On the completeness of describing an equilibrium canonical ensemble using a pair distribution function

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Abstract

It is shown that in equilibrium a canonical ensemble of particles with two-particle interaction the Gibbs distribution function may be expressed uniquely through a pair distribution function. It means, that for given values of the particle number \(N\), volume \(V\), and temperature \(T\), the pair distribution function contains as many information about the system as a full Gibbs distribution. The latter is represented as a series expansion in the pair distribution function. A recurrence relation system is constructed, which allows all terms of this expansion to be calculated successively.

1 Introduction

In classical statistical mechanics all properties of a closed system of \(N\) particles in a volume \(V\) (canonical ensemble) are described by a distribution function \(D_N(X)\), where \(X\) is a set of system phase variables. For a system with an additive particle interaction, multi-particle distribution functions are introduced [1, 2] to calculate thermodynamical values of the system. These multi-particle functions are considered to have less information than the full Gibbs \(N\)-particle distribution function. Thereby the smaller the order of the distribution function is the less information it contains [1]. But there does not exist a proof of this statement in the literature.

At the same time it is known that for an equilibrium system of \(N\) non-interacting particles in the volume \(V\) with the temperature \(T\) the GDF decomposes into a product of one-particle distribution functions. This means that all information about the system is contained in the one-particle distribution function.

In this article a problem on equivalence of describing an equilibrium system by full Gibbs distribution function (GDF) and pair (two-particle) distribution function (PDF) is considered. It is used a non-normalized GDF and particle distribution functions are defined through it. The motive for that will be explained in concluding section. As to a description of system the non-normalized GDF contains all information about it.
We shall show that for the equilibrium system of $N$ pair-interacting particles in the absence of an external field, when the energy $U_N$ equals
\begin{equation}
U_N(q_1, \ldots, q_N) = \sum_{1 \leq i < j \leq N} u_2(q_i, q_j),
\end{equation}
a Gibbs distribution may be expressed uniquely through a PDF. It means the PDF contains all information on the system under consideration.

## 2 Formulation of the problem

The equilibrium canonical ensemble is described by the Gibbs distribution function
\begin{equation}
D_N(q_1, \ldots, q_N) = \exp\{-\beta U_N(q_1, \ldots, q_N)\}, \quad q_i \in V, \quad i = 1, \ldots, N, \quad \beta = 1/kT.
\end{equation}
where $k$ is the Boltzmann constant, $V$ is a domain of three-dimensional Euclidean space. Here and below in this paper we shall suppose that the configuration variables $q$ belong to the domain $V$.

For the system of particles with the energy (1) the expression (2) may be rewritten as follows
\begin{equation}
D_N(q_1, \ldots, q_N) = \prod_{1 \leq i < j \leq N} v(q_i, q_j),
\end{equation}
where we introduce the function
\begin{equation}
v(q, q') \overset{\text{def}}{=} \exp\{-\beta u_2(q, q')\}.
\end{equation}
Due to reality of the potential $u_2$ this function is nonnegative. And by virtue of symmetry with respect to its arguments the function $v(q, q')$ is symmetrical too.

It follows from the expression (3) that statistical properties of the system under consideration are described completely by specifying a single function $v(q, q')$ of two configuration variables. Naturally, there appears an assumption that, if this system may be described entirely with the help of one function of two variables $v(q, q')$, then it may also be described with the help of another function of two variables connected with previous one, namely a pair distribution function. We prove this assumption here.

Define $s$-particle distribution functions $G_s$ starting from the non-normalized GDF (3)
\begin{equation}
G_s(q_1, \ldots, q_s) = \frac{N!}{(N-s)!V^N} \int D_N(q_1, \ldots, q_N) dq_{s+1} \cdots dq_N \quad s = 1, \ldots, N.
\end{equation}

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1. Here we consider only a configuration part of the distribution function, because its momentum part is factorized per a one-particle Maxwell function.
2. Three-dimensionality of space is not essential. It may be of any finite dimensionality. Only finiteness of the Lebegue measure (volume) of the manifold $V$ is important.
Here $V$ is a volume of the domain $\mathcal{V}$. The integration will be carried out over the $\mathcal{V}$ and the domain of integration will not be indicated explicitly.

Due to the properties of $v(q, q')$ all multi-particle distribution functions are real, nonnegative, and symmetrical. These functions are related to the ones $f_s$, introduced by Balescu \[1\], and to $F_s$ introduced by Bogoliubov \[2\] as follows: $f_s = Q_N^{-1} V^N G_s$ and $F_s = (Q_N N!)^{-1} (N - s)! V^{N-s} G_s$ respectively, where $Q_N$ is a configuration integral

$$Q_N = \int dq_1 \cdots dq_N D_N(q_1, \ldots, q_N) = \frac{V^N (N - s)!}{N!} \int dq_1 \cdots dq_s G_s(q_1, \ldots, q_s). \quad (6)$$

Write down explicitly an expression for the PDF

$$G_2(q_1, q_2) = \frac{N(N-1)}{V^N} \int \prod_{1 \leq i < j \leq N} v(q_i, q_j) dq_3 \cdots dq_N. \quad (7)$$

Let us introduce a nonlinear operator $\Phi$ as follows

$$\Phi(v) \overset{\text{def}}{=} \frac{1}{V^{N-2}} \int \prod_{1 \leq i < j \leq N} v(q_i, q_j) dq_3 \cdots dq_N. \quad (8)$$

Then the relation (7) may be considered as a nonlinear operator equation relative to $v$

$$\frac{V^2}{N(N-1)} G_2 - \Phi(v) \overset{\text{def}}{=} F(v, G_2) = 0. \quad (9)$$

If this equation has a solution

$$v = \chi(G_2), \quad (10)$$

then its substitution into (9) permits one to express the non-normalized Gibbs distribution $D_N(q_1, \ldots, q_N)$ through the two-particle function $G_2(q, q')$. Accordingly all $s$-particle functions may also be expressed via $G_2(q, q')$.

For calculating the function $v$ from equation (9) it is necessary to set an additional condition

$$G_2^{(0)} = \frac{N(N-1)}{V^2} \Phi(v^{(0)}), \quad (11)$$

where $G_2^{(0)}$ and $v^{(0)}$ are known functions. It is easy to find such a condition assuming particularly

$$v^{(0)}(q, q') = 1, \quad G_2^{(0)}(q, q') = \frac{N(N-1)}{V^2}, \quad (12)$$

which corresponds to the choice of noninteracting particle system as a reference one\[^3\].

\[^3\]Evidently, instead of unity for $v^{(0)}(q, q')$ we may take an arbitrary constant $C$. In this case an expression for $G_2^{(0)}(q, q')$ is calculated easily too.
3 Proof of the existence of a solution

In functional analysis there exist a number of theorems about implicit functions for operators of different smoothness degree (e.g. [3, 4]). We use a theorem for analytical operators in Banach spaces in the form presented in the book [3, Theorem 22.2].

**Theorem.** Let \( F(v, G_2) \) be an analytical operator in \( D_r(v^{(0)}, E_1) + D_{\rho_1}(G_2^{(0)}, E) \) with values in \( E_2 \). Let the operator \( B \equiv -\partial F(v^{(0)}, G_2^{(0)})/\partial v \) have a bounded inverse operator. Then there exist such positive numbers \( \rho_1 \) and \( r_1 \) that in the solid sphere \( D_{\rho_1}(G_2^{(0)}, E) \) there exists a unique solution (10) to equation (9). This solution is defined in the solid sphere \( D_{r_1}(v^{(0)}, E_1) \), being analytical in it and satisfying the condition \( v^{(0)} = \chi(G_2^{(0)}) \).

Here \( D_r(x_0, E) \) is a solid sphere of the radius \( r \) in the vicinity of an element \( x_0 \) of Banach space \( E \), the sign "\( \dot{+} \)" denotes an algebraic sum of manifolds, and \( \partial F/\partial v \) is a Frechet derivative [4, 5] of a nonlinear operator \( F \).

To investigate the problem of existence and uniqueness of the solution of equation (9) it is necessary to verify fulfillment of the conditions of the theorem for the system under consideration.

At first define spaces \( E_1 \) and \( E_2 \), in which the operator \( F \) is set, and space \( E_2 \) containing a range of values of this operator. We shall use a class of functions \( v(q, q') \) bounded almost everywhere in \( \mathcal{V} \). The space of such functions becomes a complete normalized (Banach) space of essentially bounded functions \( L_\infty \) if we introduce the following norm [4, 5]

\[
\|v\| = \text{vrai sup} \ |v(q, q')| ,
\]

where "vrai sup" denotes an essential exact supremum of the function on the indicated set, and \( \mathcal{V} \times \mathcal{V} \) is a direct product of the manifold \( \mathcal{V} \) by itself, that is a set of ordered pairs \( (q, q') \). Moreover \( v(q, q') \) is a symmetrical function of its arguments. Therefore \( v(q, q') \in L_\infty^s(\mathcal{V} \times \mathcal{V}) \). The index \( s \) means that we use a space of symmetrical functions. Such properties of the function \( v(q, q') \) defined by (4) are provided by any interaction potential bounded below almost everywhere in the domain \( \mathcal{V} \). All potentials conventionally used in statistical mechanics satisfy this condition.

Using the properties of real functions and their integrals [4], one can show that for any functions \( v \) from \( L_\infty^s(\mathcal{V} \times \mathcal{V}) \) both GDF and all multi-particle ones are also essentially bounded functions being defined on appropriate sets. Thus the space of functions \( v \) is \( E_1 = L_\infty^s(\mathcal{V} \times \mathcal{V}) \), the space of functions \( G_2 \) is \( E = L_\infty^s(\mathcal{V} \times \mathcal{V}) \), and the space containing the range of values of the operator \( F \) is \( E_2 = L_\infty^s(\mathcal{V} \times \mathcal{V}) \). The set of pairs \( (v, G_2) \) of nonnegative functions in the direct product \( E_1 \times E_2 \) is a range of definition \( 4 \) of the operator \( F \), with its range of values belonging to \( E_2 \). Everywhere in the range of definition the operator \( F(v, G_2) \) is analytical.

\(^4\text{In general, the requirement on nonnegativity of the functions follows from physical reasons. From the mathematical point of view the operator } F(v, G_2) \text{ is defined on a whole space } E_1 \times E.\)
Calculate an operator \( B \) defined in the theorem formulation for chosen values of the functions \( v(0) \) and \( G_2^{(0)} \). It is a linear operator acting from space \( E_1 \) to space \( E_2 \). Using the first equality in (9), we obtain the expression

\[
B = \frac{d\Phi(v(0))}{dv}.
\] (14)

Substitute into it the expression (8) for the operator \( \Phi(v) \) and use the condition \( v(0) = 1 \). We obtain the operator \( B \) as follows

\[
(Bh)(q, q') = h(q, q') + (N - 2) \left[ \bar{h}(q) + \bar{h}(q') \right] + \frac{(N - 2)(N - 3)}{2} \bar{\bar{h}},
\] (15)

where we introduce designations

\[
\bar{h}(q) = \frac{1}{V} \int h(q, q')dq', \quad \bar{\bar{h}} = \frac{1}{V^2} \int h(q, q')dq \, dq'.
\] (16)

In these expressions \( h \in E_1, Bh \in E_2 \). The range of definition of \( B \) is a whole space \( E_1 \): \( D(B) = E_1 \) and its range of values is a space \( E_2 \): \( R(B) = E_2 \). The operator \( B \) is bounded. Its norm defined by (see eg. [4, 5])

\[
\|B\| = \sup_{\{h \in E_1: \|h\| = 1\}} \|Bh\|,
\] (17)

is evaluated easily with the help of (15). Using properties of norms we obtain the following inequality

\[
\|Bh\| \leq \|h(q, q')\| + (N - 2)\|\bar{h}(q) + \bar{h}(q')\| + \frac{(N - 2)(N - 3)}{2} \|\bar{\bar{h}}\|.
\] (18)

Evaluate the norms in right hand side of (18). The value \( \bar{\bar{h}} \) is a constant. For its norm we have an estimate

\[
\|\bar{\bar{h}}\| = \left| \frac{1}{V^2} \int h(q, q')dq \, dq' \right| \leq \frac{1}{V^2} \int \operatorname{vrai sup}_{(q, q') \in V \times V} |h(q, q')| \, dq \, dq' =
\]

\[
= \frac{1}{V^2} \int \|h\| \, dq \, dq' = \|h\|.
\] (19)

For the norm of a sum of the functions \( \bar{h}(q) \) and \( \bar{h}(q') \) we obtain the following estimate

\[
\|\bar{h}(q) + \bar{h}(q')\| = \operatorname{vrai sup}_{(q, q') \in V \times V} |\bar{h}(q) + \bar{h}(q')| \leq 2 \operatorname{vrai sup}_{q \in V} \bar{h}(q) =
\]

\[
= \frac{2}{V} \operatorname{vrai sup}_{q \in V} \left| \int h(q, p)dp \right| \leq \frac{2}{V} \int \operatorname{vrai sup}_{q \in V} |h(q, p)| \, dp \leq
\]
\[
\leq \frac{2}{V} \int \text{vrai sup}_{(q,p) \in V \times V} |h(q,p)| \, dp = 2\|h\|.
\] (20)

Substituting (19) and (20) in the inequality (18), we obtain the following estimate for the norm of the function \((Bh)(q,q')\)

\[
\|Bh\| \leq \frac{N(N-1)}{2}\|h\|. \tag{21}
\]

Taking into account the definition (17), we obtain an estimate of the norm of the operator \(B\)

\[
\|B\| \leq \frac{N(N-1)}{2}. \tag{22}
\]

One can show that actually in (22) the sign of equality is valid. Indeed because there is at least one function \(h(q,q') \in E_1\) with the norm equal to unity for which \(\|Bh\|\) equals the value in the right hand side of (21), namely \(h(q,q') \equiv 1\), then the inequality (22) reduces to the equality

\[
\|B\| = \frac{N(N-1)}{2}. \tag{23}
\]

Now define a kernel (a space of zeros) of the operator \(B\), i.e. find a solution to the homogeneous equation \(Bh = 0\). Using (15), we obtain the equation

\[
h(q,q') + (N-2)[\bar{h}(q) + \bar{h}(q')] + \frac{1}{2}(N-2)(N-3)\bar{h} = 0. \tag{24}
\]

Integration of this equation over \(q'\) and then over \(q\) allows one to obtain its solution. It turns out to be trivial \(h(q,q') = 0\).

This means that the corresponding inhomogeneous equation \(Bh = g\) is uniquely solvable. Its solution has the form

\[
h(q,q') \overset{\text{def}}{=} (B^{-1}g)(q,q') = g(q,q') - \frac{N-2}{N-1}[\bar{g}(q) + \bar{g}(q')] + \frac{N-2}{N} \bar{g}, \tag{25}
\]

where \(\bar{g}(q)\) and \(\bar{g}\) are determined by the formulae analogous to (16) where \(h\) is replaced by \(g\). It is easy to see that the operator \(B^{-1}\) is specified for all \(g \in E_2\), that is indeed \(R(B) = E_2\). It is easy to verify that the conditions \(B^{-1}B = BB^{-1} = I\) are valid. In other words, the operator \(B\) is both left and right inverse. Therefore it is the inverse operator to \(B\) \cite{footnote6}.

The operator \(B^{-1}\) is bounded. For its norm an upper bound is easily evaluated just as for the operator \(B\). At once write down this estimate

\[
\|B^{-1}\| \leq \frac{3N^2 - 6N + 2}{N(N-1)}. \tag{26}
\]

Thus all conditions of the theorem are fulfilled. Hence there is a unique solution (19) expressing unambiguously the function \(v(q,q')\) in terms of PDF \(G_2(q,q')\). This
solution is determined in the solid sphere $D_{p_1}(G_2^{(0)}, E)$, analytical in it, and satisfies the conditions (11), (12). We shall not produce here a proof of this theorem. It is presented in many books in functional analysis. Indicate merely the procedure of constructing a solution to equation (9) and estimating the convergence domain size.

Introduce functions $g(q, q')$ and $h(q, q')$ by the relations

$$G_2 = \frac{N(N-1)}{V^2}(1 + g), \quad v = 1 + h.$$  \hspace{1cm} (27)

Here and below $h(q, q')$ is a Mayer function. Equation (9) with the operator $\Phi(v)$ (8) may be rewritten as

$$h = B^{-1}g + B^{-1}\sum_{s=2}^{\mathcal{N}} B_s(h),$$  \hspace{1cm} (28)

where $\mathcal{N} = N(N - 1)/2$ and $B_s(h)$ is a homogeneous power operator of the order $s$ with respect to $h$ equal to

$$B_s(h) = -\frac{1}{V^{N-2}}\int dq_3 \cdots dq_N \sum_{1 \leq K_1 < \ldots < K_s \leq \mathcal{N}} h(X_{K_1}) \cdots h(X_{K_s}).$$  \hspace{1cm} (29)

Here we introduce a generalized index $K$ varying from 1 to $\mathcal{N}$ enumerating all possible different index pairs $(i,j)$, satisfying the condition $i < j$ which can be constructed from the numbers $1, 2, \ldots, N$. Such one-to-one correspondence $K \leftrightarrow (i,j)$ is easily found. $X_K$ designates the corresponding pair of coordinates $(q_i, q_j)$. One can consider that the power operator $B_s(h)$ is generated by the $s$-linear operator $R_s(h_1, \ldots, h_s)$ \[3\]

$$R_s(h_1, \ldots, h_s) = -\frac{1}{V^{N-2}}\int dq_3 \cdots dq_N \sum_{1 \leq K_1 < \ldots < K_s \leq \mathcal{N}} h_1(X_{K_1}) \cdots h_s(X_{K_s}).$$  \hspace{1cm} (30)

Using this operator, we rewrite equation (28) as follows

$$h = B^{-1}g + B^{-1}\sum_{s=2}^{\mathcal{N}} R_s(h, \ldots, h).$$  \hspace{1cm} (31)

Its solution is found as a power series in $g$

$$h = \sum_{k=1}^{\infty} h_k(g),$$  \hspace{1cm} (32)

where $h_k(g)$ is an homogeneous power operator of the order $k$ acting from space $E_2$ to $E_1$. This means that each operator $h_k(g)$ is a function of two configuration variables.
Substituting the expression (32) into equation (31) and producing some rearrangement of terms in the right-hand side, we derive the following relationship
\[
\sum_{k=1}^{\infty} h_k(g) = B^{-1}g + B^{-1} \sum_{k=2}^{\infty} \sum_{s=2}^{k} \theta_{sN} \sum_{k_1+\ldots+k_s=k} R_s(h_{k_1}(g), \ldots, h_{k_s}(g)),
\] (33)
where
\[
\theta_{sN} = \begin{cases} 1, & s \leq N, \\ 0, & s > N. \end{cases}
\]
In accordance with the theorem on uniqueness of analytical operators [7], equating terms of identical orders on \(g\), we obtain a recurrent system for determining the operators \(h_k(g)\)
\[
h_1(g) = B^{-1}g,
\]
\[
h_k(g) = B^{-1} \sum_{s=2}^{k} \theta_{sN} \sum_{k_1+\ldots+k_s=k} R_s(h_{k_1}, \ldots, h_{k_s}), \quad k = 2, 3, \ldots
\] (34)
All expansion terms of (32) are calculated successively from this system. And thus a formal solution \(h(g)\) is derived in the form of the above series (32). It is also necessary to prove a convergence of this series. It is established, just as in [3], by Goursat method [8] of majorizing functions.

Consider a subsidiary equation
\[
\xi = \gamma \left\{ \eta + \sum_{s=2}^{N} \frac{N!}{s!(N-s)!} \xi^s \right\},
\] (35)
in which we designate \(\xi = \|h\|, \eta = \|g\|\) and \(\gamma = \|B^{-1}\|\). The right-hand side of this equation majorizes the norm of the right-hand side of (31). A solution to equation (35) can be found in the form of expansion in series with respect to \(\eta\): \(\xi = \sum_{k=1}^{\infty} \xi_k \eta^k\).

For coefficients \(\xi_k\) it is easy to derive a recurrent system analogous to the structure of (34). Just as in a classical case [8] it is easy to show that \(\|h_k(g)\| \leq \xi_k \eta^k\). Therefore if the series for \(\xi\) converges, then the series (32) absolutely converges.

On the other hand, equation (35) can be rewritten as
\[
\gamma \eta = \xi - \gamma [1 + \xi]^N - 1 - N \xi.
\] (36)
We cannot write out a solution to this equation explicitly by quadratures, but it is easy to show graphically that the solution \(\xi(\eta)\) satisfying the condition \(\xi(0) = 0\) exists in the region \(\eta < \eta_0\). This solution is unique, analytical and monotonically increasing. The value \(\xi_0\) determines the maximum point of the function in the right-hand side of equation (36) and equals \(\xi_0 = (1 + 1/\gamma N)^{N^\gamma} - 1\). The value \(\eta_0\) is defined by this function maximum and equals
\[
\eta_0 = (N - 1) \left(1 + \frac{1}{\gamma N}\right) \left[\left(1 + \frac{1}{\gamma N}\right)^{N^{\gamma-1}} - 1\right] - \frac{1}{\gamma N}.
\]
Thus the series (32) converges if \( \|g\| < \eta_0 \). Thereby, as a radius \( \rho_1 \) of the solid sphere from the theorem, any number smaller than \( \eta_0 \) may be chosen. By virtue of a monotonic increase of the function \( \xi(\eta) \), the radius \( r_1 \) may be taken equal to \( r_1 = \xi(\rho_1) \). So the solution (10) to the problem (9) exists, it is unique and may be presented in the form of the expansion (32) into an absolutely convergent series with respect to \( g = (G_2 - G_2^{(0)})/G_2^{(0)} \) at least in the region \( \|g\| < \eta_0 \).

Since the Gibbs distribution (33) is a polynomial of the order \( N \) relative to the function \( h \), then it is an analytical function of \( g \) and may be expanded in a power series with respect to \( g \\
\\ D_N = \sum_{k=0}^{\infty} d_k(g),
\\
\\ where \( d_k(g) \) is a homogeneous power operator of the order \( k \) acting from a space \( E_2 \) into a space \( E_N = L_\infty^{(N)}(\mathcal{V} \times \cdots \times \mathcal{V}) \), that is, each operator \( d_k(g) \) is a function of \( N \) configuration variables. Due to determining the PDF (7) and the theorem on uniqueness of analytical operators (7) these functions satisfy the conditions

\[
\frac{1}{\sqrt{V - 2}} \int dq_1 \cdots dq_N d_1(q_1, \ldots, q_N) = g(q_1),
\]

\[
\int dq_1 \cdots dq_N d_k(q_1, \ldots, q_N) = 0, \quad k = 2, 3, \ldots.
\]

Substituting the expansion (32) into into (33) and producing some rearrangement of terms, we can derive uniquely all terms \( d_k(g) \) of the expansion (37) in the form

\[
d_k(g) = \sum_{s=1}^{k} \sum_{k_1 + \cdots + k_s = k} \sum_{k_1 < k_2 < \cdots < k_s \leq N} h_{k_1}(X_{K_1}) \cdots h_{k_s}(X_{K_s}), \quad k = 1, 2, \ldots,
\]

where \( h_i(X_{K_i}) \) are expressed in terms of \( g \) by formulae (34). The expressions obtained determine an expansion of the GDF in series with respect to PDF.

4 Concluding remarks

Using the theorem of existence and uniqueness of the solution to equation (2), we reveal that Mayer function \( h(q, q') \) may be expressed uniquely through the PDF \( G_2(q, q') \). It permits us to present the non-normalized Gibbs distribution as a nonlinear operator function of PDF and write its expansion to power series. The procedure of successive calculation of the all terms of this series is formulated.

So knowing a pair distribution function, we can express the Gibbs distribution uniquely in terms of this function. In turn PDF is expressed uniquely via GDF.
(see [5]). This means that descriptions of equilibrium canonical ensemble by both GDF and PDF are equivalent, at least for \( h \) belonging to the solid sphere \( D_{r_1}(0, E_1) \). Hence PDF contains the same quantity of information as the full GDF, as well as all \( s \)-particle distribution functions for \( s > 2 \) which are also expressed uniquely through PDF.

The one-particle distribution function \( G_1(q) \) does not seem to contain all information about the system with two-particle interaction. If one tries to realize the above programme, then one fails to satisfy all conditions of the theorem on existence and uniqueness of the solution. The kernel (a space of zeros) of the corresponding operator \( B \) consists of arbitrary functions \( h(q, q') \in L_\infty^{(s)}(\mathcal{V} \times \mathcal{V}) \) limited only by one condition \( h(q) \equiv 0 \). Therefore it is possible for such operator \( B \) to have no inverse operator. And indeed it has not.

Note here that for the normalized GDF an analogous programme cannot be realized too. In this case the corresponding operator \( B \) has also a nontrivial kernel, namely a set of all functions constant on \( \mathcal{V} \). Mathematically, it is clear from the form of the normalized GDF in which the function \( v \) may be multiplied by an arbitrary constant without changing the distribution function. Physically, it corresponds to the fact that the interacting potential is defined with an accuracy to an arbitrary constant.

**References**

[1] R. Balescu. Equilibrium and nonequilibrium statistical mechanics. John Wiley & Sons Inc. N.Y., 1975.

[2] N.N. Bogoliubov. Problems of dynamical theory in statistical physics. In Selected proceedings, Vol. 2, pp. 99-196, 1970 (in Russian).

[3] M.M. Vainberg, and V.A. Trenogin. A Theory of branching of nonlinear equation solutions. M., Nauka, 1969, (in Russian).

[4] L.V. Kantorovich, and G.P. Akilov. Functional analysis. M., Nauka, 1977 (in Russian).

[5] B.Z. Vulikh. A brief course in the theory of functions of real variables (An introduction to the theory of integral). M., Mir, 1976.

[6] V.A. Trenogin. Functional analysis. M., Nauka, 1980 (in Russian).

[7] E. Hille, and R.S. Phillips. Functional analysis and semi-groups. Providence, 1957.

[8] E. Goursat. Course of mathematical analysis, Vol. 1. M.-L., ONTI, 1936 (in Russian); (E. Goursat. Cours d’analyse mathématique, Vol. 1. Gauthier - Villars, Paris, 1923).