Plus-construction of algebras over an operad, cyclic and Hochschild homologies up to homotopy *

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Abstract

In this paper we generalize the plus-construction given by M. Livernet for algebras over rational differential graded Koszul operads to the framework of admissible operads (the category of algebras over such operads admits a closed model category structure). We follow the modern approach of J. Berrick and C. Casacuberta defining topological plus-construction as a nullification with respect to a universal acyclic space. Similarly, we construct a universal $H_*^{Q}$-acyclic algebra $U$ and we define $A \to A^+$ as the $U$-nullification of the algebra $A$. This map induces an isomorphism on Quillen homology and quotients out the maximal perfect ideal of $\pi_0(A)$. As an application, we consider for any associative algebra $R$ the plus-constructions of $gl(R)$ in the categories of Lie and Leibniz algebras up to homotopy. This gives rise to two new homology theories for associative algebras, namely cyclic and Hochschild homologies up to homotopy. In particular, these theories coincide with the classical cyclic and Hochschild homologies over the rational.

Introduction

Quillen’s plus construction for spaces was designed so as to yield a definition of higher algebraic $K$-theory groups of rings. Indeed, for any $i \geq 1$, $K_i R = \pi_1 BGL(R)^+$, where

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GL(R) is the infinite general linear group on the ring R. The study of the additive analogue, namely the Lie or Leibniz algebra \( gl(R) \) has already produced a number of papers showing the strong link with cyclic and Hochschild homology (For classical background on these theories we refer to\cite{21} and to the survey \cite{22}). However there have always been restrictions, such as working over the rationals.

For example M. Livernet has given a plus-construction for algebras over a Koszul operad in the rational context\cite{18} by way of cellular techniques imitating the original topological construction given by D. Quillen (see also\cite{23} for a plus-construction in the context of simplicial algebras). Specializing to the category of Lie, respectively Leibniz algebras, she proved then that the homotopy groups of \( gl(R)^+ \) are isomorphic to the cyclic, respectively Hochschild homology groups of \( R \).

In the category of topological spaces plus-construction can be viewed as a localization functor, which has the main advantage to be functorial. This idea goes probably back to A.K. Bousfield and E. Dror Farjoun, but the first concrete model was given by J. Berrick and C. Casacuberta in\cite{4}. They provide a “small” universal acyclic space \( BF \) such that the nullification \( P_{BF}X \) is the plus-construction \( X^+ \).

Recently, thanks to the work of P. Hirschhorn\cite{15} it appears possible to do homotopical localization in a very general framework. In fact, one can construct localizations in any closed model category satisfying some mild extra conditions (left proper and cofibrantly generated), such as categories of algebras over admissible operads. The category of Lie algebras over an arbitrary ring is not good enough for example. One needs to take first a cofibrant replacement \( L_\infty \) of the Lie operad and can perform localization in the category of \( L_\infty \)-algebras, which we call Lie algebras up to homotopy.

This allows to define a functorial plus-construction in the category of algebras over an admissible operad as a certain nullification functor with respect to an algebraic analogue \( \mathcal{U} \) of Berrick’s and Casacuberta’s acyclic space. This extends the results of M. Livernet to the non-rational case. In the following theorem \( \mathcal{P}\pi_0(A) \) denotes the maximal \( \pi_0\mathcal{O} \)-perfect ideal of \( \pi_0A \).

**Theorem 3.1** Let \( \mathcal{O} \) be an admissible operad. Then the homotopical nullification with respect to \( \mathcal{U} \) is a functorial plus-construction in the category of algebras over \( \mathcal{O} \). It enjoys the following properties:

(i) \( A^+ \simeq Cof\left(\bigsqcup_{f \in \text{Hom}(\mathcal{U},A)} \mathcal{U} \to A\right) \)

(ii) \( \pi_0(A^+) \cong \pi_0(A) / \mathcal{P}\pi_0(A) \)
(iii) $H_*^Q(A) \cong H_*^Q(A^+)$

Of particular interest is the plus-construction in the category of Lie algebras up to homotopy. If we apply these constructions to the algebra $gl(R)$ of matrices of an associative algebra, and if we consider it as a Lie algebra up to homotopy, we obtain what we call the cyclic homology theory up to homotopy. Thus $HC_i^\infty(R)$ is defined as $\pi_i gl(R)^+$ for any $i \geq 0$. This theory corresponds to the classical cyclic homology over the rationals. We summarize in a proposition the computations of the lower homology groups (see Proposition 5.1, 5.2, 5.3). They share a striking resemblance with the low dimensional algebraic $K$-groups ($st(R)$ stands for the Steinberg Lie algebra, see Section 5).

**Proposition.** Let $k$ be a field and $R$ be an associative $k$-algebra. Then

1. $HC_0^\infty(R)$ is isomorphic to $R/[R,R]$.
2. $HC_1^\infty(R)$ is isomorphic to $Z(st(R)) \cong H_1^Q(sl(R))$.
3. $HC_2^\infty(R)$ is isomorphic to $H_2^Q(st(R))$.
4. $HC_3^\infty(R)$ is isomorphic to $H_3^Q(st(R))$.

The same kind of results hold for Hochschild homology up to homotopy, which is defined similarly using Leibniz algebras. The above computation in the case $k = \mathbb{Q}$ yields the following corollary for classical cyclic homology.

**Corollary 5.4** Let $R$ be an associative $\mathbb{Q}$-algebra. Then

1. $HC_0(R)$ is isomorphic to $R/[R,R]$.
2. $HC_1(R)$ is isomorphic to $Z(st(R)) \cong H_2^{Lie}(sl(R))$.
3. $HC_2(R)$ is isomorphic to $H_3^{Lie}(st(R))$.
4. $HC_3(R)$ is isomorphic to $H_4^{Lie}(st(R))$.

Computation (1) is well known and trivial, whereas (2), (3) and (4) are non-trivial results (for (2) and (3) we refer to [10]). The plan of the paper is as follows. First we introduce the notion of algebra over an operad and recall when and how one can do homotopy theory with these objects. In a short second section we explain what homotopical localization and nullification functors are for algebras. The main theorem about
the plus-construction appears then in Section 3 and Section 4 contains the properties of plus-construction with respect to fibrations and extensions. The final section is devoted to the computations of the low dimensional additive $K$-theory groups.

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1 Operads and algebras over an operad

We fix $R$ a commutative and unitary ring. We work in the category $\mathbf{R-dgm}$ of differential $\mathbb{Z}$-graded $R$-modules and especially with chains (the differential decreases the degree by 1). All the objects we will consider in this article will be in fact $\mathbb{N}$-graded, but for technical reasons, namely in order to use techniques of fiberwise localization, it is handy to view them as unbounded chain complexes. For classical background about operads and algebras over an operad see [11], [12], [17] and [20].

$\Sigma_*$-modules. A $\Sigma_*$-module is a sequence $\mathcal{M} = \{\mathcal{M}(n)\}_{n>0}$ of objects $\mathcal{M}(n)$ in $\mathbf{R-dgm}$ together with an action of the symmetric group $\Sigma_n$. The category of $\Sigma_*$-modules is a monoidal category. We denote by $\mathcal{M} \odot \mathcal{N}$ the product of two $\Sigma_*$-modules and by $1$ the unit of this product. The unit is defined by $1(1) = R$ and $1(i) = 0$ for $i \neq 1$.

Operads. An operad $\mathcal{O}$ is a monad in the category of $\Sigma_*$-modules. Hence we have a product $\gamma : \mathcal{O} \odot \mathcal{O} \to \mathcal{O}$ which is associative and unital. Equivalently the product $\gamma$ defines a family of composition products

$$
\gamma : \mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \ldots \otimes \mathcal{O}(i_n) \to \mathcal{O}(i_1 + \ldots + i_n)
$$

which must satisfy equivariance, associativity and unitality relations (also called May’s axioms). Moreover we suppose that $\mathcal{O}(1) = R$ and that each chain complex $\mathcal{O}(n)$ is concentrated in positive differential degrees (i.e. $\mathcal{O}(n)_p = 0$ for any $p < 0$).
There is a free operad functor:

\[ F : \Sigma - \text{modules} \longrightarrow \text{Operads} \]

which is left adjoint to the forgetful functor. It can be defined using the formalism of trees.

**Algebras over an operad.** Let us fix an operad \( \mathcal{O} \). An algebra over \( \mathcal{O} \) (also called \( \mathcal{O} \)-algebra) is an object \( A \) of \( \mathbb{R} - \text{dgm} \) together with a collection of morphisms

\[ \theta : \mathcal{O}(n) \otimes A^\otimes n \longrightarrow A \]

called evaluation products, which are equivariant, associative and unital. There is a free \( \mathcal{O} \)-algebra functor

\[ S(\mathcal{O}, -) : \mathbb{R} - \text{dgm} \longrightarrow \mathcal{O} - \text{algebras} \]

which is left adjoint to the forgetful functor. For any \( M \in \mathbb{R} - \text{dgm} \) it is given by

\[ S(\mathcal{O}, M) = \bigoplus_{n > 0} \mathcal{O}(n) \otimes_{\mathbb{R}[\Sigma_n]} M^\otimes n. \]

**Classical operads.**

a) Let \( M \) be an object of \( \mathbb{R} - \text{dgm} \). We associate to it the endomorphism operad given by:

\[ \text{End}(M)(n) = \text{Hom}_{\mathbb{R} - \text{dgm}}(M^\otimes n, M). \]

Any \( \mathcal{O} \)-algebra structure on \( M \) is given by a morphism of operads \( \mathcal{O} \longrightarrow \text{End}(M) \).

b) The operad \( \text{Com} \), defined by \( \text{Com}(n) = \mathbb{R} \). The \( \text{Com} \)-algebras are the differential graded algebras.

c) The operad \( \text{As} \) defined by \( \text{As}(n) = \mathbb{R}[\Sigma_n] \). The \( \text{As} \)-algebras are precisely the differential graded associative algebras.

d) The operad \( \text{Lie} \). A \( \text{Lie} \)-algebra \( L \) is an object of \( \mathbb{R} - \text{dgm} \) together with a bracket which is anticommutative and satisfies the Jacobi relation. If \( 2 \in \mathbb{R} \) is invertible then a \( \text{Lie} \)-algebra is a classical Lie algebra. Otherwise, the category of classical Lie algebras appears as full subcategory of the category of \( \text{Lie} \)-algebras.

e) The operad \( \text{Leib} \) which is the operad of Leibniz algebras. Leibniz algebras are algebras equipped with a bracket of degree zero that satisfies the Jacobi relation. A Lie algebra is an anti-commutative Leibniz algebra. We have an epimorphism of operads

\[ \text{Leib} \longrightarrow \text{Lie}. \]
**Homotopy of operads.** In \[14\], and \[3\], V. Hinich and C. Berger-I. Moerdijk proved that the category of operads is a closed model category. This structure is obtained from the one on the category $$\mathbf{R} - \mathbf{dgm}$$ via the free operad functor. In this category the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms. The cofibrant operads are the retracts of the quasi-free operads.

**Homotopy of algebras over an operad.** Let $$W_d$$ be the following object of $$\mathbf{R} - \mathbf{dgm}$$:

$$\ldots 0 \rightarrow R = R \rightarrow 0 \ldots$$

concentrated in differential degrees $$d$$ and $$d + 1$$ ($$d \in \mathbb{Z}$$). Using the terminology of \[3\], we say that $$\mathcal{O}$$ is admissible if the canonical morphism of $$\mathcal{O}$$-algebras:

$$A \rightarrow A \coprod S(\mathcal{O}, W_d)$$

is a quasi-isomorphism for any $$\mathcal{O}$$-algebra $$A$$ and for all $$d$$. For any admissible operad $$\mathcal{O}$$ there exists a closed model structure on the category of $$\mathcal{O}$$-algebras, which is transferred from $$\mathbf{R} - \mathbf{dgm}$$ along the free-forgetful adjunction given by $$S(\mathcal{O}, -)$$. As for operads the weak equivalences are the quasi-isomorphisms, the fibrations are the epimorphisms, and the cofibrant $$\mathcal{O}$$-algebras are the retracts of the quasi-free $$\mathcal{O}$$-algebras.

The category of $$\mathcal{O}$$-algebras is cofibrantly generated and cellular in the sense of P. Hirschhorn \[15\]. The set of generating cofibrations is

$$I = \{i_n : \mathcal{O}(x_n) \rightarrow \mathcal{O}(x_n, y_{n+1})\}$$

where $$\mathcal{O}(x_n)$$ is the free $$\mathcal{O}$$-algebra on a generator of degree $$n$$ and $$\mathcal{O}(x_n, y_{n+1})$$ is the free $$\mathcal{O}$$-algebra over the differential graded module $$R < x_n, y_{n+1} >$$ with two copies of $$R$$, one in degree $$n$$ the other in degree $$n + 1$$, the differential of $$y_{n+1}$$ being $$x_n$$. The set of generating acyclic cofibrations is

$$J = \{j_n : 0 \rightarrow \mathcal{O}(x_n)\}.$$

Notice that the free algebra $$\mathcal{O}(x_n)$$ plays the role of the sphere $$S^n$$.

Over the rational numbers all operads are admissible. This is not the case over an arbitrary ring, for example the operads $$\mathbf{Com}$$ and $$\mathbf{Lie}$$ over the integers are not. However cofibrant operads and the operad $$\mathcal{A}s$$ are always admissible.
**Homology of algebras.** Let $\mathcal{O}$ be an admissible operad, and let $A$ be an $\mathcal{O}$-algebra. An element $a \in A$ is called decomposable if it lies in the ideal $A^2$, the image of the evaluation products

$$\theta(n) : \mathcal{O}(n) \otimes A^\otimes n \rightarrow A$$

for any $n > 1$. We denote by $QA = A/A^2$ the space of indecomposables of the algebra $A$. The Quillen homology of $A$, denoted by $H^*_Q(A)$, is the homology of $QS(\mathcal{O}, V)$ where $S(\mathcal{O}, V)$ is a cofibrant replacement of $A$. This does not depend on the choice of the cofibrant replacement.

Moreover, any cofibration sequence $A \rightarrow B \rightarrow C$ of $\mathcal{O}$-algebras yields a long exact sequence in Quillen homology.

In [9] Fresse generalizes Koszul duality for operads as defined by Ginzburg and Kapranov in [12]. As the operads $\text{Lie}$ and $\text{Leib}$ are Koszul operads, over $\mathbb{Q}$ one can compute their Quillen homology by way of a nice complex:

**Lie-algebras.** Let $L$ be a $\text{Lie}$-algebra over $\mathbb{Q}$. The homology of $L$, denoted here by $H^*_\text{Lie}(L)$, is computed using the Chevalley-Eilenberg complex $CE_*(L)$. Now we can consider $L$ as a $\mathcal{L}_\infty$-algebra and compute $H^*_Q(L)$. Fresse’s results give the following isomorphism:

$$H^*_Q(L) \cong H^*_\text{Lie}(L).$$

**Leib-algebras.** The same kind of results hold for $\text{Leib}$-algebras. Consider a Leibniz algebra $L$. The homology of $L$, denoted by $H^*_\text{Leib}(L)$, is computed using a complex described in [19]. One has again a similar isomorphism:

$$H^*_Q(L) \cong H^*_\text{Leib}(L).$$

**Quillen cohomology of discrete algebras.** We refer the reader to [10] for more details and precise definitions. A discrete algebra is an $\mathcal{O}$-algebra concentrated in differential degree 0. The structure of $\mathcal{O}$-algebra reduces then in fact to a structure of $\pi_0(\mathcal{O})$-algebra.

In the case of discrete algebras there is also a notion of Quillen cohomology with coefficients. Fix a discrete $\mathcal{O}$-algebra $A$ and a discrete $A$-module $M$.

A derivation $D : A \rightarrow M$ in $\text{Der}(A, M)$ is a linear map such that for any $o \in \mathcal{O}(n)$ we have:

$$D(o(a_1, \ldots, a_n)) = \sum_{i=1}^{n} o(a_1, \ldots, D(a_i), \ldots, a_n).$$
We can define Quillen cohomology by computing the derived functors of $\text{Der}(A, M)$, that is by taking $A'$ a cofibrant replacement of $A$ in the category of $\mathcal{O}$-algebras and computing the homology of the complex $\text{Der}(A', M)$. This has also a homotopical interpretation: Consider $M$ as a trivial $\mathcal{O}$-algebra and denote by $\Sigma^n M$ the $n$-th suspension of $M$ in the category $R – \text{dgm}$. Then

$$H^n_Q(A, M) = [A, \Sigma^n M]_{\mathcal{O}_{-\text{alg}}} \cong [QA, \Sigma^n M]_{R – \text{dgm}}.$$ 

Moreover $H^1_Q(A, M)$ classifies square zero extensions of $A$ by $M$. A square zero extension is an exact sequence of modules:

$$0 \to M \to B \stackrel{p}{\to} A \to 0,$$

such that $p$ is a morphism of $\mathcal{O}$-algebras and the structure of $\mathcal{O}$-algebra on $M$ is trivial. The set of isomorphism classes of square zero extension is denoted by $Ex(A, M)$. By a classical result of Quillen [24] we have the following isomorphism:

$$H^1_Q(A, M) \cong Ex(A, M).$$

**Hurewicz Theorem.** In her thesis [18, Theorem 2.13] M. Livernet proved a Hurewicz type theorem for algebras over a Koszul operad in the rational case. A result of Getzler and Jones about the construction of a cofibrant replacement for $\mathcal{O}$-algebras, which uses the Bar-Cobar construction, extends the proof of Livernet to admissible operads.

**Theorem 1.1** Let $A$ be an $\mathcal{O}$-algebra. We suppose that the underlying chain complex of $A$ is concentrated in non-negative degrees. Then there is a Hurewicz morphism:

$$Hu : \pi_*(A) \longrightarrow H_*^Q(A)$$

induced by the projection on indecomposable elements. It satisfies the following properties:

i) If $\pi_k(A) = 0$ for $0 \leq k \leq n$ then $Hu$ is an isomorphism for $k \leq 2n + 1$ and an epimorphism for $k = 2n + 2$.

ii) If $\pi_0(A) = 0$ and $H^Q_k(A) = 0$ for $0 \leq k \leq n$ then $Hu$ is an isomorphism for $k \leq 2n + 1$ and an epimorphism for $k = 2n + 2$.

**Proof.** In the case of 0-connected chain complexes we have a Quillen adjunction between $\mathcal{O}$-algebras and $B\mathcal{O}$-coalgebras, [11] and [2]. These two functors provide for any $A$ a cofibrant replacement of the form $S(\mathcal{O}, C(B\mathcal{O}, A))$ where $C(B\mathcal{O}, A)$ is the coalgebra over
the cooperad $BO$ obtained by applying the operadic bar construction. Now Livernet’s arguments apply to $C(BO, A)$. □

**Perfect algebras over an operad.** Consider an algebra $A$ over an operad in the category of $R$-modules. The algebra $A$ is called $O$-perfect if any element in $A$ is decomposable i.e. $A = A^2$ or $QA = 0$. We define $PA$, the maximal $O$-perfect ideal of $A$, by transfinite induction.

Let $A_0$ be the ideal $A^2$. We define the ideals $A_{\alpha}$ inductively by setting $A_{\alpha} = (A_{\alpha-1})^2$ if $\alpha$ is a successor ordinal and $A_{\alpha} = \cap_{\beta<\alpha} A_{\beta}$ if $\alpha$ is a limit ordinal. Then we set $PA = \lim_{\alpha} A_{\alpha}$. This inverse system actually stabilizes for some ordinal $\beta$, hence $A^2_{\beta} = A_{\beta}$ and $PA = A_{\beta}$. Of course, if $QA = 0$ then we have $PA = A$.

Consider an epimorphism $f : O \to O'$ of operads and let $A$ be an $O'$-algebra. Then if $A$ is $O'$-perfect it is also $O$-perfect. Thus we have an inclusion $PA \subseteq P'A$ of the $O$-perfect ideal into the $O'$-perfect ideal of $A$.

For any differential graded algebra $A$ over an operad $O$, denote the $i$-th homology group of the underlying chain complex by $\pi_i(A)$. Then $\pi_0(A)$ is a $\pi_0(O)$-algebra in the category of $R$-modules. Moreover we have $H^Q_i(A) = Q\pi_0(A)$.

## 2 Homotopical localization and nullification

The theory of homological and homotopical localization of topological spaces developed by Bousfield and Dror Farjoun (see e.g. [5], [8]) has an analogue in the category of all algebras over an admissible operad $O$. This takes place in the more general framework established by Hirschhorn in [15]. Our category $O$-alg of all algebras over an admissible operad $O$ is indeed cellular and left proper. We explain first how to build mapping spaces in a model category which is not supposed to be simplicial, and recall then what is meant by homotopical localization with respect to a morphism in this context.

**Mapping spaces.** One way to construct mapping spaces up to homotopy in a model category is to find a cosimplicial resolution of the source (as in [15] Definition 18.1.1]). An elementary method to get one is provided by [6]. Let $X$ be a cofibrant $O$-algebra and define $\Delta[n] \otimes X$ as the homotopy colimit of the constant diagram with value $X$ over the simplex category $\Delta[n]$. This is simply the category of non-empty subsets of $\underline{n} = \{1, \ldots, n\}$ with
terminal object $n$. For example $\Delta[1] \otimes X$ is the homotopy colimit of the constant diagram $X \to X \leftarrow X$. A cofibrant replacement of this diagram is given by $X \to Cyl(X) \leftarrow X$ where $Cyl(X)$ is a cylinder object for $X$ (by definition the folding map factorizes as a cofibration followed by a weak equivalence $X \coprod X \hookrightarrow Cyl(X) \xrightarrow{\sim} X$). Having a functorial cofibrant replacement guarantees that $\Delta[-] \otimes X$ is a cosimplicial object. Therefore one can define $\text{map}(X, Y) = \text{mor}_{\mathcal{O}-\text{alg}}(\Delta[-] \otimes X, Y)$.

Let $\partial \Delta[n] \otimes X$ be the homotopy colimit of the constant diagram with value $X$ over the simplex category of the boundary $\partial \Delta[n]$, i.e. the category $\Delta[n]$ without the terminal object. Define $S^n \wedge X$ to be the homotopy cofiber of $\partial \Delta[n] \otimes X \hookrightarrow \Delta[n] \otimes X$. For example $S^1 \wedge X$ is simply the homotopy cofiber of $X \coprod X \hookrightarrow Cyl(X)$.

**Lemma 2.1** For any cofibrant $\mathcal{O}$-algebra $X$ we have $S^1 \wedge X \simeq \Sigma X$.

**Proof.** It is always true in a pointed model category that the homotopy cofiber of $X \coprod X \hookrightarrow Cyl(X)$ is equivalent to the suspension of $X$. □

**Proposition 2.2** In the category of $\mathcal{O}$-algebras $S^n \wedge X \simeq \Sigma^n X$.

**Proof.** The proof is by induction on $n$, the case $n = 1$ having been proved in the preceding lemma. Recall that $\partial \Delta[n] \otimes X$ can be computed by decomposing the cubical homotopy colimit as a push-out involving homotopy colimits on the front and back face (see for example [13, Lemma 0.2]): $\partial \Delta[n] \otimes X \simeq \text{hocolim}(X \leftarrow \partial \Delta[n-1] \otimes X \to X)$. Commuting homotopy colimits again we see that $S^n \wedge X$ is the homotopy push-out of the diagram ($* \leftarrow S^{n-1} \wedge X \to *$), i.e. $\Sigma(S^{n-1} \wedge X) \simeq \Sigma^n X$. □

**Homotopy groups of mapping spaces.** When $X$ is cofibrant and $Y$ fibrant, the simplicial set $\text{map}(X, Y)$ is fibrant, so its homotopy groups can be computed. There is always at least a morphism $X \to Y$, namely the trivial one. Let us denote by $\text{map}^0(X, Y)$ the component of this trivial map.

**Proposition 2.3** Let $X$ be a cofibrant and $Y$ a fibrant $\mathcal{O}$-algebra. Then $\pi_n \text{map}^0(X, Y) \cong [\Sigma^n X, Y]$. 

**Proof.** The mapping space $\text{map}(X, Y)$ is fibrant, so that the $n$-th homotopy group consists of homotopy classes of those $n$-simplices whose faces are trivial. An $n$-simplex is a morphism $f : \Delta[n] \otimes X \to Y$ and its faces are trivial if the composite $\partial \Delta[n] \otimes X \leftrightarrow$
\[ \Delta[n] \otimes X \to Y \] is so. Therefore \( f \) factorizes through \( S^n \wedge X \) and we conclude by Proposition 2.2.

\[ \square \]

**f-localization.** Let \( \mathcal{O} \) be an admissible operad. Let \( f : X \to Y \) be a morphism between two cofibrant \( \mathcal{O} \)-algebras. An \( \mathcal{O} \)-algebra \( Z \) is called \( f \)-local if the map of simplicial sets

\[ \text{map}(f, Z) : \text{map}(Y, Z) \to \text{map}(X, Z) \]

is a weak homotopy equivalence. A morphism of \( \mathcal{O} \)-algebras \( h : A \to B \) is called an \( f \)-equivalence if it induces a weak homotopy equivalence \( \text{map}(h, Z) : \text{map}(B, Z) \to \text{map}(A, Z) \) for every \( f \)-local algebra \( Z \). Theorem 4.1.1 from [15] ensures then the existence of an \( f \)-localization functor, i.e. a continuous functor \( L_f : \mathcal{O}\text{-alg} \to \mathcal{O}\text{-alg} \) together with a natural transformation \( \eta : \text{Id} \to L_f \) from the identity functor to \( L_f \), such that \( \eta_A : A \to L_f A \) is an \( f \)-equivalence and \( L_f A \) is \( f \)-local for any \( \mathcal{O} \)-algebra \( A \).

**X-nullification.** In the special case when \( f \) is of the form \( f : X \to 0 \) or \( f : 0 \to X \), the functor \( L_f \) is also denoted by \( P_X \), and called \( X \)-nullification functor. Note that an algebra \( Z \) is \( f \)-local or \( X \)-null if \( \text{map}(X, Z) \) is weakly homotopy equivalent to a point. By Proposition 2.3 this is equivalent to requiring that \( [\Sigma^k X, Z] \) be trivial for all \( k \geq 0 \).

**Example:** An interesting example is when \( X \) is the free \( \mathcal{O} \)-algebra \( \mathcal{O}(x) \) with one generator \( x \) in dimension \( n \). This plays the role of the \( n \)-dimensional sphere, hence \( \mathcal{O}(x) \)-nullification gives rise to a functorial \( n \)-Postnikov section in this category.

### 3 An additive plus-construction

A Quillen plus-construction of an algebra \( A \) over an operad \( \mathcal{O} \) is a Quillen homology equivalence \( \eta : A \to A^+ \) which quotients out the perfect radical on \( \pi_0 \), that is

\[ \pi_0(A^+) \cong \pi_0(A)/\mathcal{P}\pi_0(A). \]

If it exists then it is unique up to quasi-isomorphism. This is because of the following universal property: For any morphism \( g : A \to B \) with \( Q\pi_0(B) = \pi_0(B) \), there exists a map \( \tilde{g} : A^+ \to B \) such that \( \tilde{g}\eta = g \), and which is unique up to homotopy.
We now construct a functorial Quillen plus-construction as a nullification with respect to a universal acyclic algebra, i.e. an acyclic algebra $U$ such that the associated nullification $A \rightarrow P_U A$ is the plus-construction.

**A universal acyclic algebra.** The algebra $U$ is defined as a big coproduct

$$U = \coprod_{(T,\phi)} U_{(T,\phi)},$$

where $(T,\phi)$ ranges over all $\mathcal{O}$-trees. Every $U_{(T,\phi)}$ will be the homotopy colimit of a certain direct system of free $\mathcal{O}$-algebras $\{U_r, \varphi_r\}_{r \geq 0}$ associated to $(T,\phi)$ in a canonical way. A **rooted tree** $T = \{V(T), A(T)\}$ is a directed graph such that any vertex $v \in V(T)$ has one ingoing arrow $a_v \in A(T)$, except the root that has no ingoing arrow, and such that the following additional conditions are satisfied: Each vertex $v$ has a finite number of outgoing arrows, denoted by $\text{val}(v)$; the set $\text{suc}(v)$ of successors vertices of $v$, i.e. those which are connected to $v$ by an ingoing arrow is finite and totally ordered; and finally, the vertices $v$ of even level have at least 2 successors. The root has level 0, and inductively we say that a vertex $v$ has level $k$ if $v \in \text{suc}(u)$ for some $u$ of level $k-1$.

Let $\mathcal{O}$ be any operad. An $\mathcal{O}$-tree is a pair $(T, \phi)$ where $T$ is a rooted tree and $\phi$ is a function which associates to each vertex $v$ of odd level a multilinear operation $o_n \in \mathcal{O}(n)_0$ where $n = \text{val}(v) \geq 2$.

We next define the direct system $\{U_r, \varphi_r\}$ of free $\mathcal{O}$-algebras associated to a given $\mathcal{O}$-tree $(T,\phi)$, by induction on $r$: Let $U_0$ be the free $\mathcal{O}$-algebra on one generator $x$ in dimension 0 (corresponding to the root). Let $n = \text{val}(\text{root})$ and $\text{suc}(\text{root}) = \{v_1, \ldots, v_n\}$. For each $j = 1, \ldots, n$, let $k_j = \text{val}(v_j)$ and $o_{k_j} = \phi(v_j)$ be the multilinear operation in $\mathcal{O}(k_j)_0$ associated to the vertex $v_j$. Choose $k_j$ free generators $x_{1j1}, x_{1j2}, \ldots, x_{1jk_j}$ in dimension 0 (corresponding to the vertices in $\text{suc}(v_j)$ of level 2, the first index indicates half of the level). Let $U_1$ be the free $\mathcal{O}$-algebra on those generators. Define $\phi_1 : U_0 \rightarrow U_1$ on the generator $x$ by

$$\phi_1(x) = \sum_{j=1}^{n} \theta(o_{k_j}; x_{1j1}, x_{1j2}, \ldots, x_{1jk_j}).$$

Inductively, we define then $U_r$ as the free $\mathcal{O}$-algebra on as many generators as there are vertices of level $2r$, and $\phi_r : U_{r-1} \rightarrow U_r$ is given on each generator of $U_{r-1}$ by a similar formula as the above one for $\phi_1(x)$. Hence the homotopy colimit $U_{(T,\phi)}$ is free on generators $x_I$ of degree 0 and $y_I$ of degree 1 where $I$ is a multi-index of the form $ljs$, $l$ indicating
half of the level where these generators are created, $1 \leq s \leq k_j$, and the differential $d(y_I) = x_I - \phi_t(x_I)$.

**The cone of $\mathcal{U}$.** In order to do some computations with this acyclic algebra, we need to describe how to construct the cone of it. Let us simply describe the cone on $\mathcal{U}_{(T,\phi)}$ for a fixed tree $T$. For each generator $x_I$ in degree 0 we add a generator $\bar{x}_I$ in degree 1, and for each generator $y_I$ in degree 1 we add a generator $\bar{y}_I$ in degree 2. The differential is as follows: $dy_I = x_I - \phi_t(x_I)$, as in $\mathcal{U}_{(T,\phi)}$, $d\bar{x}_I = x_I$, so we kill $\pi_0$, and $d\bar{y}_I = y_I - \bar{x}_I - u_I$ where $u_I$ is a decomposable element of degree 1 such that $du_I = \phi_t(x_I)$. Such an element exists indeed since $\phi_t(x_I)$ is a decomposable element in degree 0, where all indecomposables are hit by a differential.

**Theorem 3.1** Let $\mathcal{O}$ be an admissible operad. Then the homotopical nullification with respect to $\mathcal{U}$ is a functorial plus-construction in the category of algebras over $\mathcal{O}$. It enjoys the following properties:

(i) $A^+ \simeq \text{Cof}(\coprod_{f \in \text{Hom}(\mathcal{U}, A)} \mathcal{U} \to A)$

(ii) $\pi_0(A^+) \cong \pi_0(A)/\mathcal{P}\pi_0(A)$

(iii) $H_*^\mathcal{Q}(A) \cong H_*^\mathcal{Q}(A^+)$

**Proof.** Consider the cofibre sequence

$$\coprod_{f \in \text{Hom}(\mathcal{U}, A)} \mathcal{U} \xrightarrow{ev} A \to B$$

Clearly $A \to B$ is a $P_\mathcal{U}$-equivalence. So it remains to show that $B$ is $\mathcal{U}$-local. Now, as the suspension of $\mathcal{U}$ is contractible by the Hurewicz Theorem [11], $B$ is $P_\mathcal{U}$-local if and only if $[\mathcal{U}, B] = 0$. This happens exactly when $\mathcal{P}\pi_0(B) = 0$. Let us thus compute $\pi_0 B$. Consider actually the more elementary cofibre $C_\alpha$ of a single map $\alpha : \mathcal{U}_{(T,\phi)} \to A$. Such a map corresponds to an element $a \in \mathcal{P}\pi_0 A$ together with a decomposition following the pattern indicated by the tree $(T, \phi)$. Let us replace $A$ by a free algebra $\mathcal{O}(V)$ and construct now $C_\alpha$ as the push-out of $\mathcal{O}(V) \leftarrow U_{(T,\phi)} \to C(U_{(T,\phi)})$. The models of these algebras we exhibited earlier show that $C_\alpha = \mathcal{O}(V) \coprod \mathcal{O}(\bar{x}_I, \bar{y}_I)$ with $d\bar{x}_I = a_I = \alpha(x_I)$ and $d\bar{y}_I = b_I - \bar{x}_I - \alpha(u_I)$. Clearly $\pi_0 C_\alpha \cong \pi_0 A/\langle a \rangle$. Likewise $\pi_0 B \cong \pi_0 A/\mathcal{P}\pi_0(A)$.

Hence $\mathcal{P}\pi_0(B) = 0$, which shows that $B \simeq A^+$ and the other two properties are now direct consequences of the first one. $\square$
Naturality. We conclude this section with a discussion of the naturality of the plus-construction with respect to the operad. We denote by \( U' \) the universal acyclic \( O' \)-algebra as constructed above and \( A'^+ = P_{U'}A \) the associated plus-construction.

**Proposition 3.2** Let \( f : O \rightarrow O' \) be a map of operads, then there is a map of \( O \)-algebras \( f : U \rightarrow U' \).

**Proof.** The map \( f \) induces a map between the directed systems \( \{ U_r, \phi_r \} \) and \( \{ U'_r, f(\phi_r) \} \). Where \( \{ U_r, \phi_r \} \) is the directed system associated to a \( O \)-tree \( (T, \phi) \) and \( \{ U'_r, f(\phi_r) \} \) is the directed system associated to the \( O' \)-tree \( (T, f(\phi)) \) where each vertex is of the form \( f(o) \). There is a natural transformation between the directed systems of \( O \)-algebras, thus also a map between their homotopy colimits. \( \square \)

**Proposition 3.3** Let \( f : O \rightarrow O' \) be a quasi-isomorphism of operads, and suppose that either we work over \( \mathbb{Q} \), or the operads \( O \) and \( O' \) are cofibrant. Then \( f : U \rightarrow U' \) is a quasi-isomorphism of \( O \)-algebras.

**Proof.** The result follows from the fact that free algebras over the operads \( O \) and \( O' \) and over the same generators are quasi-isomorphic as \( O \)-algebras. \( \square \)

As a consequence, when replacing an operad by a cofibrant one to do homotopy, the choice of this cofibrant operad does not matter.

**Corollary 3.4** Let \( A \) be an \( O' \)-algebra, and let \( f : O \rightarrow O' \) be a morphism of operads. Under the same assumptions as in the preceding proposition, the map \( A^+ \rightarrow A'^+ \) is a quasi-isomorphism of \( O \)-algebras. \( \square \)

## 4 Fibrations and the plus-construction

Let \( O \) be an admissible operad over a field \( k \). This section is devoted to the analysis of the behavior of the plus-construction with fibrations. In particular we will be interested in the fibre \( AX \) of the map \( X \rightarrow X^+ \). As one should expect it, \( AX \) is the universal acyclic algebra over \( X \) in the sense that any map \( A \rightarrow X \) from an acyclic algebra \( A \) factors through \( AX \). The most efficient tool to deal with such questions is the technique of fibrewise localization in our model category of \( O \)-algebras. To our knowledge, such a tool had not been developed up to now in any other context than spaces, and we refer therefore to the separate paper [7] for the following claim:
Theorem 4.1 Let $F \to E \to B$ be a fibration of $\mathcal{O}$-algebras. There exists then a commutative diagram

$$
\begin{array}{ccc}
F & \to & E \\
\downarrow & & \downarrow \\
F^+ & \to & E^+ \\
\end{array}
\begin{array}{ccc}
& & B \\
\downarrow & & \downarrow \\
& & B \\
\end{array}
$$

where both lines are fibrations and the map $E \to \bar{E}$ is a $P_U$-equivalence.

The main ingredient in the proof of this theorem is the fact that the category of $\mathcal{O}$-algebras satisfies the cube axiom. From the above theorem we infer that the plus-construction sometimes preserves fibrations.

Theorem 4.2 Let $F \to E \to B$ be a fibration of $\mathcal{O}$-algebras. If the basis $B$ is local with respect to the $U$-nullification then we have a fibration

$$F^+ \to E^+ \to B.$$

Proof. By Theorem 4.1 this is a direct consequence of the fact that the total space $\bar{E}$ sits in a fibration where both the fibre and the base space are $U$-local and hence is also $U$-local.

The fibre of the plus-construction. Another consequence of the fibrewise plus-construction is that the fibre $AX$ is acyclic.

Proposition 4.3 The fiber $AX$ of the plus-construction $X \to X^+$ is $H^Q_\ast$-acyclic for any $\mathcal{O}$-algebra $X$.

Proof. Consider the fibration $AX \to X \to X^+$. The plus-construction preserves this fibration by the above theorem, i.e. $(AX)^+$ is the fibre of the identity on $X^+$. Thus $(AX)^+ \simeq \ast$, which means that $AX$ has the same Quillen homology as a wedge of copies of $U$.

Cellularization. We can go a little further in the analysis of the fibre $AX$. Our next result says precisely that the map $AX \to X$ is a $CW_U$-equivalence, where $CW_U$ is Farjoun’s cellularization functor ([8, Chapter 2]). We do not know whether $AX$ is actually the $U$-cellularization of $X$.

Proposition 4.4 We have $\text{map}(U, AX) \simeq \text{map}(U, X)$ for any $\mathcal{O}$-algebra $X$.  

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Proof. We know $AX$ is acyclic by Proposition 4.3. Apply now $\text{map}(U, -)$ to the fibration $AX \to X \to X^+$ so as to get a fibration of simplicial sets

$$\text{map}(U, AX) \to \text{map}(U, X) \to \text{map}(U, X^+)$$

By definition $X^+$ is $U$-local, so that $\text{map}(U, X^+)$ is trivial. Therefore $\text{map}(U, AX) \simeq \text{map}(U, X)$. □

On the level of components, this implies we have an isomorphism $[U, AX] \simeq [U, X]$, which means that any element in the $\mathcal{O}$-perfect ideal $\mathcal{P}_{\pi_0}X$ together with a given decomposition can be lifted in a unique way to such an element in $\pi_0 AX$.

**Proposition 4.5** The fibration $AX \to X \to X^+$ is also a cofibration.

Proof. By definition $X^+$ is the homotopy cofibre of a map $\coprod U \to X$. By the above proposition this map admits a unique lift to $AX$. By considering the composite $\coprod U \to AX \to X$, we get a cofibration

$$\text{Cof}(\coprod U \to AX) \to \text{Cof}(\coprod U \to X) \to \text{Cof}(AX \to X)$$

The first cofibre is $(AX)^+$, which is contractible, and the second is $X^+$. The third is thus $X^+$ as well. □

**Preservation of square zero extensions.** Let us finally study the effect of the plus-construction on a square zero extension, as introduced at the end of Section 1. In the case of Lie or Leibniz algebras this notion coincides of course with the classical one of central extension, as exposed e.g. in [16]. Following [24], [10, chapter 5], such a square zero extension is classified by an element in the first Quillen cohomology group $H^1_Q(B; Z) \cong [B, K(Z, 1)]$. Here $K(Z, 1)$ is the delooping of $Z$, given as $\mathcal{O}$-algebra by the chain complex $Z$ concentrated in degree 1. As for group extensions, the fibre of the classifying map $B \to K(Z, 1)$ (the $k$-invariant of the extension) is precisely $E$.

**Proposition 4.6** Let $Z \hookrightarrow E \to B$ be a square zero extension of discrete $\mathcal{O}$-algebras. Then $Z \to E^+ \to B^+$ is a fibration.

Proof. Let us consider the $k$-invariant and the associated fibration $E \to B \to K(Z, 1)$. The base is 0-connected, thus $P_{\mathcal{U}}$-local. Theorem 4.2 tells us that $E^+ \to B^+ \to K(Z, 1)$ is also a fibration. Therefore so is $Z \to E^+ \to B^+$. □
5 Applications to algebras of matrices

Recollections on algebras of matrices. Let $k$ be a field and $R$ be an associative $k$-algebra. Consider $gl(R)$ the union of the $gl_n(R)$'s. This is a Lie-algebra and also a Leib-algebra for the classical bracket of matrices. The trace $tr : gl(R) \to R/[R,R]$ is a morphism of Lie and Leib-algebras, whose kernel is by definition the algebra $sl(R)$.

We define the Steinberg algebra $st(R)$ for the two operads Lie and Leib by taking the free algebras over the generators $u_{i,j}(r), r \in R$ and $1 \leq i \neq j$ with the relations
a) $u_{i,j}(m \cdot r + n \cdot s) = m \cdot u_{i,j}(r) + n \cdot u_{i,j}(s)$ for $r, s \in R$ and $m, n \in \mathbb{Z}$.
b) $[u_{i,j}(r), u_{k,l}(s)] = 0$ if $i \neq l$ and $j \neq k$.
c) $[u_{i,j}(r), u_{k,l}(s)] = u_{i,k}(rs)$ if $i \neq l$ and $j = k$.

We have the following extension of algebras (for both operads):

$$Z(st(R)) \to st(R) \to sl(R)$$

where $Z(st(R))$ is the kernel of the canonical map between $st(R)$ and $sl(R)$. Following the work of C. Kassel and J.L. Loday this is a universal square zero extension [16, Proposition 1.8].

Now we can consider all these algebras as algebras over cofibrant replacements $L_\infty$ and $Leib_\infty$ of the operads Lie and Leib.

Homology theories. In the category of $L_\infty$-algebras we define cyclic homology up to homotopy $HC^\infty$:

$$HC^\infty_*(R) = \pi_*(gl(R)^+)$$

Likewise in the category of $Leib_\infty$-algebras we define Hochschild homology up to homotopy:

$$HH^\infty_*(R) = \pi_*(gl(R)^+)$$

By Corollary 3.4 we notice that these definitions do not depend on the choice of the cofibrant replacement of the operads $Lie$ or $Leib$. These theories define two functors from the category of associative algebras to the categories of Lie and Leib graded algebras. We recall that the homotopy of a $L_\infty$-algebra (resp. a $Leib_\infty$-algebra) is a graded $Lie$-algebra (resp. a $Leib$-algebra).

When we consider these two theories over $\mathbb{Q}$, we have quasi-isomorphisms $Leib_\infty \to Leib$ and $Lie_\infty \to Lie$. Then by the results of M. Livernet [18] our theories coincide with the
classical cyclic and Hochschild homologies:

\[ HC^\infty(R) \cong HC(R), \]
\[ HH^\infty(R) \cong HH(R). \]

We do not know if these isomorphisms remain valid over \( \mathbb{Z} \). However, using the properties of our construction, we are able to compute the first four groups of \( HC^\infty \) and \( HH^\infty \). These results form perfect analogues of the classical computations in algebraic \( K \)-theory, see for example [25, Theorem 4.2.10], and [1, Theorem 3.14] for a topological approach.

**Abelianization.** In order to compute \( HH^\infty_0(R) \) and \( HC^\infty_0(R) \) we use the following fibration given by the trace:

\[ sl(R) \rightarrow gl(R) \rightarrow R/[R, R]. \]

**Proposition 5.1** Let \( R \) be an associative \( k \)-algebra. Then \( HH^\infty_0(R) \) and \( HC^\infty_0(R) \) are both isomorphic to \( R/[R, R] \).

**Proof.** By Theorem 4.2 we get a fibration

\[ sl(R)^+ \rightarrow gl(R)^+ \rightarrow R/[R, R]. \]

The commutator subgroup of \( gl(R) \) as well as \( sl(R) \) (i.e. the perfect radical in either the category of Lie or Leibniz algebras) is \( sl(R) \). Therefore so is the perfect radical in \( L_\infty \) and \( Leib_\infty \) (this is the case for any discrete algebra). Hence \( \pi_0 gl(R)^+ \cong R/[R, R] \). \( \square \)

**The center of the Steinberg algebra.** In order to compute \( HH^\infty_1(R) \) and \( HC^\infty_1(R) \), we use the Steinberg Lie, respectively the Steinberg Leibniz, algebra \( st(R) \) and the following square zero extension:

\[ Z(st(R)) \rightarrow st(R) \rightarrow sl(R). \]

This is the universal central extension of the perfect algebra \( sl(R) \). In particular \( st(R) \) is superperfect, meaning that \( H^Q_1(st(R)) = 0 \).

**Proposition 5.2** Let \( R \) be an associative \( k \)-algebra. Then the first homology group \( HC^\infty_1(R) \) (respectively \( HH^\infty_1(R) \)) is isomorphic to \( Z(st(R)) \cong H^Q_1(sl(R)) \) (respectively to the center of the Steinberg Leibniz algebra).
Proof. We have to compute $\pi_1 gl(R)^+$. From Theorem 4.2 we infer that $sl(R)^+ \to gl(R)^+ \to R/[R,R]$ is a fibration. Hence the preceding proposition tells us that $sl(R)$ is the 0-connected cover of $gl(R)^+$, so that we only need to compute $\pi_1 sl(R)^+$. By the Hurewicz Theorem, this is isomorphic to $H^Q_0(sl(R))$.

Moreover Proposition 4.6 shows that $Z(st(R)) \to st(R)^+ \to sl(R)$ is a fibration. Both $sl(R)$ and $st(R)$ are perfect algebras, so their plus-constructions are 0-connected. Actually $st(R)^+$ is even 1-connected since $H^Q_1(st(R)) = 0$. The homotopy long exact sequence allows now to conclude that $\pi_1 sl(R)^+ \cong Z(st(R))$. □

Proposition 5.3 Let $R$ be an associative $k$-algebra. Then for $2 \leq i \leq 3$, $HC^\infty_i(R)$ (respectively $HH^\infty_i(R)$) is isomorphic to $H^Q_i(st(R))$, the Quillen homology of the Steinberg Lie algebra in the category of $L_\infty$-algebras (respectively to $H^Q_2(st(R))$, the Quillen homology of the Steinberg Leibniz algebra in the category of $Leib_\infty$-algebras).

Proof. This is the same proof as the first part of the computation of $HH^\infty_2$ and $HC^\infty_2$. The computation of the third groups follows from the Hurewicz Theorem □

As explained in the first section there is an isomorphism over $\mathbb{Q}$ between the Quillen homology $H^Q_*$ and $H^\text{Lie}_{*+1}$, respectively $H^\text{Leib}_{*+1}$. Together with the fact that the theories up to homotopy coincide with their classical analogues over $\mathbb{Q}$, the three computations we made above yield the following isomorphisms.

Corollary 5.4 Let $R$ be an associative algebra over $\mathbb{Q}$ then

1. $HC_0(R) = HH_0(R) = R/[R,R]$,
2. $HC_1(R) = H^\text{Lie}_2(sl(R)), HH_1(R) = H^\text{Leib}_2(sl(R))$,
3. $HC_2(R) = H^\text{Lie}_3(st(R)), HH_2(R) = H^\text{Leib}_3(st(R))$,
4. $HC_3(R) = H^\text{Lie}_4(st(R)), HH_3(R) = H^\text{Leib}_4(st(R))$. □

Morita invariance. These theories are obviously Morita invariant since $gl(gl(R))$ is isomorphic to $gl(R)$. Hence we have $HC^\infty_*(gl(R)) \cong HC^\infty_*(R)$ and $HH^\infty_*(gl(R)) \cong HH^\infty_*(R)$.

Products. Let $R$ and $S$ be two associative $k$-algebras, and form the product in the category of associative algebras $R \times S$. We want to compute $HC^\infty_*(R \times S)$ and $HH^\infty_*(R \times S)$.
S). Observe that \( gl(R \times S) \) is isomorphic as a Lie-algebra to the product \( gl(R) \times gl(S) \). As nullifications preserve products (this is a consequence of the fiberwise localization [7]) one has:

**Proposition 5.5** Let \( R \) and \( S \) be two associative \( k \)-algebras. Then:

(i) \( HC_\ast^\infty(R \times S) \cong HC_\ast^\infty(R) \oplus HC_\ast^\infty(S) \)

(ii) \( HH_\ast^\infty(R \times S) \cong HH_\ast^\infty(R) \oplus HH_\ast^\infty(S) \)

□

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