Destruction of a metastable string by particle collisions

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Abstract

We calculate the probability of destruction of a metastable string by collisions of the Goldstone bosons, corresponding to the transverse waves on the string. We find a general formula that allows to determine the probability of the string breakup by a collision of arbitrary number of the bosons. We find that the destruction of a metastable string takes place only in collisions of even number of the bosons, and we explicitly calculate the energy dependence of such process in a two-particle collision for an arbitrary relation between the energy and the largest infrared scale in the problem, the length of the critical gap in the string.
1 Introduction

Field configurations with the properties of a metastable relativistic string arise in a number of models [1, 2, 3]. Such string can be viewed as having two phases, 1 and 2, with their respective tensions $\varepsilon_1$ and $\varepsilon_2$, and the larger tension $\varepsilon_1$ corresponding to the ‘upper’ metastable phase 1. The transition to the lower phase with $\varepsilon_2 < \varepsilon_1$ is inhibited by the energy (mass) $m$ associated with the interface between the two phases. The transition from the upper to the lower phase proceeds, similarly to the decay of false vacuum [4, 5] through formation in the initial string 1 of a gap spanned by the phase 2 and subsequent expansion of the gap occupied by the lower phase. Clearly, the classical expansion of such gap is possible only starting from a finite critical length of the gap $\ell_c = 2m/(\varepsilon_1 - \varepsilon_2)$ at which length the energy loss due to the mass of the two ends of the gap is compensated by the gain due to the phase energy difference $(\varepsilon_1 - \varepsilon_2) \ell_c$. A situation, where the string completely breaks up, such as e.g. a breakup of a QCD string with formation of a heavy quark-antiquark pair at the ends, can be viewed as the $\varepsilon_2 \to 0$ limit of a two-phase string.

The nucleation of the critical gap is a tunneling process, which can be analyzed in terms of the low energy dynamics of such string which dynamics can be described by the effective Nambu-Goto action

$$S = \varepsilon_1 A_1 + \varepsilon_2 A_2 + m P,$$

where $A_1$ ($A_2$) is the space-time area of the world sheet occupied by the phase 1 (resp. 2), and $P$ is the space-time length of the interface between the two phases. The action (1) applies at the length scales much larger than any scale for the internal structure of the string or of the interface, which typically is the thickness of the string. Using this effective action the calculation of the probability of the decay of the metastable phase is quite similar to the treatment of false vacuum decay in the so-called thin wall approximation [4, 5], and its exponential power has been calculated [1, 2, 3] as well as the pre-exponential factor [6]. The rate $\gamma_0$ of spontaneous nucleation of critical gaps per unit length per unit time in a string living in $d$ space-time dimensions is given by

$$\gamma_0 = F^{d-2} \frac{\varepsilon_1 - \varepsilon_2}{2\pi} \exp \left( -\frac{m^2}{\varepsilon_1 - \varepsilon_2} \right),$$

where $F$ is a dimensionless function of the ratio $\varepsilon_2/\varepsilon_1$, smoothly varying from $F = e/(2\sqrt{\pi}) = 0.7668\ldots$ at $\varepsilon_2 = 0$ to $F \to 1$ at $\varepsilon_2 \to \varepsilon_1$. 

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The excitations of the string, described by the action (1) are massless Goldstone bosons corresponding to transverse waves on the string. One can reasonably expect on general grounds that the presence of such waves in an excited state of the metastable phase catalyzes its decay. This catalytic effect of the Goldstone excitations gives rise to the recently calculated [7] thermal enhancement of the decay rate. It can be noted that the massless Goldstone modes are present in a thermal state at arbitrary low temperature $T$, so that the thermal effect expands in powers of $T$. In particular the leading at low $T$ behavior of the catalysis factor $K$ in the thermal decay rate $\gamma_T = K \gamma_0$ is given by

$$K = 1 + (d - 2) \frac{\pi^8}{450} \left( \frac{\varepsilon_1 - \varepsilon_2}{3\varepsilon_1 - \varepsilon_2} \right)^2 \left( \frac{\ell_c T}{2} \right)^8 + O \left[ (\ell_c T)^{12} \right],$$

and the expansion parameter is $\ell_c T$, i.e. the ratio of the size of the critical gap to the typical thermal wavelength\(^1\).

In this paper we consider the destruction of the metastable phase of the string by collisions of the Goldstone bosons. We find that this problem in fact can be solved using appropriate interpretation and extension of the thermal treatment. Indeed, the enhancement of the decay rate, described by the factor $K$, is nothing else than the contribution of the production of the critical gap in collisions of the bosons that are present in the thermal bath. In particular, we find that the first temperature dependent term in the expansion (3) is entirely due to the process of the critical gap creation in collision of two particles in the limit, where their center of mass energy $E = \sqrt{s}$ is much smaller than $1/\ell_c$. We further consider a modification of the approach of Ref.[7] by formally introducing a negative chemical potential for the bosons, and considering the decay of such a thermal state. This allows to separate the contribution of $n$-boson collisions with different $n$ in all terms of the expansion in the temperature, and thus to find the dependence of the critical gap production in those collisions for an arbitrary relation between $\ell_c$ and the energy of the bosons. One of the results of our consideration is that the string destruction takes place only in collisions of even number of bosons and is absent at odd $n$. By performing a full calculation we find an explicit expression for the (dimensionless) probability $W_2$ of the critical gap creation in two-boson collisions at arbitrary values of the parameter $E\ell_c$. Our final result for this process has especially simple form for

\(^1\)The full expression [7] in fact diverges at $\ell_c T = 1$, beyond which point the treatment in Ref.[7] is not applicable.
the decay of the string into ‘nothing’ i.e. at $\varepsilon_2 = 0$:

$$W_2 = 2\pi^2 \ell_c^2 \gamma_0 \left[ I_3 \left( \frac{E\ell_c}{2} \right) \right]^2,$$

(4)

with $I_3(x)$ being the standard modified Bessel function of the third order.

Although in the end we make no assumption about the value of $(E\ell_c)$, our treatment in the present paper is limited by the condition

$$E^2 \ll \varepsilon_1,$$

(5)

so that the exponential growth of the energy dependent factor in Eq.(4) (and in subsequent more general expressions) does not overcome the exponential suppression in the factor $\gamma_0$, described by Eq.(2).

Unlike the thermal factor $K$, which is singular [7] at $\ell_c T = 1$, the two-boson production rate does not exhibit any singularity at any value of the parameter $E\ell_c$, and we find similar smooth energy behavior for $n$-boson processes as well. We thus conclude that the ‘explosion’ of the thermal rate at $\ell_c T = 1$ is a result of infinite number of processes becoming important at this point, rather than due to a finite set of processes with a limited range of $n$ developing large probability at the energy per particle of the order of $1/\ell_c$.

The rest of the material in this paper is organized as follows. In the Section 2 we present a brief account of the Euclidean space-time calculation of the string transition, and describe some general properties of the processes of string destruction by particle collisions following from their relation to the absorption in the forward scattering amplitudes. In the Section 3 we discuss the interpretation of the thermal catalysis of the string transition as induced by collisions of particles in the thermal state, and we derive from the first term of the low-temperature expansion in Eq.(3) the low-energy limit of the two-particle string destruction probability $W_2$. The Section 4 describes the extension of the finite temperature analysis by introduction of a negative chemical potential, which allows to identify the contribution to the catalysis factor of the string destruction by individual $n$-particle collisions. The derived general expression is then used in the Section 5 to find the two-particle probability $W_2$ in all orders in the parameter $(E\ell_c)$. Finally, the Section 6 contains a discussion and summary of the results of the present paper.
2 Euclidean formulation and general considerations for particle collisions

In complete analogy with the false vacuum decay [5, 8, 9], the calculation of the string decay rate [6, 7] can be performed entirely in terms of Euclidean path integration. Considering the string in the metastable phase 1 of large length $X$ extended along the $x$ axis with the ends fixed at $x = \pm X/2$ and the transverse coordinates $z_i (i = 1, \ldots, d - 2)$ fixed at zero at the ends of the string, $z_i(\pm X/2) = 0$, one needs to calculate the Euclidean path integral $Z_b$ over the string variables with the action (1) around the bounce configuration, i.e. a configuration involving a patch of the phase 2. The decay probability in the full space-time area of the world sheet is then given by the expression

$$W_0 = 2 \frac{\text{Im} Z_b}{Z_0}$$

(6)

with $Z_0$ being the path integral calculated around the ‘flat’ configuration of the string 1, i.e. over the configuration without the patch of the lower phase.

For the spontaneous decay and the thermally enhanced decay (at $T \ell_c < 1$) the patch has the shape of a disk with the radius

$$R = \frac{\ell_c}{2} = \frac{m}{\varepsilon_1 - \varepsilon_2}.$$  

(7)

The action on the bounce, relative to the unperturbed string, is equal to $\pi m^2/(\varepsilon_1 - \varepsilon_2)$ and determines the exponential power for the spontaneous decay rate in Eq.(2) and also for all the processes discussed in the present paper.

The only difference between the calculation of the decay at finite temperature $T$ and at $T = 0$ arises at the level of calculating the pre-exponential factor due to the functional determinant of the quadratic part of the action and amounts to the standard treatment of the boundary conditions in (Euclidean) time for the fluctuations: zero boundary conditions at large time for considering zero temperature and periodic boundary conditions with period $\beta = 1/T$ at finite temperature.

The formula (6) is in fact the unitarity relation [9] between the decay rate and the imaginary part of the transition amplitude from the false vacuum to the false vacuum $\langle \text{vacuum 1}_{\text{out}} | \text{vacuum 1}_{\text{in}} \rangle$. Similarly, one can treat the probability of the string breakup by an excited state in terms of the bounce contribution to the imaginary part of its forward scattering amplitude. Proceeding to discussion of the string decay induced by
the Goldstone bosons, we readily notice that a state with just one Goldstone boson cannot induce the destruction of the string. Indeed, the total probability of such induced process is Lorentz invariant and thus can depend only on the (Lorentz) square of the particle momentum $k^2$. The Goldstone bosons are massless, so that for them $k^2 = 0$ and is fixed. Thus if a single massless boson produced an effect on the decay, this effect would have no dependence on the energy $\omega$ of the boson. In the limit $\omega \to 0$ the Goldstone boson is indistinguishable from the vacuum. (In other words, the limit $\omega \to 0$ corresponds to an overall shift of the string in transverse direction.) Thus the decay rate of a single-boson state is the same as that of the vacuum, and the presence of a single boson with any energy produces no effect.

The simplest excited state, contributing to the string destruction, is that with two particles. The probability $W_2$ of creation of the critical gap in a collision of two particles with two-momenta $k_1$ and $k_2$ can depend only on the Lorentz invariant $s = (k_1 + k_2)^2$. Obviously, for two particles colliding on a string one has $s = 4\omega_1\omega_2$ (and $s = 0$ for two particles moving in the same direction, i.e., non-colliding). Using the unitarity relation this probability can be found in terms of the imaginary part of the forward scattering amplitude $A(k_1, k_2; k_1, k_2)$:

$$W_2 = C \frac{\text{Im} A(k_1, k_2; k_1, k_2)}{\omega_1\omega_2}$$

where the factor $\omega_1\omega_2$ is the usual flux factor, and the constant $C$ does not depend on either of the energies, and is determined by specific convention about the normalization of the amplitude.

The dynamics of the Goldstone bosons on the string, including their scattering, can be considered in terms of the transverse shift $z_i(x)$ treated as a two-dimensional field and described by the Nambu-Goto action (1) as

$$S = \varepsilon_1 \int_{A_1} \sqrt{1 + (\partial_\mu z_i)^2} \, d^2x + \varepsilon_2 \int_{A_2} \sqrt{1 + (\partial_\mu z_i)^2} \, d^2x + m \int_P \sqrt{1 + (\partial z_i / \partial l)^2} \, dl,$$

where $dl$ is the element of the length of the interface $P$ between the phases.

Clearly at low energy of the Goldstone bosons one can make use of the expansion in Eq.(9) in powers of $(\partial z)$ which generates the expansion of the scattering amplitudes in the momenta of the particles with each one entering the amplitude with (at least) one power of its energy $\omega$, as is mandatory for Goldstone bosons. For the scattering in the metastable state this generates expansion in powers of $\omega / \sqrt{\varepsilon_1}$, so that in the zeroth order in this ratio, It should be emphasized that the considered probability is inclusive in the sense that we make no assumption about any additional to the critical gap products of the reaction.
that we are discussing here, it is sufficient to retain only the quadratic in \(\partial z\) terms in the action (9). It should be noted that in spite of retaining only the quadratic terms, the multi-boson scattering amplitudes do not vanish, since the necessary nonlinearity arises from the bounce configuration. In other words, the bosons scatter ‘through the bounce’. The energy expansion for these amplitudes is determined by the bounce scale \(\ell_c\), so that the parameter of such expansion is \((\omega \ell_c)\), and we do not assume this parameter to be small. Clearly, the condition for applicability of the present approach where the terms of order \(\omega/\sqrt{\varepsilon_1}\) are dropped, while those with the parameter \(\omega \ell_c\) are retained is that

\[
\frac{m^2}{\varepsilon_1 - \varepsilon_2} \gg 1 - \frac{\varepsilon_2}{\varepsilon_1},
\]

which is always true if the semiclassical tunneling can be applied at all to the string decay.

We shall now show that in the on-shell scattering through the bounce each external leg enters with at least two powers of its energy. Let us start, for the simplicity of illustration, with the binary scattering. The general two \(\rightarrow\) two scattering amplitude \(A(k_1, k_2; k_3, k_4)\) can be related in the standard application of the reduction formula to the connected 4-point Green’s function \(\langle \text{vacuum} 1 | T \{z(x_1)z(x_2)z(x_3)z(x_4)\} | \text{vacuum} 1 \rangle\), which in turn is an analytical continuation of the Euclidean connected correlator

\[
\langle \text{vacuum} 1 | z(x_1)z(x_2)z(x_3)z(x_4) | \text{vacuum} 1 \rangle = Z_0^{-1} \frac{\delta^4 Z_b[j]}{\delta j(x_1) \delta j(x_2) \delta j(x_3) \delta j(x_4)} |_{j=0} .
\]

The latter expression for the correlator implicitly assumes the conventional procedure of introducing in the action the source term \(\int j(x) z(x) d^2x\) for the Goldstone variable \(z(x)\), and \(Z_b[j]\) is the path integral around the bounce in the presence of the source.

The low-energy limit of the on-shell scattering amplitude is determined by the correlator at widely separated points \(x_1, \ldots, x_4\). The weak \(\delta\)-function sources ‘prop’ the string in the transverse direction at those points as shown in Figure 1, and generally distort the bounce located between the sources. The correlator (11) is determined by the distortion of the bounce by all four sources, so that at large separation between the points the bounce is located far (in the scale of its size \(\ell_c\)) from the sources, where the overall distortion of the world sheet for the string is small and is slowly varying on the scale \(\ell_c\). One can therefore expand the background field \(z_s\) generated by the sources at the bounce location in the Taylor

\(^3\)We temporarily suppress the spatial index of the transverse shift variable \(z_i(x)\), which is equivalent to considering just one transverse dimension, i.e. \(d = 3\).
series around an arbitrarily chosen inside the bounce point $x = 0$. Clearly the first term of this expansion $z_s(0)$ corresponds to an overall transverse shift of the string and does not alter the bounce shape and the action. Moreover, the linear term in this expansion, proportional to the gradient $\partial z_s(0)$, does not change the bounce action either. Indeed this term corresponds to a linear incline of the string in the transverse coordinates, and can be eliminated by an overall rotation of the string in the $d$ dimensional space. In other words the absence of the linear in the gradient of $z_s$ term in the action is a direct consequence of the $d$-dimensional Lorentz invariance of the string. We thus arrive at the conclusion that the expansion for the distortion of the bounce starts from the second order in the derivatives of $z_s$ (the curvature of the background world sheet), which for a connected correlator implies that in the expansion in the energy each external source enters with at least two powers of the energy. This conclusion clearly applies to the on-shell amplitudes with arbitrary number of external legs, since the generalization to multiparticle scattering is straightforward.

In particular, the binary scattering amplitude $A(k_1, k_2; k_3, k_4)$ at low energy scales as the eighth power of the energy. It follows from Eq.(8) that the probability of the string destruction by collision of two particles is proportional to the sixth power of the energy scale, or equivalently, to $s^3$ at small $s$. Applying in the same manner the unitarity condition to the forward scattering amplitude of a general $n$-particle state, one readily concludes that the corresponding probability $W_n$ of the induced string breakup scales with the overall energy scale $\omega$ as $W_n \propto \omega^{3n}$.

Figure 1: Bounce configuration distorted by the sources
3 Thermal decay and the string destruction by particle collisions

The described procedure for calculating the bounce-induced scattering amplitudes in terms of the Euclidean correlators runs into the technical difficulty of calculating the bounce distortion in the background created by the sources. Furthermore, this procedure obviously involves a great redundancy, if the final purpose is a calculation of the probabilities \( W_n \), i.e. only the absorptive parts of the forward scattering on-shell amplitudes are the quantities of interest. We find that in fact one can directly derive the probabilities \( W_n \) by an appropriate interpretation of the more readily calculable thermal decay rate in terms of the collision-induced probabilities. Such an approach is technically more tractable due to the fact that in the thermal calculation the bounce is not distorted, i.e. it is still a flat disk, and only the boundary conditions for the modes of the fluctuations are modified. We first illustrate this approach by using the first nontrivial term of the low temperature expansion in Eq.(3) for a calculation of the low energy limit of the binary probability \( W_2 \).

Indeed, the temperature dependent factors in the catalysis factor \( K \) arise from the critical gap nucleation in boson collisions weighed with the thermal number density distribution for the massless Goldstone bosons

\[
d n(k) = \frac{1}{e^{\omega/T} - 1} \frac{d k}{2\pi},
\]

where \( k \) is the spatial momentum, \( |k| = \omega \), running from \(-\infty\) to \(+\infty\). Given the low energy behavior \( W_n \propto \omega^{3n} \) found in the previous section, one readily concludes that the contribution of \( n \)-particle string destruction starts with the term \( T^{4n} \) in the low \( T \) expansion for \( K \). Thus the first term written in Eq.(3) can arise only from \( n = 2 \), i.e. from the production of the critical gap in binary collisions. Writing the expansion in \( s \) for the probability \( W_2 \) as \( W_2 = c_3 s^3 + \ldots \), one determines the coefficient \( c_3 \) of the \( s^3 \) term by comparing the result in Eq.(3) with the one calculated in terms of the two-particle collision rate using the number density distribution (12):

\[
(d - 2) c_3 \int_0^\infty \frac{d\omega_1}{e^{\omega_1/T} - 1} \int_0^\infty \frac{d\omega_2}{e^{\omega_2/T} - 1} \frac{s^3}{4\pi^2} = (d - 2) c_3 \frac{16\pi^6}{225} T^8,
\]
The factor \((d-2)\) counting the number of the transverse dimensions, corresponds to the summation over the polarizations of the Goldstone bosons. The expression for the coefficient \(c_3\) following from Eq.(13) thus determines the first term in the expansion for \(W_2\)

\[
W_2 = \gamma_0 R^2 \left[ \frac{\pi^2}{32} \left( \frac{\varepsilon_1 - \varepsilon_2}{3\varepsilon_1 - \varepsilon_2} \right)^2 R^6 s^3 + \ldots \right].
\]

(14)

(One can notice that at \(\varepsilon_2 = 0\) the \(s^3\) term coincides with the first term of expansion of the expression in Eq.(4).)

4 Thermal bath with a chemical potential

The discussed procedure for extracting the coefficients of the energy expansion for the probability of collision-induced string decay is obviously limited to only the first term in \(W_2\). In the higher terms in the temperature expansion for \(K\) the contribution of the energy expansion for \(W_n\) with different \(n\) generally gets entangled. This happens because the temperature is the only parameter and terms originating from different \(n\) can contribute in the same power in \(T\). In order to disentangle the terms of higher order in the energy in \(W_n\) with low \(n\) from similar terms originating from higher \(n\) we introduce a negative chemical potential \(\mu\) for the Goldstone bosons. Generally, such procedure would no be possible, since the number of these bosons is not conserved. However in our application this procedure is fully legitimate. Indeed, the thermal state of the string that we study here is that of collisionless bosons, in which their number is conserved. The string decay, resulting in a change in this number, is a very weak process that we consider only in the first order, which justifies averaging the rate of this process over the unperturbed state with conserved number of particles. At negative \(\mu\) the number density distribution of the bosons (12) is replaced by

\[
d n(k) = \frac{1}{e^{\frac{\omega(k)}{T}} - 1} \frac{d k}{2\pi},
\]

(15)

and by tuning the parameter \(|\mu|/T\) one can readily resolve between the contribution of \(n\)-particle processes with different \(n\).

The introduction of the chemical potential requires us to modify our previous thermal calculation[7]. Fortunately, this modification is quite straightforward, and we describe it here by first briefly recapitulating the relevant part of that previous work, where further details can be readily found.
The bounce configuration is a disk of the radius $R$, and we place the origin of the string world sheet coordinates in the Euclidean space-time $(x, t)$ at the center of the bounce. A calculation of the pre-exponential factor in the path integral $Z_b$ requires finding the functional determinant of the second variation of the action over the fluctuations of the Goldstone field $z_i(x)$. The functional determinant for the fluctuations in each transverse direction factorizes, so that it is sufficient to consider only one Goldstone polarization $z(x)$. The eigenmodes of the second variation of the action are harmonic functions of $(x, t)$, and they can thus be considered as the real and imaginary parts of holomorphic functions of the complex variable $w = t + i x$. For the problem at zero temperature the harmonics inside the bounce (the ‘inner’ ones) are generated in this way by the set of functions

$$Z_{\text{in}} = \frac{w^k}{R^k}$$

with integer $k$ starting from $k = 0$. while those regular at $|w| \to \infty$ (the ‘outer’ ones) are given by the set of functions $R^k/w^k$.

At finite temperature $T$ the path integral runs over the configurations periodic in (the Euclidean) time $t$ with the period $\beta = 1/T$. At $T < 1/\ell_c$ the bounce fits within one period and the inner part of the fluctuations is not affected by the thermal boundary conditions, while the outer part can be taken as the set of functions corresponding to periodically summed outer harmonics. For $k \geq 2^4$ these periodic harmonics can be written as

$$g_k(w) = \frac{R^k}{w^k} + \sum_{n=1}^{\infty} \left[ \frac{R^k}{(w - n\beta)^k} + \frac{R^k}{(w + n\beta)^k} \right].$$

The functions $g_k$ are expanded in powers of $w$ as

$$g_k(w) = \frac{R^k}{w^k} + \sum_{p=1}^{\infty} d_{pk} \left( \frac{w}{R} \right)^p,$$

with

$$d_{pk} = \left[ (-1)^k + (-1)^p \right] \left( \frac{R}{\beta} \right)^{k+p} \frac{(p + k - 1)!}{p!(k - 1)!} \zeta(p + k),$$

where we use the standard notation $\zeta(q) = \sum_{n=1}^{\infty} n^{-q}$ for the Riemann zeta function.

The matching of the outer eigenmodes with the inner ones given by Eq.(16) thus involves an additional (in comparison with the limit $T = 0$) term, described by the matrix $D$ with

The harmonics with $k = 0$ and $k = 1$ are found [7] to be inessential for the discussed thermal effects. This property can in fact be traced to the discussed in Sect.2 insensitivity of the bounce to a constant and linear shift of the background transverse coordinates $z$.  

\[\text{[Footnote]}\]
the elements $D_{pk} = d_{pk}$ for $p$ and $k$ starting from two. This is this additional contribution that produces the thermal effect in the string decay rate, and the catalysis factor $K$ can be written in terms of the matrix $D$ with the elements

$$D_{pk} = - \left[ (-1)^k + (-1)^p \right] (RT)^{k+p+2} \frac{p}{p+b} \frac{(p+k+1)!}{(p+1)! k!} \zeta(p+k+2),$$

where the indices $p$ and $k$ are shifted by one unit, so that their count starts at one: $p, k = 1, \ldots$, and the notation is used $b = (\varepsilon_1 + \varepsilon_2)/(\varepsilon_1 - \varepsilon_2)$. The final expression expression[7] for the thermal factor $K$ in $d$ space-time dimensions reads as

$$K = \text{Det} \left[ 1 - D^2 \right]^{-(d-2)/2}.$$

The modification of this calculation for a thermal state with a negative chemical potential, in which the number density of the bosons is given by Eq.(15), is achieved by introducing a ‘damping factor’ in the periodic sums for the outer function $s$ in Eq.(17):

$$g_k(w) \rightarrow g_k^{(\mu)}(w) = \frac{R^k}{w^k} + \sum_{n=1}^{\infty} \left[ \frac{R^k}{(w-n\beta)^k} + \frac{R^k}{(w+n\beta)^k} \right] \exp \left( -n |\mu| \beta \right).$$

One can readily see that the only net result of the so-introduced $\mu$ dependent factor in the calculation of Ref.[7] is a modification of the matrix coefficients $d_{pk}$ amounting to the replacement of the factors $\zeta(q)$ by the polylogarithm function,

$$\zeta(q) \rightarrow \text{Li}_q(e^{-|\mu|/T}).$$

In other words, the catalysis factor for a thermal state with a negative chemical potential is given by the expression

$$K(\mu, T) = \text{Det} \left[ 1 - D^2(\mu, T) \right]^{-(d-2)/2},$$

with the elements of the matrix $D(\mu, T)$ having the form

$$[D(\mu, T)]_{pk} = - \left[ (-1)^k + (-1)^p \right] (RT)^{k+p+2} \frac{p}{p+b} \frac{(p+k+1)!}{(p+1)! k!} \text{Li}_{p+k+2} \left( e^{-|\mu|/T} \right).$$

The dependence on two ‘tunable’ parameters $\mu$ and $T$ in Eq.(23) makes it possible to disentangle the contribution of processes with different number of particles from the energy behavior in each of these processes. Such a separation becomes straightforward if one notices
that each factor with the polylogarithm Li arises from the integration over the distribution (15):

\[ \int_0^\infty \frac{\omega^q}{e^{\omega q|\mu/T|} - 1} d\omega = q! T^{q+1} \text{Li}_{q+1} \left( e^{-|\mu|/T} \right). \] (25)

One therefore concludes that the number of the ‘Li factors’ in each term of the expansion of the catalysis factor in Eq.(23) directly gives the number of particles in the process, while the indices of these polylogarithmic factors give the power of the energy for each particle. Given that the matrix \( \mathcal{D}(\mu, T) \) is linear in the ‘Li factors’, one can count the number of particles contributing to each term of the expansion for \( K(\mu, T) \) by counting the power of \( \mathcal{D}(\mu, T) \). The latter counting is simplified if one rewrites the equation (23) in the equivalent form, suitable for the expansion in powers of \( \mathcal{D}(\mu, T) \):

\[ K(\mu, T) = \exp \left\{ -\frac{d-2}{2} \text{Tr} \ln \left[ 1 - \mathcal{D}(\mu, T)^2 \right] \right\} = \]

\[ \frac{d-2}{2} \text{Tr} \left[ \mathcal{D}(\mu, T)^2 \right] + \frac{d-2}{4} \text{Tr} \left[ \mathcal{D}(\mu, T)^4 \right] + \frac{(d-2)^2}{8} \left\{ \text{Tr} \left[ \mathcal{D}(\mu, T)^2 \right] \right\}^2 + O \left( \mathcal{D}^6 \right). \] (26)

The latter expression merits some observations. The first is that all the terms in the expansion in powers of \( \mathcal{D}(\mu, T) \) are positive, which is certainly the necessary condition for the consistency of our interpretation of these terms as corresponding to the probability of the destruction of the string by \( n \)-particle collisions. The second is that the string is destroyed only in collisions of \( \text{even} \) number of particles, since the expansion manifestly goes in the even powers of \( \mathcal{D}(\mu, T) \). Finally, the third observation is related to the dependence in Eq.(26) on the number of the transverse dimensions \((d-2)\). Namely, the quadratic in \( \mathcal{D}(\mu, T) \) term is proportional to \((d-2)\). This corresponds to that in two-particle collisions only the Goldstone bosons with the same transverse polarization do destroy the string. On the contrary, the quartic in \( \mathcal{D}(\mu, T) \) term has one contribution proportional to \((d-2)\) and one proportional to \((d-2)^2\). The linear in \((d-2)\) part corresponds to all the bosons in the collision having the same polarization, while the quadratic in \((d-2)\) part is necessarily contributed by the collisions, where the colliding bosons have different polarizations.
5 Destruction of the string in two-particle collisions at arbitrary \((sR)^2\)

The expression (26) for the catalysis factor \(K(\mu, T)\) together with the formulas (24) and (25) reduce the calculation of the probability of the string breakup by a collision of an arbitrary (even) number \(n\) of particles to straightforward, although not necessarily short, algebraic manipulations. In this section we consider in full the most physically transparent case of two-particle collisions. The probability in this case is found from the term in Eq.(26) with the trace \(\text{Tr}[\mathcal{D}(\mu, T)]\). Using Eq.(24), this trace can be written as a double sum:

\[
\text{Tr} \left[ \mathcal{D}(\mu, T)^2 \right] = 4 \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(p + b + 1)(k + b + 1)} \frac{[(p + k + 1)! (RT)^{p+k+2} \text{Li}_{p+k+2} \left( e^{-|\mu|/T} \right)]^2}{(p - 1)! (p + 1)! (k - 1)! (k + 1)!}
\]

One can readily recognize the expression in the straight braces here as the integral (25) with the power of the energy \(q\) given by \((p + k + 1)\), and thus identify the coefficient of the same power of \(s = 4\omega_1\omega_2\) in the expansion of the probability \(W_2(s)\) in \(s\). In this way we find the following formula for \(W_2(s)\) in terms of this expansion,

\[
W_2(s) = 8 \pi^2 \gamma_0 R^2 \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(p + b + 1)(k + b + 1)} \frac{(sR^2/4)^{p+k+1}}{(p - 1)! (p + 1)! (k - 1)! (k + 1)!} \Phi_b(\sqrt{s} R)^2,
\]

where the function \(\Phi_b(x)\) expands in a single series as

\[
\Phi_b(x) = \frac{x}{2} \sum_{p=1}^{\infty} \frac{1}{p + b + 1} \frac{x^{2p}}{(p - 1)! (p + 1)!}.
\]

It can be noted that the two-particle probability depends only on the odd powers of \(s\). This in fact is a consequence of the binary forward scattering amplitude being even in \(s\), as required by the Bose symmetry.

For integer values of the parameter \(b\), the function \(\Phi_b(x)\) has a simple expression in terms of the modified Bessel functions \(I_q(x)\) of the order \(q\) up to \(q = b + 2\). This expression is especially simple for \(b = 1\), i.e. for the case of the string decay into ‘nothing’: \(\Phi_1(x) = I_3(x)\), so that one arrives at the formula (4). We also write here, for an illustration, the corresponding expressions for the next two integer values of \(b\):

\[
\Phi_2(x) = \frac{1}{x} \left[ I_5(x) + 6I_4(x) \right]; \quad \Phi_3(x) = \frac{1}{x^2} \left[ (48 + x^2) I_5(x) + x I_6(x) \right].
\]
6 Discussion and summary

One can readily evaluate the sum in Eq.(29) at large $x$ by using the Stirling formula for the factorials, and find that independently of $b$ the function behaves as an exponent: $\Phi_b(x) \sim e^x$. This corresponds to an exponential behavior of the probability $W_2$ at large $(sR^2)$. In combination with the exponential factor in $\gamma_0$ this behavior can be written as

$$W_2(s) \sim \exp \left( -\pi \frac{m^2}{\varepsilon_1 - \varepsilon_2} + 2\sqrt{s} \frac{m}{\varepsilon_1 - \varepsilon_2} \right).$$  

(31)

Clearly, the energy growth of the probability never overcomes the large negative power due to the tunneling. Indeed, due to the condition $m^2 \gg \varepsilon_1$ for the semiclassical approximation to be valid at all and the condition $s \ll \varepsilon_1$ for the validity of our approach based on neglecting direct interactions between the Goldstone bosons, one always has $\sqrt{s} \ll m$.

It can be also noted that the power series for $\Phi_b(x)$ converges for any value of $x$, so that the probability $W_2(s)$ has no singularity at any $s$. The same conclusion also holds for the probabilities $W_n$ at any $n$. Indeed, each term of order $D^n$ with a specific power $n$ in the expansion (26) is described by an absolutely convergent series in the energy parameters. This implies that the reason for the singularity[7] at $T \ell_c = 1$ of the thermal factor $K$ in Eq.(21) is not that some finite set of collision processes gives a singular contribution, but rather that at this point an infinite multitude of processes becomes important.

In summary. We have considered the processes in which collisions of the Goldstone bosons on a metastable string induce its destruction by creating a critical gap of the stable phase with lower tension. It has been shown that at energy that is much smaller than the scale set by the tension $\varepsilon_1$ of the string the interaction of the Goldstone bosons in the bulk of the string can be neglected, and the discussed processes proceed ‘through the bounce’, which interaction mechanism has the scale set by the length $\ell_c$ of the critical gap. We have demonstrated that when the energy scale $\omega$ in a collision is much smaller than $1/\ell_c$ the probability $W_n$ of the string destruction in an $n$ particle collision is proportional to $(\omega \ell_c)^3n$. In particular the low-energy expression for $W_2$ is determined from our previous result for the string decay at finite temperature. In order to calculate the probability for an arbitrary relation between the energy scale and $\ell_c$ we have extended the calculation of the thermal decay rate by formally introducing a negative chemical potential for the bosons. The dependence of the resulting string decay rate on the chemical potential and the temperature allows to separate the contribution to this rate of individual $n$-particle collision processes. In
other words, the expression (26) provides a generating function for the probabilities $W_n$. It turns out that collisions of any odd number of the Goldstone bosons do not lead at all to the string destruction and this process occurs only in collisions of an even number of particles. As an application of the general formulas we have calculated explicitly the probability of destruction of the string in a two-particle collision as a function of their c.m. energy $\sqrt{s}$. The found expression exhibits an exponential growth at large values of the product $\sqrt{s} \ell_c$, however within the validity of the approximations used in the present treatment this growth never compensates the overall semiclassical exponential suppression of the string decay.

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