Central potentials and examples of hidden algebra structure

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Abstract

We propose two generalisations of the Coulomb potential equation of quantum mechanics and investigate the occurrence of algebraic eigenfunctions for the corresponding Schrödinger equations. Some relativistic counterparts of these problems are also discussed.

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1 Introduction

Considerable attention has been always paid to the quantum-mechanical problems which can be solved analytically. The reason for that is twofold. First, it appeared that some, from the physical point of view most important problems (like, for example, the Coulomb problem), are analytically solvable. Moreover, every such problem provides a laboratory for testing the general ideas, approximate methods, etc. These so-called exactly solvable models were analysed carefully and it was revealed that there is usually (if not always) some algebraic structure (mainly group-theoretical one) underlying the phenomenon of solvability.

It has been discovered [1-7] quite recently that there exists a class of quantum-mechanical models, which, being generically analytically nontractable, allow for partial determination of the spectrum once a fine tuning of the coupling constants has been made. Such models were named quasi exactly solvable (QES). It was also revealed that in many cases there exists a hidden $sl_2$ structure responsible for such a behaviour. For the first sight, the class of such models might seem to be very restricted. However, it has been shown recently [8] that the Coulomb correlation problem of two electrons in one external oscillator potential belongs to this class, explaining the existence of particular solutions to this problem [9,10].

In this note we first point out that the hidden $sl_2$ symmetry is also responsible for the existence of exact solutions for fraction power potentials discussed recently in Ref.[11]. We further study the occurrence of algebraic solutions for two classes of potentials which, generalizing those of Refs.[8,11], do not lead to quasi-exactly solvable equations. The discussion involves both, non-relativistic (i.e. Schrödinger) as well as relativistic equations (for instance the Klein-Gordon and the Dirac equation).
2 Schrödinger equations

The model investigated in Ref. [11] consists of a particle moving in a central potential

\[ V(r) = \frac{\alpha}{r^{1/2}} + \frac{\gamma}{r} + \frac{\beta}{r^{3/2}} \]  

(actually the \( \gamma \)-term is absent in Ref. [11] but it can be included without difficulty). The Schrödinger equation \((\hbar = 2m = 1)\)

\[ \left\{ \frac{d^2}{dr^2} + E - V(r) - \frac{l(l+1)}{r^2} \right\} \psi(r) = 0 \]  

transforms, under the following change of variable

\[ r = x^2, \quad \psi(r) = x^{2l+2} p(x) \exp(ax^2 + bx) \]  

into

\[ \left\{ \begin{array}{l}
\frac{d^2}{dx^2} + (4ax^2 + 2bx + 4l + 3) \frac{d}{dx} \\
+ 4(E + a^2)x^3 + 4(ab - a)x^2 \\
+ (b^2 + 8a(l + 1) - 4\gamma)x + (4l + 3)b \end{array} \right\} p(x) = 4\beta p(x) \]  

For bound states, \( E \) is negative, so we can choose the free parameters \( a \) and \( b \) so that the terms of highest degrees in Eq. (4) vanish:

\[ a = -\sqrt{-E}, \quad b = \frac{-\alpha}{\sqrt{-E}} \]  

Now, the differential operator \( D \) standing on the left-hand side of Eq. (4) preserves the space \( P_n \) of polynomials of degree at most \( n \), provided the following relation hold

\[ b^2 + 4a(2l + 2 + n) - 4\gamma = 0. \]  

It can then be expressed in terms of the generators of the \((n+1)\)-dimensional representation of \( sl_2 \)

\[ J^+ = x^2 \frac{d}{dx} - nx, \quad J^0 = x \frac{d}{dx} - \frac{n}{2}, \quad J^- = \frac{d}{dx} \]
We have indeed

\[ D = J^0 J^- + 4aJ^+ + 2bJ^0 + (4l + 3 + \frac{n}{2})J^- + (4l + 3 + n)b \]  

(8)

Equations (5) and (6) give the eigenvalue condition

\[ (n + 2l + 2)(\sqrt{-E})^3 + \gamma(\sqrt{-E})^2 = \frac{\alpha^2}{4} \]  

(9)

which has always one positive solution for \( \sqrt{-E} \). In the particular case \( \gamma = 0 \) we obtain Eq.(12) of Ref.[11].

Let us fix once forever \( \alpha \) and \( \beta \) and choose some natural number \( n \). Then Eq.(9) fixes uniquely the energy as a function of \( \alpha, \gamma \) and \( n \). At the same time, Eq.(4) becomes an algebraic equation for \( \beta \) in the space \( \mathcal{P}_n \). If we invoke the theorem relating the level number to the number of modes of the corresponding eigenfunction, then the following picture emerges. Given \( \alpha, \gamma \) and \( n \) and the corresponding energy \( E(\alpha, \gamma, n) \) we have a set of Hamiltonians parametrized by \( \beta_i (i = 0, 1, 2, \cdots, n) \) such that this energy corresponds to the \( i \)-th eigenstate of the Hamiltonian at \( i \)-th value of \( \beta \). All relations given between the coupling constants \( \alpha, \beta \) in Ref.[11] can be recovered in this way. This is exactly the second form of quasi exactly solvable Hamiltonians in the terminology of Turbiner [3,8].

2.1 General fractional power potentials

Let us now consider the following generalisation of Eqs.(1),(2). Take a positive natural \( N \) and put

\[ V(r) = \sum_{k=1}^{2N-1} \frac{\alpha_k}{r^{k/N}} \]  

(10)

The Schrödinger equation (2) is now rewritten in terms of \( x = r^{1/N} \) and of \( p(x) \) given by

\[ \psi(r) = x^{N(l+1)}(\exp q(x))p(x) \quad , \quad q(x) = \sum_{k=1}^{N} \beta_k x^k \]  

(11)
and reads

\[
\{ \frac{x}{N^2} \frac{d^2}{dx^2} + \left[ \frac{1-N}{N^2} + \frac{2xq'}{N^2} + \frac{2(l+1)}{N} \right] \frac{d}{dx} \\
+ \left[ E x^{2N-1} - \sum_{k=1}^{2N-1} \alpha_k x^{2N-1-k} \\
+ \frac{(1-N)q'}{N^2} + \frac{(q'^2 + q'')x}{N^2} + \frac{2(l+1)q'}{N} \right] \} p(x) = 0
\]

(12)

It is immediate to see that, for \( N > 2 \), the operator, say \( D \), on the left-hand side is not expressible in terms of the generators (7); for instance, the terms of the form \( x^s dx - cx^{s-1} \) do not preserve \( \mathcal{P}_n \) if \( s > 2 \). However this does not constitute a limitation since (it was the case also with Eq.(4)) the eigenenergy \( E \) does not appear as the spectral parameter of Eq.(12).

The construction of polynomial solutions to Eq.(12) is performed in a few steps. We first annihilate the pure power terms in the operator \( D \), (i.e. the terms proportional to \( x^{2N-1}, x^{2N-2}, \ldots x^N \)) by suitably choosing the parameters \( \beta_k \) appearing in the exponential prefactor. This step is done independently of the degree of \( p(x) \).

Eq.(12) can then be rewritten in the form

\[
D p(x) = \{(c_{-1} x dx + c'_{-1}) dx + \sum_{k=0}^{N-1} x^k (c_k x dx + c'_k)\} p(x) = 0
\]

(13)

where \( c_k, c'_k \) are constants. Let \( p(x) \) be a polynomial of degree \( n \); the equation above leads to a system of \( N + n \) homogeneous linear equations in the \( n + 1 \) coefficients of \( p(x) \). Practically, one first solve the sub-system of \( n + 1 \) equations corresponding to the \( n + 1 \) lowest powers of \( x \). This determines, in principle, \( n + 1 \) values for \( \alpha_{2N-1} \) (this coupling constant plays a role of the eigenvalue of the system) and \( n + 1 \) polynomial solutions for \( p(x) \). The \( N - 1 \) remaining equations are finally fulfilled by imposing constraints among the coupling constants and the energy \( E \).

In summary there are \( N \) relations (including the one fixing \( \alpha_{2N-1} \)) among the \( 2N \) parameters \( E, \alpha_k \). One of the constraints fixes \( E \) in terms of the coupling
constants and the \( N - 1 \) remaining ones define an \( N \)-dimensional manifold in the \( 2N - 1 \) dimensional space of the coupling constants. This manifold can be parametrized in terms of \( \alpha_1, \cdots, \alpha_N \) or, alternatively, in terms of the parameters entering in \( q(x) \).

The form of the constrained potentials reads trivially from Eq.(12) if we choose \( p(x) \) as a constant. In this case, we obtain a family of models having the same ground state energy and wave function.

### 2.2 General polynomial potentials

The reasoning developed above can be repeated, mutatis mutandis, for the generalisation of quasi exactly solvable models considered in Ref.[8]. The generalised potential is of the form

\[
V(r) = \frac{\alpha}{r} + \sum_{k=1}^{2N} \alpha_k r^k
\]

and the counterpart of Eq.(12) reads
\[
\left\{ \begin{array}{l}
\frac{d^2}{dx^2} + 2(xq' + (l + 1)) \frac{d}{dx} \\
+ \quad Ex - xV(x) + x(q'' + q'^2) + 2(l + 1)q' \quad p(r) = 0
\end{array} \right. \tag{15}
\]

This equation is not quasi exactly solvable for \( N > 1 \). However a set of algebraic solutions can be found following the lines discussed above.

In this case there are alternative circumstances under which the Schrödinger equation under consideration admits different types of algebraic solutions. First remark that the form (15) of the equation is obtained after multiplication by a power of \( x \) which makes that the coefficients standing in front of the derivatives are polynomials. Now, let the "odd" coefficients (i.e. the \( \alpha, \alpha_{2j+1}'s \)) be zero in the potential (14) and divide the full equation (15) by \( x \). If the integer \( N \) in \( V(x) \) is of the form \( 2j-1 \), then it appears that the polynomial \( q(x) \) is even also and so will be the full operator \( D/x \). Accordingly, its eigenfunctions are even or odd functions of \( x \). If we focus on the even solutions, one observes that all the (apparent) singular terms involving \( (1/x) \) naturally cancel and we conclude that new polynomial solutions exist in these cases too.

We notice that the Coulomb interaction is now absent and that the energy parameter \( E \) plays its role of eigenvalue of the operator \( D/x \). The degree of the potential is of the form \( 4j - 2 \). The case \( j = 1 \) corresponds to the harmonic oscillator (quadratic potential) which can be solved exactly. The case \( j = 2 \) corresponds to the famous example of quasi exactly solvable system (Refs.[3]) with a potential of degree six. It admits a total of \( N + 1 \) algebraic solutions if the coupling constants are suitably tuned. The potentials of higher degree, corresponding to \( j > 2 \), have a less rich set of algebraic solutions, only a single eigenvector is available if the coupling constants fulfil all the constraints.

Finally let us note that the quasi exactly solvable central potentials remain quasi exactly solvable if one adds a Dirac monopole placed at the origin. This conclusion follows immediately from the fact that the monopole interaction con-
tributes only to the centrifugal part of the radial Schrödinger equation [12]

3 Relativistic equations

In the previous section, we focused our attention on the algebraic solutions of Schrödinger equations. The techniques employed can be tentatively applied in the study of the relativistic counterparts of the Schrödinger equation, for instance the Klein-Gordon and the Dirac equations treated in the background of some radial potential \( A_0 = V(r) \).

The separation of the angular variable is effective for these equations too; in the case of the Klein-Gordon equation the condition which determines the radial part of the wave function reads

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - m^2 + (E + V(r))^2 \right] R(r) = 0 \tag{16}
\]

If the potential is of the form Coulomb + polynomial

\[
V(r) = \frac{\alpha}{r} + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_n r^n \tag{17}
\]

Eq.(16) becomes

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1) - \alpha^2}{r^2} + \frac{2\alpha E}{r} + (E^2 - m^2 + 2\alpha \alpha_1 + 2(\alpha \alpha_2 + E \alpha_1) r + \cdots + \alpha_n^2 r^{2n}) \right] R(r) = 0 \tag{18}
\]

for which the above procedure can be applied. The value of \( \mu \) in the exponential prefactor is slightly affected by a term proportional to \( \alpha^2 \) and the polynomial \( q(r) \) of the exponential prefactor is of degree \( n + 1 \).

The Dirac equation

After the separation of the angular variable, the radial part of the Dirac equation reads [13]

\[
\begin{pmatrix}
\frac{d}{dr} - \frac{\kappa}{r} \\
\frac{d}{dr} + \frac{\kappa}{r}
\end{pmatrix}
\begin{pmatrix}
m - V(r) - E \\
m + V(r) + E
\end{pmatrix}
\begin{pmatrix}
f(r) \\
g(r)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}
\]
where $\kappa$ is the eigenvalue of the operator $1 + L \cdot \sigma$.

For the Coulomb potential ($V(r) = \alpha/r$), it is well known that the bound states of Eq.(19) can be obtained by solving algebraic equations. We will indicate how the operator defining Eq.(19) can be set in a form that preserves the vector space $P(n-1,n)$, i.e. the space of couples of polynomials of respective degrees $n-1$ and $n$.

We first extract the standard prefactors from $f(r)$ and $g(r)$:

$$f(r) = p(r)r^\mu \exp(-\lambda r), \quad g(r) = q(r)r^\mu \exp(-\lambda r) \quad (20)$$

and fix them by imposing $\lambda = \sqrt{m^2 - E^2}$, $\mu = \sqrt{\kappa^2 - \lambda^2}$, so that polynomial solutions for $p(r)$, $q(r)$ are possible. The differential operator acting on $p,q$ reads now

$$D \begin{pmatrix} p(r) \\ q(r) \end{pmatrix} \equiv \begin{pmatrix} r \frac{d}{dr} - \lambda r + (\mu - \kappa) & r(m - V(r) - E) \\ r(m + V(r) + E) & r \frac{d}{dr} - \lambda r + (\mu + \kappa) \end{pmatrix} \begin{pmatrix} p(r) \\ q(r) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (21)$$

where, along with Ref.[8], we multiplied the equation by $r$. Let us now multiply the operator $D$ by $AU^{-1}$ on the left and by $UB$ on the right, with

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \sqrt{m - E} & \sqrt{m - E} \\ \sqrt{m + E} & -\sqrt{m + E} \end{pmatrix} \quad (22)$$

the matrix $U$ is nothing but the matrix which diagonalises the piece of the operator $D$ linear in $r$. The new operator, say $D'$ then reads

$$D' = -2\sqrt{m^2 - E^2} \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} - 2\alpha \sqrt{m - E \over m + E} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & r \frac{d}{dr} - n \\ 0 & 0 \end{pmatrix} \quad (23)$$

$$+ \begin{pmatrix} r \frac{d}{dr} + \mu - \kappa + \alpha \sqrt{m - E \over m + E} & 0 \\ 0 & r \frac{d}{dr} + \mu + \kappa + \alpha \sqrt{m - E \over m + E} \end{pmatrix} \quad (24)$$

where we have posed

$$n \equiv -\mu + \frac{\alpha E}{\sqrt{m^2 - E^2}} \quad (25)$$
The condition that \( n \) is an integer is equivalent to the well known quantization formula of the bound energies of the relativistic hydrogen atom. In this cases we see that the operator \( D' \) manifestly preserves the space \( P(n-1,n) \). It is known [5,6] that the operators preserving this vector space constitute a projectivised representation of the envelopping algebra of the supersymmetric algebra \( \text{osp}(2,2) \). The form (23) therefore reveals a hidden symmetry of the radial Dirac equation; it can be formulated in terms of the generators of the super-algebra \( \text{osp}(2,2) \).

Let us finally mention that we could not construct any algebraic solutions of Eq.(19) by modifying the Coulomb potential along the same lines as in the previous section.
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