Multipartite entanglement criterion via generalized local uncertainty relations

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We study the detection of multipartite entanglement based on the generalized local uncertainty relations. A sufficient criterion for the entanglement of four-partite quantum systems is presented in terms of the local uncertainty relations. Detailed examples are given to illustrate the advantages of our criterion. The approach is generalized to general multipartite entanglement cases.

Quantum entanglement is a remarkable feature in quantum physics¹ and has attracted much attention in recent years. Entangled states are recognized as the essential resources in quantum information processing, with many experimental realizations²,³ and applications in such as quantum algorithms⁴, quantum teleportation⁵, quantum cryptography⁶. Recently, it was shown that quantum entanglement is tightly connected to wave-particle duality, and it can create a wave-particle entangled state of two photons⁷. Detecting entanglement of multipartite systems is a fundamental problem in the theory of quantum entanglement. Separability criteria to determine whether a given state is separable or not are of crucial importance⁸. Enormous efforts have been dedicated to solve the separability problems⁹–³⁵. Nevertheless, the characterization and quantification of multipartite entanglement are less understood than that of bipartite case, as multipartite states can be entangled in more different ways.

There have been many efficient entanglement criteria such as local uncertainty relations (LUR)¹¹,¹², covariance matrix criterion (CMC)¹³, computable cross-norm or realignment criterion (CCNR)¹⁴, permutation separability criteria¹⁵, criterion based on Bloch representations¹⁷,¹⁸, entanglement witnesses²¹, Bell-type inequalities criteria²², and criterion based on quantum Fisher information²³. Generally, these criteria are only necessary condition for separable states and have different advantages in detect different entanglements.

The LUR criterion, the symmetric CMC criterion and the realignment criterion are usually considered as complementary to the the positive partial transposition criterion. The main advantage of LUR criterion is that it allows us to detect the entanglement of quantum states without having to fully understand them, and it can detect bound entangled states more effectively.

Recently, based on the local sum uncertainty relations, some entanglement criteria have been proposed for both discrete and continuous variable bipartite systems and three-qubit systems³¹–³³. Zhang et al. proposed a tighter form of the original LUR criterion to improve the range of entanglement detection³¹, Akbari-Kourbolagh and Azhdargalam generalized the LUR criterion to the tripartite systems³³.

This paper is structured as follows. We start by introducing the entanglement criterion based on LUR for tripartite systems and generalize the entanglement criterion to four-partite quantum systems. Some detail examples are then given to illustrate the advantages of the criterion. Then, the entanglement criterion for N-partite systems (N > 4) is discussed. Brief discussion and summary are given at last.

Results

Let \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \) be an N-partite system with \( \mathcal{H}_k \) the \( d_k \)-dimensional vector space associated with the \( k \)-th subsystem. An N-partite state \( \rho \in \mathcal{H} \) is said to be separable if \( \rho \) can be written as

\[
\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^N,
\]

(1)

where \( \rho_i^k \) are density matrices of the subsystem \( \mathcal{H}_k \), \( 0 \leq p_i \leq 1, \sum_i p_i = 1 \).

In quantum theory, the observables of a quantum system are represented by a set of Hermitian operators \( \{A_i\} \). The uncertainty principle shows that it is impossible to predict the measurement results of all observables of the system at the same time. The variance of \( A_i \) with respect to \( \rho \) is the uncertainty of an observable \( A_i \), defining as \( (\Delta A_i)^2 = \langle A_i^2 \rangle_\rho - \langle A_i \rangle^2_\rho \), where \( \langle A_i \rangle_\rho = \text{Tr}(\rho A_i) \) is the mean value. For a set of quantum observables \( \{A_i\} \), there

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exists a constant $U$ such that $\sum (\Delta A_i)^2 \geq U$. This inequality gives a universally valid limitation of the measurement outcomes. Generally, it is difficult to determine the value $U$. For the case of Pauli matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$, one has $(\Delta \sigma_{xy})^2 + (\Delta \sigma_{yx})^2 + (\Delta \sigma_{zx})^2 \geq 2^5$.

In Ref.\(^{33}\), based on the local sum uncertainty relations, an entanglement criterion has been presented for tripartite systems.

Let $\{A_i^1\}$, $\{A_i^2\}$ and $\{A_i^3\}$ be the set of local observables associated to the subsystems $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively. $U_1$, $U_2$, $U_3$ are lower bound of these local observables, such that $\sum_i \Delta(A_i^1)^2 \geq U_1$, $\sum_i \Delta(A_i^2)^2 \geq U_2$ and $\sum_i \Delta(A_i^3)^2 \geq U_3$. For any separable tripartite states, the following inequalities hold under any permutations of $\{1, 2, 3\}^{33}$:

$$F_{12}^{13} = \sum_i \Delta(A_i^1 + A_i^2 + A_i^3)^2 - (U_1 + U_2 + U_3 + M_{12}^2 + M_{123}^2) \geq 0,$$

where

$$M_{12} = \sqrt{\sum_i \Delta(A_i^1)^2 - U_1} - \sqrt{\sum_i \Delta(A_i^2)^2 - U_2}, \quad M_{123} = \sqrt{F_{12}^{13} - \sum_i \Delta(A_i^3)^2 - U_3},$$

$$F_{12}^{13} = \sum_i \Delta(A_i^1 + A_i^3)^2 - (U_1 + U_3 + M_{123}^2 + M_{1234}^2) \geq 0,$$

where

$$F = \sum_i \Delta(A_i^1 + A_i^2 + A_i^3 + A_i^4)^2 - \sum_{j=1}^4 U_j, \quad M_{1234} = \sqrt{F_{12}^{13} - \sum_i \Delta(A_i^4)^2 - U_4},$$

$$M_{1234} = \sqrt{F_{12}^{13} - \sqrt{F_{12}^{13}}}.$$

Theorem 1 provides a necessary condition of separable four-partite states. The violations of the inequalities in (1) sufficiently imply entanglement. For the four-qubit $W$ state, $\rho = |W_4\rangle\langle W_4|$ with $|W_4\rangle = \frac{1}{2}(|10100\rangle + |01010\rangle + |00101\rangle + |00011\rangle)$. Let $A_1^1 = A_1^2 = A_1^3 = -A_2^1 = \sigma_z$, $A_1^4 = A_2^3 = -A_3^1 = \sigma_y$ and $A_4^3 = A_2^1 = -A_3^2 = \sigma_x$, thus we get $\sum_i \Delta(A_i^j)^2 \geq 2$, $M_{12} = 0$, $M_{14} = 0$, $M_{123} = \sqrt{3} - \sqrt{2}$, $M_{1234} = \sqrt{\frac{7}{4} - M_{123}^2 - \sqrt{2}}$ and $M_{1234} = \sqrt{3}$, which give rise to $F_{12}^{13} = 3 - M_{123}^2 - M_{1234}^2 < 0$ and $F_{12}^{13} = 0$, which provide a violation for the inequalities (4). Therefore, the criterion identifies four-qubit $W$ state is entangled. By taking use of Theorem 1, more generally states can be detected and we consider some detailed examples for mixed states below.

Example 1 (Four-qubit $W$ state mixed with white noise) We first consider $\rho_1 = \frac{1}{16}I + (1 - \rho)|W_4\rangle\langle W_4|$, $0 \leq \rho \leq 1$. For this state, we choose $-A_1^1 = -A_1^2 = -A_1^3 = A_1^4 = \sigma_x$, $-A_2^1 = -A_2^3 = A_2^4 = A_2^2 = \sigma_y$ and $-A_3^1 = -A_3^2 = -A_3^3 = A_3^4 = \sigma_z$, hence $\sum_i \Delta(A_i^j)^2 \geq 2$, $M_{12} = 0$, $M_{14} = 0$, $M_{123} = \sqrt{3} - \sqrt{2} - \sqrt{1 - \frac{1}{4}(1 - \rho)^2}$, $M_{1234} = \sqrt{\frac{10p - 9p^2 + 1}{4} - M_{123}^2} - \sqrt{\frac{1}{4}(1 - \rho)^2}$ and $M_{1234} = \sqrt{3 - \frac{1}{2}p^2 - \sqrt{2}p - p^2 + 1}$. Then, we get $F_{12}^{13} = 10p - 4p^2 - 2 - M_{123}^2 - M_{1234}^2$ and $F_{12}^{13} = 10p - 4p^2 - 2 - M_{1234}^2$. When $p \leq 0.3605$, $F_{12}^{13} \leq 0$, so the state $\rho_1$ violates one of the inequalities (4). Therefore, the four-partite LUR criterion identifies the $\rho_1$ as an entangled state, see Fig. 1. While, $\rho_1$ is detected based on the witness $\mathcal{W} = \frac{3}{2}I - |W_4\rangle\langle W_4|$ which is proposed in Ref.\(^{27}\) when $p < 0.267$, see Fig. 2. That is to say our result detects better the entanglement than the criterion of Ref.\(^{27}\).
represent $-\frac{3}{4}$ and $0.1001 (1010 (1001))$ in Theorem 1. We can see that when $p \leq 0.3605$, state $\rho_1$ violates one of the inequalities (4), hence $\rho_1$ is entangled for $p \leq 0.3605$.

Figure 1. For the four-partite $W$ state mixed with the white noise $\rho_1$. The blue line represents $\rho_{1234}$ and the red dash line stands for $\rho_{12}$ in Theorem 1. We can see that when $p \leq 0.3605$, state $\rho_1$ violates one of the inequalities (4), hence $\rho_1$ is entangled for $p \leq 0.3605$.

Figure 2. For the four-partite $W$ state mixed with the white noise $\rho_1$. The black line represents $\text{Tr}(\rho_1 W)$ in Ref.27. We can see that $\rho_1$ is detected by the witness $\frac{1}{3} I - |W_1\rangle \langle W_4|$, thus $\rho_1$ is entangled for $p \leq 0.267$.

Example 2 (Four-qubit Dicke state mixed with white noise) Now, we take $\rho_2 = \frac{p}{12} I + (1 - p)(|D_2^1\rangle\langle D_2^1|)$, $0 \leq p \leq 1$, where $|D_2^1\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |0110\rangle + |0011\rangle + |0101\rangle + |0110\rangle + |0010\rangle)$. For this state, we choose $A_1^1 = \ldots = A_1^p = \sigma_x, A_2^1 = A_2^p = \sigma_y, A_3^1 = A_3^p = -A_1^1 = -A_2^1 = -A_3^1 = -A_4^1 = \sigma_z$. By direct calculations, we get $M_{12} = 0, M_{34} = 0, M_{123} = \sqrt{A - 2p - 1}, M_{1234} = \sqrt{\frac{35}{3} - \frac{2}{31} p - M_{123}^2 - 1}$ and $F_{p_2} = \frac{2}{3} (p - 1) + \frac{2\sqrt{2}}{3} \sqrt{2p - 2 + 3\sqrt{4 - 2p}}$ and $F_{p_2} = \frac{8p - 8 + 4\sqrt{4 - 2p}}{3}$. When $p \leq 0.437, F_{p_2} \leq 0$, and $F_{p_2} \leq 0$ for $p \leq 0.543$. It can be seen, from Fig. 3, that the $\rho_2$ violate inequalities (4) for $p \leq 0.543$. Furthermore, comparing with the result in Ref.27 which show that $\rho_2$ is entangled for $p < 0.356$ (see Fig. 4), the Theorem 1 also detects more entanglement.

For a more general case, we consider the set of local observable $\{A_{i1}^1, \ldots, A_{iN}^1\}$ associated to the subsystems $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$, respectively. Every local observable has a lower bound $U_j$ ($j = 1, 2, \ldots, N$) satisfies $\sum_q (A_j)^q \geq U_j$. In order to simplify calculation, let $i_{Nj}$ represent $\{A_{i1}^1, \ldots, A_{iN}^1\}$ and the bi-partition index $(i_1 i_2 \cdots i_K | i_{K+1} \cdots i_N)$ is denoted as $k_1 k_0$, where $k_1 = i_1 i_2 \cdots i_K$ and $k_0 = i_{K+1} i_{K+2} \cdots i_N, \frac{N}{i - 1} \leq K < N$ and $1 \leq i_1 < i_2 < \cdots < i_K \leq N$. For instance, if $N = 4$, hence $K = 2$, and $k_1 k_0 = (12) | 34, 13, 24, 14 | 23$, which represents three classes of bi-partition index of local observable set in N-body quantum system. Similar to the derivation of the Theorem 1, we obtain the following lemma and theorem.

**Lemma 2** For multipartite separable states, the following inequalities must hold:
and

\[
\sqrt{F_{12}^{12-N-1}} \sum_i \Delta(A_i^1) - \sum_i \left( \left( A_i^1 + \cdots + A_{N-1}^1 \right) \otimes A_N^1 \right) - \left( A_i^1 + \cdots + A_{N-1}^1 \right) \left( A_N^1 \right) \geq 0. \tag{5}
\]

and

\[
\sqrt{F_{12}^{12-N-1}} \sum_i \left[ \left( A_i^1 + \cdots + A_K^1 \right) \otimes \left( A_K^1 + \cdots + A_N^1 \right) \right] \geq 0.
\]

Figure 3. For the four-partite Dicke state $D_4^2$ mixed with the white noise $\rho_2$. The blue line stands for $F_{123}^{124}$ and the red dash line stands for $F_{123}^{124}$ and the red dot line stands for $F_{12}^{1234}$ in Theorem 1. When $p \leq 0.3605$, we can see that the state $\rho_2$ violates one of the inequalities (4), whence our criterion detects the entanglement of $\rho_2$ for $0 \leq p \leq 0.543$.

Figure 4. For the four-partite Dicke state $D_4^2$ mixed with the white noise $\rho_2$. The black line stands for $\text{Tr}(\rho W)$ in Ref.27. By using the witness $W$, we can see that $\rho_2$ is entangled for $p \leq 0.356$.

\[
F_{12}^{12-N-1} = \sum_i \Delta(A_i^1 + A_i^2 + \cdots + A_{N-1}^1)^2 - \left( \sum_{j=1}^{N-1} U_j + M_{12}^1 + M_{12}^2 + \cdots + M_{12-N-2}^1 \right).
\]

Theorem 2. For any multipartite separable states, the following inequalities hold under any permutations of the subsystems,

\[
F_{12}^{12-N-1} = - \left( \sum_{j=1}^{N-1} U_j + M_{12}^1 + M_{12}^2 + \cdots + M_{12-N-2}^1 \right).
\]

\[
F_{12}^{12-N-1} = - \left( \sum_{j=1}^{N-1} U_j + M_{12}^1 + M_{12}^2 + \cdots + M_{12-N-2}^1 \right).
\]

where
\[ F = \sum_{i=1}^{N} \Delta(A_1^i + A_2^i + \cdots + A_N^i)_{\rho}^2 - \sum_{j=1}^{N} U_j, \quad (7) \]

and

\[ M_{k_1|k_0} = \sqrt{F_{\rho|k_0}} - \sqrt{\sum_i \Delta(A_i^i)^2 - U_{|KN}, \text{ for } K = N-1, \quad (8) \]

\[ M_{k_1|k_0} = \sqrt{F_{\rho}} - \sqrt{F_{\rho|0}}, \text{ for } K < N-1. \]

\(A_i^i\) is an operator acting on the \(i\)-th subsystem \(\mathcal{H}_i\) with the rest subsystems as identity operators in \(N\)-partite quantum systems.

Let us consider five-partite quantum systems to illustrate the theorem. In the case of \(N = 5\), we can have

\[ \begin{cases} 
  k_1 \in \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\} \text{ and } k_0 \in \{45, 35, 34, 25, 24, 23, 15, 14, 13, 12\} & K=3; \\
  k_1 \in \{1234, 1235, 1245, 1345, 2345\} \text{ and } k_0 \in \{5, 4, 3, 2, 1\} & K=4. 
\end{cases} \]

Hence we have

\[ \begin{align*}
F_{12345}^{12345} &= F - (M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{15}^2 + M_{2345}^2), \\
F_{12345}^{12354} &= F - (M_{12}^2 + M_{13}^2 + M_{15}^2 + M_{235}^2 + M_{24}^2), \\
F_{12345}^{12453} &= F - (M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{245}^2 + M_{35}^2), \\
F_{12345}^{12543} &= F - (M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{254}^2 + M_{3}^2), \\
F_{12345}^{13452} &= F - (M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{15}^2 + M_{245}^2) = F - (M_{12}^2 + M_{13}^2 + M_{14}^2 + M_{15}^2 + M_{245}^2), \\
F_{12345}^{13542} &= F - (M_{12}^2 + M_{13}^2 + M_{15}^2 + M_{235}^2 + M_{4}^2), \\
F_{12345}^{14532} &= F - (M_{12}^2 + M_{15}^2 + M_{234}^2 + M_{3}^2), \\
F_{12345}^{14523} &= F - (M_{14}^2 + M_{24}^2 + M_{235}^2 + M_{5}^2), \\
F_{12345}^{14523} &= F - (M_{14}^2 + M_{24}^2 + M_{23}^2 + M_{15}^2), \\
F_{12345}^{14523} &= F - (M_{14}^2 + M_{24}^2 + M_{15}^2 + M_{23}^2), \\
F_{12345}^{14523} &= F - (M_{14}^2 + M_{24}^2 + M_{15}^2 + M_{23}^2), \\
\end{align*} \quad (9) \]

where \( F = \sum_i \Delta(A_i^i + A_2^i + \cdots + A_N^i)_{\rho}^2 - \sum_{j=1}^{N} U_j \), \(M_{12345} = \sqrt{F_{12345}^{12345}} - \sqrt{\sum_i \Delta(A_i^i)^2 - U_{5}} \), \(M_{12345} = \sqrt{\sum_{i} \Delta(A_i^i)^2 - U_{5}} \).

As a simple example, consider the five-qubit state \(\rho = |W_{\delta}\rangle \langle W_{\delta}| \) with \(|W_{\delta}\rangle = \frac{1}{\sqrt{5}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle)\). Let \(A_1^1 = A_2^1 = A_3^1 = A_4^1 = A_5^1 = \sigma_x + \), \(A_2^2 = A_3^2 = A_4^2 = A_5^2 = \sigma_y\), \(A_1^3 = -A_2^3 = A_3^3 = A_4^3 = A_5^3 = \sigma_z\). We have \(U_1 = U_2 = U_3 = U_4 = U_5 = 2, M_{12} = M_{34} = 0, M_{13} = 0.2161, M_{1234} = 1.218, M_{123} = 0, M_{1245} = 0.2797\) and \(M_{12345} = 0.8536\), which give rise to \(F_{12345} = 3 - M_{12}^2 - M_{23}^2 - M_{15}^2 < 0\) and \(F_{12345} < 0\), namely, the state is entangled.

**Conclusion**

We have generalized the LUR criterion for three qubit quantum systems to multiquit quantum systems, and obtained new entanglement criteria for four-partite quantum systems as well as for general multipartite systems. By detailed examples we have shown that our criteria can detect better the entanglement than some existing criteria. It is further known that in certain situations they can provide a nonlinear refinement of linear entanglement witnesses\(^{38}\), and it can be measured in experimental settings similar to those of entanglement witnesses. The effectiveness of the LUR criteria relies heavily on certain notions of information content of quantum states and choice of observables.

Quantum entanglement is fundamentally connected to the quantum steering, local uncertainty relations (LURs) are a common tool for entanglement detection, and the underlying idea can be directly generalized to steering detection\(^{29}\).

The considered system here is closed systems with no decoherence effects taken into account. Also, it would be interesting to find criteria for open quantum systems, since realistic quantum systems inevitably interact with the environment. It would be also interesting if our approach may highlight further investigations on the \(k\)-separability\(^{27}\) of multipartite systems and genuine multipartite entanglement detection.
Methods

Proof of the Theorem 1 By straightforward computation, we have
\[
\sum_i \Delta(A_i^1 + A_i^2 + A_i^3 + A_i^4) = \sum_i \Delta(A_i^1 + A_i^2 + A_i^3)^2 + \sum_i \Delta(A_i^4)^2 + 2 \sum_i \left[ \langle (A_i^1 + A_i^2 + A_i^3) \otimes A_i^4 \rangle - \langle A_i^1 + A_i^2 + A_i^3 \rangle \langle A_i^4 \rangle \right].
\]
Taking into account that for any tripartite separable states \( \rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \),
\[
\sqrt{F_{\rho}^{12}} \sum_i \Delta(A_i^3)^2 - U_3 \pm \sum_i \left[ \langle (A_i^1 + A_i^2) \otimes A_i^3 \rangle - \langle A_i^1 + A_i^2 \rangle \langle A_i^3 \rangle \right] \geq 0,
\]
where \( F_{\rho}^{12} = \sum_i \Delta(A_i^1 + A_i^2)^2 - (U_1 + U_2 + M_i^2) \), we obtain
\[
\sum_i \Delta(A_i^1 + A_i^2 + A_i^3 + A_i^4)^2 \geq U_1 + U_2 + U_3 + U_4 + M_i^2 + M_i^4 \quad \text{for any tripartite separable states} \quad \rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3.
\]

Proof of the Theorem 2 We denote the length of \( k_0 \) as \(|k_0|\). From above, one has \(|k_0| + |k_1| = N\).

When \( K = N = 1, \) one has \(|k_0| = 1, \) by straightforward computation, we have
\[
\sum_i \Delta(A_i^1 + A_i^2 + \cdots + A_i^N)^2 = \sum_i \Delta(A_i^1 + A_i^2 + \cdots + A_i^{N-1})^2 + \sum_i \Delta(A_i^N)^2 + 2 \sum_i \left[ \langle (A_i^1 + A_i^2 + \cdots + A_i^{N-1}) \otimes A_i^N \rangle - \langle A_i^1 + A_i^2 + \cdots + A_i^{N-1} \rangle \langle A_i^N \rangle \right].
\]
By Lemma 2, for any multiparticle separable states \( \rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N, \)
\[
\sqrt{F_{\rho}^{12-N-1}} \sum_i \Delta(A_i^N)^2 - U_N \pm \sum_i \left[ \langle (A_i^1 + A_i^2 + \cdots + A_i^{N-1}) \otimes A_i^N \rangle - \langle A_i^1 + A_i^2 + \cdots + A_i^{N-1} \rangle \langle A_i^N \rangle \right] \geq 0,
\]
via calculation, we obtain
\[
\sum_i \Delta(A_i^1 + A_i^2 + \cdots + A_i^N)^2 \geq \sum_{j=1}^{N} U_j + M_{12}^2 + M_{123}^2 + \cdots + M_{123\cdots N-1}^2, \]
namely, \( F_{\rho}^{12-N-1\uparrow} \geq 0 \). By relabeling the sub-indices, we have \( F_{\rho}^{k_{0}\downarrow} \geq 0 \).

When \( K < N - 1, \) one has \(|k_0| \geq 2, \)

\[\]
\[
\sum_i \Delta(A_i^1 + \cdots + A_i^N)_p^2 = \sum_i \Delta(A_i^1 + \cdots + A_i^K)^2 + \sum_i \Delta(A_{K+1}^i + \cdots + A_{2K-1}^i)^2 + 2 \sum_i [<(A_i^1 + \cdots + A_i^K) \otimes (A_{K+1}^i + \cdots + A_{2K-1}^i)>(A_i^1 + \cdots + A_i^K)(A_{K+1}^i + \cdots + A_{2K-1}^i)].
\]

By using Lemma 2, we get
\[
\sum_i \Delta(A_i^1 + A_i^2 + \cdots + A_i^N)^2_p \geq \sum_{j=1}^N U_j + (M_{12}^1 + M_{12}^2 + \cdots + M_{12}^{K+1} + M_{12}^{K+2} + \cdots + M_{2K-1}^{K+1} + M_{2K}^{2K-1} - N),
\]

namely, \(F_p^{12-K-K+1K+2-N} \geq 0\). By relabeling the sub-indices, one can show that \(F_p^{k(k)} \geq 0\).

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**Author contributions**
The first and the second authors wrote the main manuscript text and all authors reviewed the manuscript.

**Competing interests**
The authors declare no competing interests.

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