Model Theory of Ultrafinitism I: 
Fuzzy Initial Segments of Arithmetic 
(Preliminary Draft)

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Abstract
This article is the first of an intended series of works on the model theory of Ultrafinitism. It is roughly divided into two parts. The first one addresses some of the issues related to ultrafinitistic programs, as well as some of the core ideas proposed thus far. The second part of the paper presents a model of ultrafinitistic arithmetics based on the notion of fuzzy initial segments of the standard natural numbers series. We also introduce a proof theory and a semantics for ultrafinitism through which feasibly consistent theories can be treated on the same footing as their classically consistent counterparts. We conclude with a brief sketch of a foundational program, that aims at reproducing the transfinite within the finite realm.

1 Preamble
To the memory of our unforgettable friend Stanley "Stan" Tenenbaum (1927 – 2005), Mathematician, Educator, Free Spirit

As we have mentioned in the abstract, this article is the first one of a series dedicated to ultrafinitistic themes.

First papers often tend to take on the dress of manifestos, road maps, or both, and this one is no exception. It is the revised version of an invited conference talk, and was meant from the start for a quite large audience of philosophers, logicians, computer scientists, and mathematicians, who might have some interest in the ultrafinite. Therefore, neither the philosophico-historical, nor the mathematical side, are meant to be detailed investigations. Instead, a number of items, proposals, questions, etc. are raised, which will be further explored in subsequent works of the series.

Our chief hope is that readers will find the overall "flavor" a bit "Tennenbaumian". And friends of Stan, old and new, know what we mean . . .
2 Introduction: the Radical Wing of Constructivism

In their encyclopedic work on Constructivism in Mathematics ([14]), A. Troelstra and D. Van Dalen dedicate only a small section to Ultrafinitism (UF in the following). This is no accident: as they themselves explain therein, there is no consistent model theory for ultrafinitistic mathematics. It is well-known that there is a plethora of models for intuitionist logic and mathematics (realizability models, Kripke models and their generalization based on category theory, etc.). Thus, a skeptical mathematician who does not feel committed to embrace the intuitionist faith (and most do not), can still understand and enjoy the intuitionist’s viewpoint while remaining all along within the confines of classical mathematics. Model theory creates, as it were, the bridge between quite different worlds.

It would be desirable that something similar were available for the more radical positions, that go under the common banner of Ultrafinitism. To be sure, in the fifteen years since the publication of the cited book, some proposals have emerged to fill the void. We still feel, though, that nothing comparable to the sturdy structure of model theory for intuitionism is available thus far. This article is the first one in a series that aims at proposing several independent but related frameworks for UF.

Before embarking on this task, though, an obvious question has to be addressed first:

What is Ultrafinitism, really?

It turns out that to provide a satisfactory answer is no trivial task. Of course, one could simply answer: all positions in foundation of mathematics that are more radical than traditional constructivism (in its various flavors). This answer would beg the question: indeed, what makes a foundational program that radical?

There is indeed at least one common denominator for ultrafinitists, namely the deep-seated mistrust for all kinds of infinite, actual and potential alike. Having said that, it would be tempting to conclude that UF is quite simply the rejection of infinity in favor of the study of finite structures (finite sets, finite categories), a program that has been partly carried out in some quarters. For instance in Finite Model Theory (see the excellent online notes written by Jouko Vannanen [16]).

Luckily (or unluckily, depending on reader’s taste), things are not that straightforward, on account of at least two substantive points:

- First, the rejection of infinitary methods, even the ones based on the
so-called potential infinite, must be applied at all levels, including metamathematics and logical rules. Both syntax and semantics must be changed to fit the ultrafinitistic paradigm. Approaches such as Finite Model Theory are simply not radical enough for the task at hand, as they are still grounded in a semantics and syntax that is deeply entrenched with the infinite.

- Second, barring one term in the dichotomy finite-infinite, is, paradoxically, an admission of guilt: the denier implicitly agrees that the dichotomy itself stands solid. But does it? Perhaps the black-white chessboard should be replaced with various tonalities of grey.

These two points must be addressed by a convincing model theory of Ultrafinitism. This means that such a model theory (assuming that anything like it can be produced), must be able to generate classical (or intuitionistic) structures, let us call them ultrafinitistic universes, wherein an hypothetical ultrafinitist mathematician can live forever happily. The dweller of those universes should be able to treat certain finite objects as de facto infinite.

Logicians are quite used to the inside versus outside pattern of thought: it suffices to think of the minimal model of ZF to get the taste (inside the minimal model countable ordinals look and feel like enormous cardinals, to quote just one blatant effect). It seems that, if we could "squeeze" the minimal model below $\aleph_0$, we could get what we are looking for.

Only one major obstacles stands in the way: the apparently absolute character of the natural number series.

But it is now time for a bit of history . . .

## 3 Short History and Prehistory of Ultrafinitism.

*The trouble with eternity is that one never knows when it will end.*

Sam Stoppard, Rosenkranz and Guildenstern Are Dead.

Ultrafinitism has, quite ironically, a very long prehistory. Indeed, it extends into and encroaches the domains of cultural anthropology and child cognitive psychology: some "primitive" cultures, and children alike, do not seem to have even a notion of arbitrarily large numbers. To them, the natural number series looks a bit like: -One, two, three, . . . , many!-. An exploration of these alluring territories would bring us too far afield, so we shall restrict our tale to the traditional beginning of western culture, the Greeks (our version here is only a sketch. For a more exhaustive and analytical investigation, see our [17]).

Ancient Greek mathematical work does not explicitly treat the ultrafinite. It is therefore all the more interesting to note that early Greek poetry, philosophy,
and historical writing incorporate two notions that are quite relevant for the study of the ultrafinite. These notions are epitomized by two words: **Murios** (μυριος) and **Apeirōn** (απειρων).

**Murios**

The term murios has two basic senses, each of which is used in specific ways. These senses are 'very many' / 'a lot of'; and 'ten thousand.' The first sense denotes an aggregate or quantity whose exact number is either not known or not relevant; the second denotes a precise number. With some exceptions, to be detailed below, the syntax and context make clear which sense is intended in each case. It is part of the aim of this paper to draw attention to the importance of contextualized usage in understanding the ultrafinite.

The earliest occurrences of the term *murios* appear in the oldest extant Greek writing, viz., Homer's Iliad and Odyssey. In Homer, all 32 instances of forms of *murios* have the sense 'very many' / 'a lot of.' Translations often render the word as 'numberless,' 'countless,' or 'without measure.' What exactly does this mean: does *murios* refer to an indefinite number or quantity, to an infinite number or quantity, to a number or quantity that is finite and well-defined but that is not feasibly countable for some reason, or to a number or quantity that the speaker deems large but unnecessary to count? Our investigation reveals that Homer tends to use the term in the last two ways, that is, to refer to numbers or quantities for which a count or measure would be unfeasible, unnecessary, or not to the point. In general, Homer uses the word in situations where it is not important to know the exact number of things in a large group, or the exact quantity of some large mass.

Some representative examples of Homer's usages of *murios*: (a) At Iliad 2.468, the Achaeans who take up a position on the banks of the Scamander are murioi (plural adjective), "such as grow the leaves and flowers in season." The leaves and flowers are certainly not infinite in number, nor are they indefinite in number, but they are not practicably countable, and there is no reason to do so - it is enough to know that there are very many all over. (b) In the previous example, the Achaeans must have numbered at least in the thousands. *Murios* can, however, be used to refer to much smaller groups. At Iliad 4.434, the clamoring noise made by the Trojans is compared to the noise made by muriai ewes who are being milked in the courtyard of a very wealthy man (the ewes are bleating for their lambs). The number of ewes owned by a man of much property would certainly be many more than the number owned by someone of more moderate means, but that rich man's ewes - especially if they all fit in a courtyard - must number at most in the low hundreds. This suggests that the ewes are said to be muriai in number because there are a comparatively large number of them; because there is no need to count them (a man who had 120 ewes and was considered very wealthy would not cease to be considered very wealthy if he lost one or even ten of them; the Sorites paradox is later); and possibly because
it might not be practicable to count them (they might be moving around, and they all look rather like one another).

Similarly, at Odyssey 17.422 Odysseus says he had murioi slaves at his home in Ithaca before he left for the Trojan war. The word for 'slaves' in this case is *dmôes*, indicating that these are prisoners of war. Given what we know of archaic Greek social and economic structures, the number of slaves of this type that a man in his position could have held must have been in the dozens at most. The key to Odysseus' use of the term is the context. The sentence as a whole reads, "And I had murioi slaves indeed, and the many other things through which one lives well and is called wealthy." That is, the quantities of slaves and of other resources that he commanded were large enough to enable him to be considered wealthy. The exact number of slaves might have been countable, but it would have been beside the point to count them. (c) *murios* can also refer to quantities that are not such as to be counted. At Odyssey 15.452, a kidnapped son of a king is projected to fetch a *murios* price as a slave. Here *murios* must mean 'very large,' 'vast.' This is by no means to say that the price will be infinite or indefinite, for a price could not be thus. Rather, the situation is that the exact price cannot yet be estimated, and the characters have no need to estimate it (i.e., they are not trying to raise a specific amount of money). Prices are not countable, but they are of course measurable or calculable. There are also instances of *murios* in Homer that refer to kinds of things that do not seem to be measurable or calculable. At Iliad 18.88, Achilles says that his mother Thetis will suffer *murios* grief (penthos) at the death of Achilles, which is imminent. At 20.282, *murios* distress comes over the eyes of Aeneas as he battles Achilles.

The usual translation of *murios* here is 'measureless'. This translation may be somewhat misleading if it is take literally, as there is no evidence that the Greeks thought that smaller amounts of grief and distress were necessarily such as to be measurable or measured. A more appropriate translation might be 'vast' or 'overwhelming.' It is possible that Achilles means that Thetis will suffer grief so vast that she will never exhaust it nor plumb its depths even though she is immortal; but it is also possible that Homer did not consider whether grief or distress could be unending and infinite or indefinite in scope.

The epic poet Hesiod (8th-7th BCE) and the historian Herodotus (5th BCE) sometimes use *murios* in the senses in which Homer does, but they also use it to mean ten thousand. With a very few exceptions, the syntax and context make clear in each instance which meaning is present. At Works and Days 252, Hesiod says that Zeus has tris murioi immortals (i.e. divinities of various kinds) who keep watch over mortals, marking the crooked and unjust humans for punishment. *Tris* means three times or thrice, and there is no parallel in Greek for understanding *tris murioi* as "three times many" or "three times a lot." There are parallels for understanding *tris* with an expression of quantity as three times a specific number; and the specific number associated with *murios*
is ten thousand. Therefore tris murios should indicate thirty thousand. Some instances of murios in Herodotus clearly refer to quantities of ten thousand; some clearly refer to large amounts whose exact quantities are unspecified; and a few are ambiguous but do not suggest any meaning other than these two.

(d) At 1.192.3, Herodotus says that the satrap Tritantaechmes had so much income from his subjects that he was able to maintain not only warhorses but eight hundred (oktakosioi) other breeding stallions and hexakischilai kai muriai mares. Hexakischilai means six times one thousand, so that the whole expression should read six thousand plus muriai. The next line tells us that there are twenty (eikosi) mares for every stallion, so that the total number of mares must be sixteen thousand, and muriai must mean ten thousand (it is a plural adjective to agree with the noun). The case is similar at 2.142.2-3. Here Herodotus says that three hundred (triêkosiai) generations of men come to muriai years since three generations come to one hundred (hekaton) years. Clearly, muriai means ten thousand here.

(e) In some places, Herodotus cannot be using murios to mean ten thousand, and it is the context that shows this. At 2.37.3, for example, describing the activities of Egyptian priests, he says that they fulfill muriai religious rituals, hōs eipein logoi. He may in fact mean that they fulfill muriai rituals each day, since the rest of the sentence speaks of their daily bathing routines. Herodotus does not give any details about the rituals or their number, and hōs eipein logoi means "so to speak." Thus Herodotus seems to be signalling that he is not giving an exact figure, and muriai must simply mean "a great many." At 2.148.6, Herodotus reports that the upper chambers of the Egyptian Labyrinth thōma murion pareichonto, furnished much wonder, so remarkably were they built and decorated. Certainly no particular amount of wonder is being specified here.

(f) Some occurrences of murios are ambiguous in a way that is of interest for the study of the ultrafinite. At 1.126.5, Cyrus sets the Persians the enormous task of clearing an area of eighteen or twenty stadia (2 1/4 or 2 1/2 miles) on each side in one day, and orders a feast for them the next. He tells them that if they obey him, they will have feasts and muria other good things without toil or slavery, but that if they do not obey him, they will have anarithmetoi toils like that of the previous day. That is, Cyrus is contrasting muria good things with anarithmetoi bad ones. Is he asking the Persians to consider this a choice between comparable large quantities? If so, a murios amount would be anarithmîtos, which can mean either "unnumbered" or "innumerable, numberless." It is also possible that murios is supposed to mean ten thousand, so that the magnitude of the undesirable consequences of defying Cyrus is greater than the great magnitude of the advantages of obeying him. If that is the meaning, Herodotus may be using murios in a somewhat figurative sense, as when one says today that one has "ten thousand things to do today." murios, then, referred in the earliest recorded Greek thought to large numbers or amounts. When it did not refer to an exact figure of ten thousand, it referred to numbers.
or amounts for which the speaker did not have an exact count or measurement. Our analysis indicates that the speaker might lack such a count or measurement either because the mass or aggregate in question could not practicably be counted or measured under the circumstances, or because an exact count or measurement would not add anything to the point the speaker was making. In most cases it is clear that the numbers and amounts referred to as *murios* were determinate and finite, and could with appropriate technology be counted or measured. In instances where it is not clear whether that which is referred to as *murios* is supposed to be such as to admit of measuring or counting (Thetis’ grief, for example; and Cyrus’ *murios* good things if they are comparable to the *anarithmētos*), there is no evidence as to whether the *murios* thing or things are supposed to be infinite or indefinite in scope. Indeed, there is no evidence that these early writers thought about this point. (This is perhaps why *anarithmētos* can mean both "unnumbered" and "innumerable," and why it is often difficult to tell which might be meant and whether a writer has in mind any distinction between them.)

When *murios* does not mean "ten thousand," context determines the order of quantity to which it refers. Any number or amount that is considered to be "a lot" or "many" with respect to the circumstances in which it is found can be called *murios*. Leaves and flowers in summer near the Scamander number many more than those of other seasons, perhaps in the millions; but the rich man’s ewes are *muriai* too, even if they number perhaps a hundred. They are several times more than the average farmer has, and they may fill the courtyard so much that they cannot easily be counted.

**Apeiron**

Since *murios* seems to refer overwhelmingly to determinate and finite quantities, it is useful to note that Greek had ways of referring to quantities that were indeterminate, unlimited, indefinite, or infinite. The most significant of these, for philosophical and mathematical purposes, was the word *apeiros* or *apeirōn* (m., f.)/ *apeiron* (n.).

The etymology of this word is generally understood to be *peirar* or *peras*, "limit" or "boundary," plus alpha-privative, signifying negation: literally, "not limited" or "lacking boundary." Etymology alone does not tell us the range of uses of the term or the ways in which it was understood, so we must again consider its occurrences in the earliest sources. The term appears as early as Homer, in whose poems it generally refers to things that are vast in extent, depth, or intensity.

Homer uses *apeiros* most frequently of expanses of land or sea. In each case, the *apeiros*/apeirōn thing is vast in breadth or depth; whether its limits are determinable is not clear from the context, but limits do seem to be implied in these cases. Some instances may imply a surpassing of some sort of boundaries
or borders (though not necessarily of all boundaries or borders). At Iliad 24.342 and Odyssey 1.98 and 5.46, a god swiftly crosses the apeiron earth. Within the context, it is clear that the poet means that the divinity covers a vast distance quickly. There may be a further implication that the gods transcend or traverse boundaries (be these natural features or human institutions) with ease, so that the world has no internal borders for them. Similarly, in Od. 17.418, the expression *kat’ apeirona gaion,* often translated as "through[out] the boundless earth," is used to suggest that something is spread over the whole earth. What is spread covers a vast expanse, and it also crosses all boundaries on the earth. Two other Homeric examples are of interest. At Od. 7.286, a sleep is described as apeiron, meaning either that it is very deep, or unbroken, or both. At Od. 8.340, strong bonds are apeiron, surpassing limits of a god’s strength, and so unbreakable.

Hesiod also uses apeiron to describe things that extend all over the earth, but also uses the word once in reference to a number. In Shield of Heracles 472, the word refers to a large number of people from a great city involved in the funeral of a leader; the sense seems to be that there were uncountably many, and possibly that the leader’s dominion had been vast.

Herodotus (5th century BCE) uses apeiron in two cases where its meaning clearly derives from the privative of peirar. In 5.9 he uses it to refer to a wilderness beyond Thracian settlements. In 1.204 a plain is apeiron, perhaps hugely or indeterminately vast. In both cases, Herodotus knows that the lands are definite in extent (he identifies the peoples who live beyond them). The contexts suggest that he means that these lands are vast and that their exact boundaries are not known. He may also have in mind that they cannot be easily, if at all, traversed by humans.

The first and perhaps best-known philosophical use of apeiron is in the reports about the work of Anaximander in the sixth century BCE. Anaximander is reported to have held that the source of all familiar things, the fundamental generative stuff of the cosmos, was something apeiron. The testimonia report that the apeiron was eternal in duration, unlimited or indeterminate in extent, and qualitatively indeterminate. It was neither fire, nor water, nor air, nor

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1Herodotus also uses a homophone word that is derived from another root, so we have only included instances where context clearly indicates that the word is the one derived from peirar.

2Kahn (24) argues that "[t]he true sense of apeiros is therefore 'what cannot be passed over or traversed from end to end.'" He admits, however, that Homer’s gods traverse the earth and sea that Homer calls apeiron. For the term to have the sense of "untraversable," it would have to be understood as an exaggeration in Homer and Herodotus. Gods and rivers, for example, do traverse the things described as apeiron, but it is extremely difficult or impractical for humans to do so. An alternative to this interpretation would be to say that the term does not denote untraversibility, though it may connote that in some cases, and that it instead denotes vastness and often practical indeterminability.

3Anaximander’s fundamental entity is generally referred to as 'the apeiron,' since he did not give it another name and did not say what it was in any other way. Adjectives, especially
earth, nor hot, cold, light, dark, etc.; but it was that which could give rise to all of these. All of the familiar cosmos, for Anaximander, arose from the apeiron.

We may note that so far no instance of apeiron clearly means 'infinite.' Only one, Anaximander’s, could possibly involve an infinite extent, and even in that case it is not clear that the extent is infinite; it may be indefinite or inexhaustible without being infinite. Anaximander’s stuff is eternal, i.e. always in existence, but it is not at all clear that a sixth century Greek would have taken "always" to mean an infinite amount of time. Whether any Greek of the 8th to 5th centuries BCE conceived of quantities or magnitudes in a way that denoted what we would call infinity is not certain.

It is sometimes thought that Zeno of Elea (5th century) spoke of the infinite, but there is good evidence that he had quite a different focus. It is only in the arguments concerning plurality that are preserved by Simplicius that we find what may be quotations from Zeno’s work (regarding his arguments concerning motion and place we have only reports and paraphrases or interpretations). In fragments DK29 B1 and B2, Zeno argued "from saying that multiple (polla, many) things are, saying opposite things follows.” In particular, if we say that multiple things are, then we must conclude that "the same things must be so large as to be apeira (neuter plural) and so small as to lack magnitude (megethos).” Zeno was evidently interested in the claim that there are multiple things with spatial magnitude, and it appears from the fragments that he thought that the possibilities for analyzing the components of spatial magnitude were the hypothesis that a thing that has spatial magnitude must be composed of parts with positive spatial magnitude, parts of no magnitude, or some combination of these. If a thing had no magnitude, Zeno argued, it would not increase (in magnitude) anything to which it was added, nor decrease anything from which it was removed. Therefore it could not "be" at all (at least, it could not "be" as the spatial thing it was said to be). Nothing with magnitude could be composed entirely of such things. However, if we assume that the components of a spatial thing have positive magnitude, another problem arises. In measuring such a thing, we would try to ascertain the end of its "projecting part" (i.e. the outermost part of the thing). Each such projecting part would always have its own projecting part, so that the thing would have no ultimate "extreme" (eschaton). That is, the outer edge of something always has some thickness, as do the lines on any ruler we might use to measure it; and this thickness itself can always be divided. Thus the magnitude of a spatial thing,
and thus its exact limits, will not be determinable. There is nothing in this to suggest that Zeno thought that the claim that there are multiple spatial things led to the conclusion that such things must be infinitely large. Rather, his description suggests that the things would be indeterminable, and indeterminate or indefinite, in size.

In the philosophy of the fourth century BCE, and arguably as early as Zeno, an apeiron quantity could not be calculated exactly, at least as long as it was regarded from the perspective according to which it was apeiron. In fact, Aristotle's argument that a continuous magnitude bounded at both ends could be traversed in a finite amount of time - despite the fact that it "contains" an apeiron number of points, and despite Zeno's Dichotomy argument - rests precisely on the notions that the magnitude is not composed of the apeiron number of points, and that from one perspective it is bounded. Aristotle does not refute Zeno's argument, but merely argues that within the framework of his physics, the question Zeno addresses can be put into different terms. Thus where murios did not clearly refer to ten thousand, a murios quantity was generally recognized as definite but was not calculated exactly. An apeiron thing or quantity in Homer or Herodotus might be definite or not, and in later thinkers, especially in philosophy, the term came to emphasize that aspect of the thing or group or quantity that was indefinite, indeterminate, or unlimited.

Intermezzo

We shall see momentarily that subsequent reflections on the ultrafinite orbit, for the most part, around the myrios-apeiron pair, as if around a double star. Meanwhile, to keep the length of this paper a feasible one, we must skip over two thousand years of mathematical and philosophical thought (where ultrafinitistic themes crop up from time to time in the philosophical and mathematical debate), and pick up the thread once again well into the twentieth century.

Recent History of UF

The passage from prehistory to the history of UF is difficult to pin down. Perhaps a bit arbitrarily, we shall say that it starts with the radical criticism of Brouwer's Intuitionist Programme by Van Dantzig \cite{8} in 1950. \footnote{Van Dantzig himself points out that some of his ideas were anticipated by the Dutch philosopher Mannoury, and by the French mathematician Emil Borel \cite{32}. Borel observed that large finite numbers \textit{(les nombres inaccessibles)} present the same order of difficulties as the infinite.} This seminal small paper contains \textit{in nuce} most of the later motives. Chief among them that (quoting him verbatim):

\begin{quote}
- the difference between finite and infinite numbers is not an essential, but a
\end{quote}

\footnote{For an account of what Zeno showed, and what he may have been trying to show, in these arguments, see \cite{17}, pages 5-7.}
According to this view, an infinite number is a number that surpasses everything I can ever obtain. One is here reminded of a game inadvertently initiated by the Greek mathematician Archimedes in his Sand Reckoner (Ψαµµιτες. A good translation is [33]), a game that is still played nowadays: two fellows, A and B, trying to beat each other at naming huge numbers. The winner is the contestant who is able to construct a faster growing primitive recursive function. The winner is the (temporary) owner of ”infinite numbers” (see on this [1]).

A second major character in this story is the Russian logician Yessenin-Volpin. In a series of papers (see for instance his 1970 manifesto [13]) he exposes his views on UF. Unfortunately, in spite of their appeal, his views are difficult to articulate: there is simply too much there (though, to be sure, David Isles has made a serious and quite successful attempt to clarify some of Volpin’s tenets in [11]). One of the fundamental ideas put forth by Volpin is that there is no uniquely defined natural number series. Volpin’s ruthless attach unmasks the circularity behind the induction scheme, and leaves us with various not isomorphic finite natural number series.

Interestingly, a few years later, Princeton mathematician Ed Nelson had an epiphany, described in his ”confession” [5]: in a morning of the Fall 1976, in Canada, Nelson lost his ”pythagorean faith” in the natural numbers. What was left was nothing more than finite arithmetic terms, and the rules to manipulate them. Nelson’s Predicative Arithmetics ([6]), albeit an essential steps toward the re-thinking of mathematics along strictly finitist lines, seems to us not as radical as Nelson’s amazing vision: why stopping at induction over bounded formulae? If numbers are no more, and arithmetics is a concrete manipulation of symbols (a position that could be aptly called ultra-formalist), ”models” of arithmetics are conceivable, where even the successor operation is not total, and all induction is either restricted or banished altogether (for an analysis of Nelson’s arithmetics and for our proposal of an arithmetics for ultra-formalists see [20]).

Enter Parikh [7]. His celebrated ”Existence and Feasibility in Arithmetics” (1971) introduces an expanded version of Peano Arithmetic, enriched with a unary predicate F (where F(x) intended meaning is that x is a feasible number, in some unspecified sense). Mathematical induction does not apply to F, and a new axiom is added to PA saying that a very large number is not feasible. More precisely, the axiom says that the number $2^{1000}$, where $2^{0} = 1$ and $2^{k+1} = 2^{2^k}$, is not feasible. Parikh proves, among other important results, that the theory $PA + \neg F(2^{1000})$ is feasibly consistent: though obviously inconsistent from the classical standpoint, all proofs exhibiting its inconsistency are unfeasible, in the sense that the length of any such proof is a number $n \geq 2^{1000}$.

From the point of view of this history, Parikh achieves at least two goals: first, he
turns some of the claims of ultrafinitists into concrete verifiable theorems. Secondly, he paves the way to a new kind of ultrafinitistic proof theory. Parikh's approach has been improved by several authors ([?], [?]). Quite recently, Vladimir Sazonov [10], has made a serious attempt at making more explicit the structure of Ultrafinitistic Proof Theory. Also, Alessandra Carbone and Stephen Semmes, have investigated the consistency of $PA + \neg F(2^{1000})$ and similar theories from a novel proof theoretical standpoint, involving the combinatorial complexity of proofs ([30]). We shall touch upon these ideas in Section 5.

Parikh's seminal paper leaves us with a desire for more: knowing that $PA + \neg F(2^{1000})$ is feasibly consistent, there ought to be some way of saying that it has a model. In other words, the suspicion rises that, were a genuine semantics of ultrafinitistic theory available to us, the celebrated G"odel's completeness theorem (or a finitist version thereof) should hold true. But where to look for such a semantics? Models are structured sets (or objects in some category with structure). We must thus turn our sight from proof theory to set theory and category theory.

On the set-theoretical side, there are at least two major contributions. The first one is Vopenka's programme to reform Cantor's Set Theory, also known as Alternative Set Theory (AST). AST has been developed for more than three decades, so even a scanty exposition of his results is not feasible here. We can just recall the main themes: AST is a phenomenological theory of finite sets. Some sets can have subclasses that are not themselves sets. Sets of that kind are "infinite" in Vopenka's sense. Here we go back to one of the senses of the word apeiron, previously described: some sets are (or appear) infinite because they live outside our perceptual horizon. It should be pointed out that AST is not, per se, a UF framework. However, Vopenka envisioned the possibility of "witnessed universes", i.e. universes where infinite (in his sense) semisets contained in finite sets do exist. These witnessed universes would turn AST into a "universe of discourse" for ultrafinitism. To our knowledge, though, witnessed AST has not been developed beyond its initial stage.

There are other variants of set theory with some finitist flavor. For instance, Andreev and Gordon [2] describe a theory oh Hyper-finite Sets (THS), which is not incompatible with classical set theory. An interesting fact is that both AST and THS produce as a by-product a quite natural model of non-standard analysis, thereby providing at least a good reason for interest in finitistic math

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In the cited paper the absolute character of being a feasible number is asserted, on some physical ground (Sazonov's actual position on this issue is more articulate, as it appears from his recent FOM postings). We do not share this belief. As pointed out elsewhere in this paper, we think it is important to maintain the notion of contextual feasibility (after all, who knows for sure what is out there? Perhaps new discoveries in Plank level physics will show that the estimated upper bound of "particles" in the universe was way too small. Whereas logic should be able to account for physical limitations, it should not be enslaved by them.)

The very large and the very small are indeed intimately related: if one has a consistent notion of large, unfeasible number $n$, one automatically gets the "infinitesimal" $\frac{1}{n^2}$ via the
for mainstream mathematicians.

The second set-theoretical approach we are aware of, is the one described in the book of Shaughan Lavine [15]. Here, a finitistic variant of ZF is introduced, where the existence of a large number, the Zillion, is posited (again, the reader may recall, a old motive: the coming back of myrion. Zillion replaces in some way the missing $\aleph_0$)

Finally, category theory. This is, with one notable exception, a totally uncharted (but very promising) territory. The exception is a couple of works in the late seventies and early eighties by the late Jon M. Beck, on using simplicial and homotopic methods to model finite, concrete analysis [23]. As far as we understand it, Beck’s core idea is to use the simplicial category $\triangle$, truncated at a certain level $\triangle[n]$, to replace the role of the natural number series (or, because we are here in a categorical framework, the so-called natural number object that several topos possess). The truncated simplicial category has enough structure to carry out some finitistic version of recursion; moreover, its homotopy theory provides new tools to model finite flow diagrams. Beck’s approach will be discussed in a later work [20], together with other possible paths using category theory and the topos approach to realizability in particular. 11

Before moving on to our first sketch of a proposal, let us try to sum up some lessons we believe can be learned from the foregoing:

- First, the notion of feasibility should be contextual. An object (a term, a number, a set) is feasible only within a specific context, that specifies the type of resources available (functions, memory, time, etc.). Thus a full-blown model theory of UF should provide the framework for a dynamic notion of feasibility.
- As the contest changes, so does feasibility. What was unfeasible before, may become feasible now. Perhaps our notion of potential infinity came as the realization (or faith) that any contest can be transcended.
- The transcendency degrees of feasibility are not necessarily linearly ordered. One can envision contests in which what is feasible for A is not feasible for B, and viceversa.
- Last, but not least, the pair Murios-Apeiron. Every convincing approach to UF should be broad enough to encompass both terms. Even better, it should unify the two streams into a single, flexible framework.

10 As it has been pointed out in a recent posting by Michael Barr on the category list in appreciation of Beck’s mathematics, addition for the finite calculator is not an associative operation. Homotopy "repairs" the lack of associativity by providing "associativity up to homotopy" via coherence rules.

11 As it is well known, topos have an internal logic, which is intuitionistic. One can hope that, by isolating "feasible" objects in the realizability topos via a suitable definition of feasible realizability, a categorical universe of discourse for UF could be, as it were, carved out.
4 Fuzzy Initial Segments of NN: a model of ultrafinitistic arithmetics

In our first model we shall adopt a moderate position, namely we still assume that "numbers" exist, at least in the background (a more radical step, where the term model of arithmetic is de-constructed into its partial fragments, and where numbers are replaced by finite similarity classes of terms, will be investigated in [21]).

For the time being, let us imagine a concrete entity (an individual, a finite physical machine, . . .), named Amanda, trying to embrace the natural number series. We shall imagine that we have some way to gauge Amanda’s level of confidence that a particular number is within her reach. This measure will be coded by a certain function $G$ from the natural numbers into the $[0, 1]$ interval. For a particular number $n$, $G(n) = 1$ shall indicate that $n$ is definitely within reach, or strongly feasible. On the opposite side of the spectrum, $G(n) = 0$ means that $n$ is unfeasible, or totally beyond her grasp. All other values in between will be considered feasible, in some weak sense.

We shall make the assumption that the graspable number series (i.e. the subclass of natural numbers that are presently not entirely outside Amanda’s reach), is closed downward and weakly closed under some finite collection of elementary primitive recursive function $F = \{f_0, f_1, \ldots, f_n\}$. By weakly closed we mean the following: consider the algebra of closed terms that can be obtained from $F$ from 0. If $t_1, \ldots, t_n$ are terms in the algebra and $G(t_1) = \ldots = G(t_n) = 1$ and $f \in F$ is an $n$-ary function, then $G(f(t_1, \ldots, t_n)) > 0$. The motivation behind the foregoing is that it ought not to be possible to discriminate between the strongly feasible and the unfeasible using any of the functions in $F$.[12]

Let us start small: Amanda knows only $S$, the successor function (or, to be more accurate, she knows only how to iterate $S$ a number of time, depending on her actual resources. After a while, she will probably fall asleep, or, in case she is a machine, run out of RAM). So, the family $F$ is $F = \{f_0 = S\}$. The arithmetic world of Amanda is encapsulated by the following definition:

**Definition 1** A Fuzzy Initial Segment (FIS for short) of the Natural Number Series is a function $G : N \rightarrow [0, 1]$ such that

1) $G(0) = 1$ (0 is a feasible number!)

2) $\forall n \; G(n + 1) \leq G(n)$ (G is monotonically decreasing)

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[12] A guiding mental image is the following: think of a “castle” inhabited by the strongly feasible numbers. A “moat” of weakly feasible numbers separates the castle from the vast world of the unfeasible beyond. The postulate above basically says that by shooting “darts” from within the castle one never hits the unfeasible: they all fall into the muddy moat in between.
3) \( \forall n \; G(n) = 1 \Rightarrow G(n + 1) > 0 \) (no jumps allowed)

A FIS is strict iff:

4) \( \exists k \; G(k) = 0 \) (not every number is feasible)

People even scantily aware of fuzzy logic will have recognized a FIS as a special fuzzy subset of \( \mathbb{N} \). The intuition behind the foregoing definition is that a FIS is downward closed as well as weakly closed with respect to basic counting 0, S0, SS0, SSS0, . . .

The definition we just gave is a bit too general, at least for most purposes: it includes some pathological fuzzy initial segments where the rate of decrease in feasibility may temporarily slow down. \(^{13}\) We can thus restrict ourselves to more well behaved FIS:

**Definition 2** A regular FIS (RFIS) is a FIS such that the function \( R(n) = G(n) - G(n + 1) \) is not decreasing, in the feasible segment \( \{ n | G(n) > 0 \} \).

A brief note on the last definition: the function \( R(n) \) is an important gauge for a FIS: the one-step feasibility rate change. Our postulate 3) in Definition 1 rules out \( R(n) = 1 \), but it is clear that a real-life FIS should have additional requirements on its rate of feasibility loss. This topic will be elaborated elsewhere, as it eventually leads to a general classification of fuzzy initial segments of arithmetics. For the time being, though, we shall stick to the above definition.

An elementary (but nevertheless interesting) example of a RFIS is given by the the family \( \{ G_N(n) \} \):

(♣) \[ G_N(n) = \max(1 - \frac{n}{N}, 0) \]

This family represent a set of fuzzy initial segments of arithmetic where the degree of confidence decreases at the same pace \( \frac{1}{N} \), till the bottom is reached. If \( N \uparrow \infty \), then the FIS spans the entire natural numbers series; in other words, all finite natural numbers are feasible.\(^{14}\) Observe that, according to our definition, only 0 is strongly feasible in (♣). This is admittedly odd, but it can be easily fixed by starting the decay at some positive number \( n_0 \).

\(^{13}\) We are not ruling out a priori the possibility of non-monotonic feasibility. Quite to the contrary: it is perfectly conceivable that within certain contexts the rate at which feasibility goes down as numbers get bigger will decrease, or even that larger numbers may be considered more feasible (perhaps their description has low complexity whereas some smaller number hasn’t). There is here an intriguing connection between Kolmogorov complexity and feasibility, yet to be explored. Here, we are simply following the old common sense precept: easier things first.

\(^{14}\) The implicit suggestion here is that classical/intuitionistic reasoning could be recovered by "passing to the limit": from the ultrafinitistic standpoint this passage to the limit is obtained by postulating an unbounded set of computational resources.
Everybody knows that counting by "adding one" is extremely inefficient: that is one of the underlying reasons for creating more manageable notations. As in the mentioned "name the bigger number game", the key is in devising fast growing primitive recursive functions. Once such a function definition is achieved, it is possible to reach out farther than before. Thus, a new "fuzzy initial segment" is obtained. Going back to Amanda, let us assume that a family \( F = \{f_0, \ldots, f_n\} \) of function is available to her. Let us indicate with \( \mathfrak{T}(F) \) the set of closed terms generated by \( F \). The arithmetic world of Amanda is now a fuzzy initial segment of arithmetics weakly closed under \( F \):

**Definition 3** A Fuzzy Initial Segment closed under a family of functions \( F \) is a function \( G : \mathbb{N} \rightarrow [0, 1] \) such that:

1) \( G(0) = 1 \)

2) \( G \) is not increasing

3) if \( t_1, \ldots, t_k \) are in \( \mathfrak{T}(F) \) and \( f \in A \) is an \( n \)-ary function \( \forall i \ G(t_i) = 1 \Rightarrow f(t_1, \ldots, t_k) > 0 \)

4) \( \exists t \in \mathfrak{T}(F) \) such that \( G(t) = 0 \).

Given two FIS, \( G \) and \( G' \), \( G' \) dominates \( G \) if \( \forall k \ G(k) < G'(k) \).

The number \( n \) such that \( \exists t \in \mathfrak{T}(F) \) \( (n = \text{length}(t) \land G(t) > 0) \) and \( \forall t' \in \mathfrak{T}(F) \setminus \mathfrak{T}(F) \setminus (G(t) > G(t') > 0 \land \text{length}(t') < \text{length}(t)) \), is called the feasibility radius of the FIS.

The feasibility radius is a gauge of how far out one can go using the given a notation system.

Notice that a FIS in the sense of the previous definition is just a particular case of Definition 2, where \( F = \{S\} \). The feasibility radius is just the maximal \( n = SS \ldots S0 \) such that \( G(n) > 0 \).

As we have already said, any FIS models a concrete arithmetic world for a particular entity at a given time. Worlds change, and so do arithmetic worlds. Amanda can expand hers in at least three ways:

1. either by extending her feasibility radius (perhaps through memory enhancers, or yoga . . . ), or

2. by being more creative and elaborating new ways of going farther with the same resources.

3. Finally, she can move from her current \( G \) to a \( G' \) that has exactly the same radius but such that \( G' \) dominates \( G \) (in other words, her grasp on the feasible segment has gotten sharper).
Examples of 1 are easy to imagine, and we leave them to the reader. As for 2, let us go back to $G_N(n)$: it is clear that the feasibility radius of $G_N(n)$ is $N - 1$. Now, suppose Amanda realizes that the unary successor notation is quite clumsy (indeed, sticking to it, most of us would find the number 10000 unfeasible!). Let us say that she discovers the binary digit notation. Now, define a new FIS as

$$G'(n) = (\lfloor \log_2(n) \rfloor + 1)$$

**Proposition 1** Let $G$ be weakly closed under $f(m, n) = m + n + 2$. Then $G'$ above is closed under multiplication. Also, $G'$ dominates $G$.

The trivial proof uses the elementary properties of the logarithm: $G'(m \ast n) = G(\lfloor \log_2(mn) \rfloor) \geq G(\lfloor \log_2(m) \rfloor + \lfloor \log_2(n) \rfloor + 2)$. Now, use 3) in Definition 3.

In the last example, Amanda’s grasp has increased exponentially, even though her memory/storage resources are the same (this trivial example shows the power hidden in notation systems: through them we can “handle” otherwise un-graspable numbers).

As for 3, compare $G_{2N}(n)$ with $G^S_{2N}(n) = 1 - \frac{2n}{2N}$: their radius with respect to successor is the same (huge, even for small $N$!), but the shape of the FIS has changed quite a bit. In $G^S_{2N}(n)$, things start out slowly (making the first numbers almost strongly feasible), and precipitates later on. Although the two FIS have the same length, $G^S_{2N}$ clearly dominates $G_{2N}$.

Before moving on, we wish to mention *en passant* a classification of feasible numbers into “size” (small, medium and large), due to Kolmogorov (and taken up by Sazonov). Assume $G$ is a FIS weakly closed under multiplication and such that $G(2) = 1$. Define another FIS by $G^s(n) = G(2^n)$ (the superscript stands for small)

**Proposition 2** The FIS $G^s(n)$ is closed under addition.

The fuzzy set $S = \{n \mid G^s(n) > 0\}$ is the collection of $G$-small numbers. Further classifications can be obtained by refining this method.

Though the arithmetic models we have presented are by no means the most general, they are still broad enough to encompass not only the “apeiron” view of UF, but also the ”murios”. Indeed, a model of arithmetics with a largest number, such as the one proposed by Mycielski [29](see for instance the interesting article by Van Bendegem [28]), can be obtained by ”moding out” with respect to the following equivalence relation:

$$n \sim m \Leftrightarrow (G(n) > 0 \land G(m) > 0 \land n = m) \lor (G(n) = 0 \land G(m) = 0)$$

In plain words, for Amanda every number beyond the horizon is the same number, aka infinity. Actually, there is yet another equivalence relation producing
a finite arithmetics model, namely the one that identifies all numbers that are not strictly feasible. Indeed, any "defuzzification operator" amongst the ones investigated by Fuzzy Logic would do. By defuzzifying a FIS one ends up with a crisp finite arithmetical universe, which describes a concrete arithmetical world where computational resources are sharply known.

To summarize what we have seen, we can say that the basic arithmetical world may be enriched in a number of ways:

- by considering a relativized form of feasibility: as the reader has certainly observed, the numbers that are feasible are the ones that can be reached from 0 in a small number of steps by "jumping forward" with the aid of suitably devised terms.

- by studying definable or even computable/effective Fuzzy Initial Segments, i.e. by putting some restriction on the function $G$ (the reader has surely noticed that so far no such restriction has been made).

- by considering a family of FIS, parameterized by a time parameter, representing different feasibility contexts in the life of Amanda

- by introducing several "actors" (say, Amanda’s boyfriend), each equipped with his/her own FIS. This scenario appears to have some similarities with multi-agents epistemic logics, but we are unable to establish precise connections.

We shall not pursue these directions here, although we feel that each one deserves further analysis. Instead, we will show that these simple models are "enough" to provide an adequate semantics for feasibly consistent arithmetics theories. But first a preliminary analysis of proof theory from the perspective of UF is in order.

5 Dissipative Proof Theory

In this section we briefly sketch a "revision" of standard proof theory, which better accommodates the needs of UF. Ordinary proofs can be unfeasible, in the eyes of an ultrafinitistic mathematician: their length, or other complexity measures he/she may want to consider, could potentially turn them into proofs that would be qualified as "ideal", and therefore unacceptable. It is then clear that we need a way to gauge the degree of convincingness of a proof. At the same time, this should be done in the broadest generality, so as to encompass different viewpoints.

Where shall we start? A proof is a tree built out of basic logical rules. Each rule enables us to transfer the credibility of the premisses into the consequences: if I believe that $A$ is the case and I also believe $B$ is also the case, then, via
∧-introduction, I believe that $A \land V$ is true as well. In standard proof theory, I can apply a rule as many times as I like, or I can choose a different proof altogether: if the hypothesis is held valid, the final conclusion will also be considered valid, with the same degree of credibility.

We intend to replace the foregoing implicit assumption by a new proof theory that allows for dissipative deductions, i.e. deduction where some degree of convincingness may be lost at each steps. Just like classical formal arithmetics should be seen as a limit case of feasible arithmetics, standard proof theory should be viewed as a special case of Dissipative Proof Theory, where the rate of truth loss is always zero, or negligible.

What we are suggesting here, is that each logical rule is both a ”joint”, through which subproofs can be melded, and a credibility transfer operator, i.e. a channel that transforms the credibility of the subproofs into the credibility of their merger.

Concretely, we need a Truth Transfer Policy, i.e. a policy that prescribes for each logical rule the degree of credibility of the consequences a a function of the credibility of the premisses.

In the following we shall work within a Natural Deduction’s framework: $ND$ — Derivations will be the set of all correct derivation from logical and non logical axioms. Everything can be rephrased, mutatis mutandis, in other systems of proof theory. Also, as a ”credibility thermometer”, we shall choose the unit interval $I = [0, 1]$, following Zadeh’s fuzzy logic. It goes without saying, though, that quite different measures of credibility could be adopted, as long as a corresponding policy is also introduced.

**Definition 4** A Truth Transfer Policy (TTP) $\mathfrak{P}$, is an assignment, for each logical rule $R$, of a function $\mathfrak{P}_R : I^k \rightarrow I$, where $k$ is the number of premisses in the rule. $\mathfrak{P}$ will obey the following restrictions:

- The TTP of the axioms is the identity $id : I \rightarrow I$.
- $\mathfrak{P}_R(0, \ldots , 0) = 0$ for each rule.
- $\mathfrak{P}_R(1, \ldots , 1) > 0$ for each rule.
- $\mathfrak{P}_R(p_1, \ldots , p_k) \leq \min(p_1, \ldots , p_k)$

Given a TTP $\mathfrak{P}$ and a derivation $p$, we can associate an $I$-value to it, its $\mathfrak{P}$-credibility, by assigning to all the logical and non logical axioms a credibility

\[15\] For a view of natural deduction as a system of assertions, the basic reference is Martin Lõf’s analysis in [26].

\[16\] The word dissipative is suggested by the following metaphor: think of truth as some kind of ”heat”, flowing through the derivation, from the leaves (i.e. the premisses). Depending on the medium through which heat flows, there can be some loss due to dissipation.

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value of 1, and applying $P$ at every steps of the deduction. This recursive procedure produces a function

$$F_\mathfrak{P} : NP - Derivations \rightarrow [0 \ 1]$$

that will be referred to as the $\mathfrak{P}$-based credibility measure.

We shall say that $A$ is a feasible consequence of the theory $T$ under the TTP $\mathfrak{P}$ iff there is a proof $p$ of $A$ from $T$ such that $F_\mathfrak{P}(p) > 0$. If $F_\mathfrak{P}(p) = 1$, then $A$ is a strong consequence of $T$.

We shall use the notation $T \vdash_{F_\mathfrak{P}} A$ to say that $A$ is feasibly derivable from $T$ under $F$. If the derivation is strong, we shall indicate it as $T \vdash_{F_\mathfrak{P}}^! A$.

A theory $T$ is $\mathfrak{P}$-feasibly consistent under $F$ there are no feasible proofs under $\mathfrak{P}$ of $\bot$ from $T$.

Let us briefly see what consequences Definition 5 entails: we have assumed that $\mathfrak{P}$ has no credibility jumps, in order to make sure that

1) $T \vdash_{F_\mathfrak{P}} A$ and $T \vdash_{F_\mathfrak{P}} B$ implies $T \vdash_{F_\mathfrak{P}} A \land B$

2) $T \vdash_{F_\mathfrak{P}} A$ and $T \vdash_{F_\mathfrak{P}} A \Rightarrow B$ implies $T \vdash_{F_\mathfrak{P}} B$

and similarly for the other connectives. In general, it will not be true that $T \vdash_{F_\mathfrak{P}} A$ and $T \vdash_{F_\mathfrak{P}} A \Rightarrow B$ implies $T \vdash_{F_\mathfrak{P}} B$: merging already barely feasible proofs of $A$ and $B$ may go beyond the horizon, constructing an unfeasible proof.

The reader may be a bit puzzled: what is the connection of the above with the usual way of thinking about feasibility of proofs, namely their length? Our TTP encompasses this view as a very special case; indeed, looking at the length of a proof as a measure of its convincingness basically corresponds to assigning the same constant erosion factor to each logical rule. In other words, the only impact that a rule application has on the credibility of a proof in progress is simply the one that results from the increase of size of the proof itself.

To illustrate the foregoing with a rather trivial example, let us assume, for sake of simplicity, that instead of Natural Deduction we operate with an Hilbert-style system. Also, in order to simplify matters, let us imagine that only Modus Ponens has a non trivial dissipative policy. More specifically, if I have a proof of $A$ of credibility $p_A$ and I have a proof of $A \Rightarrow B$ of credibility $p_{(A \Rightarrow B)}$, then a single-step application of modus ponens will lead to a proof of $B$ of credibility

$$\mathfrak{P}(MP) = f : [0 \ 1]^2 \rightarrow [0 \ 1]$$

where $f = \max(0, \min(p_{(A \Rightarrow B)}, p_A) - E)$
and $0 < E < 1$ is the "constant credibility erosion factor".

Now, suppose that we have a basic arithmetical theory, say $Q$, with a feasibility predicate, $F()$, obeying

$$F(0), F(n) \Rightarrow F(n+1), \text{ but } \neg F(2^{1000}) \text{ (no induction on the predicate $F$!)}.$$  

It is immediate to check that posing $E = \frac{1}{n^{2^{1000}}}$, makes the theory feasibly consistent: the convincingness (obtained by iterated modus ponens) of $F(n)$ is only $1 - \frac{n}{2^{1000}}$. As trivial as this example is, it already reveals a couple of interesting things:

- first, that for small numbers the credibility is almost equal one, creating, as it were, the illusion of an indefinite series of feasible numbers. Indeed, standard Hilbert style proof theory corresponds to setting the constant erosion factor equal zero;

- secondly, it implicitly suggests where a FIS itself may come from: it is simply generated by the proof-theoretical discoveries of Amanda. Posing $G(n) = 1 - \frac{n}{2^{1000}}$ we recover the trivial FIS described in (♣) of the previous Section.

The point of view presented in the foregoing could be qualified as *local*: the overall credibility of a proof is the result of "integrating" over all the one-step credibility changes caused by applying such and such rule in the course of the proof.

It is quite possible that *non-local* measures of credibility might become relevant as well. For instance, Carbone and Semmes have advocated a measure of complexity of proofs based on the number of cycles in the logical flow graph of the proof itself (see [30]). In turn, one could take their measure, compose it with a notion of feasibility for natural numbers, and obtain a credibility measure of proofs. Such a measure is non local, at least at first sight; it is not clear to us whether it could ever be recovered as a TTP-grounded measure through some kind of roundabout procedure, but it seems quite unlikely at this stage. At any event, this order of considerations implicitly suggests the following definition:

**Definition 5** A general credibility measure (GCM) for derivations is simply a fuzzy characteristic function $F : ND - \text{Derivations} \rightarrow [0, 1]$  

A GCM $F$ is well-behaved iff $T \vdash F A \iff T, \neg A$ is $F$-consistent.  

A GCM $F$ is factorable iff there exist a complexity measure

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\footnote{We omit for brevity the definition of feasible consequence, feasible consistency, etc. They are just the same as in the case of a GCM that is generated by a TTP (see page 20).}
In his proposal for an ultrafinitistic proof theory (see again [10]), Sazonov introduces two restrictions to ordinary proof theory: proofs must be physically realizable, and normal (or, which is equivalent, cut-free if one uses Gentzen’s approach). To us, the first prescription needs to be formalized, as it is not absolute (feasibility is contextual!). This can be easily accomplished by choosing the GCM obtained from a distinguished FIS (representing here the semi-set of feasible, i.e constructible numbers on a concrete machine), and using as the complexity measure the standard “number of symbols”. As for his second restriction, namely normality of proofs, it can be absorbed by the following observation: suppose one has a GCM $F$:

\[
\begin{array}{ccc}
N & \xrightarrow{ND - Derivations} & [0, 1] \\
\end{array}
\]

Indeed, the schema above seems to underly Sazonov’s choice. Why banishing a rule in the first place? A sensible answer is that (as it has been pointed out in several quarters) the cut rule “hides” the actual complexity of a derivation. By “unwrapping” it, one has a better sense of how complicated the proof
actually is. The unwrapping is accomplished precisely by setting $T$ above as the standard cut-elimination procedure.

We have just barely introduced the bare bones of Dissipative Proof Theory. We just wish to remark that the notion of $TTP$ forces a paradigm shift even in the very way one thinks of proofs: derivations, from being _barren trees_, become _decorated_.

**Question 1** Which factorable credibility measures are of the form $F\Psi$ for a suitable $TTP\Psi$?

**Question 2** The notion of well-behaved CGM appears to be critical (see next Section). It would thus be important to know which $TTP$ are such that their generated GCM are well-behaved. Also, which measures of complexity and FIS combine to form well-behaved GCM?

What we have seen in this Section has to do with credibility, not Aristotelian-Tarskian truth. Nevertheless, it is clear that to re-establish a comfortable parallel between provability and truth, between what-can-be-proved and what-is-out-there, a careful revision of the core foundations of Model Theory is necessary. This is our goal in the next Section.

### 6 Vague Truth: a semantics à la Tarski for feasibly consistent theories

Before we begin, let us pause a moment and point out that the very distinction between syntax and semantics, the cornerstone of classical logic, is suspect from an ultrafinitistic viewpoint. To be sure, even from a more conservative constructivist perspective, people have raised a number of objections against the dualism syntax-semantics (Jean-Yves Girard has recently written on this pivotal issue in [27]). All these objections remain true _a fortiori_ for ultrafinitism (a Tarski-free semantics for arithmetics inspired by formal games will be described in [21]).

So, why even bother trying to establish a tarskian semantics for feasibly consistent theories? As we declared in the introduction, our chief goal is to provide a bridge between the classical world mathematics and the world-to-be of ultrafinitism. As Tarskian semantics, grounded in Set Theory, is pervasive, it seems reasonable to start from there. Moreover, we would like to turn the informal and intuitive models of Section 4 into formal ones, thus providing a way to talk about feasibly consistent theories without being hampered by the "bureaucracy of syntax" (we are borrowing here the colorful expression coined by Girard).

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18 A note in passing for the categorical-minded reader: this suggests that the topic of Dissipative Proof Theory could possibly benefit from an injection of ideas and method from Enriched Category Theory.
The Tarskian notion of satisfaction, upon which the entire cathedral of standard model theory hinges, is unfit to provide a semantics for UF. Indeed, as it stands, it cannot even accommodate traditional constructivism, and we are here in a territory far more "picky" than all brands of liberal constructivism (i.e., constructivism that allows for potential infinity in its arguments) can possibly be.

We thus need to revise the "rules" of satisfaction, and replace the abstract notion of truth underpinning the Tarskian definition with a more realistic one.

The key is in the TTP introduced in the last Section\[19\] the TTP for a specific connective should act as an lower bound for the "degree of truth" in a given model. Assume in a model $M$ (yet to be defined) $A$ holds with truth degree $p_A$, and $B$ with degree $p_B$ (where $p_A, p_B \in [0, 1]$). The degree of truth of $A \wedge B$ should be at least the degree of truth of its proof from the premisses $A$ and $B$ using $\wedge$-introduction. In other words, we are not ruling out the possibility that in $M$ additional credibility may be available for $A \wedge B$ through some kind of evidence; what we are saying is that its degree of truth should be at least the one inferred using a one-step proof from $A$ and $B$.

**Definition 6** Let $L$ be a first-order language and $\mathfrak{P}$ an assigned Truth Transfer Policy. A structure for $(L, \mathfrak{P})$ is an n-tuple $M = (D, c_k, f_i, P_j)$ where domain, constructors and function symbols are interpreted the usual way, whereas each n-ary predicate $P$ becomes a fuzzy subset of $(D)^n$: $P : (D)^n \rightarrow [0, 1]$.

Given an assignment $s$ of the variables, we say that $M$ satisfies the atomic predicate $P(x_1, \ldots, x_n)$ iff $P(s(x_1), \ldots, s(x_n)) > 0$. In symbols, $(M, s) \models P$. If the evaluation is 1, then we say that $M$ strongly satisfies $P$ ($(M, s) \models_! P$).

Satisfaction is extended to arbitrary predicates by posing constraints to the shape of their corresponding fuzzy subsets (we shall present satisfaction rules only for the complete set \{\wedge, \neg, \exists\}):  

1) $P(x_1, \ldots, x_n) \wedge Q(y_1, \ldots, y_m) \geq \mathfrak{T}_{\wedge\text{-intro}}(P(x_1, \ldots, x_n), Q(y_1, \ldots, y_m))$

2) $\neg P(x_1, \ldots, x_n) = 1 - P(x_1, \ldots, x_n)$

3) $\exists x P(x) \geq \mathfrak{T}_{\exists\text{-intro}}(P(c))$, where $c$ is any constant in $M$.

A $\mathfrak{T}$-model $M$ for a theory $T$ is a structure such that all axioms of $T$ are strongly true in $M$.

\[19\]In the following, we shall restrict ourselves to TTP-generated deduction systems, as those systems give us "control" on the flow of truth. Also, for sake of simplicity, we shall confine ourselves to a subset of ND which includes rules only for the set \{\wedge, \neg, \exists\} (classically, this set is complete). To which extent our arguments can be lifted to more general context is not clear at present.
A comment to the above is in order: the notion of satisfaction in the foregoing definition is (as the careful reader has certainly not failed to observe) strictly coupled with a specific TTP. This may be puzzling at first, but it suffices to see that standard first-order semantics does exactly the same thing, only in disguise. Indeed, Tarski’s truth is simply a particular case of the previous definition in which the underlying TTP is assumed to be the trivial one, i.e. zero-decay:

**within classical logic truth is preserved in its entirety at every logical step.**

Definition 6 gives ”legal status” to FIS as models of theories. Recall once again the theory $T_0 = Q + 
eg F(2^{1000})$ of the previous section. Consider the structure $M = (N, =, S, +, *, G)$, where $G$ is the FIS $G(n) = 1 - \frac{1}{1000}$ (the interpretation of equality is the usual one). $M_0$ is a model of $T_0$, according to Definition 6. Similarly, one can construct models for other arithmetical theories enriched with a feasibility predicate.

A good semantics ought to be at the very least sound. This is indeed the case:

**Proposition 3 Soundness.** Let $T$ be a first order theory. If $T \vdash_A A$, then for every $T$-model $M$ of $T$, $M \models A$

The proof is, just like its standard counterpart, by induction on the complexity of $A$. □

Notice that something more can be extracted by the last proposition: the inequality holds for each proof of $A$ from $T$.

**Corollary 1** $A(x_1, \ldots, x_n) \geq \min\{F_T(p_A)\}$, where $p_A$ denotes a proof of $A$ from $T$.

The last inequality implicitly suggests that somewhere there must be a model where equality holds, i.e. a model where the degree of truth is exactly what one gets by choosing, as it were, the ”best” proof available. This is indeed the case, when one consider the familiar construction of the term model, a key ingredient in the Henkin’s version of the completeness theorem for FOL. We are ready to establish an ultrafinitistic version of the completeness theorem.

**Proposition 4 Completeness.** Let $\Sigma$ be a TTP such that the corresponding GMC is well-behaved. If for every $\Sigma$-model $M$ of $T$, $M \models A$, then $T \vdash_A A$.

Sketch of proof: if it is not true that $T \vdash_A A$, then the theory $T = T \cup \{\neg A\}$ is $F(\Sigma)$-feasibly consistent (here the well-behavior of $\Sigma$ is needed). The goal is to show that this theory has a model, thereby contradicting the assumption. Just like in the standard completeness theorem, we can construct a special model, the term model $M_0$, of a feasibly consistent completion of the theory $T + \neg A$. The key is defining truth in $M_0$: the value of a given predicate $A^M(x_1, \ldots, x_n)$ will be the minimum credibility value over all $F(\Sigma)$-feasible
proofs of $A^M(x_1, \ldots, x_n)$ from $T$. If no such proofs exists, one can "feasibly complete" $M_0$ by adding either $A^M(x_1, \ldots, x_n)$ or its negation to it (unless, of course, $T$ feasibly proves the negation, in which case there is no need to expand $T$). In other words, the construction mimics the familiar construction of the term model, but where the role of consistent completion is taken by feasible consistency. Indeed, Henkin’s proof should be recoverable when one "passes to the classical limit". □

Before we conclude this Section, we would like to remark that the notion of feasible consistency is akin to the so-called para-consistent theories, studied by several authors ([40]). Indeed, in our Definition 6, a structure can satisfy a formula and its negation at the same time. However, in our setting, it is not the logic that has changed, as in para-consistent systems. It is the very notion of deducibility and truth that has been "upgraded", to account for more stringent requirements than either classical logic or intuitionistic one can provide. It would still be useful to see if a bridge between these two approaches can be drawn. Much remains to be done here . . .

7 Cantorian Nanotechnology: Miniaturizing Cantor’s Paradise

We have just seen that it is possible to "miniaturize" the notion of natural number series and, through a suitable emendation of syntax and semantics, view these objects as genuine nano-models of standard arithmetical theories. In a metaphorical way, this is a bit like Nanotechnology, only applied to Logic instead of Engineering: choose a macroscopic object, and strive to create a microscopic clone, all the while retaining its salient properties.

In the same spirit, one can think of the (bounded) finite as a microscopic world, and the transfinite (or even the unbounded finite) as the macroscopic one. The challenge then seems to be: to what extent can we miniaturize familiar infinitary structures? For instance, can we create an entire theory of transfinite cardinals once we have a finitary copy of $\Omega_0$ via a FIS? These cardinals would form a ladder of transcendency degrees of finiteness, just as the alephs gauge the size of standard infinite sets.

This program should by no means remain confined to structures. Instead, it should attempt to miniaturize basic mathematical and meta-mathematical results as well. For instance, as we have seen in the last Section, a coherent ultrafinitistic version of the celebrated completeness theorem can be formulated (possibly different ones: we are not making any claim that our account is the only possible route). What about the incompleteness theorems? Or the Lowenheim-Skolem theorem? Armed with a notion of satisfaction such as the one we have just described, one could start the exploration of concrete first-
order mathematical theories (such as the theory of infinite dimensional vector spaces) that do not afford conventional finite models.

In our belief, this program, which we would like to refer to as Cantorian Nanotechnology, is a viable channel of investigation. At the very least, it will give us a better sense on the real boundaries (assuming there are any) between the finite and the infinite. If successful, it could provide us with a cornucopia of new entities that may have an impact in the way we model our physical world and our very selves.

Last, but by no means least, Cantorian Nanotechnology could be an attempt to move beyond the lasting debate between constructivists of all brands and staunch defenders of Cantor’s Paradise: it may turn out, after all, that the beautiful land that Cantor created for the endless joy of mathematicians, can be faithfully reproduces at the finite scale.

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In recent months, the first author has posted a few messages on the FOM list (see for instance [13], [14]), concerning ultrafinitism and related issues. Several FOM subscribers (including, but not limited to, Vladimir Sazonov, David Isles, Stephen Simpson, and Karlis Podnieks) replied, publicly or privately, with valuable comments, criticisms, suggestions, and different viewpoints. To them all goes our gratitude.

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\[\text{A single example: quantum mechanics represents a physical system as a Hilbert space. Many quantum systems, such as the free spinless particle on the line, are associated to an infinite dimensional Hilbert space. In BN, the same particle could be modeled by a subspace } H \text{ of } C^N, \text{ where } N \text{ is a large enough finite integer; } H \text{ would appear as infinite dimensional. Incidentally, we observe en passant that such a novel perspective could be used to motivate approaches to Quantum Mechanics from the Quantum Computing’s angle.}\]
midst of the most abstruse and seemingly difficult topics, was, and is, the true driving force behind this project.

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