Strong Bounds for Evolution in Undirected Graphs

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Abstract

This work studies the generalized Moran process, as introduced by Lieberman, Hauert, and Nowak [Nature, 433:312-316, 2005], where the individuals of a population reside on the vertices of an undirected connected graph. The initial population has a single mutant of a fitness value \( r \) (typically \( r > 1 \)), residing at some vertex \( v \) of the graph, while every other vertex is initially occupied by an individual of fitness 1. At every step of this process, an individual (i.e. vertex) is randomly chosen for reproduction with probability proportional to its fitness, and then it places a copy of itself on a random neighbor, thus replacing the individual that was residing there. The main quantity of interest is the fixation probability, i.e. the probability that eventually the whole graph is occupied by descendants of the mutant. In this work we concentrate on the fixation probability when the mutant is initially on a specific vertex \( v \), thus refining the older notion of Lieberman et al. which studied the fixation probability when the initial mutant is placed at a random vertex. We then aim at finding graphs that have many “strong starts” (or many “weak starts”) for the mutant. Thus we introduce a parameterized notion of selective amplifiers (resp. selective suppressors) of evolution, i.e. graphs with at least some \( b(n) \) vertices (starting points of the mutant), which fixate the graph with large (resp. with small) probability. We prove the existence of strong selective amplifiers (i.e. for \( b(n) = \Theta(n) \) vertices \( v \) the fixation probability of \( v \) is at least \( 1 - \frac{c(r)}{n} \) for a function \( c(r) \) that depends only on \( r \)). We also prove the existence of quite strong selective suppressors. Regarding the traditional notion of fixation probability from a random start, we provide the first non-trivial upper and lower bounds: first, we demonstrate the non-existence of “strong universal” amplifiers, i.e. we prove that for any undirected graph the fixation probability from a random start is at most \( 1 - \frac{c(r)}{n^{3/4}} \). Finally we prove the Thermal Theorem, which states that for any undirected graph, when the mutant starts at vertex \( v \), the fixation probability at least \( \frac{(r-1)}{(r+ \frac{\deg(v)}{\deg_{min}})} \). This implies the first nontrivial lower bound for the usual notion of fixation probability, which is almost tight. This theorem extends the “Isothermal Theorem” of Lieberman et al. for regular graphs. Our proof techniques are original and are based on new domination arguments which may be of general interest in Markov Processes that are of the general birth-death type.

Keywords: Evolutionary dynamics, undirected graphs, fixation probability, lower bound, Markov chain.

1 Introduction

Population and evolutionary dynamics have been extensively studied [2, 6, 7, 15, 21, 24, 25], mainly on the assumption that the evolving population is homogeneous, i.e. it has no spatial structure. One of the main models in this area is the Moran Process [19], where the initial population contains a single mutant with fitness \( r > 0 \), with all other individuals having fitness 1. At every step of this process, an individual is chosen for reproduction with probability proportional to its fitness. This individual then replaces a second individual, which is chosen uniformly at random, with a copy of itself. Such dynamics as the above have been extensively studied also in the context of strategic interaction in evolutionary game theory [11, 14, 23].

In a recent article, Lieberman, Hauert, and Nowak [16] (see also [20]) introduced a generalization of the Moran process, where the individuals of the population are placed on the vertices of a connected graph (which is, in general, directed) such that the edges of the graph determine competitive interaction. In the generalized Moran process, the initial population again consists of a single mutant of fitness \( r \), placed on a vertex that is chosen uniformly at random, with each other vertex occupied by a non-mutant of fitness 1. An individual is chosen for reproduction exactly as in the standard Moran process, but now the second individual

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to be replaced is chosen among its neighbors in the graph uniformly at random (or according to some weights of the edges) \[16, 20\]. If the underlying graph is the complete graph, then this process becomes the standard Moran process on a homogeneous population \[16, 20\]. Several similar models describing infections and particle interactions have been also studied in the past, including the SIR and SIS epidemics \[10, \text{Chapter 21}\], the voter and antivoter models and the exclusion process \[1, 9, 17\]. However such models do not consider the issue of different fitness of the individuals.

The central question that emerges in the generalized Moran process is how the population structure affects evolutionary dynamics \[16, 20\]. In the present work we consider the generalized Moran process on arbitrary finite, undirected, and connected graphs. On such graphs, the generalized Moran process terminates almost surely, reaching either fixation of the graph (all vertices are occupied by copies of the mutant) or extinction of the mutants (no copy of the mutant remains). The fixation probability of a graph \( G \) for a mutant of fitness \( r \), is the probability that eventually fixation is reached when the mutant is initially placed at a random vertex of \( G \), and is denoted by \( f_r(G) \). The fixation probability can, in principle, be determined using standard Markov Chain techniques. But doing so for a general graph on \( n \) vertices requires solving a linear system of \( 2^n \) linear equations. Such a task is not computationally feasible, even numerically. As a result of this, most previous work on computing fixation probabilities in the generalized Moran process was either restricted to graphs of small size \[6\] or to graph classes which have a high degree of symmetry, reducing thus the size of the corresponding linear system (e.g. paths, cycles, stars, and cliques \[3–5\]). Experimental results on the fixation probability of random graphs derived from grids can be found in \[22\].

A very interesting recent result \[8\] shows how to construct fully polynomial randomized approximation schemes (FPRAS) for the probability of reaching fixation (when \( r \geq 1 \)) or extinction (for all \( r > 0 \)). The result of \[8\] uses a Monte Carlo estimator, i.e. it runs the generalized Moran process several times\[4\] while each run terminates in polynomial time with high probability \[8\]. Note that improved lower and upper bounds for the fixation probability on connected undirected graphs, are that \( f_r(G) \geq \frac{1}{n} \) and \( f_r(G) \leq 1 - \frac{1}{n+r} \); these are quite weak bounds and easy to derive.

Lieberman et al. \[16, 20\] proved the Isothermal Theorem, namely that the fixation probability of a “symmetric directed” graph is equal to that of the complete graph (i.e. the homogeneous population of the standard Moran process), which tends to \( 1 - \frac{1}{n} \) when the size \( n \) of the population grows. Intuitively, in the Isothermal Theorem, every vertex of the graph has a temperature which determines how often this vertex is being replaced by other individuals during the generalized Moran process. In the terminology of \[16\], symmetric directed graphs become regular graphs (i.e. graphs with overall the same vertex degree) when we consider undirected graphs. Furthermore, the complete graph (or equivalently, any regular graph) serves as a benchmark for measuring the fixation probability of an arbitrary graph \( G \): if \( f_r(G) \) is larger (resp. smaller) than that of the complete graph then \( G \) is called an amplifier (resp. a suppressor) \[16, 20\]. Until now only graphs with similar (i.e. a little larger or smaller) fixation probability than regular graphs have been identified \[3–5, 16, 18\], while no class of strong amplifiers/suppressors is known so far.

Our contribution. The structure of the graph, on which the population resides, plays a crucial role in the course of evolutionary dynamics. Human societies or social networks are never homogeneous, while certain individuals in central positions may be more influential than others \[20\]. Motivated by this, we introduce in this paper a new notion of measuring the success of an advantageous mutant in a structured population, by counting the number of initial placements of the mutant in a graph that guarantee fixation of the graph with large probability. This provides a refinement of the notion of fixation probability in undirected graphs. Specifically, we do not any more consider the fixation probability as the probability of reaching fixation when the mutant is placed at a random vertex, but we rather consider the probability \( f_r(v) \) of reaching fixation when a mutant with fitness \( r > 1 \) is introduced at a specific vertex \( v \) of the graph: \( f_r(v) \) is termed the fixation probability of vertex \( v \). Using this notion, the fixation probability \( f_r(G) \) of a graph \( G = (V, E) \) with \( n \) vertices is \( f_r(G) = \frac{1}{n} \sum_{v \in V} f_r(v) \).

We aim in finding graphs that have many “strong starts” (or many “weak starts”) of the mutant. Thus

\[1\] For approximating the probability to reach fixation (resp. extinction), one needs a number of runs which is about the inverse of the best known lower (resp. upper) bound of the fixation probability.
we introduce the notions of \((h(n), g(n))-selective amplifiers\) (resp. \((h(n), g(n))-selective suppressors\)), which include those graphs with \(n\) vertices for which there exist at least \(h(n)\) vertices \(v\) with \(f_r(v) \geq 1 - \frac{c(r)}{g(n)}\) (resp. \(f_r(v) \leq \frac{c(r)}{g(n)}\)) for an appropriate function \(c(r)\) of \(r\). We contrast this new notion of \((h(n), g(n))-selective amplifiers\) (resp. suppressors) with the notion of \(g(n))-universal amplifiers\) (resp. suppressors) which include those graphs \(G\) with \(n\) vertices for which \(f_r(G) \geq 1 - \frac{c(r)}{g(n)}\) (resp. \(f_r(G) \leq \frac{c(r)}{g(n)}\)) for an appropriate function \(c(r)\) of \(r\). For a detailed presentation and a rigorous definition of these notions we refer to Section 2.

Using these new notions, we prove that there exist strong selective amplifiers, namely \((\Theta(n), n))-selective amplifiers\) (called the \(urchin graphs\)). Furthermore we prove that there exist also strong selective suppressors, namely \((\frac{n}{\phi(n)+1}, \frac{n}{\phi(n)})-selective suppressors\) (called the \(\phi(n))-urchin graphs\) for any function \(\phi(n) = \omega(1)\) with \(\phi(n) \leq \sqrt{n}\).

Regarding the traditional measure of the fixation probability \(f_r(G)\) of undirected graphs \(G\), we provide the first non-trivial upper and lower bounds, while only the trivial bounds \(\frac{1}{n}\) and \(1 - \frac{1}{n+r}\) were known \[8\]. More specifically, first of all we demonstrate the nonexistence of "strong" universal amplifiers by showing that for any graph \(G\) with \(n\) vertices, the fixation probability \(f_r(G)\) is strictly less than \(1 - \frac{c(r)}{g(n)}\), for any \(\varepsilon > 0\). This is in a wide contrast with what happens in directed graphs, as Lieberman et al. \[16\] provided directed graphs with arbitrarily large fixation probability (see also \[20\]).

On the other hand, we provide our lower bound in the \(Thermal Theorem\), which states that for any vertex \(v\) of an arbitrary undirected graph \(G\), the fixation probability \(f_r(v)\) of \(v\) is at least \((r-1)/(r+\frac{\deg v}{\deg_{\min}})\) for any \(r > 1\), where \(\deg v\) is the degree of \(v\) in \(G\) (i.e. the number of its neighbors) and \(\deg_{\min}\) (resp. \(\deg_{\max}\)) is the minimum (resp. maximum) degree in \(G\). This result extends the Isothermal Theorem for regular graphs \[16\]. In particular, we consider here a different notion of \(temperature\) for a vertex than \[16\]: the temperature of vertex \(v\) is \(\frac{1}{\deg v}\). As it turns out, a "hot" vertex (i.e. with high temperature) affects more often its neighbors than a "cold" vertex (with low temperature). The \(Thermal Theorem\), which takes into account the vertex on which the mutant is introduced, provides immediately the first non-trivial lower bound \((r-1)/(r+\frac{\deg_{\max}}{\deg_{\min}})\) for the fixation probability \(f_r(G)\) of any undirected graph \(G\). The latter lower bound is almost tight, as it implies that \(f_r(G) \geq \frac{r-1}{r+1}\) for a regular graph \(G\), while the Isothermal Theorem implies that the fixation probability of a regular graph \(G\) tends to \(\frac{r-1}{r}\) as the size of \(G\) increases. Note that our new upper/lower bounds for the fixation probability lead to better time complexity of the FPRAS proposed in \[8\], as the \(Monte Carlo technique proposed in\) \[8\] now needs to simulate the Moran process a less number of times (to estimate fixation or extinction).

Our techniques are original and of a constructive combinatorics flavor. For the class of strong selective amplifiers (the \(urchin graphs\)) we introduce a novel decomposition of the Markov chain \(M\) of the generalized Moran process into \(n-1\) smaller chains \(M_1, M_2, \ldots, M_{n-1}\), and then we decompose each \(M_k\) into two even smaller chains \(M_k^1, M_k^2\). Then we exploit a new way of composing these smaller chains (and returning to the original one) that is carefully done to maintain the needed domination properties. For the proof of the lower bound in the \(Thermal Theorem\), we first introduce a new and simpler weighted process that bounds fixation probability from below (the generalized Moran process is a special case of this new process). Then we add appropriate dummy states to its (exponentially large) Markov chain, and finally we iteratively modify the resulting chain by maintaining the needed monotonocity properties. Eventually this results to the desired lower bound of the \(Thermal Theorem\). Finally, our proof for the non-existence of strong universal amplifiers is done by contradiction, partitioning appropriately the vertex set of the graph and discovering an appropriate independent set that leads to the contradiction.

The paper is organized as follows. Preliminaries and notation are given in Section 2. Furthermore we present our results on amplifiers and suppressors in Sections 3 and 4, respectively.

2 Preliminaries

Throughout the paper we consider only finite, connected, undirected graphs \(G = (V,E)\). Our results apply to connected graphs as, otherwise, the fixation probability is necessarily zero. The edge \(e \in E\) between two
vertices \( u, v \in V \) is denoted by \( e = uv \). For a vertex subset \( X \subseteq V \), we write \( X + y \) and \( X - y \) for \( X \cup \{y\} \) and \( X \cap \{y\} \), respectively. Furthermore, throughout \( r \) denotes the fitness of the mutant, while the value \( r \) is considered to be independent of the size \( n \) of the network, i.e. we assume that \( r \) is constant. For simplicity of presentation, we call a vertex \( v \) “infected” if a copy of the mutant is placed on \( v \). For every vertex subset \( S \subseteq V \) we denote by \( f_r(S) \) the fixation probability of the set \( S \), i.e. the probability that, starting with exactly \(|S|\) copies of the mutant placed on the vertices of \( S \), the generalized Moran process will eventually reach fixation. By the definition of the generalized Moran process \( f_r(\emptyset) = 0 \) and \( f_r(V) = 1 \), while for \( S \not\in \{\emptyset, V\} \),

\[
f_r(S) = \frac{\sum_{xy \in E, x \in S, y \notin S} \left( \frac{r}{\deg x} f_r(S + y) + \frac{1}{\deg y} f_r(S - x) \right)}{\sum_{xy \in E} \left( \frac{r}{\deg x} + \frac{1}{\deg y} \right)}
\]

Therefore, eliminating self-loops in the above Markov process,

\[
f_r(S) = \frac{\sum_{xy \in E, x \in S, y \notin S} \left( \frac{r}{\deg x} f_r(S + y) + \frac{1}{\deg y} f_r(S - x) \right)}{\sum_{xy \in E} \left( \frac{r}{\deg x} + \frac{1}{\deg y} \right)} \tag{1}
\]

In the next definition we introduce the notions of universal and selective amplifiers.

**Definition 1** Let \( \mathcal{G} \) be an infinite class of undirected graphs. If there exists an \( n_0 \in \mathbb{N} \), an \( r_0 \geq 1 \), and some function \( c(r) \), such that for every graph \( G \in \mathcal{G} \) with \( n \geq n_0 \) vertices and for every \( r > r_0 \):

- \( f_r(G) \geq 1 - \frac{c(r)}{g(n)} \), then \( \mathcal{G} \) is a class of \( g(n) \)-universal amplifiers,
- there exists a subset \( S \) of at least \( h(n) \) vertices of \( G \), such that \( f_r(v) \geq 1 - \frac{c(r)}{g(n)} \) for every vertex \( v \in S \), then \( \mathcal{G} \) is a class of \( (h(n), g(n)) \)-selective amplifiers.

Moreover, \( \mathcal{G} \) is a class of strong universal (resp. strong selective) amplifiers if \( \mathcal{G} \) is a class of \( n \)-universal (resp. \( (\Theta(n), n) \)-selective) amplifiers.

Similarly to Definition 1 we introduce the notions of universal and selective suppressors.

**Definition 2** Let \( \mathcal{G} \) be an infinite class of undirected graphs. If there exist functions \( c(r) \) and \( n_0(r) \), such that for every \( r > 1 \) and for every graph \( G \in \mathcal{G} \) with \( n \geq n_0(r) \) vertices:

- \( f_r(G) \leq \frac{c(r)}{g(n)} \), then \( \mathcal{G} \) is a class of \( g(n) \)-universal suppressors,
- there exists a subset \( S \) of at least \( h(n) \) vertices of \( G \), such that \( f_r(v) \leq \frac{c(r)}{g(n)} \) for every vertex \( v \in S \), then \( \mathcal{G} \) is a class of \( (h(n), g(n)) \)-selective suppressors.

Moreover, \( \mathcal{G} \) is a class of strong universal (resp. strong selective) suppressors if \( \mathcal{G} \) is a class of \( n \)-universal (resp. \( (\Theta(n), n) \)-selective) suppressors.

Note that \( n_0 = n_0(r) \) in Definition 2 while in Definition 1 \( n_0 \) is not a function of \( r \). The reason for this is that, since we consider the fitness value \( r \) to be constant, the size \( n \) of \( G \) needs to be sufficiently large with respect to \( r \) in order for \( G \) to act as a suppressor. Indeed, if we let \( r \) grow arbitrarily, e.g. if \( r = n^2 \), then for any graph \( G \) with \( n \) vertices the fixation probability \( f_r(v) \) tends to 1 as \( n \) grows. The next lemma follows by Definitions 1 and 2.

**Lemma 1** If \( \mathcal{G} \) is a class of \( g(n) \)-universal amplifiers (resp. suppressors), then \( \mathcal{G} \) is a class of \( (\Theta(n), g(n)) \)-selective amplifiers (resp. suppressors).
Proof. Suppose that \( \mathcal{G} \) is a class of \( g(n) \)-universal amplifiers. That is, for every \( r > r_0 \) and for every graph \( G = (V,E) \in \mathcal{G} \) with \( n \geq n_0 \) vertices, the fixation probability of \( G \) is \( f_r(G) \geq 1 - \frac{c(r)}{g(n)} \), where \( c(r) \) is some function that depends only on \( r \). Let \( S \subseteq V \) be the subset of vertices such that \( f_r(v) \geq 1 - \frac{c(r)}{g(n)} \) for some function \( c'(r) \) that depends only on \( r \). Then there exists an appropriate function \( \phi(n,r) = \omega(1) \), i.e. \( \lim_{n \to \infty} \phi(n,r) = \infty \), such that \( f_r(v) \leq 1 - \frac{\phi(n,r)}{g(n)} \) for every \( v \in V \setminus S \). Thus the fixation probability of \( G \) is

\[
f_r(G) \leq \frac{|S| \cdot 1 + (n - |S|) \cdot (\frac{\phi(n,r)}{g(n)})}{n} = 1 - \frac{(n - |S|) \cdot \phi(n,r)}{g(n)} \tag{2}
\]

Now, since \( f_r(G) \geq 1 - \frac{c(r)}{g(n)} \), it follows by (2) that \( (n - |S|) \leq n \cdot \frac{c(r)}{\phi(n,r)} \), and thus \( |S| \geq n(1 - \frac{c(r)}{\phi(n,r)}) = \Theta(n) \), since \( \phi(n,r) = \omega(1) \). Thus it follows by definition of the set \( S \) that \( \mathcal{G} \) is a class of \( (\Theta(n),g(n)) \)-selective amplifiers.

Suppose now that \( \mathcal{G} \) is a class of \( g(n) \)-universal suppressors. That is, for every \( r > r_0 \) and for every graph \( G = (V,E) \in \mathcal{G} \) with \( n \geq n_0(r) \) vertices, the fixation probability of \( G \) is \( f_r(G) \leq c(r) \), where \( c(r) \) is some function that depends only on \( r \). Let \( S \subseteq V \) be the subset of vertices such that \( f_r(v) \leq c(r) \) for some function \( c'(r) \) that depends only on \( r \). Then there exists an appropriate function \( \phi(n,r) = \omega(1) \), i.e. \( \lim_{n \to \infty} \phi(n,r) = \infty \), such that \( f_r(v) \geq \frac{\phi(n,r)}{g(n)} \) for every \( v \in V \setminus S \). Thus the fixation probability of \( G \) is

\[
f_r(G) \geq \frac{|S| \cdot 0 + (n - |S|) \cdot \frac{\phi(n,r)}{g(n)}}{n} = \frac{(n - |S|) \cdot \phi(n,r)}{g(n)} \tag{3}
\]

Now, since \( f_r(G) \leq c(r) \), it follows by (3) that \( (n - |S|) \leq n \cdot \frac{c(r)}{\phi(n,r)} \), and thus \( |S| \geq n(1 - \frac{c(r)}{\phi(n,r)}) = \Theta(n) \), since \( \phi(n,r) = \omega(1) \). Thus it follows by definition of the set \( S \) that \( \mathcal{G} \) is a class of \( (\Theta(n),g(n)) \)-selective suppressors.

The most natural question that arises by Definitions 1 and 2 is whether there exists any class of strong selective amplifiers/suppressors, as well as for which functions \( h(n) \) and \( g(n) \) there exist classes of \( g(n) \)-universal amplifiers/suppressors and classes of \( (h(n),g(n)) \)-selective amplifiers/suppressors. In Section 3 and 4 we provide our results on amplifiers and suppressors, respectively.

### 3 Amplifier bounds

In this section we prove that there exist no strong universal amplifiers (Section 3.1), although there exists a class of strong selective amplifiers (Section 3.2).

#### 3.1 Non-existence of strong universal amplifiers

**Theorem 1** For any function \( g(n) = \Omega(n^{\frac{3}{2} + \varepsilon}) \) for some \( \varepsilon > 0 \), there exists no graph class \( \mathcal{G} \) of \( g(n) \)-universal amplifiers for any \( r > r_0 = 1 \).

**Proof.** Let \( r_0 = 1 \) and \( g(n) = \Omega(n^{1-\delta}) \), where \( \delta = \frac{1}{4} - \varepsilon < \frac{1}{4} \). Suppose that \( \mathcal{G} \) is a class of \( g(n) \)-universal amplifiers. That is, for every graph \( G = (V,E) \in \mathcal{G} \) with \( n \geq n_0 \) vertices, the fixation probability of \( G \) is \( f_r(G) \geq 1 - \frac{c_0(r)}{n^{1-\delta}} \) for every \( r > 1 \), where \( c(r), c_0(r) \) are two functions that depend only on \( r \). We partition the vertex set \( V \) into two three sets \( V_1, V_2, V_3 \) such that

\[
V_1 = \{ v \in V : f_r(v) \geq 1 - \frac{c_0(r)}{n^{1-\delta}} \} \tag{4}
\]

\[
V_2 = \{ v \in V \setminus V_1 : f_r(v) \geq 1 - \frac{c_1(r)}{n^{1-2\delta}} \} \tag{5}
\]

\[
V_3 = \{ v \in V \setminus (V_1 \cup V_2) : f_r(v) \leq 1 - \frac{\phi(n,r)}{n^{1-2\delta}} \} \tag{6}
\]
where $c_1(r)$ is an appropriate function of $r$, and $\lim_{n \to \infty} \phi(n, r) = \infty$, i.e. $\phi(n, r) = \omega(1)$. Note that $V_1 \neq \emptyset$, since $f_r(G) \geq 1 - \frac{c_0(r)}{n^{1-\delta}}$ by assumption. Using (4), the fixation probability $f_r(G)$ of $G$ is upper bounded by

$$f_r(G) \leq \left( \left| V_1 \right| + \left| V_2 \right| \right) \cdot 1 + \left| V_3 \right| \cdot \left( 1 - \frac{\phi(n, r)}{n^{1-2\delta}} \right)$$

$$= 1 - \frac{|V_3|}{n} \cdot \frac{\phi(n, r)}{n^{1-2\delta}}$$

(7)

Now, since $f_r(G) \geq 1 - \frac{c_0(r)}{n^{1-\delta}}$, it follows by (7) that $1 - \frac{c_0(r)}{n^{1-\delta}} \leq 1 - \frac{|V_3|}{n} \cdot \frac{\phi(n, r)}{n^{1-2\delta}}$, and thus

$$|V_3| \leq n^{1-\delta} \cdot \frac{c_0(r)}{\phi(n, r)}$$

(8)

For an arbitrary vertex $v \in V$, we obtain an upper bound of the probability $f_r(v)$ by assuming that fixation is reached if the process reaches at least two infected vertices, when it starts with only $v$ being infected. Therefore

$$f_r(v) \leq \frac{r \cdot 1 + \sum_{x \in N(v)} \frac{1}{\deg x} \cdot 0}{r + \sum_{x \in N(v)} \frac{1}{\deg x}} = \frac{r}{r + \sum_{x \in N(v)} \frac{1}{\deg x}}$$

(9)

for every $v \in V$. It follows now by (4) and (9) that for every $v \in V_1$,

$$1 - \frac{c_0(r)}{n^{1-\delta}} \leq 1 - \frac{\sum_{x \in N(v)} \frac{1}{\deg x}}{r + \sum_{x \in N(v)} \frac{1}{\deg x}} \iff \sum_{x \in N(v)} \frac{1}{\deg x} \leq \frac{r \cdot c_0(r)}{n^{1-\delta} - c_0(r)} \leq \frac{c'(r)}{n^{1-\delta}}$$

(10)

for an appropriate function $c'(r)$ of $r$. Therefore, since $\sum_{x \in N(v)} \frac{1}{\deg x} \geq \deg(v) \cdot \frac{1}{n}$, (10) implies that

$$\deg v \leq c'(r) \cdot n^\delta$$

(11)

for every $v \in V_1$. Furthermore, since $\frac{1}{\deg u} \leq \sum_{x \in N(u)} \frac{1}{\deg x}$ for every $u \in N(v)$, (10) implies that

$$\deg u \geq \frac{n^{1-\delta}}{c'(r)}$$

(12)

for every $u \in N(v)$, where $v \in V_1$. Similarly it follows by (5) and (9) that for every $v \in V_2$,

$$\sum_{u \in N(v)} \frac{1}{\deg u} \leq \frac{r \cdot c_1(r)}{n^{1-2\delta} - c_1(r)} \leq \frac{c''(r)}{n^{1-2\delta}}$$

(13)

for some function $c''(r)$ of $r$, and that

$$\deg v \leq c''(r) \cdot n^{2\delta}$$

(14)

$$\deg u \geq \frac{n^{1-2\delta}}{c''(r)}$$

(15)

for every $v \in V_2$ and $u \in N(v)$.

Consider now a vertex $v \in V_1$ and a vertex $u \in N(v)$. Then, since $\delta < \frac{1}{4}$ by assumption, (11), (12), and (14) imply that $u \notin V_1$ and $u \notin V_2$, and thus $u \in V_3$. That is, $N(v) \subseteq V_3$ for every $v \in V_1$, and $V_1$ is an independent set.
Let now $v \in V$ such that $f_r(v)$ is maximized. Then clearly $v \in V_1$. Denote $Q_v = \sum_{x \in N(v)} \frac{1}{\deg(x)}$; note that $Q_v = \Omega(\frac{1}{n^2})$. Furthermore, denote for any neighbor $u \in N(v)$ the quantity $Q_{uv} = \sum_{x \in N(v) \setminus \{v\}} \frac{1}{\deg(x)} + \sum_{x \in N(u) \setminus \{v\}} \frac{1}{\deg(x)}$. Then it follows by (17) that

$$f_r(G) \leq \frac{2r^2}{2r^2 + Q_vQ_{uv}}$$

(cf. Theorem 1 in [18] and its proof therein. Let $u_0 \in N(v)$ be such that the right hand side of (16) is maximized, and thus

$$f_r(G) \leq \frac{2r^2}{2r^2 + Q_vQ_{u_0v}}$$

Note that $u_0 \in V_3$, since $N(v) \subseteq V_3$ for every $v \in V_1$, as we proved above. Therefore, since $f_r(G) \geq 1 - \frac{c_0(r)}{n^{1-\delta}}$, it follows by (17) that

$$1 - \frac{c_0(r)}{n^{1-\delta}} \leq 1 - \frac{Q_vQ_{u_0v}}{2r^2 + Q_vQ_{u_0v}} \Leftrightarrow n^{1-\delta}Q_vQ_{u_0v} \leq 2r^2c_0(r) + c_0(r)Q_vQ_{u_0v} \Leftrightarrow Q_vQ_{u_0v} \leq \frac{2r^2c_0(r)}{n^{1-\delta} - c_0(r)} \leq \frac{c''(r)}{n^{1-\delta}}$$

for an appropriate function $c''(r)$ of $r$.

Note that $u_0$ has at least $\Omega(n^{1-\delta})$ neighbors in $V_1 \cup V_2$ by (8) and (12), since $\phi(n,r) = \omega(1)$ by assumption. Thus it follows by (11) and (14) that

$$\sum_{x \in N(u_0) \setminus \{v\}} \frac{1}{\deg(x)} = \Omega(n^{1-3\delta})$$

Furthermore $Q_{u_0v} \geq \sum_{x \in N(u_0) \setminus \{v\}} \frac{1}{\deg(x)}$ by the definition of $Q_{u_0v}$, and thus

$$Q_{u_0v} = \Omega(n^{1-3\delta})$$

Moreover $Q_vQ_{u_0v} = \Omega(n^{-3\delta})$, since $Q_v = \Omega(\frac{1}{n})$. Therefore it follows by (18) that

$$\Omega(n^{-3\delta}) = Q_vQ_{u_0v} \leq \frac{c''(r)}{n^{1-\delta}}$$

which is a contradiction, since $\delta < \frac{1}{4}$ by assumption. Therefore there exists no class $\mathcal{G}$ of $g(n)$-universal amplifiers for any $r > r_0 = 1$, where $g(n) = \Omega(n^{1-\delta})$ for some $\delta = \frac{1}{4} - \varepsilon < \frac{1}{4}$.

The next corollary follows from Theorem 1

**Corollary 1** There exists no infinite class $\mathcal{G}$ of undirected graphs which are strong universal suppressors.

### 3.2 A class of strong selective amplifiers

In this section we present the first class $\mathcal{G} = \{G_n : n \geq 1\}$ of strong selective amplifiers, which we call the urchin graphs. Namely, the graph $G_n$ has $2n$ vertices, consisting of a clique with $n$ vertices, an independent set of $n$ vertices, and a perfect matching between the clique and the independent set, as it is illustrated in Figure 1. For every graph $G_n$, we refer for simplicity to a vertex of the clique of $G_n$ as a clique vertex of $G_n$, and to a vertex of the independent set of $G_n$ as a nose of $G_n$, respectively. We prove in this section that the class $\mathcal{G}$ of urchin graphs are strong selective amplifiers. Namely, we prove that, whenever $r > r_0 = 5$, the fixation probability of any nose $v$ of any graph $G_n$ is $f_r(v) \geq 1 - \frac{c(r)}{n}$, where $c(r)$ is a function that depends only on the fitness $r$ of the mutant.

Let $v$ be a clique vertex (resp. a nose) and $u$ be its adjacent nose (resp. clique vertex). If $v$ is infected and $u$ is not infected, then $v$ is called an isolated clique vertex (resp. isolated nose), otherwise $v$ is called a covered clique vertex (resp. covered nose). Let $k \in \{0,1, \ldots, n\}$, $i \in \{0,1, \ldots, n-k\}$, and $x \in \{0,1,2, \ldots, k\}$. Denote by $Q^{k}_{i,x}$ the state of $G_n$ with exactly $i$ isolated clique vertices, $x$ isolated noses, and $k-x$ covered noses. An example of the state $Q^{k}_{i,x}$ is illustrated in Figure 2(a). Furthermore, for every $k, i \in \{0,1, \ldots, n\}$,
we define the state \( P^k_i \) of \( G_n \) as follows. If \( i \leq k \), then \( P^k_i \) is the state with exactly \( i \) covered noses and \( k-i \) isolated noses. If \( i > k \), then \( P^k_i \) is the state with exactly \( k \) covered noses and \( i-k \) isolated clique vertices. Note that \( Q^k_{i,0} = P^k_{k+i} \) and \( Q^k_{0,x} = P^k_{k-x} \), for every \( k \in \{0,1,\ldots,n\}, i \in \{0,1,2,\ldots,n-k\} \), and \( x \in \{0,1,2,\ldots,k\} \). Two examples of the state \( P^k_i \), for the cases where \( i \leq k \) and \( i > k \), are illustrated in Figures 2(b) and 2(c) respectively.

![Diagram](image)

**Figure 2**: The state (a) \( Q^k_{i,x} \) and the state \( P^k_i \), where (b) \( i \leq k \), and (c) \( i > k \).

Let \( k \in \{1,2,\ldots,n-1\} \). For all appropriate values of \( i \) and \( x \), we denote by \( q^k_{i,x} \) (resp. \( p^k_i \)) the probability that, starting at state \( Q^k_{i,x} \) (resp. \( P^k_i \)) we eventually arrive to a state with \( k+1 \) infected noses before we arrive to a state with \( k-1 \) infected noses.

**Lemma 2** Let \( k \in \{1,2,\ldots,n-1\} \). Then \( q^k_{i,x} > q^k_{i-1,x-1} \), for every \( i \in \{1,2,\ldots,n-k\} \) and every \( x \in \{1,2,\ldots,k\} \).

**Proof.** Denote by \( \mathcal{M}_1 \) the Markov chain with starting state \( Q^k_{i,x} \). Similarly, denote by \( \mathcal{M}_2 \) the Markov chain with starting state \( Q^k_{i-1,x-1} \). Note that, initially, both Markov chains \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have the same number \( 2(k-x) + x + i \) of infected vertices. Moreover, \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) coincide initially on all their vertices except two. In particular, \( \mathcal{M}_1 \) has initially an isolated nose \( u \), which is a covered nose in \( \mathcal{M}_2 \). Furthermore, \( \mathcal{M}_1 \) has initially an isolated clique vertex \( v \), which is an unaffected clique vertex in \( \mathcal{M}_2 \). Denote by \( u' \) the (unique) clique vertex that is adjacent to \( u \) in \( G_n \). Furthermore denote by \( v' \) the (unique) nose that is adjacent to \( v \) in \( G_n \). Note that, initially, \( u' \) is uninfected in \( \mathcal{M}_1 \) and infected in \( \mathcal{M}_2 \), while \( v' \) is uninfected in both \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \).

Note that at every iteration of the processes \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), one vertex \( w \) is activated and then it replaces a neighbor \( w' \) of it by an offspring of \( w \). Thus, an equivalent way to analyze these processes is to consider that, at every iteration, one directed edge between two adjacent vertices is activated (with the appropriate probability). In order to prove that \( q^k_{i,x} > q^k_{i-1,x-1} \), we simulate the progress of \( \mathcal{M}_1 \) by the random choices made at the corresponding steps by \( \mathcal{M}_2 \). In particular, we simulate the processes \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) until they reach states \( S_1 \) and \( S_2 \), respectively, such that either \( S_1 = S_2 \), or one of \( S_1 \) and \( S_2 \) is strictly included in the other. Furthermore, during the whole simulation of \( \mathcal{M}_1 \) by \( \mathcal{M}_2 \), before we reach such states \( S_1 \) and \( S_2 \), both \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) have the same number of infected vertices at the corresponding iterations.
If a $w \notin \{u, u', v, v'\}$ is activated for reproduction in $M_2$, then we activate $w$ also in $M_1$. If $w$ places its offspring at a vertex $w' \notin \{u, u', v, v'\}$ in $M_2$, then $w$ places its offspring at the same vertex $w'$ also in $M_1$.

If the clique vertex $v$ (resp. $u'$) is activated for reproduction in $M_2$, then we activate $u'$ (resp. $v$) in $M_1$. In this case, if $v$ (resp. $u'$) places in $M_2$ its offspring at a clique vertex $w \neq u'$ (resp. $w \neq v$), then $u'$ (resp. $v$) places in $M_1$ its offspring at the same clique vertex $w$. If $v$ (resp. $u'$) places in $M_2$ its offspring at the clique vertex $u'$ (resp. $v$), then $u'$ (resp. $v$) places in $M_1$ its offspring at the clique vertex $v$ (resp. $u'$); in this case we arrive to two identical states in both $M_1$ and $M_2$. Finally, if $v$ (resp. $u'$) places in $M_2$ its offspring at its adjacent nose $v'$ (resp. $u$), then $u'$ (resp. $v$) places in $M_1$ its offspring at its adjacent nose $u$ (resp. $v'$); in this case we arrive in $M_1$ to a state, in which the infected vertices are a strict subset (resp. superset) of the infected vertices in $M_2$.

If a clique vertex $w \notin \{v, u'\}$ is activated for reproduction in $M_2$, and if $w$ places in $M_2$ its offspring at $v$ (resp. $u'$), then $w$ places in $M_1$ its offspring at $u'$ (resp. $v$). In this case, if the number of infected vertices changes in $M_2$, then we arrive to the same state in both $M_1$ and $M_2$.

Finally, if the nose $w = v'$ (resp. $w = u$) is activated for reproduction in $M_2$, then we activate the same nose also in $M_1$. In this case we arrive in $M_1$ to a state, in which the infected vertices are a strict subset (resp. superset) of the infected vertices in $M_2$.

Note now that $q_{i,x}^k > q_{i-1,x-1}^k$ if and only if, in the above simulation of $M_1$ by $M_2$, the probability that we arrive to strictly more infected vertices in $M_1$ than $M_2$ is greater or equal to the probability that we arrive to strictly less infected vertices in $M_1$ than $M_2$. Furthermore, note that we arrive in $M_1$ to a state with strictly more or strictly less infected vertices than in $M_2$ only when one of the edges $uu'$ or $vv'$ is activated (in some direction) in the process $M_2$. In particular, whenever this event occurs, $M_1$ receives strictly more infected vertices than $M_2$, if either $u$ places its offspring at $u'$ in $M_2$, or if $u'$ places its offspring at $u$ in $M_2$. Similarly, $M_1$ receives strictly less infected vertices than $M_2$, if either $v$ places its offspring at $v'$ in $M_2$, or if $v'$ places its offspring at $v$ in $M_2$. The ratio of these probabilities is

$$\frac{r \cdot \frac{1}{n} + r \cdot \frac{1}{n+1}}{\frac{1}{n}} = \frac{r}{1} > 1$$

and thus $q_{i,x}^k > q_{i-1,x-1}^k$. ■

**Corollary 2** Let $k \in \{1, 2, \ldots, n-1\}$, $i \in \{0, 1, \ldots, n-k\}$, and $x \in \{0, 1, \ldots, k\}$. Then $q_{i,x}^k > q_{i-1,x-1}^k$.

**Proof.** Suppose first that $i \geq x$. Then Lemma 2 implies that $q_{i,x}^k > q_{i-1,x-1}^k > \cdots > q_{i-1,0}^k = p_{k+i-x}^k$. Suppose now that $i < x$. Then Lemma 2 implies that $q_{i,x}^k > q_{i-1,x-1}^k > \cdots > q_{i,0}^k = p_{k+i-x}^k$. ■

Note by Corollary 2 that, in order to compute a lower bound for the fixation probability $f_r(v)$ of a nose $v$ of the graph $G_n$, we can assume that, whenever we have $k$ infected noses and $i$ infected clique vertices, we are at state $P_i^k$. That is, in the Markov chain of the generalized Moran process, we replace any transition to state $Q_{i,x}^k$ with a transition to state $P_{k+i-x}^k$. Denote this relaxed Markov chain by $M$; we will compute a lower bound of the fixation probability of state $P_0^k$ in the Markov chain $M$ (cf. Theorem 3).

In order to analyze $M$, we decompose it first into the $n-1$ smaller Markov chains $M_1, M_2, \ldots, M_{n-1}$, as follows. For every $k \in \{1, 2, \ldots, n-1\}$, the Markov chain $M_k$ captures all transitions of $M$ between states with $k$ infected noses. The state graph of $M_k$ is illustrated in Figure 3, where we denote by $F_{k-1}$ (resp. $F_{k+1}$) an arbitrary state with $k-1$ (resp. $k+1$) infected noses. Moreover, we consider $F_{k-1}$ and $F_{k+1}$, as absorbing states of $M_k$. Since we want to compute a lower bound of the fixation probability, whenever we arrive at state $F_{k+1}$ (resp. at state $F_{k-1}$), we assume that we have the smallest number of infected clique vertices with $k+1$ (resp. with $k-1$) infected noses. That is, whenever $M_k$ reaches state $F_{k+1}$, we assume that $M$ has reached state $P_{k+1}^k$ (and thus we move to the Markov chain $M_{k+1}$). Similarly, whenever $M_k$ reaches state $F_{k-1}$, we assume that $M$ has reached state $P_{k-1}^k$ (and thus we move to the Markov chain $M_{k-1}$).

### 3.2.1 A decomposition of $M_k$ into two Markov chains

In order to analyze the Markov chain $M_k$, where $k \in \{1, 2, \ldots, n-1\}$, we decompose it into two smaller Markov chains $\{M_k^1, M_k^2\}$, as they are shown in Figure.
In $\mathcal{M}_k$, we consider the state $P_{k+1}^k$ absorbing. For every $i \in \{0, 1, \ldots, k\}$ denote by $h_i^k$ the probability that, starting at state $P_i^k$ in $\mathcal{M}_k^1$, we eventually reach state $P_{k+1}^k$ before we reach state $F_{k-1}$, cf. Figure 4(a).

In this Markov chain $\mathcal{M}_k^1$, every transition probability between two states is equal to the corresponding transition probabilities in $\mathcal{M}_k$.

In $\mathcal{M}_k^2$, we denote by $s_i^k$, where $i \in \{k, k+1, \ldots, n\}$, the probability that starting at state $P_i^k$ we eventually reach state $F_{k+1}$ before we reach state $F_{k-1}$, cf. Figure 4(b). In this Markov chain $\mathcal{M}_k^2$, the transition probability from state $P_i^k$ to state $P_{k+1}^k$ (resp. to state $F_{k-1}$) is equal to $h_i^k$ (resp. $1 - h_i^k$), while all other transition probabilities between two states in $\mathcal{M}_k^2$ are the same as the corresponding transition probabilities in $\mathcal{M}_k$.

**Lemma 3** In the Markov chain $\mathcal{M}_k^1$, for any $r > 1$,

$$h_i^k \geq 1 - \frac{2}{n(r-1) + 1} = 1 - O\left(\frac{1}{n}\right) \tag{19}$$

**Proof.** For $i = 0$, the value of $h_i^k$ in the Markov chain $\mathcal{M}_k^1$ is

$$h_0^k = \frac{rk \cdot h_i^k + \frac{k}{n} \cdot 0}{rk + \frac{k}{n}} = \frac{rn}{rn + 1} \cdot h_i^k \tag{20}$$
and thus
\[ h_i^k - h_0^k = \frac{1}{rn} h_0^k \leq \frac{1}{rn} \]
(21)
Furthermore, for every \( i \in \{1, \ldots, k\} \), where \( 1 \leq k \leq n - 1 \), the value of \( h_i^k \) in \( M_i^1 \) can be computed as follows.
\[ h_i^k = \alpha_i^k \cdot h_{i+1}^k + \beta_i^k \cdot h_{i-1}^k + \gamma_i^k \cdot 0 \]
(22)
where
\[ \alpha_i^k = \frac{r((k-i)n+i(n-i))}{\sum_i^k} \]
\[ \beta_i^k = \frac{i(n-i)}{\sum_i^k} \]
\[ \gamma_i^k = \frac{k-i}{\sum_i^k} \]
(23)
and \( \sum_i^k = r \frac{(k-i)n+i(n-i)}{n} + \frac{i(n-i)}{n} + \frac{k-i}{n} \). Therefore (22) implies that
\[ h_i^{k+1} - h_i^k = \frac{\beta_i^k}{\alpha_i^k} (h_i^k - h_{i-1}^k) + \frac{\gamma_i^k}{\alpha_i^k} h_i^k \]
(24)
Furthermore, (23) implies that
\[ \frac{\beta_i^k}{\alpha_i^k} = \frac{1}{r} \cdot \frac{i(n-i)}{(k-i)n+i(n-i)} \leq \frac{1}{r} \]
(25)
\[ \frac{\gamma_i^k}{\alpha_i^k} = \frac{1}{r} \cdot \frac{k-i}{(k-i)n+i(n-i)} = \frac{1}{r} \cdot \frac{n+i(n-i)}{n} \leq \frac{1}{rn} \]
(26)
Note that the inequality \( \frac{\gamma_i^k}{\alpha_i^k} \leq \frac{1}{rn} \) in (26) holds also for \( i = k \), since \( \gamma_k^k = 0 \). Therefore it follows by (24), (25), and (26) that
\[ h_i^{k+1} - h_i^k \leq \frac{1}{r} (h_i^k - h_{i-1}^k) + \frac{1}{rn} h_i^k \]
(27)
Thus, since \( h_{k+1}^k = 1 \) by definition, it follows by (27) for \( i = k \) that
\[ 1 - h_k^k \leq \frac{1}{r} (h_k^k - h_{k-1}^k) + \frac{1}{rn} h_k^k \]
\[ \leq \frac{1}{r^2} (h_k^k - h_{k-2}^k) + \frac{1}{rn} h_k^k + \frac{1}{r} \cdot \frac{1}{rn} h_k^k \]
\[ \leq \ldots \]
\[ \leq \frac{1}{r^k} (h_k^k - h_0^k) + \frac{1}{rn} (h_k^k + \frac{1}{r} h_{k-1}^k + \ldots + \frac{1}{r^{k-1}} h_1^k) \]
Therefore, since \( h_1^k \leq h_2^k \leq \ldots \leq h_k^k \) and \( h_k^k - h_0^k \leq \frac{1}{rn} \) by (21), it follows that
\[ 1 - h_k^k \leq \frac{1}{r^{k+1}n} + \frac{1}{rn} h_k^k (1 + \frac{1}{r} + \ldots + \frac{1}{r^{k-1}}) \]
\[ \leq \frac{1}{r^{k+1}n} + \frac{1}{r} + \ldots + \frac{1}{r^{k-1}} \]
\[ = \frac{1}{r^{k+1}n} + \frac{n(r-1)}{(r-1)n} h_k^k \leq \frac{1}{r^{k+1}n} + \frac{1}{n(r-1)} h_k^k \]
Therefore
\[ h_k^k \geq \frac{n(r-1) - 1}{n(r-1) + 1} \]
\[ h_k^k \geq 1 - \frac{2}{n(r-1) + 1} \]  

**Lemma 4** In the Markov chain \( M_k^1 \), for any \( r > 1 \),

\[ h_0^k \geq 1 - \frac{k + 2}{n(r-1)} \]  

**Proof.** Let \( 1 \leq i \leq k \). Recall by (27) in the proof of Lemma 3 that

\[
\begin{align*}
    h_{i+1}^k - h_i^k &\leq \frac{1}{r} (h_i^k - h_{i-1}^k) + \frac{1}{rn} h_i^k \\
    &\leq \frac{1}{r^2} (h_i^k - h_{i-1}^k) + \frac{1}{r^2} h_{i-1}^k + \frac{1}{r^3} h_{i-2}^k + \cdots + \frac{1}{r^{i+1}} h_i^k \\
    &\leq \frac{1}{r^2} (h_i^k - h_{i-1}^k) + \frac{1}{r^2} (1 + \frac{1}{r} + \cdots + \frac{1}{r^{i-1}}) \\
    &\leq \frac{1}{r^2} (h_i^k - h_{i-1}^k) + \frac{1}{n(r-1)}
\end{align*}
\]

Therefore it follows by (27) that

\[ h_{i+1}^k - h_i^k \leq \frac{1}{r^{i+1}n} + \frac{1}{n(r-1)} \]  

(30)

Summing up (30) for every \( i = 1, 2, \ldots, k \), it follows that

\[
\begin{align*}
    1 - h_i^k &\leq \frac{1}{n} \left( \frac{1}{r^2} + \cdots + \frac{1}{r^{k+1}} \right) + \frac{k}{n(r-1)} \\
    &\leq \frac{1}{n} \frac{1}{r(r-1)} + \frac{k}{n(r-1)} \leq \frac{k + 1}{n(r-1)}
\end{align*}
\]

since \( h_{k+1}^k = 1 \), and thus

\[ h_i^k \geq 1 - \frac{k + 1}{n(r-1)} \]  

(31)

Therefore it follows now by (20) that

\[
\begin{align*}
    h_0^k &\geq \frac{rn}{rn + 1} \left( 1 - \frac{k + 1}{n(r-1)} \right) \geq \frac{n(r-1)}{n(r-1) + 1} \left( 1 - \frac{k + 1}{n(r-1)} \right) \\
    &= \frac{n(r-1) - k - 1}{n(r-1) + 1} = 1 - \frac{k + 2}{n(r-1) + 1} \geq 1 - \frac{k + 2}{n(r-1)}
\end{align*}
\]

(32)

**Lemma 5** In the Markov chain \( M_k^2 \), for any \( r > 5 \),

\[ h_k^k \geq 1 - \frac{64r}{(r - 5)(r - 1)} \cdot \frac{n}{(n - k)^2} \]
Proof. For \( i = k \), the value of \( s_i^k \) in the Markov chain \( \mathcal{M}_2^k \) is

\[
s_i^k = h_i^k \cdot s_{i+1}^k + (1 - h_i^k) \cdot 0 = h_i^k \cdot s_{i+1}^k
\]

Therefore Lemma 3 implies that

\[
s_i^k \geq (1 - \frac{2}{n(r-1)+1})s_{i+1}^k \geq (1 - \frac{2}{n(r-1)})s_{i+1}^k
\]

and thus

\[
s_{i+1}^k - s_i^k \leq \frac{2}{n(r-1)}s_{i+1}^k \leq \frac{2}{n(r-1)}
\]

Furthermore, for every \( i \in \{k+1, \ldots, n\} \), the value of \( s_i^k \) in \( \mathcal{M}_2^k \) can be computed as follows.

\[
s_i^k = \alpha_i^k \cdot s_{i+1}^k + \beta_i^k \cdot s_{i-1}^k + \gamma_i^k \cdot 1
\]

where

\[
\alpha_i^k = \frac{r \cdot i(n-i)}{\Sigma_i^k}
\]

\[
\beta_i^k = \frac{(i-k) + \frac{i(n-i)}{n}}{\Sigma_i^k} = \frac{(i-k)n + i(n-i)}{n \cdot \Sigma_i^k}
\]

\[
\gamma_i^k = \frac{r \cdot i-k}{\Sigma_i^k}
\]

and \( \Sigma_i^k = r \cdot \frac{i(n-i)}{n} + \frac{(i-k)n + i(n-i)}{n} + r \cdot \frac{i-k}{n} \). Therefore (36) implies that

\[
s_{i+1}^k - s_i^k = \frac{\beta_i^k}{\alpha_i^k} (s_i^k - s_{i-1}^k) - \frac{\gamma_i^k}{\alpha_i^k} (1 - s_i^k)
\]

Furthermore, (37) implies that

\[
\frac{\beta_i^k}{\alpha_i^k} = \frac{1}{r} \cdot (1 + \frac{(i-k)n}{i(n-i)})
\]

\[
\frac{\gamma_i^k}{\alpha_i^k} = \frac{i-k}{i(n-i)} \geq \frac{i-k}{i} \cdot \frac{1}{n}
\]

We now prove that \( \frac{\beta_i^k}{\alpha_i^k} \leq \frac{5}{r} \), whenever \( i \leq \frac{n+k}{2} \). Suppose first that \( k \leq \frac{n}{2} \). Then \( i \leq \frac{n+k}{2} \leq \frac{n+n}{2} \), i.e. \( i \leq \frac{3n}{4} \). Thus \( \frac{1}{n-i} \leq \frac{4}{n} \), and thus (39) implies that \( \frac{\beta_i^k}{\alpha_i^k} \leq \frac{1}{r} \cdot (1+4) = \frac{5}{r} \). Suppose now that \( n \geq k > \frac{n}{2} \). Then also \( i > \frac{n}{2} \), since \( i \geq k+1 \), and thus \( \frac{i}{n-i} < 2 \). Furthermore \( i-k \leq n-i \), since \( i \leq \frac{n+k}{2} \). Therefore \( \frac{(i-k)n}{i(n-i)} = \frac{i-k}{n-i} \cdot \frac{n}{n-i} < 2 \), and thus (39) implies that \( \frac{\beta_i^k}{\alpha_i^k} < \frac{1}{r} \cdot (1+2) = \frac{3}{r} \). Summarizing, for every \( k \in \{1, 2, \ldots, n-1\} \) and every \( i \in \{k+1, \ldots, \frac{n+k}{2}\} \),

\[
\frac{\beta_i^k}{\alpha_i^k} \leq \frac{5}{r}
\]

Therefore it follows by (38), (40), and (41) that

\[
s_{i+1}^k - s_i^k \leq \frac{5}{r} (s_i^k - s_{i-1}^k) - \frac{i-k}{in} (1 - s_i^k)
\]
Thus, in particular
\[
\begin{align*}
s_i^k - s_{i-1}^k &\leq \frac{5}{r}(s_{k-1}^k - s_{k-2}^k) \\
&\leq \left(\frac{5}{r}\right)^{i-k-1}(s_{k+1}^k - s_k^k) 
\end{align*}
\]
(43)

Now (42) and (43) imply that
\[
s_{i+1}^k - s_i^k \leq \left(\frac{5}{r}\right)^{i-k}(s_{k+1}^k - s_k^k) - \frac{i-k}{m} (1 - s_i^k)
\]
(44)

Note that \(s_i^k = s_k^k + (s_{k+1}^k - s_k^k) + (s_{k+2}^k - s_{k+1}^k) + \ldots + (s_i^k - s_{i-1}^k)\). Thus (43) implies that
\[
\begin{align*}
s_i^k &\leq s_k^k + (s_{k+1}^k - s_k^k) \cdot (1 + \frac{5}{r} + \ldots + \left(\frac{5}{r}\right)^{i-k-1}) \\
&\leq s_k^k + (s_{k+1}^k - s_k^k) \cdot \frac{1}{1 - \frac{5}{r}} \\
&= s_k^k + (s_{k+1}^k - s_k^k) \cdot \frac{r}{r - 5}
\end{align*}
\]
(45)

Therefore (44) and (45) imply that
\[
s_{i+1}^k - s_i^k \leq \left(\frac{5}{r}\right)^{i-k}(s_{k+1}^k - s_k^k) - \frac{i-k}{m} (1 - s_k^k - (s_{k+1}^k - s_k^k) \cdot \frac{r}{r - 5})
\]
(46)

Note that (46) holds also for \(i = k\) and that in this case it becomes an equality. Summing up (46) for every \(i \in \{k, \ldots, \frac{n+k}{2}\}\), it follows that
\[
\begin{align*}
s_{n+k+1}^k - s_k^k &\leq (1 + \frac{5}{r} + \ldots + \left(\frac{5}{r}\right)^{\frac{n+k}{2}-k})(s_{k+1}^k - s_k^k) - (1 - s_k^k - (s_{k+1}^k - s_k^k) \cdot \frac{r}{r - 5}) \sum_{i=k+1}^{\frac{n+k}{2}} \frac{i-k}{m}
\end{align*}
\]
(47)

Note now that for any positive numbers \(x, y, z, w > 0\), it holds that \(\frac{x}{y} + \frac{z}{w} > \frac{x+z}{y+w}\). Therefore, for every \(i \in \{k+1, \ldots, \frac{n+k}{2}\}\),
\[
\frac{i-k}{m} + \frac{(\frac{n+k}{2} - i + k + 1) - k}{(\frac{n+k}{2} - i + k + 1)n} > \frac{n+k - k + 1}{(\frac{n+k}{2} + k + 1)n} = \frac{n-k+2}{n(n+3k+2)} > \frac{n-k}{n(n+3k)}
\]

Thus
\[
2 \sum_{i=k+1}^{\frac{n+k}{2}} \frac{i-k}{m} > (\frac{n+k}{2} - k) \cdot \frac{n-k}{n(n+3k)} = \frac{(n-k)^2}{2n(n+3k)}
\]
i.e.
\[
\sum_{i=k+1}^{\frac{n+k}{2}} \frac{i-k}{m} > \frac{(n-k)^2}{4n(n+3k)}
\]
(48)
It follows now by (47) and (48) that
\[ 0 \leq s_{k+1}^k - s_k^k \leq \frac{r}{r-5} (s_{k+1}^k - s_k^k) - \frac{(n-k)^2}{4n(n+3k)} \left( 1 - s_k^k - (s_{k+1}^k - s_k^k) \cdot \frac{r}{r-5} \right) \]
\[ = \frac{r}{r-5} (s_{k+1}^k - s_k^k) \left( 1 + \frac{(n-k)^2}{4n(n+3k)} \right) - \frac{(n-k)^2}{4n(n+3k)} \left( 1 - s_k^k \right) \]
\[ = \frac{r}{r-5} (s_{k+1}^k - s_k^k) \frac{4n(n+3k) + (n-k)^2}{4n(n+3k)} - \frac{(n-k)^2}{4n(n+3k)} (1 - s_k^k) \]
Therefore
\[ (n-k)^2 (1 - s_k^k) \leq \frac{r}{r-5} (s_{k+1}^k - s_k^k) (4n(n+3k) + (n-k)^2) \]
and thus
\[ s_k^k \geq 1 - \frac{r}{r-5} (s_{k+1}^k - s_k^k) \left( 1 + \frac{4n(n+3k)}{(n-k)^2} \right) \quad (49) \]
Now (49) and (55) imply that
\[ s_k^k \geq 1 - \frac{2}{r-5} \frac{2}{n(r-1)} \left( 1 + \frac{4n(n+3k)}{(n-k)^2} \right) \geq 1 - \frac{2}{r-5} \frac{2}{n(r-1)} \frac{4n(n+3k)}{(n-k)^2} \]
\[ = 1 - \frac{16r}{n} \frac{n+3k}{n} \frac{n}{(n-k)^2} \geq 1 - \frac{64r}{(r-5)(r-1)} \frac{n}{(n-k)^2} \]
\[ = 1 - \frac{64r}{(r-5)(r-1)} \frac{1}{\log n} = 1 - O(\frac{1}{\log n}) \]

The next two corollaries follow now from Lemma 5 by direct substitution.

**Corollary 3** In the Markov chain $M_k^n$, for any $r > 5$ and any $k \leq \frac{n}{2}$,
\[ s_k^k \geq 1 - \frac{64r}{(r-5)(r-1)} \frac{4}{n} = 1 - O(\frac{1}{n}) \]

**Corollary 4** In the Markov chain $M_k^n$, for any $r > 5$ and any $k \leq n - \sqrt{n \log n}$,
\[ s_k^k \geq 1 - \frac{64r}{(r-5)(r-1)} \frac{1}{\log n} = 1 - O(\frac{1}{\log n}) \]

We now present an auxiliary lemma that provides a trivial lower bound for the probability $s_k^k$, for any $k \leq n - 1$.

**Lemma 6** In the Markov chain $M_k^n$, for any $r > 5$ and any $k \leq n - 1$,
\[ s_k^k \geq 1 - \frac{1}{n} \]

**Proof.** Let $1 \leq k \leq n - 1$. Recall by (36) and (37) in the proof of Lemma 5 that for $i = k + 1$,
\[ s_{k+1}^k = \frac{\alpha_{k+1}^k \cdot s_{k+2}^k + \beta_{k+1}^k \cdot s_k^k + \gamma_{k+1}^k \cdot 1}{\alpha_{k+1}^k + \beta_{k+1}^k + \gamma_{k+1}^k} \]
Therefore, since $s_{k+2}^k \geq s_{k+1}^k$, it follows that
\[ s_{k+1}^k \geq \frac{\beta_{k+1}^k \cdot s_k^k + \gamma_{k+1}^k \cdot 1}{\beta_{k+1}^k + \gamma_{k+1}^k} \quad (50) \]
In particular, it follows by (50) and (37) for \( i = k + 1 \) that
\[
    s_{k+1}^i \geq \frac{(n + (k + 1)(n - k - 1)) \cdot s_k^i + r \cdot 1}{n + (k + 1)(n - k - 1) + r}
\]
Furthermore recall by (34) in the proof of Lemma 5 that
\[
    s_{k+1}^k \leq \frac{n(r - 1)}{n(r - 1) - 2}s_k^k = (1 + \frac{2}{n(r - 1) - 2})s_k^k
\]
Thus (51) and (52) imply that
\[
(1 + \frac{2}{n(r - 1) - 2})s_k^k \geq \frac{(n + (k + 1)(n - k - 1)) \cdot s_k^k + r \cdot 1}{n + (k + 1)(n - k - 1) + r}
\]
and thus
\[
(n + (k + 1)(n - k - 1) + r)(1 + \frac{2}{n(r - 1) - 2})s_k^k \geq (n + (k + 1)(n - k - 1))s_k^k + r \iff
\]
\[
(n + (k + 1)(n - k - 1)) \frac{2}{n(r - 1) - 2}s_k^k + r(1 + \frac{2}{n(r - 1) - 2})s_k^k \geq r \iff
\]
\[
(2(n + (k + 1)(n - k - 1)) + rn(r - 1))s_k^k \geq r(n(r - 1) - 2)
\]
Therefore
\[
s_k^k \geq \frac{r(n(r - 1) - 2)}{2(n + n^2) + rn(r - 1)} \geq \frac{r(r - 1) - 2n}{2(n + 1) + r(r - 1)}
\]
Note now that \((k + 1)(n - k - 1) < n^2\), and thus the last inequality implies that
\[
s_k^k \geq \frac{r(n(r - 1) - 2)}{2(n + n^2) + rn(r - 1)} \geq \frac{r(r - 1) - 2n}{2(n + 1) + r(r - 1)}
\]
Therefore, since \( r > 5 \) and \( r < n \) by assumption, it follows that
\[
s_k^k \geq \frac{20 - 2}{2(n + 1) + 20} = \frac{9}{n + 11} > \frac{1}{n}
\]

### 3.2.2 Urchin graphs are strong selective amplifiers

In this section we conclude our analysis by combining the results of Section 3.2.1 on the two Markov chains \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). The Markov chain \( \mathcal{M} \) is illustrated in Figure 5, where the transition from state \( P_0^k \) to the states \( P_k, P_0^{k-1} \) is done through the Markov chain \( \mathcal{M}_1 \), and the transition from state \( P_k^k \) to the states \( P_{k+1}^k, P_0^{k-1} \) is done through the Markov chain \( \mathcal{M}_2 \), respectively.

![Figure 5: The Markov chain \( \mathcal{M} \), using the Markov chains \( \mathcal{M}_1^k \) and \( \mathcal{M}_2^k \), where \( k \in \{1, 2, \ldots, n - 1\} \).](image-url)
In Figure 5, the transition probability from state $P_k$ to state $P_{k+1}^k$ (resp. $P_{0}^{k-1}$) is $s_k^k$ (resp. $1 - s_k^k$). Recall that $s_k^k$ is the probability that, starting at $P_k^k$ in $M_2$ (and thus also in $M$), we reach state $F_{k+1}$ before we reach $P_{k-1}$. Furthermore, the transition probability from state $P_0^k$ to state $P_k^k$ is equal to the probability that, starting at $P_k^k$ in $M_1$, we reach $P_k^k$ before we reach $F_{k-1}$. Note that this probability is larger than $h_k^0$. Therefore, in order to compute a lower bound of the fixation probability of a nose in $G_n$, we can assume that in $M$ the transition probability from state $P_0^k$ to state $P_k^k$ (resp. $P_k^{k-1}$) is $h_k^0$ (resp. $1 - h_k^0$), as it shown in Figure 5.

Note that for every $k \in \{2, \ldots, n - 1\}$ the infected vertices of state $P_k^k$ is a strict subset of the infected vertices of state $P_k^k$. Therefore, in order to compute a lower bound of the fixation probability of state $P_0^k$ in $M$, we can relax $M$ by changing every transition from state $P_{k-1}^{k-1}$ to state $P_k^k$ and for sufficiently large $n$. This relaxed Markov chain $\mathcal{M}'$ is illustrate in Figure 6(a). After eliminating the states $P_k^k$ in $\mathcal{M}'$, where $k \in \{1, 2, \ldots, n - 1\}$, we obtain the equivalent birth-death process $\mathcal{B}_n$ that is illustrated in Figure 6(b). Denote by $p_1$ the fixation probability of state $P_0^1$ in $\mathcal{B}_n$, i.e. $p_1$ is the probability that, starting at state $P_0^1$ in $\mathcal{B}_n$, we eventually arrive to state $P_m^n$.

\begin{align*}
\mathcal{M}' : \\
&\begin{array}{c}
\begin{array}{c}
1 - s_k^k \\
0 \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
P_0^1 \\
P_1^1 \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^2 \\
P_1^2 \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^3 \\
P_1^3 \\
0 \\
\end{array} \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P_0^k \\
P_1^k \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^{k+1} \\
P_1^{k+1} \\
0 \\
\end{array} \\
\end{array} \end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P_0^{k+2} \\
P_1^{k+2} \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^{k+3} \\
P_1^{k+3} \\
0 \\
\end{array} \\
\end{array} \\
\end{array}
\end{align*}

\begin{align*}
\mathcal{B}_n : \\
&\begin{array}{c}
\begin{array}{c}
1 - h_k^1 s_k^k \\
0 \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
P_0^1 \\
P_1^1 \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^2 \\
P_1^2 \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^3 \\
P_1^3 \\
0 \\
\end{array} \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P_0^{k} \\
P_1^{k} \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^{k+1} \\
P_1^{k+1} \\
0 \\
\end{array} \\
\end{array} \end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P_0^{k+2} \\
P_1^{k+2} \\
0 \\
\end{array} \\
\begin{array}{c}
P_0^{k+3} \\
P_1^{k+3} \\
0 \\
\end{array} \\
\end{array} \\
\end{array}
\end{align*}

Figure 6: (a) The relaxed Markov chain $\mathcal{M}'$ and (b) the birth-death process $\mathcal{B}_n$ that is obtained from $\mathcal{M}'$ after eliminating the states $P_k^k$ in $\mathcal{M}'$, where $k \in \{1, 2, \ldots, n - 1\}$.

**Theorem 2** For any $r > 5$ and for sufficiently large $n$, the fixation probability $p_1$ of state $P_0^1$ in $\mathcal{B}_n$ is $p_1 \geq 1 - \frac{c(r)}{n}$, for some appropriate function $c(r)$ of $r$.

**Proof.** Denote by $\lambda_k$ the forward bias of $\mathcal{B}_n$ at state $P_0^1$, i.e. $\lambda_k = \frac{h_k^0 s_k^k}{1 - h_k^0 s_k^k}$ is the ratio of the forward over the backward transition probability at state $P_0^k$. Then the fixation probability $p_1$ of state $P_0^1$ in $\mathcal{B}_n$ is

$$p_1 = \frac{1}{1 + \frac{1}{\lambda_1} + \frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_1 \lambda_2 \lambda_3} + \ldots + \frac{1}{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_{n-1}}}$$

Note now by Lemma 4 and Corollary 3 that for every $k \leq \frac{n}{2}$,

$$\lambda_k = \frac{h_k^0 s_k^k}{1 - h_k^0 s_k^k} \geq \frac{(1 - \frac{k+2}{n(r-1)}) (1 - \frac{256r}{(r-5)(r-1)} - \frac{1}{n})}{1 - \frac{(r-1)(r-1)}{n (r-1)} (1 - \frac{256r}{(r-5)(r-1)} - \frac{1}{n})}$$

$$\geq \frac{1 - \frac{256r}{(r-5)(r-1)} - \frac{1}{n} + \frac{k+2}{n(r-1)}}{\frac{256r}{(r-5)} + (k+2)}$$

$$\geq \frac{n(r-1) - \frac{256r}{(r-5)} - (k+2)}{\frac{256r}{(r-5)} + (k+2)}$$
Therefore, since $k \leq \frac{n}{2}$ and $\frac{36r}{(r-5)} < \log n < \frac{n}{2} - 2$ for sufficiently large $n$, it follows by (54) that

$$\lambda_k > \frac{n(r-2)}{\log n + (k+2)} > \frac{n(r-2)}{2\log n + k}$$

and thus

$$\frac{1}{\lambda_k} < \frac{2\log n + k}{n(r-2)}$$

for every $k \leq \frac{n}{2}$. Furthermore, note by Lemma 4 and Corollary 4 that, whenever $\frac{n}{2} < k \leq n - \sqrt{n \log n}$,

$$\lambda_k = \frac{h_0^k 1/n}{1 - h_0^k 1/n} \geq \frac{(1 - \frac{k+2}{n(r-1)}) \cdot \frac{1}{\log n}}{(1 - (1 - \frac{k+2}{n(r-1)}) \cdot \frac{1}{\log n})} \geq \frac{1 - \frac{64r}{(r-5)(r-1)} \cdot \frac{1}{\log n} - \frac{k+2}{n(r-1)}}{\frac{64r}{(r-5)} + \frac{k+2}{n} \log n} = \frac{(r-1) \log n - \frac{64r}{(r-5)} - \frac{k+2}{n} \log n}{\log n - \frac{r-2}{r-1} - \frac{k+2}{n} \log n}$$

Therefore, since $k+2 < n$ and $\frac{64r}{(r-5)} < \frac{\log n}{r-2}$ for sufficiently large $n$, it follows that

$$\lambda_k > \frac{(r-1) \log n - \frac{\log n}{r-2} - \log n}{\log n - \frac{r-2}{r-1} + \log n} = r - 3$$

and thus

$$\frac{1}{\lambda_k} \leq \frac{1}{r - 3}$$

whenever $\frac{n}{2} < k \leq n - \sqrt{n \log n}$. Moreover, note by Lemma 4 and Lemma 6 that, whenever $n - \sqrt{n \log n} < k \leq n - 1$,

$$\lambda_k = \frac{h_0^k 1/n}{1 - h_0^k 1/n} \geq \frac{(1 - \frac{k+2}{n(r-1)}) \cdot \frac{1}{n(r-1)}}{(1 - \frac{k+2}{n(r-1)}) \cdot \frac{1}{n(r-1)}} \geq \frac{1 - \frac{2}{r-1}}{n - 1 + \frac{2}{r-1}}$$

Note now that $1 - \frac{2}{r-1} > \frac{1}{2}$ since $r > 5$ by assumption, and thus the latter inequality implies that $\lambda_k > \frac{1/2}{n}$, i.e.

$$\frac{1}{\lambda_k} < 2n$$

whenever $n - \sqrt{n \log n} < k \leq n - 1$.

Since $r > 5$ by assumption, note now by (55) that $\frac{1}{\lambda_k} < \frac{3\log n}{n(r-2)} < \frac{\log n}{n}$ whenever $k \leq \log n$, and that $\frac{1}{\lambda_k} < \frac{3k}{n(r-2)} < \frac{k}{n}$ whenever $\log n < k \leq \frac{n}{2}$. Therefore, for every $k \in \{2, 3, \ldots, \log n\},$

$$\frac{1}{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k} < \left(\frac{\log n}{n}\right)^k$$

Furthermore, for every $k \in \{\log n + 1, \ldots, \frac{n}{2}\}$,

$$\frac{1}{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_k} < \left(\frac{\log n}{n}\right)^{\log n} \prod_{i=\log n+1}^{k} \frac{i}{n}$$

$$< \left(\frac{\log n}{n}\right)^{\log n} < \left(\frac{\log n}{n}\right)^3$$
Therefore, for every \( k \in \{2, 3, \ldots, \frac{n}{2}\} \),
\[
\frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \log n \left(\frac{\log n}{n}\right)^2 + \frac{n}{2} \left(\frac{\log n}{n}\right)^3
= \frac{\log^3 n}{n^2} + \frac{\log^3 n}{2n^2} = \frac{3\log^3 n}{2n^2} < \frac{1}{n}
\]
for sufficiently large \( n \). Note furthermore by (56) that \( \frac{1}{\lambda_k} < 1 \) whenever \( \frac{n}{2} < k \leq n - \sqrt{n \log n} \), since \( r > 5 \) by assumption. Therefore, for every \( k \in \{\frac{n}{2} + 1, \ldots, n - \sqrt{n \log n}\} \),
\[
\frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_{2n/3}}
\]
and thus it follows by (60) that
\[
\sum_{k=\frac{n}{2}+1}^{n-\sqrt{n \log n}} \frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \left(\frac{n}{2} - \sqrt{n \log n}\right) \left(\frac{\log n}{n}\right)^3 \frac{\log^3 n}{2n^2} < \frac{1}{n}
\]
for sufficiently large \( n \). Let now \( n - \sqrt{n \log n} < k \leq n - 1 \). Then it follows by (57) and (59) that
\[
\frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \left(\frac{\log n}{n}\right)^{\log n} \prod_{i=\log n+1}^{n/2} \frac{i}{n} \cdot (2n)^{k-n-\sqrt{n \log n}}
\]
\[
< \left(\frac{\log n}{n}\right)^{\log n} \left(\frac{1}{n/2-\log n}\right) \cdot (2n)^{\sqrt{n \log n}}
\]
\[
= \frac{2\log n}{n\log n} \cdot (\log n)^{\log n} \cdot 2^{\sqrt{n \log n} \cdot n\sqrt{n \log n}}
\]
\[
< \frac{2\log n \log n \log n}{2^{n/2}} \cdot \frac{2\sqrt{n \log n} \cdot n\sqrt{n \log n}}{2^{n/2}}
\]
\[
= \frac{2^{\log n \log n + \sqrt{n \log n} + \log n \sqrt{n \log n}}}{2^{n/2}}
\]
However
\[
\log n \log n + \sqrt{n \log n} + \log n \sqrt{n \log n} < \frac{n}{4}
\]
for sufficiently large \( n \), and thus (63) implies that
\[
\frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \frac{1}{2^{n/4}}
\]
for every \( k \in \{n - \sqrt{n \log n} + 1, \ldots, n - 1\} \). Therefore
\[
\sum_{k=n-\sqrt{n \log n}+1}^{n-1} \frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \frac{n}{2^{n/4}} < \frac{1}{n}
\]
for sufficiently large \( n \). Thus, summing up (61), (62), and (64), it follows that
\[
\sum_{k=2}^{n-1} \frac{1}{\lambda_1\lambda_2\lambda_3\ldots\lambda_k} < \frac{3}{n}
\]
For \( k = 1 \), (54) implies that
\[
\lambda_k \geq \frac{n(r - 1) - \frac{256r}{(r-5)} - 3}{\frac{256r}{(r-5)} + 3} > n \frac{(r - 2)}{\frac{256r}{(r-5)} + 3} = \frac{n(r - 2)(r - 5)}{259r - 15}
\]
and thus
\[
\frac{1}{\lambda_k} < \frac{259r - 15}{(r - 2)(r - 5)} \cdot \frac{1}{n}
\]

(65)

Summarizing, it follows by (55), (65), and (66) that
\[
p_1 = \frac{1}{1 + \frac{1}{n} \left(3 + \frac{259r - 15}{(r - 2)(r - 5)} \right)} \geq 1 - \frac{c(r)}{n}
\]
where \( c(r) \) is an appropriate function of \( r \). This completes the proof of the theorem. ■

We are now ready to provide our main result in this section.

**Theorem 3** The class \( \mathcal{G} = \{G_n : n \geq 1\} \) of urchin graphs is a class of strong selective amplifiers.

**Proof.** Consider the urchin graph \( G_n \), where \( n \geq 1 \). Let \( v \) be a nose in \( G_n \). Then the fixation probability \( f_r(v) \) of \( v \) in the generalized Moran process is greater than or equal to the fixation probability of state \( P_0^1 \) in the Markov chain \( \mathcal{M} \) of Figure 5 (cf. Corollary 2 and the discussion after it in Section 3.2). Furthermore, the fixation probability of state \( P_0^1 \) in the Markov chain \( \mathcal{M} \) is greater than or equal to the fixation probability \( p_1 \) of state \( P_0^1 \) in the birth-death process \( \mathcal{B}_n \) in Figure 6(b). Therefore, since there exist exactly \( 2n \) noses in \( G_n \), it follows by Definition 2 that the class \( \mathcal{G} \) of urchin graphs is a class of \( (\frac{n}{2}, n) \)-selective amplifiers, and thus \( \mathcal{G} \) is a class of strong selective amplifiers. ■

## 4 Suppressor bounds

In this section we prove the first non-trivial lower bound for the fixation probability of an arbitrary undirected graph, namely the **Thermal Theorem** (Section 4.1), which generalizes the analysis of the fixation probability of regular graphs [16]. Furthermore we present for every function \( \phi(n) \), where \( \phi(n) = \omega(1) \) and \( \phi(n) \leq \sqrt{n} \), a class of \( (\frac{n}{\phi(n)+1}, \frac{n}{\phi(n)}) \)-selective suppressors in Section 4.2.

### 4.1 The Thermal Theorem

Consider a graph \( G = (V, E) \) and a fitness value \( r > 1 \). Denote by \( \mathcal{M}_r(G) \) the generalized Moran process on \( G \) with fitness \( r \). Then, for every subset \( S \notin \{\emptyset, V\} \) of vertices, the fixation probability \( f_r(S) \) of \( S \) in \( \mathcal{M}_r(G) \) is given by (1), where \( f_r(\emptyset) = 0 \) and \( f_r(V) = 1 \). That is, the fixation probabilities \( f_r(S) \), where \( S \notin \{\emptyset, V\} \), are the solution of the linear system (1) with boundary conditions \( f_r(\emptyset) = 0 \) and \( f_r(V) = 1 \).

Suppose that at some iteration of the generalized Moran process the set \( S \) of vertices is infected and that the edge \( xy \in E \) (where \( x \in S \) and \( y \notin S \)) is activated, i.e. either \( x \) infects \( y \) or \( y \) disinfects \( x \). Then (1) implies that the probability that \( x \) infects \( y \) is higher if \( \frac{1}{\deg x} \) is large; similarly, the probability that \( y \) disinfects \( x \) is higher if \( \frac{1}{\deg y} \) is large. Therefore, in a fashion similar to [16], we call for every vertex \( v \in V \) the quantity \( \frac{1}{\deg v} \) the **temperature** of \( v \): a “hot” vertex (i.e. with high temperature) affects more often its neighbors than a “cold” vertex (i.e. with low temperature).

It follows now by (1) that for every set \( S \notin \{\emptyset, V\} \) there exists at least one pair \( x(S), y(S) \) of vertices with \( x(S) \in S \), \( y(S) \notin S \), and \( x(S)y(S) \in E \) such that
\[
f_r(S) \geq \frac{r}{\deg x(S)} f_r(S + y(S)) + \frac{1}{\deg y(S)} f_r(S - x(S))
\]

(67)
Thus, solving the linear system that is obtained from (67) by replacing inequalities with equalities, we obtain a lower bound for the fixation probabilities \( f_r(S) \), where \( S \not\in \{\emptyset, V\} \). In the next definition we introduce a weighted generalization of this linear system, which is a crucial tool for our analysis in obtaining the Thermal Theorem.

**Definition 3 (the linear system \( L_0 \))** Let \( G = (V, E) \) be an undirected graph and \( r > 1 \). Let every vertex \( v \in V \) have weight (temperature) \( d_v > 0 \). The linear system \( L_0 \) on the variables \( p_r(S) \), where \( S \subseteq V \), is given by the following equations whenever \( S \not\in \{\emptyset, V\} \):

\[
p_r(S) = \frac{rd_x(S)p_r(S + y(S)) + d_y(S)p_r(S - x(S))}{rd_x(S) + d_y(S)}
\]

(68)

with boundary conditions \( p_r(\emptyset) = 0 \) and \( p_r(V) = 1 \).

With a slight abuse of notation, whenever \( S = \{u_1, u_2, \ldots, u_k\} \), we denote \( p_r(u_1, u_2, \ldots, u_k) = p_r(S) \).

**Observation 1** The linear system \( L_0 \) in Definition 3 corresponds naturally to the Markov chain \( M_0 \) with one state for every subset \( S \subseteq V \), where the states \( \emptyset \) and \( V \) are absorbing, and every non-absorbing state \( S \) has exactly two transitions to the states \( S + y(S) \) and \( S - x(S) \) with transition probabilities \( q_S = \frac{rd_x(S)}{rd_x(S) + d_y(S)} \) and \( 1 - q_S \), respectively.

**Observation 2** Let \( G = (V, E) \) be a graph and \( r > 1 \). For every vertex \( x \in V \) let \( d_x = \frac{1}{d_{\text{deg}_x}} \) be the temperature of \( x \). Then \( f_r(S) \geq p_r(S) \) for every \( S \subseteq V \), where the values \( p_r(S) \) are the solution of the linear system \( L_0 \).

Before we provide the Thermal Theorem (Theorem 4), we first prove an auxiliary result in the next lemma which generalizes the Isothermal Theorem of [16] for regular graphs, i.e. for graphs with the same number of neighbors for every vertex.

**Lemma 7** Let \( G = (V, E) \) be a graph with \( n \) vertices, \( r > 1 \), and \( d_u \) be the same for all vertices \( u \in V \). Then for every vertex \( u \in V \),

\[
p_r(u) = \frac{1 - \frac{1}{r}}{1 - \frac{1}{r^n}} \geq 1 - \frac{1}{r}
\]

**Proof.** Since \( d_u \) is the same for all vertices \( u \in V \), it follows by (68) that for every set \( S \not\in \{\emptyset, V\} \), the forward probability is \( q_S = \frac{1}{r^{|S|}} \) and the backward probability is \( 1 - q_S = \frac{1}{1 - \frac{1}{r}} \). Therefore, by symmetry, \( p_r(S) = p_r(S') \) whenever \( |S| = |S'| \). For every \( 0 \leq k \leq n \) denote by \( p_k = p_r(S) \), where \( |S| = k \). Note that \( p_0 = 0 \) and \( p_n = 1 \). Then it follows by (68) that, whenever \( 1 \leq k \leq n - 1 \),

\[
p_{k+1} - p_k = \frac{1}{r}(p_k - p_{k-1}) = \cdots = \frac{1}{r^k}(p_1 - p_0)
\]

Therefore, summing up these equations for every \( 1 \leq k \leq n - 1 \) it follows that

\[
p_n - p_1 = (p_1 - p_0)(1 + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{n-1}})
\]

and thus, since \( p_0 = 0 \) and \( p_n = 1 \),

\[
p_1 = \frac{1}{1 + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{n-1}}} = \frac{1 - \frac{1}{r}}{1 - \frac{1}{r^n}} \geq 1 - \frac{1}{r}
\]

We are now ready to provide our main result in this section which provides the first non-trivial lower bound for the fixation probability on arbitrary graphs, parameterized by the maximum ratio between two different temperatures in the graph.
Theorem 4 (Thermal Theorem) Let $G = (V, E)$ be a connected undirected graph and $r > 1$. Then $f_r(v) \geq \frac{r^{n-1}}{r + \deg_{\text{min}}}$ for every $v \in V$.

Proof. Let $G$ have $n$ vertices, i.e. $|V| = n$. Our proof is based on the linear system $L_0$ of Definition 3. Namely, we consider the linear system $L_0$ with weight $d_v = \frac{1}{\deg v}$ for every vertex $v \in V$. Note that $d_{\text{min}} = \frac{1}{\deg_{\text{max}}}$ and $d_{\text{max}} = \frac{1}{\deg_{\text{min}}}$. Recall that $M_0$ is the Markov chain that can be defined from the linear system $L_0$ (cf. Observation 1), and that every state $S \notin \{\emptyset, V\}$ of $M_0$ has exactly two transitions, namely to states $S + y(S)$ and $S - x(S)$.

We now define the Markov chain $M_0^*$ from $M_0$ as follows. Consider an arbitrary state $S \subseteq V$ such that $1 \leq |S| \leq n - 2$. Denote $x(S) = u$ and $y(S) = v$ (note that $v \notin S$). Furthermore denote $x(S + v) = x_0$ and $y(S + v) = y_0$. Then perform the following changes to the Markov chain $M_0$:

Step A. add a new dummy state $X_S$ to $M_0$,

Step B. replace the transition from $S$ to $S + v$ by a transition from $S$ to $X_S$ (with the same transition probability $q_S$),

Step C. add to state $X_S$ the transitions to states $S + v + y_0$ and $S + v - x_0$, with transition probabilities $q_{S + v}$ and $1 - q_{S + v}$, respectively.

An example of the application of the above Steps A, B, C is illustrated in Figure 7. Denote by $M_0^*$ the Markov chain obtained after applying these steps to $M_0$ for every state $S \subseteq V$ with $1 \leq |S| \leq n - 2$. Furthermore denote by $L_0^*$ the linear set that corresponds to $M_0^*$. Note that $L_0^*$ has the additional variables $\{p_r(X_S) : 1 \leq |S| \leq n - 2\}$ that do not exist in $L_0$. Moreover, for every state $X_S$ of $M_0^*$, note by the construction of $M_0^*$ that $p_r(X_S) = p_r(S + y(S))$ in the solution of $L_0^*$.

Figure 7: Parts of (a) the Markov chain $M_0$ and of (b) the Markov chain $M_0^*$ (after the execution of Steps A, B, and C).

Consider an arbitrary numbering $v_0, v_1, \ldots, v_{n-1}$ of the $n$ vertices of $G$. For every $i \in \{1, 2, \ldots, n - 1\}$, we construct iteratively the linear system $L_i^*$ (and the corresponding Markov chain $M_i^*$) from $L_{i-1}^*$ (and from the corresponding Markov chain $M_{i-1}^*$) as follows. Consider a state $S \notin \{\emptyset, V\}$, where $y(S) = v_i$ (note that in this case $v_i \notin S$). Denote $x(S) = u$, $x(S + v_i) = x_0$, and $y(S + v_i) = y_0$. Then perform the following changes to the Markov chain $M_{i-1}^*$:

Step 1. replace the transition from $X_S$ to $S + v_i + y_0$ by a transition from $X_S$ to $S + y_0$ (with the same transition probability $q_{S + v_i}$),

Step 2. replace the transition from $X_S$ to $S + v_i - x_0$ by a transition from $X_S$ to $S - x_0$ (with the same transition probability $1 - q_{S + v_i}$).

Denote by $M_i^*$ the Markov chain obtained by iteratively applying Steps 1 and 2 to $M_{i-1}^*$ for every state $S \notin \{\emptyset, V\}$ with $y(S) = v_i$. Note that for every state $S_1 \notin \{\emptyset, V\}$ and for every state $S_2 \neq V$, where $v_i \in S_2$ and $v_i \notin S_1$, there exists no transition path in $M_i^*$ from $S_1$ to $S_2$. Furthermore denote by $L_i^*$ the linear system that corresponds to $M_i^*$ (cf. Observation 1). Note that, in the above Step 2, if $x_0 = v_i \notin S$ then
Therefore the forward probability of state $S$ in $\mathcal{M}_i^+$ is (after eliminating the state $X_S$) equal to

$$\frac{q_s q_{S+v_i}(S+y_0) + (1-q_s)p^i_v(S)}{q_{S+v_i}(S+y_0) + (1-q_s)p^i_v(S)}\quad (69)$$
**Case 2:** \(x_0 \neq v_i\), cf. Figure 9. In this case we have in the linear system \(L_i^*\) that

\[
p_i^v(S) = q_s p_i^v(X_S) + (1 - q_s) p_i^v(S - u)
\]

\[
p_i^v(X_S) = q_{S+v, i}^v(S + y_0) + (1 - q_{S+v, i}) p_i^v(S - x_0)
\]

and thus

\[
p_i^v(S) = q_s q_{S+v, i}^v(S + y_0) + q_s (1 - q_{S+v, i}) p_i^v(S - x_0) + (1 - q_s) p_i^v(S - u)
\]

where

\[
q_s = \frac{r d_u}{r d_u + d_v}
\]

\[
q_{S+v, i} = \frac{r d_{x_0}^i}{r d_{x_0} + d_{y_0}}
\]

Therefore the forward probability of state \(S\) in \(M_t^*\) is (after eliminating the state \(X_S\)) equal to

\[
q_s q_{S+v, i} = \frac{r d_u}{r d_u + d_v} \cdot \frac{r d_{x_0}^i}{r d_{x_0} + d_{y_0}}
\]

(70)

It follows now by Cases 1 and 2 (cf. (69) and (70)) that the forward probability of state \(S\) in \(M_t^*\) (after eliminating the state \(X_S\)) is a monotone decreasing function of \(d_{v_i}\). Therefore, for every state \(S' \subseteq V\) with \(v_i \notin S'\), the value \(p_i^v(S')\) is also a monotone decreasing function of \(d_{v_i}\). Thus, in particular, also the value \(p_j^v(v_i)\), where \(j \neq i\), is a monotone decreasing function of \(d_{v_i}\). We now increase the value of \(d_{v_i}\) to \(d_{\max}\) in \(L_i^*\). Thus for every \(j \neq i\), the value \(p_j^v(v_i)\) decreases after this change.

Recall that \(f_r(v_0) \geq p_r^0(v_0)\) by Observation 2. Therefore, since also \(p_r^0(v_0) \geq \ldots \geq p_r^{n-2}(v_0) \geq p_r^{n-1}(v_0)\), it follows that \(f_r(v_0) \geq p_r^{n-1}(v_0)\), i.e. \(p_r^{n-1}(v_0)\) is a lower bound for the fixation probability \(f_r(v_0)\) in the Markov chain \(M_r(G)\). Furthermore \(d_{v_1} = d_{v_2} = \ldots = d_{v_{n-1}} = d_{\max}\) in the linear system \(L_{n-1}^*\). Consider now the state \(S = \{v_0\}\) in the Markov chain \(M_{n-1}^*\), and let \(y(S) = v_{i_0}\), where \(1 \leq i_0 \leq n - 1\). Note that \(x(S) = v_0\). Then the value \(p_r^{n-1}(v_0)\) equals

\[
p_r^{n-1}(v_0) = \frac{r d_{v_0} p_r^{n-1}(v_0, v_{i_0}) + d_{v_{i_0}} p_r^{n-1}(0)}{r d_{v_0} + d_{v_{i_0}}}
\]

(71)

cf. Definition 3. Recall that \(d_{v_0} = \frac{1}{\deg v_0}\) and \(d_{\max} = \frac{1}{\deg_{\min}}\) by definition. Thus, since \(f_r(v_0) \geq p_r^{n-1}(v_0)\) as we proved above, (71) implies that

\[
f_r(v_0) \geq p_r^{n-1}(v_0) \geq \frac{r}{r + \frac{\deg_{v_0}}{\deg_{\min}}} p_r^{n-1}(v_{i_0})
\]

(72)

Now, similarly to the above transformations of the linear system \(L_{i-1}^*\) to \(L_i^*\), where \(1 \leq i \leq n - 1\), we construct the linear system \(L_i^*\) (and the corresponding Markov chain \(M_t^*\)) from \(L_{n-1}^*\) (and from the corresponding Markov chain \(M_{n-1}^*\)), by applying iteratively the above Steps 1 and 2 to the states \(S \subseteq V\), where \(y(S) = v_0\) (instead of \(y(S) = v_i\) above). Furthermore we increase the value of \(d_{v_0}\) to \(d_{\max}\) in the resulting linear system \(L_i^*\). Then, similarly to the construction of \(L_i^*\), where \(1 \leq i \leq n - 1\), it follows that \(p_r^i(v_j) \leq p_r^{n-1}(v_j)\) for every \(j \neq 0\). Thus, in particular, \(p_r^i(v_{i_0}) \leq p_r^{n-1}(v_{i_0})\). Furthermore \(d_{v_1} = d_{v_2} = \ldots = d_{v_{n-1}} = d_{\max}\) in \(L_i^*\), and thus \(p_r^i(v_{i_0}) \geq 1 - \frac{1}{r}\) by Lemma 3. Therefore, since \(p_r^i(v_{i_0}) \leq p_r^{n-1}(v_{i_0})\), it follows by (72) that

\[
f_r(v_0) \geq \frac{r}{r + \frac{\deg_{v_0}}{\deg_{\min}}} p_r^i(v_{i_0}) \geq \frac{(r - 1)}{r + \frac{\deg_{v_0}}{\deg_{\min}}}
\]

(73)
Lemma 8 For every $v \in V^2_{\phi(n),n}$ and sufficiently large $n$,
\[ f_r(v) < 5r \cdot \frac{\phi(n)}{n} \]

**Proof.** Denote by $S_k$ the state, in which exactly $k \geq 0$ vertices of $V^1_{\phi(n),n}$ are infected and all vertices of $V^2_{\phi(n),n}$ are not infected. Note that $S_0$ is the empty state. Furthermore denote by $F_k$ the state where exactly $k \geq 0$ vertices of $V^1_{\phi(n),n}$ and at least one vertex of $V^2_{\phi(n),n}$ are infected. In order to compute an upper bound for the fixation probability $f_r(S_1)$ (i.e. of the fixation probability $f_r(v)$ where $v \in V^1_{\phi(n),n}$), we can set the value $f_r(S_k)$ and the values $f_r(F_k)$ for every $k \geq 1$ to their trivial upper bound 1. That is, we assume that the state $S_{\frac{n}{2}}$ as well as all states $F_k$, where $k \geq 1$, are absorbing. After performing these relaxations (and eliminating self loops), we obtain a Markov chain, whose state graph is illustrated in Figure 10(b) For any $1 \leq k \leq \frac{n}{2} - 1$ in this relaxed Markov chain,
\[ f_r(S_k) = \alpha_k f_r(S_{k+1}) + \beta_k f_r(S_{k-1}) + \gamma_k \] (74)
where
\[ \alpha_k = \frac{r k}{n^{n-k}} \sum_k \left( \frac{\phi^2(n)}{\phi(n)} + \frac{n-k}{n+\phi^2(n)-1} \right) \]
\[ \beta_k = \frac{k}{n^{n-k}} \sum_k \frac{\phi^2(n)}{\phi(n)} \]
\[ \gamma_k = \frac{r k}{n^{n-k}} \sum_k \frac{\phi^2(n)}{\phi(n)} \]

Note now by (75) that
\[ \beta_k = \frac{1}{r} \left( 1 + \frac{\phi^2(n)}{\phi(n)} \frac{(n + \phi^2(n) - 1)}{(n - k)} \right) > \frac{\phi(n)}{r} > 1 \]

Furthermore, since \( k \leq \frac{n}{2} \), it follows by (75) that
\[ \gamma_k = \frac{\phi^2(n)}{n - k} < \frac{2 \phi^2(n)}{n} \]

Now, since \( \alpha_k + \beta_k + \gamma_k = 1 \) and \( f_r(S_k) \geq f_r(S_{k-1}) \) for every \( k \), (76) implies by (77) that
\[ f_r(S_{k+1}) - f_r(S_k) = \frac{\beta_k}{\alpha_k} (f_r(S_k) - f_r(S_{k-1})) - \frac{\gamma_k}{\alpha_k} (1 - f_r(S_k)) > \frac{\phi(n)}{r} (f_r(S_k) - f_r(S_{k-1})) - 2 \frac{\phi^2(n)}{n} \]
\[ > \left( \frac{\phi(n)}{r} \right)^2 \cdot (f_r(S_1) - f_r(S_0)) - 2 \frac{\phi^2(n)}{n} \cdot \sum_{i=0}^{k-1} \left( \frac{\phi(n)}{r} \right)^i \]
\[ = \left( \frac{\phi(n)}{r} \right)^2 \cdot f_r(S_1) - 2 \frac{\phi^2(n)}{n} \cdot \left( \frac{\phi(n)}{r} \right)^k - 1 \]

Thus, since \( f_r(S_{\frac{n}{2}}) = 1 \) in the relaxed Markov chain, we have that
\[ 1 - f_r(S_1) = \sum_{k=1}^{n/2} (f_r(S_{k+1}) - f_r(S_k)) > \sum_{k=1}^{n/2} \left[ \left( \frac{\phi(n)}{r} \right)^k \cdot f_r(S_1) - 2 \frac{\phi^2(n)}{n} \cdot \left( \frac{\phi(n)}{r} \right)^k - 1 \right] \]

Therefore
\[ f_r(S_1) \sum_{k=0}^{n/2-1} \left( \frac{\phi(n)}{r} \right)^k < 1 + 2 \frac{\phi^2(n)}{n} \cdot \left( \frac{\phi(n)}{r} \right)^{n/2-1} \sum_{k=0}^{n/2-1} \left[ \left( \frac{\phi(n)}{r} \right)^k - 1 \right] \]
and thus
\[ f_r(S_1) < 2 \frac{\phi^2(n)}{n} \frac{\phi(n)}{r} - 1 + \frac{1}{\sum_{k=0}^{n/2-1} \left( \frac{\phi(n)}{r} \right)^k} \]
\[ = 2r \frac{\phi^2(n)}{n} (\phi(n) - r) + \frac{1}{\sum_{k=0}^{n/2-1} \left( \frac{\phi(n)}{r} \right)^k} \]
Therefore, since \( \phi(n) = \omega(1) \) and \( r \) is constant by assumption, it follows that \( r \leq \frac{\phi(n)}{2} \) for sufficiently large \( n \), and thus
\[
f_r(S_1) < 4r \frac{\phi(n)}{n} + \frac{1}{n} < 5r \frac{\phi(n)}{n}
\]

Using Lemma 8 we can now prove the next theorem.

**Theorem 5** For every function \( \phi(n) \), where \( \phi(n) = \omega(1) \) and \( \phi(n) \leq \sqrt{n} \), the class \( G_{\phi(n)} = \{G_{\phi(n),n} : n \geq 1\} \) of \( \phi(n) \)-urchin graphs is a class of \( (\frac{n}{\phi(n)+1}, \frac{n}{\phi(n)}) \)-selective suppressors.

**Proof.** It follows by Lemma 8 that, if \( v \in V_{\phi(n),n}^1 \), then \( f_r(v) < \frac{5r \phi(n)}{n} = \frac{5r}{n/\phi(n)} \) for any \( r > 1 \) and sufficiently large \( n \). Therefore, since \( |V_{\phi(n),n}| = \frac{n}{\phi(n)+1} \) for every graph \( G_{\phi(n),n} \), it follows by Definition 2 that the class \( G_{\phi(n)} \) of graphs is a class of \( (\frac{n}{\phi(n)+1}, \frac{n}{\phi(n)}) \)-selective suppressors. ■

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