Design of Discrete-time Matrix All-Pass Filters Using Subspace Nevanlinna Pick Interpolation

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Abstract—Unitary matrix-valued functions of frequency are matrix all-pass systems, since they preserve the norm of the input vector signals. Typically, such systems are represented and analyzed using their unitary-matrix valued frequency domain characteristics, although obtaining rational realizations for matrix all-pass systems enables compact representations and efficient implementations. However, an approach to obtain matrix all-pass filters that satisfy phase constraints at certain frequencies was hitherto unknown. In this paper, we present an interpolation strategy to obtain a rational matrix-valued transfer function from frequency domain constraints for discrete-time matrix all-pass systems. Using an extension of the Subspace Nevanlinna Pick Interpolation Problem (SNIP), we design a construction for discrete-time matrix all-pass systems that satisfy the desired phase characteristics. An innovation that enables this is the extension of the SNIP to the boundary case to obtain efficient time-domain implementations of matrix all-pass filters as matrix linear constant coefficient difference equations, facilitated by a rational (realizable) matrix transfer function. We also show that the derivative of matrix phase constraints, related to the group delay at the interpolating points, can be optimized to control the all-pass transfer matrices at the unspecified frequencies. Simulations show that the proposed technique for unitary matrix filter design performs as well as traditional DFT based interpolation approaches, including Geodesic interpolation and the popular Givens rotation based matrix parameterization.

1 INTRODUCTION

Filtering signals is among the most fundamental operations in signal processing. In general, filtering scalar signals is well understood, and there is mature theory that discusses filter design and implementation for both analog and digital scalar filters. However, with the increased interest in multiple-input multiple-output systems in several allied areas, the concept of filtering vector signals has gained importance. Designing precise filters for vector signals (wherein the signal at each time instant is a real or complex vector) under various constraints is also interesting from the point of view of several practical applications, although there has not been much work in the past in this direction. In this paper, we focus on the design of discrete-time “matrix” all-pass filters, that transform a vector signal’s phase while ensuring that their norm is not altered for all frequencies. In particular, unlike the standard practice of using frequency domain transform techniques for filtering, we present an interpolation based filter design technique that produces matrix all-pass filters for practical realizability. This idea has several applications, such as combined left and right audio signals in case of stereo audio as well as for feedback in control and communication systems etc. As an example to show the effectiveness of the proposed techniques, we consider the MIMO precoders for wireless communication systems which employ orthogonal frequency division multiplexing (OFDM). These precoders can be accurately and efficiently realized using frequency domain techniques, as opposed to the traditionally used approaches [1, 2].

Matrix filtering with a norm preservation constraint is typically accomplished using frequency domain techniques [3, 4]. Specifically, this involves computing the Fourier transform of the signal, performing the all-pass filtering on a per-frequency basis, and using the inverse Fourier transform, as is common in the case of vector communication systems [4, 5]. However, when the matrix all-pass filter lends itself to a time domain realization, this method is not ideal. In particular, when the matrix all-pass filter has an efficient linear constant coefficient difference equation (LCCDE) realization, the filter realized using frequency domain techniques will be inaccurate, and will also result in less efficient realizations. To address this, we present an interpolation based matrix all-pass filter design technique that results in a realizable filter (that can be implemented in the time domain as an LCCDE) while satisfying the frequency domain constraints. Our approach extends the classical Subspace Nevanlinna Pick Interpolation (SNIP) method [6] that is well-known in the context of control systems to the “boundary” case to obtain matrix filters that satisfy some prior constraints, while ensuring that the Fourier transform of its system function is a unitary matrix at all frequencies, thus obtaining norm-preserving (matrix all-pass) filters.

The classical Nevanlinna interpolation problem has its roots in the problem of synthesis of dynamical systems as passive electrical networks (see [2]). This, as well as all the subsequent extensions of it, however, considers only the situation wherein the interpolating frequencies and the prescribed values of the desired transfer function are strictly within the respective critical regions. For example, in the scalar version of the SNIP dealt with in [6], the polar plot of the transfer function must lie strictly within the unit disk, and the frequencies that are given lie on the open right-half of the complex plane. It is important to note that the solution of the classical SNIP crucially depends on these strictness assumptions. In this paper, we push the SNIP to its boundary: we deal with the case wherein the desired transfer function’s polar plot is on the unit disk (i.e., all-pass), and the frequencies, too, are given on the boundary (the unit circle because we consider discrete time systems).

Our key contributions in this paper are as follows:

- We present an approach to realize a discrete-time matrix
all-pass filter, when given a feasible set of frequency responses (unitary matrices) and group delay matrices for a finite set of frequencies \( \{ \omega_i \} \). Specifically, our solution yields a rational matrix \( z \)-transform for the required all-pass filter that satisfies all the given frequency domain conditions, and its transfer function matrix is unitary valued for all \( \omega \in (-\pi, \pi] \). This can be viewed as a generalization of the Blaschke interpolation based approach that is specific to \textit{scalar} all-pass filter design [8] [9] to the matrix case.

- We obtain this \textit{matrix} all-pass filter by extending the SNIP technique to the boundary case. Specifically, since the Pick matrix in the case of standard SNIP [6] becomes ill-defined when we demand a unitary valued solution, we provide a modified approach using the modified Pick (Schwarz-Pick) matrix to generalize the SNIP filter realization to the boundary case in discrete-time setting.
- Finally, we also present an optimization based approach that tunes the slopes of the matrix phase response at specific frequencies to obtain realizable filters with desirable characteristics.

The proposed approach for filtering is both novel as well as efficient in terms of implementation. In particular, prior approaches to perform all-pass matrix filtering in the frequency domain have used DFT based techniques that involve at least \( N_{\text{FFT}} \) multiplications [10] [11]. In addition, these techniques have largely relied on frequency domain interpolation of precoders interpolation on manifolds [12] [13] or interpolation of parameterized unitary matrices [14] [15], a technique that is employed in recent wireless OFDM based standards as well [11]. However, as we show in this paper, in situations where the all-pass filter has an impulse response that can be characterized using fewer coefficients, significant savings in terms of computations can be realized using the SNIP based approach, while faithfully capturing the frequency domain precoder characteristics.

The rest of the paper is organized as follows: Section 2 outlines the importance of matrix all-pass filter design problem statement. Section 3 briefly describes the classical SNIP. Section 4 describes the discrete-time matrix all-pass filter design problem and its solution. Section 5 contains the simulations results and interpretations for some practical purposes. Section 6 provides some concluding remarks and discusses future directions.

\textit{Notation:} Unless otherwise specified, bold capital symbols refer to matrices, bold smallcase symbols correspond to vectors, \( I_m \) refers to an \( m \times m \) identity matrix and \( 0_{m \times m} \) refers to an \( m \times m \) all-zero matrix. We also use the abbreviations CT for continuous-time and DT for discrete-time. For any matrix \( A \), \( A^* \) is the conjugate transpose of \( A \).

## 2 Motivation

To motivate this problem, we first pose the \textit{scalar} all-pass filter problem: if the frequency response of a discrete-time all-pass filter is given for certain (finitely many) frequencies, how can we obtain an all-pass filter that satisfies these constraints? In general, there exist an infinite number of filters that satisfy these constraints. Recent work has shown that, if the group delays are also known at the given frequencies, then a realizable all-pass filter can be obtained as a Blaschke product [8]. However, the Blaschke product based approach is only suited to the solution of the discrete-time \textit{scalar} all-pass filter design problem, and a direct extension of the same approach to the case of matrix all-pass filters is not known.

Discrete-time all-pass filters are used for phase compensation in various applications. A scalar all-pass filter can be used to correct phase distortions in scalar signals. To the best of our knowledge, this concept is yet to be extended to vector signals (MIMO systems), wherein the input and output are complex vector signals, and the transformation filter is a matrix valued all-pass filter (unitary). These matrix all-pass filters are commonly encountered in several situations, such as MIMO-OFDM systems and stereo audio systems. In MIMO-OFDM communication systems, when symbols are precoded with unitary matrices at the transmitter, if the transmitter possesses some channel state information (CSI), unitary matrix precoding is typically performed using the right singular vectors (or related unitary matrices) that are obtained from the singular value decomposition (SVD) of the channel matrix for every subcarrier (frequency band) [5]. Multiplying with a unitary matrix in the frequency domain is norm preserving, and thus, it can be thought of as a matrix all-pass filtering operation. In practice, having norm preserving matrix filters is important in order to satisfy various constraints, such as power in communication systems, or volume in the case of audio signals, while only altering the matrix-valued “phase”.

Performing such a phase transformation using the coefficients of an appropriate discrete-time matrix all-pass filter with a standard LCCDE implementation would obviate the need for frequency domain processing, and result in a more faithful realization, and can significantly reduce the precoding complexity in modern systems, such as those that use precoding for millimeter wave wireless systems [16] [17].

The block diagram shown in Fig.1 depicts an unknown matrix all pass filter with efficient LCCDE representation, with \( x[n] \) as input and \( y[n] \) as output. Our goal is to construct a system that mimics the unknown matrix all-pass filter. One approach is to use DFT based techniques, wherein output \( y_1 [n] \) is produced for the input \( x[n] \). Another method is to follow a matrix filter design technique and construct a rational matrix all pass filter with LCCDE representation, and this yields output \( y_2 [n] \) for the input \( x[n] \). In this situation only the latter method is accurate.

It is evident from the block diagram Fig.1 that when the matrix all-pass filter has an efficient LCCDE realization, the filter realized using frequency domain techniques may be inaccurate, and may also result in less efficient realizations. An advantage of using the SNIP based LCCDE filter design is that the computational complexity can be reduced with a rational (realizable) all-pass filter method that has a compact representation, being more amenable to time-domain LCCDE implementations (details of the same are discussed further in
where \( \omega \) wherein the solution is unitary valued for all frequencies (this issue and prove that we obtain a filter that satisfies the Pick matrix. We then present our modified SNIP that addresses issues in the context of the unitary matrix valued (all-pass) for continuous-time filter interpolation along with its limitations.

In the subsequent sections, we first discuss the SNIP approach and thus, SNIP is not directly applicable in our case. We, thus, present a modified SNIP that accommodates the boundary case to address our needs.

In the subsequent sections, we first discuss the SNIP approach for continuous-time filter interpolation along with its limitations in the context of the unitary matrix valued (all-pass) constraint (boundary SNIP interpolation) due to the ill-defined Pick matrix. We then present our modified SNIP that addresses this issue and prove that we obtain a filter that satisfies the required conditions.

### 3 Subspace Nevanlinna Pick Interpolation

We now briefly outline the SNIP approach to perform filter interpolation that is commonly used in the context of continuous-time control systems to obtain filters that satisfy frequency domain constraints [6]. We restrict our consideration to square matrices, since our eventual focus would be on square unitary matrix valued frequency responses.

A contractive subspace \( (V_i \subset \mathbb{C}^{2m}) \) is a subspace that satisfies the following property:

\[
\left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V_i, \text{ with } v_1, v_2 \in \mathbb{C}^{m} \text{ and } v \neq 0 \right\} = \{ \|v_1\|_2 > \|v_2\|_2 \}.
\]

We consider \( N \) distinct points \( \lambda_i \) in the open right-half complex plane, given together with \( N \) subspaces \( V_i \subset \mathbb{C}^{2m} \), where \( 1 \leq i \leq N \).

The statement of the classical SNIP is as follows: Given the \( N \) pairs \( (\lambda_i, \gamma_i) \), find an \( m \times m \) polynomial matrix \( U \) and a non-singular \( m \times m \) polynomial matrix \( Y \) such that

- \( U \) and \( Y \) are left co-prime;
- \( (U(\lambda_i) - Y(\lambda_i))v = 0 \quad \forall \ v \in V_i, 1 \leq i \leq N; \)
- \( \|Y^{-1}U\|_{\infty} < 1 \).

We assume that a full column rank matrix \( V_i \in \mathbb{C}^{2m \times m} \) is given such that \( \text{Im}(V_i) = V_i \) for every \( i \in \{1, 2, \ldots, N\} \). The Pick matrix \( P \) for the given data, for all \( j, k \in \{1, 2, \ldots, N\} \), is defined as

\[
P[(k - 1)m + 1 : km, (j - 1)m + 1 : jm] := \left( V_k^* J V_j \right)_{\lambda_k^* + \lambda_j}
\]

where \( J := \begin{bmatrix} I_m & 0_{m \times m} \\ 0_{m \times m} & -I_m \end{bmatrix} \).

(1)

Here, \( J \) is called a signature matrix, and \( P[a : b, c : d] \) is a submatrix of \( P \) obtained from contiguous rows, ranging from the \( a \)th row to the \( b \)th row, and contiguous columns ranging from the \( c \)th column to the \( d \)th column.

**Proposition 1.** There exists a solution to the Subspace Nevanlinna Interpolation Problem (SNIP) discussed above if and only if the Hermitian matrix \( P \) is positive definite.

This result is established in Theorem 4.1 of [6]. One key disadvantage with the SNIP is that the contractive subspace structure prevents it from being applicable to the case of matrix all-pass filter design (as detailed in the following sections). In Section 4 we adapt the SNIP framework to design matrix all-pass filters.

### 4 Modified SNIP for DT Matrix All-Pass Filters

In this section, we present an adaptation of SNIP to the discrete-time all-pass filter design case, wherein the filter constraints are presented for a sampled system. We remark that this case is directly applicable to several signal processing applications, such as stereo audio processing, MIMO-OFDM precoding etc.

**Note:** We refer to square matrices \( X \) and \( Y \) of size \( k \times k \) as unitarily similar if there exists an unitary \( k \times k \) matrix \( T \) such that \( Y = T^* X T \).

#### 4.1 Problem Statement

The discrete-time matrix all-pass filter design problem can be stated as follows: given data set \( \mathbb{D} = \{ (\omega_i, A_i, \Gamma_i) | i = 1, \ldots, n \} \), where \( \omega_i \in (-\pi, \pi), A_i \in \mathbb{C}^{m \times m} \) is unitary, and \( \Gamma_i \in \mathbb{C}^{m \times m} \) is a positive definite Hermitian matrix \( (\Gamma_i = \Gamma_i^*) \), we wish to obtain a rational transfer function matrix \( G(z) \in \mathbb{C}(z)^{m \times m} \) that satisfies the following conditions

\[
G(e^{j\omega_i}) = A_i \quad \forall \ i = 1, \ldots, n, \quad (2a)
\]

\[
G^*(e^{j\omega})G(e^{j\omega}) = I_m \quad \forall \ \omega \in (-\pi, \pi), \quad (2b)
\]

if \( F_i = jG^*(e^{j\omega_i}) \frac{dG(e^{j\omega_i})}{d\omega} \) then

\[
F_i \text{ and } \Gamma_i \text{ are unitarily similar matrices } \forall \ i = 1, \ldots, n \quad (2c)
\]
The condition described by equation (25) provides us an all-pass filter transfer function, that also matches the desired 
(unitary) frequency responses $A_i$, at the frequencies $\omega_i$ as constrained by equation (25). Specifying only these two 
constraints leads to infinite possible all-pass filters (as in the scalar case [8]). Therefore, to restrict the set of possible solutions, 
we further specify $\Gamma_i$ that are positive definite matrices and constrain the matrices $F_i$ at $\omega_i$ (equation (26)). The matrices $F_i$ 
correspond to the derivative of the matrix-valued phase and thus, are referred to as group delay matrices [13]. The 
group delay matrices $\{F_i\}$ are Hermitian $\forall i \in \{1,2,...n\}$. This is a direct consequence of the unitary constraint imposed in (25), 
and can be verified by differentiating (25) with respect to $\omega$:

$$G^*(e^{j\omega}) \frac{dG(e^{j\omega})}{d\omega} + \frac{dG^*(e^{j\omega})}{d\omega} G(e^{j\omega}) = 0_{m \times m}$$

$$\Rightarrow jG^*(e^{j\omega}) \frac{dG(e^{j\omega})}{d\omega} - j\frac{dG^*(e^{j\omega})}{d\omega} G(e^{j\omega}) = 0_{m \times m}.$$ 

It is evident from the above problem statement that the SNIP [6] is very similar to our problem, but the SNIP does not 
require any slope or group delay constraints (like (26) in our problem statement). Moreover, in the SNIP, the norm of the output vector is strictly smaller than the norm of the input vector, whereas in our case, they are equal.

Thus, our current all-pass filter design problem can be considered as an extension of the SNIP problem to the boundary 
case, wherein the norm of the output vector is exactly equal to the norm of the input vector. This requirement prevents 
the SNIP from being directly applicable to our problem, and motivates the formulation of a modified version that can be 
used for boundary problems as well.

4.2 Formulating the modified Pick matrix

To enable a step-wise solution to the matrix all-pass interpolation problem, we suitably modify the data set $\mathcal{D}$ to facilitate 
an inductive solution in subsequent steps. The data set $\mathcal{D}$ is altered by replacing $A_i \in \mathbb{C}^{m \times m}$ with $B_i \in \mathbb{C}^{2m \times m}$ 
that satisfy

$$\text{span}_\mathbb{C} B_i = \text{span}_\mathbb{C} [I_m \ A_i^T]^T \quad \forall \ i = 1, \ldots, n.$$ 

Now, in order to retain the unitary nature of $A_i$, we define neutral $B_i$ that satisfy

$$B_i^* J B_i = 0_{m \times m} \quad \text{where} \quad J = \begin{bmatrix} I_m & 0_{m \times m} \\ 0_{m \times m} & -I_m \end{bmatrix}. \quad (3)$$

The modified form of our data set is given by:

$$\mathcal{D} = \{ (\omega_i, B_i, \Gamma_i) | i = 1, \ldots, n \}, \quad \text{where} \quad \omega_i \in (-\pi, \pi], \ B_i \in \mathbb{C}^{2m \times m} \text{ is neutral}, \ \Gamma_i = \Gamma_i^* \in \mathbb{C}^{m \times m} \text{ is positive definite.}$$

We now set out to obtain the rational transfer function matrix $G(z) \in \mathbb{C}(z)^{m \times m}$ that satisfies the constraint equation (2), 
with $A_i$ replaced with $B_{i,1,2} B_{i,1}^{-1}$, where,

$$B_i = [B_{i,1}^T \ B_{i,2}^T]^T \quad \text{and} \quad B_{i,1,2} \in \mathbb{C}^{m \times m} \quad \forall i \in \{1,2,\ldots,n\}.$$ 

We cannot use the classical SNIP solution construction from [6] directly by substituting $B_i$ and $e^{j\omega_i}$ in place of $V_i$ 
and $\lambda_i$ respectively in the definition of the Pick matrix block $[1]$ and matrix $J$ defined in (4). This is because doing so results in 
a $\mathcal{0} \mathcal{0}$ form along the diagonal blocks of the Pick matrix due to unitary nature of matrix $A_i$:

$$P((i-1)m + 1 : im, (i-1)m + 1 : im) = \frac{B_i^* J B_i}{1 - e^{j(\omega_i - \omega_j)}} = \frac{I_m - A_i^* A_i}{\omega_i - \omega_j \mathcal{0} \mathcal{0}}.$$ 

Therefore, we modify the Pick matrix to replace the the ill-defined matrices forming the diagonal blocks, and establish a 
set of necessary and sufficient conditions on the modified Pick matrix for solvability of the DT matrix all-pass filter design problem.

The modified $nm \times nm$ Pick matrix $P$ is defined as follows:

$$P((i-1)m + 1 : im, (k-1)m + 1 : km) := \begin{cases} I_m & \text{if} \ i = k \\
B_i, B_i^* J B_i & \text{otherwise} \end{cases}, \quad (4)$$ 

for all $i, k \in \{1, \ldots, n\}$. We now state a theorem that guarantees a solution for our problem using this Pick matrix.

Theorem 1. Given the data set $\mathcal{D} = \{ (\omega_i, A_i, \Gamma_i) | i = 1, \ldots, n \}$ with $\omega_i$ distinct, a rational transfer function matrix $G(z) \in \mathbb{C}(z)^{m \times m}$ that satisfies (2), (9) and (10) exists if and only if the Pick matrix defined in (4) is positive definite.

Proof. Explicit construction of a solution is specified in Section 4.3 to prove the if part. To prove the only if part, we show that, for a given data set $\mathcal{D} = \{ (\omega_i, A_i, \Gamma_i) | i = 1, \ldots, n \}$ if we have $G(z) \in \mathbb{C}(z)^{m \times m}$ that satisfies (9), (10) and (10) then the Pick matrix for $\mathcal{D}$ defined in (4) is positive definite.

Since our interpolation problem concerns itself with the discrete-time case, both the input ($x[n]$) and output ($y[n]$) to the 
matrix all-pass filter are discrete-time vector signals. For simplicity, we represent $x[n]$ as $x$ and $y[n]$ as $y$, suppressing the 
time index. In the language of the theory of dissipative systems, the all-pass transfer function represents a lossless DT 
dynamical system with the supply rate defined as $x^*y - y^*y$. From the fundamental theorem of dissipative systems, it 
follows that lossless dissipative systems satisfy the following equation [19,21]. That is, 

$$\text{supply rate}[n] = \text{storage function}[n + 1] - \text{storage function}[n] \quad (5)$$ 

where $\{\text{supply rate}[n]\}$ is the supply rate value computed at time index $n$ and similarly $\{\text{storage function}[n]\}$ is the storage 
function value computed at time index $n$.

Therefore, since the storage function is a positive definite function of state variables [21], using the concepts of discrete-
time lossless dissipative dynamical systems explained in [20], we observe that

$$\sum_{n=-\infty}^{0} (x^*x - y^*y) \geq 0 \quad (6)$$ 

Now, consider $x = v e^{(\epsilon_i + j\omega_i)n}$, where $v$ is an arbitrary vector in $\mathbb{C}^m$, $\epsilon_i$ is an arbitrary small positive real number, 
and $\omega_i$ is the $i$-th interpolation frequency. The input to the system is $x$, and hence, the output is $y = Gx$. Thus, we
get \( y = G(e^{\epsilon_1 j\omega_1})y e^{(\epsilon_1 j\omega_1)n} \). Let \( G_i := G(e^{\epsilon_1 j\omega_1}) \). Now, substituting \( x, y \) in (6), we get
\[
\begin{align*}
0 \sum_{n=-\infty}^{0} (x^* x - y^* y) &= v^* (I_m - G_i^* G_i) v \sum_{n=-\infty}^{0} (e^{2\epsilon_1 n}) \\
&= v^* (I_m - G_i^* G_i) v \frac{1}{1 - e^{-2\epsilon_1}}.
\end{align*}
\]
Taking limit \( \epsilon_1 \to 0^+ \) in the above equation, we get
\[
\lim_{\epsilon_1 \to 0^+} v^* (I_m - G_i^* G_i) v \geq 0.
\]
Now, applying L'Hospital's rule yields
\[
\lim_{\epsilon_1 \to 0^+} v^* (I_m - G_i^* G_i) v = \lim_{\epsilon_1 \to 0^+} \frac{d}{d\epsilon_1} v^* (I_m - G_i^* G_i) v = \frac{1}{2} v^* (I_m - G_i^* G_i) v \\
= \frac{v^* (I_m - G_i^* G_i) v}{1 - e^{-2\epsilon_1}}.
\]

4.3 Induction based solution construction

As mentioned earlier in Theorem 1, if the Pick matrix (defined in (3)) for the given data set \( D \) is positive definite, then a matrix all-pass filter that satisfies the constraints (2a), (2b) and (2c) can be constructed. (Proof of the if part of Theorem 1).

We construct a solution to the problem discussed above using induction on the number of data points \( n \) as follows. First, we define
\[
B_i = [I_m \ A_i^T]^T, \quad i = 1, 2, \ldots, n.
\]

Base Step: \( n = 1, D = (\omega_1, B_1, \Gamma_1) \).

Let, \( z = e^{j\omega_1} \) and \( z_1 = e^{j\omega_1} \). We first construct the following matrix polynomials. Define,
\[
N(z) := (z - z_1) I_m + \frac{A_1 \Gamma_1^{-1} (I_m - A_1^*) z_2 (1 + z)}{(1 + z_1)}
\]
\[
D(z) := (z - z_1) I_m + \frac{\Gamma_1^{-1} (I_m - A_1^*) z_1 (1 + z)}{(1 + z_1)}.
\]

Since, \( \Gamma_1 \) is positive definite, \( N(z) \) and \( D(z) \) are well-defined. Correspondingly, we define \( G(z) := N(z) D(z)^{-1} \). We now present a brief verification that confirms that the base step of this induction indeed satisfies the required conditions. To this end, we note that \( N(z_1) D(z_1)^{-1} = A_1 \), thus satisfying (2a). We know that, \( G(z) e^{j\omega_1} G(z) = I_m \) if \( N(z) e^{j\omega_1} N(z) = D(z) e^{j\omega_1} D(z) \) for all \( \omega \in (-\pi, \pi) \), which can be easily verified by performing simple polynomial multiplications, thus satisfying (2b). Next, we see that (2c) can be verified as follows:
\[
G(z) D(z) := N(z) \Rightarrow \frac{dG(z_1)}{dz} D(z_1) + G(z_1) \frac{dD(z_1)}{dz} = \frac{dN(z_1)}{dz} \\
\Rightarrow jG(z_1) D(z_1) \frac{dG(z_1)}{dz} D(z_1) = jG(z_1) \frac{dG(z_1)}{dz} D(z_1) - j D(z_1) \frac{dG(z_1)}{dz} D(z_1) \\
\Rightarrow \left( jG(z_1) \frac{dG(z_1)}{dz} D(z_1) \right) \left( \Gamma_1^{-1} (I_m - A_1^*) z_1 \right) = (I_m - A_1^*) z_1.
\]

Therefore, we have \( jG(z_1)^{e^{j\omega_1}} \frac{dG(z_1)}{dz} = \Gamma_1 \), and thus, \( G(z) \) satisfies the constraint mentioned in (2c). Thus, the base step is verified.

Inductive Step: We assume that our problem is solvable for \( n - 1 \) points, and use this to prove that the problem is solvable for \( n \) points. First we suitably modify the given data set of \( n - 1 \) points. To this end, we define the following \( 2m \times 2m \) matrix,
\[
H(z) = \begin{bmatrix}
2(z - z_1) I_m & (z + 1) (z_1 + 1) A_1^T \\
-(z + 1) (z_1 + 1) A_1 & 2(z - z_1) I_m + (z + 1) (z_1 + 1) A_1^T A_1
\end{bmatrix}.
\]

Now, for each \( i \in \{1, 2, \ldots, n-1\} \), we define a modified data set as follows:
\[
\hat{B}_i = H(e^{j\omega_{i+1}}) [I_m \ A_i^T]^T, \quad \hat{\Gamma}_i = \Gamma_1^{-1} - \left( I - A_i A_i^T \right) \left( I - A_i^T A_i \right)^{-1} \left( (I - A_i^T A_i) \Gamma_1^{-1} \right) \left( 1 - e^{j(\omega_{i+1} - \omega_1)} \right) \left( 1 - e^{j(\omega_1 - \omega_{i+1})} \right)^{-1}.
\]

Both of the above modifications result in new data sets that also satisfy the conditions outlined in the section 4.2 since \( \hat{B}_i \)s
are neutral and \( \hat{\Gamma}_s \) are still Hermitian. Thus, the modified data set is 
\[
\hat{D} := \{ (\omega_{i+1}, B_i, \hat{\Gamma}_i) \mid i = 1, \ldots, n - 1 \}.
\]

It is important to note that the modifications in (11) are performed so that the Pick matrix of the new data set \( \hat{D} \) is the Schur complement of the Pick matrix of the original input data set \( D \) with respect to \( \Gamma_1 \). Thus, using the Schur complement property, we can argue that if the original Pick matrix for the data set \( D \) containing \( n - 1 \) points is neutral and \( N \) is positive definite, then the new Pick matrix for data set \( \hat{D} \) will also be positive definite.

Using the original induction assumption, we know that we can solve for \( \hat{N}(z), \hat{D}(z) \in \mathbb{C}^{m \times m} \) such that \( \hat{G}(z) = \hat{N}(z)\hat{D}(z) \) satisfies the matrix all-pass filter design problem for the new data set \( \hat{D} \) containing \( n - 1 \) points. Therefore, for \( z_i := e^{j\omega} \), the following holds
\[
\hat{B}_i^* \left[ \begin{array}{c} \hat{D}(z_{i+1}) \\ -\hat{N}(z_{i+1}) \end{array} \right] = 0 \quad \forall \ i = 1, \ldots, N - 1
\]

\[
\hat{B}_i = H(z_{i+1})B_{i+1} \Rightarrow \hat{B}_i^* H(z_{i+1})^* \left[ \begin{array}{c} \hat{D}(z_{i+1}) \\ -\hat{N}(z_{i+1}) \end{array} \right] = 0.
\]

We know that \( B_i^* H(z_i) = 0_{m \times 2m} \). Therefore, \( \forall \ i = 1, \ldots, N \), we have
\[
\left[ \begin{array}{c} \hat{D}(z) \\ -\hat{N}(z) \end{array} \right] := H(z)^* \left[ \begin{array}{c} \hat{D}(z) \\ -\hat{N}(z) \end{array} \right] \Rightarrow \hat{N}(z_i)\hat{D}(z_i)^{-1} = A_i.
\]

(12)

We now verify the unitary nature of \( N(e^{j\omega})D(e^{j\omega})^{-1} \) as follows:
\[
N^*(z)N(z) - D^*(z)D(z) = \left[ \begin{array}{c} N^*(z) \\ -D^*(z) \end{array} \right] \left[ \begin{array}{c} L_m \\ 0_{m \times m} \\ 0_{m \times m} \\ -L_m \end{array} \right] \left[ \begin{array}{c} N(z) \\ -D(z) \end{array} \right] = 0.
\]

(13)

The last equality can be inferred from the original induction assumption that \( \hat{N}(e^{j\omega})\hat{D}(e^{j\omega})^{-1} \) satisfies the matrix all-pass filter constraints. Therefore, \( \{ N(e^{j\omega})N(e^{j\omega}) - D^*(e^{j\omega})D(e^{j\omega}) = 0 \} \) \( \forall \omega \in (-\pi, \pi] \), which implies \( N(e^{j\omega})D(e^{j\omega})^{-1} \) is unitary for all \( \omega \in (-\pi, \pi] \). Thus, we expand the expression of the solution for the original data set of \( n \) points as follows (from equation (12):
\[
N(z) := \left[ 2(z - z_1)I - (z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)^*\hat{N}(z) + [(z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)]D(z) \right.
+ \left. [(z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)^*\hat{N}(z) + [(z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)]D(z) \right]
\]

\[
D(z) := \left[ 2(z - z_1)I + (z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)D(z) \right.
+ \left. [(z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)^*\hat{N}(z) + [(z - 1) + 1(\hat{\Gamma}_1^{-1}A_i^*)]D(z) \right]
\]

\[
G(z) := N(z)D(z)^{-1}.
\]

This \( G(z) \) satisfies the conditions in (2a), (2b), and (2c). This completes the mathematical induction steps.

Note: For a more intuitive understanding, we refer to \( \Gamma_1 \) as ‘group delay matrices’ in the sequel, while noting that they are unitarily similar to the original group delay matrices \( F_i \) specified in (2c).

The group delay matrices \( \{ \Gamma_1 \} \) may not always be available at the interpolating points in the given data set. In such situations we can obtain suitable \( \Gamma_1 \) through an optimization process described in Section 4.4.

4.4 Optimizing group delay matrices at interpolating points

Theorem 1 indicates that if the Pick matrix defined in (4) is positive definite, then an infinite number of solutions exist that satisfy (2a). These solutions are determined by specifying \( \Gamma \forall i \in \{ 1, 2, \ldots, n \} \). An interpretation of \( \Gamma_1 \) is that of a “group delay” matrix, mirroring the notion of group delay for scalar filters. In the scalar case, the group delay is roughly the delay that a signal envelope encounters when processed by a filter, and is often sought to be minimized for several real-time applications. In the matrix all-pass filtering case, as considered in discussions such as [23, 24], the group delay matrix is related to the “dispersion” among the component waveforms, representing the relative delay among them. Thus, it is prudent to minimize a function of the group delay matrices for optimal performance. Therefore, we choose to minimize the trace of these matrices, since the diagonal elements of these positive semidefinite group delay matrices directly relates to the delay of the various signal components.

Extending the consideration in [8], the trace of the Pick matrix whose block diagonals contain the group delay matrices \( \Gamma_i \) is a convex function that can be efficiently minimized. The constraints on the trace can be expressed as linear matrix inequalities (LMI), and semi-definite programming (SDP) can be used to solve it efficiently. The optimization technique is to minimize the sum of the diagonal elements in the Pick matrix, subject to the constraint that the modified Pick matrix defined in Section 4.2 is positive definite (for the selected \( \Gamma_1 \)).

minimize \( \text{Trace}(P) \)
subject to \( P > 0, \ P_{ii} = \Gamma_i > 0 \) and
\[
P_{ik} = \frac{\hat{B}_i^*JB_k}{1 - e^{j(\omega_i - \omega_k)}}, \quad i \neq k
\]
for all \( i, k \in \{ 1, 2, \ldots, n \} \)

where \( P_{ik} = P[(i - 1)m + 1 : im, (k - 1)m + 1 : km] \). The key intuition behind the formulation of this optimization is that minimizing the sum of the trace of the group delay matrices yields a filter that has smaller phase variations, and thus have better phase characteristics across frequencies without abrupt variation. Moreover, the convex nature of the problem implies that efficient tools exist that can solve the problem fast and with adequate numerical accuracy. Simulations confirm that the above optimization problem yields effective realizable matrix all-pass filters.
5 Simulation and Discussion

In this section, we use precoding in MIMO-OFDM communication systems as a sample application of the proposed SNIP approach to obtain matrix all-pass filters, though the same generalization is applicable to the case of other applications, such as audio, as well. We consider a MIMO-OFDM system that uses an $N_{\text{FFT}}$ sized IDFT on a wireless Rayleigh fading frequency selective channel. The OFDM case is particularly interesting, since the channel within each FFT sub-band (referred to as subcarrier) can be assumed to be frequency flat (i.e., flat fading). Thus, assuming that the channel in the $k$-th subcarrier is modeled as the matrix $H[k] \in \mathbb{C}^{m \times m}$, this matrix can be decomposed using the SVD as $H[k] = U[k][\Sigma[k]V^*[k]]$, where $V[k]$ is typically used to precode (pre-multiply) the data at the transmitter to enable channel parallelization. In other words, the symbol vector at every subcarrier is pre-multiplied by $V[k]$, $k = 0, 1, 2, \ldots N_{\text{FFT}}$. However, since the channel is typically estimated at the receiver, the receiver must feedback $V[k]$ for all subcarriers to the transmitter. This can lead to a significant overhead in terms of feedback requirements. In contrast, the technique from Section 4 can offer two key advantages:

- By viewing $V[k]$ as samples of $V(e^{j\omega})$, $\omega \in (-\pi, \pi]$, we can use the technique from Section 4 to obtain a rational transfer function that represents the precoding operation. Thus, we need to feedback only the coefficients of the transfer function.
- A compact rational transfer function would yield a simple system of linear constant coefficient difference equations (LCCDE) that can be implemented in the time domain, leading to a more compact representation. We can thereby compute the precoder matrix by simple matrix addition and multiplication operations, unlike the matrix exponential and logarithm operations that are used for geodesic interpolation. More importantly, these alternate approaches do not result in realizable filters, thereby making them suitable for only frequency domain processing.

To emphasize these points, consider the precoding problem in a frequency selective MIMO channel for a wireless system, wherein the matrix precoding function $V(e^{j\omega})$ has a compact representation in terms of $\omega$. For several such randomly generated channels, we compare the performance of our interpolation technique with the traditional geodesic interpolation [25] and Givens rotation based parameterization [13, 15]. We implemented the above discussed discrete-time matrix all-pass filter design method on a $2 \times 2$ MIMO system, with an input data set of 6 elements. That is, the unitary precoding filter and the corresponding group delay matrices are known at six frequencies in $(-\pi, \pi]$. The data set of unitary matrices was generated from a wireless MIMO Rayleigh fading channel ($H(e^{j\omega})$), and taking the singular value decomposition as follows:

$$H(e^{j\omega}) = U(e^{j\omega})[\Sigma(e^{j\omega})V^*(e^{j\omega})],$$

where $U(e^{j\omega})$ and $V(e^{j\omega})$ are unitary. The channels follow the ITU Vehicular A power delay profile [26]. The unitary matrices $\{V(e^{j\omega})\}$ that correspond to the right singular vectors of the channel are evaluated at $\omega \in \{-0.99\pi, -0.95\pi, -0.9\pi, -0.85\pi, -0.8\pi, -0.75\pi, 0.99\pi\}$, both for the modified SNIP approach as well as the frequency domain approaches geodesic and Givens rotation based parameterization. At frequencies other than those in data set, we interpolate the unitary matrices using the respective techniques. To quantify the accuracy of interpolation method, we plot the error between the interpolated unitary matrix $\{V(e^{j\omega})\}$ and the unitary matrix $\{H(e^{j\omega})\}$ that is realized from the channel matrix $\{H(e^{j\omega})\}$ for all $\omega$. We remark here that the frequency domain approach does not yield a realizable rational matrix all-pass filter, while our proposed approach is guaranteed to do so.

Fig. 2 and Fig. 3 present the error as measured both using the Frobenius norm, and Flag Distance in [13] as performance metrics.

The Flag distance gives a measure of distance between two matrices which are considered equivalent upon multiplication by a diagonal unitary matrices. The SVD is not unique for a matrix $H(e^{j\omega})$, since multiplying the right and left singular vectors by diagonal unitary matrices results in equivalent precoders [27]. The channel capacity can be achieved on each subcarrier by precoding with any equivalent right-singular matrix extracted from channel matrix $\{H(e^{j\omega})\}$ using the SVD [12] (due to the Flag manifold structure).

The error values in the plots are averaged over 1000 channel iterations. We can observe from the simulations that the output of the discrete-time matrix all-pass filter exactly matches the unitary matrix in data set at the frequencies in data set (referred to as interpolating points), thereby confirming that the realizable filter satisfies the specified frequency domain constraints.

While the performance of our discrete-time matrix all-pass filter design technique based on SNIP (without optimizing group delays) is comparable to geodesic interpolation and Givens rotation based parameterization, it is evident that there is still a gap in performance when compared to the geodesic and givens rotation based approaches. This can be attributed to the fact that there may exist other choices of group delay matrices $\{\Gamma_i\}$ that are unitarily similar matrices (see (2c)) which are not considered. We address this limitation using the optimization based approach to filter realization described in Section 4.4. The group delay ($\Gamma_i$) matrices are obtained using optimization, and then the matrix all-pass filter is constructed using the technique presented in 4.3.

It is evident from Fig. 2 and Fig. 3 that optimally choosing the $\Gamma_i$ (group delay) matrices results in better performance. Thus, by constraining the Pick matrix to be positive definite and optimizing $\Gamma_i$ matrices, we are able to achieve significantly improved interpolation, even while obtaining realizable (rational) matrix all-pass filters. Therefore, by using SNIP based matrix all-pass filtering technique, not only do we have computationally efficient LCCDE realizations, but we are also able to obtain performance comparable to a frequency-domain only technique like geodesic interpolation (which does not...
yield LCCDEs). Fig. 4 and Fig. 5 depicts the corresponding performances for a $4 \times 4$ system and a $8 \times 8$ system respectively, this confirms that the optimization based construction is effective for higher sizes as well and the performance compares favourably with the geodesic technique in spite of the realizability requirement.

In terms of complexity, using the SNIP approach would incur complexity in evaluating $N(z)$, $D(z)$ (from their coefficients) and finally evaluating $G(e^{j\omega}) = N(e^{j\omega})D(e^{j\omega})^{-1}$, while the geodesic based approach typically requires the computation of the matrix exponential for each subcarrier, thereby needing many more computations than the SNIP approach. To practically compare the computational complexity of the geodesic as well as the SNIP based approach of evaluating the frequency response at the intermediate frequencies, we compare the time taken and memory used to construct a precoder from the given data set. The comparisons between the the geodesic interpolation and SNIP based matrix all-pass filter technique are listed in the Table I. These values are computed on a standard Google Colab GPU free tier, consisting of a 2 core Intel Xeon 2.2 GHz CPU with 13 GB RAM.

It is evident from the simulation results that, SNIP based approach takes much less time and consumes less memory than geodesic interpolation to construct a precoder (unitary matrix). This translates to much lower complexity, especially for modern millimeter wave communication systems that employ large antenna arrays and FFT sizes.

**6 Conclusion**

All-pass filtering of vector signals is typically done in the frequency domain using the DFT. However, this could lead to incorrect filtering, and may require more complex representation of filters. In this paper, we have presented a method to obtain a realizable (rational) matrix all-pass filter by extending the SNIP to the boundary case, leading to an LCCDE implementation. If the matrix valued phase response is specified (or $\omega$), we can obtain an $n$ pole all-pass filter that exactly satisfies the phase constraints at these points. In addition, we show that, if the values of the group delay for the interpolating $\omega$ are specified, while often ensuring a compact representation. The group delay matrices at the interpolating points in the problem statement.
can be optimized or tuned to control the phase response for the remaining frequencies, which is also done in [8] for the scalar case. Simulations reveal that the method proposed can significantly outperform other optimization based approaches with much lower complexity. Future work would focus on stability of the solution under perturbation and approaches to minimize filter order as well as the problem of generating a good data sets that minimize the overfitting and undersampling for this boundary SNIP-based interpolation.

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