Angular Momentum Distribution Function
of the Laughlin Droplet

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Abstract

We have evaluated the angular-momentum distribution functions for finite numbers of electrons in Laughlin states. For very small numbers of electrons the angular-momentum state occupation numbers have been evaluated exactly while for larger numbers of electrons they have been obtained from Monte-Carlo estimates of the one-particle density matrix. An exact relationship, valid for any number of electrons, has been derived for the ratio of the occupation numbers of the two outermost orbitals of the Laughlin droplet and is used to test the accuracy of the Monte-Carlo calculations. We compare the occupation numbers near the outer edges of the droplets with predictions based on the chiral Luttinger liquid picture of Laughlin state edges and discuss the surprisingly large oscillations in occupation numbers which occur for angular momenta far from the edge.

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In the strong magnetic field limit all electrons in a two dimensional electron gas occupy only the single-particle orbitals which have the minimum quantized kinetic energy, those of the lowest Landau level. In the symmetric gauge, appropriate for a finite droplet of electrons with circular symmetry, the lowest Landau level orbitals \[ \phi_m(z) = (2\pi m!2^m)^{-1/2} z^m \exp(-|z|^2/4), m = 0, 1, 2, ... \text{ where } z_i = x_i + iy_i. \] (We use the magnetic length, \( \ell \equiv (\hbar c/eB)^{1/2} \), as the unit of length.) The orbital with angular momentum \( m \) has its radial coordinate localized within \( \sim \ell \) of \( R_m = \sqrt{2m} \ell \) while its dependence on its angular coordinate is the same as that of an orbital with momentum \( k_m = 2\pi m/R_m \) in a one-dimensional (1D) electron gas with periodic boundary conditions. The system is therefore strongly analogous to an interacting 1D electron gas except that the orbitals have only one sign of momentum and the effective ‘length’ of the 1D gas system depends on the momentum of the orbital. For orbitals near the edge of a large droplet \( R_m \) is nearly fixed and the analogy with a one dimensional gas with only one sign of momentum (a chiral Luttinger liquid) can \([2,3]\) be made precise.

In the ground state of an \( N \)-particle non-interacting electron gas the orbitals at energies below the Fermi energy are occupied and all others are empty. For an interacting gas the single-particle state occupation numbers fluctuate. The change in average momentum state occupation numbers, the momentum distribution function, is one of the qualitative changes due to electron-electron interactions. In this Rapid Communication we report on the first study of the angular-momentum distribution function (AMDF) for strongly correlated two-dimensional electron droplets in a strong magnetic field \([4]\). The most compact \( N \)-electron droplet is a single Slater determinant in which the orbitals with \( m = 0, 1, \cdots, N - 1 \) are occupied; the area of this maximum density droplet is \( A_{N}^{MDD} \sim N2\pi\ell^2 \sim \pi R_N^2 \). When the electron system occupies a larger area the average occupation numbers will be smaller than one and the ground state of the electron system will be strongly correlated. The best understood strongly correlated states are those discovered by Laughlin \([5]\):

\[
\psi_M[z] = \prod_{i<j}^N (z_i - z_j)^M \exp\left(-\sum_{k} |z_k|^2/4\right),
\] (1)
is the ground state when the droplet occupies \( MA_{N}^{MDD} \) an area \( \sim MA_{MDD} \). The \( M=1 \) Laughlin state is the maximum density droplet. We restrict our attention here to these many-body states.

We evaluate the one-body density matrix, \( n(z, z') \) defined by

\[
    n(z, z') = N \int d^{2}z_{2} \cdots \int d^{2}z_{N} \psi_{M}(z, z_{2}, \ldots z_{N}) \times \\
    \psi_{M}^{*}(z', z_{2}, \ldots z_{N})/Q_{N} 
\]  

where \( Q_{N} \) is the normalization integral for the \( N \)-electron Laughlin wave function. The second quantized form of Eq. (2) relates \( n(z, z') \) to the AMDF:

\[
    n(z, z') = M(N-1) \sum_{m=0}^{M(N-1)} \langle n_{m} \rangle \phi_{m}^{*}(z') \phi_{m}(z),  
\]  

To obtain Eq. (3) we have noted that no orbital angular momentum larger than \( M(N-1) \) is ever occupied in the \( N \)-electron Laughlin droplet and that the Laughlin state is an eigenstate of total angular momentum. \( (M_{TOT} = MN(N-1)/2) \). Eq. (3) can be inverted to express \( \langle n_{m} \rangle \) in terms of \( n(z, z') \):

\[
    \langle n_{m} \rangle = \int d^{2}z \int d^{2}z' \phi_{m}^{*}(z)n(z, z')\phi_{m}(z').  
\]  

For sufficiently small \( N \) and \( M \) the AMDF for a Laughlin droplet can be evaluated analytically. Some results \([9]\) for very small droplets are listed in Table I.

For larger droplets we have been unable to evaluate the AMDF analytically \([10]\) but some properties of the distribution function are known. One such property follows from the expansion of \( \psi_{M}[z] \) in decreasing powers of \( z_{1} \):

\[
    \psi_{M}^{(N)}(z_{1}, \cdots, z_{N}) = \\
    [z_{1}^{M(N-1)} - M(N-1)z_{1}^{M(N-1)-1} \tilde{Z} + \cdots] \times \\
    \exp \left( -|z_{1}|^{2}/4 \right) \psi_{M}^{(N-1)}(z_{2}, \cdots, z_{N}).  
\]  

In Eq. (5), \( \psi_{M}^{(N-1)}(z_{2}, \cdots, z_{N}) \) is the \( N-1 \) electron droplet, \( \tilde{Z} \) is its center of mass coordinate and only the two highest powers of \( z_{1} \) have been retained. Using Eq. (3) in Eq. (5) we obtain
where \( m_\alpha = M(N-1) \) is the angular momentum of the outermost occupied orbital. Similarly

\[
\langle n_{m_\alpha - 1} \rangle = N \frac{Q_{N-1}}{Q_N} 2^{m_\alpha - 1} (m_\alpha - 1)! M^2 (N-1)^2 \langle \bar{Z}^2 \rangle_{N-1}.
\]

It is easy to show a Laughlin droplet has definite center-of-mass angular momentum equal to zero \( \mathcal{8} \) from which it follows that \( \langle \bar{Z}^2 \rangle_{N-1} = 2 \ell^2 / (N-1) \) and hence that for any \( N \)

\[
\langle n_{m_\alpha - 1} \rangle = M \langle n_{m_\alpha} \rangle.
\]

This result can be extended to orbitals farther from the outer edge using an argument due to Wen \( \mathcal{2} \). The density, \( n(z) \equiv n(z, z) \) far outside the droplet may be determined up to a constant from Eq. (2) and Eq. (3) by using a plasma analogy \( \mathcal{5} \). The result for \( r = |z| \gg R_{m_\alpha} \) is \( \mathcal{2} \)

\[
n(r) \propto \exp(-r^2/2)(r^2/2)^{m_\alpha}(1 - R_{m_\alpha}^2/r^2)^{-m}.
\]

Expanding the right hand side of Eq.(9) and comparing with Eq.(4) gives

\[
\langle n_{m_\alpha - k} \rangle = \left( \frac{k + M - 1}{M - 1} \right) \langle n_{m_\alpha} \rangle
\]

which reduces to Eq. (8) for \( k = 1 \). We remark that while Eq. (8) is exact for any number of particles Eq. (10) (for \( k > 1 \)) becomes exact only in the limit \( N^{1/2} \gg k \). (For example, we see that the occupation numbers listed in Table I do not satisfy Eq. (10). Eq. (10) implies that for \( N^{1/2} \gg k \gg M \), \( \langle n_{m_\alpha - k} \rangle \propto k^{M-1} \). The behavior of the AMDF in this limit is thus consistent with Luttinger liquid \( \mathcal{12} \) behavior with a critical exponent related to the quantized Hall conductance as argued on more general grounds by Wen \( \mathcal{2} \). This property of the AMDF presumably holds as long as the fractional quantum Hall effect occurs and is not unique to the Laughlin state.

To examine the AMDF in the interior of the droplet and to test how well Eq. (10) is satisfied for finite size droplets it is sufficient to evaluate \( n(z, z') \) for the case of \( |z| = |z'| = r \). Using the explicit form of the one-particle orbitals Eq. (3) simplifies for this case to
\[ n(r, r; \theta) = \frac{1}{2\pi} e^{-r^2/2} \sum_{m=0}^{M(N-1)} \langle n_m \rangle \frac{1}{m!} \frac{r^{2m}}{2} e^{-i m \theta}. \]  

(11)

where \( \theta \) is the angle between the \( z \) and \( z' \). By making use of the fact that a finite number of angular momenta have finite occupation numbers in the Laughlin droplet we obtain an exact expression for the AMDF in terms of a sum over a finite number of angles at a single radius:

\[ \langle n_m \rangle = \frac{1}{f_m(r)} \sum_{j=0}^{M(N-1)} e^{i j m} n(r, r; \theta_j), \]  

(12)

where

\[ f_m(r) = \frac{1}{2\pi m!} \frac{r^{2m}}{2} e^{-r^2/2} \]  

(13)

is proportional to \( |\phi_m(r)|^2 \) and

\[ \theta_j = \frac{2\pi j}{M(N-1)+1}. \]  

(14)

We evaluate the AMDF numerically by combining a Monte-Carlo evaluation of \( n(r, r; \theta_j) \) with Eq. (12).

It is interesting to note that the full AMDF can be determined, and therefore that the full one-particle density matrix can be reconstructed, from the angular dependence of \( n(z, z') \) at any common radius, \( |z| = |z'| = r \). This property is unique \[11\] to two-dimensional systems in the strong magnetic field limit where all electrons are restricted to a single Landau level. However, the Monte-Carlo values of \( n(r, r; \theta_j) \) will inevitably have some statistical uncertainty. From Eq. (12) we see that the resulting uncertainty in the occupation number will be a minimum when \( r \) is near the maximum in \( f_m(r) \) which occurs at \( r = \sqrt{2m} = R_m \). Typical uncertainties in \( \langle n_m \rangle \) become very large unless \( r \) is near \( R_m \). For all the numerical results reported below we estimated \( \langle n_m \rangle \) values from the the angular dependence of the density matrix at \( r \) near \( R_m \).

The Monte-Carlo calculation evaluates the complex function \( n(r, r; \theta) \) by a Metropolis \[13\] sampling of the positions of particles 2 through \( N \). The weighting factor is
\[ W(z_2, ..., z_N) = \prod_{1 < l < m} |z_l - z_m|^{2M} \exp(-\sum_{k>1} |z_k|^2/2). \] (15)

With this weighting factor

\[
n(r, r; \theta) = NQ^{N-1}Q_N^{-1} e^{-r^2/2} \prod_{j>1} \left\{ (re^{-i\theta} - rje^{-i\theta_j}) \times (r - rje^{i\theta_j}) \right\}^M. \] (16)

Thus \( n(r, r; \theta) \) is determined up to a constant which is independent of both \( \theta \) and \( r \). We determine \( Q_N^{-1}/Q_N \) by requiring that the integral of the diagonal elements of the density-matrix over position equal \( N \) (\( \int d\vec{r} n(r, r; \theta = 0) = N \)). \( n(r, r; \theta_j) \) was evaluated in this way at a set of radii separated by 0.5\( \ell \). \( \langle n_m \rangle \) was estimated by averaging results over different values of \( r \) weighted by the factor \( e^{-(r-R_m)^2/2} \).

Results for the occupation numbers obtained in this way are illustrated in Fig. (1) for \( M = 3, M = 5, \) and \( M = 7 \) Laughlin droplets with \( N = 25, N = 20 \) and \( N = 15 \) respectively. Monte Carlo calculations were also carried out for smaller size droplets where comparisons with analytic results could be made. The two main features apparent in these results are the large oscillations in the occupation numbers in the interior of the droplets and the rapid decline in the occupation numbers as the outermost orbitals are approached. For the three cases illustrated \( m_o = 72, 95 \) and 98 and the occupation numbers for the outermost droplets are \( 1.07 \times 10^{-2}, 2.3 \times 10^{-4} \) and \( 2.6 \times 10^{-5} \) respectively. In Fig. (2) we compare the angular momentum state occupation numbers near the edge of the droplet with Eq. (10), which we find to be accurately satisfied for the droplet sizes studied. We conclude that Luttinger-liquid-like behavior should be visible in the low-energy properties even for quite small Laughlin droplets. For example for the \( M = 7, N = 15 \), Laughlin droplet \( m_o = 98 \), and the Monte Carlo calculation gives \( \langle n_{93} \rangle / \langle n_{98} \rangle = 423 \pm 15 \) compared to the value 462 implied by Eq. (10). For the droplet sizes illustrated Eq. (10) begins to fail badly for \( k \) larger than \( \sim 10 \). Away from the edge the occupation numbers reach a maximum value and oscillate as the interior is approached. In the limit of extremely large droplets it is known from the plasma analogy for Laughlin wave functions that the density deep in the interior approaches
\((2\pi M)^{-1}\), from which it follows that \(\langle n_m \rangle\) approaches \(M^{-1}\). Our Monte-Carlo calculations show that this limit is approached slowly as \(N\) increases. The largest occupation numbers occur near the edge of the droplet and the relative excess in this region is larger for larger \(M\). For the droplets illustrated the largest occupation numbers are \(\langle n_{50} \rangle = 0.52 \pm 0.02\) for \(M = 3\), \(\langle n_{72} \rangle = 0.47 \pm 0.04\) for \(M = 5\) and \(\langle n_{75} \rangle = 0.43 \pm 0.03\) for \(M = 7\). The oscillations in occupation number are related to the oscillations in charge density expected near the edge of a finite 2D plasma [15] but are much more pronounced since the density averages occupation numbers of orbitals with \(R_m\) near \(r\).

As we see from the plots of \(\langle n_m \rangle\), the statistical uncertainties, obtained by averaging over independent runs, are smaller closer to the edge of the droplet. We believe that this is because fewer orbitals contribute importantly to the one-particle density matrix. In fact the occupation numbers for orbitals very close to the edge are actually more reliably calculated by evaluating the angular dependence of the density-matrix at \(r\) substantially larger than \(R_m\). In closing we remark that it is in principle possible to determine the occupation numbers uniquely if the radial dependence of the particle-density is known precisely. However, this procedure is extremely ill-conditioned and we believe that it is practical only where the particle-density is known analytically. We have found it to be impossible to determine the AMDF accurately from Monte-Carlo particle densities even for quite small droplets.

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FIGURES

FIG. 1. Occupation numbers for $M = 3$, $M = 5$, and $M = 7$, Laughlin droplets with $N = 25$, $N = 20$, and $N = 15$ respectively plotted vs. $m = R_m^2/2$. The solid line is the particle density in $(2\pi\ell^2)^{-1}$ units plotted vs. $r^2/2$ calculated from the occupation numbers using Eq. (3). The dashed line shows the particle density calculated directly; the difference in these two quantities is a measure of the error introduced in extracting the occupation numbers from the density matrix.

FIG. 2. Comparison of Monte Carlo occupation numbers near the edge with the formula derived by Wen, Eq. (10).
TABLE I. Angular momentum state occupation numbers for some representative small $N$

| $M, N$ | $m = 0$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 6$ |
|--------|---------|---------|---------|---------|---------|---------|---------|
| 3,2    | 1/4     | 3/4     | 3/4     | 1/4     | 0       | 0       | 0       |
| 3,3    | 7/31    | 9/31    | 18/31   | 22/31   | 21/31   | 12/31   | 4/31    |
| 5,2    | 1/16    | 5/16    | 10/16   | 10/16   | 5/16    | 1/16    | 0       |