DIRICHLET TO NEUMANN OPERATORS AND THE $\bar{\partial}$-NEUMANN PROBLEM

DARIUSH EHSANI

ABSTRACT. We study the Dirichlet to Neumann operator of the $\bar{\partial}$-Neumann problem, and the relation between the $\bar{\partial}$-Neumann boundary conditions and the Dirichlet to Neumann operator.

1. INTRODUCTION

The $\bar{\partial}$-Neumann problem is an example of a boundary value problem involving an elliptic operator but whose boundary conditions lead to non-elliptic equations. In order to conclude Sobolev estimates for the solution to the $\bar{\partial}$-Neumann problem control (of $L^2$-norms) over derivatives in all directions must be obtained, but the boundary conditions of the problem disadvantage one direction. The boundary conditions contain the boundary value operator, the Dirichlet to Neumann operator (DNO), giving the boundary values of the outward derivative of the solution to a homogeneous Dirichlet problem. In some cases (for example the case of strictly pseudoconvex domains) the DNO allows for some control of the disadvantaged direction, in other cases of weak pseudoconvexity, the situation is more delicate. The purpose of this article therefore, is to study the DNO of related to the $\bar{\partial}$-Neumann problem with particular emphasis on the resulting boundary equations.

The DNO will be written as a pseudodifferential operator acting on a boundary distribution, and our first results are a reworking of results of Chang, Nagel, and Stein in [1]. It is well known that to highest order the DNO is given by the square root of the highest order tangential terms in the elliptic interior operator. The highest two orders of the DNO are calculated, as in [1], and reduce to those results in a special case. The approach of [1] could be used here as well to calculate the DNO, but we take another approach outlined in [4] based on pseudodifferential operators on domains with boundary, an approach which was useful in calculating the symbol of the normal derivative to the Green’s operator, as well permitting similar calculations and estimates in the situation of piecewise smooth domains [5]. Relations among operators comprising the DNO, as well as other derived boundary

2010 Mathematics Subject Classification. 32W05, 32W10, 32W25, 32W50.
value operators in the boundary conditions are essential in the construction of a solution to the $\bar{\partial}$-problem if a solution operator to $\bar{\partial}_b$ is assumed in [3].

We further demonstrate in this paper the persistence of the non-elliptic character of the $\bar{\partial}$-Neumann conditions. In Section 7, we examine what happens when a perturbation is made of the elliptic operator of the problem. This change naturally also leads to a different DNO, however as we shall see the associated boundary condition is essentially the same (and non-elliptic!). The boundary operator can be approximated by Kohn’s Laplacian, $\Box_b$. This suggests that the $\bar{\partial}$-Neumann problem can be solved by inverting the $\Box_b$ operator and that the $\bar{\partial}$-problem can be solved by using a solution operator for $\bar{\partial}_b$. This approach to $\bar{\partial}$ is taken up in [3].

Most the work presented here was undertaken while the author was at the University of Wuppertal and the hospitality of the University and its Complex Analysis Working Group is sincerely appreciated. The author particularly thanks Jean Ruppenthal for his warm and generous invitation to work with his group. A visit to the Oberwolfach Research Institute in 2013 as part of a Research in Pairs group was also helpful in the formation of this article, for which the author extends gratitude to the Institute as well as to Sönmez Şahutoğlu for helpful discussions.

2. Notation and Background

We fix some notation used throughout the article. Our notation for derivatives is $\partial_t := \frac{\partial}{\partial t}$. We also use the index notation for derivatives: with $a = (a_1, \ldots, a_n)$ a multi-index

$$\partial^a_x = \partial^{a_1}_{x_1} \cdots \partial^{a_n}_{x_n}.$$ 

Multiplication of derivatives with $-i$ come in handy when dealing with symbol expansions of pseudodifferential operators and we will use the notation $D^a_x$ to denote $-i\partial^a_x$.

We let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain and define pseudodifferential operators on $\Omega$ as in [12]:

**Definition 2.1.** We denote by $S^a(\Omega)$ the space of symbols $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ which have the property that for any given compact set, $K$, and for any $n$-tuples $k_1$ and $k_2$, there is a constant $c_{k_1,k_2}(K) > 0$ such that

$$\left| D^{k_1}_{\xi} D^{k_2}_{x} a(x, \xi) \right| \leq c_{k_1,k_2}(K) (1 + |\xi|)^{a-|k_1|} \quad \forall x \in K, \xi \in \mathbb{R}^n.$$ 

Associated to the symbols in class $S^a(\Omega)$ are the pseudodifferential operators, denoted by $\Psi^a(\Omega)$. If $u \in \mathcal{E}'(\Omega)$, we can define $u \in \mathcal{E}'(\mathbb{R}^n)$ by using an extension by 0, and then define the Fourier Transform of the extended $u$. We denote the transform of the extended distribution simply by $\hat{u}(\xi)$. The definition of pseudodifferential operators on a domain $\Omega$ is given by
Definition 2.2. We say an operator $A : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is in class $\Psi^a(\Omega)$ if $A$ can be written as an integral operator with symbol $a(x, \xi) \in S^a(\Omega)$:

$$Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} d\xi.$$  

(2.1)

In our applications in this article we will be dealing with operators defined on all of $\mathbb{R}^{2n}$ applied to functions defined on $\Omega$ (but which can be extended by 0 to the whole space). The operators on $\Omega$ will thus be the composition of the restriction to $\Omega$ operator with the pseudodifferential operators defined on $\mathbb{R}^{2n}$.

If we let $\chi_j$ be such that $\{\chi_j \equiv 1\}_j$ is a covering of $\Omega$, and let $\varphi_j$ be a partition of unity subordinate to this covering, then locally, we describe a boundary operator $A : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ in terms of its symbol, $a(x, \xi)$ according to

$$Au = \frac{1}{(2\pi)^n} \int a(x, \xi) \bar{\chi}_j \hat{u}(\xi) d\xi$$

on supp $\varphi_j$. Then we can describe the operator $A$ globally on all of $\Omega$ by

$$Au = \frac{1}{(2\pi)^n} \sum_j \varphi_j \int a(x, \xi) \bar{\chi}_j \hat{u}(\xi) d\xi.$$  

(2.2)

The difference arising between the definitions in (2.1) and (2.2) is a smoothing term $\hat{u}$, which we write as $\Psi^{-\infty}_u$, to use the notation of Definition 2.2.

While $\Psi^a(\Omega)$ will denote a class of operators, the use of $\Psi^a$ will be used to refer to any operator in class $\Psi^a(\Omega)$. Furthermore, operators defined on the boundary of a domain will be denoted with a subscript $b$. For instance, if $A \in \Psi^a(\partial \Omega)$ we write $A = \Psi^a_b$.

In our use of Fourier transforms and equivalent symbols we use cutoffs in order to make use of local coordinates, one of which being a defining function, denoted by $\rho$, for the domain. We use $-\hat{\cdot}$ to indicate transforms in tangential directions. Let $p \in \partial \Omega$ and let $(x_1, \ldots, x_{n-1}, \rho)$ be local coordinates around $p$, $(\rho < 0)$. Let $\chi(p, x, \rho)$ denote a cutoff which is $\equiv 1$ near $p$ and vanishes outside a small neighborhood of $p$ on which the local coordinates $(x, \rho)$ are valid. Then with $u \in L^2(\Omega)$ we write

$$\bar{\chi}_p \hat{u}(\xi, \eta) = \int \chi(p, x, \rho) e^{-ix \cdot \xi} e^{-i\eta \cdot \rho} dxd\rho$$

$$\bar{\chi}_p \hat{u}(\xi, \rho) = \int \chi(p, x, \rho) e^{-ix \cdot \xi} dx.$$  

We also use the $\hat{-}$ notation when describing transforms of functions supported on the boundary. With notation and coordinates as above, we let $u_b(x) \in L^2(\partial \Omega)$ and write

$$\chi(p, 0) u_b(\xi) = \int \chi(p, 0) u_b(x) e^{-ix \cdot \xi} dx.$$  

We want to apply pseudodifferential operator techniques to vector fields on a smoothly bounded domain $\Omega \subset \mathbb{C}^n$. Let $\rho$ be a smooth defining function for $\Omega$.
\( \Omega = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \), normalized so that \(|\nabla \rho| = 1\) on \(\partial \Omega\). We choose an orthonormal basis of \((1,0)\) forms, \(\omega_1, \ldots, \omega_n\) in which \(\omega_n = \sqrt{2} \partial \rho\), and we denote \(L_1, \ldots, L_n\) the vector fields respectively dual to the \(\omega_j\).

We let \(T = \frac{1}{\sqrt{2}} (L_n - T_n)\), and \(T^0 = T|_{\partial \Omega}\). If we choose a boundary point \(p\) we can choose local coordinates, as above, in a neighborhood of \(p\) such that \(L_n\) has the form

\[
L_n = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + iT
= \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + iT^0 + O(\rho)
= \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + i \frac{\partial}{\partial x_{2n-1}} + O(\rho).
\]

Similarly, in a neighborhood of \(p\), we can represent the \(L_j\) vector fields as

\[
L_j = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right) + \sum_{k=1}^{2n-1} \ell_k^j(x-p) \frac{\partial}{\partial x_k} + O(\rho),
\]

where \(\ell_k^j(x) = O(\rho)\).

In Fourier space we use \(\xi\) to denote the dual variables to the \(x\) coordinates, \(\xi_i\) corresponding to \(x_i\) for \(i = 1, \ldots, 2n-1\), and \(\eta\) dual to \(\rho\). To help distinguish the complex tangential behavior, we set \(\xi_L^2\) to be given by

\[
\xi_L^2 = \sum_{i=1}^{2n-2} \xi_i^2.
\]

We use the standard decomposition of the Fourier transform space to separate three microlocal neighborhoods (see for instance [2, 7, 8, 10]). We let \(\psi^+, \psi^0, \text{ and } \psi^-\) be a smooth partition of unity on the unit ball, \(|\xi| = 1\). We choose the functions so that \(\psi^+\) has support in \(\xi_{2n-1} > \frac{1}{4} |\xi_L|\) and \(\psi^+ \equiv 1\) in the region \(\xi_{2n-1} > \frac{3}{4} |\xi_L|\). The function \(\psi^-\) is defined symmetrically: \(\psi^-\) has support in \(\xi_{2n-1} < -\frac{1}{4} |\xi_L|\) and \(\psi^- \equiv 1\) in the region \(\xi_{2n-1} < -\frac{3}{4} |\xi_L|\). The function \(\psi^0\) has support in \(|\xi_{2n-1}| < \frac{3}{4} |\xi_L|\) and satisfies \(\psi^0 = 1 - \psi^+ - \psi^-\). We extend the functions radially, so that, in particular, they satisfy

\[
|\partial^k_\xi \psi^*| \lesssim |\xi|^{-k}
\]

outside of some compact neighborhood of \(\xi = 0\). This last property ensures that \(\psi^+, \psi^0, \text{ and } \psi^-\) are in the class of symbols, \(\mathcal{S}^0(\mathbb{R}^{2n-1})\). Cutoffs are also introduced so that (using the same notation for the functions) \(\psi^0 \equiv 1\) on a neighborhood of \(\xi = 0\) contained in \(|\xi| < 1\). The radial extensions from the unit circle together with the support of \(\psi^0\) near 0 are then to form a partition of unity of the transform space, i.e., \(\psi^+ + \psi^0 + \psi^- = 1\) for all \(\xi \in \mathbb{R}^{2n-1}\). The operators corresponding to the symbols, \(\psi^+, \psi^0, \text{ and } \psi^-\), will be denoted by \(\Psi^+, \Psi^0, \text{ and } \Psi^-\), respectively.
As mentioned above, we take an approach to calculating boundary value operators based on a pseudodifferential calculus for domains with boundary worked out in [4]. In particular, we will make use of the results detailing the behavior of functions which result from the application of pseudodifferential operators (in $\mathbb{R}^{2n}$) to distributions supported on the boundary of the domain, as well as certain operators applied to distributions with support in the whole domain (which can be thought of as a distribution on all of $\mathbb{R}^{2n}$ with an extension by 0). Let us recall here a few results from [4]. The results in [4] were stated for half-planes and these will be applied to domains $\Omega \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$, using local coordinates $(x, \rho)$ with $\rho < 0$ defining the domain.

We first define certain operators which appear in taking inverses to elliptic operators:

**Definition 2.3.** Let $A \in \Psi^{-k}(\mathbb{R}^{2n})$ for $k \geq 1$ have the property that for any $N \in \mathbb{N}$, it can be written in the form

$$A = B + \Psi^{-N},$$

where $B \in \Psi^{-k}(\mathbb{R}^{2n})$ has symbol, $\sigma(B)(x, \rho, \xi, \eta)$, which is meromorphic (in $\eta$) with poles at

$$\eta = q_1(x, \rho, \xi), \ldots, q_k(x, \rho, \xi)$$

with $q_i(x, \rho, \xi)$ themselves, as well as the imaginary parts, $\text{Im } q_i$, symbols of pseudodifferential operators of order 1 (restricted to $\eta = 0$) such that for each $\rho$, $\text{Res}_{\eta=q_i} \sigma(B) \in S^{k+1}(\mathbb{R}^{2n-1})$ with symbol estimates uniform in the $\rho$ parameter.

We call such an operator, $A$, a *decomposable* operator.

The first theorem is taken from Theorems 2.2 and 2.4 of [4].

**Theorem 2.4.** Let $g \in \mathcal{D}(\Omega)$ of the form $g(x, \rho) = g_b(x) \delta(\rho)$ for $g_b \in W^s(\partial \Omega)$. Let $A \in \Psi^k(\Omega)$, $k \leq -1$ be a decomposable operator. Then for all $s$,

$$\|Ag\|_{W^s(\Omega)} \lesssim \|g_b\|_{W^{s+k+1/2}(\partial \Omega)}.$$

Theorem 2.4 for instance is applicable for any term arising in the symbol expansion of the inverse to an elliptic differential operator. With $A$ an elliptic differential operator of order $k$, we can write for any $N \in \mathbb{N}$,

$$A^{-1} = B_{-k} + \Psi^{-N} \quad (2.4)$$

where $B_{-k}$ is a pseudodifferential operator of order $-k$ and satisfies the conditions of the $B$ operator in Definition 2.3.

Thus for instance if $\triangle$ is a second order elliptic differential operator on $\mathbb{R}^{2n}$, and for some given $s \geq 0$, $g_b \in W^s(\partial \Omega)$, with $g = g_b \times \delta(\rho)$ as above. Then $\triangle^{-1}$
satisfies the hypothesis of Theorem 2.4 and we have
\[ \|\triangle^{-1}g\|_{W^{3,2}(\Omega)} \lesssim \|g_b\|_{W^3(\partial\Omega)}. \]

We also have the following useful Lemmas.

**Lemma 2.5.** Let \( g \in \mathcal{D}(\Omega) \) be of the form \( g(x,\rho) = g_b(x)\delta(\rho) \) for \( g_b \in \mathcal{D}(\partial\Omega) \). Let \( A \in \Psi^k(\Omega) \), be a pseudodifferential operator of order \( k \). Let \( \rho \) denote the operator of multiplication with \( \rho \). Then \( \rho \circ A \) induces a pseudodifferential operator of order \( k-1 \) on \( g \):
\[ \rho A g \equiv \Psi^{k-1}g. \]

Let \( R \) denote the restriction operator, \( R : \mathcal{D}(\Omega) \to \mathcal{D}(\partial\Omega) \), given in local coordinates \((x,\rho)\) by \( R\phi = \phi|_{\rho=0} \).

**Lemma 2.6.** Let \( g \in \mathcal{D}(\Omega) \) be of the form \( g(x,\rho) = g_b(x)\delta(\rho) \) for \( g_b \in \mathcal{D}(\partial\Omega) \). Let \( A \in \Psi^k(\Omega) \), be an operator of order \( k \), for \( k \leq -2 \). Then \( R \circ A \) induces a pseudodifferential operator in \( \Psi^{k+1}_{\rho}(\partial\Omega) \) acting on \( g_b \) via
\[ R \circ Ag \equiv \Psi^{k+1}_{\rho}g_b. \]

## 3. \( \partial \)-Neumann problem

We look more closely at the \( \partial \)-Neumann problem, \( \square u = f \), where
\[ \square = \partial\partial^* + \partial^*\partial. \]

For \( f \) a \((0, q)\)-form, the equation \( \square u = f \) comprises a system of equations, and we write our equations in matrix form. We use the convention of writing indices with increasing entries: a particular index of length \( q \), \( j = (j_1, \ldots, j_q) \), is ordered according to \( j_l < j_m \) for \( l < m \). For the matrix we consider the ordering of two indices, \( J_1 = (j_{11}, j_{12}, \ldots, j_{1q}) \) and \( J_2 = (j_{21}, j_{22}, \ldots, j_{2q}) \), according to \( J_1 < J_2 \) if \( j_{1k} < j_{2k} \) for the first \( k \) such that \( j_{1k} \neq j_{2k} \), and \( J_1 = J_2 \) if \( j_{1k} = j_{2k} \) for all \( k = 1, \ldots, q \). The rows (and columns) of the matrix are in order of increasing indices. Thus, for instance, if we denote \( J_1 = (1, 2, \ldots, q) \), the \((1, 1)\)-entry of the matrix corresponds to the action on \( u_{11} \) which results in a form whose component is \( \omega_{11} \). Similarly, with \( J_2 = (1, 2, \ldots, q - 1, n) \), the \((n - q + 1, 1)\)-entry of the matrix corresponds to the action on \( u_{1n} \) which results in a form whose component is \( \omega_{1n} \), etc.

We want to calculate in general a \( J^{th} \) row of the matrix of operators describing \( \square \). We thus need to know which forms would result in a \( \omega_{1l} \) term when some input form is given into \( \square \). Let \( J = (j_1, \ldots, j_q) \) with \( j_m = k \), where \( 1 \leq k \leq n \), for some \( m \). We use the notation \( J_q \) to denote the index of length \( q - 1 \) \( (j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_q) \). We further use the set notation \( I_q \cup \{l\} \) to denote the index of length \( q \) (we assume the case \( l \neq j_i \) for \( 1 \leq i \leq q, i \neq m \)) consisting of the set \( \{j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_q, l\} \)
in the appropriate order (recall a particular index $K = (k_1, \ldots, k_q)$ is ordered if $k_r < k_s$ for $r < s$).

Since $\bar{\partial}$ increases the type of the form by 1 and $\bar{\partial}^*$ decreases it by one, we see it is when $\Box$ operates on a form of the type $u_{J_l \cup \{l\}} \bar{\omega}_{J_l \cup \{l\}}$ that a $\bar{\omega}_l$ term would result. And so we calculate $\Box u_{J_l \cup \{l\}} \bar{\omega}_{J_l \cup \{l\}}$. With abuse of notation, for a prescribed $J$, we write $u_{kl}$ in place of $u_{J_l \cup \{l\}}$, assuming $k \neq l$, with the obvious simplification in dimension 2.

We use our notation in the case $k \neq l$, although the case $k = l$ is also included in the same calculations. We start with the $\bar{\omega}_{L_k}$ components resulting from $\bar{\partial}^*(u_{kl} \bar{\omega}_{J_l \cup \{l\}})$. We use the notation

$$c^l_{J_l \cup \{m\}} = \bar{\partial}(\bar{\omega}_l) |_{\bar{\omega}_{J_l \cup \{m\}}}$$

and for $j = 1, \ldots, n$, we define $d_j$ according to an integration by parts

$$(\phi, T_j \varphi) = \left( (-L_j + d_j) \phi, \varphi \right)$$

for $\phi, \varphi \in C^0_0(\Omega)$ with support in a coordinate patch such that so that $L_j$ can be written in terms of local coordinates as in (2).

From

$$\left( \bar{\partial}^* u_{kl} \bar{\omega}_{J_l \cup \{l\}}, \varphi \bar{\omega}_k \right) = \left( u_{kl} \bar{\omega}_{J_l \cup \{l\}}, \bar{\partial}(\varphi \bar{\omega}_k) \right)$$

$$= \left( u_{kl} \bar{\omega}_{J_l \cup \{l\}}, \mathcal{T}_j \varphi \bar{\omega}_l \wedge \bar{\omega}_k + \partial_j^l \bar{\omega}_{J_l \cup \{l\}} \varphi \bar{\omega}_k \right)$$

$$= \left( u_{kl}, \mathcal{T}^k_{J_l \cup \{l\}} \varphi + \partial_j^l \bar{\omega}_{J_l \cup \{l\}} \varphi \right),$$

where we write, for some index $K = (k_1, \ldots, k_{q-1})$ and $mK = (m, k_1, \ldots, k_{q-1})$,

$$c^{mK}_{K \cup \{m\}} = \bar{\omega}_{mK} |_{\bar{\omega}_{K \cup \{m\}}}$$

we have

$$\bar{\partial} \bar{\partial}^* u_{kl} \bar{\omega}_{J_l \cup \{l\}} = \left( \mathcal{T}^k_{J_l \cup \{l\}} (-L_l + d_l) u_{kl} + \partial_j^l \bar{\omega}_{J_l \cup \{l\}} \varphi \bar{\omega}_k \right) \bar{\omega}_k \wedge \bar{\omega}_l + \cdots$$

with $\varphi$ some test function, where the $\cdots$ refers to terms whose contraction with $\bar{\omega}_k$ results in 0 (and which contain a $\bar{\omega}_l$ component). And thus

$$\bar{\partial} \bar{\partial}^* u_{kl} \bar{\omega}_{J_l \cup \{l\}} = \left( \mathcal{T}^k_{J_l \cup \{l\}} (-L_l + d_l) u_{kl} + \partial_j^l \bar{\omega}_{J_l \cup \{l\}} \varphi \bar{\omega}_k \right) \bar{\omega}_k \wedge \bar{\omega}_l$$

$$- \left( -\mathcal{T}^k_{J_l \cup \{l\}} \epsilon_j^l \mathcal{T}_k u_{kl} + \mathcal{T}_j \epsilon_j^l \mathcal{T}_k u_{kl} \right) \bar{\omega}_l + \cdots$$

$$= \left( -\mathcal{T}^k_{J_l \cup \{l\}} \epsilon_j^l \mathcal{T}_k L_k u_{kl} + \left( \mathcal{T}^k_{J_l \cup \{l\}} \epsilon_j^l \mathcal{T}_k L_k u_{kl} \right) \bar{\omega}_l + \cdots$$

(3.3)
where here the \( \cdots \) refers to terms which upon contraction with \( \bar{\omega}_J \) result in 0 as well as zero order terms.

We note that the calculations also show

\[
(3.4) \quad \bar{\delta}^* u_J \bar{\omega}_J = \sum_{l \in J} \left( - L_l u_J + (d_l + \epsilon_J^l \bar{\omega}_J) L_l u_J - \epsilon_J^l c_J^l L_l u_J \right) \bar{\omega}_J + \cdots.
\]

Similarly, to calculate \( \bar{\delta}^* \delta u_{kl} \bar{\omega}_{l \cup \{l\}} \) we start with

\[
\delta u_{kl} \bar{\omega}_{l \cup \{l\}} = \left( \epsilon_{\{l\}}^{k l} \bar{\omega}_{l \cup \{l\}} + c_{\{l\}}^{k l} \bar{\omega}_{l \cup \{l\}} \right) \bar{\omega}_{l \cup \{l\}}
\]

modulo terms whose contraction with \( \bar{\omega}_{l \cup \{l\}} \) result in 0. As in (3.2), we have

\[
\bar{\delta}^* v \bar{\omega}_{l \cup \{l\}} = \left( \epsilon_{\{l\}}^{k l} (- L_l + \bar{\omega}_l) \bar{\omega}_{l \cup \{l\}} \right) \bar{\omega}_{l \cup \{l\}}
\]

modulo terms whose contraction with \( \bar{\omega}_J \) result in 0, which when applied to \( \delta u_{kl} \bar{\omega}_{l \cup \{l\}} \) above, yields

\[
(3.5) \quad \bar{\delta}^* \delta u_{kl} \bar{\omega}_{l \cup \{l\}} = \left( - \epsilon_{\{l\}}^{k l} \epsilon_{\{l\}}^{k l} \bar{\omega}_{l \cup \{l\}} + \epsilon_{\{l\}}^{k l} \bar{\omega}_{l \cup \{l\}} \right) L_k u_{kl} + \epsilon_{\{l\}}^{k l} \bar{\omega}_{l \cup \{l\}} \bar{\omega}_{l \cup \{l\}}
\]

where the \( \cdots \) refers to terms whose contraction with \( \bar{\omega}_J \) result in 0 as well as terms of order 0. And similarly,

\[
\delta^* \delta u_k \bar{\omega}_J = \sum_{l \in J} \left( - L_l u_J + (d_l + \epsilon_J^l \bar{\omega}_J) L_l u_J \right) \bar{\omega}_J + \cdots.
\]

Adding (3.4) and (3.6) yields \( \Box u_J \bar{\omega}_J \):

\[
\Box u_J \bar{\omega}_J = - \sum_{l \in J} L_l u_J \bar{\omega}_J - \sum_{l \in J} L_l u_J \bar{\omega}_J + \left\{ \begin{array}{ll}
(-1)^{ |J| } \left( - \bar{\epsilon}_J^l \bar{\omega}_J + \epsilon_J^l \bar{\omega}_J \right) u_J \bar{\omega}_J + d_n \bar{\omega}_J \bar{\omega}_J & \text{if } n \in J \\
(-1)^{ |J| } \left( \epsilon_J^l \bar{\omega}_J \right) u_J \bar{\omega}_J + d_n \bar{\omega}_J \bar{\omega}_J & \text{if } n \notin J.
\end{array} \right.
\]

We only collect the (complex) normal derivatives, as they are enough to determine the behavior of the relevant boundary value operators in the direction of the field \( T \) (see the discussion at the end of Section 5). The terms included in the \( \cdots \) thus refer to terms which are orthogonal to \( \bar{\omega}_J \), of zero order, or involve only vector fields orthogonal to \( L_n \) or \( \bar{T}_n \).
The two cases can be combined into the single expression
\[ \Box u_j \tilde{\omega}_j = - \sum_{l \in J} L_l u_j \tilde{\omega}_j - \sum_{l \notin J} L_l \overline{\nabla} u_j \tilde{\omega}_j \]
\[ + \left( -1 \right)^{|J| \cup \{ n \}} \left( \epsilon_{J \cup \{ n \}} \nabla - \mathcal{C}_{J \cup \{ n \}} \mathcal{T}_n \right) u_j \tilde{\omega}_j + d_n \overline{\nabla} u_j \tilde{\omega}_j + \cdots. \]

We now add (3.3) and (3.5) to obtain \( \Box u_{kl} \tilde{\omega}_{k \cup \{ l \}} \). To simplify the result we note
\[ (3.7) \quad \epsilon_{J \cup \{ l \}}^{\ell} k_l \epsilon_{J \cup \{ l \}}^{k_l} = - \epsilon_{J \cup \{ l \}}^{\ell} \epsilon_{J \cup \{ l \}}^{k_l} \]
for \( k \neq l \), and as we are only interested in the complex normal derivatives, in comparing \( \epsilon_{J \cup \{ l \}}^{\ell} \epsilon_{J \cup \{ l \}}^{k_l} \) and \( \epsilon_{J \cup \{ l \}}^{\ell} \epsilon_{J \cup \{ l \}}^{k_l} \) we look at the case \( l = n \), for which we have
\[ \epsilon_{J \cup \{ l \}}^{\ell} \epsilon_{J \cup \{ l \}}^{k_l} = - \epsilon_{J \cup \{ l \}}^{n} \epsilon_{J \cup \{ l \}}^{k_l}. \]
Similarly, in comparing \( \epsilon_{J \cup \{ l \}}^{k_l} \epsilon_{J \cup \{ l \}}^{\ell} \) and \( \epsilon_{J \cup \{ l \}}^{k_l} \epsilon_{J \cup \{ l \}}^{\ell} \) we are only interested in the case \( k = n \) for which
\[ \epsilon_{J \cup \{ l \}}^{n} \epsilon_{J \cup \{ l \}}^{\ell} = - \epsilon_{J \cup \{ l \}}^{n} \epsilon_{J \cup \{ l \}}^{\ell}. \]
We thus can write, by adding (3.3) and (3.5),
\[ \Box u_{kl} \tilde{\omega}_{k \cup \{ l \}} = - \epsilon_{J \cup \{ l \}}^{k_l} \epsilon_{J \cup \{ l \}}^{\ell} \overline{\nabla}_{k_l} L_j u_{kl} \tilde{\omega}_j \]
for \( k \neq l \), modulo terms with the vector fields \( L_j \) or \( \overline{\nabla}_j \) for \( j = 1, \ldots, n-1 \), zero order terms, or forms orthogonal to \( \tilde{\omega}_j \).

We collect our results in the following proposition:

**Proposition 3.1.** Modulo the vector fields \( L_j \) or \( \overline{\nabla}_j \) acting on components of \( u \) for \( j = 1, \ldots, n-1 \), zero order terms, or forms orthogonal to \( \tilde{\omega}_j \), we have

\[ i) \quad \Box (u_j \tilde{\omega}_j) = - \sum_{l \in J} L_l u_j \tilde{\omega}_j - \sum_{l \notin J} L_l \overline{\nabla} u_j \tilde{\omega}_j \]
\[ + \left( -1 \right)^{|J| \cup \{ n \}} \left( \epsilon_{J \cup \{ n \}} \nabla - \mathcal{C}_{J \cup \{ n \}} \mathcal{T}_n \right) u_j \tilde{\omega}_j + d_n \overline{\nabla} u_j \tilde{\omega}_j + \cdots. \]

\[ ii) \quad \Box u_{kl} \tilde{\omega}_{k \cup \{ l \}} = - \epsilon_{J \cup \{ l \}}^{k_l} \epsilon_{J \cup \{ l \}}^{\ell} \overline{\nabla}_{k_l} L_j u_{kl} \tilde{\omega}_j. \]

4. THE DIRICHLET TO NEUMANN OPERATOR

The Dirichlet to Neumann operator (DNO) is the boundary value operator giving the outward normal derivative of the solution to a Dirichlet problem. We look at the DNO corresponding to the operator \( 2 \Box \). We study the solution, \( v \), which solves
\[ 2 \Box v = 0 \quad \text{in } \Omega \]
\[ v = g_\partial \quad \text{on } \partial \Omega, \]
and we obtain an expression for $\frac{\partial v}{\partial \rho}$ (modulo smooth terms) near a given point $p \in \partial \Omega$ in terms of $g_b$.

If $\chi_p$ is a smooth cutoff function with support in a small neighborhood of $p$ and $\chi'_p$ a smooth cutoff such that $\chi'_p \equiv 1$ on supp $\chi_p$, we have

\begin{equation}
2 \Box (\chi_p v) = \Psi^1 (\chi'_p v) \quad \text{on } \Omega ,
\end{equation}

\begin{equation}
\chi_p v = \chi_p g_b \quad \text{on } \partial \Omega .
\end{equation}

The term $\Psi^1 (\chi'_p v)$ above arises due to derivatives falling on the cutoff function $\chi_p$. The use of the cutoff function allows us to consider the equation locally, and is equivalent to using pseudodifferential operators with symbols defined in local coordinate patches, with one of the coordinates given by $\rho$. We thus consider $v$ to be supported in a neighborhood of a given boundary point, $p \in \partial \Omega$.

To study the operator $\Box$ and the associated boundary operators, we consider the equation $\Box v = 0$ for a $(0,q)$-form $v$. In a small neighborhood of a boundary point, which we take to be $0 \in \partial \Omega$, we write the vector fields, $L_j$ in local coordinates as in (2.3):

\begin{equation}
L_j = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right) + \sum_{k=1}^{2n-1} \ell^j_k (x) \frac{\partial}{\partial x_k} + O(\rho),
\end{equation}

where $\ell^j_k (x) = O(x)$. Also we recall from Section 2, the representation of $L_n$:

\begin{equation}
L_n = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + i \frac{\partial}{\partial x_{2n-1}} + O(\rho).
\end{equation}

We use the symbol notation

\begin{align*}
\sigma(\partial_\rho) &= i \eta \\
\sigma(\partial_{x_j}) &= i \xi_j \quad j = 1, \ldots, 2n - 1.
\end{align*}

The second order terms of the (diagonal matrix) operator in $\Box$ from Proposition 3.1 are given by

$$- \sum_{l \in \mathcal{J}} T_l L_l L_l - \sum_{l \in \mathcal{J}} L_l T_l L_l .$$

Expanding this operator using (4.2) and (4.3), we write in local coordinates

$$- \sum_{l \in \mathcal{J}} T_l L_l - \sum_{l \in \mathcal{J}} L_l T_l = - \frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} + 2 \frac{\partial^2}{\partial x_{2n-1}^2} + \sum_{j,k=1}^{2n-1} l_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right) + O(\rho),$$

where $l_{jk} = O(x)$, and modulo first order terms. We define the operator $\Gamma$ to be given by the terms without a $\rho$ factor on the right hand side:

$$\Gamma := - \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} + 2 \frac{\partial^2}{\partial x_{2n-1}^2} + \sum_{j,k=1}^{2n-1} l_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right).$$
We let $L_{bj}$ for $j = 1, \ldots, n - 1$ be defined by

$$
\sigma(L_{bj}) = \sigma(L_j) \big|_{\rho = 0}.
$$

Then we have

$$
\sigma_2(\Box) = \frac{1}{2} \eta^2 + \xi_{2n-1}^2 + \sum_{j=1}^{n-1} \sigma(L_{bj})\sigma(T_{bj}) + O(\rho)O(\xi^2)
$$

$$
= \frac{1}{2} \eta^2 + \xi_{2n-1}^2 + \frac{1}{4} \sum_{j=1}^{2n-2} \xi_j^2 + O(\rho)O(\xi^2) + O(\rho)O(\xi^2)
$$

$$
= \frac{1}{2} \sigma(\Gamma) + O(\rho)O(\xi^2).
$$

We use the Kohn-Nirenberg notation, $\sigma_j$ to denote the part of a symbol homogeneous of degree $j$ in $\xi$ and $\eta$ in its symbol expansion.

Let us now denote

$$
\Xi^2(x, \xi) = 2\xi_{2n-1}^2 + 2 \sum_{j=1}^{n-1} \sigma(L_{bj})\sigma(T_{bj})
$$

(4.4)

$$
= 2\xi_{2n-1}^2 + \frac{1}{2} \sum_{j=1}^{2n-2} \xi_j^2 + O(\rho)O(\xi^2)
$$

so that we can write

$$
\sigma(\Gamma) = \eta^2 + \Xi^2(x, \xi).
$$

We now collect the second order $O(\rho)$ terms from $2\Box$ in an operator, $\tau$, i.e.

$$
\sigma(\tau) \big|_{\rho = 0} = 2 \frac{\partial}{\partial \rho} \sigma_2(\Box) \big|_{\rho = 0},
$$

and all tangential first order operators in the expression of the operator $2\Box$ as in Proposition 3.1 into a pseudodifferential operator, denoted $A$. We also denote by the operator $S$ the zero order operator which is multiplication by the (matrix) coefficient of $\frac{\partial}{\partial \rho}$ in the operator $2\Box$.

With the notation $v|_{\rho = 0} = g_b(x)$, the equation $2\Box v = 0$ can be written locally as

$$
(4.5)
\Gamma v + \sqrt{2} S \left( \frac{\partial v}{\partial \rho} \right) + Av + \rho \tau(v) = 0.
$$

The Dirichlet to Neumann operator (DNO) is defined here as the boundary operator producing the boundary values of the outward normal derivative of the solution to the Poisson equation $2\Box v = 0$, with boundary values $v = g_b$ on $\partial \Omega$. In the equation (4.5) above, the DNO can be found by solving for $\partial_{\rho} v \big|_{\rho = 0}$.
We rewrite (4.5) using Fourier Transforms, extending (4.5) to \( \mathbb{R}^{2n} \) by 0. Let \( E \) denote the extension by 0. The term \( E \circ \Gamma v \) can be written
\[
E \circ \Gamma v = \Gamma \circ Ev - \frac{1}{(2\pi)^2n} \int \left( \partial_{\rho} \tilde{v} \bigg|_{\rho=0} + i\eta \tilde{g}_b(\xi) \right) e^{i\rho \eta} e^{ix \cdot \xi} d\xi d\eta.
\]
For ease of notation, we will disregard the extension operator, \( E \), and instead use the subscript \( \text{int} \) to signify an operator is to be applied to the extension by 0 to \( \mathbb{R}^{2n} \) of a distribution defined in \( \Omega \). With this convention, we write
\[
\Gamma v = \Gamma_{\text{int}} v - \frac{1}{(2\pi)^2n} \int \left( \partial_{\rho} \tilde{v} \bigg|_{\rho=0} + i\eta \tilde{g}_b(\xi) \right) e^{i\rho \eta} e^{ix \cdot \xi} d\xi d\eta,
\]
where \( \Gamma \) on the left-hand side is to be understood as an operator \( \Gamma : \mathcal{D}'(\Omega) \to \mathcal{D}'(\mathbb{R}^{2n}) \) via (left-side) composition with \( E \), and where \( \Gamma_{\text{int}} : \mathcal{D}'(\mathbb{R}^{2n}) \to \mathcal{D}'(\mathbb{R}^{2n}) \) has as symbol:
\[
\sigma(\Gamma_{\text{int}}) = \eta^2 + \Xi^2(x, \xi).
\]
The term \( S\left(\frac{\partial v}{\partial \rho}\right) \) can be written
\[
S\left(\frac{\partial v}{\partial \rho}\right) = \frac{1}{(2\pi)^2} \int s(x, \rho) i\eta \tilde{\vartheta}(\xi, \eta) e^{i\rho \eta} e^{ix \cdot \xi} d\xi d\eta + \frac{1}{(2\pi)^2} \int s(x, 0) \tilde{g}_b(\xi) e^{ix \cdot \xi} d\xi
\]
\[
:= S_{\text{int}} v + S_b g_b,
\]
where similarly the left-hand side is understood to be composed on the left by \( E \), and where \( S_{\text{int}} := S \circ E \). We have \( \sigma(S_{\text{int}}) = s(x, \rho) i\eta \), and \( S_b \in \Psi(\partial \Omega) \) with \( \sigma(S_b) = s(x, 0) \). From Proposition 3.1, \( s(x, \rho) \) is a diagonal matrix (of smooth functions).

We now rewrite (4.5) as
\[
\Gamma_{\text{int}} v = \frac{1}{(2\pi)^2} \int \left( \partial_{\rho} \tilde{v} \bigg|_{\rho=0} + i\eta \tilde{g}_b(\xi) \right) e^{i\rho \eta} e^{ix \cdot \xi} d\xi d\eta
\]
\[
- \sqrt{2} S_{\text{int}} v - \sqrt{2} S_b g_b - Av - \rho \tau v,
\]
where \( v \) is understood to be extended by 0 to all of \( \mathbb{R}^{2n} \).

\( \Gamma_{\text{int}} \) is an elliptic operator on \( \mathbb{R}^{2n} \) and so we can apply an inverse to \( \Gamma_{\text{int}} \):
\[
v = \frac{1}{(2\pi)^2} \Gamma_{\text{int}}^{-1} \circ \int \left( \partial_{\rho} \tilde{v} \bigg|_{\rho=0} + i\eta \tilde{g}_b(\xi) \right) e^{i\rho \eta} e^{ix \cdot \xi} d\xi d\eta
\]
\[
- \sqrt{2} \Gamma_{\text{int}}^{-1} \circ S_{\text{int}} v - \sqrt{2} \Gamma_{\text{int}}^{-1} \circ S_b g_b - \Gamma_{\text{int}}^{-1} \circ Av - \Gamma_{\text{int}}^{-1} \circ (\rho \tau v)
\]
modulo smoothing terms. The idea behind our calculations of the DNO is to write \( \partial_{\rho} v \bigg|_{\rho=0} = \Lambda_1^1 g_b + \Lambda_0^0 g_b + \cdots \), insert this expansion into the first integral on the right-hand side of (4.6), set \( \rho = 0 \) in (4.6), and equate terms with the same order, or, equivalently, of the same degree in \( \Xi(x, \xi) \) (see [1] for another approach).
We first prove a Proposition about the Poisson operator, giving the solution, \(v\), above. In order to consolidate the various smoothing terms which arise, we write \(R^{-\infty}\) to include the restriction to \(\rho = 0\) of any sum of smoothing operators in \(\Psi^{-\infty}(\Omega)\) applied to \(v\), or smoothing operators in \(\Psi^{-\infty}(\partial \Omega)\) applied to the boundary values \(g_b\) or \(\partial_\rho v|_{\rho=0}\). We also write \(R^{-\infty}\) to include any sum of smoothing operators in \(\Psi^{-\infty}(\Omega)\) applied to \(v\), smoothing operators in \(\Psi^{-\infty}(\partial \Omega)\) applied to the boundary values \(g_b\) or \(\partial_\rho v|_{\rho=0}\), or decomposable operators in \(\Psi^{-k}(\Omega)\) for \(k \geq 1\) applied to \(R^{-\infty}\) (such terms can thus be estimated in terms of smooth boundary terms, see Theorem 2.4) as well as smoothing operators in \(\Psi^{-\infty}(\Omega)\) applied to \(R^{-\infty}\). From the definitions we have \(R(R^{-\infty}) = R^{-\infty}\).

Estimates for the Poisson operator corresponding to an elliptic operator were worked out in [4]. In those results, the highest order term of the DNO was also calculated. The calculations here follow those in [4] to find the Poisson operator corresponding to \(\Box\). As the operator, \(\Box\), is slightly different than the operator considered in the author’s earlier work (namely in the first order terms), and as the Poisson operator will be used to obtain the lower order terms of the DNO, we go through the calculations in detail, obtaining first an expression for the Poisson operator, and then calculating the DNO.

We define the Poisson operator corresponding to \(\Box\) as the solution operator, \(P\), mapping \((0,q)\)-forms on \(\partial \Omega\) to \((0,q)\)-forms on \(\Omega\), to

\[
2\Box \circ P = 0 \\
R \circ P = I.
\]

(4.7)

We assume the classical results guaranteeing existence and uniqueness of a solution.

**Theorem 4.1.** Let \(g_b\) be a \((0,q)\)-form on \(\partial \Omega\); each component of \(g_b\) is a distribution supported on \(\partial \Omega\). Let \(g = g_b(x) \times \delta(\rho)\) in local coordinates. Then

\[
P_g = \Psi^{-1} g + R^{-\infty}.
\]

(4.8)

*Proof*. From (4.6), we have modulo \(R^{-\infty}\)

\[
v = \frac{1}{(2\pi)^{2n}} \int \frac{\partial_\mu \tilde{v}(\xi,0) + i\eta \tilde{g}_b(\xi)}{\eta^2 + \Xi^2(x,\xi)} e^{ix^\xi e^{i\eta \xi} d\xi d\eta} \\
- \sqrt{2} \Gamma^{-1} \circ S \left( \frac{\partial v}{\partial \rho} \right) - \Gamma^{-1} \circ A(v) - \Gamma^{-1} \circ \rho \tau(v) \\
+ \Psi^{-3} \left( \partial_\rho v|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} g + \Psi^{-2} v
\]

(4.8)

locally, in a small neighborhood of the origin; we recall, using the pseudodifferential analysis, we consider \(v\) to have compact support in a neighborhood of a
boundary point, which we take to be the origin, and the pseudodifferential operators are also composed on the left with cutoffs with support in a neighborhood of the origin; see the discussion in Section 2 as well as the discussion following (4.1). For ease of notation, we omit the writing of the cutoffs. We will also omit mention of the smooth $R^{-\infty}$ terms, inserting them again at the end of the calculations.

We note the terms $\Gamma^{-1}_{int} \circ S \left( \frac{2n}{\xi} \right)$ and $\Gamma^{-1}_{int} \circ A(v)$ contribute terms $\Psi^{-1}v$ and $\Psi^{-2}g$.

To handle the term

\begin{equation}
\Gamma^{-1}_{int} \circ \rho \tau(v),
\end{equation}

we write the operator $\tau$ using the form of its symbol

\[ \sigma(\tau) = \sum_{j,k=1}^{2n} \tau_{jk}(x, \rho) \xi_j \xi_k, \]

and we rearrange (4.8) as

\[ v = \frac{1}{(2\pi)^{2n}} \int \frac{\partial_\rho \tilde{v}(\xi,0)}{\eta^2 + \Xi^2(x, \xi)} e^{ix \xi e^{i\eta \rho} d\xi d\eta} + \frac{1}{(2\pi)^{2n}} \int \rho \sum_{j,k=1}^{2n} \tau_{jk}(x, \rho) \xi_j \xi_k \tilde{v}(\xi, \eta) e^{ix \xi e^{i\eta \rho} d\xi d\eta} + \Psi^{-3} \left( \partial_\rho v \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2}g + \Psi^{-1}v,
\]

as the terms involving the operators $S$ and $A$ are included in the last two remainder terms. We then bring the second term on the right to the left-hand side:

\begin{equation}
\frac{1}{(2\pi)^{2n}} \int \left( 1 - \rho \sum_{j,k=1}^{2n} \tau_{jk}(x, \rho) \xi_j \xi_k \right) \tilde{v}(\xi, \eta) e^{ix \xi e^{i\eta \rho} d\xi d\eta} =
\end{equation}

\[ \frac{1}{(2\pi)^{2n}} \int \frac{\partial_\rho \tilde{v}(\xi,0)}{\eta^2 + \Xi^2(x, \xi)} e^{ix \xi e^{i\eta \rho} d\xi d\eta} + \Psi^{-3} \left( \partial_\rho v \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2}g + \Psi^{-1}v.
\]

For small enough $\rho$ (which, without loss of generality, can be assumed by choosing the cutoffs defining the pseudodifferential operators appropriately small) the symbol

\begin{equation}
1 - \rho \sum_{j,k=1}^{2n} \tau_{jk}(x, \rho) \xi_j \xi_k \frac{\eta^2 + \Xi^2(x, \xi)}{\eta^2 + \Xi^2(x, \xi)}
\end{equation}

is non-zero, and so (shrinking the support of $v$ if necessary) we can apply a parametrix of the operator with symbol (4.11) to both sides of (4.10). We note the symbol of such an operator is of the form $1 + O(\rho)$, where the second term is a symbol of
class $\mathcal{S}^0(\Omega)$, which is $O(\rho)$. We obtain

$$v = \frac{1}{(2\pi)^{2n}} \int \frac{1}{(1 + O(\rho))} \frac{\partial_{\rho}\bar{v}(\xi, 0) + i\eta \bar{g}_b(\xi)}{\eta^2 + |x - \xi|^2} e^{ix\xi} e^{i\rho\eta} d\eta$$

$$+ \Psi^{-3}\left( \partial_{\rho}v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2}g + \Psi^{-1}v.$$  

From Lemma 2.5 we have that

$$\frac{1}{(2\pi)^{2n}} \int O(\rho) \frac{\partial_{\rho}\bar{v}(\xi, 0) + i\eta \bar{g}_b(\xi)}{\eta^2 + |x - \xi|^2} e^{ix\xi} e^{i\rho\eta} d\eta$$

$$= \Psi^{-3}\left( \partial_{\rho}v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2}g.$$

Returning to (4.12) we write

$$v = \frac{1}{(2\pi)^{2n}} \int \frac{\partial_{\rho}\bar{v}(\xi, 0) + i\eta \bar{g}_b(\xi)}{\eta^2 + |x - \xi|^2} e^{ix\xi} e^{i\rho\eta} d\eta$$

$$+ \Psi^{-3}\left( \partial_{\rho}v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2}g + \Psi^{-1}v.$$  

The expression above is locally confined to a neighborhood of the origin, but using coverings and a partition of unity (as in the explanation in (2.2)) we can obtain an expression for $v$ on all of $\Omega$. Then inverting an operator of the form $I - \Psi^{-1}$ gives an expression for $v$ on $\Omega$:

$$v = \Psi^{-2}\left( \partial_{\rho}v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1}g + \Psi^{-\infty}v,$$

as a vector-valued relation, with matrix-valued pseudodifferential operators.

Using the residue calculus, we can take an inverse transform in (4.13) with respect to $\eta$. For $\rho \to 0^+$, we have

$$0 = \frac{1}{(2\pi)^{2n-1}} \int \frac{1}{2i|\Xi(x, \xi)|} \partial_{\rho}\bar{v}(\xi, 0) e^{i\rho\eta} d\eta - \int \frac{|\Xi(x, \xi)|}{2i|\Xi(x, \xi)|} \bar{g}_b(\xi) e^{i\rho\eta} d\eta$$

$$+ R \circ \Psi^{-3}\left( \partial_{\rho}v \big|_{\rho=0} \times \delta(\rho) \right) + R \circ \Psi^{-2}g + R \circ \Psi^{-1}v$$

$$= \frac{1}{(2\pi)^{2n-1}} \int \frac{1}{2|\Xi(x, \xi)|} \partial_{\rho}\bar{v}(\xi, 0) - \frac{1}{2} \bar{g}_b(\xi) e^{i\rho\eta} d\eta$$

$$+ \Psi^{-2}\left( \partial_{\rho}v \big|_{\rho=0} \right) + \Psi^{-1}g + R \circ \Psi^{-1}v,$$

where we apply Lemma 2.6 to the terms with operators $R \circ \Psi^{-3}$ and $R \circ \Psi^{-2}$ in the second step. We can now invert the operator with symbol $1/2|\Xi(x, \xi)|$ and solve for $\partial_{\rho}v|_{\rho=0}$:

$$\frac{\partial v}{\partial \rho}(x, 0) = \int |\Xi(x, \xi)| \bar{g}_b(\xi) e^{i\rho\eta} d\eta + \Psi^{-1}g_b + \Psi^{-1}b \circ R \circ \Psi^{-1}v.$$  

locally, in a small neighborhood of the origin. Alternatively, we could in a similar manner use the residue calculus to take an inverse transform with respect to $\eta$ in (4.13) and calculate for $\rho \to 0^-$ with the same result.
For the term $\Psi^1_b \circ R \circ \Psi^{-1} v$ we insert (4.14) in the argument:

$$\Psi^1_b \circ R \circ \Psi^{-1} v = \Psi^1_b \circ R \circ \Psi^{-1} \left( \partial_\rho v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1} g + \Psi^{-\infty} v$$

$$= \Psi^1_b \circ R \circ \Psi^{-2} \left( \partial_\rho v \big|_{\rho=0} \times \delta(\rho) \right) + \Psi^1_b \circ R \circ \Psi^{-2} g + \Psi^1_b \circ R \circ \Psi^{-\infty} v$$

$$= \Psi^1_b \circ \Psi^{-2} \left( \partial_\rho v \big|_{\rho=0} \right) + \Psi^1_b \circ \Psi^{-1} g_b + R \circ \Psi^{-\infty} v$$

(4.15) above leads to the well-known result that the DNO is a first order operator on the boundary data, with principal term $|\Xi(x, \xi)|$:

$$\left. \frac{\partial v}{\partial \rho} \right|_{\rho=0} = \int \left| \Xi(x, \xi) \right| g_b(\xi) e^{i\xi(x, \xi) \cdot \eta d\xi + \Psi^0 g_b + \Psi^{-1} \left( \partial_\rho v \big|_{\rho=0} \right) + R_b^{-\infty} v. $$

Again, using a covering and the local expressions to obtain a global relation, and solving for (the vector) $\partial_\rho v \big|_{\rho=0}$ and absorbing extra $\Psi^{-\infty} \left( \partial_\rho v \big|_{\rho=0} \right)$ terms into the remainder term, $R_b^{-\infty}$, leads to the expression:

(4.16) $$\left. \partial_\rho v \right|_{\rho=0} = |D| g_b + \Psi^0 g_b + R_b^{-\infty}$$

where $|D|$ is defined as the first order operator with symbol locally given by $\sigma(|D|) = |\Xi(x, \xi)|$.

We can now insert (4.16) in (4.15) and obtain in a small neighborhood of the origin

$$v = \frac{1}{(2\pi)^{2n}} \int \frac{\left| \Xi(x, \xi) \right| + i\eta g_b(\xi) e^{i\xi(x, \xi) \cdot \eta d\eta + \Psi^{-2} \left( \partial_\rho v \big|_{\rho=0} \right) + \Psi^{-1} g + \Psi^{-\infty} R_b^{-\infty} \right|}{\eta + i|\Xi(x, \xi)|} e^{i\xi(x, \xi) \cdot \eta} d\xi d\eta$$

$$= \frac{i}{(2\pi)^{2n}} \int \frac{g_b(\xi)}{\eta + i|\Xi(x, \xi)|} e^{i\xi(x, \xi) \cdot \eta} d\xi d\eta$$

which we write as

(4.17) $$v = \Psi^{-1} g + \Psi^{-1} v + R^{-\infty}.$$ 

We thus obtain

$$v = \Psi^{-1} g + R^{-\infty},$$

on all of $\Omega$.

From the proof of the Theorem we also have the principal symbol of the operator $\Psi^{-1}$ acting on $g_b \times \delta(\rho)$; it is given locally by (the diagonal matrix)

(4.18) $$\frac{i}{\eta + i|\Xi(x, \xi)|}.$$
which we note for future reference. Using the representation as in (4.16),

\[
\partial_\nu v |_{\partial \Omega} = \Psi_b^1 g_b + R_b^{-\infty},
\]

we can obtain with Lemma 2.4 estimates for the Poisson operator.

We first handle the smooth terms, \( R^{-\infty} \) and \( R_b^{-\infty} \).

**Lemma 4.2.** For \( R^{-\infty} \) and \( R_b^{-\infty} \), and \( g_b \), defined as above, we have for all \( s \)

\[
\| R^{-\infty} \|_{W^s(\Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)}
\]

and

\[
\| R_b^{-\infty} \|_{W^s(\partial \Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)}.
\]

**Proof.** We note the \( L^2 \) estimates for the Poisson operator, \( \| P(g) \|_{L^2(\Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)} \) (see for instance [9]).

For \( R^{-\infty} \), we have by definition

\[
\| R^{-\infty} \|_{W^s(\Omega)} \lesssim \| u \|_{W^{-\infty}(\Omega)} + \| g_b \|_{W^{-\infty}(\partial \Omega)} + \| \partial_\nu u |_{\partial \Omega} \|_{W^{-\infty}(\partial \Omega)}
\]

\[
\lesssim \| g_b \|_{L^2(\partial \Omega)} + \| \partial_\nu u |_{\partial \Omega} \|_{W^{-\infty}(\partial \Omega)}
\]

for any \( s \geq 0 \).

We can estimate boundary values of a term, \( \partial_\nu u |_{\partial \Omega} \), by assuming support in a neighborhood of \( \partial \Omega \) intersected with \( \Omega \) and writing

\[
\partial_\nu u |_{\rho=0} = \int_{-\infty}^{0} \partial^2_\rho u d\rho = \int_{-\infty}^{0} D^2_\nu u d\rho + \int_{-\infty}^{0} (\phi_1 \partial_\nu u + \phi_2 u) d\rho,
\]

where \( D^2_\nu \) is a second order tangential operator, and \( \phi_1 \) and \( \phi_2 \) are smooth with support in the interior of \( \Omega \). From interior regularity, we have

\[
\| \phi_1 u \|_{W^2(\Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)}.
\]

Thus, applying a tangential smoothing operator to both sides and integrating yields

\[
\| \partial_\nu u |_{\partial \Omega} \|_{W^{-\infty}(\partial \Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)}.
\]

Hence,

\[
\| R^{-\infty} \|_{W^s(\Omega)} \lesssim \| g_b \|_{L^2(\partial \Omega)}.
\]

The estimates for \( R_b^{-\infty} \) follow similarly. \qed

**Theorem 4.3.** Let \( P \) be the Poisson operator on \( \Omega \) for the system (4.7). Then for \( s \geq 0 \)

\[
\| P(g) \|_{W^{s+1/2}(\Omega)} \lesssim \| g_b \|_{W^s(\partial \Omega)}.
\]
Proof. We use the representation $P(g) = \Psi^{-1}g + R^{-\infty}$ as in Theorem 4.1, where the $\Psi^{-1}$ operator is decomposable. The estimates then follow from Theorem 2.4 and Lemma 3.2.

Included in the proof of Theorem 4.1 is the calculation of the highest order term of the DNO; from (4.16) we have in particular the first component of the DNO, which we write as, $N^-$ (the $-$ superscript to denote we compute the outward pointing normal derivative):

**Theorem 4.4.**

\begin{equation}
(4.20) \quad N^- g = |D|g_b + \Psi^0_b(g_b) + R^-\infty.
\end{equation}

We now want to write out the highest order terms included in $\Psi^0_b(g)$ in (4.20). That is to say, writing $\partial_\rho v \bigg|_{\rho=0} = \Lambda_1 g_b + \Lambda_0 g_b + \cdots$, we have $\Lambda_1 = |D|$, and we want to calculate an expression for the operator $\Lambda_0^b$.

Recall in (4.6) we had the relation

\[ v = \frac{1}{(2\pi)^2} \Gamma_{int}^{-1} \circ \int \left( \partial_\rho v \bigg|_{\rho=0} + \eta \eta \tilde{g}_b(\xi) \right) e^{i\eta} e^{ix \cdot \xi} d\eta d\xi \]

\[ - \sqrt{2} \Gamma_{int}^{-1} \circ S_{int} v - \sqrt{2} \Gamma_{int}^{-1} \circ \Lambda_0 g_b - \Gamma_{int}^{-1} \circ Av - \Gamma_{int}^{-1} \circ (\rho \tau v) \]

modulo smooth terms. With $\partial_\rho v \bigg|_{\rho=0} = \Lambda_1 g_b + \Lambda_0 g_b + \cdots$, and using

\[ \frac{1}{(2\pi)^2} \Gamma_{int}^{-1} \circ \int \left( \partial_\rho v \bigg|_{\rho=0} + \eta \eta \tilde{g}_b(\xi) \right) e^{i\eta} e^{ix \cdot \xi} d\eta d\xi = \Theta^+ g + \Psi^{-2} g + R^{-\infty}, \]

we can write the relation as

\begin{equation}
(4.21) \quad v = \Theta^+ g + \Psi^{-2} g + \Gamma_{int}^{-1} \circ \Lambda_0 g_b - \sqrt{2} \Gamma_{int}^{-1} \circ S_{int} v - \sqrt{2} \Gamma_{int}^{-1} \circ \Lambda_0 g_b - \Gamma_{int}^{-1} \circ Av - \Gamma_{int}^{-1} \circ (\rho \tau v) + R^{-\infty},
\end{equation}

where $\Theta^+$ is defined by

\[ \sigma(\Theta^+) = \frac{i}{\eta + i\Xi(x, \xi)}. \]

The pseudodifferential calculus also yields the principal term of the $\Psi^{-2}$ operator in (4.21). The operator arises in the expansion of the symbol for the inverse, $\Gamma_{int}^{-1}$:

\[ \sigma \left( \Gamma_{int}^{-1} \right) = \frac{1}{\eta^2 + \Xi^2(x, \xi)} + \frac{\partial_\eta \Xi^2 \cdot D_\xi \Xi^2}{(\eta^2 + \Xi^2(x, \xi))^3} + \cdots. \]

And so the principal symbol of the $\Psi^{-2}$ operator in (4.21) is given by

\begin{equation}
(4.22) \quad \frac{\partial_\eta \Xi^2 \cdot D_\xi \Xi^2}{(\eta^2 + \Xi^2(x, \xi))^3} \left( |\Xi(x, \xi)| + i\eta \right) = \frac{\partial_\eta \Xi^2 \cdot \partial_\xi \Xi^2}{(\eta^2 + \Xi^2(x, \xi))^2(\eta + i|\Xi(x, \xi)|)}. \]

For the term $\Lambda_0^0$, we set $\rho = 0$ in (4.21) and look at the terms of order $-1$ in $\Xi(x, \xi)$. The first term, $\Theta^+$ leads to a term which is homogeneous of order $0$ in $|\Xi(x, \xi)|$. We go through the other terms individually. For the operator with symbol as in (4.22) we calculate

$$
\frac{1}{(2\pi)^{2n}} \int \frac{\partial_s \Xi^2 \cdot \partial_s \Xi^2}{(\eta^2 + \Xi^2(x, \xi))^2(\eta + i|\Xi(x, \xi)|)} \tilde{g}_b(\xi) e^{ix} d\eta d\xi
$$

$$
= -\frac{3i}{(2\pi)^{2n-1}} \int \frac{\partial_s \Xi^2 \cdot \partial_s \Xi^2}{16\Xi^4(x, \xi)} \tilde{g}_b(\xi) e^{ix} d\xi.
$$

Next, we have

$$
R \circ \Gamma^{-1}_{int} \circ \Lambda_0^0 g = \frac{1}{(2\pi)^{2n-1}} \int \frac{\Lambda_0^0 g_b(\xi)}{2|\Xi(x, \xi)|} e^{ix} d\xi + \Psi^{-2} g.
$$

For terms involving $v$ we use the expression

(4.23)

$$
v = \Theta^+ g + \Psi^{-2} g
$$

modulo smoothing terms as in Theorem 4.1.

With (4.23), and $s_0(x) := s(x, 0)$, we thus have

$$
R \circ \Gamma^{-1}_{int} \circ S_{int} v = -\frac{1}{(2\pi)^{2n}} \int s_0(x) \frac{\eta}{\eta^2 + \Xi^2(x, \xi)} \frac{\tilde{g}_b(\xi)}{\eta + i|\Xi(x, \xi)|} d\eta e^{ix} d\xi
$$

$$
= -\frac{1}{(2\pi)^{2n-1}} \int s_0(x) \frac{\tilde{g}_b(\xi)}{4|\Xi(x, \xi)|} e^{ix} d\xi,
$$

modulo lower order terms. Note that any $O(\rho)$ terms from an expansion of $s(x, \rho) = s_0(x) + O(\rho)$ lead to lower order terms by Lemma 2.5.

Next,

$$
R \circ \Gamma^{-1}_{int} \circ S_b g_b = \frac{1}{(2\pi)^{2n}} \int s_0(x) \frac{\tilde{g}_b(\xi)}{\eta^2 + \Xi^2(x, \xi)} d\eta e^{ix} d\xi
$$

$$
= \frac{1}{(2\pi)^{2n-1}} \int s_0(x) \frac{\tilde{g}_b(\xi)}{2|\Xi(x, \xi)|} e^{ix} d\xi,
$$

modulo lower order terms.

Similar to the calculation involving $\Gamma^{-1}_{int} \circ S_{int} v$ above, we have for $\Gamma^{-1}_{int} \circ A v$

$$
R \circ \Gamma^{-1}_{int} \circ A v = \frac{1}{(2\pi)^{2n-1}} \int a_0(x, \xi) \frac{\tilde{g}_b(\xi)}{4\Xi^2(x, \xi)} e^{ix} d\xi
$$

modulo lower order terms, where $a_0(x, \xi) = \sigma(A) \big|_{\rho=0}$.

For the term, $\Gamma^{-1}_{int} \circ (\rho tv)$, we use

$$
\rho tv = \rho \tau \circ \Theta^+ g + \cdots ,
$$
where the \( \cdots \) means lower order terms or smoothing terms. Hence, modulo lower order terms, we have

\[
\rho \tau v = \rho \tau \circ \Theta^+ g \\
= \rho \int_{(2\pi)^{2n}} \sum_{\ell} \int \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{\eta + i \xi(x, \xi)} \hat{g}(\xi) e^{i \eta \xi} d\eta d\xi \\
= \frac{1}{(2\pi)^{2n}} \sum_{\ell} \int \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{\eta + i \xi(x, \xi)} \hat{g}(\xi) e^{i \eta \xi} d\eta d\xi,
\]

where \( \tau_{0}^{\ell}(x) := \tau_{\ell}(x, \cdot) \), and thus

\[
\Gamma^{-1}_{inv}(\rho \tau v) = \frac{1}{(2\pi)^{2n}} \sum_{\ell} \int \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{\eta + i \xi(x, \xi)} \hat{g}(\xi) e^{i \eta \xi} d\eta d\xi,
\]

again, modulo lower order terms. Integrating over \( \eta \) and setting \( \rho = 0 \) yields

\[
R \circ \Gamma^{-1}_{inv}(\rho \tau v) = - \frac{1}{(2\pi)^{2n-1}} \frac{1}{8} \sum_{\ell} \int \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{\xi(x, \xi)} \hat{g}(\xi) e^{i \eta \xi} d\xi,
\]

modulo \( \Psi^{-2}_b g \) and smoothing terms.

We can now read off the symbols homogeneous of degree -1 with respect to \( |\xi| \) in \( (4.21) \):

\[
0 = - \frac{1}{(2\pi)^{2n-1}} \frac{3i}{16} \int \frac{\partial_x \xi^2 \cdot \partial_x \xi^2}{\xi^3(x, \xi)} \hat{g}(\xi) e^{i \eta \xi} d\xi + \frac{1}{(2\pi)^{2n-1}} \frac{1}{2} \int \frac{\Lambda_0^0 g_b}{|\xi(x, \xi)|} e^{i \eta \xi} d\xi \\
- \frac{1}{(2\pi)^{2n-1}} \frac{\sqrt{2}}{4} \int s_0(x) \hat{g}_b(\xi) e^{i \eta \xi} d\xi - \frac{1}{(2\pi)^{2n-1}} \frac{1}{4} \int a_0(x, \xi) \hat{g}(\xi) e^{i \eta \xi} d\xi \\
+ \frac{1}{(2\pi)^{2n-1}} \frac{1}{8} \sum_{\ell} \int \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{|\xi(x, \xi)|^3} \hat{g}_b(\xi) e^{i \eta \xi} d\xi.
\]

Solving for \( \sigma(\Lambda_0^0)(x, \xi) \) yields the

**Proposition 4.5.**

\[
\sigma(\Lambda_0^0) = \frac{\sqrt{2}}{2} s_0(x) + \frac{a_0(x, \xi)}{2|\xi(x, \xi)|} - \frac{1}{4} \sum_{\ell} \frac{\tau_{0}^{\ell}(x) \xi_\ell \xi_k}{|\xi^2(x, \xi)|} + \frac{3i}{8} \frac{\partial_x \xi^2 \cdot \partial_x \xi^2}{|\xi(x, \xi)|^3}.
\]

Finally, we can state the
Theorem 4.6. Modulo pseudodifferential operators of order $-1$, the symbol for $N^-$ is given by

$$\sigma(N^-)(x, \xi) = |\Xi(x, \xi)|$$

$$+ \frac{\sqrt{2}}{2} s_0(x) + \frac{a_0(x, \xi)}{2|\Xi(x, \xi)|} - \frac{1}{4} \sum_{jk} i \tau_{jk}^0 (x) \frac{\xi_j \xi_k}{\Xi^2(x, \xi)} + \frac{3i}{8} \frac{\partial_\xi \Xi^2 \cdot \partial_\xi \Xi^2}{|\Xi(x, \xi)|^3}.$$  

This is the same as Theorem 1.2 in [1].

5. The zero order term

In this section we will look at the zero order term of the DNO, and note its possible vanishing under the hypothesis of a weakly pseudoconvex domain. The vector field $(L_n - T_n)/2i$ will play a special role in the following sections and the behavior of the boundary value operators in its direction will be studied now. We use the terminology *transverse tangential* to refer to a vector field which is tangential and transverse to the complex tangent space (also called the vector field of the "missing direction" or the "bad direction" in the literature).

We start by recalling our notation used in writing $N^-$. Let $N_{1-}$ denote the operator which is given by the principal (first order) symbol of $N^-$, homogeneous of degree 1 in $|\Xi(x, \xi)|$, where $\Xi(x, \xi)$ is given in (4.4):

$$(5.1) \quad |\Xi(x, \xi)| = \sqrt{2} e^{2n-1} + 2 \sum_{k \leq n} \sigma(T_{bk}) \sigma(L_{bk}).$$

For the zero operator, we write the symbol $a_0(x, \xi)$ in Theorem 4.6 according to

$$a_0(x, \xi) = \sum_{j=1}^{2n-1} a_0^j(x) \xi_j.$$  

From Theorem 4.6 the zero order operator, denoted by $N_{0-}$, has symbol given by

$$(5.2) \quad \sigma(N_{0-}) = \sqrt{2} s_0(x) + \frac{1}{2} \sum_{j=1}^{2n-1} a_0^j(x) \xi_j - \frac{1}{4} \tau_{0j}^k (x) \xi_j \xi_k + \frac{3i}{8} \frac{\partial_\xi \Xi^2 \cdot \partial_\xi \Xi^2}{|\Xi(x, \xi)|^3},$$

in a neighborhood of a boundary point, which we assume to be $0 \in \partial \Omega$. Recall the functions $s_0, a_0^j$ and $\tau_{0j}^k$ as defined in Section 4.

According to Proposition 3.1 $s_0(x)$ is a diagonal matrix. (i.e. there are no normal derivatives of $u_I$ for $I \neq J$ which contribute to the term $f_I \bar{\omega}_J$ on the right hand side of $\square u = f_I \bar{\omega}_J$). We have $s_{0,J}$, the diagonal $(J, J)$-entry of the matrix $s_0$, to be of the form

$$(5.3) \quad s_{0,J} = -2i(-1)^{|J|} \Im(c_J^I) + d_n$$

for $n \notin J$, from Proposition 3.1(i).
Now let $A_j$ denote the $j^{th}$ row of the matrix of first order operators, $A$. We need (the vector product) $A_j \cdot u$, so as to calculate the $j^{th}$ component of $Au$, and in particular, we will need the operators with the transversal tangential, $T$, component in the expression $A_j \cdot u$ (in applying the results to the $\bar{\partial}$-Neumann condition in Section [6] we are interested in the behavior of the operators in a microlocal neighborhood determined by $\psi^\perp$).

For the contribution of the sum occurring in Proposition 3.1 to the $A$ operators, we handle the case $l = n$ separately (again, assuming $n \notin J$):

$$-L_n \bar{T}_n = \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + iT \right) \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} - iT \right)$$

$$= -\frac{1}{2} \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x_{2n-1}^2} + \frac{i}{\sqrt{2}} \left[ \frac{\partial}{\partial \rho}, T \right] + O(\rho) \Psi^2$$

$$= -\frac{1}{2} \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial x_{2n-1}^2} + \frac{i}{\sqrt{2}} T^1 + O(\rho) \Psi^2,$$

where $T^1$ is defined to be $\left[ \frac{\partial}{\partial \rho}, T \right]$ at $\rho = 0$ (see also (5.8) below). We will be interested in the transverse tangential component of the first order vector field at $\rho = 0$ of $-2L_n \bar{T}_n$, that is, in

$$(5.4) \quad \frac{1}{|T^0|} \left\langle 2i \sqrt{2} T^1, \frac{T^0}{|T^0|} \right\rangle.$$

We use $<\cdot, \cdot>$ to denote the interior product of two vector fields. To ease notation we will also use the notation of the dot product to denote the interior product in what follows. Thus

$$\frac{1}{|T^0|} \frac{2i \sqrt{2} T^1}{|T^0|} \cdot \frac{T^0}{|T^0|} := \frac{1}{|T^0|} \left\langle 2i \sqrt{2} T^1, \frac{T^0}{|T^0|} \right\rangle$$

We could calculate this term explicitly, but we will not need to; it will eventually cancel out with another term in the DNO.

For $L_k \bar{L}_k$, $k \neq n$, we recall (4.2) and write

$$\bar{T}_k|_{\rho=0} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2k-1}} + i \frac{\partial}{\partial x_{2k}} \right) + \sum_{j=1}^{2n-1} \ell^k_j(x) \frac{\partial}{\partial x_j},$$

where $\ell^k_j(x) = O(x)$. We also use the representation

$$\bar{L}_k = \sqrt{2} \sum \gamma^k_j \frac{\partial}{\partial x_j},$$
where the $\gamma_j^k$ have the property that
\[ \sum_j \gamma_j^k \gamma_j^l = \delta_{kl}, \quad 1 \leq k \leq n - 1 \]
for the delta function $\delta_{kl} = 1$ for $k = l$ and $\delta_{kl} = 0$ for $k \neq l$, and
\[ \sum_j \gamma_j^k \frac{\partial \rho}{\partial \bar{z}_j} = 0, \quad 1 \leq k \leq n - 1. \]

At the boundary point $0 \in \partial \Omega$, we can write
\[ \bar{T}_k \bigg|_{p=0} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2k-1}} + i \frac{\partial}{\partial x_{2k}} \right) \bigg|_{p=0} = \sqrt{2} \sum \gamma_j^k(0) \frac{\partial}{\partial \bar{z}_j}, \]
for $k \leq n - 1$.

As $\bar{T}_k \cdot T = 0$ we have
\[ \frac{1}{2} \ell_{2n-1}^k(x) = -\sqrt{2} \left( \sum \gamma_j^k(0) \frac{\partial}{\partial \bar{z}_j} \right) \cdot T + O(x^2) \]
\[ = -\frac{\sqrt{2}}{2i} \left( \sum \gamma_j^k(0) \frac{\partial}{\partial \bar{z}_j} \right) \cdot T_n + O(x^2) \]
\[ = i \left( \sum \gamma_j^k(0) \frac{\partial \rho}{\partial \bar{z}_j} \right) + O(x^2), \]
where we use $T \cdot \partial_{x_j} = O(x)$ for $j \neq 2n - 1$, $T \cdot T = \frac{1}{2}$ and
\[ T_n = 2\sqrt{2} \sum \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_j}. \]

Hence,
\[ \ell_{2n-1}^k(x) = 2i \sum \gamma_j^k(0) \frac{\partial \rho}{\partial \bar{z}_j} + O(x^2). \]

With
\[ L_k = \sqrt{2} \sum \gamma_j^k \frac{\partial}{\partial \bar{z}_j}, \]
we can write the coefficient of the transverse tangential vector field, $T$, in the expression from $-L_k \bar{T}_k$ as
\[ -L_k(\ell_{2n-1}^k(x)) = -2i \sum \gamma_j^k(0) L_k \left( \frac{\partial \rho}{\partial \bar{z}_j} \right) + O(x) \]
\[ = -2i \sqrt{2} \sum \gamma_j^k(0) \gamma_j^k \frac{\partial^2 \rho}{\partial \bar{z}_j^2} + O(x) \]
\[ = -i \sqrt{2} |L_k|^2 + O(x), \]
(5.5)
\[ ds^2 = \sum \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j. \]

That is,
\[-2L_k \mathcal{T}_k = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_{2k-1}^2} + \frac{\partial^2}{\partial x_{2k}^2} \right) + i 2 \sqrt{2} |L_k|_L^2 \frac{\partial}{\partial x_{2n-1}} + \cdots \]

where the \( \cdots \) refer to second order terms with coefficients in \( O(x) \) or first order terms which upon contraction with \( T \) result in \( O(x) \) functions. And similarly,
\[-2L_k \mathcal{T}_k = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_{2k-1}^2} + \frac{\partial^2}{\partial x_{2k}^2} \right) + i 2 \sqrt{2} |L_k|_L^2 \frac{\partial}{\partial x_{2n-1}} + \cdots . \]

Thus the transverse tangential component to be included in the operator \( A \) of the first order vector fields from \(-2 \sum_{k \notin j} L_k \mathcal{T}_k - 2 \sum_{k \in j} \mathcal{T}_k L_k \) written in our local coordinates is given by
\[(5.6) \quad i 2 \sqrt{2} \left( \sum_{k \in j} |L_k|_L^2 - \sum_{k \notin j} |L_k|_L^2 \right) . \]

From the first order operators in Proposition 3.1, we see there are also the \( T \) components to be included in the operator \( A \) given by
\[(5.7) \quad - (-1)^{|j|} 4j! \Re(c^j_{\mu}) - 2id_\mu. \]

We now move to the operator \( \tau \). From Proposition 3.1 \( \tau \) is a diagonal operator. Let us calculate the asymptotic behavior of the entries of the symbol of \( \tau \) for large \( |\xi_{2n-1}| \). Recall that in the \( \tau \) operator, we collected all the second order tangential derivatives with coefficients which are \( O(\rho) \).

We expand
\[(5.8) \quad T = T^0 + \rho T^1 + \rho^2 T^2 + \cdots \]
and
\[L_j = L_j^0 + \rho L_j^1 + \rho^2 L_j^2 + \cdots \]
for \( 1 \leq j \leq n-1 \).

The second order \( O(\rho) \) operators arise from
\[-2L_n \mathcal{T}_n = -2 \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + i (T^0 + \rho T^1 + \cdots ) \right) \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} - i (T^0 + \rho T^1 + \cdots ) \right) \]
\[= \cdots - 2\rho T^1 T^0 - 2\rho T^0 T^1 + \cdots \]
as well as
\[-2L_j T_j = - 2 \left( L_j^0 + \rho L_j^1 + \cdots \right) \left( T_j^0 + \rho T_j^1 + \cdots \right)
\]
\[= \cdots - 2 \rho L_j^1 T_j^0 - 2 \rho L_j^0 T_j^1 + \cdots. \]

We specifically want, \(\tau^{2n-1,2n-1}\), the coefficient of \(\partial^2\) in (the diagonal components of) \(\tau\). Thus, for instance, from the \(L_n T_n\), \(\tau^{2n-1,2n-1}\) contains the coefficient of \(-2\)

\(-2\left( T_1 \cdot T_0 \right) \frac{T_0^0}{|T_0^0|} T_0^0 - 2 T_0^0 \left( T_1 \cdot T_0 \right) \frac{T_0^0}{|T_0^0|} T_0^0\),

i.e.,

\[\tau^{2n-1,2n-1} = -4 \sqrt{2} \left( T_1, \frac{T_0^0}{|T_0^0|} \right), \tag{5.9}\]

since there are no contributions from \(-2L_j T_j\), in the form of \(-2L_j^1 T_j^0\) and \(-2L_j^0 T_j^1\), due to the property that \(L_j \cdot T = 0\). As we mentioned earlier, we will have no need to calculate explicitly the interior product.

Furthermore, we can handle the last term in (5.2) by noting that

\[\sum \partial_j \Xi^2(x, \xi) \partial_{\xi_j} \Xi^2(x, \xi) \frac{1}{|\Xi(x, \xi)|^3} = O(\xi \frac{|\xi_{2n-1}|}{|\Xi(x, \xi)|^3}) + O(\xi) \cdot O(\xi)\]

\[= O\left( \frac{|\Xi|}{|\Xi(x, \xi)|^3} \right) + O(x), \tag{5.10}\]

for large \(\xi_{2n-1}\).

Lastly, to handle the non-diagonal terms in \(\sigma(N^-)\) we consider the transverse tangential components of the terms in Proposition 3.1 (ii). Noting that \([L_j, T_k] \cdot T\) give the entries for the Levi matrix, and assuming without loss of generality that the Levi matrix is diagonal (at the given point \(0 \in \partial \Omega\)), the contributions of such components in the transverse tangential direction are \(O(x)\). The non-diagonal terms in \(\sigma(N^-)\) are thus in \(O\left( \frac{|\xi_{2n-1}|}{|\Xi(x, \xi)|^3} \right) + O(x)\).

In the expression for the zero order term of \(N^-\) we write \((b_j)\) to mean the diagonal matrix whose \((J, J)\) entry is given by \(b_j\). All terms in the expression for \(\sigma'(N^-)\), with the exception of error terms, will be diagonal matrices. Using (5.3), (5.4), (5.6), (5.7), and (5.9) in the expression for \(\sigma'(N^-)\) in (5.2), and restricting to the boundary, we have

**Proposition 5.1.** Let \(\sigma'(N^-)\) be the zero order symbol in the expansion of the DNO associated with the \(\Box\) operator. Then in a microlocal neighborhood of the boundary point
0 ∈ ∂Ω for large |ξ_{2n−1}| we have

\[ \sigma(N_0^{-}) = \frac{\sqrt{2}}{2} \left( -2i(-1)^{|J|} \Im (c_{Jn}^I) + d_n \right)_J \\
+ \left( (-1)^{|J|} 2R(c_{Jn}^I) + d_n - \left< T^1, \frac{T^0}{|T^0|} \right>_J \frac{\xi_{2n-1}}{|\Xi(x, \xi)|} \right) \\
- \sqrt{2} \left( \sum_{k \in J} |L_{bk}|^2 - \sum_{k \notin J} |L_{bk}|^2 \right) \frac{\xi_{2n-1}}{|\Xi(x, \xi)|} \\
- \sqrt{2} \left( \left< T^1, \frac{T^0}{|T^0|} \right>_J \frac{\xi_{2n-1}}{|\Xi(x, \xi)|} \right) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right) + O(x). \]

6. Boundary Equation

The \( \bar{\partial} \)-Neumann problem for a \((0, q)\)-form \( f \in L^2_{(0, q)}(\Omega) \) is to find a solution \( u \in L^2_{(0, q)}(\Omega) \) to \( \Box u = f \). As the \( \Box \) operator consists of \( \bar{\partial}^* \) operators, boundary conditions arise on \( u \) so as to fulfill conditions regarding its inclusion in the domain of \( \bar{\partial}^* \). The \( \bar{\partial} \)-Neumann is the boundary value problem

\[ \Box u = f \quad \text{in } \Omega \]

with boundary conditions

\[ u |_{\bar{\partial} \rho} = 0 \]
\[ \bar{\partial} u |_{\bar{\partial} \rho} = 0 \]
on \( \partial \Omega \).

The first boundary condition \( u |_{\bar{\partial} \rho} = 0 \) is just \( u_J = 0 \) on \( \partial \Omega \) for any \( J \) such that \( n \in J \). For the second condition involving \( \bar{\partial} u \), we note

\[ \bar{\partial} u = \sum_{J \not\in n} \left( (-1)^{|J|} \mathcal{T}_n u_J + \epsilon_{Jn}^I u_J + \epsilon_{Jn}^k \mathcal{T}_k u_J \right) \bar{\omega}_n + \cdots, \]

where \( \cdots \) refers to terms with no \( \bar{\omega}_n \) component.

Assuming the boundary condition \( u |_{\bar{\partial} \rho} = 0 \), we have \( \bar{\partial} u |_{\bar{\partial} \rho} = 0 \) is equivalent to

\[ (6.1) \quad \mathcal{T}_n u_J + (-1)^{|J|} c_{Jn}^I u_J = 0 \]
on \( \partial \Omega \) for \( J \) such that \( n \not\in J \).

We write the solution \( u \) in terms of a Green’s solution and Poisson solution:

\[ u = G(2f) + P(u_b) \]
where the operators $G$ and $P$ satisfy

\begin{align}
2\Box \circ G &= I \\
R \circ G &= 0
\end{align}

(6.2)

and

\begin{align}
2\Box \circ P &= 0 \\
R \circ P &= I,
\end{align}

(6.3)

respectively.

The $J$th component will be written $u_J = G_J(2f) + P_J(u_b)$. $\bar{\nabla}_n u_J$ can now be written on $\partial \Omega$ as

$$R \circ \bar{\nabla}_n u_J = \frac{1}{\sqrt{2}} R \circ \partial_\rho \circ G_J(2f) + \left( \frac{1}{\sqrt{2}} N^\perp - iT^0 \right) u_{b,J},$$

where $u_{b,J}$ is the $J$th component of $u_b$.

From [4] (Theorem 3.3), we use the property that

$$R \circ \partial_\rho \circ G_J \equiv R \circ \Psi^{-1}$$

modulo smoothing terms. The boundary condition (6.1) for $J \neq n$ can therefore be written as

\[
\left( \frac{1}{\sqrt{2}} N^\perp - iT^0 \right) u_{b,J} + (-1)^{|J|} c_{Jn} u_{b,J} + \frac{1}{\sqrt{2}} \left( N_0^- \right) u_{b,J} = R \circ \Psi^{-1} f,
\]

modulo lower order terms in $u_b$. As mention above in Section 5, we will concentrate on the microlocal region determined by the symbol, $\psi^-$, that is, the region in which (in local coordinates) $\xi_{2n-1}$ is large and negative. The reason is that in the other regions, estimates for $u_b$ can be obtained by inverting the operator, $1/\sqrt{2} N^\perp - iT^0$.

We will need the behavior of the operators in the microlocal neighborhood of a boundary point, $0 \in \partial \Omega$ and with support in the support of the symbol $\psi^-$. To this end, we first consider the limit of $N_0^\perp$ as $\xi_{2n-1} \to -\infty$.

Let us write

$$\left( N_0^- u_b \right)_J = N_{0,J}^- u_{b,J} + N_{0,JX}^- u_{b,J},$$

where $N_0^-_{J,J}$ is the $(J,J)$ entry in the matrix $N_0^-$ and $N_0^-_{JX}$ is the matrix consisting of the $J$th row of $N_0^-$, with the $(J,J)$ entry replaced with 0, and zeros elsewhere.

From Proposition 5.1

$$\sigma \left( N_{0,JX}^- \right) = O \left( \frac{|\xi_L|}{|\xi(x, \xi)|} \right) + O(x),$$

where

$$\sigma \left( N_{0,JX}^- \right) \equiv \sigma \left( \frac{\xi_{JX}}{\xi(x, \xi)} \right) + O(x).$$
and as $\xi_{2n-1} \rightarrow -\infty$, we see

$$
\sigma(N_{0,l}) \sim \frac{\sqrt{2}}{2} \left( -2i(-1)^{|l|} \Re(c_{J}^{l}) + e_n \right) 
- \frac{1}{\sqrt{2}} \left( (-1)^{|l|} \Im(c_{J}^{l}) + d_n - \left\langle T^1, \frac{T^0}{|T^0|} \right\rangle \right)
+ \sum_{k \leq l} |L_{bk}|^2 - \sum_{k \notin l} |L_{bk}|^2 - \frac{1}{\sqrt{2}} \left\langle T^1, \frac{T^0}{|T^0|} \right\rangle + O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right)
= - (-1)^{|l|} \sqrt{2} c_{J}^{l} + \sum_{k \leq l} |L_{bk}|^2 - \sum_{k \notin l} |L_{bk}|^2
+ O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right),
$$

(6.4)

We could also at this point proceed to calculate each of the $c_{J}^{l}$, but as we will see, these will also cancel in what follows. We will denote the zero order operator $(-1)^{|l|} c_{J}^{l} + \frac{1}{\sqrt{2}} N_{0,l}^0$ (with $c_{J}^{l}$ referring to the operator with a single diagonal entry) by $Y_{J}^0$. From above we have

$$
\sigma \left( Y_{J}^0 \right) \sim (-1)^{|l|} c_{J}^{l} + \frac{1}{\sqrt{2}} \left( (-1)^{|l|} \sqrt{2} c_{J}^{l} + \sum_{k \leq l} |L_{bk}|^2 - \sum_{k \notin l} |L_{bk}|^2 \right)
+ O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right)
= \frac{1}{\sqrt{2}} \left( \sum_{k \leq l} |L_{bk}|^2 - \sum_{k \notin l} |L_{bk}|^2 \right) + O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right),
$$

as $\xi_n \rightarrow -\infty$, recalling that $N_{0,l,x}^0 = O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right)$.

We collect our results in the following Proposition

**Proposition 6.1.** The boundary equation for the $\hat{\delta}$-Neumann problem has the form

(6.5)

$$
\left( \frac{1}{\sqrt{2}} N_{1}^0 - i T^0 \right) u_{b,j} + Y_{J}^0 u_b = R \circ \Psi^{-1} f,
$$

where

(6.6)

$$
Y_{J}^0 u_b = Y_{J}^0 u_{b,j} + \sum_{k \neq j} Y_{J,K}^0 u_{b,k},
$$

and $Y_{J}^0$ is a pseudodifferential operator of order 0, whose symbol has the property

(6.7)

$$
\sigma(Y_{J}^0) = \frac{1}{\sqrt{2}} \left( \sum_{k \leq l} |L_{bk}|^2 - \sum_{k \notin l} |L_{bk}|^2 \right) + O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right)
$$

and $Y_{J,K}^0$ is a pseudodifferential operator of order 0, whose symbol has the property

(6.8)

$$
\sigma(Y_{J,K}^0) = O(x) + O \left( \frac{|\xi_L|}{|\Xi(x, \xi)|} \right)
$$
for \( K \neq J \).

At this point, we take a moment to review how previous work on inverting the Kohn Laplacian, \( \Box_b \), defined on the boundary, could be useful in solving \((6.5)\). An inverse to \( \Box_b \) in the case of strictly pseudoconvexity was studied in \([6]\), and we first relate our boundary equation \((6.5)\) to that of \([6]\). We simplify our equation, throwing away the \( O(x) \) and \( O\left( \frac{\partial}{\partial (\xi\bar{\xi})} \right) \) terms (for the purpose of illustration only) and consider

\[
\left( \frac{1}{\sqrt{2}} N_1^\perp - iT^0 \right) u_{b,J} + \Psi^0_u u_{b,J} = R \circ \Psi^{-1} f,
\]

with

\[
\sigma(\Psi^0_u) = \frac{1}{\sqrt{2}} \left( \sum_{k \in J} \frac{|L_{b_k}|^2}{2} - \sum_{k \notin J} \frac{|L_{b_k}|^2}{2} \right).
\]

We now apply the operator \( \frac{1}{\sqrt{2}} N_1^\perp + iT^0 \) to both sides:

\[
\left( \frac{1}{\sqrt{2}} (N_1^\perp)^2 + (T^0)^2 \right) u_{b,J} + \frac{i}{\sqrt{2}} [T^0, N_1^\perp] u_{b,J} + \left( \frac{1}{\sqrt{2}} N_1^\perp + iT^0 \right) \circ \Psi^0 u_{b,J} = R \circ \Psi^0 f.
\]

(6.9)

We first note some properties of the operators involved. Consider the first order operator \([T^0, N_1^\perp]\). By expanding the symbol for \( N_1^\perp \) for large \(|\xi_{2n-1}|\), we see (for large \(|\xi_{2n-1}|\))

\[
\sigma([T^0, N_1^\perp]) = \partial_{x_{2n-1}} \sigma(N_1^\perp) = O(|\xi_L|) + O(x)O(|\xi|)
\]

modulo \( S^{-\infty}(\partial \Omega) \). We also write the operator \( \frac{1}{2} (N_1^\perp)^2 + (T^0)^2 \) in terms of the vector fields \( L_j \) and \( \bar{L}_j \). The symbol of \( (N_1^\perp)^2 \) is given by

\[
\sigma \left( (N_1^\perp)^2 \right) = \sigma(N_1^\perp) \sigma(N_1^\perp) - i \partial_\xi \sigma(N_1^\perp) \cdot \partial_x \sigma(N_1^\perp) + \cdots
\]

\[
= \Xi^2(x, \bar{\xi}) + O(|\xi_L|) + O(x)
\]

modulo \( S^{-1}(\partial \Omega) \). For the term \( \Xi^2(x, \bar{\xi}) \) we have from (4.4)

\[
\Xi^2(x, \bar{\xi}) = 2|\xi_{2n-1}|^2 + 2 \sum_j \sigma(\bar{L}_{bj}) \sigma(L_{bj})
\]

and for \( \sigma(\bar{L}_{bj}) \sigma(L_{bj}) \) we have the relations

\[
\sigma(\bar{L}_{bj} \cdot L_{bj}) = \sigma(\bar{L}_{bj}) \sigma(L_{bj}) - i \sum_j \partial_\bar{\xi} \sigma(\bar{L}_{bj}) \cdot \partial_x \sigma(L_{bj})
\]

\[
= \sigma(\bar{L}_{bj}) \sigma(L_{bj}) + \bar{L}_{bj}(\bar{\xi}_{2n-1}) (i \xi_{2n-1}) + O(x)O(|\xi|)
\]

\[
= \sigma(\bar{L}_{bj}) \sigma(L_{bj}) + \sqrt{2} |\xi_{2n-1}|^2 L_{bj}^2 + O(x)O(|\xi|),
\]
modulo $\xi_L$ terms and symbols of class $S^0$, where we use (5.5) in the last line.

Similarly, we have for $\sigma(L_{bj})\sigma(L_{bj})$

$$\sigma(L_{bj}L_{bj}) = \sigma(L_{bj})\sigma(L_{bj}) - \sqrt{2}\xi_{2n-1}|L_{bj}|^2 \xi_{2n-1} + O(|\xi|) + \cdots .$$

Then the expression for $\Xi^2(x, \xi)$ gives

$$\Xi^2(x, \xi) = 2\xi_{n-1}^2 + 2 \sum_{k \notin J} \sigma(T_{bk}L_{bk}) + 2 \sum_{k \in J} \sigma(L_{bk}T_{bk})$$

$$- 2\sqrt{2}\xi_{n-1} \sum_{k \notin J} |L_{bk}|^2 \xi_{n-1}^2 + 2\sqrt{2}\xi_{n-1} \sum_{k \in J} |L_{bk}|^2 \xi_{n-1} + O(|\xi|) + O(|\xi_L|),$$

which, combined with the expression in (6.10) above, yields (for the $(J, J)$-entry)

$$\sigma\left(\frac{1}{2}(N_1^-)^2 + (T^0)^2\right) = \sum_{k \notin J} \sigma(T_{bk}L_{bk}) + \sum_{k \in J} \sigma(L_{bk}T_{bk})$$

$$- \sqrt{2}\xi_{n-1} \sum_{k \notin J} |L_{bk}|^2 \xi_{n-1}^2 + \sqrt{2}\xi_{n-1} \sum_{k \in J} |L_{bk}|^2 \xi_{n-1} + O(|\xi|) + O(|\xi_L|).$$

Furthermore,

$$\sigma\left(\frac{1}{\sqrt{2}} N_1^- + iT^0\right) \circ \mathcal{F}_0^0 \left(\sum_{k \in J} |L_k|^2 \xi_{n-1}^2 - \sum_{k \notin J} |L_k|^2 \xi_{n-1}^2\right)$$

$$= \sqrt{2}|\xi_{n-1}^2| \left(\sum_{k \in J} |L_k|^2 \xi_{n-1}^2 - \sum_{k \notin J} |L_k|^2 \xi_{n-1}^2\right)$$

$$+ O(|\xi_L|) + O(|\xi|),$$

for $|\xi_L| \ll |\xi_{2n-1}|$, and $\xi_{2n-1} < 0$, modulo lower order symbols.

(6.9) is thus reduced to studying

$$\sum_{k \notin J} T_{bk}L_{bk} + \sum_{k \in J} L_{bk}T_{bk}$$

modulo first order operators with symbols which can be made arbitrarily small in a microlocal neighborhood of the boundary point $0 \in \partial \Omega$ for $|\xi_L| \ll |\xi_{2n-1}|$.

In the highest order, this is just the Kohn Laplacian, $\Box_b$ which, under the hypothesis of strict pseudoconvexity, can be inverted by analyzing the operator on the Heisenberg group, as in [6], or in the case of finite type by considering relations of commutators of the vector fields, $L_j$ and their conjugates, as in [11]. The problem in the case of weak pseudoconvexity is that the means to control derivatives in the direction of $T$, namely through commutators of the vector fields, $L_j$, with vector fields, $T_k$, is no longer available.

One of the immediate difficulties in using the method of applying the boundary operator $\frac{1}{\sqrt{2}} N_1^- + iT^0$ as above leading to (6.9) is that the resulting highest order symbol,

$$\sigma\left(\frac{1}{2}(N_1^-)^2 + (T^0)^2\right)$$
is not elliptic. It is missing estimates from below by the $\xi_{2n-1}$ transform variable. In other words, an estimate of the form
\[
\sigma \left( \frac{1}{2} (N_1^-)^2 + (T^0)^2 \right) \gtrsim 1 + |\xi_{2n-1}|^2
\]
for $|\xi_L| \gg 1$ does not hold. It still may be possible to obtain information of the solution to (6.5) if it were possible to obtain a lower order estimate, an estimate of the first order terms of (6.9) of the form
\[
\sigma \left[ \left( \frac{1}{\sqrt{2}} N_1^- + i T^0 \right) \circ \mathcal{Y}^0 \right] \gtrsim 1 + |\xi_{2n-1}|
\]
and use the missing first order estimate as a (weaker) substitute for an elliptic second order estimate. This idea is used in [5] to obtain (weighted) estimates of the boundary solution.

The aim of the next sections is to show how persistent the absence of ellipticity in the boundary equation is.

7. Variations of the $\square$ operator

In this section we consider operators obtained from the $\square$ operator by adding additional terms. In particular, we let $\phi$ be a function supported near the boundary and with $\square \phi = \bar{\partial}^+ \bar{\partial}^- + \bar{\partial}^- \bar{\partial}^+ (1 + \phi)$, we consider the boundary value problem:
\[
\square \phi u = f
\]
with the boundary conditions,
\[
\begin{align*}
 u |_{\partial \rho} &= 0, \\
 \bar{\partial}((1 + \phi)u) |_{\bar{\partial} \rho} &= 0,
\end{align*}
\]
holding on $\partial \Omega$. The first condition ensures $u \in \text{dom}(\bar{\partial}^+)$ and the second that $\bar{\partial}((1 + \phi)u) \in \text{dom}(\bar{\partial}^-)$.

We first look at the case $\phi$ only depends on $\rho$: $\phi = \phi(\rho)$, and $\phi(0) = 0$, and we use the notation from the previous sections. In this case the condition $\bar{\partial}((1 + \phi)u) |_{\bar{\partial} \rho} = 0$ can be written
\[
\sum_k (1 + \phi) \mathcal{T}_k u_k + (1 + \phi)(-1)^{|J|} \mathcal{T}_n u_j + (1 + \phi)(-1)^{|J|}(\mathcal{T}_n \phi) u_j + (1 + \phi)c_{Jn}^I u_j = 0.
\]
Combined with the first boundary condition, $u |_{\partial \rho} = 0$, and recalling $\phi(0) = 0$, this yields
\[
\mathcal{T}_n u_j + (\mathcal{T}_n \phi) u_j + (-1)^{|J|} c_{Jn}^I u_j = 0.
\]

At first sight, a hold on regularity appears possible, in light of the discussion at the end of Section [6] as the term $\mathcal{T}_n \phi$ allows for a strictly positive (diagonal)
addition to the $\Psi^d_j$ operator. We repeat the steps of the previous sections to obtain an expression of (7.2) in terms of the complex tangential vector fields, $L_j$; as before, the main calculation concerns the DNO.

To recall, we write $u_j$ as a sum of solutions to Dirichlet problems, the solutions written in terms of Green’s operator and a Poisson operator (for the analogues to the systems, (6.2) and (6.3), with $\square$ replaced by $\square_{\phi}$):

$$u = G^\phi(2f) + P^\phi(u_b).$$

Also, we have

$$\left.\overline{\nabla} P^\phi(u_b)\right|_{\rho=0} = \left.\left(\frac{1}{\sqrt{2}} \partial_\rho - iT\right) P^\phi(u_b)\right|_{\rho=0} = \frac{1}{\sqrt{2}} N^\psi u_b - iT^0 u_b,$$

and

$$\left.\overline{\nabla} G^\phi(2f)\right|_{\rho=0} = \left.\frac{1}{\sqrt{2}} \partial_\rho G^\phi(2f)\right|_{\rho=0} = R \circ \Psi^{-1} f.$$ 

We can now rewrite (7.2) as

$$(7.3) \quad \left(\frac{1}{\sqrt{2}} N^\psi - iT^0\right) u_{bj} + \left(\frac{1}{\sqrt{2}} \phi'(0) + (-1)^{1}|\xi|^l J_n\right) u_{bj} = R \circ \Psi^{-1} f.$$ 

As in the case with $\phi \equiv 0$, the operator $\frac{1}{\sqrt{2}} N^\psi - iT^0$ is of first order, but it is not elliptic since its principal symbol,

$$\frac{1}{\sqrt{2}} |\Xi(x, \xi)| + \xi_{2n-1}$$

tends to 0 as $\xi_{2n-1} \to -\infty$. However, a non-vanishing zero order term in the symbol expansion of $N^\psi$ would, after composition with $\frac{1}{\sqrt{2}} N^\psi + iT^0$ lead to a first order term whose symbol is non-vanishing in the support of $\psi^\phi$. We thus examine the term $\sigma_0(N^\psi)$. 

For $\phi = 0$, we have Theorem 4.6. In the case $\phi = \phi(\rho) \neq 0$ we examine the changes induced on the DNO. To highest order, the operators $\square_{\phi}$ and $\square$ are identical, so we determine the operators, $S_{\phi}$, $A_{\phi}$, and $\tau_{\phi}$, (and their corresponding symbols) with which $\square_{\phi} v$ can be written as in (4.5) in the interior of $\Omega$ as

$$\Gamma v + \sqrt{2} S_{\phi} \left(\frac{\partial v}{\partial \rho}\right) + A_{\phi} v + \rho \tau_{\phi}(v) = 0.$$ 

We write the operator $\square_{\phi}$ in local coordinates. We have

$$\square_{\phi} = \square + \delta^{\phi} \delta_{\phi}.$$
In Proposition 3.1 we examined the forms which, upon action through the □\_\_ operator would result in terms with a certain \( \tilde{\omega}_J \) component. We follow this same approach here for the operator \( \squarephi \) in order to obtain an expression for the DNO corresponding to the \( \squarephi \) operator.

We examine the term \( \tilde{\partial}^* \tilde{\partial}(\phi u) \). We have

\[
(7.4) \quad \tilde{\partial}(\phi u J\tilde{\omega}_J) = (-1)^{|J|} (\nabla_J \phi) u_j \tilde{\omega}_j \wedge \tilde{\omega}_n + \phi(\rho) \sum_j (-1)^{|J|} (\nabla_n u_j) \tilde{\omega}_j \wedge \tilde{\omega}_n + \cdots ,
\]

where the \( \cdots \) refer to terms with no \( \tilde{\omega}_n \) component, or, in the case \( n \in J \) involve only the \( \nabla_j \) operators for \( j = 1, \ldots, n-1 \).

And from

\[
\tilde{\partial}^* \nu \tilde{\omega}_j \wedge \tilde{\omega}_n = \left((-1)^{|J|} (-L_n + d_n + \epsilon \nabla_{J \cup \{n\}}) \right) \nu \tilde{\omega}_j + \cdots ,
\]

where here the \( \cdots \) denote terms which are orthogonal to \( \tilde{\omega}_J \), we get

\[
\tilde{\partial}^* \tilde{\partial}(\phi u J\tilde{\omega}_J)
\]

\[
= \left(- \phi(\rho) L_n \nabla J u_j - (\nabla_J \phi) L_n u_j - (\nabla_n \phi) \nabla_n u_j \right.
\]

\[
+ \phi(\rho) \left( d_n + (-1)^{|J|} \epsilon \nabla_{J \cup \{n\}} \right) \nabla_n u_j \left. \right) \tilde{\omega}_j + \cdots
\]

\[
= \left(- \phi(\rho) L_n \nabla J u_j - \phi'(\rho) \partial \rho u_j + \phi(\rho) \left( d_n + (-1)^{|J|} \epsilon \nabla_{J \cup \{n\}} \right) \nabla_n u_j \right) \tilde{\omega}_j + \cdots .
\]

Again, for the error terms we include all 0 order terms, terms orthogonal to \( \tilde{\omega}_J \), and terms involving only \( L_j \) and/or \( \nabla J \) for \( j = 1, \ldots, n-1 \) in the \( \cdots \).

From (3.5), we have

\[
\tilde{\partial}^* \tilde{\partial}(\phi(\rho) u_k \tilde{\omega}_{J \cup \{l\}}) = -\phi(\rho) \epsilon \nabla_{J \cup \{l\}} \epsilon \nabla_{J \cup \{l\}} \nabla_k u_k \tilde{\omega}_j + \cdots
\]

for \( l \neq n \) and \( J \neq n \), where here the \( \cdots \) refer to terms which are of the form \( O(\rho) L_j + O(\rho) \nabla J \), or are of order 0, or are terms orthogonal to \( \tilde{\omega}_J \). In the case \( l = n \) we have

\[
\tilde{\partial}^* \tilde{\partial}(\phi u_k n \tilde{\omega}_{J \cup \{n\}}) =
\]

\[
- (-1)^{|J|} \epsilon \nabla_{J \cup \{l\}} \frac{1}{\sqrt{2}} \phi(\rho) \nabla_k u_k n \tilde{\omega}_j - (-1)^{|J|} \epsilon \nabla_{J \cup \{l\}} \phi(\rho) L_n \nabla_k u_k n \tilde{\omega}_j + \cdots .
\]
From these calculations, we see that in a small neighborhood of a boundary point \( p \in \partial \Omega \), for which again we assume \( p = 0 \), the equation \( 2 \Box \phi v = 0 \) corresponding to forms for which \( v_j = 0 \) if \( n \in J \) can be written

\[
-\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} + 2 \sum_{2n-1} \frac{\partial^2}{\partial x_j \partial x_{n-1}} + \sum_{j,k=1} l_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right) v
+ \sqrt{2} S \phi \left( \frac{\partial v}{\partial \rho} \right) + A \phi v + \rho \tau \phi (v) = 0,
\]

where

\[
S \phi = S - \sqrt{2} \phi' (0) + O(\rho),
A \phi = A + O(\rho),
\]

and

\[
\tau \phi = \tau - 2 \phi (\rho) L_n \bar{L}_n + \cdots
\]

where \( S, A, \) and \( \tau \) are the operators from Section 4, and the \( \cdots \) in the expression for \( \tau \phi \) refer to second order terms which are \( O(\rho) \), and are compositions with at least one \( L_k \) or \( \bar{L}_k \) for \( k \in \{1, \ldots, n - 1\} \) (this also holds true in the case \( n \in J \), although is not needed).

We now examine the contributions from the \( \phi \) function to the DNO. Using Lemma 2.5 the \( O(\rho) \) terms of the operator \( S \phi \) and \( A \phi \) above lead to operators of order \(-1\) and lower. From Theorem 4.6 we see the symbol \( s_0 (x) \) for the DNO corresponding to \( 2 \Box \) should be replaced with \( s_0 (x) - \sqrt{2} \phi' (0) \).

For the contributions from the \( \tau \phi \) operator we expand \( \phi (\rho) = \phi' (0) \rho + O(\rho^2) \) and look at the terms

\[
-2 \phi' (0) \rho \left( \frac{1}{2} \frac{\partial^2}{\partial \rho^2} + T^2 \right)
\]

coming from \(-2 \phi (\rho) L_n \bar{L}_n \) in \( \tau \phi \). A term \( \rho \partial^2 \rho v \) can be written using transforms, assuming the support of \( v \) is contained in a small coordinate patch around \( 0 \in \partial \Omega \), as

\[
\rho \frac{1}{(2\pi)^{2n}} \int \left( -\eta^2 \bar{\phi} (\xi, \eta) + \partial_\rho \bar{\phi} (\xi, 0) + i \eta \bar{\phi} (\xi) \right) e^{ix\xi} e^{i\rho \eta} d\xi d\eta.
\]

Since \( \rho \cdot \delta(\rho) \equiv 0 \), we have

\[
\rho \int \partial_\rho \bar{\phi} (\xi, 0) e^{i\rho \eta} e^{ix\xi} d\xi d\eta \equiv 0
\]

in the term above, and

(7.5) \quad \rho \partial^2 \rho v = \rho \frac{1}{(2\pi)^{2n}} \int \left( -\eta^2 \bar{\phi} (\xi, \eta) + i \eta \bar{\phi} (\xi) \right) e^{ix\xi} e^{i\rho \eta} d\xi d\eta.
We examine first the term $\rho \int \eta^2 \overline{\varphi} e^{ix\xi} e^{ip\eta} d\eta d\xi$, recalling that $v = \Theta^+ g$ modulo lower order terms:

$$
\rho \frac{1}{(2\pi)^{2n}} \int \left(-\eta^2 \overline{\varphi}(\xi, \eta)\right) e^{ix\xi} e^{ip\eta} d\eta d\xi
$$

$$
= -\rho \frac{i}{(2\pi)^{2n}} \int \frac{\eta^2}{\eta + i|\Xi(x, \xi)|} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi
$$

$$
= -\frac{1}{(2\pi)^{2n}} \int \frac{\eta^2}{\eta + i|\Xi(x, \xi)|} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi + \frac{2\eta}{(2\pi)^{2n}} \int \frac{\eta}{\eta + i|\Xi(x, \xi)|} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi,
$$

modulo lower order terms and smooth terms (of the form $R^{-\infty}$). In the calculation of the DNO, the above term contributes

(7.6)

$$
2|\Xi(x, \xi)| \Gamma^{-1}_{\text{int}} \circ \phi'(0) \circ F.T.^{-1} \left(\eta^2 \overline{\varphi}\right)
$$

(see the calculation preceding Proposition 4.5). We thus need

$$
\phi'(0) \Gamma^{-1}_{\text{int}} \circ \rho F.T.^{-1} \left(\eta^2 \overline{\varphi}\right)
$$

$$
= \frac{\phi'(0)}{(2\pi)^{2n}} \int \frac{1}{\eta^2 + |\Xi(x, \xi)|^2} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi
$$

$$
= \frac{\phi'(0)}{(2\pi)^{2n}} \int \frac{1}{\eta + i|\Xi(x, \xi)|} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi + \frac{2\eta}{(2\pi)^{2n}} \int \frac{1}{\eta - i|\Xi(x, \xi)|} \overline{\varphi_b(\xi)} e^{ip\eta} e^{ix\xi} d\eta d\xi,
$$

again, modulo lower order terms and smooth terms. Integrating over $\eta$ and setting $\rho = 0$ yields

$$
\frac{\phi'(0)}{8} \frac{1}{(2\pi)^{2n-1}} \int \frac{\overline{\varphi_b(\xi)}}{|\Xi(x, \xi)|} e^{ix\xi} d\xi - \frac{\phi'(0)}{4} \frac{1}{2} \frac{1}{(2\pi)^{2n-1}} \int \frac{\overline{\varphi_b(\xi)}}{|\Xi(x, \xi)|} e^{ix\xi} d\xi,
$$

modulo $\Psi^2 \hat{g}_b$ and smoothing terms. When setting the terms of order $-1$ in the $|\Xi(x, \xi)|$ factors equal as we did to show Proposition 4.5, we are led to the symbols

$$
\frac{\phi'(0)}{4} - \phi'(0) = -\frac{3}{4} \phi'(0).
$$

for the contribution of (7.6) in the DNO for the operator $2\square \varphi$.

We further need the contribution of the boundary term, $g_b$ in (7.5) to the DNO. Similar to above, the contribution comes through

$$
-2|\Xi(x, \xi)| \phi'(0) \Gamma^{-1}_{\text{int}} \circ \rho F.T.^{-1} (i\eta \overline{\varphi_b(\xi)})
$$
for which we have
\[
\left. \phi'(0) \Gamma_{i\eta}^{-1} \circ \rho F.T.^{-1} (i\eta \tilde{g}_b(\xi)) \right|_{\rho=0}
\]
\[
= - \frac{\phi'(0)}{(2\pi)^{2n}} \int \frac{2\eta^2}{(\eta^2 + \Xi^2(x, \xi))} \tilde{g}_b(\xi) e^{i\xi \cdot x} d\eta d\xi
\]
\[
= - \frac{\phi'(0)}{2} \frac{1}{(2\pi)^{2n-1}} \int \frac{\tilde{g}_b(\xi)}{|\Xi(x, \xi)|} e^{i\xi \cdot x} d\eta d\xi,
\]
modulo lower order and smoothing terms.

As in the calculations of Theorem 4.6 the $-2\phi'\rho T^2$ terms lead to a term with symbol
\[
- \frac{\phi'(0) \xi_{2n-1}^2}{2 \Xi^2(x, \xi)}
\]
in the DNO.

We note the $O(\rho)$ second order terms with at least one of $L_k$ or $T_k$ with $k \in \{1, \ldots, n-1\}$ lead to terms $O \left( \frac{\xi_{2n-1}}{|\Xi(x, \xi)|} \right)$. Therefore, the contributions from the operator $\tau$ in addition to those from $\bar{\partial}$ are given by adding
\[
- \frac{3}{4} \phi'(0) + \phi'(0) \frac{\xi_{2n-1}^2}{2 \Xi^2(x, \xi)}
\]
to the DNO for $2\Box$. Note this term tends to 0 as $\xi_{2n-1} \to -\infty$.

We thus have the following description of the DNO in a microlocal neighborhood in the support of $\psi^{-}$:

**Proposition 7.1.** Modulo pseudodifferential operators of order $-1$, the symbol for $N^{\phi^-}$ is given by
\[
\sigma(N^{\phi^-})(x, \xi) = \sigma(N^-)(x, \xi) - \phi'(0) + O \left( \frac{\xi_{2n}}{|\Xi(x, \xi)|} \right).
\]

Returning to the boundary conditions, we see how the additional terms from the DNO coming from the added $\phi(\rho)$ function affect the boundary equations (7.1). The first condition, $u | \bar{\partial} \rho$ remains the same, and is equivalent to $u_J = 0$ if $n \in J$.

We recall the second condition written as in (7.2):\[ \left( \frac{1}{\sqrt{2}} N^{\phi^-} - iT^0 \right) u_{bJ} + \left( \frac{1}{\sqrt{2}} \phi'(0) + (-1)^{|J|} c_{J}^f \right) u_{bJ} = R \circ \Psi^{-1}(f). \]

From Proposition 7.1 we can write $\sigma(N^{\phi^-}) = \sigma(N^-) - \phi'(0) + O \left( \frac{\xi_{2n}}{|\Xi(x, \xi)|} \right)$, modulo lower order symbols. In particular, the $\phi'(0)$ term in the boundary equation cancels with that coming from the DNO. We can state the
Theorem 7.2. Let $\Box_{\phi} = \bar{\partial}^* + \bar{\partial} \circ (1 + \phi)$. Let $\phi = \phi(\rho)$ be a smooth function which depends only on the defining function, with the property $\phi(\rho) = O(\rho)$. The condition

$$\bar{\partial} \circ (1 + \phi)u \in \text{dom}(\partial^*)$$

equivalent to $\bar{\partial}((1 + \phi)u) \partial \rho = 0$ on $\partial \Omega$, has the form

$$\left( \frac{1}{\sqrt{2}} N^1_i - iT^0 \right) u_{b,f} + \Upsilon^0_J u_b = R \circ \Psi^{-1} f$$

as in (6.5) of Proposition 6.1, with $\Upsilon^0_J$ sharing the same properties as those of (6.6), (6.7) and (6.8).

REFERENCES

[1] D. C. Chang, A. Nagel, and E. Stein. Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$. Acta Math., 169:153–228, 1992.
[2] M. Christ. On the $\bar{\partial}$ equation for three-dimensional CR manifolds. Proc. Sympos. Pure Math., 52(3):63–82, 1991.
[3] D. Ehsani. Exact regularity of the $\bar{\partial}$-problem with dependence on the $\bar{\partial}_b$-problem on weakly pseudoconvex domains in $\mathbb{C}^2$. Preprint.
[4] D. Ehsani. Pseudodifferential analysis on domains with boundary. Preprint.
[5] D. Ehsani. Weighted estimates for the $\bar{\partial}$-neumann problem on intersection domains in $\mathbb{C}^2$. Preprint.
[6] G. Folland and E. Stein. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. Comm. Pure Appl. Math., 27:429–522, 1974.
[7] J. Kohn and A. Nicoara. The $\bar{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3. J. Funct. Anal., 230(2):251–272, 2006.
[8] J.J. Kohn. Estimates for $\bar{\partial}_b$ on pseudoconvex CR manifolds. Proc. Sympos. Pure Math., 43:207–217, 1985.
[9] J.-L. Lions and E. Magenes. Non-homogeneous boundary value problems and applications, volume I. Springer-Verlag, New York, 1972.
[10] A. Nicoara. Global regularity for $\bar{\partial}_b$ on weakly pseudoconvex CR manifolds. Adv. Math., 199:356–447, 2006.
[11] L. Rothschild and E. Stein. Hypoelliptic differential operators and nilpotent groups. Acta Math., 137:248–320, 1976.
[12] F. Treves. Introduction to Pseudodifferential and Fourier Integral Operators. The University Series in Mathematics. Plenum Press, 1980.

Hochschule Merseburg, Eberhard-Leibniz-Str. 2, D-06217 Merseburg, Germany
E-mail address: dehsani.math@gmail.com