Abstract

We extend the inner product from bosonic Fock space to a pairing between suitable antifunctionals on the symmetric algebra. Our account is illustrated by the (Gaussian) half-form pairing between positive polarizations in the form needed for geometric quantization.

Introduction

The standard framework in which to discuss a free bosonic system rests on the completion of the symmetric algebra over the underlying one-particle complex Hilbert space. In conventional approaches, one is often faced with figurative expressions which only assume an honest existence when appropriate conditions are satisfied. A specific instance of this scenario occurs in the context of the classical Shale theorem on the unitary implementation of a symplectic automorphism in the bosonic Fock representation: a conventional approach to this theorem hinges on the corresponding displaced Fock vacuum, a figurative Gaussian that lies in Fock space precisely when the symplectic automorphism and the one-particle complex structure have Hilbert-Schmidt commutator. In [6] we developed a framework in which such figurative objects have a strictly legitimate existence and thereby offered a new account of the Shale theorem. Our purpose here is to show that this framework also supports rigorously extending the Fock space inner product to a pairing between suitable formerly figurative expressions: specifically, between suitable (not necessarily bounded) antifunctionals on the symmetric algebra; such a pairing is important for the theory of geometric quantization, as discussed in [5] and [7].

For convenience, we recall familiar foundational material on bosonic Fock space, perhaps from an unfamiliar perspective. The precise formulation of-
fered here was first presented in [6] though a more traditional reference such as [1] or [2] may also be consulted for some of the details.

Let \( V \) be a complex Hilbert space with \( \langle \cdot | \cdot \rangle \) as its inner product. For the sake of simplicity, we shall suppose that \( V \) has finite complex dimension \( m \).

The symmetric algebra \( SV = \bigoplus_{d \geq 0} S^d V \) is graded by degree and carries a canonical inner product relative to which the homogeneous summands are perpendicular, the Fock vacuum \( 1 \in S^0 V = \mathbb{C} \) is a unit vector and if \( x_1, \ldots, x_d, y_1, \ldots, y_d \in V \) then
\[
\langle x_1 \cdots x_d | y_1 \cdots y_d \rangle = \sum_p \prod_{j=1}^d \langle x_j | y_{p(j)} \rangle
\]
where \( p \) runs over all permutations of \( \{1, \ldots, d\} \). In particular, if \( x, y \in V \) then
\[
\langle x^d | y^d \rangle = d! \langle x | y \rangle^d
\]
and if \( V \) has unitary basis \( (v_1, \ldots, v_m) \) then \( SV \) has unitary basis \( \{v^D : D \in \mathbb{N}^m\} \) where if \( D = (d_1, \ldots, d_m) \in \mathbb{N}^m \) then
\[
v^D = \frac{v_1^{d_1} \cdots v_m^{d_m}}{\sqrt{d_1! \cdots d_m!}}.
\]

Let \( SV' \) be the full (purely algebraic) antidual of the symmetric algebra, comprising all (not necessarily bounded) antilinear functionals \( SV \to \mathbb{C} \). The antidual \( SV' \) is naturally a commutative associative complex algebra: its product is defined by the rule that if \( \Phi, \Psi \in SV' \) then
\[
\theta \in SV \implies \langle [\Phi \Psi](\theta) = \langle [\Phi \otimes \Psi](\Delta \theta)
\]
where the cocommutative coproduct \( \Delta : SV \to SV \otimes SV \) is the composite \( SV \to S(V \oplus V) \to SV \otimes SV \) in which the first map is induced by the diagonal \( V \to V \oplus V \) and the second is the canonical isomorphism. Note that the grading on \( SV \) gives each \( \Phi \in SV' \) the structure of a formal series, thus
\[
\Phi = \sum_{d \geq 0} \Phi_d
\]
where if \( d \geq 0 \) then \( \Phi_d = \Phi | S^d V \). In the opposite direction, if to each \( d \geq 0 \) is associated an element \( \Phi_d \in S^d V' \) then the formal series above defines an element of \( SV' \) because individual elements of \( SV \) vanish in sufficiently high degree.
The canonical inner product on the symmetric algebra embeds it in its antidual: it is readily verified that the canonical map

$$SV \rightarrow SV' : \phi \mapsto \langle \cdot | \phi \rangle$$

is an injective algebra homomorphism. In these terms, bosonic Fock space $S[V] = \bigoplus_{d \geq 0} S^d[V]$ may be defined either as the Hilbert space completion of $SV$ or as the subspace of $SV'$ comprising all bounded antifunctionals. Note that if $\Phi, \Psi \in S[V]$ then their inner product is given by

$$\langle \Phi | \Psi \rangle = \sum_{d \geq 0} \langle \Phi_d | \Psi_d \rangle.$$

A proper treatment of Fock space includes a discussion of the Fock representation in terms of creators and annihilators; as such a treatment is not necessary for our purposes, we again refer to [6], [2] or [1] for details.

**Bosonic pairings**

Recall that the homogeneous summands in the symmetric algebra $SV$ are mutually perpendicular relative to its standard inner product and that the corresponding remark holds true for the bosonic Fock space $S[V]$. This being so, we are led to define a bosonic pairing $\langle \cdot : \cdot \rangle_1$ between suitable elements of the full antidual $SV'$ as follows, at least in preliminary form. For $\Phi, \Psi \in SV'$ we define

$$\langle \Phi : \Psi \rangle_1 = \sum_{d \geq 0} \langle \Phi_d | \Psi_d \rangle$$

whenever the indicated series is convergent. A little later, we shall extend this definition; for now, we consider properties of the bosonic pairing as defined in the present sense.

According to our recollection, this pairing extends the standard inner product from $SV$ through $S[V]$ to a partially-defined inner product on $SV'$. The bosonic pairing also reproduces the canonical pairing between $SV$ and $SV'$.

**Theorem 1.** If $\phi \in SV$ and $\Psi \in SV'$ then

$$\langle \phi : \Psi \rangle_1 = \Psi(\phi).$$
Proof. If $\phi = \sum_{d=0}^{D} \phi_d$ then

$$
\Psi(\phi) = \sum_{d=0}^{D} \Psi(\phi_d) = \sum_{d=0}^{D} \langle \phi_d | \Psi_d \rangle = \sum_{d \geq 0} \langle \phi_d | \Psi_d \rangle
$$

whence $\langle \phi : \Psi \rangle_1$ exists and has the indicated value. \qed

Of course, it is likewise true that if $\Phi \in SV'$ and $\psi \in SV$ then

$$
\langle \Phi : \psi \rangle_1 = \overline{\Phi(\psi)}.
$$

The bosonic pairing is defined in situations that involve naturally the number operator and its powers. Recall that the number operator $N$ is defined initially on $SV$ (where it multiplies homogeneous elements by degree) and extends antidually to $SV'$. In Fock space $S[V]$, it is then defined as a self-adjoint operator on the natural domain

$$
\mathbb{D}(N) = \{ \Phi \in SV' : \sum_{d \geq 0} d^2 \| \Phi_d \|^2 < \infty \}.
$$

More generally, if $r \in \mathbb{R}$ then the power $N^r$ has natural domain

$$
\mathbb{D}(N^r) = \{ \Phi \in SV' : \sum_{d \geq 0} d^{2r} \| \Phi_d \|^2 < \infty \}
$$

on which it has the effect

$$
\Phi \in \mathbb{D}(N^r) \Rightarrow N^r(\Phi) = \sum_{d \geq 0} d^r \Phi_d
$$

with an appropriate understanding of the $d = 0$ term.

**Theorem 2.** Let $r \in \mathbb{R}$. If $\Phi \in \mathbb{D}(N^{-r})$ and $\Psi \in \mathbb{D}(N^r)$ then

$$
\langle \Phi : \Psi \rangle_1 = \langle N^{-r} \Phi | N^r \Psi \rangle.
$$

Proof. An elementary and direct calculation: if $d > 0$ then

$$
\langle \Phi_d | \Psi_d \rangle = \langle d^{-r} \Phi_d | d^r \Psi_d \rangle
$$

whence summation concludes the argument. \qed
The bosonic pairing is also defined in situations of Hölder type. Let \( p \geq 1 \) be real and define
\[
\mathbb{H}^p[V] = \{ \Phi \in SV' : \sum_{d \geq 0} \| \Phi_d \|^p < \infty \}.
\]
In addition, define \( \mathbb{H}^\infty[V] \) to comprise all those \( \Phi \in SV' \) for which the sequence \( (\| \Phi_d \|_{d=0}^\infty) \) is bounded. As a special case, note that \( \mathbb{H}^2[V] \) is precisely Fock space \( S[V] \).

**Theorem 3.** Let the indices \( p \geq 1 \) and \( q \geq 1 \) be conjugate in the sense that \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( \Phi \in \mathbb{H}^p[V] \) and \( \Psi \in \mathbb{H}^q[V] \) then \( \langle \Phi : \Psi \rangle_1 \) is defined.

**Proof.** Assume that \( p > 1 \) and \( q > 1 \). If \( d \geq 0 \) then
\[
|\langle \Phi_d | \Psi_d \rangle| \leq \| \Phi_d \| \| \Psi_d \|
\]
by the Cauchy-Schwarz inequality, whence summation yields
\[
\sum_{d \geq 0} |\langle \Phi_d | \Psi_d \rangle| \leq \left\{ \sum_{d \geq 0} \| \Phi_d \|^p \right\}^{1/p} \sum_{d \geq 0} \| \Psi_d \|^q \right\}^{1/q}
\]
on account of the Hölder inequality. The case in which \( \{p,q\} = \{1, \infty\} \) is still more transparent. \( \square \)

As a more specific illustration, let us consider the bosonic pairing of Gaussians. Thus, let the symmetric antilinear map \( Z : V \to V \) and quadratic \( \zeta \in S^2V \) correspond as usual according to the rule
\[
x, y \in V \Rightarrow \langle y | Zx \rangle = \langle xy | \zeta \rangle
\]
and consider the associated Gaussian
\[
e^Z = \sum_{d \geq 0} \frac{\zeta_d}{d!} \in SV'.
\]

The following is Theorem 2.13 in [6] but we include a proof of it here for ease of reference.
Theorem 4. Let $Z : V \to V$ be symmetric antilinear. The Gaussian $e^Z$ lies in Fock space $S[V]$ precisely when $\|Z\| < 1$ and then

$$\|e^Z\|^2 = \text{Det}^{1/2}(I - Z^2)^{-1}.$$ 

Proof. By diagonalization, $Z$ furnishes a unitary basis $v_1, \ldots, v_m$ for $V$ and nonnegative scalars $\lambda_1, \ldots, \lambda_m$ such that if $1 \leq k \leq m$ then $Zv_k = \lambda_k v_k$ and therefore $\zeta = \frac{1}{2} \sum_{k=1}^{m} \lambda_k v_k^2$. Now, if $d \in \mathbb{N}$ then

$$\zeta^d = \sum_D \left( \frac{d}{d_1 \cdots d_m} \right) \left( \frac{\lambda_1}{2} \right)^{d_1} \cdots \left( \frac{\lambda_m}{2} \right)^{d_m} v_1^{2d_1} \cdots v_m^{2d_m}$$

whence

$$\frac{\|\zeta^d\|^2}{(d!)^2} = \sum_D \left( \frac{2d_1}{d_1} \right) \cdots \left( \frac{2d_m}{d_m} \right) \left( \frac{\lambda_1}{2} \right)^{2d_1} \cdots \left( \frac{\lambda_m}{2} \right)^{2d_m}$$

where summation takes place over all multiindices $D = (d_1, \ldots, d_m) \in \mathbb{N}^m$ for which $d_1 + \cdots + d_m = d$. It follows that

$$\sum_{d \geq 0} \frac{\|\zeta^n\|^2}{(d!)^2} = \sum_{d_1 \geq 0} \left( \frac{2d_1}{d_1} \right) \left( \frac{\lambda_1}{2} \right)^{2d_1} \cdots \sum_{d_m \geq 0} \left( \frac{2d_m}{d_m} \right) \left( \frac{\lambda_m}{2} \right)^{2d_m}$$

$$= (1 - \lambda_1^2)^{-1/2} \cdots (1 - \lambda_m^2)^{-1/2}$$

$$= \text{Det}^{1/2}(I - Z^2)^{-1}$$

provided that each of the nonnegative numbers $\lambda_1, \ldots, \lambda_m$ is strictly less than unity. In the opposite direction, if (say) $\lambda = \lambda_k \geq 1$ then

$$\sum_{d \geq 0} \left( \frac{2d}{d} \right) \left( \frac{\lambda}{2} \right)^{2d} = \infty$$

and this divergence already prohibits $e^Z$ from membership in $S[V]$. \qed

For convenience, let us denote by $\overline{D}(V)$ the set of all symmetric antilinear maps $Z : V \to V$ such that $\|Z\| \leq 1$ and let $D(V)$ comprise those $Z$ that satisfy $\|Z\| < 1$; these are versions of closed and open Siegel domains. Let us also write $G(V)$ for the set comprising all those (necessarily invertible) complex-linear maps $T : V \to V$ such that if $0 \neq v \in V$ then $\text{Re}\langle v | Tv \rangle > 0$: it is readily verified that $G(V)$ is a convex open neighbourhood of the identity in the complex general linear group on $V$; consequently, $\text{Det}$ has a holomorphic
square-root $\text{Det}^{1/2} : \mathcal{G}(V) \to \mathbb{C}$. Now, if $X$ and $Y$ lie in the closed Siegel domain $\overline{\mathcal{D}}(V)$ and $I - YX$ is invertible then in fact $I - YX \in \mathcal{G}(V)$: indeed, if $v \in V$ then

$$2\text{Re}\langle v|(I - YX)v\rangle = (\|v\|^2 - \|Xv\|^2) + (\|v\|^2 - \|Yv\|^2) + \|Xv - Yv\|^2;$$

this vanishes only when $Xv = Yv$ and $(I - Y^2)v = 0$ which forces $(I - YX)v = 0$ and therefore $v = 0$ by the supposed invertibility of $I - YX$. Of course, the invertibility of $I - YX$ is certainly guaranteed if either $X$ or $Y$ actually lies in the open Siegel domain $\mathcal{D}(V)$. After these preparatory comments, we may evaluate the inner product between a pair of Gaussians in Fock space.

**Theorem 5.** If $X, Y \in \mathcal{D}(V)$ then

$$\langle e^X|e^Y \rangle = \text{Det}^{1/2}(I - YX)^{-1}.$$

**Proof.** Both sides of the claimed formula are antiholomorphic in $X$ and holomorphic in $Y$. Equality on the diagonal of $\mathcal{D}(V) \times \mathcal{D}(V)$ is established in Theorem 4; equality on the whole of $\mathcal{D}(V) \times \mathcal{D}(V)$ follows by the principle of analytic continuation.

For a variety of purposes (notably for half-form pairings within geometric quantization; see [5] and [7] for details) this inner product formula must be extended beyond the open Siegel domain. Let $X, Y \in \overline{\mathcal{D}}(V)$ be such that $I - YX$ is invertible; in this case, recall that $I - YX \in \mathcal{G}(V)$ and therefore that $\text{Det}^{1/2}(I - YX)^{-1}$ is defined. Thus, the right side of the inner product formula in Theorem 5 makes perfectly good sense; by contrast, the left side makes sense as an inner product only when $X, Y \in \mathcal{D}(V)$ for only then do the associated Gaussians lie in Fock space. It is tempting to replace the left side by the bosonic pairing $\langle e^X : e^Y \rangle_1$ and indeed this works when $V$ is one-dimensional.

**Theorem 6.** Let $V$ be one-dimensional. If $X, Y \in \overline{\mathcal{D}}(V)$ are such that $I - YX$ is invertible then

$$\langle e^X : e^Y \rangle_1 = \text{Det}^{1/2}(I - YX)^{-1}.$$

**Proof.** For convenience, choose and fix a conjugation $\sigma$ on $V$: thus, $\sigma^2 = I$ and if $x, y \in V$ then $\langle \sigma x|\sigma y \rangle = \langle y|x \rangle$; further, let $u \in V$ be one of the two
unit vectors fixed by the conjugation. There exist complex scalars $a$ and $b$ in the closed unit disc such that $X = a\sigma$ and $Y = b\sigma$ whence $I - YX = (1 - ab)I$ is invertible iff $ab \neq 1$; the quadratics in $S^2V$ that correspond to $X$ and $Y$ are $\xi = \frac{1}{2}au^2$ and $\eta = \frac{1}{2}bu^2$ respectively. Now, if $d \in \mathbb{N}$ then $\langle \xi^d|\eta^d \rangle = (2d)! \left(\frac{ab}{4}\right)^d$ whence by summation

$$\langle e^X : e^Y \rangle_1 = \sum_{d \geq 0} \left(\frac{2d}{d}\right) (\frac{ab}{4})^d = (1 - ab)^{-1/2} = \text{Det}^{1/2}(I - YX)^{-1}.$$ Evaluation of the sum in this argument is provided by the general binomial theorem (see [4] Item 247).

Unfortunately, this naïve approach fails when $V$ has complex dimension $m > 1$. To see this by example, let $Z : V \to V$ be a conjugation and let $\zeta \in S^2V$ be the corresponding quadratic: in this case, it may be checked that if $d \geq 0$ then

$$\frac{\|\zeta^{d+1}\|^2}{(d+1)!^2} \frac{\|\zeta^d\|^2}{d!^2} = \frac{d + \frac{1}{2}m}{d + 1}$$

so the formal series

$$\langle e^Z : e^{-Z} \rangle_1 = \sum_{d \geq 0} (-1)^d \|\zeta^d\|^2/d!^2 \geq 1$$

cannot converge, although of course $I + Z^2$ is invertible and $\text{Det}^{1/2}(I + Z^2)^{-1}$ is defined since $Z^2$ is a positive operator.

Accordingly, we extend our definition of the bosonic pairing by means of a regularization. As preparation, fix $0 < t < 1$ and define $\langle \cdot : \cdot \rangle_t$ by scaling $S^dV$ by $t^d$ for each $d \geq 0$: explicitly, define

$$\Phi, \Psi \in SV' \Rightarrow \langle \Phi : \Psi \rangle_t = \sum_{d \geq 0} \langle \Phi_d|\Psi_d \rangle t^{2d}$$

whenever the indicated series is convergent.

**Definition**: The (extended) bosonic pairing is defined by the rule

$$\Phi, \Psi \in SV' \Rightarrow \langle \Phi : \Psi \rangle = \lim_{t \uparrow 1} \langle \Phi : \Psi \rangle_t$$

whenever the indicated limit exists.
We hasten to point out at once that we have indeed fashioned an extension of the original bosonic pairing.

**Theorem 7.** Let \( \Phi, \Psi \in SV' \). If \( \langle \Phi : \Psi \rangle_1 \) is defined then \( \langle \Phi : \Psi \rangle \) is defined and has the same value.

**Proof.** An immediate consequence of the Abel limit theorem for complex power series (see [4] Item 232).

Thus, all our previous evaluations of bosonic pairings carry over into this extended context.

Standard Tauberian theorems guarantee that where \( \langle : \rangle \) is defined, \( \langle : \rangle_1 \) is also defined under certain conditions. The very simplest of these theorems (due to Pringsheim: see Example B27 on page 251 of [3] and the Theorem following Item 101 in [4]) implies that if \( \Phi \in SV' \) and \( \langle \Phi : \Phi \rangle \) is defined then \( \langle \Phi : \Phi \rangle_1 \) is defined. However, this extended bosonic pairing is a strict extension of the original, as we proceed to see explicitly.

In fact, we are now able to pair Gaussians as we would wish.

**Theorem 8.** If \( X,Y \in \overline{D}(V) \) are such that \( I - YX \) is invertible then

\[
\langle e^X : e^Y \rangle = \text{Det}^{1/2}(I - YX)^{-1}.
\]

**Proof.** Let \( X,Y \) correspond to the quadratics \( \xi,\eta \) in the usual manner. Fix \( 0 < t < 1 \) and notice that \( tX,tY \in D(V) \): if \( d \geq 0 \) then of course \( \langle \xi^d|\eta^d \rangle t^{2d} = \langle (t\xi)^d|(t\eta)^d \rangle \) whence summation yields

\[
\langle e^X : e^Y \rangle_t = \text{Det}^{1/2}(I - t^2YX)^{-1}
\]

according to Theorem [5]. The continuity of \( \text{Det}^{1/2} \) on \( G(V) \) as noted prior to Theorem [5] permits us to let \( t \uparrow 1 \) and complete the proof.

We close with some remarks on the process of regularization by which we extended the bosonic pairing. On the one hand, it is already well-established as a technique for taming divergent series, under the name A-summability (after Abel: see [4] Sections 59 and 61). On the other hand, it naturally incorporates the grading of the symmetric algebra; indeed, the (extended) bosonic pairing is actually invariant under the corresponding unitary group. Explicitly, let \( U_N(SV) \) denote the group comprising all those unitary automorphisms \( U \) of \( SV \) that respect the grading in that if \( d \geq 0 \) then \( U(S^dV) = S^dV \). Naturally, each \( U \in U_N(SV) \) acts antidually on \( SV' \): thus, if \( \Phi \in SV' \) and \( \psi \in SV \) then \( [U\Phi](\psi) = \Phi[U^*\psi] \); further, if also \( d \geq 0 \) then \( (U\Phi)_d = U(\Phi_d) \).
**Theorem 9.** Let $U \in U_N(SV)$ and let $\Phi, \Psi \in SV'$. If $\langle \Phi : \Psi \rangle$ is defined then so is $\langle U\Phi : U\Psi \rangle$ and
\[
\langle U\Phi : U\Psi \rangle = \langle \Phi : \Psi \rangle.
\]

**Proof.** In fact, if $0 < t < 1$ then plainly $\langle U\Phi : U\Psi \rangle_t = \langle \Phi : \Psi \rangle_t$ and taking the limit as $t \uparrow 1$ concludes the argument. \hfill \Box

In particular, the functorial extension of each element of the unitary group $U(V)$ to a unitary automorphism of $SV \subset SV'$ preserves the bosonic pairing.

In contrast, the bosonic pairing is not invariant under the ‘full’ unitary group $U(SV)$ even when $V$ is one-dimensional. To see this, choose a unit vector $v \in V$ and for $d \geq 0$ let $v_d = v^d / \sqrt{d!}$ so that $SV$ has $(v_d : d \geq 0)$ as unitary basis: in this way, we identify $SV$ with the space $\mathfrak{s}$ comprising all finitely-nonzero complex sequences and $SV'$ with the space $\mathfrak{s}'$ comprising all complex sequences; furthermore, if the complex sequences $\lambda = (\lambda_d : d \geq 0)$ and $\mu = (\mu_d : d \geq 0)$ lie in $\mathfrak{s}'$ and are bounded then $\langle \lambda : \mu \rangle_t = \sum_{d \geq 0} (\lambda_d \mu_d) t^{2d}$ when $0 < t < 1$. Now, for example, let $U \in U(SV)$ be defined by fixing $v_0$ and interchanging $v_{2n-1}$ with $v_{2n}$ when $n \geq 1$; further, let $\lambda_d = 1$ and $\mu_d = (-1)^d$ for $d \geq 0$. If $0 < t < 1$ then
\[
\langle \lambda : \mu \rangle_t = 1 - t^2 + t^4 - t^6 + \cdots = 1/(1 + t^2)
\]
while
\[
\langle U\lambda : U\mu \rangle_t = 1 + t^2 - t^4 + t^6 - \cdots = 1 + t^2/(1 + t^2)
\]
so that $\langle \lambda : \mu \rangle = 1/2$ while $\langle U\lambda : U\mu \rangle = 3/2$.

A full investigation of the bosonic pairing, for a one-particle space of arbitrary dimension, promises to be both useful and interesting.

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