Adaptive Estimation
In High-Dimensional Additive Models
With Multi-Resolution Group Lasso

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Abstract

In additive models with many nonparametric components, a number of regularized estimators have been proposed and proven to attain various error bounds under different combinations of sparsity and fixed smoothness conditions. Some of these error bounds match minimax rates in the corresponding settings. Some of the rate minimax methods are non-convex and computationally costly. From these perspectives, the existing solutions to the high-dimensional additive nonparametric regression problem are fragmented. In this paper, we propose a multi-resolution group Lasso (MR-GL) method in a unified approach to simultaneously achieve or improve existing error bounds and provide new ones without the knowledge of the level of sparsity or the degree of smoothness of the unknown functions. Such adaptive convergence rates are established when a prediction factor can be treated as a constant. Furthermore, we prove that the prediction factor, which can be bounded in terms of a restricted eigenvalue or a compatibility coefficient, can be indeed treated as a constant for random designs under a nearly optimal sample size condition.

Keywords: Additive model; Sobolev space; Reproducing kernel Hilbert space; Model selection; Prediction; Adaptive estimation.

1 Introduction

Additive model (AM) dates back to the 1980’s when the curse of dimensionality was a major concern in nonparametric regression, as the multivariate nonparametric model

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generally fails when there are not enough observations to fit a moderately smooth but otherwise unrestricted function of multiple inputs. The AM (Friedman and Stuetzle, 1981; Stone, 1985; Hastie and Tibshirani, 1986) can be written as

\[ y_i = \sum_{j=1}^{p} f_j(x_{i,j}) + \varepsilon_i, \quad i = 1, \ldots, n, \]  

(1)

where \( \varepsilon_i \) is the noise, \( x_{i,j} \) is the \( j \)-th design variable with the \( i \)-th sample point, and each \( f_j(\cdot) \) is an unknown univariate function satisfying a certain smoothness condition. Hereafter, we will call each \( f_j \) a component function and their sum the regression function.

For fixed \( p \), Stone (1985) established a squared error rate \( n^{-2\alpha/(2\alpha+1)} \) for the spline estimates of the regression function in (1), where \( \alpha \) is the assumed degree of smoothness of the underlying functional class. During the same period, numerous advances emerged in the study of specific additive non- and semi-parametric models and their extensions (Engle et al., 1986; Wahba, 1986; Hastie and Tibshirani, 1987). Still, model (1) remains among the most widely used and intensively studied AM owing to its balance of flexibility, interpretability, and tractability. When accounting for interaction terms, (1) leads to the smoothing spline analysis of variance (SS-ANOVA) model (Wahba et al., 1995). When certain smoothness conditions are imposed on each component function, (1) is approximately the sum of several univariate smooth splines (Wahba, 1990; Wood, 2004). Along with the theoretical explorations of model (1), several algorithms have been proposed to solve the nonparametric AM problem including the alternating conditional expectations (Breiman and Friedman, 1985) and backfitting (Buja et al., 1989). These algorithms construct a general iterative framework where in each iteration the component functions are estimated by local smoothing (Hastie and Tibshirani, 1986), maximum likelihood (Hastie and Tibshirani, 1987) and other methods. When the component functions are restricted to smoothing splines, one may use generalized cross validation and generalized maximum likelihood (Craven and Wahba, 1978; Wahba, 1985; Gu, 1990; Gu and Wahba, 1991). Iteratively re-weighted least squares (Green, 1984) and restricted maximum likelihood (Wood, 2011) are available when the AM is interpreted as a generalized linear model.

Upon the emergence of the SS-ANOVA, researchers started to concern about the problem of component selection, removing the component functions that explain a little fraction of the model. The early approaches include the angle cosine between the residue vector and the components (Gu, 1992), Bayesian prior on the coefficients (Yau et al., 2003) or the component functions (Wood et al., 2002), sparse kernels (Gunn and Kandola, 2002) and likelihood basis pursuit (Zhang et al., 2004). Despite the reported efficiency in performance, these approaches are either hard to implement or short of theoretical guarantees. Component selection is also the main theme in the study of high-dimensional sparse additive models where the sample size \( n \) is much
smaller than the number of additive components $p$ but much larger than the number of non-zero components. To seek a parsimonious model with better interpretation, component functions with little explanatory contribution should be eliminated.

High-dimensional AM has been very active topic. Several different approaches have been introduced in various contexts, providing distinct interpretations. Lin and Zhang (2006) proposed the COmponent Selection and Smoothing Operator (COSSO) in the context of reproducing kernel Hilbert space (RKHS) which essentially minimizes a penalized empirical risk function with the penalty term being proportional to the sum of the RKHS norms. The COSSO has the interpretations as a smoothing spline regularized by nonnegative garrote (Breiman, 1995) as well as a generalization of the Lasso, and can be computed by iterations between fitting the smoothing spline and solving the nonnegative garrote. Both Yuan (2007) and Ravikumar et al. (2009) proposed to apply a generalization of the nonnegative garrote to the sparse AM with penalization schemes analogous to that in Lin and Zhang (2006), but different algorithms were implemented. In an selection consistency analysis, Ravikumar et al. (2009) interpreted their scheme as a functional version of the group Lasso after transforming each component function in basis expansion. Huang et al. (2010) applied group Lasso to basis expansions of the components $f_j$ and provided sufficient conditions for selection consistency.

Among more recent studies of the sparse AM, this paper is most closely related to the following papers. To abbreviate the discussion, assume here that the AM (1) holds for $f_j(\cdot) = f_j^*(\cdot)$ and iid $\varepsilon_i \sim N(0, \sigma^2)$ and that $f_j^*(\cdot)$ are uniformly $\alpha$-smooth in the sense that the $L_2$ norms of the $\alpha$-th derivative of $f_j^*(\cdot)$ (or the RKHS norms) are uniformly bounded, along with additional side conditions imposed in the respective following referenced papers. Let $f^* = f^*(x_1, \ldots, x_p) = \sum_{j=1}^p f_j^*(x_j)$ be the true regression function and $s_0 = \#\{j : f_j^*(\cdot) \neq 0\}$. Meier et al. (2009) proved that their penalized least squares estimator (LSE) of $f^*$ achieves the squared error rate $s_0 \{(\log p)/n\}^{2\alpha/(2\alpha+1)}$ under a compatibility condition, and for certain random designs the compatibility condition holds when $s_0 \{(\log p)/n\}^{2\alpha/(2\alpha+1)-1/2}$ is sufficiently small. Koltchinskii and Yuan (2010) established the squared error rate

$$s_0\left\{n^{-2\alpha/(2\alpha+1)} + (\log p)/n\right\}$$

for their penalized LSE of $f^*$ under an $\ell_\infty$ constraint on $f^*$ and its estimator. As $n^{-2\alpha/(2\alpha+1)}$ is the minimax rate for the estimation of a single $\alpha$-smooth $f_j$ and $s_0(\log p)/n$ is the minimax rate when $f_j$ are all linear, (2) is the minimax rate in the sparse AM without $\ell_\infty$ constraint. Raskutti et al. (2012) proved that the $\ell_\infty$ constraint in Koltchinskii and Yuan (2010) can be removed when the covariates are independent uniform variables on $[0, 1]$, and pointed out that when $\|f^*\|_\infty$ is bounded the squared error rate

$$(\log s_0)^{1/2}(s_0^{1/\alpha}/n)^{1/2} + (s_0/n) \log(p/s_0)$$

(3)
is also achievable so that (2) is not necessarily sharp. Suzuki and Sugiyama (2012) extended and improved the results of Koltchinskii and Yuan (2010) under $\ell_1$ and $\ell_2$ conditions on the RKHS norms of $f_j^*(\cdot)$ and an $\ell_\infty$ condition on the noise $\varepsilon_i$ in (1). Yuan and Zhou (2015) considered the LSE under a bounded $\ell_q$ constraint on the RKHS norms of $f_j^*(\cdot)$, $q \in (0, 1)$, and proved squared error rate

$$n^{-2\alpha/(2\alpha+1)} + \left\{ (\log p)/n \right\}^{1-q/2} \quad (4)$$

which is rate minimax for the $\ell_q$ class. Tan and Zhang (2019) considered regularized LSE with a sum of empirical and functional penalties and proved that the estimator achieves the minimax squared error rate

$$n^{-(2-q)/(2+(1-q)/\alpha)} + \left\{ (\log p)/n \right\}^{1-q/2} \quad (5)$$

under a bounded $\ell_1$ constraint on the RKHS norms of $f_j^*(\cdot)$ and a bounded $\ell_q$ constraint on the $L_2$ norms of $f_j^*(\cdot)$. In this literature, the penalty levels are chosen according to the function class under consideration, and the theoretical results all require additional side conditions.

While the above results are very interesting in and of themselves individually and complementary to each other as a whole, the state of art solution to the high-dimensional additive nonparametric regression problem is still fragmented. A rate minimax estimator under a certain complexity condition on the true $f^*$ would lose to another estimator under a different complexity condition. Moreover, these existing methods all require a priori information on the degree of smoothness of the underlying component functions $f_j^*(\cdot)$. Yet in practice, we often do not possess such information.

In this paper, we propose a new method, Multi-Resolution Group Lasso (MR-GL), as an adaptive solution to the high-dimensional additive nonparametric regression problem. The MR-GL estimator will be shown to simultaneously achieve or improve existing error bounds, and it requires no knowledge of either the level of sparsity or the degree of smoothness. Thus, the method is adaptive and expected to be superior over existing approaches when the sparsity and the degree of smoothness of the true regression function are unknown. Furthermore, the MR-GL is easy to compute as it can be directly implemented as a group Lasso algorithm.

In nonparametric regression, i.e. $p = 1$ in (1), adaptive estimation dates back to Efroimovich and Pinsker (1984, 1986) who proposed to apply shrinkage estimators to coefficient blocks in an orthonormal basis expansion of the unknown and proved the adaptive optimality of the method in the sense of simultaneous approximate minimaxity over a broad spectrum of Sobolev classes of regression functions. Such adaptive optimality can be also achieved with different estimators in more general classes, for example, block James-Stein estimators in Sobolev classes (Brown et al., 1997; Cai, 1999), threshold estimators for near or rate minimaxity in more general Besov classes (Donoho and Johnstone, 1994, 1998; Hall et al., 1999), and block
general empirical Bayes estimators for exact minimaxity in Besov classes (Zhang, 2005). In Fourier basis such blocks represent disjoint bands of frequency, whereas in wavelet bases such blocks represent different scales. Thus, the above adaptive estimation methods can be all viewed as multi-resolution (Mallat, 1989; Meyer, 1992) solutions to the nonparametric regression problem.

The thrust of this paper is that adaptive rate optimal estimation can be achieved in high-dimensional AM by directly applying the group Lasso in a carefully architected multi-resolution analysis, i.e. MR-GL. We write each nonparametric component \( f_j(\cdot) \) as an infinite sum \( f_j(x) = \sum_{k=k_s}^{\infty} f_{j,k}(x) \) with \( f_{j,k}(x) = \sum_{\ell=1}^{2^{(k-1)\vee k_s}} u_{j,k,\ell}(x) \beta_{j,k,\ell} \) where \( u_{j,k,\ell}(\cdot) \) are certain basis functions for \( f_j(\cdot) \). After the basis expansion, we fit the response vector \( y = (y_1, \ldots, y_n)^\top \) by the group Lasso with \( f_{j,k} = (f_{j,k}(x_{1,j}), \ldots, f_{j,k}(x_{n,j}))^\top \) as the group effect representing \( f_j(\cdot) \) at the \( k \)-th resolution level (or in the \( k \)-th block of the basis), \( k = k_s, \ldots, k^* \) and \( j = 1, \ldots, p \). We choose \( k_s \) and \( k^* \) to cover a wide spectrum of resolution levels so that adaptation is achieved for nonparametric components with heterogeneous smoothness indices \( \alpha_j > 1/2 \).

The group Lasso is not new in the analysis of the AM as we have briefly reviewed. As the group penalty can be viewed as a sum of inner-product norms of groups of coefficients in a linear system, most such penalized MLE approaches to the sparse AM use a group Lasso penalty on \( f_j \) or a sum of two such penalties. However, as shown in the nonparametric regression literature and the sparse AM literature, regularizing one or two inner product norms of \( f_j \) amounts to fixing the bandwidth for its estimation and thus are non-adaptive to its level of smoothness. Our analysis reveals that by applying group regularization to \( f_{j,k} \) in a suitable multi-resolution structure, the optimality of the group Lasso translates to adaptive optimality in the estimation of the regression function in high-dimensional sparse AM.

**Notation:** We write the data as \( (X, y) = (x_1, \ldots, x_p, y) \) with covariate vectors \( x_j = (x_{1,j}, \ldots, x_{n,j})^\top \) as columns of \( X \) and response vector \( y = (y_1, \ldots, y_n)^\top \). Moreover, we use bold face \( g_j \) to denote realizations of function \( g(x) \) in the sample vector \( x_j \) so that \( g_j = g(x_j) = (g(x_{1,j}), \ldots, g(x_{n,j}))^\top \). Similarly for any function \( g : \mathbb{R}^p \to \mathbb{R} \), \( g = g(X) \) denotes the realization of \( g \) in the sample as the vector with elements \( g(x_{i,1}, \ldots, x_{i,p}) \). Thus, the vector version of the AM (1) is \( f = \sum_{j=1}^p f_j \) and its multi-resolution approximation can be written as \( \sum_{j=1}^p \sum_{k=k_s}^{k^*} f_{j,k} \). For vectors \( u \) and \( v \in \mathbb{R}^n \), \( \langle u, v \rangle_n = u^\top v / n \) and \( \| u \|_{2,n} = \| u \|_2 / \sqrt{n} \) denote the length normalized \( l_2 \) inner product and norm. For random design \( X \) and functions \( g(\cdot) \) and \( h(\cdot) \) from \( \mathbb{R}^p \) to \( \mathbb{R} \), \( \langle g, h \rangle_{L_2} = \mathbb{E}[\langle g, h \rangle_n] \) and \( \| g \|_{L_2} = \langle g, g \rangle_{L_2}^{1/2} \) denote the population inner product and the associated \( L_2 \) norm.
2 Multi-resolution group Lasso

In two subsections, we describe the multi-resolution structure for the approximation of the regression function in AM and introduce the MR-GL as the group Lasso estimator in the structure.

2.1 Multi-resolution structure

In a multi-resolution analysis, a nonparametric function \( f_j(\cdot) \) can be written as a linear combination of basis functions in a block structure,

\[
 f_j(x) = \sum_{k=k_*}^{\infty} f_{j,k}(x), \quad f_{j,k}(x) = \sum_{\ell=1}^{2^{(k-1)\vee k_*}} \beta_{j,k,\ell} u_{j,k,\ell}(x),
\]

(6)

where the basis functions \( u_{j,k,\ell}(\cdot), 1 \leq \ell \leq 2^{(k-1)\vee k_*}, k \geq k_* \), are linearly independent in some infinite dimensional space for each \( j \). We shall choose \( \{u_{j,k,\ell}\} \) to form a complete system in the function spaces of interest to us, so that \( f_j(x) \) can be approximated by finitely many terms in the series in (6) to an arbitrary level of accuracy. For simplicity and definitiveness, we take fixed \( k_* \geq 0 \) and block size \( 2^{(k-1)\vee k_*} \) in all infinite series expansions of the component functions \( f_j(\cdot) \) throughout the paper. Under the observed design matrix \( X \) with elements \( x_{i,j} \), we write \( f_j = \sum_{k=k_*}^{\infty} f_{j,k} \) and \( f_{j,k} = \sum_{\ell=1}^{2^{(k-1)\vee k_*}} \beta_{j,k,\ell} u_{j,k,\ell} \) with \( u_{j,k,\ell} = (u_{j,k,\ell}(x_{1,j}), \ldots, u_{j,k,\ell}(x_{n,j}))^T \).

With the above block structure, we also consider truncated series of the form

\[
 \overline{f}_j(x) = \sum_{k=k_*}^{k^*} \overline{f}_{j,k}(x), \quad \overline{f}_{j,k}(x) = \sum_{\ell=1}^{2^{(k-1)\vee k_*}} \overline{\beta}_{j,k,\ell} u_{j,k,\ell}(x), \quad \overline{\beta}_{j,k,\ell} \in \mathbb{R}.
\]

(7)

We write \( \overline{f}_j = \sum_{k=k_*}^{\infty} \overline{f}_{j,k} \) and \( \overline{f}_{j,k} = \sum_{\ell=1}^{2^{(k-1)\vee k_*}} \overline{\beta}_{j,k,\ell} u_{j,k,\ell} \) with the observed \( X \).

Here and in the sequel, \( f_j \) refers to a general candidate of the \( j \)-th AM component under consideration with functional version \( f_j(\cdot) \) and realized vector version \( f_j \), and \( \overline{f}_j \) refers to a general candidate for its approximation up to a fixed resolution level \( k^* \) with functional version \( \overline{f}_j(\cdot) \) and vector version \( \overline{f}_j \). Similarly, \( f_{j,k} \), \( f_{j,k}(\cdot) \), \( \overline{f}_{j,k} \), \( \overline{f}_{j,k}(\cdot) \), \( \overline{f}_{j,k} \) are the corresponding expressions at the resolution level \( k \). Thus,

\[
 \mathcal{F}_{n,j}^{(NP)} = \left\{ \overline{f}_j : \overline{f}_j(\cdot) \text{ in (7)} \right\} = \left\{ \sum_{k=k_*}^{k^*} \sum_{\ell=1}^{2^{(k-1)\vee k_*}} b_{k,\ell} u_{j,k,\ell} : b_{k,\ell} \in \mathbb{R} \right\}
\]

(8)

is the linear subspace of \( \mathbb{R}^n \) generated by the approximation candidates in (7). In this approximation scheme, the terms with \( k > k^* \) are viewed as of ultra-high resolution and presumably ignorable.
The multi-resolution expansion (6) and the associated approximation (7) are essential in adaptive estimation of smooth functions in nonparametric analysis. For $\alpha > 0$ define Sobolev-type norms $\| f_j \|_{\alpha,2,n}$ and $\| f_j \|_{\text{Sobolev},\alpha,n}$ by

$$
\| f_j \|_{\alpha,2,n}^2 = \sum_{k=k^*+1}^{\infty} 2^{2\alpha k} \| f_j,k \|_{2,n}^2, \quad \| f_j \|_{\text{Sobolev},\alpha,n}^2 = \| f_j,k \|_{2,n}^2 + \| f_j \|_{\alpha,2,n}^2
$$

with the vector versions of $f_j$ and $f_{j,k}$ in (6). We shall say that $f_j(\cdot)$ is $\alpha$-smooth (in the $\ell_2$ and empirical senses) when $\| f_j \|_{\alpha,2,n}$ is bounded and $f_j(\cdot)$ can be expressed in the infinite series expansion in (7). Essentially, (9) implies

$$
\| f_j - \mathcal{F}_j \|_{2,n} \leq \| f_j \|_{\alpha,2,n} \left( \sum_{k>k^*} 2^{-2\alpha k} \right)^{1/2} = \frac{2^{-\alpha k^*} \| f_j \|_{\alpha,2,n}}{(4^\alpha - 1)^{1/2}}
$$

for some $\mathcal{F}_j \in \mathcal{F}_{n,j}^{(\text{NP})}$, for example with $\beta_{j,k,\ell} = \beta_{j,k,\ell}$ in (7) but not necessarily. It is reasonable to assign the smoothness index $\alpha$ in (9) as

$$
\mathbb{E} \| f_j \|_{\text{Sobolev},\alpha,n}^2 = O(1) \int \left\{ f_j^2(x) + ((d/dx)^\alpha f_j(x))^2 \right\} dx
$$

when $x_{i,j}$ are random variables supported in a fixed compact interval with uniformly bounded marginal probability density functions and $u_{j,k,\ell}(\cdot)$ form the Fourier, wavelet or spline bases for each $j$. We note that $\mathbb{E} \| f_j \|_{\alpha,2,n}^2$ does not depend on $n$ when $x_{i,j}$ are identically distributed for the given $j$.

Consider the case where the AM holds with only one non-zero component $f_j = f_j^*$ and iid noise $\varepsilon_i \sim N(0,\sigma^2)$. Suppose that we are prepossessed to estimate $f^* = \mathbb{E}[y|X]$ by the least squares projection of $y$ to $\mathcal{F}_{n,j}^{(\text{NP})}$ when the oracular knowledge of $f^* = f_j^*$ is available. This oracle LSE has the average variance $2^{k^*} \sigma^2/n$ and maximum average squared bias of the order $2^{-2\alpha k^*}$ in view of (10). Thus, the $\ell_2$-regularization of $f_j$ as a group is not expected to be rate minimax unless $2^{k^*} \asymp n^{1/(2\alpha+1)}$, as it is not expected to outperform the oracle LSE in view of the minimax rate for fixed $p$ (Stone, 1985). This forces the choice of $k^*$ to depend on $\alpha$ for rate optimality (Huang et al., 2010) and explains the general non-adaptivity in $\alpha$ in regularization schemes based on a fixed number of inner-product norms of $f_j(\cdot)$.

The situation is different if we regularize each $f_j$ at individual resolution levels $k$, or equivalently regularize the effect represented by the $f_j,k$ in (6). When a high-frequency $f_{j,k}^*$ carries little signal, its noisy expression in the data would be automatically thresholded or shrunk to reduce the variability of the estimator when $k_* \leq k \leq k^*$. However, when $k > k^*$, a great portion of the signal in the missing term $f_{j,k}$ is not recoverable even when the signal is large as it is excluded in the general approximation scheme, possibly causing excess bias and sub-optimality. These heuristics suggest the choice of a large $k^*$ to facilitate adaptation to the smoothness of the true $f_j^*(\cdot)$. The
choice of the block size $2^{(k-1)\sqrt{k_\ast}}$ for the nonparametric $f_j$ is critical to the success of the MR-GL. First, it guarantees that different resolution levels are differentiated. Second, it groups large number of basis functions together to take advantage of the sparsity at the component function level.

In the nonparametric regression setting, we still need the dimension of the approximation space $\mathcal{F}_{n,j}^{(NP)}$ to be smaller than $n$ to relate it to its population version. As the dimension of the space is $2^{k_\ast}$ in (8), we take $2^{k_\ast} < n$ to capture most of the signal in $f_j^\ast$ when $f_j^\ast(\cdot)$ is $\alpha_j$-smooth with $n^{1/(2\alpha_j+1)} \leq 2^{k_\ast} < n$, allowing all $\alpha_j > 1/2$. Similarly, we shall not pick too large a $k_\ast$ as it forces the grouping of $2^{k_\ast}$ low frequency basis functions. For smooth functions, the baseline resolution level typically contains a fixed number of (constant or polynomial) terms. Later in our theoretical study, we will use $k_\ast \approx 2\log p$ to minimize the rate of theoretical error bounds. For notational convenience, we label the baseline resolution level $k_\ast$ so that there are totally $2^k$ basis functions up to level $k$.

The AM also accommodate linear and group effects $f_j$ by allowing the first resolution in (6) and (7) to have flexible number of basis functions,

$$f_{j,k_\ast}(x) = \sum_{\ell=1}^{d_j^*} \beta_{j,k_\ast,\ell} u_{j,k_\ast,\ell}(x), \quad \overline{f}_{j,k_\ast}(x) = \sum_{\ell=1}^{d_j^*} \overline{\beta}_{j,k_\ast,\ell} u_{j,k_\ast,\ell}(x),$$

with $f_j = f_{j,k_\ast}$ and $\overline{f}_j = \overline{f}_{j,k_\ast}$ for a positive integer $d_j^*$, and $f_{j,k}(\cdot) = \overline{f}_{j,k}(\cdot) = 0$ for all $k > k_\ast$. For example, when $f_j(x) = \beta_j x$ is a linear function, we take $\beta_{j,k_\ast,1} = \beta_j$ and $u_{j,k_\ast,1}(x) = x$ with $d_j^* = 1$. Similarly, a group effect of dimension $d_j^*$ can be written as (11) when $u_{j,k_\ast,\ell}(x_{i,j})$ are viewed as covariates in the group, $\ell \leq d_j^*$. We shall call parametric components such linear and group effects in the AM.

Putting the nonparametric components (6) and the parametric components (11) together, the AM is written as

$$f = f(x_1, \ldots, x_p) = \sum_{j \in J_0} f_j(x_j) + \sum_{j \in J_1} \sum_{k = k_\ast}^\infty f_{j,k}(x_j) = \sum_{(j,k) \in \mathcal{K}} f_{j,k}(x_j),$$

where $J_0$ and $J_1$ are respectively the index sets of the parametric and nonparametric components $f_{j}(\cdot)$, and $\mathcal{K} = \{(j,k) : k = k_\ast \forall j \in J_0, k \geq k_\ast \forall j \in J_1\}$ is the index set for the parametric components represented at the nominal resolution level $k_\ast$ ($f_j = f_{j,k_\ast}$) and the nonparametric components represented at individual resolution levels $k \geq k_\ast$. In this multi-resolution structure, $f$ is approximated by

$$\overline{f} = \overline{f}(x_1, \ldots, x_p) = \sum_{j \in J_0} \overline{f}_j(x_j) + \sum_{j \in J_1} \sum_{k = k_\ast}^{k_\ast} \overline{f}_{j,k}(x_j) = \sum_{(j,k) \in \mathcal{K}^*} \overline{f}_{j,k}(x_j)$$

with $\overline{f}_{j,k}$ in (7) and (11) and $\mathcal{K}^* = \{(j,k) : k = k_\ast \forall j \in J_0, k_\ast \leq k \leq k_\ast \forall j \in J_1\}$. The noisy expression of the true $f_{j,k_\ast}^\ast$ in the data is still subject to regularization in the MR-GL for $k \leq k_\ast$, but the MR-GL does not attempt to recover signals of nonparametric
components in resolution levels \( k > k^* \). We note that \( k^* - k_s \gg \log n \) typically in this multi-resolution scheme to allow adaptation to different levels of smoothness of the nonparametric components \( f_j \).

### 2.2 Estimation through group Lasso

In the vector notation, the AM (1) can be written as

\[
y = \sum_{j=1}^{p} f_j + \varepsilon,
\]

where \( y = (y_1, \ldots, y_n)^\top \) is the response vector, \( f_j = (f_j(x_{1,j}), \ldots, f_j(x_{n,j}))^\top \) represents the contribution of the \( j \)-th covariate to the model, and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\top \) is the noise vector.

As the ultra high-frequency terms with \( k > k^* \) are automatically excluded in the approximation scheme (13), the MR-GL estimator of \( f^* \) is defined as

\[
\hat{f} = \arg \min_f \left\{ \frac{\|y - f\|_{2,n}^2}{2} + A_0 \sum_{(j,k) \in \mathcal{K}^*} \lambda_{j,k} \|f_{j,k}\|_{2,n} : f = \sum_{(j,k) \in \mathcal{K}^*} f_{j,k} \right\},
\]

where the minimum is taken over the sum \( f \) with \( f_{j,k} = (f_{j,k}(x_{1,j}), \ldots, f_{j,k}(x_{n,j}))^\top \) corresponding to the \( f_{j,k}(\cdot) \) in (6) for \( j \in J_1 \) and in (11) for \( j \in J_0 \), \( \lambda_{j,k} \) are proper threshold levels, and \( A_0 > 1 \) is a constant, e.g. \( A_0 = 1.01 \) or \( A_0 = 2 \). As \( \hat{f} \) is a regularized projection of \( y \) to a linear space through convex minimization in (15), it is uniquely defined.

We may also use the basis vectors \( u_{j,k,\ell} \) to write the MR-GL more explicitly. Let

\[
f_{j,k} = U_{j,k} \beta_{j,k}, \quad U_{j,k} \in \mathbb{R}^{d \times d_{j,k}}, \quad \beta_{j,k} \in \mathbb{R}^{d_{j,k}},
\]

where \( U_{j,k} = (u_{j,k,1}, \ldots, u_{j,k,d_{j,k}}) \in \mathbb{R}^{n \times d_{j,k}}, d_{j,k} = d_j^\star \forall j \in J_0 \) as in (11), \( d_{j,k} = 2^{(k-1)\log k_s} \forall j \in J_1 \) as in (6) and \( u_{j,k,\ell} = (u_{j,k,\ell}(x_{1,j}), \ldots, u_{j,k,\ell}(x_{n,j}))^\top \). Recall that for the nonparametric components with \( j \in J_1 \), the first \( U_{j,k} \), with \( k = k_s \), represents the design matrix for a group of low-resolution components of \( f_j \), while \( U_{j,k} \) with \( k > k_s \) blends in high-resolution components. Let

\[
\beta = (\beta_{j,k}^\top, (j, k) \in \mathcal{K}^*)^\top \in \mathbb{R}^{d^\star}
\]

be the coefficient vector of the \( f \) in (13) in the basis \( \{u_{j,k,\ell}\} \) with \( d^\star = \sum_{(j,k) \in \mathcal{K}^*} d_{j,k} \) and the \( \mathcal{K}^* \) in (13). The MR-GL estimates the coefficient vector (17) for an ideal approximate of \( f^* \) by

\[
\hat{\beta} = \arg \min_{\beta} \left\{ \frac{1}{2} \|y - \sum_{(j,k) \in \mathcal{K}^*} U_{j,k} b_{j,k} \|_{2,n}^2 + A_0 \sum_{(j,k) \in \mathcal{K}^*} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{2,n} \right\}
\]
with $\hat{\beta} = (\hat{\beta}_{j,k}^\top, (j, k) \in \mathcal{K}^*)^\top$ and $b = (b_{j,k}^\top, (j, k) \in \mathcal{K}^*)^\top$. This can be viewed as group Lasso in linear regression when the columns $u_{j,k,\ell}$ of $U_{j,k}$ are treated as design vectors and $f_{j,k}$ are treated as group effects in (16). While (18) is not necessarily unique, any solution of it would recover the unique solution of (15) via

$$\hat{f} = \sum_{j=1}^p \hat{f}_j = \sum_{(j, k) \in \mathcal{K}^*} \hat{f}_{j,k}, \quad \hat{f}_j = \sum_{k: (j, k) \in \mathcal{K}^*} \hat{f}_{j,k}, \quad \hat{f}_{j,k} = U_{j,k} \hat{\beta}_{j,k},$$

with the $U_{j,k}$ in (16). The vectors $\hat{f}_j$ and $\hat{f}_{j,k}$ have the interpretation as the estimated AM components and their representation in individual resolution levels. Thus, (15), (18) and (19) are equivalent and can be all viewed as MR-GL. The MR-GL enjoys the advantage of group Lasso, a computationally feasible and parsimonious solution, and adaptation to the smoothness of the component functions through group selection in the multi-resolution decomposition.

Finally we discuss the choice of the thresholding level $\lambda_{j,k}$. As (18) is group Lasso, the literature provides a variety of its error bounds for $\hat{f} - f^*$ when

$$\mathbb{P}\{\Omega_0\} = 1 + o(1) \quad \text{with} \quad \Omega_0 = \left\{ \max_{(j, k) \in \mathcal{K}^*} \frac{\|P_{j,k}(y - f^*)\|_2}{\lambda_{j,k}} \leq 1 \right\},$$

where $P_{j,k}$ is the orthogonal projection from $\mathbb{R}^n$ to the column space of $U_{j,k}$ in (16). When $y - f^*$ has iid $\mathcal{N}(0, \sigma^2)$ components, (20) holds when

$$\lambda_{j,k} = \begin{cases} \sigma\left(\left(d_j^*/n\right)^{1/2} + \sqrt{(2/n)\log(p/\epsilon)}\right), & j \in J_0, \\ \sigma\left(\left(2^k/n\right)^{1/2} + \sqrt{(2/n)\log(p/\epsilon)}\right), & j \in J_1, \end{cases}$$

with $\epsilon \in (0, 1]$. The above $\lambda_{j,k}$ also provides (20) when $y - f^*$ is a sub-Gaussian vector with a slightly inflated $\sigma$ (Huang and Zhang, 2010). Throughout the paper, we use $f^*_j$ and $f^*_{j,k}$ to denote the estimation target which could be the true $f_j$ and $f_{j,k}$ associated with $f^* = \mathbb{E}[y|X]$ or any $f^*$ satisfying (20) via $f^* = \sum_{j=1}^p f^*_j = \sum_{(j,k) \in \mathcal{K}^*} f^*_{j,k}$.

### 3 Theoretical Results for Fixed Designs

In this section, we carry out a theoretical analysis of the MR-GL in both fixed and random design settings. The main results are summarized by oracle inequalities stated in Theorem 1 through Theorem 4. In the fixed design setting, we prove that the MR-GL adaptively achieves and generalizes existing rate optimal error bounds for various classes of the unknown $f^*$ under an empirical compatibility condition. In the random design setting, we provide oracle inequalities in both the empirical and population error measures as well as sufficient condition on the sample size to guarantee the empirical compatibility condition under mild conditions on the design population.
In this section, we consider fixed designs in which the covariates \( x_{i,j} \) in (1) and (14) are treated as deterministic. We measure the performance of the MR-GL by the in-sample squared prediction error \( \| \hat{f} - f^* \|_{2,n}^2 \) where \( f^* \) is the estimation target. Typically \( f^* = \mathbb{E}[y] \) as \( X \) is deterministic but our theorems also apply to any \( f^* \) satisfying (20). As the MR-GL (15) can be viewed as group Lasso via (18) and (19), our results are closely related to the existing group Lasso theory (Nardi and Rinaldo, 2008; Huang and Zhang, 2010; Lounici et al., 2011; Bühlmann and van de Geer, 2011; Negahban et al., 2012; Mitra and Zhang, 2011). However, due to the choice of \( \lambda_{j,k} \) in (21) and the truncation of the nonparametric components beyond the resolution level \( k^* \), we need different error bounds based on the approximate sparsity of the group components. We discuss in separate subsections oracle inequalities for the MR-GL in general, the adaptive minimaxity of the MR-GL to the smoothness index \( \alpha \) in (9) in the nonparametric AM, and the connection of our results to the group Lasso in linear regression.

### 3.1 General error bounds

For the \( U_{j,k} \) in (16), \( b = (b_{j,k}^\top, (j,k) \in \mathcal{X}^*)^\top \in \mathbb{R}^{d^*} \) as in (18) and \( S \subset \mathcal{X}^* \), define

\[
pen_S(b) = \sum_{(i,k) \in S} \lambda_{i,k} \| U_{i,k} b_{j,k} \|_{2,n}
\]

(22)
as the partial penalty on \( b \) in the index set \( S \). For \( \xi \geq 1 \) and \( S \subset \mathcal{X}^* \), define

\[
C_{\text{pred}}(\xi, S) = \sup_b \frac{\{\text{pen}_S(b) - \xi^{-1}\text{pen}_{S^c}(b)\}^2}{\|\lambda_{S}\|_2^2 \| \sum_{(j,k) \in \mathcal{X}^*} U_{j,k} b_{j,k} \|_{2,n}^2}
\]

(23)
as a constant factor for empirical prediction error bounds, with \( S^c = \mathcal{X}^* \setminus S \), \( \|\lambda_{S}\|_2 = (\sum_{(j,k) \in S} \lambda_{j,k}^2)^{1/2} \) and the convention \( 0/0 = 0 \). This prediction factor is no greater than the reciprocal of the compatibility coefficient for group Lasso (Bühlmann and van de Geer, 2011) as we will discuss in Section 3.3.

**Theorem 1** (i) Let \( \hat{f} \) be as in (19) with \( \hat{\beta} = (\hat{\beta}_{j,k}^\top, (j,k) \in \mathcal{X}^*)^\top \in \mathbb{R}^{d^*} \) in (18). Let \( \tilde{f} = \sum_{(j,k) \in \mathcal{X}^*} \tilde{f}_{j,k} \) with \( \tilde{f}_{j,k} = U_{j,k} \hat{\beta}_{j,k} \), \( U_{j,k} \in \mathbb{R}^{n \times d_{j,k}} \) in (16) and \( \beta = (\beta_{j,k}^\top, (j,k) \in \mathcal{X}^*)^\top \) in (17). Let \( P_{j,k} \) be the orthogonal projection to the range of \( U_{j,k} \), \( \text{pen}_S(b) \) in (22) and \( C_{\text{pred}}(\xi, S) \) in (23) with \( \xi = (A_0 + 1)/(A_0 - 1) > 1 \) and \( S \subset \mathcal{X}^* \). Then,

\[
\| \hat{f} - f^* \|_{2,n}^2 \leq B_S + \overline{\Delta}_S, \quad \| \tilde{f} - \beta \|_{2,n}^2 + \| \hat{f} - f^* \|_{2,n}^2 \leq 4B_S + 2\overline{\Delta}_S, \quad \text{(24)}
\]

with \( \overline{\Delta}_S = \| \tilde{f} - f^* \|_{2,n}^2 + 4A_0\text{pen}_{S^c}(\beta) \) and \( B_S = (A_0 + 1)^2C_{\text{pred}}(\xi, S)\|\lambda_{S}\|_2^2 \), when

\[
\| P_{j,k}(y - f^*) \|_{2,n} \leq \lambda_{j,k} \quad (j,k) \in \mathcal{X}^*. \quad \text{(25)}
\]
In particular, for $S = \{(j, k) \in \mathcal{K}^*: \| \hat{f}_{j,k} \|_{2,n} \geq A_0 \lambda_{j,k} \}$,
\[
\| \hat{f} - \bar{f} \|^2_{2,n} + \| \hat{f} - \hat{f}^* \|^2_{2,n} \leq 2 \| \bar{f} - \hat{f}^* \|^2_{2,n} + C^*_\text{pred}(\xi, S) \sum_{(j,k) \in \mathcal{K}^*} \lambda_{j,k}^2 \wedge (\lambda_{j,k} \| \hat{f}_{j,k} \|_{2,n})
\]
with $C^*_\text{pred}(\xi, S) = \max \{ 8A_0^2, 4(A_0 + 1)^2 C^*_\text{pred}(\xi, S) \}$ and $\bar{f}_{j,k} = U_{j,k} \beta_{j,k}$.

(ii) Suppose $y - \hat{f}^*$ has iid $N(0, \sigma^2)$ entries. Then, (25) holds with probability at least $1 - \epsilon/\sqrt{2 \log(p/\epsilon)}$ when $\lambda_{j,k}$ are given by (21) with $0 < \epsilon \leq 1 \leq 2 \log(p/\epsilon)$.

In the above theorem, $f^*$ can be viewed as $\mathbb{E}[y]$ as the design is treated as deterministic. However, (24) and (26) apply to any vector $f^* \in \mathbb{R}^n$ satisfying (25) with high probability. In particular, $f^*$ is not required to have additive components. In Theorem 1, (26) bounds the in-sample squared prediction error $\| \hat{f} - f^* \|^2_{2,n}$ of the MR-GL by the sum of the squared approximation error $\| \bar{f} - f^* \|^2_{2,n}$ of a deterministic approximation candidate $\bar{f}$ generated by a function $\bar{f}$ in (13) and the product of the prediction factor $C^*_\text{pred}(\xi, S)$ and a normalized complexity measure of $\bar{f}$. Thus, Theorem 1 asserts that the in-sample prediction error of the MR-GL is small when the estimation target $f^*$ can be approximated by some unknown sparse $\bar{f}$, and the sparsity of $\bar{f}$ means the sparsity of its functional components as well as the decay of the signal strength at high resolution levels due to the smoothness of the individual nonparametric components, respectively in the index $j$ and in the index $k$ given $j$.

Here $\mathcal{F}$ can be any member of the approximate space
\[
\mathcal{F}_n = \left\{ \bar{f} = \sum_{j=1}^{p} \bar{f}_j : \bar{f}_j = \bar{f}_{j,k}, \quad \forall j \in J_0, \bar{f}_j \in \mathcal{F}_n^{(NP)} \forall j \in J_1 \right\}
\]
\[
= \left\{ \bar{f} = U \beta = \sum_{(j,k,\ell) \in \mathcal{L}^*} \beta_{j,k,\ell} u_{j,k,\ell} : \beta \in \mathbb{R}^{d^*}, \beta_{j,k,\ell} \in \mathbb{R} \right\},
\]
where $\mathcal{F}_n^{(NP)}$ is the approximation space in (8) for the nonparametric components, $\mathcal{L}^* = \{(j, k, \ell) : (j, k) \in \mathcal{K}^*, \ell \leq d_j^* \forall j \in J_0, \ell \leq 2^{(k-1)\wedge k} \forall j \in J_1 \}$, $d^* = |\mathcal{L}^*|$, $U$ is the $n \times d^*$ matrix composed of columns $u_{j,k,\ell} = (u_{j,k,\ell}(x_{1,j}), \ldots, u_{j,k,\ell}(x_{n,j}))^\top$ with $(j, k, \ell) \in \mathcal{L}^*$, and $\beta$ is the coefficient vector as in (17). We note that when $d^* = |\mathcal{L}^*| > n$, $\mathcal{F}_n$ typically fills the entire space $\mathbb{R}^n$ but the approximation candidates $\bar{f}$ is still meaningful through its representation $U \beta$ as in sparse linear regression. Moreover, the coefficients allow identification of the components $\bar{f}_j$ through the functional representations of $\bar{f}_j$ in (7) and (11).

The second term on the right-hand side of (26) can be viewed as a normalized complexity measure of the approximation candidate $\bar{f}$ in the following sense. Let $\delta_k = I\{k > k_\ast \}$ and $d_{j,k}$ be as in (16). For each group with index $(j, k)$ in $\mathcal{K}^*$, the quantity
\[
\frac{\lambda_{j,k}^2 \wedge (\lambda_{j,k} \| \bar{f}_{j,k} \|_{2,n})}{\sigma^2/n} = (2\delta_k d_{j,k}^{1/2} + \sqrt{2 \log(p/\epsilon)})^2 \min \left(1, \frac{\| \bar{f}_{j,k} \|_{2,n}}{\lambda_{j,k}} \right).
\]

(28)
is roughly the number of data points needed to estimate $\overline{f}_{j,k} = U_{j,k} \overline{f}_{j,k}$ as expressed in (16), as $d_{j,k}$ is the number of columns of $U_{j,k}$. For strong signals $\|\overline{f}_{j,k}\|_{2,n} \geq \lambda_{j,k}$, the nominal degrees of freedom $d_{j,k}$ is inflated to (28) to take into account of the uncertainty about its signal strength in group selection. For weaker signals, (28) discounts the complexity by the ratio between the signal strength $\|\overline{f}_{j,k}\|_{2,n}$ and penalty level $\lambda_{j,k}$.

The above connection between the error bound (26) and the complexity of the approximation of the unknown indicates its rate optimality in a broad range of settings, and thus the adaptive optimality of the MR-GL as we will discuss below.

### 3.2 Adaptive optimality in nonparametric AM

The main objective of this subsection is to present the implications of Theorem 1 in the nonparametric AM in which all the components $f_j$ are nonparametric, i.e. $J_0 = \emptyset$ and $J_1 = \{1, \ldots, p\}$, while the discussions below up to the statement of Corollary 1, including Theorem 2, are applicable to the general semi-parametric setting with possibly nonempty $J_0$. We shall focus on the theory in which the complexity of $f_j$ is measured by the Sobolev-type norm (9) with a common index $\alpha$ and possibly a second smaller common index $\alpha_0$ to facilitate more direct comparisons between our results and those in the existing literature.

We first connect the second term in the error bound (26) to the Sobolev-type norm $\|f_j\|_{\alpha,2,n}$ in (9). For real $c$ and $0 \leq q \leq 1$, define

$$J_c^{(q)}(k_1, k_2) = \begin{cases} \left( \sum_{k=k_1+1}^{k_2} 2^{ck/(1-q/2)} \right)^{1-q/2}, & c \leq 0, \\ \left( \sum_{k=0}^{k_2-k_1-1} 2^{-ck/(1-q/2)} \right)^{1-q/2}, & c > 0, \end{cases}$$

for all integers $0 \leq k_1 \leq k_2$, with $J_c^{(q)}(k,k) = 0$. Consider two smoothness indices $0 \leq \alpha_0 \leq \alpha$. By the definition of the norm $\|\overline{f}_j\|_{\alpha,2,n}$ in (9), the sum is bounded by

$$\sum_{k=k_*+1}^{k_*} \lambda_{j,k}^2 \wedge (\lambda_{j,k}\|\overline{f}_{j,k}\|_{2,n}) \geq \max_{k_* < k \leq k_*^*} \lambda_{j,k} \min \left( \lambda_{j,k}, \frac{\|\overline{f}_j\|_{\alpha_0,2,n}}{2^{\alpha_0 k}}, \frac{\|\overline{f}_j\|_{\alpha_2,n}}{2^{\alpha k}} \right).$$

When $\lambda_{j,k} \asymp 2^{k/q}$ with $2^{k/q} \asymp \log(p/\epsilon)$ in (21), the above lower bound is essentially sharp as the summations below and above the maximizing $k$ are both bounded by geometric series of the form $J_c^{(0)}(k_1, k_2)$. The following proposition refines the above argument and facilitates the use of the sequence norms of $\|\overline{f}_j\|_{\alpha,2,n}$ and $\|\overline{f}_j\|_{\alpha_0,2,n}$ to bound the sum $\sum_{j \in J_1} \sum_{k=k_*+1}^{k_*^*} \lambda_{j,k}^2 \wedge (\lambda_{j,k}\|\overline{f}_{j,k}\|_{2,n})$ in (26).

**Proposition 1** Let $\alpha \geq 1/2$, $0 \leq q \leq 1$ and $0 \leq \alpha_0 \leq \alpha$. Define

$$\gamma = \frac{(2 - q)(\alpha - 1/2) + (1 - q/2 - \alpha_0)_+}{\alpha - 1/2 + (1 - q/2 - \alpha_0)_+}$$

(29)
with $\gamma = 1$ for $\alpha = 1/2$. Let $\sigma_n > 0$ and $\lambda_k \leq \sigma_n 2^{k/2}$ for $k_* < k \leq k^*$. Then,

$$\sum_{k=k_*+1}^{k^*} \lambda_k \min \left( \| \overline{f}_{j,k} \|_{2,n}, \lambda_k \right) \leq \sigma_n^2 J_{q,\alpha,\alpha_0}(k_*, k^*) \{ \| \overline{f}_j \|_{\alpha_0,2,n}^q \}^{1-\rho} \| \overline{f}_j \|_{\alpha,2,n}^\rho, \quad (30)$$

where $\rho = (1 - q/2 - \alpha_0 q)_+ / \{ \alpha - 1/2 + (1 - q/2 - \alpha_0 q)_+ \}$ and

$$J_{q,\alpha,\alpha_0}(k_*, k^*) = \{ (1/\rho - 1)^\rho + (1/\rho - 1)^{\rho-1} \} \{ J_{1-q/2-\alpha_0q}(k_*, k^*) \}^{1-\rho} \{ J_{\alpha-1/2}(k_*, k^*) \}^\rho.$$

**Remark 1** We note that $1 \leq \gamma \leq \min\{4\alpha/(2\alpha + 1), 2 - q\}$ always holds.

$$\gamma = \begin{cases} 
\min\{4\alpha/(2\alpha + 1), 2 - q\}, & \alpha_0 = \alpha, \\
4\alpha/(2\alpha + 1), & q = 0, \\
(2 - q)/2\alpha/(2\alpha + 1 - q), & \alpha_0 = 0, \\
1, & \alpha = 1/2 \text{ or } q = 1,
\end{cases}$$

$\gamma < 2 - q$ iff $1 - q/2 - \alpha_0 q > 0$, and $\gamma < 4\alpha/(2\alpha + 1)$ if $q \neq 0$ and $\alpha_0 < \alpha$. Moreover,

$$\rho = \begin{cases} 
(2 - q - \gamma)/(1 - q), & 0 \leq q < 1, \\
2/(2\alpha + 1), & q = 0, \\
(2 - q)/(2\alpha + 1 - q), & \alpha_0 = 0, \\
1, & \alpha = 1/2, \\
(1 - 1/2 - \alpha_0)/(\alpha - 1/2 + (1 - 1/2 - \alpha_0)_+), & q = 1,
\end{cases}$$

and $\gamma + q(1 - \rho) + \rho = 2$ for the $\rho$ in (30).

**Remark 2** We note that $J^*_c(q)(k_1, k_2) \asymp \min \{ |c|^{q/2-1}, (k_2 - k_1)^{1-q/2} \}$ uniformly in $(c, q, k_1, k_2)$. For $c \neq 0$, $J^*_c(q)(k_1, k_2) \leq \{ 1 - 2^{-|c|/(1-q/2)} \}^{q/2-1}$ in Theorem 1. However, this upper bound is not accurate when $c$ is close to zero.

We need to specify an $\overline{f}$ in Theorem 1 so that the first term on the right hand side of (26) is small. When $\overline{f}^*$ arises from the AM (12), a natural way of constructing $\overline{f}$ is to truncate the ultra-high-frequency terms of $\overline{f}^*$, or equivalently to set the coefficients $\overline{\beta}_{j,k,\ell}$ in (7) equal to the true version. Formally, suppose the AM holds for the true regression function in the sense of $f_j(x) = f^*_j(x)$ and $\mathbb{E}[\xi] = 0$ in (1). When $f^*_j(x)$ has an infinite series expansion of the form (6) with coefficients $\beta_{j,k,\ell}^*$, we may simply set $\overline{\beta}_{j,k,\ell} = \beta_{j,k,\ell}^*$ in (7). This gives $\overline{f} = \sum_{(j,k) \in \mathcal{K}^*} \overline{f}_{j,k} \in \mathcal{F}_n$ as in (27) with

$$\overline{f}_{j,k} = f^*_j(x) \forall (j,k) \in \mathcal{K}^*, \quad f^*_j(x) = \sum_{\ell=1}^{d_{j,k}} \beta_{j,k,\ell}^* u_{j,k,\ell} \forall (j,k) \in \mathcal{K}, \quad (31)$$
where \( \mathbf{u}_{j,k} = (u_{j,k}(x_1), \ldots, u_{j,k}(x_n))^\top \), \( d_{j,k} \) are as in (16), and \( \mathcal{H} \) and \( \mathcal{H}^* \) are as in (12) and (13) respectively. When the nonzero components of \( \mathbf{f}^* - \mathbf{f} \) are not highly correlated, we expect that (31) would provide

\[
\| \mathbf{f}^* - \mathbf{f} \|_{2,n}^2 \lesssim \sum_{j \in J_1} \left( \sum_{k=k^*+1}^{\infty} \| \mathbf{f}^*_{j,k} \|_{2,n}^2 \right) \leq \frac{2^{-2\alpha k^*}}{4^\alpha - 1} \sum_{j \in J_1} \| \mathbf{f}^*_{j,k} \|^2_{\alpha,2,n}
\]

where \( \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n} \) is the Sobolev-type norm defined in (9) and \( J_1 \) is as in (12). Thus, the first term on the right hand side of (26) is controlled by the \( \ell_2 \) norm of \( \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n} \). Alternatively we may use the following cruder bound which always holds:

\[
\| \mathbf{f}^* - \mathbf{f} \|_{2,n} \leq \sum_{j \in J_1} \left( \sum_{k=k^*+1}^{\infty} \| \mathbf{f}^*_{j,k} \|_{2,n} \right) \leq \frac{2^{-2\alpha k^*}}{(4^\alpha - 1)^{1/2}} \sum_{j \in J_1} \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n}.
\]

Thus, the first term on the right hand side of (26) is explicitly controlled by the \( \ell_1 \) norm of \( \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n} \) without condition on the design.

We are now ready to present the adaptive optimality of the MR-GL. Define

\[
M_{\alpha,q,n}^q = \sum_{j \in J_1} \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n}^q, \quad M_{\alpha,q,n}^{q,\text{BR}} = \sum_{j=1}^p (\lambda_{j,k,j}/\lambda_0) 2^{-q} \| \mathbf{f}^*_{j,k,j} \|_{2,n}^q,
\]

with \( \lambda_0 = \sigma \sqrt{(2/n) \log(p/\epsilon)} \), respectively as the \( q \)-power of the \( \ell_q \) “norm” of the norms \( \| \mathbf{f}^*_{j,k} \|_{\alpha,2,n} = \left\{ \sum_{k=k^*+1}^{\infty} 2^{2ak} \| \mathbf{f}^*_{j,k} \|_{2,n}^2 \right\}^{1/2} \) in (9) for the nonparametric components and the \( q \)-power of a weighted \( \ell_q \) norm of the \( \ell_2 \) norms of \( \mathbf{f}^*_{j,k} \) for all parametric and nonparametric components, for all \( q \geq 0 \), with \( q = 0 \) treated as the limit at \( q = 0^+ \). While \( M_{\alpha,q,n}^{q,\text{BR}} \) describes the complexity of the representation of all the components \( f_j \) at the baseline resolution level \( k = k^* \), \( M_{\alpha,q,n}^q \) measures the complexity of the nonparametric components \( \{ f_j, j \in J_1 \} \) beyond the baseline resolution. In the nonparametric AM with \( J_1 = \{ 1, \ldots, p \} \), \( \lambda_{j,k,j}/\lambda_0 = 2^{k^*/2}/n \log(p/\epsilon) + 1 \leq 3 \).

**Theorem 2** Suppose \( \mathbf{y} - \mathbf{f}^* \) has iid \( \mathcal{N}(0, \sigma^2) \) entries with \( \mathbf{f}^* = \sum_{j=1}^p \sum_{k \geq k^*} \mathbf{f}^*_{j,k} \) where \( \mathbf{f}^*_{j,k} \) are as in (31). Let \( k^* \geq k^* \) be integers satisfying \( 2^{k^* - 1} < 2 \log(p/\epsilon) \leq 2^{k^*} \) and \( 2^{k^*} \geq n^{1/(2\alpha + 1)} \) for some \( \alpha > 0 \). Let \( \hat{\mathbf{f}} \) be the MR-GL estimator in (19) with the estimated coefficients \( \hat{\lambda}_{j,k} = \sigma_n (2^{k^*/2} + \sqrt{2 \log(p/\epsilon)}) \) with \( \sigma_n = \sigma/n^{1/2} \), and constant \( A_0 > 1 \). Let \( \tilde{\mathbf{f}} \) be as in (31), \( \{ q, q_0 \} \subset [0, 1], 0 \leq \alpha_0 \leq \alpha \) with \( \alpha > 1/2 \), \( \gamma \) as in (29), \( \rho \) as in (30) and \( q(1 - \rho)/q_2 + \rho/q_1 = 1 \) with \( q_1 \geq \rho \). Then,

\[
\left\| \hat{\mathbf{f}} - \tilde{\mathbf{f}} \right\|_{2,n}^2 + \left\| \hat{\mathbf{f}} - \mathbf{f}^* \right\|_{2,n}^2 \leq \frac{n^{-2\alpha/(2\alpha + 1)} M_{\alpha,1,n}^2}{(4^\alpha - 1)^{1/2}} + C_{\text{pred}}(\xi, S) \left[ \sigma_n^2 M_{\alpha,0,q_2}^\rho M_{\alpha,q_1,n}^\rho + \lambda_0^{2-q_0} M_{\alpha,0,n}^{q_0,\text{BR}} \right],
\]

(35)
Corollary 1 Consider the nonparametric AM with $J_1 = \{1, \ldots, p\}$, $\alpha \geq \alpha_*$ and $\log(1/\epsilon) = O(\log p)$, for example $\epsilon = p^{-a_0}$ with fixed $a_0 \geq 0$.

(i) With $M_{a,\infty} = O(1)$ in (36),

$$
\|\hat{f} - f^*\|_{2,n}^2 \lesssim s_0 n^{-2\alpha/(2\alpha+1)} M_{a,\infty,n}^2 + s_0 \sigma_n^{2\alpha/(2\alpha+1)} M_{a,\infty,n}^{2/(2\alpha+1)} + s_0 \lambda_0^2.
$$

(ii) With $\alpha_0 = \alpha$, $\gamma = \min\{4\alpha/(2\alpha+1), 2 - q\}$, $q_1 = 1$ and $q_2 = q$ in (35),

$$
\|\hat{f} - f^*\|_{2,n}^2 \lesssim n^{-2\alpha/(2\alpha+1)} M_{a,2,n}^2 + \sigma_n^{2\alpha} M_{a,q,n}^2 + \lambda_0^{2-q_0} M_{q_0,n}^{\text{BR}}.
$$

(iii) With $\alpha_0 = 0$, $\gamma = (2-q)2\alpha/(2\alpha+1-q)$, $\rho = \gamma/(2\alpha)$, $q_1 = 1$ and $q_2 = q$ in (35),

$$
\|\hat{f} - f^*\|_{2,n}^2 \lesssim n^{-2\alpha/(2\alpha+1)} M_{a,2,n}^2 + \sigma_n^{2\alpha} M_{0,q,n}^{(1-\rho)} M_{a,1,n}^\rho + \lambda_0^{2-q_0} M_{q_0,n}^{\text{BR}}.
$$

The oracle inequalities in the above theorem use three terms to bound the in-sample squared prediction error of the MR-GL. The first term represents the ultra high resolution part of $f_j^*$ ignored by the MR-GL. While $\alpha_* > 0$ depends on the tuning parameter $k^*$ via $2k^* \geq n^{1/(2\alpha_*)}$, we can typically take a small $\alpha_* \in (0, 1/2)$ so that $\alpha > \alpha_*$ and it would be reasonable to expect

$$
n^{-2\alpha/(2\alpha+1)} M_{a,1,n}^2 = o(1) \sigma_n^\gamma C_{\text{pred}}(\xi, S) J_{q,\alpha,a_0}(k_*, k^*) M_{a,2,q,n}^2 M_{a,1,n}^\rho
$$

in (35) and to remove the first terms in (36), (37) and (38). For example, for $\alpha = 2$ and $\alpha_* \leq 1/2$, $2\alpha/(2\alpha + 1) \geq 2$ so that (36) holds without requiring (32) when $M_{a,\infty} = O(1)$ because $n^{-2\alpha/(2\alpha+1)} M_{a,1,n}^2 \leq s_0 n^{-2} M_{a,\infty,n}^2 \leq n^{-1} M_{a,\infty,n}^2$ when $s_0 \leq n$.

The third term on the right-hand sides of the oracle inequalities in Theorem 2 represents the risk of estimating the $f_j$ at the baseline resolution level $k = k_*$ with model uncertainty adjustment. When $d_j^*/(\log p/\epsilon)$ is uniformly bounded as in the nonparametric AM, this term is of the same order as the in-sample squared prediction error rate for the Lasso in linear regression. In general, this term matches the squared error rate for the group Lasso.

As the first term is typically of smaller order and the third term is unavoidable, the optimality of the oracle inequalities in Theorem 2 is largely determined by the second term on the right-hand side. When $\sigma_n^\gamma \geq n^{-2\alpha/(2\alpha+1)}$, i.e. $\gamma = 4\alpha/(2\alpha + 1)$, their rate optimality is evident because the rate matches the minimax risk for the estimation of a single smooth function. The rate optimality of the MR-GL in several special cases of the nonparametric AM are discussed below and compared with the existing results.
(ii) With \( q_0 = q \) and \( M_{q,n}^\alpha \lor M_{q,n}^{BR} = O(1) \) in (37),
\[
\| \hat{f} - f^* \|_{2,n}^2 \lesssim n^{-2\alpha/(2\alpha+1)} + ((\log p)/n)^{1-q/2}.
\] (41)

(iii) With \( q_0 = q \) and \( M_{\alpha,1,n} \lor M_{q,0,n}^q \lor M_{q,n}^{BR} = O(1) \) in (38),
\[
\| \hat{f} - f^* \|_{2,n}^2 \lesssim n^{-(2-q)\alpha/(2\alpha+1-q)} + ((\log p)/n)^{1-q/2}.
\] (42)

(iv) With \( q = q_0 = 0 \) and \( \#\{j : f_j^* \neq 0\} \leq s_0 \) in (38),
\[
\| \hat{f} - f^* \|_{2,n}^2 \lesssim s_0^{(2\alpha-1)/(2\alpha+1)} n^{-2\alpha/(2\alpha+1)} M_{\alpha,1,n}^{\alpha/(2\alpha+1)} + s_0(\log p)/n.
\] (43)

(v) When \( q = 1, q_0 = 0, M_{\alpha,0,n} \lor M_{\alpha,\infty,n} = O(1) \) and \( \#\{j : f_j^* \neq 0\} \leq s_0 \) in (38),
\[
\| \hat{f} - f^* \|_{2,n}^2 \lesssim s_0^{1/(2\alpha)} n^{-1/2} + s_0(\log p)/n.
\] (44)

We note that \( n^{-\gamma/2} \leq \{(\log p)/n\}^{1-q/2} \) when \( \gamma = 2-q \) and \( p \geq 3 \) in part (ii) above, and omit the proof of Corollary 1.

Compared with the existing results discussed in Section 1, (40) is of the same form as the oracle inequality (2) in Koltchinskii and Yuan (2010) and Raskutti et al. (2012), (41) is of the same form as the oracle inequality (4) in Yuan and Zhou (2010), (42) is of the same form as the oracle inequality (5) in Tan and Zhang (2019), and (43) is of the same form as the oracle inequality in Suzuki and Sugiyama (2012). As \( \| \sum_{j=1}^p f_j^*(x_j)\|_\infty = \| \sum_{j=1}^p f_j(x_j)\|_\infty \) when the maximum is taken over \((x_1, \ldots, x_p)^\top \in [0, 1]^p\), (44) improves upon the oracle inequality (3) of Raskutti et al. (2012) by a logarithmic factor in the first term. Thus, (35) unifies all the above results with a single oracle inequality, achieves rate minimaxity in these special cases as discussed in Section 1 and elucidated in the referenced papers, and provides new error bounds for \( 0 < \alpha_0 < \alpha \) and \( q \neq 0 \). More important, while the estimators in the above existing results all use tuning parameters depending on the smoothness index \( \alpha \) of the underlying component functions \( f_j \) and at least implicitly on the regularization parameters \( q \) and \( q_0 \), the MR-GL procedure does not involve any such tuning parameters.

Of course, the above comparisons are made under parallel conditions but not always in the same setting. While Theorem 2 concerns deterministic designs and involves the prediction factor \( C_{\text{pred}}^*(\xi, S) \), the above referenced papers mainly focus on random designs. In Section 4, we will derive \( L_2 \) oracle inequalities parallel to Theorem 2 and Corollary 1 under a population compatibility condition with random designs, allowing direct comparisons.
3.3 Connection to the group Lasso theory

To describe the connection of Theorem 1 to the theory of group Lasso, we write

\[
\hat{\beta} = \arg \min_{\beta} \left\{ \frac{1}{2} \| y - \sum_{k=1}^{g^*} U_k b_k \|_{2,n}^2 + A_0 \sum_{k=1}^{g^*} \lambda_k \| U_k b_k \|_{2,n} \right\}
\] (45)

as the group Lasso in the typical setting where \( g^* \) is the total number of groups, \( U_k \in \mathbb{R}^{n \times d_k} \) is composed of design vectors in the \( k \)-th group \( GR_k \), \( d_k = |GR_k| \) and \( b = (b_1^T, \ldots, b_{g^*}^T)^T \) with \( b_j \in \mathbb{R}^{d_k} \). This matches (18) with the label change \((j,k) \rightarrow k \) and \( g^* = |\mathcal{X}^*| \). Let \( U = (U_1, \ldots, U_{g^*}) \in \mathbb{R}^{n \times d^*} \) with \( d^* = \sum_{k=1}^{g^*} d_k \).

Let \( y \sim N(f^*, \sigma^2 I_{n \times n}) \) and \( \lambda_k = \sigma n^{-1/2}(d_k/2 + t_k) \). Theorem 1 asserts that for any \( f^* \in \mathbb{R}^n, \| \beta - \beta \|_{2,n}^2 \leq 8A_0 \sum_{k \in S} \lambda_k \| U_k \beta_k \|_{2,n} + 4(A_0 + 1)^2 C_{\text{pred}}(\xi, S) \sum_{k \in S} \lambda_k^2 \)

with at least probability \( 1 - \sum_{k=1}^{g^*} P\{N(0,1) > t_k\} \), where \( \xi = (A_0 + 1)/(A_0 - 1) \) and

\[
C_{\text{pred}}(\xi, S) = \sup_{b: \| Ub \|_{2,n} > 0} \left\{ \frac{\{ \sum_{k \in S} \lambda_k \| U_k b_k \|_{2,n} - \xi^{-1} \sum_{k \in S^*} \lambda_k \| U_k b_k \|_{2,n} \|^2}{\sum_{k=1}^{g^*} \| U_k b_k \|_{2,n}^2 \sum_{k \in S} \lambda_k^2} \right\}. \] (47)

Theorem 1 further suggests the choice \( S = \{ k : \| U_k \beta_k \|_{2,n} \geq A_0 \lambda_k \} \) to achieve

\[
\| U\hat{\beta} - U\beta \|_{2,n}^2 + \| U\hat{\beta} - f^* \|_{2,n}^2 \leq 2 \| U\beta - f^* \|_{2,n}^2 + C_{\text{pred}}^*(\xi, S) \sum_{k=1}^{g^*} \lambda_k^2 \land (\lambda_k \| U_k \beta_k \|_{2,n}) \]

with \( C_{\text{pred}}^*(\xi, S) = \max \{ 8A_0^2, 4(A_0 + 1)^2 C_{\text{pred}}(\xi, S) \} \). By allowing arbitrary \( f^*, \beta \) and \( S \), the oracle inequalities (46) and (48) improve upon the more familiar existing ones under the hard group sparsity \( S = \{ k : \| \beta_k \|_2 > 0 \} \) with \( f^* = U\beta \). Under the hard group sparsity, (46) gives the squared error rate \( \sum_{k \in S} \lambda_k^2 \approx \sum_{k \in S}(d_k + t_k^2)/n \) to exploit the benefit of the group sparsity (Huang and Zhang, 2010).

Let \( \mathcal{G}(\xi, S) = \{ b : \sum_{k \in S} \lambda_k \| U_k b_k \|_{2,n} \leq \xi \sum_{k \in S} \lambda_k \| U_k b_k \|_{2,n} \} \). The prediction factor \( C_{\text{pred}}(\xi, S) \) is closely related to the groupwise compatibility coefficient (CC)

\[
\kappa(\xi, S) = \inf \left\{ \frac{\| \sum_{k=1}^{g^*} U_k b_k \|_{2,n}}{\sum_{k \in S} \lambda_k \| U_k b_k \|_{2,n}} : b \in \mathcal{G}(\xi, S) \right\}. \] (49)

with \( \| \lambda S \|_2 = \left( \sum_{k \in S} \lambda_k^2 \right)^{1/2} \). This quantity reduces to the CC for the Lasso (van de Geer and Bühlmann, 2009) when \( d_k = 1 \). It is clear by definition that
\[ C_{\text{pred}}(\xi, S) \leq 1/\kappa^2(\xi, S). \]  

Bühlmann and van de Geer (2011) defined the groupwise CC as

\[
\kappa_0(\xi, S) = \inf \left\{ \frac{\| \sum_{k=1}^{g^*} U_k b_k \|^2_{2,n} \left( \sum_{k \in S} d_k \right)^{1/2}}{\sum_{k \in S} d_k^{1/2}} : b \in \mathcal{C}_0(\xi, S) \right\}
\]

(50)

with \( \mathcal{C}_0(\xi, S) = \{ b : \sum_{S^c} d_k^{1/2} \| b_k \|_2 \leq \xi \sum_S d_k^{1/2} \| b_k \|_2 \} \) to match group Lasso penalties satisfying \( \lambda_k \propto \sqrt{d_k} \). The two versions of the groupwise CC are equivalent up to a constant in the full nonparametric AM with \( 2^{k^* - 1} < 2 \log(p/\epsilon) \leq 2^{k^*} \) as in Theorem 2 under mild side conditions. Specifically,

\[
C_{\text{pred}}(\xi, S) \leq 1/\kappa^2(\xi, S) \leq \left\{ 2 c^* / \kappa_0(\xi_0, S_0) \right\}^2
\]

(51)

when \( c^* \| b_k \|_2^2 \leq \| U_k b_k \|_{2,n}^2 \leq c^* \| b_k \|_2^2 \) and \( \sqrt{2d_k} \leq \lambda_k n^{1/2} / \sigma \leq 2 \sqrt{2d_k} \) for all \( b_k \in \mathbb{R}^{d_k} \) and \( 1 \leq k \leq g^* \), \( S = \{ k : \| U_k \beta_k \|_{2,n} \geq A_0 \lambda_k \} \), \( S_0 = \{ k : c^* \| \beta_k \|_2 \geq A_0 \lambda_k \} \) and \( \xi_0 = 2 c^* / c_* \). Inequality (51) is a consequence of \( S \subseteq S_0 \) and \( \mathcal{C}(\xi, S) \subseteq \mathcal{C}_0(\xi_0, S_0) \) and the monotonicity properties of the two versions of the groupwise CC. A similar strategy will be used in our study of CC under random design in Section 4.

### 4 Random Designs

In the random design setting, out-of-sample squared prediction error \( \| \hat{f} - f^* \|_{2,2}^2 \) will be used in our analysis to evaluate the performance of the MR-GL estimator in (15), in addition to the in-sample squared prediction error \( \| \hat{f} - f^* \|_{2,n}^2 \) considered in Section 3. Moreover, instead of the empirical groupwise compatibility condition, we will impose a theoretical groupwise compatibility condition and prove that the empirical groupwise CC in (49) can be bounded from below by its population version up to a constant factor under the sample size condition \( n \gg s (\log s)^2 (\log d^*) (\log n) \).

#### 4.1 Equivalence between the empirical and population conditions on the design

The groupwise compatibility condition is closely related to the groupwise Restricted Eigenvalue (RE) condition in the sense that the groupwise CC is always no smaller than the groupwise RE. They characterize similar desirable properties of the design matrix which ensure the performance of regularized least squares and related methods. In general, the RE is aimed at bounding the \( \ell_2 \) error rate for estimating the coefficients and the CC is aimed at the prediction error.

For \( \ell_1 \) regularized LSE, error bounds for the Lasso and Dantzig selector were established in Bickel et al. (2009), Koltchinskii (2009) and van de Geer and Bühlmann (2009) among many others. van de Geer and Bühlmann
(2009) called CC the $\ell_1$ RE. Great effort has been devoted to establishing the empirical RE-type conditions under more interpretable conditions. For instance, Bickel et al. (2009), van de Geer and Bühlmann (2009), Zhang (2009) and Ye and Zhang (2010) provided lower bounds for the RE and CC in terms of the lower and upper sparse eigenvalues of the empirical Gram matrix or its population version. Raskutti et al. (2010) and Rudelson and Zhou (2012) proved that the RE condition is guaranteed by its population version without imposing a condition on the upper sparse eigenvalue condition on the design. Based on their results, the RE condition is understood to be of a weaker form than the restricted isometry property (Candes and Tao, 2005, 2007) and the sparse Riesz condition (Zhang and Huang, 2008). Additionally, Lecué and Mendelson (2014) and van de Geer and Muro (2014) proved the empirical RE-type conditions respectively under a high-order moment condition and a higher order isotropy condition.

As we have mentioned in Section 3.3, the prediction factor used in the oracle inequalities in Theorems 1 and 2 is bounded by the reciprocal of the corresponding groupwise CC as defined in (49).

For the group Lasso, groupwise RE and compatibility conditions have been used to derive oracle inequalities (Nardi and Rinaldo, 2008; Huang and Zhang, 2010; Lounici et al., 2011; Bühlmann and van de Geer, 2011; Negahban et al., 2012; Mitra and Zhang, 2016) but little has been done on the groupwise RE-type conditions. It is understood that the groupwise RE and CC can be bounded by the lower and upper sparse eigenvalues (Mitra and Zhang, 2016). However, it is unclear whether the groupwise RE-type conditions are guaranteed by its population versions and if so the sample size required. Here we carry out a systematic study of the groupwise RE-type conditions for uniformly bounded design variables by extending the analysis in Rudelson and Vershynin (2008) from the $\ell_1$ regularization to general group regularization. The core of our analysis is the following theorem which asserts that the sample RE in a general form is bounded from below by its population version up to a constant factor.

Our general result concerns a pair of norms $\| \cdot \|_1^*$ and $\| \cdot \|_2^*$, respectively related to the group regularization and the loss function of interest, such that

$$\|b_k\|_1 \leq \|b_k\|_1^*, \quad \sum_{k \in S_0} \|b_k\|_1^* \leq s^{1/2} \|b\|_2^*,$$

(52)

for all $b$ satisfying $\sum_{k \in S_0} \|b_k\|_1^* < \xi_0 \sum_{k \in S_0} \|b_k\|_1^*$ and a constant $s > 0$, where $S_0$ is a deterministic nonempty subset of $\{1, \ldots, g^*\}$. Recall that we group elements of vectors $b \in \mathbb{R}^{d^*}$ by writing $b = (b_1^\top, \ldots, b_{g^*}^\top)^\top$ with $b_k \in \mathbb{R}^{d_k}$ and $d^* = \sum_{k=1}^{g^*} d_k$. For $\xi_0 > 0$ and the norms and $S_0$ in (52), define the general deterministic cone as

$$\mathcal{C}_0(\xi_0, S_0) = \mathcal{C}_0(\xi_0, S_0; \| \cdot \|_1^*) = \left\{ b \in \mathbb{R}^{d^*} : \sum_{k \in S_0} \|b_k\|_1^* < \xi_0 \sum_{k \in S_0} \|b_k\|_1^* \right\}.$$
and the corresponding generalized groupwise RE and its population version by

$$\text{RE}_0(\xi_0, S_0) = \inf_{b \in C_0(\xi_0, S_0) : \|b\|_1 = 1} \frac{\|Ub\|_{2,n}}{\|b\|_2}, \quad \text{RE}_0^*(\xi_0, S_0) = \inf_{b \in C_0(\xi_0, S_0) : \|b\|_1 = 1} \frac{\|Ub\|_{L_2,n}}{\|b\|_2},$$

(53)

where $U$ is the design matrix and $\|Ub\|_{L_2,n} = (E[\|Ub\|_{L_2,n}^2])^{1/2}$. This includes as special cases the groupwise RE in Lounici et al. (2011) with $\|b_k\|_1 = (\lambda_k n^{1/2}/\sigma)\|b_k\|_2$ and $\|b\|_2^* = \max_{|T|=|S_0|}(\sum_{k \in T} \|b_k\|_2^2)^{1/2}$ and the groupwise CC in (50). We may also take $\|b\|_2^* = \|b\|_2$ for groupwise RE. The following theorem provides sufficient conditions under which the generalized groupwise RE in (53) is bounded from below by its population version.

**Theorem 3** Let $\text{RE}_0(\xi_0, S_0)$ and $\text{RE}_0^*(\xi_0, S_0)$ be as in (53) with a random matrix $U \in \mathbb{R}^{n \times d}$ and norms $\| \cdot \|_1^*$ and $\| \cdot \|_2^*$ satisfying (52). Suppose $U$ has independent rows and is uniformly bounded, $\|U\|_{\text{max}} \leq L_0$ for some constant $L_0$. Then,

$$E\left\{1 - \text{RE}_0^*(\xi_0, S_0)/\text{RE}_0^2(\xi_0, S_0)\right\} \leq \eta,$$

(54)

with $\eta = C_0 L_0 (s_1/n)^{1/2}(\log d^*)/(\log n)$, where $s_1$ is a constant satisfying $s_1 \geq (1 + \xi_0)^2 s/\text{RE}_0^2(\xi_0, S_0)$ with the $s$ in (52), $d^* = \sum_{k=1}^d d_k$, and $C_0$ is a numerical constant. Moreover, for constants $c_0$ and $\epsilon_1$ in $(0, 1),$

$$\mathbb{P}\left\{ \|Ub\|_{2,n} - 1 \leq c_0 \forall b \in C_0(\xi_0, S_0), \|Ub\|_2^2 = 1 \right\} \geq 1 - \epsilon_1$$

(55)

when $5e^{-c_0/\eta} \leq \epsilon_1$ and $4\pi s_1 L_0/n \leq \eta$.

The uniform boundedness condition $\|U\|_{\text{max}} \leq L_0$ is natural in the study of the nonparametric AM when the Fourier basis functions are used to build the MR-GL. For the CC in (50), $\|b_k\|_1^* = d_k^{1/2}\|b_k\|_2$, $\|b_k\|_2^2 = \sum_{k \in S_0} d_k^{1/2}\|b_k\|_2^2/s^{1/2}$, $s = \sum_{k \in S_0} d_k$ is the total number of variables for the groups with $k \in S_0$, and Theorem 3 yields

$$\mathbb{P}\left\{ \kappa_0(\xi_0, S_0) \geq \sqrt{1 - c_0} \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{L_2,n}^{1/2}}{\sum_{k \in S_0} d_k^{1/2}\|b_k\|_2} \right\} = 1 + o(1)$$

(56)

when $(s/n)(\log s)^2(\log d^*)/(\log n) = o(1)$. As $s/n \leq 1$ is required to have $\kappa_0(\xi_0, S_0) > 0$, the sample size requirement is optimal up to a logarithmic factor. Theorem 3 also yields a similar lower bound for the RE with the $\ell_2$ loss

$$\mathbb{P}\left\{ \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{2,n}}{\|b\|_2} \geq \sqrt{1 - c_0} \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{L_2,n}}{\|b\|_2} \right\} = 1 + o(1)$$

(57)
under the same sample size condition for the same cone as in \eqref{eq:tropp consc 0}.

To bound the CC \eqref{eq:tropp consc} more directly associated with the prediction factor in \eqref{eq:tropp consc pred}, we still need to deal with random cones involving the empirical norms \(\|U_k b_k\|_{2,n}\) and possibly random \(S \subset \{1, \ldots, g^*\}\) instead of the deterministic \(S_0\). To this end, we provide in the following lemma the equivalence between the empirical norm \(\|U_k b_k\|_{2,n}\) and its population version \(\|U_k b_k\|_{L_2,n}\) with an application of the non-commutative Bernstein inequality \cite{Tropp:2012}.

\textbf{Lemma 1}  Let \(U_k \in \mathbb{R}^{n \times d_k}\) be random matrices with independent rows \(r_k^i\) satisfying 
\[
\mathbb{P}\left\{ \|r_k^i\|_2 \leq L_0 d_k^{1/2}, \forall i \leq n \right\} = 1 \text{ for some constant } L_0 > 0. \]
Let \(\nu_{+,k} = \max\|b_k\|_{2,n} = 1 \mathbb{E}[\|U_k b_k\|_{2,n}^2]\) and \(\nu_{-,k} = \min\|b_k\|_{2,n}\). Then,
\[
\mathbb{P}\left\{ \max_{1 \leq k \leq g^*} \max_{\|b_k\|_{2,n} = 1} \left( \|U_k b_k\|_{2,n}^2 - \mathbb{E}[\|U_k b_k\|_{2,n}^2] \right) > c_0 \right\} \leq \epsilon_1 \quad \text{(58)}
\]
when \(\sum_{k=1}^{g^*} 2d_k \exp\left[ - nc_0^2 / \left( 2d_k L_0^2 (\nu_{+,k} + c_0 / 3) \right) \right] \leq \epsilon_1\). Moreover,
\[
\mathbb{P}\left\{ \max_{1 \leq k \leq g^*} \max_{\mathbb{E}[\|U_k b_k\|_{2,n}] > 0} \left( \mathbb{E}[\|U_k b_k\|_{2,n}^2] - 1 \right) > c_0 \right\} \leq \epsilon_1 \quad \text{(59)}
\]
when \(\sum_{k=1}^{g^*} 2d_k \exp\left[ - nc_0^2 / \left( 2d_k (L_0^2 / \nu_{-,k}) (1 + c_0 / 3) \right) \right] \leq \epsilon_1\).

In Lemma 1, \eqref{eq:tropp consc 1} states that with probability at least \(1 - \epsilon_1\), the squared empirical norm \(\|U_k b_k\|_{2,n}^2\) and its population version \(\|U_k b_k\|_{L_2,n}^2 = \mathbb{E}[\|U_k b_k\|_{2,n}^2]\) differ by at most a small fraction of \(\|b_k\|_{2,n}^2\) simultaneously for all \(b_k\) and \(k\), and \(\mathbb{E}[\|U_k b_k\|_{2,n}] > 0\) implies
\[
\mathbb{P}\left\{ c_- \|b_k\|_{2,n}^2 \leq \|U_k b_k\|_{2,n}^2 \leq c_+ \|b_k\|_{2,n}^2 \forall b_k, k \leq g^* \right\} \geq 1 - \epsilon_1 \quad \text{(60)}
\]
with \(c_- = \nu_{-,k} - 1 - c_0\) and \(c_+ \leq \nu_{+,k} + c_0\).

With Theorem 3 and Lemma 1 we are ready to study the (empirical) groupwise CC. Given \(\xi_0 > 0\) and a deterministic subset \(S_0\) of \(\{1, \ldots, g^*\}\), define
\[
\pi(\xi_0, S_0) = \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{L_2,n} \lambda_{S_0}}{\sum_{k \in S_0} \lambda_k \|U_k b_k\|_{L_2,n}} \quad \text{(61)}
\]
as the population version of the groupwise CC in \eqref{eq:tropp consc}, with \(\nu_{S_0} = \left( \sum_{k \in S_0} \lambda_k^2 \right)^{1/2}\) and \(C_0(\xi_0, S_0) = \{b : \sum_{k \in S_0} \lambda_k \|U_k b_k\|_{L_2,n}^2 \leq \xi_0 \sum_{k \in S_0} \lambda_k \|U_k b_k\|_{L_2,n}^2\}\).

\textbf{Corollary 2} Let \(U\) be a random matrix with independent rows and \(\|U\|_{\text{max}} \leq L_0\) for some constant \(L_0\). Let \(\xi > 0\), \(S_0\) be a deterministic subset of \(\{1, \ldots, g^*\}\) and \(\nu(\xi, S_0)\) as in \eqref{eq:tropp consc} with \(S = S_0\). Let \(c_0 \in (0, 1)\), \(\xi_0 = \xi \sqrt{(1 - c_0)/(1 + c_0)}\) and \(\pi(\xi_0, S_0)\) be as in \eqref{eq:tropp consc 2}. Let \(\nu_{-,k} = \min_{1 \leq k \leq g^*} \nu_{-,k}\) with the \(\nu_{-,k}\) in Lemma 1. Then,
\[
\mathbb{P}\left\{ \nu(\xi, S_0) \geq \sqrt{(1 - c_0)/(1 + c_0)} \pi(\xi_0, S_0) \right\} \geq 1 - 2\epsilon_1 \quad \text{(62)}
\]
when \(c_0\) and \(\epsilon_1\) satisfy the conditions for \eqref{eq:tropp consc 5} and \eqref{eq:tropp consc 6} with \(s = \|\nu_{S_0}\|_{2,n}^2 / (\sigma^2 \nu_{-})\).
This can be seen as follows. In the event in (59), \( C(\xi, S_0) \subseteq C_0(\xi_0, S_0) \) so that
\[
\frac{\kappa(\xi, S_0)}{(1 + c_0)^{-1/2}} \geq \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{2,n} \lambda_{S_0}}{\lambda_{k}\|U_k b_k\|_{L_{2,n}}} = \inf_{b \in C_0(\xi_0, S_0)} \frac{\|Ub\|_{2,n} w_{S_0}}{\lambda_{k}\|U_k b_k\|_{L_{2,n}}}
\]
with \( w_k = \lambda_k n^{1/2}/(\sigma \nu_0^{1/2}) \) to guarantee \( \|b_k\|_1 \leq 1 \) \( \|b_k\|_1 \) in the application of (55) with \( \|b_k\|_1^* = w_k \|U_k b_k\|_{L_{2,n}} \) and \( \|b\|_2 = \sum_{k \in S} w_k \|U_k b_k\|_{L_{2,n}}/\|w_{S_0}\|_2 \). Here the scaling of \( w_k \) only impacts the bound through \( s = \|\lambda_{S_0}\|_2^2 n/(\sigma^2 \nu_0) \) in the conditions for (55).

### 4.2 Out-of-sample error bounds in nonparametric AM

In this subsection, we derive the \( L_2 \) prediction error bound for the MR-GL in the nonparametric AM. In this model, a response vector \( y = (y_1, \ldots, y_n)^\top \) and a random design matrix \( X = (x_{i,j})_{n \times p} \) are observed such that
\[
y_i = f^*(x_{i,1}, \ldots, x_{i,p}) + \varepsilon_i, \quad f^* = f^*(x_1, \ldots, x_p) = \sum_{j=1}^p f_j^* (x_j),
\]
where \( \varepsilon_i \) are iid \( N(0, \sigma^2) \) variables independent of \( X \). We write the components \( f_j^*(\cdot) \) in the following multi-resolution expansion in a uniformly bounded basis,
\[
f_j^*(x) = \sum_{k=k_0}^{\infty} f_{j,k}^*(x), \quad f_{j,k}^*(x) = \sum_{\ell=1}^{2^{k+\nu(k-1)}} u_{j,k,\ell}(x) \beta_{j,k,\ell}^*, \quad \|u_{j,k,\ell}\|_\infty \leq L_0,
\]
for some fixed constant \( L_0 \). For abbreviation, we write \( f_j^* = f_j^*(x_j) \) and \( f_{j,k}^* = f_{j,k}^*(x_j) \) as functions of \( (x_1, \ldots, x_p)^\top \) depending on \( x_j \) only.

The MR-GL estimator \( \hat{\beta} \) in (18) yields an estimator of the regression function \( f^* \),
\[
\hat{f} = \sum_{j=1}^p \sum_{k=k_0}^{k^*} \hat{f}_{j,k}, \quad \hat{f} = \hat{f}(x_1, \ldots, x_p), \quad \hat{f}_{j,k} = \hat{f}_{j,k}(x_j),
\]
with \( \hat{f}_{j,k}(x_j) = \sum_{\ell=1}^{2^{k+\nu(k-1)}} u_{j,k,\ell}(x_j) \beta_{j,k,\ell} \) through the basis functions \( u_{j,k,\ell}(\cdot) \) in (64).

For deterministic or random functions \( h(x_1, \ldots, x_p) \), the \( L_2 \) norm is defined as
\[
\|h\|_{L_2} = \|h\|_{L_{2,n}} = \left( \int_{R^p} h^2(x_1, \ldots, x_p) G(dx_1, \ldots, dx_p) \right)^{1/2}
\]
with \( G(x_1, \ldots, x_p) = n^{-1} \sum_{i=1}^n G_i(x_1, \ldots, x_p) \), where \( G_i \) is the joint distribution of the \( i \)-th row of the design matrix \( X \). The loss function \( \|\hat{f} - f^*\|_{L_2}^2 \) measures the out-of-sample prediction error as it is the expected difference between the predictions by the estimated \( \hat{f} \) and true \( f^* \) at an out-of-sample random point \( x \in \mathbb{R}^p \) with joint distribution \( G \). In (66) \( h \in \mathbb{R}^n \) is understood as the realization of \( h(\cdot) \) in the sample and its elements are given by \( h(x_{i,1}, \ldots, x_{i,p}) \) with the \( x_{i,j} \) in (64).
For $\mathbf{U} \in \mathbb{R}^{n \times d'}$ and $\mathbf{U}_{j,k} \in \mathbb{R}^{n \times 2^{k^\ast \vee (k-1)}}$ in (16) with the basis functions $u_{j,k,\ell}(\cdot)$ in (64) and $d' = 2^k \cdot p$ and the corresponding vectors $\mathbf{b} \in \mathbb{R}^{d'}$ and $\mathbf{b}_{j,k} \in \mathbb{R}^{2^{k^\ast \vee (k-1)}}$ with elements $b_{j,k,\ell} \in \mathbb{R}$, the vector version of the norm is useful as it gives $\| \mathbf{U} \mathbf{b} \|_{L^2,n}$ and $\|\mathbf{U}_{j,k} \mathbf{b}_{j,k}\|_{L^2,n}$ the meaning of $\| \mathbf{h} \|_{L^2,n}$ and $\| \mathbf{h}_{j,k} \|_{L^2,n}$ with

$$h(x_1, \ldots, x_p) = \sum_{j=1}^{p} \sum_{k=k_\ast}^{k^\ast} h_{j,k}(x_j), \quad h_{j,k}(x_j) = \sum_{\ell=1}^{\| \mathbf{b}_{j,k,\ell} \|_{\infty}} u_{j,k,\ell}(x_j) b_{j,k,\ell},$$

for random $\mathbf{b}$. This saves explicit definition of the function $h$ when the vector $\mathbf{b}$ is explicitly given. For deterministic $\mathbf{b}$, $\| \mathbf{U} \mathbf{b} \|_{L^2,n} = \mathbb{E}[\| \mathbf{U} \mathbf{b} \|_{2,n}^2]$ and $\|\mathbf{U}_{j,k} \mathbf{b}_{j,k}\|_{L^2,n} = \mathbb{E}[\|\mathbf{U}_{j,k} \mathbf{b}_{j,k}\|_{2,n}^2]$, in agreement with the notation in Subsection 4.1.

Parallel to (9) and (34), we define for $\alpha > 0$ the Sobolev-type norms

$$\|f_j\|_{2,\alpha}^2 = \sum_{k=k_\ast+1}^{\infty} 2^{2\alpha k} \|\beta_{j,k}^\ast\|_{2}^2, \quad \|f_j\|_{\text{Sobolev,} \alpha}^2 = \|\beta_{j,k,\alpha}^\ast\|_2^2 + \|f_j\|_{\alpha,2}^2,$$

with $\beta_{j,k}^\ast = (\beta_{j,k,\ell}^\ast, \ell \leq 2^{k^\ast \vee (k-1)})^T$ for the coefficients $\beta_{j,k,\ell}^\ast$ in (64), and we measure the overall complexity of the true $f^* = \sum_{j=1}^{p} f_j^*$ by

$$M_{\alpha,q}^q = \sum_{j=1}^{p} \|f_j^*\|^q_{\alpha,2}, \quad M_{q}^{q,\text{BR}} = \sum_{j=1}^{p} \|\beta_{j,k,\alpha}^\ast\|_2^q,$$

for all $q \geq 0$, with $q = 0$ treated as the limit at $q = 0^+$. For $\alpha > 1/2$, the finiteness of the norm $\|f_j\|_{\alpha,2}$ implies the uniform absolute convergence of the series in (64) as

$$\|f_j\|_{\infty} \leq \sum_{k=k_\ast}^{\infty} L_0 2^{(k^\ast \vee (k-1))/2} \|\beta_{j,k}^\ast\|_{\infty} \leq L_0 \left\{2^{k^\ast/2} + (4(\alpha - 1/2) - 1)^{-1/2} 2^{-\alpha(1/2)k_\ast}\right\} \|f_j^\ast\|_{\text{Sobolev,} \alpha}.$$

To relate the norms in (9) and (67), define as in Lemma 1

$$\nu_+ = \max_{(j,k) \in \mathcal{X}} \max_{\|\mathbf{b}_{j,k}\|_2 = 1} \|\mathbf{U}_{j,k} \mathbf{b}_{j,k}\|_{L^2,n}^2, \quad \nu_- = \min_{(j,k) \in \mathcal{X}} \min_{\|\mathbf{b}_{j,k}\|_2 = 1} \|\mathbf{U}_{j,k} \mathbf{b}_{j,k}\|_{L^2,n}^2,$$

with $\mathcal{X}^* = \{1, \ldots, p\} \times \{k_\ast, \ldots, k^\ast\}$. As $\mathbb{E}[\|f_j^\ast\|^2_{\alpha,2,n}] = \sum_{k=k_\ast}^{\infty} L_0 2^{2\alpha k} \mathbb{E}[\|\mathbf{U}_{j,k} \beta_{j,k}^\ast\|_{L^2,n}^2],$

$$\mathbb{E}[\|f_j^\ast\|^2_{\alpha,2,n}] \leq \nu_+ \|f_j^\ast\|^2_{\alpha,2}, \quad \mathbb{E}[\|f_j^\ast\|^2_{\text{Sobolev,} \alpha,n}] \leq \nu_+ \|f_j^\ast\|^2_{\text{Sobolev,} \alpha},$$

with the $\nu_+$ in (69), so that the norms in (9) can be bounded by those in (67). Similarly, the complexity measures in (34) can be bounded by their population version (68) through

$$\mathbb{E}[M_{\alpha,q,n}^q] \leq \nu_+^q/2 M_{\alpha,q}^q, \mathbb{E}[M_{q,n}^{q,\text{BR}}] \leq \nu_+^q/2 M_{q,\text{BR}}^q, \quad q \leq 2.$$
For constants $\xi_0 > 0$ and deterministic $S_0 \subset \mathcal{X}^*$, define the population CC as

$$\kappa(\xi_0, S_0) = \inf_{b \in \mathcal{C}(\xi_0, S_0)} \frac{\|U b\|_{L_2,n} \left(\sum_{(j,k) \in S_0} \lambda_{j,k}^2\right)^{1/2}}{\sum_{(j,k) \in S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n}}$$

(71)

where $\mathcal{X}^*$ is as in (69), $U b = \sum_{(j,k) \in \mathcal{X}^*} U_{j,k} b_{j,k}$ and

$$\mathcal{C}_0(\xi_0, S_0) = \left\{ b : \sum_{(j,k) \in \mathcal{X}^* \setminus S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n} \leq \xi_0 \sum_{(j,k) \in S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n} \right\}$$

To bound the prediction factor (23) by the population CC via Corollary 2 in the error term $B_{S_0}$ in Theorem 1, we need

$$5 e^{-c_0/n} \leq \epsilon_1, \quad 4\pi s_1 L_0/n \leq \eta, \quad \frac{n c_0^2}{2^k \nu_0 \nu_-} \geq (4/3) \log(n p/\epsilon)$$

(72)

for a fixed $c_0 \in (0, 1)$ and small $\epsilon_1 \in (0, 1)$, where $L_0$ is as in (64), $s_1$ is a constant satisfying $s_1 \geq (1 + \xi_0)^2(4s/\nu_-)/\kappa^2(\xi_0, S_0)$, $\eta = C_0 L_0 (s_1/n)^{1/2} (\log s_1) \sqrt{\log(n) \log(np)}$ and $\nu_-$ is as in (69), with $s = \sum_{(j,k) \in S_0} 2^k \nu_+ (k-1)$ and $C_0$ being the numerical constant in Theorem 3. Under (72), $P\{\Omega_1\} \geq 1 - 2\epsilon_1$ with

$$\Omega_1 = \left\{ \begin{array}{l} \|U b\|_{L_2,n} - 1 \leq c_0 \forall b \in \mathcal{C}(\xi, S_0), \|U b\|_{L_2,n}^2 = 1 \\ \|U_{j,k} b_{j,k}\|_{L_2,n} - 1 \leq c_0 \forall \|U_{j,k} b_{j,k}\|_{L_2,n} \geq 1, \|b\|_1 \leq s_1^{1/2} \\ \|U_{j,k} b_{j,k}\|_{L_2,n}^2 - 1 \leq c_0 \forall \|U_{j,k} b_{j,k}\|_{L_2,n} = 1, (j, k) \in \mathcal{X}^* \end{array} \right\}$$

(73)

by (55) and (59) as in Corollary 2, where $\xi_0 = \sqrt{(1 + c_0)/(1 - c_0)} \xi$, and

$$\kappa(\xi_0, S_0) = \inf_{b \in \mathcal{C}(\xi_0, S_0)} \frac{\|U b\|_{L_2,n} \left(\sum_{(j,k) \in S_0} \lambda_{j,k}^2\right)^{1/2}}{\sum_{(j,k) \in S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n}}$$

(74)

and $\mathcal{C}(\xi, S_0) = \left\{ b : \sum_{(j,k) \in S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n} \leq \xi \sum_{(j,k) \in S_0} \lambda_{j,k} \|U_{j,k} b_{j,k}\|_{L_2,n} \right\}$. We note that the condition for (59) holds by the last inequality in (72) due to $d^* = \sum_{(j,k) \in \mathcal{X}^*} 2^k \nu_+(k-1) = 2^k \nu_- \leq np$. Similarly, the condition for (55) holds by the first two inequalities in (72) as $\lambda_{S_0}^2/n/(\sigma^2 \nu_-) = \sum_{(j,k) \in S_0} (2^k + \sqrt{2 \log(n p/\epsilon)})^2 / \nu_- \leq 4s/\nu_-$, with $2^k \nu_- \leq 2 \log(n p/\epsilon) \leq 2^k$, as in Theorem 2.

We also need to bound the contribution of $f_{j,k}^*$ to the error term $\overline{\Delta}_{S_0}$ in Theorem 1 for $(j, k) \in S_0$. To this end, we will prove $P\{\Omega_1 \setminus \Omega_2\} \leq \epsilon_2$ with

$$\Omega_2 = \left\{ \|f^* - \overline{f}\|_{L_2,n}^2 + 4 A_0 \sum_{(j,k) \in \mathcal{X}^* \setminus S_0} \lambda_{j,k} \|U_{j,k} \beta_{j,k}^*\|_{L_2,n} \leq \frac{\sigma^2 s_2}{n} \right\}$$

(75)
with the $\mathcal{H}^*$ in (69), $\xi(x_1, \ldots, x_p) = \sum_{(j,k) \in \mathcal{H}^*} f^*_{j,k}(x_j)$ and

$$s_2 = \frac{n}{\sigma^2} \left[ 2\|f^* - \hat{f}\|_{L_2}^2 + 4A_0(1 + c_0)\nu_+^{1/2} \sum_{(j,k) \in \mathcal{H}^* \setminus S_0} \lambda_{j,k} \|\beta^*_{j,k}\|_2 ight]$$

$$+ \min \left\{ \frac{(C_{\alpha-1/2}L_0M_{\alpha,1})^2 \log \epsilon_2}{n^{2(\alpha+\alpha^*)/(2\alpha+1)}}, \|f^* - \tilde{f}\|_{L_2}^2(1/\epsilon_2 - 2) \right\},$$

where $\alpha > 1/2$, $C_{\alpha} = 1/(4^{\alpha} - 1)^{1/2}$ and $f^*$ and $\xi$ are the realized vector version of $f^*$ and $\xi$, respectively. To bound the $L_2$ norm above, we notice that

$$\|f^* - \tilde{f}\|_{L_2}^2 = \left\| \sum_{j=1}^p \sum_{k=k^*+1}^{\infty} f^*_{j,k} \right\|_{L_2}^2 \leq C_2 2^{-2\alpha} M_{\alpha,1}^2 \leq \frac{(C_{\alpha}M_{\alpha,1})^2}{n^{2\alpha/(2\alpha+1)}}$$

by (33) and (70), and we may impose parallel to (32) the condition that

$$\|f^* - \tilde{f}\|_{L_2}^2 \leq C_2 \sum_{j=1}^p \sum_{k=k^*+1}^{\infty} \left\| f^*_{j,k} \right\|_{L_2}^2 \leq C_2 2^{-2\alpha} M_{\alpha,2}^2 \leq \frac{C_2 M_{\alpha,2}^2}{n^{2\alpha/(2\alpha+1)}}$$

with some constant $C_2^*$. As (78) is imposed only on the ultra high resolution components of the true $f^*$, it is in a much weaker form than the parallel side condition $\| \sum_{k=1}^p f_j \|_{L_2}^2 \leq \sum_{k=1}^p \| f_j \|_{L_2}^2$ typically imposed in the literature as in the references discussed below Corollary 1.

We are now ready to present our main oracle inequality for the nonparametric AM with random design.

**Theorem 4** Let $f^*$ and $f^*_{j,k}$ be as in (63) and (64), $\tilde{f}$ as in (75), and $\hat{f}$ as in (65) with the penalty levels $\lambda_{j,k} = \sigma(2^{k^*/2} + \sqrt{2 \log(p/\epsilon)})/n^{1/2}$ and $\{A_0, k^*, \alpha, \alpha, 1\}$ as in Theorem 2. Let $f^* \in \mathbb{R}^n$, $\tilde{f} \in \mathbb{R}^n$ and $\hat{f} \in \mathbb{R}^n$ be respectively the realizations of $f^*$, $\tilde{f}$ and $\hat{f}$ in the sample as in Theorem 2. Let $\Omega_0$, $\Omega_1$ and $\Omega_2$ be as in (20), (73) and (75) respectively. Let $\{\xi, \xi_0, c_0, s_1\}$ be as in (72) and (13), $\mathcal{H}^*$ as in (69), $s_2$ as in (76) and $\mathcal{K}(\xi_0, S_0)$ the population CC in (71) with a deterministic $S_0 \subset \mathcal{H}^*$.

(i) Let $\xi = (A_0 + 1)/(A_0 - 1)$. In the event $\Omega_0 \cap \Omega_1 \cap \Omega_2$,

$$\left\| \hat{f} - \tilde{f} \right\|_{L_2}^2 + \left\| \hat{f} - f^* \right\|_{L_2}^2 \leq \frac{4(A_0 + 1)^2 \|\lambda_s\|^2}{\{(1-c_0)/(1+c_0)\}^2 \mathcal{K}^2(\xi_0, S_0)} + \frac{2\sigma^2 s_{2}/n}{n}.$$ (79)

(ii) Let $s_1 \geq s_2$ and $\xi = (3A_0 + 1)/(A_0 - 1)$. In the event $\Omega_0 \cap \Omega_1 \cap \Omega_2$,

$$\left\| \hat{f} - \tilde{f} \right\|_{L_2}^2 + \left\| \hat{f} - f^* \right\|_{L_2}^2 \leq C_{A_0, \nu_- c_0} \left( \|\lambda_s\|^2 / \mathcal{K}^2(\xi_0, S_0) + \sigma^2 s_{2}/n \right).$$ (80)

where $C_{A_0, \nu_- c_0}$ is a constant depending on $(A_0, \nu_-, c_0)$ only with the $\nu_-$ in (69).

(iii) Suppose (63), (64) and (72) hold with $\epsilon_1 \in (0, 1)$ and $s_2 \leq s_1$. Then,

$$P\{\Omega_0\} \leq \epsilon/(2 \log(p/\epsilon))^{1/2}, \quad P\{\Omega_1\} \leq 2\epsilon_1, \quad P\{\Omega_1 \setminus \Omega_2\} \leq \epsilon_2.$$
(iv) Suppose (63) and (64) hold with a fixed $L_0$, $1/\kappa(\xi_0, S_0) = O(1)$ and $1/\nu_- = O(1)$ in (69). Let $s = \sum_{(j,k) \in S_0} 2^{k_0 + \nu(k-1)}$. Suppose $n \gg (s + s_2)(\log(s + s_2))^2(\log n) \log(np)$ and $n \gg 2^{k^*} \log(np)$. Then,
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim \sigma^2(s + s_2)/n. \tag{81}
\]

(v) Suppose the conditions in (iv) hold with $S_0 = \{(j, k) \in \mathcal{X}^* : \|\theta_{j,k}^*\|_2 \geq \lambda_{j,k}\}$, (78) holds, $\nu_+ = O(1)$ in (69) and $\alpha_0 \neq 1/q - 1/2$. Then, the oracle inequalities (36), (37) and (38) all hold with $\|\hat{f} - f^*\|^2_{2, n}$ replaced by $\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n}$ on the left-hand side and $\{M_{\alpha,q,n}, M_{q_0,n}^{q, \text{BR}}\}$ replaced by the population version $\{M_{\alpha, q}, M_{q_0, \text{BR}}\}$ in (68) on the right-hand side, and with the respective $\{\alpha_0, q, q_0, q_1, q_2, \gamma, \rho\}$ in Theorem 2.

We note that Theorem 4 (i) and (ii) are analytical and probability is involved only in Theorem 4 (iii), (iv) and (v). In view of Theorem 2 and Corollary 1, Theorem 4 (v) implies the following corollary. Theorem 4 uses quantities $\nu_\pm$ in (69) to bound the norms $\|\theta_{j,k}\|_2$ and $\|U_{j,k}\theta_{j,k}\|_{L_2,n}$ by each other with these constant factors, largely due to the use of the $\ell_2$ norm to define the complexity measures in (67) and (68).

**Corollary 3** Suppose the conditions of Theorem 4 (v) hold with $\alpha \geq \alpha_\ast$, $\alpha_0 \neq 1/q - 1/2$ and $\log(1/\epsilon) = O(\log p)$.

(i) If $\# \{j : f_j^* \neq 0\} \leq s_0$ and $M_{\alpha, \infty} = O(1)$, then
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim s_0 n^{-2\alpha/(2\alpha + 1)} + s_0 (\log p)/n. \tag{82}
\]

(ii) If $M_{\alpha,q} \vee M_{q, \text{BR}}^q = O(1)$, then
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim n^{-2\alpha/(2\alpha + 1)} + ((\log p)/n)^{1-q/2}. \tag{83}
\]

(iii) If $M_{\alpha,1} \vee M_{q_0,q}^q \vee M_{q, \text{BR}}^q$ = $O(1)$, then
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim n^{-(2-q)\alpha/(2\alpha + 1) - q} + ((\log p)/n)^{1-q/2}. \tag{84}
\]

(iv) If $\# \{j : f_j^* \neq 0\} \leq s_0$, then
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim s_0^{(2\alpha - 1)/(2\alpha + 1)} n^{-2\alpha/(2\alpha + 1)} M_{\alpha,1}^{2/(2\alpha + 1)} + s_0 (\log p)/n. \tag{85}
\]

(v) If $M_{\alpha,1} \vee M_{\alpha, \infty} = O(1)$ and $\# \{j : f_j^* \neq 0\} \leq s_0$, then
\[
\|\hat{f} - f^*\|^2_{L_2} + \|\hat{f} - f^*\|^2_{2, n} \lesssim s_0^{1/(2\alpha)} n^{-1/2} + s_0 (\log p)/n. \tag{86}
\]

As we discussed below Corollary 1, (82), (83), (84), (85) and (86) are directly comparable respectively with (2) of Kolchinskii and Yuan (2010) and Raskutti et al. (2012), (4) of Yuan and Zhou (2015), (5) of Tan and Zhang (2019), the oracle inequality of Suzuki and Sugiyama (2012) and (3) of Raskutti et al. (2012) under population compatibility conditions for random design.
5 Appendix

Proof of Theorem 1. (i) Applying Lemma 4 of Tan and Zhang (2019) with their \(\| \cdot \|_{F,j} = 0\) to functional effects \(f_{j,k}\), we find that for any \(\beta\) and \(S \subset \mathcal{X}^*\)

\[
\frac{1}{2}\| \hat{f} - \hat{f}\|_{2,n}^2 + \frac{1}{2}\| f - f^*\|_{2,n}^2 + (A_0 - 1)\text{pen}_S(\beta - \beta) \\
\leq \frac{1}{2}\| \hat{f} - f^*\|_{2,n}^2 + 2A_0\text{pen}_S(\beta) + (A_0 + 1)\text{pen}_S(\beta - \beta).
\]

If \((A_0 + 1)\text{pen}_S(\beta - \beta) \leq (A_0 - 1)\text{pen}_S(\beta - \beta)\), we have

\[
\| \hat{f} - \hat{f}\|_{2,n}^2 + \| \hat{f} - f^*\|_{2,n}^2 \leq \| \hat{f} - f^*\|_{2,n}^2 + 4A_0\text{pen}_S(\beta) = \Delta_S.
\]

Otherwise, by the definition of \(C_{\text{pred}}(\xi, S)\),

\[
\{\text{pen}_n(\{h_{j,k}\}; S) - \xi^{-1}\text{pen}_n(\{h_{j,k}\}; S^c)\}^2 \leq C_{\text{pred}}(\xi, S)\|\lambda S\|_2^2 = B_S/(A_0 + 1)^2
\]

with \(h_{j,k} = (f_{j,k} - \hat{f}_{j,k})/\| \hat{f} - \hat{f}\|_{2,n}\), so that

\[
\frac{1}{2}\| \hat{f} - \hat{f}\|_{2,n}^2 + \frac{1}{2}\| f - f^*\|_{2,n}^2 \leq \frac{1}{2}\Delta_S + B_S^{1/2}\| \hat{f} - \hat{f}\|_{2,n}.
\]

This gives \(\| \hat{f} - f^*\|_{2,n}^2 \leq \Delta_S + B_S\). We also have \(z^2/2 \leq \Delta_S/2 + B_S^{1/2}z\) with \(z = (\| \hat{f} - \hat{f}\|_{2,n}^2 + \| \hat{f} - f^*\|_{2,n}^2)^{1/2}\), so that \(z^2 \leq (B_S^{1/2} + \sqrt{B_S + \Delta_S})^2 \leq 4B_S + 2\Delta_S\). For (26), we note that \(\| \hat{f}_{j,k}\|_{2,n} \leq A_0\lambda_{j,k}\) on \(\mathcal{X}^* \setminus S\).

(ii) When \(y - f^*\) has iid \(N(0, \sigma^2)\) entries, \(\| P_{j,k}(y - f^*)\|_{2,n}^2 n/\sigma^2\) has the chi-square distribution with \(\text{rank}(P_{j,k}) \leq d_{j,k}\) degrees of freedom. Thus, by (21) and the Gaussian concentration inequality (Borell, 1975; Kwapień, 1994)

\[
P \left\{ \sup_{(j,k) \in \mathcal{X}^*} \| P_{j,k}(y - f^*)\|_{2,n}/\lambda_{j,k} > 1 \right\}
\leq \sum_{(j,k) \in \mathcal{X}^*} P \left\{ N(0, 1) > \lambda_{j,k}\sqrt{n}/\sigma - \sqrt{d_{j,k}} \right\}
\leq \frac{\epsilon}{\sqrt{4\pi \log(p/\epsilon)}} + |J_1| \sum_{k=k_0+1}^{k_*} \exp \left[ -\frac{\{2^{k/2} - 2^{(k-1)/2} + \sqrt{2\log(p/\epsilon)}\}^2}{2} \right]
\leq \frac{\epsilon}{\sqrt{2 \log(p/\epsilon)}} \left\{ \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} e^{-(1-2^{-1/2})2^{k-1}-(1-2^{-1/2})2^k} \right\}
\leq \frac{\epsilon}{\sqrt{2 \log(p/\epsilon)}}
\]

when \(2 \log(p/\epsilon) \geq 1\). In the above, \(\lambda_{j,k}\sqrt{n}/\sigma - \sqrt{d_{j,k}} = \sqrt{2\log(p/\epsilon)}\) for \(j \in J_0\) \((k = k_*\) necessarily\) and \(\lambda_{j,k}\sqrt{n}/\sigma - \sqrt{d_{j,k}} = 2^{k/2} - 2^{(k-1)/2} + \sqrt{2 \log(p/\epsilon)}\) for \(j \in J_1\). \(\square\)
Proof of Proposition 1. Let $t_0 > 0$ and
\[
\bar{k} = \begin{cases} k_*, \\ \{k_* \vee \left\lceil \log_2 \left( \frac{\sigma_n(\gamma-1)/(\alpha-1/2)}{t_0} \right) \right\rceil \} \wedge k^*, \end{cases} \quad \gamma = 1, \quad \gamma > 1,
\]
with integers $0 \leq k_* \leq k^*$. We shall first prove
\[
\sum_{k = k_* + 1}^{k^*} \lambda_k \min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \lambda_k \right) \leq \sigma_n \gamma \left\{ \sigma_n \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \mathbf{f}_{j,k} \right\}_n \| \mathbf{f}_j \|_{\alpha_0,2,n} + \frac{t_0^{1/2}}{q} \gamma \mathbf{f}_{j,1} \| \mathbf{f}_j \|_{\alpha,2,n} \right\}.
\]
As $\lambda_k \leq \sigma_n 2^k$, we have
\[
\sum_{k = k_* + 1}^{k^*} \lambda_k \min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \lambda_k \right) \leq \sum_{k = k_* + 1}^{k^*} \sigma_n ^{2} \frac{2^k}{\min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \sigma_n 2^k \right)}.
\]
Let $\bar{r} = \sigma_n^{-1} \gamma(1)/(\alpha-1/2)$, $\bar{r} = 2^{k_*}$ for $\alpha = 1/2$. We write $\overline{k}$ as
the positive integer satisfying $k_* \leq \overline{k} \leq k^*$, $\sigma_n \overline{k} \leq \bar{r} \vee 2^{k_*}$ and $\bar{r} \wedge 2^{k_* + 1} \leq \overline{k} + 1$. For $1 - q/2 - \alpha_0 q \leq 0$, we have $\gamma = 2 - q$ and
\[
\sum_{k = k_* + 1}^{k^*} \sigma_n^{2} \frac{2^k}{\min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \sigma_n 2^k \right)} \leq \sum_{k_* < k \leq \overline{k}^*} 2(1-\alpha_0 q) \| \mathbf{f}_{j,k} \|_{2,n}^2 + \sum_{\overline{k} < k \leq k^*} \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{2,n}^2
\]
\[
\leq \sigma_n^{2} \frac{2^k}{\min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \sigma_n 2^k \right)} \leq \sum_{k_* < k \leq \overline{k}^*} \mathbf{f}_{j,k} \|_{2,n}^2 + \sum_{\overline{k} < k \leq k^*} \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{2,n}^2
\]
\[
\leq \sigma_n^{2} \frac{2^k}{\min \left( \| \mathbf{f}_{j,k} \|_{2,n}, \sigma_n 2^k \right)} \leq \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{2,n}^2 + \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{2,n}^2
\]
Thus, as $2^{-\overline{k}(1)/(\alpha-1/2)} \leq \bar{r}^{1/(1-\alpha)} = \sigma_n^{-1} \gamma t_0^{1/(1-\alpha)}$, we have
\[
\sum_{k = k_* + 1}^{k^*} \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \mathbf{f}_{j,k} \|_{2,n}^2 + \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{1/2,2,n}^2
\]
Similarly, for $1 - q/2 - \alpha_0 q > 0$, we have
\[
\sum_{k = k_* + 1}^{k^*} \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{2,n}^2 + \sigma_n \gamma \left( \frac{t_0^{-1} - \alpha_0 q}{2-\alpha_0 q} \right) \| \mathbf{f}_{j,k} \|_{1/2,2,n}^2
\]
\[
\begin{align*}
\leq & \quad \sigma_n^{2-q} \sum_{k^* < k \leq \mathcal{T}} 2^{k(1-q/2-\alpha q)} \|2^{\alpha k} f_{j,k}\|_2^n + \sum_{k < k^*} \sigma_n^{2-k(\alpha-1/2)} \|2^{\alpha k} f_{j,k}\|_2^n \\
\leq & \quad J_{1-q/2-\alpha q}(k^*, \mathcal{T}) \sigma_n^{2-q} t_{1-q/2-\alpha q} \|f_j\|_{\alpha,2,n} + J_{\alpha-1/2}(k^*, \mathcal{T}) \sigma_n^{1/2-\alpha} \|f_j\|_{\alpha,2,n}.
\end{align*}
\]

For \(\alpha > 1/2\), \((2-q) - (1-q/2 - \alpha q) + (\gamma - 1)/(\alpha - 1/2) = \gamma\) by algebra. For \(\alpha = 1/2\), \(q = 1\), \(\mathcal{T} = k^*\) and \(J_{1-q/2-\alpha q}(k^*, \mathcal{T}) = 0\). Thus, by the definition of \(\mathcal{T}\), \(\sigma_n^{2-q} t_{1-q/2-\alpha q} = \sigma_n^{0} t_0^{-(1-q/2-\alpha q)}\), so that \((89)\) also holds for \(1 - q/2 - \alpha q > 0\). Consequently, \((87)\) follows from \((88)\) and \((89)\). Finally, we obtain \((30)\) by minimizing

\[
t_0^{-(1-q/2-\alpha q)} J_{1-q/2-\alpha q}(k^*, k^*) \|f_j\|_{\alpha,2,n} + t_0^{\alpha-1/2} J_{\alpha-1/2}(k^*, k^*) \|f_j\|_{\alpha,2,n}
\]

over \(t_0 > 0\). Note that \((30)\) follows from \((87)\) directly when \(\alpha = \alpha_0 = 1/2 = q/2\). \(\square\)

**Proof of Theorem 2.** It follows from \((32)\) and \((33)\) that \(2\|f^* - \overline{f}\|_2^n\), the first term in \((26)\), is bounded by the first term on the right-hand side of \((35)\). For the second term in \((26)\), Proposition 1 implies that the summation is bounded by

\[
\sum_{(j,k) \in \mathcal{K}^*} \lambda_{j,k}^\alpha (\lambda_{j,k} \|f_{j,k}\|_2^n)
\]

\[
\leq 4 \sum_{j \in J_1} \sum_{k = k_* + 1} \left( \sigma_n^{2} 2^k \right) \lambda_{j,k}^\alpha \|f_{j,k}\|_2^n + \sum_{j = 1}^p \lambda_{j,k_*}^\alpha \|f_{j,k_*}\|_2^n
\]

\[
\leq 4 \sum_{j \in J_1} \sigma_n^{2} \lambda_{j}^\alpha \|f_{j}\|_{\alpha,2,n} \{ \|f_{j}\|_{\alpha,2,n} \}^{1-p} \|f_{j}\|_{\alpha,2,n}^p + \lambda_0^{2-q_0} \sum_{j = 1}^p w_{j,k_*}^2 \|f_{j,k_*}\|_2^n
\]

\[
\leq 4 \sigma_n^{2} \lambda_{j}^\alpha \|f_{j}\|_{\alpha,2,n} \{ \|f_{j}\|_{\alpha,2,n} \}^{1-p} \|f_{j}\|_{\alpha,2,n}^p + \lambda_0^{2-q_0} M_{q_0,n,\alpha}^{2-q_0} M_{q_0,n,\alpha}^{2-q_0}
\]

as \(f_{j,k} = f_{j,k}^*\). This gives \((35)\). When \(C_{\text{pred}}(\xi, S) = O(1), (36), (37)\) and \((38)\) follows from \((35)\) by plugging-in the respective values of \((\gamma, q, \rho, q_1, q_2)\). For \((36), M_{q_0,n,\alpha}^2 \leq s_0 M_{q_0,n,\alpha}^{2-q_0} M_{q_0,n,\alpha}^2 \leq (s_0 M_{q_0,n,\alpha}^2)^{1-\rho} (s_0 M_{q_0,n,\alpha}^2)^{\rho} = s_0 M_{q_0,n,\alpha}^{2-\gamma}.\) For \((37), M_{\alpha,1,n} \leq M_{\alpha,q,n}^2 \) and \(q(1-\rho) + \rho = 2 - \gamma\) by Remark 1. \(\square\)

**Proof of Theorem 3.** By the scale invariance of the ratios in \((53)\),

\[
\text{RE}_0(\xi, S_0) = \inf_{b \in \mathcal{C}_0(\xi, S_0)} \frac{\|Ub\|_{2,n}}{\|b\|_2^n} \geq \text{RE}_0(\xi, S_0) \inf_{b \in \mathcal{C}_0(\xi, S_0)} \frac{\|Ub\|_{2,n}}{\|b\|_2^n}
\]

with \(\mathcal{C}_0(\xi, S_0) = \{ b : b \in \mathcal{C}_0(\xi, S_0), \mathbb{E}[\|Ub\|_{2,n}^2] = 1 \} \). For \(b \in \mathcal{C}_0(\xi, S_0)\),

\[
\|b\|_1 \leq \sum_{k=1}^{g_0} \|b_k\|_1^* \leq (1 + \xi) \sum_{k \in S_0} \|b_k\|_1^* \leq (1 + \xi) s_{1/2}^2 \|b\|_2^* \leq \frac{(1 + \xi) s_{1/2}^2}{\text{RE}_0(\xi, S_0)} \leq s_{1/2}^2.
\]

Recall that \(s_1\) is a constant satisfying \(s_1 \geq (1 + \xi)^2 s/\text{RE}_0^2(\xi, S_0)\). Define

\[
M_U = \sup_{b \in \mathcal{C}_0(\xi, s_1)} \frac{\|Ub\|_{2,n}^2}{\|b\|_{2,n}^2 - 1}
\]

(90)
with $\mathcal{C}_1^*(s_1) = \{b \in \mathbb{R}^d : \|b\|_1 \leq s_1^{1/2}, \mathbb{E}[\|U b\|_{2,n}^2] = 1\}$. We have
\[
\left\{1 - \text{RE}_0^2(\xi, S_0)/\overline{\text{RE}}_0^2(\xi, S_0)\right\} \leq M_U.
\]

Let $r^i$ be the $i$-th row of $U$. To bound $M_U$, we write it as
\[
M_U = \sup_{b \in \mathcal{C}_1^*(s_1)} \left\{\frac{1}{n} \sum_{i=1}^n \left(\langle r^i, b \rangle - \mathbb{E}\langle r^i, b \rangle\right)^2 \right\}.
\]

As in Rudelson and Vershynin (2008), we shall apply the Gaussian symmetrization and Dudley’s inequality to bound the expectation $\mathbb{E}[M_U]$. By the Gaussian symmetrization [Lemma 6.3 and (4.8) in Ledoux and Talagrand (1991)],
\[
\mathbb{E}[M_U] \leq \mathbb{E}[M_U^{\text{sym}}] \quad \text{with} \quad M_U^{\text{sym}} = \sup \left\{\left\{\frac{\sqrt{2\pi}}{n} \sum_{i=1}^n z_i (\langle r^i, b \rangle)^2 : b \in \mathcal{C}_1^*(s_1)\right\} \right\},
\]
where $\{z_i, i \leq n\}$ is a set of independent $N(0, 1)$ variables independent of $U$. Let $\mathbb{E}_U$ be the conditional expectation given $U$ and $\mu_U$ the conditional expectation of $M_U^{\text{sym}}$. By Dudley’s inequality [Theorem 11.17 in Ledoux and Talagrand (1991)],
\[
\mu_U = \mathbb{E}_U[M_U^{\text{sym}}] \leq \frac{24 \sqrt{2\pi}}{n} \int_0^\infty \sqrt{\log N(\mathcal{C}_1^*(s_1), d(\cdot, \cdot), w)} \, dw
\]
where $N(\mathcal{C}_1^*(s_1), d(\cdot, \cdot), w)$ is the minimal covering number of $\mathcal{C}_1^*(s_1)$ by balls of radius $w$ in the metric $d(\cdot, \cdot)$ given by
\[
d(a, b) = \left\{\sum_{i=1}^n (\langle r^i, a \rangle - \langle r^i, b \rangle)^2\right\}^{1/2}.
\]

To bound the entropy integral in (94), we transform the distance into a simpler one as follows. As $\max_{b \in \mathcal{C}_1^*(s_1)} \frac{1}{n} \sum_{i=1}^n (\langle r^i, b \rangle)^2/n \leq M_U + 1$ by (90), for $\{a, b\} \subset \mathcal{C}_1^*(s_1)$
\[
d(a, b) \leq L_0 \left[\sum_{i=1}^n (\langle r^i, a + b \rangle)^2\right]^{1/2} \|a - b\|_U \leq 2L_0 n^{1/2}(M_U + 1)^{1/2} \|a - b\|_U,
\]
where $\|b\|_U = \max_{i \leq n} |\langle b, r^i \rangle|/L_0$. Let $\mathcal{B}_1^{\text{sym}} = \{b : \|b\|_1 \leq 1\}$ and
\[
J_U = \int_0^1 \sqrt{\log N(\mathcal{B}_1^{\text{sym}}, \|\cdot\|_U, w)} \, dw
\]
As $\|b\|_1 \leq s_1^{1/2}$ in $\mathcal{C}_1^*(s_1)$ and $d(a, b) \leq 2L_0 n^{1/2}(M_U + 1)^{1/2} \|a - b\|_U$, (94) is further bounded by
\[
\mu_U \leq \frac{24 \sqrt{2\pi}}{n^{1/2}} \left(2L_0(M_U + 1)^{1/2}\right) \int_0^\infty \sqrt{\log N(s_1^{1/2} \mathcal{B}_1^{\text{sym}}, \|\cdot\|_U, w)} \, dw
\]
\[ = C'_0 L_0(s_1/n)^{1/2}(M_U + 1)^{1/2} J_U \]

with \( C'_0 = 48\sqrt{2\pi} \). As \( \|J_U\|_\infty \leq C''_0 \log(s_1) \sqrt{(\log d^*) \log n} \) by (3.9) of Rudelson and Vershynin (2008), we have

\[ \mu_U \leq (M_U^{1/2} + 1)\eta/4 \leq M_U/4 + (\eta^2 + \eta)/4 \]  \hspace{1cm} (95)

with \( \eta = 4C'_0 L_0(s_1/n)^{1/2}C''_0 \log(s_1) \sqrt{(\log d^*) \log n} \). Thus, by Cauchy-Schwarz,

\[ \mathbb{E}[M_U] \leq \mathbb{E}[\mu_U] \leq (\eta^2 + \eta)/3 \leq \eta \text{ when } \eta < 1. \]

We note that (54) is trivial when \( \eta > 1 \). The tail probability bound can be derived in a similar symmetrization argument. Let \( \sigma^2_U = \sup_{b \in C'_1(s_1)} \sum_{i=1}^n \langle r^i, b \rangle^4 2\pi/n^2 \) As \( \mu_U \) is the conditional expectation of \( M_U^{\text{sym}} \), the Gaussian concentration inequality (Borell, 1975; Kwapien, 1994) provides

\[ \mathbb{P}_U \{ M_U^{\text{sym}} > \mu_U + \sigma_U t \} \leq \mathbb{P}\{N(0, 1) > t\}, \forall t \geq 0. \]

Thus, symmetrizing via the Jensen inequality as in (93), we have

\[ \mathbb{E}[e^{M_U/\eta}] \leq \mathbb{E}[\exp(M_U^{\text{sym}}/\eta)] \leq \mathbb{E}[e^{\mu_U/\eta}(1/2 + \exp(\sigma^2_U/(2\eta^2)))]. \]

As \( \|b\|_1 \leq s_1^{1/2} \) for \( b \in C'_1(s_1) \), we have \( \sigma^2_U \leq M_U(2\pi s_1 L_0/n) \). Thus, for \( 2\pi s_1 L_0/n \leq \eta/2 \) and \( \eta \leq 1 \), the above inequality and (95) yield

\[ \mathbb{E}[e^{M_U/\eta}] \leq \mathbb{E}[e^{M_U/(4\eta) + (1+\eta)/4}(1/2 + e^{M_U/(4\eta)})] \leq \left( \mathbb{E}[e^{M_U/\eta}] \right)^{1/4}(\sqrt{e}/2) + (\mathbb{E}[e^{M_U/\eta}])^{1/2} \sqrt{e}. \]

For \( x = (\mathbb{E}[e^{M_U/\eta}])^{1/2} \), this gives \( x \leq e^{1/2}(1/2(2\sqrt{x}) + 1) \), which implies \( x \leq \sqrt{5} \). Consequently, for \( 4\pi s_1 L_0/n \leq \eta \leq 1 \), \( \mathbb{P}(M_U > c_0) \leq e^{-c_0/\eta} \mathbb{E}e^{M_U/\eta} \leq 5e^{-c_0/\eta} \). The conclusion follows as \( c_0 \in (0, 1) \) and \( 5e^{-c_0/\eta} \leq \epsilon_1 \leq 1 \) imply \( \eta \leq 1 \). \hfill \Box

**Proof of Lemma 1.** Let \( M_i = r_k^i \otimes r_k^i - \mathbb{E}[r_k^i \otimes r_k^i] \) where \( u \otimes v = uv^\top \) for all vectors \( u \) and \( v \). We have \( \|M_i\|_{\mathcal{S}} \leq L_0^2 d_k \) and \( \sum_{i=1}^n \mathbb{E}M_i^2/n \leq L_0^2 d_k V_k \) with \( V_k = \mathbb{E}(U_k^\top U_k)/n \). By the non-commutative Bernstein Inequality (Tropp, 2012),

\[ \mathbb{P}\{\|U_k^\top U_k/n - \mathbb{E}V_k\|_{\mathcal{S}} > x\} \leq 2d_k \exp \left[ \frac{-nx^2/2}{L_0^2 d_k \|V_k\|_{\mathcal{S}} + L_0^2 d_k x/3} \right] \]

for all \( x > 0 \). This gives (58) via the union bound. For (59), we have \( \|V_k^{-1/2} r_k^i\|_2 \leq \nu^{-1/2} L_0 d_k^{-1/2} \). Applying the above inequality to \( (U_k^\top V_k^{-1/2}) \), we find that

\[ \mathbb{P}\left\{ \max_{E(\|U_k^\top U_k\|_{\mathcal{S}}^2) > 0} \frac{\|U_k^\top U_k\|_{\mathcal{S}}^2 - 1}{\mathbb{E}[\|U_k^\top U_k\|_{\mathcal{S}}^2]} > x \right\} \leq 2d_k \exp \left[ \frac{-nx^2/2}{(L_0^2/\nu) d_k (1 + x/3)} \right] \]

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as the above maximum is \( \| (U_k V_k^{-1/2})^\top (U_k V_k^{-1/2})/n - I_{d_k}\|_S \).

**Proof of Theorem 4.**

(i) In \( \Omega_0 \), (24) holds. In \( \Omega_1 \),

\[
C_{\text{pred}}(\xi, S_0) \leq 1/\kappa^2(\xi, S_0) \leq (1 + c_0)/\{(1 - c_0)\kappa^2(\xi_0, S_0)\}.
\]

By (75), \( \Omega_2 = \{ \sum_{S_0} \leq \sigma^2 s_2/n \} \) for the \( \sum_{S_0} \) in Theorem 1. Thus, (24) implies (79).

Here \( \xi = (A_0 + 1)/(A_0 - 1) \) is as in Theorem 1.

(ii) Consider the case where \( \Omega_0 = 0 \).

\( h \) in (particular, as the above maximum is \( \| (U_k V_k^{-1/2})^\top (U_k V_k^{-1/2})/n - I_{d_k}\|_S \).

As in the proof of Theorem 1 (i), in the event \( \Omega_0 \cap \Omega_2 \).

\[
\| \hat{f} - \tilde{f} \|^2_{2,n} + \| \hat{f} - f^* \|^2_{2,n} + 2(A_0 - 1)\sum_{(j,k) \in \mathcal{X}^*} \lambda_{j,k} \| U_{j,k} h_{j,k} \|_{2,n} 
\leq \sigma^2 s_2/n + 4A_0 \sum_{(j,k) \in S_0} \lambda_{j,k} \| U_{j,k} h_{j,k} \|_{2,n}.
\]

We shall consider two cases. In the first case where

\[
\sigma^2 s_2/n \leq 4A_0 \sum_{(j,k) \in S_0} \lambda_{j,k} \| U_{j,k} h_{j,k} \|_{2,n},
\quad (96)
\]

we have \( 2(A_0 - 1)\sum_{(j,k) \in S_0} \lambda_{j,k} \| U_{j,k} h_{j,k} \|_{2,n} \leq (6A_0 + 2)\sum_{(j,k) \in S_0} \lambda_{j,k} \| U_{j,k} h_{j,k} \|_{2,n} \), so that the proof of Theorem 1 proceeds verbatim with \( \xi = (3A_0 + 1)/(A_0 - 1) \). In particular, \( h \in C(\xi, S_0) \) and by the first inequality in \( \Omega_1 \) in (73),

\[
\| \hat{f} - \tilde{f} \|^2_{L_2} = \| U h \|^2_{2,n} \leq \| U h \|^2_{2,n}/(1 - c_0) = \| \hat{f} - \tilde{f} \|^2_{2,n}/(1 - c_0).
\]

In the second case where (96) does not happen, we have

\[
\| h \|_1 \leq \sum_{(j,k) \in \mathcal{X}^*} \frac{n^{1/2} \lambda_{j,k}}{\sigma} \| h_{j,k} \|_{2}
\leq \sum_{(j,k) \in \mathcal{X}^*} \frac{n^{1/2} \lambda_{j,k}}{\sigma \nu^{-1/2}_-} \| U_{j,k} h_{j,k} \|_{2,n}
\leq \sum_{(j,k) \in \mathcal{X}^*} \frac{n^{1/2} \lambda_{j,k}}{\sigma \nu^{-1/2}_-} \| U_{j,k} h_{j,k} \|_{2,n}
\leq \frac{\sigma n^{-1/2} s_2}{(A_0 - 1)\nu^{-1/2}_-(1 - c_0)}
\]

by (21), (69) and the third inequality in \( \Omega_1 \) in (73). Thus, by the second inequality in \( \Omega_1 \) with \( b = (A_0 - 1)\nu^{-1/2}_-(1 - c_0)(n/s_2)^{1/2} h \) and under the condition \( s_2 \leq s_1 \)

\[
\| \hat{f} - \tilde{f} \|_{L^2} = \begin{cases} \| U h \|^2_{L_2,n} \leq \| U h \|^2_{2,n}/(1 - c_0), & \| U b \|^2_{L_2,n} \geq 1, \\ \| U h \|^2_{L_2,n} < \sigma^2 s_2/\{(1 - c_0)^2(A_0 - 1)^2 \nu_- n\}, & \| U b \|^2_{L_2,n} < 1. \end{cases}
\]
Thus, in either cases, (79) and the above bounds for $\|\hat{f} - \mathcal{T}\|_{L^2}$ yield
\[
\|\hat{f} - \mathcal{T}\|_{L^2}^2 + \|\hat{f} - f^*\|_{L^2}^2 \\
\leq 3\|\hat{f} - \mathcal{T}\|_{L^2}^2 + 2\|\hat{f} - f^*\|_{L^2}
\leq \max \left( \frac{3\sigma^2 s_2}{(1 - c_0^2)(A_0 - 1)^2\nu_{-} n}, \frac{3\|\hat{f} - \mathcal{T}\|_{L^2}^2}{1 - c_0} \right) + \frac{\sigma^2 s_2}{n},
\leq C_{A,\nu_{-},c_0} \left( \|\lambda s\|^2/\mathcal{K}^2(\xi_0, S_0) + \sigma^2 s_2/n \right),
\]
with a constant $C_{A_0,\nu_{-},c_0}$ depending on $(A_0, \nu_{-}, c_0)$ only. This gives (80).

(iii) By (20) and Theorem 1 (ii), we have $\mathbb{P}\{\Omega_0^c\} \leq \epsilon/\sqrt{2\log(p/\epsilon)}$. By (55) and (59) we have $\mathbb{P}\{\Omega_0^c\} \leq 2\epsilon_1$ for the $\Omega_1$ in (73) when (72) hold with $\{c_0, \epsilon_1\} \subset (0, 1)$ and $s_2 \leq s_1$. It remains to prove $\mathbb{P}\{\Omega_1 \setminus \Omega_2\} \leq \epsilon_2$ for the $s_2$ in (76). Because
\[
\sum_{(j,k) \in X^c \setminus S_0} \lambda_{j,k} \|U_{j,k} \beta^*_{j,k}\|_{2,n} \leq (1 + c_0)\nu_{+}^{1/2} \sum_{(j,k) \in X^c \setminus S_0} \lambda_{j,k} \|eta^*_{j,k}\|_2
\]
by (69) and the third inequality in $\Omega_1$ in (73) and $\mathbb{P}\{\|f^* - \mathcal{T}\|_{2,n}^2 \geq \|f^* - \mathcal{T}\|_{L^2}/\epsilon_2 \} \leq \epsilon_2$ by the Markov inequality, it suffices to prove
\[
\mathbb{P}\left\{ \|f^* - \mathcal{T}\|_{2,n}^2 \geq 2\|f^* - \mathcal{T}\|_{L^2} + \frac{(C_{A,\nu_{-},c_0})^2 \log \epsilon_2}{n^{2(\alpha + \alpha^*)/(2\alpha + 1)}} \right\} \leq \epsilon_2.
\]
(97)
in view of (75) and (76). As $\|f^* - \mathcal{T}\|_{2,n}^2$ is an average of independent variables $(f^*(r^i) - \mathcal{T}(r^i))^2$ with $r^i$ being the rows of $X$, the Bernstein inequality yields
\[
\mathbb{P}\{\|f^* - \mathcal{T}\|_{2,n}^2 \geq \mu_* + \sqrt{\mu_* C_* \log \epsilon_2/\epsilon_2 + 2C_* \log \epsilon_2/(3n)} \} \leq \epsilon_2
\]
(98)
with $\mu_* = \mathbb{E}[\|f^* - \mathcal{T}\|_{2,n}^2]$ and $C_* = \|f^* - \mathcal{T}\|_{\infty}$. Here the variance bound $n\text{Var}[\|f^* - \mathcal{T}\|_{2,n}^2] \leq \mu_* C_s$ is used.

As $\|u_{j,k,\ell}(\cdot)\|_\infty \leq L_0$ and $f^* - \mathcal{T} = \sum_{j=1}^p \sum_{k > k^*} \sum_{\ell = 1}^{2k - 1} u_{j,k,\ell}(x_j) \beta^*_{j,k,\ell}$, (68) gives
\[
C_* \leq \left( \sum_{j=1}^p \sum_{k > k^*} 2^{(k-1)/2} L_0 \|\beta^*_{j,k}\|_2 \right)^2 \leq \frac{1}{2} \left( \sum_{j=1}^p \frac{C_{\alpha - 1/2} L_0 \|f^*_{\alpha,2}\|_{\alpha,2}}{2^{(\alpha - 1/2)k^*/2}} \right)^2 \leq \frac{(C_{A,\nu_{-},c_0})^2}{2n^{2(\alpha - 1/2)/(2\alpha + 1)}}
\]
with $C_{\alpha} = 1/(4^\alpha - 1)^{1/2}$, in view of the condition $2k^* \geq n^{1/(2\alpha + 1)}$. Thus,
\[
\mu_* + \sigma_* \sqrt{2\log \epsilon_2/\epsilon_2 + 2C_* \log \epsilon_2/(3n)} \leq 2\mu_* + \left( \frac{1}{2} + \frac{1}{3} \right) C_* \frac{2 \log \epsilon_2}{n}
\]
by the bounds for $\mu_*$ and $C_*$. This and (98) yield (97).

(iv) We notice that when $L_0$ and $c_0$ are fixed positive constants and both $1/\nu_{-}$ and $1/\mathcal{K}(\xi_0, S_0)$ are bounded, we are allowed to take $s_1 \leq \max\{O(s), s_2\}$ and for this $s_1$ (72) holds with $\epsilon_1 = o(1)$ when $n \gg 2^{k^*} \log(np)$ and $n \gg s_1 \log s_1^2 \log(n) \log(np)$. Thus, part (iv) follows from parts (ii) and (iii).

(v) We omit the proof of this part as it is identical to the proof of Theorem 2. \(\square\)
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