LOW REGULARITY A PRIORI ESTIMATES FOR THE FOURTH ORDER CUBIC NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the low regularity behavior of the fourth order cubic nonlinear Schrödinger equation (4NLS)
\[ i\partial_t u + \partial^4_x u = \pm |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \]
\[ u(x, 0) = u_0(x) \in H^s(\mathbb{R}). \]

In [29], the author showed that this equation is globally well-posed in \( H^s(\mathbb{R}), s \geq -\frac{1}{2} \) and mildly ill-posed in the sense that the solution map fails to be locally uniformly continuous for \(-\frac{15}{14} < s < -\frac{1}{2} \). Therefore, \( s = -\frac{1}{2} \) is the lowest regularity that can be handled by the contraction argument. In spite of this mild ill-posedness result, we obtain an a priori bound below \( s < -\frac{1}{2} \). This an a priori estimate guarantees the existence of a weak solution for \(-\frac{3}{4} < s < -\frac{1}{2} \). Our method is inspired by Koch-Tataru [17]. We use the \( U^p \) and \( V^p \) based spaces adapted to frequency dependent time intervals on which the nonlinear evolution can still be described by linear dynamics.

1. Introduction. In this paper, we study the Cauchy problem for the fourth order cubic nonlinear Schrödinger equation on \( \mathbb{R} \):
\[ i\partial_t u + \partial^4_x u = \pm |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \]
\[ u(x, 0) = u_0(x) \in H^s(\mathbb{R}), \quad (4NLS) \]
where \( u \) is a complex-valued function. This equation is called defocusing when the sign of the nonlinear term is negative and focusing when the sign is positive. In the following, we make no distinction between the defocusing or focusing nature of (4NLS) and hence we assume that it is defocusing, that is, with the + sign in (4NLS). This equation is also known as the biharmonic NLS. The (4NLS) has been studied in the context of stability of solitons in magnetic materials. For more physical background see [12, 13]. In the past twenty years, the fourth order nonlinear Schrödinger equation has been extensively studied. More precisely, see [1, 8, 28, 26, 24, 23, 25, 21, 27].

The (4NLS) is also a Hamiltonian PDE with the following Hamiltonian:
\[ H(u(t)) = \frac{1}{2} \int_{\mathbb{R}} |\partial^2_x u(t)|^2 dx + \frac{1}{4} \int_{\mathbb{R}} |u(t)|^4 dx. \] 

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Moreover, the mass \( M(u(t)) \) is defined by

\[
M(u(t)) = \int_{\mathbb{R}} |u(t)|^2 \, dx.
\] (1.2)

Both the Hamiltonian (1.1) and mass (1.2) are conserved under the (4NLS) flow. The (4NLS) is invariant with respect to the scaling \( (\lambda > 0) \)

\[
u(t, x) \rightarrow \lambda^2 u(\lambda^4 t, \lambda x).
\] (1.3)

Therefore, the scale invariant homogeneous space is \( H^{-\frac{3}{2}} \). In general, we have

\[
\|u_{0, \lambda}\|_{H^s(\mathbb{R})} = \lambda^{s + \frac{3}{2}} \|u_0\|_{H^s(\mathbb{R})}.
\] (1.4)

As in nonlinear Schrödinger equation (NLS) [31], by using Strichartz estimates and mass conservation, we can easily show that (4NLS) is globally well-posed for initial data \( u_0 \in L^2(\mathbb{R}) \) (small modification of [31]). Therefore, it is natural to ask whether the well-posedness also holds in negative Sobolev spaces between scaling critical space \( H^{-\frac{3}{2}}(\mathbb{R}) \) and \( L^2(\mathbb{R}) \).

Indeed, in [29] the author showed that \( s = -\frac{1}{2} \) is the sharp regularity threshold for which the well-posedness can be handled by Picard iteration argument. More precisely, for \( s \geq -1/2 \), the author proved that (4NLS) is globally well-posed in \( H^s \) and below \( s < -1/2 \), it is mildly ill-posed in the sense that the solution map fails to be locally uniformly continuous in \( H^s \).

Although the solution map is not locally uniformly continuous for \( s < -1/2 \), we may have well-posedness with only continuous dependence on the initial data. Therefore, our final goal is to fill the gap between \( H^{-3/2} \) and \( H^{-1/2} \). In fact, to prove the well-posedness, we need to show a priori \( H^s \) bounds for the solutions and also establish continuous dependence on the initial data. In this paper, we prove a priori estimates up to \( s > -3/4 \). As a corollary, we show the existence of a weak solution for any initial data \( u_0 \in H^s, s > -3/4 \). Our method is inspired by Koch-Tataru [17, 18], Christ-Collander-Tao [2], Christ-Holmer-Tataru [4], and Baoping Liu [22]. They proved existence (without uniqueness) of solutions to the cubic NLS, mKdV, and KdV on \( \mathbb{R} \) in negative Sobolev spaces. Later, based on the structure of the completely integrable equation, an alternative approach to show a priori bounds for low regularity solutions was developed by Koch-Tataru [19]. Here, we point out that (4NLS) is not completely integrable and hence the ideas in [19] cannot be applied.

**Remark 1.1.** Let us investigate the one-dimensional cubic NLS:

\[
\begin{aligned}
&i \partial_t u + \partial_x^2 u = \pm |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R} \\
&u(x, 0) = u_0(x) \in H^s(\mathbb{R}).
\end{aligned}
\] (NLS)

We look at the Galilean invariance: if \( u \) is a solution of (NLS) with initial data \( u_0 \), then

\[
u_N(t, x) = e^{i x N} e^{-i t N^2} u(t, x - 2 N t)
\] (1.5)

is a solution to the same equation (NLS) with initial data \( e^{i x N} u_0(x) \). As a consequence of the Galilean invariance, the solution map cannot be locally uniformly continuous in \( H^s, s < 0 \). See for example [3, 15]. In view of the failure of local uniform continuity for the solution map for (NLS) in negative Sobolev spaces, we can know that it is impossible to prove well-posedness of (NLS) via a contraction argument in negative Sobolev spaces. As for (4NLS), thanks to the lack of the Galilean symmetry, there is a hope to prove local well-posedness of (4NLS) by a
contraction argument in negative Sobolev spaces. Indeed, as we mentioned above, in [29], the author showed that (4NLS) is locally well-posed in $H^s, s \geq -\frac{1}{2}$ via a contraction argument.

**Remark 1.2.** For the renormalized cubic 4NLS on the circle $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$, Oh and Wang [25] proved global existence in $H^s(\mathbb{T}), s > -\frac{9}{20}$ via the short-time Fourier restriction norm method in [11] and showed enhanced uniqueness in $H^s(\mathbb{T}), s > -\frac{1}{3}$ by using an infinite iteration of normal form reductions. Kwak [21] applied the Takaoka and Tsutsumi’s argument [30] and proved local well-posedness of the renormalized cubic 4NLS in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{3}$.

The main results of this paper are the following an a priori estimate and the existence of weak solution. In the following, we denote Schwartz space by $S(\mathbb{R})$.

**Theorem 1.3 (A priori estimate).** Let $-\frac{3}{4} < s < -\frac{1}{2}$. Then for any $M > 0$, there exists time $T > 0$ and constant $C > 0$ so that for initial data $u_0 \in S(\mathbb{R})$ satisfying

$$\|u_0\|_{H^s} \leq M,$$

the unique solution $u \in C([0,T]; S(\mathbb{R}))$ to (4NLS) (focusing or defocusing) satisfies

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s_x} \leq C\|u_0\|_{H^s_x}. \quad (1.6)$$

Using the uniform bound (1.6) together with the uniform bound on nonlinearity

$$\|\chi_{[0,T]} u\|_{X^s} + \|\chi_{[0,T]} u^2\|_{Y^s} \lesssim \|u_0\|_{H^s},$$

which is a byproduct of our analysis in proving Theorem 1.3, one may also prove the existence of weak solution by following the similar argument as in [2]. The spaces $X^s, Y^s$ are defined in Section 2.

**Corollary 1.4 (Existence of weak solution).** Let $-\frac{3}{4} < s < -\frac{1}{2}$. For any $M > 0$ there exist time $T > 0$ and constant $C > 0$ so that for any initial data in $H^s$ satisfying

$$\|u_0\|_{H^s} \leq M,$$

there exists a weak solution $u \in C([0,T]; H^s) \cap X^s$ to (4NLS) which solves the equation in the sense of distributions and satisfies

$$\|u\|_{L^\infty_t([0,T];H^s)} + \|\chi_{[0,T]} u\|_{X^s} + \|\chi_{[0,T]} u^2\|_{Y^s} \lesssim C\|u_0\|_{H^s}.$$

The solution obtained by Corollary 1.4 is a weak limit of smooth solutions with smooth initial data approximating the given data. We call these solutions weak solutions because we do not have any uniqueness or continuous dependence in $H^s, -3/4 < s < -1/2$.

**Remark 1.5.** As in [17], we can always rescale the initial data and hence it suffices to prove Theorem 1.3 in the small data case $M \ll 1$.

In [29], by just taking advantage of dispersive smoothing effects (bilinear Strichartz estimates, nonresonant interactions), the author proved the local and global well-posedness for $s \geq -1/2$. The main part of the local well-posedness is to show that the following trilinear estimate

$$\|u_1 u_2 u_3\|_{X^{-\frac{1}{2}}_{\chi^s}} \lesssim \|u_1\|_{X^{-\frac{1}{2}}_{\chi^s}} \|u_2\|_{X^{-\frac{1}{2}}_{\chi^s}} + \|u_3\|_{X^{-\frac{1}{2}}_{\chi^s}} + \|u_1\|_{X^s_{\chi^s}} \|u_2\|_{X^s_{\chi^s}},$$

holds for $s \geq -1/2$. However, in [29], the author display an example that for $s < -1/2$, the above trilinear estimate fails because of the strong resonant interaction of high-high-high to high.
To deal with these resonant interactions, in this paper we use the short time structure. More precisely, we use functions spaces adapted to a short time interval depending on the dyadic size of spatial frequencies. Then, these high-high-to-high resonant interaction is handled so that we can prove the trilinear estimate below $s < -1/2$. A precursor of this method appears in the work of Koch and Tzvetkov [16], where localization in time was combined with the Strichartz norms.

In Remark 1.6, one can see that nonlinear solution which is localized at frequency $N$ can still be described by linear dynamics up to the time scale $N^{4s+2}$. In fact, for $s \geq -1/2$, we already have a local well-posedness result. Therefore, in view of perturbation, nonlinear solutions behave like a linear solution over a time interval which is independent of frequency $N$. For $s < -1/2$, one can observe $N^{4s+2} \ll 1$ for all large $N \gg 1$. In contrast to the case $s \geq -1/2$, it means that different time scales are required for nonlinear solution localized at frequency $N$ to follow linear dynamics.

We also use the $U_p$ and $V_p$ spaces. These spaces have been originally introduced in unpublished work of Tataru on wave maps. In Koch-Tataru [17], they also used $U_p$ and $V_p$ spaces adapted to time intervals depending on the size of spatial frequencies. In Section 2, we define the function spaces employed in our analysis.

We briefly review the difference between Picard iteration method and the short time structure method. In fact, the latter is less perturbative than the former. We consider the following evolution equation: $\partial_t u - Lu = \mathcal{N}(u)$, where $Lu$ is the linear part and $\mathcal{N}(u)$ is a homogeneous nonlinearity of degree $p$. Then the usual Picard iteration method needs to establish the following two estimates:

Linear: $\|u\|_{F^s} \lesssim \|u_0\|_{H^s} + \|\mathcal{N}(u)\|_{N^s}$,

Nonlinear: $\|\mathcal{N}(u)\|_{N^s} \lesssim \|u\|_{pF^s}$,

where $F^s$ is the space to measure the solutions and $N^s$ is the space to measure the nonlinearity. After obtaining these two estimates, we can apply the fixed point argument to obtain the local well-posedness in $H^s$. For the short time structure method, by using the gain coming from the short time scale, nonlinear estimate can be improved up to lower regularity levels compared with the previous method. However, linear estimates are worse than before (See (1.7)). To address these expense, we need to establish the additional energy estimates. In summary, we must establish the following three estimates:

Linear: $\|u\|_{X^s} \lesssim \|u_0\|_{H^s} + \|\mathcal{N}(u)\|_{N^s}$,

Nonlinear: $\|\mathcal{N}(u)\|_{Y^s} \lesssim \|u\|_{pX^s}$,

Energy: $\|u\|_{L^\infty_t H^s_x} \lesssim \|u_0\|_{H^s} + \|u\|_{pX^s}$,

where $X^s, Y^s$ and the energy space $L^\infty_t H^s_x$ are presented in Section 2. Then by using a continuity argument combining with above three estimates (1.7), (1.8), and (1.9), one can establish an a priori bound and hence prove the existence of solutions by a compactness argument. In order to obtain the energy bound (1.9), we need to use a normal form technique. In applying the normal form reduction, we need to take the expense of introducing higher order multilinear terms.
Therefore, the process of obtaining an a priori bound is divided into two main steps. One is to prove the following trilinear estimate (4.1) (nonlinear estimate (1.8)) : For \(-3/4 < s < -1/2\), we have
\[
\|u_1 u_2 u_3\|_{Y^s} \lesssim \|u_1\|_{X^s} \|u_2\|_{X^s} \|u_3\|_{X^s},
\]
where \(Y^s\) is a function space to measure the nonlinear term in (4NLS) and \(X^s\) is a function space to measure the solutions for (4NLS). These function spaces are defined in Section 2.

In the nonlinear interactions which result in high frequency \(N\), we can use the gain \(|J| = N^{4s+2}\) occurring from the short time structure and the dispersive smoothing effects (e.g. Strichartz estimates (3.5) and bilinear Strichartz estimates (3.13), (3.14)), we can obtain the trilinear estimates for all \(s < -1/2\). However, there is a trade-off of using the short time structure. One can expect a loss resulted from summation of short time intervals. More precisely, in the nonlinear interactions which result in low frequency \(N\), there is a loss of derivative originating from the interval summation. In fact, in proving trilinear estimates (4.1), the high-high-high to low interaction is the worst case in terms of interval summation losses. To address this side effect of short time structure, we need to use another dispersive smoothing effect. We observe that the high-high-high to low interaction is a nonresonant interaction in the sense that either the output or at least one of the inputs must have high modulation: under \(\xi = \xi_1 - \xi_2 + \xi_3, \tau = \tau_1 - \tau_2 + \tau_3\), we have
\[
|\tau_1 - \xi_1| + |\tau_2 - \xi_2| + |\tau_3 - \xi_3| + |\tau - \xi| \gtrsim \left( |\xi - \xi_1| (|\xi - \xi_3| \left( \xi_2^2 + \xi_3^2 + \xi_3^2 + \xi^2 + 2(\xi_1 + \xi_3)^2 \right) \right) \gtrsim M_1 M_3^3,
\]
where the size of frequencies \(\{\xi, \xi_1, \xi_2, \xi_3\}\) is \(\{M, M_1, M_2, M_3\}\) with \(|\xi| \approx M, M \ll M_1 \lesssim M_2 \approx M_3\). Furthermore, we use the local smoothing effect. Therefore by exploiting these high modulation gain in the high-high-high to low interaction and local smoothing effect, we are able to weaken the interval summation losses and hence we can prove the trilinear estimate (4.1) up to \(s > -\frac{3}{4}\). The details are presented in Lemma 4.2. We remark that by using only the bilinear smoothing effect (3.13) and (3.14) without using the above high modulation gain and local smoothing effect, we can prove trilinear estimate (4.1) up to \(s > -\frac{5}{7}\). For more details, see Remark 4.4.

In Proposition 2.5, we prove the following linear estimates (linear estimate (1.7)) : Let \(u\) be a solution of \(i\partial_t u - \partial_x^2 u = f\). Then we have
\[
\|u\|_{X^s} \lesssim \|u\|_{L^{\infty} H^s} + \|f\|_{Y^s}.
\]
Therefore, we use trilinear estimate (4.1) to control the second term on the right hand side of (1.10). Therefore, the other part is to prove the following energy estimates (5.1) (energy estimate (1.9)) to control the first term on the right hand side of (1.10): Let \(-\frac{3}{2} < s < -\frac{1}{2}\) and \(u\) be a solution to (4NLS). On the time interval \([0, 1]\), we have the following energy estimates (5.1)
\[
\|u\|_{L^{\infty} H^s} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^3.
\]
In Section 5, the energy estimate is established by using a high frequency damped multiplier. This method is a modification of the \(I\)-method introduced by Colliander-Keel-Staffilani-Takaoka-Tao \([5, 6, 7]\). In the process of obtaining energy estimates, we use the normal form technique with the function spaces relying on frequency dependent time scales. As in proving trilinear estimate (4.1), there is a loss of
We want to explain that a solution whose frequency is localized to estimate (4.1) and energy estimates (5.1). The details are presented in Section 6. The remaining part is to just use standard bootstrapping argument with trilinear metrization process is an essential tool to get rid of resonant interactions in (5.3). Therefore, we choose time scale $T$.

Therefore, we choose time scale $T$.

By observing this heuristic calculation, we will construct our function spaces to follow linear dynamics on $[0, T]$, i.e., $u \approx e^{it\partial_x^4} u_0$ on $[0, T]$, $u$ satisfies the Strichartz estimate (3.3): $\|D_x\frac{1}{2}e^{it\partial_x^4}u_0\|_{L_t^\infty L_x^6(\mathbb{R}\times\mathbb{R})} \lesssim \|u_0\|_{L_x^2(\mathbb{R})}$. Therefore, by applying the Strichartz estimate (3.3), we have $\|u\|_{L_t^3L_x^5} \approx \|e^{it\partial_x^4}u_0\|_{L_t^3L_x^5([0,T]\times\mathbb{R})} \lesssim N^{-\frac{1}{4}}\|u_0\|_{L_x^2} \approx N^{-\frac{1}{4}-s}\|u_0\|_{H^s} \approx N^{-\frac{1}{4}-s}$. To obtain (1.11), we need $T^\frac{1}{2}N^s\|u\|_{L_t^3L_x^5([0,T]\times\mathbb{R})}^3 \lesssim T^\frac{1}{2}N^s \left(N^{-\frac{1}{4}-s}\right)^3 \approx 1$. Therefore, we choose time scale $T \approx N^{4s+2} \ll 1$.

By observing this heuristic calculation, we will construct our function spaces to be adapted to time intervals whose length depends on the time scale $T = N^{4s+2}$.

Organization of paper. This paper is organized as follows. In Section 2, we introduce $U^p$ and $V^p$ spaces adapted to short time intervals. In Section 3, we collect the linear and bilinear dispersive estimates used to prove the trilinear estimate and the energy estimate. These include Strichartz estimates, bilinear Strichartz estimates, local smoothing estimates and maximal function estimates. In Section 4, the trilinear estimate is proved. To weaken the interval summation losses, we take advantage of Lemma 4.2. In Section 5, the energy estimate with a higher order correction term is established by using a variation of the I-method. Finally, in Section 6, all materials are collected to give a proof of Theorem 1.3.
Notation. We use $A \lesssim B$ if $A \leq CB$ for some $C > 0$. We use $A \approx B$ when $A \lesssim B$ and $B \lesssim A$. Moreover, we use $A \ll B$ if $A \leq \frac{1}{C}B$, where $C$ is a sufficiently large constant. We also write $A^{\pm\epsilon}$ to mean $A^{\pm\epsilon C}$ for any $\epsilon > 0$.

Given $p \geq 1$, we let $p'$ be the Hölder conjugate of $p$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. We denote $L^p = L^p(\mathbb{R}^d)$ be the usual Lebesgue space. We also define the Lebesgue space $L^q(I, L^r)$ be the space of measurable functions from an interval $I \subset \mathbb{R}$ to $L^r$ whose $L^q(I, L^r)$ norm is finite, where

$$
\|u\|_{L^q(I, L^r)} = \left( \int_I \|u(t)\|^q_{L^r} \right)^{\frac{1}{q}}.
$$

We may write $L^q(I, L^r_x(I \times \mathbb{R})$ instead of $L^q(I, L^r)$.

We denote the space time Fourier transform of $u(t, x)$ by $\widehat{u}(\tau, \xi)$ or $\mathcal{F}u$

$$
\widehat{u}(\tau, \xi) = \mathcal{F}u(\tau, \xi) = \int e^{-i\tau t - ix\xi} u(t, x) \, dt dx.
$$

On the other hand, the space Fourier transform of $u(t, x)$ is denoted by

$$
\widehat{u}(t, \xi) = \mathcal{F}_x u(t, \xi) = \int e^{-ix\xi} u(t, x) \, dx.
$$

The fractional differential operator is given via Fourier transform by

$$
\widehat{D^\alpha u}(\xi) = |\xi|^{\alpha} \widehat{u}(\xi), \quad \alpha \in \mathbb{R},
$$

and the biharmonic Schrödinger semigroup is defined by

$$
e^{itD_x^{4}} = \mathcal{F}_x^{-1} e^{it|\xi|^4} \mathcal{F}_x g \tag{1.12}
$$

for any tempered distribution $g$. Let $\varphi : \mathbb{R} \to [0, 1]$ be an even, smooth cutoff function supported on $[-2, 2]$ such that $\varphi = 1$ on $[-1, 1]$. Given a dyadic number $N \geq 1$, we set $\varphi_1(\xi) = \varphi(|\xi|)$ and

$$
\varphi_N(\xi) = \varphi \left( \frac{|\xi|}{N} \right) - \varphi \left( \frac{2|\xi|}{N} \right)
$$

for $N \geq 2$. Then we define the Littlewood-Paley projection operator $P_N$ as the Fourier multiplier operator with symbol $\varphi_N$. Moreover, we define $P_{\leq N}$ and $P_{\geq N}$ by $P_{\leq N} = \sum_{1 \leq M \leq N} P_M$ and $P_{\geq N} = \sum_{M \geq N} P_M$. They commute with the derivative operator $D^\alpha$ and the semigroup $e^{itD_x^{4}}$. We also use the notation $u_N = P_N u$ if there is no confusion.

2. Function spaces. In this section, we set up the function spaces employed in our analysis. We also go over the properties of function spaces $U^p$ and $V^p$ established by Koch, Tataru. These spaces have been used in developing well-posedness of dispersive equations at scaling critical regularities. The details are presented in Hadac-Herr-Koch [9], Herr-Tataru-Tzvetkov [10], Koch-Tataru [19] and Koch-Tataru-Visan [20].

We take a time interval $I = [a, b], -\infty \leq a < b \leq \infty$. Let $\mathcal{Z}$ be the set of partitions $a = t_0 < t_1 < \cdots < t_K = b$ of $I$. We also consider functions taking values in $L^2 = L^2(\mathbb{R})$. 

Definition 2.1. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^{K} \subset \mathbb{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2(\mathbb{R})$ with $\phi_0 = 0$ and $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$. We call the function $a : I \to L^2$ given by

$$a(t) = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)}(t)\phi_{k-1}$$

an $U^p(I; L^2)$-atom. We define the $U^p(I; L^2)$ space:

$$U^p(I; L^2) := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ is } U^p(I; L^2)\text{-atom, } \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$\|u\|_{U^p(I; L^2)} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j U^p(I; L^2)\text{ - atom} \right\}.$$

Definition 2.2. Let $1 \leq p < \infty$. We define the space $V^p(I; L^2)$ as the space of functions on $I$ such that

$$v(a) = \lim_{t \to a} v(t) \text{ exists and } v(b) := \lim_{t \to b} v(t) = 0,$$

and for such functions $v(t)$ we define the norm

$$\|v\|_{V^p(I; L^2)} = \sup_{t_k \in \mathbb{Z}} \left( \sum_{k} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}}.$$

We also use the notation $U^p = U^p(I; L^2)$ and $V^p = V^p(I; L^2)$ if there is no confusion.

Lemma 2.3 ([9],[17]). Fix an interval $I = [a, b]$.

1. Let $1 \leq p < q < \infty$. Then we have continuous embeddings $U^p \hookrightarrow U^q$ and $V^p \hookrightarrow V^q$, i.e.

$$\|u\|_{U^q} \lesssim \|u\|_{U^p} \quad \text{and} \quad \|u\|_{V^q} \lesssim \|u\|_{V^p}. \quad (2.1)$$

2. If $1 \leq p < \infty$ and $u(b) = 0$, then we have $U^p \hookrightarrow V^p$, i.e.

$$\|u\|_{V^p} \lesssim \|u\|_{U^p}.$$

3. If $1 \leq p < q \leq \infty$, $u(a) = 0$, and $u \in V^p$ is right continuous, then we have

$$\|u\|_{U^q} \lesssim \|u\|_{V^p}.$$

We define $U^p_S(I; L^2)$, $V^p_S(I; L^2)$ spaces to be the set of all functions $u : I \to L^2$ such that the following $U^p_S(I; L^2)$-norm and $V^p_S(I; L^2)$-norm are finite:

$$\|u\|_{U^p_S(I; L^2)} := \|S(-t)u\|_{U^p(I; L^2)} \quad \text{and} \quad \|u\|_{V^p_S(I; L^2)} := \|S(-t)u\|_{V^p(I; L^2)},$$

where $S(t) = e^{it\partial_x^4}$ denotes the linear propagator for (4NLS). Also we use the notation $U^p_S = U^p_S(I; L^2)$ and $V^p_S = V^p_S(I; L^2)$ if there is no confusion.

Remark 2.4. Observe that $U^p_S$ is the atomic space, where atoms are piecewise solutions to the linear equation

$$u = \sum_{k} \chi_{[t_{k-1}, t_k)}(t)e^{it\partial_x^4} \phi_{k-1}, \quad \sum_{k} \|\phi_{k-1}\|_{L^2(\mathbb{R})}^p = 1.$$
We denote by $DU^p_S$ the space of functions

$$DU^p_S = \{(i\partial_t - \partial^4_x)u; u \in U^p_S\}$$

with the induced norm. Then we have the trivial bound

$$\|u\|_{U^p_S} \lesssim \|u_0\|_{L^2_x} + \| (i\partial_t - \partial^4_x)u\|_{DU^p_S}.$$  \hfill (2.2)

Moreover we have the duality relations

$$(DU^p_S)^* = V^p_S, \quad 1 < p < \infty.$$  

More precisely, given $\phi \in V^p_S$, the mapping $f \mapsto \int (f, \phi)_{L^2_t} dt$ belongs $(DU^p_S)^*$ and this identification is a surjective isometry. In fact, the spaces $DU^p_S$ and $DV^p_S$ are characterized as the spaces for which the following norms are finite:

$$\|f\|_{DU^p_S} = \sup \left\{ \int (f, \phi)_{L^2_t} dt : \|\phi\|_{V^p_S} \leq 1, \phi \in C^\infty_c \right\} \quad (2.3)$$

$$\|f\|_{DV^p_S} = \sup \left\{ \int (f, \phi)_{L^2_t} dt : \|\phi\|_{U^q_S} \leq 1, \phi \in C^\infty_c \right\}. \quad (2.4)$$

More specifically, see for instance [17, 19].

There is another choice for estimating the solution to (4NLS). The Bourgain's $X^{s,b}$ spaces is defined by

$$\|u\|_{X^{s,b}} = \int |\hat{u}(\tau, \xi)|^2 \langle \xi \rangle^{2s} \langle \tau - \xi^4 \rangle^{2b} d\xi d\tau,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. The space $X^{s,b}$ turns out to be very useful in the study of low regularity theory. But for $b = \frac{1}{2}$, logarithmic divergences happen in several estimates. To deal with this issues we consider dyadic decompositions with respect to the modulation $\tau - \xi^4$. This leads to the additional homogeneous Besov type norms

$$\|u\|_{\dot{X}^{s,\frac{1}{2}},b} = \sum_M \left( \int_{|\tau - \xi^4| \approx M} |\hat{u}(\tau, \xi)|^2 \xi^{2s} |\tau - \xi^4| d\xi d\tau \right)^{\frac{1}{2}},$$

$$\|u\|_{\dot{X}^{s,\frac{1}{2}}_{b,\infty}} = \sup_M \left( \int_{|\tau - \xi^4| \approx M} |\hat{u}(\tau, \xi)|^2 \xi^{2s} |\tau - \xi^4| d\xi d\tau \right)^{\frac{1}{2}}.$$

These homogeneous Besov type spaces are closely related to the spaces $U^{\frac{3}{2}}_S$ and $V^{\frac{3}{2}}_S$. Combining the embedding $V^2 \hookrightarrow \dot{B}^{\frac{3}{2}}_{2,\infty}$ with duality we have

$$\dot{X}^{0,\frac{1}{2},1} \hookrightarrow U^{\frac{3}{2}}_S \hookrightarrow V^{\frac{3}{2}}_S \hookrightarrow \dot{X}^{0,\frac{1}{2},\infty}.$$  

In the following we use a Littlewood-Paley decomposition with respect to the modulation $\tau - \xi^4$ as well as a spatial Littlewood-Paley decomposition

$$1 = \sum_{N \geq 1} P_N, \quad 1 = \sum_{N \geq 1} Q_N.$$  

Both decompositions are inhomogeneous. It is easy to see that we have the uniform boundedness properties

$$P_N : U^p_S \rightarrow U^p_S, \quad Q_N : U^p_S \rightarrow U^p_S.$$
and similarly for $V_p^q$. Moreover $U_p^q$ and $V_p^q$ spaces behave well with respect to sharp time cut off. If $I$ is a time interval, then we have

$$\chi_I : U_p^q \to U_p^q, \quad \chi_I : V_p^q \to V_p^q$$

with uniform bounds with respect to $I$.

We define an energy space with a standard energy norm

$$\|u\|_{F^q_{\alpha}L^p_t H^s_x}^2 := \sum_{N \geq 1} N^{2s} \|u_N\|^2_{L^p_t L^q_x},$$

where we sum over all dyadic numbers $\geq 1$ with the obvious modification at $N = 1$. Note that $\|u\|_{L^p_t H^s_x} \leq \|u\|_{F^q_{\alpha}L^p_t H^s_x}$, but the converse is not true.

To estimate the solutions to (4NLS) we define the space $X^\ast$ with the norm

$$\|u\|_{X^\ast} := \sum_{N \geq 1} N^{2s} \sup_{|I| = N^{4s+2}} \|\chi_I u_N\|_{L^p_t L^q_x}^2$$

(2.5)

where we sum over all dyadic numbers $\geq 1$ with the obvious modification at $N = 1$ and the supremum is taken over all subintervals $I \subset [0,1]$ of length $N^{4s+2}$.

To measure the regularity of the nonlinear term we define the space $Y^\ast$ with the norm

$$\|f\|_{Y^\ast} := \sum_{N \geq 1} N^{2s} \sup_{|I| = N^{4s+2}} \|\chi_I f_N\|_{H^s_x}^2$$

(2.6)

where we sum over all dyadic numbers $\geq 1$ with the obvious modification at $N = 1$ and the supremum is taken over all subintervals $I \subset [0,1]$ of length $N^{4s+2}$.

**Proposition 2.5.** Let $u$ be a solution to $i\partial_t u - \partial^4_x u = f$ on $I$. Then we have

$$\|u\|_{X^\ast} \lesssim \|u\|_{F^q_{\alpha}L^p_t H^s_x} + \|f\|_{Y^\ast}.$$  

**Proof.** This proposition follows from (2.2)

$$\|u\|_{U^q_p} \lesssim \|u_0\|_{L^2_x} + \|f\|_{DU^q_p}.$$  

More specifically, see [4, 17] \qed

3. Strichartz, local smoothing and bilinear Strichartz estimate. In this section, we collect the standard linear and bilinear estimates.

**Lemma 3.1 ([29]).** For any $\alpha \in [0,1]$, we call a pair $(q,r,\alpha)$ admissible exponents if $r \geq 2, q \geq \frac{8}{(1+\alpha)}$ and $\frac{2}{q} + \frac{1+\alpha}{r} = \frac{1+\alpha}{2}$. Then for any admissible exponents $(q,r,\alpha)$, we have

$$\|D^{\frac{k}{2}(1 - \frac{1}{q})} e^{it\partial_t^4} u_0\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim_q \|u_0\|_{L^2_x(\mathbb{R})}$$

(3.1)

In particular, we have

$$\|D^{\frac{1}{2}} e^{it\partial_t^4} u_0\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u_0\|_{L^2_x(\mathbb{R})},$$

(3.2)

$$\|D^{\frac{1}{2}} e^{it\partial_t^4} u_0\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u_0\|_{L^2_x(\mathbb{R})},$$

(3.3)

$$\|e^{it\partial_t^4} u_0\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u_0\|_{L^2_x(\mathbb{R})}.$$  

(3.4)

**Corollary 3.2.** Let $I = [a,b]$ be an interval. Then for any admissible pair $(q,r,\alpha)$, we have

$$\|P_N u\|_{L^q_t L^r_x(I \times \mathbb{R})} \lesssim N^{-\frac{k}{2}(1 - \frac{1}{q})}\|\chi_I u\|_{U^q_p}.$$  

(3.5)

Moreover, we have the dual estimate for $q > 2$:

$$\|P_N u\|_{DU^q_p(I;L^r_x)} \lesssim N^{-\frac{k}{2}(1 - \frac{1}{q})}\|u\|_{L^q_t L^r_x(I \times \mathbb{R})}.$$  

(3.6)
By applying (3.5) and Lemma 2.3, we have

\[ \|P_N u\|_{L_t^q L_x^r} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})}. \]  

(3.7)

and show that

\[ \|P_N u\|_{L_t^q L_x^r} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})}. \]  

(3.8)

By using Strichartz estimates (3.1) we have

\[ \|P_N u\|_{L_t^q L_x^r} = \sum_{k=1}^K \|\chi_{[t_{k-1}, t_k)}(t) P_N S(t) \phi_{k-1}\|_{L_t^q L_x^r} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})} \sum_{k=1}^K \|\phi_{k-1}\|_{L_x^r}^q = N^{-\frac{q}{2}(1-\frac{q}{r})}. \]

Now we prove dual estimate. Recall that \((DU^2 (I; L^2))^* = V^2 (I; L^2)\). Therefore, we have

\[ \|P_N u\|_{D(U^2 (I; L^2))} = \sup_{\|w\|_{V^2(I; L^2)} \leq 1} \left| \int_I \int \chi \partial_t^\alpha \chi \partial_x^\beta u dx dt \right| \lesssim \|w\|_{V^2(I; L^2)} \|P_N u\|_{L_t^q L_x^r (I \times \mathbb{R})}. \]

(3.9)

By applying (3.5) and Lemma 2.3, we have

\[ \|P_N u\|_{L_t^q L_x^r (I \times \mathbb{R})} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})} \|\chi \partial_t u\|_{U^2} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})} \|\chi \partial_t u\|_{V^2}. \]

(3.10)

which complete the proof.

**Lemma 3.3** (Local smoothing, maximal function estimates [14]).

\[ \|D^{\frac{3}{2}} e^{it \partial_x^3} u_0\|_{L_t^\infty L_x^q} \lesssim \|u_0\|_{L_x^2} \]

\[ \|D^{\frac{1}{2}} e^{it \partial_x^3} u_0\|_{L_t^\infty L_x^q} \lesssim \|u_0\|_{L_x^2}. \]

(3.11)

**Proof.** In the case of local smoothing estimate, the proof is the same as Schrödinger case. In fact, it is reducible to using Plancherel theorem in \(L_t^q\). For the proof of the maximal function estimate, see Theorem 2.5 in [14].

**Corollary 3.4.** Let \(I = [a, b)\) be an interval. Then we have

\[ \|P_N u\|_{L_t^\infty L_x^q (I \times \mathbb{R})} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})} \|\chi \partial_t u\|_{U^2}, \]

(3.10)

\[ \|P_N u\|_{L_t^\infty L_x^q (I \times \mathbb{R})} \lesssim N^{-\frac{q}{2}(1-\frac{q}{r})} \|\chi \partial_t u\|_{U^2}. \]

(3.11)

**Proof.** As we proceed in the proof of Corollary 3.2, it suffices to consider \(U^2, U^4\) atoms respectively. For more details, see Proposition 2.19 in [9].

**Proposition 3.5** (Bilinear estimate). Let \(A_1, A_2 \subset \mathbb{R}\) such that

\[ |\xi_1^2 - \xi_2^2| = |\xi_1 - \xi_2| |\xi_1 + \xi_2| \approx N N_{\max}. \]

for all \(\xi_1 \in A_1, \xi_2 \in A_2\) and \(N_{\max} = \max \{|\xi_1|, |\xi_2|\}\). We define the projection operators \(P_{A_i}\) as the Fourier multiplier operators \(P_{A_i} \hat{\psi}(\xi) = \chi_{A_i} \hat{\psi}(\xi)\) for a function \(\psi\). Then we have

\[ \|P_{A_1} e^{it \partial_x^3} \phi_1 P_{A_2} e^{it \partial_x^3} \phi_2\|_{L_t^q L_x^r} \lesssim N^{-\frac{q}{2}} N_{\max}^{-1} \|P_{A_1} \phi_1\|_{L_x^r} \|P_{A_2} \phi_2\|_{L_x^r}. \]

(3.12)
Proof. Note that by duality,
\[
\left\| P_{A_1} e^{it\phi_1} P_{A_2} e^{it\phi_2} \right\|_{L^2_t(\mathbb{R} \times \mathbb{R})}
\]
\[
= \left\| \int_{\mathbb{R}} e^{-it\xi} \widehat{P_{A_1}\phi_1}(\xi_1) e^{it(\xi-\xi_1)^t} \widehat{P_{A_2}\phi_2}(\xi-\xi_1) \ d\xi_1 \right\|_{L^2_\xi(\mathbb{R} \times \mathbb{R})}
\]
\[
= \sup_{\|\psi\|_{L^2_{1,\xi}}=1} \left| \int_{\mathbb{R} \times \mathbb{R}} \int_{\mathbb{R}} e^{-it\xi} \widehat{P_{A_1}\phi_1}(\xi_1) e^{it(\xi-\xi_1)^t} \widehat{P_{A_2}\phi_2}(\xi-\xi_1) \psi(t,\xi) \ d\xi_1 \ d\xi dt \right|
\]
Hence, it suffices to consider the integral
\[
\left| \int_{\mathbb{R} \times \mathbb{R}} \int_{\mathbb{R}} e^{-it\xi} \widehat{P_{A_1}\phi_1}(\xi_1) e^{it(\xi-\xi_1)^t} \widehat{P_{A_2}\phi_2}(\xi-\xi_1) \psi(t,\xi) \ d\xi_1 \ d\xi dt \right|
\]
\[
= \left| \int_{\mathbb{R} \times \mathbb{R}} \int_{\mathbb{R}} e^{-it\xi} \widehat{P_{A_1}\phi_1}(\xi_1) e^{it(\xi-\xi_1)^t} \widehat{P_{A_2}\phi_2}(\xi_1) \psi(t,\xi_1+\xi_2) \ d\xi_1 \ d\xi_2 dt \right|
\]
\[
= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{P_{A_1}\phi_1}(\xi_1) \widehat{P_{A_2}\phi_2}(\xi_1) \mathcal{F}_t \psi(\xi_1^4+\xi_2^4,\xi_1+\xi_2) \ d\xi_1 \ d\xi_2 \right|
\]
We consider the change of variable \((\xi_1,\xi_2) \to (\eta_1(\xi_1,\xi_2),\eta_2(\xi_1,\xi_2)) = (\xi_1+\xi_2,\xi_1^4+\xi_2^4)\) with the Jacobian \(|J| = 4 (\xi_1^4 - \xi_2^4) \approx NN_{\max}^2\). Hence,
\[
|J| \ d\xi_1 \ d\xi_2 = d\eta_1 \ d\eta_2 \quad \text{or} \quad d\xi_1 \ d\xi_2 = |J|^{-1} d\eta_1 \ d\eta_2.
\]
By Hölder’s inequality, the Plancherel theorem and again change of variable, we have
\[
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{P_{A_1}\phi_1}(\xi_1) \widehat{P_{A_2}\phi_2}(\xi_2) \mathcal{F}_t \psi(\xi_1^4+\xi_2^4,\xi_1+\xi_2) \ d\xi_1 \ d\xi_2 \right|
\]
\[
\lesssim \|\mathcal{F}_t \psi\|_{L^2_{1,\eta}} \int_{\eta_1 \in \mathbb{R}} \int_{\eta_2 \in \mathbb{R}} \left| \widehat{P_{A_1}\phi_1}(\eta_1,\eta_2) \widehat{P_{A_2}\phi_2}(\eta_1,\eta_2) \right|^2 |J|^{-2} \ d\eta_1 \ d\eta_2 \|^\frac{1}{2}
\]
\[
\lesssim N^{-\frac{1}{2}} N_{\max}^{-1} \int_{\xi_1 \in \mathbb{R}} \int_{\xi_2 \in \mathbb{R}} \left| \widehat{P_{A_1}\phi_1}(\xi_1) \widehat{P_{A_2}\phi_2}(\xi_2) \right|^2 d\xi_1 d\xi_2 \lesssim N^{-\frac{1}{2}} N_{\max}^{-1} \left\| P_{A_1}\phi_1 \right\|_{L^2_\xi(\mathbb{R})} \left\| P_{A_2}\phi_2 \right\|_{L^2_\xi(\mathbb{R})}.
\]
\[\square\]
Corollary 3.6 (bilinear Strichartz estimates). Let \(N_1 \ll N_2\). Then we have
\[
\| P_{N_1} u_1 P_{N_2} u_2 \|_{L^2_t L^\infty_x (I \times \mathbb{R})} \lesssim N_2^{-\frac{3}{2}} \| \chi_1 P_{N_1} u_1 \|_{V^\frac{3}{2} L^2} \| \chi_1 P_{N_1} u_1 \|_{V^\frac{3}{2} L^2},
\]
and
\[
\| P_{N_1} u_1 P_{N_2} u_2 \|_{L^2_t L^\infty_x (I \times \mathbb{R})} \lesssim N_2^{-\frac{3}{2}} \left( \ln N_2 + 1 \right)^2 \| \chi_1 P_{N_1} u_1 \|_{V^\frac{3}{2} L^2} \| \chi_1 P_{N_1} u_1 \|_{V^\frac{3}{2} L^2}. \]
\[\text{(3.13)}\]
\[\text{(3.14)}\]
\[\text{Proof.}\] In fact (3.13) is the result of the transference principle. For more details, see Proposition 2.19 in [9]. The proof of estimate (3.14) follows from the argument in [9], Proposition 2.20, Corollary 2.21.
\[\square\]

4. Trilinear estimate. In this section, we prove the following trilinear estimates below \(s < -1/2\). As we mentioned above, we use both dispersive smoothing effects (bilinear Strichartz estimates (3.13), (3.14)) and short time structure. Our method is inspired by Koch-Tataru [17].

Proposition 4.1 (Trilinear estimate). Let \(-3/4 < s < -1/2\). Then we have
\[
\| u_1 u_2 u_3 \|_{Y^s} \lesssim \| u_1 \|_{X^s} \| u_2 \|_{X^s} \| u_3 \|_{X^s}.
\]
\[\text{(4.1)}\]
Proof. We estimate the nonlinearity $|u|^2 u$ at frequency $N$ in a $N^{4s+2}$ time interval $I$. We also consider a full dyadic decomposition of each of the factors.

\[ \|u_1 u_2 u_3\|_{Y^s} = \left( \sum_{N \geq 1} N^{2s} \sup_{|I| = N^{4s+2}} \| \chi_I P_N (u_1 u_2 u_3) \|_{DU^3}^2 \right)^{1/2} \]

\[ \lesssim \sum_{N \geq 1} N^s \sup_{|I| = N^{4s+2}} \| \chi_I P_N (u_1 u_2 u_3) \|_{DU^3} \]

and

\[ \| \chi_I P_N (u_1 u_2 u_3) \|_{DU^3} \lesssim \sum_{1 \leq N_1, N_2, N_3} \| \chi_I P_N (u_{N_1} u_{N_2} u_{N_3}) \|_{DU^3} \]

Therefore, we need to show that for an interval $|I| = N^{4s+2}$

\[ \| \chi_I P_N (u_{N_1} u_{N_2} u_{N_3}) \|_{DU^3} \lesssim N_1^{\alpha_1} N_2^{\alpha_2} N_3^{\alpha_3} \prod_{j=1}^{3} \sup_{|I_j| = N_j^{4s+2}} \| \chi_{I_j} u_{N_j} \|_{U^3_{\alpha_j}} \]

where $N_1^{\alpha_1} N_2^{\alpha_2} N_3^{\alpha_3} N^{\alpha}$ have summability with respect to $N_1, N_2, N_3, N \geq 1$. We denote

\[ \{N, N_1, N_2, N_3\} = \{M, M_1, M_2, M_3\}, \quad M_1 \leq M_2 \leq M_3, \quad N = M. \]

Note that the two largest frequencies must be comparable. Hence, we investigate the following cases of interactions:

Case 1. $M_1 \leq M_2 \leq M_3 \approx M$.

Case 2. $M_1 \leq M \ll M_2 \approx M_3$.

Case 3. $M \ll M_1 \leq M_2 \approx M_3$.

Case 1. $M_1 \leq M_2 \leq M_3 \approx M$. In this case, we observe that $|I_j| \geq |I|$ for $j = 1, 2, 3$. Hence there is no interval summation loss. At the case 1, there is no role of the complex conjugate. Therefore we drop the complex conjugate sign. By using the dual estimates (3.6), Strichartz estimate (3.3) and embedding $U^3_{\alpha_j} \hookrightarrow U^3_{\beta_j}$, we have

\[ \| \chi_I P_M (u_{M_1} u_{M_2} u_{M_3}) \|_{DU^3} \]

\[ \lesssim \|u_{M_1} u_{M_2} u_{M_3}\|_{L^1_t L^2_x(I \times \mathbb{R})} \lesssim |I|^{1/2} \|u_{M_1} u_{M_2} u_{M_3}\|_{L^2_t L^2_x(I \times \mathbb{R})} \]

\[ = M^{2s+1} \|u_{M_1} u_{M_2} u_{M_3}\|_{L^2_t L^2_x(I \times \mathbb{R})} \]

\[ \lesssim M^{2s+1} \|u_{M_3}\|_{L^2_t L^2_x(I \times \mathbb{R})} \|u_{M_2}\|_{L^2_t L^2_x(I \times \mathbb{R})} \|u_{M_1}\|_{L^2_t L^2_x(I \times \mathbb{R})} \]

\[ \lesssim M^{2s+1} M_1^{-\frac{3}{4}} M_2^{-\frac{3}{4}} M_3^{-\frac{3}{4}} \prod_{j=1}^{3} \| \chi_{I_j} u_{N_j} \|_{U^3_{\alpha_j}} \lesssim M^{2s+1} M_1^{-\frac{3}{4}} M_2^{-\frac{3}{4}} M_3^{-\frac{3}{4}} \prod_{j=1}^{3} \| \chi_{I_j} u_{M_j} \|_{U^3_{\alpha_j}}. \]

By combining above calculations, we need to focus on the following summation

\[ \sum_{M_1 \leq M_2 \leq M_3 \approx M} M^s \sup_{|I| = M^{4s+2}} \| \chi_I P_M (u_{M_1} u_{M_2} u_{M_3}) \|_{DU^3} \]

\[ \lesssim \sum_{M_1 \leq M_2 \leq M_3 \approx M} M^{2s+1} M_1^{-\frac{3}{4}} M_2^{-\frac{3}{4}} M_3^{-\frac{3}{4}} \prod_{j=1}^{3} \| u_{M_j} \|_{X^s} \]

\[ \lesssim \| u_1 \|_{X^s} \sum_{M_2 \leq M_3} M_2^{-2s-\frac{3}{4}} M_3^{2s+\frac{3}{4}} \| u_{M_2} \|_{X^s} \| u_{M_3} \|_{X^s}. \]
Therefore, by using Schur’s test, we obtain the desired result. Observe that in this interaction, trilinear estimate is satisfied for all $s < -1/2$.

Case 2. $M_1 \lesssim M \ll M_2 \approx M_3$. Similarly in case 1 there is no role of complex conjugate and hence we drop the complex conjugate sign. Observe that the $u_{M_2}, u_{M_3}$ should be estimated in norm with timescale $M_3^{4s+2}$. We divide $I$ into $|I|/|J| = (M_3/M)^{-4s-2} \gg 1$ intervals of size $|J| = M_3^{4s+2}$. By applying the duality (2.3), we have

$$
\|\chi_I P_M (u_{M_1} u_{M_2} u_{M_3})\|_{L^2_{x,t}} = \sup_{\|u\|_{L^2_x}=1} \left| \int_I \int_R u_{M_1} u_{M_2} u_{M_3} P_M u \ dx dt \right|.
$$

By using the bilinear Strichartz estimates (3.13), (3.14), we estimate

$$
\lesssim \left( \frac{M_3}{M} \right)^{-4s-2} \sup_{J \subset I, |J|=M_3^{4s+2}} \left| \int_I \int_R u_{M_1} u_{M_2} u_{M_3} u \ dx dt \right|
$$

$$
\lesssim \left( \frac{M_3}{M} \right)^{-4s-2} \sup_{J \subset I, |J|=M_3^{4s+2}} \|\chi_J u_{M_1} u_{M_2} \|_{L^2_t L^2_x} \|\chi_J u_{M_3} \|_{L^2_t L^2_x}
$$

$$
\lesssim M_3^{-3} (\log M_3)^2 \sup_{J \subset I, |J|=M_3^{4s+2}} \|\chi_J u_{M_1} u_{M_2} \|_{L^2_t L^2_x} \|\chi_J u_{M_3} \|_{L^2_t L^2_x}.
$$

By combining above calculation, we focus on the following summation:

$$
\sum_{M_1 \lesssim M \ll M_2 \approx M_3} M^s \sup_{|I|=M_3^{4s+2}} \|\chi_I P_M \chi_{M_1} u_{M_2} u_{M_3} \|_{L^2_{x,t}} \lesssim M_3^{-6s-5+ M_3^{5s+2} M_1^{-s}} \|u_{M_1}\|_{X^s} \|u_{M_2}\|_{X^s} \|u_{M_3}\|_{X^s}.
$$

We can deal with above summation by using Schwarz inequality if $s > -\frac{5}{8}$.

Case 3. $M \ll M_1 \leq M_2 \approx M_3$. Case 3 is the worst case in terms of the short time structure because the interval summation loss is the largest. However, because case 3 is also a nonresonant interaction unlike the other two cases above, it is necessary to take advantage of nonresonant interaction to weaken this interval summation loss.

To demonstrate the case 3, we need the following lemma

**Lemma 4.2.** Let $-\frac{3}{4} < s < -\frac{1}{2}$ and $I$ be a time interval with $|I| = N^{4s+2}$. Define

$$
f = \sum_{N_1, N_2, N_3 \geq N} \chi_I \chi_{M_1} (u_{N_1} u_{N_2} u_{N_3})
$$

$$
= \sum_{N_1, N_2, N_3 \geq N} \sum_{|J|=N_1^{4s+2}} P_N(\chi_J u_{N_1} \chi_J u_{N_2} \chi_J u_{N_3}) \cdot N_{\max} = \max(N_1, N_2, N_3).
$$

Here the $J$ summation is understood to be over a partition of $I$ into intervals $J$ of the indicated size. Then, we have the estimates

$$
\|Q_{\geq N} f \|_{X^{0, -\frac{1}{4} +}} \lesssim N^{-3-3s} \|u\|_{X^s}^3.
$$
and
\[ \|Q_{\leq N^4}f\|_{L^1_t L^2_x + N^{-\frac{1}{4}} L^4_x L^4_t} \leq N^{-3-3s}\|u\|_X^3. \]

**Remark 4.3.** In fact, the same estimates hold if we replace \( u_{N_2} \) by \( u_N \) and this case become easier.

**Remark 4.4.** By using only the bilinear smoothing effect without Lemma 4.2, we can prove the trilinear estimate up to \( s > -\frac{5}{4} \).

In case 3 \( M \ll M_1 \ll M_2 \approx M_3 \), we consider two subcase \( M \ll M_1 \ll M_2 \approx M_3 \) and \( M \ll M_1 \approx M_2 \approx M_3 \).

Subcase 3.a. \( M \ll M_1 \ll M_2 \approx M_3 \). By using the same method as in case 2, we need to focus on the following summation
\[ \sum_{M \ll M_1 \ll M_2 \approx M_3} M_3^{-6s-5+M_5 s+2 M_1^{-s}}\|u_{M_1}\|_{X^s} \|u_{M_2}\|_{X^s} \|u_{M_3}\|_{X^s}. \]

The summation is handled by using Schwarz inequality if \( s > -\frac{5}{4} \).

Subcase 3.b \( M \ll M_1 \approx M_2 \approx M_3 \). In order for the final output to be at frequency \( M \), the two frequency \( M_3 \) factors must be \( M_3 \) separated. We may assume that \( u_{M_2}, u_{M_3} \) are \( M_3 \) separated. Therefore we obtain
\[ \|x_j u_{M_2} u_{M_3}\|_{L^2_t L^2_x} \lesssim M_3^{-\frac{3}{2}} \|x_j u_{M_2}\|_{L^2_x} \|x_j u_{M_3}\|_{L^2_x} \]
and
\[ \|x_j u_{M_2} u_{M_3}\|_{L^2_t L^2_x} \lesssim M_3^{-\frac{3}{2}} \log M_3 \|x_j u_{M_2}\|_{L^2_x} \|x_j u_{M_3}\|_{L^2_x}. \]

Therefore we can proceed as in Case 2. We focus on the following summation
\[ \sum_{M \ll M_1 \ll M_2 \approx M_3} M_3^{-7s-5+M_5 s+2} \|u_{M_1}\|_{X^s} \|u_{M_2}\|_{X^s} \|u_{M_3}\|_{X^s}. \]

The summation is handled by using Schwarz inequality if \( s > -\frac{5}{4} \).

Observe that the embedding \( X^{0, -\frac{1}{4}} \hookrightarrow U_{2}^3 \) implies the embedding \( \hat{X}^{0, -\frac{1}{4}} \hookrightarrow DU_{2}^3 \). By using \( \hat{X}^{0, -\frac{1}{4}} \hookrightarrow DU_{2}^3 \), dual estimates (3.6) and Lemma 4.2 we have
\[ N^s \|f\|_{DU_{2}^3} \lesssim N^{-3-2s} \|u_1\|_{X^s} \|u_2\|_{X^s} \|u_3\|_{X^s}. \]

Therefore the summation with respect to \( N \) is easily handled. So we conclude the proof of Proposition 4.1.

**Proof of Lemma 4.2.** Let \( (\tau_1, \xi_1) \) be the frequencies for each factor and let \( (\tau, \xi) \) be the resulting frequency. Then we have
\[ \xi_1 - \xi_2 + \xi_3 = \xi, \quad \tau_1 - \tau_2 + \tau_3 = \tau. \] (4.3)

Under the relation (4.3), we have
\[ (\tau_1 - \xi_1^4) - (\tau_2 - \xi_2^4) + (\tau_3 - \xi_3^4) - (\tau - \xi_4^4) = (\xi - \xi_1)(\xi - \xi_2)(\xi_1^4 + \xi_2^4 + \xi_3^4 + \xi_4^4 + 2(\xi_1 + \xi_2)^2). \]

The size of frequencies \( \{\xi_1, \xi_2, \xi_3, \xi_4\} \) is \( \{M, M_1, M_2, M_3\} \) with \( M \ll M_1 \lesssim M_2 = M_3 \). Here we allow for a slight abuse of notation, as the highest \( M_j \)'s need not be equal but merely comparable. Therefore, we have
\[ |\tau_1 - \xi_1^4| + |\tau_2 - \xi_2^4| + |\tau_3 - \xi_3^4| + |\tau - \xi_4^4| \gtrsim M_1 M_3 M_3^2 \text{ if } N_2 = M_3 \]
and
\[ |\tau_1 - \xi_1^4| + |\tau_2 - \xi_2^4| + |\tau_3 - \xi_3^4| + |\tau - \xi_4^4| \gtrsim M_3 M_3 M_3^2 \text{ if } N_2 = M_1 \]
Hence at least one modulation should be large. Therefore, to take advantage of this modulation gain, we consider the following cases:

Case 1. All input factors have small modulation i.e., \( |\tau_1 - \xi_1^4|, |\tau_2 - \xi_2^4|, |\tau_3 - \xi_3^4| \ll M_1 M_3^3 \). Therefore, output has large modulation \( |\tau - \xi^4| \gtrsim M_1 M_3^4 \).

Case 2. There is at least one input which has large modulation i.e., \( |\tau_1 - \xi_1^4| \gtrsim M_1 M_3^3 \) or \( |\tau_2 - \xi_2^4| \gtrsim M_1 M_3^3 \) or \( |\tau_3 - \xi_3^4| \gtrsim M_1 M_3^3 \). Depending on which factor has the large modulation and on whether the conjugated factor \( \pi_{N_2} \) has lower frequency \( M_1 \) or not, we divide this case into six.

Case 1. This is when the input factors have small modulation and hence the output has large modulation. Depending on whether the conjugated factor has a lower frequency or not we divide this case into three.

Subcase 1.a. First, we consider the first component of \( f \) i.e., the conjugated factor \( \pi_{N_2} \) has the highest frequency \( M_3 \)

\[
f_1 = \sum_{M \ll M_1, M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} P_M \left( Q_{\ll M_1 M_3^3} (\chi J u M_3) \bar{Q}_{\ll M_1 M_3^3} (\chi J u M_3) Q_{\ll M_1 M_3^3} (\chi J u M_3) \right)
\]

Since all input factors have small modulation \( \ll M_1 M_3^3 \), we have

\[
|\tau - \xi^4| = |(\tau_1 - \xi_1^4) - (\tau_2 - \xi_2^4) + (\tau_3 - \xi_3^4)| \approx \tau_1 M_3^4.
\]

Hence \( f_1 M_1, M_3 \) is localized at modulation \( M_1 M_3^3 \gg M_4 \).

For the triple product \( v_{M_3} \bar{v}_{M_5} v_{M_1} \), by using the energy bound for \( v_{M_3} \) and the bilinear estimate for \( v_{M_3} v_{M_1} \), we have

\[
\|v_{M_3} \bar{v}_{M_5} v_{M_1}\|_{L_t^2 L_x^2} \lesssim M_3^{-\frac{2}{3}} \|v_{M_3}\|_{U_3^2} \|v_{M_5}\|_{U_3^2} \|v_{M_1}\|_{U_3^2}.
\]

Applying \( P_M \) and the Bernstein’s inequality, we obtain

\[
\|P_M (v_{M_3} \bar{v}_{M_5} v_{M_1})\|_{L_t^2 L_x^2} \lesssim M_3^{-\frac{s}{2}} M_4^7 \|v_{M_3}\|_{U_3^2} \|v_{M_5}\|_{U_3^2} \|v_{M_1}\|_{U_3^2}.
\] (4.4)

In order to bound \( f_1 \), we need to consider the interval summation losses \( M_4^{4s+2} M_3^{4s-2} \). Therefore, by using (4.4) and uniform boundedness property \( Q_N : U^p_{N} \rightarrow U^p_{N} \), we obtain

\[
\sum_{J \subset I, |J| = M_3^{4s+2}} \|f_1 M_1, M_3\|_{L_t^2 L_x} \lesssim M_4^{4s+2} M_3^{4s-2} M_1^{-s} M_3^{-2s} M_5^4 \|v_{M_3}\|_{X^s} \|v_{M_5}\|_{X^s} \|v_{M_1}\|_{X^s}.
\]

Since \( f_1 M_1, M_3 \) is localized at modulation \( M_1 M_3^3 \), we have

\[
\sum_{J \subset I, |J| = M_3^{4s+2}} \|f_1 M_1, M_3\|_{X^{0, -\frac{1}{2}}} \lesssim M_4^{6s-5} M_1^{-s-\frac{1}{2}} M_3^{4s+2} \|v_{M_3}\|_{X^s} \|v_{M_5}\|_{X^s} \|v_{M_1}\|_{X^s}.
\]

The summation with respect to the dyadic numbers \( M_1 \) and \( M_3 \) is handled if \( s \geq -\frac{11}{14} \). As a result, we conclude

\[
\|f_1\|_{X^{0, -\frac{1}{2}}} \lesssim \sum_{M \ll M_1, M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} \|f_1 M_1, M_3\|_{X^{0, -\frac{1}{2}}}.
\]
Subcase 1.b. The second component of \( f \) is that the conjugated factor \( \overline{u}_N \) has lower frequency \( M_1 \)

\[
 f_2 = \sum_{M \ll M_1} \sum_{J \subset I, |J| = M_3^{k+2}} P_M \left( Q \ll M_1 \chi_j u_{M_3} \right) \overline{Q} \ll M_1 \chi_j u_{M_3} \chi_j u_{M_3} \chi_j u_{M_3} \chi_j u_{M_3})
\]

\[
 = \sum_{M \ll M_1} \sum_{J \subset I, |J| = M_3^{k+2}} f_2^{M_1, M_3}.
\]

Then \( f_2^{M_1, M_3} \) is localized at modulation \( M_3^k \gg M^4 \).

By using Bernstein, energy bound and bilinear estimate, we have the following \( L^2_{t,x} \) bound:

\[
 \| P_M (v_{M_3} \overline{u}_1 v_{M_3}) \|_{L^2_{t} L^2_x} \lesssim M_3^{-\frac{3}{2}} M_3^{\frac{1}{2}} \| v_{M_3} \|_{U^3} \| v_{M_3} \|_{U^2} \| v_{M_1} \|_{U^2}.
\]

(4.5)

By considering the interval summation losses and using (4.5), we obtain

\[
 \sum_{J \subset I, |J| = M_3^{k+2}} \| f_2^{M_1, M_3} \|_{L^2_{t,x}} \lesssim M_3^{4s+2} M_3^{4s-2} M_3^{-s} M_3^{-2s} M_3^{\frac{1}{2}} M_3^{-\frac{3}{2}} \| u_{M_3} \|_{X^s} \| u_{M_3} \|_{X^s} \| u_{M_1} \|_{X^s}.
\]

Since \( f_2^{M_1, M_3} \) has modulation \( M_3^k \), we have

\[
 \sum_{J \subset I, |J| = M_3^{k+2}} \| f_2^{M_1, M_3} \|_{X^{0, -\frac{3}{4}}} \lesssim M_3^{-6s-\frac{11}{2}} M_3^{-s} M_3^{4s+\frac{3}{2}} \| u_{M_3} \|_{X^s} \| u_{M_3} \|_{X^s} \| u_{M_1} \|_{X^s}.
\]

which is summable with respect to dyadic number \( M_1, M_3 \) if \( s \geq -\frac{11}{12} \). After summation, we obtain

\[
 \| f_2 \|_{X^{0, -\frac{3}{4}}} \lesssim M^{-3-3s} \| u \|_{X^s}.
\]

Subcase 1.c. The third component of \( f \) is

\[
 f_3 = \sum_{M \ll M_1 = M_3} \sum_{J \subset I, |J| = M_3^{k+2}} P_M \left( Q \ll M_3^k (\chi_j u_{M_3}) \overline{Q} \ll M_3^k (\chi_j u_{M_3}) Q \ll M_3^k (\chi_j u_{M_3}) \right)
\]

\[
 = \sum_{M \ll M_1 = M_3} \sum_{J \subset I, |J| = M_3^{k+2}} f_3^{M_3}.
\]

Then \( f_3^{M_3} \) has modulation \( M_3^k \gg M^4 \).

Observe that \( \xi_1 - \xi_2 + \xi_3 = \xi \) with \( |\xi| = M, |\xi_i| = M_3, i = 1, 2, 3 \). In order for the output to be at a low frequency \( M \), two of the frequencies \( \xi_1, -\xi_2, \xi_3 \) should be \( M_3 \) separated. Therefore, we use the bilinear estimates for those two factors and the energy bound for the remaining factor to obtain

\[
 \| P_M (v_{M_3} \overline{u}_3 v_{M_3}) \|_{L^2_{t} L^2_x} \lesssim M_3^{-\frac{3}{2}} M_3^{\frac{1}{2}} \| v_{M_3} \|_{U^3} \| v_{M_3} \|_{U^2} \| v_{M_3} \|_{U^2}.
\]

Hence, we obtain as in Case 1.a, Case 1.b

\[
 \sum_{J \subset I, |J| = M_3^{k+2}} \| f_3^{M_3} \|_{X^{0, -\frac{3}{4}}} \lesssim M_3^{-7s-\frac{11}{2}} M_3^{4s+\frac{3}{2}} \| u_{M_3} \|_{X^s} \| u_{M_3} \|_{X^s} \| u_{M_3} \|_{X^s},
\]
which is summable with respect to $M_3$ if $s \geq -\frac{11}{14}$. After summation with respect to $M_3$, we obtain
\[
\|f_3\|_{X_0^{-\frac{11}{14}}} \lesssim M^{-3-3s}\|u\|_{X_s}^3.
\]

Case 2. In this case, there is at least one input which has large modulation. Depending on which factor has the large modulation and on whether the conjugated factor $\overline{\sigma}_{M_2}$ has lower frequency or not, we divide this case into six.

Subcase 2.a. We consider
\[
f_4 = \sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} P_M \left( Q_{\geq M_1 M_3^2} (\chi_J u_{M_2}) \overline{\chi_J u_{M_2}} \chi_J u_{M_1} \right)
\]
\[
= \sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} f_4^{M_1, M_3},
\]
\[
f_5 = \sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} P_M \left( \chi_J u_{M_2} Q_{\geq M_1 M_3^2} (\chi_J u_{M_2}) \chi_J u_{M_1} \right)
\]

Since the two terms $f_4$ and $f_5$ are similar, we may consider the first one $f_4$.

By using the embedding $V_3^2 \hookrightarrow X_0^{0, \frac{13}{4}}$ for the first factor and the bilinear estimates for the remaining terms, we have
\[
\| (Q_{\geq M_1 M_3^2} u_{M_3}) \overline{v_{M_2}} \|_{L_1^1 L_2^1} \lesssim M_3^{-\frac{7}{4}} M_3^{-3} \| v_{M_3} \|_{U_3} \| u_{M_3} \|_{U_2}. \tag{4.6}
\]

Low modulation output. By Bernstein’s inequality, we have
\[
\| P_M \left( Q_{\geq M_1 M_3^2} u_{M_3} \right) \overline{v_{M_2}} \|_{L_1^1 L_2^1} \lesssim M_3^\frac{7}{4} M_3^{-3} \| v_{M_3} \|_{U_3} \| u_{M_3} \|_{U_2} \| u_{M_1} \|_{U_2}.
\]

By considering the interval summation loss, we have
\[
\sum_{J \subset I, |J| = M_4^{s+2}} f_4^{M_1, M_3} \| L_1^1 L_2^1 \]
\[
\lesssim M_1^{-s} M_3^{-2s} M_3^{4s+2} M_3^{-4s-2} M_3^{-\frac{1}{2}} M_3^{-3} \| u_{M_3} \|_{X} \| u_{M_3} \|_{X} \| u_{M_1} \|_{X}.
\]

This summation with respect to $M_1, M_3$ can be dealt with $s \geq -\frac{11}{14}$. After summation with respect to $M_1, M_3$, we obtain
\[
\| Q_{\geq M_4} f_4 \|_{L_1^1 L_2^1} \lesssim \| f_4 \|_{L_1^1 L_2^1} \lesssim \sum_{M_1 \ll M_3 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} \| f_4^{M_1, M_3} \|_{L_1^1 L_2^1}
\]
\[
\lesssim M^{-3-3s} \| u \|_{X_s}^3.
\]

To estimate $\| Q_{\geq M_4} f_4 \|_{X_0^{0, -\frac{11}{14}}}$, we need to decompose $Q_{\geq M_4} f_4$ into the following intermediate modulation output and high modulation output:
\[
Q_{\geq M_4} f_4 = \sum_{M_1, M_3 : M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} Q_{\geq M_4} f_4^{M_1, M_3}
\]
\[
= \sum_{M_1, M_3 : M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} Q_{M_4 \leq \sigma \ll M_3} f_4^{M_1, M_3}
\]
\[
+ \sum_{M_1, M_3 : M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_4^{s+2}} Q_{\gg M_3} f_4^{M_1, M_3}.
\]
Intermediate modulation output. In this case, we consider the $\tilde{X}^{0,-\frac{1}{2},1}$ estimate at modulation $M^4 \leq \sigma \lesssim M_1 M_3^3$. By using (4.6) and Bernstein inequality we have

$$\|Q_\sigma P_M \left( Q_{\gtrless M_1 M_3^3} v_{M_3} \bar{v}_{M_3} v_{M_3} \right) \|_{L^2_t x} \lesssim (M\sigma)^{\frac{1}{2}} M_1^{-\frac{1}{2}} M_3^{-3} \|v_{M_3}\|_{U^3_2} \|v_{M_3}\|_{U^3_2} \|v_{M_1}\|_{U^3_2}.$$ 

By considering the interval summation, we have

$$\sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma P_M \left( Q_{\gtrless M_1 M_3^3} (\chi J u_{M_3}) \bar{\chi J u_{M_3}} \chi J u_{M_1} \right) \|_{L^2_t x} \lesssim M_3^{4s+2} M_3^{-4s-2} (M\sigma)^{\frac{1}{2}} M_1^{-\frac{1}{2}} M_3^{-3} \sup_{J \subset I, |J| = M_3^{4s+2}} \|\chi J u_{M_3}\|_{U^3_2} \|\chi J u_{M_3}\|_{U^3_2} \|\chi J u_{M_1}\|_{U^3_2}$$

$$\lesssim \sigma^{\frac{3}{2}} M_3^{4s+2} M_1^{-\frac{1}{2} - s} M_3^{-6s-5} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_1}\|_{X^s},$$

or equivalently

$$\sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma f_4^{M_1, M_3} \|_{X^{0,-\frac{1}{2},1}} \lesssim M_3^{4s+2} M_1^{-\frac{1}{2} - s} M_3^{-6s-5} \ln \left( \frac{M_3}{M} \right) \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_1}\|_{X^s}.$$ 

By summing over $M^4 \leq \sigma \lesssim M_1 M_3^3$, we obtain

$$\sum_{M^4 \leq \sigma \lesssim M_1 M_3^3} \sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma f_4^{M_1, M_3} \|_{X^{0,-\frac{1}{2},1}} \lesssim M_3^{4s+2} M_1^{-\frac{1}{2} - s} M_3^{-6s-5} \ln \left( \frac{M_3}{M} \right) \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_1}\|_{X^s}.$$ 

Hence, the summation with respect to $M_1, M_3$ can be handled with $s > -\frac{11}{14}$. After summation with respect to $M_1, M_3$, we obtain

$$\sum_{M_1 \lesssim M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} \|Q f_4^{M_1, M_3} \|_{X^{0,-\frac{1}{2},1}} \lesssim M^{-3-3s} \|u\|_{X^{0,-\frac{1}{2},1}}.$$

High modulation output. In this case, we need to estimate the output localized at modulations $\sigma \gg M_1 M_3^3$. In order to obtain such an output at least one of the inputs should have modulation at least $\sigma$. We assume that the lower frequency factor has modulation $\sigma$. This is the worst case in terms of bilinear separation.

For the product $v_{M_3} \bar{v}_{M_3} Q_\sigma v_{M_1}$, we use the embedding $V^2_3 \hookrightarrow X^{0,-\frac{1}{2},1}$ for $Q_\sigma v_{M_1}$ and the bilinear estimate for $v_{M_3} \bar{v}_{M_3}$. Observe that in order for the final output to be at frequency $M$, the two factors $v_{M_3} \bar{v}_{M_3}$ should be frequency localized in $M_1$ separated intervals of length $|\xi_1 - \xi_2| \approx M_1$. Therefore, by using the high modulation bound for $Q_\sigma v_{M_1}$ and the bilinear estimate for $v_{M_3} \bar{v}_{M_3}$ with bilinear gain $(M_1 M_3^2)^{-\frac{1}{2}}$, we have

$$\|v_{M_3} \bar{v}_{M_3} Q_\sigma v_{M_1} \|_{L^1_t L^2_x} \lesssim M_1^{-\frac{1}{2}} M_3^{-1} \sigma^{-\frac{1}{2}} \|v_{M_3}\|_{U^3_2} \|v_{M_3}\|_{U^3_2} \|v_{M_1}\|_{U^3_2}.$$ 

Applying $Q_\sigma P_M$ and the Bernstein’s inequality, we obtain

$$\|Q_\sigma P_M (v_{M_3} \bar{v}_{M_3} Q_\sigma v_{M_1}) \|_{L^2_t L^2_x} \lesssim (\sigma M)^{\frac{1}{2}} M_1^{-\frac{1}{2}} M_3^{-1} \sigma^{-\frac{1}{2}} \|v_{M_3}\|_{U^3_2} \|v_{M_3}\|_{U^3_2} \|v_{M_1}\|_{U^3_2}.$$
By considering the interval summation, we have
\[ \sum_{J \subset I, |J| = M_3^{4s+2}} \| Q_{M_1, M_3} f_M^{M_1, M_3} \|_{X^{0, -\frac{1}{2}, 1}} \]
\[ \lesssim \sum_{J \subset I, |J| = M_3^{4s+2}} \| Q_{M_1, M_3} P_M \left( Q_{M_1, M_3} (\chi_J u_{M_1}) \chi_J u_{M_1} \right) \|_{X^{0, -\frac{1}{2}, 1}} \]
\[ \lesssim \sum_{J \subset I, |J| = M_3^{4s+2}} \sum_{\sigma \geq M_1 M_3} \| Q_{\sigma} P_M \left( Q_{M_1, M_3} (\chi_J u_{M_1}) \chi_J u_{M_1} \right) \|_{X^{0, -\frac{1}{2}, 1}} \]
\[ \lesssim M_3^{4s+2} M_3^{-4s-2} (M_1 M_3)^{-\frac{1}{2}} (M_1^{-\frac{1}{2}} M_3^{-\frac{1}{2}} M_3^{-1}) M_1^{-s} M_3^{-2s} \]
\[ \times \| u_{M_3} \|_{X^s} \| u_{M_1} \|_{X^s} \| u_{M_1} \|_{X^s} \]
\[ = M_3^{-\frac{6s}{2}} M_1^{-1-s} M_3^{4s+\frac{7}{2}} \| u_{M_3} \|_{X^s} \| u_{M_3} \|_{X^s} \| u_{M_1} \|_{X^s}. \]

Therefore, the summation with respect to $M_1, M_3$ is handled if $s \geq -\frac{3}{13}$. After summation with respect to $M_1, M_3$, we obtain
\[ \sum_{M \ll M_1 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} \| Q_{M_1, M_3} f_M^{M_1, M_3} \|_{X^{0, -\frac{1}{2}, 1}} \lesssim M^{-3-3s} \| u \|_{X^s}. \quad (4.9) \]

Therefore, by combining the intermediate modulation case (4.7) and high modulation case (4.9), we have
\[ \| Q_{M_3} f_M \| \lesssim M^{-3-3s} \| u \|_{X^s}. \]

Subcase 2.b. In this case, the low frequency factor has high modulation. Here, we consider
\[ f_6 = \sum_{M \ll M_1 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} P_M \left( \chi_J u_{M_3} \chi_J u_{M_3} Q_{M_1, M_3} \left( \chi_J u_{M_1} \right) \right) \]
\[ = \sum_{M \ll M_1 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} f_6^{M_1, M_3}. \]

For the $v_{M_3} u_{M_3} Q_{M_1, M_3} u_{M_1}$, we use the embedding $V_3^2 \hookrightarrow X^{0, \frac{1}{2}, 1}$ for $Q_{M_1, M_3} u_{M_1}$ and the bilinear estimate for $v_{M_3} u_{M_3}$. Observe that in order for the final output to be at frequency $M_1$, the two factors $v_{M_3} u_{M_3}$ should be frequency localized in $M_1$ separated intervals of length $|\xi_1 - \xi_2| \approx M_1$. Therefore, by using the high modulation bound for $Q_{M_1, M_3} u_{M_1}$ and the bilinear estimate for $v_{M_3} u_{M_3}$ with bilinear gain $(M_1 M_3)^{-\frac{1}{2}}$, we have
\[ \| v_{M_3} u_{M_3} Q_{M_1, M_3} u_{M_1} \|_{L^1 L^1} \lesssim M_1^{-\frac{1}{2}} M_3^{-\frac{1}{2}} (M_1 M_3)^{-\frac{1}{2}} \| u_{M_1} \|_{U^s_2} \| v_{M_3} \|_{U^s_2} \| v_{M_3} \|_{U^s_2} \]
\[ = M_1^{-\frac{1}{2}} M_3^{-\frac{1}{2}} \| v_{M_3} \|_{U^s_2} \| v_{M_3} \|_{U^s_2} \| v_{M_3} \|_{U^s_2}. \]

Low modulation output. By Bernstein’s inequality, we have
\[ \| P_M \left( v_{M_3} u_{M_3} \left( Q_{M_1, M_3} u_{M_1} \right) \right) \|_{L^1 L^2} \]
\[ \lesssim M_1^{-\frac{1}{2}} M_3^{-\frac{1}{2}} \| v_{M_3} \|_{U^s_2} \| v_{M_3} \|_{U^s_2} \| v_{M_3} \|_{U^s_2}. \quad (4.10) \]

By considering the interval summation loss, we have
\[ \sum_{J \subset I, |J| = M_3^{4s+2}} \| f_6^{M_1, M_3} \|_{L^1 L^2} \]
\[
\lesssim M_1^{-s} M_3^{-2s} M_3^{4s+2} M_3^{-4s-2} M_1^{\frac{1}{2}} M_1^{-\frac{1}{2}} M_3^\frac{3}{2} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|
\]

\[
= M_3^{4s+\frac{1}{2}} M_3^{-6s-\frac{5}{2}} M_1^{-s-1} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} .
\]

This summation with respect to \( M_1, M_3 \) can be dealt with \( s \geq -\frac{5}{4} \). After summation with respect to \( M_1, M_3 \), we obtain

\[
\|Q_{\leq M^4} f_6\|_{L_t^1 L_x^2} \lesssim \|f_6\|_{L_t^1 L_x^2} \lesssim \sum_{M < M_3 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} \|f_6^M_{M_3, M_3}\|_{L_t^1 L_x^2} \lesssim M^{-3-3s} \|u\|_{X^s}^3 .
\]

To estimate \( \|Q_{\geq M^4} f_6\|_{X^0-\frac{1}{2}^s} \), we need to decompose \( Q_{\geq M^4} f_6 \) into the following intermediate modulation output and high modulation output:

\[
Q_{\geq M^4} f_6 = \sum_{M_1, M_3: M < M_1 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} Q_{\geq M^4} f_6^{M_1, M_3}
\]

\[
= \sum_{M_1, M_3: M < M_1 < M_3} \sum_{J \subset I, |J| = M_3^{4s+2}} Q_{M^4 \leq \sigma} \lesssim M_3^3 M_3^{1-1} M_3^{-\frac{5}{2}} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} .
\]

Intermediate modulation output. We consider the \( X^0-\frac{1}{2}^s \) estimate at modulation \( M^4 \leq \sigma \lesssim M_1 M_3^3 \). By using (4.10) and Bernstein's inequality we have

\[
\|Q_\sigma P_M \left( \chi_{J} u_{M_3} \chi_{J} u_{M_3} \right) \|_{L_t^2_x} \lesssim (M \sigma)^{\frac{1}{2}} M_3^{1-1} M_3^{\frac{3}{2}} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} .
\]

By considering the interval summation, we have

\[
\sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma f_6^{M_1, M_3}\|_{L_t^2_x} \lesssim \sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma P_M \left( \chi_{J} u_{M_3} \chi_{J} u_{M_3} \right) \|_{L_t^2_x} \lesssim M^{4s+\frac{1}{2}} M_3^{-4s-\frac{5}{2}} M_3^{1-1} M_3^{-\frac{5}{2}} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} \|u_{M_3}\|_{L_t^2} .
\]

or equivalently

\[
\sum_{J \subset I, |J| = M_3^{4s+2}} \|Q_\sigma f_6^{M_1, M_3}\|_{X^0-\frac{1}{2}^s} \lesssim M^{4s+\frac{1}{2}} M_3^{-1-s} M_3^{-6s-\frac{5}{2}} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} .
\]

By summing over \( M^4 \leq \sigma \lesssim M_1 M_3^3 \), we obtain

\[
\sum_{J \subset I, |J| = M_3^{4s+2}} \sum_{M_1, M_3: M_1 M_3^3 \leq \sigma} \|Q_\sigma f_6^{M_1, M_3}\|_{X^0-\frac{1}{2}^s} \lesssim M^{4s+\frac{1}{2}} M_3^{-1-s} M_3^{-6s-\frac{5}{2}} \ln \left( \frac{M_3}{M} \right) \times \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} \|u_{M_3}\|_{X^s} .
\]
Hence, the summation with respect to $M_1, M_3$ can be handled with $s > -\frac{3}{4}$. After summation with respect to $M_1, M_3$, we obtain
\[
\sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_3^{\delta s + 2}} \| Q_{M \leq \sigma \leq M_1 M_3^3} f_6^{M_1, M_3} \|_{X^{0, -\frac{3}{4}}_s} \lesssim M^{-3-3s} \| u \|_{X^3}^3. \tag{4.11}
\]

High modulation output. In this case, we need to estimate the output localized at modulations $\sigma > M_1 M_3^3$. In order to obtain such an output at least one of the inputs should have modulation at least $\sigma$. We assume that the lower frequency factor has modulation $\sigma$. This is the worst case in terms of bilinear separation.

For the product $v_{M_3} v_{M_3} Q_M v_{M_1}$, we use the embedding $V_3^2 \hookrightarrow X^{0, \frac{1}{4} - 1}$ for $Q_M v_{M_1}$ and the bilinear estimate for $v_{M_3} v_{M_3}$. Observe that in order for the final output to be at frequency $M$, the two factors $v_{M_3} v_{M_3}$ should be frequency localized in $M_1$ separated intervals of length $|\xi_1 - \xi_2| \approx M_1$. Therefore, by using the high modulation bound for $Q_M v_{M_1}$ and the bilinear estimate for $v_{M_3} v_{M_3}$ with bilinear gain $(M_1 M_3^3)^{-\frac{1}{2}}$, we have
\[
\| v_{M_3} v_{M_3} Q_M v_{M_1} \|_{L^4 I_1} \lesssim M_1^{-\frac{1}{8}} M_3^{-1} \sigma^{-\frac{1}{2}} \| v_{M_3} \|_{U^3_1} \| v_{M_3} \|_{U^3_1} \| v_{M_1} \|_{U^3_1}.
\]

Applying $Q_M P_M$, Bernstein’s inequality, and considering the interval summation, we have
\[
\sum_{J \subset I, |J| = M_3^{\delta s + 2}} \| Q_{M \gg M_1 M_3^3} f_6^{M_1, M_3} \|_{L^4_{t,x}} \lesssim \sum_{J \subset I, |J| = M_3^{\delta s + 2}} \| Q_{M \gg M_1 M_3^3} P_M \left( \chi_J u_{M_3} \overline{\chi_J u_{M_3}} Q_{M \gg M_1 M_3^3} (\chi_J u_{M_1}) \right) \|_{L^4_{t,x}} \lesssim \sum_{J \subset I, |J| = M_3^{\delta s + 2}} \sum_{\sigma \geq M_1 M_3^3} \| Q_{M \gg M_1 M_3^3} \|_{L^4_{t,x}},
\]
or equivalently
\[
\sum_{J \subset I, |J| = M_3^{\delta s + 2}} \| Q_{M \gg M_1 M_3^3} f_6^{M_1, M_3} \|_{X^{0, -\frac{3}{4}}_s} \lesssim M^{\delta s + 2} M_3^{-4s - 2} (M_1 M_3^3)^{-\frac{1}{8}} \left( M_1^2 M_1^{-\frac{1}{4}} M_3^{\frac{1}{2}} \right) M_1^{-s} M_3^{-2s} \| u_{M_3} \| X^s \| u_{M_3} \| X^s \| u_{M_1} \| X^s.
\]

Therefore, the summation with respect to $M_1, M_3$ is handled if $s \geq -\frac{3}{4}$. After summation with respect to $M_1, M_3$, we obtain
\[
\sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_3^{\delta s + 2}} \| Q_{M \gg M_1 M_3} f_6^{M_1, M_3} \|_{X^{0, -\frac{3}{4}}_s} \lesssim M^{-3-3s} \| u \|_{X^3}^3. \tag{4.12}
\]

Therefore, by combining the intermediate modulation case (4.11) and high modulation case (4.12), we have
\[
\| Q_{M \gg M_1} f_6 \| \lesssim M^{-3-3s} \| u \|_{X^3}^3.
\]

**Subcase 2.c** In this case, the low frequency factor is conjugated but does not have high modulation. We consider
\[
f_7 = \sum_{M \ll M_1 \ll M_3} \sum_{J \subset I, |J| = M_3^{\delta s + 2}} P_M \left( Q_{M \gg M_3} (\chi_J u_{M_3}) \chi_J u_{M_3} \overline{\chi_J u_{M_1}} \right)
\]
If $M_1 \ll M_3$, then the last two factors are $M_3$ separated in frequency. Although if $M_1 \approx M_3$, in order for the resulting frequency to be localized at frequency $M$, the two last factors should be still $|\xi_3 - \xi_2| \approx M_3$ separated. Therefore, by using the bilinear estimate and high modulation bound, we obtain the trilinear estimate:

$$
\left\| (Q_{M_1, M_3} v_{M_3}) \overline{v_{M_3}} v_{M_3} \right\|_{L^1_t L^1_x} \lesssim M_1^{-\frac{3}{2}} M_3^{-3} \|v_{M_3}\|_{L^3_{t,x}} \|v_{M_3}\|_{L^6_{t,x}} \|v_{M_3}\|_{L^\infty_{t,x}}.
$$

Therefore, the rest of the argument proceeds as in Subcase 2.a without any significant changes.

Subcase 2.d. In this case, the low frequency factor is conjugated and has high modulation. We consider

$$
f_8 = \sum_{M \ll M_1 \ll M_3, J \subset I, |J| = M_3^{4s+2}} \sum_{M_1, M_3} P_M \left( \chi_J u_{M_1} \chi_M \overline{Q_{M_1, M_3}} (\chi_J u_{M_3}) \right)
$$

In order for the resulting frequency to be localized at frequency $M$, the two frequency $M_3$ factors should be still $M_3$ separated. Therefore, by applying the bilinear estimate and high modulation bound, we obtain the trilinear estimate

$$
\left\| v_{M_3} Q_{M_1, M_3} v_{M_3} \right\|_{L^1_{t,x}} \lesssim M_3^{-\frac{3}{2}} \|v_{M_3}\|_{L^3_{t,x}} \|v_{M_3}\|_{L^6_{t,x}} \|v_{M_3}\|_{L^\infty_{t,x}}.
$$

Therefore, we can argue as in Subcase 2.a with better gains.

Subcase 2.e. In this case, all frequencies are equal and the conjugated factor has high modulation. We consider

$$
f_9 = \sum_{M \ll M_1 \ll M_3, J \subset I, |J| = M_3^{4s+2}} \sum_{M_1, M_3} P_M \left( \chi_J u_{M_1} \chi_M \overline{Q_{M_1, M_3}} (\chi_J u_{M_3}) \right)
$$

This is the worst case since we cannot have any frequency separation among the two unconjugated factors and hence we cannot depend on bilinear gains. Therefore, to obtain summability for $M_3$, we use the local smoothing estimates. By using the local smoothing estimates (3.10), maximal function estimates (3.11) and high modulation bound, we have

$$
\left\| v_{M_3} Q_{M_1, M_3} v_{M_3} \right\|_{L^1_{t,x}} \lesssim \left\| v_{M_3} \right\|_{L^2_{t,x} L^2_t} \left\| Q_{M_1, M_3} v_{M_3} \right\|_{L^2_{t,x} L^2_t} \left\| v_{M_3} \right\|_{L^4_{t,x} L^\infty_t} \lesssim M_3^{-\frac{3}{4}} \|v_{M_3}\|_{L^3_{t,x}} \|v_{M_3}\|_{L^6_{t,x}} \|v_{M_3}\|_{L^\infty_{t,x}}.
$$

Low modulation output. By considering the interval summation, we obtain

$$
\sum_{J \subset I, |J| = M_3^{4s+2}} \left\| f_9^M \right\|_{L^4_{t,x} L^1_{t}} \lesssim (M_3^{4s+2} M_3^{-8s-2}) M_3^{-5s} M_3^{-\frac{3}{4}} \left\| u_{M_3} \right\|_{X^s} \lesssim M_3^{4s+2} M_3^{-7s-\frac{13}{2}} \left\| u_{M_3} \right\|_{X^s}.
$$
which is summable with respect to $M_3$ if $s \geq -\frac{3}{4}$. After summation with respect to $M_3$, we obtain
\[
\|Q_{\geq M^4} f_0\|_{L^4_t L^\infty_x} \lesssim \|f_0\|_{L^4_t L^\infty_x} \lesssim M^{-3s - \frac{12s}{13}} \|u\|_{X^s}^3.
\]
To estimate $\|Q_{\geq M^4} f_0\|_{X^{0, -\frac{3}{4}}}^3$, we need to decompose $Q_{\geq M^4} f_0$ into the following intermediate modulation output and high modulation output:
\[
Q_{\geq M^4} f_0 = \sum_{M_3 : M \ll M_3, J \subset I, |J| = M_3^{4+2}} \sum_{Q_{\geq M^4} f_0} Q_{M^4} f_0 \sum_{M_3 : M \ll M_3, J \subset I, |J| = M_3^{4+2}} \sum_{Q_{\geq M^4} f_0} Q_{M^4} f_0.
\]
Intermediate modulation output. By using Bernstein’s inequality and (4.13), we have
\[
\|Q_{\sigma} P_M \left( v_{M_3} Q_{X^3, M_3} v_{M_3} \right) \|_{L^2_x L^2_t} \lesssim \sigma^\frac{1}{2} M^\frac{4}{3} M_3^{-\frac{13}{14}} \|v_{M_3}\|_{U^2} \|v_{M_3}\|_{U^2} \|v_{M_3}\|_{U^2}.
\]
Hence, by considering the interval summation, we obtain
\[
\sum_{J \subset I, |J| = M_3^{4+2}} \|Q_{\sigma} f_0 \|_{L^2_x L^2_t} \lesssim \left( M^{4s + 2} M_3^{-4s - 2} \right) \sigma^\frac{1}{2} M^\frac{4}{3} M_3^{-\frac{12}{13}} M_3^{-3s} \|u_{M_3}\|_{X^s},
\]
or equivalently
\[
\sum_{J \subset I, |J| = M_3^{4+2}} \|Q_{\sigma} f_0 \|_{X^{0, -\frac{3}{4}}} \lesssim M^{4s + \frac{2}{3}} M_3^{-7s - \frac{24}{13}} \sigma^\frac{1}{2} \|u_{M_3}\|_{X^s}.
\]
By summing over $M^4 \leq \sigma \leq M_3^4$, we obtain
\[
\sum_{M^4 \leq \sigma \leq M_3^4} \sum_{J \subset I, |J| = M_3^{4+2}} \|Q_{\sigma} f_0 \|_{X^{0, -\frac{3}{4}}} \lesssim M^{4s + \frac{2}{3}} M_3^{-7s - \frac{24}{13}} \ln \left( \frac{M_3}{M} \right) \|u_{M_3}\|_{X^s}.
\]
Hence, the summation with respect to $M_3$ can be handled with $s > -\frac{3}{4}$. After summation with respect to $M_3$, we obtain
\[
\sum_{M^4 \leq \sigma \leq M_3^4} \sum_{J \subset I, |J| = M_3^{4+2}} \|Q_{M^4} f_0 \|_{X^{0, -\frac{3}{4}}} \lesssim M^{-3s - 3} \|u\|_{X^s}^3.
\]
High modulation output. In this case, we need to estimate the output localized at modulations $\sigma \gg M_3^4$. In order to obtain such an output at least one of the inputs should have modulation at least $\sigma$. We assume that the conjugated factor has modulation $\sigma$. This is the worst case since, for the remaining case, we can use the bilinear estimates. By using Bernstein’s inequality, we have
\[
\|Q_{\sigma} P_M \left( v_{M_3} Q_{\sigma} v_{M_3} v_{M_3} \right) \|_{L^2_x L^2_t} \lesssim \sigma^\frac{1}{2} M^\frac{4}{3} M_3^{-\frac{13}{14}} \|v_{M_3} Q_{\sigma} v_{M_3} v_{M_3}\|_{L^4_t L^1_x}.
\]
By applying local smoothing estimate, maximal function estimate and high modulation bound, we have
\[
\|Q_{\sigma} P_M \left( v_{M_3} Q_{\sigma} v_{M_3} v_{M_3} \right) \|_{L^2_x L^2_t} \lesssim \sigma^\frac{1}{2} M^\frac{4}{3} M_3^{-\frac{13}{14}} \|v_{M_3}\|_{U^2} = M^\frac{4}{3} M_3^{-\frac{13}{14}} \|v_{M_3}\|_{U^2}.
\]
By considering the interval summation, we obtain
\[ \sum_{J \subset I, |J| = M_4^s + 2} \|Q \gg M_3^3 f_9\|_{X^{0,-\frac{1}{2}}} \lesssim \sum_{J \subset I, |J| = M_4^s + 2} \|Q a f_9\|_{X^{0,-\frac{1}{2}}} \lesssim (M_4^{4s+2} M_3^{-4s-2}) M_3^{\frac{13}{8}} \|u_{M_3}\|_{X^{s}}^{3} \]
which is summable with respect to $M_3$ if $s \geq -\frac{3}{4}$. After summation with respect to $M_3$, we obtain
\[ \sum_{M \ll M_3} \sum_{J \subset I, |J| = M_4^s + 2} \|Q_M f_9\|_{X^{0,-\frac{1}{2}}} \lesssim M_3^{-3-3s} \|u\|_{X^s}^{3}. \quad (4.15) \]
Therefore, by combining the intermediate modulation case (4.14) and high modulation case (4.15), we have
\[ \|Q \gg M_3 f_9\| \lesssim M_3^{-3-3s} \|u\|_{X^s}^{3}. \]

5. Conservation of the $H^s$ energy. In this section, we want to show the conservation of the $H^s$ energy. We are inspired by the method that is analogous to that in Colliander-Keel-Staffilani-Takaoka-Tao [7] and follow the argument in Koch-Tataru [17]. To obtain the energy estimate, we use the $I$-method with correction term.

We first define the $H^s$ energy:
\[ E_0 (u) = \langle a (D) u, u \rangle. \]

For the $H^s$ energy conservation, we want to choose the symbol $a (\xi) = (1 + \xi^2)^s$, but as in [17], we allow a slightly larger class of symbols.

**Definition 5.1.** Let $\epsilon > 0, s \in \mathbb{R}$. Then $S^s_\epsilon$ is the class of spherically symmetric symbols with the following properties:
(i) Slowly varying condition: For $|\xi| \approx |\xi'|$, we have
\[ a (\xi) \approx a (\xi'). \]
(ii) Symbol regularity,
\[ |\partial^\alpha a (\xi)| \lesssim a (\xi) (1 + \xi^2)^{-\frac{3}{2}}. \]
(iii) Decay at infinity,
\[ s - \epsilon \leq \frac{\log a (\xi)}{\log (1 + \xi^2)} \leq s + \epsilon. \]
Here $\epsilon$ is a small parameter.

The main goal of this section is to obtain the following energy bound.

**Proposition 5.2.** Let $-\frac{3}{4} < s < -\frac{1}{2}$ and $u$ be a solution to (4NLS) with
\[ \|u\|_{L^\infty_t H^s} \ll 1. \]
Then we have
\[ \|u\|_{L_t^\infty H^s} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^{3}. \quad (5.1) \]
We define the energy functional
\[ E_0 (u) = \| u \|^2_{H^s} = \langle a(D) u, u \rangle_{L^2}. \]

By differentiating this energy under the (4NLS) flow, we have
\[ \frac{d}{dt} E_0 (u) = 2\Re \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} i a (\xi_1) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_4). \]

By symmetrizing above integral, we have
\[ \frac{1}{2} \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} i (a (\xi_1) - a (\xi_2) + a (\xi_3) - a (\xi_4)) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_4) \] (5.2)

We want to cancel this term by adding the correction term \( E_1 (u) \), where \( E_1 (u) \) has the form
\[ E_1 (u) = \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} b_4 (\xi_1, \ldots, \xi_4) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_4), \]
where the function \( b_4 \) is symmetric under the even \( \xi_j \) indices, or of the odd \( \xi_j \) indices. The \( b_4 \) will be determined later. The role of \( b_4 \) is to make a cancelation. Observe that
\[ \frac{d}{dt} E_1 (u) = \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} ib_4 (\xi_1, \ldots, \xi_4) (\xi_4^4 - \xi_2^4 + \xi_3^4 - \xi_4^4) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_4) \]
\[ + 4\Re \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} ib_4 (\xi_1, \xi_2, \xi_3, \xi_4) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_4) |u|^2 \bar{u} (\xi_4). \]

To cancel the first integral in \( \frac{d}{dt} E_1 (u) \), we choose \( b_4 \) as follows:
\[ b_4 (\xi_1, \ldots, \xi_4) = \frac{-a (\xi_1) - a (\xi_2) + a (\xi_3) - a (\xi_4)}{i (\xi_4^4 - \xi_2^4 + \xi_3^4 - \xi_4^4), \text{ on } P_4 = \{ \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0 \}. \] (5.3)

In this situation, a resonant interaction does not appear. Later in Proposition 5.3, we will show that the multiplier \( b_4 \) should be fully nonresonant.

Therefore by using the above calculations, we have
\[ \Lambda_0 (u(t)) \] (5.4)
\[ := \frac{d}{dt} (E_0 + E_1) (u) \]
\[ = 4\Re \int_{\xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5 = \xi} ib_4 (\xi_1, \xi_2, \xi_3, \xi_5) \bar{u} (\xi_1) \bar{u} (\xi_2) \bar{u} (\xi_3) \bar{u} (\xi_5) \bar{u} (\xi_6). \] (5.5)

Before we prove the energy estimate, we prove the following multiplier estimate (5.8) that shows the multiplier \( b_4 \) is fully nonresonant. In order to estimate the correction term \( E_1 (u) \) and the derivative of modified energy \( \frac{d}{dt} (E_0 + E_1) \), we need to obtain the size of \( b_4 \). Originally, \( b_4 \) is defined only on the diagonal \( \{ \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0 \} \).

In order to separate variables, we want to extend it off diagonal in a smooth way.

Before stating the Lemma 5.3, we recall the following two mean value formulas: if \(|\eta|, |\lambda| \ll |\xi|\), then
\[ |a(\xi + \eta) - a(\xi)| \lesssim |\eta| \sup_{|\xi'| = |\xi|} |a'(\xi')|, \] (5.6)
and

\[ |a(\xi + \eta + \lambda) - a(\xi + \eta) - a(\xi + \lambda) + a(\xi)| \lesssim |\eta||\lambda| \sup_{|\xi'| = |\xi|} |a''(\xi')|. \]  

(5.7)

**Proposition 5.3.** Let \( a \) be a multiplier in \( S^s_x \). Then for each dyadic \( M_1 \leq M_2 \leq M_3 \) there is an extension of \( b_4 \) from the diagonal set

\[ \{(\xi_1, \xi_2, \xi_3, \xi_4) \in P_4, |\xi_1| \approx M_1, |\xi_2| \approx M_2, |\xi_3|, |\xi_4| \approx M_3 \} \]

to the full dyadic set

\[ \{|\xi_1| \approx M_1, |\xi_2| \approx M_2, |\xi_3|, |\xi_4| \approx M_3 \} \]

which satisfies the size and regularity conditions

\[ \left| \partial^{\beta_1}_1 \partial^{\beta_2}_2 \partial^{\beta_3}_3 \partial^{\beta_4}_4 b_4(\xi_1, \xi_2, \xi_3, \xi_4) \right| \lesssim a(M_1) M_2^{-1} M_3^{-3} M_1^{-\beta_1} M_2^{-\beta_2} M_3^{-\beta_3} \]  

(5.8)

The implicit constants are independent of \( M_1, M_2, M_3 \).

The proof of Proposition 5.3 is analogous to the proof of the Proposition 5.2 in [17]. The only difference is that the stronger dispersion produces an extra smoothing effect as much as \( M_3^2 \). For reader’s convenience, we present the proof in detail.

**Proof.** Observe that on \( P_4 \) resonance function admits the following factorization

\[ \xi_1^4 - \xi_2^4 + \xi_3^4 - \xi_4^4 = (\xi_4 - \xi_1)(\xi_4 - \xi_3) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right) \]

along with all versions of it due to the symmetries of \( P_4 \). For the proof, see [24].

We consider several cases:

i) \( M_1 \ll M_2 \leq M_3 \). Then the extension of \( b_4 \) is defined using the formula

\[ b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{a(\xi_1) - a(\xi_2) + a(\xi_3) - a(\xi_4)}{(\xi_4 - \xi_1)(\xi_4 - \xi_3) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right)} \]

and its size and regularity properties are easily followed from \( |\xi_4 - \xi_1| \approx M_3 \) and \( |\xi_1 - \xi_2| \approx M_2 \).

ii) \( M_1 \approx M_2 \ll M_3 \). Then the extension of \( b_4 \) is defined by

\[ b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{a(\xi_1) - a(\xi_2)}{(\xi_1 - \xi_2)(\xi_4 - \xi_1) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right)} \]

\[ - \frac{a(\xi_3) - a(\xi_4)}{(\xi_3 - \xi_4)(\xi_1 - \xi_3) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right)}. \]

Observe that \( |\xi_4 - \xi_1| \approx M_3 \) and the remaining quotients exhibits cancellation properties. More precisely, by using the mean value formula (5.6), we have

\[ \left| \frac{a(\xi_1) - a(\xi_2)}{\xi_1 - \xi_2} \right| \lesssim \frac{a(M_1)}{M_1} \quad \text{and} \quad \left| \frac{a(\xi_3) - a(\xi_4)}{\xi_3 - \xi_4} \right| \lesssim \frac{a(M_3)}{M_3}. \]

(iii) \( M_1 \approx M_2 \approx M_3 \). Then the extension of \( b_4 \) is defined by

\[ b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{-a(\xi_4 - (\xi_4 - \xi_3) - (\xi_4 - \xi_1)) - a(\xi_4 - (\xi_4 - \xi_3)) - a(\xi_4 - (\xi_4 - \xi_1)) + a(\xi_4)}{(\xi_4 - \xi_1)(\xi_4 - \xi_3) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right)} \]

\[ - \frac{a(\xi_4 - (\xi_4 - \xi_3) - (\xi_4 - \xi_1)) - a(\xi_4 - (\xi_4 - \xi_3)) - a(\xi_4 - (\xi_4 - \xi_1)) + a(\xi_4)}{(\xi_3 - \xi_4)(\xi_3 - \xi_1) \left( \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + 2(\xi_1 + \xi_3)^2 \right)}. \]
In this case, the resonant interaction is the most serious. But we can also use the cancellation properties. More precisely, by using the double mean value theorem (5.7), we have
\[
\left| a(\xi_1 - (\xi_2 - \xi_3) - (\xi_4 - \xi_5)) - a(\xi_4 - (\xi_2 - \xi_3)) - a(\xi_4 - (\xi_1 - \xi_5)) + a(\xi_4) \right| \lesssim \frac{a(M_3)}{M_3^2}
\]
\[
\square
\]

The effect of $E_1$ to the modified energy is easily controlled by $E_0$.

**Proposition 5.4.** Let $a \in S^s$ with $s + \epsilon < -\frac{1}{2}$. Then we have
\[
|E_1(u)| \lesssim E_0(u)^2.
\]

**Proof.** We may assume the functions $\tilde{u}_j$ are nonnegative. By using the Lemma 5.3, we have
\[
|E_1(u)| = \left| \int_{P_4} b_4(\xi_1, \ldots, \xi_4) \tilde{u}(\xi_1) \overline{\tilde{u}}(\xi_2) \tilde{u}(\xi_3) \overline{\tilde{u}}(\xi_4) \right|
\]
\[
\lesssim \sum_{1 \leq N_1 \leq N_2 \leq N_3 \approx N_4} \frac{a(N_1)}{N_2 N_3^2} \|u_{N_1} u_{N_2} u_{N_3} u_{N_4}\|_{L^4}.
\]

We may assume $N_1 \leq N_2 \leq N_3 \approx N_4$ by using the symmetry. Therefore, we focus on the summation:
\[
\lesssim \sum_{1 \leq N_1 \leq N_2 \leq N_3 \approx N_4} \frac{a(N_1)}{N_2 N_3^2} \|u_{N_1} \|_{L^4} \|u_{N_2} \|_{L^8} \|u_{N_3} \|_{L^8} \|u_{N_4} \|_{L^8}.
\]

By using Bernstein’s inequality, we have
\[
\lesssim \sum_{1 \leq N_1 \leq N_2 \leq N_3 \approx N_4} \frac{a(N_1)^{\frac{7}{4}}}{N_2 N_3^2} \|u_{N_1} \|_{L^4} \|u_{N_2} \|_{L^8} \|u_{N_3} \|_{L^8} \|u_{N_4} \|_{L^8}.
\]

The remaining summation with respect to $N_1, N_2, N_3$ is easily handled. In fact, it is enough to assume $s > -\frac{7}{6}$.

**Proposition 5.5.** Let $a \in S^s$ with $s + \epsilon < -\frac{1}{2}$ and $s > -\frac{3}{4}$. Then we have
\[
\left| \int_0^1 R_a(u) \, dt \right| \lesssim \|u\|_{X^s}^6.
\]

**Proof.** We consider a dyadic decomposition and represent the above integral in the frequency side as a dyadic sum of terms of the form
\[
\int_0^1 \int_{|\xi_1 - \xi_2 + \xi_3 = \xi_4 - \xi_5 + \xi_6 = \xi, |\xi| \approx N} b_4(\xi_1, \xi_2, \xi_3, \xi) \tilde{u}_{N_1}(\xi_1) \overline{\tilde{u}_{N_2}(\xi_2)} \tilde{u}_{N_3}(\xi_3) \varphi_N(\xi) \times \left( \tilde{u}_{N_4}(\xi_4) \overline{\tilde{u}_{N_5}(\xi_5)} \tilde{u}_{N_6}(\xi_6) \right) \, dt.
\]

Here $\varphi_N$ is the Fourier multiplier for $P_N$. There are two cases to consider:
Case 1: $N \ll N_4, N_5, N_6$. Then for the frequency $N$ factor we take advantage of Lemma 4.2. We denote

$$\{N, N_1, N_2, N_3\} = \{M_1, M_2, M_3, M_3\}, \quad M_1 \leq M_2 \leq M_3,$$

and

$$f_N = \chi I \sum_{N, N_1, N_2, N_3} P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right), \quad |I| = N^{4s+2}.$$ 

Since $b_4$ is smooth in each variable on the corresponding dyadic scale, we can expand it into a rapidly convergent Fourier series. This allows us to separate variables and reduce the problem to the case when $b_4$ is of product type

$$b_4 (\xi_1, \xi_2, \xi_3, \xi) = \frac{a(M_1)}{M_2 M_3} \chi^1 (\xi_1) \chi^2 (\xi_2) \chi^3 (\xi_3) \chi^0 (\xi),$$

where $\chi^i$’s are unit size bump functions which are smooth on the respective dyadic scales. Since the symbol $\chi^j (D)$ are bounded in $U^2_{3/2}$ space, we can discard $\chi^1, \chi^2$ and $\chi^3$ and incorporate $\chi^0$ into $P_N$. In the following, we drop the complex conjugate sign. Therefore, we have reduced the problem to the case

$$\sum_{\{N, N_1, N_2, N_3\} = \{M_1, M_2, M_3, M_3\}} \frac{a(M_1)}{M_2 M_3} \int_0^1 \int \sum_{N, N_1, N_2, N_3} \chi I \sum_{N, N_1, N_2, N_3} P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right) dx dt$$

$$= \sum_{\{N, N_1, N_2, N_3\} = \{M_1, M_2, M_3, M_3\}} \frac{a(M_1)}{M_2 M_3} \sum_{I \subseteq [0, 1], |I| = N^{4s+2}} \int \int \sum_{N, N_1, N_2, N_3} \chi I \sum_{N, N_1, N_2, N_3} P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right) dx dt$$

$$= \chi I \sum_{N, N_1, N_2, N_3} \chi I P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right)$$

We decompose $f_N$ into low modulation output and high modulation output

$$Q_{\leq M_3^3} \sum_{N \ll N_4, N_5, N_6} \chi I P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right) + Q_{\geq M_3^3} \sum_{N \ll N_4, N_5, N_6} \chi I P_N \left( \frac{\tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3}}{N_4, N_5, N_6, N \ll N_4, N_5, N_6} \right)$$

Subcase 1a. $N = M_3$. First, we consider the low modulation output. For the $L_t^1 L_x^2$ term in $f_N$, we estimate $\tilde{u}_{M_1}, \tilde{u}_{M_2}$ in $L_t^\infty$ and $\tilde{u}_{M_3}$ in $L_t^\infty L_x^2$ by using the Bernstein’s inequality and Lemma 4.2:
Therefore, we consider the summation
\[
\sum_{M_1 \leq M_2 \leq M_3} M_3^{8s-8} a(M_1) M_1^{s+\frac{1}{2}} M_2^{\frac{1}{2}-s} \|u_{M_1}\|_{X^s} \|u_{M_2}\|_{X^s} \|u_{M_3}\|_{X^s} \|u\|_X^{3s}.
\]
This summation can be dealt with \(s \geq \frac{17}{18}\).

For the \(L_2^\frac{5}{3} L_1^1\) term in \(f_N\), we estimate \(u_{M_1}, u_{M_2}\) in \(L^\infty\) and \(u_{M_3}\) in \(L_2^\frac{5}{3} L_\infty^\infty\). Then by using Bernstein’s inequality, maximal function estimate and Lemma 4.2, we obtain
\[
|\Lambda_1| \lesssim a(M_1) M_1^{4s-2} \left( \frac{M_2}{M_3} \right)^{\frac{s}{2}} \left( \frac{M_3}{M_2} \right)^{s+\frac{1}{2}} \left( \frac{M_3}{M_1} \right)^{s+\frac{1}{2}} \left( \frac{M_3}{M_1} \right)^{\frac{s}{2}},
\]
which gives the same result as in the previous case. Therefore, the summation with respect to \(M_1, M_2, M_3\) can be dealt with \(s \geq \frac{17}{18}\).

For the high modulation part of \(f_N\) at modulation \(\sigma \gg M_3^\frac{4}{3}\), we observe that at least one of three factors \(\chi_I u_{M_1}, \chi_I u_{M_2}, \chi_I u_{M_3}\) must have modulation at least \(\sigma \gg M_3^\frac{4}{3}\). We may assume \(Q_\sigma (\chi_I u_{M_1})\). This is the worst case. We bound \(Q_\sigma \chi_I u_{M_1}\) in \(L^2\) and the other two \(u_{M_2}, u_{M_3}\) in \(L^\infty\). Observe that
\[
|\Lambda_1| \lesssim a(M_1) \frac{M_1^{4s-2}}{M_2 M_3^\frac{4}{3}} \left( \frac{M_2}{M_3} \right)^{s+\frac{1}{2}} \left( \frac{M_3}{M_2} \right)^{\frac{s}{2}} \left( \frac{M_3}{M_1} \right)^{\frac{s}{2}},
\]
which is the same as before. Therefore, the summation with respect to \(M_1, M_2, M_3\) can be done with \(s \geq \frac{17}{18}\).

Subcase 1b. \(N = M_2 \ll M_3\). First, we consider the low modulation output. For the \(L_1^1 L_2^\frac{5}{3}\) term in \(Q_{\leq M_3} f_N\), we estimate \(u_{M_1}, u_{M_2}, u_{M_3}\) in \(L_1^1 L_2^\frac{5}{3}\). By considering the interval summation loss and using the Bernstein inequality, Lemma 4.2, we have
\[
|\Lambda_1| \lesssim a(M_1) \frac{M_1^{4s-2}}{M_2 M_3^\frac{4}{3}} \left( \frac{M_2}{M_3} \right)^{s+\frac{1}{2}} \left( \frac{M_3}{M_2} \right)^{\frac{s}{2}} \left( \frac{M_3}{M_1} \right)^{\frac{s}{2}},
\]
Therefore, the summation with respect to $M_1, M_2, M_3$ is handled if $s \geq -\frac{3}{4}$.

Next, we consider the $L^\frac{4}{3} L^1_t$ term in $f_N$. By considering the interval summation loss and using the Bernstein’s inequality, Lemma 4.2, we have

\[
|A_1| \lesssim a(M_1) M_2^{-4s-2} \sup_{I \subset [0,1], |I'| = M_2^{s+2}} \left| \int_I \int_R \chi_I u_{M_1} u_{M_3} Q \chi_X f_N \, dx \, dt \right|
\]

\[
\lesssim a(M_1) M_2^{-4s-2} \sup_{I \subset [0,1], |I'| = M_2^{s+2}} \| \chi_I u_{M_1} u_{M_3} \|_{L^\infty_T L^2_x} \| Q \chi_X f_N \|_{L^\infty_T L^2_x}
\]

\[
\lesssim a(M_1) M_2^{-4s-2} \left( M_2^{s+2} M_3^{-4s-2} \right)^\frac{1}{2}
\]

\[
\times \sup_{J \subset [0,1], |J'| = M_2^{s+2}} \| \chi_J u_{M_1} u_{M_3} \|_{L^\infty_T L^2_x} \| Q \chi_X f_N \|_{L^\infty_T L^2_x}
\]

\[
\lesssim a(M_1) M_2^{-4s-2} \left( M_2^{s+2} M_3^{-4s-2} \right)^\frac{1}{2}
\]

\[
\times \sup_{J \subset [0,1], |J'| = M_2^{s+2}} \| \chi_J u_{M_1} u_{M_3} \|_{L^\infty_T L^2_x} \| Q \chi_X f_N \|_{L^\infty_T L^2_x}
\]

Hence, by using the bilinear estimates and Bernstein inequality, we have

\[
|A_1| \lesssim a(M_1) M_2^{-4s-2} \left( M_2^{s+2} M_3^{-4s-2} \right)^\frac{1}{2} \| u_{M_3} \|_{X'} \| u_{M_3} \|_{X'} \| u \|_{X'}. \]

This summation with respect to $M_1, M_2, M_3$ is also handled if $s \geq -\frac{3}{4}$.

For the high modulation part of $f_N$ at modulation $\sigma \gg M_4^2$, we observe that one of three factors $\chi_I u_{M_1}, \chi_I u_{M_3}, \chi_I u_{M_3}$ must have modulation at least $\sigma$. We may assume $Q_\sigma (\chi_I u_{M_1})$. This is the worst case. We bound $Q_\sigma (\chi_I u_{M_1})$ in $L^2$ and the other two $\chi_I u_{M_2}, \chi_I u_{M_3}$ in $L^\infty$. Observe that

\[
|A_1| \lesssim a(M_1) M_2^{-4s-2} \sup_{I \subset [0,1], |I'| = M_2^{s+2}} \left| \sum_{\sigma \gg M_4^2} \int_R \int_R \chi_I u_{M_1} \chi_I u_{M_3} \chi_I u_{M_3} Q_\sigma f_N \, dx \, dt \right|
\]

\[
\lesssim a(M_1) M_2^{-4s-2}
\]

\[
\times \sup_{I \subset [0,1], |I'| = M_2^{s+2}} \sum_{\sigma \gg M_4^2} \| Q_\sigma (\chi_I u_{M_1}) \chi_I u_{M_3} \chi_I u_{M_3} \|_{L^\infty_T L^2_x} \| Q_\sigma f_N \|_{L^\infty_T L^2_x}
\]

\[
\lesssim a(M_1) M_2^{-4s-2}
\]
\[
\times \sup_{I \subset [0,1], |J|=M_2^{+2s} \gg M_3^4} \left( \sum_{J \subset I, |J|=M_3^{4s+2}} \left\| Q_\sigma(\chi_J u_{M_3}) u_{M_3} u_{M_3} \right\|_{L_2^2 L_2^2(J \times \mathbb{R})} \right)^{\frac{1}{2}} \\
\times \left\| Q_\sigma f_N \right\|_{L_2^2 L_2^2} \\
\lesssim a \left( M_1 \right) \frac{M_2 M_3}{M_2^3 M_3^3} \left( M_2^{-2s-1} M_3^{-2s-1} \right) \left( M_1 M_2^{-s} M_2^{-s} M_3^{-s} \right) M_3 M_2^{-3s-3s} \left\| u_{M_3} \right\|_{X} \left\| u_{M_3} \right\|_{X} \left\| u \right\|_{X}^4,
\]

Hence, by using Bernstein inequality, we have

\[
|A_1| \lesssim a \left( M_1 \right) \frac{M_2 M_3}{M_2^3 M_3^3} \left( M_2^{-2s-1} M_3^{-2s-1} \right) \left( M_1 M_2^{-s} M_2^{-s} M_3^{-s} \right) M_3 M_2^{-3s-3s} \left\| u_{M_3} \right\|_{X} \left\| u_{M_3} \right\|_{X} \left\| u \right\|_{X}^4.
\]

Hence, the summation with respect to \( M_1, M_2, M_3 \) is also handled if \( s \geq -\frac{3}{4} \).

Subcase 1.c. \( N = M_1 \ll M_2 \). First, we consider the low modulation output. For the \( L_2^2 L_2^2 \) term in \( f_N \), we estimate \( u_{M_2} u_{M_3} u_{M_3} \) in \( L_2^2 \). By considering the interval summation loss with square summability and using Lemma 4.2, we have

\[
|A_1| \lesssim a \left( M_1 \right) \frac{M_2 M_3}{M_2^3 M_3^3} \left( M_2^{-4s-2} \right) \left( M_1 M_2^{-4s-2} M_3^{-4s-2} \right) M_3^{M_2^{-4s-2}} \left\| u_{M_3} \right\|_{X} \left\| u_{M_3} \right\|_{X} \left\| u \right\|_{X}^4.
\]

Observe that even if \( M_2 \approx M_3 \) as two of the \( M_3 \) sized frequencies should be \( M_3 \) separated in order for output frequency to be localized at \( N \). Therefore, by using the bilinear estimates and Lemma 4.2, we have

\[
|A_1| \lesssim a \left( M_1 \right) \frac{M_2 M_3}{M_2^3 M_3^3} \left( M_2^{-2s-1} M_3^{-2s-1} \right) \left( M_1 M_2^{-s} M_2^{-s} M_3^{-s} \right) M_3 M_2^{-3s-3s} \left\| u_{M_3} \right\|_{X} \left\| u_{M_3} \right\|_{X} \left\| u \right\|_{X}^4.
\]

Therefore, the summation with respect to \( M_1, M_2, M_3 \) is handled if \( s \geq -\frac{3}{4} \).

For the \( L_2^2 L_2^2 \) term in \( f_N \), we can proceed as in Subcase 1.b. In this case, the summation with respect to \( M_1, M_2, M_3 \) is handled if \( s \geq -\frac{3}{4} \).

For the high modulation part of \( f_N \) at modulation \( \sigma \gg M_3^4 \), we observe that one of three factors \( u_{M_2}, u_{M_3}, u_{M_3} \) must have modulation at least \( \sigma \). We may assume \( Q_\sigma u_{M_2} \). This is the worst case. We bound \( Q_\sigma u_{M_2} \) in \( L^2 \) and the other two \( u_{M_3}, u_{M_3} \) in \( L^\infty \). By proceeding as in Subcase 1.b, we obtain

\[
|A_1| \lesssim a \left( M_1 \right) \frac{M_2^{-5s-4} M_2^{-1s} M_3^{-4s-3} \left\| u_{M_3} \right\|_{X} \left\| u_{M_3} \right\|_{X} \left\| u \right\|_{X}^4.
\]

Hence, the summation with respect to \( M_1, M_2, M_3 \) is also handled if \( s \geq -\frac{3}{4} \).
Case 2. $N \geq \min \{N_1, N_5, N_6\}$. Without loss of generality we may assume
\[ N_1 \leq N_2 \leq N_3, \quad N_4 \leq N_5 \leq N_6. \]
Then we must have
\[ N_4 \lesssim N \lesssim N_3. \]
We denote
\[ \{N, N_1, N_2, N_3\} = \{M_1, M_2, M_3, M_4\}, \quad M_1 \leq M_2 \leq M_3. \]
We may expand the Fourier multiplier $\varphi_N$ for $P_N$ into a Fourier integral. For a Schwartz function $\rho_N$, we have
\[ \varphi_N(\xi) = \int \rho_N(y) e^{i\xi y} dy = \int \rho_N(y) e^{i\xi_1 y} e^{-i\xi_2 y} e^{i\xi_3 y} dy, \quad \xi = \xi_1 - \xi_2 + \xi_3. \quad (5.10) \]
Here we can separate the exponential into three factors since in the domain of integration we have $\xi = \xi_1 - \xi_2 + \xi_3$. The complex exponentials are bounded symbols and thus bounded on $U^S$. Therefore it can be harmlessly absorbed into $u_{N_4}, u_{N_5}, u_{N_6}$. Moreover we have $||\rho_N||_{L^1} \lesssim 1$ uniformly in $N$. Plugging in the expression (5.8) and absorbing the factors originating from (5.10) into the $\tilde{u_i}$, we are left with estimating
\[
\sum_{N,N_1,N_2,N_3,N_4,N_5} \frac{a(M_1)}{M_2 M_3^2} \int_0^1 \int_R u_{N_1} u_{N_2} u_{N_3} u_{N_4} u_{N_5} u_{N_6} dx dt \\
= \sum_{N,N_1,N_2,N_3,N_4,N_5} A_2
\]
Subcase 2.a. $N = M_3$. In this case we have $N_4 \lesssim N = M_3, N = M_3 \lesssim N_6$.
Subcase 2.a.i. $N_5 \approx N_6 \gg M_3$. We use the bilinear estimates for the products $u_{M_5} u_{N_5}$ and $u_{N_4} u_{N_6}$ and the $L^\infty$ bound for $u_{M_1}, u_{M_2}$. Then by considering the interval summation loss $N_6^{-4s-2}$, we have
\[
|A_2| \lesssim \frac{a(M_1)}{M_2 M_3} N_6^{-4s-2} (M_1^{-s} M_2^{-s} M_3^{-s} N_4^{-s} N_6^{-2s}) \left( M_1^{-\frac{3}{2}} M_2^{-\frac{3}{2}} \right) \left( N_6^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right) \\
\times \prod_{j=1}^{3} ||u_{M_j}||_{X^s} ||u_{N_4}||_{X^s} ||u_{N_5}||_{X^s} \\
\lesssim a(M_1) M_1^{-\frac{3}{2}} M_2^{-\frac{3}{2}} M_3^{-3s} N_4^{-s} N_6^{-6s-5} \prod_{j=1}^{3} ||u_{M_j}||_{X^s} ||u_{N_4}||_{X^s} ||u_{N_5}||_{X^s}.
\]
Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{5}{6}$.
Subcase 2.a.ii. $N_5 \leq N_6 \approx M_3$ and $M_2 \ll M_3$. Then we use the bilinear estimates for $u_{M_5} u_{M_3}$ and $L^\infty$ bound for $u_{M_1}$ and the $L^6$ Strichartz estimate for the $u_{N_4}, u_{N_5}, u_{N_6}$. By considering the interval summation loss, we have
\[
|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^2} M_3^{-4s-2} (M_1^{-s} M_2^{-s} M_3^{-s} N_4^{-s} N_5^{-s} M_3^{-s}) \left( M_1^{-\frac{3}{2}} M_2^{-\frac{3}{2}} \right) \left( N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} M_3^{-\frac{3}{2}} \right) \\
\times \prod_{j=1}^{3} ||u_{M_j}||_{X^s} ||u_{N_4}||_{X^s} ||u_{N_5}||_{X^s} ||u_{M_3}||_{X^s} \\
\lesssim a(M_1) M_1^{-\frac{3}{2}} M_2^{-\frac{3}{2}} M_3^{-6s-\frac{49}{2}} N_4^{-s} N_5^{-s} M_3^{-\frac{3}{2}} ||u_{M_3}||_{X^s} ||u_{M_3}||_{X^s} ||u||_{X^s}.
\]
Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{15}{16}$. 
Subcase 2.a.iii $N_5 \leq N_6 \approx M_3$ and $M_2 \approx M_3$. In this case, we use the $L^6$ Strichartz estimate for all the factors. Then we have

$$|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^3} M_3^{-4s-2} (M_1^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} M_3^{-s})$$

$$\times \left( M_1^{-\frac{1}{2}} M_3^{-\frac{1}{2}} N_4^{-\frac{1}{4}} N_5^{-\frac{1}{4}} M_3^{-\frac{3}{2}} \right)^6 \| u_{M_3} \|_{X^*}.$$

$$\lesssim a(M_1) M_1^{-s-\frac{1}{2}} N_4^{-s-\frac{1}{2}} N_5^{-s-\frac{1}{2}} M_3^{-7s-7} \| u_{M_3} \|_{X^*} \| u_{M_3} \|_{X^*} \| u \|_{X^*}^6.$$

Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{23}{27}$.

Subcase 2.b. $N = M_2 < M_3$, in this case we have $N_4 \lesssim N = M_2$.

Subcase 2.b.i. $N_6 \approx N_5 \gg M_3$. In this case, we use the bilinear estimates for the products $u_{M_1} u_{N_5}$ and $u_{N_5} u_{N_6}$ and the $L^\infty$ bound for $u_{M_1}, u_{M_3}$. Then we have

$$|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^3} N_6^{-4s-2} (M_1^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} M_3^{-s}) \left( M_1^{\frac{1}{2}} M_3^{\frac{1}{2}} \right) \left( N_6^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \| u \|_{X^*} \| u_{N_6} \|_{X^*}$$

$$\lesssim a(M_1) M_1^{-s-\frac{1}{2}} M_2^{-1} M_3^{-2s-\frac{1}{2}} N_4^{-s-\frac{1}{2}} N_5^{-s-\frac{1}{2}} N_6^{-6s-8} \| u \|_{X^*} \| u_{N_6} \|_{X^*}.$$

Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{5}{6}$.

Subcase 2.b.ii. $N_5 \leq N_6 \approx M_3$. Then we use the bilinear estimates for $u_{M_1} u_{M_3}$ and $L^\infty$ bound for $u_{M_3}$ and the $L^6$ Strichartz estimate for the $u_{N_4}, u_{N_5}, u_{N_6}$. Then we have

$$|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^3} M_3^{-4s-2} (M_1^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} M_3^{-s})$$

$$\times M_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{4}} M_3^{-\frac{3}{2}} \| u \|_{X^*} \| u_{N_6} \|_{X^*}$$

$$\lesssim a(M_1) M_1^{-s-\frac{1}{2}} N_2^{-1} M_3^{-7s-4} N_4^{-s-\frac{1}{2}} N_5^{-s-\frac{1}{2}} \| u_{M_3} \|_{X^*} \| u_{M_3} \|_{X^*} \| u \|_{X^*}.$$n

Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{5}{6}$.

Subcase 2.b.iii. $N_5 \leq N_6 \ll M_3$. In this case, we use the bilinear estimates for the products $u_{M_1} u_{N_5}$ and $u_{M_3} u_{N_6}$ and the $L^\infty$ bound for $u_{M_1}, u_{M_3}$. Then we have

$$|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^3} M_3^{-4s-2} (M_1^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} N_6^{-s})$$

$$\left( M_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \left( M_3^{\frac{1}{2}} M_3^{\frac{3}{2}} \right) \| u \|_{X^*} \| u_{M_3} \|_{X^*}^2.$$

$$\lesssim a(M_1) M_1^{-s-\frac{1}{2}} N_2^{-1} N_4^{-s-\frac{1}{2}} N_5^{-s-\frac{1}{2}} M_3^{-6s-8} \| u_{M_3} \|_{X^*} \| u_{M_3} \|_{X^*} \| u \|_{X^*}.$$n

Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{17}{18}$.

Subcase 2.c. $N = M_1 < M_2$. In this case we have $N_4 \lesssim N = M_1$.

Subcase 2.c.i. $N_6 \approx N_5 \gg M_3$. We use the bilinear estimates for the products $u_{M_3} u_{N_5}$ and $u_{M_2} u_{N_6}$ and the $L^\infty$ bound for $u_{M_2}, u_{N_6}$. Then we have

$$|A_2| \lesssim \frac{a(M_1)}{M_2 M_3^3} N_6^{-4s-2} (M_2^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} N_6^{-s})$$

$$\left( M_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \left( M_3^{\frac{1}{2}} M_3^{\frac{3}{2}} \right) \| u \|_{X^*} \| u_{N_6} \|_{X^*}^2.$$

$$\lesssim a(M_1) M_2^{-\frac{1}{2}} N_4^{-s-\frac{1}{2}} N_3^{-2s-3} N_6^{-6s-5} \| u \|_{X^*} \| u_{N_6} \|_{X^*}.$$n

Hence the summation with respect to $N, N_1, \ldots, N_6$ is handled if $s \geq -\frac{5}{6}$.

Subcase 2.b.ii. $N_5 \leq N_6 \approx M_3$. Then we use the bilinear estimates for $u_{N_4} u_{M_3}$ and $L^\infty$ bound for $u_{M_2}$ and the $L^6$ Strichartz estimate for the $u_{M_3}, u_{N_5}, u_{N_6}$. Then
we have

\[ |A_2| \lesssim \frac{a(M_1)}{M_2 M_3^2} M_3^{-4s-2} (M_2^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} M_3^{-s}) \]

\[ \times M_3^{-\frac{3}{2}} M_2^{\frac{1}{2}} \left( M_3^{-\frac{1}{2}} N_5^{-\frac{1}{2}} M_3^{-\frac{1}{4}} \right) \|u\|_{X^s}^4 \|u_{N_0}\|_{X^s}^4, \]

\[ \lesssim a(M_1) M_2^{-\frac{1}{2} - s} M_3^{-7s - \frac{45}{4}} N_4^{-s} N_5^{-s - \frac{1}{2}} \|u_{M_1}\|_{X^s} \|u_{N_0}\|_{X^s} \|u\|_{X^s}^4. \]

Hence the summation with respect to \(N, N_1, \ldots, N_6\) is handled if \(s \geq -\frac{8}{9}\).

**Subcase 2.b.iii** \(N_5 \leq N_6 \ll M_3\). In this case, we use the bilinear estimates for the products \(u_{M_1} u_{N_6}\) and \(u_{M_2} u_{N_0}\) and the \(L^\infty\) bound for \(u_{M_3} u_{N_4}\). Then we have

\[ |A_2| \lesssim \frac{a(M_1)}{M_2 M_3^2} M_3^{-4s-2} (M_2^{-s} M_3^{-2s} N_4^{-s} N_5^{-s} N_6^{-s}) \left( M_2^{\frac{1}{2}} M_3^{-\frac{3}{2}} \right) (M_3^{-\frac{1}{2}} M_3^{-\frac{3}{2}}) \|u\|_{X^s}^4 \|u_{M_3}\|_{X^s}^4 \|u_{M_2}\|_{X^s}^2 \|u\|_{X^s}^4. \]

Hence the summation with respect to \(N, N_1, \ldots, N_6\) is handled if \(s \geq -\frac{17}{18}\). \(\square\)

To finish the proof of energy estimate, we need to choose suitable symbol \(a(\xi)\) in the previous sections. As in [17], we need the following sequence:

\[ \beta_N = \frac{N^{2s} \|u_{0,N}\|_{L^2}^2}{\|u_0\|_{H^s}^2}, \]

\[ \beta_N = \sum_M 2^{-\frac{1}{2} \log N - \log M} \beta_M^0. \]

These \(\beta_N\) satisfy the following property:

(i) \(N^{2s} \|u_{0,N}\|_{L^2}^2 \lesssim \beta_N \|u_0\|_{H^s}^2\),

(ii) \(\sum \beta_N \lesssim 1\),

(iii) \(\beta_N\) is slowly varying in the sense that

\[ |\log_2 \beta_N - \log_2 \beta_M| \lesssim \frac{\xi}{2} |\log_2 N - \log_2 M|. \]

We want to show that

\[ \sup_t N_0^s \|u_{N_0}(t)\|_{L^2} \lesssim \beta_N^{\frac{1}{2}} \left( \|u_0\|_{H^s} + \|u\|_{X^s}^2 \right). \quad (5.11) \]

Then by using the property (ii) we can conclude Proposition 5.2

To prove (5.11) for some frequency \(N_0\) we choose

\[ a_N = N^{2s} \max \left\{ 1, \beta_N^{-1 \frac{1}{2} - \epsilon} \log_2 N_0 \right\} \]

Correspondingly we take a function \(a(\xi) \in S_c^s\) so that

\[ a(\xi) \sim a_N, \quad |\xi| \approx N. \]

Then from the slowly varying condition, we obtain

\[ \sum_N a_N \|u_{0,N}\|_{L^2}^2 \lesssim \sum_N N^{2s} \|u_{0,N}\|_{L^2}^2 + 2^{-\epsilon} \log_2 N_0 N^{2s} \beta_N^{-1} \|u_{0,N}\|_{L^2}^2 \lesssim \|u_0\|_{H^s}^2. \]

From \(\|u\|_{L^\infty_t L^\infty_x H^s} \ll 1\), we have \(\sup_t E_0(u(t)) \ll 1\). Recall that

\[ \frac{d}{dt} (E_0 + E_1)(u) = \Lambda_6(u(t)). \]
From Proposition 5.4 the contribution of $E_1$ to the energy is controlled by $E_0$. Also, we use the energy estimate in Proposition 5.5 for this choice of $a$. Therefore, we obtain
\[
\left( \sum_N a(N) \|u_N(t)\|_{L^2_x}^2 \right)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^3.
\]
At fixed frequency $N = N_0$, we obtain (5.11).

6. Proof of Theorem 1.3. In this section we prove our main Theorem 1.3. The remaining part is just to do standard bootstrapping argument with trilinear estimate (4.1) and energy estimate (5.1). Our method is similar to the argument in Koch-Tataru [17]. Before we prove Theorem 1.3, we collect ingredients we need:

**Linear:**
\[
\|u\|_{X^s} \lesssim \|u\|_{\ell^2_t L^\infty_x H^s} + \|u^2 u\|_{Y^s} \quad (2.5),
\]

**Nonlinear:**
\[
\|u^2 u\|_{Y^s} \lesssim \|u\|_{X^s}^3 \quad (4.1),
\]

**Energy:**
\[
\|u\|_{\ell^2_t L^\infty_x H^s} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s}^3 \quad (5.2).
\]

As we mentioned in Remark 1.5, by rescaling the problem we consider small initial data. Let $\epsilon > 0$ be a small constant and suppose $\|u_0\|_{H^s(R)} < \epsilon$. Take a small $\delta$ so that $\epsilon \ll \delta \ll 1$. We denote by $A$ the set
\[
A = \left\{ T \in [0, 1]; \|u\|_{\ell^2_t L^\infty_x H^s([0,T] \times R)} \leq 2\delta, \|u\|_{X^s([0,T] \times R)} \leq 2\delta \right\}.
\]
We want to show that $A = [0, 1]$. Clearly $A$ is not empty and $0 \in A$. We need to prove that it is closed and open. From the definition, the norms used in $A$ are continuous with respect to $T$ and hence $A$ is closed.

Let $T \in A$. By using Proposition 5.2, we have
\[
\|u\|_{\ell^2_t L^\infty_x H^s([0,T] \times R)} \lesssim \epsilon + \delta^3,
\]
and by Proposition 2.5 and 4.1, we have
\[
\|u\|_{X^s([0,T] \times R)} \lesssim \epsilon + \delta^3.
\]
Hence by taking $\epsilon$ and $\delta$ sufficiently small, we conclude that
\[
\|u\|_{\ell^2_t L^\infty_x H^s([0,T] \times R)} \leq \delta, \quad \|u\|_{X^s([0,T] \times R)} \leq \delta.
\]
Since the norms are continuous with respect to $T$, it follows that a neighborhood of $T$ is in $A$. Therefore $A = [0, 1]$ and hence we prove Theorem 1.3.

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REFERENCES

[1] M. Ben-Artzi, H. Koch and J. C. Saut, Dispersion estimates for fourth order Schrödinger equations, C. R. Acad. Sci. Paris Sér. I Math., 330 (2000), 87–92.
[2] M. Christ, J. Colliander and T. Tao, A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order, J. Funct. Anal., 254 (2008), 368–395.
[3] M. Christ, J. Colliander and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Am. J. Math., 125 (2003), 1235–1293.
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[4] M. Christ, J. Holmer and D. Tataru, Low regularity a priori bounds for the modified Korteweg-de Vries equation, \textit{Lib. Math.}, \textbf{32} (2012), 51–75.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness for Schrödinger equations with derivative, \textit{SIAM J. Math. Anal.}, \textbf{33} (2001), 649–669.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation, \texttt{arXiv:math/0203218}.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness for KdV and modified KdV on R and T, \texttt{arXiv:math/0110045}.

[8] B. Guo and B. Wang, The global Cauchy problem and scattering of solutions for nonlinear Schrödinger equations in $H^s$, \textit{Differ. Integral Equ.}, \textbf{15} (2002), 1073–1083.

[9] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, \textit{Ann. Inst. H. Poincaré Anal. Non Linéaire}, \textbf{26} (2009), 917–941.

[10] S. Herr, D. Tataru and N. Tzvetkov, Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(T^3)$, \textit{Duke Math. J.}, \textbf{159} (2011), 329–349.

[11] S. Herr, D. Tataru and N. Tzvetkov, Global well-posedness of the KP-I initial-value problem in the energy space, \textit{Invent. Math.}, \textbf{173} (2008), 265–304.

[12] V. I. Karpman, Lyapunov approach to the soliton stability in highly dispersive systems. I. Fourth order nonlinear Schrödinger equations, \textit{Phys. Lett. A}, \textbf{215} (1996), 254–256.

[13] V. I. Karpman and A. G. Shagalov, Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion, \textit{Phys. D}, \textbf{144} (2000), 194–210.

[14] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, \textit{Indiana Univ. Math. J.}, \textbf{40} (1991), 33–69.

[15] C. E. Kenig, G. Ponce and L. Vega, On the ill-posedness of some canonical dispersive equations, \textit{Duke Math. J.}, \textbf{106} (2001), 617–633.

[16] H. Koch and N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(R)$, \textit{Int. Math. Res. Not.}, (2003), 1449–1464.

[17] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces, \textit{Int. Math. Res. Not.}, (2007), 1073–7928.

[18] H. Koch and D. Tataru, Energy and local energy bounds for the 1-d cubic NLS equation in $H^{-\frac{1}{2}}$, \textit{Ann. Inst. H. Poincaré Anal. Non Linéaire}, \textbf{29} (2012), 955–988.

[19] H. Koch and D. Tataru, Conserved energies for the cubic nonlinear Schrödinger equation in one dimension, \textit{Duke Math. J.}, \textbf{167} (2018), 3207–3313.

[20] H. Koch, D. Tataru and M. Vişan, \textit{Dispersive Equations and Nonlinear Waves}, Birkhäuser Seminars, Birkhäuser/Springer, Basel, 2014.

[21] C. Kwak, Periodic fourth-order cubic NLS: Local well-posedness and non-squeezing property, \textit{J. Math. Anal. Appl.}, \textbf{461} (2018), 1327–1364.

[22] B. Liu, A priori bounds for KdV equation below $H^{-\frac{1}{2}}$, \textit{J. Funct. Anal.}, \textbf{268} (2015), 501–554.

[23] T. Oh, P. Sosoe and N. Tzvetkov, An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation, \textit{J. Éc. polytech. Math.}, \textbf{5} (2018), 793–841.

[24] T. Oh and N. Tzvetkov, Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation, \textit{Comm. Math. Phys.}, \textbf{368} (2019), 1121–1168.

[25] T. Oh and Y. Wang, Global well-posedness of the periodic cubic fourth order NLS in negative Sobolev spaces, \texttt{arXiv:1707.02013}.

[26] B. Pausader, The cubic fourth-order Schrödinger equation, \texttt{arXiv:0807.4916}.

[27] Roberto A. Capistrano-Filho, Márcio Cavalcante, Fernando A. Gallego, Lower regularity solutions of the biharmonic Schrödinger equation in a quarter plane, \texttt{arXiv:1812.11079}.

[28] J. I. Segata, Modified wave operators for the fourth-order non-linear Schrödinger-type equation with cubic non-linearit, \textit{Math. Methods Appl. Sci.}, \textbf{29} (2006), 1785–1800.

[29] K. Seong, Well-Posedness and Ill-Posedness for the Fourth order cubic nonlinear Schrödinger equation in negative Sobolev spaces, \texttt{arXiv:1911.02253}.

[30] H. Takaoka and Y. Tsutsumi, Well-posedness of the Cauchy problem for the modified KdV equation with periodic boundary condition, \textit{Int. Math. Res. Not.}, \textbf{56} (2004), 3099–3040.

[31] Y. Tsutsumi, $L^2$-solutions for nonlinear Schrödinger equations and nonlinear groups, \textit{Funkcial. Ekvac.}, \textbf{30} (1987), 115–125.

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