) \) layer-by-layer. The \( k \)-th layer should be the plabic graph \( G' \). By Lemma 3.6 the adjacent layers are determined up to triangulation by one color. Each subgraph \( G_i \) is the \( k_i \)-th cross-section of some fine
tiling of $Z(d_i, 3)$. All of the vertices of a this copy of $Z(d_i, 3)$ are in the adjacent layers, but some of the edges may depend on the triangulation. Triangulate the adjacent layers in a way such that all of the tilings of the $Z(d_i, 3)$ are matched. Continue extending up and down in this manner, triangulating so that $\Delta$ agrees with the tilings of the $Z(d_i, 3)$, and otherwise triangulating arbitrarily. The resulting tiling $\Delta$ respects the boundaries of the $Z(d_i, 3)$. Then the sub-tilings of these copies of $Z(d_i, 3)$ can all be replaced with individual copies of $Z(d_i, 3)$. The new tiling $\Delta'$ has the Grassmannian graph $G$ as its $k$-th cross-section; the effect of replacing the tilings of $Z(d_i, 3)$ with single tiles in the $k$-th cross-section is to contract the graphs $G_i$ to the single vertices $v_i$. □

Having Lemma 6.2 at our disposal, we interpret Theorem 4.6 in terms of the GBP to give an evidence in favor of the affirmative answer to Question 6.1.

**Theorem 6.3.** If $n \geq 6$, the poset $\omega(k, n, 2) - \hat{1}$ is simply connected.

**Proof.** By Lemma 6.2 we can work with the zonotopal sections instead of Grassmannian graphs.

The poset of Grassmannian graphs is not graded, but we still can introduce the rank function as the length of the longest chain of covering relations finishing at a plabic trivalent graph. Then the rank 0 graphs are just plabic trivalent graphs; the rank 1 graphs are almost plabic in the notation of [12] and correspond to plabic moves; the rank 2 graphs correspond to the 2-cells of the complex $X_\pi$ in Theorem 4.6.

Consider the sub-poset $\omega_{\leq 2}$ of $\omega(k, n, 2) - \hat{1}$ consisting of those three types of graphs of rank at most 2. The condition $n \geq 6$ implies that the excluded singleton-graph $\hat{1}$ was of rank higher than 2.

The nerve (or the order complex) of $\omega_{\leq 2}$ is isomorphic to the barycentric subdivision of $X_{\pi}$, as simplicial complexes. This is done by a straightforward identification of the low rank graphs with the cells of $X_{\pi}$, as above. By Theorem 4.6, $\omega_{\leq 2}$ is simply connected.

Consider the 2-skeleton $\omega^{(2)}$ of the nerve of $\omega(k, n, 2) - \hat{1}$. It contains $\omega_{\leq 2}$. We show that $\omega^{(2)}$ is simply connected. Let $L$ be a (simplicial) loop in $\omega^{(2)}$. We would like to pull it down (in the sense of rank) along triangles of $\omega^{(2)}$, to a loop in $\omega_{\leq 2}$, and then contract it there. We start by breaking $L$ into intervals starting and finishing at local minima (in the sense of rank). For each local minimum $G_i$ pull it down to a rank 0 graph $\hat{G}_i$, in an arbitrary fashion. For each interval $G_1 - \ldots - G_m$ between two consecutive minima $G_1$ and $G_m$, find its maximum $G_i$. Connect the graphs $\hat{G}_1$ and $\hat{G}_m$ inside the poset $\{G_i, \text{ of rank } 0 \text{ or } 1\}$ by a chain $\hat{G}_1 = H_1 - \ldots - H_t = \hat{G}_m$ (it can be done by [12] Proposition 11.4). Note that the chain $G_1 - \ldots - G_m$ can be pulled down to the chain $\hat{G}_1 - H_1 - \ldots - H_t - \hat{G}_m$ along the triangles $G_1 G_2 \hat{G}_1, \ldots, G_{i-1} G_i G_1, H_1 G_1 H_2, \ldots, H_{t-1} G_i H_t, \hat{G}_m G_i G_{i+1}, \ldots, \hat{G}_m G_{m-1} G_m$. Repeating this over all intervals between the local minima, we homotope $L$ to a loop inside $\omega^{(2)}$. But $\omega^{(2)}$ is simply connected, so $\omega(k, n, 2) - \hat{1}$ is simply connected as well. □

7. Open Questions

(1) A (single) wiring diagram is a way to write the completely inverted permutation $w_0$ as a product of $n \choose 2$ elementary transpositions $s_i$ in $S_n$, and the flip (or mutation) operation is the Coxeter move, $s_is_{i+1}s_i \leftrightarrow s_{i+1}s_is_{i+1}$, which can connect any two wiring diagrams. There is
one-to-one correspondence between wiring diagrams and the rhombus tilings of the regular 2n-gon, so the proof of Henriques and Speyer in [6] applies to wiring diagrams as well and shows that simple cycles of wiring diagram mutations are of length 4 and 8.

We can ask the same question for double wiring diagrams, introduced in [4], which are formed by two interlaced single wiring diagrams. The moves shown in Figure 6 can relate any two double wiring diagrams, and are akin to the square move in plabic graphs. Any double wiring diagram on n strands can be realized as a plabic graph with strand connectivity π(2n,n), but the converse is false for n > 3. Corollary 3.5 suggests that there is a simply connected complex of double wiring diagrams with 2-cells glued along 4- and 5-cycles. But because not all plabic graphs are double wiring diagrams, we can also see 8-cycles like the ones in single wiring diagrams, as can be seen in the flip graph for n = 4. We conjecture that filling in these 8-cycles is enough to make the complex simply connected.

Figure 6. Double wiring diagram flips, with red coxeter move omitted

(2) In the appendix below we reprove a major case of Theorem 4.6 by extending plabic graphs to zonotopal tilings, and relating the cycles of plabic moves to the cycles of zonotopal flips. Can one prove the general case of Theorem 4.6 in a similar way, by relating general plabic graphs to the zonotopal tilings of some tileable subdomains of Z(n,3)? The uniqueness of maximal/minimal tilings, as in Theorem 2.5, no longer holds for general tileable domains.

(3) When defining $X_\pi$ in Section 4, we glued certain 2-cells, corresponding to the relations between plabic moves, to the flip graph of plabic configurations. It seems that the complexes $X_\pi(i), 1 \leq i \leq 5$, are just 2-spheres. One can proceed and glue 3-cells to $X_\pi$ along the following 2-spheres:

- every copy of $X_\pi(i), 1 \leq i \leq 5$, occurring inside $X_\pi$;
- every copy of $\partial (X_\pi(5,i) \times X_\pi(4,j))$, $1 \leq i \leq 4, 1 \leq j \leq 3$;
- every copy of $\partial (X_\pi(4,i) \times X_\pi(4,j) \times X_\pi(4,\ell))$, $1 \leq i, j, \ell \leq 3$.

If the 3-complex gotten from $X_\pi(7,i), 1 \leq i \leq 6$, happens to be a 3-sphere (which we expect), we can proceed by gluing 4-cells in a similar fashion, and so on.

**Question 7.1.** Is the n-complex, obtained from $X_\pi(n+4,i), 1 \leq i \leq n + 3$, by gluing cells up to dimension n as above, homeomorphic to the n-sphere?

This question seems to be closely related to the generalized Baues problem 6.1.
(4) Is there any nice combinatorial interpretation for the cross-sections of higher zonotopes $Z(n, \geq 4)$? An analogue of plabic triangulation in the 3-dimensional sections of $Z(n, 4)$ is a certain type of tesselations by tetrahedra and octahedra, while the counterpart of the square move switches between two specific subdivisions of a certain 9-vertex polytope. Methods used in this paper allow to describe the cycles in the flip graph of such configurations.

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9. Appendix

The proof of Theorem 4.6 in Section 4 relies on the base case, which can be reformulated as follows. Consider the polyhedral 2-complex $U_n$, defined as follows

- The vertices of $U_n$ are the triangulations of a convex $n$-gon.
- The edges of $U_n$ are the flips between triangulations differing inside a quadrilateral.
- Every five triangulations differing inside a pentagon give rise to a 5-cycle of flips, along which we glue a 2-cell in $U_n$.
- Every two commuting triangulations occurring in non-overlapping quadrilaterals give rise to a 4-cycle of flips, along which we glue a 2-cell in $U_n$.

Fact 9.1. The complex $U_n$ is simply connected.

This is equivalent to the case $k = 1$ of Theorem 4.6. Fact 9.1 is well-known, and goes back at least to Stasheff’s work [15], where he constructs his celebrated associahedron. The formulation as above (but in greater generality) could be found, for example, in [8, Theorem 7]. An independent proof of this fact follows from the argument below.

We give a different proof of Theorem 4.6 for the case of cyclic connectivity $\pi^c = \pi(n, k)$. An advantage of this proof that it doesn’t rely on the base case $k = 1$, unlike the one in Section 4.

Our strategy is to consider a cycle of plabic moves as a cycle of zonotopal tiling flips, and apply theorem 2.6. We would perform each plabic move by doing flips in tiling containing the graph as a cross-section. Unfortunately the appropriate flip isn’t always available, but luckily we can set it up without changing the relevant layer. Let $\Delta$ be a zonotopal tiling and $G_k$ be a plabic graph formed by a cross-section of $\Delta$.

Lemma 9.2. Suppose $M$ is a possible black (resp. white) trivalent move in $G_k$. Then there exists a finite sequence of flips $(S_1, S_2, \ldots, S_m)$ in $\Delta$, such that $G_\ell$ is unchanged by each of the first $m - 1$ mutations for any $\ell$ at least (resp. at most) $k$, but the move $M$ occurs on the last mutation.

Proof. Complementing all of the labels of the vertices doesn’t change the structure of the available flips but does change the colors of all of the regions, so it suffices to prove the result when $M$ is a black trivalent move. We proceed by induction on $k$. When $k \leq 2$, there are no legal black trivalent moves, so the claim holds vacuously. Now, the black trivalent move corresponds to two black triangles in $\Sigma_k$, which by Lemma 3.6 creates two white triangles in $\Sigma_{k-1}$, which are forced to border two black regions. If the black regions are triangulated such
that a square move is legal using the white triangles, then perform the corresponding flip and we’re done. Otherwise, there exists a sequence of triangulation flips in the black regions which would make the square move legal. By the inductive hypothesis, each of these flips can be done through a finite sequence of mutations, each of which (except the last) leave $G_\ell$ unchanged for all $\ell \geq k - 1$. The last flip in each sequence performs a black trivalent move in $\Sigma_{k+1}$, so also leaves $S_k$ unchanged. Therefore we can set up the square move in $\Sigma_{k-1}$ without changing $\Sigma_k$ at all, so the induction is complete. $\square$

In order to properly embed cycles as cyclic zonotopal flips, we also need to match up tilings which share a cross-section. Once we’ve done that, we are ready to prove the result for cyclic connectivities.

**Lemma 9.3.** Let $\Delta$ and $\Delta'$ be two fine zonotopal tilings of $Z(n,3)$ which are identical on $G_k$ for some fixed $k$. Then there exists a series of flips, none of which alter $G_k$, which transform $\Delta$ into $\Delta'$.

**Proof.** It suffices to find such a sequence of moves which make $\Delta$ match $\Delta'$ on $\Sigma_{k+1}$ (without ever changing $G_k$ or any lower layer) and $\Sigma_{k-1}$ (without ever changing $G_k$ or any higher layer). Once this is done we can recursively match all of the layers to transform $\Delta$ into $\Delta'$. By Lemma 3.6, $\Delta$ and $\Delta'$ already agree up to white (resp. black) triangulation on $\Sigma_{k+1}$ (resp. $\Sigma_{k-1}$). By the flip connectivity of triangulations, there exists a sequence of white (resp. black) trivalent flips in $G_{k+1}$ (resp. $G_{k-1}$) which transform $\Delta$ to completely match $\Delta'$ on $\Sigma_{k+1}$ (resp. $\Sigma_{k-1}$). By Lemma 9.2, for each of these flips there exists a finite sequence of flips which perform only this move in $G_{k+1}$ (resp. $G_{k-1}$), none of which change $G_{\ell}$ for any $\ell \leq k$ (resp. $\ell \geq k$). Therefore all of these triangulation moves can be performed without ever changing $G_k$ or any lower (resp. higher) layer, as desired. $\square$

**Proof of Theorem 4.6 for cyclic permutations.** Fix any cyclic permutation $\pi = \pi(n,k)$. Let $\gamma = M_1 M_2 \cdots M_m$ be a loop in $X_n$ connecting plabic graphs $G_k, G_k^2, \ldots, G_k^{m+1} = G_k^1$ with connectivity $\pi(n,k)$. We will construct a loop $Z(\gamma)$ in $Z_{n,3}$ such that the flips in $Z(\gamma)$ cause exactly the moves $M_1, M_2, \ldots, M_m$ to occur in $G_k$, in that order. For $i = 0, 1, \ldots, m-1$, there exists $\Delta_i$ whose cross-section at height $k$ is exactly $G_k^{i+1}$. By Lemma 3.5 (if $M_{i+1}$ is a square move) and Lemma 9.2 (if $M_{i+1}$ is a black or white trivalent move), there exists a sequence of moves starting from $\Delta_i$ which performs only the move $M_{i+1}$ in $G_k^{i+1}$. The resulting tiling $\Delta_i'$ from this sequence of moves is identical to $\Delta_{i+1}$ at height $k$, so by Lemma 9.3, there exists another sequence flips, none of which cause a move in $G_k$, which turns $\Delta_i'$ into $\Delta_{i+1}$, where $i+1$ is considered modulo $m$. Concatenating all these sequences of moves results in our loop $Z(\gamma)$ with the desired properties.

By Theorem 2.6, the loop $Z(\gamma)$ is contractible to a point by moving it across the 2-cells in $Z_{n,3}$. We will show that these 2-cells correspond to 2-cells in $X_{\pi(n,k)}$ nicely, so that we can also contract $\gamma$ to a point.

The quadrilaterals in $Z_{n,3}$ are formed by two commuting flips in $Z(n,k)$, which result in either two moves in separate parts of $G_k$ (a quadrilateral in $X_{\pi(n,k)}$), one move being performed twice in $G_k$ (an edge in $X_{\pi(n,k)}$), or no moves in $G_k$ (a point in $X_{\pi(n,k)}$). In all cases, when $Z(\gamma)$ is moved across a quadrilateral, the image of the quadrilateral in $X_{\pi(n,k)}$ is a vertex, edge, or 2-cell which $\gamma$ can also be moved across.

The only other 2-cells in $Z_{n,3}$ are decagons whose vertices correspond to the ten refinements of an instance of $Z(5,3)$ inside $Z(n,3)$. Depending on where the plane $x = k$ intersects the
copy of $Z(5,3)$, one of five things could happen in $G_k$ as the ten flips in the decagon are performed (see Figure 4).

1. If $x = k$ does not intersect the copy of $Z(5,3)$ or only touches the top or bottom vertex, no moves occur in $G_k$ and the image of the decagon in $X_{\pi(n,k)}$ is a vertex.

2. If $x = k$ intersects the copy of $Z(5,3)$ at relative height 1, then five white trivalent moves occur in a subgraph of $G_k$ with connectivity $\pi(5,1)$. The image of the decagon in $X_{\pi(n,k)}$ is a pentagon.

3. If $x = k$ intersects the copy of $Z(5,3)$ at relative height 2, then five square moves and five white trivalent moves occur in a subgraph of $G_k$ with connectivity $\pi(5,2)$. The image of the decagon in $X_{\pi(n,k)}$ is another decagon.

4. If $x = k$ intersects the copy of $Z(5,3)$ at relative height 3, then five square moves and five black trivalent moves occur in a subgraph of $G_k$ with connectivity $\pi(5,3)$. The image of the decagon in $X_{\pi(n,k)}$ is another decagon.

5. If $x = k$ intersects the copy of $Z(5,3)$ at relative height 4, then five black trivalent moves occur in a subgraph of $G_k$ with connectivity $\pi(5,4)$. The image of the decagon in $X_{\pi(n,k)}$ is a pentagon.

In all cases, when $Z(\gamma)$ is moved across the decagon, the image of the decagon is a vertex or 2-cell in $X_{\pi(n,k)}$ which $\gamma$ can be moved across.

Finally, let $Z(\gamma)'$ be a deformation of $Z(\gamma)$ by moving it across a 2-cell. We have considered all possible 2-cells in $Z_{n,3}$ and shown that there always exists a cell in $X_{\pi(n,k)}$ which $\gamma$ can be moved across to create $\gamma'$ such that $Z(\gamma') = Z(\gamma)'$. Therefore by contracting $Z(\gamma)$ to a point in $Z_{n,3}$ step-by-step while adjusting $\gamma$ along the way, $\gamma$ is also contracted to a point. We can conclude that $X_{\pi(n,k)}$ is simply connected. \(\square\)

This method of proof cannot be straightforwardly applied to the more general statement of Theorem 4.6. Although a loop of flips for any connectivity $\pi'$ could still be included in a loop of zonotopal tiling flips, the contraction of the loop might not stay inside the graph with connectivity $\pi'$ (see the second question in Section 7).
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