Crossing the superfluid-supersolid transition of an elongated dipolar condensate

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We provide a theoretical characterization of the dynamical crossing of the superfluid-supersolid phase transition for a dipolar condensate confined in an elongated trap, as observed in the recent experiment by G. Biagioni et al. [Phys. Rev. X 12, 021019 (2022)]. By means of the extended Gross-Pitaevskii theory, which includes the Lee-Huang-Yang quantum fluctuation correction, we first analyze the ground state configurations of the system as a function of the interparticle scattering length, for both trap configurations employed in the experiment. Then, we discuss the effects of the ramp velocity, by which the scattering length is tuned across the transition, on the collective excitations of the system in both the superfluid and supersolid phases. We find that, when the transverse confinement is sufficiently strong and the transition has a smooth (continuous) character, the system essentially displays a (quasi) 1D behavior, its excitation dynamics being dominated by the axial breathing modes. Instead, for shallower transverse trapping, when the transition becomes discontinuous, the collective excitations of the supersolid display a coupling with the transverse modes, signalling the onset of a dimensional crossover.

I. INTRODUCTION

Supersolids are an exotic phase of matter combining superfluid properties (phase coherence and frictionless flow [1, 2]) with the translational symmetry breaking that characterizes crystalline structures [3–5]. Firstly predicted in the 50s [6, 7], supersolids have gained the interest of the scientific community as a consequence of their recent experimental realizations in dipolar condensates [8–16] and in other ultracold atomic systems [17–19]. In particular, Ref. [20] has recently reported the experimental investigation of the superfluid-supersolid quantum phase transition in an elongated dipolar condensate, driven by tuning the interparticle interactions (by means of Feshbach resonances). Remarkably, it has been shown that the character of the transition can be changed from continuous to discontinuous simply by tuning the transverse confinement (or the atom number), therefore providing a dimensional crossover between second-order transitions in 1D [21, 22] and first-order transitions in 2D [8, 11, 23–26].

In the present work we provide a complementary theoretical characterization of the equilibrium and dynamical properties of the dipolar condensate in the two trap configurations, $V_C$ and $V_D$, employed in the experiment of Ref. [20]. This analysis is carried out within the standard framework of the extended Gross-Pitaevskii theory [27], including both contact and dipolar interactions [28], as well as the Lee-Huang-Yang quantum correction [29]. We first consider the equilibrium properties of this system and we show that the transition characterized by a tighter transverse confinement ($V_C$) presents a smooth/continuous transition between the superfluid and supersolid phases, whereas for shallower trapping potentials ($V_D$) a discontinuous character clearly shows up. Then, we thoroughly discuss how the velocity of the ramp employed to experimentally tune the $s$–wave scattering length across the transition affects the dynamical response of the system, commenting also on the role of the formation time of the supersolid. Remarkably, we find that – when the system enters the supersolid phase – the collective modes in the two traps present distinctive behaviors: in the trap $V_C$, where the system can be considered effectively quasi 1D, the excitation dynamics is dominated by the axial breathing modes; instead, in the trap $V_D$, the axial excitations of the supersolid display a clear coupling to the collective transverse modes. This provides a signature of the onset of a dimensional crossover, in agreement with the discussion in Ref. [20].

The paper is organized as follows. In Section II we introduce the system parameters and the general framework of the extended Gross-Pitaevskii theory for dipolar condensates. Then, in Section III we analyze the equilibrium properties of the condensate in the two trap configurations, $V_C$ and $V_D$, and we characterize the corresponding superfluid-supersolid transition as a function of the $s$-wave scattering length. Section IV is instead devoted to the dynamical crossing of the transition. We first address, in Sec. IV A, the effect of the ramp velocity and then, in Sec. IV B we discuss how the formation time of the supersolid affects the crossing of the transition. Finally, in Sec. IV C we examine the collective oscillation of the system that arise due to the excess of energy acquired during the ramp across the transition, in both the supersolid and superfluid phases. Concluding remarks are drawn in Section V.
II. SYSTEM

We consider the typical experimental configuration of Ref. [20]. A dipolar condensate composed by \( N = 3 \times 10^4 \) magnetic atoms of \(^{162}\)Dy – with tunable s-wave scattering length \( a_s \) and dipolar length \( a_{dd} = 130 a_0 \) (\( a_0 \) being the Bohr radius) – is trapped by a harmonic potential with frequencies \( \omega = 2 \pi \times (\nu_x, \nu_y, \nu_z) \). As in the experiment, we consider two different trap configurations, namely \( \omega_C = 2 \pi \times (15, 101, 94) \) Hz, and \( \omega_D = 2 \pi \times (20, 67, 102) \) Hz, where the labels \( C/D \) refer to the continuous/discontinuous character of the transition (see Sec. III), in line with the notations employed in Ref. [20]. Accordingly, we indicate the corresponding harmonic potentials as \( V_C \) and \( V_D \).

This system can be described in terms of a generalized Gross-Pitaevskii (GP) theory including dipolar interactions [28] and the Lee-Huang-Yang correction accounting for quantum fluctuations (within the local density approximation) [29]. The energy functional can be written as

\[
E = E_{GP} + E_{dd} + E_{LHY}
\]

with

\[
E_{GP} = \int \left[ \frac{\hbar^2}{2m} \nabla \psi(r) \right]^2 + V_{C/D}(r)n(r) + \frac{g}{2} n^2(r) \, dr,
\]

\[
E_{dd} = C_{dd} \int \int n(r)V_{dd}(r - r')n(r')d^3r d^3r',
\]

\[
E_{LHY} = g_{LHY} \int n^{5/2}(r)dr,
\]

where \( E_{GP} = E_k + E_{ho} + E_{int} \) is the standard GP energy functional including the kinetic, potential, and contact interaction terms, \( V(r) = (m/2) \sum_{\alpha=x,y,z} \omega_{\alpha}^2 r_{\alpha}^2 \) is the harmonic trapping potential, \( n(r) = |\psi(r)|^2 \) represents the condensate density (normalized to the total number of atoms \( N \)), \( g = 4\pi \hbar^2 a_s / m \) is the contact interaction strength, \( V_{dd}(r) = (1 - 3 \cos^2 \theta) / (4r^3) \) the (bare) dipole-dipole potential, \( C_{dd} \equiv \mu_0 \mu^2 \) its strength, \( \mu \) the modulus of the dipole moment \( \mu \), and \( r \) the distance between the dipoles, and \( \theta \) the angle between the vector \( r \) and the dipole axis, \( \cos \theta = \mu \cdot r / (\mu r) \). As in Ref. [20] we consider the magnetic dipoles to be aligned along the \( z \) direction by a magnetic field \( B \).

Finally, the LHY correction is obtained from the expression for homogeneous 3D dipolar condensates under the local-density approximation [29, 30]. The LHY coefficient is \( g_{LHY} = 256\sqrt{\pi} \hbar^2 a_s^{3/2} / (15m) \left( 1 + 3 c_{dd}^2 / 2 \right) \), with \( c_{dd} = \mu_0 \mu^2 N / (3g) \).

III. GROUND STATE

We compute the ground state of the system by minimizing numerically the energy functional \( E[\psi] \) by means of a conjugate algorithm, see, e.g., Refs. [31, 32]. In the numerical code the double integral appearing in Eq. (1) is mapped into Fourier space where it can be conveniently computed by means of fast Fourier transform (FFT) algorithms, after regularization [28, 33]. The behavior of the ground-state energy for the two potentials is shown in Fig. 1 as a function of the s-wave scattering length \( a_s \), in the range \( a_s \in [90, 96]a_0 \), along with some representative images of the density distributions in the supersolid (SS) and superfluid (SF) phases [34]. Notice that in the supersolid phase both traps \( V_C \) and \( V_D \) exhibit two almost degenerate configurations, characterized by either a maximum or a minimum at the center of the trap [35]. These two states are those that – for symmetry reasons – survive in the presence of the trap among the infinite equivalent configuration that would be possible in a (infinite) uniform system [36].

Let us start by considering the case of the potential \( V_C \) (characterized by the tighter confinement along the \( y \) direction), shown in the left panel of Fig. 1. For this trap, the transition takes place at \( a_s^c \approx 94.4a_0 \) and it exhibits a continuous behavior: the superfluid and the supersolid states morph continuously one into the other,
as well as their energies [see the inset of Fig. 1 (left)]. The critical point can be identified, for instance, by the slope change in the first derivative of the energy with respect to the $s$-wave scattering length, see Fig. 2. The continuity of $\partial E/\partial a_s$ at the critical point confirms the continuous character of the transition in this case (within the numerical precision).

In the case of the potential $V_D$, which is characterized by a weaker transverse confinement, the transition takes place at lower value of the scattering length, namely $a_s^c \simeq 92.4a_0$. Remarkably, in this case the two SS states manifest a different behavior in the vicinity of the transition, see the inset of Fig. 1b. In particular, the configuration with a maximum at the trap center is the one with lower energy at the boundary with the SF phase, so that $a_s^c$ is actually defined by the crossing of the energy of such a state with that of the SF state. The transition clearly exhibits a discontinuous jump in the first derivative of the energy with respect to the $s$-wave scattering length, see Fig. 2. As discussed in Ref. [20], this discontinuous behavior of the SF–SS transition is reminiscent of that expected for trapped supersolids with 2D lattice structures [37–40], and it is due to the fact that even in the case of a single row supersolid the background density may exhibit a triangular structure. Even if not visible in the snapshots in Fig. 1, this structure is enhanced when the system is out-of-equilibrium. Indeed, a clear 2D modulation of the background density can be observed during the dynamics of the supersolid discussed in the following section.

**IV. DYNAMICAL STUDY OF THE TRANSITION**

We now turn to the dynamical study of the phase transition, following a protocol similar to the one employed in the experiment of Ref. [20]: the system is initially prepared in a stationary superfluid or supersolid state, at a certain scattering length $a_s^i$, and then the value of $a_s$ is tuned along a linear ramp with constant velocity $\dot{a}_s = v_a$ towards a final value $a_s^f$ in the other phase. The ramp scheme and the simulation timing are shown in Fig. 3. For conceptual clarity, here we consider $a_s^i$ and $a_s^f$ to be in specular position with respect to the critical point $a_s^c$, namely $a_s^{i/f} = a_s^c \pm \delta a_s$. In the following, we shall consider $\delta a_s = 1.5a_0$ and three ramps with different velocities: i) a “quench”, $v_a = \infty$; ii) $v_a = 0.5a_0/\text{ms}$, a lower velocity that allows for a quasi adiabatic crossing of the supersolid-supersuperfluid transition in the trap $V_C$ (see Sec. IV A), as discussed in Ref. [20]. This latter choice permits to reproduce a scenario similar to that of the above mentioned experiment, without having to introduce dissipation effects in the theoretical modeling (which are instead present in the experiment [20]) [41].

Therefore, in the following we shall restrict the discussion to the dissipationless scenario, obtained by solving the GP equation [2]

$$i\hbar \partial_t \psi = \frac{\delta E[\psi, \psi^*]}{\delta \psi^*},$$

(2)

where the energy functional $E[\psi, \psi^*]$ is the one in Eq. (1) [42]. Regarding the two supersolid configurations discussed in the previous section, we notice that during the dynamics across the SF-SS transition the system is likely to select spontaneously the configuration with a maximum at the center of the trap, so that we have used such a configuration also for the initial state of the ramp in the opposite direction, for the sake of simplicity.

**FIG. 2.** Derivative of the total energy of the ground state for the two trap configurations, $V_C$ and $V_D$. The latter exhibits a discontinuity (vertical dashed line) at the transition point (red dots).

**FIG. 3.** Scheme of the ramp employed in the numerical simulations: the system is prepared in the ground state either in the superfluid or in the supersolid phase, at $a_s^i = a_s^c \pm \delta a_s$, and then the scattering length is varied along a linear ramp – during a time $t_R \equiv 2\delta a_s/v_a$ – towards a final value in other phase, $a_s^f = a_s^c \mp \delta a_s$. Then, the system is kept at the final value $a_s^f$ for a variable time $T$. The time spent in the SS/SF phase after crossing the critical value of the scattering length is indicated as $\tau$ (see Sec. IV B).
end of each ramp, namely at \( t = t_R \), are shown in panels (a-c) for the SF-SS transition and in (d-f) for the SS-SF transition.

It is interesting to notice that, when crossing the transition in the downward direction, from SF to SS, both the energy variation and the final density distribution are weakly affected by the ramp velocity. In the case of a quench, this has to be so because the system is “projected” instantaneously in the other quantum phase without changing its density distribution, so that in this case the final configurations exactly coincide with the initial ones. The case at \( v_a = 0.5 a_0/\text{ms} \) turns out to be almost equivalent to a quench (contrarily to what happens during the SS–SF transition, see below). Only in the case of the slowest ramp at \( v_a = 0.05 a_0/\text{ms} \) in the trap \( V_D \) can a slight modulation superimposed to the initial state be appreciated. The origin of this behavior has to do with the formation time of the supersolid (see, e.g., Ref. [9]), that will be discussed in Sec. IV B.

In the opposite direction, when crossing the SS to SF transition, the behavior is quite different: the energy grows linearly if the scattering length is quenched, while it follows the ground state energy almost adiabatically if the scattering length is slowly varied (at \( v_a = 0.05 a_0/\text{ms} \)). In addition, it is evident both from the energy behavior and from the final configuration in Fig. 4f that for the trap \( V_C \) such a ramp is sufficiently slow to bring the SS state close to the SF ground-state, with a small excitation energy embedded in a density deformation that

A. Effect of the ramp

Let us now discuss how the system gets modified while varying the scattering length. In particular, we shall first consider how the different ramp velocities affect the energy of the system, and which is the final density distribution of the condensate at the end of each ramp (the dynamics following the end of the ramp will be discussed in Sec. IV C). This is shown in Figs. 4 and 5, for the traps \( V_C \) and \( V_D \), respectively. In the top panel we show the behavior of the energy of the system as a function of the scattering length \( a_s(t) \) along the three ramps across the SF-SS transition (blue lines) and in the opposite direction, from SS to SF (red lines). In the case of the quench, the line is simply a guide to the eye that connects the initial and final value of the scattering length. The insets represent the initial configurations in the SF and SS phases. The density distributions obtained at the
is reminiscent of the initial state. Remarkably, such a deformation is significantly larger in trap $V_D$ (compare Figs. 4f and 5f). Moreover, the residual energy on the superfluid side is larger in trap $V_D$ than in trap $V_C$ for each of the three ramps (see also the discussion in Ref. [20]). These observations are consistent with the continuous/discontinuous character of the transition in the two cases.

In order to get further insight on the behaviour of the total energy along the ramp it is convenient to rewrite the energy functional (1) in the following form, which makes the dependence on $a_s$ explicit:

$$E[\psi; a_s] = E_k[\psi] + E_{ho}[n] + a_s E^{int}[n] + E_{dd}[n]$$

$$+ a_s^{5/2} \left[ 1 + \frac{3}{2} \left( \frac{a_{dd}}{a_s} \right)^2 \right] E^{LHY}[n],$$

with $E^{int}[n]$ and $E^{LHY}[n]$ functionals that depend on the density only. Thus the dependence on the scattering length is explicit for the mean-field interaction energy and the LHY correction, whereas all the other terms depend on $a_s$ only indirectly, through the condensate density (with the exception of the kinetic term, that is sensitive also to the wave-function phase).

The behavior of the system during the ramp across the transition is therefore characterized by two time scales: the duration of the ramp, $t_R = 2\delta a_s/v_a$, and the timescale required for variations in the condensate density to appear. Remarkably, the latter strongly depends on the density only. The dependence on the scattering length is explicit for the mean-field interaction energy and the LHY correction, whereas all the other terms depend on $a_s$ only indirectly, through the condensate density (with the exception of the kinetic term, that is sensitive also to the wave-function phase).

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In the present case, since the phase transition mainly affects the density distribution along the axial direction of the trap, the major contribution comes from the so-called axial breathing mode [11]. The latter can be conve-
nently characterized by considering the width along the x-direction, which corresponds to the axial direction of the supersolid. In order to do so, we define a normalized relative width as

\[ w_\alpha(t) \equiv [\sigma_\alpha(t) - \sigma^{eq}_\alpha]/\sigma^{eq}_\alpha, \]  

(4)

where \( \sigma^2_\alpha(t) \equiv \langle x^2 \rangle - \langle x \rangle^2 = (1/N) \int x^2 \rho(r,t) dr \), \( \alpha \) indicates the SF and SS states, and \( \sigma^{eq}_\alpha \) the corresponding equilibrium widths at \( \Delta^*_0 \pm \delta a_s \), respectively. In the following, we shall consider especially the behavior of the width as a function of the time \( \tau \) elapsed from the crossing of the SF-SS transition, namely \( w_\alpha(\tau) \). This quantity is shown in the bottom panels of Figs. 6, 7 for the SS case, and in Fig. 8 for the SF case. Overall, the behavior of \( w_\alpha(\tau) \) provides an additional characterization of the (non)adiabaticity of the various ramps and of the character of the transition for the two trap configurations.

Let us first consider the SS case, in Figs. 6, 7. First of all, we notice that the ramp at \( v_a = 0.5 a_0/\text{ms} \), corresponding to the nominal value employed in the experimental protocol of Ref. [20], is almost indistinguishable from a quench. Instead, the slowest ramp at \( v_a = 0.05 a_0/\text{ms} \) presents a distinctive feature in the fact that the value of the width decreases gradually along the ramp, \( -30 \text{ ms} < \tau < 30 \text{ ms} \), indicating that the system is able to smoothly adjust its shape to the changing value of the scattering length, from \( a_t^0 \) to \( a_s^0 \).

As discussed in Ref. [11], the excitation dynamics of an elongated supersolid is characterized by a doubling of the axial breathing mode of a dipolar condensate (in the SF regime, see later on). The two modes that appear in the SS phase, for the trap \( V_C \), are associated to the onset of a transverse oscillation mode shown in the inset (see text). It is also worth noticing that in this trap the supersolid starts to form slightly earlier than as in Fig. 6, at \( \tau \approx 25-30 \text{ ms} \) (see the top panels).
similar result is obtained from the fit of the other two
lines. In the transverse directions we do not see any
significant oscillation, as one can see from the behavior
of transverse width along the $y$ direction shown in the
inset. In the figure we also show some snapshot of the
density distribution, at selected times: when the supersolid
structure starts to emerge clearly (top row, discussed pre-
viously), at $w_{SS}(\tau) = 0$ (middle row), and when $w_{SS}(\tau)$
first reaches a minimum of the oscillation (bottom row).
These snapshots well represent the qualitative behavior
along the whole dynamics considered here (also when $w_{SS}(\tau)$
gets to an oscillation maximum), which can be indeed
fully characterized by the deformation of the sup-
ersolid structure discussed previously, affecting the am-
plitude and the spacing along the axial direction.

Instead, the case of the trap $V_D$ presents a distinctive
behavior associated with the emergence of a characteris-
tic pattern in the background density distribution. This
is visible in the top panels of Fig. 7 (middle and bot-
tom rows). Remarkably, this pattern is reminiscent of the
triangular lattice structure expected for 2D supersol-
soids [37–40], see also the discussion in Ref. [20]. We
find that the corresponding transverse excitation mode
(shown in the inset) is characterized by a relatively high
frequency, $\nu_\perp \approx 60\text{Hz}$, which couples with the axial
breathing modes. This accounts for the “fast” oscillations
that are visible in the continuous line at $v_a = 0.05$
a$_0/\text{ms}$ in Fig. 7. As a matter of fact, a clean sinusoidal
fit (with two or even three frequencies) is not possible in
this case.

Finally, let us consider the excitation produced by the
ramps in the opposite direction, when the system
is driven into the SF phase. In this case we find that
the condensate oscillations are dominated by a single
excitation mode, namely the axial breathing mode of a
dipolar condensate discussed previously, see Fig. 8.
This holds for both trap configurations, $V_C$ and $V_D$.
The corresponding frequency is expected to be slightly
below the mean-field solution for the breathing mode
frequency of a superfluid without dipolar interactions,

$$\omega = \sqrt{5/2} \omega_c,$$

see again Ref. [11]. In particular, we find $(\nu/\nu_c)_C \approx 1.51$ ($\epsilon_{dd}^C = 1.35$) and $(\nu/\nu_c)_D \approx 1.47$
($\epsilon_{dd}^D = 1.37$).

V. CONCLUSIONS

We have presented a theoretical discussion – within
the framework of the extended Gross-Pitaevskii theory
including Lee-Huang-Yang quantum corrections – of the
superfluid-supersolid transition of an elongated dipolar
condensate as reported in the recent experiment by G.
Biagioni et al. [Phys. Rev. X 12, 021019 (2022)]. We
have considered both trapping potentials employed in the
experiment, providing a characterization of the equilib-
rium and dynamical properties of the system as a func-
tion of the inter-particle scattering length, which is the
parameter that is varied experimentally for driving the
transition. Although both traps display a one row sup-
ersolid (for $a_s < a_s^c$), already at the level of the ground
state the two traps present a distinctive behavior. For a
sufficiently strong transverse confinement (the trap $V_C$)
the SF-SS transition has a smooth continuous character,
as it is expected for (quasi) 1D systems, with the su-
perfluid and the supersolid states morphing continuously
one into the other, as well as their energies. Instead, in
the case of the potential $V_D$, which is characterized by
a weaker transverse confinement, the transition clearly
exhibits a discontinuous jump in the first derivative of the
energy with respect to the $s$-wave scattering length,
as it is expected for trapped supersolids with 2D lat-
tice structures [37–40]. These properties reflects in the
collective oscillations of the system, when the scatter-
ing length is dynamically ramped across the transition,
from one phase to the other. In particular, we find that
when the system is driven quasi adiabatically into the
superfluid phase the system performs clean axial breath-
ing oscillations, in both traps. In the opposite direction,
the situation is quite different: in the trap $V_C$ the ex-
citation dynamics is still dominated by the doubling of
the axial breathing modes, whereas when the transition becomes discontinuous, in the trap $V_p$, the collective excitations of the supersolid display a coupling with the transverse modes, signalling the onset of a dimensional crossover. These findings provide further insights on the transverse modes, signalling the onset of a dimensional crossover. These findings provide further insights on the transverse modes, signalling the onset of a dimensional crossover.

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[35] Numerically, these two states are obtained by using different initial trial wave functions, exploiting the fact that the conjugate gradient algorithm used in the minimization procedure is sensitive to the initial conditions.

[36] Without an external trapping potential, a change of the lattice phase costs zero energy, therefore in the infinite system the phase of the supersolid lattice is undefined. This is no longer the case in trapped systems, where among all the possible phases, only two minimize the cost in trapping energy. These two states corresponds to the two almost degenerate configurations, characterized by either a maximum or a minimum at the center of the trap.

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[43] A similar argument holds also in the SS phase once the supersolid has been formed.

[44] Notice that the formation time observed experimentally in Ref. [20] is shorter than the one we find from the present numerical analysis, likely due to finite temperature effects and three-body losses, see also Refs. [9, 48].

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