Complex Time Solutions with Nontrivial Topology and Multi Particle Scattering in Yang-Mills Theory

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Abstract

A classical solution to the Yang-Mills theory is given a new semiclassical interpretation in terms of particle scattering. It solves the complex time boundary value problem, which arises in the semiclassical approximation to a multi particle transition probability in the one-instanton sector at fixed energy. The imaginary part of the action of the solution on the complex time contour and its topological charge obey the same relation as the self-dual Euclidean configurations. Hence the solution is relevant for the problem of tunneling with fermion number violation in the electroweak theory. It describes transitions from an initial state with a smaller number of particles to a final state with a larger number of particles. The implications of these results for multi particle production in the electroweak theory are also discussed.

\textsuperscript{1}Supported in part by the National Science Foundation under grant PHY-90-9619.
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An intriguing feature of the Yang-Mills gauge theory is the periodic structure of its vacuum [1, 2]. In the semiclassical approximation, the topology of finite energy solutions leads to a classification of all gauge-inequivalent vacua in the theory. The discovery of this rich structure has had a profound impact on our understanding of non-perturbative aspects of the theory, notably a low energy phenomenon like the solution of the famous U(1) problem in QCD [3]. However, the role of the vacuum in the dynamics of particle scattering has not yet been as deeply understood. This deficit in our understanding has been confronted in recent years with the study of so-called “instanton-induced” cross-sections [4, 6, 7].

The simplest semiclassical estimate of the contribution of the BPST [5] instanton to a total inclusive two particle cross-section in electroweak theory implies a result which grows exponentially with center-of-mass energy [4, 6]. It has been shown that the instanton is the basis of a systematic perturbative expansion of the final state radiative corrections to the cross-section [6], which determines its leading semiclassical behavior, neglecting initial state radiative corrections,

\[ \sigma_{tot}(x) \sim \exp \left[ \frac{16\pi^2}{g^2} F(x) + o(\alpha^0) \right] \]  

as an expansion in powers of a small parameter \( x \equiv E/E_0 \), the ratio of the center-of-mass energy \( E \) and a mass scale of order the electroweak sphaleron mass, \( E_0 \simeq M_w/\alpha_w \simeq 10 \text{ TeV} \). The so-called “Holy Grail function”, \( F(x) \), is approximately \(-1\) for small \( x \), reflecting the severe 'tHooft suppression factor, \( \exp \left[ -\frac{16\pi^2}{g^2} \right] \approx 10^{-127} \), due to the large instanton action. The fact that the Holy Grail function is an increasing function of \( x \) for small \( x \) has led many to speculate about the possibility of overcoming the severe exponential suppression factor at energies of order \( E_0 \). The possibility of strong multi-particle scattering in electroweak theory at multi-TeV energies has led to an enormous effort to understand the behavior of multi-particle cross-sections in the sphaleron energy regime [7].

Most of these studies, however, are based on semiclassical expansions around configurations which are not influenced by external sources. Indeed, the instantons obey vacuum boundary conditions, and as such are relevant to this problem only in the approximation in which external sources are neglected. While the final state corrections can be accounted for in the perturbative expansion in \( x \), the initial state corrections are more subtle. These involve radiative corrections to hard particles which are not \textit{a priori} calculable semiclassically. However, there have been some indications [8] that the contributions to the leading
semiclassical result due to corrections involving hard initial legs may also be calculable in a
semiclassical manner. It may then be possible to calculate the entire leading order semiclass-
sical exponent in a saddle point approximation. What is needed is a new technique which
accounts for external sources to make the semiclassical behavior of the total cross-section
manifest.

A strategy for out-flanking the problem of initial state corrections was recently proposed
by Rubakov and Tinyakov [8]. The basic idea is to consider transitions from states of a
fixed large number of particles, say $N_{\text{in}} = \nu/g^2$. The instanton-like transition probability
from a multi-particle initial state is then calculable semiclassically, in the limit $g \to 0$ with
$\nu$ fixed. Its leading semiclassical behavior is determined by the solution to a boundary value
problem. The boundary conditions imposed at initial and final times correctly account for
the energy transfer from the initial multiparticle state to the final multiparticle state. The
leading semiclassical behavior of the $N_{\text{in}}$ particle transition probability has a form similar to
(1). It is a rigorous upper bound on the inclusive two-particle cross section and is related
to a lower bound under less rigorous assumptions [10]. Since the $N_{\text{in}}$ particle transition
probability contains all initial state corrections for the $N_{\text{in}} = \nu/g^2$ particle transition, it is
hoped that it reproduces the leading semiclassical behavior of a two particle transition when
$\nu$ is small, including initial and final state corrections. Indications from explicit calculations
of initial and final state corrections are that the limit of $\nu \to 0$ is smooth [8], so that the
contribution to a semiclassical transition probability from the solution of the boundary value
problem contains the initial and final state corrections. The boundary value problem posed
in this way also holds the promise of being amenable in principle to numerical computation
of multiparticle transitions. It would now be useful to have some analytical examples to
guide future efforts in this direction [11, 12, 13].

The use of a Minkowski or Euclidean time contour for the semiclassical calculation
of transition amplitudes in the one-instanton sector is too restrictive. Recall that com-
puting tunneling contributions to fixed-energy (i.e. time-independent) Green functions
in quantum mechanics can be performed in the WKB approximation only on a complex
time contour, chosen to lie in Minkowski directions at early and late times, with a pe-
riod of Euclidean evolution inserted at an intermediate time. They give the dominant
WKB-contribution to classically forbidden processes. In the present case of quantum field
theory, we will similarly be interested only in time-independent transition probabilities.
In nonabelian gauge theories, the transition amplitude between vacua with different topological number is known to be maximized by instantons. They and, in fact, any finite action Euclidean solution have vacuum asymptotics at infinity. For transitions involving many-particle initial and final states, vacuum boundary conditions are clearly not the correct ones. Considering solutions on a complex time contour, $C_T$, (fig.1) provides a natural description of the initial and final states in Minkowski space in terms of the free wave asymptotics of the solution at $|\text{Re } t| \to \infty$.

A few such solutions on a complex time contour have already been investigated. The periodic instanton is a solution of the complex time boundary value problem which arises from the semiclassical approximation to the inclusive transition probability from all initial states at fixed energy, or a microcanonical distribution [10]. It has two turning points on the complex time contour. It has been shown to determine the maximal probability for transition in the one-instanton sector from states of fixed energy [10, 9]. The periodic instanton in electroweak theory has so far been constructed only in a low energy approximation, and the resulting transition probability is determined in a perturbative expansion similar to that in [1]. It has been found to describe transitions between states of equal number of particles which is large in the semiclassical limit, $N_{\text{in}} = N_{\text{fin}} \sim 1/g^2$. So, this solution is irrelevant for describing $2 \to n$ scattering processes at high energies, though it does play a role in determining the rate of tunnelling, and anomalous baryon number violation, at finite temperature [14].

Fig. 1
A solution which describes transitions from a state of smaller number of particles to a state with a larger number of particles has also only been constructed in a low energy expansion \cite{9}. Similarly, it determines the maximum transition probability from states of fixed energy and particle number. However, it remains to construct solutions which describe such processes in general. This is a formidable task, requiring a solution of the Yang-Mills equations with arbitrary boundary conditions on a complex time contour. In this paper, we pursue more modest goals. We consider the SO(4) conformally invariant Minkowski time solutions of Lüscher and Schechter \cite{16}, analytically continued to a complex time contour (fig. 1). The solution in Minkowski-time describes an energy density which evolves from early times as a thin collapsing spherical shell, bounces at an intermediate time, and expands outward again at late times. As yet, the role of these solutions in scattering problems has not been fully developed \cite{15}.

Our aim is the calculation of many-particle transition amplitudes in the one-instanton sector. Therefore, we will consider only a subclass of these solutions which have integer topological charge on the complex time contour $C_T$. Only the solutions with a turning point at say, $t = 0$ for all $\vec{x}$, have this property, as we will show. The Lüscher-Schechter solutions are real in Minkowski time. The turning point condition assures that their analytic continuation to the Euclidean time axis is real as well. Note that in general the fields will be complex on the Re $t < 0$ part of the contour, since $t = iT$ is not a turning point of the solution\cite{16}.

Lüscher and Schechter have shown that the most general solution for which a SO(4)-conformal transformation can be compensated by a global SU(2)-gauge transformation is parameterized by a single function $q(\phi)$, where $2r \cosh \phi = (1 + r^2 + \tau^2) \cos \omega$, $r \sinh \phi = \tau \cos \omega$ and $2r \tan \omega = r^2 + \tau^2 - 1$. Its Euclidean action is \cite{16}:

$$S = \frac{i}{4g^2} \int d^4x F^a_{\mu \nu} \tilde{F}^a_{\mu \nu} = i \frac{12\pi}{g^2} \int_{0}^{\infty} dr \int_{-\infty}^{+\infty} d\tau \frac{\cos^4 \omega}{r^2} \left[ \frac{1}{2} \dot{q}^2 + \frac{1}{2} \left( q^2 - 1 \right)^2 \right], \quad (2)$$

where $\dot{q} \equiv \frac{d}{d\phi} q(\phi)$, $\tau \equiv it$ is the Euclidean time and $r \equiv |\vec{x}|$.

The topological charge in terms of the Lüscher-Schechter Ansatz becomes:

$$Q \equiv \frac{1}{32\pi^2} \int d^4x F^a_{\mu \nu} \tilde{F}^a_{\mu \nu} = \frac{1}{2\pi} \int_{0}^{\infty} dr \int_{-\infty}^{+\infty} d\tau \frac{\cos^4 \omega}{r^2} \dot{q} \left( 3q^2 - 3 \right). \quad (3)$$

\footnote{It is easy to show that the SO(4)-conformally invariant solutions can have at most one turning point.}
A solution with a turning point is easy to find by considering the one-dimensional double-well problem, following from (3). It represents oscillatory motion in the well between $q = 1$ and $q = -1$ of the potential $V(q) = \frac{1}{2} (q^2 - 1)^2$. The turning point condition\footnote{One can verify, using the explicit formulae relating $q$ to gauge potentials [3], that the condition $\dot{q}(\phi = 0) = 0$ corresponds to a turning point of the gauge potentials at $\tau = 0$ for all $\vec{x}$.} at $\phi = 0$ leaves one free parameter: the “energy” $\epsilon$ ($\epsilon < 1/2$), or equivalently the initial coordinate, $q_- = \sqrt{1 - \sqrt{2} \epsilon}$, of the particle in the well. This solution is explicitly given in terms of the Jacobian elliptic sine:

$$q(\phi(r, \tau)) = q_- \operatorname{sn}(q_+\phi(r, \tau) + K, k).$$  (4)

Now, the imaginary part of $S_{CT}$ is the quantity entering the WKB-exponent of a transition probability dominated by this solution. Since the solution is real on the Minkowski time axis, the contribution to the action from the real time axis is purely real. So, the residue at the singularity between the complex time contour and the Minkowski time axis alone determines $\operatorname{Im} S_{CT}$:

$$\operatorname{Im} S_{CT} = -\frac{24\pi^2}{g^2} \int_0^\infty dr \operatorname{Im} \sum_{nm} \left. \left\{ \cos^4 \frac{w}{r^2} \left[ \frac{1}{2} \dot{q}^2 + \frac{1}{2} \left( q^2 - 1 \right)^2 \right] \right\} \right|_{\tau_{nm}(r)}.$$  (5)

Consider now the topological charge $Q$ (3) on the closed contour $C_T + C_M$ where $C_M$ runs along the Minkowski time axis. Our solution (4) is an even function of $\phi$, therefore the integral for $Q$ on the Minkowski time axis vanishes. Thus, $Q$ on $C_T$ is determined by the residues at the singularity as well:

$$Q = 3i \int_0^\infty dr \sum_{nm} \left. \left\{ \cos^4 \frac{w}{r^2} \dot{q} \left( q^2 - 1 \right) \right\} \right|_{\tau_{nm}(r)}.$$  

Explicit calculation yields for the imaginary part of the action and the topological charge:

$$\operatorname{Im} S = N \frac{8\pi^2}{g^2}, \quad Q = N,$$  (6)

where $N$ is the number of singularity lines between the complex time contour and the Minkowski time axis. It should be stressed that the relation (6) is far from trivial on the complex time contour. The usual arguments for establishing the Bogomol’nyi bound do not seem to hold here, since the fields take complex values on the contour [3].
point condition at $\tau = 0$ is crucial for (6) to hold. As was shown in [15], the Minkowski time topological charge vanishes only for solutions with a turning point. The Lüscher-Schechter solutions without a turning point have fractional topological charge on the contour $C_T$.

This gauge field configuration gives the dominant contribution to an inclusive transition probability from a fixed initial state, in the saddle point approximation [11]. The initial state, and the most probable final state for transition from this initial state, are characterized by the free-wave asymptotics of the solution at the ends of the complex time contour $C_T$. The total transition probability from an initial coherent state, $|\{a(\mathbf{k})\}\rangle$, projected onto fixed center-of-mass energy $E$, is:

$$\sigma_E (\{a(\mathbf{k})\}) = \sum_f |\langle f | \hat{S} P_Q P_E |\{a(\mathbf{k})\}\rangle|^2.$$  \hspace{1cm} (7)

$P_E$ is a projection operator onto states of fixed center-of-mass energy, $E$. The probability is unity unless the initial state is projected also onto a subspace which does not commute with the Hamiltonian; a projection operator $P_Q$ onto states of fixed winding number $Q$ is implicit in our choice of a classical field with this property. Furthermore, the inclusive sum is over all final states built above a neighboring sector of the periodic vacuum.

This quantity is relevant to the study of multiparticle cross-sections because, when summed over all initial states,

$$\sigma_E = \sum_a \sigma_E (\{a(\mathbf{k})\})$$  \hspace{1cm} (8)

it gives the “microcanonical” transition probability in the one-instanton sector; the probabilities of transition from all states of energy $E$ are equally weighted in this sum. When evaluated in the saddle point approximation, $\sigma_E$ yields the maximal transition probability among all states with energy $E$. It sets therefore an upper bound on the two-particle inclusive cross-section in the one-instanton sector [10].

By writing (7) as a path integral of an exponential, its saddle point nature in the limit $g \to 0$ can be made evident, provided all terms in the exponent are $\sim 1/g^2$. This is the case if both the number of particles in the initial and final states is $O(1/g^2)$. The complex time contour $C_T$ provides a natural way of incorporating non-vacuum boundary conditions at the initial and final times. In the semiclassical approximation, the initial and final states

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3The charge on the contour $C_T$ is in this case the sum of an (integer) residue and a (fractional [15]) Minkowski time contour contribution.
are coherent states of the form:

\[ | \{ d(k) \} \rangle = \exp \left[ \int d k d(k) \hat{a}^\dagger(k) \right] | 0 \rangle \tag{9} \]

The creation operator is \( \hat{a}^\dagger(k) \) and all color and polarization indices have been suppressed. The complex amplitudes \( d(k) \) are determined by the free-field asymptotics of the solution at the ends of the contour \( C_T \).\(^6\)\(^,\)\(^1\)\(^1\)\(^,\)\(^1\)\(^3\).

In order to calculate the Fourier transforms of the gauge fields at large Minkowski time, we note that at large \( t \) the solution (4) represents a thin shell of energy, expanding with the speed of light. Then, the surface energy density decreases like \( 1/r^2 \sim 1/t^2 \) and we expect the nonlinear terms to become subdominant in the infinite time limit. Hence, as \( t \to \infty \), the solution reduces to a solution of the free equations of motion. The calculation of the Fourier transforms of the fields at initial time, on the complex part of the contour \( C_T \), is less straightforward, since at large early times the contour is trapped between two singularities of the solution. However, the Fourier transforms of the fields at the initial and final times differ only by the residues of the solution (4) at its poles in the complex-\( r \) plane, as a result of its analytic structure \(^{13}\). This allows us to calculate the complex amplitudes of the initial and most probable final state.

We find for the total average number of particles in the final state

\[ \bar{N}_{\text{fin}} = \frac{3 \epsilon \pi^2}{g^2}, \tag{10} \]

which exactly coincides with the energy of the classical solution \(^{13}\)\(^,\)\(^1\)\(^3\). The saddle point conditions arising from the integration over the initial values of the fields determine the initial coherent state in terms of the asymptotics of the solution \(^9\). The average number of particles in the initial state is found to be \(^{13}\):

\[ \bar{N}_{\text{in}} \sim \epsilon^{1/7} \bar{N}_{\text{fin}}. \]

Our solution describes therefore a transition from a state with a smaller number of particles, \( \bar{N}_{\text{in}} \), to a state with a larger number of particles, \( \bar{N}_{\text{fin}} \), their ratio being controlled by the small parameter, \( \epsilon \).\(^1\)\(^1\)\(^,\)\(^1\)\(^3\).

However, our solution does not maximize the microcanonical transition probability (8). It does not give the maximum transition probability at a given energy. Thus, it can not be used to provide an upper bound on the \( 2 \to n \) process cross-section.
We have found that a subclass of the SO(4)-conformally invariant solutions found by Lüscher and Schechter exhibits a number of remarkable properties on a suitably chosen complex time contour:

1. The semiclassical suppression is equal to the action of the BPST instanton, \( \text{Im } S = \frac{8\pi^2}{g^2} \). This quantity controls the semiclassical exponential dependence of a transition probability between coherent states.

2. The topological charge of the solution is equal to the BPST instanton charge, \( Q = 1 \). Thus, the solution may have a direct interpretation for fermion number violating processes [15].

3. It solves the boundary value problem for the transition probability in the one-instanton sector from a coherent state with a smaller number of particles, to a state with a larger number of particles. This property makes the solution interesting for the investigation of \( 2 \to n \) processes with fermion number violation at high energies [9].

Thus, the Lüscher-Schechter solution considered in this paper provides an analytical benchmark for future numerical computations of many particle transition amplitudes in Yang-Mills theory.

The assumption of conformal symmetry may allow a straightforward extension of the ideas presented here to a few more complicated field equations, coupled to the Yang-Mills equations. Minkowski time solutions of the field equations for a scalar triplet and fermion fields coupled to gauge fields have already appeared in the literature [19]. It may be interesting to investigate the properties of these solutions on the complex time contour, with an eye towards incorporating the additional fields of the Standard Model in this formalism. In particular, it may be possible to understand the process of fermion number or chirality violation in the Dirac-Yang-Mills system on the complex time contour.

However, the high degree of symmetry assumed here clearly limits the scope of the results. The spherical symmetry \( (SO(3)_{\text{rot}} \subset SO(4)_{\text{conf}}) \) of the solution has led us astray from the problem of high-energy \( 2 \to n \) processes. The solution in Minkowski time has the form of a spherical shell of energy, which collapses from infinity, then, at \( t = 0 \), bounces back and expands with the speed of light. Clearly, such a classical field configuration is a poor approximation to an initial state of two highly energetic colliding particles. Physical intuition
would lead one to believe that a solution with only a cylindrical symmetry might be a better candidate.

The assumption of conformal symmetry has also made less transparent an important application of this formalism: the electroweak theory. The mass scale $v \simeq 246$ GeV of electroweak theory explicitly breaks the classical conformal invariance of the pure gauge theory. In the case that the center-of-mass energy $E$ greatly exceeds the scale of symmetry breaking $v$, the Yang-Mills theory considered here may correctly describe the classical behavior of the gauge sector of the electroweak theory. Then, the results of this paper have direct relevance to the behavior in this energy region $[11, 13]$.

The complex-time solution presented here may also be considered the “core” of a constrained solution in a spontaneously-broken gauge theory $[20]$. In the Euclidean approach, there are no exact finite-action solutions to the electroweak gauge-Higgs field equations. In this case, the constrained expansion is a device to obtain the approximate solutions which provide the dominant semiclassical contribution to scattering amplitudes. In the complex time approach however, nothing prevents the existence of an exact solution to the Minkowski gauge-Higgs equations, which would represent an additional saddle point contributions to some transition amplitudes. The effect of symmetry-breaking on the solution presented in this paper has yet to be explored.

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