A COUNTEREXAMPLE TO THE ARAKELYAN CONJECTURE

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Abstract. A “self–similar” example is constructed that shows that a conjecture of N. U. Arakelyan on the order of decrease of deficiencies of an entire function of finite order is not true.

1. Introduction

Let \( f \) be an entire function and \( \delta(a, f) \) denotes the Nevanlinna deficiency of \( f \) at the point \( a \in \mathbb{C} \). Standard references are [8, 9]. (No knowledge of Nevanlinna theory is necessary to understand this paper. We really deal with a problem of potential theory.) Since \( \delta(a, f) \geq 0 \) and the deficiency relation of Nevanlinna states that

\[
\sum_{a \in \mathbb{C}} \delta(a, f) \leq 2,
\]

it follows that the set of deficient values, that is, \( \{a : \delta(a, f) > 0\} \), is at most countable. We denote the sequence of deficiencies by \( \{\delta_n\} \). In 1966 Arakelyan [2] (see also [8] or [7]) constructed the first example of an entire function of finite order having infinitely many deficient values. In this example the deficiencies satisfy

\[
\sum_{n=1}^{\infty} \frac{1}{\log(1/\delta_n)} < \infty,
\]

(1.1)

and he conjectured that (1.1) is true for every entire function of finite order. Another method of constructing such examples was proposed in [3], but the function in [3] also satisfies (1.1).

For meromorphic functions of finite order Weitsman [11] proved

\[
\sum_{n=1}^{\infty} \delta_n^{1/3} < \infty,
\]

(1.2)

and this is known to be best possible [9, 4]. The only known improvement of (1.2) for entire functions is due to Lewis and Wu [10]:

\[
\sum_{n=1}^{\infty} \delta_n^{1/3-\epsilon_0} < \infty,
\]

(1.3)

\[\delta_n\] denotes the \( n \)-th deficiency of \( f \).
where $\epsilon_0$ is an absolute constant. In fact, the value $\epsilon_0 = 2^{-264}$ is given in [10].

In this note we will give a construction that produces an entire function of finite order having infinitely many deficiencies $\delta_n$ with the property

$$\delta_n \geq c^{-n},$$

where $c > 1$ is a constant. Thus Arakelyan’s conjecture (1.1) fails.

Of course a substantial gap still remains between the theorem of Lewis and Wu and our example. It is natural to ask whether

$$\sum_{n=1}^{\infty} \frac{1}{\log^{1+\epsilon}(1/\delta_n)} < \infty$$

is true with arbitrary $\epsilon > 0$ for entire functions of finite order.

It is more or less well known that the problem of estimating deficiencies for entire functions of finite order is equivalent to a problem of potential theory. Namely, the following statements are equivalent:

A. Given any $\rho > 1/2$ and a sequence of complex numbers $a_n$, there exists an entire function $f$ of order $\rho$ with the property $\delta(a_n, f) \geq c\delta_n$ with some constant $c > 0$.

B. There exist a bounded subharmonic function $u$ in the annulus $A = \{z : 1 < |z| < 2\}$ and disjoint open sets $E_n \subset A$, $1 \leq n < \infty$ with the following properties:

(i) Each $E_n$ is a union of some components of the set $\{z \in A : u(z) < 0\}$;
(ii) for every $r \in [1, 2]$ $\int_{\{\theta : re^{i\theta} \in E_n\}} u(re^{i\theta}) d\theta \leq -\delta_n$.

We indicate briefly how to prove the equivalence. To prove $A \rightarrow B$ we take a sequence of Pólya peaks [9, p. 101] $r_k$ for $\log M(r, f)$ and consider the sequence of subharmonic functions

$$u_k(z) = \frac{\log |f(r_kz)|}{\log M(r_k, f)}, \quad |z| < 2.$$ 

This sequence is precompact in an appropriate topology and we may take a subsequence that converges to a subharmonic function $u$. If $f$ has deficient values then $u$ satisfies (i) and (ii). See [1, 6] for details.

To prove $B \rightarrow A$ we apply the construction from [3] that involves an extension of $u$ to a subharmonic function in $C$ with the property of self-similarity: $u(2z) = ku(z)$, $k = \text{const} > 0$, approximation of $u$ by the logarithm of modulus of an entire function $g$ and performing a quasi-conformal modification on the function $g$ that produces the entire function $f$ satisfying $A$. It is also plausible that Arakelyan’s original method could be applied directly as soon as a subharmonic function with the properties (i) and (ii) is constructed.

Remark. The above-mentioned paper of Lewis and Wu contains also the solution of a problem of Littlewood on the upper estimate of mean spherical derivative of a polynomial. The connection between the two problems seems somewhat obscure. An example that gives a lower estimate in the Littlewood’s problem was constructed in [5] using some self-similar sets arising in the iteration theory of polynomials. It is interesting that the example we are going to construct now also has the property
of self-similarity. Instead of iteration of a polynomial here the crucial role is played by a semigroup of Möbius transformations of the plane.

2. The example

Consider the semigroup $\Gamma$ generated by $z \mapsto z \pm 1$ and $z \mapsto z/2$. We have

$$\Gamma = \{ \gamma_{n,k} : n = 0, 1, 2, \ldots; k = 0, \pm 1, \pm 2, \ldots \},$$

where $\gamma_{n,k}(z) = 2^{-n}(z + k)$.

Denote by $S_{0,0}^+$ the square

$$S_{0,0}^+ = \{ z : |\Re z| \leq \frac{4}{10}, |\Im z - 1| \leq \frac{4}{10} \}$$

and set $S_{n,k}^+ = \gamma_{n,k}(S_{0,0}^+)$. It is easy to see that the squares $S_{n,k}^+$ are disjoint.

Consider the domain $D_0 = \{ z : 0 < \Im z < 4/3 \} \setminus \bigcup_{n,k} S_{n,k}^+$. The boundary $\partial D_0$ consists of the real axis, boundaries of some squares, and the horizontal line $l_0 = \{ z : \Im z = 4/3 \} \subset D_0$. The domain $D_0$ is $\Gamma$-invariant and the transformation $z \mapsto z/2$ maps $D_0$ onto

$$D_1 = \{ z : 0 < \Im z < 2/3 \} \setminus \bigcup_{n,k} S_{n,k}^+ \subset D_0.$$ 

The boundary of $D_1$ consists of the real axis, boundaries of some squares, and the horizontal line $l_1 = \{ z : \Im z = 2/3 \} \subset D_0$.

Let $u$ be the harmonic function in $D_0$ that solves the Dirichlet problem

$$u(z) = 1, \quad z \in l_0,$$

$$u(z) = 0, \quad z \in \partial D_0 \setminus l_0.$$ 

This Dirichlet problem has a unique solution. So we conclude from translation invariance that

$$u(z + 1) = u(z), \quad z \in D_0. \quad (2.6)$$

It follows that the function $u$ has a positive minimum $M^{-1} < 1$ on the line $l_1 \subset D_0$. Comparing $u(z)$ and $u(2z)$ on $\partial D_1$ and using the maximum principle, we conclude that $u(2z) \leq Mu(z)$, $z \in D_1$, which is equivalent to

$$u(z) \leq Mu(z/2), \quad z \in D_0. \quad (2.7)$$

It follows from (2.6) and (2.7) that

$$u(\gamma_{n,k}(z)) \geq M^{-n}u(z), \quad z \in D_0. \quad (2.8)$$

Now we are going to extend $u$ to the strip

$$S^+ = \{ z : 0 < \Im z < 4/3 \},$$

that is, to define $u$ in the squares. We start by defining $u$ in $S_{0,0}^+$. The normal derivative (in the direction of the outward normal to the boundary of the square) of $u$ has positive infimum on $\partial S_{0,0}^+$; it tends to $+\infty$ as we approach a corner of the
square. Denote by $G > 0$ the Green function for $S^+_{0,0}$ with the pole at the point $i = \sqrt{-1}$. It is clear that the normal derivative of $G$ on the boundary of the square is bounded (it tends to zero as we approach a corner). Set
\[ u(z) = -tG(z), \quad z \in S^+_{0,0}, \]
where $t > 0$. If $t$ is small enough we obtain a subharmonic extension of $u$ into $S^+_{0,0}$, because the jump of the normal derivative will be positive as we cross the boundary of the square from inside. Fix such $t$, and extend $u$ to the remaining squares by the formula
\[ u(\gamma_{n,k}(z)) = M^{-n}u(z), \quad z \in S^+_{0,0}. \]

It follows from (2.8) that the normal derivative always has a positive jump as we cross the boundary of $S^+_{n,k}$, so the extended function is subharmonic in $S^+$.

Now consider the smaller squares $\{ z : |\Re z| \leq \frac{1}{2}, |\Im z - 1| \leq \frac{2}{3} \} \subset S^+_{0,0}$, and extend $u$ to the remaining squares by the formula
\[ u(z) = -\beta M^{-n}, \quad z \in K^+_{n,k}, \]
for some $\beta > 0$ and all $n$ and $k$.

Now we are going to extend $u$ to the strip $S = S^+ \cup S^-$ where
\[ S^- = \{ z : -\frac{3}{4} < \Re z < 0 \}. \]

To do this we repeat the above construction starting with the square $S^+_0 = \{ z : |\Re z - \frac{1}{2}| \leq \frac{3}{20}, |\Im z + 1| \leq \frac{7}{20} \}$ and using the same semigroup $\Gamma$. We obtain the squares $K^-_{n,k} = \gamma_{n,k}(K^-_{0,0})$ and $K^-_{n,k} = \gamma_{n,k}(K^-_{0,0})$, where
\[ K^-_{0,0} = \{ z : |\Re z - \frac{1}{2}| \leq \frac{2}{3}, |\Im z + 1| \leq \frac{7}{3} \}, \]
and the function $u_1$ subharmonic in $S^-$ that satisfies the inequality similar to (2.10):
\[ u_1(z) = -\beta_1 M^{-n}, \quad z \in K^-_{n,k}, \]
with some $\beta_1 > 0$ and $M_1 > 1$.

Extend $u$ to $S = S^+ \cup S^-$ by setting $u(z) = u_1(z)$, $z \in S^-$ and $u(x) = 0$, $x \in \mathbb{R}$. The extended function $u$ is continuous in $S$. We will prove that it is subharmonic in $S$.

Consider the strips $\Pi_n = \{ z : |\Im z| < \frac{4}{3}2^{-n} \}$. Define the functions $v_n$ in the following way: $v_n(z) = u(z)$, $z \in S \setminus \Pi_n$; $v_n$ are continuous in $S$ and harmonic in $\Pi_n$. Then $v_n(x) > 0$, $x \in \mathbb{R}$, and it follows from the maximum principle (applied to $\Pi_n^+ = \{ z : 0 < \Re z < \frac{4}{3}2^{-n} \}$) that $v_n \geq u$ in $S$. We conclude that $v_n$ are subharmonic because the sub-mean value property holds in every point of $S$. Furthermore, it is evident that $v_n \rightarrow u$ uniformly in $S$ as $n \rightarrow \infty$, so $u$ is subharmonic in $S$. 

\[ \text{ folks } \]
Denote \( K_n^+ = \bigcup_k K_{n,k}^+ \) and \( K_n^- = \bigcup_k K_{n,k}^- \) and remark that each vertical line \( \Re z = x_0 \) intersects \( K_n^+ \cup K_n^- \). Indeed the projection of \( K_n^+ \) onto the real axis is
\[
\bigcup_{k \in \mathbb{Z}} \{x : |x - 2^{-n}k| \leq \frac{2}{7} 2^{-n}\}
\]
and the projection of \( K_n^- \) is
\[
\bigcup_{k \in \mathbb{Z}} \{x : |x - 2^{-n}(k - \frac{1}{2})| \leq \frac{2}{7} 2^{-n}\}.
\]
It is clear that the union of these two sets is the whole real axis.

Now set \( E_n = \bigcup_k (S_{n,k}^+ \cup S_{n,k}^-) \) and \( K_n = K_n^+ \cup K_n^- \). Then each \( E_n \) is a union of some components of the set \( \{z \in S : u(z) < 0\} \) and for every vertical line \( l = \{z : \Re z = x_0\} \) the length of intersection \( l \cap K_n \) is at least \( b^{-n} \) for some \( b > 1 \).

So we have in view of (2.10) and (2.11):
\[
\int_{l \cap E_n} u(x_0 + iy) \, dy \leq \int_{l \cap K_n} u(x_0 + iy) \, dy \leq -c^{-n}
\]
with some constant \( c > 1 \).

It remains to make a change of variable \( z = \frac{4}{15} \log \zeta, \zeta \in Q = \{\zeta : 1 < |\zeta| < 2, |\arg \zeta| < \varepsilon\} \), and to extend the function \( u(z(\zeta)) \) (in any desired manner) from \( Q \) to a subharmonic function in the annulus \( \{1 < |\zeta| < 2\} \). The extended function will have the properties (i) and (ii) of \( A \) with \( \delta_n \geq c^{-n}, c > 1 \).

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