Diabatic quantum annealing by counter-diabatic driving

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We introduce a two-parameter approximate counter-diabatic term into the Hamiltonian of the transverse-field Ising model for quantum annealing to accelerate convergence to the solution, generalizing an existing single-parameter approach. The protocol is equivalent to unconventional diabatic control of the longitudinal and transverse fields in the transverse-field Ising model and thus makes it more feasible for experimental realization than an introduction of new terms such as non-stoquastic catalysts toward the same goal of performance enhancement. We test the idea for the p-spin model with p = 3, which has a first-order quantum phase transition, and show that our two-parameter approach leads to significantly larger ground-state fidelity and lower residual energy than those by traditional quantum annealing as well as by the single-parameter method. We also find a scaling advantage in terms of the time to solution as a function of the system size in a certain range of parameters as compared to the traditional methods.

I. INTRODUCTION

Quantum annealing is a metaheuristic for combinatorial optimization problems [1–6] and has often been analyzed theoretically in the framework of adiabatic quantum computing [7–10]. A serious bottleneck in this approach originates in the exponential closing of the energy gap as a function of the system size between the ground state and the first excited state, typically at a first-order quantum phase transition, by which the computation time explodes exponentially according to the adiabatic theorem of quantum mechanics (see, e.g., [9]). One of the promising candidates to circumvent this difficulty is diabatic quantum annealing [11], in which one ingeniously drives the system out of the ground state to avoid the problem of closing energy gap and thus reach the final ground state with high fidelity. There have been attempts to design protocols to control the system variables based on this idea [11], and shortcuts to adiabaticity [12–15] is one of the strong candidates, providing a systematic way toward this goal.

Among these shortcuts-to-adiabaticity methods [16–22], counter-diabatic (CD) driving [21, 23–27] is one of the most promising approaches. The underlying idea of CD driving is to speed up an originally-adiabatic process by additionally applying a CD Hamiltonian (adiabatic gauge potential) that suppresses transitions between eigenstates.

However, for many-body quantum systems, finding the exact CD Hamiltonian requires a priori knowledge of eigenstates at all times during the dynamics [17], which is unfeasible practically. Recently, Sels, Polkovnikov, and collaborators [25, 26, 28] have developed a variational approach where a simple and local, but approximate, CD Hamiltonian is introduced, which makes the formulation and realization much simpler not just theoretically but experimentally as well [27, 29]. See also [30] for a related development. The price to pay is that the enhancement of performance is often limited.

In the present contribution, we propose a method to identify an approximate but enhanced local CD Hamiltonian. We introduce a second adiabatic gauge potential taking advantage of a second time-dependent parameter of the Hamiltonian and minimize the operator distance between the exact and approximate CD Hamiltonians to maximize the performance of the latter. We test the idea for the p-spin model with p = 3 as the problem Hamiltonian, which is known to be a simple model, yet a hard problem to solve by traditional quantum annealing [31–36]. We demonstrate that our approximate two-parameter CD Hamiltonian leads to clearly enhanced final ground state fidelity and reduced residual energy compared to traditional quantum annealing and the existing method of approximate single-parameter CD Hamiltonian. We further show a scaling advantage of the method compared to its traditional counterparts in a certain parameter range. Our two-parameter CD Hamiltonian improves the ground state fidelity and residual energy for both,
short and longer, annealing times and thus decreases the
time-to-solution considerably. We note that the modified CD
Hamiltonian used in this approach involves only local
\(\sigma_i^x\) operators, where \(i\) is the site index, and can thus be
rotated in the spin space at each site such that the result
consists only of \(\sigma_i^x\) in addition to the original
transverse-field Ising Hamiltonian. This is nothing but
the usual transverse-field Ising model but with unconven-
tional diabatic control of the transverse and longitudinal
fields, making it feasible for experimental realization.

The paper is structured as follows. In Sec. II, we
introduce the method of finding the two-parameter CD
protocol and apply the formulation to the \(p\)-spin model.
Numerical tests are presented in Sec. III for the \(p\)-spin
model with \(p = 3\), and Sec. IV discusses and concludes the
paper.

II. METHOD

Quantum annealing is a metaheuristic that aims at
solving combinatorial optimization problems. The basic
idea is to find the lowest-energy eigenstate of a problem
Hamiltonian \(\mathcal{H}_p\) – that encodes a combinatorial optimization
problem that we want to solve as an Ising model [37]
– by adiabatically transferring the easy-to-prepare ground
state of the driver Hamiltonian

\[
\mathcal{H}_d = -\gamma \sum_{i=1}^{N} \sigma_i^x
\]

with \(\gamma\) the time-independent transverse magnetic field
strength and \(N\) the number of sites (qubits) in the system,
into the ground state of \(\mathcal{H}_p\). The annealing schedule is
often chosen as

\[
\mathcal{H}_0(t) = [1 - \lambda(t)]\mathcal{H}_d + \lambda(t)\mathcal{H}_p,
\]

where \(\lambda(t)\) is a time-dependent driving function that
fulfills the boundary conditions \(\lambda(t = 0) = 0\) and
\(\lambda(t = \tau) = 1\) with \(\tau\) the total annealing time. Reaching
the exact ground state of \(\mathcal{H}_p\) generally requires adiabatic-
ity, and the time necessary to satisfy this condition grows
exponentially as a function of \(N\) if the energy gap be-
tween the ground state and the first excited state closes
exponentially, which is the case in most of the interesting
combinatorial optimization problems [9].

To overcome this bottleneck, one can implement a so-
called counter-diabatic Hamiltonian \(\mathcal{H}_{CD}(t)\) to suppress
transitions between eigenstates. The full Hamiltonian
then reads

\[
\mathcal{H}(t) = \mathcal{H}_0(t) + \mathcal{H}_{CD}(t),
\]

where \(\mathcal{H}_{CD}(t) = \dot{\lambda}(t)\mathcal{A}_\lambda(t)\) is the additional counter-
diabatic Hamiltonian, \(\mathcal{A}_\lambda(t)\) is the exact time-dependent
adiabatic gauge potential [25, 26, 28] with respect to the
driving function \(\lambda(t)\) in Eq. (2) and \(\lambda(t)\) its time deriva-
tive.

Finding the exact adiabatic gauge potential is a chal-
lenging task and generally requires a priori knowledge of
the eigenstates of the system for the whole annealing
time [17], which is impossible in practice. To overcome
this difficulty, one can employ an approximate adiabatic
gauge potential, denoted as \(\mathcal{A}^\gamma_\lambda(t)\), which includes only
local single-spin terms involving \(\{\sigma_i^y\}\) [25]. Notice that
the adiabatic gauge potential should be chosen imaginary
in the computational basis and thus only a linear combina-
tion of \(\sigma_i^y\) can appear in the simplest formulation where
only single-body Pauli matrices are accepted.

Following the variational principle of Ref. [25], one finds
the best possible approximate adiabatic gauge potential by
defining a Hermitian operator \(G_\lambda(A_\lambda) \equiv \partial_t \mathcal{H}_0 + i[A_\lambda, \mathcal{H}_0]\)
and minimizing the operator distance

\[
D^2(A_\lambda^\gamma) = \text{Tr}[(G_\lambda(A_\lambda) - G_\lambda(A_\lambda^\gamma))^2]
\]

between the exact, \(A_\lambda\), and approximate, \(A_\lambda^\gamma\), adiabatic
gauge potentials with respect to the parameters in \(A_\lambda^\gamma\).
This is equivalent to minimizing the action

\[
S(A_\lambda^\gamma) = \text{Tr}[G_\lambda^2(A_\lambda^\gamma)]
\]

with respect to its parameters, symbolically written as \(\delta S(A_\lambda^\gamma)/\delta A_\lambda^\gamma = 0\), as detailed in Refs. [25, 26,
28] and Appendix A. In this general approach, the driver,
\(\mathcal{H}_d\), and problem, \(\mathcal{H}_p\), Hamiltonian are kept intact.

Now, we notice that the time-dependent coefficients of
the two terms in the Hamiltonian \(\mathcal{H}_0(t)\), Eq. (2), can
be chosen independently – not necessarily in a single-
parameter form as in Eq. (2) – as long as the initial
Hamiltonian is \(\mathcal{H}_d\) and the final Hamiltonian is \(\mathcal{H}_p\). We
take advantage of this degree of freedom and choose to
write the coefficients using two time-dependent parame-
ters \(\lambda(t)\) and \(\gamma(t)\) as

\[
\mathcal{H}_0(t) = [1 - \lambda(t)]\mathcal{H}_d + \lambda(t)\mathcal{H}_p,
\]

where \(\lambda(t)\) satisfies the same boundary conditions as be-
fore, \(\lambda(0) = 0\) and \(\lambda(\tau) = 1\), and \(\gamma(t)\) is an arbitrary
function satisfying \(\gamma(0) \neq 0\). Since we have an addi-
tional function \(\gamma(t)\), it is natural to introduce a corre-
sponding additional adiabatic gauge potential \(\mathcal{A}^\gamma\). We therefore
employ an approximate local two-parameter CD Hamilto-
nian

\[
\mathcal{H}_0^{\lambda,\gamma}(t) = \dot{\lambda}(t)\mathcal{A}_\lambda(t) + \dot{\gamma}(t)\mathcal{A}_\gamma^\gamma(t),
\]

where \(\mathcal{A}_\lambda^\gamma(t)\) is also a linear combination of \(\sigma_i^y\) but with a
different coefficient than in \(\mathcal{A}_\lambda^\gamma(t)\).

One may wonder if the same linear combination of
\(\sigma_i^y\) operators as in \(\mathcal{A}_\lambda^\gamma(t)\) with just a different coefficient
would lead to different results. As we will see in the
next section, it indeed leads to an improvement of the
annealing performance in several measures, thanks to the
enhanced space of search for variational optimization of
the coefficients as functions of time.

As shown in Appendix A, finding the optimal coeffi-
cients in the two adiabatic gauge potentials \(\mathcal{A}_\lambda^\gamma\) and \(\mathcal{A}_\gamma^\gamma\)
is equivalent to minimizing the two-parameter action

\[
S = \text{Tr}[G_\lambda^2(A_\lambda^\gamma)] + \text{Tr}[G_\gamma^2(A_\gamma^\gamma)]
\]
with respect to the parameters in the two adiabatic gauge potentials $\mathcal{A}_\lambda^\gamma$ and $\mathcal{A}_\gamma^\lambda$, i.e.,

$$\frac{\delta S}{\delta \mathcal{A}_\lambda^\gamma} = 0, \quad \frac{\delta S}{\delta \mathcal{A}_\gamma^\lambda} = 0,$$

(8)

where $\mathcal{G}_\gamma(\mathcal{A}_\gamma^\lambda) \equiv \partial_t \mathcal{H}_0 + i [\mathcal{A}_\gamma^\lambda, \mathcal{H}_0]$ is the additional Hermitian operator with respect to $\gamma(t)$.

As we will see later, the introduction of an additional time dependence for the transverse magnetic field strength, $\gamma(t)$, and thus the emergence of the additional adiabatic gauge potential $\mathcal{A}_\gamma^\lambda(t)$ has significant consequences for local CD driving. The operator distance and thus the corresponding action, Eq. (7), can be analytically determined for a given set of functions $\lambda(t)$ and $\gamma(t)$. A detailed derivation of Eqs. (6) and (7) can be found in Appendix A and the application of the latter on the easy single-body Landau-Zener model in Appendix B.

Although it is desirable to find the best possible functional forms of $\lambda(t)$ and $\gamma(t)$, we work with simple conventional forms of those functions as illustrated in the next section and delegate the optimization of those functions to a future project.

### p-spin model

Our method can be applied to any problem Hamiltonian $\mathcal{H}_p \gamma$. In the present paper, we test our approach by using the $p$-spin model with $p = 3$ as the problem Hamiltonian since it is a hard problem for traditional quantum annealing due to a first-order quantum phase transition, though the flat ground state is trivially known to be ferromagnetic [31–35]. Another advantage of the $p$-spin model is that the total spin quantum number is conserved, which makes it possible to study very large system sizes numerically by restricting ourselves to the subspace of a fixed spin quantum number corresponding to the ground state as we shall see in the next section.

The total Hamiltonian reads

$$\mathcal{H}_0^{\lambda, \gamma}(t) = -[1 - \lambda(t)] \gamma(t) \sum_{i=1}^N \sigma_i^z - \lambda(t) N \left( \frac{1}{N} \sum_{i=1}^N \sigma_i^z \right)^3.$$  

(9)

Throughout this work, we will use the driving functions

$$\lambda(t) = \sin^2 \left[ \frac{\pi}{2} \sin^2 \left( \frac{\pi t}{2 \tau} \right) \right],$$

$$\gamma(t) = \gamma_{\text{init}} + \lambda(t),$$

(10)

where we have chosen the function $\lambda(t)$ following Ref. [25].

The above form of $\gamma(t)$ is chosen arbitrarily and its optimization is a future task as mentioned before. We have checked numerically that small variations of the value of $\gamma_{\text{init}}$ do not lead to noticeable changes of results.

Since any local adiabatic gauge potential should be a linear combination of $\sigma_i^y$, we write for the latter

$$\mathcal{A}_\lambda^\gamma = \sum_{i=1}^N \alpha \sigma_i^y, \quad \mathcal{A}_\gamma^\lambda = \sum_{i=1}^N \beta \sigma_i^y.$$  

(11)

Minimizing the corresponding action $S$, Eq. (7), with respect to the coefficients $\alpha$ and $\beta$, as detailed in Appendix C, we obtain the optimal solutions and thus the CD Hamiltonian, Eq. (6), as

$$\mathcal{H}_\text{CD}^{\lambda, \gamma}(t) = \sum_{i=1}^N (\lambda \alpha + \gamma \beta) \sigma_i^y, \quad \alpha = -\kappa \gamma, \quad \beta = \kappa (1 - \lambda) \lambda,$$

$$\kappa = \frac{1}{2} (1 - \lambda)^2 \gamma^2 N^4 + \lambda^2 (27 N^2 - 66 N + 40).$$  

(12)

It is noticed that $\kappa$ is proportional to $1/N$ for large $N$ and thus the CD Hamiltonian $\mathcal{H}_\text{CD}^{\lambda, \gamma}(t)$ becomes small for very large $N$ [38]. We therefore expect that the effect of the adiabatic gauge potentials is seen most prominently for relatively small to moderate $N$. This also means that, as long as the $p$-spin model is concerned, the present method does not lead to a drastic scaling advantage that reduces the computational complexity from exponential to polynomial in the limit of very large $N$, although significant improvements will be observed numerically even for moderately-large $N$, as we will see in the next section. The corresponding full Hamiltonian is

$$\mathcal{H}^{\lambda, \gamma}(t) = \mathcal{H}_0^{\lambda, \gamma}(t) + \sum_{i=1}^N (\lambda \alpha + \gamma \beta) \sigma_i^y.$$  

(13)

To facilitate experimental implementation, we eliminate the $\sigma_i^y$ terms by rotating this full Hamiltonian around the $z$-axis in spin space, i.e., applying the unitary transformation

$$U(t) = \exp \left( \frac{i \theta(t)}{2} \sum_i \sigma_i^z \right)$$  

(14)

over the angle $\theta(t) = \arctan(Y/X)$ with $X = -(1 - \lambda) \gamma$ and $Y = \lambda \alpha + \gamma \beta$. The resulting effective Hamiltonian has the form

$$\mathcal{H}_\text{eff}^{\lambda, \gamma}(t) = \sum_{i=1}^N \sqrt{X^2 + Y^2} \sigma_i^z - \lambda(t) \frac{6}{N^2} \sum_{i<j<k} \sigma_i^z \sigma_j^z \sigma_k^z$$

$$- \sum_{i=1}^N \left[ \frac{1}{2} X \sigma_i^x + \lambda(t) \frac{3 N - 2}{N^2} \right] \sigma_i^z.$$  

(15)

(see Appendix C and Ref. [25] for additional details). This Hamiltonian consists only of $\sigma_i^x$ and $\sigma_i^z$ terms which makes it more feasible for experimental realization than Eq. (13) with $\sigma_i^y$. 
Figure 1. **Ground state fidelity and residual energy.** (a)-(c) Final ground state fidelity and (d)-(f) residual energy for (i) traditional quantum annealing (diamonds, blue solid line), (ii) single-parameter CD drive (squares, orange dashed line), and (iii) two-parameter CD drive (circles, green dash-dotted line) as functions of annealing time $\tau$. The system sizes are $N = 4$ [panels (a),(d)], $N = 30$ [panels (b),(e)], and $N = 50$ [panels (c),(f)], where $\gamma_{\text{init}} = 0.1$ for all panels. Time ranges are color-coded as: Short-time regime (green-shaded areas) where the fidelity is approximately $1/2N$ for traditional quantum annealing, long-time regime (yellow-shaded areas) where transient behavior is observed and the two-parameter CD drive shows a clear advantage, and adiabatic regime (gray-shaded areas) where $F(\tau) > 0.99$.

### III. NUMERICAL VERIFICATION

We next present numerical results for our method for the $p$-spin model with $p = 3$. To this end, we compute the final ground state fidelity $F(\tau) = |\langle \psi(\tau) | \phi_0 \rangle|^2$ with $|\psi(\tau)\rangle$ and $|\phi_0\rangle$ the states at the end of annealing and the true ground state of the problem Hamiltonian, respectively, and residual energy $\Delta E = E(\tau) - E_0$ with $E(\tau)$ and $E_0$ the energy at the end of annealing and the true ground state energy. We compare three protocols: (i) traditional quantum annealing with the original Hamiltonian $[H_0(t), \text{Eq. (9)}]$, (ii) the existing method of single-parameter CD driven Hamiltonian [Eq. (12) with $\gamma(t) = \gamma_{\text{init}}$ and thus $\beta = 0$]; square, orange dashed line], and (iii) the two-parameter CD Hamiltonian $[H_{\text{eff}}(t), \text{Eq. (15)}]$. We tested a wide range of annealing time $\tau$ from $0.1$ to $10^5$ and a number of system sizes up to $N = 100$ using the spin symmetry of the problem.

We numerically solved the Schrödinger equation for the Hamiltonian dynamics and computed the fidelity, residual energy, and the time-to-solution, which is a measure of effective annealing time to reach the solution with probability $p_r$ [39], i.e.,

$$\text{TTS}(\tau) = \begin{cases} \tau \frac{\ln(1-p_r)}{\ln[1-F(\tau)]} & \text{for } F(\tau) < 1, \\ \tau & \text{for } F(\tau) = 1, \end{cases} \quad (16)$$

where we have set $p_r = 0.99$ as the success probability threshold. For our numerical computations we used QuTip 4.5 [40].

#### Dependence on annealing time

Figure 1 depicts the final ground state fidelity $F(\tau)$ [panels (a)-(c)] and residual energy $\Delta E$ [panels (d)-(f)] as functions of annealing time $\tau$ for system sizes $N = 4$ [panels (a),(d)], $N = 30$ [panels (b),(e)], and $N = 50$ [panels (c),(f)].

For the original Hamiltonian without CD term $[H_0(t), \text{Eq. (9)}]$; diamond with blue solid line in the figure], we see that the final state is far away from the ground state for short annealing time $\tau$ [green-shaded areas, where $F(\tau) \approx 1/2N$ for $H_0(t)$] and reach the final ground state in the adiabatic regime [gray-shaded areas, where $F(\tau) > 0.99$] for very long annealing time.

The existing method of single-parameter CD driven Hamiltonian [Eq. (12) with $\gamma(t) = \gamma_{\text{init}}$ and thus $\beta = 0$; square, orange dashed line] reaches a considerably higher final ground state fidelity and lower residual energy, respectively, especially for short annealing time (green-shaded areas), yet converges towards their original counterparts for longer annealing time (yellow-shaded areas).
Figure 2. **Time to solution.** Time to solution $TTS(\tau)$ for (i) traditional quantum annealing (diamonds, blue solid line), (ii) single-parameter CD drive (squares, orange dashed line), and (iii) two-parameter CD drive (circles, green dash-dotted line) for (a) $N = 20$ and (b) $N = 100$. For the latter, the data between $\tau = 1$ to about 10 are not shown because the values are too large to achieve reasonable numerical precision. Minimal time to solution for (c) short time region ($\tau \lesssim 1$) and (d) long time region ($\tau \gtrsim 10$). Other parameters are the same as in Fig. 1.

due to the fact that $\dot{\lambda} \propto 1/\tau$ [cf. Eq. (10)]. Consequently, the counter-diabatic Hamiltonian $H_{\text{CD}}^{\lambda,\gamma}(t)$ naturally converges towards zero for longer annealing time, in particular in the adiabatic limit (gray-shaded areas), and thus does not yield any further speedup.

On the other hand, for the two-parameter CD driven Hamiltonian $[H_{\text{CD}}^{\lambda,\gamma}(t)$, Eq. (15)], where $\gamma(t) = \gamma_{\text{init}} + \lambda(t)$; circle, green dash-dotted line], it is observed that we reach considerably higher final ground state fidelity and lower residual energy compared to traditional quantum annealing and single-parameter CD driving for long annealing time regime (yellow-shaded areas). This is important since the asymptotic adiabatic regime (gray-shaded areas) starts at later times for larger system sizes as seen in Fig. 1(c), meaning that the system performance in the long, but not yet adiabatic, time regime (yellow-shaded areas) becomes more and more important for larger systems. Stated otherwise, we come closer to the adiabatic regime more quickly, thus performing much better (around an order of magnitude reduction in annealing time to reach the same values of fidelity and residual energy) compared to its traditional quantum annealing and single-parameter CD driving counterparts.

**Time to solution**

We further studied the time-to-solution $TTS(\tau)$, a central measure of annealing time necessary to reach the solution with high probability [39], for different system sizes $N$. Figures 2(a) and (b) depict the time-to-solution $TTS(\tau)$, Eq. (16), for fixed system sizes $N = 20$ and 100, respectively. It is observed that the minimum is located at the shortest annealing time we studied, $\tau = 0.1$, except for the case of traditional quantum annealing with $N = 20$ and $N = 100$. We did not study even shorter time ranges because the time derivative of $\lambda(t)$ becomes anomalous for small $\tau$ and also experimental implementation may be difficult for too short annealing time. We further find a local minimum of $TTS(\tau)$ at a longer time $\tau \approx 10^3$.

Figure 2(c) depicts the system size dependence of the minimal $TTS$ at the shortest annealing time studied, $\tau = 0.1$. We see that the existing single-parameter CD method and our two-parameter method have a scaling advantage over traditional quantum annealing. Figure 2(d) depicts the $TTS$ at the local minimum $\tau \approx 10^3$ as a function of the system size. Our two-parameter approach has the same scaling behavior (the same slope) as the other two methods but has a constant speedup of the order of around 10. The same scaling behavior for large $N$ is not
very surprising because the adiabatic gauge potential is proportional to $1/N$ and will disappear for $N \gg 1$. But the non-negligible constant speedup was an unexpected result. We notice here that this asymptotic vanishing of the adiabatic gauge potential is a special property of the $p$-spin model, and the advantage of the present method is expected to remain finite for large system size and large annealing time in other models, for which we have very preliminary evidence. The comparison of Figs. 2(c) and (d) reveals that it is more advantageous to repeat very short annealing processes many times than to run a single long annealing, at least for the present problem.

Our preliminary data for a few other problem Hamiltonians indicate the possibility that the absolute minimum at the shortest annealing time may be a finite-size effect. If this proves to be true, the $p$-spin model is peculiar in the sense that finite-size effects persist for system sizes as large as $N = 100$. Whether or not this behavior is shared by other problems is an interesting future topic of research.

Behavior of coefficients

Figure 3(a) depicts the time dependence of the coefficient of each term of the Hamiltonian, i.e.,

$$H_x(t) = \sum_{i=1}^{N} \sqrt{X^2 + Y^2} \sigma_i^z,$$

$$H_{xZZ} = -\lambda(t) \frac{6}{N^2} \sum_{i<j<k} \sigma_i^z \sigma_j^z \sigma_k^z,$$

$$H_{z}(t) = -\sum_{i=1}^{N} \left[ \frac{1}{2} \frac{X Y - Y X}{X^2 + Y^2} + \frac{3N - 2}{N^2} \right] \sigma_i^z$$

in the rotated frame [$H_{\text{eff}}(t)$, Eq. (15)] for annealing time $\tau = 10$, system size $N = 30$ and other parameters as in Fig. 1. Figure 3(b) depicts the coefficients of the adiabatic gauge potentials $A_i^x(t)$ and $A_i^z(t)$ and the corresponding coefficients $\alpha(t)$ and $\beta(t)$, Eq. (12), in the inset. The maximal corresponding strength of the additional magnetic field in $y$ direction in the original frame [panel (b)] and in the rotated frame [reflected in the coefficients of $H_x(t)$ and $H_z(t)$ in panel (a)] is not (overwhelmingly) larger than the original parameters in $H_{xZZ}(t)$ for this annealing time regime, which makes this approach attractive for experimental realization.

Energy spectrum of two-parameter CD drive

It is useful to see how the wave function is spread over the instantaneous eigenstates of the Hamiltonian, Eq. (15), during the present two-parameter CD drive.

Figure 4 depicts the occupation probability of each instantaneous eigenstate expressed by the thickness of red lines for the system size $N = 30$ and annealing time $\tau = 300$, corresponding to Fig. 1(b) and (e), where the two-parameter CD drive shows a clear advantage over traditional QA and single-parameter CD driving. We observe that Fig. 4(a) and (b) share a very similar eigenspectrum, and the wavefunction is spread over many excited states after $t/\tau \approx 0.3$ via a cascade of avoided level crossings. In contrast, in the two-parameter CD case [panel (c)], the structure of the eigenspectrum has significantly changed and the system is driven downward in the spectrum around $t/\tau \approx 0.3$, which results in the high probabilities in low energy eigenstates to the end of the annealing process. We emphasize that such an ingenious protocol has emerged naturally from the two-parameter variational approach to suppress undesirable diabatic transitions observed in panels (a) and (b).
IV. DISCUSSION AND CONCLUSION

We have proposed and tested a method to find an efficient local CD Hamiltonian that outperforms its original adiabatic computing and single-parameter approximate CD counterparts with respect to enhanced final ground state fidelity and reduced residual energy as well as time to solution. The method introduces an additional term in the adiabatic gauge potential by taking advantage of the degree of freedom of choosing the strength of the time-dependent transverse magnetic field. The CD Hamiltonian in this approach is local and can be expressed, after a rotation in spin space, just in terms of the usual transverse-field Ising model but with unconventional diabatic control of field strengths. It may thus be implemented in quantum annealing devices on various platforms more easily than other approaches, which introduce more involved terms into the Hamiltonian such as two-body $\sigma_i^x \sigma_j^x$ interactions, for the goal of performance improvement.

We have tested the idea using the $p$-spin model with $p = 3$ because it is possible to solve the Schrödinger dynamics numerically for very large system sizes for this model due to its special symmetry of conserved total quantum spin number. We have derived the analytical expression of the two-parameter CD Hamiltonian and numerically demonstrated a considerable increase in final ground state fidelity and reduction in residual energy and time to solution compared to traditional quantum annealing and the single-parameter CD Hamiltonian approach. We further demonstrated a scaling advantage of the approximate single- and two-parameter CD methods in the short-time region and a constant speedup of the two-parameter method in the long-time region. The lack of scaling advantage in the latter time region may originate in the $1/N$ scaling of the coefficients of the CD Hamiltonian for the $p$-spin model, which is a special property of this multi-body mean-field-like problem. We may expect better scaling behavior in many other problems where those coefficients of the CD Hamiltonian generally stay finite in the large-$N$ limit. We have also illustrated how the two-parameter CD Hamiltonian resolves the problem of excitation to higher energy states by showing the modification of the energy eigenspectrum that eliminates a cascade of avoided level crossings toward higher energy states. It is an interesting future problem to identify problems in which this mechanism leads to a clear scaling advantage even for very large system sizes. Such examples may well exist because of the special disadvantageous property of the $p$-spin model as described above, i.e., that the coefficients of the CD Hamiltonian tends to vanish for larger system size.

We note that there exist other approaches to optimize the time dependence of coefficients in quantum annealing, e.g., from the viewpoint of optimal control theory and related ideas often under the context of quantum approximate optimization algorithm [29, 41–43] (see also Ref. [44] for a related idea of inverse engineering). It is not clear a priori whether our two-parameter CD Hamiltonian is better or not in comparison with these approaches since the criteria of optimality are different. The comparison in terms of relevant physical quantities such as fidelity, residual energy and the time-to-solution will be the best way to measure the performance of different protocols. It can happen that one is better than the other in some problems and the other way round in other problems, which reveals an interesting future topic to be studied.

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Appendix A: Derivation of adiabatic gauge potentials

In this Appendix, we derive the two adiabatic gauge potentials \( A_\lambda \) and \( A_\gamma \), by considering a quantum state \( |\psi\rangle \) evolving under the time-dependent Hamiltonian \( H_0^{\lambda,\gamma}(t) \).

The effective Schrödinger equation \( i\partial_t |\psi(t)\rangle = H_0^{\lambda,\gamma}(U|\psi(t)\rangle) \) in the moving frame by applying the unitary transformation \( U = U(\lambda, \gamma) \) with \( |\psi(t)\rangle = U^\dagger |\psi\rangle \) and thus \( |\psi\rangle = U|\psi(t)\rangle \) is written as

\[
\begin{align*}
  i\partial_t (U|\psi(t)\rangle) &= H_0^{\lambda,\gamma}(U|\psi(t)\rangle) \\
  i(\partial_t U^\dagger \lambda + \partial_\gamma U^\dagger \gamma)|\psi(t)\rangle + iU\partial_t |\psi(t)\rangle &= H_0^{\lambda,\gamma} U |\psi(t)\rangle.
\end{align*}
\]

(A1)

If we apply \( U^\dagger \) from the left, we have

\[
\begin{align*}
  i(U^\dagger \partial_\lambda U^\dagger \lambda + U^\dagger \partial_\gamma U^\dagger \gamma)|\psi(t)\rangle + iU^\dagger U\partial_t |\psi(t)\rangle &= U^\dagger H_0^{\lambda,\gamma} U |\psi(t)\rangle \\
  \Rightarrow U^\dagger &= H_0^{\lambda,\gamma} U |\psi(t)\rangle
\end{align*}
\]

(A2)

or

\[
\begin{align*}
  i\partial_t |\psi(t)\rangle &= \tilde{H}_0^{\lambda,\gamma}|\psi(t)\rangle - i(U^\dagger \partial_\lambda \lambda + U^\dagger \partial_\gamma \gamma)|\psi(t)\rangle
\end{align*}
\]

(A3)

which we write as

\[
\begin{align*}
  i\partial_t |\psi(t)\rangle &= \tilde{H}_0^{\lambda,\gamma}|\psi(t)\rangle - (\lambda \tilde{\lambda}_\lambda + \gamma \tilde{\lambda}_\gamma)|\psi(t)\rangle
\end{align*}
\]

(A4)

where \( \tilde{H}_0^{\lambda,\gamma}(\lambda, \gamma) = U^\dagger H_0^{\lambda,\gamma} U \) is diagonal in its instantaneous eigenbasis, and \( A_\lambda = U^\dagger i\partial_\lambda U \) and \( A_\gamma = U^\dagger i\partial_\gamma U \) are the corresponding adiabatic gauge potentials in the moving frame with respect to the two time-dependent driving parameters \( \lambda(t) \) and \( \gamma(t) \), respectively.

The counter-diabatic Hamiltonian with respect to these two adiabatic gauge potentials that suppresses any transitions between the eigenstates back in the laboratory frame can consequently be written as

\[
H_0^{\lambda,\gamma}(t) = \lambda(t) A_\lambda(t) + \gamma(t) A_\gamma(t).
\]

(A5)

It is straightforward to verify that the two adiabatic gauge potentials fulfill the relations

\[
\begin{align*}
  [A_\lambda, H_0^{\lambda,\gamma}] &= i\partial_\lambda H_0^{\lambda,\gamma} - iM_\lambda \\
  [A_\gamma, H_0^{\lambda,\gamma}] &= i\partial_\gamma H_0^{\lambda,\gamma} - iM_\gamma
\end{align*}
\]

(A6)

where the operators \( M_\lambda = \sum_n |n\rangle\langle n| \partial_\lambda H_0^{\lambda,\gamma}|n\rangle\langle n| \) and \( M_\gamma = \sum_n |n\rangle\langle n| \partial_\gamma H_0^{\lambda,\gamma}|n\rangle\langle n| \) are diagonal in the instantaneous eigenbasis \( |n(\lambda, \gamma)\rangle \). As \( [H_0^{\lambda,\gamma}, M_\lambda] = [H_0^{\lambda,\gamma}, M_\gamma] = 0 \), we can rewrite the conditions, Eq. (A6), as

\[
\begin{align*}
  [H_0^{\lambda,\gamma}, i\partial_\lambda H_0^{\lambda,\gamma} - [A_\lambda, H_0^{\lambda,\gamma}]] &= 0 \\
  [H_0^{\lambda,\gamma}, i\partial_\gamma H_0^{\lambda,\gamma} - [A_\gamma, H_0^{\lambda,\gamma}]] &= 0
\end{align*}
\]

(A7)

The exact solution for the adiabatic gauge potentials, i.e., \( A_\lambda \) and \( A_\gamma \), generally requires a priori knowledge of the system’s eigenstates, i.e., \( M_\lambda \) and \( M_\gamma \), during the whole annealing time through \( |n\rangle = |n(\lambda(t), \gamma(t))\rangle \). To generate the latter, \( A_\lambda \) and \( A_\gamma \) have complicated many-body interacting terms of all combinations of the operators \( \sigma_i^x, \sigma_i^y, \text{ and } \sigma_i^z \) (cf. Ref. [46] in the case of quantum criticality).

To circumvent this difficulty, we follow Ref. [25] and define the Hermitian operators \( G_\lambda(A_\lambda) = \partial_\lambda H_0^{\lambda,\gamma} + i[A_\lambda, H_0^{\lambda,\gamma}] \) and \( G_\gamma(A_\gamma) = \partial_\gamma H_0^{\lambda,\gamma} + i[A_\gamma, H_0^{\lambda,\gamma}] \) and insert a suitable ansatz \( A_\lambda^* \) and \( A_\gamma^* \), respectively, to solve Eqs. (A7) approximately. Notice that inserting the exact solutions into the Hermitian operators leads to the expressions \( G_\lambda(A_\lambda) = -M_\lambda \) and \( G_\gamma(A_\gamma) = -M_\gamma \).

We aim to approximate the exact solutions for the adiabatic gauge potentials as faithfully as possible. To
measure the distance between our approximate \( \mathcal{A}_\lambda \) and \( \mathcal{A}_\gamma \) and exact \( \mathcal{A}_\lambda \) and \( \mathcal{A}_\gamma \) adiabatic gauge potentials, it is convenient to introduce the operator distance as the Frobenius norm. The two-parameter operator distance can be written as

\[
D^2 := \text{Tr}[(G_\lambda(\mathcal{A}_\lambda^\dagger) + M_\lambda)^2] + \text{Tr}[(G_\gamma(\mathcal{A}_\gamma^\dagger) + M_\gamma)^2]
\]

\[
= \text{Tr}[G_\lambda^2(\mathcal{A}_\lambda^\dagger)] + \text{Tr}[G_\gamma^2(\mathcal{A}_\gamma^\dagger)] - \text{Tr}[M_\lambda^2] - \text{Tr}[M_\gamma^2]
\]

(A8)

where we use the fact that \( \mathcal{H}_{0}^{\lambda,\gamma} \) commutes with \( M_\lambda \) and \( M_\gamma \), respectively, and \( \text{Tr}[M_\lambda \partial_\lambda \mathcal{H}_0^{\lambda,\gamma}] = -\text{Tr}[M_\lambda^2] \).

As the generalized forces \( M_\lambda \) and \( M_\gamma \) do not depend on \( \mathcal{A}_\lambda \) and \( \mathcal{A}_\gamma \), we can minimize the two-parameter operator distance, Eq. (A8), by minimizing the two-parameter action

\[
S = \text{Tr}[G_\lambda^2(\mathcal{A}_\lambda^\dagger)] + \text{Tr}[G_\gamma^2(\mathcal{A}_\gamma^\dagger)]
\]

(A9)

with respect to the parameters of our ansatz for the adiabatic gauge potentials, \( \mathcal{A}_\lambda \) and \( \mathcal{A}_\gamma \), as \( \delta S/\delta \mathcal{A}_\lambda = 0 \), \( \delta S/\delta \mathcal{A}_\gamma = 0 \).

Appendix B: Landau-Zener model

In this Appendix, we illustrate the method of our two-parameter CD drive for the Landau-Zener model. Its original Hamiltonian reads

\[
\mathcal{H}_{LZ,0}(t) = -(1 - \lambda(t))\gamma(t)\sigma^y - \lambda(t)h\sigma^z,
\]

where the driving functions are

\[
\lambda(t) = \sin^2 \left[ \frac{\pi}{2} \sin^2 \left( \frac{\pi t}{2\tau} \right) \right],
\]

\[
\gamma(t) = 1 - \lambda(t).
\]

We have followed Ref. [25] in choosing the functional form of \( \lambda(t) \). We employ the ansatz \( \mathcal{A}_\lambda \equiv \alpha \sigma^y \) and \( \mathcal{A}_\gamma \equiv \beta \sigma^x \) for the adiabatic gauge potentials with respect to \( \lambda \) and \( \gamma \), respectively, and calculate the two Hermitian operators \( G_\lambda(\mathcal{A}_\lambda^\dagger) = \partial_\lambda \mathcal{H}_{LZ,0} + i[\mathcal{A}_\lambda, \mathcal{H}_{LZ,0}] \) and \( G_\gamma(\mathcal{A}_\gamma^\dagger) = \partial_\gamma \mathcal{H}_{LZ,0} + i[\mathcal{A}_\gamma, \mathcal{H}_{LZ,0}] \) and then minimize the corresponding two-parameter action \( S \), Eq. (A9), with respect to the coefficients \( \alpha \) and \( \beta \). The Hermitian operators turn out to be

\[
G_\lambda(\mathcal{A}_\lambda^\dagger) = (\gamma + 2\lambda \hbar)\sigma^x - [h + 2(1 - \lambda)\gamma] \sigma^z,
\]

\[
G_\gamma(\mathcal{A}_\gamma^\dagger) = [2\lambda \beta - (1 - \lambda)]\sigma^x - 2(1 - \lambda)\gamma \beta \sigma^z.
\]

(B3)

Pauli matrices are traceless and thus calculating the trace of the square of the Hermitian operators is equivalent to adding up squares of the coefficients in front of every Pauli matrix. Therefore, the action \( S \), Eq. (A9), reads

\[
S = (\gamma + 2\lambda \hbar)^2 + [h + 2(1 - \lambda)\gamma] \alpha^2
\]

\[
+ [2\lambda \beta - (1 - \lambda)]^2 + 4(1 - \lambda)^2.
\]

(B4)

By calculating the derivatives of this action with respect to \( \alpha \) and \( \beta \), i.e., solving the system of equations \( \{ \delta S/\delta \alpha = 0 \}, \delta S/\delta \beta = 0 \}, \) we obtain the optimal solution for the CD Hamiltonian \( \mathcal{H}_{LZ,CD}(t) = (\lambda \alpha + \gamma \beta)\sigma^y \), Eq. (6) from the main text, as

\[
\alpha = -\frac{1}{2} \frac{h\gamma(t)}{X^2 + Y^2 \sigma^x + 2 h\lambda(t) \sigma^z}
\]

\[
\beta = \frac{1}{2} \frac{[1 - \lambda(t)] \lambda(t) h}{X^2 + Y^2 \sigma^x + 2 h\lambda(t) \sigma^z}.
\]

(B5)

It turns out that this solution reduces to the exact CD term, cf. Ref. [20], when \( \gamma(t) \) is constant – as \( \dot{\gamma}(t) \) then becomes zero and consequently we are left with the solution for \( \alpha \) in Eq. (B5), with \( \gamma(t) = \gamma \), alone, i.e., \( \beta = 0 \).

The full Hamiltonian in the rotated frame reads

\[
\mathcal{H}_{LZ,\sigma}(t) = \sqrt{X^2 + Y^2 \sigma^x} - \frac{1}{2} \frac{X Y - \dot{X} Y}{X^2 + Y^2} + h\lambda(t) \sigma^z
\]

(B6)

with \( X = -[1 - \lambda(t)] \gamma(t) \) and \( Y = \lambda(t) \alpha(t) + \dot{\gamma}(t) \beta(t) \).

Figure 5 depicts the coefficients of the driver and problem Hamiltonian

\[
\mathcal{H}_{LZ,\sigma}(t) = \sqrt{X^2 + Y^2 \sigma^x},
\]

\[
\mathcal{H}_{LZ,\sigma}(t) = -\frac{X Y - \dot{X} Y}{2(X^2 + Y^2)} + h\lambda(t) \sigma^z,
\]

(B7)

for two-parameter CD driving, Eq. (B6) with \( \gamma(t) = \gamma_{\text{init}} + \lambda(t) \) for annealing time of \( \tau = 1 \). Other parameter: \( h = 0.1 \).
Appendix C: \(p\)-spin model

In this Appendix, we derive the solutions of the optimal two-parameter CD Hamiltonian, \(H_{\text{CD}}^\lambda(t) = \sum_{i=1}^{N}(\lambda\alpha + \gamma\beta)\sigma_i^y\), Eq. (12) from the main text, for the \(p\)-spin model with \(p = 3\) and original Hamiltonian \(H_0(t), \text{Eq. (9)}\), with driving functions \(\lambda(t)\) and \(\gamma(t), \text{Eq. (10)}\). For the latter, we can rewrite the original Hamiltonian, Eq. (9), into the form

\[
H_0^\lambda(t) = - [1 - \lambda(t)] \sum_{i=1}^{N} \gamma(t)\sigma_i^z
- \lambda(t) \frac{1}{N^2} \left[ 6 \sum_{i<j<k} \sigma_i^z \sigma_j^z \sigma_k^z + (3N - 2) \sum_{i=1}^{N} \sigma_i^z \right].
\]  

(C1)

For this many-body case, we employ the ansatz \(A_\lambda^x = \sum_{i=1}^{N} \alpha \sigma_i^z\) and \(A_\gamma^x = \sum_{i=1}^{N} \beta \sigma_i^z\) for the corresponding adiabatic gauge potentials. Calculating the Hermitian operators \(G_\lambda(A_\lambda^x)\) and \(G_\gamma(A_\gamma^x)\) requires the commutators

\[
i[A_\lambda^x, H_0^\lambda] = \sum_{i=1}^{N} \frac{2\lambda(3N - 2)}{N^2} \alpha \sigma_i^z - 2(1 - \lambda) \alpha \gamma \sigma_i^z
+ \frac{12\lambda}{N^2} \sum_{i<j<k} \alpha(\sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z).
\]

(C2)

\[
i[A_\gamma^x, H_0^\lambda] = \sum_{i=1}^{N} \frac{2\gamma(3N - 2)}{N^2} \beta \sigma_i^z - 2(1 - \lambda) \beta \gamma \sigma_i^z
+ \frac{12\lambda}{N^2} \sum_{i<j<k} \beta(\sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z).
\]

Adding the two partial derivatives \(\partial_\lambda H_0^\lambda\) and \(\partial_\gamma H_0^\lambda\), respectively, leads to the Hermitian operators

\[
G_\lambda(A_\lambda^x) = \sum_{i=1}^{N} \left[ \gamma + \frac{2\alpha \lambda(3N - 2)}{N^2} \right] \sigma_i^z
- \frac{N}{2} \sum_{i<j<k} \alpha(\sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z).
\]

\[
G_\gamma(A_\gamma^x) = \sum_{i=1}^{N} \left[ \frac{3N - 2}{N^2} + 2\beta(1 - \lambda) \gamma \right] \sigma_i^z
- \frac{N}{2} \sum_{i<j<k} \beta(\sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z + \sigma_i^z \sigma_j^z \sigma_k^z).
\]

Consequently the action \(S = \text{Tr}[G_\lambda^2(A_\lambda^x)] + \text{Tr}[G_\gamma^2(A_\gamma^x)], \text{Eq. (A9)},\) can be written as

\[
\frac{S}{2N} = \sum_{i=1}^{N} \left[ \gamma + \frac{2\alpha \lambda(3N - 2)}{N^2} \right]^2 + \left[ \frac{3N - 2}{N^2} + 2\alpha(1 - \lambda) \gamma \right]^2
+ \frac{72\lambda^2}{N^4} (N - 1)(N - 2) \alpha_i^2
+ \sum_{i=1}^{N} \left[ \frac{2\beta \lambda(3N - 2)}{N^2} - (1 - \lambda) \right]^2 + \left[ \frac{3N - 2}{N^2} + 2\beta(1 - \lambda) \gamma \right]^2
+ \frac{72\lambda^2}{N^4} (N - 1)(N - 2) \beta_i^2
- \frac{12(N - 1)(N - 2)}{N^2}
\]

(C4)

and minimizing this action with respect to each coefficient \(\alpha\) and \(\beta\) leads to the solutions, Eq. (12) from the text.