PROJECTIVE NORMALITY OF MODEL VARIETIES
AND RELATED RESULTS

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ABSTRACT. We prove that the multiplication of sections of globally generated line bundles on a model wonderful variety $M$ of simply connected type is always surjective. This follows by a general argument which works for every wonderful variety and reduces the study of the surjectivity for every couple of globally generated line bundles to a finite number of cases. As a consequence, the cone defined by a complete linear system over $M$ or over a closed $G$-stable subvariety of $M$ is normal. We apply these results to the study of the normality of the compactifications of model varieties in simple projective spaces and of the closures of the spherical nilpotent orbits. Then we focus on a particular case proving two specific conjectures of Adams, Huang and Vogan on an analog of the model orbit of the group of type $E_8$.

INTRODUCTION.

Let $G$ be a complex linear algebraic group, semisimple and simply connected. A $G$-variety $M$ is called wonderful of rank $n$ if it satisfies the following conditions:

- $M$ is smooth and projective;
- $M$ possesses an open orbit whose complement is a union of $n$ smooth prime divisors (the boundary divisors) with non-empty transversal intersections;
- any orbit closure in $M$ equals the intersection of the prime divisors which contain it.

Examples of wonderful varieties are the flag varieties, which are the wonderful varieties of rank zero, and the complete symmetric varieties introduced by C. De Concini and C. Procesi [16], which we will rather call adjoint symmetric wonderful varieties. Wonderful varieties were then considered in full generality by D. Luna, who started a program of classification in terms of combinatorial invariants [31].

Consider the following.

Question. Let $M$ be a wonderful variety and, for $\mathcal{L}, \mathcal{L}' \in \text{Pic}(M)$, consider the multiplication map

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(M, \mathcal{L}) \otimes \Gamma(M, \mathcal{L}') \rightarrow \Gamma(M, \mathcal{L} \otimes \mathcal{L}').$$

Is $m_{\mathcal{L},\mathcal{L}'}$ surjective for all globally generated $\mathcal{L}, \mathcal{L}'$?

In the case of a flag variety, the answer to the previous question is affirmative, indeed by the Borel-Weil theorem $\Gamma(M, \mathcal{L})$ is a simple $G$-module for all $\mathcal{L} \in \text{Pic}(M)$. In the case of an adjoint symmetric wonderful variety a still affirmative answer was given by R. Chirivi and the third named author in [13] with an inductive argument. In the special case of the wonderful compactification of an adjoint group (regarded as a symmetric $G \times G$-variety) the same was obtained by S.S. Kannan [23] with a more direct method. In general, when all the above-mentioned multiplication maps are surjective, it follows that the image of $M$ in the dual projective space of the complete linear system associated to any globally generated line bundle is projectively normal.

Another remarkable class of wonderful varieties is that of the model wonderful varieties, introduced by Luna [32]. Given a central subgroup $\Gamma \subset G$, a model homogeneous space for the algebraic group $G_{\Gamma} := G/\Gamma$ is a quasi-affine homogeneous space $G/H$ such that $\Gamma \subset H$ and the coordinate ring $\mathbb{C}[G/H]$ is a model of the representations of $G_{\Gamma}$ in the sense of I.M. Gel’fand (see [3], [20], [21]), that is, $\mathbb{C}[G/H]$ is isomorphic as a $G$-module to the direct sum of all the irreducible representations of $G_{\Gamma}$. The model homogeneous spaces for $G_{\Gamma}$ were classified in [32]. Luna constructed a variety $M_{G_{\Gamma}}^{\text{mod}}$, which is wonderful for the action of $G$, whose orbits parametrize the model homogeneous spaces for $G_{\Gamma}$. Varieties of the shape $M_{G_{\Gamma}}^{\text{mod}}$ for some central group $\Gamma \subset G$ are called model wonderful varieties. Given a model wonderful variety $M_{G_{\Gamma}}^{\text{mod}}$, we say that it is of simply connected type if $M_{G_{\Gamma}}^{\text{mod}}$ and $M_{G_{\Gamma}}^{\text{mod}}$ are $G$-equivariantly isomorphic. For $G$ almost simple, it follows by Luna’s description that $M_{G_{\Gamma}}^{\text{mod}}$ is not of simply connected type if and only if $G_{\Gamma} = \text{SO}(2r + 1)$ (in which case $G = \text{Spin}(2r + 1)$ and $\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$ is the center of $G$).

We will prove the following.
Theorem A. Let $M$ be a model wonderful variety of simply connected type. The multiplication of global sections

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(M, \mathcal{L}) \otimes \Gamma(M', \mathcal{L}') \rightarrow \Gamma(M, \mathcal{L} \otimes \mathcal{L}')$$

is surjective for all globally generated line bundles $\mathcal{L}, \mathcal{L}'$ on $M$.

This is false if $M$ is a model wonderful variety not of simply connected type. Indeed, in case $M = M_{SO(2r+1)}^\text{mod}$, the multiplication of global sections $m_{\mathcal{L},\mathcal{L}'}$ is surjective for all globally generated line bundles $\mathcal{L}, \mathcal{L}'$ on $M$ if and only if $r < 4$ (see Section 9.1). This is essentially a consequence of the fact that the tensor product of an almost simple group of type $B_r$ does not satisfy the saturation property in the sense of A. Klyachko (see [27] and [29]).

Theorem A will follow from a general argument which works for every wonderful variety $M$ and reduces the surjectivity of the maps $m_{\mathcal{L},\mathcal{L}'}$ for all globally generated line bundles $\mathcal{L}, \mathcal{L}'$ on $M$ to the surjectivity of a finite number of couples. In order to better explain this general reduction, we introduce some further notation.

Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Denote by $\Sigma$ the set of $G$-stable prime divisors of $M$ and by $\Delta$ the set of $B$-stable prime divisors of $M$ which are not $G$-stable: since $M$ possesses an open $B$-orbit, $\Delta$ is a finite set. Then the Picard group Pic$(M)$ is freely generated by the line bundles $\mathcal{L}_D$ with $D \in \Delta$ and the free group $\mathbb{Z} \Sigma$ embeds in Pic$(M)$, so that we may regard $\Sigma \Delta$ as a sublattice of $\mathbb{Z} \Delta$. Moreover, via the isomorphism Pic$(M) \simeq \mathbb{Z} \Delta$, the semigroup of globally generated line bundles is identified with the semigroup $\mathbb{N} \Delta$. We denote by $\mathcal{L}_E$ the globally generated line bundle associated to an element $E \in \mathbb{N} \Delta$. Define the following partial order relation on $\mathbb{N} \Delta$:

$$E \leq_{\Delta} F \quad \text{if} \quad E - F \in \mathbb{N} \Sigma.$$

This partial order is tightly related to the isotypic decomposition of the spaces of global sections of the globally generated line bundles on $M$, which we may always assume linearized (see Proposition 13).

In case $M$ is the wonderful compactification of the adjoint group $G_{ad}$ regarded as a $G \times G$-variety, then $\Sigma$ is naturally identified with the basis of the root system of $G$, while $\Delta$ is naturally identified with the set of the fundamental weights of $G$. More generally, this is true whenever $M$ is a non-exceptional symmetric wonderful variety, in which case it always exists a root system $\Phi_{\Sigma}$ (the reduced root system) such that $\Sigma$ is a basis of $\Phi_{\Sigma}$ and $\Delta$ is the corresponding set of fundamental weights. This is no longer true in the case of a general wonderful variety: while it always exists a root system $\Phi_{\Sigma} \subset \mathbb{Z} \Delta$ with $\Sigma$ as set of simple roots, the fundamental weights of $\Phi_{\Sigma}$ associated to $\Sigma$ may differ from $\Delta$. Therefore we may think the couple $(\Sigma, \Delta)$ as a generalization of a root system.

Suppose that $E, F \in \mathbb{N} \Delta$ are such that $E \leq_{\Delta} F$ and there is no $D$ with $E \leq_{\Delta} D <_{\Sigma} F$: then we say that $F - E \in \mathbb{N} \Sigma$ is a covering difference. The set of the covering differences is finite and in the case of an usual root system it was studied by J.R. Stembridge in [36].

For an element $E = \sum_{D \in \Delta} n_D D \in \mathbb{Z} \Delta$, define the positive part $E^+ = \sum_{n_D > 0} n_D D$ and the height $ht(E) = \sum_{D \in \Delta} n_D$. We prove the following.

Lemma B (see Lemma 24). Let $M$ be a wonderful variety and let $n$ be such that $ht(\gamma^+) \leq n$ for every covering difference $\gamma$. If the multiplication map

$$m_{\mathcal{L}_E,\mathcal{L}_F} : \Gamma(M, \mathcal{L}_E) \otimes \Gamma(M, \mathcal{L}_F) \rightarrow \Gamma(M, \mathcal{L}_E \otimes \mathcal{L}_F)$$

is surjective for all $E, F \in \mathbb{N} \Delta$ with $ht(E + F) \leq n$, then it is surjective for all $E, F \in \mathbb{N} \Delta$.

We will use Lemma B to prove Theorem A. We will first study the covering relation in the case of a model wonderful variety proving that $ht(\gamma^+) \leq 2$ for all covering differences $\gamma$, then we will study the multiplication maps $m_{\mathcal{L}_E,\mathcal{L}_F}$ in the fundamental cases $E, F \in \Delta$. The fact that $ht(\gamma^+) \leq 2$ for all covering differences $\gamma$ is an easy exercise in case the couple $(\Sigma, \Delta)$ corresponds to a root system, and as far as we know it could be a general fact which holds for all wonderful varieties.

Proceeding inductively on the partial order $\leq_{\Delta}$, it is easy to reduce the surjectivity of the multiplication map $m_{\mathcal{L}_E,\mathcal{L}_F}$ for every $E, F \in \mathbb{N} \Delta$ to the fact that some special submodules of $\Gamma(M, \mathcal{L}_{E+F})$ occur in the image of $m_{\mathcal{L}_E,\mathcal{L}_F}$. This leads to the definition of low triple (see Definition 23), which was already introduced in [13] to treat the case of an adjoint symmetric wonderful variety. To prove Lemma B we will show that it is possible to treat inductively (w.r.t. the height) the low triples of $M$.

The first part of the paper is entirely devoted to the proof of Theorem A and Lemma B. In Section 1 we fix the notation and recall some results about the wonderful varieties and their line bundles. In Section 2 we define the low triples and prove Lemma B. In Section 3 and in Section 4 we focus on the case of a model wonderful variety, first classifying the covering differences and then classifying the
low fundamental triples and studying the associated inclusions. To check the inclusions arising in the exceptional group cases we use the computer.

In the second part we have collected various consequences of Theorem A and Lemma B in different directions.

In Section 5 we prove the surjectivity of the multiplication for a very special class of wonderful varieties, the *comodel* wonderful varieties, whose colors and spherical roots can be seen in a parallel with those of the model wonderful varieties for groups of simply-laced type (Theorem 5.1).

In Section 6 we explain how in general the surjectivity of the multiplication map can give information on the normality of the closure of a spherical orbit in the projective space of a simple $G$-module.

In Section 7 we use our results (see Theorem 7.1) to study the normality or the non-normality (which are already well-known) of spherical nilpotent orbit closures, as they are cones over orbit closures in the projective space of a simple module. In particular, we reobtain the normality of the model orbit closure of $E_8$ (see [1]).

In Section 8 we concentrate on this model orbit of $E_8$. Following J. Adams, J-S. Huang and D.A. Vogan Jr. in [1], we also consider an analog this model orbit. More explicitly, we consider a $K$-orbit, where $K$ is the complexification of the maximal compact subgroup of the split real form of $E_8$. Then $K$ is the fixed point subgroup of an involution of $E_8$, this involution passes to the Lie algebra and $K$ acts on the eigenspace $p$ of eigenvalue $−1$. Our $K$-orbit here is just the intersection of the model orbit with $p$. Moreover, in this very special case, it turns out to be a comodel orbit, that is, related to the comodel wonderful $K$-variety in the same way in which a model $G$-orbit is related to the model wonderful $G$-variety. For the closure of this comodel orbit we prove the normality and describe the coordinate ring (Theorem 8.6). Furthermore, we describe the space of $K$-finite vectors of the unitary representation of the split real form of $E_8$ that should be associated to this comodel $K$-orbit via the so-called orbit method (Theorem 8.10). Both descriptions were already present in [1] as consequences of some conjectures, which as far as we know are still open.

In Section 9 we give the above mentioned counterexample to the surjectivity of the multiplication in the case of a model wonderful variety of not simply connected type, and this leads us to discuss some general properties of the multiplication map.

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1. Generalities.

Let $G$ be a simply connected semisimple algebraic group over an algebraically closed field of characteristic zero, fix a maximal torus $T$ of $G$ and a Borel subgroup $B ⊇ T$. For any group $K$ we denote by $χ(K)$ the multiplicative characters of $K$. We denote also by $χ(T)^+$ the set of dominant characters w.r.t. $B$ and if $λ ∈ χ(T)^+$ we denote by $V(λ)$ an irreducible representation of highest weight $λ$ and by $−λ^+$ the lowest weight of $V(λ)$. We denote by $S$ the set of simple roots.

Let $M$ be a wonderful $G$-variety with (unique) closed $G$-orbit $Y$. By [30], $M$ is spherical, i.e. it possesses an open $B$-orbit, say $B · x_0 ⊂ G · x_0 ⊂ M$. We denote by $H$ the stabilizer of $x_0$ in $G$.

1.1. Colors and spherical roots. Since $B · x_0$ is affine, $G · x_0 ∼ B · x_0$ is a union of finitely many $B$-stable divisors and we denote by $\Delta$ the set of their closures in $M$:

$$\Delta = \{D ⊂ M : D is a B-stable prime divisor, D ∩ G · x_0 ≠ ∅\}.$$ 

The elements of $\Delta$ are called the *colors* of $M$.

Denote by $B^-$ the opposite Borel subgroup of $B$ and let $y_0 ∈ Y$ be the unique $B^-$-fixed point of $M$. The normal space of $Y$ in $M$ at $y_0$, $T_{y_0}M/T_{y_0}Y$, is a multiplicity-free $T$-module. Set

$$\Sigma = \{T$-weights of $T_{y_0}M/T_{y_0}Y\} :$$

the elements of $\Sigma$ are called the *spherical roots* of $M$ and they naturally correspond to the local equations of the boundary divisors of $M$, which are $G$-stable. If $σ ∈ Σ$, we denote by $M^σ$ the associated boundary divisor of $M$ such that $T_{y_0}M/T_{y_0}M^σ$ is the 1-dimensional $T$-module of weight $σ$. 


1.2. Picard group and Cartan pairing. Recall that every line bundle on $M$ or on $Y$ has a unique $G$–linearization.

We may identify $\text{Pic}(Y)$ with a sublattice of $\mathcal{X}(T)$ and $\text{Pic}(G \cdot x_0)$ with $\mathcal{X}(H)$ (see [20]): we identify $\mathcal{L} \in \text{Pic}(Y)$ with the character of $T$ acting on the fiber of $\mathcal{L}$ over $y_0$, and we identify $\mathcal{L} \in \text{Pic}(G \cdot x_0)$ with the character of $H$ acting on the fiber over $x_0$. Consider now the maps $\omega : \text{Pic}(M) \to \mathcal{X}(T)$ and $\xi : \text{Pic}(M) \to \mathcal{X}(H)$ defined by the restriction to the closed and to the open orbit. We may regard $\text{Pic}(M)$ as a sublattice of $\mathcal{X}(T) \times \mathcal{X}(H)$ by identifying $\mathcal{L} \in \text{Pic}(M)$ with the couple $(\omega(\mathcal{L}), \xi(\mathcal{L}))$ (see [10]). Moreover, we have the following exact sequence

$$0 \to \mathbb{Z}\Sigma \to \text{Pic}(M) \to \mathcal{X}(H) \to 0.$$  

As a group, $\text{Pic}(M)$ is freely generated by the equivalence classes of line bundles $\mathcal{L}_D := \mathcal{O}(D)$, for $D \in \Delta$ (see [9] Proposition 2.2). For all $E \in \mathbb{Z}\Delta$, the associated line bundle $\mathcal{L}_E := \mathcal{O}(E)$ is globally generated (resp. ample) if and only if $E$ is a non-negative (resp. positive) combination of colors. We set $\omega_E = \omega(\mathcal{L}_E)$, $\xi_E = \xi(\mathcal{L}_E)$.

There exists a natural $\mathbb{Z}$–bilinear pairing (called the Cartan pairing of $M$)

$$c : \mathbb{Z}\Delta \times \mathbb{Z}\Delta \to \mathbb{Z}$$

which maps the couple $(D, \sigma)$ to the coefficient of $[M^\sigma]$ along $[D]$. So, regarding $\mathbb{Z}\Sigma$ as a sublattice of $\mathbb{Z}\Delta$, for $\sigma \in \mathbb{Z}\Sigma$ we have

$$\sigma = \sum_{D \in \Delta} c(D, \sigma)D.$$

1.3. Global sections and multiplication. We now recall the description of the space of global sections $\Gamma(M, \mathcal{L})$ of a line bundle $\mathcal{L}$. Notice first that since $M$ is spherical the decomposition of $\Gamma(M, \mathcal{L})$ into $G$-modules is multiplicity free for all $\mathcal{L} \in \text{Pic}(M)$. If $E \in \mathbb{N}\Delta$ then $\mathcal{L}_E$ is generated by global sections. In particular $\Gamma(M, \mathcal{L}_E)$ must contain a copy of $V(\omega_E)$ (which is the space of sections of $\mathcal{L}_E$ on $Y$), hence $\omega_E$ is dominant. We denote by $V_E$ the unique simple $G$-submodule of $\Gamma(M, \mathcal{L}_E)$ of highest weight $\omega_E$.

Notice also that the image of $x_0$ in $\mathbb{P}(\Gamma(M, \mathcal{L}_E)^*)$ is a point fixed by $H$. In particular, since $BH \subset G$ is open, it follows that the space of spherical vectors

$$V(\omega_E)^{(\mathcal{L}_E)} = \{v \in V(\omega_E) : hv = \xi_E(h)v \quad \forall h \in H\}$$

has dimension one and we denote by $h_E$ a generator of this line.

If $\gamma = \sum a_\sigma \sigma \in \mathbb{N}\Sigma$, we denote by $s^\gamma \in \Gamma(M, \mathcal{L}_M^\gamma)$ a section whose divisor is equal to $M^\gamma = \sum a_\sigma M^\sigma$. Notice that this section is $G$-invariant. Recall, as defined in the introduction, that we say that $F \leq \Sigma E$ if $E - F \in \mathbb{N}\Sigma$. If $E \in \mathbb{N}\Delta$ and $E \leq \Sigma F$ the multiplication by $s^{F-E}$ induces a $G$-equivariant map from the sections of $\mathcal{L}_E$ to the sections of $\mathcal{L}_F$, in particular we have $s^{F-E}V_E \subset \Gamma(M, \mathcal{L}_F)$.

Proposition 1.1 ([10] Proposition 2.4)]. Let $F \in \mathbb{Z}\Delta$. Then

$$\Gamma(M, \mathcal{L}_F) = \bigoplus_{E \in \mathbb{N}\Delta : E \leq \Sigma F} s^{F-E}V_E.$$

If $E, F \in \mathbb{N}\Delta$, consider the multiplication of sections

$$m_{E,F} : \Gamma(M, \mathcal{L}_E) \otimes \Gamma(M, \mathcal{L}_F) \to \Gamma(M, \mathcal{L}_{E+F}).$$

A way to translate the description of this map into a problem on spherical vectors is the following.

Lemma 1.2 ([12] Lemma 19)]. Let $D, E, F \in \mathbb{N}\Delta$ be such that $D \leq E + F$. Then $s^{E+F-D}V_D \subset V_EV_F$ if and only if the projection of $h_E \otimes h_F \in V(\omega_E^*) \otimes V(\omega_F^*)$ onto the isotypic component of highest weight $\omega_D^*$ is non-zero.

1.4. Distinguished sets of colors. Let $M'$ be a wonderful $G$-variety together with a surjective equivariant morphism $\phi : M \to M'$ with connected fibers and denote $\Delta_\phi \subset \Delta$ the set of colors which map dominantly onto $M'$. Then the semigroup

$$(\mathbb{N}\Sigma) / \Delta_\phi = \{\gamma \in \mathbb{N}\Sigma : c(D, \gamma) = 0 \quad \forall D \in \Delta_\phi\}$$

is free and its basis, which we denote by $\Sigma / \Delta_\phi$, coincides with the set of spherical roots of $M'$, while the set of colors of $M'$ is identified with $\Delta \times \Delta_\phi$.

Conversely, if $\Delta_0 \subset \Delta$, then there exists a wonderful variety $M'$ (unique up to isomorphism) together with a surjective equivariant morphism $\phi : M \to M'$ with connected fibers if and only if $\Delta_0$ is distinguished ([11] Theorem 3.1]), that is, there exists an element $D \in \mathbb{N}_{\geq 0}\Delta_0$ such that $c(D, \sigma) \geq 0$ for every $\sigma \in \Sigma$. 

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If $\Delta_0 \subset \Delta$ is distinguished, then we say that the associated wonderful variety $M'$ is the quotient of $M$ by $\Delta_0$, denoted by $M/\Delta_0$.

Recall the following general fact.

**Proposition 1.3.** Let $X,Y$ be normal varieties and suppose that $\phi : X \to Y$ is a surjective proper morphism with connected fibers. If $L \subseteq \text{Pic}(Y)$, then $\Gamma(Y,L) = \Gamma(X,\phi^*L)$.

If $E = \sum_{D \in \Delta} a_D D \in \mathbb{Z} \Delta$ we define $\text{Supp}(E)$, the support of $E$, as the set of colors $D$ such that $a_D \neq 0$.

**Corollary 1.4.** Let $E \in \mathbb{N} \Delta$ and suppose that $\Delta_0$ is a distinguished subset of $\Delta$ such that $\Delta_0 \cap \text{Supp}(E) = \emptyset$. Then $\Gamma(M,L_E) = \Gamma(M/\Delta_0,L_E)$.

### 1.5. Parabolic induction: spherical roots and colors.

We describe now a standard way to construct a wonderful variety for the group $G$ from a wonderful variety for a Levi subgroup of $G$.

Let $L$ be a proper Levi subgroup of $G$ which is standard w.r.t. the choice of $S$ as set of simple roots. Let $Q$ be the parabolic subgroup associated to $L$ and containing $B$, and let $Q^-$ be the opposite parabolic subgroup. Denote by $R_Q$ and $R_Q^-$ the solvable radicals and by $U_Q$ and $U_Q^-$ the unipotent radicals of $Q$ and $Q^-$, respectively. Finally, let $L^s$ be the semisimple part of $L$ and denote by $L_{ad} = L^s/Z(L^s) = L/Z(L)$ the adjoint quotient of $L$.

Let $N$ be a wonderful variety for the group $L_{ad}$, in particular it is a wonderful variety for $L$ where $Z(L)$ acts trivially. Extend the action of $L$ on $N$ to $Q^-$, with $U_Q^-$ acting trivially, and define the parabolic induction of $N$ as

$$M = G \times_{Q^-} N.$$

We have a projection $\pi : M \to G/\text{Q}^-$, $N$ is the fiber over $Q^-$ and moreover it is the set of points of $M$ fixed by $U_Q^-$. One can easily check the following properties.

1. $M$ is a wonderful $G$-variety.
2. The intersection of a subset of $M$ with $N$ induces a bijection from the set of $G$-stable subsets of $M$ to the set of $L$-stable subsets of $N$. For every spherical root $\sigma$ of $M$ we denote by $N^\sigma$ the intersection $M^\sigma \cap N$.
3. The restriction of line bundles from $M$ to $N$ induces an isomorphism $\rho$ between the class groups of linearized line bundles $\text{Pic}_G(M)$ and $\text{Pic}_L(N)$.
4. We have an injective map $\varepsilon$ from the set $\Delta(N)$ of colors of $N$ to the set $\Delta(M)$ of colors of $M$ given by $\varepsilon(D) = B \cdot D$ for all $D \in \Delta(N)$.
5. The exact sequence $0 \to \mathcal{X}(L/L^s) \to \text{Pic}_L(N) \to \text{Pic}_{L^s}(N) \to 0$ has a natural splitting given by $[D] \mapsto [\varepsilon(D)]$ for every color $D$ of $N$.
6. If $D \in \Delta(M) \setminus \varepsilon(\Delta(N))$ then the restriction of $\mathcal{O}(D)$ to $N$ is a trivial line bundle on $N$.
7. The set of colors $\varepsilon(\Delta(N))$ is distinguished and $M/\varepsilon(\Delta(N)) \simeq G/\text{Q}^-$. 
8. Since $L^s$ is simply connected we have $\text{Pic}(N) \simeq \text{Pic}_{L^s}(N)$.

Although wonderful varieties have been defined for semisimple groups, in this case it is convenient to look at $N$ as an $L$-variety and to consider $L$-linearized line bundles on $N$. The definitions given for a wonderful variety can be extended to this situation. We define

$$\Sigma = \{T\text{-weights of } T_{y_0} N / T_{y_0} L \cdot y_0 \},$$

where we recall that $y_0$ is the point of $M$ fixed by $B^-$, hence $L \cdot y_0$ is the unique closed orbit of $N$. The set $\Sigma$ is in correspondence with the set of $L$-stable divisors and in particular $\Sigma$ is the $T$-weight of $T_{y_0} N / T_{y_0} N^o$. Moreover, we have that $\Sigma$ equals the set of spherical roots of $M$ and every element of $\Sigma$ is a sum of simple roots of $L$.

By the above properties (3) and (4) for every color $D$ in $\Delta(N)$ we have a canonical choice of a linearization of the associated line bundle and we have a natural decomposition

$$\text{Pic}_L(N) \simeq \mathcal{X}(L/L^s) \oplus \bigoplus_{D \in \Delta(N)} \mathbb{Z}[D].$$

For all $\sigma \in \Sigma$, $[N^\sigma] = \sum_{D \in \Delta(M)} \varepsilon(D,\sigma) [\mathcal{X}(D)]$. If $L, L' \in \text{Pic}_L(N)$ we define $L \geq_{\Sigma} L'$ if, using the additive notation, $L - L' = \sum a_\sigma [N^\sigma]$ with $a_\sigma \geq 0$ for all $\sigma$, similarly to what we have done in Section 1.3. In this way Proposition 1.4 holds without any change.

It is easy to see when a $G$-wonderful variety $M$ can be obtained by parabolic induction. Let $P^-$ be the stabilizer of $y_0$ and let $S^p$ the set of simple roots that are in the set of roots of $P^-$. Assume that $S' = S^p \cup \text{Supp}_L \Sigma \neq S$ and let $L$ be the standard Levi subgroup associated to $S'$. Let
Q, Q^-, R_Q, R_{Q^*}, U_Q, U_{Q^*}, L^a, L_{ad} be defined as above. Set \( \Delta = \{ D \in \Delta : \text{if } P_\alpha \cdot D \neq D \text{ then } \alpha \in S' \} \), then \( \Delta \) is a distinguished set and \( M/\Delta \cong G/Q^- \). Moreover, \( N = M^{U_Q} \) is a wonderful variety for \( L_{ad} \) and \( M \) is obtained by parabolic induction from \( N \) as above.

### 1.6. Parabolic induction: global sections.

Let now \( M \) be obtained by parabolic induction from \( N \) as above. Let \( M \) be a line bundle on \( M \) and denote by \( N \) its restriction to \( N \). We want to compare the sections of \( M \) and \( N \). The restriction of sections induces a map \( r_M : \Gamma(M,M) \rightarrow \Gamma(N,N) \). Notice that \( U \cdot N \) is a dense subset of \( M \), hence the restriction of \( r_M \) to \( \Gamma(M,M) \) is injective. We first describe the kernel of \( r_M \).

Let \( M = L_E \), then \( \Gamma(M,M) = \bigoplus sE V_E \) where the sum is over all \( E \in \mathbb{N}\Delta \) such that \( F \backslash \Sigma \). For each \( E \in \mathbb{N}\Delta \) set \( W_E = V_E^{U_Q} \). This is an irreducible \( L \)-submodule with the same highest weight of \( V_E \).

Let \( I_E \) be the \( L \)-stable complement of \( W_E \) in \( V_E \).

#### Lemma 1.5.

We have
\[
\ker r_M = \bigoplus_{E \in \mathbb{N}\Delta : E \nsubseteq \Sigma} sE \cdot I_E.
\]

**Proof.** By the above discussion it is enough to prove that \( r_{L_E}(I_E) = 0 \). Let \( v \) be a highest weight vector (for the action of \( L \)) in \( I_E \) of weight \( \lambda \). Then \( \omega(E) - \lambda = \sum_{\alpha \in S} a_\alpha \omega_\alpha \) with \( a_\alpha \geq 0 \). If \( S' \) is the set of simple roots for \( L \) notice that, since \( v \in I_E \), there exists \( \alpha \in S \setminus S' \) such that \( a_\alpha \neq 0 \). On the other hand, by the generalization of Proposition [13] for \( N \) discussed in the previous section, the weights of the highest weight vectors in \( \Gamma(N,L_E) \) are of the form \( \omega(E) - \beta \) with \( \beta \in \mathbb{N}S' \). In particular we must have \( r_{L_E}(v) = 0 \).

Since the restriction commutes with the multiplication we deduce the following.

#### Proposition 1.6 ([13] Proposition 2.9).

For all \( E, F \in \mathbb{N}\Delta \) and \( \gamma \in \mathbb{N}\Sigma \), \( s^\gamma V_{E+F-\gamma} \subseteq V_{E} V_{F} \) if and only if \( s^\gamma W_{E+F-\gamma} \subseteq W_{E} W_{F} \).

In particular, using the property (6) of the previous section, we have the following.

#### Corollary 1.7.

For all \( D \in \mathbb{N}(\Delta \setminus \Delta) \) and \( E \in \mathbb{N}\Delta \), \( V_D V_E = V_{D+E} \).

### 2. Projective normality and the covering relation.

#### 2.1. The covering relation.

Let \( \{ \omega_\alpha : \alpha \in S \} \) be the set of fundamental weights w.r.t. the simple roots \( S \).

For all \( \lambda = \sum k_\alpha \omega_\alpha \in \mathcal{X}(T) \), denote by \( \text{Supp}(\lambda) \) the set of \( \alpha \in S \) such that \( k_\alpha \neq 0 \) and define its positive part \( \lambda^+ \), resp. its negative part \( \lambda^- \), as the dominant weights
\[
\lambda^+ = \sum_{k_\alpha > 0} k_\alpha \omega_\alpha \quad \text{and} \quad \lambda^- = \lambda^+ - \lambda.
\]

If \( \lambda \in \mathcal{X}(T)^+ \), define also the height of \( \lambda \) as the number \( \text{ht}(\lambda) = \sum_{\alpha \in S} k_\alpha \).

Suppose that \( \lambda \) and \( \mu \) are dominant weights with \( \lambda < \mu \) (w.r.t. the usual dominance order) and suppose that there is no dominant weight \( \nu \) such that \( \lambda < \nu < \mu \): then one says that \( \mu \) covers \( \lambda \) and we call \( \mu - \lambda \) a covering difference in \( \mathcal{X}(T)^+ \). Notice that an element \( \gamma \in \mathbb{N}S \) is a covering difference in \( \mathcal{X}(T)^+ \) if and only if \( \gamma^+ \) covers \( \gamma^- \). Although the following proposition is an immediate consequence of [36] Theorem 2.6), here we give an easy independent proof.

#### Proposition 2.1.

If \( \gamma \in \mathbb{N}S \) is a covering difference in \( \mathcal{X}(T)^+ \), then \( \text{ht}(\gamma^+) \leq 2 \).

**Proof.** Let \( \gamma \in \mathbb{N}S \) be a covering difference and suppose that \( \alpha \in S \) is such that \( \langle \gamma^+, \alpha^\vee \rangle \geq 2 \): then \( \gamma^+ - \alpha \in \mathcal{X}(T)^+ \), hence it must be \( \gamma = \alpha \) and \( \text{ht}(\gamma^+) = 2 \). Hence we may assume that \( \langle \gamma^+, \alpha^\vee \rangle \leq 1 \) for every \( \alpha \in S \). Suppose that \( \text{ht}(\gamma^+) \geq 3 \), (up to reindexing the simple roots) we can take \( \alpha_1, \alpha_2 \in \text{Supp}S(\gamma^+) \) with \( i < j \) such that \( \{\alpha_i, \ldots, \alpha_j\} \) generates an irreducible subsystem of type \( A \): then \( \gamma^+ - (\alpha_i + \cdots + \alpha_j) \in \mathcal{X}(T)^+ \) and it follows \( \gamma = \alpha_1 + \cdots + \alpha_j \) and \( \text{ht}(\gamma^+) = 2 \) (a contradiction).

We now consider the covering relation in the more general context of wonderful varieties. Let \( M \) be a wonderful variety with set of spherical roots \( \Sigma \) and with set of colors \( \Delta \). For all \( E = \sum_{D \in \Delta} k_D D \in \mathbb{Z}\Delta \), define its positive part \( E^+ \) and its negative part \( E^- \) as
\[
E^+ = \sum_{k_D > 0} k_D D \quad \text{and} \quad E^- = E^+ - E.
\]
If $E \in \Delta$, define the height of $E$ as the number $\text{ht}(E) = \sum_{D \in \Delta} k_D$. On the other hand, for all $\gamma = \sum a_\sigma \sigma \in \Sigma$ define its $\Sigma$-height as $\text{ht}_\Sigma(\gamma) = \sum a_\sigma$. Let $E$ and $F$ be in $\Delta$ with $E <_\Sigma F$ and suppose that there is no $D \in \Delta$ such that $E <_\Sigma D <_\Sigma F$; then we say that $F$ covers $E$ and we call $F - E$ a covering difference in $\Delta$. Again, $\gamma \in \Sigma$ is a covering difference in $\Delta$ if and only $\gamma^+ \text{ covers } \gamma^-$. 

Remark 2.2. Notice that there are only finitely many covering differences. Indeed, consider the lattice $X = \{(D, E) \in \mathbb{Z} \Delta \times \mathbb{Z} \Delta : E - D \in \mathbb{Z} \Sigma\}$ and the rational cone $X^+ = \{(D, E) \in \Delta \times \Delta : D - E \in \Sigma\}$. If $\gamma$ is a covering difference then $(\gamma^+, \gamma^-)$ is an indecomposable element of $X^+$. 

Finally, the number of indecomposable elements in $X^+$ is finite and can be controlled as follows: let $\ell_1, \ldots, \ell_t$ be the half-lines generating the cone, let $v_i = (D_i, E_i)$ be a generator of $\ell_i \cap X^+$ and let $n_i = \text{ht}(D_i) + \text{ht}(E_i)$. Then if $(D, E)$ is indecomposable $\text{ht}(D) + \text{ht}(E) < \sum_i n_i$. 

As already said in the introduction, in the case of a non-exceptional symmetric wonderful variety, there exists a root system $\Phi_\Sigma$ (the restricted root system) which is generated by the spherical roots and such that $\Delta$ is naturally identified with the set of fundamental weights of $\Phi_\Sigma$ and the pairing between $\Sigma$ and $\Delta$ is the Cartan pairing of $\Phi_\Sigma$. Although $\Sigma$ is always the basis of a root system $\Phi_\Sigma$, in the general case it is not possible to identify $\Delta$ with the fundamental weights of $\Phi_\Sigma$: in particular, it may happen that $\mathbb{Z} \Sigma \subset \mathbb{Z} \Delta$ is not a sublattice of finite index and that the semigroup of radical dominant weights of $\Phi_\Sigma$ is not even contained in $\Delta$. However, one can consider the property of Proposition 2.3 without modifications in this context: 

(2-h) If $\gamma \in N \Sigma$ is a covering difference in $\Delta$, then $\text{ht}(\gamma^+) \leq 2$. 

Notice that by the above discussion the property (2-h) holds if $M$ is a non-exceptional symmetric wonderful variety. In the case of the exceptional symmetric wonderful varieties the same argument works without serious complications. In the subsequent section we will show that it is true also in the case of a model wonderful variety (of simply connected type). We have also checked many other examples and, as far as we know, it is possible that it holds for all wonderful varieties. 

In analogy with the case of a root system, we say that an element $D \in \Delta$ is minuscule if it is minimal w.r.t. the partial order $\leq$. 

2.2. Projective normality. The notion of low triple has been introduced in [13] in the case of a symmetric wonderful variety. Here we will use the same terminology with a slightly weaker definition. 

Definition 2.3. Let $M$ be a wonderful variety with set of spherical root $\Sigma$ and with set of colors $\Delta$. Let $D, E, F \in \Delta$ be such that $F \leq \Sigma D + E$. The triple $(D, E, F)$ is called a low triple if: 

for all $D' \leq \Sigma D$ and $E' \leq \Sigma E$ such that $F \leq \Sigma D' + E'$, it holds $D' = D$ and $E' = E$. 

The triple $(D, E, F)$ is called a fundamental triple if $D, E \in \Delta$. 

Notice that if $(D, E, F)$ is a low triple, then $s^{D+E-F}V_F \subset \Gamma(M, L_D)\Gamma(M, L_E)$ if and only if $s^{D+E-F}V_F \subset V_DV_E$. 

Lemma 2.4. Let $M$ be a wonderful variety and let $n$ be such that $\text{ht}(\gamma^+) \leq n$ for every covering difference $\gamma$. If 

\begin{equation}
 s^{D+E-F}V_F \subset V_DV_E,
\end{equation}

for all low triples $(D, E, F)$ with $\text{ht}(D + E) \leq n$, then the multiplication map 

\begin{equation}
 m_{D, E} : \Gamma(M, L_D) \otimes \Gamma(M, L_E) \longrightarrow \Gamma(M, L_{D+E})
\end{equation}

is surjective for all $D, E \in \Delta$. 

Proof. For any $D \in \Delta$ let $\Gamma_D = \Gamma(M, L_D)$. For any triple $(D, E, F)$ we will show that $s^{D+E-F}V_F \subset \Gamma_D\Gamma_E$. We proceed by induction first on $\text{ht}_\Sigma(D + E - F)$ and then on $\text{ht}(D + E)$. If $\text{ht}_\Sigma(D + E - F) = 0$ the claim is trivial. 

If $(D, E, F)$ is not a low triple then there exist $D' \leq \Sigma D$ and $E' \leq \Sigma E$ such that $F \leq D' + E'$ and $\text{ht}_\Sigma(D' + E' - F) < \text{ht}_\Sigma(D + E - F)$. Hence the claim is true for $(D', E', F)$, and 

\begin{equation}
 s^{D+E-F}V_F = s^{D-D'+E-E'}s^{D'+E'-F}V_F \subset s^{D-D'+E-E'}\Gamma_D\Gamma_E \subset \Gamma_D\Gamma_E.
\end{equation}

If $(D, E, F)$ is a low triple and $\text{ht}(D + E) \leq n$ then the claim is true by assumption.
Assume now that $\text{ht}(D + E) > n$ and that $(D, E, F)$ is a low triple. Let $F_1$ be a divisor covered by $D + E$ and such that $F_1 \supseteq F$. Since $\text{ht}(D + E) > \text{ht}(\gamma)$ for every covering difference $\gamma \in \mathbb{N}\Sigma$, it follows that $\text{Supp}(F_1) \cap \text{Supp}(D + E) \neq \emptyset$. Fix $D_0 \in \text{Supp}(F_1) \cap \text{Supp}(D + E)$ and set

$$(D_1, E_1) = \begin{cases} (D - D_0, E) & \text{if } D_0 \in \text{Supp}(E) \\ (D, E - D_0) & \text{otherwise} \end{cases}$$

Choose $F_2$ in $\mathbb{N}\Delta$ minimal w.r.t. $\leq \Sigma$ such that $F - D_0 \leq \Sigma F_2 \leq \Sigma F_1 - D_0$. Notice that $\text{ht}_\Sigma(F_2) < \text{ht}_\Sigma(D + E - F)$. Moreover (if there exist $F_2' \leq F_2$ and $D_0 \leq D_0$ in $\mathbb{N}\Delta$ such that $F \leq D_0 + F_2'$ then

$$(F_2) \leq \Sigma D_0' + F_2' \leq \Sigma D_0' + D_1 + E_1 \leq \Sigma D + E.$$ Since $(D, E, F)$ is a low triple, we get $D_0' = D_0$, while since $F_2$ is minimal we get $F_2' = F_2$. Hence, by induction, it follows that $s^{D_0 + F_2 - F} V_F \subset V_{D_0} V_{F_2}$.

Notice that we have $\text{ht}_\Sigma(D_1 + E_1 - F_2) \leq \text{ht}_\Sigma(D + E - F)$ and that $\text{ht}(D_1 + E_1) = \text{ht}(D + E) - 1$, hence we can apply the inductive hypothesis to $(D_1, E_1, F_2)$. So

$$s^{D + E - F} V_F = s^{D_1 + E_1 - F_2} s^{D_0 + F_2 - F} V_F \subset s^{D_1 + E_1 - F_2} V_{D_0} V_{F_2} V_{D_0} \subset \Gamma D_1 \Gamma E_1 \Gamma V_{D_0} \subset \Gamma D \Gamma E.$$

Together with Remark 2.2, the lemma implies that to prove the surjectivity of $m_{D, E}$ for all $D, E \in \mathbb{N}\Delta$ it is enough to check a finite number of cases. In particular if the property (2-ht) holds, it is enough to check the above inclusion (11) only for the low fundamental triples.

In Section 5.4 we will show that there exist wonderful varieties possessing low fundamental triples $(D, E, F)$ such that $s^{D + E - F} V_F \not\subset V_D V_E$: in particular the multiplication of sections of line bundles on a wonderful variety is not necessarily surjective.

3. The covering relation for model wonderful varieties.

As said in the introduction, a model homogeneous space for a reductive group $G$ (not necessarily simply connected) is a quasi affine $G$-homogeneous space whose coordinate ring is a model representation of $G$: each irreducible representation of $G$ appears exactly with multiplicity one. By [32], for every group $G$ there exists a wonderful variety $M$ such that for every model homogenous space $G/H_0$ there exists a point in $M$ whose stabilizer is equal to $N_G(H_0)$ and, viceversa, if $H$ is the stabilizer of a point in $M$ and we set $H_0$ to be the intersections of the kernels of the multiplicative characters of $H$ then $G/H_0$ is a model homogeneous space. The variety $M$ is called the model wonderful variety of $G$.

Notice that $H_0$ has no characters, hence the set of spherical vectors (i.e., the eigenvectors of $\omega$) of simply connected type, and we will prove that the property (2-ht) holds in this case. For model wonderful varieties of simply connected type the set of colors $\Delta$ is in bijection, via $\omega$, with the set of fundamental weights $\Sigma$ of $G$. Hence, in the case of a model wonderful variety, we may reduce to the case of $G$ almost simple. Moreover, if $\gamma = \sum_{\alpha \in \Delta} a_\alpha \alpha \in \Sigma \Delta$ we denote by $\text{Supp}_S(\gamma)$ the set of simple roots $\alpha$ such that $a_\alpha \neq 0$. Notice that if $\gamma$ is a covering difference in $\Sigma \Delta$ w.r.t. $\Sigma$, then $\text{Supp}_S(\gamma)$ is a connected subset of $\Sigma$ and $\gamma$ is a covering difference for the model variety of the simply connected group associated to the root subsystem generated by $\text{Supp}_S(\gamma)$. Therefore, we will only classify the covering differences $\gamma$ such that $\text{Supp}_S(\gamma) = S$.

In the following two sections, $r$ will denote the semisimple rank of the group and $\Delta = \{D_1, \ldots, D_r\}$ will be the set of colors, labelled in such a way that $\omega_{D_i}$ is the fundamental weight $\omega_i$. For simplicity we also set $D_i = 0$ for all $i \leq 0$ and for all $i > r$. 

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3.1. Type $A_r$. $r \geq 2$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{r-1}\}$ be the set of spherical roots where $\sigma_i = \alpha_i + \alpha_{i+1}$.

**Proposition 3.1.** Let $\gamma \in \text{N} \Sigma$ be a covering difference in $\text{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then either

1. $r$ is even and $\gamma = \sum_{i=1}^{r/2} 2\sigma_{2i-1} = D_1 + D_r$, or
2. $r$ is odd and $\gamma = \sum_{i=1}^{r-1} \sigma_i = D_2 + D_{r-1}$.

**Proof.** Set $\gamma = \sum_{i=1}^{r-1} a_i \sigma_i = \sum_{i=1}^r c_i D_i$. Clearly, $a_1 > 0$.

Assume first that $a_2 = 0$. Take the maximum integer $k$ such that $a_i > 0$ for all $i \leq k$. Then $c_j \geq 0$ for all $3 \leq j \leq k + 2$ otherwise $\gamma^+ + \sum_{i=2}^{j-1} \sigma_i \in \text{N} \Delta$.

If $k$ was $< r - 1$, then $c_{k+1} + c_{k+2} = -a_{k-1} - a_{k+3}$ would be $< 0$, a contradiction. Therefore, $\text{Supp}_2(\gamma) = \Sigma$.

Now, since $\gamma^+ + \sum_{i=1}^{r-1} \sigma_i \in \text{N} \Delta$, $\gamma$ must necessarily be equal to $\sum_{i=1}^{r-1} \sigma_i$, which is a covering difference if and only if $i$ is odd; indeed, if $i$ is even, $\sum_{i=1}^{r/2} \sigma_{2i-1} \in \text{N} \Delta$.

Assume now that $a_2 = 0$. Take the maximum odd integer $k$ such that $a_i = 0$ for all even $i < k$. Then $c_k \geq 0$ otherwise $\gamma^- + \sum_{i=1}^{(k-1)/2} 2\sigma_{2i-1} \in \text{N} \Delta$. Furthermore, $c_{k-1} \geq 0$ otherwise, reasoning as above, $a_j > 0$ for all $j \geq k$ and $\gamma^- + \sum_{i=1}^{k-1} \sigma_i \in \text{N} \Delta$.

If $k$ was $< r - 1$, then $c_{k-1} + c_k = -a_{k+1}$ would be $< 0$, a contradiction. Therefore, $a_i > 0$ if $i$ is odd, and $r$ is even.

Now, since $\gamma^- + \sum_{i=1}^{r/2} 2\sigma_{2i-1} \in \text{N} \Delta$, $\gamma$ must necessarily be equal to $\sum_{i=1}^{r/2} \sigma_{2i-1}$ which is a covering difference.

3.2. Type $B_r$. $r \geq 2$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{r-1}\}$ be the set of spherical roots where $\sigma_i = \alpha_i + \alpha_{i+1}$.

**Proposition 3.2.** Let $\gamma \in \text{N} \Sigma$ be a covering difference in $\text{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then $r$ is even and $\gamma = \sum_{i=1}^{r/2} \sigma_{2i-1} = D_1$.

**Proof.** Set $\gamma = \sum_{i=1}^{r-1} a_i \sigma_i = \sum_{i=1}^r c_i D_i$. Clearly, $a_{r-1} > 0$.

Suppose that $k < r$ is such that $r - k$ is odd and $c_i \leq 0$ for every $k < i < r$ with $r - i$ odd. Then we have the inequalities (where $a_i = 0$ if $i \leq k$)

$$a_{r-1} \leq a_{r-3} - a_r - 2 \leq a_{r-5} - a_r - 4 \leq \ldots \leq a_{k+2} - a_r - 4k \leq a_k - a_{k+1}$$

and it follows that $a_j \geq a_{r-1} > 0$ for every $j \geq k$ with $r - j$ odd. In particular, $k$ must be $> 0$.

Let $k$ be maximal with $r - k$ odd and $c_k > 0$. Denote $\gamma_0 = \sum_{i=0}^{(r-k)/2} \sigma_{k+2i} = -c_k - 1 + c_k$; then $\gamma_0 \leq \gamma$ and $\gamma^+ - \gamma_0 \in \text{N} \Delta$, hence it must be $\gamma = \gamma_0$, $k = 1$ and $r$ even.

3.3. Type $C_r$. $r \geq 3$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{r-1}\}$ be the set of spherical roots where $\sigma_i = \alpha_i + \alpha_{i+1}$.

**Proposition 3.3.** Let $\gamma \in \text{N} \Sigma$ be a covering difference in $\text{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then $\gamma = \sum_{i=1}^{r-1} \sigma_i = D_2$.

**Proof.** Set $\gamma = \sum_{i=1}^{r-1} a_i \sigma_i = \sum_{i=1}^r c_i D_i$. Clearly, $a_{r-1} > 0$.

Let $k$ be maximal such that $c_k > 0$. Then we have the following inequalities (for $k < r$)

$$a_{k-1} = a_k \geq \ldots \geq a_{r-1}.$$ 

It follows that $a_i > 0$ for every $k - 1 \leq i < r$. Therefore, $\gamma^+ = \sum_{i=k-1}^{r-1} \sigma_i = \gamma^+ + c_k - 2 \sigma_k \in \text{N} \Delta$ and we get $k = 2$ with $\gamma = \sum_{i=1}^{r-1} \sigma_i$.

3.4. Type $D_r$. $r \geq 4$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{r-1}\}$ be the set of spherical roots where $\sigma_i = \alpha_i + \alpha_{i+1}$ if $i < r - 1$ and $\sigma_{r-1} = \alpha_r + \alpha_r$.

**Proposition 3.4.** Let $\gamma \in \text{N} \Sigma$ be a covering difference in $\text{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then $r$ is odd and $\gamma = \sum_{i=1}^{(j-1)/2} \sigma_{2i-1} + \sum_{i=(j+1)/2}^{(r-3)/2} 2\sigma_{2i-1} + \sigma_r + \sigma_{r-1} = D_1 - D_{j-1} + D_j$ for $j$ odd $\leq r - 2$.

**Proof.** Set $\gamma = \sum_{i=1}^{r/2} a_i \sigma_i = \sum_{i=1}^r c_i D_i$. Clearly, $a_{r-2}$ and $a_{r-1}$ must be $> 0$.

One has $c_{r-2} \leq 1$, otherwise $\gamma^- - (2D_{r-3} + 2D_{r-2}) \in \text{N} \Delta$, thus $\gamma = -2D_{r-3} + 2D_{r-2} = \sigma_{r-2} + \sigma_{r-1}$, but $\text{Supp}_2(\gamma) \not\subseteq S$. Since $c_{r-2} = -a_{r-4} + a_{r-3} + a_{r-2} + a_{r-1}$ we get $a_{r-4} > 0$ and moreover, if $c_{r-2} \leq 0$, $a_{r-4} > 1$.

We go on this way step-by-step. Let $k = (r - 1)/2$. Assume $a_{r-2i} > 0$ for all $i \leq k$ and moreover if, for some $j < k$, $c_{r-2i} \leq 0$ for all $i \leq j$ then $a_{r-2j} - 2 > 1$ for all $i \leq j$. One has $\sum_{i=1}^k c_{r-2i} \leq 1$, otherwise there would exist $1 \leq i_1 \leq i_2 \leq k$ such that $\gamma^+ = (-D_{r-2i_2-1} + D_{r-2i_2} - D_{r-2i_1-1} + D_{2-2i_1}) \in \text{N} \Delta$, ...
Proposition 3.5. Let \( \gamma \in \Sigma \) be a covering difference in \( \nabla \Delta \) with \( \text{Supp}_S(\gamma) = S \).

If \( r = 6 \),

1. either \( \gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 = D_1 - D_2 \)
2. or \( \gamma = 2\sigma_1 + 2\sigma_2 + \sigma_3 + \sigma_4 + 2\sigma_5 = D_1 + D_6 \).

If \( r = 7 \),

3. \( \gamma = \sigma_1 + \sigma_2 + \sigma_4 + \sigma_5 + \sigma_6 = D_1 + D_6 - D_3 \).

If \( r = 8 \),

4. either \( \gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + 2\sigma_5 + \sigma_6 + \gamma = D_6 - D_2 \)
5. or \( \gamma = 2\sigma_1 + 2\sigma_2 + \sigma_3 + \sigma_4 + 2\sigma_5 + \gamma = D_1 + D_6 - D_7 \).

Proof. Set \( \gamma = \sum_{i=1}^{r-1} \alpha_i \sigma_i = \sum_{i=1}^{r-1} c_i D_i \) for all \( i \).

Let \( r \) equal 6. Clearly, \( a_1, a_2, a_5 \) are \( > 0 \). Moreover, \( c_1 + c_4 + c_6 = a_2 = a_6 \).

If \( c_2 > 0 \), then \( c_1, c_4, c_6 \) would all be \( \leq 0 \). Therefore, \( c_2 \leq 0 \) and \( a_2 + a_4 > 0 \). By symmetry, we can assume \( a_3 > 0 \).

Assume \( c_4 > 0 \). Then \( c_3 \leq 0 \). If \( a_4 \) was zero, \( c_2 + c_3 \) would be equal to \( a_1 \) (which is \( > 0 \)). Therefore, \( a_4 > 0 \) and \( \gamma = D_1 - D_2 \).

Assume now \( c_4 < 0 \). We can also assume \( c_6 > 0 \) (by symmetry, if \( a_4 > 0 \)). Then both \( c_3 \) and \( c_6 \) are \( < 0 \), hence \( a_4 \) is \( > 0 \). Since \( c_2 + c_3 + c_4 + c_5 = 0 \), all \( c_i = 0 \), for \( 2 \leq i \leq 5 \). This implies \( a_1, a_2, a_5 > 2 \), \( c_6 \leq 1 \) (since \( (2\sigma_1 + \sigma_2 + \sigma_4)^+ = 2D_1 \) and \( c_1 > 0 \). Therefore, \( \gamma = D_1 + D_6 \).

Let \( r \) equal 7. Clearly, \( a_1, a_2, a_5 \) are \( > 0 \). Moreover, \( c_1 + c_4 + c_6 = a_2 = a_6 \).

Assume \( a_3 > 0 \). We get \( c_1 > 0 \). Then both \( c_2 \) and \( c_3 \) are \( \leq 0 \), hence \( a_4 \) is \( > 0 \). Then \( c_5 \leq 0 \), hence \( a_5 > 0 \). Moreover, \( c_6 \leq 0 \) (since \( (\sigma_1 + \sigma_2 + \sigma_4)^+ = D_1 + D_4 \), \( c_1 \leq 1 \) (since \( (2\sigma_1 + \sigma_2 + \sigma_4)^+ = 2D_1 \)) and \( c_6 > 0 \). Therefore, \( \gamma = D_1 + D_6 - D_3 \).

Assume \( a_1 > 0 \). If \( c_4 \) was \( > 0 \), then both \( c_2 \) and \( c_3 \) would be \( \leq 0 \), hence \( a_4 \) is \( > 0 \). This would imply \( c_7 \leq 0 \), hence \( a_5 \geq 0 \), which is impossible (since \( (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)^+ = D_4 \)). Then \( c_4 \leq 0 \).

If \( c_2 \) was \( > 0 \), then both \( c_2 \) and \( c_3 \) would be \( < 0 \), hence \( a_4 \) is \( > 0 \). This would imply \( c_7 \leq 0 \) hence \( a_5 \geq 0 \), and \( c_4 \leq 1 \) (since \( (2\sigma_1 + \sigma_2 + \sigma_4)^+ = 2D_1 \)) hence \( c_6 > 0 \), which is impossible (since \( (\sigma_1 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6)^+ = D_1 + D_6 \)). Then \( c_1 \leq 0 \).

If \( c_4 \leq 0 \), we would get \( c_7 \leq 0 \), \( a_5 > 0 \) and \( c_6 \leq 1 \), which is impossible (since \( c_1 + c_4 + c_6 = a_2 + a_6 > 2 \)).

Let \( r \) equal 8. Clearly, \( a_1, a_2, a_7 \) are \( > 0 \). Moreover, \( c_1 + c_4 + c_6 + c_8 = a_2 \).

If \( c_4 \) was \( > 0 \), we would get \( c_2 \leq 0 \), \( a_3 > 0 \) and \( c_6 > 0 \), actually \( a_3 > 0 \) (\( a_5 = 0 \) would imply \( c_1 > 0 \), but \( a_4 > 0 \) and \( c_1 \leq 0 \), hence \( c_3 \leq 0 \), \( a_4 > 0 \), \( c_6 \leq 1 \) and \( c_1 \leq 0 \), therefore \( a_5 > 0 \) (since \( a_2 + a_4 - a_5 = a_1 + a_4 \leq 1 \), which is impossible (since \( (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5)^+ = D_4 \)). Then \( c_4 \leq 0 \).

Assume \( c_6 \leq 0 \). Then \( a_5 > 0 \) (since \( a_6 > 0 \) implies \( c_7 < 0 \)), hence both \( c_2 \) and \( c_4 \) are \( \leq 0 \).

If \( a_3 \) was \( > 0 \), we would get \( c_1 > 0 \), \( a_2 > 0 \), \( a_6 > 0 \) (since \( a_5 = a_4 = c_2 + c_5 \leq 0 \), which is impossible (since \( (\sigma_1 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6)^+ = D_1 + D_6 \)). Then \( a_3 > 0 \), hence \( c_1 \leq 0 \) and \( a_4 > 0 \).

If \( a_5 \) was zero, since \( c_2 + c_3 + c_4 + c_5 = -a_6 \), all \( c_i \) would be \( < 0 \), for \( 2 \leq i \leq 5 \). This would imply \( a_5 \geq 2 \) (since \( -2a_2 + a_5 = c_2 + c_5 = 0 \), on the other hand \( c_8 \) would be \( > 0 \), which is impossible (since \( (\sigma_2 + \sigma_3 + 2\sigma_5 + \sigma_7)^+ = D_6 + D_8 \)). Then \( a_5 > 0 \), hence \( c_7 \leq 0 \) and \( a_5 \geq 2 \). Therefore, \( \gamma = D_6 - D_2 \).

Assume now \( c_6 \leq 0 \) and \( c_1 > 0 \). Then both \( c_2 \) and \( c_3 \) are \( < 0 \), hence \( a_4 > 0 \) and \( c_5 \leq 0 \). If \( a_5 \) was \( > 0 \), we would get \( c_7 \leq 0 \), \( a_5 > a_6 \), \( a_3 > 0 \) (since \( -2a_2 + a_5 - a_6 = c_2 + c_5 \leq 0 \)), this would imply \( a_1 \geq 2 \) hence \( c_1 \leq 1 \) (since \( (2\sigma_1 + \sigma_3 + \sigma_4)^+ = 2D_1 \)) and on the other hand \( a_2 \geq 2 \) hence \( c_8 > 0 \) (since \( c_1 + c_4 + c_6 = a_2 \)), but this is impossible (since \( (2\sigma_1 + 2\sigma_2 + \sigma_3 + \sigma_4 + 2\sigma_5 + \sigma_7)^+ = D_1 + D_6 \)). Then \( a_6 = 0 \), hence \( c_7 > 0 \) and \( c_8 > 0 \). Furthermore, since \( c_1 = 0 \) for all \( 2 \leq i \leq 5 \), \( a_3 > 0 \) and \( a_1, a_2, a_5 \geq 2 \). Therefore, \( \gamma = D_1 + D_8 - D_7 \).
Finally, assume both $c_1$ and $c_6 \leq 0$. Recall that $c_4 \leq 0$, therefore $c_8 \geq a_2 > 0$. Furthermore, $a_3 > 0, c_7 \leq 0, a_5 > 0, c_2 \leq 0, c_3 \leq 0$ and $a_4 > 0$. Then $a_2 \geq 2$, hence $c_8 \geq 2, a_7 \geq 2$ and $a_5 \geq 2$, which is impossible (since $(\sigma_2 + \sigma_3 + 2\sigma_5 + 2\sigma_7)^2 = 2D_3$).

3.6. Type $F_4$. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of spherical roots where $\sigma_i = \alpha_i + \alpha_{i+1}$.

**Proposition 3.6.** Let $\gamma \in \mathbb{N} \Delta$ be a covering difference in $\mathbb{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then either

1. $\gamma = \sigma_1 + \sigma_2 + 2\sigma_3 = D_4$, or
2. $\gamma = \sigma_1 + \sigma_3 + D_1 + D_4 - D_3$.

**Proof.** Set $\gamma = a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = c_1 D_1 + c_2 D_2 + c_3 D_3 + c_4 D_4$. Clearly, $a_1, a_3 > 0$.

Since $\sigma_2^2 = D_2, c_2 \leq 0$. Then $c_4 > 0$ (since $c_2 + c_4 = 1$), hence $a_3 > a_2$.

If $a_2 > 0$, $\gamma = D_4$. If $a_2 = 0, c_1 > 0$ and $\gamma = D_1 + D_4 - D_3$. □

3.7. Type $G_2$. Let $\Sigma = \{\sigma\}$ be the set of spherical roots where $\sigma = \alpha_1 + \alpha_2$.

**Proposition 3.7.** Let $\gamma \in \mathbb{N} \Delta$ be a covering difference in $\mathbb{N} \Delta$ with $\text{Supp}_S(\gamma) = S$. Then $\gamma = \sigma = D_2 - D_1$.

The proof is trivial.

4. Low fundamental triples for model wonderful varieties.

In this section we will classify all the low fundamental triples for the model wonderful varieties of simply connected type, and prove that these triples all satisfy the condition [11] of Lemma [7,4]. As in the previous section, we can restrict ourselves to the case of an almost simple group $G$. As we will see in the proof of Theorem A at the end of the section, it is enough to consider only low fundamental triples $(D, E, F)$ such that $\text{Supp}_S(D + E - F) = S$.

We keep the notation of the previous section. We denote by $H$ the stabilizer of a point $x_0$ in the open orbit of $M$ and by $H_0$ the intersection of the kernels of the multiplicative characters of $H$.

4.1. Type $A_r$.

**Lemma 4.1.** Let $(D_p, D_q, F)$ be a low fundamental triple with $\text{Supp}_S(D_p + D_q - F) = S$. Then $F = 0$ and $p + q = r + 1$. If moreover $r$ is odd, then $p$ and $q$ are even.

**Proof.** Notice that every fundamental triple is low: indeed, by Proposition [8,1] for every $\eta \in \mathbb{N} \Delta$ covering difference in $\mathbb{N} \Delta$ one has $h(\eta^+)$ $= 2$, hence every color is minimal in $\mathbb{N} \Delta$ w.r.t. $\leq \Sigma$. Therefore, we only need to compute the fundamental triples $(D_p, D_q, F)$ with $\text{Supp}_S(D_p + D_q - F) = S$.

Take a sequence

$$ F = F_n < \Sigma F_{n-1} < \Sigma \ldots < \Sigma F_0 = D_p + D_q $$

such that $F_i \in \mathbb{N} \Delta$ and $\gamma_i = F_{i-1} - F_i$ is a covering difference for every $i \leq n$. By Proposition [8,1] it follows that if $\gamma_i^+ = F_{i-1} + D_i$ (with $1 \leq i \leq r$) then $\gamma_i^- = F_i = D_{s+r} + D_{r+j}$ (with $j$ equal to 1 or 2).

Therefore, $\text{Supp}_S(\gamma_n) = \text{Supp}_S(D_p + D_q - F) = S, F = \gamma_n = 0$ and $q = r + 1 - p$.

If $r$ is odd, all the covering differences $\gamma_i = F_{i-1} - F_i$ are of type Proposition [8,1](2), then $p$ is even. □

**Proposition 4.2.** Let $r$ be odd and let $(D, E, F)$ be a low fundamental triple with $\text{Supp}_S(D + E - F) = S$. Then $s^{D+E-F}V_F \subset V_DV_E$.

**Proof.** By the previous lemma, $(D, E, F) = (D_p, D_{r+1-p}, 0)$ and $p$ is even (as well as $r + 1 - p$).

Set $\Delta_{\text{odd}} = \{D_i \in \Delta: i \text{ is odd}\}$; the subset $\Delta_{\text{odd}} \subset \Delta$ is distinguished and the quotient $M' = M/\Delta_{\text{odd}}$ is a symmetric wonderful variety (with spherical roots $\alpha_{2k-1} + 2\alpha_{2k} + \alpha_{2k+1}$).

By Corollary [1,4] together with the surjectivity of the multiplication map in the symmetric case

$$ \Gamma(M, \mathcal{L}_D) \Gamma(M, \mathcal{L}_E) = \Gamma(M', \mathcal{L}_D) \Gamma(M', \mathcal{L}_E) = \Gamma(M', \mathcal{L}_{D+E}) = \Gamma(M, \mathcal{L}_{D+E}) $$

□

**Proposition 4.3.** Let $r$ be even and let $(D, E, F)$ be a low fundamental triple with $\text{Supp}_S(D + E - F) = S$. Then $s^{D+E-F}V_F \subset V_DV_E$.

**Proof.** By Lemma [3,1] we have $F = 0$ and $V(\omega_E) \simeq V(\omega_D)^*$, hence $V(\omega_D)^* \oplus V(\omega_E)^* \simeq \text{End}(V(\omega_D))$.

If $r$ is even, the stabilizer $H$ of a point in the open $G$-orbit of $M$ is the normalizer in $G$ of $\text{Sp}(r)$ and in particular is reductive (see [32]). Therefore, the one-dimensional $H$-submodules of $V(\omega_D)^*$ and of $V(\omega_E)^*$ associated respectively with $D$ and $E$ are dual to each other, hence we may choose the $H$-eigenvectors $h_D \in V(\omega_D)^*$ and $h_E \in V(\omega_E)^* \simeq V(\omega_D)$ in such a way that $h_D(h_E) = 1$. If we complete
Proposition 4.4. There are no low triples \((D, E, F)\) with \(\sigma_{r-1} \in \text{Supp}_\Sigma(D + E - F)\).

Proof. Let \(\Delta = \{D_i \in \Delta : r - i \text{ is even}\}\)
\(\Delta^{\text{odd}} = \{D_i \in \Delta : r - i \text{ is odd}\}\)

If \(F < \Sigma D + E\) with \(\sigma_{r-1} \in \text{Supp}_\Sigma(D + E - F)\), then \(\text{Supp}(D + E) \cap \Delta^{\text{odd}} \neq \emptyset\). Indeed, if \(\text{Supp}(D + E)\) was included in \(\Delta^{\text{even}}\), take a sequence
\(F = F_0 < \Sigma F_{n-1} < \Sigma \ldots < \Sigma F_0 = D + E\)
of coverings in \(\mathbb{N}\Delta\): every covering difference \(\gamma_i = F_i-1 - F_i\) would necessarily be of type Proposition 3.3 (2), hence \(\sigma_{r-1} \notin \text{Supp}_\Sigma(D + E - F)\).

Let \(k < r\) be the maximum such that \(D_k \in \text{Supp}(E + F) \cap \Delta^{\text{odd}}\). Set \(D + E - F = \gamma = \sum_{i=1}^{r-1} a_i \sigma_i\).
Since \(c(\gamma, D_i) \leq 0\) for every \(k < i < r\) with \(r - i\) odd, as in the proof of Proposition 3.2 it follows that \(a_j \geq a_{r-1} > 0\) for every \(j \geq k\) with \(r - j\) odd.
Denote \(\gamma_0 = \sum_{i=0}^{(r-k-1)/2} \gamma_{k+2i} = -D_{k-1} + D_k\) and \(F' = D + E - \gamma_0\). Then \(F' \in \mathbb{N}\Delta\) and \(F \leq \Sigma F' < \Sigma D + E\). Set
\((D', E') = \begin{cases} (D - \gamma_0, E) & \text{if } D_k \in \text{Supp}(D) \\ (D, E - \gamma_0) & \text{otherwise} \end{cases}\)
then \(D' \leq \Sigma D, E' \leq \Sigma E\) and \(F \leq \Sigma D' + E' < \Sigma D + E\), hence \((D, E, F)\) is not a low triple. \(\square\)

4.3. Type \(\mathbb{C}_r\).

Proposition 4.5. There are no low triples \((D, E, F)\) with \(\sigma_{r-1} \in \text{Supp}_\Sigma(D + E - F)\).

Proof. Let \(D, E, F \in \mathbb{N}\Delta\) with \(F < \Sigma D + E\) and \(\sigma_{r-1} \in \text{Supp}_\Sigma(D + E - F)\). Denote \(k \leq r\) the maximum such that \(D_k \in \text{Supp}(D + E)\). Reasoning as in the proof of Proposition 3.3 it follows that \(\sigma_i \in \text{Supp}_\Sigma(D + E - F)\) for every \(k-1 \leq i < r\). Therefore, if we set \(\gamma_0 = \sum_{i=k-1}^{r-1} \gamma_i = -D_{k-2} + D_k\) and \(F' = D + E - \gamma_0\), then \(F' \in \mathbb{N}\Delta\) and \(F \leq \Sigma F' < \Sigma D + E\). Set
\((D', E') = \begin{cases} (D - \gamma_0, E) & \text{if } D_k \in \text{Supp}(E) \\ (D, E - \gamma_0) & \text{otherwise} \end{cases}\)
then \(D' \leq \Sigma D, E' \leq \Sigma E\) and \(F \leq \Sigma D' + E' < \Sigma D + E\), hence \((D, E, F)\) is not a low triple. \(\square\)

4.4. Type \(\mathbb{D}_r\).

Lemma 4.6. Let \((D_p, D_q, F)\) be a low fundamental triple with \(\text{Supp}_S(D_p + D_q - F) = S\). Then \(p, q, r\) are odd, \(p, q \leq r - 2\) and either
(1) \(p + q \leq r - 1\) with \(F = D_{p+q-2}\), or
(2) \(p + q = r + 1\) with \(F = D_{r-1} + D_r\).

Proof. Notice that, as in type A, every fundamental triple is low. Therefore, we only need to compute the fundamental triples \((D_p, D_q, F)\) with \(\text{Supp}_S(D_p + D_q - F) = S\). Assume \(p \leq q\).
Take a sequence
\(F = F_0 < \Sigma F_{n-1} < \Sigma \ldots < \Sigma F_0 = D_p + D_q\)
such that \(F_i \in \mathbb{N}\Delta\) and \(\gamma_i = F_{i-1} - F_i\) is a covering difference for every \(i \leq n\). Recall the classification of covering differences of Propositions 3.1 and 3.3
(1) \(p, q \leq r - 2\). If \(q\) was equal to \(r - 1\) then \((r - 1) - p\) should be non-zero and odd, thus \(\gamma_1 = -D_{p-1} + D_0 + D_{r-1} - D_r\) and \(F_1 = D_{p-1} + D_r\), which is impossible since the distance between the vertices \(r\) and \(p\) is even. By symmetry, the same argument works if \(q = r\).
(2) \(q - p\) is even. If \(q - p\) was odd, there would exist \(i\) such that \(F_i = D_{p'} + 2D_{q'}\) with \(q'\) equal to \(r - 1\) or \(r\) and \((r - 1) - p'\) even, which is impossible.
(3) \( r - p \) is even (as well as \( r - q \)). Here again if \( r - p \) and \( r - q \) were odd, there would exist \( i \) such that \( F_i \) is \( D_{q'} + 2D_{q''} \) with \( q' \) equal to \( r-1 \) or \( r \) and \( (r-1) - p' \) even.

Therefore, there exists \( i \) such that \( \gamma_j \) is of type Proposition 3.1(2) for every \( j < i \) and \( \gamma_i \) is either of type Proposition 3.1(1) or of type \( D \).

In the first case, \( F_{i-2} = D_{p'} + D_{q'} \) with \( q' = r - 2 \) and \( p' + q' = p + q \), \( F_{i-1} = D_{p'-2} + 2D_{q''} \) where \( q'' \) is equal to \( r-1 \) or \( r \). Then necessarily \( p'-2 = 1 \) and \( F_i = F_n = D_{r-1} + D_r \).

In the second case, \( F_{i-1} = D_{p'} + D_{q'} \) with \( p' + q' = p + q \) and \( F_i = D_{p'-1} + D_{p'-1} \). Then \( \gamma_i \) is of type Proposition 3.1(2) for every \( j > i \) until \( F_{i-1} = D_{p'-1-2k} + 2D_{q'-1+2k} \) where \( p'-1-2k = 2(k = n-1-i) \) and \( F_{i-n} \) is equal to either \( D_{p'-1+2k+2} \) if \( q'-1+2k \) is even or \( D_{p'-1+2k} \) if \( q'-1+2k = r-3 \).

**Proposition 4.7.** Let \((D, E, F)\) be a low fundamental triple with \( \text{Supp}_S(D + E - F) = S \), then \( s^{D + E - F} V_F \subset V_D V_E \).

**Proof.** We need an explicit computation.

Denote by \( U = \mathbb{C}^{2r} \) the first fundamental representation of \( G \) (that is, the standard representation of \( \text{SO}(2r) \)) and \( h \in S^2 U \) a \( G \)-invariant non-degenerate symmetric 2-form. If \( W \subset U \) is a maximal isotropic subspace, we get then a decomposition \( U = W \oplus W^* \). Fix a non-zero vector \( e_0 \in W \) and consider the corresponding decomposition \( U = V \oplus \mathbb{C} e_0 \oplus V^* \oplus \mathbb{C} e_0^* \), where \( V \subset W \) is a complement of the line \( \mathbb{C} e_0 \) and where \( e_0^* \in W^* \) is defined by \( e_0^*(e_0) = 1 \), \( e_0^*|_V = 0 \).

Then (see [22]) the Lie algebra \( \mathfrak{h}_0 \) of \( H_0 \) can be described as

\[ \mathfrak{h}_0 = \mathfrak{sp}(V) \oplus V^* \oplus \text{Skew}(V, V^*), \]

where \( \text{Skew}(V, V^*) \subset \text{Hom}(V, V^*) \) denotes the subspace of skew-symmetric linear maps and where \( \mathfrak{h}_0 \) is embedded in \( \mathfrak{so}(U) \) as follows (here we denote by \( u = (e, \lambda e_0, \psi, \mu e_0) \) a generic element in \( U = V \oplus \mathbb{C} e_0 \oplus V^* \oplus \mathbb{C} e_0^* \)):

- if \( f \in \mathfrak{sp}(V) \), then \( f(u) = f(v) + f \cdot \psi \),
- if \( \phi \in V^* \), then \( \phi(u) = \phi(v)e_0 - (\lambda + \mu)\phi(\psi)e_0^* \),
- if \( \Phi \in \text{Skew}(V, V^*) \), then \( \Phi(u) = \Phi(v) \).

As already mentioned at the beginning of Section 3 every simple \( G \)-module possesses a unique \( \mathfrak{h}_0 \)-invariant element. In particular, if we denote \( h_1 = e_0 - e_0^* \in U \) and \( h_2 \in \Lambda^2 V^* \) a \( \mathfrak{sp}(V) \)-invariant non-degenerate 2-form, then \( h_1 \in U^{\mathfrak{h}_0} \) and \( h_2 \in (\Lambda^2 U)^{\mathfrak{h}_0} \). In this way we may describe the \( \mathfrak{h}_0 \)-invariant vectors in every exterior power \( \Lambda^k U \) with \( i \leq r - 1 \). Set indeed

\[ h_i = \begin{cases} h_2^{2k} & \text{if } i = 2k \text{ is even} \\ h_1 \wedge h_2^{2k} & \text{if } i = 2k + 1 \text{ is odd} \end{cases} \]

then \( h_i \in (\Lambda^k U)^{\mathfrak{h}_0} \).

Set \( \omega_0 = 0 \) and recall that if \( i \geq 0 \) then

\[ \Lambda^k U = \begin{cases} V(\omega_i) & \text{if } i \leq r - 2 \\ V(\omega_{r-1} + \omega_i) & \text{if } i = r - 1 \end{cases} \]

To conclude the proof, by Lemma 4.8, we only need to show that, if \( i, j \) are odd integers with \( i + j \leq r + 1 \), then there exists an equivariant projection \( \pi : \Lambda^k U \otimes \Lambda^j U \to \Lambda^{i+j-2} U \) such that \( \pi(h_i \otimes h_j) \neq 0 \). Define \( \pi \) as follows:

\[ \pi\left((u_1 \wedge \cdots \wedge u_i) \otimes (w_1 \wedge \cdots \wedge w_j)\right) = \sum_{h,k} (-1)^{h+k} b(u_h, w_k) u_1 \wedge \cdots \wedge \hat{u}_h \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge w_j. \]

Suppose that \( i, j \) are odd with \( i + j \leq r + 1 \) and set \( k = (i + j - 2)/2 \). Notice that \( \pi(h_i \otimes h_j) = b(h_1, h_1) h_2^{2k} + q \), where \( q \in \Lambda^{i+j-2} U \) is linearly independent with \( h_2^{2k} \): since \( b(h_1, h_1) \neq 0 \) and since \( h_2^{2k} \neq 0 \), it follows then \( \pi(h_i \otimes h_j) \neq 0 \).

4.5. Type \( E_r \).

**Lemma 4.8.** The low fundamental triples \((D, E, F)\) with \( \text{Supp}_S(D + E - F) = S \) are the following:

- If \( r = 6 \): \( (D_1, D_3, D_2) (D_1, D_5, D_3) (D_1, D_6, 0) (D_1, D_6, D_5) (D_1, D_6, D_2) \),
- If \( r = 7 \): \( (D_1, D_6, D_3) (D_6, D_6, D_2 + D_7) \),
- If \( r = 8 \): \( (D_1, D_1, D_2) (D_1, D_5, 2D_2) (D_1, D_7, D_3) (D_1, D_8, D_7) (D_3, D_8, D_5) (D_5, D_8, D_2 + D_7) (D_7, D_8, D_2) \).

After Proposition 3.5, the proof of this lemma is a quite long but trivial case-by-case computation, which we do not report here.
Proposition 4.9. Let \((D, E, F)\) be a low fundamental triple with \(\text{Supp}_S(D + E - F) = S\), then \(s^{D + E - F}V_F \subset V_DV_E\).

Let us first deal with the triple \((D_1, D_0, 0)\) of \(E_6\). The set \(\Delta_0 = \{D_2, D_3, D_4, D_5\}\) is distinguished and the quotient \(M' = M/\Delta_0\) is a symmetric wonderful variety with spherical roots \(\{2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_8\}\). Therefore, we can conclude by Corollary 4.4 and the surjectivity of the multiplication map in the symmetric case.

For all the other triples of Lemma 4.8 we have to use Lemma 4.2. Since the dimension of the involved representations is quite high, we have used the computer and, more precisely, GAP [19], a software for computations which contains built-in functions to construct and deal with representations of simple Lie algebras (see also [17]).

Although the dimension of some of the involved representations is very high, we have succeeded to make the computation accessible with a currently available home computer. A quite convenient tool is the quadratic Casimir operator \(c\), which acts as the scalar \((\lambda + 2\mu, \lambda)\) on every irreducible representation \(V(\lambda)\). Let \((D, E, F)\) be a low fundamental triple, once the irreducible representations \(V^*_D\) and \(V^*_E\) are constructed, and the vectors \(h_D\) and \(h_E\) are explicitly found, it is typically enough to project \(h_D \otimes h_E\) onto the eigenspace of \(c\) relative to \((\omega_+^*_F + 2\rho, \omega_+^*_F)\), and there is no need to construct the whole tensor product \(V_D \otimes V_E\).

Sometimes for \(E_8\) the dimension is so high that it is even quite costly to construct the irreducible representation \(V_D\) itself \((V_{D_8}\) has dimension 146,325,270). Here we use a further escamotage. The set \(\Delta_0 = \{D_1, D_4, D_5, D_6\}\) is distinguished and the quotient \(M' = M/\Delta_0\) is the parabolic induction of a symmetric wonderful \(\text{SL}(8)\)-variety with spherical roots \(\{\alpha_1 + 2\alpha_3 + \alpha_4, \alpha_2 + 2\alpha_5 + \alpha_6, \alpha_2 + 2\alpha_7 + \alpha_8\}\). This means that, if \(D\) equals \(D_3, D_5\) or \(D_7\), then \(h_D\) can be chosen to be the \(\text{Sp}(8)\)-invariant vector in the simple \(\text{SL}(8)\)-submodule \(W_D \subset V_D\) of highest weight \(\omega_D\). Furthermore, \(V_{D_3}\) and \(V_{D_5}\) are still accessible (the former has dimension 3875 and the latter is the adjoint representation), \(V_{D_3} \subset \Lambda^2 V_{D_4}, V_{D_5} \subset \Lambda^2 V_{D_6}, V_{D_7} \subset \Lambda^2 V_{D_8}\), and respectively \(W_{D_3} \subset \Lambda^3 V_{D_4}, W_{D_5} \subset \Lambda^4 V_{D_6}, W_{D_7} \subset \Lambda^4 V_{D_8}\). Therefore, the vectors \(h_D\) can be explicitly determined if \(D\) equals \(D_1, D_3, D_5, D_7\) or \(D_8\), and notice that this is enough to treat all the above triples.

4.6. Type \(F_4\).

Lemma 4.10. The only low fundamental triple \((D, E, F)\) with \(\text{Supp}_S(D + E - F) = S\) is \((D_1, D_4, D_3)\).

The proof is trivial, after Proposition 4.6.

Proposition 4.11. \(s^{D_1 + D_4 - D_3}V_{D_3} \subset V_{D_1}V_{D_4}\).

This can easily be checked via computer.

4.7. Type \(G_2\).

Proposition 4.12. There are no low triples \((D, E, F)\) with \((D + E - F) \neq 0\).

The proof is trivial.

4.8. Projective normality of model wonderful varieties. We are now ready to prove that the multiplication of sections on a model wonderful variety of simply connected type is surjective.

A localization of a wonderful variety \(M\) is a \(G\)-stable subvariety of \(M\), which is a wonderful \(G\)-variety by itself. Notice that we have a bijective correspondence between localizations of \(M\) and subsets of \(\Sigma\). More precisely, for all subsets \(\Sigma' \subset \Sigma\), the intersection of the boundary divisors \(M^\sigma\) for \(\sigma \in \Sigma'\) gives a wonderful variety with \(\Sigma \setminus \Sigma'\) as set of spherical roots.

Proof of Theorem A. By the classification of the covering differences given in the previous section, it follows that the model wonderful varieties of simply connected type satisfy the property (2-ht). Hence by Lemma 2.4 it is enough to show the inclusion \(s^{D + E - F}V_F \subset V_DV_E\) for all the low fundamental triples \((D, E, F)\). Here we only need to show that we can reduce to the low fundamental triples \((D, E, F)\) such that \(\text{Supp}_S(D + E - F) = S\), whose case has been proved above in this section.

Let \((D, E, F)\) be a low fundamental triple, denote \(\gamma = D + E - F\) and \(S' = \text{Supp}_S(\gamma)\) and suppose that \(S' \neq S\). Let \(Q\) be the parabolic subgroup associated to \(S'\) and set \(G' = Q/R_Q\). Since \((D, E, F)\) is a low triple, it follows that \(S'\) is connected, hence \(G'\) is an almost simple group. Consider the localization \(M'\) of \(M\) with \(\Sigma' = \{\sigma \in \Sigma : \text{Supp}_S(\sigma) \subset S'\}\) as set of spherical roots. Then \(M'\) is the parabolic induction of the model wonderful \(G'\)-variety \(N\) of simply connected type.
The restrictions \( \text{Pic}(M) \to \text{Pic}(M') \) and \( \text{Pic}(M') \to \text{Pic}(N) \) identify \( \text{Pic}(N) \) with a sublattice of \( \text{Pic}(M) \). More precisely, if \( \Delta \subset \Delta' \) is the set of colors \( D' \) such that \( \omega_D = \omega_{
abla} \) for some \( \alpha \in S' \), then \( \text{Pic}(N) \) is identified with \( \mathbb{Z} \Delta' \). Moreover, an element of \( \mathbb{Z} \Delta' \) induces a globally generated line bundle on \( N \) if and only if its coefficients are non-negative. If \( \tilde{D} \in \mathbb{N} \Delta, \) set \( W_{\tilde{D}} = V_{\tilde{D}}^\mathbb{Q} \).

Notice that \( D, E \in \Delta \) and consider the difference \( \tilde{F} = D + E - \gamma \in \mathbb{Z} \Delta \), where \( \gamma \) is regarded as sum of spherical roots of \( N \) as an element of \( \mathbb{Z} \Delta \). Since \( (D, E, F) \) is a low triple in \( \mathbb{N} \Delta \), it follows that \( (D, E, F) \) is a low triple in \( \mathbb{N} \Delta \). Since \( \text{Supp}_\gamma' = S' \), we have that \( s^\gamma W_{\tilde{F}} \subset W_D W_E \), therefore \( \text{Proposition } \text{[1.14]} \) implies that \( s^\gamma V_{\tilde{F}} \subset V_D V_E \) with respect to the multiplication map \( m_{D, E}^* : \Gamma(M', \mathcal{L}_D) \otimes \Gamma(M', \mathcal{L}_E) \to \Gamma(M', \mathcal{L}_{D+E}) \). This concludes the proof, since the latter is just the restriction of the multiplication map \( m_{D, E} : \Gamma(M, \mathcal{L}_D) \otimes \Gamma(M, \mathcal{L}_E) \to \Gamma(M, \mathcal{L}_{D+E}) \).

Let \( M \) be a wonderful variety and let \( N \) be a quotient of \( M \). Then the pull-back of line bundles identifies \( \text{Pic}(N) \) with a sublattice of \( \text{Pic}(M) \) and a line bundle \( \mathcal{L} \in \text{Pic}(N) \) is generated by global sections if and only if its pull-back (which we still denote by \( \mathcal{L} \)) is. Moreover, by Corollary [4.3] we have \( \Gamma(N, \mathcal{L}) = \Gamma(M, \mathcal{L}) \). It follows that if \( \mathcal{L}, \mathcal{L}' \in \text{Pic}(N) \) are generated by global sections and if the multiplication \( \Gamma(M, \mathcal{L}) \otimes \Gamma(M, \mathcal{L}') \to \Gamma(M, \mathcal{L} \otimes \mathcal{L}') \) is surjective, then the multiplication \( \Gamma(N, \mathcal{L}) \otimes \Gamma(N, \mathcal{L}') \to \Gamma(N, \mathcal{L} \otimes \mathcal{L}') \) is also surjective.

Let now \( N \) be a localization of \( M \) and let \( \mathcal{L}, \mathcal{L}' \in \text{Pic}(M) \) be generated by global sections. Then the restriction of sections to \( N \) is surjective, therefore if the multiplication \( \Gamma(M, \mathcal{L}) \otimes \Gamma(M, \mathcal{L}') \to \Gamma(M, \mathcal{L} \otimes \mathcal{L}') \) is surjective, then the multiplication \( \Gamma(N, \mathcal{L}) \otimes \Gamma(N, \mathcal{L}') \to \Gamma(N, \mathcal{L} \otimes \mathcal{L}') \) is also surjective. If moreover \( M \) is a model wonderful variety of simply connected type, then the restriction of line bundles induces an identification \( \text{Pic}(N) = \text{Pic}(M) \) for every localization \( N \subset M \), and the line bundles generated by global sections are also identified. The same is true whenever \( M' \) is the quotient of a localization of a model wonderful variety of simply connected type and \( N' \) is a localization of \( M' \). Therefore proceeding inductively we get the following corollary of Theorem A.

**Corollary 4.13.** Let \( N \) be a wonderful variety obtained by a model wonderful variety of simply connected type via operations of localization and quotient by distinguished sets of colors. Then the multiplication map

\[
m_{\mathcal{L}, \mathcal{L}'} : \Gamma(N, \mathcal{L}) \otimes \Gamma(N, \mathcal{L}') \to \Gamma(N, \mathcal{L} \otimes \mathcal{L}')
\]

is surjective for all the line bundles \( \mathcal{L}, \mathcal{L}' \in \text{Pic}(N) \) generated by global sections.

5. Projective normality of comodel wonderful varieties.

Motivated by an application that we will illustrate below in Section 8 here we will study the surjectivity of the multiplication of sections in another class of wonderful varieties, which we call **comodel**.

Let \( G \) be simply connected of simply-laced type, and fix \( T \) and \( B \) as usual. Let \( M \) be a model wonderful variety of \( G \), with set of colors \( \Delta \) (in bijection with the set of fundamental weights), set of spherical roots \( \Sigma \) and Cartan pairing \( c : \Delta \times \Sigma \to \mathbb{Z} \). Let \( G^\vee \) be a simply connected group whose root system is isomorphic to \( \Phi_\Sigma \), the root system generated by \( \Sigma \). Once fixed \( T^\vee \) and \( B^\vee \), its set of simple roots \( S^\vee \) is thus in bijective correspondence with \( \Sigma \). Then there exists a comodel wonderful variety of \( G^\vee \), that is, a wonderful variety \( M^\vee \) for the action of the group \( G^\vee \) whose set of spherical roots \( \Sigma^\vee \) is equal to the set of simple roots \( S^\vee \) of \( G^\vee \), its set of colors \( \Delta^\vee \) is in bijective correspondence with \( \Delta \) and, under these correspondences, its Cartan pairing \( c^\vee : \Delta^\vee \times \Sigma^\vee \to \mathbb{Z} \) equals the Cartan pairing \( c \) of the model wonderful variety \( M \). In this case the type of \( G \) will be called the **cotype** of \( G^\vee \).

Forgetting the model wonderful variety, in the following \( M \) will be a comodel wonderful variety of \( G \), \( H \) will denote the stabilizer of a point \( x_0 \) in the open orbit of \( M \), and \( H_0 \) the kernel of its multiplicative characters; \( h_0 \) will denote the Lie algebra of \( H_0 \) in the Lie algebra \( \mathfrak{g} \) of \( G \).

The comodel wonderful varieties correspond to the following cases in [4].

- Cotype \( A_{2m} \): type \( A_{m-1} \times A_m \), case S-5.
- Cotype \( A_{2m+1} \): type \( A_m \times A_{m+1} \), case S-4.
- Cotype \( D_{2m} \): type \( A_{m-1} \times D_m \), case S-10.
- Cotype \( D_{2m+1} \): type \( A_{m-1} \times D_{m+1} \), case S-11.
- Cotype \( E_6 \): type \( A_5 \), case S-50.
- Cotype \( E_7 \): type \( A_6 \), case S-49.
- Cotype \( E_8 \): type \( D_7 \), case S-58.
An explicit description of the corresponding subgroups $H$ can be found in [5].

The colors of $M$ will be enumerated as in the corresponding model wonderful variety. Similarly, we denote by $h_i \in V^{\ast}_i$ the $H$-seminvariant associated to $D_i$. This vector will be invariant under $H_0$.

More explicitly, set the map $\omega$ as follows.

- Cotype $A_{2m}$: $\omega(D_1) = \omega'_1$, $\omega(D_2) = \omega_1 + \omega'_1$ for $i < m$, $\omega(D_{2i-1}) = \omega_i + \omega'_i$ for $i > 1$, $\omega(D_{2m}) = \omega'_m$.

- Cotype $A_{2m+1}$: $\omega(D_1) = \omega_1$, $\omega(D_2) = \omega_1 + \omega'_1$, $\omega(D_{2i-1}) = \omega_i + \omega'_i$ for $1 < i < m$, $\omega(D_{2m}) = \omega'_m$.

- Cotype $D_{2m}$: $\omega(D_1) = \omega_1$, $\omega(D_{2i-1}) = \omega_i + \omega'_i$ for $1 < i < m$, $\omega(D_{2m}) = \omega'_m$.

- Cotype $D_{2m+1}$: $\omega(D_1) = \omega'_1$, $\omega(D_2) = \omega_1 + \omega'_1$, $\omega(D_{2i-1}) = \omega_i + \omega'_i$ for $1 < i < m$, $\omega(D_{2m}) = \omega'_m$, $\omega(D_{2m+1}) = \omega'_m$.

- Cotype $E_6$: $\omega(D_1) = \omega_2$, $\omega(D_2) = \omega_3$, $\omega(D_3) = \omega_2 + \omega_3$, $\omega(D_4) = \omega_1 + \omega_4$, $\omega(D_5) = \omega_1 + \omega_5$, $\omega(D_6) = \omega_1$.

- Cotype $E_7$: $\omega(D_1) = \omega_2$, $\omega(D_2) = \omega_3$, $\omega(D_3) = \omega_2 + \omega_3 + \omega_6$, $\omega(D_4) = \omega_2 + \omega_4 + \omega_6$, $\omega(D_5) = \omega_2 + \omega_5$, $\omega(D_6) = \omega_1 + \omega_5$, $\omega(D_7) = \omega_1 + \omega_7$, $\omega(D_8) = \omega_7$.

Since the Cartan pairing of the comodel wonderful varieties is the same as that of the model varieties, the classification of the covering differences is also the same and the property $(2,ht)$ holds. As we will see at the end of the section, in order to apply Lemma 2.4 it is enough to test the surjectivity on the same low fundamental triples arising in Section 4 in the model case.

In the computations below we use the following conventions. We denote by $e_i, \ldots, e_n$ the standard basis of $V = \mathbb{C}^n$ and by $\varphi_1, \ldots, \varphi_n$ the dual basis. We also denote by $e_i \wedge e_j \wedge \cdots e_k$ the vector $e_i \wedge e_j \wedge \cdots e_k \in \Lambda^k V$ and similarly for $\varphi_i \varphi_j \varphi_k$. On $V \otimes V^*$ is defined the symmetric bilinear form $(u, \varphi) = (v, \psi)$.

For a vector space $U$ we have contraction maps $\kappa^{ij}_U: \Lambda^i U \otimes \Lambda^j U^* \rightarrow \Lambda^{i+j} U$ given by

$$\kappa^{ij}_U(u_1 \wedge \cdots \wedge u_i \otimes \varphi_1 \wedge \cdots \wedge \varphi_j) = \sum_{k,l} (-1)^{k+l} \varphi_l(u_k) u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge u_i \otimes \varphi_1 \wedge \cdots \wedge \hat{\varphi_l} \wedge \cdots \wedge \varphi_j.$$

In particular we set $\kappa^{ij}_U = \kappa^{ij}_{\Lambda^1}$.

Similarly, for a vector space $U$ with a symmetric bilinear form $(,)$ we have $\tilde{\kappa}^{ij}_U: \Lambda^i U \otimes \Lambda^j U \rightarrow \Lambda^{i+j-2} U$ such that

$$\tilde{\kappa}^{ij}_U(u_1 \wedge \cdots \wedge u_i \otimes v_1 \wedge \cdots \wedge v_j) = \sum_{k,l} (-1)^{k+l} (u_k, v_l) u_1 \wedge \cdots \wedge \hat{u}_k \wedge \cdots \wedge u_i \wedge v_1 \wedge \cdots \wedge \hat{v}_l \wedge \cdots \wedge v_j.$$

5.1. Cotype $A_r$. We test the triples of Lemma 4.4 $(D_p, D_{r+1-p}, 0)$ where, if $r$ is odd, $p$ is even.

If $r$ is odd, the set $\Delta_{odd}$ of odd-indexed colors is distinguished and the quotient is a symmetric wonderful variety, so the surjectivity follows as in Proposition 4.2.

If $r$ is even, $H$ is reductive and the surjectivity follows as in Proposition 4.3.

5.2. Cotype $D_r$. We test the triples of Lemma 4.6 $(D_p, D_q, F)$ where $p, q, r$ are odd, $p, q \leq r - 2$ and either

- $p + q \leq r - 1$ with $F = D_{p+q-2}$, or
- $p + q = r + 1$ with $F = D_{r-1} + D_r$.

Let $m > 1$ and set $r = 2m + 1$. Let $V = \mathbb{C}^m$, and set $W = V \oplus \mathbb{C} e \oplus V^* \oplus \mathbb{C} \varepsilon$. On $W$ is defined a quadratic form such that $V$ and $V^*$ are orthogonal to $e$ and $\varepsilon$, moreover $(e, \varepsilon) = 0$.
Let $G = \text{SL}(V) \times \text{Spin}(W)$. Notice that we have a natural immersion of $\text{SL}(V)$ in $\text{Spin}(W)$. Consider the action of $\Lambda^2 V$ on $W$ which is zero on $V$, $e$ and $\varepsilon$ and whose action on $V^*$ is given by the identification of $\Lambda^2 V$ with the antisymmetric maps from $V^*$ to $V$. Consider also the action of $V$ on $W$ given as follows: if $v, u \in V$ and $\varphi \in V^*$ then
\[
v \cdot u = 0 \quad v \cdot e = v \quad v \cdot \varphi = -\varphi(v)(e + \varepsilon) \quad v \cdot \varepsilon = v.
\]
This defines an embedding of $\mathfrak{h}_0 = \mathfrak{sl}(V) \oplus \Lambda^2 V \oplus V$ into $\mathfrak{g}$ such that $\mathfrak{h}_0$ turn to be a subalgebra of $\mathfrak{g}$.

Let now $D_i = D_i$ for $i \neq 0$ for $i \neq 2m$ while let $D_{2m} = D_{2m} + D_{2m+1}$. We have
\[
V^*_i = \Lambda^i V^* \otimes \Lambda^i W \quad V^*_{2i+1} = \Lambda^i V^* \otimes \Lambda^{i+1} W.
\]
The $\mathfrak{h}_0$-invariants in $V^*_{2i+1}$ are generated by the vector $h_{2i+1}$, corresponding to the identity element in $\Lambda^i V^* \otimes \Lambda^i V \subset \Lambda^i V^* \otimes \Lambda^i W$. Similarly, the $\mathfrak{h}_0$-invariants in $V^*_i$ are generated by the vector $h_{2i+1} = h_{2i} \wedge (e - \varepsilon)$. So that we have
\[
h_{2i} = \sum_{j_1 < \cdots < j_i} \varphi_{j_1 \cdots j_i} \otimes e_{j_1 \cdots j_i} \quad \text{and} \quad h_{2i+1} = \sum_{j_1 < \cdots < j_i} \varphi_{j_1 \cdots j_i} \otimes (e_{j_1 \cdots j_i} \wedge (e - \varepsilon)).
\]
Finally if $p = 2t + 1$ and $q = 2s + 1$ then the projection $\Phi: V^*_{2p} \otimes V^*_{2q} \rightarrow V^*_{2p+q-2}$ is given by
\[
\Phi((x \otimes w) \otimes (y \otimes z)) = x \wedge y \otimes h_{2t+1}^{t+1+1}(w \otimes z)
\]
for $x \in \Lambda^i V^*$, $y \in \Lambda^j V^*$, $w \in \Lambda^{i+1} W$ and $z \in \Lambda^{j+1} W$. A direct computation shows that
\[
\Phi(h_{2p} \otimes h_{2q}) = (-1)^{t+s+1} 2\binom{t+s}{t} h_{2p+q-2}.
\]

### 5.3. Cotype $E_6$

Let $V = \mathbb{C}^3$ and $W = V \oplus V^*$. Let $\mathfrak{g} = \mathfrak{sl}(W)$ and $\mathfrak{h}_0 = \mathfrak{sl}(V) \otimes \mathbb{C}^2 V$ where the action of $\mathfrak{sl}(V)$ is the natural one and the action of $\mathbb{C}^2 V$ is given by $b \cdot (v, \varphi) = (b(\varphi), 0)$ while the action on $W^* = V^* \oplus V$ is given by $b \cdot (v, \varphi, e) = (0, -b(\varphi))$.

We analyze the triples of Lemma 5.2 (2) $(D_1, D_3, D_2)$, $(D_4, D_5, D_3)$, $(D_4, D_6, 0)$, $(D_3, D_6, D_3)$, $(D_3, D_6, D_2)$.

The set of colors $\{D_2, D_4, D_1, D_3\}$ is distinguished and the associated quotient is a symmetric wonderful variety, so the third triple follows as in Proposition 5.2 Therefore, by symmetry it is enough to analyze the last two triples. We need to compute $h_3$, $h_5$, $h_6$.

We have $V^*_{2n} = \Lambda^2 V$. Looking at the action of $\mathfrak{sl}(V)$ we find only one invariant in $V \otimes V^* \subset \Lambda^2 V$ corresponding to the identity in $\text{End}(V^*)$. Hence we get $h_6 = e_1 \wedge \varphi_1 + e_2 \wedge \varphi_2 + e_3 \wedge \varphi_3$.

The representation $V^*_{2n}$ is contained in $\Lambda^2 W \otimes W^*$ and it is the kernel of $\kappa_W$. It contains two invariants under the action of $\mathfrak{sl}(V)$: $x \in \Lambda^2 V \otimes V \simeq \text{End}(V)$ and $y \in \Lambda^2 V^* \otimes V^* \simeq \text{End}(V^*)$. Notice that $\Lambda^2 W \otimes W^* \simeq V^*_{2n} \otimes W$ so both $x, y \in V^*_{2n}$. Moreover the action of $\mathbb{C}^2 V$ on $x$ is clearly trivial while it is not on $y$. So we have $h_3 = e_{13} \otimes \varphi_3 - e_{12} \otimes \varphi_2 + e_{23} \otimes \varphi_1$.

Similarly we have $V^*_{2n} \simeq \ker \kappa_W \subset \Lambda^2 W^* \otimes W$ and $h_3 = e_{12} \otimes e_3 - e_{13} \otimes e_2 + e_{23} \otimes e_1$.

### 5.3.1. Analysis of the triple $(D_5, D_6, D_2)$

We have $V^*_{2n} = \Lambda^2 V$. Consider the map $\Phi: \Lambda^2 W \otimes \Lambda^2 W \otimes W^* \rightarrow \Lambda^2 W$ given by
\[
\Phi(x \otimes y \otimes \varphi) = \kappa_W(x \otimes \varphi) \wedge y
\]
for $x, y \in \Lambda^2 W$ and $\varphi \in W^*$. A direct computation shows that $\Phi(h_6 \otimes h_5) = 3e_{123} \neq 0$.

### 5.3.2. Analysis of the triple $(D_3, D_6, D_5)$

Consider the map $\Phi: \Lambda^2 W \otimes \Lambda^2 W \otimes W^* \rightarrow \Lambda^2 W$ given by
\[
\Phi((x \otimes y) \otimes (\varphi \otimes \psi) \otimes w) = (\kappa_W(x \otimes y \otimes \varphi) \otimes w) \otimes \psi - (\kappa_W(x \otimes y \otimes \psi) \otimes w) \otimes \varphi
\]
for $x, y, w \in W$ and $\varphi, \psi \in W^*$. A direct computation shows that $\Phi(h_6 \otimes h_5) = 2h_5$.

### 5.4. Cotype $E_7$

Let $V = \mathbb{C}^3$ and set $W = V \oplus V^* \oplus \mathbb{C} e$, $\mathfrak{h}_0 = \mathfrak{sl}(V) \otimes \mathfrak{S}^2 V \oplus \Lambda^2 V \oplus V$ and $\mathfrak{g} = \mathfrak{sl}(W)$. Recall that we identify $\mathfrak{S}^2 V \oplus \Lambda^2 V$ with the decomposition of $\text{Hom}(V^*, V)$ into symmetric and antisymmetric matrices. We have an action of $\mathfrak{h}_0$ on $W$ given as follows. Let $(a, b, \omega, u) \in \mathfrak{h}_0$ and $(v, \varphi, \lambda) \in W$ then
\[
a \cdot v = a(v) \quad b \cdot v = 0 \quad \omega \cdot v = 0 \quad u \cdot v = 0
\]
\[
a \cdot \varphi = -a'(\varphi) \quad b \cdot \varphi = b(\varphi) \quad \omega \cdot \varphi = \omega(\varphi) \quad u \cdot \varphi = 0
\]
\[
a \cdot e = 0 \quad b \cdot e = 0 \quad \omega \cdot e = \gamma(\omega) \quad u \cdot e = u
\]
where $\gamma = \gamma_2: \Lambda^2V \rightarrow V^*$ is defined as above. This defines an immersion of $\mathfrak{h}_0$ into $\mathfrak{sl}(W)$ whose image is closed under the Lie bracket. Indeed, if $(a, b, \omega, u), (a', b', \omega', u') \in \mathfrak{h}_0$ then:

$$
[a, a'] = aa' - a'a; \quad [a, b] = ab - ba; \quad [a, \omega] = a\omega + \omega a; \quad [a, u] = a(u); \quad [b, b'] = 0; \quad [b, \omega] = b(\gamma(\omega)); \quad [b, u] = 0; \quad [\omega, \omega'] = 0; \quad [\omega, u] = 0; \quad [u, u'] = 0.
$$

It is also useful to write the action on $W^*$ which is given as follows. We have $W^* = V^* \oplus V \oplus \mathbb{C}e^*$, then

$$
a \cdot \varphi = -a'(\varphi) \quad b \cdot \varphi = -b(\varphi) \quad \omega \cdot \varphi = \omega(\varphi) \quad u \cdot \varphi = -\varphi(u)e^* \\
a \cdot v = a(v) \quad b \cdot v = 0 \quad \omega \cdot v = -\varphi(u)e^* \quad u \cdot v = 0 \\
a \cdot e^* = 0 \quad b \cdot e^* = 0 \quad \omega \cdot e^* = \gamma(\omega) \quad u \cdot e^* = u
$$

The triples of Lemma 4.8 are $((D_1, D_6), D_3)$ and $((D_0, D_6), D_2 + D_7)$.

5.4.1. Computation of $h_1$. The representation associated to $D_1$ is $\Lambda^4W$. Looking at the action of $\mathfrak{sl}(V)$ we get three invariants: the vector $x \in \Lambda^2V \oplus \Lambda^2\mathbb{C}e$ corresponding to the identity in $\text{End}(\Lambda^2V)$, the vector $y = e_{123} \wedge c \in \Lambda^3V \otimes \mathbb{C}e$ and the vector $z = \varphi_{123} \wedge c \in \Lambda^3\mathbb{V}^* \otimes \mathbb{C}e$. Hence the invariant is a linear combination of these vectors. A small computation shows that

$$
h_1 = 2y - x = 2e_{123} \wedge c - e_{12} \wedge \varphi_{12} - e_{13} \wedge \varphi_{13} - e_{23} \wedge \varphi_{23}.
$$

5.4.2. Computation of $h_6$. The representation associated to $D_6$ is the kernel of the map $\kappa_W$ in $\Lambda^2W \otimes W^*$. In $\Lambda^4W \otimes W^*$ there are five invariant vectors under $\mathfrak{sl}(V)$: $x_1 \in \Lambda^2V \otimes V$, $x_2 \in \Lambda^2V^* \otimes V^*$, $y_1 \in (V \wedge \mathbb{C}e) \otimes V^*$, $y_2 \in (V^* \wedge \mathbb{C}e) \otimes V$ and $z \in (V \wedge V^*) \otimes \mathbb{C}e^*$. A small computation shows that

$$
h_6 = 2x_1 + z = 2e_{12} \otimes \varphi_2 + 2e_{23} \otimes \varphi_3 + (e_1 \wedge \varphi_1) \otimes e^* + (e_2 \wedge \varphi_2) \otimes e^* + (e_3 \wedge \varphi_3) \otimes e^*.
$$

5.4.3. Analysis of the wedge product $W \otimes \Lambda^3V \rightarrow \Lambda^3V^*$. We consider the map $\Psi: \Lambda^4W \otimes \Lambda^4W \otimes W^* \rightarrow W \otimes \Lambda^4W$ given by

$$
\Phi \left( u \otimes (v_1 \wedge v_2) \otimes \varphi \right) = v_1 \otimes (\kappa_W^4(u \otimes \varphi) \wedge v_2) - v_2 \otimes (\kappa_W^4(u \otimes \varphi) \wedge v_1).
$$

A direct computation shows that

$$
\Phi(h_1 \otimes h_6) = -4\left( e_1 \otimes (e_{123} \wedge \varphi_1) + e_2 \otimes (e_{123} \wedge \varphi_2) + e_3 \otimes (e_{123} \wedge \varphi_3) \right) \neq 0.
$$

5.4.4. Analysis of the wedge product $\Lambda^3W \otimes W^* \otimes W^* \rightarrow \Lambda^3W$. We consider the map $\Psi: (\Lambda^2W \otimes W^*) \otimes (\Lambda^2W \otimes W^*) \rightarrow \Lambda^3W \otimes W^*$ given by

$$
\Psi \left( (u \otimes \varphi) \otimes (v \otimes \psi) \right) = (u \otimes (\kappa_W(v \otimes \varphi))) \otimes \psi.
$$

A direct computation shows that

$$
\Psi(h_6 \otimes h_6) = -6e_{123} \otimes e^* \in V_{D_2 + D_7}.
$$

5.5. Cotype $E_8$. Let $V = \mathbb{C}^4$, and set $W = V \otimes \Lambda^2V \otimes V^* = X \oplus Y \oplus Z$. On $W$ it is defined a non-degenerate symmetric bilinear form such that $V \otimes V^*$ is orthogonal to $\Lambda^2V$ and such that restricted to $V \otimes V^*$ and to $\Lambda^2V$ is the one introduced at the beginning of this section.

Let $A$ be the kernel of the wedge product $X \otimes \Lambda^2V \rightarrow \Lambda^3V$ and set $\mathfrak{h}_0 = \mathfrak{sl}(V) \oplus A \oplus \Lambda^2V$. There is an action of $\mathfrak{h}_0$ on $W$ given as follows. Let $(a, b, \alpha) \in \mathfrak{h}_0$ and $(v, \omega, \varphi) \in W$ then

$$
a \cdot v = a(v) \quad b \cdot v = 0 \quad \alpha \cdot v = 0 \\
a \cdot \omega = a(\omega) \quad b \cdot \omega = \gamma(b \otimes \omega) \quad \alpha \cdot \omega = 0 \\
a \cdot \varphi = -a'(\varphi) \quad b \cdot \varphi = \delta(b \otimes \varphi) \quad \alpha \cdot \varphi = \kappa_V(\alpha \otimes \varphi)
$$

where $\gamma, \delta$ are defined as follows.

$$
\gamma: V \otimes \Lambda^2V \otimes \Lambda^2V \rightarrow V \\
\delta: V \otimes \Lambda^2V \otimes V^* \rightarrow \Lambda^2V
$$

$$
\gamma(v \otimes \alpha \otimes \alpha') = (\alpha, \alpha') v \\
\delta(v \otimes \alpha \otimes \varphi) = \varphi(v) \alpha
$$

\[18\]
The action of $h_0$ on $W$ defines an immersion of $h_0$ into $\mathfrak{so}(W)$, whose image is closed under the Lie bracket and more explicitly for $a, a' \in \mathfrak{sl}(V)$, $b, b' \in A$ and $\alpha, \alpha' \in \Lambda^2V$ we have:

$$
[a, a'] = aa' - a'a \quad |[a, b] = a(b) \quad |[a, \alpha] = a(\alpha)
$$

$$
[b, b'] = -\zeta(b \otimes b') \quad |[b, \alpha] = 0 \quad |[\alpha, \alpha'] = 0
$$

where $\zeta$ is the map

$$
\zeta: (V \otimes \Lambda^2V) \otimes (V \otimes \Lambda^2V) \rightarrow \Lambda^2V \quad \zeta((u \otimes \alpha) \otimes (u' \otimes \alpha')) = (\alpha, \alpha') u \wedge u'.
$$

The triples of Lemma [6,3] are $(D_1, D_1, D_2), (D_1, D_5, 2D_2), (D_1, D_7, D_3), (D_1, D_8, D_7), (D_1, D_8, D_5), (D_5, D_8, D_2 + D_7)$ and $(D_7, D_8, D_2)$.

5.5.1. Computation of $h_1$. The representation associated to $D_1$ is $\Lambda^3W$. Looking at the action of $\mathfrak{sl}(V)$ we get two invariants: $x \in \Lambda^2X \otimes Y \simeq \Lambda^2V \otimes \Lambda^2V$ and $y \in Y \otimes \Lambda^2Z \simeq \Lambda^2V \otimes \Lambda^2V^*$. We notice now that $X$ is invariant also by the action of $A$ and $\Lambda^2V$. Indeed if $b \in A$ then $b \cdot x \in \Lambda^3X \simeq \Lambda^3V$, in particular we get a $\mathfrak{sl}(V)$-equivariant map from $A$ to $\Lambda^3V$, which must be zero. A similar argument proves that $\alpha \cdot x = 0$ for $\alpha \in \Lambda^2V$. While it is easy to check that if $b = e_1 \wedge e_{12}$, which belongs to $A$, then $b \cdot y \neq 0$. Hence

$$
h_1 = x = e_1 \wedge e_2 \wedge e_{34} - e_1 \wedge e_3 \wedge e_{24} + e_1 \wedge e_4 \wedge e_{23} + e_2 \wedge e_3 \wedge e_{14} - e_2 \wedge e_4 \wedge e_{13} + e_3 \wedge e_4 \wedge e_{12}.
$$

5.5.2. Computation of $h_3, h_5, h_7$. Let $P$ be the parabolic of Spin($W$) defined by $g(V) \subset V$. Notice that $H \subset P$. Let $U$ be the unipotent radical of $P$, $L$ its Levi subgroup, and $L^m$ its semisimple part. Notice that $L^m \simeq SL(4) \times SL(4)$. Let $T \subset G$ the maximal torus of elements acting diagonally on $W$ with respect to the basis $e_1, \ldots, e_4, e_{12}, e_{13}, e_{14}, e_{23}, -e_{24}, e_{34}, \varphi_4, \ldots, \varphi_1$ and $B \subset G$ the subgroup of elements whose action on $W$ is upper triangular with respect to this basis. The natural action of $SL(V)$ on $W$ induces an embedding $SL(V) \rightarrow L$ on diagonal elements takes the form

$$
(t_1, t_2, t_3, t_4) \mapsto (t_1, t_2, t_3, t_4, t_1t_2, t_1t_3, t_1t_4, t_2t_3, t_2t_4, t_3t_4, t_1^{-1}, t_2^{-1}, t_3^{-1}, t_4^{-1}).
$$

The set of colors $\{D_1, D_4, D_6, D_8\}$ is distinguished, and $K_0 = H_0U$ is the intersection of the kernels of the multiplicative characters of the stabilizer $K$ of a point in the open orbit of the quotient. Therefore, $h_3, h_5, h_7$ must be $K_0$-invariant vectors in $V_{D_3}^*$. Let $W_1 = (V_{D_3})^\vee$. This is an irreducible representation of $L$ of the same highest weight of $V_{D_3}^*$: $\omega_3 + \omega_5$ for $i = 3$, $\omega_2 + \omega_5$ for $i = 5$, and $\omega_1 + \omega_5$ for $i = 7$. When we restrict this representations to $SL(V)$ we get

$$
W_3 \simeq \Lambda^3V \otimes V \quad W_5 \simeq \Lambda^2V \otimes \Lambda^2V \quad W_7 \simeq V \otimes \Lambda^3V
$$

in particular there is an invariant element under $h_0$.

For notational convenience, here and below, set

$$
e_5 = e_{12}, e_6 = e_{13}, e_7 = e_{14}, \varphi_7 = e_{23}, \varphi_6 = -e_{24}, \varphi_5 = e_{34}.
$$

Let $U \subset W$ be the subspace spanned by $e_1, \ldots, e_7$, so that $W$ becomes $U \oplus U^*$. Now we need to describe the spin representations. Consider the whole exterior algebra $\Lambda^*U^*$. It decomposes into odd and even degree parts $\Lambda^\text{odd}U^* \oplus \Lambda^\text{even}U^*$. Since the $G$-action we are going to define is not the natural one, we stress the difference by using a different notion: set $\psi_{i_1 \cdots i_k} = \varphi_{i_1 \cdots i_k}$. Define the map $\sigma: W \otimes \Lambda^*U^* \rightarrow \Lambda^*U^*$ such that

$$
\sigma(e_i \otimes \psi_{i_1 \cdots i_k}) = \delta_{i_1}^{i} \psi_{i_1 \cdots i_k} \otimes e_i
$$

$$
\sigma(\varphi_i \otimes \psi_{i_1 \cdots i_k}) = \varphi_i \otimes \varphi_{i_1 \cdots i_k},
$$

and more generally the map $\sigma^n: \otimes^n W \otimes \Lambda^*U^* \rightarrow \Lambda^*U^*$ such that

$$
\sigma^n(w_1 \otimes \cdots \otimes w_n \otimes y) = \sigma(w_1 \otimes \sigma(\cdots \otimes \sigma(w_n \otimes y) \cdots))
$$

which we can restrict to $\Lambda^nW$ if we think $w_1 \wedge \cdots \wedge w_n$ as the corresponding antisymmetric tensor, with coefficient $\frac{1}{n!}$.

To get the spin representations we can just take the map $\sigma^2$, indeed notice that $\Lambda^2W$ identifies with $\mathfrak{so}(W)$ through $w_1 \wedge w_2 = (w_2, w_1) - (w_1, w_2)$. We thus have that the vector $\psi_{i_1 \cdots i_k}$ has weight

$$
\frac{1}{2} \left( \sum_{i \not\in \{i_1, \ldots, i_k\}} \varepsilon_i - \sum_{i \in \{i_1, \ldots, i_k\}} \varepsilon_i \right),
$$

$V(\omega_6) \simeq \Lambda^\text{odd}U^*$ and $V(\omega_7) \simeq \Lambda^\text{even}U^*$.
We get the following expressions of the $H_5$-invariants:

\[ h_3 = e_{123} \otimes \psi_{65} + e_{124} \otimes \psi_{75} + e_{134} \otimes \psi_{76} - e_{234} \otimes \psi_8, \]
\[ h_5 = e_{12} \otimes e_{1234} \otimes \varphi_5 + e_{13} \otimes e_{1234} \otimes \varphi_6 + e_{14} \otimes e_{1234} \otimes \varphi_7 + e_{23} \otimes e_{1234} \otimes \varphi_8 + e_{123} \otimes e_{1234} - e_{24} \otimes e_{1234} + e_{34} \otimes e_{1234} + e_{12} \otimes e_{1234} + e_{13} \otimes e_{1234} + e_{14} \otimes e_{1234} + e_{23} \otimes e_{1234} - e_{123} \otimes e_{1234}. \]
\[ h_7 = e_1 \otimes \psi_{765} - e_2 \otimes \psi_5 - e_3 \otimes \psi_6 - e_4 \otimes \psi_7. \]

5.5.3. Computation of $h_8$. The representation $V_{h_8}^*$ is the spin representation of highest weight $\omega_6$. By direct computation one can show that the only $H_5$-invariant is given by

\[ h_8 = \psi_1 + \psi_{762} - \psi_{753} + \psi_{654}. \]

5.5.4. Analysis of the triple $(D_1,D_1,D_2)$. The representation associated with $D_1$ is $\Lambda^4 W$. The $H_5$-invariant $h_2$ is $e_{1234}$. Indeed, the set of colors $\Delta \setminus \{D_2\}$ is distinguished and the quotient is homogeneous, hence a $H$-semiinvariant in $V_{D_2}^*$ must be $P$-semiinvariant.

Here we get

\[ \tilde{k}_{W}^3(h_1 \otimes h_1) = 3h_2. \]

5.5.5. Analysis of the triple $(D_1,D_3,2D_2)$. Consider the map $\Phi : \Lambda^3 W \otimes \Lambda^2 W \otimes \Lambda^5 W \rightarrow \Lambda^4 W \otimes \Lambda^4 W$ such that

\[ \Phi((w_1 \wedge w_2 \wedge w_3) \otimes x \otimes y) = (w_1 \wedge w_2 \wedge x) \otimes \kappa_W^5(y \otimes w_3) - (w_1 \wedge w_3 \wedge x) \otimes \kappa_W^5(y \otimes w_2) + (w_2 \wedge w_3 \wedge x) \otimes \kappa_W^5(y \otimes w_1). \]

We have

\[ \Phi(h_1 \otimes h_3) = 6h_2 \otimes h_2. \]

5.5.6. Analysis of the triple $(D_1,D_7,D_3)$. Consider the map $\Phi : \Lambda^3 W \otimes W \otimes \Lambda^{\text{odd}} U^* \rightarrow \Lambda^3 W \otimes \Lambda^{\text{even}} U^*$ such that

\[ \Phi((w_1 \wedge w_2 \wedge w_3) \otimes w \otimes \psi) = (w_1 \wedge w_2 \wedge w) \otimes \sigma(w_3 \otimes \psi) - (w_1 \wedge w_3 \wedge w) \otimes \sigma(w_2 \otimes \psi) + (w_2 \wedge w_3 \wedge w) \otimes \sigma(w_1 \otimes \psi). \]

We get

\[ \Phi(h_1 \otimes h_7) = 3h_3. \]

5.5.7. Analysis of the triple $(D_1,D_8,D_7)$. Consider the map $\Phi : \Lambda^3 W \otimes \Lambda^{\text{odd}} U^* \rightarrow W \otimes \Lambda^{\text{odd}} U^*$ such that

\[ \Phi((w_1 \wedge w_2 \wedge w_3) \otimes \psi) = w_1 \otimes \sigma^2((w_2 \wedge w_3) \otimes \psi) - w_2 \otimes \sigma^2((w_1 \wedge w_3) \otimes \psi) + w_3 \otimes \sigma^2((w_1 \wedge w_2) \otimes \psi). \]

We get

\[ \Phi(h_1 \otimes h_8) = -3h_7. \]

5.5.8. Analysis of the triple $(D_3,D_8,D_5)$. On $\Lambda^6 W$ there is a symmetric bilinear form $(\cdot, \cdot)$ naturally induced by the given form on $W$. On the other hand, $V(\omega_5)$ and $V(\omega_7)$ are reciprocally dual, so there is $(\cdot, \cdot)$ natural non-degenerate pairing. Consider the map $\Psi : \Lambda^{\text{odd}} U^* \otimes \Lambda^{\text{even}} U^* \rightarrow \Lambda^6 W$ such that

\[ (u, \Psi(\psi \otimes \psi')) = (\sigma^6(u) \psi, \psi'), \]

and the map $\Phi : \Lambda^3 W \otimes \Lambda^{\text{odd}} U^* \otimes \Lambda^{\text{even}} U^* \rightarrow \Lambda^2 W \otimes \Lambda^5 W$ such that

\[ \Phi((w_1 \wedge w_2 \wedge w_3) \otimes \psi \otimes \psi') = (w_1 \wedge w_2) \otimes \tilde{k}_W^2(\Psi(\psi \otimes \psi') \otimes w_3) - (w_1 \wedge w_3) \otimes \tilde{k}_W^2(\Psi(\psi \otimes \psi') \otimes w_2) + (w_2 \wedge w_3) \otimes \tilde{k}_W^2(\Psi(\psi \otimes \psi') \otimes w_1). \]

We get

\[ \Phi(h_3 \otimes h_8) = h_5. \]
5.5.9. Analysis of the triple \((D_5, D_8, D_2 + D_7)\). Consider the map \(\Phi: \Lambda^2 W \otimes \Lambda^5 W \otimes \Lambda^{\text{odd}} U^* \rightarrow \Lambda^4 W \otimes W \otimes \Lambda^{\text{odd}} U^*\) such that

\[
\Phi((w_1 \wedge w_2) \otimes (z_1 \wedge \cdots \wedge z_5) \otimes \psi) =
\sum_i (-1)^{i+1}(z_1 \wedge \cdots \wedge z_i \wedge \cdots \wedge z_5) \otimes w_2 \otimes \sigma^2((z_i \wedge w_1) \otimes \psi)
- \sum_i (-1)^{i+1}(z_1 \wedge \cdots \wedge z_i \wedge \cdots \wedge z_5) \otimes w_1 \otimes \sigma^2((z_i \wedge w_2) \otimes \psi)
\]

We get

\[
\Phi(h_5 \otimes h_8) = -3 h_2 \otimes h_7.
\]

5.5.10. Analysis of the triple \((D_7, D_8, D_2)\). Consider the map \(\Phi: W \otimes \Lambda^{\text{odd}} U^* \otimes \Lambda^{\text{odd}} U^* \rightarrow \Lambda^{\text{odd}} U^* \otimes \Lambda^{\text{even}} U^*\) such that

\[
\Phi(w \otimes \psi \otimes \psi') = \sigma(w \otimes \psi') \otimes \psi.
\]

Here we get that \(\Phi(h_7 \otimes h_8)\) is a \(U\)-invariant vector of weight \(\omega_4\).

5.6. Projective normality of comodel wonderful varieties.

**Theorem 5.1.** Let \(M\) be a comodel wonderful variety and let \(L, L'\) be line bundles generated by global sections. Then the multiplication map \(m_{L, L'}\) is surjective.

**Proof.** As in the case of the model wonderful varieties, by Lemma 2.4, we are reduced to study the low fundamental triples.

Recall that in the model case we have classified, for every \(G\) of connected Dynkin type, the low fundamental triples \((D, E, F)\) of the model wonderful \(G\)-variety (of simply connected type) such that \(\text{Supp}_S(D + E - F) = S\).

In the comodel case, these correspond to the low fundamental triples \((D, E, F)\) of the comodel wonderful varieties of connected Dynkin type such that \(\text{Cosupp}(D + E - F) = \Delta\), where

\[
\text{Cosupp} \gamma = \{ D \in \Delta : \langle \alpha', \omega_D \rangle \neq 0 \text{ for some } \alpha \in \text{Supp} \gamma \}.
\]

Let us go back to the model case. Let \(S\) be the set of simple roots. Let \(M\) be a model wonderful variety with set of colors \(\Delta\). Recall that for every low fundamental triple \((D, E, F)\), \(\text{Supp}_S(D + E - F) = S'\) is connected. Hence \((D, E, F)\) corresponds to a low fundamental triple \((D, E, \tilde{F})\), of a model wonderful variety \(N\) (with set of colors \(\tilde{\Delta} \subset \Delta\)) of connected Dynkin type \(\gamma\) (with \(S'\) as set of simple roots) such that \(M\) is parabolic induction of \(N\), with \(D, E \in \tilde{\Delta}, D + E - F = D + E - \tilde{F}\) and \(\text{Cosupp}_D(D + E - \tilde{F}) = S'\).

This can be translated into the comodel case. Every low fundamental triple \((D, E, F)\), of a comodel wonderful variety \(M\) (with set of colors \(\Delta\)), corresponds to a low fundamental triple \((D, E, \tilde{F})\), of a comodel wonderful variety \(N\) of connected Dynkin type \(\gamma\) (with set of colors \(\tilde{\Delta} \subset \Delta\)) such that \(M\) is parabolic induction of \(N\), with \(D, E \in \tilde{\Delta}, D + E - F = D + E - \tilde{F}\) and \(\text{Cosupp}(D + E - \tilde{F}) = \tilde{\Delta}\).

Since we have already checked the surjectivity for all such low fundamental triples, we can conclude by applying Proposition 1.6 as in the proof of Theorem A (see Section 4.3).

6. On the normality of spherical orbit closures in simple projective spaces.

Let \(M\) be a wonderful \(G\)-variety with set of spherical roots \(\Sigma\) and set of colors \(\Delta\). Denote by \(H\) the stabilizer of a point \(x_0\) in the open \(G\)-orbit.

If \(D\) is in \(N\Delta\) and \(h_D \in (V^*_D)_{\xi_D}\) is the associated \(H\)-eigenvector, consider the orbit closure

\[
X_D = G \cdot [h_D] \subset P(V^*_D),
\]

which is a simple (possibly non-normal) spherical variety. We have a natural morphism \(\phi_D : M \rightarrow X_D\) such that \(\phi_D^*O(1) = L_D\).

By [25 Corollary 7.6] and [2 Corollary 2.4.2.2], every spherical orbit in a simple projective space always admits a wonderful compactification, so the described situation is absolutely general. As a consequence of the results of the previous sections, here we will show that under some special assumptions on \(M\), the variety \(X_D\) is always normal.

The variety \(X_D\) was studied by G. Pezzini in [25] when \(D\) is ample, that is, \(D \in N_{>0}\Delta\). Under this assumption, either \(X_D\) is isomorphic to \(M\) or it is not even normal. In case \(X_D \simeq M\), then \(M\) is called strict: this is equivalent to the conditions \(H = N_G(H)\) and \(\Sigma \cap S = \emptyset\). There are essentially two main
classes of examples of strict wonderful varieties: the adjoint symmetric wonderful varieties and the model wonderful varieties.

When $D$ is not ample, the variety $X_D$ was then studied in [32] in the symmetric case and in [18] in general. More precisely, the orbit structure of $X_D$ and that of its normalization $\tilde{X}_D$ were analyzed. In particular, it was proved that the normalization $\tilde{X}_D \to X_D$ is always bijective if $M$ is adjoint symmetric or if $G$ is of simply laced type and $M$ is strict, while the main counterexamples where bijectivity fails, in the strict case, arise with the model wonderful varieties for groups of not simply laced type.

We say that $D \in \mathbb{N}_{\Delta}$ is a faithful divisor on $M$ if $\phi_D$ restricts to an open embedding of $G/H$, i.e. if $H$ equals the stabilizer of $[h_D]$. The formalism of distinguished sets of colors allows to characterize combinatorially the faithful divisors. For simplicity, we restrict to the case of a strict wonderful variety.

**Proposition 6.1** (see [7 Proposition 2.4.3]). Let $M$ be a strict wonderful variety and let $D \in \mathbb{N}_{\Delta}$. Then $D$ is faithful if and only if every distinguished subset of $\Delta$ intersects $\text{Supp}(D)$.

Let $D \in \mathbb{N}_{\Delta}$, denote $\hat{A}(D) = \bigoplus_{n \in \mathbb{N}} \Gamma(M, L_n D)$ and denote $A(D) \subset \hat{A}(D)$ the subalgebra generated by $V_D \subset \Gamma(M, L_D)$. Let $\tilde{X}_D$ be the image of $M$ in $\mathbb{P}(\Gamma(M, L_D)^*)$ via the morphism $\hat{\phi}_D$ associated to the complete linear system of $D$. Then $\tilde{A}(D)$ is identified with the projective coordinate ring of $\tilde{X}_D$, whereas $\hat{A}(D)$ is identified with the projective coordinate ring of $X_D$. Notice that we have a natural projection $\beta_D : \tilde{X}_D \to X_D$ such that $\beta_D = \hat{\phi}_D \circ \beta_D$. Since $M$ is smooth, $\hat{A}(D)$ is an integrally closed algebra, therefore $\tilde{X}_D$ is normal. Moreover, we have the following.

**Proposition 6.2** (see [11 Proposition 2.1]). The algebra $\hat{A}(D)$ is integral over $A(D)$.

It follows that $\beta_D : \tilde{X}_D \to X_D$ is the normalization if and only if it is birational. Clearly this is the case if $D$ is minuscule or faithful. On the other hand, $\beta_D$ is not necessarily birational: if $M$ is the model wonderful variety of $C_4$, then $A(D_2) \subset \hat{A}(D_2)$ have not the same quotient field. However, we have the following.

**Proposition 6.3.** Suppose that $M$ is adjoint symmetric or that $G$ is simply laced and $M$ is strict. Then $\hat{A}(D)$ and $A(D)$ have the same field.

**Proof.** Under the given hypotheses, using the formalism of the distinguished sets of colors, one can see that every morphism $\phi : M \to M'$ to a second wonderful variety $M'$ has connected fibers, i.e. $M'$ is the quotient of $M$ by a distinguished set of colors. Consider now the morphism $\phi_D : M \to \mathbb{P}(V_D^*)$ and denote $M'$ the wonderful compactification of the orbit $G \cdot [h_D]$. Then $D$ can be identified with a faithful divisor on $M'$ and $\phi_D$ factors through $M'$, which is a quotient of $M$. Therefore by Corollary [11, Corollary 1.4] we get $\Gamma(M, L_n D) \simeq \Gamma(M', L_n D')$ for every $n \geq 0$ and we are reduced to the case of a faithful divisor. □

In the adjoint symmetric case, the above proposition was proved in [11, Theorem 2.6]. If $D_1, \ldots, D_m \in \mathbb{N}_{\Delta}$, consider the variety

$$X_{D_1, \ldots, D_m} = G \cdot ([h_{D_1}] \times \cdots \times [h_{D_m}]) \subset \mathbb{P}(V_{D_1}^*) \times \cdots \times \mathbb{P}(V_{D_m}^*)$$

and denote by $\phi_{D_1, \ldots, D_m} : M \to X_{D_1, \ldots, D_m}$ the map such that $\phi_{D_1, \ldots, D_m} = \phi_{D_1}(x), \ldots, \phi_{D_m}(x)$.

**Proposition 6.4** ([6 Proposition 1.2]). Let $M$ be a wonderful variety and let $D, E \in \mathbb{N}_{\Delta}$.

1. If $\text{Supp}(\omega_D) \cap \text{Supp}(\omega_E) = \emptyset$, then $X_{D+E} \simeq X_{D,E}$.
2. If $M$ is strict and $\text{Supp}(D) = \{D_1, \ldots, D_m\}$, then $X_D \simeq X_{D_1, \ldots, D_m}$. In particular, if $M$ is strict, $X_D \simeq X_E$ if and only if $\text{Supp}(D) = \text{Supp}(E)$.

**Proof.** i) Since $\text{Supp}(\omega_D) \cap \text{Supp}(\omega_E) = \emptyset$, by [6 Lemma 1.1] we have a closed equivariant embedding $\psi_{D,E} : \mathbb{P}(V_D^*) \times \mathbb{P}(V_E^*) \to \mathbb{P}(V_{D+E}^*)$, and since $\psi_{D,E}([h_D] \times [h_E]) = [h_{D+E}]$ the isomorphism $X_{D+E} \simeq X_{D,E}$ follows.

ii) Since $M$ is strict, by the description of the restriction $\omega : \text{Pic}(M) \to X(T)$ (see [37, Lemma 30.24]) it follows that $\text{Supp}(D) \cap \text{Supp}(E) = \emptyset$ if and only if $\text{Supp}(\omega_D) \cap \text{Supp}(\omega_E) = \emptyset$. Therefore, by [6 Lemma 1.1] we have a closed equivariant embedding $\psi_D : \mathbb{P}(V_{D_1}^*) \times \cdots \times \mathbb{P}(V_{D_m}^*) \to \mathbb{P}(V_D^*)$ such that $\psi_{D}([h_{D_1}], \ldots, [h_{D_m}]) = [h_D]$ and it follows $X_D \simeq X_{D_1, \ldots, D_m}$ □

Assume now that the multiplication of sections is surjective for every couple of globally generated line bundles. In particular $\hat{A}(D)$ is generated by its degree one component $\Gamma(M, L_D)$ and it follows that $A(D) = \hat{A}(D)$ if and only if $\Gamma(M, L_D) = V_D$ if and only if $D$ is minuscule. Since $\hat{A}(D)$ is integrally closed, we get the following proposition which we need for later use.

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Proposition 6.5. Let $M$ be a wonderful variety and suppose that the multiplication of sections is surjective for every couple of globally generated line bundles. Let $D \in \mathbb{N}\Delta$.

i) If $D$ is minuscule, then $X_D$ is projectively normal.

ii) If $\beta_D$ is birational and $X_D$ is projectively normal, then $D$ is minuscule.

If $D_1, \ldots, D_m \in \mathbb{N}\Delta$ and if $V_1, \ldots, V_m$ are $G$-modules of sections such that $V_{D_i} \subset V_i \subset \Gamma(M, L_{D_i})$, consider the associated morphisms $\phi_{V_i} : M \rightarrow \mathbb{P}(V^*_i)$ and denote $X_{V_i} = \phi_{V_i}(M)$. Also, we denote by $X_{V_1, \ldots, V_m} \subset \mathbb{P}(V^*_1 \times \cdots \times V^*_m)$ the image of $M$ via the map $\phi_{V_1, \ldots, V_m}(x) = (\phi_{V_1}(x), \ldots, \phi_{V_m}(x))$ and by $X_{V_1 \otimes \cdots \otimes V_m} \subset \mathbb{P}(V^*_1 \otimes \cdots \otimes V^*_m)$ the image of $X_{V_1, \ldots, V_m}$ via the Segre embedding.

Lemma 6.6. Let $D_1, \ldots, D_m \in \mathbb{N}\Delta$ and denote $D = \sum_{i=1}^m D_i$. If $V_1, \ldots, V_m$ are $G$-modules such that $V_{D_i} \subset V_i \subset \Gamma(M, L_{D_i})$, then the projective coordinate ring of $X_{V_1 \otimes \cdots \otimes V_m}$ is the subalgebra $A(V_1, \ldots, V_m) \subset \mathbb{A}(D)$ generated by the product $V_1 \cdots V_m \subset \Gamma(M, L_D)$.

Proof. Consider the map $\phi_{V_1 \otimes \cdots \otimes V_m} : M \rightarrow X_{V_1 \otimes \cdots \otimes V_m}$. The lemma follows by noticing that $\phi^*_{V_1 \otimes \cdots \otimes V_m} \mathcal{O}(1) = \mathcal{L}_D$ and that $\phi^*_{V_1 \otimes \cdots \otimes V_m} : V_1 \otimes \cdots \otimes V_m \rightarrow \Gamma(M, \mathcal{L}_D)$ is the multiplication map. $\Box$

Proposition 6.7. Let $M$ be a wonderful variety and suppose that the multiplication of sections is surjective for every couple of globally generated line bundles. Let $D_1, \ldots, D_m \in \mathbb{N}\Delta$ and denote $\Gamma = \Gamma(M, L_D)$. Then the variety $X_{\Gamma_1 \otimes \cdots \otimes \Gamma_m} \subset \mathbb{P}(\Gamma^*_1 \otimes \cdots \otimes \Gamma^*_m)$ is projectively normal.

Proof. Denote $D = \sum_{i=1}^m D_i$. Since the multiplication of sections is surjective, $\Gamma_1 \cdots \Gamma_m = \Gamma(M, L_D)$, hence by the previous lemma $A(\Gamma_1, \ldots, \Gamma_m) = \mathbb{A}(D)$ and $X_{\Gamma_1 \otimes \cdots \otimes \Gamma_m}$ is a projectively normal variety. $\Box$

Corollary 6.8. Let $M$ be a wonderful variety and suppose that the multiplication of sections is surjective for every couple of globally generated line bundles.

i) Let $D, E \in \mathbb{N}\Delta$ be such that $\text{Supp}(\omega_D) \cap \text{Supp}(\omega_E) = \emptyset$. If $X_D, X_E$ are normal, then $X_{D+E}$ is normal as well.

ii) If $M$ is strict, then $X_D$ is normal for all $D \in \mathbb{N}\Delta$ if and only if it is normal for all $D \in \Delta$.

Proof. i) Denote $\Gamma = \Gamma(M, L_D)$ and $\Gamma_E = \Gamma(M, L_E)$: by the previous proposition, we have that $X_{\Gamma_1 \otimes \cdots \otimes \Gamma_m}$ is a normal variety. On the other hand since $X_D$ and $X_E$ are normal, we have that $X_{\Gamma_1 \otimes \cdots \otimes \Gamma_m}$ is $\mathbb{A}(D)$, while by Proposition 6.4 we have that $\Gamma_{D+E} \simeq \Gamma_{D,E}$. ii) Since $M$ is strict, it follows by the description of $\omega : \text{Pic}(M) \rightarrow \mathcal{X}(T)$ that $\text{Supp}(\omega_D) \cap \text{Supp}(\omega_E) = \emptyset$ if and only if $\text{Supp}(D) \cap \text{Supp}(E) = \emptyset$. Therefore the claim follows straightforwardly from i). $\Box$

Corollary 6.9. Suppose that $M$ is a symmetric variety with reduced root system of type $A$ or that it is a model wonderful variety for a connected semisimple group of type $A\Delta$. Then $X_D$ is normal for all $D \in \mathbb{N}\Delta$.

Proof. By the description of the covering relation, it follows that under the assumptions on $M$ it holds $\text{ht}(\gamma^+) = 2$ for every covering difference $\gamma$ in $\mathbb{N}\Delta$. Therefore every $D \in \Delta$ is minuscule and $\Gamma(M, L_D) = V_D$. Therefore, $X_D$ is projectively normal for all $D \in \Delta$ and it follows by the previous corollary that $X_D$ is normal for all $D \in \mathbb{N}\Delta$. $\Box$

7. On the normality of cones and nilpotent orbits.

Following the same approach of [15] and [11], we can apply Theorem A to study the normality of cones over model varieties. In particular, as pointed out by Luna some years ago, we can apply our theory to study the normality of the closure of spherical nilpotent orbits in the Lie algebra of $G$.

Let $M$ be a wonderful variety with set of colors $\Delta$ and set of spherical roots $\Sigma$ and assume that the multiplication of sections is surjective for every couple of globally generated line bundles. Let $D \in \mathbb{N}\Delta$ and denote by $C_D \subset V_D^*$ the cone over the variety $X_D$ introduced in the previous section. Analogously, denote by $\tilde{C}_D \subset \Gamma(M, L_D)^*$ the cone over the variety $\tilde{X}_D$. Then the coordinate ring of $C_D$ is identified with $A(D)$, whereas that of $\tilde{C}_D$ is identified with $\tilde{A}(D)$, which is an integrally closed ring. This yields a map $\alpha_D : \tilde{C}_D \rightarrow C_D$ that is birational if and only if $\beta_D : \tilde{X}_D \rightarrow X_D$ is birational. As already recalled in the previous section, $\tilde{A}(D)$ is the integral closure of $A(D)$ if and only if $\alpha_D$ is birational, whereas $A(D) = \tilde{A}(D)$ if and only if $D$ is minuscule.
In the case of the model wonderful varieties of simply connected type we have the following classification of minuscule weights, where \( a, b, c \in \mathbb{N} \).

Case \( A_r \), \( r \) even: \( D_1, D_2, \ldots, D_r, aD_1, aD_r, aD_1 + D_2 \) with \( d \) odd, \( D_m + bD_r \) with \( m \) even;

Case \( A_r \), \( r \) odd: \( D_1, D_2, \ldots, D_r, aD_1, aD_r, aD_1 + D_2 \) with \( d \) odd, \( D_m + bD_r \) with \( m \) odd, \( aD_1 + bD_r \);

Case \( B_r \), \( r \) even: \( aD_1 + D_r, aD_1 + D_m + bD_r \) with \( m \) even;

Case \( B_r \), \( r \) odd: \( aD_1 + bD_r, aD_1 + D_m + bD_r \) with \( m \) odd;

Case \( C_r \): \( D_1, D_2, \ldots, D_r, aD_1 + bD_r \);

Case \( D_r \), \( r \) even: \( D_1, D_2, \ldots, D_r, aD_1 + bD_r - 1 + cD_r, aD_1 + bD_r - 1 + D_m + cD_r \) with \( m \) odd;

Case \( D_r \), \( r \) odd: \( D_1, D_2, \ldots, D_r, aD_1 + bD_r - 1 + cD_r, aD_1 + bD_r - 1 + D_m + bD_r \) with \( m \) even;

Case \( E_6 \): \( D_1, D_6, aD_2, aD_2 + D_3, aD_2 + D_5 \);

Case \( E_7 \): \( D_1, D_6, aD_2 + bD_7, aD_2 + bD_7 + D_3, aD_2 + bD_7 + D_5 \);

Case \( E_8 \): \( D_1, D_6, D_7, D_8, aD_2, aD_2 + D_3, aD_2 + D_5 \);

Case \( F_4 \): \( aD_1, D_1 + D_2, D_3 + D_1, D_2, D_3, D_4 \);

Case \( G_2 \): \( aD_1, D_2 \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). If \( \mathfrak{g} \) is not simple, then its nilpotent orbits are products of the nilpotent orbits of its simple factors, and so are their closures. Therefore we may assume that \( \mathfrak{g} \) is simple.

Let \( e \in \mathfrak{g} \) be a non-zero nilpotent element, let \( \mathcal{O} \) be its adjoint orbit, consider an \( \mathfrak{sl}(2) \)-triple \( (e, h, f) \). Choose a maximal toral subalgebra \( t \) of \( \mathfrak{g} \) containing \( h \) and a Borel subalgebra \( \mathfrak{b} \) containing \( t \) and \( e \) and such that \( \alpha(h) \geq 0 \) for every \( \alpha \in S \), where we denote by \( S = \{ \alpha_1, \ldots, \alpha_r \} \) the set of simple roots defined by the choice of \( t \) and \( \mathfrak{b} \). The string \( (\alpha_1(h), \ldots, \alpha_r(h)) \) is called the Kostant–Dynkin diagram of \( \mathcal{O} \) and it uniquely determines the orbit \( \mathcal{O} \). Moreover, every \( \alpha_i(h) \) is 0, 1 or 2. Let \( \theta \) be the highest root corresponding to the choice of \( S \) and define the height of \( \mathcal{O} \) as \( \text{height}(\mathcal{O}) = \theta(h) \). The height does not depend on the various choices we have made (see [14]); furthermore, \( \mathcal{O} \) is spherical if and only if \( \text{height}(\mathcal{O}) \leq 3 \) (see [34]). Notice that this last condition is equivalent to say that \( \mathcal{O} \) is \( \{0\} \) or it has height equal to 2 or to 3, see again [14].

By making use of the projective normality of the symmetric wonderful varieties, in [11] it has been proved that the closure \( \overline{\mathcal{O}} \) is normal if \( \text{height}(\mathcal{O}) = 2 \), which is originally due to W. Hesselink [22]. We now study the normality of \( \overline{\mathcal{O}} \) in the case of \( \text{height}(\mathcal{O}) = 3 \) (see [34] Table 2) by making use of the projective normality of the model wonderful varieties.

Denote by \( U \simeq G/H \) the orbit of the line \([e] \in \mathbb{P}(\mathfrak{g}) = \mathbb{P}(V(\theta)) \), namely the image of \( \mathcal{O} \) via the natural projection. As every spherical orbit in the projective space of a simple \( G \)-module, \( U \) possesses a wonderful compactification, which we will denote by \( M_{\mathcal{O}} \). In [5] we can find a description of the stabilizer of \([e]\) as well as the associated Luna diagram.

I) Type \( B_{2n+1} \), Kostant–Dynkin diagram \((10 \ldots 01)\), case \((13)\) in [5];

II) Type \( B_{2n+1+1} \), Kostant–Dynkin diagram \((10 \ldots 010 \ldots 0)\) with \( \alpha_{2n+1}(h) = 1 \), case \((18)\) in [5];

III) Type \( D_{2n+2} \), Kostant–Dynkin diagram \((10 \ldots 01)\), case \((41)\) in [5];

IV) Type \( D_{2n+1+2} \), Kostant–Dynkin diagram \((10 \ldots 010 \ldots 0)\), with \( \alpha_{2n+1}(h) = 1 \), case \((43)\) in [5];

V) Type \( E_6 \), Kostant–Dynkin diagram \((0001000)\), case \((53)\) in [5];

VI) Type \( E_7 \), Kostant–Dynkin diagram \((00100000)\), case \((54)\) in [5];

VII) Type \( E_7 \), Kostant–Dynkin diagram \((0100001)\), case \((51)\) in [5];

VIII) Type \( E_8 \), Kostant–Dynkin diagram \((00000010)\), case \((52)\) in [5];

IX) Type \( E_8 \), Kostant–Dynkin diagram \((01000000)\), case \((51)\) in [5];

X) Type \( F_4 \), Kostant–Dynkin diagram \((0100)\), case \((60)\) in [5];

XI) Type \( G_2 \), Kostant–Dynkin diagram \((10)\), case \((66)\) in [5].

In particular, \( M_{\mathcal{O}} \) turns out to be a strict wonderful variety, and in particular the restriction of line bundles to the closed orbit \( \text{Pic}(M_{\mathcal{O}}) \to \mathcal{X}(B) \) is always injective. Therefore, we may regard \( \theta \) as an element of \( \mathfrak{N} \Delta \) and we have \( \overline{U} = X_\theta \) and \( \overline{\mathcal{O}} = C_\theta \).

In order to study the normality of \( \overline{\mathcal{O}} \), we will need the following.

**Theorem 7.1.** Let \( \mathcal{O} \subset \mathfrak{g} \) be a nilpotent orbit and let \( M_{\mathcal{O}} \) be the associated wonderful variety. Then the multiplication map

\[
m_{\mathcal{L}, \mathcal{L}'} : \Gamma(M_{\mathcal{O}}, \mathcal{L}) \otimes \Gamma(M_{\mathcal{O}}, \mathcal{L}') \to \Gamma(M_{\mathcal{O}}, \mathcal{L} \otimes \mathcal{L}')
\]

is surjective for all globally generated line bundles \( \mathcal{L}, \mathcal{L}' \) on \( M_{\mathcal{O}} \).
The rest of the section will be essentially devoted to the proof of this theorem. Notice that by construction \( \theta \) is identified with a faithful divisor on \( M_\mathcal{O} \). Therefore the normality (resp. the non-normality) of \( \overline{\mathcal{O}} = G_\theta \) will follow case-by-case by noticing that \( \theta \) is minuscule (resp. non-minuscule) and applying Proposition 7.2.

7.1. Cases I, III, VII, IX, XI: model orbits. By the description in \([5]\), in these cases we have that \( M_\mathcal{O} \) is the model wonderful variety of the simply connected group \( G \). Notice that \( \theta \) is always minuscule in \( N\Delta \) but in the cases \( B_{2n+1} \) and \( G_2 \) it follows that \( \overline{\mathcal{O}} \) is normal in the cases III, VII, IX, whereas it is not normal in the cases I, XI.

7.2. Cases IV (m even), VI, VIII: localization of model wonderful varieties. In this case \( M_\mathcal{O} \) is not a model wonderful variety, however it is a quotient of a localization of a model wonderful variety and we still may proceed as in the case of a model orbit thanks to Corollary 7.4 and Corollary 7.11. In order to conclude the argument, we now describe which localizations and quotients we have to take into account in each of the considered cases. We denote by \( M \) the model wonderful variety of \( G \).

7.2.1. Case IV (m even). Let \( N \) be the boundary divisor corresponding to the spherical root \( \sigma_{2n+1} = \alpha_{2n+1} + \alpha_{2n+2} \). Then \( M_\mathcal{O} \) is the quotient of \( N \) by the distinguished subset of colors \( \Delta' = \{D_{2n+2}, D_{2n+3}, \ldots, D_{2n+m+2}\} \). Since \( \theta = D_2 \) is minuscule, it follows that \( \overline{\mathcal{O}} \) is normal.

7.2.2. Case VI. Let \( N \) be the boundary divisor corresponding to the spherical root \( \sigma_3 = \alpha_3 + \alpha_4 \). Then \( M_\mathcal{O} \) is the quotient of \( N \) by the distinguished subset \( \Delta' = \{D_2, D_5, D_7\} \). Being \( \theta = D_1 \) minuscule, it follows that \( \overline{\mathcal{O}} \) is normal.

7.2.3. Case VIII. Let \( N \) be the boundary divisor corresponding to the spherical root \( \sigma_6 = \alpha_6 + \alpha_7 \). Then \( M_\mathcal{O} \) is the quotient of \( N \) by the distinguished subset \( \Delta' = \{D_2, D_3, D_4, D_5\} \). Since \( \theta = D_8 \) is minuscule, \( \overline{\mathcal{O}} \) is normal.

7.3. Cases II, IV. In these cases we need to prove the surjectivity of the multiplication for other classes of wonderful varieties. Let \( G = \text{Spin}(k) \), let \( r \) be the semisimple rank of \( G \) (i.e., \( k = 2r + 1 \) if \( k \) odd or \( k = 2r \) if \( k \) even) and let \( 2 \leq s \leq (k-3)/2 \). Consider the wonderful variety \( M \) corresponding in \([5]\) to the case (18) when \( \theta \) is odd and to the case (43) when \( \theta \) is even.

Its spherical roots and colors are given as follows: \( \Sigma = \{\sigma_1, \ldots, \sigma_s\} \) and \( \Delta = \{D_1, \ldots, D_{s+1}\} \), where \( \sigma_i = \alpha_i + \alpha_{i+1} = D_1 + D_{i+1} - D_{i-1} - D_{i+2} \) for \( i = 1, \ldots, s-1 \) and

\[
\sigma_s = \begin{cases} 
2(\alpha_{s+1} + \cdots + \alpha_r) = 2D_{s+1} - 2D_s & \text{if } k \text{ is odd} \\
2(\alpha_{s+1} + \cdots + \alpha_{s-r}) + \alpha_{s-1} + \alpha_r = 2D_{s+1} - 2D_s & \text{if } k \text{ is even}
\end{cases}
\]

Notice that the Cartan pairing of \( M \) does not depend on the parity of \( k \). Also, notice that \( \omega(D_i) = \omega_i \) for \( i = 1, \ldots, s+1 \).

First we need to classify the covering differences for \( M \). We omit the proof of the following proposition, which is essentially the same of Proposition 7.4.

**Proposition 7.2.** Let \( \gamma \in \mathbb{N} \Sigma \) be a covering difference in \( \mathbb{N} \Delta \) with \( \text{Supp}_S(\gamma) = S \). Then \( s \) is even and \( \gamma = \sum_{i=1}^{(s-1)/2} \sigma_{2i-1} + \sum_{i=(s+1)/2}^s \sigma_{2i-1} \). Let \( s = d_1 + \cdots + d_j \) for \( j \) odd \( \leq s + 1 \).

In particular, it follows that every covering difference for \( M \) satisfies the property (2-h). Therefore by Lemma 6.4 in order to prove the surjectivity of the multiplication of sections of globally generated line bundles on \( M \), we are reduced to the study of the low fundamental triples. Here, as in Lemma 4.3 we have the following.

**Lemma 7.3.** Let \( (D_p, D_q, F) \) be a low fundamental triple with \( \text{Supp}_S(D_p + D_q - F) = S \). Then \( s \) is even, \( p, q \) are odd, \( p + q \leq s + 2 \) and \( F = D_{p+q-2} \).

**Proposition 7.4.** Let \( (D, E, F) \) be a low fundamental triple. Then \( sD + E - F \) is a torsion in \( V_D V_E \).

**Proof.** As in the proof of Theorem A, if \( \text{Supp}(D + E - F) \neq S \) we can proceed by localization and parabolic induction. Therefore it is enough to consider the triples of Lemma 7.3.

Denote by \( U = \mathbb{C}^k \) the standard representation of \( \text{Spin}(k) \) with the invariant symmetric bilinear form \( b \). Let \( V \subset U \) be a totally isotropic subspace of dimension \( s \), let \( \omega_V \in \Lambda^2 V \) be a symplectic form on \( V \) and fix \( e_0 \in U \setminus (V \oplus V^*) \) with \( b(e_0, e_0) = 1 \).

Let \( H \) be the generic stabilizer of \( M \). Then \( H \) contains the center of \( \text{Spin}(k) \) and its image \( \overline{H} \) in \( \text{SO}(k) \) is described as follows:

\[
\overline{H} = \{ g \in \text{SO}(k) : g e_0 = e_0, \ g V \subset V, \ g \omega_V \in \mathbb{C} \omega_V \}.
\]
If $j \leq s + 1$, then we have $V_{D_j} = V(\omega_j) = \Lambda^j U$. Let $h_j$ be a non-zero $H$-semi-invariant vector in $V_{D_j}$. If $2j \leq s$, then up to a scalar factor we have $h_{2j} = \omega_{2j}^+$ and $h_{2j+1} = e_0 \wedge h_{2j}$. The projection $\pi : \Lambda^p U \otimes \Lambda^q U \to \Lambda^{p+q-2} U$ is given by contraction as follows:

$$
\pi((v_1 \wedge \cdots \wedge v_p) \otimes (u_1 \wedge \cdots \wedge u_q)) = \sum_{i,j} (-1)^{i+j} b(v_i, u_j) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_q.
$$

Therefore, if $(D_{p}, D_{q}, D_{p+q-2})$ is as in Lemma [7.3] we get $\pi(h_p \otimes h_q) = h_{p+q-2} \neq 0$, and by Lemma [1.2] it follows that $s^{D_{p+q-2}} V_{D_{p+q-2}} \subset V_D V_{D_4}$.

If $k = 4n + 2n + 3$ and $s = 2n + 1$, then $M = M_\mathcal{O}$ is the wonderful variety corresponding to the nilpotent orbit $\mathcal{O}$ of the case II, while if $k = 4n + 2n + 4$ and $s = 2n + 1$, then $M = M_\mathcal{O}$ is the wonderful variety corresponding to the nilpotent orbit $\mathcal{O}$ of the case IV. By the previous proposition, the multiplication of sections is surjective for every couple of globally generated line bundles on $M_\mathcal{O}$. Since $\theta = D_2$ is minuscule in $\mathbb{N}_L$, it follows that $\mathcal{O}$ is normal.

7.4. Case V. The variety $M_\mathcal{O}$ corresponds to the case (53) in [5]. We have $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ where

$$
\sigma_1 = \sigma_1 + \alpha_6 = 2 D_1 - D_3;
$$
$$
\sigma_2 = \sigma_2 + \alpha_3 = D_2 - D_3 + D_4;
$$
$$
\sigma_3 = \sigma_3 + \alpha_6 = -D_1 + 2D_3 - 2D_4;
$$

and $\omega(D_1) = \omega_1 + \omega_6$, $\omega(D_2) = \omega_2$, $\omega(D_3) = \omega_3 + \omega_5$, $\omega(D_4) = \omega_4$. The covering differences are

$$
\sigma_1, \sigma_2, \sigma_3, \sigma_1 + \sigma_3, \sigma_2 + \sigma_3, 2 \sigma_2 + \sigma_3, \sigma_1 + \sigma_2 + \sigma_3,
$$

therefore we have $\text{ht}(\sigma^+) = 2$ for every covering difference. Correspondingly, we get the following low fundamental triples:

$$(D_1, D_1, D_3), (D_2, D_1, D_3), (D_3, D_1, D_2 + 2D_4), (D_1, D_3, 2D_4)$$

$$(D_2, D_3, D_1 + D_4), (D_2, D_2, D_1), (D_1, D_2, D_4).$$

We need to show that $s^{E+D-F} V_D \subset V_D V_E$ for all low fundamental triples $(D, E, F)$. As in the proof of Theorem A, if $\text{Supp}(D + E - F) \neq S$ we can proceed by localization and parabolic induction. In this way, the triples $(D_1, D_1, D_3)$, $(D_3, D_3, D_1 + 2D_4)$, $(D_1, D_3, 2D_4)$ reduce to the case of a symmetric wonderful variety, the triple $(D_2, D_4, D_3)$ reduce to a model wonderful variety and the triples $(D_2, D_3, D_1 + D_4)$, $(D_2, D_2, D_1)$ reduce to a wonderful variety studied in the case IV. So we are left to study the case $D = D_1, E = D_2$ and $F = D_4$, which can easily be checked via computer.

7.5. Case X. The variety $M_\mathcal{O}$ corresponds to the case (60) in [5]. We have $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ where

$$
\sigma_1 = 2 \alpha_4 = 2D_1 - D_3;
$$
$$
\sigma_2 = \sigma_1 + \alpha_2 = D_2 - D_3 + D_4;
$$
$$
\sigma_3 = 2 \alpha_3 = -D_1 + 2D_3 - 2D_4;
$$

and $\omega(D_1) = 2\omega_4$, $\omega(D_2) = \omega_2$, $\omega(D_3) = 2\omega_3$, $\omega(D_4) = \omega_1$. Since the Cartan pairing of $M_\mathcal{O}$ is the same as that of the previous case, it follows that the covering differences and the low fundamental triples are also the same.

In order to prove that $s^{E+D-F} V_D \subset V_D V_E$ for all low fundamental triples $(D, E, F)$, if $\text{Supp}(D + E - F) \neq S$ we can proceed by localization and parabolic induction. In this way, the triples $(D_1, D_1, D_3)$, $(D_3, D_3, D_1 + 2D_4)$, $(D_1, D_3, 2D_4)$ reduce to the case of a symmetric wonderful variety, the triple $(D_2, D_4, D_3)$ reduce to a model wonderful variety and the triples $(D_2, D_3, D_1 + D_4)$, $(D_2, D_2, D_1)$ reduce to a wonderful variety studied in the case III. The only remaining case, $D = D_1, E = D_2$ and $F = D_4$, can easily be checked via computer.

8. Application to the real model orbit of type $\mathfrak{E}_8$

In [1], Adams, Huang and Vogan study the model orbit in the Lie algebra of type $\mathfrak{E}_8$. Their study is motivated by the so-called orbit method to construct representations of reductive Lie groups. In their case the group is the complex algebraic group of type $\mathfrak{E}_8$ considered as a Lie group. In particular they describe the decomposition into irreducible modules of the coordinate ring of the nilpotent orbit of type $\mathfrak{X}$ (see Section 7) and prove it is indeed a model orbit. In the same paper they also make some conjectures about another orbit which is the analogue of the model orbit for the split real form of $\mathfrak{E}_8$. 26
We start with some general preliminaries, we refer to [1] and to [38] for the motivation of these constructions coming from the representation theory of Lie groups. Let $G_R$ be a real form of a connected and complex algebraic semisimple group $G$ and let $\sigma$ be the associated Galois involution of $G$. There exists a complex algebraic involution $\theta$ of $\hat{G}$ which commutes with $\sigma$ such that the subgroup $K_\theta$ of points of $G_R$ fixed by $\theta$ is a maximal compact subgroup of $G_R$. Then the subgroup $K$ of points of $G$ fixed by $\theta$ is a complexification of $K_\theta$. The Lie algebra $\hat{g}$ of $\hat{G}$ decomposes as $\hat{\mathfrak{g}} \oplus \mathfrak{p}$ where $\hat{\mathfrak{g}}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is the eigenspace of eigenvalue $-1$ of $\theta$. An analogue of the nilpotent cone $N$ is defined as
\[ N_\theta = N \cap \mathfrak{p}. \]

Fix a point $e \in N_\theta$, let $O$ be its $K$-orbit and $K(e)$ its stabilizer. Consider the multiplicative character $\gamma_e$ of $K(e)$ given by $\gamma_e(g) = \det(Ad_{\hat{g}}(g)^{-1}) = \det(Ad_{\hat{g}}(g))^{-1}$. If $\chi : K(e) \to \mathbb{C}^*$ is any multiplicative character we can consider the algebraic line bundle on $O$ given by $\mathcal{V}_\chi = K \times_{K(e)} \mathbb{C}_\chi$. As in [1], the pair $(e, \chi)$ is said to be admissible if $\mathcal{O} \cap \mathcal{O}$ has codimension at least two in $\mathcal{O}$ and $\chi^2(g) = \gamma_e(g)$ for all $g$ in the identity component of $K(e)$. In this paper we need a slightly more general definition of admissible pair.

Let $G$ be a double cover of $K$ and $G(e)$ be the inverse image of $K(e)$ in $G$ so that $O \cong G/G(e)$. Let $\gamma_e$ denote the character of $G(e)$ induced by $\gamma_e$. Given a character $\chi$ of $G(e)$ we can construct the line bundle $\mathcal{V}_\chi$ as above. We say that the pair $(e, \chi)$ is admissible if $\mathcal{O} \cap \mathcal{O}$ has codimension at least two in $\mathcal{O}$ and $\chi^2(g) = \gamma_e(g)$ for all $g$ in the identity component of $G(e)$. Let also $G_R$ be the inverse image of $K_R$ in $G$. Notice that $G$ is the complexification of $G_R$. The coverings of $G_R$ are in correspondence with the coverings of $G_R$ hence there exists a Lie group $\hat{G}_R$ which is a double cover of $\hat{G}_R$ whose maximal compact subgroup is $G_R$.

According to the orbit method it should be possible to associate with any admissible pair $(e, \chi)$ an irreducible unitary representation $R(e, \chi)$ of $\hat{G}_R$ such that the decomposition of the $G_R$-finite vectors of $R(e, \chi)$ into irreducible submodules equals the decomposition into irreducible $G$-submodules of the space of algebraic sections $\Gamma(O, \mathcal{V}_\chi)$ (see [1] Conjecture 2.9).

8.1. The case of the complex model orbit. In [1] the geometric side of a particular case of this construction is analyzed. Let $G_R$ be the complex algebraic group of type $E_8$. Hence $G = G_R \times G_R$ and $G = K$ is the diagonal subgroup. So $\mathfrak{p}$ is isomorphic to the Lie algebra of $G$ and one can consider the nilpotent orbit with Kostant-Dynkin diagram equal to $(0, 1, 0, 0, 0, 0, 0, 0)$ (case IX of Section 6). We now introduce some notation, recall some properties of this orbit and deduce Adams, Huang and Vogan’s result from the projective normality of model varieties.

We fix a maximal torus $T$, and a Borel subgroup of $G$ containing $T$. We denote by $\Phi$ the set of roots and by $S$ the set of simple roots determined by these choices. We denote also by $\varepsilon_1, \ldots, \varepsilon_8$ an orthonormal basis of $\mathcal{X}(T)$ such that $\Phi$ and $S$ have the following description (with respect to the choice given in [1] we have changed the sign of $\varepsilon_1$):

\[ \Phi = A \cup B \text{ where } A = \{ \pm \varepsilon_i \pm \varepsilon_j : i \neq j \} \text{ and } \]

\[ B = \{ \frac{1}{2} \sum_{i=1}^{8} a_i \varepsilon_i : a_i = \pm 1 \text{ and } \prod_{i=1}^{8} a_i = 1 \}; \]

\[ S = \{ \alpha_1, \ldots, \alpha_8 \} \text{ where } \alpha_1 = -\varepsilon_1 - \varepsilon_2, \]

\[ \alpha_2 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8) \text{ and } \]

\[ \alpha_i = \varepsilon_{i-1} - \varepsilon_i \text{ for } i = 3, \ldots, 8. \]

For each root $\alpha$ choose also an $\mathfrak{sl}(2)$-triple $x_\alpha, y_\alpha, h_\alpha$ where $x_\alpha$ has weight $\alpha$ and $y_\alpha$ has weight $-\alpha$. Let $\beta_1 = -\varepsilon_1 + \varepsilon_2$, $\beta_2 = \varepsilon_3 + \varepsilon_4$, $\beta_3 = \varepsilon_5 + \varepsilon_6$, $\beta_4 = \varepsilon_7 + \varepsilon_8$ and define

\[ e_0 = x_{\beta_1} + x_{\beta_2} + x_{\beta_3} + x_{\beta_4}, \quad f_0 = y_{\beta_1} + y_{\beta_2} + y_{\beta_3} + y_{\beta_4}, \]

\[ h_0 = \beta_1^\vee + \beta_2^\vee + \beta_3^\vee + \beta_4^\vee = -\varepsilon_1^\vee + \varepsilon_2^\vee + \cdots + \varepsilon_8^\vee. \]

These elements are an $\mathfrak{sl}(2)$-triple and it is clear that $e_0$ is an element with associated Kostant-Dynkin diagram equal to $(0, 1, 0, 0, 0, 0, 0, 0)$. Let $O$ be its orbit. As we have already recalled in Section 6 the orbit of the line spanned by $e_0$ in $\mathbb{P}(g)$ is the open orbit of the model wonderful variety of $E_8$. Notice also that the stabilizer $G(e_0)$ of $e_0$ is connected in this case and has Levi factor isomorphic to $Sp(8)$ and that the character $\gamma_{e_0}$ is trivial (see also [1]). Hence we are interested in the space of regular functions on $O$. 

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The colors of the model wonderful variety of $E_8$ are $D_1, \ldots, D_8$ with $\omega(D_i) = \omega_i$, where $\omega_1, \ldots, \omega_8$ are the fundamental weights w.r.t. the simple system $\alpha, \ldots, \alpha_8$. Notice also that, in the notation of the previous section, we have $\mathcal{O} = C_{D_8}$. Finally the spherical roots of $M$ are

$$
\sigma_1 = \alpha_6 + \alpha_7, \quad \sigma_2 = \alpha_4 + \alpha_5, \quad \sigma_3 = \alpha_1 + \alpha_3, \quad \sigma_4 = \alpha_2 + \alpha_4,
$$

$$
\sigma_5 = \alpha_5 + \alpha_6, \quad \sigma_6 = \alpha_3 + \alpha_4, \quad \sigma_7 = \alpha_7 + \alpha_8.
$$

Here we have ordered them so that it is clear that they are a basis of a root system of type $D_7$. Notice also that, since in this case $\omega$ is injective, the list above determines also the Cartan pairing $c$ of the wonderful variety $M$.

**Theorem 8.1** ([8] Theorem 1.1). The variety $\mathcal{O}$ is normal and

$$
\mathbb{C}[O] = \mathbb{C}[\mathcal{O}] \simeq \bigoplus_{\lambda \in X(T)^+} V(\lambda).
$$

**Proof.** We know that $\mathcal{O}$ is normal by the discussion in Section 7, in particular we have $\mathcal{O} = C_{D_8} = C_8$. Moreover each adjoint orbit has even dimension so the first equality follows by normality. Since $\mathcal{O} = C_{D_8}$ we have also

$$
\mathbb{C}[\mathcal{O}] \simeq \bigoplus_{n \geq 0} \Gamma(M, L_{nD_8}) \simeq \bigoplus_{n \geq 0, \gamma \in \Sigma^\circ} s^3 V_{nD_8-\gamma}.
$$

Now notice that $D_8$ and the spherical roots are linearly independent and that $\omega$ is injective in this case, so all irreducible $G$-representations occur with multiplicity at most one. Moreover, since the variety is irreducible, if $V(\lambda)$ and $V(\mu)$ occur in this decomposition then also $V(\lambda + \mu)$ occurs. Finally, we have

$$
D_1 = 2D_8 - (\sigma_4 + 2\sigma_5 + \sigma_6 + 2\sigma_7)
$$

$$
D_2 = 4D_8 - (\sigma_1 + 2\sigma_2 + 3\sigma_3 + 4\sigma_4 + 6\sigma_5 + 3\sigma_6 + 5\sigma_7)
$$

$$
D_3 = 5D_8 - (\sigma_1 + 2\sigma_2 + 3\sigma_3 + 5\sigma_4 + 7\sigma_5 + 3\sigma_6 + 6\sigma_7)
$$

$$
D_4 = 7D_8 - (\sigma_1 + 2\sigma_2 + 4\sigma_3 + 6\sigma_4 + 9\sigma_5 + 4\sigma_6 + 8\sigma_7)
$$

$$
D_5 = 6D_8 - (\sigma_1 + 2\sigma_2 + 4\sigma_3 + 6\sigma_4 + 8\sigma_5 + 4\sigma_6 + 7\sigma_7)
$$

$$
D_6 = 4D_8 - (\sigma_2 + 2\sigma_3 + 3\sigma_4 + 4\sigma_5 + 2\sigma_6 + 4\sigma_7)
$$

$$
D_7 = 3D_8 - (\sigma_2 + 2\sigma_3 + 3\sigma_4 + 4\sigma_5 + 2\sigma_6 + 3\sigma_7)
$$

$$
D_8 = D_8 - 0
$$

hence $V(\omega_i)$ occurs in the decomposition of $\mathbb{C}[\mathcal{O}]$, for all $i$. $\square$

Notice that our proof of the normality of $\mathcal{O}$ relies on Theorem A for which we used a computer.

8.2. **From complex to real orbits: general considerations.** Let $\tilde{G}, K, \tilde{\mathfrak{g}}, \mathfrak{p}$ be as at the beginning of this section. Let $O \subset \tilde{\mathfrak{g}}$ be a nilpotent adjoint orbit of the complex algebraic group $\tilde{G}$. We want to make some general remarks on the intersection $O \cap \mathfrak{p}$. Fix $e \in O \cap \mathfrak{p}$ and let $\tilde{G}(e)$ and $K(e)$ be the stabilizers of $e$ in $\tilde{G}$ and $K$, respectively. The subgroup $\tilde{G}(e)$ is stable under $\theta$ and we define $Z = \{z \in \tilde{G}(e) : \theta(z) = z^{-1}\}$. We have an action of $\tilde{G}(e)$ on $Z$ by $g \cdot z = g\theta(g)^{-1}$ and we define $\mathbb{H}^1$ as $Z$ modulo the action of $\tilde{G}(e)$.

**Lemma 8.2.**

i) Every connected component of $O \cap \mathfrak{p}$ is a single $K$-orbit;

ii) The map $g : x \mapsto g^{-1}b(g)$ induces a bijection from the set of $K$-orbits in $O \cap \mathfrak{p}$ to $\mathbb{H}^1$;

iii) Every connected component of $Z$ is a single $\tilde{G}(e)^0$-orbit (where $\tilde{G}(e)^0$ is the identity component of $\tilde{G}(e)$).

**Proof.** Set $\eta = -\theta$. First notice that $O \cap \mathfrak{p} = O^\eta$. Being $O$ smooth and $\eta$ an involution we deduce that $O^\eta$ is smooth. Take $e \in O^\eta$. We prove that

$$
(\tilde{\mathfrak{g}} \cdot e) \cap \mathfrak{p} = \mathfrak{k} \cdot e.
$$

Let $\mathfrak{g}(e)$ be the annihilator of $e$ in $\tilde{\mathfrak{g}}$ and let $y \in \mathfrak{g}$ be such that $y \cdot e \in \mathfrak{p}$. Then $\theta(y \cdot e) = -y \cdot e$, hence $z = \theta(y) - y \in \mathfrak{g}(e)$. Take $u = \frac{1}{2}z$ then $u \in \mathfrak{g}(e)$ is such that $v = y + u \in \mathfrak{k}$ and $v \cdot e = y \cdot e$.

This proves that any $K$-orbit in $\tilde{G} \cdot e \cap \mathfrak{p}$ is open. This implies i).

Point iii) can be proved similarly and ii) is trivial. $\square$
In particular notice that if we prove that \( Z \) is connected then it follows that \( O \cap p \) is a single \( K \)-orbit.

We refine this lemma, using the Jordan decomposition. Choose an \( sl(2) \)-triple, \( e, h, f \) such that \( \theta(h) = h \) and \( \theta(f) = -f \), this is always possible, see (4). Let \( U \) be the unipotent radical of \( \tilde{G}(e) \) and \( L = \{ g \in \tilde{G}(e) : g \cdot h = h \} \). Then \( G(e) = L \cdot U \) is a Levi decomposition of \( G(e) \) (see [2 Proposition 2.4]). Notice also that \( L \) and \( U \) are stable under the action of \( \theta \). Define \( Z_L = Z \cap L \) and \( H_L \) as \( Z_L \) mod the action of \( L \) given by \( g \cdot z = g z \theta(g)^{-1} \).

**Lemma 8.3.**

i) The inclusion \( Z_L \subset Z \) induces a bijection from \( H_L^1 \) to \( H^1 \);

ii) If \( Z_L \) is connected then \( O \cap p \) is a single \( K \)-orbit.

**Proof.** Let \( x, y \in Z_L \) and assume that \( x = h y \theta(h)^{-1} + h \ell u, \ell \in L \) and \( u \in U \). It follows immediately that \( x = h y \theta(h)^{-1} \). This prove the injectivity of the map \( H_L^1 \rightarrow H^1 \).

Hence the map \( H^1 \rightarrow H^1 \) is surjective. This proves i). Now ii) follows from Lemma 8.2. \( \square \)

### 8.3. The case of the real model orbit of \( E_8 \).

In the last two sections of [1] a real version of the model orbit is considered. We now recall, from [1], some of the structural results about this orbit. We also prove Proposition 8.3 which is probaly well known and it is somehow implicit (even if not necessary) in the discussion in [1].

Let \( \tilde{G} \) be the complex algebraic group of type \( E_8 \) and let \( \tilde{G}_R \) be its split real form. Then \( K \) is isomorphic to a quotient 2 to 1 of Spin(16), more precisely we can coniugate \( \tilde{G}_R \) so that \( f \) is the Lie algebra spanned by \( t \), the Lie algebra of \( T \), and by the vectors \( x_\alpha \) with \( \alpha \in A \). With this choice \( p \) is spanned by the vectors \( x_\alpha \) with \( \alpha \in B \).

As a simple system for the root system of \( f \simeq so(16) \) we choose the usual basis, but we enumerate it starting from zero, that is

\[
\tau_i = \epsilon_{i+1} - \epsilon_{i+2} \text{ for } i = 0, \ldots, 6 \text{ and } \tau_7 = \epsilon_7 + \epsilon_8.
\]

We denote by \( \omega_0^D \) the associated fundamental weights. In particular we obtain that \( p \) is the spin representation associated to the weight \( \omega_0^D \). Moreover, since \( \tilde{G} \) is simply connected the subgroup \( K \) is connected. Finally notice that \( \omega_0^D \not\in \mathcal{A}(T) \) while \( \omega_0^D \in \mathcal{A}(T) \). Hence the group \( K \) is the quotient 2 to 1 of Spin(16). We set also \( G = \text{Spin}(16) \). Notice that in [1] \( K \) is claimed to be isomorphic to Spin(16), but this does not seriously affect any of their arguments.

In order to prove that the roots \( \tau_1, \ldots, \tau_7 \) are coniugated to the roots \( \sigma_1, \ldots, \sigma_7 \) introduced in Section 8.1 we introduce a new simple system of the root system of type \( E_8 \). The vectors

\[
\zeta_1 = \frac{1}{2}(-\epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5), \quad \zeta_2 = \frac{1}{2}(\epsilon_1 + \epsilon_6 + \epsilon_7 - \epsilon_8), \\
\zeta_3 = \frac{1}{2}(\epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5), \quad \zeta_4 = \frac{1}{2}(\epsilon_1 + \epsilon_6 - \epsilon_7 - \epsilon_8), \\
\zeta_5 = \frac{1}{2}(\epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5), \quad \zeta_6 = \frac{1}{2}(\epsilon_1 - \epsilon_6 + \epsilon_7 + \epsilon_8), \\
\zeta_7 = \frac{1}{2}(\epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5), \quad \zeta_8 = \frac{1}{2}(\epsilon_1 - \epsilon_6 - \epsilon_7 - \epsilon_8),
\]

form an orthonormal basis of \( t^* \), and \( \pm \epsilon_i \pm \epsilon_j \in \Phi \) for \( i \) odd and \( j \) even. Notice also that

\[
\gamma_2 = \frac{1}{2}(\zeta_1 - \zeta_2 - \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_7 + \zeta_8) = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8) \in \Phi, \\
\gamma_1 = -\epsilon_1 - \epsilon_2, \text{ and } \gamma_i = \epsilon_{i-1} - \epsilon_i, \text{ for } i = 3, \ldots, 8, 
\]

form a simple system of \( \Phi \) (since they are elements of \( \Phi \) with the right scalar products). Hence there exists an element \( w \) in the Weyl group such that \( w(\epsilon_i) = \zeta_i \), for all \( i \). Finally, notice that we have

\[
\tau_1 = \gamma_6 + \gamma_7, \quad \tau_2 = \gamma_4 + \gamma_5, \quad \tau_3 = \gamma_1 + \gamma_3, \quad \tau_4 = \gamma_2 + \gamma_4, \\
\tau_5 = \gamma_5 + \gamma_6, \quad \tau_6 = \gamma_3 + \gamma_4, \quad \tau_7 = \gamma_7 + \gamma_8
\]

so that \( w(\sigma_i) = \tau_i \). Notice also that

\[
\tau_0 = \frac{1}{2}(\zeta_1 - \zeta_3 - \zeta_5 + \zeta_7 + \zeta_2 + \zeta_4 + \zeta_6 + \zeta_8) = \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5.
\]
We denote by $\Lambda$ the weight lattice of Spin(16) and by $\Lambda_\mathbb{R} = \mathcal{X}(T)$ the sublattice given by the weights of $E_8$. We denote also by $\Lambda^+$ the dominant weights of $\Lambda$ w.r.t. the simple system $\gamma_0, \ldots, \gamma_7$ and by $\Lambda_\mathbb{R}^+$ the dominant weights of $\Lambda_\mathbb{R}$ w.r.t. the simple system $\gamma_0, \ldots, \gamma_8$. We have $\Lambda_\mathbb{R}^+ \subset \Lambda^+$. We denote by $\omega_i^\gamma$ the fundamental weights of $\Lambda_\mathbb{R}$ w.r.t. the simple system $\gamma_1, \ldots, \gamma_8$. Notice that $\omega_0^\gamma = \omega_7^\gamma$.

Let now $O$ be the adjoint orbit of $\tilde{g}$ considered in Section 5.5.3, (notice that what was in that section denoted by $G$ is now denoted by $\tilde{G}$). We want to study the intersection $O \cap p$. Let us first choose an element $e$ of $O \cap p$.

Consider the semisimple part $K_0$ of the standard Levi factor of the maximal parabolic subgroup $P_0$ of $K$ associated with $\tau_0$. This is a group isogeneous to Spin(14), we denote its Lie algebra by $\mathfrak{k}_0$. Let $\mathfrak{p}_0$ be the subspace of $\mathfrak{p}$ spanned by the root vectors of weight of the form $\frac{1}{2}(\sum a_i e_i)$ with $a_1 = -1$. Its highest weight is $\frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8)$. Hence it is an irreducible Spin(14)-submodule of $\mathfrak{p}$ isomorphic to the module $V_{\frac{1}{2} \mathfrak{h}}$ of Section 5.5.3.

**Lemma 8.4.** The vector $h_8$ of Section 5.5.3 belongs to $O$.

**Proof.** The vectors

\[
\begin{align*}
\eta_1 &= \frac{1}{2}(-e_1 - e_6 - e_7 - e_8), \\
\eta_2 &= \frac{1}{2}(-e_2 + e_3 + e_4 + e_5), \\
\eta_3 &= \frac{1}{2}(-e_1 + e_6 - e_7 - e_8), \\
\eta_4 &= \frac{1}{2}(-e_2 + e_3 + e_4 + e_5), \\
\eta_5 &= \frac{1}{2}(-e_1 - e_6 + e_7 - e_8), \\
\eta_6 &= \frac{1}{2}(e_2 + e_3 - e_4 + e_5), \\
\eta_7 &= \frac{1}{2}(e_1 - e_6 + e_7 + e_8), \\
\eta_8 &= \frac{1}{2}(e_2 + e_3 + e_4 - e_5).
\end{align*}
\]

form an orthonormal basis of $\mathfrak{t}$, and $\pm u_i \pm u_j \in \Phi$ if $i$ is odd and $j$ is even. Notice also that

\[
\begin{align*}
\tilde{\gamma}_2 &= \frac{1}{2}((\eta_1 - \eta_2 - \eta_3 + \eta_4 + \eta_5 + \eta_6 + \eta_7 + \eta_8) = \epsilon_2 - \epsilon_6, \\
\tilde{\gamma}_1 &= -\eta_1 - \eta_2, \text{ and } \tilde{\gamma}_i = \eta_i - 1 - \eta_i, \text{ for } i = 3, \ldots, 8,
\end{align*}
\]

is a simple system of $\Phi$. Hence there exists an element $w$ of the Weyl group such that $w(\epsilon_i) = \eta_i$, for all $i$. Choose a representative $\tilde{w}$ of $w$ in $\tilde{G}$. Define $e = \tilde{w}(e_0)$, $h = w(h_0) = -\eta_1^\gamma + \eta_2^\gamma + \ldots + \eta_8^\gamma$ and $f = \tilde{w}(f_0)$. Then $\theta(h) = h$ and $\theta(f) = -f$. More explicitly, we have $h = -2\epsilon_1^\gamma + \epsilon_2^\gamma + \epsilon_3^\gamma + \epsilon_4^\gamma + \epsilon_5^\gamma$, $e = x_3 + x_4 + x_5 + x_8$, where $w(\beta) = \delta_1 = -\eta_1 + \eta_2$, $w(\beta_2) = \delta_2 = \eta_3 + \eta_4$, $w(\beta_3) = \delta_3 = \eta_5 + \eta_6$, $w(\beta_4) = \delta_4 = \eta_7 + \eta_8$ and similarly for $f$.

Notice that $e \in \mathfrak{p}_0$. Moreover, the two vectors $e$ and $h_8$ are linear combinations of vectors of the same weights $\delta_1, \delta_2, \delta_3, \delta_4$ and these weights are linearly independent, so they are conjugated under the action of the maximal torus.

As shown in the proof we can choose a representative $\tilde{w}$ of $w$ such that $\tilde{w}(e_0) = h_8$. Set $e = h_8$ and $h = w(h_0)$. In particular the stabilizer of $e$ is the parabolic induction of the stabilizer of $h_8$ in Spin(14).

More explicitly, we have

\[
\mathfrak{t}(e) = \mathbb{C} \omega_0^\gamma \oplus \mathfrak{u}_0^\gamma \oplus \mathfrak{h}_0
\]

where $\mathfrak{u}_0^\gamma$ is the Lie algebra of the unipotent radical of the parabolic subgroup opposite to $P_0$. $\mathfrak{h}_0$ is the annihilator of $h_8$ in $\mathfrak{k}_0$ and $\omega_0^\gamma$ is orthogonal to $\tau_1, \ldots, \tau_7$. In particular the Levi factor of $\mathfrak{t}(e)$ is isomorphic to $\mathfrak{gl}(4)$.

We now want to describe in some detail the stabilizer $K(e)$ and apply Lemma 8.3 to prove that $O \cap p$ is a single $K$-orbit. As recalled in Section 5.5.3, the Levi factor $L$ of $\tilde{G}(e)$ is Sp(8) and $L = \tilde{L}^d$. Furthermore, notice that there is only one involution of Sp(8) such that the Lie algebra of the fixed point subgroup is isomorphic to $\mathfrak{gl}(4)$. This involution can be described as follows. Define the $8 \times 8$ matrices

\[
J = \begin{pmatrix}
0 & I_4 \\
-I_4 & 0
\end{pmatrix}, \quad \rho = \begin{pmatrix}
I_4 & 0 \\
0 & -I_4
\end{pmatrix}
\]

and Sp(8) as the matrices preserving the form $J$. Then Sp(8) is stable under the conjugation by the matrix $\rho$, which is an involution. Moreover $L$ is isomorphic to GL(4).

**Proposition 8.5.** If $O$ is the model orbit of $E_8$ and $\tilde{G}_R$ is the split real form of $E_8$, then $O \cap p$ is a single $K$-orbit.

**Proof.** By Lemma 8.3 and the above discussion, it is enough to prove that

\[
Z_L = \{ z \in \text{Sp}(8) : \theta(z) = z^{-1} \}
\]
is connected. Let \( z = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) where \( A, B, C, D \) are \( 4 \times 4 \) matrices. The condition \( z \in Z_L \) is equivalent to

\[
A^2 = I_4 + BC, \quad D = A^\tau, \quad B = B^\tau, \quad C = C^\tau, \quad AB = BA^\tau \text{ and } A^T C = CA.
\]

Acting by \( g \in \text{Sp}(8) \), via \( g \cdot z = g \theta(g)^{-1} \), we remain in the same connected component. Using \( g \) of the form \( \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) \) with \( \alpha \in \text{GL}(4) \) we see that \( B \) is of the form \( \left( \begin{array}{cc} b & 0 \\ 0 & c \end{array} \right) \), and \( A = \left( \begin{array}{cc} a & d \\ b & c \end{array} \right) \), and using equations (1) we see that \( c = 0, \quad a = a^\tau \) and \( d^2 = 1 \). Then using \( g \) of the form \( \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta^{-1} \end{array} \right) \) and \( \beta = \left( \begin{array}{cc} 0 & \gamma \\ 0 & 0 \end{array} \right) \) we can also assume that \( d \) is diagonal. Now we choose \( g \) of the form \( \left( \begin{array}{cc} s & 0 \\ 0 & 1 \end{array} \right) \). We get

\[
g \cdot z = \left( \begin{array}{cc} A + sC & B + s(A + A^\tau) + s^2C \\ C & A^\tau + sC \end{array} \right).
\]

If we compute the determinant of \( B + s(A + A^\tau) + s^2C \) we see that it is a polynomial in \( s \) and its lowest degree term is 4 det(\( d \)). Hence there exists \( s \) such that \( B + s(A + A^\tau) + s^2C \) is invertible. So we can assume \( B \) invertible and arguing as before we can assume \( B = I_4 \). Now, for \( B = I_4 \), the equations in (1) take the form \( A = A^\tau = C, \quad C = -I + A^2 \). Such equations define an algebraic subset which is isomorphic to an affine space and, in particular, connected. Therefore \( Z_L \) is connected.

8.4. The coordinate ring of the real model orbit. Here we describe the coordinate ring of \( O \cap p \). In [1] it is shown that this description follows from a vanishing result which in the case of the real model orbit is conjectural (see Conjecture 3.13 and Theorem 7.13 in [1]).

Let \( M_\theta \) be the wonderful comodel variety of cotype \( E_8 \). This is a wonderful variety for the group \( \text{Spin}(14) \). Consider the parabolic induction \( M \) of \( M_\theta \). This is a wonderful variety with spherical roots equal to \( \tau_1, \ldots, \tau_7 \) and colors \( c_0', \ldots, c_4' \), where \( \omega(c_0') = \omega_0' \) and, for \( i > 0 \), \( c_i' \) is induced by the respective color of \( M_\theta \) (see Section 5).

Let \( O_p = O \cap p \). Then \( X_p = \mathbb{P}(O_p) \subset \mathbb{P}(p) \) is the open orbit of \( M \), and \( O_p \) is the cone \( C_D \) for \( D = D_8' \). Indeed \( e = h_s \in V_\omega(D_8') = V_\omega^{\alpha} = V_2^\alpha \simeq p \) and \( D_8' \) is faithful.

**Theorem 8.6.** The cone \( O_p \) is normal and we have the following isomorphisms of \( K \)-modules

\[
\mathbb{C}[O_p] \cong \bigoplus_{\lambda \in \Lambda^+_8} V(\lambda).
\]

**Proof.** Notice that the combinatorics of colors and spherical roots is essentially the same of that of the model wonderful variety of type \( E_8 \). In particular, since \( D_8' \) is minuscule, from Theorem 5.1 it follows that \( O_p \) is normal. Hence, as a \( K \)-module we have that its coordinate ring is the sum of all the modules \( V_{nD_8' - \tau} \) for \( n \geq 0 \), \( \tau \in \mathbb{N}[\tau_1, \ldots, \tau_7] \) and \( nD_8' - \tau \in \mathbb{N}[D_8', \ldots, D_8'] \). Moreover, with \( \tau \) as above, the latter condition is equivalent to \( nD_8' - \tau \in \mathbb{N}[D_8', \ldots, D_8'] \). Now notice that \( \omega(D_8') \) and \( \tau_1, \ldots, \tau_7 \) are linearly independent (this is not true for \( M_\theta \) but it is true for \( M \) because of the presence of the extra color \( D_8' \)).

So we obtain that the coordinate ring of \( O_p \) is the sum of all modules \( V_{nD_8' - \tau} \) where \( n \) and \( \tau \) are as above. Finally, the computation is exactly the same of that given in the proof of Theorem 5.1 for the model orbit of type \( E_8 \), since the two situations are conjugated by an element of the Weyl group.

Notice that our proof of the normality of \( O_p \) via Theorem 5.1 did not require any computer calculation. Moreover, the description of the coordinate ring of the normalization of \( O_p \) is independent of Theorem 5.1.

The combinatorics of distinguished subsets of colors allows to describe completely the \( K \)-orbits in the closure of \( O_p \) (see for example [18]), and in particular to prove that \( O_p \setminus O_p \) has codimension at least two in \( O_p \). Indeed, one sees that this property depends only on the combinatorics of colors and spherical roots, and in this case the combinatorics is the same of that of the complex model orbit, whose boundary has codimension at least two.

Here we can avoid such an argument. Below we will prove that \( \mathbb{C}[O_p] = \mathbb{C}[O_p] \) and this will also imply that \( O_p \setminus O_p \) has codimension at least two.

8.5. Computation of the space of sections of a line bundle associated to an admissible pair: general considerations. For the next lemma we put ourself in a general setting. Let \( G \) be simply connected and let \( M \) be a wonderful compactification of \( G/H \). Let \( E \in \mathbb{N}_{\Delta} \). Let \( \mathcal{C} h \) be a line in \( V(\omega_E^\lambda) \) where \( H \) acts with character \( \xi_E \). Assume that the stabilizer of \( \mathcal{C} h \) is equal to \( H \) and let \( H_0 \) be the stabilizer of \( h \). Furthermore, assume that \( \xi_E \) induces an isomorphism of \( H/H_0 \) with \( C^* \); we identify \( H/H_0 \) with \( C^* \) using this isomorphism. Finally, notice that \( H/H_0 \) acts on the right on \( G/H_0 \).

---

1 Here \( \text{codim}_{O_p}(O \setminus O) = 16 \) and \( \text{codim}_{O_p}(O_p \setminus O_p) = 8 \).
Lemma 8.7. Let $D \in \mathbb{Z}\Delta$ and let $\chi$ denote the restriction of $\xi_D$ to $H_0$. We have the following isomorphism of $G$-modules

$$\Gamma(G/H_0, \mathcal{V}_\chi) \simeq \bigoplus_{n \in \mathbb{Z}, \sigma \in \mathbb{Z}\Sigma} V_{D+nE-\sigma}.$$

Proof. We have

$$\Gamma(G/H_0, \mathcal{V}_\chi) \simeq \langle \mathbb{C}[G] \otimes \mathbb{C}_{-\chi} \rangle^{H_0} \simeq \bigoplus V(\lambda^*) \otimes (V(\lambda) \otimes \mathbb{C}_{-\chi})^{H_0}.$$

Now notice that $H$ acts on $(V(\lambda) \otimes \mathbb{C}_{-\chi})^{H_0}$ and so the latter decomposes according to the action of a character of the form $\xi_D + n\xi_E$. Moreover, all these eigenspaces will be of dimension one, since $H$ is spherical. Hence we have

$$\Gamma(G/H_0, \mathcal{V}_\chi) \simeq \bigoplus_{\lambda \in \mathcal{X}(T)^+, n \in \mathbb{Z}} V(\lambda^*) \otimes (V(\lambda) \otimes \mathbb{C}_{-\xi_D-n\xi_E})^H.$$

Now recall that the space of spherical vectors $V(\omega^*_F)^{(H)}$ is nonzero if and only if $F \in \mathbb{N}\Delta$, and $\xi_F = \xi_{D+nE}$ if and only if $F = D + nE - \sigma$ for $\sigma \in \mathbb{Z}\Sigma$. \hfill \Box

Let us specialize this identity to the case $D = 0$ and compare the coordinate ring of $G/H_0$ with the coordinate ring of the normalization of its closure in $V(\omega^*_F)$. Since

$$\mathbb{C}[\tilde{C}_E] = \bigoplus_{n \geq 0} \Gamma(M, L_{nE}) \simeq \bigoplus_{n \geq 0, \sigma \in \mathbb{N}\Sigma; nE-\sigma \in \mathbb{N}\Sigma} V_{nE-\sigma},$$

we have the following.

Lemma 8.8. The equality $\mathbb{C}[G/H_0] = \mathbb{C}[\tilde{C}_E]$ is equivalent to the fact that, for all $n \in \mathbb{Z}$ and $\sigma \in \mathbb{Z}\Sigma$, if $nE - \sigma \in \mathbb{N}\Delta$ then $n \geq 0$ and $\sigma \in \mathbb{N}\Sigma$.

We now apply this lemma to our special case in which $G$ is $\text{Spin}(16)$, $H_0 = G(e)$, $M$ is the wonderful compactification of $X_F$ and $E = D'_8$.

Lemma 8.9. For all $n \in \mathbb{Z}$ and $\sigma \in \mathbb{Z}\Sigma$, if $nD'_8 - \sigma \in \mathbb{N}\Delta$ then $n \geq 0$ and $\sigma \in \mathbb{N}\Sigma$.

Proof. Write $nD'_8 - \sigma = \sum_{i=0}^8 a_iD'_i$ and notice that the conditions $a_1, \ldots, a_8 \geq 0$ are equivalent to the conditions obtained from Theorem 8.7 by applying Lemma 8.8 to the complex model orbit of type $E_8$. \hfill \Box

From this and Theorem 8.7 by applying Lemma 8.8 to the real model orbit, we get that

$$\mathbb{C}[O_F] = \mathbb{C}[O_{\tilde{F}}].$$

8.6. Sections of the line bundle associated to the admissible pair for the real model orbit.

We now want to compute the characters $\gamma_e$ and $\gamma'_e$. For this we further analyze the stabilizer $T(e)$ of $e$ in $T$. Let $A_8$ be the lattice dual to $A_8$. Recall the definition of the vectors $\eta_i$ given in the equations (2) and denote by $\eta'_i$ the vectors of the dual basis. Let $x_1^i = \eta_1^i + \eta_2^i$ and $x_2^i = \eta_{2i-1}^i - \eta_{2i}^i$ for $i = 2, 3, 4$ and define $R^\epsilon$ as the sublattice of $A_8^\epsilon$ generated by $x_1^1, \ldots, x_4^1$ and $R_{0}^{\epsilon}$ as the sublattice generated by $x_1^1, x_2^1, x_3^1, x_4^1, x_1^2, x_2^2, x_3^2, x_4^2$. Then $R^\epsilon$ and $R_{0}^{\epsilon}$ are direct factors of $A_8^\epsilon$. Finally, let $T_0$ be the maximal torus of $K_0$ (the subgroup of $\text{Spin}(14)$ introduced above) contained in $T$ and $T_0(e) = T(e) \cap T_0$. We have

$$T = A_8^\epsilon \otimes_{\mathbb{Z}} \mathbb{C}^*, \quad T(e) = R^\epsilon \otimes_{\mathbb{Z}} \mathbb{C}^* \quad \text{and} \quad T_0(e) = R_{0}^{\epsilon} \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

We already know that the Levi factor of $K(e)$ is isomorphic to $\text{GL}(4)$ so that any character is a power of the determinant. We describe now the center of $\text{GL}(4)$ and the determinant as an element of dual lattice of $R^\epsilon$. Let $z^* = x_1^1 - x_2^1 - x_3^1 - x_4^1 = 2\epsilon_1^* + \epsilon_2^* + \epsilon_3^* + \epsilon_4^*$, then using the description of $b_0$ given in Section 8.5 it is easy to see that $z^*$ is a central cocharacter. In particular, if $R_{0}^{\epsilon} = \mathbb{Z}z^*$ then

$$T_Z = R_{0}^{\epsilon} \otimes_{\mathbb{Z}} \mathbb{C}^*$$

is the center of $L \simeq \text{GL}(4)$. Now we compute the character $\gamma_e$. We have already noticed that it is enough to compute its restriction to $T$. Moreover its restriction to $T_0(e)$ must be trivial since $\text{SL}(4)$ has no characters so we compute its restriction to $T_Z$. Using the description of the stabilizer $T(e)$ in (2) and the description of $b_0$ given in Section 8.5 it is easy to prove that $(\gamma_e, z^*) = 4$. Hence we see that $\gamma_e$ restricted to the Levi of $K(e)$, which is isomorphic to $\text{GL}(4)$, equals the determinant and

$$\gamma_e = x_1 - x_2 - x_3 - x_4.$$
In particular it is not the square of a character. However, on the cover \( G \) of \( K \) we can consider the character \( -\omega^0_0 = -\varepsilon_1 \), and we denote by \( \chi \) its restriction to \( G(\varepsilon) \). We have
\[
\chi = \frac{1}{2} (x_1 - x_2 - x_3 - x_4) = \frac{1}{2} \gamma.'
\]
Notice that \( \xi_{D_1} = \chi \). Indeed, \( \omega^0_{D_1} = \omega^0_0 \) and \( G(\varepsilon) \) contains the unipotent radical \( U_0^+ \) of \( P_0^+ \). Hence the only \( G(\varepsilon) \)-semiinvariant in \( V(\omega^0_0) \) is the lowest weight vector which has weight \( -\omega^0_0 \).

We now describe the space of sections \( \Gamma(O_p, V_\chi) \). In \([1]\) the same description follows from a vanishing result which is not proved in this case (see Conjecture 8.6 and Theorem 8.10 in \([1]\)).

**Theorem 8.10.** We have the following isomorphism of \( G \)-modules
\[
\Gamma(O_p, V_\chi) \simeq \bigoplus_{\lambda \in \Lambda^+_E} V(\omega^0_0 + \lambda).
\]

**Proof.** We have seen above that \( \xi_{D_0} = \chi \). Hence we can apply Lemma 8.7 with \( D = D_0' \) and \( E = D_8' \). We obtain
\[
\Gamma(O_p, V_\chi) \simeq \bigoplus_{n \in \mathbb{Z}, \sigma \in \Sigma} V(D_0' + nD_8' - \sigma).
\]
Write \( D_0' + nD_8' - \sigma = \sum_{i=0}^8 a_i D_i' \). Notice that as in the proof of Lemma 8.9 this implies \( a_1, \ldots, a_8 \geq 0 \). Hence we obtain \( n \geq 0 \) and \( \sigma \in \Sigma \). In particular the condition \( a_0 \geq 0 \) is automatically satisfied. As already noticed in the proof of Theorem 8.6, \( \Lambda^+_E \) consists of the weights of the form \( \omega(nD_8' - \sigma) \) for \( n \geq 0 \), \( \sigma \in \Sigma \) and \( nD_8' - \sigma \in \Delta \).

9. Degeneracy of the multiplication.

Here we give a counterexample to the surjectivity of the multiplication of sections of line bundles, on wonderful varieties, generated by local sections.

9.1. Counterexample. Let \( G \) be \( \text{SO}(2r+1) \) and let \( M \) be the model wonderful variety for the group \( G \). It is not isomorphic to the model wonderful variety for \( \text{Spin}(2r+1) \). Its set of colors \( \Delta = \{ D_1, \ldots, D_r \} \) is in bijection, via \( \omega \), with the set \( \{ \omega_1, \ldots, \omega_{r-1}, 2\omega_r \} \). Its set of spherical roots is \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{r-1} + \alpha_r, 2\alpha_r \} \). If \( r = 2 \) or \( r = 3 \), then the multiplication of sections is surjective. Set \( r = 4 \) and consider the lower triple \( (D_2, D_2, D_1) \): then \( V(\omega_1) \not\subset V(\omega_2) \), hence \( s^{2D_2 - D_1} V_{D_1} \not\subset V_{D_2} \). In particular, the multiplication of sections is not surjective, for all \( r \geq 4 \).

9.2. Degeneracy of the multiplication. Roughly speaking, in its nature the previous example does not express a lack of the multiplicity, but rather a lack of the tensor product. Indeed \( V(\omega_1) \not\subset V(\omega_2) \) but \( V(2\omega_1) \subset V(2\omega_2) \), so that it expresses the fact that the saturation property does not hold for groups of type \( B \): similar things cannot happen if \( G \) is of type \( A \) and, conjecturally, whenever \( G \) is simply laced. It is worth noticing that the same situation holds if we consider the multiplication of the wonderful variety considered in previous example: \( s^{2D_2 - D_1} V_{D_2} \not\subset V_{D_2} \), but \( s^{2D_2 - 2D} V_{D_2} \subset V_{D_2} \).

We briefly recall what the saturation property is. Let \( G \) be a simply connected almost simple algebraic group. We say that the saturation property holds for \( G \) if, whenever \( d > 0 \) and \( \lambda, \mu, \nu \in \Lambda^+ \) are such that \( \nu \leq \lambda + \mu \) and \( V(d\nu) \subset V(d\lambda) \otimes V(d\mu) \), then it holds also \( V(\nu) \subset V(\lambda) \otimes V(\mu) \). In [27] A. Knutson and T. Tao shown that the saturation property holds if \( G \) is of type \( A \), while in [24] M. Kapovich and J. Millson conjectured that the saturation property holds whenever \( G \) is simply laced.

We want to consider the saturation property in the more general context of the multiplicity law, the classical case corresponding to the wonderful compactification of an adjoint group ([23, Lemma 3.1]). We say that the saturation property holds for a wonderful variety \( M \) with set of colors \( \Delta \) and set of spherical roots \( \Sigma \) if, whenever \( d > 0 \) and \( D, E, F \in \Delta \) are such that \( D \leq \Sigma E + F \) and \( s^{d(E+F-D)} V_{dE} \subset V_{dE} V_{dF} \), then it holds also the inclusion \( s^{dE + F - D} V_D \subset V_{dE} V_{dF} \).

Suppose that \( M \) is a wonderful variety and let \( E, F \in \Delta \). Then the following inclusion holds
\[
V_{E} V_{F} \subset \bigoplus_{D \in \Delta \setminus \{ D \leq \Sigma E + F \}} s^{E+F-D} V_{D}.
\]
If the equality holds in the previous inclusion, then we say that the product \( V_{E} V_{F} \) is non-degenerate, while if the equality holds for every \( E, F \in \Delta \) then we say that the multiplication of \( M \) is non-degenerate.

It is easy to show that if the multiplication of \( M \) is non-degenerate and if the saturation property holds for \( G \), then the latter holds for \( M \) as well.
Another equivalent description of the multiplication follows by identifying sections of line bundles on \( M \) with \( H \)-semi-invariant functions on \( G \), \( H \) acting on the right. More precisely, given \( E \in \mathbb{N}\Delta \), we may identify \( \Gamma(M, \mathcal{L}_E) \) with a submodule of \( \mathbb{C}[G]^{(H)} \). This yields a decomposition
\[
\mathbb{C}[G]^{(H)} = \bigoplus_{E \in \mathbb{N}\Delta} V_E
\]
which is compatible with the multiplication of sections, therefore we may also regard the product \( V_E V_F \) as the restriction of the natural multiplication in \( \mathbb{C}[G]^{(H)} \).

**Example 9.1.** Let \( G = \text{SL}(2) \) and consider the basic case of the rank one wonderful variety \( M = \mathbb{P}^1 \times \mathbb{P}^1 \), whose generic stabilizer is the maximal torus \( T \). Then \( \Delta = D^+, D^- \), where \( D^+ = (1, 0) \times \mathbb{P}^1 \) and \( D^- = [1, 0] \times \mathbb{P}^1 \). Moreover \( \Delta = D^+ + D^- \) and \( \omega_{D^+} = \omega_{D^-} = 1 \) equal the fundamental weight of \( G \).

Given \( n \in \mathbb{N} \), identify the simple \( G \)-module \( V(n) \) of highest weight \( n \) with the set of homogeneous polynomials of degree \( n \) in two variables. Given \( aD^+ + bD^- \in \mathbb{N}\Delta \), the corresponding \( T \)-eigenvector is \( h(a, b) = x^a y^b \).

The projection \( \pi_m : V(a + b) \otimes V(c + d) \to V(a + b + c + d - 2m) \) is described on the basis of \( T \)-eigenvectors as follows:
\[
x^a y^b \otimes x^c y^d \mapsto \sum_{u + v = m} (-1)^u u!v! \begin{pmatrix} a \\ u \\ v \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} d \\ v \end{pmatrix} x^{a+c+m} y^{b+d-m}.
\]

As shown by the following two examples, the multiplication of \( M \) is degenerate.

- Consider \( D^+ + D^- \leq \Sigma 2(D^+ + D^-) \). Then \( V(2) \subset V(2)^{\otimes 2} \) but \( \pi_2(h(1, 1) \otimes h(1, 1)) = 0 \), so that \( V_{D^+ + D^-} \not\subset V_{2(D^+ + D^-)} \). This can also be explained since \( V(2) \) is not contained in the second symmetric power of \( V(2) \).
- Consider \( 2D^+ + D^- \leq (3D^+ + D^-) + (D^+ + 2D^-) \). Then \( V(3) \subset V(4) \otimes V(3) \) but \( \pi_2(h(3, 1) \otimes h(1, 2)) = 0 \), so that \( V_{2D^+ + D^-} \not\subset V_{3D^+ + D^-} - V_{D^+ + 2D^-} \).

More generally, the multiplication is degenerate whenever \( \Sigma \cap S \neq \emptyset \). This can be reduced to the basic case of \( \text{SL}(2)/T \) as shown in the following proposition.

**Proposition 9.2.** Suppose that \( S \cap \Sigma \neq \emptyset \). Then the multiplication is degenerate.

**Proof.** Let \( \alpha \in S \cap \Sigma \) and suppose that the multiplication is non-degenerate. Consider the rank one wonderful subvariety \( M' \) defined by intersecting all the \( G \)-stable prime divisors \( M' \) with \( \sigma \in \Sigma \setminus \{ \alpha \} \), denote \( \Delta' \) its set of colors. Then \( M' \) is the parabolic induction of a wonderful variety for \( \text{SL}(2) \) isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), whose generic stabilizer is a maximal torus of \( \text{SL}(2) \) and whose set of colors is identified with \( \Delta'(\alpha) = \{ D \in \Delta' : P_\alpha D \neq D \} \), where \( P_\alpha \) denotes the minimal parabolic associated to \( \alpha \). It follows then by Corollary 17 that, for every \( D' \in \mathbb{N}\Delta' \setminus \Delta'(\alpha) \), \( V_{D'} V_{E'} = V_{D'+E'} \) for all \( E' \in \mathbb{N}\Delta' \).

Let \( D', E', F' \in \mathbb{N}\Delta' \) be such that \( D' \leq \Sigma, E' \leq F' \) and \( V(\omega_{D'}) \subset V(\omega_{E'}) \otimes V(\omega_{F'}) \). Denote \( \rho : \text{Pic}(M') \to \text{Pic}(M) \) the restriction. From the combinatorics of colors and spherical roots it follows that there exist \( E'_0, F'_0 \in \mathbb{N}[\Delta' \setminus \Delta'(\alpha)] \) and \( E, F \in \mathbb{N}\Delta \) such that \( E' + E'_0 = \rho(E) \) and \( F' + F'_0 = \rho(F) \). Let \( D = E + F - (E' + F'_0 - D') \) and notice that \( D \in \mathbb{N}\Delta \) and \( D' + E'_0 + F'_0 = \rho(D) \). Moreover the inclusion \( V(\omega_{D'}) \subset V(\omega_{E'}) \otimes V(\omega_{F'}) \) implies that \( V(\omega_D) \subset V(\omega_E) \otimes V(\omega_F) \), hence the non-degeneracy of the multiplication of \( M \) implies that \( V_D \subset V_E V_F \) and it follows \( V_{D' + E'_0 + F'_0} \subset V_{E' + E'_0} V_{F' + F'_0} \). On the other hand, by Corollary 18, \( V_{D' + E'_0 + F'_0} \subset V_{E' + E'_0} V_{F' + F'_0} V_{E'_0 + F'_0} = V_P V_P V_{E'_0 + F'_0} \) if and only if \( V_D \subset V_E V_F \).

Therefore, we have deduced that the multiplication of \( M' \) is non-degenerate, but this is a contradiction by the previous example together with Proposition 10. \( \square \)

Given a spherical subgroup \( H \subset G \), define its spherical closure \( \overline{H} \) as the kernel of the action of the normalizer \( N_G(H) \) on \( X(H) \). If \( H \) is equal to its spherical closure, then \( X(H) \) is also spherical closed. By a theorem of F. Knop 24, Corollary 7.6, if \( H \) is spherical closed then \( G/H \) admits a wonderful compactification. If \( G \) is not simply laced, then not spherically closed spherical subgroups \( H \) such that \( G/H \) admits a wonderful compactification exist. The projection \( G/H \to G/H \) canonically identifies the colors of \( G/H \) with those of \( G/H \).

Generally speaking, if \( M \) is the wonderful compactification of \( G/H \) where \( H \) is not spherically closed, then the multiplication may be degenerate. Indeed, it is easy to show that \( \mathbb{C}[G]^{(H)} \cong \mathbb{C}[\overline{G}] \), therefore, if \( \Sigma \) is the set of spherical colors of \( G/\overline{H} \) and if \( D \in \mathbb{N}\Delta \) is such that \( V_D \subset V_E V_F \), then it must be
\( D \leq E + F \). In this way we may construct examples of non-spherically closed spherical subgroups \( H \) of \( G \) (possessing no simple spherical roots) whose associated multiplication is degenerate.

**Example 9.3.** Consider the non-adjoint symmetric wonderful variety \( M \) for \( \text{Sp}(8) \) with spherical roots \( \sigma_1 = \alpha_1 + 2\alpha_2 + \alpha_3, \sigma_2 = \alpha_3 + 4\alpha_4 \). Then \( M \) possesses two colors \( D_2 \) and \( D_4 \), where \( \omega_{D_2} = \omega_2 \) and \( \omega_{D_4} = \omega_4 \). Then we have \( D_2 < \sigma \cdot 2D_2 \) and \( V(\omega_2) \subset V(\omega_2)^{\otimes 2} \), on the other hand \( \Sigma = \{ \sigma_1, 2\sigma_2 \} \) and \( 2D_2 - D_2 = \sigma_1 + \sigma_2 \not\in \mathbb{N}\Sigma \), therefore it cannot be \( V_{D_2} \subset V_{D_2}^2 \).

Suppose now that \( M \) is a strict wonderful variety and suppose that \( E, F \in \mathbb{N}\Delta \) are such that \( E + F \) is not faithful. The following example shows that the product \( V_EV_F \) may be degenerate, essentially reducing to a not spherically closed case.

**Example 9.4.** Let \( M \) be the model wonderful variety of type \( C_4 \). Then \( D_2 < \sigma \cdot 2D_2 \) and \( V(\omega_2) \subset V(\omega_2)^{\otimes 2} \) (more precisely, \( V(\omega_2) \) is also contained in the second symmetric power of \( V(\omega_2) \)). Notice that \( 2D_2 \) is not faithful: the maximal distinguished subset of \( \Delta \) which does not intersect the support of \( 2 \) \( \omega_2 \) (more precisely, \( \omega_2 \) \( \omega_2 \) \( \omega_2 \)) \( \omega_2 \) \( \omega_2 \). Then we may identify \( \Gamma(M, L_{D_2}) = \Gamma(M', L_{D_2}) \) and \( \Gamma(M, L_{2D_2}) = \Gamma(M', L_{2D_2}) \), so that we may regard the product \( V_{D_2}^2 \) inside \( \Gamma(M, L_{2D_2}) \) and by Example 9.3 it follows that \( V_{D_2} \not\subset V_{D_2}^2 \).

**Question.** Let \( M \) be a strict wonderful variety and let \( E, F \in \mathbb{N}\Delta \) be such that \( E + F \) is a faithful divisor. Is the product \( V_EV_F \) non-degenerate?

Suppose that \( G \) is simply laced: then the class of the strict wonderful varieties is stable with respect to the operation of quotient by a distinguished set of colors, so phenomena as that in Example 9.4 cannot happen. Therefore, if the answer to the previous question was affirmative, proceeding by induction on the rank of \( M \), it would follow that the multiplication is non-degenerate whenever \( M \) is a strict wonderful variety for a simply laced group.

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