INITIAL COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF MERO MORPHIC BI-UNIVALENT FUNCTIONS

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Abstract. In this paper we extend the concept of bi-univalent to the class of meromorphic functions. We propose to investigate the coefficient estimates for two classes of meromorphic bi-univalent functions. Also, we find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in these new classes. Some interesting remarks and applications of the results presented here are also discussed.

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1. Introduction

Let $A$ denote the class of functions of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$.

It is well known that every function $h \in S$ has an inverse $h^{-1}$, defined by

$$h^{-1}(h(z)) = z, \quad (z \in U)$$

and

$$h(h^{-1}(w)) = w, \quad (|w| < r_0(h); \ r_0(h) \geq \frac{1}{4}),$$

where

$$h^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.$$  \hspace{1cm} (1.2)

A function $h \in S$ is said to be bi-univalent in $U$ if both $h(z)$ and $h^{-1}(z)$ are univalent in $U$. Let $\Sigma_B$ denote the class of bi-univalent functions in $U$ given by (1.1).

In 1967, Lewin [14] investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$. On the other hand, Brannan and Clunie [2] (see also [3, 4, 23]) and Netanyahu [15] made an attempt to introduce various subclasses of the bi-univalent function class $\Sigma_B$ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n| (n \in \mathbb{N} \setminus \{1, 2\}; \ \mathbb{N} := \{1, 2, 3, \ldots\})$ is still an open problem. Following Brannan and Taha [4], many researchers (see [1, 3, 7, 8, 10, 16, 20, 22, 24, 26]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class $\Sigma_B$ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. 
Let $\Sigma$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n},$$

which are meromorphic univalent functions defined in

$$\mathcal{V} := \{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}.$$ 

It is well known that every function $f \in \Sigma$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{V})$$

and

$$f^{-1}(f(w)) = w \quad (M < |w| < \infty, M > 0).$$

Furthermore, the inverse function $f^{-1}$ has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n},$$

where $M < |w| < \infty$.

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, for example [6, 12, 18]). In 1951, Springer [21] conjectured on the coefficient of the inverse of meromorphic univalent functions, latter the problem was investigated by many researchers for various subclasses (see, for details [11, 12, 13, 19, 25]).

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic bi-univalent if both $f$ and $f^{-1}$ are meromorphic univalent in $\mathcal{V}$. We denote by $\Sigma_M$ the class of all meromorphic bi-univalent functions in $\mathcal{V}$ given by (1.3).

A function $f$ in the class $\Sigma$ is said to be meromorphic bi-univalent starlike of order $\alpha$ ($0 \leq \alpha < 1$) if it satisfies the following inequalities

$$f \in \Sigma_M, \quad \Re \left( z \frac{f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{V}) \quad \text{and} \quad \Re \left( \frac{wg'(w)}{g(w)} \right) > \alpha \quad (w \in \mathcal{V}),$$

where $g(w) = f^{-1}(w)$ is the inverse of $f(z)$ whose series expansion is given by (1.4), a simple calculation shows that

$$g(w) = w - b_0 - \frac{b_1}{w} \frac{b_2 + b_0b_1}{w^2} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} + \ldots. \quad (1.5)$$

We denote by $\Sigma_M^*(\alpha)$ the class of all meromorphic bi-univalent starlike functions of order $\alpha$. Similarly, a function $f$ in the class $\Sigma$ is said to be meromorphic bi-univalent strongly starlike of order $\alpha$ ($0 < \alpha \leq 1$) if it satisfies the following conditions

$$f \in \Sigma_M, \quad \left| \arg \left( z \frac{f'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \mathcal{V}) \quad \text{and} \quad \left| \arg \left( \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \mathcal{V}),$$

where $g(w)$ is given by (1.5). We denote by $\widetilde{\Sigma}_M^*(\alpha)$ the class of all meromorphic bi-univalent strongly starlike functions of order $\alpha$. The classes $\Sigma_M^*(\alpha)$ and $\widetilde{\Sigma}_M^*(\alpha)$ were introduced and studied by Halim et al. [9].

Motivated by the works of Halim et al. [9] we define the following general subclasses $\Sigma_M^*(\alpha, \mu, \lambda)$ and $\widetilde{\Sigma}_M^*(\alpha, \mu, \lambda)$ of the function class $\Sigma$. 

Definition 1.1. A function $f$ given by (1.3) is said to be in the class $\Sigma_M^*(\alpha, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_M, \ \Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \alpha \ (\mu \geq 0, \lambda \geq 1, \lambda > \mu; z \in \mathcal{V})$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \alpha \ (\mu \geq 0, \lambda \geq 1, \lambda > \mu; w \in \mathcal{V})$$

(1.6)

for some $\alpha (0 \leq \alpha < 1)$, where $g$ is given by (1.5).

Definition 1.2. A function $f$ given by (1.3) is said to be in the class $\tilde{\Sigma}_M^*(\alpha, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_M, \ \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha \pi}{2} \ (\mu \geq 0, \lambda \geq 1, \lambda > \mu; z \in \mathcal{V})$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right| < \frac{\alpha \pi}{2} \ (\mu \geq 0, \lambda \geq 1, \lambda > \mu; w \in \mathcal{V})$$

(1.8)

for some $\alpha (0 < \alpha \leq 1)$, where $g$ is given by (1.5).

It is interesting to note that, for $\lambda = 1$ and $\mu = 0$ the classes $\Sigma_M^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_M^*(\alpha, \mu, \lambda)$ respectively, reduces to the classes $\Sigma_M^*(\alpha)$ and $\tilde{\Sigma}_M^*(\alpha)$ introduced and studied by Halim et al. [9].

The object of the present paper is to extend the concept of bi-univalent to the class of meromorphic functions defined on $\mathcal{V}$ and find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the above-defined classes $\Sigma_M^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_M^*(\alpha, \mu, \lambda)$ of the function class $\Sigma_M$ by employing the techniques used earlier by Halim et al. [9].

In order to derive our main results, we shall need the following lemma.

Lemma 1.3. (see [17]) If $\varphi \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k$, where $\mathcal{P}$ is the family of all functions $\varphi$, analytic in $\mathbb{U}$, for which

$$\Re\{\varphi(z)\} > 0 \quad (z \in \mathbb{U}),$$

where

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

2. Coefficient Bounds for the Function Classes $\Sigma_M^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_M^*(\alpha, \mu, \lambda)$

We begin this section by finding the estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\Sigma_M^*(\alpha, \mu, \lambda)$.

Theorem 2.1. Let the function $f(z)$ given by (1.3) be in the following class:

$$\Sigma_M^*(\alpha, \mu, \lambda) \quad (0 \leq \alpha < 1; \lambda \geq 1; \mu \geq 0; \lambda > \mu).$$
Then
\[ |b_0| \leq \frac{2(1 - \alpha)}{\lambda - \mu} \]  
(2.1)
and
\[ |b_1| \leq 2(1 - \alpha) \sqrt{\frac{(1 - \mu)^2(1 - \alpha)^2}{(\lambda - \mu)^4} + \frac{1}{(2\lambda - \mu)^2}}. \]  
(2.2)

Proof. It follows from (1.6) and (1.7) that
\[ (1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu - 1} = \alpha + (1 - \alpha)p(z) \]  
(2.3)
and
\[ (1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu - 1} = \alpha + (1 - \alpha)q(w), \]  
(2.4)
where \( p(z) \) and \( q(w) \) are functions with positive real part in \( V \) and have the following forms:
\[ p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots \]  
(2.5)
and
\[ q(z) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \cdots, \]  
(2.6)
respectively. Now, equating coefficients in (2.3) and (2.4), we get
\[ (\mu - \lambda)b_0 = (1 - \alpha)p_1, \]  
(2.7)
\[ (\mu - 2\lambda)(b_1 + (\mu - 1)\frac{b_0^2}{2}) = (1 - \alpha)p_2, \]  
(2.8)
\[ (\lambda - \mu)b_0 = (1 - \alpha)q_1 \]  
(2.9)
and
\[ (2\lambda - \mu)(b_1 - (\mu - 1)\frac{b_0^2}{2}) = (1 - \alpha)q_2. \]  
(2.10)
From (2.7) and (2.9), we get
\[ p_1 = -q_1 \]  
(2.11)
and
\[ b_0^2 = \frac{(1 - \alpha)^2(p_1^2 + q_1^2)}{2(\lambda - \mu)^2}. \]  
(2.12)
Since \( \Re\{p(z)\} > 0 \) in \( V \), the function \( p(1/z) \in P \) and hence the coefficients \( p_n \) and similarly the coefficients \( q_n \) of the function \( q \) satisfy the inequality in Lemma 1.3, we get
\[ |b_0| \leq \frac{2 - 2\alpha}{\lambda - \mu}. \]
This gives the bound on \( |b_0| \) as asserted in (2.1).

Next, in order to find the bound on \( |b_1| \), we use (2.8) and (2.10), which yields,
\[ (1 - \mu)^2(2\lambda - \mu)^2b_0^4 - 4(1 - \alpha)^2p_2q_2 = 4(2\lambda - \mu)^2b_1^2. \]  
(2.13)
It follows from (2.13) that
\[ b_1^2 = \frac{(1 - \mu)^2b_0^4}{4} - \frac{(1 - \alpha)^2}{(2\lambda - \mu)^2}p_2q_2. \]
Substituting the estimate obtained (2.12), and applying Lemma 1.3 once again for the coefficients $p_2$ and $q_2$, we readily get

$$|b_1| \leq 2(1 - \alpha) \sqrt{\frac{(1 - \mu)^2(1 - \alpha)^2}{(\lambda - \mu)^4} + \frac{1}{(2\lambda - \mu)^2}}.$$  

This completes the proof of Theorem 2.1.

Next we estimate the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\tilde{\Sigma}(\alpha, \mu, \lambda)$.

**Theorem 2.2.** Let the function $f(z)$ given by (1.1) be in the following class:

$$\tilde{\Sigma}(\alpha, \mu, \lambda) \quad (0 < \alpha \leq 1; \quad \lambda \geq 1; \quad \mu \geq 0; \quad \lambda > \mu).$$

Then

$$|b_0| \leq \frac{2\alpha}{\lambda - \mu} \quad (2.14)$$

and

$$|b_1| \leq 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{(\lambda - \mu)^4}}. \quad (2.15)$$

**Proof.** It follows from (1.8) and (1.9) that

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^\mu \left( \frac{1}{\lambda - \mu} \right)^{\mu-1} = [p(z)]^\alpha \quad (2.16)$$

and

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^\mu \left( \frac{1}{\lambda - \mu} \right)^{\mu-1} = [q(w)]^\alpha, \quad (2.17)$$

where $p(z)$ and $q(w)$ have the forms (2.5) and (2.6), respectively. Now, equating the coefficients in (2.16) and (2.17), we get

$$\mu - \lambda)b_0 = \alpha p_1, \quad (2.18)$$

$$(\mu - 2\lambda)(b_1 + (\mu - 1)b_0^2) = \frac{1}{2} \left[ \alpha(\alpha - 1)p_1^2 + 2\alpha p_2 \right], \quad (2.19)$$

and

$$- (\lambda - \mu)b_0 = \alpha q_1 \quad (2.20)$$

and

$$(2\lambda - \mu)(b_1 - (\mu - 1)b_0^2) = \frac{1}{2} \left[ \alpha(\alpha - 1)q_1^2 + 2\alpha q_2 \right]. \quad (2.21)$$

From (2.18) and (2.20), we find that

$$p_1 = -q_1 \quad (2.22)$$

and

$$b_0^2 = \frac{\alpha^2(p_1^2 + q_1^2)}{2(\lambda - \mu)^2}. \quad (2.23)$$

As discussed in the proof of Theorem 2.1 applying Lemma 1.3 for the coefficients $p_2$ and $q_2$, we immediately have

$$|b_0| \leq \frac{2\alpha}{\lambda - \mu}. \quad (2.24)$$

This gives the bound on $|b_0|$ as asserted in (2.14).
Next, in order to find the bound on $|b_1|$, by using (2.19) and (2.21), we get
\[ 2(2\lambda - \mu)^2 b_1^2 + (2\lambda - \mu)^2 (1 - \mu)^2 b_0^2 = \frac{\alpha^2 (\alpha - 1)^2 (p_1^4 + q_1^4)}{4} + \alpha^2 (p_2^2 + q_2^2) + \alpha^2 (\alpha - 1) (p_1^2 p_2 + q_1^2 q_2). \]

It follows from (2.24) and (2.23) that
\[ 2(2\lambda - \mu)^2 b_1^2 = \frac{\alpha^2 (\alpha - 1)^2 (p_1^4 + q_1^4)}{4} + \alpha^2 (p_2^2 + q_2^2) + \alpha^2 (\alpha - 1) (p_1^2 p_2 + q_1^2 q_2) \]
\[ - \frac{(2\lambda - \mu)^2 (1 - \mu)^2 \alpha^4}{8(\mu - \lambda)^4} (p_1^4 + q_1^4). \]

Applying Lemma 1.3 once again for the coefficients $p_1$, $p_2$, $q_1$ and $q_2$, we readily get
\[ |b_1| \leq 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{(\lambda - \mu)^4}}. \]

This completes the proof of Theorem 2.2.

Remark 2.3. For $\lambda = 1$ and $\mu = 0$ the bounds obtained in Theorems 2.1 and 2.2 are coincidence with outcome of [9, Theorem 1 and Theorem 2]. Similarly, various interesting corollaries and consequences could be derived from our results, the details involved may be left to the reader.

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