Force driven nonlinear vibrations of a thin plate with 1:1:1 internal resonance in a fractional viscoelastic medium

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Abstract. In the present paper, the dynamic behaviour of a nonlinear plate embedded into a fractional derivative viscoelastic medium is studied by the generalized method of multiple time scales when the plate is subjected to the conditions of the one-to-one-to-one internal resonance. Damping features of the surrounding medium are defined by the fractional derivative Kelvin-Voight model with a fractional parameter changing from zero to one. The influence of viscosity on the energy exchange mechanism between interacting nonlinear modes has been analyzed. For the 1:1:1 internal resonance, the nonlinear set of resolving equations in terms of amplitudes and phase differences has been obtained, and a comparative analysis of numerical calculations for free and forced vibrations are presented.

1. Introduction
It is well known that plates are used as structural elements in different fields of industry and technology, therefore the need to study forced nonlinear vibrations of plates is of paramount importance [1]. The formulation of the equations governing the fundamental dynamic behavior of thin plates, which is frequently used in engineering practice, is attributed to Mushtari, Donell and Vlasov [2]. A comprehensive survey of the methods used for the analysis of nonlinear free and forced vibrations of isotropic plates could be found in [3, 4, 5]. Thus, a one-term solution for free and forced vibrations of the rectangular plates using Galerkin’s method has been presented in [6]. Lin [7] studied the response of a nonlinear flat panel to periodic and randomly varying loadings. Nonlinear forced oscillations of a rectangular orthotropic plate subjected to uniform harmonic excitation have been examined using the method of multiple scales in [8, 9]. Stochastic response of a plate on a generalized foundation driven by random excitation has been studied in [10]. Geometrically nonlinear vibrations of rectangular plates subjected to radial harmonic excitation in the spectral neighborhood of the lowest resonances have been investigated in [4].

It is known that nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [11]. The internal resonance could be found within a certain combination of natural frequencies of one and the same type of vibrations, when vibratory motions are described by one equation [12], or between natural frequencies belonging to different types of vibrations, when two or more equations are used for the description of dynamic behaviour of a structure [13].
Nonlinear free damped vibrations of a rectangular plate described by three nonlinear differential equations have been studied in [14], wherein the procedure resulting in decoupling linear parts of equations has been proposed with the further utilization of the generalized method of multiple scales [15, 16] for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales.

In the present paper, the approaches proposed in [14] for free vibrations of plates and in [17] for forced vibrations of a nonlinear oscillator with weak fractional damping have been extended to the force driven vibrations of a thin plate under the one-to-one-to-one internal resonance with the force frequency approximately equal to a certain natural frequency of vertical vibrations.

2. Problem formulation and the method of solution
Let us consider the dynamic behavior of a free supported nonlinear thin rectangular plate (figure 1), forced vibrations of which in a viscoelastic fractional derivative medium are described by the following three differential equations in the dimensionless form (free damped equations presented in [14] are supplemented herein by the vertical harmonic force applied at the point with the coordinates \(x_0, y_0\):

\[
\begin{align*}
    u_{xx} + \frac{1 - \nu}{2} \beta_1^2 u_{yy} + \frac{1 + \nu}{2} \beta_1 v_{xy} + w_x \left( w_{xx} + \frac{1 - \nu}{2} \beta_1^2 w_{yy} \right) + \frac{1 + \nu}{2} \beta_1^2 w_y w_{xy} &= \ddot{u} + \omega_1^2 u, \quad (1) \\
    \beta_1^2 v_{yy} + \frac{1 - \nu}{2} v_{xx} + \frac{1 + \nu}{2} \beta_1 u_{xy} + \beta_1 w_y \left( \beta_1^2 w_{yy} + \frac{1 - \nu}{2} w_{xx} \right) + \frac{1 + \nu}{2} \beta_1 w_x w_{xy} &= \ddot{v} + \omega_2^2 v, \quad (2) \\
    \frac{\beta_1^2}{12} \left( w_{xxxx} + 2 \beta_1^2 w_{xxyy} + \beta_1^4 w_{yyyy} \right) - w_{xx} \left( u_x + \nu \beta_1 v_y \right) - w_x \left( u_{xx} + \nu \beta_1 v_{xy} \right) - \frac{1 - \nu}{2} \beta_1 \left( w_{xy} \left( \beta_1 u_y + v_x \right) + w_y \left( \beta_1 u_y + v_x \right) \right) - \beta_1^2 \left( w_{yy} \left( \nu u_x + \beta_1 v_y \right) + w_y \left( \nu u_x + \beta_1 v_y \right) \right) - \frac{1 - \nu}{2} \beta_1 \left( w_{xy} \left( \beta_1 u_y + v_x \right) + w_x \left( \beta_1 u_y + v_x \right) \right) - F \cos(\Omega t) \delta(x - x_0) \delta(y - y_0) &= -\ddot{w} - \omega_3^2 D^\gamma w, \quad (3)
\end{align*}
\]

subjected to the initial

\[
\begin{align*}
    u\big|_{t=0} &= v\big|_{t=0} = w\big|_{t=0} = 0, \quad (4) \\
    \ddot{u}\big|_{t=0} &= \varepsilon U_0(x, y), \quad \ddot{v}\big|_{t=0} = \varepsilon V_0(x, y), \quad \ddot{w}\big|_{t=0} = \varepsilon W_0(x, y), \quad (5)
\end{align*}
\]

as well as the boundary conditions

\[
\begin{align*}
    w\big|_{x=0} = w\big|_{x=1} = 0, \quad u\big|_{x=0} = u\big|_{x=1} = 0, \quad u_x\big|_{x=0} = u_x\big|_{x=1} = 0, \quad w_{xx}\big|_{x=0} = w_{xx}\big|_{x=1} = 0, \quad (6) \\
    w\big|_{y=0} = w\big|_{y=1} = 0, \quad u\big|_{y=0} = u\big|_{y=1} = 0, \quad v_y\big|_{y=0} = v_y\big|_{y=1} = 0, \quad w_{yy}\big|_{y=0} = w_{yy}\big|_{y=1} = 0, \quad (7)
\end{align*}
\]

where \(u = u(x, y, t)\), \(v = v(x, y, t)\) and \(w = w(x, y, t)\) are the displacements of points located in the plate’s middle surface in the \(x-, y-,\) and \(z-\)directions, respectively, \(\nu\) is the Poisson’s ratio, \(\beta_1 = a/b\) and \(\beta_2 = h/a\) are the parameters defining the dimensions of the plate, \(a\) and
Figure 1. Scheme of a freely supported rectangular plate.

3. The method of solution

An approximate solution of (8)-(10) for small but finite amplitudes weakly varying with time can be represented by an expansion in terms of different time scales in the form

\[ X_i = \varepsilon X_{i1}(T_0, T_2, \ldots) + \varepsilon^2 X_{i2}(T_0, T_2, \ldots) + \varepsilon^3 X_{i3}(T_0, T_2, \ldots) + \ldots \]  (11)

where \( i = 1, 2, 3 \), \( \varepsilon \) is a small parameter of the same order as amplitudes, \( T_0 = t \) is a fast time characterizing oscillatory motions with eigenfrequencies, and \( T_2 = \varepsilon^2 t \) as a slow scale characterizing the modulation of amplitudes and phases of nonlinear vibrations.
Recall that the first, the second and fractional derivatives [15] are expanded as follows

\[
\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left(D_1^2 + 2D_0 D_2\right) + \ldots, \quad (12)
\]

\[
\left( \frac{d}{dt} \right)^\gamma = \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots\right)^\gamma
\]

\[= D_0^\gamma + \varepsilon \gamma D_0^{\gamma-1} D_1 + \frac{1}{2} \varepsilon^2 \gamma \left[(\gamma - 1)D_0^{\gamma-2} D_1^2 + 2D_0^{\gamma-1} D_2\right] + \ldots, \quad (13)
\]

where \(D_n = \partial/\partial t_n, D_0^\gamma\), and \(D_0^{\gamma-1}, D_0^{\gamma-2}, \ldots\) are the Riemann–Liouville fractional time derivatives.

For further treatment we will consider that viscosity of the medium surrounding the plate under consideration and the external force are small values, that is, \(\varepsilon\) is small.

Substituting (11)-(13) in (8)-(10), after equating the coefficients at like powers of \(\varepsilon\) to zero, we are led to a set of recurrence equations to various orders of \(\varepsilon\):

**To order \(\varepsilon\)**

\[D_0^\gamma X_{11} + \omega_1^2 X_{11} = 0, \quad (14)\]
\[D_0^\gamma X_{21} + \omega_2^2 X_{21} = 0, \quad (15)\]
\[D_0^\gamma X_{31} + \omega_3^2 X_{31} = 0; \quad (16)\]

**To order \(\varepsilon^2\)**

\[D_0^\gamma X_{12} + \omega_1^2 X_{12} = -\zeta_1 X_{31}, \quad (17)\]
\[D_0^\gamma X_{22} + \omega_2^2 X_{22} = -\zeta_2 X_{31}, \quad (18)\]
\[D_0^\gamma X_{32} + \omega_3^2 X_{32} = -\zeta_3 X_{11} X_{31} - \zeta_3 X_{21} X_{31}. \quad (19)\]

**To order \(\varepsilon^3\)**

\[D_0^\gamma X_{13} + \omega_1^2 X_{13} = -2D_0 D_2 X_{11} - 2\zeta_1 X_{31} X_{32} - \mu_1 \tau_1^2 D_0^\gamma X_{11}, \quad (20)\]
\[D_0^\gamma X_{23} + \omega_2^2 X_{23} = -2D_0 D_2 X_{21} - 2\zeta_2 X_{31} X_{32} - \mu_2 \tau_2^2 D_0^\gamma X_{21}, \quad (21)\]
\[D_0^\gamma X_{33} + \omega_3^2 X_{33} = -2D_0 D_2 X_{31} - \zeta_3 (X_{11} X_{32} + X_{12} X_{31}) - \zeta_3 (X_{21} X_{32} + X_{22} X_{31}) - \mu_3 \tau_3^2 D_0^\gamma X_{31} + 4f \cos(T_0 T_0). \quad (22)\]

To solve the sets of equations (14)-(22), it is necessary to specify the action of the fractional derivative \(D_0^\gamma\) on the functions \(X_{ij}\). That is, to calculate \(D_0^\gamma e^{i\omega_j ft}\). It has been shown in [17] that

\[
D_0^\gamma e^{i\omega_j ft} = (i\omega_j)^\gamma e^{i\omega_j ft} + \frac{\sin \pi \gamma}{\pi} \int_0^\infty \frac{u^\gamma}{u + i\omega_j} e^{-u t} du.
\]

The first term in (23) is equivalent to the action of the fractional derivative with the low limit tending to \(-\infty\), the application of which for the exponential function is reduced to

\[
D_0^\gamma e^{i\omega_j ft} \approx (i\omega_j)^\gamma e^{i\omega_j ft}.
\]

It has been revealed [15] that nonlinear vibrations of the plate could be accompanied by different types of the internal resonance when two or more modes could be coupled. Moreover,
its type depends on the order of smallness of the viscosity involved into consideration. Thus, at the \( \varepsilon^2 \)-order, damped vibrations could be accompanied by the following types of the internal resonance:

the one-to-one internal resonance

\[
\begin{align*}
\omega_1 &= \omega_2 \quad (\omega_3 \neq \omega_1, \, \omega_3 \neq \omega_2), \\
\omega_1 &= \omega_3 \quad (\omega_2 \neq \omega_1, \, \omega_2 \neq \omega_3), \\
\omega_2 &= \omega_3 \quad (\omega_1 \neq \omega_2, \, \omega_1 \neq \omega_3), \\
\end{align*}
\]

the one-to-one-to-one internal resonance

\[
\omega_1 = \omega_2 = \omega_3 \quad (1:1:1)
\]

the combinational resonance of the additive-difference type

\[
\omega_1 = \omega_2 + 2\omega_3, \quad \omega_1 = 2\omega_3 - \omega_2, \quad \omega_1 = \omega_2 - 2\omega_3,
\]

where \( \omega_1 \) and \( \omega_2 \) are the frequencies of certain modes of in-plane vibrations along the \( x \)- and \( y \)-axes, respectively, and \( \omega_3 \) is the frequency of a certain mode of vertical vibrations.

3.1. One-to-one-to-one internal resonance, when \( \omega_1 = \omega_2 = \omega_3 = \omega \)

Now let us consider the case of the one-to-one-to-one internal resonance \( \omega_1 \approx \omega_2 \approx \omega_3 = \omega \) accompanied by the external resonance \( \omega \approx \Omega_F \).

Using the system of solvability equations to eliminate secular terms in the case of free vibrations [14], and adding the external resonance, we obtain the following solvability equations for the case of force driven vibrations:

\[
\begin{align*}
2i\omega_1 D_2 A_1 + \mu_1(\omega_1 \tau_1)^2 A_1 + 2\zeta_1 (k_5 + k_7) A_1 A_3 \tilde{A}_3 + 2\zeta_1 (k_6 + k_8) A_2 A_3 \tilde{A}_3 \\
&+ 2\zeta_1 k_7 A_1 A_3^2 + 2\zeta_1 k_8 A_2 A_3^2 = 0, \\
2i\omega_2 D_2 A_2 + \mu_2(\omega_2 \tau_2)^2 A_2 + 2\zeta_2 (k_6 + k_8) A_2 A_3 \tilde{A}_3 + 2\zeta_2 (k_5 + k_7) A_1 A_3 \tilde{A}_3 \\
&+ 2\zeta_2 k_7 A_2 A_3^2 + 2\zeta_2 k_8 A_1 A_3^2 = 0, \\
2i\omega_3 D_2 A_3 + \mu_3(\omega_3 \tau_3)^2 A_3 + [\zeta_13 (k_1 + 2k_2) + \zeta_23 (k_3 + 2k_4)] A_2 A_3 \tilde{A}_3 \\
&+ \zeta_13 (k_5 + k_7) A_1 A_3 \tilde{A}_3 + \zeta_23 (k_6 + k_8) A_2 A_3 \tilde{A}_3 + \zeta_13 k_7 A_1 A_3 \tilde{A}_3 + \zeta_23 k_8 A_2 A_3 \tilde{A}_3 \\
&+ (\zeta_13 k_6 + \zeta_23 k_5) A_1 A_2 A_3 + (\zeta_13 k_8 + \zeta_23 k_5) A_1 A_2 A_3 \\
&+ (\zeta_13 k_6 + \zeta_23 k_5) A_1 A_2 A_3 - 2\tilde{f} = 0,
\end{align*}
\]

where \( A_{j,mn}(T_2) \) and \( \tilde{A}_{j,mn}(T_2) \)(\( j = 1, 2, 3 \)) are complex conjugate functions to be found, \( \zeta_1, \zeta_2, \zeta_13, \) and \( \zeta_23 \) are coefficients depending on the plate dimensions and numbers of excited modes [14], while coefficients \( k_i \) (\( i = 1, 2, \ldots, 8 \)) have the form

\[
\begin{align*}
k_1 &= \frac{\zeta_1}{4\omega_3^2 - \omega_1^2}, \quad k_2 = -\frac{\zeta_1}{\omega_1^2}, \quad k_3 = \frac{\zeta_2}{4\omega_3^2 - \omega_2^2}, \quad k_4 = -\frac{\zeta_2}{\omega_2^2}, \quad k_5 = \frac{\zeta_13}{\omega_1(\omega_1 + 2\omega_3)}, \\
k_6 &= \frac{\zeta_23}{\omega_2(\omega_3 + 2\omega_3)}, \quad k_7 = \frac{\zeta_13}{\omega_1(\omega_1 - 2\omega_3)}, \quad k_8 = \frac{\zeta_23}{\omega_3(\omega_3 - 2\omega_3)}.
\end{align*}
\]

Let us multiply equations (28)–(30), respectively, by \( A_1, \tilde{A}_2, \) and \( A_3 \) and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from another, and representing the functions \( A_i(T_2) \) in the polar form \( a_i(T_2)e^{i\varphi_i(T_2)} \)
(i = 1, 2, 3), where \( a_i \) and \( \varphi_i \) are amplitudes and phases of nonlinear vibrations, as a result we arrive at a set of six coupled nonlinear differential equations

\[
\begin{align*}
(a_1^2)' + s_1a_1^2 &= -2\omega^{-1}\xi_1[k_7a_1^2\sin 2(\varphi_3 - \varphi_1) + (k_6 + k_8)a_1a_2\sin(\varphi_2 - \varphi_1)] \\
&\quad + k_8a_1a_2\sin(2\varphi_3 - \varphi_2 - \varphi_1)]a_3^2, \\
(a_2^2)' + s_2a_2^2 &= -2\omega^{-1}\xi_2[k_8a_2^2\sin 2(\varphi_3 - \varphi_2) - (k_5 + k_7)a_1a_2\sin(\varphi_2 - \varphi_1)] \\
&\quad + k_7a_1a_2\sin(2\varphi_3 - \varphi_2 - \varphi_1)]a_3^2, \\
(a_3^2)' + s_3a_3^2 &= \omega^{-1}\left[\xi_3k_7a_1^2\sin 2(\varphi_3 - \varphi_1) + \xi_23k_8a_2^2\sin 2(\varphi_3 - \varphi_2)ight]a_3^2 \\
&\quad + \omega^{-1}a_1a_2(\xi_3k_6 + \xi_23k_7)\sin(2\varphi_3 - \varphi_2 - \varphi_1)a_3^2 - 2f\omega^{-1}a_3\sin(\varphi_2 - \varphi_1), \\
\dot{\varphi}_1 &= \frac{1}{2}\sigma_1 + \omega^{-1}\xi_1[k_5 + k_7 + k_7\cos 2(\varphi_3 - \varphi_1) + (k_6 + k_8)a_1a_2\cos(\varphi_2 - \varphi_1)] \\
&\quad + k_8a_1a_2\cos(2\varphi_3 - \varphi_2 - \varphi_1)]a_3^2, \\
\dot{\varphi}_2 &= \frac{1}{2}\sigma_2 + \omega^{-1}\xi_2[k_6 + k_8 + k_8\cos 2(\varphi_3 - \varphi_2) + (k_5 + k_7)a_1a_2\cos(\varphi_2 - \varphi_1)] \\
&\quad + k_7a_1a_2\cos(2\varphi_3 - \varphi_2 - \varphi_1)]a_3^2, \\
\dot{\varphi}_3 &= \frac{1}{2}\sigma_3 + \frac{1}{2}\omega^{-1}\left[\xi_3(k_5 + k_7)a_1^2 + \xi_3k_6 + k_8)a_2^2 + (\xi_3(k_1 + k_2k_1) + \xi_3(k_3 + k_4))a_3^2ight] \\
&\quad + \frac{1}{2}\omega^{-1}\left[\xi_3k_7a_1^2\cos 2(\varphi_3 - \varphi_1) + \xi_3k_8a_2^2\cos(\varphi_3 - \varphi_2)ight] \\
&\quad + \frac{1}{2}\omega^{-1}[a_1a_2(\xi_3k_6 + \xi_3k_7 + \xi_3k_8 + \xi_3k_5)\cos(\varphi_2 - \varphi_1)] \\
&\quad + \frac{1}{2}\omega^{-1}a_1a_2(\xi_3k_6 + \xi_3k_7)\cos(2\varphi_3 - \varphi_1 - \varphi_2) - f(\omega_3a_3)^{-1}\cos \varphi_3.
\end{align*}
\]  

where an overdot denotes the differentiation with respect to \( T_s \), \( s_i = \mu_i\tau_i^\gamma\omega_i^{-1}\sin \psi \), \( \sigma_i = \mu_i\tau_i^\gamma\omega_i^{-1}\cos \psi \), and \( \psi = \pi\gamma/2 \) (i = 1, 2, 3).

4. Numerical investigations

Equations (31)–(36) have been solved numerically using the Runge-Kutta fourth-order algorithm.

The results of calculations for the cases of free (\( f = 0 \)) vibrations are presented in figure 2 for different magnitudes of the fractional parameter \( \gamma \): 0 (the case of undamped vibrations), 0.25, 0.3, 0.5, 0.75 and 0.9 (fractionally damped vibrations), wherein viscosity takes on two magnitudes: 0.25 and 0.5. From figure 2, wherein the \( T_s \)-dependence of the amplitudes \( a_1 \) (dashed lines), \( a_2 \) (solid lines), and \( a_3 \) (dotted lines) is shown for the initial amplitudes \( a_0 = 0.5 \), it is seen that the presence of small viscosity results in damping of the energy exchange between coupled two-in-plane modes and one out-of-plane mode of vibrations due to the one-to-one-to-one internal resonance \( \omega_{1m_1n_1} \approx \omega_{2m_2n_2} \approx \omega_{3m_3n_3} = \omega \). Reference to figure 2 shows that the increase in the fractional parameter \( \gamma \) and in the viscosity coefficient \( \alpha \) results in the increase in the period of damping vibrations and acceleration of the process of energy dissipation.

The influence of the external force on the amplitudes of nonlinear vibrations could be traced from figure 3. It is evident that in the case under consideration, when the frequency of the external vertical harmonic force is close to the frequency of plate’s vertical vibrations \( \Omega_F \approx \omega_3 \) which, in its turn, is involved in the internal resonance \( \omega_1 \approx \omega_2 \approx \omega_3 \approx \omega \), the external force increases all three amplitudes of plate’s vibrations which are coupled due to the internal
Figure 2. The $T_2$-dependence of the amplitudes $a_1$ (dashed lines), $a_2$ (solid lines) and $a_3$ (dotted lines) in the case of free vibrations at $\nu = 0.27$, $\beta_1 = 4.75$, $\beta_2 = 0.0386$, $m_1 = m_2 = 1$, $n_1 = 3$, $n_2 = 5$, $m_3 = 7$, $n_3 = 4$, $\omega_{1m_1n_1} = 44.88$, $\omega_{2m_2n_2} = 45.11$ and $\omega_{3m_3n_3} = 45.36$: (a) $\alpha = 0.25$, $\gamma = 0$ and $\gamma = 0.25$; (b) $\alpha = 0.5$, $\gamma = 0$ and $\gamma = 0.25$; (c) $\alpha = 0.25$, $\gamma = 0.3$ and $\gamma = 0.5$; (d) $\alpha = 0.5$, $\gamma = 0.3$ and $\gamma = 0.5$; (e) $\alpha = 0.25$, $\gamma = 0.75$ and $\gamma = 0.9$; (f) $\alpha = 0.5$, $\gamma = 0.75$ and $\gamma = 0.9$. 
Figure 3. The $T_2$-dependence of the amplitudes $a_1$ (dashed lines), $a_2$ (solid lines), and $a_3$ (dotted lines) in the case of force driven ($f \neq 0$) vibrations at $\nu = 0.27$, $3$, $\beta_2 = 0.0386$, $m_1 = m_2 = 1$, $n_1 = 3$, $n_2 = 5$, $m_3 = 7$, $n_3 = 4$, $\omega_{1m_1n_1} = 44.88$, $\omega_{2m_2n_2} = 45.11$ and $\omega_{3m_3n_3} \approx \Omega_F = 45.36$: (a) $f = 1$, (b) $f = 10$, (c) $f = 25$, (d) $f = 50$.

resonance. With the increase in the magnitude of force amplitude, the energy exchange process has been amplified, and the periods of energy interchange cycles have been decreased.

The impact of the plate’s dimensions on the amplitudes of the coupled modes could be seen in figures 4–6, wherein the $T_2$-dependence of the amplitudes are presented for free and forced vibrations at the same level of the force amplitude for three plates: a squared plate with $\beta_1 = 1$ (figure 4), a rectangular plate with $\beta_1 = 4.75$ (figure 5), and a rather long narrow rectangular plate with $\beta_1 = 15$ (figure 6). Reference to figures 4–6 shows that the squared plate is more stable to the external force action than rectangular plates. The more the level of the coefficient $\beta_1$, the more the amplification of the amplitudes of coupled modes.

5. Conclusion
In the present paper, nonlinear force driven vibrations of thin plates in a viscoelastic medium have been studied, when the motion of the plate is described by a set of three coupled nonlinear differential equations subjected to the conditions of the one-to-one-to-one internal resonance, resulting in the interaction of three modes. The nonlinear set of resolving equations has been obtained in terms of amplitudes and phases by the generalized method of multiple time scales.
The resulting set of six nonlinear equations has been solved numerically by the Runge-Kutta fourth-order algorithm.

The influence of the fractional parameter, viscosity coefficient, the level of the force amplitude, and plate’s dimensions on the amplitudes of the nonlinear vibration modes coupled by the one-to-one-to-one internal resonance has been studied.

Unlike external resonance, which can be eliminated by changing the frequency of external influence, internal resonance is not removable, because it is no longer ready to alter, and when designing it is impossible to predict the presence of internal resonance in the construction.
Figure 6. The $T_2$-dependence of the amplitudes $a_1$ (dashed lines), $a_2$ (solid lines) and $a_3$ (dotted lines) for a rectangular plate with $\beta_1 = 15$ at $\nu = 0.27$, $\beta_2 = 0.01$, $m_1 = 10$, $m_2 = 5$, $m_3 = 1$, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $\omega_{1m_1n_1} = 56.64$, $\omega_{2m_2n_2} = 57.73$, and $\omega_{3m_3n_3} \approx \Omega_F = 57.74$: (a) free vibrations, and (b) forced vibrations for $f = 50$.

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References
[1] Breslavsky I D, Amabili M and Legrand M 2014 Physically and geometrically non-linear vibrations of thin rectangular plates International Journal of Non-Linear Mechanics 58 30–40
[2] Volmir A 1972 Nonlinear dynamics of plates and shells (Nauka, Moscow) p. 432
[3] Nayfeh A H and Mook D T 1995 Applied Nonlinear Dynamics (Wiley Interscience) p. 704
[4] Amabili M 2004 Nonlinear vibrations of rectangular plates with different boundary conditions: Theory and experiments Computers and Structures 31-32 (82) 2587–2605
[5] Amabili M 2018 Nonlinear damping in nonlinear vibrations of rectangular plates: derivation from viscoelasticity and experimental validation Journal of the Mechanics and Physics of Solids 118 275–292
[6] Yamaki N 1961 Influence of large amplitudes on flexural vibrations of rectangular elastic plates Zeitschrift fur Angewandte Mathematik and Mechanik 41 501–510
[7] Lin Y K 1962 Response of a nonlinear flat panel to periodic and randomly-varying loadings Journal of the Aerospace Sciences 29 (1066) 1029–1033
[8] Eslami H and Kandil O A 1989 Two-mode nonlinear vibration of orthotropic plates AIAA Journal 27 (7) 961–967
[9] Shooshtari A and Khadem S E 2007 A multiple scale method solution for the nonlinear vibration of rectangular plates Scientia Iranica 14 (1) 64–71
[10] Hosseinkhani A, Younesian D and Farhangdoust S 2018 Dynamic analysis of a plate on the generalized foundation with fractional damping subjected to random excitation Mathematical Problems in Engineering 2018 ID 3908371
[11] Nayfeh A H 2000 Nonlinear Interaction: Analytical, Computational, and Experimental Methods (New York: Wiley) p. 685
[12] Ribeiro P and Petyt M 2000 Non-linear free vibration of isotropic plates with internal resonance International Journal of Non-Linear Mechanics 35 (2) 263–78
[13] Rossikhin Yu A and Shitikova M V 2003 Free damped nonlinear vibrations of a viscoelastic plate under two-to-one internal resonance Materials Science Forum 440–441 29–36
[14] Rossikhin Yu A, Shitikova M V and Ngenzi J Cl 2015 A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium Shock and Vibration 2015 795606
[15] Rossikhin Yu A and Shitikova M V 1998 Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges ASCE Journal of Engineering Mechanics 124 (9) 1029–1036
[16] Rossikhin Yu A and Shitikova M V 2012 On fallacies in the decision between the Caputo and Riemann–Liouville fractional derivatives for the analysis of the dynamic response of a nonlinear viscoelastic oscillator Mechanics Research Communications 45 22–27
[17] Rossikhin Yu A, Shitikova M V and Shcheglova T A 2009 Forced vibrations of a nonlinear oscillator with weak fractional damping Journal of Mechanics of Materials and Structures 4 (9) 1619–36
[18] Rossikhin Yu A and Shitikova M V 2010 Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results Applied Mechanics Reviews 63 (1) 010801
[19] Samko S G, Kilbas A A and Marichev O I 1993 Fractional integrals and derivatives. Theory and applications (English translation by Gordon and Breach Science Publishers, Amsterdam from the original Russian version of Nauka i Tekhnika, Minsk, Belarus, 1988) p. 1006