An elementary proof of uniqueness of particle trajectories for solutions of a class of shear-thinning non-Newtonian 2D fluids

Luigi C Berselli$^1$ and Luca Bisconti$^2$

$^1$ Dipartimento di Matematica Applicata ‘U. Dini,’ Università di Pisa, Via F. Buonarroti 1/c, I-56127, Pisa, Italy
$^2$ Dipartimento di Sistemi e Informatica, Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy

E-mail: berselli@dma.unipi.it and luca.bisconti@unifi.it

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Abstract
We prove some regularity results for a class of two-dimensional non-Newtonian fluids. Then, by applying results from Dashti and Robinson (2009 *Nonlinearity* 22 735–46), we show uniqueness of particle trajectories.

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1. Introduction

In this paper we consider the following system of partial differential equations

\[ \begin{align*}
  u_t - v_0 \Delta u - v_1 \text{div} \mathcal{S}(D\mathbf{u}) + (u \cdot \nabla) u + \nabla \pi &= f & \text{in } [0, T] \times \Omega, \\
  \text{div} u &= 0 & \text{in } [0, T] \times \Omega, \\
  u(0) &= u_0 & \text{in } \Omega,
\end{align*} \tag{1.1} \]

where $\Omega$ denotes either a two-dimensional bounded domain or a two-dimensional flat torus, the vector field $u = (u_1, u_2)$ is the velocity, the scalar $\pi$ is the kinematic pressure, the vector field $f = (f_1, f_2)$ is the external body force, $u_0$ is the initial velocity and $v_0, v_1$ are positive constants. We denote by

\[ D\mathbf{u} := \frac{1}{2} (\nabla u + (\nabla u)^T) = \frac{1}{2} (\partial_j u_i + \partial_i u_j) \quad \text{for } i, j = 1, 2, \]

the symmetric part of $\nabla u$, the convective term is $(u \cdot \nabla) u := \sum_{k=1}^2 u_k \partial_k u$ and $\mathcal{S}$ denotes the extra stress-tensor, defined by

\[ \mathcal{S}(D\mathbf{u}) := (\delta + |D\mathbf{u}|^2)^{p-2} D\mathbf{u}, \quad p \in [1, 2), \tag{1.2} \]
where $\delta$ is a non-negative constant. System (1.1) describes a shear-thinning homogeneous fluid; for an introduction to the mathematical theory see Málek et al (1995). We mainly study the problem endowed with homogeneous Dirichlet boundary conditions

$$u_{|\Gamma} = 0 \quad \text{where} \quad \Gamma = \partial \Omega,$$

but we also remark on the space-periodic case.

The main goal of this paper is to study the problem of uniqueness for the particle trajectories (or characteristics), which are solutions of the following Cauchy problem for ordinary differential equations:

$$\begin{cases}
\frac{dX(t)}{dt} = u(X(t), t) & t \in [0, T], \\
X(0) = x \in \Omega,
\end{cases}$$

where $u$ is the fluid velocity from (1.1). For the 3D Navier–Stokes equations the problem of the existence of particle trajectories and the Lagrangian representation of the flow first appeared in the work of Foias et al (1985) and the related results of regularity in $\mathbb{R}^d$ are proved in Chemin and Lerner (1995) by means of the Littlewood–Paley decomposition. The question of uniqueness of particle trajectories has been recently addressed by elementary tools and in a more general context in Dashiti and Robinson (2009), Robinson and Sadowski (2009a), Robinson and Sadowski (2009b) and it is strictly related with uniqueness for linear transport equations. We consider here the same problem, in the case of shear-thinning fluids, described by (1.1). To this end, we will study certain regularity properties of the solutions of (1.1), investigating whether velocity verifies the appropriate hypotheses for uniqueness results.

In particular, classical results concerning Lipschitz continuous fields $u$ (which can generally be verified by checking that $\nabla u$ is bounded in the space variables) are not easily applicable here, since such a regularity is very difficult to prove, even in the two-dimensional case for the system (1.1). We recall that, restricting to the two-dimensional case, some $C^{1,\gamma}$-results are obtained in Kaplický et al (1997), Kaplický et al (1999) in the stationary case. Early results in the time-dependent case (but not up-to-the-boundary) are those in Seregin (1997), while results in the space-periodic time-dependent case have been obtained in Kaplický et al (2002). We observe that essentially all the above results require the extra stress-tensor $S$ to be slightly smoother than that defined in (1.2). In particular, it is requested that the stress-tensor is replaced, for instance, by $S(Du) := (\delta + |Du|^2)^{\frac{p-2}{2}}Du$. We wish also to mention the results in Bothe and Prüss (2007), where local-in-time existence of smooth solutions is proved, with the latter more regular stress-tensor and under certain conditions of smoothness of the data e.g., initial data with two derivatives in certain Lebesgue spaces.

In any case we study regularity up-to-the-boundary with non-smooth initial data and our results, proved in an elementary way, are original. Moreover, the difficulties appearing in the 3D case do not seem to be completely explained by current mathematical knowledge for such fluids; this is why we restrict our study to the two-dimensional case.

Since we want to have elementary proofs (in order to possibly extend the results to the widest possible class of solutions and stress-tensors) we will work with classical energy-type methods. Concerning uniqueness of particle trajectories, there have been some recent improvements strictly related with the Osgood criterion and with Log-Lipschitz properties of Sobolev functions $W^{s+1,2}(\mathbb{R}^d)$ in the case of limiting Sobolev exponents such that $q = \frac{d}{r}$. In particular we will use the result below, proved in Dashiti and Robinson (2009, theorem 2.1).

**Theorem 1.1.** Let $\Omega$ be either the whole space $\mathbb{R}^d$, $d \geq 2$, a periodic $d$-dimensional domain or an open bounded subset of $\mathbb{R}^d$ with a sufficiently smooth boundary. Assume that for some $r > 1$

$$u \in L^r(0, T; W^{\frac{d}{r} - 1,2}(\Omega)) \quad \text{and} \quad \sqrt{r} u \in L^2(0, T; W^{\frac{d}{r} - \frac{1}{2},2}(\Omega)),$$

the result holds.
with \( u_T = 0 \), when \( \Omega \) is a domain with a boundary. Then, the Cauchy problem (1.4) has a unique solution in \([0, T]\).

The latter results show that certain (slightly weaker than \( C^{1,\gamma} \)) results of Sobolev space-regularity can be used to obtain uniqueness for (1.4). On the other hand, the \( W^{2,2}(\mathbb{R}^2) \) regularity for fluids with shear-dependent viscosity is another non-trivial task (while in 3D proving \( u \in W^{3/2,2}(\mathbb{R}^3) \) seems at the moment out of sight). Some recent results (in the stationary case) for second-order space-derivatives appeared in Beirão da Veiga (2009), Berselli (2009), Crispo and Grisanti (2008) even if the square integrability of second-order derivatives is not reached in general domains, or if certain limitations on the smallness of the force are not satisfied. For the non-stationary case, we recall the result in the space-periodic setting (obtained uniformly in \( \delta \geq 0 \)) from Berselli et al (2010), Diening and Růžička (2005).

We also point out that one of the main technical obstructions is represented by the pressure and the associated divergence-free constraint. In the case of p-Laplacian systems, in fact, the results in Beirão da Veiga and Crispo (2012) show that \( u \in W^{2,q}(\Omega) \) for arbitrary \( q \), if \( f \) is smooth and under certain restrictions in the range of \( p \in (1, 2) \). Two latter results are proved in the stationary case and there is no counterpart for the \( p \)-Stokes result. Results of higher regularity for the parabolic p-Laplacian in the singular case have been recently obtained in Crispo and Maremonti (2012). At present it seems that these results are not enough to apply the same machinery we use here to show uniqueness of particle trajectories for the singular problem (especially if the initial data are only in \( L^2(\Omega) \)), unless some extra smoothness and compatibility conditions are assumed at \( t = 0 \), cf. the proof of theorem 1.3 in Crispo and Maremonti (2012).

We highlight that in the case of non-Newtonian fluids many features of the problem are critical: the type of boundary conditions, the range of \( p \) and whether the parameter \( \delta \) is strictly larger than zero. We will discuss later on some of the technical issues of the problem and we will explain why we have to reduce to the 2D case with \( v_0, \delta > 0 \). We start by considering the easier case of the space-periodic setting where \( \Omega \) is the flat 2D torus \( \mathbb{T}^2 := \mathbb{R}^2 / 2\pi \mathbb{Z} \) and we will prove the following result.

**Proposition 1.1.** Let \( v_0 > 0, \delta \geq 0 \) and \( p \in (1, 2] \). Let be given \( u_0 \in L^2(\mathbb{T}^2) \) such that \( \text{div } u_0 = 0 \) and \( f \in L^2(0, T; L^2(\mathbb{T}^2)) \). Then, weak solutions to (1.1) satisfy \( \sqrt{t} u \in L^2(0, T; W^{2,2}(\mathbb{T}^2)) \) and hence problem (1.4) admits a unique solution for all \( x \in \Omega \).

We emphasize that the assumption \( v_0 > 0 \) is crucial to our method. When \( v_0 = 0 \) it is possible to prove a regularity result that, although not useful to the application of theorem 1.1, seems interesting by itself. See remark 3.1, cf. Kost (2010).

In the Dirichlet case the problem of regularity is more delicate. We will consider problem (1.1) in a domain with a flat boundary. We will first prove a regularity result, by using techniques similar to those used in Crispo (2009) and formerly introduced for the case \( p > 2 \) in Beirão da Veiga (2005). With smooth data, we have the following result.

**Proposition 1.2.** Let \( \delta > 0, v_0 > 0, v_1 \in \left[ \frac{3}{2}, 2 \right], u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \) with \( \text{div } u_0 = 0 \) and \( f \in W^{1,2}(0, T; L^2(\Omega)) \). Then, weak solution to problems (1.1)–(1.3) satisfy

\[
\|u_t\|_{L^\infty(0, T; L^1)} + \|\nabla u\|_{L^\infty(0, T; L^3)} + \|\nabla \pi\|_{L^2(0, T; L^2)} + \|\nabla u_t\|_{L^2(0, T; L^2)} + \|D^2 u\|_{L^2(0, T; L^2)} \leq C,
\]

where \( C \) depends on \( p, \delta, v_0, v_1, \|f\|_{W^{1,2}(0, T; L^2)}, \|u_0\|_{2,2}, T \) and \( \Omega \).

Some hypotheses can be relaxed, since the time regularity is unnecessary for the proof of uniqueness of particle trajectories, but the arguments used to prove proposition 1.2 will play
a fundamental role to demonstrate our main uniqueness criterion for the problem (1.4). The main result of this paper reads as follows:

**Theorem 1.2.** Let $\delta > 0$, $\nu_0 > 0$, $p \in [\frac{3}{2}, 2]$, $u_0 \in L^2(\Omega)$ with $\text{div} u_0 = 0$, such that $(u_0 \cdot n)_{\Gamma} = 0$ and $f \in L^2(0, T; L^2(\Omega))$. Then, weak solution to (1.1)–(1.3) satisfy $\sqrt{\delta} u \in L^2(0, T; W^{2,2}(\Omega))$ and consequently (1.4) admits a unique solution for all $x \in \Omega$.

**Outline of the paper.** In section 2 we introduce the notation and we give some preliminary results. In section 3 we consider the space-periodic setting and we prove proposition 1.1. Thereafter, in section 4, we prove a preliminary space-time regularity result for the solutions of (1.1)–(1.3) and then we demonstrate proposition 1.2. Finally, in section 5, we give the proof of theorem 1.2.

2. Preliminaries and basic results

Let us introduce the notation related especially to the problem (1.1) with Dirichlet boundary conditions. The needed assumptions or changes for the space-periodic case are specified in section 3.

Throughout the article, when speaking of $\Omega$ of a bounded domain we denote a two-dimensional cube $\Omega := [-1, 1]^2$ and we also define by $\Gamma$ the two opposite sides in the $x_2$ direction

$$
\Gamma := \{ x = (x_1, x_2): |x_1| < 1, x_2 = -1 \} \cup \{ x = (x_1, x_2): |x_1| < 1, x_2 = 1 \}.
$$

We use the following boundary conditions

$$
\begin{cases}
    u_{\Gamma} = 0, \\
    u 
	ext{ is } 2 -\text{periodic w.r.t } x_1.
\end{cases}
$$

(2.1)

Here, $x_1$ represents the tangential direction to $\Gamma$ and this idealized setting of a ‘periodic strip’ corresponds to the half-space with vanishing Dirichlet boundary conditions, but without the complications at infinity.

Given $q \geq 1$, by $L^q(\Omega)$ we indicate the usual Lebesgue space with norm $\| \cdot \|_q$. Moreover, by $W^{k,q}(\Omega)$, $k$ a non-negative integer and $q$ as before, we denote the usual Sobolev space with norm $\| \cdot \|_{k,q}$. We also denote by $W^{1,q}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$ and by $W^{-1,q}_0(\Omega)$, $q' = q/(q - 1)$, the dual of $W^{1,q}_0(\Omega)$ with norm $\| \cdot \|_{-1,q'}$. Let $X$ be a real Banach space with norm $\| \cdot \|_X$. We will use the customary spaces $W^{k,q}(0, T; X)$, with norm denoted by $\| \cdot \|_{W^{k,q}(0, T; X)}$, recalling that $W^{0,q}(0, T; X) = L^q(0, T; X)$. We will also use the notation $\Omega_T := \Omega \times (0, T)$, we will not distinguish between scalar and vector fields and the symbol $(\cdot, \cdot)$ will indicate a duality pairing. Here and in the sequel, we will denote by $C$ positive constants that may assume different values, even in the same line. We also define

$$
V_q := \{ v \in W^{1,q}(\Omega): \nabla \cdot v = 0, v_{\Gamma} = 0, v \text{ is } 2 -\text{periodic w.r.t } x_1 \},
$$

with dual space $V'_q$. Since the extra stress-tensor $S$ is a function not of the gradient, but of the deformation tensor (in order to have frame-invariant equations) we recall a Korn-type inequality, see Crispo (2009).

**Lemma 2.1.** There exists a positive constant $C = C(\Omega)$ such that

$$
\| v \|_q + \| \nabla v \|_q \leq C \| Dv \|_q, \quad \text{for each } v \in V_q.
$$

We write $f \simeq g$, if there exist $c_0$, $c_1 > 0$ such that $c_0 f \leq g \leq c_1 f$. When considering the tensor $S(D) = (\delta + |D|^{\text{sym}})^{p/2} D^{\text{sym}}$, as that introduced in (1.2) (where $D$ is a second-order
tensor and $D^{sym}$ its symmetric part), it can be easily checked that for any second-order tensor $C$ the following relations are verified

\[ \sum_{i,j,k,l=1}^{2} \partial_{kl} S_{ij}(D) C_{ij} C_{kl} \geq (p - 1)(\delta + |D^{sym}|)^{p-2} |C|^2, \quad (2.2a) \]

\[ |\partial_{kl} S_{ij}(D)| \leq (3 - p)(\delta + |D^{sym}|)^{p-2}. \quad (2.2b) \]

The symbol $\partial_{kl} S_{ij}$ represents the partial derivative $\partial S_{ij}/\partial D_{kl}$ of the $(i, j)$-component of $S$ with respect to the $(k, l)$-component of the underlying space of $2 \times 2$ matrices. Monotonicity and growth properties of $S$ are characterized in the following standard lemma.

**Lemma 2.2.** Assume that $p \in (1, \infty)$ and $\delta \in [0, \infty)$. Then, for all $A, B \in \mathbb{R}^{2 \times 2}$ there holds

\[ (S(A) - S(B)) : (A^{sym} - B^{sym}) \simeq (\delta + |B^{sym}| + |A^{sym}|)^{p-2} |A^{sym} - B^{sym}|^2, \]

\[ |S(A) - S(B)| \simeq (\delta + |B^{sym}| + |A^{sym}|)^{p-2} |A^{sym} - B^{sym}|, \]

where the constants $c_0, c_1 > 0$ depend only on $p$ and are independent of $\delta \geq 0$.

From the elementary inequality $a^p \leq a^2 b^{p-2} + b^p$, valid for all $0 \leq a, 0 < b$, and $p \in [1, 2]$, we get the relation

\[ \delta + t \simeq (\delta + t)^{\frac{p-2}{p}} t + \delta^\frac{2}{p}, \quad \delta, t \geq 0, \quad (2.3) \]

with constants depending only on $p$, (Berselli et al 2010 corollary 2.19).

Since in the Dirichlet case we need to handle in a different way tangential and normal derivatives, we denote by $D^2 u$ the set of all the second-order partial derivatives of $u$. In addition, the symbol $D^3 u$ denotes all partial derivatives $\partial^2_{ij} u_{j}$, except for the derivative $\partial^3_{ij} u_{1}$, namely

\[ |D^3 u|^2 := |\partial_{22} u| + \sum_{i,j,k=1}^{2} |\partial^3_{ij} u_{j}|^2. \]

We introduce the following quantities strictly related to the stress tensor $S$ and appearing naturally in the problem when using the techniques introduced in Beirão da Veiga (2009), Diening and Růžička (2005), Málek et al (1995):

\[ I_1(u) := \int_{\Omega} (\delta + |Du|)^{p-2} |\partial_1 Du|^2 \, dx, \quad (2.4a) \]

\[ I(u) := \int_{\Omega} (\delta + |Du|)^{p-2} |\nabla Du|^2 \, dx, \quad (2.4b) \]

\[ J(u) := \int_{\Omega} (\delta + |Du|)^{p-2} |Du_1|^2 \, dx, \quad (2.4c) \]

where $I$ is obtained by integration by parts when testing the extra stress-tensor $S$ with $-\Delta u$ (and this is possible in the space-periodic case); a multiple of $I_1$ is obtained with $-\partial_1 u$ and the calculations are possible in the flat domain; finally, a multiple of $J$ is obtained testing with $u_1$, and calculations are valid also in the Dirichlet case for a generic domain.

We will also use this classical result, see Nečas (1966).

**Lemma 2.3.** If $g \in L^1(\Omega)$ and it holds $\nabla g = \text{div} \, G$, for some $G \in (L^q(\Omega))^{2 \times 2}$, for $1 < q < +\infty$ then

\[ \|g - \int_{\Omega} g(x) \, dx\|_q \leq c \|G\|_q. \]
Let us recall the definition of a weak solution to the problems (1.1)–(2.1).

**Definition 2.1.** Let \( T > 0 \) and assume that \( f \in L^2(0, T; V'_2) \). We say that \( u \) is a weak solution of problem (1.1) if:

\[
u \in L^2(0, T; V'_2) \cap L^\infty(0, T; L^2(\Omega)), \quad u_t \in L^2(0, T; V'_2),\]

\[
\int_\Omega u(t) \varphi \, dx + v_0 \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \varphi \, dx \, ds + v_1 \int_0^t \langle S(Du(s)), D\varphi \rangle \, ds 
- \int_0^t \int_\Omega (u(s) \cdot \nabla) \varphi \, u(s) \, dx \, ds = \int_\Omega u_0 \varphi \, dx + \int_0^t \langle f(s), \varphi \rangle \, ds \quad \forall \varphi \in V_2.
\]

Due to the fact that \( v_0 > 0 \), the existence of weak solutions follows for all \( p > 1 \) in a standard way and one has not to resort to very sophisticated tools as in Diening et al (2010). We will return to this for the motivation on this assumption on \( v_0 \). In particular, we do not have any further restriction on \( p \) and the proof follows the same lines of the classical work on monotone operators, as summarized in Lions (1969). The result below is part of the folklore associated with non-Newtonian fluids. We will give a sketch of the proof since some of the calculations will be used many times in the sequel.

**Theorem 2.1.** Let be given \( v_0, v_1 > 0, p \in [1, 2], u_0 \in L^2(\Omega) \) with \( \text{div}u_0 = 0 \) and \( (u_0, n)|_{\Gamma} = 0 \), and \( f \in L^2(0, T, V'_2) \). Then, there exists a unique weak solution \( u \) (in the sense of definition 2.1) to (1.1)–(2.1). Moreover, the following estimates are verified

\[
\|u\|_{L^\infty(0,T;L^2)}^2 + v_0 \|\nabla u\|_{L^2(0,T;L^2)}^2 \leq \|u_0\|_{L^2}^2 + \frac{1}{v_0} \|f\|_{L^2(0,T;V'_2)}^2 + C(p) T v_1 \delta^p,
\]

\[
\|u_t\|_{L^2(0,T;V'_2)}^2 \leq C,
\]

where \( C = C(p, \delta, v_0, v_1, \|f\|_{L^2(0,T;V'_2)}, \|u_0\|_2, T, \Omega) \).

**Proof.** We deduce the a priori estimates on which the existence of weak solutions to (1.1)–(2.1) is based. More properly, one should consider approximate Galerkin solutions defined as follows. Let \( \{\phi_r\} \), with \( r \in \mathbb{N} \), be the eigenfunctions of the Stokes operator and let \( \{\lambda_r\} \) be the corresponding eigenvalues; we define \( X_m := \text{span}\{\phi_1, \ldots, \phi_m\} \) and \( P_m \) is the orthogonal projection operator over \( X_m \). We will seek approximate functions \( u^m(t, x) = \sum_{r=1}^{m} c_r(t) \phi_r(x) \) as solutions of the system of ordinary differential equations below, for all \( 1 \leq r \leq m, t \in [0, T] \)

\[
\int_\Omega \left[ u^m \phi_r + v_0 \nabla u^m \cdot \nabla \phi_r + v_1 S(Du^m)D\phi_r + (u^m \cdot \nabla) u^m \phi_r \right] \, dx = \langle f, \phi_r \rangle,
\]

\[
u^m(0) = P_m u_0.
\]

Taking the \( L^2 \) product of (1.1) with \( u^m \), using suitable integrations by parts and by Young’s inequality we get

\[
\frac{1}{2} \frac{d}{dt} \|u^m\|_{L^2}^2 + v_0 \|\nabla u^m\|_{L^2}^2 + \frac{v_1}{2} \int_\Omega (\delta + |Du^m|)^{p-2} |Du^m|^2 \, dx \leq \frac{v_0}{2} \|\nabla u^m\|_{L^2}^2 + \frac{1}{2v_0} \|f\|_{V'_2}^2.
\]

Using (2.3) and integrating in time we arrive at the following inequality

\[
\|u^m(t)\|_{L^2}^2 + v_0 \int_0^t \|\nabla u^m(s)\|_{L^2}^2 \, ds + C v_1 \int_0^t \|Du^m(s)\|_{L^2}^2 \, ds
\leq \|u_0\|_{L^2}^2 + \frac{1}{v_0} \int_0^t \|f(s)\|_{V'_2}^2 \, ds + C(p) v_1 \delta^p,
\]
for a.e. \( t \in [0, T] \). We estimate, by comparison, the time derivative. The only term which requires some care is the extra stress-tensor \( S \). Since \( p \leq 2 \) we get

\[
\int_0^T \langle S(Du^m), D\varphi \rangle \, ds \leq \| S(Du^m) \|_{L^2(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)} \\
\leq \| Du^m \|_{L^{2+1}(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)} \\
\leq C(T, \Omega) \| \nabla u^m \|_{L^{2+1}(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)}.
\]

For \( \varphi \in V_2 \) we get from (1.1)

\[
\int_0^T \langle u_t, \varphi \rangle \, ds = -\int_0^T v_0 \langle \nabla u, \varphi \rangle + \langle S(Du^m), D\varphi \rangle + \langle (u \cdot \nabla) u, \varphi \rangle - \langle f, \varphi \rangle \, ds \\
\leq v_0 \| \nabla u \|_{L^2(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)} + C(T, \Omega) \| \nabla u^m \|_{L^{2+1}(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)} \\
+ C(\Omega) \| u \|_{L^{\infty}(L^2(\Omega_T))} \| \nabla u \|_{L^2(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)} + \| f \|_{L^2(\Omega_T)} \| \nabla \varphi \|_{L^2(\Omega_T)},
\]

where we used the Gagliardo–Nirenberg type inequality

\[
\| u \|_4 \leq C \| u \|_2^{1/2} \| \nabla u \|_2^{1/2},
\]

(2.5)

which is valid in two-space dimensions for functions belonging to \( V_2 \), see Lions (1969, chapter I, section 6.2), to estimate the convective term. The above standard calculations and the definition by duality of the norm of \( V_2 \) imply then by comparison

\[
\int_0^T \| u^m(s) \|_{V^2}^2 \, ds \leq C < +\infty,
\]

for a constant \( C \) depending on \( p, v_0, v_1, \| f \|_{L^{2+1}(0, T; V_2)}, \| u_0 \|_2, T \) and \( \Omega \). In particular, the dependence on the various data comes through the energy inequality. This proves that if \( u^m \) is a Galerkin approximate solution then, uniformly in \( m \in \mathbb{N}, \)

\[
u^m \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V_2) \quad \text{and} \quad u^m_t \in L^2(0, T; V_2').
\]

Note that we can extract sub-sequences converging weakly to some \( u \in L^2(0, T; V_2) \), weakly* in \( L^\infty(0, T; L^2(\Omega)) \) and, by the Aubin–Lions theorem, strongly in \( L^2(\Omega_T) \), and a.e. in \( \Omega_T \). We have enough regularity to pass to the limit in the convective term. Moreover, since \( S(Du^m) \) is bounded uniformly in \( L^2(\Omega_T) \), it follows that \( S(Du^m) \rightharpoonup A \) for some \( A \in L^2(\Omega_T) \). (Observe that without the Laplacian term we would have only a bound in \( L^2(\Omega_T) \).) We have now to check that \( A = S(Du) \). This is obtained with the monotonicity trick, see e.g. Lions (1969, section 2–5.2). By usual Sobolev embeddings (since we are in two dimensions) the function \( t \mapsto \int_{\Omega} (u \cdot \nabla) u \, dx \in L^1(0, T) \), hence we can write the energy equality between any couple \( 0 \leq s_0 \leq s \leq T \)

\[
\frac{1}{2} \| u(s) \|^2_2 + v_0 \int_{s_0}^s \| \nabla u \|^2_2 \, d\tau + v_1 \int_{s_0}^s \langle A, Du \rangle \, d\tau = \frac{1}{2} \| u(s_0) \|^2_2 + \int_{s_0}^s (f, u) \, d\tau.
\]

(2.6)

Defining for \( \phi \in L^2(0, T; V_2) \) (a test function with the same regularity of \( u \))

\[
\mathcal{X}_s^m := v_1 \int_0^s \langle S(Du^m) - S(D\phi), Du^m - D\phi \rangle \, d\tau + v_0 \int_0^s \| \nabla u^m \|^2_2 \, d\tau + \frac{1}{2} \| u^m(s) \|^2_2^2,
\]

it follows, by using that monotone \( S \) and by semi-continuity of the norm, that

\[
\lim_{m \to +\infty} \mathcal{X}_s^m \geq v_0 \int_0^s \| \nabla u \|^2_2 \, d\tau + \frac{1}{2} \| u(s) \|^2_2
\]

and also that

\[
\lim_{m \to +\infty} \mathcal{X}_s^m = \int_s^t (f, u) + \frac{1}{2} \| u_0 \|^2_2 - v_1 \int_0^t \langle A, D\phi \rangle \, d\tau - v_1 \int_0^t \langle S(D\phi), Du - D\phi \rangle \, d\tau.
\]
Hence, by using the equality (2.6) we get
\[ v_1 \int_0^t (A - S(D\phi), Du - D\phi) \, dt \geq 0 \quad a.e. \ s \in [0, T]. \]

We fix \( \phi = u - \lambda \psi \) for \( \psi \in L^2(0, T; V_2) \) and \( \lambda > 0 \). Finally, letting \( \lambda \to 0^+ \) the thesis follows.

It is important to point out that the weak solution constructed above is unique. Let us suppose that we have two solutions \( u_1 \) and \( u_2 \) corresponding to the same data. Since \( U := u_1 - u_2 \in L^2(0, T; V_2) \) is allowed as a test function, and due also to the fact that \( U \in L^2(0, T; V_2) \), we obtain the following equality
\[ \int_0^t (U(t), U') \, dt + v_0 \int_0^t \| \nabla U(s) \|_{L^2}^2 \, ds + v_1 \int_0^t \langle S(Du_1) - S(Du_2), Du_1 - Du_2 \rangle \, ds = \int_0^t \int_\Omega (U \cdot \nabla) u_2 \, U \, dx \, ds. \]

Since \( S \) is monotone (cf lemma 2.2) the integral involving the extra stress-tensor is non-negative and we also observe that \( \int_0^t (U(t), U') \, dt = \frac{1}{2} \| U(t) \|_{L^2}^2 \), due to the fact that \( U(0) = 0 \), see Constantin and Foias (1988, section 10). By using again the inequality (2.5) we estimate the right hand side by \( C \int_0^t \| \nabla u_2(s) \|_2 \| U(s) \|_2 \| \nabla U(s) \|_2 \, ds \) and with a further application of Young’s inequality we get
\[ \| U(t) \|_{L^2}^2 + v_0 \int_0^t \| \nabla U(s) \|_{L^2}^2 \, ds \leq \frac{C}{v_0} \int_0^t \| \nabla u_1(s) \|_2 \| U(s) \|_2 \, ds. \]

Using the Gronwall lemma and the energy estimate one finally obtains \( U \equiv 0 \). \( \Box \)

This latter result is very relevant since it allows to conclude that all the sequence \( \{ u^n \} \) converges to \( u \). Moreover, if we have other a priori estimates on \( u^n \), the extra-regularity is inherited by weak solutions directly. This will be used in the proof of theorem 1.1. Observe also that, at the moment, we do not have any information on the pressure apart from that it exists as a distribution, by using the De Rham theorem.

3. The space-periodic case

In this section we are concerned with the space-periodic case, that is \( \Omega = \mathbb{T}^2 \). Each considered function \( u \) will satisfy \( u(x + 2\pi e_i) = u(x), \ i = 1, 2, \) where \( \{ e_1, e_2 \} \) is the canonical basis of \( \mathbb{R}^2 \). We also require all functions to have a vanishing mean value to ensure the validity of the Poincaré inequality. We prove some regularity results and we will show why the hypothesis \( v_0 > 0 \) seems necessary in many arguments. We define \( V_{\text{per}}(\Omega) \) as the space of vector-valued functions on \( \Omega \) that are smooth, divergence-free and space-periodic with zero mean value. For \( 1 < q < \infty \) and \( k \in \mathbb{N} \), set
\[ W_{\text{div}}^{k,q}(\Omega) := \left\{ \text{closure of } V_{\text{per}}(\Omega) \text{ in } W^{k,q}(\Omega) \right\}, \]
endowed with the usual norms.

In the space-periodic setting many calculations are simpler since we can use \( -\Delta u \) as a test function (now formally but the procedure goes through the Galerkin approximation). Since in the 2D space-periodic case \( \int_{\Omega} (u \cdot \nabla) u \, \Delta u \, dx = 0 \), if \( f \in L^2(\Omega_T) \) we get
\[ \frac{d}{dt} \| \nabla u \|_{L^2}^2 + v_0 \| \Delta u \|_{L^2}^2 + v_1 \| I(u) \|_{L^2}^2 \leq C \| f \|_{L^2}^2. \] (3.1)
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hence, if we are able to construct such a solution (this is not trivial at all due to some technical issues when passing to the limit in \( \int_0^T I(u(t)) \, ds \), for a fixed \( T > 0 \)) and if \( v_0 = 0 \), we obtain a higher-order estimate

\[
\int_0^T I(u) \, dt = \int_0^T \int_{\mathbb{T}^2} (\delta + |D u|)^{p-2} |\nabla D u| \, dx \, dt < +\infty.
\]

We recall the following lemma, which is an adaption of Berselli et al (2010, lemma 4.4) of the two-dimensional case.

**Lemma 3.1.** Let \( p \in (1, 2], \delta \in (0, \infty) \) and \( \ell \in [1, 2) \). Then, for all sufficiently smooth functions \( u \) with a vanishing mean value over \( \Omega \), the following relations hold true

\[
\|u\|_{2, \ell}^p \leq c(I(u) + \delta^p).
\]

Hence, the information on the regularity in the space variable which we can extract from (3.1), in the case \( v_0 = 0 \), could be at most

\[
u \in W^{2, \ell}(\mathbb{T}^2) \quad \forall \ell < 2, \quad a.e. \ t \in [0, T].
\]

This is not enough to employ theorem 1.1 and explains the introduction of the hypothesis \( v_0 > 0 \).

**Proof of proposition 1.1.** In light of the above observations the proof follows as in the 2D Navier–Stokes equations, see Dashti and Robinson (2009). We test the equations by \( -t \Delta u^m \) and we have

\[
\frac{d}{dt}(t \| \nabla u^m \|_2^2) + v_0 t \| \Delta u^m \|_2^2 + v_1 t I(u^m) \leq C t \| f \|_2^2 + \| \nabla u \|_2^2.
\]

Hence, no matter of the non-negative term coming from the extra stress-tensor, integrating in time over \([0, T]\) we have that \( \sqrt{t} u^m \in L^2(0, T; W^{2,2}(\mathbb{T}^2)) \). Due to uniqueness of the solution the whole sequence \( \{u^m\} \) converges to \( u \) and by lower-semicontinuity of the norm we obtain that \( \sqrt{t} u \in L^2(0, T; W^{2,2}(\mathbb{T}^2)) \).

**Remark 3.1.** For the sake of completeness, we recall that in the periodic 2D case, with \( v_0 = 0 \) it is possible to prove the existence of regular solutions, see Kost (2010), by adapting results for the 3D case in Berselli et al (2010). In the absence of the Laplacian also the existence and uniqueness of solutions are more delicate and the limit process on Galerkin solutions requires some care. The following result, which is of interest by itself, is not enough for our purposes of studying uniqueness for solutions to (1.4).

Let be given \( \delta \in [0, \delta_0] \), for some \( \delta_0 > 0 \), set \( v_0 = 0 \), \( v_1 > 0 \) and let \( p \in (1, 2] \). Given \( T > 0 \), assume that \( f \in L^\infty(0, T; W^{1,2}(\mathbb{T}^2)) \cap W^{1,2}(0, T; L^2(\mathbb{T}^2)) \). Let \( u_0 \in W^{2,2}(\mathbb{T}^2) \) be such that \( \text{div} u_0 = 0 \) and \( \text{div} S(D u_0) \in L^2(\mathbb{T}^2) \). Then, there is a time \( 0 < T' \leq T \) (depending on the data of the problem) such that the system (1.1), has a strong solution \( u \) on \([0, T']\) satisfying, for \( r \in (4/3, 2) \),

\[
u \in L^q(0, T'; W^{2,r}(\mathbb{T}^2)) \cap C(0, T'; W^{1,q}(\mathbb{T}^2)), \quad \forall q < \infty.
\]

One can obtain further regularity results for \( u \) and also for \( \nabla u \) (the latter if \( \delta > 0 \)).
4. Space-time regularity in the Dirichlet case

In this section we consider the time evolution problem with Dirichlet boundary conditions and we prove a result of regularity for smooth data. Then, we will relax some of the assumptions to prove the main result of the paper. We start by showing a first regularity result for the time derivative of the solutions to the problem (1.1) with Dirichlet boundary conditions. We prove now some results by using as test functions first- and second-order time-derivatives of the velocity. These are legal test functions, since if \( u \) is divergence-free and \( u|_{\partial D} = 0 \), then \( \frac{du}{dt} \) shares the same two properties for all \( k \in \mathbb{N} \). In particular, the following result is valid in any smooth and bounded domain, while the hypothesis of a flat boundary will be used for the \( W^{2,2}(\Omega) \)-regularity.

**Lemma 4.1.** Let \( p \in (1, 2] \), \( \delta > 0 \), \( f \in W^{1,2}(0, T; L^2(\Omega)) \), \( u_0 \in W^{2,2}(\Omega) \cap V_2 \) and let \( u \) be a weak solution of problems (1.1)–(2.1). Then,

\[
\|u_t\|_{L^2(0,T;L^2)}^2 + \|\nabla u\|_{L^2(0,T,L^2)}^2 + v_0 \|\nabla u_t\|_{L^2(0,T,L^2)}^2 + v_1 \|J(u)\|_{L^1(0,T)} \leq C,
\]

where the constant \( C \) depends on \( \delta, v_0, v_1, \|f\|_{W^{1,2}(0,T,L^2)}, \|u_0\|_{L^2}, T \) and \( \Omega \).

As in the previous result we only prove the \emph{a priori} estimates. A complete proof can be obtained through a Galerkin approximation and for the remainder of this section we drop the superscript 'm'. We also define the following quantity:

\[
M(t) := \int_0^t (\delta + s)^{p-2} \, ds \geq 0, \quad \text{for } t \geq 0.
\]

Observe that \( M(t) \simeq (\delta + t)^{p-2} t^2 \) and also \((\delta + t)^{p-2} t^2 \leq t^p\), with \( 1 \leq p \leq 2 \). This shows that

\[
0 \leq M(u) := \int_{\Omega} M(\|Du\|) \, dx \leq C(p)\|Du\|_{L^p}^p, \quad \text{with } 1 \leq p \leq 2.
\]

**Proof of lemma 4.1.** First, we multiply (1.1) by \( u_t \) and integrate by parts. We observe that taking the duality product of \(-\text{div} S(Du)\) against \( u_t \), we get

\[
\langle \text{div} S(Du), u_t \rangle = \|S(Du), Du_t \| = \frac{d}{dt} \mathcal{M}(u).
\]

By suitable integrations (since \( \text{div} \, u = 0 \)) we obtain

\[
\|u_t\|_{L^2}^2 + \frac{v_0}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + v_1 \frac{d}{dt} \mathcal{M}(u) = \int_{\Omega} (f u_t - (u \cdot \nabla) u_t) \, dx.
\]

By using the Hölder and Gagliardo–Nirenberg inequality (2.5), with the boundedness of the kinetic energy, we get, for all \( \varepsilon > 0 \)

\[
\left| \int_{\Omega} (u \cdot \nabla) u_t \, dx \right| \leq \|u\|_4 \|\nabla u\|_2 \|u_t\|_4
\]

\[
\leq C \|u\|_2^\frac{1}{2} \|\nabla u\|_2^\frac{1}{2} \|u_t\|_2^\frac{1}{2} \|\nabla u_t\|_2^\frac{1}{2}
\]

\[
\leq C \|u\|_2^\frac{1}{2} \|\nabla u\|_2^\frac{1}{2} \|u_t\|_2^\frac{1}{2} + \|u_t\|_2^\frac{1}{2} \|\nabla u_t\|_2^\frac{1}{2} + \|f\|_2^\frac{1}{2} + \|u_t\|_2^\frac{1}{2} \|\nabla u_t\|_2^\frac{1}{2}.
\]

Thus, we obtain the following differential inequality

\[
\|u_t\|_{L^2}^2 + \frac{d}{dt} \left( v_0 \|\nabla u\|_{L^2}^2 + v_1 \mathcal{M}(u) \right) \leq C \|u\|_2^\frac{1}{2} \|\nabla u\|_2^\frac{1}{2} \|u_t\|_2^\frac{1}{2} + \|u_t\|_2^\frac{1}{2} \|\nabla u_t\|_2^\frac{1}{2} + \|f\|_2^\frac{1}{2} + \|u_t\|_2^\frac{1}{2} \|\nabla u_t\|_2^\frac{1}{2}
\]

which we clearly cannot use directly due to the lack of control for \( \nabla u_t \).

**Remark 4.1.** Another path will be that of using improved estimates for \( \nabla u \) to estimate the convective term (see the last section).
We take now the time derivative of (1.1), multiply by $u_t$ and integrate by parts (recalling that $\int_\Omega (u \cdot \nabla) u_t \, dx = 0$) to obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|u_t\|^2 + v_0 \|\nabla u_t\|^2 + v_1 \beta_j(Du_t), Du_t \right) \leq \left| \int_\Omega \left( (u_t \cdot \nabla) u_t + f_t, u_t \right) \, dx \right|.
\end{equation}
By (2.2a) the term involving $S$ in (4.5) is non-negative being estimated from below by a multiple of $\mathcal{J}(u) \geq 0$. Let us focus on the right hand side of (4.5). By using the Hölder inequality, (2.5), and the energy estimate we get, for each $\eta > 0$,
\begin{equation}
\left| \int_\Omega (u_t \cdot \nabla) u_t, dx \right| = \left| \int_\Omega (u_t \cdot \nabla) u_t, dx \right| \\
\leq \|u_t\|_2 \|\nabla u_t\|_2 \|u_t\|_4 \\
\leq C \|u_t\|_2 \|\nabla u_t\|_2 \|u_t\|_2 \|\nabla u\|_2 \\
\leq c_0 \|\nabla u\|_2 \|u_t\|_2 + \eta \|\nabla u_t\|_2^2,
\end{equation}
hence, by choosing $\eta > 0$ to be small enough, we get
\begin{equation}
\frac{d}{dt} \left( \|u_t\|^2 + v_0 \|\nabla u_t\|^2 + v_1 \mathcal{J}(u) \right) \leq C \left( \|\nabla u\|_2 \|u_t\|_2 + \|f_t\|_2^2 \right).
\end{equation}
Summing up (4.4)–(4.6) and choosing $\epsilon > 0$ to be small enough we get finally
\begin{equation}
\frac{d}{dt} \left( \|u_t\|^2 + v_0 \|\nabla u_t\|^2 + v_1 \mathcal{J}(u) \right) + \|u_t\|_2 + v_0 \|\nabla u_t\|_2 + v_1 \mathcal{J}(u) \\
\leq C \left( \|\nabla u\|_2 \|u_t\|_2 + \|\nabla u\|_2 + \|f\|_2 \right).
\end{equation}
To integrate over $[0, T]$ we need to make sense of $\|u_t(0, \cdot)\|_2$. From the assumptions on the data, the fact that $\delta > 0$ and $u^m(0) = P^m u_0$ we easily get (see Berselli et al (2010, section 5))
\begin{equation}
\|u^m(0)\|_2 \leq C \left( \|u_0\|_2 + \|f^m(0)\|_2 \right).
\end{equation}
Recall that we are working on the finite-dimensional approximation $u^m$ and taking the limit $m \to +\infty$. With the Gronwall lemma and by using the fact that $\nabla u \in L^2(0, T; L^2(\Omega))$, we get for a.e $t \in [0, T]$
\begin{equation}
\|u_t(t)\|^2 + v_0 \|\nabla u(t)\|^2 + M(u(t)) + \int_0^t \left( \|u_t(s)\|^2 + v_0 \|\nabla u_t(s)\|^2 + \mathcal{J}(u(s)) \right) \, ds \\
\leq C(\delta, v_1, T, \|f\|_{W^{1,2}(\Gamma_1)}, \|u_0\|_{L^2}, \Omega),
\end{equation}
hence the thesis. □

**Remark 4.2.** The hypotheses on the external force can be slightly relaxed, but this is inessential in our treatment.

We now prove proposition 1.2. For the reader’s convenience we split the proof into two parts. First, we perform a preliminary study of the system obtained by removing the convective term $(u \cdot \nabla) u$ from (1.1).

\begin{align}
\frac{u_t - v_0 u}{\Delta} - v_1 \Delta u + \nabla \pi &= f & \text{in } [0, T] \times \Omega, \\
\text{div } u &= 0 & \text{in } [0, T] \times \Omega, \\
\frac{u}{\alpha} &= 0 & \text{in } [0, T] \times \Gamma, \\
\frac{u(0)}{\alpha} &= u_0 & \text{in } \Omega,
\end{align}
and focusing on the role of the nonlinear stress-tensor. The system (4.7a)–(4.7d) can be treated similarly to a steady state problem if we have good enough \textit{a priori} estimates on $u_t$. We will then address the full problem (1.1)–(2.1), by adding suitable estimates for the convective term.
Lemma 4.2. Let \( v_0 > 0, \delta > 0 \) and \( p \in \left[ \frac{2}{3}, 2 \right] \). Given \( T > 0 \), assume that \( u_0 \in W^{2,2}(\Omega) \cap V_2 \) and \( f \in W^{1,2}(0,T; L^2(\Omega)) \). Then, problem (4.7a)–(4.7d)–(4.1) admits a unique solution, such that (1.5) holds true.

Proof. We adapt to the time-dependent case a technique with three intermediate steps taken from Beirão da Veiga (2009), Crispo (2009): in the first step we bound the tangential derivative of velocity and pressure; in the second step we estimate the normal derivative of the velocity field; in the last step we estimate the normal derivative of the pressure.

Again we merely prove the \textit{a priori} estimates. Observe that for this simpler problem without convection, the same existence proved in theorems 2.1 and regularity from lemma 4.1 clearly hold true (this is particularly relevant for what concerns \( u_t \)).

\textbf{Step 1.} We first prove that the following estimates, concerning the tangential derivatives, hold true

\[
\begin{aligned}
&v_0 \| \nabla_1 u \|_{L^2(0,T;L^2)}^2 + v_0 \| \partial^{2}_{22} u_2 \|_{L^2(0,T;L^2)}^2 + \| \partial_1 \pi \|_{L^2(0,T;L^2)} \leq C, \\
\end{aligned}
\]

(4.8)

where \( C \) depends on \( p, \delta, v_0, \| f \|_{W^{1,2}(0,T;L^2)}, \| u_0 \|_{L^2}, T \) and \( \Omega \).

We now use the particular features of the flat domain. Multiplying equation (4.7a) by \(-\partial^{2}_{11} u\) and integrating by parts, it follows that

\[
\frac{1}{2} \frac{d}{dt} \| \partial_1 u(t) \|_2^2 + v_0 \nabla_1 u(t) \cdot f \leq C \left( \| \nabla u_0 \|_2^2 + \frac{1}{v_0} \int_0^t \| f(s) \|_2^2 ds \right),
\]

(4.9)

and, since \( \text{div} u = 0 \), \( \partial^{2}_{22} u_2 = -\partial^{2}_{11} u_1 \) and the estimate on \( \partial^{2}_{22} u_2 \) follows.

Let us focus on the pressure term. Differentiating the equation (4.7a) with respect to the tangential direction \( x_1 \), one has that

\[
\nabla_1 \pi = v_0 \text{div} \partial_1 \nabla u + v_1 \text{div} \partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] - \partial_1 u_t + \partial_1 f \quad \text{a.e. in } \Omega_T.
\]

We observe that \( \partial_1 u_t = \text{div} \left( \begin{array}{c} \frac{\delta \partial_1 u}{\partial x_2} \\ \partial_1 f \end{array} \right) \) and \( \partial_1 f = \text{div} \left( \begin{array}{c} 0 \\ \partial_1 f \end{array} \right) \). Hence to apply lemma 2.3 to estimate \( \partial_1 \pi \), we only have to bound the term \( \partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] \). A direct computation gives

\[
\partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] = (\delta + |Du|)^{p-2} \partial_1 Du + (p - 2)(\delta + |Du|)^{p-3} (Du \cdot \partial_1 Du) \frac{Du}{|Du|},
\]

and consequently

\[
\| \partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] \|_2 \leq (3 - p)(\delta + |Du|)^{p-2} |\partial_1 Du| \quad \text{a.e. in } \Omega_T.
\]

Therefore, by comparison \( \partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] \in L^2(\Omega) \) and it follows that

\[
\int_\Omega \left| \partial_1 \left[ (\delta + |Du|)^{p-2} Du \right] \right|^2 \leq c \delta^{p-2} I_1(u) \quad \text{a.e. } t \in [0, T],
\]

By applying lemma 2.3 we have that

\[
\| \partial_1 \pi \|_2^2 \leq \| u_0 \|_2^2 + v_0 \| \partial_1 \nabla u \|_2^2 + v_1 C \delta^{p-2} I_1(u) + \| f \|_2^2 \quad \text{a.e. } t \in [0, T],
\]

from which, integrating in time over \([0, T]\), using (4.9) and recalling the bounds previously proved on \( u_t, \partial_1 \nabla u \) and \( I_1 \), then (4.8) follows.
Under the same hypotheses as before, but for \( p \) \( \in \left( \frac{3}{2}, 2 \right) \), we have
\[
\|a_{22}^2 u_1\|_{L^2(0,T;L^2)} \leq C,
\]
where the constant \( C \) depends on \( p, \delta, v_0, v_1, \|f\|_{W^{3,2}(0,T;L^2)}, \|u_0\|_{2,2}, T \) and \( \Omega \).

We follow the main lines established in the proof of Crispo (2009, lemma 3.3). By calculating \( \partial_2 \left( (\delta + |Du|)^{p-2} Du \right) \), the first equation in (4.7a) can be written as
\[
\alpha_1 a_{22}^2 u_1 = -F_1 - f_1 + \partial_1 u_1 + \partial_1 \pi,
\]
where
\[
\alpha_1 := v_0 + \frac{v_1}{2} (\delta + |Du|)^{p-2} + v_1 (p - 2) (\delta + |Du|)^{p-3} \frac{|Du|}{|Du|} (Du)_{12} (Du)_{12},
\]
and
\[
F_1 := \left[ v_0 + v_1 (\delta + |Du|)^{p-2} \right] a_{22}^2 u_1 + \frac{v_1}{2} (\delta + |Du|)^{p-2} a_{12}^2 u_2 + v_1 (p - 2) (\delta + |Du|)^{p-3} \frac{|Du|}{|Du|} \left( \sum_{k,l=1}^2 (Du)_{kl} \partial_1 (Du)_{kl} (Du)_{11} + a_{12}^2 u_2 (Du)_{22} (Du)_{12} \right).
\]
By direct calculation it can be easily seen that
\[
|F_1| \leq C \left[ v_0 + v_1 \left( p - \frac{3}{2} \right) (\delta + |Du|)^{p-2} \right] |Du| \quad \text{a.e. in } \Omega_T
\]
and by using \( p \geq \frac{3}{2} \) we get
\[
\alpha_1 \geq \left[ v_0 + v_1 \left( p - \frac{3}{2} \right) (\delta + |Du|)^{p-2} \right] \geq v_0 > 0.
\]
Division of both sides of (4.10) by \( \alpha_1 \) is then legitimate and we infer that
\[
|a_{22}^2 u_1| \leq C \left( |Du| + \frac{1}{v_0} (|\partial_1 \pi| + |\partial_1 u_1| + |f_1|) \right) \quad \text{a.e. in } \Omega_T.
\]
Therefore, squaring and integrating over \( \Omega_T \) we get
\[
\int_0^T \|a_{22}^2 u_1(s)\|^2 \, ds \leq C \int_0^T \left( \|Du(s)\|^2 + \|\partial_1 \pi(s)\|^2 + \|\partial_1 u_1(s)\|^2 + \|f_1(s)\|^2 \right) \, ds,
\]
which, by the previous results is finite. This finally shows that \( D^2 u \in L^2(\Omega_T) \).

Step 3. The final step, which is not strictly required for particle trajectory uniqueness, is the regularity of the normal derivative of pressure. Nevertheless, we include it for the sake of completeness. Under the same hypotheses as step 2 we have
\[
\|a_{22}^2 u_1\|_{L^2(0,T;L^2)} \leq C,
\]
where the constant \( C \) depends on \( p, \delta, v_0, v_1, \|f\|_{W^{3,2}(0,T;L^2)}, \|u_0\|_{2,2}, T \) and \( \Omega \).

By using the second equation in (4.7a), one can write
\[
|\partial_2 \pi| \leq c \left( v_0 + v_1 (p - 2) (\delta + |Du|)^{p-2} \right) |D^2 u| + |\partial_2 u_1| + |f_2| \quad \text{a.e. in } \Omega_T.
\]
Hence, straightforward calculations lead to
\[
\int_0^T \|\partial_2 \pi(s)\|^2 \, ds \leq c \int_0^T \left( \left[ v_0 + v_1 \delta^{2(p-2)} \right] |D^2 u(s)|^2 + \|\partial_2 u_1(s)\|^2 + \|f_2(s)\|^2 \right) \, ds,
\]
and the assertion follows as a consequence of the previous results. \( \square \)
We finally prove the same regularity results also in the presence of the convective term. We use a perturbation argument, treating \((u \cdot \nabla) u\) as the right hand side in equation (1.1).

**Proof of proposition 1.2.** Here, we use the *a priori* estimates obtained for the problem (4.7a)–(4.7d) with external body force

\[ F := -(u \cdot \nabla) u + f. \]

In the derivation of estimates for \(u_t\), we use that \(\|f\|_{W^{r,2}(T;L^2)}\), while in lemma 4.2 the estimates depend essentially on the \(L^2(\Omega_T)\)-norm of the external force. Hence, by using lemma 4.1 it is then sufficient to estimate \(\|(u \cdot \nabla) u\|_{L^2(0,T;L^2)}\) in terms of second-order derivatives of \(u\), to follow the same calculations in steps 1–3 of the previous result.

By applying the Hölder, Gagliardo-Nirenberg and Young inequalities and the energy estimate, we get for each \(\varepsilon > 0\)

\[
\|(u \cdot \nabla) u\|_2 \leq \|u\|_4 \|
abla u\|_4 \leq c \|u\|_4^{3/2} \|\nabla u\|_2^{1/2} \|
abla u\|_2 \|D^2 u\|_2^{1/2} \\
\leq c \|\nabla u\|_2^2 + \varepsilon \|D^2 u\|_2.
\]

(4.11)

By using the same calculations as the previous proposition and the *a priori* estimates in (4.1)—especially that \(\nabla u \in L^\infty(0, T; L^2(\Omega))\)—we have

\[
\int_0^T (\|u\|_2^2 + \|\tau\|_2^2) \, ds \leq C \int_0^T (\|f\|_2^2 + \|u_t\|_2^2 + \|(u \cdot \nabla) u\|_2^2) \, ds \\
\leq C(p, \delta, v_0, v_1, \|f\|_{W^{r,2}(0,T;L^2)}, \|u_0\|_{2,2}, T, \Omega, \varepsilon) + \varepsilon \int_0^T \|D^2 u\|_2^2 \, ds,
\]

and, by choosing \(\varepsilon > 0\) to be small enough, we end the proof.

As a consequence of the above result we have full \(L^2\)-space-time regularity of the solution up to second-order space-derivatives, hence the uniqueness of particle trajectories. The result is not optimal in view of application to uniqueness of trajectories, in the sense that some of the hypotheses can be slightly relaxed. For instance \(f_t \in L^2(\Omega_T)\) and \(u_0 \in W^{2,2}(\Omega)\) can be removed (at the price of less regularity on \(u_t\)) by following a slightly different path as we do in the next section.

**5. Proof of theorem 1.2**

In this section we finally address the problem of the uniqueness of particle trajectories under ‘minimal’ assumptions on the data. We will show how the previous regularity result, together with theorem 1.1, allow us to prove theorem 1.2.

**Proof of theorem 1.2.** In the same way as in the proof of lemma 4.2, we perform separately the *a priori* estimates for the normal and tangential derivative of the time-weighted \(\sqrt{t} u^m\) (which we call \(\sqrt{t} u\)). In particular, here we do not use a lot of regularity on \(u_t\), but we have to deal with a non-smooth \(u_0\). By adapting standard weighted estimates, we multiply the equation (1.1) by \(-t \partial_1^2 u\). Integrating by parts, and with Young’s inequality we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( t \|\partial_1 u\|_2^2 \right) + v_0 t \|\nabla \partial_1 u\|_2^2 + (p - 1)v_1 t \mathcal{I}_1(u) \\
\leq \frac{v_0}{2} t \|\partial_1^2 u\|_2^2 + \frac{C}{v_0} t \left( \|\partial_1 u\|_2^2 + \|f\|_2^2 \right) + \|\partial_1 u\|_2^2.
\]
Integrating in time and using the energy estimate to bound \( \int_0^t \| \partial_1 u \|^2 \, ds \), it follows

\[
t \| \partial_1 u(t) \|^2 + v_0 \int_0^t s \| \nabla \partial_1 u(s) \|^2 \, ds + v_1 \int_0^t s J_1(u(s)) \, ds \\
\leq \frac{C}{v_0} \left[ \| u_0 \|^2 + \int_0^t s \left( \| (u(s) \cdot \nabla) u(s) \|^2 + \| f(s) \|^2 \right) \, ds \right] \quad \text{a.e. in } [0, T]. \tag{5.1}
\]

We take now the \( L^2 \)-inner product of (1.1) with \( u_t \). By suitable integrations by parts, and using (4.3)–(4.2) we reach

\[
t \| u \|^2 + \frac{v_0}{2} \frac{d}{dt} (t \| u \|^2) + v_1 \frac{d}{dt} (t \mathcal{M}(u)) \\
\leq t \left( \int_{\Omega} (u \cdot \nabla) u u_t \, dx \right) + v_0 \| \nabla u \|^2 \mathcal{M}(u) \\
\leq \frac{t}{4} \left( \| u \cdot \nabla u \|^2 + \| f \|^2 \right) + \frac{v_0}{2} \| u \|^2 + v_0 \| \nabla u \|^2 + C v_1 \| \mathcal{D}u \|^p.
\]

Integrating this inequality in time, by appealing to the energy inequality and recalling that \( \mathcal{M}(u) \geq 0 \), it follows that for a.e. \( t \in [0, T] \)

\[
v_0 t \| \nabla u(t) \|^2 + \int_0^t s \| u_t(s) \|^2 \, ds \\
\leq C \left[ \| u_0 \|^2 + \int_0^t s \left( \| (u(s) \cdot \nabla) u(s) \|^2 + \| f(s) \|^2 \right) \, ds \right], \tag{5.2}
\]

where \( C \) depends on \( p, \delta, v_0, v_1, T \) and \( \Omega \).

Let us now focus on the normal derivatives of \( u \). Arguing as in step 2 of the proof of lemma 4.2, and replacing \( f \) with \( f + (u \cdot \nabla) u \), we infer that

\[
|\partial_{\nu}^2 u_{t1}| \leq C \left( |D^2 u| + \frac{1}{2v_0} \left[ |\partial_1 u| + |u_t| + |(u \cdot \nabla) u| + |f| \right] \right) \quad \text{a.e. in } \Omega_T.
\]

Then, squaring, multiplying by \( t \) and integrating over \( (0, t) \times \Omega \), we find

\[
\int_0^t s \| \partial_{\nu}^2 u_{t1}(s) \|^2 \, ds \leq \frac{C}{v_0} \int_0^t s \left( \| D^2 u(s) \|^2 + \| \partial_1 \nabla u(s) \|^2 + \| u_t(s) \|^2 + \| (u(s) \cdot \nabla) u(s) \|^2 \right) \, ds.
\]

To control \( \int_0^t s \| \partial_1 \nabla u(s) \|^2 \, ds \) we proceed again as in step 2 of the proof of lemma 4.2. Thus, for a.e. \( t \in [0, T] \), the following inequality holds true

\[
\int_0^t s \| \partial_1 \nabla u(s) \|^2 \, ds \leq \frac{C}{\delta} \int_0^t s \left( \| u_t(s) \|^2 + \| \partial_1 \nabla u(s) \|^2 \right) \, ds \\
+ \delta^{p-2} J_1(u(s)) + \| f(s) \|^2 + \| (u(s) \cdot \nabla) u(s) \|^2 \, ds \\
\leq \frac{C}{\delta} \left[ \| u_0 \|^2 + \int_0^t s \left( \| (u(s) \cdot \nabla) u(s) \|^2 + \| f(s) \|^2 \right) \, ds \right],
\]

where we have used relations (5.1) and (5.2). Once again we apply (5.1), so that relation (5.3) gives, for a.e. \( t \in [0, T] \)

\[
\int_0^t s \| \partial_{\nu}^2 u_{t1}(s) \|^2 \, ds \leq C \left[ \| u_0 \|^2 + \int_0^t s \left( \| (u(s) \cdot \nabla) u(s) \|^2 + \| f(s) \|^2 \right) \, ds \right],
\]

where \( C \) depends on \( p, \delta, v_0, v_1, T \) and \( \Omega \). Summing up the above inequality with (5.1) and (5.2), we get for a.e. \( t \in [0, T] \)

\[
t \| \nabla u(t) \|^2 + v_0 \int_0^t s \| D^2 u(s) \|^2 \, ds \\
\leq C \left[ \| u_0 \|^2 + \int_0^t s \left( \| (u(s) \cdot \nabla) u(s) \|^2 + \| f(s) \|^2 \right) \, ds \right].
\]
with $C$ depending on $p$, $\delta$, $v_0$, $v_1$, $T$ and $\Omega$. The convective term can be estimated as in (4.11) and, by choosing $\varepsilon > 0$ to be small enough, we get for a.e. $t \in [0, T]$

$$t \| \nabla u(t) \|^2_2 + v_0 \int_0^t s \| D^2 u(s) \|^2_2 \, ds \leq c_\varepsilon \int_0^t \left( s \| \nabla u(s) \|^2_2 \right) \| \nabla u(s) \|^2_2 \, ds + C(p, \delta, v_0, v_1, \| f \|_{L^2(0, T; L^p)}, \| u_0 \|_2, T, \Omega).$$

Hence, by using Gronwall inequality over $[\lambda, T]$ (for any $\lambda > 0$), letting $\lambda \to 0^+$, and by using the energy inequality we get

$$\int_0^T t \| D^2 u(t) \|^2_2 \, dt \leq C(p, \delta, v_0, v_1, \| f \|_{L^2(0, T; L^p)}, \| u_0 \|_2, T, \Omega).$$

Then, the assertion follows by means of theorem 1.1. $\square$

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