A four component cubic peakon (4CH) equation

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Abstract
Two different four component Camassa–Holm (4CH) systems with cubic nonlinearity are proposed. The Lax pair and Hamiltonian structure are defined for both CH systems. The first 4CH system includes as a special case the 3CH system considered by Xia, Zhou and Qiao, while the second contains the two-component generalization of Novikov system considered by Geng and Xiu.

Keywords: peakons, Lax representations, Hamiltonian structure

1. Introduction

Solitons and integrable models are very attractive objects in nonlinear sciences, originally found in experiments on shallow water propagations a century and a half ago. In recent years, the Camassa–Holm (CH) [1] equation

\[ m_t + 2mu_x + m_u = 0, \quad m = u - u_{xx}, \]  

(1)

was derived with the aid of an asymptotic approximation to the Hamiltonian for the Green–Naghdi equations [2] and has attracted much attention. The most interesting feature of the CH equation is the admittance of a peaked soliton, the so called peakon solution. A peakon is a weak solution in some Sobolev space with a corner at its crest. Subsequently there are a large number of papers where various properties of these equation are established. For example, it is shown that CH equation is integrable by the inverse scattering transformation, it possesses the Lax pair formalism, multisoliton solitons and multipeakon solutions [3–13].

In 1998, Degasperis and Procesi [14] proposed a different equation

\[ m_t + 3u_x m + m_u = 0, \quad m = u - u_{xx}, \]  

(2)

which admits the peakon solution as well.
The solutions of the equation (2) are not mere abstractizations: the peakons replicate a feature that is characterizing for the waves of great height-waves of of largest amplitude that are exact solutions of the governing equations for irrotational water waves [15, 16]. The Degasperi–Procesi equation (2) is also considered as the horizontal component of the fluid velocity at the time in the spatial x-direction [17] with momentum density $$y$$, but evaluated at the different level line of the fluid domains.

Both the Comassa–Holm and Degasperi–Procesi equation are third order equations with quadratic nonlinearity. In 1996 Olver and Rosenau [4] proposed a general approach to the construction of Camassa–Holm like equations with the peakon solution. In particular, the following equation appears

$$m_t = -[m(u^2 - u_x^2)]_x, \quad m = u - u_{xx}$$

and has been considered by Fokas [18, 19] and Qiao [20] referred as Fokas–Olver–Rosenau–Qiao’s (FORQ) equation.

In 2008, Novikov [21], in his symmetry classification study, proposed several different generalizations of the CH equation to the peakon equations with cubic nonlinearity. One of them

$$m_t = -u^2m_x - 3uu_xm,$$ (4)

referred to as the Novikov equation, has been thoroughly considered by scientists from a different point of view.

The Fokas–Olver–Rosenau–Qiao and Novikov equations, referred to sometimes as the cubic CH equations, are integrable equations, which possess the Lax representation and bi-Hamiltonian structures [22–24]. For example, equation (4) arises as a zero curvature equation

$$F_t - G_x + [F, G] = 0$$ (5)

while equation (3) arises as a zero-curvature equation

$$U_t - V_x + [U, V] = 0$$

where

$$U = -\frac{1}{2} \left( \lambda m \begin{pmatrix} 0 & \lambda \frac{1}{2} \lambda^2 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{\lambda^2} - u_x \lambda - \frac{\lambda u^2}{2} & -\frac{\lambda u^2}{2} - \frac{\lambda^2 u^2}{2} \\ \frac{1}{\lambda^2} - u_x \lambda & -\frac{\lambda u^2}{2} \end{pmatrix}, \quad V = \begin{pmatrix} \lambda^{-2} \left(u + u_x\right) & -\lambda^{-1} \left(u - u_x\right) - \frac{1}{2} \lambda m \left(u^2 - u_x^2\right) \\ -\lambda^{-1} \left(u + u_x\right) & -\lambda^{-1} \left(u - u_x\right) \end{pmatrix} \right) \right).$$ (6)

The CH and cubic CH equations have several different generalizations to the two-component case. In 2012 Xia, Qiao and Zhou [25, 26] generalized the Olver–Rosenau–Qiao equation (3) to the two-component case

$$m_t = [m(uv - u_xv_x)]_x - m(uv_x - u_xv)$$
$$n_t = [n(uv - u_xv_x)]_x + n(uv_x - u_xv)$$
$$m = u - u_{xx}, \quad n = v - v_{xx}$$ (7)
and presented its Bi-hamiltonian structure and Lax representation. The spectral part of the Lax representation for equation (7) is

\[ U = -\frac{1}{2} \begin{pmatrix} 1 & -\lambda m \\ \lambda m & -1 \end{pmatrix} \]  

The Novikov equation (4) has been extended to the two-component case

\[ m_{1,i} = u_1 u_2 m_{1,i} - 3u_2 u_{1,x} m_1, \quad m_{2,i} = -u_1 u_2 m_{2,x} - 3u_1 u_{2,x} m_2, \]

where \( m_i = u_i - u_{i,x}, \quad i = 1, 2 \) by Geng and Xue [27]. This system has Bi-Hamiltonian formulations and possesses the Lax representation [28].

Further generalizations to the multicomponent cubic CH equation have been considered in the literature. For example, Qia [29] generalized the Lax representation (8) in which \( n, m \) are the \( N \) dimensional vector function. On the other side Xia, Zhou and Qia [30] generalized the spectral problems for the CH type equations are contained for this.

In this paper, we would like to investigate the equations of motion and Hamiltonian structure obtained from the matrix Lax representation (11). By matrix representation we mean that in the Lax representation (11) \( M_i \) are now \( N \) dimensional matrices. We consider the case...
of two-dimensional matrices for which as we will show, it is possible to obtain from the Lax representation two different four-component cubic peakon equations. We refer to these equations as 4CH equations.

The first 4CH system includes, as a special case, the 3CH system considered by Xia, Zhou and Qiao [30], while the second contains the two-component generalizations of the Novikov system considered by Geng and Xue [27]. For these equations, we have constructed the Hamiltonian operators.

The paper is organized as follows. In the first section, we present the general form of the matrix cubic peakon system obtained from the matrix Lax representation (11). In the next sections, we consider the special reduced version, where we deal with the two-dimensional matrices $m_i$. In appendix A, we consider the case $m_3 = m_2, m_4 = m_1$, while, in appendix B, the case $m_3 = m_4 = 0$. For both cases, the Hamiltonian structure is constructed. The paper contains two appendices, in which we prove that our Hamiltonian operators satisfy the Jacobi identities. The last section contains concluding remarks.

2. Matrix peakon equation

Let us consider the matrix version of the Lax representation equations (11), in which $m_i = u_i - u_{i+1}, i = 1, 2, 3, 4$ are $N$ dimensional matrices, $\Gamma$ is an arbitrary scalar function, not a matrix.

The integrability condition $\Psi_{tx} = \Psi_{tx}$ forces the following equation of motion

\[
\begin{align*}
\dot{m}_{1j} &= -(m_1 \Gamma)_x + m_2 \Gamma - m_1 (g_2 f_2 + f_1 g_1) - g_1 (g_2 m_4 + f_1 m_1), \\
\dot{m}_{2j} &= -(m_2 \Gamma)_x - m_3 \Gamma + m_2 (g_2 g_2 + m_3 g_1) + (g_2 f_2 + f_1 g_1) m_2, \\
\dot{m}_{3j} &= -(m_3 \Gamma)_x - m_2 \Gamma + (f_1 g_1 + g_2 f_2) m_3 + (m_2 f_2 + m_3 g_1) f_1, \\
\dot{m}_{4j} &= -(m_4 \Gamma)_x + m_1 \Gamma - f_2 (f_1 m_1 + g_2 m_4) - m_4 (f_1 g_1 + g_2 f_2) 
\end{align*}
\]

(12)

where $f_1, f_2, g_1, g_2$ are defined in the same manner as in the scalar case but now $u_i, i = 1, 2, 3, 4$ are matrices.

Our equations (12) depend on four arbitrary $N$ dimensional matrix functions $m_i, i = 1, 2, 3, 4$, and, hence constitute the system of $4N$ equations.

The Hamiltonian structure for our equations, if such exists, depends on these arbitrary functions as well. In the non-matrix case, we can assume a special reduction of the function $m_i, i = 1, 2, 3, 4$, fix the $\Gamma$ function, and then find the Hamiltonian structure. In the next section, we will show that for the two-dimensional matrices it is also possible to assume special reduction of matrices $m_i$, fix $\Gamma$ function, and finally find the Hamiltonian structure.

3. 4CH equations

In this section, we consider the case where $m_i$ are two-dimensional matrices only. In this situation, we have 16 arbitrary functions which parametrize the matrices $m_i$, and the equations (12) constitute 16 equations. It appears that, when we restrict the considerations to the four arbitrary functions, it is possible to fix the $\Gamma$ function and find the Hamiltonian formulations for the system of equations (12).
3.1. First 4CH equation $m_0 = m_2, m_4 = m_1$

For this case our equation (12) reduces to

$$m_{1,2} = - (m_1 \Gamma)_x + m_1 \Gamma$$

$$m_{2,3} = - (m_2 \Gamma)_x - m_2 \Gamma$$

and constitutes the system of eight equations in which $\Gamma$ is an arbitrary function.

Further, we assume that $m_2 = m_1^*, u_2 = u_1^*$ where $^*$ denotes the hermitean conjugation, and we parametrize our matrices $u_i$ and $m_i, i = 1, 2,$ as

$$u_1 = u_{1,0} + iu_{1,1} \sigma_1 + iu_{1,2} \sigma_2 + iu_{1,3} \sigma_3,$$

$$u_2 = u_{1,0} - iu_{1,1} \sigma_1 - iu_{1,2} \sigma_2 - iu_{1,3} \sigma_3,$$

$$m_k = u_k + u_{k,xx}, \quad m_{k,j} = u_{k,j} - u_{k,j,xx}, \quad k = 1, 2, j = 0, 1, 2, 3$$

where $\sigma_1$ are Pauli matrices.

From the assumption $m_{2,j} = m_1^*$ it follows that $\Gamma = 4 \sum_{k=0}^{3} (u_{1,k}^2 - u_{1,k,xx}^2)$.

Now, our equations (12) reduce to the system of four equations

$$m_{1,j} = -(\Gamma m_{1,j})_x + \sum_{k=0}^{3} (u_{1,k} u_{1,j} - u_{1,k} u_{1,j,xx}) m_{1,k}.$$  \(15\)

It is our first four component cubic 4CH system.

Apparently, this system reduces to the matrix peakon equation (10), when

$$u_{1,0} = 0, \quad u_{1,1} = v_1, \quad u_{1,2} = \frac{v_2 + v_3}{2}, \quad u_{1,3} = -i \frac{v_2 - v_3}{2}.$$  \(16\)

It can be further reduced to the CH equation (1), when $v_3 = 0, v_1 = 0$, or to the cubic CH equation, when $v_3 = v_4 = 0$.

In order to obtain the Hamiltonian formulation of the first cubic peakon 4CH system, let us introduce new a parametrization of $u_{1,j}, m_{1,j}$ as

$$u_{1,0} = \frac{1}{2} (v_1 + v_2), \quad u_{1,1} = -i \frac{1}{2} (v_1 - v_2),$$

$$u_{1,2} = \frac{1}{2} (v_3 + v_4), \quad u_{1,3} = -i \frac{1}{2} (v_3 - v_4),$$

$$n_j = v_j - v_{j,xx}, \quad \Gamma = 4 (v_1 v_2 + v_3 v_4 - v_{1,xx} v_{2,xx} - v_{3,xx} v_{4,xx}),$$  \(17\)

where $j = 1, 2, 3, 4$ and our equation (15) transforms to

$$n_{i,j} = -(n_i \Gamma)_x + 4 v_i (v_{1,i} n_2 + v_{2,i} n_1 + v_{3,i} n_4 + v_{4,i} n_3)$$

$$- 4 v_{i,xx} (v_{1,i} n_2 + v_{2,i} n_1 + v_{3,i} n_4 + v_{4,i} n_3).$$  \(18\)
These equations could be formulated as the Hamiltonian system,

\[ H = \frac{1}{2} \int dx \ (v_1 n_2 + v_3 n_4), \mathcal{L} = \mathcal{L}_1 - \mathcal{L}_2 \]

\[ \mathcal{L}_{1,j} = -\partial_n \partial^{-1} n_j \partial + \delta_j n_i \partial^{-1} n_i \]

\[ \mathcal{L}_2 = \begin{pmatrix}
0 & n_3 \partial^{-1} n_3 + n_1 \partial^{-1} n_2 & -n_3 \partial^{-1} n_1 & -n_3 \partial^{-1} n_1 \\
0 & n_2 \partial^{-1} n_2 & -n_2 \partial^{-1} n_2 & -n_2 \partial^{-1} n_2 \\
0 & n_1 \partial^{-1} n_1 & 0 & n_1 \partial^{-1} n_1 \\
0 & n_2 \partial^{-1} n_2 & n_1 \partial^{-1} n_1 & 0
\end{pmatrix} \]

\[ \begin{pmatrix}
\frac{1}{2} & 1 & 4 \mathcal{L}
\end{pmatrix}
\]

\[ H_n = \frac{\delta n}{\delta x}, \text{ and} \]

The proof that the Jacobi identity for the \( \mathcal{L} \) operator holds is delegated to appendix A.

3.2. Second 4CH equation \( m_3 = m_4 = 0 \)

Similarly to the first cubic peakon 4CH system we assume that \( m_2 = m_1^* \) and parametrize the matrices \( u_i \) and \( m_i, i = 1, 2 \) as

\[ u_1 = r_1 + i r_2 \sigma_1 + i r_3 \sigma_2 + i r_4 \sigma_3, \quad u_2 = r_1 - i r_2 \sigma_1 - i r_3 \sigma_2 - i r_4 \sigma_3 \]

\[ m_1 = p_1 + i p_2 \sigma_1 + i p_3 \sigma_2 + i p_4 \sigma_3, \quad m_2 = p_1 - i p_2 \sigma_1 - i p_3 \sigma_2 - i p_4 \sigma_3 \]

\[ m_i = u_i - u_{i,xx}, \quad i = 1, 2, \quad p_j = r_j - r_{j,xx}, \quad j = 1, 2, 3, 4. \]

The assumption \( m_{3,i} = m_{4,i} = 0 \) forces that \( \Gamma = r_1^2 + r_2^2 + r_3^2 + r_4^2 \).

Because of this, the equations (12) become

\[ m_{1,i} = m_1 (u_{2,x} u_1 - u_{2} u_{1,x}) - u_{1,x} u_{2} m_1 - (u_{1} u_{2} m_1)_x = 0 \]

\[ m_{2,i} = (u_{2} u_{1,x} - u_{2,x} u_1) m_2 - m_2 u_{2,x} - (m_2 u_{1} u_2)_x = 0 \]

and can be rewritten as

\[ p_{ij} = -\left( p_i \Gamma \right)_x - \frac{1}{2} p_i \Gamma_x - \sum_{j,k,x=1}^4 \epsilon_{ijk,x} p_k r_{j,x} r_{k,x} + 3 \sum_{k=1}^4 p_k (r_k r_{k,x} - r_{k,x} r_k). \]
where $\epsilon_{i,j,k,l}$ is the antisymmetric tensor such that $\epsilon_{1,2,3,4} = 1$.

This is our second four component cubic 4CH system.

The system of equation (23) allows for further reduction. For example, in the case when $p_3 = p_4 = r_3 = r_4 = 0$ and assuming that $p_1 = a_1 + a_2$, $p_2 = i(a_1 - a_2)$ the system (23) reduces to the two-component Novikov equation considered by Geng, Xiu, Li and Liu [27, 28].

The system of equations (23) is a hamiltonian system,

\[
\begin{pmatrix}
  p_1 \\
  p_2 \\
  p_3 \\
  p_4 \\
\end{pmatrix}
= \mathcal{L}
\begin{pmatrix}
  H_{p_1} \\
  H_{p_2} \\
  H_{p_3} \\
  H_{p_4} \\
\end{pmatrix}
\]  

(24)

where

\[
\mathcal{L}_{j,k} = (3p_j \partial + 2p_{j,\lambda})(\partial_{xxx} - 4\partial)^{-1}(3p_k \partial + p_{k,\lambda})
\]

\[
+ \sum_{s,r=1}^4 \epsilon_{j,k,s,r} p_s \partial^{-1} p_r - 3 \delta_{j,k} \sum_{s=1}^4 p_s \partial^{-1} p_s + 3 p_j \partial^{-1} p_j
\]

(25)

\[
H = \frac{1}{2} \int dx \, (p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4).
\]

The proof that the operator satisfies the Jacobi identity is described in appendix B.

4. Conclusion

In this paper we studied two different four component Camassa–Holm (4CH) systems with cubic nonlinearity. The Lax pairs and Hamiltonian structure have been proposed for these two different systems. The first 4CH system include as a special case the 3CH system considered by Xia, Zhou and Qiao while the second contains the two-component generalizations of Novikov system considered by Geng and Xiu. Our Lax pair is a matrix generalization of the Lax pair for the 4CH type hierarchy considered in [31].

Our matrix Lax representation produces a huge number of different cubic CH type equations and it will be interesting to investigate these further, especially to study the existence of infinitely many conservation laws.

Appendix A

We use the traditional manner to verify the Jacobi identity [33]. In order to prove that the operator $\mathcal{L}$, defined in (20), satisfies the Jacobi identity, we utilize the standard form of the Jacobi identity

\[
\text{Jacobi} = \int dx \mathcal{L}^*_B C + \text{cyclic}(A,B,C) = 0,
\]

(A.1)

where $A$, $B$ and $C$ are the test vector functions for example $A = (a_1, a_2, a_3, a_4)$ while $*$ denotes the Gato derivative along the vector $\mathcal{L}(B)$. We check this identity utilizing the computer algebra Reduce and package Susy2 [34]. We will briefly explain our procedures used during the verification of the Jacobi identity (A.1).
In the first stage, we remove the derivatives from the test functions $a_i, b_i$ and $c_i$ in the Jacobi identity, using the rule

$$f_x = \frac{\partial}{\partial x} f - f \frac{\partial}{\partial \partial x}$$  \hspace{1cm} (A.2)

where $f$ is an arbitrary function.

Because of this, the Jacobi identity can be split into three segments. The first and second segments contain terms in which the integral operator $\partial^{-1}$ appears twice and once respectively. The last segment does not contain any integral operators. We consider each segment separately.

The first segment is a combination of the following expressions:

$$\int d^n c \partial^{-1} a_1 \partial^{-1} b_1 + \int d^n \tilde{c} \partial^{-1} \tilde{a}_1 \partial^{-1} \tilde{b}_1 + \text{cyclic}(a, b, c)$$

where $n_a$ denotes $n_j a_i$ or $n_{j, x} a_i$, $i, j = 1, 2, 3, 4$ and similarly for $n_i, n_t, \tilde{n}_a, \tilde{n}_b, \tilde{n}_c$.

Here $\partial^{-1}$ is an integral operator, and, therefore, each ingredient could be rewritten as, for example,

$$\int d^n x n_i \partial^{-1} n_i \partial^{-1} n_b = - \int d^n x n_i (\partial^{-1} n_i) (\partial^{-1} n_b).$$  \hspace{1cm} (A.3)

Now, we replace $n_a$ in the last formula by

$$n_a = \frac{\partial}{\partial x} (\partial^{-1} n_a) - (\partial^{-1} n_a) \frac{\partial}{\partial x}.$$

Hence, the expression (A.3) transforms to

$$\int d^n x n_i \partial^{-1} n_i \partial^{-1} n_b = \int d^n x n_i (\partial^{-1} n_i) (\partial^{-1} n_b) + \int d^n x n_i (\partial^{-1} n_i) (\partial^{-1} n_c).$$

Now, repeating this procedure for $n_a$ and $\tilde{n}_a$ in the first segment, it appears that this segment reduces to zero.

The second segment is constructed from the combinations of the following terms:

$$\int d^n \Lambda_a \Lambda_c \partial^{-1} \Lambda_b + \int d^n \tilde{\Lambda}_b \partial^{-1} \tilde{\Lambda}_b \tilde{\Lambda}_c + \text{cyclic}(a, b, c).$$

Here $\Lambda_a$ takes values in $\{n_{a, i}, n_{i, x} a_i, n_{j, x} a_i, n_{j, a, i} a_i, n_{j, a, x} a_i\}, i, j = 1, 2, 3, 4$. In a similar manner the $\Lambda_b, \Lambda_c, \tilde{\Lambda}_a, \tilde{\Lambda}_b, \tilde{\Lambda}_c$ are defined. These terms are rewritten as

$$\int d^n \Lambda_a \Lambda_c (\partial^{-1} \Lambda_b) - \int d^n \tilde{\Lambda}_a \tilde{\Lambda}_c (\partial^{-1} \tilde{\Lambda}_b) + \text{cyclic}(a, b, c).$$

Next, using rule (A.2), we replace once more the derivatives in $a_{i, x}$ and $b_{i, x}$ in the second segment. After this replacement, it appears that the second segment contains no term with an integral operator. Therefore, we add this segment to the third segment.

Now, it is easy to check that this last segment vanishes. Indeed, it is enough to use the rule (A.2) in order to remove the derivatives from $a_{i, x}$ in the last segment.

This finishes the proof.

**Appendix B**

We use the idea of the decompression of the Hamiltonian operator [35] in order to prove that the operator $L$ defined in (25) satisfies the Jacobi identity. To this end, let us consider a new operator $\Lambda$.
$$\Lambda = \begin{pmatrix}
(\partial^3 - 4v\partial - 2v), & 3p_1\partial + p_{1,x}, & 3p_2\partial + p_{2,x}, & 3p_3\partial + p_{3,x}, & 3p_4\partial + p_{4,x} \\
p_1\partial + 2p_{1,x}, & 3p_2\partial + 2p_{2,x}, & 3p_3\partial + 2p_{3,x}, & 3p_4\partial + 2p_{4,x}, & \Theta
\end{pmatrix}$$

where $\Theta$ is the matrix with the entries

$$\Theta_{jk} = \sum_{s,n=1}^{4} \epsilon_{j,k,s,n}p_s\partial^{-1}p_n - 3\delta_{jk}\sum_{s=1}^{4} p_s\partial^{-1}p_s + 3p_j\partial^{-1}p_j.$$

As we see, the Hamiltonian operator $\mathcal{L}$, defined in (25), is a Dirac reduced version of the $\Lambda$ operator when $v = 1$. From this, it is enough to prove that the operator $\Lambda$ satisfies the Jacobi identity. We verified the Jacobi identity using the same procedure as in appendix A.

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