HARMONIC SUPERPOTENTIALS AND SYMMETRIES IN GAUGE THEORIES WITH EIGHT SUPERCHARGES

Boris Zupnik

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, Moscow Region, 141980, Russia;
e-mail: zupnik@thsun1.jinr.ru

Abstract

Models of interactions of $D$-dimensional hypermultiplets and supersymmetric gauge multiplets with $\mathcal{N}=8$ supercharges ($D \leq 6$) can be formulated in the framework of harmonic superspaces. The effective Coulomb low-energy action for $D=5$ includes the free and Chern-Simons terms. We consider also the non-Abelian superfield $D=5$ Chern-Simons action. The biharmonic $D=3,\mathcal{N}=8$ superspace is introduced for a description of $l$ and $r$ supermultiplets and the mirror symmetry. The $D=2,(4,4)$ gauge theory and hypermultiplet interactions are considered in the triharmonic superspace. Constraints for $D=1,\mathcal{N}=8$ supermultiplets are solved with the help of the $SU(2) \times Spin(5)$ harmonics. Effective gauge actions in the full $D\leq 3,\mathcal{N}=8$ superspaces contain constrained (harmonic) superpotentials satisfying the $(6-D)$ Laplace equations for the gauge group $U(1)$ or corresponding $(6-D)p$-dimensional equations for the gauge groups $[U(1)]^p$. Generalized harmonic representations of superpotentials connect equivalent superfield structures of these theories in the full and analytic superspaces. The harmonic approach simplifies the proofs of non-renormalization theorems.

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$^1$On leave of absence from the Institute of Applied Physics, Tashkent State University, Uzbekistan
1 Introduction

The harmonic superspace \((HS)\) has firstly been introduced for the off-shell description of matter, gauge and supergravity superfield theories with the manifest \(D=4, N_s=2\) supersymmetry \([1, 2]\). The \(SU(2)/U(1)\) harmonics \(u_i^\pm\) and corresponding harmonic derivatives \(\partial^+, \partial^-\) and \(\partial^0\) are used for the consistent solution of the superfield constraints in the \(N_s=2\) superspace. The basic relations for the harmonics are

\[
[\partial^+, \partial^-] = \partial^0, \quad [\partial^0, \partial^{\pm\pm}] = \pm 2\partial^{\pm\pm},
\]
\[
\partial^+ u_i^+ = 0, \quad \partial^+ u_i^- = u_i^+,
\]
\[
\partial^- u_i^- = 0, \quad \partial^- u_i^+ = u_i^-.
\]

The \(HS\) approach has also been applied to consistently describe hypermultiplets and vector multiplet in \(D=6, N_s=1\) supersymmetry \([3, 4]\). It is convenient to use the total number of supercharges \(N\) for the classification of all these models in different dimensions \(D\) instead of the number of spinor representations for supercharges \(N_D\). Let us review briefly the basic aspects of the \(D=6, N=8\) harmonic gauge theory. The harmonics \(u_i^\pm\) are used to construct the analytic 6D coordinates \(\zeta=(\bar{x}_\alpha^\rho, \theta^{\alpha\pm})\) and the additional spinor coordinate \(\theta^{\alpha-}\), where \(\alpha, \beta, \rho\ldots\) are the 4-spinor indices of the \((1,0)\) representation of the \(Spin(5,1)\) group and \(\theta^{\pm \alpha}=u_i^\pm \theta^\alpha\). The harmonized spinor derivatives and harmonic derivatives have the following form in these coordinates:

\[
D_\alpha^+ = \partial_\alpha^+, \quad D_\alpha^- = -\partial_\alpha^+-i\theta^\gamma \hat{\partial}_\alpha^\gamma,
\]
\[
D^{++} = \partial^{++} + \frac{i}{2} \theta^{\alpha+} \theta^{\gamma+} \hat{\partial}_\alpha^\gamma + \theta^{\alpha+} \partial^+_\alpha,
\]
\[
D^{--} = \partial^{--} + \frac{i}{2} \theta^{\alpha-} \theta^{\gamma-} \hat{\partial}_\alpha^\gamma + \theta^{\alpha-} \partial^-_\alpha,
\]

where \(\hat{\partial}_\alpha^\gamma = \partial/\partial x^{\alpha\gamma}\).

The Grassmann analyticity condition in \(HS\) is \(D_\alpha^+ \omega=0\). Superfield constraints of \(D=6\) \(SYM\) in the ordinary superspace (central basis or \(CB\)) are equivalent to the integrability conditions preserving this analyticity. The Yang-Mills prepotential \(V^{++}(\zeta, u)\) in the analytic basis \((AB)\) describes the 6D vector multiplet \((A_\alpha^\rho, \lambda^\alpha_i, X^{ik})\) and possesses the gauge transformation with the analytic matrix parameter \(\lambda(\zeta, u)\)

\[
\delta_\lambda V^{++} = D^{++} \lambda + [V^{++}, \lambda] = \nabla^{++} \lambda .
\]

The action of the \(D=6 SYM\) theory has the form of integral over the full superspace

\[
S(V^{++}) = \frac{1}{g_s^2} \sum_{n=1}^\infty \frac{(-1)^n}{n} \int d^6x d^8\theta du_1 \ldots du_n \frac{\text{Tr} V^{++}(z, u_1) \ldots V^{++}(z, u_n)}{(u_1^+ u_2^+) \ldots (u_n^+ u_1^+)}
\]

where \(g_s\) is the coupling constant of dimension \(d=1\). The harmonic distribution \(1/(u_1^+ u_2^+)\) satisfies the relation

\[
\partial_1^{++} \frac{1}{(u_1^+ u_2^+)} = \delta^{(1, -1)}(u_1, u_2),
\]

where \(\delta^{(1, -1)}\) is the harmonic \(\delta\)-function.
The gauge variation of this action
\[ \delta_{\lambda}S(V^{++}) = \frac{1}{g^2_6} \int d^6x d^8\theta du Tr \nabla^{++} \lambda V^{--} = -\frac{1}{g^2_6} \int d^6x d^8\theta du Tr \lambda D^{--} V^{++} = 0 \] (1.10)
vanishes due to the analyticity of the parameter and prepotential. We have used here the harmonic zero-curvature equation
\[ D^{++} V^{--} - D^{--} V^{++} + [V^{++}, V^{--}] = 0 , \]
where \( V^{--} \) is the connection for the harmonic derivative \( D^{--} \). Note that the superfield density of the gauge actions in the full superspace is not invariant for any \( D \) in contrast to the chiral density of the \( D=4 \) gauge action.

Reality conditions for the harmonic connections include the special conjugation of harmonics preserving the \( U(1) \)-charges \[ \bar{u}^\pm_i = u^{i\pm}, \quad (V^{\pm\pm})^\dagger = -V^{\pm\pm} . \] (1.12)

The physical fields of the hypermultiplet \( f^i \) and \( \psi_\alpha \) and the infinite number of auxiliary fields are components of the analytic 6D superfield \( q^+(\zeta, u) \). The interaction of the hypermultiplet and gauge field can be written in the analytic superspace
\[ S(q^+, V^{++}) = \int d\zeta^{(-4)} du \bar{q}^+ (D^{++} + V^{++}) q^+ , \] (1.13)
where \( d\zeta^{(-4)} = d\theta_A (D^-)^4 \) is the analytic measure in \( HS \).

Universality of harmonic superspaces is connected with the possibility of constructing \( N=8 \) models in \( D<6 \) by a dimensional reduction. The \( HS \) analysis of the \( D=4 \) low-energy effective actions has been considered for the gauge superfields \[ \text{[7]} \] and for the hypermultiplets \[ \text{[8]} \]. The manifestly supersymmetric calculations in \( HS \) are in a good agreement with the basic ideas of the Seiberg-Witten theory \[ \text{[6]} \], however, the \( HS \) geometry allows us to rewrite the chiral-superspace Coulomb action as the integral in the full superspace
\[ S_4 = i \int d^4x d^8\theta F(W) + \text{c.c.} = \int d^4x d^8\theta du V^{++} V^{--} [F(W) + \text{c.c.}] , \] (1.14)
where \( F(W) = - i W^{-2} F(W) \) is the holomorphic part of the superpotential in this representation and \( W = (\bar{D}^+)^2 V^{--} \). We have used the following decompositions of the chiral and full Grassmann measures in terms of the harmonized spinor derivatives:
\[ d\theta = (D^+)^2 (D^-)^2 (D^-)^2 , \quad d\bar{\theta} = (D^+)^2 (D^-)^2 , \] (1.15)
where \( (D^\pm)^2 = (1/2) D^\alpha^\pm D^\alpha^\pm \) and \( (\bar{D}^\pm)^2 = (1/2) \bar{D}^\dot{\alpha}^\pm \bar{D}^\dot{\alpha}^\pm \).

It should be stressed that the superpotential \( f(W, \bar{W}) = [F(W) + \text{c.c.}] \) in the full-superspace representation satisfies the constraints
\[ D^+_\alpha \bar{D}^+_\dot{\alpha} f(W, \bar{W}) = 0 \rightarrow \partial_w \bar{\partial}_{\dot{w}} f(W, \bar{W}) = 0 , \] (1.16)
which follow from the gauge invariance
\[ \delta_{\lambda}S_4 \sim \int d^4x d^8\theta du \lambda D^{--} V^{++} f(W, \bar{W}) \]
\[ \sim \int d^8(x^-)^4 du \lambda \partial^\alpha \bar{\partial}_{\dot{\alpha}} V^{++} D^+_\alpha \bar{D}^+_{\dot{\alpha}} f(W, \bar{W}) = 0 . \] (1.17)
Representations of the action in the full, analytic and chiral superspaces are also important for the HS interpretation of the 4D electric-magnetic duality [9].

The holomorphic action $S_4$ can be reduced to lower dimensions, however, this reduction does not produce the general effective action. The $\mathcal{N}=8$ supersymmetries have some specific features for each dimension based on differences in the structure of Lorentz groups $L_D$, maximum automorphism groups $R_D$ and the set of central charges $Z_D$. The main result of this work is a construction of the Coulomb effective actions for the dimensions $D=1, 2, 3$ and 5 in the full $\mathcal{N}=8$ superspace

$$S_D = \int d^D x \, d^8 \theta \, du \, V^{++} V^{--} f_D(W) \ ,$$

where $f_D(W)$ is the superpotential and $W$ is the constrained $(6-D)$-component superfield strength for the $U(1)$ gauge prepotential $V^{++}$. The gauge invariance of this action implies the $(6-D)$-dimensional Laplace equation for the general superpotential

$$\Delta^{w_D} f_D(W) = 0 \ ,$$

which generalizes the 2D-Laplace equation (1.14). The $(6-D)$-harmonic solutions of this equation can be used for a description of non-perturbative solutions in the $\mathcal{N}=8$ gauge theories. We discuss harmonic-integral representations of the $D\leq 3$ superpotentials which allow us to construct the equivalent analytic-superspace representations of $S_D$.

It should be remarked that the function $f_D$ determines $\sigma$-model structures and interactions of the $(6-D)$-dimensional scalar field with fermion and vector fields.

Renormalization theorems in this approach are connected with the $R_D$-invariant solutions of Eq. (1.19)

$$f^R_D(w_D) = g_D^{-2} + k_D w_D^{D-4} \ , \quad D \neq 4 \ ,$$

where the invariant superfield $w_D$ can be interpreted as a length in the $(6-D)$-moduli space, and $g_D$ and $k_D$ are coupling constants.

The effective actions of the $[U(1)]^p$ gauge theories are considered also by these methods. The matrix superpotentials of these theories satisfy the $(6-D)\! p$-dimensional Laplace-type equations which are evident for $D=4$ and 5, and also for the $D\leq 3$ superpotentials in the harmonic-integral representations. The harmonic structures of moduli spaces for the $D\leq 3$, $\mathcal{N}=8$ theories arising in connection with the equations for superpotentials generalize the original $SU(2)$-harmonic structure of the $D\geq 4$, $\mathcal{N}=8$ theories. These structures are necessary to classify various $D\leq 3$ supermultiplets. All alternative Grassmann analyticities are compatible with basic supersymmetries and reflect the rich HS geometry of these supersymmetric theories.

Sect. 2 is devoted to the 5-dimensional supersymmetric gauge theories. The HS approach is natural for the perturbative and nonperturbative analysis of these theories. The unique effective Abelian action of the $D=5$ theory in HS contains the free terms and the cubic Chern-Simons terms. This uniqueness is the symmetry basis of the quantum stability of these theories. We also construct the non-Abelian superfield Chern-Simons term.

In Sect. 3, we consider the biharmonic superspace ($BHS$) using harmonics of the automorphism group $SU_l(2) \times SU_r(2)$ in the $D=3$, $\mathcal{N}=8$ models. The 3-dimensional $l$-vector multiplet can be described in terms of the $SU_l(2)$ harmonics, however, the $SU_r(2)$ harmonics arise in the integral representation of the general $D=3$ low-energy superpotential.
f_\alpha$. The Grassmann $r$-analyticity generalizes the holomorphicity in the $HS$ description of the 3D low-energy actions. The $l$-analytic gauge prepotentials and hypermultiplets have their mirror partners in the $r$-analytic superspace.

The $D=2$, $(4,4)$ models in the triharmonic $SU_c(2) \times SU_l(2) \times SU_r(2)$ superspace ($THS$) are discussed in Sect.4. We underline the importance of the $(4,4)$ gauge theory and derive the formula for the effective action in the full superspace using the $SU_c(2)$ harmonics. The integral representation of the $D=2$ superpotential $f_z$ in the $SU_l(2) \times SU_r(2)$ harmonic space contains the $rl$-analytic function of the primary analytic superfield. The full-superspace effective action is equivalent to the action in the $rl$-analytic superspace. The $c$-, $l$- and $r$-analytic superspaces are convenient in classifying the $(4,4)$ representations and duality relations.

The one-dimensional models with 8 supercharges are used in Matrix theory describing the D0-D4 brane interactions. These models have been intensively studied in the field-component formalism and the $\mathcal{N}=4$ superspace. An adequate superfield description of the $D=1, \mathcal{N}=8$ theories requires the use of harmonics for the automorphism group $R_1=SU_c(2) \times Spin(5)$. We define the corresponding $BHS$ gauge and hypermultiplet models in Sect.5.

Problems of the $\mathcal{N}=8$ gauge theories have earlier been discussed in the framework of the component-field formalism or the formalism with $\mathcal{N}=4$, $D=1,2,3$ superfields (see e.g. [12, 14, 36, 41]). In particular, the $(6-D)$ Laplace equations have been considered in the $\mathcal{N}=4$ superfield formalism of the $\mathcal{N}=8$ gauge theories and in the formalism of the corresponding $\sigma$-models. Nevertheless, it should be stressed that the manifestly covariant $HS$ approach provides the most adequate and universal methods to solve the problems of the $\mathcal{N}=8$ theories in all dimensions. A short discussion of these ideas has also been presented in [11].

It should be remarked that the general mathematical formalism for low-dimensional harmonic superspaces has been considered in ref. [11]. This approach treats $HS$ as homogeneous spaces of corresponding complex superconformal groups. We do not discuss the superconformal transformations in this paper and use harmonic variables to find covariant separations of the spinor coordinates in the superfield theories. Our formulations of different harmonic superspaces are connected with the alternative off-shell representations of low-dimensional supermultiplets, their interactions and duality relations.

2 Five-dimensional harmonic gauge theories

The consistent non-anomalous five-dimensional supersymmetric gauge theories have been discussed in refs. [12, 13]. The Coulomb phase of these theories contains the cubic 5D Chern-Simons terms for the gauge fields and cubic interaction of the scalar fields. We shall consider the $D=5$, $\mathcal{N}=8$ $HS$-formalism which is very natural for the perturbative and nonperturbative analysis of quantum problems in these theories.

Let us consider firstly the harmonic superspace with the $D=5$, $\mathcal{N}=8$ supersymmetry. The general five-dimensional superspace has the coordinates $z=(x^m, \theta_i^\alpha)$, where $m$ and $\alpha$ are the 5-vector and 4-spinor indices of the Lorentz group $L_5=SO(4,1)$, respectively, and $i$ is the 2-spinor index of the automorphism group $R_5=SU(2)$. The spinors of $L_5$ are equivalent to the pair of the $SL(2,C)$ spinors: $\Psi^{\alpha}=(\psi^\alpha, \bar{\psi}^{\bar{\alpha}})$.

The invariant symplectic matrices $\Omega_{\alpha\beta}$ and $\Omega^{\alpha\beta}$ can be constructed in terms of the
SL(2, C) ε-symbols

\[ \Omega_{\alpha \rho} = \begin{pmatrix} \varepsilon_{\alpha \rho} & 0 \\ 0 & \varepsilon_{\tilde{\alpha} \tilde{\rho}} \end{pmatrix}, \quad \Omega_{\alpha \rho} \Omega^{\rho \sigma} = \delta_{\alpha}^{\sigma}. \] (2.1)

These matrices connect spinors with low and upper indices.

The antisymmetric traceless representation of the Γ-matrices contains the 4D Weyl matrices \( \sigma_m \) and ε-symbols

\[ (\Gamma_m)_{\alpha \beta} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha \beta} \\ - (\sigma_m)_{\beta \alpha} & 0 \end{pmatrix}, \quad (\Gamma_4)_{\alpha \beta} = \begin{pmatrix} i \varepsilon_{\alpha \beta} & 0 \\ 0 & - i \varepsilon_{\tilde{\alpha} \tilde{\beta}} \end{pmatrix}. \] (2.2)

The corresponding representation of the 5D Clifford algebra has the following form:

\[ (\Gamma_m)_{\alpha \beta} (\Gamma_n)_{\beta \gamma} + (\Gamma_n)_{\alpha \beta} (\Gamma_m)_{\beta \gamma} = -2 \delta_{\gamma}^{\gamma} \eta_{mn}, \] (2.3)

where \( (\Gamma_n)_{\beta \gamma} = \Omega_{\beta \rho} \Omega_{\gamma \sigma} (\Gamma_n)_{\rho \sigma} \) and \( \eta_{mn} \) is the metric of the (4,1) space.

The 5-vector projector in the spinor space is

\[ (\Pi_5)_{\alpha \gamma}^{\rho \sigma} = \frac{1}{4} (\Gamma_m)_{\alpha \gamma}^{\rho \sigma} = \frac{1}{2} (\delta_{\rho}^{\alpha} \delta_{\sigma}^{\gamma} - \delta_{\sigma}^{\alpha} \delta_{\rho}^{\gamma}) + \frac{1}{4} \Omega_{\rho \sigma} \Omega_{\alpha \gamma}. \] (2.4)

Consider also the relations between the antisymmetric 4-spinor symbol \( \mathcal{E} \) and the matrices \( \Omega \) and \( \Gamma \)

\[ \mathcal{E}_{\alpha \rho \mu \nu} = \Omega_{\alpha \rho} \Omega_{\mu \nu} + \Omega_{\alpha \mu} \Omega_{\nu \rho} + \Omega_{\alpha \nu} \Omega_{\rho \mu} = - \frac{1}{2} (\Gamma_m)_{\alpha \rho} (\Gamma_m)_{\mu \nu} + \frac{1}{2} \Omega_{\alpha \rho} \Omega_{\mu \nu}. \] (2.5)

It is convenient to use the bispinor representation of the 5D coordinates and partial derivatives

\[ x^{\alpha \rho} = \frac{1}{2} (\Gamma_m)^{\alpha \rho} x^m, \quad \partial_{\alpha \rho} = \frac{1}{2} (\Gamma_m)^{\alpha \rho} \partial_m. \] (2.6)

The C-conjugation rules for the Spin(4,1) objects are similar to the corresponding rules for (1,0) spinors in the 6D space

\[ \overline{\theta}^{\alpha} \equiv \varepsilon_{ik} C \gamma^k (\theta^i)^* = \theta^{\alpha}, \quad (C^2)^{\alpha}_{\gamma} = - \delta_{\gamma}^{\alpha}, \] (2.7)

\[ \overline{\Omega}^{\alpha \rho} = - \Omega_{\alpha \rho}, \quad \overline{x}^{\alpha \rho} = x^{\alpha \rho}, \quad \overline{\partial}_{\alpha \rho} = - \partial_{\alpha \rho}. \] (2.8)

The basic relations between the spinor derivatives of the D=5, N=8 superspace have the following form:

\[ \{ D_{\alpha}^k, D_{\gamma}^l \} = i \varepsilon^{kl} (\partial_{\alpha \gamma} + \frac{1}{2} \Omega_{\alpha \gamma} Z), \] (2.9)

where Z is the real central charge. We shall consider the basic superspace with Z=0 and introduce the central charges via the interaction of gauge superfields satisfying the constraints

\[ \{ \nabla_{\alpha}^k, \nabla_{\gamma}^l \} = i \varepsilon^{kl} (\nabla_{\alpha \gamma} + \frac{1}{2} \Omega_{\alpha \gamma} W), \] (2.10)

where W is the real superfield.
The spinor $SU(2)/U(1)$ harmonics $u_\pm^i$ can be used to construct the $R_5$-invariant $HS$ coordinates $\zeta=(x_A^m, \theta^\alpha)$, $\theta^{\alpha-}$, spinor derivatives $D_\alpha^\pm$ and harmonic derivatives by analogy with Eqs.(1.4-1.6)

$$D_\alpha^+ = \partial_\alpha^+ , \quad D_\alpha^- = -\partial_\alpha^- - i\theta^{\gamma+} \partial_\alpha^\gamma , \quad (2.11)$$

$$D^{++} = \partial^{++} + i\frac{1}{2} \theta^{\alpha+} \theta^{\gamma+} \partial_\alpha^\gamma + \theta^{\alpha+} \partial_\alpha^\gamma . \quad (2.12)$$

We shall use the following notation for degrees of the spinor derivatives:

$$D^{(\pm 2)} = \frac{1}{4} D^{\alpha\pm} D_\alpha^\pm , \quad D^{(\pm 2)} = (\Pi_5)_{\alpha\gamma} D_\rho^{\pm} D^{\pm} , \quad (2.13)$$

$$D^{(\pm 2)} = D_\alpha^\pm D^{(\pm 2)} , \quad D^{(4)} = 2 D^{(\pm 2)} D^{(\pm 2)} \quad (2.14)$$

and the important identities

$$D^{(2)} D^{(\pm 2)} = 0 , \quad D^{(2)} D^{(\pm 2)} = -2 (\Pi_5)_{\alpha\gamma} \rho\sigma D^{(4)} , \quad (2.15)$$

$$D^{(4)} D^{(\pm 2)} = -2 \partial^{m} \partial_{m} D^{(4)} . \quad (2.16)$$

The analytic Abelian prepotential $V^{++}(\zeta, u)$ describes the 5D vector supermultiplet. In the $WZ$-gauge, this harmonic superfield contains the real scalar field $\Phi$, the Maxwell field $A_m$, the isodoublet of spinors $\lambda_\alpha$ and the auxiliary isotriplet $X^{ij}$

$$V^{++}_{WZ} = i \Theta^{(2)} \Phi(x_\alpha) + \Theta^{(2)} \rho A_{\rho}(x_\alpha) + i \Theta^{(2)} \mu \nu u_\nu \lambda^{\alpha}(x_\alpha) + i \Theta^{(2)} u_k u_j X^{k\nu}(x_\alpha) , \quad (2.17)$$

where

$$\Theta^{(2)} = \frac{1}{4} \theta^{\alpha+} \theta^{\alpha-} , \quad \Theta^{(2)} = (\Pi_5)_{\mu\nu} \theta^{\mu+} \theta^{\nu+} . \quad (2.18)$$

The superfield strength of this theory can be written in terms of the harmonic connection $V^{--}(V^{++})$ (see Eqs. (1.11) and (2.29))

$$W_A = -2i D^{(2)} V^{--} , \quad W_A^\dagger = W_A . \quad (2.19)$$

This superfield satisfies the following constraints:

$$\nabla^{++} W_A = D^{++} W_A + [V^{++}, W_A] = 0 , \quad (2.20)$$

$$D^{(2)} W_A = 0 , \quad (2.21)$$

where the relations (1.11) and (2.13) are used. The Abelian superfield $W_A=W$ does not depend on harmonics.

The 5D SYM action has the universal form (1.8) in the full harmonic superspace. The SYM equations $D^{(4)} V^{--}=0$ have the vacuum Abelian solution $v^{\pm} = i \Theta^{(2)} Z$ where $Z$ is the linear combination of the Cartan generators of the gauge group (see the analogous $D=4$ solution in ref.[8]). This vacuum solution spontaneously breaks the gauge symmetry, but it conserves the $D=5$ supersymmetry with the central charge and produces $BPS$ masses of the $Z$-charged fields. The harmonic supergraphs of this theory can be constructed by the analogy with refs.[2][3]. We do not analyze here the one-loop contribution to the low-energy effective action of this non-renormalizable theory, however, it is not
difficult to consider the general symmetric framework for the description of such actions in HS.

Chiral superspaces are not Lorentz-covariant in the case $D=5$, so one can use the full or analytic superspaces only. It is readily to construct the most general low-energy effective $U(1)$-gauge action in the full $\mathcal{N}=8$ harmonic superspace

$$S_5 = \int d^5x d^8\theta du \, V^{++}V^{-+}\left[g_5^{-2} + k_5 W\right], \quad (2.22)$$

where $g_5$ is the coupling constant of dimension $1/2$, and $k$ is the dimensionless constant of the 5D Chern-Simons interaction. Note that the next-to-leading order effective Abelian 5D action can be written via the manifestly gauge invariant function $H(W)$, but we do not consider these terms.

The linear superpotential $f_5 = g_5^{-2} + k_5 W$ is a solution of the constraints

$$D^{\pm\pm}f_5 = 0, \quad D^{(+2)}_{\alpha\rho}f_5 = 0, \quad (2.23)$$

which arise from the gauge invariance of $S_5$.

It is evident in the HS approach that the unique effective action $f_5$ cannot be renormalized by any consistent calculations preserving the supersymmetry and $U(1)$-gauge symmetry.

Note that the $R_5$ invariance of the effective action can be broken by the Fayet-Iliopoulos term in the analytic superspace

$$S_{FI} = \int d\zeta^{(-4)}du \, \xi^{ik}u^+_{ik}u^+_kV^{++}, \quad (2.24)$$

which implies also the spontaneous breaking of supersymmetry.

The gauge-invariant Chern-Simons term for the group $[U(1)]^p$ contains the following cubic interactions of the Abelian prepotentials $V^{++}_B$ and corresponding constrained superfields $V^{-+}_B$ and $W_B$

$$\int d^5xd^8\theta du \, k_{BCD}V^{++}_B V^{-+}_C W_D, \quad (2.25)$$

where $k_{BCD}$ are coupling constants and $B, C, D = 1 \ldots p$.

It is not difficult to construct the non-Abelian 5D Chern-Simons term $S_{CS}^5$ starting from the following formula of its variation $\delta$:

$$\delta S_{CS}^5 = k_5 \int d^5xd^8\theta du \, Tr \delta V^{++}\{V^{-+}, D^{(+2)}V^{-+}\}$$

$$= k_5 \int d\zeta^{(-4)}du \, Tr \delta V^{++}D^{(+4)}\{V^{-+}, D^{(+2)}V^{-+}\}, \quad (2.26)$$

which guarantees the gauge invariance taking into account Eqs. (1.7, 1.11, 2.20) and (2.21)

$$\delta\lambda S_{CS}^5 = k_5 \int d\zeta^{(-4)}du \, Tr \lambda D^{(+4)}\{D^{(+2)}V^{-+}, \nabla^{++}V^{-+}\}$$

$$= k_5 \int d\zeta^{(-4)}du \, Tr \lambda D^{(+4)}\{D^{(+2)}V^{-+}, D^{-+}V^{++}\} = 0. \quad (2.27)$$

Stress that the analogous term with $[V^{-+}, D^{(+2)}V^{-+}]$ in $\delta S_{CS}^5$ vanishes as an integral of the total spinor derivative.

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\textsuperscript{2}See Note added in the replaced version.
The scale-invariant non-Abelian action $S_{CS}$ has the following form:

$$S_{CS} = \frac{k}{3} \int d^5xd^8\theta du \text{ Tr } V^{++}\{V^{-+}(V^{++}), \; D^{(+2)}V^{-+}(V^{++})\} + \ldots \quad (2.28)$$

where higher-order terms are omitted and the linear approximation of the perturbative solution for $V^{-+}$ is used

$$V^{-+}(V^{++}) = \int du_{1} \frac{V^{++}(z,u_{1})}{(u_{1}^{+})^2} . \quad (2.29)$$

3 Three-dimensional biharmonic superspace

Three-dimensional supersymmetric gauge theories have been intensively studied in the framework of new nonperturbative methods [14, 15, 16]. Superfield description of the $D=3$ theories and various applications have earlier been discussed in refs. [17]-[21]. Three-dimensional harmonic superspaces were considered in refs.[22, 23]. The most interesting features of the $D=3$ theories are connected with the Chern-Simons terms for gauge fields and also with the mirror symmetry between vector multiplets and hypermultiplets.

The $D=3$, $\mathcal{N}=8$ gauge theory can be constructed in the superspace with the automorphism group $R_{3}=SU_{l}(2) \times SU_{r}(2)$. Coordinates of the corresponding general superspace are $z=(x^{\alpha\beta}, \theta^{a}_{\alpha \ia})$. We use here the two-component indices $\alpha$, $\beta$... for the space-time group $SL(2,\mathbb{R})$, $i$, $k$... for the group $SU_{l}(2)$ and $a$, $b$... for $SU_{r}(2)$, respectively.

The relations between basic spinor derivatives are

$$\{D_{\alpha}^{ka}, D_{\beta}^{lb}\} = i\epsilon^{kl}\epsilon^{ab}\partial_{\alpha\beta} + i\epsilon^{kl}\zeta_{\alpha\beta}Z^{ab} , \quad (3.1)$$

where $\partial_{\alpha\beta} = \partial/\partial x^{\alpha\beta}$ and $Z^{ab}$ are the central charges which commute with all generators except for the generators of $SU_{r}(2)$. These central charges can be interpreted as covariantly constant Abelian gauge superfields by analogy with [8].

The superfield constraints of the $\mathcal{N}=8$ SYM theory in the central basis can be written as follows:

$$\{\nabla_{\alpha}^{ka}, \nabla_{\beta}^{lb}\} = i\epsilon^{kl}\epsilon^{ab}\nabla_{\alpha\beta} + i\epsilon_{\alpha\beta}\zeta^{kl} W^{ab} , \quad (3.2)$$

where $\nabla_{M}$ are covariant derivatives with superfield connections and $W^{ab}$ is the constrained superfield of the SYM theory ($l$-vector supermultiplet)

$$\nabla_{\alpha}^{ka} W^{bc} + \nabla_{\alpha}^{kb} W^{ca} + \nabla_{\alpha}^{kc} W^{ab} = 0 . \quad (3.3)$$

Note that gauge transformations in $CB$ have the standard form

$$\delta\nabla_{\alpha}^{ka} = [\tau(z), \nabla_{\alpha}^{ka}] , \quad \delta W^{ab} = [\tau(z), W^{ab}] , \quad (3.4)$$

where $\tau(z)$ is the matrix gauge parameter.

The simplest constraints of the $l$-hypermultiplet are

$$\nabla_{\alpha}^{ia} q^{k} + \nabla_{\alpha}^{ka} q^{i} = 0 . \quad (3.5)$$

It is evident that one can consider the mirror $r$-versions of superfield constraints for the vector multiplet and hypermultiplets changing the roles of $SU_{l}(2)$ and $SU_{r}(2)$ indices

$$\nabla_{\alpha}^{ia} W_{r}^{kl} + \nabla_{\alpha}^{ka} W_{r}^{li} + \nabla_{\alpha}^{la} W_{r}^{ik} = 0 , \quad (3.6)$$

$$\nabla_{\alpha}^{ia} q_{r}^{b} + \nabla_{\alpha}^{ib} q_{r}^{a} = 0 . \quad (3.7)$$
We shall define the general 3D biharmonic superspace which has simple properties with respect to the exchange $l \leftrightarrow r$. The mirror symmetry connects $l$-vector multiplets with $r$-hypermultiplets and vice versa.

Let us consider the $l$-harmonics $u_l^\pm \equiv u_l^{(\pm l,0)}$ of the group $SU_l(2)$ and the analogous $r$-harmonics $v_r^{(0,\pm 1)}$ of the group $SU_r(2)$. The notation of charges in $BHS$ is $(q_l, q_r)$. The constraints of the $l$-vector multiplet and $l$-hypermultiplet can be solved with the help of the $l$-harmonics only, so we shall use also the notation with the one charge for the $l$-harmonic superspace $HS_l$ which is analogous to the $4D$ $HS$. The $r$-harmonic structures arise in the geometric description of the low-energy self-interaction of the $l$-vector multiplets and dualities between $l$- and $r$-type supermultiplets.

The spinor and $l$-harmonic derivatives have the following form in the $l$-analytic coordinates $\zeta_l=(x_{l\alpha\beta}, \theta^{a+})$ and $\theta^{a-}_l$:

$$D^{b+}_{a} = u_l^{b+}D^{b}_{a} = \partial^{b+}_{a}, \quad D^{b-}_{a} = u_l^{b-}D^{b}_{a} = -\partial^{b-}_{a} + i\theta^{b\beta}D^{\beta}_{a},$$

$$D^{b+}_{l} = \partial^{b+}_{l} - \frac{i}{2}\theta^{a+}\theta^{b+}\partial^{\beta}_{l} + \theta^{a+}\partial^{a+}.$$  \hspace{1cm} (3.8)

The following relations will be used in $HS_l$:

$$\{D^{a+}, D^{b-}\} = -i\varepsilon^{a\beta}\partial^{b\beta}, \quad [D^{b-}, D^{a+}] = D^{a-},$$

$$D^{(1/2)}_{a\beta}D^{ab+} = 0, \quad D^{(1/2)}_{ab}D_{cd}^{(1/2)} = (\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{bc}\varepsilon_{ad})(D^{+})^4,$$ \hspace{1cm} (3.10)

where

$$D^{(1/2)}_{a\beta} = \frac{1}{2}D^{+a}_{a\beta}, \quad D^{ab+} = \frac{1}{2}D^{a+}D^{b+}. \hspace{1cm} (3.12)$$

The $l$-harmonic superspace is adequate to the solution of the constraints (3.2)

$$u_l^{+}u_k^{+}\{\nabla^{ia}_{\alpha}, \nabla^{kb}_{\beta}\} \equiv \{\nabla^{a+}_{\alpha}, \nabla^{b+}_{\beta}\} = 0,$$ \hspace{1cm} (3.13)

$$\nabla^{a+}_{\alpha} = g^{-1}(z,u)D^{a+}_{\alpha}g(z,u),$$ \hspace{1cm} (3.14)

where $g(z,u)$ is the bridge matrix $[\mathbb{I}]$. The $l$-analytic prepotential of the SYM theory is

$$V^{++}_{+}(\zeta_l, u) = (D^{++}g)g^{-1}, \quad D^{a+}V^{++}_l = 0,$$ \hspace{1cm} (3.15)

$$\delta g = \lambda g - g\tau(z), \quad \delta_{\lambda}V^{++}_l = D^{++}\lambda + [V^{++}_l, \lambda].$$ \hspace{1cm} (3.16)

The components of this superfield can be determined in the WZ gauge

$$(V^{++}_l)_{wz} = \theta^{a+}\theta^{b+\alpha}_{\alpha} + \theta^{a\alpha}\theta^{b+\alpha\beta}_{\alpha\beta}A_{\alpha\beta}(x_l) + \theta^{a+}\theta^{b+\alpha}_{\alpha}u_k^{b+\lambda}_{\lambda\beta}(x_l) + i(\theta^{+})^4u_k^{b+}\lambda^{kj}_{\beta}(x_l).$$ \hspace{1cm} (3.17)

The superfield strength of the $D=3, N=8$ gauge theory in the analytic basis contains the corresponding harmonic connection $V^{++}_l(V^{++}_l)$

$$W^{ab}_{A} = -iD^{ab+}V^{++}_l \equiv gW^{ab}g^{-1},$$ \hspace{1cm} (3.18)

$$\delta_{\lambda}W^{ab}_{A} = [\lambda, W^{ab}_{A}], \quad \delta_{\lambda}V^{++}_l = \nabla^{++}\lambda.$$ \hspace{1cm} (3.19)

It satisfies the following constraints:

$$D^{a+}W^{bc}_{\alpha\beta} + D^{b+}W^{ca}_{\alpha\beta} + D^{c+}W^{ab}_{\alpha\beta} = 0, \Rightarrow D^{(1/2)}_{a\beta}W^{bc}_{A} = 0,$$ \hspace{1cm} (3.20)

$$D^{a+}W^{ab}_{\alpha\beta} + [V^{++}_l, W^{ab}_{A}] = 0.$$ \hspace{1cm} (3.21)
which are equivalent to the CB-constraints (3.3).

The Abelian superfield $W^{a\beta}_{a\beta} \equiv W_{ab}$ does not depend on harmonics ($D^{\pm\pm} W_{ab} = 0$). The vacuum Abelian solution of the SYM theory

$$V_{++} \equiv v_{++} = i \theta^{a\gamma} \dot{\theta}^{b\pm} Z_{ab} , \quad (D^+) v_{--} \sim D^{(+)2}_{ab} W_{ab} = 0$$

(3.22)
is covariant with respect to the supersymmetry with central charges $Z_{ab}$ by analogy with the case $D=4$.

The $l$-analytic hypermultiplet $q^\pm(\zeta, u) \equiv q^{(1,0)}$ has the standard minimal interaction with $V_{++} \equiv V^{(2,0)}$ (see (1.13)). By analogy with refs. $\ddagger$, $\ddagger$, one can construct the free $HS$ propagator for this superfield in the covariantly constant background (3.22)

$$i \langle q^+(1) | q^+(2) \rangle = - \frac{1}{\Box_1^2} (D^+) (D^+)^4 (v^2 - v_1) \delta^{11}(z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^3} ,$$

(3.23)

where $\Box^2 = \partial^{a\beta} \partial_{a\beta} + Z_{ab} Z_{ab}$ and $D^{++} v = v_{++}$. The manifestly supersymmetric perturbation theory is the important advantage of the $HS$ approach.

One can consider also the alternative 3D hypermultiplet $\omega_l(\zeta, u)$ and the $l$-linear multiplet $L^{(2,0)}(\zeta, u)$ satisfying the harmonic condition $D^{(2,0)}_l L^{(2,0)} = 0$.

We do not discuss here the harmonic-supergraph calculations of the perturbative effective action and consider the general symmetry framework for these constructions in the full $l$-harmonic superspace. The low-energy $U(1)$ effective action can be expressed in terms of the superpotential $f(W_{ab})$ which does not depend on $u^\pm$

$$S_3 = \int d^3 x d^3 \theta d u \ V_{++} V_{--} f_3(W_{ab}) .$$

(3.24)
The corresponding bosonic-field Lagrangian contains the nonlinear $\sigma$-model interaction of the 3-component scalar field $\Phi_{ab}$ (1.17) and the non-minimal interaction of this field with the Abelian gauge field. All interactions of the field $\Phi_{ab}$ are determined via derivatives of the function $f_3(\Phi_{ab})$.

The gauge invariance produces the following constraint:

$$\delta_\lambda S_3 = -2 \int d^3 x d^3 \theta d u \lambda D^{--} V_{++} f_3(W_{ab})$$

$$\sim \int d^3 x (D^-)^4 d u \lambda \partial^{a\beta} V_{++} D^{(+)2}_{a\beta} f_3(W_{ab}) = 0 ,$$

(3.25)

where the analyticity of $V_{++}$ and relations (1.11) and $d^3 \theta = (D^-)^4 (D^+)^4$ are used.

This constraint on the superpotential is equivalent to the 3D Laplace equation

$$D^{(+)2}_{a\beta} f_3(W_{ab}) = 0 \rightarrow \frac{\partial}{\partial W_{ab}} \frac{\partial}{\partial W_{ab}} f_3(W_{ab}) = 0 .$$

(3.26)

The general solution of this equation breaks the $SU_r(2)$ invariance. The $R_3$-invariant superpotential has the following form:

$$f_3^R(w_3) = g_3 w_3^{-2} + k_3 w_3^{-1} , \quad w_3 = \sqrt{W_{ab} W_{ab}} ,$$

(3.27)

where $g_3$ is the coupling constant of dimension $d = -1/2$, and $k_3$ is the dimensionless constant of the $N=8$ WZNW-type interaction of the vector multiplet. This superpotential is singular at the point $Z_{ab} = 0$ of the moduli space. The field model is well defined
in the shifted variables \( \hat{W}_{ab} = W_{ab} - Z_{ab} \) for nonvanishing central charges. It should be remarked that the superfield interactions of the 3D-vector multiplets with dimensionless constants (Chern-Simons terms) have earlier been constructed for the case \( \mathcal{N}=4 \) \cite{18} and \( \mathcal{N}=6 \) \cite{3, 4}.

The effective action (3.27) is an example of the \( D=3 \) non-perturbative calculation based on the \( \mathcal{N}=8 \) supersymmetry and the \( R_3 \)-invariance. Stress that the HS approach simplifies the proof of this non-renormalization theorem.

The general solution of Eq. (3.26) can be written in the \( v \)-integral harmonic representation
\[
    f_s(W^{ab}) = \int dv F_3[\nu^{(0,2)}], v_a^{(0,\pm 1)}], \quad W^{(0,2)} = v_a^{(0,1)} v_b^{(0,1)} W^{ab}, \tag{3.28}
\]
where \( F_s \) is an arbitrary function with \( q = (0,0) \), and \( W^{(0,2)} \) is the \( r \)-harmonic projection of the basic superfield. The proof is based on the following \( r \)-harmonic representation of the Laplace operator:
\[
    \varepsilon_{ac} \varepsilon_{bd} \frac{\partial}{\partial W_{ab}} \frac{\partial}{\partial W_{cd}} \sim \frac{\partial}{\partial W^{(0,2)}} \frac{\partial}{\partial W^{(0,-2)}} - \left( \frac{\partial}{\partial W^{(0,0)}} \right)^2, \tag{3.29}
    \varepsilon_{ac} = v_a^{(0,1)} v_c^{(0,-1)} - v_c^{(0,1)} v_a^{(0,-1)}. \tag{3.30}
\]

This solution is a covariant form of the well-known integral representation of the 3D-harmonic functions \cite{24}. The functions \( F_3 \) do not depend on the projections \( W^{(0,-2)} \) and \( W^{(0,0)} \) of the superfield \( W^{ab} \) while the holomorphic functions \( F(W_{11}) \) of the chiral superfield \( W_{11} \) are independent of the components \( W_{22} \) and \( W_{12} \). The \( v \)-integral representation of \( F(W_{11}) \) can depend on the 1-st harmonic component \( v_1^{(0,\pm 1)} \) only, so the representation \( (3.28) \) is more general than the holomorphic representation of superpotential. We shall show that the function \( F_3 \) as well as the superfield \( W^{(0,2)} \) satisfy the condition of the Grassmann \( r \)-analyticity.

Let us introduce the following definitions and relations for the spinor derivatives in the biharmonic superspace:
\[
    D^{(\pm 1,\pm 1)}_\alpha = u_i^{(\pm 1,0)} D^{i\alpha}_a, \quad D^{(\pm 2,\pm 2)}_\alpha = D^{(\pm 1,\pm 1)}_\alpha D^{(\pm 1,\pm 1)}_\alpha, \tag{3.31}
\]
\[
    [D^{(\pm 2,0)}_i, D^{(\pm 1,\pm 1)}_\alpha] = D^{(\pm 1,\pm 1)}_\alpha, \quad [D^{(0,\pm 2)}_r, D^{(\pm 1,\pm 1)}_\alpha] = D^{(\pm 1,\pm 1)}_\alpha, \tag{3.32}
\]
\[
    [D^{(\pm 2,0)}_i, D^{(\pm 1,\pm 1)}_\alpha] = [D^{(0,\pm 2)}_r, D^{(\pm 1,\pm 1)}_\alpha] = 0, \tag{3.33}
\]
\[
    D^{(\pm 4,0)} = D^{(\pm 2,2)} D^{(\pm 2,-2)}, \quad D^{(0,4)} = D^{(2,2)} D^{(-2,-2)}. \tag{3.34}
\]

Introduce the \( r \)-analytic coordinates
\[
    x^{\alpha \beta}_r = x^{\alpha \beta} + \frac{i}{2} [\theta^{(0,1)}_k \theta^{(0,-1)}_l + (\alpha \leftrightarrow \beta)], \tag{3.35}
\]
\[
    \theta^{(0,1)}_\alpha = v_a^{(0,1)} \theta^{(0,1)}_\alpha. \tag{3.36}
\]

These coordinates are natural for the \( r \)-analytic superfields \( \Phi_r(x^{(0,1)}_\alpha, \theta^{(0,1)}_\alpha, v_a^{(0,\pm 1)}) \) which describe the alternative representations of the \( D=3, \mathcal{N}=8 \) supersymmetry.

The spinor and harmonic derivatives in these coordinates have the following form:
\[
    D^{(0,1)}_\alpha = v_a^{(0,1)} D^{ka}_\alpha = \partial^{(0,1)}_\alpha, \tag{3.37}
\]
\[
    D^{(0,-1)}_\alpha = v_a^{(0,-1)} D^{ka}_\alpha = -\partial^{(0,-1)}_\alpha + i \theta^{(0,-1)}_k \partial^{(0,-1)}_\alpha \tag{3.38}
\]
\[
    D^{(0,2)}_r = \partial^{(0,2)}_r - \frac{i}{2} \theta^{(0,1)}_k \theta^{(0,1)} \partial^{(0,1)}_\alpha + \theta^{(0,1)}_k \partial^{(0,1)}_\alpha. \tag{3.39}
\]
The constraints \((3.20)\) are equivalent to the \(r\)-analyticity condition

\[
D^k(\zeta, v) = 0
\]

and the following \(r\)-harmonic conditions:

\[
D_r(\zeta, v) = 0 .
\] (3.41)

One can consider the component decomposition of this representation of the \(l\)-vector multiplet which is equivalent to the \(r\)-linear analytic multiplet

\[
W(r, 2) = -iD(2)\bar{V}(-2, 0) = -i\int duD(-2, 2)\bar{V}(2, 0) .
\] (3.43)

It is clear that this representation of the \(l\)-vector multiplet is equivalent to the representation \((3.13)\).

Consider the \(r\)-harmonic decomposition of the full spinor measure

\[
d^\theta = D(0, -4)D(0, 4) .
\] (3.44)

Using this decomposition and Eqs.\((3.24, 3.28)\) and \((3.43)\) we can construct an equivalent form of the effective action in the \(r\)-analytic superspace

\[
S_3 = \int d^3x D(0, -4)dv[W(0, 2)]^2F_3[W(0, 2), v(0, \pm 1)] .
\] (3.45)

It should be underlined that this action with the gauge-invariant analytic Lagrangian can be generalized to the case of non-Abelian SYM theory.

Let us consider now the set of \(l\)-analytic prepotentials \(V_{lB}^{(2, 0)}\) in the \([U(1)]p\) gauge theory and the corresponding \(r\)-analytic superfields \(W_{lB}^{(0, 2)}(V_{lB}^{(2, 0)})\). The effective action of this theory in the \(r\)-analytic superspace is

\[
S_p^3 = \int d^3x D(0, -4)dv \sum_{B, C=1}^p W_{lB}^{(0, 2)}W_{lC}^{(0, 2)}f_{BC}^3[W_{lB}^{(0, 2)}, v(0, \pm 1)] ,
\] (3.46)

where \(F_{BC}\) are real \(q=(0, 0)\) functions of the superfields \(W_{lB}^{(0, 2)}, \ldots W_{lB}^{(0, 2)}\) and \(v\)-harmonics. The corresponding effective action in the full superspace contains the matrix superpotential of the \([U(1)]p\) gauge theory

\[
S_p^3 = \sum_{B, C=1}^p \int d^3x d^\theta dv V_{lB}^{(2, 0)}V_{lC}^{(-2, 0)}f_{BC}^3(W_{lB}^{(0, 2)}, \ldots W_{lB}^{(0, 2)}) ,
\] (3.47)

\[
f_{BC}^3(W_{lB}^{0, 2}, \ldots W_{lB}^{0, 2}) = \int dvF_{BC}^3[W_{lB}^{(0, 2)}, v(0, \pm 1)] .
\] (3.48)
The $v$-integral representation of the matrix superpotential satisfies the following conditions:

\[
\frac{\partial}{\partial W_M^{cd}} \frac{\partial}{\partial W_N^{ef}} f_{BC}^3 \equiv \Delta_{MN}^3 f_{BC}^3 = 0 ,
\]

(3.49)

\[
D_{\alpha \beta}^{(+2)} f_{BC}^3 = 0 .
\]

(3.50)

This can be proved with the help of the harmonic decomposition of the $SU_r(2)$-invariant operator $\Delta_{MN}^3$ by analogy with (3.29)

\[
\Delta_{MN}^3 \sim \frac{\partial}{\partial W_M^{(0,2)}} \frac{\partial}{\partial W_N^{(0,-2)}} - \frac{\partial}{\partial W_M^{(0,0)}} \frac{\partial}{\partial W_N^{(0,0)}} .
\]

(3.51)

These conditions guarantee the gauge invariance of $S_{p3}$ in the full superspace.

The $r$-forms of the hypermultiplet constraints have been discussed in ref.[23]. Consider the superfield constraints for these hypermultiplets in the framework of $BHS$

\[
D_{\alpha}^{(+1.1)} q_{r}^{a(0,1)} = 0 , \quad q_{r}^{a(0,1)} = (q_{r}^{(0,1)}, \bar{q}_{r}^{(0,1)}) ,
\]

(3.52)

\[
D_{\alpha}^{(+1.1)} \omega_{r} = 0 , \quad D_{l}^{(\pm 2,0)} (\omega_{r}, q_{r}^{a(0,1)}) = 0 ,
\]

(3.53)

where $D_{l}^{(\pm 2,0)}$ are the $l$-harmonic derivatives.

These hypermultiplets are dual to each other and also to the $r$-linear analytic multiplet

\[
q_{r}^{a(0,1)} = v^{a(0,1)} \omega_{r} + v^{a(0,-1)} L^{(0,2)} .
\]

(3.54)

The duality relation between the $\omega_{r}$ and $r$-linear multiplet is described by the action

\[
\int d^3 x D^{(0,-4)} dv \{ \omega_{r} [D^{(0,2)} L^{(0,2)}] + F^{(0,4)} [L^{(0,2)}, v^{(0,\pm 1)}] \} ,
\]

(3.55)

where $F^{(0,4)}$ is an arbitrary $r$-analytic function.

It is clear that the $l$-analytic hypermultiplets $q_{l}^{(1,0)}$ and $\omega_{l}$ are dual to the alternative $r$-version of the vector multiplet which can be described by the $r$-analytic prepotential $V_{r}^{(0,2)}$.

Thus, the $l$-analyticity allows us to solve the constraints of the $l$-vector multiplet and $l$-hypermultiplets, while the $r$-analyticity generalizes the holomorphicity and chirality in the HS description of low-energy gauge actions and duality symmetries.

4 Two-dimensional (4,4) harmonic superspaces

The $D=2$, (4,4)-supersymmetric field theories describe 1-branes probing a background with 5-branes in M-theory [28, 36, 37]. The two-dimensional (4,4) and (4,0) $\sigma$-models have been discussed in the field-component formalism and in the framework of the ordinary or harmonic superspaces [25]-[34]. The 2D mirror symmetry and the (4,4) gauge theory has been considered in the component formalism and in the (2,2) superspace [35, 36, 37]. We shall study the geometry of this theory in the manifestly covariant harmonic formalism which is convenient for the superfield quantum calculations. Three types of Grassmann analyticities will be used to classify the (4,4) supermultiplets, their interactions and 2D duality symmetries.
The maximum automorphism group of the (4,4) superspace is $SO_l(4) \times SO_r(4)$; however, we shall mainly use the group $R_c = SU_c(2) \times SU_l(2) \times SU_r(2)$. Let us choose the left and right coordinates in the (4,4) superspace

$$z_l = (y, \theta^{\alpha}) , \quad z_r = (\bar{y}, \bar{\theta}^{\alpha}) ,$$  \hspace{1cm} (4.1)

where $y = (1/\sqrt{2})(t + x)$ and $\bar{y} = (1/\sqrt{2})(t – x)$ are the light-cone 2D coordinates; and the following types of 2-spinor indices are used: $i, k, \ldots$ for $SU_c(2)$; $\alpha, \beta, \ldots$ for $SU_l(2)$ and $a, b, \ldots$ for $SU_r(2)$, respectively. The $SO(1,1)$ weights of coordinates are $(1, 1/2)$ for $z_l$ and $(-1, -1/2)$ for $z_r$. The algebra of spinor derivatives in this superspace contains the central charges $Z_{ab}$.

The $CB$-geometry of the (4,4) SYM theory is described by the superfield constraints

$$\{ \nabla_{ka}, \nabla_{l\beta} \} = i \varepsilon_{kl} \varepsilon_{\alpha \beta} \nabla_y , \quad (4.2)$$
$$\{ \bar{D}_{ka}, \bar{D}_{lb} \} = i \varepsilon_{kl} \varepsilon_{\alpha \beta} \bar{\nabla}_y , \quad (4.3)$$
$$\{ D_{ka}, D_{lb} \} = i \varepsilon_{kl} Z_{ab} \quad (4.4)$$

where $\nabla_M = D_M + A_M$ is the covariant derivative for the corresponding coordinate. The gauge-covariant superfield $W_{ab}$ satisfies the constraints of the (4,4) vector multiplet which are equivalent to the constraints of the so-called twisted multiplet [25, 29].

The authors of refs. [31, 32, 33] have discussed three types of harmonics: $u^\pm_i = u^{(\pm 1,0,0)}_i$ for $SU_c(2)/U_c(1)$; $l^\pm_\alpha = l^{(0,\pm 1,0)}_\alpha$ for $SU_l(2)/U_l(1)$; and $r^\pm_\alpha = r^{(0,0,\pm 1)}_\alpha$ for $SU_r(2)/U_r(1)$. We use the notation with 3 charges in the triharmonic superspace (THS) and the standard notation in the $c$-harmonic superspace $HS_c$. The basic geometric structures of the gauge theory are mainly connected with the $c$-harmonics $u^c_i$ and the corresponding analytic coordinates

$$\zeta_c = (y_c, \theta^{\alpha+}), \quad \bar{\zeta}_c = (\bar{y}_c, \bar{\theta}^{\alpha+}) , \quad (4.8)$$

The $HS_c$ spinor derivatives and harmonic derivatives have the following form in the case of vanishing central charges:

$$D^+_\alpha = \partial^{\alpha+}_\alpha , \quad D^\pm_\alpha = \partial^\pm_\alpha = -\partial^\mp_\alpha - i\theta^\mp_\alpha \partial^c_\alpha \quad (4.9)$$
$$\bar{D}^+_a = \partial^{\alpha+}_a , \quad \bar{D}^\pm_a = \partial^\pm_a = -\partial^\mp_a - i\bar{\theta}^\mp_a \partial^c_a \quad (4.10)$$
$$D^{+a} = \partial^{+a} + i/2 \theta^{\alpha+} \theta^a \partial^c + i/2 \bar{\theta}^{\alpha+} \bar{\theta}^a \partial^c$$

The basic combinations of the spinor derivatives are

$$(D^\pm)^2 = 1/2 D^\pm_\alpha D^\pm_\alpha , \quad (\bar{D}^\pm)^2 = 1/2 \bar{D}^\pm_a \bar{D}^\pm_a , \quad (D^\pm)^4 = (D^\pm)^2(\bar{D}^\pm)^2$$ \hspace{1cm} (4.12)

The $c$-harmonic projections of the constraints [4.5, 4.7] are equivalent to the integrability conditions of the $c$-analyticity by analogy with the $D \geq 3$, $N=8$ theories

$$\{ \nabla^\pm_\alpha, \nabla^\pm_\beta \} = \{ \nabla^+_a, \nabla^+_b \} = \{ \nabla^+_a, \nabla^+_b \} = 0, \quad (4.13)$$
where $\nabla^+_a = u^+_a \nabla^i_a$ and $\nabla^+_a = u^+_a \nabla^i_a$.

The prepotential of the (4,4) gauge theory in $HS_c$ is the $c$-analytic harmonic connection $V^{++}_c(\zeta_c, \tilde{\zeta}_c, u) \equiv V^{(2,0,0)}_c$ which determines the second harmonic connection $V^{--}_c \equiv V^{(-2,0,0)}_c$. The $WZ$ gauge for this prepotential has the following form:

$$(V^{++}_c)_{WZ} = \theta^{\alpha^+} \bar{\theta}^{\alpha^+} \Phi_{\alpha \beta}(y_c, \bar{y}_c) + (\theta^{\alpha^+})^2 \bar{A}(y_c, \bar{y}_c) + (\bar{\theta}^{\alpha^+})^2 A(y_c, \bar{y}_c) + (\bar{\theta}^{\alpha^+})^2 \theta^{\alpha^+} \bar{\lambda}_i\delta(y_c, \bar{y}_c) + i(\theta^{\alpha^+})^2 (\bar{\theta}^{\alpha^+})^2 u_k^\nu \bar{u}^\nu_k X^{kj}(y_c, \bar{y}_c)$$

where the components of the 2D vector multiplet are defined.

The gauge-covariant Abelian superfield strength can be constructed by analogy with $D=3$

$$W_{ab} \equiv (\sigma^m)_{ab} W_m = -i D_\alpha^+ \bar{D}^+_{\nu} V^{--}_c$$

where $(\sigma^m)_{ab}$ are the Weyl matrices for $SU_1(2) \times SU_r(2)$ and $W_m$ is the 4-vector representation of this superfield.

The Abelian superfield $W_{ab}$ does not depend on harmonics, and the constraints for this superfield are

$$D^a_{\alpha} W_{b\beta} = \frac{1}{2} \varepsilon_{\alpha \beta} D^{+\rho} W_{\rho \delta}, \quad \bar{D}^a_{\alpha} W_{b\beta} = \frac{1}{2} \varepsilon_{\alpha \beta} \bar{D}^{+\rho} W_{\rho \delta}.$$  \hspace{1cm} (4.16)

We shall use also the following consequences of these relations:

$$(D^+)^2 W_{aa} = (\bar{D}^+)^2 W_{aa} = 0.$$  \hspace{1cm} (4.17)

The $c$-analytic (4,4) hypermultiplets $q^+_c$ and $\omega_c$ have the minimal interactions with $V^{++}_c$. The corresponding $HS_c$ Feynmann rules can be formulated by analogy with ref. [2]. The $HS_c$ perturbative methods can be useful in the analysis of the vector-hypermultiplet Matrix models with (8,8) supersymmetry, however, we shall discuss here the general symmetry framework for such calculations.

The universal harmonic construction of the $U(1)$ effective action with 8 supercharges has the following form in the case $D=2$:

$$S_2 = \int d^2xd\theta du V^{++}_c V^{--}_c f_2(W_m),$$

where $d^2x = dtdx \equiv dyd\bar{y}$. The gauge invariance imposes the following constraints:

$$(D^+)^2 f_2(W_m) = (\bar{D}^+)^2 f_2(W_m) = 0.$$ \hspace{1cm} (4.19)

Using the Eqs. (4.17) one can prove that the (4,4) superpotential satisfies the 4D Laplace equation

$$\Delta^u_1 f_2(W_m) = 0, \quad \Delta^u_1 = \left( \frac{\partial}{\partial W_m} \right)^2.$$ \hspace{1cm} (4.20)

The analogous 4D Laplace equation in the (4,4) $\sigma$-models has been discussed, for instance, in refs. [28, 34].

The $R_x$-invariant solution of this equation determines uniquely the exact superpotential of the (4,4) gauge theory

$$f_2^R(w_2) = g_2^{-2} + k_2 w_2^{-2}, \quad w_2 = \sqrt{W^{aa} W_{aa}}.$$ \hspace{1cm} (4.21)
The same function of the (2,2) superfields generates the $R_\sigma$-invariant K"ahler potential of the $D=2$, $(4,4)$ gauge theory \cite{30}. Note that the K"ahler potential of the (2,2) formalism is gauge-invariant by definition, and the 4D Laplace equation arises in this approach from the restrictions of the $(4,4)$ supersymmetry; while in our formulation the analogous condition on the $(4,4)$ superpotential (4.20) follows from the gauge invariance. The manifestly (4,4) covariant formalism of the harmonic gauge theory simplifies the proof of the non-renormalization theorem.

The $THS$ projections of the 2D spinor derivatives are
\begin{equation}
D^{(\pm 1,\pm 1,0)} = u^{(\pm 1,0,0)}_\alpha l^{(0,\pm 1,0)}_\alpha D^{\alpha\bar{\alpha}} , \quad \bar{D}^{(\pm 1,\pm 0,1)} = u^{(\pm 1,0,0)}_\alpha r^{(0,0,\pm 1)}_\alpha \bar{D}^{\alpha\bar{\alpha}} .
\end{equation}

The $rl$-version of the $c$-vector multiplet (4.15) has the following form:
\begin{equation}
W^{(0,1,1)} = -i D^{(1,1,0)} \bar{D}^{(1,0,1)} V_c^{(-2,0,0)} = -i \int du D^{(-1,1,0)} \bar{D}^{(-1,0,1)} V_c^{(2,0,0)} .
\end{equation}

By construction, this superfield satisfies the conditions of the $rl$-analyticity
\begin{equation}
D^{(\pm 1,1,0)} W^{(0,1,1)} = 0 , \quad \bar{D}^{(\pm 1,0,1)} W^{(0,1,1)} = 0
\quad \text{and the harmonic conditions}
\end{equation}
\begin{equation}
D^{(\pm 2,0,0)}_c W^{(0,1,1)} = D^{(0,2,0)}_l W^{(0,1,1)} = D^{(0,0,2)}_r W^{(0,1,1)} = 0 .
\end{equation}

The analogous constraints on the $rl$-harmonic superfield $q^{(1,1)}$ have been considered in ref.\cite{31} (this notation does not indicate the $U_r(1)$ charge). Note that the $c$-vector multiplet (4.23) contains the field-strength of the 2D vector field instead of the auxiliary scalar component in the superfield $q^{(1,1)}$.

The $c$-analyticity become manifest in the coordinates (1.8). Let us consider the analogous $rl$-analytic coordinates which help to solve the conditions (1.24)
\begin{equation}
\zeta_l = (y_l, \theta^{(\pm 1,1,0)}) , \quad \theta^{(\pm 1,1,0)} = u^{(\pm 1,0,0)}_\alpha l^{(0,\pm 1,0)}_\alpha \theta^{\alpha\bar{\alpha}} ,
\end{equation}
\begin{equation}
y_l = y + \frac{i}{2}[\theta^{(1,1,0)} \theta^{(-1,1,0)} - \theta^{(-1,1,0)} \theta^{(1,1,0)}] ,
\end{equation}
\begin{equation}
\bar{\zeta}_r = (\bar{y}_r, \bar{\theta}^{(\pm 1,0,1)}) , \quad \bar{\theta}^{(\pm 1,0,1)} = u^{(\pm 1,0,0)}_\alpha r^{(0,0,\pm 1)}_\alpha \bar{\theta}^{\alpha\bar{\alpha}} ,
\end{equation}
\begin{equation}
y_r = y + \frac{i}{2}[\bar{\theta}^{(1,0,1)} \bar{\theta}^{(-1,1,0)} - \bar{\theta}^{(-1,0,1)} \bar{\theta}^{(1,1,0)}] .
\end{equation}

The spinor and harmonic derivatives have the following form in these coordinates:
\begin{equation}
D^{(\pm 1,1,0)} = \pm \partial^{(\pm 1,1,0)} , \quad D^{(\pm 1,1,-1)} = \mp \partial^{(\pm 1,1,-1)} + i \theta^{(1,1,0)} \partial_y ,
\end{equation}
\begin{equation}
\bar{D}^{(\pm 1,0,1)} = \pm \bar{\partial}^{(\pm 1,0,1)} , \quad \bar{D}^{(\pm 1,0,1)} = \mp \bar{\partial}^{(\pm 1,0,1)} + i \bar{\theta}^{(1,1,0)} \bar{\partial}_y ,
\end{equation}
\begin{equation}
D^{(0,2,0)}_l = \partial^{(0,2,0)} + i \theta^{(1,1,0)} \theta^{(-1,1,0)} \partial_y + \theta^{(1,1,0)} \theta^{(-1,1,0)} \partial_y + \theta^{(-1,1,0)} \theta^{(1,1,0)} ,
\end{equation}
\begin{equation}
\bar{D}^{(0,0,2)}_c = \bar{\partial}^{(0,0,2)} + i \bar{\theta}^{(1,1,0)} \bar{\theta}^{(-1,1,0)} \bar{\partial}_y + \bar{\theta}^{(1,1,0)} \bar{\partial}_y + \bar{\theta}^{(-1,1,0)} \bar{\theta}^{(1,1,0)} .
\end{equation}

The $c$-analytic coordinates in the $THS$ notation are
\begin{equation}
\zeta_c = (y_c, \theta^{(1,1,0)}) , \quad y_c = y + \frac{i}{2}[\theta^{(-1,1,0)} \theta^{(1,1,0)} + \theta^{(1,1,0)} \theta^{(-1,1,0)}] ,
\end{equation}
\begin{equation}
\bar{\zeta}_c = (\bar{y}_c, \bar{\theta}^{(1,0,\pm 1)}) , \quad \bar{y}_c = \bar{y} + \frac{i}{2}[\bar{\theta}^{(-1,0,\pm 1)} \bar{\theta}^{(1,0,\pm 1)} + \bar{\theta}^{(1,0,\pm 1)} \bar{\theta}^{(-1,0,\pm 1)}] .
\end{equation}
It is important that all coordinates $\zeta_c, \bar{\zeta}_c, \zeta_l$ and $\bar{\zeta}_r$ are separately real with respect to the corresponding conjugation. Of course, all these sets of coordinates are irreducible with respect to the supersymmetry transformations.

The solution of the 4D Laplace equation (4.20) has the simple harmonic representation

$$ f_2(W_{\alpha \beta}) = \int dldr F_2[W^{(0,1,1)}, l, r] , \quad (4.36) $$

where $F_2$ is the real function and $W^{(0,1,1)} = l_{\alpha}^{(0,1,0)} r_a^{(0,1,0)} W^{\alpha a} (4.23)$. The proof is based on the THS decomposition of the 4D Laplace operator

$$ \frac{\partial}{\partial W^{ab}} \frac{\partial}{\partial W_{ab}} \sim \frac{\partial}{\partial W^{(0,1,1)}} \frac{\partial}{\partial W_{(0,1,1)}} - \frac{\partial}{\partial W^{(0,1,0)}} \frac{\partial}{\partial W_{(0,1,0)}} . \quad (4.37) $$

Note that the formal change of the density in (4.36)

$$ F_2[W^{(0,1,1)}, l, r] \rightarrow F'[W^{(0,1,1)}, W^{(0,1,-1)}, l, r] \quad (4.38) $$

does not produce more general superpotentials. This can be easily shown for the polynomial solutions of Eq.(4.20).

Consider the THS decomposition of the Grassmann measure

$$ d^4 \theta = D^{(0,-2,-2)} D^{(0,2,2)} , \quad (4.39) $$

$$ D^{(0,\pm 2,\pm 2)} = D^{(1,\pm 1,0)} D^{(-1,\pm 1,0)} \bar{D}^{(1,0,\pm 1)} \bar{D}^{(-1,0,\pm 1)} . \quad (4.40) $$

Using this decomposition and Eqs.(4.13,4.23) we can obtain the following equivalent representation of the effective (4,4) action in the rl-analytic superspace:

$$ S_2 = \int dldr d^2 x D^{(0,-2,-2)} [W^{(0,1,1)}]^2 F_2[W^{(0,1,1)}, l, r] . \quad (4.41) $$

One can construct the effective (4,4) action for the gauge group $[U(1)]^p$ in the rl-analytic and full superspaces by analogy with the case $D=3 (3.40,3.41)$.

An analogous action of the $q^{(1)}$ multiplet and dual superfields $\omega^{(\pm 1,\pm 1)}$ has been considered in refs.[31,32,33]. The relation between the c-analytic gauge superfield and rl-analytic hypermultiplets is a specific manifestation of the 2D mirror symmetry [34]. Consider the rl-analytic superfield $Q^{(0,1,1)}$ in our notation. The r- and l-harmonic constraints (4.25) can be introduced via the rl-analytic Lagrange multipliers

$$ S(Q, \omega) = \int dldr d^2 x D^{(0,-2,-2)} [F^{(0,2,2)}(Q^{(0,1,1)}, r, l) + \omega^{(0,1,-1)} D^{(0,2,0)} Q^{(0,1,1)} + \omega^{(0,-1,1)} D^{(0,0,2)} Q^{(0,1,1)}] . \quad (4.42) $$

The triharmonic superspace is convenient for the classification of the (4,4) supermultiplets. Let us consider, for instance, the cr-analytic superfield $Q^{(1,0,1)}_{cr}(\zeta_c, \bar{\zeta}_r, u, r)$ satisfying the subsidiary harmonic conditions

$$ D^{(2,0,0)}_c Q^{(1,0,1)}_{cr} = 0 , \quad D^{(0,0,2)}_r Q^{(1,0,1)}_{cr} = 0 , \quad (4.43) $$

where the analytic coordinates (4.34) and (4.29) are used. The cl-analytic superfield $Q^{(1,1,0)}_{cl}(\zeta_c, \zeta_l, u, l)$ can be defined analogously.

Thus, the alternative HHS structures and their embedding to the general triharmonic superspace are natural for the off-shell geometric description of the (4,4) supersymmetric theories.
5 One-dimensional harmonic superspaces

The one-dimensional $\sigma$-models have been considered in the component formalism and also in the framework of the superspaces with $\mathcal{N}=1, 2$ and 4 \cite{33, 38, 40}. Recently, the $\mathcal{N}=4$ superspace has been used for the proof of the non-renormalization theorem in the $\mathcal{N}=8$ gauge theory \cite{41}. This quantum-mechanical model describes D0-probes moving in different D4-brane backgrounds.

It is interesting to study these models in the framework of the manifestly supersymmetric harmonic approach. We shall consider the $D=1$, $\mathcal{N}=8$ superspace which is based on the maximum automorphism group $R=SU(c)\times Spin(5)$ and has coordinates $z=(t, \theta^\alpha_i)$ ($i, k, l, \ldots$ are the 2-spinor indices and $\alpha, \beta, \rho, \ldots$ are the 4-spinor indices of the group $Spin(5)=USp(4)$). The algebra of spinor derivatives is

$$\{D^k_\alpha, D^l_\rho\} = i\varepsilon^{kl}\Omega_{\alpha\rho}\partial_t + i\varepsilon^{kl}Z_{\alpha\rho}, \quad (5.1)$$

where $Z_{\alpha\rho}$ are central charges and $\Omega_{\alpha\rho}$ is the antisymmetric $Spin(5)$ metric.

Conjugation rules in the group $Spin(5)$ differ from the corresponding rules in $Spin(4,1)$ \cite{2.9}

$$\bar{\theta}_i^\alpha = \theta^i_\alpha, \quad \bar{\Omega}_{\alpha\rho} = -\Omega^{\alpha\rho}, \quad \bar{Z}_{\alpha\rho} = Z^{\alpha\rho}. \quad (5.2)$$

The CB-geometric superfield constraints of the $\mathcal{N}=8$ SYM theory are

$$\{\nabla^k_\alpha, \nabla^l_\rho\} = i\varepsilon^{kl}\Omega_{\alpha\rho}(\partial_t + A_t) + i\varepsilon^{kl}W_{\alpha\rho}, \quad (5.3)$$

where a traceless bispinor (or 5-vector) superfield representation of the 1D vector multiplet $W_{\alpha\rho}(z)$ is defined.

The harmonics $u_t^\pm$ can be used for a construction of the $D=1$ c-analytic coordinates $\zeta_c = (t_c, \theta^{+\alpha})$

$$t_c = t + \frac{i}{2}\theta^{0k}_\alpha \theta^k_\alpha u^+_k + u^-_k, \quad \theta^{+\alpha} = u^+_k \theta^k_\alpha. \quad (5.4)$$

The algebra of the c-harmonized 1D spinor derivatives resembles the corresponding algebra of the 5D derivatives \cite{2.13-2.15} with $Spin(5)$ indices instead of the $Spin(4,1)$ indices. In the case of vanishing central charges we have

$$D^{+\alpha}_\alpha = \partial^{+\alpha}_\alpha, \quad D^{-\alpha}_\alpha = -\partial^{-\alpha}_\alpha + i\theta^{-\alpha}_\alpha \partial^+_t, \quad (5.5)$$

$$D^{++}_c = \partial^{++}_c - \frac{i}{2}\theta^{0\alpha}_\alpha \theta^\alpha_c \partial^+_t + \theta^{\alpha+}_\alpha \partial^+_t. \quad (5.6)$$

The constraints \cite{5.3} correspond to the integrability conditions of the c-analyticity

$$\{\nabla^{+\alpha}_c, \nabla^{+\alpha}_c\} = 0, \quad \nabla^{+\alpha}_c = u^+_t \nabla^{+\alpha}_c. \quad (5.7)$$

The c-analytic prepotential $V^{++}_c(\zeta_c, u)$ describes the 1D vector multiplet (or 8+8 $\sigma$-model) and contains the pure gauge one-dimensional field $A$

$$(V^{++}_c)_w z = \Theta^{(2)}(t_c) + \Theta^{(2)\alpha\rho}_\Phi(t_c) + \frac{1}{2}[\Theta^{(2)}]_u_\alpha u^+_k X^{kj}(t_c), \quad (5.8)$$

where the notation \cite{2.18} is used. Of course, one can use the subsidiary gauge condition $A(t_c)=0.$
The basic superfield in the $D=1, \mathcal{N}=8$ Abelian gauge theory has the following form:

$$W_{\alpha \rho} \equiv \frac{1}{2} (\Gamma^m)_{\alpha \rho} W_m = -i D^{(+2)} V^{--}, \quad \Omega^{\alpha \rho} W_{\alpha \rho} = 0 , \quad (5.9)$$

where the $\Gamma$ matrices of $Spin(5)$ are introduced.

The constraints for this superfield are satisfied by construction

$$D^{\pm \alpha} W_{\beta \gamma} = \frac{2}{5} \Omega^{\alpha \beta} D^{+\sigma} W_{\sigma \gamma} - \frac{2}{5} \Omega^{\alpha \gamma} D^{+\sigma} W_{\sigma \beta} + \frac{1}{5} \Omega^{\beta \gamma} D^{+\sigma} W_{\sigma \alpha} , \quad (5.10)$$

$$D^{(+2)} W_{\alpha \rho} = 0 , \quad D^{\pm \mp} W_{\alpha \rho} = 0 . \quad (5.11)$$

These constraints are equivalent to the conditions of different (twisted) chiralities for the superfields $W_{13}, W_{14}, W_{23}$ and $W_{24}$, e.g.

$$D^{\pm 1} W_{13} = D^{\pm 3} W_{13} = 0 , \quad D^{\pm 1} W_{14} = D^{\pm 4} W_{14} = 0 . \quad (5.12)$$

The $c$-analytic hypermultiplets $q^+ (\zeta_c, u)$ and $\omega_c (\zeta_c, u)$ can be introduced by analogy with $HS$ of higher dimensions. These superfields have the $R_1$-invariant minimal interactions with the prepotential $V^{++}_c$. We do not consider here the $HS$ perturbative analysis of this model which can describe the $D0$-$D4$ interactions in Matrix theory and restrict ourselves to the study of a general symmetry framework for these calculations.

The $D=1$ low-energy $U(1)$-gauge action has the following universal form:

$$S_1 = \int dt d\theta du \ V^{++} V^{--} f_1 (W_m) . \quad (5.13)$$

Using the constraint (5.10) one can prove that the gauge invariance of $S_1$ is equivalent to the $5D$ Laplace equation for the superpotential

$$D^{(+2)} f_1 (W_m) = 0 \rightarrow \Delta^w f_1 (W_m) = 0 . \quad (5.14)$$

The $R_1$-invariant $D=1$ superpotential

$$f_1^R (w_1) = g_i^{-2} + k_i w_1^{-3} , \quad w_1 = (W^{\rho \sigma} W_{\rho \sigma})^{1/2} \quad (5.15)$$

is the unique solution of this equation. The non-renormalizability of this superpotential is protected by the $Spin(5)$-invariance and the $\mathcal{N}=8$ supersymmetry. Note that the same function determines the Kähler potential of the $D=1$ gauge theory in the $\mathcal{N}=4$ superfield formalism [41].

By analogy with the cases $D=2$ and 3, the geometric description of the $D=1, \mathcal{N}=8$ models requires the use of harmonic variables for the whole group $SU(2) \times Spin(5)$. Let us introduce now the biharmonic 1D-superspace using the $SU_c(2)$ harmonics $u_i^{(+1,0,0)} = u_i^+$ and harmonics $v^{(0,\pm 1,0)}_\alpha, v^{(0,0,\pm 1)}_\alpha$ of the group $USp(4)$ [12]. The basic relations for the $v$-harmonics are

$$\Omega^{\alpha \rho} v^{(0,0,a)}_\alpha = \delta^{ab} , \quad \Omega^{\alpha \rho} v^{(0,0,b)}_\alpha = \delta^{ab} , \quad \Omega^{\alpha \rho} v^{(0,0,0)}_\alpha = \delta^{ab} . \quad (5.15)$$

$$\Omega^{\alpha \rho} v^{(0,a)}_\alpha = \delta^{a} , \quad \Omega^{\alpha \rho} v^{(0,0,a)}_\alpha = \delta^{a} , \quad \Omega^{\alpha \rho} v^{(0,0,0)}_\alpha = \delta^{a} . \quad (5.18)$$
where \(a, b = \pm 1\) and \(\delta_{ab}\) is the Kronecker symbol. These harmonics determine the 8-dimensional coset space \(H_8 = USp(4)/U(1) \times U(1)\).

The harmonic derivatives \(D_{\nu}^{(0, \pm 2, 0)}\), \(D_{\nu}^{(0, 0, \pm 2)}\) and \(D_{\nu}^{(0, 0, 0, \pm 1)}\) are defined in ref. [12]

\[
\begin{align*}
D_{\nu}^{(0, \pm 2, 0)} v_{\alpha}^{(0, \pm 1, 0)} & = v_{\alpha}^{(0, \pm 1, 0)}, & D_{\nu}^{(0, 0, \pm 2)} v_{\alpha}^{(0, 0, \pm 1)} & = v_{\alpha}^{(0, 0, \pm 1)}, \\
D_{\nu}^{(0, \pm 1, \pm 1)} v_{\alpha}^{(0, \pm 1, \pm 1)} & = v_{\alpha}^{(0, \pm 1, \pm 1)}, & D_{\nu}^{(0, \pm 1, \pm 1)} v_{\alpha}^{(0, \pm 1, \pm 1), 0} & = v_{\alpha}^{(0, 0, \pm 1)}, \\
D_{\nu}^{(0, \pm 2, 0)} v_{\alpha}^{(0, \pm 1, 0)} & = D_{\nu}^{(0, 0, \pm 2)} v_{\alpha}^{(0, 0, \pm 1)} = D_{\nu}^{(0, 0, \pm 2)} v_{\alpha}^{(0, 0, \pm 1)} = 0, \\
D_{\nu}^{(0, 0, \pm 1)} v_{\alpha}^{(0, 0, \pm 1)} & = D_{\nu}^{(0, 0, \pm 1)} v_{\alpha}^{(0, 0, \pm 1), 0} = 0. 
\end{align*}
\]

The algebra of harmonic derivatives on \(H_8\) contains also the \(U(1)\)-charges \(D^{0}_{\nu 2}\) and \(D^{0}_{\nu 3}\). The harmonic derivatives on the coset \(SU_c(2)/U_c(1)\) are \(D^{(\pm 2, 0, 0)}_{c}\) and \(D^{0}_{c}\).

The \(\nu\)-harmonic representation of the general 1D superpotential (5.14) is

\[
f_{\nu}(W_{\alpha \rho}) = \int dvF_{\nu}[W^{(0, 1, 1)}, v_{\alpha}],
\]

where the real function \(F_{\nu}\) of the single harmonic projection \(W^{(0, 1, 1)}\) and all components of the \(\nu\)-harmonics determines the general solution of the 5D Laplace equation. The proof is based on the \(\nu\)-harmonic decomposition of the operator \(\Delta_{\nu}^{\nu}\) using the \(\nu\)-harmonic completeness relation

\[
\Omega_{\alpha \rho} = v_{\alpha}^{(0, -1, 0)} v_{\rho}^{(0, 1, 0)} + v_{\alpha}^{(0, 0, -1)} v_{\rho}^{(0, 0, 1)} - (\alpha \leftrightarrow \rho).
\]

Partial solutions can contain the restricted density functions \(F_{\nu}\) of some harmonic components \(v_1, \ldots\) and correspond, for instance, to the holomorphic functions of \(W_{13}\) and/or \(W_{14}\).

The BHS spinor derivatives are

\[
D^{(\pm 1, 1, 0)} = u_{i}^{(\pm 1, 0, 0)} v_{\alpha}^{(0, \pm 1, 0)} D_{i \alpha}, & \quad D^{(\pm 1, 0, \pm 1)} = u_{i}^{(\pm 1, 0, 0)} v_{\alpha}^{(0, 0, \pm 1)} D_{i \alpha}.
\]\n
The \(\nu\)-projection (5.24) of the basic gauge superfield (5.9) can be written in terms of the \(c\)-harmonic connections

\[
W^{(0, 1, 1)} = -iD^{(1, 1, 0)} D^{(1, 0, 1)} V^{(-2, 0, 0)} = -i\int dv D^{(-1, 1, 0)} D^{(-1, 1, 0)} V^{(2, 0, 0)}.
\]

By construction, this superfield is \(\nu\)-analytic

\[
D^{(\pm 1, 1, 0)} W^{(0, 1, 1)} = 0, \quad D^{(\pm 1, 0, 1)} W^{(0, 1, 1)} = 0
\]

and also satisfies the harmonic constraints

\[
D^{(\pm 2, 0, 0)} W^{(0, 1, 1)} = 0, \quad D^{A}_{\nu} W^{(0, 1, 1)} = 0,
\]

where \(D^{A}_{\nu}\) is the triplet of harmonic derivatives conserving the \(\nu\)-analyticity (5.28)

\[
D^{A}_{\nu} = (D^{(0, 1, 1)}_{\nu}, D^{(0, 2, 0)}_{\nu}, D^{(0, 0, 2)}_{\nu})\),
\]

\[
[D_{A}, D^{(\pm 1, 1, 0)}] = [D_{A}, D^{(\pm 1, 0, 1)}] = 0.
\]
The $v$-analytic coordinates $\zeta_v = (t_v, \theta^{(1,1,0)}, \theta^{(1,0,1)})$ can be defined by analogy with (5.4)

\[
t_v = t + \frac{1}{2} \left[ \theta^{(-1,-1,0)}(1,1,0) - \theta^{(-1,1,0)}(1,1,0) - \theta^{(1,0,-1)}(1,0,1) + \theta^{(-1,0,1)}(1,0,1) \right],
\] (5.32)

\[
\theta^{(\pm 1,0)} = u_i^{(0,1,0)} v_{\alpha}^{(0,1,0)} \theta^{\alpha}, \quad \theta^{(\pm 1,0)} = u_i^{(0,1,0)} v_{\alpha}^{(0,0,1)} \theta^{\alpha}.
\] (5.33)

These coordinates are convenient for the $v$-analytic superfields.

Using Eqs. (5.24, 5.23) we can obtain the $v$-analytic representation of the 1D effective action (5.13)

\[
S_i = \int dv dt D^{(0,-2,-2)} [W^{(0,1,1)}(\zeta_v)]^2 F_1 [W^{(0,1,1)}(\zeta_v), v_{\alpha}],
\] (5.34)

where the corresponding Grassmann measure is

\[
D^{(0,-2,-2)} = D^{(1,-1,0)} D^{(-1,-1,0)} D^{(1,0,-1)} D^{(-1,0,-1)}.
\] (5.35)

The invariant effective action for an arbitrary gauge group can be constructed immediately in the $v$-analytic superspace. The $v$-integral representation of the matrix superpotential for the gauge group $[U(1)]^p$ in the full superspace satisfies the following conditions:

\[
\frac{\partial}{\partial W_{\beta \gamma}^{\sigma}} \frac{\partial}{\partial W_{\gamma \sigma}^{\gamma N}} f_{BC}^1 (W_{\gamma D}^{\alpha \rho}, \ldots W_{\gamma D}^{\alpha \rho}) = 0, \quad D^{(+2)} f_{BC}^1 = 0.
\] (5.36)

The proof is analogous to the proof of the relations (3.3) in the case $D=3$.

The duality for the $N=8$ vector multiplet can be formulated in the $v$-analytic superspace. It is not difficult to define the triplet of $v$-analytic superfields which is dual to the superfield $W^{(0,1,1)}$

\[
\omega_v = (\omega_v^{(0,1,1)}, \omega_v^{(0,1,1)}, \omega_v^{(0,0,0)}),
\] (5.37)

\[
D^{(\pm 1,0)} \omega_v = D^{(\pm 1,0)} \omega_v = D^{(\pm 2,0)} \omega_v = 0.
\] (5.38)

These superfields have an infinite number of auxiliary components.

The interpolating term for the duality relation has the following form:

\[
\int dv dt D^{(0,-2,-2)} [W^{(0,1,1)} D_v^{(0,1,1)} \omega_v^{(0,0,0)} + W^{(0,1,1)} D_v^{(0,2,0)} \omega_v^{(0,-1,1)} + W^{(0,1,1)} D_v^{(0,0,2)} \omega_v^{(0,1,-1)}],
\] (5.39)

where $W^{(0,1,1)}$ is treated as an unrestricted $v$-analytic superfield.

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Note added in the replaced version

In this new e-print version, I have corrected the formulas (2.26-2.29) in connection with the recent results on the component non-Abelian superconformal-invariant 5D action: T. Kugo and K. Ohashi, Prog. Theor. Phys. 105 (2001) 323; [hep-ph/0010288].
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