Eco-epidemiological model and analysis of potato leaf roll virus using fractional differential equation

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ABSTRACT
In this paper, mathematical models for transmission dynamics of Potato leaf roll virus using integer and fractional order differential equations are developed. The models considered potato as well as Vector populations. The potato population is sub-divided as susceptible \( (S_p(t)) \) and infected \( (I_p(t)) \). The vector population also sub-divided as susceptible \( (S_v(t)) \) and infected \( (I_v(t)) \). Firstly, we proposed the integer order of potato leaf roll virus (PLRV) model and then we extend into fractional order due to the reason that fractional order possesses memory and other benefit in modeling of real-life phenomena. Secondly, the qualitative behavior of the model including invariant region, positivity of future solution and equilibrium points are analyzed in both approaches. Finally, numerical simulation is done to investigate the effect of each parameter on the control of the disease for both integer and fractional orders. The results obtained from numerical simulation indicate that increasing elimination rate of infected potato and contact rate have a great contribution in combating potato leaf roll virus in the specified period of time.

1. Introduction
One of the top starch rich food crop in the world is potato and originated from Andes of South America (FAO Food & Agricultural Organization of United Nations, 2008). However, Potato is vulnerable to different diseases caused by various microorganisms; but virus is responsible in most cases. From viral disease, different types of potato virus like potato leaf-roll virus (PLRV) are the dominant one throughout the world (Astarini, Margareth, & Temaja, 2018).

Potato leaf roll virus (PLRV) belongs to genus polaro virus and family of luteo viridae (CIP International Potato Center, 1998). This disease is most risky global, mainly overwhelming in countries with inadequate supply and organization. It can be in charge for separate plant harvest sufferers of over 50% and accountable for yearly worldwide yield damage of 20 million tons (Wales, Platt, & Cattin, 2008).

Primary infection of Potato occurs after it is bitten by virus caring aphid. This aphid got the pathogen from infected plants (Davis, 2008). Once infected with leaf roll virus, the plant aphid can transmit the virus for life. Moreover, secondary infection can occur when infected tuber is planted and then Aphid T spread the infection further in neighboring plants (Moller, 2008).

Mathematical model provides a powerful device that helps us to understand the dynamics of infectious disease. Recently, a number of mathematical models for transmission dynamics of plant virus diseases to study the blow-out of disease and analyzed in controlling strategies. For example, Windarto and Putri (2019) developed a maize disease model. They analyzed the model by using ordinary differential equation. The other study done by Rida, Khalil, Hosham, and Gadellah (2016) also consider vector as a cause of plant disease. They described the system of the model using fractional differential equation to analyze the stability of equilibrium point. The other study by Magoyo, Irunde, and Kuznetsov (2019) was developed to determine the model for cassava mosaic disease. In their studies, whitefly vector is considered to be the cause. Banana Xanthomonas wilt deterministic model also proposed by Horub and Julius (2017). This model was formulated and analyzed with system of ordinary differential equation. Recently, Haileyesus, Makinde, and Theuri (2019) developed a mathematical model of maize streak virus and analyzed its transmission dynamics. Nowadays, different authors prefer to model different infectious disease using fractional order. Some of the works are done by Kumar, Singh, Al Qurashi, and Baleanu (2019), Kumar, Singh, and Baleanu (2020), Kumar et al. (2020a), Kumar, Kumar, Singh, Nisar, and Kumar (2020b), Lopes and Tenreiro Machado (2014),
Singh (2020), Singh, Kumar, and Baleanu (2020) and Nemecek (1993).

Fractional differential equation is a powerful tool to understand the dynamics of different life situation in fractional order and it is applicable in different scenarios, for example, Ahmad, Khan, Ahmad, Stanimirović, and Chu (2020) used for reaction–diffusion model, Ahmad, Akgül, Khan, Stanimirović, and Chu (2020) used in deriving conventional solutions of the nonlinear partial differential equation, Ahmad, Khan, Stanimirović, Chu, and Ahmad (2020) used in developing of diffusion models, Singh et al. (2020) applied to obtain hyperbolic type solutions for the couple Boiti–Leon–Pempinelli system and Ahmad, Seadawy, Khan, and Thounthong (2020) also used to solve parabolic dynamical wave equation.

However, to the best of our knowledge, there is no study done using fractional differential equation to model PLRV transmission. Therefore, this paper developed a fractional order mathematical model of PLRV by considering aphid vectors. The results indicate that decreasing the infection rate (a) and contact rate (ô) have a great contribution for controlling PLRV disease. Moreover, increasing elimination rate of infected potato (a2) through uprooting and burning of the infected potato from field has a great benefit in controlling the disease.

This paper is organized as follows. Section 2 introduces formulation and description about proposed model. Section 3 is devoted on the analysis of the model. Section 4 discuss about numerical simulation of the model. Finally, Section 5 contains summary and conclusions.

2. Mode description and formulation

The PLRV model considered potato and vector populations. These populations have susceptible and infected subclasses. The susceptible potato denotes by $S_p$ and infected potato by $I_p$. Similarly, susceptible vector is also represented by $S_v$ and infected by $I_v$. The model considered recruitment rate of susceptible vectors by $\pi_2$ and move to infected vectors $(I_v)$ with $\delta$ rate after consuming ill plants or potatoes. The susceptible potato also replanted at rate $\pi_1$ and the diseases spread to potato, when infected vectors $(I_v)$ react with susceptible potato $(S_p)$ at rate of $a$ through eating. Potato once become infected not ever mends and gives yield or produce very low yield of potato. The model also assumes that $\gamma_p$ is the natural death rate for potato population, and $\gamma_v$ is natural death rate for vector population. In the model, $a_1$ is disease causing death rate and $a_2$ is eliminating rate of infected plant from uninfected to control the disease.

The above description of the model is plotted in Figure 1 below.

From Figure 1 and the model description we generated the following model as four system of differential equation:

$$
\begin{align*}
\frac{dS_p}{dt} &= \pi_1 - aS_pI_v - \gamma_p S_p, \\
\frac{dI_p}{dt} &= aS_pI_v - (\alpha_1 + \alpha_2 + \gamma_p)I_p, \\
\frac{dS_v}{dt} &= \pi_2 - \delta S_v I_p - \gamma_v S_v, \\
\frac{dI_v}{dt} &= \pi_2 - \delta S_v I_p - \gamma_v I_v,
\end{align*}
$$

(1)

with initial condition,

$$
S_p(0) = S_{p0} \geq 0, \quad I_p(0) = I_{p0} \geq 0, \quad S_v(0) = S_{v0} \geq 0 \quad \text{and} \quad I_v(0) = I_{v0} \geq 0.
$$

Currently, theory and application of fractional differential equation have become useful and important in modeling of biological processes. Because fractional order possess memory and minimize errors from different reasons (Miao, Abdurahman, Teng, &
Kang, 2017; Rostamy & Mottaghi, 2016) and (Zeb, Zaman, Chohan, Momani, & Erturk, 2013).

Therefore, we converted Equation (1) in to fractional order for 0 < α ≤ 1 and become:
\[
\begin{aligned}
D^\alpha N_p &= \pi_1 - aS_p I_v - \gamma_p S_p,
D^\alpha I_p &= aS_p I_v - (\alpha_1 + \alpha_2 + \gamma_p)I_p,
D^\alpha S_v &= \pi_2 - \delta S_p V_v - \gamma_v S_v,
D^\alpha I_v &= \pi_3 - \delta I_p V_v - \gamma_v I_v,
\end{aligned}
\]  
(2)
where \( S_p(0) = S_{p0} \geq 0, I_p(0) = I_{p0} \geq 0, S_v(0) = S_{v0} \geq 0 \) and \( I_v(0) = I_{v0} \geq 0 \).

3. Qualitative analysis of the model
3.1. Invariant region
In this section, we obtain a region in which the solution of Equation (1) is bounded.

**Theorem 1.** The feasible solution set \( \{S_p, I_p, S_v, I_v\} \) of the system equation of the model enter and bounded in the region \( \Omega : \{S_p, I_p, S_v, I_v\} \in \mathbb{R}_+^4 : 0 \leq N_p \leq \frac{S_{p0}}{\pi_1}, \quad 0 \leq N_v \leq \frac{S_{v0}}{\pi_3} \} \).

**Proof.** To prove this, we considered potato and vector population separately. The potato population is given by \( N_p = S_p + I_p \) and vector population is given by \( N_v = S_v + I_v \). Then, we differentiate the two populations with respect to time by considering fractional order.

\[
D^\alpha N_p = D^\alpha (S_p + I_p)
\]
and
\[
D^\alpha N_v = D^\alpha (S_v + I_v).
\]
(3)
(4)
After we substitute expressions of \( D^\alpha N_p \) and \( D^\alpha I_p \) from Equations (2) in to (3), we obtain:
\[
D^\alpha N_p = \pi_1 - \gamma_p S_p - (\alpha_1 + \alpha_2 + \gamma_p)I_p.
\]
(5)
In the absence of death rate of potato due to infection \( (\alpha_1 = 0) \) and elimination \( (\alpha_2 = 0) \), Equation (5) become
\[
D^\alpha N_p(t) \leq \pi_1 - \gamma_p S_p - \gamma_p I_p.
\]
Taking \( N_p = S_p + I_p \), Equation (6) become:
\[
D^\alpha N_p(t) \leq \pi_1 - \gamma_p N_p.
\]
(7)
Now take Laplace transform \( \mathcal{L}\{D^\alpha N_p(t)\} + \mathcal{L}\{\gamma_p N_p\} \geq \mathcal{L}\{\pi_1\} \)
\[
\mathcal{L}\{S_{p0}(s)\} - D^{-1(\alpha)}N_p(0) + \gamma_p N_p(s) \geq \frac{\pi_1}{s}
\]
Thus, take \( D^{-1(\alpha)}N_p(0) = 0 \) at \( t = 0 \)
\[
(S^\alpha + \gamma_p)N_p(s) \geq \frac{\pi_1}{s}.
\]
Then we have \( N_p(s) \leq \frac{\pi_1}{S(s + \gamma_p)} \).
(8)

Now by finding the Laplace inverse transform of \( N_p(s) \) and using Mittag-Leffler function gives
\[
N_p(t) \leq \frac{\pi_1}{\gamma_p} (1 - E_\alpha(-\gamma_p t^\alpha)).
\]
Then as \( t \to \infty \) and since \( \gamma_p > 0 \) thus \( N_p(t) \to \frac{\pi_1}{\gamma_p} \geq 0 \). Therefore,
\[
0 \leq N_p(t) \leq \frac{\pi_1}{\gamma_p}.
\]
(9)
After re-arranging equation (9), we obtained:
\[
\Omega_p = \{ (S_p, I_p) \in \mathbb{R}_+^2 : 0 \leq N_p(t) \leq \frac{\pi_1}{\gamma_p} \}.
\]
(10)
By the same fashion, the population of vector which is \( N_v = S_v + I_v \) and we obtain:
\[
\Omega_v = \{ (S_v, I_v) \in \mathbb{R}_+^2 : 0 \leq N_v \leq \frac{\pi_3}{\gamma_v} \}.
\]
(11)
In general, from Equations (10) and (11), the invariant region of the systems (1) and (2) of the models is
\[
\Omega = \Omega_p \times \Omega_v = \{ (S_p, I_p, S_v, I_v) \in \mathbb{R}_+^4 : 0 \leq N_p, \quad 0 \leq N_v \leq \frac{\pi_1}{\gamma_p}, \quad \frac{\pi_3}{\gamma_v} \}.
\]
(12)

3.2. Positivity of the solution
In this section, we showed all the solution of the models Equations (1) and (2) remains positive for future time if their respective initial values are positive.

**Theorem 2.** If \( S_{p0} > 0, I_{p0} > 0, S_{v0} > 0, \) and \( I_{v0} > 0, \) then all the solution set \( \{S_p(t), I_p(t), S_v(t), I_v(t)\} \) of system (1) and (2) are positive for future time.

**Proof.** Let us take the first equation of the model in Equation (2), \( D^\alpha S_p = \pi_1 - aS_p I_v - \gamma_p S_p \), for \( 0 < \alpha \leq 1 \).
Since \( \pi_1 \) is a positive it is true that
\[
D^\alpha S_p \geq -S_p (aI_v + \gamma_p).
\]
(12)
Consider Laplace transform in both sides of Equation (12) gives
\[
\mathcal{L}\{D^\alpha S_p\} \geq -(aI_v + \gamma_p) \mathcal{L}\{S_p\}.
\]
(13)
Evaluating Equation (13) with suitable techniques gives us:
\[
S_p(s) \geq \frac{S_{p0}}{S^\alpha + (aI_v + \gamma_p)}.
\]
(14)
The Laplace inverse transform of Equation (14) using Mittag-Leffler function application is:
\[
S_p(t) \geq S_{p0} t^{\alpha-1} E_{\alpha,\sigma}(-(aI_v + \gamma_p) t^\alpha) > 0 \quad \text{for } t > 0.
\]
By the same fashion, all the rest equations of the system of the model become:
\[
I_p(t) \geq I_{p0} t^{\alpha-1} E_{\alpha,\sigma}(-k t^\alpha) \geq 0 \quad \text{for } t > 0,
\]
\[
S_v(t) \geq S_{v0} t^{\alpha-1} E_{\alpha,\sigma}(-(\delta_p + \gamma_v) t^\alpha) \geq 0 \quad \text{for } t > 0,
\]
\[
I_v(t) \geq I_{v0} t^{\alpha-1} E_{\alpha,\sigma}(-\gamma_v t^\alpha) \geq 0 \quad \text{for } t > 0.
\]
Therefore, all the solution sets of system of the model are positive for future time.
3.3. Disease free equilibrium points (DFE)

To obtain DFE point we consider the state where there is no infection, i.e. ($I_p = I_v = 0$). Consider the initial value problem of Equation (2) for $0 < \sigma \leq 1$.

To obtain the equilibrium point: Let us equate initial value problem of Equation (2) for $0 < \sigma \leq 1$.

Then by the principle of next generation matrix, first let us take the two infective compartments of the model of equation (2):

$D^0S_p = 0$, $D^0I_p = 0$, $D^0S_v = 0$, and $D^0I_v = 0$.

Therefore, the model has DFE given by

$$E^0 \left( S_p^0, I_p^0, S_v^0, I_v^0 \right) = \left( \frac{\pi_1}{Y_p}, 0, \frac{\pi_2}{Y_v}, 0 \right).$$

3.4. Basic reproduction number ($R_o$)

In this subsection, we obtained the basic reproduction number of the system of the PLRV model by using the next generation matrix method.

By the principle of next generation matrix, first let us take the two infective compartments of the model of equation (2):

$D^0I_p = aS_pI_v - (Y_p + \alpha_1 + \alpha_2)I_p,$

$D^0I_v = \delta S_v I_p - Y_v I_v.$

Then by the principle of next generation matrix,

$$G = FV^{-1} \quad (15)$$

where $F \left( aS_pI_v \right)$ and $V \left( \frac{\pi_1}{Y_p}, 0, \frac{\pi_2}{Y_v}, 0 \right)$,

for $k = Y_p + \alpha_1 + \alpha_2$.

Now we obtained the Jacobian matrix of $F$ and $V$ with respect to $I_p$ and $I_v$ at the disease-free equilibrium point $\left( \frac{\pi_1}{Y_p}, 0, \frac{\pi_2}{Y_v}, 0 \right)$.

$$J(E^0) = \begin{bmatrix} 0 & a\frac{\pi_1}{Y_p} \\ \delta \frac{\pi_2}{Y_v} & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} k \\ Y_v \end{bmatrix} \quad \text{but} \quad V^{-1} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$  

Therefore,

$$FV^{-1} = \begin{bmatrix} 0 & \frac{a\pi_1}{Y_p} \\ \delta \frac{\pi_2}{Y_v} \end{bmatrix}.$$  

The eigenvalues of Equation (15) become:

From this $\lambda_1 = \sqrt{\frac{a\pi_1}{Y_p}}$ and $\lambda_2 = -\sqrt{\frac{a\pi_2}{Y_v}}$.

Since, basic reproduction number is the dominant or the maximum of Eigenvalues. Therefore $R_o = \sqrt{\frac{a\pi_1}{Y_p}}.$

This $R_o$ shows that two generations are required for transmission of PLRV. That is from an infectious plant (potato) to susceptible vectors and then from an infectious vectors to susceptible potatoes. That is why the square roots found in $R_o$.

This implies that

$$R_o = \sqrt{\frac{a\pi_1}{kY_p}} = R_{op} \times R_{ov},$$

where

$$R_{op} = \sqrt{\frac{a\pi_1}{kY_p}} \quad \text{and} \quad R_{ov} = \sqrt{\frac{\delta\pi_2}{Y_v^2}}.$$ (when potato plants infect the vector) and

(when vector population infects potato plants).

3.5. Local stability of disease-free equilibrium (DFE)

3.5.1. Local stability of disease-free equilibrium for integer order

Theorem 3. The disease-free equilibrium (DFE) point is locally asymptotically stable if

$$R_o < 1 \quad \text{and unstable if} \quad R_o > 1.$$  

Proof. To prove this theorem first, we must obtain the Jacobian matrix of the system of the model of equation (1). That is:

$$J = \begin{bmatrix} -\left( a\frac{Y_v}{Y_p} \right) & 0 & 0 & -aS_p \\ \delta & -k & 0 & aS_p \\ 0 & -\delta Y_v & -\left( Y_v + \delta \right) & 0 \\ 0 & \delta \frac{\pi_2}{Y_v} & -Y_v & 0 \end{bmatrix}.$$ (17)

where $k = Y_p + \alpha_1 + \alpha_2$.

Evaluating Equation (17) at $E^0 = \left( \frac{\pi_1}{Y_p}, 0, \frac{\pi_2}{Y_v}, 0 \right)$ gives us:

$$J(E^0) = \begin{bmatrix} -Y_p & 0 & 0 & -a\frac{\pi_1}{Y_p} \\ 0 & -k & 0 & a\frac{\pi_1}{Y_p} \\ 0 & -\delta \frac{\pi_2}{Y_v} & -Y_v & 0 \\ 0 & \delta \frac{\pi_2}{Y_v} & -Y_v & 0 \end{bmatrix}. - \lambda I = 0.$$ (19)

From Equation (18), the Eigenvalues of the $J(E^0)$ can be obtained by

$$|J(E^0) - \lambda I| = 0.$$ (20)
Evaluating Equation (20) gives us
\[
(-Y_p - \lambda)(-k - \lambda)
\left[(-Y_v - \lambda)(-Y_v - \lambda)
\right]
+ \frac{\alpha_1}{Y_p}
\left[-\frac{\delta\pi}{Y_v}ight]
\left[(-Y_v - \lambda)\right] = 0. \tag{21}
\]
From Equation (21), we have
\[
(-Y_p - \lambda)(-Y_v - \lambda)(-k - \lambda)(-Y_v - \lambda)
+ \frac{\alpha_1}{Y_p}
\left[-\frac{\delta\pi}{Y_v}\right] = 0. \tag{22}
\]
From Equation (22), we obtained:
\[
\lambda_1 = -Y_p < 0, \lambda_3 = -Y_v < 0
\]
\[
< 0 \text{ and } (-k - \lambda)(-Y_v - \lambda) + \frac{\alpha_1}{Y_p}
\left[-\frac{\delta\pi}{Y_v}\right] = 0 \tag{23}
\]
Now from Equation (23), we can obtained, a characteristic polynomial given by:
\[
\lambda^2 + b_1\lambda + b_2 = 0 \tag{24}
\]
where \(b_1 = k + Y_v\) and \(b_2 = Y_vk - \frac{\alpha_1\delta\pi}{Y_pY_v}\).

Since \(\lambda_1\) and \(\lambda_3\) < 0 but the sign of the rest two Eigenvalues (\(\lambda_2\) and \(\lambda_4\)) are not known. Let us determine from characteristic polynomial of Equation (24) by applying the Routh–Hurwitz criteria, Therefore, according to Routh–Hurwitz criteria, Equation (24) has strictly negative real root if and only if \(b_1 > 0\) and \(b_2 > 0\).

Clearly \(b_1 > 0\) because it is the sum of positive parameters. But \(b_2 > 0\) if
\[
Y_vk - \frac{\alpha_1\delta\pi}{Y_pY_v} > 0. \tag{25}
\]
After re-arranging Equation (25), we can get
\[
\frac{\alpha_1\delta\pi}{Y_pkY_v^2} < 1. \tag{26}
\]
Equation (26) shows that
\[
R_0^2 < 1, \text{ from this } R_0 < 1.
\]
Therefore, the disease-free equilibrium \(E^0\) is locally asymptotically stable if \(R_0 < 1\).

3.5.2. Local stability of disease-free equilibrium for fractional order

**Theorem 4.** The disease-free equilibrium point of the system (2) is locally asymptotically stable if all the eigenvalues of the J(E0) satisfy the condition \(\text{arg}(\lambda_i) > \frac{\pi}{2}\), where \(i = 1, 2, 3, 4\).

**Proof.** In Section 3.5.1, we obtained
\[
\lambda_1 = -Y_p < 0 \text{ and } \lambda_3 = -Y_v < 0.
\]
Hence, these two eigenvalues satisfy the condition \(\text{arg}(\lambda) > \frac{\pi}{2}\).

Since \(\lambda_1, \lambda_3 < 0\), for the rest two eigenvalues \(\lambda_2\) and \(\lambda_4\) we determine from expansion
\[
\lambda^2 + b_1\lambda + b_2 = 0.
\]
Then let D(p) is the discriminant of the polynomial
\[
p(x) = x^2 + b_1x + b_2.
\]
Hence, these two eigenvalues satisfy the condition \(\text{arg}(\lambda) > \frac{\pi}{2}\).

Clearly \(b_1 > 0\), because it is the sum of positive parameters. The disease-free equilibrium, \(E^0\) is stable if \(D(p) > 0\).

Substituting \(b_1\) and \(b_2\) into \(D(p)\) gives us:
\[
(k + Y_v)^2 - 4\left(Y_vk - \frac{\alpha_1\delta\pi}{Y_pY_v}\right) > 0. \tag{28}
\]
Simplifying Equation (28) and substituting expression of \(R_0\) gives us:
\[
(R_0^2 - 1)Y_vk < 0.
\]
Since \(D(p) > 0\) if \(R_0 < 1\), it follows that the two remaining Eigenvalues (\(\lambda_2\) and \(\lambda_4\)) will have negative values, if they satisfy the condition \(\text{arg}(\lambda) > \frac{\pi}{2}\).

Then \(\text{arg}(\lambda_{2,4}) > \frac{\pi}{2}\) for all \(0 < \sigma < 1\).
Therefore, we can conclude that the disease-free equilibrium \(E^0\) is locally asymptotically stable if \(R_0 < 1\).

3.6. Global stability of disease free equilibrium (DFE)

**Theorem 4.** The disease-free equilibrium (DFE) point is globally stable if \(R_0 < 1\).

**Proof.** To prove this theorem, we consider the following Lyapunov function:
\[
V(S_p, I_p, S_v, I_v) = \left( S_p - S_0 - S_0 \ln \frac{S_p}{S_0} \right) + \left( I_p - I_0 - I_0 \ln \frac{I_p}{I_0} \right)
\]
\[
+ \left( S_v - S_0 - S_0 \ln \frac{S_v}{S_0} \right) + \left( I_v - I_0 - I_0 \ln \frac{I_v}{I_0} \right).
\]
Thus \(V > 0\) for all values of \((S_p, I_p, S_v, I_v) \neq (S_0^0, I_0^0, S_0^0, I_0^0)\) and \(V = 0\) if and only if \((S_p, I_p, S_v, I_v) = (S_0^0, I_0^0, S_0^0, I_0^0)\).

By calculating the \(\sigma\) order of \(V(S_p, I_p, S_v, I_v)\) we can \(D^\sigma V \leq 0\), which is:
\[
D^\sigma V \leq \left( \frac{S_p - S_0}{S_p} \right) D^\sigma S_p + \left( \frac{I_p - I_0}{I_p} \right) D^\sigma I_p
\]
\[
+ \left( \frac{S_v - S_0}{S_v} \right) D^\sigma S_v + \left( \frac{I_v - I_0}{I_v} \right) D^\sigma I_v. \tag{29}
\]
Equation (29) further simplified as:

\[ D^n V \leq \left( \frac{S_p - S^0_p}{S_p} \right) D^n S_p + D^n l_p + \frac{S_v - S^0_v}{S_v} D^n S_v + D^n I_v. \]

By equilibrium condition, \( \pi_1 = S^0_p Y_p \) and \( \pi_2 = S^0_v Y_v. \)

Equation (29) can be written as:

\[ D^n V \leq \left( \frac{S_p - S^0_p}{S_p} \right) \left( S^0_p Y_p - a S_p I_v - Y_p S_p \right) + a S_p I_v \\
- k l_p + \left( \frac{S_v - S^0_v}{S_v} \right) \left( S^0_v Y_v - \delta S_v I_p - Y_v S_v \right) + \delta S_v I_p - Y_v l_v. \]

After further simplification we obtain:

\[ D^n V \leq -Y_p \left( \frac{S_p - S^0_p}{S_p} \right)^2 + a S^0_p I_v \\
- k l_p - Y_v \left( \frac{S_v - S^0_v}{S_v} \right)^2 + \delta S^0_v I_p - Y_v l_v. \]  \quad (30)

After replacing equilibrium points \( S^0_p = \frac{\pi_1}{Y_p} \) and \( S^0_v = \frac{\pi_2}{Y_v} \), Equation (30) become:

\[ D^n V \leq -Y_p \left( \frac{S_p - \frac{\pi_1}{Y_p}}{S_p} \right)^2 - Y_v \left( \frac{S_v - \frac{\pi_2}{Y_v}}{S_v} \right)^2 \\
+ l_v Y_v \left( \frac{a \pi_1 \pi_2}{k Y_p Y_v} \right) - 1. \]  \quad (31)

After replacing expression \( R_0 \) in Equation (31) gives us:

\[ D^n V \leq -Y_p \left( \frac{S_p - \frac{\pi_1}{Y_p}}{S_p} \right)^2 - Y_v \left( \frac{S_v - \frac{\pi_2}{Y_v}}{S_v} \right)^2 \\
+ l_v Y_v \left( R^2_0 - 1 \right). \]  \quad (32)

Therefore, Equation (32) shows that \( D^n V \leq 0 \) if \( R_0 < 1 \) for \( \forall (S_p, l_p, I_p, I_v) \in \mathbb{R}^+ \) and

\[ D^n V = 0 \] if and only if \( S_p = S^0_p, I_p = \frac{\pi_1}{S^0_p}, S_v = S^0_v, I_v = \frac{\pi_2}{S^0_v}. \)

Therefore, the disease-free equilibrium \( E^0 \) is globally stable if \( R_0 < 1. \)

### 3.7. The endemic equilibrium points

In the presence of PLRV disease, the model has an equilibrium point called endemic equilibrium point, denoted by \( E^* = (S^*_p, l^*_p, S^*_v, I^*_v) \). Endemic equilibrium \( (E^*) \) is the steady state solution where the PLRV disease persist in the population of potato and vector. It can be obtained by equating each equation of the model equal to zero. Therefore:

\[ E^* = \left( \frac{\pi_1}{a l^*_p Y_p}, \frac{\pi_1 l^*_v}{a l^*_p Y_p}, \frac{\pi_2}{k (l^*_p + Y_p)}, \frac{\pi_3 k (l^*_p + Y_p)}{a l^*_p (l^*_v + Y_v) k Y_p} \right). \]

### 3.8. Global stability of the endemic equilibrium points

In this section, we proved the global stability of the endemic equilibrium \( E^* = (S^*_p, l^*_p, S^*_v, I^*_v) \) of the fractional order model.

**Theorem 5.** Let \( \sigma \in (0, 1) \) and \( R_0 > 1 \). Then the endemic equilibrium \( (EE) \) of fractional order model is globally stable in the interior of \( \Omega \).

**Proof.** Consider the following Lyapunov function

\[ V(S_p, l_p, S_v, I_v) = \left( S_p - S^*_p \right) + \left( I_p - l^*_p \right) + \left( S_v - S^*_v \right) + \left( I_v - l^*_v \right). \]

Function \( V \) is defined as continuous and positive definite for all \( S_p > 0, l_p > 0, S_v > 0 \) and \( I_v > 0 \).

Here we calculate the \( \sigma \) order of \( V(S_p, l_p, S_v, I_v) \) to show \( D^n V \leq 0 \) at the endemic equilibrium point.

Then we have

\[ D^n V \leq \left( \frac{S_p - S^*_p}{S_p} \right) D^n S_p + \left( \frac{l_p - l^*_p}{l_p} \right) D^n l_p \\
+ \left( \frac{S_v - S^*_v}{S_v} \right) D^n S_v + \left( \frac{l_v - l^*_v}{l_v} \right) D^n l_v. \]  \quad (33)

By the endemic equilibrium conditions \( \pi_1 = S^*_p a l_v, \pi_2 = S^*_v Y_p \) and \( \pi_2 = S^*_v \delta I_v + S^*_v Y_v \) and letting \( l^*_p k = S^*_p a l_v \) and \( l^*_v Y_v = S^*_v \delta l_p \), Equation (31) become:

\[ D^n V \leq -Y_p \left( \frac{S_p - \frac{\pi_1}{Y_p}}{S_p} \right)^2 - k l_p - Y_v \left( \frac{S_v - \frac{\pi_2}{Y_v}}{S_v} \right)^2 \\
+ l_v Y_v \left( R^2_0 - 1 \right). \]  \quad (34)

Thus from Equation (34) \( D^n V \leq 0 \) and \( D^n V = 0 \) if and only if \( S_p = S^*_p, l_p = l^*_p, S_v = S^*_v, I_v = I^*_v. \)

Therefore, \( D^n V = 0 \) \( \forall \left( S^*_p, l^*_p, S^*_v, l^*_v \right) \in \Omega \) is the singleton EE. This shows that EE is globally asymptotically stable in \( \Omega \).
Table 1. Sensitivity index table.

| Parameter symbol | Parameter description | Sensitivity index |
|------------------|-----------------------|------------------|
| $\pi_1$          | Replanting rate of potato | +ve |
| $\pi_2$          | Recruit rate of vector | +ve |
| $\alpha$         | Infection rate of potato | +ve |
| $\Delta$         | Infection rate of vectors | +ve |
| $\gamma_1$       | Virus induced death rate | -ve |
| $\gamma_2$       | Death rate of elimination infection | -ve |
| $\gamma_v$       | Natural causing death rate of vector | -ve |
| $\gamma_p$       | Natural causing death rate of potato | -ve |

3.9. Sensitivity analysis

This is done to determine and check the parameter that can impact the basic reproduction number ($R_0$).

For this purpose, we used a formula defined as $\Pi_{R_0}^a = \frac{e^a}{R_0} \frac{\partial}{\partial a}$ where $X_i$ is any parameter in $R_0$. Then for example, the sensitivity index of $R_0$ to $a$ is

$$\frac{\partial R_0}{\partial a} = \left( \frac{1}{2} \frac{\partial \pi_1 \pi_2}{\partial a} \frac{1}{\gamma_v \gamma_p} \right) \frac{a}{2} \frac{1}{2} > 0.$$

By the same procedure we calculate the sensitivity index of the remaining parameters.

Theses parameters in Table 1 are ordered from most sensitive to least. These parameters that have positive indices are ($\pi_1, \pi_2, a, \delta$). This implies that when these parameters are increased by keeping other parameters constant, they have a great contribution to expand (persist) the disease in population. Due to the reason that the basic reproduction number increases as their values increase. This also shows that the average number of secondary cases of infection increases in population. While the parameter $(\alpha_v, \gamma_v, \gamma_p)$ have a negative sensitivity indices. This shows that decrease in the value of basic reproduction number when they are increased while keeping the other parameters constant. This implies that they decrease the endemicity of the disease or eliminating the disease from population as they have negative indices.

4. Numerical results and discussion

In this section, numerical simulation of the PLRV model carried out to support the theoretical analysis and illustrate the behavior of the model for different integer order and fractional order $0 < \sigma \leq 1$. This model simulated with a set of parameters values namely $a, \delta, \alpha_v$ and initial condition of the system that are given in Table 2. The simulation results are displayed in graphs from Figures 2–5.

Table 2. Parameter values for PLRV model.

| Variables and parameters | Values | Sources |
|--------------------------|--------|---------|
| $S_V$                    | 600    | Assumed |
| $I_V$                    | 200    | Assumed |
| $S_P$                    | 100    | Assumed |
| $I_P$                    | 10     | Assumed |
| $\pi_1$                  | 0.8    | Assumed |
| $\pi_2$                  | 0.19   | Kumar et al. (2020b) |
| $a$                      | 0.00022| Ahmad, Seadawy, et al. (2020) |
| $\Delta$                 | 0.0025 | Kumar et al. (2020) |
| $\alpha_v$               | 0.033  | Nemecek (1993) |
| $\gamma_v$               | 0.01   | Assumed |
| $\gamma_p$               | 0.0028 | Wales et al. (2008) |
| $\gamma_p$               | 0.04   | Assumed |

4.1. Comparison of integer and fractional order trend of PLRV model

Figure 2 show that the simulation results of the integer order (right) and fractional order (left) trends of the transmission dynamics of PLRV in potato population. We observe that the solution of the integer order model equation converges to respective equilibrium point, whereas the fractional order model equation converges to respective equilibrium with zigzag manner. This shows that fractional order model is better to indicate real-life phenomena than integer order.

4.2. Effect of contact rate of infected vector and susceptible potato on infected potato population

Figure 3 shows that the numerical result obtained by varying the value of contact rate ($a$) of infected vector and susceptible potato while keeping other parameters constant. The value of $a$ is increased from 0.00022 to 0.02 then the number of infected potatoes is increased. At $a = 0.03$, the number of infected potatoes is higher than others. In general, the numerical results show that increasing the value of contact rate cause to increase the number of infected potatoes. Then, we conclude that the virus spread in the potato become harsh when the contact rate is high. Therefore, all concerned body and policy makers have to think in minimizing the contact rate of infected vector and susceptible potato to control the spread of the disease.

4.3. Impact of contact rate of susceptible vector and infected potato on infected potato population

Figure 4 depicts the numerical simulation results obtained by varying the value of contact rate of susceptible vector and infected potato ($\delta$) while keeping other parameters constant. These numerical results show that the graphs go up when we increase the value contact rate $\delta$ in both fractional order (left)
Figure 2. Graph of fractional order (left) and integer order (right) of PLRV model.

Figure 3. Graph of effect of contact rate of infected vector and susceptible potato (a) on infected potato for fractional order (left) and integer order (right).

Figure 4. Graph of impact of contact rate (δ) on infected potato for fractional order (left) and integer order (right).
and integer order (right). This indicates that when the value of $\delta$ increases, the number of infected potatoes also increases. This happens due to the reason that the susceptible vector become infected vector that increases the number of infected potatoes in the field. Therefore, the government and agricultural expert have to advise potato producers to eliminate infected potato from the field once it develops a symptom.

### 4.4. Impact of elimination rate of infected potato

Figure 5 shows the effect of elimination rate ($\alpha_2$) of infected potato on the dynamics of PLRV. This numerical result is obtained by varying the value of $\alpha_2$ while keeping other parameters fixed. The results show that the number of infected potatoes decreases as increased the value of the elimination rate in both fractional order (left) and integer order (right). From this result, we can conclude that the increasing value of elimination rate contributes to a great role in eradicating the disease from the potato population.

### 5. Summary and conclusion

In this paper we developed an integer order mathematical model of PLRV transmission dynamics and then it is extended to fractional order. In Section 2, we have described and developed the two models. In Section 3, we have studied the qualitative behaviors of the model by obtaining the feasible region, positivity of the solution, equilibrium points and analyzed their local and global stability and basic reproduction number of the model. The sensitivity analysis of the basic reproduction number has been determined to identify the parameters that have a great impact on the control of PLRV. In Section 4, the numerical simulation results are performed and analyzed by comparing the integer order and fractional order. In this numerical simulation, we have investigated the impact of parameters $a$, $\delta$ and $\alpha_2$ on both integer and fractional order model. The numerical results show that as the value of $\sigma$ order of fractional order increases, the behavior of the fractional order model solution approaches to integer order model. And as decrease in the values of $\sigma$ order of fractional order, the solution of the model converges to respective equilibrium more slowly to show more real-life phenomena. In conclusion, from this study, we understand that increasing the infection rate of infected vector ($a$) and contact rate of susceptible vector and infected potato ($\delta$) have a great contribution for spread of PLRV disease in potato population. Thus, from the results of this study, PLRV disease management is concerned with reducing the disease infection of potato population. In this case, reduction in the disease infection of potato should be made by reducing the infection rate ($a$) as well as contact ($\delta$) by increasing the elimination rate of infected potato ($\alpha_2$) through uprooting and burning of the infected potato from field.

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### Availability of the date

The data we used for this study is from respective published articles that are cited.
Disclosure statement

The authors declare that they have no conflict of interests.

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