LONGTIME BEHAVIOR OF COMPLETELY POSITIVELY CORRELATED SYMBIOTIC BRANCHING MODEL

PATRIC KARL GLÖDE, LEONID MYTNIK

Abstract. We study the longtime behavior of a continuous state Symbiotic Branching Model (SBM). SBM can be seen as a unified model generalizing the Stepping Stone Model, Mutually Catalytic Branching Processes, and the Parabolic Anderson Model. It was introduced by Etheridge and Fleischmann [EF04]. The key parameter in these models is the local correlation $\rho$ between the driving Brownian Motions. The longtime behavior of all SBM exhibits a dichotomy between coexistence and non-coexistence of the two populations depending on the recurrence and transience of the migration and also in many cases on the branching rate. The most significant gap in the understanding of the longtime behavior of SBM is for positive correlations in the transient regime. In this article we give a precise description of the longtime behavior of the SBM with $\rho = 1$ with not necessarily identical initial conditions.

1. Introduction and Results

1.1. Symbiotic Branching Dynamics. The Symbiotic Branching Model (SBM) is a continuous state model of a two type branching population living and migrating in a geographic space. The branching rate of each of the two populations locally depends on the size of the other population. While the SBM is a continuous state model the dynamics are best understood by having an informal look at the discrete particle approximation. Both the continuous and the discrete model depend essentially on the branching rate $b > 0$ and the correlation parameter $\rho \in [-1, 1]$. Geographic space is represented by $\mathbb{Z}^d$. So the model describes the evolution in time of the number of particles of each population at each site $k \in \mathbb{Z}^d$ and let us denote the process by $(X_t, Y_t)_{t \geq 0}$. We refer to the particles of the two populations as particles of type 1 and 2, respectively. The evolution of the particle model is as follows:

- At each site, each pair of particles of opposite type share an exponential clock of rate $b|\rho|$. When the clock rings, both particles die and the following happens depending on the parameter $\rho$:
  - If $\rho \in [-1, 0)$, with equal probability either the type 1 particle has two offspring and the type 2 particle has no offspring or vice versa.
  - If $\rho \in (0, 1]$, with equal probability either both particles have zero or two offspring.

- At time $t > 0$, each particle of type 1 living at site $k$ has an exponential clock of rate $b(1 - |\rho|)Y_t(k)$, each particle of type 2 at site $k$ has an exponential clock of rate $b(1 - |\rho|)X_t(k)$. When the clock rings, the particle dies and with equal probability has either zero or two offspring.

- All particles perform simple continuous, rate 1, random walks independently from each other.

- The migration and the branching dynamics are independent from each other.

The particle system described above has a scaling limit. It can be shown that if each particle is assigned mass $1/n$ and the initial mass is of order $n$, then the rescaled particle system converges to
a system of interacting diffusions \((u, v) = (u_i, v_i)_{t \geq 0}\) which satisfy the following stochastic differential equations:

\[
\begin{align*}
    du_i(t) &= \Delta u_i(t) \, dt + \sqrt{bu_i(t)u_i(t)} \, dW^u_i(t), \quad t \geq 0, \\
    dv_i(t) &= \Delta v_i(t) \, dt + \sqrt{bv_i(t)v_i(t)} \, dW^v_i(t), \quad t \geq 0,
\end{align*}
\]

where \(i \in \mathbb{Z}^d\) and \(\{(W^u(i), W^v(i)) : i \in \mathbb{Z}^d\}\) is an independent field of locally correlated planar Brownian motions and the spatial correlation is given by \(\rho\). This model was introduced Etheridge and Fleischmann \cite{EF04} and is known as the Symbiotic Branching Model (SBM).

The SBM generalizes a couple of famous particle models. If \(\rho = -1\), the Brownian motions \(W^u(i)\) and \(W^v(i)\) are totally anti-correlated for each \(i \in \mathbb{Z}^d\). Under the additional condition that \(u_0 + v_0 = 1\) for all \(k \in \mathbb{Z}^d\), the process \(u_t + v_t\) solves the heat equation and hence since the initial conditions are constant, \(u_t + v_t \equiv 1\) for all \(t \geq 0\). Therefore \(v_t = 1 - u_t\) for all \(t \geq 0\). One can therefore rewrite \(1.4\) to obtain the well known system of differential equations for the stepping stone model (also known as interacting Fisher-Wright diffusions), see Shiga \cite{Shi80}. If \(\rho = 0\), the Brownians motions \(W^u(i)\) and \(W^v(i)\) are independent for each \(i \in \mathbb{Z}^d\). This leads to the well known mutually catalytic branching model, studied by Dawson and Perkins \cite{DP98}, Cox, Dawson and Greven \cite{CDG04}, Cox, Klenke and Perkins \cite{CKP00} among others. In continuous space, in dimension \(d = 2\), this models was studied in a series of papers by Dawson et. al. \cite{DEF+02a, DEF+02b, DFM+03}. If \(\rho = 1\), the Brownian motions \(W^u(i)\) and \(W^v(i)\) are totally positively correlated for each \(i \in \mathbb{Z}^d\). Under the additional assumption that the identical initial conditions are the same for both populations, that is, \(u_0 = v_0\), the SBM coincides with the Parabolic Anderson Model (PAM). For more information about the PAM see Carmona and Molchanov \cite{CM94}, Greven and den Hollander \cite{GdH07}, Glöö \cite{Glo06}.

Let us note, that after certain limiting procedures, the above models give a rise to the so called infinite rate mutually catalytic and symbiotic models, that were studied extensively in the recent years, see Klenke and Mytnik \cite{KM10, KM12a, KM12b, KM20}, Blath, Hammer and Ortgiese \cite{BHO16}, Hammer, Ortgiese and Florian \cite{HOV18}.

In our paper we are interested in the longtime behavior of the SBM with \(\rho = 1\) in the case of not necessarily equal initial conditions — the case that has not been studied in the literature. The main question is whether both populations can survive forever or whether just one population will survive while the other population will die out. If there is a positive probability that both populations will survive we will say coexistence is possible. Otherwise we will say coexistence is impossible.

The longtime behavior of the SBM has been thoroughly studied for different correlation parameters \(\rho\). For correlations \(\rho \in (-1,0]\) it has been proved that there is a clear dichotomy: coexistence is possible if and only if the migration is transient. See Blath, Döring and Etheridge \cite{BDE11}, Dawson and Perkins \cite{DP98}, and Döring and Mytnik \cite{DM13}. For \(\rho = -1\), Shiga \cite{Shi80} has proved that in the recurrent regime coexistence is impossible for the particular case of \(u_0 + v_0 = 1\). For correlation \(\rho \in (0,1)\), Blath, Döring, and Etheridge \cite{BDE11} have proved that if the migration is recurrent, coexistence is impossible.

For \(\rho = 1\) in the case of identical initial conditions (the PAM) the phase transition between survival and non-survival (note that in the PAM there is only one population so it does not make sense to speak of coexistence of two populations) occurs in the transient regime, that is, survival is impossible if the migration is recurrent. If the migration is transient then there is a critical branching parameter \(b_{\ast}\) such that for \(b > b_{\ast}\) survival is impossible while for \(b < b_{\ast}\) survival is possible. See Theorem 1.5 below (which is essentially the result of Greven and den Hollander \cite{GdH07}).

For correlation \(\rho \in (0,1)\), Blath, Döring, and Etheridge \cite{BDE11} have proved that if the migration is recurrent, coexistence is impossible. It is believed that for positive correlation there is a critical branching parameter such for all \(b\) larger than this parameter coexistence is impossible also in the transient regime (like in the case of the PAM). This conjecture however has not yet been proved.

Thus, note that for \(\rho \in (-1,0]\) and for the PAM the longtime behavior is fully characterized. Also, in the recurrent regime the longtime behavior of the SBM is fully understood for all \(\rho\). However, in the transient regime there are gaps in the understanding of what happens for \(\rho = -1\), with \(u_0 + v_0 \neq 1\),
for ρ ∈ (0, 1), and for ρ = 1 with non-identical initial conditions. In this paper our aim is to contribute towards a more complete understanding of the longtime behavior for positive correlations.

For ρ ∈ (0, 1] one of the main open questions related to the longtime behavior of SBM is:

If the migration is transient, is there a critical parameter b#(ρ) such that for b < b#(ρ) coexistence is possible while for b > b#(ρ) it is impossible.

Our main result, Theorem 1.6 gives a partial answer to this question: we characterize coexistence/non-coexistence dichotomy for SBM for the case of ρ = 1 and with initial conditions that are not necessarily equal.

The paper is organised as follows. In Section 1.2 we formally introduce the SBM and related concepts that will be investigated. Section 1.3 states existing relevant results on the longtime behavior of the PAM which is of profound importance for studying the long time behavior of the SBM with correlation ρ = 1. In Section 1.4 we present our main result for the longtime behavior of the SBM with correlation ρ = 1. The proof of our result is split into two parts. In Section 2 we treat the regime when the coexistence is possible while in Section 3 we deal with the regime when the coexistence is impossible.

1.2. Definitions. In this section we formally introduce the SBM. For a rigorous definition of the processes we need to specify an appropriate state space. Let \( \varphi_\lambda : \mathbb{Z}^d \to \mathbb{R}, \varphi_\lambda(k) = e^{\lambda |k|}, \lambda \in \mathbb{R} \) and define

\[
E_{\text{tem}} := \{ \phi : \mathbb{Z}^d \to \mathbb{R}_+ : \langle \phi, \varphi_\lambda \rangle < \infty \text{ for all } \lambda < 0 \},
\]

\[
E_{\text{fin}} := \{ \phi : \mathbb{Z}^d \to \mathbb{R}_+ : \sum_{k \in \mathbb{Z}^d} \phi(k) < \infty \},
\]

\[
E_{\text{cpt}} := \{ \phi : \mathbb{Z}^d \to \mathbb{R}_+ : \phi(k) = 0 \text{ for all but finitely many } k \in \mathbb{Z}^d \}.
\]

Following definitions on page 1091 of [DP98], for any \( \lambda \in \mathbb{R} \) and \( u, v : \mathbb{Z}^d \to \mathbb{R} \), set \( |u - v|_\lambda := \langle |u - v|, \varphi_\lambda \rangle \), and thus we define metric on \( E_{\text{tem}} \) as follows:

\[
d_{\text{tem}}(u, v) := \sum_{n=1}^{\infty} 2^{-n} (|u - v| - \lambda_n \wedge 1),
\]

where \( \lambda_n \downarrow 0 \). Also \( E_{\text{fin}} \) is topologized by the \( l^1 \)-norm, \( \|u - v\|_1 := \sum_{k \in \mathbb{Z}^d} |u(k) - v(k)| \), and \( E_{\text{cpt}} \) is topologized by the \( l^\infty \)-norm.

Now \( \Omega_{\text{tem}} \) (resp. \( \Omega_{\text{fin}} \)) is the space of \( E_{\text{tem}} \)-valued (resp. \( E_{\text{fin}} \)-valued) continuous paths on \( \mathbb{R}_+ \) with the compact-open topology.

Whenever \( x : \mathbb{Z}^d \to \mathbb{R}_+ \) and \( x(i) = \theta \geq 0 \) for all \( i \in \mathbb{Z}^d \) we use bold letters and write \( x = \theta \). We refer to this situation as a flat configuration.

In the following we denote the discrete Laplace operator by \( \Delta \), that is, for \( \phi : \mathbb{Z}^d \to \mathbb{R}_+ \) we set

\[
\Delta \phi(i) := \frac{1}{2d} \sum_{j \sim i} (\phi(j) - \phi(i)), \quad i \in \mathbb{Z}^d,
\]

where \( j \sim i \) means \( j \) is a neighbor of \( i \).

We denote the semigroup corresponding to the continuous-time rate 1 simple symmetric random walk on \( \mathbb{Z}^d \) by \( (P_t)_{t \geq 0} \) and the transition probabilities by \( p_t(i, j), t \geq 0, i, j \in \mathbb{Z}^d \). That is, if \( (Z_t)_{t \geq 0} \) is a simple symmetric random walk on \( \mathbb{Z}^d \) and \( \phi : \mathbb{Z}^d \to \mathbb{R} \) is a bounded function, then \( P_t \phi(i) = \mathbb{E}[\phi(Z_t) | Z_0 = i] \) and \( p_t(i, j) = \mathbb{P}\{Z_t = j | Z_0 = i\} \), where \( t \geq 0 \) and \( i, j \in \mathbb{Z}^d \). Also, we denote the Green’s function of a simple symmetric random walk on \( \mathbb{Z}^d \) (\( d \geq 3 \)) by \( g(\cdot, \cdot) \), that is,

\[
g(x, y) = \int_0^\infty P_s \mathbb{1}_{\{y\}}(x) \, ds = \int_0^t p_s(x, y) \, ds, \quad x, y \in \mathbb{Z}^d.
\]

Moreover if \( X, Y \) are stochastic processes defined on the same probability space, we denote their quadratic co-variation by \([X, Y]\), and quadratic variation by \([X]_t\). Equality in distribution is denoted by \( \overset{d}{=} \).
Definition 1.1. Let $b > 0$, $\rho = 1$, and $(x, y) \in E^2_{tem}$. We say that a stochastic process $(u, v) = (u_t, v_t)_{t \geq 0}$ is a Symbiotic Branching Model SBM$(1, b, x, y)$ on a filtered probability space $(\Omega, A, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ if the sample paths of $(u, v)$ lie in $\Omega_{tem}$, and there exists a family of planar Brownian motions $\{\tilde{W}(i) = (W(i)(t))_{t \geq 0} : i \in \mathbb{Z}^d\}$ adapted to the filtration $\mathcal{F}$ such that the following is satisfied: $(u, v)$ solves the following system of interacting stochastic differential equations

$$
\begin{align*}
(u_0, v_0) &= (x, y), \\
\mathrm{d}u_t(i) &= \Delta u_t(i) \mathrm{d}t + \sqrt{b u_t(i) v_t(i)} \mathrm{d}W_t(i), \quad t \geq 0, i \in \mathbb{Z}^d, \\
\mathrm{d}v_t(i) &= \Delta v_t(i) \mathrm{d}t + \sqrt{b u_t(i) v_t(i)} \mathrm{d}W_t(i), \quad t \geq 0, i \in \mathbb{Z}^d. 
\end{align*}
$$

(1.4)

In what follows, we assume throughout the paper that

$\rho = 1$.

Also, let us note, that for simplicity, whenever confusion is impossible we omit some or all of the parameters in the abbreviation SBM$(1, b, u_0, v_0)$.

Next proposition provides the existence and uniqueness result for (1.4).

Proposition 1.2. Let $b > 0$, and $(x, y) \in E^2_{tem}$. The system of stochastic differential equations (1.4) has a unique strong solution with sample paths in $\Omega_{tem}$.

Proof. Existence of a weak solution to (1.4) is proved in Proposition 3.1 of [BDE11].

Now let us show the pathwise uniqueness. First note that $\eta_t = v_t - u_t$ is deterministic and solves the discrete heat equation, that is $\mathrm{d}\eta_t = \Delta \eta_t \mathrm{d}t$. Hence $(u, v)$ is uniquely determined if $u$ and $\eta$ are. For $\eta$ existence and uniqueness are known of course. For $u$ we have the following system of locally 1-dimensional stochastic differential equations:

$$
\begin{align*}
\mathrm{d}u_t(i) &= \Delta u_t(i) \mathrm{d}t + \sqrt{b u_t(i)^2 + b v_t(i)\eta_t(i)} \mathrm{d}W_t^u(i), \\
&\quad i \in \mathbb{Z}^d. 
\end{align*}
$$

(1.5)

For (1.5) pathwise uniqueness follows by the argument similar to the one used in the proof of Theorem 3.2 in Shiga and Shimizu [SS80]. Note that in [SS80] the diffusion coefficient does not depend on time. However, the Yamada-Watanabe argument used in [SS80] obviously works just as well for our time-dependent diffusion coefficient since $\eta_t(i) < \infty$ for all $t \geq 0$. Also [SS80] assumes that at each site the process takes values in $[0,1]$. It is easy to seen however that for the argument in [SS80] to work it is sufficient to have $\sup_{t \in [0,1]} \mathbb{E}[|u_t(i) - u'_t(i)|] < \infty$ for all $T \geq 0$, for any two solutions $u, u'$ to (1.5). This can be immediately seen as follows. Recall that $(P_t)_{t \geq 0}$ denotes the semigroup corresponding to the simple symmetric random walk on $\mathbb{Z}^d$. Then, by Proposition 3.1 of [BDE11], $\mathbb{E}[u_t(i)] = P_t u_0(i)$ for every $i \in \mathbb{Z}^d$ and thus $\sup_{t \geq 0} \mathbb{E}[u_t(i)] < \infty$.

By Yamada-Watanabe theorem weak existence and pathwise uniqueness imply that there exists unique strong solution to (1.5) (see e.g. Theorem 2.2 in [SS80] for the analogous result). \hfill \Box

In the study of the SBM an important role is played by the total mass process and many results can be deduced from the behavior of this simpler process. The total mass of an element $x : \mathbb{Z}^d \mapsto \mathbb{R}_+$, is denoted by

$$
\bar{x} = \langle x, 1 \rangle = \sum_i x(i).
$$

Proposition 1.3 (Proposition 3.2 of [BDE11]). If $u_0, v_0 \in E_{\text{fin}}$, then the total mass processes $\bar{u} = (\bar{u}_t)_{t \geq 0}$ and $\bar{v} = (\bar{v}_t)_{t \geq 0}$ are non-negative, continuous, square integrable martingales and

$$
\begin{align*}
\mathrm{d}\bar{u}_t &= \sqrt{b(u_t, v_t)} \mathrm{d}\tilde{W}_t, \quad t \geq 0, \\
\mathrm{d}\bar{v}_t &= \sqrt{b(u_t, v_t)} \mathrm{d}\tilde{W}_t, \quad t \geq 0,
\end{align*}
$$

where $\tilde{W}$ is a Brownian motion with the variance $[\tilde{W}, \tilde{W}]_t = t$.

The lemma also implies that

$$
[\bar{u}, \bar{v}]_t = b \int_0^t \langle u_s, v_s \rangle \mathrm{d}s, t \geq 0.
$$
Another consequence of the lemma is that by the martingale convergence theorem, the limits
\[ \lim_{t \to \infty} \bar{u}_t = \bar{u}_\infty \quad \text{and} \quad \lim_{t \to \infty} \bar{v}_t = \bar{v}_\infty, \]
exist, almost surely.

What happens in the case of flat initial conditions, that is, for \((u_0, v_0) = (\theta_1, \theta_2)\) with \(\theta_1, \theta_2 \geq 0\)? In this case, the existence of limit of \((u_t, v_t)\) in \(E_{\text{tem}}^2\) is known for the case of \(\rho \in (-1, 1)\) (see Proposition 4.1 in [BDE11]). As for the case of \(\rho = 1\), since expectations of \(\{u_t, \varphi_t\}\) are constant in \(t\) and thus bounded for any \(\lambda > 0\), we can immediately get that the family \(\{(u_t, v_t), t \geq 0\}\) is tight in \(E_{\text{tem}}^2\) (see Lemma 2.3(c) of [DP98] and its proof for analogous argument). Thus there exist weak limit points \((u_\infty, v_\infty) \in E_{\text{tem}}^2\).

**Definition 1.4 (coexistence).**

1. **Assume that** \((u_0, v_0) \in E_{\text{fin}}^2\). \(\) We say that coexistence is possible if
\[ \mathbb{P}\{\bar{u}_\infty \bar{v}_\infty > 0\} > 0. \]
We say that coexistence is impossible if
\[ \mathbb{P}\{\bar{u}_\infty \bar{v}_\infty > 0\} = 0. \]

2. **Assume that** \((u_0, v_0) = (\theta_1, \theta_2)\) with \(\theta_1, \theta_2 \geq 0\). \(\) We say that coexistence is possible if for any weak limit point \((u_\infty, v_\infty)\) of \(\{(u_t, v_t), t \geq 0\}\) there exists \((\phi, \psi) \in E_{\text{cpt}}^2\) such that
\[ \mathbb{P}\{\langle u_\infty, \phi \rangle \langle v_\infty, \psi \rangle > 0\} > 0. \]
We say that coexistence is impossible if for any weak limit point \((u_\infty, v_\infty)\) of \(\{(u_t, v_t), t \geq 0\}\) and for all \(\phi, \psi \in E_{\text{cpt}}\)
\[ \mathbb{P}\{\langle u_\infty, \phi \rangle \langle v_\infty, \psi \rangle > 0\} = 0. \]

**Remark 1.1.** In fact Greven and den Hollander [GdH07] prove only \(b_2 \leq b_*\). They conjectured that in fact \(b_2 < b_*\). The fact that \(b_2 < b_*\) follows from results of Birkner and Sun [BS10] for dimensions \(d \geq 4\) and [BS11] for dimension \(d = 3\). [GdH07] also show that the second moments of the PAM at each site \(i \in \mathbb{Z}^d\) are bounded in time if and only if \(b < b_2\).

**Proof of Theorem 1.5.** For flat initial conditions, the result is contained in Theorems 1.2, 1.3, 1.4, and 1.5 of [GdH07]. For summable initial conditions the longtime behavior can be carried over from the flat setting using the self-duality from Lemma 2.1 as follows: for all \(\theta > 0\) and \(u_0 \in E_{\text{cpt}}\)
\[ \mathbb{E}_{u_0} [e^{-\theta \bar{u}_\infty}] = \lim_{t \to \infty} \mathbb{E}_{u_0} [e^{-\theta \langle 1, u_t \rangle}] = \lim_{t \to \infty} \mathbb{E}_{\theta} [e^{-\theta \bar{u}_t, u_0}], \]
\[ (1.6) \]
where $(\tilde{u}_t)_{t \ge 0}$ is PAM($b, \theta$). By Theorem 1.3 and 1.4 of [GdH07], the limiting law of $\tilde{u}_t$ exists, it is translation invariant and this implies that the right hand side of (1.6) equals 1 if and only if $\lim_{t \to \infty} \tilde{u}_t = 0$. Here and elsewhere $\text{w-lim}$ denotes weak (in probability sense) limit.

1.4. Our Main Result. Recall the parameter $b_*$ from Theorem 1.5. We establish the longtime behavior of the SBM for the case of completely positive correlations ($\rho = 1$). Recall that, as we mentioned above, if it is not stated otherwise, we assume that $\rho = 1$ in what follows.

**Theorem 1.6** (Longtime behavior for summable initial conditions). Let $(u_0, v_0) \in E_{\text{fin}}^2$ such that $\bar{u}_0 \bar{v}_0 > 0$.

(i) Let $d \ge 3$. Then for all $b \in (0, b_*)$ coexistence for $\text{SBM}(1, b, u_0, v_0)$ is possible.

(ii) Let

(a) $d \in \{1, 2\}$ and $b > 0$,

or

(b) $d \ge 3$ and $b > b_*$.

Then coexistence for $\text{SBM}(1, b, u_0, v_0)$ is impossible. Moreover, in both cases (a) and (b), if $\bar{u}_0 \le \bar{v}_0$, then

$$\bar{u}_t \to 0,$$

and

$$\bar{v}_t \to \bar{v}_0 - \bar{u}_0,$$

almost surely, as $t \to \infty$.

**Theorem 1.7** (Longtime behavior for flat initial conditions). Assume that $(u_0, v_0) = (\theta_1, \theta_2)$ with $\theta_1, \theta_2 > 0$.

(i) Let $d \ge 3$. Then for all $b \in (0, b_*)$ coexistence for $\text{SBM}(1, b, u_0, v_0)$ is possible.

(ii) Let

(a) $d \in \{1, 2\}$ and $b > 0$,

or

(b) $d \ge 3$ and $b > b_*$.

Then coexistence for $\text{SBM}(1, b, u_0, v_0)$ is impossible. Moreover, in both cases (a) and (b), if $\theta_1 \le \theta_2$, then for any $\phi \in E_{\text{cpt}}$

$$\langle u_t, \phi \rangle \to 0,$$

and

$$\langle v_t, \phi \rangle \to (\theta_2 - \theta_1) \langle 1, \phi \rangle,$$

in probability as $t \to \infty$.

Let us comment on the above results. First of all, as we see, the transition threshold between possible coexistence and non-coexistence for the $\text{SBM}(1, b, u_0, v_0)$ is exactly the same as for the PAM model which is, in fact, $\text{SBM}(1, b, u_0, v_0 = u_0)$ model. This may sound like as a pretty much well expected result, however we found that the proof of it is surprisingly not very straightforward. It does use close connection between $\text{SBM}(1)$ and PAM models, however in the regime of non-coexistence with summable non-monotone initial conditions (that is, $u_0 \not\le v_0$ and $v_0 \not\le u_0$) on top of comparison with the PAM, one uses non-trivial decomposition of the SBM and some interesting PDE results (see Section 3.2 for this argument). We hope that our proofs with give an additional motivation for studying the open question of existence of the phase transition in the transient regime for $\rho \in (0, 1)$.

2. Proof of Theorems 1.6(i) and 1.7(i): Coexistence Possible

2.1. Preparations. In Section 2 we prove the coexistence parts of Theorems 1.6, 1.7. The actual proof will be carried out in Subsection 2.2. In this subsection we will review and prove a couple of results for the PAM which we will need for the proof of Theorems 1.6(i), 1.7(i).

Let $w$ be $\text{PAM}(b, w_0)$, that is, $w$ satisfies the following equation:

$$dw_t(i) = \Delta w_t(i) dt + \sqrt{bw_t^2(i)} dW_t(i), \quad t \ge 0, i \in \mathbb{Z}^d.$$
The PAM exhibits the following simple but very usefull self-duality property, see Section 2 of Cox, Klenke and Perkins [CKP00]: let \( w \) and \( \tilde{w} \) be PAM(b) processes such that \( w_0 = \theta \) and \( \tilde{w}_0 = \phi \in E_{\text{fin}} \). Then

\[
E_{\tilde{w}_0}[e^{-\lambda(\tilde{w}, \theta)}] = E_{\tilde{w}_0}[e^{-(\tilde{w}, \lambda\theta)}] = E_{\lambda\theta}[e^{-(\tilde{w}_0, u\theta)}] = E_{\theta}[e^{-\lambda(\phi, w\theta)}], \quad \forall t \geq 0.
\]

From the self-duality we immediately obtain the following statement.

**Lemma 2.1 (self-duality).** Let \( w \) and \( \tilde{w} \) be PAM(b) such that \( w_0 = \theta \) and \( \tilde{w}_0 = \phi \in E_{\text{fin}} \). Then,

\[
\langle \tilde{w}_t, \theta \rangle \overset{d}{=} \langle \phi, w_t \rangle, \quad \forall t \geq 0.
\]

We now prove a result which allows us to bound the moments of appropriate functions of SBM by those of PAM. Since we know a lot of information about PAM this will be very useful.

**Proposition 2.2 (comparison).** Let \( (u, v) \) be the solution of SBM(1, b, u_0, v_0). Let \( w \) be a PAM(b, w_0) such that \( w_0 = u_0 + v_0 \). Let \( \Phi(t) \) be arbitrary non-negative non-decreasing convex function on \( \mathbb{R}_+ \).

(i) If \( (u_0, v_0) \in E_{\text{fin}} \), then

\[
E_{(u_0, v_0)}[\Phi(u_t, 1) + \langle v_t, 1 \rangle] \leq E_{w_0}[\Phi(\langle w_t, 1 \rangle)], \quad \forall t \geq 0. \tag{2.1}
\]

(ii) If \( (u_0, v_0) = (\theta_1, \theta_2) \) for \( \theta_1, \theta_2 \geq 0 \), then for all \( \phi \in E_{\text{fin}} \) such that \( \phi \geq 0 \) one has

\[
E_{(u_0, v_0)}[\Phi((u_t, \phi) + \langle v_t, \phi \rangle)] \leq E_{w_0}[\Phi(\langle w_t, \phi \rangle)], \quad \forall t \geq 0.
\]

**Proof.** We start with proving (i). The idea is to use comparison result of Greven, Klenke and Wakolbinger [GWK02] (the earlier version of this result for the homogeneous case is due to Cox, Fleischmann, and Greven [CFG96]): since the conditions of the result in [GWK02] on function \( \Phi \) are not satisfied in the proposition we will use the usual technique of approximation. Let \( \Lambda = [-N, N]^d \) be a torus, and \( \Delta \) acting on functions on the torus will be Laplacian with periodic boundary conditions. Let \( (u^N, v^N) \) be a solution of

\[
\begin{cases}
\text{du}^N_t(i) = \Delta u^N_t(i) \, dt + \sqrt{bu^N_t(i)v^N_t(i)} \left(1 - \frac{u^N_t(i)}{N}\right) \left(1 - \frac{v^N_t(i)}{N}\right) \, dW_t(i), \quad t \geq 0, i \in \Lambda_N, \\
\text{dv}^N_t(i) = \Delta v^N_t(i) \, dt + \sqrt{bu^N_t(i)v^N_t(i)} \left(1 - \frac{u^N_t(i)}{N}\right) \left(1 - \frac{v^N_t(i)}{N}\right) \, dW_t(i), \quad t \geq 0, i \in \Lambda_N.
\end{cases}
\]

with \( u^N_0(i) = u_0(i), v^N_0(i) = v_0(i), i \in \Lambda_N \). The solutions of this equation take values in \([0, N]^{\Lambda_N} \times [0, N]^{\Lambda_N} \).

In the following it is understood that \( i \in \Lambda_N \) and \( t \geq 0 \). Let

\[
\eta^N_t(i) := v^N_t(i) - u^N_t(i)
\]

and

\[
\xi^N_t(i) := u^N_t(i) + v^N_t(i) = 2u^N_t(i) + \eta^N_t(i) \text{ so that } u^N_t(i) = \frac{1}{2} (\xi^N_t(i) - \eta^N_t(i)).
\]

Then

\[
u^N_t(i)v^N_t(i) = \frac{1}{4} \left( (\xi^N_t(i))^2 - \eta^N_t(i)^2 \right).
\]

Therefore, we get that \( \xi^N_t(i) \) satisfies the following system of equations:

\[
d\xi^N_t(i) = \Delta \xi^N_t(i) \, dt + \sqrt{b(\xi^N_t(i))^2 - \eta^N_t(i)^2} \left(1 - \frac{\xi^N_t(i) - \eta^N_t(i)}{2N}\right) \left(1 - \frac{\xi^N_t(i) + \eta^N_t(i)}{2N}\right) \, dW_t(i), \quad t \geq 0, i \in \Lambda_N.
\]

Clearly since the noises for \( u^N \) and \( v^N \) are the same we get that \( \eta^N_t(i) \) is deterministic that solves the following system of equations:

\[
d\eta^N_t(i) = \Delta \eta^N_t(i) \, dt, \quad t \geq 0, i \in \Lambda_N.
\]
Let now \( w^N = (w^N_t)_{t \geq 0} \) be an approximate \( \text{PAM}(b, w_0) \). That is, for an independent family of Brownian motions \( \{W^w(i) : i \in \mathbb{Z}^d\} \), let \( w^N \) on \( \Lambda_N \) satisfy
\[
dw^N_t(i) = \Delta w^N_t(i) dt + \sqrt{b w^N_t(i)^2 \left( 1 - \frac{w^N_t(i)}{2N} \right)} \, dW^w_t(i), \quad t \geq 0, \, i \in \Lambda_N.
\]
with \( w^N_0(i) = w_0(i), \, i \in \Lambda_N \). Let
\[
g^N_k(i, x, t) := b(x^2 - \eta^N_k(i)^2) \left( 1 - \frac{x - \eta^N_k(i)}{2N} \right) \left( 1 - \frac{x + \eta^N_k(i)}{2N} \right), \quad x, \, t \geq 0, \quad i \in \Lambda_N,
\]
\[
g^N_\xi(i, x) := b x^2 \left( 1 - \frac{x}{2N} \right), \quad x, \, t \geq 0, \quad i \in \Lambda_N.
\]
Then, for all \( x, \, t \geq 0, \, i \in \Lambda_N \)
\[
g^\xi(i, x, t) \leq g^N_\xi(i, x).
\]
The respective generators of \( \xi^N \) and \( w^N \) are given by the closure of the following operators
\[
G^\xi_{N,i} f(\phi) = \sum_{i \in \Lambda_N} \Delta \phi(i) \partial_i f(\phi) + \frac{1}{2} \sum_{i \in \Lambda_N} g^\xi_k(\phi(i), t) \partial_i^2 f(\phi), \quad \phi \in (\mathbb{R}^+)^{\Lambda_N},
\]
\[
G^w_N f(\phi) = \sum_{i \in \Lambda_N} \Delta \phi(i) \partial_i f(\phi) + \frac{1}{2} \sum_{i} g^w_N(\phi(i)) \partial_i^2 f(\phi), \quad \phi \in (\mathbb{R}^+)^{\Lambda_N},
\]
acting on functions \( f : (\mathbb{R}^+)^{\Lambda_N} \mapsto \mathbb{R} \) such that all their partial derivatives up to order 2 exist and are continuous. Now let \( D_N \) consist of functions \( f \) as specified in the previous sentence with the additional requirement that \( \partial_i \partial_j f \geq 0 \) for all \( i, j \in \Lambda_N \). Let \( C_{bc}^2 \) be the set of convex functions \( F : \mathbb{R}_+ \mapsto \mathbb{R} \) with bounded on the compacts continuous partial derivatives of orders \( m = 0, 1, 2 \). Fix arbitrary \( F \in C_{bc}^2 \).

For \( B_r \) the open ball of radius \( r \) in \( \mathbb{Z}^d \), let \( F_r : (\mathbb{R}^+)^{\mathbb{Z}^d} \mapsto \mathbb{R}_+ \), be defined by
\[
F_r(x) = F\left( \sum_{i \in B_r} x(i) \right).
\]
Then, for all \( r \in (0, N) \), we get that \( F_r \in D_N \), and taking into account (2.2) we can apply Theorem 2 of [GKW02] to obtain that for each \( r \in (0, N) \),
\[
\mathbb{E}_{u^N_0 + v^N_0}[F_r(\xi^N)] \leq \mathbb{E}_{u^N_0 + v^N_0}[F_r(w^N)], \quad \forall t \geq 0.
\]
It is easy to check that as \( N \to \infty \), \( (\xi^N, \eta^N) \) converges weakly (in the space of continuous \( E^2_{\text{tem}} \)-valued paths) to \( (\xi, \eta) \), where \( \xi = u + v \) and \( \eta = v - u \). Moreover \( w^N \) converges weakly (in the space of continuous \( E_{\text{tem}} \)-valued paths) to \( w \) which solves \( \text{PAM}(b, w_0 = u_0 + v_0) \). Thus, letting \( N \to \infty \), passing to the limit in (2.3) and using continuity of \( F_r \) on \( E_{\text{tem}} \), we get
\[
\mathbb{E}_{u_0 + v_0}[F_r(\xi)] \leq \mathbb{E}_{u_0 + v_0}[F_r(w_t)], \forall t \geq 0.
\]
Taking the limit \( r \to \infty \) on both sides and using the monotone convergence theorem yields
\[
\mathbb{E}_{(u_0, v_0)}[F(\langle u_t, 1 \rangle + \langle v_t, 1 \rangle)] \leq \mathbb{E}_{u_0 + v_0}[F(\langle w_t, 1 \rangle)], \forall t \geq 0.
\]
Then by approximating a non-negative non-decreasing convex function \( \Phi \) by functions from \( C_{bc}^2 \) we can easily finish the proof of (2.1) by passing to the limit.

The proof of (ii) goes along the same lines with
\[
F_r(x) = F\left( \sum_{i \in B_r} x(i) \phi(i) \right)
\]
and will thus be omitted. \( \square \)

**Corollary 2.3** (Uniform integrability). Let \( d \geq 3 \). Let \( b_* \) be as in Theorem 1.3 and let \( b \in (0, b_*) \). Let \( (u, v) \) be the solution of \( \text{SBM}(1, b, u_0, v_0) \).

(a) If \( (u_0, v_0) \in E^*_\text{tem} \), then \( \{\bar{u}_t, t \geq 0\} \) and \( \{\bar{v}_t, t \geq 0\} \) are uniformly integrable.
(b) If \((u_0, v_0) = (\theta_1, \theta_2)\) for \(\theta_1, \theta_2 \geq 0\), then for all \(\phi \in \mathbb{E}_{\text{fin}}\)

\[
\{\langle u_t, \phi \rangle, t \geq 0 \} \text{ and } \{\langle v_t, \phi \rangle, t \geq 0 \} \text{ are uniformly integrable.}
\]

**Proof.** Let \(w\) be a \(\text{PAM}(b, w_0)\) such that \(w_0 = u_0 + v_0\).

Let us first prove (b), that is, the case of flat initial conditions. In this case the uniform integrability of \(\{\langle u_t, \phi \rangle, t \geq 0 \}\), for \(\phi \in \mathbb{E}_{\text{fin}}\), follows easily from Theorems 1.3, 1.4, of [GdH07]. Then by necessary and sufficient criterion for uniform integrability, for any \(\phi \in \mathbb{E}_{\text{fin}}\), we get the existence of non-negative non-decreasing convex function \(\Phi(t)\) on \(\mathbb{R}_+\) such that \(\lim_{t \to \infty} \frac{\Phi(t)}{t} < \infty\)

\[
\sup_{t \geq 0} \mathbb{E}[\Phi(\langle w_t, \phi \rangle)] = \infty.
\]

Then Proposition 2.2(ii) and again the criterion for uniform integrability imply that both \(\{\langle u_t, \phi \rangle, t \geq 0 \}\) and \(\{\langle v_t, \phi \rangle, t \geq 0 \}\) are uniformly integrable for any \(\phi \in \mathbb{E}_{\text{fin}}\).

For the proof of (a) we use the self-duality of \(\text{PAM}\) proved in Lemma 2.1 and again Theorems 1.3, 1.4, of [GdH07] to get uniform integrability of \(\{\tilde{u}_t, t \geq 0\}\) and then we use Proposition 2.2(i) and follow the lines of the proof for the case (b). \(\square\)

### 2.2. Proof of Theorem 1.6(i).

With the preparations from the previous section we are now in a position to prove Theorem 1.6(i), the coexistence result for summable initial conditions.

We need to show that

\[
P\{\tilde{u}_\infty \bar{v}_\infty > 0\} > 0. \tag{2.4}
\]

Without loss of generality assume that \(\bar{u}_0 \leq \bar{v}_0\). Note that \(\tilde{\eta}_t \equiv \tilde{v}_t - \tilde{u}_t = \bar{v}_0 - \bar{u}_0 \geq 0\) for all \(t\) since \(\eta\) solves the heat equation and its total mass remains constant. Therefore, by sending \(t\) to infinity we get

\[
\bar{v}_\infty \geq \bar{u}_\infty, \quad \mathbb{P} - \text{a.s.} \tag{2.5}
\]

By Corollary 2.3(a) \(\{\tilde{u}_t, t \geq 0\}\) is uniformly integrable, and thus we get

\[
\mathbb{E}_{u_0}\tilde{u}_\infty = \mathbb{E}_{w_0}\lim_{t \to \infty} \tilde{u}_t = \lim_{t \to \infty} \mathbb{E}_{u_0}\tilde{u}_t = \bar{u}_0 > 0.
\]

Thus \(\mathbb{P}\{\tilde{u}_\infty > 0\} > 0\) and using (2.5) we get (2.4). \(\square\)

### 2.3. Proof of Theorem 1.7(i).

The proof of the coexistence result for flat initial conditions goes along the similar lines as the proof for summable initial conditions. Let \((u_\infty, v_\infty)\) be an arbitrary weak limit point of \(\{u_t, v_t\}, t \geq 0\). We will show a little bit more than required in the definition of coexistence. Let \((\phi, \phi)\) be an arbitrary element in \(E^2_{\text{cpt}}\). We will show that

\[
P\{\langle u_\infty, \phi \rangle\langle v_\infty, \phi \rangle > 0\} > 0. \tag{2.6}
\]

Without loss of generality assume that \(\theta_1 \leq \theta_2\). Note that \(\eta_t(x) \equiv v_t(x) - u_t(x) \equiv \theta_2 - \theta_1 \geq 0\) for all \(t \geq 0\) and \(x \in \mathbb{Z}^d\) since \(\eta\) solves the heat equation starting at constant initial condition \(\theta_2 - \theta_1\).

Therefore, by sending \(t\) to infinity we get that for an arbitrary weak limit point \((u_\infty, v_\infty)\) the following holds:

\[
v_\infty(x) \geq u_\infty(x), \forall x \in \mathbb{Z}^d, \quad \mathbb{P} - \text{a.s.} \tag{2.7}
\]

By Corollary 2.3(b) \(\{u_t, \phi\}, t \geq 0\) is uniformly integrable, and thus we get

\[
\mathbb{E}[\langle u_\infty, \phi \rangle] = \lim_{t \to \infty} \mathbb{E}_{(\theta_1, \theta_2)}[\langle u_t, \phi \rangle] = \theta_1\langle 1, \phi \rangle > 0.
\]

Thus \(\mathbb{P}\{\langle u_\infty, \phi \rangle > 0\} > 0\) and using (2.7) we get (2.6). \(\square\)
3. Proof of Theorems 1.6(ii) and 1.7(ii): Coexistence Impossible.

3.1. Proof of Theorem 1.7(ii). We are now turning our attention to the proof of Theorem 1.7(ii) which states that coexistence is impossible for flat initial conditions in the recurrent case \( d \leq 2 \) for arbitrary \( b > 0 \) and in the transient case \( d \geq 3 \) for arbitrary \( b > b_* \).

If in addition, we assume \( u_0 = v_0 = \theta > 0 \), then the SBM coincides with the PAM, whose longtime behavior has been studied extensively in \([ \text{GdH07}]\), see Theorem 1.5. In this case we immediately get that the coexistence is impossible, and both populations do not survive.

Now, without loss of generality assume that \( \theta_1 \leq \theta_2 \). In what follows, we consider both cases of \( d \leq 2 \), \( b > 0 \) and \( d \geq 3 \), \( b > b_* \), simultaneously. Note that \( \eta_t(i) \equiv v_t(i) - u_t(i) \equiv \theta_2 - \theta_1 \geq 0 \) for all \( t \geq 0 \) and \( i \in \mathbb{Z}^d \), since \( \eta \) solves the heat equation starting at constant initial condition \( \theta_2 - \theta_1 \) and thus it remains constant. Note that we can re-write the equation for \( u \) as follows:

\[
du_t(i) = \Delta u_t(i) \, dt + \sqrt{b} u_t(i)(u_t(i) + (\theta_2 - \theta_1)) \, dW_t(i), \quad i \in \mathbb{Z}^d.
\]

Thus, the diffusion coefficient in the generator for \( u \) is strictly larger than the diffusion coefficient for the PAM(b). We can therefore again use a comparison result for interacting diffusions by Greven, Klenke and Wakolbinger \([ \text{GKW02}]\). If \( w \) is PAM(b) with \( w_0 = \theta_1 \), Theorem 2 in \([ \text{GKW02}]\) immediately implies

\[
E_{\theta_1, \theta_2}[e^{-\langle u_t, \phi \rangle}] \geq E_{\theta_1}[e^{-\langle w_t, \phi \rangle}], \quad \forall t \geq 0,
\]

for any \( \phi \in \mathcal{E}_{\text{opt}} \). By Theorem 1.6(i), the right hand side of the above equation converges to 1 as \( t \to \infty \). Therefore the left hand side also converges to 1, and thus we get that

\[
\langle u_t, \phi \rangle \to 0,
\]

in probability as \( t \to \infty \). Since \( v_t = u_t + (\theta_2 - \theta_1) \) we immediately get

\[
\langle v_t, \phi \rangle \to (\theta_2 - \theta_1)\langle 1, \phi \rangle,
\]

in probability as \( t \to \infty \), and this finishes the proof. \( \square \)

Remark 3.1. Morally, the comparison result implies that “more noise kills”. In other words, if a system of (locally 1-dimensional) interacting diffusions weakly dies out as time tends to infinity, then so does a system which has a larger diffusion coefficient. Intuitively this makes sense since locally 0 is a trap for the system. Very roughly speaking, the process gets close to 0 at some time almost surely, and the higher the fluctuations the higher the chance that the process will actually hit zero once it gets close. Therefore, when initial conditions are monotonic, the comparison result implies that the non-coexistence regime of the PAM immediately carries over to the SBM. Actually it implies not only non-coexistence but also that it is the population which has been smaller at the beginning is the one which will suffer extinction.

3.2. Proof of Theorem 1.6(ii). In the case of monotone initial conditions \( v_0(\cdot) \geq u_0(\cdot) \), to get the result, we can argue as in the proof of Theorem 1.7(ii) while again using the comparison theorem from Greven, Klenke and Wakolbinger \([ \text{GKW02}]\).

Thus, let us consider the general case when initially one population does not necessary dominate another one. Again, in what follows, we consider both cases of \( d \leq 2 \), \( b > 0 \) and \( d \geq 3 \), \( b > b_* \), simultaneously. Recall that we assume without loss of generality that \( \overline{u}_0 = \langle u_0, 1 \rangle \leq \langle v_0, 1 \rangle = \overline{v}_0 \).

Define

\[
\eta_t := v_t - u_t, \quad t \geq 0.
\]

Then \( d\eta_t = \Delta \eta_t \, dt, \ t \geq 0 \). Note that we can re-write the equation for \( u \) as follows:

\[
du_t(i) = \Delta u_t(i) \, dt + \sqrt{b} u_t(i)(u_t(i) + \eta_t(i)) \, dW_t(i), \quad t \geq 0, i \in \mathbb{Z}^d.
\]

(3.1)

For our general initial conditions in \( E_{\text{fin}} \) the perturbation \( \eta_t \) in the diffusion coefficient in (3.1) can be positive or negative. Hence, the comparison argument used above fails since it is no longer true that \( u_t(i)(u_t(i) + \eta_t(i)) \geq u_t(i)^2 \) for all \( t \geq 0, i \in \mathbb{Z}^d \). Nonetheless, we will again use comparison techniques with the PAM. We will quickly explain the main idea.
We will need the following notation: for any $\phi : \mathbb{Z}^d \mapsto \mathbb{R}$ we denote its positive and negative part by

$$\phi^+ := \max\{\phi, 0\}, \quad \phi^- := \max\{-\phi, 0\}.$$  

The idea is to decompose $u_t$ into the mass where it exceeds $\nu$ and the minimum of $u$ and $\nu$. Define the minimum process $w = (w_t)_{t \geq 0}$ by setting

$$w_t(i) := \min\{u_t(i), \nu_t(i)\}, \quad t \geq 0, i \in \mathbb{Z}^d.$$  

Then, clearly

$$u_t(i) = w_t(i) + \eta_t^-(i), \quad t \geq 0, i \in \mathbb{Z}^d.$$  

We will show that the total mass of the negative part of the heat equation, $\eta_t^-$, vanishes as $t \to \infty$. Moreover it turns out that the minimum process $w$ exhibits an “approximate” duality with the PAM which allows us to deduce its extinction from the extinction of the PAM. Then, clearly if the excess $u$ has over $\nu$ vanishes and the minimum of $u$ and $\nu$ also goes to 0, then $u$ will eventually suffer extinction!

Before we proceed to the proof of the non-coexistence result we need a brief digression on the heat equation.

In order not to confuse with the difference process $v_t - u_t$, for the following general arguments we use the generic notation $\zeta$ for the solution of the heat equation. So, for $f : \mathbb{Z}^d \mapsto \mathbb{R}$, let

$$d\zeta_f(t,i) = \Delta \zeta_f(t,i) dt, \quad \zeta(0,i) = f(i), \quad t \geq 0, i \in \mathbb{Z}^d.$$  

If $M > 0$ and $f = M \mathbb{1}_{\{0\}}$, we write $\zeta^M$ for the corresponding solution. In the following, for a differentiable function $g : \mathbb{R}_+ \mapsto \mathbb{R}$, we denote the derivative by $\partial_h g$.

For the proof Theorem 1.6(ii) we will need to understand the behavior of the negative part of $\zeta$. For example, we will prove and use the fact that if at time $t = 0$ there is an overall excess of “positive heat”, that is, $\langle \zeta^f, 1 \rangle \geq \langle \zeta^- f, 1 \rangle$, the negative heat will eventually disappear: $\langle \zeta^f(t), 1 \rangle \to 0$, as $t \to \infty$.

Note that $\langle \zeta_f(t), 1 \rangle = \langle f, 1 \rangle$ is constant.

Some of the facts about the discrete heat equation which we will state and prove in the following are probably well-known in the literature. For completeness, we present anyway these results including our proofs.

For $T > 0$ and $i \in \mathbb{Z}^d$, let $Z_f(T,i)$ be the set of zeros of $\zeta_f(t,i)$ in $[0,T]$, that is,

$$Z_f(T,i) := \{ t \in [0,T] : \zeta_f(t,i) = 0 \}.$$  

In what follows for any $A \subset \mathbb{R}$, $|A|$ will denote the number of points in $A$.

**Proposition 3.1** ($\zeta$ analytic). Let $f : \mathbb{Z}^d \mapsto \mathbb{R}$ be bounded. Then $\zeta_f(\cdot,i)$ is (real) analytic for each $i \in \mathbb{Z}^d$. If $f \neq 0$, then $|Z_f(T,i)| < \infty$ for each $T > 0$, $i \in \mathbb{Z}^d$.

**Proof.** In the following we consider $i \in \mathbb{Z}^d$ to be fixed. Recall that $p_t(\cdot, \cdot)$ denotes the transition probability of a continuous-time, rate 1, simple symmetric random walk on $\mathbb{Z}^d$ and let $p^n(i,j)$ be the probability that a discrete-time simple symmetric random walk on $\mathbb{Z}^d$ jumps from $i$ to $j$ in $n$ steps. Then, for $t \geq 0$,

$$\zeta_f(t,i) = \sum_j p_t(i,j)f(j) = e^{-t} \sum_j \left( \sum_{n=0}^{\infty} \frac{p^n(i,j)}{n!} \right) f(j) = e^{-t} \sum_n \frac{1}{n!} \left( \sum_j p^n(i,j)f(j) \right) t^n \quad (3.2)$$

and thus $\zeta_f(\cdot,i)$ has a representation as a power series whose radius of convergence is infinite. Note that if $f$ is bounded, then

$$\sum_n \sum_j \frac{t^n}{n!} p^n(i,j) |f(j)| < \infty$$

and hence we can re-order the series in (3.2).

To prove the second statement, note that it is immediate from (3.2) that $\zeta_f(\cdot,i)$ can be extended to an analytic function on the complex plane. Assume that for some $T > 0$, $|Z_f(T,i)| = \infty$. Then $Z_f(T,i)$ has an accumulation point. It is well-known that if the zero set of an analytic function has an accumulation point then the function is identically zero on its entire domain (see for example Theorem 4.3.7 in [Cont]). But this is a contradiction to the fact that $f \neq 0$. \qed
Corollary 3.2 (Differentiability of $\zeta^+$). Let $f : \mathbb{Z}^d \to \mathbb{R}$ be bounded, $i \in \mathbb{Z}^d$, and $T > 0$. There are only finitely many points in $[0, T]$ where $\zeta^+_f (\cdot, i)$ is not differentiable.

Proof. Clearly the set of points in $[0, T]$ where $\zeta^+_f (\cdot, i)$ is not differentiable is a subset of the set of points in $[0, T]$ where $\zeta (\cdot, i)$ changes its sign. But the latter set is a subset of $\mathcal{Z}_f (T, i)$. Therefore the claim is a consequence of Proposition 3.1.

Proposition 3.3 (Properties of $\zeta^-$). Let $f : \mathbb{Z}^d \to \mathbb{R}$. There is a function $q_f : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{R}$ such that

$$\zeta^-_f (t, i) = f^- (i) + \int_0^t \Delta \zeta^-_f (s, i) \, ds - q_f (t, i), \quad t \geq 0, \, i \in \mathbb{Z}^d. \tag{3.3}$$

For each $i \in \mathbb{Z}^d$, the function $q_f (\cdot, i) : \mathbb{R}_+ \to \mathbb{R}$ has the following properties:

(i) $q_f (\cdot, i)$ is non-negative and non-decreasing.

(ii) For $T > 0$, define $\mathcal{Q}_f (T, i) := \{ t \in [0, T] : \text{the derivative of } q(t, i) \text{ exists in } t \}$. Then $|T \setminus \mathcal{Q}_f (T, i)| < \infty$.

(iii) For any $t > 0$ define

$$\partial q_f (t, i) := \begin{cases} \text{the derivative of } q(\cdot, i) \text{ at } t, & \text{if } t \in \bigcup_{T > 0} \mathcal{Q}_f (T, i), \\ 0, & \text{if } t \notin \bigcup_{T > 0} \mathcal{Q}_f (T, i). \end{cases}$$

Then,

$$q_f (t, i) = \int_0^t \partial q_f (s, i) \, ds, \quad t \geq 0, \tag{3.4}$$

that is, $q_f (\cdot, i)$ is absolutely continuous.

For simplicity we will omit the subindex $f$ whenever confusion is impossible.

Proof of Proposition 3.3. Since $f$ is fixed we drop the subindex $f$ in what follows in the proof of the proposition. Also in the following we consider $i \in \mathbb{Z}^d$ to be fixed and $\zeta$ as a function of $t$ only. Define

$$q(t, i) := -\zeta^+(t, i) + \zeta^+(0, i) + \int_0^t \Delta \zeta^+(s, i) \, ds, \quad t \geq 0. \tag{3.5}$$

Then, by writing $\zeta^- (t, i) = \zeta^+ (t, i) - \zeta (t, i)$ and re-arranging terms in the discrete heat equation we obtain

$$\zeta^- (t, i) = \zeta^- (0, i) + \int_0^t \Delta \zeta^- (s, i) \, ds - q(t, i), \quad t \geq 0.$$

Corollary 3.2 and the definition of $q$ in (3.5) clearly imply that $|T \setminus \mathcal{Q}_f (T, i)| < \infty$ for each $T > 0$. Since in addition we know that $\zeta (\cdot, i)$ is analytic, it is easy to see that (3.1) holds true.

We still have to prove that $q(t, i)$ is non-decreasing in $i$. As $(T \setminus \mathcal{Q}(T, i)) \subset \mathcal{Z}_f (T, i)$ and $|\mathcal{Z}_f (T, i)| < \infty$ for every $T > 0$ it is clearly enough to prove that $\partial q(t, i) \geq 0$ for $t \in \bigcup_{T > 0} (T \setminus \mathcal{Z}_f (T, i))$. So, let $t \in \bigcup_{T > 0} (T \setminus \mathcal{Z}_f (T, i))$. Then (3.5) implies

$$\partial q(t, i) = -\partial \zeta^+ (t, i) + \Delta \zeta^+ (t, i). \tag{3.6}$$

Now, assume that $\zeta (t, i) > 0$. Then $\zeta^+(t, i) = \zeta (t, i)$ and $\partial \zeta^+ (t, i) = \zeta (t, i)$. Therefore, since $\partial \zeta (t, i) = \Delta \zeta (t, i)$ we get from (3.6) that

$$\partial q(t, i) = \Delta \zeta^+ (t, i) - \Delta \zeta (t, i) \geq 0.$$

Alternatively if $\zeta (t, i) < 0$, we have that $\zeta^+ (t, i) = 0$ and $\partial \zeta^+ (t, i) = 0$. Therefore (3.6) implies

$$\partial q(t, i) = \Delta \zeta^+ (t, i) \geq 0.$$
Moreover, which, as can be easily checked, for each
\[q(t, i) = 0 \text{ this implies also } q(t, i) \geq 0.\]
Thus we have proved all claims. \(\square\)

**Lemma 3.4.** Let \(f \in E_{\text{fin}}\) and denote \(M := \langle f, 1 \rangle\). Then,
\[
\lim_{t \to \infty} \langle |\zeta(t) - \zeta(t)|, 1 \rangle = 0.
\]

For a related result for the heat equation on \(\mathbb{R}^d\) see (1.10) in [EZ91]. We are grateful to Yehuda Pinchover who pointed us to this reference.

**Proof.** Recall that \(p_t(\cdot, \cdot)\) denotes the transition probability of a simple symmetric random walk on \(\mathbb{Z}^d\) with the generator \(\Delta\). Note that then \(\zeta(t, i) = \sum_j p_t(i, j) f(j) = \sum_i p_t(0, i) f(j)\) and thus \(\zeta(t, i) = M p_t(0, i) = \sum_j f(j) p_t(0, i)\). Therefore,
\[
\langle \zeta(t) - \zeta(t), 1 \rangle = \sum_i \left| \sum_j (p_t(0, i) - p_t(0, i)) f(j) \right| \\
\leq \sum_j |f(j)| \sum_i (p_t(0, i) - p_t(0, i), t \geq 0.
\]

Now since \(\sum_i (p_t(0, i) - p_t(0, i)) \leq 2\) and \(f \in E_{\text{fin}}\) the dominated convergence theorem implies that it is enough to show that, for each \(j,\)
\[
\lim_{t \to \infty} \sum_i |p_t(0, i) - p_t(0, i)| = 0.
\]

Let \(p^n(0, i)\) be the probability that a discrete time simple random walk on \(\mathbb{Z}^d\) jumps from 0 to \(i\) in \(n\) steps. Then
\[
\sum_i |p_t(0, i) - p_t(0, i)| \leq e^{-t} \sum_{n \geq 0} \frac{n}{n!} \sum_i |p^n(0, i) - p^n(0, i)|, \quad t \geq 0.
\]

By Proposition 2.4.1 in Lawler and Limic [LL10], one has for all \(j \in \mathbb{Z}^d\)
\[
\sum_i |p^n(0, i) - p^n(0, i)| \leq c|j|^{-1/2},
\]
for some constant \(c\). Hence,
\[
\sum_i |p_t(0, i) - p_t(0, i)| \leq c|j|e^{-t} \sum_{n \geq 0} \frac{n}{n!} \frac{1}{n^{1/2}},
\]
which, as can be easily checked, for each \(j\) tends to 0 as \(t \to \infty\) and we are done. \(\square\)

The above lemma implies the following proposition.

**Proposition 3.5** (longtime behavior of \(\zeta^-\) and \(\zeta^+\)). Let \(f \in E_{\text{fin}}\) and denote \(M := \langle f, 1 \rangle\). Assume \(M \geq 0\). Then,
\[
\lim_{t \to \infty} \langle |\zeta(t)|, 1 \rangle = \lim_{t \to \infty} \langle |\zeta^+(t)|, 1 \rangle = M, \quad \lim_{t \to \infty} \langle |\zeta^-(t)|, 1 \rangle = 0. \quad (3.7)
\]

Moreover,
\[
\lim_{t \to \infty} \langle q(t), 1 \rangle = \langle f^-, 1 \rangle.
\]

**Proof.** Recall that \(\langle \zeta(t), 1 \rangle = M\) for all \(t \geq 0\). Thus by the reverse triangle inequality (applied to the \(L^1\) norm on \(E_{\text{fin}}\)),
\[
\langle |\zeta(t)|, 1 \rangle - M \geq \|\zeta(t) - \zeta(t)\|_1 = \|\zeta(t) - \zeta(t)\|_1 \leq \|\zeta(t) - \zeta(t)\|_1 = \langle |\zeta(t) - \zeta(t)|, 1 \rangle, \quad t \geq 0.
\]
By Lemma 3.4, \( \lim_{t \to \infty} \langle \zeta_f(t) - \zeta^M(t), 1 \rangle = 0 \). Now we use
\[
\langle \zeta_f(t), 1 \rangle = \frac{1}{2}(\langle |\zeta_f(t)|, 1 \rangle \pm \langle \zeta_f(t), 1 \rangle) = \frac{1}{2}(\langle |\zeta_f(t)|, 1 \rangle \pm M), \quad t \geq 0.
\]
to get (3.7). Finally, (3.3) and the fact that \( \lim_{t \to \infty} \langle \zeta_f(t), 1 \rangle = 0 \) imply \( \lim_{t \to \infty} \langle q_f(t), 1 \rangle = \langle f^-, 1 \rangle \). □

We are now ready to prove the non-coexistence result for SBM(1).

Proof of Theorem 1.6 (ii). Recall that \( \eta_t = v_t - u_t \) and \( u_t = w_t + \eta_t^- \), where \( w_t = \min\{u_t, v_t\} \). Since we assume that \( \langle u_0, 1 \rangle \leq \langle v_0, 1 \rangle \), we have \( \langle \eta_0, 1 \rangle \geq 0 \). Therefore, Proposition 3.5 implies that
\[
\lim_{t \to \infty} \langle \eta^+_t, 1 \rangle = 0.
\]
Hence, if we can also prove that
\[
\lim_{t \to \infty} \langle w_t, 1 \rangle = 0, \tag{3.8}
\]
then \( \lim_{t \to \infty} \langle u_t, 1 \rangle = \lim_{t \to \infty} (\langle w_t, 1 \rangle + \langle \eta^-_t, 1 \rangle) = 0 \) and we are done. So we are left with the task to show (3.8). The idea is to use the fact that \( w \) is “approximately” dual to a PAM as \( t \to \infty \).

Note again that \( \eta_t = \zeta_{u_0}(t) \) is the solution of the discrete heat equation started in \( v_0 - u_0 \). Hence, since \( w_t = u_t - \eta_t^- \), by Proposition 3.3, we get that \( w \) satisfies the following equation
\[
w_t(i) = w_0(i) + \int_0^t \Delta w_s(i) \, ds + \int_0^t \sqrt{b_{u_s}(i)v_s(i)} \, dW_s(i) + q(t, i), \quad t \geq 0, \, i \in \mathbb{Z}^d. \tag{3.9}
\]
Here, actually \( q(t, i) = q_{v_0-u_0}(t, i) \) but for simplicity we omit the subindex. By (3.9) and the properties of \( q \) proved in Proposition 3.3, \( w(i) \) is a semimartingale, for every \( i \in \mathbb{Z}^d \). Note that
\[
w_t^2(i) \leq u_t(i)v_t(i), \quad t \geq 0, \, i \in \mathbb{Z}^d, \tag{10.10}
\]
an observation which will be essential for the comparison between \( w \) and the PAM.

Now, fix \( \varepsilon > 0 \) arbitrary small. Recall that we denote \( \bar{q}(t) = \sum_i q(t, i) \). By Proposition 3.3, \( \lim_{t \to \infty} \bar{q}(t) = \bar{q}^\infty \) exists and is finite. Define
\[
T^* := \inf\{t \geq 0 : 0 \leq \bar{q}^\infty - \bar{q}(t) \leq \varepsilon\}. \tag{11.11}
\]

Now let \( \tilde{w} = (\tilde{w}_t)_{t \geq 0} \) be a PAM starting at \( \tilde{w}_0 \) and independent of \( w \). That is, it satisfies
\[
\tilde{w}_t(i) = \bar{w}_0(i) + \int_0^t \Delta \tilde{w}_s(i) \, ds + \sqrt{b\tilde{w}_s(i)} \, d\tilde{W}_s(i), \quad t \geq 0, \, i \in \mathbb{Z}^d,
\]
where \( \tilde{W}(i), i \in \mathbb{Z}^d \), are independent Brownian motions which are assumed to be also independent of \( \{W(i), i \in \mathbb{Z}^d\} \).

Now recall the following result which is stated in Lemma 4.4.10 of Ethier and Kurtz [EK86]. Let \( f : [0, \infty) \times [0, \infty) \to \mathbb{R} \) be a function such that \( f(\cdot, t) \) is absolutely continuous for each \( t \) and \( f(s, \cdot) \) is absolutely continuous for each \( s \) and \( \int_0^T \int_0^T |f_1(s, t) + f_2(s, t)| \, ds \, dt < \infty, T \geq 0 \). Here \( (f_1, f_2) = \nabla f \). Then, for every \( R \geq 0 \),
\[
f(T, 0) - f(R, T - R) = \int_R^T (f_1(s, T - s) - f_2(s, T - s)) \, ds, \quad \text{for almost every } T \geq R. \tag{12.12}
\]

Lemma 4.4.10 [EK86] actually states this formula only for \( R = 0 \) but it can easily verified that it holds for all \( R \geq 0 \). Now we would like to apply (12.12) to the function
\[
f(a, b) = \mathbb{E}[\exp(-\langle w_a, \tilde{w}_b \rangle)], \quad a, b \geq 0, \tag{13.13}
\]
with \( R = T^* \). Note that Itô’s formula implies that \( f \) is absolutely continuous in \( a \) for each fixed \( b \) and it is absolutely continuous in \( b \) for each fixed \( a \). Hence the conditions of Lemma 4.4.10 in [EK86] are satisfied. Now we use Itô’s formula to calculate \( f_1(s, t) \) and \( f_2(s, t) \). Then, for all non-negative \( \phi \in E_{\text{fin}} \), we have
\[
\exp(-\langle w_s, \phi \rangle)
\]
\[
\begin{align*}
&= \exp(-\langle w_0, \phi \rangle) - \sum_i \int_0^s e^{-(w_r, \phi)} \phi(i) \, dw_r(i) \\
&\quad + \frac{1}{2} \sum_i \int_0^s e^{-(w_u, \phi)} \phi(i)^2 \, d[w(i)], \\
&= \exp(-\langle w_0, \phi \rangle) - \int_0^s e^{-(w_r, \phi)} \sum_i \phi(i) \Delta w_r(i) \, dr \\
&\quad - \int_0^s e^{-(w_r, \phi)} \sum_i \phi(i) \sqrt{bu_r(i)v_r(i)} \, dW_r(i) - \sum_i \int_0^s e^{-(w_r, \phi)} \phi(i) \, dq(r, i) \\
&\quad + \frac{1}{2} \int_0^s e^{-(w_r, \phi)} \sum_i \phi(i)^2 bu_r(i)v_r(i) \, dr, \quad s \geq 0.
\end{align*}
\]

It is easy to check that \( s \mapsto \int_0^s e^{-(w_r, \phi)} \sum_i \phi(i) \sqrt{bu_r(i)v_r(i)} \, dW_r(i) \) is a martingale and not just the local martingale (see Proposition 3.2 of Blath, Döring and Etheridge [BDE11] for a relevant result). Thus, taking expectation on both sides yields

\[
\mathbb{E}[\exp(-\langle w_s, \phi \rangle)] = \exp(-\langle w_0, \phi \rangle) - \mathbb{E} \left[ \int_0^s e^{-(w_r, \phi)} (\Delta w_r, \phi) \, dr \right] - \mathbb{E} \left[ \langle \int_0^s e^{-(w_r, \phi)} \, dq(r, \phi) \rangle \right] + \mathbb{E} \left[ \frac{1}{2} \int_0^s e^{-(w_r, \phi)} (bu_r, \phi^2) \, dr \right], \quad s \geq 0.
\]

Taking derivative with respect to \( s \) (it exits for almost every \( s \) for which \( \partial_s q(s) \) exists, see Proposition 3.3) yields

\[
\partial_s \mathbb{E}[\exp(-\langle w_s, \phi \rangle)] = -\mathbb{E} \left[ e^{-(w_s, \phi)} (\Delta w_s, \phi) \right] - \mathbb{E} \left[ e^{-(w_s, \phi)} (\partial_s q(s), \phi) \right] + \mathbb{E} \left[ \frac{1}{2} e^{-(w_s, \phi)} (bu_s, \phi^2) \right], \quad s \geq 0.
\]

Now assume that \( \hat{w}_0 \in E_{\text{fin}} \). In exactly the same way as above we calculate for every \( \psi \in E_{\text{fin}} \) (use that the Laplacian is self-adjoint):

\[
\begin{align*}
\partial_t &\mathbb{E}[\exp(-\langle \psi, \hat{w}_t \rangle)] \\
&= -\mathbb{E} \left[ e^{-(\psi, \hat{w}_t)} (\Delta \hat{w}_t, \psi) \right] + \mathbb{E} \left[ \frac{1}{2} e^{-(\psi, \hat{w}_t)} (\psi^2, b\hat{w}_t^2) \right] \\
&= -\mathbb{E} \left[ e^{-(\psi, \hat{w}_t)} (\Delta \psi, \hat{w}_t) \right] + \mathbb{E} \left[ \frac{1}{2} e^{-(\psi, \hat{w}_t)} (b\psi^2, \hat{w}_t^2) \right], \quad t \geq 0.
\end{align*}
\]

For \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \) define \( |x|_\infty = \sup_{i=1, \ldots, d} |x_i| \) and \( B_n = \{ x \in \mathbb{Z}^d, |x|_\infty \leq n \} \). Fix arbitrary \( \theta > 0 \). Let \( \hat{w}_{0T}^\theta = \theta 1_{B_n} \) and \( \hat{w}^n \) be the solution to the PAM starting at \( \hat{w}_{0T}^\theta \). Now by (3.12), (3.13), (3.14) and (3.15), for any \( T > T^* \), we have

\[
\begin{align*}
\mathbb{E} &\left[ \exp(-\langle w_T, \hat{w}_T^n \rangle) \right] - \mathbb{E} [\exp(-\langle w_T, \hat{w}^\theta \rangle)] \\
&= \int_T^{T^*} \mathbb{E} \left[ e^{-(w_s, \hat{w}_T^n)} (\Delta w_s, \hat{w}_T^n) \right] + \mathbb{E} \left[ \frac{1}{2} e^{-(w_s, \hat{w}_T^n)} (bu_s^2, (\hat{w}_T^n)^2) \right] \, ds \\
&\quad + \mathbb{E} \left[ e^{-(w_s, \hat{w}_T^n)} (\Delta w_s, \hat{w}_T^n) \right] + \mathbb{E} \left[ e^{-(w_s, \hat{w}_T^n)} (\partial_s q(s), \hat{w}_T^n) \right] \\
&\quad - \mathbb{E} \left[ \frac{1}{2} e^{-(w_s, \hat{w}_T^n)} (bu_s, (\hat{w}_T^n)^2) \right] \\
&= \int_T^{T^*} \mathbb{E} \left[ \frac{1}{2} e^{-(w_s, \hat{w}_T^n)} (b(u_s^2 - u_s v_s), (\hat{w}_T^n)^2) \right] \, ds + \mathbb{E} \left[ e^{-(w_s, \hat{w}_T^n)} (\partial_s q(s), \hat{w}_T^n) \right].
\end{align*}
\]
we can take $n$ sufficiently large such that
\[
\int_0^T \mathbb{E} \left[ e^{-\langle w_r^x, \bar{w}_{T-s}^x \rangle} \langle \partial_s q(s), \bar{w}_{T-s}^x \rangle \right] \, ds \leq \int_0^T \langle \partial_s q(s), \mathbb{E} [\bar{w}_{T-s}^x] \rangle \, ds \leq \int_0^T \langle \partial_s q(s), P_{T-s} \bar{w}_0^x \rangle \, ds.
\] (3.16)

This implies (3.8) since $\varepsilon$ was arbitrary small. Since $w_T = \theta$, for any $T > T^*$ such that
\[
\mathbb{E}[\exp(-\langle w_{T^*}, \bar{w}_{T-T^*}^x \rangle)] - 1 \leq \varepsilon, \text{ for } T > T^*.
\] (3.17)

where the last inequality follows by (3.11). We know from Theorem 1.5 that for any compactly supported function $\phi$, $w - \lim_{n \to \infty} \langle \phi, \bar{w}_n \rangle = 0$ if $b > b_\ast$. Note that $w_{T^*}$ is not necessarily compactly supported, so we need an additional simple argument. Fix $\varepsilon > 0$ arbitrary small. Since $P_{T^*} \bar{w}_0 \in E_{\bar{w}_0}$ we can take $n$ sufficiently large such that
\[
\mathbb{E} \left[ \langle w_{T^*} \cdot \mathbf{1}_{B^x_{\varepsilon}}, \bar{w}_{T-T^*} \rangle \right] = \theta \langle \langle P_{T^*} \bar{w}_0 \rangle \cdot \mathbf{1}_{B^x_{\varepsilon}}, \mathbf{1} \rangle = \theta \langle P_{T^*} \bar{w}_0, \mathbf{1}_{B^x_{\varepsilon}} \rangle \leq \varepsilon.
\] (3.18)

On the other hand, by Theorem 1.5
\[
\langle w_{T^*} \cdot \mathbf{1}_{B^x_{\varepsilon}}, \bar{w}_{T-T^*} \rangle \Rightarrow 0,
\] (3.21)

at $T \to \infty$. Since $\varepsilon$ was arbitrary small we get from (3.20), (3.21)
\[
\langle w_{T^*} \cdot \mathbf{1}_{B^x_{\varepsilon}}, \bar{w}_{T-T^*} \rangle \Rightarrow 0,
\] (3.22)

at $T \to \infty$. Hence $\lim_{T^* \to \infty} \mathbb{E}[\exp(-\langle w_{T^*}, \bar{w}_{T-T^*}^x \rangle)] = 1$. So we can find $T^* > T^*$ such that
\[
\mathbb{E}[\exp(-\langle w_{T^*}, \bar{w}_{T-T^*} \rangle)] - 1 \leq \varepsilon, \text{ for } T > T^*.
\] (3.23)

Hence for $t > T^*$, we get by (3.19), (3.23)
\[
\mathbb{E}[\exp(-\theta(w_T, 1))] \geq 1 - \varepsilon(1 + \theta).
\] (3.24)

This implies (3.8) since $\varepsilon$ was arbitrary small.

\[ \square \]

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Technion—Israel Institute of Technology, Faculty of Industrial Engineering & Management, Haifa 32000, Israel

Email address: Patric.Gloede@t-online.de, leonid@ie.technion.ac.il