AN ELEMENTARY PROOF OF RAMANUJAN’S FORMULA FOR $\zeta(2m+1)$

SARTH CHAVAN

Abstract. In this article, we present an elementary proof of a special case of Ramanujan’s formula for $\zeta(2m+1)$ in terms of hyperbolic cotangent sums using Mittag-Leffler expansion of hyperbolic cotangent and Euler’s formula for $\zeta(2m)$. We also show that our approach produces an elementary proof of the more involved Ramanujan’s formula for $\zeta(2m+1)$.

1. A Brief History and Introduction

It’s well known that many number theoretic properties of $\zeta(2n+1)$ are nowadays still unsolved mysteries, such as the rationality, transcendence and existence of some closed-form functional equation that are satisfied by $\zeta(2n+1)$. Only in 1978 did Apéry [6] famously proved that $\zeta(3)$ is irrational. This was later reproved in a variety of ways by several authors, in particular Beukers [7] who devised a simple approach involving certain intergrals over $[0,1]^3$. The reader should consult Fischler’s very informative Bourbaki Seminar [8] for more details and references. In the early 2000s, an important work of Rivoal [9] and Ball and Rivoal [10] determined that infinitely many values of $\zeta$ at odd integers are irrational, and the work of Zudilin [11] proved that at least one among $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational. Despite of these advances, to this day no value of $\zeta(2n+1)$ with $n \geq 2$ is known to be irrational.

One should mention that Brown [12] has in the past few years outlined a geometric approach to understand the structures involved in Beukers’s proof of the irrationality $\zeta(3)$ and how this may generalize to other zeta values, see also the recent work of Dupont [13] on it.

Ramanujan made many beautiful and elegant discoveries in his short life of 32 years. One of the most remarkable formulas suggested by Ramanujan is the following intriguing identity involving the odd values of the Riemann zeta function [2, 1.2]:

**Theorem 1.1** (Ramanujan’s Formula for $\zeta(2m+1)$). If $\alpha$ and $\beta$ are positive numbers such that $\alpha\beta = \pi^2$ and if $m$ is a positive integer, then we have

$$\alpha^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{+\infty} \frac{1}{n^{2m+1} (e^{2\alpha n} - 1)} \right\} - (-\beta)^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{+\infty} \frac{1}{n^{2m+1} (e^{2\alpha n} - 1)} \right\}$$

$$= 2^m \sum_{k=0}^{m+1} (-1)^{k-1} \frac{B_{2k} B_{2m-2k+2}}{(2k)! (2m-2k+2)!} \alpha^{m-k+1} \beta^k,$$

where $B_m$ denotes the $m$-th Bernoulli number and $\zeta$ represents the Riemann zeta function.

This theorem appears as Entry 21 in Chapter 14 of Ramanujan’s second notebook [1, 173]. It also appears in a formerly unpublished manuscript of Ramanujan that was published in its original handwritten form with his lost notebook [4, formula (28), page 318–322].

Key words and phrases. Riemann zeta function, Ramanujan’s Formula for $\zeta(2m+1)$.  

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The first proof of Theorem 1.1 is due to S.L. Marulkar [14]. B. Berndt and A. Straub in their very recent article [2, 3.1] show that identity (1.1) is equivalent to

\[
\alpha - m \sum_{n=1}^{+\infty} \frac{\coth(\alpha n)}{n^{2m+1}} + \beta \sum_{n=1}^{+\infty} \frac{\coth(\beta n)}{n^{2m+1}} = -2^m \sum_{k=0}^{m+1} \frac{(-1)^{k-1} B_{2k} B_{2m+2-2k}}{(2k)! (2m+2-2k)!} \alpha^{m+1-k} \beta^k
\]

which is also known as Ramanujan’s formula for \( \zeta(2m+1) \) in terms of hyperbolic cotangent sums. Letting \( \alpha = \beta = \pi \) and replacing \( m \) with \( 2m + 1 \), we get the following [2, 3.2]

**Proposition 1.2** (Special case of Ramanujan’s formula for \( \zeta(2m+1) \)).

\[
\sum_{n=1}^{+\infty} \frac{\coth(\pi n)}{n^{4m+3}} = 2 (2\pi)^{4m+3} \sum_{k=0}^{m+2} \frac{(-1)^{k+1} B_{2k} B_{4m+4-2k}}{4 (2k)! (4m+4-2k)!}
\]

where as before \( m \in \mathbb{N} \) and \( B_n \) denotes the \( n \)-th Bernoulli number.

This variation was apparently first established by M. Lerch [15]. Later proofs were given by G.N. Watson [23], H.F. Sandham [21], J.R. Smart [18] and F.P. Sayer [20].

In the present work, we present an elementary proof of theorem 1.1 and proposition 1.2. Our proof of proposition 1.2 solely relies on the Mittag–Leffler expansion of hyperbolic cotangent and Euler’s formula for \( \zeta(2m) \) whereas in the proof of theorem 1.1, we make use of an identity involving the quasisymmetric zeta function. Let us first walk through the preliminaries.

2. Preliminaries

**Proposition 2.1** (Euler’s formula for \( \zeta(2m) \)). Let \( B_m \) denote the \( m \)-th Bernoulli number, then for \( m \geq 1 \), we have

\[
\zeta(2m) = (2\pi)^{2m} \frac{(-1)^{m-1} B_{2m}}{2 (2m)!}.
\]

Euler’s formula for \( \zeta(2m) \) not only provides an elegant formula for evaluating \( \zeta(2m) \) for any integer \( m \geq 1 \), but it also gives us information about the arithmetical nature of \( \zeta(2m) \). For a fascinating account of the history and proof of this formula, we refer the reader to [19].

**Proposition 2.2.** Let \( \alpha \in \mathbb{R}^+ \) and \( k \in \mathbb{N} \), then we have the Mittag–Leffler expansion

\[
\frac{1}{e^{\alpha k} - 1} = -\frac{1}{2} + \frac{1}{2 \alpha k} + \sum_{m=1}^{+\infty} \frac{\alpha k}{\pi^2 m^2 + \alpha^2 k^2}.
\]

**Proof.** Recall the Fourier series expansion of \( \cos(\alpha x) \) [17, 5.4.5.2] where \( \alpha \notin \mathbb{Z} \) and \( x \in \mathbb{R} \):

\[
\cos(\alpha x) = \frac{\sin(\alpha \pi)}{\alpha \pi} \left( 1 + 2 \alpha^2 \sum_{m=1}^{+\infty} \frac{(-1)^{m-1} \cos(mx)}{m^2 - \alpha^2} \right)
\]

Substituting \( \alpha \to i\alpha k/\pi \) and \( x \to \pi \) everywhere in identity (2.3) we find that

\[
\cos(i\alpha k) = \frac{\sin(i\alpha k)}{i\alpha k} \left( 1 - 2 \alpha^2 k^2 \sum_{m=1}^{+\infty} \frac{1}{\pi^2 m^2 + \alpha^2 k^2} \right) = \frac{\sin(i\alpha k)}{i\alpha k} - 2 \frac{\sin(i\alpha k)}{i} \sum_{m=1}^{+\infty} \frac{\alpha k}{\pi^2 m^2 + \alpha^2 k^2}.
\]

Expanding \( \cos(i\alpha k), \sin(i\alpha k) \) and simplifying the right-hand side produces the desired result.

\[\blacksquare\]
Corollary 2.3 (Mittag-Leffler expansion of hyperbolic cotangent). Let $z \in \mathbb{C} \setminus \{0\}$, then

\begin{equation}
\frac{\pi \coth(\pi z)}{2z} - \frac{1}{2z^2} = \sum_{k=1}^{\infty} \frac{1}{z^2 + k^2}.
\end{equation}

This is a direct consequence of proposition 2.2.

\section{Special Case of Ramanujan’s Formula for $\zeta(2m+1)$}

\textbf{Theorem 3.1} (Special case of Ramanujan’s formula for $\zeta(2m+1)$).

\begin{equation}
\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{4m+3}} = 2 \left(2\pi\right)^{4m+3} \sum_{k=0}^{2m+2} (-1)^{k+1} \frac{B_{2k} B_{4m+4-2k}}{4(2k)! (4m + 4 - 2k)!}
\end{equation}

where $B_n$ denotes the $n$-th Bernoulli number and $m \in \mathbb{N}$.

\textbf{Proof.} Replacing $z \mapsto n$ and dividing both sides of corollary 2.3 by $n^{4m+2}$ and summing over $n$ produces

\[
\frac{1}{2} \sum_{n=1}^{\infty} \frac{\pi \coth(\pi n)}{n^{4m+3}} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{4m+4}} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{4m+2} (k^2 + n^2)}.
\]

Since the above double sum is absolutely convergent, we may switch the order of summation to get

\[
\frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{\pi \coth(\pi n)}{n^{4m+3}} - \sum_{n=1}^{\infty} \frac{1}{n^{4m+4}} \right) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} (k^2 + n^2)}.
\]

Next, we simply notice that

\[
\frac{1}{n^{4m+2} (k^2 + n^2)} = \frac{1}{n^{4m} k^2} \left( \frac{1}{n^2} - \frac{1}{k^2 + n^2} \right)
\]

so that

\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} k^2} - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m} k^2 (k^2 + n^2)} = \zeta(2) \zeta(4m+2) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} k^2 (k^2 + n^2)}.
\]

Similarly we have

\[
\frac{1}{n^{4m} k^2 (k^2 + n^2)} = \frac{n^2}{n^{4m} k^4} \left( \frac{1}{n^2} - \frac{1}{k^2 + n^2} \right)
\]

so that

\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m} k^4} - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-2} k^4 (k^2 + n^2)} = \zeta(4) \zeta(4m) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-2} k^4 (k^2 + n^2)}.
\]

It turns out that this recursive pattern that produces even zeta values continues as follows:

\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-2} k^4 (k^2 + n^2)} - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-4} k^6 (k^2 + n^2)} = \zeta(6) \zeta(4m - 2) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-4} k^6 (k^2 + n^2)}
\]

\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-4} k^6 (k^2 + n^2)} - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-6} k^8 (k^2 + n^2)} = \zeta(8) \zeta(4m - 4) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m-6} k^8 (k^2 + n^2)}
\]

and so on.
and so on. At the end, we get
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^2 k^{4m} (k^2 + n^2)} = \zeta(4m + 2) \zeta(2) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{4m+2} (k^2 + n^2)}.
\]

Therefore we obtain the final result
\[
(3.2) \quad \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} (k^2 + n^2)} = \zeta(2) \zeta(4m + 2) - \zeta(4) \zeta(4m) + \cdots + \zeta(4m + 2) \zeta(2)
\]

which can be easily proved by induction on \(m\). Next, we observe that, by symmetry,
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} (k^2 + n^2)} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{4m+2} (k^2 + n^2)}.
\]

Therefore we have
\[
2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{4m+2} (k^2 + n^2)} = (-1)^m \zeta(2m + 2) \zeta(2m + 2) + 2 \sum_{k=0}^{m-1} (-1)^k \zeta(4m + 2 - 2k) \zeta(2k + 2).
\]

Putting all things together produces
\[
\sum_{n=1}^{\infty} \frac{\pi \coth(\pi n)}{n^{4m+3}} = \zeta(4m + 4) + (-1)^m \zeta(2m + 2) \zeta(2m + 2) + 2 \sum_{k=1}^{m} (-1)^{k-1} \zeta(4m - 2k + 4) \zeta(2k)
\]

where we have replaced \(k\) by \(k + 1\) in the summand. Thus it suffices to prove that
\[
\zeta(4m + 4) + (-1)^m \zeta(2m + 2) \zeta(2m + 2) + 2 \sum_{k=1}^{m} (-1)^{k-1} \zeta(4m - 2k + 4) \zeta(2k)
\]

\[
= (2\pi)^{4m+4} \sum_{k=0}^{2m+2} (-1)^{k-1} \frac{B_{2k} B_{4m+4-2k}}{4 (2k)! (4m + 4 - 2k)!}
\]

After replacing \(m\) with \(m - 1\) everywhere in the above equation, it suffices to prove that
\[
(3.3) \quad \zeta(4m) + (-1)^{m-1} \zeta(2m) \zeta(2m) + 2 \sum_{k=1}^{m-1} (-1)^{k-1} \zeta(4m - 2k) \zeta(2k)
\]

\[
= (2\pi)^{4m} \sum_{k=0}^{2m} (-1)^{k-1} \frac{B_{2k} B_{4m-2k}}{4 (2k)! (4m - 2k)!}
\]

which is indeed just a matter of substitution. Using Euler’s formula for \(\zeta(2m)\) (2.1) we get
\[
\zeta(4m) + (-1)^{m-1} \zeta(2m) \zeta(2m) + 2 \sum_{k=1}^{m-1} (-1)^{k-1} \zeta(4m - 2k) \zeta(2k)
\]

\[
(3.4) \quad = (2\pi)^{4m} \left\{ - \frac{B_{4m}}{2 (4m)!} + (-1)^{m-1} \frac{B_{2m} B_{2m}}{4 (2m)! (2m)!} + 2 \sum_{k=1}^{m-1} (-1)^{k-1} \frac{B_{2k} B_{4m-2k}}{4 (2k)! (4m - 2k)!} \right\}
\]
The summand in equation (3.3) is invariant under the substitution \( k \mapsto 2m - k \), so if we split the summand as \( k \in \{0, 2m\}, k \in \{1, 2, \ldots, m - 1, m + 1, \ldots, 2m - 1\} \) and \( k = m \), we get

\[
\sum_{k=0}^{2m} \frac{(-1)^{k+1} B_{2k} B_{4m-2k}}{4(2k)! (4m - 2k)!} = \frac{B_{4m}}{2(4m)!} + \frac{(-1)^{m-1} B_{2m} B_{2m}}{4(2m)! (2m)!} + 2 \sum_{k=1}^{m-1} \frac{(-1)^{k-1} B_{2k} B_{4m-2k}}{4(2k)! (4m - 2k)!}
\]

Multiplying both sides of the above expression by \((2\pi)^4\) we get

\[
(2\pi)^4 \sum_{k=0}^{2m} \frac{(-1)^{k+1} B_{2k} B_{4m-2k}}{4(2k)! (4m - 2k)!} = (2\pi)^4 \left\{ -\frac{B_{4m}}{2(4m)!} + \frac{(-1)^{m-1} B_{2m} B_{2m}}{4(2m)! (2m)!} + 2 \sum_{k=1}^{m-1} \frac{(-1)^{k-1} B_{2k} B_{4m-2k}}{4(2k)! (4m - 2k)!} \right\}
\]

(3.5)

Combining equations (3.3), (3.4) and (3.5) we get the desired result, completing the proof.  

4. Proof of Theorem 1.1

Dividing both sides of proposition 2.2 by \( \alpha^n k^{2n+1} \) and summing over \( k \) produces

\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2nk} - 1)} + \frac{1}{2} \zeta(2n+1) \right\} = \frac{\zeta(2n+2)}{2\alpha^{n+1}} + \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\alpha^2 k^2)^n (\pi^2 m^2 + \alpha^2 k^2)}
\]

where we have simply used the fact that

\[
\alpha^{-n} \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{k^{2n} (\pi^2 m^2 + \alpha^2 k^2)} = \alpha^{n+1} \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\alpha^2 k^2)^n (\pi^2 m^2 + \alpha^2 k^2)}.
\]

Next we notice that, for two arbitrary and non-vanishing sequences \( \{x_k\} \) and \( \{z_m\} \), we have

\[
\sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{x_k (x_k + z_m)} = \sum_{k=1}^{+\infty} \frac{1}{x_k} \sum_{m=1}^{+\infty} \frac{1}{z_m} - \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{z_m (x_k + z_m)},
\]

assuming that the sums involved are convergent. Moreover, for an arbitrary positive integer \( n \) such that the considered sums converge, this identity can be easily generalized to

(4.1)

\[
\sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{x_k^n (x_k + z_m)} = \sum_{k=1}^{+\infty} \frac{1}{x_k^n} \sum_{m=1}^{+\infty} \frac{1}{z_m} - \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{z_m^n (x_k + z_m)}
\]

Using the same recursive technique as in the previous section, we get

(4.2)

\[
\sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{x_k^n (x_k + z_m)} = \sum_{k=1}^{+\infty} \frac{1}{x_k^n} \sum_{m=1}^{+\infty} \frac{1}{z_m} - \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{z_m^n (x_k + z_m)}
\]

\[
+ \sum_{k=1}^{+\infty} \frac{1}{x_k^n} \sum_{m=1}^{+\infty} \frac{1}{z_m^3} + \cdots + (-1)^{n-1} \sum_{k=1}^{+\infty} \frac{1}{x_k} \sum_{m=1}^{+\infty} \frac{1}{z_m^n} + (-1)^n \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{z_m^n (x_k + z_m)}.
\]

For notational convenience, let

\[
\sum_{k=1}^{+\infty} \frac{1}{x_k^n} = \zeta_x(n), \quad \sum_{k=1}^{+\infty} \frac{1}{z_m^n} = \zeta_z(n).
\]
These are also known as *quasisymmetric zeta function* (the correct analog of a Riemann zeta function in the ring of quasisymmetric functions). Identity (4.2) can now be rewritten as

\[
(4.3) \quad \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{x_k^n (x_k + z_m)} = \sum_{p=0}^{n-1} (-1)^p \zeta(n - p) \zeta(p + 1) + (-1)^n \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{z_m^n (x_k + z_m)}
\]

Substituting \(x_k \mapsto \alpha^2 k^2\) and \(z_m \mapsto \pi^2 m^2\) in identity (4.3) produces

\[
(4.4) \quad \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\alpha^2 k^2)^n (\alpha^2 k^2 + \pi^2 m^2)} = \sum_{p=0}^{n-1} (-1)^p \frac{\zeta(2n - 2p)}{\alpha^{2n-2p}} \frac{\zeta(2p + 2)}{\pi^{2p+2}} + (-1)^n \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\pi^2 m^2)^n (\alpha^2 k^2 + \pi^2 m^2)}.
\]

Euler’s formula for \(\zeta(2m)\) (2.1) allows us to transform the zetas into Bernoullis as follows

\[
\sum_{p=0}^{n-1} (-1)^p \frac{\zeta(2n - 2p)}{\alpha^{2n-2p}} \frac{\zeta(2p + 2)}{\pi^{2p+2}} = (-1)^n 2^{2n} \sum_{p=0}^{n-1} (-1)^p \left( \frac{\pi}{\alpha} \right)^{2n-2p} \frac{B_{2n-2p}}{(2n - 2p)!} \frac{B_{2p}}{(2p + 2)!}.
\]

Moreover, using \(\alpha \beta = \pi^2\) and replacing \(p \mapsto p + 1\) everywhere in the summand produces

\[
\sum_{p=0}^{n-1} (-1)^p \frac{\zeta(2n - 2p)}{\alpha^{2n-2p}} \frac{\zeta(2p + 2)}{\pi^{2p+2}} = (-1)^n 2^{2n} \sum_{p=1}^{n} (-1)^p \left( \frac{\beta}{\alpha} \right)^{n-p+1} \frac{B_{2n-2p+2}}{(2n - 2p + 2)!} \frac{B_{2p}}{(2p)!}.
\]

Substituting this expression in identity (4.4) we get

\[
\sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\alpha^2 k^2)^n (\alpha^2 k^2 + \pi^2 m^2)} = (-1)^n 2^{2n} \sum_{p=1}^{n} (-1)^p \left( \frac{\beta}{\alpha} \right)^{n-p+1} \frac{B_{2n-2p+2}}{(2n - 2p + 2)!} \frac{B_{2p}}{(2p)!} + (-1)^n \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\pi^2 m^2)^n (\alpha^2 k^2 + \pi^2 m^2)}.
\]

Putting all things together produces

\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2ak} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\} = (-1)^n \alpha^{n+1} \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{(\pi^2 m^2)^n (\alpha^2 k^2 + \pi^2 m^2)}
\]

\[
+ \frac{\zeta(2n + 2)}{2\alpha^{n+1}} + \alpha^{n+1} \left\{ (-1)^n 2^{2n} \sum_{p=1}^{n} (-1)^p \left( \frac{\beta}{\alpha} \right)^{n-p+1} \frac{B_{2n-2p+2} B_{2p}}{(2n - 2p + 2)!} (2p)! \right\}.
\]

Next, we notice that the remaining zeta term

\[
\frac{\zeta(2n + 2)}{2\alpha^{n+1}} = \frac{1}{\alpha^{n+1}} \frac{(-1)^n (2\pi)^{2n+2} B_{2n+2}}{2 (2n + 2)!} = (-1)^n 2^{2n} \beta^{n+1} \frac{B_{2n+2}}{(2n + 2)!}
\]

is the missing \(p = 0\)-th term in the summand. Putting all things together produces

\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2ak} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\} - \alpha^{n+1} \left\{ \frac{(-1)^n}{\pi^2 n} \sum_{k=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{1}{m^{2n} (\alpha^2 k^2 + m^2 \pi^2)} \right\}
\]
Now notice that, using $\alpha \beta = \pi^2$ we obtain
\[
\sum_{k=1}^{+\infty} \frac{1}{m^{2n} (\alpha^2 k^2 + m^2 \pi^2)} = \frac{1}{\alpha} \sum_{k=1}^{+\infty} \frac{1}{m^{2n} (\alpha k^2 + \beta m^2)} = \frac{1}{\alpha} \sum_{k=1}^{+\infty} \frac{1}{k^{2n} (\alpha^2 m^2 + \beta k^2)},
\]
where the last equality is obtained simply by replacing $k$ by $m$ and $m$ by $k$ so that we get
\[
\sum_{k=1}^{+\infty} \frac{1}{m^{2n} (\alpha^2 k^2 + m^2 \pi^2)} = \frac{\beta}{\alpha} \sum_{k=1}^{+\infty} \frac{1}{k^{2n} (\beta^2 k^2 + m^2 \pi^2)}
\]
Dividing both sides of Proposition 2 by $k^{2n+1}$ and summing over $k$ produces
\[
\beta \sum_{k=1}^{+\infty} \frac{1}{k^{2n} (\beta^2 k^2 + m^2 \pi^2)} = \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\beta k} - 1)} + \frac{1}{2} \zeta(2n + 1) - \frac{1}{2\beta} \zeta(2n + 2)
\]
where we have replaced $\alpha$ with $\beta$. Therefore we deduce that
\[
\frac{\beta}{\alpha} \sum_{k=1}^{+\infty} \frac{1}{k^{2n} (\beta^2 k^2 + m^2 \pi^2)} = \frac{1}{\alpha} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\beta k} - 1)} + \frac{1}{2} \zeta(2n + 1) - \frac{1}{2\beta} \zeta(2n + 2) \right\}.
\]
Finally substituting this expression in identity (4.6) we get
\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\alpha k} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\} - (-1)^n 2^{2n} \sum_{p=0}^{n} (-1)^p \frac{B_{2n-2p+2} B_{2p}}{(2n - 2p + 2)! (2p)!} \alpha^p \beta^{n-p+1}
\]
\[
= \frac{(-1)^n \alpha^{n+1} (-1)^n 2^{2n+2} \pi^{2n+2} B_{2n+2}}{4 (2n + 2)!} = (-1)^{2n+1} 2^{2n} \frac{B_{2n+2}}{(2n + 2)!} \alpha^{n+1}
\]
is the $p = n+1$–st term in the summand. Therefore we deduce that
\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\alpha k} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\} - (-\beta)^{-n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\beta k} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\}
\]
\[
= (-1)^n 2^{2n} \sum_{p=0}^{n+1} (-1)^p \frac{B_{2n-2p+2} B_{2p}}{(2n - 2p + 2)! (2p)!} \alpha^p \beta^{n-p+1}.
\]
Replacing $p$ with $n - p + 1$ in the above summand produces
\[
\frac{1}{\alpha^n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\alpha k} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\} - (-\beta)^{-n} \left\{ \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1} (e^{2\beta k} - 1)} + \frac{1}{2} \zeta(2n + 1) \right\}
\]
\[ 2^{2n} \sum_{p=0}^{n+1} \left( -1 \right)^{p-1} \frac{B_{2n-2p+2} B_{2p}}{(2n-2p+2)! (2p)!} \alpha^{n-p+1} \beta^p. \]

as desired. This completes the proof of Ramanujan’s formula for \( \zeta(2m+1) \).

5. Acknowledgements

The author would like to thank Christophe Vignat for his guidance and support throughout the completion of this work and for taking the time to read a draft of this paper. The author would also like to thank Bruce Berndt and Atul Dixit for endorsing him on arXiv.

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