RING OBJECTS IN THE EQUIVARIANT DERIVED SATAKE
CATEGORY ARISING FROM COULOMB BRANCHES

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Appendix B by Gus Lonergan

Abstract. This is the second companion paper of [Part II]. We consider the morphism
from the variety of triples introduced in [Part II] to the affine Grassmannian. The direct
image of the dualizing complex is a ring object in the equivariant derived category on
the affine Grassmannian (equivariant derived Satake category). We show that various
constructions in [Part II] work for an arbitrary commutative ring object.

The second purpose of this paper is to study Coulomb branches associated with star
shaped quivers, which are expected to be conjectural Higgs branches of 3d Sicilian theories
in type $A$ by [BTX10].

1. Introduction

This is the second companion paper of [Part II], where we give a mathematical definition
of the Coulomb branch $\mathcal{M}_C$ of a 3d SUSY gauge theory associated with a complex reductive
group $G$ and its symplectic representation $\mathbf{M}$ of a form $\mathbf{N} \oplus \mathbf{N}^*$. Recall that $\mathcal{M}_C$ is defined
as an affine algebraic variety whose coordinate ring is the equivariant Borel Moore homology
group $H^G_{\mathfrak{O}}(\mathcal{R})$ of a certain space $\mathcal{R}$, called the variety of triples. The product is given by
the convolution. Here $\mathfrak{O}$ is the $\mathbb{C}[[z]]$-valued points of $G$.

By its definition, we have a projection $\pi: \mathcal{R} \to \text{Gr}_G$, where $\text{Gr}_G$ is the affine Grassmannian for $G$. Therefore we have a natural object $\mathcal{A}$ in an appropaaate Ind-completion
$D_G(\text{Gr}_G)$ of the derived $G_0$-equivariant constructible category on $\text{Gr}_G$ defined by
$\pi_* \omega_{\mathcal{R}}[-2 \dim \mathfrak{O}_G]$, where $\omega_{\mathcal{R}}$ is the dualizing complex on $\mathcal{R}$. We can recover $H^G_{\mathfrak{O}}(\mathcal{R})$
as $H^G_{\mathfrak{O}}(\text{Gr}_G, \mathcal{A})$. Moreover the construction of the convolution product gives us a ho-
momorphism $\mathfrak{m}: \mathcal{A} \star \mathcal{A} \to \mathcal{A}$, where $\star$ is the convolution product on $D_G(\text{Gr}_G)$. It is an
associative multiplication on $\mathcal{A}$. Then we have an induced multiplication on $H^G_{\mathfrak{O}}(\text{Gr}_G, \mathcal{A})$
from $\mathfrak{m}$, which is the same as the product on $H^G_{\mathfrak{O}}(\mathcal{R})$ defined in [Part II]. We also prove
that it is a commutative object in $D_G(\text{Gr}_G)$, and hence the induced multiplication on
$H^G_{\mathfrak{O}}(\text{Gr}_G, \mathcal{A})$ is commutative. It is the second proof of the commutativity of the product
on $H^G_{\mathfrak{O}}(\mathcal{R})$, which is more conceptual than the first computational proof in [Part II].

In view of the original proposal in [Nak16], we expect that this construction can be
generalized to the case when $\mathbf{M}$ is not necessarily of the form $\mathbf{N} \oplus \mathbf{N}^*$.

Anyhow if we have a commutative ring object $\mathcal{A}$ in $D_G(\text{Gr}_G)$, we get a commutative ring
structure on $H^G_{\mathfrak{O}}(\text{Gr}_G, \mathcal{A})$, and hence the ‘Coulomb branch’ as its spectrum.

Our reformulation of the definition of the Coulomb branch via $(\mathcal{A}, \mathfrak{m})$ reminds us a
construction of the nilpotent cone and its Springer resolution via a perverse sheaf $\mathcal{A}_R$.
[ABG04]. Here $\mathcal{A}_R$ is a perverse sheaf corresponding to the regular representation $\mathbb{C}[G^\vee]$ of the Langlands dual group $G^\vee$ under the geometric Satake correspondence, and hence is a commutative ring object in $\text{Perv}_{G_\mathcal{O}}(\text{Gr}_G)$. Let us call it the regular sheaf. It is given by $\bigoplus_{\lambda} (V_G^\lambda)^\vee \otimes \text{IC}(\text{Gr}_G^\lambda)$, where $(V_G^\lambda)^\vee$ is the dual of the irreducible representation of $G^\vee$ with the highest weight $\lambda$ and $\text{Gr}_G^\lambda$ is the closure of the $G_\mathcal{O}$-orbit of $z^\lambda$ in $\text{Gr}_G$. We prove that $\mathcal{A}_R$ is realised as a variant of the above $\mathcal{A}$ for a quiver gauge theory in type $A$. (We consider the framed quiver gauge theory of type $A_N$ with $\dim V = (N-1, N-2, \ldots, 1)$, $\dim W = (N, 0, \ldots, 0)$ and consider the pushforward to $\text{Gr}_{\text{PGL}(N)}$. See §2(v) for more detail.) This construction might be generalized to type $BCD$, once we can generalize our definition to the case when $\mathcal{M}$ is not necessarily of a form $N \oplus N^*$ (cotangent type). However we do not expect $\mathcal{A}_R$ arises in a similar way for exceptional types. Hence we have more examples of commutative ring objects in $D_G(\text{Gr}_G)$ than our construction.

Once we have a collection $\{\mathcal{A}_i\}$ of commutative ring objects in $D_G(\text{Gr}_G)$, we can construct a new commutative ring object as $\bigoplus_{\Delta}(\boxtimes \mathcal{A}_i)$, where $i_{\Delta}: \text{Gr}_G \to \prod_i \text{Gr}_G$ is the diagonal embedding. We call this the gluing construction. It is motivated by [CHMZ14a]. (See [Nak16, 5(i)] for a quick review and links to other physics literature.)

The second purpose of this paper is to study Coulomb branches associated with a star shaped quiver. It is regarded as an example of the gluing construction of a ring objects from those for legs. It is expected that Coulomb branches of star shaped quiver gauge theories are conjectural Higgs branches of 3$d$ Sicilian theories in type $A$ [BTX10]. (See [Nak16, 3(iii)] for a review for a mathematician.) Expected properties of these Higgs branches are listed in [MT12]. Recently Ginzburg-Kazhdan [GK] construct holomorphic symplectic varieties satisfying (most of) these properties for any type. The construction of $\mathcal{A}_R$ as $\mathcal{A}$ implies that Coulomb branches of star shaped quiver gauge theories are isomorphic to Ginzburg-Kazhdan varieties in type $A$ via [Bap15]. We check two among the remaining properties, which identify Ginzburg-Kazhdan varieties of type $A_1$, $A_2$ with $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the minimal nilpotent orbit of $E_6$ respectively.

We do not expect Ginzburg-Kazhdan varieties for exceptional groups are Coulomb branches of gauge theories. This is compatible with physicists’ expectation that 3$d$ Sicilian theories are not lagrangian theories. Nevertheless 3$d$ Sicilian theories are accepted as well-defined quantum field theories. And there are many such examples. It is compatible with our observation that

1. We have examples of ring objects on $D_G(\text{Gr}_G)$, which may not arise from any pair $(G, N)$.
2. We have manipulations on ring objects, such as the gluing construction and hamiltonian reduction (see §5(viii) for the latter).

We thus hope that ring objects are useful to study non-lagrangian theories.

There is an appendix §A, which discusses a result of independent interest. We construct a complex reductive group hamiltonian action on the Coulomb branch of a framed quiver gauge theory by integrating hamiltonian vector fields of functions introduced in [Quiver, Appendix B]. This extends a torus action constructed in [Part II, §3(v)] by grading on
The regular sheaf $\mathcal{A}_R$ has the $G^\prime$-action, which is identified with this group action for the framed quiver gauge theory mentioned above.

The other parts of the paper are organized as follows. In §2 we show that $\pi_*\omega_R[-2 \dim N_O]$ and its cousin for gauge theory with a flavor symmetry group are ring objects. We observe that $\text{Ext}^\bullet_{D_G}(\mathcal{I}_{G_{\mathcal{G}}}, \mathcal{A})$ is a commutative ring for a commutative ring object $\mathcal{A}$ in $D_G(G_{\mathcal{G}})$, where $\mathcal{I}_{G_{\mathcal{G}}}$ is the skyscraper sheaf at the base point in $G_{\mathcal{G}}$. Considering skyscraper sheaves at other points, we construct line bundles over a partial resolution of $\text{Spec} \text{Ext}^\bullet_{D_G}(\mathcal{I}_{G_{\mathcal{G}}}, \mathcal{A})$. We follow [ABG04] for these constructions. The gluing construction is explained in §2(viii). In §3 we give a proof of commutativity of $m$. The idea is well-known: we use Beilinson-Drinfeld Grassmannian to deform a situation where the product is manifestly symmetric. Then we use nearby cycle functors and dual specialization homomorphisms. In §4 we show that the regular sheaf $\mathcal{A}_R$ arises as a pushforward in a framed quiver gauge theory in type $A$. In §5 we study Coulomb branches associated with star shaped quivers. Since §§4, 5 depend crucially on the construction in §A, the authors recommend the reader to go to §A before visiting §§4, 5.

In §B written by Gus Lonergan, we give another proof of the commutativity of the convolution product. This proof is more direct than the proof in the main text. A key ingredient is a global version of the convolution diagram for the variety of triples $\mathcal{R}$.

**Notation.** We basically follow the notation in [Part II] and [Quiver]. The Weyl group is denoted by $W$ in order to distinguish a vector space $W$ used for a quiver.

Sections, equations, Theorems, etc in [Part II] (resp. [Quiver]) will be referred with ‘II.’ (resp. ‘Q.’) plus the numbering, such as Theorem II.5.26 (resp. Theorem Q.3.10).

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2. Complexes on the affine Grassmannian

In this section we interpret the convolution product $*$ in terms of a complex on the affine Grassmannian. Our goal is to construct a commutative ring object in $D_G(G_{\mathcal{G}})$, an appropriate Ind-completion of the $G_{\mathcal{G}}$-equivariant derived constructible category on $G_{\mathcal{G}}$. Here the multiplication is given by the product $*$ appearing in geometric Satake correspondence [MV07].

The construction of this section, except §2(viii), is motivated by the work of Arkhipov, Bezrukavnikov and Ginzburg [ABG04], where the nilpotent cone $N$ of the Langlands dual group is constructed from the regular sheaf $\mathcal{A}_R$ on $G_{\mathcal{G}}$. 


The construction of §2(viii) is motivated by [CHMZ14a], as we have mentioned already in Introduction.

2(i). **Categorical generalities.** Let $X$ be a scheme of finite type over $\mathbb{C}$. Then we denote by $D(X)$ the ind-completion of the bounded derived category of constructible sheaves on $X$; same definition applies to the equivariant derived category $D_G(X)$ where $G$ is a (pro)algebraic group acting on $X$. It is obvious that for a $G$-equivariant morphism $f: X \to Y$ the derived direct images $f_!, f_*: D_G(X) \to D_G(Y)$ are well-defined. The same thing is true for the inverse images $f^!, f^*: D_G(Y) \to D_G(X)$.

Assume that $G$ has finitely many orbits on $X$. Then a morphism $\mathcal{F} \to \mathcal{G}$ in $D_G(X)$ is an isomorphism if and only if it is an isomorphism on all $!$-stalks (the assumption that $G$ acts with finitely many orbits is needed in order to guarantee that there is an open dense subset of $X$ on which both $\mathcal{F}$ and $\mathcal{G}$ are locally constant).

Let now $X$ be an ind-scheme which is a filtered inductive limit of schemes of finite type over $\mathbb{C}$ with respect to closed embeddings. For simplicity we shall assume that $X$ is just the union of closed subschemes $X_0 \subset X_1 \subset \cdots$ where each $X_i$ is a scheme of finite type over $\mathbb{C}$ and each inclusion $X_i \subset X_{i+1}$ is a closed embedding; we denote this embedding by $\sigma_i$. We shall call such ind-schemes **good**. Assume that a (pro)algebraic group $G$ acts on each $X_i$ and this action commutes with $\sigma_i$’s. Then we shall say that $X$ is a good $G$-scheme.

For a good $G$-ind-scheme $X$ we define the category $D_G(X)$ whose objects are systems $(\mathcal{F}_i, \kappa_i)_{i=0}^\infty$ where

- $\mathcal{F}_i \in D_G(X_i)$
- $\kappa_i: \sigma_i^*\mathcal{F}_{i+1} \to \mathcal{F}_i$ is an isomorphism.

A morphism $\alpha: (\mathcal{F}_i, \kappa_i) \to (\mathcal{F}_i', \kappa_i')$ is collection of morphisms $\mathcal{F}_i \to \mathcal{F}_i'$ for each $i$ which commute with the $\kappa_i$’s. It is easy to see that $D_G(X)$ is a triangulated category. Assume that $G$ acts with finitely many orbits on each $X_i$; in this case we shall say that $X$ is a very good $G$-ind-scheme. Then again a morphism $\mathcal{F} \to \mathcal{G}$ in $D_G(X)$ is an isomorphism if and only if it is an isomorphism on all $!$-stalks.

Let $X, Y$ be two good $G$-ind-schemes and let $f: X \to Y$ be a $G$-equivariant morphism. Then we can define the functor $f_*: D_G(X) \to D_G(Y)$ (but a priori not the functor $f_!$). It is defined in the following way. Given an object $(\mathcal{F}_i, \kappa_i)$ of $D_G(X)$ we need to define an object $(\mathcal{G}_j, \eta_j)$ of $D_G(Y)$. Let $Z_j = f^{-1}(Y_j)$. This is again a good $G$-ind-scheme – it is the inductive limit of $Z_{i,j} = X_i \cap f^{-1}(Y_j)$. Let $\mathcal{F}_{i,j}$ denote the $!$-restriction of $\mathcal{F}_i$ to $Z_{i,j}$. Let also $f_{i,j}: Z_{i,j} \to Y_j$ denote the natural morphism. Since $(\sigma_i^!)$ is right adjoint to $(\sigma_i)_*$, the isomorphism $\kappa_i$ gives rise to a map $(\sigma_i)_*\mathcal{F}_i = (\sigma_i)_*\mathcal{F}_i \to \mathcal{F}_{i+1}; !$-restricting this to $Z_j$ we get a morphism $(\sigma_i)_*\mathcal{F}_{i,j} \to \mathcal{F}_{i+1,j}$ which gives rise to a natural map $(f_{i,j})_*\mathcal{F}_{i,j} \to (f_{i+1,j})_*\mathcal{F}_{i+1,j}$. Hence the inductive limit of $(f_{i,j})_*\mathcal{F}_{i,j}$’s (with respect to $i$) makes sense and we denote it by $\mathcal{G}_j$. The construction of isomorphisms $\eta_j$ between the $!$-restriction of $\mathcal{G}_{j+1}$ and $\mathcal{G}_j$ is immediate from the usual base change.

In what follows we are going apply it for example to $X$ being $\text{Gr}_G$ for some reductive group $G$. In this case we can talk about the equivariant derived category $D_G(\text{Gr}_G)$ which as before we shall simply denote by $D_G(\text{Gr}_G)$ (a priori it depends on a choice of $X_i$’s above; to simplify the discussion we are going to make this choice, although it is not difficult to
see that the resulting category is independent of that choice; it is clear that (for any choice of $X_i$) $\text{Gr}_G$ is a very good $G_C$-ind-scheme. The above general discussion also shows that given two objects $\mathcal{F}, \mathcal{G} \in D_G(\text{Gr}_G)$ we can define their convolution $\mathcal{F} \ast \mathcal{G} \in D_G(\text{Gr}_G)$.

2(ii). **Pushforward to the affine Grassmannian.** Let $\mathcal{N}$ be a finite dimensional representation of a complex reductive group $G$. Let $\mathcal{R}$ be the variety of triples as in [Part II], and $\omega_\mathcal{R}$ its dualizing complex.

**Proposition 2.1.** Let $\pi : \mathcal{R} \to \text{Gr}_G$ be the projection and $\mathcal{A} \overset{\text{def.}}{=} \pi_\ast \omega_\mathcal{R}[-2 \dim \mathcal{N}_\mathcal{O}] \in D_G(\text{Gr}_G)$.

1. There exists a natural multiplication homomorphism
   $$m : \mathcal{A} \ast \mathcal{A} \to \mathcal{A},$$
   where the left hand side is the convolution product of $\mathcal{A}$ with itself given by the diagram (II.3.1).

2. Let $1_{\text{Gr}_G}$ denote the skyscraper sheaf at the base point in $\text{Gr}_G$. Recall that it is the unit element in $D_G(\text{Gr}_G)$, i.e., we have natural isomorphisms $1_{\text{Gr}_G} \ast \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \ast 1_{\text{Gr}_G}$. We have a homomorphism $1 : 1_{\text{Gr}_G} \to \mathcal{A}$ such that
   $$\mathcal{A} \cong \mathcal{A} \ast 1_{\text{Gr}_G} \overset{\text{id} \ast 1}{\longrightarrow} \mathcal{A} \ast \mathcal{A} \overset{m}{\longrightarrow} \mathcal{A}, \quad \mathcal{A} \cong 1_{\text{Gr}_G} \ast \mathcal{A} \overset{1 \ast \text{id}}{\longrightarrow} \mathcal{A} \ast \mathcal{A} \overset{m}{\longrightarrow} \mathcal{A}$$
   are both $\text{id}_\mathcal{A}$.

3. Under the natural associativity isomorphism $\mathcal{A} \ast (\mathcal{A} \ast \mathcal{A}) \cong (\mathcal{A} \ast \mathcal{A}) \ast \mathcal{A}$, we have
   $$m \circ (m \ast \text{id}) = m \circ (\text{id} \ast m).$$

4. The product on $H^*_G(\text{Gr}_G, \mathcal{A}) \cong H^G(\mathcal{R})$ induced by $m$ is the same as the convolution product $\ast$.

5. (1)~(4) remain true for the $G_\mathcal{O} \times \mathbb{C}^\times$-equivariant setting.

The product in (4) is defined as follows: Let $x, y \in H^*_G(\mathcal{A}) = \text{Ext}^*_D(\text{Gr}_G)(\mathcal{C}_{\text{Gr}_G}, \mathcal{A})$. Then $x \ast y \in \text{Ext}^*_D(\mathcal{C}_{\text{Gr}_G}, \mathcal{C}_{\text{Gr}_G}, \mathcal{A} \ast \mathcal{A})$. We have a natural homomorphism $\mathcal{C}_{\text{Gr}_G} \to \mathcal{C}_{\text{Gr}_G} \ast \mathcal{C}_{\text{Gr}_G}$ from the adjunction homomorphism $\mathcal{C}_{\text{Gr}_G} \to m_\ast m^\ast \mathcal{C}_{\text{Gr}_G}$. Therefore we combine it with $m : \mathcal{A} \ast \mathcal{A} \to \mathcal{A}$, we get $x \ast y \in \text{Ext}^*_D(\mathcal{C}_{\text{Gr}_G})(\mathcal{C}_{\text{Gr}_G}, \mathcal{A})$.

**Proof.** Let us combine two diagrams (II.3.1) and (II.3.2):

$\mathcal{R} \times \mathcal{R} \overset{\tilde{p}}{\longrightarrow} p^{-1}(\mathcal{R} \times \mathcal{R}) \overset{\tilde{q}}{\longrightarrow} q(p^{-1}(\mathcal{R} \times \mathcal{R})) \overset{\tilde{m}}{\longrightarrow} \mathcal{R}$

$\mathcal{T} \times \mathcal{R} \overset{p}{\longrightarrow} G_K \times \mathcal{R} \overset{q}{\longrightarrow} G_K \times G_\mathcal{O} \mathcal{R} \overset{m}{\longrightarrow} \mathcal{T}$

$\text{Gr}_G \times \text{Gr}_G \overset{\tilde{p}}{\longrightarrow} G_K \times \text{Gr}_G \overset{\tilde{q}}{\longrightarrow} \text{Gr}_G \times \text{Gr}_G \overset{\tilde{m}}{\longrightarrow} \text{Gr}_G,$

where we have changed the notation for morphisms in the bottom row putting `bar`. We also denote $\pi \circ i$ simply by $\pi$ for brevity.
The restriction with support homomorphism (II.3.7) induces
\[ \mathcal{A} \boxtimes \mathcal{A} = (\pi \times \pi)_{*}(\omega_{\mathcal{R} \times \mathcal{R}})[-4 \dim \mathcal{N}_{\mathcal{O}}] \]
\[ \rightarrow (\pi \times \pi)_{*}q^!(\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathcal{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]) \]
\[ \cong \tilde{p}^{*}(\text{id}_{G_{\mathcal{K}}} \times \pi)_{*}i_{s}^!\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathcal{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]). \]

By adjunction, we get
\[ \tilde{p}^{*}(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow (\text{id}_{G_{\mathcal{K}}} \times \pi)_{*}i_{s}^!\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathcal{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]). \]

Since \( \tilde{q} \) is the quotient by \( G_{\mathcal{O}} \), the right hand side is
\[ (\text{id}_{G_{\mathcal{K}}} \times \pi)_{*}i_{s}^!\tilde{q}^{*}\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathcal{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]) \cong \tilde{q}^{*}\pi_{s}^{*}\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathcal{N}_{\mathcal{O}}]. \]

Applying \((\tilde{q}^{*})^{-1}\), we get a homomorphism
\[ (2.3) \quad \mathcal{A} \boxtimes \mathcal{A} = (\tilde{q}^{*})^{-1}\tilde{p}^{*}(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \pi_{s}^{*}\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathcal{N}_{\mathcal{O}}]. \]

We further apply \( \tilde{m}_{*}^{\dagger} \):
\[ \mathcal{A} \star \mathcal{A} = \tilde{m}_{*}^{\dagger}(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \pi_{s}^{*}\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathcal{N}_{\mathcal{O}}]. \]

The left hand side is nothing but the convolution product \( \mathcal{A} \star \mathcal{A} \) defined by the diagram (II.3.1).

Since \( \tilde{m} \) is proper, we have a natural homomorphism \( \tilde{m}_{*}\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathcal{N}_{\mathcal{O}}] \rightarrow \omega_{\mathcal{R}}[2 \dim \mathcal{N}_{\mathcal{O}}]. \] Thus we obtain the homomorphism in (1).

Proofs of (2),(3) are already given in the proof of Theorem II.3.10. Note that the associativity isomorphism is given by the \( \text{Gr}_{G} \)-version of the big square diagram appearing in the proof of Theorem II.3.10. See [MV07, Prop. 4.6].

Taking hypercohomology groups, one can check (4). We omit the detail. \( \square \)

Remarks 2.4. (1) By [BF08, Theorem 5], \( \mathcal{A} \in D_{G_{\mathcal{O}} \times \mathcal{C} \times}(\text{Gr}_{G}) \) corresponds to a certain differential graded Harish-Chandra bimodule of \( G^{\vee} \). We do not know anything about it except the example just below.

(2) Let us denote by \( \mathcal{A}_{R} \) the regular sheaf, i.e., the perverse sheaf corresponding to the regular representation \( \mathbb{C}[G^{\vee}] \) of the Langlands dual group \( G^{\vee} \) under the geometric Satake correspondence. It was denoted by \( \mathcal{R} \) in [ABG04], but it conflicts with our notation for the space \( \mathcal{R} \). It is endowed with a natural morphism \( \mathbf{m}: \mathcal{A}_{R} \star \mathcal{A}_{R} \rightarrow \mathcal{A}_{R} \) with properties listed in Proposition 2.1. The nilpotent cone \( \mathcal{N} \) of \( G^{\vee} \) and its Springer resolution \( \hat{\mathcal{N}} \) were constructed from \( \mathcal{A}_{R} \) in [ABG04]. Since it is more natural to compare \( \mathcal{A}_{R} \) with \( \mathcal{A} \) arising in the framework of a flavor symmetry group, more detail will be given \( \S 2(v) \). Finally, the dg-Harish-Chandra bimodule corresponding to \( \mathcal{A}_{R} \) is the ring \( U_{h}^{[1]} \times \mathbb{C}[G^{\vee}] \) of \( h \)-differential operators on \( G^{\vee} \).

The construction in this and the subsequent subsections shows that it is enough to have \( \mathcal{A} \) with \( \mathbf{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A} \), i.e., a ring object in \( D_{G}(\text{Gr}_{G}) \) to define the Coulomb branch \( \mathcal{M}_{C} \). For example, \( \mathcal{A}_{R} \). Since \( \mathcal{A}_{R} \) for an exceptional group is unlikely to arise from any gauge theory \((G, \mathcal{N})\), it is interesting to find other recipes to construct such an \((\mathcal{A}, \mathbf{m})\). We give one example of such a recipe in \( \S 2(viii) \) below.
2(iii). **Commutativity.** In this subsection we forget the loop rotation.

Let $\Theta: \mathcal{A} \star \mathcal{A} \to \mathcal{A} \star \mathcal{A}$ be the commutativity constraint of the convolution product. Its construction, following [MV07, §5] and also [Gai01], will be recalled in §3(i).

**Theorem 2.5.** We have $m \circ \Theta \cong m$ as homomorphism $\mathcal{A} \star \mathcal{A} \to \mathcal{A}$.

It means that $(\mathcal{A}, m)$ is a commutative ring object in $(D_G(\text{Gr}_G), \star)$. We give a proof in §3.

Our proof is indirect. We construct another multiplication $m^\psi: \mathcal{A} \star \mathcal{A} \to \mathcal{A}$ using nearby cycle functors and dual specialization. We have $m^\psi \circ \Theta \cong m^\psi$. Therefore $(\mathcal{A}, m^\psi)$ is commutative, but we cannot check $m^\psi$ is associative directly.

Next we show $m^\psi = m$ for $N = 0$. This implies that $m^\psi = m$ holds after the fixed point localization for general $N$. We do not have torsion where $m$ and $m^\psi$ live, hence this is enough.

2(iv). **A complex on the affine Grassmannian of the flavor symmetry group.** We suppose that $N$ is a representation of a larger group $\tilde{G}$ containing $G$ as a normal subgroup as in §II.3(viii), §II.3(ix). Let $G_F = \tilde{G}/G$. We are going to construct a ring object in $D_{G_F}(\text{Gr}_{G_F})$.

Let us denote $\mathcal{T}_{G,N}, \mathcal{R}_{G,N}$ by $\tilde{\mathcal{T}}, \tilde{\mathcal{R}}$ respectively for short as before. Composing $\tilde{\mathcal{T}} \to \text{Gr}_{\tilde{G}}$ or $\tilde{\mathcal{R}} \to \text{Gr}_{\tilde{G}}$ with the morphism $\text{Gr}_{\tilde{G}} \to \text{Gr}_{G_F}$, we have

$$m^\psi \circ \Theta \cong m^\psi.$$ 

(2.6) Let us denote the fiber over $\lambda \in \text{Gr}_{G_F}$ by $\tilde{\mathcal{R}}^\lambda$, where $\lambda$ is a coweight of $G_F$ regarded as a point in $\text{Gr}_{G_F}$. (In §II.3(ix) it was denoted by $\lambda_F$.)

As in Proposition 2.1, we consider a pushforward of the dualizing sheaf $\omega_{\tilde{\mathcal{R}}}$. Here we consider the dualizing sheaf of the larger space $\tilde{\mathcal{R}}$, and take the pushforward $\mathcal{A} \equiv \tilde{\pi}_* \omega_{\tilde{\mathcal{R}}}[\dim \mathcal{N}_{\mathcal{O}}]$ to $\text{Gr}_{G_F}$. We consider it as an object in $D_{\tilde{G}}(\text{Gr}_{G_F})$, an appropriate Ind-completion of the $\tilde{G}$-equivariant derived constructible category of $\text{Gr}_{G_F}$. We also have $Q_{\tilde{\pi}_*} Q_{\tilde{\mathcal{R}}} \cong Q_{\text{id}_*} \mathcal{A}$, which is a $(G_F)_0$-equivariant object on $\text{Gr}_{G_F}$. Here ‘id’ is the identity of $\text{Gr}_{G_F}$ and the general pushforward functor $Q_{\text{id}_*}, Q_{\tilde{\pi}_*}$ changes the equivariance group from $\tilde{G}_0$ to $(G_F)_0$. See [BL94, §6].

In the same way as in Proposition 2.1, we have natural homomorphisms

$$m: \lambda \star \mathcal{A} \to m: \mathcal{A} \star \mathcal{A} \to \mathcal{A}, \quad Q_{\text{id}_*} \mathcal{A} \star Q_{\text{id}_*} \mathcal{A} \to Q_{\text{id}_*} \mathcal{A},$$

(2.7) that satisfy the unit and associativity properties. It also satisfies the commutativity.

Let us give a small remark for the construction of the homomorphisms: When we define the convolution product $\mathcal{A} \star \mathcal{A}$, we use $(q^*)^{-1}$ for $\text{Gr}_{G_F}$. For this, we only need the $(G_F)_0$-equivariant structure, therefore we can replace the second factor $\mathcal{A}$ by $Q_{\text{id}_*} \mathcal{A}$. However in the definition of the first homomorphism $m$, we need to go back to the space $\tilde{\mathcal{R}}$, hence we need the $\tilde{G}_0$-equivariant structure. The second homomorphism $m$ is induced from the first by applying $Q_{\text{id}_*}$ and using the smooth base change.

Let $1_{\text{Gr}_{G_F}}$ be the skyscraper sheaf at the base point in $\text{Gr}_{G_F}$. As in [ABG04, §7.2] we have an algebra structure on $\text{Ext}^*_{\tilde{G}}(\text{Gr}_{G_F})(1_{\text{Gr}_{G_F}}, \mathcal{A})$: Let $x \in \text{Ext}^*_{\tilde{G}}(\text{Gr}_{G_F})(1_{\text{Gr}_{G_F}}, \mathcal{A}), y \in$
Ext_{D_G(G_{G_F})}^j(1_{G_{G_F}}, \mathcal{A}). We consider \( x \star y \in \text{Ext}_{D_G(G_{G_F})}^{i+j}(1_{G_{G_F}} \star 1_{G_{G_F}}, \mathcal{A} \star \mathcal{A}). \) We compose

\[ 1: 1_{G_{G_F}} \cong 1_{G_{G_F}} \star 1_{G_{G_F}} \text{ and } m: \mathcal{A} \star \mathcal{A} \to \mathcal{A}, \]

we get \( m(x \star y)1 \in \text{Ext}_{D_G(G_{G_F})}^{i+j}(1_{G_{G_F}}, \mathcal{A}). \)

Note that ext-groups in \( D_{G_F} \) and \( D_{G_F} \) are isomorphic:

\[
\text{Ext}^*_{D_G(G_{G_F})}(1_{G_{G_F}}, Q_{1d} \star \mathcal{A}) \cong \text{Ext}^*_{D_G(G_{G_F})}(1_{G_{G_F}}, \mathcal{A}),
\]

where the right hand side is regarded as \( H^*_G(\text{pt}) \)-module via \( H^*_G(\text{pt}) \to H^*_G(\text{pt}) \). See [BL94, §13.5]. Thus the difference between \( \mathcal{A} \) and \( Q_{1d} \star \mathcal{A} \) is not essential, we omit \( Q_{1d} \) hereafter.

Since the fiber of \( \tilde{\pi}: \tilde{\mathcal{R}} \to \text{Gr}_{G_F} \) at the base point is our original \( \mathcal{R} \), we have a natural isomorphism

\[ \text{Ext}^*_{D_G(G_{G_F})}(1_{G_{G_F}}, \mathcal{A}) \cong H^*_G(\mathcal{R}) \]

of \( H^*_G(\text{pt}) \)-modules.

The definition of the multiplication on \( \text{Ext}^*_{D_G(G_{G_F})}(1_{G_{G_F}}, \mathcal{A}) \) uses \( \tilde{G} \) (or \( G_F \)) equivariance, as we use the descent \((q^*)^{-1}\). On the other hand, the multiplication on the right hand side given in Proposition II.3.22 descends to \( H^*_G(\mathcal{R}) \). In fact, we will see that a simple modification of the definition gives a multiplication on the left hand side with the group changed from \( \tilde{G} \) to \( G \) in §2(vi).

**Lemma 2.9.** The isomorphism (2.8) respects the multiplication. The same is true for \( \tilde{G}_\mathcal{O} \times \mathbb{C}^* \)-equivariant groups.

**Proof.** Let us consider a modification of the commutative diagram (2.2):

\[
\begin{array}{c}
\tilde{T} \times \tilde{\mathcal{R}} & \leftarrow & \tilde{G}_K \times \tilde{\mathcal{R}} & \rightarrow & \tilde{G}_K \times \tilde{\mathcal{R}} & \rightarrow & \tilde{\mathcal{R}} \\
\xi \times \tilde{\pi} & \downarrow & \tilde{G}_K \times \tilde{\mathcal{R}} & \rightarrow & \tilde{G}_K \times \tilde{\mathcal{R}} & \rightarrow & \tilde{\mathcal{R}} \\
\text{Gr}_{G_F} \times \text{Gr}_{G_F} & \rightarrow & (G_F)_K \times \text{Gr}_{G_F} & \rightarrow & \text{Gr}_{G_F} \times \text{Gr}_{G_F} & \rightarrow & \text{Gr}_{G_F},
\end{array}
\]

where \( \xi: \tilde{G}_K \to (G_F)_K \) is a morphism induced from \( \tilde{G} \to G_F \), and all other maps are given by replacing \( G, \mathcal{R}, ... \) by \( \tilde{G}, \tilde{\mathcal{R}}, ..., \) and composing \( \text{Gr}_{G_F} \to \text{Gr}_{G_F} \), etc. We omit the first row for brevity.

Let \([1_{G_F}]\) denote the base point in \( \text{Gr}_{G_F} \). We take the inverse images of \([1_{G_F}] \times [1_{G_F}], (G_F)_\mathcal{O} \times [1_{G_F}], [1_{G_F}] \star [1_{G_F}], [1_{G_F}] \) in the first row. They are \( \tilde{\mathcal{T}} \times \tilde{\mathcal{R}}, \tilde{G}_K^{\mathcal{O}} \times \tilde{\mathcal{R}}, \tilde{G}_K^{\mathcal{O}} \times \tilde{\mathcal{R}}, \mathcal{T} \) respectively. Here \( \tilde{G}_K^{\mathcal{O}} = \xi^{-1}((G_F)_\mathcal{O}) \) is the group introduced in §II.3(viii). Thus we recover the diagram (II.3.23). Now the assertion is easy to check, and hence we omit the detail. \( \square \)

---

\(^1\)We thank Roman Bezrukavnikov for a clarification of this point.
An alternative construction of a regular sheaf. Consider a quiver gauge theory of type $A_{N-1}$ with $\dim V = (N-1, N-2, \ldots, 1)$, $\dim W = (N, 0, \ldots, 0)$ with $G = \text{GL}(V) = \prod_{i=1}^{N-1} \text{GL}(i)$, $\hat{G} = \text{(GL}(V) \times \text{GL}(W))/Z$, where $Z \cong \mathbb{C}^\times$ is the diagonal central subgroup. We have $G_F = \text{PGL}(W) = \text{PGL}(N)$ and apply the above construction to define $\mathcal{A}$. It is a complex on $\text{Gr}_{\text{PGL}(N)}$. We also know that the Coulomb branch $\mathcal{M}_C$ of this quiver gauge theory is the nilpotent cone in $\mathfrak{sl}(N)$. (We know that $\mathcal{M}_C$ is a transversal slice in the affine Grassmannian by §Q.3 for a quiver gauge theory of type ADE. And in this case the transversal slice in the affine Grassmannian is the nilpotent cone by [Lus81]. See also [MV03].) Recall $\mathcal{A}_R$ in Remark 2.4(2). We take $G = \text{PGL}(N)$. Then $\text{Ext}^*_{\mathcal{D}(\text{Gr}_{\text{PGL}(N)})}(1_{\text{Gr}_{\text{PGL}(N)}}, \mathcal{A}_R)$ gives also the nilpotent cone [ABG04, 7.3.1]. This is not a coincidence. We have

**Theorem 2.11.** $\mathcal{A}_R$ and $\mathcal{A}$ are isomorphic as ring objects in $\mathcal{D}_{\text{PGL}(N)}(\text{Gr}_{\text{PGL}(N)})$.

The proof will be given in §4.

Line bundles via homology groups of fibers. We now return back to a general situation: we are given a commutative ring object in $\mathcal{D}_G(\text{Gr}_G)$, i.e., we are given $\mathcal{A} \in \mathcal{D}_G(\text{Gr}_G)$ with $1: 1_{\text{Gr}_G} \to \mathcal{A}$, $\mathfrak{m}: \mathcal{A} \circ \mathcal{A} \to \mathcal{A}$ satisfying the unit and associativity properties in Proposition 2.1(2) and the commutativity as in Theorem 2.5. The object constructed in §2(ii), as well as the object $\mathcal{A}$ or $Q_{\text{id}} \ast \mathcal{A}$ in §2(iv) is an example when we regard $G_F$ as $G$. In fact, the latter is our primary example.

Let $D(\text{Gr}_G)$ denote an appropriate Ind-completion of the constructible derived category on $\text{Gr}_G$ (without $G_\mathcal{O}$-equivariance structure). Let For: $\mathcal{D}_G(\text{Gr}_G) \to D(\text{Gr}_G)$ be the forgetful functor.

**Remark 2.12.** In the setting of §2(iv), we could consider $\mathcal{D}_G(\text{Gr}_{G_F})$, an appropriate Ind-completion of the $G_\mathcal{O}$-equivariant constructible derived category on $\text{Gr}_{G_F}$. Note that $G_\mathcal{O}$ acts trivially on $\text{Gr}_{G_F}$. Let $\text{Res}_{G_\mathcal{O}, \hat{G}_\mathcal{O}}$ be the restriction functor $D_G(\text{Gr}_{G_F}) \to D_G(\text{Gr}_{G_F})$ restricting the group action from $G_\mathcal{O}$ to $G_\mathcal{O}$. Then we could consider $\mathcal{A}^{\text{res}} = \text{Res}_{G_\mathcal{O}, \hat{G}_\mathcal{O}} \mathcal{A} \in \mathcal{D}_G(\text{Gr}_{G_F})$. This allows us to consider $\text{Ext}^*_{\mathcal{D}_G(\text{Gr}_{G_F})}(1_{\text{Gr}_{G_F}}, \mathcal{A}^{\text{res}})$, but the difference between this Ext group and $\text{Ext}^*_{\mathcal{D}(\text{Gr}_{G_F})}(1_{\text{Gr}_{G_F}}, Q_{\text{id}} \ast \mathcal{A})$ is not essential as we have remarked above. Therefore we do not keep two groups $G, G_F$, and just consider the above situation for brevity of the notation.

Let $\mathcal{A}^{\text{for}} \overset{\text{def}}{=} \text{For} \mathcal{A}$. Note that $\mathcal{A}^{\text{for}} \ast \mathcal{A}^{\text{for}}$ is not defined as we do not have $(q^*)^{-1}$ for non $G_\mathcal{O}$-equivariant objects. However we still have $\text{For}: \mathcal{D}(\mathcal{A} \ast \mathcal{A}) \to \mathcal{A}^{\text{for}} = \text{For} \mathcal{A}$.

Viewing a coweight $\lambda$ of $G$ as a point in $\text{Gr}_G$, we denote the embedding by $i_\lambda: \{\lambda\} \to \text{Gr}_G$.

Recall $m: \text{Gr}_G \ast \text{Gr}_G \to \text{Gr}_G$. For a coweight $\chi$, let $\text{Gr}^2_\chi \overset{\text{def}}{=} m^{-1}(\chi)$ and denote the embedding $\text{Gr}^2_\chi \to \text{Gr}_G \ast \text{Gr}_G$ by $j_\chi$. We have the base change $i_\chi^* m_* = m_* j_\chi^!$. Recall $\mathcal{A} \ast \mathcal{A} = m_*(q^*)^{-1} p^* (\mathcal{A} \boxtimes \mathcal{A})$. Let us set $\mathcal{A} \boxtimes \mathcal{A} = (q^*)^{-1} p^* (\mathcal{A} \boxtimes \mathcal{A})$. As the forgetful functor commutes with $m_*$, we have $\text{For}(\mathcal{A} \ast \mathcal{A}) = m_* \text{For}(\mathcal{A} \boxtimes \mathcal{A})$. We have

$$m_* j_\chi^! \text{For}(\mathcal{A} \boxtimes \mathcal{A}) = i_\chi^! m_* \text{For}(\mathcal{A} \boxtimes \mathcal{A}) = i_\chi^! \text{For}(\mathcal{A} \ast \mathcal{A}) \overset{i_\chi^* \text{For} m}{\longrightarrow} i_\chi^! \mathcal{A}^{\text{for}}.$$
Claim. The embedding \( \{\lambda\} \times \{\mu\} \to \Gr_{\lambda+\mu}^2 \) induces a natural homomorphism

\[
(2.14) \quad H^*(i^!_{\lambda\mu} \mathcal{A}_{\text{for}}) \otimes H^*(i^!_{\mu\lambda} \mathcal{A}_{\text{for}}) \to H^*(j^!_{\lambda+\mu} \text{For}(\mathcal{A} \boxtimes \mathcal{A})).
\]

Proof. Let us regard \( \lambda \) as an element in \( G_K \) and denote the embedding \( \{\lambda\} \to G_K \) by \( i_\lambda \).

The morphism \( q(i_\lambda \times i_\mu): \{\lambda\} \times \{\mu\} \to \Gr_G \times \Gr_G \) factors through \( \Gr_{\lambda+\mu}^2 \). Let us write the embedding \( k_{\lambda+\mu}^!: \{\lambda\} \times \{\mu\} \to \Gr_{\lambda+\mu}^2 \).

We note \( i_{\lambda+\mu}^!j^!_{\lambda+\mu} = (i_\lambda \times i_\mu)^!q^* = (i_\lambda \times i_\mu)^!q^*[2 \dim G_O] \). Since the forgetful functor commutes with pull back homomorphisms [BL94, §3.4], we get

\[
(2.15) \quad k_{\lambda+\mu}^!j^!_{\lambda+\mu} \text{For}(A \boxtimes A) = (i_\lambda \times i_\mu)^!q^!(A_{\text{for}} \boxtimes A_{\text{for}})[2 \dim G_O] = \left( i_\lambda \times i_\mu \right)^!q^!(A_{\text{for}} \boxtimes A_{\text{for}}) = (i_\lambda \times i_\mu)^!q^!(A_{\text{for}} \boxtimes A_{\text{for}}).
\]

Since \( k_{\lambda+\mu} \) is proper, we have a homomorphism \( k_{\lambda+\mu}^!k^!_{\lambda+\mu} = k_{\lambda+\mu}k^!_{\lambda+\mu} \to \text{id} \). Now the assertion is clear. \( \square \)

Combining (2.13) with \( \chi = \lambda + \mu \) and (2.14), we obtain a multiplication

\[
(2.16) \quad H^*(i^!_{\lambda\mu} \mathcal{A}_{\text{for}}) \otimes H^*(i^!_{\lambda\mu} \mathcal{A}_{\text{for}}) \to H^*(i^!_{\lambda+\mu} \mathcal{A}_{\text{for}}).
\]

Remarks 2.17. (1) Note that the embedding \( i_\lambda \) is \( T_O \)-equivariant. Therefore we can use the restriction functor \( \text{Res}_{T_O,G_O} \) from \( G_O \) to \( T_O \) instead of the forgetful functor \( \text{For} \). Then the same construction gives a multiplication

\[
(2.18) \quad H^*_{T_O}(i^!_{\lambda\mu} \text{Res}_{T_O,G_O} \mathcal{A}) \otimes H^*_{T_O}(i^!_{\lambda\mu} \text{Res}_{T_O,G_O} \mathcal{A}) \to H^*_{T_O}(i^!_{\lambda+\mu} \text{Res}_{T_O,G_O} \mathcal{A}).
\]

(2) Suppose \( G = T \). Then \( \Gr_T = \bigsqcup_{\lambda \in Y} \{\lambda\} \), hence \( H^*_{T_O}(\Gr_T, \mathcal{A}) = \bigoplus_{\lambda \in Y} H^*_{T_O}(i^!_{\lambda\mu} \mathcal{A}) \).

The multiplication explained after Proposition 2.1 is \( Y \)-graded, hence gives \( H^*_{T_O}(i^!_{\lambda\mu} \mathcal{A}) \otimes H^*_{T_O}(i^!_{\lambda+\mu} \mathcal{A}) \to H^*_{T_O}(i^!_{\lambda+\mu} \mathcal{A}) \). It is clear that this multiplication is same as (2.18).

Suppose \( \lambda = \mu = 0 \). We have a commutative diagram

\[
\begin{array}{ccc}
\Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}) \otimes \Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}) & \xrightarrow{\text{m}(\text{m})(1)} & \Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}) \\
| & \downarrow \text{For} & | \\
H^*(i^!_{0\lambda} \mathcal{A}_{\text{for}}) \otimes H^*(i^!_{0\mu} \mathcal{A}_{\text{for}}) & \xrightarrow{(2.16)} & H^*(i^!_{0\mu} \mathcal{A}_{\text{for}})
\end{array}
\]

via the isomorphism \( \Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}_{\text{for}}) \cong H^*(i^!_{0\mu} \mathcal{A}_{\text{for}}) \).

In fact, the only place we need to check is the commutativity of

\[
\begin{array}{ccc}
\Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}) \otimes \Ext^*_{D(G,G)}(1_{Gr^G}, \mathcal{A}) & \xrightarrow{(q^*)^{-1}p^*} & \Ext^*_{D(G,G) \times D(G,G)}(C[1_G], \mathcal{A} \boxtimes \mathcal{A}) \\
| & \downarrow \text{For} & | \\
H^*(i^!_{0\lambda} \mathcal{A}_{\text{for}}) \otimes H^*(i^!_{0\mu} \mathcal{A}_{\text{for}}) & \xrightarrow{(2.16)} & H^*(j^!_{0} \text{For}(\mathcal{A} \boxtimes \mathcal{A})).
\end{array}
\]
where the right vertical arrow is defined as the embedding of \([1_G] \ast [1_G]\) into \(\text{Gr}_G \times \text{Gr}_G\) factors through \(\text{Gr}_G^2\). This commutativity is clear from (2.15).

In the setting of the previous subsection, the upper row of (2.19) is the same as the multiplication on \(H^*_G(\mathcal{R})\) by Lemma 2.9, hence the lower row is also the same as \(\ast\) on \(H^*_G(\mathcal{R})\). In this sense the multiplication in (2.16) is a generalization of \(\ast\).

Thus \(\bigoplus H^*(i^q_0 A^\text{for})\) is an algebra graded by the coweight lattice of \(G\). For \(\lambda = 0\), we have a subalgebra \(H^*(\tilde{\iota}_0^! A^\text{for})\), which is isomorphic to \(H^*_G(\mathcal{R})\) in the setting of the previous subsection. One can also take a direct sum over dominant coweights \(\lambda\) of \(G\).

For a fixed coweight \(\lambda\), we consider the direct sum of \(H^*(i_{n\lambda}^! A^\text{for})\) with degrees \(n\lambda\) (\(n \in \mathbb{Z}_{\geq 0}\)). It is an algebra graded by \(\mathbb{Z}_{\geq 0}\). Its \(\text{Proj}(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! A^\text{for}))\) has a natural projective morphism to \(\text{Spec}(H^*(i_{\lambda}^! A^\text{for}))\). We have a natural line bundle \(O(1)\) on \(\text{Proj}(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! A^\text{for}))\) such that \(H^*(i_{n\lambda}^! A^\text{for})\) is identified with the space of sections of \(O(n) = O(1)^{\otimes n}\). Under some circumstances we expect \(\text{Proj}(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! A^\text{for}))\) is a (partial) resolution of \(\text{Spec}(H^*(i_{\lambda}^! A^\text{for}))\).

In the example in Remark 2.4, \(\mathcal{A}_R\) gives the Springer resolution of the nilpotent cone \(\mathcal{N}\) of \(G^\vee\), the Langlands dual group of \(G\). See [ABG04, 8.5.2].

See [Nak16, §5.1] (and also Remark II.3.26) for a physical origin of this construction.

**Remark 2.20.** In view of Remark 2.17(2), the construction in §II.3(ix) and the above construction is the same for \(\mathcal{A}\) in §II(iv). Here the construction in §II.3(ix) is as follows: Let us suppose \(G \leftarrow \tilde{G}\) as in §II(iv) and further assume \(G_F = \tilde{G}/G\) is a torus. Let us write \(T_F = G_F\). The Coulomb branch \(\mathcal{M}_{\mathcal{C}}(\tilde{G}, \mathcal{N})\) for the larger group \(\tilde{G}\) has an action of \(\pi_1(T_F)^\vee = T_F^\vee\), and Proposition II.3.18 says that \(\mathcal{M}(G, \mathcal{N})\) is the Hamiltonian reduction of \(\mathcal{M}(\tilde{G}, \mathcal{N})\) by \(T_F^\vee\). Let us denote the moment map by \(\mu_T\). The hamiltonian reduction more precisely means the affine algebro-geometric quotient \(\mu_{T_F^\vee}^{-1}(0)/T_F^\vee\). If we have a cocharacter \(\lambda_F\) of \(T_F\), we view it as a character of \(T_F^\vee\) and consider the GIT quotient \(\mu_{T_F^\vee}^{-1}(0)/\lambda_F T_F^\vee\).

2(vii). **Wakimoto sheaves.** The original definition of the multiplication (2.16) in [ABG04] was given by Wakimoto sheaves, and the above definition is taken from the proof of [ABG04, Th. 8.5.2]. Although it is unnecessary, let us review the construction for the sake of the reader.

Let \(I\) be the Iwahori subgroup of \(G_K\) and let \(\text{Fl}_G = G_K/I\) be the affine flag variety. We have a smooth proper morphism \(\varpi: \text{Fl}_G \to \text{Gr}_G\) of ind-schemes. Let \(\mathcal{W}_\lambda\) be the Wakimoto sheaf on \(\text{Fl}_G\) for \(G\) corresponding to a coweight \(\lambda\). See [ABG04, §8] for the definition (due to Mirković). By [ABG04, §8.4], we have a 'multiplication'

\[
(2.21) \quad \mathbf{E}_\lambda \boxtimes \mathbf{E}_\mu \to \mathbf{E}_{\lambda + \mu}, \quad \mathbf{E}_\lambda = \text{Ext}^*_\text{Fl}(\mathcal{D}(\text{Gr}_G))(1_{\text{Gr}_G}, \mathcal{W}_\lambda \ast \mathcal{A}),
\]

where \(\ast\) is the convolution product on \(I\)-equivariant complexes on \(\text{Fl}_G\) and \(\text{Gr}_G\). Let \(x \in \mathbf{E}_\lambda, y \in \mathbf{E}_\mu\). We consider the composite

\[
y \cdot x: 1_{\text{Gr}_G} \xrightarrow{\varpi} \mathcal{W}_\mu \ast \mathcal{A} = \mathcal{W}_\mu \ast 1_{\text{Gr}_G} \ast \mathcal{A} \xrightarrow{\mathcal{W}_\mu \ast \ast} \mathcal{W}_\mu \ast \mathcal{W}_\lambda \ast \mathcal{A} \ast \mathcal{A} = \mathcal{W}_{\lambda + \mu} \ast \mathcal{A} \ast \mathcal{A} \xrightarrow{m} \mathcal{W}_{\lambda + \mu} \ast \mathcal{A}.
\]
Note that $\mathcal{W}_\mu \star x$ is well-defined as $x$ is an $I$-equivariant homomorphism, and hence $\mathcal{W}_\mu \boxtimes x$ descends for the morphism $q$.

We have an isomorphism $\mathbb{E}_\lambda \cong H^*_G (i_\lambda^* \operatorname{Res}_{T_G, G} A)$ (see [ABG04, (8.7.2)]), and the above multiplication is the same as (2.18).

2(viii). **Gluing construction.** One of motivations of [CHMZ14a] extending the monopole formula from the Hilbert series of the coordinate ring of the Coulomb branch $\mathcal{M}_G$ to the character of the space of sections of a line bundle (see Remark II.3.26) is to write down the Hilbert series of a complicated Coulomb branch from simpler ones. We use the machinery prepared in earlier subsections to introduce the corresponding construction at the level of commutative ring objects in $D_G(\text{Gr}_G)$.

The setting in [CHMZ14a] is as follows. Suppose that we have a finite collection $\{(G_i, N_i)\}$ ($i = 1, 2, \ldots$) of gauge theories sharing the common flavor symmetry group, i.e., $N_i$ is a representation of a larger group $\tilde{G}_i$ containing $G_i$ as a normal subgroup with $G_F = \tilde{G}_i / G_i$, independent of $i$. Then we define $G$ as the fiber product of $\tilde{G}_i$ over $G_F$, and $N = \bigoplus N_i$. The monopole formula for the Hilbert series of the Coulomb branch of $(G, N)$ is given by extended monopole formula for $(G_i, N_i)$. See also [Nak16, §5(i)] for a review.

An example is a star shaped quiver gauge theory, which is the 3d mirror of the Sicilian theory of type $A_{N-1}$, reviewed in [Nak16, §3(iii)]. See Figure 1. We have three copies of type $A_{N-1}$ quiver gauge theory with $\dim V = (N-1, N-2, \ldots, 1)$, $\dim W = (N, 0, \ldots, 0)$ as in §2(v). We divide the group $\text{GL}(V) = \prod \text{GL}(V_i)$ by the diagonal central subgroup $Z$ and take it as the gauge group. The common flavor symmetry group is $G_F = \text{PGL}(N)$.

![Figure 1. A star shaped quiver gauge theory](image)

The variety $\mathcal{R}_{G, N}$ is the fiber product of $\mathcal{R}_{\tilde{G}_i, N_i}$ over $\text{Gr}_G$. Let us denote the natural projections $\mathcal{R}_{G, N} \to \text{Gr}_G$ and $\mathcal{R}_{\tilde{G}_i, N_i} \to \text{Gr}_G$ by $\pi$ and $\pi_i$ respectively. Then

$$\pi_* \omega_{\mathcal{R}_{G, N}}[-2 \dim N_O] = i_\Delta^* \left( \boxtimes \pi_i^* \omega_{\mathcal{R}_{\tilde{G}_i, N_i}}[-2 \dim (N_i)_O] \right),$$

where $i_\Delta : \text{Gr}_G \to \prod_\Delta \text{Gr}_G$ is the diagonal embedding. Note that $\pi_i^* \omega_{\mathcal{R}_{\tilde{G}_i, N_i}}[-2 \dim (N_i)_O]$ is the commutative ring object in $D_{G_F}(\text{Gr}_G)$, considered in §2(iv).
Motivated by the above example, we consider the following setting. (We use the convention in §2(vi), i.e., replace $G_P$ by $G$.) Suppose that we have a finite collection $\{A_i\}$ of commutative ring objects in $D_G(\text{Gr}_G)$. Let $i_\Delta: \text{Gr}_G \to \prod_i \text{Gr}_G$ be the diagonal embedding. Then the following is clear:

**Proposition 2.22.** $A \overset{\text{def}}{=} i_\Delta^! (\boxtimes A_i)$ is a commutative ring object in $D_G(\text{Gr}_G)$. In particular, we can consider the affine scheme $\text{Spec} H_G^*(\text{Gr}_G, A)$.

In fact, we have $\boxtimes m: (\boxtimes A_i) \otimes (\boxtimes A_i) = \boxtimes (A_i \otimes A_i) \to \boxtimes A_i$ from $m: A_i \otimes A_i \to A_i$. Then we apply $i_\Delta^!$. We claim that there is a natural homomorphism

$$i_\Delta^!(\boxtimes A_i) \otimes i_\Delta^!(\boxtimes A_i) \to i_\Delta^!(\boxtimes (A_i \otimes A_i)),$$

hence its composition with $i_\Delta^!(\boxtimes m)$ gives the desired multiplication homomorphism of $i_\Delta^!(\boxtimes A_i)$. We prove the claim by comparing the convolution diagrams (II.3.1) for $\text{Gr}_G$ and $\prod_i \text{Gr}_G$. Since $p, q$ are smooth, $p^*, q^*$ commute with $i_\Delta^!$. The last part of the convolution diagram for $G$ and $\prod_i G$ is

$$\begin{array}{ccc}
\text{Gr}_G \times \text{Gr}_G & \overset{m}{\longrightarrow} & \text{Gr}_G \\
i_\Delta \downarrow & & \downarrow i_\Delta \\
\prod_i \text{Gr}_G \times \text{Gr}_G = \text{Gr}_{\prod_i G} \times \text{Gr}_{\prod_i G} & \overset{\prod_i m}{\longrightarrow} & \text{Gr}_{\prod_i G} = \prod_i \text{Gr}_G,
\end{array}$$

where we denote the diagonal embedding of the left column by $i_\Delta$ to distinguish it from the right column. Let $\boxtimes (A_i \boxtimes A_i)$ denote the complex on $\text{Gr}_{\prod_i G} \times \text{Gr}_{\prod_i G}$ obtained in the course of the convolution product for $\prod_i G$. We define the homomorphism as

$$m_*i_\Delta^!(\boxtimes (A_i \boxtimes A_i)) = m_* \boxtimes (A_i \boxtimes A_i) \to \boxtimes m_*(A_i \boxtimes A_i) = i_\Delta^!(\prod_i m)_* (\boxtimes A_i \boxtimes A_i))$$

by the natural homomorphism [KS90, (2.6.24) or the dual of (2.6.22)].

See §5 for an application of the gluing construction.

3. **Proof of commutativity**

We denote $\text{Gr}_G$ by Gr for brevity in this section. In this section we closely follow [MV07, §5], [Gai01] and [BeiDr, §5.3].

3(i). **Commutativity constraint.** Let us give a definition of the commutativity constraint $\Theta$.

Let us choose a smooth curve $X$. We define $\text{Gr}_X$ the moduli space of triples $(x, P, \varphi)$ of a point $x \in X$, a $G$-bundle $P$ on $X$ and its trivialization $\varphi$ over $X \setminus \{x\}$. We also have a group scheme $G_{X, \Theta}$, the global analog of $G_\Theta$.

More generally, we introduce an ind-scheme $\text{Gr}_{X^n}$ as the moduli space of $(x_1, \ldots, x_n, P, \varphi)$ of $n$ ordered points in $X$, a $G$-bundle $P$ on $X$ and its trivialization $\varphi$ over $X \setminus \{x_i\}$. We also have $G_{X^n, \Theta}$, which is the moduli space of $(x_1, \ldots, x_n, P, \kappa_{x_1, \ldots, x_n})$ where $(x_1, \ldots, x_n) \in X^n$, $P$ the trivial $G$-bundle on $X$, and $\kappa_{x_1, \ldots, x_n}$ is a trivialization of $P$ on $X_{x_1, \ldots, x_n}$. 
Then we define the convolution product of \( A, B \in D_{G_{X,O}}(Gr_X) \) as before, using the global version of the diagram (II.3.1):

\[
(3.1) \quad Gr_X \times Gr_X \xleftarrow{p_X} Gr_X \times Gr_X \xrightarrow{q_X} Gr_X \times Gr_X \xrightarrow{m_X} Gr_X^2.
\]

Here \( Gr_X \times Gr_X \) is the moduli space of \((x_1, x_2, \mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2)\), a pair of points \((x_1, x_2) \in X^2\), two \( G \)-bundles \( \mathcal{P}_1, \mathcal{P}_2 \) and their trivializations \( \varphi_i \) over \( X \setminus \{x_i\} \) together with a trivialization \( \kappa \) of \( \mathcal{P}_1 \) on the formal neighborhood of \( x_2 \). The twisted product \( Gr_X \times Gr_X \) is the moduli space of \((x_1, x_2, \mathcal{P}_1, \varphi_1, \mathcal{P}, \eta)\) as above, but \( \eta: \mathcal{P}_1|_{X \setminus x_2} \cong \mathcal{P}|_{X \setminus x_2} \) instead of \( \varphi_2 \) and \( \kappa \). The morphism \( q_X \) is given by defining \( \mathcal{P} \) as the gluing of \( \mathcal{P}_1|_{X \setminus x_2} \) and \( \mathcal{P}_2|_{X \setminus x_2} \) by \( \varphi_2^{-1} \circ \kappa \) over \((X \setminus x_2) \cap \hat{x}_{x_2} = \hat{x}_{x_2} \setminus x_2 \). (When \( X = D \), the formal disk, \( \mathcal{P} \) and \( \mathcal{P}_2 \) are isomorphic. Hence this construction was omitted before.) The definitions of morphisms \( p_X, m_X \) are as before, and are omitted. (See [MV07, \$5].) Note that \( p_X \) is a \( G_{X,O} \)-torsor by the action changing \( \kappa \). The second projection \( q_X \) is also a \( G_{X,O} \)-torsor by the action changing \( \kappa \) and \( \varphi_2 \) simultaneously.

The diagram (3.1) gives a \( G_{X,O} \)-equivariant object defined on \( Gr_X \) by \( A_X \star_X B_X \) def. \( m_X(q_X^*)^{-1}p_X^*(A_X \boxtimes B_X) \) for \( A_X, B_X \in D_{G_{X,O}}(Gr_X) \).

We take \( X = \mathbb{A}^1 \). We have \( Gr_X \cong X \times Gr \) thanks to a choice of a global coordinate on \( \mathbb{A}^1 \). In particular, we have a projection \( \tau: Gr_X \to Gr \). For an object \( A \in D_G(Gr) \), we can attach \( A_X \in D_{G_{X,O}}(Gr_X) \) by \( \tau^*A[1] \). In fact, we can do more generally if we use the \( \text{Aut}(O) \)-bundle over \( X \) parametrizing all choices of local coordinates and consider \( \text{Aut}(O) \)-equivariant objects as in [BeiDr, Gai01].

Let \( \Delta \) denote the diagonal in \( X^2 \) and \( U \) denote the complement \( X^2 \setminus \Delta \). The restrictions of \( Gr_{X_2} \) to \( \Delta \) and \( U \) are isomorphic to \( Gr_X \) and \( (Gr_X \times Gr_X)|_U \) respectively. In fact, the restriction to \( \Delta \) is obvious. For a given \((x_1, x_2, \mathcal{P}, \varphi)\) with \( x_1 \neq x_2 \), we define \( \mathcal{P}_i \) by gluing \( \mathcal{P}_i|_{X \setminus x_i} = (X \setminus x_i) \times G \) and \( \mathcal{P}_i|_{X \setminus x_{3-i}} = \mathcal{P}|_{X \setminus x_{3-i}} \) by \( \varphi \) on \( X \setminus \{x_1, x_2\} \). Hence we have the diagram

\[
\begin{array}{ccc}
Gr_X & \xrightarrow{i} & Gr_{X_2} \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & X^2 \\
\end{array}
\]

(3.2)

We consider the nearby cycle functor

\[
\psi_{Gr_X}: D_{(G_{X,O} \times G_{X,O})|_U}((Gr_X \times Gr_X)|_U) \to D_{G_{X,O}}(Gr_X).
\]

See [KS90, \$8.6], where we change the source domain to objects defined on \( (Gr_X \times Gr_X)|_U \), and shift by \(-1\), following the convention in [Gai01].

Then an argument in [Gai01, Proposition 6] shows there is a natural isomorphism

\[
\psi_{Gr_X}((A_X \boxtimes B_X)|_U) \cong (A \star B)_X.
\]

(3.3)

We have the isomorphism \((A_X \boxtimes B_X)|_U \cong (B_X \boxtimes A_X)|_U\) exchanging the factors. Therefore together with (3.3) it gives us an isomorphism \( A \star B \cong B \star A \). This is the definition of the commutativity constraint \( \Theta \) used in Theorem 2.5.
Let us briefly explain how (3.3) is constructed. For a later purpose, we give a slightly different explanation from [Gai01].

By the definition of the nearby cycle functor, we have a natural homomorphism
\[
(3.4) \quad \psi_{\text{Gr}_X}((A_X \ast_X B_X)|_U) \to \iota^!(A_X \ast_X B_X)[1].
\]
It is the dual of the specialization homomorphism. See [KS90, (8.6.7)]. We restrict the diagram (3.1) to the diagonal to see that
\[
(3.5) \quad (A \ast B)_X \cong \iota^!(A_X \ast_X B_X)[1].
\]
Therefore we need to check
\[
\text{Claim.}
\]
\[
(3.6a) \quad \text{We have a natural isomorphism } (A_X \ast_X B_X)|_U \cong (A_X \boxtimes B_X)|_U.
\]
\[
(3.6b) \quad \text{ps in (3.4) is an isomorphism.}
\]

\textbf{Proof.} Let us denote the restrictions of \(p_X, q_X, m_X\) to inverse images of \(U\) by \(p_U, q_U, m_U\) respectively.

Over \(U\), we have a natural commutative diagram
\[
(3.7) \quad \begin{array}{ccc}
(\text{Gr}_X \times \text{Gr}_X)|_U & \xleftarrow{p_U} & (\text{Gr}_X \times \text{Gr}_X)|_U \\
\downarrow \cong & & \downarrow \cong \\
(\text{Gr}_X \times \text{Gr}_X)|_U & \xrightarrow{q_U} & (\text{Gr}_X \times \text{Gr}_X)|_U \\
\downarrow \cong & & \downarrow \cong \\
(\text{Gr}_X \times \text{Gr}_X)|_U & \xrightarrow{m_U} & (\text{Gr}_X \times \text{Gr}_X)|_U
\end{array}
\]
where \(\text{Gr}_X \times \text{Gr}_X \to X\) in the bottom middle term is through the projection \(X \times X \to X\) to the second factor. Here the second vertical isomorphism is given by regarding \(\kappa\) as a trivialization of the trivial bundle over \(\check{X}_x\) via the trivialization \(\varphi_1: \mathcal{P}_1|_{\check{X}_x} \overset{\cong}{\to} \check{X}_x \times G\). The third vertical isomorphism is given by considering \(\eta\) as a trivialization of \(\mathcal{P}\). The lower left arrow is given by forgetting \(G_{X,0}\). The lower right arrow is given by the action of \(G_{X,0}\) on the second factor of \(\text{Gr}_X \times \text{Gr}_X\). Since we are considering equivariant objects, we have a canonical isomorphism \((q_U^*)^{-1}p_U^!((A_X \boxtimes B_X)|_U) \cong (A_X \boxtimes B_X)|_U\). We now apply \(m_{U*}\) and observe that \(m_{U*}(q_U^*)^{-1}p_U^!((A_X \boxtimes B_X)|_U) = (A_X \ast_X B_X)|_U\). Thus we have checked (a).

Let us turn to the assertion (b). The idea is to consider nearby cycle functors for four spaces in (3.1).

Let us start with \(m_X\). Since nearby cycle functors commute with proper morphisms, we have
\[
\psi_{\text{Gr}_X}(m_{U*}(A_X \boxtimes B_X)|_U) \cong m_{\Delta*}\psi_{\text{Gr}_X \times \text{Gr}_X}((A_X \boxtimes B_X)|_U)
\]
where \(m_{\Delta}\) is the restriction of \(m\) to \(\Delta\).

Next consider \(p_X\) and \(q_X\). They are both smooth ([MV07, p.114]), and hence commute with nearby cycle functors. Therefore
\[
\psi_{\text{Gr}_X \times \text{Gr}_X}((A_X \boxtimes B_X)|_U) \cong (q_X^*)^{-1}\psi_{\text{Gr}_X}(p_U^*(A_X \boxtimes B_X)|_U) \cong (q_X^*)^{-1}p_{\Delta*}\psi_{\text{Gr}_X}(u(A_X \boxtimes B_X)|_U),
\]
where $p_\Delta, q_\Delta$ are restrictions of $p_X, q_X$ to $\Delta$. Hence

$$\psi_{\text{Gr}_X^2}((A_X \ast X B_X)|_U) \cong m_{\Delta*}(q_\Delta^*)^{-1}p_\Delta^*\psi_{\text{Gr}_X}((A_X \boxtimes B_X)|_U).$$

Now $\text{Gr}_X \times \text{Gr}_X = X \times X \times \text{Gr} \times \text{Gr}$, hence $\psi_{\text{Gr}_X \times \text{Gr}_X}((A_X \boxtimes B_X)|_U)$ is just $(A \boxtimes B)_X$. More precisely, the isomorphism is given by the dual specialization homomorphism

$$\psi_{\text{Gr}_X \times \text{Gr}_X}((A_X \boxtimes B_X)|_U) \xrightarrow{\cong} \iota^!(A_X \boxtimes B_X)[1] = (A \boxtimes B)_X,$$

thanks to vanishing of the vanishing cycle functor $\varphi_{\text{Gr}_X \times \text{Gr}_X}(A_X \boxtimes B_X)$. Thus

$$\psi_{\text{Gr}_X^2}((A_X \ast X B_X)|_U) \xrightarrow{m_{\Delta*}(q_\Delta^*)^{-1}p_\Delta^*\psi_{\text{Gr}_X^2}} m_{\Delta*}(q_\Delta^*)^{-1}p_\Delta^*\psi_{(A \boxtimes B)_X}.$$

Notice that the restriction of (3.1) to the diagonal is just the product of $X$ and the diagram (II.3.1). Therefore the right hand side is $(A \ast B)_X$. Now one can check that dual specialization homomorphisms commute with proper pushforward and smooth pull-backs so that they are compatible with the commutation of nearby cycle functors. (See the argument in the proof of Lemma 3.10 below.) Therefore $m_{\Delta*}(q_\Delta^*)^{-1}p_\Delta^*\psi_{\text{Gr}_X^2}$ is equal to $\psi_{\text{Gr}_X^2}$ over $\text{Gr}_X^2$. Thus (b) is checked. \hfill $\Box$

3(ii). **Factorization version of $R$**. We define a global version of the variety of triples $\mathcal{R}$ in this subsection.

Let us assume that we are given a smooth connected curve $X$, an algebraic group $G$ and a representation $N$ of $G$ and a finite set $I$. Consider a functor Schemes$/\mathbb{C} \to \text{Sets}$ which sends a scheme $S$ to the following data: 1) A map $f: S \to X^I$. We shall think about $f$ as a collection of maps $f_i: S \to X$ for $i \in I$ and we denote by $\Gamma$ the union of graphs of $f_i$ — this is a closed subscheme of $S \times X$.

2) A $G$-bundle $\mathcal{P}$ on $S \times X$.

3) A trivialization $\varphi$ of $\mathcal{P}$ over $S \times X \setminus \Gamma$.

4) A section $s$ of the associated bundle $\mathcal{P}_N$ over the formal neighbourhood of $\Gamma$ in $S \times X$ and a section $s'$ of the trivial $N$-bundle over the same formal neighbourhood which are equal on the “formal punctured neighbourhood” (this makes sense because of 3). These notions (formal neighbourhood, formal punctured neighbourhood) are explained in [KV04].

Now we claim that this functor is representable by an ind-scheme. Moreover, this ind-scheme has a natural closed embedding into $\text{Gr}_{X^I} \times_{\mathcal{J}_{N,X^I}} \mathcal{J}_{N,X^I}$ where

a) $\text{Gr}_{X^I}$ is the factorization (a.k.a. Beilinson-Drinfeld) Grassmannian over $X^I$

b) $\mathcal{J}_{N,X^I}$ is the Kapranov-Vasserot factorization version of the $N$-jet space over $X^I$.

Indeed it is enough to construct this closed embedding (as a closed subfunctor of an ind-scheme is also an ind-scheme). But an $S$-point of $\text{Gr}_{X^I,G,BD}$ is precisely the data of 1), 2), 3) and an $S$-point of $\mathcal{J}_{N,X^I}$ is the data of 1), 2) and $s'$ from 4). Since $s$ is obviously uniquely determined by all the data and since the existence of $s$ is a closed condition on the other data we get the above closed embedding.

Let us denote the above ind-scheme by $\mathcal{R}_{X^I}$. Then obviously from 1) we get a morphism $\pi_{X^I}: \mathcal{R}_{X^I} \to X^I$ and it is clear that the restriction of $\mathcal{R}_{X^I}$ to the complement $U$ of all the diagonals in $X^I$ is naturally isomorphic to the similar restriction of $(\mathcal{R}^{(1)})^I$. 

On the other hand, assume that we are given a surjective morphism $I \to J$ of finite sets. Such a morphism defines a closed embedding $X^J \hookrightarrow X^I$ (as a partial diagonal) and it follows that the restriction of $\mathcal{R}_{X^I}$ to $X^J$ is naturally isomorphic to $\mathcal{R}^{(J)}$.

Similarly, we can define a factorization version of the bundle $\mathcal{T}$ over $\text{Gr}$. By definition an $S$-point of $\mathcal{T}_{X^I}$ is a quadruple $(f, \mathcal{P}, \varphi, s)$ as above (i.e. no $s'$).

We claim again that this functor is representable by an ind-scheme. For this it is enough to show that the morphism $\mathcal{T}_{X^I} \to \text{Gr}_{X^I}$ (which corresponds to forgetting $s$) is representable. This can be done by a word-by-word repetition of the proof of the fact that the factorization version of the jet scheme is representable by a scheme (cf. again Section 3 of [KV04]).

In what follows we shall only need the above spaces when $I = \{1, 2\}$.

3(iii). **Definition of another multiplication.** We consider the space $\mathcal{R}_{X^2}$, its dualizing complex $\omega_{\mathcal{R}_{X^2}}$ and the pushforward $\pi_*\omega_{\mathcal{R}_{X^2}}$. Its restriction to $U$ is isomorphic to $(\pi_*\omega_{\mathcal{R}_X} \boxtimes \pi_*\omega_{\mathcal{R}_X})|_U$ under $j$ in (3.2). We consider two dual specialization homomorphisms

$$
\psi_{\text{Gr}_X^2}(\pi_*\omega_{\mathcal{R}_X^2}|_U) \xrightarrow{\text{ps}} \iota! \pi_*\omega_{\mathcal{R}_X^2}[1]
$$

where $\iota: \text{Gr}_X \to \text{Gr}_X^2$ is the inclusion, and the vertical arrow is given by $\pi_*\omega_{\mathcal{R}_X^2}|_U \cong (\pi_*\omega_{\mathcal{R}_X} \boxtimes \pi_*\omega_{\mathcal{R}_X})|_U \cong (\pi_*\omega_{\mathcal{R}_X} \boxtimes \pi_*\omega_{\mathcal{R}_X})|_U$. (See (3.6a).) The lower homomorphism is an isomorphism thanks to (3.6b). Note that $\pi_*\omega_{\mathcal{R}_X}$ is $(\pi_*\omega_{\mathcal{R}})_X[1]$. Therefore the right bottom term is $(\pi_*\omega_{\mathcal{R}} \boxtimes \pi_*\omega_{\mathcal{R}})_X[2]$ by (3.5). Note also $\iota! \pi_*\omega_{\mathcal{R}_X^2}[1] = \pi_*\omega_{\mathcal{R}_X}[1] = (\pi_*\omega_{\mathcal{R}})_X[2]$. Therefore we obtain a homomorphism

$$
m^\psi: \mathcal{A} \star \mathcal{A} \to \mathcal{A}, \quad \mathcal{A} = \pi_*\omega_{\mathcal{R}}[-2 \dim N_O],
$$

by specializing the dotted arrow at a point in $X$.

The degree shift should be checked by going back to finite dimensional approximation of $\mathcal{R}$. We have shifts by $\dim N_O/z_{d_1} N_O$ and $\dim N_O/z_{d_2} N_O$ for two factors in $\psi_{\text{Gr}_X^2}((\pi_*\omega_{\mathcal{R}_X} \boxtimes \pi_*\omega_{\mathcal{R}_X})|_U)$. Then we have a shift $\dim N_O/z_{d_1+d_2} N_O$ for $\pi_*\omega_{\mathcal{R}_X}$.

Now our goal is to check two properties:

(i) $m^\psi = m$,

(ii) $m^\psi$ is invariant under the exchange of factors of $\mathcal{A}_X \star \mathcal{A}_X$. (More precisely, exchange after going back to $(\mathcal{A} \boxtimes \mathcal{A})|_U$.)

The property (ii) is clear as the diagram (3.2) is invariant under the exchange of two factors of $X^2 = X \times X$.

We will check (i) for $N = 0$ in the next subsection. We have a difficulty to check (i) directly for general $N$, so we will argue indirectly by reduction to the case $N = 0$.

---

2Note that if we instead only choose $s'$ and do not choose $s$ then the resulting functor is represented by $\text{Gr}_{X^I} \times J_{N,X^I}$.
3(iv). The case $N = 0$. We first consider the case $N = 0$.

We consider the dual specialization homomorphism for $\omega_{\tilde{Gr}_X \tilde{Gr}_X}$:

$$\psi_{\tilde{Gr}_X \tilde{Gr}_X}(\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) \to i'_*\omega_{\tilde{Gr}_X \tilde{Gr}_X}[1] = \omega_{\tilde{Gr}_X \tilde{Gr}_X}[1]$$

where $\iota: X \times \tilde{Gr}_X = (\tilde{Gr}_X \times \tilde{Gr}_X)|_\Delta \to \tilde{Gr}_X \times \tilde{Gr}_X$ is the embedding from the definition of the nearby cycle functor. Here we have used $\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U \cong (\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U)$ from (3.7).

The following two assertions identify (3.9) with the pull-back of $m$ under (3.3), hence we obtain the property (i) for $N = 0$.

**Lemma 3.10.** (1) $\psi$ is equal to the composition of $m_{\Delta*} \psi$, and the natural morphism $m_{\Delta*} m_{\Delta!} = m_{\Delta!} m_{\Delta*} \to \id$.

(2) The homomorphism $\psi$, coincides with the homomorphism $\omega_{\tilde{Gr}_X \tilde{Gr}_X} \to \omega_{\tilde{Gr}_X \tilde{Gr}_X}$ constructed in (2.3), pull-backed by $\tilde{Gr}_X \times \tilde{Gr}_X \to \tilde{Gr}_X \times \tilde{Gr}_X$. It is an isomorphism.

**Proof.** (1) Recall the definition of the nearby cycle functor and the dual specialization morphism ([KS90, §8.6]). We have $f: \tilde{Gr}_X \to X^2 = \mathbb{C}^2 \to \mathbb{C}$, where the second map is $(x_1, x_2) \mapsto x_1 - x_2$. We then consider $p: \mathbb{C}^\times \to \mathbb{C}$, the composition of the universal covering $\mathbb{C}^\times \to \mathbb{C}^\times$ and the inclusion $\mathbb{C}^\times \to \mathbb{C}$. We then pull back $p$ by $f$ to get $\tilde{p}: \tilde{Gr}_X \to \tilde{Gr}_X$. Then

$$\psi_{\tilde{Gr}_X}(\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) = \tilde{p}_* \tilde{p}^* \omega_{\tilde{Gr}_X \tilde{Gr}_X}[1] \cong \tilde{p}_* \Hom(f^* p|_{\mathbb{C}^\times}, \omega_{\tilde{Gr}_X \tilde{Gr}_X}[1])$$

and $\psi$ is defined from $\mathbb{C}[0] \to p|_{\mathbb{C}^\times}[2]$.

Let us write the identification $(\tilde{Gr}_X \times \tilde{Gr}_X)|_U \cong \tilde{Gr}_X \times \tilde{Gr}_X|_U \cong (\tilde{Gr}_X \times \tilde{Gr}_X)|_U$ explicitly as $m_U$, the restriction of $m_X$ to $U$. The commutativity of

$$\begin{array}{ccc}
\psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) & \xrightarrow{\psi} & \psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) \\
\cong & & \cong \\
m_{\Delta*} \psi_{\tilde{Gr}_X \tilde{Gr}_X}(\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) & \xrightarrow{m_{\Delta*} \psi} & m_{\Delta*} \psi_{\tilde{Gr}_X \tilde{Gr}_X}(\omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U)
\end{array}$$

is clear as both vertical arrows are given by base change and adjunction. This property has been already used in the construction of the commutativity constraint above. Next the commutativity of

$$\begin{array}{ccc}
\psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) & \xrightarrow{\psi} & \psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) \\
\psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U) & \xrightarrow{\psi} & \psi_{\tilde{Gr}_X}(m_{U*} \omega_{\tilde{Gr}_X \tilde{Gr}_X}|_U)
\end{array}$$

is also clear from the definition. Therefore we get the assertion.
(2) The following diagram is commutative:

\[
q_\Delta^* \psi_{\Gr_X \times \Gr_X} \left( \omega_{(\Gr_X \times \Gr_X)|u} \right) \xrightarrow{q_\Delta^* \iota^*_\mu} q_\Delta^* \iota/\omega_{\Gr_X \times \Gr_X}[1] \\
\cong \downarrow \cong
\psi_{\Gr_X \times \Gr_X} \left( q_\Delta^* \omega_{(\Gr_X \times \Gr_X)|u} \right) \xrightarrow{p_\Delta^* \iota^*_m} p_\Delta^* \iota^*_m/\omega_{\Gr_X \times \Gr_X}[1],
\]

where \( \iota^*_\mu: G_K \times \Gr_X \to \Gr_X \times \Gr_X \) is the inclusion given by \( \Gr_X \times \Gr_X|_\Delta \cong G_K \times \Gr_X \), and \( p_\mu \) is the dual specialization given for \( \omega_{\Gr_X \times \Gr_X} = q_\Delta^* \omega_{\Gr_X \times \Gr_X} \). In fact, the left vertical arrow is given by the composition of

\[
q_\Delta^* \psi_{\Gr_X \times \Gr_X} \left( \omega_{\Gr_X \times \Gr_X}|_u \right) \cong q_\Delta^* \iota^*_\mu \Hom (m_\times^* f^* p_1^1 \mathcal{C} \times, \omega_{\Gr_X \times \Gr_X}[-1]) \\
\cong \iota^*_\mu \Hom (m_\times^* f^* p_1^1 \mathcal{C} \times, \omega_{\Gr_X \times \Gr_X}[-1]) \\
\cong \iota^*_\mu \Hom (q_\Delta^* m_\times^* f^* p_1^1 \mathcal{C} \times, q_\Delta^* \omega_{\Gr_X \times \Gr_X}[-1]) \cong \psi_{\Gr_X \times \Gr_X} \left( q_\Delta^* \omega_{\Gr_X \times \Gr_X}|_u \right)
\]

where we have used [KS90, Prop. 3.1.13] and \( q_\Delta^* = q_\Delta^*[2 \dim G \mathcal{O}] \) for the third isomorphism. Now we apply \( \mathbb{C}(0) \to p_1^1 \mathcal{C} \times [2] \).

In the same way, we have another commutative diagram

\[
\psi_{\Gr_X \times \Gr_X} \left( p_\Delta^* \omega_{(\Gr_X \times \Gr_X)|u} \right) \xrightarrow{p_\Delta^* \iota^*_m} p_\Delta^* \iota^*_m/\omega_{\Gr_X \times \Gr_X}[1] \\
\cong \downarrow \cong
p_\Delta^* \psi_{\Gr_X \times \Gr_X} \left( \omega_{(\Gr_X \times \Gr_X)|u} \right) \xrightarrow{p_\Delta^* \iota^*_m} p_\Delta^* \iota^*_m/\omega_{\Gr_X \times \Gr_X}[1],
\]

where \( \iota^*_m: X \times \Gr \to \Gr_X \times \Gr_X \) is the inclusion given by \( \Gr_X \times \Gr_X|_\Delta \cong X \times \Gr \times \Gr \). Now the assertion is proved. Note \( p_\Delta^* \iota^*_m \) is an isomorphism, hence so is \( p_\mu \).

Remark 3.11. We have a difficulty to generalize the argument in §3(iv) to \( N \neq 0 \) since we lack an \( \mathcal{R} \)-version of \( \Gr_X \times \Gr_X \), as a well-defined ind-scheme. This difficulty will be overcome in §B written by Gus Lonergan.

3(v). Completion of the proof. Let \( z_{X^2}: \Gr_{X^2} \to \mathcal{T}_{X^2} \) be the factorization version of the embedding \( z: \Gr \to \mathcal{T} \) discussed in §II.5(iv). It factors as \( z_{X^2} = i \circ \tilde{z}_{X^2} \), where \( \tilde{z}_{X^2}: \Gr_{X^2} \to \mathcal{R}_{X^2} \), and \( i: \mathcal{R}_{X^2} \to \mathcal{T}_{X^2} \) is the embedding. Since \( \mathcal{T}_{X^2} \to \Gr_{X^2} \) is a vector bundle, we have \( z_{X^2}^* \omega_{\mathcal{T}_{X^2}} \to \omega_{\Gr_{X^2}}[2 \dim \mathcal{N} \mathcal{O}] \), and also \( \omega_{\mathcal{R}_{X^2}}[-2 \dim \mathcal{N} \mathcal{O}] \to \tilde{z}_{X^2}^* \omega_{\Gr_{X^2}} \) by the pull-back with support. We apply \( \pi_* \) to obtain

\[
z_{X^2}^* : \pi_* \omega_{\mathcal{R}_{X^2}}[-2 \dim \mathcal{N} \mathcal{O}] \to \omega_{\Gr_{X^2}}.
\]

We now apply the nearby cycle functor \( \psi_{\Gr_{X^2}} \) and the dual specialization homomorphisms:

\[
\psi_{\Gr_{X^2}} (\pi_* (\omega_{\mathcal{R}_{X^2}}[-2 \dim \mathcal{N} \mathcal{O}]|_U)) \xrightarrow{ps} \pi_* \omega_{\mathcal{R}_{X^2}}[-2 \dim \mathcal{N} \mathcal{O} + 1] \\
\cong \downarrow \cong
\psi_{\Gr_{X^2}} (\omega_{\Gr_{X^2}}|_U) \xrightarrow{ps} \omega_{\Gr_{X}}[1].
\]
This is a commutative diagram by the argument in the proof of Lemma 3.10. Removing the unnecessary factor \( X \), we get

\[
\begin{array}{c}
\mathcal{A} \ast \mathcal{A} \\
z^{\ast} \ast z^{\ast} = z_{X^2}^{\ast} \\
\omega_{Gr} \ast \omega_{Gr} \\
m_{Gr}^{\ast}
\end{array} \xrightarrow{m_{\mathcal{A}}^{\ast}} \begin{array}{c} \mathcal{A} \\
z^{\ast} \\
\omega_{Gr}
\end{array}
\]

(3.12)

where we put the superscript \( \psi \) to emphasize that the multiplication is defined via nearby cycle functors. Since the restriction of \( z_{X^2}^{\ast} \) to \( U \) is \( z^{\ast} \otimes z^{\ast} \), the left vertical arrow is equal to \( z^{\ast} \ast z^{\ast} \).

Let us view \( m, m^{\psi} \) as elements of \( \text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \mathcal{A}) \). It is a module over \( H_{T_{\mathcal{G}}}^{\ast}(pt) \). We consider the restriction functor \( \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \) from the \( G_{\mathcal{O}} \)-equivariant derived category to the \( T_{\mathcal{O}} \)-equivariant one. Then we have

\[
\text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \mathcal{A}) \rightarrow \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A} \ast \mathcal{A}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A}),
\]

and the latter is a module over \( H_{T_{\mathcal{O}}}^{\ast}(pt) = \mathbb{C}[t] \).

We have

(i) \( m^{\psi} \) and \( m \) are equal in \( \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A} \ast \mathcal{A}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A})) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \). Hence \( m \) is commutative up to an element which vanishes in \( \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A} \ast \mathcal{A}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A})) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \).

This statement recovers (i), as \( \text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \mathcal{A}) \) is the Weyl group invariant part of \( \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A} \ast \mathcal{A}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A})) \), and \( \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A} \ast \mathcal{A}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{A})) \) is a free \( \mathbb{C}[t] \)-module: More generally, for \( \mathcal{F}, \mathcal{G} \in D_{G}(\text{Gr}) \), \( \text{Ext}^{\ast}_{D_{T}(G)}(\text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{F}, \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \mathcal{G})) \) is a free \( \mathbb{C}[t] \)-module. Indeed, by devissage it reduces to the case of irreducible perverse \( \mathcal{F}, \mathcal{G} \) where it is well known, see e.g. [Gin91].

Let us suppress \( \text{Res}_{T_{\mathcal{O}},G_{\mathcal{O}}} \) hereafter.

Let us consider the commutative diagram (3.12). We have the corresponding diagram for \( m_{\mathcal{R}} \), the multiplication constructed in Proposition 2.1, where the lower arrow is \( m_{Gr}^{\ast} \), cf. Lemma II.5.11. We compose \( z^{\ast} \) to get

\[
\text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \mathcal{A}) \xrightarrow{z^{\ast}} \text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \omega_{Gr}).
\]

The commutativity of the diagram and \( m_{Gr}^{\ast} = m_{Gr}^{\psi} \) (§3(iv)) imply that \( z^{\ast} m_{\mathcal{R}}^{\psi} \) and \( z^{\ast} m_{\mathcal{R}} \) are equal in \( \text{Ext}^{\ast}_{D_{\mathcal{G}}(G)}(\mathcal{A} \ast \mathcal{A}, \omega_{Gr}) \). Therefore it is enough to check that

\[
\text{Ext}^{\ast}_{D_{T}(G)}(\mathcal{A} \ast \mathcal{A}, \mathcal{A}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \xrightarrow{z^{\ast}} \text{Ext}^{\ast}_{D_{T}(G)}(\mathcal{A} \ast \mathcal{A}, \omega_{Gr}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)
\]

is an isomorphism. The argument is almost same as one in the proof of Lemma II.5.13.

By the definition of \( z^{\ast} \), it factors through

\[
\text{Ext}^{\ast}_{D_{T}(G)}(\mathcal{A} \ast \mathcal{A}, \pi_{\ast} \omega_{T}[-2 \dim N_{\mathcal{O}}]) \otimes_{\mathbb{C}[t]} \mathbb{C}(t).
\]
Since $\mathcal{T} \to \text{Gr}$ is a vector bundle of rank $2 \dim \mathcal{N}$, we have $\pi_* \omega_\mathcal{T}[-2 \dim \mathcal{N}] \cong \omega_{\text{Gr}}$. Therefore it is enough to check that

$$\text{Ext}^*_\mathcal{T}(\mathcal{A} \star \mathcal{A}, \mathcal{A}) \otimes \mathbb{C}[t] C(t) \overset{i_*}{\to} \text{Ext}^*_\mathcal{T}(\mathcal{A} \star \mathcal{A}, \pi_* \omega_\mathcal{T}) \otimes \mathbb{C}[t] C(t)$$

given by the closed embedding $i:\mathcal{R} \to \mathcal{T}$ is an isomorphism. Let $j: \mathcal{T}\setminus \mathcal{R} \to \mathcal{T}$ be the inclusion of the complement. We have the distinguished triangle $ii' \omega_\mathcal{T} \to \omega_\mathcal{T} \to j_*j^* \omega_\mathcal{T}$. From the associated long exact sequence, it is enough to show that $\text{Ext}^*_\mathcal{T}(\mathcal{A} \star \mathcal{A}, \pi_* j_* j^* \omega_\mathcal{T}) \otimes \mathbb{C}[t] C(t)$ vanishes. But $\text{Ext}^*_\mathcal{T}(\mathcal{A} \star \mathcal{A}, \pi_* j_* j^* \omega_\mathcal{T}) = \text{Ext}^*_\mathcal{T}(\mathcal{T}\setminus \mathcal{R})(j^* \pi^*(\mathcal{A} \star \mathcal{A}), \omega_\mathcal{T}\setminus \mathcal{R})$ is an equivariant cohomology group over $\mathcal{T}\setminus \mathcal{R}$ which does not contain $T$-fixed points by Lemma II.5.1. Therefore it is torsion and vanishes once we take a tensor product with $\mathbb{C}(t)$.

4. Proof of Theorem 2.11

In this section we prove Theorem 2.11.\(^3\)

During the proof, the $\mathbb{C}^+$-action on the Coulomb branch will play an important role. The $\mathbb{C}^+$-action is given by the homological grading, shifted according to the convention in Remark II.2.8(2). Then the monopole formula in Proposition II.2.7 is modified to

$$(4.1) \quad P_{t}^{\text{mod}}(\mathcal{R}) = \sum_{\lambda} t^{2\Delta(\lambda)} P_{\mathcal{C}}(t; \lambda).$$

As mentioned in Remark II.2.8(2), this modification is harmless as the difference $d_{\lambda} - 2\langle \rho, \lambda \rangle - \Delta(\lambda)$ depends only on connected components of $\mathcal{R}$. Nevertheless we will see that this convention is a correct choice.

4(i). Characters of global sections of line bundles on Kleinian surfaces. Recall that $\mathcal{S}_N$ is the hypersurface in $\mathbb{A}^3$ given by the equation $zy = w^N$. It is the categorical quotient $\mathbb{A}^2//\mathbb{Z}/\mathbb{N}^\mathbb{Z}$ where $\zeta \in \mathbb{Z}/\mathbb{N}^\mathbb{Z}$ takes $(u, v) \in \mathbb{A}^2$ to $(\zeta u, \zeta^{-1}v)$. We consider the following action of $\mathbb{C}^+ \times \mathbb{C}^+$ on $\mathbb{A}^2$: $(x, t) \cdot (u, v) = (t^{-1}x^{-1}u, t^{-1}xv)$. This action descends to $\mathcal{S}_N$. The action of the second $\mathbb{C}^+$ $t: (u, v) = (t^{-1}u, t^{-1}v)$ is a restriction of an SU(2)-action on $\mathbb{A}^2 = \mathbb{R}^4$ rotating hyper-Kähler structures. Hence it is natural in view of Remark II.2.8.

We are interested in characters of certain $\mathbb{C}^+ \times \mathbb{C}^+$-equivariant sheaves on $\mathcal{S}_N$. The tautological characters of $\mathbb{C}^+ \times \mathbb{C}^+$ will be denoted by $x$ and $t$. We denote by $\pi: \tilde{\mathcal{S}}_N \to \mathcal{S}_N$ the minimal resolution of $\mathcal{S}_N$. The action of $\mathbb{C}^+ \times \mathbb{C}^+$ lifts to $\tilde{\mathcal{S}}_N$. We recall the well known facts about the $\mathbb{C}^+ \times \mathbb{C}^+$-fixed points in $\tilde{\mathcal{S}}_N$.

We will denote these points by $p_0, \ldots, p_{N-1}$, so that the exceptional divisor $E \subset \tilde{\mathcal{S}}_N$ consists of projective lines $E_1, \ldots, E_{N-1}$, and $E_r$ contains $p_{r-1}, p_r$. The character of the tangent space $T_{p_r} \tilde{\mathcal{S}}_N$ is $t^{N-2r-2}x^{-N} + t^{2r-N}x^N$. The Picard group $\text{Pic}(\tilde{\mathcal{S}}_N)$ is canonically identified with the weight lattice of $\text{SL}(N)$. Namely, $\tilde{\mathcal{S}}_N \subset T^* \mathcal{B}$ is the preimage of a subregular (Slodowy) slice in the Springer resolution $T^* \mathcal{B} \to \mathcal{N}$ of the nilpotent cone for $\text{SL}(N)$. For an $\text{SL}(N)$-weight $\lambda$ the corresponding line bundle $L_{\lambda}$ on $\tilde{\mathcal{S}}_N$ is the restriction

\(^3\)The second named author thanks Roman Bezrukavnikov for his numerous explanations about the Andersen-Jantzen sheaves on Kleinian surfaces and nilpotent cones.
to $\tilde{S}_N \subset T^*B$ of the pullback to $T^*B$ of the line bundle $O(\lambda)$ on the flag variety $B$. The line bundle $L_{\omega_i}$ corresponding to the fundamental weight $\omega_i$, $1 \leq i \leq N-1$, admits a natural $\mathbb{C}^\times \times \mathbb{C}^\times$-equivariant structure such that the character of its fiber at $p_r$ is $t^{N-i}x_i t^{-N}$ provided $0 \leq r \leq i-1$, and $t^i x^i$ provided $i \leq r \leq N-1$. This is defined so that $\Gamma(\tilde{S}_N, L_{\omega_i})$ is the space of semi-invariants $\mathbb{C}[\mathbb{A}^2]^\chi$ where $\chi_i(\zeta) = \zeta^i$. Under the above identification, $O_{\tilde{S}_N}(-E_r)$ is nothing but $L_{\alpha_r}$ where $\alpha_r$ is the $r$-th simple root of $SL(N)$.

We will write a dominant SL($N$)-weight $\lambda$ as a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ defined up to simultaneous shift of all $\lambda_i$. In other words, $\lambda = \sum_{i=1}^{N-1} (\lambda_i - \lambda_{i+1}) \omega_i$. Then $L_\lambda$ admits a natural $\mathbb{C}^\times \times \mathbb{C}^\times$-equivariant structure (as a tensor product of fundamental line bundles) such that the character of its fiber at $p_r$, $0 \leq r \leq N-1$, is $t^{\sum_{i=1}^N (\lambda_i - \lambda_{i+1})} \chi^N_{\sum_{i=1}^N (\lambda_i - \lambda_{i+1})}$. If $\lambda$ is not necessarily dominant, we get the character $t^{\sum_{i=1}^N (\lambda_i - \lambda_{i+1})} + \sum_{i=r+1}^N (\lambda_i - \lambda_{i+1}) \chi^N_{\sum_{i=1}^N (\lambda_i - \lambda_{i+1})}$.

**Lemma 4.2.** For dominant $\lambda$, the character of $\Gamma(\tilde{S}_N, L_\lambda)$ equals

$$\sum_{m \geq 2} x^{\sum_{i=1}^N (\lambda_i - m)} t^{\sum_{i=1}^N (\lambda_i - m)} (1 + t^2 + t^4 + \ldots).$$

**Proof.** We compute the above expression as

$$\frac{x^{\sum_{i=1}^N \lambda_i}}{1 - t^2} \left( \sum_{m=\lambda_1}^{\lambda_1-1} x^{-Nm} t^{-\sum_{i=1}^N (\lambda_i - m)} + \sum_{m=\lambda_2}^{\lambda_2-1} x^{-Nm} t^{-\sum_{i=2}^N (\lambda_i - m)} + \ldots \right)$$

$$= \frac{x^{\sum_{i=1}^N \lambda_i}}{1 - t^2} \left( \frac{x^{-N\lambda_1 t^{-\sum_{i=1}^N (\lambda_i - 1)}}}{1 - x^{-N t N}} - \frac{x^{-N\lambda_1 t^{-\sum_{i=1}^N (\lambda_i - 1)}}}{1 - x^{-N t N-2}} + \frac{x^{-N\lambda_2 t^{-\sum_{i=1}^N (\lambda_i - 2)}}}{1 - x^{-N t N-2}} + \ldots \right).$$

We combine $(2r-1)th$ and $(2r)th$ terms $(1 \leq r \leq N)$ to get

$$x^{\sum_{i=1}^N (\lambda_i - \lambda_r) t^{\sum_{i=1}^N (\lambda_i - \lambda_r) - \sum_{i=r+1}^N (\lambda_i - \lambda_r)}} \left( \frac{1}{1 - x^{-N t N-2r+2}} - \frac{1}{1 - x^{-N t N-2r}} \right)$$

$$= \frac{x^{\sum_{i=1}^N (\lambda_i - \lambda_r) t^{\sum_{i=1}^N (\lambda_i - \lambda_r) - \sum_{i=r+1}^N (\lambda_i - \lambda_r)}}}{(1 - x^{-N t N-2r+2})(1 - x^{-N t N-2r})}.$$  

This is the contribution of $p_{r-1}$ to the Lefschetz fixed point formula to the Euler characteristic of $L_\lambda$. (The denominator is $\Lambda_{-1} T_{p_{r-1}}^* \tilde{S}_N$, and the numerator is $(L_\lambda)_{p_{r-1}}$.) Since $\lambda$ is dominant, higher cohomology vanishes. Hence this is the character of $\Gamma(\tilde{S}_N, L_\lambda)$.  

4(ii). **Pushforwards of line bundles on Kleinian surfaces.** For dominant $\lambda$ we denote by $F_\lambda$ the torsion free sheaf $R\pi_* L_\lambda = \pi_* L_\lambda$ on $S_N$. We also set $\tilde{\lambda} = \omega_{|\lambda| (\mod N)}$ where $\omega_0 := 0$. 

Lemma 4.3. For dominant weight λ let \( \mathcal{F} \) be a \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariant torsion-free sheaf on \( S_N \) such that the character of \( \Gamma(S_N, \mathcal{F}) \) coincides with the character of \( \Gamma(S_N, \mathcal{F}_\lambda) \). Then

(a) The restriction \( \mathcal{F}|_{S_N^c} \) is a line bundle, isomorphic to \( \mathcal{F}_\lambda|_{S_N^c} \). Here \( S_N^c := S_N \setminus \{0\} \).

(b) An isomorphism in (a) is defined uniquely up to multiplication by a scalar, even if one forgets the \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariance.

(c) The composition of isomorphisms \( \mathcal{F}|_{S_N^c} \cong \mathcal{F}_\lambda|_{S_N^c} \) gives an isomorphism \( \mathcal{F}|_{S_N^c} \overset{\sim}{\longrightarrow} \mathcal{F}_\lambda|_{S_N^c} \) which extends to an isomorphism \( \mathcal{F} \overset{\sim}{\longrightarrow} \mathcal{F}_\lambda \).

Proof. An automorphism of a line bundle on \( S_N^c \) is given by multiplication by an invertible function on \( S_N^c \). Any invertible function on \( S_N^c \) is constant. Indeed, it lifts to \( \mathbb{A}^2 \setminus \{0\} \) where all the invertible functions are constant. Hence uniqueness in (b).

A torsion free sheaf \( \mathcal{F} \) is locally free on the complement of \( S_N \) to finitely many points. Due to the \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariance, \( \mathcal{F} \) is locally free on \( S_N^c \). Let us denote by \( j: S_N^c \hookrightarrow S_N \) the open embedding. Then \( \mathcal{F} \hookrightarrow j_*(\mathcal{F}|_{S_N^c}) \). Since we know the character of \( \Gamma(S_N, \mathcal{F}) \), we conclude that \( \mathcal{F} \) is generically of rank one, i.e. \( \mathcal{F}|_{S_N^c} \) is a line bundle. Now Pic(\( S_N^c \)) = \( \mathbb{Z}/N\mathbb{Z} \), and any line bundle on \( S_N^c \) is isomorphic to \( \mathcal{F}_\mu|_{S_N^c} \) for \( \mu \in \{0, \omega_1, \ldots, \omega_{N-1}\} \). Thus \( \mathcal{F} \hookrightarrow j_*(\mathcal{F}_\mu|_{S_N^c}) \) (if we disregard the \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariant structure). But any two \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariant structures on the line bundle \( \mathcal{F}_\mu|_{S_N^c} \) are isomorphic up to twist by a character \( \chi \) of \( \mathbb{C}^x \times \mathbb{C}^x \). So we have a \( \mathbb{C}^x \times \mathbb{C}^x \)-equivariant embedding \( \mathcal{F} \otimes \chi \hookrightarrow j_*(\mathcal{F}_\mu|_{S_N^c}) \). We claim that \( \mu \) is congruent to \( \lambda \) modulo the root lattice, that is \( \mu = \omega|\lambda| \mod N = \lambda \). Indeed, we take a sufficiently negative \( m \) in the formula of Lemma 4.2 for the character of \( \Gamma(S_N, \mathcal{F}) \), so that \( \lambda_i - m > 0 \) for any \( i \). Then \( \sum_i(\lambda_i - m) = -Nm + \sum_i \lambda_i \), and so \( |\lambda| \mod N \) is determined from the character of \( \Gamma(S_N, \mathcal{F}) \).

However, \( \mathcal{F}_\lambda \overset{\sim}{\longrightarrow} j_*(\mathcal{F}_\lambda|_{S_N^c}) \) (see Lemma 4.6 below and restrict to a subregular slice \( S_N \subset N \)). Thus we have \( \mathcal{F} \otimes \chi \hookrightarrow \mathcal{F}_\lambda \hookrightarrow \mathcal{F}_\lambda \), and we have to check that the images of \( \mathcal{F} \otimes \chi \) and \( \mathcal{F}_\lambda \) inside \( \mathcal{F}_\lambda \) coincide, and \( \chi = 1 \). But the character of (global sections of) \( \mathcal{F}_\lambda \) is multiplicity free, and the characters of \( \mathcal{F}_{\lambda_1} \otimes \chi_{\lambda_1} \), \( \mathcal{F}_{\lambda_2} \otimes \chi_{\lambda_2} \) coincide if and only if \( \lambda_1 = \lambda_2 \), \( \chi_1 = \chi_2 \), so the equality of characters of \( \mathcal{F} \) and \( \mathcal{F}_\lambda \) guarantees \( \chi = 1 \) and the coincidence of the images of \( \mathcal{F} \) and \( \mathcal{F}_\lambda \) in \( \mathcal{F}_\lambda \). \( \Box \)

4(iii). Line bundles on Kleinian surfaces via homology groups of fibers. Recall the setup of §2(iv) and §2(vi). We consider the quiver gauge theory of type \( A_1 \) with \( \dim V = 1 \), \( \dim W = N \) with \( G = \text{GL}(V) = \mathbb{C}^x \), \( \hat{G} = \text{GL}(V) \times \text{GL}(W)/\mathbb{Z} \), \( G_F = \text{PGL}(W) = \text{PGL}(N) \), and varieties of triples \( \mathcal{R}, \hat{\mathcal{R}} \) for \( (G, N) \), \( (\hat{G}, N) \) and the corresponding complex \( \mathcal{A} \) on \( \text{Gr}_{\text{PGL}(N)} \). See §2(iv). We are interested in its costalks at the points \( \lambda \in \text{Gr}_{\text{PGL}(N)} \) where \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N) \) is a dominant coweight of \( \text{PGL}(N) \). According to (2.16), the costalk \( i^\lambda_\mathcal{A} \) forms a module over the algebra \( i^\lambda_0 \mathcal{A} \). The algebra \( i^\lambda_0 \mathcal{A} \) is nothing but the Coulomb branch \( H^\lambda_{\text{co}}(\mathcal{R}) \cong \mathbb{C}[S_N] \) where \( N = \text{Hom}(W, V) \) by §II.4(iv). The costalk \( i^\lambda_\mathcal{A} \) is nothing but \( H^\lambda_{\text{co}}(\hat{\mathcal{R}}^\lambda) \) where \( \hat{\mathcal{R}}^\lambda \) is the fiber of \( \hat{\pi}: \hat{\mathcal{R}} \rightarrow \text{Gr}_{G_F} = \text{Gr}_{\text{PGL}(N)} \) over \( \lambda \in \text{Gr}_{G_F} \), see (2.6).

Lemma 4.4. The \( i^\lambda_0 \mathcal{A} \)-module \( i^\lambda_\mathcal{A} \) is torsion free.
Proposition 4.5. Under the identification as it is invariant under simultaneous shift of all \( m \pi \) the Hilbert series of the bigraded module \( H \) By the monopole formula of Proposition II.2.7 with the convention Remark II.2.8(2),

Proof. Both \( i_0^! A_{\text{for}} \) and \( i_\lambda^! A_{\text{for}} \) are free \( H^*_G(\text{pt}) \)-modules. So if \( i_\lambda^! A_{\text{for}} \) had torsion, then it would still have torsion after the base change to \( H^*_T(\text{pt}) \) and localization to the generic point of \( H^*_T(\text{pt}) \). However, this is impossible since after this localization, \( i_\lambda^! A_{\text{for}} \) becomes a free (rank \( 1 \)) \( i_0^! A_{\text{for}} \)-module by the Localization Theorem.

Recall that \( H^*_G(\mathcal{R}) \) has an additional grading induced from \( \pi(\text{Gr}_G) = \pi_1(G) = \pi_1(\mathbb{C}^*) \cong \mathbb{Z} \) compatible with the convolution product (see §II.3(v)). We also have an additional grading on \( H^*_G(\mathcal{R}^\lambda) \) compatible with the \( H^*_G(\mathcal{R}) \)-module structure from \( \pi(\text{Gr}_G) = \pi_1(\tilde{G}) \) in the same way. We choose \( \pi(\text{Gr}_G) \rightarrow \mathbb{Z} \) so that the connected component of \( \mathcal{R}^\lambda \) corresponding to the \( m \)-th component of \( \text{Gr}_G \) goes to \( \sum_{i=1}^N (\lambda_i - m) \). This is well-defined as it is invariant under simultaneous shift of all \( \lambda_i \) and \( m \).

Proposition 4.5. Under the identification \( i_0^! A_{\text{for}} = H^*_G(\mathcal{R}) \cong \mathbb{C}[S_N] \), the \( i_0^! A_{\text{for}} \)-module \( i_\lambda^! A_{\text{for}} = H^*_G(\mathcal{R}^\lambda) \) is isomorphic to the \( \mathbb{C}[S_N] \)-module \( \Gamma(S_N, F^\lambda) \). More precisely,

(a) The localization of \( i_\lambda^! A_{\text{for}} \) to \( S_N \) is a line bundle isomorphic to \( F^\lambda|_{S_N} \).

(b) An isomorphism in (a) is uniquely up to multiplication by a scalar.

(c) An isomorphism in (a) extends to an isomorphism \( i_\lambda^! A_{\text{for}} \stackrel{\sim}{\rightarrow} \Gamma(S_N, F^\lambda) \).

Proof. By the monopole formula of Proposition II.2.7 with the convention Remark II.2.8(2), the Hilbert series of the bigraded module \( H^*_G(\mathcal{R}^\lambda) \) is \( \sum_{m \in \mathbb{Z}} t^{\sum_{i=1}^N (\lambda_i - m)} t^{\sum_{i=1}^N |\lambda_i - m|} (1 + t^2 + t^4 + \ldots) \). By Lemma 4.4, \( H^*_G(\mathcal{R}^\lambda) \) is a torsion-free \( H^*_G(\mathcal{R}) \)-module. Comparing its Hilbert series with the formula of Lemma 4.2 and applying the criterion of Lemma 4.3 we obtain the desired result.

Let us write down the isomorphism more concretely when \( \lambda \) is the \( n \)-th fundamental coweight \( \omega_n \).

Recall \( w, y, z \) are identified with elements in \( H^*_G(\mathcal{R}) \) as follows (see §II.4(iv)):

- \( w \) is the generator of \( H^*_G(\text{pt}) \).
- \( y \) is the fundamental class of the fiber \( \pi^{-1}(1) \), where \( \pi: \mathcal{R} \rightarrow \text{Gr}_G \cong \mathbb{Z} \).
- \( z \) is the fundamental class of the fiber \( \pi^{-1}(-1) \).

The space \( \Gamma(S_N, L_{\omega_n}) \) of sections of the line bundle corresponding to \( \omega_n \) is identified with the space of semi-invariants \( \mathbb{C}[\mathbb{A}^2]^{\lambda_n} \) where \( \chi_n(\zeta) = \zeta^n \). It has a linear basis

\[ w^m z^k, \quad v^{N-n} y^m w^k \quad (m, k \in \mathbb{Z}_{\geq 0}), \]

where \( w = uv, z = u^N, y = v^N \).

Let us consider a coweight \( (m, 1, \ldots, 1, 0, \ldots, 0) \) \( (m \in \mathbb{Z}) \) of \( \tilde{G} \), where the first \( m \) is a coweight of \( G \). Let \( r^m \) denote the fundamental class of the corresponding fiber for the projection \( \tilde{R} \rightarrow \text{Gr}_G \). Note that the pairing between the coweight above and weights of \( \text{Hom}(W, V) \) are \( m - 1, \ldots, m - 1 \) \( (n \text{ times}) \) and \( m, \ldots, m \) \( (N - n \text{ times}) \). Thus we have \( n \) negative terms if \( m = 0 \), \( N - n \) positive terms if \( m = 1 \), all negative or all positive
otherwise. Therefore
\[ yr^m = \begin{cases} 
  r^{m+1} & \text{if } m > 0, \\
  w_n r^{m+1} & \text{if } m = 0, \\
  w_N r^{m+1} & \text{if } m < 0,
\end{cases} \quad zr^m = \begin{cases} 
  r^{m-1} & \text{if } m \leq 0, \\
  w^{N-n} r^{m-1} & \text{if } m = 1, \\
  w^N r^{m-1} & \text{if } m > 1,
\end{cases} \]

by §II.4. (Note that we can replace \( \tilde{G} \) by \( GL(V) \times T(W)/Z \) where \( T(W) \subset GL(W) \) is a maximal torus of \( GL(W) \) as in §II.3(ix). Hence we can use computation in §II.4.) Now we get an isomorphism \( i_{\omega_n}^! A_{\text{for}} \cong \mathbb{C}[\mathbb{A}^2]^\omega \) of \( \mathbb{C}[\mathcal{M}_C] = \mathbb{C}[y, z, w]/(yz = w^N) \)-modules by setting
\[ w^k r^m = \begin{cases} 
  v^{N-n} y^{m-1} w^k & \text{if } m > 0, \\
  u^n z^{-m} w^k & \text{if } m \leq 0.
\end{cases} \]

4(iv). **Andersen-Jantzen sheaves on a nilpotent cone.** We denote by \( \mathcal{N} \) the nilpotent cone of \( \mathfrak{sl}_N \). We denote by \( \mathcal{B} \) the flag variety of \( \mathfrak{sl}_N \), and by \( T^* \mathcal{B} \) its cotangent bundle. We denote by \( \pi: T^* \mathcal{B} \to \mathcal{N} \) the Springer resolution. We denote by \( j: \mathcal{O}_{\text{reg}} \hookrightarrow \mathcal{N} \) the embedding of the regular nilpotent orbit. For a dominant weight \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_N) \) we denote by \( \mathcal{O}(\lambda) \) the line bundle on \( T^* \mathcal{B} \) obtained by the pullback of the corresponding line bundle on \( \mathcal{B} \). It is known that \( \mathcal{J}_\lambda := \pi_\ast \mathcal{O}(\lambda) = R\pi_\ast \mathcal{O}(\lambda) \) is a torsion-free sheaf on \( \mathcal{N} \) (an Andersen-Jantzen sheaf, see e.g. \([BK05, \text{Theorem 5.2.1}]\)).

**Lemma 4.6.** For \( \lambda \in \{0, \omega_1, \ldots, \omega_{N-1}\} \) we have \( \mathcal{J}_\lambda = j_\ast (\mathcal{J}_\lambda|_{\text{reg}}) \).

**Proof.** We have to check that \( \mathcal{J}_\lambda \) is Cohen-Macaulay. It follows from the fact that its Grothendieck-Serre dual \( R\pi_\ast (\mathcal{O}(-\lambda)) \) has no higher cohomology by \([BK05, \text{Theorem 5.2.1}]\). \( \square \)

Recall that according to \([Lus81] \), \( \mathcal{N} \) is isomorphic to the transversal slice \( \overline{W}_0^{\omega_1} \) in the affine Grassmannian \( \text{Gr}_{GL(N)} \). Recall the factorization morphism \( \Pi := \pi_{N\omega^1} \circ s_\mu: \mathcal{N} = \overline{W}_0^{N\omega_1} \to \mathbb{A}^{N\omega_1^\ast} \) of Lemma Q.2.7 (it is also called the Gelfand-Tsetlin integrable system).

**Lemma 4.7.** The morphism \( \Pi \circ \pi: T^* \mathcal{B} \to \mathbb{A}^{N\omega_1^\ast} \) is flat.

**Proof.** It suffices to prove that all the fibers of \( \Pi \circ \pi \) have the same dimension \( N(N-1)/2 \). We recall the proof of Lemma Q.2.7. There the dimension estimate on the fibers of \( \Pi \) followed from the semismallness of the convolution morphism \( q \). Under the identification \( \mathcal{N} = \overline{W}_0^{N\omega_1} \), the Springer resolution \( \pi: T^* \mathcal{B} \to \mathcal{N} \) corresponds to the iterated convolution morphism \( m: \text{Gr}_{GL(N)}^{\omega_1} \times \ldots \times \text{Gr}_{GL(N)}^{\omega_1} \to \text{Gr}_{GL(N)} \) restricted to the slice \( \overline{W}_0^{N\omega_1} \subset \text{Gr}_{GL(N)} \). Now the convolution morphism \( m \) is semismall, and moreover, its composition with \( q \) is semismall as well, so the proof of Lemma Q.2.7 goes through in the present situation as well. \( \square \)

4(v). **Andersen-Jantzen sheaves via homology groups of fibers.** We change the setup of §4(iii) to that of §2(v). According to (2.16), the costalk \( i_{\lambda}^! A_{\text{for}} \) forms a module over the algebra \( i_0^! A_{\text{for}} \). The algebra \( i_0^! A_{\text{for}} \) is nothing but the Coulomb branch \( H^G_C(\mathcal{R}) \).
constant. The constant $i^\lambda_0A^\text{for}$ is nothing but $H_\bullet^G\circ(\tilde{R}_\lambda)$ where $\tilde{\pi}: \tilde{R} \to \Gr_G = \Gr_{\PGL(N)}$ and $\tilde{R}_\lambda = \tilde{\pi}^{-1}(\lambda)$, see (2.6).

We have the $L^\text{bal} = \PGL(N)$-action on the Coulomb branch $H_\bullet^G\circ(\mathcal{R})$ by Proposition A.3 and example A.6. By example A.5 it coincides with the standard action on $\mathcal{N}$.

**Theorem 4.8.** Under the identification $i^0_0A^\text{for} = H_\bullet^G\circ(\mathcal{R}) \simeq \mathbb{C}[\mathcal{N}]$, the $i^0_0A^\text{for}$-module $i^\lambda_0A^\text{for} = H_\bullet^G\circ(\tilde{R}_\lambda)$ is isomorphic to the $\mathbb{C}[\mathcal{N}]$-module $\Gamma(\mathcal{N}, J_\lambda)$.

**Proof.** The $\mathbb{C}[\mathcal{N}]$-module $i^\lambda_0A^\text{for}$ is torsion free generically of rank 1, see Lemma 4.4. By Proposition A.3 and example A.6, we have an action of $\tilde{L}^\text{bal} = \SL(N)$ on $i^\lambda_0A^\text{for}$. The $i^0_0A^\text{for}$-module $i^\lambda_0A^\text{for}$ is $\SL(N)$-equivariant (under the natural projection $\SL(N) \to \PGL(N)$).

Hence, the restriction of the associated coherent sheaf $(i^\lambda_0A^\text{for})_\text{loc}$ to $\mathcal{O}_\text{reg} \subset \mathcal{N}$ is a line bundle. Now $\text{Pic}(\mathcal{O}_\text{reg}) = \mathbb{Z}/N\mathbb{Z}$, and any line bundle on $\mathcal{O}_\text{reg}$ is isomorphic to $\mathcal{J}_\mu|\mathcal{O}_\text{reg}$ for $\mu \in \{0, \omega_1, \ldots, \omega_{N-1}\}$. Thus we obtain an embedding $i^\lambda_0A^\text{for} \hookrightarrow \Gamma(\mathcal{N}, j_*((\mathcal{J}_\mu)|\mathcal{O}_\text{reg})) = \Gamma(\mathcal{N}, \mathcal{J}_\mu)$. We claim that $\bar{\mu} = \omega|\lambda| \pmod{N} = \bar{\lambda}$. Indeed, $\SL(N)$-module $i^\lambda_0A^\text{for}$ has the same central character as $V^\lambda$.

Thus we obtain an embedding $i^\lambda_0A^\text{for} \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_\lambda)$. Similarly, we have an embedding $\Gamma(\mathcal{N}, \mathcal{J}_\lambda) \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\lambda}})$. In other words, denoting $i^\lambda_0A^\text{for}|_{\mathcal{O}_\text{reg}}$ the restriction of $(i^\lambda_0A^\text{for})_\text{loc}$ to $\mathcal{O}_\text{reg}$, we obtain an isomorphism of line bundles $i^\lambda_0A^\text{for}|_{\mathcal{O}_\text{reg}} \simeq \mathcal{J}_\mu|\mathcal{O}_\text{reg}$. Note that this isomorphism is defined uniquely up to a scalar multiplication since the automorphism group of any line bundle on $\mathcal{O}_\text{reg}$ is $\Gamma(\mathcal{O}_\text{reg}, \mathcal{O}^\times) = \mathbb{C}^\times$. Indeed, an invertible function on $\mathcal{O}_\text{reg}$ extends to a regular function on $\mathcal{N}$ due to normality of $\mathcal{N}$. This extended function is still invertible since otherwise its zero divisor would intersect $\mathcal{O}_\text{reg}$. Its lift to $T^*\mathcal{B}$ is invertible and hence constant on each fiber of $T^*\mathcal{B} \to \mathcal{B}$. So it is lifted from $\mathcal{B}$ and hence constant.

We will show that the above isomorphism extends to $\mathcal{N}$. To this end we use the factorization morphism $\Pi: \mathcal{N} \to \mathbb{A}^{N\omega_1^*} = t(V)/\mathcal{W}$ as in Theorem II.5.26 and Remark II.5.27, where $t(V)$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$, and $\mathcal{W}$ is the Weyl group of $(\mathfrak{gl}(V), t(V))$. The condition $\Pi_*\mathcal{J}_\lambda = \Pi_*\pi_*\mathcal{O}(\lambda) \hookrightarrow j_*\Pi_*\pi_*\mathcal{O}(\lambda)|_{T^*\mathcal{B}} = j_*\Pi_*\mathcal{J}_\lambda|_{T^*\mathcal{B}}$ of Remark II.5.27 is satisfied since the complement of $T^*\mathcal{B}$ in $T^*\mathcal{B}$ is of codimension 2 by Lemma 4.7. So it suffices to check the regularity of our rational isomorphism after the base change $t(V) \to t(V)/\mathcal{W}$ and localizations at general points of the root hyperplanes. Moreover, since we already know that our isomorphism is regular at $\mathcal{O}_\text{reg}$, it remains to check the regularity at the localizations at general points of the coordinate hyperplanes $w_{1,r} = 0$, $r = 1, \ldots, N-1$, cf. the proof of Theorem Q.3.10. By an application of the Localization Theorem, just as in loc. cit., the comparison reduces to Proposition 4.5. Namely, let $t$ be a general point of the hyperplane $w_{1,r} = 0$, and let $x$ be a point of the subregular nilpotent orbit above $t$. Then there is a slice $\mathcal{S}_t \subset \mathcal{N}$ through $x$ such that the isomorphism of $\mathcal{J}_\mu|\mathcal{O}_\text{reg}$ and $i^\lambda_0A^\text{for}|_{\mathcal{O}_\text{reg}}$ restricted to $\mathcal{S}_t$ extends to the localization $(\mathcal{S}_t)_t$ (by Proposition 4.5). Due to the $\SL(N)$-equivariance, the pullback of the above isomorphism to $\SL(N) \times \mathcal{S}_t \xrightarrow{\text{act}} \mathcal{O}_\text{reg}$ extends to $(\SL(N) \times \mathcal{S}_t)_t$. By the faithfully flat descent, the above isomorphism extends to $\mathcal{S}_t$, and hence to the whole of $\mathcal{N}$.

□
Remark 4.11. Let us write down the modified monopole formula (4.1) in our case explicitly. (This appeared first in [CHMZ14a, (3.9)].) It is

\[(4.9) \quad P_t^{\text{mod}}(i_\lambda^! A_{\text{for}}) = \sum_{\tilde{\lambda}} t^{2\Delta(\tilde{\lambda})} P_{GL(V)}(t, \tilde{\lambda})\]

(the sum over the dominant coweights \(\tilde{\lambda} = (\lambda^1, \ldots, \lambda^{N-1})\) of \(GL(N - 1) \times \ldots \times GL(1)\)), where

\[2\Delta(\lambda, \lambda^1, \ldots, \lambda^{N-1}) := \sum_{j=1}^{N-1} \sum_{i,i'} |\lambda^j_i - \lambda^j_{i'}| - 2 \sum_{j=1}^{N-1} \sum_{i<i'} |\lambda^j_i - \lambda^j_{i'}|;\]

and we set for convenience \(\lambda^0 := \lambda\). We also set \(n(\lambda) = \sum_{i=1}^{N} (i-1)\lambda_i\). Then \(\dim \text{Gr}_{\text{PGL}(N)}^\lambda = \langle 2\rho_{\text{PGL}(N)}, \lambda \rangle = (N-1)|\lambda| - 2n(\lambda)\).

Lemma 4.10. \(i_\lambda^! A_{\text{for}}\) lives in (modified) degrees \(\geq \dim \text{Gr}_{\text{PGL}(N)}^\lambda\), and its component of this degree has the same dimension as the irreducible \(SL(N)\)-module \(V^\lambda\).

Proof. We have to compute

\[t^{2n(\lambda) - (N-1)|\lambda|} \sum_{\tilde{\lambda}} t^{2\Delta(\tilde{\lambda})} P_{GL(V)}(t, \tilde{\lambda})|_{t=0} = t^{2n(\lambda) - (N-1)|\lambda|} \sum_{\tilde{\lambda}} t^{2\Delta(\tilde{\lambda})}|_{t=0}.\]

One checks that this is the sum of \(1\)'s over the set of \((N-1)\)-tuples \(\tilde{\lambda}\) which interlace, i.e. \(\lambda^j_i \geq \lambda^j_{i+1} \geq \lambda^j_i, 0 \leq j \leq N-2, 1 \leq i \leq N - j - 1\) (recall that \(\lambda^0 = \lambda\)). In other words, this is the cardinality of the set of Gelfand-Tsetlin patterns of shape \(\lambda\), that is \(\dim V^\lambda\).

Remark 4.11. Characters of \(\Gamma(\mathcal{N}, J_\lambda)\) are given by Hall-Littlewood polynomials by computation of Euler characteristic [Hes80, Bry89] and the vanishing theorem [Bro93]. Therefore (4.9) gives a combinatorial expression of Hall-Littlewood polynomials. We asked several people (including mathoverflow [Nak17]) whether it is known or not. But we could not find earlier appearance. In view of the argument in the special case \(t = 0\) in Lemma 4.10, there should be a purely combinatorial proof.

4(vi). Modified homological grading. Let us write down the modified monopole formula (4.1) in our case explicitly. (This appeared first in [CHMZ14a, (3.9)].) It is

\[(4.9) \quad P_t^{\text{mod}}(i_\lambda^! A_{\text{for}}) = \sum_{\tilde{\lambda}} t^{2\Delta(\tilde{\lambda})} P_{GL(V)}(t, \tilde{\lambda})\]

\(4(vi). \) Modified grading of Andersen-Jantzen modules. We have the dilatation action of \(C^\times\) on \(T^*B\) and the natural \(C^\times\)-equivariant structure on \(O(\lambda)\); hence a grading on \(\Gamma(T^*B, O(\lambda)) = \Gamma(\mathcal{N}, J_\lambda)\) starting in degree 0 with \(\Gamma(B, O(\lambda)) = (V^\lambda)^\vee\). We modify the grading by doubling all the degrees and shifting it by \((N-1)|\lambda| - 2n(\lambda)\). From now on we consider \(\Gamma(\mathcal{N}, J_\lambda)\) with this modified grading only.

Theorem 4.12. The isomorphism of \(C[\mathcal{N}]\)-modules \(i_\lambda^! A_{\text{for}} \simeq \Gamma(\mathcal{N}, J_\lambda)\) of Theorem 4.8 is a graded isomorphism.

Proof. For \(\lambda = 0\) the claim is nothing but Remark Q.3.13. Clearly, \(\Gamma(\mathcal{N}, J_\lambda)\) is a graded \(SL(N) \times C[\mathcal{N}]\)-module; \(i_\lambda^! A_{\text{for}}\) is also a graded \(SL(N) \times C[\mathcal{N}]\)-module by construction of \(\mathcal{A}\) (see example A.6). Both embeddings \(i_\lambda^! A_{\text{for}} \rightarrow \Gamma(\mathcal{N}, J_\lambda)\) and \(\Gamma(\mathcal{N}, J_\lambda) \rightarrow \Gamma(\mathcal{N}, J_\lambda)\) are compatible with the gradings up to a shift since the structure of a \(SL(N)\)-equivariant line

\[4\text{We learned this observation in [Gor17].}\]
bundle on $J_{\lambda}|_{\mathcal{O}_{\mathbb{R}^g}}$ extends to a $\text{SL}(N) \times \mathbb{C}^\times$-equivariant structure uniquely up to tensoring with a character of $\mathbb{C}^\times$. The above shifts match because both the grading of $T_\lambda A^\text{for}$ and the grading of $\Gamma(N, J_\lambda)$ start in the same degree $(N - 1)|\lambda| - 2n(\lambda)$. \hfill $\square$

4(viii). The regular sheaf. By Lemma 4.10, $\mathcal{A} \in \mathcal{D}_{\ell}^{\geq 0} (\text{Gr}_{\text{PGL}(N)})$, and $\mathcal{H}^0(\mathcal{A}) \cong \bigoplus_\lambda (V^\lambda)^{\vee} \otimes \text{IC}(\text{Gr}_{\text{PGL}(N)^\text{reg}}) =: \mathcal{A}_R$.

**Theorem 4.13.** The natural morphism $\sigma: \mathcal{A}_R = \mathcal{H}^0(\mathcal{A}) \rightarrow \mathcal{A}$ is an isomorphism of ring objects.

**Proof.** First we prove that $\sigma$ is an isomorphism disregarding the ring structure. We have to check $\tau_{>0} \mathcal{A} = \text{Cone}(\sigma) = 0$. Note that all the costalks of $\text{IC}(\text{Gr}_{\text{PGL}(N)^\text{reg}})$ live in the degrees of the same parity as $|\lambda|$, see [Lus83]. We will call this phenomenon *parity vanishing.* The parity vanishing for $\mathcal{A}$ also holds true (on a given connected component of $\text{Gr}_{\text{PGL}(N)}$, all the costalks of $\mathcal{A}^\text{for}$ live in the same parity as all the costalks of any IC sheaf on this component, see (4.9)). This implies that $\mathcal{H}^0(\mathcal{A}) = 0$, and hence $\mathcal{H}^0(\tau_{>0} \mathcal{A}) = 0$. Now the Hilbert series of $\mathcal{H}_\lambda^0$ and $\mathcal{H}_\lambda^0(\tau_{>0} \mathcal{A})$ coincide by Theorem 4.12 and the comparison of [Bry89] and [Lus83]. Hence if $\sigma$ were not an isomorphism, its costalk $\sigma_\lambda$ would have both kernel and cokernel for some $\lambda$. Thus, $\text{Cone}(\sigma)$ would have a costalk of wrong parity at $\lambda$. This would contradict the parity vanishing for $\tau_{>0} \mathcal{A} = \text{Cone}(\sigma)$. We conclude that $\sigma$ is an isomorphism.

Now we compare the ring structures. Since both $\mathcal{A}$ and $\mathcal{A}_R$ are perverse, it suffices to check that the fiber functor $H^*(\sigma)$ induces an isomorphism of the rings $H^*(\text{Gr}_{\text{PGL}(N)}, \mathcal{A}_R)$ and $H^*(\text{Gr}_{\text{PGL}(N)}, \mathcal{A}^\text{for})$. It is enough to check the assertion for $\text{GL}(N)$ instead of $\text{PGL}(N)$, as $\text{Gr}_{\text{GL}(N)}$ is the union of copies of $\text{Gr}_{\text{PGL}(N)}$. We have $H^*(\text{Gr}_{\text{GL}(N)}, \mathcal{A}_R) \cong \mathbb{C}[\text{GL}(N)]$ by geometric Satake equivalence. On the other hand, the cohomology $H^*(\text{Gr}_{\text{GL}(N)}, \mathcal{A}^\text{for})$ is the quotient of the equivariant cohomology $H^*_{\ell}(\text{Gr}_{\text{GL}(N)}, \mathcal{A})$ modulo the augmentation ideal of $H^*_{\ell}(\text{GL}(N)^{\text{pt}})$. And $H^*_{\ell}(\text{Gr}_{\text{GL}(N)}, \mathcal{A}) = H^*_{\ell}(\text{GL}(V) \times \text{GL}(W))^0 \odot (\text{R}_{\text{GL}(V) \times \text{GL}(W)}^{\text{GL}(N)})$ where $\text{GL}(V) \times \text{GL}(W), N)$ is the quiver gauge theory obtained from $(\text{GL}(V), N)$ by turning $\text{GL}(W)$ to a gauge group. By Theorem Q.3.1, $H^*_{\ell}(\text{GL}(V) \times \text{GL}(W))^0 \odot (\text{R}_{\text{GL}(V) \times \text{GL}(W)}^{\text{GL}(N)}) \cong \mathbb{C}[Z_{\text{PGL}(N+1)}]$ where $\alpha = N\alpha_1 + (N - 1)\alpha_2 + \ldots + \alpha_N$. By [BP08, Theorem 1], $Z_{\text{PGL}(N+1)} \cong \text{GL}(W) \times W$, and its projection to Spec $H^*_{\ell}(\text{GL}(N)^{\text{pt}})$ is nothing but the projection of $\text{GL}(W) \times W$ to $W$. Hence the zero fiber of this projection is isomorphic to $\text{GL}(W) = \text{GL}(N)$. \hfill $\square$

5. Mirrors of Sicilian theories

In the first half of this section, we study examples of Coulomb branches $\mathcal{M}_C$ of star shaped quiver gauge theories as in Figure 1. As explained at the end of Introduction, they are conjectural Higgs branches of Sicilian theories.

Let us briefly review [MT12] on expected properties of Higgs branches of Sicilian theories. It is conjectured that there exists a functor from the category of 2-bordisms to a category HS of holomorphic symplectic varieties with Hamiltonian group actions. For the latter, objects
are complex algebraic semisimple groups. A homomorphism from $G$ to $G'$ is a holomorphic symplectic variety $X$ with a $\mathbb{C}^\times$-action scaling the symplectic form with weight 2 together with hamiltonian $G \times G'$ action commuting with the $\mathbb{C}^\times$-action. For $X \in \text{Hom}(G', G)$, $Y \in \text{Hom}(G, G'')$, their composition $Y \circ X \in \text{Hom}(G'', G')$ is given by the symplectic reduction of $Y \times X$ by the diagonal $G$-action. The identity $\in \text{Hom}(G, G)$ is the cotangent bundle $T^*G$ with the left and right multiplication of $G$.

Let us fix a complex semisimple group $G$. Physicists associate a 3d Sicilian theory to $G$ and a Riemann surface with boundary, and consider its Higgs branch. It depends only on the topology of the Riemann surface, and gives a functor as above. We associate $S^1$ with $G$, and a cylinder with $T^*G$. Since $T^*G$ is the identity in HS, it is one of requirements.

Physical argument shows that the variety associated with a disk is $G \times S$, where $S$ is the Kostant slice to the regular nilpotent orbit.

Let $W \equiv W_G$ be the variety associated with $S^2$ three disks removed. This is a fundamental piece as other varieties are obtained by reductions of products of its copies. It has an action of $S_3 \rtimes G^3$. It is expected that

- $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ if $G = \text{SL}(2)$.
- $W$ is the minimal nilpotent orbit of $E_6$ if $G = \text{SL}(3)$.

For other groups, $W$ is unknown.

Recently Ginzburg and Kazhdan [GK] construct a functor, and check most of properties, in particular show that the gluing of Riemann surfaces corresponds to the hamiltonian reduction with respect to the diagonal action. Via a result of [Bap15] their symplectic variety associated with $S^2$ minus $b$ disks is defined as

$$W^b \overset{\text{def}}{=} \text{Spec} H^*_G(\text{Gr}_G, i^!\Delta^!(\mathbb{R}^h_{k=1}(A_R)_k)),$$

where $(A_R)_k$ is a copy of the regular sheaf on $\text{Gr}_G$. Here the complex symplectic group taken as the object of the target category is $G^\vee$, the Langlands dual group. (E.g., $b = 2$ gives $T^*G^\vee$.)

By Theorem 2.11 together with §2(viii) we immediately get the following:

Theorem 5.1. The symplectic variety $W^b$ of Ginzburg-Kazhdan for $G^\vee = \text{SL}(N)$ is isomorphic to the Coulomb branch of the star shaped quiver gauge theory in Figure 1 with $b$ legs instead of 3.

More precisely, as we divide $\text{GL}(V) \times \text{GL}(W)$ by $\mathbb{C}^\times$ in §2(v), we also divide $\text{GL}(V)$ for the star shaped quiver gauge theory also by the diagonal central subgroup $Z \cong \mathbb{C}^\times$. If we replace the central $\text{GL}(N)$ by $\text{SL}(N)$ instead of taking the quotient by $Z$, we get $W$ for $G^\vee = \text{PGL}(N)$.

We could consider the Coulomb branch for more general quiver gauge theory associated with a Riemann surface with boundary as in [Nak16, 3(iii) Figure 5] (namely we have $b$ legs, as well as $g$ loops at the central vertex), which is the Higgs branch of a 3d Sicilian theory, obtained by compactifying 6d $\mathcal{N} = (2,0)$ theory of type A by

\[5\text{We thank Yuji Tachikawa for an explanation of this procedure.}\]
$S^1 \times$ (punctured Riemann surface). Ginzburg-Kazhdan construction is also generalized. See §5(x). We conjecture that Theorem 5.1 is generalized.

**Conjecture 5.2.** Let $W^{g,b} \overset{\text{def}}{=} \text{Spec} H^*_G(\text{Gr}_G, A^b \otimes \mathcal{B}^g)$ as in §5(x) for $G^\vee = \text{SL}(N)$. It is isomorphic to the Coulomb branch of the gauge theory associated with the quiver [Nak16, 3(iii) Figure 5].

By §2(viii) it is enough to show that the complex $\mathcal{B} = \mathcal{B}^{g=1}$ introduced in §5(x) is isomorphic to the object $A = \pi_* \omega_{\mathcal{R}}[-2 \dim N_G]$ associated with $(G, N) = (\text{PGL}(N), \mathfrak{pgl}(N))$. We conjecture that this is true for general $G$ and its adjoint representation $N = \mathfrak{g}$. Note that we prove that $\text{Spec} H^*_G(\text{Gr}_G, \mathcal{B}) = (T^* \times t)/W$ (using Losev’s result in §5(xiii)), which coincides with the Coulomb branch for the adjoint representation. (See §II.3(x)(b).)

The remainder of this section is as follows. In the first five subsections, we study examples of $\mathcal{M}_C$, in particular check two cases $\text{SL}(2)$, $\text{SL}(3)$ above. These are basically applications of [Quiver] and §A, and we will not use the sheaf $A$. In the subsequent five subsections, we show the gluing property and also $W^{b=2} = T^*G^\vee$. They were shown in [BK], but we give proofs for completeness. They are direct consequences of [BF08]. In §§5(xi), 5(xii) we explain similarities between the gluing property and hamiltonian reduction.

Let us use the following notation as in [Quiver]. Let $Q$ be a quiver with sets $I$, $\Omega$ of vertices and arrows respectively. We take an $I$-graded vector space $V = \bigoplus V_i$ with dimension vector $\alpha = (\dim V_i)_{i \in I}$. We set $\text{GL}(V) = \prod \text{GL}(V_i)$, $N = N^\alpha = \bigoplus \text{Hom}(V_i, V_j)$, where the sum is over the arrows $i \to j \in \Omega$. We also take the diagonal central subgroup $Z = \mathbb{C}^x \subset \text{GL}(V)$ and set $\text{PGL}(V) = \text{GL}(V)/Z$. We consider $\mathcal{R} = \mathcal{R}_{\text{PGL}(V), N}$ and $\mathcal{M}_C = \text{Spec}(H^*_{\text{PGL}(V)^G}((\mathcal{R}_{\text{PGL}(V), N})))$.

**Remark 5.3.** Consider the regular sheaf $A_R$ on $\text{Gr}_G$. In type $A$, it arises as the ring object associated with a quiver gauge theory by Theorem 2.11. Using Sp/SO quiver as in [Nak16, App.A.2], we can conjecture that $A_R$ for classical groups is constructed in a similar way, once we can generalize our definition to the case when $M$ is not necessarily of cotangent type. For exceptional groups, we do not expect that $A_R$ appears in this way, as argued in [Nak16, §3(i)]. Nevertheless it is expected that $A_R$ arises from the $3d$ $\mathcal{N} = 4$ quantum field theory $T(G)$, which was introduced in Gaiotto-Witten [GW09]. This theory is not a usual gauge theory nor a lagrangian theory for an exceptional group, hence is difficult to understand from a mathematical point of view. But it has a $G \times G^\vee$-symmetry, and its Higgs/Coulomb branches are nilpotent cones $\mathcal{N}$ and $\mathcal{N}^\vee$ of $G$ and $G^\vee$ respectively. The Sicilian theory $S_{G^\vee}(g, b)$ associated with $b$ punctured genus $g$ Riemann surface $C$ considered above is constructed from $T(G)$ by ‘gauging’ quantum field theories up to $3d$ mirror:

$$S_{G^\vee}(g, b) \overset{3d \text{ mirror}}{\leftrightarrow} T[G]^b \times \text{Hyp}(\mathfrak{g} \oplus \mathfrak{g})^g \# G_{\text{diag}},$$

where we use the notation $\#$ for the gauging in [Tac]. (See also [Tac17].) This observation was given in [BTX10]. Note that we ignore the parameter $\tau$ in [Tac, §2.6]. The deformation parameter, which corresponds to the complex structure of $C$, is not relevant to Higgs branches as complex symplectic varieties. Hence we can safely write $S_{G^\vee}(g, b)$ instead of $S_{G^\vee}(C)$, and understand that the Higgs branch of $S_{G^\vee}(g, b)$ is the Coulomb branch of the
right hand side. A similarity between (5.4) and the definition $W^{g,b} = \text{Spec} H^*_G(G,B \otimes \mathcal{B})$ is clear. We identify $T[G]$ with $A$, Hyp$(g \oplus g^*)$ with $B$, and $\#G_{\text{diag}}$ with taking $H^*_G(G,B \otimes \mathcal{B})$ after the $!$-restriction to the diagonal subgroup. See §3.5(xi) for a further discussion. We thank Davide Gaiotto and Yuji Tachikawa for this remark.

5(i). Cylinder. Consider the two legs star shaped quiver gauge theory instead of three legs in Figure 1. It is a quiver gauge theory of type $A_{2n-1}$ with $\dim V = (1, 2, \ldots, N-1, N, N-1, \ldots, 2, 1)$. We first consider the Coulomb branch for $\text{GL}(V)$. By §Q.3(i), $\mathcal{M}_C(\text{GL}(V), N)$ is the moduli space $\hat{Z}^\alpha_{\text{PGL}(2N)}$ of based maps from $\mathbb{P}^1$ to a flag variety of type $A_{2n-1}$ with degree $\alpha = \dim V$. By [BP08, Th. 7.2], it is isomorphic to $T^* \text{GL}(N)$.

Note that $\alpha = 2\omega_N$. By Remark Q.3.12, we have an isomorphism $\hat{Z}^\alpha_{\text{PGL}(2N)} \xrightarrow{\sim} S_\alpha \cap \overline{W}_0^\alpha$ and the natural action of $\text{Stab}_{\text{PGL}(2N)}(\alpha) = \text{PGL}(N, N) := \text{GL}(N) \times \text{GL}(N)/C^\times$ on $\hat{Z}^\alpha_{\text{SL}(2N)}$, where $C^\times$ is the diagonal central subgroup of $\text{GL}(N) \times \text{GL}(N)$. It coincides with the natural action of $\text{GL}(N) \times \text{GL}(N)$ on $T^* \text{GL}(N)$ through the quotient homomorphism $\text{GL}(N) \times \text{GL}(N) \rightarrow \text{PGL}(N, N)$. By example A.5 this action coincides with the one given in §A. More precisely the $\text{PGL}(N, N)$ action on $\overline{W}_0^\alpha$ coincides with the one given in §A, and the embedding $S_\alpha \cap \overline{W}_0^\alpha \rightarrow \overline{W}_0^\alpha$ is equivariant for both actions, as it is given by Remark Q.3.11 as Coulomb branches.

By Remark Q.3.5 $H^*_{\text{GL}(V)^0}(\mathcal{R}_{\text{GL}(V), N}) \rightarrow H^*_{\text{GL}(V)^0}(\mathcal{R}_{\text{PGL}(V), N})$ is nothing but the restriction to the level set $F^{-1}_\alpha(1)$, where $F_\alpha$ is the boundary function (see §Q.2(i)). In this particular case, we have $F_\alpha = c \det$ for $c \in C^\times$: all the invertible regular function on $\hat{Z}^\alpha$ are of the form $cE^k_\alpha$, $k \in \mathbb{Z}$, $c \in C^\times$ [BDF16, Lemma 5.4]. Now by degree reasons, $\det = cF_\alpha$. Therefore $H^*_{\text{GL}(V)^0}(\mathcal{R}_{\text{PGL}(V), N}) \cong \mathfrak{gl}(N) \times \text{SL}(N)$. Moreover, $H^*_{\text{PGL}(V)^0}(\mathcal{R}_{\text{PGL}(V), N}) \rightarrow H^*_{\text{GL}(V)^0}(\mathcal{R}_{\text{PGL}(V), N})$ is nothing but the projection $\mathfrak{gl}(N) \times \text{SL}(N) \rightarrow \text{Lie PGL}(N) \times \text{SL}(N)$. Identifying Lie PGL with $\mathfrak{sl}(N)^*$ via the Killing form, we get $\mathcal{M}_C(\text{GL}(V), N) \cong T^* \text{SL}(N)$. This is the symplectic variety associated with a cylinder as expected.

Let us check how the action in §A is affected by the replacement $\text{GL}(V) \rightarrow \text{PGL}(V)$. The semisimple Lie algebra $\mathfrak{p}^\text{ss}_{\text{bal}}$ remains the same: the variety $\mathcal{R}_{\text{PGL}(V), N}$ is obtained from $\mathcal{R}_{\text{GL}(V), N}$ by identifying isomorphic connected components. Therefore the construction of Appendix Q.B applies. On the other hand $\pi_1(\text{GL}(V)) \cong \bigoplus_{i \in \mathbb{Q}_0} Z \alpha_i$ is replaced by $\pi_1(\text{PGL}(V)) \cong \pi_1(\text{GL}(V))/Z(\sum_{i \in \mathbb{Q}_0} \dim V_i \alpha_i)$. The root datum is $R^\text{bal} \subset \pi_1(\text{PGL}(V))$, $R^\text{bal} \subset \pi_1(\text{PGL}(V))^\vee$. Thus $\text{PGL}(N, N)$ is replaced by its subgroup $\text{PGL}(N, N)'''$ defined by $\{[g_1, g_2] \mid \det g_1 = \det g_2\}$. We have $\text{SL}(N) \times \text{SL}(N) \rightarrow \text{PGL}(N, N)'''$ with kernel $\mathbb{Z}/N\mathbb{Z}$, the diagonal central subgroup. The standard action on $T^* \text{SL}(N)$ coincides with the one given in §A.

On the other hand, if we replace the central $\text{GL}(N)$ by $\text{SL}(N)$, the corresponding Coulomb branch is the hamiltonian reduction of $T^* \text{GL}(N)$ with respect to the $C^\times$-action corresponding to $\pi_1(\text{GL}(N)) \cong \mathbb{Z}$. (See Proposition II.3.18.) In this case $C^\times$-action is the scalar multiplication on $T^* \text{GL}(N)$ (Remark Q.3.2), hence the reduction is $T^* \text{PGL}(N)$ as expected.
5(ii). Disk. The variety for the disk is calculated as for the cylinder. We consider a quiver gauge theory of type $A$ with $\dim V = (1, 2, \ldots, N)$. As we remarked in the proof of Theorem 4.13, the Coulomb branch is $\mathbb{P}_{\text{PGL}(N+1)}^\alpha \simeq \text{GL}(N) \times \mathbb{C}^N$, where $\mathbb{C}^N$ is identified with the Kostant slice for $\text{GL}(N)$, and $\alpha = (N + 1)\omega_N$. By Remark Q.3.12, we have an isomorphism $\mathbb{P}_{\text{PGL}(N+1)}^\alpha \overset{\sim}{\to} S_\alpha \cap \overline{W}_0^\alpha$ and the natural action of $\text{Stab}_{\text{PGL}(N+1)}(\alpha) = \text{GL}(N)$ on $\mathbb{P}_{\text{PGL}(N+1)}^\alpha$ coinciding with the natural action of $\text{GL}(N)$ on $\text{GL}(N) \times \mathbb{C}^N$ (trivial on $\mathbb{C}^N$ and by left shifts on $\text{GL}(N)$). By example A.5 this action coincides with the one given in §A.

The modification to cases $\text{SL}(N)$, $\text{PGL}(N)$ are similar to the above.

5(iii). $S^2$ with three punctures for $\text{SL}(2)$. We next consider the Higgs branch of the Sicilian theory of type $\text{SL}(2)$ associated with $S^2$ with three punctures. The mirror quiver gauge theory is of type $D_4$.

We consider the $D_4$ quiver with the central vertex 1 and other vertices 2, 3, 4. We orient the edges from the central vertex. We take $V_1 = \mathbb{C}^2$, $V_2 = V_3 = V_4 = \mathbb{C}$. The diagonal central subgroup $Z = \mathbb{C}^\times \subset \text{GL}(V)$ acts trivially on $N = \bigoplus_{i=2}^4 \text{Hom}(V_i, V_i)$, so the action of $\text{GL}(V)$ factors through $\text{PGL}(V) := \text{GL}(V)/Z$. We will prove $\mathcal{M}_C(\text{PGL}(V), N) \simeq \mathbb{A}^8$.

According to Theorem Q.3.1, $\mathcal{M}_C(\text{GL}(V), N) \simeq \mathbb{Z}^\alpha$, the moduli space of degree $\alpha$ based maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}$ of the simply connected group $G = \text{Spin}(8)$ of type $D_4$. Here $\alpha = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is the highest coroot. Note that $\alpha = \omega_1$ is a fundamental coweight. We also consider the transversal slice $s_0^\alpha: \overline{W}_{G,0}^\alpha \to Z^\alpha$ (see §Q.2(ii); note that $-w_0 = \text{Id}$ for our $G$). It is the moduli space of the data $(\mathcal{P}_{\text{triv}} \overset{\sigma}{\to} \mathcal{P})$ where $\mathcal{P}_{\text{triv}}$ is the trivialized $G$-bundle on $\mathbb{P}^1$, and $\sigma$ is an isomorphism on $\mathbb{P}^1 \setminus \{0\}$ with a trivial $G$-bundle $\mathcal{P}$ possessing a degree $\alpha$ pole at $0 \in \mathbb{P}^1$. We consider an open subset $U \subset \overline{W}_{G,0}^\alpha$ formed by the data $(\mathcal{P}_{\text{triv}} \overset{\sigma}{\to} \mathcal{P})$ such that the transformation of the (unique) degree 0 complete flag in $\mathcal{P}$ with value $B_-$ at $\infty \in \mathbb{P}^1$ viewed as a generalized $B$-structure in $\mathcal{P}_{\text{triv}}$ acquires no defect at $0 \in \mathbb{P}^1$. We have $s_0^\alpha: U \overset{\sim}{\to} \mathbb{Z}^\alpha$. We also have another open subset $U' \subset \overline{W}_{G,0}^\alpha$ formed by the data $(\mathcal{P}_{\text{triv}} \overset{\sigma}{\to} \mathcal{P})$ such that the transformation of the (unique) degree 0 complete flag in $\mathcal{P}_{\text{triv}}$ with value $B_-$ at $\infty \in \mathbb{P}^1$ viewed as a generalized $B$-structure in $\mathcal{P}$ acquires no defect at $0 \in \mathbb{P}^1$. This open subset $U'$ is nothing but the intersection of $\overline{W}_{G,0}^\alpha$ with the semiinfinite orbit $T_{-\alpha} \subset \text{Gr}_G$. Since the trivialization of $\mathcal{P}$ at $\infty \in \mathbb{P}^1$ (arising from $\sigma$) uniquely extends to the trivialization of $\mathcal{P}$ over the whole of $\mathbb{P}^1$, we obtain an involution $\iota: \overline{W}_{G,0}^\alpha \overset{\sim}{\to} \overline{W}_{G,0}^\alpha$ reversing the roles of $\mathcal{P}$ and $\mathcal{P}_{\text{triv}}$ and replacing $\sigma$ by $\sigma^{-1}$. We have $\iota: U \overset{\sim}{\to} U'$. Thus we obtain an isomorphism $s_0^\alpha \circ \iota: U' \overset{\sim}{\to} \mathbb{Z}^\alpha$.

Since $\alpha$ is the highest coroot, $\overline{W}_{G,0}^\alpha$ is isomorphic to the minimal nilpotent orbit closure $\mathcal{N}_{\text{min}} = 0 \cup \mathcal{O}_{\text{min}} \subset \mathfrak{g}$, see [BeiDr, 4.5.12, page 182] or [MOV05, Lemma 2.10] for a published account. The projectivization of $\mathcal{N}_{\text{min}}$ is the partial flag variety $\mathcal{N}_\alpha = G/P_{234}$: the quotient with respect to a submaximal parabolic subgroup. Thus we have a $\mathbb{C}^\times$-bundle $p: \mathcal{O}_{\text{min}} \to \mathcal{B}_\alpha$. The big Bruhat cell (the open $B$-orbit) $C \subset \mathcal{B}_\alpha$ is the free orbit of the unipotent radical $U_{-234}^-$ of $P_{-234}^-$. Via the exponential map, $U_{-234}^- \simeq \mathfrak{u}_{-234}$, the nilpotent radical of the Lie algebra of $P_{-234}^-$. For the Levi subgroup $L_{234} \subset P_{-234}^-$ we have $[L_{234}, L_{234}] \simeq \text{SL}(2) \times \mathbb{A}^6$. 


SL(2)_3 × SL(2)_4, and u_234 as a [L_{234}, L_{234}]-module is isomorphic to C^2_p ⊗ C^3_p ⊗ C^4_p ⊗ C^5_p ⊗ C^6_p. Note that the center ˜Z(SL(2)_2 × SL(2)_3 × SL(2)_4) = (Z/2Z)^3 has a natural projection onto Z/2Z (the sum of coordinates). The kernel K of this projection, as a subgroup of Spin(8), coincides with the center Z(Spin(8)). The action of SL(2)_2 × SL(2)_3 × SL(2)_4 on u_234 factors through the action of ˜L_{234} := (SL(2)_2 × SL(2)_3 × SL(2)_4)/K.

Finally, we have U' = p^{-1}(C). Thus we obtain a projection (a C^∞-bundle) p ◦ (s_0^α ◦ υ)^{-1} : ˜Z^α → C. This action of C^∞ is nothing but the composition of the natural T-action (Cartan torus T = B ∩ B_−) with the cocharacter α_1 : C^∞ → T. The boundary equation F^α : ˜Z^α → C^∞ has weight 1 with respect to C^∞ → T [BF14, Proposition 4.4]. It follows that (p ◦ (s_0^α ◦ υ)^{-1}, F^α) : ˜Z^α ↠ C × C^∞. The action of ˜L_{234} on ˜Z^α = C × C^∞ is via the above action on C. Note that ˜L_{234} is nothing but [L_{bal}, L_{bal}] ⊂ PSO(8) of §A, and the action of ˜L_{234} on M_C(GL(V), N) coincides with the action constructed in §A by example A.5.

Now the surjection
\[ \mathbb{C}[\hat{Z}^α] = \mathbb{C}[M_C(GL(V), N)] = H^{GL(V)}_*(\mathcal{R}_{GL(V), N}) \to H^{GL(V)}_*(\mathcal{R}_{PGL(V), N}) \]
is nothing but the restriction to the level set F^{-1}_α(1) (see Remark Q.3.5), hence Spec H^{GL(V)}_*(\mathcal{R}_{PGL(V), N}) ∼ C. Furthermore, the embedding
\[ \mathbb{C}[M_C(PGL(V), N)] = H^{PGL(V)}_*(\mathcal{R}_{PGL(V), N}) \to H^{GL(V)}_*(\mathcal{R}_{PGL(V), N}) = \mathbb{C}[F^{-1}_α(1)] \]
is nothing but the embedding of the ring of functions invariant with respect to the translations action of G_a on ˜Z^α. Here we view G_a as a subgroup of automorphisms of P^1 preserving ∞ ∈ P^1; its action on ˜Z^α preserves the boundary equation F^α and its level set F^{-1}_α(1). In terms of the identification F^{-1}_α(1) ∼ C ∼ u_234 ∼ C_2^p ⊗ C_3^p ⊗ C_4^p ⊗ C, the action of G_a is nothing but the action of the last summand C, and hence M_C(PGL(V), N) = F^{-1}_α(1)/G_a ∼ C_2^p ⊗ C_3^p ⊗ C_4^p.

The above action of ˜L_{234} = [L_{bal}, L_{bal}] on M_C(GL(V), N) induces its action on M_C(PGL(V), N). One can also see that ˜L_{234} is the reductive group corresponding to the root datum R^{bal} ⊂ π_1(PGL(V)), R^{bal} ⊂ π_1(PGL(V)) via π_1(GL(V)) → π_1(PGL(V)) as in §5(i).

**Remark 5.5.** Let us give another argument, which the third named author was taught by Amihay Hanany. Let us consider functions E^{(1)}_1, F^{(1)}_1 for the middle vertex 1 by Remark A.7. Since ⟨α, α_1⟩ = 1, we have the action of G_a^2 by integrating hamiltonian vector fields H^{E^{(1)}_i}, H^{F^{(1)}_i}. We combine it with the action of SL(2)_2 × SL(2)_3 × SL(2)_4. Let us consider the Lie subalgebra of C[M_C(PGL(V), N)] generated by E^{(1)}_i, F^{(1)}_i (i = 1, 2, 3, 4). Viewing (PGL(V), N) as a framed quiver gauge theory of type A_3 with dim V = 1 2 1, dim W = 0 1 0, we see that μ = dim W − C dim V satisfies the condition ⟨μ, α⟩ ≥ −1 for any positive root α. Hence elements in the Lie subalgebra have either degree 1, 1/2, or 0, and the degree 0 part consists of constant functions by Remark Q.B.20. Since the Poisson bracket is of degree −1, {f, g} is a constant if f, g are of degree 1/2.

Commutator relations in Appendix Q.B imply that E^{(1)}_1 (resp. F^{(1)}_1) is a lowest (resp. highest) weight vector in the tensor product C_2^p ⊗ C_3^p ⊗ C_4^p of vector representations. Hence
we have a factorization \( \mathcal{M}_C(\text{PGL}(V), N) \cong A^8 \times \mathcal{M}'_C \) by Remark A.7. But \( \mathcal{M}'_C \) must be a point as \( \mathcal{M}_C(\text{PGL}(V), N) \) is 8-dimensional. (Both \( E_1^{(1)} \) and \( F_1^{(1)} \) live in the same representation, as \( \mathcal{M}_C(\text{PGL}(V), N) \) would have a factor \( A^{16} \) otherwise.)

The same argument shows that the Coulomb branch \( \mathcal{M}_C(\text{PGL}(V), N) \) for \( \dim V = 1 \ 2 \ 3 \ 4 \ \ldots \ N-1 \ N \ N-1 \ \ldots \ 2 \ 1 \) is \( \text{Hom}(C^N_i, C^N_r) \oplus \text{Hom}(C^N_i, C^N_r) \), where we have an \( \text{SL}(N)_l \times \text{SL}(N)_r \)-action from the balanced vertices in the left and right legs, and \( C^N_i, C^N_r \) are its vector representations. See [MT12, (4.6)].

5(iv). \( S^2 \) with three punctures for \( \text{SL}(3) \). We next consider type \( \text{SL}(3) \). The mirror quiver gauge theory is of type affine \( \text{E}_6 \). We start with a simple observation. Let us denote by 0 (resp. 6) a special vertex (resp. a vertex adjacent to 0) of the affine quiver of type \( \text{E}_6 \). This choice breaks \( \mathfrak{S}_3 \) symmetry of the quiver. We have an isomorphism \( \prod_{i \neq 0} \text{GL}(V_i) \cong \text{PGL}(V) \cong \text{GL}(V)/Z \). Therefore we can view \( \text{PGL}(V), N \) as a framed quiver gauge theory of finite type \( \text{E}_6 \) with \( \dim V = 1 \ 2 \ 3 \ 2 \ 1 \), \( \dim W = 0 \ 0 \ 0 \ 0 \ 0 \). Therefore its Coulomb branch is \( \mathcal{W}_{\nu_6} \) by Theorem Q.3.10, where \( G \) is the group \( \text{E}_6 \) of adjoint type. By [BeiDr, 4.5.12, page 182] (see [MOV05, Lemma 2.10] for a published account), this is isomorphic to the closure \( \mathcal{N}_{\min} \) of the minimal nilpotent orbit \( \mathcal{O}_{\min} \). We have the action of \( G \) by Proposition A.3, which is identified with the standard one by example A.5.

The action of \( \text{SL}(3)^2 \) corresponding to two legs not containing the chosen special vertex 0 is coming from the standard inclusion \( \text{SL}(3)^2 \subset \text{E}_6 \). The remaining \( \text{SL}(3) \) action for the leg containing 0 is given as follows.

First let us note that the Lie algebra \( \mathfrak{l} \) of degree 1 elements in \( \mathbb{C}[\mathcal{M}_C] \) is \( \mathfrak{e}_6 \), as we already know \( \mathcal{M}_C = \mathcal{N}_{\min} \).

Returning back to the original gauge group \( \text{PGL}(V) = \text{GL}(V)/Z \), we have degree 1 elements \( E_0^{(1)}, F_0^{(1)}, H_0^{(1)} \in \mathfrak{l} \) corresponding to the special vertex 0 by Lemma A.2. The variety of triples \( \mathcal{R}_{\text{PGL}(V), N} \) is obtained from \( \mathcal{R}_{\text{GL}(V), N} \) by identifying isomorphic connected components. Therefore the construction of Appendix Q.B applies. The computation of Poisson brackets \( \{H_0^{(1)}, E_0^{(1)}\}, \{H_0^{(1)}, F_0^{(1)}\} \) remains unchanged by the replacement \( \text{GL}(V) \to \text{PGL}(V) \), hence we conclude that \( E_0^{(1)}, F_0^{(1)} \) are root vectors corresponding to the highest weight of \( \mathfrak{l} = \mathfrak{e}_6 \). It also follows that \( E_0^{(1)}, F_0^{(1)} \) together with \( E_6^{(1)}, F_6^{(1)} \) generate an additional \( \mathfrak{sl}(3) \), and \( \text{SL}(3) \).

We have the \( \mathfrak{S}_3 \)-action on \( \mathcal{M}_C \) induced by permutation of three legs. From the above consideration, it is clear that it corresponds to \( \mathfrak{S}_3 \) of automorphisms of \( \mathfrak{e}_6 \) exchanging root subspaces corresponding to the highest weight and two remaining special vertices. (See [Kac90, Th. 8.6] for the detail of the construction of automorphisms.)

5(v). Torus with one puncture for \( \text{SL}(3) \). We consider the Higgs branch of the Sicilian theory of type \( \text{SL}(3) \) associated with a torus with one puncture. According to Conjecture 5.2 the mirror quiver gauge theory is \( 1 \to 2 \to 3 \to 1 \), where numbers are dimensions (and we use them also for indices of vertices). Note that we have an edge loop at the vertex 3. Let us denote the Coulomb branch of this quiver gauge theory by \( \mathcal{M} \). The following result is informed to the third-named author by Amihay Hanany. (It is based on an earlier observation in [GR12, §2.1], [CHMZ14b, (3.3.2)].)
Proposition 5.6. \( \mathcal{M} \) is isomorphic to the subregular orbit closure \( \mathcal{O}_{\text{subreg}} \subset \mathfrak{g}_2 \).

Proof. Let us first construct the action of \( G_2 \).

We consider operators \( E_i^{(1)}, F_i^{(1)}, H_i^{(1)} \) \((i = 1, 2, 3)\) as in Lemma A.2. The vertices 1, 2 are balanced, while 3 is not. But we still have \( \text{deg} E_3^{(1)}, F_3^{(1)} = 1 \) by \((Q.A.4)\). Let us consider the Lie subalgebra \( \mathfrak{g} \) of \( \mathbb{C} [\mathcal{M}] \) generated by these operators. By \((Q.A.2(i))\) \( E_i^{(1)}, F_i^{(1)}, H_i^{(1)} \) \((i = 1, 2)\) define the Lie algebra \( \mathfrak{sl}(3) \), and the corresponding hamiltonian vector fields are integrated to an \( \text{SL}(3) \)-action on \( \mathcal{M} \).

The proof of commutation relations \( \{ E_i^{(1)}, F_i^{(1)} \} = 0 = \{ F_i^{(1)}, E_i^{(1)} \}, \{ H_i^{(1)}, E_i^{(1)} \} = 0 = \{ H_i^{(1)}, F_i^{(1)} \}, \{ H_2^{(1)}, E_3^{(1)} \} = -E_3^{(1)}, \{ H_2^{(1)}, F_3^{(1)} \} = F_3^{(1)} \) in Appendix Q.B remains to work even though 3 has an edge loop. Similarly to \([KN16, \text{Lemma} 6.8]\) we calculate \( \{ E_3^{(1)}, F_3^{(1)} \} = 2(3w_{2,1} + 3w_{2,2} - 2w_{3,1} - 2w_{3,2} - 2w_{3,3}) \). Since \( H_1^{(1)} = 2w_{1,1} - w_{2,1} - w_{2,2} \) and \( H_2^{(1)} = 2w_{2,1} + 2w_{2,2} - w_{1,1} - w_{3,1} - w_{3,2} - w_{3,3} \), we conclude that \( \{ E_3^{(1)}, F_3^{(1)} \} = 4H_2^{(1)} + 2H_1^{(1)} \).

Hence \( \mathfrak{g} \) is simple of type \( G_2 \) with Cartan subalgebra \( \mathfrak{h} \) spanned by \( H_1^{(1)} \) and \( H_2^{(1)} \), and \( \mathfrak{sl}(3) \) is spanned by \( \mathfrak{h} \) and the long roots. Note that \( E_3^{(1)} \) and \( F_3^{(1)} \) generate the fundamental representations \( V \) and \( V^* \) of \( \mathfrak{sl}(3) \), and \( \mathfrak{g} = \mathfrak{sl}(3) \oplus V \oplus V^* \).

More concretely, \( F_3^{(1)} = (w_{3,1} + u_{3,2} + u_{3,3}) \) and \( E_3^{(1)} = (w_{3,1} - w_{2,1})u_{3,1}^{-1} + (w_{3,2} - w_{2,1})u_{3,2}^{-1} + (w_{3,3} - w_{2,1})u_{3,3}^{-1} \) and \( E_2^{(1)} = (w_{2,1} - w_{1,1})(w_{2,2} - w_{2,1})u_{2,1}^{-1} + (w_{2,2} - w_{1,1})(w_{2,2} - w_{2,1})u_{2,2}^{-1} \), and \( F_2^{(1)} = (w_{2,1} - w_{3,1})(w_{2,1} - w_{3,2})u_{2,1}^{-1} + (w_{2,2} - w_{3,1})(w_{2,2} - w_{3,2})u_{2,2}^{-1} \), while \( F_1^{(1)} = (w_{3,1} - w_{2,1})(w_{3,2} - w_{2,1})u_{3,1}^{-1} + (w_{3,2} - w_{2,1})(w_{3,3} - w_{2,1})u_{3,2}^{-1} \) and \( E_1^{(1)} = u_{1,1}^{-1} \).

Now consider an auxiliary quiver gauge theory of type \( D_4 \) with 1-dimensional framing at the middle vertex numbered by 2, and the outer vertices numbered by \((3,1), (3,2), (3,3)\). We take \( \dim V_{3,1}'' = \dim V_{3,2}'' = \dim V_{3,3}'' = 1, \dim V_2'' = 2 \) (and \( \dim V_1'' = 1 \)). We know that \( \mathcal{M}_C(\text{GL}(V''), \mathbb{N}'') = \mathcal{N}_{\text{min}} \) is the closure of the minimal orbit in \( \mathfrak{so}_8 \). On the other hand, we consider a quiver gauge theory of the affine type \( \mathcal{D}_4 \) with the extra vertex numbered by 1, \( \dim V_1 = 1 \), and all the other dimensions as before, but no framing. We denote the corresponding graded vector space by \( V' = V'' \oplus V_1 \), and the corresponding representation of \( \text{GL}(V') \) by \( \mathbb{N}' \). Then \( \mathcal{M}_C(\text{GL}(V''), \mathbb{N}'') = \mathcal{M}_C(\text{PGL}(V'), \mathbb{N}') \). We have an embedding \( \mathbb{C}[\mathcal{M}_C(\text{PGL}(V'), \mathbb{N}')] \hookrightarrow \mathbb{C}(w_{1,1}, w_{2,1}, w_{2,2}, w_{3,2}, w_{3,3}, u_{1,1}, u_{2,1}, u_{2,2}, u_{3,1}, u_{3,2}, u_{3,3})^{S_2} \) where the symmetric group \( S_2 \) acts by permuting \((w_{2,1}, u_{2,1})\) and \((w_{2,2}, u_{2,2})\). Also we have an embedding \( \mathbb{C}[\mathcal{M}] \hookrightarrow \mathbb{C}(w_{1,1}, w_{2,1}, w_{2,2}, w_{3,2}, w_{3,3}, u_{1,1}, u_{2,1}, u_{2,2}, u_{3,1}, u_{3,2}, u_{3,3})^{S_2 \times S_3} \) where the symmetric group \( S_3 \) acts by permuting \((w_{3,1}, u_{3,1})\), \((w_{3,2}, u_{3,2})\), and \((w_{3,3}, u_{3,3})\). By inspection of \((Q.A.3), (Q.A.5)\), Theorem Q.B.18 we check \( F_2^{(1)} = F_2^{(1)}, E_2^{(1)} = E_2^{(1)}, F_3^{(1)} = T_{3,1}^{(1)} + T_{3,2}^{(1)} + F_{3,3}^{(1)}, E_3^{(1)} = E_3^{(1)} + T_{3,2}^{(1)} + E_{3,3}^{(1)} \) where \( E, F \) refer to the generators of \( \mathfrak{so}_8 \) in \( \mathbb{C}[\mathcal{M}_C(\text{PGL}(V'), \mathbb{N}') \), while \( E, F \) refer to the generators of \( \mathfrak{g}_2 \) in the previous paragraph. Since the projection \( \mathcal{M}_C(\text{PGL}(V'), \mathbb{N}') = \mathcal{N}_{\text{min}} \rightarrow \mathfrak{so}_8 \) is an embedding, we conclude that the projection \( \mathcal{M} \rightarrow \mathfrak{g}_2 \) is generically an embedding. Hence the differential of the \( G_2 \)-action on \( \mathcal{M} \) is generically surjective, so \( \mathcal{M} \) has an open \( G_2 \)-orbit \( \mathbb{O} \subset \mathcal{M} \) which is a nonramified cover of its image adjoint orbit \( \mathbb{O} \subset \mathfrak{g}_2 \).
Now the monopole formula for $\mathcal{M}$ gives degrees in $\mathbb{N}$. Indeed, the contribution of a dominant coweight $\lambda = (\lambda_{1,1}, \lambda_{2,1}) \geq (\lambda_{2,2}, \lambda_{3,1}) \geq (\lambda_{3,2}, \lambda_{3,3})$ of $(\text{GL}(1) \times \text{GL}(2) \times \text{GL}(3))/\mathbb{Z}$ equals $\Delta(\lambda) = \lambda_{2,2} - \lambda_{2,1} + \frac{1}{2}|\lambda_{2,1} - \lambda_{1,1}| + \frac{1}{2}|\lambda_{2,2} - \lambda_{1,1}| + \frac{1}{2} \sum_{r,s} |\lambda_{2,r} - \lambda_{3,s}|$ which is easily seen to be nonnegative and integral. We conclude that $\mathcal{M}$ is conical, and hence its image $\overline{\mathcal{O}} \subset \mathfrak{g}_2$ is conical as well. It follows that $\mathcal{O}$ is a nilpotent orbit. But $\mathfrak{g}_2$ has a unique 10-dimensional nilpotent orbit: the subregular one. Hence $\mathcal{O} = \mathcal{O}_{\text{subreg}}$.

Now we have to identify the cover $\overline{\mathcal{O}} \twoheadrightarrow \mathcal{O}$. It is known that the universal cover of $\mathcal{O}_{\text{subreg}}$ is an open piece of the minimal nilpotent orbit $\mathcal{O}_{\text{min}} \subset \mathfrak{so}_8$, see e.g. [BK94], and the Galois group of this cover is $S_3$. Moreover, the degree 1 functions on the universal cover constitute the Lie algebra $\mathfrak{so}_8$. It follows that if the cover $\overline{\mathcal{O}} \twoheadrightarrow \mathcal{O}$ corresponds to a subgroup $\pi_1 \subset S_3$, then the degree 1 functions on $\overline{\mathcal{O}}$ constitute $\mathfrak{so}_8^{\pi_1}$. Since we know that the degree 1 functions on $\mathcal{M}$ constitute $\mathfrak{g}_2 = \mathfrak{so}_8^{S_3}$, we conclude that $\pi_1 = S_3$, so that $\overline{\mathcal{O}} \twoheadrightarrow \mathcal{O}_{\text{subreg}}$. Finally, the normality property of the orbit closure $\overline{\mathcal{O}}_{\text{subreg}}$ guarantees that $\mathcal{M} \twoheadrightarrow \overline{\mathcal{O}}_{\text{subreg}}$.

Note that a torus with one puncture is obtained from $S^2$ with three punctures by gluing two punctures. We have computed the Higgs branch associated with the latter in §5(iv). The Higgs branch is the closure $\mathcal{N}_{\text{min}}(\mathfrak{e}_6)$ of the minimal nilpotent orbit of $\mathfrak{e}_6$. Therefore the Higgs branch $\mathcal{M}$ for a torus with one puncture is the Hamiltonian reduction $\mathcal{N}_{\text{min}}(\mathfrak{e}_6) \sslash \Delta_{\text{SL}(3)}$ with respect to the diagonal $\text{SL}(3)$ in $\text{SL}(3) \times \text{SL}(3)$ corresponding to two legs which are glued. Therefore we have an action of the centralizer of $\Delta_{\text{SL}(3)}$ in $E_6$, which is $G_2$. (See e.g., [Rub08, §3.2] and the references therein.) Combining with Proposition 5.6, we should have $\mathcal{N}_{\text{min}}(\mathfrak{e}_6) \sslash \Delta_{\text{SL}(3)} \cong \overline{\mathcal{O}}_{\text{subreg}}(\mathfrak{g}_2)$, the closure of the subregular nilpotent orbit of $\mathfrak{g}_2$. We do not have a proof of this statement, though it might be known to an expert.

5(vi). Recollections on derived Satake equivalence. We consider a reductive group $G$ with Langlands dual group $G^\vee$ and its Lie algebra $\mathfrak{g}^\vee$. We have a commutative ring object $A_R = \bigoplus \lambda \mathbb{IC}(\text{Gr}^\lambda) \otimes (V^\lambda)^* \subset D_G(\text{Gr}_G)$.

Let $e, h, f \in \mathfrak{g}^\vee$ be a principal $\mathfrak{sl}_2$-triple such that $f$ is lower triangular, and $e$ is upper triangular. We consider the Kostant slice $e + h(f)$ to the regular nilpotent orbit. Let $\Sigma$ be the image of $e + h(f)$ under a $G^\vee$-invariant isomorphism $\mathfrak{g}^\vee \simeq (\mathfrak{g}^\vee)^*$. Let $\Upsilon$ be the image of $e + h^\vee$. (Borel subalgebra) under a $G^\vee$-invariant isomorphism $\mathfrak{g}^\vee \simeq (\mathfrak{g}^\vee)^*$. We have canonical isomorphisms $\Sigma = t/\mathbb{W} = \Upsilon/\mathbb{U}_\vee$ (unipotent subgroup) by the compositions $\Sigma \hookrightarrow (\mathfrak{g}^\vee)^* \rightarrow (\mathfrak{g}^\vee)^*/G^\vee \cong t/\mathbb{W}$ and $\Sigma \hookrightarrow \Upsilon \rightarrow \Upsilon/\mathbb{U}_\vee$.

According to [BF08, Theorem 5], there is an equivalence of monoidal triangulated categories $\Psi: D^{G^\vee}(\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee)) \rightarrow D_G(\text{Gr}_G)$. Recall that $D_G(\text{Gr}_G)$ stands for the Ind-completion of the bounded derived equivariant constructible category on $\text{Gr}_G$. Accordingly, $D^{G^\vee}(\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee))$ stands for the Ind-completion of the triangulated category $D^{G^\vee}_{\text{perf}}(\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee))$ formed by the $G^\vee$-equivariant perfect dg-modules over $\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee)$: the graded symmetric algebra of $\mathfrak{g}^\vee$ where any element of $\mathfrak{g}^\vee$ is assigned degree 2 (with trivial differential). The monoidal structure on $D_G(\text{Gr}_G)$ is given by the convolution $\ast$, and the monoidal structure on $D^{G^\vee}(\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee))$ is $M_1, M_2 \mapsto M_1 \otimes_{\text{Sym}^\mathbb{I}(\mathfrak{g}^\vee)} M_2$. The algebra $\mathbb{C}[\Sigma] = \text{Sym}(\mathfrak{g}^\vee)^{G^\vee}$ acts on
we have a canonical isomorphism
Lemma 5.8.

Proof. For any group $H$, the convolution operation $\mathcal{F}_1 \ast \mathcal{F}_2 = m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ on $D(H)$ has rigidity $\mathcal{F} \mapsto \mathcal{C}_H \circ D\mathcal{F}$ where $\mathcal{C}_H$ is induced by the automorphism $h \mapsto h^{-1}, H \mapsto H$. Namely, $\text{RHom}(1_H, \mathcal{F}_1 \ast \mathcal{F}_2) = \mathcal{F}_1(\mathcal{C}_H, \nabla^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)) = \text{RHom}(\mathcal{C}_H, \mathcal{C}_H \mathcal{F}_1 \boxtimes \mathcal{F}_2)$, where $\nabla : H \mapsto H \times H$ is the antidiagonal embedding $h \mapsto (h^{-1}, h)$.

We apply this to the category of $G_{\mathbb{Q}}$-left-right equivariant sheaves on $H = G_K$.

More formally, let us use the six operations for constructible derived categories on Artin stacks. There is a reference [LO08] for $\mathbb{Q}_r$-coefficients. We choose an isomorphism $\mathbb{Q}_r \cong \mathbb{C}$ and use it for complex coefficients. Our stack is $\mathcal{X} := G_{G \setminus G_{\mathbb{Q}}}$. It is the moduli stack of pairs $\mathcal{P}_1, \mathcal{P}_2$ of $G$-bundles on the formal disc $D$ equipped with an isomorphism $\eta : \mathcal{P}_1|_{D^*} \sim \mathcal{P}_2|_{D^*}$. There is an involution $i : \mathcal{X} \to \mathcal{X}$ induced by the inversion $g \mapsto g^{-1}$ of $G_K$. In modular terms, $i(\mathcal{P}_1, \mathcal{P}_2, \eta) = (\mathcal{P}_2, \mathcal{P}_1, \eta^{-1})$. Recall that $G_{\mathbb{R}}$ is the moduli space of $G$-bundles $\mathcal{P}$ on $D$ equipped with an isomorphism $\sigma : \mathcal{P}_{\text{triv}}|_{D^*} \sim \mathcal{P}|_{D^*}$. We have a projection $\text{pr}_2 : G_{\mathbb{R}} \to \mathcal{X}$ sending $(\mathcal{P}, \sigma)$ to $(\mathcal{P}_{\text{triv}}, \mathcal{P}, \sigma)$. Similarly, we define $G_{G_{\mathbb{Q}}}$ as the moduli space
of $G$-bundles $\mathcal{P}$ on $D$ equipped with an isomorphism $\tau: \mathcal{P}|_{D^*} \xrightarrow{\cong} \mathcal{P}_{\text{triv}}|_{D^*}$. We have a projection $\text{pr}_1 : \mathfrak{g}\mathfrak{r} G \to \mathcal{X}$ sending $(\mathcal{P}, \tau)$ to $(\mathcal{P}, \mathcal{P}_{\text{triv}}, \tau)$. We have isomorphisms $\iota: \text{Gr}_G \xrightarrow{\cong} \mathfrak{g}\mathfrak{r} G$ and $\iota: \mathfrak{g}\mathfrak{r} G \xrightarrow{\cong} \text{Gr}_G$ sending $\sigma$ to $\tau = \sigma^{-1}$ and $\sigma$ to $\tau = \sigma^{-1}$. Obviously, $\iota \text{pr}_1 = \text{pr}_2 \iota$ and $\iota \text{pr}_2 = \text{pr}_1 \iota$. There is a morphism $m: \mathfrak{g}\mathfrak{r} G \times \text{Gr}_G \to \mathcal{X}$ induced by the multiplication in $G_K$. In modular terms, $m(\mathfrak{p}_1, \tau; \mathfrak{p}_2, \sigma) = (\mathfrak{p}_1, \mathfrak{p}_2, \sigma \circ \tau)$. The convolution on $D(\mathcal{X})$ is defined as $\mathcal{F}_1 \ast \mathcal{F}_2 := m_*(\text{pr}_1^* \mathcal{F}_1 \boxtimes \text{pr}_2^* \mathcal{F}_2)$. The unit object $1$ is $\mathbb{C}_{\text{Gr}_G \setminus \text{Gr}_G}$. We claim that the rigidity is $i^* \circ \mathcal{D}$. Indeed, $\text{RHom}_X(1, \mathcal{F}_1 \ast \mathcal{F}_2) = i_0^!(\mathcal{F}_1 \ast \mathcal{F}_2) = \text{RHom}_{\mathbb{C}_{\text{Gr}_G \setminus \text{Gr}_G}}(\mathbb{C}_{\text{Gr}_G}, \nabla^!(\text{pr}_1^* \mathcal{F}_1 \boxtimes \text{pr}_2^* \mathcal{F}_2)) = \text{RHom}_{\mathbb{C}_{\text{Gr}_G \setminus \text{Gr}_G}}(\mathbb{C}_{\text{Gr}_G}, \iota^* \text{pr}_1^* \mathcal{F}_1 \boxtimes \text{pr}_2^* \mathcal{F}_2) = \text{RHom}_{\mathbb{C}_{\text{Gr}_G \setminus \text{Gr}_G}}((\mathbb{C}_{\text{Gr}_G}, (\text{pr}_1^* \mathcal{F}_1, \text{pr}_2^* \mathcal{F}_2) = \text{RHom}_{\mathbb{C}_{\text{Gr}_G \setminus \text{Gr}_G}}(\iota^* \circ \mathcal{D} \text{pr}_1^* \mathcal{F}_1, \text{pr}_2^* \mathcal{F}_2) = \text{RHom}_{X}(\iota^* \circ \mathcal{D} \mathcal{F}_1, \mathcal{F}_2)$, where $\nabla: \text{Gr}_G \to \mathfrak{g}\mathfrak{r} G \times \text{Gr}_G$ is $(i, \text{id})$. $\square$

5(vii). **Regular sheaf and derived Satake equivalence.** Under the equivalence $\Psi^{-1}$, the ring object $\mathcal{A}_R \in D_G(\text{Gr}_G)$ corresponds to the $G^\vee$-equivariant free $\text{Sym}^\bullet(\mathcal{g}^\vee)$-module $\mathbb{C}[G^\vee] \otimes \text{Sym}^\bullet(\mathcal{g}^\vee)$ which will be denoted $\mathbb{C}[T^*G^\vee]$. The $G^\vee$-action comes from the left action of $G^\vee$ on $T^*G^\vee = G^\vee \times (\mathcal{g}^\vee)^*$, $g_1 \cdot (g_2, \xi) = (g_1 g_2, \xi)$. And the action of $\text{Sym}^\bullet(\mathcal{g}^\vee)$ on $\mathbb{C}[T^*G^\vee]$ comes from the morphism $\mu_1: T^*G^\vee \to (\mathcal{g}^\vee)^*$, $(g, \xi) \mapsto \text{Ad}_g \xi$ (the moment map of the left action). Recall that $\mathcal{A}_R$ is equipped with an action of $G^\vee$. Under the equivalence $\Psi^{-1}$, this action goes to the action on $\mathbb{C}[T^*G^\vee]$ coming from the right action of $G^\vee$ on $T^*G^\vee = G^\vee \times (\mathcal{g}^\vee)^*$, $g_1 \cdot (g_2, \xi) = (g_2 g_1^{-1}, \text{Ad}_{g_1} \xi)$. For this reason the action of $G^\vee$ on $\mathcal{A}_R$ will be called the right action. The moment map of the right $G^\vee$-action on $T^*G^\vee$ is $\mu_r: T^*G^\vee \to (\mathcal{g}^\vee)^*, (g, \xi) \mapsto \xi$.

Also note that $\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)$ is a formal dg-algebra (since e.g. $\mathbb{C}[T^*G^\vee]$ is a free $\text{Sym}^\bullet(\mathcal{g}^\vee)$-module), so $\mu_r$ gives rise to a $G^\vee$-equivariant morphism of dg-algebras

$$(5.9) \quad \text{Sym}^\bullet(\mathcal{g}^\vee) \to \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R).$$

Altogether we have the action of $G^\vee \times \text{Sym}^\bullet(\mathcal{g}^\vee)$ on $\mathcal{A}_R$ that will be called the right action.

**Remark 5.10.** For $\mathcal{F}_1, \mathcal{F}_2 \in D_G(\text{Gr}_G)$ we distinguish $\text{Ext}^*_D(\mathcal{F}_1, \mathcal{F}_2)$ and $\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{F}_1, \mathcal{F}_2)$. They are isomorphic in $\mathcal{D}(\text{Vect})$, the derived category of vector spaces, which is equivalent to $\mathcal{D}(\mathcal{g}^\text{gr})$, the category of graded vector spaces. But when we consider additional structures, such as a dg-algebra structure or a structure of a dg-module over $G^\vee \times \text{Sym}^\bullet(\mathcal{g}^\vee)$, they are not isomorphic. We thus understand $\text{Ext}^*_D(\mathcal{F}_1, \mathcal{F}_2) = H^*(\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{F}_1, \mathcal{F}_2))$.

**Definition 5.11.** The morphism $\text{(5.9)} \quad \text{Sym}^\bullet(\mathcal{g}^\vee) \to \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)$ induces, for any $\mathcal{F} \in D_G(\text{Gr}_G)$, the composed morphism

$$\text{Sym}^\bullet(\mathcal{g}^\vee) \to \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R) \to \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R \otimes^\mathbb{L} \mathcal{F}, \mathcal{A}_R \otimes \mathcal{F})$$

of dg-algebras. Also, the morphism $\text{(5.9)}$ induces, for any $\mathcal{F} \in D_G(\text{Gr}_G)$, the composed morphism

$$\text{Sym}^\bullet(\mathcal{g}^\vee) \otimes \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R \otimes \mathcal{F}) \to \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R) \otimes \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R \otimes \mathcal{F})$$
of complexes of vector spaces. This morphism is $G^\vee$-equivariant for the $G^\vee$-action on $\text{RHom}_{D_C(Gr_G)}(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \mathcal{F})$ induced by the right $G^\vee$-action on $\mathcal{A}_R$. Thus, the complex $\text{RHom}_{D_C(Gr_G)}(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \mathcal{F})$ acquires the structure of an object of $D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee))$, and $\text{Ext}^*_D(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \bullet)$ gets upgraded to the functor

$$\text{RHom}_{D_C(Gr_G)}(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \bullet): D_C(Gr_G) \to D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee)).$$

Similarly, $\text{RHom}_{D_C(Gr_G)}(\mathbb{D} \mathcal{A}_R, \mathcal{F}), \text{RHom}_{D_C(Gr_G)}(1_{Gr_G}, \mathbb{D} \mathcal{A}_R \star \mathcal{F}), \text{RHom}_{D_C(Gr_G)}(\mathcal{A}_R, \mathcal{F})$, etc. all acquire the structures of objects of $D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee))$, and $\text{Ext}^*_D(\mathcal{A}_R, \bullet)$ gets upgraded to the functor $\text{RHom}_{D_C(Gr_G)}(\mathcal{A}_R, \bullet): D_C(Gr_G) \to D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee))$.

**Lemma 5.12.** From Definition 5.11 we obtain an action of $(G^\vee)^b \times \text{Sym}^\bullet(\mathfrak{g}^\vee)\otimes \mathfrak{h}$ on $\text{RHom}_{D_C(Gr_G)}(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \mathfrak{h})$. The resulting dg-module over $(G^\vee)^b \times \text{Sym}^\bullet(\mathfrak{g}^\vee)\otimes \mathfrak{h}$ is formal.

**Proof.** Using [BF08, Proposition 5] we reformulate the claim for $D_C(Gr_G)$ replaced by $D_{Gr_G}(Gr_G)$. Then $\mathcal{A}_R$ is pointwise pure (meaning that all its costalks are pure with respect to the Frobenius action). Hence $\mathcal{A}_R \otimes \mathfrak{h}$ is also pointwise pure. Then the Cousin spectral sequence for the Schubert stratification of $Gr_G$ shows that $H^*_D(Gr_G, \mathcal{A}_R \otimes \mathfrak{h})$ is pure. Also, $\text{Ext}^*_D(Gr_G, \mathcal{A}_R, \mathcal{A}_R)$ is pure. Now the argument of [BF08, Section 6.5] proves that the dg-algebra $\text{RHom}_{D_C(Gr_G)}(\mathcal{A}_R, \mathcal{A}_R)$ is formal as well as its $b$-th tensor power, and the dg-module $\text{RHom}_{D_C(Gr_G)}(\mathcal{C}_{Gr_G}, \mathcal{A}_R \otimes \mathfrak{h})$ over $\text{RHom}_{D_C(Gr_G)}(\mathcal{A}_R, \mathcal{A}_R) \otimes \mathfrak{h}$ is formal. □

The Kostant-Whittaker (hamiltonian) reduction of $T^*G^\vee$ with respect to the right action is $T^*G^\vee/_{U^\vee, \psi} := \mu^\psi(\mathcal{Y})/U^\vee$. (We use $/\!/$ for a hamiltonian reduction in order to avoid a conflict with $/\!/$ for a GIT quotient.) At the level of dg-modules, $\kappa^\psi(\mathbb{C}[T^*G^\vee][\mathfrak{h}]) := (\mathbb{C}[\mu^\psi(\mathcal{Y})][\mathfrak{h}])/U^\vee = \mathbb{C}[\mu^\psi(\mathcal{Y})][\mathfrak{h}] := \mathbb{C}[T^*G^\vee][\mathfrak{h}] \otimes \text{Sym}^\bullet(\mathfrak{g}^\vee)[\mathcal{C}[\mathcal{Y}][\mathfrak{h}]$, tensor product with respect to the action of the right copy of $\text{Sym}^\bullet(\mathfrak{g}^\vee)$.\(^6\) (We have an isomorphism $U^\vee \times \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}$ given by the action of $U^\vee$ on $\mathcal{Y}$. Hence $\mathbb{C}[\mathcal{Y}][\mathfrak{h}] \cong \mathbb{C}[\mathcal{Y}]$. Moreover, for any $U^\vee$-equivariant sheaf $\mathcal{F}$ on $\mathcal{Y}$, we have $\mathcal{F}/_{U^\vee} = \mathcal{F}|\mathcal{Y}$. Similarly, for a $U^\vee$-equivariant dg-module $M$ over $\mathbb{C}[\mathcal{Y}][\mathfrak{h}]$ we have $M/U^\vee = M \otimes \mathbb{C}[\mathcal{Y}][\mathfrak{h}]$. This is an object of $D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee))$ (with respect to the residual left action of $G^\vee$) corresponding under the equivalence $\Psi^{-1}$ to the dualizing complex $\omega_{Gr_G}$ [BF08, Proposition 4]. (In fact, [BF08, Proposition 4] is proved for the extra equivariance under the loop rotations.) Instead of $\omega_{Gr_G} = \Psi(\kappa^\psi(\Psi^{-1} \mathcal{A}_R))$, we will write $\omega_{Gr_G} = \kappa^\psi(\mathcal{A}_R)$ for short.

Under the dualities $D, \mathbb{D}$ we have $\mathbb{D} \mathbb{C}[T^*G^\vee][\mathfrak{h}] = \mathbb{C}[T^*G^\vee][\mathfrak{h}]$, while $\mathbb{D} \mathcal{A}_R = \mathcal{C}_G\mathcal{A}_R$.

We define $\Phi := \mathcal{C}_G \circ \Psi^{-1}: D_C(Gr_G) \to D^{G^\vee}(\text{Sym}^\bullet(\mathfrak{g}^\vee))$. We have $\Phi(\mathcal{A}_R) = \mathbb{C}[T^*G^\vee][\mathfrak{h}]$.\(^6\)Passing to cohomology, we obtain the usual hamiltonian reduction: $H^*(\kappa^\psi(\mathbb{C}[T^*G^\vee][\mathfrak{h}])) = \mathbb{C}[T^*G^\vee] \otimes \text{Sym}^\bullet(\mathfrak{g}^\vee) \mathbb{C}[\mathcal{Y}] =: \kappa^\psi(\mathbb{C}[T^*G^\vee])$. We use the same notation $\kappa^\psi$ for the hamiltonian reduction of the usual modules and of dg-modules. It is clear from the context which one is used in what follows.
Lemma 5.13. (a) Let us define a functor $\text{Ext}^*_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \bullet): D_G(\text{Gr}_G) \to D(\text{Vect}) = \text{Vect}^{gr}$ by

$$\text{Ext}^*_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{F}) := \bigoplus_\lambda \text{Ext}^*_{D_G(\text{Gr}_G)}(\text{IC}(\mathcal{G}^\lambda), \otimes (V^\lambda)^*, \mathcal{F}).$$

It is canonically isomorphic to $\text{Ext}^*_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \mathcal{A}_R \bullet \bullet)$ by the rigidity together with $C_G D A_R = \mathcal{A}_R$ (see Lemma 5.8). Then both $\text{Ext}^*_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \bullet)$ and $\text{Ext}^*_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \mathcal{A}_R \bullet \bullet)$ are canonically isomorphic to the composition $\text{Forg} \circ \Phi^{-1}$, where $\text{Forg}$ is the forgetful functor $D^G_* (\text{Sym}^{\underline{n}}(\mathfrak{g}^*)) \to D(\text{Vect}) = \text{Vect}^{gr}$.

Their upgrades $\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \bullet)$ and $\text{RHom}_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \mathcal{A}_R \bullet \bullet): D_G(\text{Gr}_G) \to D^G_*(\text{Sym}^{\underline{n}}(\mathfrak{g}^*))$ (see Definition 5.11) are canonically isomorphic to $\Psi^{-1}$.

(b) For $\mathcal{F}_1, \mathcal{F}_2 \in D_G(\text{Gr}_G)$, there are canonical isomorphisms $H_G^0(\text{Gr}_G, \mathcal{F}_1 \otimes \mathcal{F}_2) = \text{Ext}^*_{D_G(\text{Gr}_G)}(\text{C}_G, \mathcal{F}_1 \otimes \mathcal{F}_2) \cong \text{Ext}^*_{D_G(\text{Gr}_G)}(\text{D}_1, \mathcal{F}_1, \mathcal{F}_2) \cong \text{Ext}^*_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \text{C}_G \mathcal{F}_1 \bullet \mathcal{F}_2)$ in $D(\text{Vect}) = \text{Vect}^{gr}$.

(c) Let us define a functor $\text{Ext}^*_{D_G(\text{Gr}_G)}(\mathbb{D} \mathcal{A}_R, \bullet): D_G(\text{Gr}_G) \to D(\text{Vect}) = \text{Vect}^{gr}$ by

$$\text{Ext}^*_{D_G(\text{Gr}_G)}(\mathbb{D} \mathcal{A}_R, \mathcal{F}) := \bigoplus_\lambda \text{Ext}^*_{D_G(\text{Gr}_G)}(\text{IC}(\mathcal{G}^\lambda), \otimes V^\lambda, \mathcal{F}).$$

It is canonically isomorphic to $\text{Ext}^*_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \text{C}_G \mathcal{A}_R \bullet \bullet)$ by the rigidity together with $C_G \mathcal{A}_R = \mathbb{D} \mathcal{A}_R$ (see Lemma 5.8). We have canonical isomorphisms

F $\Phi(\bullet) \cong (\text{Ext}^*_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \text{C}_G \mathcal{A}_R \bullet \bullet) \cong (\text{Ext}^*_{D_G(\text{Gr}_G)})(\mathbb{D} \mathcal{A}_R, \bullet)$

and

Thus, the upgraded functors (see Definition 5.11)

$\Phi \cong \text{RHom}_{D_G(\text{Gr}_G)}(1_{\text{Gr}_G}, \text{C}_G \mathcal{A}_R \bullet \bullet)$

$\cong \text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{D} \mathcal{A}_R, \bullet)$

are isomorphic as functors from $D_G(\text{Gr}_G)$ to $D^G_*(\text{Sym}^{\underline{n}}(\mathfrak{g}^*))$.

Proof. (a) We consider $\mathbb{C}[T^*G^\vee]^{\underline{n}}$ as a $G^\vee \times G^\vee$-equivariant $(\text{Sym}^{\underline{n}}(\mathfrak{g}^*))$ bimodule by the left and right action. We have the canonical matrix coefficient morphisms

$$M \xrightarrow{\varphi_M} \text{RHom}_{D^G_*(\text{Sym}^{\underline{n}}(\mathfrak{g}^*))}(\text{Sym}^{\underline{n}}(\mathfrak{g}^*), \mathbb{C}[T^*G^\vee]^{\underline{n}} \otimes M),$$

where $G^\vee \times (\text{Sym}^{\underline{n}}(\mathfrak{g}^*))$ acts on $\mathbb{C}[T^*G^\vee]^{\underline{n}}$ by the left action, and the right hand side is regarded as an object in $D^G_*(\text{Sym}^{\underline{n}}(\mathfrak{g}^*))$ by the residual right action on $\mathbb{C}[T^*G^\vee]^{\underline{n}}$. Here $\varphi_M$ is defined as follows: Given a $G^\vee$-module $M$, we have $\varphi_M: M \otimes M^* \to \mathbb{C}[G^\vee]$ by $\varphi_M(m \otimes m^*)(g) := \langle gm, m^* \rangle$. It is a morphism of $G^\vee \times G^\vee$-modules. By swapping $M^*$ to the target, $\varphi_M$ can be viewed as a morphism $\varphi_M: M \to \text{Hom}_{G^\vee}(C, \mathbb{C}[G^\vee] \otimes M)$. The morphism $\varphi_M$ is an isomorphism. One can think about it as the usual fiber functor $\text{Forg}$ on $\text{Rep}(G^\vee)$ being represented by $\mathbb{C}[G^\vee]$. Its inverse $\varphi_M^{-1}$ is given by the evaluation
Be a \( G \)-valued \( \mathbb{C} \)-scheme. We consider this over \( \text{Sym}[G'] \) to get (5.14). We now apply the derived Satake equivalence. We get

\[ \Psi^{-1}(\mathcal{F}) \xrightarrow{\varphi_{\Psi^{-1}(\mathcal{F})}} \text{RHom}_{D^G(G')}([\text{Sym}[G']], \mathbb{C}[T^*G'] \otimes \Psi^{-1}(\mathcal{F})) \]

\[ \xrightarrow{\sim} \text{RHom}_{D_G(G)}(\mathbb{C}[G], \mathcal{A}_R \otimes \mathcal{F}). \]

The second isomorphism holds since the construction of \( \Psi^{-1} \) actually passes through dg-categories (as opposed to being defined at the level of derived categories), and is compatible with the action of the formal dg-algebra \( \text{Sym}[G'] \). We have an isomorphism \( \text{RHom}_{D_G(G)}(\mathbb{C}[G], \mathcal{A}_R \otimes \mathcal{F}) \) by the rigidity plus \( \mathcal{A}_R = \mathcal{C}_G \circ \mathbb{D}_R \).

(b) The first isomorphism is the rigidity for the monoidal category \((D_G(G), \otimes')\), while the second is for \((D_G(G), \otimes)\). See Lemma 5.8.

(c) The first isomorphism is a consequence of (a) together with \( \mathbb{D}_R = \mathcal{C}_G \mathcal{A}_R \). The second and third are nothing but (b). In order to see that the second and third isomorphisms are upgraded to \( \text{RHom} \), we observe that quasi-isomorphisms

\[ \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \xrightarrow{\sim} \text{RHom}_{D_G(G)}(\mathbb{D}_R, \mathcal{F}) \xrightarrow{\sim} \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \]

are compatible with the action of \( G' \times \text{Sym}[G'] \).

Let us suppose \( \mathcal{F} \in D_G(G) \) is a ring object, i.e. it is equipped with a commutative multiplication homomorphism \( m_{\mathcal{F}}: \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \). Then \( \Psi^{-1}(\mathcal{F}) \in D^G(G') \) is also a ring object, i.e. it is equipped with \( \Psi^{-1}(m_{\mathcal{F}}): \Psi^{-1}(\mathcal{F}) \otimes \text{Sym}[G'] \to \Psi^{-1}(\mathcal{F}) \). The same is true for \( \Phi \). On the other hand, \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, A' \otimes \mathcal{F}) = \text{RHom}_{D_G(G)}(\mathcal{F}, \mathcal{F}) \) in Lemma 5.13(b) is equipped with a multiplication by \( m_{\mathcal{F}} \) and \( m_{\mathcal{A}_R} \) (equivalently, a coproduct \( C_G \mathbb{D}_m_{\mathcal{A}_R} : \mathcal{A}_R \to \mathcal{A}_R \otimes \mathcal{A}_R \)). Similarly, \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathbb{C}_G \mathcal{A}_R \otimes \mathcal{F}) = \text{RHom}_{D_G(G)}(\mathbb{D}_R, \mathcal{F}) \) in Lemma 5.13(c) is equipped with a multiplication by \( m_{\mathcal{F}} \) and \( C_G \mathbb{D}_m_{\mathcal{A}_R} : C_G \mathcal{A}_R \otimes \mathcal{A}_R \to C_G \mathcal{A}_R \) (equivalently, a coproduct \( \mathbb{D}_m_{\mathcal{A}_R} : \mathcal{A}_R \to \mathcal{A}_R \otimes \mathcal{A}_R \)). Finally, a multiplication on \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \) is defined as in Proposition 2.22.

**Proposition 5.15.** (a) Multiplications on \( \Psi^{-1}(\mathcal{F}) \), \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \) and \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \) are equal under the isomorphism in Lemma 5.13(a).

(b) The same is true for \( \Phi(\mathcal{F}) \), \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \), \( \text{RHom}_{D_G(G)}(\mathbb{D}_R, \mathcal{F}) \), and \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \), under the isomorphisms of Lemma 5.13(c).

**Proof.** (a) The isomorphism \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathcal{A}_R \otimes \mathcal{F}) \) is given by the rigidity, and respects the multiplication by definition. Therefore it is enough to check the compatibility under the isomorphism \( \Psi^{-1}(\mathcal{F}) \) in (5.14) under the derived Satake equivalence. Therefore it is enough to check that \( \varphi_M \) respects the multiplication when \( M \) is an algebra in the category \( D^G(G') \). This is trivial as \( \varphi_M^{-1} \) is given by the evaluation \( e_1: \mathbb{C}[G'] \to \mathbb{C} \).

(b) Applying (a) to \( \mathcal{F} \), we see that the multiplications on \( \Phi(\mathcal{F}) \) and \( \text{RHom}_{D_G(G)}(\mathbb{C}_G, \mathbb{C}_G \mathcal{A}_R \otimes \mathcal{F}) = \text{RHom}_{D_G(G)}(\mathbb{D}_R, \mathcal{F}) \) are equal under the isomorphisms of Lemma 5.13(c).
It remains to compare them with the multiplication on \( \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, A_R \otimes^I F) \) defined in Proposition 2.22 as the composition

\[
\mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, A_R \otimes^I F) \otimes \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, A_R \otimes^I F) \rightarrow \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, (A_R \otimes^I F) \star (A_R \otimes^I F)) \rightarrow \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, A_R \otimes^I F).
\]

(2.23)

Note that \( \mathcal{RHom}_{DG(G)}(\mathbb{D}A_R, F) \cong \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, \mathcal{RHom}(\mathbb{D}A_R, F)) \). Recall that the convolution product \( \star \) is defined as \( m_\ast(q^*)^{-1}p^* \), see (2.2), where we omit \( \neg \) for brevity. Since \( p \) and \( q \) are smooth with fibers both \( G_\circ \), we have \( (q^*)^{-1}p^* = (q^\dagger)^{-1}p^\dagger \). By [KS90, (2.6.24)] for \( m_\ast \) and [KS90, Prop. 3.1.13] for \( p^\dagger, q^\dagger \), we have a natural homomorphism

(5.16) \( \mathcal{RHom}(\mathbb{D}A_R, F) \rightarrow \mathcal{RHom}(\mathbb{D}A_R \ast \mathbb{D}A_R, F \star F) \).

Hence we have the multiplication on \( \mathcal{RHom}_{DG(G)}(\mathbb{C}_{Gr}, \mathcal{RHom}(\mathbb{D}A_R, F)) \) by \( m_F \) and \( \mathbb{D}m_{A_R} \). Then the isomorphism \( \mathcal{RHom}_{DG(G)}(\mathbb{D}A_R, F) \cong \mathcal{RHom}(\mathbb{D}A_R, F) \) is compatible with the multiplication.

Now our remaining task is to check that the isomorphism \( A_R \otimes^I F \cong \mathcal{RHom}(\mathbb{D}A_R, F) \) (see [KS90, Prop. 3.4.6]) is compatible with the multiplication. Since the following diagram commutes:

\[
\begin{array}{ccc}
(A_R \otimes^I F) \otimes (A_R \otimes^I F) & \xrightarrow{m_{A_R} \otimes m_F} & A_R \otimes^I F \\
\downarrow & & \downarrow \\
\mathcal{RHom}(\mathbb{D}A_R \star \mathbb{D}A_R, F \star F) & \xrightarrow{m_F \circ \mathbb{D}m_{A_R}} & \mathcal{RHom}(\mathbb{D}A_R, F),
\end{array}
\]

the proof is reduced to the commutativity of

(5.17) \( \mathcal{RHom}(\mathbb{D}A_R, F) \ast \mathcal{RHom}(\mathbb{D}A_R, F) \rightarrow \mathcal{RHom}(\mathbb{D}A_R \ast \mathbb{D}A_R, F \star F) \).

Recall that horizontal arrows in (5.17) are defined as composite of homomorphisms for \( p^\dagger, q^\dagger \), \( m_\ast \) under \( \star = m_\ast(q^\dagger)^{-1}p^\dagger \). Thus the commutativity follows from compatibilities of homomorphisms for \( p^\dagger, q^\dagger \), \( m_\ast \) under the isomorphism [KS90, Prop. 3.4.6]. We leave the reader to check the detail.

\( \square \)

5(viii). Hamiltonian reduction. The right Kostant-Whittaker reduction of \( A_R = \Psi(\mathbb{C}[T^*G^\vee]^I) \) equipped with \( G^\vee \)-action is a particular case of the following construction.

Let \( \mathfrak{g} \) be a commutative ring object of \( D^{G^\vee}(\text{Sym}^I(\mathfrak{g}^\vee)) \) equipped with an action of an algebraic group \( H \). Let \( \mathfrak{h} \) be the Lie algebra of \( H \). Let \( \text{Sym}^I(\mathfrak{h}) \) be the symmetric algebra of \( \mathfrak{h} \) equipped with a nonnegative grading (not necessarily the standard one, nor the doubled standard one) and viewed as a dg-algebra with a trivial differential. Let \( \mu^\ast : \text{Sym}^I(\mathfrak{h}) \rightarrow \).
\[ \text{RHom}_D G^*(\text{Sym}^\bullet(g^*)) (\mathcal{G}, \mathcal{G}) \] be an \( H \)-equivariant homomorphism of dg-algebras such that the multiplication morphism \( m : \mathcal{G} \otimes_{\text{Sym}^\bullet(g^*)} \mathcal{G} \to \mathcal{G} \) is \( H \)-equivariant and \( \text{Sym}^\bullet(h) \)-linear.

In all the examples below the following property holds: after applying the forgetful functor and taking cohomology and their spectrum, \( \text{Spec } H^*(\text{Forg } \mathcal{G}) \) is equipped with an \( H \)-invariant symplectic form, and \( \mu^* \) is compatible with a moment map \( \mu : \text{Spec } H^*(\text{Forg } \mathcal{G}) \to \mathfrak{h}^* \).

Given an \( H \)-invariant subvariety \( X \subset \mathfrak{h}^* \) such that the projection \( \text{Sym}(\mathfrak{h}) \to \mathbb{C}[X] \) is compatible with the grading \( \text{Sym}^\bullet(\mathfrak{h}) \) and induces the grading \( \mathbb{C}[X]^\bullet \), we define \( \mathcal{G} \parallel (H, X) := (\mathcal{G} \otimes_{\text{Sym}^\bullet(\mathfrak{h})} \mathbb{C}[X]^\bullet)^H \). This is a commutative ring object of \( D^G (\text{Sym}^\bullet(g^*)) \). If \( X = \{ 0 \} \subset \mathfrak{h}^* \), we simply write \( \mathcal{G} \parallel H \) for \( \mathcal{G} \parallel (H, \{ 0 \}) \).

5(ix). Leg amputation. Following Proposition 2.22, we consider a commutative ring object \( \mathcal{A}^b := i^b_\Delta(\boxtimes_{k=1}^b (\mathcal{A}_R)_k) \) (in particular, the ring object associated with \( S^2 \) with three punctures is \( \mathcal{A}^3 \) in our present notation). According to \( \S A, \mathcal{A}^b \) is equipped with an action of \( \text{SL}(N)^b = (G^*)^b \). More generally, we consider a commutative ring object \( \mathcal{A}^b := i^b_\Delta(\boxtimes_{k=1}^b (\mathcal{A}_R)_k) \) on \( \text{Gr}_G \) equipped with an action of \( (G^*)^b \) for a reductive flavor group \( G \).

We set \( W_G := \text{Spec } H^*_{\text{Gr}_G} (\text{Gr}_G, \mathcal{A}^b) \). We conjecture that \( H^*_{\text{Gr}_G} (\text{Gr}_G, \mathcal{A}^b) \) is finitely generated, which we checked so far only in type \( A \). We assume it hereafter. Then \( W_G \) is an affine variety with Poisson structure equipped with a hamiltonian action of \( (G^*)^b \). In particular, \( W_G^3 \) is \( W \) of the beginning of current \( \S 5 \).

Also, \( W_G^3 = T^*G^* \) since \( \mathbb{C}[W_G^3] = \text{Forg } \circ \Phi (\mathcal{A}_R) = \mathbb{C}[T^*G^*] \), see Lemma 5.13(c).

According to Definition 5.11, we have the action of \( b \) copies of \( \text{Sym}^\bullet(g^*) \) on \( \mathcal{A}^b \). We can consider its Kostant-Whittaker reduction \( \kappa_b^* (\mathcal{A}^b) = \mathcal{A}^b \otimes_{\text{Sym}^\bullet(\text{new}(g^* ))} \mathbb{C}[\Sigma]^\bullet \) with respect to the last copy of \( G^* \) in \( (G^*)^b \) (cf. \( \S 5(viii) \)). More precisely we apply \( \Psi \) to \( \kappa_b^* (\Psi^{-1} \mathcal{A}^b) = (\Psi^{-1} \mathcal{A}^b) \otimes_{\text{Sym}^\bullet(\text{new}(g^* ))} \mathbb{C}[\Sigma]^\bullet \).

Lemma 5.18. \( \kappa_b^* (\mathcal{A}^b) = \mathcal{A}^{b-1} \).

**Proof.** We have \( \kappa_b^* (\mathcal{A}^b) = \kappa_b^* (i^b_\Delta(\boxtimes_{k=1}^b (\mathcal{A}_R)_k)) = \kappa_b^* (i_\Delta^b (\mathcal{A}^{b-1} \boxtimes \mathcal{A}_R)) = i_\Delta^b (\mathcal{A}^{b-1} \boxtimes \kappa^* (\mathcal{A}_R)) = i_\Delta^b (\mathcal{A}^{b-1} \boxtimes \omega_{\text{Gr}_G}) = \mathcal{A}^{b-1} \). Indeed, \( \kappa^*_b (\bullet) = \bullet \otimes_{\text{Sym}^\bullet(\text{new}(g^*))} \mathbb{C}[\Sigma]^\bullet \) (with respect to the action of the \( b \)-th copy of \( \text{Sym}^\bullet(\text{new}(g^*)) \)). In the third equality we use that for \( \mathcal{F} = \mathcal{A}^{b-1} \in D_G (\text{Gr}_G) \) and \( \mathcal{F}' = \mathcal{A}_R \in D_G (\text{Gr}_G) \) with a dg-algebra \( A = \text{Sym}^\bullet(g^*) \) equipped with a homomorphism to \( \text{RHom}_{D_G (\text{Gr}_G)} (\mathcal{F}', \mathcal{F}') \), and a dg-module \( M = \mathbb{C}[\Sigma]^\bullet \) over \( A \), we have \( (\mathcal{F} \otimes \mathcal{F}') \otimes_A M = \mathcal{F} \otimes (\mathcal{F}' \otimes_M M) \) by associativity of tensor product. This equality is compatible with the commutative ring structures by the construction in Proposition 2.22 (the reduction \( \kappa_b^* (\mathcal{A}^b) \) carries the induced ring structure by the explanation in \( \S 5(viii) \) since the multiplication \( m : \mathcal{A}_R \ast \mathcal{A}_R \to \mathcal{A}_R \) is \( \text{Sym}^\bullet(g^*) \)-linear for the right action of \( \text{Sym}^\bullet(g^*) \) on \( \mathcal{A}_R = \Psi (\mathbb{C}[T^*G]^\bullet) \).)

Corollary 5.19. \( \kappa_b (W_G^b) = W_G^{b-1} \).

**Proof.** We have to check that \( \kappa_b \) commutes with \( H^*_{\text{Gr}_G} (\text{Gr}_G, \bullet) \). After applying the derived Satake equivalence we can check that \( \kappa_b^* \) commutes with \( \kappa^* \). Recall that \( \kappa^* (\bullet) = \bullet \otimes_{\text{Sym}^\bullet(\text{new}(g^*))} \mathbb{C}[\Sigma]^\bullet \).
\( H^*(\bullet \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0) \) (with respect to the action of the left copy of \( \text{Sym}_{\text{new}}(g^r) \)), while
\[ \kappa^b_r(\bullet) = \bullet \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0 \) (with respect to the action of the \( b \)-th right copy of \( \text{Sym}_{\text{new}}(g^r) \)).

We have \( \Psi^{-1}(A^b) \in D^{(G^r)+1}(\text{Sym}^l((g^r)^{\otimes b+1}) \) (one left structure and \( b \) right structures).

We assign number 0 to the left structure. Then
\[
\mathbb{C}[W_G^{-1}] = H^* \left( (\Psi^{-1}(A^b) \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0) \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0 \right)
\]
\[
= H^* \left( (\Psi^{-1}(A^b) \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0) \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0 \right)
\]
\[
= H^* \left( (\Psi^{-1}(A^b) \otimes_{\text{Sym}_{\text{new}}(g^r)} \mathbb{C}[\Sigma]^0) \otimes_{\text{Sym}(g^r)} \mathbb{C}[\Sigma] = \kappa_b(\mathbb{C}[W_G^0]). \right)
\]

The third equality (commutation of taking cohomology and tensor product with a \( b \) \( \text{Sym}_{\text{new}}(g^r) \)-module) is clear for free modules, and then for perfect complexes by devissage, and then for Ind-perfect complexes since cohomology commutes with direct images. \( \square \)

5(\( x \)). **General surfaces for arbitrary reductive groups and fusion.** First we study the case of cylinder and give another explanation of the identification \( W_G^0 = T^*G^r \).

We consider the equivalence
\[
\Psi \otimes \Psi: D^{G^r \times G^r}(\text{Sym}^l((g^r \oplus g^r)) \to D_{G \times G}(\text{Gr}_G \times \text{Gr}_G).
\]

Under this equivalence, the ring object \( A_R \otimes A_R \in D_{G \times G}(\text{Gr}_G \times \text{Gr}_G) \) corresponds to the \( G^r \times G^r \)-equivariant free \( \text{Sym}^l((g^r \oplus g^r)) \)-module \( \mathbb{C}[G^r \times G^r] \otimes \text{Sym}^l((g^r \oplus g^r)) \) which will be denoted \( \mathbb{C}[T^{G^r} \times T^*G^r]^l \). It is equipped with the right action of \( G^r \times G^r \) with the right moment map \( (\mu_r, \mu_r) \). The Hamiltonian reduction with respect to the diagonal right action
\[
(T^*G^r \times T^*G^r) / \Delta_{G^r} := \text{Spec} \left( \mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(g^r)*})]^{\Delta_{G^r}} \right) = (\mu_r, \mu_r)^{-1}(\Delta_{(g^r)*}) / \Delta_{G^r}
\]
(the categorical quotient is the set-theoretical object, as it is with respect to the free action of \( G^r \) is nothing but \( T^*G^r \) equipped with the residual left action of \( G^r \times G^r \): \( (h_1, h_2)(g, \xi) = (h_2gh_1^{-1}, Ad_{h_1} \xi) \), and the equivariant morphism to \( (g^r)^* \oplus (g^r)^* \): \( (g, \xi) \mapsto (\xi, Ad_g \xi) \). Note that the natural projection \( \mathbb{C}[T^*G^r \times T^*G^r] \to \mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(g^r)*})] \) is compatible with the grading of \( \mathbb{C}[T^*G^r \times T^*G^r] \), and so it induces a grading on the target, to be denoted \( \mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(g^r)*})] \). This in turn induces a grading on the \( \Delta_{G^r} \)-invariant subalgebra, to be denoted \( \mathbb{C}[T^*G^r \times T^*G^r / \Delta_{G^r}]^l \). Viewing it as a \( G^r \times G^r \)-equivariant graded module over \( \text{Sym}^l((g^r \oplus g^r)) \) (with zero differential) and taking its free resolution, we obtain the same named object of \( D^{G^r \times G^r}(\text{Sym}^l((g^r \oplus g^r))) \). We will denote \( \Psi \otimes \Psi(\mathbb{C}[T^*G^r \times T^*G^r / \Delta_{G^r}]^l) \) by \( A_R \otimes A_R / \Delta_{G^r} \in D_{G \times G}(\text{Gr}_G \times \text{Gr}_G) \) for short, cf. \( 5(\text{viii}) \).

Now \( \Psi^{-1}(A_R \otimes A_R) = \mathbb{C}[T^*G^r]^l \otimes_{\text{Sym}^l(g^r)} \mathbb{C}[T^*G^r]^l \), and \( \Psi^{-1}(\text{IC}(\text{Gr}_G^0)) = \text{Sym}^l(g^r) \).

Hence, \( W_G^2 = \text{Spec} H_{\text{Gr}_G^0}(\text{Gr}_G, A^2) = (T^*G^r \times T^*G^r) / \Delta_{G^r} = T^*G^r \). The action \( G^r \times G^r \) on \( W_G^2 \) is the natural action of \( G^r \times G^r \) on \( T^*G^r : (h_1, h_2)(g, \xi) = (h_2gh_1^{-1}, Ad_{h_1} \xi) \); in particular, the diagonal action of \( \Delta_{G^r} \) is the adjoint action \( h(g, \xi) = (hgh_1, Ad_{h} \xi) \).

We denote \( A^2 / \Delta_{G^r} \) by \( B \in D_{G}(\text{Gr}_G) \). We have \( H_{\text{Gr}_G^0}(\text{Gr}_G, B) = H_{\text{Gr}_G^0}(\text{Gr}_G, A^2) / \Delta_{G^r} = \mathbb{C}[(T^*G^r) / \Delta_{G^r}] = \mathbb{C}[T^* \times \mathfrak{t}]^W \). Here the last equality is a multiplicative analog of the
isomorphism \((g^\vee \times g^\vee) \sslash \Delta_{G^\vee} = (t^\vee \times t^\vee)/W\) due to I. Losev. Its proof is given in §5(xiii) below. More generally, we have \(H_{G^\vee}^*(Gr_G; i_X^*(A^b \boxtimes B)) = \mathbb{C}[W_G^b] \sslash \Delta_{G^\vee}^{b+1,b+2} = \mathbb{C}[(\mu_{b+1}^1, \mu_{b+2}^1) - 1(\Delta_{G^\vee})^*_r] \Delta_{G^\vee}^{b+1,b+2}\) where \(\Delta_{G^\vee}^{b+1,b+2}\) stands for the diagonal in the product of the last two copies in \((G^\vee)^{b+2}\).

We denote \(B^g := i^1_{\Delta} (\boxtimes_{k=1} B_k) \in D_G(Gr_G).\) Then \(Sp^* H_{G_G}^*(Gr_G; A^b \boxtimes B^g)\) is an object of \(H\)S associated with a surface of genus \(g\) with \(b\) punctures. Now we turn to the study of fusion of surfaces.

**Proposition 5.20.** Let \(\Delta_{G^\vee}^{b_1,b_2}\) denote the diagonal action of the \(b_1\)-st and \(b_2\)-nd copy of \(G^\vee\) on \(W_G^b_1 \times W_G^b_2\). Then \(W_G^{b_1+b_2-2} = (W_G^b_1 \times W_G^b_2) \sslash \Delta_{G^\vee}^{b_1,b_2}\).

**Proof.** We have

\[
\mathbb{C}[W_G^{b_1+b_2-2}] = H_{G^\vee}^*(Gr_G; A^b_1 - 1 \boxtimes A^b_2) = \text{Ext}_{D_G(Gr_G)}^*(1_{Gr_G}; C_G A^b_1 \boxtimes A^b_2)
\]

\[
= \text{Ext}_{D^b_G}^*(\text{Sym}[[g^\vee]]; C^\vee_G; \Psi^{-1}(A^b_1) \boxtimes \text{Sym}[[g^\vee]]; \Psi^{-1}(A^b_2))
\]

\[
= \text{Ext}_{D^b_G}^*(\text{Sym}[[g^\vee]]; \text{Sym}[[g^\vee]; C^\vee_G; \Phi(A^b_1) \boxtimes \text{Sym}[[g^\vee]; C^\vee_G; \Phi(A^b_2))
\]

(the second equality is Lemma 5.13(b)).

Now \(\text{Ext}_{D^b_G}^*(\text{Sym}[[g^\vee]]; \text{Sym}[[g^\vee]; C^\vee_G; \Phi(A^b_1) \boxtimes \text{Sym}[[g^\vee]; C^\vee_G; \Phi(A^b_2))\text{ is the hamiltonian reduction (}\Phi(A^b_1) \boxtimes C^\vee_G; \Phi(A^b_2))\text{ of }\Phi(A^b_1) \boxtimes C^\vee_G; \Phi(A^b_2)\text{ with respect to the diagonal (left) action of }G^\vee.\) According to Lemma 5.13(c) and Lemma 5.12,

\[\Phi(A^b_1) = H_{G^\vee}^*(Gr_G; A^b_1) = H_{G^\vee}^*(Gr_G; A^b_2) = \mathbb{C}[W_G^b],\]

and the left \(G^\vee \times \text{Sym}[[g^\vee]\) -module structure in the LHS coincides with the right \(G^\vee \times \text{Sym}[[g^\vee]\) -module structure in the RHS (with respect to the last copy of \(G^\vee \times \text{Sym}[[g^\vee]\)). This completes the proof.

**Remark 5.21.** The same argument shows that

\[H_{G^\vee}^*(Gr_G; A_1 \boxtimes A_2) \simeq H_{G^\vee}^*(Gr_G; A_R \boxtimes A_R) \boxtimes C_G; H_{G^\vee}^*(Gr_G; A_R \boxtimes A_R),\]

for ring objects \(A_1, A_2 \in D_G(Gr_G).\)

The natural action of \(H_{G^\vee}^*(pt) \boxtimes H_{G^\vee}^*(pt) = \mathbb{C}[\Sigma] \boxtimes \mathbb{C}[\Sigma]\) on the RHS factors through the multiplication homomorphism \(\mathbb{C}[\Sigma] \boxtimes \mathbb{C}[\Sigma] \rightarrow \mathbb{C}[\Sigma],\) and the resulting action of \(H_{G^\vee}^*(pt) = \mathbb{C}[\Sigma]\) in the RHS coincides with its natural action in the LHS.

**Remark 5.22.** The results of §5(vi)–§5(x) have their quantum counterparts if we consider an extra equivariance with respect to the loop rotations. They are based on the equivalence of monoidal triangulated categories \(D^{b\vee}(U_h[[g^\vee]]) \xrightarrow{\sim} D_{G^\vee \times C^\vee}(Gr_G)\) [BF08, Theorem 5] (recall that \(D_G(Gr_G)\) is a shorthand for the \(G^\vee\)-equivariant derived category \(D_G(Gr_G)\)). In particular, the regular sheaf \(A_R \in D_{G^\vee \times C^\vee}(Gr_G)\) corresponds to \(U_h[[g^\vee]] \times \mathbb{C}[G^\vee].\) For a loop rotation equivariant ring object \(A \in D_{G^\vee \times C^\vee}(Gr_G)\) one can consider the loop rotation
equivariant cohomology ring \( H^*_\mathbb{G} \times \mathfrak{C} (\text{Gr}_G, \mathcal{A}) \). Similarly to Remark 5.21, we have

\[
H^*_\mathbb{G} \times \mathfrak{C} (\text{Gr}_G, \mathcal{A}_1 \otimes \mathcal{A}_2) \cong H^*_\mathbb{G} \times \mathfrak{C} (\text{Gr}_G, \mathcal{A}_R \otimes \mathcal{A}_2) \otimes \mathfrak{C}_G \cdot H^*_\mathbb{G} \times \mathfrak{C} (\text{Gr}_G, \mathcal{A}_R \otimes \mathcal{A}_1) \parallel \Delta_G.
\]

(*quantum Hamiltonian reduction*) for ring objects \( \mathcal{A}_1, \mathcal{A}_2 \) in \( D_{\mathbb{G} \times \mathfrak{C}} (\text{Gr}_G) \).

In particular, we set \( \mathbb{C}[W^b] := H^*_\mathbb{G} \times \mathfrak{C} (\text{Gr}_G, \mathcal{A}^b) \), a quantization of \( \mathbb{C}[W^b] \). Then \( \mathbb{C}[W^b] = (\mathbb{C}[W^b] \otimes \mathbb{C}[W^b]) \parallel \Delta_G^\bullet \).

5(xi). **Gluing construction vs hamiltonian reduction.** Let us slightly change the point of view to our gluing construction §2(viii) so that it formally looks similar to a hamiltonian reduction.

Let \( \mathcal{A} \) be a commutative ring object on \( \text{Gr}_G \). Let \( G' \) be a subgroup of \( G \), which is also reductive. We have an inclusion \( i: \text{Gr}_G \to \text{Gr}_G \).

The \( ! \)-pull back \( i^! \mathcal{A} \) is a ring object on \( \text{Gr}_{G'} \).

When \( \mathcal{A} \) arises as \( \pi_* (\omega_R [-2 \dim \mathcal{N}_O]) \) from a representation \( \mathcal{N}_O \), \( i^! \mathcal{A} \) is the ring object associated with \( \mathcal{N}_O \) viewed as a representation of \( G' \).

Next suppose we have a homomorphism \( G \to G'' \) to another reductive group \( G'' \). We consider the induced morphism \( p: \text{Gr}_G \to \text{Gr}_{G''} \), which is equivariant under the induced group homomorphism \( G_O \to G''_O \). Then

The pushforward \( Q_{p^!} \mathcal{A} \) is a ring object on \( \text{Gr}_{G''} \).

Here \( Q_{p^!} \) is the general pushforward as in §2(iv). The construction of §2(iv) is an example of the pushforward, where \( G, G'' \) here are \( \hat{G}, G' \), there, and \( \mathcal{A} \in D_G (\text{Gr}_G) \) here is the ring object on \( D_G (\text{Gr}_G) \) associated with a representation \( \mathcal{N}_O \) of \( \hat{G} \) there. When \( G'' \) is the trivial group, the pushforward is nothing but taking the cohomology \( H^*_\mathbb{G} (\text{Gr}_G, \mathcal{A}) \). In physics terminology this operation corresponds to the *gauging* with respect to the kernel of the homomorphism \( G \to G'' \).

Note that this construction is formally similar to a hamiltonian reduction: suppose that we have a hamiltonian \( G \) space \( X \). We take a hamiltonian reduction \( X \parallel G' \) with respect to a normal subgroup \( G' \triangleleft G \). Then \( X \parallel G' \) is a hamiltonian \( G'' = G / G' \) space. This is not just an analogy if we consider gauging in quantum field theories: The Higgs branch of a gauge theory associated with \( (G, \mathcal{N}_O) \) is the hamiltonian reduction \( \mathcal{N}_O \oplus \mathcal{N}_O^* \parallel G \). (See [Tac] for a review for mathematicians.)

As an example of the similarity, let us consider (5.4) which we regard as a quantum field theory *upgrade* of the definition \( W^{g,b} = \text{Spec} H^*_\mathbb{G}_G (\text{Gr}_G, \mathcal{A}^b \otimes \mathcal{B}^g) \). Let us consider the Coulomb branch of the left hand side, which should be equal to the Higgs branch of the right hand side. Under the gauging \( \parallel \), the Higgs branch is replaced by the symplectic reduction as we have just mentioned. Hence we get

\[
\mathcal{M}_C(S_{G''} (C)) = \mathcal{N}_G^b \times (\mathfrak{g} \oplus \mathfrak{g}^*)^\parallel G_{\text{diag}},
\]

where \( \mathcal{N}_G \) is the nilpotent cone of \( G \), and \( \mathfrak{g} \oplus \mathfrak{g}^* \) is symplectic by the natural pairing. Thus the Coulomb branch \( \mathcal{M}_C(S_{G''} (g, b)) \) is the ‘additive version’ of the \( G \)-character variety on the punctured Riemann surface \( C \), where the monodromy around punctures sit in the regular unipotent orbit. When \( G \) is of type \( A \), this is the Higgs branch of the quiver gauge
Let us identify a proof.

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It is nothing but the following: 

Field theories (due to Gaiotto [Gai12]), we have the corresponding property also for \( D_{G} \).

Hence it is

Gluing in the Higgs branch side. Let us pursue the analogy between the gluing construction and hamiltonian reduction further. Let us consider a ring object associated with \( S_{G}(g, b) \) in the Coulomb branch side instead of the Higgs branch side. It is the Higgs branch ring object associated with the right hand side of (5.4) after exchanging \( G \) and \( G' \).

Hence it is

\[
\mathcal{A}_{S_{G}(g, b)} = \mathcal{D}_{k=1}^{b} (A_{R})_{k} \boxtimes \mathcal{D}_{l=1}^{g} \text{Sym}(g^{' \oplus (g^{'})^*)/\Delta_{G'} ,
\]

where \( \text{Sym}(g^{' \oplus (g^{'})^*) \) is considered as a ring object on the affine Grassmannian Gr\(_{r} \) for the trivial group \( \{ e \} \) with the diagonal \( G'\)-action. Therefore \( \mathcal{A}_{S_{G}(g, b)} \) is a ring object in \( D_{G'}(\text{Gr}_{G'}) \).

Since Proposition 5.20 is a consequence of an upgraded equality in quantum field theories (due to Gaiotto [Gai12]), we have the corresponding property also for \( \mathcal{A}_{S_{G}(g, b)} \).

It is nothing but the following:

**Proposition 5.23.**

\[
p_{*} i_{1,1}^{b} (A_{S_{G}(g_{1}, b_{1})} \boxtimes A_{S_{G}(g_{2}, b_{2})}) = A_{S_{G}(g_{1}+g_{2}, b_{1}+b_{2}-2)},
\]

where (a) \( i_{1,1} : \text{Gr}_{G}^{b_{1}+b_{2}-1} \to \text{Gr}_{G}^{b_{1}} \times \text{Gr}_{G}^{b_{2}} \) is the product of the evident map \( \text{Gr}_{G}^{b_{1}+b_{2}-1} \to \text{Gr}_{G}^{b_{1}} \times \text{Gr}_{G}^{b_{2}} \) and the diagonal embedding \( \text{Gr}_{G} \to (\text{Gr}_{G})^{2} \) of the last factor to the product of the \( b_{1} \)st and the \( b_{2} \)nd factors, and (b) \( p : (\text{Gr}_{G})^{b_{1}+b_{2}-1} \to (\text{Gr}_{G})^{b_{1}+b_{2}-2} \) is the projection given by forgetting the last factor.

**Proof.** Let us identify \( g^{'} \) and \( (g^{'})^* \) by a non-degenerate invariant form. Let us consider

\[
(T^* G')^{b} \times (g^{' \oplus (g^{'})^*)^g / \Delta_{G'} = T^* G^{b} \times \cdots \times T^* G^{b} \times (g^{' \oplus (g^{'})^*) \times \cdots \times (g^{' \oplus (g^{'})^*) / \Delta_{G'},
\]

where \( \Delta_{G'} \)

is the diagonal subgroup acting on \( T^* G^{b} \times \cdots \times T^* G^{b} \) by the right action, and on \( (g^{' \oplus (g^{'})^*) \times \cdots \times (g^{' \oplus (g^{'})^*) \) by the adjoint action. The dg-version of its coordinate ring is \( \Psi^{-1} \mathcal{A}_{S_{G}(g, b)} \).

It is isomorphic to \( (T^* G^{b})^{b-1} \times (g^{' \oplus (g^{'})^*)^g \) by

\[
[g_{1}, \xi_{1}, \ldots, g_{b}, \xi_{b}, \eta_{1}, \xi_{1}, \ldots, \eta_{g}, \xi_{g}] \mapsto (g_{1}', \xi_{1}', \ldots, g_{b-1}', \xi_{b-1}', \eta_{1}', \xi_{1}', \ldots, \eta_{g}', \xi_{g}')
\]

\[
g_{k}' = g_{k} g_{b}^{-1}, \quad \xi_{k}' = \text{Ad}_{g_{b}} \xi_{k} \quad (k = 1, \ldots, b-1), \quad \eta_{l}' = \text{Ad}_{g_{b}} \eta_{l}, \quad \xi_{l}' = \text{Ad}_{g_{b}} \xi_{l} \quad (l = 1, \ldots, g).
\]

The left \( (G^{'})^{b-1} \)-action on \( (T^* G^{b})^{b} \times (g \times g)^{g} / \Delta_{G} \) is identified with the left \( G' \)-action (and the trivial action on \( (g^{' \oplus (g^{'})^*)^g \)) for the first \( (b-1) \) factors, but the last factor acts by \( (T^* G^{b})^{b-1} \in (g_{1}', \xi_{1}', \ldots, g_{b-1}', \xi_{b-1}') \mapsto (g_{1}' h_{1}^{-1}, \text{Ad}_{h_{b}} \xi_{1}', \ldots, g_{b-1}' h_{b-1}^{-1}, \text{Ad}_{h_{b}} \xi_{b-1}', \text{Ad}_{h_{b}} \eta_{l}', \text{Ad}_{h_{b}} \xi_{l}') \) for \( h_{b} \in G \).

The corresponding moment map is also the standard one for the first \( (b-1) \)-factors, and the last one is

\[
-\xi_{1}' - \cdots - \xi_{b-1}' = \sum_{l=1}^{g} [\eta_{l}', \xi_{l}']
\]

This is nothing but the restriction to the diagonal subgroup of the product of the right action and the adjoint action.
Now by Lemma 5.13(b) and Lemma 5.12, 
P_{*}\hat{i}_{\Delta}^{!}(A_{S_{G}(g_{1},b_{1})} \boxtimes A_{S_{y}(g_{2},b_{2})})
goesto
\text{Ext}^{*}_{D^{G}(\text{Sym}(g^{\vee}))}(\text{Sym}^{[\cdot]}(g^{\vee}), C_{G^{\vee}} \Psi^{-1}A_{S_{G}(g_{1},b_{1})} \boxtimes \text{Sym}^{[\cdot]}(g^{\vee}) \Psi^{-1}A_{S_{G}(g_{2},b_{2})}),
under the derived Satake equivalence. Here \(g^{\vee}\) is the Lie algebra of the diagonal subgroup in the product of last factors of \((G^{\vee})^{b_{1}}\) and \((G^{\vee})^{b_{2}}\). It is equal to
\[\mathbb{C}[((T^{*}G^{\vee})^{b_{1}+b_{2}-2} \times (g^{\vee} \times g^{\vee})^{g_{1}+g_{2}}//\Delta_{G^{\vee}})]]\]
by the above computation. This is nothing but \(\Psi^{-1}A_{S_{G}(g_{1}+g_{2},b_{1}+b_{2}-2)}\).

5(xiii). Hamiltonian reduction of \(T^{*}G\) with respect to the adjoint action. Let \(G\) be a connected reductive group over \(\mathbb{C}\) and let \(g\) be its Lie algebra. Consider the adjoint action of \(G\) on itself and the induced Hamiltonian action of the cotangent bundle \(T^{*}G\). Using a non-degenerate invariant form we identify \(g\) with \(g^{\vee}\), this gives rise to an identification \(T^{*}G \cong G \times g\) (with the diagonal action of \(G\)). The moment map \(\mu: T^{*}G \to g\) becomes \((g,x) \mapsto \text{Ad}_{g}x - x\). It follows that \(\mu^{-1}(0) = \{(g,x)| \text{Ad}_{g}x = x\}\). Consider the Hamiltonian reduction \(\mu^{-1}(0)//G\) with the reduced scheme structure.

Now consider \(T^{*}T = T \times t\). We have a natural morphism of varieties \(\psi: (T \times t)//W \to \mu^{-1}(0)//G\) induced from \(T \times t \hookrightarrow G \times g\).

Proposition 5.24 (I. Losev). The morphism \(\psi: (T \times t)//W \to \mu^{-1}(0)//G\) is an isomorphism of varieties.

We can consider the analogous situation for the Lie algebras: we have the moment map \(\mu: g^{2} \to g, (x,y) \mapsto [x,y]\). In this situation, a direct analog of Proposition 5.24 is known thanks to [Jos97]: we have \(t^{2}/W \xrightarrow{\sim} \mu^{-1}(0)//G\). In particular, the variety \(\mu^{-1}(0)//G\) is normal.

Proof. The proof is in several steps.

Step 1. Let us show that \(\psi\) is a bijection. The variety \(\mu^{-1}(0)//G\) parameterizes the closed \(G\)-orbits in \(\mu^{-1}(0) = \{(g,x)| \text{Ad}_{g}x = x\}\). It follows easily from the Hilbert-Mumford theorem that the orbit \(G(g,x)\) is closed if and only if both \(g,x\) are semisimple. Also any \(G\)-orbit of semisimple commuting elements \((g,x)\) intersects \(T \times t\) in a single \(W\)-orbit. The claim in the beginning of the step follows.

Step 2. We claim that it is enough to show that \(\mu^{-1}(0)//G\) is a normal algebraic variety. Indeed, any bijective morphism to a normal variety is an isomorphism. The normality of \(\mu^{-1}(0)//G\) will follow if we check that the formal neighborhood of every point in \(\mu^{-1}(0)//G\) is normal. In order to do that we will describe the formal neighborhood using a version of a slice theorem for Hamiltonian actions on affine symplectic varieties, see, e.g., [Los06] (in that paper complex analytic neighborhoods were considered, but the result carries over to the formal neighborhood in a straightforward way).

Step 3. Let us recall the slice theorem. Let \(Y\) be a smooth affine symplectic variety equipped with a Hamiltonian action (with moment map \(\mu\)) of a reductive group \(G\) and let \(y \in Y\) be a point with closed \(G\)-orbit. Let us write \(H\) for the stabilizer of \(y\) in \(G\). The normal space \(T_{y}Y/T_{y}Gy\) can be decomposed as \(\mathfrak{h}^{\perp} \oplus V\), where \(V\) is a symplectic vector space with \(H\) acting on \(V\) by linear symplectomorphisms. Then the formal neighborhood
of \( G_y \) in \( Y \) is \( G \)-equivariantly isomorphic to the formal neighborhood of the zero section in \( G \times H \). An isomorphism can be chosen to be compatible with symplectic forms and moment maps. In particular, the moment map \( \mu' : G \times H (\mathfrak{h}^\perp \oplus \mathfrak{v}) \to \mathfrak{g} \) is the unique \( G \)-equivariant map that on the fiber \( \mathfrak{h}^\perp \oplus \mathfrak{v} \) over \( 1H \) is given by \( \mu(z, v) = z + \mu_H(v) \), where \( \mu_H : V \to \mathfrak{h} \) is the standard moment map for a linear symplectic action. In particular, we see that the formal neighborhood of \( G_y \) in \( \mu^{-1}(0) \) is isomorphic to the formal neighborhood of \( 0 \) in \( \mu^{-1}(0) / H \).

**Step 4.** Consider \( Y = G \times \mathfrak{g} \) and \( y = (g, 0) \) for a semisimple element \( g \in G \). We can identify \( T_yGy \) with \( \{ \text{Ad} \, x - x | x \in \mathfrak{g} \} \) so \( T_yY / T_yGy = \mathfrak{h} \oplus \mathfrak{g} \) and \( H = Z_C(g) \). We conclude that \( V \simeq \mathfrak{h} \oplus \mathfrak{h} \) with diagonal action of \( H \). By [Jos97], we see that \( \mu_H^{-1}(0) / H \) is normal. So the formal neighborhood of \( G(g, 0) \) in \( \mu^{-1}(0) / G \) is normal, equivalently, \( G(g, 0) \) is a normal point.

**Step 5.** To finish the proof note that \( \mathbb{C}^\times \) acts on \( \mu^{-1}(0) / G \), the action is induced from the dilation action on \( \mathfrak{g} \). This action contracts \( \mu^{-1}(0) / G \) to \( G / G \). Since the points in the latter are normal, \( \mu^{-1}(0) / G \) is a normal algebraic variety.

**Appendix A. Group action on the Coulomb branch**

In this section we give a proof of the expected property [Nak16, §4(iii)(d)], using an idea of Namikawa [Nam16]. See also [CHLZ17].

A(i). **The degree 1 subspace.** Let us consider the \( \mathbb{C}^\times \)-action on the Coulomb branch \( \mathcal{M}_C \) given by \( \Delta(\lambda) \) as in Remark II.2.8(2). Recall that the \( \mathbb{C}^\times \)-action is shifted from one given by the homological degree by a hamiltonian action. In particular, the Poisson bracket \( \{ , \} \) is of degree \(-1\) as in §II.3(vi).

Consider the subspace \( I \) of degree 1 elements in \( \mathbb{C}[\mathcal{M}_C] \). It forms a Lie subalgebra under the Poisson bracket \( \{ , \} \). Then \( \mathbb{C}[\mathcal{M}_C] \) can be considered as a representation of this Lie algebra \( I \) by the Poisson bracket: \( \{ f, \bullet \} (f \in I) \). If we restrict it to the regular locus of \( \mathcal{M}_C \), it is nothing but the hamiltonian vector field \( H_f \) associated with \( f \in I \) by the symplectic form. The action preserves the Poisson bracket and the degree. In more geometric term, \( H_f \) preserves the symplectic form and commutes with the \( \mathbb{C}^\times \)-action.

**Remark A.1.** Namikawa [Nam16] shows that \( \mathcal{M}_C \) is the closure of a nilpotent orbit if \( \mathbb{C}[\mathcal{M}_C] \) is generated by \( I \), under the assumption that \( \mathcal{M}_C \) has symplectic singularities. In this case \( I \) is the Lie algebra of \( \text{Aut}^{\mathbb{C}^\times} (\mathcal{M}_C, \omega) \), the group of \( \mathbb{C}^\times \)-equivariant symplectic automorphisms of \( \mathcal{M}_C \). We conjecture that this statement is true for general \( \mathcal{M}_C \). Namikawa’s argument works in much more general cases without the assumption that the coordinate ring is generated by \( I \). But we are not sure as we do not know \( \mathcal{M}_C \) has symplectic singularities, and the \( \mathbb{C}^\times \)-action is not conical in general. These seem essential in Namikawa’s argument.

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Footnotes:

1. The third named author thanks Amihay Hanany for his explanation of the idea to use the Lie algebra of degree 1 subspace.
2. The third named author thanks Yoshinori Namikawa for explanation.
A(ii). Balanced vertices in quiver gauge theories. Let us take a quiver $Q = (Q_0, Q_1)$ and two $Q_0$-graded vector spaces $V = \bigoplus V_i$, $W = \bigoplus W_i$. We consider the associated quiver gauge theory $(GL(V), N)$ as in [Nak16, §2(iv)] and §Q.3, i.e.,
\[
GL(V) = \prod_{i \in Q_0} GL(V_i), \quad N = \bigoplus_{h \in Q_1} \text{Hom}(V_{h(h)}, V_{\tilde{h}(h)}) \oplus \bigoplus_{i \in Q_0} \text{Hom}(W_i, V_i).
\]
In order to treat a group action on a line bundle in §II.3(ix) and §2(vi), we also consider a larger symmetry group $\hat{G} = GL(V) \times GL(W)/\mathbb{C}^\times$ with $\hat{G}/GL(V) = PGL(W) = \prod_{i \in Q_0} GL(W_i)/\mathbb{C}^\times$, where both $\mathbb{C}^\times$ are diagonal scalar subgroups.

Recall $\mathbb{C}[\mathcal{M}_C]$ has a grading parameterized by $\pi_1(GL(V))$ (§II.3(v)). In our situation, we have $\pi_1(GL(V)) = \bigoplus_{i \in Q_1} \pi_1(GL(V_i)) \cong \mathbb{Z}^{Q_0}$. For the larger symmetry group, we have $\pi_1(\hat{G}) \cong \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{\{i \in Q_0 | W_i \neq 0\}}/\mathbb{Z}$, where $\mathbb{Z}$ is embedded into $\mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{\{i \in Q_0 | W_i \neq 0\}}$ by $1 \mapsto (\dim V_i, \dim W_i)$. We have the corresponding action of $\pi_1(GL(V))^\leftarrow \cong (\mathbb{C}^\times)^{Q_0}$ on $\mathcal{M}_C$ and $\pi_1(\hat{G})^\leftarrow \cong (\mathbb{C}^\times)^{\mathbb{Z}}$ (modulo finite groups) on a line bundle in §II.3(ix), §2(vi). Here $^\leftarrow$ is the Pontryagin dual. We will not be interested in the action of finite groups, hence we replace $\pi_1(\hat{G})$ by its free part $\pi_1(\hat{G})_{\text{fr}}$ hereafter. We have the corresponding space $H_{GL(V)}(pt)$ (or $H_{\mathbb{C}}(pt)$ for $\pi_1(\hat{G})^\leftarrow$, which consists of degree 1 elements.

A vertex $i$ is balanced if there is no edge loop at $i$, and the corresponding coweight $\mu$ satisfies $\langle \mu, \alpha_i \rangle = 0$, i.e., $2 \dim V_i = \dim W_i + \sum_j a_{ij} \dim V_j$, where $a_{ij}$ is the number of edges (either in $Q_1$ or its opposite) between $i$ and $j$. We consider the subquiver $Q^{\text{bal}}$ of $Q$ consisting of balanced vertices and edges among them. By a well-known result (e.g. [Kac90, Th. 4.3]), $Q^{\text{bal}}$ is a union of finite $ADE$ quivers, unless $Q^{\text{bal}}$ is a union of connected components of $Q$ of affine type with $W = 0$ on them. We suppose it is not the latter case.

We consider elements $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ from the shifted Yangian considered in Appendix Q.B. Looking at relations therein, we see that their Poisson brackets satisfy the relations of $\mathfrak{sl}_2$ as $H_i^{(p)} = 0$ ($p < 0$), $H_i^{(0)} = 1$ as $\langle \mu, \alpha_i \rangle = 0$. Moreover if both $i$ and $j$ are balanced, $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}, E_j^{(1)}, F_j^{(1)}, H_j^{(1)}$ satisfy the relations of $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ according to whether $i$ and $j$ are connected in the quiver or not. We then have the corresponding semisimple Lie algebra $\mathfrak{psl}^{\text{bal}}$ generated by $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i \in Q_0^{\text{bal}}$).

From the definition of $H_i(z)$ in Appendix Q.B, $H_i^{(1)}$ is the coefficient of $z^{-1}$ in $Z_i(z) \prod_j W_j(z)^{a_{ij}}/W_i(z)^2$, where $Z_i(z) = \prod_{k: i_k = i} (z - z_k)$, $W_i(z) = \prod_r (z - w_{i,r})$. (Note that we set $\hbar = 0$.) Here $z_k, w_{i,r}$ are equivariant variables for $\prod GL(W_i)$ and $GL(V_i)$ respectively. Therefore $H_i^{(1)}$ is $-\sum_{k: i_k = i} z_k - \sum_{j,s} a_{ij} w_{j,s} + 2 \sum r w_{i,r}$. This is nothing but $-c_1(W_i) - \sum_j a_{ij} c_1(V_j) + 2c_1(V_i)$ if we regard $V_i, W_i$ as representations of $GL(V_i), GL(W_i)$ respectively. Now we apply Lemma II.3.20. The Poisson bracket $\{H_i^{(1)}, \cdot\}$ is given by $\gamma_i^W + \sum_j a_{ij} \gamma_j - 2 \gamma_i$ on the component with grading $\gamma = (\gamma_j, \gamma_j^W) \in \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{Q_0}/\mathbb{Z}$. In particular, the action of $H_i^{(1)}$ is lifted to $\pi_1(GL(V))^\leftarrow \cong (\mathbb{C}^\times)^{Q_0}$ for $\mathcal{M}_C$, and to $\pi_1(\hat{G})_{\text{fr}}$ for line bundles.

**Lemma A.2.** $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i \in Q_0^{\text{bal}}$) are of degree 1.

Therefore the Lie algebra $\mathfrak{l}_1$ in the previous subsection at least contains the semisimple Lie algebra $\mathfrak{psl}^{\text{bal}}$ above.
Note that $H_1^{(1)}$ is in $H^2_{\text{GL}(V)}(pt)$ or $H^2_{\tilde{G}}(pt)$ when we consider the larger group $\tilde{G}$. Let $t^{\text{bal}}$ (resp. $\tilde{t}^{\text{bal}}$) be the Lie subalgebra of $I$ generated by $t^{\text{bal}}_{ss}$ and $H^2_{\text{GL}(V)}(pt)$ (resp. $H^2_{\tilde{G}}(pt)$).

\textbf{Proof.} We have already checked the assertion for $H_1^{(1)}$.

Looking at the definition of the homomorphism in Theorem Q.B.18, we see that $E_{i}^{(1)}$, $F_{i}^{(1)}$ are fundamental classes $[\mathcal{R}_{\pm\varpi_i}]$ up to sign. By the formula for $\Delta(\pm\varpi_i)$ in (Q.A.4), their degree is 1 as $i$ is a balanced vertex.

Recall $\{H_i^{(1)}, \cdot\}$ defines an element of $\pi_1(\tilde{G})_{fr}$ by $\gamma \mapsto \gamma_i^W + \sum_j a_{ij} \gamma_j - 2\gamma_i$. More generally $H_\alpha$ corresponding to a root $\alpha$ of $t^{\text{bal}}_{ss}$ defines an element of $\pi_1(\tilde{G})_{fr}$. Thus we regard naturally coroots $\in R^{\text{bal}_\nu}$ of $t^{\text{bal}}_{ss}$ as elements in $\pi_1(\tilde{G})_{fr}$. If we disregard the flavor symmetry, we consider the restriction to $\gamma_i^W = 0$, hence we still have $R^{\text{bal}_\nu} \subset \pi_1(G)_{\gamma}$.

On the other hand, $\pi_1(\prod_{i \in Q^{\text{bal}}_0} \text{GL}(V_i)) \cong \mathbb{Z}^{Q^{\text{bal}}_0}$ is naturally identified with the root lattice of the Lie algebra $t^{\text{bal}}_{ss}$ by sending the $i$-th coordinate vector to the $i$-th simple root $\alpha_i$. Thus we consider roots $\in R^{\text{bal}}$ as elements of $\pi_1(\prod_{i \in Q^{\text{bal}}_0} \text{GL}(V_i)) \subset \pi_1(\text{GL}(V)) \subset \pi_1(\tilde{G})_{fr}$.

We regard $R^{\text{bal}} \subset \pi_1(\text{GL}(V))$, $R^{\text{bal}_\nu} \subset \pi_1(\text{GL}(V))_{\nu}$ as a root datum, and consider the corresponding reductive group $L^{\text{bal}}$. For line bundles, we consider $R^{\text{bal}} \subset \pi_1(\tilde{G})_{fr}$, $R^{\text{bal}_\nu} \subset \pi_1(\tilde{G})_{fr}$. We denote the corresponding reductive group by $\tilde{L}^{\text{bal}}$.

\textbf{Proposition A.3.} The actions of $\{E_{i}^{(1)}, \cdot\}$, $\{F_{i}^{(1)}, \cdot\}$ are locally nilpotent. Hence the action of $t^{\text{bal}}$ (resp. $\tilde{t}^{\text{bal}}$) is lifted to $L^{\text{bal}}$ (resp. $\tilde{L}^{\text{bal}}$).

\textbf{Proof.} When the $\mathbb{C}^\times$-action on $\mathcal{M}_C$ is conical, this is clear as subspaces of $\mathbb{C}[\mathcal{M}_C]$ with given degree are finite dimensional, and $E_{i}^{(1)}$, $F_{i}^{(1)}$ preserve them. In order to deal with general cases, we modify the argument.

Consider a closed subvariety $\mathcal{R}_{\leq \lambda}$ as in §II.2(i). Since $\text{GL}(V) = \prod_{i \in Q_0} \text{GL}(V_j)$, we can modify it by imposing the constraint at $j \neq i$, but not on $i$. Let us denote the resulted closed subvariety by $\mathcal{R}_{< \lambda}$. It is still true that $H^{\text{GL}(V)}_{\ast}(\mathcal{R})$ is the limit of $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{\leq \lambda})$. Operators $\{E_{i}^{(1)}, \cdot\}$, $\{F_{i}^{(1)}, \cdot\}$ are well-defined on $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{\leq \lambda})$, as we do not impose the constraint at $i$.

Let $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{\leq \lambda})[d]$ denote the subspace of $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{\leq \lambda})$ of degree $d$ elements. It is enough to show that it is finite dimensional, as $\{E_{i}^{(1)}, \cdot\}$, $\{F_{i}^{(1)}, \cdot\}$ preserve this subspace.

Suppose that an element in $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{\leq \lambda})[d]$ is contained in $H^{\text{GL}(V)}_{\ast}(\mathcal{R}_{< \mu})[d]$. If we decompose $\mu$ as $(\mu^j)$ according to $j \in Q_0$, the component $\mu^j$ with $j \neq i$ is less than or equal to the component $\lambda^j$ of $\lambda$ by the definition of $\leq$.

In order to bound the remaining component $\mu^i$, Let us look at the formula of $\Delta(\mu)$:

$$\Delta(\mu) = -\sum_{\alpha \in \Delta^+} |\langle \alpha, \mu \rangle| + \frac{1}{2} \sum_{\chi} |\langle \chi, \mu \rangle| \dim N(\chi).$$
We have $\Delta(\mu) \leq d$ by our assumption. Let us look at terms involving $\mu^i$:

$$(A.4) \quad - \sum_{a < b} |\mu^i_a - \mu^i_b| + \frac{1}{2} \left( \sum_j a_{ij} \sum_{a,b} |\mu^i_a - \mu^i_b| + \dim W_i \sum_a |\mu^i_a| \right),$$

where we write $\mu^i = (\mu^i_1, \mu^i_2, \ldots)$. This is bounded by a constant from above, as we have bounds on $\mu^i_a$ ($j \neq i$). Since $\mu^i_b$ is bounded, the middle term can be replaced by $\sum_j a_{ij} \dim V_j \sum_a |\mu^i_a|$. Now by the assumption $2 \dim V_i = \dim W_i + \sum a_{ij} \dim V_j$, the first term can be absorbed in the middle and last term, so that we still have a bound on $\sum_a |\mu^i_a|$. Thus $\mu$ is bounded by a constant depending on $\lambda$ and $d$. Hence $H^*_G(V) (R_{\leq \lambda})[d]$ is finite dimensional.

This argument works also for the case of $\tilde{L}^{\text{bal}}$.

Note that the comoment map of the $L^{\text{bal}}$-action on $M_C$ is the natural homomorphism

$$\mathbb{C}[\mathfrak{t}^{\text{bal}}] = \text{Sym}(\mathfrak{t}^{\text{bal}}) \to \mathbb{C}[M_C]$$

by the definition of the action.

Example A.5 (cf. Remark Q.3.12). Consider a framed quiver gauge theory of type $ADE$. Let us define two coweights $\lambda = \sum_{i \in Q_0} \dim W_i \omega_i$, $\mu = \lambda - \sum_{i \in Q_0} \dim V_i \alpha_i$. Then the Coulomb branch is the generalized slice $W^\lambda_{\mu^*}$ for the adjoint group $G$ of type $Q$, where $\lambda^* = -w_0(\lambda)$, $\mu^* = -w_0(\mu)$ (Theorem Q.3.10). Then the group $L^{\text{bal}}$, acting on $M_C$ is $\text{Stab}_G(\mu^*)$, as $\pi_1(\text{GL}(V))$ is the weight lattice of $G$. The action is the standard one, at least when $\mu$ is dominant. The following argument is explained to the authors by Joel Kamnitzer:

First consider the case $\mu = 0$. Then $\overline{W}^\lambda_{0^*}$ is the intersection of $\overline{G}_G^\lambda$ and $\overline{G}_G^0 = G_1[[t^{-1}]]$, where $G_1[[t^{-1}]]$ is the first congruence subgroup of $G[[t^{-1}]]$ as in [KWWY14]. Then the assertion follows from a computation of Poisson brackets on $\mathbb{C}[G_1[[t^{-1}]]]$ in [KWWY14, Prop. 2.13]. For a general dominant $\mu$, we replace $G_1^0$ by $G_{1^0}^\mu$, the orbit of $G_1[[t^{-1}]]$ through $\mu^*$. But $\mathbb{C}[G_{1^0}^\mu]$ is a Poisson subalgebra of $\mathbb{C}[G_G^0]$ preserved by the action of $\text{Stab}_G(\mu^*)$ (see [KWWY14, a paragraph before Lemma 2.19]). Hence the assertion follows from the $\mu = 0$ case.

Example A.6. Let us consider the quiver gauge theory in §2(v). All vertices are balanced in this case. We have

$$\pi_1(\tilde{G}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N, N - 1, \cdots, 2, 1) \mathbb{Z} \cong \mathbb{Z}^{N-1},$$

where the isomorphism is given by $[\lambda_1, \ldots, \lambda_N] \mapsto (\lambda_1 - N\lambda_N, \ldots, \lambda_{N-1} - 2\lambda_N)$. We have an exact sequence

$$0 \to \pi_1(G) \cong \mathbb{Z}^{N-1} \to \pi_1(\tilde{G}) \cong \mathbb{Z}^{N-1} \to \pi_1(\text{PGL}(N)) \cong \mathbb{Z}/NZ \to 0,$$

where the first inclusion is given by $\lambda_1 = 0$, and the last projection is $[\lambda_1, \ldots, \lambda_N] \mapsto \lambda_1 \text{ mod } N$. It is clear that $\pi_1(G)$ is the weight lattice of $\text{PGL}(N)$, while $\pi_1(\tilde{G})$ is that of $\text{SL}(N)$. Therefore $L^{\text{bal}} = \text{PGL}(N)$, but $\tilde{L}^{\text{bal}} = \text{SL}(N)$. 

Note that the comoment map of $L^{\text{bal}}$-action on $M_C$ is the natural homomorphism

$$\mathbb{C}[\mathfrak{t}^{\text{bal}}] = \text{Sym}(\mathfrak{t}^{\text{bal}}) \to \mathbb{C}[M_C]$$

by the definition of the action.
Remark A.7. Let us consider the case $\langle \mu, \alpha_i \rangle = -1$ instead of 0. Then $H^{(p)}_i = 0$ ($p \leq 0$), $H^{(1)}_i = 1$. Thus we have $\{E^{(1)}_i, F^{(1)}_i\} = 1$. (Note also $\Delta(\pm \omega_{i,1}) = 1/2$ by (Q.A.4).) Looking at the argument in the proof of Proposition A.3, we see that we only need a bound $\dim V_i - 1 \leq \dim W_i + \sum a_{ij} \dim V_j$ to derive a bound on $\sum a_i |\mu_i|$ from (A.4). In particular, the proof of Proposition A.3 works in the case $\langle \mu, \alpha_i \rangle = -1$, hence $\{E^{(1)}_i, \bullet\}$, $\{F^{(1)}_i, \bullet\}$ are locally nilpotent, and the corresponding hamiltonian vector fields $H_{E^{(1)}_i}, H_{F^{(1)}_i}$ are integrable. Moreover $[H_{E^{(1)}_i}, H_{F^{(1)}_i}] = 0$ as $\{E^{(1)}_i, F^{(1)}_i\} = 1$. Therefore we have an action of $G^2_\mu$. Let $\Phi = (F^{(1)}_i, -E^{(1)}_i): \mathcal{M}_C \to \mathbb{A}^2$. Then $\Phi$ is $G^2_\mu$-equivariant, and the action map $G^2_\mu \times \Phi^{-1}(0) \cong \mathcal{M}_C$ is an isomorphism. See also [Nam16, Th.(i)].

Suppose further that $j$ is balanced, i.e., $\langle \mu, \alpha_j \rangle = 0$. Commutation relations in Appendix Q.B imply that $\{E^{(1)}_j, E^{(1)}_i\} = 0$, $\{H^{(1)}_j, E^{(1)}_i\} = -\langle \alpha_i \cdot \alpha_j \rangle E^{(1)}_i$ and $\{E^{(1)}_j, F^{(1)}_i\} = 0$, $\{H^{(1)}_j, F^{(1)}_i\} = \langle \alpha_i \cdot \alpha_j \rangle F^{(1)}_i$. Thus $E^{(1)}_j$ (resp. $F^{(1)}_j$) is a lowest (resp. highest) weight vector of an $\mathfrak{sl}(2)_j = \langle E^{(1)}_j, F^{(1)}_j, H^{(1)}_j \rangle$ module with the highest (resp. lowest) weight $\mp (\alpha_i \cdot \alpha_j)$.

**APPENDIX B. A GLOBAL CONVOLUTION DIAGRAM FOR THE VARIETY OF TRIPLES**

By Gus Lonergan

The aim of this appendix is to give another proof of the commutativity of the Coulomb branch by constructing a global convolution diagram for $\mathcal{R}$. This is a direct generalization of the traditional proof of the case $N = 0$, which uses the Beilinson-Drinfeld global convolution diagram for $\text{Gr}_G$.

B(i). Preliminaries on arc-spaces and loop-spaces.

(a). In this section, we recall certain standard constructions and facts of [BeiDr, Chapters 4-5].

(b). Let $X$ be a smooth complex curve and let $S$ be a finite set. Given a commutative ring $R$ and an $R$-point $x$ of $X^S$, we denote the coordinates of $x$ by $x_s$ ($s \in S$), and write $\Delta_S(x)$ for the formal neighborhood of the union of the graphs of $x_s$ ($s \in S$). For notational simplicity, we frequently remove commas and braces from $S$, and also drop the part $(x)$, when it is clear which point we refer to. So for example the expression:

$\Delta_{\{1,2\}}(x)$

becomes:

$\Delta_{12}$. 
(c). Now fix an affine algebraic group $A$ over $\mathbb{C}$. Consider the following functor from commutative rings to groups over $X^S$:

$$A_S(R) := \{(x, f)|x \in X^S(R), f : \Delta_S \to A\}.$$ 

Then $A_S$ is represented by the limit of a projective system of smooth affine group schemes over $X^S$:

$$A_S = \lim_{\longleftarrow}(\ldots \to (A_S)_2 \to (A_S)_1)$$

such that each transition morphism is a smooth homomorphism. In particular, $A_S$ is a formally smooth affine group scheme (of countably infinite type) over $X^S$, but this is not so important for us. Recall that in the definition of the Coulomb branch as a convolution algebra formal homological shifts such as

$$[2 \dim A(O)]$$

appear (for $A = G, N$). Similarly, in the global situation formal homological shifts such as

$$[2 \dim A_S]$$

will appear. For example, in the case where the underlying space is $A_S$, for each $d$ we have $\omega_{(A_S)_d} \cong C_{(A_S)_d}[2 \dim(A_S)_d]$. These complexes are compatible in the natural way under $!$-pullbacks along the transition morphisms. We thus consider $\omega_{A_S}$ as the formal homological shift

$$\omega_{A_S} \cong C_{A_S}[2 \dim A_S],$$

where both sides are to be understood by evaluating on smooth quotients of $A_S$ and ‘piecing together’ using $!$-pullbacks. Likewise we have a formal expression

$$\omega_{A_S}[-2 \dim A_S] \cong C_{A_S}$$

where both sides are to be understood by evaluating on the smooth quotients $(A_S)_d$ of $A_S$ and ‘piecing together’ using $\ast$-pullbacks.

(d). Let $\theta : S' \to S$ be a morphism of finite sets. It induces a map $X^S \to X^{S'}$. Given an $R$-point $x$ of $X^S$, this map determines an $R$-point $x'$ of $X^{S'}$, and an embedding $\Delta_{S'}(x') \to \Delta_S(x)$. Hence by restriction along this embedding we obtain a map

$$\rho^\theta : A_S \to A_{S'}.$$ 

This induces a homomorphism

$$q^\theta : A_S \to A_{S'} \times_{X^{S'}} X^S$$

over the base $X^S$. If $\theta$ is surjective, then $q^\theta$ is an isomorphism. If $\theta$ is injective, then $q^\theta$ seems strange at first sight. For instance if $\theta'$ is a section of $\theta$ then $q^\theta$ is an isomorphism over the resulting copy of $X^{S'} \subset X^S$, whereas over a typical point of $X^S$, $q^\theta$ takes the form of a projection map

$$A(O)^S \to A(O)^{S'}.$$ 

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Footnote:

9Only for $S$ of cardinality 1 or 2; but it clarifies the picture and simplifies the exposition to work more generally at this point.
However, this is misleading: $q^\theta$ is pro-smooth when $\theta$ is injective. What we mean by this is that the projective systems of smooth affine group schemes over $X^S$ with smooth transition morphisms

$((A_s)_d)_{d \in \mathbb{N}}$

underlying $A_s$ may be taken, simultaneously for all $S$, to be compatible with all $q^\theta$, i.e. so that $q^\theta$ is the limit of a morphism

$(q^\theta_d): (A_s)_d \to (A_{s'})_d \times_{X^{s'}_d} X^S_d$\n
of projective systems, where each map $q^\theta_d$ is a smooth homomorphism over $X^S$. Thus, it makes sense to write (and is true that):

$(q^\theta_*)^* \omega_{A_{s'} \times_{X^{s'}_d} X^S_d}[-2 \dim A_{s'} - 2(|S| - |S'|)] = \omega_{A_s}[-2 \dim A_s],$

et cetera, where the formula should be understood as a statement about complexes on smooth quotients over $X^S$, compatible under $*$-pullbacks.

(e). Example. Consider the case $S = \{1\}$. Then $A_1$ is a Zariski-locally trivial $A(\mathcal{O})$-bundle over $X$. Then, the formal homological shift $[2 \dim A(\mathcal{O})]$ also makes sense in this context, and we have $[2 \dim A(\mathcal{O})] = [2 \dim A_1 - 2]$.

(f). Example. Consider for instance the case $A = \mathbb{C}$ and $s = \{1, 2\}$. Then $A_{12}$ should be thought of as a deformation of the first following projective system into the second:

$((\mathbb{C}[t]/t^{2d})_d \hookrightarrow (\mathbb{C}[t]/t^d \times \mathbb{C}[t]/t^d)_d$

while $A_1$ should be thought of as a trivial deformation:

$((\mathbb{C}[t]/t^d)_d \hookrightarrow (\mathbb{C}[t]/t^d)_d$

and we have the morphism of deformations of projective systems:

$((\mathbb{C}[t]/t^{2d})_d \hookrightarrow (\mathbb{C}[t]/t^d \times \mathbb{C}[t]/t^d)_d$

where the first downward arrow is the quotient map, and the second downward arrow is the projection map (to the first factor), both of which halve dimension in the $d^{th}$ approximation. It just happens that the limit of the first downward arrow is an isomorphism, while the limit of the second downward arrow is a non-trivial projection.

(g). From now on, we assume $\theta$ is an injection, and identify $S'$ with its image under $\theta$. Now, in addition to the formal neighborhood $\Delta_S$ we have the punctured formal neighborhood

$\Delta_{S'}^S(x) := \Delta_S(x) - \cup_{s \in S'} x_s$

where in this formula we conflate the point $x_s$ with its graph. The general notational paradigm\(^{10}\) here is that subscripts determine discs and superscripts determine punctures.

\(^{10}\)Warning: this doesn’t apply to $X$!
Consider the functor
\[ A_S^S(R) := \{ (x, f) | x \in X^S(R), f : \Delta^S_S(x) \to A \}. \]

Then \( A_S^S \) is represented by an ind-scheme, formally smooth over \( X^S \). It is a group in ind-schemes (over \( X^S \)), but not an inductive limit of groups. Nonetheless, it is an *ind-locally nice, reasonable* ind-scheme in the sense of [Dri06], meaning that it is a direct limit of closed embeddings with finitely generated ideals:
\[
(A_S^S)^1 \to (A_S^S)^2 \to \ldots
\]
of schemes over \( X^S \), each of which is locally nice, meaning that Zariski-locally\(^{11}\) it is the product of a finite-type scheme with an affine space (of countable dimension). We shall call such an ind-scheme *reasonably nice*. The subgroup \( A_S \) may be taken as the first subscheme \( (A_S^S)^1 \) in this inductive structure. The left- and right-regular actions of the subgroup \( A_S \) preserve the inductive structure, meaning that each \( (A_S^S)^c \) has an action on both sides by \( A_S \) over \( X^S \), even though it is not itself a group. Moreover the quotient \( (A_S^S)^c / A_S \) is of finite-type over \( X^S \), and flat, although generally quite singular. The result is that the quotient
\[ A_S^S / A_S \]
has the structure of ind-finite-type flat ind-scheme over \( X^S \).

**Lemma B.1.**  
(1) \( A_S^S / A_S \) is ind-projective if and only if \( A \) is reductive.
(2) \( A_S^S / A_S \) is reduced if and only if \( A \) has non no-trivial characters.

**Remark B.2.** \( A_S^S \) is the Beilinson-Drinfeld grassmannian (on \( |S| \) points).

(h). For any chain of inclusions \( S'' \overset{q'}{\to} S' \overset{q}{\to} S \) we have natural maps
\[
p^\theta : A_S^{S''} \to A_S^{S'},
q^\theta : A_S^{S''} \to A_S^{S''} \times_{X^{S'}} X^S,
\]
defined as in subsection §(d). Then \( q^\theta : A_S^{S''} \to A_S^{S''} \times_{X^{S'}} X^S \) has as a subgroup \( q^\theta : A_S \to A_S \times_{X^{S'}} X^S \), and the resulting map
\[
A_S^{S''} / A_S \to (A_S^{S''} / A_S') \times_{X^{S'}} X^S
\]
is an isomorphism.

(i). **Warning.** Observe that \( A_S^{S''} \) is an ind-\( A_S \)-torsor over the ind-scheme \( (A_S^{S''} / A_S') \times_{X^{S'}} X^S \), and the homomorphism \( q^\theta : A_S^{S''} \to A_S^{S''} \times_{X^{S'}} X^S \) is surjective. It is tempting therefore to try to view \( A_S^{S''} \) as being in some sense a torsor over \( A_S^{S''} \times_{X^{S'}} X^S \) for some group \( \ker q^\theta \). However, the kernel of the projective system
\[
((A_S)_{d} \to (A_S)'_{d} \times_{X^{S'}} X^S)_{d \in \mathbb{N}}
\]
of subsection §(d) is not Mittag-Leffler. We are not sure how to overcome this issue, so do not attempt to take this point of view.

\(^{11}\)In [Dri06] this is relaxed to ‘Nisnevich-locally’.
B(ii). **Global convolution diagram for $R$.**

(a). For a finite set $S$, we put

$$\mathcal{T}_S^{s'}(R) = \{(x, \mathcal{E}, f, \bar{v})\}/\sim$$

where $x \in X^S(R)$, $\mathcal{E}$ is a principal $G$-bundle on $\Delta_S$, $f$ is a trivialization of $\mathcal{E}$ on $\Delta_S^{s'}$, and $\bar{v}$ is an $N$-section of $\mathcal{E}$, taken up to equivalence. This is the same as the balanced product

$$\mathcal{T}_S^{s'} = G_S^{s'} \times_{X^S} N_S.$$

Thus, $\mathcal{T}_S^{s'}$ is represented by a reasonably nice ind-scheme with an ind-pro-smooth map to the Beilinson-Drinfeld grassmannian $G_S^{s'}/G_S$. In particular it is formally smooth. Multiplication gives us a map

$$\mathcal{T}_S^{s'} \to N_S^{s'}$$

and we define $R_S^{s'}$ to be the fiber product

$$R_S^{s'} := \mathcal{T}_S^{s'} \times_{N_S^{s'}} N_S.$$

Over any closed $X^S$-subscheme of $G_S^{s'}/G_S$, the embedding $R_S^{s'} \to \mathcal{T}_S^{s'}$ has finite codimension. Therefore $R_S^{s'}$ is also a reasonably nice ind-scheme, mapping to $G_S^{s'}/G_S$, and of ind-finite codimension in $\mathcal{T}_S^{s'}$. Note that $R_S^{s'}$ is not formally smooth, and in particular the map $R_S^{s'} \to G_S^{s'}/G_S$ is no longer ind-pro-smooth. As a functor we have

$$R_S^{s'}(R) = \{(x, \mathcal{E}, f, v)\}/\sim$$

where $x, \mathcal{E}, f$ are as in $\mathcal{T}_S^{s'}$, and $v$ is an $N$-section of $\mathcal{E}$ such that $f(v)$ extends to $\Delta_S$. We define the shifted dualizing complex on $R_S^{s'}$, $R_S^{s'}$ as for $\mathcal{T}, R$. Namely:

1. On each closed subscheme $(\mathcal{T}_S^{s'})^c$ of $(\mathcal{T}_S^{s'})^c$, pro-smooth over $(G_S^{s'}/G_S)^c$ we set

$$\omega_{(\mathcal{T}_S^{s'})^c}[-2 \dim N_S + 2|S|]$$

    to be the pullback of the dualizing complex of $(G_S^{s'}/G_S)^c$, i.e. the collection of its pullbacks to each formally smooth quotient $(\mathcal{T}_S^{s'})^c \hat{\to} (\mathcal{T}_S^{s'})^c$ smooth over $(G_S^{s'}/G_S)^c$, compatible under $*$-pullback;

2. Since $\mathcal{T}_S^{s'}$ is a reasonably nice ind-scheme, we can apply the $!$-pullback to such a collection of complexes on $(\mathcal{T}_S^{s'})^c$, and obtain one on $(\mathcal{T}_S^{s'})^c$. In this way, the collections $\omega_{(\mathcal{T}_S^{s'})^c}[-2 \dim N_S + 2|S|]$ are compatible under $!$-pullbacks. The resulting compatible collection is called $\omega_{\mathcal{T}_S^{s'}}[-2 \dim N_S + 2|S|]$.

3. Using the ind-finite codimensionality of the embedding $i: R_S^{s'} \to \mathcal{T}_S^{s'}$, we form a $!$-compatible collection of $*$-compatible collections of complexes

$$\omega_{R_S^{s'}}[-2 \dim N_S + 2|S|] := i'_! \omega_{\mathcal{T}_S^{s'}}[-2 \dim N_S + 2|S|].$$

---

\(^{12}\)It is a priori defined only on $\Delta_S^{s'}$. The extension is necessarily unique.
(b). We will apply the abbreviations of subsection §B(i)(b) to our spaces $\mathcal{R}, \mathcal{T}$ etc. so that for instance
\[ \mathcal{R}^{[2]}_{1,2} \]
becomes
\[ \mathcal{R}^2_{12}. \]
We will also write $X^S$ as $\Pi_{s \in S} X_s$, e.g. $X^{(1,2)} = X_1 \times X_2$. The obvious starting point for the global convolution diagram is $\mathcal{R}^1_1 \times \mathcal{R}^2_2$, a Zariski-locally trivial $\mathcal{R}$-bundle over $X_1 \times X_2$. Consider the following space:
\[ \mathcal{R}_{1+2}(R) = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2)\}/\sim \]
where $x_1, x_2$ are $R$-points of $X$, each $\mathcal{E}_i$ a principal $G$-bundle on $\Delta_{12}$, $f_i$ is a trivialization of $\mathcal{E}_i$ on $\Delta_{12}$, and $v_i$ is a $\mathbb{N}$-section of $\mathcal{E}_i$ such that $f_i(v_i)$ extends to $\Delta_{12}$. It is constructed as
\[ \mathcal{R}_{1+2} = \mathcal{R}^1_{12} \times_{x_1 \times x_2} \mathcal{R}^2_{12}, \]
a reasonably nice ind-scheme over $X_1 \times X_2$. It is of ind-finite codimension in the formally smooth reasonably nice ind-scheme
\[ \mathcal{T}_{1+2} = T^1_{12} \times_{x_1 \times x_2} T^2_{12} = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, \tilde{v}_1, \tilde{v}_2)\}/\sim. \]
There is a map
\[ \alpha: \mathcal{R}_{1+2} \to \mathcal{R}^1_1 \times \mathcal{R}^2_2 \]
given by restricting $\mathcal{E}_i, f_i, v_i$ to $\Delta_i \subset \Delta_{12}$. Over the diagonal $X_0 \subset X_1 \times X_2$, this map $\alpha$ is an isomorphism. But on the complement $U$ of the diagonal, we have a canonical isomorphism
\[ \mathcal{R}_{1+2}|_U = (\mathcal{R}^1_1 \times \mathcal{R}^2_2)|_U \times_U (\mathcal{N}^1_1 \times \mathcal{N}^2_2)|_U \]
and $\alpha$ is just the projection. Nonetheless, $\alpha$ is \textit{ind-pro-smooth}. Indeed, it is the product over $X_1 \times X_2$ of maps
\[ \mathcal{R}^1_{12} \to \mathcal{R}^1_1 \times X_2, \]
\[ \mathcal{R}^2_{12} \to \mathcal{R}^2_2 \times X_1; \]
so it suffices to see that the former is ind-pro-smooth. But note that we can write
\[ T^1_1 \times X_2 = G^1_1 \times_{G^1_1} \mathcal{N}^1_1 \times X_2 = G^1_{12} \times_{G^1_{12}} \mathcal{N}^1 \]
where $G_{12}$ acts on $\mathcal{N}^1$ via the homomorphism $G_{12} \to G_1$. Then, the natural map
\[ T^1_{12} \to T^1_1 \times X_2 \]
is that associated to the pro-smooth map $\mathcal{N}_{12} \to \mathcal{N}^1$, so is ind-pro-smooth. The fact that the diagram
\[ \begin{array}{ccc} \mathcal{R}^1_{12} & \to & \mathcal{R}^1_1 \times X_2 \\ \downarrow & & \downarrow \\ T^1_{12} & \to & T^1_1 \times X_2 \end{array} \]
is Cartesian gives the result. We have:
\[ (B.3) \quad \alpha^* \omega_{\mathcal{R}^1_1 \times \mathcal{R}^2_2} \sim [-2 \dim \mathcal{N}^1_1 \times \mathcal{N}^2_2] \cong \omega_{\mathcal{R}_{1+2}} \sim [-2 \dim \mathcal{N}_{12} \times x_1 \times x_2 \mathcal{N}_{12}]. \]
Note that $\mathcal{R}_1 \times \mathcal{R}_2$, $\mathcal{T}_1 \times \mathcal{T}_2$ are acted on factor-wise by $G_1 \times G_2$, which receives the factor-wise map from $G_{12} \times X_1 \times X_2 G_{12}$. This latter group also acts in the natural way on $\mathcal{R}_{1+2}$, $\mathcal{T}_{1+2}$, and the diagram

\[
\begin{array}{c}
\mathcal{R}_{1+2} \rightarrow \mathcal{R}_1 \times \mathcal{R}_2 \\
\downarrow \\
\mathcal{T}_{1+2} \rightarrow \mathcal{T}_1 \times \mathcal{T}_2 
\end{array}
\]

is $G_{12} \times X_1 \times X_2 G_{12}$-equivariant. This action preserves the inductive structure of the diagram, and also the locally nice structure of each closed piece, which allows us to view the appropriately shifted dualizing complex on each space as $G_{12} \times X_1 \times X_2 G_{12}$-equivariant. We may thus define the shifted equivariant Borel-Moore homologies:

\[
H^G_{* - 2 \dim N_1 \times N_2} (\mathcal{R}_1 \times \mathcal{R}_2),
\]

\[
H^G_{* - 2 \dim N_1 \times N_2} (\mathcal{R}_1 \times \mathcal{R}_2),
\]

\[
H^G_{* - 2 \dim N_{12} \times X_1 \times X_2 N_{12}} (\mathcal{R}_{1+2}),
\]

as the colimits of the equivariant cohomologies of the appropriately shifted dualizing complexes on the various finite-dimensional approximations. We have maps

\[
H^G_{* - 2 \dim N_1 \times N_2} (\mathcal{R}_1 \times \mathcal{R}_2) \rightarrow H^G_{* - 2 \dim N_1 \times N_2} (\mathcal{R}_1 \times \mathcal{R}_2) \rightarrow H^G_{* - 2 \dim N_{12} \times X_1 \times X_2 N_{12}} (\mathcal{R}_{1+2}).
\]

The first map is the restriction of the equivariant structure, while the second is induced by $\alpha^*$, using equation (B.3). This is the first step of our global convolution story.

(c). Let’s define the remaining parts of the global convolution diagram. We set

\[
\tilde{\mathcal{R}}_{1+2} = \{(x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2, g_1) / \sim \}
\]

where $x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2$ are as in $\mathcal{R}_{1+2}$, and $g_1$ is a trivialization of $\mathcal{E}_1$ (on $\Delta_{12}$) required to satisfy:

\[
g_1 v_1 = f_2 v_2.
\]

Note that $v_1$ is determined by the rest of the data as $v_1 = g_1^{-1} f_2 v_2$. That is, $\tilde{\mathcal{R}}_{1+2}$ is related to

\[
\tilde{\mathcal{T}}_{1+2} := \{(x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_2, g_1) / \sim = G^L_{12} \times X_1 \times X_2 \mathcal{R}^2_{12}
\]

by the Cartesian square

\[
\begin{array}{c}
\tilde{\mathcal{R}}_{1+2} \rightarrow \tilde{\mathcal{T}}_{1+2} \\
\downarrow \\
\mathcal{R}_{12} \rightarrow \mathcal{T}_{12}
\end{array}
\]

where the rightmost downward arrow is the composition

\[
\tilde{\mathcal{T}}_{1+2} = G^L_{12} \times X_1 \times X_2 \mathcal{R}^2_{12} \rightarrow G^L_{12} \times X_1 \times X_2 N_{12} \rightarrow G^L_{12} \times X_1 \times X_2 \frac{N_{12}}{G_{12}} = \mathcal{T}^L_{12}.
\]
We have factor-wise actions of \( G_{12} \times \times X_1 \times X_2 \) \( G_{12} \) on \( \mathcal{R}_{1+2} \), \( \mathcal{T}_{1+2} \), such that the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{R}_{1+2} & \xrightarrow{\beta} & \mathcal{R}_{1+2} \\
\downarrow & & \downarrow \\
\mathcal{T}_{1+2} & \xrightarrow{\rho} & \mathcal{T}_{1+2} \times X_1 \times X_2 \end{array}
\]

is equivariant. In terms of points, the left-hand \( G_{12} \) acts by changing the trivialization \( f_1 \), while the right-hand factor acts by changing simultaneously the trivializations \( g_1, f_2 \); \( \beta \) is the map which simply forgets \( g_1 \). The right-hand \( G_{12} \) acts freely, and the quotient space is

\[
\mathcal{R}_{1+2} = \left\{ ((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2) \right\} / \sim
\]

where \( x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, v_1 \) are as in \( \mathcal{R}_{1+2} \), while \( g_1^{-1} f_2 \) is an isomorphism from \( \mathcal{E}_2 \) to \( \mathcal{E}_1 \) over \( \Delta_{12}^2 \), and \( v_2 \) is an \( \mathbb{N} \)-section of \( \mathcal{E}_2 \) such that \( g_1^{-1} f_2 v_2 \) extends to \( \Delta_{12} \) and is equal to \( v_1 \) there (again \( v_1 \) is determined by the rest of the data). We write

\[
\gamma: \mathcal{R}_{1+2} \to \mathcal{R}_{1+2}
\]

for the projection. It is ind-pro-smooth. Finally, we have a natural map

\[
\delta: \mathcal{R}_{1+2} \to \mathcal{R}_{1+2}^{12} = \left\{ ((x_1, x_2), \mathcal{E}, f, v) \right\} / \sim
\]

\[
((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2) \mapsto ((x_1, x_2), \mathcal{E}_2, f_1 g_1^{-1} f_2, v_2).
\]

Note that \( \delta \) factors as \( \delta = \delta' \delta'' \) where \( \delta' : \mathcal{R}_{1+2} \to \bullet \) is an ind-closed embedding of finite codimension and \( \delta' : \bullet \to \mathcal{R}_{12}^{12} \) is defined by the Cartesian square

\[
\begin{array}{ccc}
\mathcal{R}_{1+2} & \xrightarrow{\delta'} & \mathcal{R}_{12}^{12} \\
\downarrow & & \downarrow \\
G_{12} \times X_1 \times X_2 \mathcal{G}_{12}^2 / G_{12} & \xrightarrow{d} & G_{12}^2 / G_{12}
\end{array}
\]

where the bottom row is simply the top row for \( \mathbb{N} = 0 \), and the vertical maps forget \( v_1, v_2, v \). It is well-known that \( d \) is ind-projective; this fact shows up already in [MV07] and essentially follows from Lemma B.1. It follows that \( \delta \) is also ind-projective, meaning that in each piece of the inductive structure, \( \delta \) is Zariski-locally of the form

\[
Y \times \mathbb{A} \xrightarrow{f \times \text{id}} Z \times \mathbb{A}
\]

for \( f : Y \to Z \) a projective map between schemes of finite type, and \( \mathbb{A} \) some affine space of countable dimension. In fact, \( \delta \) is an isomorphism over \( U \), while over the diagonal its fibers are products of closed subvarieties of affine Grassmannians. Furthermore, \( \delta \) is \( G_{12} \)-equivariant.

(d). The global convolution diagram is

\[
\mathcal{R}_{1} \times \mathcal{R}_{2}^{2} \xleftarrow{\alpha} \mathcal{R}_{1+2} \xleftarrow{\beta} \mathcal{R}_{1+2} \xrightarrow{\gamma} \mathcal{R}_{1+2} \xrightarrow{\delta} \mathcal{R}_{1+2}^{12}.
\]
As we have explained, \( \alpha, \beta \) are \( G_{12} \times X_1 \times X_2 \) \( G_{12} \)-equivariant, \( \gamma \) is the quotient map by the free action of the right-hand \( G_{12} \), and \( \delta \) is equivariant for the remaining copy of \( G_{12} \). We have already explained how \( \alpha \) defines a map
\[
\alpha^* : H_{* - 2 \dim N_1 \times N_2}^{G_{12} \times G_2} (\mathcal{R}_{1+2}^{1} \times \mathcal{R}_{2}^{2}) \to H_{* - 2 \dim N_1 \times N_2 \times N_{12}}^{G_{12} \times X_1 \times X_2 G_{12}} (\mathcal{R}_{1+2}).
\]

Everything else works out essentially as in the main paper, as we now indicate. First, recall the \( G_{12} \times X_1 \times X_2 \) \( G_{12} \)-equivariant Cartesian diagram (B.4):
\[
\widetilde{\mathcal{R}}_{1+2} \xrightarrow{\beta} \mathcal{R}_{1+2} \\
\downarrow \\
\widetilde{T}_{1+2} \xrightarrow{b} T_{12}^{1} \times X_1 \times X_2 \mathcal{R}_{12}^{2}
\]
and recall that \( \widetilde{T}_{1+2} \) is nothing other than \( G_{12}^{1} \times X_1 \times X_2 \mathcal{R}_{12}^{2} \). Thus we may write
\[
b = \text{pr}_1 b \times X_1 \times X_2 \text{pr}_2 b \\
\text{pr}_1 b = \psi \phi \\
\text{pr}_2 b = \text{pr}_2
\]
where we have factored \( \text{pr}_1 b \) as
\[
G_{12}^{1} \times X_1 \times X_2 \mathcal{R}_{12}^{2} \xrightarrow{\phi} G_{12}^{1} \times X_1 \times X_2 N_{12} \xrightarrow{\psi} T_{12}^{1}.
\]

It follows that
\[
b^* \omega_{T_{12}^{1} \times X_1 \times X_2 \mathcal{R}_{12}^{2}} [-2 \dim N_{12} \times X_1 \times X_2 N_{12}] \approx \omega_{\widetilde{R}_{1+2}} [-2 \dim N_{12} \times X_1 \times X_2 G_{12}]
\]
and hence by base change we have have a map
\[
(\mathbf{B.5}) \quad \beta^* \omega_{\mathcal{R}_{1+2}} [-2 \dim N_{12} \times X_1 \times X_2 N_{12}] \to \omega_{\widetilde{R}_{1+2}} [-2 \dim N_{12} \times X_1 \times X_2 G_{12}].
\]
This map is equivariant, and it therefore determines a ‘pullback with support’ map:
\[
\beta^* : H_{* - 2 \dim N_{12} \times X_1 \times X_2 N_{12}}^{G_{12} \times X_1 \times X_2 G_{12}} (\mathcal{R}_{1+2}) \to H_{* - 2 \dim N_{12} \times X_1 \times X_2 G_{12} G_{12}} (\mathcal{R}_{1+2}).
\]
Since it is a \( G_{12} \)-torsor, \( \gamma \) induces an isomorphism
\[
\gamma^* : H_{* - 2 \dim N_{12}}^{G_{12}} (\mathcal{R}_{1+2}) \approx H_{* - 2 \dim N_{12} \times X_1 \times X_2 G_{12} G_{12}} (\mathcal{R}_{1+2}).
\]
Finally since it is ind-proper and equivariant, \( \delta \) induces a map
\[
\delta^* : H_{* - 2 \dim N_{12}}^{G_{12}} (\mathcal{R}_{1+2}) \to H_{* - 2 \dim N_{12}}^{G_{12}} (\mathcal{R}_{12}^{12}).
\]
(e) Recall that (dual) specialization maps commute with pullbacks along smooth maps and pushforwards along proper maps, and are compatible with equivariance with respect to smooth group schemes. Therefore, since every space in sight is a reasonably nice ind-scheme and the groups \( G_S \) are pro-smooth over \( X^S \), we have (dual) specialization maps to
the diagonal $X_0 \subset X_1 \times X_2$:

$$
\begin{align*}
\text{s1: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \\
\text{s2: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \\
\text{s3: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \\
\text{s4: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \\
\text{s5: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \\
\text{s6: } & H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right)
\end{align*}
$$

Here $\tilde{R}_0$, $R_0$ are respectively locally trivial $p^{-1}(R \times R)$, $q(p^{-1}(R \times R))$-bundles over $X_0$ in the notations of diagram (3.2). In fact, the restriction of the convolution diagram to $X_0$ induces the following maps between the targets of the specialization maps:

$$
H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \xrightarrow{id} H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right) \xrightarrow{id} H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right)
$$

I claim that the maps $\alpha^*, \beta^*, (\gamma^*)^{-1}, \delta_*$ are intertwined with $id, \beta_0^*, (\gamma_0^*)^{-1}, (\delta_0)_*$ by the (dual) specialization maps. For $\alpha^*$, $(\gamma^*)^{-1}$ it is a consequence of ind-pro-smoothness of $\alpha, \gamma$ (and also pro-smoothness of $G_{12}$). For $\delta_*$ it is a consequence of ind-properness. For $\beta^*$, it is because the map

$$
(\beta_0)^* \omega_{R_0^0 \times X_0 N_0} \left[ -2 \dim N_0 \times X_0 N_0 \right] \to \omega_{\tilde{R}_0} \left[ -2 \dim N_0 \times X_0 G_0 \right]
$$

defined using the Cartesian square:

$$
\begin{array}{ccc}
\tilde{R}_0 & \xrightarrow{\beta_0} & R_0^0 \\
\downarrow & & \downarrow \\
\tilde{T}_0 & \xrightarrow{b_0} & T_0^0 \times X_0 R_0^0
\end{array}
$$

obtained by restricting diagram (B.4) to $X_0$, factors as:

$$
(\beta_0)^* \omega_{R_0^0 \times X_0 N_0} \left[ -2 \dim N_0 \times X_0 N_0 \right] \xrightarrow{\cong} (\beta_0)^* i_1^* \omega_{R_{12}^0} \left[ -2 \dim N_{12} \times X_1 \times X_2 N_{12} + 2 \right] \xrightarrow{\cong} i_2^* \beta^* \omega_{R_{12}^0} \left[ -2 \dim N_{12} \times X_1 \times X_2 N_{12} + 2 \right]
$$

Here $can$ is the canonical map arising from the base change isomorphism, $(B.5)$ denotes the map of equation (B.5), and $i_1, i_2$ denote the appropriate inclusions of the diagonal subspaces. The consequence is the following formula:

$$
s_6 \delta_*(\gamma^*)^{-1} \beta^* \alpha^* = (\delta_0)_*(\gamma_0^*)^{-1} \beta_0^* s_1: H^G_{G_1 \times G_2} \left( R_1^1 \times R_2^2 \right) \to H^G_{G_0 \times X_0} \left( R_0^0 \times X_0 \right).
$$
(f). Now each (dual) specialization map $s_n$ factors as $s'_n j^*_n$ where $j^*_n$ is the restriction map to the equivariant Borel-Moore homology of the part lying over $U$, and $s'_n$ is some other map. Furthermore, the restriction of the convolution diagram to $U$ induces the following maps between the targets of the restriction maps:

$$\begin{align*}
H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^1)|_U &

\to

H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U \\
H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U &

\to

H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U \\
H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U &

\to

H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U \\
H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U &

\to

H_{s-2\dim(N_1 \times N_2)}^{(G_1 \times G_2)}|_U (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U
\end{align*}$$

Let us explain what each map does:

1. The first map views any $(G_1 \times G_2)|_U$-equivariant class as also equivariant for the trivial actions of the left-hand copy of $G_2$, and the right-hand copy of $G_1$, in $(G_1 \times G_2)|_U (G_1 \times G_2)|_U$.
2. The second map pulls this back along the $(N_2 \times N_1)|_U$-bundle map (i.e. multiplies fiberwise by the equivariant fundamental class of $N(O) \times N(O)$).
3. The third map starts by rewriting $(\mathcal{R}_1^1 \times N_2)|_U (N_1 \times \mathcal{R}_2^2)|_U$ as $((\mathcal{R}_1^1 \times N_1) \times (N_2 \times \mathcal{R}_2^2))|_U$, and rewriting the action of $(G_1 \times G_2)|_U (G_1 \times G_2)|_U$ as one of $((G_1 \times G_1) \times (G_2 \times G_2))|_U$. By definition, $\tilde{\mathcal{R}}_1$ is the locally trivial $p^{-1}(\mathcal{R} \times \mathcal{R})$-bundle on $X_1$ given as

$$\tilde{\mathcal{R}}_1 = N_1 \times N_1^1 (G_1 \times G_1).$$

The $G_1 \times G_1$-equivariant map from here to $\mathcal{R}_1^1 \times \mathcal{R}_2^1$ is given as the product (over $X_1$) of the quotient by the right-hand copy of $G_1$ with the projection to the right-hand copy of $N_1$. The ‘pullback with support’ map

$$H_{s-2\dim N_1 \times N_1}^{G_1 \times G_1} (\mathcal{R}_1^1 \times N_1) \to H_{s-2\dim N_1 \times N_1}^{G_1 \times G_1} (\tilde{\mathcal{R}}_1)$$

corresponds to the composition of usual ‘pullback with support’ (spread out over $X_1$) with multiplication by $H_{G_1}^*(X_1)$ under the identification

$$H_{s-2\dim N_1 \times N_1}^{G_1 \times G_1} (\mathcal{R}_1^1 \times N_1) = H_{s-2\dim N_1}^*(\mathcal{R}_1^1) \otimes H^*(X_1) H_{G_1}^*(X_1).$$

Meanwhile, the ‘pullback with support’ (actually, here no support is required) map

$$H_{s-2\dim N_2 \times N_2}^{G_2 \times G_2} (N_2 \times N_2) (G_2 \times G_2) \to H_{s-2\dim G_2 \times N_2}^{G_2 \times G_2} (G_2 \times G_2)$$

is isomorphic simply to the multiplication map

$$H_{G_2}^* (X_2) \otimes H^*(X_2) H_{s-2\dim N_2}^{G_2} (\mathcal{R}_2^2) \to H_{s-2\dim N_2}^{G_2} (\mathcal{R}_2^2).$$

4. The fourth map is the isomorphism, and the fifth is the identity.
The result is that the composition of all these maps is the identity. On the other hand, since the restriction maps $j^*_n$ intertwine these maps with the corresponding maps on the $X_1 \times X_2$ level, we have the following:

$$(\delta_0)_*(\gamma^*)_1^{-1}\beta^*_0s_1 = s_0\delta_*(\gamma^*)_1^{-1}\beta^*_0\alpha^*_s = s_0'j^*_0\delta_*(\gamma^*)_1^{-1}\beta^*_0\alpha^*_s = s_0'j^*_1.$$ 

(g). Finally, note that this last map $s_0'j^*_1$ is symmetric with respect to the automorphism $\tau$ of $H_{* - 2\dim N_1 \times N_2}^{G_1 \times G_2}(R_1^1 \times R_2^2)$ induced by the degree 2 automorphisms of $G_1 \times G_2$, $R_1^1 \times R_2^2$ which switch the factors (and also exchange 1 with 2). Therefore, $(\delta_0)_*(\gamma^*)_1^{-1}\beta^*_0s_1$ has the same property. But, taking $X = \mathbb{C}$, we identify the domain

$$H_{* - 2\dim N_1 \times N_2}^{G_1 \times G_2}(R_1^1 \times R_2^2) = H_{* - 2\dim N(O)}^{G(O)}(\mathcal{R}) \otimes H_{* - 2\dim N(O)}^{G(O)}(\mathcal{R})$$

and the target

$$H_{* - 2\dim N_0}^{G_0}(R_0^1) = H_{* - 2\dim N(O)}^{G(O)}(\mathcal{R}).$$

The map $(\delta_0)_*(\gamma^*)_1^{-1}\beta^*_0s_1$ is the usual convolution ($s_1$ is an isomorphism) while $\tau$ is the standard twist. Therefore, the Coulomb branch is commutative as claimed.

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