A note on stochastic Fubini’s theorem and stochastic convolution

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Abstract

We provide a version of the stochastic Fubini’s theorem which does not depend on the particular stochastic integrator chosen as far as the stochastic integration is built as a continuous linear operator from an $L^p$ space of Banach space-valued processes (the stochastically integrable processes) to an $L^p$ space of Banach space-valued paths (the integrated processes). Then, for integrators on a Hilbert space $H$, we consider stochastic convolutions with respect to a strongly continuous map $R : (0, T] \rightarrow L(H)$, not necessarily a semigroup. We prove existence of predictable versions of stochastic convolutions and we characterize the measurability needed by operator-valued processes in order to be convoluted with $R$. Finally, when $R$ is a $C_0$-semigroup and the stochastic integral provides continuous paths, we show existence of a continuous version of the convolution, by adapting the factorization method to the present setting.

Key words: stochastic Fubini’s theorem, stochastic convolution.

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1 Introduction

In this note we prove a stochastic Fubini’s theorem and apply it to obtain existence of predictable/continuous versions of stochastic convolutions. We do not choose any particular stochastic integrator. We look at the stochastic integration simply as a linear and continuous operator $\mathcal{L}$ from an $L^p$ space of Banach space-valued processes, the stochastically integrable processes, to another $L^p$ space, containing functions whose values are the paths of the stochastic integrals. The paths do not need to be continuous. Within this setting, the continuity assumption on $\mathcal{L}$ plays the role of Itô’s isometry or of the Burkholder-Davis-Gundy inequality in the standard construction of stochastic integrals with respect to square integrable continuous martingales.

For such an operator $\mathcal{L}$, we prove the stochastic Fubini’s theorem (Theorem 2.3). The result can be applied e.g. to stochastic integration in infinite dimensional spaces with respect to $L^p$-integrable martingales ([10, Ch. 8]) or more general martingale-valued measures (for the finite dimensional case, see e.g. [2, Ch. 4]), generalizing standard results as [4, Theorem 4.33], [8, Theorem 2.8], [10, Theorem 8.14].

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1.1 Consider the convolution process

We assume that

By using the stochastic Fubini's theorem, we show that \( (\mathcal{I}(1_{[0,t]}(\cdot)R(t-\cdot)\Phi))_t \), \( t \in [0,T] \).

(1.1)

By using the stochastic Fubini's theorem, we show that (1.1) admits a jointly measurable version (Theorem 3.5). The joint measurability of the stochastic convolution is of interest e.g. when its paths must be integrated, as it happens in the factorization formula ([4, Theorem 5.10]). We also provide a characterisation of the measurability needed by functions \( \Phi: \Omega \times [0,T] \rightarrow L(U,H) \) in order that \( 1_{[0,t]}(\cdot)R(t-\cdot)\Phi \) has the necessary measurability required by the operator \( \mathcal{I} \) (Theorem 3.10). This measurability result turns out to be useful e.g. in order to understand what are the most general measurability conditions for coefficients of stochastic differential equations in Hilbert spaces for which mild solutions are considered.

Finally, in case \( \mathcal{I} \) takes values in a space of processes with continuous paths and \( R = S \) is a \( C_0 \)-semigroups, by adapting the factorization method to the present setting, we show that (1.1) admits a continuous version (Theorem 3.13).

2 Stochastic Fubini’s theorem

Throughout this section, \( (G,\mathcal{G},\mu) \) and \( (D_2,\mathcal{D}_2,\nu_2) \) are positive finite measure spaces, \( (D_1,\mathcal{D}_1) \) is a measurable space, and \( \nu_1 \) is a kernel from \( D_2 \) to \( D_1 \), i.e.

\[ \nu_1: \mathcal{D}_1 \times D_2 \rightarrow \mathbb{R}^+ \]

is such that

(i) \( \nu_1(A,\cdot) \) is \( \mathcal{D}_2 \)-measurable, for all \( A \in \mathcal{D}_1 \);

(ii) \( \nu_1(\cdot,x) \) is a positive measure, for all \( x \in D_2 \).

We assume that

\[ C := \int_{D_2} \nu_1(D_1,x)\nu_2(dx) < \infty. \]

Let \( D := D_1 \times D_2 \). On \( (D,\mathcal{D}_1 \otimes \mathcal{D}_2) \), we define the measure \( \nu \) by

\[ \nu(A) := \int_{D_2} \left( \int_{D_1} 1_A(x_1,x_2)\nu_1(dx_1,x_2) \right) \nu_2(dx_2), \quad \forall A \in \mathcal{D}_1 \otimes \mathcal{D}_2. \]

Notice that \( \nu(D) = C \) is finite.

Let \( \mathcal{D} \) be a given sub-\(\sigma\)-algebra of \( \mathcal{D}_1 \otimes \mathcal{D}_2 \). When we consider measurability or integrability with respect to \( G \) (resp. \( D_1, D_2, D_1 \times D_2, D \)), we always mean it with respect to the space \( (G,\mathcal{G},\mu) \) (resp. \( (D_1,\mathcal{D}_1,\nu_1), (D_2,\mathcal{D}_2,\nu_2), (D,\mathcal{D},\nu) \)). According to that, if we write, for example \( L^1(D,V) \), for some Banach space \( V \), we mean \( L^1((D,\mathcal{D},\nu),V) \), and similarly for other spaces of integrable functions on \( G, D_1, D_2, D \).

Let \( E \) be a given Banach space. For \( p, q \in [1,\infty) \), we denote by \( L^{p,q}_{\mathcal{D}}(E) \) the space of measurable functions \( f: (D,\mathcal{D}) \rightarrow E \) such that
(i) there exists $N \in \mathcal{D}$ such that $\nu(N) = 0$ and $f(G \setminus N)$ is separable;

(ii) the following integrability condition holds:

$$|f|_{p,q} := \left( \int_{D_2} \left( \int_{D_1} |f(x,y)|_p^q \nu_1(dx,y) \right)^{q/p} \nu_2(dy) \right)^{1/q} < \infty.$$  

It is not difficult to see that $(L^{p,q}_{\mathcal{E}}(E), |\cdot|_{L^{p,q}_{\mathcal{E}}(E)})$ is a Banach space, with the usual identification $f = g$ if and only if $f = g$ $\nu$-a.e.. Indeed, if $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^{p,q}_{\mathcal{E}}(E)$, then it is Cauchy also in $L^1((D, \mathcal{D}, \nu), E)$. Passing to a subsequence if necessary, we may assume that $f_n \to f$ $\nu$-a.e., for some $f \in L^1((D, \mathcal{D}, \nu), E)$. Now Fatou’s lemma gives $f \in L^{p,q}_{\mathcal{E}}(E)$ and $f_n \to f$ in $L^{p,q}_{\mathcal{E}}(E)$.

Finally, we use the short notation $L^1(D \times G, E)$ for the space

$$L^1((D \times G, \mathcal{D} \otimes \mathcal{G}, \nu \otimes \mu), E).$$

We will prove the stochastic Fubini’s theorem first for simple functions and then for the general case through approximation. We need the following preparatory lemma.

**Lemma 2.1.** Let $p, q \in [1, \infty)$ and $f \in L^1(G, L^{p,q}_{\mathcal{G}}(E))$. If $q > 1$, assume that

$$C(p, q) := \left( \int_{D_2} (\nu_1(D_1, x))^{q-1} \nu_2(dx) \right)^{q/p-1} < \infty. \quad (2.1)$$

If $q = 1$ and $p > 1$, assume that

$$C(p, 1) := \sup_{x \in D_2} (\nu_1(D_1, x))^{p-1} < \infty. \quad (2.2)$$

Define $C(1, 1) := 1$. Then there exist measurable functions

$$\tilde{f} : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E \quad (2.3)$$

$$\tilde{f}_n : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E, \ n \in \mathbb{N} \quad (2.4)$$

such that

$$\tilde{f}(\cdot, y) \in L^{p,q}_{\mathcal{G}}(E), \ \forall y \in G, \quad (2.5)$$

$$G \to L^{p,q}_{\mathcal{G}}(E), \ y \to \tilde{f}(\cdot, y) \text{ is measurable} \quad (2.6)$$

$$\tilde{f}(\cdot, y) = f(y) \text{ in } L^{p,q}_{\mathcal{E}}(E) \mu\text{-a.e. } y \in G, \quad (2.7)$$

$$\tilde{f}_n(\cdot, y) \in L^{p,q}_{\mathcal{G}}(E), \ \forall y \in G, \ \forall n \in \mathbb{N}, \quad (2.8)$$

$$G \to L^{p,q}_{\mathcal{G}}(E), \ y \to \tilde{f}_n(\cdot, y) \text{ is a simple function, } \forall n \in \mathbb{N}, \quad (2.9)$$

$$\lim_{n \to \infty} \int_G \left( \int_{D_2} \left( \int_{D_1} |\tilde{f}_n(x_1, x_2), y) - \tilde{f}(x_1, x_2, y)|_E^q \nu_1(dx_1, x_2) \right)^{q/p} \nu_2(dx_2) \right)^{1/q} \mu(dy) = 0 \quad (2.10)$$
\[ \tilde{f}(x, \cdot) \in L^1(G, E), \forall x \in D, \quad (2.11) \]
\[ D \to L^1(G, E), x \mapsto \tilde{f}(x, \cdot), \text{ belongs to } L^{p,q}(D, L^1(G, E)) \quad (2.12) \]
\[ \tilde{f}_n(x, \cdot) \in L^1(G, E), \forall x \in D, \forall n \in \mathbb{N}, \quad (2.13) \]
\[ D \to L^1(G, E), x \mapsto \tilde{f}_n(x, \cdot), \text{ belongs to } L^{p,q}(D, L^1(G, E)) \quad (2.14) \]
\[ \lim_{n \to \infty} \int_{D_2} \left( \int_{D_1} \left| \tilde{f}_n((x_1, x_2), \cdot) - \tilde{f}((x_1, x_2), \cdot) \right|_{L^1(G, E)}^p \nu_1(d x_1, x_2) \right)^{q/p} \nu_2(d x_2) = 0. \quad (2.15) \]

**Proof.** Since \( f \) is Bochner integrable, without loss of generality we can assume that \( f(G) \) is separable. Then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) of \( L^{p,q}_{\mathcal{G}}(E) \)-valued simple functions such that

\[ \lim_{n \to \infty} f_n(y) = f(y) \text{ in } L^{p,q}_{\mathcal{G}}(E), \forall y \in G, \quad (2.16) \]
\[ \lim_{n \to \infty} |f_n - f|_{L^1(G, L^{p,q}_{\mathcal{G}}(E))} = 0. \quad (2.17) \]

Each \( f_n \) can be written in the form

\[ f_n(y) = \sum_{i=1}^{M(n)} 1_{A^n_i}(y) \varphi_{n,i} \quad \forall y \in G, \quad (2.18) \]

where \( M(n) \in \mathbb{N} \), \( A^n_i \in \mathcal{G} \), and \( \varphi_{n,i} \) is a fixed representant of its equivalence class in \( L^{p,q}_{\mathcal{G}}(E) \). For \( n \in \mathbb{N} \), define

\[ \tilde{f}_n : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E, (x, y) \mapsto f_n(y)(x). \]

By using (2.18), we have the measurability of (2.4), and (2.8), (2.9), (2.13), (2.14), are immediately verified.

We claim that the sequence \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \) is Cauchy in \( L^1(D \times G, E) \). Indeed, since \( \varphi_{n,i} \in L^{p,q}_{\mathcal{G}}(E) \), we have \( \tilde{f}_n \in L^1(D \times G, E) \), for every \( n \in \mathbb{N} \). Moreover, by Hölder’s inequality,

\[
\int_{D \times G} |\tilde{f}_n - \tilde{f}_m|_E d(\nu \otimes \mu) = \\
= \int_G \left( \int_{D_2} \left( \int_{D_1} |\tilde{f}_n((x_1, x_2), y) - \tilde{f}_m((x_1, x_2), y)|_E v_1(d x_1, x_2) \right) v_2(d x_2) \right) \mu(d y) \\
\leq C(p, q) \int_G \left( \int_{D_2} \left( \int_{D_1} |\tilde{f}_n((x_1, x_2), y) - \tilde{f}_m((x_1, x_2), y)|_E^p v_1(d x_1, x_2) \right)^{q/p} v_2(d x_2) \right)^{1/q} \mu(d y) \\
= C(p, q) \|f_n - f_m\|_{L^1(G, L^{p,q}_{\mathcal{G}}(E))},
\]

and the last member tends to 0 as \( n \) and \( m \) tend to \( \infty \), by (2.17). Then there exists \( \tilde{f} \in L^1(D \times G, E) \) such that, after replacing \( \{f'_n\}_{n \in \mathbb{N}} \) by a subsequence if necessary,

\[ \lim_{n \to \infty} f_n(x, y) = \tilde{f}(x, y) \quad \forall (x, y) \in (D \times G) \setminus N \quad (2.19) \]
\[ \lim_{n \to \infty} \tilde{f}_n = \tilde{f} \quad \text{in } L^1(D \times G, E), \quad (2.20) \]

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where $N$ is a $\nu \otimes \mu$-null set. We redefine $\tilde{f}$ on $N$ by $\tilde{f}(x,y) := 0$ for $(x,y) \in N$. After such a redefinition, the partial results of the theorem till now proved still hold true.

By (2.19), since we can assume that each $\varphi_{n,i}$ has separable range, we see that the range of $\tilde{f}$ is separable. By measurability of sections of real-valued measurable functions and by Pettis’s measurability theorem (use the fact that the range of $\tilde{f}$ is separable and then use Hahn-Banach theorem to extend continuous linear functionals on the space generated by the range of $\tilde{f}$ to the whole space $E$), we have that

$$\int_G \liminf_{m \to \infty} |\tilde{f}(\cdot,y) - \tilde{f}_m(\cdot,y)|_{p,q} \mu(dy)$$

are measurable, for all $y' \in G$ and all $x' \in D$. Since

$$\leq \lim_{m \to \infty} \int_G \left( \int_{D_2} \left( \int_{D_1} |\tilde{f}(x_1,x_2),y) - \tilde{f}_m(x_1,x_2),y)|^p_{E} v_1(dx_1,x_2) \right)^{\frac{q}{p}} v_2(dx_2) \right)^{\frac{1}{q}} \mu(dy)$$

$$\leq \lim_{m \to \infty} \liminf_n |f_n - f_m|_{L^1(G,L^p,q(E))} = 0,$$

we have

$$\liminf_{m \to \infty} |\tilde{f}(\cdot,y) - \tilde{f}_m(\cdot,y)|_{p,q} = 0 \quad \mu\text{-a.e. } y \in G. \tag{2.22}$$

By recalling that $\tilde{f}(\cdot,y) \in L^p,q(E)$ for all $y \in G$, (2.22) shows that the map

$$D \to E, \ x \mapsto \tilde{f}(x,y)$$

belongs to $L^p,q(E)$ for all $y \in G \setminus N'$, where $N'$ is a $\mu$-null set. We redefine $\tilde{f}$ on $N'$ by $\tilde{f}(x,y) := 0$ for $(x,y) \in D \times N'$. Again, we notice that the partial results of the theorem till now proved still hold true after the redefinition on $D \times N_1$. In addition, \[
\forall y \in G, \text{ the map } D \to E, \ x \mapsto \tilde{f}(x,y), \text{ belongs to } L^p,q(E). \]

This provides (2.5). Moreover, since $N'$ can be chosen such that (2.22) holds for all $y \in G \setminus N'$ and since $G \to L^p,q(E)$, $y \mapsto \tilde{f}_n(\cdot,y) = f_n(y)$, is measurable, for all $n \in \mathbb{N}$, also (2.6) is proved. From the last inequality of (2.21), (2.10) follows. From (2.16) and (2.22), (2.7) follows as well.

By Hölder’s inequality, we have $|\tilde{f}|_{L^1(D \times G,E)} \leq C(p,q)|\tilde{f}|_{L^1(G,L^p,q(E))} < \infty$. Then, after redefining $\tilde{f}$ on a set $N'' \times G$, where $N''$ is a $\nu$-null set, by $\tilde{f}(x,y) := 0$ for $(x,y) \in N'' \times G$, we have

$$\forall x \in D, \text{ the map } G \to E, \ y \mapsto \tilde{f}(x,y), \text{ belongs to } L^1(G,E).$$

This provides (2.11). By applying Minkowski’s inequality for integrals twice (see [7, p. 194, 6.19]), we have

$$\lim_{n \to \infty} \left( \int_{D_2} \left( \int_{D_1} \left( \int_{G} |\tilde{f}(x_1,x_2),y) - \tilde{f}_n(x_1,x_2),y)|^p_{E} v_1(dx_1,x_2) \right)^{\frac{q}{p}} v_2(dx_2) \right)^{\frac{1}{q}} \mu(dy)$$

$$\leq \lim_{n \to \infty} \left( \int_{D_2} \left( \int_{D_1} \left( \int_{G} |\tilde{f}(x_1,x_2),y) - \tilde{f}_n(x_1,x_2),y)|^p_{E} v_1(dx_1,x_2) \right)^{\frac{1}{p}} \mu(dy) \right)^{\frac{q}{p}} v_2(dx_2) \right)^{\frac{1}{q}} \mu(dy).$$
Since the latter member tends to 0 because of the second inequality in (2.21), the estimate above provides (2.12) and (2.15), after redefining \( \tilde{f} \) on a set \( N'' \times G \), where \( N'' \) is a suitably chosen \( \nu \)-null set, by \( f(x, y):= 0 \) for \( (x, y) \in N'' \times G \).

Let \( T > 0 \) and let \( B_T \) be a short notation for the Borel \( \sigma \)-algebra \( B_{[0,T]} \) on \([0,T] \). We recall that, if \( \mathcal{T} \) is a topological space, then \( B_{\mathcal{T}} \) denotes the Borel \( \sigma \)-algebra of \( \mathcal{T} \)\(^1\). Let \( (\Omega,\mathcal{F},\mathbb{P}) := \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}\} \) be a complete filtered probability space. We endow the product space \( \Omega_T := \Omega \times [0,T] \) with the \( \sigma \)-algebra \( \mathcal{P}_T \) of predictable sets associated to the filtration \( \mathbb{F} \) and the measurable space \((\Omega_T, \mathcal{P}_T)\) with the product measure \( \mathbb{P} \otimes m \), where \( m \) denotes the Lebesgue’s measure. We need to introduce some further notation.

- \( F \) is a Banach space;
- \( \mathbb{T} \subset B_b([0,T], F) \) is a closed subspace (with respect to the norm \(| \cdot |_\infty \)) such that
  \[
  \mathbb{T} \times [0,T] \to F, \quad (x, t) \mapsto x(t);
  \]  
  (2.23)
  is Borel measurable, when \( \mathbb{T} \times [0,T] \) is endowed with the product \( \sigma \)-algebra \( \mathcal{B}_{\mathbb{T}} \otimes \mathcal{B}_{[0,T]} \) (and not just with the Borel \( \sigma \)-algebra of the product topology!).
- \( \mathcal{P}' \) is a given sub-\( \sigma \)-algebra of \( \mathcal{F}_T \otimes \mathcal{B}_T \) such that, for all \( A \in \mathcal{F}_T \) with \( \mathbb{P}(A) = 0 \), \( A \times [0,T] \in \mathcal{P}' \).
- \( \mathcal{L}^0_{\mathcal{P}'}(\mathbb{T}) \) is the vector space of measurable functions
  \[
  X : (\Omega_T, \mathcal{P}') \to F
  \]
  such that, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), the path
  \[
  X(\omega) : [0,T] \to F, \quad t \mapsto X_t(\omega)
  \]
  belongs to \( \mathbb{T} \), and the \( \mathbb{P} \)-a.e. defined map
  \[
  (\Omega, \mathcal{F}_T) \to \mathbb{T}, \quad \omega \mapsto X(\omega)
  \]  
  (2.24)
  is measurable, when \( \mathbb{T} \) is endowed with the Borel \( \sigma \)-algebra induced by the norm \(| \cdot |_{\infty} \).
- For \( r \in [1,\infty) \), \( \mathcal{L}^r_{\mathcal{P}'}(\mathbb{T}) \) denotes the space of (equivalence classes of) \( X \in \mathcal{L}^0_{\mathcal{P}'}(\mathbb{T}) \) such that (2.24) has separable range and
  \[
  |X|_{\mathcal{L}^r_{\mathcal{P}'}(\mathbb{T})} := \left( \mathbb{E} \left[ |X|_\infty^r \right] \right)^{1/r} < \infty.
  \]
  Then \( (\mathcal{L}^r_{\mathcal{P}'}(\mathbb{T}), | \cdot |_{\mathcal{L}^r_{\mathcal{P}'}(\mathbb{T})}) \) is a Banach space.

**Remark 2.2.** The space \( \mathbb{T} \) can be e.g. \( C_b([0,T], F) \), because in such a case (2.23) is continuous, hence measurable. This permits also to consider \( \mathbb{T} \) as the space of left-limited right-continuous functions, because, if \( \varphi \) is real valued and continuous with support \([0,1]\) and if \( \varphi_\varepsilon(t) = \varepsilon^{-1} \varphi(\varepsilon^{-1} t) \), then \( \varphi_\varepsilon \circ \mathbb{x} \) converges pointwise to \( \mathbb{x} \) everywhere on \([0,T]\) as \( \varepsilon \to 0^+ \), after extending \( \mathbb{x} \) by continuity beyond \( T \). We finally observe that (2.23) is measurable whenever \( \mathbb{T} \) is separable: this comes from a straightforward application of [1, Lemma 4.51].

\(^1\)No topological space will be denoted by \( T \), hence there will not be any confusion with \( B_T \).
We now provide the main result of this section.

**Theorem 2.3** (Stochastic Fubini’s theorem). Let $p, q, r \in [1, \infty)$, $g \in L^1(G, L^{p,q}_\mathcal{D}(E))$. Let

$$\mathcal{L} : L^{p,q}_\mathcal{D}(E) \to \mathcal{L}^r_{\mathcal{D}}(\mathbb{T})$$

be a linear and continuous operator. Then there exist measurable functions

$$X_1 : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E$$
$$X_2 : (\Omega_T \times G, (\mathcal{F}_T \otimes \mathcal{B}_T) \otimes \mathcal{G}) \to F$$

such that

$$X_1(x, \cdot) \in L^1(G, E), \forall x \in D, \text{ and } X_2((\omega, t), \cdot) \in L^1(G, F), \forall (\omega, t) \in \Omega_T$$

$$D \to L^1(G, E), x \mapsto X_1(x, \cdot) \in L^{p,q}_\mathcal{D}(L^1(G, E))$$

$$(\Omega_T, \mathcal{F}_T) \to L^1(G, F), (\omega, t) \mapsto X_2((\omega, t), \cdot), \text{ is measurable}$$

$$X_1(\cdot, y) \in L^{p,q}_\mathcal{D}(E), \forall y \in G$$

$$G \to L^{p,q}_\mathcal{D}(E), y \mapsto X_1(\cdot, y) \in L^1(G, L^{p,q}_\mathcal{D}(E))$$

$$X_1(\cdot, y) = g(y) \text{ in } L^{p,q}_\mathcal{D}(E) \text{ for } \mu\text{-a.e. } y \in G$$

$$X_2(\cdot, y) \in \mathcal{L}^r_{\mathcal{D}}(\mathbb{T}), \forall y \in G$$

$$X_2(\cdot, y) = \mathbb{L}g(y) \text{ in } \mathcal{L}^r_{\mathcal{D}}(\mathbb{T}), \mu\text{-a.e. } y \in G \quad (2.25a)$$

and such that

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, (\mathbb{L}Y)(\omega, t) = \int_G X_2((\omega, t), y)\mu(dy), \forall t \in [0, T], \quad (2.26)$$

where

$$Y(x) := \int_G X_1(x, y)\mu(dy), \forall x \in D.$$  

**Proof.** By Lemma 2.1, there exist measurable functions

$$\tilde{f} : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E,$$
$$\tilde{f}_n : (D \times G, \mathcal{D} \otimes \mathcal{G}) \to E, \quad n \in \mathbb{N}$$

satisfying (2.11)–(2.15). For $n \in \mathbb{N}$, $\tilde{f}_n$ has the form

$$\tilde{f}_n(x, y) = \sum_{i=1}^{M(n)} 1_{A_i^n}(y)\varphi_{n,i}(x) \quad \forall x \in D, \forall y \in G,$$

where $\varphi_{n,i}$ is a fixed representant of its class in $L^{p,q}_\mathcal{D}(E)$. For all $n \in \mathbb{N}$, the function $\tilde{f}_n^{(\mu)}$ defined by

$$\tilde{f}_n^{(\mu)} : D \to E, x \mapsto \int_G \tilde{f}_n(x, y)\mu(dy) = \sum_{i=1}^{M(n)} \varphi_{n,i}(x)\mu(A_i^n)$$

belongs to $L^{p,q}_\mathcal{D}(E)$. Then, if we define

$$\tilde{f}^{(\mu)} : D \to E, x \mapsto \int_G \tilde{f}(x, y)\mu(dy),$$

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due to (2.15), we have
\[
\lim_{n \to \infty} \tilde{f}_n^{(\mu)} = \tilde{f}^{(\mu)} \text{ in } L_{\| \cdot \|}^{p,q}(E).
\] (2.27)

By linearity of \( \mathcal{L} \), we have
\[
\mathcal{L} \tilde{f}_n^{(\mu)} = \sum_{i=1}^{M(n)} \mu(A_i^n) \mathcal{L} \varphi_{n,i} \text{ in } L_{\| \cdot \|}^p(\mathbb{T}).
\] (2.28)

By continuity of \( \mathcal{L} \), (2.27) and (2.28) give
\[
\lim_{n \to \infty} \sum_{i=1}^{M(n)} \mu(A_i^n) \mathcal{L} \varphi_{n,i} = \mathcal{L} \tilde{f}^{(\mu)} \text{ in } L_{\| \cdot \|}^p(\mathbb{T}).
\] (2.29)

For \( n \in \mathbb{N} \), we now consider the measurable function
\[
\tilde{f}_n^{(\Omega)} : (\Omega_T \times G, \mathcal{D} \otimes \mathcal{G}) \to F, ((\omega, t), y) \mapsto \sum_{i=1}^{M(n)} \mathbf{1}_{A_i^n}(y) (\mathcal{L} \varphi_{n,i})(\omega, t)
\]
where here \( \mathcal{L} \varphi_{n,i} \) is a fixed representant of its class in \( L_{\| \cdot \|}^p(\mathbb{T}) \). For all \( y \in G \), \( \tilde{f}_n^{(\Omega)}(\cdot, y) \) is a representant of the class of \( \mathcal{L} \{ \tilde{f}_n(\cdot, y) \} \) in \( L_{\| \cdot \|}^p(\mathbb{T}) \). Moreover,
\[
\int_G \tilde{f}_n^{(\Omega)}((\omega, t), y) \mu(dy) = \sum_{i=1}^{M(n)} \mu(A_i^n) (\mathcal{L} \varphi_{n,i})(\omega, t) \quad \forall (\omega, t) \in \Omega_T, \forall n \in \mathbb{N}.
\]

By (2.28), we obtain
\[
\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \int_G \tilde{f}_n^{(\Omega)}((\omega, t), y) \mu(dy) = (\mathcal{L} \tilde{f}_n^{(\mu)})(\omega, t) \forall t \in [0, T].
\] (2.30)

We now show that we can pass to the limit in (2.30). By (2.10),
\[
\lim_{n \to \infty} \int_G |\tilde{f}_n(\cdot, y) - \tilde{f}(\cdot, y)|_{L_{\| \cdot \|}^{p,q}(E)} \mu(dy) = 0,
\]

hence, by continuity of \( \mathcal{L} \),
\[
\lim_{n \to \infty} \int_G \left| \mathcal{L} \{ \tilde{f}_n(\cdot, y) \} - \mathcal{L} \{ \tilde{f}(\cdot, y) \} \right|_{L_{\| \cdot \|}^p(\mathbb{T})} \mu(dy) = 0.
\] (2.31)

Since \( L_{\| \cdot \|}^p(\mathbb{T}) \) is a closed subspace of
\[
L^r(\Omega, \mathbb{T}) := L^r(\mathbb{R}^d, (\mathcal{D}, \mathcal{G}), (\mathbb{P}, |\cdot|_\infty)),
\]
the map
\[
(G, \mathcal{D}) \to L^r(\Omega, \mathbb{T}), \ y \mapsto \mathcal{L} \{ \tilde{f}(\cdot, y) \}
\] (2.32)

is measurable and integrable (the range of (2.32) is separable). By applying Lemma 2.1 again, now to (2.32), we have that there exists a measurable function
\[
g : (\Omega \times G, \mathcal{D} \otimes \mathcal{G}) \to \mathbb{T}
\] (2.33)
such that, for some $A \in \mathcal{B}$ with $\mu(A^c) = 0$,

$$g(\cdot, y) = \mathcal{L}(\tilde{f}(\cdot, y)) \text{ in } L'(\Omega, \mathcal{T}), \forall y \in A. \quad (2.34)$$

Define

$$X_2((\omega, t), y) := \begin{cases} g(\omega, y)(t) & \forall((\omega, t), y) \in \Omega_T \times A \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, since $\mathcal{L}(\tilde{f}(\cdot, y))$ is $\mathcal{P}'$-measurable for all $y \in G$ (by definition of $\mathcal{L}$) and since $\mathcal{P}'$ contains the sets $N \times [0, T]$ when $N \in \mathcal{F}_T$ and $\mathbb{P}(N) = 0$, we have, by (2.34), that $X_2(\cdot, y)$ is $\mathcal{P}'$-measurable for all $y \in G$. Moreover, since the evaluation map (2.23) is assumed to be measurable, by measurability of (2.33) and by definition of $X_2$ we have that

$$X_2: (\Omega_T \times G, (\mathcal{F}_T \otimes \mathcal{B}_G) \otimes \mathcal{F}) \rightarrow F$$

is measurable. By (2.31), we can write

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left( \int_{G} \sup_{t \in [0, T]} \left| \tilde{f}_n((\omega, t), y) - X_2((\omega, t), y) \right|_F \mu(dy) \right) \mathbb{P}(d\omega) \leq \lim_{n \rightarrow \infty} \int_G \left( \int_{[0, T]} \sup_{\Omega} \left| \tilde{f}_n((\omega, t), y) - X_2((\omega, t), y) \right|_F \mathbb{P}(d\omega) \right)^{1/r} \mu(dy) \quad (2.35)$$

where the measurability of $||\tilde{f}_n((\omega, \cdot), y) - X_2((\omega, \cdot), y)||_\infty$, jointly in $(\omega, y)$, is due to the measurability of (2.33), to the definition of $X_2$, and to the definition of $\tilde{f}_n$. By (2.35), by considering a subsequence if necessary, it follows that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \int_G \tilde{f}_n((\omega, t), y) \mu(dy) - \int_G X_2((\omega, t), y) \mu(dy) \bigg|_F = 0 \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (2.36)$$

By (2.28), (2.29), (2.30), and (2.36), we conclude that, for $\mathbb{P}$-a.e. $\omega \in \Omega,$

$$(\mathcal{L}(\tilde{f}(\omega))(\omega, t) = \int_G X_2((\omega, t), y) \mu(dy), \forall t \in [0, T],$$

which provides (2.26), after defining $X_1 := \tilde{f}.$

\section{Stochastic convolution}

One of the contents of Theorem 2.3 is the existence of the jointly measurable function $X_2$, whose sections $X_2(\cdot, y)$ coincide with the “stochastic integral” $\mathcal{L}g(y)$, for a.e. $y$. This fact permits to obtain a jointly measurable version of a stochastic convolution, as we will explain in the present section.

Let us recall/introduce the following notation. We consider separable Hilbert spaces $H$ and $U$, with scalar product $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_U$, respectively.
• $L_2(U,H)$ denotes the space of Hilbert-Schmidt linear operators from $U$ into $H$.

Let $E$ be a Banach space.

• If $E$ is a Banach space, $L^0_{\mathcal{P}_T}(E)$ denotes the space of $E$-valued $\mathcal{P}_T/\mathcal{B}_E$-measurable processes $\Phi: \Omega_T \rightarrow E$, for which there exists $N \in \mathcal{P}_T$ with $\mathbb{P} \otimes m(N) = 0$ such that $X(\Omega_T \setminus N)$ is separable. Two processes are equal in $L^0_{\mathcal{P}_T}(E)$ if they coincide $\mathbb{P} \otimes m$-a.e.. The space $L^0_{\mathcal{P}_T}(E)$ is a complete metrizable space when endowed with the topology induced by the convergence in measure (see [9, Sec. 5.2]).

• $L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ denotes the space of (equivalence classes of) $E$-valued $\mathcal{P}_T \otimes \mathcal{B}_T/\mathcal{B}_E$-measurable processes $\zeta: \Omega_T \times [0,T] \rightarrow E$, with separable range, up to a modification on a $(\mathbb{P} \otimes m) \otimes m$-null set if necessary. Two processes are equal in $L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ if they coincide $(\mathbb{P} \otimes m) \otimes m$-a.e. $L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ is endowed with the metrizable complete vector topology induced by the convergence in measure.

• For $p, q \in [1, \infty)$, $L^{p,q}_{\mathcal{P}_T}(E)$ denotes the subspace of $L^0_{\mathcal{P}_T}(E)$ whose members $X$ satisfy

$$|X|_{p,q} = \left( \int_0^T \mathbb{E} \left[ |X_t|^p \right] dt \right)^{1/p} < \infty.$$ 

($L^{p,q}_{\mathcal{P}_T}(E), |\cdot|_{p,q}$) is a Banach space. The space $L^{p,q}_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ is defined similarly to $L^{p,q}_{\mathcal{P}_T}(E)$, after replacing $\mathcal{P}_T$ by $\mathcal{P}_T \otimes \mathcal{B}_T$. We use the notation $L^p_{\mathcal{P}_T}(E), L^p_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$, for $L^{p,q}_{\mathcal{P}_T}(E), L^{p,q}_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$, respectively.

• For $p, q, r \in [1, \infty)$. $L^{p,q,r}_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ denotes the space containing those $\zeta \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(E)$ such that

$$|\zeta|_{p,q,r} := \left( \int_0^T \left( \int_0^T \mathbb{E} \left[ |\zeta((\omega,s),t)|_E^p \right] ds \right)^{q/r} dt \right)^{1/r} < \infty. \quad (3.1)$$

($L^{p,q,r}_{\mathcal{P}_T \otimes \mathcal{B}_T}(E), |\cdot|_{p,q,r}$) is a Banach space.

### 3.1 Jointly measurable version

In this section we employ Theorem 2.3 to obtain jointly measurable versions of stochastic integrals (represented, as in the previous section, by a generic continuous linear operator $\mathcal{J}$) depending on parameter.

We will often need to consider sections of measurable functions and their measurability with respect to some codomains. We begin with the following lemma.

**Lemma 3.1.** Let $\zeta \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(L_2(U,H))$. Then

$$f_\zeta: [0,T] \rightarrow L^0_{\mathcal{P}_T}(L_2(U,H)), \ t \mapsto \zeta(\cdot,t) \quad (3.2)$$

is measurable.

**Proof.** Let us first suppose that $U = H = \mathbb{R}$, hence $L_2(U,H) = \mathbb{R}$. Define

$$\mathcal{C} := \{ A \in \mathcal{P}_T \otimes \mathcal{B}_T \text{ s.t. } f_{1_A} \text{ is measurable} \}.$$
It is clear that the rectangles of the form \( B \times C \), with \( B \in \mathcal{P}_T \) and \( C \in \mathcal{B}_T \), belong to \( \mathcal{C} \), because \( f_{1_{B \times C}} \) assumes only the two values 0 and 1 on \( \Omega_T \setminus C \) and on \( B \), respectively. If \( A \in \mathcal{C} \), \( B \in \mathcal{C} \), \( B \subset A \), then \( f_{1_A \setminus B} = f_{1_A} - f_{1_B} \) is measurable, and then \( A \setminus B \in \mathcal{C} \). If \( \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{C} \) is an increasing sequence, then \( f_{1_{\cup_{n \in \mathbb{N}} A_n}}(t) = \lim_{n \to \infty} f_{1_{A_n}}(t) \) in \( L^0_{\mathcal{P}_T}(\mathbb{R}) \) for all \( t \in [0,T] \), hence \( \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{C} \). This shows that \( \mathcal{C} \) is a \( \lambda \)-class containing the rectangles \( B \times C \), with \( B \in \mathcal{P}_T \) and \( C \in \mathcal{B}_T \), hence \( \mathcal{P}_T \otimes \mathcal{B}_T \subset \mathcal{C} \). By linearity and by monotone convergence, we have that \( f_\omega \) is measurable for all \( \omega \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(\mathbb{R}) \).

Now let \( U, H \), be generic separable Hilbert spaces and let \( \{\varphi_n\}_{n \in \mathbb{N}} \) be an orthonormal basis for \( L_2(U,H) \) (we consider the case \( \dim L_2(U,H) = \infty \); the case \( < \infty \) is similar). If \( \zeta \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(L_2(U,H)) \), then, for all \( t \in [0,T] \),

\[
 f_\zeta(t)(\omega,s) = \sum_{n \in \mathbb{N}} \langle \varphi_n, \zeta((\omega,s),t) \rangle_{L_2(U,H)} \varphi_n = \sum_{n \in \mathbb{N}} f_{\langle \varphi_n, \zeta \rangle_{L_2(U,H)}}(t)(\omega,s) \quad \forall (\omega,s) \in \Omega_T.
\]

From the first part of the proof, \( f_{\langle \varphi_n, \zeta \rangle_{L_2(U,H)}} \) is measurable, after the identification \( L^0_{\mathcal{P}_T}(\mathbb{R}) = L^0_{\mathcal{P}_T}(\mathbb{R}) \varphi_n \) and the continuous, hence measurable, embedding

\[
 L^0_{\mathcal{P}_T}(\mathbb{R}) \varphi_n \subset L^0_{\mathcal{P}_T}(L_2(U,H)).
\]

We conclude that \( f_\zeta \) is measurable, because it is the pointwise limit of the sequence

\[
 \left\{ \sum_{n=0}^{N} f_{\langle \varphi_n, \zeta \rangle_{L_2(U,H)}} \varphi_n : [0,T] \to L^0_{\mathcal{P}_T}(L_2(U,H)) \right\}_{N \in \mathbb{N}}.
\]

**Remark 3.2.** If \( p,q \in [1,\infty) \), the map

\[
 |\cdot|_{p,q} : L^0_{\mathcal{P}_T}(L_2(U,H)) \to [0,\infty], \ \xi \mapsto |\xi|_{p,q}
\]

is lower-semicontinuous (Fatou’s Lemma). By Lemma 3.1, if \( \zeta \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(L_2(U,H)) \), then

\[
 f_\zeta : [0,T] \to L^0_{\mathcal{P}_T}(L_2(U,H)), \ t \mapsto \zeta(\cdot,t)
\]

is measurable. By combining \( f_\zeta \) with \( |\cdot|_{p,q} \), we have that the set

\[
 B_\zeta := \left\{ t \in [0,T] : \zeta(\cdot,t) \in L^{p,q}_{\mathcal{P}_T}(L_2(U,H)) \right\}
\]

is a Borel set.

Clearly the set \( B_\zeta \) defined in Remark 3.2 depends on the representant of \( \zeta \) chosen in \( L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(L_2(U,H)) \). Hereafter, whenever a notion associated to some function \( f \) belonging to some quotient space of measurable functions is pointwise dependent, we mean that the notion is actually associated to a chosen representant \( f \).

**Notation.** In what follows, we will always use the notation \( B_\zeta \) for the set defined by (3.4), when \( \zeta \in L^0_{\mathcal{P}_T \otimes \mathcal{B}_T}(L_2(U,H)) \). In the notation, we omit the dependence of \( B_\zeta \) on \( p,q \), as it will be always clear from the context.
The next result is a variant of Lemma 3.1 for \( L_{\mathcal{D}^q}^{p,q}(L_2(U,H)) \). It will be used to derive jointly measurable versions of stochastic convolutions.

**Lemma 3.3.** Let \( p,q,r \in [1,\infty) \) and let \( \zeta \in L_{\mathcal{D}^q}^{p,q,r}(L_2(U,H)) \). Let \( B_\zeta \) be the Borel set defined by (3.4). Then \( m([0,T] \setminus B_\zeta) = 0 \) and

\[
f_\zeta : B_\zeta \to L_{\mathcal{D}^q}^p(L_2(U,H)), \ t \mapsto \zeta(\cdot,t)
\]

is Borel measurable.

**Proof.** It is clear that \( m([0,T] \setminus B_\zeta) = 0 \), because \( \zeta \in L_{\mathcal{D}^q}^{p,q,r}(L_2(U,H)) \) and then \( \zeta(\cdot,t) \in L_{\mathcal{D}^q}^{p,q}(L_2(U,H)) \) for m-a.e. \( t \in [0,T] \). By redefining \( \zeta((\omega,s),t) := 0 \) for \( ((\omega,s),t) \in \Omega_T \times [0,T] \), \( t \in [0,T] \setminus B_\zeta \), we can assume that \( B_\zeta = [0,T] \). In such a case, to show that \( f_\zeta \) is Borel measurable, we argue as in the proof of Lemma 3.1, after replacing \( L_0^{q,r}(\mathcal{D}^q) \) by \( L_0^{q,r}(\mathcal{D}^q) \) and \( L_0^{q,r} \) by \( L_0^{p,q} \).

For \( p,q,r \in [1,\infty) \), let

\[
\mathcal{J} : L_{\mathcal{D}^q}^{p,q,r}(L_2(U,H)) \to L_{\mathcal{F}^q}^{r,r}(\mathbb{T})
\]

be a linear and continuous operator, where \( L_{\mathcal{F}^q}^{r,r}(\mathbb{T}) \) is defined as in Section 2 (p. 6), with \( \mathcal{D}' = \mathcal{F} \times \mathcal{B} \).

Let \( \zeta \in L_{\mathcal{D}^q}^{0,r}(L_2(U,H)) \) be a given representant of its class. Our aim is to show that there exists a \((\omega,t)\)-jointly measurable version of the family of random variables

\[
\gamma_\zeta^t := (\mathcal{J}(\zeta(\cdot,t)))_{t \in B_\zeta},
\]

where \( B_\zeta \) is defined by (3.4).

**Remark 3.4.** Definition 3.6 depends on the chosen representant \( \zeta \). If \( \zeta = \zeta' \) in the space \( L_{\mathcal{D}^q}^{0,r}(L_2(U,H)) \), then \( m(B_\zeta \cap B_{\zeta'}) = 0 \), and, due to the fact that \( \mathcal{J} \) has values in \( L_{\mathcal{F}^q}^{r,r}(\mathbb{T}) \), we have \( \gamma_\zeta^t = \gamma_{\zeta'}^t \) \( \mathbb{P} \)-a.e., for all \( t \in B_\zeta \cap B_{\zeta'} \).

**Theorem 3.5.** Let \( p,q,r \in [1,\infty) \), let \( \zeta \in L_{\mathcal{D}^q}^{p,q,r}(L_2(U,H)) \), and let \( B_\zeta \) be the set defined by (3.4). Then there exists a process

\[
\Sigma^\zeta \in L_{\mathcal{F}^q}^{r}(\mathbb{F})
\]

such that

\[
\text{for m-a.e. } t \in B_\zeta, \quad \Sigma^\zeta_t(\omega) = (\mathcal{J}(\zeta(\cdot,t)))_t(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

Moreover, the map

\[
\mathcal{J} : L_{\mathcal{D}^q}^{p,q,r}(L_2(U,H)) \to L_{\mathcal{F}^q}^{r}(\mathbb{F}), \ \zeta \mapsto \Sigma^\zeta
\]

is linear, continuous, uniquely determined by (3.7), (3.8). The operator norm of \( \mathcal{J} \) is bounded by the operator norm of \( \mathcal{J} \).

**Proof.** We apply Theorem 2.3, with the following data:
Moreover, let $X_2$ be the process provided by application of the theorem. Then

$$X_2(\cdot, t) = \mathcal{I}(\cdot, t)) \text{ in } L^p(\mathcal{F}_T \otimes \mathcal{B}_T), \ P\text{-a.e. } t \in B_\zeta. \quad (3.10)$$

Define

$$\Sigma^\zeta_t(\omega) := X_2((\omega, t), t) \quad \forall (\omega, t) \in \Omega_T, \ t \in [0, T].$$

Then $\Sigma^\zeta$ is jointly measurable in $(\omega, t)$, and, by (3.10), for $m\text{-a.e. } t \in B_\zeta$,

$$\Sigma^\zeta_t(\omega) = X_2((\omega, t), t) = (\mathcal{I}(g(t)), \tau(\omega) = (\mathcal{I}(\cdot, t))_t(\omega)) \quad P\text{-a.e. } \omega \in \Omega.$$

Moreover,

$$\int_0^T \mathbb{E}\left[|\Sigma^\zeta_t|_F^r\right] dt = \int_0^T \mathbb{E}\left[|X_2((\cdot, t), t)|_F^r\right] dt$$

$$\leq \int_0^T \mathbb{E}\left[\sup_{s \in [0, T]} |X_2((\cdot, s), t)|_F^r\right] dt$$

$$= \int_0^T |X_2(\cdot, t)|_{L^p(\mathcal{F}_T \otimes \mathcal{B}_T)}^r dt$$

$$= (by \ (3.10)) = \int_0^T |\mathcal{I}(\cdot, t)|_{L^p(\mathcal{F}_T \otimes \mathcal{B}_T)}^r dt$$

$$\leq C \int_0^T |\zeta(\cdot, t)|_{L^p(\mathcal{F}_T \otimes \mathcal{B}_T)(L^2(U, H))}^r dt = C |\zeta|_{L^p(\mathcal{F}_T \otimes \mathcal{B}_T)(L^2(U, H))}^r.$$

This shows (3.7).

Now, if $\Sigma_1$ and $\Sigma_2$ satisfy (3.7) and (3.8), with respect to the same $\zeta$, then they belong to the same class in $L^p(\mathcal{F}_T \otimes \mathcal{B}_T)$, because $m([0, T] \setminus B_\zeta) = 0$. Similarly, if $\zeta_1 = \zeta_2$ in $L^p(\mathcal{F}_T \otimes \mathcal{B}_T)(L^2(U, H))$, then, as noticed in Remark 3.4, for $m\text{-a.e. } t \in [0, T]$, $\mathcal{I}^{\zeta_1}(\omega) = \mathcal{I}^{\zeta_2}(\omega)$ $P\text{-a.e. } \omega \in \Omega$. Then (3.8) entails $\Sigma^{\zeta_1} = \Sigma^{\zeta_2}$ for $P \otimes m\text{-a.e. } (\omega, t) \in \Omega_T$. This shows that (3.9) is well-defined. Linearity is clear. Continuity comes from (3.11).

In general, we cannot hope to have versions of $\mathcal{I}^\zeta$ with a better measurability than the one provided by Theorem 3.5, without further assumptions on $\mathcal{I}$ (observe that our assumptions on $\mathcal{I}$ do not take in consideration any progressive measurability of the values of $\mathcal{I}$).
We now address the case when $\zeta \in L^{p,q,r}_{\mathcal{D} \otimes \mathcal{B}}(L_2(U,H))$ has the form
\[
\zeta((\omega,s),t) = R(t-s)\Phi_s(\omega) = :\Phi^r_R((\omega,s),t) \quad \forall (\omega,s) \in \Omega_T, \ t \in (s,T],
\]
where $R : (0,T] \to L(H)$ is strongly continuous and $\Phi \in L(U,H)^{\Omega_T}$ is a function.

Under a technical assumption on $R$, we characterize those functions $\Phi \in L(U,H)^{\Omega_T}$ for which $\Phi^r_R$ belongs to $L^{0}_{\mathcal{D} \otimes \mathcal{B}}(L_2(U,H))$. This fact is of interest because it is the minimal requirement in order to define the family $\mathcal{Y}_{\Phi^r_R} = \{ \mathcal{Y}^\Phi_{\Phi^r_R} \}_{t \in \mathbb{B}_R}$ by (3.6) (with $\zeta = \Phi_R$), and to obtain the joint measurability of $\mathcal{Y}_{\Phi^r_R}$ through Theorem 3.5.

**Assumption 3.6.** The function $R : (0,T] \to L(H)$ is strongly continuous and there exists a sequence $\{ t_n \}_{n \in \mathbb{N}} \subset (0,T]$ converging to 0 such that, if $C \subset H$ is closed, convex, and bounded, then $u \in C$ if and only if $\exists m \in \mathbb{N} : R(t_n)u \in R(t_n)C \ \forall n \geq m$.

**Remark 3.7.** Due to the fact that the closed convex sets in $H$ are the same in the weak and in the strong topology, then, if the following implication holds for some $\{ t_n \}_{n \in \mathbb{N}} \subset (0,T]$ converging to 0:
\[
\{ x_n \}_{n \in \mathbb{N}} \subset H \text{ bounded such that } \{ R(t_n)x_n \}_{n \in \mathbb{N}} \text{ is definitely null } \implies x_n \to 0, \quad (3.12)
\]
Assumption 3.6 holds true. To see it, let us suppose that there exists $m \in \mathbb{N}$ such that $R(t_n)u \in R(t_n)C$ for $n \geq m$. This means that $R(t_n)(u-c_n) = 0$ for $n \geq m$. By (3.12), $c_n \to u$, hence $u$ belongs to $C$.

In particular, we notice that (3.12) is satisfied whenever $R : \mathbb{R}^+ \to L(H)$ is a $C_0$-semigroup on $H$. In such a case, $R^*$ is a $C_0$-semigroup (see [5, pp. 43–44, Section 5.14]), and then we can write, if $\{ t_n \}_{n \in \mathbb{N}}$ is any bounded sequence converging to 0 and if $\{ x_n \}_{n \in \mathbb{N}}$ is such that $\{ R(t_n)x_n \}_{n \in \mathbb{N}}$ is definitely null,
\[
\lim_{n \to \infty} \langle x_n, y \rangle = \lim_{n \to \infty} \langle x_n, R^*(t_n)y \rangle = \lim_{n \to \infty} \langle R(t_n)x_n, y \rangle = 0 \quad \forall y \in H.
\]

In what follows, we denote by $\overline{\mathcal{D}_T}$ the completion of $\mathcal{D}_T$ with respect to $\mathbb{P} \otimes m$. If $\Phi \in L(U,H)^{\Omega_T}$, we denote by $\Phi_R$ the map defined by
\[
\Phi_R : \Omega_T \times [0,T] \to L(U,H), \ ((\omega,s),t) \mapsto 1_{[0,t)}(s)R(t-s)\Phi_s(\omega). \quad (3.13)
\]
By saying that $\Phi \in L(U,H)^{\Omega_T}$ is strongly measurable, we mean that
\[
(\Omega_T, \mathcal{D}_T) \to H, \ (\omega,t) \mapsto \Phi_t(\omega)u
\]
is measurable, for all $u \in U$. Similarly, if $\Phi \in L(U,H)^{\Omega_T}$, then $\Phi_R$ is strongly measurable if $\Phi_R(\cdot)u$ is $\mathcal{D}_T \otimes \mathcal{B}_H$-measurable, for all $u \in U$.

**Proposition 3.8.** Let $R : (0,T] \to L(H)$ be strongly continuous and let $\Phi \in L(U,H)^{\Omega_T}$.

(i) If $\Phi$ is strongly measurable, then $\Phi_R$ is strongly measurable.

(ii) Suppose that $R$ satisfies Assumption 3.6. If $\Phi_R$ is strongly measurable, then there exists $\hat{\Phi} \in L(U,H)^{\Omega_T}$ and a $\mathbb{P} \otimes m$-null set $A \in \mathcal{D}_T$ such that $\Phi = \hat{\Phi}$ on $\Omega_T \setminus A$ and $\hat{\Phi}$ is strongly measurable.
Proof. (i) Let \( \Phi \in L(U, H)^{\Omega_T} \) be strongly measurable. Let
\[
\rho := \{0 = t_0 < \ldots < t_k = T\} \subset [0, T].
\]
Denote \( \delta(\rho) := \sup_{i=0,\ldots,k-1} |t_{i+1} - t_i| \). Define the function
\[
\Phi_{R,\rho} : (\Omega_T \times [0, T], \mathcal{P}_T \otimes \mathcal{B}_T) \to L(U, H)
\]
by
\[
\Phi_{R,\rho}((\omega, s), t) := \sum_{i=0}^{k-1} 1_{[t_i, t_{i+1})} (t) 1_{[0, t_i)} (s) R(t_i - s) \Phi_s(\omega) + 1_{[T]} (t) 1_{[0, T)} (s) R(T - s) \Phi_s(\omega).
\]
For all \( t \in [0, T] \) and \( h \in H \), the map
\[
(\Omega_T, \mathcal{P}_T) \to H, \ (\omega, s) \mapsto 1_{[0, t)} (s) R^*(t-s) h
\]
is measurable, by strong continuity of \( R \) and Pettis’s measurability theorem. Moreover, for \( u \in U \),
\[
(\Omega_T, \mathcal{P}_T) \to H, \ (\omega, s) \mapsto \Phi_s(\omega) u
\]
is measurable by assumption, we conclude that, for \( u \in U \) and \( t \in [0, T] \),
\[
(\Omega_T, \mathcal{P}_T) \to \mathbb{R}, \ (\omega, s) \mapsto \langle 1_{[0, t)} (s) R(t-s) \Phi_s(\omega) u, h \rangle_H
\]
is measurable. Then, again by Pettis’s measurability theorem,
\[
(\Omega_T, \mathcal{P}_T) \to H, \ (\omega, s) \mapsto 1_{[0, t)} (s) R(t-s) \Phi_s(\omega) u
\]
is measurable, for every \( u \in U \) and \( t \in [0, T] \). Hence \( \Phi_{R,\rho} \) is strongly measurable. By strong continuity of \( R \), we have
\[
\lim_{\delta(\rho) \to 0} \Phi_{R,\rho}((\omega, s), t) u = \Phi_R((\omega, s), t) u \quad \forall ((\omega, s), t) \in \Omega_T \times [0, T],
\]
for every \( u \in U \). This shows that \( \Phi_R \) is strongly measurable.

(ii) Suppose that \( \Phi_R \) is strongly measurable. Let \( u \in U \) and let \( C \subset H \) be closed, convex, and bounded. Let \( \{t_n\}_{n \in \mathbb{N}} \) be as in Assumption 3.6. For \( n \in \mathbb{N} \), define
\[
\Delta_n := \{(\omega, s), t) \in \Omega_T \times [0, T] : t - s = t_n\}
\]
\[
B_n := \{(\omega, s), t) \in \Omega_T \times [0, T] : \Phi_R((\omega, s), t) u \in R(t_n)C\}
\]
\[
F_n := \{(\omega, s) \in \Omega_T : R(t_n)\Phi_s(\omega) u \in R(t_n)C\}.
\]
It is clear that \( \Delta_n \in \mathcal{P}_T \otimes \mathcal{B}_T \). By weak compactness of \( C \), \( R(t_n)C \) is closed. Then, by strong measurability of \( \Phi_R, B_n \in \mathcal{P}_T \otimes \mathcal{B}_T \), hence \( B_n \cap \Delta_n \in \mathcal{P}_T \otimes \mathcal{B}_T \).

Let \( \pi_{\Omega_T} : \Omega_T \times [0, T] \to \Omega_T \) be the projection defined by
\[
\pi_{\Omega_T}((\omega, s), t) := (\omega, s).
\]
By the projection theorem (see [3, p. 75, Theorem III-23]), \( \pi_{\Omega_T}(B_n \cap \Delta_n) \in \mathcal{P}_T \). Notice that
\[
\pi_{\Omega_T}(B_n \cap \Delta_n) = ((\omega, s) \in \Omega_T : s + t_n \leq T \text{ and } R(t_n)\Phi_s(\omega) u \in R(t_n)C) = F_n \cap (\Omega \times [0, T - t_n]).
\]
(3.14)
By Assumption 3.6 and by recalling that \((t_n)_{n∈N} ⊂ (0, T)\) converges to 0, we have
\[
((ω,s) ∈ Ω_T : Φ_s(ω) u ∈ C, s < T) = \bigcup_{m∈N} \left( F_n \cap (Ω ⋂ (0, T − t_n)) \right).
\] (3.15)

By (3.14) and (3.15), we conclude \(((ω,s) ∈ Ω_T : Φ_s(ω) u ∈ C, s < T) ∈ \mathcal{P}_T\). The slice \((ω,T) ∈ Ω_T : Φ_T(ω) u ∈ C\) is a \(\mathcal{P} \ast m\)-null set. Then
\[
((ω,s) ∈ Ω_T : Φ_s(ω) u ∈ C) ∈ \mathcal{P}_T.
\]

Since this holds for every closed, convex, bounded set \(C\), hence for balls, and since \(H\) is separable, we have that \(Φu\) is \(\mathcal{P}_T / \mathcal{B}_H\)-measurable, for every \(u ∈ U\).

Now let \((u_n)_{n∈N}\) be a dense subset of \(U\). Since \(\mathcal{P}_T\) is the completion of \(\mathcal{P}_T\) with respect to \(\mathcal{P} \ast m\), and since \(H\) is separable, for every \(n ∈ N\) there exists \(A_n ∈ \mathcal{P}_T\) such that \(\mathcal{P} \ast m(A_n) = 0\) and \(1_{A_n} Φu_n\) is \(\mathcal{P}_T / \mathcal{B}_H\)-measurable. Let \(A := \bigcup_{n∈N} A_n\). Then \(A ∈ \mathcal{P}_T\), \(\mathcal{P} \ast m(A) = 0\), and \(1_A Φu_n\) is \(\mathcal{P}_T / \mathcal{B}_H\)-measurable for every \(n ∈ N\). Since \(Φ_s(ω) ∈ L(U,H)\) for every \((ω,s) ∈ Ω_T\), by density of \((u_n)_{n∈N}\) we conclude that \(1_A Φu\) is \(\mathcal{P}_T / \mathcal{B}_H\)-measurable for every \(u ∈ U\). This concludes the proof of (ii) and of the proposition. ■

We will make use of the following lemma, whose proof can be found in [6, Ch. 1].

**Lemma 3.9.** Let \((G, Λ)\) be a measurable space. Let \(f : (G, Λ) → L_2(U,H)\). Then \(f(·)u\) is \(\Lambda / \mathcal{B}_L(U,H)\)-measurable, for all \(u ∈ U\), if and only if \(f\) is \(\Lambda / L_2(U,H)\)-measurable.

Under Assumption 3.6, the following theorem characterizes those functions \(Φ ∈ L(U,H)^{Ω_T}\) for which \(Φ_R\) belongs to \(L^0_{\mathcal{P}_T \ast \mathcal{B}_T}(L_2(U,H))\).

**Theorem 3.10.** Let \(R : (0,T) → L(H)\) be strongly continuous and let \(Φ ∈ L(U,H)^{Ω_T}\).

(i) If \(Φ\) is strongly measurable and if \(1_{[0,T]}(s)R(t−s)Φ_s(ω) ∈ L_2(U,H)\) for all \(((ω,s),t) ∈ Ω_T \times [0,T]\), then \(Φ_R\) is measurable as an \(L_2(U,H)\)-valued map (that is when \(L_2(U,H)\) is endowed with its Borel \(σ\)-algebra).

(ii) Suppose that \(R\) satisfies Assumption 3.6. If \(Φ_R\) has values in \(L_2(U,H)\) and if it is measurable as an \(L_2(U,H)\)-valued map, then there exists \(Φ ∈ L(U,H)^{Ω_T}\) and a \(\mathcal{P} \ast m\)-null set \(A ∈ \mathcal{P}_T\) such that \(Φ = Φ \setminus A\), \(Φ\) is strongly measurable, and \(1_{[0,T]}(s)R(t−s)Φ_s(ω) ∈ L_2(U,H)\) for all \(((ω,s),t) ∈ Ω_T \times [0,T]\).

**Proof.** Apply Proposition 3.8 and Lemma 3.9. ■

**Example 3.11.** Let \(Q\) be a positive self-adjoint operator of trace class in \(H\) and let \(W\) be a \(U\)-valued \(Q\)-Wiener process with respect to \((Ω, \mathcal{F}, F, P)\). Let \(U_0 := Q^{1/2}(U)\) be the Hilbert space isometric to \(U\) through \(Q^{-1/2} : U_0 → U\). By [4, p. 114, Theorem 4.37], for \(p ≥ 2\), the stochastic integral is a linear and continuous map

\[
\mathcal{I}_W : L^{p,2}_{\mathcal{P}_T}(L_2(U_0,H)) → L^p_{\mathcal{P}_T}(C([0,T],H)), Ψ → Ψ ⋅ W := \int_0^T Ψ_s dW_s.
\]

Let \(R\) be as in Assumption 3.6. Let \(Φ ∈ L(U_0,H)^{Ω_T}\) be strongly measurable and such that \(R(t−s)Φ_s(ω) ∈ L_2(U_0,H)\) for \((ω,s) ∈ Ω_T, t ∈ (s,T]\). Then, by Theorem 3.10(i), \(Φ_R ∈ L^{0}_{\mathcal{P}_T \ast \mathcal{B}_T}(L_2(U_0,H))\). If \(|Φ_R|_{p,2,p} < ∞\), then we can apply Theorem 3.5, according to which the process
\[
\left\{ \int_0^t R(t−s)Φ_s dW_s \right\}_t,
\]
which is well-defined for a.e. \(t ∈ [0,T]\), has an \(\mathcal{P}_T \ast \mathcal{B}_T\)-jointly measurable version.
3.2 Continuous version

In this section we review the factorization method used to show existence of continuous version of stochastic convolutions made with respect to a $C_0$-semigroup.

**Notation.** Throughout this section

- $S$ denotes a strongly continuous semigroup on $H$ and $M := \sup_{t \in [0,T]} |S_t|_{L(H)}$;
- $\mathbb{W} := C([0,T],H)$;
- for $\beta \in (0,1)$, $c_\beta$ denotes the number $c_\beta := \left( \int_0^1 (1 - w)^{\beta-1} w^{-\beta} \, dw \right)^{-1}$.

As noticed in Remark 3.7, $S$ verifies Assumption 3.6.

The factorization method relies on the semigroup property of $S$ and on the fact that continuous linear operator commutes with stochastic integral. We rephrase this commutativity assumption in our setting through the following

**Assumption 3.12.** Let $p, q, r \in [1, \infty)$, and let
\[
\mathcal{J} : L^{p,q}_{\mathcal{F}_r} (L_2(U,H)) \to L^{r}_{\mathcal{F}_T \otimes \mathcal{B}_T} (\mathbb{W})
\]
be a linear and continuous operator such that
\[
Q(\mathcal{J} \Phi) = \mathcal{J}(Q \Phi) \text{ in } L^{r}_{\mathcal{F}_T \otimes \mathcal{B}_T} (\mathbb{W}) (^2) \quad \forall Q \in L(H). \tag{3.16}
\]

For $p, q, r \in [1, \infty)$ and $\beta \in [0,1)$, $\Lambda^{p,q,r}_{\mathcal{F}_T,S,\beta} (L(U,H))$ denotes the vector space of equivalence classes of strongly measurable functions $\Phi \in L(U,H)^{\Omega_T}$ such that
\[
\left( \int_0^T \left( \int_0^t (t-s)^{-\beta q} \left[ E \left| S(t-s) \Phi \right|_{L_2(U,H)}^p \right]^{q/p} \, ds \right)^{r/q} \, dt \right)^{1/r} < \infty. \tag{3.17}
\]

Two functions $\Phi_1$, $\Phi_2$, are in the same class if the quantity (3.17) is 0 for $\Phi = \Phi_1 - \Phi_2$. This implies, for all $u \in U$, for $\mathbb{P} \otimes m$-a.e. $(\omega, s) \in \Omega_T$,
\[
1_{[0,t]}(s) S(t-s) (\Phi_1)_s u = 1_{[0,t]}(s) S(t-s) (\Phi_2)_s u \quad \text{m.a.e. } t \in [0,T],
\]

hence, by strong continuity of $S$, for all $u \in U$,
\[
(\Phi_1)_s u = (\Phi_2)_s u \quad \mathbb{P} \otimes m$-a.e. $(\omega, s) \in \Omega_T.
\]

By separability of $U$ we conclude that $\Phi_1 = \Phi_2$ in $\Lambda^{p,q,r}_{\mathcal{F}_T,S,\beta} (L(U,H))$ if and only if $(\Phi_1)_s(\omega) = (\Phi_2)_s(\omega)$ in $L(U,H)$ $\mathbb{P} \otimes m$-a.e. $(\omega, s) \in \Omega_T$.

For $\Phi \in \Lambda^{p,q,r}_{\mathcal{F}_T,S,\beta} (L(U,H))$, we define, for all $(\omega, s) \in \Omega_T$ and $t \in [0,T]$,
\[
\Phi_{S,\beta}(\omega, s, t) := 1_{[0,t]}(s) (t-s)^{-\beta} S(t-s) \Phi_s(\omega)
\]
\[
\Phi_S(\omega, s, t) := 1_{[0,t]}(s) S(t-s) \Phi_s(\omega),
\]

\(^2Q\) applied to a process $\Phi$ means the pointwise composition $Q(\Phi_t(\omega))$, for $(\omega, t) \in \Omega_T$. 

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By Theorem 3.10(i), $\Phi_{S,\beta} \in L^0_{\mathcal{P} \otimes \mathcal{B}_T} (L_2(U, H))$, and (3.17) can be written as

$$|\Phi_{S,\beta}|_{p,q,r} < \infty. \quad (3.18)$$

Then, through the well-defined map

$$\Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H)) \to L_{\mathcal{P} \otimes \mathcal{B}_T}^{p,q,r} (L_2(U, H)), \; \Phi \mapsto \Phi_{S,\beta},$$

$\Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H))$ is identified with a subspace of $L_{\mathcal{P} \otimes \mathcal{B}_T}^{p,q,r} (L_2(U, H))$. In particular, the map

$$\Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H)) \to \mathbb{R}^+, \; \Phi \mapsto |\Phi_{S,\beta}|_{p,q,r}$$

is a norm. In what follows we always consider $\Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H))$ endowed with the norm $|\Phi_{S,\beta}|_{p,q,r}$.

Again by Theorem 3.10(i), $\Phi_S \in L^0_{\mathcal{P} \otimes \mathcal{B}_T} (L_2(U, H))$. Moreover, for all $t' \in [0,T]$, we have, by applying Minkowski’s inequality for integrals (see [7, p. 194, 6.19]),

$$|\Phi_S(\cdot,t)|_{p,q} = c_\beta \left( \int_0^{t'} \left( \int_s^{t'} (t' - t)^{\beta - 1} (t' - s)^{-\beta} \left( E [S(t' - s)\Phi_s]^{1/p}_{L_2(U,H)} \right)^q \right)^{1/q} ds \right)^{1/q} \leq c_\beta \left( \int_0^{t'} (t' - t)^{\beta - 1} \left( \int_0^{t'} (t' - s)^{-\beta q} \left( E [S(t' - s)\Phi_s]^{1/p}_{L_2(U,H)} \right)^q \right) ds \right)^{1/q} dt. \quad (3.19)$$

Now, if we take $r > 1$ and $\beta \in (1/r, 1)$, by applying Hölder’s inequality to the last term and writing $S(t' - s) = S(t' - t)S(t' - s)$,

$$|\Phi_S(\cdot,t')|_{p,q} \leq c_\beta M \left( \int_0^T w^{(r-1)/r} dw \right)^{(r-1)/r} |\Phi_{S,\beta}|_{p,q,r} < \infty. \quad (3.20)$$

This shows that

$$\Phi_S(\cdot,t') \in L_{\mathcal{P} \otimes \mathcal{B}_T}^{p,q,r}(L_2(U, H)), \; \forall t' \in [0,T]. \quad (3.20)$$

**Theorem 3.13.** Let $p, q \in [1,\infty)$, $r \in (1,\infty)$, $\beta \in (1/r, 1)$. Let $\mathcal{J}$ be as in Assumption 3.12. Then there exists a unique linear and continuous function

$$\mathcal{C}: \Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H)) \to L_{\mathcal{P} \otimes \mathcal{B}_T}^{p,q,r}(\mathbb{W}) \quad (3.21)$$

such that, for all $\Phi \in \Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H))$, for all $t \in [0,T]$,

$$\mathcal{J} \left( \{ \mathbf{1}_{[0,t]}(\cdot)S(\cdot - \cdot)\Phi \} \right)_t = (\mathcal{C}(\Phi))_t \quad \mathbb{P}\text{-a.e..} \quad (3.22)$$

The operator norm of $\mathcal{C}$ is bounded by a constant depending only on $\beta$, $r$, $T$, $M$, and on the operator norm of $\mathcal{J}$.

**Proof.** Let $\Phi \in \Lambda_{\mathcal{P},T,S,\beta}^{p,q,r}(L(U, H))$. First notice that the left-hand side of (3.22) is meaningful because of (3.20). We now construct $\mathcal{C}(\Phi)$. Fix $t' \in [0,T]$, and define

$$\Phi_{S,\beta}^{(t')}((\omega,s), t) := c_\beta \mathbf{1}_{[0,t']}(t')(t' - t)^{\beta - 1} \mathbf{1}_{[0,t]}(s)(t' - s)^{-\beta} S(t' - s)\Phi_s(\omega) \quad (\omega,s) \in \Omega_T, \; t \in [0,T].$$
By Theorem 3.10(i), $\Phi^{(t')}_{S,\beta} \in L^0_{\mathcal{P} \circ \mathcal{B}_T} (L^2(U,H))$. Moreover,
\[
|\Phi^{(t')}_{S,\beta}|_{p,q,1} = c_\beta \int_0^t (t' - t)^{\beta - 1} \left( \int_0^t (t-s)^{-q/2} \left[ E \left[ |S(t' - s)\Phi^{(p)}_{s}|_{L^2(U,H)}^q \right] \right] ds \right)^{1/q} dt \leq c_\beta M \left( \int_0^T w(t) dt \right)^{(r-1)/r} |\Phi^{(t')}_{S,\beta}|_{p,q,r} < \infty.
\]
Then $\Phi^{(t')}_{S,\beta} \in L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T} (L^2(U,H))$. By Lemma 3.3, the map
\[
g : B_0 \rightarrow L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T} (L^2(U,H)), \quad t \mapsto \Phi^{(t')}_{S,\beta}(\cdot, t), \quad (3.23)
\]
where $B_0$ is the set of $t$ such that $|\Phi^{(t')}_{S,\beta}(\cdot, t)|_{p,q} < \infty$, is Borel measurable. Let us define $g = 0$ on $[0,T] \setminus B_0$. By $\Phi^{(t')}_{S,\beta} \in L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T} (L^2(U,H))$ and by measurability of (3.23), we have $g \in L^1([0,T], L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T}(L^2(U,H)))$. We can then apply Theorem 2.3, with the following data:
\[
\begin{align*}
&\bullet \quad G = [0,T], \mathcal{G} = \mathcal{B}_T, m = \mu; \\
&\bullet \quad D_1 = \Omega, D_2 = [0,T], D = \Omega_T, \mathcal{D} = \mathcal{B}_T, v_1 = \mathbb{P}, v_2 = m; \\
&\bullet \quad E = L^2(U,H); \\
&\bullet \quad F = H; \\
&\bullet \quad \mathcal{L} = \mathcal{I}; \\
&\bullet \quad g \text{ as above.}
\end{align*}
\]
The theorem provides measurable functions
\[
X_1 : (\Omega_T \times [0,T], \mathcal{P} \circ \mathcal{B}_T) \rightarrow L^2(U,H) \quad X_2 : (\Omega_T \times [0,T], (\mathcal{P} \circ \mathcal{B}_T) \circ \mathcal{B}_T) \rightarrow H
\]
such that, for m-a.e. $t \in [0,T]$,
\[
\begin{align*}
X_1(\cdot, t) = g(t) \text{ in } L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T} (L^2(U,H)) \\
X_2(\cdot, t) = \mathcal{I}(g(t)) \text{ in } L^{p,q,1}_{\mathcal{P} \circ \mathcal{B}_T}(\mathbb{W}),
\end{align*}
\]
and
\[
\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, (\mathcal{I}Y)_t(\omega) = \int_0^T X_2((\omega, t), s) ds \quad \forall t \in [0,T], \quad (3.25)
\]
where
\[
Y_t(\omega) = \int_0^T X_1((\omega, t), s) ds, \quad \forall (\omega, t) \in \Omega_T. \quad (3.26)
\]
By (3.24), by definition of $g$, and by joint measurability of $X_1$ and $\Phi^{(t')}_{S,\beta}$, we have
\[
X_1((\omega, s), t) = \Phi^{(t')}_{S,\beta}(\omega, s, t) \quad (\mathbb{P} \circ m) \circ m\text{-a.e. } ((\omega, s), t) \in \Omega_T \times [0,T]. \quad (3.27)
\]
Then (3.26) becomes
\[
Y_t(\omega) = \int_0^T \Phi^{(t')}_{S,\beta}(\omega, s, t) ds = 1_{[0,t')}((\omega, t)S(t' - t)\Phi_t(\omega) \quad \mathbb{P} \circ m\text{-a.e. } (\omega, t) \in \Omega_T, \quad (3.28)
\]
hence, in particular,
\[(3Y)_t(\omega) = \mathcal{J}(1_{[0,t')}S(t' - \cdot)\Phi)'_t(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3.29)\]

On the other hand, for \(m\text{-a.e. } t \in [0,T],\)
\[g(t) = c_\beta 1_{[0,t')} (t') \beta - 1 S(t') \Phi S_{\beta}(\cdot, t) \text{ in } L^{p,q}_{(\mathcal{F}_t)}(L_2(U,H)).\]

Then, by assumption on \(\mathcal{J},\) we have, for \(m\text{-a.e. } t \in [0,T],\)
\[\mathcal{J}(g(t)) = c_\beta 1_{[0,t')} (t') \beta - 1 S(t') \mathcal{J}(\Phi S_{\beta}(\cdot, t)) \text{ in } \mathcal{L}^p_{(\mathcal{F}_t)}(\mathbb{W}), \quad (3.30)\]
hence, in particular, for \(m\text{-a.e. } t \in [0,T],\)
\[(\mathcal{J}(g(t)))_t(\omega) = c_\beta 1_{[0,t')} (t') \beta - 1 S(t') \mathcal{J}(\Phi S_{\beta}(\cdot, t))_t(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3.31)\]

Now our aim is to replace the last factor in (3.31) with a process jointly measurable in \((\omega, t).\) We noticed in (3.18) that \(\Phi S_{\beta} \in L^{p,q}_{(\mathcal{F}_t \otimes \mathcal{B}_T)}(L_2(U,H)).\) We can then apply Theorem 3.5. Let
\[\Sigma^{\Phi S_{\beta}} := J(\Phi S_{\beta}) \in L^r_{(\mathcal{F}_t \otimes \mathcal{B}_T)}(H) \quad (3.32)\]
be the process obtained by applying the map (3.9) to \(\Phi S_{\beta}.\) We know that \(\Sigma^{\Phi S_{\beta}}(\omega)\) is \(\mathcal{F}_T \otimes \mathcal{B}_T\)-measurable in \((\omega, t) \in \Omega_T\) and that, for \(m\text{-a.e. } t \in [0,T],\)
\[\mathcal{J}(\Phi S_{\beta}(\cdot, t))_t(\omega) = \Sigma^{\Phi S_{\beta}}_t(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3.33)\]

By (3.31) and (3.33), we can write, for \(m\text{-a.e. } t \in [0,T],\)
\[(\mathcal{J}(g(t)))_t(\omega) = c_\beta 1_{[0,t')} (t') \beta - 1 S(t') \Sigma^{\Phi S_{\beta}}(\omega) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3.34)\]

Then, by (3.24) and taking into account the joint measurability of \(X_2\) and \(\Sigma^{\Phi S_{\beta}},\)
\[X_2((\omega, t'), t) = c_\beta 1_{[0,t')} (t') \beta - 1 S(t') \Sigma^{\Phi S_{\beta}}(\omega) \quad \mathbb{P} \otimes m\text{-a.e. } (\omega, t) \in \Omega_T. \quad (3.35)\]

By (3.25), (3.35), (3.29), we finally obtain, for \(\mathbb{P}\text{-a.e. } \omega \in \Omega,\)
\[\mathcal{J}(1_{[0,t')}S(t' - \cdot)\Phi)'_t(\omega) = (3Y)_t(\omega) = \int_0^T \int_0^{t'} X_2((\omega, t'), s)ds \quad (3.36)\]
\[= c_\beta \int_0^{t'} (t' - s) \beta - 1 S(t' - t) \Sigma^{\Phi S_{\beta}}(\omega)dt. \quad (3.37)\]

Now define the process \(C(\Phi)\) by
\[C(\Phi)_t(\omega) := \begin{cases} c_\beta \int_0^t (t - s) \beta - 1 S(t - s) \Sigma^{\Phi S_{\beta}}(\omega)ds & \text{if } \Sigma^{\Phi S_{\beta}}(\omega) \in L^r([0,T], H) \\ 0 & \text{otherwise} \end{cases} \quad (3.37)\]
for all \((\omega, t) \in \Omega_T.\) By [4, p. 129, Proposition 5.9], \(C(\Phi)\) is well-defined and pathwise continuous. By Hölder’s inequality,
\[|C(\Phi)(\omega)|_\infty \leq C_{\beta,r,T,M} |\Sigma^{\Phi S_{\beta}}(\omega)|_{L^r([0,T], H)} \quad \forall \omega \in \Omega, \quad (3.38)\]
where \( C_{\beta,r,T,M} \) depends only on \( \beta, r, T, M \). Hence, by recalling (3.32),
\[
|C(\Phi)|_{L^p_{F,T,S,\beta}(\mathbb{W})} \leq C_{\beta,r,T,M} |\Sigma^{\Phi S,\beta}|_{L^p_{F,T,S,\beta}(H)} \leq C_{\beta,r,T,M,|\mathcal{I}|} |\Phi S,\beta|_{p,q,r}.
\]
(3.39)
where \( C_{\beta,r,T,M,|\mathcal{I}|} \) depends only on \( \beta, r, T, M \), and on the operator norm \( |\mathcal{I}| \) of \( \mathcal{I} \). Moreover, since \( t' \in [0,T] \) was arbitrary chosen, and since the choice of \( \Sigma^{\Phi S,\beta} \) does not depend on \( t' \), (3.36) gives, for all \( t' \in [0,T] \),
\[
\mathcal{I}(\mathbf{1}_{[0,t']} S(t' - \cdot) \Phi)_{t'} = (C(\Phi))_{t'} \quad \mathbb{P}\text{-a.e.}
\]
(3.40)
It is clear that the process \( C(\Phi) \) is uniquely identified by (3.40) in \( L^p_{F,T,S,\beta}(\mathbb{W}) \), because it is continuous. Moreover, if \( \Phi = \Phi' \) in \( \Lambda^{p,q,r}_{F,T,S,\beta}(L(U,H)) \), then \( C(\Phi) = C(\Phi') \) in \( L^p_{F,T,S,\beta}(\mathbb{W}) \). Linearity of \( C \) is clear as well. This concludes the proof that the map (3.21) is well-defined on \( \Lambda^{p,q,r}_{F,T,S,\beta}(L(U,H)) \), linear, and that (3.22) is satisfied. Continuity with operator norm bounded by a constant depending only on \( \beta, r, T, M, |\mathcal{I}| \) is due to (3.39).

We remark that the joint measurability of \( X_1, X_2, \Sigma^{\Phi S,\beta} \), provided by Theorem 2.3 and Theorem 3.5, play a central role in order to obtain the factorization formula (3.37).

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