Sherali-Adams gaps, flow-cover inequalities and generalized configurations for capacity-constrained Facility Location

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Abstract

Metric facility location is a well-studied problem for which linear programming methods have been used with great success in deriving approximation algorithms. The capacity-constrained generalizations, such as capacitated facility location (CFL) and lower-bounded facility location (LbFL), have proved notorious as far as LP-based approximation is concerned: while there are local-search-based constant-factor approximations, there is no known linear relaxation with constant integrality gap. According to Williamson and Shmoys devising a relaxation-based approximation for CFL is among the top 10 open problems in approximation algorithms.

This paper advances significantly the state-of-the-art on the effectiveness of linear programming for capacity-constrained facility location through a host of impossibility results for both CFL and LbFL. We show that the relaxations obtained from the natural LP at $\Omega(n)$ levels of the Sherali-Adams hierarchy have an unbounded gap, partially answering an open question of [27, 6]. Here, $n$ denotes the number of facilities in the instance. Building on the ideas for this result, we prove that the standard CFL relaxation enriched with the generalized flow-cover valid inequalities [1] has also an unbounded gap. This disproves a long-standing conjecture of [24]. We finally introduce the family of proper relaxations which generalizes to its logical extreme the classic star relaxation and captures general configuration-style LPs. We characterize the behavior of proper relaxations for CFL and LbFL through a sharp threshold phenomenon.

1 Introduction

Facility location is one of the most well-studied problems in combinatorial optimization. In the uncapacitated version (UFL) we are given a set $F$ of facilities and set $C$ of clients. We may open facility $i$ by paying its opening cost $f_i$ and we may assign client $j$ to facility $i$ by paying the connection cost $c_{ij}$. We are asked to open a subset $F' \subseteq F$ of the facilities and assign each client to an open facility. The goal is to minimize the total opening and connection cost. A $\rho$-approximation algorithm, $\rho \geq 1$, outputs in polynomial time a feasible solution with cost at most $\rho$ times the optimum. The approximability of general UFL is settled by an $O(\log |C|)$-approximation [15] which is asymptotically best possible, unless $P = NP$. In metric UFL the service costs satisfy the following variant of the triangle inequality: $c_{ij} \leq c_{ij'} + c_{i'j'} + c_{i'j}$ for any $i, i' \in F$ and $j, j' \in C$. This very natural special case of UFL is approximable within a constant-factor, and many improved results have been published over the years. In those, LP-based methods, such as filtering, randomized rounding and the primal-dual method have been particularly prominent (see, e.g., [33]). After a long series of papers the currently best approximation ratio for metric UFL is 1.488 [26], while the

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best known lower bound is 1.463, unless P = NP (17 and Sviridenko 32). In this paper we focus
on two generalizations of metric UFL: the capacitated facility location (CFL) and the lower-bounded
facility location (LBFL).

CFL is the generalization of metric UFL where every facility \( i \) has a capacity \( u_i \) that specifies
the maximum number of clients that may be assigned to \( i \). In uniform CFL all facilities have the
same capacity \( U \). Finding an approximation algorithm for CFL that uses a linear programming
lower bound, or even proving a constant integrality gap for an efficient LP relaxation, are notorious
open problems. Intriguingly, the following rare phenomenon occurs. The natural LP relaxations
have an unbounded integrality gap and the only known \( O(1) \)-approximation algorithms are based
on local search, with the currently best ratios being 5 [9] for the non-uniform and 3 [4] for the
uniform case respectively. In the special case where all facility costs are equal, CFL admits an
LP-based approximation [24]. Comparing the LP optimum against the solution output by an LP-
based algorithm establishes a guarantee that is at least as strong as the one established a priori
by worst-case analysis. In contrast, when a local search algorithm terminates, it is not at all clear
what the lower bound is. According to Williamson and Shmoys 33 devising a relaxation-based
algorithm for CFL is one of the top 10 open problems in approximation algorithms.

A lot of effort has been devoted to understanding the quality of relaxations obtained by an
iterative lift-and-project procedure. Such procedures define hierarchies of successively stronger
relaxations, where valid inequalities are added at each level. After at most \( n \) levels, where \( n \) is
the number of variables, all valid inequalities have been added and thus the integer polytope is
expressed. Relevant methods include those developed by Balas et al. 8, Lovász and Schrijver
28 (for linear and semidefinite programs), Sherali and Adams 2, Lasserre 22 (for semidefinite
programs). See 23 for a comparative discussion.

The seminal work of Arora et al. 7, studied integrality gaps of families of relaxations for
Vertex Cover, including relaxations in the Lovász-Schrijver (LS) hierarchy. This paper introduced
the use of hierarchies as a restricted model of computation for obtaining LP-based hardness of
approximation results. Proving that the integrality gap for a problem remains large after many
levels of a hierarchy is an unconditional guarantee against the class of relaxation-based algorithms
obtainable through the specific method. At the same time, if an LP relaxation maintains a gap of
\( g \) after a linear number of levels, one can take this as evidence that polynomially-sized relaxations
are unlikely to yield approximations better than \( g \) (see also 24). In fact, the former belief is now
a theorem for maximum constraint satisfaction problems: in terms of approximation, LPs of size
\( n^k \), are exactly as powerful as \( O(k) \)-level Sherali-Adams relaxations 11.

LBFL is in a sense the opposite problem to CFL. In an LBFL instance every facility \( i \) comes with
a lower bound \( b_i \) which is the minimum number of clients that must be assigned to \( i \) if we open it.
In uniform LBFL all the lower bounds have the same value \( B \). LBFL is even less well-understood
than CFL. The first approximation algorithm for the uniform case had a performance guarantee of
448 31, which has been improved to 82.6 5. Both use local search.

Apart from some work of the authors 20 21 there has been no systematic theoretical study
of the power of linear programming for approximating CFL. In 20 we show an unbounded gap
for CFL at \( \Omega(n) \) levels of the LS and the semidefinite mixed-LS+ hierarchies, \( n \) being the number
of facilities. In 21 we show that linear relaxations in the classic variables require at least an
exponential number of constraints to achieve a bounded integrality gap. Note that it is well-known
that hierarchies may produce an exponential number of inequalities already after one round. For
related problems there are some recent interesting results. Improved approximations were given
for \( k \)-median 27 and capacitated \( k \)-center 14, 6, problems closely related to facility location.
For both, the improvements are obtained by LP-based techniques that include preprocessing of
the instance in order to defeat the known integrality gap. For \( k \)-median, the authors of 27 state
that their \((1 + \sqrt{3} + \epsilon)\)-approximation algorithm can be converted to a rounding algorithm on an \(O(\frac{1}{\epsilon})\)-level LP in the Sherali-Adams (SA) lift-and-project hierarchy. They propose exploring the direction of using SA for approximating \(C_{fl}\). In [6] the authors raise as an important question to understand the power of lift-and-project methods for capacitated location problems, including whether they automatically capture relevant preprocessing steps.

**Our results.** We give impossibility results on arguably the most promising directions for strengthening linear relaxations for \(C_{fl}\) and \(L_{bfl}\) and in doing so we answer open problems from the literature. Our contribution is threefold.

First, we show that the LPs obtained from the natural relaxations for \(C_{fl}\) and \(L_{bfl}\) at \(\Omega(n)\) levels of the SA hierarchy have an unbounded gap on an instance where \(|F| = \Theta(n)\) and \(|C| = \Theta(n^3)\). This result answers the questions of [27] and [6] stated above as far as the natural LP is concerned and moreover it is asymptotically tight. In the instances we consider clients have unit demands and it is well known that in this case the integer polytope and the mixed-integer (where fractional client assignments are allowed) polytope are the same. Since SA extends to mixed-integer programs as well [13, 8], the mixed-integer polytope is obtained after at most \(n\) levels. Thus at most that many levels are needed also by the stronger, full-integer, SA procedure we employ, which in the lifting stage multiplies also with assignment variables. From a qualitative aspect, we give the first, to our knowledge, SA bounds for a relaxation where variables have more than one type of semantics, namely the facility opening and the client assignment type. Compare this, for example, with the Knapsack and Max Cut LPs that contain each one type of variable. The lifting of the assignment variables raises obstacles in the proof that we managed to overcome as discussed in Section 3.

We use the local-to-global method which was implicit in [7] for local-constraint relaxations and was then extended to the SA hierarchy in [15]. See also [16] for an explicit description and [12] for applications to Max Cut and other problems. In this approach, the feasibility of a solution for the \(t\)-level SA relaxation is established through the design of a set of appropriate distributions over feasible integer solutions for each constraint such that these global distributions agree with each other locally on relevant events. To prove Theorem 3.1 for \(C_{fl}\) we devise first in Lemma 3.2 an intuitive method to construct an initial set of distributions for a constraint. These initial distributions are inadequate for constraints where all facilities appear as indices. An alteration procedure, explained in Propositions 3.1–3.3, produces the final set of distributions. Theorem 3.1 extends significantly our earlier result on the LS hierarchy for \(C_{fl}\) [20] to the stronger SA hierarchy. It turns out that in both cases we can start from the same bad instance. It should be noted that the methodology in the two proofs is completely different – in [20] the result was obtained via an inductive construction of protection matrices.

Our second contribution (cf. Theorem 4.1) is that the effective capacity inequalities introduced in [1, 2] for \(C_{fl}\) fail to reduce the gap of the classic relaxation to constant. These constraints generalize the flow-cover inequalities for \(C_{fl}\). Thus we disprove the long-standing conjecture of [21] that the addition of the latter to the classic LP suffices for a constant integrality gap. Our proof deviates from standard integrality gap constructions by applying the local-global method. The bad solution fools every inequality \(\pi\) because its part that is “visible” to \(\pi\) can be extended to a solution \(s^\pi\) that is a convex combination of feasible integer solutions. Our ideas can be extended to even more general families such as the submodular inequalities [1], cf. Theorem 13.1 in the Appendix. All results in this paper make no time-complexity assumptions. To our knowledge no efficient separation algorithm for the effective capacity inequalities is known.

We finally introduce the family of proper relaxations which are configuration-like linear programs. The so-called Configuration LP was used by Bansal and Sviridenko [10] for the Santa Claus problem and has yielded valuable insights, mostly for resource allocation and scheduling problems.
(e.g., [30]). The analogue of the Configuration LP for facility location already exists, it is the star relaxation (see, e.g., [19]). We take the idea of a star to its logical extreme by introducing classes. A class consists of a set with an arbitrary number of facilities and clients together with an assignment of each client to a facility in the set. A proper relaxation for an instance is defined by a collection C of classes and a decision variable for every class. We allow great freedom in defining C: the only requirement is that the resulting formulation is symmetric and valid. The complexity α of a proper relaxation is the maximum fraction of the available facilities that are contained in a class of C. In Theorem 5.1 we characterize the behavior of proper relaxations for CFL and LBFL through a threshold result: anything less than maximum complexity results in unboundedness of the integrality gap, while there are proper relaxations of maximum complexity with a gap of 1.

Our results disqualify the so far most promising approaches for an efficient LP relaxation for CFL. Moreover, we advance drastically the state-of-the-art for the little understood LBFL. Whether a fundamentally new approach may succeed for either problem remains as an open question.

For lack of space, some proofs and all material on LBFL are in the Appendix.

2 Preliminaries

Given an instance I(F, C) of CFL or LBFL, we use n, m to denote |F| and |C| respectively. We will show our negative results for uniform, integer, capacities and lower bounds. Each client can be thought of as representing one unit of demand. It is well-known that in such a setting the splittable and unsplittable versions of the problem are equivalent. The following 0-1 IP is the standard valid formulation of uncapacitated facility location with unsplittable unit demands.

\[
\begin{align*}
\min \{& \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c_{ij} x_{ij} \mid x_{ij} \leq y_i, \forall i \in F, \forall j \in C, y_i, x_{ij} \in \{0,1\}, \forall i \in F, \forall j \in C \}
\end{align*}
\]

The linear relaxation results from the above IP by replacing the integrality constraints with: 0 ≤ yi ≤ 1, 0 ≤ xij ≤ 1, ∀i ∈ F, ∀j ∈ C. To obtain the standard LP relaxations for uniform CFL (and LBFL) with capacity U (lower bound B) the following constraints are added respectively:

\[
\begin{align*}
\sum_j x_{ij} & \leq U y_i, \forall i \in F \quad \text{and} \quad \sum_j x_{ij} & \geq B y_i, \forall i \in F.
\end{align*}
\]

We will slightly abuse terminology by using the term (LP-classic) for both LPs. It will be clear from the context to which problem, CFL or LBFL, we refer.

We proceed to define the Sherali-Adams hierarchy [3]. Consider a polytope P ⊆ Rd defined by the linear constraints Ax − b ≤ 0, 0 ≤ xi ≤ 1, i = 1, . . . , d. We define the polytope SAk(P) ⊆ Rd as follows. For every constraint π(x) ≤ 0 of P, for every set of variables U ⊆ {xi | i = 1, . . . , d} such that |U| ≤ k, and for every W ⊆ U, consider the valid constraint: π(x) \prod_{x_i \in U \setminus W} x_i \prod_{x_i \in W} (1 − x_i) ≤ 0. Linearize the system obtained this way by replacing (i) x_i^2 with x_i for all i and (ii) \prod_{x_i \in I} x_i with x_I for each set I ⊆ {xi | i = 1, . . . , d}. SAk(P) is the projection of the resulting linear system onto the singleton variables. We call SAk(P) the polytope obtained from P at level k of the SA hierarchy. Given a cost vector c ∈ Rd, the relaxation obtained from P at level k of SA is \(\min\{c^T x \mid x \in SA^k(P)\}\).

3 Sherali-Adams gap for CFL

Consider an instance of metric CFL with a total of 2n facilities, n with opening cost 0 which we call cheap (and denote the corresponding set by Cheap) and n with opening cost 1 which we call costly (and denote by Costly). The capacity U = n^3 and we have a total of nU + 1 clients. All
connection costs are 0. We will show that the following bad solution $s$ to the instance survives a number of SA levels, which is linear in the number $2n$ of facilities. On the other hand, it is known that at level $2n$ the relaxation obtained expresses the integral polytope. Let $\alpha = n^{-2}$. For all $i \in \text{Cheap}$ and for all $j \in C$, $y_{ij} = 1$ and $x_{ij} = \frac{1}{n}$, and for all $i \in \text{Costly}$ and for all $j \in C$, $y_{ij} = \frac{1}{\alpha}$ and $x_{ij} = \frac{1}{n}$. Theorem 3.1 below indicates that, as often with hierarchies, simple valid inequalities are generated after many rounds. The reader who is further interested in the robustness of SA for Cfl may consult Section A.2 in the Appendix.

The following lemma, which is implicit in previous work [15, 16] gives sufficient conditions for a solution to be feasible at level $k$ of the SA hierarchy.

**Lemma 3.1** [15, 16] Let $s$ be a feasible solution to the relaxation and let $v(\pi, z)$ be the set of variables appearing in a lifted constraint obtained from $\pi$ multiplied by $z$. Solution $s$ survives $k$ levels of SA if for every constraint $\pi$ and each multiplier $z$ with at most $k$ distinct variables there is:

1. A solution $s' = s_{\pi, z}$ which agrees with $s$ on $v(\pi, z)$ such that $s'$ is a convex combination $E_d$ of integer solutions (and thus $E_d$ defines a distribution on integer solutions) and

2. For any two sets $v(\pi_1, z_1)$ and $v(\pi_2, z_2)$, let $x_1 x_2 \ldots x_l, l \leq k+1$, be a product appearing in both lifted constraints obtained from $\pi_1$ and $\pi_2$ multiplied with $z_1$ and $z_2$ respectively. Then the probability $P[x_1 = 1 \land x_2 = 1 \land \ldots \land x_l = 1]$ is the same in both distributions $E_{d_1}$ and $E_{d_2}$ associated with $v(\pi_1, z_1)$ and $v(\pi_2, z_2)$ respectively.

First consider a constraint $\pi$: $\sum_j x_{ij} \leq U y_{is}$ and a multiplier $z$. After multiplying by $z$ and expanding, we obtain a linear combination of monomials (products). Then, for the $k < n - 1$ levels we consider there must be some costly facility $i_b \notin v(\pi, z)$. We construct a solution $s_{\pi, z} = (y', x')$ by setting $y_{ib}' = 1 - \sum_{i \in \text{Costly} - \{i_b\}} y_i$ and letting all other variables the same as in the original bad solution $s$. We say that facility $i_b$ takes the blame. We will prove that $s_{\pi, z}$ can be obtained as a convex combination $E_d$ of a set of integer solutions satisfying constraint $\sum_{i \in \text{Costly}} y_i = 1$. While $s_{\pi, z}$ can be obtained as a convex combination $E_d$ in a variety of ways, we require that the assignments of clients to the cheap facilities are indistinguishable in $E_d$ and the same must be true for the assignments to costly facilities other than $i_b$. In the upcoming definition, we use the product $p = z_1 z_2 \ldots z_l$ as an abbreviation of the event $E_p := \bigwedge_{i=1}^{l} z_i = 1$.

**Definition 3.1** Let $i_b$ be the facility that takes the blame. We say that a distribution $E_d$ is assignment-symmetric if the following are true:

1. $P_{E_d}[x_{i_1 j_1} \ldots x_{i_t j_t} y_{i_{t+1}} \ldots y_{i_{h}}],$ with $t + l \leq k + 1$ is the same if we exchange all occurrences of cheap facility $i_r$ by cheap facility $i_r'$ (in other words relabeling facilities). Note that we allow repetitions of facilities and clients in the description of the event.

2. $P_{E_d}[x_{i_1 j_1} \ldots x_{i_t j_t} y_{i_{t+1}} \ldots y_{i_{h}}]$ is the same if we exchange all occurrences of client $j_q$ by client $j_{q'}$.

3. $P_{E_d}[x_{i_1 j_1} \ldots x_{i_t j_t} y_{i_{t+1}} \ldots y_{i_{h}}]$ is the same if we exchange all occurrences of costly facility $i_1$ by costly facility $i_2$, $i_1, i_2 \neq i_b$.

We can always obtain $s_{\pi, z}$ from such an assignment-symmetric distribution $E_d$ as shown in the following lemma.

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1. The reader should notice that any similarity with Knapsack is superficial. Theorem 3.1 is about the Cfl polytope. Moreover, it is easy to embed our instance in a slightly larger one, with a non-trivial metric, so that the projection of the bad Cfl solution to the $y$-variables, is in the integral polytope of the “underlying” knapsack instance.
Lemma 3.2 Solution \( s_{\pi,z} \) is a convex combination \( E_d \) of integer solutions which defines an assignment-symmetric distribution.

Proof. We describe a probabilistic experiment which induces an assignment-symmetric distribution \( E_d \) over integer solutions satisfying \( \sum_{i \in \text{Costly}} y_i = 1 \).

Fix costly facility \( i_b \). Let \( w^1_{i_b} = \frac{\sum_j x_{i_b,j}}{\chi_{i_b}} \) be the desired number of clients assigned to facility \( i_b \) in the integer solutions in \( E_d \) where facility \( i_b \) is opened. To simplify the presentation let us assume that \( w^1_{i_b} \) and the \( w \) values we subsequently define are integers (we discuss in the Appendix how to handle fractional \( w \)'s). Let \( w^1_{i_{ch}} = \frac{|C|-w_{i_b}}{|\text{Cheap}|} \) be the number of clients assigned to facility \( i_{co}, i_b \neq i_b \) and \( i_{co} \) is opened. Let \( w^2_{i_{ch}} = \frac{|C|-w_{i_{co}}}{|\text{Cheap}|} \) be the number of clients assigned to facility \( i_{co}, i_b \neq i_b \) and \( i_{co} \) is opened. Observe that all the defined \( w \)'s are less than \( U \). The following procedure produces the assignment-symmetric distribution \( E_d \).

Pick costly facility \( i_b \) with probability \( y_{i_b}' \). If \( i_b = i_c \) \( (i_b \neq i_b) \) then consider \( n \) bins corresponding to the \( n \) cheap facilities each one having \( w^1_{i_b} (w^2_{i_b}) \) slots and 1 bin corresponding to \( i_{co} \) having \( w^1_{i_b} (w^2_{i_b}) \) slots. Randomly distribute \( |C| \) balls to the slots of the \( n+1 \) bins, with exactly one ball in each slot. Note that the above experiment induces a distribution over feasible integer solutions satisfying \( \sum_{i \in \text{Costly}} y_i = 1 \) since all the defined bin capacities are less than \( U \) and every client is assigned to exactly one opened facility in each outcome and exactly \( 1 \) costly facility is opened. Moreover the induced distribution \( E_d \) is assignment-symmetric and the expected \( (y,x) \) vector with respect to \( E_d \) is solution \( s_{\pi,z} \).

Clearly, \( s_{\pi,z} \) is the convex combination induced by \( E_d \) and \( E_d \) is assignment-symmetric: the cheap facilities are always open, and the costly are open a fraction of the time that is equal to the value of their corresponding \( y \) variable. The expected demand assigned to each \( i_{co} \in \text{Costly} \) is \( y_{i_{co}} w_{i_{co}} \) which is the total demand assigned to \( i_{co} \) by \( s_{\pi,z} \). Since the clients have the same probability of being tossed in the bin corresponding to \( i_{co} \), the expected assignment of each client \( j \) to \( i_{co} \) is the same as in \( s_{\pi,z} \). Similarly we can prove that the expected assignments to the cheap facilities are as required, see the Appendix for details.

We set the product-variables \( x_I \) appearing in constraint \( \pi \) multiplied by multiplier \( z \) to \( P_{E_d}[I] \).

Constraints \( x_{ij} \leq y_i, x_{ij} \leq 1, y_i \leq 1 \), are handled in the exact same way; the set of variables appearing in them is a subset of those appearing in the more complex constraints.

The second and more challenging case is when constraint \( \pi \) is \( \sum_i x_{ij} = 1 \) for some client \( j \). Let again \( z \) be a multiplier of level \( k \). Observe now that all facilities in \( F \) appear in \( v(\pi, z) \) as indexes of at least the \( x_{ij} \) variables. We select one facility \( i_b \) not appearing in \( z \) to take the blame. Let \( s_{\pi,z} = (y',x') \) be the corresponding extended solution that can be written as a convex combination/assignment-symmetric distribution \( E_d \) of integer solutions; the existence of \( E_d \) is ensured by Lemma 3.2. In this case there is a major obstacle to the agreement of the products \( x_I \) conditioning on the event \( x_{i_b,j} \) the probability of an event \( x_{i_j,j} \), \( i \in \text{Cheap} \) for some \( j \neq j' \) is higher than it would be if we were to condition on the event \( x_{i,j'} \), \( i \in \text{Cheap} \) \( \{i_b\} \). The same is true for more complex events involving assignments to cheap facilities conditioning on an assignment of facility \( i_b \) compared to the analogous event conditioning on some other costly facility. This can be problematic since facility \( i_b \) takes the blame in some distributions but does not in some others and thus there is the danger of violating the consistency required by the 2nd condition of Lemma 3.1. We overcome this difficulty by making alterations to \( E_d \) and constructing a distribution \( E_f \) where the probabilities of the aforementioned events are the same.
We now devise the altered distribution $E_f$. We first display the intuition in the following example: consider the event $A$: $x_{ib} = 1 \land x_{i\cdot ch'} = 1$ and the event $B$: $x_{ic\cdot o} = 1 \land x_{i\cdot ch'} = 1$ with $i\cdot o \in \text{Costly} - \{i_b\}$ and $i\cdot ch \in \text{Cheap}$. The probability of $A$ is $P[A] = P[x_{ib} = 1]P[x_{i\cdot ch'} = 1 | x_{ib} = 1] = x_{ib}^\prime w_{ib}^{-1}$ and the probability of $B$ is $P[B] = P[x_{ic\cdot o} = 1]P[x_{i\cdot ch'} = 1 | x_{ic\cdot o} = 1] = x_{ic\cdot o}^\prime w_{ch}^{-1}$.

Note that $P[A] \approx P[B](1 + 1/n)$ so $P[A]$ is only slightly greater. We nullify the difference between those probabilities by performing an alteration step to distribute $E_d$ that we call transfusion of probability. We pick some measure of an integer solution $s_1$ for which $x_{i\cdot ch'} = 1 \land x_{ib} = 1 \land x_{i\cdot j''} = 0$ for some client $j''$. We pick the same quantity of measure of some integer solution (or of some set of solutions) $s_2$ for which $x_{i\cdot ch'} = 0 \land x_{ib} = 0 \land x_{i\cdot j''} = 1$ and we exchange the values of the assignments $x_{ib}, x_{i\cdot j''}$ of the solutions. Let that quantity be $P[A] - P[B]$, it is easy to see that each set of solutions has enough measure to perform the transfusion. The resulting distribution $E_f$ now has $P[A] = P[B]$. In general, when transfusing probabilistic measure for complex events, we must be careful not to change the probability of events involving only assignments to cheap facilities, as opposed to the simplified example above.

Now let $p$ be a product appearing in constraint $\pi$ after having multiplied by multiplier $z$. We only consider products where exactly one variable $x_{ib}$ appears. Recall we chose $i_b$ so that it does not appear in $z$; thus we cannot have $y_i$ or more than one assignments of $i_b$ appearing in a product $p$. We may also assume that there is no $y_i$ variable in $p$, since if there is for some $i \in \text{Costly} - \{i_b\}$ the probability of $E_p$ is simply 0 and if $i \in \text{Cheap}$ then we can ignore the effect of $y_i$ since it is always true. Likewise we assume that there is no assignment variable of another costly facility.

We shall make corrections of the probability of all such events $E_p$ in a top-down manner: at step $i$ we fix the probability of all the events $x_{ib} = 1 \land x_{i\cdot j} = 1 \land \ldots \land x_{ia_k\cdot j_{k+1} \cdot i\cdot j_{k+1}} = 1$ where $x_{ib}x_{ia_1\cdot j_1} \cdots x_{ia_k\cdot j_{k+1} \cdot i\cdot j_{k+1}}$ is a product $p$ appearing in constraint $\pi$ multiplied by $z$. In other words, we fix the probabilities in decreasing order of the cardinality of the set of variables appearing in $p$. The following proposition relates the probability of $E_p$ with that of $E_p = E_p x_{ij}$, an event with the additional requirement that $x_{ij} = 1$.

**Proposition 3.1** Let $p = x_{ib}x_{ia_1\cdot j_1}x_{ia_2\cdot j_2} \ldots x_{ia_k\cdot j_k}$ and let $p' = px_{ia_{k+1}\cdot j_{k+1}}$. Then in $E_d$, $(1 - o(1))P[E_p]/n \leq P[E_{p'}]/n$. Let $E_p$ be equal to $P[E_{p/fixed}] = P[x_{ij} = 1 \land x_{ia_1\cdot j_1} = 1 \land \ldots \land x_{ia_k\cdot j_{k+1} \cdot i\cdot j_{k+1}} = 1]$ in $E_d$ for $i^* \in \text{Costly} - \{i_b\}$. We bound the ratio $P[E_{p/fixed}]/P[E_p]$. The probability of events of previous iterations affect the probabilities of the events of the current iteration of the procedure that constructs $E_f$. We bound this effect on the probability of an event $E_p$ of the current iteration $i$ by considering the corrections of the events $E_{p'} = E_p \setminus x_{ij} = 1$, with $x_{ij}$ in the set of variables appearing in $z$ and $x_{ij} \notin E_p$, of the previous iteration and using the union bound. There are exactly $i$ events needed to be taken into consideration for each such $E_p$ of the current step $i$. The amount of the effect of the correction of the previous iteration is by Proposition [3.2] at most $i((1 + 1 + o(1))/n)^{k-i-2} - 1)P[E_{p'/fixed}]$ while the measure of the needed

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2Notice that any effect of iteration $j < i - 1$ on $P[E_p]$, originates from events that are subsets of $E_{p'}$ and has therefore been accounted for.
correction for $E_p$ is at least $((1 + (1 - o(1)) \frac{1}{n})^{k+1} - 1)P[E_p^{\text{fixed}}]$ which by Proposition 3.1 and by the number of rounds we consider is higher, in particular $((1 + (1 - o(1)) \frac{1}{n})^{k+1} - 1)P[E_p^{\text{fixed}}] \geq n((1 - o(1))((1 + (1 - o(1)) \frac{1}{n})^{k+1} - 1)P[E_p^{\text{fixed}}] > i((1 + (1 + o(1)) \frac{1}{n})^{k+2} - 1)P[E_p^{\text{fixed}}]$.

To subtract from $P[E_p^{\text{fixed}}]$ the rest of the probabilistic measure required from the correction, say a measure of $\mu$, we do the following transfusion step: pick a measure $\mu$ of solutions from distribution $E_d$ such that $x_{i,j} = 0$, $x_{i,j'} = 1$ for any $j'$ such that $x_{i,j'} \notin v(\pi, z)$, all the other events of $E_p$ are false, and so are all the remaining events corresponding to assignments in $z$. Likewise pick an equal measure of solutions from $E_d$ such that $x_{i,j} = 1$, $x_{i,j'} = 0$ with $x_{i,j'} \notin v(\pi, z)$, all the other events of $E_p$ are true, and all the remaining events corresponding to assignments in $z$ are false. Now exchange the values of the assignments $x_{i,j}$ and $x_{i,j'}$ of the solutions of the two sets. The resulting distribution has the probability of $E_p$ fixed and moreover, by the choice of the sets of solutions on which we perform the transfusion step, the probability of the events fixed in previous iterations was not altered and neither was the probability of events containing only assignments of cheap facilities. Clearly, the solution $s_{\pi,z}$ is still obtained in expectation. It remains to show that the transfusion step can be performed, i.e., that there is enough measure $\mu$ in the involved sets of integer solutions.

**Proposition 3.3** The probabilistic transfusion step of the above iterative procedure can always be performed.

**Theorem 3.1** There is a family of CFL instances with $2n$ facilities and $n^4 + 1$ clients such that the relaxations obtained from (LP-classic) at $\Omega(n)$ levels of the Sherali-Adams hierarchy have an integrality gap of $\Omega(n)$.

**Proof.** For each lifted constraint $\pi$ multiplied by multiplier $z$ at level $t$, the corresponding distribution $E_d$ or $E_f$ is clearly a distribution over integer solutions, so the first condition of Lemma 3.1 is satisfied. For the second condition, observe that if an event $E_p$ involves more than one costly facility, it has 0 probability in all distributions. If an event $E_p$ involves only cheap facilities, it has the same probability in all distributions $E_f$ and $E_d$, since in the construction of a distribution $E_f$ we took care not to change the probability of such events. An event $E_p$ that involves more than one assignment of a costly facility (but no other costly) has in every distribution $E_f$ the same probability (which is the same as in every $E_d$) since in the construction of $E_f$ we did not alter the probabilities of such events. And lastly, when an event $E_p$ involves exactly one assignment of some costly facility $i_x$, note that in some cases $i_x$ takes the blame but in other cases it does not, depending on $v(\pi, z)$. But due to the iterative procedure of probabilistic transfusion, the probability of event $E_p$ in a distribution in which $i_x$ is not the facility that takes the blame is equal to the probability of the same event in the distributions that $i_x$ takes the blame. So Lemma 3.1 holds. It is easy to see that bad solution has cost $\Theta(n^{-1})$ while any feasible solution to the instance has cost $\Omega(1)$.

### 4 Fooling the effective capacity inequalities for CFL

In this section we show that the (LP-classic) for CFL with the addition of the effective capacity inequalities proposed in [1] has unbounded gap.

Consider the general case where facility $i$ has capacity $u_i$ and client $j$ has demand $d_j$. For a set $J$ of clients, we denote their total demand by $d(J) = \sum_{j \in J} d_j$. Let $J \subseteq C$ be a set of clients, let $I \subseteq F$ be a set of facilities, and let $J_i \subseteq J$ be a set of clients for each facility $i \in I$. Given a facility $i$, we denote the effective capacity of $i$ with respect to $J_i$ by $\bar{u}_i = \min\{u_i, d(J_i)\}$. $I$ is a cover with respect to $J$ if $\sum_{i \in I} \bar{u}_i = d(J) + \lambda$ with $\lambda > 0$. $\lambda$ is called the excess capacity. Let
\((x)^+ = \max\{x, 0\}\). In the case where \(J_i = J\) for all \(i \in I\) the following inequalities called flow-cover inequalities were introduced for CFL in [1].

\[\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} (u_i - \lambda)^+(1 - y_i) \leq d(J)\]

If \(\max_{i \in I}(\tilde{u}_i) > \lambda\), the following inequalities, called the effective capacity inequalities are valid and strengthen the flow-cover inequalities [1].

\[\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} (\tilde{u}_i - \lambda)^+(1 - y_i) \leq d(J)\]

The proof of the following theorem uses some of the ideas we introduced earlier for Theorem 3.1. In the appendix we give Theorem B.1 which strictly generalizes Theorem 4.1 to the so-called submodular inequalities.

**Theorem 4.1** The integrality gap of the relaxation obtained from (LP-classic) with the addition of the effective capacity inequalities is unbounded, even for uniform CFL with unit demands.

**Proof.** Consider an instance with \(n\) cheap and \(n + 2\) costly facilities and \(U n + 1\) clients, \(U = n^3\). Define the bad solution \(s\), similarly to Section 3, s.t. for every \(ch \in Cheap, co \in Costly\), and client \(j\), \(y_{ch} = 1, x_{chj} = \frac{1 - \alpha}{n}, y_{co} = 10/n^2, x_{coj} = \frac{\alpha}{n + 2}\). Recall that \(\alpha = n^{-2}\). We add a set of \(n + 2\) facilities \(a_i, 1 \leq i \leq n + 2\), all with 0 opening costs, on the same point at distance 1 from the rest (an instance of the so-called facility location on a line). In the bad solution \(s\) we additionally set \(y_{a_i} = 1\) and \(x_{a_i, j} = 0\) for all \(i\) and for all clients \(j\).

We will prove that in every cover \(I\) with respect to some client set \(J\) and to the \(J_i\) client sets for each \(i\), there must always be a number of at least \(2n^3\) clients whose assignment variables to some costly and to some cheap, \(co\), and thus no costly facility is opened, the capacity of the corresponding bin is \(\bar{y}_{I_{co'}} = \frac{\alpha}{n + 2}\) and the capacity of the bin of \(a_i\) is \(\frac{|C| - w^2_{co'}}{n}\). We randomly select some \(w^2_{co'}\) clients that do not belong to \(J^*\) to be tossed in the bin of \(i_{co'}\); we randomly distribute the balls corresponding to the remaining clients to the slots of the cheap facilities. When \(a_{i'}\) is active, and thus no costly facility is opened, the capacity of the corresponding bin is \(w^1_{a_{i'}} = \frac{\sum_j x'_{a_{i'}, j}}{1 - \sum_{ch \in Cheap} y_{ch}}\) and the capacity of the cheap is \(\frac{|C| - w^1_{a_{i'}}}{n}\). We select randomly some \(w^1_{a_{i'}}\) clients in \(J^*\) and we toss the corresponding balls in the bin of \(a_{i'}\). We randomly toss the remaining balls to the slots of the bins of the cheap facilities.

Note that the above experiment induces a distribution over feasible integer solutions since all the defined bin capacities are less than \(U\) (this is by the choice of the size of \(J^*\)) and every client is assigned to exactly one opened facility in each outcome. We do not need this distribution to be assignment-symmetric. Observe that the expected vector with respect to the latter distribution is
solution \( s' \). Finally, note that we once again treated the capacities \( w \) of the bins as being integral. For fractional bin capacities (which is actually always the case for the defined \( w \)'s) we can define the experiment in a similar way to the proof of Lemma 3.2.

\section{Proper Relaxations}

In this section we present the family of proper relaxations and characterize their strength. Consider a 0-1 \((y, x)\) vector on the set of variables of the classic relaxation (LP-classic) such that \( y_i \geq x_{ij} \) for all \( i \in F, j \in C \). The meaning of \( y_i = 1 \) is the usual one that we open facility \( i \). Likewise, the meaning of \( x_{ij} = 1 \) is that we assign client \( j \) to facility \( i \). We call such a vector a \textit{class}. Note that the definition is quite general and a class can be defined from any such \((y, x)\), which may or may not have a relationship to a feasible integer solution. We denote the vector corresponding to a class \((y, x)_{cl}\) as \((y', x')_{cl}\). We associate with class \( cl \) the \textit{cost of the class} \( c_{cl} = \sum_{i\in\{y_i=1\}(y,x)_{cl}} f_i + \sum_{i,j\in\{x_{ij}=1\}(y,x)_{cl}} c_{ij} \).

Let the assignments of class \( cl \) be defined as \( Agn_{cl} = \{(i, j) \in F \times C | x_{ij} = 1 \text{ in } (y, x)_{cl}\} \). We say that \( cl \) \textit{contains} facility \( i \), if the corresponding entry \( y_i \) in the vector \((y, x)_{cl}\) equals 1. The set of facilities contained in \( cl \) is denoted by \( F(cl) \).

\textbf{Definition 5.1 (Constellation LPs)} Let \( C \) be a set of classes defined for an instance \( I(F, C) \) of CFL or LBFL. Let \( x_{cl} \) be a variable associated with class \( cl \in C \). The constellation LP with class set \( C \), denoted \( LP(C) \), is defined as \( \min \{ \sum_{cl\in C} c_{cl} x_{cl} | \sum_{cl\in C} \sum_{\exists (i,j)\in Agn_{cl}} x_{cl} = 1 \forall j \in C, \sum_{cl\in F(cl)} x_{cl} \leq 1 \forall i \in F, x_{cl} \geq 0 \forall cl \in C \} \).

We refer simply to a \textit{constellation LP} when \( C \) is implied from the context. We define the \textit{projection} \( s' = (y', x') \) of solution \( s = (x^s_{ij})_{cl\in C} \) of \( LP(C) \) to the facility opening and assignment variables \((y, x)\) as \( y'_s = \sum_{cl\in C} x^s_{cl} \) and \( x'_{ij} = \sum_{cl\in C} x^s_{cl} \). We restrict our attention to constellation LPs that satisfy a symmetry property that is very natural for uniform capacities and unit demands.

\textbf{Definition 5.2 (\( P_1 \): Symmetry)} We say that property \( P_1 \) holds for the constellation linear program \( LP(C) \) if for every class \( cl \in C \), all classes resulting from a permutation that relabels the facilities and/or the clients of \( cl \) are also in \( C \).

\textbf{Definition 5.3 (Proper Relaxations)} We call proper relaxation for CFL (LBFL) a constellation LP that is valid and satisfies property \( P_1 \).

A simple example of a constellation LP is the well-known \((LP\text{-star})\) (see, e.g., [19]) where \( C \) corresponds to the set of all \textit{stars}: a facility and a set of at most \( U \) (or at least \( B \) for LBFL) clients assigned to it. Obviously \((LP\text{-star})\) is a proper relaxation, while \((LP\text{-classic})\) is equivalent to \((LP\text{-star})\). Therefore proper relaxations generalize the known natural relaxations for CFL and LBFL. In order to characterize the strength of a proper LP we need the notion of complexity.

\textbf{Definition 5.4 (Complexity of proper relaxations)} Given an instance \( I(F, C) \) of CFL (LBFL) let \( F' \) be a maximum-cardinality set of open facilities in an integral feasible solution. The complexity \( \alpha \) of a proper relaxation \( LP(C) \) for \( I \) is defined as the \( \sup_{cl\in C} (|F(cl)|/|F'|) \).

The complexity of a proper LP represents the maximum fraction of the total number of feasibly openable facilities that is allowed in a single class. A complexity of nearly 1 means that there are classes that take each into consideration almost the whole instance at once. Low complexity means that all classes consider the assignments of a small fraction of the instance at a time. By increasing the complexity of a proper LP for a given instance we can produce strictly stronger proper relaxations, an example is given in the Appendix.
Theorem 5.1 Every proper relaxation for uniform CFL (LBFL) with complexity $\alpha < 1$ has an unbounded integrality gap. There is a proper relaxation for CFL (LBFL) of complexity 1 whose projection to $(y, x)$ expresses the integral polytope.

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A Appendix to Section

Omitted part of the proof of Lemma 3.2

Proof. First we explain how to handle fractional bin capacities. To handle the case where the \( w \)'s are not integers, we simply do the following: each time costly facility \( i_b \) \((i_c \neq i_b)\) is picked, we set the number of slots of the corresponding bin to \( |w_{ib}^1| \left( |w_{ico}^2| \right) \) with probability \( 1 - (|w_{ib}^1| - |w_{ib}^1|) \left( 1 - (|w_{ico}^2| - |w_{ico}^2|) \right) \), otherwise set the slots to \( |w_{ib}^1| \left( |w_{ico}^2| \right) \). If the number of slots of \( i_b \) \((i_c)\) is set to \( |w_{ib}^1| \left( |w_{ico}^2| \right) \) then we pick some \( n \left( \frac{|C| - |w_{ib}^1|}{n} \right) - \left( \frac{|C| - |w_{ib}^1|}{n} \right) \) (\( n \left( \frac{|C| - |w_{ico}^2|}{n} \right) - \left( \frac{|C| - |w_{ico}^2|}{n} \right) \)) cheap facilities at random and set their corresponding number of slots to \( \frac{|C| - |w_{ib}^1|}{n} \left( \frac{|C| - |w_{ico}^2|}{n} \right) \) and the number of slots of the rest of the cheap facilities to \( \frac{|C| - |w_{ib}^1|}{n} \left( \frac{|C| - |w_{ico}^2|}{n} \right) \). Otherwise pick some \( n \left( \frac{|C| - |w_{ib}^1|}{n} \right) - \left( \frac{|C| - |w_{ib}^1|}{n} \right) \) (\( n \left( \frac{|C| - |w_{ico}^2|}{n} \right) - \left( \frac{|C| - |w_{ico}^2|}{n} \right) \)) cheap facilities at random and set their corresponding number of slots to \( \frac{|C| - |w_{ib}^1|}{n} \left( \frac{|C| - |w_{ico}^2|}{n} \right) \) and the number of slots of the rest to \( \frac{|C| - |w_{ib}^1|}{n} \left( \frac{|C| - |w_{ico}^2|}{n} \right) \). Note than in every case the expected number of slots per facility is as in the previous experiment.

As for the expected assignments to the cheap facilities, observe that in every outcome of the experiment the demand not assigned to costly facilities is exactly the demand assigned to cheap. Since we have proved that the expected assignments to the costly facilities are those of the bad solution, by linearity of expectation we get that the total assignments to all cheap facilities are \( \sum_{i \in \text{Cheap}} \sum_{j} x'_{ij} \) (the total assignment of each client adds up to 1 by the constraints of the LP).

Proof of Proposition 3.1

Proof. Since the considered distribution is assignment-symmetric, event \( \mathcal{E}_p \) is equivalent to the event of randomly distributing \( l + 1 \) balls to the slots of \( n + 1 \) bins, with at most one ball in each slot, each bin having \( w_{ib}^1 \) identical slots, asking that ball \( j \) is tossed in the bin of \( i_b \) and ball \( j_i \) is...
always pick the required amount of measure. Since there are \( \Theta(n^3) \) slots in each bin and the balls are at most \( n \), it is easy to see that \((1 - o(1)) \frac{P[\mathcal{E}_p]}{n} \leq P[\mathcal{E}_p] \leq (1 + o(1)) \frac{P[\mathcal{E}_p]}{n} \).

Proof of Proposition 3.2

**Proof.** Consider again the random experiment of the proof of Proposition 3.1. Recall that, ignoring constant factors, \( w_{ch}^1 = n^3 - 1 \) and \( w_{ch}^2 = n^3 - n^2 \). \( P[\mathcal{E}_p] = x_{i,j} P[x_{i,1 \to j_1} = 1 \land x_{i,2 \to j_2} = 1 \land \ldots \land x_{i,k \to j_n} = 1 | x_{i,j} = 1] \) and \( P[\mathcal{E}_{p/fixed}] = x_{i,j} P[x_{i,1 \to j_1} = 1 \land x_{i,2 \to j_2} = 1 \land \ldots \land x_{i,k \to j_n} = 1 | x_{i,j} = 1 \). If and since \( x_{i,j} = x_{i,j} \) we can compute the ratio of the probability of success of the tossing of \( k - i + 1 \) balls when \( x_{i,j} = 1 \), and thus the capacity of the bins corresponding to the facilities is \( w_{ch}^1 \), to the probability of success of the tossing of \( k - i + 1 \) balls when \( x_{i,j} = 1 \) and thus the capacity of the bins corresponding to the facilities is \( w_{ch}^2 \). When tossing the ball \( j_b \) given the successful tossing of balls \( j_b \) with \( q < r \), the probability of success is \( \frac{w_{ch}^1 - a}{w_{ch}^1 - r + 1} \) and \( \frac{w_{ch}^2 - a}{w_{ch}^2 - r + 1} \) respectively, where \( 0 \leq a \leq r \) is the number of balls already placed in some slot of the bin corresponding to the facility \( a \). We have that \( 1 + o(1) \)\( n \) \( \leq \frac{w_{ch}^1 - a}{w_{ch}^1 - r + 1} \leq \frac{w_{ch}^1 - a}{w_{ch}^2 - a} \leq (1 + (1 + o(1))) \frac{n}{n} \). So \( (1 + (1 + o(1))) \frac{n}{n} \leq \frac{P[\mathcal{E}_p]}{P[\mathcal{E}_{p/fixed}]} \leq (1 + (1 + o(1))) \frac{n}{n} \leq 1 + o(1) \), and using that \( \lim_{x \to \infty} (1 + d/x)^x = e^d \).

Proof of Proposition 3.3

**Proof.** The intuition behind the proof is that the “donor” event that supplies the required measure is much more likely to occur than the events that require the transfusion.

Consider the measure \( t \) in \( E_d \) of the set of integer solutions satisfying \( y_{i_b} = 1 \) and all events encountered at any iteration being false, namely \( x_{i,j} = 0 \land x_{i,j} = 0 \land x_{i,j} = 0 \land \ldots \land x_{i,j} = 0 \). Then, by the random experiment of the construction of \( E_d \), this event is equivalent to the event that facility \( a \) is picked, \( x_{i,j} = 0 \) and the \( k \) facilities corresponding to the clients of the rest of the events are not tossed in their corresponding bins. Using again that both \( w_{ch}^1, w_{ch}^2 \) are \( \Theta(n^3) \) and \( k < n \), we can bound the probability of the \( k \) balls by that of \( k \) Bernoulli trials with probability of success \( 2/n \) (we are once again very generous). Then the probability that all events fail is at least \( (1 - 2/n)^k > \lim_{n \to \infty} (1 - 2/n)^n = 1/e^2 \). Thus measure \( t \) is at least \( (y_{i_b} - x_{i,j}) \) which is constant.

On the other hand the measure required by the transfusion step for each event \( E_p \) of iteration \( i \) that needs to be fixed is at most \( (e^2 - 1) \frac{P[\mathcal{E}_{p/fixed}]}{P[\mathcal{E}_p]} = \Theta(1/n^i) \). There are \( \frac{k+1}{k-i+1} \) such events of iteration \( i \), and summing over all the iterations of our construction we get \( \sum_{i=1}^{k} \frac{k+1}{k-i+1} \Theta(1/n^i) \) which quantity is less than \( (y_{i_b} - x_{i,j})/e^2 \) for the \( k = n/10 \) levels of SA we consider, so we can always pick the required amount of measure.

A.1 SA gap for LBFL

A similar result to Theorem 3.1 can be proved for LBFL. Consider an instance with \( n \) facilities, lower bound \( B = n^3 \) and a total of \( n(B - 1) \) clients. The metric space here is more intriguing than the one for the CFL case. Consider a regular \((n - 1)\)-dimensional simplex with edge length 1. On each of the \( n \) vertices of the simplex a facility along with some \( B - 1 \) clients are located. All opening costs are 0. Clearly every integer solution has a cost of at least \( B - 1 \) since we can open at most \( n - 1 \) of the facilities, and so at least \( B - 1 \) clients will have to be assigned to some facility other than the one on the same vertex. We call a client \( j \) that is located on the same vertex with facility \( i \), exclusive client of \( i \). We denote by Exclusive\((i)\) the set of clients that are exclusive to facility \( i \). On the other hand we can show that the following bad solution \( s \) is feasible at \( \Omega(n) \) levels of the SA hierarchy. For all \( i \in F, y_i = 1 - n^{-2} \); for a client \( j \in C, x_{ij} = 1 - 10n^{-2}, \) if \( j \in \text{Exclusive}(i) \), and \( x_{ij} = \frac{10n^{-2}}{n-1} \) for all other facilities. Solution \( s \) incurs a cost of \( o(B) \).
Theorem 4.1: we have

It therefore suffices to include in the body of the paper the simplest possible proof for the theorem.

Theorem 3.1 is to demonstrate that the SA hierarchy is inadequate for such preprocessing purposes.

impossible without some yet unknown form of preprocessing. In fact part of the motivation behind

Identifying “areas” of a fractional solution where the demand exceeds the available capacity is

demand. This is a well-known facet-inducing constraint for our instance, see, e.g., [25, p. 283]. Of

Sketch of proof of Theorem A.1.

Consider a constraint \( \pi \): \( \sum_j x_{ij} \geq By_{ix} \) and a multiplier \( z \) at level \( k \) and let \( v(\pi, z) \) be the set of variables appearing in the multiplied constraint. We pick a facility \( i_b \) not in \( v(\pi, z) \) to take the blame. We construct a solution \( s' \) where we set \( y_{ib}' = n - 1 - \sum_{i \neq i_b} y_i \) and for each \( j \in \text{Exclusive}(i_b) \) we set \( x_{ibj}' = y_{ib}' = \frac{1 - 1/\pi}{n} \) and we distribute the remaining demand that was assigned to \( i_b \) to each facility from a constant-size set \( I_b \) of facilities not appearing in \( v(\pi, z) \). Solution \( s' \) can be obtained as a convex combination of integer solutions by constructing a distribution similarly to Lemma 3.2. This time the distribution satisfies that exactly \( n - 1 \) facilities are opened in each outcome of the experiment. Note that we do not require the underlying distribution to be assignment symmetric, because facilities have to treat differently their exclusive clients. We set the values of the linearized products appearing in the multiplied constraint equal to the probability of the corresponding events with respect to the aforementioned distribution. No product involving variables of \( i_b \cup I_b \) appear in the constraint. For constraints \( 0 \leq x_{ij}, y_i \leq 1 \) and \( x_{ij} \leq y_i \) the construction of the distribution is the same. The distributions constructed so far are locally consistent as required by Lemma 3.1.

The case where the constraint is \( \pi \): \( \sum_i x_{ij} = 1 \) is once again more complicated. We choose a facility \( i_b \notin z \) and moreover \( j^\pi \notin \text{Exclusive}(i_b) \) to take the blame and the set \( I_b \) is defined as before except we also require that \( j^\pi \) is not exclusive to any of them. Solution \( s' \) is constructed like in the previous case. All products take the value of the corresponding events in the distribution except those in which the unique variable involving \( i_b \) appears, namely \( x_{ibj} \) and those involving facilities in \( I_b \). We perform a transfusion step so that the probabilities of all the events whose corresponding products appear in the lifted constraint become consistent with the distributions of the previous case: this time we need to fix the probabilities of the events involving facility \( i_b \) or some facility \( i \in I_b \).

A.2 Robustness of the SA gap

In this section we explain to the interested reader how adding simple valid inequalities does not affect our arguments on the SA hierarchy.

As an example we address the valid inequality \( \sum_i y_i \geq \lceil D/U \rceil \), where \( D \) is the total amount of demand. This is a well-known facet-inducing constraint for our instance, see, e.g., [25, p. 283]. Of course this inequality is rendered useless by slight modifications to the instance and the bad solution. Identifying “areas” of a fractional solution where the demand exceeds the available capacity is impossible without some yet unknown form of preprocessing. In fact part of the motivation behind Theorem 4.1 is to demonstrate that the SA hierarchy is inadequate for such preprocessing purposes. It therefore suffices to include in the body of the paper the simplest possible proof for the theorem.

We modify the family of "bad" instances by using the same trick we used in the proof of Theorem 4.1 we have \( n \) cheap and \( n \) costly facilities and \( Un + 1 \) clients, and the bad solution in which for every \( ch \in \text{Cheap}, co \in \text{Costly}, \) and client \( j \), \( y_{ch} = 1, x_{chj} = \frac{1 - \alpha}{n}, y_{co} = 10/n^2, x_{coj} = \frac{\alpha}{n} \) and additionally we add a set of \( n \) dummy facilities \( a_i \), \( 1 \leq i \leq n \), all with 0 opening costs, on the same point at distance 1 from the rest. In the bad solution \( s \) we additionally set \( y_{ai} = 1 \) and \( x_{aij} = 0 \) for all \( i \) and for all clients \( j \). The inequality is obviously satisfied.
In the design of the locally consistent distributions, now we must give a distribution for the case where the constraint \( \pi \) is the new one \( \sum_i y_i \geq \lceil D/U \rceil \), and verify that the “visible” part of the distribution agrees with the visible part of all other distributions of the proof. In this case there must be some dummy facility \( a_d \) not appearing as an index in the multiplier \( z \) of the constraint (although its \( y \) variable does appear in \( \pi \)). Additionally there must be a costly facility \( i' \) for which the assignments of clients to \( i' \) do not appear in \( v(\pi, z) \) – this is ensured by the number of rounds we consider. We modify the solution \((y, x)\) to obtain \((y', x')\) where the facilities \( i' \) and \( a_d \) exchange the values of their corresponding assignments. We define now the random experiment similarly to the proof of Lemma 3.2 with facility \( a_d \) taking the blame. The only difference is that while \( a_d \) is opened 100% of the time, it is not assigned any demand when a costly facility other than \( i' \) is opened. In the terminology of Theorem 3.1 \( a_d \) is always open but it is inactive when some \( i \in Costly, i \neq i' \), is opened. It is easy to see that the distribution obtained is consistent with all the other distributions defined for this modified instance, as required by Lemma 3.1.

**B Appendix to Section 4**

**B.1 How to fool submodular inequalities**

Here we show that the classic relaxation strengthened by the submodular inequalities has unbounded gap. The submodular inequalities introduced in [1] are even stronger than the effective capacity inequalities. We limit our discussion to uniform CFL where all clients have unit demands.

Choose a subset \( J \subseteq C \) of clients, and let \( I \subseteq F \) be a subset of facilities. For each facility \( i \in I \) choose a subset \( J_i \subseteq J \). Consider a 3-level network \( G \) with a source \( s \), a set of nodes corresponding to the facilities, a set of nodes corresponding to the clients and a sink \( t \). The source \( s \) is connected by an edge of capacity \( \lceil U, |J_i| \rceil \) to each facility node \( i \). That node is connected by an edge of unit capacity to each node corresponding to client \( j, j \in J_i \). Each node corresponding to some client is connected by an edge of unit capacity to the sink \( t \).

Define \( f(I) \) as the maximum \( s-t \) flow value in \( G \). Define \( f(I \setminus \{i\}) \) as the maximum flow when facility \( i \) is closed, i.e., when the capacity of edge \((s,i)\) is set to zero. The difference in maximum flow when all facilities in \( I \) are open, and when all facilities except facility \( i \) are open, is called the *increment* function and is defined as \( \text{increment}(I \setminus \{i\}) = f(I) - f(I \setminus \{i\}) \).

For any choice of \( I \subseteq F \), \( J \subseteq C \), and \( J_i \subseteq J \), for all \( i \), the following inequalities, called the *submodular inequalities*, are valid for CFL [1]. The name reflects the fact that the function \( f(I) \) is submodular.

\[
\sum_{i \in I} \sum_{j \in J_i} x_{ij} + \sum_{i \in I} \text{increment}(I \setminus \{i\})(1 - y_i) \leq f(I)
\]

**Theorem B.1** The integrality gap of (LP-classic) remains unbounded even after the addition of the submodular inequalities.

**Proof.** Consider the instance and bad solution \( s \) that we used in Theorem 3.1 for the SA result. To prove that \( s \) is feasible for the classic relaxation strengthened by the submodular inequalities we take the idea of fooling local constraints a little further: either the constraint is local enough that we can use the ideas from our previous proofs (define \( s' \) that is a convex combination of integer solutions and has the same visible part as \( s \) with respect to the constraint), or we can define another instance \( I' \) and solution \( s' \) for which the inequality in question is true with respect to \( s' \) and again \( s' \) has the same visible part as \( s \) with respect to the constraint. Note that our arguments include two different instances as opposed to all our other proofs so far.
Consider the submodular inequality $\pi$ for some $I$, $J$ and some selection of $J_i$'s. If not all the costly facilities appear in the constraint the proof is similar to that of Lemma 3.2. If at least $n$ assignment variables to cheap facilities do not appear in $\pi$ we do the following: we add one more facility $a$ to the instance. We construct a solution $s'$ for the new instance $I'$ as follows. We transfer the demand corresponding to the missing assignments of the cheap to $a$, and we set $y_a = 1$. Observe that $\pi$ is valid for $I'$. Now we can show that $s'$ is a convex combination of integer solutions similarly to the proof of Theorem 4.1, where the role of $J^*$ is played by those clients whose assignments were transferred from the Cheap to $a$. Facility $a$ will be active only when no costly facilities are open. Because, in the fractional solution $s'$, $a$ is assigned a total demand of at least $1 - 1/n^2$, in each outcome of the random experiment in which $a$ is active, it will be assigned at least one client. By the convex combination produced, the inequality is satisfied by $s'$. Thus the same inequality for the original instance is satisfied by $s$.

Now consider the case where less than $n$ assignments to cheap facilities are missing from $\pi$. We will show that it cannot be the case that all $y_i$ variables of costly facilities appear in the constraint as well. Consider the quantity $\rho_i(I \setminus \{i\})$ for some costly facility $i$. If $\rho_i(I \setminus \{i\}) > 0$, then $J_i$ is not empty. We will show that the set of nodes $(Cheap \cap I) \cup \{i\}$ in $G$ has enough incident edges so that the flow originating from them is equal to the total client demand $|J|$ in $G$. We first give some properties of graph $G$.

**Claim B.1** If less than $n$ assignments to cheap facilities are missing from $\pi$, then $(Cheap \cap I) = Cheap$ and $J = C$.

*Proof of Claim.* To see that $(Cheap \cap I) = Cheap$, notice that if a cheap facility is missing from $I$, at least $|C| = n^4 + 1$ assignment variables will be missing from $\pi$, a contradiction. For the second part of the claim, if a client $j$ is missing from $J$, then all the corresponding $n$ edges that would connect $j$ to a cheap facility cannot be in $G$. Therefore at least $n$ assignment-to-cheap variables are missing from $\pi$, a contradiction. The proof of the claim is complete.

We return to proving that $Cheap \cup \{i\}$ has enough incident edges so that the flow originating from them is equal to the total client demand $|C|$ in $G$. “Assign” one client $j \in J_i$ to facility $i$ and for the remaining $|C| - 1$ clients do the following: assign each client $j'$ involved in the set of variables of assignments-to-cheap that are missing from $\pi$ to a cheap facility $i'$ such that $j' \in J_{i'}$. There is always such a cheap facility $i'$ since the missing edges from the client-nodes in $G$ to the cheap-facility nodes are less than $n$. Assign the remaining clients arbitrarily to the cheap facilities respecting the capacities, since all the edges from cheap to those clients are included in the network. Thus it must be the case that $\rho_{i'}(I \setminus \{j'\}) = 0$ for any other costly facility $i' \neq i$. Since the $y_{i'}$ variable of such a facility $i'$ has 0 coefficient in the constraint, it can take the blame and the proof is similar to that of Lemma 3.2.

**C Appendix to Section 5**

**Example C.1** An increased complexity allows strictly stronger proper relaxations.

First we show how one can construct any integer solution using classes that open the same number of facilities. Consider an integer solution $s$ with opened facilities $1, \ldots, t$. We will use the following classes in which exactly $r < t$ facilities are opened: For any set of $t$ consecutive classes in a cyclic ordering, namely $(1, \ldots, r), (2, \ldots, r + 1), \ldots, (t, \ldots, r - 1)$, define a class that opens those facilities and makes the same assignments to them as $s$. Then the integer solution is obtained if for every $cl$ we set $x_{cl} = 1/r$. Observe that the latter solution is feasible for the proper relaxation.
We give a toy example showing that by increasing the complexity, we can get strictly stronger relaxations. Consider an LBFL instance with 4 facilities 2 sets $S_1, S_2$ of 13 clients each and 2 sets $S_3, S_4$ of 9 clients each and $B = 10$. For the star relaxation (complexity $\alpha = 1/4$ for this instance) there is a feasible solution $\bar{s}$ whose projection to $(y, x)$ is the following ($\bar{y}, \bar{x}$): for facility 1, $\bar{y}_1 = 1$ and is assigned $S_1$ integrally, for facility 2, $\bar{y}_2 = 1$ and is assigned $S_2$ integrally, for facility 3, $\bar{y}_3 = 9/10$ and is assigned each client of $S_3$ with a fraction of 9/10 and each of $S_4$ with 1/10, and similarly for facility 4, $\bar{y}_4 = 9/10$ and is assigned each client of $S_4$ with a fraction of 9/10 and each of $S_3$ with 1/10. Actually a direct consequence of Theorem 5.1 is that for any proper relaxation of the same complexity as the star relaxation, the above solution is feasible.

Now consider the following proper relaxation: all characteristic vectors of integer solutions with at most 3 facilities are classes plus all the vectors of solutions with 4 facilities restricted in any 3 facilities (3/4 parts of integer solutions that open all four facilities). It is symmetric and valid by the previous discussion and has complexity $\alpha = 3/4$. In any assignment of values to the class variables that projects to $(\bar{y}, \bar{x})$ the following are true: since classes with less than 3 facilities are integer solutions, they contain assignments for all the clients and thus if we were to use a non-zero measure of such classes we would make non-zero assignment that does not exist in the support of $(\bar{y}, \bar{x})$. If we use classes with exactly 3 facilities, then exactly one of facilities 3, 4 must be present, since no integer solution opens them both with just the clients in $S_3 \cup S_4$. So we have to use at least $\bar{y}_3 + \bar{y}_4 = 18/10$ measure of such classes. So each one of facilities 1, 2 must be present in more than a unit of classes, which would make the solution infeasible.

Proof of Theorem 5.1

We first prove the easy part, that there are proper relaxations for CFL and LBFL with complexity 1 that express the integral polytope. For a given instance let $C$ consist of a class for each distinct integral solution. The resulting $LP(C)$ is clearly proper. Let $x$ be any feasible solution of $LP(C)$ and let $S$ be the support of the solution. For every $cl \in S$, and for every client $j \in C$, there is an $i \in F$, such that $(i, j) \in Assignments_{cl}$. Therefore

$$\sum_{cl \in S} x_{cl} = 1.$$  

This implies that $x$ is a convex combination of integral solutions. By the boundedness of the feasible region of $LP(C)$, the corresponding polytope is integral. Clearly not every LP with complexity 1 has an integrality gap of 1 since it might contain weak classes together with strong ones.

In the next two subsections, we prove the first part of Theorem 5.1 for LBFL and CFL respectively.

C.1 Proof of Theorem 5.1 for LBFL

Our proof includes the following steps. We define an instance $I$ and consider any proper relaxation $LP(C)$ for $I$ that has complexity $\alpha < 1$. Given $\alpha$, we use the validity and symmetry properties to show the existence of a specific set of classes in $C$. Then we use these classes to construct a desired feasible fractional solution, relying again on symmetry. In the last step we specify the distances between the clients and the facilities, so that the instance is metric and the constructed solution has an unbounded integrality gap.
C.1.1 Existence of a certain type of classes

Let us fix for the remainder of the section an instance \( I \) with \( n + 1 \) facilities, where \( n \) is sufficiently large to ensure that \( \alpha n \leq n - c_0 \) where \( c_0 \) is a constant greater than or equal to 2. Let the bound \( B = n^2 \), and let the number of clients be \( n^3 \). Notice that there are enough clients to open \( n \) facilities, with exactly \( n^2 \) clients assigned to each one that is opened. The facility costs and the assignment costs will be defined later. Recall that the space of feasible solutions of a proper relaxation is independent of the costs.

We assume that the facilities are numbered \( i = 1, 2, \ldots, n + 1 \). For a solution \( p \) we denote by \( \text{Clients}_p(i) \) the set of clients that are assigned to facility \( i \) in solution \( p \), and likewise for a class \( cl \) we denote by \( \text{Clients}_cl(i) \) the set of clients that are assigned to facility \( i \). Consider an integral solution \( s \) to the instance where facilities \( 1, \ldots, n \) are opened. Since our proper relaxation is valid, it must have a feasible solution \( s' = (x_{cl})_{cl \in \mathcal{C}} \) whose projection to \((y, x)\) gives the characteristic vector of \( s \). We prove the existence of a class \( cl_0 \), with some desirable properties, in the support of \( s' \).

By Definition 3.1 \( s' \) can only be obtained as a positive combination of classes \( cl \) such that for every facility \( i \) we have \( \text{Clients}_cl(i) \subseteq \text{Clients}_s(i) \). Otherwise, if the variables of a class \( cl \) with \( \text{Clients}_cl(i) \setminus \text{Clients}_s(i) \neq \emptyset \) have non-zero value, then in \( s' \) there will be some client assigned to some facility with a positive fraction, while the projection of \( s' \), namely \( s \), does not include the particular assignment. Moreover, since exactly \( B \) clients are assigned to each facility in \( s \), for every facility \( i \) that is contained in such a class \( cl \), \( \text{Clients}_cl(i) = \text{Clients}_s(i) \). To see why this is true, since in \( s \) we have \( y_i = 1 \), for all \( i \leq n \), it follows that for every facility \( i \leq n \), \( \sum_{cl' \ni (i,j) \in \text{Agn}_{cl'}} x_{cl'} = 1 \). But then we have that \( |\text{Clients}_s(i)| = B = \sum_{cl' \ni (i,j) \in \text{Agn}_{cl'}} x_{cl'} |\text{Clients}_cl(i)| \). We have already established that \( x_{cl'} > 0 \iff |\text{Clients}_cl(i)| \leq B \). Then \( B \) is a convex combination of quantities less than or equal to \( B \), so for all such classes \( cl \) we have \( |\text{Clients}_cl(i)| = B \).

Therefore in the class set of any proper relaxation for \( I \), there is a class \( cl_0 \) that assigns exactly \( B \) clients to each of the facilities in \( F(cl_0) \). By the value of \( \alpha \), \( |F(cl_0)| \leq n - c_0 \). The following lemma has been proved.

**Lemma C.1** Given the specific instance \( I \), any proper relaxation of complexity \( \alpha \) for \( I \) contains in its class set a class \( cl_0 \) that assigns \( B \) clients to each of \( n - c \) facilities, for some integer \( c \geq 2 \).

C.1.2 Construction of a bad solution

In the present section we will use the class \( cl_0 \) along with the symmetric classes to construct a solution to the proper LP with the following property: there are some \( q \) facilities that are almost integrally opened while the number of distinct clients assigned to them will be less than \( Bq \).

Recall that by property \( P_1 \) every class that is isomorphic to \( cl_0 \) is also a class of our proper relaxation. This means that every set of \( n - c \) facilities and every set of \( B(n - c) \) clients assigned to those facilities so that each facility is assigned exactly \( B \) clients, defines a class, called admissible, that belongs to the set of classes defined of a proper relaxation for the instance \( I \).

Let us turn again to the solution \( s \) to provide some more definitions. For every facility \( i, i = 1, \ldots, n - 1 \), we choose arbitrarily a client \( j' \) assigned to it by \( s \). For each such facility \( i \) we denote by \( \text{Exclusive}(i) \) the set of clients \( \text{Clients}_s(i) \setminus \{j'\} \), i.e., the set of clients assigned to \( i \) by \( s \) after we discard \( j' \) (we will also call them the exclusive clients of \( i \)). For facilities \( n, n + 1 \) the sets \( \text{Exclusive}(n), \text{Exclusive}(n + 1) \) are identical and defined to be equal to the union of \( \text{Clients}_s(n) \) with all the discarded clients from the other facilities. In the fractional solution that we will construct below, the clients in \( \text{Exclusive}(i) \) will be almost integrally assigned to \( i \) for \( i = 1, \ldots, n - 1 \).
We are ready to describe the construction of the fractional solution. We will use a subset $S$ of admissible classes that do not contain both $n$ and $n + 1$. $S$ contains all such classes $cl_i$ that assign to each facility $i \leq n - 1$ in the class the set of clients $Exclusive(i)$ plus one more client selected from the sets $Exclusive(i')$ for those facilities $i' \leq n - 1$ that do not belong to $cl_i$ (there are at least $c - 1$ of them). As for facility $n$ (resp. $n + 1$), if it is contained in $cl_i$, then it is assigned some set of $B$ clients out of the total $B + n - 1$ in $Exclusive(n)$ (resp. $Exclusive(n + 1)$). All classes not in $S$ will get a value of zero in our solution. We will distinguish the classes in $S$ into two types: the classes of type $A$ that contain facility $n$ or $n + 1$ but not both, and classes of type $B$ that contain neither $n$ nor $n + 1$.

We consider first classes of type $A$. We give to each such class a very small quantity of measure $\epsilon$. Let $\phi$ be the total amount of measure used. We call this step $Round_A$. The following lemma shows that after $Round_A$, the partial fractional solution induced by the classes has a convenient and symmetric structure:

**Lemma C.2** After $Round_A$, each client $j \in Exclusive(i)$, $i \leq n - 1$, is assigned to $i$ with a fraction of $\frac{n-c-1}{n-1}\phi$ and is assigned to each other facility $i'$, $i' \neq i$, $i' \leq n - 1$, with a fraction of $\frac{n-c-1}{(n-1)(n-2)(n-1)}\phi$. Each client $j \in Exclusive(n)$ ($= Exclusive(n+1)$) is assigned to $n$ and to $n+1$ with a fraction of $\frac{n^2}{2(n^2+n-1)}\phi$.

**Proof.** Consider a facility $i, i \leq n - 1$. Since exactly one of facilities $n,n+1$ is present in all the classes of type $A$ and each class contains $n - c$ facilities, $i$ is present in the classes of $Round_A$ $\frac{n-c-1}{n-1}$ of the time due to symmetry of the classes. Each time $i$ is present in a class $cl_i$ that class $cl_i$ assigns all $j \in Exclusive(i)$ to $i$. So client $j$ is assigned to $i$ with a fraction of $\frac{n-c-1}{n-1}\phi$. When $i$ is not present in class $cl_i$, which happens $\frac{c}{n-1}$ of the time, then its exclusive clients along with the exclusive clients of all the other $c - 1$ facilities that are also not present in $cl_i$ are used to help the $n - c - 1$ facilities $i \leq n - 1$, reach the bound $B$ of clients (recall that the number of exclusive clients of each such facility is equal to $B - 1$). Each time this happens, the $n - c - 1$ facilities in $cl_i$ need $n - c - 1$ additional clients, while the exclusive clients of the $c$ facilities that are not present in $cl_i$ are $c(n^2 - 1)$ in total. Due to symmetry once again, a specific client $j \in Exclusive(i)$ is assigned to one of those $n - c - 1$ facilities $\frac{n-c-1}{c(n^2-1)}$ of the time of those cases. So in total this happens $\frac{c}{n-1} \times \frac{n-c-1}{c(n^2-1)} = \frac{n-c-1}{(n-1)(n^2-1)}$ of the time, so it follows that client $j$ is assigned to a specific facility $i'$, $i' \neq i$, $i' \leq n - 1$, $\frac{n-c-1}{(n-1)(n-2)(n^2-1)}$ of the time. The fraction with which $j$ is assigned to $i'$ after $Round_A$ is $\frac{n-c-1}{(n-1)(n-2)(n^2-1)}\phi$.

For the proof of the second part of the lemma, consider facilities $n,n+1$. Each one of those is present in the classes of type $A$ an equal fraction $1/2$ of the time. The only clients that are assigned to them are their exclusive clients. Each class $cl_i$ assigns exactly $B = n^2$ out of those $n^2 + n - 1$ clients. So, due to symmetry, each client $j \in Exclusive(n)$ is present in $cl_i \frac{n^2}{n^2+n-1}$ of the time, so $j$ is assigned to $n$ and $n+1$ with a fraction of $\frac{n^2}{2(n^2+n-1)}\phi$ to each.

Note that after $Round_A$ each facility $i, i \leq n - 1$, has a total amount $\frac{(n-c-1)B}{(n-1)}\phi$ of clients (since it is present in a class $\frac{(n-c-1)}{(n-1)}$ of the time and when this happens it is given $B$ clients). Similarly, facilities $n,n+1$ after $Round_A$ have a total amount $B\phi/2$ each.

Now we can explain the underlying intuition for distinguishing between the two types of classes. The feasible fractional solution $(y^*, x^*)$ we intend to construct is the following: for each facility $i \leq n - 1$, its exclusive clients are assigned to it with a fraction of $\frac{n^2-1}{n^2}$ each, while they are assigned with a fraction of $\frac{1}{(n^2)(n-2)}$ to each other facility $i' \leq n - 1$. As for facilities $n,n+1$, all of their exclusive clients are assigned with a fraction of $1/2$ to each. If we project the solution to $(y, x)$,
the $y$ variables will be forced to take the values $y_i^* = \frac{n^2 - 1}{n}$, for $i \leq n - 1$, and $y_n^* = y_{n+1}^* = \frac{n^2 + n - 1}{2n}$. Observe as we give some amount of measure to Round$_A$, the variables concerning the assignments to facilities $n, n+1$ tend to their intended values in the solution we want to construct “faster” than the variables concerning the assignments to the other facilities. This is because, by Lemma C.2 after Round$_A$ each exclusive client of $n, n+1$ is assigned to each of them with a fraction of $\frac{n}{2(n^2 + n - 1)}$ which is $\frac{n^2}{n^2 + n - 1}$ of the intended value. At the same time, every exclusive client of each other facility is assigned to it with a fraction of $\frac{n - c - 1}{n - 1}$ which is $\frac{n^2}{n^2 + n - 1}$ of the intended value. For sufficiently large instance $I$, as $n$ tends to infinity, the assignments to $n$ and $n+1$ will reach their intended values while there will be some fraction of every other client left to be assigned. Subsequently we have to use classes of type $B$, to achieve the opposite effect: the variables concerning the assignments of the first $n - 1$ facilities should tend to their intended values “faster” than those of $n$ and $n+1$ (since $n$ and $n+1$ are not present in any of the classes of type $B$, the corresponding speed will actually be zero).

We proceed with giving the details of the usage of type $B$ classes. As before, we give to each such class a very small quantity of measure $\epsilon$. Let $\xi$ be the total amount of measure used. We call this step Round$_B$.

**Lemma C.3** After Round$_B$, each client $j \in$ Exclusive($i$), $i \leq n - 1$, is assigned to $i$ with a fraction of $\frac{n - c}{n - 1}\xi$ and is assigned to each other facility $i'$, $i' \neq i$, $i' \leq n - 1$, with a fraction of $\frac{n - c}{(n - 1)(n - 2)(n^2 - 1)}\xi$.

**Proof.** The proof follows closely that of Lemma C.2 A facility $i, i \leq n - 1$, is present in a class of type $B$ $\frac{n - c}{n - 1}$ of the time (since $c \geq 2$ this fraction is less than 1). Each such time, every client $j \in$ Exclusive($i$) is assigned to it (again this is due to the definition of classes of type $B$). So after Round$_B$, $j$ is assigned to $i$ with a fraction of $\frac{n - c}{n - 1}\xi$. Also, when $i$ is present in a class, it is assigned exactly one client which is exclusive to a facility not in the class. Since in total there are $(n - 2)(B - 1)$ such candidate clients, and by symmetry, after round $B$ each one of them is picked an equal fraction of the time to be assigned to $i$, we have that each client $j$ is assigned to a facility for which $j$ is not exclusive with a fraction $\frac{n - c}{(n - 1)(n - 2)(n^2 - 1)}\xi$.

To construct the aforementioned fractional solution $(y^*, x^*)$, set $\phi = \frac{n^2 + n - 1}{n}$ and $\xi = (\frac{n^2 + 1}{n^2} - \frac{n - c - 1}{n - 1}\phi)\frac{n - 1}{n - c}$, and add the fractional assignments of the two rounds.

It is easy to check that the facility and assignment variables of facilities $n, n+1$ take the value they have in $(y^*, x^*)$. Same is true for the facility variables for $i \leq n - 1$ and the assignment variables of the clients to the facilities they are exclusive. To see that the same goes for the non-exclusive assignments, observe that since every class assign exactly $B$ clients to its facilities we have that $\sum_j x_{ij} = By_i$. So each $i \leq n - 1$ takes exactly $1 - 1/n^2$ demand from non-exclusive clients which are $(n - 2)(B - 1)$ in total. Thus, by symmetry of the construction, each one them is assigned to $i$ with a fraction of $\frac{B - 1}{n(n - 2)(B - 1)} = \frac{1}{n^2(n - 2)}$.

**C.1.3 Proof of unbounded integrality gap of the constructed solution**

In the present subsection, we manipulate the costs of instance $I$, which we left undefined, so as to create a large integrality gap while ensuring that the distances form a metric.

Set each facility opening cost to zero. As for the connection costs (distances) consider the $(n - 2)$-dimensional Euclidean space $\mathbb{R}^{n - 2}$. Put every facility $i, i \leq n - 1$, together with its exclusive clients on a distinct vertex of an $(n - 2)$-dimensional regular simplex with edge length $D$. Put facilities $n, n+1$ together with their exclusive clients to a point far away from the simplex, so the minimum distance from a vertex is $D' >> D$. Setting $D' = \Omega(nD)$ is enough.
Since the distance between a facility and one of its exclusive clients is 0, the cost of the fractional solution we constructed is $O(nD)$. This cost is due to the assignments of exclusive clients of facility $i$, $i \leq n - 1$, to facilities $i'$ with $i' \neq i$, $i' \leq n - 1$. As for the cost of an arbitrary integral solution, observe that since the $n^2 + n - 1$ exclusive clients of $n, n + 1$ are very far from the rest of the facilities, using $n$ of them to satisfy some demand of those facilities and help to open all of them, incurs a cost of $\Omega(nD') = \Omega(n^2D)$. On the other hand, if we do not open all of the $n - 1$ facilities on the vertices of the simplex (since they have in total $(n - 1)(B - 1)$ exclusive clients which is not enough to open all of them), there must be at least one such facility not opened in the solution, thus its $B - 1 = \Theta(n^2)$ exclusive clients must be assigned elsewhere, incurring a cost of $\Omega(n^2D)$.

This concludes the proof of Theorem 5.1.

C.2 Proof of Theorem 5.1 for CFL

The proof is similar to that for LBFL. We prove that the relaxation must use a specific set of classes and then we use these classes to construct a desired feasible solution. In the last step we define appropriately the costs of the instance.

C.2.1 Existence of a specific type of classes

Consider an instance $I$ with $n$ facilities, where $n$ is sufficiently large to ensure that $cn \leq n - c_0$ where $c_0$, is a constant greater than or equal to 1. Let the capacity be $U = n^2$, and let the number of clients be $(n - 1)U + 1$. Notice that in every integer solution of the instance we must open at least $n$ facilities. The facility costs and the assignment costs will be defined later.

We assume, like before, that the facilities are numbered $1, 2, \ldots, n$. Consider an integral solution $s$ for $I$ where all the facilities are opened, and furthermore facilities $1, \ldots, n - 1$ are assigned $U$ clients each and facility $n$ is assigned one client. Since our proper relaxation is valid, there must be a solution $s'$ in the space of feasible solutions of the proper relaxation whose $(y, x)$ projection is the characteristic vector of $s$. By Definition 5.1, it is easy to see that $s'$ can only be obtained as a positive combination of classes $cl$ such that for every facility $i$ we have $\text{Clients}_{cl}(i) \subseteq \text{Clients}_{s}(i)$. Recall that since the complexity of our relaxation is $\alpha$, the classes in the support of any solution have at most $n - c_0 \leq n - 1$ facilities.

Now consider the support of $s'$. We will distinguish the classes $cl$ for which variable $x_{cl}$ is in the support of $s'$ into 2 sets. The first set consists of the classes that assign exactly one client to facility $n$; call them type A classes. The second set consists of the classes that do not assign any client to facility $n$; call those type B classes. By the discussion above those sets form a partition of the classes in the support of $s'$, and moreover they are both non-empty: this is by the fact that at most $n - c_0$ facilities are in any class, and by the fact that in $s$ all $n$ facilities are opened integrally. Notice also that no class of type B can contain facility $n$ even though the definition of a class does not exclude the possibility that a class contains a facility to which no clients are assigned.

We call density of a class $cl$ the ratio $d(cl) = \frac{\sum_{x_{cl} \neq 0} |\text{Clients}_{s}(i)|}{|F(cl) - \{n\}|}$. By the discussion above we have that $d(cl) \leq U$ for all $cl$ in the support of $s'$. The following holds:

**Lemma C.4** All classes in the support of $s'$ have density $U$.

**Proof.** The amount of demand that a class $cl$ contributes to the demand assigned to the set of the first $n - 1$ facilities by $s'$ is $d(cl)|F(cl) - \{n\}|x_{cl}$. We have $\sum_{cl} d(cl)|F(cl) - \{n\}|x_{cl} = (n - 1)U$. Observe that by the projection of $s'$ on $(y, x)$ and by the fact that for $i = 1, \ldots, n - 1$, $y_i = 1$ in $s$, we have $\sum_{cl} |F(cl) - \{n\}|x_{cl} = n - 1$. Setting $m_{cl} = \frac{x_{cl}|F(cl) - \{n\}|}{n - 1}$ we have from the above
∑_{cl} m_{cl} = 1 and ∑_{cl} m_{cl} d(cl) = U. The latter together with the fact that d(cl) ≤ U we have that d(cl) = U for all classes cl in the support of s'.

The following corollary is immediate from the above:

**Corollary C.1** There is a type B class in the support of s' that has density U.

So far we have proved that in the class set of any proper relaxation for I, there is a class cl_0 of type B with density d(cl_0) = C. Let |F(cl_0)| = t ≤ n − 1.

### C.2.2 Construction of a bad solution

Consider the symmetric classes of cl_0 for all permutations of the n facilities and for all permutations of the clients. Those classes are not necessarily in the support of s'. Take a quantity of measure ε and distribute it equally among all those classes. Since class cl_0 has density U, all those symmetric classes assign on average U clients to each of their facilities. Due to symmetry, each facility is in a class ε/\(\frac{n}{4}\) of the time and is assigned ε/\(\frac{n}{4}\)U demand. Each client is assigned to each facility ε/\(\frac{n}{4}\) of the time. We call that step of our construction round A.

Now consider the symmetric classes of cl_0 for all permutations of the first n − 1 facilities and for all permutations of the clients (those classes are well defined since t ≤ n − 1). Again distribute a quantity of measure ε equally among all those classes. Similarly to the previous, each facility is in a class ε/\(\frac{n-1}{4}\) of the time and is assigned ε/\(\frac{n-1}{4}\)U demand. Each client is assigned to each facility ε/\(\frac{n-1}{4}\) of the time. We call that step of our construction round B.

Spending φ = 1/\(\frac{1}{n}\) measure in round A and ξ = \(\frac{(n-1)(1-1/n^2)}{t}\) measure in round B we construct a solution s_b whose projection to (y, x) is the following (y^*, x^*): y^*_i = 1 for i = 1, . . . , n − 1, y^*_n = 1/n^2, and for every client j, x^*_nj = \(\frac{U/n^2}{(n-1)U+1}\) and x^*_ij = \(\frac{1-x^*_nj}{n-1}\) for i = 1, . . . , n − 1. It is easy to see that s_b is a feasible solution for our proper relaxation.

Now simply set all distances to 0, and define the facility opening costs as f_n = 1 and f_i = 0 for i ≤ n − 1. It is easy to see that the integrality gap of the proper relaxation is Ω(n^2).