A Log-space Algorithm for Canonization of Planar Graphs*

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Abstract

Planar graph canonization is known to be hard for \(L\); this directly follows from \(L\)-hardness of tree-canonization [Lin92]. We give a log-space algorithm for planar graph canonization. This gives completeness for log-space under \(AC^0\) many-one reductions and improves the previously known upper bound of \(AC^1\) [MR91].

A planar graph can be decomposed into biconnected components. We give a log-space procedure for the decomposition of a biconnected planar graph into a triconnected component tree. The canonization process is based on these decomposition steps.

1 Introduction

The general graph isomorphism problem (GI) is a well studied problem in computer science. Given two graphs, find a bijection between the sets of vertices of both graphs such that the adjacencies are preserved. The problem is contained in \(NP\), but not known to be complete for \(NP\). In fact, if GI is complete for \(NP\), then the polynomial time hierarchy collapses to its second level. On the other hand, no polynomial time algorithm is known. In general, GI is known to be hard for \(DET\) [Tor04], which is the class of problems \(NC^1\)-reducible to the determinant, defined by Cook [Coo85].

GI restricted to certain graph classes has been studied. The isomorphism problem for tournament graphs is also hard for \(DET\) [Wag07]. For trees, \(L\) and \(NC^1\) hardness is known, depending on the encoding of the input [JT98]. For trees [Lin92], [Bus97], or vertex-coloured graphs with bounded colour multiplicity [Luk86], \(NC\) algorithms are known. We are interested in the case where the graphs under consideration are planar graphs. Weinberg [Wei66] presented an \(O(n^2)\) algorithm for testing isomorphism of 3-connected planar graphs. Hopcroft and Tarjan [HT74] extended this for general planar graphs, improving the time complexity to \(O(n \log n)\).

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Hopcroft and Wong [HW74] further improved it to give a linear time algorithm. Recently Kukluk, Holder, and Cook [KHC04] gave an $O(n^2)$ algorithm for planar graph isomorphism, which is suitable for practical applications.

Its parallel complexity was first considered by Miller and Reif [MR91] and Ramachandran and Reif [RR90]. They gave an upper bound of $AC^1$. Thierauf and Wagner [TW08] improved it to $UL \cap \text{coUL}$ for canonization of 3-connected planar graphs. They also proved that this problem is hard for $L$. Recently, Datta, Limaye and Nimbhorkar [DLN08] gave a log-space algorithm for the same problem. We give a log-space algorithm for biconnected planar graph canonization and planar graph canonization. This improves the earlier upper bound of $AC^1$ [MR91]. Basically, our algorithm can be given in five steps:

- **Decomposition into biconnected components**: Given a planar graph, we decompose it into its biconnected components. This can be done in log-space [ADK08]. Two such components share at most one common vertex. The resulting graph structure (according to these adjacencies) is the *biconnected component tree*.

- **Decomposition into triconnected components**: Given a biconnected planar graph $G$, we decompose it into its triconnected components. Hopcroft and Tarjan [HT73] presented a sequential algorithm for this problem. We show that this step can be accomplished in log-space. This decomposition into triconnected components is unique [Mac37].

- **Canonization of triconnected planar graphs**: Triconnected components consist of 3-connected components, cycles, and bonds. A *bond* is a multi-graph on two vertices with one or more edges [HT73]. We canonize each 3-connected component using the log-space algorithm for canonizing planar 3-connected graphs [DLN08]. Here we give a log-space procedure for canonizing cycles and bonds. These triconnected components along with the pairs of vertices that separate them from each other form a tree. We call this the *triconnected component tree*.

- **Canonization of biconnected planar graphs**: Given two triconnected component trees, we find whether the corresponding graphs are isomorphic by comparing these trees in log-space. This requires a canonical ordering of the nodes of these trees, using ideas from Lindell’s algorithm for tree canonization [Lin92]. A log-space transducer outputs the canon of such a tree, with the canonical ordering algorithm as an oracle. Thus canonization of such a tree can be done in log-space.

- **Canonization of planar graphs**: We traverse a biconnected component tree by comparing the biconnected components on the tree-nodes with the given ordering procedure. Once we can order them, we output their canons (which are children of the same parent to be canonized) with the given canonization procedure in log-space.

The rest of the paper is structured as follows: Section 2 contains preliminary definitions. In Section 3, we give a log-space procedure for decomposition of a biconnected planar graph into its triconnected components, to get the triconnected component tree. In Section 4, we give a log-space algorithm for the isomorphism ordering of triconnected component trees. Using this, we give a log-space procedure that outputs the canon of a biconnected planar graph. In Section 5 we extend this to the canonization of planar graphs.

# 2 Preliminaries

In this section, we recall some basic graph theoretic and complexity theoretic notions.
A graph \( G = (V, E) \) is connected if there is a path between any two vertices in \( G \). A vertex \( v \in V \) is an articulation point if \( G(V \setminus \{v\}) \) is not connected. A pair of vertices \( u, v \in V \) is a separating pair if \( G(V \setminus \{u, v\}) \) is not connected. A biconnected graph contains no articulation points. A 3-connected graph contains no separating pairs. A triconnected graph is either a 3-connected graph or a cycle or a 3-bond. A \( k \)-bond is a graph consisting of two vertices joined by \( k \) edges.

A pair of vertices \( (a, b) \) is said to be 3-connected if there are three or more vertex-disjoint paths between them. In particular, a separating pair \( \{a, b\} \) that spans a face \( f \) is called 3-connected if there are at least three vertex-disjoint paths between \( a, b \) i.e. there is a path between \( a, b \) in \( \hat{G} \) which is vertex-disjoint from the boundary of \( f \).

Let \( E_v \) be the set of edges incident to \( v \). A rotation scheme for a graph \( G \) is a set \( \rho \) of permutations, \( \rho = \{ \rho_v \mid v \in V \} \), where \( \rho_v \) is a cyclic permutation on \( E_v \) that has only one cycle (which is called a rotation). Let \( \rho^{-1} \) be the set of inverse rotations, \( \rho^{-1} = \{ \rho_v^{-1} \mid v \in V \} \). A rotation scheme \( \rho \) describes an embedding of graph \( G \) in the plane. If the embedding is planar, we call \( \rho \) a planar rotation scheme. Note that in this case \( \rho^{-1} \) is a planar rotation scheme as well. Allender and Mahajan [AM00] showed that a planar rotation scheme for a planar graph can be computed in log-space.

Two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are said to be isomorphic \( (G_1 \cong G_2) \) if there is a bijection \( \phi : V_1 \rightarrow V_2 \) such that \((u, v) \in E_1 \) if and only if \((\phi(u), \phi(v)) \in E_2 \). Graph isomorphism (GI) is the problem of deciding whether two given graphs are isomorphic.

A planar graph \( G \), along with its planar embedding (given by \( \rho \)) is called a plane graph \( \hat{G} = (G, \rho) \). A plane graph divides the plane into regions. Each such region is called a face. Let Planar-GI be the special case of GI when the given graphs are planar. The biconnected (respectively, 3-connected) planar GI is a special case of Planar-GI when the graphs are biconnected (3-connected) planar graphs.

Let \( \mathcal{G} \) be a class of graphs. Let \( f : \mathcal{G} \rightarrow \{0, 1\}^* \) be a function such that for all \( G, H \in \mathcal{G} \) we have \( G \cong H \iff f(G) = f(H) \). Then \( f \) computes a complete invariant for \( \mathcal{G} \). If \( f \) computes for \( G \) a graph \( f(G) \) such that \( G \cong f(G) \) then we call \( f(G) \) the canon for \( G \).

By \( L \) we denote the languages computable by a log-space bounded Turing machine.

### 3 Decomposition of Biconnected Planar Graphs

In this section, we prove the following theorem.

**Theorem 3.1** The decomposition of biconnected planar graphs into triconnected components is in log-space.

Intuitively, the triconnected components of a graph \( G \) are the parts of \( G \) obtained by cutting \( G \) along separating pairs in pieces, called components, such that a copy of a separating pair \( (u, v) \) occurs in each component, where \( u \) and \( v \) is contained. These separating pairs like \( u, v \) are connected by a designated virtual edge. In the case that a separating pair is connected by an edge in \( G \), this edge is represented in an own 3-bond component. An exception from the above is made for cycles, these are not decomposed. We describe a log-space algorithm for such a decomposition of a biconnected planar graph. We start with definitions and then prove some properties of separating pairs.

**Definition 3.2** A separating pair \( \{a, b\} \) spans a face \( f \) if both its endpoints \( a, b \) lie on the boundary of \( f \). Let \( v_0, v_1, \ldots, v_k \) be a face boundary. Two separating pairs \( \{v_i, v_j\}, \{v_i', v_j'\} \) are called intersecting if \( i < i' < j < j' \), and non-intersecting otherwise.
Lemma 3.3 Every separating pair spans some face.

To see this, note that in a plane graph $\hat{G}$, a split component of a separating pair is embedded in some face. This can be considered as the spanned face.

Lemma 3.4 In a plane graph, 3-connected separating pairs which span the same face are non-intersecting.

Proof: Suppose $\{a, c\}$ and $\{b, d\}$ are two 3-connected intersecting separating pairs in $S_f$ and let $\rho$ be a path outside $f$ from $b$ to $d$. In particular, $\rho$ doesn’t pass through $a$ or $c$.

As the pair $b, d$ is 3-connected, it cannot be separated from the rest of the graph by any other separating pair. Let $v$ be a vertex that gets separated from $b$ and $d$ when $a$ and $c$ are removed from the graph. Since $v$ lies outside $f$, there is a path outside $f$ from $a$ via $v$ to $c$. Since the graph is planar, this path must intersect $\rho$. Thus there is a path from $v$ to $b$ and $d$ that doesn’t pass through $a$ or $c$. This contradicts the choice of $v$. ■

Definition 3.5 Call a set of vertices $V' \subseteq V(\hat{G})$ inseparable if there does not exist a 3-connected separating pair $\{a, b\}$ in $V(\hat{G})$ such that the removal of $\{a, b\}$ divides $V'$ into different connected components. Otherwise $V'$ is called separable. Given an inseparable triple $\tau = \{u, v, w\}$, define $C_{\tau} = \{x \mid \{u, v, w, x\} \text{ is inseparable}\}$.

With inseparable triples we can compute triconnected components. If a triple of vertices is inseparable, then it is a part of the same triconnected component. For distinct $\tau_1, \tau_2$, the sets $C_{\tau_1}$ and $C_{\tau_2}$ are either disjoint or identical. This allows us to identify any such $C_{\tau}$ with the lexicographically smallest $\tau_0$ (considering the labels of vertices in $\tau$ lexicographically sorted) such that $C_{\tau} = C_{\tau_0}$.

Two vertices lie in the same 3-connected component, if and only if, on the removal of any single separating pair not containing either vertex, the vertices belong to the same connected component. This allows us to identify the $C_{\tau}$ above with the triconnected components.

Lemma 3.6 Let $G$ be a biconnected planar graph. If $G$ is not 3-connected and not a cycle then $G$ has a 3-connected separating pair.

Proof: Let $G$ be neither 3-connected nor a cycle and let $a, b$ be a separating pair of $G$. If $a$ and $b$ are 3-connected we are done. So assume that $a$ and $b$ are not 3-connected.

Let $f$ be a face spanned by $a$ and $b$. Then $a$ and $b$ are connected by two vertex-disjoint paths, say $P_1$ and $P_2$, which form the boundary of $f$, and the removal of $(a, b)$ separates these two paths. Since $G$ is not a single cycle, it has more faces apart from $f$. Therefore $f$ shares some of its edges with another face, say $f'$. Consider the common boundary between $f$ and $f'$. The endpoints of this boundary, say $(u, v)$ have three vertex-disjoint paths between them, and hence are 3-connected.

Both $u$ and $v$ lie on $P_1$ or both lie on $P_2$, since otherwise $P_1$ and $P_2$ will not be separated on the removal of $(a, b)$. Without loss of generality, assume that $u, v \in P_1$. Let $P_1 = \{a = v_1, v_2, \ldots, v_k = b\}$ and consider all 3-connected pairs $(v_i, v_j)$ of vertices that lie on $P_1$. Pick a pair, say $(v_i, v_j)$, that is maximally apart on $P_1$. We claim that $(v_i, v_j)$ is a separating pair: if not, there exists a path outside $f$ from $v_{i'}$ to $v_j$ for some $i' < i$, or from $v_{j'}$ to $v_i$ for some $j' > j$. In the first case, $v_{i'}, v_j$ is 3-connected pair that is further apart than $(v_i, v_j)$, in the second case $(v_i, v_{j'})$. But this contradicts the choice of $(v_i, v_j)$. ■
Log-space algorithm for the decomposition. Hopcroft and Tarjan [HT73] presented a sequential algorithm for the decomposition of a planar biconnected graph into its triconnected components. We now describe the changes to this algorithm. 3-connected components and 3-bonds will be the same as in [HT73]. We are not considering the separating pairs that do not have a path from outside, hence a cycle will not be split into two or more components in our log-space procedure, as we will see soon. Although the sequential algorithm in [HT73] splits such cycles, it also combines its parts in a later step. This decomposition of a biconnected graph into triconnected components namely, 3-bonds, 3-connected components and cycles, is unique [Mac37].

Consider Algorithm 1. First, the algorithm computes all 3-connected separating pairs in the set \( S \). From these, we get all the 3-bonds. The for-loop from line 8 on computes the 3-connected components \( C_\tau \): In line 9, we search for the first inseparable triple \( \tau \not\in \{ C_{\tau_h} \mid 1 \leq h < i \} \) that can be separated from all previous ones. In lines 11 and 12, we search for all \( \tau_j \subseteq C_\tau \). By Lemma 3.6, it suffices to consider the pairs in \( S \) to check whether a set is separable or not. The set \( C_i \) finally equals \( C_{\tau_i} \). In line 13 we compute the triconnected component induced by \( C_{\tau_i} \).

Algorithm 1 Algorithm to decompose a graph into triconnected components.

**Input:** Biconnected planar graph \( G = (V, E) \).

**Output:** The triconnected components of \( G \).

1: fix a planar embedding \( \hat{G} \) of \( G \)
2: for all faces \( f \) of \( \hat{G} \) do
3: Compute \( S_f = \{ (u, v) \mid (u, v) \) is a 3-connected separating pair that spans \( f \} \)
4: \( S = \bigcup_{f \in \hat{G}} S_f \) the set of 3-connected separating pairs
5: for all \( (u, v) \in S \) do
6: if \( (u, v) \in E \) then output a 3-bond for \( (u, v) \)
7: compute the set of all inseparable triples \( \tau_1, \ldots, \tau_k \)
8: for \( i \leftarrow 1 \) to \( k \) do
9: if \( \forall h < i \tau_i \cup \tau_h \) is separable then
10: \( C_i \leftarrow \tau_i \)
11: for \( j \leftarrow i + 1 \) to \( k \) do
12: if \( \tau_i \cup \tau_j \) is an inseparable set then \( C_i \leftarrow C_i \cup \tau_j \)
13: output the induced subgraph on \( C_i \) without edges corresponding to 3-bonds, including virtual edges \( \{ s \in S \mid s \subseteq C_i \} \)

The algorithm can be implemented in log-space: a combinatorial embedding for planar graphs can be computed in log-space [AM00]. Separating pairs, inseparable triples and the triconnected components can be computed in log-space making oracle queries to undirected reachability [Rei05].

An example is provided in Figure 1. \( \hat{G} \) is a planar biconnected graph with its triconnected components \( G_1, \ldots, G_3 \) and the corresponding triconnected component tree \( T \), which is defined later. In \( \hat{G} \), \( (a, b) \) and \( (c, d) \) are 3-connected separating pairs. The inseparable triples are \( \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, f\}, \) and \( \{c, d, e\} \). Hence the triconnected components are induced graphs on \( \{a, b, f\}, \{c, d, e\}, \{a, b, c, d\} \) as shown in Figure 1. Since \( c, d \) is a 3-connected separating pair and an edge in \( \hat{G} \) we also get \( \{c, d\} \) as triple-bond. The virtual edges corresponding to the 3-connected separating pairs are drawn with dashed lines.

The triconnected component tree \( T \) Construct a graph \( T \) such that its node set corresponds to triconnected components and separation pairs, see Figure 1. There is an edge between a triconnected component node and a separating pair node if and only if the vertices of the sep-
Figure 1: The decomposition of a biconnected planar graph.

Lemma 3.7  The graph $T$ is a tree.

Proof: To see this, if $T$ has a cycle it will be an alternating cycle of separating pairs and triconnected components. Let $p_1, c_1, p_2, c_2, \ldots, p_r, c_r, p_1$ be one such cycle. Then on removing a separating pair $p_i$, triconnected components $c_i$ and $c_{i-1}$ remain connected through other elements of the cycle, contradicting the assumption that $p_i$ separates them.

We call $T$ the triconnected component tree. The following properties of $T$ are obvious:

1. All the leaves of $T$ are triconnected components.
2. Each path in $T$ has alternating occurrences of separating pairs and triconnected components. Hence, a path between two leaves is always of odd length.
3. Thus, it follows that $T$ has a unique centre $R$.
4. A 3-bond is introduced as a child of a separating pair only as an indicator that the vertices of the separating pair have an edge between them in the original graph. Hence a 3-bond is always a leaf.

4  Canonization of Biconnected Planar Graphs

In this section, we give a log-space algorithm to canonize biconnected planar graphs. For canonization of a biconnected planar graph $G$, we obtain its triconnected component tree $T$, as described in Section 3 and canonize this tree, similar to that in Lindell’s tree canonization algorithm. We first give a brief overview of Lindell’s algorithm and then describe our canonization procedure.

4.1  Overview of Lindell’s Algorithm

Lindell [Lin92] gave a log-space algorithm for tree-canonization. Given two trees $S$ and $T$, canonical ordering for them is defined as follows: $S <_T T$ if

1. $|S| < |T|$, or

2. $|S| = |T|$ but $\#s < \#t$, where $s$ is the root of $S$ and by $\#s$ we mean number of children of $s$, or
3. $|S| = |T|$ and $\#s = \#t$, but $(S_1 \leq_T \ldots \leq_T S_k) <_T (T_1 \leq_T \ldots \leq_T T_k)$ lexicographically, where we inductively assume $S_1 \leq_T \ldots \leq_T S_k$ and $T_1 \leq_T \ldots \leq_T T_k$ are ordered subtrees of $S$ and $T$ rooted at their children.

Comparison in steps 1 and 2 can be made in log-space. Lindell proved that even the third step can be performed in log-space using two-pronged depth-first search, which is briefly described below:

- Find the number of minimal size children of $s$ and $t$. If these numbers are unequal then the tree with larger number of minimal children is declared to be smaller. If equality is found, then remember the minimal size and check for the next size. This process is continued till an inequality in the sizes is detected or all children of $s$ and $t$ are exhausted.

- If $s$ and $t$ have the same number of children of each size, we can assume that the children of $s$ and $t$ are partitioned into size-classes (referred to as blocks in [Lin92]) in the increasing order of the the sizes of the subtrees rooted at them. Then compare children in each size-class recursively as follows:

  Let $k$ be the number of children in one of the size-classes.

  - $k = 0$: $s$ and $t$ have no children. They are isomorphic as all one node trees are isomorphic.

  - $k = 1$: Here, only one recursive call is made without using any space while making the call.

  - $k \geq 2$: Here, each subtree is of size $N/k$ by equicardinality condition. For each node in the size-class, we compute its order profile. The order profile of any node consists of three counts: $c_\prec, c_\succ, c_\equiv$. These counts indicate the number of children in the corresponding size-class of the other subtree which are respectively smaller than, greater than, or equal to the node under consideration. The counts are updated by making cross-comparisons. The children in the size-class of $S$ and $T$, which is currently being considered, with a count $c_\succ = 0$ form the first isomorphism-class. The size of this isomorphism-class is compared across the trees by comparing the values of the $c_\equiv$ variables. If these values match, this means both the trees have same number of minimal children. Now to compare larger children in the same size-class, the value of $c_\equiv$ in the last step is used as a threshold $h$. This threshold is used to search for the next minimal children of $s$ and $t$ by ignoring $h c_\succ$’s in their order profile. The process is then repeated as above and the threshold is incremented each time until reaching $k$, at which point we proceed to the next size-class. If we reach the end of all the size-classes without uncovering an inequality, the trees must be isomorphic.

4.2 Isomorphism Ordering of Triconnected Component Trees

We describe an isomorphism ordering procedure for two triconnected component trees $S$ and $T$, corresponding to two biconnected planar graphs $G$ and $H$, respectively. Both $S$ and $T$ are rooted at separating pairs, say $s = (a, b)$ and $t = (a', b')$, as described in Section 3. Therefore $S$ and $T$ have separating pairs at odd levels and triconnected components at even levels, see Figure 2. Our canonical ordering procedure is considerably more complex than Lindell’s algorithm, because each node of the tree is a separating pair or a triconnected component. Thus, unlike in the case of Lindell’s algorithm, two leaves in a triconnected component tree are not always isomorphic.
We first describe the comparison steps briefly, then we show how to compare two leaf-nodes, and then we go into the details of the comparison steps.

We define the isomorphism ordering for $S$ and $T$ (denoted by $<_T$, $>_T$, and $=T$) by first comparing their sizes, then the number of children of $s$ and $t$, these two steps being exactly the same as in Lindell’s algorithm. If equality is found in these two steps, then in the third step we make recursive comparisons of the subtrees of $S$ and $T$. We describe its details later in this section. Even if equality is detected in this step, we need a further comparison step to ensure that $G$ and $H$ are indeed isomorphic. This additional step is required for the following reason:

Suppose we detect equality in the first three steps, thus subtrees of $S$ and $T$ are pairwise isomorphic. Now we can assume that they are partitioned into equivalence classes based on their isomorphism. We refer to these classes as isomorphism-classes. In particular, the graphs, say $G_i$ and $H_i$ represented by the subtrees $S_i$ and $T_i$ are isomorphic. To ensure that $s$ and $t$ are mapped to each other in the isomorphism of $G_i$ and $H_i$, we can give a special colour to the virtual edge corresponding to $s$ and $t$ in $G_i$ and $H_i$, respectively. However, it is possible that the isomorphism from $G_i$ to $H_i$ maps vertices $a$ and $b$ from $s$ to vertices $a'$ and $b'$ of $t$, respectively, whereas the isomorphism of some other graphs, say $G_j$ and $H_j$ maps $a$ to $b'$ and $b$ to $a'$. In this case, the isomorphism among all the $G_i$’s and $H_i$’s can not be extended to get an isomorphism between $G$ and $H$.

To get around this problem, after completing the recursive comparisons of the subtrees of $S$ and $T$ and detecting an equality, we need to check that the isomorphism among the subtrees can indeed be extended to an isomorphism between $G$ and $H$. For this, we introduce the notion of orientation which is given by a subtree rooted at a triconnected component to its parent separating pair, and also collectively by all the children of a separating pair. This is the next step in the isomorphism ordering of $S$ and $T$. We describe details of this step later in this section.

**Isomorphism ordering of two leaf-nodes** Recall that leaves of triconnected component trees are always triconnected components. Thus two leaves, say $G_i$ and $H_j$, can be of three types: 3-bonds, or cycles, or 3-connected graphs. If the types of $G_i$ and $H_j$ are different, we immediately detect an inequality as $G_i$ and $H_j$ then certainly differ in the number of vertices,
or in the number of edges. If they are of the same type, then we have following cases:

1. When $G_i$ and $H_j$ are 3-bonds: In this case, we can immediately conclude that $G_i \cong H_j$ as all 3-bonds are isomorphic.

2. When $G_i$ and $H_j$ are cycles or 3-connected components: In this case, we construct the canons of $G_i$ and $H_j$ and compare them bit by bit. To canonize a cycle, we traverse it starting from the virtual edge that corresponds to its parent, and then going along the entire cycle along the edges encountered. There are two possible traversals depending on which direction of the starting edge is chosen for the traversal. Thus a cycle has two possible canons.

To canonize a 3-connected component $G_i$, we use the log-space algorithm from [DLN08], which needs as input a starting edge $e = \{u, v\}$ of $G_i$, an end-point $u$ of $e$ as the starting vertex (which can also be interpreted as the direction of $e$ provided as input is $u \rightarrow v$). We use this terminology in the rest of the discussion and a combinatorial embedding $\rho$ of $G_i$. We always provide the virtual edge corresponding to $G_i$’s parent as the starting edge. Then there are two choices for the direction of this edge. Further, a 3-connected graph has two planar combinatorial embeddings [Whi33]. Hence there are four possible ways to canonize $G_i$.

We start canonization of $G_i$ and $H_j$ in all the possible ways (two if they are cycles and four if they are 3-connected components), compare these canons bit-by-bit, and eliminate the larger ones. If all the canons of $H_i$ (or $G_i$) get eliminated then we conclude that $G_i <_T H_j$ (respectively, $H_j <_T G_i$), otherwise we conclude that $G_i \cong H_j$.

If we find that the canon(s) of $G_i$ (or $H_j$) corresponding to one choice of direction of the parent, say $(a \rightarrow b)$, $((a' \rightarrow b')$, respectively) are smaller than those obtained by the other choice, we say that $G_i$ ($H_j$) gives orientation $a \rightarrow b$ (respectively, $(a' \rightarrow b')$) to its parent.

If we obtain minimum canons for both the choices of directions, we say that $G_i$ and $H_j$ are symmetric about their parent, and thus they do not give any orientation to the parent.

**Computation of the orientation given to a separating pair by its children:** Consider a separating pair $(a, b)$ which has children $(G_1, \ldots, G_k)$. For the sake of simplicity, assume that all of them are leaves. The case when they are not leaves, is also similar. We partition the children into isomorphism-classes through their pairwise comparisons, as described above, and find the orientation given by each of the $G_i$s to $(a, b)$. Note that some $G_i$s may not give any orientation. Now we consider the orientation collectively given to $(a, b)$ by an isomorphism-class.

We say that an isomorphism-class gives an orientation to the parent separating pair $(a, b)$ as $a \rightarrow b$ ($b \rightarrow a$, respectively) if the number of components that give $a \rightarrow b$ ($b \rightarrow a$) orientation is larger than those which give $b \rightarrow a$ ($a \rightarrow b$) orientation. If these numbers are equal, or if each component in this class is symmetric about $(a, b)$ then no orientation is given by this class, and the class is said to be symmetric about $(a, b)$. Note that in an isomorphism-class, either all or none of the components are symmetric about the parent.

We call the orientation given by the smallest isomorphism-class as the reference orientation. (If the smallest isomorphism-class is symmetric about $(a, b)$, we consider the next smallest class, and so on.) Reference orientation is considered as the orientation collectively given by all the children of $(a, b)$.

**Isomorphism ordering of two subtrees rooted at triconnected components:** We consider isomorphism ordering of two subtrees $S_i$ and $T_j$ rooted at triconnected components $G_i$ and $H_j$, respectively. First we compare $|S_i|$ and $|T_j|$ and then the number of children of $G_i$.
and $H_j$. If we find equality, we proceed further. $G_i$ and $H_j$ can be either 3-bonds or cycles or 3-connected components. As in the case of leaves, if they are of different types, we immediately detect an inequality, and do not have to look at their children.

If both are 3-bonds, they are leaves, and we report an equality.

If both $G_i$ and $H_j$ are cycles (or 3-connected graphs), we start constructing two (respectively four) canons for them, as in the case of leaves. In addition to their bit-by-bit comparisons, we also have the following:

1. We check for virtual edges encountered during the construction of the canons. If, at some point of time, a virtual edge is traversed in the construction of canon $C_x$, but not in the construction of another canon $C_y$, we conclude that $C_x > C_y$ and eliminate $C_x$.

2. If two (or more) canons, say $C_x$ and $C_y$ encounter different virtual edges $(u,v)$ and $(w,z)$ corresponding to a child of $G_i$ and $H_j$, we need to recursively compare the subtrees rooted at $(u,v)$ and $(w,z)$. If, in the recursion, we find that the subtree rooted at $(u,v)$ is lexicographically smaller than the subtree rooted at $(w,z)$, we conclude $C_x < C_y$ and eliminate $C_y$.

3. If we find that the subtrees rooted at $(u,v)$ and $(w,z)$ are isomorphic, we check the orientations given to $(u,v)$ and $(w,z)$ by their children, say $(u \rightarrow v)$ and $(w \rightarrow z)$. If $(u,v)$ is traversed in the construction of $C_x$ in the same direction as of this orientation, but $(w,z)$ is traversed in the reverse direction in the construction of $C_y$, we conclude that $C_x < C_y$ and eliminate $C_y$.

In the end, the canons that are not eliminated are the minimum canons. If we have minimum canons for both $G_i$ and $H_j$, we conclude that $S_i \cong T_j$. We also look at the choice of the direction of parents of $G_i$ and $H_j$ which led to the minimum canons, and return this direction as the orientation given by $S_i$ and $T_j$ to their parents. If the minimum canons are obtained for both choices of directions of the parents, we say that $S_i$ and $T_j$ are symmetric about their parent, and thus do not give any orientation.

**Isomorphism ordering of two subtrees rooted at separating pairs:** Consider two subtrees $S_i$ and $T_j$ rooted at $(a,b)$ and $(a',b')$, respectively. To find the isomorphism ordering between them, we first compare their sizes and the number of children of their roots. If we find equality in both these steps, we proceed as follows:

We first order the subtrees rooted at the children of $(a,b)$ and $(a',b')$. We use a similar procedure as in Lindell’s algorithm, with additional steps for comparing each pair of subtrees, as described above. If we find an inequality, we conclude $S_i <_T T_j$ or $S_i >_T T_j$ accordingly.

If we find that the subtrees of $S_i$ and $T_j$ are pairwise isomorphic, we assume that they are partitioned into, say $p$ isomorphism classes $C_1 <_T \ldots <_T C_p$ and $C'_1 <_T \ldots <_T C'_p$, ordered in the isomorphism ordering, and that, for $1 \leq x \leq p$, $C_x$ and $C'_x$ contain the same number of subtrees. We also obtain the reference orientation given to $(a,b)$ and $(a',b')$ by all their children, as described above.

Now, for each isomorphism-class, say $C_x$ and $C'_x$ of subtrees of $S_i$ (and $T_j$), we compute an ordered pair $O_x = (c^\text{ref}_x, c^\text{rev}_x)$ (respectively $O'_x = (d^\text{ref}_x, d^\text{rev}_x)$), where $c^\text{ref}_x$ ($d^\text{ref}_x$) is the number of subtrees from $C_x$ (respectively from $C'_x$) which give the same orientation to the parent as the reference orientation (we refer to it as the correct orientation), whereas $c^\text{rev}_x$ ($d^\text{rev}_x$) is the number of subtrees from $C_x$ ($C'_x$) that give the other (reverse) orientation to the parent. For each $x$ from 1 to $p$, we compare the pairs $(c^\text{ref}_x, c^\text{rev}_x)$ and $(d^\text{ref}_x, d^\text{rev}_x)$ till an inequality is detected or till all the pairs are compared. If we find an $x$ such that $c^\text{ref}_x < d^\text{ref}_x$ ($c^\text{ref}_x > d^\text{ref}_x$), we say that
$S_i <_T T_j$ (respectively $S_i >_T T_j$). For an example, see Figure 3. The graphs $G$ and $H$ are planar and contain three triconnected components. The 3-bonds form one isomorphism class and the other two components form the second isomorphism class, as they all are pairwise isomorphic. However, note that the graphs $G$ and $H$ are not isomorphic. The non-isomorphism is found by comparing the directions given to the parent separating pair. Here, $p = 2$ and $O_1 = O'_1 = (0, 0)$ whereas $O_2 = (2, 0)$, and $O'_2 = (1, 1)$ and hence $O_2 \neq O'_2$.

Figure 3: Isomorphism ordering given by the orientation of subgraphs

**Summary of the steps in the isomorphism ordering:** Thus, we have the following steps for isomorphism ordering of two triconnected component trees:

Define $S <_T T$ if

Step 1: $|S| < |T|$ or

Step 2: $|S| = |T|$ but $#s < #t$ or

Step 3: $|S| = |T|$, $#s = #t = k$, but $(S_1, \ldots, S_k) <_T (T_1, \ldots, T_k)$ where we inductively assume that $(S_1 \leq_T \ldots \leq_T S_k)$ and $(T_1 \leq_T \ldots \leq_T T_k)$ are the ordered subtrees of $S$ and $T$, respectively.

Step 4: $|S| = |T|$, $#s = #t = k$, $(S_1 \leq_T \ldots \leq_T S_k) =_T (T_1 \leq_T \ldots \leq_T T_k)$ but $(O_1, \ldots, O_p) < (O'_1, \ldots, O'_p)$ where $O_j$ (and $O'_j$) is the orientation given by the $j^{th}$ isomorphism-class of $S_i$’s (and $T_i$’s) to the parent separating pair $s$ (respectively $t$).

We say that two triconnected component trees $S$ and $T$ are equal according to the isomorphism ordering above (denoted by $S =_T T$) if neither $S <_T T$ nor $T <_T S$ holds.

### 4.3 Correctness and Complexity of the Isomorphism Ordering Algorithm

We show that the algorithm is correct, and analyse its space-complexity. Lemma 4.1 shows that the algorithm outputs $S =_T T$ if and only if the corresponding biconnected planar graphs $G$ and $H$ are isomorphic.

**Lemma 4.1** $G$ and $H$ are isomorphic if and only if there is a choice of separating pairs $e, e'$ in $G$ and $H$ such that $S =_T T$ when rooted at $e$ and $e'$, respectively.
Proof: First, we show $S = T T \Rightarrow G \cong H$. Let $S = T T$. We prove this by induction on the depth of the trees. For the base case assume that the depth is 2 with $s$ and $t$ as root and triconnected components $G_1, \ldots, G_k$ and $H_1, \ldots, H_k$ as children, respectively. As $S = T T$ we know that $(G_1 \leq \ldots \leq G_k) = T (H_1 \leq \ldots \leq H_k)$. Thus for $1 \leq i \leq k$, $G_i \cong H_i$. This gives one or two mappings between $G_i$ and $H_i$ (eventually permuting separating pair vertices). From Step 4 above, we fix one mapping for $G_i$ and $H_i$. This bijection can be extended for whole of $G$ and $H$ in a straightforward way.

Assume that the induction holds at depth at most $2d$. Consider the case when the depth is $2d+2$. As $S = T T$, $(G_1 \leq \ldots \leq G_k) = T (H_1 \leq \ldots \leq H_k)$ and recursively, $S_1, \ldots, S_l$ are mapped to $T_1, \ldots, T_l$ according to the minimum canons of $G_i$ and $H_i$ for all $i$ by induction hypothesis. Because the isomorphism-classes are of the same size and $O_j = O'_j$ for all isomorphism-classes $j$, we can extend the isomorphism among the biconnected subgraphs (rooted at $G_i$’s and $H_i$’s in $S$ and $T$) to $G$ and $H$.

Consider the other direction. Now $G \cong H$. We have to show that $S = T T$. Let $\phi$ be the isomorphism between $G$ and $H$ which describes a mapping of triconnected components of $G$ to triconnected components of $H$.

Restricting $\phi$ to each of the triconnected components, we get a one to one correspondence between triconnected components of $G$ and $H$. Further, if $\phi$ maps a separating pair $(a, b) \in G$ to a separating pair $(c, d)$ in $H$, then each of the triconnected components containing $a, b$ will be mapped onto triconnected components containing $c, d$. To prove $S = T T$, we induct on the number of separating pairs in both trees.

Base case: Consider the case when there is only one separating pair $(a, b)$ in $G$ and $(c, d)$ in $H$. In this case, the triconnected component trees $S$ and $T$ will be rooted at $(a, b)$ and $(c, d)$. Also, $\phi$ will map $(a, b)$ onto $(c, d)$. Their children, $(G_1 \leq \ldots \leq G_k)$ and $(H_1 \leq \ldots \leq H_k)$, are triconnected components, which are pairwise isomorphic. Hence equality for $S$ and $T$ holds up to Step 3 in the ordering procedure. As $G_i \cong H_i$ they give the same orientation to $(a, b)$ and $(c, d)$. This implies equality in Step 4. Hence $S = T T$.

Induction step: Let the hypothesis hold when the number of separating pairs in each of the trees is at most $r - 1$. Consider the case when $G$ and $H$ have $r$ separating pairs. Suppose the isomorphism between $G$ and $H$ maps separating pair $e$ of $G$ to $e'$ of $H$. Let the trees $S$ and $T$ be rooted at $e$ and $e'$, respectively. Let $(G_1 \leq \ldots \leq G_k)$ and $(H_1 \leq \ldots \leq H_k)$ be their children, which are pairwise isomorphic. As each of them has at most $r - 1$ separating pairs, induction hypothesis holds. According to the canons of $G_i$ and $H_i$ the subtrees, say $S_1, \ldots, S_r$ and $T_1, \ldots, T_r$ rooted at their children are pairwise equal. Also, the orientations given by them to the parent separating pair match. Hence $S = T T$.


Complexity analysis: Now we analyse the complexity of the algorithm. We show that the algorithm needs a work-tape of size $O(\log n)$ and that it can be implemented using a stack of size $O(\log n)$.

We denote the size of a triconnected component tree (i.e. number of nodes in it) by $N$ and the size of an individual component $G_i$ corresponding to a node in the tree by $n_{G_i}$. Let $k_{G_i}$ be the number of equal sized components among the siblings of $G_i$.

We make recursive comparisons of two subtrees only if they are of the same size. Thus $O(\log k_{G_i})$ space suffices for storing the order profile of $G_i$ currently being considered, exactly as in Lindell’s algorithm. In addition to this, each triconnected component $G_i$ of size $n_{G_i}$ can be canonized in space $O(\log n_{G_i})$ and we canonize only two components at any point of time. So, the space used on the work-tape never exceeds $O(\log n_{G_i} + \log k_{G_i})$. Now we analyse the needed stack-space.
Comparing two subtrees $S_i$ and $T_j$ rooted at triconnected components $G_i$ and $H_j$:

As described in the previous section, we start constructing and comparing all the possible canons of $G_i$ and $H_j$, eliminating the larger ones, and making recursive comparisons whenever the canons encounter virtual edges simultaneously. We can keep track of the not eliminated canons in $O(1)$ space. While making a recursive call for two children, say $(a, b)$ and $(a', b')$, for two canons $C_x$ and $C_y$, we can not store the entire work-tape contents on the stack. However, it suffices to store the information about uneliminated canons and about which canons encountered the virtual edges corresponding to $(a, b)$ and $(a', b')$, and also the direction in which the virtual edges $(a, b)$ and $(a', b')$ were encountered. This needs only $O(1)$ bits on stack.

When a recursive call is completed, we need to find the point where the recursive call was made from and resume the computation from that point. For this, we ensure that we make a recursive comparison between $(a, b)$ and $(a', b')$ only if the comparison was not done earlier for two canons $C_x$ and $C_y$, for the same direction of traversal of $(a, b)$ and $(a', b')$. (Otherwise we know that the comparison must have resulted in an equality.) This can be done as follows:

During the construction of the canons, we keep a counter $cnt$ for the number of bits compared so far. This needs $O(\log n_{G_i})$ space, as the number of bits in each canon is polynomial in $n_{G_i}$, the size of $G_i$. When we encounter separating pairs $(a, b)$ and $(a', b')$ in $C_x$ and $C_y$, we freeze this count and the entire work-tape contents, and start another recomputation of $C_x$ and $C_y$ with a fresh counter $cnt'$. This recomputation is continued till we encounter $(a, b)$ and $(a', b')$ simultaneously in the same direction as that in the frozen computation. If the current location is the first occurrence of such a pair $(a, b)$ and $(a', b')$ then we go into recursion, (at this point, both $cnt$ and $cnt'$ can be forgotten). Else $(a, b)$ and $(a', b')$ have occurred simultaneously before. We do not go into recursion during this recomputation, as all the comparisons must have been done earlier and must have resulted in an equality (else one of $C_x$ and $C_y$ would have been eliminated earlier). When we traverse $(a, b)$ and $(a', b')$ in this recomputation of $C_x$ and $C_y$, both in the same direction as that in the frozen computation, we check if $cnt = cnt'$. If so, we can conclude that we need to recurse on the subtrees otherwise $cnt' < cnt$ and we know that they were compared earlier and found to be equal.

Now, after returning from the recursive call, we can again recompute $C_x$ and $C_y$ till $(a, b)$ and $(a', b')$ are encountered simultaneously for the first time, in the same direction as the one stored on the stack.

Thus $O(1)$ bits of stack space suffice for making a recursive call from two triconnected components.

However, to limit the stack space even to $O(\log n)$, we can not even store $O(1)$ bits on the stack if the subtrees for which the recursive call is made have size larger than $\frac{|S_i|}{\pi} = \frac{|T_j|}{\pi}$. (We refer to such a child as large child.) To get around this problem, we check whether $G_i$ and $H_j$ have a large child, before starting the construction and comparison of their canons. If so, we compare the large children a priori and store the result of their recursive comparison. Note that a node can have at most one large child, and hence this needs only $O(1)$ additional bits. Now, whenever the virtual edges corresponding to the large children from $S_i$ and $T_j$ are encountered simultaneously in a canon of $G_i$ and $H_j$, the stored result can be used, thus avoiding a recursive call.

Comparing two subtrees $S_i$ and $T_j$ rooted at separating pairs $s_i = (a, b)$ and $t_j = (a', b')$: 

• Comparing two subtrees $S_i$ and $T_j$ rooted at triconnected components $G_i$ and $H_j$: 

As described in the previous section, we start constructing and comparing all the possible canons of $G_i$ and $H_j$, eliminating the larger ones, and making recursive comparisons whenever the canons encounter virtual edges simultaneously. We can keep track of the not eliminated canons in $O(1)$ space. While making a recursive call for two children, say $(a, b)$ and $(a', b')$, for two canons $C_x$ and $C_y$, we can not store the entire work-tape contents on the stack. However, it suffices to store the information about uneliminated canons and about which canons encountered the virtual edges corresponding to $(a, b)$ and $(a', b')$, and also the direction in which the virtual edges $(a, b)$ and $(a', b')$ were encountered. This needs only $O(1)$ bits on stack.

When a recursive call is completed, we need to find the point where the recursive call was made from and resume the computation from that point. For this, we ensure that we make a recursive comparison between $(a, b)$ and $(a', b')$ only if the comparison was not done earlier for two canons $C_x$ and $C_y$, for the same direction of traversal of $(a, b)$ and $(a', b')$. (Otherwise we know that the comparison must have resulted in an equality.) This can be done as follows:

During the construction of the canons, we keep a counter $cnt$ for the number of bits compared so far. This needs $O(\log n_{G_i})$ space, as the number of bits in each canon is polynomial in $n_{G_i}$, the size of $G_i$. When we encounter separating pairs $(a, b)$ and $(a', b')$ in $C_x$ and $C_y$, we freeze this count and the entire work-tape contents, and start another recomputation of $C_x$ and $C_y$ with a fresh counter $cnt'$. This recomputation is continued till we encounter $(a, b)$ and $(a', b')$ simultaneously in the same direction as that in the frozen computation. If the current location is the first occurrence of such a pair $(a, b)$ and $(a', b')$ then we go into recursion, (at this point, both $cnt$ and $cnt'$ can be forgotten). Else $(a, b)$ and $(a', b')$ have occurred simultaneously before. We do not go into recursion during this recomputation, as all the comparisons must have been done earlier and must have resulted in an equality (else one of $C_x$ and $C_y$ would have been eliminated earlier). When we traverse $(a, b)$ and $(a', b')$ in this recomputation of $C_x$ and $C_y$, both in the same direction as that in the frozen computation, we check if $cnt = cnt'$. If so, we can conclude that we need to recurse on the subtrees otherwise $cnt' < cnt$ and we know that they were compared earlier and found to be equal.

Now, after returning from the recursive call, we can again recompute $C_x$ and $C_y$ till $(a, b)$ and $(a', b')$ are encountered simultaneously for the first time, in the same direction as the one stored on the stack.

Thus $O(1)$ bits of stack space suffice for making a recursive call from two triconnected components.

However, to limit the stack space even to $O(\log n)$, we can not even store $O(1)$ bits on the stack if the subtrees for which the recursive call is made have size larger than $\frac{|S_i|}{\pi} = \frac{|T_j|}{\pi}$. (We refer to such a child as large child.) To get around this problem, we check whether $G_i$ and $H_j$ have a large child, before starting the construction and comparison of their canons. If so, we compare the large children a priori and store the result of their recursive comparison. Note that a node can have at most one large child, and hence this needs only $O(1)$ additional bits. Now, whenever the virtual edges corresponding to the large children from $S_i$ and $T_j$ are encountered simultaneously in a canon of $G_i$ and $H_j$, the stored result can be used, thus avoiding a recursive call.

Comparing two subtrees $S_i$ and $T_j$ rooted at separating pairs $s_i = (a, b)$ and $t_j = (a', b')$: 

In this case, apart from the order profile of a child, we need to store the reference orientation given to \((a, b)\) and \((a', b')\) by their children, which needs two bits. For the isomorphism-class \(C_x\) and \(C'_x\) of children of \(s_i\) and \(t_j\) being considered at any point of time, we need to store two ordered pairs \(O_x\) and \(O'_x\) of counts of the number of children which give the same orientation to the parent as the reference orientation, and those which give reverse orientation. If \(k\) children of \(s_i\) and \(t_j\) each have the subtrees of the same size, the orientation counts need at most \(O(\log k)\) bits on the stack, and each of the subtrees has size at most \(\frac{|S_i|}{k} = \frac{|T_j|}{k}\).

However, if \(s_i\) and \(t_j\) have a large child, no stack space can be used while making a recursive call for their comparison. Hence even the reference orientation can not be stored. To get around this problem, we recurse on large children a priori and store the result in \(O(1)\) bits. Then we process other subtrees of \(S_i\) and \(T_j\). When we reach the size-class of the large child, we know the reference orientation, if any. Now we use the stored result to compare the orientations given by the large children to their respective parent, and return the result accordingly.

**Theorem 4.2** The algorithm for isomorphism ordering of biconnected planar graphs is in log-space.

**Proof:** As seen above, while comparing two trees of size \(N\), the algorithm uses no stack space for making a recursive call for a subtree of size larger than \(\frac{N}{k}\), and it uses \(O(\log k)\) stack space if the subtrees are of size at most \(\frac{N}{k}\), where \(k \geq 2\). Hence we get the following recurrence for the stack-space \(S(N)\):

\[
S(N) = S\left(\frac{N}{k}\right) + O(\log k), \quad k \geq 2
\]

Thus \(S(N) = O(\log N)\). □

### 4.4 The Canon of a Biconnected Planar Graph

From Section 4.2, we know that the isomorphism ordering for triconnected component trees is computable in log-space. Now we give a log-space procedure which makes oracle queries to the isomorphism ordering procedure and outputs the canonical list of edges (along with some delimiters for subtrees of the same node) of the corresponding biconnected planar graph \(G\) in log-space. For this, we need \(T\) to be a rooted tree. Further, our algorithm for canonization of a triconnected component tree needs \(T\) to be rooted at a separating pair. If the centre \(R\) of \(T\) is a separating pair, we root \(T\) at \(R\). Otherwise we root it at an arbitrary separating pair. As there are only polynomially many separating pairs, a log-space transducer can cycle through all of them and can determine the separating pair which, when chosen as the root, leads to the minimum canon of \(G\).

The steps in this procedure are as follows:

1. Start the canonization process from the root of the tree, say \((s, t)\). Query the orientation of \((s, t)\) to the oracle. If the orientation is \(s \rightarrow t\), output \((s, t)\). Otherwise if the orientation is \(t \rightarrow s\), output \((t, s)\). In case the subtrees give no orientation, choose an orientation arbitrarily.

2. Recursively output the canonical list of edges for the subtrees of \(T\) starting from the smallest in the isomorphism order. Consider canonization of a subtree rooted at a triconnected component \(G_1\). If \(G_1\) is a 3-bond, output its canon as \((s, t)\) or \((t, s)\), depending upon the orientation of its parent. If \(G_1\) is a cycle, it has a unique canonical list for the
given orientation of its parent. If $G_1$ is a 3-connected component, then it has two possible canons for the given orientation of its parent, one for each of the two embeddings. Query the oracle for the embedding that leads to the lexicographically smaller canon and output it.

3. Recursively output the list of canonical edges of the subtrees of $G_1$ in the order in which the corresponding virtual edges are encountered during the construction of the above canon of $G_1$. For a child $(a, b)$ of $G_1$, query the orientation of $(a, b)$ to the oracle and output $(a, b)$ or $(b, a)$, accordingly. If the subtree rooted at $(a, b)$ does not give any orientation to $(a, b)$, then take that orientation for $(a, b)$, in which it is encountered during the construction of the above canon for $G_1$.

Note that we canonize a triconnected component $G_i$ according to the reference orientation of its parent and it is independent of the orientation given by $G_i$ to its parent. Among isomorphic siblings, those which give the correct orientation to the parent are considered before those which give the reverse orientation.

For example, the canonical list of edges for the first tree in Figure 2 is:

$$[(s, t)[l(G_1)[l(S_1)] \ldots [l(S_{j_1})] \ldots [l(G_k)[l(S_l)]]]$$

where $l(\cdot)$ denotes the canonical list of edges of a component or a subtree. Whenever there are more than one choices for the orientation of a separating pair or for the embedding of a 3-connected component, which lead to the same canonical list, an arbitrary choice can be made.

Now the final canon can be obtained by a log-space transducer which relabels the vertices in the order of their first occurrence in this canonical list and outputs the list using these new labels. The new labels are the canonical labels of the vertices of $G$.

Note that the canonical list of edges contains virtual edges as well, which are not a part of $G$. However, this is not a problem as the virtual edges can be distinguished from real edges because of the presence of 3-bonds.

Thus we have the following result:

**Theorem 4.3** A biconnected planar graph can be canonized in log-space.

## 5 Canonization of Planar Graphs

In this section, we give a log-space algorithm for canonization of connected planar graphs. As in the case of biconnected planar graphs, we first decompose a planar graph into its biconnected components and then construct a tree on these biconnected components and articulation points. We refer to this tree as the biconnected component tree. This tree is unique and can be constructed in log-space [ADK08]. Similar to triconnected component trees, we put a copy of an articulation point $a$ into each of the components formed by the removal of $a$. In the discussion, we refer to a copy of an articulation point in a biconnected component $B$ as an articulation point in $B$. It is clear that $B$ does not have an articulation point, and hence an articulation point in $B$ always refers to a copy of an articulation point. Although an articulation point has at most one copy in each of the biconnected components, the corresponding triconnected component tree can have many copies of the same articulation point.

Note that, canonization of connected planar graphs can be easily extended to planar graphs, which may have more than one connected component. This is because the components can be canonized individually and their canons can be output in lexicographically increasing order. Hence in the rest of the discussion, we assume that the given planar graph is connected.
Given a planar graph $G$, we root its biconnected component tree at an articulation point. During the isomorphism ordering of two such trees $S$ and $T$, we can fix the root of $S$ arbitrarily and check for equality for all choices of roots for $T$. As there are only $O(n)$ articulation points, a log-space transducer can cycle through all of them for the choice of the root of $T$.

In the foregoing discussion, we denote the size of a biconnected component tree (i.e. number of nodes in it) by $N$ and size of an individual component $B$ corresponding to a node in the tree by $n_B$.

### 5.1 Isomorphism ordering of Biconnected Component Trees

The isomorphism ordering for biconnected component trees proceeds in a way similar to that for triconnected component trees. However, we note two main differences:

1. Step 4 of the isomorphism ordering of triconnected component trees is not required in the case of biconnected component trees, as there is no notion of orientation for an articulation point. In other words, we colour the copy of the parent articulation point in a biconnected component with a distinct colour, and then the pairwise isomorphism among the subtrees of $S$ and $T$ can be extended to the isomorphism between the corresponding planar graphs $G$ and $H$ in a straightforward way.

2. In case of triconnected component trees, when we compare two subtrees rooted at triconnected components $G_i$ and $H_j$, we simultaneously construct and compare all possible canons of $G_i$ and $H_j$. This takes only $O(1)$ space on stack during recursion, as there are at most four canons for each of $G_i$ and $H_j$. However, while comparing two subtrees rooted at biconnected components $B$ and $B'$, we cannot construct and compare all possible canons simultaneously. This is because the canons of $B$ and $B'$ depend on the roots (separating pair nodes) chosen for their respective triconnected component trees. As there are $O(n_B)$ possibilities (where $n_B$ is the size of $B$ and $B'$), we need $O(n_B \log n_B)$ space to simultaneously construct all of them. Even if we construct and compare only two canons at a time, we need $O(\log n_B)$ stack-space to keep track of them during recursive calls.

Thus we cannot limit the stack-space to $O(\log N)$ (where $N$ is the size of each of the trees $S$ and $T$), if we naively cycle through the choices of roots for $B$ and $B'$. Later in this section, we show how to limit the number of possible choices of roots so that the algorithm takes only $O(\log N)$ stack-space.

We define the isomorphism ordering on two biconnected component trees $S$ and $T$ rooted at articulation points $s$ and $t$ as follows:

Define $S <_B T$ if

- **Step 1:** $|S| < |T|$ or
- **Step 2:** $|S| = |T|$ but $\#s < \#t$ or
- **Step 3:** $|S| = |T|$, $\#s = \#t = k$, but $(S_1, \ldots, S_k) <_B (T_1, \ldots, T_k)$ where we inductively assume that $(S_1 \leq_B \ldots \leq_B S_k)$ and $(T_1 \leq_B \ldots \leq_B T_k)$ are the ordered subtrees of $S$ and $T$.

We say that two biconnected component trees are equal, denoted by $S =_B T$ if neither of $S <_B T$ and $T <_B S$ holds.

The inductive ordering of the subtrees of $S$ and $T$ proceeds exactly as in Lindell’s algorithm, by partitioning them into size-classes and comparing the children in the same size-class recursively. The book-keeping required (e.g. order profile of a node, the number of nodes in
a size-class that have been compared so far) is the same as in Lindell’s algorithm. We discuss how to compare two such subtrees $S_i$ and $T_j$, rooted at biconnected components $B$ and $B'$, respectively.

**Recursive comparison of two subtrees rooted at biconnected component nodes:** We construct the triconnected component trees $S'$ and $T'$ for $B$ and $B'$, respectively. For now, we assume that the number of choices for roots, and hence the number of possible canons of $B$ and $B'$ can be bounded by a suitable number $k$, so as to limit the total stack space to $O(\log N)$. The value of $k$ depends on the children of $B$ and $B'$. We describe what the value of $k$ is, and how to limit the number of possible choices of the root to $k$ and prove that the stack-space used is $O(\log N)$ later in Theorem 5.6.

While canonizing $B$ and $B'$, we distinctly colour the copy of their parent articulation point, to ensure that the parent articulation points are always mapped to each other. Now we want to find out whether the minimum of the $k$ possible canons of $B$ is smaller, larger or equal to the minimum of the $k$ possible canons of $B'$. This can be done through pairwise cross-comparisons of the canons. The required book-keeping needs $O(\log k)$ space. As only two canons are constructed and compared at a time, $O(\log n_B)$ space suffices for the construction.

During the construction of canons $C$ and $C'$ of $B$ and $B'$ respectively, whenever the copies of their children $a$ and $a'$ are encountered for the first time, we need to check the subtrees rooted at them, recursively. For this, we need to push on the stack $O(\log k)$ bits of book-keeping required to cycle through all the canons and finding the minimum one, and the information about the two canons which are being compared currently (which also takes $O(\log k)$ bits, as there are only $O(k)$ canons to be compared). During recursion, if we detect an inequality, we return the result and move to the next pair of possible canons. If we detect an equality, then we need to pop the information from the stack, recompute $C$ and $C'$ till we find the first occurrence of $a$ and $a'$ together and continue the construction of $C$ and $C'$ from that point.

To detect whether a pair of articulation points is encountered for the first time in the construction of $C$ and $C'$, and thus to decide whether to make a recursive call for them, we follow the same procedure as in the case of triconnected component trees.

**Limiting the number of possible choices for the root:** Let $T$ be the triconnected component tree of a biconnected planar graph $B$. Consider $B$ as a biconnected component in a biconnected component tree $S$. We partition the children of $B$ (in $S$) into size-classes. If there are at least $k \geq 2$ subtrees in each of the size-classes, then we know that all the subtrees are of size at most $\frac{N}{k}$, where $N$ is the size of the subtree rooted at $B$. Thus, while making any recursive call, we can always use $O(\log k)$ stack-space. Therefore, if we can limit the number of possible canons of $B$ to $O(k)$ (i.e. linear in the size of the smallest size-class of the children of $B$), we can limit the stack-space used for the subtree rooted at $B$ to $O(\log N)$. Note that if $B$ is a leaf, then we do not need to make any recursive call from $B$ and thus can consider all the separating pairs as the choices for the root of $T$. Therefore, in the rest of the discussion, we assume that $B$ is not a leaf.

If the smallest size-class contains one child which is large, i.e. the subtree rooted at it has size more than $\frac{N}{2}$, then we consider it separately as we did for a triconnected component tree. We compare it before canonizing the parent and the other subtrees, and store the result till the canonization of the parent is completed.

We first note the following properties of articulation points:

1. A biconnected component tree has a unique centre $R$, similar to a triconnected component tree.
2. If an articulation point \( u \) appears in a separating pair node \( s \) in \( T \), then it appears in all the triconnected components which are adjacent to \( s \) in the tree.

3. If an articulation point \( u \) appears in two nodes \( C \) and \( D \) in \( T \), it appears in all the nodes that lie on the path between \( C \) and \( D \) in \( T \). Hence, for an articulation point \( u \), we find the triconnected component nearest to the centre of \( T \) that contains \( u \) and associate \( u \) with it. Thus we can uniquely associate each articulation point contained in \( B \) with a triconnected component in \( T \).

Now, we consider the following cases about the structure of \( T \) and for each of the cases, describe how to construct the set of separating pairs which are possible choices for the root of \( T \). As described above, we partition the children of \( B \) into size-classes according to the sizes of the subtrees rooted at them, and number these size-classes in increasing order of these sizes.

We colour an articulation point appearing in \( B \) with colour \( i \), if it belongs to size-class \( i \). Let \( B \) be connected to an articulation point \( a \) in \( S \), \( a \) is called the parent articulation point. Let \( k \) members be contained in the smallest size-class.

**Case 1:** The centre of \( T \), say \( R \), is a separating pair: We choose \( R \) as the root of \( T \). Thus we have only one choice for the root, and the subtree rooted at \( B \) can be canonized in a unique way. For the remaining cases, we assume that \( R \) is a triconnected component.

**Case 2:** \( R \) is a triconnected component and \( a \) is not associated with \( R \): Let \( a \) be associated with a triconnected component \( C \). We find the path from \( C \) to \( R \) in \( T \) and find the separating pair closest to \( R \) on this path. This serves as the unique choice for the root of \( T \). In the rest of the cases, we assume that \( a \) is associated with \( R \).

**Case 3:** \( a \) is associated with \( R \) and \( R \) is a cycle: We canonize \( R \) for the two edges incident to \( a \) as starting edges, and \( a \) as the starting vertex. We construct these canons till a virtual edge is encountered in one or both of them. We choose the separating pair(s) corresponding to the first virtual edge(s) encountered in the canon(s) as root of \( T \). Thus we get at most two choices for root of \( T \). In rest of the cases, we assume that \( R \) is a 3-connected component.

**Case 4:** \( a \) is associated with \( R \), and \( k = 1 \):

i. We consider the following two cases:

   i. One or more of the articulation points from these size-classes is associated with a triconnected component \( C \) other than \( R \): In this case, we choose a root separating pair uniquely as in Case 2, but for the articulation point in the smallest colour class (instead of \( a \)), which is not associated with \( R \).

   ii. There are at least two such size-classes, and the articulation points in them are all in \( R \): In this case, \( R \) has at least three vertices that are fixed by all its automorphisms. In this case, \( R \) has only one non-trivial automorphism, and hence has only two choices of starting edge, starting vertex and embedding to get the minimum canon. We take the separating pair(s) corresponding to the virtual edge(s) first encountered in these minimum canons as the choice for the root of \( T \).

   (b) There is only one size-class of size 1: In this case, treat this child of \( B \) as large child, process it a priori, and store the result. Then consider the next smallest size-class, which has at least two subtrees in it. Proceed for this size-class as in Case 5.

**Case 5:** \( a \) is associated with \( R \) and \( k \geq 2 \):
(a) One or more articulation points from the smallest size-class are associated with components other than $R$: We can choose candidate separating pairs as in Case 2. As the $k$ points can be associated with at most $k$ distinct components, there can be at most $k$ such pairs.

(b) All the articulation points of the smallest size-class are associated with $R$: In this case, we prove that $R$ can have at most $O(k)$ automorphisms that preserve the colours of these vertices. (See Lemma 5.1.) Thus, as in Case 4, we have $O(k)$ ways of choosing the root of $T$.

Case 6: $T$ consists of a single node which is a triconnected component. In this case, if there is no child articulation point, we can find the minimum canon for it using the triconnected planar graph canonization algorithm. If there is only one child, the comparison for such a child can be done and stored a priori. If there are at least two children and $k = 1$, there is at most one non-trivial automorphism and hence at most two minimum canons for $T$. If $k \geq 2$, the number of automorphisms and hence the minimum number of canons for $T$ is $O(k)$. In both these cases, make recursive calls only for the minimum canons, which are obtained by just working with the colours of the articulation points.

The following lemma gives a relation between the size of the smallest colour class and the number of automorphisms for a 3-connected graph, which has one distinctly coloured vertex:

**Lemma 5.1** Let $G$ be a 3-connected planar graph with colours on its vertices such that one vertex $a$ is coloured distinctly, and let $k \geq 2$ be the size of the smallest colour class apart from the one which contains $a$. $G$ has $O(k)$ automorphisms.

To prove Lemma 5.1, we refer to the following results:

**Lemma 5.2** [Bab95](P. Mani) Every triconnected planar graph $G$ can be embedded on the 2-sphere as a convex polytope $P$ such that the automorphism group of $G$ coincides with the automorphism group of the convex polytope $P$ formed by the embedding.

**Lemma 5.3** [AD04, Bab95, Art96] For any convex polytope other than tetrahedron, octahedron, cube, icosahedron, dodecahedron, the automorphism group is the product of its rotation group and $(1, \tau)$ where $\tau$ is a reflection. The rotation group is either $C_k$ or $D_k$, where $C_k$ is the cyclic group of order $k$ and $D_k$ is the dihedral group of order $2k$.

**Proof:** (Lemma 5.1) Let $H$ be the subgroup of the rotation group, which permutes the vertices of the smallest colour class among themselves. Then $H$ is cyclic since the rotation group is cyclic. Let $H$ be generated by a permutation $\pi$.

Notice that a non-trivial rotation of the sphere fixes exactly two points of the sphere viz. the end-points of the axis of rotation. Then, the following claim holds.

**Claim 5.4** In the cycle decomposition of $\pi$ each non-trivial cycle has the same length.

**Proof:** (of Claim 5.4) Suppose $\pi_1, \pi_2$ are two non-trivial cycles of lengths $p_1 < p_2$ respectively in the cycle decomposition of $\pi$. Then $\pi^{p_1}$ fixes all elements of $\pi_1$ but not all elements of $\pi_2$. Thus $\pi^{p_1} \in H$ cannot be a rotation of the sphere which contradicts the definition of $H$. □

As a consequence, the order of $H$ is bounded by $k$, since the length of any cycle containing one of the $k$ coloured points is at most $k$. □

This leads to the following corollary, which justifies Case 4(a)ii.
Corollary 5.5 Let $G$ be a 3-connected planar graph with at least 3 coloured vertices, each having a distinct colour. Then $G$ has at most one non-trivial automorphism.

Proof: An automorphism of $G$ has to fix all the coloured vertices. Consider the embedding of $G$ on a 2-sphere. The only possible symmetry is a reflection about the plane containing the coloured vertices, which leads to exactly one non-trivial automorphism.

Note: In cases 5b and 6, if the triconnected component $R$ is one of the exceptions stated in Lemma 5.3, it implies that $R$ has $O(1)$ size. Thus we do not have to limit its number of possible minimum canons.

5.2 Correctness and Complexity of the Ordering Algorithm

The isomorphism ordering proceeds exactly as in Lindell’s algorithm, with the additional stack-space. In the last section, we discussed about how to limit the stack space to $O(\log N)$, we get the following:

Theorem 5.6 The algorithm for isomorphism ordering of planar graphs works in log-space.

Proof: At an articulation point, the book-keeping required to recursively canonize its subtrees is exactly as in Lindell’s algorithm. At a biconnected component $B$ (of size $n_B$), the space used on the work-tape is limited to $O(\log n_B)$ since a biconnected component can be canonized in log-space. To compare two of its $k$ canons, we need $O(\log k)$ additional space. While making recursive calls during the comparison of two canons, only these $O(\log k)$ bits need to be stored on the stack. After returning from the recursive calls, we use these bits to find out the point from where the recursive call was made. This is done exactly as in the case of triconnected component trees. We need $O(\log k)$ bits on the stack, only when the size of each subtree is bounded by $\frac{N}{k}$. As in the case of triconnected component trees, we get the following recurrence for the stack space:

$$S(N) = S\left(\frac{N}{k}\right) + O(\log k)$$

Hence $S(N) = \log N$.

We prove now the correctness of our isomorphism ordering algorithm.

Theorem 5.7 Given two connected planar graphs $G$ and $H$, and their biconnected component trees $S$ and $T$, $G \cong H$ if and only if $S =_B T$ according to the isomorphism ordering steps, for some choice of the root of $T$.

Proof: First we prove that if $G \cong H$ then the isomorphism ordering gives $S =_B T$ for some choice of the root of $T$. We prove this by induction on the number of articulation points $k$ in $G$ and $H$. Let $\phi : G \rightarrow H$ be an isomorphism. Clearly $\phi$ maps articulation points of $G$ to those of $H$. Let an articulation point $s$ be the root of $S$. Then consider the choice of the root of $T$ to be $t = \phi(s)$.

Base case $k = 1$: In this case $s$ and $t$ are the only articulation points in $G$ and $H$ respectively. As $\phi(s) = t$, $\phi$ maps the biconnected components $(B_1, \ldots, B_i)$ of $G$ to the biconnected components $(B'_1, \ldots, B'_i)$ of $H$. Our algorithm detects this isomorphism to get $(B_1, \ldots, B_i) =_B (B'_1, \ldots, B'_i)$ using the isomorphism ordering for biconnected planar graphs. Thus, in $S$ and $T$, the children of $s$ and $t$ are equal and hence the algorithm declares $S =_B T$.

Induction step: Let $G \cong H \Rightarrow S =_B T$ hold for some value $l$ of the number of articulation points in each of them. Let $G$ and $H$ have $l+1$ articulation points. As before, the articulation points of $G$ should be mapped to articulation points of $H$ by any isomorphism $\phi$ from $G$ to $H$. In particular, let $\phi(s) = t$ for the root $s$ of $S$. Then we consider $t$ as the root of $T$. As $\phi(s) = t$, the
components obtained by removal of \(s\) and \(t\) must be isomorphic. Let \((S_1, \ldots, S_i)\) and \((T_1, \ldots, T_i)\) be the subtrees corresponding to these components, such that the component corresponding to \(S_j\) is mapped to the component corresponding to \(T_j\) by \(\phi\), for \(1 \leq j \leq i\). Now, each of the biconnected components in \(S_j\) is isomorphic to those of \(T_j\). As \(\phi(s) = t\), the biconnected component \(B_j\) of \(S_j\) containing \(s\) should be isomorphic to the biconnected component \(B'_j\) of \(T_j\) containing \(t\) and hence they should be the roots of \(S_j\) and \(T_j\) respectively. Also, \(\phi\) should map the other articulation points in \(B_j\) to those in \(B'_j\). Thus, using the isomorphism ordering for biconnected planar graphs, our isomorphism ordering algorithm for planar graphs gets a one-to-one correspondence among the subtrees of \(S_j\) and \(T_j\). As the graphs \(G_j\) and \(H_j\) corresponding to \(S_j\) and \(T_j\) are isomorphic, and \(B_j \cong B'_j\), restricting \(\phi\) to the vertices of \(G_j \setminus B_j\) and \(H_j \setminus B'_j\) gives an isomorphism between the subtrees of \(S_j\) and \(T_j\). As \(G_j \setminus B_j\) and \(H_j \setminus B'_j\) contain at most \(l\) articulation points, by induction hypothesis the subtrees of \(S_j\) are pairwise equal to the subtrees of \(T_j\), according to the isomorphism ordering. Further, \(B_j \cong B'_j\). Hence \(S'_j = B T_j\). This holds for all \(j\). Thus all the subtrees of \(S\) and \(T\) are pairwise equal, and hence the isomorphism ordering of \(S\) and \(T\) gives \(S = T\).

Now we prove that if \(S = T\) according to the isomorphism ordering, then \(G \cong H\), and that such an isomorphism can be obtained inductively from the one-to-one correspondence of the nodes of \(S\) and \(T\), given by their isomorphism ordering. We prove this by induction on the depth of \(S\) and \(T\).

Base case: Let depth of \(S\) and \(T\) be 2. Let \(s\) and \(t\) be the roots of \(S\) and \(T\), and all their children be leaves. As \(S = T\), the leaves should be pairwise equal. The corresponding biconnected components should be pairwise isomorphic, from Lemma 4.1. Let the children of \(s\) be \((B_1, \ldots, B_i)\) which are respectively isomorphic to \((B'_1, \ldots, B'_i)\) of \(t\). For \(1 \leq j \leq i\), let \(\phi_j : B_j \rightarrow B'_j\) be the isomorphisms. As \(s\) and \(t\) are coloured distinctly in each of the children, \(\phi_j(s) = t\) for all \(j\). Thus the isomorphisms among the leaves of the trees can be combined in a straightforward way to get an isomorphism \(\phi : G \rightarrow H\).

Induction step: Let the hypothesis hold for depth at most \(2d\). Now consider the case when the isomorphism ordering gives \(S = T\), their depth being \(2d + 2\). Consider the subtrees of \(S\) and \(T\) rooted at biconnected components \(B\) and \(B'\). Let the subtrees rooted at their children be \((S_1, \ldots, S_i) = (T_1, \ldots, T_i)\). Their depth is at most \(2d\) and hence the corresponding graphs are isomorphic. Let \(\phi_j : G_j \rightarrow H_j\), where \(1 \leq j \leq i\) and \(G_j\) and \(H_j\) are the graphs corresponding to \(S_j\) and \(T_j\) respectively. Also, \(B \cong B'\) by the definition of equality of \(S\) and \(T\). Let \(\phi' : B \rightarrow B'\). Further, the algorithm ensures that the articulation points in \(B\) and \(B'\) that get the same canonical labels have equal subtrees rooted at them. Thus, \(\phi_j, 1 \leq j \leq i\) and \(\phi'\) can be combined to get an isomorphism between the graphs corresponding to the subtrees rooted at \(B\) and \(B'\). This can be done for all the subtrees of \(S\) and \(T\). Further, as \(s\) and \(t\) are coloured distinctly in each of them, it is ensured that all such isomorphisms map \(s\) to \(t\).

Therefore we get an isomorphism \(\phi : G \rightarrow H\) in a straightforward way.

5.3 The Canon of a Planar Graph

From Theorem 5.6, we know that the isomorphism ordering of biconnected component trees can be done in log-space. Using this algorithm, we show that the canon of a planar graph can be output in log-space.

The canonization proceeds exactly as in the case of triconnected component trees. A log-space procedure traverses the biconnected component tree, makes oracle queries to the isomorphism ordering algorithm and outputs a canonical list of edges, along with delimiters to separate the lists for siblings. A log-space transducer then renames the vertices according to their first occurrence in this list, to get the final canon. This canon depends upon the choice of the root
of the biconnected component tree. Another log-space transducer can cycle through all the articulation points as roots and output the minimum canon among them.

Compute for all connected components of the given planar graph their canons as described and sort them lexicographically. This can be done by another log-space transducer. This proves the main theorem.

**Theorem 5.8** A planar graph can be canonized in log-space.

## 6 Conclusion

In this note, we improve the known upper bound for isomorphism and canonization of planar graphs from \( \text{AC}^1 \) to \( \text{L} \). This implies \( \text{L} \)-completeness for this problem, thereby settling its complexity. An interesting question is to extend it to other important classes of graphs.

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