Soliton solutions of a gauged O(3) sigma model with interpolating potential

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Soliton modes in a gauged sigma model with interpolating potential have been investigated. Numerical solutions using a fourth order Runge–Kutta method are discussed. By tuning the interpolation parameter the transition from symmetrybreaking to the symmetric phase is highlighted.

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I. INTRODUCTION

The O(3) nonlinear sigma model has long been the subject of intense research due to its theoretical and phenomenological basis. This theory describes classical (anti) ferromagnetic spin systems at their critical points in Euclidean space, while in the Minkowski one it delineates the long wavelength limit of quantum antiferromagnets. The model exhibits solitons, Hopf instantons and novel spin and statistics in 2+1 space-time dimensions with inclusion of the Chern-Simons term.

The soliton solutions of the model exhibit scale invariance which poses difficulty in the particle interpretation on quantization. A popular means of breaking this scale invariance is to gauge a U(1) subgroup of the O(3) symmetry of the model by coupling the sigma model fields with a gauge field through the corresponding U(1) current. This class of gauged O(3) sigma models in three dimensions have been studied over a long time. Initially the gauge field dynamics was assumed to be dictated by the Maxwell term. Later the extension of the model with the Chern–Simons coupling was investigated. A particular form of self-interaction was required to be included in these models in order to saturate the Bogomol'nyi bounds. The form of the assumed self-interaction potential is of crucial importance. The minima of the potential determine the vacuum structure of the theory. The solutions change remarkably when the vacuum structure exhibits spontaneous breaking of the symmetry of the gauge group. Thus it was demonstrated that the observed degeneracy of the solutions is lifted when potentials with symmetry breaking minima were incorporated. The studies of the gauged O(3) sigma model is important due to their intrinsic interest and also due to the fact that the soliton solutions of the gauged O(3) Chern-Simons model may be relevant in planar condensed matter systems. Recently gauged nonlinear sigma model was considered in order to obtain self-dual cosmic string solutions. This explains the continuing interest in such models in the literature.

A particular aspect of the gauged O(3) sigma models where the gauge field dynamics is governed by the Maxwell term can be identified by comparing the results of the vacuum is symmetric and the \( n = 1 \) soliton solution does not exist, \( n \) being the topological charge. Here, solutions exist for \( n = 2 \) onwards. Moreover, these soliton solutions have arbitrary magnetic flux. When we achieve symmetrybreaking vacuum by choosing the potential appropriately, soliton solutions are obtained for \( n = 1 \). These solutions have quantized magnetic flux and qualify as magnetic vortices. It will be interesting to follow the solutions from the symmetrybreaking to the symmetric phase. This is the motivation of the present paper.

We will consider a generalisation of the models with an adjustable real parameter \( r \) in the expression of the self-interaction potential which interpolates between the symmetric and the symmetrybreaking vacua. This will in particular allow us to investigate the soliton solutions in the entire regime of the symmetrybreaking vacuum structures and also to follow the collapse of the \( n = 1 \) soliton as we move from the assymmetric to the symmetric phase. The solitons of the model are obtained as the solutions of the self–dual equation obtained by saturating the Bogomol'nyi bounds. Unfortunately, these equations fall outside the Liouville class even after assuming a rotationally symmetric ansatz. Thus exact analytical solutions are not obtainable and numerical methods are to be invoked.

The organisation of the paper is as follows. In the following section we present a brief review of the O(3) nonlinear sigma model. This will be helpful in presenting our work in the proper context. In section 3 our model is introduced. General topological classifications of the soliton solutions of the model have been discussed here. In section 4 the saturation of the self-dual limits has been examined and the Bogomol'nyi equations have been written down. Also the analytical form of the Bogomol'nyi equations has been worked out assuming a rotationally symmetric ansatz. These equations, even in the rotationally symmetric scenario, are not exactly integrable. A numerical solution has been performed to understand the details of the solution. A fourth order Runge–Kutta algorithm is adopted with provision of tuning the potential appropriately. In section 5 the numerical method and some results are presented. We conclude in section 6.
II. O(3) NONLINEAR SIGMA MODELS

It will be useful to start with a brief review of the nonlinear O(3) sigma model\textsuperscript{14}. The lagrangian of the model is given by,

\begin{equation}
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi
\end{equation}

Here $\phi$ is a triplet of scalar fields constituting a vector in the internal space with unit norm

\begin{equation}
\phi_a = n_a \cdot \phi, \quad (a = 1, 2, 3) \quad (2)
\end{equation}

\begin{equation}
\phi \cdot \phi = \phi_a \phi_a = 1 \quad (3)
\end{equation}

The vectors $n_a$ constitute a basis of unit orthogonal vectors in the internal space. We work in the Minkowskian space-time with the metric tensor diagonal, $g_{\mu \nu} = (1, -1, -1)$.

The finite energy solutions of the model (1) satisfies the boundary condition

\begin{equation}
\lim \phi^a = \phi^a_{(0)}
\end{equation}

at physical infinity. The condition (4) corresponds to one point compactification of the physical infinity. The physical space $\mathbb{R}^2$ becomes topologically equivalent to $S^2$ due to this compactification. The static finite energy solutions of the model are then maps from this sphere to the internal sphere. Such solutions are classified by the homotopy\textsuperscript{15}.

\begin{equation}
\Pi_2(S^2) = \mathbb{Z}
\end{equation}

We can construct a current

\begin{equation}
K_\mu = \frac{1}{8\pi} \epsilon_{\mu \nu \lambda} \phi \cdot (\partial^\nu \phi \times \partial^\lambda \phi)
\end{equation}

which is conserved irrespective of the equation of motion. The corresponding charge

\begin{align*}
T &= \int d^2 x K_0 \\
&= \frac{1}{8\pi} \int d^2 x \epsilon_{ij} \phi \cdot (\partial^i \phi \times \partial^j \phi)
\end{align*}

(7)

gives the winding number of the mapping (5)\textsuperscript{16}.

III. OUR MODEL - TOPOLOGICAL CLASSIFICATION OF THE SOLITON SOLUTIONS

In the class of gauged models of our interest here a U(1) subgroup of the rotation symmetry of the model (1) is gauged. We chose this to be the SO(2) [U(1)] subgroup of rotations about the 3 - axis in the internal space. The Lagrangian of our model is given by

\begin{equation}
\mathcal{L} = \frac{1}{2} D_\mu \phi \cdot D^\mu \phi - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + U(\phi)
\end{equation}

$D_\mu \phi$ is the covariant derivative given by

\begin{equation}
D_\mu \phi = \partial_\mu \phi + A_\mu n_3 \times \phi
\end{equation}

The SO(2) (U(1)) subgroup is gauged by the vector potential $A_\mu$ whose dynamics is dictated by the Maxwell term. Here $F_{\mu \nu}$ are the electromagnetic field tensor,

\begin{equation}
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\end{equation}

$U(\phi)$ is the self - interaction potential required for saturating the self - dual limits. We chose

\begin{equation}
U(\phi) = -\frac{1}{2} (v - \phi_3)^2
\end{equation}

(11)
where \( v \) is a real parameter. Substituting \( v = 0 \) we get back the model of \( \phi^2 \), whereas \( v = 1 \) gives the model of \( \phi^1 \).

We observe that the minima of the potential arise when,

\[
\phi_3 = v
\]  
(12)

which is equivalent to the condition

\[
\phi_1^2 + \phi_2^2 = 1 - v^2
\]  
(13)

on account of the constraint (9). The values of \( v \) must be restricted to

\[
|v| \leq 1
\]  
(14)

The condition (13) denotes a latitudinal circle (i.e. circle with fixed latitude) on the unit sphere in the internal space. By varying \( v \) from -1 to +1 we span the sphere from the south pole to the north pole. It is clear that the finite energy solutions of the model must satisfy (19) at physical infinity. For \( v \neq 1 \) this boundary condition corresponds to the spontaneous breaking of the symmetry of the gauge group and in the limit

\[
|v| \to 1
\]  
(15)

the asymmetric phase changes to the symmetric phase. We call the potential (11) interpolating in this sense. In the asymmetric phase the soliton solutions are classified according to the homotopy

\[
\Pi_1(S_1) = Z
\]  
(16)

instead of (5). In the symmetric phase, however, this new topology disappears and the solitons are classified according to (5) as in the usual sigma model (1). A remarkable fallout of this change of topology is the disappearance of the soliton with unit charge. The fundamental solitonic mode \( n = 1 \) (\( n \) being the vorticity) ceases to exist in the symmetric phase. The modes corresponding to \( n = 2 \) onwards still persist but the magnetic flux associated with them ceases to remain quantised.

In the asymmetric phase the vorticity is the winding number i.e. the number of times by which the infinite physical circle winds over the latitudinal circle (13). Associated with this is a unique mapping of the internal sphere where the degree of mapping is usually fractional. By inspection we construct a current

\[
K_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} [\phi \cdot D^\nu \phi \times D^\lambda \phi - F_{\nu\lambda}(v - \phi_3)]
\]  
(17)

generalising the topological current (6). The current (17) is manifestly gauge invariant and differs from (6) by the curl of a vector field. The conservation principle

\[
\partial_\mu K^\mu = 0
\]  
(18)

thus automatically follows from the conservation of (6). The corresponding conserved charge is

\[
T = \int d^2x K_0
\]  
(19)

Using (17) and (19) we can write

\[
T = \int d^2x \left[ \frac{1}{8\pi} \epsilon_{ij} \phi \cdot (\partial^i \phi \times \partial^j \phi) \right] - \frac{1}{4\pi} \int_{\text{boundary}} (v - \phi_3) A_\theta r d\theta
\]  
(20)

where \( r, \theta \) are polar coordinates in the physical space and \( A_\theta = e_\theta \cdot A \). Using the boundary condition (12) we find that \( T \) is equal to the degree of the mapping of the internal sphere. Note that this situation is different from (6) where the topological charge usually differs from the degree of the mapping. In this context it is interesting to observe that the current (17) is not unique because we can always add an arbitrary multiple of

\[
\frac{1}{8\pi} \epsilon_{\mu\nu\lambda} F^{\nu\lambda}
\]

with it without affecting its conservation. We chose (17) because it generates proper topological charge.
IV. SELF DUAL EQUATIONS IN THE ROTATIONALLY SYMMETRIC ANSATZ

In the previous section we have discussed the general topological classification of the solutions of the equations of motion following from (8). In the present section we will discuss the solution of the equations of motion. The Euler-Lagrange equations of the system (8) is derived subject to the constraint (3) by the Lagrange multiplier technique

\[ D_\nu (D^\nu \phi) = [D_\nu (D^\nu \phi) \cdot \phi] \phi + n_3 (v - \phi_3) \]
\[ + (v - \phi_3) \phi_3 \phi \]
\[ \partial_\nu F^{\mu\nu} = j^\mu \] (21)

where

\[ j^\mu = -n_3 \cdot J^\mu \text{ and } J^\mu = \phi \times D^\mu \phi \] (23)

Using (21) we get

\[ D_\mu J^\mu = -(v - \phi_3) (n_3 \times \phi_3) \phi_3 \] (24)

From (22) we find, for static configurations

\[ \nabla^2 A_0 = -A_0 (1 - \phi_3^2) \] (25)

From the last equation it is evident that we can choose

\[ A_0 = 0 \] (26)

As a consequence we find that the excitations of the model are electrically neutral.

The equations (21) and (22) are second order differential equations in time. As is well known, first order equations which are the solutions of the equations of motion can be derived by minimizing the energy functional in the static limit. Keeping this goal in mind we now construct the energy functional from the symmetric energy-momentum tensor following from (8). The energy

\[ E = \frac{1}{2} \int d^2 x \left[ D_0 \phi \cdot D_0 \phi - D_i \phi \cdot D_i \phi + (v - \phi_3)^2 - 2 (F_0^\sigma F_0^\sigma - \frac{1}{4} F_{\sigma\rho} F^{\rho\sigma}) \right]. \] (27)

For static configuration and the choice \( A_0 = 0 \), \( E \) becomes

\[ E = \frac{1}{2} \int d^2 x [(D_i \phi) \cdot (D_i \phi) + F_{ij}^2 + (v - \phi_3)^2] \] (28)

Several observations about the finite energy solutions can be made at this stage from (28). By defining

\[ \psi = \phi_1 + i \phi_2 \] (29)

we get

\[ D_i \phi \cdot D_i \phi = |(\partial_i i A_i) \psi|^2 + (\partial_i \phi_3)^2 \] (30)

The boundary condition (13) dictates that

\[ \psi \approx (1 - v^2) \frac{n}{r} e^{in\theta} \] (31)

at infinity. From (28) we observe that for finite energy configurations we require

\[ A = e^{\frac{n}{r}} \] (32)

on the boundary. This scenario is exactly identical with the observations of \( \omega \) and leads to the quantisation of the magnetic flux

\[ \Phi = \int B d^2 x = \int_{\text{boundary}} A_\theta r d\theta = 2\pi n \] (33)
The basic mechanism leading to this quantisation remains operative so far as \( v \) is less than 1. At \( v = 1 \), however, the gauge field \( A \) becomes arbitrary on the boundary except for the requirement that the magnetic field \( B \) should vanish on the boundary. Remember that not all the vortices present in the broken phase survives this demand. Specifically, the \( n = 1 \) vortex becomes inadmissible.

Now the search for the self-dual conditions proceed in the usual way. We rearrange the energy functional as

\[
E = \frac{1}{2} \int d^2x \left[ \frac{1}{2} (D_i \phi \pm \epsilon_{ij} \phi \times D_j \phi)^2 + (F^{12} \mp (v - \phi_3))^2 \right] \pm 4\pi T
\]

Equation (34) gives the Bogomol’nyi conditions

\[
D_i \phi \pm \epsilon_{ij} \phi \times D_j \phi = 0 \quad (35)
\]

\[
F^{12} \mp (v - \phi_3) = 0 \quad (36)
\]

which minimize the energy functional in a particular topological sector, the upper sign corresponds to \(+ve\) and the lower sign corresponds to \(-ve\) value of the topological charge.

We will now turn towards the analysis of the self-dual equations using the rotationally symmetric ansatz

\[
\phi_1(r, \theta) = \sin g(r) \cos n\theta \\
\phi_2(r, \theta) = \sin g(r) \sin n\theta \\
\phi_3(r, \theta) = \cos g(r) \\
A(r, \theta) = -e^\theta \frac{na(r)}{r}
\]

From (12) we observe that we require the boundary condition

\[
g(r) \to \cos^{-1} v \quad \text{as} \quad r \to \infty
\]

and equation (32) dictates that

\[
a(r) \to -1 \quad \text{as} \quad r \to \infty
\]

Remember that equation (32) was obtained so as the solutions have finite energy. Again for the fields to be well defined at the origin we require

\[
g(r) \to 0 \quad \text{or} \quad \pi \quad \text{and} \quad a(r) \to 0 \quad \text{as} \quad r \to 0
\]

Substituting the Ansatz (37) into (35) and (36) we find that

\[
g'(r) = \pm \frac{n(a + 1)}{r} \sin g, \quad (41)
\]

\[
a'(r) = \pm \frac{r}{a} (v - \cos g) \quad (42)
\]

where the upper sign holds for \(+ve\) \(T\) and the lower sign corresponds to \(-ve\) \(T\). Equations (41) and (42) are not exactly integrable. In the following section we will discuss the numerical solution of the boundary value problem defined by (41) and (42) with (38) to (40).

Using the Ansatz (37) we can explicitly compute the topological charge \( T \) by performing the integration in (19). The result is

\[
T = -\frac{n}{2} \left[ \cos g(\infty) - \cos g(0) \right] - \frac{1}{2} [v - \cos g(\infty)] \quad (43)
\]

The second term of (43) vanishes due to the boundary condition (38). Also, when \( g(0) = 0, \)

\[
T = \frac{n}{2} (1 - v) \quad (44)
\]

and, when \( g(0) = \pi, \)

\[
T = \frac{n}{2} (1 + v) \quad (45)
\]
It is evident that $T$ is in general fractional. Due to (20) it is equal to the degree of mapping of the internal sphere. This can also be checked explicitly.

From the above analysis we find that $g(0) = 0$ corresponds to $+ve$ $T$ and $g(0) = \pi$ corresponds to $-ve$ $T$. We shall restrict our attention on negative $T$ which will be useful for comparison of results with those available in the literature. The boundary value problem of interest is then

\begin{align*}
g'(r) &= -\frac{n(a + 1)}{r} \sin g \\
a'(r) &= -\frac{r}{n}(v - \cos g)
\end{align*}

with

\begin{align*}
g(0) &= \pi, a(0) = 0 \\
g(\infty) &= \cos^{-1} v, a(\infty) = 0
\end{align*}

In addition we require $a'(r) \to 0$ as $r \to \infty$. This condition follows from (46), (47) and (48) and should be considered as a consistency condition to be satisfied by their solutions.

V. NUMERICAL SOLUTION

The simultaneous equations (46) and (47) subject to the boundary conditions (48) are not amenable to exact solution. They can however be integrated numerically. We have already mentioned the quenching of the $n = 1$ solution in the limit $v \to 1$. This is connected with the transition from the symmetry breaking to the symmetric phase. The numerical solution is thus interesting because it will enable us to see how the solutions change as we follow them from the deep asymmetric phase ($v = 0$) to the symmetric phase ($v = 1$). In the following we provide the results of numerical solution to highlight these issues.

Let us note some details of the numerical method. A fourth order Runge–Kutta method was employed. The point $r = 0$ is a regular singular point of the equation. So it was not possible to start the code from $r = 0$. Instead, we start it from a small value of $r$. The behaviour of the functions near $r = 0$ can be easily derived from (46) and (47)

\begin{align*}
g(r) &\approx \pi + Ar^n \\
a(r) &\approx -\frac{r^2}{2n}(1 + v)
\end{align*}

Here $A$ is an arbitrary constant which fixes the values of $g$ and $a$ at infinity. In the symmetry breaking phase the numerical solution depends sensitively on the value of $A$.\textsuperscript{23} There is a critical value of $A$, $A = A^{crit}$ for which the boundary conditions are satisfied. If the value of $A$ is larger than $A^{crit}$ the conditions at infinity are overshooted, whereas, if the value is smaller than the critical value $g(r)$ vanishes asymptotically after reaching a maximum. The situation is comparable with similar findings elsewhere.\textsuperscript{19} The values of $g$ and $a$ were calculated at a small value of $r$ using (49) and (50). The parameter $A$ was tuned to match boundary conditions at the other end. Interestingly, this matching is not obtainable when $n = 1$ and $v = 1$. This is consistent with the quenching of the $n = 1$ mode in the symmetric vacuum situation.

After the brief discussion of the numerical technique we will present a summary of the results. As may be recalled, the purpose of the paper is to study the solutions throughout the asymmetric phase with an eye to the disappearance of the $n = 1$ mode. Accordingly profiles of $g$ and $a$ will be given for $n = 1$, for different values of $v$. In figures 1 and 2 these profiles are shown for $v = 0$, 2, 4, 6, 8. The corresponding magnetic field distributions are given in figure 3. Another interesting issue is the change of the matter and the gauge profiles with the topological charge. In figures 4 and 5 this is demonstrated for different $n$ values for a constant $v$. 

FIG. 1: The matter profile. The function $g(r)$ is plotted against $r$ for $n = 1$ and different $v$ values, indicated on the right hand top corner.

FIG. 2: The gauge field profile. The function $a(r)$ is plotted against $r$ for $n = 1$ and different $v$ values.

FIG. 3: The magnetic field profiles. Plot of the function $B(r)$ against $r$.

VI. CONCLUSION

The O(3) sigma model in (2+1) dimensional space – time with its U(1) subgroup gauged was mooted as a possible mechanism to break the scale – invariance of the solutions of the original 3 - dimensional O(3) sigma model. The
model finds possible applications in such diverse areas such as planar condensed matter physics, gravitating cosmic strings, and as such is being continuously explored in the literature. An interesting aspect of the gauged O(3) sigma models is the qualitative change of the soliton modes in the symmetric and symmetry-breaking vacuum scenario, as can be appreciated by a comparison of solutions given in [1,5]. In this paper we have considered a gauged O(3) sigma model with the gauge field dynamics determined by the Maxwell term as in [1,5]. An interpolating potential was included to investigate the solutions in the entire symmetry-breaking regime. This potential depends on a free parameter, the variation of which effects transition from the asymmetric to symmetric phase. We have discussed the transition of the associated topology of the soliton solutions. The Bogomol’nyi bound was saturated to give the self–dual solutions of the equation of motion. The self–dual equations are, however, not exactly solvable. They were studied numerically to trace out the solutions in the entire asymmetric phase with particular emphasis on the $n = 1$ mode. Our analysis may be interesting from the point of view of applications, particularly in condensed matter physics.

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21. This is different from the minimal coupling via the topological current discussed previously.
22. The disappearance of the \( n = 1 \) soliton has been shown to follow from general analytical method in .
23. This should be contrasted with the symmetric vacuum solution where \( A \) may have arbitrary values.