Antenna subtraction method for jet calculations at NNLO

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We describe the antenna subtraction method for treating real emission singularities in the calculation of jet observables at NNLO accuracy, in particular in view of the computation $e^+e^- \rightarrow 3$ jets at NNLO.

1. Introduction

Using experimental data on jet production observables to extract QCD parameters, especially the QCD coupling constant $\alpha_s$, one finds that the dominant source of error on these parameters is very often the uncertainty inherent to the QCD calculations of the observables. These calculations are at present available to next-to-leading (NLO) accuracy, their precision can be improved only by calculating corrections to the next perturbative order: next-to-next-to-leading order (NNLO).

In the recent past, many ingredients to NNLO calculations of collider observables have been derived (see [1] and references therein): the massless two-loop $2 \rightarrow 2$ and $1 \rightarrow 3$ matrix elements relevant to NNLO jet production have been computed and are now available for many processes of phenomenological relevance. The one-loop corrections to $2 \rightarrow 3$ and $1 \rightarrow 4$ matrix elements have been known for longer and form part of NLO calculations of the respective multi-jet observables. These NLO matrix elements naturally contribute to NNLO jet observables of lower multiplicity if one of the partons involved becomes unresolved (soft or collinear). In these cases, the infrared singular parts of the matrix elements need to be extracted and integrated over the phase space appropriate to the unresolved configuration to make the infrared pole structure explicit. As a final ingredient, the tree level $2 \rightarrow 4$ and $1 \rightarrow 5$ processes also contribute to ($2 \rightarrow 2$)- and ($1 \rightarrow 3$)-type jet observables at NNLO. These contain double real radiation singularities corresponding to two partons becoming simultaneously soft and/or collinear. To compute the contributions from single unresolved radiation at one-loop and double real radiation at tree level, one has to find subtraction terms which coincide with the full matrix elements in the unresolved limits and are still sufficiently simple to be integrated analytically in order to cancel their infrared pole structure with the two-loop virtual contributions. It is the aim of this talk to present a systematic method, named antenna subtraction, to construct NNLO subtraction terms and to illustrate its applications.

2. Antenna subtraction

An $m$-jet cross section at NLO is obtained by summing contributions from $(m+1)$-parton tree level and $m$-parton one-loop processes:

$$d\sigma_{NLO} = \int d\Phi_{m+1} (d\sigma_{NLO}^R - d\sigma_{NLO}^S) + \left[ \int d\Phi_{m+1} d\sigma_{NLO}^S + \int d\Phi_m d\sigma_{NLO}^V \right].$$

The cross section $d\sigma_{NLO}^R$ is the $(m+1)$-parton tree-level cross section, while $d\sigma_{NLO}^V$ is the one-loop virtual correction to the $m$-parton Born cross section $d\sigma^B$. Both contain infrared singularities, which are explicit poles in $1/\epsilon$ in $d\sigma_{NLO}^S$, while becoming explicit in $d\sigma_{NLO}^R$ only after integration over the phase space. In general, this integration involves the (often iterative) definition of the jet observable, such that an analytic integration is not feasible (and also not appropriate). Instead,
one would like to have a flexible method that can be easily adapted to different jet observables or jet definitions. Therefore, the infrared singularities of the real radiation contributions should be extracted using infrared subtraction terms. One introduces $d\sigma^R$ which is a counter-term for $d\sigma^S$ having the same unintegrated singular behaviour as $d\sigma^R$ in all appropriate limits. Their difference is free of divergences and can be integrated over the $(m+1)$-parton phase space numerically. The subtraction term $d\sigma^S$ has to be integrated analytically over all singular regions of the $(m+1)$-parton phase space. The resulting cross section added to the virtual contribution yields an infrared finite result. Several methods for constructing NLO subtraction terms system-

atically were proposed in the literature [2,3,4,5]. For some of these methods, extension to NNLO was discussed [7] and partly worked out. We focus on the antenna subtraction method [2,3], which we extend to NNLO.

The basic idea of the antenna subtraction approach at NLO is to construct the subtraction term $d\sigma^S$ from antenna functions. Each antenna function encapsulates all singular limits due to the emission of one unresolved parton between two colour-connected hard partons (tree-level three-parton antenna function). This construction exploits the universal factorisation of phase space and squared matrix elements in all unresolved limits, depicted in Figure 1. The individual antenna functions are obtained by normalising three-parton tree-level matrix elements to the corresponding two-parton tree-level matrix elements.

At NNLO, the $m$-jet production is induced by final states containing up to $(m+2)$-partons, including the one-loop virtual corrections to $(m+1)$-parton final states. As at NLO, one has to introduce subtraction terms for the $(m+1)$- and $(m+2)$-parton contributions. Schematically the NNLO $m$-jet cross section reads,

$$d\sigma_{NNLO} = \int_{d\Phi_{m+2}} (d\sigma^R_{NNLO} - d\sigma^S_{NNLO}) + \int_{d\Phi_{m+1}} (d\sigma^V_{NNLO} - d\sigma^VS_{NNLO})$$

$$+ \int_{d\Phi_{m+2}} d\sigma^S_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^VS_{NNLO} + \int_{d\Phi_m} d\sigma^V_{NNLO},$$

where $d\sigma^S_{NNLO}$ denotes the real radiation subtraction term coinciding with the $(m+2)$-parton tree level cross section $d\sigma^R_{NNLO}$ in all singular limits [8]. Likewise, $d\sigma^VS_{NNLO}$ is the one-loop virtual subtraction term coinciding with the one-loop $(m+1)$-parton cross section $d\sigma^V_{NNLO}$ in all singular limits [9]. Finally, the two-loop correction to the $m$-parton cross section is denoted by $d\sigma^V_{NNLO}$.

Both types of NNLO subtraction terms can be constructed from antenna functions. In $d\sigma^S_{NNLO}$, we have to distinguish four different types of un-

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**Figure 1.** Illustration of NLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+1)$-particle phase space. The term in square brackets represents both the antenna function and the antenna phase space.

**Figure 2.** Illustration of NNLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+2)$-particle phase space when the unresolved particles are colour connected. The term in square brackets represents antenna function and phase space.
Figure 3. Illustration of NNLO antenna factorisation representing the factorisation of both the one-loop “squared” matrix elements (represented by the white blob) and the \((m+1)\)-particle phase space when the unresolved particles are colour connected.

resolved configurations: (a) One unresolved parton but the experimental observable selects only \(m\) jets; (b) Two colour-connected unresolved partons (colour-connected); (c) Two unresolved partons that are not colour connected but share a common radiator (almost colour-unconnected); (d) Two unresolved partons that are well separated from each other in the colour chain (colour-unconnected). Among those, configuration (a) is properly accounted for by a single tree-level three-parton antenna function like used already at NLO. Configuration (b) requires a tree-level four-parton antenna function (two unresolved partons emitted between a pair of hard partons) as shown in Figure 2 while (c) and (d) are accounted for by products of two tree-level three-parton antenna functions.

In single unresolved limits, the one-loop cross section \(d\sigma^\text{V,1}_{\text{NLO}}\) is described by the sum of two terms \([9]\): a tree-level splitting function times a one-loop cross section and a one-loop splitting function times a tree-level cross section. Consequently, the one-loop single unresolved subtraction term \(d\sigma^\text{V,1}_{\text{NNLO}}\) is constructed from tree-level and one-loop three-parton antenna functions, as sketched in Figure 3. Several other terms in \(d\sigma^\text{V,1}_{\text{NLO}}\) cancel with the results from the integration of terms in the double real radiation subtraction term \(d\sigma^\text{S}_{\text{NLO}}\) over the phase space appropriate to one of the unresolved partons, thus ensuring the cancellation of all explicit infrared poles in the difference \(d\sigma^\text{V,1}_{\text{NLO}} - d\sigma^\text{V,1}_{\text{NNLO}}\).

Finally, all remaining terms in \(d\sigma^\text{S}_{\text{NLO}}\) and \(d\sigma^\text{V,1}_{\text{NNLO}}\) have to be integrated over the four-parton and three-parton antenna phase spaces. After integration, the infrared poles are rendered explicit and cancel with the infrared pole terms in the two-loop squared matrix element \(d\sigma^\text{V,2}_{\text{NNLO}}\).

3. Derivation of antenna functions

The subtraction terms \(d\sigma^\text{S}_{\text{NLO}}, d\sigma^\text{S}_{\text{NNLO}}\) and \(d\sigma^\text{V,1}_{\text{NNLO}}\) require three different types of antenna functions corresponding to the different pairs of hard partons forming the antenna: quark-antiquark, quark-gluon and gluon-gluon antenna functions. In the past \([2,3]\), NLO antenna functions were constructed by imposing definite properties in all single unresolved limits (two collinear limits and one soft limit for each antenna). This procedure turns out to be impractical at NNLO, where each antenna function must have definite behaviours in a large number of single and double unresolved limits. Instead, we derive these antenna functions in a systematic manner from physical matrix elements known to possess the correct limits. The quark-antiquark antenna functions can be obtained directly from the \(e^+e^- \rightarrow 2j\) real radiation corrections at NLO and NNLO \([10]\). For quark-gluon and gluon-gluon antenna functions, effective Lagrangians are used to obtain tree-level processes yielding a quark-gluon or gluon-gluon final state. The antenna functions are then obtained from the real radiation corrections to these processes. Quark-gluon antenna functions were derived \([11]\) from the purely QCD (i.e. non-supersymmetric) NLO and NNLO corrections to the decay of a heavy neutralino into a massless gluino plus partons \([12]\), while gluon-gluon antenna functions \([13]\) result from the
QCD corrections to Higgs boson decay into partons [14].

All tree-level three-parton and four-parton antenna functions and three-parton one-loop antenna functions are listed in [1], where we also integrate them, using the phase space integration techniques described in [15].

4. Application to $e^+e^- \rightarrow 3$ jets

To illustrate the application of antenna subtraction on a non-trivial example, we derived in [16] the $1/N^2$-contribution to the NNLO corrections to $e^+e^- \rightarrow 3$ jets. This colour factor receives contributions from $\gamma^* \rightarrow q\bar{q}qqq$ and $\gamma^* \rightarrow q\bar{q}qq$ at tree-level [17], $\gamma^* \rightarrow q\bar{q}g$ and $\gamma^* \rightarrow q\bar{q}g$ at one-loop [18] and $\gamma^* \rightarrow q\bar{q}g$ at two-loops [19]. The four-parton and five-parton final states contain infrared singularities, which are extracted using the antenna subtraction formalism.

In this contribution, all gluons are effectively photon-like, and couple only to the quarks, but not to each other. Consequently, only quark-antiquark antenna functions appear in the construction of the subtraction terms.

Starting from the program EERAD2 [2], which computes the four-jet production at NLO, we implemented the NNLO antenna subtraction method for the $1/N^2$ colour factor contribution to $e^+e^- \rightarrow 3j$. EERAD2 already contains the five-parton and four-parton matrix elements relevant here, as well as the NLO-type subtraction terms.

The implementation contains three channels, classified by their partonic multiplicity: (a) in the five-parton channel, we integrate $d\sigma_{\text{NLO}}^5 - d\sigma_{\text{NNLO}}^5$; (b) in the four-parton channel, we integrate $d\sigma_{\text{NNLO}}^4 - d\sigma_{\text{NLO}}^{V,1}$; (c) in the three-parton channel, we integrate $d\sigma_{\text{NLO}}^{V,2} + d\sigma_{\text{NLO}}^{S,1}$. The numerical integration over these channels is carried out by Monte Carlo methods.

By construction, the integrands in the four-parton and three-parton channel are free of explicit infrared poles. In the five-parton and four-parton channel, we tested the proper implementation of the subtraction by generating trajectories of phase space points approaching a given single or double unresolved limit. Along these trajectories, we observe that the antenna subtraction terms converge locally towards the physical matrix elements, and that the cancellations among individual contributions to the subtraction terms take place as expected. Moreover, we checked the correctness of the subtraction by introducing a lower cut (slicing parameter) on the phase space variables, and observing that our results are independent of this cut (provided it is chosen small enough). This behaviour indicates that the subtraction terms ensure that the contribution of potentially singular regions of the final state phase space does not contribute to the numerical integrals, but is accounted for analytically.

Finally, we noted in [1] that the infrared poles of the two-loop (including one-loop times one-loop) correction to $\gamma^* \rightarrow q\bar{q}g$ are cancelled in all colour factors by a combination of integrated three-parton and four-parton antenna functions. This highly non-trivial cancellation clearly illustrates that the antenna functions derived here correctly approximate QCD matrix elements in all infrared singular limits at NNLO. They also outline the structure of infrared cancellations in $e^+e^- \rightarrow 3j$ at NNLO, and indicate the structure of the subtraction terms in all colour factors.

5. Outlook

In this talk, we presented a new method for the subtraction of infrared singularities in the calculation of jet observables at NNLO. We introduced subtraction terms for double real radiation at tree level and single real radiation at one loop based on antenna functions. These antenna functions describe the colour-ordered radiation of unresolved partons between a pair of hard (radiator) partons. All antenna functions at NLO and NNLO can be derived systematically from physical matrix elements. To demonstrate the application of our new method on a non-trivial example, we implemented the NNLO corrections to the subleading colour contribution to $e^+e^- \rightarrow 3$ jets.

An immediate application of the method presented here is the calculation of the full NNLO corrections to $e^+e^- \rightarrow 3$ jets [20]. The antenna subtraction method can be further generalised to NNLO corrections to jet production in lepton-hadron or hadron-hadron collisions. In these
kinematical situations, the subtraction terms are constructed using the same antenna functions, but in different phase space configurations: instead of the $1 \to n$ decay kinematics considered here, $2 \to n$ scattering kinematics are required, which can also contain singular configurations due to single or double initial state radiation. These require new sets of integrated antenna functions, accounting for the different phase space configurations in these cases.

Acknowledgements

This research was supported in part by the Swiss National Science Foundation (SNF) under contracts PMPD2-106101 and 200020-109162 and by the UK Particle Physics and Astronomy Research Council.

REFERENCES

1. A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, JHEP 0509 (2005) 056.
2. J. Campbell, M.A. Cullen and E.W.N. Glover, Eur. Phys. J. C 9 (1999) 245.
3. D.A. Kosower, Phys. Rev. D 57 (1998) 5410; Phys. Rev. D 71 (2005) 045016.
4. S. Catani and M.H. Seymour, Nucl. Phys. B 485 (1997) 291; 510 (1997) 503(E).
5. W.T. Giele and E.W.N. Glover, Phys. Rev. D 46 (1992) 1980; Z. Kunszt and D.E. Soper, Phys. Rev. D 46 (1992) 192; S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B 467 (1996) 399; Z. Nagy and Z. Trocsanyi, Nucl. Phys. B 486 (1997) 189.
6. D.A. Kosower, Phys. Rev. D 67 (2003) 116003.
7. S. Weinzierl, JHEP 0303 (2003) 062; W.B. Kilgore, Phys. Rev. D 70 (2004) 031501; M. Grazzini and S. Frixione, JHEP 0506 (2005) 010; G. Somogyi, Z. Trocsanyi and V. Del Duca, JHEP 0506 (2005) 024.
8. A. Gehrmann-De Ridder and E.W.N. Glover, Nucl. Phys. B 517 (1998) 269; J. Campbell and E.W.N. Glover, Nucl. Phys. B 527 (1998) 264; S. Catani and M. Grazzini, Phys. Lett. B 446 (1999) 143; Nucl. Phys. B 570 (2000) 287; F.A. Berends and W.T. Giele, Nucl. Phys. B 313 (1989) 595; V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. B 568 (2000) 211.
9. Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B 425 (1994) 217; D.A. Kosower, Nucl. Phys. B 552 (1999) 319; D.A. Kosower and P. Uwer, Nucl. Phys. B 563 (1999) 477; Z. Bern, V. Del Duca and C.R. Schmidt, Phys. Lett. B 445 (1998) 168; Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, Phys. Rev. D 60 (1999) 116001.
10. A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Nucl. Phys. B 691 (2004) 195.
11. A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Phys. Lett. B 612 (2005) 36.
12. H.E. Haber and D. Wyler, Nucl. Phys. B 323 (1989) 267.
13. A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Nucl. Phys. B 682 (2004) 265.
14. F. Wilczek, Phys. Rev. Lett. 39 (1977) 1304; M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Lett. B 78 (1978) 443.
15. A. Gehrmann-De Ridder, T. Gehrmann and G. Heinrich, Nucl. Phys. B 652 (2002) 265.
16. A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Nucl. Phys. Proc. Suppl. 135 (2004) 97.
17. K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B 313 (1989) 850; F.A. Berends, W.T. Giele and H. Kuifj, Nucl. Phys. B 321 (1989) 39; N.K. Falck, D. Graudenz and G. Kramer, Nucl. Phys. B 328 (1989) 317.
18. Z. Bern, L.J. Dixon, D.A. Kosower and S. Weinzierl, Nucl. Phys. B 489 (1997) 3; Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B 513 (1998) 3; E.W.N. Glover and D.J. Miller, Phys. Lett. B 396 (1997) 257; J.M. Campbell, E.W.N. Glover and D.J. Miller, Phys. Lett. B 409 (1997) 503; Z. Nagy and Z. Trocsanyi, Phys. Lett. B 414 (1997) 187.
19. L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Kouvoutsakis and E. Remiddi, Nucl. Phys. B 627 (2002) 107 and 642 (2002) 227.
20. A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, work in progress.