GENERAL DRAWDOWN BASED DIVIDEND CONTROL WITH FIXED TRANSACTION COSTS FOR SPECTRALLY NEGATIVE LÉVY RISK PROCESSES

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Abstract. For spectrally negative Lévy risk processes we consider a generalized version of the De Finetti’s optimal dividend problem with fixed transaction costs, where the ruin time is replaced by a general drawdown time in the framework. We identify a condition under which a band-type impulse dividend strategy is optimal among all admissible impulse strategies. As a consequence, we are able to extend the previous results on ruin time based impulse dividend optimization problem to those on drawdown time based impulse dividend optimization problems. A new type of drawdown function is proposed at end, and various numerical examples are presented to illustrate the existence of those optimal impulse dividend strategies under different assumptions.

1. Introduction. The risk model with dividend payment was first introduced into risk theory by [13], where the insurer can choose to pay out dividends to its shareholders until the surplus drops below zero (i.e. ruin occurs). It is well-known in the literature that the dividends should be paid in an optimal way which maximizes the expected discounted total dividends paid up to time of ruin. Under a discrete model, De Finetti showed that the optimal strategy should follow the so-called barrier dividend strategy, that is, there exists a non-negative constant barrier such that the excess amount of the surplus above the barrier will be paid out as dividend to the shareholders. Recently, there have been a lot of progress in studying the optimal dividend problem under spectrally one-sided Lévy processes. For spectrally negative Lévy processes, [4] gave an analytical and explicit description of the optimal strategy within the set of barrier dividend strategies using fluctuation theory. They also investigated the scenarios when certain barrier dividend strategy is optimal among all admissible ones and identified a sufficient condition for that. Later, [24] pointed out that the aforementioned sufficient condition can be satisfied if the corresponding Lévy measure has a completely monotone density. The related work on finding the expressions for the $n$-th moment of the discounted

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cumulative dividends with barrier strategy can be found in [19] and [31]. On the other hand, under spectrally positive Lévy processes, [9] showed that the barrier dividend strategy is again optimal among all admissible ones, where the optimal barrier was characterized using the functional inverse of a scale function.

However, the obtained optimal barrier dividend strategies usually lead to ruin with probability one. The reason is that the aforementioned dividend optimization framework only considered the maximization of the shareholders’ return (in terms of dividends received) without taking into account any related solvency issues. There exist several improvements in the literature that tackle such problem. [36] introduced a component to the objective function that penalizes early ruin of the controlled risk process, such that their value function took into account both expected dividend payments and timing of ruin. They identified the optimal dividend strategies for both Cramér–Lundberg model and diffusion model, which are barrier strategies for unbounded dividend intensity and threshold strategies for bounded dividend intensity. [25] further considered such optimal dividend problems with a real valued terminal payment at ruin under spectrally negative Lévy process and showed that, under the assumption that the Lévy measure has a completely monotone density, the barrier strategy is still optimal except the case when the terminal value is greater than the perpetuity of the premium drift under Cramér–Lundberg model. In addition, [27] further illustrated the optimality of a barrier strategy and/or the take-the-money-and-run strategy when there exists an affine penalty payment at ruin for the dividend problem. Similar results under spectrally positive Lévy processes were obtained by [42], where a barrier strategy is proved to be optimal as well. More recently, [5] considered such dividend-penalty control under spectrally negative Lévy process with a general increasing penalty function at ruin. A necessary and sufficient condition for the optimality of single- or multiple-band dividend strategy was determined. Under diffusion model, [37] studied the optimal dividend problem with exponential and linear penalty payments at ruin and showed that barrier strategy is still optimal under their scenario. Other improvements regarding to the above-mentioned solvency problem in the literature are imposing ceiling rate to the dividend payment ([2]) and introducing capital injections which are injected by shareholders at certain time points to protect the insurance company from ruin ([14]). We omit the detailed introduction here, interested readers can see e.g. [24], [32], [18], [29], [43], [41] and references therein.

In the meanwhile, practical factors related to insurance business activities have been considered in the literature as well. For example, in [26], the transaction costs associated with each dividend payment was taken into account when the surplus process follows a spectrally negative Lévy process. It showed that a \((z_1, z_2)\)-type impulse dividend strategy is the optimal strategy that dominates all admissible impulse ones, if the Lévy measure has a log-convex density. The \((z_1, z_2)\)-type impulse strategy pays out dividends in a way that the reserve is reduced to a certain level \(z_1\) whenever it is above another level \(z_2\) \((z_2 > z_1)\).

However, even with the aforementioned modifications, such as penalty payments or capital injections, the model frameworks are still restricted to ruin scenario, which only take into account the most extreme case (i.e. ruin) in the dividend decision making processes. Therefore, the insurer usually arrives at certain aggressive dividend strategies, like the take-the-money-and-run strategy or barrier strategies with considerably low barriers. They are optimal when the decision maker focuses on risk of ruin, but they may not be optimal when more restricted solvency risks need
to be considered. Hence, an alternative modification we made to the De Finetti’s optimal dividend problem is to replace the ruin time by a general drawdown time. Unlike risk of ruin which is defined as the risk when surplus process drops below zero, the risk of drawdown is defined as the risk when surplus drops certain units below its historical high. Drawdown time found its interest in finance industry as certain kind of performance risk measure of the decline in value from a historical peak, which has the advantage to provide path-dependent information comparing to those traditional risk measures (see e.g. [11] and [33]). We refer to [35], [34], [1], [3] and [12] for more detailed applications of the classical drawdown times in finance. The joint Laplace transform of the classical drawdown time and the running maximum at the drawdown time under Brownian motion framework was first studied in [35]. Later, [22] extended the study to a time homogeneous diffusion process and proposed a general drawdown time under the diffusion framework, where the drawdown unit is a function of the running maximum instead of a constant unit in classical definition. Note that general drawdown times also find interesting applications in [7] in solving the Skorokhod embedding problem. For spectrally negative Lévy processes, [6] studied the Lévy tax process with a drawdown exit level which is a linear function of the running maximum process, where the expressions for the expected present values of the amount of tax are derived. More recently, [23] proved few important results regarding to a similarly defined general drawdown time for spectrally negative Lévy processes. With the help of an approximation method, they derived the Laplace transforms for two-side exit problems involving various related quantities including the general drawdown time, the hitting and creeping time over a maximum related drawdown level using scale functions; an associated potential measure was also expressed in terms of those scale functions. For other results on (general) drawdown time under insurance risk related processes can consult to e.g. [20], [21], [40], [39] and more. Therefore, by considering general drawdown time in De Finetti’s optimal dividend problem, the insurer has the flexibility to choose various forms of the drawdown functions when finding optimal strategies with different level of solvency objectives. For example, it can retrieve the classical framework when the risk of ruin is considered; It can also take into account the solvency risks with (positive) constant or path-dependent solvency levels; The classical or more general drawdown risks that frequently considered in finance can be covered as well. Details can refer to the discussions in Section 5.

As discussed above, the De Finetti’s dividend framework with modifications in the literature only considered the risk of ruin when finding the optimal dividend strategies. Then with general drawdown time, we are able to cover various level of solvency related objectives within one unified framework, especially for those objectives with more conservative solvency requirements, since the general drawdown can choose to focus more on the degree to which the surplus apart from expected levels rather than simply on the extreme adverse scenarios as ruin does. Therefore, under spectrally negative Lévy risk processes, we extend the original optimal dividend problem with fixed transaction costs from ruin time based framework to a general drawdown time based one. Note that one can retrieve the results in [24] and [26] by setting our general drawdown time equal to the ruin time.

The rest of the paper is organized as follows: In Section 2, we state some preliminary results regarding to the spectrally negative Lévy processes as well as the setup of our dividend optimization problem. We express the performance function under a \((z_1, z_2)\) impulse dividend strategy using the scale functions associated with the
spectrally negative Lévy processes in Section 3, where the optimal impulse dividend strategy among all \((z_1, z_2)\) impulse dividend strategies is identified by analyzing the form of the performance function. In Section 4, we prove a verification lemma, which verifies that the solution of the associated Hamilton-Jacobi-Bellmen (HJB) inequity coincides with the value function. Then, we develop a condition under which a \((z'_1, z'_2)\) impulse dividend strategy is the solution to the HJB inequity, i.e. the optimal strategy. In Section 5, the optimal impulse dividend strategies and corresponding value functions are obtained numerically under various scenarios and assumptions. Finally, conclusion remarks are given in Section 6.

2. Mathematical formulation of the optimization problem. To formulate our optimal dividend problem, we first let the process \(X = \{X(t); t \geq 0\}\) be a spectrally negative Lévy process with probability laws \(\{P_x; x \in [0, \infty)\}\) and natural filtration \(\{\mathcal{F}_t; t \geq 0\}\), and define the running maximum as \(\overline{X}(t) := \sup_{0 \leq s \leq t} X(s)\) for \(t \geq 0\). Note that the case with negative of a subordinator or pure increasing linear drift is excluded here. We assume that the uncontrolled risk process (no dividends are deducted from the surplus) evolves as \(\{X(t); t \geq 0\}\). We further denote the dividend process associated with an impulse dividend strategy \(D\) by \(\{L^D(t); t \geq 0\}\) which is a non-decreasing, left-continuous and \(\mathcal{F}_t\)-adapted pure jump process starts at 0. For any fixed \(t\), the random variable \(L^D(t)\) represents the cumulative dividends the shareholders received up to time \(t\). The impulse dividend strategy can be expressed explicitly as

\[
(\tau_1^D, \tau_2^D, ..., \tau_n^D, ..., \eta_1^D, \eta_2^D, ..., \eta_n^D, ...),
\]

where \(\tau_n^D\) and \(\eta_n^D\) are the time and amount of \(n\)-th dividend lump sum payment. Hence, \(L^D(t)\) can be written as

\[
L^D(t) = \sum_{n=1}^{\infty} \eta_n^D \mathbf{1}_{(\tau_n^D \leq t)}, \quad t \geq 0,
\]

and the corresponding controlled surplus process is

\[
X^D(t) = X(t) - L^D(t), \quad t \geq 0.
\]

Before we state the optimization problem, let us first introduce a general type of drawdown time. We define a general drawdown function, denoted by \(\xi\), as a continuous function satisfying: \(\xi(x) = 0\) for all \(x \in (-\infty, 0)\), \(0 = \xi(0) \leq \xi(u) < u\) for all \(u \in (0, \infty)\), and \(\xi(x) := x - \xi(x)\) is an increasing function in \(x\) with \(\lim_{x \to \infty} \xi(x) = \infty\). Then the general drawdown time with the drawdown function \(\xi\) for \(X^D(t)\) can be defined as

\[
\tau^D_\xi := \inf\{t \geq 0; X^D(t) < \xi(\overline{X}^D(t))\},
\]

where \(\overline{X}^D(t) = \sup_{\tau_n^D(t) \leq s \leq t} X^D(s)\) with \(\tau_n^D(t) = \max\{\tau_n^D; n \geq 1, \tau_n^D \leq t\}\) is a local running supremum of the controlled surplus process \(X^D(t)\) based on the impulse dividend strategy \(D\), with the convention that \(\inf \emptyset = \infty\) and \(\max \emptyset = 0\). Note that the general drawdown time we introduce here is a generalization of the usual definition of general drawdown time in the literature (see e.g. [23]). To be specific, in stead of involving the running supremum from time zero, our general drawdown time depends on the local running supremum, which only traces back from current time to a build-in stopping time in the underlying problems (the dividend payment time \(\tau^D\) in our problem). The introduction of drawdown time in finance or insurance
context aims to depict the risks of falling far from previous running maximum; however, the long-term peak of the surplus process in the history might not be a good targeting level for detecting future drawdowns, especially under a time-varying risk environment in most of the practical problems. Hence, the above definition is more reasonable when we are working on problems over a long time horizon. Then, naturally in our problem, we can recursively define the local running maximum according to the sequence of the stopping times $\tau^D_i$ in the impulse dividend strategy $D$ as expressed above. We remark that in general the dividend will only be paid in sound scenarios and the drawdown risks only need to be evaluated periodically between the consecutive dividend paying time points. Interested readers can also refer to [8] for a similar consideration of future drawdowns.

In addition, since our general drawdown time is defined based on the impulse dividend strategy, it will reduce to usual general drawdown time for the uncontrolled process without dividend payment. On the other hand, the classical ruin time can be recovered if one chooses $\xi \equiv 0$ in (1), while our general drawdown time provides more risk and surplus level related information than ruin time. In this sense, such general drawdown time shall be a more efficient and suitable tool for characterizing different levels of risk in optimal dividend problem from the risk management’s point of view.

Next, let us define an admissible impulse dividend strategy in our problem.

**Definition 2.1.** An impulse dividend strategy $D = \{\tau^D_i, \eta^D_i\}_{i \in \mathbb{N}}$ is called admissible if the following conditions are satisfied:

- $0 \leq \tau^D_i < \tau^D_{i+1}$ a.s. for all $i = 1, 2, \ldots$, and $\tau^D_i$ is a stopping time with respective to the filtration $\mathcal{F}_t$ for all $t \geq 0$.
- $\eta^D_i$ is measurable with respective to $\mathcal{F}_{\tau^D_i}$.
- $\mathbb{P}(\lim_{i \to \infty} \tau^D_i \leq T) = 0$ for all $T \geq 0$.
- The drawdown time can not be reached due to dividend payment.

Note that the last condition can be expressed as following:

$$L^D(t^+) - L^D(t) \leq [X^D(t) - \xi(X^D(t))] \vee 0, \quad t \in [0, \infty),$$

or, equivalently

$$0 \leq \eta^D_n \leq [X^D(\tau^D_n) - \xi(X^D(\tau^D_n))] \vee 0, \quad n \geq 1.$$

Additionally, we set $L^D(t) = 0$ for $t < 0$. Let $\mathcal{D}$ denotes the set of all admissible dividend strategies satisfying Definition 2.1. For any $D \in \mathcal{D}$, we define the performance function $V_D$ as

$$V_D(x) = \mathbb{E}_x \left( \sum_{n=1}^{\infty} e^{-q\tau^D_n} (\eta^D_n - \beta) 1_{\{\tau^D_n < \tau^D\}} \right), \quad x \in [0, \infty),$$

where $q > 0$ is a discounting factor and $\beta \in (0, \infty)$ is the fixed transaction cost associated with each lump sum dividend payment. Our goal is to find the optimal value function defined as

$$V(x) := \sup_{D \in \mathcal{D}} V_D(x) = \sup_{D \in \mathcal{D}} \mathbb{E}_x \left( \sum_{n=1}^{\infty} e^{-q\tau^D_n} (\eta^D_n - \beta) 1_{\{\tau^D_n < \tau^D\}} \right), \quad x \in [0, \infty), \tag{2}$$

and the corresponding optimal dividend strategy $D^*$ (if it exists) such that $V(x) = V_{D^*}(x)$ for all $x \in [0, \infty)$. 
We end this section by introducing some preliminaries on spectrally negative Lévy process. Let the Laplace exponent of $X$ be given by

$$
\psi(\theta) = \ln \left( E_x \left( e^{\theta X(t)} \right) \right) = \gamma \theta + \frac{1}{2} \sigma^2 \theta^2 - \int_{(0, \infty)} (1 - e^{-\theta x} - \theta x 1_{(0,1)}(x)) \, v(dx),
$$

where $v$ is the so-called Lévy measure and $\int_{(0, \infty)} (1 \wedge x^2) \, v(dx) < \infty$. On the other hand, $X$ can be decomposed as

$$
X(t) = \gamma t + \sigma B(t) - \int_0^t xN(ds, dx) - \int_0^t \int_{[1, \infty)} xN(ds, dx), \quad t \geq 0,
$$

where $N(ds, dx)$ is a Poisson random measure, $B(t)$ is the standard Brownian motion and $\overline{N}(ds, dx)$ is the corresponding compensated Poisson random measure. In addition, $\psi(\theta)$ is finite, strictly convex and infinitely differentiable for $\theta \in [0, \infty)$. For each $q \geq 0$, we introduce the $q$-scale functions $W^{(q)} : [0, \infty) \to [0, \infty)$ which is defined through the following Laplace transform

$$
\int_0^\infty e^{-\theta x} W^{(q)}(x) \, dx = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Psi(q),
$$

where $\Psi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$. Let $W^{(q)}(x) = 0$ for $x \in (-\infty, 0)$, and use $W$ for $0$-scale function $W^{(0)}$ in order to simplify the notation. For a detailed introduction of scale functions can refer to [10]. In addition, we assume $\psi'(0+) > -\infty$ instead of the usual safety loading condition in the literature (i.e. $\psi'(0+) \geq 0$).

Note that with the exponential change of measure for a spectrally negative Lévy process as defined below,

$$
\frac{\mathbb{P}_x^\theta}{\mathbb{P}_x} \big|_{\mathcal{F}_t} = e^{\theta (X(t) - x) - \psi(\theta) t}, \quad x \in \mathbb{R}, \ \theta \geq 0
$$

$X$ remains as a spectrally negative Lévy process under the new probability measure $\mathbb{P}_x^\theta$. Furthermore, let $W_x^{(q)}$ and $W_\theta$ be the corresponding $q$-scale and $0$-scale functions for $X$ under $\mathbb{P}_x^\theta$ respectively.

Lastly, let us define an operator $A$ acting on piecewise twice continuously differentiable function $g$ as follows

$$
Ag(x) := \gamma g(x) + \frac{1}{2} \sigma^2 g''(x)
$$

$$
+ \int_{(0, \infty)} \left( g(x - y) - f(x) + g'(x) y 1_{(0,1)}(y) \right) \, v(dy), \quad x \in [0, \infty) \setminus \{d_i; 1 \leq i \leq m_0\}.
$$

By saying that a function $g$ is piecewise continuously differentiable or piecewise twice continuously differentiable over $[0, \infty)$, we mean that $g$ is continuously differentiable or twice continuously differentiable over $[0, \infty) \setminus \{d_i; 1 \leq i \leq m_0\}$ with $0 \leq d_1 < \cdots < d_{m_0} < \infty$ and $m_0$ being a non-negative integer.

3. Value function under band-type impulse dividend strategy. In this section, we first review some preliminary results on the two-sided exit problem related to general drawdown times followed by some useful properties regarding to the value function. We define the general drawdown time with respect to a drawdown function $\xi$ and the first up-crossing time of the uncontrolled surplus process $X$ as follows,

$$
T_\xi := \inf\{t > 0 \mid X(t) < \xi(\overline{X}(t))\} \quad \text{and} \quad T^+_a := \inf\{t > 0 \mid X(t) > a\}, \quad (3)
$$

with the convention that $\inf \emptyset = \infty$. Recall that $\overline{\xi}(x) := x - \xi(x)$.
Lemma 3.1. For \( a \in [0, \infty) \) and \( x \in [0, a) \), we have
\[
\mathbb{E}_x \left( e^{-q_+ T_+^\varepsilon} \mathbf{1}_{\{T_+^\varepsilon < T_\varepsilon^\theta\}} \right) = \exp \left( - \int_x^a \frac{W'(q') \left( \xi(y) \right)}{W(q') \left( \xi(y) \right)} dy \right). \tag{4}
\]

Proof. Refer to [23] and [40]; see also [39, Proposition 1]. \( \square \)

Proposition 1. The value function \( V(x) = \sup_{D \in \mathcal{D}} V_D(x) \) is continuous, non-negative and non-decreasing over \([0, \infty)\). In addition,
\[
V(x) - V(y) \geq x - y - \beta \quad \text{for } x \geq y \geq \xi(x) \geq 0.
\]

Proof. By the definition of admissible strategy (cf., Definition 2.1) and the non-decreasing property of \( \xi \), any admissible impulse dividend strategy associated with initial surplus \( y \) also serves as an admissible impulse dividend strategy for initial surplus \( x \geq y \). Then according to the definition of the value function in (2), we have that \( V(x) \geq V(y) \) for all \( x \geq y \geq 0 \). By considering a particular dividend strategy \( L_D(t) \equiv 0 \) for all \( t \in [0, \infty) \), one finds that \( V(y) \geq 0 \) for all \( y \in [0, \infty) \). In addition, for any \( \varepsilon > 0 \) and \( x > y \), we can find an \( \varepsilon \)-optimal dividend strategy \( D_\varepsilon^y \) such that
\[
V(x) \leq V_{D_\varepsilon^y}(x) + \varepsilon. \tag{5}
\]

Define a new admissible dividend strategy as
\[ L^{T_\varepsilon^y}(t) = \left( L^{D_\varepsilon^y \circ \theta_{T_\varepsilon^y}} \right)(t), \quad t \in [0, \infty), \]
where \( T_\varepsilon^y \) is the up-crossing time defined by (3) and \( \theta \) is a time-shift operator. Then, \( D_\varepsilon^y \) is indeed an admissible impulse dividend strategy for initial surplus \( y \). Hence we have
\[
V(y) = \sup_{D \in \mathcal{D}} V_D(y) \geq V_{D_\varepsilon^y}(y) = \mathbb{E}_y \left( \int_{T_\varepsilon^y}^{\tau^y_\varepsilon} e^{-q t} dL^{D_\varepsilon^y}(t) \mathbf{1}_{\{T_\varepsilon^y < \tau^y_\varepsilon\}} \right)
= \mathbb{E}_y \left( e^{-q T_\varepsilon^y} \mathbf{1}_{\{T_\varepsilon^y < \tau^y_\varepsilon\}} \int_{T_\varepsilon^y}^{\tau^y_\varepsilon} e^{-q(t-s)} d\left( L^{D_\varepsilon^y \circ \theta_{T_\varepsilon^y}} \right)(t) \mathbf{1}_{\{T_\varepsilon^y < \tau^y_\varepsilon\}} \right)
= \mathbb{E}_y \left( \int_0^{\tau^y_\varepsilon} e^{-q t} dL^{D_\varepsilon^y}(t) \mathbf{1}_{\{T_\varepsilon^y < \tau^y_\varepsilon\}} \right) V_{D_\varepsilon^y}(x)
\geq \exp \left( - \int_y^x \frac{W'(q') \left( \xi(z) \right)}{W(q') \left( \xi(z) \right)} dz \right) V(x) - \varepsilon, \tag{6}
\]
where we have used (5) and Lemma 3.1. By non-decreasing property of \( V \) and (6) we can get
\[
0 \leq V(x) - V(y) \leq \left( 1 - \exp \left( - \int_y^x \frac{W'(q') \left( \xi(z) \right)}{W(q') \left( \xi(z) \right)} dz \right) \right) V(x) + \varepsilon.
\]
Then by letting \( \varepsilon \downarrow 0 \) followed by letting \( x \downarrow y \) and \( y \uparrow x \) respectively in the above inequality, one arrives at the continuity property of the function \( V(x) \).
For any \( \varepsilon > 0 \) and \( x \geq y \geq \xi(x) \geq 0 \) (\( \xi(x) \geq 0 \) a priori by definition of the drawdown function), let \( D^v_y \) be an admissible impulse dividend strategy associated to initial surplus \( y \) such that \( V_{D^v_y}(y) \geq V(y) - \varepsilon \). Without loss of generality, \( D^v_y \) can be expressed as

\[
(t_1^{D^v_y}, \ldots, t_n^{D^v_y}, \eta_1^{D^v_y}, \ldots, \eta_n^{D^v_y}, \ldots),
\]

with \( t_n^{D^v_y} \) and \( \eta_n^{D^v_y} \) being the time and amount of \( n \)-th dividend payout respectively. Then, let us consider a new dividend strategy \( D^x_z \) for initial surplus \( x \) characterized as follows:

\[
(0, \tau_1^{D^x_z}, \ldots, \tau_n^{D^x_z}, \ldots; x - y, \eta_1^{D^x_z}, \ldots, \eta_n^{D^x_z}, \ldots), \quad \text{for } \tau_1^{D^x_z} > 0 \text{ a.s.,}
\]

and

\[
(0, \tau_1^{D^x_z}, \ldots, \tau_n^{D^x_z}, \ldots; x - y + \eta_1^{D^x_z}, \ldots, \eta_n^{D^x_z}, \ldots), \quad \text{for } \tau_1^{D^x_z} = 0 \text{ a.s.,}
\]

which is admissible due to the increasing property of \( \xi \) and \( y \geq \xi(x) \). Applying \( D^x_z \), we get

\[
V(x) = \sup_{D \in \mathcal{D}} V_D(x) \geq V_{D^x_z}(x) = x - y - \beta + V_{D^v_y}(y) \geq x - y - \beta + V(y) - \varepsilon,
\]

which then yields \( V(x) - V(y) \geq x - y - \beta \) by letting \( \varepsilon \downarrow 0 \). \( \square \)

In the following, we consider an important type of impulse dividend strategies, namely the band–type impulse dividend strategies \( D^z_{z_1} \) with \( z_1 < z_2 \), whenever the surplus is above the upper level \( z_2 \) of the band, a lump sum amount dividend is paid such that the resulting surplus level will reduce to the \( z_1 \); while no dividend is paid when the surplus level is below \( z_2 \). For notational convenience, we shall write such type of impulse dividend strategy \( D^z_{z_1} \) as \((z_1, z_2)\). In Proposition 2 below, we express the performance function under the strategy \((z_1, z_2)\) in terms of the aforementioned scale functions. However, to make sure that the \((z_1, z_2)\) impulse dividend strategy is an admissible one, we should have \( z - z_1 \leq z - \xi(z) \) for all \( z \geq z_2 \).

**Proposition 2.** Suppose that \( W(q) \) is continuously differentiable over \((0, \infty)\). Let \( \beta > 0, z_1 \geq \xi(z) \) for all \( z \geq z_2 \), and \( z_1 + \beta \leq z_2 < \infty \). The performance function under the impulse dividend strategy \((z_1, z_2)\) is given by

\[
V_{z_2}^{z_1}(x) = \begin{cases} 
\exp \left( f_{z_2}^{z_1} \frac{W(q)(\xi(s))}{W(q)(\xi(s))} ds \right) \left( z_2 - z_1 - \beta \right), & x \in [0, z_2), \\
\exp \left( f_{z_1}^{z_2} \frac{W(q)(\xi(s))}{W(q)(\xi(s))} ds \right) - 1, & x \in [z_2, \infty). 
\end{cases}
\]  

(7)

**Proof.** For \( x \in [0, z_2) \), by (4) one has

\[
V_{z_2}^{z_1}(x) = \mathbb{E}_x \left( \int_{x}^{T_{z_2}^{q,T_2}} e^{-q_t} dL_{T_2}^{D_{z_1}^{z_2}}(t) \right) = \mathbb{E}_x \left( \mathbb{1}_{\{T_{z_2}^{q,T_2} < \tau_1^{D_{z_1}^{z_2}}\}} \int_{T_{z_2}^{q,T_2}}^{T_1^{T_2}} e^{-q_t} dL_{T_2}^{D_{z_1}^{z_2}}(t) \right)
\]

\[
= \mathbb{E}_x \left( e^{-q_T_2} \mathbb{1}_{\{T_{z_2}^{q,T_2} < \tau_1^{D_{z_1}^{z_2}}\}} \right) V_{z_2}^{z_1}(z_2)
\]

\[
= \exp \left( - \int_x^{z_2} \frac{W(q)(\xi(s))}{W(q)(\xi(s))} ds \right) V_{z_2}^{z_1}(z_2)
\]

\[
= \exp \left( - \int_x^{z_2} \frac{W(q)(\xi(s))}{W(q)(\xi(s))} ds \right) \left( z_2 - z_1 - \beta + V_{z_2}^{z_1}(z_1) \right) .
\]  

(8)
In particular, one obtains
\[ V^{\beta}_{z_1}(z_1) = \exp \left( -\int_{z_1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) (z_2 - z_1 - \beta + V^{\beta}_{z_1}(z_1)), \]
which implies
\[ V^{\beta}_{z_1}(z_1) = \frac{\exp \left( -\int_{z_1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) (z_2 - z_1 - \beta)}{1 - \exp \left( -\int_{z_1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right)}. \]
Then, (9) combined with (8) gives
\[ V^{\beta}_{z_1}(x) = \exp \left( -\int_{x}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) \frac{\exp \left( \int_{z_1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) (z_2 - z_1 - \beta)}{\exp \left( \int_{z_1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) - 1}, \]
for \( x \in [0, z_2]. \)

For \( x \geq z_2, \) one has
\[ V^{\beta}_{z_1}(x) = x - z_1 - \beta + \frac{z_2 - z_1 - \beta}{\exp \left( \int_{x}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) - 1}. \]
Combining (10) and (11) one arrives at (7).

**Remark 1.** Let \( \xi \equiv 0, \) then \( \left( \int_{x}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) = \exp \left( \int_{x}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) = \frac{W(q)(z_2)}{W(q)(x)}, \) which together with (7) reduces to the results in [26, Proposition 2].

Next, we characterize the optimal strategy among all \((z_1, z_2)\) impulse dividend strategies with \( z_1 + \beta \leq z_2 < \infty \) and \( z_1 \geq \xi(z) \) for all \( z \geq z_2. \)

**Definition 3.2.** Define an auxiliary bivariate function as follows
\[ \zeta(z_1, z_2) := \frac{z_2 - z_1 - \beta}{\exp \left( \int_{1}^{z_2} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) - \exp \left( \int_{1}^{z_1} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right)}, \quad 0 \leq z_1, z_1 + \beta \leq z_2 < \infty. \]

According to the above Definition 3.2, one has
\[ V^{\beta}_{z_1}(x) = \zeta(z_1, z_2) \exp \left( -\int_{x}^{1} \frac{W(q)'(\xi(s))}{W(q)(\xi(s))} ds \right) \]
for all \( x \in [0, z_2]. \)

**Proposition 3.** Let \((z_1^*, z_2^*)\) be a maximizer of the bivariate function \( \zeta(z_1, z_2) \) defined in (12).
(a) If $z_1^\ast + \beta < z_2^\ast$ and $z_1^\ast \neq 0$, then the following two equations hold true.

\[
\frac{W^{(q)}(\tilde{\xi}(z_1^\ast))}{W^{(q)}(\tilde{\xi}(z_1^\ast))} = \frac{z_2^\ast - z_1^\ast - \beta}{\exp\left(\int_{z_1^\ast}^{z_2^\ast} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) - 1},
\]  

and

\[
\frac{W^{(q)}(\tilde{\xi}(z_2^\ast))}{W^{(q)}(\tilde{\xi}(z_2^\ast))} = \frac{(z_2^\ast - z_1^\ast - \beta)\exp\left(\int_{z_1^\ast}^{z_2^\ast} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right)}{\exp\left(\int_{z_1^\ast}^{z_2^\ast} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) - 1}.
\]  

(b) If $z_1^\ast + \beta < z_2^\ast$ and $z_1^\ast = 0$, then (14) holds true.

**Proof.** By (12) we get

\[
\lim_{z_1 \to \infty} \zeta(z_1, z_2) = \lim_{z_1 \to \infty} \frac{z_2 - z_1 - \beta}{\exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) - 1} = \lim_{z_1 \to \infty} \frac{z_2 - z_1 - \beta}{\exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right)} - 1
\]

\[
\leq \lim_{z_1 \to \infty} \frac{1}{\exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right)} \int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy
\]

\[
\leq \lim_{z_1 \to \infty} \frac{1}{\exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right)} \Psi(q)
\]

\[
= 0,
\]

where we have used the facts that $e^z \geq 1 + z$ for all $z \geq 0$, the functions $\tilde{\xi}(z)$ is increasing with $\lim_{x \to \infty} \tilde{\xi}(x) = \infty$, the function $\frac{W^{(q')}(z)}{W^{(q)}(z)} = \frac{e^{\Psi(q)z}W_{\Psi(q)}(z)}{e^{\Psi(q)z}W_{\Psi(q)}(z)} = \Psi(q) + n_{\Psi(q)}(z \geq 0)$ ($n_{\Psi(q)}$ denotes the Itô excursion measure of the spectrally negative Lévy process reflected at the running maximum $X - X$ under $\mathbb{P}^\Psi(q)$) is decreasing for $z \in [0, \infty)$, and

\[
\lim_{z_1 \to \infty} \frac{W^{(q')}(\tilde{\xi}(z_1))}{W^{(q)}(\tilde{\xi}(z_1))} = \lim_{z_1 \to \infty} \frac{W^{(q')}(z_1)}{W^{(q)}(z_1)} = \Psi(q) > 0, \text{ for } q > 0,
\]

which implies

\[
\lim_{z_1 \to \infty} \int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy \leq \lim_{z_1 \to \infty} \int_{z_1}^{z_2} \frac{z_2 - z_1}{\Psi(q) s} = 1
\]

and

\[
\lim_{z_1 \to \infty} \exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) = \infty.
\]

In addition, by the L’Hôpital’s rule and (16), we have

\[
\lim_{z_2 \to \infty} \zeta(z_1, z_2) = \lim_{z_2 \to \infty} \frac{z_2 - z_1 - \beta}{\exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) - 1} = \lim_{z_2 \to \infty} \exp\left(\int_{z_1}^{z_2} \frac{W^{(q')}(\tilde{\xi}(y))}{W^{(q)}(\tilde{\xi}(y))} dy\right) W^{(q)}(\tilde{\xi}(z_2)) = 0.
\]
Combining (15) and (17) one can derive that, if there exist global maximum points of the bivariate function \( \zeta(z_1, z_2) \) of \((z_1, z_2) \in \{(x, y)|0 \leq x < x + \beta \leq y < \infty\} \), then all global maximum points should lie in some bounded region as \( \{(z_1, z_2)|0 \leq z_1, z_2 \leq \alpha_0, z_1 + \alpha \leq z_2\} \) for some \( \alpha_0 \in [0, \infty) \). That is

\[
\sup_{0 \leq z_1, z_2 < \infty, z_1 + \beta \leq z_2} \zeta(z_1, z_2) = \sup_{0 \leq z_1, z_2 \leq \alpha_0, z_1 + \beta \leq z_2} \zeta(z_1, z_2).
\]

Meanwhile, the bivariate function \( \zeta(z_1, z_2) \) is continuous in the bounded and closed region \( \{(z_1, z_2)|z_1, z_2 \in [0, \infty], z_1 + \beta \leq z_2\} \). Hence, the supremum value of the bivariate function \( \zeta(z_1, z_2) \) should be attained in \( \{(z_1, z_2)|z_1, z_2 \in [0, \infty], z_1 + \beta \leq z_2\} \). Let \( \mathcal{M} \) be the set of maximizers of the bivariate function \( \zeta(z_1, z_2) \), which is denoted as

\[
\mathcal{M} := \{(z_1^*, z_2^*)|z_1^* \geq 0, z_1^* + \beta \leq z_2^*, \inf_{z_1 \geq 0, z_1 + \beta \leq z_2} (\zeta(z_1^*, z_2^*) - \zeta(z_1, z_2)) \geq 0\},
\]

then \( \mathcal{M} \neq \emptyset \) and \( \mathcal{M} \subseteq \{(z_1, z_2)|z_1, z_2 \in [0, \infty], z_1 + \beta \leq z_2\} \). On the other hand, for any \((z_1, z_2)\) such that \( z_2 = z_1 + \beta \), we always have \( \zeta(z_1, z_2) \equiv 0 \); while, for any other \((z_1^*, z_2^*)\) such that \( z_2^* > z_1^* + \beta \), one should have \( \zeta(z_1^*, z_2^*) > 0 \). Thus, any global maximizer of the bivariate function \( \zeta(z_1, z_2) \) of \((z_1, z_2)\) shall not lie on the line \( z_2 = z_1 + \beta \), for \( z_1 \geq 0 \). Therefore, the following two cases need to be considered for \((z_1^*, z_2^*)\) being a maximizer of the bivariate function \( \zeta(z_1, z_2) \) of \((z_1, z_2)\),

(a) \( z_1^* + \beta < z_2^*, z_1^* \neq 0 \), i.e., \((z_1^*, z_2^*)\) is an interior point of \( \{(z_1, z_2)|z_1, z_2 \in [0, \infty], z_1 + \beta \leq z_2\} \). In this case we must have \( \frac{\partial}{\partial z_1} \zeta(z_1^*, z_2^*) = 0 \) and \( \frac{\partial}{\partial z_2} \zeta(z_1^*, z_2^*) = 0 \), which can be written respectively as (13) and (14).

(b) \( z_1^* + \beta < z_2^*, z_1^* = 0 \). In this case we should only have \( \frac{\partial}{\partial z_2} \zeta(z_1^*, z_2^*) = 0 \), which is (14).

The proof is complete. \( \square \)

Consider a dividend strategy \((z_1, z_2) \in \mathcal{M}\) defined via (18) with \( z_1 \geq \xi(z) \) for all \( z \geq z_2 \), then with the help of (14) and (7), we can derive an alternative expression for the performance function as follows:

\[
V_{z_1}^{z_2}(x) = \begin{cases} 
W^{\zeta}(\xi_1(x)) & x < z_2 \\
W^{\zeta}(\xi_1(x)) \exp \left( -\int_{z_2}^x \frac{W^{\zeta}(\xi_1(y))}{W^{\zeta}(\xi_1(y))} dy \right) , & x \in [0, z_2] \\
x - z_2 + \frac{W^{\zeta}(\xi_1(x))}{W^{\zeta}(\xi_1(x))} , & x \in [z_2, \infty).
\end{cases}
\]

(19)

It is interesting to observe that, \( V_{z_1}^{z_2} \) given in (19) is independent of \( z_1 \), while it is not true for the case we considered in Proposition 2.

Let us define

\[
\zeta(x) := \frac{W^{\zeta}(\xi(x))}{W^{\zeta}(\xi(x))} \exp \left( -\int_1^x \frac{W^{\zeta}(\xi(y))}{W^{\zeta}(\xi(y))} dy \right) , \quad x \in [0, \infty),
\]

(20)

which is non-negatively valued. If \( \xi \) and \( W^{\zeta}(q) \) are differentiable and twice differentiable respectively, we have

\[
\zeta'(x) = e^{-\int_1^x \frac{W^{\zeta}(\xi(y))}{W^{\zeta}(\xi(y))} dy} \left[ -1 + \frac{W^{\zeta}(\xi(x))^2}{W^{\zeta}(\xi(x))^2} \right] , \quad x \geq 0.
\]

Then, for any \((z_1, z_2) \in \mathcal{M}\) with \( z_1 \geq \xi(z) \) for all \( z \geq z_2 \), we have

\[
V_{z_1}^{z_2}(x) = \zeta(z_2) \exp \left( -\int_x^{z_2} \frac{W^{\zeta}(\xi(y))}{W^{\zeta}(\xi(y))} dy \right) , \quad x \in [0, z_2],
\]
with $c$ defined as in (20). Hence, if $\xi$ and $W^{(q)}$ are differentiable and twice differentiable respectively, we get

\[
\frac{\partial}{\partial z_2} V_{z_2}(x) = \exp\left( -\int_x^{z_2} W^{(q)}(\xi(y)) dy \right) \left( z_2 \right), \quad \text{for } x \in [0, z_2],
\]

\[
\frac{\partial}{\partial z_2} V_{z_2}(x) = \exp\left( \int_1^{z_2} W^{(q)}(\xi(y)) dy \right) \left( z_2 \right), \quad \text{for } x \in (z_2, \infty).
\]

On the other hand, given $(z_1, z_2) \in \mathcal{M}$ with $z_1 \geq \xi(z)$ for all $z \geq z_2$, if $W^{(q)}$ is continuously differentiable, by taking derivative with respect to $x$ in (19), one arrives at

\[
V_{z_1}^{z_2}(x) = \left\{ \begin{array}{ll}
\exp\left( -\int_x^{z_2} W^{(q)}(\xi(y)) dy \right) \xi(z_2), & x \in [0, z_2) \\
1, & x \in (z_2, \infty).
\end{array} \right.
\]

Hence, it is obvious that $V_{z_1}^{z_2}(x)$ is continuous on $[0, \infty)$, in particularly, $V_{z_1}^{z_2}(x)$ is continuous at $z_2$ according to (20). If further $\xi$ and $W^{(q)}$ are continuously differentiable and twice continuously differentiable respectively, one can further verify that

\[
V_{z_1}^{z_2}(x) = \left\{ \begin{array}{ll}
\exp\left( \int_1^{z_2} W^{(q)}(\xi(y)) dy \right) \xi(z_2) - \exp\left( -\int_x^{z_2} W^{(q)}(\xi(y)) dy \right) \xi(z_2), & x \in [0, z_2), \\
0, & x \in (z_2, \infty).
\end{array} \right.
\]

(21)

From (21), one can obtain that $V_{z_1}^{z_2}(x)$ is continuous over $[0, z_2)$ and $(z_2, \infty)$. However, it is not evident whether $V_{z_1}^{z_2}(x)$ is continuous at $z_2$ or not. In fact, the twice differentiability at $z_2$ is unknown, even we have that $\xi$ and $W^{(q)}$ are continuously differentiable and twice continuously differentiable respectively. Furthermore, if $\xi$ and $W^{(q)}$ are only assumed to be piecewise continuously differentiable and piecewise twice continuously differentiable respectively (as is assumed in the Lemma 4.1 and Theorem 4.2 below), then $V_{z_1}^{z_2}(x)$ is also piecewise well-defined and continuous over $[0, \infty)$.

**Proposition 4.** Let $(z_1, z_2) \in \mathcal{M}$ satisfying $z_1 \geq \xi(z)$ for all $z \geq z_2$. Then $V_{z_1}^{z_2}(x)$ is continuous, non-negatively valued and nondecreasing over $[0, \infty)$. In addition, we have

\[
V_{z_1}^{z_2}(y_2) - V_{z_1}^{z_2}(y_1) \geq y_2 - y_1 - \beta,
\]

for all $0 \leq y_1 \leq y_2 < \infty$.

**Proof.** $V_{z_1}^{z_2}(x)$ is continuous, non-negatively valued and nondecreasing over $[0, \infty)$ according to (19). The nondecreasing property of $V_{z_1}^{z_2}$ gives (22) for $0 \leq y_1, y_2 < \infty$ with $y_1 + \beta > y_2$.

By the fact that $(z_1, z_2) \in \mathcal{M}$ and the definition of $\mathcal{M}$ and $\xi(z_1, z_2)$ in (18) and (12) respectively, we have

\[
\frac{\exp\left( \int_1^{z_2} W^{(q)}(\xi(y)) dy \right) - \exp\left( \int_1^{z_1} W^{(q)}(\xi(y)) dy \right)}{z_2 - z_1 - \beta} \leq \frac{\exp\left( \int_1^{y_2} W^{(q)}(\xi(y)) dy \right) - \exp\left( \int_1^{y_1} W^{(q)}(\xi(y)) dy \right)}{y_2 - y_1 - \beta},
\]

\[
0 \leq y_1, y_1 + \beta \leq y_2 < \infty.
\]

(23)
Therefore, by (19) and (14) we obtain that
\[
V_{z_1}^{z_2}(y_2) - V_{z_1}^{z_2}(y_1) = \frac{W^{(q)}(\xi(z_2))}{W^{(q)}(\xi(z_2))} \left( \exp \left( - \int_{y_2}^{z_2} \frac{W^{(q)}(\xi(y))}{W^{(q)}(\xi(y))} dy \right) - \exp \left( - \int_{y_1}^{z_2} \frac{W^{(q)}(\xi(y))}{W^{(q)}(\xi(y))} dy \right) \right) - y_2 - y_1 - \beta.
\]

Lastly, for \( y_2 \geq z_2 \geq y_1 \) such that \( y_1 + \beta \leq y_2 \), with the help of (23) once again we have that
\[
V_{z_1}^{z_2}(y_2) - V_{z_1}^{z_2}(y_1) = y_2 - y_1 \geq y_2 - y_1 - \beta.
\]

The proof is completed.

4. Verification lemma and optimal dividend strategy. According to the above analysis, even if continuously differentiability and twice continuously differentiability are imposed on \( \xi \) and \( W^{(q)} \) respectively, we still cannot obtain twice differentiability of \( V_{z_1}^{z_2} \) at \( z_2 \) in general, not to mention the continuity of \( V_{z_1}^{z_2}'' \) at \( z_2 \). However, let us relax such conditions and only assume that \( \xi \) and \( W^{(q)} \) are piecewise continuously and twice continuously differentiable respectively, which are sufficient for us to arrive at a piecewise well-defined and piecewise continuous function \( V_{z_1}^{z_2}(x) \) over \([0, \infty)\). Hence, to obtain the optimality of certain impulse dividend strategy \((z_1, z_2) \in \mathcal{M}\), when the value function is not twice differentiable at only finite many points, we need a modified verification lemma. Note that since \( V_{z_1}^{z_2} \) is not twice continuously differentiable on \((0, \infty)\), the Itô’s formula cannot be
applied directly in the derivation, then we circumvent this difficulty by using the following Lemma 4.1.

**Lemma 4.1. (Verification Lemma)** Let $D^* \in \mathcal{D}$ be an admissible strategy such that $V_{D^*}(x) \in C^1((0, \infty)) \cap C^2((0, \infty) \setminus \{d_1, \ldots, d_m\})$ and $\max\{\lim_{x \uparrow d_i} |V_{D^*}''(x)| \vee \lim_{x \downarrow d_i} |V_{D^*}''(x)|\} < \infty$ with $\{d_1, \ldots, d_m\} \subseteq [0, \infty)$ and $m_0$ being a nonnegative integer. Suppose that

$$\begin{cases}
AV_{D^*}(x) - qV_{D^*}(x) \leq 0, & \text{for } x \in [0, \infty) \setminus \{d_1, \ldots, d_m\}, \\
V_{D^*}(x) \geq 0, & \text{for } x \in (0, \infty).
\end{cases} \quad (24)$$

Suppose further that $V_{D^*}(x)$ is a continuous and non-decreasing function of $x$, and

$$V_{D^*}(x_2) - V_{D^*}(x_1) \geq x_2 - x_1 - \beta,$$

for all $x_1, x_2 \in [0, \infty)$ with $x_1 \leq x_2$. Then $D^*$ is optimal among all admissible impulse dividend strategies, that is, $V_{D^*}(x) = \sup_{D \in \mathcal{D}} V_D(x)$ for all $x \in [0, \infty)$.

**Proof.** We first define $g(x) := V_{D^*}(x) \in C^2((0, \infty))$. Consider any arbitrary admissible dividend strategy $D$, let $\{X^D(t); t \geq 0\}$ denotes the continuous part of the controlled surplus process $\{X^D(t); t \geq 0\}$. And define further a sequence of localization stopping times $\{T_{m,n}; m \geq 1, n \geq 1\}$ as

$$T_{m,n} = \inf \left\{ t \geq 0 \mid X^D(t) > n \text{ or } X^D(t) < \xi(X^D(t)) + \frac{1}{m} \right\}.$$ 

According to [17, Theorem 4.57], we have, for $x \in (0, \infty)$

$$e^{-q(t \wedge \tau_\xi^D \wedge T_{m,n})} g(X^D(t \wedge \tau_\xi^D \wedge T_{m,n}))$$

$$= g(x) - \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs} g(X^D(s))ds + \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs} g'(X^D(s))dX^D(s)$$

$$+ \frac{1}{2} \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs} g''(X^D(s))d\{X^D(\cdot), X^D_s(\cdot)\}$$

$$+ \sum_{s \leq t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs} \left( g(X^D(s^-)) + \Delta X^D(s) - g(X^D(s^-)) - g'(X^D(s^-))\Delta X^D(s) \right)$$

$$+ \sum_{s \in \Pi^D(t \wedge \tau_\xi^D \wedge T_{m,n})} e^{-qs} \left( g(X^D(s^+)) - g(X^D(s)) + g'(X^D(s))\left( L^D(s^+) - L^D(s) \right) \right)$$

$$= g(x) + \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs}(A - q)g(X^D(s))ds + \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} e^{-qs} g'(X^D(s))dB(s)$$

$$+ \int_{0^-}^{t \wedge \tau_\xi^D \wedge T_{m,n}} \int_0^\infty e^{-qs} \left[ g(X^D(s^-) - y) - g(X^D(s^-)) \right] (N(ds, dy) - v(dy)ds)$$

$$+ \sum_{s \in \Pi^D(t \wedge \tau_\xi^D \wedge T_{m,n})} e^{-qs} \left[ g(X^D(s^+)) - g(X^D(s^+) + (L^D(s^+) - L^D(s))) \right], \quad (26)$$

where $\Delta X^D(s) = X^D(s) - X^D(s^-)$. Since for all $s \in \left[0, t \wedge \tau_\xi^D \wedge T_{m,n}\right]$ we have (cf. (25))

$$g(X^D(s^+)) - g(X^D(s^+) + (L^D(s^+) - L^D(s))) + L^D(s^+) - L^D(s) - \beta \leq 0. \quad (27)$$

Therefore, by (26) and (27) we have, for $x \in (0, \infty)$
\[e^{-q(t\wedge\tau^D_{t,m,n})}g(X^D_{t\wedge\tau^D_{t,m,n}})\]
\[
geq g(x) + \int_0^{t\wedge\tau^D_{t,m,n}} e^{-qs}(A - q)g(X^D(s))ds
\]
\[
+ \int_{t\wedge\tau^D_{t,m,n}}^\infty e^{-qs}[g(X^D(s^-) - y) - g(X^D(s^-))](N(ds, dy) - v(dy)ds)
\]
\[
+ \int_{0^-}^{t\wedge\tau^D_{t,m,n}} e^{-qs}g'(X^D(s))dB(s)
\]
\[
- \sum_{s\leq t\wedge\tau^D_{t,m,n}} e^{-qs}(L^D(s^+) - L^D(s) - \beta).
\] (28)

In addition, it follows from [16, p. 62] that
\[t \mapsto \int_0^{t\wedge\tau^D_{t,m,n}} \int_0^\infty e^{-qs}[g(X^D(s^-) - y) - g(X^D(s^-))] (N(ds, dy) - v(dy)ds)\]
is an \((\mathcal{F}_t)\)-martingale with zero mean. Similarly, the following stochastic integration
\[t \mapsto \int_0^{t\wedge\tau^D_{t,m,n}} \sigma e^{-qs}g'(X^D(s))dB(s),\]
is also a zero mean \((\mathcal{F}_t)\)-martingale. Then take expectation on both sides of (28) and with the fact that \((A - q)g(x) \leq 0\), we have
\[g(x) \geq \mathbb{E}_x \left( e^{-q(t\wedge\tau^D_{t,m,n}\wedge\tau^D_{t,m,n})}g(X^D_{t\wedge\tau^D_{t,m,n}\wedge\tau^D_{t,m,n}}) \right)
\]
\[+ \mathbb{E}_x \left( \sum_{s\leq t\wedge\tau^D_{t,m,n}} e^{-qs} (L^D(s^+) - L^D(s) - \beta) \right), \quad x \in (0, \infty), \quad (29)\]

Finally, by letting \(t, m\) and \(n\) approach to \(\infty\) in (29) and noting that \(g(x) \geq 0\) for all \(x \in (0, \infty)\), we arrive at
\[g(x) \geq \mathbb{E}_x \left( \sum_{s\leq \tau^D_{t,m,n}} e^{-qs} (L^D(s^+) - L^D(s) - \beta) \right) = V_D(x), \quad \forall x \in (0, \infty).\]

Note that strategy \(D\) is arbitrary, hence we have
\[g(x) \geq \sup_{D \in \mathcal{D}} V_D(x), \quad \forall x \in (0, \infty).\]

Then with the continuity of \(g(x)\) and \(\sup_{D \in \mathcal{D}} V_D(x)\) given in Proposition 1, we obtain
\[g(x) \geq \sup_{D \in \mathcal{D}} V_D(x), \quad \forall x \in [0, \infty),\]
that is
\[V_{D^*}(x) \geq \sup_{D \in \mathcal{D}} V_D(x), \quad \forall x \in [0, \infty).\] (30)

For the general case \(V_{D^*_i}(x) \in C^1(0, \infty) \cap C^2(\{0, \infty\} \setminus \{d_1, \ldots, d_m\})\) and \(\max_i \lim_{x \uparrow d_i} |V_{D^*_i}(x)|, \lim_{x \downarrow d_i} |V_{D^*_i}(x)| < \infty\), one can adopt the mollifying arguments
(see [38, Lemma 4.3]) to get a sequence of twice continuous differentiable version of $V_D^*$, denoted by $(g_n)_{n \geq 1}$, such that

\[
\begin{cases}
A g_n(x) - q g_n(x) \leq 0, & \text{for } x \in [0, \infty), \\
g_n(x) \geq 0, & \text{for } x \in [0, \infty),
\end{cases}
\]

$g_n$ satisfies (25) for all $x_1, x_2 \in [0, \infty)$ and $n \geq 1$ with $x_1 \leq x_2$, and \( \lim_{n \to \infty} g_n(x) = V_D^*(x) \) almost everywhere. Hence, by replicating the arguments above, one can get $g_n(x) \geq \sup_{D \in \mathcal{D}} V_D^*(x)$ over $[0, \infty)$, which yields the desired result (30) by letting $n \to \infty$. Recalling that the reversed inequality of (30) is trivial, one completes the proof.

**Remark 2.** By Lemma 4.1, absence of twice differentiability of the function $V_D^*(x)$ at finitely many points will not affect the optimality verification arguments for the candidate optimal dividend strategy $D^*$.

Next, in Theorem 4.2, we follow a similar argument adopted in [24] and derive the condition under which certain $(z_1, z_2)$--type dividend strategy is optimal among all admissible impulse dividend strategies.

**Theorem 4.2.** Suppose $\xi$ and $W^{(q)} \in C^1(0, \infty)$ are piecewise continuously differentiable and piecewise twice continuously differentiable over $(0, \infty)$ respectively. Let $(z_1, z_2) \in \mathcal{M}$ such that $z_1 \geq \xi(z)$ for all $z \geq z_2$, and the function $\varsigma$ is decreasing over $[z_2, +\infty)$, i.e.,

\[
\varsigma(x) \geq \varsigma(y), \quad \text{for all } z_2 \leq x \leq y < \infty.
\]

Then, the impulse dividend strategy $(z_1, z_2)$ is the optimal dividend strategy among all admissible impulse dividend strategies.

**Proof.** With Proposition 4 and Lemma 4.1, we only need to show

\[
A V_{z_2}^{z_2}(x) - q V_{z_1}^{z_2}(x) \leq 0,
\]

holds true for $x \in [0, \infty) \setminus \{d_1, \ldots, d_{m_0}\}$, where the $\{d_1, \ldots, d_{m_0}\}$ with $d_0 := 0 \leq d_1 < \cdots < d_{m_0} < \infty := d_{m_0+1}$ are those points at which the continuously differentiability (twice continuously differentiability, respectively) does not hold true for $\xi(\cdot)$ ($W^{(q)}(\xi(\cdot))$, respectively). Given any $x \in (0, z_2)$, we are safe to assume $x \in (d_i, d_{i+1}) \cap (0, z_2)$ for some $0 \leq i \leq m_0$. Let $\tau^* := \tau_{d_i}^- \wedge \tau_{d_{i+1} \wedge z_2}^+ \wedge \tau_{\xi}^{D_{z_2}^2}$ with $\tau_{d_i}$ and $\tau_{d_{i+1} \wedge z_2}^+$ defined via

\[
\tau_{d_{i+1} \wedge z_2}^+ := \inf \left\{ t > 0 \left| X_{D_{z_2}^2}(t) > d_{i+1} \wedge z_2 \right. \right\}, \quad \tau_{d_i}^- := \inf \left\{ t > 0 \left| X_{D_{z_2}^2}(t) \leq d_i \right. \right\}.
\]
According to the strong Markov property of the process \( X \), we have

\[
\mathbb{E}_x \left( \int_0^{\tau^{D_{z_1}^2}_{z_1}} e^{-qu} dL^{D_{z_1}^2}_{z_1}(t) \right | \mathcal{F}_{s \land \tau^*}) = \mathbb{E}_x \left( \int_0^{\tau^{D_{z_1}^2}_{z_1} - s \land \tau^*} e^{-qu} dL^{D_{z_1}^2}_{z_1}(u + s \land \tau^*) \right | \mathcal{F}_{s \land \tau^*})
\]

\[
= e^{-q(s \land \tau^*)} \mathbb{E}_{X(s \land \tau^*)} \left( \int_0^{\tau^{D_{z_1}^2}_{z_1}} e^{-qu} dL^{D_{z_1}^2}_{z_1}(u) \right)
\]

\[
= e^{-q(s \land \tau^*)} V^{\tau^*}_{z_1} \left( X^{D_{z_1}^2}_{z_1}(s \land \tau^*) \right), \quad s \geq 0.
\]

Note that there are no dividend payments during the time interval \([0, \tau^+_z] \supseteq [0, s \land \tau^*)\); i.e., \( X^{D_{z_1}^2}_{z_1}(s \land \tau^*) = X(s \land \tau^*) \) for all \( s \leq \tau^+_z \). Then the above equation implies that the process

\[
\left( e^{-q(s \land \tau^*)} V^{\tau^*}_{z_1} \left( X^{D_{z_1}^2}_{z_1}(s \land \tau^*) \right) \right)_{s \geq 0}
\]

is a martingale, which further implies that

\[
\mathcal{A} V^{\tau^*}_{z_1}(x) - q V^{\tau^*}_{z_1}(x) = 0, \quad x \in (0, z_2) \setminus \{d_1, \ldots, d_{m_0}\}. \quad (32)
\]

Indeed, for \( x \in (d_i, d_{i+1}) \cap (0, z_2) \) and \( \tau^* = \tau_{d_i}^+ \land \tau_{d_{i+1}^-} \land \tau_{D_{z_1}^2}^+ \). Itô’s formula gives

\[
e^{-q(s \land \tau^*)} V^{\tau^*}_{z_1} \left( X^{D_{z_1}^2}_{z_1}(s \land \tau^*) \right) - V^{\tau^*}_{z_1}(x)
\]

\[
= \int_0^{s \land \tau^*} e^{-qu}(\mathcal{A} - q) V^{\tau^*}_{z_1}(X^{D_{z_1}^2}_{z_1}(u)) du + \int_0^{s \land \tau^*} \sigma e^{-qu} V^{\tau^*}_{z_1}(X^{D_{z_1}^2}_{z_1}(u)) dB(u)
\]

\[
+ \int_0^{s \land \tau^*} \int_0^{\infty} e^{-qu}[V^{\tau^*}_{z_1}(X^{D_{z_1}^2}_{z_1}(u^-)) - y] (N(du, dy) - v(dy) du), \quad s \geq 0.
\]

It is obvious that except the first term on right hand side of the above equation, one arrives at zero expectations after localization, then

\[
0 = \mathbb{E}_x \left( \int_0^{s \land \tau^* \land T_{m,n}} e^{-qu}(\mathcal{A} - q) V^{\tau^*}_{z_1}(X^{D_{z_1}^2}_{z_1}(u)) du \right), \quad s \geq 0, \quad (33)
\]

where \( \{T_{m,n}; m, n \geq 1\} \) is the sequence of localizing stopping times given in Lemma 4.1. Then (32) can be proved by first dividing \( \mathbb{E}_x(s \land \tau^* \land T_{m,n}) \) on both sides of (33) and then letting \( s \downarrow 0 \). We refer to [15, Proposition 2.1] for a detailed proof of (32). Letting \( x \downarrow 0 \), we can conclude that (32) holds true for \( x = 0 \) as well, given that \( d_i \neq 0 \) for all \( 1 \leq i \leq m_0 \). Therefore, we obtain (32) for \( x \in [0, z_2) \setminus \{d_1, \ldots, d_{m_0}\} \), i.e.,

\[
\mathcal{A} V^{\tau^*}_{z_1}(x) - q V^{\tau^*}_{z_1}(x) = 0, \quad x \in [0, z_2) \setminus \{d_1, \ldots, d_{m_0}\}.
\]

Then, we further show that

\[
\mathcal{A} V^{\tau^*}_{z_1}(x) - q V^{\tau^*}_{z_1}(x) \leq 0, \quad x \in [z_2, \infty) \setminus \{d_1, \ldots, d_{m_0}\}. \quad (34)
\]
Let \( V_x(y) \) be the value function when barrier strategy with barrier level \( x \) is applied when initial surplus equals \( y \), then,

\[
V_x(y) = \begin{cases} 
  \frac{w^{(q)}(\xi(x))}{\bar{w}^{(q)}\left(\frac{y}{x}\right)} \exp \left(-\int_y^x \frac{w^{(q)}(\xi(z))}{\bar{w}^{(q)}(\xi(z))} \, dz \right), & y \in [0, x], \\
  y - x + \frac{w^{(q)}(\xi(x))}{\bar{w}^{(q)}(\xi(x))}, & y \in [x, \infty).
\end{cases}
\]  

(35)

similar to the proof of (32), we arrive at

\[
\lim_{y \to x} (AV_x(y) - qV_x(y)) = 0, \quad y \in [0, x) \setminus \{d_1, \ldots, d_m\}, \quad x \in (0, \infty),
\]

which implies that

\[
\lim_{y \uparrow x} (AV_x(y) - qV_x(y)) = 0, \quad x \in (z_2, \infty) \setminus \{d_1, \ldots, d_m\}.
\]  

(36)

In the meanwhile, by the continuity of the function \( AV_{z_1} - qV_{z_1} \) at \( x \not\in \{d_1, \ldots, d_m\} \), we have

\[
\lim_{y \uparrow x} (AV_{z_1} - qV_{z_1}(y)) = AV_{z_1}(x) - qV_{z_1}(x), \quad x \in (z_2, \infty) \setminus \{d_1, \ldots, d_m\}.
\]  

(37)

(36) and (37) above indicate that (34) holds true if the following inequality can be proved:

\[
\lim_{y \uparrow x} (A[V_{z_1} - V_x(y)] - q[V_{z_1} - V_x]) \leq 0, \quad x \in (z_2, \infty) \setminus \{d_1, \ldots, d_m\}.
\]

For \( x \in (z_2, \infty) \setminus \{d_1, \ldots, d_m\} \), the dominated convergence theorem implies that

\[
\lim_{y \uparrow x} (A[V_{z_1} - V_x(y)] - q[V_{z_1} - V_x])
\]

\[
= \gamma \left( [V_{z_1}'(x) - V_x'(x)] \right) + \frac{\sigma^2}{2} \left[ V_{z_1}''(x) - \lim_{y \uparrow x} V_x''(y) \right] - q [V_{z_1}'(x) - V_x(x)]
\]

\[
+ \int_{(0, \infty)} \left( [V_{z_1}'(x-y) - V_x(x-y)] - [V_{z_1}'(x) - V_x(x)] \right) \nu(dy)
\]

\[
+ \left( [V_{z_1}'(x) - V_x'(x)] \right) \nu_{(0,1)}(y) \nu(dy)
\]

\[
= -\frac{\sigma^2}{2} \lim_{y \uparrow x} V_x''(y) - q [V_{z_1}'(x) - V_x(x)]
\]

\[
+ \int_{(0, \infty)} \left( [V_{z_1}'(x-y) - V_x(x-y)] - [V_{z_1}'(x) - V_x(x)] \right) \nu(dy),
\]  

(38)

where the last equality holds true since \( V_{z_1}'(x) = V_x'(x) = 1 \) and \( V_{z_1}''(x) = 0 \) for all \( x > z_2 \).

By (31), (35) and the same arguments of (21), we have that

\[
\lim_{y \uparrow x} V_x''(y) \geq 0, \quad x \in (z_2, \infty) \setminus \{d_1, \ldots, d_m\}.
\]  

(39)

By the definition of \( \zeta \) given in (20) and the mean value theorem, we obtain

\[
V_{z_1}(x) - V_x(x) = x - z_2 + \frac{w^{(q)}(\xi(z_2))}{w^{(q)}(\xi(x))} - \frac{w^{(q)}(\xi(z_2))}{w^{(q)}(\xi(x))}
\]

\[
= \sum_{i=0}^{m_0} \left( x_{i+1} - x_i + \frac{w^{(q)}(\xi(x_i))}{w^{(q)}(\xi(x_{i+1}))} - \frac{w^{(q)}(\xi(x_i))}{w^{(q)}(\xi(x_{i+1}))} \right)
\]

\[
= -\int_{\xi(x_i)}^{\xi(x_{i+1})} \frac{w^{(q)}(y)}{w^{(q)}(\xi(y))} \, dy \zeta'(\theta_i) (x_i - x_{i+1}) \geq 0, \quad x \geq z_2.
\]  

(40)
where \( x_0 = z_2, x_{m_0+1} = x, x_i = (z_2 \lor d_i) \land x \) for \( i \in \{1, \cdots, m_0\} \), \( \theta_i \in (x_i, x_{i+1}) \) and \( \varsigma'(\theta_i) \leq 0 \) (cf. (31)) for \( \theta_i \in (x_i, x_{i+1}) \subseteq (z_2, x) \) whenever \( x_i < x_{i+1} \), and \( \varsigma'(\theta) \) is additionally defined to be 0 whenever \( x_i = x_{i+1} \).

In addition, for \( x > z_2 \), let \( f(z) := V^x_{z_1^*}(z) - V^x_{z_2}(z) \), for \( z \in [0, \infty) \). By (31), we claim that

\[
\begin{cases}
\frac{w(\varsigma'(\xi(z))))}{w(\varsigma'(\xi(y)))} \exp \left(-\int_z^y \frac{w(\varsigma'(\xi(t))))}{w(\varsigma'(\xi(t)))} dt \right) \varsigma'(\xi(z)) \left(1 - \frac{\varsigma(z)}{\varsigma(z_2)}\right) \\
1 - \frac{w(\varsigma'(\xi(z))))}{w(\varsigma'(\xi(y)))} \exp \left(-\int_z^y \frac{w(\varsigma'(\xi(t))))}{w(\varsigma'(\xi(t)))} dt \right) \varsigma'(\xi(z)) \left(1 - \frac{\varsigma(z)}{\varsigma(z_2)}\right) \\
1 - 1 = 0,
\end{cases}
\]

whenever \( \varsigma(z) < \varsigma(z_2) \).

That is to say, \( f(z) \) is a nondecreasing function of \( z \). Hence, we have

\[
[V^x_{z_1}(x - y) - V^x_{z_2}(x - y)] - [V^x_{z_1^*}(x) - V^x_{z_2}(x)] = f(x - y) - f(x) \leq 0, \quad y \in [0, \infty). \quad (42)
\]

Putting (39), (40) and (42) together, we conclude that the right hand side of (38) is non-positive, which verifies (34) for \( x \in (z_2, +\infty) \). The equality in (34) for \( x = z_2 \) can be obtained by letting \( x \downarrow z_2 \), if \( d_i \neq z_2 \) for all \( 1 \leq i \leq m_0 \). Therefore, according to Lemma 4.1, we have proved that the impulse dividend strategy \((z_1, z_2)\) is optimal among all admissible impulse dividend strategies.

5. **Numerical examples.** In order to illustrate the existence of the optimal impulse dividend strategy \((z_1^*, z_2^*)\), in this section, we compute the optimal strategies and corresponding value functions under several examples of spectrally negative Lévy process, namely Cramér-Lundberg model, Brownian motion with drift and Jump-diffusion process. In addition, we also examine how the fixed transaction costs affect the resulting optimal impulse dividend strategies. Before that, we shall first introduce a particular type of drawdown function. As discussed in Remark 1 and Remark 2 in [6], it can be linked to different problems in insurance or finance when the drawdown function takes different forms. For examples, when \( \xi(x) \equiv 0 \) it retrieves ruin time in risk theory; the classical drawdown time can be obtained by setting \( \xi(x) = x - d \), which has many applications in finance; a linear drawdown function of the form \( \xi(x) = kx - d \) for \( k < 1 \) is also interesting under the risk management context, which refers to fixed units drop from a fraction of current maximum. For more details see [6] and examples in [23]. Nonlinear drawdown functions, which may not be frequently adopted in insurance or financial problems, can be used to provide a solution of the Skorohod embedding problem (see e.g. [7], [30] and [28]).

However, in our examples, we introduce a new type of drawdown function defined as

\[
\xi(x) := \min\{kx, \alpha\}, \quad x \geq 0, \quad (43)
\]

where \( k \in [0, \infty) \) and \( \alpha > 0 \). As mentioned in Section 1, the current modifications on De Finetti’s dividend problem only took into account the risk of ruin; then, by considering a general drawdown time with drawdown function defined in (43), we are able to cover various solvency related objectives when finding the optimal dividend strategy. (a) When \( \alpha \to \infty \) and \( k \in [0, 1) \), the interpretation of \( k \) in (43) is the same as the one in the linear drawdown function (for simplicity, we omit the constant \( d \) in the linear form, but the results can be easily extended to case with \( d \)), indicating that the insurer focuses on the linear drawdown risks rather than ruin, which is more conservative in terms of solvency requirements. However, any finite value of \( \alpha \) can be interpreted as a sufficient solvency level, such that the risk of linear drawdown above \( \alpha \) will not be considered in the optimal dividend problem in
order to avoid over-weighting on solvency comparing to profitability; but note that the level $\alpha$ will not play a role when the previous running maximum is sufficiently low. (b) When $k \to \infty$ and $\alpha > 0$, we end up with a constant solvency level $\alpha$, this scenario is interesting if the solvency risk related to certain minimum solvency level need to be considered. (c) When $k = 0$, we retrieve the original ruin case. The sample paths (in terms of Cramér-Lundberg model) of the aforementioned classical, linear drawdown times and ruin time are illustrated in Figure 1 and the two scenarios of general drawdown time based on (43) are showed in Figure 2 respectively.

**Figure 1.** Illustration of various drawdown times

**Figure 2.** General drawdown times based on Eq.(43)

**Remark 3.** The general drawdown function given in (43) is piecewise continuously differentiable which is the least assumption we needed in Theorem 4.2. In addition, since $\xi(x)$ in (43) is upper bounded by $\alpha$, the condition $z_1^* \geq \xi(z)$ for all $z \geq z_2^*$ can be easily satisfied.

5.1. Cramér-Lundberg model. Let $X(t) = x + ct - \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a homogeneous Poisson process with intensity $\lambda$, $\{Y_i\}_{i \geq 1}$ are i.i.d. exponential r.v.s. with mean $1/\gamma$. And $c > 0$ is the premium rate. We shall assume that $c > \lambda/\gamma$
for the safety loading condition. Then we can derive the explicit expression for the scale function \( W(q)(x) \) for any \( q > 0 \),

\[
W(q)(x) = \frac{A_+ e^{\theta_+ x} - A_- e^{\theta_- x}}{c}, \quad x \geq 0,
\]

where

\[
A_+ = \gamma + \theta_+ , \quad A_- = \gamma + \theta_- ,
\]

and

\[
\theta_\pm = q + \lambda - c\gamma \pm \sqrt{(q + \lambda - c\gamma)^2 + 4cq\gamma}.
\]

Example 5.1. In this example, we assume that the underlying risk process is the Cramér-Lundberg model with parameters defined as follows: let \( c = 1.5, \gamma = 1.0, \lambda = 1.0, \) the discounting factor \( q = 0.05 \) and the fixed transaction cost \( \beta = 0.5 \). We further assume that the drawdown function is defined by (43) with \( \alpha = 2.0, 1.5, 1.2, 1.0 \) and \( k = \infty, 0.6, 0.4, 0.2 \) respectively. Then by numerically maximizing the function \( \zeta(z_1, z_2) \) given in (12) with the constraints that \( z_1 + \beta < z_2, \ z_1 \geq 0 \) and \( z_1 \geq \xi(z) \) for all \( z \geq z_2 \), we obtain the optimal impulse dividend strategies \((z_1^*, z_2^*)\). Results are given in Table 1.

| \((z_1^*, z_2^*)\) | \(k = \infty\) | \(k = 0.6\) | \(k = 0.4\) | \(k = 0.2\) |
|-----------------|----------------|----------------|----------------|----------------|
| \(\alpha = 2.0\) | (3.9689, 11.6898) | (3.9689, 11.6902) | (2.0000, 11.2002) | (2.0000, 10.1503) |
| \(\alpha = 1.5\) | (3.4688, 11.1898) | (3.4688, 11.1893) | (1.5000, 10.0581) | (1.5330, 10.3557) |
| \(\alpha = 1.2\) | (3.1688, 10.8898) | (3.1680, 10.8917) | (3.1688, 10.8898) | (1.5369, 10.6025) |
| \(\alpha = 1.0\) | (2.9688, 10.6898) | (2.9687, 10.6901) | (2.9688, 10.6894) | (2.9688, 10.6904) |

Table 1. Cramér-Lundberg model

The values of \( \zeta(z) \) defined in (20) for \( z \geq z_2^* \) are illustrated in Figure 3, which verifies the optimality of the resulting impulse dividend strategies. Then, the corresponding value functions \( V_{z_2^*}^{z_1^*} \) are given in Figure 4. We also calculate the value functions for ruin time scenarios as a comparison.

Example 5.2. We examine the influence of the fixed transaction costs \( \beta \) on the optimal impulse dividend strategy in this example. Assume that \( c = 1.5, \gamma = 1.0, \lambda = 1.0 \) and the discounting factor \( q = 0.05 \). The resulting optimal strategies for various values of \( \alpha \) and \( k \) and ruin time scenarios are listed in the Table 2.

| \((z_1^*, z_2^*)\) | \(\alpha = 2.0, k = \infty\) | \(\alpha = 1.5, k = 0.6\) | \(\alpha = 1.0, k = 0.2\) | Ruin Time |
|-----------------|----------------|----------------|----------------|----------------|
| \(\beta = 0.5\) | (4.5855, 10.5305) | (4.0855, 10.0305) | (2.6070, 9.3218) | (2.5855, 8.5305) |
| \(\beta = 1.0\) | (3.9689, 11.6898) | (3.4688, 11.1893) | (1.7244, 10.3557) | (1.5369, 10.6025) |
| \(\beta = 1.5\) | (3.5369, 12.6025) | (3.0369, 12.1025) | (1.0605, 11.9132) | (1.5369, 10.6025) |
| \(\beta = 2.0\) | (3.1911, 13.3968) | (1.5000, 12.8618) | (1.1912, 11.3968) | (1.5369, 10.6025) |
| \(\beta = 2.5\) | (2.8968, 14.1194) | (1.5000, 13.5114) | (0.8968, 12.1194) | (1.5369, 10.6025) |

Table 2. Cramér-Lundberg model: \((z_1^*, z_2^*)\) vs. \(\beta\)
Figure 3. Cramér-Lundberg model: $\varsigma(z)$

In Table 1, we can observe that in general both $z_1^*$ and $z_2^*$ are decreasing when $\alpha$ and/or $k$ decrease, but the impulse amount $(z_2^* - z_1^*)$ is roughly stable. However, we also observe different scenarios when $k$ is small and $\alpha$ is large, where lower level $z_1^*$ in the optimal impulse dividend strategies are restricted by the solvency level $\alpha$. In Figure 4, we can observe that the value functions under our general drawdown times are less than the one under ruin time for any initial surplus, which indicates that as a measure of risk in optimal dividend problem, drawdown is more conservative than ruin. The differences are larger for larger $k$ and $\alpha$, where the cases with fixed solvency level $\alpha$ (i.e., $k \to \infty$) are in general the most conservative scenarios here.

In addition, Table 2 shows that when the fixed transaction costs $\beta$ increases, the upper level $z_2^*$ in the band-type impulse dividend strategy will increase accordingly, and the amount for the lump sum dividend will also increase; however, decreasing trends are found in $z_1^*$. This observation is reasonable since when transaction cost is large, the insurer will try to reduce the possible number of transactions for paying out dividends, which in turn results in a higher level of impulse amount each time.

5.2. Brownian motion with drift. Next, we assume that $X(t) = ut + B(t)$, where $B(t)$ is a standard Brownian Motion and $u$ is a drift constant. Then the $q$-scale function is given by

$$W(q)(x) = \frac{1}{\sqrt{u^2 + 2q}} \left( e^{\mu+ x} - e^{\mu- x} \right), \quad x \geq 0,$$

where

$$\mu_+ = -u + \sqrt{u^2 + 2q}, \quad \mu_- = -u - \sqrt{u^2 + 2q}.$$
Example 5.3. In this example, we calculate the optimal impulse dividend strategies and corresponding value functions when the underlying risk process is a Brownian motion with drift $u = 1.5$, the discounting factor $q = 0.05$ and the fixed transaction cost $\beta = 1.0$. Similar to the example in Cramér-Lundberg model, we consider the piecewise continuously differentiable drawdown function $\xi(x)$ for various values of $\alpha$ and $k$. The results are listed in Table 3, Figure 5 and Figure 6 respectively.

| $(z_1^*, z_2^*)$ | $k = \infty$ | $k = 0.6$ | $k = 0.4$ | $k = 0.2$ |
|-------------------|--------------|------------|------------|------------|
| $\alpha = 2.0$    | (3.8442, 12.3162) | (3.8441, 12.3176) | (3.0519, 11.7522) | (2.2989, 10.8580) |
| $\alpha = 1.5$    | (3.3442, 11.8162) | (3.3441, 11.8162) | (3.0575, 11.6979) | (2.2989, 10.8580) |
| $\alpha = 1.2$    | (3.0442, 11.5162) | (3.0441, 11.5162) | (3.0397, 11.5094) | (2.2989, 10.8580) |
| $\alpha = 1.0$    | (2.8442, 11.3162) | (2.8439, 11.3162) | (2.8441, 11.3166) | (2.2989, 10.8580) |

Table 3. Brownian Motion with drift

Example 5.4. We calculate the optimal impulse dividend strategies for different values of fixed transaction costs $\beta$ under Brownian motion with drift $u = 1.5$, and the discounting factor $q = 0.05$. The same drawdown functions (including ruin time case) are chosen as in Example 5.2. The results are listed in Table 4.

Figure 4. Cramér-Lundberg model: $V_{z_1^*}^{z_2^*}(x)$
According to Table 4, under Brownian motion with drift risk model, the increase of the fixed transaction cost $\beta$ will also increase the amount for the lump sum dividend payment. However, an interesting observation here is that unlike the case under Cramér-Lundberg model where the influence on $z_1^*$ and $z_2^*$ are roughly balanced, the case under Brownian motion with drift model are unbalanced, where the change of $\beta$ has only limited effect on $z_1^*$ comparing to $z_2^*$. This also indicates the difference between insurance type risk models like Cramér-Lundberg model and diffusion processes that frequently used in financial modeling.

5.3. Jump-diffusion process. In this subsection, we let $X$ be a jump-diffusion process defined as follows:

$$X(t) = x + ct + \sigma B(t) - \sum_{i=1}^{N(t)} Y_i,$$

where

$$\begin{align*}
(z_1^*, z_2^*) & \quad \alpha = 2.0, k = \infty \\
\beta = 0.5 & \quad (3.9695, 9.9869) \\
\beta = 1.0 & \quad (3.4411, 11.8162) \\
\beta = 1.5 & \quad (3.7664, 14.1668) \\
\beta = 2.0 & \quad (3.7087, 15.7657) \\
\beta = 2.5 & \quad (3.6623, 17.2027)
\end{align*}$$

Table 4. Brownian motion with drift: $(z_1^*, z_2^*)$ vs. $\beta$
where \( \sigma > 0 \), \( B(t) \) is a standard Brownian motion, \( N(t) \) is a Poisson process with intensity \( \lambda \); \( c \) and \( x \) are the premium rate and initial surplus respectively. Furthermore, we follow the same assumption on \( \{Y_i\}_{i \geq 1} \) in [24], namely they are i.i.d. Erlang\((2, \gamma)\) random variables. Then the scale function \( W^{(q)}(x) \) for any \( q > 0 \) can be expressed as
\[
W^{(q)}(x) = \sum_{j=1}^{4} D_j e^{\theta_j x}, \quad x \geq 0,
\]
where
\[
D_j = \frac{(\gamma + \theta_j)^2}{\sigma^2 \prod_{i=1, i \neq j}^{4} (\theta_j - \theta_i)},
\]
and \( (\theta_j)_{j=1}^{4} \) are the (possibly complex) distinct zeros of the polynomial \( (\psi(\cdot) - q)(\gamma + \cdot)^2 \), where \( \psi(\cdot) \) is the corresponding Laplace exponent of the jump-diffusion process \( X \). For details one can refer to [24].

**Example 5.5.** Consider the above jump-diffusion process with \( \sigma = 2.0 \), \( \lambda = 1.0 \), \( c = 1.5 \), \( q = 0.05 \) and \( \gamma = 2.0 \). We calculate the optimal impulse dividend strategies for \( \alpha = 2.0, 1.5, 1.2, 1.0 \) and \( k = \infty, 0.6, 0.4, 0.2 \) respectively; the results are listed in Table 5. The value functions for various general drawdown times and the ruin time case are given in Figure 8. The corresponding results for \( \zeta(z) \) are illustrated in Figure 7 as well.
Table 5. Jump-diffusion process

Example 5.6. In this last example, we compare the optimal impulse dividend strategies for various values of fixed transaction costs $\beta$ under the jump-diffusion process. The optimal impulse dividend strategies for three different drawdown functions as well as ruin time case are calculated. The results are listed in Table 6.

| $(z_1^*, z_2^*)$ | $k = \infty$ | $k = 0.6$ | $k = 0.4$ | $k = 0.2$ |
|------------------|---------------|---------------|---------------|---------------|
| $\alpha = 2.0$ | (5.4394, 14.7877) | (5.4394, 14.7879) | (5.4393, 14.7878) | (2.0000, 13.4730) |
| $\alpha = 1.5$ | (4.9394, 14.2876) | (4.9394, 14.2876) | (4.9394, 14.2875) | (2.1303, 13.7094) |
| $\alpha = 1.2$ | (4.6394, 13.9876) | (4.6394, 13.9876) | (4.6386, 13.9915) | (2.7529, 13.7678) |
| $\alpha = 1.0$ | (4.4394, 13.7876) | (4.4394, 13.7865) | (4.4394, 13.7877) | (3.5068, 13.7285) |

Table 6. Jump-diffusion process: $(z_1^*, z_2^*)$ vs. $\beta$
Figure 8. Jump-diffusion process: $V_{z_1}^{z_2}(x)$

The graphs of value functions under the three different risk models, especially for the drawdown functions with high solvency level, show that the value functions for small initial surplus under our general drawdown times are considerably different comparing to those with ruin time. This differences can also be observed directly by comparing the expression of the performance function given in (7) to the corresponding results under ruin time case in [24]. This may indicate the necessity of considering drawdown risk rather than ruin risk in De Finetti’s optimal dividend problem when the surplus level is considerable low, especially under the risk management point of view.

6. Conclusion. This paper investigates a general drawdown time based De Finetti’s dividend problem with fixed transaction costs under spectrally negative Lévy risk processes, where a general drawdown time associated with the dividend problem is introduced to replace the ruin time in the framework. Such modification plays an important role in finding balance between the profitability and solvency in the De Finetti’s dividend problem. We express the performance functions under $(z_1, z_2)$-type impulse dividend strategies in terms of the scale functions, and identify the condition under which certain $(z_1^*, z_2^*)$ impulse dividend strategy is optimal among all admissible impulse dividend strategies. We further proposed a new type of drawdown function, with which we arrive at a general drawdown time based optimal dividend framework with more flexible balance between profitability and solvency comparing to ruin time based ones. We also calculate the optimal strategies and corresponding value functions for Cramér–Lundberg model, Brownian motion
with drift and Jump-diffusion process respectively; and investigate how the drawdown functions and the fixed transaction costs affect the resulting optimal dividend strategies through various numerical examples. The difference (in terms of value functions) between the ruin time based and our general drawdown time based framework are illustrated as well, from which we can conclude that by considering the risk of drawdown with well selected drawdown functions in De Finetti’s optimal dividend problem, we are able to obtain less aggressive optimal dividend strategies and remedy, to some extent, the unbalance between profitability and solvency in the original framework.

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