Rotation number of integrable symplectic mappings of the plane

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Symplectic mappings are discrete-time analogs of Hamiltonian systems. They appear in many areas of physics, including, for example, accelerators, plasma, and fluids. Integrable mappings, a subclass of symplectic mappings, are equivalent to a Twist map, with a rotation number, constant along the phase trajectory. In this letter, we propose a succinct expression to determine the rotation number and present two examples. Similar to the period of the bounded motion in Hamiltonian systems, the rotation number is the most fundamental property of integrable maps and it provides a way to analyze the phase-space dynamics.

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For a one degree-of-freedom time-independent system, the Hamiltonian function, $H(p,q,t) = E$, is the integral of the motion. If the motion is bounded, it is also periodic and the period of oscillations can be determined by integrating

$$T = \oint \left( \frac{\partial H}{\partial p} \right)^{-1} dq,$$

where $p = p(E,q)$. Similarly, a map $(q',p') = M(q,p)$ in the plane is called integrable, if there is a non-constant real-valued continuous function $K(q,p)$, which is invariant under $M$. The function $K(q,p)$ is called integral. In this paper, we are describing the case, for which the level sets $K = \text{const}$ are compact closed curves (or sets of points) and for which the identity

$$K(q',p') = K(q,p)$$

holds for all $(q,p)$. There are many examples of integrable mappings, including the famous McMillan mapping \cite{1}, described below. The dynamics is in many ways similar to that of a continuous system, however, Eq. (1) is not directly applicable since the integral $K(q,p)$ is not the Hamiltonian function.

The Arnold-Liouville theorem for maps \cite{2,3} states that in action-angle variables, consecutive iterations of map $M$ lie on nested circles of radius $J$ and that the map can be written in the form of a Twist map

$$J_{n+1} = J_n, \quad \theta_{n+1} = \theta_n + 2\pi \nu(J) \mod 2\pi,$$

where $|\nu(J)| \leq 0.5$ is the rotation number, $\theta$ is the angle variable and $J$ is the action variable, defined by the mapping $M$ as

$$J = \frac{1}{2\pi} \oint p \, dq.$$

For integrable mappings, $K(q,p) = K(J)$ is a function of the action variable. In what follows, we present a simple analytical expression to calculate the rotation number, $\nu(K)$, without constructing an action-angle transformation. This is useful, when, for example, the action variable (4) is not known explicitly but an integral $K(q,p)$ is.

**Theorem (Danilov):**

$$\nu(K) = \int_q^{q'} \left( \frac{\partial K}{\partial p} \right)^{-1} dq \div \oint \left( \frac{\partial K}{\partial p} \right)^{-1} dq,$$

where both integrals are taken along the invariant curve, $K(q,p)$.

**Proof:** Consider the following system of differential equations:

$$\frac{dq}{dt} = \frac{\partial K}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial K}{\partial q},$$

such that $K(q,p)$ does not change along a solution of the system. Define a new map, $\tilde{M}(q,p)$ (see Fig. 1)

$$(q',p') = \tilde{M}(q,p) = (q(\tau), p(\tau))$$

with

$$q = q(0) \quad \text{and} \quad p = p(0),$$

where $q(t)$ and $p(t)$ are the solutions of the system \cite{4} and $\tau$ is the discrete time step. For a given value of $K$,
which is an integral of both $M$ and $\tilde{M}$, and is identical to map $M$.

section of the continuous flow of the system (red curve) which is identical to map $M$.

$\tau$ is defined as

\[ \tau = \nu(K)T(K). \]

Let us now calculate $\nu(K)$:

\[ \nu(K) = \frac{\tau}{T} = \frac{\int_0^T dt}{\int_0^T dq} = \frac{\int_0^T \left( \frac{dq}{dt} \right)^{-1} dt}{\int_0^T \left( \frac{dq}{dt} \right)^{-1} dq} = \frac{\int_0^T \left( \frac{dq}{dp} \right)^{-1} dt}{\int_0^T \left( \frac{dq}{dp} \right)^{-1} dq}. \]

Q.E.D.

In order to employ the Danilov theorem, we will use the following parametrization:

\[ a - d = 2\alpha \sin(2\pi \nu), \]
\[ b = \beta \sin(2\pi \nu), \]
\[ c = -\gamma \sin(2\pi \nu). \]

The symplecticity condition gives $\beta \gamma - \alpha^2 = 1$. With this parametrization, an integral of mapping (11) can be written as

\[ K = \gamma q^2 + 2\alpha qp + \beta p^2. \]

To calculate the rotation number, we first express $p$ through $K$ and $q$:

\[ p = -\alpha q + \sqrt{\alpha^2 q^2 - \beta(\gamma q^2 - K)} / \beta = -\alpha q + \sqrt{\beta K - q^2}. \]

Now \[ \left( \frac{\partial K}{\partial p} \right)^{-1} = \frac{1}{2(\alpha q + \beta p)} = \pm 1 / 2\sqrt{\beta K - q^2}. \]

We will use \[ (q, p) = (0, \sqrt{K/\beta}) \] and

\[ (q', p') = (b \sqrt{K/\beta}, d \sqrt{K/\beta}) \]

to evaluate the integral in the numerator:

\[ \int_0^b \sqrt{\beta K - q^2} \frac{dq}{2\sqrt{\beta K - q^2}} = \int_0^{k/\beta} \frac{dx}{2\sqrt{1 - x^2}} = \frac{\pi}{2} \arcsin \frac{b}{\beta} = \pi \nu. \]

As our second example, we will consider the so-called McMillan map [1],

\[ \begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \]

with $ad - bc = 1$ and $|a + d| \leq 2$. This mapping is very common in accelerator physics and has been described in [1]. The rotation number for this mapping is well known:

\[ \nu = \frac{1}{2\pi} \arccos \frac{a + d}{2}. \]
while at large amplitudes, the rotation number becomes 0.25. Again, we first express $p$ through $\mathcal{K}$ and $q$ and evaluate the integrand in (5):

$$\left( \frac{\partial \mathcal{K}}{\partial p} \right)^{-1} = \frac{1}{2p(q^2 + \Gamma)} - 2\epsilon q = \frac{\pm 1}{2\sqrt{-\Gamma q^4 + \delta q^2 + \lambda \mathcal{K}}}.$$  

(24)

where $\delta(\mathcal{K}) = \epsilon - \Gamma^2 + \mathcal{K}$. Let us define a parameter,

$$k(\mathcal{K}) = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1 + \frac{\delta}{\sqrt{\delta^2 + 4\mathcal{K}\Gamma^2}}},}$$  

(25)

which spans from 0 to 1. Also, define $k' = \sqrt{1 - k^2}$. Then, the rotation number can be expressed through Jacobi elliptic functions as follows:

$$\nu(\mathcal{K}) = \frac{1}{4K(k)} \text{arcds} \left( \sqrt{\frac{k k' \Gamma}{\sqrt{\mathcal{K}}}}, k \right),$$  

(26)

where $K(k)$ is the complete elliptic integral of the first kind and the inverse Jacobi function, $\text{arcds}(x, k)$, is defined as follows

$$\text{arcds}(x, k) = \int_x^{\infty} \frac{dt}{\sqrt{(t^2 + k^2)(t^2 - k^2)}}.$$  

(27)

Figure 2 shows an example of the rotation number, for the case of $\epsilon = 0.8$ and $\Gamma = 1$ ($\nu(0) \approx 0.102$), as a function of integral, $\mathcal{K}$. These two examples demonstrate that the Danilov theorem is a powerful tool. The McMillan map is a classic example of a nonlinear integrable discrete-time system. It is a typical member of a wide class of area-preserving transformations called a Twist map [5]. In this Letter we demonstrated a general and exact method on how to find a Poincaré rotation number. It complements the discrete Arnold-Liouville theorem for maps [2, 3] and permits the analysis of the system dynamics. In conclusion, we would like to point out that for cases when the integral $\mathcal{K}$ is also known as a function of action, $J$, one would be able to express the rotation number as a function of action, as well as the Hamilton’s function, $H(J)$, for mapping since

$$\nu(J) = \frac{dH}{dJ}.$$  

(28)

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