Mapping trapped atomic gas with spin-orbit coupling to quantum Rabi-like model

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We construct a connection of the ultracold atomic system in a harmonic trap with Raman-induced spin-orbit coupling to the quantum Rabi-like model. By mapping the trapped atomic system to a Rabi-like model, we can get the exact solution of the Rabi-like model following the methods to solve the quantum Rabi model. The existence of such a mapping implies that we can study the basic model in quantum optics by using trapped atomic gases with spin-orbit coupling.

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I. INTRODUCTION

Recently, many attentions have been paid to the cold atomic system with synthetic gauge field and spin-orbit (SO) coupling, which have been successfully realized in ultracold Bose gases [1, 2] and Fermi gases [3, 4]. The experimental progress stimulated the intensive theoretical studies, including exploring schemes to create general gauge fields [10, 11] and studying various interesting phases in these novel atomic systems with SO coupling [12–18]. While many theoretical works focused on the uniform systems with isotropic Rashba-type SO coupling, trapped systems with an external harmonic trap have also been addressed [10, 20]. It was shown that the presence of a confining potential may qualitatively change the physical properties of atomic gases with SO coupling.

In this work, we shall consider the anisotropic one-dimensional (1D) SO coupling realized in current experiments and scrutinize the problem of cold atomic system with SO coupling in a harmonic trap. We find that even the single particle problem of the trapped atomic system is highly nontrivial and not easy to be solved analytically when both the SO coupling term and Raman-coupling term exist. As we shall show in the context, the trapped atomic system with Raman-induced spin-orbit coupling is equivalent to a quantum Rabi-like model. As a paradigm for modeling the simplest light-atom interacting quantum system [21, 22], the quantum Rabi model [21] was surprisingly not able to be analytically solved and only very recently has an analytical solution been found by Braak [23]. The theoretical progress has renewed the interest in the study of the quantum Rabi model [24, 25]. Particularly, Braak’s solution can be also re-derived [24] within the extended coherent stats [29, 30] in a more straightforward way.

Within the same scheme of solving the quantum Rabi model [22, 24], we can solve the Rabi-like model and get an analytical solution in terms of Braak’s transcendental functions. By using the analytical solution, we can calculate the energy spectrum and dynamics of the Rabi-like model exactly. As the Rabi model has been widely applied to different fields of physics, including quantum optics, the cavity and circuit quantum electrodynamics [31, 52] and semiconductor systems [33], et al., we expect that the mapping of the trapped atomic system with spin-orbit coupling to quantum Rabi-like model paves the way to study physical phenomena related to quantum Rabi model by using cold atomic systems.

II. THE TRAPPED ATOMIC GAS WITH SO COUPLING

The Hamiltonian of a single atom with SO coupling takes form

\[ \hat{H}_{3d} = \frac{\hat{p}^2}{2m} + \hat{V}_{SO} + \hat{V}(\vec{r}), \]

(1)

with \( \hat{V}(\vec{r}) = \frac{1}{4}m(\omega_{x}^{2}x^{2} + \omega_{y}^{2}y^{2} + \omega_{z}^{2}z^{2}) \) the trap potential, \( m \) the atomic mass, and \( \hat{p} \) the atomic momentum operator. The term of \( \hat{V}_{SO} \) describes the SO coupling. For effective spin-1/2 systems considered in Refs. [3, 4], one has

\[ \hat{V}_{SO} = 2k_{r}\hat{p}_{x}\hat{\sigma}_{x} + \frac{\Omega}{2}\hat{\sigma}_{x} + \frac{\delta}{2}\hat{\sigma}_{z}, \]

(2)

where \( k_{r} \) is the recoil momentum, \( \Omega \) is the Raman-coupling strength, \( \delta \) is the two-photon detuning and \( \hat{\sigma} \) represents the Pauli operators. For the SO coupling described by Eq. (2), the Hamiltonian can be separated into \( \hat{H}_{3d} = \hat{H}(x) + \hat{H}_{2d}(y,z) \) with the effective 1D Hamiltonian given by

\[ \hat{H} = \frac{\hat{p}_{x}^{2}}{2m} + \frac{1}{2}m\omega_{x}^{2}x^{2} + \hat{V}_{SO}, \]

(3)

where we have used \( \omega = \omega_{x} \) for brevity. The model of \[ has a deceptively simple form. However, as we shall show in the next calculation, the model is not easy to be solved except the special cases with either \( \omega = 0 \) or \( \Omega = 0 \).

III. MAPPING TO THE RABI-LIKE MODEL

To make a connection of the SOC model in the harmonic trap to the well known quantum Rabi model, we...
use the representation of ladder operators, i.e., $a$ and its adjoint $a^\dagger$, to rewrite the Eq. (3) as

$$\hat{H} = \hbar \omega (a^\dagger a + \frac{1}{2}) + \delta \hat{\sigma}_z / 2 + \Omega \hat{\sigma}_x / 2 + ig(a^\dagger - a)\hat{\sigma}_z,$$  

where $g = k_s \sqrt{2m\hbar \omega}$ and $\hat{\sigma}_x = i \sqrt{m\hbar \omega / 2} (a^\dagger - a)$ is used.

To simplify the above equation and compare with Rabi model, we make a shift $H \rightarrow H - \hbar \omega / 2$ and take $\hbar = 1$.

Now the Hamiltonian can be rewritten as

$$\hat{H} = \omega a^\dagger a + \delta \frac{1}{2} \hat{\sigma}_z + \frac{\Omega}{2} \hat{\sigma}_x + ig(a^\dagger - a)\hat{\sigma}_z.$$  

(5)

We note that a generalized quantum Rabi model can be represented as

$$\hat{H}_R = \omega a^\dagger a + \delta \frac{1}{2} \hat{\sigma}_z + \frac{\Omega}{2} \hat{\sigma}_x + g(a^\dagger - a)\hat{\sigma}_z,$$  

(6)

with $\delta = 0$ corresponding to the quantum Rabi model.

Comparing Eq. (6) with the generalized Rabi model, we find that the only difference is the spin-boson coupling term. While the coupling term in the generalized Rabi model describes a spin-space coupling, the coupling term in the Rabi-like model of the trapped atomic gas describes a spin-momentum coupling.

The resemblance of these two models suggests us that the Rabi-like model can be solved by applying similar methods to solve the quantum Rabi model. Here we shall use the method of Chen et al. to derive our solution of Rabi-like model. For brevity, we give results and key steps of derivation in the main text, but leave some details of derivation in the appendix. To diagonalize the Rabi-like model, it is instructive to span the Hamiltonian in the spin space with the following matrix form

$$\hat{H} = \begin{pmatrix} a^\dagger a + ig(a^\dagger - a) + \delta \frac{1}{2} & \frac{\Omega}{2} a^\dagger a - ig(a^\dagger - a) - \frac{\delta}{2} \\ \frac{\Omega}{2} a^\dagger a - ig(a^\dagger - a) - \frac{\delta}{2} & a^\dagger a + ig(a^\dagger - a) + \delta \frac{1}{2} \end{pmatrix}.$$  

(7)

Here, in order to keep consistent with Ref. 24, we take the matrix form in units of $\omega = 1$. By introducing the Bogoliubov transformations $A = a + ig$ and $B = a - ig$, the diagonal matrix element $H_{11}$ and $H_{22}$ can be diagonalized as $A^\dagger A$ and $B^\dagger B$, respectively.

In terms of operator $A$, we can rewrite the Hamiltonian as

$$\hat{H} = \begin{pmatrix} A^\dagger A - \alpha & \frac{\Omega}{2} A^\dagger A - 2ig(A^\dagger - A) + \beta \\ \frac{\Omega}{2} A^\dagger A - 2ig(A^\dagger - A) + \beta & A^\dagger A - \alpha \end{pmatrix},$$  

(8)

where

$$\alpha = g^2 - \frac{\delta}{2}, \quad \beta = 3g^2 - \frac{\delta}{2}.$$  

To diagonalize the Hamiltonian, we take the wavefunction as

$$|\psi\rangle = \left( \sum_{n=0}^{\infty} \sqrt{n}! J_n^+ |n\rangle_A \right) \left( \sum_{n=0}^{\infty} \sqrt{n}! K_n^- |n\rangle_A \right)^T,$$  

(9)

where $|n\rangle_A = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle_A$ is the extended coherent state with $|0\rangle_A = e^{-\frac{\delta}{2}g^2 - ig\tau} |0\rangle_a$. The expansion coefficients $J_n^+$ and $K_n^-$ can be determined self-consistently by solving the eigenvalue equation $H|\psi\rangle = E|\psi\rangle$, which gives some restricted relations to coefficients $J_n^+$ and $K_n^-$ (see appendix).

Similarly, in terms of operator $B$, the Hamiltonian is written as

$$\hat{H} = \left( B^\dagger B + 2ig (B^\dagger - B) + \beta' \frac{\Omega}{2} B^\dagger B - \alpha' \right),$$  

(10)

where

$$\alpha' = g^2 + \frac{\delta}{2}, \quad \beta' = 3g^2 + \frac{\delta}{2}.$$  

The wavefunction can be also represented in terms of $B$ as

$$|\psi'\rangle = \left( \sum_{n=0}^{\infty} (-1)^n \sqrt{n}! K_n^+ |n\rangle_B \right) \left( \sum_{n=0}^{\infty} (-1)^n \sqrt{n}! J_n^+ |n\rangle_B \right)^T.$$  

(11)

where $|n\rangle_B = \frac{(B^\dagger)^n}{\sqrt{n!}} |0\rangle_B$ is the extended coherent state with $|0\rangle_B = e^{-\frac{\delta}{2}g^2 + ig\tau} |0\rangle_a$, and $K_n^+$ and $J_n^+$ are the coefficients to be determined by solving the eigenvalue equation $H|\psi'\rangle = E|\psi'\rangle$.

Since both wavefunctions $|\psi\rangle$ and $|\psi'\rangle$ are eigenfunctions for the same eigenvalue $E$, they should be different by a complex constant if they are not degenerate. This requirement leads to a self-consistent condition between the two sets of coefficients, i.e.,

$$\sum_{n=0}^{\infty} J_n^- (-ig)^n \sum_{n=0}^{\infty} J_n^+ (-ig)^n = \sum_{n=0}^{\infty} K_n^- (-ig)^n \sum_{n=0}^{\infty} K_n^+ (-ig)^n,$$  

(12)

The above equation can be rewritten as a compact form $G_\delta(x) = 0$ with the transcendental function defined as

$$G_\delta(x) = \left( \frac{\Omega}{2} \right)^2 \tilde{R}^+(x) \tilde{R}^-(x) - R^+(x) R^-(x),$$  

(13)

where $x = E + g^2$,

$$R^\pm(x) = \sum_{n=0}^{\infty} K_n^\pm(x) (-ig)^n,$$  

$$\tilde{R}^\pm(x) = \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x - n \pm \frac{\delta}{2}} (-ig)^n.$$  

The $K_n^\pm(x)$ are defined recursively,

$$nK_n^\pm = f_n^\pm(x) (K_n^\pm - 1) + K_{n-2}^\pm,$$  

(14)

with the initial condition $K_0^\pm = 1$, $K_1^\pm = f_0^\pm(x)$, and

$$f_n^\pm(x) = 2ig - \frac{1}{2ig} \left( n - x + \frac{\delta}{2} - \frac{(\Omega/2)^2}{4(x - n \mp \delta/2)} \right).$$  

(15)
The eigenenergies can be determined by the zeros of the transcendental function $G_\delta(x)$ [23]. For $\delta = 0$, $G_\delta(x) = 0$ can be reduced to $G_0^\pm(x) = 0$ with

$$G_0^\pm(x) = \sum_{n=0}^{\infty} K_n(x) \left( 1 \pm \frac{\Omega/2}{x-n} \right) (-i g)^n.$$ (15)

**IV. RESULTS AND DISCUSSIONS**

The spectrum of the Rabi-like model can be determined from the zeros of transcendental function [12] or [16]. As illustrated in Fig.1, we can read the eigenenergy from the intersection points of G-function with horizontal axis $(x_n)$ via $E = x_n - g^2$. Using the above G-function, it is quite easy to get the relation of energy spectrum as function of coupling constant. To give a concrete example, we plot the energy spectrum for the system with $\Omega = 1.4$ in Fig.2. As shown in the figure, the ground state energy decreases with increasing coupling constant $g$. When $\delta/2$ is not a multiple of $\omega/2$, no level crossings appear. This is consistent with Braak’s results [23].

We note that the G function contains all the information of our system, including both eigenvalues and eigenfunctions. As the eigenvalues are determined, coefficients of the corresponding eigenfunctions can be obtained by recursion relations [13]. Although the G-function is represented as a summation of infinite power series, undoubtedly practical calculations of the zeros of G function require truncations of coherent state orbits. Nevertheless, the compact form of our solution implies that we can get desired accurate results in our scheme. A connection with the quantum Rabi model inspires us to investigate the dynamical behavior of the Rabi-like model, which may stimulate the study of quantum dynamics related to quantum Rabi model by using cold atomic systems.

In order to study the dynamics of Rabi-like model, we first prepare the bosonic atoms in a harmonic trap and suppose that the interaction between atoms is very weak and the system can be treated as noninteracting bosons. In principle, one can tune the interaction between atoms to be zero by the Feshbach resonance. We take the initial state as $|0\rangle_a \otimes (1,0)^T$, which is the ground state of the system in a harmonic trap. At time $T = 0$, we turn on the Raman-induced SO coupling. The system will evolve under the Hamiltonian [5] and the time-dependent wavefunction reads

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle = \sum_n e^{-iE_n t} |\psi_n\rangle \frac{\langle \psi_n | \psi(0) \rangle}{\langle \psi_n | \psi_n \rangle},$$ (16)

where $|\psi_n\rangle$ is the eigenstate of the Rabi-like model corresponding to the eigenvalue $E_n$. The analytical solution of the Rabi-like model enables us to calculate the time evolution of the expectation value of $\sigma_z$ in a numerically exact way [28, 34].

The expectation value of $\sigma_z$ is defined as $P_z(t) = \langle \psi(t) | \sigma_z | \psi(t) \rangle$, which describes the population difference of two component atoms. The left panels of Fig.3 show the time-dependent population difference $P_z(t)$ for $\Omega = 0.5$, $\delta = 0$ under various coupling strengths. For the weak coupling strength with $g = 0.1$, $P_z(t)$ oscillates between 1 and −1, which indicates the periodic population swapping between different component atoms. With increasing the coupling strength, the typical Rabi oscillation in the weak coupling regime breaks down. For $g = 1.5$, $P_z(t)$ displays a sharp steplike decay with similar dynamic behavior as in the quantum Rabi model [28]. In the right panels of Fig.3, we show the time-dependent population difference for system with $\Omega = 0.5$ and $\delta = 0.4$. It is obvious that the oscillation amplitude of $P_z(t)$ is suppressed by the nonzero term of $\delta$. From the Hamiltonian [5], given the initial state as $(1,0)^T$ in $\sigma_z$ representation, one may understand the competition between the $\Omega$ term and $\delta$ term. While the nondiagonal term of $\Omega$ flips the spin, the diagonal term of $\delta$ tends to stabilize the initial state. The effect of coupling $g$ is
somewhat like the diagonal term while in a more subtle way.

In order to compare with the well-known quantum Rabi model, we take $\Omega$ with the same energy order as $\omega$, which is generally fulfilled for cases of quantum optical systems. However, the energy scale of cold atomic systems is quite different from quantum optical systems. In quantum optical systems, $\Omega \approx \omega$ and the coupling $g$ is the smallest quantity, while in cold atom systems, the harmonic trap is usually not deep enough and generally $\Omega \gg \omega$. For example, in the recent experiment of USTC group [7], one has $\omega = (2\pi)50kHz$, $E_{\text{recoil}} = (2\pi)2.21kHz$, $\delta = 4E_{\text{recoil}}$, $\Omega \approx E_{\text{recoil}}$, from which we can get $\Omega \approx 40$ and $\delta \approx 160$ in units of $\omega$. To see the effect of a large $\Omega$, in Fig.4, we give an example for the case with $\Omega = 10$. Comparing with Fig.3, we can clearly see the oscillation frequency increases with increasing the Raman coupling strength. The oscillation amplitude is strongly suppressed by the $\delta$ term.

The mapping of the trapped atomic gas with spin-orbit coupling to quantum Rabi-like model may stimulate the study of interesting phenomena related to the basic quantum optical model by using the cold atomic system. Although the parameters of trapped cold atomic systems in current experiment are much different from the general Rabi model, the good tunability of cold atomic systems may make it be possible to access the parameter regime of traditional quantum Rabi model by increasing the frequency of trap potential. One possible way is using the 1D deep optical lattice to produce a series of two-dimensional (2D) Bose gases. In the direction of 1D optical lattice, each 2D gas can be viewed to be tightly trapped by an effective harmonic trap. By this way, the frequency of the effective harmonic trap may be tuned to the same order of Raman-coupling strength [32]. As quantum Rabi model in the quantum optical system and SO coupled atomic gases can be viewed as two limit cases in parameter spaces, one may also explore some novel properties of the Rabi-like model by using cold atomic systems in the parameter regime which is not accessible in traditional optical systems.

V. SUMMARY

In conclusion, we show that the ultracold atomic system in a harmonic trap with Raman-induced SO coupling is equivalent to a Rabi-like model and the closed form of the exact solution is obtained. The connection of the trapped cold atoms with SO coupling to the Rabi-like model enables us to explore a variety of properties of the cold atomic system by using traditional methods in quantum optics. We want to point out that this connection is rather implicit as the harmonic trap here provide a source of bosonic field, which was only considered as a confinement in previous studies. Another reason of this implicitness is the different energy scales in quantum optics and cold atoms. Rapid progress in cold atomic experiments gives the possibility to prepare our system in quantum optics limit. Our study also provides a new vision for understanding the trapped cold atomic systems with SO coupling.

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Appendix A: Derivation of solution.

For simplicity, we shall use the method of extended coherent states developed by Chen et al., which was used to re-derive the Braak's solution to the quantum Rabi model. Following the procedure for the Rabi model, we can get the solution of our Rabi-like model.

Left-multiplying $\langle n | A$ to the eigenvalue equation $H | \psi \rangle = E | \psi \rangle$, we can get the following restricted relations to the coefficients $J_n^-$ and $K_n^-$:

$$ (n - \alpha - E) J_n^- = -\frac{\Omega}{2} K_n^-, \quad (n + \beta - E) K_n^- + 2ig (n + 1) K_{n+1}^- - 2ig K_{n-1}^- = -\frac{\Omega}{2} J_n^-.$$  

From the above equations, we can determine the coefficient $f_n^-$ recursively,

$$J_n^- = -\frac{\Omega}{2 (n - \alpha - E)} K_n^-, \quad (A1)$$

$$n K_n^- = f_{n-1}^- K_{n-1}^- + K_{n-2}^-, \quad (A2)$$

$$f_n^- = \frac{-1}{2ig} \left[ (n + \beta - E) - \frac{\Omega^2}{4 (n - \alpha - E)} \right] (A3)$$

with $K_0^- = 1$ and $K_1^- = f_0^-$. Similarly, proceeding as before, we get the relations for two coefficients $K_n^+$ and $J_n^+$:

$$ (n - \alpha' - E) J_n^+ = \frac{\Omega}{2} K_n^+$$

$$ (n + \beta' - E) K_n^+ + 2ig (n + 1) K_{n+1}^+ - 2ig K_{n-1}^+ = -\frac{\Omega}{2} J_n^+,$$

then we have

$$J_n^+ = -\frac{\Omega}{2 (n - \alpha' - E)} K_n^+ \quad (A4)$$

$$n K_n^+ = f_{n-1}^+ K_{n-1}^+ + K_{n-2}^+, \quad (A5)$$

$$f_n^+ = \frac{-1}{2ig} \left[ (n + \beta' - E) - \frac{\Omega^2}{4 (n - \alpha' - E)} \right] (A6)$$

with $K_0^+ = 1$ and $K_1^+ = f_0^+$. As alternative representations of a non-degenerate state with the eigenvalue $E$, the wavefunction $\langle n |$ and $\langle 11 |$ should be equivalent, i.e., $| \psi \rangle = r | \psi' \rangle$, where $r$ is a constant. The above equivalent requirement leads to the following relations:

$$\sum_{n=0}^{\infty} \sqrt{n!} J_n^- | n \rangle_A = \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} K_n^+ | n \rangle_B, \quad (A7)$$

$$\sum_{n=0}^{\infty} \sqrt{n!} K_n^- | n \rangle_A = \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} J_n^+ | n \rangle_B. \quad (A8)$$

Left multiplying the vacuum state $\langle 0 |$ to both sides of the above equations and making use of the relation of $\sqrt{n!} | 0 \rangle_A = (-1)^n \sqrt{n!} | 0 \rangle_B = e^{-s^2/2} (\pm i)^n$, we can eliminate the ratio constant $r$ and get the following relation

$$\sum_{n=0}^{\infty} J_n^- (-ig)^n \sum_{n=0}^{\infty} J_n^+ (-ig)^n = \sum_{n=0}^{\infty} K_n^- (-ig)^n \sum_{n=0}^{\infty} K_n^+ (-ig)^n. \quad (A9)$$

Making substitutions in terms of Eqs. (A1) and (A4), we obtain

$$\sum_{n=0}^{\infty} \frac{\Omega}{n - \alpha - E} K_n^- (-ig)^n \sum_{n=0}^{\infty} \frac{\Omega}{n - \alpha' - E} K_n^+ (-ig)^n$$

$$\sum_{n=0}^{\infty} K_n^- (-ig)^n \sum_{n=0}^{\infty} K_n^+ (-ig)^n. \quad (A10)$$

In terms of the the transcendental function $G_\delta(x)$ defined by Eq.(12) in the main text, the above equation is nothing else but $G_\delta(x) = 0$ with $x = E + g^2$.
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