Generalized Dedekind’s theorem and integer group determinants

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Abstract

In this paper, we give a refinement of a generalized Dedekind’s theorem, which is a generalization of Laquer’s theorem. In addition, we show that all possible values of an integer group determinant are also possible values of the integer group determinant of any its abelian subgroup.

1 Introduction

For a finite group $G$, let $x_g$ be an indeterminate for each $g \in G$, and let $\mathbb{Z}[x_g]$ be the multivariate polynomial ring in $x_g$ over $\mathbb{Z}$. The group determinant $\Theta_G(x_g)$ of $G$ was defined by Dedekind as

$$\Theta_G(x_g) := \det (x_{gh^{-1}})_{g,h \in G} \in \mathbb{Z}[x_g].$$

For a finite group $G$, let $\hat{G}$ be a complete set of representatives of the equivalence classes of irreducible representations of $G$ over $\mathbb{C}$. For the case of $G$ is abelian, Dedekind gave the irreducible factorization of $\Theta_G(x_g)$ over $\mathbb{C}$ (see e.g., [1]): Let $G$ be a finite abelian group. Then

$$\Theta_G(x_g) = \prod_{\chi \in \hat{G}} \sum_{g \in G} \chi(g)x_g.$$ 

This called Dedekind’s theorem. Frobenius gave the irreducible factorization of the group determinant of any finite groups [2], [3]; see also [5]. This is a well-known generalization of Dedekind’s theorem. On the other hand, another generalization of Dedekind’s theorem was given in [12]: Let $G$ be a finite abelian group and let $H$ be a subgroup of $G$. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = |G/H|$ and

$$\Theta_G(x_g) = \prod_{\chi \in \hat{H}} \sum_{h \in H} \chi(h)A_h = \Theta_H(A_h). \quad (1)$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$. 

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A group determinant called an integer group determinant when its variables are integers. For a finite group \( G \), let

\[ S(G) := \{ \Theta_G(x_g) \mid x_g \in \mathbb{Z} \}. \]

Determining \( S(G) \) is an open problem. For some types of groups, this problem was solved in \([7, 9, 10, 11]\), et al. In particular, the problem called Olga Taussky-Todd’s circulant problem when \( G \) is a cyclic group. Even Olga Taussky-Todd’s circulant problem remains as an open problem.

Let \( \mathbb{Z}/n\mathbb{Z} = \{ 0, 1, \ldots, n-1 \} \) be the cyclic group of order \( n \). Laquer \([9]\) gave the following factorization to determine \( S(\mathbb{Z}/n\mathbb{Z}) \), where \( p \) is a odd prime: Let \( n = rs \), where \( r \) and \( s \) are relatively prime, and let \( x_j := x_{j+1} \) for any \( 1 \leq j \leq n \). Then

\[
\Theta_{\mathbb{Z}/n\mathbb{Z}}(x_j) = \prod_{l=0}^{s-1} \Theta_{\mathbb{Z}/r\mathbb{Z}}(y_j^l), \quad y_j^l := \sum_{k=0}^{s-1} \zeta_s^{l(kr+j-1)} x_{kr+j},
\]

where \( \zeta_s \) is a primitive \( s \)-th root of unity.

As mentioned in \([7, \text{Lemma 3.6}]\), the following is known: Let \( n, q \geq 1 \). If \( q \mid n \), then

\[ S(\mathbb{Z}/n\mathbb{Z}) \subset S(\mathbb{Z}/q\mathbb{Z}). \]  \( (3) \)

In this paper, we give a refinement of (1), which is a generalization of (2). Also, we give a generalization of (3). The following are results of this paper.

**Theorem 1.1.** Let \( G \) be a finite abelian group, let \( H \) be a subgroup of \( G \) and let

\[ \hat{G}_H := \left\{ \chi \in \hat{G} \mid \chi(h) = 1, h \in H \right\}, \quad G = \bigcup_{t \in T} tH, \quad \hat{G} = \bigcup_{\chi \in X} \chi \hat{G}_H. \]

Then we have

\[
\Theta_G(x_g) = \prod_{\chi \in X} \Theta_{G/H}(y_{tH}^\chi) = \Theta_H(z_h),
\]

where

\[
y_{tH}^\chi := \sum_{h \in H} \chi(th)x_{th}, \quad z_h := \frac{1}{|H|} \sum_{\chi \in X} \chi(h^{-1}) \Theta_{G/H}(y_{tH}^\chi).\]

**Theorem 1.1** is a refinement of (1) since \( \{ \chi \mid H \mid \chi \in X \} = \hat{H} \) holds and we can take \( A_h = z_h \) in (1). In addition, Theorem 1.1 is a generalization of \([14, \text{Theorem 2}]\), and hence also a generalization of (2) (for details see \([14]\)).

For any \( f(x_g) \in \mathbb{Z}[x_g] \), we denote by \( f(x_g)_h \) the sum of all monomials \( cx_{a_1}x_{a_2} \cdots x_{a_k} \) in \( f(x_g) \) satisfying \( a_1a_2 \cdots a_k = h \) (see Example 2.3). The following theorem gives another expression for \( z_h \) in Theorem 1.1.
Theorem 1.2. Let $y_{tH} := \sum_{h \in H} x_{th}$. Then we have

$$z_h = \Theta_{G/H}(y_{tH})_h \in \mathbb{Z}[x_g].$$

Theorem 1.1 implies that $\Theta_G(x_g)$ is obtained from $\Theta_H(x_h)$ and $\Theta_{G/H}(x_{tH})$. When calculating $z_h$, the expression for $z_h$ in Theorem 1.2 might be more useful than one in Theorem 1.1 (see Example 2.3).

We obtain the following as corollary of Theorems 1.1 and 1.2:

Corollary 1.3. Let $G$ be a finite abelian group and let $H$ be a subgroup of $G$. Then

$$S(G) \subseteq S(H).$$

Furthermore, Corollary 1.3 is generalized as the following:

Theorem 1.4. Let $G$ be a finite group and let $H$ be an abelian subgroup of $G$. Then

$$S(G) \subseteq S(H).$$

Theorem 1.4 is a generalization of (3).

2 Proofs of Theorems 1.1 and 1.2

For a finite group $G$ and a commutative ring $R$, let $x_g$ be an indeterminate for each $g \in G$, let $R[x_g]$ be the multivariate polynomial ring in $x_g$ over $R$, let $RG$ the group algebra of $G$ over $R$, and let $R[x_g]G := R[x_g] \otimes RG = \left\{ \sum_{g \in G} A_g g \mid A_g \in R[x_g] \right\}$ be the group algebra of $G$ over $R[x_g]$. Also, for a finite abelian group $G$ and a subgroup $H$ of $G$, let

$$\hat{G}_H := \left\{ \chi \in \hat{G} \mid \chi(h) = 1, h \in H \right\}.$$

It is easily verified that

$$\hat{G}_H = \left\{ \varphi \circ \pi \mid \varphi \in \hat{G/H} \right\},$$

where $\pi : G \to G/H$ is the canonical homomorphism. To prove Theorem 1.1 we use the following lemma:

Lemma 2.1 ([12, Lemma 3.6]). Let $G$ be a finite abelian group and let $H$ be a subgroup of $G$. For every $h \in H$, there exists a homogeneous polynomial $A_h \in \mathbb{C}[x_g]$ such that $\deg A_h = |G/H|$ and

$$\prod_{\chi \in \hat{G}_H} \sum_{g \in G} \chi(g) x_g g = \sum_{h \in H} A_h h \in \mathbb{C}[x_g]H.$$

If $H = G$, we can take $A_h = x_h$ for each $h \in H$. 

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**Proof of Theorem 1** From Dedekind’s theorem, we have

\[
\Theta_G(x_g) = \prod_{\chi \in \hat{X}} \sum_{g \in G} \chi(g)x_g = \prod_{\chi \in X} \prod_{\chi' \in \hat{X}/H} \prod_{t \in T} \sum_{h \in H} (\chi \chi')(th)x_{th} = \prod_{\chi \in X} \prod_{\chi' \in \hat{X}/H} \prod_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th} = \prod_{\chi \in X} \prod_{\chi' \in \hat{X}/H} \prod_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th} = \prod_{\chi \in X} \Theta_{G/H}(y_{tH}^\chi).
\]

Next, we show that there exists \( A_h \in \mathbb{C}[x_g] \) satisfying

\[
\Theta_{G/H}(y_{tH}^\chi) = \sum_{h \in H} \chi(h)A_h
\]

for any \( \chi \in X \). For any \( \chi \in X \), let \( F_\chi : \mathbb{C}[x_g]G \to \mathbb{C}[x_g]G \) be the \( \mathbb{C}[x_g] \)-algebra homomorphism defined by \( F_\chi(g) = \chi(g)g \). For any \( \chi \in X \), from Lemma 2.1 there exists \( A_h \in \mathbb{C}[x_g] \) satisfying

\[
\sum_{h \in H} \chi(h)A_h = F_\chi\left( \sum_{h \in H} A_hh \right) = F_\chi\left( \prod_{\chi' \in \hat{X}/H} \sum_{g \in G} \chi'(g)x_g \right) = F_\chi\left( \prod_{\chi' \in \hat{X}/H} \prod_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th} \right) = \prod_{\chi' \in \hat{X}/H} \sum_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th}.
\]

Let \( F : \mathbb{C}[x_g]G \to \mathbb{C}[x_g] \) be the \( \mathbb{C}[x_g] \)-algebra homomorphism defined by \( F(g) = 1 \). Applying \( F \) to the both sides of the above, we have

\[
\sum_{h \in H} \chi(h)A_h = \prod_{\chi' \in \hat{X}/H} \prod_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th} = \prod_{\chi' \in \hat{X}/H} \sum_{t \in T} \sum_{h \in H} \chi'(t)\chi(h)x_{th} = \Theta_{G/H}(y_{tH}^\chi).
\]

From the above, it follows that there exists \( A_h \in \mathbb{C}[x_g] \) satisfying

\[
\Theta_G(x_g) = \prod_{\chi \in X} \Theta_{G/H}(y_{tH}^\chi) = \prod_{\chi \in X} \sum_{h \in H} \chi(h)A_h = \prod_{\chi \in X} \sum_{h \in H} \chi(h)A_h = \Theta_H(A_h).
\]
Finally, we show that $A_h$ in (4) is expressed as

$$A_h = \frac{1}{|H|} \sum_{\chi \in \chi} \chi(h^{-1}) \Theta_{G/H}(y_{tH}^x)$$

for any $h \in H$. From orthogonality relations for characters, for any $h \in H$, we have

$$\sum_{\chi \in \chi} \chi(h^{-1}) \Theta_{G/H}(y_{tH}^x) = \sum_{\chi \in \chi} \chi(h^{-1}) \sum_{h' \in H} \chi(h') A_{h'}$$

$$= \sum_{\chi \in \chi} \sum_{h' \in H} \chi(h^{-1}h') A_{h'}$$

$$= \sum_{h' \in H} \sum_{\chi \in \chi} \chi(h^{-1}h') A_{h'}$$

$$= |H| A_h.$$

\[\square\]

**Remark 2.2.** From the proof of Theorem 1.1, $A_h$ in Lemma 2.1 equals to $z_h$ in Theorem 1.1.

**Proof of Theorem 1.2.** From Lemma 2.1 and Remark 2.2 we have

$$\sum_{h \in H} z_h = \prod_{g \in G} \sum_{g \in G} \chi(g)x_g = \prod_{g \in G} \sum_{g \in G} \chi(gH)x_g = \prod_{h \in H} \sum_{h' \in H} \sum_{\chi \in \chi} \chi(tH)x_{th'th'}.$$ 

Therefore, we have

$$z_h = \left( \prod_{\chi \in \chi} \sum_{h' \in H} \chi(tH) \sum_{h' \in H} x_{th'} \right) = \Theta_{G/H}(y_{th})\in \mathbb{Z}[x_g].$$ 

\[\square\]

From Theorems 1.1 and 1.2, $\Theta_G(x_g) = \Theta_H(z_h)$ with $z_h \in \mathbb{Z}[x_g]$ holds. Therefore, Corollary 1.3 is obtained.

**Example 2.3.** Using Theorems 1.1 and 1.2 we calculate the group determinant of cyclic group of order 4. Let $G = \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$ and $H = \{0, 2\}$. Then, $G/H = \{0H, 1H\}$. From Theorem 1.2 we have

$$z_0 = \Theta_{G/H}(y_{tH})_0 = x_0^2 + x_2^2 - 2x_1x_3,$$

$$z_2 = \Theta_{G/H}(y_{tH})_2 = 2x_0x_2 - x_1^2 - x_3^2$$

since $y_{0H} = x_0 + x_2$, $y_{1H} = x_1 + x_3$ and

$$\Theta_{G/H}(y_{tH}) = y_{0H}^2 - y_{1H}^2 = (x_0^2 + 2x_0x_2 + x_2^2) - (x_1^2 + 2x_1x_3 + x_3^2).$$

Therefore, from Theorem 1.1 we have

$$\Theta_G(x_g) = \Theta_H(z_h) = z_0^2 - z_2^2 = (x_0^2 + x_2^2 - 2x_1x_3)^2 - (2x_0x_2 - x_1^2 - x_3^2)^2.$$
3 Proof of Theorem 1.4

Theorem 1.4 is immediately obtained from the following lemma, which is essentially provided in [13 Theorem 1.4].

Lemma 3.1. Let $G$ be a finite group and $H$ be an abelian subgroup of $G$. There exists a homogeneous polynomial $A_h \in \mathbb{Z}[x_g]$ such that $\deg A_h = [G : H]$ and

$$\Theta_G(x_g) = \Theta_H(A_h),$$

where $[G : H]$ is the index of $H$ in $G$.

The proof in [13] is not concise. We give a brief proof of Lemma 3.1. For the purpose, we use the following [6 p. 82, Theorem 2.6]; see also [4, 8]: Given the block matrix $M$ of the form

$$
\begin{pmatrix}
M_{11} & M_{12} & \cdots & M_{1n} \\
M_{21} & M_{22} & \cdots & M_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
M_{n1} & M_{n2} & \cdots & M_{nn}
\end{pmatrix},
$$

where the matrices $M_{ij}$ are pairwise commuting of size $r \times r$ then

$$\det M = \det \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{n\sigma(n)} \right).$$

The matrix $(x_{g,h^{-1}})_{g,h \in G}$ is called a group matrix of $G$, which is a matrix form of $\sum_{g \in G} x_g g$. For details, see e.g., [4].

Proof of Lemma 3.1. Let $G = \{g_1, g_2, \ldots, g_{mn}\}$, let $H = \{h_1, h_2, \ldots, h_m\}$ and let $G = t_1 H \sqcup t_2 H \sqcup \cdots \sqcup t_n H$, where $g_i = t_k h_i \in G$, $i = (k-1)m + l$ for $1 \leq k \leq n$ and $1 \leq l \leq m$. Then, the group matrix $\left( x_{g,h^{-1}} \right)_{1 \leq i,j \leq mn}$ of $G$ can be expressed as the block matrix:

$$\left( x_{g,h^{-1}} \right)_{1 \leq i,j \leq mn} = (M_{kl})_{1 \leq k,l \leq n},$$

where $M_{kl}$ is the matrix obtained by replacing each $x_{h_i,h_j^{-1}}$ in the group matrix $\left( x_{h_i,h_j^{-1}} \right)_{1 \leq i,j \leq m}$ of $H$ to $x_{(t_k h_i)(t_j h_j)^{-1}}$. That is, $M_{kl} = (x_{(t_k h_i)(t_j h_j)^{-1}})_{1 \leq i,j \leq m}$. Since $H$ is abelian, $M_{kl}$ are pairwise commuting. Therefore, there exists $A_h \in \mathbb{Z}[x_g]$ satisfying

$$\Theta_G(x_g) = \det \left( x_{g,h^{-1}} \right)_{1 \leq i,j \leq mn} = \det \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{n\sigma(n)} \right) = \Theta_H(A_h)$$

since $\sum_{\sigma \in S_n} \text{sgn}(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{n\sigma(n)}$ is also of the form of a group matrix of $H$. \qed
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