New classes of infinitely divisible distributions related to the Goldie–Steutel–Bondesson class

Takahiro Aoyama · Alexander Lindner · Makoto Maejima

(Running head: New classes of infinitely divisible distributions)

Abstract Recently, many classes of infinitely divisible distributions on $\mathbb{R}^d$ have been characterized in several ways. Among others, the first way is to use Lévy measures, the second one is to use transformations of Lévy measures, and the third one is to use mappings of infinitely divisible distributions defined by stochastic integrals with respect to Lévy processes. In this paper, we are concerned with a class of mappings, by which we construct new classes of infinitely divisible distributions on $\mathbb{R}^d$. Then we study a special case in $\mathbb{R}^1$, which is the class of infinitely divisible distributions without Gaussian parts generated by stochastic integrals with respect to a fixed compound Poisson processes on $\mathbb{R}^1$. This is closely related to the Goldie–Steutel–Bondesson class.

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1. INTRODUCTION

Throughout this paper, $\mathcal{L}(X)$ denotes the law of an $\mathbb{R}^d$-valued random variable $X$ and $\hat{\mu}(z), z \in \mathbb{R}^d$, denotes the characteristic function of a probability distribution $\mu$ on $\mathbb{R}^d$. Also $I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions

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on \( \mathbb{R}^d \), \( I_{\text{sym}}(\mathbb{R}^d) = \{ \mu \in I(\mathbb{R}^d) : \mu \text{ is symmetric on } \mathbb{R}^d \} \), \( I_{\log}(\mathbb{R}^d) = \{ \mu \in I(\mathbb{R}^d) : \int |x| > 1 \log |x| \mu(dx) < \infty \} \) and \( I_{\log^m}(\mathbb{R}^d) = \{ \mu \in I(\mathbb{R}^d) : \int |x| > 1 (\log |x|)^m \mu(dx) < \infty \} \), where \( |x| \) is the Euclidean norm of \( x \in \mathbb{R}^d \). Let \( C_\mu(z), z \in \mathbb{R}^d \), be the cumulant function of \( \mu \in I(\mathbb{R}^d) \). That is, \( C_\mu(z) \) is a continuous function with \( C_\mu(0) = 0 \) such that \( \hat{\mu}(z) = \exp \{ C_\mu(z) \}, z \in \mathbb{R}^d \).

We use the generating triplet \((A, \nu, \gamma)\) for \( \lambda \) and satisfies
\[
\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty.
\]

The polar decomposition of the Lévy measure \( \nu \) of \( \mu \in I(\mathbb{R}^d) \), with \( 0 < \nu(\mathbb{R}^d) \leq \infty \), is the following: There exist a measure \( \lambda \) on \( S = \{ \xi \in \mathbb{R}^d : |\xi| = 1 \} \) with \( 0 < \lambda(S) \leq \infty \) and a family \( \{ \nu_\xi : \xi \in S \} \) of measures on \((0, \infty)\) such that \( \nu_\xi(B) \) is measurable in \( \xi \) for each \( B \in \mathcal{B}((0, \infty)) \), \( 0 < \nu_\xi((0, \infty)) \leq \infty \) for each \( \xi \in S \),
\[
(1.1) \quad \nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]

Here \( \lambda \) and \( \{ \nu_\xi \} \) are uniquely determined by \( \nu \) up to multiplication by a measurable function \( c(\xi) \) and \( c(\xi)^{-1} \) with \( 0 < c(\xi) < \infty \). The measure \( \nu_\xi \) is a Lévy measure on \((0, \infty)\) for \( \lambda\)-a.e. \( \xi \in S \). We say that \( \nu \) has the polar decomposition \((\lambda, \nu_\xi)\) and \( \nu_\xi \) is called the radial component of \( \nu \). (See, e.g., Lemma 2.1 of [3] and its proof.)

The classes which we are going to study in this paper are the following.

**Definition 1.1.** (Class \( E_\alpha(\mathbb{R}^d), \alpha > 0 \).) We say that \( \mu \in I(\mathbb{R}^d) \) belongs to the class \( E_\alpha(\mathbb{R}^d) \) if \( \nu = 0 \) or \( \nu \neq 0 \) and, in case \( \nu \neq 0 \), \( \nu_\xi \) in (1.1) satisfies
\[
\nu_\xi(dr) = r^{\alpha - 1}g_\xi(r^{\alpha})dr, \ r > 0,
\]
for some function \( g_\xi(r) \), which is completely monotone in \( r \in (0, \infty) \) for \( \lambda\)-a.e. \( \xi \), is measurable in \( \xi \) for each \( r > 0 \) and satisfies
\[
\int_0^\infty (r^{\alpha + 1} \wedge r^{\alpha - 1})g_\xi(r^{\alpha})dr < \infty, \ r > 0, \ \lambda\text{-a.e. } \xi.
\]

The following four known classes are needed in our discussion.
(1) Class $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class): $\mu \in I(\mathbb{R}^d)$ belongs to the class $B(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in this case, $\nu_\xi$ in (1.1) satisfies

$$
\nu_\xi(dr) = g_\xi(r)dr,
$$

where $g_\xi(r)$ is completely monotone in $r \in (0, \infty)$ for $\lambda$-a.e. $\xi$ and is measurable in $\xi$ for each $r > 0$. Hence $E_1(\mathbb{R}^d) = B(\mathbb{R}^d)$.

(2) Class $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions): $\mu \in I(\mathbb{R}^d)$ belongs to the class $L(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in this case, $\nu_\xi$ in (1.1) satisfies

$$
\nu_\xi(dr) = r^{-1}k_\xi(r)dr,
$$

where $k_\xi(r)$ is nonincreasing in $r \in (0, \infty)$ for $\lambda$-a.e. $\xi$ and is measurable in $\xi$ for each $r > 0$.

(3) Class $M(\mathbb{R}^d)$: $\mu \in I(\mathbb{R}^d)$ belongs to the class $M(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in this case, $\nu_\xi$ in (1.1) satisfies

$$
\nu_\xi(dr) = r^{-1}g_\xi(r^2)dr,
$$

where $g_\xi(r)$ is completely monotone in $r \in (0, \infty)$ for $\lambda$-a.e. $\xi$ and is measurable in $\xi$ for each $r > 0$.

(4) Class $T(\mathbb{R}^d)$ (the Thorin class): $\mu \in I(\mathbb{R}^d)$ belongs to the class $T(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in this case, $\nu_\xi$ in (1.1) satisfies

$$
\nu_\xi(dr) = r^{-1}g_\xi(r)dr,
$$

where $g_\xi(r)$ is completely monotone in $r \in (0, \infty)$ for $\lambda$-a.e. $\xi$ and is measurable in $\xi$ for each $r > 0$.

We introduce four mappings from $I(\mathbb{R}^d)$ (or $I_{\log}(\mathbb{R}^d)$) into $I(\mathbb{R}^d)$, which are related to the classes above. Throughout this paper, $\{X_t^{(\mu)}\}$ denotes a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

**Definition 1.2.** (1) For $\alpha > 0$, $E_\alpha(\mu) = \mathcal{L}\left(\int_0^1 (\log t^{-1})^{1/\alpha} dX_t^{(\mu)}\right)$, $\mu \in I(\mathbb{R}^d)$.
(2) $\Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-t}dX_t^{(\mu)}\right)$, $\mu \in I_{\log}(\mathbb{R}^d)$.
(3) $\mathcal{M}(\mu) = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right)$, $\mu \in I_{\log}(\mathbb{R}^d)$, where $m(x) = \int_x^\infty u^{-1}e^{-u^2}du, x > 0$, and $m^*(t)$ is its inverse function in the sense that $m(x) = t$ if and only if $x = m^*(t)$.
(4) $\Psi(\mu) = \mathcal{L}\left(\int_0^\infty e^*(t)dX_t^{(\mu)}\right)$, $\mu \in I_{\log}(\mathbb{R}^d)$, where $e(x) = \int_x^\infty u^{-1}e^{-u}du, x > 0$, and $e^*(t)$ is its inverse function in the sense that $e(x) = t$ if and only if $x = e^*(t)$.
Only the mapping $\mathcal{E}_\alpha$ (for $\alpha \neq 1$) is new. It is known that $\mathcal{D}(\Phi) = \mathcal{D}(\Psi) = I_{\log}(\mathbb{R}^d)$, where the domain $\mathcal{D}(\ast)$ means the set of infinitely divisible distributions $\mu$ on $\mathbb{R}^d$ on which the $\ast$-mapping is definable, in the sense of improper integrals with respect to independently scattered random measures on $\mathbb{R}^d$, as in Definitions 2.3 and 3.1 of Sato [15]. (For the determination of $\mathcal{D}(\Phi)$, $\mathcal{D}(\Psi)$ and $\mathcal{D}(\mathcal{M})$, see [16], [3] and [2], respectively.) For $\mathcal{E}_\alpha$, we have $\mathcal{D}(\mathcal{E}_\alpha) = I(\mathbb{R}^d)$, as shown in Proposition 2.1 below.

**Remark 1.3.** $\mathcal{E}_1$ is known as the Upsilon mapping (denoted by $\Upsilon$ in the literature) and it is known that $\mathcal{D}(\Upsilon) = I(\mathbb{R}^d)$ and $\Upsilon(I(\mathbb{R}^d)) = B(\mathbb{R}^d)$. Recall that $E_1(\mathbb{R}^d) = B(\mathbb{R}^d)$. Hence

$$E_1(\mathbb{R}^d) = \mathcal{E}_1(I(\mathbb{R}^d)).$$

The paper is organized as follows. In Section 2, we show several properties of the mapping $\mathcal{E}_\alpha$. In Section 3, we show that $E_\alpha(\mathbb{R}^d) = \mathcal{E}_\alpha(I(\mathbb{R}^d))$, $\alpha > 0$. This gives us stochastic integral representations of the elements of the class $E_\alpha(\mathbb{R}^d)$. In Section 4, we consider the composition $\mathcal{N}_\alpha$, say, of two mappings $\Phi$ and $\mathcal{E}_\alpha$, and show in particular that $\mathcal{M}(=\mathcal{N}_2)$ is the composition of $\Phi$ and $\mathcal{E}_2$. Then as an application of this equality, we show that the limit of certain subclasses of $N_\alpha(\mathbb{R}^d) := \mathcal{N}_\alpha(I_{\log}(\mathbb{R}^d))$, constructed by the iteration of the mapping $\mathcal{N}_\alpha$, is the closure of the class of the stable distributions as Maejima and Sato [10] showed for other mappings. In Section 5, we restrict ourselves to the case $d = 1$ and characterize

$$E_\alpha^0(\mathbb{R}^1) := \{\mu \in E_\alpha(\mathbb{R}^1) : \mu \text{ has no Gaussian part}\}$$

and certain subclasses of $E_\alpha^0(\mathbb{R}^1)$ which correspond to Lévy processes of bounded variation with zero drift, by (essential improper) stochastic integrals with respect to some compound Poisson processes. This gives us a new sight of the Goldie–Steutel–Bondesson class in $\mathbb{R}^1$.

**2. Several properties of the mapping $\mathcal{E}_\alpha$**

We start with showing several properties of the mapping $\mathcal{E}_\alpha$.

**Proposition 2.1.** Let $\alpha > 0$.

(i) $\mathcal{E}_\alpha(\mu)$ can be defined for any $\mu \in I(\mathbb{R}^d)$ and is infinitely divisible, and we have
\[
\int_0^1 |C_\mu(z(\log t^{-1})^{1/\alpha})| \, dt < \infty \quad \text{and} \\
C_{E_\alpha(\mu)}(z) = \int_0^1 C_\mu(z (\log t^{-1})^{1/\alpha}) \, dt, \quad z \in \mathbb{R}^d.
\]

(ii) The generating triplet \((\tilde{A}, \tilde{\nu}, \tilde{\gamma})\) of \(\bar{\mu} = E_\alpha(\mu)\) can be calculated from \((A, \nu, \gamma)\) of \(\mu\) by

\[
\tilde{A} = \Gamma(1 + 2/\alpha) A,
\]

\[
\tilde{\nu}(B) = \int_0^\infty \nu(u^{-1}B)\alpha u^{\alpha - 1}e^{-u^\alpha} \, du, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),
\]

\[
\tilde{\gamma} = \Gamma(1 + 1/\alpha) \gamma + \int_0^\infty \alpha u^{\alpha - 1} \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |ux|^2} - \frac{1}{1 + |x|^2} \right) \nu(dx) \, du.
\]

(iii) The mapping \(E_\alpha : I(\mathbb{R}^d) \to I(\mathbb{R}^d)\) is one-to-one.

(iv) Let \(\mu_n \in I(\mathbb{R}^d), \ n = 1, 2, \ldots\) If \(\mu_n\) converges weakly to \(\mu \in I(\mathbb{R}^d)\) as \(n \to \infty\), then \(E_\alpha(\mu_n)\) converges weakly to \(E_\alpha(\mu)\) as \(n \to \infty\). Conversely, if \(E_\alpha(\mu_n)\) converges weakly to \(\bar{\mu}\) for some distribution \(\bar{\mu}\) as \(n \to \infty\), then \(\bar{\mu} = E_\alpha(\mu)\) for some \(\mu \in I(\mathbb{R}^d)\) and \(\mu_n\) converges weakly to \(\mu\) as \(n \to \infty\). In particular, the range \(E_\alpha(I(\mathbb{R}^d))\) is closed under weak convergence.

(v) For any \(\mu \in I(\mathbb{R}^d)\) we also have

\[
E_\alpha(\mu) = \mathcal{L} \left( \int_0^1 \left( \log \frac{1}{1 - t} \right)^{1/\alpha} \, dX_t^{(\mu)} \right) = \mathcal{L} \left( \lim_{s \downarrow 0} \int_s^1 \frac{1}{\alpha t} (\log t^{-1})^{1/\alpha - 1} X_t^{(\mu)} \, dt \right),
\]

where the limit is almost sure.

Proof. (The proof follows along the lines of Proposition 2.4 of [3]. However, we give the proof for the completeness of the paper.)

(i) The function \(f(t) = (\log t^{-1})^{1/\alpha} 1_{[0,1]}(t)\) is clearly square integrable, hence the result follows from Sato [14], see also Lemma 2.3 in Maejima [8].

(ii) By a general result (see Lemma 2.7 and Corollary 4.4 of Sato [13]) and a change of variable, we have

\[
\tilde{A} = \left( \int_0^1 (\log t^{-1})^{2/\alpha} \, dt \right) A = \left( \int_0^\infty u^{2/\alpha} e^{-u} \, du \right) A = \Gamma(1 + 2/\alpha) A,
\]

\[
\tilde{\nu}(B) = \int_0^1 \nu((\log t^{-1})^{-1/\alpha} B) \, dt = \int_0^\infty \nu(u^{-1}B) \alpha u^{\alpha - 1} e^{-u^\alpha} \, du,
\]

\[
\tilde{\gamma} = \int_0^1 (\log t^{-1})^{1/\alpha} \left( \gamma + \int_{\mathbb{R}^d} x \left( \frac{1}{1 + |(\log t^{-1})^{1/\alpha} x|^2} - \frac{1}{1 + |x|^2} \right) \nu(dx) \right) \, dt
\]
Hence we conclude that for each \( C \in \mathbb{R}^d \),
\[
C_{\mathcal{E}_\alpha(\mu)}(z) = \int_0^1 C_\mu(z(\log t^{-1})^{1/\alpha}) \, dt = \int_0^\infty C_\mu(z v^{1/\alpha}) e^{-v} \, dv.
\]
Hence we conclude that for each \( z \in \mathbb{R}^d \),
\[
\frac{1}{u} C_{\mathcal{E}_\alpha(\mu)}(u^{-1/\alpha} z) = \int_0^\infty \frac{1}{u} C_\mu \left( \left( \frac{v}{u} \right)^{1/\alpha} z \right) e^{-v} \, dv = \int_0^\infty C_\mu(w^{1/\alpha} z) e^{-uw} \, dw.
\]
Hence we see that for each \( z \in \mathbb{R}^d \), the function \((0, \infty) \rightarrow \mathbb{R}, u \mapsto u^{-1} C_{\mathcal{E}_\alpha(\mu)}(u^{-1/\alpha} z)\) is the Laplace transform of \((0, \infty) \rightarrow \mathbb{R}, w \mapsto C_\mu(w^{1/\alpha} z)\). Hence for each fixed \( z \in \mathbb{R}^d \), \( C_\mu(w^{1/\alpha} z) \) is determined by \( \mathcal{E}_\alpha(\mu) \) for almost every \( w \in (0, \infty) \), and by continuity for every \( w > 0 \). In particular for \( w = 1 \), we see that \( C_\mu(z) \) is determined by \( \mathcal{E}_\alpha(\mu) \) for every \( z \in \mathbb{R}^d \).

(iv) Apart from minor adjustments, the proof is the same as that of Proposition 2.4 (v) in Barndorff-Nielsen et al. [3] and hence omitted.

(v) The first equality is clear by duality (see Sato [12], Proposition 41.8). For the second, observe that \( \int_s^1 (\log t^{-1})^{1/\alpha} \, dX_t(\mu) \) converges almost surely to \( \int_0^1 (\log t^{-1})^{1/\alpha} \, dX_t(\mu) \) as \( s \downarrow 0 \) by the independently scattered random measure property of \( X_t(\mu) \). Using partial integration, we conclude
\[
\int_s^1 (\log t^{-1})^{1/\alpha} \, dX_t(\mu) = -X_s(\mu)(\log s^{-1})^{1/\alpha} - \int_s^1 X_t(\mu) \, d(\log t)^{1/\alpha}.
\]
But \( \lim_{s \uparrow 0} X_s(\mu)(\log s^{-1})^{1/\alpha} = 0 \) a.s. (see Sato [12], Proposition 47.11), and the claim follows. \( \square \)

**Corollary 2.2.** Let \( \alpha > 0 \). Then a distribution \( \mu \) is symmetric if and only if \( \mathcal{E}_\alpha(\mu) \) is symmetric.

**Proof.** It is well known that an infinitely divisible distribution \( \mu \) with generating triplet \((A, \nu, \gamma)\) is symmetric if and only if \( \nu \) is symmetric and \( \gamma = 0 \). Symmetry of \( \mu \) hence implies symmetry of \( \mathcal{E}_\alpha(\mu) \) by (2.1) and (2.2). Conversely, suppose that \( \tilde{\mu} = \mathcal{E}_\alpha(\mu) \) with triplet \((\tilde{A}, \tilde{\nu}, \tilde{\gamma})\) is symmetric. Then
\[
\int_0^\infty \nu(u^{-1} B) \alpha u^{\alpha-1} e^{-u^\alpha} \, du = \int_0^\infty \nu(-u^{-1} B) \alpha u^{\alpha-1} e^{-u^\alpha} \, du
\]
for every \( B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \). In particular,
\[
\int_0^\infty \nu(u^{-1} t^{-1} B) u^{\alpha-1} e^{-u^\alpha} \, du = \int_0^\infty \nu(-u^{-1} t^{-1} B) u^{\alpha-1} e^{-u^\alpha} \, du \quad \forall \, t > 0,
\]
so that
\[
\int_0^\infty \nu(w^{-1}B)w^{\alpha-1}e^{-w^\alpha/t^\alpha} \, dw = \int_0^\infty \nu(-w^{-1}B)w^{\alpha-1}e^{-w^\alpha/t^\alpha} \, dw \quad \forall \, t > 0.
\]
By the uniqueness theorem for the Laplace transform, it follows that for fixed \(B\), \(\nu(w^{-1}B) = \nu(-w^{-1}B)\) for almost every \(w > 0\). Now let \(B\) be of the form
\[
B = \{ x \in \mathbb{R}^d : |x| > r \text{ and } x/|x| \in U \}
\]
for some \(r > 0\) and some \(U \in \mathcal{B}(S)\). Then both \(u \mapsto \nu(uB)\) and \(v \mapsto \nu(-uB)\) are càdlàg, and we conclude equality of \(\nu(B)\) and \(\nu(-B)\). This shows that \(\nu\) is symmetric, which then also shows \(\gamma = \tilde{\gamma} = 0\) by (2.2). \(\square\)

3. Stochastic integral characterization of the classes \(E_\alpha(\mathbb{R}^d)\)

We start with stating the following known result. In what follows, for two mappings \(\Phi_1\) and \(\Phi_2\), \(\Phi_1 \circ \Phi_2\) means their composition \((\Phi_1 \circ \Phi_2)(\mu) = \Phi_1(\Phi_2(\mu))\).

**Theorem 3.1.** (1) \(L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))\). ([16] and others.)
(2) \(M(\mathbb{R}^d) = \mathcal{M}(I_{\log}(\mathbb{R}^d))\).
(3) \(\Phi \circ \mathcal{E}_1 = \Psi\) and \(T(\mathbb{R}^d) = \Psi(I_{\log}(\mathbb{R}^d))\). ([3].)

In [2], \(M(\mathbb{R}^d) \cap I_{\text{sym}}(\mathbb{R}^d)\) is studied. The statement (2) above can be shown by exactly the same way as in [2].

Now we want to prove the following two theorems.

**Theorem 3.2.** For any \(0 < \alpha < \beta\),
\[
E_\alpha(\mathbb{R}^d) \subset E_\beta(\mathbb{R}^d).
\]

The following is an extension of (1.2) in Remark 1.3 for general \(\alpha > 0\).

**Theorem 3.3.** For \(\alpha > 0\),
\[
E_\alpha(\mathbb{R}^d) = \mathcal{E}_\alpha(I(\mathbb{R}^d)).
\]

**Proof of Theorem 3.2.** Let \(0 < \alpha < \beta\). Then if \(\mu \in E_\alpha(\mathbb{R}^d)\), \(\nu_\xi\) of \(\mu\) is
\[
\nu_\xi(dr) = r^{\alpha-1}g_\xi(r^\alpha)dr = r^{\beta-1}g_\xi\left((r^{\alpha/\beta})^\beta\right) = r^{\beta-1}g_\xi\left((r^{\alpha/\beta})^\beta\right).\]
Let
\[
h_\xi(x) = g_\xi(x^{\alpha/\beta}) x^{(\beta-\alpha)/\beta}.
\]
Note that if $g$ is completely monotone and $\psi$ a nonnegative function such that $\psi'$ is completely monotone, then the composition $g \circ \psi$ is completely monotone (see, e.g., Feller [6], page 441, Corollary 2), and if $g$ and $f$ are completely monotone then $gf$ is completely monotone. Thus $g_\xi(x^{\alpha/\beta})$ is completely monotone and then $h_\xi(x)$ is also completely monotone, and we have

$$\nu_\xi(dr) = r^{\beta-1}h_\xi(r^\beta).$$

Hence $\mu \in E_\beta(\mathbb{R}^d)$. □

**Proof of Theorem 3.3.**

(i) (Proof for that $E_\alpha(\mathbb{R}^d) \supset E_\alpha(I(\mathbb{R}^d))$.) Let $\tilde{\mu} \in E_\alpha(I(\mathbb{R}^d))$. Then $\tilde{\mu} = \mathcal{L} \left( \int_0^1 (\log t^{-1})^{1/\alpha} dX_t(\mu) \right)$ for some $\mu \in I(\mathbb{R}^d)$, and hence

$$\nu(B) := \int_B x^{\alpha} \nu_\xi(dx) = \int_0^\infty \nu_\xi(dx) \int_0^\infty \mathbf{1}_B(u) x^{\alpha} \nu_\xi(dx),$$

where $\nu$ is the Lévy measure of $\mu$ and below $\nu_\xi$ is the radial component of $\nu$. Thus, the spherical component $\tilde{\lambda}$ of $\tilde{\nu}$ is equal to the spherical component $\lambda$ of $\nu$, and the radial component $\tilde{\nu}_\xi$ of $\tilde{\nu}$ satisfies that, for $B \in \mathcal{B}((0, \infty))$

$$\tilde{\nu}_\xi(B) := \alpha \int_0^\infty u^{\alpha-1} e^{-u} du \int_0^\infty \mathbf{1}_B(xu) \nu_\xi(dx)$$

where

$$\tilde{\nu}_\xi(B) := \alpha \int_0^\infty \nu_\xi(dx) \int_0^\infty \mathbf{1}_B(y/x) x^{\alpha-1} e^{-y/x} x^{-1} dy$$

and

$$\tilde{\nu}_\xi(B) := \alpha \int_0^\infty \mathbf{1}_B(y) y^{\alpha-1} \tilde{g}_\xi(y^{\alpha}) dy,$$

with the measure $\tilde{Q}_\xi$ being defined by

$$\tilde{Q}_\xi(B) = \alpha \int_0^\infty \mathbf{1}_B(x^{-\alpha}) x^{-\alpha} \nu_\xi(dx), \quad B \in \mathcal{B}((0, \infty)).$$

We conclude that $\tilde{g}_\xi(\cdot)$ is completely monotone. Thus,

$$\tilde{\nu}_\xi(dy) = y^{\alpha-1} \tilde{g}_\xi(y^{\alpha}) dy$$

for some completely monotone function $\tilde{g}_\xi$. This concludes that $\tilde{\mu} \in E_\alpha(\mathbb{R}^d)$.

(ii) (Proof for that $E_\alpha(\mathbb{R}^d) \subset E_\alpha(I(\mathbb{R}^d))$.) Let $\tilde{\mu} \in E_\alpha(\mathbb{R}^d)$ with Lévy measure $\tilde{\nu}$ of the form

$$\tilde{\nu}(B) = \int_0^\infty \tilde{\lambda}(r) \int_0^\infty \mathbf{1}_B(r) r^{\alpha-1} \tilde{g}_\xi(r^{\alpha}) dr, \quad B \in \mathcal{B}((0, \infty)).$$
where \( g_\xi(r) \) is completely monotone in \( r \) and measurable in \( \xi \). For each \( \xi \), there exists a Borel measure \( \tilde{Q}_\xi \) on \([0, \infty)\) such that \( \tilde{g}_\xi(r) = \int_{[0,\infty)} e^{-rt} \tilde{Q}_\xi(dt) \) and \( \tilde{Q}_\xi(B) \) is measurable in \( \xi \) for each \( B \in \mathcal{B}([0, \infty)) \) (see the proof of Lemma 3.3 in Sato [11]).

For \( \nu \) to be a Lévy measure, it is necessary and sufficient that

\[
\int_\mathbb{R} \left( \frac{1}{\alpha} \right) \nu(d\xi) \int_0^1 t^{-1-2/\alpha} \tilde{Q}_\xi(dt) < \infty, \quad \int_\mathbb{R} \left( \frac{1}{\alpha} \right) \nu(d\xi) \int_1^\infty t^{-1-2/\alpha} \tilde{Q}_\xi(dt) < \infty.
\]

In part (i) we have defined \( \tilde{Q}_\xi = U(\rho_\xi) \) as the image measure of \( \rho_\xi \) under the mapping \( U : (0, \infty) \rightarrow (0, \infty), r \mapsto r^{-\alpha} \), where \( \rho_\xi \) has density \( r \mapsto \alpha r^{-\alpha} \) with respect to \( \nu_\xi \).

Denoting by \( V : r \mapsto r^{-1/\alpha} \), the inverse of \( U \), it follows that \( \rho_\xi \) is the image measure of \( \tilde{Q}_\xi \) under the mapping \( V \). Hence, given \( \tilde{Q}_\xi \), we define \( \nu_\xi \) as having density \( r \mapsto \alpha^{-1} r^{\alpha} \) with respect to the image measure \( V(\tilde{Q}_\xi) \) of \( \tilde{Q}_\xi \) under \( V \), i.e.

\[
\nu_\xi(B) = \alpha^{-1} \int_0^\infty 1_B(r^{-1/\alpha}) r^{-1} \tilde{Q}_\xi(dr), \quad B \in \mathcal{B}((0, \infty)).
\]

Define further a measure \( \nu \) to have spherical component \( \lambda = \tilde{\lambda} \) and radial parts \( \nu_\xi \), i.e.

\[
\nu(B) = \int_\mathbb{R} \tilde{\lambda}(d\xi) \int_0^\infty 1_B(r\xi) \nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\]

Then \( \nu \) is a Lévy measure, since

\[
\int_s \tilde{\lambda}(d\xi) \int_0^\infty (r^2 \wedge 1) \nu_\xi(dr) \\
\leq \int_s \tilde{\lambda}(d\xi) \int_0^1 r^2 \nu_\xi(dr) + \int_s \tilde{\lambda}(d\xi) \int_1^\infty \nu_\xi(dr)
\]

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which is finite by (3.1). If \( \nu \) can be chosen such that \( E \) the transformation of the generating triplet in Proposition 2.1 we see that 

Proof. Let 

and only if 

Proposition 4.1. Let \( \alpha > 0, m \in \{1, 2, \ldots\} \) and \( \mu \in I(\mathbb{R}^d) \). Then \( \mu \in I_{\log^m}(\mathbb{R}^d) \) if and only if \( E(\mu) \in I_{\log^m}(\mathbb{R}^d) \).

Proof. Let \( \nu \) and \( \bar{\nu} \) denote the Lévy measures of \( \mu \) and \( E(\mu) \), respectively. By (2.1), we conclude that

\[
\int_{\mathbb{R}^d} \varphi(x) \bar{\nu}(dx) = \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \varphi(ux)\alpha u^{-1}e^{-u^\alpha} du
\]

for every measurable nonnegative function \( \varphi : \mathbb{R}^d \to [0, \infty] \). In particular, we have

\[
\int_{|x|>1} (\log |x|)^m \bar{\nu}(dx) = \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty (\log (u|x|))^m \alpha u^{\alpha-1}e^{-u^\alpha} du
\]

\[
= \int_{\mathbb{R}^d} \nu(dx) \sum_{n=0}^m \binom{m}{n} (\log |x|)^{m-n} \int_{1/|x|}^\infty (\log u)^n \alpha u^{\alpha-1}e^{-u^\alpha} du
\]

\[
=: \int_{\mathbb{R}^d} h(x)\nu(dx), \text{ say.}
\]

Then it is easy to see that \( h(x) = o(|x|^2) \) as \( |x| \downarrow 0 \) and that \( \lim_{|x| \to \infty} h(x)/(\log |x|)^m = \int_0^\infty \alpha u^{\alpha-1}e^{-u^\alpha} du = 1 \). Hence, \( \int_{|x|>1} (\log |x|)^m \bar{\nu}(dx) < \infty \) if and only if \( \int_{|x|>1} (\log |x|)^m \nu(dx) < \infty \), giving the claim.

\[\square\]

Theorem 4.2. Let \( \alpha > 0 \) and

\[ n_\alpha(x) = \int_x^\infty u^{-1}e^{-u^\alpha} du, \quad x > 0. \]

Let \( x = n_\alpha^*(t), t > 0, \) be its inverse function, and define the mapping \( N_\alpha : I_{\log}(\mathbb{R}^d) \to I(\mathbb{R}^d) \) by

\[
N_\alpha(\mu) = \mathcal{L} \left( \int_0^\infty n_\alpha^*(t) dX_t^{(\mu)} \right), \quad \mu \in I_{\log}(\mathbb{R}^d).
\]
It then holds
\[(4.1) \quad \Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi = \mathcal{N}_\alpha,\]
including the equality of the domains. In particular, we have
\[(4.2) \quad \Phi \circ \mathcal{E}_2 = \mathcal{E}_2 \circ \Phi = \mathcal{M}.\]

**Remark 4.3.** A more general mapping than $\mathcal{N}_\alpha$-mapping is already defined in [9] and it is shown that $\mathcal{D}(\mathcal{N}_\alpha) = I_{\log}(\mathbb{R}^d)$ in Theorem 2.4 of [9].

**Proof of Theorem 4.2.** We remark that the equation $\Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi$ including the equality of domains can be concluded from Proposition 4.1 and the general theory of Upsilon transformations in [4], which could also be used to show that $\Phi \circ \mathcal{E}_\alpha$ and $\mathcal{N}_\alpha$ transform the Lévy measure of the underlying $\mu$ in the same way. To obtain the transformation of the generating triplet, however, we give the following proof, which does not refer to the general theory of Upsilon transformations.

It follows from Proposition 4.1 that both $\Phi \circ \mathcal{E}_\alpha$ as well as $\mathcal{E}_\alpha \circ \Phi$ are well defined on $I_{\log}(\mathbb{R}^d)$ and that they have the same domain. Note that

\[C_{\mathcal{E}_\alpha(\mu)}(z) = \int_0^1 C_{\mu}((\log t^{-1})^{1/\alpha}z) \, dt\]

and

\[C_{\Phi(\mu)}(z) = \int_0^\infty C_{\mu}(e^{-t}z) \, dt.\]

Then, if we are allowed to exchange the order of the integrals by Fubini’s theorem, we have

\[C_{(\mathcal{E}_\alpha \circ \Phi)(\mu)}(z) = \int_0^1 dt \int_0^\infty C_{\mu}((\log t^{-1})^{1/\alpha}e^{-s}z) \, ds\]

\[= \int_0^1 dt \int_0^1 C_{\mu}((\log t^{-1})^{1/\alpha}uz) u^{-1} \, du\]

\[= \int_0^1 u^{-1}du \int_0^1 C_{\mu}((\log t^{-1})^{1/\alpha}uz) \, dt\]

\[= \int_0^1 u^{-1}du \int_0^\infty C_{\mu}(vuz)\alpha v^{\alpha-1}e^{-v} \, dv\]

\[= \int_0^\infty \alpha v^{\alpha-1}e^{-v} \, dv \int_0^1 C_{\mu}(vuz)u^{-1} \, du\]

\[= \int_0^\infty \alpha v^{\alpha-1}e^{-v} \, dv \int_0^v C_{\mu}(sz)s^{-1} \, ds\]

\[= \int_0^\infty C_{\mu}(sz)s^{-1} \, ds \int_s^\infty \alpha v^{\alpha-1}e^{-v} \, dv\]

\[= \int_0^\infty C_{\mu}(sz)s^{-1} \, ds \int_s^\infty \alpha v^{\alpha-1}e^{-v} \, dv\]
\[= \int_0^\infty C_\mu(sz)s^{-1}e^{-s^\alpha}ds\]
\[= \int_0^\infty C_\mu(n_\alpha(t)z)dt,\]
and the same calculation can be carried out for \(C_{(\Phi \circ \zeta_\alpha)}(z) = \int_0^\infty C_\mu(n_\alpha(t)z)dt.\)

In order to assure the exchange of the order of the integrations by Fubini’s theorem, it is enough to show that
\[(4.3) \quad \int_0^1 u^{-1}du \int_0^\infty |C_\mu(\nu u z)|v^{\alpha-1}e^{-v^\alpha}dv < \infty.
\]
We have
\[|C_\mu(z)| \leq 2^{-1}(\text{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d}|g(z, x)|\nu(dx),\]
where
\[g(z, x) = e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle(1 + |x|^2)^{-1}.
\]
Hence
\[|C_\mu(\nu u z)| \leq 2^{-1}(\text{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d}|g(z, uwx)|\nu(dx)
\]
\[\quad + \int_{\mathbb{R}^d}|g(\nu uz, x) - g(z, uwx)|\nu(dx) =: I_1 + I_2 + I_3 + I_4,
\]
say. The finiteness of \(\int_0^1 u^{-1}du \int_0^\infty (I_1 + I_2)v^{\alpha-1}e^{-v^\alpha}dv\) is trivial. Noting that \(|g(z, x)| \leq c_z|x|^2(1 + |x|^2)^{-1}\) with a positive constant \(c_z\) depending on \(z\), we have
\[\int_0^1 u^{-1}du \int_0^\infty I_3v^{\alpha-1}e^{-v^\alpha}dv
\]
\[\leq c_z \int_{\mathbb{R}^d}\nu(dx) \int_0^1 u^{-1}du \int_0^\infty \frac{(uv|x|)^2}{1 + (uv|x|)^2}v^{\alpha-1}e^{-v^\alpha}dv
\]
\[= c_z \left( \int_{|x| \leq 1} \nu(dx) + \int_{|x| > 1} \nu(dx) \right) \int_0^1 u^{-1}du \int_0^\infty \frac{(uv|x|)^2}{1 + (uv|x|)^2}v^{\alpha-1}e^{-v^\alpha}dv
\]
\[=: I_{31} + I_{32},
\]
say, and
\[I_{31} \leq c_z \int_{|x| \leq 1} |x|^2\nu(dx) \int_0^1 u du \int_0^\infty v^{\alpha+1}e^{-v^\alpha}dv < \infty.
\]
As to \(I_{32}\), we have
\[I_{32} = c_z \int_{|x| > 1} \nu(dx) \left( \int_0^{1/|x|^2} + \int_{1/|x|^2}^1 \right) u^{-1}du \int_0^\infty \frac{(uv|x|)^2}{1 + (uv|x|)^2}v^{\alpha-1}e^{-v^\alpha}dv
\]
\[=: I_{321} + I_{322},
\]
say, and
\[ I_{321} \leq c_z \int_{|x| > 1} \nu(dx) \int_0^{1/|x|^2} du \int_0^{\infty} v^{\alpha+1} e^{-v^\alpha} dv < \infty, \]
\[ I_{322} \leq c_z \int_{|x| > 1} \nu(dx) \int_1^{1/|x|^2} u^{-1} ds \int_0^{\infty} v^{\alpha-1} e^{-v^\alpha} dv = 2c_z \int_{|x| > 1} \log |x| \nu(dx) \int_0^{\infty} v^{\alpha-1} e^{-v^\alpha} dv < \infty, \]
because \( \mu \in I_{\log}(\mathbb{R}^d) \). As to \( I_4 \), note that for \( a \in \mathbb{R} \),
\[ |g(az, x) - g(z, ax)| = \frac{|\langle az, x \rangle| |x|^2 |1 - a^2|}{(1 + |x|^2)(1 + |a|^2)} \leq \frac{|z||x|^3(|a| + |a|^3)}{(1 + |x|^2)(1 + |a|^2)} \leq \frac{|z||x|^2(1 + |a|^2)}{2(1 + |x|^2)}, \text{ (since } |b|(1 + b^2)^{-1} \leq 2^{-1}). \]
\[ \leq \frac{1}{2} |z|(1 + |a|^2). \]
Then
\[ \int_0^1 u^{-1} du \int_0^{\infty} |v|^{\alpha-1} e^{-v^\alpha} dv = \left( \int_{|x| \leq 1} \nu(dx) + \int_{|x| > 1} \nu(dx) \right) \int_0^1 u^{-1} du \int_0^{\infty} |g(uz, x) - g(z, ux)|v^{\alpha-1} e^{-v^\alpha} dv =: I_{41} + I_{42}, \]
say. We have
\[ I_{41} \leq |z| \int_{|x| \leq 1} \nu(dx) \int_0^1 u^{-1} du \int_0^{\infty} uv(1 + (uv)^2)|x|^3 v^{\alpha-1} e^{-v^\alpha} dv \leq |z| \int_{|x| \leq 1} |x|^3 \nu(dx) \int_0^{\infty} (1 + v^2)v^\alpha e^{-v^\alpha} dv < \infty. \]
Also,
\[ I_{42} \leq |z| \int_{|x| > 1} \nu(dx) \left( \int_0^{1/|x|^3} + \int_0^{1/|x|^3} \right) u^{-1} du \int_0^{\infty} \frac{vu(1 + (vu)^2)|x|^3}{(1 + |x|^2)(1 + |ux|^2)} v^{\alpha-1} e^{-v^\alpha} dv =: I_{421} + I_{422}, \]
say. We have
\[ I_{421} \leq |z| \int_{|x| > 1} |x|^3 \nu(dx) \int_0^{1/|x|^3} du \int_0^{\infty} (1 + v^2)v^\alpha e^{-v^\alpha} dv \]
Lemma 4.5. That Proposition 4.1, we have

\[ N \]

Now we prove that

\[ D \]

By the absolute convergence of the above integrals, we see that \( \int_0^\infty n_\alpha^a(t) \, dX_t^{(\mu)} \) is indeed definable for every \( \mu \in I_{\log}(\mathbb{R}^d) \) and that

\[
C_{(\varphi \circ \mathcal{E}_\alpha)(\mu)}(z) = C_{(\mathcal{E}_\alpha \circ \varphi)(\mu)}(z) = \int_0^\infty C_{\mu}(n_\alpha^a(t)z) \, dt = C_{\mathcal{N}_\alpha(\mu)}(z), \quad z \in \mathbb{R}^d,
\]

(see Sato [15], Theorem 3.5), and we must have \( \Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \varphi = \mathcal{N}_\alpha \). Since \( \mathcal{N}_2 = \mathcal{M} \), this shows in particular [12].

An immediate consequence of Theorem 4.2 is the following.

**Theorem 4.4.** Let \( \alpha > 0 \). Then

\[
\Phi(E_\alpha(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = \mathcal{E}_\alpha(L(\mathbb{R}^d)) = \mathcal{N}_\alpha(I_{\log}(\mathbb{R}^d)).
\]

We conclude this section with an application of the relation (4.1) to characterize the limit of certain subclasses obtained by the iteration of the mapping \( \mathcal{N}_\alpha \). We need some lemmas. In the following, \( \mathcal{N}_\alpha^m \) is defined recursively as \( \mathcal{N}_\alpha^{m+1} = \mathcal{N}_\alpha^m \circ \mathcal{N}_\alpha \).

**Lemma 4.5.** Let \( \alpha > 0 \). For \( m = 1, 2, \ldots \), we have

\[
\mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{N}_\alpha^m = \Phi^m \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha^m \circ \Phi^m.
\]

**Proof.** By Proposition 4.1 we have \( \mu \in I_{\log}(\mathbb{R}^d) \) if and only if \( \mathcal{E}_\alpha(\mu) \in I_{\log}(\mathbb{R}^d) \).

As shown in the proof of Lemma 3.8 in [10], we also have that \( \mu \in I_{\log}(\mathbb{R}^d) \) if and only if \( \mu \in I_{\log}(\mathbb{R}^d) \) and \( \Phi(\mu) \in I_{\log}(\mathbb{R}^d) \), and thus \( \mathfrak{D}(\Phi^m) = I_{\log}(\mathbb{R}^d) \). Since \( \mathcal{N}_\alpha = \Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi \), we conclude that

\[
(4.4) \quad \mu \in I_{\log}(\mathbb{R}^d) \quad \text{if and only if} \quad \mu \in I_{\log}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{N}_\alpha(\mu) \in I_{\log}(\mathbb{R}^d).
\]

Now we prove \( \mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log}(\mathbb{R}^d) \) inductively. For \( m = 1 \) this is known, so assume that \( \mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log}(\mathbb{R}^d) \) for some \( m \geq 1 \). If \( \mu \in \mathfrak{D}(\mathcal{N}_\alpha^{m+1}) \), then \( \mathcal{N}_\alpha^{m+1}(\mu) = \mathcal{N}_\alpha^m(\mathcal{N}_\alpha(\mu)) \) is well-defined. Thus, \( \mathcal{N}_\alpha(\mu) \in \mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log}(\mathbb{R}^d) \) by assumption, so that \( \mu \in I_{\log}(\mathbb{R}^d) \) by (4.4). Conversely, if \( \mu \in I_{\log}(\mathbb{R}^d) \), then \( \mu \in I_{\log}(\mathbb{R}^d) \) and \( \mathcal{N}_\alpha(\mu) \in I_{\log}(\mathbb{R}^d) \) by (4.4), so that \( \mathcal{N}_\alpha^m(\mathcal{N}_\alpha(\mu)) \) is well-defined by assumption. This
shows \( \mathcal{D}(N_{\alpha}^{m+1}) = I_{\log^{m+1}(\mathbb{R}^d)} \). That \( N_{\alpha}^{m} = \Phi^{m} \circ \mathcal{E}_{\alpha}^{m} = \mathcal{E}_{\alpha}^{m} \circ \Phi^{m} \) for every \( m \) then follows easily from (4.1), Proposition 4.1 and \( \mathcal{D}(\Phi^{m}) = I_{\log^{m}(\mathbb{R}^d)} \).

Let \( S(\mathbb{R}^d) \) be the class of all stable distributions on \( \mathbb{R}^d \), and for \( m = 0, 1, \ldots \) denote \( L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}(\mathbb{R}^d)}) \), \( L_\infty(\mathbb{R}^d) = \cap_{m=0}^{\infty} L_m(\mathbb{R}^d) \), \( N_{\alpha,m}(\mathbb{R}^d) = N_{\alpha}^{m+1}(I_{\log^{m+1}(\mathbb{R}^d)}) \) and \( N_{\alpha,\infty}(\mathbb{R}^d) = \cap_{m=0}^{\infty} N_{\alpha,m}(\mathbb{R}^d) \). It is known (cf. Sato, [11]) that \( L_\infty(\mathbb{R}^d) = S(\mathbb{R}^d) \), where the closure is taken under weak convergence and convolution. In order to show that also \( N_{\alpha,\infty}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)} \), we need two further lemmas.

**Lemma 4.6.** For \( \alpha > 0 \), \( \mathcal{E}_{\alpha} \) maps \( S(\mathbb{R}^d) \) bijectively onto \( S(\mathbb{R}^d) \), namely
\[
\mathcal{E}_{\alpha}(S(\mathbb{R}^d)) = S(\mathbb{R}^d).
\]

This is an immediate consequence of Proposition 2.1 (ii).

**Lemma 4.7.** Let \( \alpha > 0 \). For \( m = 0, 1, \ldots \), \( N_{\alpha,m}(\mathbb{R}^d) \) is closed under convolution and weak convergence, and
\[
S(\mathbb{R}^d) \subset N_{\alpha,m}(\mathbb{R}^d) = \mathcal{E}_{\alpha}^{m+1}(L_m(\mathbb{R}^d)) \subset L_m(\mathbb{R}^d).
\]

**Proof.** By Lemma 4.5
\[
N_{\alpha,m}(\mathbb{R}^d) = N_{\alpha}^{m+1}(I_{\log^{m+1}(\mathbb{R}^d)}) = (\mathcal{E}_{\alpha}^{m+1} \circ \Phi^{m+1})(I_{\log^{m+1}(\mathbb{R}^d)}) = \mathcal{E}_{\alpha}^{m+1}(L_m(\mathbb{R}^d)),
\]
hence \( S(\mathbb{R}^d) \subset N_{\alpha,m}(\mathbb{R}^d) \) by Lemma 4.6 and the fact that \( S(\mathbb{R}^d) \subset L_m(\mathbb{R}^d) \). Further,
\[
N_{\alpha,m}(\mathbb{R}^d) = (\Phi^{m+1} \circ \mathcal{E}_{\alpha}^{m+1})(I_{\log^{m+1}(\mathbb{R}^d)}) \subset \Phi^{m+1}(I_{\log^{m+1}(\mathbb{R}^d)}) = L_m(\mathbb{R}^d).
\]
Next observe that \( \mathcal{E}_{\alpha} \) and hence \( \mathcal{E}_{\alpha}^{m+1} \) clearly respect convolution. Since \( L_m(\mathbb{R}^d) \) is closed under convolution and weak convergence (see the proof of Theorem D in [3]), it follows from (4.5) and Proposition 2.1 (iv) that \( N_{\alpha,m}(\mathbb{R}^d) \) is closed under convolution and weak convergence, too.

We can now characterize \( N_{\alpha,\infty}(\mathbb{R}^d) \) as the closure of \( S(\mathbb{R}^d) \) under convolution and weak convergence:

**Theorem 4.8.** Let \( \alpha > 0 \). It holds
\[
L_\infty(\mathbb{R}^d) = N_{\alpha,\infty}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}.
\]

**Proof.** By (4.3) we have
\[
\overline{S(\mathbb{R}^d)} = L_\infty(\mathbb{R}^d) \supset N_{\alpha,\infty}(\mathbb{R}^d) \supset S(\mathbb{R}^d).
\]
But since each \( N_{\alpha,m}(\mathbb{R}^d) \) is closed under convolution and weak convergence, so must be the intersection \( N_{\alpha,\infty}(\mathbb{R}^d) = \cap_{m=0}^{\infty} N_{\alpha,m}(\mathbb{R}^d) \), and the assertion follows.
5. Characterization of subclasses of $E_\alpha(\mathbb{R}^1)$ by stochastic integrals with respect to some compound Poisson processes

For any Lévy process $Y = \{Y_t\}_{t \geq 0}$, denote by $L_{(0, \infty)}(Y)$ the class of locally $Y$-integrable functions on $(0, \infty)$ (cf. Sato [15], Definition 2.3), and let

$$\text{Dom}(Y) = \left\{ h \in L_{(0, \infty)}(Y) : \int_0^\infty h(t) dY_t \text{ is definable} \right\},$$

$$\text{Dom}^1(Y) = \{ h \in \text{Dom}(Y) : h \text{ is a left-continuous and decreasing function such that } \lim_{t \to \infty} h(t) = 0 \}.$$ Here, following Sato [15], Definition 3.1, by saying that the (improper stochastic integral) $\int_0^\infty h(t) dY_t$ is definable we mean that $\int_p^q h(s) \gamma_Y + \int_\mathbb{R} h(s)x \left( \frac{1}{1 + |h(s)x|^2} - \frac{1}{1 + |x|^2} \right) \nu_Y(dx) ds < \infty$

for all $0 < p < q < \infty$ and

$$\lim_{p \downarrow 0, q \to \infty} \int_p^q \left( h(s) \gamma_Y + \int_\mathbb{R} h(s)x \left( \frac{1}{1 + |h(s)x|^2} - \frac{1}{1 + |x|^2} \right) \nu_Y(dx) \right) ds \text{ exists in } \mathbb{R}.$$ In this case, $\int_0^\infty h(t) dY_t$ is infinitely divisible without Gaussian part and its Lévy measure $\nu_{Y,h}$ is given by

$$\nu_{Y,h}(B) = \int_0^\infty ds \int_\mathbb{R} 1_B(h(s)x) \nu_Y(dx), \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$ If $\nu_Y$ is symmetric and $\gamma_Y = 0$, then (5.2) and (5.3) are automatically satisfied, so that $h \in \text{Dom}(Y)$ if and only if (5.1) is satisfied, in which case $\gamma_Y$ in the generating triplet of $\int_0^\infty h(t) dY_t$ is 0.

Recall the definitions of $E_\alpha^0(\mathbb{R}^1)$ and $E_\alpha^{0, \text{sym}}(\mathbb{R}^1)$ from (1.3) and (1.4). The next theorem characterizes $E_\alpha^{0, \text{sym}}(\mathbb{R}^1)$ as the class of distributions which arise as improper stochastic integrals over $(0, \infty)$ with respect to some fixed symmetric compound Poisson process.
Theorem 5.1. Let $\alpha > 0$ and denote by $Y^{(\alpha)} = \{Y_t^{(\alpha)}\}_{t \geq 0}$ a compound Poisson process on $\mathbb{R}$ with Lévy measure $\nu_{Y^{(\alpha)}}(dx) = |x|^\alpha e^{-|x|} dx$ (without drift). Then

\begin{align}
E_{\alpha}^{0,\text{sym}}(\mathbb{R}^1) &= \left\{ \mathcal{L} \left( \int_0^\infty h(t) dY_t^{(\alpha)} \right) : h \in \text{Dom}(Y^{(\alpha)}) \right\} \\
&= \left\{ \mathcal{L} \left( \int_0^\infty h(t) dY_t^{(\alpha)} \right) : h \in \text{Dom}^1(Y^{(\alpha)}) \right\}.
\end{align}

Proof. Let $\mu \in E_{\alpha}^{0,\text{sym}}(\mathbb{R}^1)$. By definition, the Lévy measure $\nu$ of $\mu$ has the polar decomposition $(\lambda, \nu_\xi)$ given by

\begin{align}
\nu_\xi(dr) &= r^{\alpha-1} g_\xi(r^\alpha) dr, \quad r > 0, \, \xi \in \{-1, 1\},
\end{align}

and

\begin{align}
\lambda(dx) &= (\delta_{\{-1\}} + \delta_{\{1\}})(dx),
\end{align}

where $g_1 = g_{-1}$ are completely monotone and $\delta_x$ denotes Dirac measure at $x$ (If $\mu = \delta_0$ we define $g_\xi = 0$ and shall also call $(\lambda, \nu_\xi)$ a polar decomposition, even if $\nu_\xi$ is not strictly positive here). In the following, we drop the subscript $\xi$ of $g_\xi$ and $\nu_\xi$.

Since $g$ is completely monotone, there exists a Borel measure $Q$ on $[0, \infty)$ such that $g(y) = \int_{[0,\infty)} e^{-yt} Q(dt)$. By \[\text{(3.1)},\] in order for $\nu$ to satisfy $\int_0^\infty (x^2 \wedge 1) \nu(dx) < \infty$, it is necessary and sufficient that

\begin{align}
Q(\{0\}) &= 0, \quad \int_0^1 t^{-1} Q(dt) < \infty \quad \text{and} \quad \int_1^\infty t^{-1-2/\alpha} Q(dt) < \infty.
\end{align}

Observe that under this condition, we have for each $r > 0$,

\begin{align}
\nu([r, \infty)) &= \int_r^\infty y^{\alpha-1} g_\alpha(y^\alpha) dy = \int_0^\infty (\alpha t)^{-1} Q(dt) \int_r^\infty \alpha y^{\alpha-1} e^{-yt} dy \\
&= \int_0^\infty (\alpha t)^{-1} e^{-rt} Q(dt).
\end{align}

Next, observe that $Y^{(\alpha)}$ is symmetric without Gaussian part, so that by \[\text{(5.1)}\] a measurable function $h$ is in $\text{Dom}(Y^{(\alpha)})$ if and only if

\begin{align}
\int_0^\infty ds \int_{\mathbb{R}} (|h(s)x|^2 \wedge 1) |x|^\alpha e^{-|x|^\alpha} dx < \infty,
\end{align}

in which case $\int_0^\infty h(t) dY_t^{(\alpha)}$ is infinitely divisible with the generating triplet $(A_{Y,h} = 0, \nu_{Y,h}, \gamma_{Y,h} = 0)$ and the Lévy measure $\nu_{Y,h}$ is symmetric and by \[\text{(5.4)}\] satisfies

\begin{align}
\nu_{Y,h}([r, \infty)) &= \int_0^\infty ds \int_{r/|h(s)|}^\infty x^{\alpha-1} e^{-x^\alpha} dx = \alpha^{-1} \int_0^\infty e^{-r^\alpha/|h(s)|^{\alpha}} ds
\end{align}

for every $r > 0$. Hence, in order to prove \[\text{(5.5)}\] and \[\text{(5.6)}\], it is enough to prove the following:
(a) For each Borel measure $Q$ on $[0, \infty)$ satisfying (5.8) there exists a function $h \in \text{Dom}^↓(Y(\alpha))$ such that

\begin{equation}
\int_0^{\infty} t^{-1} e^{-r^\alpha t} Q(dt) = \int_0^{\infty} e^{-r^\alpha/h(s)\alpha} ds \quad \forall \ r > 0.
\end{equation}

(b) For each $h \in \text{Dom}(Y(\alpha))$ there exists a Borel measure $Q$ on $[0, \infty)$ satisfying (5.8) such that (5.11) holds.

To show (a), let $Q$ satisfy (5.8), and denote

$$F(x) := \int_{(0,x]} t^{-1} Q(dt), \quad x \in [0, \infty),$$

and by

$$F^-(t) = \inf\{y \geq 0 : F(y) \geq t\}, \quad t \in [0, \infty),$$

its left-continuous inverse, with the usual convention $\inf \emptyset = +\infty$. Now define

$$h = h_Q : (0, \infty) \to [0, \infty), \quad t \mapsto (F^-(t))^{-1/\alpha}.$$

Then $h$ is left-continuous, decreasing, and satisfies $\lim_{t \to \infty} h(t) = 0$. Denote Lebesgue measure on $(0, \infty)$ by $m_1$, and consider the function

\begin{equation}
T : (0, \infty) \to (0, \infty), \quad s \mapsto h(s)^{-\alpha} = F^-(s).
\end{equation}

Then $(T(m_1))_{|(0,\infty)}$, the image measure of $m_1$ under the mapping $T$, when restricted to $(0, \infty)$, satisfies

\begin{equation}
(T(m_1))_{|(0,\infty)}(dt) = t^{-1} Q_{|(0,\infty)}(dt).
\end{equation}

Hence it follows that for every $r > 0$,

$$\int_{(0,\infty)} e^{-r^\alpha/h(s)\alpha} m_1(ds) = \int_{(0,\infty)\cap\{s:T(s)\neq\infty\}} e^{-r^\alpha T(s)} m_1(ds) = \int_{(0,\infty)} e^{-r^\alpha t} (T(m_1))(dt),$$

yielding (5.11). To show (5.9), namely that $h \in \text{Dom}(Y(\alpha))$, observe that

\begin{align*}
\int_0^{\infty} ds \int_{\mathbb{R}} \left( |h(s)x|^2 \wedge 1 \right) |x|^\alpha - 1 e^{-|x|^\alpha} dx & = 2 \int_0^{\infty} x^{\alpha+1} e^{-x^\alpha} dx \int_{\{s:h(s)\leq 1/x\}} h(s)^2 ds + 2 \int_0^{\infty} ds \int_{1/h(s)}^{\infty} x^{\alpha-1} e^{-x^\alpha} dx \\
& = 2 \int_0^{\infty} x^{\alpha+1} e^{-x^\alpha} dx \int_{\{s:T(s)\geq 1/x\}} T(s)^{-2/\alpha} ds + 2 \alpha^{-1} \int_0^{\infty} e^{-T(s)} ds \\
& = 2 \int_0^{\infty} x^{\alpha+1} e^{-x^\alpha} dx \int_{\{t \geq x^\alpha\}} t^{-1-2/\alpha} Q(dt) + 2 \alpha^{-1} \int_0^{\infty} e^{-t} t^{-1} Q(dt)
\end{align*}
by \([5.13]\). The second of these terms is clearly finite by \([5.8]\). To estimate the first, observe that
\[
\int_0^\infty x^{\alpha+1}e^{-x^\alpha}dx \int_0^\infty t^{-1-2/\alpha}Q(dt) = \int_1^\infty x^{\alpha+1}e^{-x^\alpha}dx \int_1^\infty t^{-1-2/\alpha}Q(dt) + \int_0^1 x^{\alpha+1}dx \int_1^\infty t^{-1-2/\alpha}Q(dt),
\]
and the first two summands are finite by \([5.8]\), while the last summand is equal to
\[
\int_0^1 t^{-1-2/\alpha}Q(dt) \int_0^{t^{1/\alpha}} x^{\alpha+1}dx = (\alpha + 2)^{-1} \int_0^1 t^{1+2/\alpha}t^{-1-2/\alpha}Q(dt),
\]
and hence also finite. This shows \([5.9]\) for \(h\) and hence \((a)\).

To show \((b)\), let \(h \in \text{Dom}(Y^{(\alpha)})\) and assume first that \(h\) is nonnegative. Let \(T: (0, \infty) \rightarrow (0, \infty)\) be defined by \(T(s) = h(s)^{-\alpha}\) as in \([5.12]\), and consider the image measure \(T(m_1)\). Define the measure \(Q\) on \([0, \infty)\) by \(Q(\{0\}) = 0\) and equality \([5.13]\). Since \(\int_0^\infty h(t) dY_t^{(\alpha)}\) is automatically infinitely divisible with Lévy measure \(\nu_{Y,h}\) given by \([5.10]\), we have as in the proof of \((a)\) for every \(r > 0\)
\[
\int_{(0, \infty)} e^{-r^\alpha t(\alpha t)^{-1}}Q(dt) = \alpha^{-1} \int_0^\infty e^{-r^\alpha/h(s)^\alpha} ds = \nu_{Y,h}([r, \infty)).
\]
In particular, \(Q\) must be a Borel measure and \([5.11]\) holds. Since the left hand side of this equation converges and the right hand side is known to be the tail integral of a Lévy measure, it follows that \([5.8]\) must hold. Hence we have seen that \(\mathcal{L}(\int_0^\infty h(t) dY_t^{(\alpha)}) \in E_0^{\alpha, \text{sym}}(\mathbb{R}^1)\) for nonnegative \(h \in \text{Dom}(Y^{(\alpha)})\). For general \(h \in \text{Dom}(Y^{(\alpha)})\), write \(h = h^+ - h^-\) with \(h^+ := h \vee 0\) and \(h^- := (-h) \vee 0\). Then \(h^+, h^- \in \text{Dom}(Y^{(\alpha)})\) by \([5.9]\), and Equation \([5.4]\) and the discussion following it show that \(\int_0^\infty h(t) dY_t^{(\alpha)}\) has no Gaussian part, gamma part 0 and satisfies \(\nu_{Y,h} = \nu_{Y,h^+} + \nu_{Y,h^-}\). The corresponding Borel measure \(Q\) is given by \(Q = Q^+ + Q^-\), where \(Q^+\) and \(Q^-\) are constructed from \(h^+\) and \(h^-\), respectively, completing the proof of \((b)\).

Next, we ask whether every distribution in \(E_0^\alpha(\mathbb{R}^1)\) can be represented as a stochastic integral with respect to the compound Poisson process \(Z^{(\alpha)}\) having Lévy measure \(\nu_{Z^{(\alpha)}}(dx) = x^{\alpha-1}e^{-x^\alpha}1_{(0, \infty)}(x)\ dx\) (without drift) plus some constant. We shall prove that such a statement is true e.g. for those distributions in \(E_0^\alpha(\mathbb{R}^1)\) which correspond to Lévy processes of bounded variation, but that not every distribution in
$E_0^0(\mathbb{R}^1)$ can be represented in this way. However, every distribution in $E_0^0(\mathbb{R}^1)$ appears as an essential limit of locally $Z^{(\alpha)}$-integrable functions. Following Sato [15], Definition 3.2, for a Lévy process $Y = \{Y_t\}_{t \geq 0}$ and a locally $Y$-integrable function $h$ over $(0, \infty)$ we say that the essential improper stochastic integral on $(0, \infty)$ of $h$ with respect to $Y$ is definable if for every $0 < p < q < \infty$ there are real constants $\tau_{p,q}$ such that $\int_p^q h(t) dY_t - \tau_{p,q}$ converges in probability as $p \downarrow 0, q \to \infty$. We write $\text{Dom}_{es}(Y)$ for the class of all locally $Y$-integrable functions $h$ on $(0, \infty)$ for which the essential improper stochastic integral with respect to $Y$ is definable, and for each $h \in \text{Dom}_{es}(Y)$ we denote the class of distributions arising as possible limits $\int_p^q h(t) dY_t - \tau_{p,q}$ as $p \downarrow 0, q \to \infty$ by $\Phi_{h,es}(Y)$ (the limit is not unique, since different sequences $\tau_{p,q}$ may give different limit random variables). As for $\text{Dom}(Y)$, the property of belonging to $\text{Dom}_{es}(Y)$ can be expressed in terms of the characteristic triplet $(A_Y, \nu_Y, \gamma_Y)$ of $Y$. In particular, if $A_Y = 0$, then a function $h$ on $(0, \infty)$ is in $\text{Dom}_{es}(Y)$ if and only if $h$ is measurable and (5.1) and (5.2) hold, and in that case $\Phi_{h,es}(Y)$ consists of all infinitely divisible distributions $\mu$ with characteristic triplet $(A_{Y,h} = 0, \nu_{Y,h}, \gamma)$, where $\nu_{Y,h}$ is given by (5.4) and $\gamma \in \mathbb{R}$ is arbitrary (cf. [15], Theorems 3.6 and 3.11).

Denote

$$
E_0^+ (\mathbb{R}^1) := \{ \mu \in E_0(\mathbb{R}^1) : \mu((\infty, 0)) = 0 \},
$$
$$
E_0^{-,0}(\mathbb{R}^1) := \{ \mu \in E_0^+(\mathbb{R}^1) : \{X^{(\mu)}_t\} \text{ has zero drift} \},
$$
$$
E_0^{BV}(\mathbb{R}^1) := \{ \mu \in E_0(\mathbb{R}^1) : \{X^{(\mu)}_t\} \text{ is of bounded variation} \},
$$
$$
E_0^{BV,0}(\mathbb{R}^1) := \{ \mu \in E_0^{BV}(\mathbb{R}^1) : \{X^{(\mu)}_t\} \text{ has zero drift} \}.
$$

We then have:

**Theorem 5.2.** Let $\alpha > 0$ and denote by $Z^{(\alpha)} = \{Z^{(\alpha)}_t\}_{t \geq 0}$ a compound Poisson process on $\mathbb{R}$ with Lévy measure $\nu_{Z^{(\alpha)}}(dx) = x^{\alpha-1} e^{-x^\alpha} 1_{(0,\infty)}(x) dx$ (without drift). Then it holds:

(i) The class of distributions arising as limits of essential improper stochastic integrals with respect to $Z^{(\alpha)}$ is $E_0^0(\mathbb{R}^1)$:

$$
E_0^0(\mathbb{R}^1) = \bigcup_{h \in \text{Dom}_{es}(Z^{(\alpha)})} \Phi_{h,es}(Z^{(\alpha)}).
$$

(ii) Distributions in $E_0^{BV,0}(\mathbb{R}^1)$ and $E_0^{+,0}(\mathbb{R}^1)$ can be expressed as improper stochastic integrals over $(0, \infty)$ with respect to $Z^{(\alpha)}$. More precisely

$$
E_0^{+,0}(\mathbb{R}^1) = \left\{ \mathcal{L} \left( \int_0^\infty h(t) dZ^{(\alpha)}_t \right) : h \in \text{Dom}(Z^{(\alpha)}), h \geq 0 \right\},
$$

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(5.16) \[ E_{a}^{BV,0}(\mathbb{R}^1) = \left\{ \mathcal{L} \left( \int_{0}^{\infty} h(t) dZ^{(a)}_{t} \right) : h \in \text{Dom}(Z^{(a)}) \text{ such that} \right\} \left[ \int_{0}^{\infty} ds \int_{\mathbb{R}} (|h(s)x| \wedge 1)\nu_{Z^{(a)}}(dx) < \infty. \right\} \]

In particular,

(5.17) \[ E_{a}^{+}(\mathbb{R}^1) = \left\{ \mathcal{L} \left( \int_{0}^{\infty} h(t) dZ^{(a)}_{t} + b \right) : h \in \text{Dom}(Z^{(a)}), h \geq 0, b \in [0, \infty) \right\}. \]

(iii) Not every distribution in \( E_{a}^{0}(\mathbb{R}^1) \) can be represented as an improper stochastic integral over \((0, \infty)\) with respect to \( Z^{(a)} \) plus some constant. It holds

(5.18) \[ E_{a}^{BV}(\mathbb{R}^1) \cup E_{a}^{0, \text{sym}}(\mathbb{R}^1) \subset \not\subseteq \left\{ \mathcal{L} \left( \int_{0}^{\infty} h(t) dZ^{(a)}_{t} + b \right) : b \in \mathbb{R}, h \in \text{Dom}(Z^{(a)}) \right\} \subset \not\subseteq E_{a}^{0}(\mathbb{R}^1). \]

Proof. (i) Let \( h \in \text{Dom}_{a}(Z^{(a)}) \) and \( \mu \in \Phi_{h,\text{es}}(Z^{(a)}) \) and write \( h = h^{+} - h^{-} \) with \( h^{+} \) and \( h^{-} \) being the positive and negative parts of \( h \), respectively. Then \( \mu \) is infinitely divisible without Gaussian part and by (5.4) its Lévy measure \( \nu_{Z,h} \) satisfies

\[ \nu_{Z,h,1}([r, \infty)) := \nu_{Z,h}([r, \infty)) = \alpha^{-1} \int_{0}^{\infty} e^{-r^{\alpha}/h^{+}(s)^{\alpha}} ds, \]

\[ \nu_{Z,h,-1}([r, \infty)) := \nu_{Z,h}((-\infty, -r]) = \alpha^{-1} \int_{0}^{\infty} e^{-r^{\alpha}/h^{-}(s)^{\alpha}} ds \]

for every \( r > 0 \). Define the mappings \( T_{1}, T_{-1} : (0, \infty) \rightarrow (0, \infty) \) by \( T_{1}(s) = (h^{+}(s))^{-\alpha} \) and \( T_{-1}(s) = (h^{-}(s))^{-\alpha} \) and the measures \( Q_{1} \) and \( Q_{-1} \) on \([0, \infty)\) by

\[ Q_{\xi}(\{0\}) = 0 \quad \text{and} \quad (T_{\xi}(m_{1}))(0, \infty)(dt) = t^{-1} Q_{\xi}(0, \infty)(dt), \quad \xi \in \{-1, 1\}. \]

Then as in the proof of Theorem 5.11,

\[ \int_{(0, \infty)} e^{-r^{\alpha} \xi (\alpha t)^{-1} Q_{\xi}(dt)} = \nu_{Z,h,\xi}([r, \infty)), \quad r > 0, \quad \xi \in \{-1, 1\}, \]

and \( Q_{1} \) and \( Q_{-1} \) satisfy (5.3) and we conclude that \( \nu_{Z,h,\xi}(dr) = r^{\alpha-1} g_{\xi}(r^{\alpha}) dr \) for completely monotone functions \( g_{1} \) and \( g_{-1} \), so that \( \Phi_{h,\text{es}}(Z^{(a)}) \subset E_{a}^{0}(\mathbb{R}^1) \), giving the inclusion \( \overset{\subset}{\subset} \) in equation (5.14).

Now let \( \mu \in E_{a}^{0}(\mathbb{R}^1) \) with Lévy measure \( \nu \), and define the Lévy measures \( \nu_{1} \) and \( \nu_{-1} \) supported on \([0, \infty)\) by

(5.19) \[ \nu_{1}(B) := \nu(B), \quad \nu_{-1}(B) := \nu(-B), \quad B \in \mathcal{B}([0, \infty)). \]

Then

(5.20) \[ \nu_{\xi}([r, \infty)) = \int_{0}^{\infty} (\alpha t)^{-1} e^{-r^{\alpha} \xi Q_{\xi}(dt)}, \quad r > 0, \quad \xi \in \{-1, 1\}, \]

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for some Borel measures \( Q_1 \) and \( Q_{-1} \) satisfying (5.8). As in the proof of (a) in Theorem 5.1, we find nonnegative and decreasing functions \( h_1, h_{-1} : (0, \infty) \to [0, \infty) \) such that (5.9) (i.e. (5.1) with \( \nu_{Z^{(a)}} \) in place of \( \nu_Y \)) and (5.11) hold. Since \( h_1, h_{-1} \) are bounded on compact subintervals of \( (0, \infty) \) and since \( Z^{(a)} \) has bounded variation, it follows that \( h_1 \) and \( h_{-1} \) satisfy also (5.22), so that \( h_1, h_{-1} \in \text{Dom}_{\text{es}}(Z^{(a)}) \) and the Lévy measures of \( \tilde{\mu}_1 \in \Phi_{h_1, \text{es}}(Z^{(a)}) \) and \( \tilde{\mu}_{-1} \in \Phi_{h_{-1}, \text{es}}(Z^{(a)}) \) are given by \( \nu_1 \) and \( \nu_{-1} \), respectively. Now define the function \( h : (0, \infty) \to \mathbb{R} \) by

\[
(5.21) \quad h(t) = \begin{cases} 
  h_1(t-n), & t \in (2n, 2n+1], \quad n \in \{1, 2, \ldots\}, \\
  h_{-1}(t-n-1), & t \in (2n+1, 2n+2], \quad n \in \{1, 2, \ldots\}, \\
  h_1(t-2^{-k-1}), & t \in (2^{-k}, 2^{-k} + 2^{-k-1}], \quad k \in \{1, 2, \ldots\}, \\
  h_{-1}(t-2^{-k}), & t \in (2^{-k} + 2^{-k-1}, 2^{-k+1}], \quad k \in \{1, 2, \ldots\}. 
\end{cases}
\]

Then also \( h \in \text{Dom}_{\text{es}}(Z^{(a)}) \) and any \( \tilde{\mu} \in \Phi_{h, \text{es}}(Z^{(a)}) \) has Lévy measure \( \nu \), showing the inclusion “\( \subseteq \)” in equation (5.14).

(ii) Let \( h \in \text{Dom}(Z^{(a)}) \). Then \( \int_0^\infty h(t) \, dZ_t^{(a)} \in E_0^0(\mathbb{R}^1) \) by (a). Further, by Theorem 3.15 in Sato [15], \( \int_0^\infty h(t) dZ_t^{(a)} \) is the distribution at time 1 of a Lévy process of bounded variation if and only if

\[
(5.22) \quad \int_0^\infty ds \int_\mathbb{R} (|h(s)x| \wedge 1) \nu_{Z^{(a)}}(dx) < \infty,
\]

in which case this Lévy process will have zero drift. Since \( \mathcal{L}(\int_0^\infty h(t) dZ_t^{(a)}) \) has trivially support contained in \( [0, \infty) \) if \( h \geq 0 \), this gives the inclusion “\( \supseteq \)” in (5.15) and (5.16).

Now suppose that \( \mu \in E_{\alpha}^{BV,0}(\mathbb{R}^1) \) with Lévy measure \( \nu \), define \( \nu_1 \) and \( \nu_{-1} \) by (5.19) and choose Borel measures \( Q_1 \) and \( Q_{-1} \) such that (5.20) holds. Then it can be shown in complete analogy to the proof leading to (3.1) that for \( \xi \in \{-1, 1\} \), \( \nu_{\xi} \) satisfies \( \int_0^\infty (1 \wedge x) \nu_{\xi}(dx) < \infty \) if and only if

\[
(5.23) \quad Q_{\xi}(\{0\}) = 0, \quad \int_1^\infty t^{-1} Q_{\xi}(dt) < \infty \quad \text{and} \quad \int_1^\infty t^{-1-\alpha} Q_{\xi}(dt) < \infty.
\]

For \( \xi \in \{-1, 1\} \) and \( x \in [0, \infty) \) define \( F_\xi(x) := \int_{[0, x]} t^{-1} Q_{\xi}(dt) \), \( h_\xi = (F_\xi^-)^{-1/\alpha} \) and \( T_\xi = (h_\xi)^{-\alpha} = F_\xi^- \). Then it follows in complete analogy to the proof of (a) of Theorem 5.1 using (5.23), that (5.11) and (5.22) hold for \( h_\xi \) and \( Q_{\xi} \). By Theorem 3.15 in Sato [15] this then shows that \( h_\xi \in \text{Dom}(Z^{(a)}) \) for \( \xi \in \{-1, 1\} \). Now if \( \mu \in E_{\alpha}^{+,0}(\mathbb{R}^1) \), define \( h(t) := h_1(t) \), and for general \( \mu \in E_{\alpha}^{BV,0} \), define \( h(t) \) by (5.21). In each case \( h \) satisfies (5.22), \( h \in \text{Dom}(Z^{(a)}) \), and \( \mu = \mathcal{L}(\int_0^\infty h(t) dZ_t^{(a)}) \), giving the inclusions “\( \supseteq \)” in (5.15) and (5.16).

(iii) Let \( \mu \in E_{\alpha}^{0,\text{sym}}(\mathbb{R}^1) \). By Theorem 5.1 there exists \( f \in \text{Dom}^1(Y^{(a)}) \) such that \( \mu = \mathcal{L}(\int_0^\infty f(t) dY_t^{(a)}) \). Write \( h_1 = h_{-1} := f \) and define the function \( h : (0, \infty) \to \mathbb{R} \).
by (5.21). We claim that \( h \in \text{Dom}(Z^{(a)}) \). To see this, observe that \( h \) clearly satisfies (5.1) with respect to \( \nu_{Z^{(a)}} \) since \( f \) has the corresponding property with respect to \( \nu_{Y^{(a)}} \). Next, since \( |h(s)x|(1 + |h(s)x|^2)^{-1} \) is bounded by \( 1/2 \) and \( \nu_{Z^{(a)}}(\mathbb{R}) \) is finite, it follows that
\[
(5.24) \quad \int_0^q \left| \int_0^\infty \frac{h(s)x}{1 + |h(s)x|^2} x^{a-1}e^{-x^a} \, dx \right| \, ds < \infty \quad \forall \ q > 0.
\]
But since \( Z^{(a)} \) has the generating triplet
\[
\left( A_{Z^{(a)}} = 0, \ \nu_{Z^{(a)}}, \ \gamma_{Z^{(a)}} = \int_0^\infty \frac{x}{1 + x^2} x^{a-1}e^{-x^a} \, dx \right),
\]
(5.24) shows that (5.2) is satisfied for \( h \) with respect to \( \nu_{Z^{(a)}} \). Finally, by the definition of \( h \), for
\[
\gamma_{Z,h,0,q} := \int_0^q \left( \int_0^\infty \frac{h(s)x}{1 + |h(s)x|^2} x^{a-1}e^{-x^a} \, dx \right) ds, \quad q > 0,
\]
we have \( \gamma_{Z,h,0,q} = 0 \) for \( q = 2, 4, 6, \ldots \), and since \( \lim_{t \to \infty} h(t) = 0 \) it follows that \( \lim_{q \to \infty} \gamma_{Z,h,0,q} \) exists and is equal to 0. We conclude that (5.3) is satisfied, so that \( h \in \text{Dom}(Z^{(a)}) \). By (5.4) we clearly have \( \mathcal{L}\left( \int_0^\infty h(t)dZ_t^{(a)} \right) = \mathcal{L}\left( \int_0^\infty f(t)dY_t^{(a)} \right) = \mu \). Together with (5.14) and (5.16) and this shows (5.18) apart from the fact that the inclusions are proper.

To show that the first inclusion in (5.18) is proper, let \( \mu \in E_{a,\alpha,d}^{0,\text{sym}}(\mathbb{R}^1) \setminus E_{a,\alpha}^{B\mathcal{V}}(\mathbb{R}^1) \). The latter set is nonempty since by (5.8) and (5.23) it suffices to find a Borel measure \( Q \) on \([0, \infty)\) such that (5.8) holds but \( \int_1^\infty t^{-1-1/\alpha}Q(dt) = \infty \). As already shown, there exists \( h \in \text{Dom}(Z^{(a)}) \) such that \( \mu = \mathcal{L}(\int_0^\infty h(t)dZ_t^{(a)}) \). Then \( h + 1_{[1,2]} \in \text{Dom}(Z^{(a)}) \), and \( \mathcal{L}(\int_0^\infty (h(t) + 1_{[1,2]})(t)dZ_t^{(a)}) \) is clearly neither symmetric nor of finite variation.

To see that the second inclusion in (5.18) is proper, let \( \mu \in E_{a,\alpha}^0(\mathbb{R}^1) \) with Lévy measure \( \nu \) being supported on \([0, \infty)\) such that \( \int_0^1 x \nu(dx) = \infty \). Suppose there are \( b \in \mathbb{R} \) and \( h \in \text{Dom}(Z^{(a)}) \) such that \( \mu = \mathcal{L}(\int_0^\infty h(t)dZ_t^{(a)} + b) \). Since \( \nu \) is supported on \([0, \infty)\), we must have \( h \geq 0 \) Lebesgue almost surely, so that we can suppose that \( h \geq 0 \) everywhere. Then we have from (5.11) and (5.3) that
\[
\int_0^\infty ds \int_0^\infty (|h(s)x|^2 \wedge 1) \nu_{Z^{(a)}}(dx) < \infty
\]
and
\[
\int_0^\infty ds \int_0^\infty \frac{h(s)x}{1 + h(s)x} \nu_{Z^{(a)}}(dx) < \infty.
\]
Together these two equations imply
\[
\int_0^\infty ds \int_0^\infty (|h(s)x| \wedge 1) \nu_{Z^{(a)}}(dx) < \infty,
\]

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so that $\mu \in E_{B^V}^\alpha (\mathbb{R}^1)$ by (5.16), contradicting $\int_0^1 x \nu(dx) = \infty$. This completes the proof of (5.18). \ \ \ \ \square

In the following, we shall call a class $F$ of distributions in $\mathbb{R}^1$ closed under scaling if for every $\mathbb{R}^1$-valued random variable $X$ such that $\mathcal{L}(X) \in F$ it also holds that $\mathcal{L}(cX) \in F$ for every $c > 0$. If $F$ is a class of infinitely divisible distributions in $\mathbb{R}^1$ and satisfies that $\mu \in F$ implies $\mu^{**} \in F$ for any $s > 0$, where $\mu^{**}$ is the distribution with characteristic function $(\hat{\mu}(z))^s$, we shall call $F$ closed under taking of powers. Recall that a class $F$ of infinitely divisible distributions on $\mathbb{R}^1$ is called completely closed in the strong sense if it is closed under convolution, weak convergence, scaling, taking of powers, and additionally contains $\mu \ast \delta_b$ for any $\mu \in F$ and $b \in \mathbb{R}$. We can now characterize $E_{\alpha}(\mathbb{R}^1)$ and certain subclasses as smallest classes which satisfy certain properties.

**Theorem 5.3.** Let $\alpha > 0$ and $Y^{(\alpha)}$ and $Z^{(\alpha)}$ be defined as in Theorems 5.1 and 5.2, respectively. Then it holds:

(i) The class $E_{\alpha}(\mathbb{R}^1)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^1$ which is completely closed in the strong sense and contains $\mathcal{L}(Z_1^{(\alpha)})$ and $\mathcal{L}(\mathcal{Z}_1^{(\alpha)})$.

(ii) The class $E^+_{\alpha}(\mathbb{R}^1)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^1$ which is closed under convolution, weak convergence, scaling, taking of powers and contains $\mathcal{L}(Z_1^{(\alpha)})$.

(iii) The class $E^\text{sym}_{\alpha}(\mathbb{R}^1) := E_{\alpha}(\mathbb{R}^1) \cap I_{\text{sym}}(\mathbb{R}^1)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^1$ which is closed under convolution, weak convergence, scaling, taking of powers and contains $\mathcal{L}(Y_1^{(\alpha)})$.

**Proof.** From the definition it is clear that all the classes under consideration are closed under convolution, scaling, and taking of powers. The class $E_{\alpha}(\mathbb{R}^1)$ is closed under weak convergence by Proposition 2.1(iv) and Theorem 3.3 and hence so are $E^+_{\alpha}(\mathbb{R}^1)$ and $E^\text{sym}_{\alpha}(\mathbb{R}^1)$. Further, all the given classes contain the specified distributions which can be seen by taking $h = 1_{[0,1]}$ (or also $-1_{[0,1]}$ for (i)) in Theorems 5.1 and 5.2, respectively. So it only remains to show that the given classes are the smallest classes among all classes with the specified properties.

(iii) Let $F$ be a class of infinitely divisible distributions which is closed under convolution, weak convergence, scaling, taking of powers and which contains $\mathcal{L}(Y_1^{(\alpha)})$. Since $F$ is closed under taking of powers it contains also $\mathcal{L}(Y_t^{(\alpha)})$ for all $t > 0$, and by
the scaling and convolution property also
\[
\mathcal{L} \left( \int_{(0,\infty)} h_n(t) \, dY_t^{(\alpha)} \right) = \mathcal{L} \left( \sum_{j=1}^{k_n} \beta_{j,n} (Y_{t_j,n}^{(\alpha)} - Y_{t_{j-1,n}}^{(\alpha)}) \right)
\]
for every simple function \( h_n = \sum_{j=1}^{k_n} \beta_{j,n} 1_{(t_{j-1,n},t_{j,n}]} \) with \( \beta_{j,n} \geq 0 \) and \( 0 < t_{1,n} < \ldots < t_{k_n,n} \). Now let \( h \) be an arbitrary function in \( \text{Dom}^1(Y^{(\alpha)}) \) and choose a sequence \( (h_n)_{n \in \mathbb{N}} \) of such simple functions such that \( h_n \) converges pointwise from below to \( h \). Then \( \int_{[a,b]} h_n(t) \, dY_t^{(\alpha)} \) converges in probability (even almost surely) to \( \int_{[a,b]} h(t) \, dY_t^{(\alpha)} \) for all \( 0 < a < b < \infty \), from which it follows that \( \mathcal{L} \left( \int_{[a,b]} h(t) \, dY_t^{(\alpha)} \right) \in F \) for all \( 0 < a < b < \infty \) and hence \( \mathcal{L} \left( \int_0^\infty h(t) \, dY_t^{(\alpha)} \right) \in F \). Together with Theorem 5.1 this proves \( E^{0,\text{sym}}_\alpha(\mathbb{R}^1) \subset F \). It still remains to show that the Gaussian distribution is in \( F \). Since \( \mathcal{E}_\alpha \) is a bijection from \( I(\mathbb{R}^1) \) onto \( E_\alpha(\mathbb{R}^1) \) and since \( E^{0,\text{sym}}_\alpha(\mathbb{R}^1) \subset F \), it follows that
\[
\{ \mu \in I_{\text{sym}}(\mathbb{R}^1) : \mu \text{ has Gaussian part } 0 \} = \mathcal{E}_\alpha^{-1}(E^{0,\text{sym}}_\alpha(\mathbb{R}^1)) \subset \mathcal{E}_\alpha^{-1}(F),
\]
where we used Proposition 2.1 and Corollary 2.2. Let \( \zeta \) be a standard normal distribution and consider the compound Poisson distribution \( \mu_n \) with Lévy density \( n \sqrt{2\pi} e^{-nx^2/2} \, dx \). Then \( \mu_n \) converges weakly to \( \zeta \) as \( n \to \infty \) as shown in the proof of Theorem 8.1(i) of Sato [12], pp.44-45. Also observe that \( \mu_n \) has Gaussian part 0 and is symmetric, so that \( \mu_n \in \mathcal{E}_\alpha^{-1}(F) \). Hence \( \mathcal{E}_\alpha(\mu_n) \in F \) and \( \mathcal{E}_\alpha(\mu_n) \) converges weakly to \( \mathcal{E}_\alpha(\zeta) = \mathcal{L}(\sqrt{\Gamma(1 + 2/\alpha)} \zeta) \), which is in \( F \) since \( F \) is closed under weak convergence. By the scaling property, also \( \mathcal{L}(\zeta) \in F \), so that \( E^{\text{sym}}_\alpha(\mathbb{R}^1) \subset F \), giving (iii).

(ii) Let \( F \) be a class of infinitely divisible distributions on \( \mathbb{R}^1 \) which is closed under convolution, weak convergence, scaling and taking of powers, and contains \( \mathcal{L}(Z_1^{(\alpha)}) \). Then it follows in complete analogy to the proof of (iii) that \( E^{+,0}_\alpha(\mathbb{R}^1) \subset F \), now using Theorem 5.2(ii) instead of Theorem 5.1. Since \( \mathcal{L}(t^{-1}Z_t^{(\alpha)}) \in F \) for every \( t > 0 \) by the power and scaling property and since \( Z_1^{(\alpha)} \) has finite and positive expectation, it follows from the strong law of large numbers that \( t^{-1}Z_t^{(\alpha)} \) converges almost surely and hence in distribution to \( E(Z_1^{(\alpha)}) > 0 \) as \( t \to \infty \), implying that \( \delta_{E(Z_1^{(\alpha)})} \in F \). Together with the scaling and convolution property this shows \( E^{+,\text{sym}}_\alpha(\mathbb{R}^1) \subset F \), giving (ii).

(i) Let \( F \) be a class of infinitely divisible distributions which is closed under convolution, weak convergence, scaling and taking of powers and contains \( \mathcal{L}(Z_1^{(\alpha)}) \) and \( \mathcal{L}(-Z_1^{(\alpha)}) \). By (ii) we have \( \delta_1 \in F \) and similarly \( \delta_{-1} \in F \), and using Theorem 5.2(i) we then obtain similarly to the proof of (iii) that \( E^{0}_\alpha(\mathbb{R}^1) \subset F \). Using (iii) we see further
see that $F$ must contain the Gaussian distributions, and it follows that $E_\alpha(\mathbb{R}^1) \subset F$, finishing the proof. \hfill \Box

**Remark 5.4.** $E_1^+(\mathbb{R}^1) = B(\mathbb{R}_+) \text{ and } E_1(\mathbb{R}^1) = B(\mathbb{R}^1)$ are the Goldie–Steutel–Bondesson classes on $\mathbb{R}_+$ and $\mathbb{R}^1$, respectively. Both are themselves the smallest classes of infinitely divisible distributions closed under convolution and weak convergence and contain the distributions of all (elementary, resp.) mixtures of exponential variables; see Bondesson [4] for $B(\mathbb{R}_+)$ and Barndorff-Nielsen et al. [3] for $B(\mathbb{R})$. So, Theorems 5.1 and 5.2 give us another interpretation of (subclasses) of $E_\alpha(\mathbb{R}^1)$ as (essential limits) of stochastic integrals with respect to some fixed compound Poisson processes. Observe that also Theorem 5.3 gives a new interpretation of $B(\mathbb{R}^1)$ and $B(\mathbb{R}_+)$, since it is based on containing the law of some compound Poisson process, which is not an exponential distribution.

**Remark 5.5.** Recently, the authors had a chance to look at the paper by James et al. [7]. In their paper, the authors introduced the Wiener-Gamma integrals, which are stochastic integrals with respect to the standard gamma process that is a subordinator without drift with Lévy measure $x^{-1}e^{-x}1_{(0,\infty)}(x)dx$, and studied generalized gamma convolutions, which are related to the Thorin class. Actually, the Thorin class $T(\mathbb{R}^1)$ in one dimension, is the smallest class that contains all gamma distributions and is closed under convolution and weak convergence. Distributions in the class $T(\mathbb{R}^1)$ are named generalized gamma convolutions. As shown in [3], $T(\mathbb{R}^d)$ is characterized as $\Psi(I_{\log}(\mathbb{R}^d))$, and $B(\mathbb{R}^d)$ is characterized as $\Upsilon(I(\mathbb{R}^d))$. In this paper, we characterized $E_1^0(\mathbb{R}^1)$, as mentioned in Remark 5.4, in a different way by using compound Poisson processes. From this point of view, the way of using gamma process in the paper by James et al. [7] has a similar fashion.

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**References**

[1] Aoyama, T.: Nested subclasses of the class of type $G$ selfdecomposable distributions on $\mathbb{R}^d$. To appear in *Probab. Math. Statist.* (2008)
[2] Aoyama T., Maejima M. and Rosiński J.: A subclass of type $G$ selfdecomposable distributions. J. Theor. Probab. 21, 14–34 (2008)

[3] Barndorff-Nielsen, O.E., Maejima, M. and Sato, K.: Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. Bernoulli 12, 1–33 (2006)

[4] Barndorff-Nielsen, O.E., Rosiński, J. and Thorbjørnsen, S.: General $\Upsilon$-transformations. ALEA Lat. Am. J. Prob. Math. Stat. 4, 131–165 (2008)

[5] Bondesson, L.: Classes of infinitely divisible distributions and densities. Z. Wahrsch. Verw. Gebiete 57, 39–71 (1981); Correction and addendum, 59, 277 (1982)

[6] Feller W.: An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed. John Wiley & Sons (1966)

[7] James, L.F., Roynette, B. and Yor, M.: Generalized Gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. Probab. Surv. 5, 346–415 (2008)

[8] Maejima, M.: Subclasses of Goldie-Steutel-Bondesson class of infinitely divisible distributions on $\mathbb{R}^d$ by $\Upsilon$-mapping. ALEA Lat. Am. J. Prob. Math. Stat. 3, 55-66 (2007)

[9] Maejima, M. and Nakahara, G.: A note on new classes of infinitely divisible distributions on $\mathbb{R}^d$. Submitted (2008)

[10] Maejima, M. and Sato, K.: The limits of nested subclasses of several classes of infinitely divisible distributions are identical with the closure of the class of stable distributions. To appear in Probab. Theory Related Fields (2008)

[11] Sato, K.: Class $L$ of multivariate distributions and its subclasses. J. Multivar. Anal. 10, 207–232 (1980)

[12] Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)

[13] Sato, K.: Stochastic integrals in additive processes and application to semi-Lévy processes. Osaka J. Math. 41, 211-236 (2004)

[14] Sato, K.: Additive processes and stochastic integrals. Illinois J. Math. 50, 825–851 (2006)

[15] Sato, K.: Transformations of infinitely divisible distributions via improper stochastic integrals. ALEA Lat. Am. J. Prob. Math. Stat. 3, 67–110 (2007)

[16] Wolfe, S.J.: On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. Stoch. Proc. Appl. 12, 301-312 (1982)