Recursion Operators for Multidimensional Integrable PDEs

Artur Sergyeyev

Received: 11 January 2022 / Accepted: 11 August 2022 / Published online: 20 September 2022
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Abstract
We present a novel construction of recursion operators for integrable second-order multidimensional PDEs admitting isospectral scalar Lax pairs with Lax operators being first-order scalar differential operators linear in the spectral parameter. Our approach, illustrated by several examples and applicable to many other PDEs of the kind in question, employs an ansatz for the sought-for recursion operator of the equation under study based on the Lax pair for the latter.

Keywords Recursion operators · Lax pairs · Integrable systems · Symmetries

1 Introduction

Symmetries are well known to play an important role in the study of partial differential equations (PDEs) and the search for their exact solutions, cf. e.g. [5, 6, 10, 13, 15, 17, 18, 21, 28, 30, 35] and references therein. In particular, the PDEs that are integrable in the sense of soliton theory have rich symmetry algebras as such PDEs usually belong to integrable hierarchies which are essentially infinite countable families of pairwise compatible integrable systems that can be thought of as symmetries of each other, cf. e.g. [17, 30].

An integrable hierarchy associated with a given PDE is usually constructed using a recursion operator which, roughly speaking, maps any given symmetry of the PDE under study into a (new) symmetry, see the seminal paper [29] and e.g. [1, 6, 11, 12, 19, 21–24, 30, 32, 35, 36, 38] and references therein.

In what follows we restrict ourselves to considering a single second-order PDE in \( d \) independent variables \( x^1, \ldots, x^d \) for a single unknown function \( u \),

\[
F(\bar{x}, u, u_{x^1}, \ldots, u_{x^d}, u_{x^1x^1}, u_{x^1x^2}, \ldots, u_{x^dx^d}) = 0.
\]  

(1)

Here \( \bar{x} = (x^1, \ldots, x^d) \), and, as usual, the subscripts refer to partial derivatives; \( d > 1 \) is a natural number.

All functions here and below are tacitly assumed sufficiently smooth for all expressions and computations to make sense; this can be formalized using the language of differential algebra, cf. e.g. [37] and references therein.

✉ A. Sergyeyev
artur.sergyeyev@math.slu.cz

1 Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, Opava, 74601, Czech Republic
The class of PDEs (1) contains plenty of integrable equations, e.g. various heavenly equations [7, 34], the Boyer–Finley equation [4], the dispersionless Kadomtsev–Petviashvili equation, see e.g. [14], also known as Khokhlov–Zabolotskaya equation [40], the ABC equation [41], etc., and many of those equations are of relevance for applications.

To the best of our knowledge, for nonlinear equations of the form (1) in three or more independent variables their recursion operators known to date usually turn out to be Bäcklund auto-transformations for the linearized versions of these equations rather than genuine operators mapping one symmetry to another, see [11, 17, 22, 23, 28, 32, 35] for details, references and history. Note that for some integrable systems in more than two independent variables there exist recursion operators of a different type, namely bilocal ones, see e.g. [10] and references therein, but this case is beyond the scope of the present work.

To date, several methods for construction of recursion operators being Bäcklund auto-transformations for the linearized versions of equations under study were discovered, see [16, 20, 23, 25–27, 35, 39] and references therein for details. However, most of these methods either involve rather heavy computations (those of [23], [25–27], and, to an extent, [16, 35]) or have a somewhat restricted range of applicability (those of [20, 39]).

Below we introduce an approach to the construction of recursion operators in question which, while inspired by the ideas from [35], is significantly more straightforward and less computationally demanding yet apparently not really that much narrower than that of [35] in the range of applicability as far as equations of the form (1) are concerned. The idea of the proposed approach can be roughly described as constructing the recursion operator using an ansatz based on a twist of the Lax operators for the PDE under study, see Section 3 below for details.

The rest of the paper is organized as follows. In Section 2 we briefly recall the basic definitions we will need, and in Section 3 we present our approach to construction of recursion operators for equations of the form (1). Section 4 contains examples, and Section 5 gives conclusions and discussion.

2 Preliminaries

Introduce the standard multiindex notation for derivatives (cf. e.g. [23, 30])

\[ u_{k_1 \ldots k_d} = \partial^{k_1 + \ldots + k_d} u / \partial (x^1)^{k_1} \ldots \partial (x^d)^{k_d}, \quad k_1, \ldots, k_d = 0, 1, 2, \ldots, \]

so \(u_{00 \ldots 0} \equiv u, \ u_{x1} \equiv u_{10 \ldots 0}, \) etc.

Following the standard approach of the geometric theory of PDEs, see e.g. [17, 30], we consider \(x^i\) and \(u_{k_1 \ldots k_d}\) as independent entities; one can think of them as coordinates on the suitable jet space.

In this setting a local function is a function of \(x^i, u\) and of finitely many partial derivatives of \(u\), and the operators of partial derivatives \(\partial / \partial x^i\) are modeled on the jet space by the operators of total derivatives

\[ D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{k_1, \ldots, k_d = 0}^{\infty} u_{k_1 \ldots k_{j-1} \bar{k}_j k_{j+1} \ldots k_d} \frac{\partial}{\partial u_{k_1 \ldots k_d}}, \]  \hspace{1cm} (2)

where \(\bar{k}_j = k_j + 1\). Geometrically \(D_{x^j}\) are vector fields on the jet space.
We also need the operator of linearization of the left-hand side $F$ of (1), cf. e.g. [17],

$$\ell_F = \frac{\partial F}{\partial u} + \sum_{i=1}^{d} \frac{\partial F}{\partial u_{x_i}} D_{x_i} + \sum_{i=1}^{d} \sum_{j=i}^{d} \frac{\partial F}{\partial u_{x_i x_j}} D_{x_i} D_{x_j}.$$ 

The system

$$D^{k_1} \cdots D^{k_d} F = 0, \quad k_1, \ldots, k_d = 0, 1, 2, \ldots,$$

which consists of (1) and all differential consequences thereof, defines an (infinite-dimensional) submanifold $\mathcal{F}$ of the jet space; this $\mathcal{F}$ is often referred to as the diffiety associated with our PDE (1), cf. e.g. [17] and references therein.

Below we tacitly assume that the operators of total derivatives are restricted to $\mathcal{F}$ while retaining for them the same notation as above. Moreover, following [23, 35], we retain the same notation $D_{x_i}$ for the operators of total derivatives further extended to nonlocal variables (i.e., lifted to a covering in the terminology of [17] and references therein), as in our setup this does not cause confusion.

Recall, see e.g. [17, 30], that a local function $U$ is a (characteristic of a) symmetry for (1) if we have $\ell_F(U) = 0$ on $\mathcal{F}$ (in other words, modulo (1) and its differential consequences). If $U$ depends not just on $x^i, u$, and finitely many derivatives $u$ but also on a finite number of nonlocal variables and satisfies $\ell_F(U) = 0$ modulo (1), determining equations for nonlocal variables, and differential consequences of both (1) and of the said determining equations, we shall, cf. e.g. [30, 35] and references therein, refer to $U$ as to a nonlocal symmetry of (1).

Note that a number of authors, see e.g. [17] and references therein, refer to nonlocal symmetries as defined above as to shadows of nonlocal symmetries but we shall not use this terminology here because in what follows we shall not encounter nonlocal symmetries in the sense employed by these authors; see also [31] and references therein for a different approach to nonlocal symmetries.

3 The Construction of Recursion Operators

Suppose now that (1) admits an isospectral Lax representation with first-order scalar Lax operators linear in the spectral parameter $\lambda$.

More precisely, we assume (cf. e.g. [17, 35] and references therein) that there exist nonzero operators of the form

$$\mathcal{X}^{-1}_i = \lambda \mathcal{X}^{-1}_i - \mathcal{X}^0_i,$$  

where

$$\mathcal{X}^s_i = X_{i s 0} + \sum_{j=1}^{d} X_{i j}^s D_{x_j}, \quad i = 1, 2, \quad s = 0, 1,$$  

and $X_{i k}^s$ for all $i, s, k$ are local functions, such that we have

$$[\mathcal{X}^1_i, \mathcal{X}^2_i] = 0$$

modulo (1) and its differential consequences; geometrically, this means that (5) must hold on the diffiety $\mathcal{F}$ associated with (1).
Notice that having (5) holding modulo (1) and differential consequences thereof implies that the equations

\[ \mathcal{X}_i \psi = 0, \quad i = 1, 2 \]  

(6)

define a Lax pair for (1).

We now present our ansatz for a recursion operator of (1).

**Theorem 1** Suppose that for equation (1) there exist nontrivial \( \mathcal{X}_i \) of the form given by (3) and (4) that satisfy (5) modulo (1) and its differential consequences.

Further assume that there exist local functions \( f_i^j \), \( i = 1, 2, j = 0, 1 \), such that for any nonlocal symmetry \( U \) of (1) the system

\[ \mathcal{X}_i^{1}(\tilde{U}) + f_i^1 \tilde{U} = f_i^0 U + \mathcal{X}_i^0(U), \quad i = 1, 2 \]  

(7)

for \( \tilde{U} \) is compatible and \( \tilde{U} \) defined by this system is again a nonlocal symmetry for (1).

Then the relations (7) define a recursion operator for (1).

**Proof** As per Section 1, here and below a recursion operator for (1) is essentially defined as a Bäcklund auto-transformation for the linearized version \( \ell_F(U) = 0 \) of (1), cf. e.g. [11, 22, 23, 32, 35] for more details on this definition.

Now we are to show that under our assumptions (7) indeed defines a recursion operator for (1). To this end it suffices to observe that requiring (7), seen as a system for \( \tilde{U} \), to be compatible for any nonlocal symmetry \( U \) of (1) along with requiring \( \tilde{U} \) defined by (7) to be a nonlocal symmetry for (1) whenever so is \( U \) means that (7) indeed defines a Bäcklund auto-transformation, relating \( \tilde{U} \) to \( U \), for the linearized version of (1), and the result follows.

The conditions from Theorem 1 give rise to a (in general overdetermined) system of equations for \( f_i^j \), and a solution of this system, if such a solution exists, yields a recursion operator for (1) defined via (7).

Writing down and solving the said system of equations for \( f_i^j \) is quite straightforward and can be readily done using the existing computer algebra software like e.g. Jets [3] which we used for handling the examples given below.

Note that the ansatz (7) for a recursion operator is based on the Lax pair (6) with \( \lambda \psi \) formally replaced by \( \tilde{U} \) and \( \psi \) formally replaced by \( U \), with an additional twist thanks to the terms involving \( f_i^j \).

Thus, we propose the following procedure for finding a recursion operator for (1):

1. Given an equation (1) with a Lax pair of the form (6) with \( \mathcal{X}_i \) of the form (3)–(4) satisfying (5) modulo (1) and its differential consequences, write down the associated system (7) and require that the following conditions are satisfied:
   a) system (7), as a system for \( \tilde{U} \), is compatible whenever \( U \) is a (nonlocal) symmetry for (1);
   b) \( \tilde{U} \) defined by (7) is a (in general nonlocal) symmetry for (1) if so is \( U \).
2. Solve the system of equations for \( f_i^j \) resulting from the requirements from the previous step. A solution of this system for \( f_i^j \), if such a solution exists, by Theorem 1 yields a recursion operator for (1) defined via (7).

Note that in practice it is often sufficient to look for solutions \( f_i^j \) of the system in question that depend on the derivatives of \( u \) up to at most third order.
In principle such solutions are by no means obliged to exist but it appears that for all integrable equations (1) in more than two independent variables known to date that possess Lax representations with the Lax operators of the form given by (3) and (4) the situation is as follows: whenever such an equation at all admits a recursion operator being a Bäcklund auto-transformation for the linearized equation (cf. the discussion in Introduction and in Section 5), then such a recursion operator is of the form (7) for some $f_j^i$.

4 Examples

**Example 1** Let $d = 4$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$; consider the PDE [8]

$$D_z(u_y/u_x) - D_x(u_t/u_x) = 0$$

so in this example $F = D_z(u_y/u_x) - D_x(u_t/u_x)$.

Eq. (8) has [8] a Lax pair from the class (6) that can be written as

$$\lambda \psi_z + (u_t/u_x)\psi_x - \psi_t = 0, \quad (\lambda + u_y/u_x)\psi_x - \psi_y = 0,$$

so, comparing this Lax pair with the general formula (6) and taking into account (3) and (4), we get $\mathcal{X}_0^0 = D_t - (u_t/u_x)D_x$, $\mathcal{X}_0^2 = D_y - (u_y/u_x)D_x$, $\mathcal{X}_1^1 = D_z$, $\mathcal{X}_2^1 = D_x$; it is immediate that in this case (5) holds modulo (8) and its differential consequences.

Thus, according to the procedure from Section 3, we are to look for a recursion operator for (8) defined by the relations of the form (7) with the above $\mathcal{X}_j^i$, that is,

$$\tilde{U}_z + f_1^1 \tilde{U} = U_t - \frac{u_t}{u_x} U_x + f_1^0 U, \quad \tilde{U}_x + f_1^2 \tilde{U} = U_y - \frac{u_y}{u_x} U_x + f_2^0 U.$$  

(9)

Here and below we slightly abuse the notation and write $U_x$ instead of $D_x U$ etc. wherever this does not cause confusion.

According to Section 3 we need to find local functions $f_j^i$ such that

1) whenever $U$ satisfies the equation $\ell_F(U) = 0$, i.e.,

$$D_z \left( \frac{U_y}{u_x} - \frac{u_y U_z}{u_x^2} \right) - D_x \left( \frac{U_t}{u_x} - \frac{u_t U_x}{u_x^2} \right) = 0,$$

modulo (8) and its differential consequences (and the determining equations for whatever nonlocal variables involved in $U$, if any, and differential consequences thereof), then the same holds for $\tilde{U}$ defined by (9).

2) system (9) for $\tilde{U}$ is compatible, i.e., $(\tilde{U}_z)_x = (\tilde{U}_x)_z$, modulo (1), (9), (10), and differential consequences thereof.

More explicitly, the condition 1) means that $\tilde{U}$ defined via (9) should satisfy

$$D_z \left( \frac{\tilde{U}_y}{u_x} - \frac{u_y \tilde{U}_z}{u_x^2} \right) - D_x \left( \frac{\tilde{U}_t}{u_x} - \frac{u_t \tilde{U}_x}{u_x^2} \right) = 0$$

modulo (8), (9), (10) and differential consequences of these equations.

According to the remark in the previous section, we shall restrict ourselves to looking for $f_j^i$ that depend at most on $x$, $y$, $z$, $t$, $u$, and first-, second- and third-order derivatives of $u$. 

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It is easily seen that \( f^j_i \) such that the conditions 1) and 2) hold, and thus Theorem 1 applies, indeed exist and have the form

\[
\begin{align*}
f^1_1 &= -\frac{u_{xz}}{u_x}, \quad f^1_2 = -\frac{u_{xx}}{u_x}, \quad f^0_1 = 0, \quad f^0_2 = 0.
\end{align*}
\]

Thus, the relations

\[
\begin{align*}
\tilde{U}_z - \frac{u_{xz}}{u_x} \tilde{U} &= U_i - \frac{u_i}{u_x} U_x, \\
\tilde{U}_x - \frac{u_{xx}}{u_x} \tilde{U} &= U_y - \frac{u_y}{u_x} U_x,
\end{align*}
\]

(11)
define a recursion operator for (8) which produces a new (nonlocal) symmetry \( \tilde{U} \) for (8) from a known one, \( U \), and, to the best of our knowledge, this recursion operator has not yet appeared in the literature.

\textbf{Example 2} Let \( d = 3, x^1 = x, x^2 = y, x^3 = t \). Consider the \( ABC \) equation \[41\]

\[
Au_xu_yt + Bu_yu_xt + Cu_xtu_{xy} = 0 \tag{12}
\]

where \( A, B, C \) are nonzero constants such that \( A + B + C = 0 \).

Equation (12) has a Lax pair of the form \[41\]

\[
\begin{align*}
\psi_y &= \frac{\lambda u_y}{u_x} \psi_x, \\
\psi_t &= \frac{\mu u_t}{u_x} \psi_x,
\end{align*}
\]

where \( \mu = (A + B)\lambda/(A\lambda + B) \).

We can readily rewrite this Lax pair in the form which is linear in \( \lambda \) and thus belongs to the class (6):

\[
\psi_y - \frac{\lambda u_y}{u_x} \psi_x = 0, \quad (A\lambda + B)\psi_t - \frac{(A + B)\lambda u_t}{u_x} \psi_x = 0,
\]

Moreover, it is clear that in order to obtain a simpler recursion operator it is convenient to replace \( \lambda \) by \( 1/\lambda \) and multiply the result by (new) \( \lambda \) so that the operators \( X^j_i \) are simpler than \( X^0_i \).

Thus, we arrive at the following Lax pair for (12):

\[
\begin{align*}
\lambda \psi_y - \frac{u_y}{u_x} \psi_x = 0, \quad (A + B\lambda)\psi_t - \frac{(A + B)u_t}{u_x} \psi_x = 0.
\end{align*}
\]

(13)

Comparing (13) with (6) and taking into account (3) and (4), we readily see that in the case under study \( X^j_i \) have the form

\[
\begin{align*}
X^0_1 &= \frac{u_y}{u_x} D_x, \quad X^0_2 = AD_t - \frac{(A + B)u_t}{u_x} D_x, \quad X^1_1 = D_y, \quad X^1_2 = BD_t,
\end{align*}
\]

and the associated \( X^j_i \) satisfy (5) modulo (12) and its differential consequences.

We now look for a recursion operator for (12) defined by the associated relations (7), i.e.,

\[
\begin{align*}
\tilde{U}_y + f^1_1 \tilde{U} &= \frac{u_y}{u_x} U_x + f^0_1 U, \\
B \tilde{U}_t + f^1_2 \tilde{U} &= -AU_t + \frac{(A + B)u_t}{u_x} U_x + f^0_2 U.
\end{align*}
\]

(14)
By analogy with the previous example, we assume that \( f_j^i \) depend on \( x, y, t, u \) and on first-, second-, and third-order derivatives of \( u \).

We readily see that Theorem 1 applies again and (14) indeed defines a recursion operator for (12) if

\[
f_1^0 = f_1^1 = -\frac{u_{xy}}{u_x}, \quad f_2^0 = f_2^1 = -\frac{Bu_{xt}}{u_x}.
\]

Since \( B \neq 0 \) by assumption, putting \( \kappa = A/B \), dividing the second equation of (14) by \( B \), and using the above formulas for \( f_j^i \) we find that the equations

\[
\begin{align*}
\tilde{U}_y - \frac{u_{xy}}{u_x} \tilde{U} &= \frac{u_y}{u_x} U_x - \frac{u_{xy}}{u_x} U, \\
\tilde{U}_t - \frac{u_{xt}}{u_x} \tilde{U} &= -\kappa U_t + \frac{(\kappa + 1)u_t}{u_x} U_x - \frac{u_{xt}}{u_x} U
\end{align*}
\]

(15)

define a recursion operator for (12) which is equivalent to a special case of the one found in [23].

**Example 3** Let \( d = 2n + 5 \) and \( \bar{x} = (x, y, z, s, t, q_1, \ldots, q_n, r^1, \ldots, r^n) \), where \( n \) is a non-negative integer (for \( n = 0 \) we assume that \( \bar{x} = (x, y, z, s, t) \)).

Consider the \((2n + 5)\)-dimensional equation

\[
\begin{align*}
&u_{xz} - u_{yt} + u_tu_{zs} - u_zu_{st} \\
&\quad + \sum_{i=1}^{n} \left( u_{zq_i} (u_{r^i} + q_iu_{s}) - u_{tq_i} (u_{zr^i} + q_iu_{zs}) \right) = 0
\end{align*}
\]

(16)

which has a Lax pair of the form

\[
\begin{align*}
\lambda \psi_t - \psi_x - u_t \psi_s + \sum_{i=1}^{n} \left( q_iu_{tq_i} \psi_x + u_{tq_i} \psi_{r^i} - (u_{r^i} + q_iu_{s}) \psi_{q_i} \right) &= 0, \\
\lambda \psi_z - \psi_y - u_z \psi_s + \sum_{i=1}^{n} \left( q_iu_{zq_i} \psi_x + u_{zq_i} \psi_{r^i} - (u_{zr^i} + q_iu_{zs}) \psi_{q_i} \right) &= 0.
\end{align*}
\]

(17)

It is understood here and below that for \( n = 0 \) all sums \( \sum_{i=1}^{n} \cdots \) identically vanish.

Notice that unlike the other examples in the present paper equation (16) is not translation invariant – it explicitly involves \( q_i \).

Equation (16) has a number of remarkable reductions and special cases. For instance, if \( u_s = 0 \), we recover a special case of the \((2n + 4)\)-dimensional equation (4) from [21]; in particular, for \( n = 1 \) and \( u_s = 0 \) we have the six-dimensional heavenly equation from [33], cf. also [7] and references therein. On the other hand, (16) for \( n = 0 \) becomes a five-dimensional equation studied e.g. in [1]. It could be of interest to study other reductions for (16) as well.

Moreover, (16) admits a geometric interpretation: it is nothing but the anti-self-dual Yang–Mills equation on a 4-manifold with local coordinates \( x, y, z, t \) upon a suitable choice of the gauge (cf. e.g. Example 5 in [35] and references therein) for the case when the associated gauge Lie algebra is that of functions with respect to the contact bracket on \( \mathbb{R}^{2n+1} \) with coordinates \( s, q_1, \ldots, q_n, r^1, \ldots, r^n \) and contact form \( ds - \sum_{i=1}^{n} q_idr^i \); the Lax pair (17) is
written in terms of associated contact vector fields; cf. e.g. [37] and references therein for further details on contact geometry.

According to the general procedure outlined in the previous section we can readily identify the $\mathcal{X}^i_j$ associated with (17) and verify that the relevant $\mathcal{X}_i$ satisfy (5) modulo (16) and its differential consequences, and thus we are to look for a recursion operator for (16) defined by the formulas

\[
\tilde{U}_t + f^1_1 \tilde{U} = U_s + u_t U_s + f^0_1 U - \sum_{i=1}^n (q_i U_{q_i} U_s + u_{t q_i} U_i) - (u_{t r^i} + q_i u_{t s}) U_q, \\
\tilde{U}_z + f^2_2 \tilde{U} = U_y + u_z U_s + f^0_2 U - \sum_{i=1}^n (q_i U_{z q_i} U_s + u_{z q_i} U_i) - (u_{z r^i} + q_i u_{z s}) U_q,
\]

(18)

where $f^i_j$ are local functions of $\vec{x}$, $u$ and of derivatives of $u$ up to order three that we should determine.

Just as in the previous examples, we readily find that there indeed exist $f^i_j$ such that Theorem 1 applies and (18) defines a recursion operator for (16), namely,

\[
f^1_1 = f^2_2 = 0, \quad f^0_1 = -u_{t s}, \quad f^0_2 = -u_{z s}.
\]

(19)

The recursion operator for (16) defined via (18) and (19) is a special case of the well-known recursion operator for the anti-self-dual Yang–Mills equations, see Example 5 in [35] and references therein, and can be recovered from the latter recursion operator if the gauge Lie algebra in question is specified as above.

**Example 4** Consider the six-dimensional ($d = 6$) equation

\[
us uz - u_z us - u_s us_x + u_y us_x - u_y ur_z + u_z ur_y = 0,
\]

(20)

derived in [35] from a certain first-order system from [9]; we now have $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, $x^5 = r$, $x^6 = s$.

Eq.(20) has [35] a Lax pair of the form (6)

\[
\psi_x - u_s \psi_s - \lambda \psi_x + \lambda u_s \psi_r = 0, \\
\psi_y - u_s \psi_s - \lambda \psi_t + \lambda u_s \psi_r = 0.
\]

To simplify the form of the sought-for recursion operator it is convenient to replace $\lambda$ by $1/\lambda$ and multiply the result by (new) $\lambda$ just as we did in Example 2. Identifying the relevant $\mathcal{X}^i_j$ etc. is left as an exercise for the reader.

Thus, we are to look for a recursion operator for (20) defined by the formulas

\[
\tilde{U}_y - \frac{uy}{us} \tilde{U}_s + f^1_1 \tilde{U} = -\frac{uy}{us} U_r + U_r + f^0_1 U, \\
\tilde{U}_z - \frac{uz}{us} \tilde{U}_s + f^2_2 \tilde{U} = -\frac{uz}{us} U_r + U_r + f^0_2 U,
\]

where $f^i_j$ depend at most on $x, y, z, r, s, t, u$ and derivatives of $u$ up to order 3.
Again, it is readily checked that Theorem 1 applies, a recursion operator for (20) does exist, and it has the form

\[
\begin{align*}
\tilde{U}_y - \frac{u_y}{u_s} \tilde{U}_s &= - \frac{u_y}{u_s} U_r + U_f + \frac{(u_{xy} - u_{xt})}{u_s} U, \\
\tilde{U}_z - \frac{u_z}{u_s} \tilde{U}_s &= - \frac{u_z}{u_s} U_r + U_x + \frac{(u_{xz} - u_{xs})}{u_s} U,
\end{align*}
\]

(21)

so in this case

\[
\begin{align*}
f^0_1 &= \frac{(u_{xy} - u_{xt})}{u_s}, \\
f^0_2 &= \frac{(u_{xz} - u_{xs})}{u_s}, \\
f^1_1 &= 0, \\
f^1_2 &= 0,
\end{align*}
\]

and thus we have recovered the recursion operator for (20) found earlier in [35].

**Example 5** For our final example let \( d = 4, x^1 = x, x^2 = y, x^3 = z, x^4 = t \), as in Example 1, and consider another equation from [8],

\[
D_t(m) + \alpha m D_x(n) - D_z(n) - \alpha n D_x(m) = 0,
\]

(22)

where \( m = (u_y - u_z)/u_x \), \( n = (u_z - u_t)/u_x \), and \( \alpha \) is an arbitrary constant.

This equation has [8] a Lax pair of the form

\[
\psi_y = c(m + \lambda n) \psi_x + \lambda^2 \psi_t, \\
\psi_z = cn \psi_x + \lambda \psi_t,
\]

where \( c = 1 + \alpha - \lambda \alpha \).

This Lax pair is quadratic in \( \lambda \) but it can, upon expressing \( \lambda \psi_t \) from the second equation and substituting the result into the first one, be rewritten in the form which is linear in \( \lambda \) and thus belongs to the class (6):

\[
\lambda \psi_z + (1 + \alpha - \lambda \alpha) m \psi_x - \psi_y = 0, \\
(1 + \alpha - \lambda \alpha) n \psi_x + \lambda \psi_t - \psi_z = 0.
\]

Identifying \( X^j_i \) from the above Lax pair, checking that the associated \( X^j_i \) satisfy (5) modulo (22) and its differential consequences, and applying the procedure from Section 3 in the same fashion as for the previous examples reveals that Theorem 1 again applies and yields an apparently hitherto unknown recursion operator for (22) of the general form (7), namely,

\[
\begin{align*}
\tilde{U}_z - \alpha \frac{u_z}{u_s} \tilde{U}_s &= (1 + \alpha) \frac{(u_z - u_t)}{u_x} U_x + U_y + f^0_1 U, \\
\tilde{U}_t + \alpha \frac{u_t}{u_s} \tilde{U}_s &= (1 + \alpha) \frac{(u_t - u_z)}{u_x} U_x + U_z + f^0_2 U,
\end{align*}
\]

(23)

where now

\[
\begin{align*}
f^0_1 &= f^1_1 = \frac{(u_{xz} - u_{xt}) - u_{xt}}{u_x}, \\
f^0_2 &= f^1_2 = \frac{(u_{xz} - u_{xt}) - u_{xt}}{u_x}.
\end{align*}
\]

In closing note that yet another example of a recursion operator obtained using our construction presented in Section 3 can be found in [2].
5 Concluding Remarks

In the present article we gave a novel construction for recursion operators for multidimensional integrable second-order scalar PDEs admitting isospectral scalar Lax pairs with Lax operators being first-order differential operators linear in the spectral parameter $\lambda$. This construction was successfully applied to several PDEs, the recursion operators for two of which (Examples 1 and 5) are, to the best of our knowledge, new.

Note that the presence of a Lax pair with the Lax operators of the form (3)–(4), especially with $X_{i,0} \equiv 0$, is typical for the case when the PDE (1) under study is dispersionless, i.e., it can be written as a first-order homogeneous quasilinear system, cf. e.g. [8, 35, 37] and references therein for details.

In this connection it should be pointed out that there is a lot of dispersionless PDEs of the form (1). For example, any quasilinear PDE of the form

$$
\sum_{i=1}^{d} \sum_{j=1}^{d} h_{ij}(\vec{x}, u_{x^1}, \ldots, u_{x^d}) u_{x^i x^j} = 0
$$

(24)

can (cf. e.g. [35]) be turned into a quasilinear homogeneous first-order system for the vector function $v = (u_{x^1}, \ldots, u_{x^d})$ upon being supplemented by the compatibility conditions

$$(v_i)_{x^j} = (v_j)_{x^i}, \quad i, j = 1, \ldots, d, \quad i < j.$$

Thus, any equation of the form (24) is indeed dispersionless.

In fact, all our examples in Section 4 are dispersionless; moreover, all of them belong to the class (24). An example of integrable second-order PDE not of the form (24) is provided e.g. by the general heavenly equation [34] (cf. also [7]), and it is readily verified that its recursion operator, first found in [35], can also be readily obtained using our construction from Section 3.

Of course, dispersionless second-order PDEs that are integrable are rather exceptional, and some of them admit Lax pairs with Lax operators of a significantly more general form than (3)–(4), for instance, involving higher powers of $\lambda$ and/or derivatives w.r.t. $\lambda$, cf. e.g. [20, 35, 37] and references therein.

Notice, however, that in quite a few cases the Lax pairs that are nonlinear in $\lambda$ (but do not involve the derivatives w.r.t. $\lambda$) can still be rewritten in the form which is linear in $\lambda$, cf. e.g. Examples 2 and 3 in Section 4.

It appears that our construction, in spite of its simplicity, allows to reproduce all known to date recursion operators for equations of the form (1) in more than two independent variables, including e.g. those for the six-dimensional heavenly equation, see e.g. [23] and references therein, integrable four-dimensional symplectic Monge–Ampère equations listed in [7], including the general heavenly equation [34], the Martínez Alonso–Shabat equation [27] and its modified counterpart [2, 28], etc.

In fact, it would be very interesting to find out whether there exists an equation of the form (1) in more than two independent variables whose recursion operator while being a Bäcklund autotransformation for the linearized equation would not be reproducible within our approach and, in particular, cannot be written in the form (7) for suitable $\lambda_i^j$ and $f_i^j$; so far we failed to find such an example.

In closing note that our method for construction of recursion operators has no inherent restriction on the order of the equation under study: in principle, it should be applicable not
just to second-order PDEs (1) but also to more general equations of the form

\[ F(\vec{x}, u, (k)u) = 0, \]

where \((k)u\) denotes the derivatives of \(u\) up to order \(k\), and \(k \geq 2\), if the equations in question admit Lax pairs with the Lax operators of the form (3)–(4) where now \(X_j^{i,j}\) can depend on \(\vec{x}, u, (k)u\), and \(f_j^i\) can depend on \(\vec{x}, u, (k+1)u\).

Acknowledgements

This research was supported in part under RVO funding for IČ47813059, and by the Grant Agency of the Czech Republic (GAČR) under grant P201/12/G028.

A part of work on this paper was performed in the course of the author’s visit to Matej Bel University at Banská Bystrica supported by the Slovak Research and Development Agency under the contract No. APVV-15-0439.

Support from the above sources of funding is gratefully acknowledged.

I am pleased to thank Matej Bel University and especially Lubomír Snoha for the warm hospitality extended to me in the course of the above visit.

Many computations in the article were performed using the software Jets [3] for Maple® whose use is gratefully acknowledged.

It is my pleasure to thank Igor Leite Freire for helpful comments and the anonymous referees for useful suggestions.

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