Bulk Viscosity of a Gas of Massless Pions

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Abstract

In the hadronic phase, the dominant configuration of QCD with two flavors of massless quarks is a gas of massless pions. We calculate the bulk viscosity (ζ) using the Boltzmann equation with the kinetic theory generalized to incorporate the trace anomaly. We find that the dimensionless ratio ζ/s, s being the entropy density, is monotonic increasing below $T = 120$ MeV, where chiral perturbation theory is applicable. This, combined with previous results, shows that ζ/s reaches its maximum near the phase transition temperature $T_c$, while η/s, η being the shear viscosity, reaches its minimum near $T_c$ in QCD with massless quarks.
I. INTRODUCTION

Transport coefficients of Quantum Chromodynamics (QCD) are of high interests recently. This was triggered by the discovery that quark gluon plasma (QGP) has a viscosity close to the conjectured universal minimum bound [1], indicating that QGP is close to a "perfect fluid" [2, 3, 4] just above the deconfinement temperature. This bound, $\eta/s \geq 1/4\pi$, $s$ being the entropy density, is motivated by the uncertainty principle and is found to be saturated for a large class of strongly interacting quantum field theories whose dual descriptions in string theory involve black holes in anti-de Sitter space [5, 6, 7, 8]. There are some debates about whether the minimum bound on $\eta/s$ is truly universal [9, 10, 11] and the RHIC data might be better fit with $\eta/s < 1/4\pi$ [12, 13] (lattice results for gluon plasma, however, is still consistent with the bound [14, 15]). In any case, smaller $\eta$ implies stronger interparticle interaction (here $\eta$ is normalized by the density) and the smallness of QGP $\eta$ indicating an intriguing strongly interacting state is reached near the deconfinement temperature.

In general, the minimum of $\eta/s$ is found near the phase transition temperature $T_c$ or when the system goes through a fast crossover. This behavior was observed [1, 16, 17] in all the materials, including N, He, and H$_2$O, with data available in the NIST and CODATA websites [18, 19]. Surprisingly, it is also observed in QCD at zero chemical potential [17, 20], near the nuclear liquid gas phase transition [16, 21], and in cold fermionic atom systems at the limit with two-body scattering length tuned to infinity [22]. Using weakly-coupled real scalar field theories, in which perturbation is reliable, the same $\eta/s$ behaviors in first-, second-order phase transitions and crossover also emerge as in the liquid-gas transitions in N, He, and H$_2$O and essentially all the matters with data available in the NIST database mentioned above [23]. This agreement is expected to hold when the theory is generalized to $N$ components with an $O(N)$ symmetry. Thus, these behaviors might be general properties of fluid and might be used to probe the QCD critical end point [24].

Less well studied is the bulk viscosity ($\zeta$) of QCD. In general, bulk viscosity vanishes when a system is conformally invariant such that the system is invariant under a uniform expansion (dilatation). For a non-interacting non-relativistic or ultrarelativistic system (assuming the interaction is turned off after thermal equilibrium), the system is conformally invariant and hence has zero bulk viscosity. When the interaction is turned on, conformal symmetry could be broken to give a finite bulk viscosity. (A notable exception
is the infinite scattering length limit where conformal symmetry is preserved \[25, 26, 27\].

In QCD with heavy quarks integrated out and with the light quark masses set to zero, conformal symmetry is broken in the quantum level. In the perturbative region, up to some logarithmic corrections, \(\zeta /s \propto \alpha_s^{-2} (1/3 - v_s^2)^2 \propto \alpha_s^2\) [28] while \(\eta /s \propto \alpha_s^{-2}\) [29, 30]. Thus, \(\zeta\) is smaller than \(\eta\) in the perturbative regime. When the temperature is reduced, \(\eta /s\) reaches its minimum near \(T_c\), while \(\zeta /s\) rises sharply near \(T_c\) [31, 32, 33]. It will be interesting to see whether the maximum of \(\zeta /s\) is also reached near \(T_c\) from below, which is the main purpose of this work. We will focus on the case with two flavors of massless quarks such that below \(T_c\) the dominant degrees of freedom are massless pions.

II. LINEARIZED BOLTZMANN EQUATION AND THE GENERALIZED KINEMATIC THEORY

The bulk viscosity of a system is defined by the Kubo formula

\[
\zeta = \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int d^3 r e^{i\omega t} \langle [T_\mu(x), T_\nu(0)] \rangle ,
\]

(1)

with \(T_\mu\) the trace of the energy momentum tensor. The Kubo formula involves an infinite number of diagrams at the leading order (LO) even in the weakly-coupled \(\phi^4\) theory [34]. However, it is proven that the summation of LO diagrams in a weakly coupled \(\phi^4\) theory [34] or in hot QED [35, 36] is equivalent to solving the linearized Boltzmann equation with temperature-dependent particle masses and scattering amplitudes. Since the proofs do not use properties restricted to scalar theories, the conclusion is expected to hold for more general theories with weak couplings, including QCD in the perturbative regime [28, 29, 30]. Here, we assume the equivalence between the Kubo formula and the Boltzmann equation also applies to massless pions.

The Boltzmann equation describes the evolution of the isospin averaged pion distribution function \(f = f(x, p, t) \equiv f_p(x)\) (a function of space, time and momentum)

\[
\frac{p^\mu}{E_p} \partial_\mu f_p(x) = \frac{g_\pi}{2} \int_{123} d\Gamma_{12;3p} \left\{ f_1 f_2 (1 + f_3) (1 + f_p) - (1 + f_1) (1 + f_2) f_3 f_p \right\} ,
\]

(2)

where \(E_p = \sqrt{p^2 + m_\pi^2}\), \(p = |p|\) and \(g_\pi = 3\) is the degeneracy factor for three pions,

\[
d\Gamma_{12;3p} = \frac{1}{2E_p} |T|^2 \prod_{i=1}^3 \frac{d^3k_i}{(2\pi)^3(2E_i)} \times (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - p) ,
\]

(3)
and where $\mathcal{T}$ is the scattering amplitude for particles with momenta $1, 2 \rightarrow 3, p$. In chiral perturbation theory ($\chi$PT), which is a low-energy effective field theory of QCD, the LO isospin averaged $\pi\pi$ scattering amplitude in terms of Mandelstam variables ($s, t,$ and $u$) is

$$|\mathcal{T}|^2 = \frac{1}{9f_\pi^4} \left\{ 9s^2 + 3(t - u)^2 \right\} ,$$

where $f_\pi = 88.3$ MeV is the pion decay constant in the chiral limit. The pions remain massless below $T_c$ and the temperature dependence of the scattering amplitude is of higher order and will be neglected.

In local thermal equilibrium, the distribution function $f_p^{(0)}(x) = \left( e^{\beta(x)V_\mu(x) p^\mu} - 1 \right)^{-1}$, where $\beta(x) = 1/T(x)$ is the inverse temperature and $V^\mu(x)$ is the four velocity of the fluid at the space-time point $x$. A small deviation of $f_p$ from local equilibrium is parametrized as

$$f_p(x) = f_p^{(0)}(x) + \delta f_p(x) ,$$
$$\delta f_p(x) = -f_p^{(0)}(x) \left[ 1 + f_p^{(0)}(x) \right] \chi_p(x) .$$

In kinetic theory, the energy momentum tensor in a weakly interacting system is

$$T_{\mu\nu}(x) = g_\pi \int \frac{d^3p}{(2\pi)^3} \frac{f_p(x)}{E_p} p_\mu p_\nu .$$

It is the sum of the energy momentum tensor of each particle with inter-particle interactions neglected. This is usually a good approximation when the interparticle spacing is much larger than the range of interaction such that the potential energy is negligible.

The conservation of energy momentum tensor, $\partial^\mu T_{\mu\nu} = 0$, is automatically satisfied by the Boltzmann equation. We will decompose $T_{\mu\nu}$ as

$$T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta T_{\mu\nu} ,$$

where $\delta T_{\mu\nu}$ is the deviation from the thermal equilibrium part $T_{\mu\nu}^{(0)}$.

$$T_{\mu\nu}^{(0)} = (\epsilon + P) V_\mu V_\nu - P g_{\mu\nu} ,$$

where $\epsilon$ is the energy density and $P$ is the pressure.

We will work at the $V(x) = 0$ frame for the point $x$. This implies $\partial_\mu V^\mu = 0$ after taking a derivative on $V_\mu(x) V^\mu(x) = 1$. The conservation law at local thermal equilibrium, $\partial^\mu T_{\mu\nu}^{(0)} = 0$, implies

$$\partial_t \epsilon + (\epsilon + P) \nabla \cdot V = 0 ,$$
$$\partial_\nu V + (\epsilon + P)^{-1} \nabla P = 0 .$$
Then using the thermal dynamic relation
\[ \epsilon + P = T \frac{\partial P}{\partial T} , \] (10)
one has
\[ \beta \partial_t V - \nabla \beta = 0 , \]
\[ \partial_t \beta - \beta v_s^2 \nabla \cdot V = 0 , \] (11)
where \( v_s^2 = \frac{\partial P}{\partial \epsilon} \) is the speed of sound.

The shear and bulk viscosity are defined by the small deviation away from equilibrium:
\[ \delta T_{ij} = -2\eta \left( \frac{\nabla_i V_j(x) + \nabla_j V_i(x)}{2} - \frac{1}{3} \delta_{ij} \nabla \cdot V(x) \right) - \zeta \delta_{ij} \nabla \cdot V(x) , \] (12)
where \( i \) and \( j \) are spacial indexes and Eq.(11) is used to replace the time derivatives \( \partial_t \beta \) and \( \partial_t V \) by spacial derivatives \( \nabla \cdot V \) and \( \nabla \beta \). Also, \( \delta T_{0i}(x) = 0 \), since the momentum density at point \( x \) is zero in the \( V(x) = 0 \) frame. Furthermore, if there is no viscosity, the energy density at the same point will only be a function of \( T \) governed by thermodynamics, which implies \( \delta T_{00} = 0 \). Viscosity could generate heat during the perturbation. However, the amount of heat generated should be time reversal even, because heat will be generated no matter whether the system is expanding or contracting. However, there is no first derivative term which is even under time reversal. Thus, at this order,
\[ \delta T_{00} = 0 . \] (13)

It is easy to see why \( \zeta \simeq 0 \) for ultrarelativistic and monatomic non-relativistic systems based on Eqs. (6) and (13). For ultrarelativistic systems, \( p^2 \simeq 0 \); therefore, \( T_{\mu}^{\mu} \simeq 0 \) by Eq. (6). For non-relativistic systems, if the particle number for each species is conserved, then \( \delta T_{i}^{i} = 2\delta T_{0}^{0} = 0 \) and, hence, \( \zeta = 0 \). These are general results of the kinetic theory which assumes the potential energy from short-range interactions is negligible in a dilute system. They can be traced back to the conformal symmetry of non-interacting ultrarelativistic and non-relativistic systems. When interactions are turned on and the conformal symmetry is broken, Eq.(6) has to be modified to include the effect of interaction in order to give the leading non-vanishing \( \zeta \) result.

For pions in the chiral limit, they always satisfy the dispersion relation \( p^2 = 0 \) even at finite \( T \). This is because their goldstone boson nature prevents them from generating
thermal masses. However, this does not imply that the system is traceless. Direct computation using \( \chi \)PT shows that trace anomaly first appears at the order of three loops \[37\]. This is the manifestation of the gluon trace anomaly operator of QCD. In the expression of Eq. (6), \( T^\mu_\mu = 0 \) once \( p^2 = 0 \). Thus, it needs to be generalized to have non-zero \( T^\mu_\mu \).

In principle, one could add two-pion, three-pion... distribution amplitudes to take into account the pion interaction associate with the loop diagrams. However, one can integrate out the medium effect and sum up the effective one-pion contributions to \( T^{\mu \nu} \)

\[
T^{\mu \nu} = \sum_i \left< \pi_i \left| \hat{T}^{\mu \nu} \right| \pi_i \right>,
\]

where \( \hat{T}^{\mu \nu} \) is the energy momentum operator. Note that Eq. (6) is just the leading order effect of the above equation which takes into account the free pion contribution to \( T^{\mu \nu} \) only. Using symmetries, \( T^{\mu \nu} \) has the general form:

\[
T^{\mu \nu}(x) = g_\pi \int \frac{d^3p}{(2\pi)^3} f_p(x) \left[ p_\mu p_\nu (1 + g_1(x)) + \frac{g_2(x) g_{\mu \nu}}{\beta(x)^2} + \frac{g_3(x) V_\mu(x) V_\nu(x)}{\beta(x)^2} \right].
\]

Here Lorentz symmetry is broken down to \( O(3) \) symmetry by the temperature, and \( g_{1-3} \) are dimensionless functions of \( \beta(x) \) and \( f_\pi \). In \( \chi \)PT, \( g_{1-3} = \mathcal{O}(T^4/(4\pi f_\pi)^4) \)[37]. The structure \((p^\mu V^\nu + V^\mu p^\nu)\) is not allowed because the \( \pi^+ \) and \( \pi^- \) matrix elements should be the same by charge conjugation or isospin symmetry. Thus, \( \left< \pi_i(p) \left| \hat{T}^{\mu \nu} \right| \pi_i(p) \right> \) should be invariant under crossing symmetry \((p^\mu \rightarrow -p^\mu)\). In equilibrium, \( T^{\mu \nu(0)} = \epsilon - 3P \) and

\[
c \equiv 4g_2 + g_3 = \frac{\epsilon - 3P}{\beta^2 \int \frac{d^3p}{(2\pi)^3} f_p^{(0)}}.
\]

Note that energy momentum conservation is not a problem with the new terms in Eq. (15). In equilibrium, one just has to replace \( \nu_s^2 \) in Eq. (11) by the new value to obtain \( \partial^{\mu} T^{\mu(0)}_{\nu} = 0 \). Away from equilibrium, the net effect of \( \zeta \) is to replace \( P \rightarrow P - \zeta \nabla \cdot V \) in Eq. (9) which will induce second spacial derivative terms in Eq. (11). Thus, as long as Eq. (15) gives the correct \( T^{\mu \nu} \), energy momentum conservation can be satisfied.

Working to the first order in a derivative expansion, \( \chi_\rho(x) \) can be parametrized as

\[
\chi_\rho(x) = \beta(x) A(p) \nabla \cdot V(x) + \beta(x) B(p) \left( \hat{p}_i \hat{p}_j - \frac{1}{3} \delta_{ij} \right) \left( \frac{\nabla_i V_j(x) + \nabla_j V_i(x)}{2} - \frac{1}{3} \delta_{ij} \nabla \cdot V(x) \right),
\]

where \( A \) and \( B \) are functions of \( x \) and \( p \). But we have suppressed the \( x \) dependence.

Substituting (17) into the Boltzmann equation and using Eq. (11), one obtains one
linearized equation for $A$ (associated with the $\nabla \cdot \mathbf{V}$ structure):

$$
\frac{1}{3} p^2 - v_s E_p^2 = \frac{g_x E_p}{2} \int_{123} d\Gamma_{12;3p} (1 + n_1)(1 + n_2) n_3 (1 + n_p)^{-1} \\
\times [A(p) + A(k_3) - A(k_2) - A(k_1)],
$$

(18)

where at point $x$, $f_i^{(0)}(x)$ is written as $n_i = (e^{\beta E_i} - 1)^{-1}$. There is also a linearized equation for $B$ (associated with the $(\nabla_i V_j + \nabla_j V_i - \text{trace})$ structure) that is related to the shear viscosity $\eta$. The computation of $\eta$ of the pion gas has been discussed in Ref. [20]. We will focus on solving $\zeta$ in this work.

### III. VARIATIONAL CALCULATION

Equation (18) only determines $A(p)$ up to a combination $a_1 + a_2 E_p$, where $a_1$ and $a_2$ are constants [34]. These “zero modes” ($a_1$ and $a_2 E_p$) only appear in the analysis of bulk viscosity but not shear viscosity. We will discuss their effects in this section.

The variation of Eq. (15) yields

$$
\delta T_{\mu\nu} = g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ p_\mu p_\nu (1 + g_1) + \frac{g_2 g_{\mu\nu}}{\beta^2} + \frac{g_3 V_\mu V_\nu}{\beta^2} \right] \\
+ f_p \left[ p_\mu p_\nu \delta g_1 + \frac{\delta g_2 g_{\mu\nu}}{\beta^2} + \frac{\delta g_3 V_\mu V_\nu}{\beta^2} \right] \right\}.
$$

(19)

Note that $g_{1-3}$ represent loop corrections of the energy momentum tensor, thus they are functionals of $f_p$. To compute $\zeta$, we need

$$
\delta T_{ii} = g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ p^2 (1 + g_1) - \frac{3g_2}{\beta^2} \right] + f_p \left[ p^2 \delta g_1 - \frac{3\delta g_2}{\beta^2} \right] \right\}.
$$

(20)

This can be simplified using the constraint,

$$
0 = \delta T_{00} = g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ p^2 (1 + g_1) + \frac{g_2 + g_3}{\beta^2} \right] + f_p \left[ p^2 \delta g_1 + \frac{\delta g_2 + \delta g_3}{\beta^2} \right] \right\}.
$$

(21)

After eliminating the $g_2/\beta^2$ term in $\delta T_{ii}$ using the constraint, we have

$$
\delta T_{ii} = g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ 4p^2 (1 + g_1) \frac{4g_2 + g_3}{g_2 + g_3} \right] \\
+ f_p \left[ p^2 \delta g_1 \frac{4g_2 + g_3}{g_2 + g_3} + \frac{3(g_2 \delta g_3 - g_3 \delta g_2)}{(g_2 + g_3)^2} \right] \right\} \\
\approx 4g_\pi d \int \frac{d^3 p}{(2\pi)^3} p \delta f_p \left( 1 + \mathcal{O}\left( \frac{T^4}{(4\pi f_\pi)^4} \right) \right),
$$

(22)
where \( d = \frac{(4g_2 + g_3)}{(g_2 + g_3)} \) and the pion remains massless in the chiral limit even at finite \( T \), so we have used \( p^2 = E_p^2 \). The above expression for \( \delta T_{ii} \) implies

\[
\zeta = \frac{4}{3} g_\pi \beta d \int \frac{d^3 p}{(2\pi)^3} E_p n_p \left(1 + n_p\right) A(p) .
\]

(23)

Then using Eq. (18) and the symmetry property of the scattering amplitude,

\[
\zeta = \frac{g_2^2 \beta d}{2(1 - 3v_s^2)} \int \prod_{i=1,2,3,p} \frac{d^3 k_i}{(2\pi)^3(2E_i)} |T|^2 (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - p) \\
\times (1 + n_1)(1 + n_2)n_3 n_p [A(p) + A(k_3) - A(k_2) - A(k_1)]^2 .
\]

(24)

Note that equating Eqs. (23) and (24) is equivalent to taking a projection of Eq. (18).

It can be shown that any ansatz satisfying Eqs. (23) and (24) gives a lower bound on \( \zeta \). Thus, one can solve \( \zeta \) variationally, i.e. finding an ansatz \( A(p) \) that gives the biggest \( \zeta \).

It is known that if one uses the ansatz \( A(p) = a_1 + a_2 E_p \), then it will not contribute to the \( 2 \to 2 \) scattering on the right-hand side of Eq. (18) (the \( a_2 \) terms cancel by energy conservation). In fact, this ansatz will not contribute to all the particle number conserving processes but can contribute to particle number changing processes, such as \( 2 \leftrightarrow 4 \) scattering, which we have not shown. As we know from Eqs. (18) and (23), \( \zeta \) is proportional to the size of \( A(p) \) which is inversely proportional to rate of scattering. Thus, if the \( 2 \to 2 \) scattering has a bigger rate than the \( 2 \leftrightarrow 4 \) scattering, then this ansatz gives a bigger \( \zeta \) by bypassing the faster \( 2 \to 2 \) scattering. In \( \phi^4 \) theory, it was found that \( \zeta \) is indeed set by the \( 2 \leftrightarrow 4 \) scattering [34]. However, in perturbative QCD (PQCD), the soft particle number changing bremsstrahlung is faster than the \( 2 \to 2 \) scattering [28]. Thus, \( \zeta \) is governed by \( 2 \to 2 \) scattering.

In the case with massless pions, however, \( 2 \to 2 \) scattering is still the dominant process. While using the ansatz \( A(p) = a_1 + a_2 E_p \), the \( \delta T_{00} = 0 \) constraint in Eq.(21) demands \( a_1/a_2 = 0 \) because \( n_p \propto 1/p \) as \( p \to 0 \). Since \( A(p) \) parametrizes a small deviation of \( f_p \) away from thermal equilibrium, \( a_1/a_2 = 0 \) gives \( a_1 = 0 \) instead of \( a_2 \to \infty \) and \( a_1 \) finite. Thus, to maximize \( \zeta \), we use the ansatz \( A(p) = a_2 E_p + a_3 E_p^2 + \ldots \) without the \( a_1 \) term. The point is, \( 2 \to 2 \) scattering cannot be bypassed and it will be the dominant process in our calculation.

To compute \( \zeta \), it is easier to eliminate the \((1 + g_1)\) term in Eq.(20) using Eq.(21):
\[ 
\delta T_{ii} = -g_\pi \int \frac{d^3p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ \frac{4g_2 + g_3}{\beta^2} \right] 
+ f_p \left[ \frac{4\delta g_2 + \delta g_3}{\beta^2} \right] \right\} .
\]

(25)

Note that \( g_2 \) and \( g_3 \) terms at \( \mathcal{O}(T^4/(4\pi f_\pi)^4) \) arise from three-loop diagrams and from two-loop diagrams with insertions of higher order counterterms and each loop integral has one power of \( f_\pi \) in the integrand. Thus, we will make an approximation here to assume the \((4\delta g_2 + \delta g_3)\) term is proportional to the \( \delta f_p \) term with a proportional constant \((l - 1)\), where \( l \) means the power of \( f_\pi \) (or the number of loops) in \( T_{ii} \). Since \( l \) is between 2 and 3, we take the mean value \( l = 2.5 \) and associate the uncertainty of \( l \) to the error estimation of \( \zeta \).

The trace anomaly for massless pions appears from three-loop diagrams and from two-loop diagrams with insertions of higher order counterterms [all are \( \mathcal{O}(T^8/f_\pi^4) \)] \[37\]. Thus, for two-loop diagrams, the associated \( l \) factor is 2 while for three-loop diagrams, the associated \( l \) factor is 3. Here, without distinguishing the contribution from each diagram, we take the mean value \( l = 2.5 \) and associate the uncertainty of \( l \) to the error estimation of \( \zeta \). Thus,

\[ 
\zeta = -\frac{g_\pi lc}{3\beta} \int \frac{d^3p}{(2\pi)^3 E_p} \frac{1}{n_\rho} (1 + n_\rho) A(p) .
\]

(26)

Note that \( A(p) \propto g_\pi^{-1} \left( \frac{1}{3} - v_s^2 \right) f_\pi^4 \) from Eq. \([18]\). Thus, for massless pions,

\[ 
\zeta = hl (\epsilon - 3\mathcal{P}) \left( \frac{1}{3} - v_s^2 \right) \frac{f_\pi^4}{T^5} ,
\]

(27)

where \( T^5 \) is given by dimensional analysis and \( h \) is a dimensionless constant. To find the numerical solution for \( h \), we neglect the higher-order \( g_{1-3} \) terms in Eq. \([21]\) and use the ansatz \( A(p) = \sum_{n=1}^m c_n p^n \). We find

\[ 
h \simeq 65 .
\]

(28)

Using the \( \chi \)PT result of Ref. \[37\] for \( \epsilon \) and \( \mathcal{P} \), we obtain

\[ 
\zeta \simeq 0.15 \left( \frac{l}{2.5} \right) \left( \ln \frac{\Lambda_\rho}{T} - \frac{1}{4} \right) \left( \ln \frac{\Lambda_\rho}{T} - \frac{3}{8} \right) \frac{T^7}{f_\pi^4} ,
\]

(29)

where \( \Lambda_\rho \simeq 275 \text{ MeV} \). As expected, the bulk viscosity vanishes as \( f_\pi \to \infty \) or when the coupling between pions vanishes.
The leading order contribution for pion entropy density $s$ is just the result for a free pion gas:

$$s = \frac{2\pi^2 g_\pi T^3}{45}. \quad (30)$$

The trace anomaly for massless pions appears from three-loop diagrams and from two-loop diagrams with insertions of higher order counterterms [all are $\mathcal{O}(T^8/f_\pi^4)$] \cite{37}. Thus, for two-loop diagrams, the associated $l$ factor is 2 while for three-loop diagrams, the associated $l$ factor is 3. Here, without distinguishing the contribution from each diagram, we take the mean value $l = 2.5$ and associate the uncertainty of $l$ to the error estimation of $\zeta$.

$$\delta T_{\mu\nu} \simeq g_\pi l \int \frac{d^3 p}{(2\pi)^3 E_p} \delta f_p \left[ p_\mu p_\nu (1 + g_1) + \frac{g_2 g_{\mu\nu}}{\beta^2} + \frac{g_3 V_\mu V_\nu}{\beta^2} \right].$$

Using Eqs. (15) and (12),

$$\delta T_{ii}(x) = -g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ \frac{4g_2 + g_3}{\beta(x)^2} \right] + f_p \left[ \frac{4\delta g_2 + \delta g_3}{\beta(x)^2} \right] \right\}. \quad (31)$$

$$\delta T_{ii}(x) = g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left\{ \delta f_p \left[ 4p^2 (1 + g_1)(4g_2 + g_3) \right] + f_p(x) \left[ (4g_2 + g_3)p^2 \delta g_1 + \frac{3(g_2 \delta g_3 - g_3 \delta g_2)}{\beta^2} \right] \right\}. \quad (32)$$

where the factor $l$ appears because in the computation of $\delta T_{ii}$ from Eq. (15), not only $f_p$ but also $g_{1-3}$ depend on $\delta f_p$. In $\chi$PT, the trace anomaly for massless pions appears from three-loop diagrams and from two-loop diagrams with insertions of higher order counterterms [all are $\mathcal{O}(T^8/f_\pi^4)$] \cite{37}. For two-loop diagrams, the associated $l$ factor is 2 while for three-loop diagrams, the associated $l$ factor is 3. Here, without distinguishing the contribution from each diagram, we take the mean value $l = 2.5$ and associate the uncertainty of $l$ to the error estimation of $\zeta$.

The above-mentioned constraint, $\delta T_{00} = 0$, yields,

The dimensionless combination $\zeta/s$ is shown in Fig. 1. The solid line below $T_c$ is the leading order massless pion gas result (we have used $l = 2.5$, explained below Eq. (20), and the lattice result, $T_c \approx 200$ MeV, for 2+1 flavors of improved staggered fermion as an estimation \cite{39}). The error on this curve is estimated to be 30%-40% from $l$ and higher-order corrections. But the monotonic increasing behavior should be robust. The solid
FIG. 1: (Color online) $\zeta/s$ shown as a function of $T/T_c$. The solid line below $T_c$ is the massless pion gas result ($T_c \simeq 200$ MeV and $l = 2.5$, explained below Eq. (20), are used). The error on this curve is estimated to be 30%-40%. The points are the lattice results for gluon plasma [33]. The solid and dashed lines above $T_c$ give the central values and the error band from the QGP sum rule result of Ref. [32].

The points are the lattice results for gluon plasma [33]. The QGP curves above $T_c$ (the solid line gives the central values and the dashed lines give the estimated errors) are based on an exact sum rule, a lattice result for the equation of state, and a spectral function ansatz with massive quarks [32]. Since the light quark mass dependence in the QGP curve is expected to be small, Fig. 1 shows that, in the chiral limit, QCD $\zeta/s$ reaches its maximum while $\eta/s$ reaches its minimum around $T_c$ as mentioned above. The same $\zeta/s$ behavior is also seen in molecular-dynamics simulations of Lennard-Jones model fluids [40].

A recent massive pion gas calculation shows that $\zeta$ has two peaks [43], one is near 10 MeV and the other is near $T_c$. They are corresponding to breaking of the conformal symmetry by the pion mass and the anomaly, respectively. The behavior near the higher temperature peak is similar to what we have found here for the massless pion case. It is also similar to the $\zeta$ behavior of [44] near $T_c$ with Hagedorn states included. The behavior near the lower temperature peak is similar to earlier results of [45, 46]. The massless pion calculation of [43] also conforms our qualitative behavior of $\zeta$. 
In the large \( N_c \) (the number of colors) limit,

\[
\frac{\zeta}{s} \propto \frac{1}{N_c^2 N_f^2} \text{ for massless pion gas,} \tag{35}
\]

and

\[
\frac{\zeta}{s} \propto \frac{\alpha_s^2}{N_c^2} \propto \frac{1}{N_c^4} \text{ for PQCD,} \tag{36}
\]

where we have used the scaling \( f_\pi \propto \sqrt{N_c} \), \( g_\pi \propto N_f^2 \), \( \alpha_s^2 \propto 1/N_c \) and \( N_f \) is the number of light quark flavors. Also, for massless pions,

\[
\frac{\zeta}{\eta} \simeq 180 \left( \frac{l}{2.5} \right) \left( \frac{1}{3} - \frac{\mathcal{P}}{c} \right) \left( \frac{1}{3} - v_s^2 \right). \tag{37}
\]

This is similar to \( \zeta/\eta \sim 15 (1/3 - v_s^2)^2 \), which is obtained for a photon gas coupled to hot matter \([41]\) and is also parametrically correct for PQCD \([28]\). This is because in those cases, \( 2 \rightarrow 2 \) scattering is the dominant process in both \( \zeta \) and \( \eta \) computations. It is not the case, however, in \( \phi^4 \) theory in which \( (1/3 - v_s^2)^{-2} \zeta/\eta \) has large \( T \) dependence because \( \zeta \) is dominated by \( 2 \leftrightarrow 4 \) scattering while \( \eta \) is dominated by \( 2 \rightarrow 2 \) scattering. The scaling is also different from \( \zeta/\eta \propto (1/3 - v_s^2) \) for strongly coupled \( \mathcal{N} = 2^* \) gauge theory using AdS/CFT \([42]\).

\textbf{IV. CONCLUSIONS}

We have computed the bulk viscosity for a gas of massless pions using the Boltzmann equation with the kinetic theory generalized to incorporate the trace anomaly. The resulting \( \zeta/s \), together with the corresponding results of gluon plasma \([33]\) and quark gluon plasma \([31]\) indicates \( \zeta/s \) reaches its maximum near \( T_c \) while \( \eta/s \) reaches its minimum near \( T_c \). If the \( \zeta/s \) behavior is unchanged for massive pions, then the hadronization of the fire ball in heavy ion collisions would imply large entropy production \([31, 33]\) and slow equilibration. It would be interesting to explore the implications of the possible large bulk viscosity near a phase transition in cosmology if the phase transition above the TeV scale is based on some strongly interacting mechanism.

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