Subexponential densities of infinitely divisible distributions on the half-line

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Abstract. We show that, under the long-tailedness of the densities of normalized Lévy measures, the densities of infinitely divisible distributions on the half-line are subexponential if and only if the densities of their normalized Lévy measures are subexponential. Moreover, we prove that, under a certain continuity assumption, the densities of infinitely divisible distributions on the half-line are subexponential if and only if their normalized Lévy measures are locally subexponential.

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1 Introduction and main results

The subexponentiality of infinitely divisible distributions on the half-line was characterized by Embrechts et al. [5] and on the real line by Pakes [16, 17] and Watanabe [25]. The subexponentiality of a density is a stronger and more difficult property than the subexponentiality of a distribution. Some infinitely divisible distributions on the half-line such as Pareto, lognormal, and Weibull (with parameter less than 1) distributions have subexponential densities. Watanabe and Yamamuro [29] proved that the density of a self-decomposable distribution on the real line is subexponential if and only if the density on $(1, \infty)$ of its normalized Lévy measure is subexponential. The purpose of this paper is to characterize the subexponential densities of absolutely continuous infinitely divisible distributions on the half-line under some additional assumptions.

In what follows, we denote by $\mathbb{R}$ the real line and by $\mathbb{R}_+$ the half-line $[0, \infty)$. We denote by $\mathbb{N}$ the set of positive integers. The symbol $\delta_a(dx)$ stands for the delta measure at $a \in \mathbb{R}$. Let $\eta$ and $\rho$ be finite measures on $\mathbb{R}$. We denote by $\eta \ast \rho$ the convolution of $\eta$ and $\rho$ and by $\rho^{n\ast}$ the $n$th convolution power of $\rho$ with the understanding that $\rho^{0\ast}(dx) = \delta_0(dx)$. The characteristic function of a distribution $\rho$ is denoted by $\hat{\rho}$, that is, for $z \in \mathbb{R}$,

$$\hat{\rho}(z) := \int_{-\infty}^{\infty} e^{izx} \rho(dx).$$
Let \( f \) and \( g \) be probability density functions on \( \mathbb{R} \). We denote by \( f \otimes g \) the convolution of \( f \) and \( g \) and by \( f^n \) the \( n \)-th convolution power of \( f \) for \( n \in \mathbb{N} \). For positive functions \( f_1 \) and \( g_1 \) on \([A, \infty)\) for some \( A \in \mathbb{R} \), we define the relation \( f_1(x) \sim g_1(x) \) by \( \lim_{x \to \infty} f_1(x)/g_1(x) = 1 \). We use the symbols \( \mathcal{L} \) and \( \mathcal{S} \) in the sense of long-tailedness and subexponentiality, respectively. The subscripts \( \text{d}, \text{ac}, \) and \( \text{loc} \) mean density, absolutely continuity, and locality, respectively.

**Definition 1.**

(i) A nonnegative and eventually positive measurable function \( g \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L} \) if \( g(x + a) \sim g(x) \) for every \( a \in \mathbb{R} \).

(ii) A probability density function \( g \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L}_d \) if \( g \in \mathcal{L} \). A distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L}_{ac} \) if there is \( g \in \mathcal{L}_d \) such that \( \rho(dx) = g(x) \, dx \).

(iii) A probability density function \( g \) on \( \mathbb{R} \) belongs to the class \( \mathcal{S}_d \) if \( g \in \mathcal{L}_d \) and \( g^{2\otimes}(x) \sim 2g(x) \). A distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{S}_{ac} \) if there is \( g \in \mathcal{S}_d \) such that \( \rho(dx) = g(x) \, dx \).

**Definition 2.**

(i) Let \( \Delta := (0, c] \) with \( c > 0 \). A distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L}_\Delta \) if \( \rho((x, x + c]) \in \mathcal{L} \).

(ii) A probability distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L}_{\Delta, \text{loc}} \) if \( \rho \in \mathcal{L}_\Delta \) for each \( \Delta := (0, c] \) with \( c > 0 \).

(iii) Let \( \Delta := (0, c] \) with \( c > 0 \). A distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{S}_\Delta \) if \( \rho \in \mathcal{L}_\Delta \) and \( \rho^{2\ast}(x, x + c] \sim 2\rho(x, x + c] \). A distribution \( \rho \) on \( \mathbb{R} \) belongs to the class \( \mathcal{S}_{\Delta, \text{loc}} \) if \( \rho \in \mathcal{S}_\Delta \) for each \( \Delta := (0, c] \) with \( c > 0 \).

A probability distribution \( \rho \) on \( \mathbb{R} \) is called subexponential if \( \rho((x, \infty)) \in \mathcal{L} \) and \( \rho^{2\ast}(x, \infty) \sim 2\rho(x, \infty) \).

The class of all subexponential distributions on \( \mathbb{R} \) is denoted by \( \mathcal{S} \). The class \( \mathcal{S} \) was introduced by Chistyakov [3] and studied by Embrecht et al. [6], Korshnov [12], Pitman [18], Rogozin [19], and Teugels [23]. Functions in the class \( \mathcal{L} \) are called long-tailed functions. Closure properties of the class \( \mathcal{S} \) and the class of long-tailed distributions are found in Leslie [14] and Xu et al. [32]. Probability density functions in the classes \( \mathcal{L}_d \) and \( \mathcal{S}_d \) are called long-tailed densities and subexponential densities, respectively. The class \( \mathcal{S}_d \) was introduced by Chover et al. [4] and studied by Finkelshtein and Tkachov [7], Foss and Zahary [9], Klüppelberg [11], Korshnov [13], and Wang and Wang [24]. Note that if \( f \in \mathcal{L}_d \), then \( \lim_{x \to \infty} f(x) = 0 \) and \( \lim_{x \to \infty} e^{sx} f(x) = \infty \) for every \( s > 0 \); see Foss et al. [8]. Distributions in the classes \( \mathcal{S}_\Delta \) and \( \mathcal{S}_{\Delta, \text{loc}} \) are called \( \Delta \)-subexponential and locally subexponential, respectively. The classes \( \mathcal{S}_\Delta \) and \( \mathcal{S}_{\Delta, \text{loc}} \) were introduced by Asmussen et al. [1] and Borovkov and Borovkov [2]. Infinitely divisible distributions in the classes \( \mathcal{S}_\Delta \) and \( \mathcal{S}_{\Delta, \text{loc}} \) are found in Watanabe and Yamamuro [28] and Shimura and Watanabe [22]. Second-order subexponentiality in the classes \( \mathcal{S}_{\text{loc}} \) and \( \mathcal{S}_{ac} \) were discussed by Lin [15] and Watanabe [27]. The class \( \mathcal{S} \) includes the classes \( \mathcal{S}_\Delta \) and \( \mathcal{S}_{ac} \).

Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \). Then its characteristic function \( \hat{\mu} \) is represented as

\[
\hat{\mu}(z) = \exp \left( \int_0^\infty \left( e^{izx} - 1 \right) \nu(dx) + i\gamma_0 z \right),
\]

where \( \gamma_0 \in \mathbb{R}_+ \), and \( \nu \) is a measure on \( \mathbb{R}_+ \) satisfying \( \nu(\{0\}) = 0 \) and

\[
\int_0^\infty (1 \wedge x) \nu(dx) < \infty.
\]

The measure \( \nu \) is called the Lévy measure of \( \mu \). Denote by \( \mu^t \) the \( t \)-th convolution power of \( \mu \) for \( t > 0 \). Then \( \mu^t \) is a distribution of a certain Lévy process on \( \mathbb{R}_+ \). The distribution \( \mu \) is called nontrivial if it is not a delta measure on \( \mathbb{R}_+ \). A nontrivial infinitely divisible distribution \( \mu \) on \( \mathbb{R}_+ \) is called semistable if, for some \( a > 1 \), there exist \( b > 1 \) and \( c \in \mathbb{R} \) such that, for \( z \in \mathbb{R} \),

\[
\hat{\mu}(z)^a = \hat{\mu}(bz)e^{icz}.
\]

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For a semistable distribution on $\mathbb{R}_+$, the span $b$, and the index $\alpha := \log a / \log b$ $(0 < \alpha < 1)$ are important. A span is not unique, and the index is independent of the choice of span. See Sato [20]. Shimura and Watanabe [21] proved that a semistable distribution on $\mathbb{R}_+$ is subexponential if and only if its Lévy measure is continuous. Watanabe and Yamamuro [30] studied the tail behaviors of semistable distributions on $\mathbb{R}$.

Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $\nu$. Throughout the paper, we assume that $\gamma_0 = 0$ and that the tail $\nu((e, \infty))$ is positive for all $e > 0$. Define the normalized distribution $\nu_{(1)}$ as

$$\nu_{(1)}(dx) := 1_{(1, \infty)}(x) \frac{\nu(dx)}{\nu((1, \infty))}.$$ 

Here the symbol $1_{(1, \infty)}$ stands for the indicator function of the set $(1, \infty)$. It is known that if

$$\int_{-\infty}^{\infty} |\hat{\mu}(z)| \, dz < \infty, \quad (1.1)$$

then $\mu$ is absolutely continuous with bounded and continuous density on $\mathbb{R}$. It is also known that if $\mu$ is an infinitely divisible distribution on $\mathbb{R}_+$ with absolutely continuous Lévy measure $\nu$ satisfying $\nu((0, 1)) = \infty$, then $\mu$ is absolutely continuous. See Theorem 27.7 of Sato [20]. First, we have the following basic result.

**Proposition 1.** Let $\mu$ be an absolutely continuous infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $\nu$, and let $p$ be a density of $\mu$. Then we have:

(i) Assume that $p \in L_\Delta$. Then, $p \in S_\Delta$ if and only if $\nu_{(1)} \in S_{\text{loc}}$.

(ii) The following are equivalent:

(a) $p \in S_\Delta$,

(b) $\nu_{(1)} \in L_{\text{loc}}$, and $p(x) \sim \nu((x, x+1])$.

(c) $\nu_{(1)} \in L_{\text{loc}}$, and there is $C \in (0, \infty)$ such that $p(x) \sim C\nu((x, x+1])$.

Next, we characterize the case where $\nu_{(1)}$ is absolutely continuous with long-tailed density. Note that Corollary 1.1 of Jiang et al. [10] is analogous to Theorem 1, but there is a gap in the proof of Corollary 1.1 from [10] since it is not proved that the density of $H$ is long-tailed.

**Theorem 1.** Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}_+$ with absolutely continuous Lévy measure $\nu(dx) = g(x) \, dx$ satisfying $\nu((0, 1)) = \infty$. Let $g_1$ be a density of $\nu_{(1)}$. Let $f_1$ be a density of the infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $1_{(0,1)}(x)\nu(dx)$. Assume that, for some $b_0 > 0$,

$$\lim_{x \to \infty} \exp(b_0 x) f_1(x) = 0$$

and that $g_1$ is bounded and $g_1 \in L_\Delta$. Then we can choose a density $p$ of $\mu$ such that the following are equivalent:

(1) $p \in S_\Delta$.

(2) $g_1 \in S_\Delta$.

(3) $p(x) \sim C_0 g_1(x)$ with $C_0 := \int_1^{\infty} g(x) \, dx$.

(4) There is $C \in (0, \infty)$ such that $p(x) \sim C g_1(x)$.

**Remark 1.** Assume that $\nu(dx) = 1_{(0,1)}(x)x^{-1}k(x) \, dx + 1_{(1, \infty)}(x)h(x) \, dx$, where $k$ is nonnegative and decreasing with $0 < k(0+) \leq \infty$ and $\int_0^1 k(x) \, dx < \infty$, and $h$ is nonnegative, bounded, and integrable on $\mathbb{R}_+$. Then we can choose a density $f_1$ of the infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $1_{(0,1)}(x)\nu(dx)$ such that, for any $b > 0$,

$$\lim_{x \to \infty} \exp(bx) f_1(x) = 0.$$
Rem. 2. Theorem 1 includes all self-decomposable cases on $\mathbb{R}_+$ of Watanabe and Yamamuro [29]. We see from Theorem 53.6 of Sato [20] that there is a self-decomposable case on $\mathbb{R}_+$ of Watanabe and Yamamuro [29], which does not satisfy condition (1.1). Hence Theorem 2 does not include Theorem 1.

We apply Theorem 1 to the tail asymptotic behavior of the density of the distribution of a Lévy process on $\mathbb{R}_+$.

**Corollary 1.** Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}_+$ with absolutely continuous Lévy measure $\nu(dx)$ satisfying $\nu((0, 1)) = \infty$. Let $g_1$ be a density of $\nu(1)$. Let $f_1^t$ be a density of the infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $t \mathbf{1}_{(0, 1)}(x)\nu(dx)$. Assume that, for every $t > 0$, there is $b_0(t) > 0$ such that

$$
\lim_{x \to \infty} \exp(b_0(t)x)f_1^t(x) = 0
$$

and that $g_1$ is bounded and $g_1 \in \mathcal{L}_d$. Then we can choose a density $p^t$ of $\mu^t*$ such that the following hold:

(i) If $p^t \in \mathcal{S}_d$ for some $t > 0$, then $p^t \in \mathcal{S}_d$ for all $t > 0$, and

$$p^t(x) \sim tp^1(x) \quad \text{for all } t > 0.
$$

(ii) If $p^1(x) \in \mathcal{L}_d$ and, for some $t \in (0, 1) \cup (1, \infty)$, there is $C(t) \in (0, \infty)$ such that

$$p^t(x) \sim C(t)p^1(x),
$$

then $C(t) = t$ and $p^1(x) \in \mathcal{S}_d$.

Under assumption (1.1), we can characterize the subexponential density of an infinitely divisible distribution on $\mathbb{R}_+$ as follows.

**Theorem 2.** Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $\nu$. Assume that (1.1) holds. Let $p$ be the continuous density of $\mu$. Then the following are equivalent:

1. $p \in \mathcal{S}_d$.
2. $\nu(1) \in \mathcal{S}_{loc}$.
3. $\nu(1) \in \mathcal{L}_{loc}$ and $p(x) \sim \nu((x, x + 1])$.
4. $\nu(1) \in \mathcal{L}_{loc}$ and there is $C \in (0, \infty)$ such that $p(x) \sim C\nu((x, x + 1])$.

**Remark 3.**

(i) Assume that $\nu(dx) = g(x)dx$ with $\lim\inf_{x \to 0^+} xg(x) > 1$. Then we have (1.1). See Lemma 53.9 of Sato [20]. Some sufficient conditions in order that certain infinitely divisible distributions $\mu$ on $\mathbb{R}_+$ satisfy (1.1) are found in Sato [20] and Watanabe [26].

(ii) Let $\mu$ be a semistable distribution on $\mathbb{R}_+$ with Lévy measure $\nu$. By Proposition 24.20 of Sato [20] we have (1.1). Thus we see from the theorem that $\mu \in \mathcal{S}_{ac}$ if and only if $\nu(1) \in \mathcal{S}_{loc}$.

(iii) Let $1 < x_0 < b$ and choose $\delta \in (0, 1)$ satisfying $\delta < (x_0 - 1) \wedge (b - x_0)$. We take a continuous periodic function $h(x)$ on $\mathbb{R}$ with period $\log b$ such that $h(\log x) > 0$ for $x \in [1, x_0] \cup (x_0, b]$ and

$$h(\log x) := \begin{cases} 
0 & \text{for } x = x_0, \\
\frac{1}{\log |x - x_0|} & \text{for } 0 < |x - x_0| < \delta.
\end{cases}
$$

Let $\Phi(x) := x^{-\alpha-1}h(\log x)$ on $(0, \infty)$ with $\alpha \in (0, 1)$. Then $\nu$ defined by

$$
\nu(dx) := \mathbf{1}_{(0, \infty)}(x)\Phi(x) \, dx
$$


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is the Lévy measure of a certain semistable distribution on $\mathbb{R}_+$ with span $b$ and index $\alpha$. Watanabe and Yamamuro proved in the proof of Theorem 1 of [31] that $\nu(1) \in \mathcal{S}_{\text{loc}} \setminus \mathcal{S}_{\text{ac}}$. Thus we find from (ii) that in this case, $\mu \in \mathcal{S}_{\text{ac}}$ but $\nu(1) \notin \mathcal{S}_{\text{ac}}$, and hence Theorem 1 does not include Theorem 2. Note that the continuous density $p(x)$ of this semistable distribution $\mu$ is not almost decreasing because

$$\lim_{n \to \infty} \frac{p(b^n(x_0 + \delta))}{p(b^n x_0)} = \lim_{n \to \infty} \frac{\nu((b^n(x_0 + \delta), b^n(x_0 + \delta) + 1])}{\nu((b^n x_0, b^n x_0 + 1])} = \infty.$$ 

Finally, we can apply the theorem to the tail asymptotic behavior of the density of the distribution of a Lévy process on $\mathbb{R}_+$.

**Corollary 2.** Let $\mu$ be an infinitely divisible distribution on $\mathbb{R}_+$. Assume that $\int_{-\infty}^{\infty} |\hat{\mu}(z)|^t \, dz < \infty$ for every $t > 0$. Let $p^t$ be the continuous density of $\mu^{*t}$ for $t > 0$. Then we have:

(i) If $p^t \in \mathcal{S}_{\text{d}}$ for some $t > 0$, then $p^t \in \mathcal{S}_{\text{d}}$ for all $t > 0$ and

$$p^t(x) \sim tp^1(x) \quad \text{for all } t > 0.$$

(ii) If $p^1 \in \mathcal{L}_{\text{d}}$ and, for some $t \in (0, 1) \cup (1, \infty)$, there is $C(t) \in (0, \infty)$ such that

$$p^t(x) \sim C(t)p^1(x),$$

then $C(t) = t$ and $p^1 \in \mathcal{S}_{\text{d}}$.

The organization of this paper is as follows. In Section 2, we explain the basic results on the classes $\mathcal{L}_{\text{d}}$, $\mathcal{S}_{\text{d}}$, and $\mathcal{S}_{\text{loc}}$ as preliminaries. In Section 3, we prove the main results.

## 2 Preliminaries

In this section, we give several fundamental results on the classes $\mathcal{L}_{\text{d}}$, $\mathcal{S}_{\text{d}}$, and $\mathcal{S}_{\text{loc}}$.

**Lemma 1.** Let $f$ and $g$ be probability density functions on $\mathbb{R}$.

(i) If $f, g \in \mathcal{L}_{\text{d}}$, then $f \otimes g \in \mathcal{L}_{\text{d}}$.

(ii) Let $f \in \mathcal{L}_{\text{d}}$ and define a distribution $\rho$ on $\mathbb{R}$ by

$$\rho(dx) := f(x) \, dx.$$ 

Then $\rho \in \mathcal{S}_{\text{loc}}$ if and only if $f \in \mathcal{S}_{\text{d}}$, that is, $\mathcal{S}_{\text{ac}} = \mathcal{S}_{\text{loc}} \cap \mathcal{L}_{\text{d}}$.

**Proof.** Statement (i) is due to Theorem 4.3 of Foss et al. [8]. Next, we prove statement (ii). Assume that $f \in \mathcal{L}_{\text{d}}$. Then, by (i), $f^{2\otimes} \in \mathcal{L}_{\text{d}}$, and hence $f(x + u) \sim f(x)$ and $f^{2\otimes}(x + u) \sim f^{2\otimes}(x)$ uniformly in $u \in [0, c]$ with $c > 0$. Thus we have, for $x > 0$,

$$\rho((x, x + c]) = \int_0^c f(x + u) \, du \sim cf(x)$$

and

$$\rho^{2\otimes}((x, x + c]) = \int_0^c f^{2\otimes}(x + u) \, du \sim cf^{2\otimes}(x).$$

Hence we see that $\rho \in \mathcal{S}_{\text{loc}}$ if and only if $f^{2\otimes}(x) \sim 2f(x)$, that is, $f \in \mathcal{S}_{\text{d}}$. □
Lemma 2. Let $f$ and $g$ be probability density functions on $\mathbb{R}_+$.

(i) If $f \in S_d$ and $g(x) \sim cf(x)$ with $c \in (0, \infty)$, then $g \in S_d$.

(ii) Assume that $f \in S_d$ and $f$ is bounded on $\mathbb{R}_+$. Then, for any $\epsilon > 0$, there are $x_0(\epsilon) > 0$ and $C(\epsilon) > 0$ such that, for all $x > x_0(\epsilon)$ and all $n \in \mathbb{N}$,

$$f^{n\otimes}(x) \leq C(\epsilon)(1 + \epsilon)^n f(x).$$

(iii) If $f \in S_d$, then, for all $n \in \mathbb{N}$,

$$f^{n\otimes}(x) \sim nf(x).$$

Proof. Statements (i), (ii), and (iii) are due to Theorem 4.8, Theorem 4.11, and Corollary 4.10 of Foss et al. [8], respectively. □

Lemma 3.

(i) If a distribution $\rho$ on $\mathbb{R}_+$ belongs to $L_{\text{loc}}$, then $\rho((x, x+c]) \sim c\rho((x, x+1])$ for all $c > 0$.

(ii) Let $q$ be a continuous probability density function on $\mathbb{R}_+$ such that $\lim_{x \to \infty} e^{\gamma_1 x} q(x) = 0$ for some $\gamma_1 > 0$. Let $\rho$ be a distribution on $\mathbb{R}_+$ and define the probability density function $p$ on $\mathbb{R}_+$ as

$$p(x) := \int_{0-}^{x+} q(x-u) \rho(du). \quad (2.1)$$

Then, $\rho \in L_{\text{loc}}$ implies $p \in L_d$.

Proof. Statement (i) is proved as (2.6) in Theorem 2.1 of Watanabe and Yamamuro [29]. Next, we prove statement (ii). Suppose that $\rho \in L_{\text{loc}}$ and a probability density function $q$ on $\mathbb{R}_+$ is continuous on $\mathbb{R}_+$ and $\lim_{x \to \infty} e^{2\gamma x} q(x) = 0$ for some $\gamma > 0$. Let $p$ be a probability density function on $\mathbb{R}_+$ defined by (2.1). Let $N \in \mathbb{N}$ and $x > 2N$. We have $p(x) = \sum_{j=1}^3 I_j(x)$, where

$$I_1(x) := \int_{(x-N)+}^{x+} q(x-u) \rho(du), \quad I_2(x) := \int_{0-}^{N+} q(x-u) \rho(du),$$

and

$$I_3(x) := \int_{N+}^{(x-N)+} q(x-u) \rho(du).$$

For $M \in \mathbb{N}$, there are $\delta(M) \geq 0$ and $a_n = a_n(M) \geq 0$ for all $n \in \mathbb{N}$ such that

$$a_n \leq q(x) \leq a_n + \delta(M)$$

for $M^{-1}(n-1) \leq x \leq M^{-1}n$ and $1 \leq n \leq MN$. Define

$$J(x; M, N) := \sum_{n=1}^{MN} a_n \rho((x-M^{-1}n, x-M^{-1}(n-1))).$$

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Then we see from (i) that
\[ J(x; M, N) \sim \rho((x, x+1]) M N \sum_{n=1}^{M N} a_n M^{-1} \]
and
\[ J(x; M, N) \leq I_1(x) \leq J(x; M, N) + \delta(M) \rho((x - N, x]). \]

Since we can make \( \lim_{M \to \infty} \delta(M) = 0 \) and
\[ \lim_{M \to \infty} \sum_{n=1}^{M N} a_n M^{-1} = \int_0^N q(x) \, dx, \]
we find that
\[ I_1(x) \sim \rho((x, x+1]) \int_0^N q(x) \, dx. \] (2.2)

For \( x \in \mathbb{R} \), the symbol \([x]\) stands for the largest integer not exceeding \( x \). We can take \( N \in \mathbb{N} \) sufficiently large such that \( q(y) \leq e^{-2\gamma y} \) for \( y > N \) and, for some \( C_1 > 0 \) and for integers \( n \) in \( 0 \leq n \leq [x] - 2N + 1 \) with \( x > 2N \),
\[ \rho((N + n, N + n + 1]] \leq C_1 e^{\gamma(x-N-n)} \rho((x, x+1]). \]

Thus we have
\[ I_2(x) \leq \int_{0^-}^{[x]-2N+1} e^{-2\gamma(x-u)} \rho(du) \leq e^{-2\gamma(x-N)} = o(\rho((x, x+1])). \] (2.3)

Moreover, we obtain that, for some \( C_2 > 0 \),
\[ I_3(x) \leq \int_{N}^{(x-N)+} e^{-2\gamma(x-u)} \rho(du) \leq \sum_{n=0}^{[x]-2N+1} e^{-2\gamma(x-N-n-1)} \rho((N + n, N + n + 1]) \leq e^{2\gamma} C_1 \sum_{n=0}^{[x]-2N+1} e^{-\gamma(x-N-n)} \rho((x, x+1]) \leq C_2 e^{-\gamma N} \rho((x, x+1]). \]

Thus we see that
\[ \lim_{N \to \infty} \limsup_{x \to \infty} \frac{I_3(x)}{\rho((x, x+1])} = 0. \] (2.4)

Hence we find from (2.2), (2.3), and (2.4) that
\[ p(x) \sim \rho((x, x+1]) \int_0^\infty q(x) \, dx = \rho((x, x+1]), \]
and thereby \( p \in \mathcal{L}_d \). \( \square \)
Watanabe and Yamamuro [29] used the results of Watanabe [25] on the convolution equivalence of infinitely divisible distributions on \( \mathbb{R} \) to prove the following lemmas. Our main results essentially depend on those two results.

**Lemma 4.** (See [29, Thm. 1.1] ) Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu \). Then the following are equivalent:

1. \( \mu \in \mathcal{S}_{\text{loc}} \).
2. \( \nu(1) \in \mathcal{S}_{\text{loc}} \).
3. \( \nu(1) \in L_{\text{loc}} \) and \( \mu((x, x + c]) \sim \nu((x, x + c]) \) for all \( c > 0 \).
4. \( \nu(1) \in L_{\text{loc}} \) and there is \( C \in (0, \infty) \) such that \( \mu((x, x + c]) \sim C\nu((x, x + c]) \) for all \( c > 0 \).

**Lemma 5.** (See [29, Thm. 1.2] ) Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu \). Then we have:

(i) If \( \mu^{x} \in \mathcal{S}_{\text{loc}} \) for some \( t > 0 \), then \( \mu^{tx} \in \mathcal{S}_{\text{loc}} \) for all \( t > 0 \), and

\[
\mu^{tx}((x, x + c]) \sim t\mu((x, x + c])
\]

for all \( t > 0 \) and all \( c > 0 \).

(ii) If \( \mu \in \mathcal{S}_{\text{loc}} \) and, for some \( t \in (0, 1) \cup (1, \infty) \), there is \( C(t) \in (0, \infty) \) such that

\[
\mu^{tx}((x, x + c]) \sim C(t)\mu((x, x + c])
\]

(2.5)

for all \( c > 0 \), then \( C(t) = t \) and \( \mu \in \mathcal{S}_{\text{loc}} \).

Finally, we give some results of Sato [20]. For the definition of the submultiplicativity of the function on \( \mathbb{R}_+ \), see Sato [20]. Note that \( e^{sx} \) with \( s > 0 \) is submultiplicative on \( \mathbb{R}_+ \).

**Lemma 6.** (See [20, Thm. 25.3] ) Let \( g \) be a submultiplicative, locally bounded, and measurable function on \( \mathbb{R}_+ \). Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu \). Then, \( \mu \) has a finite \( g \)-moment if and only if \( \{ \nu \} \{ x > 1 \} \) has a finite \( g \)-moment.

**Lemma 7.** (See [20, Ex. 33.15] ) Let \( \eta > 0 \). Let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu \) satisfying \( \int_{1}^{\infty} e^{\eta x} \nu(dx) < \infty \). Then \( e^{\eta x} \mu(dx)/ \int_{0}^{\infty} e^{\eta x} \mu(dx) \) is an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( e^{\eta x} \nu(dx) \).

**Lemma 8.** (See [20, Thm. 53.1] ) Every self-decomposable distribution on \( \mathbb{R} \) is unimodal.

**Lemma 9.** (See [20, Remark 53.7] ) Let \( \mu \) be a self-decomposable distribution on \( \mathbb{R}_+ \). Then its density is continuous on \((0, \infty)\).

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**3 Proofs of the main results**

In this section, let \( \mu \) be an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu \), and let \( \mu_1 \) be a compound Poisson distribution on \( \mathbb{R}_+ \) with Lévy measure \( \nu_1(dx) := 1_{(c_1, \infty)}(x) \nu(dx) \) for \( c_1 > 0 \). We define an infinitely divisible distribution \( \zeta_1 \) on \( \mathbb{R}_+ \) by \( \mu = \mu_1 \ast \zeta_1 \).

**Lemma 10.**

(i) We have \( \int_{-\infty}^{\infty} |\widehat{\mu}(z)| \, dz < \infty \) if and only if \( \int_{-\infty}^{\infty} |\widehat{\zeta_1}(z)| \, dz < \infty \) for all sufficiently large \( c_1 > 0 \).

(ii) Suppose that \( \int_{-\infty}^{\infty} |\widehat{\zeta_1}(z)| \, dz < \infty \) for some \( c_1 > 0 \). Let \( q \) be the continuous density of \( \zeta_1 \) on \( \mathbb{R}_+ \). Then, for any \( \gamma > 0 \), \( \lim_{x \to \infty} e^{\gamma x} q(x) = 0 \).
Proof. First, we prove statement (i). Since

\[ |\tilde{\mu}(z)| = |\tilde{\mu}_1(z)| |\hat{\gamma}_1(z)| \quad \text{and} \quad |\tilde{\mu}_1(z)| \leq 1, \]

we have \( |\tilde{\mu}(z)| \leq |\hat{\gamma}_1(z)| \). Thus, if \( \int_\gamma^\infty |\hat{\gamma}_1(z)| \, dz < \infty \) for some \( c_1 > 0 \), then \( \int_{-\infty}^{\infty} |\tilde{\mu}(z)| \, dz < \infty \). Let \( \lambda := \nu((c_1, \infty)) \) and define the constant

\[ C_\nu := \inf\{ e > 0 : \nu((e, \infty)) < \log 2 \}. \]

Since \( \mu_1 \) is a compound Poisson distribution and

\[ |\tilde{\mu}_1(z) - e^{-\lambda}| = |\tilde{\mu}_1(z) - e^{-\lambda}\delta_0(z)| \leq 1 - e^{-\lambda}, \]

we see that, for all \( c_1 > C_\nu \),

\[ |\tilde{\mu}_1(z)| \geq e^{-\lambda} - |\tilde{\mu}_1(z) - e^{-\lambda}| \geq 2e^{-\lambda} - 1 > 0. \]

Hence we have \( |\tilde{\mu}(z)| \geq C_0|\hat{\gamma}_1(z)| \) with some \( C_0 > 0 \) for all \( c_1 > C_\nu \). Thus, if \( \int_{-\infty}^{\infty} |\tilde{\mu}(z)| \, dz < \infty \), then \( \int_\gamma^\infty |\hat{\gamma}_1(z)| \, dz < \infty \) for all \( c_1 > C_\nu \). Next, we prove statement (ii). Suppose that \( \int_{-\infty}^{\infty} |\hat{\gamma}_1(z)| \, dz < \infty \) for some \( c_1 > 0 \). It is clear that \( q \) is bounded and continuous on \( \mathbb{R} \). Since, for all \( \gamma > 0 \),

\[ (c_1, 1) \to \int_{1-} e^{\gamma x} \nu(dx) < \infty, \]

we see from Lemma 6 that, for all \( \gamma > 0 \),

\[ \int_{0-}^{\infty} e^{\gamma x} \zeta_1(dx) < \infty. \]

Define the exponential tilt \( \xi \) of the distribution \( \zeta_1 \) on \( \mathbb{R}_+ \) as

\[ \xi(dx) := \frac{e^{\gamma x} q(x)}{\int_{0-}^{\infty} e^{\gamma x} \zeta_1(dx)} \, dx. \]

We find from Lemma 7 that \( \xi \) is an infinitely divisible distribution on \( \mathbb{R}_+ \) with Lévy measure \( 1_{(0, c_1]}(x) e^{\gamma x} \times \nu(dx) \). Thus we have

\[ |\hat{\xi}(z)| = |\hat{\zeta}_1(z)| \exp\left( \int_0^{c_1} (\cos zx - 1)(e^{\gamma x} - 1) \, \nu(dx) \right) \leq |\hat{\zeta}_1(z)|. \]

Hence \( \int_\gamma^\infty |\hat{\xi}(z)| \, dz < \infty \), and by the Riemann–Lebesgue lemma we have that, for any \( \gamma > 0 \), the limit

\[ \lim_{x \to \infty} e^{\gamma x} q(x) = 0. \]

Proof of Proposition 1. We see from Lemma 4 that the following four conditions are equivalent: (1) \( \mu \in \mathcal{S}_{\text{loc}} \); (2) \( \nu((1) \in \mathcal{S}_{\text{loc}} \); (3) \( \nu((1) \in \mathcal{L}_{\text{loc}}, \) and \( \mu((x, x + c]) \sim \nu((x, x + c]) \) for all \( c > 0 \); and (4) \( \nu((1) \in \mathcal{L}_{\text{loc}} \), and there is
$C \in (0, \infty)$ such that $\mu((x, x + c)) \sim C \nu((x, x + c))$ for all $c > 0$. First, we prove statement (i). Let $p \in \mathcal{L}_d$. If $p \in \mathcal{S}_d$, then $p \in \mathcal{S}_{loc}$, and hence $\nu(1) \in \mathcal{S}_{loc}$. Conversely, if $\nu(1) \in \mathcal{S}_{loc}$, then $\mu \in \mathcal{S}_{loc}$, and thereby $p \in \mathcal{S}_d$ by (ii) of Lemma 2. Next, we prove statement (ii). We obtain from (ii) of Lemma 2 that (a), (b), and (c) are respectively equivalent to (1) and $p \in \mathcal{L}_d$, (3) and $p \in \mathcal{L}_d$, and (4) and $p \in \mathcal{L}_d$. Hence we find that (a)–(c) are equivalent. □

**Proof of Theorem 1.** Assume that, for some $b_0 > 0$,

$$\lim_{x \to \infty} \exp(b_0x)f_1(x) = 0$$

and that $g_1$ is bounded and $g_1 \in \mathcal{L}_d$. We see from (i) of Proposition 1 and (ii) of Lemma 2 that (1) implies (2). We obtain from (ii) of Proposition 1 and $g_1 \in \mathcal{L}_d$ that (1), (3), and (4) are equivalent. Finally, we prove that (2) implies (1). Let $\nu(dx) = \eta_1(dx) + \eta_2(dx)$, where $\eta_1(dx) := 1_{(0,1)}(x)g(x)\,dx$ and $\eta_2(dx) := 1_{(1,\infty)}g(x)\,dx$.

Let $\xi_j(dx)$ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure $\eta_j(dx)$ for $j = 1, 2$. Then $\xi_1(dx) = f_1(x)\,dx$. By Lemma 6 we have, for any $b > 0$,

$$\int_{0^+}^{\infty} \exp(bx)f_1(x)\,dx < \infty.$$ 

Note that $\nu_{(1)}(dx) = \eta_2(dx)/\eta_2((1, \infty))$ has the bounded density $g_1$ belonging to $\mathcal{S}_d$ and $\xi_2(dx)$ is a compound Poisson distribution. Thus we have

$$\xi_2(dx) = c\delta_0(dx) + (1 - c)f_2(x)\,dx,$$

where $c := \exp(-\eta_2((1, \infty))) \in (0, 1)$. Moreover, by (ii) and (iii) of Lemma 2 with $c_1 := \eta_2((1, \infty))$ we have

$$(1 - c)f_2(x) := \sum_{n=1}^{\infty} c_1^n g_1^{n\otimes}(x) n! \sim c_1 g_1(x).$$

Thus $f_2$ is bounded, and $f_2 \in \mathcal{S}_d$ by (i) of Lemma 2. Define

$$p(x) := cf_1(x) + (1 - c)f_1 \otimes f_2(x).$$

Then $p$ is the density of $\mu(dx) = \xi_1 \ast \xi_2(dx)$. Note that

$$\lim_{x \to \infty} \frac{f_1(x)}{f_2(x)} = \lim_{x \to \infty} \frac{\exp(b_0x)f_1(x)}{\exp(b_0x)f_2(x)} = 0.$$ 

Since $f_2$ is bounded and $f_2 \in \mathcal{S}_d$, for $0 \leq u \leq x$ and for sufficiently large $x > 0$, there are $C > 0$ and $b_1 > 0$ such that

$$\frac{f_2(x - u)}{f_2(x)} \leq C \exp(b_1u).$$

Since $f_2 \in \mathcal{S}_d$, by Fatou’s lemma we have

$$\lim_{x \to \infty} \inf \frac{f_1 \otimes f_2(x)}{f_2(x)} \geq \lim_{N \to \infty} \int_{0^+}^{N} \lim_{x \to \infty} \frac{f_2(x - u)}{f_2(x)} f_1(u)\,du = 1.$$
and

\[
\limsup_{x \to \infty} \frac{f_1 \otimes f_2(x)}{f_2(x)} \leq \lim_{N \to \infty} \limsup_{x \to \infty} \int_0^N \frac{f_2(x-u)}{f_2(x)} f_1(u) \, du + \lim_{N \to \infty} \limsup_{x \to \infty} \int_N^\infty \frac{f_2(x-u)}{f_2(x)} f_1(u) \, du
\]

\[
\leq \lim_{N \to \infty} \int_0^N \limsup_{x \to \infty} \frac{f_2(x-u)}{f_2(x)} f_1(u) \, du + \lim_{N \to \infty} \int C \exp(b_1 u) f_1(u) \, du
\]

\[
= 1.
\]

Thus \( p(x) \sim (1-c)f_1 \otimes f_2(x) \sim (1-c)f_2(x) \), and hence \( p \in S_d \) by (i) of Lemma 2, that is, (1) holds. □

**Proof of Remark 1.** Let \( p_1 \) be the self-decomposable density on \( \mathbb{R}_+ \) with Lévy measure \( 1_{(0,1)}(x)x^{-1}k(x) \, dx \). By Lemma 6 we have, for any \( b > 0 \),

\[
\int_{0-}^\infty \exp(bx)p_1(x) \, dx < \infty.
\]

Note that by Lemma 9 \( p_1 \) is continuous on \((0, \infty)\). Hence from the unimodality of \( p_1(x) \, dx \) by Lemma 8 we see that, for any \( b > 0 \),

\[
\lim_{x \to \infty} \exp(bx)p_1(x) \leq \lim_{x \to \infty} \int_{x-}^x \exp(2bu)p_1(u) \, du = 0.
\]

Let \( \mu_2(dx) := c\delta_0(dx) + (1-c)p_2(dx) \, dx \) be the compound Poisson distribution on \( \mathbb{R}_+ \) with Lévy measure \( 1_{(0,1)}(x)h(x) \, dx \). Define \( c_1h_1(x) := 1_{(0,1)}(x)h(x) \) with \( c_1 = \int_0^1 h(x) \, dx \) Then \( c := \exp(-\int_0^1 h(x) \, dx) \) and

\[
(1-c)p_2(x) := e^{\sum_{n=1}^\infty \frac{c_1^n h_1^{(n)}(x)}{n!}}.
\]

Since \( h \) is bounded, \( p_2 \) is bounded as well. For \( b > 0 \), we define the exponential tilt \( (\mu_2)_b \) of \( \mu_2 \) as

\[
(\mu_2)_b := \frac{\exp(bx)}{\int_{0-}^\infty \exp(bx)\mu_2(dx)} \mu_2(dx).
\]

Then we find from Lemma 7 that \((\mu_2)_b \) is the compound Poisson distribution with Lévy measure

\[
\eta(dx) := 1_{(0,1)}(x) \exp(bx)h(x) \, dx.
\]

Note that the support of \( \eta^{\mu_2} \) is included in the interval \([0, n]\) and \( \exp(bx)h(x) \) is bounded on \([0, 1]\). Thus, with \( C_0, C_1 > 0 \), we have, for any \( b > 0 \),

\[
\lim_{x \to \infty} \exp(bx)p_2(x) \leq C_0 \lim_{x \to \infty} \sum_{n \geq x} \frac{C_1^n}{n!} = 0.
\]
We have
\[ p_1 \otimes p_2(2x) \leq \sup_{x \leq u \leq 2x} p_2(u) \int_0^x p_1(u) \, du + \sup_{x \leq u \leq 2x} p_1(u) \int_x^{2x} p_2(2x - u) \, du. \]

We find that
\[ \lim_{x \to \infty} \exp(2bx) \sup_{x \leq u \leq 2x} p_2(u) \int_0^x p_1(u) \, du = 0 \]
and, by the unimodality of \( p_1(x) \, dx \),
\[ \lim_{x \to \infty} \exp(2bx) \sup_{x \leq u \leq 2x} p_1(u) \int_x^{2x} p_2(2x - u) \, du = 0. \]

Thus we see that, for any \( b > 0 \),
\[ \lim_{x \to \infty} \exp(2bx)p_1 \otimes p_2(2x) = 0, \]
and hence, for any \( b > 0 \),
\[ \lim_{x \to \infty} \exp(bx)f_1(x) = \lim_{x \to \infty} \exp(bx)(cp_1(x) + (1-c)p_1 \otimes p_2(x)) = 0. \]

We have proved the remark. \( \Box \)

**Proof of Corollary 1.** Assume that, for every \( t > 0 \), there is some \( b_0(t) > 0 \) such that
\[ \lim_{x \to \infty} \exp\left(b_0(t)x\right)f_1^t(x) = 0 \]
and that \( g_1 \) is bounded and \( g_1 \in \mathcal{L}_d \). First, we prove statement (i). Suppose that \( p^t \in \mathcal{S}_d \) for some \( t > 0 \). Then we obtain from Theorem 1 that \( g_1 \in \mathcal{S}_d \), and hence \( p^t \in \mathcal{S}_d \) for all \( t > 0 \). Moreover, we find again from Theorem 1 that, for all \( t > 0 \),
\[ p^t(x) \sim tC_0 g_1(x) \sim t p^1(x) \]
with \( C_0 := \int_1^\infty \nu(dx) \). Next, we prove statement (ii). Suppose that \( p^t \in \mathcal{L}_d \) and, for some \( t \in (0, 1) \cup (1, \infty) \), there is \( C(t) \in (0, \infty) \) such that (1.2) holds. Then we have that \( \mu \in \mathcal{L}_{\text{loc}} \) and (2.5) holds. Thus we obtain from Lemma 5 that \( C(t) = t \) and \( \mu \in \mathcal{S}_{\text{loc}} \), and hence, by \( p^1 \in \mathcal{L}_d \) and (ii) of Lemma 2, \( p^1 \in \mathcal{S}_d \). \( \Box \)

**Proof of Theorem 2.** Assume that \( \int_{-\infty}^{\infty} |\tilde{\mu}(z)| \, dz < \infty \). We see from (ii) of Proposition 1 that (1), (3), and (4) are equivalent. Further, we find from (i) of Proposition 1 that (1) implies (2). Finally, we prove that (2) implies (1). Suppose that (2) holds. We see from Lemma 4 that \( \mu_1 \in \mathcal{S}_{\text{loc}} \subset \mathcal{L}_{\text{loc}} \) for every \( c_1 > 0 \). By (i) of Lemma 10 we can choose sufficiently large \( c_1 > 0 \) so that \( \int_{-\infty}^{\infty} |\tilde{\zeta}_1(z)| \, dz < \infty \). Let \( q \) be the continuous density of \( \zeta_1 \). Then we obtain from (ii) of Lemma 10 that, for every \( \gamma > 0 \), \( \lim_{x \to \infty} e^{\gamma x}q(x) = 0 \). Since
\[ p(x) = \int_{-\infty}^{\infty} q(x-u) \, \mu_1(du), \]
we see from (ii) of Lemma 3 that \( p \in \mathcal{L}_d \) and hence from (i) of Proposition 1 that \( p \in \mathcal{S}_d \). \( \Box \)
Proof of Corollary 2. Assume that \( \int_{-\infty}^{\infty} |\bar{\mu}(z)|^t \, dz < \infty \) for every \( t > 0 \). First, we prove statement (i). Suppose that \( p^t \in S_d \) for some \( t > 0 \). Then from Theorem 2 we obtain that \( \nu_{(1)} \in S_{\text{loc}} \), and hence \( p^t \in S_d \) for all \( t > 0 \). Moreover, again from Theorem 2 we find that, for all \( t > 0 \),

\[
p^t(x) \sim t \nu((x, x+1)) \sim tp^1(x).
\]

Next, we prove statement (ii). Suppose that \( p^1 \in \mathcal{L}_d \) and, for some \( t \in (0, 1) \cup (1, \infty) \), there is \( C(t) \in (0, \infty) \) such that (1.3) holds. Then we have that \( \mu \in \mathcal{L}_{\text{loc}} \) and (2.5) holds. Thus from Lemma 5 we obtain that \( C(t) = t \) and \( \mu \in S_{\text{loc}} \), and hence, since \( p^1 \in \mathcal{L}_d \), \( p^1 \in S_d \) by (ii) of Lemma 2. \( \square \)

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