Lorentz symmetry violation in the fermion number anomaly with the chiral overlap operator

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Recently, Grabowska and Kaplan proposed a four-dimensional lattice formulation of chiral gauge theories on the basis of a chiral overlap operator. We compute the classical continuum limit of the fermion number anomaly in this formulation. Unexpectedly, we find that the continuum limit contains a term which is not Lorentz invariant. The term is, however, proportional to the gauge anomaly coefficient, and thus the fermion number anomaly in this lattice formulation automatically restores the Lorentz-invariant form when and only when the anomaly cancellation condition is met.

Subject Index   B01, B05, B31

1. Introduction

It is important to give a non-perturbative definition of chiral gauge theories. Recently, Grabowska and Kaplan constructed a five-dimensional domain-wall lattice formulation of chiral gauge theories [1]. More recently, they proposed a four-dimensional lattice formulation on the basis of the so-called chiral overlap operator, which is derived from the above domain-wall formulation [3,4]. Their four-dimensional formulation contains left- and right-handed fermions and, in the tree-level approximation, the left-handed component couples only to the original gauge field and the right-handed component couples only to a gauge field evolved by the gradient flow [5–8] for infinite time. The right-handed Weyl fermion is called the fluffy mirror fermion or fluff. Okumura and Suzuki [9] argued that the fermion number anomaly in this formulation possibly has phenomenological implications for the strong CP problem, baryogenesis, and the dark matter problem. They also conjectured the form of the classical continuum limit of the fermion number anomaly, but the explicit calculation was not carried out in Ref. [9].

In the present paper, we complete the calculation of the classical continuum limit of the fermion number anomaly in the formulation of Refs. [3,4]; the correct expression turns out to be more complicated than the simple expression conjectured in Ref. [9]. Rather unexpectedly, we find that the anomaly contains a term which is not Lorentz invariant. The term is proportional to the gauge anomaly coefficient and thus the fermion number anomaly in this lattice formulation automatically restores the Lorentz-invariant form when and only when the anomaly cancellation condition is met. The physical meaning of this finding is not immediately obvious; however, remembering that the fermion number anomaly is a very basic property of chiral gauge theories and any sensible

1 For a six-dimensional domain-wall formulation related to their formalism, see Ref. [2].
formulation of chiral gauge theories must fail when the anomaly cancellation condition is not met, our finding appears interesting and quite suggestive.

2. Basic formulation

In the formulation of Ref. [3,4], there are two gauge fields, \( A \) and \( A^\star \). \( A \) couples to the physical left-handed fermion while \( A^\star \) is given from \( A \) by the gradient flow for infinite flow time and couples to the would-be invisible right-handed fermion, the fluffy mirror fermion. This formulation manifestly preserves the gauge invariance. If we regard the gauge fields as non-dynamical external fields, the partition function is given by

\[
\int D\psi D\bar{\psi} \exp \left[ -a^4 \sum_x \bar{\psi}(x) D_\chi \psi(x) \right],
\]

where \( a \) is the lattice spacing and \( D_\chi \) denotes the chiral overlap operator,

\[
aD_\chi = 1 + \gamma_5 \left[ 1 - (1 - \epsilon_\star) \frac{1}{\epsilon\epsilon_\star + 1} (1 - \epsilon) \right].
\]

Here we have used the sign functions

\[
\epsilon = \frac{H_w[A]}{\sqrt{H_w[A]^2}}, \quad \epsilon_\star = \frac{H_w[A_\star]}{\sqrt{H_w[A_\star]^2}},
\]

of the Hermitian Wilson Dirac operator

\[
H_w = \gamma_5 \left[ \frac{1}{2} \nabla_\mu (\nabla_\mu + \nabla^*_\mu) - \frac{1}{2} a \nabla_\mu \nabla^*_\mu - m \right],
\]

where \( \nabla_\mu \) is the forward gauge-covariant lattice derivative and \( \nabla^*_\mu \) is the backward one,

\[
\nabla_\mu[A]f(x) = \frac{1}{a} \left[ U(x, \mu)f(x + a\hat{\mu}) - f(x) \right]
\]

\[
= \left[ D_\mu + \frac{a}{2} D_\mu D_\mu + \frac{a^2}{6} D_\mu D_\mu D_\mu + O(a^3) \right] f(x),
\]

\[
\nabla^*_\mu[A]f(x) = \frac{1}{a} \left[ f(x) - U(x - a\hat{\mu}, \mu)^\dagger f(x - a\hat{\mu}) \right]
\]

\[
= \left[ D_\mu - \frac{a}{2} D_\mu D_\mu + \frac{a^2}{6} D_\mu D_\mu D_\mu + O(a^3) \right] f(x).
\]

In Eqs. (2.5) and (2.7), the link variable is given by

\[
U(x, \mu)[A] = P \exp \left[ a \int_0^1 dt A_\mu(x + ta\hat{\mu}) \right],
\]

where \( P \) denotes the path-ordered product and \( \hat{\mu} \) is the unit vector in the direction of \( \mu \); in Eqs. (2.6) and (2.8), \( D_\mu = \partial_\mu + A_\mu \). For \( \nabla_\mu[A_\star] \) and \( \nabla^*_\mu[A_\star] \), \( D_\mu \) is replaced by \( D_\star_\mu = \partial_\mu + A_\star_\mu \). The sign functions satisfy

\[
\epsilon^2 = \epsilon_\star^2 = 1, \quad \left[ 1 - (1 - \epsilon_\star) \frac{1}{\epsilon\epsilon_\star + 1} (1 - \epsilon) \right]^2 = 1
\]
and, as a consequence, the Ginsparg–Wilson relation \[10\]

\[\gamma_5 \mathcal{D}_X + \mathcal{D}_X \gamma_5 = a \mathcal{D}_X \gamma_5 \mathcal{D}_X\]  

(2.11)

holds. It is then natural to introduce a modified \(\gamma_5\) \([9,11,12]\)

\[
\hat{\gamma}_5 \equiv \gamma_5(1 - a \mathcal{D}_X),
\]  

(2.12)

which satisfies

\[
(\hat{\gamma}_5)^2 = 1, \quad \mathcal{D}_X \hat{\gamma}_5 = - \gamma_5 \mathcal{D}_X.
\]  

(2.13)

Note that \(\hat{\gamma}_5\) is not Hermitian in this formulation. Using modified chiral projection operators

\[
\hat{P}_\pm \equiv \frac{1}{2} (1 \pm \hat{\gamma}_5),
\]  

(2.14)

the chiral components of the fermion can be defined as

\[
\hat{P}_- \psi(x) = \psi_L(x), \quad \psi_L(x) \hat{P}_+ = \psi_L(x),
\]  

(2.15)

\[
\hat{P}_+ \psi(x) = \psi_R(x), \quad \psi_R(x) \hat{P}_- = \psi_R(x).
\]  

(2.16)

Owing to the second relation in Eq. (2.13), the action is decomposed into left- and right-handed components as

\[
a^4 \sum_x \bar{\psi}(x) \mathcal{D}_X \psi(x) = a^4 \sum_x [\bar{\psi}_L(x) \mathcal{D}_X \psi_L(x) + \bar{\psi}_R(x) \mathcal{D}_X \psi_R(x)].
\]  

(2.17)

### 3. The classical continuum limit of the fermion number anomaly

The fermion number anomaly on the lattice associated with the left-handed fermion in Eq. (2.15) is given by \([9]\)

\[
\mathcal{A}^{(a)}_L(x) \equiv \langle \partial_{\mu} j_{\mu,L}(x) \rangle = \text{tr} \left[ \hat{P}_-(x,x) - \hat{P}_+ \frac{1}{a^4} \delta(x,x) \right] = - \frac{1}{2} \text{tr} \hat{\gamma}_5(x,x),
\]  

(3.1)

where \(\text{tr}\) stands for the trace over the spinor and gauge indices and we have used \(\text{tr} \gamma_5 = 0\) to obtain the last expression.\(^2\) In what follows, we compute the classical continuum limit, \(\mathcal{A}_L \equiv \lim_{a \to 0} \mathcal{A}^{(a)}_L\), for a smooth gauge field configuration.

Let us first determine a general form of \(\mathcal{A}_L\), by assuming that it is Lorentz invariant. The following argument is helpful to simplify the explicit tedious calculation of \(\mathcal{A}_L\). First, using Eq. (2.10), we decompose \(\mathcal{A}^{(a)}_L\) into the parity-odd part \(\mathcal{A}^{(a,\text{odd})}_L\) and the parity-even part \(\mathcal{A}^{(a,\text{even})}_L\), as \(\mathcal{A}^{(a)}_L = \mathcal{A}^{(a,\text{odd})}_L + \mathcal{A}^{(a,\text{even})}_L\), where

\[
\mathcal{A}^{(a,\text{odd})}_L(x) = \frac{1}{2} \text{tr} \frac{2}{\epsilon + \epsilon^*}(x,x), \quad \mathcal{A}^{(a,\text{even})}_L(x) = \frac{1}{2} \text{tr}(\epsilon - \epsilon^*)(x,x).
\]  

(3.2)

Then it is obvious that, under the exchange of \(A\) and \(A^*\),

\[
\mathcal{A}^{(a,\text{odd})}_L[A,A^*] = + \mathcal{A}^{(a,\text{odd})}_L[A,A^*], \quad \mathcal{A}^{(a,\text{even})}_L[A,A^*] = - \mathcal{A}^{(a,\text{even})}_L[A,A^*].
\]  

(3.3)

\(^2\) We use the notation \(O(x,y) \equiv a^{-4} O_{xy}\) for any matrix \(O\).
As the second property, we note when \( A_\star = A \),
\[
\mathcal{A}_L^{(a)}(x)[A, A] = \frac{1}{2} \text{tr} \epsilon(x, x).
\] (3.4)

Finally, the integral of \( \mathcal{A}_L^{(a)}(x) \) over four-dimensional spacetime is given by [9]
\[
a^4 \sum_x \mathcal{A}_L^{(a)}(x) = \frac{1}{2} a^4 \sum_x \text{tr} \epsilon(x, x),
\] (3.5)

and [13–16]
\[
\frac{1}{2} \text{tr} \epsilon(x, x) \xrightarrow{a \to 0} -\frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu} F_{\rho\sigma}],
\] (3.6)

for \( 0 < ma < 2 \).

Now, for convenience, we introduce
\[
C_\mu(x) \equiv A_\star \mu(x) - A_\mu(x),
\] (3.7)

which transforms as the adjoint representation under the gauge transformation on \( A \) and \( A_\star \).

We note that \( \mathcal{A}_L \) is a dimension 4 gauge-invariant local polynomial of \( A \) and \( A_\star \). Then, by examining the above properties, we find that the most general form of \( \mathcal{A}_L \) is given by
\[
\mathcal{A}_L = \mathcal{A}_L + d_1 \partial_\mu \text{tr} [C_\mu C_\nu C_\nu] + \frac{1}{2} d_2 \partial_\mu \text{tr} [C_\nu \{F_{\mu\nu} + F_{\star\mu\nu}\}]
\] + (Lorentz symmetry violating part),
\] (3.8)

where
\[
\mathcal{A}_L \equiv -\frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} \{\text{tr} [F_{\mu\nu} F_{\rho\sigma} + F_{\star\mu\nu} F_{\star\rho\sigma}] - b \partial_\mu \text{tr} [C_\nu D_\rho C_\sigma + C_\nu D_\star\rho C_\sigma]\}
\] (3.9)

with \( D_\rho = \partial_\rho + [A_\rho, \cdot \] and \( D_\star\rho = \partial_\rho + [A_\star\rho, \cdot \]. The coefficients \( d_1, d_2, \) and \( b \) cannot be determined from the above argument alone. In the first line of Eq. (3.8), \( \mathcal{A}_L \) (3.9) arises from the parity-odd part and the other terms from the parity-even part. The second term \( \partial_\mu \text{tr} [C_\nu D_\rho C_\sigma + C_\nu D_\star\rho C_\sigma] \) is proportional to the gauge anomaly coefficient and thus it vanishes for anomaly-free cases. The following explicit calculation shows that \( d_2 = 0 \) in the third term. As we will show below, there actually exists a Lorentz symmetry violating term in \( \mathcal{A}_L \). Note that, generally speaking, the restoration of the Lorentz symmetry is not automatic with the lattice regularization.

Let us describe how the explicit calculation of \( \mathcal{A}_L \) proceeds.\(^4\) We first note that
\[
\text{tr} \hat{\gamma}_5(x, x) = \sum_y \text{tr} \hat{\gamma}_5(x, y) \delta_{y,x}.
\] (3.10)

In this expression, we use
\[
\delta_{y,x} = \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} e^{i p (y-x)/a} = \int_p e^{ip(y-x)/a}.
\] (3.11)

\(^3\)In the effective action, there could be gauge-invariant relevant operators in terms of \( C_\mu \) such as the mass term \( (1/a)^2 \text{tr} C_\mu C_\mu \). These terms would require fine-tuning toward the correct continuum limit. We would like to thank the referee for a comment on this point.

\(^4\)We used the Mathematica package NCAlgebra for this calculation.
From Eq. (2.4), we have
\[ \sum_y aH_n(x,y)[A]e^{ipy/af(y)} = e^{ipx/af(x,y)} \left[ \sum_y \gamma_y \mu (s_\mu - iaQ_\mu) - \sum_\mu (c_\mu - 1) - aR - ma \right] \]
\[ (x,y)f(y), \quad (3.12) \]
where
\[ s_\mu = \sin p_\mu, \quad c_\mu = \cos p_\mu. \quad (3.13) \]
\[ Q_\mu = \frac{1}{2} \left( e^{ip_\mu} \nabla_\mu + e^{-ip_\mu} \nabla^*_\mu \right), \quad R = \frac{1}{2} \sum_\mu \left( e^{ip_\mu} \nabla_\mu - e^{-ip_\mu} \nabla^*_\mu \right). \quad (3.14) \]

\( Q_\mu \) and \( R_\mu \) are defined similarly. Thus, \( A_{(a)}^L \) can be written in terms of operators \( Q_\mu, R, Q_\mu^*, R_\mu^* \). Next, we expand \( A_{(a)}^L \) into the power series of the lattice spacing \( a \) up to \( O(a^0) \), noting that \( ma \sim O(a^0) \). For this, we need the following expansions:
\[ Q_\mu = c_\mu D_\mu + \frac{a}{2} is_\mu D_\mu D_\mu + \frac{a^2}{6} c_\mu D_\mu D_\mu D_\mu + O(a^3), \quad (3.15) \]
\[ R = \sum_\mu \left( is_\mu D_\mu + \frac{a}{2} c_\mu D_\mu D_\mu + \frac{a^2}{6} is_\mu D_\mu D_\mu D_\mu + O(a^3) \right). \quad (3.16) \]

Carrying out the explicit calculation, we find that only four covariant derivatives with the same spacetime indices appear in the Lorentz symmetry violating terms. Therefore, taking into account the above properties of the fermion number anomaly, the Lorentz symmetry violating part in the general form (3.8) must be
\[ (\text{Lorentz symmetry violating part}) = d'_1 \partial_\mu \text{tr}\left[ C_\mu C_\nu C_\nu \right], \quad (3.17) \]
with a to-be-determined coefficient \( d'_1 \).

Then, the explicit tedious expansion of \( A_{(a)}^L \) can be exactly combined into the form of Eqs. (3.8) and (3.17). After this calculation, we finally obtain
\[ A_{(a)}^L(x) = \tilde{A}_{(a)}^L(x) + d'_1 \partial_\mu \text{tr}\left[ C_\mu C_\nu C_\nu \right] + d'_1 \partial_\mu \text{tr}\left[ C_\mu C_\nu C_\nu \right], \quad (3.18) \]
where \( \tilde{A}_{(a)}^L(x) \) is given by Eq. (3.9) with the correct overall factor,\(^5\) and the coefficients are
\[ b = \frac{2}{3}, \quad (3.19) \]
\[ d'_1 (ma) = \frac{1}{128} \int \frac{1}{t^2}c_\rho c_\sigma + \frac{1}{8} \int \frac{1}{t^2} - \frac{1}{32} \int \frac{1}{t^2} \rho^2 \sigma^2 + \frac{3}{16} \int \frac{1}{t^2}(c - c_\rho)(c - c_\sigma)c_\rho c_\sigma, \quad (3.20) \]

\(^5\) The momentum integral in this factor is known in the context of the axial anomaly with the usual overlap operator. See, for example, Ref. [17] and the references cited therein.
and $d_2 = 0$ as mentioned above, where

$$c \equiv \sum_{\mu} (c_\mu - 1) + ma, \quad t \equiv \sum_{\mu} s_\mu^2 + c^2. \quad (3.22)$$

In the integrals in Eq. (3.20) and Eq. (3.21), indices $\rho$ and $\sigma$ are arbitrary as long as they differ from each other. The coefficients $d_1(ma)$ and $d'_1(ma)$ as functions of $ma$ are plotted in Figs. 1 and 2. As already announced, the last term in Eq. (3.18) is not Lorentz invariant. However, this term is proportional to the gauge anomaly coefficient. Thus, in anomaly-free chiral gauge theories, this Lorentz symmetry violating term [and the second term of Eq. (3.18)] vanishes; only $\bar{A}_L$ provides the fermion number anomaly.

4. Conclusion

In the present paper, we computed the classical continuum limit of the fermion number anomaly in the lattice formulation of chiral gauge theories by Grabowska and Kaplan. The anomaly consists of two parts: One is the parity-odd part being proportional to the epsilon tensor, $\bar{A}_L$. The other is the parity-even part, which is proportional to the gauge anomaly coefficient. The latter contains a Lorentz symmetry violating term. In anomaly-free cases, only the former $\bar{A}_L$ contributes, which
is Lorentz invariant. It appears quite interesting and suggestive that the Lorentz symmetry and the
gauge anomaly are linked in this way in this lattice formulation on the basis of the fluffy mirror
fermion.

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Note added
In the present paper, we considered only the fermion number $U(1)$. As is discussed in Ref. [18],
however, the gauge anomaly cancellation condition does not necessarily imply the vanishing of the
$d'_1$ term of Eq. (3.18) for more general $U(1)$ charges.

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