Diffractive Nonlinear Geometrical Optics
for Variational Wave Equations
and the Einstein Equations

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Abstract

We derive an asymptotic solution of the vacuum Einstein equations
that describes the propagation and diffraction of a localized, large-
amplitude, rapidly-varying gravitational wave. We compare and con-
trast the resulting theory of strongly nonlinear geometrical optics for
the Einstein equations with nonlinear geometrical optics theories for
variational wave equations.

1 Introduction

Geometrical optics\(^1\) and its generalizations, such as the geometrical theory
of diffraction, are a powerful approach to the study of wave propagation,
for both linear and nonlinear waves. In this paper, we develop a theory
of strongly nonlinear geometrical optics for gravitational wave solutions of
the vacuum Einstein equations. Specifically, we derive asymptotic equa-
tions that describe the diffraction of large-amplitude, rapidly-varying grav-
itational waves.

\(^1\)Here, we use the term ‘geometrical optics’ to refer to any asymptotic theory for the
propagation of short-wavelength, high-frequency waves, irrespective of its area of physical
application.
These equations are a generalization of the straightforward non-diffractive, nonlinear geometrical optics equations for large-amplitude gravitational waves derived in \cite{2}. This strongly nonlinear theory differs fundamentally from the weakly nonlinear theory for small-amplitude gravitational waves obtained by Choquet-Bruhat \cite{4} and Isaacson \cite{15}, because it captures the direct nonlinear self-interaction of the waves.

The Einstein equations may be derived from a variational principle, and, when written with respect to a suitable gauge, they form a system of wave equations for the gravitational field. In order to explain the structure of nonlinear geometrical optics theories for gravitational waves, and to motivate the form of our asymptotic expansion, it is useful to consider first such theories for a general class of variational wave equations.

We describe straightforward and diffractive geometrical optics theories for variational wave equations in Section 2. In the weakly nonlinear theory for waves with periodic waveforms, the amplitude-waveform function $a(\theta, v)$ of the wave depends on a ray variable $v$ and periodically on a ‘fast’ phase variable $\theta$. The wave-amplitude $a$ satisfies the Hunter-Saxton equation (2.12),

\[
\left\{ a_v + \left( \frac{1}{2} \Lambda a^2 \right)_\theta + Na \right\}_\theta = \frac{1}{2} \Lambda \left\{ a^2_\theta - \langle a^2_\theta \rangle \right\},
\]

where the angular brackets denote an average with respect to $\theta$. The coefficient $\Lambda$ of the nonlinear terms in (1.1) may be interpreted as a derivative of the wave speed with respect to the wave amplitude. In addition, the wave-amplitude is coupled with a slowly-varying mean field.

An application of this expansion to the Einstein equations leads to the theory of Choquet-Bruhat \cite{4} and Isaacson \cite{15}. In that case, the nonlinear coefficients corresponding to $\Lambda$ in the equations for the amplitudes of the gravitational waves are identically zero. This theory therefore describes a nonlinear interaction between a high-frequency, oscillatory gravitational wave and a slowly-varying mean gravitational field, but does not describe the direct nonlinear self-interaction of the gravitational wave itself.

In Section 2.4, we make a distinction between ‘genuinely nonlinear’ wave-fields in variational wave equations, for which $\Lambda$ is never zero, and ‘linearly degenerate’ wave-fields for which $\Lambda$ identically zero (see Definition 1). All wave-fields in the Einstein equations are linearly degenerate. As observed in \cite{6,7,9,11,21}, for example, this fact reflects a fundamental degeneracy in the nonlinearity of the Einstein equations in comparison with general wave equations.
A property of the Einstein equations related to their linear degeneracy is that they possess an exact solution for non-distorting, large-amplitude, plane waves, the Brinkman solution \[3, 19\]. One can use this solution to derive a strongly nonlinear geometrical optics theory for large-amplitude gravitational waves. We outline the resulting non-diffractive theory \[2\] in Section 2.5.

We derive our diffractive, strongly nonlinear geometrical optics solution of the vacuum Einstein equations in Section 3. This solution describes the propagation and diffraction of a thin, large-amplitude gravitational wave, such as a pulse or ‘sandwich’ wave. The simplest, and basic, case is that of a plane-polarized gravitational wave diffracting in a single direction. We summarize the resulting asymptotic equations here.

We suppose that the polarization of the wave is aligned with the diffraction direction. Then, with respect to a suitable coordinate system \((u, v, y, z)\), the metric \(g\) of the wave adopts the form

\[
g = -2e^{-M}(du - \varepsilon Y dy)dv + e^{-U}(e^V dy^2 + e^{-V}dz^2) + O(\varepsilon^2). \tag{1.2}\]

Here, \(\varepsilon\) is a small parameter. The leading-order metric component functions \((U, V, M)\), and the first-order function \(Y\) depend upon a ‘fast’ phase variable \(\theta\), an ‘intermediate’ transverse variable \(\eta\), and the ‘slow’ ray variable \(v\), where

\[
\theta = \frac{u}{\varepsilon^2}, \quad \eta = \frac{y}{\varepsilon}.
\]

The phase \(u\) is light-like and the transverse coordinate \(y\) is space-like.

This metric describes a gravitational wave whose wavefronts are close to the null-hypersurface \(u = 0\). The wave is plane-polarized in the \((y, z)\)-directions and diffracts in the \(y\)-direction.

To write equations for the metric component functions in a concise form, we define a derivative \(D_\eta\) and functions \(\phi, \psi\) by

\[
D_\eta = e^U(\partial_\eta + Y\partial_\theta), \quad \phi = D_\eta M - e^U Y\theta, \quad \psi = D_\eta(U + V). \tag{1.3, 1.4, 1.5}\]

We note that \(D_\eta\) has the geometrical significance that

\[
D_\eta = \frac{1}{\varepsilon}e^V(dy)\uparrow,
\]

where \(\uparrow\) denotes the ‘raising’ operator from one-forms to vector fields.
Then \((U, V, M, Y)\) satisfy the following system of PDEs:

\[
U_{\theta\theta} - \frac{1}{2} \left( U_\theta^2 + V_\theta^2 \right) + U_\theta M_\theta = 0, \tag{1.6}
\]

\[
(\phi + \psi)_\theta = \psi (U + V)_\theta, \tag{1.7}
\]

\[
U_{\theta v} - U_\theta U_v = \frac{1}{2} e^{-(U + V + M)} \left\{ D_\eta \phi + D_\eta \psi - \frac{1}{2} \phi^2 - \phi \psi - \frac{1}{2} \psi^2 \right\}, \tag{1.8}
\]

\[
V_{\theta v} - \frac{1}{2} \left( U_\theta V_v + U_v V_\theta \right) = \frac{1}{2} e^{-(U + V + M)} \left\{ -D_\eta \phi + \frac{1}{2} \phi^2 \right\}, \tag{1.9}
\]

\[
M_{\theta v} + \frac{1}{2} \left( U_\theta U_v - V_\theta V_v \right) = \frac{1}{2} e^{-(U + V + M)} \left\{ -D_\eta \psi - \frac{1}{2} \phi^2 + \psi^2 \right\}. \tag{1.10}
\]

This system is the main result of our analysis. It is an asymptotic reduction of the full vacuum Einstein equations to a \((1 + 2)\)-dimensional system of PDEs. The system provides a model nonlinear wave equation for general relativity, and should be useful, for example, in studying the focusing of nonplanar gravitational waves and the effect of diffraction on the formation of singularities. We plan to study these topics in future work.

We also derive asymptotic equations for the diffraction of a gravitational wave of general polarization in two transverse directions. These equations are given by (3.11)–(3.13), (C.1), and (C.3)–(C.5), but they are much more complicated than the ones for plane-polarized waves written out above.

We study the structure of (1.6)–(1.10) in Section 4. The constraint equation (1.6) is a nonlinear ODE in \(\theta\) in which \(\eta, v\) appear as parameters. As shown in Proposition 11, this constraint is preserved by the remaining equations. Equation (1.7) is a linear, nonhomogeneous ODE in \(\theta\) for \(Y\), whose coefficients depend on \(\theta\)- and \(\eta\)-derivatives of \((U, V, M)\). It may therefore be regarded as determining \(Y\) in terms of \((U, V, M)\). Equations (1.8)–(1.10) are a system of evolution equations for \((U, V, M)\) which are coupled with \(Y\). The main part of the system of evolution equations consists of a \((1+2)\)-dimensional wave equation in \((\theta, \eta, v)\) for \((U + V)\), in which \((\theta, v)\) are characteristic coordinates.

When \(Y = 0\) and all functions are independent of \(\eta\), equations (1.6)–(1.10) reduce to the colliding plane wave equations, without the \(v\)-constraint equation (see Section 2.5). When all functions depend on \((\xi, \eta)\) with \(\xi = \theta - \lambda v\) for some constant \(\lambda\), we get a system of PDEs in two variables that is studied further in 11. This system describes space-times that are stationary with respect to an observer moving close to the speed of light.

4
2 Geometrical optics

The Einstein equations do not form a hyperbolic system of PDEs because of their gauge-covariance, but their properties are closely related to those of hyperbolic PDEs. In order to develop and interpret geometrical optics solutions for the Einstein equations, it is useful to begin by studying geometrical optics solutions for hyperbolic systems of variational wave equations.

We remark that there is a close correspondence between geometrical optics theories for hyperbolic systems of conservation laws \cite{12} and variational wave equations. For example, the inviscid Burgers equation \cite{5, 13} is the analog of the Hunter-Saxton equation \cite{21, 4}, and the unsteady transonic small disturbance equation \cite{11} is the analog two-dimensional Hunter-Saxton equation \cite{2, 15}. One can also derive large-amplitude geometrical optics theories for linearly degenerate waves in hyperbolic conservation laws (see \cite{23, 24}, for example) that are analogous to the large-amplitude theories described here for the Einstein equations.

There are other nonlinear geometrical optics theories for dispersive waves, most notably Whitham’s ‘averaged Lagrangian method’ \cite{28, 29} for large-amplitude dispersive waves. This theory has a different character from the ones for nondispersive hyperbolic waves. Nonlinear dispersive waves have specific waveforms — given by traveling wave solutions — in which the effects of dispersion and nonlinearity balance, whereas nondispersive hyperbolic wave equations, and the Einstein equations, have traveling wave solutions with arbitrary waveforms, which may distort as the waves propagate.

In straightforward theories of geometrical optics, waves are locally approximated by plane waves and propagate along rays. In linear geometrical optics, the wave amplitude satisfies an ODE (the transport equation) along a ray; in nonlinear geometrical optics, the amplitude-waveform function typically satisfies a nonlinear PDE in one space dimension (a generalization of the transport equation) along a ray. This difference may be understood as follows: linear hyperbolic waves propagate without distortion, so one requires only an ODE along each ray to determine the change in the wave amplitude; by contrast, wave-steepening and other effects typically distort the waveform of nonlinear hyperbolic waves, so one requires a PDE along each ray to determine the change in the wave amplitude and the waveform.

Straightforward geometrical optics, and a local plane-wave approximation, break down when the effects of wave-diffraction become important; for example, this occurs when a high-frequency wave focuses at a caustic, or when a wave beam of large (relative to its wavelength), but finite, transverse...
extent spreads out. The effects of diffraction on a high-frequency wave may be described by the inclusion of additional length-scales in the straightforward geometrical optics asymptotic solution.

Perhaps the most basic asymptotic solution that incorporates the effect of wave diffraction leads to the ‘parabolic approximation’, described below for wave equations. In Section 3, we will derive analogous asymptotic solutions of the Einstein equations for the diffraction of large-amplitude gravitational waves.

2.1 The wave equation

We begin by recalling geometrical optics theories for the linear wave equation

\[ g_{tt} = \nabla \cdot (c_0^2 \nabla g) \]  \hspace{1cm} (2.1)

Here, \( g(t, x) \) is a scalar function, the wave speed \( c_0(t, x) \) is a given smooth function, and \( x \in \mathbb{R}^d \). Although well-known, these theories provide a useful background to our analysis of nonlinear variational wave equations and the Einstein equations.

We look for a short-wavelength asymptotic solution \( g = g^\varepsilon \) of (2.1), depending on a small parameter \( \varepsilon \), of the form

\[ g^\varepsilon(t, x) \sim a \left( \frac{u(t, x)}{\varepsilon^2}, t, x \right) \quad \text{as} \quad \varepsilon \to 0. \]

The solution depends on a ‘fast’ phase variable\(^2\)

\[ \theta = \frac{u}{\varepsilon^2} \]

and the ‘slow’ space-time variables \((t, x)\).

One finds that the scalar-valued phase function \( u(t, x) \) satisfies the eikonal equation

\[ u_t^2 = c_0^2 |\nabla u|^2. \]  \hspace{1cm} (2.2)

The amplitude-waveform function \( a(\theta, t, x) \) satisfies the equation

\[ a_{\theta\nu} + Na_\theta = 0, \]  \hspace{1cm} (2.3)

where

\[ \partial_{\nu} = u_t \partial_t - c_0^2 \nabla u \cdot \nabla \]  \hspace{1cm} (2.4)

\(^2\)Here, we use \( u/\varepsilon^2 \) as a phase variable, rather than \( u/\varepsilon \), for consistency with the diffractive expansion below.
is a derivative along the rays associated with the phase \( u \), and \( N(t, x) \) is
given by

\[
N = \frac{1}{2} \left\{ u_{tt} - \nabla \cdot \left( c_0^2 \nabla u \right) \right\}.
\]  

Equation (2.3) has solutions of the form

\[
a(\theta, t, x) = A(t, x) F(\theta),
\]

where \( A(t, x) \) is a wave-amplitude, which satisfies an ODE along a ray (the transport equation of linear geometrical optics)

\[
A_v + NA = 0,
\]  

and \( F(\theta) \) is an arbitrary function that describes the waveform of the wave. For example, if \( F(\theta) = e^{i\theta} \), then the solution describes an oscillatory harmonic wave; if

\[
F(\theta) = \begin{cases} 
\theta^n & \theta > 0, \\
0 & \theta \leq 0,
\end{cases}
\]

then the solution describes a wavefront across which the normal derivative of \( u \) of order \( n \) jumps; and if \( F(\theta) \) has compact support, then the solution describes a localized pulse.

The term \( NA \) in the transport equation (2.6) describes the effect of the ray geometry on the wave amplitude. The coefficient \( N \) becomes infinite on caustics where the rays focus. The straightforward geometrical optics solution breaks down when this happens, and diffractive effects must then be taken into account.

There are many ways in which diffraction modifies straightforward geometrical optics. Here, we consider one of the simplest diffractive expansions, given by

\[
g^\varepsilon(t, x) \sim a \left( \frac{u(t, x)}{\varepsilon^2}, \frac{y(t, x)}{\varepsilon}, t, x \right) \quad \text{as } \varepsilon \to 0.
\]

This asymptotic solution depends upon an additional ‘intermediate’ variable

\[
\eta = \frac{y}{\varepsilon},
\]

where \( y(t, x) \) is a scalar-valued transverse phase.

One finds that \( u \) satisfies the eikonal equation, as before, and

\[
y_v = 0,
\]
meaning that \( y \) is constant along the rays associated with \( u \). Moreover, the amplitude-waveform function \( a(\theta, \eta, t, x) \) satisfies the equation

\[
a_{\theta v} + Na_{\theta} + \frac{1}{2} Da_{\eta\eta} = 0,
\]

(2.7)

where the coefficient \( D(t, x) \) of the diffractive term is given by

\[
D = y_t^2 - c_0^2 |\nabla y|^2.
\]

(2.8)

For harmonic solutions, we have

\[
a(\theta, \eta, t, x) = A(\eta, t, x)e^{i\theta},
\]

and equation (2.7) reduces to a Schrödinger equation

\[
i \{ A_v + NA \} + \frac{1}{2} DA_{\eta\eta} = 0.
\]

This ‘parabolic approximation’ and its generalizations are widely used in the study of wave propagation [27].

In the simplest case of the diffraction of plane wave solutions of the two-dimensional wave equation with wave speed \( c_0 = 1 \),

\[
g_{tt} = g_{xx} + g_{yy},
\]

we may choose

\[
u = \frac{t - x}{\sqrt{2}}, \quad y = y, \quad v = \frac{t + x}{\sqrt{2}}.
\]

Equation (2.7) is then

\[
a_{\theta v} = \frac{1}{2} a_{\eta\eta}.
\]

(2.9)

This equation describes waves that propagate in directions close to the \( x \)-direction, and is a wave equation in which \( \theta \) and \( v \) are characteristic coordinates. We will see that equations with a similar structure to (2.9) in their highest-order derivatives arise from the Einstein equations.

### 2.2 A variational wave equation

Next, we consider the following nonlinear, scalar wave equation [14]

\[
g_{tt} - \nabla \cdot (c^2(g)\nabla g) + c(g)c'(g)|\nabla g|^2 = 0.
\]

(2.10)
We assume that the wave speed \( c : \mathbb{R} \to \mathbb{R}^+ \) is a smooth, non-vanishing function, and a prime denotes the derivative with respect to \( g \). This equation is derived from the variational principle

\[
\delta \int \left\{ \frac{1}{2}g_t^2 - \frac{1}{2}c^2(g)|\nabla g|^2 \right\} \, dt dx = 0.
\]

The structure of the nonlinear terms in (2.10) resembles that of the Einstein equations, although, as we shall see, the effects of nonlinearity are qualitatively different because of a ‘linear degeneracy’ in the Einstein equations.

We look for an asymptotic solution of (2.10) of the form \([9, 12, 14]\)

\[ g^\varepsilon(t, x) \sim g_0(t, x) + \varepsilon^2 a\left(\frac{u(t, x)}{\varepsilon^2}, t, x\right) \quad \text{as} \quad \varepsilon \to 0. \]

This solution represents a small-amplitude, high-frequency perturbation of a slowly-varying field \( g_0 \). The amplitude and the wavelength of the perturbation are chosen to be of the same order of magnitude because this leads to a balance between the effects of weak nonlinearity and the ray geometry.

First, we suppose that the amplitude-waveform function \( a(\theta, t, x) \) is a periodic function of the phase variable \( \theta \). We assume, without loss of generality, that its mean with respect to \( \theta \) is zero.

We find that the phase \( u(t, x) \) satisfies the linearized eikonal equation (2.2), with \( c_0 = c(g_0) \). The mean-field \( g_0 \) satisfies the nonlinear wave equation

\[ g_{tt} - \nabla \cdot \left( c_0^2 \nabla g_0 \right) + c_0 c'_0 |\nabla g_0|^2 + |\nabla u|^2 c_0 c'_0 \langle a^2_\theta \rangle = 0, \quad (2.11) \]

where the angular brackets denote an average with respect to \( \theta \) over a period, and \( c'_0 = c'(g_0) \). Equation (2.11) has the same form as the original wave equation, with an additional source term proportional to the mean energy of the wave-field.

The amplitude-waveform function \( a(\theta, t, x) \) satisfies the periodic Hunter-Saxton equation (1.1),

\[
\left\{ a_v + \left( \frac{1}{2} \Lambda a^2 \right)_{\theta} + Na \right\}_{\theta} = \frac{1}{2} \Lambda \left\{ a^2_{\theta} - \langle a^2_\theta \rangle \right\},
\]

where \( \partial_v \) is the ray derivative defined in (2.4), \( N \) is given by (2.5), and

\[ \Lambda = -|\nabla u|^2 c_0 c'_0. \quad (2.13) \]

The coefficient \( \Lambda \) is proportional to the derivative of the wave-speed with respect to the wave amplitude, and it provides crucial information about
the effect of nonlinearity on the waves; for a given $g_0$, we obtain a nonlinear PDE for the amplitude-waveform function only if $\Lambda \neq 0$.

The expansion summarized above for waves with periodic waveforms is uniformly valid for $(t, x) = O(1)$ and $u = O(\varepsilon^{-1})$ as $\varepsilon \to 0$. If we consider localized waves, then we may obtain an asymptotic expansion that is valid near\(^3\) the wave-front $u = 0$, in which we neglect the mean-field effects. Thus, the background field $g_0(x)$ is a solution of the original wave equation, and $a(\theta, t, x)$ satisfies

\[
\left\{ a_v + \left( \frac{1}{2} \Lambda a^2 \right)_\theta + Na \right\}_\theta = \frac{1}{2} \Lambda a^2. \tag{2.14}
\]

It is straightforward to include diffractive effects in the expansion for localized waves. The asymptotic solution has the form

\[
g^\varepsilon(t, x) \sim g_0(x) + \varepsilon^2 a \left( \frac{u(t, x)}{\varepsilon^2}, \frac{y(t, x)}{\varepsilon}, t, x \right) \quad \text{as } \varepsilon \to 0.
\]

Here, $g_0$ is a solution of the original wave equation, the phase $u$ satisfies the eikonal equation at $g_0$, the transverse function $y$ is constant along the rays associated with $u$, and $a(\theta, \eta, t, x)$ satisfies a $(1 + 2)$-dimensional generalization of the Hunter-Saxton equation (2.14),

\[
\left\{ a_v + \left( \frac{1}{2} \Lambda a^2 \right)_\theta + Na \right\}_\theta + \frac{1}{2} D a_{\eta\eta} = \frac{1}{2} \Lambda a^2, \tag{2.15}
\]

where $D$ is given by (2.8).

### 2.3 Systems of variational wave equations

In this section, we consider a class of hyperbolic systems of nonlinear wave equations that are derived from variational principles of the form\(^{[14]}\)

\[
\delta \int A_{pq}^{\alpha\beta}(g) \frac{\partial g^p}{\partial x^\alpha} \frac{\partial g^q}{\partial x^\beta} \, dx = 0. \tag{2.16}
\]

Here, $x = (x^0, \ldots, x^d) \in \mathbb{R}^{d+1}$ are the space-time variables,

\[
g = (g^1, \ldots, g^m) : \mathbb{R}^{d+1} \to \mathbb{R}^m
\]

are the dependent variables, $A_{pq}^{\alpha\beta} : \mathbb{R}^m \to \mathbb{R}$ are smooth coefficient functions, and we use the summation convention. We assume that $A_{pq}^{\alpha\beta} = A_{qp}^{\alpha\beta} = A_{\alpha\beta}$.

\(^3\)Specifically, the expansion is valid when $\theta = O(1)$, or $u = O(\varepsilon^2)$, and $t = O(1)$.\]
The Euler-Lagrange equations associated with (2.16) are

\[ G_p [g] = 0, \]  \hspace{1cm} (2.17)

where

\[ G_p[g] = \frac{\partial}{\partial x^\alpha} \left\{ A_p^{\alpha \beta} (g) \frac{\partial g^q}{\partial x^{\beta'}} \right\} - \frac{1}{2} \frac{\partial A_{q r}^{\alpha \beta}}{\partial g^{\alpha'}} (g) \frac{\partial g^q}{\partial x^\alpha} \frac{\partial g^{\alpha'}}{\partial x^{\beta'}}. \]  \hspace{1cm} (2.18)

We assume that (2.17) forms a hyperbolic system of PDEs. The scalar wave equation considered in the previous section is the simplest representative of this class of equations.

The weakly nonlinear geometrical optics solution of (2.17) has the form \[ g^\varepsilon (x) \sim g_0 (x) + \varepsilon^2 a \left( \frac{u(x)}{\varepsilon^2}, x \right) R(x) \] as \( \varepsilon \to 0, \] \hspace{1cm} (2.19)

where \( g_0 : \mathbb{R}^{d+1} \to \mathbb{R}^m \) is a slowly-varying function, \( u : \mathbb{R}^{d+1} \to \mathbb{R} \) is a phase function, \( a : \mathbb{R} \times \mathbb{R}^{d+1} \to \mathbb{R} \) is an amplitude-waveform function, and \( R : \mathbb{R}^{d+1} \to \mathbb{R}^m \) is a suitable vector field. We find that the phase \( u \) satisfies the eikonal equation

\[ \det C = 0, \]  \hspace{1cm} (2.20)

where the \( m \times m \) matrix \( C(x) \) has components

\[ C_{pq} = u_{x^\alpha} u_{x^{\beta'}} A_{p q}^{\alpha \beta} (g_0). \]  \hspace{1cm} (2.21)

The vector \( R = (R^1, \ldots, R^m)^T \) in (2.19) is a right null-vector of \( C \), so that \( C_{pq} R^q = 0 \). Here, and in (2.19), we assume that we are dealing with a simple characteristic, meaning that the null-space of \( C \) is one-dimensional.

We write the scalar-valued amplitude-waveform function \( a(\theta, x) \) as a function of the ‘fast’ phase variable \( \theta = u/\varepsilon^2 \), and the slow variables \( x \). If \( a(\theta, x) \) is a periodic function of \( \theta \) with zero mean, then the mean-field \( g_0(x) \) satisfies the equation

\[ G_p [g_0] = \frac{1}{2} H_p \left\langle a_\theta^2 \right\rangle, \]  \hspace{1cm} (2.22)

where the angular brackets denote an average with respect to \( \theta \), and

\[ H_p = u_{x^\alpha} u_{x^{\beta'}} \frac{\partial A_{q r}^{\alpha \beta}}{\partial g^{\alpha'}} (g_0) R^q R^r. \]
Equation \([2.22]\) has the same form as the original equation for \(g\) with an additional source term proportional to \(\langle a_n^2 \rangle\).

The equation for \(a\) is \((2.12)\) with
\[
\partial_v = 2u_x A_{pq}^\alpha (g_0) R^p R^q \partial_x^\alpha,
\]
\[
\Lambda = u_x A_{pq}^\alpha \frac{\partial A_p^\alpha}{\partial g^q} (g_0) R^p R^q R^r,
\]
\[
N = \frac{\partial}{\partial x^\alpha} \left\{ u_x A_{pq}^\alpha (g_0) R^p R^q \right\} - u_x \frac{\partial A_p^\alpha}{\partial g^q} (g_0) \frac{\partial g^q}{\partial x^\alpha} R^p R^q R^r.
\]

For localized solutions, the mean-field interactions may be neglected, and we can obtain diffractive versions of this expansion as before.

These equations generalize easily to the case when equation \((2.17)\) has a multiple characteristic of constant multiplicity \(n \geq 2\), say. In that case, one obtains a mean-field equation whose source term is a sum of averages of products of \(\theta\)-derivatives of the wave amplitudes, and an \(n \times n\) coupled system of Hunter-Saxton equations for the wave amplitudes. In particular, if the vectors \(\{R_1, \ldots, R_n\}\) form a basis of the null-space of the matrix \(C\) defined in \((2.21)\), then the coefficients \(\Lambda_{ijk}\) of the nonlinear terms in the system of equations for the wave amplitudes are given by
\[
\Lambda_{ijk} = u_x A_{pq}^\alpha \frac{\partial A_p^\alpha}{\partial g^q} (g_0) R^p_i R^q_j R^r_k, \quad 1 \leq i, j, k \leq n.
\]

2.4 Linearly degenerate wave equations

Motivated by the corresponding definition for hyperbolic conservation laws introduced by Lax [20], we make the following definition.

**Definition 1** A simple characteristic of a hyperbolic system of wave equations \((2.17) - (2.18)\) is genuinely nonlinear (respectively, linearly degenerate) if, for every \(g_0 \in \mathbb{R}^m\) and every non-zero \(du \in \mathbb{R}^{d+1}\) that satisfies \((2.20)\), the quantity \(\Lambda\) defined in \((2.23)\) is non-zero (respectively, zero). A multiple characteristic of constant multiplicity \(n\) is linearly degenerate if all coefficients \(\Lambda_{ijk}\) defined in \((2.24)\) are zero. We say that the system is genuinely nonlinear (respectively, linearly degenerate) if all of its characteristics are genuinely nonlinear (respectively, linearly degenerate).

Thus, for linearly degenerate wave equations, the amplitude-waveform function in the weakly nonlinear theory corresponding to the ansatz \((2.19)\)
always satisfies a linear PDE. Interactions with a mean-field may still occur, however.

For example, from (2.13), the nonlinear wave equation (2.10) is genuinely nonlinear if \( c'(g) \neq 0 \) for all \( g \in \mathbb{R} \), and linearly degenerate if \( c \) is constant, when it reduces to the linear wave equation.

Even if a wave-field is not linearly degenerate, a loss of genuine nonlinearity may occur at a particular point \( g_0 \in \mathbb{R}^m \) and direction \( du \in \mathbb{R}^{d+1} \). A loss of genuine nonlinearity is related to the null condition for nonlinear wave equations introduced by Klainerman [17]. The Einstein equations do not satisfy the Klainerman null condition completely because of the non-zero coupling between gravitational waves and a mean gravitational field [6]. Nevertheless, it is the vanishing of the nonlinear \( \Lambda \)-coefficients in the equations for the amplitude-waveform functions at the Minkowski metric, for example, that permits the existence of global smooth, small-amplitude perturbations [8, 18, 22]. A similar result would not be true for \((1+3)\)-dimensional variational wave equations in which \( \Lambda \neq 0 \).

We emphasize that this definition for wave equations is analogous to but different from the definition for hyperbolic systems of conservation laws. If a genuinely nonlinear wave equation were rewritten as a first-order hyperbolic system, it would be classified as a linearly degenerate first-order hyperbolic system, since its wave speeds are independent of the derivative \( \nabla g \).

Hyperbolic systems of conservation laws with linearly degenerate wave-fields possess non-distorting plane wave solutions, and one can derive a large-amplitude geometrical optics theory for them [23, 24]. Linearly degenerate variational systems of wave equations possess non-distorting plane wave solutions, and a large-amplitude geometrical optics theory, only if they satisfy certain additional degeneracy conditions. We will not analyze these issues here, however, and instead discuss the analogous theory for the Einstein equations.

### 2.5 The Einstein equations

The Einstein equations do not fall exactly into the class of variational wave equations considered in Section 2.3 because of their gauge-covariance. They can be derived from a variational principle of the form (2.16), obtained after an integration by parts in the Einstein-Hilbert action, but the resulting Euler-Lagrange equations are not hyperbolic. Nevertheless, as is well-known, they become hyperbolic when written with respect to a suitable gauge, such as the harmonic gauge. The equation for the metric \( g \), with components \( g_{\mu\nu} \)
and determinant \( g \), then adopts a similar form to (2.17)–(2.18), namely

\[
\frac{\partial}{\partial x^\alpha} \left\{ g^{\alpha\beta} \sqrt{-g} \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right\} - H_{\mu\nu}(g) (\partial g, \partial g) = 0,
\]

where \( H_{\mu\nu} \) is a quadratic form in the metric derivatives \( \partial g \) with coefficients depending on \( g \). (See [22], for example, for an explicit expression.)

The weakly nonlinear expansion described in Section 2.3 for variational systems of wave equations corresponds to the expansion of Choquet-Bruhat [4] and Isaacson [15] for the Einstein equations. The Einstein equations are not strictly hyperbolic, and all of the nonlinear coefficients in the asymptotic equations for the amplitude-waveform functions are zero. Thus, in this generalized sense, the Einstein equations are a linearly degenerate system of variational wave equations.

The Einstein equations possess a non-distorting, plane wave solution for large-amplitude gravitational waves, which forms the basis of a strongly nonlinear geometrical optics theory for large-amplitude gravitational waves. It does not, however, appear possible to obtain a self-consistent theory for oscillatory large-amplitude waves, since the mean energy-momentum associated with an extended wave-packet would generate a very strong background curvature of space-time. This restriction is related to the fact that mean-field interactions with oscillatory waves already occur at leading order in the small-amplitude theory. We therefore consider localized waves.

An asymptotic theory for the propagation of localized, large-amplitude, rapidly varying gravitational waves into slowly varying space-times was developed in [2]. In this theory, the metric of a plane-polarized wave may be written, with respect to a suitable coordinate system \((u, v, y, z)\), as

\[
g = -2e^{-M} du dv + e^{-U}(e^V dy^2 + e^{-V} dz^2) + O(\varepsilon^2). \tag{2.25}
\]

Here, the metric component functions \((U, V, M)\), depend on \((\theta, v)\), where the ‘fast’ phase variable \(\theta\) is given by

\[
\theta = \frac{u}{\varepsilon^2}.
\]

The functions \((U, V, M)\) satisfy the following PDEs:

\[
U_{\theta\theta} - \frac{1}{2} \left( U_{\theta}^2 + V_{\theta}^2 \right) + U_{\theta} M_{\theta} = 0, \quad \tag{2.26}
\]
\[
U_{\theta v} - U_{\theta} U_{v} = 0, \quad \tag{2.27}
\]
\[
V_{\theta v} - \frac{1}{2} \left( U_{\theta} V_{v} + U_{v} V_{\theta} \right) = 0, \quad \tag{2.28}
\]
\[
M_{\theta v} + \frac{1}{2} \left( U_{\theta} U_{v} - V_{\theta} V_{v} \right) = 0. \quad \tag{2.29}
\]
Equations (2.27)–(2.29) are wave equations for \((U, V, M)\) in characteristic coordinates \((\theta, v)\), and (2.26) is a constraint which is preserved by (2.27)–(2.29).

These equations correspond to a well-known exact solution of the Einstein equations, the colliding plane wave solution \([10, 16, 25, 26]\), without the usual constraint equation in \(v\),

\[
U_{vv} - \frac{1}{2} (U_v^2 + V_v^2) + U_v M_v = 0.
\]

This constraint need not be satisfied if the slowly-varying space-time into which the wave propagates is not that of a counter-propagating gravitational wave. If it is not satisfied, then the resulting metric is an asymptotic solution of the Einstein equations but not an exact solution.

The asymptotic equations we derive in this paper are a diffractive generalization of (2.26)–(2.29).

3 The asymptotic expansion

In this section, we outline our asymptotic expansion of the Einstein equations, and specialize it to the case of plane-polarized gravitational waves that diffract in a single direction. Our goal is to explain the structure of the expansion and the resulting perturbation equations. The detailed algebra is summarized in the appendices.

3.1 The general expansion

Let \(g\) be a Lorentzian metric. We denote the covariant components of \(g\) with respect to a local coordinate system \(x^\alpha\) by \(g_{\alpha\beta}\). The connection coefficients \(\Gamma^\lambda_{\alpha\beta}\) and the covariant components \(R_{\alpha\beta}\) of the Ricci curvature tensor associated with \(g\) are defined by

\[
\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left( \frac{\partial g_{\beta\mu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right),
\]

\[
R_{\alpha\beta} = \frac{\partial \Gamma^\lambda_{\alpha\beta}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda_{\beta\lambda}}{\partial x^\alpha} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} - \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\beta\mu}.
\]

Here, and below, Greek indices \(\alpha, \beta, \lambda, \mu\) run over the values 0, 1, 2, 3. The vacuum Einstein equations may be written as

\[
R_{\alpha\beta} = 0.
\]
We look for asymptotic solutions of (3.3) with metrics of the form
\[ g = g\left(\frac{u(x^\alpha)}{\varepsilon^2}, \frac{y^a(x^\alpha)}{\varepsilon}, x^\alpha; \varepsilon\right), \]  
and
\[ g(\theta, \eta^a, x^\alpha; \varepsilon) = g(\theta, \eta^a, x^\alpha) + \varepsilon g(\theta, \eta^a, x^\alpha) + \varepsilon^2 g(\theta, \eta^a, x^\alpha) + O(\varepsilon^3). \]

Here, \( \varepsilon \) is a small parameter, \( u \) is a phase, \( y^a \) with \( a = 2, 3 \) are spacelike transverse variables, and
\[ \theta = \frac{u}{\varepsilon^2}, \quad \eta^a = \frac{y^a}{\varepsilon} \]
are ‘stretched’ variables. The ansatz in (3.4) corresponds to a metric that varies rapidly and strongly in the \( u \)-direction, with less rapid variations in the \( y^a \)-directions, and slow variations in \( x^\alpha \).

We remark that the form of this ansatz is relative to a class of coordinate systems, since an \( \varepsilon \)-dependent change of coordinates can alter the way in which the metric depends on \( \varepsilon \). It would be desirable to give a geometrically intrinsic characterization of such an ansatz, and to carry out the expansion in a coordinate-invariant way, but we do not attempt to do so here.

Using (3.4) in the Einstein equations (3.3), we find that, to get a non-trivial solution at the leading orders, the phase \( u \) must be null to leading order, and the transverse variables \( y^a \) must be constant along the rays associated with \( u \) to leading order. Moreover, the leading-order metric must have the form of the colliding plane wave metric.

To carry out the expansion in detail, we use the gauge-covariance of the Einstein equations to make a choice of coordinates that is adapted to the metric in (3.4). We assume that \( u \) is approximately null up to the order \( \varepsilon^2 \), that \( y^2, y^3 \) are constant along the rays associated with \( u \) to the order \( \varepsilon \). Furthermore, we assume that we can extend \((u, y^2, y^3)\) to a local coordinate system
\[ (x^0, x^1, x^2, x^3) = (u, v, y^2, y^3). \]
Then, as shown in Appendix A, we can use appropriate gauge transformations, which involve a near identity transformation of the phase and the transverse variables, to write a general metric (3.4) whose leading-order term has the form of the colliding plane wave metric, as
\[
\begin{align*}
g &= 2 g_{01} dx^0 dx^1 + g_{ab} dx^a dx^b \\
&\quad + \varepsilon \left\{ 2 g_{1a} dx^1 dx^a + \frac{1}{2} g_{ab} dx^a dx^b \right\} + \varepsilon^2 g_{ij} dx^i dx^j + O(\varepsilon^3). \tag{3.5}
\end{align*}
\]
In (3.5), and below, indices $a,b,c,\ldots$ take on the values 2, 3, while indices $i,j,k,\ldots$ take on the values 1, 2, 3. We raise and lower indices using the leading order metric components; for example, we write

$$
\left(0 \ g_{\alpha\beta}\right)^{-1}, \quad h^\alpha = 0 \ g_{\alpha\beta} h_{\beta}.
$$

We have one additional gauge freedom in (3.5), involving a nonlinear transformation of the phase, which we can use to set either

$$
0 \ g_{01} = 1
$$

or one of the three components

$$
1 \ g_{12}, \quad 1 \ g_{13}, \quad 2 \ g_{11}
$$

equal to zero. We will exploit this gauge freedom later, since it is convenient in formulating a variational principle for the asymptotic equations (see Appendix D).

Using the method of multiple scales, we expand derivatives of a function $f$ with respect to $x^\mu$ as

$$
\frac{\partial}{\partial x^\mu} f \left( \frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x \right) = \frac{1}{\varepsilon^2} f_{,\theta} u_{\mu} + \frac{1}{\varepsilon} f_{,\bar{a}} y_{a \mu} + f_{,\mu}, \quad (3.6)
$$

and then treat $(\theta, \eta^a, x)$ as independent variables. In (3.6), in the appendices, and below, we use the shorthand notation

$$
f_{,\theta} = \frac{\partial f}{\partial \theta} \bigg|_{\theta, x}, \quad f_{,\bar{a}} = \frac{\partial f}{\partial \eta^a} \bigg|_{\theta, x}, \quad f_{,\mu} = \frac{\partial f}{\partial x^\mu} \bigg|_{\theta, \eta^a},
$$

$$
u_{\mu} = \frac{\partial u}{\partial x^\mu}, \quad y_{a \mu} = \frac{\partial y^a}{\partial x^\mu}.
$$

We use (3.5)–(3.6) in (3.1)–(3.2) and expand the result with respect to $\varepsilon$. We find that

$$
\Gamma^\lambda_{\alpha\beta} = \frac{1}{\varepsilon^2} \Gamma^\lambda_{\alpha\beta} - \frac{2}{\varepsilon} \Gamma^\lambda_{\alpha\beta} + \frac{1}{\varepsilon} \Gamma^\lambda_{\alpha\beta} + 0 \ g_{\alpha\beta} + O(\varepsilon),
$$

$$
R_{\alpha\beta} = \frac{1}{\varepsilon^4} R_{\alpha\beta} + \frac{1}{\varepsilon^3} R_{\alpha\beta} + \frac{1}{\varepsilon^2} R_{\alpha\beta} + O(\varepsilon^{-1}), \quad (3.7)
$$

where explicit expressions for the terms in the above expansions are given in Appendix B. The nonzero components of the Ricci tensor up to the order $\varepsilon^{-2}$ are summarized in Appendix C.
Using (3.7) in (3.3) and equating coefficients of \( \varepsilon^{-4} \), \( \varepsilon^{-3} \) and \( \varepsilon^{-2} \) to zero, we get

\[
\begin{align*}
-4 R_{\alpha\beta} &= 0, \\
-3 R_{\alpha\beta} &= 0, \\
-2 R_{\alpha\beta} &= 0.
\end{align*}
\] (3.8) (3.9) (3.10)

These perturbation equations lead to a closed set of equations for the leading-order and first-order components of the metric, as we now explain.

The only component of (3.8) that is not identically satisfied is

\[
\begin{align*}
-4 R_{00} &= 0.
\end{align*}
\] (3.11)

From (3.11) and (C.1), it follows that \( g_{\alpha\beta} \) satisfies

\[
-\frac{1}{2}(g_{ab} g_{ab,\theta}) + \frac{1}{2} g_{01} g_{01,\theta} g_{ab,\theta} - \frac{1}{4} g_{ac} g_{bc,\theta} g_{bd,\theta} = 0.
\]

This equation is a constraint on the leading order metric which involves only \( \theta \)-derivatives, and has the same form as the constraint equation for a single gravitational plane wave.

The nonzero components of the Ricci tensor at the next order in \( \varepsilon \) are

\[
\begin{align*}
-3 R_{00} &= 0, \\
-3 R_{0a} &= 0.
\end{align*}
\] (3.12)

The condition

\[
-3 R_{0a} = 0,
\] (3.12)

yields equations for \( g_{1a} \) and their derivatives with respect to \( \theta \), with non-homogeneous terms depending on \( \theta \)- and \( \eta \)-derivatives of the leading order metric. Thus, (3.12) provides two equations relating the first-order perturbation in the metric to the leading order metric.

The equation

\[
-3 R_{00} = 0
\]

is a single equation that is homogeneous in the first-order components \( g_{ab} \), as can be seen from (C.2). A nonzero solution of this equation corresponds physically to a free, small-amplitude gravitational wave of strength of the order \( \varepsilon \) propagating in the space-time of the large-amplitude gravitational wave.
wave. Retaining a nonzero solution of this homogenous equation would not change our final equations for the large-amplitude wave, and for simplicity we assume that

$$g_{ab}^1 = 0.$$ 

This assumption is also consistent with the higher-order perturbation equations.

At the next order in $\varepsilon$, the nonzero components of the Ricci curvature are

$$-2R_{00}, -2R_{01}, -2R_{0a}, -2R_{ab}.$$ 

The corresponding perturbation equations for $-2R_{00}$ and $-2R_{0a}$ in (3.10) give non-homogenous equations for the second-order metric components

$$2g_{ab}, 2g_{1a},$$

with source terms depending on the lower-order components of the metric. These equations are satisfied by a suitable choice of the second-order metric components, and they are decoupled from the equations for the leading-order and first-order metric components. We therefore do not consider them further here.

The remaining perturbation equations are

$$-2R_{01} = 0, -2R_{ab} = 0.$$ (3.13)

By use of the gauge freedom mentioned at the beginning of this section, we may set

$$2g_{11} = 0.$$ 

In that case, (3.13) provides a set of four equations relating the leading-order and first-order metric components, and their derivatives.

We remark that, in general, one cannot eliminate secular terms from the asymptotic solution that are unbounded as the ‘fast’ phase variable $\theta$ tends to infinity. As a result, the validity of the asymptotic equations is restricted to a thin layer of thickness of the order $\varepsilon^2$ about the hypersurface $u = 0$, where $\theta = O(1)$. A global asymptotic solution can be obtained by matching the ‘inner’ solution inside this layer with appropriate ‘outer’ solutions, such as slowly varying space-times on either side of the wave. In this paper, however, we focus on the construction of asymptotic solutions for localized gravitational waves, and do not consider any matching problems.
Summarizing these results, we find that the asymptotic solution is given by

\[ g = 2 \int_0^0 dx^0 dx^1 + \int_0^0 g_{ab} dx^a dx^b + \varepsilon \left\{ \int_0^1 dx^1 dx^a \right\} + O(\varepsilon^2), \quad (3.14) \]

where the six metric components

\[ g_{01}, \quad g_{ab}, \quad g_{1a} \]

satisfy a system of seven equations \((3.11)–(3.13)\). Equation \((3.11)\) is an ODE in \(\theta\), and, in the case of plane-polarized waves, we show that it is a gauge-type constraint that is preserved by the remaining equations \((3.12)–(3.13)\).

The explicit form of the equations follows from the expressions in \((C.1)–(C.5)\) for the corresponding components in the expansion of the Ricci tensor that appear in \((3.11)–(3.13)\). In general, these equations are very complicated, but they can be simplified considerably in special cases.

### 3.2 Plane-polarized gravitational waves

In this section, we specialize our asymptotic solution to the case of a plane-polarized gravitational wave that diffracts in a single direction, and write out the resulting equations explicitly.

We choose coordinates

\[ (x^0, x^1, x^2, x^3) = (u, v, y, z) \quad (3.15) \]

in which the metric has the form \((3.5)\). We suppose that the metric depends on the variables \((\theta, \eta, v)\), where the phase variable \(\theta\) and the transverse variable \(\eta\) are defined by

\[ \theta = \frac{u}{\varepsilon^2}, \quad \eta = \frac{y}{\varepsilon}, \]

and is independent of the second transverse variable \(\zeta = z/\varepsilon\). This means that the wave diffracts only in the \(y\)-direction. We could also allow the metric to depend on \(z\), which would appear in the final equations as a parameter.

We consider a leading-order metric that has the form of the colliding plane wave metric for a plane-polarized wave, polarized in the \((y, z)\)-directions. We have seen that we need only retain the higher order components

\[ g_{1a}, \quad g_{11}. \]
One component of (3.12) is homogeneous in $g_{13}$, since there is no dependence on $\zeta$, and without loss of generality we can take

$$g_{13} = 0.$$  

We therefore take the special form of the metric (3.5) given by

$$g = -2e^{-M}\left(du - \varepsilon Y dy - \frac{1}{2}\varepsilon^2 T dv\right)dv + e^{-U}(e^V dy^2 + e^{-V}dz^2) + O(\varepsilon^2),$$  

(3.16)

where the functions $(U, V, M, Y, T)$ depend on $(\theta, \eta, v)$. We recall that we have the gauge-freedom to set either $M$, $Y$, or $T$ equal to zero. In writing the equations, we choose to set $T = 0$.

The metric (3.16) must satisfy equations (3.11)–(3.13). First, using (3.16) in (3.11), and simplifying the result, we obtain, after some algebra, the $\theta$-constraint equation (1.6).

Next, using (3.16) and (C.3), we find that the only nontrivial component of (3.12) is

$$\frac{-3}{R_{02}} = 0.$$  

(3.17)

Furthermore, after the introduction of $\phi$, $\psi$ defined in (1.4)–(1.5) and some algebra, we find that (3.17) may be written as equation (1.7).

Finally, using (3.16) and (C.4)–(C.5), we find that the only nontrivial components of (3.13) are

$$\frac{-2}{R_{01}} = 0, \quad \frac{-2}{R_{22}} = 0, \quad \frac{-2}{R_{33}} = 0.$$  

(3.18)

After some algebra, these equations may be written as (1.8)–(1.10), where the derivative $D_\eta$ is defined in (1.3).

### 3.3 The variational principle

Equations (3.11)–(3.13) can be derived from a variational principle, which is obtained by expanding the variational principle for the Einstein equations. In particular, the variational principle provides a check on the algebra used to derive the equations. (We have also checked the final results by use of MAPLE.)

In order to formulate a variational principle, it is necessary to retain the component

$$\frac{2}{g_{11}}.$$  

21
Variations with respect to this component yield the gauge-type constraint (3.11), and it may be set to zero after taking variations with respect to it.

The general form of the asymptotic variational principle is given in Appendix D. Here, we specialize it to the case of a plane-polarized gravitational wave considered in Section 3.2. Using the metric (3.16) in (D.4) we find, after some algebra, that the variational principle (D.5) for (1.6)–(1.10) is

\[ \delta S^{(1)} = 0, \quad S^{(1)} = \int L^{(-2)} d\theta dv d\eta, \]

where the Lagrangian \( L^{(-2)} \) may be written as

\[
L^{(-2)} = e^{-U} \left\{ 2M_{\theta v} + 4U_{\theta v} - V_\theta V_v - 3U_\theta U_v \\
- T_{\theta \theta} + T_\theta (M_\theta + 2U_\theta) + T(M_{\theta \theta} + 2U_{\theta \theta} - \frac{3}{2}U_\theta^2 - \frac{1}{2}V_\theta^2) \\
+ e^{-(U+V+M)} \left[ -2D_\eta \phi - D_\eta \psi + \frac{3}{2} \phi^2 + 2\phi \psi + \psi^2 \right] \right\}. \tag{3.19}
\]

Variations of \( S^{(1)} \) with respect to the second-order metric component \( T \) lead to the \( \theta \)-constraint equation (1.6). Variations with respect to the first-order metric component \( Y \), which appears in \( \phi \), \( \psi \), and \( D_\eta \), lead to equation (1.7). Variations with respect to \( U \), \( V \), \( M \) lead to the evolution equations (1.8)–(1.10), after we set \( T = 0 \).

4 Properties of the equations

In this section, we study some properties of the equations for plane-polarized waves that we have derived in the previous section. We write out the structure of the highest-order derivatives that appear in the equations, and use this to formulate a reasonable IBVP for them. We also show that the \( \theta \)-constraint equation is preserved by the evolution in \( v \), and that the linearized equations are consistent with the linearized equations for gravitational waves in the parabolic approximation.

4.1 Structure of equations and an IBVP

In this subsection, we consider the structure of equations (1.6)–(1.10) in more detail. The first equation, (1.6), is an ODE with respect to \( \theta \) relating \( (U, V, M) \). As we show in the next section, this equation is a gauge-type constraint which holds for all \( v \) if it holds for \( v = 0 \), say. We may therefore neglect this equation provided that the initial data at \( v = 0 \) is compatible with it.
The remaining equations (1.7)–(1.10) form a system of equations for \((U, V, M, Y)\). In order to exhibit their structure, we rewrite them in a way that shows explicitly how the highest, second-order, derivatives appear.

Using (1.3)–(1.5), we may rewrite equation (1.7) as
\[
Y_{\theta\theta} - \left\{ (V + M)_{\theta} Y + \left\{ \frac{1}{2} U_{\theta}^2 - U_{\theta} V_{\theta} - \frac{1}{2} V_{\theta}^2 \right\} Y \right\}_{\theta} = (U + V + M)_{\theta \eta} + M_{\eta} U_{\theta} - (U + V)_{\eta} V_{\theta}. \tag{4.1}
\]

If \((U, V, M)\) are assumed known, then this equation is a linear ODE in \(\theta\) for \(Y\).

We may rewrite equations (1.8)–(1.10) as
\[
(U + V + M)_{\theta v} - \frac{1}{2} (U + V)_{\theta} (U + V)_v = \frac{1}{2} e^{-(U+V+M)} \left\{ -\frac{1}{2} \phi^2 - \phi \psi \right\}, \tag{4.2}
\]
\[
(U + V)_{\theta v} - \frac{1}{2} U_{\theta} (U + V)_v - \frac{1}{2} U_v (U + V)_\theta = \frac{1}{2} e^{-(U+V+M)} \left\{ D_{\eta}^2 (U + V) - \phi \psi - \psi^2 \right\} \tag{4.3}
\]
\[
U_{\theta v} - U_{\theta} U_v = \frac{1}{2} e^{-(U+V+M)} \left\{ D_{\eta}^2 (U + V + M) - D_{\eta} (e^{U} Y_{\theta}) - \frac{1}{2} \phi^2 - \phi \psi - \psi^2 \right\}. \tag{4.4}
\]

Examining the terms that involve second order derivatives, we see that these equations consist of a \((1 + 2)\)-dimensional wave equation in \((\theta, \eta, v)\) for \((U + V)\), and two \((1 + 1)\)-dimensional wave equations in \((\theta, v)\), for \((U + V + M)\) and \(U\), in which \((\theta, v)\) are characteristic coordinates. An additional second-order derivative term, proportional to \(D_{\eta} Y_{\theta}\), appears in the equation for \(U\). The function \(Y\) is also coupled with the evolution equations through the dependence of the transverse derivative \(D_{\eta}\), given in (1.3), on \(Y\).

In order to specify a unique solution of these equations, we expect that we need to supplement the ODE (4.1) with data for \(Y\) and \(Y_{\theta}\) on \(\theta = 0\), say, and the evolution equations (4.2)–(4.4) with characteristic initial data for \((U, V, M)\) on \(\theta = 0\) and \(v = 0\). Thus, a reasonable IBVP for (4.1)–(4.4) in the region \(\theta > 0, -\infty < \eta < \infty, \) and \(v > 0\) is
\[
(U, V, M) = (U_0, V_0, M_0) \quad \text{on} \ v = 0, \\
(U, V, M) = (U_1, V_1, M_1) \quad \text{on} \ \theta = 0, \\
(Y, Y_{\theta}) = (Y_0, Y_1) \quad \text{on} \ \theta = 0.
\]
Here, \((U_0, V_0, M_0)\) are given functions of \((\theta, \eta)\) that satisfy the constraint

\[
U_{\theta\theta} - \frac{1}{2} (U_{\theta}^2 + V_{\theta}^2) + U_{\theta} M_{\theta} = 0,
\]

and \((U_1, V_1, M_1, Y_0, Y_1)\) are given functions of \((\eta, v)\). This data may be interpreted as initial data for the state of the wave on the hypersurface \(v = 0\), and boundary data on the leading wavefront \(\theta = 0\).

### 4.2 The constraint equations

In this section, we show that the constraint equation (1.6) is preserved.

**Proposition 1** Suppose that \((U, V, M, Y)\) are smooth functions that satisfy (1.7)–(1.10). Let

\[
F = U_{\theta\theta} - \frac{1}{2} (U_{\theta}^2 + V_{\theta}^2) + U_{\theta} M_{\theta}.
\]

Then

\[
F_v = U_v F.
\]

**Proof.** We write the evolution equations (1.8)–(1.10) as

\[
U_{\theta v} - U_{\theta} U_v = \frac{1}{2} e^{-(U+V+M)} A, 
\]

\[
V_{\theta v} - \frac{1}{2} (U_{\theta} V_v + U_v V_{\theta}) = \frac{1}{2} e^{-(U+V+M)} B, 
\]

\[
M_{\theta v} + \frac{1}{2} (U_{\theta} U_v - V_v V_{\theta}) = \frac{1}{2} e^{-(U+V+M)} C, 
\]

where, using the notation defined in (1.3)–(1.5),

\[
A = D_{\eta} \phi + D_{\eta} \psi - \frac{1}{2} \phi^2 - \phi \psi - \psi^2,
\]

\[
B = -D_{\eta} \phi + \frac{1}{2} \phi^2,
\]

\[
C = -D_{\eta} \psi - \frac{1}{2} \phi^2 + \psi^2.
\]

Differentiating (4.5) with respect to \(v\), and using (4.7)–(4.9) to replace \(U_{\theta v}, V_{\theta v}, M_{\theta v}\) and \(U_{\theta\theta}\) in the result, we find that

\[
F_v = U_v F + \frac{1}{2} e^{-(U+V+M)} D,
\]
where
\[ D = A_\theta - (U + V)_\theta A - V_\theta B + U_\theta C. \] (4.14)

Differentiating equation (4.14) for \( A \) with respect to \( \theta \), introducing the commutator \([\partial_\theta, D_\eta]\) of \( \partial_\theta \) and \( D_\eta \), and using equations (1.3) and (1.5), we compute that
\[
A_\theta = \partial_\theta \left\{ D_\eta(\phi + \psi) - \frac{1}{2} \phi^2 - \phi \psi - \psi^2 \right\}
\]
\[
= D_\eta(\phi + \psi)_\theta + [\partial_\theta, D_\eta] (\phi + \psi) - (\phi + \psi) \phi_\theta - (\phi + 2\psi) \psi_\theta
\]
\[
= D_\eta \{ \psi(U + V)_\theta \}
\]
\[
= - (\phi + \psi) \phi_\theta - (\phi + 2\psi) \psi_\theta + [\partial_\theta, D_\eta] (\phi + \psi)
\]
\[
= (U + V)_\theta D_\eta \psi + \psi D_\eta (U + V)_\theta
\]
\[
= (U + V)_\theta D_\eta \psi - (\phi + \psi) \phi_\theta - (\phi + 2\psi) \psi_\theta + [\partial_\theta, D_\eta] (U + V)
\]
\[
= (U + V)_\theta D_\eta \psi - \psi D_\eta (U + V)_\theta
\]
\[
= (U + V)_\theta \{ D_\eta \psi - \psi(\phi + \psi) \}
\]
\[
+ [\partial_\theta, D_\eta] (\phi + \psi) - \psi [\partial_\theta, D_\eta] (U + V).
\] (4.15)

From the definition of \( D_\eta \) in (1.3), it follows that
\[
[\partial_\theta, D_\eta] = U_\theta D_\eta + e^U Y_\theta \partial_\theta.
\]

Hence, using (1.4)–(1.5) and (1.7), we get
\[
[\partial_\theta, D_\eta] (\phi + \psi) = U_\theta D_\eta (\phi + \psi) + e^U Y_\theta (\phi + \psi)_\theta
\]
\[
= U_\theta D_\eta (\phi + \psi) + \psi e^U Y_\theta (U + V)_\theta,
\]
\[
[\partial_\theta, D_\eta] (U + V) = U_\theta D_\eta (U + V) + e^U Y_\theta (U + V)_\theta
\]
\[
= U_\theta \psi + e^U Y_\theta (U + V)_\theta.
\]

Using these equations in (4.15), and simplifying the result, we find that
\[
A_\theta = (U + V)_\theta \{ D_\eta \psi - \phi \psi - \psi^2 \} + U_\theta \{ D_\eta \phi + D_\eta \psi - \psi^2 \}.
\]

Finally, using this equation and (4.10)–(4.12) in (4.14), and simplifying the result, we find that \( D = 0 \). It follows from (1.3) that \( F \) satisfies (1.6). □
4.3 Linearization

We consider the small-amplitude limit of (1.6)–(1.10) in which 

\[ U, V, M, Y \to 0. \]

From the constraint equation (1.6), we have \( U = O(V^2) \), so \( U \) is of higher order in a linearized approximation and can be neglected completely. From (1.3)–(1.4), we also have in this approximation that

\[
D_\eta = \partial_\eta,
\]

\[
\phi = M_\eta - Y_\theta,
\]

\[
\psi = V_\eta.
\]

Linearization of (1.7) yields

\[
(\phi + \psi)_\theta = 0,
\]

while linearization the evolution equations (1.8)–(1.10) gives

\[
0 = \frac{1}{2} (\phi + \psi)_\eta,
\]

\[
V_\theta v = -\frac{1}{2} \phi_\eta,
\]

\[
M_\theta v = -\frac{1}{2} \psi_\eta.
\]

Neglecting some arbitrary functions of integration for simplicity, we find that these equations are satisfied if \( M = -V \) and \( Y = 0 \). In that case, \( V \) satisfies an equation of the form (2.9):

\[
V_\theta v = \frac{1}{2} V_\eta.
\]

The corresponding linearized metric is given by

\[
g = 2(1 - V)du dv + (1 + V)dy^2 + (1 - V)dz^2.
\]

Linearization of the Einstein equations, with a suitable choice of gauge, leads to a set of linear wave equations for the metric components. One can verify that this linearization of our asymptotic solution agrees with what is obtained by an application of the parabolic approximation, described in Section 2.3, to the linearized Einstein equations.
A Gauge transformations

We consider a Lorentzian metric $g$, and local coordinates $x^\alpha$, in which

$$ g = g_{\alpha\beta} dx^\alpha dx^\beta. $$

We denote the contravariant form of $g$ by $g^\sharp$.

We suppose that the metric $g$ depends upon a small parameter $\varepsilon$, and there exist independent functions $u(x), y^a(x)$, $a = 2, 3$, such that as $\varepsilon \to 0$ we have

$$ g^\sharp(du, du) = O(\varepsilon^2), \quad (A.1) $$

$$ g^\sharp(du, dy^a) = O(\varepsilon), \quad a = 2, 3. \quad (A.2) $$

The first condition states that $du$ is an approximate null form up to the order $\varepsilon^2$. The second condition states that $y^a$ is approximately constant along the rays associated with $u$.

In this appendix, we reduce such a metric to the standard asymptotic form, given in (3.5). It is convenient to consider two different types of coordinate transformations. First, we show that it is possible to make a near-identity change of coordinates, to a local coordinate system $x^\alpha$ with

$$ x^0 = u + O(\varepsilon^3), \quad x^a = y^a + O(\varepsilon^2), $$

such that

$$ g = 2 \delta_{01} dx^0 dx^1 + g_{ab} dx^a dx^b + \varepsilon \left( 2 \delta_{1a} dx^1 dx^a + \delta_{ab} dx^a dx^b \right) + \varepsilon^2 \delta_{ij} dx^i dx^j + O(\varepsilon^3). \quad (A.3) $$

Second, we show that the form of the metric can be further simplified by use of a coordinate $x^0$ such that

$$ x^0 = u + O(\varepsilon^2). $$

As before, indices $\alpha, \beta, \ldots$ take on the values $0, 1, 2, 3$; indices $a, b, c, \ldots$ take on the values $2, 3$; and indices $i, j, k, \ldots$ take on the values $1, 2, 3$.

Let $x^\alpha$ be any local coordinate system in which

$$ x^0 = u, \quad x^a = y^a. $$

Then the conditions (A.1), (A.2) are equivalent to

$$ g^{00} = g^{10} = 0, \quad g^{0a} = 0. \quad (A.4) $$
These conditions imply immediately that
\[ g_{11} = 0, \quad g_{1a} = 0. \]  
(A.5)

Moreover, we have
\[ \det(g_{ab}) > 0, \]  
which says that \( y^a \) are space-like independent coordinates.

The expansion (3.4), with the additional restriction (A.5), is invariant under a coordinate transformation of the form
\[ x^\alpha \rightarrow x^\alpha + \epsilon^2 \Psi^\alpha \left( \frac{x^0}{\epsilon}, \frac{x^a}{\epsilon}; \epsilon \right), \]  
(A.7)

where, as usual,
\[ \theta = \frac{x^0}{\epsilon}, \quad \eta^a = \frac{x^a}{\epsilon}. \]

To begin with, we perform a change of coordinates of the form
\[ x^0 \rightarrow x^0, \quad x^i \rightarrow x^i + \epsilon \psi^i (\theta, \eta^c, x). \]  
(A.8)

At order zero in \( \epsilon \), the metric components transform according to
\[
\begin{align*}
g_{00} &\rightarrow g_{00} + 2 \psi^k g_{0k} + \psi^i \psi^j g_{ij}, \\
g_{0i} &\rightarrow g_{0i} + \psi^k g_{ik}, \\
g_{ij} &\rightarrow g_{ij}.
\end{align*}
\]

Recalling (A.6), the matrix \( (g_{ab}) \) is invertible, and we can choose
\[ \psi^i \]  
so that
\[ g_{00} = 0, \quad g_{0a} = 0. \]  
(A.9)

Thus, the zero order metric \( g \) takes the form given in (A.3).
Next, we consider the effect of the coordinate transformation (A.7) on the metric components at order $\varepsilon$. Under the action of the transformation

$$x^\alpha \to x^\alpha + \varepsilon^3 \frac{1}{2} \Psi^\alpha (\theta, \eta^c, x),$$

(A.10)

the form (A.3) of the metric is unchanged at order zero. At order one, the components transforms according to

$$\begin{align*}
\frac{1}{2} g_{00} &\to \frac{1}{2} g_{00} + 2 \frac{1}{2} \Psi_{,0}^0 g_{01}, \\
\frac{1}{2} g_{01} &\to \frac{1}{2} g_{01} + \frac{1}{2} \Psi_{,0}^1 g_{01}, \\
\frac{1}{2} g_{0a} &\to \frac{1}{2} g_{0a} + \frac{1}{2} \Psi_{,b}^a g_{ab}, \\
\frac{1}{2} g_{ij} &\to \frac{1}{2} g_{ij}.
\end{align*}$$

Choosing appropriately $\frac{1}{2} \Psi^\alpha$, these transformations can be used to make

$$\frac{1}{2} g_{0a} = 0.$$  \hspace{1cm} (A.11)

Moreover, the condition (A.4) implies

$$\frac{1}{2} g_{11} \equiv - \left( \frac{0}{2} g_{01} \right) = \frac{1}{2} \Psi^0 g_{00} = 0.$$  \hspace{1cm} (A.12)

Next, we simplify the metric components at order $\varepsilon^2$. Under the action of the transformation

$$x^\alpha \to x^\alpha + \varepsilon^4 \frac{2}{3} \Psi^\alpha (\theta, \eta^c, x),$$

(A.13)

the form of the metric is unchanged at order zero and one. At order two, the metric components transform according to

$$\begin{align*}
\frac{2}{3} g_{00} &\to \frac{2}{3} g_{00} + 2 \frac{2}{3} \Psi_{,0}^0 g_{01}, \\
\frac{2}{3} g_{01} &\to \frac{2}{3} g_{01} + \frac{2}{3} \Psi_{,0}^1 g_{01}, \\
\frac{2}{3} g_{0a} &\to \frac{2}{3} g_{0a} + \frac{2}{3} \Psi_{,b}^a g_{ab}, \\
\frac{2}{3} g_{ij} &\to \frac{2}{3} g_{ij}.
\end{align*}$$

Choosing $\frac{2}{3} \Psi^\alpha$ appropriately, these transformations can be used to make

$$\frac{2}{3} g_{0a} = 0.$$  \hspace{1cm} (A.14)
and the metric can be written as in (A.3).

If we allow for changes of order \( \varepsilon^2 \) in \( x^0 \), then we still have the freedom to use a transformation of the form

\[
x^0 \rightarrow \varepsilon^2 \Psi(\theta, \eta^c, x).
\]

The substantial difference between (A.13) and (A.15) is that, unlike the former one, the latter involves a nonlinear transformation of the phase \( \theta \). This transformation can be interpreted as a transformation to characteristic coordinates in the space \((\theta, \eta^a, x)\). The gauge (A.15) acts only on the following components:

\[
0^g_{01} \rightarrow \Psi, 0^g_{01}, \quad 1^g_{1a} \rightarrow g_{1a} + \Psi, 0^g_{1a}, \quad 2^g_{11} \rightarrow g_{11} + 2\Psi, 0^g_{01}.
\]

We can use (A.15) to set \( 0^g_{01} = 1 \) or one of the three components

\[
\frac{1}{g_{12}}, \quad \frac{1}{g_{13}}, \quad \frac{2}{g_{11}}
\]
equal to zero. More generally, it is possible to choose \( \Psi \) so that an appropriate relationship involving all of these components is satisfied.

The contravariant form of the metric tensor in (A.3) is

\[
\tilde{g}^2 = 2 \tilde{g}^{01}_0 \partial_0 \partial_1 + \tilde{g}^{ab}_0 \partial_a \partial_b \quad \varepsilon \left\{ 2 \tilde{h}^{0a}_0 \partial_0 \partial_a + \tilde{h}^{ab}_0 \partial_a \partial_b \right\}
\]

\[
+ \varepsilon^2 \left\{ 2 \tilde{h}^{00}_2 \partial^2_0 + 2 \tilde{h}^{0a}_2 \partial_0 \partial_a + \tilde{h}^{ab}_2 \partial_a \partial_b \right\} + O(\varepsilon^3), \quad (A.16)
\]

with

\[
\tilde{h}^{00}_2 = -(\tilde{g}^{00} - \tilde{g}_{cd}\tilde{g}^{0c}_d), \quad \tilde{h}^{0a}_2 = -(\tilde{g}^{0a} - \tilde{g}_{cd}\tilde{g}^{0c}_d), \quad \tilde{h}^{ab}_2 = -(\tilde{g}^{ab} - \tilde{g}_{cd}\tilde{g}^{ac}_d).
\]

In (A.16), \( 0^{g01} \) and \( 0^{ab} \) are defined by

\[
0^{g01}_0 0^{g01}_0 = 1, \quad 0^{ac}_g 0^{gb}_c = \delta^g_0.
\]

Moreover, \( 0^{g01} \) is used to raise the sub-index 1, and \( 0^{ab} \) is used to raise the sub-indices 2, 3. Thus, for example, we have

\[
1^{g00}_1 = (0^{g01}_0)^2 1^{g11}_1, \quad 2^{g0a}_2 = 0^{g01}_0 0^{g0b}_2 2^{g1b}_1.
\]

It follows from (A.16) that the transformed phase \( u = x^0 \) and transverse variables \( y^a = x^a \) satisfy (A.1)–(A.2).
B Expansion of connection coefficients and Ricci tensor

We look for a multiple-scale expansion as $\varepsilon \to 0$ of the metric components of the form

$$g_{\alpha\beta} = g_{\alpha\beta}^{0} \left( \frac{u(x)}{\varepsilon^2}, \frac{y^a(x)}{\varepsilon} \right),$$

(B.1)

$$g_{\alpha\beta}(\theta, \eta^{a}, x; \varepsilon) = g_{\alpha\beta}^{0}(\theta, \eta^{a}, x) + \varepsilon g_{\alpha\beta}^{1}(\theta, \eta^{a}, x) + \varepsilon^2 g_{\alpha\beta}^{2}(\theta, \eta^{a}, x) + O(\varepsilon^3).$$

The contravariant metric components $g^{\alpha\beta}$ satisfy

$$g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}_{\beta}.$$  

Expansion of this equation in a power series in $\varepsilon$ gives

$$g^{\alpha\beta} = g^{\alpha\beta}^{0} + \varepsilon g^{\alpha\beta}^{1} + \varepsilon^2 g^{\alpha\beta}^{2} + O(\varepsilon^3)$$

:=

$$g^{\alpha\beta} = g^{\alpha\beta}^{0} - \varepsilon^2 \left( \frac{1}{2} g^{\alpha\beta} - g_{\mu\nu} g^{\alpha\mu} g^{\beta\nu} \right) + O(\varepsilon^3),$$

(B.2)

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$. In (B.2) and below, we use the leading order metric components to raise indices, so that

$$\frac{1}{2} g^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu}, \quad \frac{1}{2} g^{\mu\nu} = g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu}.$$  

(B.3)

With this notation, note that, for instance, the first order term in the expansion of the covariant metric components $g_{\alpha\beta}$ with respect to $\varepsilon$ is not $\frac{1}{2} g^{\alpha\beta}$ but $\frac{1}{2} g^{\alpha\beta} = - \frac{1}{2} g_{\alpha\beta}$.

The expansion of the connection coefficients (or Christoffel symbols) is

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{\varepsilon^2} \Gamma^\lambda_{\alpha\beta}^{0} + \frac{1}{\varepsilon} \Gamma^\lambda_{\alpha\beta}^{1} + \Gamma^\lambda_{\alpha\beta}^{2} + O(\varepsilon),$$

where

$$\Gamma^\lambda_{\alpha\beta}^{0} = \frac{1}{2} g^{\lambda\mu} \left( g_{\beta\mu,\theta} u_{\alpha} + g_{\alpha\mu,\theta} u_{\beta} - g_{\alpha\beta,\theta} u_{\mu} \right),$$

$$\Gamma^\lambda_{\alpha\beta}^{1} = \frac{1}{2} g^{\lambda\mu} \left( g_{\beta\mu,\theta} y^a_{\alpha} + g_{\alpha\mu,\theta} y^a_{\beta} - g_{\alpha\beta,\theta} y^a_{\mu} \right) + \frac{1}{2} g^{\lambda\mu} \left( g_{\beta\mu,\theta} u_{\alpha} + g_{\alpha\mu,\theta} u_{\beta} - g_{\alpha\beta,\theta} u_{\mu} \right)$$

and

$$\Gamma^\lambda_{\alpha\beta}^{2} = \frac{1}{2} g^{\lambda\mu} \left( g_{\beta\mu,\theta} y^a_{\alpha} + g_{\alpha\mu,\theta} y^a_{\beta} - g_{\alpha\beta,\theta} y^a_{\mu} \right) + \frac{1}{2} g^{\lambda\mu} \left( g_{\beta\mu,\theta} u_{\alpha} + g_{\alpha\mu,\theta} u_{\beta} - g_{\alpha\beta,\theta} u_{\mu} \right).$$
\[ 0^\lambda_{\alpha\beta} = \frac{1}{2} g^\lambda_{\mu}(g_{\beta\mu,\theta} u_\alpha + g_{\alpha\mu,\theta} u_\beta - g_{\alpha\beta,\theta} u_\mu), \]

The expansion of the Ricci tensor is

\[ R_{\alpha\beta} = \frac{1}{\varepsilon^4} - \frac{4}{\varepsilon^3} R_{\alpha\beta} + \frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} R_{\alpha\beta} + \frac{1}{\varepsilon} - 2 + O(\varepsilon^{-1}), \]

where

\[ -4 R_{\alpha\beta} = -2 \Gamma^\mu_{\alpha\beta,\theta} u_\mu - 2 \Gamma^\mu_{\beta\mu,\theta} u_\alpha + 2 \Gamma^\mu_{\alpha\beta} - 2 \nu_{\mu\nu} - 2 \mu_{\alpha\nu} - 2 \nu_{\beta\mu}, \]

\[ -3 R_{\alpha\beta} = -1 \Gamma^\mu_{\alpha\beta,\theta} u_\mu + 1 \Gamma^\mu_{\beta\mu,\theta} u_\alpha + 2 \Gamma^\mu_{\alpha\beta,\bar{a}} y_{\bar{a}} - 2 \Gamma^\mu_{\beta\mu,\bar{a}} y_{\bar{a}} + 2 \nu_{\mu\nu} - 2 \nu_{\alpha\nu} - 2 \nu_{\beta\mu} - 2 \nu_{\alpha\nu} - 2 \nu_{\beta\mu}, \]

\[ -2 R_{\alpha\beta} = 0 \Gamma^\mu_{\alpha\beta,\theta} u_\mu - 0 \Gamma^\mu_{\beta\mu,\theta} u_\alpha + 0 \Gamma^\mu_{\alpha\beta,\bar{a}} y_{\bar{a}} - 0 \Gamma^\mu_{\beta\mu,\bar{a}} y_{\bar{a}} + 2 \nu_{\mu\nu} - 2 \nu_{\alpha\nu} - 2 \nu_{\beta\mu}, \]

C Nonzero connection coefficients and Ricci tensor components

It this appendix, we write out expressions for the nonzero connection coefficients and Ricci tensor components for a metric with the form given in (3.5).
Nonzero connection coefficients at order $\varepsilon^{-2}$:

\[
\begin{align*}
\Gamma_{00}^0 &= g_{00}^{01} g_{01,0}^0, & \Gamma_{10}^0 &= -\frac{1}{2} g_{00}^{01} g_{ab,0}^0, & \Gamma_{20}^0 &= \frac{1}{2} g_{ac}^0 g_{bc,0}^0.
\end{align*}
\]

Nonzero connection coefficients at order $\varepsilon^{-1}$:

\[
\begin{align*}
\Gamma_{0a}^0 &= \frac{1}{2} g_{00}^{01} g_{01,a}^0 + \frac{1}{2} g_{ab,0}^0 g_{ab,0}^0, \\
\Gamma_{1a}^0 &= -\frac{1}{2} g_{00}^{01} g_{01,a}^0, & \Gamma_{ab}^0 &= -\frac{1}{2} g_{00}^{01} g_{ab,0}^0, \\
\Gamma_{a0}^0 &= -\frac{1}{2} g_{ab,0}^0 g_{bc,0}^0 - \frac{1}{2} g_{ac}^0 g_{bc,0}^0, & \Gamma_{bc}^0 &= \frac{1}{2} g_{ab}^0 g_{bc,0}^0 - \frac{1}{2} g_{ac}^0 g_{bc,0}^0.
\end{align*}
\]

Nonzero connection coefficients at order $\varepsilon^0$:

\[
\begin{align*}
\Gamma_{00}^0 &= g_{00}^{01} g_{01,0}^0, & \Gamma_{01}^0 &= \frac{1}{2} g_{00}^{01} g_{01,c}^0 - \frac{1}{2} g_{01}^0 g_{11,0}^0, \\
\Gamma_{0a}^0 &= \frac{1}{2} g_{00}^{01} g_{01,a}^0 + \frac{1}{2} g_{ab,0}^0 g_{ab,0}^0, & \Gamma_{0b}^0 &= \frac{1}{2} g_{ab}^0 g_{bc,0}^0 - \frac{1}{2} g_{ac}^0 g_{bc,0}^0, \\
\Gamma_{1a}^0 &= \frac{1}{2} g_{ab,0}^0 g_{bc,0}^0 - \frac{1}{2} g_{ac}^0 g_{bc,0}^0, & \Gamma_{1b}^0 &= \frac{1}{2} g_{ab,0}^0 g_{bc,0}^0 - \frac{1}{2} g_{ac}^0 g_{bc,0}^0.
\end{align*}
\]

Nonzero component of the Ricci curvature tensor at order $\varepsilon^{-4}$:

\[
\begin{align*}
\bar{R}_{00} &= -\frac{1}{2} g_{ab}^0 g_{ab,0}^0 + \frac{1}{2} g_{00}^{01} g_{01,a}^0 g_{ab,0}^0 - \frac{1}{4} g_{ab}^0 g_{bc,0}^0 g_{bd,0}^0 g_{ad,0}^0.
\end{align*}
\]
Nonzero components of the Ricci tensor at order $\varepsilon^{-3}$:

\begin{equation}
-\frac{3}{2} R_{00} = -\frac{1}{2} g_a^{\theta} g_{a,\theta} + \frac{1}{2} g^{01} g_{01,\theta} + \frac{1}{2} g^{01} g_{01,\theta} g_a^{\theta} - \frac{1}{2} g^{bd} g_{bc,\theta} g_d^{\theta}\,.
\end{equation}

\begin{equation}
-\frac{3}{2} R_{0a} = \frac{1}{2} g_{ab} g^{01} g_{1,b,\theta} + \frac{1}{4} g^{cd} g_{cd,\theta} g_{ab} g^{01} g_{1,b,\theta} + \frac{1}{2} g_{bc,\theta} g_{a,\theta} + \frac{1}{4} g_{bc,\theta} g_{a,\theta} g_{de,\theta} g_{de,\theta} + \frac{1}{4} g^{01} g_{01,a} g^{cd} g_{cd,\theta} g_{a,\theta} - \frac{1}{4} g^{bd} g_{bc,\theta} g_{a,\theta} g_{de,\theta} g_{de,\theta}.
\end{equation}

The nonzero components of the Ricci curvature at the order $\varepsilon^{-2}$ are $R_{00}$, $R_{01}$, $R_{0a}$ and $R_{ab}$. The components $\tilde{R}_{00}$ and $\tilde{R}_{0a}$ are the analogs of $\tilde{R}_{0a}$ and $\tilde{R}_{0a}$ for the higher order components of the metric, and will not be listed here. The remaining components of the Ricci tensor are listed below:

\begin{equation}
-\frac{2}{2} R_{01} = -\left( g^{01} g_{01,\theta} + \frac{1}{2} g^{01} g_{01,\theta} g_{cd,\theta} + \frac{1}{4} g^{ab} g_{ab,\theta} g^{bd} g_{bd,\theta}\right) - \frac{1}{2} g^{cd} g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{ab,\theta} g_{cd,\theta} + \frac{1}{2} g_{ab,\theta} + R_{ab} - \frac{1}{4} g^{01} g_{cd,\theta} g_{ab,\theta} + g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{cd,\theta} g_{ab,\theta} + \frac{1}{2} g_{cd,\theta} g_{ab,\theta}.
\end{equation}
\[ + \frac{1}{2} g^c_a g^0 e g^c d g_{ac, d \theta} - \frac{1}{2} g^0 0 g^c_1 g^0 0 g^d_1 g_{bd, \theta}. \]  

Here, \( R_{ab} \) are the components of the Ricci tensor of the zero order metric \( g_{\alpha \beta} \) regarded as a function of the variables \( \eta^a \) only,

\[
\begin{align*}
R_{ab} &= \frac{1}{2} \left( 0 \; g^{cd} (g_{bd, \bar{a}} + g_{ad, \bar{b}} - g_{ab, \bar{d}}) \right), \\
&+ \frac{1}{2} g^{cd} (g_{bd, \bar{a}} + g_{ad, \bar{b}} - g_{ab, \bar{d}}) (g_{01}^0 g_{01, \bar{b}} + \frac{1}{2} g_{01}^0 g_{01, \bar{c}}) \\
&- (g_{01}^0 g_{01, \bar{a}} + \frac{1}{2} g^{cf} g_{cd, \bar{a}}) - \frac{1}{2} g_{01}^0 g_{01, \bar{a}} g_{01}^0 g_{01, \bar{b}} \\
&- \frac{1}{4} g^{cf} (g_{de, \bar{a}} + g_{ae, \bar{d}} - g_{ad, \bar{e}}) g_{bf, \bar{c}} g_{cf, \bar{b}} - g_{cb, \bar{f}}. 
\end{align*}
\]

\[ \text{D Variational principle} \]

The variational principle for the vacuum Einstein field equations is

\[
\delta S = 0, \quad S = \int L d^4 x, \\
L = R \sqrt{- \det g},
\]

where \( R \) is the scalar curvature,

\[ R = g^{\alpha \beta} R_{\alpha \beta}. \]

Using (B.1), (B.2), and (3.7) to expand the scalar curvature, we obtain

\[
\begin{align*}
R &= \frac{1}{\varepsilon^4} R + \frac{1}{\varepsilon^3} R - \frac{1}{\varepsilon^2} R + O(\varepsilon^{-1}), \\
-4 R &= g^{\alpha \beta} R_{\alpha \beta}, \\
-3 R &= g^{\alpha \beta} R_{\alpha \beta} - g^{\alpha \beta} R_{\alpha \beta}, \\
-2 R &= g^{\alpha \beta} R_{\alpha \beta} - g^{\alpha \beta} R_{\alpha \beta} + \frac{2}{n^\alpha} R_{\alpha \beta}.
\end{align*}
\]

For a metric of the form \( A.3, (5.15) \), with \( g_{11} = T \), we find that

\[
\begin{align*}
-3 R &= R = 0, \\
-2 R &= 2 g_{01}^0 R_{01} - g^{ab} R_{ab} - 2 g^{0a} R_{0a} + h_{00} R_{00}.
\end{align*}
\]
We use (D.3) in (D.1) and expand the result with respect to \( \varepsilon \). This gives
\[
L = \frac{1}{\varepsilon^2} L^{(-2)} + O(\varepsilon^{-1}),
\]
with
\[
L^{(-2)} = \left\{ 2^{-2} R_{01} + 0_{g_{01}}^0 - 2^{-2} 1_{g_{11}}^1 - 3^{-2} R_{0a} + 2^0 1_{g_{ab}}^a - 2^1 1_{R_{ab}}^1 - \right\} \sqrt{\det g_{ab}}. \tag{D.4}
\]
We make the change of variables in the integration
\[
d^4x = dudvdydz = \varepsilon^3 d\theta dv d\eta dz,
\]
and omit the integration with respect to the parametric variable \( z \). The leading order asymptotic variational principle then becomes
\[
\delta S^{(1)} = 0, \quad S^{(1)} = \int L^{(-2)} d\theta dv d\eta. \tag{D.5}
\]
Variations of \( S^{(1)} \) with respect to the second order metric component \( g_{11} \) give the constraint \( (3.11) \). Variations with respect to the first order metric components \( g_{0a} \) give the equation \( (3.12) \). Variations with respect to \( 0_{g_{01}}^0 \) and \( g_{0b} \) give the evolution equations \( (3.13) \).

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