The classical Neumann problem for a class of mixed Hessian equations

Chuanqiang Chen | Li Chen | Xinquin Mei | Ni Xiang

1 School of Mathematics and Statistics, Ningbo University, Zhejiang, People's Republic of China
2 Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan, People's Republic of China
3 School of Mathematical Sciences, University of Science and Technology of China, Hefei, People's Republic of China

Abstract
In this paper, we establish global $C^2$ estimates for a class of mixed Hessian equations with the Neumann boundary condition and obtain the existence theorem of $k$-admissible solutions for the classical Neumann problem of these mixed Hessian equations.

KEYWORDS
a priori estimate, mixed Hessian equation, Neumann problem

1 | INTRODUCTION

In this paper, we consider the classical Neumann problem for the following mixed Hessian equations:

$$\sigma_k(D^2u) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u), \text{ in } \Omega,$$

(1)

where $k \geq 2$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $D^2u$ is the Hessian matrix of the function $u$, $\alpha_{k-1}(x) \geq 0$ and $\alpha_l(x) > 0$ in $\overline{\Omega}$ with $l = 0, 1, \ldots, k - 2$, and for any $m = 1, \ldots, n$,

$$\sigma_m(D^2u) = \sigma_m(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m},$$
with \( \lambda(D^2 u) = (\lambda_1, ..., \lambda_n) \) being the eigenvalues of \( D^2 u \). We also set \( \sigma_0 = 1 \). Recall that the Gårding's cone is defined as

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}.
\]

If \( \lambda(D^2 u) \in \Gamma_k \) for any \( x \in \Omega \), we say \( u \) is a \( k \)-admissible function.

Equation (1) is a general class of the mixed Hessian equation. Specially, it is the Monge-Ampère equation when \( k = n \), \( \alpha_0(x) > 0 \), and \( \alpha_1(x) = \cdots = \alpha_{n-1}(x) \equiv 0 \), \( k \)-Hessian equation when \( \alpha_0(x) > 0 \) and \( \alpha_1(x) = \cdots = \alpha_{k-1}(x) \equiv 0 \), and the Hessian quotient equation when \( \alpha_m(x) > 0 \) \((k-1 \geq m > 0)\) and \( \alpha_0(x) = \cdots = \alpha_{m-1}(x) = \alpha_{m+1}(x) = \cdots = \alpha_{k-1}(x) \equiv 0 \). This kind of equations is motivated from the study of many important geometric problems. For example, the problem of prescribing convex combination of area measures was proposed in Ref. 1, which leads to mixed Hessian equations of the following form:

\[
\sigma_k(\nabla^2 u + uI_n) + \sum_{i=0}^{k-1} \alpha_i \sigma_i(\nabla^2 u + uI_n) = \phi(x), \ x \in \mathbb{S}^n.
\]

The special Lagrangian equation introduced by Harvey and Lawson\(^2\) in the study of calibrated geometries is also a mixed-type Hessian equation

\[
\text{Im} \det(I_{2n} + \sqrt{-1} D^2 u) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \sigma_{2k+1}(D^2 u) = 0.
\]

Another important example is the Fu–Yau equation in Refs. 3,4 arising from the study of the Hull–Strominger system in theoretical physics, which is an equation that can be written as the linear combination of the first and the second elementary symmetric functions

\[
\sigma_1(i \partial \bar{\partial} (e^u + \alpha' e^{-u})) + \alpha' \sigma_2(i \partial \bar{\partial} u) = 0.
\]

For the Dirichlet problem of elliptic equations in \( \mathbb{R}^n \), many results are well known. For example, the Dirichlet problem of the Laplace equation was studied in Ref. 5. Caffarelli et al.\(^6\) and Ivochkina\(^7\) solved the Dirichlet problem of the Monge–Ampère equation. Caffarelli et al.\(^8\) solved the Dirichlet problem of the \( k \)-Hessian equation. For the general Hessian quotient equation, the Dirichlet problem was solved by Trudinger in Ref. 9.

Also, the Neumann or oblique derivative problem of partial differential equations has been widely studied. For a priori estimates and the existence theorem of the Laplace equation with the Neumann boundary condition, we refer to the book\(^5\). Also, we can see the recent book written by Lieberman\(^10\) on the Neumann or oblique derivative problem of linear and quasi-linear elliptic equations. In 1986, Lions et al. solved the Neumann problem of the Monge–Ampère equation in the celebrated paper.\(^11\) Related results on the Neumann or oblique derivative problem for some class of fully nonlinear elliptic equations can be found in Urbas.\(^12,13\) For the Neumann problem of \( k \)-Hessian equations, Trudinger\(^14\) established the existence theorem when the domain is a ball, and Ma and Qiu\(^15\) and Qiu and Xia\(^16\) solved the strictly convex domain case. Chen and Zhang\(^17\) solved the Neumann problem of general Hessian quotient equations. Jiang and Trudinger,\(^18–20\) studied the general oblique boundary problem for augmented Hessian equations with some regular conditions and concavity conditions.
Krylov in Ref. 21 considered the Dirichlet problem of \( (1) \) with \( \alpha_l(x) \geq 0 \) for \( 0 \leq l \leq k - 1 \), and he observed that the natural admissible cone to make equation elliptic is also the \( \Gamma_k \). Recently, Guan and Zhang in Ref. 22 considered the \( (k - 1) \)-admissible solution without the sign of \( \alpha_{k-1} \) and obtained the global \( C^2 \) estimates.

Naturally, we want to know more about the classical Neumann problem of \( (1) \). In this paper, we obtain the existence theorem as follows.

**Theorem 1.** Suppose that \( \Omega \subset \mathbb{R}^n \) is a \( C^4 \) strictly convex domain, \( 2 \leq k \leq n \), \( \nu \) is the outer unit normal vector of \( \partial \Omega \), \( \alpha_{k-1}(x) \geq 0 \) and \( \alpha_l(x) > 0 \) \( (0 \leq l \leq k - 2) \) in \( \Omega \) are \( C^2 \) functions, and \( \varphi \in C^3(\partial \Omega) \). Then there exists a unique constant \( c \), such that the classical Neumann problem

\[
\begin{align*}
\sigma_k(D^2u) &= \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u), \quad \text{in} \quad \Omega, \\
u \cdot u &= c + \varphi(x), \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

has solutions \( u \in C^{3,\alpha}(\overline{\Omega}) \), which are unique up to a constant.

**Remark 1.** For the classical Neumann problem of mixed Hessian equations \( (2) \), it is easy to know that a solution plus any constant is still a solution. Thus we cannot obtain a uniform bound for the solutions of \( (2) \) and cannot use the method of continuity directly to get the existence. As in Lions et al.\(^{11}\) (see also Qiu and Xia),\(^{16}\) we consider the \( k \)-admissible solution \( u^\varepsilon \) of the approximation equation

\[
\begin{align*}
\sigma_k(D^2u) &= \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u), \quad \text{in} \quad \Omega, \\
u \cdot u &= -\varepsilon u + \varphi(x), \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

for any small \( \varepsilon > 0 \). We need to establish a priori estimates of \( u^\varepsilon \) independent of \( \varepsilon \), and then we can obtain a solution of \( (2) \) by letting \( \varepsilon \to 0 \) and a perturbation argument. The uniqueness holds from the maximum principle and Hopf lemma.

The rest of this paper is organized as follows. In Section 2, we give some definitions and important lemmas. In Section 3, we prove global \( C^2 \) estimates of \( (3) \). In Section 4, we give the proof for the existence, that is, Theorem 1.

### 2 | Preliminaries

In this section, we give some basic properties of elementary symmetric functions, which could be found in Ref. 23 and establish some key lemmas.

#### 2.1 | Basic properties of elementary symmetric functions

First, we denote by \( \sigma_m(\lambda|i) \) the symmetric function with \( \lambda_1 = 0 \) and \( \sigma_m(\lambda|i) \) the symmetric function with \( \lambda_1 = \lambda_j = 0 \).
Proposition 1. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $m = 1, \ldots, n$, then

$$
\sigma_m(\lambda) = \sigma_m(\lambda|i) + \lambda_i \sigma_{m-1}(\lambda|i), \quad \forall \ 1 \leq i \leq n,
$$

$$
\sum_i \lambda_i \sigma_{m-1}(\lambda|i) = m \sigma_m(\lambda),
$$

$$
\sum_i \sigma_m(\lambda|i) = (n - m) \sigma_m(\lambda).
$$

We also denote by $\sigma_m(W|i)$ the symmetric function with $W$ deleting the $i$-row and $i$-column and $\sigma_m(W|i\ j)$ the symmetric function with $W$ deleting the $i, j$-rows and $i, j$-columns. Then we have the following identities.

Proposition 2. Suppose $W = (W_{ij})$ is diagonal, and $m$ is a positive integer, then

$$
\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} 
\sigma_{m-1}(W|i), & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
$$

Recall that the Gårding’s cone is defined as

$$
\Gamma_m = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq m \}. 
$$

Proposition 3. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_m$ and $m \in \{1, 2, \ldots, n\}$. Suppose that

$$
\lambda_1 \geq \cdots \geq \lambda_m \geq \cdots \geq \lambda_n,
$$

then we have

$$
\sigma_{m-1}(\lambda|n) \geq \sigma_{m-1}(\lambda|n-1) \geq \cdots \geq \sigma_{m-1}(\lambda|m) \geq \cdots \geq \sigma_{m-1}(\lambda|1) > 0; 
$$

$$
\lambda_1 \geq \cdots \geq \lambda_m > 0, \quad \sigma_m(\lambda) \leq C_n^m \lambda_1 \cdots \lambda_m;
$$

$$
\lambda_1 \sigma_{m-1}(\lambda|1) \geq \frac{m}{n} \sigma_m(\lambda),
$$

$$
\sigma_{m-1}(\lambda|m) \geq c(n, m) \sigma_{m-1}(\lambda),
$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

Proof. All the properties are well known. For example, see Ref. 24 or Ref. 23 for a proof of (5), Ref. 25 for (6), Ref. 26 or Ref. 27 for (7), and Ref. 28 for (8).

The generalized Newton–MacLaurin inequality is as follows, which will be used all the time.
**Proposition 4.** For \( \lambda \in \Gamma_m \) and \( m > l \geq 0, r > s \geq 0, m \geq r, l \geq s \), we have

\[
\left[ \frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{1}{m-l}} \leq \left[ \frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{1}{r-s}}.
\]

**Proof.** See Ref. 29. ■

### 2.2 Key lemmas

In the establishment of the a priori estimates, the following inequalities and properties play an important role.

For the convenience of notations, we will denote

\[
G_k(D^2 u) := \frac{\sigma_k(D^2 u)}{\sigma_{k-1}(D^2 u)}, \quad G_l(D^2 u) := -\frac{\sigma_l(D^2 u)}{\sigma_{k-1}(D^2 u)}, \quad 0 \leq l \leq k - 2,
\]

\[
G(D^2 u, x) := G_k(D^2 u) + \sum_{l=0}^{k-2} \alpha_l(x) G_l(D^2 u),
\]

and

\[
G^{ij} := \frac{\partial G}{\partial u_{ij}}, \quad 1 \leq i, j \leq n.
\]

**Lemma 1.** If \( u \) is a \( C^2 \) function with \( \lambda(D^2 u) \in \Gamma_k \), and \( \alpha_l(x) (0 \leq l \leq k - 2) \) are positive, then the operator \( G \) is elliptic and concave.

**Proof.** The lemma holds for \( \lambda(D^2 u) \in \Gamma_{k-1} \) (see the proof in Ref. 22). ■

**Lemma 2.** If \( u \) is a \( k \)-admissible solution of (1), \( \alpha_{k-1}(x) \geq 0 \) and \( \alpha_l(x) > 0 \) \( (0 \leq l \leq k - 2) \), then

\[
0 < \frac{\sigma_l(D^2 u)}{\sigma_{k-1}(D^2 u)} \leq C(n, k, \inf \alpha_l), \quad 0 \leq l \leq k - 2;
\]

\[
0 < \frac{\sigma_k(D^2 u)}{\sigma_{k-1}(D^2 u)} \leq C(n, k, \sum_{l=0}^{k-1} \sup \alpha_l).
\]

**Proof.** The left-hand sides of (12) and (13) are easy to prove. In the following, we prove the right-hand sides.

First, if \( \frac{\sigma_k}{\sigma_{k-1}} \leq 1 \), then we get from Equation (1)

\[
\alpha_l \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_k}{\sigma_{k-1}} \leq 1, \quad 0 \leq l \leq k - 2.
\]
Second, if $\frac{\sigma_k}{\sigma_{k-1}} > 1$, that is $\frac{\sigma_{k-1}}{\sigma_k} < 1$, we can get for $0 \leq l \leq k - 2$ by the Newton–MacLaurin inequality,

$$\frac{\sigma_l}{\sigma_{k-1}} \leq \frac{(C_n^k)^{k-1-l}C_n^l}{(C_n^{k-1})^{k-1-l}} \leq C(n, k),$$

and

$$\frac{\sigma_k}{\sigma_{k-1}} = \sum_{l=0}^{k-1} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \leq C(n, k) \sum_{l=0}^{k-1} \sup_{\lambda} \alpha_l.$$

Lemma 3. If $u$ is a $k$-admissible solution of (1), $\alpha_{k-1}(x) \geq 0$ and $\alpha_l(x) > 0$ ($0 \leq l \leq k - 2$), then

$$\frac{n - k + 1}{k} \leq \sum G_{ii} \leq n - k - 1;$$

$$0 < \sum G_{ij} u_{ij} \leq C \left( n, k, \sum_{l=0}^{k-1} \sup_{\lambda} \alpha_l \right).$$

Proof. By direct computations, we can get

$$\sum G_{ii} \geq \sum \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) = \sum \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2}$$

$$= \frac{(n - k + 1)\sigma_{k-1}^2 - (n - k + 2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}$$

$$\geq \frac{n - k + 1}{k},$$

and

$$\sum G_{ii} = \sum \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) - \sum_{l=0}^{k-2} \alpha_l \sum \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_l}{\sigma_{k-1}} \right)$$

$$= \sum \frac{\sigma_{k-1}(\lambda|i)\sigma_{k-1} - \sigma_k \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2} - \sum_{l=0}^{k-2} \alpha_l \sum \frac{\sigma_{l-1}(\lambda|i)\sigma_{k-1} - \sigma_l \sigma_{k-2}(\lambda|i)}{\sigma_{k-1}^2}$$

$$= \frac{(n - k + 1)\sigma_{k-1}^2 - (n - k + 2)\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2}$$

$$+ \sum_{l=0}^{k-2} \alpha_l \frac{(n - k + 2)\sigma_l \sigma_{k-2} - (n - l + 1)\sigma_{l-1} \sigma_{k-1}}{\sigma_{k-1}^2}.$$
\[
\leq (n - k + 1) - \left( n - k + 2 \right) \frac{\sigma_{k-2}}{\sigma_{k-1}} - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}}
\]

\[
\leq n - k + 1,
\]

hence (14) holds. Also, we can get

\[
\sum G'u_{ij} = \sum \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_{k-1}} \lambda_i - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \lambda_i \right)
\]

\[
= \frac{\sigma_k}{\sigma_{k-1}} + \sum_{l=0}^{k-2} (k - l) \alpha_l \frac{\sigma_l}{\sigma_{k-1}},
\]

hence (15) holds.

The following lemmas play an important role in the proof of a priori estimates. The idea of the proof for these lemmas comes from Ref. 17.

**Lemma 4.** Suppose \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k, k \geq 2, \) and \( \lambda_1 < 0. \) Then we have

\[
\frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k(n-k+2)} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i},
\]

where \( G(\lambda) : = \frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)} - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)} \) with positive \( \alpha_l. \)

**Proof.** The lemma holds from Lemma 2.5 in Ref. 17 and the following fact:

\[
\frac{\partial G}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) + \sum_{l=0}^{k-2} \alpha_l \left( \frac{\sigma_{k-1}}{\sigma_l} \right)^2 \frac{\partial}{\partial \lambda_i}. \]

**Lemma 5.** Suppose \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k, k \geq 2, \) and \( \lambda_2 \geq \cdots \geq \lambda_n. \) If \( \lambda_1 > 0, \lambda_n < 0, \lambda_1 \geq \delta \lambda_2, \) and \( -\lambda_n \geq \varepsilon \lambda_1 \) for small positive constants \( \delta \) and \( \varepsilon, \) then we have

\[
\frac{\partial G}{\partial \lambda_1} \geq c_2 \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i},
\]

where \( c_2 = \frac{n}{k(n-k+2)^2} \) with \( c_1 = \min \left\{ \frac{\varepsilon^2 \delta^2}{2(n-2)(n-1)}, \frac{\varepsilon^2 \delta}{4(n-1)} \right\}. \)
**Proof.** The lemma holds from Lemma 2.7 in Ref. 17 and the following fact:

\[
\frac{\partial G}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_k}{\sigma_{k-1}} \right) + \sum_{l=0}^{k-2} \alpha_l \left( \frac{\sigma_{k-1}}{\sigma_l} \right)^2 \frac{\partial}{\partial \lambda_i} \left( \frac{\sigma_{k-1}}{\sigma_l} \right). \tag{22}
\]

\[\blacksquare\]

3  |  A PRIORI ESTIMATES OF THE APPROXIMATION EQUATION \( (3) \)

In this section, we prove the \( C^2 \) a priori estimates of \( k \)-admissible solutions of the approximation equation \( (3) \), including the \( C^0 \) estimate, the global gradient estimate, and the global second-order derivatives estimate.

3.1  |  \( C^0 \) estimate

The \( C^0 \) estimate is easy. For completeness, we produce a proof here following the idea of Lions et al.\(^{11}\)

**Theorem 2.** Suppose \( \Omega \subset \mathbb{R}^n \) is a \( C^1 \)-bounded domain, \( \alpha_{k-1}(x) \geq 0 \) and \( \alpha_l(x) > 0 \) \((0 \leq l \leq k-2)\) in \( \bar{\Omega} \) are \( C^0 \) functions and \( \varphi \in C^0(\partial \Omega) \), and \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) is the \( k \)-admissible solution of Equation \( (3) \) with \( \varepsilon \in (0, 1) \), then we have

\[
\sup_{\Omega} |\varepsilon u| \leq M_0, \tag{23}
\]

where \( M_0 \) depends on \( n, k, \text{diam}(\Omega), \max_{\partial \Omega} |\varphi| \) and \( \sum_{l=0}^{k-1} \sup_{\Omega} \alpha_l \).

**Proof.** First, since \( u \) is subharmonic, the maximum of \( u \) is attained at some boundary point \( x_0 \in \partial \Omega \). Then we can get

\[
0 \leq u(x_0) = -\varepsilon u(x_0) + \varphi(x_0). \tag{24}
\]

Hence

\[
\max_{\Omega} (\varepsilon u) = \varepsilon u(x_0) \leq \varphi(x_0) \leq \max_{\partial \Omega} |\varphi|. \tag{25}
\]

For a fixed point \( x_1 \in \Omega \), and a positive constant \( A \) large enough, we have

\[
G(D^2(u)(x_1 - x_1^2), x) = 2A \frac{C_n^k}{C_{n-1}^k} - \sum_{l=0}^{k-2} \alpha_l (2A)^{-(k-1-l)} \frac{C_n^l}{C_{n-1}^k} \geq \sup_{\Omega} \alpha_{k-1} \geq \alpha_{k-1}(x) = G(D^2u, x). \tag{26}
\]
By the comparison principle, we know \( u - A|x - x_1|^2 \) attains its minimum at some boundary point \( x_2 \in \partial \Omega \). Then

\[
0 \geq (u - A|x - x_1|^2)_\nu(x_2) = u_\nu(x_2) - 2A(x_2 - x_1) \cdot \nu
\]

\[
= -\varepsilon u(x_2) + \varphi(x_2) - 2A(x_2 - x_1) \cdot \nu
\]

\[
\geq -\varepsilon u(x_2) - \max_{\partial \Omega} |\varphi| - 2\text{Adiam}(\Omega).
\]  

(27)

Hence

\[
\min_{\Omega}(\varepsilon u) \geq \varepsilon \min_{\Omega}(u - A|x - x_1|^2) \geq \varepsilon u(x_2) - A|x_2 - x_1|^2
\]

\[
\geq -\max_{\partial \Omega} |\varphi| - 2\text{Adiam}(\Omega) - \text{Adiam}(\Omega)^2.
\]  

(28)

\[3.2\quad \text{Global gradient estimate}\]

In this subsection, we prove the global gradient estimate (independent of \( \varepsilon \)), using a similar argument of the complex Monge–Ampère equation in Li.\(^2\)

**Theorem 3.** Suppose \( \Omega \subset \mathbb{R}^n \) is a \( C^3 \) strictly convex domain, \( \alpha_{k-1}(x) \geq 0 \) and \( \alpha_l(x) > 0 \) \((0 \leq l \leq k - 2)\) in \( \overline{\Omega} \) are \( C^1 \) functions and \( \varphi \in C^2(\partial \Omega) \), and \( u \in C^3(\Omega) \cap C^2(\overline{\Omega}) \) is the \( k \)-admissible solution of Equation (3) with \( \varepsilon > 0 \) sufficiently small, then we have

\[
\sup_{\Omega} |Du| \leq M_1,
\]

(29)

and

\[
\sup_{\Omega} \left| u - \frac{1}{|\Omega|} \int_{\Omega} u \right| \leq M_0,
\]

(30)

where \( M_1 \) and \( M_0 \) depend on \( n, k, \Omega, |\varphi|_{C^2}, \inf \alpha_l \) \((0 \leq l \leq k - 2)\), and \( |\alpha_l|_{C^1} \) \((0 \leq l \leq k - 1)\).

**Proof.** We just need to prove (29), and then (30) holds directly from (29).

To prove (29), it suffices to prove

\[
D_\xi u(x) \leq M_1, \quad \forall (x, \xi) \in \overline{\Omega} \times S^{n-1}.
\]

(31)

For any \((x, \xi) \in \overline{\Omega} \times S^{n-1}\), denote

\[
W(x, \xi) = D_\xi u(x) - \langle \nu, \xi \rangle (-\varepsilon u + \varphi(x)) + \varepsilon^2 u^2 + K|x|^2,
\]

(32)
where $K$ is a large constant to be determined later, and $\nu$ is a $C^2(\bar{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$.

Assume $W$ achieves its maximum at $(x_0, \xi_0) \in \bar{\Omega} \times S^{n-1}$. It is easy to see $D_{\xi_0}^t u(x_0) > 0$, otherwise we will complete the proof. We claim $x_0 \in \partial\Omega$. If $x_0 \in \Omega$, we will get a contradiction in the following.

First, we rotate the coordinates such that $D^2 u(x_0)$ is diagonal. It is easy to see $\{G^{ij}\}$ is diagonal. For fixed $\xi = \xi_0$, $W(x, \xi_0)$ achieves its maximum at the same point $x_0 \in \Omega$ and we can easily get at $x_0$

$$0 \geq G^{ii} \partial_i W = G^{ii} \left[ u_{ii} \xi_0 - \langle \nu, \xi_0 \rangle (-\varepsilon u + \varphi) - \langle \nu, \xi_0 \rangle (-\varepsilon u_{ii} + \varphi_i) 
- 2 \langle \nu, \xi_0 \rangle (-\varepsilon u_i + \varphi_i) + 2\varepsilon^2 u_i^2 + 2\varepsilon^2 u u_{ii} + K \right]$$

$$= D_{\xi_0} \alpha_{k-1} + \sum_{l=0}^{k-2} D_{\xi_0} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} + G^{ii} \left[ 2\varepsilon^2 u_i^2 + 2\langle \nu, \xi_0 \rangle \varepsilon u_i \right]$$

$$+ G^{ii} \left[ K - \langle \nu, \xi_0 \rangle \langle -\varepsilon u + \varphi \rangle - \langle \nu, \xi_0 \rangle \varphi_i - 2 \langle \nu, \xi_0 \rangle \varphi_i \right]$$

$$\geq - |D\alpha_{k-1}| - \sum_{l=0}^{k-2} |D\alpha_l| C(n, k, \inf \alpha_l) - (n - k + 1) |D\langle \nu, \xi_0 \rangle|^2$$

$$- C(n, k, \sup \alpha_l) [1 + 2M_0]$$

$$+ \frac{n - k + 1}{k} \left[ K - |D^2 \langle \nu, \xi_0 \rangle| (M_0 + |\varphi|) - |D^2 \varphi| - 2 |D \langle \nu, \xi_0 \rangle| |D\varphi| \right]$$

$$> 0,$$

where $K$ is large enough, depending only on $n, k, \Omega, M_0, |\varphi|_{C^2}$, and $\alpha_l$. This is a contradiction.

Thus $x_0 \in \partial\Omega$. Then we continue our proof in the following three cases.

(a) If $\xi_0$ is normal at $x_0 \in \partial\Omega$, then

$$W(x_0, \xi_0) = \varepsilon^2 u^2 + K|x_0|^2 \leq C.$$ 

Then we can easily obtain (31).

(b) If $\xi_0$ is nontangential at $x_0 \in \partial\Omega$, then we can write $\xi_0 = \alpha \tau + \beta \nu$, where $\tau \in S^{n-1}$ is tangent at $x_0$, that is, $\langle \tau, \nu \rangle = 0, \alpha = \langle \xi_0, \tau \rangle > 0, \beta = \langle \xi_0, \nu \rangle < 1$, and $\alpha^2 + \beta^2 = 1$. Then we have 

$$W(x_0, \xi_0) = \alpha D_t u + \varepsilon^2 u^2 + K|x_0|^2$$

$$\leq \alpha W(x_0, \xi_0) + (1 - \alpha) (\varepsilon^2 u^2 + K|x_0|^2),$$

so

$$W(x_0, \xi_0) \leq \varepsilon^2 u^2 + K|x_0|^2 \leq C.$$ 

Then we can easily get (31).
(c) If $\xi_0$ is tangential at $x_0 \in \partial \Omega$, we may assume that the outer normal direction of $\Omega$ at $x_0$ is $(0, \ldots, 0, 1)$. By a rotation, we assume that $\xi_0 = (1, \ldots, 0) = e_1$. Then we have

$$0 \leq D_\nu W(x_0, \xi_0) = D_\nu D_1 u - D_\nu (\nu, \xi_0)(-\varepsilon u + \varphi) + 2u \cdot D_\nu u + KD_\nu |x_0|^2$$
$$\leq D_\nu D_1 u + C_1$$
$$= D_1 D_\nu u - D_1 \nu_k D_k u + C_1. \quad (35)$$

By the boundary condition, we know

$$D_1 D_\nu u = D_1 (-\varepsilon u + \varphi) \leq D_1 \varphi. \quad (36)$$

Following the argument of Ref. 30, we can get

$$-D_1 \nu_k D_k u \leq -\kappa_{\text{min}} W(x_0, \xi_0) + C_2, \quad (37)$$

where $\kappa_{\text{min}}$ is the minimum principal curvature of $\partial \Omega$. Thus

$$W(x_0, \xi_0) \leq \frac{C_1 + |D\varphi| + C_2}{\kappa_{\text{min}}}. \quad (38)$$

Then we can conclude (31). ■

### 3.3 Global second derivatives estimates

In this subsection, we prove the global second derivatives estimate (independent of $\varepsilon$), following the ideas of Lions et al.\textsuperscript{11}, Ma and Qiu,\textsuperscript{15} and Chen and Zhang.\textsuperscript{17}

**Theorem 4.** Suppose $\Omega \subset \mathbb{R}^n$ is a $C^4$ strictly convex domain, $\alpha_{k-1}(x) \geq 0$ and $\alpha_l(x) > 0$ ($0 \leq l \leq k - 2$) in $\overline{\Omega}$ are $C^2$ functions and $\varphi \in C^3(\partial \Omega)$, and $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is the $k$-admissible solution of Equation (3) with $\varepsilon > 0$ sufficiently small, then we have

$$\sup_{\Omega} |D^2 u| \leq M_2, \quad (39)$$

where $M_2$ depends on $n, k, \Omega, |\varphi|_{C^3}, \inf \alpha_l (0 \leq l \leq k - 2)$, and $|\alpha_l|_{C^2} (0 \leq l \leq k - 1)$.

**Proof.** In the following, we divide the proof of Theorem 4 into three steps. In step one, we reduce global second derivatives to double normal second derivatives on the boundary, then we prove the lower estimate of double normal second derivatives on the boundary in step two, and, at last, we prove the upper estimate of double normal second derivatives on the boundary.

**Step 1.** Prove $\sup_{\Omega} |D^2 u| \leq C(1 + \max_{\partial \Omega} |u_{\nu\nu}|)$.

Following the idea of Lions et al.\textsuperscript{11}, we assume $0 \in \Omega$ and consider the auxiliary function

$$v(x, \xi) = u_{\xi \xi} - v'(x, \xi) + K|x|^2 + |Du|^2, \quad (40)$$
where $v'(x, \xi) = 2(\xi \cdot \nu)\xi'(D\phi - \varepsilon Du - u_mDv_m) = a^m u_m + b, \nu = (\nu^1, ..., \nu^n) \in S^{n-1}$ is a $C^3(\bar{\Omega})$ extension of the outer unit normal vector field on $\partial \Omega$, $\xi' = \xi - (\xi \cdot \nu)\nu$, $a^m = 2(\xi \cdot \nu)(-\varepsilon \xi' = \xi - (\xi \cdot \nu)\nu, a^m = 2(\xi \cdot \nu)(-\varepsilon \xi(m - \xi''D_1v_m)$, $b = 2(\xi \cdot \nu)\xi''m \varphi_m$, and $K > 0$ is to be determined later.

For any $x \in \Omega$, we rotate the coordinates such that $D^2u(x)$ is diagonal, and then $\{G^{ij}\}$ is diagonal. For any fixed $\xi \in S^{n-1}$, we have

$$G^{ii}v_{ii} = G^{ij}[u_{ii}\xi - D_{ii}a^m u_m - 2D_1a^m u_{mi} - a^m u_{iim} - D_{ii}b + 2K + 2u^2_m + 2u_m u_{iim}]$$

$$= (\alpha_{k-1})^{ii}_\xi - 2 \sum_{l=0}^{k-2} (\alpha_l)G^{ii}_l u_{ii} - 2 \alpha \sum_{l=0}^{k-2} (\alpha_l)G^{ll}_l - G^{i,j,r}u_{ij} u_{rs}\xi$$

$$+ G^{ii}[2K - D_{ii}a^m u_m - D_{ii}b] + G^{ii}[2u^2_m - 2D_{ii}a^i u_{ii}]$$

$$+ (-a^m + 2u_m)(\alpha_{k-1})m - \sum_{l=0}^{k-2} (\alpha_l)G^{ll}_l$$

$$\geq G^{ii}[2K - C_2] - C_1 - 2 \sum_{l=0}^{k-2} (\alpha_l)G^{ii}_l u_{ii} - G^{i,j,r}u_{ij} u_{rs}\xi$$

$$\geq \frac{n - k + 1}{k}[2K - C_2] - C_1 - C_3 > 0,$$  \hspace{1cm} (41)

where $K$ is large enough, and we used the fact

$$- 2 \sum_{l=0}^{k-2} (\alpha_l)G^{ii}_l u_{ii} - G^{i,j,r}u_{ij} u_{rs}\xi \geq - 2 \sum_{l=0}^{k-2} (\alpha_l)G^{ii}_l u_{ii} - \sum_{l=0}^{k-2} \alpha_l G^{i,j,r}u_{ij} u_{rs}\xi$$

$$= - 2 \sum_{l=0}^{k-2} (\alpha_l)G^{ii}_l u_{ii}$$

$$- \sum_{l=0}^{k-2} \alpha_l \left[ \frac{k - 1 - l}{k - 1 - l} \frac{\partial^2 (\sigma_{k-1})}{\partial u_{ij}\partial u_{rs}} - \frac{k - l}{k - 1 - l} G^{i,j,r} G^{ll}_l \right] u_{ij} u_{rs}\xi$$

$$\geq \sum_{l=0}^{k-2} \frac{k - 1 - l}{k - l} \frac{(\alpha_l)^2}{\alpha_l} G^{ii}_l$$

$$\geq - C_3. $$

Thus $v(x, \xi)$ attains its maximum on $\partial \Omega$. We can assume $\max_{\Omega \times S^{n-1}} v(x, \xi)$, which attains at $(x_0, \xi_0) \in \partial \Omega \times S^{n-1}$.

Then we continue our proof in the following two cases following the idea of Ref. 30.
Case a: $\xi_0$ is tangential to $\partial \Omega$ at $x_0$.
By the Hopf lemma, we have
\[
0 \leq v_\nu = u_{\xi_0 \xi_0} - D_\nu a^m u_m - a^m u_{m\nu} - b_\nu + 2K(\cdot \cdot \cdot) + 2u_m u_{m\nu}
\leq u_{\xi_0 \xi_0} + (2u_m - a^m)u_{m\nu} + C. \tag{42}
\]
Following the argument in Ref. 15, we can get
\[
u_{\xi_0 \xi_0} \leq -\kappa_{\min} u_{\xi_0 \xi_0} + C(1 + |u_{\nu\nu}|), \tag{43}
\]
and
\[
|u_{m\nu}| \leq C, \quad m = 1, \ldots, n. \tag{44}
\]
Therefore, we have
\[
u_{\xi_0 \xi_0} \leq C(1 + |u_{\nu\nu}|). \tag{45}
\]
Case b: $\xi_0$ is nontangential to $\partial \Omega$ at $x_0$.
We write $\xi = \hat{\alpha} \tau + \hat{\beta} \nu$, where $\hat{\alpha} = \xi \cdot \tau$, $\tau \cdot \nu = 0$, $|\tau| = 1$, $\hat{\beta} = \xi \cdot \nu \neq 0$, and $\hat{\alpha}^2 + \hat{\beta}^2 = 1$. Then we have
\[
u(x_0, \xi) = \hat{\alpha}^2 \nu(x_0, \tau) + \hat{\beta}^2 \nu(x_0, \nu) \leq \hat{\alpha}^2 \nu(x_0, \xi) + \hat{\beta}^2 \nu(x_0, \nu), \tag{46}
\]
\[
u(x_0, \xi) \leq \nu(x_0, \nu) \leq C(1 + \max_{\partial \Omega} |u_{\nu\nu}|). \tag{47}
\]
\textbf{Step 2. Prove} $\min_{\partial \Omega} u_{\nu\nu} \geq -C$.
We assume $\min_{\partial \Omega} u_{\nu\nu} < 0$. Also if $-\min_{\partial \Omega} u_{\nu\nu} < \max_{\partial \Omega} u_{\nu\nu}$, that is $\max_{\partial \Omega} |u_{\nu\nu}| = \max_{\partial \Omega} u_{\nu\nu}$, we deal this case in the next step. Thus we assume $-\min_{\partial \Omega} u_{\nu\nu} \geq \max_{\partial \Omega} u_{\nu\nu}$, that is $\max_{\partial \Omega} |u_{\nu\nu}| = -\min_{\partial \Omega} u_{\nu\nu}$. Denote $M := -\min_{\partial \Omega} u_{\nu\nu} > 0$ and $\bar{x}_0 \in \partial \Omega$ such that $\min_{\partial \Omega} u_{\nu\nu} = u_{\nu\nu}(\bar{x}_0)$.
We consider the following test function in $\Omega_\mu = \{x \in \Omega : 0 < d(x) < \mu\}$ ($d$ is the distance function of $\Omega$, and $\mu$ is a small universal constant)
\[
\nu(x) = (1 + \beta d)[Du \cdot (-Dd) + \epsilon u - \varphi(x)] - (A + \frac{1}{2} M)d, \tag{48}
\]
where $\beta$ and $A$ are positive constants to be determined later.
On $\partial \Omega$, $P(x) = 0$, and on $\partial \Omega_\mu \setminus \partial \Omega$, we have $d = \mu$ and
\[
P(x) \leq (1 + \beta \mu)[|Du| + |\varepsilon u| + |\varphi(x)|] - A\mu \leq 0,
\]
because we take $A$ big enough. Thus on $\partial \Omega_\mu$, we have $P \leq 0$. In the following, we want to prove $P$ attains its maximum only on $\partial \Omega$. Then we can get for any $x_0 \in \partial \Omega$
\[
0 \leq P_y(x_0) = [u_{\varphi y}(x_0) - \sum \limits_m u_m d_{my} + \varepsilon u_y - \varphi_y] + \left( A + \frac{1}{2}M \right)
\leq u_{\varphi y}(x_0) + |Du||D^2d| + \varepsilon|D\varphi| + A + \frac{1}{2}M,
\]
which finishes the proof of Step 2.

To prove $P$ attains its maximum only on $\partial \Omega$, we assume $P$ attains its maximum at some point $\bar{x}_0 \in \Omega_\mu$ by contradiction. Rotating the coordinates, we can assume $D^2u(\bar{x}_0)$ is diagonal, and then so is $\{G^{ij}\}$. In the following, all the calculations are at $\bar{x}_0$.

First, we have
\[
0 = P_i(\bar{x}_0) = -(1 + \beta d)u_{ii}d_i - \left( A + \frac{1}{2}M \right) d_i + O(1),
\]
and
\[
0 \geq G^{ii}P_{ii}(\bar{x}_0)
\]
\[
= G^{ii} \left[ -2\beta u_{ii}d_i^2 + (1 + \beta d)[-u_{mii}d_m + 2u_{ii}d_{ii}] + \varepsilon u_{ii} \right] - \left( A + \frac{1}{2}M \right) d_{ii} + O(1)
\]
\[
\geq -2\beta G^{ii}u_{ii}d_i^2 - 2(1 + \beta d)G^{ii}u_{ii}d_{ii} + \left( A + \frac{1}{2}M \right) c_0 - C,
\]
where we used the fact
\[
-G^{ii}d_{ii} \geq \frac{\partial \left( \frac{\sigma_k}{\sigma_{k-1}} \right)}{\partial u_{ii}}(-d_{ii})
\]
\[
\geq c(n, k)\kappa_{min} \sum \limits_{i \neq m_0} \frac{\partial \left( \frac{\sigma_k}{\sigma_{k-1}} \right)}{\partial u_{ii}} \geq c_0(n, k, \kappa_{min}) > 0,
\]
for some $1 \leq m_0 \leq n$.

Denote $B = \{i : \beta d_i^2 < \frac{1}{n}, 1 \leq i \leq n\}$ and $G = \{i : \beta d_i^2 \geq \frac{1}{n}, 1 \leq i \leq n\}$. We choose $\beta \geq \frac{1}{\mu} > 1$, thus
\[
d_i^2 < \frac{1}{n} = \frac{1}{n}|Dd|^2, \quad i \in B.
\]
It holds $\sum_{i \in B} d_i^2 < 1 = |Dd|^2$, and $G$ is not empty. Hence for any $i \in G$, it holds

$$d_i^2 \geq \frac{1}{n\beta}.$$  

From (50), we have

$$u_{ii} = -\frac{1}{1 + \beta d_i} \left( A + \frac{1}{2} M \right) + \frac{O(1)}{(1 + \beta d) d_i}. \quad (54)$$

Thus when $A$ is large enough, we can get

$$u_{ii} \leq -\frac{A + M}{5}, \quad \forall \quad i \in G. \quad (55)$$

Also there is an $i_0 \in G$ such that

$$d_{i_0}^2 \geq \frac{1}{n} |Dd|^2 = \frac{1}{n}. \quad (56)$$

From (51), we have

\[
0 \geq G^i P_{ii}(\bar{x}_0) \\
\geq -2\beta \sum_{i \in G} G^i u_{ii} d_i^2 - 2\beta \sum_{i \in B} G^i u_{ii} d_i^2 - 2(1 + \beta d) \sum_{u_{ij} < 0} G^i u_{ij} d_{ij} \\
+ \left( A + \frac{1}{2} M \right) c_0 - C \\
\geq -\frac{2\beta}{n} G^{i_0} u_{i_0} - C + \left( \frac{2}{n} + 4\chi_{\text{max}} \right) \sum_{u_{ij} < 0} G^i u_{ii}, \quad (57)
\]

where we used the facts

$$-2\beta \sum_{i \in G} G^i u_{ii} d_i^2 \geq -2\beta G^{i_0} u_{i_0} d_{i_0}^2 \geq -\frac{2\beta}{n} G^{i_0} u_{i_0}, \quad (58)$$

and

$$-2\beta \sum_{i \in B} G^i u_{ii} d_i^2 \geq -\frac{2\beta}{n} \sum_{i \in B, u_{ii} > 0} G^i u_{ii} d_i^2 \geq -\frac{2}{n} \sum_{i \in B, u_{ii} > 0} G^i u_{ii}$$

\[
\geq -\frac{2}{n} \sum_{u_{ij} > 0} G^i u_{ii} = -\frac{2}{n} \left[ \sum_{u_{ij} < 0} G^i u_{ii} - \sum_{u_{ij} > 0} G^i u_{ii} \right] \\
\geq \frac{2}{n} \sum_{u_{ij} < 0} G^i u_{ii}. \quad (59)
\]
Therefore, we have

\[
0 \geq G^{ii}P_i(x_0)
\]

\[
\geq \frac{2\beta}{n} \left( \frac{A + M}{5} \right) c(n, k) \sum_{i=1}^{n} G^{ii} - C - \left( \frac{2}{n} + 4\kappa_{\text{max}} \right) \sum_{u_i < 0} G^{ii} |D^2u|
\]

\[
\geq \frac{2\beta}{n} \left( \frac{A + M}{5} \right) c(n, k) \frac{n-k+1}{k} - C - \left( \frac{2}{n} + 4\kappa_{\text{max}} \right) (n-k+1)(1+M)
\]

\[
> 0,
\]

by taking $\beta$ big enough. This is a contradiction. Thus $P$ attains its maximum only on $\partial \Omega$.

**Step 3. Prove** $\max_{\partial \Omega} u_{yy} \leq C$.

Similar with Step 2, we can assume $\max_{\partial \Omega} u_{yy} > 0$, and $-\min_{\partial \Omega} u_{yy} \leq \max_{\partial \Omega} u_{yy}$, that is $\max_{\partial \Omega} |u_{yy}| = \max_{\partial \Omega} u_{yy}$. Denote $M := \max_{\partial \Omega} u_{yy} > 0$ and $x_0 \in \partial \Omega$ such that $\max_{\partial \Omega} u_{yy} = u_{yy}(x_0)$.

We consider the following test function in $\Omega_\mu = \{x \in \Omega : 0 < d(x) < \mu\}$ ($d$ is the distance function of $\Omega$, and $\mu$ is a small universal constant)

\[
\tilde{P}(x) = (1 + \beta d)[D u \cdot (-D d) + \epsilon u - \varphi(x)] + \left( A + \frac{1}{2}M \right) d,
\]

(60)

where $\beta$ and $A$ are positive constants to be determined later.

On $\partial \Omega$, $\tilde{P}(x) = 0$, and on $\partial \Omega_\mu \setminus \partial \Omega$, we have $d = \mu$ and

\[
\tilde{P}(x) \geq -(1 + \beta \mu)[|D u| + |\epsilon u| + |\varphi(x)|] + A\mu \geq 0,
\]

because we take $A$ big enough. Thus on $\partial \Omega_\mu$, we have $\tilde{P} \geq 0$. In the following, we want to prove $\tilde{P}$ attains its minimum only on $\partial \Omega$. Then we can get for any $x_0 \in \partial \Omega$

\[
0 \geq \tilde{P}_y(x_0) = [u_{yy}(x_0) - \sum_m u_m d_{my} + \epsilon u_y - \varphi_y] - \left( A + \frac{1}{2}M \right)
\]

\[
\geq u_{yy}(x_0) - |D u||D^2d| - \epsilon |D u| - |D \varphi| - \left( A + \frac{1}{2}M \right),
\]

(61)

which finishes the proof of Step 3.

To prove $\tilde{P}$ attains its minimum only on $\partial \Omega$, we assume $\tilde{P}$ attains its minimum at some point $\tilde{x}_0 \in \Omega_\mu$ by contradiction. Rotating the coordinates, we can assume $D^2u(\tilde{x}_0)$ is diagonal, and then so is $\{G^{ij}\}$. In the following, all the calculations are at $\tilde{x}_0$.

First, we have

\[
0 = \tilde{P}_i(\tilde{x}_0) = -(1 + \beta d)u_i d_i + \left( A + \frac{1}{2}M \right) d_i + O(1),
\]

(62)
and
\[
0 \leq G^{ii} \tilde{P}_{ii}(\tilde{x}_0)
\]
\[
= G^{ii} \left[ -2\beta u_{ii}d_i^2 + (1 + \beta d)(-u_{mi}d_m + 2u_{ii}d_{ii}) + \varepsilon u_{ii} \right] + \left( A + \frac{1}{2} M \right) d_{ii} + O(1)
\]
\[
\leq -2\beta G^{ii} u_{ii}d_i^2 - 2(1 + \beta d)G^{ii} u_{ii}d_{ii} - \left( A + \frac{1}{2} M \right) c_0 + C.
\] (63)

Denote \( B = \{ i : \beta d_i^2 < \frac{1}{n}, 1 \leq i \leq n \} \) and \( G = \{ i : \beta d_i^2 \geq \frac{1}{n}, 1 \leq i \leq n \} \). We choose \( \beta \geq \frac{1}{\mu} > 1 \), so
\[
d_i^2 < \frac{1}{n} = \frac{1}{n} |Dd|^2, \quad i \in B.
\] (64)

It holds \( \sum_{i \in B} d_i^2 < 1 = |Dd|^2 \), and \( G \) is not empty. Hence for any \( i \in G \), it holds
\[
d_i^2 \geq \frac{1}{n\beta},
\] (65)

and from (62), we have
\[
u_{ii} = \frac{1}{1 + \beta d}(A + \frac{1}{2} M) + \frac{O(1)}{(1 + \beta d)d_i}.
\] (66)

Thus when \( A \) is large enough, we can get
\[
\frac{3A}{5} + \frac{2M}{5} \leq \nu_{ii} \leq \frac{6A}{5} + \frac{M}{2}, \quad \forall \quad i \in G.
\] (67)

Also there is an \( i_0 \in G \) such that
\[
d_{i_0}^2 \geq \frac{1}{n} |Dd|^2 = \frac{1}{n}.
\] (68)

From (63), we have
\[
0 \leq G^{ii} \tilde{P}_{ii}(\tilde{x}_0)
\]
\[
\leq -2\beta \sum_{i \in G} G^{ii} u_{ii}d_i^2 - 2\beta \sum_{i \in B} G^{ii} u_{ii}d_i^2 - 2(1 + \beta d) \sum_{u_{ii} > 0} G^{ii} u_{ii}d_{ii}
\]
\[
- \left( A + \frac{1}{2} M \right) c_0 + C
\]
\[
\leq -\frac{2\beta}{n} G^{ii} u_{i_0} + \left( \frac{2}{n} + 4\chi_{\text{max}} \right) \sum_{u_{ii} > 0} G^{ii} u_{ii} - \left( A + \frac{1}{2} M \right) c_0 + C,
\] (69)
where we used the facts

\[-2\beta \sum_{i \in G} G^{ii} u_i d_i^2 \leq -2\beta G^{lo_0} u_{i_0} d_{i_0}^2 \leq -\frac{2\beta}{n} G^{lo_0} u_{i_0}, \tag{70}\]

and

\[-2\beta \sum_{i \in B} G^{ii} u_i d_i^2 \leq -2\beta \sum_{i \in B, u_i < 0} G^{ii} u_i d_i^2 \leq -\frac{2}{n} \sum_{i \in B, u_i < 0} G^{ii} u_i^2 \]

\[\leq -\frac{2}{n} \sum_{i \in B, u_i < 0} G^{ii} u_i = -\frac{2}{n} \left[ \sum_{i \in B, u_i < 0} G^{ii} u_i - \sum_{u_i > 0} G^{ii} u_i \right] \]

\[\leq \frac{2}{n} \sum_{u_i > 0} G^{ii} u_i.\]

In the following, we divide into three cases to prove the result. Without loss of generality, we can assume that \(i_0 = 1 \in G\), and \(u_{22} \geq \cdots \geq u_{nn}\).

Case I: \(u_{nn} > 0\).

In this case, we have

\[0 \leq G^{ii} \bar{P}_{ii}(\bar{x}_0)\]

\[\leq -\frac{2\beta}{n} G^{lo_0} u_{i_0} + \left( \frac{2}{n} + 4\kappa_{\text{max}} \right) \sum_{i \in B, u_i < 0} G^{ii} u_i - \left( A + \frac{1}{2} M \right) c_0 + C\]

\[\leq \left( \frac{2}{n} + 4\kappa_{\text{max}} \right) C \left( n, k, \sum \sup \alpha_i \right) - \left( A + \frac{1}{2} M \right) c_0 + C\]

\[< 0, \tag{71}\]

by taking \(A\) large enough. This is a contradiction.

Case II: \(u_{nn} < 0\) and \(-u_{nn} < \frac{c_0}{10n(4\kappa_{\text{max}} + \frac{2}{n})} u_{11}\).

\[\left( 4\kappa_{\text{max}} + \frac{2}{n} \right) \sum_{u_i > 0} G^{ii} u_i = \left( 4\kappa_{\text{max}} + \frac{2}{n} \right) \left[ \sum_{i=1}^{n} G^{ii} u_i - \sum_{u_i > 0} G^{ii} u_i \right]\]

\[\leq \left( 4\kappa_{\text{max}} + \frac{2}{n} \right) \left[ \sum_{i=1}^{n} G^{ii} u_i - u_{nn} \sum_{i=1}^{n} G^{ii} \right]\]

\[\leq C + \frac{c_0}{10n} u_{11} \sum_{i=1}^{n} G^{ii}\]

\[\leq C + \frac{c_0}{10} \left( \frac{6A}{5} + \frac{M}{2} \right). \tag{72}\]
Hence combining (69) and (72), we have

\[ 0 \leq G^{ii} \bar{P}_{ii} (\bar{x}_0) \]

\[ \leq \left( \frac{2}{n} + 4\kappa_{\max} \right) \sum_{u_{ii} > 0} G^{ii} u_{ii} - \left( A + \frac{1}{2} M \right) c_0 + C \]

\[ \leq \frac{c_0}{10} \left( \frac{6A}{5} + \frac{M}{2} \right) - \left( A + \frac{1}{2} M \right) c_0 + C \]

\[ < 0, \]

(73)

by taking $A$ large enough. This is a contradiction.

Case III: $u_{nn} < 0$ and $-u_{nn} \geq \frac{c_0}{10n(4\kappa_{\max} + \frac{2}{n})} u_{11}$.

We have $u_{11} \geq \frac{3A}{5} + \frac{2M}{5}$ and $u_{22} \leq C(1 + M)$. So $u_{11} \geq \frac{2}{5C} u_{22}$. Let $\delta = \frac{2}{5C}$ and $\varepsilon = \frac{c_0}{10n(4\kappa_{\max} + \frac{2}{n})}$.

by Lemma 5, we have

\[ G^{11} \geq c_2 \sum_{i=1}^{n} G^{ii}. \]

(74)

Hence from (69) and (74), we have

\[ 0 \leq G^{ii} \bar{P}_{ii} (\bar{x}_0) \]

\[ \leq -\frac{2\beta}{n} \frac{c_2}{n} \left( \frac{3A}{5} + \frac{2M}{5} \right) \sum_{i=1}^{n} G^{ii} + \left( 4\kappa_{\max} + \frac{2}{n} \right) C(1 + M) \sum_{i=1}^{n} G^{ii} \]

\[ \leq -\frac{2\beta}{n} \frac{c_2}{n} \left( \frac{3A}{5} + \frac{2M}{5} \right) \sum_{i=1}^{n} G^{ii} \]

\[ < 0, \]

(75)

by taking $\beta$ and $A$ big enough. This is a contradiction.

The proof is finished. ■

4 | EXISTENCE

In this section, we prove Theorem 1.

First, we prove the existence of the $k$-admissible solution of the approximation equation (3) for any small $\varepsilon > 0$.

For the Neumann problem of approximation equation (3), we have established the $C^2$ estimates in Section 3. By the global $C^2$ a priori estimates, we obtain that Equation (3) is uniformly elliptic in $\Omega$. Owing to the concavity of the operator $G$, we can get the global Hölder estimates of second
derivative following the discussions in Ref. 31, that is, we can get

$$\left| u \right|_{C^{2,\alpha}(\Omega)} \leq C,$$  \tag{76}

where $C$ and $\alpha$ depend on $n$, $k$, $\Omega$, $\varepsilon$, $\inf \alpha_l$, $|\alpha_l|_{C^2}$, and $|\varphi|_{C^3}$.

Applying the method of continuity (see Ref. 5, Theorem 17.28), we can show the existence of the classical solution and the solution is unique by the Hopf lemma. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the high-order regularity.

Now, we start to prove Theorem 1.

By the above argument, we know there exists a unique $k$-admissible solution $u^\varepsilon \in C^{3,\alpha}(\Omega)$ to (3) for any $\varepsilon > 0$. Let $v^\varepsilon = u^\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} u^\varepsilon$, and it is easy to know $v^\varepsilon$ satisfies

$$\begin{cases}
\sigma_k(D^2v^\varepsilon) = \sum_{l=0}^{k-1} \alpha_l \sigma_l(D^2v^\varepsilon), & \text{in } \Omega, \\
(v^\varepsilon)_\nu = -\varepsilon v^\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} u^\varepsilon + \varphi(x), & \text{on } \partial\Omega.
\end{cases} \tag{77}$$

By the global gradient estimate (29), it is easy to know $\varepsilon \sup |Du^\varepsilon| \to 0$. Hence there is a constant $c$ and a function $v \in C^2(\Omega)$, such that $-\varepsilon u^\varepsilon \to c$, $-\varepsilon v^\varepsilon \to 0$, $-\frac{1}{|\Omega|} \int_{\Omega} u^\varepsilon \to c$ and $v^\varepsilon \to v$ uniformly in $C^2(\Omega)$ as $\varepsilon \to 0$. It is easy to verify that $v$ is a $k$-admissible solution of

$$\begin{cases}
\sigma_k(D^2v) = \sum_{l=0}^{k-1} \alpha_l \sigma_l(D^2v), & \text{in } \Omega, \\
v_{\nu} = c + \varphi(x), & \text{on } \partial\Omega.
\end{cases} \tag{78}$$

If there is another function $v_1 \in C^2(\Omega)$ and another constant $c_1$ such that

$$\begin{cases}
\sigma_k(D^2v_1) = \sum_{l=0}^{k-1} \alpha_l \sigma_l(D^2v_1), & \text{in } \Omega, \\
(v_1)_{\nu} = c_1 + \varphi(x), & \text{on } \partial\Omega.
\end{cases} \tag{79}$$

Applying the maximum principle and Hopf lemma, we can know $c = c_1$ and $v - v_1$ is a constant. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the high-order regularity.

**ACKNOWLEDGMENTS**

Research of the first author was supported by NSFC No.11771396, and research of the second and the fourth authors was supported by funds from Hubei Provincial Department of Education Key Projects D20171004, D20181003, and the National Natural Science Foundation of China No. 11971157. The authors would like to express sincere gratitude to Prof. Xi-Nan Ma for the constant encouragement in this subject, and the third author would also thank Bin Deng and Xiao-Han Jia for their helpful discussion and encouragement.

**ORCID**

Xinqun Mei https://orcid.org/0000-0003-2092-7834
REFERENCES

1. Schneider R. Convex Bodies: The Brunn–Minkowski Theory. Cambridge, UK: Cambridge University Press; 1993.
2. Harvey R, Lawson B. Calibrated geometries. Acta Math. 1982;148:47-157.
3. Fu JX, Yau ST. A Monge–Ampère type equation motivated by string theory. Commun Anal Geom. 2007; 15:29-76.
4. Fu JX, Yau ST. The theory of superstring with flux on non-Kähler manifolds and the complex Monge–Ampère equation. J Differ Geom. 2008;78:369-428.
5. Gilbarg D, Trudinger N. Elliptic Partial Differential Equations of Second Order. Grundlehren der Mathematischen Wissenschaften, Vol. 224. New York: Springer-Verlag, 1977.
6. Caffarelli L, Nirenberg L, Spruck J. The Dirichlet problem for nonlinear second order elliptic equations, I: Monge–Ampère equations. Commun Pure Appl Math. 1984;37:369-402.
7. Ivochkina N. Solutions of the Dirichlet problem for certain equations of Monge-Ampère type (in Russian). Mat Sb., 1985;128:403-415: English translation in Math. USSR Sb. 56(1987).
8. Caffarelli L, Nirenberg L, Spruck J. The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian. Acta Math. 1985;155:261-301.
9. Trudinger NS. On the Dirichlet problem for Hessian equations. Acta Math. 1995;175:151-164.
10. Lieberman G. Oblique Boundary Value Problems for Elliptic Equations. Singapore: World Scientific, 2013.
11. Lions PL, Trudinger N, Urbas J. The Neumann problem for equations of Monge-Ampère type. Commun Pure Appl Math. 1986;39:539-563.
12. Urbas J. Nonlinear oblique boundary value problems for Hessian equations in two dimensions. Ann Inst Henri Poincaré-Anal Non Linéaire. 1995;12(5):507-575.
13. Urbas J. Nonlinear oblique boundary value problems for two-dimensional curvature equations. Adv. Diff. Equ. 1996;1(3):301-336.
14. Trudinger NS. On degenerate fully nonlinear elliptic equations in balls. Bull Aust Math Soc. 1987;35:299-307.
15. Ma XN, Qiu GH. The Neumann problem for Hessian equations. Commun Math Phys. 2019;366:1-28.
16. Qiu GH, Xia C. Classical Neumann problems for Hessian equations and Alexandrov-Fenchel inequalities. Int Math Res Not. 2019;2019:6285-6303.
17. Chen CQ & Zhang DK The Neumann problem of Hessian quotient equations. Preprint, 2016.
18. Jiang FD, Trudinger NS. Oblique boundary value problems for augmented Hessian equation II. Commun Partial Differ Equ. 2019;44:708-748.
19. Jiang F, Trudinger NS. Oblique boundary value problems for augmented Hessian equations I. Bull Math Sci. 2018;8:353-411.
20. Jiang F, Trudinger NS. Oblique boundary value problems for augmented Hessian equations II. Nonlinear Anal. 2017;154:148-173.
21. Krylov NV. On the general notion of fully nonlinear second order elliptic equation. Trans Am Math Soc. 1995;3:857-895.
22. Guan PF & Zhang XW. A class of curvature type equations. Pure and Applied Quarterly. 3(2021), 865-907.
23. Lieberman G. Second-Order Parabolic Differential Equations. Singapore: World Scientific; 1996.
24. Huisken G, Sinestrari C. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. 1999;183:45-70.
25. Li YY. Interior gradient estimates for solutions of certain fully nonlinear elliptic equations. J Diff Equ. 1991;90:172-185.
26. Chou KS, Wang XI. A variation theory of the Hessian equation. Commun Pure Appl Math. 2001;54:1029-1064.
27. Hou ZL, Ma XN, Wu DM. A second order estimate for complex Hessian equations on a compact Kähler manifold. Math Res Lett. 2010;17(3):547-561.
28. Lin M, Trudinger NS. On some inequalities for elementary symmetric functions. Bull Aust Math Soc. 1994;50:317-326.
29. Spruck J. Geometric aspects of the theory of fully nonlinear elliptic equations. Clay Math Proc. 2005; (2):283-309.
30. Li SY. On the Neumann problems for Complex Monge–Ampère equations. Indiana Univ Math J. 1994;43:1099-1122.
31. Lieberman G, Trudinger N. Nonlinear oblique boundary value problems for nonlinear elliptic equations. *Trans Am Math Soc.* 1986;295(2):509-546.

**How to cite this article:** Chen C, Chen L, Mei X, Xiang N. The classical Neumann problem for a class of mixed Hessian equations. *Stud Appl Math.* 2022;148:5–26. https://doi.org/10.1111/sapm.12429