Note on the helicity decomposition of spin and orbital optical currents

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Abstract
In the helicity representation, the Poynting vector (current) for a monochromatic optical field, when calculated using either the electric or the magnetic field, separates into right-handed and left-handed contributions, with no cross-helicity contributions. Cross-helicity terms do appear in the orbital and spin contributions to the current. But when the electric and magnetic formulas are averaged (‘electric–magnetic democracy’), these terms cancel, restoring the separation into right-handed and left-handed currents for orbital and spin separately.

Keywords: Poynting, polarization, spin, energy flux, momentum, nonparaxial

1. Introduction

This reports a small development and slight correction to the largely well-understood\cite{1} representation of the current (=time–averaged energy flux =\(c^2\times\text{momentum density}\)) by the Poynting vector\cite{2}, for a monochromatic optical field in empty space. This can be separated in two ways: into orbital and spin currents, or into positive and negative helicities, and each representation can be expressed in terms of the electric or magnetic field, in turn representable as a superposition of plane waves. Our aim is to clarify the interrelations between these different descriptions, and in particular express the helicity decomposition in a simpler way.

The real time- and space-varying electric field vector \(E_{\text{real}}\) with frequency \(\omega = ck\), is conveniently expressed in terms of a complex vector \(E\) depending only on position \(r = (x, y, z)\):

\[
E_{\text{real}}(r, t) = \text{Re} \left[ E(r) \exp(-i\omega t) \right],
\]
and similarly for the magnetic field \(H\). The Poynting vector is

\[
P = \frac{1}{2} \text{Re} \left[ E^* \times H \right],
\]
where here and hereafter we do not indicate the \(r\) dependence explicitly. We are here considering fully three-dimensional fields, unrestricted by paraxiality.

From Maxwell’s equations, each field can be expressed in terms of the other. In SI units,

\[
\mathbf{H} = -\frac{i}{\omega \mu_0} \nabla \times \mathbf{E}, \quad \mathbf{E} = \frac{i}{\omega \epsilon_0} \nabla \times \mathbf{H},
\]

(1.3)
giving the equivalent electric and magnetic representations of the current, conveniently written as

\[
P = \frac{c^2}{2\omega} P_E = \frac{c^2}{2\omega} P_H,
\]
(1.4)
in which

\[
P_E = \epsilon_0 \text{Im} \left[ E^* \times (\nabla \times E) \right] = \mu_0 \text{Im} \left[ H^* \times (\nabla \times H) \right].
\]
(1.5)

We are here concerned with momentum density, but all results apply, \textit{mutatis mutandis}, to the angular momentum density, obtained from \(P\) by vector-multiplying by \(r\), and to characterizations of helicity\cite{3–6}. 

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2. Spin and orbital split

By an elementary vector identity, the electric and magnetic currents can be split into parts naturally interpreted as orbital and spin [1]:

\[
P_{\text{orb}E} = \frac{\varepsilon_0}{2} \text{Im} \left[ \nabla^* \cdot (\nabla) E \right], \\
P_{\text{sp}E} = \frac{\mu_0}{2} \text{Im} \left[ \nabla \times (\nabla^* E) \right], \\
P_{\text{orb}H} = \mu_0 \text{Im} \left[ \nabla^* \cdot (\nabla) H \right], \\
P_{\text{sp}H} = \frac{\mu_0}{2} \text{Im} \left[ \nabla \times (\nabla^* H) \right].
\]

(2.1)

where for the orbital currents we use the notation

\[
A \cdot (\nabla)B = A_x \nabla B_x + A_y \nabla B_y + A_z \nabla B_z.
\]

(2.2)

Although \( P_{E} = P_{H} \), the separate spin and orbital contributions in \( P_{E} \) and \( P_{H} \) are equal only for paraxial fields. In general, they are different:

\[
P_{\text{orb}E} \neq P_{\text{orb}H}, \quad P_{\text{sp}E} \neq P_{\text{sp}H}.
\]

(2.3)

This led to the ‘electric–magnetic democracy’ proposal, in which orbital and spin currents are defined by the average

\[
P_{\text{orb}} = \frac{1}{2} (P_{\text{orb}E} + P_{\text{orb}H}), \quad P_{\text{sp}} = \frac{1}{2} (P_{\text{sp}E} + P_{\text{sp}H}).
\]

(2.4)

a stratagem supported by more general considerations [7–10].

3. Helicity split

A useful separation of each field is into two components with opposite helicity,

\[
E = E_+ + E_- , \quad H = H_+ + H_-.
\]

(3.1)

defined by

\[
\nabla \times E_\pm = \pm k E_\pm, \quad \nabla \times H_\pm = \pm k H_\pm.
\]

(3.2)

These are eigenstates of the momentum projection of the three-dimensional spin operator, through the identity, for any vector \( A \),

\[
\nabla \times A = \left( \mathbf{\hat{p}} \cdot \mathbf{\hat{S}} \right) A, \quad \mathbf{\hat{p}} = -i \nabla, \quad \mathbf{\hat{S}} = \text{vector of spin–1 matrices}
\]

(3.3)

(these particular spin matrices, given explicitly in [1], are uniquely defined by the curl). The electric and magnetic helicity components are related by

\[
H_\pm = \pm \frac{1}{\mu_0 c} E_\pm, \quad \text{i.e. } H = -\frac{1}{\mu_0 c} (E_+ - E_-).
\]

(3.4)

The separate contributions correspond to the Riemann–Silberstein vectors [11, 12]

\[
\sqrt{\varepsilon_0} E_\pm = \pm i \sqrt{\mu_0} H_\pm = \frac{1}{2} \left( \sqrt{\varepsilon_0} E \pm i \sqrt{\mu_0} H \right).
\]

(3.5)

When the fields are represented as superpositions of plane waves, the helicities correspond to right- and left-
circular polarizations:

\[
E_{kz} = e_z \exp(ik \cdot r), \quad e_z = \frac{1}{\sqrt{2}} (e_1 \pm i e_2)
\]

\[
e_1 \cdot k = 0, \quad e_2 = \frac{k}{k} \times e_1.
\]

(3.6)

In singular optics, general helicity eigenstates (i.e. single-helicity superpositions) have the interesting property [12] that their lines of pure circular polarization (C lines [13–15]) for the electric field coincide with those of the magnetic field, and also with the Riemann–Silberstein vortex lines [11].

An advantage of this representation [1, 16] is that the current separates into contributions from the two helicities, i.e.

\[
P_{E} = P_{H} = P_{+} + P_{-}.
\]

(3.7)

in which

\[
P_{\pm} = \pm \varepsilon_0 k \text{ Im} \left[ E_{\pm}^* \times E_{\pm} \right] = \pm \mu_0 k \text{ Im} \left[ H_{\pm}^* \times H_{\pm} \right].
\]

(3.8)

The cross terms, anticipated because \( P \) is a quadratic combination of the fields, cancel because of the identity

\[
\text{Im} \left[ E_{+}^* \times E_{-} - E_{-}^* \times E_{+} \right] = \text{Im} \left[ E_{+}^* \times E_{-} + E_{+} \times E_{-} \right]
\]

\[
= \text{Im} \left[ 2 \text{ Re} \left( E_{+}^* \times E_{-} \right) \right] = 0.
\]

(3.9)

4. Combined spin–orbit and helicity separation

Contrary to what has been implied by one of us (sentence after equation (3.25) in [1]), the spin and orbital currents associated with \( E \) and \( H \) do not separate into positive- and negative-helicity contributions. In general there are cross terms, i.e.

\[
P_{\text{orb}E} = P_{\text{orb}E+} + P_{\text{orb}E-} + P_{\text{orb}E+-},
\]

(4.1)

where

\[
P_{\text{orb}E+} = \varepsilon_0 \text{ Im} \left[ E_{+}^* \cdot (\nabla) E_{+} \right],
\]

\[
P_{\text{orb}E-} = \varepsilon_0 \text{ Im} \left[ E_{-}^* \cdot (\nabla) E_{-} \right],
\]

\[
P_{\text{orb}E+-} = \varepsilon_0 \text{ Im} \left[ E_{+}^* \cdot (\nabla) E_{-} + E_{-}^* \cdot (\nabla) E_{+} \right] \neq 0.
\]

(4.2)

Similarly for spin:

\[
P_{\text{sp}E} = P_{\text{sp}E+} + P_{\text{sp}E-} + P_{\text{sp}E+-},
\]

(4.3)
plane wave travelling in the z direction. Choosing units such that \( k = 1 \), with unit direction vectors \( \mathbf{e}_x \), \( \mathbf{e}_y \), \( \mathbf{e}_z \), and ignoring factors \( \epsilon_0 \) and \( \mu_0 \), this field is

\[
E = E_+ + E_-,
\]

\[
E_+ = \frac{1}{\sqrt{2}} (\mathbf{e}_x + i\mathbf{e}_y) \exp (iz),
\]

\[
E_- = \frac{1}{\sqrt{2}} (\mathbf{e}_x + i\mathbf{e}_z) \exp (iy).
\]

(5.1)

The analogous magnetic contributions are

\[
P_{\text{q}E+} = P_{\text{q}E+} + P_{\text{q}E-} + P_{\text{q}E++},
\]

\[
P_{\text{q}E-} = P_{\text{q}E-},
\]

\[
P_{\text{q}E++} = P_{\text{q}E++},
\]

\[
P_{\text{q}E--} = -P_{\text{q}E--},
\]

\[
P_{\text{q}H+} = P_{\text{q}H+} + P_{\text{q}H-} + P_{\text{q}H++},
\]

\[
P_{\text{q}H-} = P_{\text{q}H-} + P_{\text{q}H--}.
\]

(4.5)

related to the corresponding electric currents by (see (3.5))

\[
P_{\text{orb}H+} = P_{\text{orb}H+},
\]

\[
P_{\text{orb}H-} = P_{\text{orb}H-},
\]

\[
P_{\text{orb}H++} = P_{\text{orb}H++},
\]

\[
P_{\text{orb}H--} = -P_{\text{orb}H--},
\]

\[
P_{\text{orb}E+} = P_{\text{orb}E+} + P_{\text{orb}E+} + P_{\text{orb}E--} + P_{\text{orb}E--}.
\]

(4.6)

We see that the cross-helicity terms have opposite signs. This leads to the main point we wish to make: the separation into positive and negative helicities of the full current survives the separation into spin and orbital currents if we apply electric–magnetic democracy:

\[
P_{\text{orb}} = \frac{1}{2} (P_{\text{orb}H+} + P_{\text{orb}H+} + P_{\text{orb}E+} + P_{\text{orb}E--}).
\]

\[
P_{\text{sp}} = \frac{1}{2} (P_{\text{q}E+} + P_{\text{q}H+} + P_{\text{q}E+} + P_{\text{q}H--}).
\]

(4.7)

Similar results have been obtained before [17, 18], expressed as double Fourier superpositions of plane waves, but our derivation is simpler, and also more general because it allows superpositions that include evanescent waves.

5. Example

The simplest illustration of the foregoing general argument is a field composed of a right-circularly polarized plane wave travelling in the z direction and a left-circularly polarized plane wave travelling in the y direction. Choosing units such that \( k = 1 \), with unit direction vectors \( \mathbf{e}_x \), \( \mathbf{e}_y \), \( \mathbf{e}_z \), and ignoring factors \( \epsilon_0 \) and \( \mu_0 \), this is

From (2.1) the orbital and spin currents associated with \( E \) and \( H \) are

\[
P_{\text{orb}E} = (\mathbf{e}_z + e_i)\left( 1 + \frac{1}{2} \cos(z - y) \right),
\]

\[
P_{\text{q}E} = -\frac{1}{2} (\mathbf{e}_z + e_i) \cos(z - y),
\]

\[
P_{\text{orb}H} = (\mathbf{e}_z + e_i) \left( 1 - \frac{1}{2} \cos(z - y) \right),
\]

\[
P_{\text{q}H} = \frac{1}{2} (\mathbf{e}_z + e_i) \cos(z - y),
\]

(5.2)

so that

\[
P_{\text{orb}E} + P_{\text{q}H} = \mathbf{e}_z + e_i.
\]

(5.3)

The orbital and spin currents are different for \( E \) and \( H \), and application of electric–magnetic democracy (2.4) gives

\[
P_{\text{orb}} = \frac{1}{2} (P_{\text{orb}H} + P_{\text{orb}H} + P_{\text{orb}E} + P_{\text{orb}E}) = \mathbf{e}_z + e_i,
\]

\[
P_{\text{sp}} = \frac{1}{2} (P_{\text{q}E} + P_{\text{q}H}) = 0.
\]

(5.4)

In the helicity representation, the contributions (4.2), (4.4) and (4.5) are

\[
P_{\text{orb}H} = P_{\text{orb}H} = \mathbf{e}_z,
\]

\[
P_{\text{orb}E} = P_{\text{orb}E} = \mathbf{e}_i,
\]

\[
P_{\text{q}E} = \frac{1}{2} (\mathbf{e}_z + e_i) \cos(y - z),
\]

\[
P_{\text{q}H} = \frac{1}{2} (\mathbf{e}_z + e_i) \cos(y - z),
\]

\[
P_{\text{q}H++} = - \frac{1}{2} (\mathbf{e}_z + e_i) \cos(y - z).
\]

(5.5)

As expected, there are non-zero cross-helicity contributions, but these cancel with the electric–magnetic democracy formula (4.7).

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