Vexillary Grothendieck Polynomials via Bumpless Pipe Dreams

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Abstract

In their recent work on the Castelnuovo-Mumford regularity of the matrix Schubert variety, Pechenik, Speyer, and Weigandt introduced a formula for the degree of any Grothendieck polynomial. We give a new proof of this formula in the special case of vexillary permutations and characterize the set of bumpless pipe dreams which contribute maximal degree terms to the Grothendieck polynomial in this case. We also conjecture a generalization of this characterization to bumpless pipe dreams for non-vexillary permutations. Furthermore, we use bumpless pipe dreams to prove new results about the support of vexillary Grothendieck polynomials, addressing special cases of conjectures of Mészáros, Setiabrata, and St. Dizier.

1 Introduction

Introduced by Lascoux and Schützenberger [5], Grothendieck polynomials are a family of polynomials, indexed by the set of permutations $S_n$, which represent K-theoretic classes of the complete flag variety.

Define the divided difference operator $\partial_i$ by $\partial_i(f) = \frac{f - s_if}{x_i - x_{i+1}}$ where $f$ is a polynomial in $x_1, \ldots, x_{n+1}$ and $s_i$ acts on $f$ by interchanging the variables $x_i$ and $x_{i+1}$. Let $\pi_i(f) = \partial_i(f - x_{i+1}f)$. The Grothendieck polynomial associated to a permutation $\omega \in S_n$ is then given by

$$G_\omega(x_1, \ldots, x_n) = x_{n-1}^{n-1}x_{n-2}^{n-2} \cdots x_1$$

where $\omega_0 = n(n-1)(n-2)\ldots 1$ is the permutation in $S_n$ with maximal length, and

$$\mathcal{G}_\omega s_i = \pi_i \mathcal{G}_\omega(x_1, \ldots, x_n)$$

where $s_i = (i, i+1)$ is a simple transposition such that $\omega s_i$ has one fewer inversion than $\omega$.

Various combinatorial models for Grothendieck polynomials include those described by [2, 6, 11] as well as some which apply specifically to Grothendieck polynomials for vexillary permutations (e.g. [3]). One such model, bumpless pipe dreams (see Section 2 for definitions) were introduced by Lam, Lee, and Shimozono [4] to give a formula for Schubert polynomials and extended to a formula for Grothendieck polynomials by Weigandt [11]. Specifically, $\mathcal{G}_\omega$ can be expressed as a sum over the set of marked bumpless pipe dreams associated to $\omega$ [11]. This article gives a characterization of the marked bumpless pipe dreams which contribute monomials of maximal degree to this sum in the case where $\omega$ is a vexillary permutation. If $(r_1, \ldots, r_n)$ is the Rajchgot code of a permutation $\omega$ and $(c_1, \ldots, c_n)$ is the Lehmer code (again, see Section 2), we will show the following.

**Theorem 1.1.** Suppose that $P$ is a bumpless pipe dream corresponding to a vexillary permutation $\omega$. Then the $i$th pipe has at most $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows. In particular, if the marked bumpless pipe dream $(P, U(P))$ corresponds to a maximal degree term of $\mathcal{G}_\omega$, the $i$th pipe will have exactly $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows (i.e. $\square$ tiles).
We also use bumpless pipe dreams to prove new results about the support of Grothendieck polynomials. In particular, Mészáros, Setiabrata, and St. Dizier\textsuperscript{7} conjectured the following.

**Conjecture 1.1** (\textsuperscript{7}, Conjecture 1.2). Let $\omega \in S_n$, and let $x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$ be any monomial with a nonzero coefficient and non-maximal degree in $G_\omega(x_1,\ldots,x_n)$. There exists a monomial $x_1^{j_1}x_2^{j_2}\ldots x_n^{j_n}$ with nonzero coefficient in $G_\omega(x_1,\ldots,x_n)$ such that $j_k = i_k + 1$ for some $1 \leq k \leq n$ and $j_l = i_l$ for all other indices $l$.

**Conjecture 1.2** (\textsuperscript{7}, Conjecture 1.3). Let $\omega \in S_n$, and suppose that $p_1(x_1,\ldots,x_n)$ and $p_2(x_1,\ldots,x_n)$ are monomials with nonzero coefficient in $G_\omega$ such that $p_1|p_2$. Then any monomial $q(x_1,\ldots,x_n)$ satisfying $p_1|q|p_2$ must also have nonzero coefficient in $G_\omega$.

We give proofs of these conjectures in the special case where $\omega$ is vexillary (in Proposition 4.1 and Theorem 4.4, respectively).

Furthermore, we address the following question posed by Weigandt\textsuperscript{12}:

“Can you find ... permutation codes that tell you the lex last monomials in each degree of a Grothendieck polynomial?”

Recall that the lexicographically last monomial in each homogeneous component $G^{(k)}_\omega(x)$ is the leading term in any monomial order satisfying $x_1 < x_2 < \cdots < x_n$. When $\omega$ is vexillary, we prove the following formula for each such leading term.

**Theorem 1.2.** Suppose that $\omega$ is vexillary, and let $m_k(x) = x_1^{e_1}\cdots x_n^{e_n}$ be the monomial such that the leading term of $G^{(k)}_\omega(x)$ in any term order with $x_1 < x_2 < \cdots < x_n$ is a scalar multiple of $m_k(x)$. If $i$ is the largest possible index such that $e_i < r_i$, the leading term of $G^{(k+1)}_\omega(x)$ will be a scalar multiple of $m_{k+1} = m_k(x)x_i$.

Section 2 contains basic definitions and some existing results regarding bumpless pipe dreams and the degree of the Grothendieck polynomial. In Section 3, we give the proof of Theorem 1.1 as well as some consequences, and we conjecture a possible generalization of that result. Section 4 includes facts about the support of vexillary Grothendieck polynomials including Theorem 1.2 and the vexillary cases of Conjectures 1.1 and 1.2.

## 2 Background

A bumpless pipe dream (BPD) is a tiling of the $n \times n$ grid with the tiles

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+ + + + + +
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such that they form a network of $n$ pipes, each running from the bottom edge of the grid to the right edge \textsuperscript{4} \textsuperscript{11}. To any such tiling $P$, there is an associated permutation $\omega$ given by labeling the pipes 1 through $n$ along the bottom edge and then reading off the labels on the right edge, ignoring any crossings after the first between each pair of pipes. In other words, we replace any redundant crossings with bump tiles \textsuperscript{12} before tracing the path of each pipe to the right edge — see Figure 2.1. For a fixed permutation $\omega$, we notate the set of all BPDs associated to $\omega$ as Pipes($\omega$). A bumpless pipe dream is called reduced if each pair of pipes crosses at most once, and the set of reduced BPDs is denoted $\text{RPipes}(\omega)$. 


For a given $\omega$, define the Rothe bumpless pipe dream $\text{RP}_{\omega}$ to be the BPD which has down-elbows (tiles) in the squares $(i,\omega(i))$ for every $i$ and no up-elbows (tiles). It is known that every $P \in \text{RP}_{\omega}$ can be reached from the Rothe BPD by a series of droops, i.e. moves of the form

$$
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
$$

where the region shown contains no other (unpicted) elbow tiles and that every $P \in \text{P}_{\omega}$ can be reached by a series of droops and K-theoretic droops, i.e. moves of the following form.

A marked bumpless pipe dream is an ordered pair $(P,S)$ where $P$ is a bumpless pipe dream and $S$ is some subset of the set $U(P)$ of up-elbow tiles in $P$. Note that if $MP_{\omega}$ is the set of all marked BPDs for $\omega$, then the Grothendieck polynomial associated to $\omega$ is given by

$$
G_{\omega}(x_1,\ldots,x_n) = \sum_{(P,S) \in MP_{\omega}} (-1)^{|D(P)|+|S|-\ell(\omega)} \left( \prod_{(i,j) \in D(P) \cup S} x_i \right)
$$

where $D(P)$ is the set of all blank tiles in $P$. 
The Rajchgot code [8] of a permutation \( \omega \in S_n \) is defined to be \((r_1, r_2, \ldots, r_n)\) where \( r_i \) is the minimum number of elements which must be removed from \( \omega(i), \omega(i+1), \ldots, \omega(n) \) to form an increasing sequence beginning with \( \omega(i) \). The Lehmer code is defined to be \((c_1, \ldots, c_n)\) where \( c_i = |\{ j : i < j, \omega(i) > \omega(j) \}| \). Note that it is always true that \( r_i \geq c_i \). Pechenik, Speyer, and Weigandt proved the following result [8].

**Theorem 2.1** ([8], Theorem 1.1). The degree of the Grothendieck polynomial \( G\omega(x_1, \ldots, x_n) \) is given by \( \sum_{i=1}^{n} r_i \).

Other combinatorial degree formulas have been given for certain special cases of Grothendieck polynomials. In particular, Rajchgot, Ren, Robichaux, St. Dizier, and Weigandt [9] proved a formula for the degree of symmetric Grothendieck polynomials, and this result was extended to vexillary Grothendieck polynomials by Rajchgot, Robichaux, and Weigandt [10]. These formulas rely on the description of vexillary Grothendieck polynomials in terms of flagged set-valued tableux given in [3]. See [11] for the relationship between these tableux and marked bumpless pipe dreams for vexillary permutations.

### 3 Vexillary Bumpless Pipe Dreams

Recall that a vexillary permutation is one that is 2143-avoiding.

**Lemma 3.1** ([11], Lemma 7.2). A permutation is vexillary if and only if the permutation has no non-reduced bumpless pipe dreams.

In particular, we have the following specialization of droop moves.

**Lemma 3.2** ([11], Lemma 7.4). For a vexillary permutation \( \omega \), \( \text{Pipes}(\omega) \) is connected by the following set of local moves.

![Diagram](insert-diagram-here)

**Lemma 3.3.** Let \( \omega \) be a vexillary permutation on \( n \) elements, and let \( P \in \text{Pipes}(\omega) \).

(i) For any \( i, j \) such that \( \omega^{-1}(i) < \omega^{-1}(j) \) or \( i < j \), the \( i^{th} \) pipe will never contain any elbow tiles which are southeast of the \( j^{th} \) pipe.

(ii) \( P \) can be constructed from the Rothe BPD for \( \omega \) by using droop moves to position pipe \( \omega(n) \), then pipe \( \omega(n-1) \), and so on through pipe \( \omega(1) \). Alternatively, we can construct \( P \) by first positioning pipe \( n \), then pipe \( n-1 \), and so on through pipe 1.

**Proof.** Follows immediately from the fact that \( \text{Pipes}(\omega) \) is connected by the local moves of Lemma 3.2.

We can now prove Theorem 1.1 beginning with the following lemma.
Lemma 3.4. For any vexillary $\omega \in S_n$, there exists a BPD such that each pipe $i$ has at least $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows.

Proof. Consider the Rothe BPD $P_R$ for $\omega$. For each pipe $i$, we can define a diagram $D_i$ as follows. Begin with an $m \times m$ grid where $m = n - \omega^{-1}(i) - c_{\omega^{-1}(i)}$, and shade the squares corresponding to the set of empty squares underneath pipe $i$ in $P_R$, pushed as far to the top-left corner as possible. Fill each remaining square above the main anti-diagonal with an X, and leave any square on or below the main anti-diagonal blank (see Figure 3.1).

We will show that whenever there is a column in $D_i$ containing $k$ Xs, there must be an increasing sequence in the one-line notation of $\omega$ beginning with $i$ and having length at least $k + 2$. Suppose that column $j$ contains $k$ Xs (and by definition, $i$ blank squares). This means that there are $k + j$ rows $l > \omega^{-1}(i)$ of $P_R$ which contain fewer than $j$ blank squares underneath pipe $i$ but which satisfy $\omega(l) > i$. Consider the sequence $\omega^{-1}(i_1), \omega^{-1}(i_2), \ldots, \omega^{-1}(i_{k+j})$ of all such rows, ordered from top to bottom. Let $\omega^{-1}(i_a)$ be the first of these rows such that $i_a < i_{a-1}$, and let $r$ be the number of indices $l$ in between $i$ and $i_{a-1}$ such that $\omega^{-1}(l) > \omega^{-1}(i_{a-1})$. Note that there can be at most $j - 1$ such indices since each would correspond to a blank square in position $(l, \omega^{-1}(i_{a-1}))$.

Now, let $i'_1, \ldots, i'_b$ be the sub-sequence of $i_a, \ldots, i_{k+j}$ consisting of the indices $i_t$ that satisfy $i_t > i_{a-1}$, and note that since $|\{i_a, \ldots, i_{k+j}\} - \{i'_1, \ldots, i'_b\}|$ must be smaller than $j$, we know that $b \geq k - a + 2$. Furthermore, we know that $i'_1 < i'_2 < \ldots < i'_b$ (otherwise, we would contradict the assumption that $\omega$ is vexillary), so $i,i_1, \ldots, i_{a-1}, i'_1, \ldots, i'_b$ forms an increasing sequence in $\omega$ of length $k + 2$.

For any pipe $i$, the length of the longest anti-diagonal in the shaded portion of $D_i$ is given by $m - 1 - k_{\text{max}}$ where $k_{\text{max}}$ is the maximal number of Xs in any column of $D_i$. By the above argument, we conclude that this anti-diagonal has at least length $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$.

We can now construct the desired BPD. First, consider the largest $i$ such that $D_i$ has any shaded squares, i.e. the largest $i$ such that $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ is nonzero. In $P_R$, the set of empty squares underneath pipe $i$ will be a partition shape matching the shaded portion of $D_i$, and we can perform droop moves so that pipe $i$ has up-elbows in each of the (at least) $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ squares of the longest anti-diagonal. Note that this moves all the blank squares that were on or above the longest anti-diagonal above pipe $i$ such that the set of blank tiles in between pipe $i$ and the closest pipe northwest of it form a new partition shape. In particular, if we consider a pipe $i'$ such that for every $i > i'$, the number of up-elbows in the $i'$th pipe is the length of the longest diagonal in $D_i$, then the set of blank squares that are below pipe $i'$ but above every lower pipe will form a partition shape with an anti-diagonal as long as the longest anti-diagonal of $D_{i'}$. We can then droop pipe $i'$ so that it has up-elbows in each square of this anti-diagonal.

Repeating this process for every pipe, thus, generates a BPD in which every pipe $i$ has at least $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows.
Figure 3.2: Labeling of the down-elbows in proof of Theorem 1.1

**Theorem 1.1.** Suppose that $P$ is a bumpless pipe dream corresponding to a vexillary permutation $\omega$. Then the $i^{th}$ pipe has at most $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows. In particular, if the marked bumpless pipe dream $(P,U(P))$ corresponds to a maximal degree term of $G_{\omega}$, the $i^{th}$ pipe will have exactly $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows.

*Proof.* Let $\omega$ be a vexillary permutation on $n$ elements, and let $P \in \text{Pipes}(\omega)$. Beginning with pipe $\omega(n)$ and working upwards, we will define an upper label and a side label for every down-elbow in the following way (see Figure 3.2). For each pipe $i$, the first down elbow (starting from the bottom) will be given side label $i$, and the last down-elbow will be given top label $i$. Now, for every down-elbow which does not yet have a top label, pipe $i$ will contain an up-elbow to the right of it. This unlabeled down-elbow will take the same top label as the closest down-elbow tile beneath this up-elbow. Similarly, every down-elbow which does not yet have a side label has an up-elbow below it, and this down-elbow will take the same side label as the closest down-elbow tile to the right of that up-elbow.

We now associate with each up-elbow tile the ordered pair $(a,b)$ where $a$ is the side label of the down-elbow immediately above it and $b$ is the top label of the down elbow immediately to the left. Notice that for every up-elbow in pipe $i$, the ordered pair $(a,b)$ has the properties:

1. $i < b < a$
2. $\omega^{-1}(a), \omega^{-1}(b) > \omega^{-1}(i)$

and that no two up-elbows from the same pipe can have ordered pairs which share the same first coordinate or the same second coordinate.

Let $(a_1, b_1), \ldots, (a_k, b_k)$ be the labels of all the up-elbows on some pipe $i$. By the properties above, we can sort the labels into disjoint lists of the form $(a_{j_1}, b_{j_1}), \ldots, (a_{j_m}, b_{j_m})$ where each $b_{j_i} = a_{j_{i+1}}$ and $a_{j_1}, a_{j_2}, \ldots, a_{j_m}, b_{j_m}$ is a decreasing list of $m + 1$ numbers which are larger than $i$ and follow $i$ in the one-line notation of the permutation. In order to create an increasing sequence in $\omega$ beginning with $i$, we would, thus, have to remove at least $m$ elements from each such list. This implies that we would have to remove at least $k$ such elements in total, hence, $k \leq r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$.

By Lemma 3.4, there exist BPDs in which every pipe has $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows, so these must be precisely the BPDs which correspond to maximal degree terms of $G_{\omega}$.

*Remark.* Theorem 1.1 also serves as an alternate proof of Theorem 2.1 in the special case where $\omega$ is vexillary.
Proposition 3.5. For a fixed vexillary permutation $\omega$, consider the set of all bumpless pipe dreams $P$ such that the marked BPD $(P, U(P))$ corresponds to a top degree term of $G_\omega$. In the $i$th pipe, the $j$th up-elbow will either be in the same row for all top degree BPDs or in the same column for all top degree BPDs.

Proof. Let $m$ be as large as possible such that $r_m - c_m > 0$. By Lemma 3.3, we can construct any BPD from the Rothe BPD by first positioning pipe $\omega(m)$, then pipe $\omega(m - 1)$ and so on, and we know that for any top-degree BPD, pipe $\omega(m)$ will have precisely $r_m - c_m$ up-elbows. Furthermore, if pipe $\omega(m)$ has $r_m - c_m$ up-elbows, the $j$th up-elbow will either be in the same row for all possible BPDs or in the same column for all possible BPDs. To see this, consider any two BPDs $P_1$ and $P_2$ such that pipe $\omega(m)$ has $r_m - c_m$ up-elbows and (without loss of generality) no other pipes have any up-elbows. Let $j$ be the first index such that the $j$th up-elbow of pipe $\omega(m)$ in $P_1$ is in both a different row and column from the corresponding up-elbow in $P_2$, and let $(x_1, y_1)$ and $(x_2, y_2)$ be the coordinates of this $j$th up-elbow in $P_1$ and $P_2$. We can perform local moves on $P_1$ and $P_2$ to slide the first $j - 1$ up-elbows to the lowest/leftmost position of the two without changing the remainder of the pipe (after the $j$th elbow). Call the new BPDs $P'_1$ and $P'_2$. Supposing without loss of generality that $x_1 < x_2$, there are now two cases.

Case 1: $y_1 > y_2$

In this case, note that $P'_1$ will have an empty tile in position $(x_2, y_2)$ and the down-elbow immediately to the left of the $j$th up-elbow can be drooped into that square to form another up-elbow. This contradicts the assumption that our pipe had the maximum number of up-elbows.

Case 2: $y_1 < y_2$

In this case, $P'_2$ will have an empty tile in positions $(x_1, y_1 + 1)$ and $(x_2, y_1)$. We can use a local move to slide the $j$th up-elbow down to position $(x_2, y_1)$. Now, if our pipe does not have an up-elbow in column $y_1 + 1$, we can drop the down-elbow immediately above the $j$th up-elbow into square $(x_1, y_1 + 1)$ giving a contradiction. If up-elbow $j + 1$ in $P'_2$ is in the same row as up-elbow $j + 1$ in $P'_1$, we can up-droop the $j$th elbow of $P'_2$ to square $(x_1, y_1)$, increasing the number of up-elbows and again yielding a contradiction. Otherwise, if none of the above cases hold, $j + 1$ is now the first index where our pipe has a corresponding pair of elbows that share neither a row nor a column. We can then repeat the above argument until one of the other cases is reached.

We, thus, conclude that no such up-elbow $j$ can exist.

Now, set pipe $\omega(m)$ to the position where each up-elbow is shifted as far downward/rightward as possible and repeat the above argument for pipe $\omega(m - 1)$. Note that with pipe $\omega(m)$ positioned in this way, all positions of pipe $\omega(m - 1)$ which have a maximal number of up-elbows will still be possible. Iterating this process for all the pipes gives the result. 

Corollary 3.6. For a fixed vexillary permutation $\omega$, the set of all bumpless pipe dreams $P$ such that the marked BPD $(P, U(P))$ corresponds to a top degree term of $G_\omega$ is connected by the following local moves (and their inverses)

which do not change the total number of up-elbows.
Proof. Begin with the top-degree bumpless pipe dream $P$ in which all the up-elbows have been pushed as far downward/rightward as possible (as constructed above). For any other top-degree bumpless pipe dream $P'$, we can now use the appropriate local moves to slide the up-elbows of pipe $\omega(1)$ from their positions in $P$ to their positions in $P'$. We can then repeat this for pipes $\omega(2), \ldots, \omega(n)$. \qed

Remark. Applying a move of the form above either does not change the monomial corresponding to the marked BPD $(P, U(P))$ (in the case of the first move) or decreases the exponent of $x_i$ and increases the exponent of $x_{i-1}$ by one (in the case of the second). Accordingly, the top-degree bumpless pipe dream $P$ in which all the up-elbows have been pushed as far downward/rightward as possible corresponds to the leading term of $G_\omega$ in a term order with $x_1 > x_2 > \ldots$

Example 3.1. Let $\omega = 1275463$. The set of BPDs $P \in \text{Pipes}(\omega)$ such that $(P, U(P))$ corresponds to a maximal degree term is shown below. A pair of BPDs are connected by an edge if and only if they differ by precisely one of the moves from Corollary 3.6.

We conjecture the following generalization of Theorem 1.1 for Grothendieck polynomials corresponding to all permutations. Note that this would yield an alternative proof of Pechenik, Speyer, and Weigandt’s result for the degree of the Grothendieck polynomial [8].

Conjecture 3.1. Let $\omega$ be any permutation, and let $P \in \text{Pipes}(\omega)$. If $P$ is non-reduced, replace the redundant crossings with bump tiles (as shown in Figure 2.1). Now, pipe $i$ can have at most $r_{\omega^{-1}(i)} - c_{\omega^{-1}(i)}$ up-elbows, including the up-elbow portions of any bump tiles.

This has been verified by computer for all permutations in $S_6$; however, the technique from the proof of Theorem 1.1 does not apply, even for reduced BPDs corresponding to non-vexillary permutations.
4 Support of Vexillary Grothendieck Polynomials

In this section, we address some properties of vexillary Grothendieck polynomials which can be proved using bumpless pipe dreams. We begin with the vexillary case of Conjecture 1.1.

**Proposition 4.1.** Let \( \omega \in S_n \) be a vexillary permutation, and let \( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \) be any monomial with a nonzero coefficient and non-maximal degree in \( \Theta_\omega(x_1, \ldots, x_n) \). There exists a monomial \( x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \) with nonzero coefficient in \( \Theta_\omega(x_1, \ldots, x_n) \) such that \( j_k = i_k + 1 \) for some \( 1 \leq k \leq n \) and \( j_l = i_l \) for all other indices \( l \).

**Proof.** Let \( (P, S) \in MPipes(\omega) \) be a marked bumpless pipe dream corresponding to the monomial \( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \). If \( S \neq U(P) \), then there is some up-elbow tile \((i, j)\) in \( P \) which is not in \( S \), and the marked BPD \( (P, S \cup \{(i, j)\}) \) will correspond to a monomial \( x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \) with the desired properties.

Otherwise, if \( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \) corresponds to a marked BPD \( (P, U(P)) \), there must be at least one pipe in \( P \) which has fewer than \( r_{\omega^{-1}(k)} - c_{\omega^{-1}(k)} \) up-elbows (since our monomial has non-maximal degree). Select the largest \( k \) such that pipe \( k \) has fewer than \( r_{\omega^{-1}(k)} - c_{\omega^{-1}(k)} \) up-elbows. Note that by Theorem 1.1, there must be some BPD in \( Pipes(\omega) \) in which pipe \( k \) has more up-elbows than in \( P \).

Define a BPD \( P' \in Pipes(\omega) \) by leaving pipes \( k \) through \( n \) in their positions from \( P \) and resetting pipes 1 through \( k - 1 \) to their positions from the Rothe BPD for \( \omega \). By Lemma 3.3 every pipe which contains any elbow tiles southeast of pipe \( k \) must have the maximal number of up-elbows, so since pipe \( k \) does not have the maximal number of elbow tiles, we can perform a move of one of the following forms on pipe \( k \) in the BPD \( P' \):

![Diagrams](image.png)

i.e. a local droop move, a reverse local droop move, or several repetitions of one of the following local moves...
followed by a local droop or a reverse local droop. Now, we perform the equivalent move on pipe $k$ in our
original BPD $P$. If the output is a valid BPD, we will denote the new BPD as $P_1$. Otherwise, we must have
performed one of the following two types of moves:

where at least one of the gray squares must contain an up-elbow from some other pipe $k'$. Furthermore,
since $\omega$ is vexillary (and, thus, $\text{Pipes}(\omega)$ is connected by local moves), all up-elbows which are in the gray
squares must belong to a single pipe. Consider the rightmost grey square containing one of these up-elbows
from pipe $k'$, and note that the tile immediately above it must be a vertical pipe. We can, therefore, perform
one of the following local moves

where the shaded square may or may not contain a tile from another pipe, in order to move our rightmost
shaded up-elbow out of the way. We can similarly perform one of the above two moves on pipe $k'$ to move
the next shaded up-elbow from the right and repeat this for each shaded up-elbow in pipe $k'$.

Again, if the output is a valid BPD, we will denote it $P_1$. Otherwise, at least one of the moves we per-
formed on pipe $k'$ must have moved it into a tile which was already occupied by an up-elbow from a different
pipe, and we repeat the process above until our output is indeed a BPD $P_1$.

Note that moves of the forms
either do not change the associated monomial in \( G_\omega(x_1, \ldots, x_n) \) or add 1 to exactly one of the exponents without changing the others. The monomial associated to \((P_1, U(P_1))\) is, thus, divisible by our original monomial.

Now, suppose the \( l^{th} \) pipe is the last pipe on which we performed either a local droop or reverse local droop move in the procedure above. We can construct another BPD \( P_2 \) by starting with \( P_1 \) and undoing all moves performed on pipes \( l' > l \) as well as all but one droop or reverse droop from pipe \( l \). \( P_2 \), thus, differs from \( P \) by exactly one droop / reverse droop along with moves which did not affect the corresponding monomial, and \((P_2, U(P_2))\) corresponds to the desired monomial \( x_{j_1}^1 x_{j_2}^2 \ldots x_{j_n}^n \).

The following similar result is well known for Grothendieck polynomials corresponding to all permutations. For reference, we include a new proof using BPDs.

**Lemma 4.2.** Let \( \omega \in S_n \) be any permutation, and let \( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \) be any monomial with a nonzero coefficient and non-minimal degree in \( G_\omega(x_1, \ldots, x_n) \). There exists a monomial \( x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \) with nonzero coefficient in \( G_\omega(x_1, \ldots, x_n) \) such that \( j_k = i_k - 1 \) for some \( 1 \leq k \leq n \) and \( j_l = i_l \) for all other \( l \).

**Proof.** Let \( (P, S) \in MPipes(\omega) \) be a marked bumpless pipe dream corresponding to the monomial \( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \). If \( S \) is non-empty, then \((P, S - \{(a, b)\})\) corresponds to a monomial with the desired property for any \((a, b) \in S \). We will, therefore, assume that \( S = \emptyset \). Note that if \( P \) is reduced, then \((P, \emptyset)\) will correspond to a monomial with minimal degree, so we can also assume that \( P \) is non-reduced.

By definition, this means that there is some pair of pipes in \( P \) which cross each other more than once. Select some such pair, and replace the topmost crossing between them with a bump tile. Let \((i_1, j_1)\) be the position of this bump tile, and let \( j_2 \) be the closest column to the left of \( j_1 \) such that the up-elbow portion of the bump tile could be reverse drooped into some blank or up-elbow tile in column \( j_2 \). If the pipe can be reverse drooped into a blank square \((i_2, j_2)\), then the resulting BPD \( P_2 \) will have the same number of empty squares as \( P \) in every row besides \( i_2 \) and one fewer in row \( i_2 \). Otherwise, if we reverse droop the pipe into an up-elbow tile, then the above process can be repeated with the new bump tile until a valid bumpless pipe dream is reached. Again, denote the resulting BPD \( P_2 \), and note that \( P_2 \) will still have one empty square fewer than \( P \) in some row \( k \) and the same number of empty squares as \( P \) in all other rows.

We thus have a marked BPD \((P_2, \emptyset) \in MPipes(\omega)\) which corresponds to a monomial with the desired property.

**Theorem 4.3.** Let \( f(x_1, \ldots, x_n) \) be any monomial in \( G_\omega(x_1, \ldots, x_n) \) for some vexillary permutation \( \omega \). There exist monomials \( f_1, f_2 \) in \( G_\omega \) such that \( f_1 \) has minimal degree, \( f_2 \) has maximal degree, and \( f_1 | f | f_2 \).
Proof. Follows by induction from Proposition 4.1 and Lemma 4.2.

We also resolve Conjecture 1.2 in the case of vexillary permutations.

**Theorem 4.4.** Let $\omega$ be a vexillary permutation, and suppose that $p_1(x_1,\ldots,x_n)$ and $p_2(x_1,\ldots,x_n)$ are monomials with nonzero coefficient in $G_\omega$ such that $p_1|p_2$. Then any monomial $q(x_1,\ldots,x_n)$ satisfying $p_1|q|p_2$ must also have nonzero coefficient in $G_\omega$.

**Proof.** Suppose that $(P_1,S_1)$ and $(P_2,S_2)$ are marked bumpless pipe dreams in $MPipes(\omega)$ corresponding, respectively, to the monomials $p_1$ and $p_2$. Note that if $p_1$ and $p_2$ have equal degree, then $p_1 = p_2$, and the result becomes trivial. We can, thus, assume without loss of generality that $p_1$ has strictly lower degree than $p_2$, and in particular, $p_2$ does not have minimal degree in $G_\omega$. Since vexillary permutations have only reduced BPDs, this implies that $S_2$ must be non-empty.

Consider any $i$ such that $x_i$ has a lower exponent in $p_1$ than in $p_2$. If $S_2$ contains a square $(i,j)$ in row $i$, then the marked BPD $(P_2,S_2-\{(i,j)\}) \in MPipes(\omega)$ corresponds to the monomial $p_2/x_i$. Otherwise, there must be some square in row $i$ which is blank in $P_2$, but not in $P_1$. It must, thus, be possible to perform one of the following types of moves on the BPD $P_2$ (where shaded squares represent squares in the set $S_2$).

Select a move of one of these forms such that the resulting pipes will be closer to their positions in $P_1$. Note
that the first six of these possibilities result in a new marked BPD corresponding to the monomial $p_2/x_i$. For the remaining options, the new marked BPD corresponds to $p_2 x_i'$. In the latter case, the exponent of $x_i'$ is now greater than in $p_1$, so we can either remove a row $i'$ square from $S_2$ to give a marked BPD corresponding to $p_2/x_i$, or we can again perform one of the above moves to remove a blank square from row $i'$.

Since every step moves the pipes closer to their positions in $P_1$ and $S_2$ is non-empty, this process will eventually end with a marked BPD corresponding to $p_2/x_i$. The result then follows by induction.

We now address Weigandt’s question regarding the leading terms in each degree of the Grothendieck polynomial [12]. Let $\mathfrak{G}_\omega^{(k)}(x)$ be the homogeneous component of $\mathfrak{G}_\omega(x)$ with degree $k$, and consider a term order with $x_1 < x_2 < \cdots < x_n$. The following results are known.

**Proposition 4.5** ([1], Corollary 3.9). The leading term of the Schubert polynomial $S_\omega(x)$ is given by $x_1^{c_1} \cdots x_n^{c_n}$.

**Theorem 4.6** ([8], Theorem 1.1). The leading term of the Grothendieck polynomial $G_\omega(x)$ is a scalar multiple of $x_1^{r_1} \cdots x_n^{r_n}$.

In particular, we know the leading terms of the lowest and highest degree homogeneous components of $\mathfrak{G}_\omega(x)$. Furthermore, for the component of second lowest degree, the leading term is given as follows.

**Proposition 4.7.** For any permutation $\omega$, the leading term of $\mathfrak{G}_\omega^{(\ell(\omega)+1)}(x)$ is given by a scalar multiple of $(x_1^{c_1} \cdots x_n^{c_n})x_i$ where $i$ is the largest possible index such that $c_i < r_i$.

**Proof.** Let $(P, S)$ be a marked BPD in $M_{\text{Pipes}}(\omega)$ corresponding to the leading term of $\mathfrak{G}_\omega^{(\ell(\omega)+1)}(x)$, and recall that $P$ can be reached from the Rothe BPD $P_R$ for $\omega$ by a series of droops and K-theoretic droops. In particular, since the corresponding monomial has degree $\ell(\omega) + 1$, it must be true that either this sequence contains no K-theoretic droop moves and $|S| = 1$ or this sequence contains exactly one K-theoretic droop and $|S| = 0$.

Consider first the case where our sequence contains a K-theoretic droop. Note that performing a droop move on a BPD will always generate a monomial of the same degree which is smaller in our term order, so the K-theoretic droop must be the last move of our sequence. Now, suppose that instead of performing one of the following K-theoretic droops,

![Diagram](image1)

we add the up-elbow in the lower right corner to $S$. This new BPD would correspond to a larger monomial in our term order, contradicting the assumption that $(P, S)$ corresponded to the leading term.
We can, thus, conclude that \( P \) must be reduced and \( S \) must have size 1. By the same argument as in the proof of Theorem 1.2, we can see that \( P \) will have no up-elbows besides the one which is in \( S \). To get such a BPD corresponding to the largest possible monomial, we take the lowest pipe in \( P_R \) which can be drooped and droop it to the closest open square. This lowest pipe will be pipe \( \omega(i) \) where \( i \) is the largest index with \( c_i < r_i \), so \((P,S)\) will correspond to \((x_1^{c_1} \cdots x_n^{c_n})x_i\).

We conjecture the following description for the leading term of each intermediate degree and give a proof of this description in the vexillary case.

**Conjecture 4.1.** For any permutation \( \omega \), let \( m_k(\mathbf{x}) = x_1^{c_1} \cdots x_n^{c_n} \) be the monic monomial such that the leading term of \( \Theta_{\omega}^{(k)}(\mathbf{x}) \) is a scalar multiple of \( m_k(\mathbf{x}) \). The leading term of \( \Theta_{\omega}^{(k+1)}(\mathbf{x}) \) will be a scalar multiple of \( m_{k+1} = m_k(\mathbf{x})x_i \) where \( i \) is the largest possible index such that \( e_i < r_i \).

**Theorem 1.2.** Suppose that \( \omega \) is vexillary, and let \( m_k(\mathbf{x}) = x_1^{c_1} \cdots x_n^{c_n} \) be the monomial such that the leading term of \( \Theta_{\omega}^{(k)}(\mathbf{x}) \) is a scalar multiple of \( m_k(\mathbf{x}) \). If \( i \) is the largest possible index such that \( e_i < r_i \), the leading term of \( \Theta_{\omega}^{(k+1)}(\mathbf{x}) \) will be a scalar multiple of \( m_{k+1} = m_k(\mathbf{x})x_i \).

**Proof.** Let \((P,S)\) be a marked BPD corresponding to the leading term of \( \Theta_{\omega}^{(k)}(\mathbf{x}) \), and note that \( S \) will have size \( k - \ell(\omega) \) since \( \ell(\omega) = \sum_i c_i \) is the degree of \( \Theta_{\omega}(\mathbf{x}) \). First, suppose that \( S \) is a proper subset of \( U(P) \) and choose \((i,j) \in U(P) - S \) such that \( i \) is as small as possible. If \( S \) contains some \((i',j') \) such that \( i' < i \), then \((P, (S - (i',j')) \cup \{(i,j)\}) \) corresponds to a larger monomial in \( \Theta_{\omega}^{(k)} \) giving a contradiction. Similarly, if \( P \) contains no up-elbows above row \( i \), then un-drooping the elbow in \((i,j)\) gives a marked BPD corresponding to a larger monomial, providing our contradiction. We, therefore, know that our leading monomial must correspond to a marked BPD of the form \((P, U(P))\).

We now know that \( P \) must have exactly \( k - \ell(\omega) \) up-elbows and that pipe \( \omega(i) \) can have at most \( r_i - c_i \) up-elbows, so we can see that for \((P, U(P))\) to correspond to the leading term, \( P \) must be constructed as follows. For the largest \( i \) such that \( r_i > c_i \), we droop pipe \( \omega(i) \) so that it has \( \min\{r_i - c_i, k - \ell(\omega)\} \) up-elbows pushed as far left as possible. Now, set \( r \) to be the difference between \( k - \ell(\omega) \) and the total number of up-elbows in our BPD. If \( r = 0 \), we are done. Otherwise, consider the next largest \( i' \) such that \( r_{i'} > c_{i'} \), and add \( \min\{r_{i'} - c_{i'}, r\} \) up-elbows to pipe \( \omega(i') \) (again, pushed as far left as possible), repeating this process until \( r = 0 \). By construction, the resulting BPD will correspond to the desired monomial \( m_k \), as described above, so we can conclude that this \( m_k \) is indeed the leading monomial of \( \Theta_{\omega}^{(k)}(\mathbf{x}) \) (up to multiplication by a scalar).

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