Perturbation of Perron roots and The max-plus spectral theorem

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Abstract

In this paper, we consider the Perron theorem over the real Puiseux field. We introduce a recursive method for calculating Perron roots and Perron vectors of positive Puiseux matrices (which satisfy some condition of genericness) by means of combinatorics based on the tropical linear algebra.

1 Introduction

A matrix, and in particular a vector, is said to be positive (resp. nonnegative) if all its entries are positive (resp. nonnegative). The first important discovery on the spectral property of positive matrices was made by O. Perron [20] in 1907. Perron’s results were extended to nonnegative matrices by G. Frobenius [9, 10, 11], who proved the following:

Theorem 1.1 (The Perron-Frobenius theorem). Let $A \in \mathbb{R}^{N \times N}$ be an irreducible and nonnegative matrix and $\rho(A)$ be its spectral radius. Then, there exist a natural number $d \geq 1$ (called the period of $A$) such that the matrix $A$ satisfies the following properties (i–iv): (i) $\rho(A) \exp(2\pi \sqrt{-1} \cdot \frac{a}{d})$ is a simple eigenvalue for each $0 \leq a < d$. (ii) A corresponding eigenvector to $\rho(A)$ can be chosen to be entry-wise positive. (iii) Any nonnegative eigenvector of $A$ corresponds to $\rho(A)$. (iv) No eigenvalue $\lambda$, except for $\rho(A) \exp(2\pi \sqrt{-1} \cdot \frac{a}{d})$, satisfies $|\lambda| = \rho(A)$.

The positive eigenvalue $\rho(A)$ is called the Perron root of $A$, and its corresponding eigenvector which is entry-wise positive is called a Perron vector of $A$.

In this paper, we will be concerned with an analogue of Perron’s result for positive matrices over the real Puiseux field $F := \bigcup_{p=1}^{\infty} \mathbb{R}((t^{1/p}))$. It is well-known that the analogue of the Perron-Frobenius theorem holds over any real closed field (especially, over $F$). (To my best knowledge, the first (non-constructive) proof was given by Eaves, Rothblum and Schneider in 1995 [7]). On the other hand, it is also known that it is a perplexing problem to give an explicit, constructive formulas to compute all the Puiseux series coefficients of $\rho(A)$. This kind of problem is one of the most important concerns of perturbation theory (see, for example, the standard texts [5, 16, 17] of analytic perturbation theory).

In the past 10 years, a new tropical-algebraic approach to estimate perturbed eigenvalues and eigenvectors has been developed by the experts in the field of tropical algebra. It was shown by Akian, Bapat and Gaubertin [1, 2] that generically, the valuations of the eigenvalues of a Puiseux matrix (= a matrix whose entries are Puiseux series) coincide with the eigenvalues of the associated tropical matrix. They

1 In [7], they found that the Perron-Frobenius theorem can be expressed by a sentence in the language of ordered fields (see Appendix A). From the Tarski-Steidenberg principle [21, 22], they concluded that the Perron-Frobenius theorem is provable in the language of ordered fields.
also gave a generalization of the classical Lidskiĭ-Višik-Ljusternik theorem [18, 23], which provides the leading terms of eigenvalues and eigenvectors of a linearly perturbed matrix \( Y = Y_0 + tY_1 \). (For other recent developments on this topic, see [3, 13].)

Inspired by the work [1], we ask whether there exists a recursive method to compute Perron roots and vectors of a generally perturbed matrix

\[
Y = Y_0 + tY_1 + t^2Y_2 + t^3Y_3 + \cdots
\]

by means of tropical mathematics. In this paper, we introduce a combinatorial and constructive proof of the Perron theorem over \( F \) under the condition of genericness (§6.4)

For the first example, consider the real Puiseux matrix

\[
Y = \begin{pmatrix}
1-t & 1 + t \\
1 + t^2 & 1 - t
\end{pmatrix}
\]

It is not hard to find the Perron root \( \lambda_0 = 2 \) and its corresponding Perron vector \( v_0 = T(1,1) \) of the leading term \( Y_{\text{top}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). As 2 is a simple eigenvalue of \( Y_{\text{top}} \), we soon obtain the pair

\[
\lambda = 2 - \frac{1}{2}t + \frac{3}{8}t^2 + \cdots, \quad v = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 5/8 \\ 0 \end{pmatrix} t^2 + \cdots,
\]

with \( Yv = \lambda v \) by applying the recursive formula (7). Since both \( \lambda \) and \( v \) are positive, we conclude that \( \lambda \) is the Perron root of \( Y \).

When the leading term of the matrix is not irreducible, for example \( Y = \begin{pmatrix} 1 - t & 1 + t \\ t^2 & 1 - t \end{pmatrix} \) or

\[
Y = \begin{pmatrix}
1 - t & 1 + t & t^2 \\
1 + t^2 & 1 - t & t \\
t & t^2 & 2t^2
\end{pmatrix}
\]

the situation is much complicated (see Example 2.17). The key of our strategy is to find a diagonal matrix \( \delta \) such that \( \delta^{-1}Y\delta \) has a simpler structure.

Our method to find such \( \delta \) is split into two steps.

By using the max-plus spectral theorem (Proposition 3.6), which is a tropical analog of the Perron-Frobenius theorem, one can find some diagonal matrix \( \delta_1 \) such that the matrix \( \delta_1^{-1}Y\delta_1 \) is a flat-slanted form (Section 4). Roughly speaking, under the transformation \( Y \mapsto Y' := \delta_1^{-1}Y\delta_1 \), the unevenness of the valuations is fixed.

For further calculations, we have to take into account the magnitudes of coefficients. By using a combinatorial methods on weighted graphs, one can find a diagonal matrix \( \delta_2 \) such that \( \delta_2^{-1}Y'\delta_2 \) is a gently-slanted form (Section 5), that is a positive Puiseux matrix which admits a recursive method for calculating the Perron root.

The main theorem of this article is:

**Theorem 1.2.** Let \( Y \) be a square matrix whose entries are positive Puiseux series. If \( Y \) satisfies the condition of genericness (§6.4), we have the recursive method (Algorithm 6.9) to calculate the Perron root and a Perron vector of \( Y \). Especially, we obtain a constructive proof of the Perron theorem for some class of Puiseux matrices.

The contents of this paper are as follows: In Section 2 we introduce a recursive method to calculate the eigenvalues and eigenvectors of a complex Puiseux matrix whose leading term has a semi-simple eigenvalue. Although there are numerous literatures and sophisticated results on this topic (see, for example [5, 14, 18, 23, 24]), we shall use an alternative method which is suitable for dealing with positivity. In Sections 3–5 we discuss about the combinatorial aspects of real Puiseux matrices. The main aim in these sections is to introduce two canonical forms of real Puiseux matrices, one of which is a flat-slanted form (Section 4) and the other is a gently-slanted form (Section 5). In Section 6 we give a construction of the Perron root/vector of a positive Puiseux matrix. We also give some examples and applications of our method in Section 7.

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2Any linearly perturbed matrix \( Y_0 + tY_1 \) satisfies this condition.
2 Reviews on spectral problem of complex Puiseux matrices

In this section, we briefly review recursive techniques of calculating eigenvalues and eigenvectors of complex Puiseux matrices whose leading term have semi-simple eigenvalues. For readers who are interested in general methods in this field, we recommend the textbooks \(5, 16, 17\), and the references therein.

2.1 Module of approximate eigenvectors

Let \( K := \bigcup_{p=1}^{\infty} \mathbb{C}((t^{1/p})) \) be the complex Puiseux series field and \( \mathcal{T} := \mathbb{Q} \cup \{+\infty\} \) be the tropical semifield. Define the valuation map \( \vartheta : K \to \mathcal{T} \) by \( \vartheta(c) = 0 \) \((c \in \mathbb{C}^\times)\), \( \vartheta(t) = 1 \) and \( \vartheta(0) = +\infty \). Let \( R := \{ x \in K \mid \vartheta(x) \geq 0 \} \) be the valuation ring of \( K \) and \( I := \{ x \in K \mid \vartheta(x) > 0 \} \) be the unique maximal ideal of \( R \). For \( \Lambda \in \mathbb{Q} \), set

\[
I[\Lambda] := \{ x \in K \mid \vartheta(x) \geq \Lambda \}, \quad I(\Lambda) := \{ x \in K \mid \vartheta(x) > \Lambda \}.
\]

Let \( Y = (a_{i,j}) \in K^{M \times N} \) be a Puiseux matrix. The valuation of \( Y \) is a rational number \( v(Y) := \min \{ \vartheta(a_{i,j}) \} \). As the number of entries are finite, there uniquely exists a rational number \( q(Y) := \min \{ r \mid a_{i,j} \in \mathbb{C}((t^{1/r})) \} \). Therefore, \( Y \) admits the expansion

\[
Y = Y_0 t^v + Y_{1/q} t^{v+1/q} + Y_{2/q} t^{v+2/q} + \cdots, \quad Y_v \neq 0, \ Y_r \in \mathbb{C}^{M \times N}.
\]

The matrix \( Y_{\top} \) is called the leading term of \( Y \).

For a square Puiseux matrix \( Y \in K^{N \times N} \), let

\[
W^{(\Lambda)}(Y) = \{ v \in (R/I(\Lambda))^N \mid Yv \equiv 0 \mod (v(Y) + \Lambda) \}
\]

be the set of \( \Lambda \)-th approximate null vectors of \( Y \). For \( \lambda \in K \), we call

\[
W^{(\Lambda)}(Y; \lambda) := W^{(\Lambda)}(Y - \lambda \cdot \text{id})
\]

the set of \( \Lambda \)-th approximate eigenvectors associated with \( \lambda \). If \( W^{(\Lambda)}(Y; \lambda) \neq \{0\} \), \( \lambda \) is called a \( \Lambda \)-th approximate eigenvalue of \( Y \). \( W^{(\Lambda)}(Y; \lambda) \) has the structure of left \( R/I(\Lambda) \)-module.

**Lemma 2.1.** There exist finitely many elements \( v^1, \ldots, v^g \in W^{(\Lambda)}(Y; \lambda) \) such that

\[
W^{(\Lambda)}(Y; \lambda) = \left\{ (R/I(\Lambda)) \cdot v^1 \oplus \cdots \oplus (R/I(\Lambda)) \cdot v^r \right\} \oplus \left\{ (I/I(\Lambda)) \cdot v^{r+1} \oplus \cdots \oplus (I/I(\Lambda)) \cdot v^g \right\}.
\]

These vectors satisfy \( 1 \leq i \leq r \Rightarrow v(v^i) = 0 \).

**Proof.** For sufficiently large \( d \), \( v \in \mathbb{N} \), all entries of \( (Y - \lambda \cdot \text{id}) \) are contained in \( t^{-v} \cdot \mathbb{C}[[t^{1/d}]] \). Because \( \mathbb{C}[[t^{1/d}]] \) is a PID, there exist invertible matrices \( P, Q \) such that

\[
Y - \lambda \cdot \text{id} = P \cdot \text{diag}(t^{m_1 \cdots}, \ldots, t^{m_m \cdots}, t^{m_{m+1} \cdots}, \ldots, t^{m_i \cdots}, t^v, \ldots, t^v) \cdot Q, \quad (+\infty \geq m_1 \geq \cdots \geq m_g).
\]

(We regard \( t^{+\infty} = 0 \)). It is enough to define \( v^i := Q^{-1} e_i \), where \( e_i \in \mathbb{R}^N \) is the \( i \)-th fundamental vector.

**Definition 2.2.** In the situation of \(1\), we denote \( W^{(\Lambda)}(Y; \lambda) = \langle v^1, \ldots, v^r, t^0 v^{r+1}, \ldots, t^0 v^g \rangle \), where \( t^0 \) means “an element of infinitesimally small valuation”.

Although the set \( \{ v^1, \ldots, v^g \} \) is not always a \( R/I(\Lambda) \)-basis of \( W^{(\Lambda)}(Y; \lambda) \), it should sound less absurd if we use the term “quasi-basis” to refer it.
Lemma 2.4. Proof. Comparing the leading terms on both sides, we obtain \( (v, \ldots , t^g v^g) \).

Next assume \( \{v, \ldots , t^g v^g\} \) be a quasi basis of \( W(\Lambda)(Y; \lambda) \). From the proof of Lemma 2.4, we can soon conclude that the numbers \( r \) and \( g \) do not depend on the choice of quasi basis.

Lemma 2.5. Ker(\( Y_\theta - \lambda \cdot \text{id} \)) = \( \mathbb{C} v^t_{1\text{top}} \oplus \cdots \oplus \mathbb{C} v^g_{1\text{top}} \), where \( v = v(Y) \).

Proof. Let \( \bar{v} := v(Y - \lambda \cdot \text{Id}) \). By definition, we have \( (Y - \lambda \cdot \text{id})v^i \equiv 0 \mod I(\bar{v} + \Lambda) \). Comparing the leading terms on both sides, we obtain \( (Y - \lambda \cdot \text{id})v^t_{1\text{top}} = 0 \), which implies \( v^t_{1\text{top}} \in \text{Ker}(Y_\theta - \lambda \cdot \text{id}) \).

2.1.1 Restriction to \( \mathbb{C}(t^1/q) \)

Let \( M = \langle v^1, \ldots , v^r, t^g v^g \rangle \) be a \( R/I(\Lambda) \)-module. Set \( L_q := \mathbb{C}[t^{1/q}] / (\mathbb{C}[t^{1/q}] \cap I(\Lambda)) \). From the inclusion \( L_q \hookrightarrow R/I(\Lambda) \), the module \( M \) can be regarded as a \( L_q \)-module by restriction. Especially, \( M \cap L_q^N \) has a structure of \( L_q \)-module. Although \( M \) is not finitely generated in general, \( M \cap L_q^N \) is always finitely generated as a \( L_q \)-module for sufficiently large \( q \). We have

\[
M \cap L_q^N = \left\{ L_q \cdot v^1 + \cdots + L_q \cdot v^r \right\} \oplus \left\{ L_q \cdot (t^{1/q} v^{r+1}) + \cdots + L_q \cdot (t^{1/q} v^g) \right\}.
\] (2)

2.1.2 Transposed matrix

Let \( T Y \) be the transposed matrix of \( Y \). Note the equation \( \text{rk} W(\Lambda)(Y; \lambda) = \text{rk} W(\Lambda)(T Y; \lambda) \), which follows from the proof of Lemma 2.5. Let

\[
W(\Lambda)(Y; \lambda) = \left\{ x^1, \ldots , x^r, t^g x^{r+1}, \ldots , t^g x^g \right\}, \quad W(\Lambda)(T Y; \lambda) = \left\{ y^1, \ldots , y^r, t^g y^{r+1}, \ldots , t^g y^g \right\}.
\]

As all Puiseux vectors \( x^1, \ldots , x^g, y^1, \ldots , y^g \) are contained in \( L_q^N \) for sufficiently large \( q \), they expand as

\[
x^i = x^i_0 + x^i_1 t^{1/q} + x^i_2 t^{2/q} + \cdots, \quad y^i = y^i_0 + y^i_1 t^{1/q} + y^i_2 t^{2/q} + \cdots, \quad (1 \leq i \leq r)
\]

\[
t^{1/q} x^i = x^i_1 t^{1/q} + x^i_2 t^{2/q} + \cdots, \quad t^{1/q} y^i = y^i_1 t^{1/q} + y^i_2 t^{2/q} + \cdots, \quad (r < i \leq g).
\] (3)

Lemma 2.5. Assume \( x^i, y^i \in L_q^N \) and \( v(Y - \lambda \cdot \text{id}) + \Lambda = k/q \). Then, for each \( i, j \),

\[
\sum_{s+u=k+1} T y^0_{1/q} (Y_{1/q} - \lambda \zeta_q) x^j_{1/q} = \sum_{s+u=k+1} T y^0_{1/q} (Y_{1/q} - \lambda \zeta_q) x^j_{1/q}.
\] (4)

Proof. In this proof, we simply write \( x = \left\{ \begin{array}{ll} x^j & (1 \leq j \leq r) \\ x^{j+1/q} & (r < j \leq g) \end{array} \right\} \), \( y = \left\{ \begin{array}{ll} y^i & (1 \leq i \leq r) \\ y^{i+1/q} & (r < i \leq g) \end{array} \right\} \). Because \( x \in W(\Lambda)(Y; \lambda) \) and \( y \in W(\Lambda)(T Y; \lambda) \), we have

\[
T y(Y - \lambda) x = T y \cdot \left( \sum_{s+u=k+1} (Y_{1/q} - \lambda \zeta_q) x^{s+u+1/q} + o(t^{s+u+1/q}) \right)
= \left( \sum_{s+u=k+1} y^0_{1/q} (Y_{1/q} - \lambda \zeta_q) t^{s+u+1/q} + o(t^{s+u+1/q}) \right) x.
\]

Comparing the coefficients of \( t^{s+u+1/q} \) on both sides, we find the desired equation. \( \square \)
2.1.3 Eigen-quadruple

Consider a quadruple

\[ X := (Y; \lambda; \{x^1, \ldots, x^r, t^0 x^{r+1}, \ldots, t^0 x^g\}, \{y^1, \ldots, y^r, t^0 y^{r+1}, \ldots, t^0 y^g\}), \quad Y \in K^{N \times N}, \lambda \in K, \]
\[ x^i, y^j \in (R/I(\lambda))^N. \]

(5)

Definition 2.6. A quadruple (5) is called an eigen-quadruple (or an EQ, shortly) of depth \( \Lambda \) if it satisfies:

1. \( \{x^1, \ldots, x^r, t^0 x^{r+1}, \ldots, t^0 x^g\} \) is a quasi basis of \( W(\Lambda)(Y; \lambda) \).
2. \( \{y^1, \ldots, y^r, t^0 y^{r+1}, \ldots, t^0 y^g\} \) is a quasi basis of \( W(\Lambda)(TY; \lambda) \).

Lemma 2.7. Let (5) be an EQ. Then, \( v(x^i) = v(y^j) \) for \( i = 1, \ldots, g \).

Proof. It follows from the proof of Lemma 2.1. \qed

For an EQ \( X \) (5), we call the number \( r \) in Definition 2.6 the rank of \( X \), and call the number \( g \) the size of \( X \). We denote \( \text{rk}(X) := r \) and \( \text{sz}(X) := g \).

Define two matrices \( P(X) = (P_{i,j}) \), \( Q(X) = (Q_{i,j}) \) of size \( g \) by

\[ P_{i,j} := \begin{cases} y^i_{\text{top}}, & \text{if } \lambda > \text{rk}(X) \text{ implies } x^j_0 = 0, \\
2y^i_{\text{top}}, \sum_{k=1}^{\Lambda_G} (Y_{\lambda+v+\frac{1}{4} - \frac{k}{n}} - \lambda_{\lambda+v+\frac{1}{4} - \frac{k}{n}}) x^j_k, & \text{if } \lambda > \text{rk}(X) \text{ implies } y^j_0 = 0, \text{ and } \lambda_{\lambda+v+\frac{1}{4} - \frac{k}{n}} \in \mathbb{C}^{k(\lambda)} \times \text{rk}(X), \\
\Lambda_G, & \text{otherwise}. \end{cases} \]

(6)

where \( x^i \) and \( y^j \) are the complex vectors in (3). Because (i) \( j > \text{rk}(X) \) implies \( x^j_0 = 0 \), (ii) \( i > \text{rk}(X) \) implies \( y^j_0 = 0 \), and (iii) \( i \leq \text{rk}(X) \) implies \( y^i_{\text{top}} = y^j_0 \), the matrices \( P(X) \) and \( Q(X) \) are of the form

\[ P(X) = \begin{pmatrix} \Delta & O \\ A & B \end{pmatrix}, \quad Q(X) = \begin{pmatrix} \Omega & O \\ A & B \end{pmatrix}, \quad \Delta, \Omega, \quad A, B, \quad \Gamma \in \mathbb{C}^{(\text{rk}(X) \times \text{rk}(X))}, \]
\[ \text{sz}(X) - \text{rk}(X) \times \text{rk}(X), \quad \text{sz}(X) - \text{rk}(X) \times \text{rk}(X). \]

We call the submatrices \( \Delta = \Delta(X) \) and \( \Omega = \Omega(X) \) the principal matrix of \( P(X) \) and \( Q(X) \), respectively.

Example 2.8. Let \( X = \begin{pmatrix} 2 + t \\ -t \end{pmatrix} \). We have \( W^{(0)}(Y; \lambda) = \{e_1, e_2\} \) if \( \lambda = 2 + o(t^0) \), and otherwise \( W^{(0)}(Y; \lambda) = \{0\} \). Let \( X_0 := (Y; \{e_1, e_2\}) \) be an EQ of depth 0. It follows that \( \text{rk}(X_0) = \text{sz}(X_0) = 2 \) and \( P(X_0) = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \) and \( Q(X_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

If \( X = (Y; \lambda; \{x^i\}, \{y^j\}) \) is an EQ of depth \( \Lambda \), the quadruple \( T X = (TY; \lambda; \{y^j\}, \{x^i\}) \) is also an EQ of depth \( \Lambda \).

Lemma 2.9. \( \Delta(T X) = T \Delta(X), \quad \Omega(T X) = T \Omega(X) \).

Proof. Let \( P(T X) = (\widehat{P}_{i,j}) \) and \( Q(T X) = (\widehat{Q}_{i,j}) \). If \( 1 \leq i \leq \text{rk}(X) = \text{rk}(TY), \) then \( x^i_{\text{top}} = x^i_0 \) and \( y^j_{\text{top}} = y^j_0 \) by definition. Therefore, from (4), we have

\[ P_{i,j} = \begin{cases} y^i_1 Y_{\lambda+v+rac{1}{4}} x^j_0 + \sum_{k=1}^{\Lambda_G} (Y_{\lambda+v+rac{1}{4} - \frac{k}{n}} - \lambda_{\lambda+v+rac{1}{4} - \frac{k}{n}}) x^j_k, & \text{if } \lambda > \text{rk}(X) \text{ implies } x^j_0 = 0, \\
y^i_1 Y_{\lambda+v+rac{1}{4}} x^j_0 + \sum_{k=1}^{\Lambda_G} y^j_0 (Y_{\lambda+v+rac{1}{4} - \frac{k}{n}} - \lambda_{\lambda+v+rac{1}{4} - \frac{k}{n}}) x^j_k, & \text{otherwise}. \end{cases} \]

and \( Q_{i,j} = y^j_0 x^i_0 = \widehat{Q}_{j,i} \) for \( 1 \leq i, j \leq \text{rk}(X) \). \qed
2.2 Simple eigenvalues

We start with the simplest case where \( Y_{\text{top}} = Y \) has a simple eigenvalue \( \lambda \). Let \( q := q(Y) \), \( s := t^{1/q} \), \( Z_n := Y_{n/q} \), \( \mu_n := \lambda_{n/q} \) and \( V := v q \).

By assumption, there exist two vectors \( x_0, w \in \mathbb{C}^N \) such that \( (Z_V - \mu_V)x_0 = 0 \), \( (T_z Z_V - \mu_V)w = 0 \), \( T_z w \cdot x_0 = 1 \) and \( T_w \cdot z = 0 \iff z \in \text{Im}(Z_V - \mu_V) \), \( z \in \mathbb{C}^N \).

Let \( f : \text{Im}(Z_V - \mu_V) \to \mathbb{C}^N \) be a linear map which satisfies \( (Z_V - \mu_V) \circ f = \text{id}_{\text{Im}(Z_V - \mu_V)} \). We recursively define countably many complex vectors \( x_1, x_2, \ldots \) and complex numbers \( \mu_{V+1}, \mu_{V+2}, \ldots \) by the following formulas (these objects are determined as: \( \mu_{V+1} \to x_1 \to \mu_{V+2} \to x_2 \to \cdots \))

\[
\mu_{V+n} := T_z w \left( Z_{V+n} x_0 + \sum_{i=1}^{n-1} (Z_{V+i} - \mu_{V+i}) x_{n-i} \right), \quad x_n := -f \left( \sum_{i=1}^{n} (Z_{V+i} - \mu_{V+i}) x_{n-i} \right). \tag{7}
\]

We should prove that this algorithm is well-worked.

Lemma 2.10. \( \sum_{i=1}^{n} (Z_{V+i} - \mu_{V+i}) x_{n-i} \in \text{Im}(Z_V - \mu_V) \).

Proof. It suffices to prove \( \sum_{i=1}^{n} T_z w(\mu_{V+i} - \mu_{V+i}) x_{n-i} = 0 \). We prove it by induction on \( n \geq 1 \). If \( n = 1 \), we have \( T_z w(Z_1 - \mu_1)x_0 = T_z w(Z_1 - T_z w Z_1 x_0) x_0 = 0 \). Let \( n > 1 \) and assume that \( x_1, \ldots, x_{n-1} \) are already defined. We have

\[
\sum_{i=1}^{n} T_z w(Z_i - \mu_i) x_{n-i} = \sum_{i=1}^{n-1} T_z w(Z_i - \mu_i) x_{n-i} + T_z w(Z_n - \mu_n) x_0
\]

\[
= \sum_{i=1}^{n-1} T_z w(Z_i - \mu_i) x_{n-i} + T_z w \left( Z_n - (T_z w Z_n x_0 + \sum_{i=1}^{n-1} T_z w(Z_i - \mu_i) x_{n-i}) \right) x_0 = 0.
\]

The second equality is derived from the definition of \( \mu_n \). \( \square \)

Let \( x := \sum_{i=0}^{\infty} x_i s^i \). By direct calculations, we obtain the equation

\[
\sum_{n=0}^{\infty} (Z_{V+n} - \mu_{V+n}) s^{V+n} x = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (Z_{V+i} - \mu_{V+i}) x_{n-i} s^{V+n}
\]

\[
= \sum_{n=0}^{\infty} \left( (Z_V - \mu_V) x_n + \sum_{i=1}^{n} (Z_{V+i} - \mu_{V+i}) x_{n-i} \right) s^{V+n} = 0.
\]

(The third equality is derived from the definition of \( x_n \)). This implies \( Z x = \mu x \), where \( \mu = \sum_{i=0}^{\infty} \mu_{V+i} s^{V+i} \).

2.3 Semi-simple eigenvalues

Next consider the case when \( \lambda_v = \mu_v \) is an semi-simple eigenvalue of \( Y_v = Z_V \). We note the relation \( \text{Ker}(T_z Z_V - \mu_V) \cdot z = 0 \iff z \in \text{Im}(Z_V - \mu_V) \), \( z \in \mathbb{C}^N \).

Lemma 2.11. Let \( \mathcal{X} = (Z; \mu; \{x^i\}, \{y^j\}) \) be an EQ of depth \( \Lambda \), rank \( r \) and size \( g \). The submatrix \( \Gamma \) of \( P(\mathcal{X}) \) in (10) is invertible.

Proof. If \( \Gamma \) is not invertible, there must exist some nonzero vector \( (0, \ldots, 0, c_p, c_{p+1}, \ldots, c_g) \in \mathbb{C}^g \) \( (r < p \leq g, c_p \neq 0) \) such that the vector \( z := \sum_{i=p}^{g} c_i x^i = z_1 s + z_2 s^2 + \cdots + z_\Lambda s^\Lambda \) satisfies

\[
T_y y_{\text{top}} \sum_{k=1}^{\Lambda} (Z_{\Lambda+V+1-k} - \mu_{\Lambda+V+1-k}) z_k = 0
\]
for \(i = 1, \ldots, g\). From Lemma \([\text{2.11}]\), this is equivalent to \(\text{Ker}(T_{Z_V - \mu_V}) \cdot \sum (Z_{\Lambda + V + 1 - k} - \mu_{\Lambda + V + 1 - k}) z_k = 0\).

Therefore, there exists some complex vector \(v\) such that

\[(Z_V - \mu_V)v + \sum_{k=1}^{\Lambda} (Z_{\Lambda + V + 1 - k} - \mu_{\Lambda + V + 1 - k}) z_k = 0. \tag{8}\]

Set \(z^* := z_1 + z_2 s + \cdots + z_{\Lambda} s^{\Lambda - 1} + v s^\Lambda\). By \((Z - \mu)z \equiv 0 \mod I(\Lambda + V)\) and \((8)\), it is directly checked that \((Z - \mu)z^* \equiv 0 \mod I(\Lambda + V)\). Therefore, \(z^* \in W^{(\Lambda)}(\Lambda; \mu)\). By construction, we have \(z^*_\text{top} = c_p x^p \Lambda + \cdots + c_g x^g\). On the other hand, because \(\{x^i\}\) is a quasi basis of \(W^{(\Lambda)}(\Lambda; \mu)\), we have \(z^* = a_1 x^1 + \cdots + a_g x^g\) for some \(a_1, \ldots, a_g \in R\). As \(1 \leq i \leq r \leftrightarrow v(x^i) = 0\), the equation \(z^*_\text{top} = (a_1 x^1)\text{top} + \cdots + (a_r x^r)\text{top}\) holds. Comparing these two equations, we obtain \(c_p x^p \Lambda + \cdots + c_g x^g\), which contradicts the fact that \(\{x^i\text{top}\}\) is linearly independent.

**Corollary 2.12.** Let \(\zeta \in \mathbb{C}\) be a complex number. The correspondence

\[
\left\{ c = \left( \begin{array}{c} v \\ v' \end{array} \right) \in \mathbb{C}^g; \{P(\mathcal{X}) - \zeta \cdot Q(\mathcal{X})\} c = 0 \right\} \rightarrow \{ v \in \mathbb{C}^r; \{\Delta(\mathcal{X}) - \zeta \cdot \Omega(\mathcal{X})\} v = 0 \}, \quad \left( \begin{array}{c} v \\ v' \end{array} \right) \rightarrow v
\]

is a linear isomorphism.

**Proof.** It is straightforward from Lemma \([\text{2.11}]\) and \((6)\). \(\square\)

Let \(L(\zeta) := \{ c \in \mathbb{C}^g; \{P(\mathcal{X}) - \zeta \cdot Q(\mathcal{X})\} c = 0 \}\) and \(\{x^1, \ldots, x^\rho\}\) be a basis of \(L(\zeta)\) \((0 \leq \rho = \text{dim}_C L(\zeta) \leq r)\). Put \((X^1, \ldots, X^\rho) := (x^1, \ldots, x^\rho) \cdot (c^1, \ldots, c^\rho)\). The vector \(X^i = X^i_0 + X^i_1 s^1 + \cdots + X^i_{\Lambda} s^\Lambda\) satisfies the equation

\[T y^i\text{top}(Z_{\Lambda + V + 1} - \zeta) X^j_0 + \sum_{k=1}^{\Lambda} T y^i\text{top}(Z_{\Lambda + V + 1 - k} - \mu_{\Lambda + V + 1 - k}) X^j_k = 0\]

\((\forall i, j)\), which is equivalent to

\[\text{Ker}(T_{Z_V - \mu_V}) \cdot \left\{ (Z_{\Lambda + V + 1} - \zeta) X^j_0 + \sum_{k=1}^{\Lambda} (Z_{\Lambda + V + 1 - k} - \mu_{\Lambda + V + 1 - k}) X^j_k \right\} = 0.\]

Therefore, there exists some complex vector \(X^j_{\Lambda + 1}\) such that

\[(Z_{\Lambda + V + 1} - \zeta) X^j_0 + \sum_{k=1}^{\Lambda} (Z_{\Lambda + V + 1 - k} - \mu_{\Lambda + V + 1 - k}) X^j_k + (Z_V - \mu_V) X^j_{\Lambda + 1} = 0.\]

This means that the vector \(\overline{x}^j := X^j_0 + X^j_1 s^1 + \cdots + X^j_{\Lambda + 1} s^{\Lambda + 1}\) and an element \(\overline{\mu} := \mu_V s^V + \mu_{V + 1} s^{V + 1} + \cdots + \mu_{\Lambda + V} s^{V + \Lambda} + \zeta s^{V + \Lambda + 1}\) satisfy \(\overline{x}^j \in W^{(\Lambda + 1)}(\Lambda; \overline{\mu})\).

**Algorithm 2.13.** We define the new EQ

\[
\overline{X} = (Z; \overline{\mu}; \{\overline{x}^i\}, \{\overline{y}^i\}) \tag{9}
\]

of depth \(\Lambda + 1\) by the following manner:

\(\overline{X}_{\Lambda + 1}\) is calculated by the formula

\[X^j_{\Lambda + 1} = -f((Z_{\Lambda + V + 1} - \zeta) X^j_0 + (Z_{\Lambda + V} - \mu_{\Lambda + V}) X^j_1 + \cdots + (Z_{\Lambda + 1} - \mu_{\Lambda + 1}) X^j_{\Lambda}) , \]

where \(f : \text{Im}(Z_V - \mu_V) \rightarrow \mathbb{C}^N\) is a linear map satisfying \((Z_V - \mu_V) \circ f = \text{Id}\).
(1). Define $\vec{\mu}, \vec{x}^1, \ldots, \vec{x}^p$ as above.
(2). Pick up $\vec{x}^p+1, \ldots, \vec{x}^r \in \{x^1, \ldots, x^r\}$ so that $\vec{x}^1_{\text{top}}, \ldots, \vec{x}^r_{\text{top}}$ are linearly independent.
(3). Define $\vec{x}^i := x^i \cdot s$ for $r < i \leq g$.
(4). Then, we obtain the new quasi basis $W^{(A+1)}(Y; \vec{\lambda}) = \langle \vec{x}^1, \ldots, \vec{x}^p, s^0 \vec{x}^p+1, \ldots, s^0 \vec{x}^g \rangle$.
(5). The Puiseux vectors $\vec{y}^i$ can be defined by similar manner.

Remark 2.14. We must note the fact that the matrix equation \{P(\lambda) - \zeta \cdot Q(\lambda)\}c = 0 does not always have a solution ($\zeta, \alpha, (c \neq 0)$. This is a typical case of failure where the naive algorithm is not applicable. (See Examples 2.17 and Remark 2.18). We will avoid this difficulty by conjugating the matrix $Y$ by some diagonal matrix.

2.4 The case if $\text{rk}(\lambda) = 1$

If $\text{rk}(\lambda) = 1$, we can make our algorithm simpler. Let $\lambda = (Z; \mu; \{x^1, s^0 x^2, \ldots, s^0 x^g\}, \{y^1, s^0 y^2, \ldots, s^0 y^q\})$ be an EQ of rank 1. Let $f : \text{Im}(Z_{\mu} - \mu V) \to \mathbb{C}^N$ be a linear map such that $(Z_{\mu} - \mu V) \circ f = \text{Id}$.

Because $\mu V$ is semi-simple, we can assume $^T y^1_0 \cdot x^1_0 = 1$ without loss of generality. The matrices $P(\lambda), Q(\lambda)$ are of the form:

$$P(\lambda) = \begin{pmatrix} P_{1,1} & 0 \\ p & \Gamma \end{pmatrix}, \quad Q(\lambda) = \begin{pmatrix} 1 & 0 \\ q & \Omega \end{pmatrix}$$

Obviously, the matrix equation \{P(\lambda) - \zeta \cdot Q(\lambda)\}c = 0 has a solution $\zeta = P_{1,1}$ and $c = ^T (1, \delta)$, where $\delta = \Gamma^{-1}(P_{1,1}q - p)$. Set

$$\vec{x}^1 := x^1 + (x^2 s, \ldots, x^g s) \cdot \delta - f \left( (Z_{A+V+1} - \zeta)\vec{x}^1_1 + (Z_{A+V} - \mu A+V)\vec{x}^1 + \cdots + (Z_{V+1} - \mu V+1)\vec{x}^1_\lambda \right),$$

where $x^1 + (x^2 s, \ldots, x^g s) \cdot \delta = \vec{x}^1_0 + \vec{x}^1 s^1 + \cdots + \vec{x}^1_\lambda s^A$. Therefore, the quintuple

$$\vec{\lambda} = (Z; \mu + \zeta s^{A+1}; \{\vec{x}^1_0, s^{1+\theta} x^2, \ldots, s^{1+\theta} x^g\}, \{\vec{y}^1_0, s^{1+\theta} y^2, \ldots, s^{1+\theta} y^q\})$$

is an EQ whose depth is greater than that of $\lambda$. Obviously, we have $^T \vec{x}^1_{\text{top}} \cdot y^1_{\text{top}} = ^T x^1_0 \cdot y^1_0 = 1$, which admits us to repeat this procedure.

2.5 Examples

Example 2.15 (Simple eigenvalue). Let $Y = \begin{pmatrix} 1 + t^2 & 2t & 2 \\ 1 + t & 2 - t & 2t \\ 2 & t^2 & 1 + t^2 \end{pmatrix} =: Y_0 + Y_1 t + Y_2 t^2$. The leading term of $Y$ is $Y_0 = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, which has a simple eigenvalue $\lambda_0 = 4$ and its corresponding eigenvector $x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The transposed matrix $^T Y_0$ has also an eigenvector $w = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ corresponding to 4.

Fix a linear map $f : \text{Im}(Y_0 - 4\text{Id}) \to \mathbb{C}^3$ which satisfies $(Y_0 - 4\text{Id}) \circ f = \text{Id} \cdot \text{Im}(Y_0 - 4\text{Id})$. For example, define $f(e_1 - e_3) := -\frac{1}{2} e_1 - \frac{1}{2} e_2$ and $f(e_2) := -e_2$.  


From (7), we have $\lambda_1 = T w Y_1 x_0 = 1$, $x_1 = -f((Y_1 - 1)x_0) = \frac{1}{2} e_1 + \frac{3}{2} e_2$, $\lambda_2 = T w \{Y_2 x_0 + (Y_0 - 1)x_1\} = \frac{9}{7}$, $x_2 = -f((Y_2 - \frac{9}{7})x_0 + (Y_1 - 1)x_1) = \frac{5}{7} e_1 - \frac{33}{7} e_2$. Finally, we obtain an eigenvector

$$x = \begin{pmatrix} 1 \\ 1/2 \\ 3/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5/8 \\ -33/8 \\ 0 \\ 0 \end{pmatrix} t^2 + \cdots$$

corresponding to an eigenvalue $\lambda = 4 + t + \frac{9}{7} t^2 + \cdots$.

**Example 2.16 (Semi-simple eigenvalue).** Let $Y = \begin{pmatrix} 1 & 1 - t & t \\ t & 2 & 2t \\ 2t & t^2 & 2 + t \end{pmatrix}$ be $Y_0 = Y_1 t + Y_2 t^2$. The leading term $Y_0$ has a semi-simple eigenvalue $\lambda_0 = 2$, whose corresponding eigenspace is generated by the two vectors $x^1 = T (1, 1, 0)$ and $x^2 = T (0, 0, 1)$. The transposed matrix $T^* Y$ has eigenvectors $y^1 = T (0, 1, 0)$ and $y^2 = T (0, 0, 1)$. Fix a linear map $f : \text{Im}(Y_0 - 2 \text{Id}) \rightarrow \mathbb{C}^3$ such that $(Y_0 - 2 \text{Id}) \circ f = \text{Id}$, for example, define $f(e_1) := e_2$. We obtain a sequence of EQs $X_0, X_1, \ldots$ by the following step-by-step method:

- The quintuple $X_0 = (Y; 2; \{x^1, x^2\}, \{y^1, y^2\})$ is an EQ of depth 0 with $P(X_0) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$. The matrix equation $\{P(X_0) - \zeta Q(X_0)\} c = 0$ has a solution $\zeta = 3$ and $c = T (1, 1, 1)$, and the transposed equation $\{P^*(X_0) - \zeta Q(X_0)\} d = 0$ has a solution $\zeta = 3, d = \frac{1}{2} T (1, 1, 1)$.

- Define $\tilde{x}_0 := (x^1, x^2)c = T (1, 1, 1), \tilde{x}_1 := -f((Y_1 - 3)\tilde{x}_0) = T (0, 3, 0)$, and $\tilde{x}_1 := \tilde{x}_0 + \tilde{x}_1t$. Also define $\tilde{y}_0 := (y^1, y^2)d = \frac{1}{2} T (0, 1, 1)$. Then, there exists some $\tilde{y}_1$ such that the quintuple $X_1 = (Y; 2 + 3t, \{\tilde{x}_1, t^6 x^2, \{y^1, t^{\theta} y^2\}\}), (\tilde{y}_1 = \tilde{y}_0 + \tilde{y}_1t)$ is an EQ of depth 1 with $P(X_1) = \begin{pmatrix} -4 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ and $Q(X_1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The matrix equation $\{P(X_1) - \zeta Q(X_1)\} c = 0$ has a solution $\zeta = -4$ and $c = T (2, 5)$.

- Set $\tilde{x}_2 := (\tilde{x}_1, \tilde{x}_2)t := T (2, 2, 2) + T (0, 6, 5)t, \tilde{x}_2 := -f((Y_2 + 4)\tilde{x}_3) + (Y_1 - 3)\tilde{x}_2) = T (0, -7, 0)$ and $\tilde{x}_2 := \tilde{x}_0 + \tilde{x}_1t + \tilde{x}_2t^2$. On the other hand, there exists some $\tilde{y}_1 = \tilde{y}_0 + \cdots$ such that the quintuple $X_2 = (Y; 2 + 3t - 4t^2, \{\tilde{x}_1, t^{1+\theta} x^2\}, \{y^1, t^{1+\theta} y^2\})$ is an EQ of depth 2.

Repeating this procedure, we obtain a sequence of EQs $X_3, X_4, \ldots$. In other words, we can calculate $\Lambda$-th approximate eigenvector/value for arbitrarily large $\Lambda$. In fact, we soon obtain an eigenvector

$$x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ 5 \\ 0 \\ 0 \end{pmatrix} t - \begin{pmatrix} 0 \\ 7 \\ 0 \\ 0 \end{pmatrix} t^2 + \cdots$$

and its corresponding eigenvalue $\lambda = 2 + 3t - 4t^2 + \cdots$.

**Example 2.17 (Case of failure).** Let $Y = \begin{pmatrix} 1 & 1 - t & t \\ t & 2 & 2t \\ 2t & t^2 & 2 + t \end{pmatrix}$ be $Y_0 = Y_1 t + Y_2 t^2$. By similar procedures as last examples, we soon obtain an EQ $A_1 = (Y; 2 + t, \{x^1, t^6 x^2\}, \{y^1, t^{\theta} y^2\})$ of depth 1, where $x^1 = T (0, 0, 1) - T (0, 1, 0)t, x^2 = T (0, 0, 1), y^1 = T (0, 1, 0) + \cdots$ and $y^2 = T (0, 1, 0)$. We have $P(X_1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $Q(X_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. As the matrix equation $\{P(X_1) - \zeta Q(X_1)\} c = 0$ has no nonzero solution $c$ for any $\zeta$, we fail to obtain an EQ of depth 2. This is a typical case of failure. See the following Remark 2.18.\footnote{We do not need to calculate $\tilde{y}_1$ here.}
Remark 2.18. Let \( \delta := \text{diag}(1, t^{1/2}, 1) \) and \( Z := \delta^{-1}Y\delta = \begin{pmatrix} t^{1/2} & t^{1/2} - t^{3/2} \\ 2t & 2t^{3/2} \\ 2 + t \end{pmatrix} \). Then \( Z_{1/2}t^{1/2} + Z_1t + \cdots + Z_5t^{5/2} \), where \( Y \) is the Puiseux matrix defined in Example 2.17. Unlike the matrix \( Y \), the matrix \( Z \) admits the recursive procedure:

- Let \( f: \text{Im}(Z_0 - 2\text{Id}) \to \mathbb{C}^3 \) be the linear map defined by \( f(e_1) = -e_1 \).
- \( X_{1/2} = (Z; 2; \{x^1, x^2\}, \{y^1, y^2\}) \), \( x^1 = T(t^{1/2}, 1, 0) \) and \( x^2 = T(0, 0, 1) \). \( y^1 = T(0, 1, 0) + \cdots, \) \( y^2 = T(0, 0, 1) + \cdots \) and \( P(X_{1/2}) = Q(X_{1/2}) = \text{Id} \). The matrix equation \( \{P(T_{X_{1/2}}) - \mu Q(X_{1/2})\}c = 0 \) has a solution \( \mu = 1 \) and \( c = \text{arbitrary} \), and the transposed equation \( \{P(T_{X_{1/2}}) - \mu Q(T_{X_{1/2}})\}d = 0 \) also has the same solution.
- \( X_1 = (Z; 2 + t; \{x^1, x^2\}, \{y^1, y^2\}) \), \( x^1 = T(t^{1/2}, 1, 0) \) and \( x^2 = T(0, 1, 0) + \cdots, \) \( y^1 = T(0, 1, 0) + \cdots \) and \( P(X_1) = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \), \( Q(X_1) = \text{Id} \). The matrix equation \( \{P(T_{X_1}) - \mu Q(X_1)\}c = 0 \) has a solution \( \mu = \sqrt{6} \) and \( d = \frac{1}{2\sqrt{6}}T(\sqrt{2}, \sqrt{2}) \), and the transposed equation \( \{P(T_{X_1}) - \mu Q(T_{X_1})\}d = 0 \) has a solution \( \mu = \sqrt{6} \) and \( d = \frac{1}{2\sqrt{6}}T(\sqrt{2}, \sqrt{2}) \).
- \( X_{3/2} = (Z; 2 + t + \sqrt{6}t^{3/2}; \{x^1, x^2\}, \{y^1, y^2\}) \), \( x^1 = T(\sqrt{3}t^{1/2} + \sqrt{2}t, 2\sqrt{3}t^{1/2} + \sqrt{2}t, \sqrt{2}, \sqrt{2}) \), \( x^2 = T(0, 0, 1) + \cdots, \) \( y^2 = T(0, 0, 1) + \cdots \). Finally, we obtain the eigenvector of \( Z \):

\[
x = \begin{pmatrix} 0 \\ \sqrt{3} \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} t^{1/2} + \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2\sqrt{3} \\ 0 \\ 0 \end{pmatrix} t^{3/2} + \cdots
\]

corresponding to \( \lambda = 2 + t + \sqrt{6}t^{3/2} + \cdots \).

3 Short reviews on graph theory and tropical linear algebra

Combinatorial aspects of positive matrices, complex matrices, Puiseux matrices, etc. reflect on their \textit{(weighted) adjacency graphs}. In this section, we discuss about \textit{weighted directed graphs} and their properties.

3.1 Weighted digraphs

For a digraph \((= \text{directed graph}) G\), we denote its node set by \( \text{node}(G) \), and its arc set by \( \text{arc}(G) \). When there is no chance of confusion, we simply write "\( i \in A \) instead of "\( i \in \text{node}(A) \)" and "\( (i \rightarrow j) \in A \)" instead of "\( (i \rightarrow j) \in \text{arc}(A) \)".

If a \textit{weighted digraph} \( G \) contains an arc of weight \( X \) from a node \( i \) to a node \( j \), we simply say "\( G \) contains an arc \( i \prec X \rightarrow j \)". The weight of an arc \( (i \rightarrow j) \in G \) is written as \( w_G(i \rightarrow j) \).

A \textit{(directed) path} is a sequence of finitely many arcs: \( i \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow j \). The \textit{length} of a path is a number of arcs contained in the path, and the \textit{weight} of a path is the sum of weights of the arcs. A \textit{closed path}, or a \textit{loop}, is a directed path from a node to itself. A \textit{simple loop} is a loop of length 1.

Two nodes \( i, j \) are said to be \textit{strongly connected} if there exist paths both from \( i \) to \( j \) and from \( j \) to \( i \). A \textit{strongly connected digraph} is a digraph in which any two nodes are strongly connected. For two nodes \( i, j \), we denote \( i \rightsquigarrow j \) if a digraph contains \( (i \rightarrow j) \) or \( (j \rightarrow i) \). Two nodes \( i, j \) are said to be \textit{connected} if there exists a sequence \( i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \cdots \rightsquigarrow j \).

The \textit{smallest weight} of a weighted digraph \( G \) is the smallest number \( s(G) \) among the weights of arcs in \( G \). If \( G \) contains no arc, \( s(G) := +\infty \). A \textit{truncated graph of} \( G \) is the subgraph \( G \leq \mu \subset G \) \((\mu \in \mathbb{Q})\)
which is gotten by removing arcs whose weight is greater than \( \mu \). The digraph \( G_{\leq \mu} \) is called the leading term of \( G \). Denote \( G_{\text{top}} := G_{\leq \mu} \).

**Definition 3.1.** A homomorphism between weighted digraphs \( G \) and \( G' \) is a map \( f : \text{node}(G) \rightarrow \text{node}(G') \) which satisfies the following property:

\[
(i \xrightarrow{W} j) \in G \land (f(i) \neq f(j)) \implies ((f(i) \xrightarrow{W'} f(j)) \in G') \land (W \geq W')
\]

**Remark 3.2.** If \( f : G \rightarrow G' \) be a homomorphism between weighted digraphs, \( f \) induces a restricted homomorphism \( f_{\leq \mu} : G_{\leq \mu} \rightarrow G'_{\leq \mu} \) for any \( \mu \).

Let \( G \) be a weighted digraph, and let \( \sim \) be an equivalence relation on \( \text{node}(G) \). The quotient graph \( G/\sim \) is the weighted digraph defined by \( \text{node}(G/\sim) := \text{node}(G)/\sim \) and

\[
\text{arc}(G/\sim) := \{ (x \xrightarrow{W} y) \mid x \neq y, W = \min_{i \in x, j \in y} \{ w \mid (i \xrightarrow{w} j) \in G \} < +\infty \}.
\]

There uniquely exists a natural homomorphism \( \pi : G \rightarrow G/\sim \) of weighted digraphs such that \( \pi(i) = x \iff i \in x \).

Let \( B \subset G \) be a subgraph. By identifying all nodes in \( B \) to a single equivalence class and regarding each node outside of \( B \) as an equivalence class with one element, we define the equivalence relation \( \sim_B \) on \( \text{node}(G) \). We will simply denote \( G/B := G/\sim_B \).

**Example 3.3.** Let \( G = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 3 & 6 & \\ \circ & \circ & \circ & \end{pmatrix} \). Define the equivalence relation \( \approx \) on \( \text{node}(G) \) by

\[ i \approx j \iff i - j \text{ is even} \]. Then \( (G/\approx) = \begin{pmatrix} 1, 3 & \circ \\ \circ & \circ & \circ & \end{pmatrix} \).

Let \( S \) be a subset of \( \text{node}(G) \). A subgraph induced by \( S \) is a subgraph \( H \subset G \) such that \( \text{node}(H) = S \) and \( \text{arc}(H) = \{ (i \rightarrow j) \in \text{arc}(G) \mid i, j \in S \} \). The induced subgraph is written as \( H = G[S] \).

A digraph \( T \) is said to be a directed tree (or simply, tree) if \( T \) contains a node \( i \) such that, for each node \( j \in T \), \( T \) contains exactly one path from \( i \) to \( j \). A forest is a disjoint union of trees.

A digraph is said to be acyclic if it contains no closed path. Any acyclic digraph admits a semi-order \( \succeq \) defined by \( i \succeq j \iff \) there exists a path from \( i \) to \( j \). We say \( G \) to be an acyclic digraph on \( x \) if \( G \) contains an unique maximal node \( x \) with respected to the semi-order \( \succeq \).

Let \( G \) be a digraph, and \( E = \{ x_1, \ldots, x_k \} \) be a proper subset of \( \text{node}(G) \). \( G \) is said to be an \( E \)-forest if each connected component of \( G \) (i) is acyclic, (ii) is on \( x_i \) for some \( i \) and (iii) \( E \cap G = \{ x_i \} \).

**Example 3.4** (\( E \)-forest). The following digraph is an example of \( E \)-forest. The sign \( \circ \) represents a node in \( E \). The graph consists of four connected components.
We define the eigenvalue of a weighted digraph $G$ by the formula
\[
\Lambda(G) := \min \left\{ \frac{\text{weight}(\gamma)}{\text{length}(\gamma)} \Bigg| \gamma \text{ is a closed path on } G \right\}.
\] (10)

Of course, $s(G) \leq \Lambda(G)$. The equality holds if and only if $G_{\text{top}}$ contains a closed path.

### 3.2 Tropical algebra

We shortly introduce some basic facts on tropical algebra. For details, see [19], for example.

A tropical matrix is a matrix over the tropical semi-field $\mathbb{T} := \mathbb{Q} \cup \{+\infty\}$. For two tropical matrices $C = (C_{i,j}) \in \mathbb{T}^{N \times M}$ and $D := (D_{i,j}) \in \mathbb{T}^{M \times L}$, we define the tropical product $C \odot D \in \mathbb{T}^{N \times L}$ by $C \odot D = (X_{i,j}), X_{i,j} := \min_{k=1}^{M} \{C_{i,k} + D_{k,j}\}$. For $a \in \mathbb{T}$ and $C = (C_{i,j}) \in \mathbb{T}^{N \times M}$, $a + C := (a + C_{i,j})$.

Let $C = (X_{i,j}) \in \mathbb{T}^{N \times N}$ be a square tropical matrix. The adjacency graph of $C$ is the weighted digraph $G(C) = (\text{node}(G(C)), \text{arc}(G(C)))$, where $\text{node}(G(C)) = \{1, \ldots, N\}$ and $\text{arc}(G(C)) = \{(j, X_{i,j} \approx +\infty) \mid i \in X_{i,j} \approx +\infty\}$. A tropical matrix is said to be irreducible if its adjacency graph is strongly connected.

**Remark 3.5.** The $1 \times 1$ matrix $(+\infty) \in \mathbb{T}^{1 \times 1}$ is an irreducible matrix, whose adjacency graph of is the simple node graph.

Denote $+\infty := T(\infty, \cdots, +\infty) \in \mathbb{T}^{N}$.

**Proposition 3.6 (Max-plus spectral theorem).** Let $C \in \mathbb{T}^{N \times N}$ be a tropical matrix.

1. There exist a tropical number $\Lambda \in \mathbb{T}$ and a tropical vector $V \in \mathbb{T}^{N}$ ($V \neq +\infty$) such that $C \odot V = \Lambda + V$. The number $\Lambda$ is called a tropical eigenvalue of $C$, and the vector $V$ is called a tropical eigenvector corresponding with $\Lambda$.

2. If $C \neq (+\infty)$ is irreducible, its tropical eigenvalue is unique. In this case, the tropical eigenvalue of $C$ coincides with the eigenvalue of the graph $G(C)$.

3. If $C \neq (+\infty)$, there exists a tropical eigenvector corresponding with $\Lambda$ which is contained in $\mathbb{Q}^{N}$.

**Proof.** See, for example, [3] [12] and the references therein.

Let $D(N) := \{(X_{i,j}) \in \mathbb{T}^{N \times N} \mid i \neq j \Leftrightarrow X_{i,j} = +\infty\}$ be the set of tropical diagonal matrices of size $N$. We write the tropical diagonal matrix $C \in D(X)$ as $C = \text{diag}[T_{1}, \ldots, T_{N}]$, where $T_{i}$ is the $(i,i)$-entry of $C$.

**Corollary 3.7.** Let $C \in \mathbb{T}^{N \times N}$ be an irreducible matrix except for $(+\infty)$, and let $\Lambda$ be the eigenvalue of $C$. Then, there exists a diagonal matrix $\Gamma \in D(N)$ such that the matrix $C' := (-\Gamma) \odot C \odot \Gamma = (X'_{i,j})_{i,j}$ satisfy the equation $\Lambda = \min_{j=1,2,\ldots,N} [X'_{i,j}]$ for each $i = 1, 2, \ldots, N$.

**Proof.** Set $C = (X_{i,j})_{i,j}$. Let $V = (V_{i})$ be a tropical eigenvector of $C$. Then, we have $C \odot V = \Lambda + V$, or equivalently, $\min_{j} [X_{i,j} + V_{j} - V_{i}] = \Lambda$, $(i = 1, 2, \ldots, N)$. It suffices to define $\Gamma := \text{diag}[V_{1}, \ldots, V_{N}]$.

### 4 Flat-slanted form of weighted digraphs

#### 4.1 Condensation map and Strong-condensation map

Let $G$ be a weighted digraph. Introduce two equivalent relations $\approx_{w}$ and $\approx_{s}$ on node($G$) by:

- $i \approx_{w} j \iff$ nodes $i$ and $j$ are connected in $G_{\text{top}}$.
- $i \approx_{s} j \iff$ nodes $i$ and $j$ are strongly connected in $G_{\text{top}}$. 


Of course, \( i \sim_s j \) implies \( i \approx_w j \).

We call the quotient graph \( G/\sim_w \) (resp. \( G/\sim_s \)) the \textit{condensation graph} of \( G \) (resp. \textit{strong-condensation graph} of \( G \)). The quotient homomorphism \( G \to G/\sim_w \) (resp. \( G \to G/\sim_s \)) will be called \textit{condensation map} (resp. \textit{strong-condensation map}). We note a few of fundamental facts which can be proved immediately:

- If \( s(G) < +\infty \), then \( s(G/\sim_w) > s(G) \).
- If \( s(G) < +\infty \), then \( (G/\sim_s)_{\leq s(G)} \) is a forest.
- Any condensation graph contains no simple loop.

**Lemma 4.1.** Let \( G \) be a weighted digraph, and \( \mu \) be a rational number. Introduce an equivalence relation \( \sim_\mu \) on node\((G)\) by \( i \sim_\mu j \iff i \) and \( j \) are connected in \( G/\sim_\mu \). Then, the homomorphism \( \pi : G \to G/\sim_\mu \) is a composition of finitely many condensation maps. \( \square \)

**Definition 4.2.** A composition of finitely many condensation maps is called a \textit{multi-condensation map}.

We denote "\( G_1 \triangleright G_2 \)" if \( G_1 \to G_2 \) is a condensation map and "\( G_1 \trianglerightarrow G_2 \)" if \( G_1 \to G_2 \) is a multi-condensation map.

### 4.2 Similarity translation of weighted graphs

Let \( G \) be a weighted digraph with \( N \) nodes. For a rational number \( \nu \in \mathbb{Q} \) and a node \( k \in G \), we define a new weighted digraph \( S_k(\nu) \cdot G \):

- by replacing \( k \xrightarrow{X} i \) with \( k \xrightarrow{X+\nu} i \) for \( \forall i \in G \) and
- by replacing \( j \xrightarrow{X} k \) with \( j \xrightarrow{X-\nu} k \) for \( \forall j \in G \).

For any tropical matrix \( C \) and its adjacency graph \( G(C) \), the following equation holds:

\[
S_k(\nu) \cdot G(C) = G((-\gamma) \circ C \circ \gamma), \quad \gamma = \text{diag}[0, \ldots, 0, \hat{\nu}, 0, \ldots, 0]. \tag{11}
\]

A \textit{similarity translation of weighted digraphs} is a composition of finitely many translations: \( G \to S_k(\mu) \cdot G \)\((k \in G, \mu \in \mathbb{Q})\). We denote \( G_1 \sim G_2 \) if there exists a similarity transformation \( G_1 \to G_2 \). The relation \( \sim \) is an equivalence relation, which we will call the \textit{similarity equivalence relation}.

Let \( f : G \to H \) be a homomorphism of weighted digraphs. For \( \nu \in H \) and \( \mu \in \mathbb{Q} \), we define the \textit{pull back of the similarity transformation} \( S_i(\mu) \) by \( f \) by

\[
f^* S_i(\mu) := S_{i_j}(\mu) \circ \cdots \circ S_{i_k}(\mu), \quad \text{where } f^{-1}(i) = \{j_1, \ldots, j_k\}.
\]

(Note that \( S_i(\mu) \circ S_j(\nu) = S_j(\nu) \circ S_i(\mu) \)). For a general similarity transformation \( S = S_{i_1}(\mu_1) \circ \cdots \circ S_{i_k}(\mu_k) \), we set \( f^* S := f^* S_{i_1}(\mu_1) \circ \cdots \circ f^* S_{i_k}(\mu_k) \). There naturally exists an induced homomorphism \( f^* S \cdot G \to S \cdot H \) whose restriction to the node sets coincides with that of \( f : G \to H \). In other words,

\[
\begin{array}{ccc}
G & \sim & f^* S \cdot G \\
H & \sim & S \cdot H
\end{array}
\]

implies

\[
\begin{array}{ccc}
G & \sim & f^* S \cdot G \\
H & \sim & S \cdot H
\end{array}
\] \tag{12}

**Remark 4.3.** If \( J \subset G \) is a subgraph such that \( f(J) = \text{(a single node)} \), \( J \) is invariant under the similarity transformation \( G \sim f^* S \cdot G \).

**Lemma 4.4.** Let \( A \) be a weighted digraph with at least one arc, \( B \) be the condensation or strong-condensation graph of \( A \), and \( D \) be a digraph which is similarity equivalent to \( B \). Consider the diagram

\[
\begin{array}{c}
A \sim \exists C \\
\downarrow & \downarrow
\end{array}
\]

\[
\begin{array}{c}
B \sim D \\
\downarrow & \downarrow
\end{array}
\]

which is derived from (12). In this situation, the following (1–3) hold:
Therefore, by using (13) repeatedly, we can find the diagram\
\[ A \sim C \]
A diagram \(B \sim D\) induces the diagram \(A \sim C\).

\[ (s(A) < s(D)) \implies (s(A) = s(C)). \quad (13) \]

More generally, we have the following lemma:

**Lemma 4.5.** A diagram \(B \sim D\) induces the diagram \(A \sim C\), where \(s(A) < s(D)\) and \(s(A) = s(C)\).

**Proof.** Because \(A \triangleright B\) is a multi-condensation map, there exists a sequence of condensation maps: \(A \triangleright A_1 \triangleright \cdots \triangleright A_K \triangleright B\). By definition of condensation maps, we have \(s(A) < s(A_1) < \cdots < s(A_K) < s(B)\).

\[ A_n \sim C \]
Therefore, by using (13) repeatedly, we can find the diagram \(A_{n+1} \sim C_{n+1}\) for all \(n\).

**4.3 Definition of flat-slanted graph**

**Lemma 4.6.** Let \(A\) be a weighted digraph whose leading term contains at least one loop. There exists a digraph \(C\) such that (i) \(A \sim C\) and (ii) \(C_{\text{top}}\) is a disjoint union of strongly connected components.

**Proof.** Set \(s = s(A)\). Let \(\pi : A \rightarrow (A/\approx_s) =: B\) be the strong-condensation map. Because \(B_{\leq s}\) is a forest (see [14, 15]), we can define the semi-order \(\triangleright\) on node\((B)\) by \(x \triangleright y \iff\) there exists a path from \(x\) to \(y\) in \(B_{\leq s}\). Fix an order-preserving bijection \(\varphi : \text{node}(B) \rightarrow \{1, 2, \ldots, n\}\) and set \(D := \{S_{x_1}(\epsilon \cdot \varphi(x_1)) \cdots S_{x_n}(\epsilon \cdot \varphi(x_n))\} \cdot B\) for a rational number \(\epsilon > 0\). If \(\epsilon > 0\) is sufficiently small, we can assume \(s(D) > s(A)\). Let \(C\) be the weighted digraph uniquely defined by (12) as \(B \sim D\). From Lemma 4.4 (3), \(C_{\text{top}} = C_{\leq s} = (\text{the disjoint union of strongly connected components of } A_{\text{top}})\).
Proof. If $G = *$ (the simple node graph), there is noting to prove. Assume $G \neq *$. Let $C \neq (+\infty)$ be the adjacency matrix of $G$. From Corollary 3.7 there exists a diagonal tropical matrix $\Gamma$ such that the matrix $C' = (\Gamma) \circ C \circ \Gamma = (X'_{i,j})_{i,j}$ satisfies $\Lambda = \min_i [X'_{i,j}]$ for all $i$. Define $J := G(C')$. By the equation above, the digraph $J$ satisfies the following properties: $\bullet$ $G \sim J$, $\bullet$ $s(J) = \Lambda$ and $\bullet$ for each node $i \in J$, there exists at least one arc $i \rightarrow j$. By pigeonhole principle, the digraph $J_{\text{top}} = \cup_{i} J^i$ contains at least one loop. By Lemma 4.5 there exists a weighted digraph $H$ such that (i) $J \sim H$ and (ii) $H_{\text{top}}$ is a disjoint union of strongly connected components. \qed

Definition 4.8. A weighted digraph whose leading term is a disjoint union of strongly connected components is said to be flat-slanted.

The results in this section can be summarized as follows:

Proposition 4.9. Any strongly connected weighted digraph is similarity equivalent to a flat-slanted graph. \qed

Let $H$ be a flat-slanted graph with at least one arc. Because $s(H) = \Lambda(H)$, we have $H \triangleright H' \Rightarrow \Lambda(H) < s(H') \leq \Lambda(H')$.

Proposition 4.10. If $D$ is flat slanted, the diagram $\nabla$ implies $\nabla \nabla$. $B \sim D, B \sim D$

Proof. The statement follows from $s(A) < s(B) \leq \Lambda(B) = \Lambda(D) = s(D)$ and Lemma 4.5 \qed

Example 4.11. Let $C := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 3 & 2 \end{bmatrix} \in \mathbb{T}^{3 \times 3}$ be the tropical matrix, whose adjacency graph is $G = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 \\ 3 & 3 \end{pmatrix}$. The tropical eigenvalue of $C$ is $\Lambda = 1/2$. The vector $V = T[1/2, 0, 5/2]$ is a corresponding eigenvector. Set $J := \{S_1(1/2) \circ S_2(0) \circ S_3(5/2)\} \cdot G = \begin{pmatrix} 1 & 1/2 \\ 2 & 1/2 \\ 3 & 1/2 \end{pmatrix}$. The leading term $J_{\text{top}} = \cup_{1/2}$ contains two strongly connected components $x = \{1, 2\}, y = \{3\}$. These strongly connected components are connected by one arc $(2 \rightarrow 3)$. Next define $H := \{S_1(\epsilon) \circ S_2(\epsilon)\} \cdot J$ for sufficiently small $\epsilon > 0$. Then, the graph $H$ is flat-slanted. For example, if $\epsilon = 1/4$, we have $H = \begin{pmatrix} 1 & 1/2 \\ 7/4 & 1/2 \\ 2 & 3 \end{pmatrix}$. The condensation graph of $H$ is is expressed as $H' = (x \rightarrow y)$. Obviously, $\Lambda(G) = \Lambda(J) = \Lambda(H) = 1/2 < \Lambda(H') = 5/4$. 

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5 Gently-slanted form of weighted digraphs

In this section, we introduce the gently-slanted graph form of weighed digraphs, obtained by "perturbing" weights of flat-slanted graphs.

5.1 Topology on the set of weighted digraphs

To make precise our claims, we need a topology on a set of weighted digraphs. Let \( U \) be an unweighted digraph and \( \mathcal{X}_U \) be the set of weighted digraphs whose underlying graph \( (i.e. \) the unweighted graph obtained by forgetting weights) is \( U \). Let \( \text{arc}(U) = \{a_1, \ldots, a_A\} \) be the arc set of \( U \). We introduce the coarsest topology on \( \mathcal{X}_U \) such that the map

\[
\mathcal{X}_U \to \mathbb{Q}^A; \quad G \mapsto (w_G(a_i))_{i=1,2,\ldots,A}
\]

is continuous. Obviously, the map \( \mathcal{X}_U \times \mathbb{Q} \to \mathcal{X}_U; (G, \mu) \mapsto S_{i}(\mu) \cdot G \) is continuous.

5.2 Definition of the gently-slanted form

Lemma 5.1. Let \( A \) be a weighted digraph without simple loops whose leading term is strongly connected. For any proper and non-empty subset \( E \subseteq \text{node}(A) \), there exist a positive number \( \epsilon \) and a continuous map

\[
B_E : (s(A) - \epsilon, s(A)) \to \mathcal{X}_U; \quad u \mapsto B_E(u)
\]

such that (i) \( A \sim B_E(u) \), (ii) \( B_E(u)_{\text{top}} \) is a \( E \text{-} \)forest, (iii) \( u = s(B_E(u)) \) and (iv) \( \lim_{u \uparrow s(A)} B_E(u) = A \).

Proof. Let \( \delta := s(A) - u > 0 \). Write \( E = \{i_1, \ldots, i_n\} \), \( (\epsilon := \frac{\delta}{2}) \). For a path \( \gamma \in A \) whose starting node is \( i_k \), define \( l(\gamma) := \text{length}(\gamma) + \frac{k-1}{n} \delta \). Define the rational number valued function \( d : \text{node}(A) \to \frac{1}{\epsilon} \cdot \mathbb{Z}_{\geq 0} \) by \( d(j) := \min_{\gamma \in E}\{l(\gamma) \mid \gamma \text{ is a path from } i \text{ to } j \text{ in } A_{\text{top}}\} \). (\( d \) is well-defined because \( A_{\text{top}} \) is strongly connected). Obviously, \( d(i_1) = 0, d(i_2) = 1/\epsilon, \ldots, d(i_n) = (n-1)/\epsilon \). Label the nodes of \( A \) as \( i_1, \ldots, i_n \) such that \( 1 \leq d(i_{r+1}) \leq d(i_{r+2}) \leq \cdots \leq d(i_n) \). Let \( X_{m,n} := \{w_A(i_n \to i_m), \text{ if } (i_n \to i_m) \in A \} \).

Define the rational sequence \( \alpha_1, \alpha_2, \ldots, \alpha_N \) recursively by

\[
\begin{align*}
\alpha_1 &= 0, \quad \alpha_2 = \frac{1}{\epsilon} \delta, \quad \alpha_3 = \frac{2}{\epsilon} \delta, \quad \ldots, \quad \alpha_n = \frac{n-1}{\epsilon} \delta, \\
\alpha_{k+1} &= \min_{i=1}^k\{X_{k+1,i} + \alpha_i\} - u, \quad (k \geq e).
\end{align*}
\]

If the number \( \delta > 0 \) is sufficiently small, \( \alpha_k \) is equal to \( d(i_k) \cdot \delta \). (It is proved by induction). We will prove that the graph

\[
B_E(u) := \{S_{i_1}(\alpha_1) \circ S_{i_2}(\alpha_2) \circ \cdots \circ S_{i_N}(\alpha_N)\} \cdot A
\]

(14)

is a desired one. (i) and (iv) are obvious. (iii): Write \( B = B_E(u) \). By \( (\mathbb{Z}, \mathbb{Z}) \), we have

\[
w_B(i_m \to i_n) = w_A(i_m \to i_n) + \alpha_m - \alpha_n = X_{n,m} + \alpha_m - \alpha_n \geq \text{small } X_{n,m} + \{d(i_m) - d(i_n)\} \delta.
\]

On the other hand, by definition of \( d \), we have \( (\mathbb{Z}) \): \( X_{n,m} = s(A) \Rightarrow d(i_m) - d(i_n) \geq -1 \), and \( (\mathbb{Z}) \): \( d(i_m) \geq 1 \Rightarrow \exists m < n ((X_{n,m} = s(A)) \And (d(i_m) - d(i_n)) \neq -1) \). We obtain \( u \geq s(B_E(u)) \) from \( (\mathbb{Z}) \), and \( u \leq s(B_E(u)) \) from \( (\mathbb{Z}) \). Hence, (iii) holds. (ii): Because \( (i_m \to i_n) \in B_E(u)_{\text{top}} \iff d(i_m) - d(i_n) = -1 \), the digraph \( B_E(u)_{\text{top}} \) is decomposed into \( e \) connected components. Each component is an acyclic graph on \( i_n \) \((n = 1, 2, \ldots, e)\). \( \square \)
**Proposition 5.2.** Let $H$ be a flat-slanted graph without simple loops and $H_{\text{top}} = h^1 \sqcup \cdots \sqcup h^n$ be the strongly connected component decomposition. Fix a proper subset $E \subseteq \text{node}(H)$ such that $E \cap h^i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Then, there exist a positive number $\epsilon > 0$ and a continuous map

$$F_E : (s(H) - \epsilon, s(H)) \to \mathcal{X}_U; \quad u \mapsto F_R(u)$$

such that (i) $H \sim F_E(u)$, (ii) $F_E(u)_{\text{top}}$ is a $E$-forest, (iii) $u = s(F_E(u))$ and (iv) $\lim_{u \uparrow s(H)} F_E(u) = H$.

**Proof.** Set $E^i := E \cap h^i$. Let $I := \{i \mid E^i \subseteq h^i\} \neq \emptyset$. From Lemma 5.1 there exists a similarity transformation $S^i(u)$, $(i \in I)$ such that the digraph $B_{E^i}(u) = S^i(u) \cdot h^i$ satisfies $\bullet \cdot u = s(B_{E^i}(u))$. Define $F_E(u) := \{S^{i_1}(u) \circ \cdots \circ S^{i_m}(u)\} \cdot H$ for $I = \{i_1, \ldots, i_m\}$. If the difference $s(H) - u > 0$ is sufficiently small, we have $F_E(u)_{\text{top}} = B_{E^{i_1}}(u)_{\text{top}} \sqcup \cdots \sqcup B_{E^{i_m}}(u)_{\text{top}} \sqcup \ast \sqcup \cdots \sqcup \ast$ ($\ast$ is a separated node). Because a disjoint union of $E^i$-forests is a $(E^1 \sqcup \cdots \sqcup E^n)$-forest, we conclude the claim. \hfill \Box

**Definition 5.3.** Let $F$ be a digraph, and $E \subseteq \text{node}(F)$ be a proper subset. A weighted digraph $F$ is said to be $E$-gently-slanted if $F_{\text{top}}$ is a $E$-forest.

**Proposition 5.2** (i), (ii) implies that $H$ is similarity equivalent to a $E$-gently-slanted graph.

**Example 5.4.** Let $C := \begin{bmatrix} +\infty & 1 & 0 \\ 0 & +\infty & 2 \\ 3 & 3 & +\infty \end{bmatrix}$ be the tropical matrix, whose adjacency graph is $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \\ 3 & 3 \end{pmatrix}$. By similar method in Example 4.11, we obtain the flat-slanted graph $H = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \\ 7/4 & 5/4 \\ 3/4 & 3 \end{pmatrix}$ such that $G \sim H$. The leading term $H_{\text{top}} = H_{\leq 1/2}$ contains two strongly connected components $x = \{1, 2\}$, $y = \{3\}$ and two arcs. Therefore, there are two possibilities: (i) $E = \{1, 3\}$ or (ii) $E = \{2, 3\}$.

(i) When $E = \{1, 3\}$, $F = S_2(\epsilon) \cdot H$ is a $E$-gently-slanted graph for sufficiently small $\epsilon > 0$. In fact,

$$F = \begin{pmatrix} 1 & 1/2+\epsilon \\ 7/4 & 5/4+\epsilon \\ 3 & 1/2+\epsilon \end{pmatrix}$$

is $E$-gently-slanted.

(ii) When $E = \{2, 3\}$, $F = S_1(\epsilon) \cdot H$ is a $E$-gently-slanted graph for sufficiently small $\epsilon > 0$. In fact,

$$F = \begin{pmatrix} 1 & 1/2+\epsilon \\ 7/4-\epsilon & 5/4+\epsilon \\ 3 & 3 \end{pmatrix}$$

is $E$-gently-slanted.
6 The Perron theorem for positive Puiseux matrices

6.1 Real Puiseux field and its subtraction free part

Let $F = \bigcup_{p=1}^{\infty} \mathbb{R}((t^{1/p}))$ be the real Puiseux series field. The positive part $F_>$ of $F$ is the subset of positive elements (i.e., elements whose leading term is positive). We call the set $F_\geq := F_> \cup \{0\}$ the nonnegative part of $F$. $F$ admits the total order $\geq$ defined by $x \geq y \iff x - y \in F_\geq$. It is known that the ordered field $F$ is a real closed field (see Section A).

The subtraction-free part of $F$ is the subset $\mathfrak{sf}$ of $F_>$ defined by

$$\mathfrak{sf} := \{a_k/q t^{k/q} + a_{k+1}/q t^{(k+1)/q} + \cdots \mid k \in \mathbb{Z}, \ q \in \mathbb{Z}_{>0}, \ a_r \geq 0 \ (\forall r), \ a_{k/q} > 0\}.$$

A real Puiseux matrix is said to be positive (resp. nonnegative, subtraction-free) if all its entries are contained in $F_>$ (resp. $F_\geq, \mathfrak{sf}$). An element of $F^N$ (resp. $F_\geq^N, F_\geq^N$) is called a real Puiseux vector (resp. a positive, nonnegative Puiseux vector).

The following is an analog of the Perron theorem over the real Puiseux field.

**Theorem 6.1** (The Perron theorem over $F$). A positive Puiseux matrix $Y$ satisfies the following properties (i–iv): (i) Its spectral radius $\rho(Y)$ is a simple and positive eigenvalue. (ii) A corresponding eigenvector with $\rho(Y)$ can be chosen to be entry-wise positive. (iii) Any nonnegative eigenvector corresponds with $\rho(Y)$. (iv) No eigenvalue $\lambda$, except for $\rho(Y)$, satisfies $|\lambda| = \rho(Y)$.

We will refer to $\rho(Y)$ as the Perron root of $Y$. A Perron vector is a positive Puiseux eigenvector corresponding to the Perron root.

In the sequel, we give a constructive proof of Theorem 6.1 for some generic class of Puiseux matrices (see Section 6.4) by demonstrating the method of calculating Perron roots and Perron vectors of positive Puiseux matrices.

**Remark 6.2.** Any positive Puiseux matrix $Y \in F_\geq^{N \times N}$ can be decomposed as $Y = kY'$, where $k \in F_>$ and $Y' \in \mathfrak{sf}_\geq^{N \times N}$. Hereafter, we can restrict ourselves on the subtraction-free matrices only.

6.2 Perron-Frobenius data

**Definition 6.3.** An EQ $\mathcal{X} = (Y; \lambda; \{x\}, \{y\})$ of depth $\Lambda$ is said to have the Perron-Frobenius property if it satisfies the following (i–iv):

(i). $\lambda \equiv \rho(Y) \mod I(v(Y - \lambda \cdot \text{id}) + \Lambda)$,

(ii). The quasi basis $\{x\}$ is expressed as $\{x\} = \{x^1, t^0 x^2, \ldots, t^0 x^N\}$, and the quasi-basis $\{y\}$ is expressed as $\{y\} = \{y^1, t^0 y^2, \ldots, t^0 y^N\}$, where $x^1 \in F_\geq^N$, $y^1 \in F_\geq^N$ and $T(y^1)_{\text{top}} : (x^1)_{\text{top}} = 1$.

(iii). Any nonnegative $\Lambda$-th approximate eigenvector of $Y$ corresponds with $\rho(Y)$.

(iv). No $\Lambda$-th approximate eigenvalue $\lambda$ of $Y$, except for $\rho(Y)$, satisfies $|\lambda| \equiv \rho(Y) \mod I(v(Y - \lambda \cdot \text{id}) + \Lambda)$.

We call the vector $x^1$ a $\Lambda$-th approximate Perron vector.

**Remark 6.4.** If $\mathcal{X} = (Y; \lambda; \{x\}, \{y\})$ is an EQ with Perron-Frobenius property of depth 0, then $\lambda$ is the Perron root of the real matrix $Y_{\text{top}}$. The quasi-basis $\{x\}$ consists of one element $\{x\} = \{x^1\}$, where $x^1 \in \mathbb{R}^N$ is a (usual) Perron vector of $Y_{\text{top}}$.

For a real Puiseux matrix $X \in F^{M \times N}$ and subsets $\alpha \subset \{1, 2, \ldots, M\}$, $\beta \subset \{1, 2, \ldots, N\}$, we denote $X[\alpha, \beta]$ the submatrix of $X$ whose rows are indexed by $\alpha$ and whose columns are indexed by $\beta$. If $M = N$ and $\alpha = \beta$, write $X[\alpha] := X[\alpha, \alpha]$. 


**Definition 6.5.** Let $Y \in \mathbb{S}^{N \times N}$ be a subtraction-free matrix and $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a partition of $\{1, 2, \ldots, N\}$ (i.e., a collection of subsets $\alpha_i \subset \{1, 2, \ldots, N\}$ such that $\alpha_i \cap \alpha_j = \emptyset$ and $\bigcup_{i=1}^m \alpha_i = \{1, 2, \ldots, N\}$). A tuple

$$A := (Y; \mathcal{X}(\alpha_1), \ldots, \mathcal{X}(\alpha_m)),$$

where $\mathcal{X}(\alpha_i) = (Y[\alpha_i]; \lambda(\alpha_i); \{x(\alpha_i)\}, \{y(\alpha_i)\})$ (15) is said to be a **quasi Perron-Frobenius data** (or shortly, a **quasi PF-data**) associated with $Y$. The adjacency graph of a quasi PF-data $A$ is the weighted digraph $G(A) = (\text{node}(A), \text{arc}(A))$, which is defined as follows:

$$\text{node}(A) := \{\alpha_1, \ldots, \alpha_m\}, \quad (\alpha_i \xrightarrow{W} \alpha_j) \in \text{arc}(A) \iff W = v(Y[\alpha_j, \alpha_i]).$$

**Definition 6.6.** A quasi PF-data (15) is said to be **singular** if $\lambda(\alpha_1) = \lambda(\alpha_2) = \cdots = \lambda(\alpha_m)$.

The **adjacency graph of a quasi PF-data $A$** is the weighted digraph $G(A) = (\text{node}(A), \text{arc}(A))$ which is defined as follows:

$$\text{node}(A) := \{\alpha_1, \ldots, \alpha_m\}, \quad (\alpha_i \xrightarrow{W} \alpha_j) \in \text{arc}(A) \iff W = v(Y[\alpha_j, \alpha_i]).$$

**Definition 6.7.** A quasi PF-data $A$ (15) is said to be a **Perron-Frobenius data** (or shortly, a **PF-data**) if the Puiseux matrix $Y$ is decomposed as

$$Y \equiv Y[\alpha_1] \oplus \cdots \oplus Y[\alpha_m] + R^s(G(A)) \mod I(s(G(A))),$$

where $R \in \mathbb{R}^{N \times N}$.

**Remark 6.8.** From (16), we have $G(A) = G/\sim_{s(G(A))}$ if $A$ is a PF-data. See Lemma 4.1

### 6.3 Outline of the construction

In the sequel, we will construct a PF-data associated with a subtraction-free matrix of arbitrarily large depth. The construction consists of the following three processes:

1. **Process A:**
   - Input: A pair $(Y, A)$, where $Y$ is a subtraction-free matrix, $\alpha$ is a partition of $\{1, \ldots, N\}$ and $A$ is a PF-data associated with $(Y, \alpha)$.
   - Output: A pair $(Z, B)$, where $Z$ is a subtraction-free matrix and $B$ is a PF-data associated with $(Z, \alpha)$ such that: (i) $Y \sim Z$, (ii) $G(A) \sim G(B)$, (iii) $G(B)$ is flat-slanted and (iv) depth($A$) > depth($B$).

2. **Process B:**
   - Input: A pair $(Y, A)$, where $Y$ is a subtraction-free matrix, $\alpha$ is a partition of $\{1, \ldots, N\}$ and $A$ is a PF-data associated with $(Y, \alpha)$ such that (i) $A$ is non-singular and (ii) $G(A)_{\text{top}}$ is strongly-connected.
   - Output: A pair $(Y, B)$, where $B$ is a PF-data associated with $(Z, \beta)$, $\beta$ is a courser partition than $\alpha$ such that depth($B$) > depth($A$).

3. **Process C:**
   - Input: A pair $(Y, A)$, where $Y$ is a subtraction-free matrix, $\alpha$ is a partition of $\{1, \ldots, N\}$ and $A$ is a PF-data associated with $(Y, \alpha)$ such that (i) $A$ is singular, (ii) $A$ is of depth 0 and (iii) $G(A)_{\text{top}}$ is strongly-connected.
   - Output: A pair $(Y, B)$, where $B$ is a PF-data associated with $(Z, \{1, \ldots, N\})$ such that depth($B$) > depth($A$).

The rough flow of the construction is as follows. From Proposition 4.9 it is sufficient to consider a subtraction-free matrix $Y$ whose adjacent matrix is flat-slanted.
Algorithm 6.9. Let $Y$ be a subtraction-free matrix whose adjacent matrix is flat-slanted.

0-a. Since the adjacent matrix of $Y$ is flat-slanted, $Y$ is decomposed as $Y = Y[α_1] ⊕ ⋯ ⊕ Y[α_n] + o(t^β)$, where each $Y[α_i]$ is irreducible. Define $α := (α_1, ..., α_n)$ and $X(α_i) := (Y[α_i]; λ(α_i), \{x\}, \{y\})$, where $λ(α_i)$ is the (usual) Perron root of $Y[α_i]$, $x$ (resp. $y$) is a Perron vector of $Y[α_i]$ (resp. $t Y[α_i]$) such that $^t y \cdot x = 1$.

0-b. Define $\mathfrak{A} := (Y; X(α_1), ..., X(α_m))$, that is a PF-data associated with $(Y, α)$ of depth 0.

1. Operate Process A on the pair $(Y, \mathfrak{A})$ and obtain a new pair $(Z, \mathfrak{B})$. Then, update $(Y, \mathfrak{A}) ← (Z, \mathfrak{B})$.

2. Since $G(\mathfrak{A})$ is flat-slanted, $Y$ is re-decomposed as

$$Y = Y[α_1] ⊕ ⋯ ⊕ Y[α_n] + R(t^s(G(\mathfrak{A}))) + o(t^s(G(\mathfrak{A}))) = Y[β_1] ⊕ ⋯ ⊕ Y[β_l] + o(t^s(G(\mathfrak{A}))), \quad (17)$$

where $β = (β_1, ..., β_l)$ is a coarser partition than $α$. Define $\mathfrak{A}(β_i) := (Y[β_i]; X(α_k) | α_k ∈ β_i)$.

3. For each $β_i$, operate Process B on $(Y[β_i], \mathfrak{A}(β_i))$ if $\mathfrak{A}(β_i)$ is non-singular, or operate Process C if $\mathfrak{A}(β_i)$ is singular of depth 0. Then, obtain a new PF-data $\mathfrak{B}(β_i)$ and update $\mathfrak{A}(β_i) ← \mathfrak{B}(β_i)$.

4. Update $\mathfrak{A} ← (Y; X(β_1), ..., X(β_n))$.

5. Repeat 1–4.

6.4 The condition of genericness

Before proceeding to the proof of Theorem 6.1, we note the fact that Algorithm 6.9 fails if the decomposition (17) in the step 2 contains some $β_i$ such that $G(β_i)$ is singular and of depth greater than 0. Unfortunately, we would not deal with the case here because singular PF-datum of higher depth possess quite complicated behavior. In other words, we assume the following condition:

Condition of genericness: At each step of Algorithm 6.9 no singular PF-data of depth greater than 0 is derived.

Note that any linearly perturbed matrix $Y_0 + t Y_1$ satisfies this condition.

6.5 Process A

Let $Y = (x_{i,j}) ∈ S^{N×N}$ be a square subtraction-free matrix. The valuation matrix of $Y$ is the tropical matrix defined by $θ(Y) := (θ(x_{i,j})) ∈ T^{N×N}$. If there is no chance of confusion, we use the abbreviation $G(Y) = G(θ(Y))$, where $G(θ)$ is the adjacency graph of $θ(Y)$.

Assume $G(Y) ∼ H$ for some digraph $H$. By definition, there exists some $r_1, ..., r_N ∈ Q$ such that $H = \{S_1(r_1) ∘ ⋯ ∘ S_N(r_N)\} · G(Y)$. We define a subtraction-free matrix $Y^{H/G(Y)} := δ^{-1} Y δ$, where $δ = \text{diag}(t^{r_1}, ..., t^{r_N})$. It is easily proved that $G(Y^{H/G(Y)}) = H$.

Lemma 6.10. Let $G = G(Y)$ be the adjacency matrix of $θ(Y)$ and $G(\mathfrak{A})$ be the adjacency graph of a PF-data $\mathfrak{A}$ associated with $Y$. Then, there exists a multi-condensation map $G ▶ G(\mathfrak{A})$ of weighted digraphs.

Proof. Define the map $π : \text{node}(G) → \text{node}(G(\mathfrak{A}))$ by $π(x) = α_i$ $⇔$ $x ∈ α_i$. From (16), the map $π$ induces the quotient map $G → G(\mathfrak{A}) = G/∼_{s(G(\mathfrak{A}))}$ (see Lemma 4.1). Therefore, this is a multi-condensation map.

Consider a PF-data $\mathfrak{A} = (Y; X(α_1), ..., X(α_m))$. Let $H$ be a weighted digraph with $G(\mathfrak{A}) ≃ H.$

Therefore, there exists a digraph $H$ obtained from the diagram (12): $↓ \quad \quad \downarrow$ Define $\mathfrak{A}^{H/G(\mathfrak{A})} := (Y^{H/G}; X(α_1), ..., X(α_m))$. Because $Y^{H/G}[α_i] = Y[α_i]$ (see Remark 4.3), the tuple $\mathfrak{A}^{H/G(\mathfrak{A})}$ satisfies the conditions in (15).
Lemma 6.11. If \( s(G) < s(\mathcal{H}) \), then \( \mathfrak{A}^{\mathcal{H}/G(\mathfrak{A})} \) is a PF-data whose adjacency matrix is \( \mathcal{H} \).

Proof. By Lemmas 6.5 and 6.10, the map \( H \to \mathcal{H} \) is a multi-condensation map. This implies \( Y^{H/G} - Y[\alpha_1] \oplus \cdots \oplus Y[\alpha_m] = o(t(s(\mathcal{H}))) \).

Corollary 6.12. For any subtraction-free square matrix \( Y \) and a PF-data \( \mathfrak{A} \) associated with \( Y \), there exists a new subtraction free matrix \( Z \) and a PF-data \( \mathfrak{B} \) associated with \( Z \) such that: (i) \( Y \sim Z \), (ii) \( G(\mathfrak{A}) \sim G(\mathfrak{B}) \), (iii) \( G(\mathfrak{B}) \) is flat-slanted and (iv) \( \Lambda(\mathfrak{A}) = \Lambda(\mathfrak{B}) \).

Proof. It is enough to define \( \mathfrak{B} := \mathfrak{A}^{\mathcal{H}/G(\mathfrak{A})} \), where \( \mathcal{H} \) is flat-slanted.

6.6 Process B

Let \( \mathfrak{A} = (Y; \mathcal{X}(\alpha_1), \ldots, \mathcal{X}(\alpha_m)) \) be a PF-data where \( G(\mathfrak{A})_{\text{top}} \) is strongly connected. Without loss of generality, we can assume \( \lambda(\alpha_1) = \cdots = \lambda(\alpha_k) > \lambda(\alpha_{k+1}) \geq \cdots \geq \lambda(\alpha_m) \).

Let \( E := \{\alpha_1, \ldots, \alpha_m\} \). From Proposition 5.2 there exists an \( E \)-gently-slanted digraph \( \mathcal{H} \) with \( G(\mathfrak{A}) \sim \mathcal{H} \). Define \( (Z, \mathfrak{B}) := (Y^{\mathcal{H}/G(\mathfrak{A})}, \mathfrak{A}^{\mathcal{H}/G(\mathfrak{A})}) \). Remember that the weights of the gently-slanted graph depend on a small parameter \( \epsilon \) and \( s(G(\mathfrak{B})) = s(G(\mathfrak{A}))/\epsilon \), \( \lim_{\epsilon \to 0} = G(\mathfrak{A}) \).

By definition of \( E \)-forest, \( k \) is the number of connected components of \( G(\mathfrak{B})_{\text{top}} \). The matrix \( Z \) is decomposed as \( Z = Z_1 \oplus \cdots \oplus Z_k + o(t(s(G(\mathfrak{B})))) \), where each submatrix \( Z_i \) is associated with the connected component \( C_i \) of \( G(\mathfrak{B})_{\text{top}} \) with \( \alpha_i \in C_i \).

The submatrix \( Z_i \) is decomposed as

\[
Z_i = \begin{pmatrix}
Y[\alpha_1] & Y[\alpha_{z_1}] & \cdots & Y[\alpha_{z_{k'}}]
\end{pmatrix} + R_i t^{s(G(\mathfrak{A}))} - \epsilon + o(t(s(G(\mathfrak{A}))))
\]

where \( C_i = \{\alpha_1, \alpha_{z_1}, \ldots, \alpha_{z_{k'}}\} \) and \( R_i \neq O \) is a non-negative real matrix.

Lemma 6.13. Let \( R_i[\alpha, \beta] \) be the submatrix of \( R_i \) associated with index sets \( \alpha, \beta \). Then, \( R_i[\alpha_i, \alpha_{z_{u}}] = O \) for all \( u \).

Proof. This is a consequence of the fact that \( G(\mathfrak{B})_{\text{top}} \) contains no arrow whose end node is \( \alpha_i \).

In other words, \( R_i \) is expressed as

\[
R_i = \begin{pmatrix}
O & \vdots \\
\vdots & O
\end{pmatrix}
\]

We denote \( W_i := Y[\alpha_{z_1}] \oplus \cdots \oplus Y[\alpha_{z_{k'}}] \). Since all the eigenvalues of \( Y[\alpha_{z_i}] \) are smaller than \( \lambda(\alpha_i) \), the matrix \( (W - \lambda(\alpha_i) \cdot \text{Id}) \) is regular. Define

\[
p_i := x^1(\alpha_i) \oplus (\lambda(\alpha_i) - W)^{-1} P_i x^1(\alpha_i) t^{s(G(\mathfrak{A}))},
\]

where \( x^1(\alpha_i) \) be the approximate Perron vector of \( Y[\alpha_i] \). It is soon checked that the vector \( p_i \) satisfies

\[
Z_i p_i = \lambda(\alpha_i) p_i + o(t(s(G(\mathfrak{A})))), \tag{18}
\]

From general arguments about real closed fields, we can prove the matrix \( (\lambda(\alpha_i) - W)^{-1} \) and the vector \( p_i \) are nonnegative. (See Lemma 5.4.)

Similarly, we can prove

\[
^T Z_i q_i = \lambda(\alpha_i) q_i + o(t(s(G(\mathfrak{A})))), \tag{19}
\]

where \( q_i = y^1(\alpha_i) \oplus 0 \).
Define the partition $\beta = (\beta_1, \ldots, \beta_k)$ by $\beta_i := \alpha_i \cup \alpha_{z1} \cup \cdots \cup \alpha_{zq}$. Among $\beta_i$ approximate eigenvectors of $Y[\alpha_1] \oplus W$, pick up $(\beta_i - 1)$ vectors $x^2(\beta_i), x^3(\beta_i), \ldots, x^{\beta_i}(\beta_i)$ arbitrarily, and similarly, among $\beta_i$ approximate eigenvectors of $T Y[\alpha_1] \oplus T W$, pick up $(\beta_i - 1)$ vectors $y^2(\beta_i), y^3(\beta_i), \ldots, y^{\beta_i}(\beta_i)$ arbitrarily.

It is directly checked that the following quintuple is an EQ of depth $\Lambda + \epsilon$:

$$X(\beta_i) := (Z_i; \lambda(\alpha_i), \{x(\beta_i)\}, \{y(\beta_i)\}),$$

where $\{x(\beta_i)\} := \{p_i, t^* + \theta x^2(\beta_i), \ldots, t^* + \theta x^{\beta_i}(\beta_i)\}$ and $\{y(\beta_i)\} := \{q_i, t^* + \theta y^2(\beta_i), \ldots, t^* + \theta y^{\beta_i}(\beta_i)\}$.

From \(\mathfrak{B}\), the tuple $\mathfrak{B} := (Z; X(\beta_1), \ldots, X(\beta_k))$ is a PF-data associated with the pair $(Z, \beta)$ of depth $\text{depth}(\mathfrak{A}) + \epsilon$.

### 6.7 Process C

Let $\mathfrak{A} = (Y; X(\alpha_1), \ldots, X(\alpha_m))$ be a PF-data of depth 0, where $G(\mathfrak{A})_{\text{top}}$ is strongly connected and $\lambda(\alpha_1) = \cdots = \lambda(\alpha_m)(:= \lambda)$.

Since $X(\alpha_i)$ is of depth 0, we have $X(\alpha_i) = (Y[\alpha_i], \lambda; \{x^i\}, \{y^i\})$, where $x^i, y^i \in \mathbb{R}^{m_i}$ are entry-wise positive vectors (see Remark 6.4). Define the piece-wise nonnegative vectors $X^i$ and $Y^i$ by

$$X^i = 0 \oplus \cdots \oplus 0 \oplus x^i \oplus 0 \oplus \cdots \oplus 0, \quad (1 \leq i \leq m),$$

$$Y^i = 0 \oplus \cdots \oplus 0 \oplus y^i \oplus 0 \oplus \cdots \oplus 0, \quad (1 \leq i \leq m).$$

Then, the quadruple

$$X := (Y; \lambda; \{X\}, \{Y\}),$$

is an EQ of depth 0.

**Lemma 6.14.** The principal matrices $\Delta(X)$ and $\Omega(X)$ are of the form:

$$\Delta(X) = \text{(an irreducible matrix whose entries except for diagonal elements are non-negative)},$$

$$\Omega(X) = \text{Id.}$$

**Proof.** Without loss of generality, the valuation of $Y$ can be assumed to be 0. The matrix $Y$ is expanded as

$$Y = Y_0 + t^{s(G(\mathfrak{A}))} Y' + \cdots.$$

For $i \neq j$, the $(i, j)$-entry of $\Delta(X)$ is calculated as $T y^i Y'[\alpha_j, \alpha_i] x^i$, which is nonnegative. Moreover, it follows that

$$T y^i Y'[\alpha_j, \alpha_i] x^i \neq 0 \iff Y'[\alpha_j, \alpha_i] \neq O \iff G(\mathfrak{A})_{\text{top}} \text{ contains } (i \to j).$$

This relation implies that $\Delta(X)$ is irreducible if and only if $G(\mathfrak{A})_{\text{top}}$ is strongly connected. This implies the first equation. The second equation follows from the definition of EQs.\[\square\]

By the classical Perron-Frobenius theorem (Theorem 1.1), the matrix equations $\Delta(X)c = \mu \Omega(X)c$ and $T \Delta(X)d = \mu T \Omega(X)d$ have the Perron root $\mu^* > 0$ and Perron vectors with $T d \cdot c = 1$. By Corollary 2.12 we obtain the new EQ

$$X_+ = (Y; \lambda + \mu^* t^{s(G(\mathfrak{A}))}; \{x^1, t^0 x^2, \ldots, t^0 x^g\}, \{y^1, t^0 y^2, \ldots, t^0 y^g\}),$$

where

$$(x^1)_{\text{top}} = (x^1, \ldots, x^m) \cdot c, \quad (y^1)_{\text{top}} = (y^1, \ldots, y^m) \cdot d.$$ 

It is soon proved that $X$ has the Perron-Frobenius property, and therefore, the pair $(Y, X)$ is a PF-data of depth $s(G(\mathfrak{A})) > 0$.  

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7 Examples

7.1 Example I

Let us consider the positive Puiseux matrix $Y = \begin{pmatrix} a & bt & ct^2 \\ dt^2 & et^3 & f \\ g & h & kt \end{pmatrix}$, where $a, b, c, d, e, f, g, h, k > 0$, whose valuation matrix is $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Its adjacency matrix is $G = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 3 & 0 \end{pmatrix}$. Define $H := S_1(1/2) \cdot G = \begin{pmatrix} 0 & \frac{5}{2} & 2 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$. Hence, the digraph $H$ is flat-slanted, whose leading term is a disjoint union of strongly connected components $\alpha_1 = \{1\}$ and $\alpha_2 = \{2, 3\}$.

Let $Z := Y^{H/G} = \begin{pmatrix} a & bt^{1/2} & ct^{3/2} \\ dt^{5/2} & et^3 & f \\ g t^{1/2} & h & kt \end{pmatrix}$. We have $Z[\alpha_1] = (a)$ and $Z[\alpha_2] = \left( \begin{array}{cc} ct^3 & f \\ h & kt \end{array} \right)$. From the classical Perron-Frobenius theorem, we can calculate EQs $\mathcal{X}[\alpha_1], \mathcal{X}[\alpha_2]$ as $\mathcal{X}[\alpha_1] = (a; a; \{1\}, \{1\}; 0)$ and $\mathcal{X}[\alpha_2] = \left( Z[\alpha_2]; \sqrt{f h}; \left\{ \left( \frac{\sqrt{h}}{\sqrt{f}} \right) \right\}, \left\{ \frac{1}{2\sqrt{f h}} \left( \frac{\sqrt{h}}{\sqrt{f}} \right) \right\}; 0 \right)$. Then, $A = (Y; \mathcal{X}[\alpha_1], \mathcal{X}[\alpha_2])$ is a PF-data of depth 0.

The behavior of a PF-data of greater depth depends on the magnitudes of $a$ and $\sqrt{f h}$.

7.1.1 The case if $a = \sqrt{f h}$

Since $T^1[\alpha_2] Y[\alpha_2, \alpha_1] x^1[\alpha_1] = \frac{g}{2\sqrt{h}} t^{1/2} + \frac{d}{2\sqrt{f}} t^{5/2}$ and $T^1[\alpha_1] Y[\alpha_1, \alpha_2] x^1[\alpha_2] = b\sqrt{f} t^{1/2} + c\sqrt{h} t^{3/2}$, we have $G(\mathfrak{A}) = \left( \alpha_1 \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}, \alpha_2 \right)$. Because $G(\mathfrak{A})$ is already flat-slanted, we only need to consider one strongly connected component $\beta = \{\alpha_1, \alpha_2\}$.

Let $\varphi : \text{Im}(Y_0 - a \cdot \text{Id}) \to \mathbb{C}^3$ be the linear map defined by $e_2 - \sqrt{\frac{f}{h}} e_3 \mapsto \frac{1}{2} e_3$. Define

$\mathcal{X} = X(\beta) = \left( Z[\alpha_1] \oplus Z[\alpha_2]; a; \left\{ (1) \oplus 0, (0) \oplus \left( \frac{\sqrt{f}}{\sqrt{h}} \right) \right\}, \left\{ (1) \oplus 0, (0) \oplus \frac{1}{2\sqrt{f h}} \left( \frac{\sqrt{h}}{\sqrt{f}} \right) \right\} \right)$. Then we have $P(\mathcal{X}) = \begin{pmatrix} 0 & b\sqrt{f} \\ \frac{g}{2\sqrt{h}} & 0 \end{pmatrix}$ and $Q(\mathcal{X}) = \text{Id}$. By the classical Perron-Frobenius theorem, the matrix equation $P(\mathcal{X}) c = \mu Q(\mathcal{X}) c$ has the Perron root $\mu = \sqrt{\frac{g b}{2}\sqrt{f h}}$. It is soon checked that the vector $c = T(\sqrt{2b\sqrt{f h}}, \sqrt{g})$ is a corresponding eigenvector. On the other hand, the transposed equation
The case if $a > \sqrt{f\lambda}$.

Recall that $G(\Delta) = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right)$. According to the inequality $a > \sqrt{f\lambda}$, we define $E = \{\alpha_1\}$.

Through the similarity transformation $F = S_{\alpha_1}(-1/4) \cdot G(\Delta) = \left( \begin{array}{c} \alpha_1 \\ \frac{3}{4} \alpha_2 \end{array} \right)$, we obtain the E-gently-slanted digraph $F$. Let $F$ be the digraph defined by the diagram $\Downarrow \Downarrow$. (To be more precise, $F := \left( \begin{array}{ccc} 0 & 9/4 & 2 \\ 7/4 & 3/4 & 1 \\ 1/4 & 0 & 3 \end{array} \right)$.)

Set $W := Y^{F/G} = \left( \begin{array}{c} a \\ bt^{3/4} \\ gt^{1/4} \end{array} \right)$. Then, $(W; \mathcal{X}(\alpha_1), \mathcal{X}(\alpha_2))$ is a PF-data. Because $a > \sqrt{f\lambda}$, we have $(a - W[\alpha_2])^{-1} = 1/a \left( \frac{W[\alpha_2]}{a} + \frac{(W[\alpha_2])^2}{a^2} + \cdots \right) = \left( \frac{\alpha}{a} \right) t^{1/4} + \cdots > 0$. Therefore, the quintuple $\mathcal{X} = (W; a; \{x^1, t^0 x^2\}, \{y^1, t^0 y^2\})$, where

\[ x^1 = T(1, f \frac{g t^{1/4}}{a}, \frac{g}{a} t^{1/4}), \quad x^2 = T(0, \sqrt{f}, \sqrt{h}), \quad y^1 = T(1, 0, 0) + \cdots, \quad y^2 = \frac{1}{2\sqrt{f\lambda}} T(0, \sqrt{h}, \sqrt{f}) \]

is a EQ with Perron-Frobenius property, and the pair $(W; \mathcal{X})$ is a PF-data of depth 1/4.
In most cases, our method is applicable for non-negative Puiseux matrices. Let

$$G := S_{\alpha_2}(-1/4) \cdot G(\mathfrak{A}) = \left( \begin{array}{c} \alpha_1 \frac{3/4}{1/4} \alpha_2 \end{array} \right).$$

Define $F$ by the diagram

$$G \sim F \Downarrow \Downarrow \sim F.$$ 

Set $W := Y^{F/G} = \left( \begin{array}{ccc} a & b t_{1/4} & c t_{5/4} \\ dt_{11/4} & e t_{3} & f \\ g t_{3/4} & h & k t \end{array} \right).$ Then, $(W; \mathcal{X}(\alpha_1), \mathcal{X}(\alpha_2))$ is a PF-data. Because $a < \sqrt{f h}$, we have $(\sqrt{f h} - a)^{-1}(b t_{1/4}, c t_{5/4}) \cdot (\sqrt{f h}) = \frac{b \sqrt{f} t_{1/4}}{\sqrt{f h} - a} + \cdots > 0$. The quintuple $\mathcal{X} = (W; \sqrt{f h}; (x^1, t^0 x^2), (y^1, t^0 y^2))$, where

$$x^1 = T \left( \frac{b \sqrt{f}}{\sqrt{f h} - a} t_{1/4}, \sqrt{f h}, \sqrt{f h} \right), \quad x^2 = e_1, \quad y^1 = \frac{1}{2 \sqrt{f h}} T(0, \sqrt{f}, \sqrt{f}) + \cdots, \quad y^2 = e_1$$

is an EQ with Perron-Frobenius property, and the pair $(W; \mathcal{X})$ is a PF-data of depth $1/4$.

### 7.2 Example II

In most cases, our method is applicable for non-negative Puiseux matrices. Let

$$Y = \begin{pmatrix} \lambda_0 & 1 & \lambda_0 & 1 \\ t^{A_1} & \lambda_0 & 1 \\ t^{A_2} & \lambda_0 & \cdots & \lambda_0 \\ \vdots & \ddots & \ddots & 1 \\ t^{A_{N-1}} & \lambda_0 \end{pmatrix}, \quad 0 < A_1 < A_2 < \cdots < A_{N-1}, \quad \lambda_0 > 0.$$ 

Then, the digraph $G := G(Y)$ is expressed as

$$6 \bigcirc 1 \bigcirc \begin{array}{c} A_1 \bigcirc 2 \bigcirc A_2 \bigcirc 3 \bigcirc A_3 \bigcirc \cdots \bigcirc A_{N-1} \bigcirc N \bigcirc 0 \end{array}.$$ 

Define $H := \{S_2(\alpha_2) \circ S_3(\alpha_3) \circ \cdots \circ S_N(\alpha_N)\} \cdot G$, where $\alpha_n = \frac{3}{2} - \frac{2}{2} A_1 + (A_2 + A_3 + \cdots + A_{N-1}) - (n-2)\epsilon$ and $\epsilon > 0$. The digraph $H$ is expressed as

$$0 \bigcirc 1 \bigcirc \begin{array}{c} \frac{A_1}{2} \bigcirc 2 \bigcirc \frac{A_1 + \epsilon}{2} \bigcirc 3 \bigcirc \frac{A_1 + \epsilon}{2} \bigcirc \cdots \bigcirc \frac{A_1 + \epsilon}{2} \bigcirc N \bigcirc 0 \end{array}.$$ 

The digraph $H$ is a flat-slanted whose leading term is a disjoint union of strongly connected components $\alpha_1 = \{1, 2\}, \alpha_2 = \{3\}, \ldots, \alpha_{N-1} = \{N\}$. Let

$$Z := Y^{H/G} = \begin{pmatrix} \lambda_0 & t^{A_1} & \lambda_0 & t^{A_2 - \frac{A_1}{2} - \epsilon} \\ t^{\frac{A_1}{2}} & \lambda_0 & t^{A_2 - \frac{A_1}{2} - \epsilon} & \lambda_0 \\ \frac{A_1 + \epsilon}{2} & \lambda_0 & \cdots & \lambda_0 \\ \frac{A_1 + \epsilon}{2} & \cdots & \frac{A_1 + \epsilon}{2} & \lambda_0 \end{pmatrix}.$$ 

Consider the quintuples

$$\mathcal{X}[\alpha_1] = (Z[\alpha_1]; \lambda_0 + t^{\frac{A_1}{2}}; \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), t^0 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} 1 \\ 2 \end{array} \right), t^0 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}).$$

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and $\mathcal{X}[\alpha_n] := (\lambda_0; \lambda_0; \{1\}, \{1\}) \ (n \geq 2)$. Then, $\mathfrak{A} := (Z; \mathcal{X}(\alpha_1), \ldots, \mathcal{X}(\alpha_N))$ is a PF-data. Because $\lambda_0 + t^{\alpha_1}$ is strictly greater than $\lambda_0$, $\mathfrak{A}$ is non-singular.

Let $E = \{\alpha_1\}$. The adjacency matrix $G(\mathfrak{A})$ is expressed as

$$
\begin{pmatrix}
\alpha_1 & \frac{\alpha_1}{A_2-\frac{\alpha_1}{A_2-\epsilon}} & \alpha_2 & \frac{\alpha_2}{A_3-\frac{\alpha_1}{A_2-\epsilon}} & \cdots & \frac{\alpha_2}{A_{N-1}-\frac{\alpha_1}{A_2-\epsilon}} & \alpha_{N-1}
\end{pmatrix}.
$$

Note that $G(\mathfrak{A})$ is already a $E$-gently-slanted.

Let $\beta_1 := \{\alpha_1\}, \beta_2 := \{\alpha_2, \ldots, \alpha_{N-1}\}$ If $\epsilon$ is sufficiently small, we have

$$
((\lambda_0 + t^{\alpha_1}) - Z[\beta_2])^{-1} \begin{pmatrix} 0 & t^{\alpha_1+\epsilon} \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} t^\epsilon \\ t^{2\epsilon} \\ \vdots \\ t^{(N-2)\epsilon} \end{pmatrix} \mod I((N-2)\epsilon).
$$

Let

$$
\begin{align*}
{x}^1 & := T(1, 1, t^\epsilon, t^{2\epsilon}, \ldots, t^{(N-2)\epsilon}), \\
{x}^2 & := e_2 t^{(N-2)\epsilon}, \ldots, \\
{x}^N & := e_N t^{(N-2)\epsilon}, \\
{y}^1 & := T(\frac{1}{2}, 1, 0, \ldots, 0) + \cdots, \\
{y}^2 & := e_2 t^{(N-2)\epsilon}, \ldots, \\
{y}^N & := e_N t^{(N-2)\epsilon}.
\end{align*}
$$

By the arguments in (2.4), the quintuple $\mathcal{X} = (Z; \lambda_0 + t^{\alpha_1}; \{x^1, t^\theta x^1\}, \{y^1, t^\theta y^1\})$ is an EQ with Perron-Frobenius property. Finally we obtain the PF-data $(Z; \mathcal{X})$ of depth $(N-2)\epsilon$.

### 8 Conclusion and future problems

In this paper, we have constructed a combinatorial method to calculate the Perron roots/vectors of positive Puiseux matrices. Our method is applicable for the large class of positive Puiseux matrices with the condition of genericness (6.3). We expect our result to evoke new problems about the structure of the ring of real Puiseux matrices, tropical matrices, Puiseux matrices over a real closed field, etc. The following is a list of future problems:

**Non-negative real Puiseux matrices**

As introduced in (7.2), the Perron root/vector of non-negative matrix can be calculated by our method in most cases. It should be interesting to find an algorithm to calculate them for general non-negative matrices.

**Weak Perron-Frobenius property**

A real (might be negative) square matrix is said to possess the weak Perron-Frobenius property [15] if its spectral radius is a positive eigenvalue corresponding to non-negative left and right eigenvectors. Our method can construct many examples of these matrices. For example, the matrix $Y$ in Example 2.16 possesses the weak Perron-Frobenius property for $t = -\epsilon$ ($0 < \epsilon \ll 1$). Many of topological properties of the set of matrices with the weak Perron-Frobenius property have been derived by perturbation methods [8]. It is expected that more detailed shape of this set will be determined.

### A The Tarski-Seidenberg principle

#### A.1 General facts about real closed fields

In this section, several basic facts about real close fields are introduced. The contents of this section are based on the textbook [3 §1,2,3].
A field $k$ is said to be real closed if (i) there exists a total order $\geq$ so that $k = (k, \geq)$ to be an ordered field, and (ii) the intermediate value theorem holds for all polynomials over $k$. The real field $\mathbb{R}$ is a typical example of real closed fields. Any real closed field contains $\mathbb{R}$ as a subfield. An element $x \in k$ is positive if $x > 0$.

Let $k$ be a real closed field. It is known that $\sqrt{-1} \notin k$. Especially, $k$ is not algebraically closed. It is also well-known that the field $K := k + k\sqrt{-1}$ is algebraically closed. By the intermediate value theorem, any positive element in $k$ is a square element. Therefore, the sign $\sqrt{\cdot}$ has a meaning over $k$.

Any element of $x \in K$ is expressed as $x = \alpha + \beta\sqrt{-1}$ $(\alpha, \beta \in k)$ uniquely. We define the norm of $x$ by $|x| := \sqrt{\alpha^2 + \beta^2}$.

Let $D$ be an ordered ring. We define the language of ordered fields with coefficients in $D$ as follows [4, p.58]. By induction, we give the definition of formulas and the set $\text{Free}(\Phi)$ of free variables of formula $\Phi$:

1. An atom is $(P > 0)$ or $(P = 0)$, where $P \in D[X_1, \ldots, X_k]$. An atom is a formula. Set $\text{Free}(P > 0) = \text{Free}(P = 0) := \{X_1, \ldots, X_k\}$.
2. If $\Phi_1$ and $\Phi_2$ are formulas, then $\Phi_1 \land \Phi_2$ and $\Phi_1 \lor \Phi_2$ are formulas. Define $\text{Free}(\Phi_1 \land \Phi_2) = \text{Free}(\Phi_1 \lor \Phi_2) := \text{Free}(\Phi_1) \cup \text{Free}(\Phi_2)$.
3. If $\Phi$ is a formula, then $\neg \Phi$ is a formula with $\text{Free}(\neg \Phi) := \text{Free}(\Phi)$.
4. If $\Phi$ is a formula and $X \in \text{Free}(\Phi)$, then $(\forall X)\Phi$ and $(\exists X)\Phi$ are formulas with $\text{Free}((\forall X)\Phi) = \text{Free}((\exists X)\Phi) := \text{Free}(\Phi) \setminus \{X\}$.

For two formulas $\Phi_1$ and $\Phi_2$, $\Phi_1 \rightarrow \Phi_2$ is the formula $(\neg \Phi_1) \lor \Phi_2$. A formula is said to be quantifier free if it contains no quantifier $(\forall, \exists)$. A sentence is a formula without free variable.

Let $\Phi$ be a formula in the language of ordered fields with coefficients in $D$ and $k$ be an ordered field such that $D \subset k$. The realization of $\Phi$ in $k$ is the formula which is obtained from $\Phi$ by replacing "$\forall X$" to "$\forall X \in k$" and "$\exists X$" to "$\exists X \in k$". Denote $\Phi|_k$ the realization of $\Phi$ in $k$. A sentence $\Phi$ is said to be true in $k$ if $\Phi|_k$ is true. A proof of $\Phi$ in $k$ is a sequence of formulas ($\Phi_1, \ldots, \Phi_n$) such that, for all $i$, (i) $\Phi_i|_k$ is an axiom of real closed field $k$ or (ii) $\Phi_i|_k$ can be derived from $\Phi_1|_k, \ldots, \Phi_{i-1}|_k$ by inference rule. A formula $\Phi$ is provable in $k$ if there exists a proof of $\Phi$ in $k$.

The following theorems are quite important [4, Theorems 2.77 and 2.80]:

**Theorem A.1** (Quantifier elimination). Let $k$ be a real closed field and $D$ be an ordered subring of $k$. If a sentence $\Phi$ in the language of ordered fields with coefficients in $D$ is true in $k$, then it is provable in $k$.

**Theorem A.2** (Tarski-Seidenberg principle). Let $k$ and $k'$ be real closed fields and $D$ be an ordered ring such that $D \subset k \subset k'$. If $\Phi$ is a sentence in the language of ordered fields with coefficients in $D$, then

$$\Phi \text{ is true in } k \iff \Phi \text{ is true in } k'.$$

**Corollary A.3.** Let $\Phi$ be a sentence in the language of ordered fields with coefficients in $\mathbb{Z}$. If $\Phi$ is true in $\mathbb{R}$, then $\Phi$ is true in any real closed field.

**A.2 An example how to use the Tarski-Seidenberg principle**

The following is a typical example of the application of the Tarski-Seidenberg principle.

**Lemma A.4.** Let $X \in k^{N \times N}$ be a square matrix all the entries of which are nonnegative. Denote $\lambda_1, \ldots, \lambda_N \in k$ the eigenvalues of $X$. If $\mu > |\lambda_n|$ for all $n$, all the entries of the matrix $(\mu \cdot \text{Id} - X)^{-1}$ are nonnegative.
Proof. Let $X_{i,j}$, $Y_{i,j}$ ($i, j = 1, \ldots, N$), $A$ and $B$ denote intermediates. Let $f(A, B, X), g(A, B, X) \in \mathbb{Z}[A, B, X_{i,j}]$ be polynomials such that $\det((A + B\sqrt{-1})\text{Id} - X) = f(A, B, X) + g(A, B, X)\sqrt{-1}$. Consider the formula $\Phi$ which is described as follows:

$$\forall X_{i,j}, \forall Y_{i,j}, \forall \mu, \forall A, \forall B \left( \bigwedge_{i,j} (X_{i,j} \geq 0) \land ((f(A, B, X) = 0 \land g(A, B, X) = 0) \rightarrow (\mu^2 > A^2 + B^2)) \land \bigwedge_{i,k} \left( \sum_{j=1}^{N} Y_{i,j} (\mu\delta_{j,k} - X_{j,k}) = \delta_{i,k} \right) \right) \rightarrow \bigwedge_{i,j} (Y_{i,j} \geq 0).$$

The realization of $\Phi$ in $k$ is nothing but the statement of the lemma. Because $\Phi$ is a sentence in the language of ordered fields with coefficients in $\mathbb{Z}$, $\Phi$ is true in $k$ if and only if $\Phi$ is true in $\mathbb{R}$. As it is soon proved that $\Phi$ is true in $\mathbb{R}$, $\Phi$ is also true in any $k$. □

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