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Differentiating along rectangles
in lacunary directions

Laurent Moonens

Abstract. We show that, given some lacunary sequence of angles \( \theta = (\theta_j)_{j \in \mathbb{N}} \) not converging too fast to zero, it is possible to build a rare differentiation basis \( \mathcal{B} \) of rectangles parallel to the axes that differentiates \( L^1(\mathbb{R}^2) \) while the basis \( \mathcal{B}_\theta \) obtained from \( \mathcal{B} \) by allowing its elements to rotate around their lower left vertex by the angles \( \theta_j, j \in \mathbb{N} \), fails to differentiate all Orlicz spaces lying between \( L^1(\mathbb{R}^2) \) and \( L \log L(\mathbb{R}^2) \).

1. Introduction

Assume that \( \theta = (\theta_j)_{j \in \mathbb{N}} \subseteq (0, 2\pi) \) is a lacunary sequence going to zero and denote by \( \mathcal{B}_\theta \) the set of all rectangles in \( \mathbb{R}^2 \), one of whose sides makes an angle \( \theta_j \) with the horizontal axis, for some \( j \in \mathbb{N} \). It follows from results by Córdoba and Fefferman [2] (for \( p > 2 \)) and Nagel, Stein and Wainger [7] (for all \( p > 1 \)) that for every \( f \in L^p(\mathbb{R}^2) \), one has:

\[
\lim_{\text{diam } R \to 0} \frac{1}{|R|} \int_R f(x) = f(x)
\]

for almost every \( x \in \mathbb{R}^2 \) (we say, in this case, that \( \mathcal{B}_\theta \) differentiates \( L^p(\mathbb{R}^n) \)). This is often equivalent, according to Sawyer-Stein principles (see e.g. García [3, Chapter 1]), to the fact that the associated maximal operator \( M_{\mathcal{B}_\theta} \), defined for measurable functions \( f \) by:

\[
M_{\mathcal{B}_\theta}f(x) := \sup_{R \ni x} \frac{1}{|R|} \int_R |f|,
\]

satisfies a weak \((p,p)\) inequality, i.e. verifies:

\[
|\{ M_{\mathcal{B}_\theta} f > \alpha \}| \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^2} |f|^p,
\]

for all \( \alpha > 0 \) and all \( f \in L^p(\mathbb{R}^2) \). By interpolation, of course, such a property for all \( p > 1 \) implies that \( M_{\mathcal{B}_\theta} \) sends boundedly \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for all \( p > 1 \).

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Since then, the $L^p$ ($p > 1$) behaviour of the operators $M_{\theta_{\ell}}$ has been studied when the lacunary sequence $\theta$ is replaced by some Cantor sets (see e.g. Katz [5] and Hare [4]); recently, Bateman [1] obtained necessary and sufficient (geometrical) conditions on $\theta$ providing the $L^p$ boundedness of $M_{\theta_{\ell}}$.

In this paper we explore the behaviour of some maximal operators associated to rare differentiation bases of rectangles oriented in a lacunary set of directions $\theta = \{\theta_j : j \in \mathbb{N}\}$, provided that the sequence $(\theta_j)$ does not converge too fast to zero. More precisely, we prove the following theorem.

**Theorem 1.** Given a lacunary sequence $\theta = (\theta_j)_{j \in \mathbb{N}} \subseteq (0, 2\pi)$ satisfying:

$$0 < \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} \leq \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} < 1,$$

there exists a differentiation basis $\mathcal{B}$ of rectangles parallel to the axes satisfying the two following properties:

(i) $M_{\mathcal{B}}$ has weak type $(1, 1)$ (in particular $\mathcal{B}$ differentiates $L^1(\mathbb{R}^2)$);

(ii) if we denote by $\mathcal{B}_\theta$ the differentiation basis obtained from $\mathcal{B}$ by allowing its elements to rotate around their lower left corner by any angle $\theta_j$, $j \in \mathbb{N}$, then for any Orlicz function $\Phi$ (see below for a definition) satisfying $\Phi = o(t \log t)$ at $\infty$, the maximal operator $M_{\theta_{\ell}}$ fails to have weak type $(\Phi, \Phi)$ (in particular $\mathcal{B}_\theta$ fails to differentiate $L^\Phi(\mathbb{R}^n)$).

**Remark 2.** The differentiation basis $\mathcal{B}$ we shall construct in the proof of Theorem 1 is rare: it will be obtained as the smallest translation-invariant basis containing a countable family of rectangles with lower left corner at the origin (see section 3 for a more precise statement).

Our paper is organized as follows: we first discuss some easy geometrical facts concerning rectangles and rotations along lacunary sequences, following with a proof of Theorem 1.

2. Some basic geometrical facts

In the sequel we always call standard rectangle in $\mathbb{R}^2$ a set of the form $Q = [0, L] \times [0, \ell]$ where $L > 0$ and $\ell > 0$ are real numbers; we then let $Q_+ := [L/2, L] \times [0, \ell]$. For $\theta \in [0, 2\pi)$ we also denote by $r_\theta$ the (counterclockwise) rotation of angle $\theta$ around the origin.

**Lemma 3.** Fix real numbers $0 \leq \vartheta < \theta < \frac{\pi}{2}$ and $0 < 2\ell < L$ and let $Q := [0, L] \times [0, \ell]$. If moreover one has $\tan(\theta - \vartheta) \geq 1/\sqrt{\frac{1}{4} \left(\frac{\ell}{L}\right)^2 - 1}$, then $r_{\vartheta}Q_+$ and $r_{\theta}Q_+$ are disjoint.

**Proof.** To prove this lemma, we can assume, without loss of generality, that one has $\vartheta = 0$ (for otherwise, apply $r_{-\vartheta}$ to $r_{\vartheta}Q_+$ and $r_{\theta}Q_+$). Let
Figure 1. The rectangles $Q$, $Q_+$, $r_\theta Q$ and $r_\theta Q_+$

$m := \tan \theta$. Observe then that the lines $y = \ell$ and $y = mx$ intersect at $x_0 = \ell/m \leq \ell \sqrt{\frac{1}{4} \left( \frac{L}{\ell} \right)^2 - 1} \leq \frac{L}{2}$ and $y_0 = \ell$. Since we also have:

$$|(x_0, y_0)| \leq \ell \sqrt{\frac{1}{4} \left( \frac{L}{\ell} \right)^2} = \frac{L}{2},$$

this shows indeed that $Q_+$ and $r_\theta Q_+$ are disjoint (see Figure 1). \qed

Lemma 4. Assume that the sequence $(\theta_j)_{j \in \mathbb{N}} \subseteq (0, \pi/2)$ is such that one has:

$$(2) \quad 0 < \lambda < \frac{\lim_{j \to \infty} \theta_{j+1}}{\theta_j} \leq \frac{\lim_{j \to \infty} \theta_{j+1}}{\theta_j} < \mu < 1.$$ 

Let $\theta := \{\theta_j : j \in \mathbb{N}\}$.

There exists constants $d(\mu) > c(\mu) > 0$ depending only on $\mu$ such that, for each $\varepsilon > 0$ and each integer $k \in \mathbb{N}^*$, one can find a standard rectangle $Q_k = [0, L_k] \times [0, \ell_k]$ and a subset $\theta_k \subset \theta$ satisfying $\#\theta_k = k$ such that the following hold:

(i) $0 \leq 2\ell_k \leq L_k \leq \varepsilon$;

(ii) $c(\mu)\lambda^{-k} \leq \frac{L_k}{\ell_k} \leq d(\mu)\lambda^{-k}$;

(iii) $\left| \bigcup_{\theta \in \theta_k} r_\theta Q_k \right| \geq \frac{k}{2}|Q_k|.$

Proof. To prove this lemma, observe first that letting $m_j := \tan \theta_j$ for all $j \in \mathbb{N}$, one clearly has:

$$\lim_{j \to \infty} \frac{m_j}{\theta_j} = 1,$$

so that (2) also holds for the sequence $(m_j)_{j \in \mathbb{N}}$. There hence exists an index $j_0 \in \mathbb{N}$ such that, for all $j \geq j_0$, one has $\lambda \leq \frac{m_{j+1}}{m_j} \leq \mu$ (we may also and will
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assume that one has \(m_{j_0} \leq 1\). For the sake of clarity, we shall now consider that \(j_0 = 0\) and compute, for an integer \(0 \leq j < k\):

\[
\tan(\theta_j - \theta_k) = \frac{m_j - m_k}{1 + m_j m_k} \geq \frac{1}{2} (m_j - m_k).
\]

Since we also have, for every integer \(0 \leq j < k\):

\[
\lambda^{k-j} m_j \leq m_k \leq \mu^{k-j} m_j,
\]

we obtain under the same assumptions on \(j\):

\[
\tan(\theta_j - \theta_k) \geq \frac{1}{2} (m_j - m_k) \geq \frac{1}{2} \lambda^{k-j} (\mu^{-1} - 1)m_0.
\]

Now choose real numbers \(0 \leq 2\ell \leq L \leq \varepsilon\) (we write \(L\) and \(\ell\) instead of \(L_k\) and \(\ell_k\) here, for the index \(k\) remains constant all through the proof) satisfying:

\[
\left(\frac{L}{\ell}\right)^2 = 4 + \lambda^{-2k} [(\mu^{-1} - 1)m_0]^2.
\]

It is clear that one has:

\[
\frac{L}{\ell} = \lambda^{-k} \sqrt{4\lambda^{2k} + \left[(\mu^{-1} - 1)m_0\right]^2},
\]

so that (ii) holds if we take, for example, \(c(\mu) := \sqrt{[\mu^{-1} - 1)m_0]^2}\) and \(d(\mu) := \sqrt{4 + \left[(\mu^{-1} - 1)m_0\right]^2}\). On the other hand, (i) is clearly satisfied by assumption.

In order to show (iii), define \(Q := [0, L] \times [0, \ell]\) and observe that one has

\[
\tan(\theta_j - \theta_k) \geq \frac{1}{\sqrt{\frac{1}{4} \left(\frac{L}{\ell}\right)^2 - 1}},
\]

for all integers \(j\) satisfying \(j < k\). According to Lemma 3, this ensures that the family \(\{r_{\theta_j} Q_+ : j \in \mathbb{N}, j < k\}\) consists of pairwise disjoint sets; in particular we get:

\[
\left| \bigcup_{j=0}^{k-1} r_{\theta_j} Q \right| \geq \left| \bigcup_{j=0}^{k-1} r_{\theta_j} Q_+ \right| = k \cdot \frac{|Q|}{2},
\]

(we used \(\sqcup\) to indicate a disjoint union) and the lemma is proved.

We now turn to studying maximal operators associated to families of standard rectangles.

3. Maximal operators associated to lacunary sequences of directions

From now on, given a family \(\mathcal{R}\) of standard rectangles and a set \(\theta \subseteq [0, 2\pi)\), we let \(r_{\theta} \mathcal{R} := \{r_{\theta} Q : Q \in \mathcal{R}, \theta \in \theta\}\), and we define, for \(f : \mathbb{R}^2 \to \mathbb{R}\)
measurable:
\[ M_\mathcal{R} f(x) := \sup \left\{ \frac{1}{|Q|} \int_{\tau(Q)} |f| : Q \in \mathcal{R}, \tau \text{ translation}, x \in \tau(Q) \right\}, \]
and:
\[ M_{\mathcal{R}_\theta} f(x) := \sup \left\{ \frac{1}{|R|} \int_{\tau(R)} |f| : R \in \mathcal{R}_\theta, \tau \text{ translation}, x \in \tau(R) \right\}. \]

Notice, in particular, that in case one has \( \inf \{ \text{diam } R : R \in \mathcal{R} \} = 0 \), \( M_\mathcal{R} \) and \( M_{\mathcal{R}_\theta} \) are the maximal operators associated to the translation-invariant differentiation bases \( \mathcal{R} \) and \( \mathcal{R}_\theta \) defined respectively by:
\[ \mathcal{R} := \{ \tau(Q) : Q \in \mathcal{R}, \tau \text{ translation} \} \]
and
\[ \mathcal{R}_\theta := \{ \tau(\theta Q) : Q \in \mathcal{R}, \theta \in \Theta, \tau \text{ translation} \}. \]

The next proposition will be useful in order to study the maximal operator \( M_{\mathcal{R}_\theta} \). Observe that it has the flavour of Stokolos’ [8, Lemma 1].

**Proposition 5.** Assume that \( (\theta_j)_{j \in \mathbb{N}} \subseteq (0,2\pi) \) satisfies:
\[ 0 < \lambda < \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} \leq \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} < \mu < 1, \]
and let \( \Theta := \{ \theta_j : j \in \mathbb{N} \} \). There exists a (countable) family \( \mathcal{R} \) of standard rectangles in \( \mathbb{R}^2 \) which is totally ordered by inclusion, verifies \( \inf \{ \text{diam } R : R \in \mathcal{R} \} = 0 \) and satisfies the following property: for any \( k \in \mathbb{N}^* \), there exists sets \( \Theta_k \subseteq \mathbb{R}^2 \) and \( Y_k \subseteq \mathbb{R}^2 \) satisfying the following conditions:

(i) \( |Y_k| \geq \kappa(\mu) \cdot k\lambda^{-k} |\Theta_k| \);  
(ii) for any \( x \in Y_k \), one has \( M_{\mathcal{R}_\theta} |\Theta_k| \).

here, \( \kappa(\mu) > 0 \) and \( \kappa'(\mu) > 0 \) are two constants depending only on \( \mu \).

**Proof.** Define \( \mathcal{R} := \{ Q_k : k \in \mathbb{N}^* \} \) where the sequence \( (Q_k)_{k \in \mathbb{N}^*} \) is defined inductively as follows. We choose \( Q_1 = [0, L_1] \times [0, \ell_1] \) and \( \theta_1 \subseteq \Theta \) associated to \( k = 1 \) and \( \epsilon = 1 \) according to Lemma 4. Assuming that \( Q_1, \ldots, Q_k \) have been constructed, for some integer \( k \in \mathbb{N}^* \), we choose \( Q_{k+1} = [0, L_{k+1}] \times [0, \ell_{k+1}] \) and \( \theta_{k+1} \) associated to \( k+1 \) and \( \epsilon = \min(\ell_k, 1/k) \) according to Lemma 4. Since the sequence \( (Q_k)_{k \in \mathbb{N}^*} \) is a nonincreasing sequence of rectangles, it is clear that \( \mathcal{R} \) is totally ordered by inclusion. It is also clear by construction that one has \( \inf \{ \text{diam } R : R \in \mathcal{R} \} = 0 \).

Now fix \( k \in \mathbb{N}^* \) and define \( \Theta_k := B(0, \ell_k) \) and \( Y_k := \bigcup_{\theta \in \Theta_k} \theta Q_k \). Compute hence, using [Claim 4 (ii) and (iii)]:
\[ |Y_k| \geq \frac{1}{2} kL_k \ell_k = \frac{1}{2\pi} k \frac{L_k}{\ell_k} \pi \ell_k^2 \geq \frac{c(\mu)}{2\pi} \cdot k\lambda^{-k} |\Theta_k|, \]
so that (i) is proved in case one lets \( \kappa(\mu) := \frac{c(\mu)}{2\pi} \).
For \( x \in Y_k \), choose \( \theta \in \theta_k \) for which one has \( x \in r_\theta Q_k \) and observe that one has (see Figure 2):\[
M_{r_\theta \Theta_k} \chi_{\Theta_k}(x) \geq \frac{|\Theta_k \cap r_\theta Q_k|}{|Q_k|} = \frac{1}{4} \cdot \frac{\pi \ell_k^2}{L_k \ell_k} = \frac{\pi}{4} \cdot \frac{\ell_k}{L_k} \geq \frac{\pi}{4d(\mu)} \chi_k^k,\]
which finishes the proof of (ii) if we let \( \kappa'(\mu) := \frac{\pi}{4d(\mu)} \). \( \square \)

For our purposes, an *Orlicz function* is a convex and increasing function \( \Phi : [0, \infty) \rightarrow [0, \infty) \) satisfying \( \Phi(0) = 0 \); we then let \( L^\Phi(\mathbb{R}^2) \) denote the set of all measurable functions \( f \) in \( \mathbb{R}^2 \) for which \( \Phi(|f|) \) is integrable (for \( \Phi(t) = t^p \), \( p \geq 1 \) this yields the usual Lebesgue space \( L^p(\mathbb{R}^2) \), while for \( \Phi(t) = \Phi_0(t) := t(1 + \log t) \) we get the Orlicz space \( L^{\Phi_0}(\mathbb{R}^2) : = L^{\Phi_0}(\mathbb{R}^2) \). Recall that a sublinear operator \( T \) is said to be of weak type \( (\Phi, \Phi) \) in case there exists a constant \( C > 0 \) such that, for all \( f \in L^\Phi(\mathbb{R}^2) \) and all \( \alpha > 0 \), one has:\[
|\{x \in \mathbb{R}^2 : Tf(x) > \alpha\}| \leq \int_{\mathbb{R}^2} \Phi\left(\frac{|f|}{\alpha}\right).
\]
Whenever \( \Phi(t) = t^p \) for \( p \geq 1 \), we shall say that \( T \) has weak type \( (p, p) \).

The next result specifies the announced Theorem 1. It is mainly a consequence of the preceding proposition and some standard techniques as developed in Moonens and Rosenblatt [6].

**Theorem 6.** Assume that \( (\theta_j)_{j \in \mathbb{N}} \subseteq (0, 2\pi) \) satisfies:
\[
0 < \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} \leq \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} < 1,
\]
and let $\theta := \{\theta_j : j \in \mathbb{N}\}$. There exists a (countable) family $\mathcal{R}$ of standard rectangles in $\mathbb{R}^2$ with $\inf\{\text{diam } R : R \in \mathcal{R}\} = 0$, satisfying the following conditions:

(i) $M_{\mathcal{R}}$ has weak type $(1, 1)$, and hence the associated differentiation basis $\mathcal{B}$ differentiates $L^1(\mathbb{R}^2)$;

(ii) for any Orlicz function $\Phi$ satisfying $\Phi = o(\Phi_0)$ at $\infty$, $M_{\mathcal{R}_0\mathcal{R}}$ fails to be of weak type $(\Phi, \Phi)$. In particular, $M_{\mathcal{R}_0\mathcal{R}}$ fails to have weak type $(1, 1)$, and hence the associated differentiation basis $\mathcal{B}_\theta$ fails to differentiate $L^1(\mathbb{R}^2)$.

Proof. Begin by choosing real numbers $0 < \lambda < \mu < 1$ such that one has:

$0 < \lambda < \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} \leq \lim_{j \to \infty} \frac{\theta_{j+1}}{\theta_j} < \mu < 1$,

and keep the notations of Proposition 5.

Let now $\mathcal{R}$ be the family of rectangles given by Proposition 5. Observe first that, since $\mathcal{R}$ is totally ordered by inclusion, it follows e.g. from [9, Claim 1] that $M_{\mathcal{R}}$ satisfies a weak $(1, 1)$ inequality.

In order to show (ii), define, for $k$ sufficiently large, $f_k := \left[1/\kappa'(\mu)\right] \cdot \lambda^{-k} \chi_{\Theta_k}$, where $\Theta_k$ and $Y_k$ are associated to $k$ and $\mathcal{R}$ according to Proposition 5.

Claim 1. For each sufficiently large $k$, we have:

$\left|\{x \in \mathbb{R}^2 : M_{\mathcal{R}} f_k(x) \geq 1\}\right| \geq c_1(\lambda, \mu) \int_{\mathbb{R}^2} \Phi_0(f_k)$,

where $c_1(\lambda, \mu) := \frac{2 \log \frac{1}{\lambda}}{\kappa'(\mu) \cdot \kappa'(\mu)}$ is a constant depending only on $\lambda$ and $\mu$.

Proof of the claim. To prove this claim, one observes that for $x \in Y_k$ we have $M_{\mathcal{R}} f_k(x) \geq 1$ according to [Proposition 5 (ii)]. Yet, on the other hand, one computes, for $k$ sufficiently large:

$$\int_{\mathbb{R}^2} \Phi_0(f_k) \leq \frac{1}{\kappa'(\mu)} \cdot \lambda^{-k} |\Theta_k| \left[1 - \log \lambda \cdot \kappa'(\mu) + k \log \frac{1}{\lambda}\right] \leq \frac{2 \log \frac{1}{\lambda}}{\kappa'(\mu)} \cdot k \lambda^{-k} |\Theta_k| \leq c_1(\lambda, \mu) \cdot |Y_k|,$$

and the claim follows. \qed

Claim 2. For any $\Phi$ satisfying $\Phi = o(\Phi_0)$ at $\infty$ and for each $C > 0$, we have:

$$\lim_{k \to \infty} \int_{\mathbb{R}^2} \Phi_0(|f_k|) = \infty.$$
Remark 7. If we are solely interested in the weak $(1, 1)$ behaviour of the maximal operators $M_{\mathcal{R}}$ and $M_{r_0 \mathcal{R}}$, observe that Theorem [6] in particular applies to $\Phi(t) = t$, ensuring that the maximal operator $M_{r_0 \mathcal{R}}$ also fails to have weak type $(1, 1)$.

Moreover, as pointed out by the referee, the construction, given a sequence of distinct angles $\theta = (\theta_j)_j \subseteq (0, \pi/2)$, of a countable family $\mathcal{R}$ of rectangles for which $M_{\mathcal{R}}$ is of weak type $(1, 1)$ while $M_{r_0 \mathcal{R}}$ is not, can be done almost immediately from Lemma [3] — and does not require a growth condition on the sequence $\theta$.

To see this, observe that for each $k$, it is easy, according to Lemma [3], and making $L_k/\ell_k \gg 1$ large enough, to construct a rectangle $Q_k = [0, L_k] \times [0, \ell_k]$ such that the rectangles $r_\theta Q_{k'}$, $0 \leq j \leq k$ are pairwise disjoint. We can also inductively construct $(Q_k)$ such that one has $Q_{k+1} \subseteq Q_k$ for all $k \in \mathbb{N}$. Hence $\mathcal{R} := \{Q_k : k \in \mathbb{N}\}$ is totally ordered by inclusion, ensuring that $M_{\mathcal{R}}$ has weak type $(1, 1)$.

On the other hand, define for $k \in \mathbb{N}$ a function $f_k := |Q_k|_\Phi \chi_{[B(0, \ell_k)]}$. For all $x \in Y_k := \bigcup_{j=0}^k r_\theta Q_k$, choose an integer $0 \leq j \leq k$ for which one has $x \in r_\theta Q_k$ and compute (see Figure [2] again):

$$M_{r_0 \mathcal{R}} f_k(x) \geq \frac{|Q_k|}{|B(0, \ell_k)|} \frac{|B(0, \ell_k) \cap r_\theta Q_k|}{|Q_k|} = \frac{1}{4}.$$
It hence follows that one has:
\[
(k + 1)\|f_k\|_1 = (k + 1)|Q_k| = 2(k + 1)|Q_{k,+}|
\]
\[
\leq 2|Y_k| \leq 2 \left| \left\{ x \in \mathbb{R}^2 : M_{\mathcal{B}_g} f_k(x) \geq \frac{1}{4} \right\} \right|,
\]
so that \(M_{\mathcal{B}_g}\) cannot have weak type \((1, 1)\).

**Remark 8.** In [Theorem 6, (ii)], it is not clear to us whether or not the space \(L \log L(\mathbb{R}^2)\) is sharp; we don’t know, for example, whether or not \(\mathcal{B}_g\) differentiates \(L \log^{1+\varepsilon} L(\mathbb{R}^2)\) for \(\varepsilon > 0\).

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