THE IMPACT OF THE BOHR TOPOLOGY ON SELECTIVE PSEUDOCOMPACTNESS

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Abstract. Recall that a space \( X \) is selectively pseudocompact if for every sequence \( \{U_n : n \in \mathbb{N}\} \) of non-empty open subsets of \( X \) one can choose a point \( x_n \in U_n \) for all \( n \in \mathbb{N} \) such that the resulting sequence \( \{x_n : n \in \mathbb{N}\} \) has an accumulation point in \( X \). This notion was introduced under the name strong countable compactness by García-Ferreira and Ortiz-Castillo; the present name is due to Dorantes-Aldama and the first author. In 2015, García-Ferreira and Tomita constructed a pseudocompact Boolean group that is not strongly pseudocompact. In this paper we establish a fairly general result involving the Bohr topology on all countable subgroups of a given topological group from which we deduce that many known examples in the literature of pseudocompact Boolean groups automatically fail to be selectively pseudocompact. In particular, all strongly self-dual pseudocompact Boolean groups constructed by Tkachenko in 2009 and all reflexive pseudocompact Boolean groups with property \( \sharp \) constructed by Galindo and Macario in 2011 are not selectively pseudocompact. It follows that, under the Singular Cardinal Hypothesis, every pseudocompact Boolean group admits a pseudocompact reflexive group topology which is not selectively pseudocompact.

All topological groups in this paper are assumed to be Hausdorff.

A group \( G \) is Boolean if \( x^2 = e \) for every \( x \in G \), where \( e \) is the identity element of \( G \). It is known (and easy to see) that all Boolean groups are abelian.

As usual, \( \mathbb{N} \) denotes the set of natural numbers.

1. Introduction

Recall that a point \( x \) of a topological space \( X \) is an accumulation point of a sequence \( \{x_n : n \in \mathbb{N}\} \) of points of \( X \) provided that the set \( \{n \in \mathbb{N} : x_n \in V\} \) is infinite for every neighbourhood \( V \) of \( x \) in \( X \).

Definition 1.1. (i) Let \( \mathcal{U} = \{U_n : n \in \mathbb{N}\} \) be a sequence of sets. If \( x_n \in U_n \) for every \( n \in \mathbb{N} \), we shall call the sequence \( \{x_n : n \in \mathbb{N}\} \) a selection for \( \mathcal{U} \).

(ii) A topological space \( X \) is selectively pseudocompact if every sequence \( \{U_n : n \in \mathbb{N}\} \) of non-empty open subsets of \( X \) admits a selection \( \{x_n : n \in \mathbb{N}\} \) which has an accumulation point in \( X \).

This notion was introduced by García-Ferreira and Ortiz-Castillo [4] under the name “strong pseudocompactness”. The present name and an equivalent reformulation of the property given in item (ii) of Definition 1.1 is due to Dorantes-Aldama and the first author [2, Theorem 2.1 and Definition 2.2]. One easily sees that

\[
\text{countably compact} \rightarrow \text{selectively pseudocompact} \rightarrow \text{pseudocompact}.
\]

In 2015, García-Ferreira and Tomita constructed a Boolean pseudocompact group that is not selectively pseudocompact [5, Example 2.4]. In this paper we establish a fairly general result
involving the Bohr topology on all countable subgroups of a given topological group (Theorem 3.3) from which we deduce that many known examples in the literature of pseudocompact Boolean groups automatically fail to be selectively pseudocompact. In particular, each of strongly self-dual pseudocompact groups constructed by Tkachenko [8, Theorem 3.3] in 2009 is not selectively pseudocompact either. Therefore, even strongly self-dual pseudocompact Boolean groups need not be selectively pseudocompact (Corollary 4.8). Furthermore, pseudocompact group topologies with property \( \mathcal{P} \) on Boolean groups constructed by Galindo and Macario [3] in 2011 are also not selectively pseudocompact. Based on this, we show that, under the Singular Cardinal Hypothesis \( \text{SCH} \), every pseudocompact Boolean group admits a pseudocompact group topology which fails to be selectively pseudocompact (Corollary 4.4).

2. An auxiliary construction

This section is inspired by the proof of [4, Lemma 2.1].

Even though all results in this section hold for arbitrary (not necessarily commutative) groups \( G \), we shall use abelian notations denoting the group operation of \( G \) by \( + \) and its identity element by \( 0 \).

**Lemma 2.1.** Let \( G \) be a non-discrete topological group and \( V_{-1} = G \). Then there exist a sequence of points \( \{g_n : n \in \mathbb{N}\} \subseteq G \setminus \{0\} \) and a sequence \( \{V_n : n \in \mathbb{N}\} \) of open neighborhoods of 0 such that the following conditions hold for every \( n \in \mathbb{N} \):

\[
\begin{align*}
(\alpha_n) & \quad 0 \not\in g_n + V_n + V_n \subseteq V_{n-1}, \\
(\beta_n) & \quad V_n \subseteq V_{n-1}.
\end{align*}
\]

**Proof.** By induction on \( n \in \mathbb{N} \), we shall select \( g_n \in G \setminus \{0\} \) and an open neighbourhood \( V_n \) of 0 satisfying \( (\alpha_n) \).

Fix \( n \in \mathbb{N} \). Suppose that \( g_i \in G_i \setminus \{0\} \) and an open neighbourhood \( V_i \) of 0 satisfying \( (\alpha_i) \) and \( (\beta_i) \) have already been chosen for every \( i = 0, \ldots, n-1 \). Since \( V_{n-1} \) is an open neighbourhood of 0 and \( G \) is non-discrete, we have \( W \neq \{0\} \). Then \( U = W \setminus \{0\} \) is a non-empty open subset of \( G \). Thus, we can select some \( g_n \in U \). Using the continuity of the group operation of \( G \), we can find an open neighbourhood \( V_n \) of 0 such that \( g_n + V_n + V_n \subseteq U \) and \( V_n \subseteq V_{n-1} \). Now \( (\alpha_n) \) and \( (\beta_n) \) hold.

**Lemma 2.2.** Under the assumptions of Lemma 2.1,

\[
\sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + V_{i_k} \subseteq g_{i_1} + V_{i_1} + V_{i_1}
\]

for every strictly increasing finite sequence \( i_1, \ldots, i_k \in \mathbb{N} \).

**Proof.** We prove this lemma by induction on the length \( k \) of the sequence. Note that (1) trivially holds for sequences of length 1. Suppose that (1) has been verified for all strictly increasing sequences of length \( k \), and let \( i_1, \ldots, i_{k+1} \in \mathbb{N} \) be a strictly increasing sequence. Observe that

\[
\sum_{j=1}^{k+1} (g_{i_j} + V_{i_j}) + V_{i_{k+1}} \subseteq \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + g_{i_{k+1}} + V_{i_{k+1}} + V_{i_{k+1}} \subseteq \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + V_{i_{k+1}}
\]

by \( (\alpha_{i_{k+1}}) \). Since \( i_k < i_{k+1} \), from \((\beta_{i_{k+1}}), (\beta_{i_{k+2}}), \ldots, (\beta_{i_{k+1}-1})\) we obtain

\[
V_{i_k} \supseteq V_{i_{k+1}} \supseteq \cdots \supseteq V_{i_{k+1}-1}.
\]

By inductive hypothesis, (1) holds. Combining (1), (2) and (3), we get

\[
\sum_{j=1}^{k+1} (g_{i_j} + V_{i_j}) + V_{i_{k+1}} \subseteq \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + V_{i_{k+1}-1} \subseteq \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + V_{i_k} \subseteq g_{i_1} + V_{i_1} + V_{i_1}.
\]
This finishes the inductive step. □

**Corollary 2.3.** Each non-discrete group $G$ has a sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of $G$ such that

$$0 \not\in \sum_{j=1}^{k} U_{i_j}$$

for every strictly increasing finite sequence $i_1, \ldots, i_k \in \mathbb{N}$.

**Proof.** Consider the sequences $\{g_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ as in the conclusion of Lemma 2.1. Let $U_n = g_n + V_n$ for every $n \in \mathbb{N}$. Clearly, $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a sequence of non-empty open subsets of $G$.

Let $i_1, \ldots, i_k \in \mathbb{N}$ be a strictly increasing finite sequence. Then

$$\sum_{j=1}^{k} U_{i_j} = \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) \subseteq \sum_{j=1}^{k} (g_{i_j} + V_{i_j}) + V_k \subseteq g_{i_1} + V_i + V_i$$

by $0 \in V_i$ and Lemma 2.2. Since $0 \not\in g_{i_1} + V_i + V_i$ by $(\alpha_{i_1})$, this implies (4). □

### 3. Main result

A topological group $G$ is *precompact*, or *totally bounded*, provided that for every open neighbourhood $U$ of the identity of $G$ one can find a finite subset $F$ of $G$ such that $G = FU$. It is well known that a topological group $G$ is precompact if and only if it is a subgroup of some compact group. Every abelian group $G$ has the maximal precompact group topology on $G$ called its *Bohr topology*. This topology is simply the initial topology with respect to the family of all homomorphisms from $G$ to the circle group $\mathbb{T}$. We shall use $G^\#$ to denote the abelian group $G$ endowed with its Bohr topology.

A subset $X$ of an abelian group $G$ is *independent* if $0 \not\in X$ and $\langle A \rangle \cap (X \setminus A) = \{0\}$ for every subset $A$ of $X$, where $\langle A \rangle$ denotes the smallest subgroup of $G$ containing $A$.

A straightforward proof of the following lemma is left to the reader.

**Lemma 3.1.** A subset $X$ of a Boolean group is independent if and only if $\sum_{a \in A} a \neq 0$ for every non-empty finite subset $A$ of $X$.

We shall need the following result from Hart and van Mill shown in [6, Lemma 1.4].

**Lemma 3.2.** Every independent subset of an abelian group $G$ is closed and discrete in $G^\#$.

Now we are ready to state our main result.

**Theorem 3.3.** Let $G$ be a Boolean topological group such that the subspace topology on every countable subgroup $H$ of $G$ coincides with the Bohr topology of $H$. Then $G$ is not strongly pseudocompact.

**Proof.** Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be the sequence of non-empty open subsets of $G$ as in Corollary 2.3. According to item (ii) of Definition 1.1 in order to prove that $G$ is not selectively pseudocompact, it suffices to show that every selection $\{x_n : n \in \mathbb{N}\}$ for $\mathcal{U}$ does not have an accumulation point in $G$. We fix an arbitrary selection $\{x_n : n \in \mathbb{N}\}$ for $\mathcal{U}$.

**Claim 1.** $X = \{x_n : n \in \mathbb{N}\}$ is a faithfully indexed independent subset of $G$.

**Proof.** First, we check that $X$ is faithfully indexed. Let $m, n \in \mathbb{N}$ and $m < n$. Then $0 \not\in U_m + U_n$ by Corollary 2.3. On the other hand, $x_m + x_n \in U_m + U_n$, so $x_m + x_n \neq 0$. Since $G$ is a Boolean group, this implies $x_n \neq x_m$.

Next, we check that $X$ is independent. By Lemma 3.1 it suffices to show that $\sum_{j=1}^{k} x_{i_j} \neq 0$ for every faithfully indexed finite subset $\{i_j : j = 1, \ldots, k\}$ of $\mathbb{N}$. Without loss of generality, we
may assume that the sequence \(i_1, \ldots, i_k\) is strictly increasing. Since \(\sum_{j=1}^k x_{i_j} = \sum_{j=1}^k U_{i_j}\), from Corollary 2.3 we conclude that \(\sum_{j=1}^k x_{i_j} \neq 0\). \(\square\)

Let \(g \in G\) be arbitrary. The subgroup \(H\) generated by the set \(X \cup \{g\}\) is countable. Since \(X\) is an independent subset of \(G\) by Claim 4.1 it is also an independent subset of \(H\), as \(H\) is a subgroup of \(G\) containing \(X\). By Lemma 3.2, \(X\) is closed and discrete in \(H^\#\). Since \(g \in H\), we can find an open neighbourhood \(U\) of \(g\) in \(H^\#\) such that \(U \cap X \subseteq \{g\}\). Since the subspace topology on \(H\) coincides with its Bohr topology by our assumption on \(G\), we conclude that \(H^\#\) is a subspace of the space \(G\). Thus, there exists an open subset \(V\) of \(G\) such that \(V \cap H = U\). Since \(X \subseteq H\), we get \(V \cap X = V \cap H \cap X = U \cap X \subseteq \{g\}\). Since the set \(X\) is faithfully indexed by Claim 4.1 this means that \(g\) cannot be an accumulation point of the sequence \(\{x_n : n \in \mathbb{N}\}\). Since \(g \in G\) was taken arbitrarily, this means that the sequence \(\{x_n : n \in \mathbb{N}\}\) does not have an accumulation point in \(G\). \(\square\)

4. Applications

**Definition 4.1.** [7] A subgroup \(H\) of a topological group \(G\) is said to be \(h\)-embedded in \(G\) if every homomorphism from \(H\) to the circle group \(\mathbb{T}\) is a restriction of some continuous group homomorphism from \(G\) to \(\mathbb{T}\).

The following fact is folklore and easy to prove.

**Fact 4.2.** Suppose that a subgroup \(H\) of a precompact abelian group \(G\) is \(h\)-embedded in \(G\). Then the subgroup topology \(H\) inherits from \(G\) coincides with the Bohr topology of \(H\).

From this fact and Theorem 3.3 one obtains the following

**Corollary 4.3.** Let \(G\) be a precompact Boolean group all countable subgroups of which are \(h\)-embedded in \(G\). Then \(G\) is not selectively pseudocompact.

For an abelian topological group \(G\), we denote by \(G^\wedge\) the group of all continuous group homomorphisms from \(G\) to \(\mathbb{T}\) endowed with the compact-open topology. Recall that \(G^\wedge\) is called the Pontryagin dual of \(G\). For each \(g \in G\), the map \(\psi_g : G^\wedge \to \mathbb{T}\) defined by \(\psi_g(h) = h(g)\) for every \(\psi_g \in G^\wedge\); that is, \(\psi_g\) is an element of the second dual \(G^{\wedge\wedge}\) of \(G\). Consider the map \(\alpha_G : G \to G^{\wedge\wedge}\) defined by \(\alpha_G(g) = \psi_g\) for \(g \in G\). The group \(G\) is called (Pontryagin) reflexive when \(\alpha_G\) is a topological isomorphism between \(G\) and \(G^{\wedge\wedge}\).

**Corollary 4.4.** A pseudocompact Boolean group \(G\) admits a pseudocompact reflexive group topology which is not selectively pseudocompact in each of the following cases:

(i) \(|G| \leq 2^{2^{\aleph_0}}\);
(ii) the Singular Cardinal Hypothesis SCH holds.

**Proof.** By [3] Corollary 5.6 and Theorem 5.8, \(G\) admits a pseudocompact group topology \(\tau\) such that every countable subgroup of \((G, \tau)\) is \(h\)-embedded in it (topological groups with this property are said to have property \(\mathcal{F}\) in [3] Definition 2.2). Since pseudocompact groups are precompact, it follows from Corollary 4.3 that \((G, \tau)\) is not selectively pseudocompact. Finally, the reflexivity of \((G, \tau)\) follows from [3] Lemma 2.3 and Theorem 6.1. \(\square\)

It follows from this corollary that, under the assumption of SCH, every pseudocompact Boolean group \(G\) can be equipped with a (reflexive) pseudocompact group topology which fails to be selectively pseudocompact. Moreover, for “small” groups \(G\) the assumption of SCH is superfluous.

Note that SCH is “rather mild” additional set-theoretic assumption beyond ZFC, the Zermelo-Fraenkel axioms of set theory augmented by the Axiom of Choice. Indeed, the failure of SCH implies the existence of a large cardinal [1].
An abelian topological group $G$ is called \textit{self-dual} if $G$ is topologically isomorphic to its Pontryagin dual $G^\wedge$.

**Corollary 4.5.** A \textit{self-dual} pseudocompact Boolean group cannot be selectively pseudocompact.

\textit{Proof.} Let $G$ be a self-dual pseudocompact Boolean group. By \cite[Theorem 2.3]{8}, all countable subgroups of $G$ are $h$-embedded in $G$. Since pseudocompact groups are precompact, $G$ is not selectively pseudocompact by Corollary 4.3. \hfill \Box

We refer the reader to \cite{8} for the definition of strong self-duality. By \cite[Proposition 2.2]{8}, strongly self-dual abelian topological groups are reflexive.

Finally, we recall the result of Tkachenko shown in \cite[Theorem 3.3]{8}.

**Theorem 4.6.** Let $\kappa$ be an infinite cardinal with $\kappa^{\omega} = \kappa$. Then there exists a pseudocompact strongly self-dual Boolean group $G$ satisfying $|G| = w(G) = \kappa$.

Combining Corollary 4.5 and Theorem 4.6, we obtain the following

**Corollary 4.7.** For every infinite cardinal $\kappa$ satisfying $\kappa^{\omega} = \kappa$, there exists a pseudocompact non-selectively pseudocompact strongly self-dual Boolean group $G$ such that $|G| = w(G) = \kappa$.

**Corollary 4.8.** A strongly self-dual pseudocompact Boolean group need not be selectively pseudocompact.

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