A remark on boundary estimates on unbounded $Z(q)$ domains in $\mathbb{C}^n$

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ABSTRACT
The goal of this note is to explore the relationship between the Folland–Kohn basic estimate and the $Z(q)$-condition. In particular, on unbounded domains, we prove that the Folland–Kohn basic estimate is equivalent to a uniform $Z(q)$ condition. As a corollary, we observe that despite the Siegel upper half space being strictly pseudoconvex and biholomorphic to the unit ball, it fails to satisfy uniform strict pseudoconvexity and hence the Folland–Kohn basic estimate fails.

1. Introduction
Let $\bar{\partial}$ denote the Cauchy–Riemann operator acting on $(0, q)$-forms and $\bar{\partial}^\ast$ denote its Hilbert space adjoint with respect to a weighted $L^2$ inner product $(f, g)_t = \int_{\Omega} f\bar{g}e^{-t|z|^2} \, dV$. The $L^2$ method for solving the inhomogeneous Cauchy–Riemann equation $\bar{\partial} u = f$ involves constructing a solution operator from the solution operator to the $\bar{\partial}$-Neumann problem $\bar{\partial} \bar{\partial}^\ast + \bar{\partial}^\ast \bar{\partial} u = f$. Unfortunately, the boundary condition associated to $\text{Dom}(\bar{\partial}^\ast)$ is non-coercive, so the solution operator to the $\bar{\partial}$-Neumann problem fails to be elliptic. Kohn’s celebrated solution [1,2] was to show that on a reasonable class of domains, the solution operator gained one half of the derivatives that would be expected in the elliptic case, and this subelliptic estimate was sufficient to obtain existence and regularity for the inhomogeneous Cauchy–Riemann equations.

The key tool in Kohn’s proof is to show that on a bounded strictly pseudoconvex domain, the weighted boundary $L^2$ norm of a $(0, q)$-form $f$ can be estimated by the weighted $L^2$ norms of $\bar{\partial} f$, $\bar{\partial} \bar{\partial}^\ast f$. We call such an estimate a basic estimate in the sense of Folland and Kohn [3]. Our goal in this note is to understand the Folland–Kohn basic estimate in the case of unbounded domains. In this context, we will see that a uniformity condition needs to be added to the usual definition of pseudoconvexity. We prove that on an unbounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the basic estimate holds in the sense of Folland and Kohn if and only if $\Omega$ is uniformly pseudoconvex.

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In Theorems 3.2.1 and 3.2.4 in [4], Hörmander showed that the basic estimate on $(0, q)$-forms in the sense of Folland and Kohn is locally equivalent to a geometric property known as $Z(q)$, with a global equivalence on bounded domains (see [4, Corollary 3.2.3]). The motivation for this paper is to show that this equivalence fails on unbounded domains. However, Hörmander’s Theorem 3.2.1 actually implies a stronger condition than $Z(q)$ on unbounded domains. As in the pseudoconvex case, the $Z(q)$ condition must be replaced with a uniform $Z(q)$ condition. These two conditions are equivalent on bounded domains. In the non-pseudoconvex case, we show that $\Omega_1$ satisfies the Folland–Kohn basic estimate on $(0, q)$-forms if and only if $\Omega_1$ satisfies a uniform $Z(q)$ condition, provided that the boundary of $\Omega$ is at least $C^4$ with uniform bounds on the fourth-order derivatives.

The classic example of an unbounded strictly pseudoconvex domain is the Siegel upper half space. When $q = 1$, the uniform $Z(q)$ condition is a uniform strict pseudoconvexity condition which we will show that the Siegel upper half space fails to satisfy. As a corollary, we obtain a result that is strikingly different than the bounded case, namely: strict pseudoconvexity is no longer sufficient for the Folland–Kohn basic estimate to hold.

Gansberger was the first to investigate closed range of $\bar{\partial}$ in (weighted) $L^2$ on unbounded domains [5], but his focus was on compactness of the $\bar{\partial}$-Neumann operator on pseudoconvex domains. In [6], we began our investigation of sufficient conditions for closed range of $\bar{\partial}$ on $(0, q)$-forms for a fixed $q$, $0 < q < n$. This led to a generalization of $Z(q)$ in [7], suitable for $C^3$ domains in a Stein manifold. It was the definition of weak $Z(q)$ in [7] that we modified for unbounded domains in [8]. Herbig and McNeal [9] also explore closed range of $\bar{\partial}$ on pseudoconvex domains that are not necessarily bounded.

The outline of this note is as follows: we state definitions and formulate the main results in Section 2 and prove the main theorems in Section 3. Some technical results from linear algebra are contained in Appendix 1.

## 2. Definitions and results

### 2.1. Definitions

To continue the discussion, we need to introduce some terminology. Our definitions follow the set-up in [8,10].

A function $\rho : \mathbb{C}^n \to \mathbb{R}$ is called a defining function for $\Omega$ if $\Omega = \{ z : \rho(z) < 0 \}$ and $d\rho \neq 0$ on $\partial \Omega$. The Levi form of $\Omega$ is the restriction of the complex Hessian of a defining function $\rho$ to the maximal complex tangent space (locally, an $(n - 1) \times (n - 1)$ matrix). The induced CR-structure on $\partial \Omega$ at $z \in \partial \Omega$ is

$$T^{1,0}_z(\partial \Omega) = \{ L \in T^{1,0}(\mathbb{C}) : \partial \rho(L) = 0 \}.$$ 

If $\rho$ is a $C^m$ function with $m \geq 1$, let $T^{1,0}(\Omega)$ be the space of $C^{m-1}$ sections of $T^{1,0}_z(\partial \Omega)$ and $T^{0,1}(\Omega) = T^{1,0}(\Omega)$. We denote the exterior algebra generated by the dual forms by $\Lambda^{p,q}(\partial \Omega)$. If we normalize $\rho$ so that $|d\rho| = 1$ on $\partial \Omega$, then the normalized Levi form $\mathcal{L}$ is the real element of $\Lambda^{1,1}(\partial \Omega)$ defined by

$$\mathcal{L}( -iL \wedge \bar{L} ) = i\partial \bar{\partial} \rho ( -iL \wedge \bar{L} )$$
for any \( L \in T^{1,0}(b\Omega) \). We will denote \( \rho_j = \frac{\partial \rho}{\partial z_j} \) and \( \rho_{jk} = \frac{\partial^2 \rho}{\partial z_j \partial z_k} \). In this notation, if \( L = \sum_{j=1}^n \tau_j \frac{\partial}{\partial z_j} \), then \( L \in T_{z}^{1,0}(\mathbb{C}) \) if and only if \( \sum_{j=1}^n \tau_j \rho_j(z) = 0 \), and the Levi form is given by

\[
\mathcal{L}(-iL \wedge \bar{L}) = \sum_{j,k=1}^n \tau_j \rho_{jk}(z) \bar{\tau}_k.
\]

We now introduce our key condition. On bounded domains, \( Z(q) \) is known to be equivalent to the Folland–Kohn basic estimate on \((0, q)\)-forms, while on unbounded domains, we will show that uniform \( Z(q) \) is equivalent to the Folland–Kohn basic estimate on \((0, q)\)-forms.

**Definition 2.1:** Let \( \Omega \subset \mathbb{C}^n \) be a domain with a \( C^m \) boundary, \( m \geq 2 \). We say that \( \Omega \) satisfies condition \( Z(q) \) if the Levi form has at least \((n - q)\)-positive or at least \((q + 1)\)-negative eigenvalues. We say that \( \Omega \) satisfies the uniform \( Z(q) \) condition if for some \( \lambda > 0 \), the normalized Levi form has at least \((n - q)\) eigenvalues greater than \( \lambda \) or at least \((q + 1)\) eigenvalues smaller than \(-\lambda\).

On unbounded domains, we have shown in [10] that a \( C^m \) defining function may not be sufficient for analysis, so we introduced the concept of a uniformly \( C^m \) defining function.

**Definition 2.2:** Let \( \Omega \subset \mathbb{C}^n \) be a domain with \( C^m \) boundary \( b\Omega \), \( m \geq 3 \), and let \( \rho \) be a \( C^m \) defining function for \( \Omega \) defined on a neighbourhood \( U \) of \( b\Omega \) such that

1. \( \text{dist}(b\Omega, bU) > 0 \),
2. \( \|\rho\|_{C^m(U)} < \infty \),
3. \( \inf_U |\nabla \rho| > 0 \).

We say that such a defining function is uniformly \( C^m \). If \( \rho \) on \( U \) is uniformly \( C^m \) for all \( m \in \mathbb{N} \), we say \( \rho \) is uniformly \( C^\infty \).

In [10], we show that we may assume \( |\nabla \rho| = 1 \) on \( U \) without loss of generality. In fact, we show that the existence of any uniformly \( C^m \) defining function implies that the signed distance function is uniformly \( C^m \).

We denote the weighted \( L^2 \)-inner product on \( L^2(\Omega, e^{-t|z|^2}) \) by

\[
(f, g)_t = \int_\Omega f \bar{g} e^{-t|z|^2} dV,
\]

where \( dV \) is Lebesgue measure on \( \mathbb{C}^n \). We denote \( d\sigma \) as the induced surface area measure on \( b\Omega \). Associated to the inner product \( (\cdot, \cdot)_t \) is a weighted norm, denoted by \( |f|_t^2 = \int_\Omega |f|^2 e^{-t|z|^2} dV \).

Let \( \tilde{\partial} : L^2_{0,q}(\Omega, e^{-t|z|^2}) \rightarrow L^2_{0,q+1}(\Omega, e^{-t|z|^2}) \) denote the maximal closure of the Cauchy–Riemann operator, and let \( \tilde{\partial}_t^* : L^2_{0,q+1}(\Omega, e^{-t|z|^2}) \rightarrow L^2_{0,q}(\Omega, e^{-t|z|^2}) \) denote the adjoint with respect to \( (\cdot, \cdot)_t \). This means for \((0, q)\)-forms, \( \text{Dom} (\tilde{\partial}) = \{ u \in L^2_{0,q}(\Omega) : \tilde{\partial} u \in L^2_{0,q+1}(\Omega) \} \), and for \((0, q + 1)\)-forms, \( \text{Dom} (\tilde{\partial}_t^*) = \{ v \in L^2_{0,q+1}(\Omega) : |(v, \tilde{\partial} u)| \leq C|u|_t \} \), for all \( u \in \text{Dom}(\tilde{\partial}) \). Since \( \tilde{\partial}_t^* \) is computed using integration by parts, the domain will induce a boundary condition; see Lemma 4.2.1 in [11] for details.

With this background in place, we are ready to formally state the Folland–Kohn basic estimate.
Definition 2.3: Let $\Omega \subset \mathbb{C}^n$ be a domain of class $C^1$ and $1 \leq q \leq n - 1$. We say that $\Omega$ satisfies the Folland–Kohn basic estimate on $(0, q)$-forms if there exist $t > 0$ and a constant $C_t > 0$ so that for all $(0, q)$-forms $f \in \text{Dom} (\partial) \cap \text{Dom} (\partial^*_f)$,

$$
\int_{b\Omega} |f|^2 e^{-t|z|^2} \, d\sigma \leq C_t \left( \|\partial f\|_t^2 + \|\partial^*_f\|_t^2 \right).
$$

(1)

Finally, when invoking estimates from [8], we will need some notation for working with $(0, q)$-forms. Let $\mathcal{I}_q = \{(i_1, \ldots, i_q) \in \mathbb{N}^n : 1 \leq i_1 < \cdots < i_q \leq n\}$. For $K \in \mathcal{I}_{q-1}$, $J \in \mathcal{I}_q$, and $1 \leq j \leq n$, let $\epsilon^J = (-1)^{|\sigma|}$ if $|j| \cup K = J$ be sets and $\sigma$ is the length of the permutation that takes $|j| \cup K$ to $J$. Set $\epsilon^J = 0$ otherwise. We use the standard notation that if $u = \sum_{J \in \mathcal{I}_q} u_J d\bar{z}_J$, then

$$
u = \sum_{J \in \mathcal{I}_q} \epsilon^J u_J.
$$

We will also need to denote the adjoint of a vector field $\frac{\partial}{\partial z_j}$ with respect to our weighted $L^2$ inner product, so we set $L^t_j = \frac{\partial}{\partial z_j} - t\bar{z}_j = e^{t|z|^2} \frac{\partial}{\partial z_j} e^{-t|z|^2}$.

2.2. Results

The next two results comprise the main new theorems that we prove in this note.

Theorem 2.4: Let $\Omega \subset \mathbb{C}^n$ be a $C^2$ domain and $1 \leq q \leq n - 1$. If $\Omega$ satisfies the Folland–Kohn basic estimate for $(0, q)$-forms, then $\Omega$ satisfies $Z(q)$ uniformly.

Conversely, if $\Omega$ is an unbounded domain admitting a uniformly $C^4$ defining function and satisfying $Z(q)$ uniformly, then there exists $t_0 \geq 0$ so that whenever $t > t_0$, there exists $C_t$ so that $\Omega$ satisfies the Folland–Kohn basic estimate on $(0, q)$-forms for all $t \geq t_0$.

Our primary (non)example is the Siegel upper half space

$$
\Sigma = \{(z, w) \in \mathbb{C}^{n+1} : \text{Im} w > |z|^2\}.
$$

The boundary of the Siegel upper half space is the Heisenberg group, $\mathbb{H}^n$. Given the description of $\Sigma$ as a global graph, it is natural to use the defining function $r(z, w) = |z|^2 - \frac{w - w_0}{2t}$, but $r$ is not a uniformly $C^4$ defining function since $|dr| = 1 + 4|z|^2$ is not bounded. However, $\rho(z, w) = \frac{r(z, w)}{|dr(z, w)|}$ is a uniformly $C^\infty$ defining function.

Theorem 2.5: The Siegel upper half space $\Sigma$ is a strictly pseudoconvex domain that admits a uniformly $C^\infty$ defining function. However, it is not uniformly strictly pseudoconvex, so the Folland–Kohn basic estimate (1) fails to hold on $(0, q)$-forms for any $1 \leq q \leq n - 1$ and $t \geq 0$.

3. Proofs and examples

In this section, we prove Theorems 2.4 and 2.5 and provide an example of a uniformly $Z(q)$ domain.
3.1. Proof of Theorem 2.4

The proof that the Folland–Kohn basic estimate (1) implies the uniform $Z(q)$ condition can be taken nearly verbatim from the argument that establishes [4, Corollary 3.2.3]. Hörmander’s arguments reduce to a local argument that renders irrelevant both the weight and unboundedness.

The proof of the converse direction follows from the basic estimates in [8]. Let $f$ be a $(0,q)$ form in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*_q)$. Via a partition of unity, we may use Theorem 3.2.4 in [4] to show that $f$ has a local $L^2$ boundary trace near each boundary point. Using Lemma 3.2 in [8], we obtain a sequence of bounded $C^4$ domains $\{\Omega_j\}$ and functions $f_j \in C^3(\overline{\Omega_j})$ such that $\Omega_j \cap B(0,j + 2) = \Omega \cap B(0,j + 2), f_j \equiv 0$ on $\Omega \setminus B(0,j + 2), f_j|_{\Omega_j} \in \text{dom} \bar{\partial}^*_q$, and

\[
\| \bar{\partial} f_j \|_{L^2(\Omega_j, e^{-t|z|^2})} + \| \bar{\partial}^*_q f_j \|_{L^2(\Omega_j, e^{-t|z|^2})} + \| f_j \|_{L^2(\Omega_j, e^{-t|z|^2})} \\
\to \| \bar{\partial} f \|_{L^2(\Omega, e^{-t|z|^2})} + \| \bar{\partial}^*_q f \|_{L^2(\Omega, e^{-t|z|^2})} + \| f \|_{L^2(\Omega, e^{-t|z|^2})}.
\]

If we can show that the Folland–Kohn estimate holds on each $\Omega_j$ with a constant that is uniform on $j$, then we will have a uniform bound on $\int_{\partial \Omega_j} |f_j|^2 e^{-t|z|^2} d\sigma$. Once we have this uniform bound, on any compact subset $K$ of $b\Omega$, we will be able to show that the boundary traces of the $f_j$ have a weak limit on $K$ satisfying this uniform bound. Since we already know that $f$ has a local boundary trace in $L^2$, this weak limit must agree with $f$. Hence, the boundary trace of $f$ is uniformly bounded in $L^2$ on every compact subset of the boundary. We may conclude that the boundary trace of $f$ is globally in $L^2(\partial b\Omega, e^{-t|z|^2})$, and the Folland–Kohn estimate will follow. It remains to show that we have uniform bounds on each $\Omega_j$.

In what follows, we will make frequent use of Lemma 1 to understand the local behaviour of the eigenvalues and eigenvectors of the Levi form. Although Lemma 1 is stated for Hermitian matrices parameterized by a flat space, our uniformity assumptions guarantee that we may locally flatten our boundary in a uniform way and let the Levi form inherit the relevant estimates. We note that $k$th derivatives of the Levi form are uniformly bounded by $\|\rho\|_{C^{k+2}(U)}$, where $U$ is given by Definition 2.2.

Enumerate the eigenvalues of the Levi form in increasing order by $\mu_1, \ldots, \mu_{n-1}$, and set $\mu_0 = \min\{0, \mu_1\}$ and $\mu_n = \max\{\mu_{n-1}, 0\}$. Suppose first that the normalized Levi form has at least $n - q$ eigenvalues greater than $\lambda$, or equivalently, $\mu_q > \lambda$ on $b\Omega$. At each point $p \in b\Omega$, there must exist at least one $1 \leq \ell_p \leq q$ such that $\mu_{\ell_p}(p) > \lambda/(q + 1)$ and $\mu_{\ell_p}(p) > \mu_{\ell_p - 1}(p) + \lambda/(q + 1)$. Otherwise, there must exist some point $p \in b\Omega$ at which $\mu_q(p) = \mu_k(p) + \sum_{j=k+1}^q (\mu_j(p) - \mu_{j-1}(p)) \leq \frac{q-k+1}{q+1}\lambda$, where $0 \leq k < q$ is the largest integer for which $\mu_k(p) \leq \lambda/(q + 1)$, a contradiction. Since (A1) implies that the Lipschitz constants of $\mu_j(p)$ are uniformly bounded by some multiple of $\|\rho\|_{C^3(U)}$, there must exist a neighbourhood $O_p$ of $p$ with radius proportional to $\frac{\lambda}{(q+1)\|\rho\|_{C^3(U)}}$ so that $\mu_{\ell_p}(p) > 0$ and $\mu_{\ell_p}(p) > \mu_{\ell_p - 1}(p) + \lambda/(2(q + 1))$ on $b\Omega \cap O_p$. Let $P_p$ denote the projection onto the span of the eigenspaces corresponding to the $\ell_p - 1$ smallest eigenvalues of the Levi form at each point in $b\Omega \cap O_p$ (with $P_p = 0$ when $\ell_p = 1$). Using (A2) and (A3), derivatives of $P_p$ are bounded by $O\left(\frac{q+1}{\lambda}\|\rho\|_{C^3(U)}\right)$ and second derivatives are bounded by $O\left(\frac{(q+1)^2}{\lambda^2}\|\rho\|_{C^2(U)}\right) + O\left(\frac{q+1}{\lambda}\|\rho\|_{C^4(U)}\right)$. As a projection, every eigenvalue of $P_p$ lies in the interval $[0, 1]$. The trace of $P_p$ is equal to $\ell_p - 1$, which is at most $q - 1$. Finally, we
have the inequality

$$\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^{n} (P_p)_{kj} \rho_{jk} = \mu_{\ell_p} + \cdots + \mu_q > \lambda$$

on $b\Omega \cap \mathcal{O}_p$. Since each $\mathcal{O}_p$ is of uniform size, $b\Omega$ is locally compact, and each $P_p$ satisfies uniform estimates on $b\Omega \cap \mathcal{O}_p$, we may use a locally finite subcover indexed by $\mathcal{P} \subset b\Omega$ and a partition of unity $\{\chi_p\}_{p \in \mathcal{P}}$ subordinate to this subcover to define a global Hermitian matrix $\Upsilon = \sum_{p \in \mathcal{P}} \chi_p P_p$. We can easily check that $\Upsilon$ inherits from $P_p$ the properties that $\Upsilon$ maps the complex normal to the zero vector, each eigenvalue of $\Upsilon$ lies in the interval $[0, 1]$, the trace of $\Upsilon$ is at most $q - 1$ and

$$\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^{n} \Upsilon_{kj} \rho_{jk} > \lambda.$$

When the normalized Levi form has at least $(q + 1)$ eigenvalues smaller than $-\lambda$, the construction of $\Upsilon$ is similar, so we will highlight the differences. This time, we have $\mu_{q+1} < -\lambda$, so we may find $q + 1 \leq \ell_p \leq n - 1$ such that $\mu_{\ell_p} < -\frac{\lambda}{n-q}$ and $\mu_{\ell_p} < \mu_{\ell_p+1} - \frac{\lambda}{n-q}$. Otherwise, we would have $\rho \in b\Omega$ at which $\mu_{q+1}(\rho) = \mu_k(\rho) + \sum_{j=q+1}^{k-1} (\mu_j(\rho) - \mu_{j+1}(\rho)) \geq -\frac{k-q}{n-q}\lambda$, where $q + 1 < k \leq n$ is the smallest integer for which $\mu_k(\rho) \geq -\lambda/(n-q)$, a contradiction. We can construct $\mathcal{O}_p$ so that $\mu_{\ell_p} < 0$ and $\mu_{\ell_p} < \mu_{\ell_p+1} - \frac{\lambda}{2(n-q)}$ on $b\Omega \cap \mathcal{O}_p$. On $b\Omega \cap \mathcal{O}_p$, $P_p$ will denote the projection onto the span of the eigenspaces corresponding to the $\ell_p$ smallest eigenvalues of the Levi form. This time, the trace of $P_p$ will be $\ell_p$, which is at least $q + 1$, and we will have the inequality

$$\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^{n} (P_p)_{kj} \rho_{jk} = -\mu_{q+1} - \cdots - \mu_{\ell_p} > \lambda$$

on $b\Omega \cap \mathcal{O}_p$. After carrying out our patching argument, we will once again have a global Hermitian matrix $\Upsilon$ with uniform bounds on its first and second derivatives such that $\Upsilon$ maps the complex normal to the zero vector, each eigenvalue of $\Upsilon$ lies in the interval $[0, 1]$, the trace of $\Upsilon$ is uniformly bounded away from $q$ and

$$\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^{n} \Upsilon_{kj} \rho_{jk} > \lambda.$$

As shown in Lemma 3.1 in [8], we can extend $\Upsilon$ to $\mathbb{C}^n$ in a uniform way without sacrificing any of our properties. Turning our attention to the bounded domains $\Omega_j$, we see that for each point in $b\Omega_j$, either $\Upsilon$ is defined in a neighbourhood of the point or $f_j = 0$ in a neighbourhood of the point. Using Proposition 3.4 and Lemma 3.5 in [8], we obtain

$$\|\partial f_j\|^2_t + \|\partial_t^\ast f_j\|^2_t = \frac{1}{2} \sum_{j \in \mathcal{T}_q} \sum_{\ell,k=1}^{n} \left( (I_{\ell k} - \Upsilon_{\ell k}) \frac{\partial f_j}{\partial z_k} \frac{\partial f_j}{\partial z_\ell} \right)_t.$$
\[
\frac{1}{2} \sum_{j \in \mathcal{I}_q} \sum_{\ell, k = 1}^n \left( \gamma^{\ell \ell} L^\ell_j(f_j), L^\ell_k(f_j) \right)_t \\
+ \sum_{K \in \mathcal{I}_{q-1}} \sum_{\ell, k = 1}^n \int_{b\Omega_j} \langle \rho_{\ell k}(f_j)^\ell, (f_j)^k \rangle e^{-t|z|^2} \, d\sigma \\
- \sum_{j \in \mathcal{I}_q} \sum_{\ell, k = 1}^n \int_{b\Omega_j} \langle \gamma^{\ell \ell} \rho_{\ell k}(f_j)^\ell, (f_j)^k \rangle e^{-t|z|^2} \, d\sigma \\
+ \sum_{j \in \mathcal{I}_q} t((q - \text{Tr}(\Upsilon))(f_j), (f_j))_t + O(\|f_j\|^2),
\]

where \( I \) is the identity matrix. Note that [8] gives a more precise error estimate of the form \( O(\|f_j\|^2) \leq C(\|\Upsilon\|_{C^1} + \|\Upsilon\|_{C^2})\|f_j\|^2. \)

Now, all of the terms involving derivatives of \( f_j \) in (2) are non-negative since \( \Upsilon \) and \( I - \Upsilon \) are both positive semidefinite. If \( q > \text{Tr}(\Upsilon) \), then we may choose \( t > 0 \) independent of \( j \) to obtain a non-negative coefficient in front of the \( \|f_j\|^2 \) terms. If \( q < \text{Tr}(\Upsilon) \), then we instead choose \( t < 0 \). Finally, Lemma 4.7 in [12] implies that on \( b\Omega_j \cap b\Omega \)

\[
\sum_{K \in \mathcal{I}_{q-1}} \sum_{\ell, k = 1}^n \langle \rho_{\ell k}(f_j)^\ell, (f_j)^k \rangle - \sum_{j \in \mathcal{I}_q} \sum_{\ell, k = 1}^n \langle \gamma^{\ell \ell} \rho_{\ell k}(f_j)^\ell, (f_j)^k \rangle \geq \left( \mu_1 + \cdots + \mu_q - \sum_{\ell, k = 1}^n \gamma^{\ell \ell} \rho_{\ell k} \right) |f_j|^2 > \lambda |f_j|^2.
\]

On \( b\Omega_j \setminus b\Omega, f_j = 0 \), so this estimate is trivial. Hence, for \( |t| \) sufficiently large, (2) gives us

\[
\|\tilde{\partial} f_j\|^2 + \|\tilde{\partial}^* f_j\|^2 \geq \int_{b\Omega_j} \lambda |f_j|^2 e^{-t|z|^2} \, d\sigma.
\]

Since this gives us a uniform bound on \( \int_{b\Omega_j} |f_j|^2 e^{-t|z|^2} \, d\sigma \), the weak limiting procedure discussed at the beginning of the proof can be used to obtain this same uniform bound on \( \int_{b\Omega} |f|^2 e^{-t|z|^2} \, d\sigma \), and the Folland–Kohn basic estimate follows. \( \square \)

### 3.2. Proof of Theorem 2.5

The definition of the Levi form required \( |d\rho| = 1 \) on \( b\Sigma \). The standard defining function for the Siegel upper half space \( \Sigma = \{ (z, w) : \text{Im} \, w = |z|^2 \} \) is

\[
r(z, w) = |z|^2 - \frac{w - \bar{w}}{2i}.
\]

This function is clearly unsuitable to use to compute the normalized Levi form. Instead, we use the signed distance function, \( \tilde{\delta}(z) \), defined by

\[
\tilde{\delta}(z) = \begin{cases} 
\text{dist} (z, b\Sigma) & \text{if } z \notin \Sigma \\
- \text{dist} (z, b\Sigma) & \text{if } z \in \Sigma.
\end{cases}
\]
If \( p \in \text{b} \Sigma \) and local coordinates \((y_1, \ldots, y_4)\) satisfy \( \frac{\partial r(p)}{\partial y_j} = 0 \) for \( j = 1, \ldots, 3 \), then

\[
\frac{\partial^2 \delta(p)}{\partial y_j \partial y_k} = |\nabla r|^{-1} \frac{\partial^2 r(p)}{\partial y_j \partial y_k},
\]

(see e.g. [10, (2.9)]). Since \( T_{1,0}^1(\text{b} \Omega) \) is spanned by \( L = \frac{\partial}{\partial z} + 2i\bar{z} \frac{\partial}{\partial w} \), the normalized Levi form can be computed by

\[
L( -iL \wedge \bar{L} ) = -|\nabla r|^{-1} = -(1 + 4|z|^2)^{-1/2}.
\]

Since \( |L|^2 = \frac{1}{2} + 2|z|^2 \), we have

\[
|L|^{-2} L( -iL \wedge \bar{L} ) = -2(1 + 4|z|^2)^{-3/2},
\]

and hence the only eigenvalue of the Levi form is \(-2(1 + 4|z|^2)^{-3/2}\). Since this approaches 0 as \( |z| \to \infty \) for \( z \in \text{b} \Sigma \), it follows that the Siegel upper half space is \( Z(1) \) but not uniformly \( Z(1) \). Thus, (1) does not hold on \( \Sigma \) by Theorem 2.4. This completes the proof of Theorem 2.5. \( \Box \)

### 3.3. A positive example

Let

\[
\Omega = \{(z', z_n) \subset \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + 2 \Re(z_n)^2 < \frac{1}{2}\}.
\]

In this case,

\[
\rho(z', z_n) = |z'|^2 + 2 \Re(z_n)^2 - \frac{1}{2}.
\]

Since \( |d\rho|^2 = 4|z'|^2 + 16 \Re(z_n)^2 \), we have \( |d\rho|^2 = 8 \Re(z_n)^2 + 2 \geq 2 \) and \( |d\rho|^2 = -4|z'|^2 + 4 \leq 4 \) on \( \Omega \). Since \( i\partial \bar{\partial} \rho = I \), the identity matrix, the Levi form is also an identity matrix with eigenvalues bounded between \( \frac{1}{2} \) and \( \frac{1}{\sqrt{2}} \). Thus, \( \Omega \) satisfies \( Z(q) \) uniformly for all \( 1 \leq q \leq n-1 \). Since \( \Omega \) is also pseudoconvex, we can use the Morrey–Kohn–Hörmander formula to prove the Folland–Kohn basic estimate, Equation (1), for \( t = 0 \).

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We would like to dedicate this note to our colleague John Ryan on the occasion of his 60th birthday.

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Appendix 1. Linear algebra facts

In all that follows, we will equip the $\mathbb{R}$-linear space of Hermitian $m \times m$ matrices with the sup-norm $\left| (a_{jk})_{1 \leq j, k \leq m} \right| = \sup_{1 \leq j, k \leq m} |a_{jk}|$. This guarantees that a sequence $\{A_t\}$ converges to $A$ in norm if and only if every element of $\{A_t\}$ converges to the corresponding element of $A$. In particular, if $A_t$ is a family of Hermitian matrices parameterized by $t \in \mathbb{R}$, then $A_t$ is differentiable if and only if every element of $A_t$ is differentiable. $\|A_t\|_{C^1}$ will denote the supremum of the $C^1$ norms of each element of $A_t$, and $\|A_t\|_{C^2}$ will denote the supremum of the $C^2$ norms.

Lemma A.1: Let $M(x)$ be a family of $m \times m$ Hermitian matrices parameterized with respect to $x \in \mathbb{R}^n$. Let $\lambda_1(x), \ldots, \lambda_m(x)$ denote the eigenvalues of $M(x)$ in increasing order. Let $U \subset \mathbb{R}^n$ be an open neighbourhood, and suppose the elements of $M(x)$ are $C^1$ functions on $\overline{U}$. Then, for every $x, y \in \overline{U}$ and $1 \leq j \leq m$, $|\lambda_j(x) - \lambda_j(y)| \leq \|M\|_{C^1(\overline{U})} |x - y|$ (A1)

Let $\{J, K\}$ be a partition of $\{1, \ldots, m\}$ such that $\lambda_j(x) \neq \lambda_k(x)$ whenever $j \in J$, $k \in K$, and $x \in \overline{U}$. Let $P_j(x)$ denote the orthogonal projection with respect to $\mathbb{R}^n$ onto the span of the eigenspaces of all $\lambda_j(x)$ with $j \in J$. $P_j$ has $C^1$ elements satisfying

$$\|P_j\|_{C^1(\overline{U})} \leq O \left( \inf_{x \in U, j \in J, k \in K} |\lambda_j(x) - \lambda_k(x)|^{-1} \right)^{-1} \frac{1}{\|M\|_{C^1(\overline{U})}}.$$ (A2)
If the elements of $M(x)$ are $C^2$ functions on $\overline{U}$, then $P_j$ also has $C^2$ elements satisfying

$$\|P_j\|_{C^2(\overline{U})} \leq O \left( \inf_{x \in U} |\lambda_j(x) - \lambda_k(x)|^{-2} \|M\|_{C^1(\overline{U})}^2 \right) + O \left( \inf_{x \in U} |\lambda_j(x) - \lambda_k(x)|^{-1} \|M\|_{C^2(\overline{U})} \right).$$  \hspace{1cm} (A3)

**Remark A.2:** We note that (A1) is sharp in the sense that $\lambda_j$ may not be $C^1$ even if $M$ has smooth coefficients. Consider $M(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$. This has smooth coefficients, but the eigenvalues are not $C^1$ at the origin, so Lipschitz eigenvalues are the best that we can hope for. We further note that this matrix does not have continuous eigenvectors on any neighbourhood of the origin, so great care will have to be taken in the following proof.

**Proof:** We begin by showing that $\lambda_j$ and $P_j$ are continuous. Fix $x \in \overline{U}$ and let $\{x_\ell\}$ be any sequence in $\overline{U}$ converging to $x$. Let $v_{j,\ell}$ be a unit-length eigenvector for $M(x_\ell)$ corresponding to $\lambda_j(x_\ell)$. Note that $\{\lambda_j(x_\ell)\}$ is uniformly bounded since $M(x_\ell)$ is uniformly bounded. Let $\tilde{v}_j, \tilde{\lambda}_j$ be any limit point of the sequence of pairs $(v_{j,\ell}, \lambda_j(x_\ell))$, and restrict to a subsequence for which we have convergence. Since $(M(x_\ell) - \tilde{\lambda}_j(x_\ell))v_{j,\ell}$ converges to $(M(x) - \tilde{\lambda}_j(x))\tilde{v}_j$, $\tilde{v}_j$ must be an eigenvector of $M(x)$ with eigenvalue $\tilde{\lambda}_j$. Having established that eigenvalues are continuous, order considerations will imply that $\tilde{\lambda}_j = \lambda_j(x)$. We have also established that the limit point of a sequence of eigenvectors is also an eigenvector. Consider the projection $P_{j,x}(x_\ell) = \sum_{k: \lambda_k(x) = \lambda_j(x)} v_{k,\ell}(\tilde{v}_{k,\ell})^T$. Since $P_{j,x}(x_\ell)$ is uniformly bounded, we may pick any limit point $\tilde{P}_{j,x}$ of $\{P_{j,x}(x_\ell)\}$ and restrict to a subsequence on which we have convergence. After further restriction, we may assume $\{v_{k,\ell}\}$ converges to some $\tilde{v}_k$ whenever $\lambda_k(x) = \lambda_j(x)$. We have shown that each $\tilde{v}_k$ is an eigenvector corresponding to $\lambda_k(x)$, and since $\{\tilde{v}_k\}$ must be an orthonormal set, $\tilde{P}_{j,x}$ must be the unique orthogonal projection onto the eigenspace corresponding to $\lambda_j(x)$. In other words, $\tilde{P}_{j,x} = P_{j,x}(x)$. Since $\tilde{P}_{j,x}$ was an arbitrary limit point, we conclude that $P_{j,x}$ is continuous in a neighbourhood of $x$.

Having shown that $\lambda_j$ is continuous, we are now ready to show that it is in fact Lipschitz. Fix $1 \leq j \leq m$. We first assume $U$ is convex. Suppose that there exist points $a_0, b_0 \in \overline{U}$ and $C > \|M\|_{C^1(\overline{U})}$ such that $|\lambda_j(a_0) - \lambda_j(b_0)| > C|a_0 - b_0|$. If we set $a_1 = \frac{a_0 + b_0}{2}$, then by setting either $b_1 = a_0$ or $b_1 = b_0$, the triangle inequality gives us $|\lambda_j(a_1) - \lambda_j(b_1)| > C|a_1 - b_1|$. Continue in this way to inductively construct a sequence in $\overline{U}$ satisfying $|\lambda_j(a_k) - \lambda_j(b_k)| > C|a_k - b_k|$ and $|a_k - b_k| \to 0$. Let $v_{j,\ell,a}$ be a unit-length eigenvector of $M(a_\ell)$ corresponding to $\lambda_j(a_\ell)$, with a similar definition for $v_{j,\ell,b}$. By restricting to a subsequence, we may assume that $c \in \mathbb{R}^m$ is the limit of $\{a_\ell\}$ and $\{b_\ell\}$, $v_{j,\ell,a}$ is the limit of $\{v_{j,\ell,a}\}$ and $v_{j,\ell,b}$ is the limit of $\{v_{j,\ell,b}\}$. Since $a_\ell \neq b_\ell$, either $a_\ell \neq c$ or $b_\ell \neq c$ for each $\ell$. After possibly exchanging these sequences, we may further assume that $a_\ell \neq c$ for every $\ell$. Since $M(a_\ell)v_{j,\ell,a} = \lambda_j(a_\ell)v_{j,\ell,a}$, we have $(\tilde{v}_{j,a})^T v_{j,\ell,a} \lambda_j(a_\ell) = (\tilde{v}_{j,a})^T M(a_\ell)v_{j,\ell,a}$. Continuity of $M$ can be used to show that we also have $M(c)v_{j,a} = \lambda_j(c)v_{j,a}$, so we obtain $(\tilde{v}_{j,a})^T v_{j,\ell,a} \lambda_j(c) = (\tilde{v}_{j,a})^T M(c)v_{j,a}$. Since $M$ is Hermitian and $\lambda_j$ is real, we may subtract these and obtain

$$(\tilde{v}_{j,a})^T v_{j,\ell,a} (\lambda_j(a_\ell) - \lambda_j(c)) = (\tilde{v}_{j,a})^T (M(a_\ell) - M(c)) v_{j,\ell,a}.$$ 

Dividing this by $|a_\ell - c|$ and taking limits, we obtain $C \leq \|M\|_{C^1(\overline{U})}$, a contradiction, so (A1) follows when $U$ is convex. If $U$ is not convex, we use the fact that $U$ is locally convex with a uniform Lipschitz constant on each convex neighbourhood to obtain (A1).

We now turn our attention to the projection $P_j$. Let $v_1(x), \ldots, v_m(x)$ be unit-length eigenvectors for $M(x)$ corresponding to $\lambda_1(x), \ldots, \lambda_m(x)$. Fix $x \in U$, and let $u$ be any unit-length vector in $\mathbb{R}^m$. For $h > 0$, we define the difference quotient $D_{u,h}f(x) = h^{-1}(f(x + hu) - f(x))$. Since $P_j$ is a
projection, we have \( P_j^2 - P_j = 0 \) on \( U \). Taking the difference quotient of this identity, we have

\[
0 = (D_{u,h}P_j(x))P_j(x + hu) + P_j(x)(D_{u,h}P_j(x)) - D_{u,h}P_j(x).
\]

If we left multiply by \( P_j(x) \) and right multiply by \( P_j(x + hu) \), we have

\[
P_j(x)(D_{u,h}P_j(x))P_j(x + hu) = 0,
\]

while if we left multiply by \( I - P_j(x) \) and right multiply by \( I - P_j(x + hu) \), we have

\[
(I - P_j(x))(D_{u,h}P_j(x))(I - P_j(x + hu)) = 0.
\]

Since \( P_j \) maps onto the span of eigenspaces of \( M \), \( MP_j \) must map into the span of these same eigenspaces, so \((I - P_j)MP_j = 0 \) on \( U \). Taking the difference quotient of this identity, we have

\[
0 = -(D_{u,h}P_j(x))M(x + hu)P_j(x + hu) + (I - P_j(x))(D_{u,h}M(x))P_j(x + hu)
+ (I - P_j(x))M(x)(D_{u,h}P_j(x)).
\]

For \( j \in J \) and \( k \in K \), we may left multiply by \((\bar{v}_k(x))^T\) and right multiply by \( v_j(x + hu) \) to obtain

\[
0 = -\lambda_j(x + hu)(\bar{v}_k(x))^T(D_{u,h}P_j(x))v_j(x + hu) + (\bar{v}_k(x))^T(D_{u,h}M(x))v_j(x + hu)
+ \lambda_k(x)(\bar{v}_k(x))^T(D_{u,h}P_j(x))v_j(x + hu).
\]

Solving for \((\bar{v}_k(x))^T(D_{u,h}P_j(x))v_j(x + hu)\), we obtain

\[
(\bar{v}_k(x))^T(D_{u,h}P_j(x))v_j(x + hu) = (\lambda_j(x + hu) - \lambda_k(x))^{-1}(\bar{v}_k(x))^T(D_{u,h}M(x))v_j(x + hu)
\]

Similarly, \( P_jM(I - P_j) = 0 \) can be used to obtain

\[
(\bar{v}_j(x))^T(D_{u,h}P_j(x))v_k(x + hu) = (\lambda_j(x) - \lambda_k(x + hu))^{-1}(\bar{v}_j(x))^T(D_{u,h}M(x))v_k(x + hu)
\]

for all \( j \in J \) and \( k \in K \). Combining (A4), (A5), (A6) and (A7) gives us

\[
D_{u,h}P_j(x)
= \sum_{j \in J, k \in K} (\lambda_j(x + hu) - \lambda_k(x))^{-1} v_k(x)(\bar{v}_k(x))^T(D_{u,h}M(x))v_j(x + hu)(\bar{v}_j(x + hu))^T
+ \sum_{j \in J, k \in K} (\lambda_j(x) - \lambda_k(x + hu))^{-1} v_j(x)(\bar{v}_j(x))^T(D_{u,h}M(x))v_k(x + hu)(\bar{v}_k(x + hu))^T.
\]

Although our choices of eigenvectors \( \{v_\ell\} \) are unlikely to be continuous, we have already shown that for any eigenvalue \( \lambda \) at \( x \) the projection \( \sum_{\ell \in J(x) = x} v_\ell(x + hu)(\bar{v}_\ell(x + hu))^T \) is both independent of our choice of coordinates and continuous in \( h \). Hence, we may take limits and conclude that \( D_uP_j(x) \) exists and satisfies

\[
D_uP_j = \sum_{j \in J, k \in K} (\lambda_j - \lambda_k)^{-1} (v_k(\bar{v}_k)^T(D_uM)v_j(\bar{v}_j)^T + v_j(\bar{v}_j)^T(D_uM)v_k(\bar{v}_k)^T)
\]

in \( U \). We obtain (A2) using (A8) to compute the directional derivatives of \( P_j \) at every point \( x \in U \). For a fixed \( k' \in K \), we can show that \( \sum_{k \in K : \lambda_k(x) = \lambda_{k'}(x)} (\lambda_j - \lambda_k)^{-1} v_k(\bar{v}_k)^T \) is continuous near \( x \).
using the same argument that we used for the projection \( \sum_{k \in K; \lambda_k(x) = \lambda_k} v_k(\bar{v}_k)^T \). Hence, (A8) decomposes \( D_u P_j \) into continuous components, so \( P_j \) is in \( C^1(\overline{U}) \).

Note that if \( \lambda \neq \lambda_k(x) \) for any \( k \in K \), then

\[
N_{\lambda,K}(x) = \sum_{k \in K} (\lambda_k(x) - \lambda)^{-1}(v_k(x))(\bar{v}_k(x))^T
\]

is the unique Hermitian matrix characterized by the relations \( (M(x) - \lambda I)N_{\lambda,K}(x) = I - P_j(x) \) and \( P_j(x)N_{\lambda,K}(x) = 0 \). Uniqueness can be checked by diagonalizing \( M(x) \), which will simultaneously diagonalize \( P_j(x) \). We also define \( N_{\lambda,f}(x) \) in a similar fashion. We will first show that \( N_{\lambda,K}(x) \) and \( N_{\lambda,j}(x) \) are differentiable, and then show that this implies that \( P_j(x) \) is twice differentiable.

First, we take the difference quotient of \( (M(x) - \lambda I)N_{\lambda,K}(x) = I - P_j(x) \) to obtain

\[
(D_{u,h}M(x))N_{\lambda,K}(x + hu) + (M(x) - \lambda I)D_{u,h}N_{\lambda,K}(x) = -D_{u,h}P_j(x).
\]

Left multiplying by \( N_{\lambda,K}(x) \) and solving for \( (I - P_j(x))D_{u,h}N_{\lambda,K}(x) \) gives us

\[
(I - P_j(x))D_{u,h}N_{\lambda,K}(x) = -N_{\lambda,K}(x)D_{u,h}P_j(x) - N_{\lambda,K}(x)(D_{u,h}M(x))N_{\lambda,K}(x + hu).
\]  

(A9)

Next, we take the difference quotient of \( P_j(x)N_{\lambda,K}(x) = 0 \) and solve for \( P_j(x)D_{u,h}N_{\lambda,K}(x) \) to obtain

\[
P_j(x)D_{u,h}N_{\lambda,K}(x) = -(D_{u,h}P_j(x))N_{\lambda,K}(x + hu).
\]  

(A10)

If we add (A9) and (A10) and take the limit as \( h \to 0 \), we see that \( N_{\lambda,K}(x) \) is differentiable and satisfies

\[
D_u N_{\lambda,K} = -(D_u P_j)N_{\lambda,K} - N_{\lambda,K}D_u P_j - N_{\lambda,K}(D_u M)N_{\lambda,K}.
\]  

(A11)

Having already established that derivatives of \( P_j \) are continuous, (A11) now tells us that \( N_{\lambda,K} \) is \( C^1 \) on any neighbourhood of a point at which \( \lambda \neq \lambda_k \) for any \( k \in K \).

Let \( u_1 \) and \( u_2 \) be unit-length vectors. Having established differentiability of \( N_{\lambda,K} \), we use (A8) to compute

\[
D_{u_1}D_{u_2}P_j
= -\sum_{j \in J} \left( (D_{u_1}N_{\lambda,K})|_{\lambda = \lambda_j} (D_{u_2}M)v_j(\bar{v}_j)^T + v_j(\bar{v}_j)^T (D_{u_2}M)(D_{u_1}N_{\lambda,K})|_{\lambda = \lambda_j} \right)
+ \sum_{j \in J, k \in K} (\lambda_j - \lambda_k)^{-1} (v_k(\bar{v}_k)^T (D_{u_1}D_{u_2}M)v_j(\bar{v}_j)^T + v_j(\bar{v}_j)^T (D_{u_1}D_{u_2}M)v_k(\bar{v}_k)^T)
+ \sum_{k \in K} (v_k(\bar{v}_k)^T (D_{u_2}M)(D_{u_1}N_{\lambda,j})|_{\lambda = \lambda_k} + (D_{u_1}N_{\lambda,j})|_{\lambda = \lambda_k}(D_{u_2}M)v_k(\bar{v}_k)^T).
\]

Observe that this decomposes the second derivatives of \( P_j \) into continuous components, so we may conclude that \( P_j \) is in \( C^2(\overline{U}) \). From (A8) and (A11), we have

\[
|D_{u_1}N_{\lambda,K}|_{\lambda = \lambda_j} \leq O \left( \left( \inf_{x \in U, j \in J, k \in K} |\lambda_j(x) - \lambda_k(x)| \right)^{-2} ||M||_{C^1(\overline{U})} \right),
\]

with an equivalent error term for \( |D_{u_1}N_{\lambda,j}|_{\lambda = \lambda_k} \), so (A3) follows. \( \square \)