Edge-critical subgraphs of Schrijver graphs II: The general case

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Abstract

We give a simple combinatorial description of an \((n − 2k + 2)\)-chromatic edge-critical subgraph of the Schrijver graph \(SG(n, k)\), itself an induced vertex-critical subgraph of the Kneser graph \(KG(n, k)\). This extends the main result of [J. Combin. Theory Ser. B 144 (2020) 191–196] to all values of \(k\), and sharpens the classical results of Lovász and Schrijver from the 1970s.

1 Introduction

Given integers \(k \geq 1\) and \(n \geq 2k\), the Kneser graph \(KG(n, k)\) is defined as follows: the vertices are all the \(k\)-element subsets of \([n] = \{1, \ldots, n\}\), and the edges are the pairs of disjoint subsets. A famous conjecture of Kneser \[6\], proved by Lovász \[8\], states that \(KG(n, k)\) is \((n − 2k + 2)\)-chromatic. Schrijver \[12\] sharpened the result by identifying the elements of \([n]\) with the vertices of the \(n\)-cycle \(C_n\), and showing that the Schrijver graph \(SG(n, k)\) — the subgraph of \(KG(n, k)\) induced by the vertices containing no pair of adjacent elements of \(C_n\) — is also \((n − 2k + 2)\)-chromatic. Moreover, Schrijver proved that \(SG(n, k)\) is \(vertex\text{-}critical\), i.e., the removal of any vertex decreases the chromatic number.
There is a stronger (and arguably, more natural) notion of criticality: a graph is said to be edge-critical, or simply critical, if the removal of any edge decreases the chromatic number — in other words, if any proper subgraph (not necessarily induced) has a smaller chromatic number than the graph itself.

The Schrijver graph $SG(n,k)$ is not edge-critical, unless $k = 1$ or $n = 2k + 1$. This prompts the following natural question: can we give a simple combinatorial description of an $(n-2k+2)$-chromatic edge-critical subgraph of $SG(n,k)$?

In a recent paper [5], such a construction was given for the case $k = 2$. Here we extend the construction to all values of $k$, thereby sharpening Schrijver’s theorem.

An edge $AB$ of $SG(n, k)$ is said to be interlacing if the elements of $A$ and $B$ alternate as we go round $C_n$. Simonyi and Tardos [13] recently proved that any edge of $SG(n, k)$ whose removal decreases the chromatic number is interlacing. Thus, a tempting candidate for an $(n-2k+2)$-chromatic edge-critical subgraph of $SG(n, k)$ might be the spanning subgraph formed by the interlacing edges. However, Litjens et al. [7] have shown that this graph has chromatic number $\lceil n/k \rceil$, so interlacing edges are much too restrictive.

We introduce instead the notion of almost-interlacing edges (we postpone the definition to Section 3), and define $XG(n,k)$ to be the spanning subgraph of $SG(n,k)$ formed by the almost-interlacing edges. The main result of this paper is the following theorem:

**Theorem 1.1.** For every $k \geq 1$ and every $n \geq 2k$, $\chi(XG(n,k)) = n-2k+2$. Moreover, $XG(n,k)$ is edge-critical.

We remark that the definition of almost-interlacing edges is particularly simple for the case $k = 2$. Indeed, almost-interlacing edges of $SG(n, 2)$ correspond to crossing and transverse edges defined in [5], so the graph $XG(n, 2)$ is precisely the graph $G_n$ studied in [5].

In a forthcoming paper, we will relate the graph $XG(n,k)$ to the graphs studied in [4], and show that $XG(n,k)$ is a quadrangulation of $\mathbb{RP}^{n-2k}$ (see [3] for a definition). In conjunction with the results from [3], this gives a new proof of the first part of Theorem 1.1.

For terminology not defined here, we refer the reader to Bondy and Murty [1].

The paper is structured as follows. Preliminary definitions and observations are collected in Section 2. Section 3 gives the definition of the graph $XG(n,k)$. The chromatic number of this graph is determined in Section 4 and the graph is shown to be edge-critical in Section 5.
2 Preliminaries

Let $C_n$ be the $n$-cycle with vertex set $[n] = \{1, \ldots, n\}$ and edges between consecutive integers as well as between 1 and $n$. The vertices of the Schrijver graph $SG(n, k)$ mentioned in Section 1 are independent sets in $C_n$ of size $k$; two such sets are adjacent in $SG(n, k)$ if they are disjoint.

We usually visualise $C_n$ in such a way that the vertices $1, \ldots, n$ appear clockwise in the given order. The vertices of $C_n$ will be referred to as elements to distinguish them from the vertices of $SG(n, k)$ or of the graph $XG(n, k)$ we will shortly define. Any arithmetic operations with the elements are performed modulo the equality $n + 1 = 1$.

Our arguments frequently use intervals in $C_n$. For $a, b \in [n]$, the interval $[a, b]$ is the set $\{a, a+1, \ldots, b\}$. Thus, $[a, b]$ consists of $a$ and the elements following $a$ clockwise up to $b$. In case $b = a - 1$, the interval $[a, b]$ contains all elements of $[n]$. By a slight abuse of this notation, we will also write $[0, n]$ for the set $\{0, \ldots, n\}$.

Open or half-open versions of intervals, namely $(a, b)$, $[a, b)$ or $(a, b]$, are defined as expected: for instance, $[a, b) = [a, b - 1]$. All of the following definitions are modified for these other versions of intervals in a straightforward way.

If $X \subseteq [n]$, it will be convenient to let $[a, b]_X = [a, b] \cap X$. The set carries a natural ordering given by the interval; thus, for instance, the first element of $[a, b]_X$ is the element of this set encountered first when moving clockwise from $a$ to $b$.

To distinguish ordered pairs from open intervals, we use the notation $(a, b)$ for an ordered pair consisting of elements $a$ and $b$. For a set $X \subseteq [n]$, we say that the pair $(a, b)$ is $X$-consecutive if $a, b \in X$ are distinct and $(a, b)_X = \emptyset$.

If $I$ is an interval in $[n]$, we say that disjoint subsets $A$, $B$ of $[n]$ alternate on $I$ if the elements of $A$ alternate with those of $B$ as we follow $C_n$ from the start to the end of $I$. Sets $A$, $B$ which alternate on $[n]$ are said to form an interlacing pair.

A crucial notion for our construction is that of an admissible interval. For disjoint subsets $A$, $B$ of $[n]$, an interval $[d, c]$ is weakly $AB$-admissible if

$$|[d, c]_A| = |[d, c]_B| = c.$$ 

Furthermore, a weakly $AB$-admissible interval $[d, c]$ is $AB$-admissible if

$$c, d \notin A \cup B.$$

We extend these notions to open or half-open intervals such as $(d, c)$ or $(d, c]$ in precisely the same way, just replacing $[d, c]$ with the interval in question.
Let us examine some basic properties of weakly \(AB\)-admissible intervals, where \(A, B\) are disjoint subsets of \([n]\), each of size \(k\). It is not yet required at this point that \(A\) and \(B\) be independent in \(C_n\), so we may view \(AB\) as an edge of the Kneser graph \(KG(n, k)\).

**Observation 2.1.** If \(AB\) is an edge of \(KG(n, k)\) and \([d, c]\) is a weakly \(AB\)-admissible interval, then \(c \leq k < d\).

**Proof.** Note first that \(c \leq k\) follows directly from the definition of weakly \(AB\)-admissible interval. Since \(A\) and \(B\) are disjoint, we have \(|[d, c]_{A\cup B}| = 2c\). It follows that \(d > c\), for otherwise \(2c \leq |[d, c]| \leq c\), leading to a contradiction as \(c \geq 1\). Now

\[
2k = |A \cup B| = |[d, c]_{A\cup B}| + |(c, d)_{A\cup B}| \leq 2c + (d - c - 1) = c + d - 1,
\]

and since \(c \leq k\), we must have \(d > k\). \(\square\)

Another basic property of weakly \(AB\)-admissible intervals is that they are nested, as shown by the first part of the following lemma:

**Lemma 2.2.** Let \(AB\) be an edge of \(KG(n, k)\) and let \([d, c]\), \([d', c']\) be weakly \(AB\)-admissible intervals. Then the following hold:

(i) \([d', c'] \subseteq [d, c]\) or vice versa,

(ii) if \([d', c'] \subseteq [d, c]\) and \(c' < c\), then the set \([d, d']_{A\cup B}\) is nonempty; if, moreover, \([d, c]\) is \(AB\)-admissible, then \(|(d, d')_{A\cup B}| \geq 2\).

**Proof.** (i) Suppose that the claim does not hold. By Observation 2.1 and by symmetry, we may assume that \(c < c' < d < d'\). Then

\[
2c' = |[d', c']_{A\cup B}| = |[d', c]_{A\cup B}| + |(c, c']_{A\cup B}| \leq 2c + (c' - c)
\]

implying \(c' \leq c\), a contradiction.

(ii) Our assumptions imply \(c' < c < d \leq d'\). We have

\[
2c = |[d, c]_{A\cup B}| = |[d, d']_{A\cup B}| + |[d', c]_{A\cup B}| + |(c', c]_{A\cup B}|
\leq |[d, d']_{A\cup B}| + 2c' + (c - c')
\tag{1}
\]

so \(|[d, d']_{A\cup B}| \geq c - c' \geq 1\).

If \(c, d \notin A \cup B\), then the \(c - c'\) term in (1) improves to \((c - c' - 1)\) and furthermore, we can write \((d, d')_{A\cup B}\) in place of \([d, d']_{A\cup B}\). The second assertion follows. \(\square\)
We conclude this section by the definition of switching, used in Section 3 to introduce the graph $XG(n, k)$. Suppose that $c, d \in [n]$. Switching at $[d, c]$ is the operation transforming any pair $AB$ of subsets of $[n]$ to another such pair $A'B'$ defined as follows:

$$A' = A \triangle [d, c]_{A \cup B},$$
$$B' = B \triangle [d, c]_{A \cup B},$$

where $\triangle$ denotes symmetric difference. The pair $A'B'$ is the result of the switching.

It is easy to see that if $AB$ is an edge of $KG(n, k)$, then the result of switching $AB$ at a weakly $AB$-admissible interval is again an edge of $KG(n, k)$. A similar statement holds for $SG(n, k)$ and switching at an $AB$-admissible interval.

Switching along a sequence $([d_i, c_i])_{i \in [m]}$ of intervals means switching at $[d_1, c_1], \ldots, [d_m, c_m]$ in this order. (Switching along an empty sequence is the identity operation on pairs.)

Under an admissibility assumption, switching along a sequence of intervals maps any edge of the Schrijver graph $SG(n, k)$ to an edge:

**Observation 2.3.** Let $AB$ be an edge of $SG(n, k)$ and let $A'B'$ be the pair obtained by switching $AB$ along a sequence $S$ of $AB$-admissible intervals. The following holds:

(i) $A'B'$ is again an edge of $SG(n, k)$,

(ii) any weakly $AB$-admissible interval is weakly $A'B'$-admissible and vice versa.

## 3 Definition of $XG(n, k)$

In this section, we define the graph $XG(n, k)$. Let $k \geq 1$ and $n \geq 2k$. The vertex set of $XG(n, k)$ coincides with that of $SG(n, k)$, so the vertices of $XG(n, k)$ are all $k$-element independent sets of $C_n$. The edges of $XG(n, k)$ are all the almost-interlacing pairs, defined as follows.

A pair $AB$ of vertices, where $A \cap B = \emptyset$, is almost-interlacing if there exists a set $X = C \cup D \subseteq [n]$ such that $C = \{c_1, \ldots, c_m\}$ and $D = \{d_1, \ldots, d_m\}$, with the following properties:

1. $1 \leq c_1 < c_2 < \cdots < c_m \leq k - 1$,
2. $k + 1 \leq d_m < d_{m-1} < \cdots < d_1 \leq n$, 


Figure 1: Vertices $A = \{4, 9, 12, 15\}$ (black dots) and $\{6, 8, 13, 16\}$ (white dots) of $XG(16, 4)$ forming an almost-interlacing pair. The elements not in $A \cup B$ are shown as tick marks. Dotted lines mark the control pairs of the $AB$-alternator $\{2, 3, 7, 11\}$. Similar conventions are used in the other figures.

(3) each interval $[d_i, c_i]$ is $AB$-admissible,

(4) switching along the sequence $([d_i, c_i])_{i \in [m]}$ changes $AB$ to an interlacing pair.

Any set $X$ satisfying this definition is called an $AB$-alternator. We often write it as $C \cup D$, with $C$ and $D$ as in the definition. The elements in $C$ are the control elements of the $AB$-alternator, the elements $c_i$ and $d_i$ ($i \in [m]$) correspond to each other, and pairs $\langle c_i, d_i \rangle$ ($i \in [m]$) are the control pairs of the $AB$-alternator.

Observe that $XG(n, k)$ is a spanning subgraph of $SG(n, k)$. Any pair of vertices $AB$ that is an interlacing pair is an edge of $XG(n, k)$, since in this case the empty set is trivially an $AB$-alternator.

Another example is shown in Figure 1 depicting an edge $AB$ of $SG(16, 4)$. The set $\{2, 3, 7, 11\}$ is an $AB$-alternator, so $A$ and $B$ are adjacent in $XG(16, 4)$. There is only one other $AB$-alternator, namely $\{2, 3, 7, 10\}$.

Let us consider the special case of the definition for $k = 2$. (See Figure 2 for an illustration.) Let $AB$ be an edge of $SG(n, 2)$. We may assume that $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, where $a_1 < a_2$, $b_1 < b_2$ and $a_1 < b_1$. Possible $AB$-alternators are $\emptyset$ (in which case $AB$ is an interlacing pair), or a set $\{1, d\}$, disjoint from $A \cup B$, such that $[d, 1]_{A\cup B} = \{a_2, b_2\}$ (which is easily seen to be equivalent to $1 < a_1 < b_1 < b_2 < a_2$). In the paper [5], pairs of these two types are referred to as crossing and transverse pairs, respectively, and they coincide with the edges of the graph studied in that paper (denoted by $G_n$).
Thus, as noted in Section 1, the present definition specialises to the one of \[5\] for \(k = 2\).

Let us add some comments on the definition of edges of \(XG(n,k)\). Note that in condition (1), the bound \(c_m \leq k\) is trivial (and stated in Observation 2.1), so (1) just strengthens this bound by one. Furthermore, the bound \(k + 1 \leq d_m\) in condition (2) is actually superfluous (though we include it for clarity) as it also follows from Observation 2.1. Using Lemma 3.1(i) below, the bounds in condition (2) can be strengthened to \(d_m \geq k + 2\) and \(d_1 \leq n - 2\).

We will now describe an algorithm that finds an \(AB\)-alternator \(C \cup D\) if it exists, where \(AB\) is an edge of \(SG(n,k)\). It may be helpful to consult Figure 1 for an illustration. First we need another lemma.

**Lemma 3.1.** Let \(C \cup D\) be an \(AB\)-alternator for an edge \(AB\) of \(SG(n,k)\) such that \(AB\) is not interlacing. The following hold:

(i) if \(\langle x, y \rangle\) is a \((D \cup \{k, n\})\)-consecutive pair other than \(\langle n, k \rangle\), then the size of \([x, y]_{A \cup B}\) is at least 2,

(ii) if \(\langle x, y \rangle\) is an \((A \cup B)\)-consecutive pair, then \(\lfloor (x, y)_{C \cup D} \rfloor\) is odd if and only if \(x, y \in A\) or \(x, y \in B\).

**Proof.** Let \(D = \{d_1, \ldots, d_m\}\) with \(d_1 > \cdots > d_m\). Since \(AB\) is not interlacing, we have \(m \geq 1\). For \(i \in [m]\), let \(c_i\) be the control element corresponding to \(d_i\).

(i) If \(x, y \in D\), then the assertion follows from Lemma 2.2(ii) and the fact that each of the intervals \([d_i, c_i]\) is \(AB\)-admissible.

For the pair \(\langle k, d_m \rangle\), we can write

\[
2k = |A \cup B| = |[d_m, c_m]_{A \cup B}| + |(c_m, k)_{A \cup B}| + |[k, d_m]_{A \cup B}| \\
\leq 2c_m + (k - c_m - 1) + |[k, d_m]_{A \cup B}|,
\]

Figure 2: Examples of edges in \(XG(8, 2)\) (left and center) and a non-edge in \(XG(8, 2)\) (right). The dotted line in the center picture shows the only control pair of the \(AB\)-alternator \(\{1, 5\}\).
so \(|[k,d_m]_{AB}| \geq k - c_m + 1 \geq 2\) since \(c_m \leq k - 1\).

Similarly, for the pair \(\langle d_1, n \rangle\), we have

\[
2c_1 = |[d_1, n]_{AB}| + |[1,c_1]_{AB}| \leq |[d_1, n]_{AB}| + (c_1 - 1)
\]

(using the fact that \(c_1 \notin A \cup B\)), and we find that \(|[d_1, n]_{AB}| \geq c_1 + 1 \geq 2\).

(ii) Let us say that a subset of \([n]\) is \emph{separating} if it contains exactly one of \(x\) and \(y\). Let \(s\) be the number of intervals \([d_i,c_i]) (i \in [m])\) which are separating. Observe that \(s\) has the same parity as \(|(x,y)_{C \cup D}|\).

For \(0 \leq j \leq m\), let \(A_j B_j\) be the pair obtained from \(AB\) by switching along \(([d_i,c_i]) \in [j]\); in particular, \(A_0 B_0 = AB\). For \(j > 0\), it is not hard to see that \(A_j\) is separating if and only if exactly one of \(A_{j-1}\) and \([d_j,c_j]\) is separating. Now since \(A_mB_m\) is an interlacing pair, \(A_m\) is not separating. It follows that either \(A\) is separating and \(s\) is even, or \(A\) is not separating and \(s\) is odd. Since \(A\) is not separating if and only if \(x,y \in A\) or \(x,y \in B\), and by the above observation on the parity of \(s\), this implies part (ii). \qed

Let us return to the task of finding an \(AB\)-alternator for a given edge \(AB\) of \(SG(n,k)\). Consider any \((A \cup B)\)-consecutive pair \(\langle x,y \rangle\) with \(x,y \in [k,n]\). If \(x,y \in A\) or \(x,y \in B\), then by Lemma 3.1(ii), our set \(D\) needs to contain an element in \((x,y)\). The latter interval is nonempty since each of \(A\) and \(B\) is independent in \(C_n\). Furthermore, by Lemma 3.1(i), \(D\) must contain exactly one element from this interval. The choice of the element from \((x,y)\) is arbitrary; in fact, we will see that this is the only choice we have in the process. In the example of Figure 4, the set \(D\) must include the element 7 and one element from \(\{10,11\}\).

Similarly to the above, Lemma 3.1(ii) and (i) implies that if exactly one of \(x,y\) is in \(A\), then \((x,y)_D\) must be empty, because its size is even and at most one. Finally, by Lemma 3.1(i), \(D\) contains no element between \(k\) and the first element of \([k,n]_{AB}\), nor between the last element of the latter set and \(n\).

Summing up, \(D\) is obtained by choosing exactly one element in each interval \((x,y)\) with \((x,y)\) an \((A \cup B)\)-consecutive pair with \(x,y \in [k,n]\) and either \(x,y \in A\) or \(x,y \in B\). Let \(D = \{d_1,\ldots,d_m\}\) for some such choice. (Thus, for the pair in Figure 4, \(D\) equals \(\{7,10\}\) or \(\{7,11\}\).)

We will show that this determines the set \(C\) whenever there exists an \(AB\)-alternator. The following lemma provides a tool.

**Lemma 3.2.** Let \(d \in [k,n]\) and let \(X\) be a vertex of \(SG(n,k)\). There is at most one element \(c \in [k-1]\) such that \(|[d,c]_X| = c\) and \(c \notin X\).

**Proof.** For \(x \in [k-1]\), let

\[
f(x) = |[d,x]_X| - x.
\]
The function $f$ is non-increasing. For each $x \in [k-2]$, we have
\[
    f(x + 1) = \begin{cases} 
    f(x) & \text{if } x + 1 \in X, \\
    f(x) - 1 & \text{otherwise.}
    \end{cases}
\]
Thus, if $f(x) = f(x + 1)$ and $x \leq k - 3$, then $f(x + 1) > f(x + 2)$ by the independence of $A$. It follows that we have $f(x) = 0$ for at most two values of $x$. Supposing (for the sake of a contradiction) that the lemma does not hold, there are two such values, say $c$ and $c + 1$, where $c \in [k - 2]$. Since $f(c + 1) = f(c)$, we have $c + 1 \in X$, so $c + 1$ does not satisfy the conditions, a contradiction.

For each $i \in [m]$, $C$ has to contain an element $c_i$ such that $[d_i, c_i]$ is $AB$-admissible. Since $c_i$ has to satisfy the condition of Lemma 3.2 with $X = A$, there is at most one such element. Furthermore, $c_i$ is independent of the choice of $d_i$: more precisely, if $\langle x, y \rangle$ is the $(A \cup B)$-consecutive pair such that $d_i \in \langle x, y \rangle$, and if $d'_i \in \langle x, y \rangle$, then $[d'_i, c]_{A \cup B} = [d_i, c]_{A \cup B}$ for any $c \in [k - 1]$. It follows that if an $AB$-alternator does exist, then each element of $C$ is uniquely determined by Lemma 3.2. Our algorithm returns $C \cup D$ when this is the case, and reports that there is no $AB$-alternator otherwise. (In the example of Figure 1, we have $c_1 = 2$ and $c_2 = 3$, so one of the sets $\{2, 3, 7, 10\}$ or $\{2, 3, 7, 11\}$ is returned.)

To obtain a unique choice for the $AB$-alternator when it exists, we impose the extra condition that for each $i \in [m]$, $d_i + 1 \in A \cup B$. This amounts to choosing the largest possible element for each $d_i$. The resulting $AB$-alternator is called standard. Speaking of the control elements or control pairs for the edge $AB$, we mean the control elements or pairs of the standard $AB$-alternator.

4 Chromatic number

In this section, we prove the first part of Theorem 1.1 — namely, that $\chi(XG(n, k)) = n - 2k + 2$ for every $k \geq 1$ and every $n \geq 2k$. It is enough to prove the inequality $\chi(XG(n, k)) \geq n - 2k + 2$, the other inequality being a direct consequence of the fact that $XG(n, k)$ is a subgraph of $KG(n, k)$.

The case $k = 2$ of Theorem 1.1 was proved in [5] using the so-called Mycielski construction. Here we prove the general case using the same idea, but rely instead on the generalised Mycielski construction, introduced by Stiebitz [14] (see also [2, 11]).

Given a graph $G = (V, E)$ and an integer $r \geq 1$, the graph $M_r(G)$ has vertex set $(V \times [0, r-1]) \cup \{z\}$, and there is an edge $(u, 0)(v, 0)$ and $(u, i)(v, i +
Figure 3: The generalised Mycielski construction applied to $C_7$ (bold) resulting in the graph $M_3(C_7)$.

1) (for every $i \in [0, r - 2]$) whenever $uv \in E$, and an edge $(u, r - 1)z$ for all $u \in V$. The construction is illustrated in Figure 3.

For every integer $t \geq 2$, we denote by $M_t$ the set of all ‘generalised Mycielski graphs’ obtained from $K_2$ by $t - 2$ iterations of $M_r(\cdot)$, where the value of $r$ can vary from iteration to iteration. That is, $H \in M_t$ if and only if there exist integers $r_1, r_2, \ldots, r_{t-2} \geq 1$ such that

$$H \cong M_{r_{t-2}}(M_{r_{t-3}}(\ldots M_{r_2}(M_{r_1}(K_2))\ldots)).$$

Using topological methods, Stiebitz [14] (see also [2, 9]) proved the following result. A ‘discrete’ proof, based on a combinatorial lemma of Fan, can be found in [10].

**Theorem 4.1** (Stiebitz [14]). If $G \in M_t$, then $\chi(G) = t$.

We now come to the key lemma of this section.

**Lemma 4.2.** For every $k \geq 1$ and every $n \geq 2k$, $M_k(XG(n - 1, k))$ is homomorphic to $XG(n, k)$.

**Proof.** We shall explicitly describe a homomorphism $f$ from $M_k(XG(n - 1, k))$ to $XG(n, k)$. Let $A$ be a vertex of $XG(n - 1, k)$ and let $(A, 0), \ldots, (A, k - 1)$ be its copies in $M_k(XG(n - 1, k))$. In order to keep all vertex names capitalised, we choose to denote the vertex $z$ in the generalised Mycielski construction by $Z$.

Suppose that $A = \{a_1, \ldots, a_k\}$, where $a_1 < \cdots < a_k$. Let $0 \leq i \leq k$. We define the set $\Lambda_{n,i} \subseteq [n]$ as follows:

$$\Lambda_{n,i} = \begin{cases} 
\{n - i + 1, n - i + 3, \ldots, n\} \cup \{2, 4, \ldots, i - 1\} & \text{if } i \text{ is odd,} \\
\{n - i + 1, n - i + 3, \ldots, n - 1\} \cup \{1, 3, \ldots, i - 1\} & \text{if } i \text{ is even.}
\end{cases}$$
Thus, for instance, \( \Lambda_{n,0} = \emptyset \), \( \Lambda_{n,1} = \{n\} \) and \( \Lambda_{n,2} = \{1, n - 1\} \).

We will now define a map \( f : V(M_k(XG(n-1,k))) \rightarrow V(XG(n,k)) \). Given a vertex \( A \) of \( XG(n-1,k) \) and an integer \( j \in [k] \), let \( A^j = [d, j]_A \), where \( d \) is the maximum integer such that \( |[d, j]_A| = j \). Furthermore, let \( A^0 = \emptyset \). We set

\[
\begin{align*}
f : (A, j) &\mapsto (A \setminus A^j) \cup \Lambda_{n,j}, \quad \text{where } 0 \leq j \leq k - 1, \\
z &\mapsto \Lambda_{n,k}.
\end{align*}
\]

Note that the image of \( f \) is contained in the vertex set of \( XG(n,k) \). Informally, \( f(A, j) \) can be seen as the result of the following process: viewing \( A \) as a subset of \( V(C_n) \), \( A^j \) consists of the \( j \) elements of \( A \) that are closest to \( j \) counterclockwise; push them clockwise in such a way that the first one stops at \( j \) and the remaining ones are tightly packed (still forming an independent set), and rotate them back by one element. The other \( k - j \) elements of \( A \) are not affected.

To verify that \( f \) is a homomorphism, it is enough to check that \( f \) maps edges of \( M_k(XG(n-1,k)) \) to edges of \( XG(n,k) \). Fix an arbitrary edge \( AB \) of \( XG(n-1,k) \), and let \( C \cup D \subseteq [n-1] \) be the standard \( AB \)-alternator. Let \( \{\langle c_i, d_i \rangle : i \in [m]\} \) be its set of control pairs.

We will show that \( f \) maps the edges \((A, 0)(B, 0), (A, j)(B, j+1)\) (for any \( j \in [0, k-1] \)), as well as \((A, k-1)Z\), to edges of \( XG(n,k) \), by finding an appropriate alternator \( C' \cup D' \).

First, consider the edge \((A, 0)(B, 0)\) of \( M_k(XG(n-1,k)) \). Since \( f((A, 0)) = A \) and \( f((B, 0)) = B \), the required alternator is obtained by taking \( C' = C \) and \( D' = D \). (Note that the definition is still satisfied if \( A \) and \( B \) are viewed as vertices of \( XG(n,k) \) rather than \( XG(n-1,k) \).)

Edges of type \((A, k-1)Z\) are another easy case: we have \( f(Z) = \Lambda_{n,k} \) and \( f((A, k-1)) \) contains \( \Lambda_{n,k-1} \) as a subset, which means that \( f((A, k-1))f(Z) \) must actually be an interlacing pair, and hence an edge of \( XG(n,k) \) (with empty alternator).

It remains to consider the edge \((A, j)(B, j+1)\), where \( j \in [0, k-1] \). Let \( A' = f((A, j)) \) and \( B' = f((B, j+1)) \). The sets \( A' \) and \( B' \) are disjoint since \( A \cap B = \emptyset \) and \( \Lambda_{n,j} \cap \Lambda_{n,j+1} = \emptyset \).

Given \( r \in [0, m] \), let \( A_rB_r \) be the pair obtained from \( AB \) by switching along \( ([d_i, c_i])_{i \in [r]} \). Since \( A_mB_m \) is interlacing, there is \( d \in [k+1, n] \) such that \( [d, j+1] \) is weakly \( A_mB_m \)-admissible. Choose \( d \) as maximal with this property. By Observation \( \ref{obs:interlacing} \)(ii), \([d, j+1] \) is weakly \( AB \)-admissible.

For any \( i \in [m] \), we have \([d_i, c_i] \subseteq [d, j+1]\) or vice versa by Lemma \( \ref{lem:interlacing} \)(i). If there is \( t \in [m] \) such that \( c_t < j + 1 \), then let \( t \) be maximal with this property; otherwise, let \( t = 0 \).
We now aim to show that the pair $A'B'$ is, in a sense, not too different from $A_tB_t$.

Let $X,Y$ be disjoint vertices of the graph $H$ (that is, vertices such that $X \cap Y = \emptyset$), where $H$ is either $XG(n-1,k)$ or $XG(n,k)$. Let $I$ be the interval $[d,j+1] \subseteq V(C_n)$, where $d$ is as above. (Thus, $n \in I$ even if $H$ is $XG(n-1,k)$.)

Let us say that the pair $XY$ is nice if the following hold:

(N1) $X \setminus I = A \setminus I$ and $Y \setminus I = B \setminus I$,

(N2) the sets $X$ and $Y$ alternate on $I$ and the first element of $I \cap (X \cup Y)$ belongs to $X$ if and only if the first element of $I \cap (A \cup B)$ belongs to $A$.

**Claim 1.** The pair $A_tB_t$ is nice.

Condition (N1) in the definition follows from the fact that for each of the intervals $[d_i,c_i]$ with $i \leq t$, we have $c_i < j + 1$ and therefore $[d_i,c_i] \subseteq I$ by Lemma 2.2(i). Thus, switching at such intervals does not affect the elements outside $I$.

Let us verify condition (N2). Since $A_mB_m$ is an interlacing pair and $I \subseteq [d_i,c_i]$ for any $i > t$, $A_t$ and $B_t$ must alternate on $I$. For the rest of condition (ii), we may assume that $t > 0$. Let $x$ be the first element of $I \cap (A \cup B)$; since $A \cup B = A_t \cup B_t$, this is also the first element of $I \cap (A_t \cup B_t)$. By Lemma 2.2(ii), $x$ is not contained in $[d_t,c_t]$ (nor in any $[d_i,c_i]$ with $i < t$), and therefore $x \in A_t$ if and only if $x \in A$. This concludes the proof of the claim.

**Claim 2.** Any nice pair $XY$ of disjoint vertices of $XG(n,k)$ forms an edge of $XG(n,k)$.

It is clear from the definition of nice pair that $XY$ can be obtained from (the nice pair) $A_tB_t$ by first extending the underlying cycle $C_{n-1}$ to $C_n$ (just inserting the element $n$) and then moving the elements of $X \cup Y$ within $I$ without changing their order on $C_n$.

It follows that switching along $([d_{t+1},c_{t+1}], \ldots, [d_m,c_m])$ changes $XY$ to an interlacing pair, just as in the case of $A_tB_t$. (Recall that $I$ is a subset of each of these intervals by the choice of $t$.) Summing up, $\{c_{t+1}, \ldots, c_m\} \cup \{d_{t+1}, \ldots, d_m\} \subseteq [n]$ is an $XY$-alternator.

The following claim relates the above observations to $A'B'$.

**Claim 3.** One of the following conditions holds:

(i) $A'B'$ is a nice pair,
(ii) the interval $I$ is $A'B'$-admissible and the pair $A''B''$ obtained by switching $A'B'$ at $I$ is nice.

First of all, observe that since $I$ is weakly $AB$-admissible, both $A'$ and $B'^{j+1}$ are contained in $I$. Furthermore, both $\Lambda_{n,j+1}$ and $\Lambda_{n,j}$ are contained in $I$: indeed, the weakly $A'B'$-admissible interval $I = [d, j + 1]$ must satisfy $d \leq n - j$, while at the same time $\Lambda_{n,j} \cup \Lambda_{n,j+1} = [n - j, j]$. This proves condition (N1) for both of the pairs involved in (i) and (ii).

We have in fact $B^{j+1} = I \cap B$ and $A^j = (I \cap A) \setminus \{a\}$ for some $a \in I \cap A$. There are essentially three possibilities for $a$, illustrated in Figure [4] if $j + 1 \notin A$, then $a$ is the first element of $[d, j + 1]_A$ and it may or may not equal $d$, while if $j + 1 \in A$, then $a = j + 1$.

All the elements of $B^{j+1}$ are replaced in $B'$ by $\Lambda_{n,j+1}$; similarly, all the elements of $A'$ are replaced in $A'$ by $\Lambda_{n,j}$. Hence, $A'$ and $B'$ alternate on $[n - j, j]$, and therefore they alternate on $I$ regardless of the position of the remaining element $a$ of $[d, j + 1]_{A' \cup B'}$.

If condition (N2) holds for $A'B'$, then we are done. Assume thus that this is not the case. We have $a \neq d$, for otherwise $a$ would be the first element of both $I \cap (A \cup B)$ and $I \cap (A' \cup B')$ while $a \in A \cap A'$, implying (N2). For a similar reason (using the fact that $A'B'$ alternates in $I$), we find $a \neq j + 1$. Consequently, neither $d$ nor $j + 1$ belong to $A'$. They do not belong to $B'$ either: this is clear in the case of $j + 1$, and $d \in B'$ would only be possible if $d = n - j$, but then $|d, j + 1|_A = j + 1$ would force $j + 1 \in A$ and hence $a = j + 1$, a contradiction. We have proved that $[d, j + 1]$ is $A'B'$-admissible.

Let $x$ be the first element of $[d, j + 1]_{A' \cup B'} = [d, j + 1]_{A' \cup B'}$ and note that $x$ belongs to $A'$ if and only if it belongs to $B''$. Thus, condition (N2) is satisfied for exactly one of the pairs $A'B'$ and $A''B''$. This proves the claim.

Let us finish the proof of the lemma. If condition (i) of Claim 3 holds, then $A'B'$ is an edge of $XG(n, k)$ by Claim 2. If condition (ii) holds, then we obtain an $A'B'$-alternator by setting $C' = C \cup \{j + 1\}$ and $D' = D \cup \{d\}$, completing the discussion for edges of type $(A,j)(B,j + 1)$ as well as the whole proof.

We are now ready to prove that $\chi(XG(n, k)) \geq n - 2k + 2$. First, observe that if $G$, $H$ are graphs such that $G$ is homomorphic to $H$, then $M_k(G)$ is homomorphic to $M_k(H)$. Hence, by repeated applications of Lemma 4.2 the graph

$$H = M_k(M_k(M_k(\ldots M_k(XG(2k,k))\ldots)))$$

where $M_k(\cdot)$ is applied $n - 2k$ times, is homomorphic to $XG(n, k)$. Since $XG(2k,k)$ is isomorphic to $K_2$, $H \in M_{n-2k}$, so using Theorem 4.1 we conclude that $\chi(XG(n, k)) \geq n - 2k + 2.$
I
(a) $a$ differs from both $d$ and $j + 1$.

I
(b) $a = d$.

I
(c) $a = j + 1$.

Figure 4: Possible cases in the proof of Claim 3 shown for $k = 4$, $j = 2$ and $AB$ interlacing. The figures on the left show the pair $AB$, those on the right show $A'B'$. Black dots represent $A$ or $A'$, white dots represent $B$ or $B'$, the interval $I = [d, j + 1]$ is shown gray. The cases are distinguished by the position of the element $a$ of $(I \cap A) \setminus A'$. The pair $A'B'$ is interlacing except in (a), in which case a switch at $[d, j + 1]$ is needed to make it interlacing.
5 Criticality

In this section, we prove the second part of Theorem 1.1, namely that $XG(n,k)$ is edge-critical. Let $AB$ be an edge of $XG(n,k)$ and let $G = XG(n,k) - AB$. We show that $G$ is $(n-2k+1)$-colourable.

Let $C \cup D$ be the standard $AB$-alternator, where $C = \{c_1, \ldots, c_m\}$, $D = \{d_1, \ldots, d_m\}$ and

$$c_1 < c_2 < \cdots < c_m \leq k - 1 < d_m < d_{m-1} < \cdots < d_1.$$ 

The sets $A$, $B$, $C$, $D$ are pairwise disjoint and for $j \in [m]$, $|[d_i, c_j]_A| = |[d_i, c_j]_B| = c_i$.

Let $W = A \cup B \cup C \cup D$. We call a vertex of $G$ essential if it is contained in $W$ and inessential otherwise. In our analysis, it will be sufficient to concentrate on essential vertices since each inessential one will get a colour special to one of its elements outside $W$, and it will be easy to see that these colour classes are independent sets in $G$.

**Lemma 5.1.** Suppose that $X, Y$ are disjoint vertices of $G$, $c \in [k-1]$ and $d \in [n]$ such that $c \neq d$. The pair $XY$ is not an edge of $G$ if one of the following conditions holds:

(i) $|[d, c]_X| > c$ and $|[d, c]_Y| < c$, or

(ii) $|[d, c]_X| > c$ and $|[d, c]_Y| < c$.

**Proof.** Assume condition (i). For the sake of a contradiction, assume that $XY$ is an edge of $G$, and consider the standard $XY$-alternator $C' \cup D'$, where $C' = \{c_1', \ldots, c_{\ell}'\}$ and $D' = \{d_1', \ldots, d_{\ell}'\}$.

Suppose first that

for each $j \in [\ell]$, $[d_j', c_j'] \subseteq [d, c]$ or vice versa. \hspace{1cm} (2)

For $0 \leq j \leq \ell$, let $X_jY_j$ be the result of switching $XY$ along $([d_i', c_i']_{i \in [j]}).$

Let

$$\beta(X_jY_j) = ||[d, c]_X_j| - |[d, c]_Y_j||.$$ 

Since $X_jY_j$ is an interlacing pair, we have $\beta(X_jY_j) \leq 1$. We claim that for $j > 0$, it holds that $\beta(X_jY_j) = \beta(X_{j-1}Y_{j-1})$. This is clear if $[d, c] \subseteq [d_j', c_j']$, for then the effect of the switch at $[d_j', c_j']$ within $[d, c]$ is just to interchange membership in $X_{j-1}$ and $Y_{j-1}$. On the other hand, if $[d_j', c_j'] \subseteq [d, c]$, then $[d_j', c_j']$ is $X_{j-1}Y_{j-1}$-admissible by Observation 2.3(ii), and therefore $[d, c]_{X_j} = [d, c]_{X_{j-1}}$ and similarly $[d, c]_{Y_j} = [d, c]_{Y_{j-1}}$. The claim follows.
Then there exists \( i \). Proposition 5.2. Let \( X \) and \( X \) is \( [1] \) balanced on \( \sigma \) and \( c_j \) is \( \sigma \) on \( \sigma \), respectively. These notions are also defined for intervals \( (d, c) \). Let \( i \) is \( X \) with condition (i) implies that our assumption (2) does not hold. Since \( X \) is \( X \) with condition (i) implies that our assumption (2) does not hold. Thus, let \( j \) be the least index such that \( |[d, c] \cap \{ c_j, d_j \}| = 1 \).

Suppose that \( d_j' \notin [d, c] \). Since \( |[d, c]| > c \) and \( c_j' \in [d, c] \), we have \( |[d', c_j]| > c \). On the other hand, \( |[d', c_j]| = c_j \), and thus

\[
| (c_j', c_j)| = c + 1 - c_j' .
\]

(3)

Since \( X \) is an independent set in \( C_n \), we have \( |(c_j', c_j)| \leq (c - c_j' + 1)/2 \). Combining this with (3), we derive \( c < c_j', \) a contradiction with the assumption that \( c_j' \in I \).

The argument for the case \( c_j' \notin [d, c] \) is similar. Analogously to (3), we find that \( |(c, c_j)| \geq c_j - c + 1 \). On the other hand, \( (c, c_j) \) is independent and thus its size is at most \( (c_j' - c)/2 \), an improvement by \( 1/2 \) coming from the fact that \( c_j' \notin Y \) as \( c_j' \) is a control element for \( XY \). As a consequence, the resulting bound \( c > c_j' + 1 \) is even stronger than its analogue in the preceding case.

A similar computation works for condition (ii). \( \square \)

Throughout the following discussion, let \( X \) be a \( k \)-element subset of \( W \). Let \( i \leq m \). We say that \( X \) is \( \text{heavy} \) on \( [d_i, c_i] \) if \( |[d_i, c_i]| > c_i \). Furthermore, \( X \) is \( \text{light} \) or \( \text{balanced} \) on \( [d_i, c_i] \) if \( |[d_i, c_i]| \) is smaller than or equal to \( c_i \), respectively. These notions are also defined for intervals \( (d_i, c_i) \) or \( [d_i, c_i] \) in the obvious way.

We say that \( X \) is \( \text{balanced} \) if it is balanced on every interval \( [d_i, c_i] \) and \( (d_i, c_i] \), where \( 1 \leq i \leq m \). The set \( X \) is \( \text{regular} \) if it is balanced and contained in \( A \cup B \). Note that \( A \) and \( B \) are regular.

Let us say that \( X \) is \( \text{min-heavy} \) on \( [d_i, c_i] \) \( (1 \leq i \leq m) \) if it is heavy on \( [d_i, c_i] \) and not heavy on any interval \( [d_j, c_j] \) nor \( (d_j, c_j] \) with \( j < i \). Similarly, \( X \) is \( \text{max-light} \) on \( [d_i, c_i] \) if it is light on this interval and not light on any \( [d_j, c_j] \) nor \( (d_j, c_j] \) with \( j > i \). Being min-heavy or max-light on the interval \( (d_i, c_i] \) is defined in an analogous manner.

A balanced pair in \( X \) is a pair \( \{ c_i, d_i \} \) \( (1 \leq i \leq m) \) such that \( \{ c_i, d_i \} \subseteq X \) and \( X \) is balanced on \( [d_i, c_i] \) (and therefore also on \( (d_i, c_i] \)).

**Proposition 5.2.** Let \( X \) be a \( k \)-element subset of \( W \). If \( X \) is not regular, then there exists \( i \in [m] \) satisfying one of the following:

(a) \( \{ c_i, d_i \} \) is a balanced pair in \( X \),

(b) \( X \) is min-heavy on \( [d_i, c_i] \) or on \( (d_i, c_i] \),

(c) \( X \) is max-light on \( [d_i, c_i] \) or on \( (d_i, c_i] \).
Proof. Suppose that $X$ is not regular. If there exists $j \in [m]$ such that $X$ is heavy or light on $[d_j, c_j]$ or $(d_j, c_j)$, then an index $i$ satisfying (b) or (c) can be obtained by making an appropriate extremal choice of $j$. We can thus assume that $X$ is balanced.

Since $X$ is not regular, it contains an element from $C \cup D$ — say, $d_\ell \in X$. (A symmetric argument works in the other case.) Being balanced, $X$ contains $c_\ell$ elements of $[d_\ell, c_\ell)$, and therefore $|(d_\ell, c_\ell)_X| = c_\ell$. Since $|(d_\ell, c_\ell)_X| = c_\ell$, we have $c_\ell \in X$. Thus, $\{c_\ell, d_\ell\}$ is a balanced pair in $X$. \hfill $\square$

For convenience, we set $d_0 = c_1$, $c_0 = d_1$, $d_{m+1} = c_m$ and $c_{m+1} = d_m$. For $i \in [m + 1]$, we define

$$U_i = (d_i, d_{i-1})_W \cup (c_{i-1}, c_i)_W.$$ 

Note that for each $i$, $U_i \subseteq A \cup B$ and $|U_i|$ is even, namely

$$|U_i| = \begin{cases} 2c_1 & \text{if } i = 1, \\ 2(k - c_m) & \text{if } i = m + 1, \\ 2(c_i - c_{i-1}) & \text{otherwise}. \end{cases}$$ 

**Proposition 5.3.** Let $X \subseteq W$ and $i \in [m]$.

(i) If $X$ is min-heavy on $[d_i, c_i]$ (respectively, $(d_i, c_i)$), then either $i > 1$ and $\{c_{i-1}, d_{i-1}\}$ is a balanced pair in $X$, or $X$ contains more than half of the elements in the set $U_i \cup \{d_i\}$ (respectively, $U_i \cup \{c_i\}$).

(ii) If $X$ is max-light on $[d_i, c_i]$ (respectively, $(d_i, c_i)$), then either $i < m$ and $\{c_{i+1}, d_{i+1}\}$ is a balanced pair in $X$, or $X$ contains more than half of the elements in the set $U_i \cup \{c_i\}$ (respectively, $U_{i+1} \cup \{d_i\}$).

Proof. We prove (i) only for the case of $X$ min-heavy on $[d_i, c_i]$ since the other case is completely analogous. For $i = 1$, the claim is trivially true since $X$ is heavy on $[d_1, c_1] = U_1 \cup \{d_1\}$. Suppose then that $i > 1$.

Since $X$ is heavy on $[d_i, c_i]$, $|[d_i, c_i)_X| \geq c_i + 1$. Let us assume that $X$ contains less than half of the elements of the (odd-sized) set $U_i \cup \{d_i\}$ — that is, $|X \cap (U_i \cup \{d_i\})| \leq c_i - c_{i-1}$. Hence

$$|[d_{i-1}, c_{i-1})_X| \geq c_{i-1} + 1. \quad (4)$$ 

On the other hand, $X$ is heavy on neither $[d_{i-1}, c_{i-1}]$ nor $(d_{i-1}, c_{i-1}]$, so $|[d_{i-1}, c_{i-1})_X| \leq c_{i-1}$ and $|(d_{i-1}, c_{i-1}]_X| \leq c_{i-1}$. Comparing with (4), we see that $c_{i-1}, d_{i-1} \in X$. Furthermore, $|[d_{i-1}, c_{i-1})_X| = c_{i-1}$, so $\{c_{i-1}, d_{i-1}\}$ is a balanced pair in $X$. 

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The proof of (ii) is similar and we only comment on the case of $X$ max-light on $[d_i,c_i]$ and $i < m$. We have $|[d_i,c_i)_X| \leq c_i - 1$. If $X$ contains less than half of the elements in $U_{i+1} \cup \{c_i\}$, then $|(d_{i+1},c_{i+1})_X| \leq (c_i - 1) + (c_{i+1} - c_i) = c_{i+1} - 1$. However, $X$ is not light on $[d_{i+1},c_{i+1})$ nor on $(d_{i+1},c_{i+1}]$, so $c_{i+1},d_{i+1} \in X$ and $|[d_{i+1},c_{i+1})_X| = c_{i+1}$. It follows that $\{c_{i+1},d_{i+1}\}$ is a balanced pair in $X$.

For $i \in [m+1]$, a set $X \subseteq W$ is skew at $d_i$ if $X$ contains the largest element of $(d_{i+1},d_i)_W$ and the second smallest element of $(d_i,d_{i-1})_W$. (By Lemma 3.1(i), each of the latter two sets contains at least two elements.)

Let $X \subseteq W$ be a vertex of $G$ and let $d \in [k,n]$. By Lemma 3.2 there is at most one element $c \in [k - 1]$ such that $|[d,c)_X| = c$ and $c \notin X$. If such an element exists, we call it the \textit{depth of $d$ in $X$} and define $\delta(X,d) = c$; otherwise, we let $\delta(X,d) = k$.

This notion will only be used for vertices $X$ containing a $W$-consecutive pair. For such a vertex, let $(s,s')$ be a $W$-consecutive pair in $X$ with $s$ as large as possible. (The choice is not really essential, but we specify it to make the definition unambiguous.) The \textit{depth $\delta(X)$ of $X$} is defined as $\delta(X,s')$.

\textbf{Lemma 5.4.} If $XY$ is an edge of $G$ with $X,Y \subseteq W$ and $X$ contains a $W$-consecutive pair, then $\delta(X)$ is one of the control elements for $XY$. In particular, $\delta(X) < k$ and $\delta(X) \notin X \cup Y$.

\textbf{Proof.} Let $(s,s')$ be a $W$-consecutive pair in $X$ with $\delta(X) = \delta(X,s')$. Let $\{[c'_i,d'_i) : i \in [\ell]\}$ be the set of control pairs of the standard $XY$-alternator. By Lemma 3.1(ii) (applied to the edge $XY$), we must have $d'_i = s' - 1$ for some $i \in [\ell]$, in which case $|[s',c'_i)_X| = |[d'_i,c'_i)_X| = c'_i$. Furthermore, $c'_i \notin X$, so $c'_i$ is the depth of $s'$ in $X$, i.e., $c'_i = \delta(X)$. The lemma follows. \hfill \Box

\textbf{Lemma 5.5.} Let $X,Y \subseteq W$ be vertices of $G$ such that $X \cap Y = \emptyset$, each of $X$ and $Y$ contains a $W$-consecutive pair, and $\delta(X) = \delta(Y)$. Then $X$ and $Y$ are non-adjacent in $G$.

\textbf{Proof.} Let $\delta = \delta(X)$. By Lemma 5.4 we may assume that $\delta \leq k - 1$. Thus let $(s,s')$ be a $W$-consecutive pair in $X$ with $\delta(X) = \delta(X,s')$, and similarly let $(t,t')$ be such a pair in $Y$. By symmetry, we may assume that $t' < s'$, and therefore $t' < s$. By the definition of depth, $|[s',\delta)_X| = |[t',\delta)_Y| = \delta$. Since $t' < s$, we have $|[s,\delta)_Y| \leq \delta - 1$ while $|[s,\delta)_X| = \delta + 1$. Lemma 5.1 implies that $X$ and $Y$ are non-adjacent. \hfill \Box

We are now ready to define a colouring of $G$ using the following set of colours:

$$\{[1] : i \in [n] \setminus (A \cup B)\} \cup \{[0]\}.$$
Since $|A| = |B| = k$, the total number of colours is $n - 2k + 1$.

From this point on, we drop the assumption that $X \subseteq W$. A vertex $X$ of $G$ is assigned a colour by the following rules, in the stated order of precedence:

(R1) If $X$ is inessential and therefore contains an element of $[n] \setminus W$, then it gets colour $j$, where $j$ is the least such element.

(R2) If $X$ contains a balanced pair, then $X$ gets colour $c_i$, where $i \in [m]$ is least such that $\{c_i, d_i\}$ is a balanced pair in $X$.

(R3) If $X$ is min-heavy or max-light on $(d_i, c_i)$ for some $i \in [m]$, then $X$ gets colour $c_i$.

(R4) If $X$ is min-heavy or max-light on $(d_i, c_i)$ for some $i \in [m]$, then $X$ gets colour $d_i$.

(R5) If $X$ contains a $W$-consecutive pair and $\delta(X) = j$, then $X$ gets colour $j$ if $j \in [k - 1] \setminus (A \cup B)$, and colour $0$ otherwise (that is, if $j = k$ or $j \in [k - 1] \cap (A \cup B)$).

(R6) If $X$ is skew at $d_i$ for some $i \in [m]$, then $X$ gets colour $d_i$, where $i$ is least with this property.

(R7) If none of the above applies, $X$ gets colour $0$.

We will now show that each colour class of this colouring is an independent set in $G$.

**Proposition 5.6.** Rules (R1)–(R7) determine a valid colouring of $G$.

**Proof.** We will discuss each colour class in turn. If $j \in [n] \setminus (W \cup [k - 1])$, then colour $j$ is only assigned by Rule (R1), namely to those vertices that contain element $j$. The colour class is therefore independent.

**Claim 1.** If $j \in [k - 1] \setminus W$, then the colour class of $j$ is independent.

Colour $j$ may be assigned to $X$ by Rule (R1) (if $j \in X$) or by Rule (R5) (if $X$ contains a $W$-consecutive pair and $\delta(X) = j$). Suppose that vertices $X,Y$ both get colour $j$.

Suppose that $j \in X$. If $j \in Y$, then $XY$ is not an edge. If $Y$ contains a $W$-consecutive pair and $\delta(Y) = j$, then $\delta(Y) \in X \cup Y$, so $XY$ is not an edge by Lemma 5.4.

The last remaining case is that $X,Y$ both contain a $W$-consecutive pair and $\delta(X) = \delta(Y) = j$. In this case, $X$ and $Y$ are non-adjacent by Lemma 5.5.
At this point, it remains to consider all the colours \[ j \] with \( j \in C \cup D \) and the colour \[ 0 \]. This is done in the following three claims.

**Claim 2.** For \( i \in [m] \), the colour class of \([c_i]\) is independent.

Colour \([c_i]\) is assigned by Rules (R2), (R3) and (R5) to essential vertices \( X \) satisfying one of the following:

- \( X \) contains a balanced pair \( \{c_i, d_i\} \),
- \( X \) contains no balanced pair and \( X \) is min-heavy on \( (d_i, c_i] \),
- \( X \) contains no balanced pair and \( X \) is max-light on \( (d_i, c_i] \),
- \( X \) contains a \( W \)-consecutive pair and \( \delta(X) = c_i \).

Let \( X \) and \( Y \) be vertices of \( G \) assigned colour \([c_i]\). We prove that \( XY \) is not an edge of \( G \). If both \( X \) and \( Y \) contain \( \{c_i, d_i\} \), then they are intersecting and therefore non-adjacent in \( G \). If both are min-heavy or both are max-light on \( (d_i, c_i] \), they intersect by Proposition 5.3.

Suppose that \( X \) is min-heavy on \( (d_i, c_i] \). If \( Y \) is max-light on \( (d_i, c_i] \), then \( |(d_i, c_i]_X| > c_i \) and \( |(d_i, c_i]_Y| < c_i \), so Lemma 5.1(ii) (with \( c = c_i \) and \( d = d_i + 1 \)) shows that \( X, Y \) are non-adjacent.

If \( \{c_i, d_i\} \) is a balanced pair in \( Y \), then we may suppose that \( c_i \notin X \). Thus, \( |(d_i, c_i]_X| > c_i \). On the other hand, \( |(d_i, c_i]_Y| = c_i + 1 \), so Lemma 5.1(ii) (with \( c = c_i \) and \( d = d_i + 1 \)) implies that \( X, Y \) are non-adjacent.

It remains to consider the case that one of \( X \) and \( Y \), say \( X \), contains a \( W \)-consecutive pair. By the position of Rule (R5), it may be assumed that \( X \) is regular; in particular, \( |(d_i, c_i]_X| = c_i \). Let \( \langle s, s' \rangle \) be a \( W \)-consecutive pair contained in \( X \) with \( s \) as large as possible. Since \( \delta(X) = c_i \), we have \( s' \in (d_i, c_i] \) and \( s \notin (d_i, c_i] \), so \( s = d_i \).

If \( Y \) also contains a \( W \)-consecutive pair, then the same argument applies; therefore, \( d_i \in X \cap Y \) and we are done. If \( Y \) is min-heavy on \( (d_i, c_i] \), then we may assume that \( s' \notin Y \) (otherwise \( X \) and \( Y \) intersect). Thus \( |(s', c_i]_Y| = |(d_i, c_i]_Y| > c_i \), while \( |(s', c_i]_X| = c_i - 1 \). Lemma 5.1(ii) implies that \( X \) and \( Y \) are non-adjacent. Finally, if \( Y \) is max-light on \( (d_i, c_i] \), then it may be assumed that \( d_i \notin Y \) (otherwise, \( X \) and \( Y \) are intersecting), so \( |(d_i, c_i]_Y| < c_i \). Since \( |(d_i, c_i]_X| = c_i + 1 \), \( X \) and \( Y \) are non-adjacent by Lemma 5.1. This finishes the proof of Claim 2.

**Claim 3.** For \( i \in [m] \), the colour class of \([d_i]\) is independent.

Note that since \( d_i > k \) by the definition of \( AB \)-alternator, Rule (R5) does not assign colour \([d_i]\). Thus, colour \([d_i]\) is only assigned by Rules (R4) and (R6) to vertices \( X \) satisfying one of the following:

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• $X$ contains no balanced pair and is min-heavy on $[d_i, c_i)$,

• $X$ contains no balanced pair and is max-light on $[d_i, c_i)$,

• $X$ is regular and skew at $d_i$.

Suppose that $X, Y$ are disjoint vertices of $G$ assigned colour $[d_i]$. We prove that $X$ and $Y$ are non-adjacent in $G$.

Suppose first that both $X$ and $Y$ are min-heavy or max-light on $[d_i, c_i)$. Let $X$ be min-heavy on $[d_i, c_i)$. By Proposition 5.3(i), $Y$ is not min-heavy on $[d_i, c_i)$, for otherwise $X$ and $Y$ would intersect. Thus we may assume that $Y$ is max-light on $[d_i, c_i)$, but then $|d_i, c_i)_y| < c_i$ and $|d_i, c_i)_x| > c_i$, so $XY$ is not an edge by Lemma 5.1. A symmetric argument applies in case $X$ is max-light on $[d_i, c_i)$.

Assume thus that $X$ is regular and skew at $d_i$. We will also assume that $i > 1$, the $i = 1$ case being analogous. By the position of Rule (R5), $X$ contains no $W$-consecutive pair. Consider the set $S = U_i \cup \{d_i\}$ and recall that $|S| = 2(c_i - c_{i-1}) + 1$. Since $X$ is regular, $|X \cap S| = c_i - c_{i-1}$.

We now distinguish several cases according to the type of $Y$. If $Y$ is skew at $d_i$, then $X$ and $Y$ both contain the second smallest element of $(d_i, d_{i-1})_W$, a contradiction.

Assume next that $Y$ is min-heavy on $[d_i, c_i)$. By Proposition 5.3, $|Y \cap S| = c_i - c_{i-1} + 1$, so the sets $X \cap S$ and $Y \cap S$ partition $S$. Now $d_i \notin X$ (since $X$ is regular), so $d_i \in Y$. Since $C \cup D$ is the standard $AB$-alternator, the smallest element of $(d_i, d_{i-1})_W$ is $d_i + 1$. We have $d_i + 1 \notin X$ as $X$ is skew at $d_i$ and contains no $W$-consecutive pair. It follows that $\{d_i, d_i + 1\} \subseteq Y$, a contradiction with the independence of $Y$ in $C_n$.

It remains to consider the case that $Y$ is max-light on $[d_i, c_i)$. Let $d^{-}$ be the last element of $(d_i+1, d_i)_W$. Since $d^- \in X$, we have $d^- \notin Y$. Furthermore, $|[d^-, c_i)_x| = c_i + 1$ as $X$ is regular on $[d_i, c_i)$, while $|[d^-, c_i)_y| < c_i$. Lemma 5.1 implies that $X$ and $Y$ are non-adjacent in $G$. The proof of Claim 3 is complete.

Claim 4. The colour class of $[0]$ is independent.

Colour $[0]$ is assigned by Rule (R5) to vertices $X$ containing a $W$-consecutive pair and having depth in the set $[1, k - 1]_{U \cup B} \cup \{k\}$, and by Rule (R7) to vertices satisfying none of the conditions in Rules (R1)-(R6). Let us say that $X$ is of type (R5) or (R7) accordingly. All of these vertices are regular (hence subsets of $W$) by Proposition 5.2. Additionally, type (R7) vertices contain no $W$-consecutive pair, and are not skew at any $d_i$ ($i \in [m]$).

Suppose that $X$ is a vertex of type (R5) and $Y$ is any vertex coloured $[0]$ with $X \cap Y = \emptyset$. If $\delta(X) = k$ then $X$ is not adjacent to $Y$ by Lemma 5.4.
We may thus assume that $\delta(X) \in [1, k-1]_{A \cup B}$. Since $X$ and $Y$ are regular, we have $X \cup Y = A \cup B$ and therefore $\delta(X) \in X \cup Y$. Lemma 5.4 implies that $XY$ is not an edge.

This leaves us with the case that $X, Y$ are vertices of type (R7). We intend to show that $\{X, Y\} = \{A, B\}$. The set $(d_1, c_1)_W$ consists of $2c_1$ elements and each of $X$ and $Y$ contains $c_1$ of them. Since none of $X$ and $Y$ contains a $W$-consecutive pair, we may assume by symmetry that $(d_1, c_1)_X = (d_1, c_1)_A$. We prove by induction that for each $i \in [m]$, $(d_1, d_i)_{X} = (d_1, d_i)_A$; recalling the convention that $c_1 = d_0$, the base case is established. Consider $i \geq 2$. Let $d^-$ be the largest element of $(d_{i+1}, d_i)_W$ and let $d', d''$ be the smallest and second smallest element of $(d_i, d_{i-1})_W$, respectively. (Recall that each of these sets has size at least 2 by Lemma 3.1(i).) By the induction hypothesis and the assumption that $X$ is not skew at $d_i$, we have

$$d^- \in X \iff d'' \notin X \iff d'' \notin A \iff d' \in A \iff d^- \in A.$$ 

Since each element of $(d_{i+1}, d_i)_W$ belongs to $X$ or $Y$, and since $X$ and $Y$ contain no $W$-consecutive pair, this implies that $(d_{i+1}, d_i)_X = (d_{i+1}, d_i)_A$ as required.

The above implies that $[k, n]_X = [k, n]_A$. Since $Y$ is regular, $[k, n]_Y = [k, n]_B$.

To prove that $X = A$, it remains to show that $(c_i, c_{i+1})_X = (c_i, c_{i+1})_A$ for each $i \in [m]$. By Lemma 3.1(ii), $d_i$ is contained in a control pair for the edge $XY$. Since $X$ and $Y$ are regular, it follows that the control pair must be $(c_i, d_i)$. Let $d$ be the largest element of $(d_{i+1}, d_i)_W$ and let $c$ be the smallest element of $(c_i, c_{i+1})_W$. By the definition of the edge set of $G$,

$$c \in X \iff d \notin X \iff d \notin A \iff c \in A.$$ 

Since $X$ and $Y$ contain no $W$-consecutive pair and cover $(c_i, c_{i+1})_W$, this determines $(c_i, c_{i+1})_X$ and shows that this set equals $(c_i, c_{i+1})_A$. The proof that $X = A$ (and $Y = B$) is complete. Since the edge $AB$ does not exist in $G$, this finishes the proof of the claim.

The proof of Proposition 5.6 is complete. 

The reference to Proposition 5.6 is complete.

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