Berry Phase and Fidelity in the Dicke model with $A^2$ term

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The instability, so-called the quantum-phase-like transition, in the Dicke model with a rotating-wave approximation for finite $N$ atoms is investigated in terms of the Berry phase and the fidelity. It can be marked by the discontinuous behavior of these quantities as a function of the atom-field coupling parameter. Involving an additional field $A^2$ term, it is observed that the instability is not eliminated beyond the characteristic atom-field coupling parameter even for strong interaction of the bosonic fields, contrarily to the previous studies.

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I. INTRODUCTION

The Dicke model (DM)\cite{1} describes an ensemble artificial two-level atoms coupling with a cavity device. It has been attracted considerable attentions recently, mainly due to the fact that the Dicke model is closely related to many recent interesting fields in quantum optics and condensed matter physics, such as the superradiant behavior by an ensemble of quantum dots \cite{3} and Bose-Einstein condensates \cite{4}, coupled arrays of optical cavities used to simulate and study the behavior of strongly correlated systems\cite{4}, and superconducting charge qubits\cite{5,6}. It is known from the previous studies\cite{5,6,4} that the full DM undergoes the second-order quantum phase transition \cite{10}.

As claimed in Ref. \cite{11}, a sequence of instabilities, so-called quantum-phase-like transitions, is involved in the problem of an ensemble of two-level atoms system interacting with a bosonic field in the rotating-wave approximation (RWA)\cite{12,13,14}. As addressed Ref. \cite{13}, the absence of field $A^2$ term from the minimal coupling Hamiltonian in the approximation of the DM leads to the possibility of the instability. In the presence of $A^2$ term, the classical thermodynamic properties have been studied previously \cite{16,17,18}. Whether the instabilities disappear when the interaction of the bosonic field $A^2$ term is taken into account is a long-standing issue and remains very controversial to date \cite{12,13,10}.

It is known that quantum critical phenomena exhibits deep relations to the Berry phase (BP) \cite{10,20,21,22} and the fidelity \cite{23,24,25,26,27,28,29,30}. The BP has been extensively studied by the geometric time evolution of a quantum system, providing means to detect the quantum effects and critical behavior, such as quantum jumps and collapse \cite{31,32,33}. A recent proposal is to use the fidelity in identifying the quantum phase transition\cite{22,23}. As a consequence of the dramatic changes in the structure of the ground states, the fidelity should drop at critical points. In our opinion, the quantum-phase-like transitions might also be studied in terms of the BP and the GS fidelity.

In this paper, we calculate the ground state BP and fidelity in the RWA DM with and without $A^2$ term to quantify phenomena of the quantum-phase-like transitions in finite system. Without $A^2$, a exact solution to the DM is given explicitly. We solve the RWA DM with an additional $A^2$ term by a exact diagonalization in the Fock space of the bosonic operators. The paper is organized as follows. In Sec.II, we review the RWA DM and the Hamiltonian with the $A^2$ term to obtain exact solutions respectively. In Sec.III, we study the instability of the RWA DM by measuring the BP and the ground state fidelity. The behaviors of these two quantities as a function of the interaction strength of the field for the RWA DM with $A^2$ term are also evaluated. Finally, we present the conclusion in Sec.IV.

II. MODEL

A. Exact solution to the RWA DM

Let us consider DM of $N$ two-level atoms with energy level $\omega_0$, interacting with a single-mode bosonic field with the frequency $\omega$. In the RWA DM, ignoring the counter-rotating term, the corresponding Hamiltonian is given by

$$H = \omega a^\dagger a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}(a^\dagger J_- + a J_+).$$

where $a^\dagger$ and $a$ are the photonic creation and annihilation operators, $J_k (k = z, \pm)$ denotes the collective spin-1/2 atomic operators, $\lambda$ is the atom-field coupling strength, and $\hbar$ is set unity.

Motivated by the exact technique of the Jaynes-Cummings model with RWA, we present a detailed numerical diagonalization procedures to solve a set of closed equations to the DM with RWA, which was also briefly discussed in Ref. \cite{12}. Since the Hamiltonian (1)
commutes with the total excitation number operator \( \hat{L} = a^\dagger a + J_z + \frac{1}{2} \), the subspace of the Hilbert space consists of a sum of subspaces labeled by different number of excitations \( L \). The Hilbert space of the collective algebra is spanned by the kets \( \{ j, m \}; m = \pm j, \pm j + 1, \cdots, j \). By adapting Schwinger’s representation of spin in terms of harmonic oscillators \([34, 35]\), \( |j, m \rangle \) can be expressed as \( |j, N - n \rangle \), which is a Dike state of \( N - n \) spin-up atoms and \( n \) spin-down atoms, \( n = 0, 1, \ldots, N \). In this work, \( j \) takes its maximal value \( N/2 \). \( |N/2, N - n \rangle \) is also known as the eigenstates of \( J_z \) and \( J^2 \) with \( J_z |\frac{N}{2}, N - n \rangle = (\frac{N}{2} - n) |\frac{N}{2}, N - n \rangle \). The action of the corresponding raising and lowering operators on this state gives

\[
J_+ |\frac{N}{2}, N - n \rangle = \sqrt{(N - n + 1)n} |\frac{N}{2}, N - n + 1 \rangle
\]

\[
J_- |\frac{N}{2}, N - n \rangle = \sqrt{(N - n)(n + 1)} |\frac{N}{2}, N - n - 1 \rangle
\]

In the subspace of \( L = N + k \) excitations the wave function is supposed as

\[
|\psi \rangle = \sum_{n=0}^{N} c_n |n + k \rangle f \otimes |N/2, N - n \rangle_a
\]

where \( c_n \)'s are coefficients, \( |n + k \rangle f \) is a Fock state of the bosonic field with an alterative number \( k \), ranging from \( -N \) to infinity. \( k \) is equal to \( -N \) in the weak coupling regions corresponding to 0 excitations and then increases with the coupling parameter \( \lambda \). When the number of excitations \( L \) is larger than the number of atoms \( N \), i.e. \( k > 0 \), the ground state of \( H \) lies in the subspace spanned by \( N + 1 \) vectors. In this way, the Hamiltonian is expressed by a tridiagonal \((N + k + 1) \times (N + k + 1)\) matrix. From Eq.\((2)\) we obtain the exact expression of the \( m \)-th row of the Schrödinger equation

\[
Ec_m = \frac{\lambda}{\sqrt{N}} \sqrt{m + k} \sqrt{(N - m + 1)mc_{m-1}}
\]

\[
+ \frac{\lambda}{\sqrt{N}} \sqrt{m + k + 1} \sqrt{(N - m)(m + 1)c_{m+1}}
\]

\[
+ [\omega(m + k) + \omega_0(N/2 - m)]c_m
\]

where

\[
m = \begin{cases} 
0,1,\ldots,N+k : & k \leq 0 \\
0,1,\ldots,N : & k > 0
\end{cases}
\]

Note that the above equations are closed and the set of linear equations for \( c \)'s takes a tridiagonal form. Solutions for a given \( k \) are readily obtained through Gaussian elimination and back substitution. Finally the chosen \( k \) corresponds to the lowest energy among eigenvalues of the solutions for a fixed coupling strength \( \lambda \). It is interesting to find that the excitation number \( L \) is added step by step and keeps a constant in a coupling parameter interval \([\lambda_l, \lambda_l]\), where \( \lambda_l(\lambda_l) \) is a quantum-phase-like transition point, as shown in Fig. 1. The first transition point is denoted as \( \lambda_0 \). The sensitive quantities like the ground state BP and the fidelity will be calculated to quantify the discontinuities, so called instability, in the finite DM with RWA.

B. Numerical exact diagonalization to the RWA DM with \( A^2 \) term

It is interesting to discuss the effect of the interacting bosonic field in the atom-field system. As the interactions vector potential \( A \), caused by the longitudinal part of the bosonic field, are taken into account, the Hamiltonian in the RWA DM can be evaluated with an additional term \( A^2 \) \([16, 17] \). The additional term \( A^2 \) has been discussed classically about thermodynamic properties by Rzazewski et al. \([17, 18]\). To extensively quantify the contributions of the \( A^2 \) term, we employ the quantum information tools such as the BP and the ground state fidelity to detect the quantum-phase-like transitions.

In terms of the bosonic operators, the \( A^2 \) term is given by \( \varepsilon(a^\dagger + a)^2 \), where \( \varepsilon \) is the interacting strength of the bosonic field. The overall Hamiltonian of the ensemble of two-level atoms interacting with the bosonic field is expressed as

\[
H_A = \omega a^\dagger a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}} (a^\dagger J_- + a J_+) + \varepsilon(a^\dagger + a)^2\]

(5)

In order to obtain the numerical exact solution, we perform a standard Bogoliubov transformation by introducing bosonic annihilation (creation) operator \( b(b^\dagger) \), such that \( b^\dagger = \mu a + \nu a^\dagger \) and \( \mu^2 - |\nu|^2 = 1 \). After substituting \( a, a^\dagger \) into Eq.\([3] \) the total Hamiltonian is diagonalized
as
\[
H_A = \sqrt{\omega^2 + 4\omega \varepsilon} b \dagger b + \omega_0 J_z + \frac{1}{2}(\sqrt{\omega^2 + 4\omega \varepsilon} - \omega)
+ \frac{\lambda}{\sqrt{N}}[\mu(b \dagger J_+ + b J_-) - \nu(b \dagger J_+ + b J_-)]
\]
where
\[
\mu^2 = \frac{1}{2}(\frac{\omega + 2\varepsilon}{\sqrt{\omega^2 + 4\omega \varepsilon}} + 1), \nu^2 = \frac{1}{2}(\frac{\omega + 2\varepsilon}{\sqrt{\omega^2 + 4\omega \varepsilon}} - 1)
\]
Note that a counter-rotating term is included in the modified Hamiltonian (6), which may plays an essential role in the following discussion. Because the \(A^2\) term breaks the gauge invariance of the Hamiltonian (1) in the DM with RWA, it was argued in Ref. [15] the instability would be then eliminated.

We now consider the wave functions of the total Hamiltonian with \(N\) atoms, which are of the form
\[
|\varphi\rangle_A = \sum_{n=0}^{N} \sum_{m=0}^{N_{tr}} d_{nm}|m\rangle_f |N/2,n\rangle_a
\]
where \(N_{tr}\) is the maximum photonic number in the artificially truncated Fock space, and \(d_{nm}\) are coefficients. \(|m\rangle_f\) is a Fock state with \(m\) photons, \(|N/2,n\rangle_a\) is a Dicke state in Schrödinger’s representation of spin with \(n\) atoms in excited state. The \(m\)-th row of the Schrödinger equation reads
\[
Ed_{nm} = [\omega \varepsilon m + \Delta(n - N/2) + \frac{1}{2}(\omega - \varepsilon)]d_{nm}
+ \frac{\lambda \mu}{\sqrt{N}}\sqrt{(m + 1)(N - n + 1)}d_{n-1,m+1}
+ \frac{\lambda \mu}{\sqrt{N}}\sqrt{m(n + 1)(N - n)}d_{n+1,m-1}
- \frac{\nu}{\sqrt{N}}\sqrt{(m + 1)(n + 1)(N - n)}d_{n+1,m+1}
- \frac{\nu}{\sqrt{N}}\sqrt{m(N - n + 1)}d_{n-1,m-1}
\]
The eigenvalues and eigenfunctions can be obtained numerically by diagonalizing a \((N + 1) \times (N_{tr} + 1)\) matrix. The BP and the fidelity can be calculated through these eigenfunctions.

To be complete, we also briefly review the contribution of the \(A^2\) term in the DM model without RWA, which yields some unimportant corrections. The Hamiltonian of the full DM with the \(A^2\) term reads
\[
H_{DM} = \omega a \dagger a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}(a \dagger + a)J_x + \varepsilon(a \dagger + a)^2
\]
With a rotation around an \(y\) axis by an angle \(\frac{\pi}{2}\) and the same Bogoliubov transformation, the modified Hamiltonian \(H_{DM}\) is rewritten as
\[
H_{DM} = \sqrt{\omega^2 + 4\omega \varepsilon} b \dagger b - \omega_0 J_x + \frac{1}{2}(\sqrt{\omega^2 + 4\omega \varepsilon} - \omega)
+ \frac{2\lambda}{\sqrt{N}}(\mu - \nu)(b \dagger + b)J_z
\]

III. GROUND STATE PROPERTY

A. Instability in the RWA DM

Berry’s adiabatic geometric phase describes a phase factor of the wave functions in a time-dependent quantum system. The interesting paths of evolution for generating a BP are those for which the ground state of the system can evolve around a region of criticality. We first measure a nontrivial BP circulating a region of "criticality" corresponding to a abrupt change. The BP \(\gamma_1\) generated after the system undergoing the time-dependent unitary transformation \(U(T) = \exp[-i\phi(t)J_z]\), varying the angle \(\phi(t)\) adiabatically from 0 to \(2\pi\), can be evaluated as a function of the atom-field coupling parameter \(\lambda\)
\[
\gamma_1 = i \int_0^{2\pi} \langle \psi_0 | U^\dagger(t) \frac{d}{d\phi} U(t) | \psi_0 \rangle d\phi = 2\pi \langle \psi_0 | J_z | \psi_0 \rangle
\]
where \(|\psi_0\rangle\) is the ground-state wave function of Hamiltonian (11) of the RWA DM. As shown in Fig. 2, the first phase-transition-like occurs at the "critical" value of the coupling parameter \(\lambda_0 = 1\) for arbitrary atom number \(N\), which recover the result in the thermodynamical limit [20]. The average BP \(\gamma_1/N\) is equal to \(\pi\) at \(\lambda < 1\) and increases abruptly at discontinuous "critical" points when \(\lambda > 1\). Note that the plateau is formed clearly for \(N = 1, 2, 4\), , the width of the plateau becomes narrower and narrower with the increasing \(N\), which are quite different from the phenomenon of the quantum phase transition in the full DM [7, 8, 9]. A clear picture of the
instability in the ground state of the RWA DM is given in terms of the BP with $N = 64$ atoms shown in the inset of Fig. 2. We now have a general expression for the BP related to the photonic number, which is driven by fields. We plot behaviors of $\gamma_2/N$ in units of $2\pi$ as a function of the atom-field coupling parameter $\lambda$ for different number of atoms $N$ in Fig. 3. The BP $\gamma_2/N$ is 0 in the weak coupling region for $\lambda \leq 1$ and then increases discontinuously as $\lambda$ increases. As shown in Fig. 3 when $N$ increases the interval of the "critical" values $\lambda$ become smaller, leading to the curve with more steps. The inset of Fig. 3 shows that there actually exist many phase-transition-like "critical" points beyond $\lambda = 1$ for large $N = 64$.

An interesting interest has been drawn in the role of the ground state fidelity in detecting the quantum phase transitions for various many-body system, with a narrow drop at the transition point. Below we propose to use this quantum tool to identify the quantum-phase-like transitions, where the GS fidelity drops to 0 in the RWA DM. It is defined as the overlap between two ground states $|\psi_0(\lambda)\rangle$ and $|\psi_0(\lambda + \delta\lambda)\rangle$ [20, 21], where $\delta\lambda$ is a tiny perturbation parameter, i.e.

$$F(\lambda, \delta\lambda) = |\langle \psi_0(\lambda) | \psi_0(\lambda + \delta\lambda) \rangle|$$

Note that $F$ is a function of both $\lambda$ and $\delta\lambda$. Based on the normalized and orthogonalized wave function in Eq. (2) for the RWA DM, the ground state fidelity can be derived analytically $|\sum_{n,m=0}^{N} c_n(\lambda) c_m(\lambda + \delta\lambda) \delta_{n,m} \delta_{n+k,m+k'}|$, and then can be simplified as

$$F(\lambda, \delta\lambda) = \begin{cases} 0 & : \ n = m, k \neq k' \\ 1 & : \ n = m, k = k' \end{cases}$$

At each transition point in RWA DM, the alternative number $k$ is changed abruptly and $F$ is then equal to 0. Beyond "critical" points, $F$ should be constant 1. The energy gap $\Delta$ between the first excited and ground state in the RWA DM as a function of the coupling parameter of $\lambda/\lambda^0$ for different number of atoms $N = 1, 2, 4, \infty$.

FIG. 3: The average Berry phase $\gamma_2/N$, i.e. the average phonon number, in units of $2\pi$ of the RWA DM as a function of the coupling parameter $\lambda$ for different number of atoms $N$. The inset gives a discontinuous picture of instability for 64 atoms.

FIG. 4: Ground state fidelity $F$ and energy gap $\Delta$ between the energies of the first excited and ground state in the RWA DM as a function of the coupling parameter of $\lambda/\lambda^0$ for different number of atoms $N = 1, 2, 4, \infty$. 

B. Behaviors of the Berry phase and fidelity in the RWA DM with $A^2$ term

We next turn to study the RWA DM with $A^2$ term by using the above quantum information tools. By manse of the general BP formula Eqs. (11) and (12), the ground state BP of the RWA DM with $A^2$ term can be evaluated as a function of the atom-field coupling $\lambda$ and the
interaction strength of the field $\varepsilon$, i.e.,

$$
\gamma_{1A} = 2\pi \langle \varphi_A | J_z | \varphi_A \rangle,
$$

$$
\gamma_{2A} = 2\pi \langle \varphi_A | b^\dagger b | \varphi_A \rangle
$$

where $| \varphi_A \rangle$ is the ground state of the Hamiltonian $\mathcal{H}$.

We plot the average BP $\gamma_{1A}/N$ in Fig. 5(a) and $\gamma_{2A}/N$ in units of $2\pi$ in Fig. 5(b) as a function of the coupling $\lambda$ for different interaction strengths $\varepsilon = 0, 0.5, 1$ for $N = 4$ atoms. We can observe that, for both $\gamma_{1A}/N$ and $\gamma_{2A}/N$, the abrupt jump occurs at the same coupling parameter $\lambda$ for the same interaction strengths $\varepsilon$. We denote the coupling parameter $\lambda$ where the first abrupt jump occurs as a characteristic parameter $\lambda_c$. It is interesting to observe that the characteristic $\lambda_c$ increases with the interaction strength $\varepsilon$. The stability claimed in Ref. [15] is only found in the smooth curves for $\lambda \leq \lambda_c$. Thus, there still exist quantum-phase-like transitions when the interaction of the bosonic field $\varepsilon$ is strong.

According to the ground state wave function $| \varphi_A \rangle$, the fidelity of the RWA DM with $A^2$ term can be also calculated $F_A(\lambda, \delta \lambda) = | \langle \varphi_A(\lambda) | \varphi_A(\lambda + \delta \lambda) \rangle |$. The numerical results for the different $\varepsilon$ with $N = 4$ atoms are exhibited in Fig. 6(c). The singularities at the "critical" points for $\varepsilon = 0, 0.5, 1$ are demonstrated by a sudden drop of $F_A$. It is clearly shown that the characteristic $\lambda_c$ moves towards the right regime with the increasing $\varepsilon$, providing the evidence of the quantum-phase-like transitions even for a strong interaction of the field.

To show the instabilities more obviously, for a more wide range of interacting strengths $\varepsilon = 0, 10^{-3}, 10^{-2}, 10^{-1}, 1$, we plot the ground state fidelity $F_A$, Berry phase $\gamma_{1A}/N$ and its first derivative $\partial \gamma_{1A}/\partial \lambda$ as a function of the scaled coupling parameter $\lambda/\lambda_c$ in Fig. 6(b). When the interaction strength of the field increases the fidelity $F$ still drops to 0 at the characteristic $\lambda_c$, where the first derivative $N^{-1} \partial \gamma_{1A}/\partial \lambda$ also changes abruptly. This is another piece of evidence that the instability of the RWA DM does not vanish in the RWA DM including the $A^2$ term.

It is illustrated that the contribution of the $A^2$ term does not eliminates the instability of the RWA DM, contrary to the previous studies by Rzążewski et al. [15]. For strong interaction strength of the bosonic field $\varepsilon$, a characteristic parameter $\lambda_c$ becomes larger than $\lambda_c^0 = 1$ in the absence of $A^2$ term. A sequence of the ground state stability reported previously only appears for $\lambda \leq \lambda_c$. 

FIG. 5: The average Berry phase (a) $\gamma_{1A}/N$, (b) $\gamma_{2A}/N$ in units of $2\pi$, as well as (c) ground state fidelity $F_A$ of the RWA DM with $A^2$ term versus $\lambda$ for different interacting strengths of the field $\varepsilon = 0, 0.5, 1$ for $N = 4$.

FIG. 6: Ground state fidelity $F_A$ (a), Berry phase $\gamma_{1A}/N$ (b) and its first derivative $\partial \gamma_{1A}/\partial \lambda$ (c) of the RWA DM with $A^2$ term as a function of the coupling of $\lambda/\lambda_c$ for different interacting strengths of the field $\varepsilon = 0, 10^{-3}, 10^{-2}, 10^{-1}, 1$. 

IV. CONCLUSIONS

In summary, we have investigated the instability of the RWA DM by quantum information tools such as the BP and the ground state fidelity. An obvious discontinuous behaviors of these quantities with finite $N$ atoms are observed. It is demonstrated that the quantum-phase-like transitions occur beyond the characteristic $\lambda_c$ for strong interaction of fields. We propose that the instability would not be eliminated by involving the $A^2$ term of the DM with RWA. Previous observed instability may be limited to the coupling regime $\lambda \leq \lambda_c$. It should be pointed out that the quantum information tools are very sensitive quantities to detect quantum phase (or like) transition.

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