Multisymplectic Maxwell Theory

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Abstract. This note provides a detailed treatment of the Multisymplectic Maxwell theory through the general setting developed in [24] [26] [27]. In particular we explore the DeDonder-Weyl theory, the question of algebraic and dynamical observable forms, the copolarization process related to the good search of canonical forms. Finally, we give - for the two dimensional case - some indications for the construction of the higher Lepage-Dedecker correspondence, in the context of the underlying Grassmannian viewpoint.

1 Introduction

This note is dedicated to the application of Multisymplectic Geometry (MG) to field theory, in particular to the Maxwell theory [44]. We apply the formalism developed by F. Hélein and J. Kouneiher [21] [23] [24] [25] [26] [27] to the Maxwell variational problem. As an introduction, we briefly make some remarks about (MG) in the section 1.1, then we offer some comments about the derivation of the Euler-Lagrange equations or Maxwell’s equations in 1.2 and finally, we draw the outline of the paper in 1.3.

1.1 Multisymplectic Geometry

Within the context of covariant canonical quantization Multisymplectic Geometry (MG) is a generalization of symplectic geometry for field theory. It allows us to construct a general framework for the calculus of variations with several variables. Historically (MG) was developed in three distinct steps. Its origins are connected with the names of C. Carathéodory [4] (1929) and H. Weyl [53] (1935) on one hand and T. De Donder [8] [9] (1935) on the other. We make this distinction since the motivations involved were different: Carathéodory and later Weyl, were involved with the generalization of the Hamilton-Jacobi equation to several variables and the line of development stemming from their work is concerned with the solution of variational problems in the setting of the action functional. On the other hand, E. Cartan [5] (1922) recognized the crucial importance of developing an invariant language for differential geometry not dependent on local coordinates. T. DeDonder carried this development further. The two approaches merged in the so-called DeDonder-Weyl (DDW) theory based on the multisymplectic manifold $\mathcal{M}_{\text{DDW}}$. The second step arose with the work of T. Lepage and P. Dedecker. As was first noticed by T. Lepage [12] [43], the DeDonder-Weyl setting is a special case of the more general multisymplectic theory that we refer as the Lepage-Dedecker (LD) theory. The geometrical tools permitting a fully general treatment were provided by P. Dedecker [6] [7]. The next step was taken by the Warsaw school in the seventies which further developed the geometric setting. W. Tulczyjew [46] [47], J. Kijowski [37] [38], K. Gawedski [17] and W. Szczyrba [40] [41] all formulated important steps. We find already in their work the notion of algebraic form, and in the work of J. Kijowski...
a corresponding formulation of the notion of dynamical observable emerges. We emphasize, for the full geometrical multisymplectic approach, two fundamental points: the generalized Legendre correspondence - introduced by T. Lepage and P. Dedecker - and the issue of observable and Poisson structure, two cornerstones within the universal Hamiltonian formalism developed by F. Hélein, [21] [23] and F. Hélein and J. Kouneiher [24] [25] [26] [27]. Hence, for field theory, we are led to think of the solutions of variational problems as \( n \)-dimensional submanifolds \( \Sigma \) embedded in a \((n+k)\)-dimensional manifold \( \mathcal{X} \). One observes the key role of the Grassmannian bundle as the analogue of the tangent bundle for variational problems for field theory.

1.2 Maxwell Theory

Let us recall the expression of the Euler-Lagrange equations for the Maxwell theory. We are first interested with the vacuum Maxwell action, given by:

\begin{align}
(i) \quad & \mathcal{L}_\text{Maxwell}^0 = \frac{1}{2} \int \mathcal{X} \mathbf{F} \wedge \star \mathbf{F}, \\
(ii) \quad & \mathcal{L}_\text{Maxwell}^0[x, A, dA] = -\frac{1}{4} \int \mathcal{X} \mathbf{F}_{\mu\nu} \sqrt{-g} d\mathbf{y}. \quad (1.1)
\end{align}

The Lagrangian density is \( \mathcal{L}(A) = -\frac{1}{4} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} \). We denote \( \text{vol}_\mathcal{X}(g) \) a Riemannian volume form such that \( \text{vol}_\mathcal{X}(g) = \sqrt{-g} d^4x \). In the case where \( \mathcal{X} \) is the Minkowski space-time we obtain then \( \sqrt{-g} = 1 \) and then \( L(A) = -\frac{1}{4} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} \). We have Maxwell vacuum equations:

\( d\mathbf{F} = 0 \quad \text{and} \quad d \star \mathbf{F} = 0. \quad (1.2) \)

Considering the current of matter \( \mathbf{J}^\mu(x) \) over \( \mathcal{X} \) the Lagrangian is written:

\( \mathcal{L}_\text{Maxwell}[x, A, dA] = \mathcal{L}_\text{Maxwell}^0[x, A, dA] - \mathbf{J}^\mu(x) A_\mu, \quad (1.3) \)

and the Euler-Lagrange equations are written:

\( d\mathbf{F} = 0 \quad \text{and} \quad d \star \mathbf{F} = \star \mathbf{J}. \quad (1.4) \)

The curvature form is written:

\( \mathbf{F} = \frac{1}{2} \mathbf{F}_{\mu\nu} dx^\mu \wedge dx^\nu \), with \( \mathbf{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Thus, we write the Hodge star \( \star \mathbf{F} \) in components:

\( \star \mathbf{F} = \star \left( \frac{1}{2!} \mathbf{F}_{\mu\nu} dx^\mu \wedge dx^\nu \right) = \frac{1}{2!} \left( \star \mathbf{F} \right)_{\rho\sigma} dx^\rho \wedge dx^\sigma, \)

\( = \frac{1}{2!} \frac{\sqrt{-g}}{2!} \varepsilon_{\rho\sigma} \mathbf{F}_{\mu\nu} dx^{\rho} \wedge dx^{\sigma} = \frac{\sqrt{-g}}{4} \varepsilon_{\rho\sigma} \mathbf{F}_{\mu\nu} dx^{\rho} \wedge dx^{\sigma} \)

\( = \frac{\sqrt{-g}}{4} g_{\alpha\mu} g_{\beta\nu} \varepsilon_{\rho\sigma} \mathbf{F}^{\alpha\beta} dx^{\rho} \wedge dx^{\sigma}. \)

We see the equivalence between the two actions (1.1)(i) and (1.1)(ii).

\[\text{Proof} \quad \text{We make a straightforward computation which involves the Hodge duality.} \]

\( \mathbf{F} \wedge \star \mathbf{F} = \left[ \frac{1}{2} \mathbf{F}_\lambda^\mu d\mathbf{x}^\lambda \wedge d\mathbf{x}^\sigma \right] \wedge \left[ \frac{\sqrt{-g}}{4} g_{\alpha\mu} g_{\beta\nu} \varepsilon_{\rho\sigma} \mathbf{F}^{\alpha\beta} dx^{\rho} \wedge dx^{\sigma} \right] \)

\( = \frac{1}{8} \left[ \mathbf{F}_\lambda^\mu \sqrt{-g} g_{\alpha\mu} g_{\beta\nu} \varepsilon_{\rho\sigma} \mathbf{F}^{\alpha\beta} \right] d\mathbf{x}^\lambda \wedge d\mathbf{x}^\sigma \wedge d\mathbf{x}^\rho \wedge d\mathbf{x}^{\sigma}. \)

\[\text{For an introductory synthesis of the principal motives and results presented in their work, see D. Vey [49] [50].} \]
Since \( \text{vol}_V(g) = \sqrt{g} \text{d}g = \frac{1}{4} \varepsilon_{\lambda_1 \cdots \lambda_n} \text{d}x^\lambda \wedge \text{d}x^\xi \wedge \text{d}x^\sigma \wedge \text{d}x^\tau \), we obtain:

\[
\mathbf{F} \wedge *\mathbf{F} = \frac{1}{8} \left[ \mathbf{F}_{\lambda} \sqrt{g} \mu \nu \rho \sigma \varepsilon_{\mu \nu \rho \sigma} \mathbf{F}^{\Lambda} \right] \text{d}x^\lambda \wedge \text{d}x^\xi \wedge \text{d}x^\sigma \wedge \text{d}x^\tau = \frac{1}{8} \mathbf{F}_{\lambda} \mathbf{F}^\lambda \varepsilon_{\alpha \beta \gamma} \varepsilon^\lambda_\rho \sqrt{g} \text{d}g
\]

\[
= -\frac{1}{2} \delta^\lambda_\xi \delta^\xi_\rho \mathbf{F}_{\lambda} \mathbf{F}^\rho \sqrt{g} \text{d}g = -\frac{1}{2} \left[ \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} - \mathbf{F}_\beta \mathbf{F}^{\alpha \beta} \right] \sqrt{g} \text{d}g = -\frac{1}{2} \mathbf{F}_{\alpha} \mathbf{F}^{\alpha \beta} \sqrt{g} \text{d}g
\]

where we have used the identity \( \varepsilon_{\alpha \beta \gamma} \varepsilon^\lambda_\rho = -2!2! \delta^\lambda_\alpha \delta^\rho_\beta \) in the last line.

The Euler-Lagrange equations for the Maxwell theory are written \((1.5)\)

\[
\frac{\partial}{\partial t} L_{\text{Maxwell}} = \frac{\partial}{\partial (\partial_\mu A_\nu)} L_{\text{Maxwell}}.
\]

We recover the Maxwell’s equations:

\[
\mathbf{J}^\mu (t) = -\frac{\partial}{\partial x^\mu} \mathbf{F}^{\mu \nu} (x).
\]

**Proof** We compute \([\mathbb{I}]\) and \([\mathbb{II}]\). The first term is:

\[
\mathbb{I} = \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial_\mu A_\nu)} = \mathbf{J}^\mu (x).
\]

The second leads to the following calculation:

\[
\mathbb{II} = \frac{\partial}{\partial t} \left( -\frac{1}{4} \left( \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} \right) \right) = -\frac{1}{4} \frac{\partial}{\partial t} \left( \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} + \mathbf{F}_\beta \mathbf{F}^{\alpha \beta} \right)
\]

\[
= -\frac{1}{4} \frac{\partial}{\partial t} \left( \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} \right) + \frac{1}{4} \frac{\partial}{\partial t} \left( \mathbf{F}_\beta \mathbf{F}^{\alpha \beta} \right)
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} \right) + \frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{F}_\beta \mathbf{F}^{\alpha \beta} \right)
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{F}_\alpha \mathbf{F}^{\alpha \beta} \right) - \frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{F}_\beta \mathbf{F}^{\alpha \beta} \right) = -\frac{1}{2} \frac{\partial}{\partial t} \left( \mathbf{F}^{\mu \nu} - \mathbf{F}^{\nu \mu} \right) = -\frac{1}{2} \mathbf{F}^{\mu \nu}
\]

Then we obtain Maxwell’s equations \((1.6)\).

### 1.3 Outline

First, we are interested in the treatment of the DeDonder-Weyl theory for the Maxwell equations, see section \((2)\). In particular, we focus on the Maxwell multisymplectic manifold and the related Dirac constraint set \((2.1)\). We obtain the multisymplectic Hamilton equations \((2.2)\). Finally, we present the Maxwell theory in the setting of an \(n\)-phase space \((2.3)\).

In section \((3)\), we investigate observables and canonical variables for \(4\)D Maxwell theory in the (DDW) setting. First, we give some simple example of algebraic 3-forms \((3.1)\) - both position and momenta that are defined on \(\mathcal{M}_{\text{Maxwell}} \subset \mathcal{M}_{\text{DDW}}\) - with their related Poisson bracket structure \((3.2)\). The following section \((3.3)\) is dedicated to the expression of all algebraic \((n-1)\)-forms and their related infinitesimal symplectomorphisms. This result of this search is given by proposition \((3.2)\). Then, we describe dynamical observables \((3.4)\) and algebraic observables in the pre-multisymplectic setting \((3.5)\). We describe observable functionals in \((3.6)\) - their kinematical and dynamical aspects. Finally, we obtain the Poisson bracket structure. The latter is a result already written by J. Kijowski and W. Szczyrba \((38)\) \((40)\). In particular, we describe generalized positions and momenta observable 3-forms in the pre-multisymplectic case, where the pre-multisymplectic manifold is \(\mathcal{M}_{\text{Maxwell}}^0 \subset \mathcal{M}_{\text{Maxwell}}\). In this case, the point of interest is that
any algebraic 3-form is a dynamical observable form. Finally, we offer in section (3.7) some remarks about the stress-energy tensor.

The following section (4) is dedicated to dynamical equations and canonical variables. In particular, we make a brief review of the use of graded structures and Grassman variables in (4.1). This is related to various works, for example the ones of I.V. Kanatchikov [30] [31] [32], M. Forger, C. Paufler and H. Römer [12] [13] [14], F. Hélein and J. Kouneiher [24] or S. Hrabak [28] [29]. In section (4.2), we show the interplay between superforms, Grassman variables and dynamical equations, and the relationship with the external bracket that appear in F. Hélein and J. Kouneiher [24]. The section (4.3) is dedicated to the copolarization setting developed by F. Hélein and J. Kouneiher in the serie of papers [25] [26] [27]. Finally we recall a possible polarization for the Maxwell multisymplectic manifold - already found in [25] [26] [27]. We give - see section (4.4) - the detailed calculation about the impossibility to include \( C_1 \) in \( \mathcal{P}_1 \mathcal{T}^\star \mathcal{M}_{\text{Maxwell}} \), in the 2D-case for practical purposes. This mathematical result is strongly connected to the physical fact that the gauge potential is not observable by itself. Then, the components \( A_\mu \) of the 1-forms \( A = A_\mu dx^\mu \) are not observables from which we can describe a good copolarization. However, thinking in terms of differential forms directly, we can build a canonical pair of forms \( (A, \pi) \) and a well-defined Poisson bracket between observable functionals related to these canonical forms, see the formula (4.9).

In section (5) we describe the Lepage-Dedecker correspondence for Maxwell theory in the two dimensional case. We describe in details the Lepage-Dedecker correspondence as well as the calculation of the Hamiltonian respectively in (5.1) and (5.2). Then we derive the generalized Hamilton-Maxwell equations in (5.3). Notice that in such a context, the Legendre correspondence is non-degenerate. We recall the definition of the enlarged pseudofiber and the pseudofiber - see [25] [26] [27] - in the last section (5.4) and use this notion in the context of the Maxwell 2D-case.

## 2 Multisymplectic DeDonder-Weyl-Maxwell theory

First, let us describe the geometrical setting and precise the notations for the four dimensional case. We consider \( \mathcal{X} \) to be the spacetime manifold with \( \dim(\mathcal{X}) = n = 4 \). Let \( A \in T^\star \mathcal{X} \), be the potential 1-form. The space of interested is \( \mathfrak{3} = T^\star \mathcal{X} \). As noticed in [25] [26], the more naive approach is to work in a local trivialization of a bundle over \( \mathcal{X} \), since a connection is not a section of a bundle. This is the chosen path here. A point \( (x, A) \) in \( \mathfrak{3} \) is in the position configuration space. Any choice \( (x, A) \) is equivalent to the data of an \( n \)-dimensional submanifold in \( \mathfrak{3} = T^\star \mathcal{X} \xrightarrow{\pi} \mathcal{X} \) described as a section of the fiber bundle over \( \mathcal{X} \). Let us consider the map \( \mathfrak{3}_A : \mathcal{X} \rightarrow \mathfrak{3} = T^\star \mathcal{X} \) described by \( (2.1) \),

\[
\mathfrak{3}_A : \begin{cases}
\mathcal{X} & \rightarrow \mathfrak{3} = T^\star \mathcal{X} \\
x & \mapsto A(x) = A_\mu(x) dx^\mu.
\end{cases}
\]  

(2.1)

which is simply some section of the related bundle. We associate with \( A \), the bundle \( \mathcal{P}^A = A^\star T^3 \mathcal{X} \). The useful quantities to describe \( dA \) the differential of the map \( A \) as sections of the bundle \( \mathcal{P}^A \) over \( \mathcal{X} \) are now introduced. We denote the exterior covariant derivative on the 1-form \( A \) by \( d^D A \):

\[
d^D A = [d^D A]_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{with} \quad [d^D A]_{\mu\nu} = \partial_{[\mu} A_{\nu]}.
\]  

(2.2)
We denote $v_{\mu\nu} = \partial_\mu A_\nu$ so that $d^DA = v_{[\mu\nu]}$. The space of interest, the analogue for tangent space is $\Lambda^nT(x,e,\omega)\mathfrak{T}$ the fiber bundle of $n$-vector fields of $\mathfrak{T}$ over $\mathfrak{X}$. For any $(x^\mu, A_\mu) \in \mathfrak{T}$, the fiber $\Lambda^nT(x,A)(T^*\mathfrak{X}) = \Lambda^nT(x,A)\mathfrak{T}$ can be identified with $\mathcal{P} = A^*\mathfrak{T} \otimes \mathfrak{T}^*\mathfrak{X}$ via the diffeomorphism:

$$
\mathcal{P} \cong A^*\mathfrak{T} \otimes \mathfrak{T}^*\mathfrak{X} \quad \sum_{\mu,\nu} [d^DA]_{\mu\nu} dx^\mu \otimes dx^\nu \quad \mapsto \quad \Lambda^nT(x,A)(T^*\mathfrak{X}) \quad \mapsto \quad z = z_1 \lor \ldots \lor z_n,
$$

(2.3)

where $\forall 1 \leq \eta \leq n$, $z_\eta = \frac{\partial}{\partial x^\eta} + \sum_{1 \leq \beta \leq n} v_{\eta\beta} \frac{\partial}{\partial A_\beta}$.

In order to fix ideas we stress that we have local coordinates $(x^\mu, A_\mu, v_{\mu\nu})$ for the configuration bundle $\mathfrak{T}$. The data of the local coordinates $(x^\mu, A_\mu, v_{\mu\nu})$ - or equivalently $(x^\mu, A_\mu, A_{\mu\nu})$ - can be thought as coordinates on $\mathcal{P}$ or $\Lambda^nT(x,e,\omega)\mathfrak{T}$). We identify $\mathcal{P} \cong \Lambda^nT(x,e,\omega)\mathfrak{T}$.

In this section we develop three points. First, we describe the setting of the (DDW)-Maxwell theory, in section (2.1). In particular we consider the Dirac primary constraint set and the related Maxwell multisymplectic manifold $\mathcal{M}_{\text{Maxwell}}$, see (2.1). Then we derive the generalized Hamilton equations respectively in the multisymplectic (2.2) and in the pre-multisymplectic (2.3) settings. In the latter, we observe the connection with the covariant phase space.

### 2.1 Multisymplectic DeDonder-Weyl-Maxwell theory

**Generalized Legendre correspondence.** The generalized Legendre correspondence is constructed on $\mathcal{M} = \mathfrak{T}^*\mathfrak{T}$. For all $(q,p) \in \mathcal{M}$ we introduce the local coordinates on the bundle $\mathcal{M}$. Let us denote $(q^\mu)_1 \leq \mu \leq 2n$ the local coordinates on $\mathfrak{T}^*\mathfrak{T}$ and $p_{\mu_1 \ldots \mu_n}$ the local coordinates on $\Lambda^n\mathfrak{T}^*\mathfrak{T}$ in the basis $(dq^{\mu_1} \land \ldots \land dq^{\mu_n})_1 \leq \mu_1 \ldots \leq \mu_n \leq 2n$, completely antisymmetric in $(\mu_1 \ldots \mu_n)$. The canonical Poincaré-Cartan $n$-form is written in local coordinates (here $n = k = 4$):

$$
\theta = \sum_{1 \leq \mu_1 < \ldots < \mu_n \leq 2n} p_{\mu_1 \ldots \mu_n} dq^{\mu_1} \land \ldots \land dq^{\mu_n}.
$$

We consider the (DDW) submanifold $\mathcal{M}_{\text{DDW}} \subset \mathcal{M}$:

$$
\mathcal{M}_{\text{DDW}} = \left\{(x,A,p)/x \in \mathfrak{X}, A \in T^*\mathfrak{X}, p \in \Lambda^n(T^*\mathfrak{X}) \text{ such that } \partial A_\mu \land \partial A_\nu \lor p = 0 \right\}.
$$

We restrict and adapt our notations to the case $\mathcal{M}_{\text{DDW}} \subset \mathcal{M}$. All the components $p_{\mu_1 \ldots \mu_n}$ are taken equal to zero, except for $p_{1 \ldots n} = e$ and for the multimomenta $p_{1 \ldots (n-1)(A_\mu)(\nu+1) \ldots n}$ denoted $\Pi^A_{\mu\nu}$. We define a Legendre correspondence:

$$
\Lambda^n(T^*\mathfrak{X}) \land \mathbb{R} \land \Lambda^n(T^*\mathfrak{X}) = \Lambda^n\Pi_{\mu\nu}^A : (q,v,w) \leadsto (q,v,p) \mapsto \langle p,v \rangle - L(q,v).
$$

**Maxwell multisymplectic manifold $\mathcal{M}_{\text{Maxwell}}$.** Let us describe the general construction for the De Donder-Weyl multisymplectic manifold. We consider $\theta_{(q,p)}^{\text{DDW}}$, the Poincaré-Cartan $n$-form:

$$
\theta_{(q,p)}^{\text{DDW}} := \omega + \Pi^A_{\mu\nu} dA_\mu \land dw_\nu,
$$

(2.4)
where \( d\eta = dx^1 \wedge \ldots \wedge dx^n \) is a volume \( n \)-form defined on \( \mathcal{X} \) and we also denote \( d\eta_{\beta} := \partial_{\beta} \cdot d\eta \). Due to the Legendre correspondence construction, the equivalence relation between \((q, v)\) and \((q, p)\) is written:

\[
(q, v) \mathcal{G} (q, p) \iff \frac{\partial (p, v)}{\partial v} = \frac{\partial L(q, v)}{\partial v}. \tag{2.5}
\]

The term \((p, v)\) is understood as the following expression \( \langle p, v \rangle = \theta^\text{DOW}_p(Z) \). With \( Z = Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_4 \) and where \( \forall \alpha \in \mathcal{A} = \frac{\partial}{\partial p^\alpha} + Z_{\alpha \mu} \frac{\partial}{\partial A_\mu} \). We gives the straightforward calculation with \( Z_{\alpha \mu} = \partial_{\alpha} A_\mu \):

\[
\langle p, v \rangle = \theta^\text{DOW}_p(Z) = cd\eta(Z) + \Pi A_{\mu} \, dA_\mu \wedge \, d\eta_\nu(Z).
\]

We have the expression \( \langle p, v \rangle = \theta^\text{DOW}_p(Z) \):

\[
\langle p, v \rangle = c + \Pi A_{\mu} \, Z_{\nu \mu} = c + \Pi A_{\mu} \, \partial_\nu A_\mu. \tag{2.6}
\]

**Proof:** We denote

\[
Z_v = \frac{\partial}{\partial v^\mu} + \sum_{1 \leq \mu \leq n} Z_{\nu \mu} \frac{\partial}{\partial A_\mu} = \sum_{1 \leq \mu \leq n} Z_\nu \frac{\partial}{\partial \eta^\mu},
\]

We have \( q^\mu = x^\mu \) if \( 1 \leq \mu = \nu \leq n \) and \( q^\mu = A_{\mu - n} = A_\mu \) if \( 1 \leq \mu - n = \mu \leq n \). The bold index \( 1 \leq \mu \leq 2n \) is a multi-index such that \( Z_\nu = \delta_\nu^\mu \) for \( 1 \leq \mu \leq n \) and \( Z_\nu = \delta_{\nu, \mu} \) for \( n + 1 \leq \mu \leq 2n \).

\[
Z = Z_1 \wedge Z_2 \wedge Z_3 \wedge Z_4 = \sum_{\mu_1 < \ldots < \mu_4} Z_{\mu_1 \ldots \mu_4} \frac{\partial}{\partial q^{\mu_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{\mu_4}} = \sum_{\mu_1 < \ldots < \mu_4} Z_{\mu_1 \ldots \mu_4} \frac{\partial}{\partial q^{\mu_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{\mu_4}}.
\]

We expand the expression:

\[
Z = Z_{1234}^{1234} \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4 + \sum_{n < \mu_4} Z_{1234}^{1234} \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \frac{\partial}{\partial q^{\mu_4}} + \sum_{n < \mu_4} Z_{1234}^{1234} \partial_1 \wedge \partial_2 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}} + \sum_{3 < \mu_4} Z_{1234}^{1234} \partial_1 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}} + \sum_{3 < \mu_4} Z_{1234}^{1234} \partial_1 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}}
\]

Now we detail the different terms involved:

\[
Z_{1234}^{1234} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Z_{1}^{1} & Z_{2}^{1} & Z_{3}^{1} & Z_{4}^{1} \end{vmatrix}
\]

\[
Z_{1234}^{1234} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Z_{1}^{1} & Z_{2}^{1} & Z_{3}^{1} & Z_{4}^{1} \end{vmatrix}
\]

Therefore we obtain:

\[
[I] = Z_{1}^{1} \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \frac{\partial}{\partial q^{\mu_4}}, \quad [II] = -Z_{1}^{1} \partial_1 \wedge \partial_2 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}}, \quad [III] = Z_{1}^{1} \partial_1 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}}
\]

\[
[IV] = -Z_{1}^{1} \partial_2 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial q^{\mu_4}}, \quad \text{Then we obtain the expression for } Z:
\]

\[
Z = \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4 + Z_{1} \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \frac{\partial}{\partial A_{\mu}} - Z_{2} \partial_1 \wedge \partial_2 \wedge \partial_4 \wedge \frac{\partial}{\partial A_{\mu}} + Z_{3} \partial_1 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial A_{\mu}} - Z_{4} \partial_2 \wedge \partial_3 \wedge \partial_4 \wedge \frac{\partial}{\partial A_{\mu}}
\]
In the Maxwell case, the Dirac set are compatibility conditions that allows us to recover a correspondence. We introduce two different spaces. The first is the submanifold \( MMM \), defined by (2.10) (with the imposed constraints \( \Pi^A \mu = F^\mu = - F^\mu \)). The equivalence (2.5) is now written (2.8):

\[
\Pi^A \mu = \eta^{\mu \lambda} \eta^{\nu \sigma} F_{\lambda \sigma} = F^\mu.
\]

The expression of the multimomenta is given by (2.7).

The degenerate feature is related with:

\[
\Pi^A \mu = \eta^{\mu \lambda} \eta^{\nu \sigma} F_{\lambda \sigma} = F^\mu.
\]

The equivalence (2.5) is now written (2.8):

\[
(q, v) \circ (q, p) \iff \Pi^A \mu = \eta^{\mu \lambda} \eta^{\nu \sigma} F_{\lambda \sigma}.
\]

Notice that the Legendre transformation is degenerated. We cannot find a unique correspondence between the multivelocities \( v \) and the multimomenta \( p \). Given a \( v \in T \mathbb{R} \otimes_3 T^* (T^* \mathcal{X}) \) the equation (2.9) has a solution \( p \in M_{DDW} \) if and only if \( p \in M_{Deg} \) with:

\[
M_{Deg} = \left\{ (x, A, \mathfrak{d} \eta + \eta^{\mu \lambda} \eta^{\nu \sigma} F_{\lambda \sigma} d A_\mu \wedge d \eta_\nu) \mid (x, A) \in T^* \mathcal{X}, \mathfrak{c} \in \mathbb{R} \right\} \subset M_{DDW}. \tag{2.9}
\]

Notice that \( M_{Deg} \subset M_{DDW} \) is a vector sub-bundle of \( M_{DDW} \). The degenerate feature is related to the constraint \( \Pi^A \mu = F^\mu = - F^\mu \). The Legendre transform is recovered if we impose the compatibility conditions:

\[
\Pi^A_\mu + \Pi^A_\mu = 0. \tag{2.10}
\]

In the Maxwell case, the Dirac set are compatibility conditions that allows us to recover a Legendre transform. The (DDW) theory setting is concerned rather with the Legendre correspondence. We introduce two different spaces. The first is the (DDW) submanifold \( M_{DDW} \) on which we consider the canonical Cartan-Poincaré 4-form (2.4):

\[
\Theta^\text{DDW} = \mathfrak{d} \eta + \sum \Pi^A \mu d A_\mu \wedge d \eta_\nu, \quad \Omega^\text{DDW} = \mathfrak{d} \mathfrak{c} \wedge d \eta + d \Pi^A \nu \wedge d A_\mu \wedge d \eta_\nu.
\]

The second is \( M_{Maxwell} \), defined by (2.10) (with the imposed constraints \( \Pi^A \mu + \Pi^A_\mu = 0 \)).

Notice that \( \mathfrak{d} \eta_1 = \partial_1 \mathfrak{d} \eta = (-1)^{1+1} dx^2 \wedge dx^3 \wedge dx^4 = dx^2 \wedge dx^3 \wedge dx^4 \) as well as \( \mathfrak{d} \eta_2 = - dx^1 \wedge dx^3 \wedge dx^4 \), also \( \mathfrak{d} \eta_3 = dx^4 \wedge dx^2 \wedge dx^3 \) and finally \( \mathfrak{d} \eta_4 = - dx^4 \wedge dx^2 \wedge dx^3 \).

The important point concerns the restriction related to those constraints on the allowed vector fields on the multisymplectic space. In such a context, the vector fields on \( M_{Maxwell} \) must be written with the term \( \gamma^A_{\mu} \left( \frac{\partial}{\partial \Pi^A \nu} - \frac{\partial}{\partial \Pi^A_\mu} \right) \) rather than with the term \( \gamma^A_{\mu} \frac{\partial}{\partial \Pi^A_\nu} \) in the expression (2.13) of \( X \in \Lambda^4 T M_{Maxwell} \).
2.2 Hamilton-Maxwell equation in the DeDonder-Weyl framework

We compute the Hamiltonian function of the Maxwell theory in the (DDW) case:

$$H_{\text{deg}}(q,p) = \langle p, v \rangle - L(q,v) = \langle p, v \rangle + \frac{1}{4} \left( \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \right).$$

Making use of relation (2.6) we find:

$$H_{\text{deg}}(q,p) = \epsilon + \Pi A^\mu_\nu Z_{\nu\mu} + \frac{1}{4} \left( \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \right).$$

Then, the Hamiltonian function (2.11) is given by:

$$H_{\text{deg}}(q,p) = \epsilon - \frac{1}{4} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi A^\mu_\nu \Pi A^\rho_\nu \quad \text{with} \quad \Pi A^\mu_\nu = \eta^{\mu\lambda} \eta^{\nu\sigma} F_{\lambda\sigma} = F^{\mu\nu} \quad (2.11)$$

In order to obtain the generalized Hamilton equations $\mathbf{X} \cdot \Omega_{\text{DDW}} = (-1)^n d\mathcal{H}$, we need to compute $dH_{\text{deg}}(q,p)$, the differential of the Hamiltonian function. Since we work with a degenerate Legendre transform a naive use of the general method leads to incorrect equations of motion. We have:

$$dH_{\text{deg}}(q,p) = d\epsilon - \frac{1}{4} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi A^\mu_\nu \Pi A^\rho_\nu = d\epsilon - \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi A^\rho_\nu d\Pi A^\mu_\nu,$$

which describes the right hand side of the Hamilton equations (2.12).

$$X \cdot \Omega_{\text{DDW}} = (-1)^n d\mathcal{H} \quad (2.12)$$

Let us denote $\forall 1 \leq \alpha \leq 4$:

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + \Theta_\alpha_\mu \frac{\partial}{\partial A^\mu_\nu} + \Upsilon_\alpha \frac{\partial}{\partial \epsilon} + \Sigma_\alpha A^\mu_\nu \frac{\partial}{\partial \Pi A^\rho_\nu}. \quad (2.13)$$

Then we consider a n-vector field $X = X_1 \wedge X_2 \wedge X_3 \wedge X_4 \in \Lambda^4 T^* \mathcal{M}_{\text{DDW}}$.

**Lemma 2.1.** Let $X$ be a $(n-1)$-vector field and let $\{d\rho_i\}_{1 \leq i \leq n}$ be a set of $n$ 1-forms. We have:

$$X \cdot (\bigwedge_{1 \leq i \leq n} d\rho_i) = X \cdot d\rho_1 \wedge \cdots \wedge d\rho_n = \sum_j (-1)^{j+1} (d\rho_1 \wedge \cdots \wedge d\rho_{j-1} \wedge d\rho_{j+1} \wedge \cdots \wedge d\rho_n)(X)d\rho_j$$

Thanks to lemma (2.1), the left side of the Hamilton equations (2.12) is written:

$$X \cdot \Omega_{\text{DDW}} = X \cdot (d\epsilon \wedge d\eta + d\Pi A^\mu_\nu \wedge dA^\mu_\nu \wedge dA^\rho_\nu)$$

$$= d\eta(X)d\epsilon - (d\epsilon \wedge d\eta_\rho)(X)dx^\rho + (dA^\mu_\nu \wedge d\eta_\nu)(X)d\Pi A^\mu_\nu$$

$$- (d\Pi A^\mu_\nu \wedge d\eta_\nu)(X)dA^\mu_\nu + (d\Pi A^\mu_\nu \wedge dA^\mu_\nu \wedge d\eta_\rho_\nu)(X)dx^\rho$$

So that we obtain:

$$X \cdot \Omega_{\text{DDW}} = d\epsilon - \Upsilon_\rho_\mu dx^\rho + \Theta_\nu_\mu d\Pi A^\mu_\nu - \Sigma_\mu A^\rho_\nu dA^\mu_\nu + (\Sigma_\mu A^\rho_\nu \Theta_\nu_\mu - \Sigma_\nu A^\rho_\nu \Theta_\mu_\nu)dx^\rho \quad (2.14)$$
The decomposition on the different forms $d\Pi^{A,\nu}$, $d\epsilon$, $dA_\mu$ and $dx^\rho$ gives:

$$
\begin{align*}
-\Theta_{\nu\mu} &= -\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\Pi^{A,\sigma} \\
-\gamma_{\nu}^{A,\mu} &= 0
\end{align*}
$$

and

$$
-\gamma_{\rho} + \left( \gamma_{\rho}^{A,\nu}\Theta_{\nu\mu} - \gamma_{\nu}^{A,\mu}\Theta_{\rho\mu} \right) = 0.
$$

that is equivalent to:

$$
\begin{align*}
\partial_\nu A_\nu &= \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\Pi^{A,\sigma} \\
\partial_\mu \Pi^{A,\nu} &= 0
\end{align*}
$$

The second line of the previous system gives the half of the Maxwell equations. Notice that the Legendre degenerate transform implies $\Pi^{A,\nu} = F^{\mu\nu}$ so that $\partial_\nu \Pi^{A,\nu} = \partial_\nu F^{\mu\nu} = 0$. However we cannot recover the full set of Maxwell’s equations, since

$$
\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\Pi^{A,\sigma} = \frac{1}{2}F^{\mu\nu} \neq \partial_\mu A_\nu.
$$

We are not recovering the usual Euler-Lagrange equations precisely because we work on the degenerate space. Now let us consider rather the space $\mathcal{M}_{\text{Maxwell}}$. The constraint $\Pi^{A,\nu} + \Pi^{A,\mu} = 0$ selects the authorized directions for the vector fields and the ones we are not allowed to described. In this context, the vector fields involved in the contraction with the multisymplectic form are given by (2.15). We denote $\forall 1 \leq \alpha \leq 4$:

$$
X_\alpha = \frac{\partial}{\partial x^\alpha} + \Theta_{\alpha\mu}\frac{\partial}{\partial A_\mu} + \gamma_{\alpha}\frac{\partial}{\partial \epsilon} + \gamma_{\alpha}^{A,\nu}\left( \frac{\partial}{\partial \Pi^{A,\nu}} - \frac{\partial}{\partial \Pi^{A,\mu}} \right)
$$

(2.15)

The Hamilton equations (2.12) becomes:

$$
X \mathcal{J} \Omega^{\text{DW}} = (-1)^n d\mathcal{H}.
$$

$$
X \mathcal{J} \Omega^{\text{DW}} = X \mathcal{J} \left( (d\epsilon \wedge d\eta) + d\Pi^{A,\nu} \wedge dA_\mu \wedge d\eta_\nu \right)
$$

$$
= d\eta(X) d\epsilon - (d\epsilon \wedge d\eta_\nu)(X) dx^\rho + (dA_\mu \wedge d\eta_\nu)(X) d\Pi^{A,\nu}
$$

$$
- (d\Pi^{A,\nu} \wedge d\eta_\nu)(X) dA_\mu + (d\Pi^{A,\nu} \wedge dA_\mu \wedge d\eta_\nu)(X) dx^\rho.
$$

Then, we obtain:

$$
X \mathcal{J} \Omega^{\text{DW}} = d\epsilon - \gamma_{\rho} d\rho^\rho + (\Theta_{\nu\mu} - \Theta_{\mu\nu}) d\Pi^{A,\nu} - (\gamma_{\nu}^{A,\nu} - \gamma_{\nu}^{A,\mu}) dA_\mu
$$

$$
+ \left( (\gamma_{\rho}^{A,\nu} \Theta_{\nu\mu} - \gamma_{\nu}^{A,\mu} \Theta_{\rho\mu}) + (\gamma_{\rho}^{A,\nu} \Theta_{\nu\mu} - \gamma_{\nu}^{A,\mu} \Theta_{\rho\mu}) \right) dx^\rho.
$$

The decompositions along $d\Pi^{A,\nu}$ and $dA_\mu$, gives:

$$
\begin{align*}
(\Theta_{\nu\mu} - \Theta_{\mu\nu}) &= -\eta_{\mu\rho}\eta_{\nu\sigma}\Pi^{A,\sigma} \\
-\gamma_{\nu}^{A,\mu} &= 0
\end{align*}
$$

and

$$
\begin{align*}
\partial_\mu A_\nu - \partial_\nu A_\mu &= F_{\mu\nu} \\
\partial_\nu (\Pi^{A,\nu} - \Pi^{A,\mu}) &= 0
\end{align*}
$$

(2.16)

Hence, the second line of equation (2.16) gives Maxwell’s equations:

$$
\frac{1}{2}\partial_\nu (\Pi^{A,\nu} - \Pi^{A,\mu}) = \partial_\nu \Pi^{A,\nu} = \partial_\nu F^{\mu\nu} = 0.
$$

(2.17)

**Remark.** We give a detailed calculation in the next section [1.4], - with the example of the 2D-case - for the expression $X \mathcal{J} \Omega^{\text{DW}}$, where we will consider respectively the case $X \in \forall X, \overline{X} \in \Lambda^n T\mathcal{M}_{\text{Maxwell}}$ and $X \in \forall X, \overline{X} \in \Lambda^n T\mathcal{M}_{\text{DW}}$. 


2.3 Maxwell theory as an \( n \)-phase space

We refer to the work of J. Kijowski [37] for the treatment of Maxwell’s theory in the setting of a \( n \)-phase space. Due to the abelian feature of the Maxwell gauge theory, this treatment is essentially the same that the one exposed in the previous section. the notion of a \( n \)-phase space, inspired by J. Kijowski and W. Szczyrba [40], and developed further by F. Hélein [22] [23].

**Definition 2.3.1.** A \( n \)-phase space is a triple \((\mathcal{M}, \Omega, \beta)\) where \(\mathcal{M}\) is a smooth manifold, \(\Omega\) is a closed \((n + 1)\)-form and \(\beta\) is an everywhere non-vanishing \(n\)-form.

For a \(n\)-phase space \((\mathcal{M}, \Omega, \beta)\), a Hamiltonian \(n\)-curve is pictured as an oriented \(n\)-submanifold which satisfies:

\[
\forall m \in \Gamma, \forall X \in \Lambda^n T_m \Gamma \quad X \cdot \Omega_m = 0 \quad \text{and} \quad \forall m \in \Gamma, \exists X \in \Lambda^n T_m \Gamma \quad X \cdot \beta_m \neq 0.
\]

The last condition is an independence condition. We can canonically construct \(n\)-phase space data by means of the hypersurface of a multisymplectic manifold. We recall that a premultisymplectic \(n\)-form is closed but may be degenerate. In the general picture of a \(n\)-phase space we express dynamics on a level set of \(H\). We recover the dynamical equations in the premultisymplectic case (2.18) - see F. Hélein [21] [22] [23].

\[
\forall \Xi \in C^\infty(\mathcal{M}, T_m \mathcal{M}), \quad (\Xi \cdot \Omega)|_{\Gamma} = 0 \quad \text{and} \quad \beta|_{\Gamma} \neq 0. \quad (2.18)
\]

The canonical premultisymplectic form is given by:

\[
\theta_{(q,p)}^{\text{pre-multi. Maxwell}} := \theta_{(q,p)}^{\text{Maxwell}} \big|_{H=0} = \epsilon d\eta + \Pi^{A_{\mu} \nu} \, dA_{\mu} \wedge d\eta_{\nu} \big|_{H=0} \quad (2.19)
\]

We have \(H(q, p) = \epsilon - 1/4 \eta_{\mu \rho} \eta_{\sigma \nu} \Pi^{A_{\mu} \nu} \Pi^{A_{\rho} \sigma}\) with \(\Pi^{A_{\mu} \nu} = \eta^{\mu \lambda} \eta^{\nu \sigma} \Gamma_{\lambda \sigma} = F^{\mu \nu}\). The imposition of the Hamiltonian constraint \(H = 0\) leads us to consider \(\epsilon = 1/4 \eta_{\mu \rho} \eta_{\sigma \nu} \Pi^{A_{\mu} \nu} \Pi^{A_{\rho} \sigma} = -H(A_{\mu}, A_{\nu}, \Pi^{A_{\mu} \nu})\). Hence, the premultisymplectic canonical forms \(\theta_{(q,p)}^{\text{pre-multi}}\) and \(\Omega_{(q,p)}^{\text{pre-multi}}\) are respectively written:

\[
\theta_{(q,p)}^{\text{pre-multi}} = \frac{1}{4} \eta_{\mu \rho} \eta_{\sigma \nu} \Pi^{A_{\mu} \nu} \Pi^{A_{\rho} \sigma} \, d\eta + \Pi^{A_{\mu} \nu} \, dA_{\mu} \wedge d\eta_{\nu}
\]

and:

\[
\Omega_{(q,p)}^{\text{pre-multi}} = d\theta_{(q,p)}^{\text{pre-multi}}
\]

\[
= \frac{1}{2} \eta_{\mu \rho} \eta_{\sigma \nu} \Pi^{A_{\rho} \sigma} \, d\Pi^{A_{\mu} \nu} \wedge d\eta + d\Pi^{A_{\mu} \nu} \wedge dA_{\mu} \wedge d\eta_{\nu}.
\]

We denote, in order to simplify the notations: \(\theta_{(q,p)}^{\text{pre-multi}} = \theta_{(q,p)}^{\mathcal{O}}\) and \(\Omega_{(q,p)}^{\text{pre-multi}} = \Omega_{(q,p)}^{\mathcal{O}}\). Therefore, we consider the theory on the premultisymplectic Maxwell space \(\mathcal{M}_{\text{Maxwell}}^{\mathcal{O}}\):

\[
\mathcal{M}_{\text{Maxwell}}^{\mathcal{O}} = \left\{ (x, A, p) \in \mathcal{M}_{\text{BDW}} \mid \Pi^{A_{\mu} \nu} + \Pi^{A_{\nu} \mu} = 0 \quad \text{and} \quad \epsilon = \frac{1}{4} \eta_{\mu \rho} \eta_{\sigma \nu} \Gamma^{A_{\mu} \nu} \Pi^{A_{\rho} \sigma} \right\} \quad (2.20)
\]

\(^6\)We can construct canonically a \(n\)-premultisymplectic manifold \((\mathcal{M}, \Omega|_{\mathcal{M}^0}, \beta = \eta \cdot \Omega|_{\mathcal{M}^0})\). Here the \(\Omega|_{\mathcal{M}^0} = H^{-1}(0) := \{ (q, p) \in \Omega|_{\mathcal{M}} \mid H(q, p) = 0 \}\) and \(\eta\) is a vector field such that \(dH(\eta) = 1\). In this case we observe the connection between relativistic dynamical systems and the treatment of the Hamiltonian constraint.
We observe the following inclusion of spaces: \( \mathcal{M}^\circ_{\text{Maxwell}} \subset \mathcal{M}_{\text{Maxwell}} \subset \mathcal{M}_{\text{DDW}} \). The generalized Hamilton equations are given with the calculation of \( X \perp \Omega^\circ \):

\[
X \perp \Omega^\circ = X \perp (1/2\eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma} \wedge \Pi^A_{\nu\nu} \wedge d\eta) + X \perp (d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\nu)
\]

\[
= 1/2\eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma} d\eta(X) d\Pi^A_{\mu\nu} - (1/2\eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma} \wedge \eta_\rho)(X) dx^\rho
\]

\[
+ (dA_\mu \wedge d\eta_\nu)(X) d\Pi^A_{\mu\nu} - (d\Pi^A_{\mu\nu} \wedge d\eta_\nu)(X) dA_\mu + (d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\rho)(X) dx^\rho
\]

So that :

\[
X \perp \Omega^\circ = dx - \Upsilon_\rho dx^\rho + (\Theta_{\nu\mu} - \Theta_{\mu\nu} + \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma}) d\Pi^A_{\mu\nu} - (\Upsilon^A_{\mu\nu} - \Upsilon^A_{\nu\mu}) dA_\mu
\]

\[
+ (\Upsilon^A_{\rho\mu} \Theta_{\nu\mu} - \Upsilon^A_{\nu\mu} \Theta_{\rho\mu}) - (\Upsilon^A_{\rho\mu} \Theta_{\nu\mu} - \Upsilon^A_{\nu\mu} \Theta_{\rho\mu}) - \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma} \gamma^A_{\rho\mu}
\]

Once again, the decompositions along \( d\Pi^A_{\mu\nu} \) and \( dA_\mu \) gives :

\[
\begin{align*}
(\Theta_{\nu\mu} - \Theta_{\mu\nu} + \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^A_{\mu\sigma}) &= 0 \\
-(\Upsilon^A_{\rho\mu} - \Upsilon^A_{\nu\mu}) &= 0
\end{align*}
\]

We recover (2.16) and then the same conclusions.

### 3 Algebraic Observables and observable functionals

We being this section with the definition (3.0.2) of algebraic observable \((n-1)\)-forms and the set \( \mathfrak{sp}_n(\mathcal{M}) \) of infinitesimal symplectomorphisms of the related multisymplectic manifold \((\mathcal{M}, \Omega)\).

**Definition 3.0.2.** Let \((\mathcal{M}, \Omega)\) be an \(n\)-multisymplectic manifold. A \((n-1)\)-form \( \varphi \) is called an algebraic observable \((n-1)\)-form if and only if there exists \( \Xi_{\varphi} \) such that \( \Xi \perp \Omega + d\varphi = 0 \).

We denote \( \mathfrak{v}^{n-1}_\circ(\mathcal{M}) \) the set of all algebraic observable \((n-1)\)-forms. This reflects the symmetry point of view. It is the natural analogue to the question of the Poisson bracket for classical mechanics. Then, \( \forall \varphi, \rho \in \mathfrak{v}^{n-1}_\circ(\mathcal{M}) \), we define the Poisson bracket (3.1):

\[
\{ \varphi, \rho \} = \Xi_{\rho} \wedge \Xi_{\varphi} \perp \Omega = -\Xi_{\rho} \perp d\varphi = \Xi_{\varphi} \perp d\rho
\]

where, \( \{ \varphi, \rho \} \in \mathfrak{v}^{n-1}_\circ(\mathcal{M}) \) and the bracket (3.1) satisfy the antisymmetry property :

\[
\{ \varphi, \rho \} + \{ \rho, \varphi \} = 0,
\]

and Jacobi structure modulo an exact term \( \forall \varphi, \rho, \eta \in \mathfrak{v}^{n-1}_\circ(\mathcal{M}) \):

\[
\{ \{ \varphi, \rho \} \eta \} + \{ \{ \rho, \eta \} \varphi \} + \{ \{ \eta, \varphi \} \rho \} = d(\xi_{\varphi} \wedge \xi_{\rho} \wedge \xi_{\eta} \perp \Omega).
\]

Since we have defined an infinitesimal symplectomorphism of \((\mathcal{M}, \Omega)\) to be a vector field \( \Xi \in \Gamma(\mathcal{M}, T\mathcal{M}) \) such that \( \mathcal{L}_{\Xi} \Omega = 0 \), using the Cartan formula, we obtain :

\[
\mathcal{L}_{\Xi} \Omega = d(\Xi \perp \Omega) + \Xi \perp d\Omega = 0.
\]
Now, since the multisymplectic \((n+1)\)-form is closed \(d\Omega = 0\), this relation is equivalent to \(d(\Xi \cup \Omega) = 0\). We are looking for vector fields \(\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})\) such that \(d(\Xi \cup \Omega) = 0\). We denote by \(\mathfrak{sp}_c(\mathcal{M})\) the set of infinitesimal symplectomorphisms of the multisymplectic manifold \((\mathcal{M}, \Omega)\):

\[
\mathfrak{sp}_c(\mathcal{M}) = \left\{ \Xi \in \Gamma(\mathcal{M}, T\mathcal{M}) \mid d(\Xi \cup \Omega) = 0 \right\}.
\]

(3.2)

### 3.1 Some algebraic observable \((n - 1)\)-forms

We are interested in the algebraic observable \((n - 1)\)-forms and their related infinitesimal symplectomorphisms on the multisymplectic manifold \((\mathcal{M}_{\text{Maxwell}}, \Omega^{\text{DDW}})\). First we take some simple examples and we enter in the general setting step by step. We find two types of algebraic observable \((n-1)\)-forms: the (generalized) positions \((n-1)\)-forms and the (generalized) momenta \((n-1)\)-forms. Let us begin with the following algebraic observable \((n-1)\)-forms:

\[
P^\mu = dx^\mu \wedge \pi, \quad P^\mu_\phi = \phi(x) dx^\mu \wedge \pi, \quad Q_{\mu \nu} = A \wedge d\eta_{\mu \nu} \quad \text{and} \quad Q^0_{\mu \nu} = \psi(x) A \wedge d\eta_{\mu \nu}.
\]

We denote the Faraday \((n-2)\)-form \([24, 30, 31]\) by:

\[
\pi = \frac{1}{2} \Pi^A_{\mu \nu} d\eta_{\mu \nu} = \frac{1}{2} \sum_{\mu, \nu} \Pi^A_{\mu \nu} \frac{\partial}{\partial x^\mu} \left( \frac{\partial}{\partial x^\nu} \right) d\eta
\]

and the potential 1-form \(A = A_\mu dx^\mu\). The couple of variables \((A, \pi)\) depicts the canonical variables for the Maxwell theory \([24, 25, 26, 27, 30, 31]\). Notice that the Faraday \((n-2)\)-form is also written \(\star dA = \eta^\mu{}^\lambda \eta^\nu{}^\sigma (\partial_\mu A_\nu - \partial_\nu A_\mu) d\eta_{\lambda \sigma}\). First, let us focus on \(P^\mu = dx^\mu \wedge \pi\). We have:

\[
P^\rho = dx^\rho \wedge \pi = dx^\rho \wedge \left( \frac{1}{2} \Pi^A_{\mu \nu} d\eta_{\mu \nu} \right) = \frac{1}{2} \Pi^A_{\mu \nu} \left( \delta^\rho_\mu d\eta_{\nu} - \delta^\rho_\nu d\eta_{\mu} \right) = \frac{1}{2} \left( \Pi^A_{\mu \nu} d\eta_{\nu} - \Pi^A_{\mu \nu} d\eta_{\mu} \right)
\]

Using the constraint \(\Pi^A_{\mu \nu} = -\Pi^A_{\nu \mu}\), we obtain: \(P^\mu = \Pi^A_{\mu \nu} d\eta_{\nu}\). Now we compute the exterior derivative \(dP^\mu = d(dx^\rho \wedge \pi) = d(\Pi^A_{\mu \nu} d\eta_{\nu}) = d\Pi^A_{\mu \nu} \wedge d\eta_{\nu}\). If we consider \(\Xi(P^\mu) = \partial / \partial A_\mu\), we have \(dP^\mu = -\Xi(P^\mu) \land \Omega^{\text{DDW}}\) as shown by the following straightforward calculation:

\[
\Xi(P^\mu) \land \Omega^{\text{DDW}} = \frac{\partial}{\partial A_\mu} \left( d\epsilon \land d\eta + d\Pi^A_{\mu \nu} \land dA_\mu \land d\eta_{\nu} \right) = -d\Pi^A_{\mu \nu} \land d\eta_{\nu}.
\]

We prefer to consider \(P_\phi = \phi_\mu(x) \Pi^A_{\mu \nu} d\eta_{\nu}\). The exterior derivative \(dP_\phi\) is given by:

\[
dP_\phi = d\left( \phi_\mu(x) \Pi^A_{\mu \nu} \right) \land d\eta_{\nu} = \left( \Pi^A_{\mu \nu} \frac{\partial \phi_\mu}{\partial x^\alpha}(x) dx^\alpha + \phi_\mu(x) d\Pi^A_{\mu \nu} \right) \land d\eta_{\nu} = \Pi^A_{\mu \nu} \frac{\partial \phi_\mu}{\partial x^\nu}(x) d\eta_{\nu} + \phi_\mu(x) d\Pi^A_{\mu \nu} \wedge d\eta_{\nu}.
\]

The related infinitesimal symplectomorphism is denoted by \(\Xi(P_\phi)\):

\[
\Xi(P_\phi) = \phi_\mu(x) \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \phi_\mu}{\partial x^\nu}(x) \Pi^A_{\mu \nu} \right) \frac{\partial}{\partial \epsilon}
\]
Let us compute the contraction $\Xi(\mathbf{P}_\phi) \lrcorner \Omega^{\text{DDW}}$:

$$
\Xi(\mathbf{P}_\phi) \lrcorner \Omega^{\text{DDW}} = \left( \phi_\mu(x) \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \phi_\mu(x)}{\partial x^\nu} A^{\nu}_{\mu} \right) \frac{\partial}{\partial \xi} \right) \lrcorner \left( d\xi \wedge d\eta + dA^{\nu}_{\mu} \wedge A_\mu \wedge d\eta_{\nu} \right)
$$

$$
= - \left( \frac{\partial \phi_\mu(x)}{\partial x^\nu} A^{\nu}_{\mu} \right) d\eta - \phi_\mu(x) dA^{\nu}_{\mu} \wedge d\eta_{\nu} = -d\mathbf{P}_\phi
$$

Now we focus on some algebraic position $(n-1)$-forms: $Q^\psi = \frac{1}{2} \psi^{\mu\nu}(x) A \wedge d\eta_{\mu\nu}$ with $\psi^{\mu\nu}(x)$ a real function which is antisymmetric in the indices $\mu, \nu$.

$$
Q^\psi = \frac{1}{2} \psi^{\mu\nu}(x) A_\mu dx^\rho \wedge d\eta_{\mu\nu} = \frac{1}{2} \psi^{\mu\nu}(x) A_\mu \left( \delta_\mu^\rho d\eta_{\nu} - \delta_\nu^\rho d\eta_{\mu} \right) = \frac{1}{2} \psi^{\mu\nu}(x) \left( A_\mu d\eta_{\nu} - A_\nu d\eta_{\mu} \right)
$$

Since $\psi^{\mu\nu} = -\psi^{\nu\mu}$ then $Q^\psi = \psi^{\mu\nu}(x) A_\mu d\eta_{\nu}$. We compute $dQ^\psi$:

$$
dQ^\psi = d \left( \psi^{\mu\nu}(x) A_\mu d\eta_{\nu} \right) A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\sigma}(x) dx^\sigma \wedge d\eta_{\nu} + \psi^{\mu\nu}(x) dA_\mu \wedge d\eta_{\nu}
$$

$$
= A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x) d\eta + \psi^{\mu\nu}(x) dA_\mu \wedge d\eta_{\nu}
$$

The related infinitesimal symplectomorphisms are denoted $\Xi(Q^\psi)$ and are given by:

$$
\Xi(Q^\psi) = - \left( (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \right) \frac{\partial}{\partial \xi} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi^{\mu\nu}} \right)
$$

Let us compute $\Xi(Q^\psi) \lrcorner \Omega^{\text{DDW}}$:

$$
\Xi(Q^\psi) \lrcorner \Omega^{\text{DDW}} = - \left( (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \right) \frac{\partial}{\partial \xi} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi^{\mu\nu}} \right) \lrcorner \left( d\xi \wedge d\eta + dA^{\nu}_{\mu} \wedge dA_\mu \wedge d\eta_{\nu} \right)
$$

$$
= - (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x) d\eta - \psi^{\mu\nu}(x) dA_\mu \wedge d\eta_{\nu} = -dQ^\psi
$$

We summarize the results relating the algebraic observable $(n-1)$-forms $\mathbf{P}_\phi, Q^\psi$ and their related infinitesimal symplectomorphisms $\Xi(\mathbf{P}_\phi), \Xi(Q^\psi)$:

$$
\begin{align*}
\mathbf{P}_\phi &= \phi_\mu(x) A^{\nu}_{\mu} d\eta_{\nu} & \Xi(\mathbf{P}_\phi) &= \phi_\mu(x) \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \phi_\mu(x)}{\partial x^\nu} A^{\nu}_{\mu} \right) \frac{\partial}{\partial \xi} \\
Q^\psi &= \frac{1}{2} \psi^{\mu\nu}(x) A \wedge d\eta_{\mu\nu} & \Xi(Q^\psi) &= - (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial \xi} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi^{\mu\nu}} )
\end{align*}
$$

(3.3)

Let notice that if we work in the pre-multisymplectic case $(\mathcal{M}_0, \Omega^0)$ we have:

$$
\begin{align*}
\mathbf{P}_\phi^0 &= \Pi^{\nu}_{\mu} d\eta_{\nu} & \Xi(\mathbf{P}_\phi^0) &= \phi_\mu(x) \frac{\partial}{\partial A_\mu} \\
Q^\psi_0 &= \frac{1}{2} \psi^{\mu\nu}(x) A \wedge d\eta_{\mu\nu} & \Xi(Q^\psi_0) &= - \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi^{\mu\nu}}
\end{align*}
$$

(3.4)

We need a more embracing view to describe more precisely the general conditions on the functions $\phi_\mu(x)$ and $\psi^{\mu\nu}(x)$ and also to consider more general choice of functions. In doing so we provide a deeper description of the infinitesimal symplectomorphisms $\Xi(Q^\psi), \Xi(\mathbf{P}_\phi), \Xi(Q^\psi)$ and $\Xi(\mathbf{P}_\phi^0)$. It is the subject of the following sections (3.5) - (3.7). Before going to that step, we give in the next section (3.2) the Poisson bracket structure in this simple case. The objects of interest are $\{Q^\psi, Q^\phi\}, \{P_\phi, P_{\phi^*}\}$ and $\{Q^\psi, P_\phi\}$. 

**Multisymplectic Maxwell Theory**
3.2 Poisson Bracket for algebraic \((n - 1)\)-forms

We begin with the following proposition:

**Proposition 3.1.** Let \(\phi_\mu(x), \tilde{\phi}_\mu(x)\) and \(\psi^{\mu\nu}(x), \tilde{\psi}^{\mu\nu}(x)\) smooth functions with \(\psi^{\mu\nu}(x) = -\psi^{\nu\mu}(x)\) and \(\tilde{\psi}^{\mu\nu}(x) = -\tilde{\psi}^{\nu\mu}(x)\). For \((\mathcal{M}_{\text{Maxwell}}, \Omega^{\text{DOW}})\) the set of canonical Poisson brackets is given by:

\[
\{Q^\phi, Q^{\tilde{\phi}}\} = 0,
\]

\[
\{P_\phi, P_{\tilde{\phi}}\} = 0
\]

\[
\{Q^\psi, P_\phi\} = -\psi^{\mu\nu}(x)\phi_\mu(x)d\eta_\nu.
\]

This corresponds to the mathematical setting of the traditional Poisson bracket for algebraic \((n - 1)\)-forms: \(\mathcal{P}^{n-1}_\circ(\mathcal{M}) \times \mathcal{P}^{n-1}_\circ(\mathcal{M}) \rightarrow \mathcal{P}^{n-1}_\circ(\mathcal{M})\). Let us consider two algebraic position observable \((n - 1)\)-forms \(Q^\psi\) and \(Q^{\tilde{\psi}}\) given by (3.3):

\[
Q^\psi = \psi^{\mu\nu}(x)A_\mu(x)d\eta_\nu, \quad \text{and} \quad Q^{\tilde{\psi}} = \tilde{\psi}^{\mu\nu}(x)A_\mu(x)d\eta_\nu.
\]

We compute the bracket:

\[
\{Q^\psi, Q^{\tilde{\psi}}\} = \Xi(Q^\psi) \cup \Xi(Q^{\tilde{\psi}}) \cup \Omega^{\text{DOW}}
\]

\[
= -\Xi(Q^\psi) \cup \left( (A_\mu \frac{\partial \tilde{\psi}^{\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial \epsilon} + \tilde{\psi}^{\mu\nu}(x) \frac{\partial}{\partial \Pi A_\mu} ) \cup (d\epsilon \wedge d\eta + d\Pi A_\mu \wedge dA_\mu \wedge d\eta_\nu) \right)
\]

\[
= \left( (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial \epsilon} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi A_\mu} ) \cup (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} )d\eta + \psi^{\mu\nu}(x)dA_\mu \wedge d\eta_\nu \right) = 0
\]

Now we compute \(\{P_\phi, P_{\tilde{\phi}}\}\) where the algebraic \((n - 1)\)-forms \(P_\phi\) and the related infinitesimal symplectomorphisms \(\Xi(P_\phi)\) are given by (3.3). Hence we have the internal bracket:

\[
\{P_\phi, P_{\tilde{\phi}}\} = \Xi(P_\phi) \cup \Xi(P_{\tilde{\phi}}) \cup \Omega^{\text{DOW}}
\]

\[
= -\Xi(P_\phi) \cup \left( -\tilde{\phi}_\mu(x) d\Pi A_\mu \wedge d\eta_\nu - (\frac{\partial \tilde{\phi}_\mu(x)}{\partial x^\nu} \Pi A_\mu) d\eta_\nu \right) = 0.
\]

Finally, we compute the last bracket \(\{Q^\psi, P_\phi\}\):

\[
\{Q^\psi, P_\phi\} = -\Xi(Q^\psi) \cup \left( -\phi_\mu(x) d\Pi A_\mu \wedge d\eta_\nu - \frac{\partial \phi_\mu(x)}{\partial x^\nu} \Pi A_\mu d\eta_\nu \right)
\]

\[
= -\left( (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial \epsilon} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi A_\mu} ) \cup (\phi_\mu(x) d\Pi A_\mu \wedge d\eta_\nu - (\frac{\partial \phi_\mu(x)}{\partial x^\nu} \Pi A_\mu) d\eta_\nu \right)
\]

\[
= -\psi^{\mu\nu}(x)\phi_\mu(x)d\eta_\nu.
\]

Finally,

\[
\{Q^\psi, P_\phi\} = \Xi(Q^\psi) \cup dP_\phi
\]

\[
= -\left( (A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu} \frac{\partial}{\partial \epsilon} + \psi^{\mu\nu}(x) \frac{\partial}{\partial \Pi A_\mu} ) \cup (\Pi A_\mu \frac{\partial \phi_\mu(x)}{\partial x^\nu} + \phi_\mu(x) d\Pi A_\mu \wedge d\eta_\nu) \right)
\]
So that \( \{ Q^\psi, P_\phi \} = -\psi^{\mu\nu}(x)\phi_\mu(x)d\eta_\nu \). We summarize our results and recover the proposition \( 3.1 \):

\[
\{ Q^\psi, Q^{\psi^*} \} = \{ P_\phi, P_{\phi^*} \} = 0 \quad \text{and} \quad \{ Q^\psi, P_\phi \} = -\psi^{\mu\nu}(x)\phi_\mu(x)d\eta_\nu
\]

### 3.3 All algebraic observable \((n - 1)\)-forms

In this section we describe the set of all algebraic \((n - 1)\)-forms and their related infinitesimal symplectomorphisms \( \Xi \in \Gamma(\mathcal{M}_{\text{Maxwell}}, T\mathcal{M}_{\text{Maxwell}}) \). First we introduce the notations. We consider \( \zeta \in \mathfrak{f} \) and we denote:

\[
\zeta = X^\nu(x, A)\frac{\partial}{\partial x^\nu} + \Theta_\mu(x, A)\frac{\partial}{\partial A_\mu}
\]

with \( X^\nu : \mathfrak{f} \to \mathbb{R} \) and \( \Theta_\mu : \mathfrak{f} \to \mathbb{R} \) are smooth functions on \( \mathfrak{f} \). Hence, we denote \( \Xi_{\text{DDW}} \in \Gamma(\mathcal{M}_{\text{DDW}}, T\mathcal{M}_{\text{DDW}}) \):

\[
\Xi_{\text{DDW}} = X^\nu(q, p)\frac{\partial}{\partial x^\nu} + \Theta_\mu(q, p)\frac{\partial}{\partial A_\mu} + \Upsilon(q, p)\frac{\partial}{\partial \epsilon} + \Upsilon^A_{\mu\nu}(q, p)\frac{\partial}{\partial \Pi^A_{\mu\nu}}.
\]

We also denote vector fields \( \Xi = \Xi_{\text{Maxwell}} \in \Gamma(\mathcal{M}_{\text{Maxwell}}, T\mathcal{M}_{\text{Maxwell}}) \):

\[
\Xi_{\text{DDW}} = X^\nu(q, p)\frac{\partial}{\partial x^\nu} + \Theta_\mu(q, p)\frac{\partial}{\partial A_\mu} + \Upsilon(q, p)\frac{\partial}{\partial \epsilon} + \Upsilon^A_{\mu\nu}(q, p)\left(\frac{\partial}{\partial \Pi^A_{\mu\nu}} - \frac{\partial}{\partial \Pi^A_{\nu\mu}}\right).
\]

The objects \( X^\nu(q, p), \Theta_\mu(q, p), \Upsilon(q, p) \) and \( \Upsilon^A_{\mu\nu}(q, p) \) are smooth functions on \( \mathcal{M}_{\text{Maxwell}} \subset \mathcal{M}_{\text{DDW}} \subset \Lambda^n(T^*\mathcal{X}) \), with values in \( \mathbb{R} \). Now we evaluate the expression \( \Xi \cdot \Omega_{\text{DDW}}^\square \) :

\[
\Xi \cdot \Omega_{\text{DDW}}^\square = \Upsilon d\eta - X^\nu d\epsilon \wedge d\eta_\nu + \Upsilon^A_{\mu\nu} dA_\mu \wedge d\eta_\nu - \Theta_\mu d\Pi^A_{\mu\nu} \wedge d\eta_\nu + X^\rho d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\rho\nu.
\]

We lift relations from the definition of a symplectomorphism \( d(\Xi \cdot \Omega_{\text{DDW}}^\square) = 0 \). We make the following calculation :

\[
d(\Xi \cdot \Omega_{\text{DDW}}^\square) = d\Upsilon \wedge d\eta - dX^\nu \wedge d\epsilon \wedge d\eta_\nu + d\Upsilon^A_{\mu\nu} dA_\mu \wedge d\eta_\nu - d\Theta_\mu \wedge d\Pi^A_{\mu\nu} \wedge d\eta_\nu + dX^\rho \wedge d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\rho\nu.
\]

Then, we write this expression in the form of a sum \( d(\Xi \cdot \Omega_{\text{DDW}}^\square) = \sum_i \iota_i \) with each terms \( \iota_i \) given by:

\[
\begin{align*}
\iota_1 &= d\Upsilon \wedge d\eta \\
\iota_2 &= -dX^\nu \wedge d\epsilon \wedge d\eta_\nu \\
\iota_3 &= d\Upsilon^A_{\mu\nu} dA_\mu \wedge d\eta_\nu \\
\iota_4 &= -d\Theta_\mu \wedge d\Pi^A_{\mu\nu} \wedge d\eta_\nu \\
\iota_5 &= dX^\rho \wedge d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\rho\nu.
\end{align*}
\]

Since \( d\Upsilon = \frac{\partial \Upsilon}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon}{\partial A_\beta} dA_\beta + \frac{\partial \Upsilon}{\partial \epsilon} d\epsilon + \frac{\partial \Upsilon}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \), the first term \( \iota_1 \) is written:

\[
\iota_1 = \frac{\partial \Upsilon}{\partial A_\beta} dA_\beta \wedge d\eta + \frac{\partial \Upsilon}{\partial \epsilon} d\epsilon \wedge d\eta + \frac{\partial \Upsilon}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \wedge d\eta
\]
Moreover, since $dX^\rho = \frac{\partial X^\rho}{\partial x^\alpha} dx^\alpha + \frac{\partial X^\rho}{\partial A_\beta} dA_\beta + \frac{\partial X^\rho}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha}$, the term $\iota_2$ is written:

$$
\iota_2 = -\frac{\partial X^\nu}{\partial x^\alpha} dx^\alpha \wedge de \wedge d\eta_\nu - \frac{\partial X^\nu}{\partial A_\beta} dA_\beta \wedge de \wedge d\eta_\nu - \frac{\partial X^\nu}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge de \wedge d\eta_\nu.
$$

(3.9)

whereas the term $\iota_5$ is written:

$$
\iota_5 = \frac{\partial X^\rho}{\partial x^\alpha} dx^\alpha \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu + \frac{\partial X^\rho}{\partial A_\beta} dA_\beta \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu
$$

$$
+ \frac{\partial X^\rho}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu
$$

(3.10)

due to $d\Upsilon^{A_\mu \nu} = \frac{\partial \Upsilon^{A_\mu \nu}}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial A_\beta} dA_\beta + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha}$ we expand the term $\iota_3$:

$$
\iota_3 = -(\frac{\partial \Upsilon^{A_\mu \nu}}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial A_\beta} dA_\beta + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha}) dA_\mu \wedge d\eta_\nu
$$

$$
+ (\frac{\partial \Upsilon^{A_\mu \nu}}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial A_\beta} dA_\beta + \frac{\partial \Upsilon^{A_\mu \nu}}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha}) dA_\mu \wedge d\eta_\nu
$$

(3.11)

and finally $d\Theta_\mu = \frac{\partial \Theta_\mu}{\partial x^\alpha} dx^\alpha + \frac{\partial \Theta_\mu}{\partial A_\beta} dA_\beta + \frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha}$ gives the last term $\iota_4$

$$
\iota_4 = \frac{\partial \Theta_\mu}{\partial x^\alpha} d\Pi^{A_\beta \alpha} \wedge d\eta - \frac{\partial \Theta_\mu}{\partial A_\beta} dA_\beta \wedge d\Pi^{A_\beta \alpha} \wedge d\eta_\nu - \frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge dA_\mu \wedge d\eta_\nu
$$

$$
- \frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\beta \alpha} \wedge d\eta_\nu.
$$

(3.12)

The decomposition of the terms (3.8)-(3.12) on the different $(n+1)$-forms involves $de \wedge d\eta$, $d\Pi^{A_\mu \nu} \wedge d\eta$, $dA_\mu \wedge d\eta$, $de \wedge dA_\mu \wedge d\eta_\nu$, $d\Pi^{A_\beta \alpha} \wedge dA_\mu \wedge d\eta_\nu$, $de \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu$, $d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge d\eta_\nu$. We now describe more precisely the different terms. The decomposition involves the following terms $j_1 - j_{12}$:

- $j_1$ is the term related to the decomposition on $[d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge d\eta_\nu]$ so that:

$$
\iota_1 = -\frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge d\eta_\nu
$$

- $j_2$ is the term related to the decomposition on $[d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu]$

$$
\iota_2 = \frac{\partial X^\rho}{\partial \Pi^{A_\beta \alpha}} d\Pi^{A_\beta \alpha} \wedge d\Pi^{A_\mu \nu} \wedge dA_\mu \wedge d\eta_\rho \wedge d\eta_\nu
$$

- $j_3$ is the term related to the decomposition on $[de \wedge d\eta]$

$$
\iota_3 = \frac{\partial X^\rho}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon^{A_\beta \alpha}}{\partial x^\alpha} dx^\alpha \wedge de \wedge d\eta
$$
— \( j_4 \) is the term related to the decomposition on \([d\Pi \wedge d\eta]\)
\[
j_4 = \frac{\partial \Theta^\mu}{\partial x^\nu} d\Pi A^\mu_\alpha \wedge d\eta + \frac{\partial \Upsilon}{\partial \Pi A^\beta_\alpha} d\Pi A^\beta_\alpha \wedge d\eta
\]

— \( j_5 \) is the term related to the decomposition on \([dA \wedge d\eta]\)
\[
j_5 = \frac{\partial \Upsilon}{\partial A_\beta} dA_\beta \wedge d\eta - \left( \frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \Theta^\mu}{\partial x^\nu} \right) dA_\mu \wedge d\eta
\]

— \( j_6 \) is the term related to the decomposition on \([dX \wedge dA \wedge d\eta]\)
\[
j_6 = \left( \frac{\partial \Upsilon}{\partial e} - \frac{\partial \Theta^\mu}{\partial e} \right) de \wedge dA_\mu \wedge d\eta - \frac{\partial X^\rho}{\partial A_\beta} dA_\beta \wedge de \wedge d\eta
\]

— \( j_7 \) is the term related to the decomposition on \([dA_\beta \wedge dA_\mu \wedge d\eta_\nu]\)
\[
j_7 = \left( \frac{\partial \Upsilon}{\partial A_\alpha} - \frac{\partial \Theta^\mu}{\partial A_\alpha} \right) dA_\beta \wedge dA_\mu \wedge d\eta_\nu
\]

— \( j_8 \) is the term related to the decomposition on \([dA \wedge d\Pi \wedge d\eta_\nu]\)
\[
j_8 = \frac{\partial X^\rho}{\partial x^\alpha} dx^\alpha \wedge d\Pi A^\mu_\nu \wedge dA_\mu \wedge d\eta_\rho_\nu - \frac{\partial \Theta^\mu}{\partial A_\beta} dA_\beta \wedge d\Pi A^\nu_\mu \wedge d\eta_\nu + \frac{\partial X^\rho}{\partial \Pi A^\alpha_\beta} d\Pi A^\beta_\beta \wedge dA_\mu \wedge d\eta_\nu
\]

— \( j_9 \) is the term related to the decomposition on \([de \wedge d\Pi A^\mu_\nu \wedge d\eta_\nu]\)
\[
j_9 = \frac{\partial \Theta^\mu}{\partial e} de \wedge d\Pi A^\mu_\nu \wedge d\eta_\nu
\]

— \( j_{10} \) is the term related to the decomposition on \([d\Pi A^\alpha_\beta \wedge de \wedge d\eta_\nu]\)
\[
j_{10} = - \frac{\partial X^\rho}{\partial \Pi A^\alpha_\beta} d\Pi A^\beta_\alpha \wedge de \wedge d\eta_\nu
\]

— \( j_{11} \) is the term related to the decomposition on \([dA \wedge d\Pi \wedge dA \wedge d\eta_\rho_\nu]\)
\[
j_{11} = \frac{\partial X^\rho}{\partial A_\beta} dA_\beta \wedge d\Pi A^\mu_\nu \wedge dA_\mu \wedge d\eta_\rho_\nu
\]

— \( j_{12} \) is the term related to the decomposition on \([de \wedge d\Pi \wedge dA \wedge d\eta_\rho_\nu]\)
\[
j_{12} = \frac{\partial X^\rho}{\partial e} de \wedge d\Pi A^\mu_\nu \wedge dA_\mu \wedge d\eta_\rho_\nu
\]

(3.13)

The decomposition of \( d(\Xi \cdot J \Omega^{BOW}) \) gives us information about the dependence of the involved functions. Hence from \( j_{13} \)-\( j_{12} \), we conclude that \( X^\rho \) and \( \Theta^\mu \) are independent of variables \( \Pi A^\beta_\alpha \). Then, from the terms \( j_9-j_{10} \) and \( j_{13} \)-\( j_{12} \) we observe that \( \Theta^\mu \) are independent of the variable \( e \). From \( j_{13} \)-\( j_{11} \) we find that \( X^\rho \) is independent of the variables \( A_\beta \). From
We consider the further condition \( \Upsilon \Upsilon \Upsilon = \Upsilon \Upsilon \Upsilon (x, A) \) together with the set of equations involving more than two terms (3.13)-j. From (3.13)-j11, we observe \( \mathbf{X}^\nu = \mathbf{X}^\nu (x, A) \), so that, due to (3.13)-j6, we conclude that: \( \Upsilon^{A_\mu \nu} = \Upsilon^{A_\mu \nu} (x, \Pi) \). We don’t have any extra information on \( \Upsilon = \Upsilon (x, A, \epsilon, \Pi) \).

The functions \( \mathbf{X}^\nu, \Theta_\mu, \Upsilon, \Upsilon^{A_\mu \nu} \) are smooth functions on \( \mathcal{M}_{\text{Weyl}} \subseteq \Lambda^n T^* \mathfrak{3} \), with values in \( \mathbb{R} \) satisfy the following coordinate dependence:

\[
\mathbf{X}^\nu = \mathbf{X}^\nu (x) \quad , \quad \Theta_\mu = \Theta_\mu (x, A) \quad , \quad \Upsilon = \Upsilon (x, A, \epsilon, \Pi) \quad , \quad \Upsilon^{A_\mu \nu} = \Upsilon^{A_\mu \nu} (x, \Pi) \quad \text{(3.14)}
\]

We consider the further condition \( \Upsilon^{A_\mu \nu} (q, p) = -\Upsilon^{A_\mu \nu} (q, p) \) so that we are left with equations (3.13)-j3-j4-j5-j6:

\[
\begin{align*}
\frac{\partial \mathbf{X}^\nu}{\partial x^\mu} + \frac{\partial \Upsilon}{\partial A_\mu} &= 0 \\
\frac{\partial \Theta_\mu}{\partial x^\nu} - \frac{\partial \Upsilon^{A_\mu \nu}}{\partial A_\mu} &= 0
\end{align*}
\]

(3.15)

together with the set of equations involving more than two terms (3.13)-j8. We have the following proposition:

**Proposition 3.2.** Let \( \Xi \in \Gamma (\mathcal{M}, T\mathcal{M}) \) then \( \Xi \) satisfies \( d(\Xi \iota \Omega_{\text{DOW}}) = 0 \) if and only if \( \Xi \) is written \( \Xi = \zeta + \chi \) with

\[
\zeta = X^\nu \frac{\partial}{\partial x^\nu} + \Theta_\mu \frac{\partial}{\partial A_\mu} - \left( \epsilon \frac{\partial X^\nu}{\partial x^\nu} + \Theta_\mu \Pi^{A_\mu \nu} \right) \frac{\partial}{\partial \epsilon}
\]

\[
-\left( \Pi^{A_\mu \nu} \delta_\rho^\mu \left( \left( \frac{\partial X^\nu}{\partial x^\sigma} \right) - \delta_\sigma^\nu \frac{\partial X^\lambda}{\partial x^\lambda} \right) + \left( \frac{\partial \Theta_\mu}{\partial A_\nu} \right) - \epsilon \left( \frac{\partial X^\nu}{\partial A_\mu} \right) \right) \frac{\partial}{\partial \Pi^{A_\mu \nu}}
\]

and \( \chi = \Upsilon \frac{\partial}{\partial \epsilon} + \Upsilon^{A_\mu \alpha} \frac{\partial}{\partial \Pi^{A_\mu \alpha}} \) with \( \chi: \mathfrak{3} \rightarrow \mathbb{R} \) and \( \Upsilon^{A_\mu \alpha}: \mathfrak{3} \rightarrow \mathbb{R} \) smooth functions on \( \mathfrak{3} \) such that:

\[
\frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \Upsilon^{A_\mu \nu}}{\partial x^\nu} = 0
\]

(3.17)

Proposition (3.2) is the application of the result due to F. Hélein and J. Kouneiher [26], which is recalled in proposition 3.3. The latter describes the more general search for all algebraic observable \((n-1)\)-forms. Any infinitesimal symplectomorphism \( \Xi \in \mathfrak{sp}_n (\mathcal{M}) \) can be written in the form \( \Xi = \chi + \zeta \).

**Proposition 3.3.** If \( \mathcal{M} \) is an open subset of \( \Lambda^n T^* \mathfrak{3} \), then the set of all infinitesimal symplectomorphisms \( \Xi \) on \( \mathcal{M} \) are of the form \( \Xi = \chi + \zeta \), where

\[
\chi := \sum_{\beta_1 < \cdots < \beta_n} \chi_{\beta_1 \cdots \beta_n} (q) \frac{\partial}{\partial p_{\beta_1 \cdots \beta_n}} \quad \text{and} \quad \zeta := \sum_{\alpha} \zeta^\alpha (q) \frac{\partial}{\partial q^\alpha} - \sum_{\alpha, \beta} \frac{\partial \zeta^\alpha}{\partial q^\beta} (q) \Pi_\beta^\alpha,
\]

(3.18)

with,

1. the coefficients \( \chi_{\beta_1 \cdots \beta_n} \) are such that \( d(\chi \iota \Omega) = 0 \).
2. \( \zeta := \sum_{\alpha} \zeta^\alpha (q) \frac{\partial}{\partial q^\alpha} \) is an arbitrary vector field on \( \mathfrak{3} \).
We denote:
\[\chi = (\chi_\varepsilon - \chi_\zeta) = (\chi_{\alpha_1...\alpha_n}(q,p) = (\chi(q,p), \chi^A_{\mu}(q,p))\right\}, \quad \Xi_{\alpha_1...\alpha_n}(q,p) = \left\{\chi(q,p), \chi^A_{\mu}(q,p)\right\}. \] (3.19)

We denote:
\[\chi = \Xi - \zeta = X^\nu \frac{\partial}{\partial x^\nu} + \Theta^\mu \frac{\partial}{\partial A_\mu} + \Upsilon^\mu + \chi^A_{\mu} \frac{\partial}{\partial \Pi_{\alpha}^A_{\mu}} - \zeta. \]

We consider the expression (3.16) so that we obtain an expression for \(\chi = \Xi - \zeta\)
\[\chi = (\Upsilon - (\epsilon(\frac{\partial X^\nu}{\partial x^\nu} + \frac{\partial \Theta^\mu}{\partial x^\nu} A_\mu))) \frac{\partial}{\partial \varepsilon} + \left(\chi^A_{\mu} - \Pi^A_{\mu} \delta_\rho^\mu \left[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta_\nu^\mu \left(\frac{\partial X^\nu}{\partial x^\lambda} \right) + (\frac{\partial \Theta^\sigma}{\partial A_\mu}) \right] - \epsilon(\frac{\partial X^\nu}{\partial A_\mu}) \right) \frac{\partial}{\partial \Pi^A_{\mu}}. \] (3.20)

As announced in the proposition (3.3), we have coefficients of \(\chi\) such that \(d(\chi \cdot \Omega^{DDW}) = 0\).

Since \(\chi = \chi^\varepsilon \frac{\partial}{\partial \varepsilon} + \chi^\Pi \frac{\partial}{\partial \Pi_{\alpha}^A_{\mu}},\) the interior product of \(\chi\) with \(\Omega^{DDW}\) is simply written:
\[\chi \cdot \Omega^{DDW} = (\chi^\varepsilon \frac{\partial}{\partial \varepsilon} + \chi^\Pi \frac{\partial}{\partial \Pi_{\alpha}^A_{\mu}}) \cdot (\delta \varepsilon \wedge d\eta + dA_\mu \wedge d\eta_\nu) = \chi^\varepsilon d\eta + \chi^\Pi dA_\mu \wedge d\eta_\nu. \]

Now we compute \(d(\chi \cdot \Omega^{DDW}) = d(\chi^\varepsilon \wedge d\eta + dA_\mu \wedge d\eta_\nu).
\[d(\chi \cdot \Omega^{DDW}) = d(\Upsilon - (\epsilon(\frac{\partial X^\nu}{\partial x^\nu} + \frac{\partial \Theta^\mu}{\partial x^\nu} A_\mu))) \wedge d\eta \]
\[+ d\left(\chi^A_{\mu} - \left(\Pi^A_{\mu} \delta_\rho^\mu \left[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta_\nu^\mu \left(\frac{\partial X^\nu}{\partial x^\lambda} \right) + (\frac{\partial \Theta^\sigma}{\partial A_\mu}) \right] - \epsilon(\frac{\partial X^\nu}{\partial A_\mu}) \right) \right) \wedge dA_\mu \wedge d\eta_\nu, \]
due to the expression of the exterior derivatives \(d\Upsilon\) and \(d\chi^A_{\mu},\) we obtain:
\[d(\chi \cdot \Omega^{DDW}) = (\frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \chi^A_{\mu}}{\partial x^\nu}) dA_\mu \wedge d\eta_\nu + (\frac{\partial \Upsilon}{\partial \varepsilon} - \frac{\partial X^\nu}{\partial x^\nu}) d\varepsilon \wedge d\eta \]
\[+ \left(\frac{\partial \chi^A_{\mu}}{\partial \Pi_{\alpha}^A_{\mu}} - \frac{\partial \Theta^\mu}{\partial x^\nu} \right) d\Pi_{\alpha}^A_{\mu} \wedge d\eta + \left(\frac{\partial \chi^A_{\mu}}{\partial A_\beta} \right) dA_\beta \wedge dA_\mu \wedge d\eta_\nu \]
\[+ \left(\frac{\partial \chi^A_{\mu}}{\partial \varepsilon} - \frac{\partial X^\nu}{\partial A_\mu} \right) d\varepsilon \wedge dA_\mu \wedge d\eta_\nu \]
\[+ \left(\frac{\partial \chi^A_{\mu}}{\partial \Pi_{\alpha}^A_{\mu}} - \delta_\rho^\mu \left[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta_\nu^\mu \left(\frac{\partial X^\nu}{\partial x^\lambda} \right) + (\frac{\partial \Theta^\sigma}{\partial A_\mu}) \right] - \epsilon(\frac{\partial X^\nu}{\partial A_\mu}) \right) d\Pi_{\alpha}^A_{\mu} \wedge dA_\mu \wedge d\eta_\nu. \]
Now we are interested in terms in which $\Upsilon$ is involved: the first three terms in the last equation are concerned. Let notice that, if we denote $q = \{x, A\}$ then, $\Upsilon = \Upsilon(q, \epsilon, \Pi)$ and the first two terms in the last equation give:

$$\frac{\partial \Upsilon}{\partial \epsilon}(q, \epsilon, \Pi) - \frac{\partial X^\nu}{\partial x^\nu}(q) = 0,$$

(3.21)

$$\frac{\partial \Upsilon}{\partial \Pi^\mu\nu}(q, \epsilon, \Pi) - \frac{\partial \Theta_{\mu}}{\partial x^\nu}(q) = 0.$$

Hence, it exists $\Upsilon(q) = \Upsilon(x, A)$. So that we write:

$$\Upsilon(q, \epsilon, \Pi) = \Upsilon(q) + \left(\frac{\partial \Theta_{\mu}}{\partial x^\nu}\right)\Pi^\mu\nu + \epsilon\left(\frac{\partial X^\nu}{\partial x^\nu}\right).$$

On the other side, $\Upsilon^\alpha\nu(q, p) = \Upsilon^\alpha\nu(x, \Pi)$, therefore the interesting information is contained in the set of equations:

$$\frac{\partial \Upsilon^\alpha\nu}{\partial x^\alpha}(x, \Pi) - \left(\frac{\partial X^\nu}{\partial A_\mu}\right)(q) = 0$$

(3.22)

$$\left[\frac{\partial \Upsilon^\alpha\nu}{\partial \Pi^\mu\sigma}(q, \epsilon, \Pi) - \sigma^\mu\nu\left(\frac{\partial X^\nu}{\partial x^\sigma} - \frac{\partial X^\lambda}{\partial x^\sigma}\right)\right] - \left(\frac{\partial \Theta_{\mu}}{\partial A_\sigma}\right) d\Pi^\mu\sigma \wedge dA_\mu \wedge d\eta_\nu = 0.$$

Hence, it exists $\Upsilon^\alpha\mu(q, p) = \Upsilon^\alpha\mu(x, A)$. So that:

$$\Upsilon^\alpha\mu(q, p) = \Upsilon^\alpha\nu(x, A) - \Pi^\mu\sigma\sigma^\alpha\nu\left(\frac{\partial X^\nu}{\partial x^\sigma} - \frac{\partial X^\lambda}{\partial x^\sigma}\right) \left(\frac{\partial \Theta_{\mu}}{\partial A_\sigma}\right) + \epsilon\left(\frac{\partial X^\nu}{\partial x^\nu}\right).$$

The set of infinitesimal symplectomorphisms $sp_0(\mathcal{M}_{\text{Maxwell}})$ of $(\mathcal{M}_{\text{Maxwell}}, \Omega^{\text{Dow}})$ is described by vector fields $\Xi = \Xi_{\mathcal{M}_{\text{Maxwell}}} = \zeta + \chi$ with $\zeta$ described by (3.15) and $\chi = \Upsilon\frac{\partial}{\partial \epsilon} + \Upsilon^\alpha\nu\frac{\partial}{\partial \Pi^\mu\nu}. \quad \text{(3.16)}$

Here, $X^\nu, \Theta_{\mu}, \Upsilon, \Upsilon^\alpha\mu$ are defined on $\mathcal{M}$ and not anymore on the full multisymplectic manifold $\mathcal{M}_{\text{Maxwell}}$.

**Proposition 3.4.** If we assume that $dx^\mu(\Xi) = 0$ - we throw away the $X^\mu$ which correspond to parts of the stress-energy-tensor - the proposition (3.2) gives:

$$\Xi = (\Upsilon^\alpha\nu - \Pi^\mu\sigma\sigma^\alpha\nu\left(\frac{\partial \Theta_{\sigma}}{\partial A_\nu}\right)) \frac{\partial}{\partial \Pi^\mu\nu} + (\Upsilon - \frac{\partial \Theta_{\mu}}{\partial \Pi^\mu\nu}\Pi^\mu\nu) \frac{\partial}{\partial \epsilon} + \Theta_{\mu} \frac{\partial}{\partial A_{\mu}}.$$

$\Upsilon^\alpha\mu, \Upsilon$ and $\Theta_{\mu}$ are smooth arbitrary functions of $(x, A)$ with $\Upsilon^\alpha\mu(q) = -\Upsilon^\nu\mu(q)$, they satisfy the condition:

$$\frac{\partial \Upsilon}{\partial A_{\mu}} - \frac{\partial \Upsilon^\alpha\nu}{\partial x^\nu} = 0.$$

**Proposition 3.5.** Let $\varphi \in \Gamma(\mathcal{M}_{\text{Maxwell}}, \Lambda^{n-1}\pi^\star\mathcal{M}_{\text{Maxwell}})$. The $(n-1)$-form $\varphi$ is an algebraic observable if and only if $\varphi$ is written $\varphi = \varphi_X + \varphi_A + \varphi_\chi$ where

$$\varphi_X = \epsilon X^\rho d\eta_{\rho} - \Pi^\mu\nu X^\rho dA_{\mu} \wedge d\eta_{\nu},$$

$$\varphi_A = \Pi^\mu\nu \Theta_{\mu} d\eta_{\nu}$$

where $X^\mu, \Theta_{\mu} : \mathcal{M} \to \mathbb{R}$ are arbitrary smooth functions on $\mathcal{M}$ and $\varphi_\chi$ is a $(n-1)$-form such that

$$d\varphi_\chi = \Upsilon d\eta + \Upsilon^\alpha\mu dA_{\mu} \wedge d\eta_{\nu},$$

(3.23)
with $\Upsilon$ and $\Upsilon^A_{\mu\nu}$ such that $\Upsilon^A_{\mu\nu} = -\Upsilon^A_{\nu\mu}$. The functions $\Upsilon$ and $\Upsilon^A_{\mu\nu}$ satisfy

$$\frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \Upsilon^A_{\mu\nu}}{\partial x^\nu} = 0$$

(3.25)

We notice that $\varphi_X + \varphi_A$ are the so-called generalized algebraic momenta $(n - 1)$-forms. Recall that an arbitrary vector field on $\mathfrak{g}$ is written \( \mathcal{J} \),

$$\zeta := \sum_\alpha \zeta^\alpha(q) \frac{\partial}{\partial q^\alpha} = X^\nu(x, A) \frac{\partial}{\partial x^\nu} + \Theta_\mu(x, A) \frac{\partial}{\partial A_\mu}$$

Let us denote $P_\zeta = \zeta \mathcal{J} \Theta$ so that

$$P_\zeta = \zeta \mathcal{J} \left( e\eta + \Pi^A_{\mu\nu} dA_\mu \wedge d\eta_\nu \right) = e\eta(\zeta) + \Pi^A_{\mu\nu} \left( (\zeta \mathcal{J} dA_\mu) \wedge d\eta_\nu - dA_\mu \wedge (\zeta \mathcal{J} d\eta_\nu) \right)$$

Since, $\zeta \mathcal{J} dA_\mu = \Theta_\mu$ and $\zeta \mathcal{J} d\eta_\nu = (X^\rho(x, A) \frac{\partial}{\partial x^\rho}) \mathcal{J} d\eta_\nu = X^\rho d\eta_\rho$, we obtain:

$$P_\zeta = eX^\rho d\eta_\rho + \Pi^A_{\mu\nu} \Theta_\mu d\eta_\nu - \Pi^A_{\mu\nu} X^\rho dA_\mu \wedge d\eta_\rho = \varphi_X + \varphi_A.$$ $P_\zeta$ are the generalized momenta $(n - 1)$-form. We have $dP_\zeta = -\zeta \mathcal{J} \Omega^{\text{now}}$. The canonical symplectomorphism associated to $P_\zeta$ is denoted $\Xi(P_\zeta) = \zeta$. We evaluate the exterior derivative $dP_\zeta = d[\varphi_X + \varphi_A]$:

$$dP_\zeta = d\left( eX^\nu + \Pi^A_{\mu\nu} \Theta_\mu \right) \wedge d\eta_\nu - d\left( \Pi^A_{\mu\nu} X^\rho \right) \wedge dA_\mu \wedge d\eta_\rho$$

$$= \underbrace{X^\nu \, d\epsilon \wedge d\eta_\nu + eX^\nu \, d\eta_\nu + \Pi^A_{\mu\nu} \, d\Theta_\mu \wedge d\eta_\nu + \Theta_\mu \, d\Pi^A_{\mu\nu} \wedge d\eta_\nu}_{t_1} \wedge d\eta_\nu$$

$$- \underbrace{X^\rho \, d\Pi^A_{\mu\nu} \wedge dA_\mu \wedge d\eta_\rho - \Pi^A_{\mu\nu} \, dX^\rho \wedge dA_\mu \wedge d\eta_\rho}_{t_6}.$$ Now we expand the objects $dX^\rho$, $d\Theta_\mu$ so that:

$$\nu_2 = eX^\nu \wedge d\eta_\nu = e\left( \frac{\partial X^\nu}{\partial x^\alpha} d\alpha + \frac{\partial X^\nu}{\partial A^\beta} dA^\beta + \frac{\partial X^\nu}{\partial \epsilon} d\epsilon + \frac{\partial X^\nu}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \right) \wedge d\eta_\nu,$$

$$\nu_3 = \Pi^A_{\mu\nu} \Theta_\mu \wedge d\eta_\nu = \Pi^A_{\mu\nu} \left( \frac{\partial \Theta_\mu}{\partial x^\alpha} d\alpha + \frac{\partial \Theta_\mu}{\partial A^\beta} dA^\beta + \frac{\partial \Theta_\mu}{\partial \epsilon} d\epsilon + \frac{\partial \Theta_\mu}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \right) \wedge d\eta_\nu.$$ Then,

$$dP_\zeta = \underbrace{X^\nu \, d\epsilon \wedge d\eta_\nu + eX^\nu \, d\eta_\nu + \Theta_\mu \, d\Pi^A_{\mu\nu} \wedge d\eta_\nu}_{t_1} \wedge d\eta_\nu$$

$$+ \underbrace{\Pi^A_{\mu\nu} \, \frac{\partial \Theta_\mu}{\partial x^\alpha} d\alpha \wedge d\eta_\nu + \Pi^A_{\mu\nu} \, \frac{\partial \Theta_\mu}{\partial A^\beta} dA^\beta \wedge d\eta_\nu + \Pi^A_{\mu\nu} \, \frac{\partial \Theta_\mu}{\partial \epsilon} d\epsilon \wedge d\eta_\nu}_{t_7} \wedge d\eta_\nu$$

$$+ \underbrace{\Pi^A_{\mu\nu} \, \frac{\partial \Theta_\mu}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \wedge d\eta_\nu - \Pi^A_{\mu\nu} \, \frac{\partial X^\rho}{\partial x^\alpha} d\alpha \wedge dA^\beta \wedge d\eta_\rho - \Pi^A_{\mu\nu} \, \frac{\partial X^\rho}{\partial \Pi^A_{\beta\alpha}} d\Pi^A_{\beta\alpha} \wedge dA^\beta \wedge d\eta_\rho}_{t_9} \wedge d\eta_\rho.$$
Since, $X = X(x)$ and $\Theta_\mu = \Theta_\mu(x, A)$ - see (3.14), we obtain vanishing contributions from the terms $\xi_9, \xi_{10}, \xi_{12}, \xi_{13}$ and $\xi_{14}$. Therefore :
\[
\begin{align*}
\text{d}P_\zeta &= \left[ X^\nu \text{d}e \wedge \text{d}n^\nu + \Theta_\mu \text{d} \Pi^\beta_\mu \wedge \text{d}n_\beta - \text{d} \Pi^\alpha_\mu X^\rho \text{d}A_\mu \wedge \text{d}n_\rho \right] \\
&\quad + \Pi^\alpha_\mu \left( \frac{\delta \Theta_\mu}{\partial x^\alpha} \right) \text{d} x^\alpha \wedge \text{d}n_\mu + \Pi^\alpha_\mu \left( \frac{\delta \Theta_\mu}{\partial A_\beta} \right) \text{d} A_\beta \wedge \text{d}n_\mu - \Pi^\alpha_\mu \left( \frac{\delta X^\rho}{\partial x^\alpha} \right) \text{d} x^\alpha \wedge \text{d}A_\mu \wedge \text{d}n_\rho
\end{align*}
\]

On the other hand, the general expression for a canonical symplectomorphism is :
\[
\zeta \mathcal{J} \Omega^\text{DDW} = \zeta \mathcal{J} \text{d}e \wedge \text{d}n + \zeta \mathcal{J} \left[ \text{d} \Pi^\mu_\nu \wedge \text{d}A_\mu \wedge \text{d}n_\nu \right]
\]
\[
= - \left( \epsilon \left( \frac{\partial X^\nu}{\partial x^\nu} \right) + \frac{\partial \Theta_\mu}{\partial x^\mu} \Pi^\mu_\nu \right) \text{d}n - X^\nu \text{d}e \wedge \text{d}n_\nu \\
+ \Pi^\alpha_\mu \left( \frac{\delta \Theta_\mu}{\partial \alpha} \right) \text{d} A_\mu \wedge \text{d}n_\nu - \Pi^\mu_\nu \left( \frac{\delta \Theta_\mu}{\partial \nu} \right) \text{d} A_\mu \wedge \text{d}n_\nu + \text{d}x^\rho (\zeta) \Pi^\alpha_\mu \wedge \text{d}A_\mu \wedge \text{d}n_\rho
\]
\[
= - \epsilon \left( \frac{\partial X^\nu}{\partial x^\nu} \right) \text{d}n - \frac{\partial \Theta_\mu}{\partial x^\mu} \Pi^\mu_\nu \text{d}n - X^\nu \text{d}e \wedge \text{d}n_\nu + \Pi^\mu_\nu \left( \frac{\delta \Theta_\mu}{\partial A_\nu} \right) \text{d} A_\mu \wedge \text{d}n_\nu
\]
\[
- \Pi^\mu_\nu \left( \frac{\delta \Theta_\mu}{\partial \nu} \right) \text{d} A_\mu \wedge \text{d}n_\nu + \text{d} \Pi^\mu_\nu (\zeta) \text{d} A_\mu \wedge \text{d}n_\nu
\]
\[
- \text{d} A_\mu (\zeta) \Pi^\mu_\nu \wedge \text{d}n_\nu + X^\rho \text{d} \Pi^\mu_\nu \wedge \text{d} A_\mu \wedge \text{d}n_\rho
\]
\[
- \text{d} A_\mu (\zeta) \Pi^\mu_\nu \wedge \text{d}n_\nu + X^\rho \text{d} \Pi^\mu_\nu \wedge \text{d} A_\mu \wedge \text{d}n_\rho
\]
\[
\text{Since, d} A_\mu (\zeta) = \Theta_\mu \text{ we observe :}
\]
\[
- \text{d} A_\mu (\zeta) \Pi^\mu_\nu \wedge \text{d}n_\nu = - \Theta_\mu \Pi^\mu_\nu \wedge \text{d}n_\nu = - \xi_4.
\]

Let us denote, $[\mathcal{I}] = \xi_1 + \xi_2 + \xi_4 + \xi_5 + \xi_7$. We remark that
\[
\zeta \mathcal{J} \Omega^\text{DDW} = - [\mathcal{I}] + \text{d} \Pi^\mu_\nu (\zeta) \text{d} A_\mu \wedge \text{d}n_\nu.
\]

Let also notice that $\text{d} \Pi^\mu_\nu (\zeta) = \zeta^{\Pi}$, with
\[
\zeta = X^\nu \frac{\partial}{\partial x^\nu} + \Theta_\rho \frac{\partial}{\partial A_\rho} + \zeta^\alpha \frac{\partial}{\partial x^\alpha} + \zeta^{\Pi} \frac{\partial}{\partial \Pi^\alpha_\nu},
\]
we are left with the term $\zeta^{\Pi}$ :
\[
\zeta^{\Pi} = \Pi^\alpha_\nu \left[ \frac{\partial X^\nu}{\partial x^\alpha} - \delta^\alpha_\nu \left( \frac{\partial X^\lambda}{\partial x^\nu} \right) + \frac{\partial \Theta_\sigma}{\partial \nu} \right] - \epsilon \left( \frac{\partial X^\nu}{\partial A_\nu} \right),
\]
so that : $\zeta \mathcal{J} \Omega^\text{DDW} = - [\mathcal{I}] + \left[ \zeta^{\Pi} \right]^\nu_\mu \text{d} A_\mu \wedge \text{d}n_\nu$. Finally we shall denote the remaining terms $[\Pi] = \xi_8 + \xi_{11}$, hence we can write the equality $\text{d}[\Phi_X + \Phi_A] = [\mathcal{I}] + [\Pi]$. Therefore in order to prove the equality $\zeta \mathcal{J} \Omega^\text{DDW} = - [\mathcal{I}] + [\Pi]$, we only need to prove that $\text{d} \Pi^\mu_\nu (\zeta) \text{d} A_\mu \wedge \text{d}n_\nu = - [\Pi]$. Since $\text{d} x^\alpha \wedge \text{d}n_\nu = \delta^\alpha_\nu \text{d}n_\nu - \delta^\alpha_\nu \text{d}n_\rho$,
\[
\begin{align*}
\xi_{11} &= - \Pi^\mu_\nu \left( \frac{\partial X^\rho}{\partial x^\sigma} \right) \text{d} x^\alpha \wedge \text{d} A_\mu \wedge \text{d}n_\rho = \Pi^\mu_\nu \left( \frac{\partial X^\rho}{\partial x^\sigma} \right) \text{d} A_\mu \wedge \left[ \delta^\alpha_\nu \text{d}n_\nu - \delta^\alpha_\nu \text{d}n_\rho \right] \\
&= \Pi^\mu_\nu \left( \frac{\partial X^\rho}{\partial x^\sigma} \right) \text{d} A_\mu \wedge \text{d}n_\nu - \Pi^\mu_\nu \left( \frac{\partial X^\rho}{\partial x^\sigma} \right) \text{d} A_\mu \wedge \text{d}n_\rho = \Pi^\mu_\nu \left[ \frac{\partial X^\nu}{\partial x^\sigma} - \delta^\alpha_\nu \left( \frac{\partial X^\lambda}{\partial x^\nu} \right) \right]
\end{align*}
\]
and also, on the same vein,

\[ \iota_\delta = \Pi^A_{\nu \sigma} \left( \frac{\partial \Theta_\mu}{\partial A_\sigma} \right) - \epsilon \left( \frac{\partial X_\nu}{\partial A_\mu} \right) \]

so that we found the wanted result.

### 3.4 Dynamical observable \((n-1)\)-forms

Let us continue the investigation with the following propositions: \((3.6)\) and \((3.7)\).

**Proposition 3.6.** Let \( \Xi \in \Gamma(M, T^*M) \) then \( \Xi \) satisfies \( d(\Xi \cup \Omega^{dow}) \) and \( dH(\Xi) = 0 \) if and only if \( \Xi \) is written \( \Xi = \Xi_X + \Xi_A \) with

\[ \Xi_X = X^\mu \partial_\mu \cup \theta, \]

a vector field on the Minkowski space-time \( X \) - a generator of the action of the Poincaré group - and

\[ \Xi_A = \Theta_\mu \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \Theta_\mu}{\partial x^\nu} \Pi^A_{\nu \sigma} \right) \frac{\partial}{\partial \Gamma^{A \sigma}_{\nu \sigma}} + \gamma^{A \mu \nu} \frac{\partial}{\partial \Pi^A_{\mu \nu}}. \] (3.27)

**Proposition 3.7.** Let \( \rho \in \Gamma(M_{\text{Maxwell}}, \Lambda^{n-1}T^*M_{\text{Maxwell}}) \). The \((n-1)\)-form \( \rho \) is a dynamical observable if and only if \( \rho \) is written \( \rho = \rho_X + \rho_A \) where

\[ \begin{align*}
\rho_X & = \epsilon X^\rho d\eta_\rho - \Pi^A_{\nu \sigma} X^\rho dA_\mu \wedge d\eta_{\rho \nu} \\
\rho_A & = \Pi^A_{\nu \sigma} \Theta_\mu d\eta_\nu + \gamma^{A \mu \nu} A_\mu d\eta_\nu
\end{align*} \] (3.28)

\( X^\mu, \Theta_\mu, \gamma^{A \mu \nu} : \mathcal{M} \to \mathbb{R} \) are arbitrary smooth functions such that

\[ \gamma^{A \mu \nu} = -\gamma^{A \nu \mu} \quad \text{with} \quad \frac{\partial \gamma^{A \mu \nu}}{\partial x^\nu} = 0. \]

Since \( dH_{\text{Maxwell}}(q, p) = dH(q, p) = d\epsilon - \frac{1}{2} \eta_{\mu \nu} \Pi^A_{\mu \sigma} d\Pi^A_{\nu \sigma} \) we consider \( dH(\Xi) \) as a polynomial expression depending on the variables \((\epsilon, \Pi^A_{\mu \nu})\). We have:

\[ dH(\Xi) = dH(X + \zeta) \]

\[ = d\epsilon \left( \frac{\partial}{\partial \epsilon} \right) - \frac{1}{2} \eta_{\mu \nu} \eta_{\sigma \rho} \Pi^A_{\mu \sigma} d\Pi^A_{\nu \rho} \left( \gamma^{A \beta}_{\sigma \nu} \frac{\partial}{\partial \Gamma^{A \beta}_{\nu \sigma}} \right) - d\epsilon \left( \left( \frac{\partial X^\nu}{\partial \xi^\nu} \right) + \frac{\partial \Theta_\mu}{\partial x^\nu} \Pi^A_{\mu \nu} \right) \frac{\partial}{\partial \epsilon}, \]

\[ - \frac{1}{2} \eta_{\mu \nu} \eta_{\sigma \rho} \Pi^A_{\mu \sigma} d\Pi^A_{\nu \rho} \left( \left( \Pi^A_{\mu \nu} \gamma^{A \mu}_{\nu \sigma} \left[ \left( \frac{\partial X^\nu}{\partial x^\nu} \right) - \frac{\partial \Theta_\mu}{\partial A_\mu} \right] \right) - \frac{\partial \gamma^{A \nu \rho}}{\partial \Gamma^{A \rho}_{\nu \sigma}} \right) \]

So that we obtain:

\[ dH(\Xi) = \gamma \left[ \left[ - \left( \frac{\partial X^\nu}{\partial x^\nu} \right) + \left[ \left[ - \frac{1}{2} \eta_{\mu \nu} \eta_{\sigma \rho} \gamma^{A \mu}_{\nu \sigma} \left[ - \frac{\partial \Theta_\mu}{\partial x^\nu} \right] \right] \right] \right] + 2 \epsilon \left[ \left[ - \frac{1}{2} \eta_{\mu \nu} \eta_{\sigma \rho} \gamma^{A \mu}_{\nu \sigma} \left[ - \frac{\partial \Theta_\mu}{\partial x^\nu} \right] \right] \right]. \]
Thanks to (3.29) we have $\Upsilon = 0$ and the following relations (3.30) as coefficient expression respectively of $e$, $\Pi^A_{\mu \nu}$, $e\Pi^A_{\lambda \kappa}$ and $\Pi^\lambda_{\mu \nu}$:

$$
0 = \left[ - \left( \frac{\partial X^\nu}{\partial x^\nu} \right) \right] \\
0 = \left[ - \frac{1}{2} \eta_{\rho \mu} \eta_{\sigma \nu} \Upsilon^A_{\nu \sigma} - \frac{\partial \Theta_\mu}{\partial x^\nu} \right] \\
0 = \left[ \frac{1}{2} \eta_{\mu \lambda} \eta_{\nu \kappa} \left( \frac{\partial X^\nu}{\partial x^\nu} \right) \right] \\
0 = \left[ - \frac{1}{2} \eta_{\mu \lambda} \eta_{\nu \kappa} \delta^\rho_\mu \left( \left( \frac{\partial X^\nu}{\partial x^\nu} \right) - \delta^\nu_\sigma \left( \frac{\partial X^\lambda}{\partial x^\lambda} \right) + \left( \frac{\partial \Theta_\kappa}{\partial A_\sigma} \right) \right) \right].
$$

The relation (3.30) - $[e\Pi^A_{\lambda \kappa}]$ leads to $X^\nu(x,A) = X^\nu(x)$. From (3.30) - $[\Pi^A_{\mu \nu}]$ we obtain the following relation:

$$
(1/2) \eta_{\rho \mu} \eta_{\sigma \nu} \Upsilon^A_{\nu \sigma} = -\partial_\nu \Theta_\mu \\
-(1/2) \eta_{\sigma \nu} \eta_{\rho \mu} \Upsilon^A_{\sigma \rho} = \partial_\mu \Theta_\nu.
$$

If we sum the last two equations (3.35) we obtain:

$$
\eta_{\rho \mu} \eta_{\sigma \nu} (1/2) (\Upsilon^A_{\nu \sigma} - \Upsilon^A_{\sigma \rho}) = \eta_{\rho \mu} \eta_{\sigma \nu} \Upsilon^A_{\nu \sigma} = \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu.
$$

Finally, due to relations (3.30), we obtain:

$$
\zeta = X^\nu \frac{\partial \Theta_\mu}{\partial A_\mu} + \Theta_\mu \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \Theta_\mu}{\partial x^\nu} \Pi^A_{\mu \nu} \right) \frac{\partial}{\partial \Theta_\nu},
$$

and $\chi = \Upsilon^A_{\mu \alpha} \frac{\partial}{\partial \Pi^A_{\mu \alpha}}$ with $\Upsilon : \mathcal{M} \to \mathbb{R}$ and $\Upsilon^A_{\mu \alpha} = \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu : \mathcal{M} \to \mathbb{R}$ smooth functions on $\mathcal{M}$ such that:

$$
\frac{\partial \Upsilon^A_{\mu \nu}}{\partial x^\nu} = 0
$$

Hence $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $d(\Xi \cup \Omega^{DDW})$ and $dH(\Xi) = 0$ if and only if $\Xi$ is written as announced in proposition (3.6):

$$
\Xi = \Xi_X + \Xi_A = X^\mu \frac{\partial}{\partial x^\mu} + \Theta_\mu \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \Theta_\mu}{\partial x^\nu} \Pi^A_{\mu \nu} \right) \frac{\partial}{\partial \Theta_\nu} + \left[ \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu \right] \frac{\partial}{\partial A_\mu}.
$$

### 3.5 Algebraic observable $(n - 1)$-forms in the pre-multisymplectic case

We enter into some details, considering the pre-multisymplectic case. For the (DDW) theory and without taking into account the decomposition on the space-time variables - so that we forget the stress-energy tensor part. We focus on the following infinitesimal symplectomorphisms, $\Xi^0 \in \Gamma(\mathcal{M}^{DDW}_0, T\mathcal{M}^{DDW}_0)$, written in the form:

$$
\Xi^0_{DDW} = \Theta_\mu (q,p) \frac{\partial}{\partial A_\mu} + \Upsilon^A_{\mu \nu} (q,p) \frac{\partial}{\partial \Pi^A_{\mu \nu}}.
$$
Notice that due to the Dirac primary constraint set, we must consider the following object \( \Xi^0 \in \Gamma(\mathcal{M}_\text{Maxwell}^\ast, T\mathcal{M}_\text{Maxwell}^\ast) \) which is given by the interplay of some forbidden directions:

\[
\Xi^0 = \Theta_\mu(q,p) \frac{\partial}{\partial A_\mu} + \Upsilon^{A_\mu\nu}(q,p) \left( \frac{\partial}{\partial \Pi^{A_\mu\nu}} - \frac{\partial}{\partial \Pi^{A_\nu\mu}} \right). \tag{3.34}
\]

\( \Theta_\mu(q,p) \) and \( \Upsilon^{A_\mu\nu}(q,p) \) are smooth functions on \( \mathcal{M}_\text{Maxwell}^\ast \subset \mathcal{M}_\text{Maxwell} \subset \mathcal{M}_{\text{DDW}} \subset \Lambda^n T^\ast(T^\ast \mathcal{X}) \), with values in \( \mathbb{R} \). We express the evaluation \( \Xi^0 \wedge \Omega^0 \):

\[
\Xi^0 \wedge \Omega^0 = \Xi^0 \left( \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A_\mu\nu} \wedge d\eta + \Xi^0 \left( d\Pi^{A_\mu\nu} \wedge dA_\mu \wedge d\eta \right) \right)
\]

\[
= \left( \Upsilon^{A_\mu\nu}(q,p) - \Upsilon^{A_\nu\mu}(q,p) \right) dA_\mu \wedge d\eta - \Theta_\mu d\Pi^{A_\mu\nu} \wedge d\eta + \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A_\mu\nu}(q,p) \left( \Upsilon^{A_\mu\nu}(q,p) - \Upsilon^{A_\nu\mu}(q,p) \right) d\eta.
\]

Now due to the definition of the symplectomorphism via the formula \( d(\Xi^0 \wedge \Omega^0) = 0 \), we make the following calculation:

\[
d(\Xi^0 \wedge \Omega^0) = d(\Upsilon^{A_\mu\nu}(q,p) \wedge dA_\mu \wedge d\eta - \Theta_\mu d\Pi^{A_\mu\nu} \wedge d\eta)
\]

\[
+ \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A_\mu\nu} d(\Upsilon^{A_\mu\nu}(q,p) \wedge d\eta)
\]

\[
+ \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \left( \Upsilon^{A_\mu\nu}(q,p) - \Upsilon^{A_\nu\mu}(q,p) \right) d\Pi^{A_\mu\nu} \wedge d\eta.
\]

Using the decomposition of \( d\Theta_\mu \) and \( d\Upsilon^{A_\mu\nu} \):

\[
d\Theta_\mu = \frac{\partial \Theta_\mu}{\partial x^\alpha} dx^\alpha + \frac{\partial \Theta_\mu}{\partial A_\beta} dA_\beta + \frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta\alpha}} d\Pi^{A_\alpha\beta}, \quad d\Upsilon^{A_\mu\nu} = \frac{\partial \Upsilon^{A_\mu\nu}}{\partial x^\alpha} dx^\alpha + \frac{\partial \Upsilon^{A_\mu\nu}}{\partial A_\beta} dA_\beta + \frac{\partial \Upsilon^{A_\mu\nu}}{\partial \Pi^{A_\beta\alpha}} d\Pi^{A_\alpha\beta}
\]

The different decompositions of the involved \((n + 1)\)-forms are written:

— \( j_1 \) is the term related to the decomposition on \([d\Pi^{A_\beta\alpha} \wedge d\Pi^{A_\mu\nu} \wedge d\eta]\) so that:

\[
j_1 = - \frac{\partial \Theta_\mu}{\partial \Pi^{A_\beta\alpha}} d\Pi^{A_\beta\alpha} \wedge d\Pi^{A_\mu\nu} \wedge d\eta
\]

— \( j_2 \) is the term related to the decomposition on \([dA \wedge dA \wedge d\eta]\)

\[
j_2 = \left( \frac{\partial \Upsilon^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \Upsilon^{A_\nu\mu}}{\partial A_\beta} \right) dA_\beta \wedge dA_\mu \wedge d\eta
\]

— \( j_3 \) is the term related to the decomposition on \([dA \wedge d\eta]\)

\[
j_3 = \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A_\rho\sigma} \left( \frac{\partial \Upsilon^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \Upsilon^{A_\nu\mu}}{\partial A_\beta} \right) dA_\beta \wedge d\eta + \left( \frac{\partial \Upsilon^{A_\mu\nu}}{\partial x^\sigma} - \frac{\partial \Upsilon^{A_\nu\mu}}{\partial x^\sigma} \right) dA_\mu \wedge d\eta
\]

— \( j_4 \) is the term related to the decomposition on \([d\Pi \wedge d\eta]\)

\[
j_4 = \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A_\rho\sigma} \left( \frac{\partial \Upsilon^{A_\mu\nu}}{\partial \Pi^{A_\beta\alpha}} - \frac{\partial \Upsilon^{A_\nu\mu}}{\partial \Pi^{A_\beta\alpha}} \right) d\Pi^{A_\alpha\beta} \wedge d\eta + \frac{\partial \Theta_\mu}{\partial x^\nu} d\Pi^{A_\mu\nu} \wedge d\eta
\]

\[
+ \frac{1}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \left( \Upsilon^{A_\mu\nu}(q,p) - \Upsilon^{A_\nu\mu}(q,p) \right) d\Pi^{A_\rho\sigma} \wedge d\eta
\]
The mathematical requirement on the infinitesimal symplectomorphic \(d(\Xi^{\text{skt048}}) = 0\) allows us to precise the conditions on the functions \(\Theta^{\mu}\) and \(\Upsilon^{A^{\mu\nu}}\). The equation (3.35)-1 gives that \(\Theta^{\mu}\) is independent of momenta, \(\Theta^{\mu} = \Theta^{\mu}(x, A)\). The equation (3.35)-2 gives \(\Upsilon^{A^{\mu\nu}} = \Upsilon^{A^{\mu\nu}}(x, \Pi)\). Since we have equation (3.35)-3 we obtain the following condition:

\[\partial_{\nu}(\Upsilon^{A^{\mu\nu}} - \Upsilon^{A^{\mu\nu}}) = 0\] (3.36)

We recover from relations that emerge from (3.35)-j1-j5 the results of J. Kijowski [38] and J. Kijowski and W. Szczyrba [40].

### 3.6 Observable functionals

First, we recall the general setting for describing the kinematical and dynamical observable functionals. Then we construct the dynamical observable functional for Maxwell theory.

**Kinematical observable functionals.** The important objects for the needs of physics are observable functionals. This provides a bridge with the classical or quantum observables of field theory. We describe a multisymplectic manifold \((\mathcal{M}, \Omega)\) together with a Hamiltonian \(H\). We denote by \(\mathcal{E}^H\) the set of Hamiltonian \(n\)-curves. This picture is the generalization of an Hamiltonian system \((\mathcal{M}, \Omega, \mathcal{H})\) to the \(n\)-dimensional case where the dynamical data are \((\mathcal{M}, \Omega, H)\). Before giving the definition of an observable functional, we introduce the notion of slice. The quantities of physical interest are functionals on the set of Hamiltonian \(n\)-curves \(\mathcal{E}^H\). We construct such observable functionals by integration of an algebraic observable \((n-1)\)-form over a submanifold \(\Sigma \subset \Gamma\) of codimension 1 of a Hamiltonian \(n\)-curve \(\Gamma\). Here we recover the picture of observable functionals in the classical (quantum) field theory as smeared integrals over a spacelike hypersurface.

**Definition 3.6.1.** A slice of codimension 1 is a submanifold \(\Sigma \subset \mathcal{M}\) such that \(T_m \mathcal{M} / T_m \Sigma\) is smoothly oriented with respect to \(m\) and, such that for any \(\Gamma \in \mathcal{E}^H\), \(\Sigma\) is transverse to \(\Gamma\).

This definition allows us to give an orientation on \(\Sigma \cap \Gamma\). If \(\Sigma\) is a slice of codimension 1 and \(\rho\) is a \((n-1)\)-form on \(\mathcal{M}\), namely \(\rho \in \Gamma(\mathcal{M}, \Lambda^{n-1}T^*\mathcal{M})\), we define the concept of functional \(F_\rho := \int_{\Sigma} \rho\). This object is described as \(\int_{\Sigma} \rho : \mathcal{E}^H \rightarrow \mathbb{R}\) on the set of Hamiltonian \(n\)-curves by means of:

\[F_\rho := \int_{\Sigma} \rho : \Gamma \mapsto \int_{\Sigma \cap \Gamma} \rho.\] (3.37)

We can integrate the \((n-1)\)-form \(\rho\) on \(\Sigma \cap \Gamma\). To reach the object of interest, we pass from those functionals to observable functionals whose form \(\rho\) is an algebraic observable.
Definition 3.6.2. Let $\Sigma$ be a slice of codimension 1 and let be $\varphi$ an algebraic observable $(n-1)$-form. An observable functional $F_{\varphi} = \int_\Sigma \varphi$ defined on the set of $n$-dimensional submanifolds $\mathcal{E}^H$ is given by the map:

$$F_{\varphi} = \int_\Sigma \varphi : \?^{\mathcal{E}^H} \rightarrow \mathbb{R}$$

(3.38)

Then for any $\varphi, \eta \in \mathcal{P}^{n-1}_0(M)$ the Poisson bracket - which coincides with the standard bracket for field theory - between two observable functionals $\int_\Sigma \varphi$ and $\int_\Sigma \eta$ is defined such that $\forall \Gamma \in \mathcal{E}^H$ we have

$$\{\int_\Sigma \varphi, \int_\Sigma \eta\}(\Gamma) := \int_{\Sigma \cap \Gamma} \{\varphi, \eta\}.$$ (3.39)

This Poisson bracket satisfies the Jacobi identity. Let us consider $\varphi, \rho, \eta \in \mathcal{P}^{n-1}_0(M)$. From previous considerations, we know that

$$\{\{\varphi, \rho\}, \eta\} + \{\{\rho, \eta\}, \varphi\} + \{\{\eta, \varphi\}, \rho\} = d(\xi_{\varphi} \wedge \xi_{\rho} \wedge \xi_{\eta} \wedge \Omega)$$

which gives by antisymmetry:

$$\{\varphi, \{\rho, \eta\}\} + \{\eta, \{\varphi, \rho\}\} + \{\rho, \{\eta, \varphi\}\} = -d(\xi_{\varphi} \wedge \xi_{\rho} \wedge \xi_{\eta} \wedge \Omega).$$

Therefore, restricting ourselves to the study of functional observables along Hamiltonian $n$-curves $\Gamma$ such that $\partial \Gamma = \emptyset$ we have the Jacobi identity:

$$\left\{\int_{\Sigma_0} \varphi, \left\{\int_{\Sigma} \theta, \int_{\Sigma} \eta\right\}\right\} + \left\{\int_{\Sigma} \eta, \left\{\int_{\Sigma} \varphi, \int_{\Sigma} \theta\right\}\right\} + \left\{\int_{\Sigma} \theta, \left\{\int_{\Sigma} \eta, \int_{\Sigma} \varphi\right\}\right\} = 0.$$ (3.40)

Dynamical observable functionals. The question of dynamical observable functionals hold the key to a fully covariant theory. In the perspective of a covariant theory, we would like to define a bracket over two different slices $\Sigma_0$ and $\Sigma_0$. The bracket defined previously in (3.39) depends on the choice of the given slice $\Sigma$. Given, $\varphi, \eta \in \mathcal{P}^{n-1}_0(M)$ we define the following bracket:

$$\left\{\int_{\Sigma_0} \varphi, \int_{\Sigma_0} \eta\right\}(\Gamma) := \int_{\Sigma_0 \cap \Gamma} \{\varphi, \eta\}.$$ (3.41)

Therefore, we are interested in dynamical observable functionals. It is precisely for dynamical observables that we can construct a fully covariant bracket (3.41). We consider an algebraic observable $(n-1)$-form $\varphi \in \mathcal{P}^{n-1}_0(M)$ via its related infinitesimal symplectomorphism $\Xi_{\varphi} \in C^\infty(M, T^*M)$ which is the unique vector field such that $\Xi_{\varphi} \wedge \Omega = -d\varphi$. The algebraic observable $(n-1)$-form becomes a dynamical observable if we have the additional condition:

$$\Xi_{\varphi} \wedge d\mathcal{H} = d\mathcal{H}(\Xi_{\varphi}) = 0.$$ (3.41)

This condition reflects a homological feature: if $\Gamma$ is a Hamiltonian $n$-curve, then this functional $F(\Gamma)$ depends only on the homology class of $\Sigma$. More precisely, following F. Hélein we show that this result follows from:
Proposition 3.8. Let \( \rho \in \Gamma(\mathcal{M}, \Lambda^{n-1} T^* \mathcal{M}) \) be a dynamical \((n-1)\)-form. Let \( \Sigma_0 \) and \( \Sigma_* \) be two slices such that there exists an open subset \( D \subset \mathcal{M} \) which verifies \( \partial D = \Sigma_0 - \Sigma_* \). We have the following equality:

\[
\int_{\Sigma_0} \rho = \int_{\Sigma_*} \rho.
\]

**Time slice and Minkowski space.** A slice of codimension 1 is thought to be as a slice of time: an hypersurface of type \( \Sigma_0 = \{ t = \tau_0 \} \) where \( \tau_0 \) is a constant. In such a context, we consider the spacetime manifold \( \mathcal{X}_{\text{Mink}} = \mathcal{X} \) to be the flat Minkowski space. In this case we denote \( \mathcal{X} = \mathbb{R}^{1,3} \) endowed with a constant metric \( \eta_{\mu\nu} \), given in the canonical basis by the matrix \( \text{diag}(1,-1,-1,-1) \). We denote the coordinates on \( \mathcal{M}_{\text{DDW}} \) by \( (x^\mu, A_\mu, \epsilon, \Pi^{A\mu\nu}) = (t, x^1, x^2, x^3, A_\mu, \epsilon, \Pi^{A\mu\nu}) \). For any \( \tau_0 \in \mathbb{R} \),

\[
\Sigma_0 := \{ (x^\mu, A_\mu, \epsilon, \Pi^{A\mu\nu}) \in \mathcal{M}_{\text{DDW}} / t = \tau_0 \}
\]

are slices of codimension 1. The slices \( \Sigma_0 \) are oriented by the following condition:

\[
\partial_0 \bigwedge \partial \eta |_{\Sigma_0} > 0
\]

Notice that in such a setting we also denote \( x = (t, \mathbf{x}) \) where \( \mathbf{x} = (x^\mu)_{1 \leq \mu \leq n-1} \) and \( t = x^0 \).

**Dynamical observable functionals for the Maxwell theory.** We follow the method developed by D.R. Harrivel [19]. Let \( A : T^* \mathcal{X} \to \mathbb{R} \) a smooth application and let \( h \in \mathbb{R} \). We associate to \( (A, h) \) the application \( \mathcal{G}_{A,h} : \mathcal{X} \to \mathcal{M} \) defined \( \forall x \in \mathcal{X} \) by:

\[
\mathcal{G}_{A,h}(x) := (x, A(x), \mathcal{E}_{A,h}(x) \partial \eta + \Pi(x, A(x), dA(x))) \cong (x, A(x), \mathcal{E}_{A,h}(x), \Pi^{A\mu\nu}(x)) \in \mathcal{M},
\]

where \( \Pi(x, A(x), dA(x)) = \Pi^{A\mu\nu} dA_\mu \wedge \partial \eta_{\nu}(x, A(x), dA(x)) \) and the function \( \mathcal{E}_{A,h} : \mathcal{X} \to \mathbb{R} \) is defined by:

\[
\mathcal{E}_{A,h}(x) : h - \mathcal{H}(x, A(x), 0, \Pi(x, A(x), dA(x))) = h + \frac{1}{4} \eta_{\mu\rho} \eta_{\nu\sigma} \Pi^{A\mu\nu} \Pi^{A\rho\sigma}
\]

The graph of the application \( \mathcal{G}_{A,h}(x) \) is written:

\[
\mathcal{G}[\mathcal{G}_{A,h}] = (x, A(x), \mathcal{E}_{A,h}(x) \partial \eta + \Pi(x, A(x), dA(x)))
\]

Then we consider \( \Gamma_{A,h} \subset \mathcal{M} \) the image of the application \( \mathcal{G}_{A,h} \). Notice that since we have:

\[
\mathcal{H}|_{\Gamma_{A,h}} \text{ constant and equal to } h \text{ on the graph of an Hamiltonian function. Notice that on the graph of an Hamiltonian function we have for any } A \text{ and a map } x \mapsto p(x) \text{ such that } (x, A(x), dA(x)) \mathcal{E}(x, A(x), p(x)) \text{ the following relations:}
\]

\[
\begin{align*}
\Pi^{A\mu\nu}(x) &= \frac{\partial L}{\partial (\partial_\nu A_\mu)}(x, A(x), dA(x)) \\
\epsilon(x) &= h + L(x, A(x), dA(x)) - \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \Pi^{A\mu\nu} \frac{\partial A_\mu}{\partial x^\nu}(x)
\end{align*}
\]
equivalently:
\[
\begin{align*}
\Pi^{\mu\nu}(x) &= F^{\mu\nu}(x) \\
\epsilon(x) &= h + \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma} F^{\mu\nu}(x) F^{\rho\sigma}(x).
\end{align*}
\] (3.45)

Now we consider a slice \(\Sigma\) of codimension 1. We describe, following [19], slices of type \(\Sigma = (x \circ \pi^X)^{-1}(0)\), where \(\pi^X\) is the natural projection \(\pi^X : \mathcal{M}_{\text{Maxwell}} \to X\) and where \(x : X \to \mathbb{R}\) is a smooth function without any critical point, such that \(x^{-1}(0) \neq 0\).

Thanks to proposition (3.7), the dynamical observable \((n-1)\)-forms are described by \(\rho = \rho_X + \rho_A\) where \(\rho_X\) and \(\rho_A\) are described by relations (3.28). Since \(\rho_X = c X^\rho d\eta - \Pi^{\lambda\nu} X^\rho dA_\mu \wedge d\eta_{\rho\nu}\),

\[
\int_{\Sigma \cap \Gamma_{A,h}} \rho_X = \int_{\pi^X(\Sigma)} \left[ h + \frac{1}{4}\eta_{\mu\alpha}\eta_{\nu\beta} F^{\mu\alpha}(x) F^{\nu\beta}(x) \right] X^\lambda \frac{\partial A_\mu}{\partial x^\rho} d\eta_{\rho\nu} - \frac{1}{4} \frac{\partial A_\mu}{\partial x^\rho} d\eta_{\rho\nu}.
\]

we obtain:

\[
\int_{\Sigma \cap \Gamma_{A,h}} \rho_X = \int_{\pi^X(\Sigma)} \left[ h + \frac{1}{4}\eta_{\mu\alpha}\eta_{\nu\beta} F^{\mu\alpha}(x) F^{\nu\beta}(x) \right] X^\lambda \frac{\partial A_\mu}{\partial x^\rho} d\eta_{\rho\nu}.
\]

On the other side \(\rho_A = (\Pi^{A\mu\nu}\Theta_\mu + \Upsilon^{A\mu\nu} A_\mu) d\eta_{\rho\nu}\), so that:

\[
\int_{\Sigma \cap \Gamma_{A,h}} \rho_A = \int_{\pi^X(\Sigma)} \left[ \Pi^{A\mu\nu}\Theta_\mu + \Upsilon^{A\mu\nu} A_\mu \right] d\eta_{\rho\nu} = \int_{\pi^X(\Sigma)} \left[ F^{\mu\nu}\Theta_\mu + \left( \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu \right) A_\mu \right] d\eta_{\rho\nu}.
\]

Now we focus on the second observable functional (3.47). We integrate on a time slice \(\Sigma_0\), with \(h = 0\) which in turn is equivalent to consider the functional \(\int_{\Sigma_0 \cap \Gamma_{A,0}} \varphi_A\):

\[
\int_{\Sigma_0 \cap \Gamma_{A,0}} \varphi_A = \int_{\Sigma_0} \left( F^{\mu\nu}\Theta_\mu + \Upsilon^{A\mu\nu} A_\mu \right) d\eta_{\rho\nu}.
\]

As noticed by J. Kijowski [38] and J. Kijowski and W. Szczyrba [40] if we consider \(\Theta_\mu |_{\Sigma_0} = -\delta_i^k \cdot \varsigma\) and \(\Upsilon^{A\mu\nu} |_{\Sigma_0} = 0\) - with these conditions, the observable functional (3.48) is now denoted \(D^k(\varsigma)\) and is the \(k\)-th component of the electric field \(\mathbf{E}^k\) smeared with the test function \(\varsigma\) - we obtain:

\[
D^k(\varsigma) = \int_{\Sigma_0} \left( F^{i\alpha}(\delta_i^k \cdot \varsigma) - \Upsilon^{A\alpha} A_i \right) d\eta_{\rho\nu} = \int_{\Sigma_0} \mathbf{E}^{k\alpha} \cdot \varsigma \ d\eta_{\rho\nu}
\]

(3.49)

On the other side, if we consider the conditions \(\Theta_\mu |_{\Sigma_0} = 0\) and \(\Upsilon^{A\mu\nu} |_{\Sigma_0} = -\varepsilon^{ijk} \partial_j \varsigma\), the observable functional (3.48) is now denoted \(B^k(\varsigma)\) and is the \(k\)-th component of the magnetic field \(\mathbf{B}^k\) smeared with the test function \(\varsigma\).

\[
B^k(\varsigma) = \int_{\Sigma_0} \left( F^{i\alpha}(\delta_i^k \cdot \varsigma) - \Upsilon^{A\alpha} A_i \right) d\eta_{\rho\nu} = -\frac{1}{2} \varepsilon^{ijk} \int_{\Sigma_0} F_{ij} \cdot \varsigma \ d\eta_{\rho\nu}.
\]

(3.50)
The objects $\mathbf{E}^k(\zeta)$ and $\mathbf{B}^k(\zeta)$ are observable-valued distributions $\zeta \in C_0^\infty(\Sigma_0) \rightarrow \mathbf{E}^k(\zeta), \mathbf{B}^k(\zeta)$ [38] [40]. From (3.49) and (3.50) we obtain the following Poisson brackets:

$$\left\{ \mathbf{E}^k(\zeta_1), \mathbf{E}^i(\zeta_2) \right\} = \left\{ \mathbf{B}^k(\zeta_1), \mathbf{B}^i(\zeta_2) \right\} = 0,$$

$$\left\{ \mathbf{B}^k(\zeta_1), \mathbf{E}^i(\zeta_2) \right\} = -\varepsilon^{kij} \int_{\Sigma_0} (\partial_j \zeta_1) \zeta_2 d\eta_0. \tag{3.51}$$

We obtain the set of equal-time Poisson bracket, using observable valued distribution $\mathbf{E}^k(x)$ and $\mathbf{B}^k(x)$, see J. Kijowski and W. Szczyrba [38] [40]:

$$\left\{ \mathbf{E}^k(x), \mathbf{E}^i(x) \right\} = 0,$$

$$\left\{ \mathbf{B}^k(x), \mathbf{B}^i(y) \right\} = 0,$$

$$\left\{ \mathbf{B}^k(x), \mathbf{E}^i(y) \right\} = -\varepsilon^{kij} \partial_j (x - y). \tag{3.52}$$

### 3.7 Stress-energy tensor

**Prolegomena** We next examine the relation with the stress energy tensor $\mathfrak{G}_{\mu\nu}$, see relations (3.54). First, we focus on some preparatory work. Let us denote:

$$\begin{align*}
\mathbf{P}_{\partial_\phi} &= \partial_\phi \int \mathbf{\theta}^{\text{DW}} = \partial_\phi \int \mathbf{\theta}^{\text{DW}} \\
\mathbf{\eta}_0 &= -\partial_\phi \left( \mathbf{\theta}^{\text{DW}} - \mathcal{H}(q,p) d\eta \right) \tag{3.53}
\end{align*}$$

**Lemma 3.1.** We denote $d\eta = dx^0 \wedge \cdots \wedge dx^{n-1}$, $d\eta_\mu = \partial_\mu \int d\eta$ and $\Pi = \Pi^{\mu\nu} dA_\mu \wedge d\eta_\nu$. We have the following relation:

$$\partial_\phi \int \Pi = \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi^{\mu\nu} dA_\mu \wedge \left( \partial_\nu \int d\eta_\phi \right)$$

$$= \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi^{\mu\nu} dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dA_\mu \wedge dx^{\nu+1} \wedge \cdots \wedge dx^{n-1}. \tag{3.54}$$

**Proof.** Since, $\partial_\phi \int dA_\mu = 0$ and $\partial_\phi \int d\eta_\nu = \partial_\phi \int \partial_\nu \int d\eta = (\partial_\phi \wedge \partial_\nu) \int d\eta = -\partial_\nu \int d\eta_\phi$. We have :

$$\partial_\phi \int \Pi = \partial_\phi \int \Pi^{\mu\nu} dA_\mu \wedge d\eta_\nu = \Pi^{\mu\nu} \left( (\partial_\nu \int dA_\mu) \wedge d\eta_\nu - dA_\mu \wedge (\partial_\nu \int d\eta_\nu) \right)$$

$$= -\Pi^{\mu\nu} dA_\mu \wedge (\partial_\nu \int d\eta_\nu) = \Pi^{\mu\nu} dA_\mu \wedge (\partial_\nu \int d\eta_\phi).$$

Now, since $d\eta_\phi = dx^1 \wedge \cdots \wedge dx^{n-1}$ we obtain :

$$\partial_\phi \int \Pi = \Pi^{\mu\nu} dA_\mu \wedge (\partial_\nu \int (dx^1 \wedge \cdots \wedge dx^{n-1}))$$

$$= \Pi^{\mu\nu} dA_\mu \wedge (-1)^{\nu-1} (dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dx^{\nu+1} \wedge \cdots \wedge dx^{n-1})$$

$$= \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi^{\mu\nu} dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dA_\mu \wedge dx^{\nu+1} \wedge \cdots \wedge dx^{n-1}. \tag{3.55}$$
From lemma (3.1), we have the expression of $P_{\partial_\alpha}$ and $\eta_\alpha$:

$$P_{\partial_\alpha} = \partial_\alpha \mathcal{J}(\epsilon \delta \eta + \Pi A_{\mu} dA_\mu \wedge d\eta_\nu) = \partial_\alpha \mathcal{J}(\epsilon \delta \eta + \Pi)$$

$$= \epsilon \delta \eta_\alpha + \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi A_{\mu} dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dA_\mu \wedge dx^{\nu+1} \cdots \wedge dx^{n-1},$$

and also:

$$\eta_\alpha = \mathcal{H}(q, p)\delta \eta_\alpha - \partial_\alpha \mathcal{J} \theta^{Dw} = \mathcal{H}(q, p)\delta \eta_\alpha - P_{\partial_\alpha}$$

$$= \mathcal{H}(q, p)\delta \eta_\alpha - \left(\epsilon \delta \eta_\alpha + \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi A_{\mu} dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dA_\mu \wedge dx^{\nu+1} \cdots \wedge dx^{n-1}\right).$$

We consider the Hamiltonian curve $\Gamma := \{(x^\nu, A_\mu(x), \epsilon(x), \Pi A_{\mu}(x))\} \subset \mathcal{M}_{\text{Maxwell}}$ and the instantaneous slices $\Sigma_\circ = \Gamma \cap \{x^\circ = t\}$, so that:

$$\int_{\Sigma_\circ} \eta_\alpha = \int_{\Sigma_\circ} \mathcal{H}(q, p)\delta \eta_\alpha$$

$$- \int_{\Sigma_\circ} \left(\epsilon \delta \eta_\alpha + \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi A_{\mu} dx^1 \wedge \cdots \wedge dx^{\nu-1} \wedge dA_\mu \wedge dx^{\nu+1} \cdots \wedge dx^{n-1}\right)$$

$$= \int_{\Sigma_\circ} \mathcal{H}(q, p)\delta \eta_\alpha - \int_{\Sigma_\circ} \epsilon \delta \eta_\alpha + \sum_{\mu=0}^{n-1} \sum_{\nu=1}^{n-1} \Pi A_{\mu} dA_\mu \wedge \left(\partial_\nu \mathcal{J} d\eta_\alpha\right).$$

**Stress-energy tensor.** The canonical stress energy tensor $\mathcal{E}_{\mu\nu}$ and the symmetric stress-energy tensor $\overline{\mathcal{E}}^{\mu\nu}$ for the electromagnetic field are described by:

$$\mathcal{E}_{\mu\nu} = -(F_{\mu\lambda}\partial_\nu A^\lambda - \frac{1}{4} \eta_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}),$$

$$\overline{\mathcal{E}}^{\mu\nu} = \overline{\mathcal{E}}^{\nu\mu} + F^{\mu\lambda} \partial_\nu A^\lambda = -(F^{\mu\lambda} F_{\lambda}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}).$$

The symmetric stress-energy tensor $\mathcal{E}_{\mu\nu}$ is obtained by adding a term $\partial_\lambda \kappa^{\mu\nu\lambda}$ with $\kappa^{\mu\nu\lambda} = -\kappa^{\nu\mu\lambda}$.

$$\overline{\mathcal{E}}_{\mu\nu} = \mathcal{E}_{\mu\nu} + \partial_\lambda \kappa^{\mu\nu\lambda}.$$ (3.55)

The relation (3.55) is known as the Belinfante-Rosenfeld formula [3] [15]. The canonical stress energy tensor $\mathcal{E}_{\mu\nu}$ associated to $A : T^* \mathcal{X} \to \mathbb{R}$ is written:

$$\mathcal{E}_{\mu\nu} = \delta_\beta (L(x, A(x), dA(x)) - \frac{\partial L}{\partial v_\lambda}(x, A(x), dA(x)) \frac{\partial A_\lambda}{\partial x^\beta}(x)).$$ (3.56)

Hence,

$$\mathcal{E}_{\mu\nu} = \delta_\beta (-\frac{1}{4} F_{\mu\rho} F^{\nu\rho}) - \eta^{\lambda\rho} \eta^{\mu\sigma} F_{\rho\sigma} \frac{\partial A_\lambda}{\partial x^\beta} = \delta_\beta (-\frac{1}{4} F_{\mu\rho} F^{\nu\rho}) - (F^{\lambda\alpha} \frac{\partial A_\lambda}{\partial x^\beta}).$$
so that we find back \([\text{(3.54)}]\). For example we notice the term \(\Theta^\nu_\beta = (-1/4 F_{\mu\nu} F^{\mu\nu}) - (F^{\lambda\nu} \partial A_{\lambda})\). We describe, see \([\text{21}]\) the Hamiltonian counterpart of the stress-energy tensor described as the Hamiltonian tensor:

\[
\mathcal{H}(q,p) = \sum_{\alpha,\beta} \mathcal{F}_{\beta}(q,p) \partial_{\alpha} \otimes dx^\beta \quad \text{with} \quad \mathcal{F}_{\beta}(q,p) = \frac{\partial L}{\partial v_{\mu\alpha}}(q,\mathcal{V}(q,p)) v_{\nu\beta} - \delta^\nu_{\beta} L(q,\mathcal{V}(q,p)) \quad \text{(3.57)}
\]

Notice that if \((x, A(x), dA(x), h) \square (q,p)\) then \(\mathcal{H}_{\beta}(q,p) = -\mathcal{F}_{\beta}(x)\), so that:

\[
\mathcal{H}_{\beta}(q,p) = \delta^\nu_{\beta} \mathcal{H}(q,p) - \left\langle p, Z_1(q,p) \wedge \cdots \wedge Z_{\mu-1}(q,p) \wedge \frac{\partial}{\partial x^\nu} \wedge Z_{\mu+1}(q,p) \wedge \cdots \wedge Z_n(q,p) \right\rangle
\]

\[
= \delta^\nu_{\beta} \mathcal{H}(q,p) - \frac{\partial \langle p, z \rangle}{\partial z^\nu} \bigg|_{z=Z(q,p)}.
\]

First, we are interested in the component \(\mathcal{H}_{\alpha}(x, A)\):

\[
\mathcal{H}_{\alpha}(x, A) = \mathcal{H}(q,p) - \left( p, \frac{\partial}{\partial x^\alpha} \wedge z_1 \wedge \cdots \wedge z_{n-1} \right).
\]

Let us evaluate:\n
\[
\langle p, Z \rangle = \theta^{dW \alpha}(z) = \mathcal{E} \mathcal{H}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
= \mathcal{E} \mathcal{H}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
+ \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
= \mathcal{E} \mathcal{H}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
+ \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z) + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
\langle p, Z \rangle = \mathcal{E} + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
\langle p, Z \rangle = \mathcal{E} + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

\[
\langle p, Z \rangle = \mathcal{E} + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)
\]

we finally find: \(\langle p, Z \rangle = \mathcal{E} + \Pi A_{\mu} dA_{\mu} \wedge \eta_{\nu}(z)\) is written:

\[
\mathcal{H}_{\alpha}(x, A) = \mathcal{H}(q,p) - \left( \mathcal{E} + \sum_{\mu=0}^{n-1} \sum_{\alpha=1}^{n-1} \Pi A_{\mu} \partial_{\alpha} A_{\mu} \right).
\]

and we finally notice that if we integrate the corresponding \((n - 1)\)-forms on a slice of time defined by \([\text{3.54]}\):

\[
\int_{\Sigma_0} \mathcal{F}_{\alpha} d\eta_0 = \int_{\Sigma_0} \eta_0
\]

Notice that we also recover the other component of the stress energy tensor \([\text{(3.57)}]\) via the study of the following terms:

\[
\mathcal{H}_{\alpha}(x, A) = -\langle p, z_0 \wedge z_1 \wedge \cdots \wedge z_{n-1} \wedge \partial_{\alpha} \wedge z_{n+1} \wedge \cdots \wedge z_{n-1} \rangle
\]

\[
\mathcal{H}_{\beta}(x, A) = -\langle p, \partial_{\beta} \wedge z_1 \wedge \cdots \wedge z_{n-1} \rangle
\]

\[
\mathcal{H}_{\gamma}(x, A) = \delta^\gamma_\beta \mathcal{H}(q,p) - \langle p, z_0 \wedge z_1 \wedge \cdots \wedge z_{n-1} \wedge \partial_{\gamma} \wedge z_{n+1} \wedge \cdots \wedge z_{n-1} \rangle
\]
Notice that some related work on the Noether theorem for covariant field theory is found in [15] [16].

4 Dynamical equations and canonical variables

4.1 Graded structures and Grassman variables

In this section, we heuristically illustrate the tension between the graded structure and the copolarization process. One of the fundamental interest for field theory is the search for the good Poisson structure. The copolarization process and the modern classification concerning the distinction between algebraic observable forms (AOF) and observable forms (OF) appear in the work of F. Hélein and J. Kouneiher [26] [27]. This work emerge from the question of graded structure and their non-uniqueness feature for \((p-1)\)-forms of various degree. Before we give some aspects of the copolarization process in [43], we emphasize this search by means of some remark on Graded structure, Grassman variables and the notion of superform found in [24].

Graded structures. They appear in the traditional (DDW) setting as the algebraic structures related to algebraic forms of arbitrary degree. We emphasize two main group of references. The first is found in the work of I.V. Kanatchikov [30] [31] where very interesting ideas on the graded setting are developed, in particular in connection with dynamical evolution for forms of lower degrees. The second concerns the closely related work of M. Forger, C. Paufler and H. Römer [12] [13] [14]. The equation under consideration is \(X_{\varphi} \Omega = d^{n-r} \varphi \) so we say that the Hamiltonian multivector field \(X_{\varphi} \) is associated with the Hamiltonian form \(n-r\varphi\). Neither \(X_{\varphi}\) nor \(n-r\varphi\) are uniquely defined. Equivalently, the kernel of \(\Omega\) on multivector fields is non-trivial. This simple fact reflects a non unique correspondence between Hamiltonian multivector fields and Hamiltonian forms [12] [13].

In the context of (MG), we concentrate on the example of I.V. Kanatchikov bracket with main focus on graded antisymmetric bracket:

\[
\{\varphi, \theta\} = (-1)^{(n-r-1)(n-s-1)} \{\theta, \varphi\},
\]

where,

\[
J_{\text{ham}}^r(\mathcal{M}) \times J_{\text{ham}}^s(\mathcal{M}) \rightarrow J_{\text{ham}}^{r+s-n+1}(\mathcal{M})
\]

\[
(\varphi, \theta) \mapsto \{\varphi, \theta\} = (-1)^{n-r} X_{\varphi} \varphi \}
\]

Notice that, as told before, we have a fundamental ambiguity in the search for Poisson structure for forms of arbitrary degree \(p\) and \(q\). To be more precise, the ambiguity lays in the choice of the objects themselves, namely in the choice of the Hamiltonian multivectors fields \(X_{\varphi}\) and \(X_{\theta}\). This ambiguity takes its origin in the pairing of forms and vector fields via the study of the equation \(d\varphi = -\Xi_{\varphi} \Omega\), and beyond by the study of the map:

\[
\begin{array}{ccc}
\Lambda^p T_m \mathcal{M} & \rightarrow & \Lambda^{n+1-p} T_m \mathcal{M} = \Lambda^{r+1} T_m^* \mathcal{M} \\
\Xi & \mapsto & \Xi \Omega \\
\end{array}
\]

\(^{8}\) (We denote \(r = n - p\) and \(s = n - q\)
Grassman variables. They appear in the BRST and BV formalisms with the use of ghosts, anti-ghosts ... Here we feel the connection to the conceptual setting of the supersymmetric landscape [10] - where additional virtual matter degree of freedom is related to the notion of ghost. From the mathematical perspective, the graded scenario and the Gerstenhaber algebra [18] are important geometric structure. In the multisymplectic landscape, we notice the use of Grassman-odd variables in the work of F. Hélein and J. Kouneiher [24] or of S. Hrabak [28] [29].

We notice the early attempt of F. Hélein and J. Kouneiher concerning the distinction of the internal, the external and the \( ^{5}p \)-bracket - see [24]. These considerations are connected to the good expression of the dynamics. We delimitate two directions in connection with this. The first is the relation between the dynamical equations and the external bracket \( \{ \mathcal{H}\mathfrak{d}\mathfrak{y}, \lambda \} \) [24] - see also the dynamical evolution equations given by I.V. Kanatchikov [30] [31]. The second is the introduction of Grassman extra variables in [24] - which makes connection with the work of S. Hrabak [28] [29]. We write dynamical equations in the form [3]:

\[
d A = \{ \mathcal{H}\mathfrak{d}\mathfrak{y}, A \} \quad \text{and} \quad d \pi = \{ \mathcal{H}\mathfrak{d}\mathfrak{y}, \pi \}, \quad (4.1)
\]

This is related to the definition of a Poisson bracket between \( \mathcal{H}\mathfrak{d}\mathfrak{y} \in \Gamma(\mathcal{M}, \Lambda^n T^*\mathcal{M}) \) and \((p - 1)\)-forms, with \( 1 \leq p \leq n - 1 \). We adopt here the terminology developed in [24] where we find the following different brackets: the external \( p \)-brackets, the internal \( p \)-brackets and the \( ^{5}p \)-bracket.

(i) Internal \( p \)-bracket. If \( \lambda, \kappa \in \mathcal{P}_0^{n-1}(\mathcal{M}) \) with \( \mathcal{P}_0^{n-1}(\mathcal{M}) \) the set of all algebraic observable \((n - 1)\)-forms, we define the internal \( p \)-bracket on \( \mathcal{P}_0^{n-1}(\mathcal{M}) \):

\[
\{ \lambda, \kappa \} = \Xi(\kappa) \downarrow \Xi(\lambda) \downarrow \Omega. \quad (4.2)
\]

The internal bracket is basically defined on algebraic \((n - 1)\)-forms.

(ii) External \( p \)-bracket. Now we extend the previous definition to the case where \( \varphi \in \Gamma(\mathcal{M}, \Lambda^p T^*\mathcal{M}) \), with \( 1 \leq p \leq n \) and \( \lambda \in \mathcal{P}_0^{n-1}(\mathcal{M}) \) we obtain the external \( p \)-bracket:

\[
\Gamma(\mathcal{M}, \Lambda^p T^*\mathcal{M}) \times \mathcal{P}_0^{n-1}(\mathcal{M}) \rightarrow \Gamma(\mathcal{M}, \Lambda^p T^*\mathcal{M})
\]

\[
(\varphi, \lambda) \mapsto \{ \lambda, \varphi \} = -\Xi(\lambda) \downarrow d\varphi.
\]

Notice that \( \{ \varphi, \lambda \} = -\{ \lambda, \varphi \} = \Xi(\lambda) \downarrow d\varphi \). The interesting case for dynamical evolution is when \( \varphi = \mathcal{H}\mathfrak{d}\mathfrak{y} \). Then we notice that for any \( \lambda, \in \mathcal{P}_0^{n-1} \) we have the following relation \( \{ \mathcal{H}\mathfrak{d}\mathfrak{y}, \lambda \} = -\Xi(\lambda) \downarrow d\mathcal{H} \wedge \mathfrak{d}\mathfrak{y} \).

(iii) \( ^{5}p \)-bracket. The method developed in [24] for the construction of a bracket between \((p - 1)\) forms for \( p \) of arbitrary degree \((1 \leq p \leq n)\) is roughly speaking the following. F. Hélein and J. Kouneiher introduced anticommuting Grassman variables \( \tau_1 \cdots \tau_n \) that behave under change of coordinates like \( \partial_1 \cdots \partial_n \). In this case, an arbitrary form \( \varphi \in \Gamma(\mathcal{M}, \Lambda^p T^*\mathcal{M}) \) depends on the set of variables [10] (\( \tau_\alpha, x^\alpha, A_\mu, \epsilon, \Pi^\lambda\mu \)). Grassman variables \( \tau_\alpha \) are related to the notion of superform in [24]. For any \( \lambda \in \mathcal{P}_0^{n-1}(\mathcal{M}) \) such that for all \( 1 \leq \alpha_1 \leq \cdots \leq \alpha_{n-p} \leq n \) we have:

\[
^{5}\lambda = \sum_{\alpha_1 < \cdots < \alpha_{n-p}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \lambda.
\]

\(^{5}\lambda\) where \( d \) is the differential along a graph \( \Gamma \) of a solution of the Hamilton equations.

\(^{5}\lambda\) For a more detailed presentation of these Grassmannian variables \( \tau_\alpha \) - and an intrinsic geometrical picture - see [24].
We define also a \( \mathfrak{s}p \)-bracket for \( \varphi \in \Gamma(\mathcal{M}, \Lambda^n T^*\mathcal{M}) \),
\[
\{ \varphi, \lambda \}_s = -\Xi(\lambda) \mathcal{J} d\varphi = - \sum_{\alpha_1 < \cdots < \alpha_{n-p}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} \Xi(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \lambda) \mathcal{J} d\varphi.
\]

Let \( \lambda \) be an admissible form \( [24] \), and let \( \Gamma \) be a \( n \)-dimensional submanifold of \( \mathcal{M} \) which is a graph over \( \mathcal{X} \), then for any oriented \( \Sigma_p \subset \Gamma \) with \( \dim(\Sigma_p) = p \) we have:
\[
\int_{\Sigma_p} \{ \mathcal{H} d\eta, \mathfrak{s}\lambda \}_s = \int_{\Sigma_p} \{ \mathcal{H} d\eta, \lambda \}.
\]

However, we do not insist on this notions of the \( \mathfrak{s}p \)-bracket since for good treatment of the dynamics, we will choose for adequate bracket a slightly different object\(^{11}\). The philosophy which underlines the \( \mathfrak{s}p \)-bracket is strongly connected to the one found in the work of S. Hrabak \( [28] [29] \) and concerns the multisymplectic formulation of the classical BRST symmetry for first order field theories\(^{12}\).

### 4.2 Dynamical equations

In this subsection we recover the dynamical equations with two methods. First with the tool of superforms and the \( \mathfrak{s}p \)-bracket, then with external brackets: \( \{ \mathcal{H} d\eta, P_\phi \} \) and \( \{ \mathcal{H} d\eta, Q_\psi \} \).

**Superforms, Grassman variables and dynamical equations.** In the context of Maxwell theory - we refer to \( [24] \) for detailed calculation - the 1-form \( A \) and the Faraday \( (n-2) \)-form \( \pi \) lead, via the use of the superform \( \mathfrak{s}A \) and \( \mathfrak{s}\pi \), to the following dynamical equations:

\( \mathfrak{i} \) \( dA = \{ \mathcal{H} d\eta, A \} \)

\( \mathfrak{ii} \) \( d\pi = \{ \mathcal{H} d\eta, \pi \} \)

\( \mathfrak{i} \) \( dA = \sum_{\alpha < \beta} \eta_{\alpha \beta} \Pi^{\mu \nu} dx^\alpha \wedge dx^\beta \)

\( \mathfrak{ii} \) \( d\pi = J^\alpha d\eta_\alpha \)

Canonical bracket is described via the computation of the \( \mathfrak{s}p \)-bracket \( \{ \mathfrak{s}\pi, \mathfrak{s}A \}_s \). The important point is that the additional Grassman variables are only a tool, as in the ghost and anti-ghost case, they disappear at the end of the calculation.

**External bracket and dynamical equations.** We also recover the dynamical equations using the following external brackets: \( \{ \mathcal{H} d\eta, P_\phi \} \) and \( \{ \mathcal{H} d\eta, Q_\psi \} \). Let us compute the bracket \( \{ \mathcal{H} d\eta, P_\phi \} \). By definition,
\[
\{ \mathcal{H} d\eta, P_\phi \} = -\Xi(P_\phi) \mathcal{J} d\mathcal{H} \wedge d\eta
\]
\[
= -\left[ \phi_\mu(x) \frac{\partial}{\partial A_\mu} - \left( \frac{\partial \phi_\mu}{\partial x^\nu}(x) \Pi^{A_\mu \nu} \right) \frac{\partial}{\partial t} \right] \mathcal{J} d\mathcal{H} \wedge d\eta
\]
\[
= \left[ -\phi_\mu(x) \frac{\partial \mathcal{H}}{\partial A_\mu} + \frac{\partial \phi_\mu(x)}{\partial t} \Pi^{A_\mu \nu} \right] d\eta
\]

\(^{11}\)The construction based on copolarization of the multisymplectic manifold allows us to define observable forms of any degree collectively. Then in the next section we find good bracket described by F. Hélein and J. Kouneiher without this superform artifact.

\(^{12}\)Here lay the connection with the conceptual setting of huge domain of modern investigation of mathematical physics. This concerns the ghosts and the anti-ghosts in the BRST formalism developed by C. Becchi, A. Rouet, R. Stora, I.V. Tyutin \( [2] [48] \) and the related BV setting of I.A. Batalin and G.A. Vilkovisky \( [1] \).
So, along the graph of a solution we have : \( \{ \mathcal{H} \mathfrak{d} \mathfrak{e}, P_{\phi} \} \big|_\Gamma = \left[ -\phi \mu J^\mu + \partial \nu \phi \mu F^{\mu \nu} \right] \mathfrak{d} \mathfrak{e} \). On the other side, since \( d(P_{\phi}) = \left[ F^{\mu \nu} \partial \phi \mu \partial x^\nu d\eta + \phi \mu(x) d\Pi^{\mu \nu} \wedge d\eta \right] \), we obtain:

\[
d(P_{\phi}) \big|_\Gamma = F^{\mu \nu} \partial \phi \mu \partial x^\nu d\eta + \phi \mu(x) \frac{\partial F^{\mu \nu}}{\partial x^\nu} (x) d\eta = \left[ F^{\mu \nu} \partial \phi \mu + \phi \mu \partial \nu F^{\mu \nu} \right] d\eta.
\]

So that we finally observe the first set of dynamical evolution equations, along a graph of generalized Hamilton equations:

\[
d(P_{\phi}) \big|_\Gamma = \{ \mathcal{H} \mathfrak{d} \mathfrak{e}, P_{\phi} \} \big|_\Gamma \iff \partial \nu F^{\mu \nu} = J^\mu. \quad (4.4)
\]

Now we are interested in the second bracket:

\[
\{ \mathcal{H} \mathfrak{d} \mathfrak{e}, Q^\psi \} = \Xi(Q^\psi) \mathfrak{J} d\mathcal{H} \wedge d\eta
\]

\[
= -\left[ (\Lambda^\mu \frac{\partial \psi^{\mu \nu}}{\partial x^\nu} (x) \frac{\partial \psi^{\mu \nu}}{\partial \eta} + \psi^{\mu \nu} (x) \frac{\partial \psi^{\mu \nu}}{\partial \Pi_{A^\mu}} \right] \mathfrak{J} d\mathcal{H} \wedge d\eta
\]

\[
= \left[ (\Lambda^\mu \frac{\partial \psi^{\mu \nu}}{\partial x^\nu}) + \psi^{\mu \nu} (x) \frac{\partial \mathcal{H}}{\partial \Pi_{A^\mu}} \right] d\eta.
\]

So that along a graph \( \Gamma \) of a solution of the Hamilton equations we find the following relation:

\[
\{ \mathcal{H} \mathfrak{d} \mathfrak{e}, Q^\psi \} \big|_\Gamma = \left[ (\Lambda^\mu \frac{\partial \psi^{\mu \nu}}{\partial x^\nu}) + \psi^{\mu \nu} (x) \frac{\partial \mathcal{H}}{\partial \Pi_{A^\mu}} \right] d\eta. \quad \text{Whereas the expression of} \ d(Q^\psi) \ \text{is written}:
\]

\[
d(Q^\psi) = \left[ (\Lambda^\mu \frac{\partial \psi^{\mu \nu}}{\partial x^\nu}) \right] d\eta + \psi^{\mu \nu} (x) dA^\mu \wedge d\eta = \left[ (\Lambda^\mu \frac{\partial \psi^{\mu \nu}}{\partial x^\nu} (x) + \psi^{\mu \nu} (x) \partial \nu A^\mu \right] d\eta
\]

So that we finally observe:

\[
d(Q^\psi) \big|_\Gamma = \{ \mathcal{H} \mathfrak{d} \mathfrak{e}, Q^\psi \} \big|_\Gamma \iff \ F^{\mu \nu} = \partial \nu A^\mu - \partial \nu A^\mu \quad (4.5)
\]

The dynamical evolution is encapsulated by the relations \( (4.4) \) and \( (4.5) \):

\[
\left| d(Q^\psi) \big|_\Gamma = \{ \mathcal{H} \mathfrak{d} \mathfrak{e}, Q^\psi \} \big|_\Gamma \\
d(P_{\phi}) \big|_\Gamma = \{ \mathcal{H} \mathfrak{d} \mathfrak{e}, P_{\phi} \} \big|_\Gamma
\]

### 4.3 Copolarization and observables \((p - 1)\)-forms

**Copolarization.** The copolarization process corresponds to the collective definition of observable forms and emerges from *Relativity Principle* and *dynamics*. For details we refer to F. Hélein and J. Kouneiher [25] [26] [27]. Here we set out some key features of the notion of copolarization. In particular we give the general definition:

**Definition 4.3.1.** Let \( (\mathcal{M}, \Omega) \) be a multisymplectic manifold. A copolarization on \( (\mathcal{M}, \Omega) \) is a smooth vector sub-bundle \( P^*_m T^* \mathcal{M} \subset \Lambda^* T^* \mathcal{M} \) which satisfies:

1. \( P^*_m T^* \mathcal{M} = \bigoplus_{1 \leq i \leq n} P^*_i T^* \mathcal{M} \)
2. Locally, for any \( m \in \mathcal{M} \), \( (P^*_m T^* \mathcal{M}, +, \wedge) \) is a subalgebra of \( (\Lambda^m T^*_m \mathcal{M}, +, \wedge) \)
3. \( \forall m \in \mathcal{M}, \forall \phi \in \Lambda^m T^*_m \mathcal{M}, \phi \in P^*_m T^*_m \mathcal{M} \) if and only if \( \forall X, \tilde{X} \in \mathcal{O}_m, X \mathfrak{J} \Omega = \tilde{X} \mathfrak{J} \Omega \Rightarrow \phi(X) = \phi(\tilde{X}) \).
We say that a multisymplectic manifold \((\mathcal{M},\Omega)\) is equipped with the copolarization \(P^*_\omega T^*\mathcal{M}\). The notion of copolarization intrinsically defines for any \(1 \leq p \leq n\) the set \(P^*_{\omega} \mathcal{M}^{p-1}\), namely the set of observable \((p - 1)\)-forms \(\varphi\) by \(\forall m \in \mathcal{M}, d\varphi_m \in P^*_{\omega} T^*_m \mathcal{M}\). We refer to \([24, 25, 26, 27]\) for the construction of the standard copolarization. The copolarization is the natural geometrical setting to describe the canonical forms for field theory based on canonical variables such as a potential 1-form.

In the case of Maxwell theory, the two canonical forms are the potential 1-form \(A = A_\mu dx^\mu\) and the so-called Faraday 2-form (in the 4 dimensional case):

\[
\pi = 1/2 \Pi^A_{\mu\nu} d\eta_{\mu\nu} = 1/2 \sum_{\mu,\nu} \Pi^A_{\mu\nu} \partial_\mu \eta \partial_\nu \eta.
\]

In a more general perspective - for gravity\(^{13}\) or non-abelian Yang-Mills theories - canonical forms are described by a couple \((\omega, \varpi)\). The general setting allows us to construct a well-defined Poisson bracket between observable functionals related to the canonical forms \((\omega, \varpi)\):

\[
\left\{ \int_{\Sigma \cap \gamma^c_\omega} \varpi, \int_{\Sigma \cap \gamma^c_\omega} \omega \right\}(\Gamma) = \sum_{m \in \Sigma \cap \gamma^c_\omega \cap \Sigma \cap \Gamma} \mathcal{G}(m).
\]

We give more details later about the bracket \((4.9)\) and the related geometrical objects \(\Sigma \cap \gamma^c_\omega\), \(\Sigma \cap \gamma^c_\omega\) and \(\Sigma \cap \gamma^c_\omega \cap \gamma^c_\omega \cap \gamma^c_\omega \cap \Gamma\), as well as the counting object \(\mathcal{G}(m)\). Notice that the study of \((p - 1)\)-forms involves analogous definitions for \(slices\) in this case. We have the following:

**Definition 4.3.2.** A slice of codimension \((n - p + 1)\) is a submanifold \(\Sigma \subset \mathcal{M}\) of codimension \((n - p + 1)\) such that \(T_m \mathcal{M}/T_m \Sigma\) is smoothly oriented with regard to \(m\) and, such that for any \(\Gamma \in \mathcal{E}^H\), \(\Sigma\) is transverse to \(\Gamma\).

We refer to \([24, 27]\) for the question of the orientation of the intersection \(\Sigma \cap \Gamma\). The straightforward analogue of definition \((4.3.3)\) for the case of arbitrary \((p - 1)\)-forms is:

**Definition 4.3.3.** Let \(\Sigma\) be a slice of codimension \((n - p + 1)\) and let \(\varphi\) be an algebraic observable \((p - 1)\)-form. An observable functional \(F = \int_\Sigma \varphi\) defined on the set of \(n\)-dimensional submanifolds \(\mathcal{E}^H\) given by the map:

\[
F_\varphi = \int_\Sigma \varphi : \left\{ \begin{array}{c} \mathcal{E}^H \rightarrow \mathbb{R} \\ \Gamma \mapsto F(\Gamma) \end{array} \right\} = \int_{\Sigma \cap \Gamma} \varphi
\]

The notion of copolarization definitively emerges from the philosophy of (GR). This highlights the fact that we cannot evaluate \(d\varphi\) along a Hamiltonian \(n\)-vector \(X\). If \(1 \leq p < n\) then an arbitrary \((p - 1)\)-form is necessarily of maximum degree \((n - 2)\). Indeed, the interesting question resides on the interpretation of the object \(d\varphi|_X\). This lack is supply precisely through the notion of copolarization. We construct a set of \(n\) 0-forms \(\{\rho_i\}_{1 \leq i \leq n}\). These \(n\) 0-forms are found in the copolarization of the multisymplectic manifold \((\mathcal{M}, \Omega)\). These are observables 0-forms: \(\forall 1 \leq i \leq n, \rho_i \in P^*_{\omega} \mathcal{M}\). Locally we write for \(m \in \mathcal{M}\) \(\forall 1 \leq i \leq n, d\rho_i \in P^*_{\omega} T^*_m \mathcal{M}\). Hence, we reach the full dynamical duality: the evaluation of \(\bigwedge_{1 \leq i \leq n} d\rho_i\) on a Hamiltonian vector field \(X\).

The fact that \(\bigwedge_{1 \leq i \leq n} d\rho_i(X)\) only depends on \(d\mathcal{H}_m\) means that \(\bigwedge_{1 \leq i \leq n} d\rho_i = d\rho_1 \wedge \cdots \wedge \rho_n(X)\) is a copolar form.

\(^{13}\)see the companion paper on Palatini Gravity \([51]\).
In the philosophy of (GR) this is fully acceptable since we never measure an observable per se but we only compare observable quantities between each others. Following [24, 25, 26, 27], the \((p - 1)\)-bracket is related to an equivalence class\(^{14}\) of (decomposable) Hamiltonian vector fields \([X]^{\Lambda_\mathcal{H}}\). We also have the notion of algebraic copolarization. This involves the same set of rules but with the replacement of \(P^nT^*_n\mathcal{M}\) by \(P^n_{\partial}T^*_n\mathcal{M}\) and where the point \([3]\) in (4.3.1) is replaced by \([3]\)):

\[\forall m \in \mathcal{M}, \forall \phi \in \Lambda^nT^*_m\mathcal{M}, \phi \in P^n_{\partial}T^*_n\mathcal{M} \text{ if and only if } \forall X, \tilde{X} \in \Lambda^nT^*_m\mathcal{M}, X \cup \Omega = \tilde{X} \cup \Omega \Rightarrow \phi(X) = \phi(\tilde{X}).\]

### 4.4 Copolarization and canonical variables

We recall the result obtained by F. Hélein and J. Kouneiher [24], they give a possible copolarization of \(\mathcal{M}_{\text{Maxwell}}, \Omega^{\text{DDW}}\) for Maxwell theory with \(\Omega^{\text{DDW}} = dx \wedge dy + d\sigma \wedge dA\):

\[
\begin{align*}
P^1T^*\mathcal{M}_{\text{Maxwell}} &= \bigoplus_{0 \leq \mu \leq 3} dx^\mu \\
P^2T^*\mathcal{M}_{\text{Maxwell}} &= \bigoplus_{0 \leq \mu_1 < \mu_2 \leq 3} dx^{\mu_1} \wedge dx^{\mu_2} + dA \\
P^3T^*\mathcal{M}_{\text{Maxwell}} &= \bigoplus_{0 \leq \mu_1 < \mu_2 < \mu_3 \leq 3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} + \bigoplus_{0 \leq \mu \leq 3} dx^\mu \wedge dA + d\pi \\
P^4T^*\mathcal{M}_{\text{Maxwell}} &= \text{d}y + \bigoplus_{0 \leq \mu \leq 3} \frac{\partial}{\partial x^\mu} \omega^{\text{DDW}} \bigoplus_{0 \leq \mu_1 < \mu_2 \leq 3} \omega^{\text{DDW}} + \bigoplus_{0 \leq \mu_1 < \mu_2 \leq 3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dA + d\pi.
\end{align*}
\]

The notion of copolarization describe the data for forms of various degrees. We can find several copolarizations for a given theory. The construction of the previous copolarization (4.8) allows us to construct a well-defined Poisson bracket between observable functionals related to the canonical forms \((A, \pi)\):

\[
\left\{ \int_{\Sigma \cap \gamma_n} \pi, \int_{\Sigma \cap \gamma_c} A \right\}(\Gamma) = \sum_{m \in \Sigma \cap \gamma_n \cap \gamma_c \cap \Gamma} \mathcal{G}(m). \tag{4.9}
\]

In [26, 27], it is emphasized that there is strong reasons to not include \(C^1T^*\mathcal{M}_{\text{Maxwell}} = \bigoplus_{0 \leq \mu \leq 3} dA_\mu\) in \(P^1T^*\mathcal{M}_{\text{Maxwell}}\). However, let us consider this set of 1-forms \(C^1T^*\mathcal{M}_{\text{Maxwell}}\) and construct the set \(P^1T^*\mathcal{M}_{\text{Maxwell}} = P^1T^*\mathcal{M}_{\text{Maxwell}} \oplus C^1T^*\mathcal{M}_{\text{Maxwell}}\). Then, we consider the smooth vector sub-bundle: \(\mathcal{P}T^*\mathcal{M}_{\text{Maxwell}} = \bigoplus_{1 \leq i \leq 4} P^iT^*\mathcal{M} \subset \Lambda^*T^*\mathcal{M}_{\text{Maxwell}}\), as a copolarization candidate of \((\mathcal{M}_{\text{Maxwell}}, \Omega^{\text{DDW}})\). Here,

\[
\mathcal{P}T^*\mathcal{M}_{\text{Maxwell}} = P^1T^*\mathcal{M}_{\text{Maxwell}} \oplus C^1T^*\mathcal{M}_{\text{Maxwell}} \tag{4.10}
\]

with, \(C^2T^*\mathcal{M}_{\text{Maxwell}}\), \(C^3T^*\mathcal{M}_{\text{Maxwell}}\) and \(C^1T^*\mathcal{M}_{\text{Maxwell}}\) are respectively written

\[
\begin{align*}
C^2T^*\mathcal{M}_{\text{Maxwell}} &= \bigoplus_{0 \leq \mu_1 < \mu_2 \leq 3} dx^{\mu_1} \wedge dA_{\mu_2} \oplus \bigoplus_{0 \leq \mu_1 < \mu_2 \leq 3} dA_{\mu_1} \wedge dA_{\mu_2}
\end{align*}
\]

\(^{14}\)If \(X \sim \tilde{X}\), we have for any \(1 \leq p \leq n\) and \(\phi \in P^pT^*_n\mathcal{M}, X \lrcorner \phi \sim \tilde{X} \lrcorner \phi\) so that we define the equivalence class \([X] \lrcorner \phi = [X \lrcorner \phi] \in P^n_{\partial}T^*_n\mathcal{M} \).
In fact, there are several obstructions for $\mathbf{P}^* T^* \mathcal{M}^\text{Maxwell}$ to describe a good copolarization. We focus, for a given $1 \leq \mu \leq n$ on the following form $\rho_\mu = dA_\mu \wedge d\pi$. The exterior derivative $d\rho_\mu$ is written:

$$d\rho_\mu = d(dA_\mu \wedge d\pi) = d^2 A_\mu \wedge d\pi - dA_\mu \wedge d\pi = -dA_\mu \wedge d\left(\frac{1}{2} \Pi A_\nu^\mu d\eta_{\rho\nu}\right) = -\frac{1}{2} dA_\mu \wedge d\Pi A_\nu^\mu \wedge d\eta_{\rho\nu}.$$ 

The form $\rho_\mu$ is not an $(OF \ (n-1))$-form : $\rho_\mu \not\in \mathcal{V}^{n-1}_q(\mathcal{M}^\text{Maxwell})$ so that $\rho_\mu$ is not an algebraic observable $(n-1)$-form (AOF), $\rho_\mu \not\in \mathcal{V}^{n-1}_0(\mathcal{M}^\text{Maxwell})$. Let us consider two decomposable vector fields $X, \bar{X} \in \Lambda^n T\mathcal{M}^\text{Maxwell}$. The $n$-vector $X \in \mathcal{D}_n^\infty \mathcal{M} \subset \Lambda^n T(q,p) \mathcal{M}$ is written $X = X_1 \wedge \ldots \wedge X_n$ and $\forall \nu = 1 \ldots n$,

$$X_\alpha = \frac{\partial q(x), p(x)}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \Theta_{\alpha\mu} \frac{\partial}{\partial A_\mu} + \Upsilon_{\alpha} \frac{\partial}{\partial t} + \Upsilon_{\alpha} A_\nu^\mu \frac{\partial}{\partial \Pi A_\nu^\mu},$$

where $\forall 1 \leq \alpha \leq n$ and $\Upsilon_{\alpha \nu} = -\Upsilon_{\alpha} A_\nu^\mu$. From (2.14), we have the expression of $X \llcorner \Omega^\text{DDW}$ and $\bar{X} \llcorner \Omega^\text{DDW}$:

$$X \llcorner \Omega^\text{DDW} = d\pi - \Theta_{\rho\mu} dx^\rho + \Theta_{\nu\mu} d\Pi A_\nu^\mu - A_\nu^\mu dA_\mu + (\Upsilon_{\rho} A_\nu^\mu \Theta_{\nu\mu} - \Upsilon_{\nu} A_\nu^\mu \Theta_{\rho\mu}) dx^\rho,$$

$$\bar{X} \llcorner \Omega^\text{DDW} = d\pi - \bar{\Theta}_{\rho\mu} dx^\rho + \bar{\Theta}_{\nu\mu} d\Pi A_\nu^\mu - \bar{A}_\nu^\mu dA_\mu + (\bar{\Upsilon}_{\rho} A_\nu^\mu \Theta_{\nu\mu} - \bar{\Upsilon}_{\nu} A_\nu^\mu \Theta_{\rho\mu}) dx^\rho.$$ 

so that $X \llcorner \Omega^\text{DDW} = \bar{X} \llcorner \Omega^\text{DDW}$ gives us the following relations:

$$\Theta_{\rho\mu} = \bar{\Theta}_{\rho\mu} \quad \text{and} \quad -\Upsilon_{\rho} + (\Upsilon_{\rho} A_\nu^\mu \Theta_{\nu\mu} - \Upsilon_{\nu} A_\nu^\mu \Theta_{\rho\mu}) = -\bar{\Upsilon}_{\rho} + (\bar{\Upsilon}_{\rho} A_\nu^\mu \Theta_{\nu\mu} - \bar{\Upsilon}_{\nu} A_\nu^\mu \Theta_{\rho\mu}).$$

(4.11)

Given the relations (4.11), we want to know if we observe $d\rho_\mu (X) = d\rho_\mu (\bar{X}) \forall X, \bar{X} \in \Lambda^n T\mathcal{M}^\text{Maxwell}$. In the following we treat the example of copolarization for Maxwell theory. The four dimensional case is a straightforward application of the calculation exposed below.
First, we focus on the 2D-case without the imposition of Dirac constraint set (we work on $\mathcal{M}^{\text{DDW}}$). In that case with $X, \mathcal{X} \in D^*_0 \mathcal{M}^{\text{DDW}} \subset \Lambda^* \mathcal{T} \mathcal{M}^{\text{DDW}},$

\begin{align*}
X_1 &= \frac{\partial}{\partial x^1} + \Theta_1 \mu \frac{\partial}{\partial A_\mu} + \gamma_1 \frac{\partial}{\partial \epsilon} + \gamma_1^{A_\mu \nu} \frac{\partial}{\partial \Pi^{A_\mu \nu}}, \\
X_2 &= \frac{\partial}{\partial x^2} + \Theta_2 \mu \frac{\partial}{\partial A_\mu} + \gamma_2 \frac{\partial}{\partial \epsilon} + \gamma_2^{A_\mu \nu} \frac{\partial}{\partial \Pi^{A_\mu \nu}}. 
\end{align*}

(4.12)

We denote $\frac{\partial}{\partial x^\mu} = \partial_\mu, \frac{\partial}{\partial A_\mu} = \partial^A_\mu, \frac{\partial}{\partial \epsilon} = \partial_\epsilon$ and finally $\frac{\partial}{\partial \Pi^{A_\mu \nu}} = \partial_{A_\mu \nu}.$

\begin{align*}
X &= \partial_1 \wedge \partial_2 + \Theta_2 \partial_1 \wedge \partial^A_1 + \gamma_2 \partial_1 \wedge \partial_\epsilon + \gamma_2^{A_\mu \nu} \partial_1 \wedge \partial_{A_\mu \nu} + \Theta_1 \partial^A_\mu \wedge \partial_\epsilon + \gamma_2^{A_\mu \nu} \partial_{A_\mu \nu} \\
&\quad + \gamma_1 \partial_\epsilon \wedge \partial_2 + \gamma_1^{A_\mu \nu} \partial_{A_\mu \nu} \wedge \partial_2 + \gamma_2^{A_\mu \nu} \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma} + \text{terms} \partial_\epsilon \wedge \partial_{A_\mu \nu} \text{ and } \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma}.
\end{align*}

Then we expand the indices :

\begin{align*}
X &= \partial_1 \wedge \partial_2 + \Theta_2 \partial_1 \wedge \partial^A_1 + \Theta_2 \partial_1 \wedge \partial^A_2 + \Theta_2 \partial_2 \wedge \partial_\epsilon + \theta_2 \partial_2 \wedge \partial_{A_\mu} + \gamma_2 \partial_2 \wedge \partial_{A_\mu} + \gamma_2^{A_\mu \nu} \partial_2 \wedge \partial_{A_\mu \nu} \\
&\quad + \gamma_2^{A_\mu \sigma} \partial_{A_\mu \sigma} \wedge \partial_2 + \gamma_2^{A_\mu \nu} \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma} + \text{terms} \partial_\epsilon \wedge \partial_{A_\mu \nu} \text{ and } \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma}.
\end{align*}

Then we work on a subset of the (DDW) multisymplectic manifold, we only keep track of the concerned terms :

\begin{align*}
X &= \partial_1 \wedge \partial_2 + \Theta_2 \partial_2 \wedge \partial^A_2 + \Theta_1 \partial_2 \wedge \partial^A_1 + \gamma_1 \partial_2 \wedge \partial_{A_\mu} + \gamma_1 \partial_2 \wedge \partial_{A_\mu} + \gamma_2 \partial_1 \wedge \partial_{A_\mu} + \gamma_2 \partial_1 \wedge \partial_{A_\mu} + \gamma_2^{A_\mu \nu} \partial_2 \wedge \partial_{A_\mu \nu} \\
&\quad + \gamma_2^{A_\mu \sigma} \partial_{A_\mu \sigma} \wedge \partial_2 + \gamma_2^{A_\mu \nu} \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma} + \text{terms} \partial_2 \wedge \partial_{A_\mu \nu} \text{ and } \partial_{A_\mu \nu} \wedge \partial_{A_\mu \sigma}.
\end{align*}

Therefore we explicitly have the following calculation :

\begin{align*}
X \jmath \Omega^{\text{DDW}} &= X \jmath (dA \wedge dx^1 \wedge dx^2 + d\Pi^{A_\mu \nu} \wedge dA_1 \wedge d\eta_\mu + d\Pi^{A_\mu \nu} \wedge dA_2 \wedge d\eta_\mu) \\
&= X \jmath (dA \wedge dx^1 \wedge dx^2) + X \jmath (d\Pi^{A_\mu \nu} \wedge dA_1 \wedge d\eta_\mu + d\Pi^{A_\mu \nu} \wedge dA_2 \wedge d\eta_\mu) \\
&\quad + X \jmath (d \Pi^{A_\mu \nu} \wedge A_1 \wedge d\eta_\mu + d \Pi^{A_\mu \nu} \wedge A_2 \wedge d\eta_\mu) \\
&= X \jmath (dA \wedge dx^1 \wedge dx^2 - d \Pi^{A_\mu \nu} \wedge dA_1 \wedge dx^1) \\
&\quad + X \jmath (d \Pi^{A_\mu \nu} \wedge A_2 \wedge dx^1 - d \Pi^{A_\mu \nu} \wedge A_2 \wedge dx^1),
\end{align*}

\begin{align*}
X \jmath \Omega^{\text{DDW}} &= d\eta(X) dA - (dA \wedge d\eta_1)(X) dx^1 - (dA \wedge d\eta_2)(X) dx^2 \\
&\quad + (dA_1 \wedge dx^1)(X) dA^1 - (dA_1 \wedge dx^2)(X) dA_1 + (dA_2 \wedge dx^1)(X) dA_2 \\
&\quad + (dA_2 \wedge dx^2)(X) dA_2 + (dA_1 \wedge dx^1)(X) dA_2 + (dA_2 \wedge dx^2)(X) dA_1 \\
&\quad = dA - (A_1 \wedge dx^1)(X) dA_1 \wedge dx^1 + (A_2 \wedge dx^2)(X) dA_2 \\
&\quad + (A_1 \wedge dx^1)(X) dA_2 \wedge dx^1 + (A_2 \wedge dx^2)(X) dA_1 - (A_1 \wedge dx^2)(X) dA_2 \\
&\quad + (A_2 \wedge dx^1)(X) dA_2 \wedge dx^1 + (A_1 \wedge dx^2)(X) dA_1 - (A_2 \wedge dx^1)(X) dA_2 \\
&\quad = dA - (A_1 \wedge dx^1)(X) dA_1 \wedge dx^1 + (A_2 \wedge dx^2)(X) dA_2 \\
&\quad + (A_1 \wedge dx^1)(X) dA_2 \wedge dx^1 + (A_2 \wedge dx^2)(X) dA_1.
\end{align*}
Then, the relations \( (4.11) \) are written:

\[
\begin{align*}
\Theta_{11} &= \Theta_{11} \\
\Theta_{12} &= \Theta_{12} \\
\Theta_{21} &= \Theta_{21} \\
\Theta_{22} &= \Theta_{22} \\
\gamma_{A1}^{1} + \gamma_{A2}^{2} &= \gamma_{A1}^{1} + \gamma_{A2}^{2} \\
\gamma_{A1}^{1} + \gamma_{A2}^{2} &= \gamma_{A1}^{1} + \gamma_{A2}^{2}
\end{align*}
\]

(4.14)

and

\[
X \cdot \Omega^{\text{DDW}}|_{x=1} = \overline{X} \cdot \Omega^{\text{DDW}}|_{x=1} \quad \text{and} \quad X \cdot \Omega^{\text{DDW}}|_{x=2} = \overline{X} \cdot \Omega^{\text{DDW}}|_{x=2}
\]

(4.15)

with,

\[
\begin{align*}
X \cdot \Omega^{\text{DDW}}|_{x=1} &= -\gamma_{1} + (\gamma_{A1}^{1} \Theta_{21} + \gamma_{A2}^{2} \Theta_{22} - \Theta_{11} \gamma_{A2}^{1} - \Theta_{12} \gamma_{A2}^{2}) \\
X \cdot \Omega^{\text{DDW}}|_{x=2} &= -\gamma_{2} + (\gamma_{A1}^{1} \Theta_{11} + \gamma_{A2}^{2} \Theta_{12} - \Theta_{11} \gamma_{A1}^{1} - \Theta_{12} \gamma_{A2}^{2}) \\
\overline{X} \cdot \Omega^{\text{DDW}}|_{x=1} &= -\overline{\gamma}_{1} + (\overline{\gamma}_{A1}^{1} \overline{\Theta}_{21} + \overline{\gamma}_{A2}^{2} \overline{\Theta}_{22} - \overline{\Theta}_{11} \overline{\gamma}_{A2}^{1} - \overline{\Theta}_{12} \overline{\gamma}_{A2}^{2}) \\
\overline{X} \cdot \Omega^{\text{DDW}}|_{x=2} &= -\overline{\gamma}_{2} + (\overline{\gamma}_{A1}^{1} \overline{\Theta}_{11} + \overline{\gamma}_{A2}^{2} \overline{\Theta}_{12} - \overline{\Theta}_{11} \overline{\gamma}_{A1}^{1} - \overline{\Theta}_{12} \overline{\gamma}_{A2}^{2})
\end{align*}
\]

We write the previous conditions in matrix notations, we denote

\[
\Theta = \Theta_{\nu \mu} = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\
\Theta_{12} & \Theta_{22} \end{pmatrix}, \quad \text{and} \quad \overline{\Theta} = \overline{\Theta}_{\nu \mu} = \begin{pmatrix} \overline{\Theta}_{11} & \overline{\Theta}_{12} \\
\overline{\Theta}_{12} & \overline{\Theta}_{22} \end{pmatrix}
\]

Let us denote for \( 1 \leq \mu \leq 2, \)

\[
\gamma_{A}^{\nu} = \begin{pmatrix} \gamma_{A1}^{\nu} \\
\gamma_{A2}^{\nu} \end{pmatrix}, \quad \text{and} \quad \overline{\gamma}_{A}^{\nu} = \begin{pmatrix} \overline{\gamma}_{A1}^{\nu} \\
\overline{\gamma}_{A2}^{\nu} \end{pmatrix},
\]

so that \( \gamma_{A}^{\nu} = \overline{\gamma}_{A}^{\nu} \) is written in matrix notation for \( 1 \leq \mu \leq 2 : \)

\[
\text{tr}(\gamma_{A}^{\nu}) = \sum_{\nu} \gamma_{A}^{\nu} = \sum_{\nu} \overline{\gamma}_{A}^{\nu} = \text{tr}(\overline{\gamma}_{A}^{\nu}).
\]

Hence relations \( (4.14) \) are written:

\[
\Theta = \overline{\Theta}, \quad \text{and} \quad \text{tr}(\gamma_{A}^{\nu}) = \text{tr}(\overline{\gamma}_{A}^{\nu}).
\]

(4.16)

Finally we look at the last relation \( (4.15) \). Let us denote for \( 1 \leq \mu \leq 2 \)

\[
\gamma^{\nu} = \gamma_{A}^{\nu} = \begin{pmatrix} \gamma_{A1}^{\nu} \\
\gamma_{A2}^{\nu} \end{pmatrix}, \quad \text{and} \quad \overline{\gamma}^{\nu} = \overline{\gamma}_{A}^{\nu} = \begin{pmatrix} \overline{\gamma}_{A1}^{\nu} \\
\overline{\gamma}_{A2}^{\nu} \end{pmatrix}.
\]

Since,

\[
\begin{pmatrix} \gamma_{A1}^{1} \\
\gamma_{A2}^{1} \\
\gamma_{A1}^{2} \\
\gamma_{A2}^{2} \end{pmatrix} \begin{pmatrix} -\Theta_{21} & -\Theta_{22} \\
\Theta_{11} & \Theta_{12} \end{pmatrix} = \begin{pmatrix} -\gamma_{A1}^{1} \Theta_{21} + \gamma_{A2}^{1} \Theta_{11} & -\gamma_{A1}^{1} \Theta_{21} + \gamma_{A2}^{1} \Theta_{11} \\
-\gamma_{A2}^{1} \Theta_{21} + \gamma_{A2}^{1} \Theta_{11} & -\gamma_{A1}^{1} \Theta_{21} + \gamma_{A2}^{1} \Theta_{11} \end{pmatrix},
\]

the relations \( (4.15) \) are written, with \( \gamma_{A}^{\nu} = \begin{pmatrix} 0 & 0 \\
0 & -\gamma_{A}^{\nu} \end{pmatrix} \) and \( \Sigma = \begin{pmatrix} -\Theta_{21} & -\Theta_{22} \\
\Theta_{11} & \Theta_{12} \end{pmatrix} :\)

\[
\begin{align*}
\text{tr}(\gamma_{A}^{\nu} + \gamma_{A}^{\nu} \Sigma) &= \text{tr}(\gamma_{A}^{\nu} + \gamma_{A}^{\nu} \Sigma), \\
\text{tr}(\gamma_{A}^{\nu} + \gamma_{A}^{\nu} \Sigma) &= \text{tr}(\gamma_{A}^{\nu} + \gamma_{A}^{\nu} \Sigma).
\end{align*}
\]

(4.17)
Now, we evaluate \( d\rho_\mu(X) = -1/2dA_\mu \land d\Pi^{A\mu} \land d\eta^{\mu\nu}(X) = 1/2d\Pi^{A\mu} \land dA_\mu \land d\eta^{\mu\nu}(X) \). We obtain via a straightforward calculation:

\[
d\rho_\mu = 1/2d\Pi^{A_1^2} \land dA_\mu \land d\eta_{12} + 1/2d\Pi^{A_2^1} \land dA_\mu \land d\eta_{21} - 1/2d\Pi^{A_1^2} \land dA_\mu + 1/2d\Pi^{A_2^1} \land dA_\mu
\]
or equivalently,

\[
d\rho_1 = -1/2d\Pi^{A_1^2} \land dA_1 + 1/2d\Pi^{A_2^1} \land dA_1,
\]

\[
d\rho_2 = -1/2d\Pi^{A_1^2} \land dA_2 + 1/2d\Pi^{A_2^1} \land dA_2.
\]

Then,

\[
d\rho_1(X) = \nabla^{A_2^1}\Theta_{11} - \nabla^{A_1^2}\Theta_{21} - \nabla^{A_1^{12}}\Theta_{11} + \nabla^{A_1^{12}}\Theta_{21},
\]

\[
d\rho_2(X) = \nabla^{A_2^1}\Theta_{12} - \nabla^{A_1^2}\Theta_{22} - \nabla^{A_1^{12}}\Theta_{12} + \nabla^{A_1^{12}}\Theta_{22}.
\]

We clearly see that the information contained in the comparison of the contraction of two vector fields \( X, X \in D_m^a\mathcal{M}^\mathrm{DDW} \subset \Lambda^n\mathcal{T}\mathcal{M}^\mathrm{DDW} \) with the multisymplectic form \( \Omega^\mathrm{DDW} \) - that is \( X \land \Omega^\mathrm{DDW} = X \land \Omega^\mathrm{DDW} \) - equivalent to

\[
\begin{vmatrix}
\Theta \\
\text{tr}(\Lambda_{[\mu,\nu]}) = \text{tr}(\Lambda_{\nu}^{\Lambda_{\mu}})
\end{vmatrix}
\text{ and and }
\begin{vmatrix}
\text{tr}(\Lambda_{[\mu,\nu]} + \Lambda_{[\sigma,\mu]}\Sigma) = \text{tr}(\Lambda_{\nu}^{\Lambda_{\mu}} + \Lambda_{\nu}^{\Lambda_{\mu}}\Sigma)
\end{vmatrix},
\]

is not sufficient to conclude that \( d\rho_1(X) = d\rho_1(X) \) or \( d\rho_2(X) = d\rho_2(X) \). In addition, in the case of the Maxwell theory, we consider the antisymmetry of multimomenta due to the Dirac primary constraint set. In that case we prefer to consider the following vector fields \( X, X \in D_m^a\mathcal{M}^\text{Maxwell} \subset \Lambda^n\mathcal{T}\mathcal{M}^\text{Maxwell} \). We denote,

\[
X_1 = \frac{\partial}{\partial x^1} + \Theta_1 \frac{\partial}{\partial A_\mu} + \nabla_1 \frac{\partial}{\partial e},
\]

\[
X_2 = \frac{\partial}{\partial x^2} + \Theta_2 \frac{\partial}{\partial A_\mu} + \nabla_2 \frac{\partial}{\partial e},
\]

so that \( X = X_1 \land X_2 \in D_m^a\mathcal{M}^\text{Maxwell} \subset \Lambda^n\mathcal{T}\mathcal{M}^\text{Maxwell} \) is written:

\[
X = (\partial_1 + \Theta_1 \partial_\mu \partial A_\mu + \nabla_1 \partial_\tau + \partial_1 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu)) \\
\land (\partial_2 + \Theta_2 \partial_\mu \partial A_\mu + \nabla_2 \partial_\tau + \partial_2 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu))
\]

\[
= \partial_1 \land (\partial_2 + \Theta_2 \partial_\mu \partial A_\mu + \nabla_2 \partial_\tau + \partial_2 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu))
\]

\[
\land \text{ terms involving } \partial_\tau \land \partial_\mu \land A_\nu \text{ and } \partial_\nu A_\mu \land \partial_\mu A_\sigma.
\]

Since we work on a subset of the (DDW) multisymplectic manifold, we only keep track of the concerned terms:

\[
X = \partial_1 \land \partial_2 + \Theta_2 \partial_1 \land \partial A_\mu + \nabla_2 \partial_1 \land \partial_\tau + \partial_2 A_\mu \nu \partial_1 \land \partial_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu)
\]

\[
+ \Theta_1 \partial_\mu \partial A_\mu \land \partial_2 + \Theta_2 \partial_\mu \partial A_\mu \land \partial_2 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu)
\]

\[
+ \nabla_1 \partial_\tau \land \partial_2 + \partial_2 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu) \land \partial_2 + \partial_1 A_\mu \nu (\partial A_\mu \nu - \partial A_\nu \mu) \land \Theta_2 \partial_\mu A_\nu
\]

\[
+ \text{ terms involving } \partial_\tau \land \partial_\mu \land \partial_\nu A_\mu \land \partial_\mu A_\sigma.
\]
Therefore we explicitly have the following calculation:

\[
X \mathcal{J} \Omega^{\text{DOW}} = X \mathcal{J} \left( \partial \wedge \partial x^1 \wedge \partial x^2 + d \Pi^{A_1^2 \nu} \wedge d A_1 \wedge d \eta_\rho + d \Pi^{A_2^2 \nu} \wedge d A_2 \wedge d \eta_\rho \right)
\]

Since, \( \Pi^{A_1^2} = \Pi^{A_2^2} = 0 \),

\[
X \mathcal{J} \Omega^{\text{DOW}} = d \eta(X) d e - (d e \wedge d \eta_1)(X)d x^1 - (d e \wedge d \eta_2)(X)d x^2
\]

Then, the relations \( X \mathcal{J} \Omega^{\text{DOW}} = \overline{X} \mathcal{J} \Omega^{\text{DOW}} \) are written:

\[
\left| \begin{array}{c}
\Theta_{12} = \overline{\Theta}_{12} \\
\Theta_{21} = \overline{\Theta}_{21}
\end{array} \right|
\quad \left| \begin{array}{c}
\gamma_{A_1^2} - \gamma_{A_2^2} = \overline{\gamma}_{A_1^2} - \overline{\gamma}_{A_2^2} \\
\gamma_{A_1^2} - \gamma_{A_2^2} = \overline{\gamma}_{A_1^2} - \overline{\gamma}_{A_2^2}
\end{array} \right| \tag{4.23}
\]

and

\[
\left( \begin{array}{c}
\Theta_{A_1^2} \Theta_{12} - \gamma_{A_1^2} \gamma_{12} \\
\Theta_{A_2^2} \Theta_{21} - \gamma_{A_2^2} \gamma_{12}
\end{array} \right) = \left( \begin{array}{c}
\overline{\Theta}_{A_1^2} \overline{\Theta}_{12} - \overline{\gamma}_{A_1^2} \overline{\gamma}_{12} \\
\overline{\Theta}_{A_2^2} \overline{\Theta}_{21} - \overline{\gamma}_{A_2^2} \overline{\gamma}_{12}
\end{array} \right)
\]

We set \( \Upsilon_{[1]} = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_1 \end{pmatrix} = \Upsilon_{[2]} = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_2 \end{pmatrix} = 0 \) since it do not disturb the general result. We also introduce the following matrix notation:

\[
\left| \begin{array}{c}
\Theta_{\nu \mu}^{\text{Maxwell}} = \begin{pmatrix} 0 & \Theta_{21} \\ \Theta_{12} & 0 \end{pmatrix} \\
(\Upsilon_{A_1^2 \nu}^{\text{Maxwell}}) = \begin{pmatrix} \gamma_{A_1^2} & 0 \\ 0 & -\gamma_{A_2^2} \end{pmatrix}
\end{array} \right|
\quad \left| \begin{array}{c}
\overline{\Theta}_{\nu \mu}^{\text{Maxwell}} = \begin{pmatrix} 0 & \overline{\Theta}_{21} \\ \overline{\Theta}_{12} & 0 \end{pmatrix} \\
(\overline{\Upsilon}_{A_1^2 \nu}^{\text{Maxwell}}) = \begin{pmatrix} \overline{\gamma}_{A_1^2} & 0 \\ 0 & -\overline{\gamma}_{A_2^2} \end{pmatrix}
\end{array} \right|
\]
so that the relation (4.23) and (4.24) are written:

\[ \Theta_{\text{Maxwell}} = \Theta_{\text{Maxwell}}^\ast \quad \text{and} \quad \text{tr}(\mathbf{Y}_{[p]}^{A_{\mu,\nu}})_{\text{Maxwell}} = \text{tr}(\mathbf{Y}_{[p]}^{A_{\mu,\nu}})_{\text{Maxwell}}. \]  

(4.25)

Due to (4.18), we have the expression of \( d_\rho_1(X) \) and \( d_\rho_2(X) \)

\[\begin{align*}
    d_\rho_1(X) &= \mathbf{T}_2^{A_12}\Theta_{11} - \mathbf{T}_2^{A_21}\Theta_{11} - \mathbf{T}_1^{A_12}\Theta_{21} + \mathbf{T}_1^{A_21}\Theta_{21} \\
    &\quad + \mathbf{T}_2^{A_12}\Theta_{11} - \mathbf{T}_2^{A_21}\Theta_{11} - \mathbf{T}_1^{A_12}\Theta_{21} + \mathbf{T}_1^{A_21}\Theta_{21}, \\
    d_\rho_2(X) &= \mathbf{T}_2^{A_12}\Theta_{12} - \mathbf{T}_2^{A_21}\Theta_{22} - \mathbf{T}_1^{A_21}\Theta_{12} + \mathbf{T}_1^{A_12}\Theta_{22} \\
    &\quad + \mathbf{T}_2^{A_12}\Theta_{12} - \mathbf{T}_2^{A_21}\Theta_{22} - \mathbf{T}_1^{A_21}\Theta_{12} + \mathbf{T}_1^{A_12}\Theta_{22}. \\
\end{align*}\]

(4.26)

Once again, this is not sufficient to conclude that \( d_\rho_1(X) = d_\rho_1(X) \) or \( d_\rho_2(X) = d_\rho_2(X) \) since we do not have necessarily \( \Theta_{\mu,\nu} = \Theta_{\mu,\nu}^\ast \) for \((\mu, \nu) = (1,1) \) or \((\mu, \nu) = (2,2) \).

5 Lepage-Dedecker for two dimensional Maxwell theory

In the sections (5.1)-(5.3) - as opposed to the last one (5.4) - we work with indices notation, in particular with the tedious but straightforward computation of the Hamiltonian. It is just to emphasize the huge amount of calculations for (LD) theories - even in a simple case \( n = 2 \) - for the setting of Maxwell theory. We refer to H.A. Katstrup [36], F. Hélin and J. Kouneiher [26] [27] for some aspects of the two dimensional Lepage-Dedecker Maxwell theory.

5.1 Lepage-Dedecker correspondence

Now we perform a Lepage-Dedecker correspondence for the Maxwell 2D theory. First we express the Lagrangian density \( L(x, A, dA) = -\frac{1}{4} \eta^{\mu\lambda\nu\sigma} F_{\mu\nu} F_{\lambda\sigma} \) so that:

\[ L(A) = -\frac{1}{4} (\eta^{11} \eta^{2\sigma} F_{12} F_{2\lambda} + \eta^{21} \eta^{1\sigma} F_{21} F_{1\lambda}) = -\frac{1}{4} (\eta^{11} \eta^{22} (F_{12})^2 + \eta^{22} \eta^{11} (F_{21})^2) = \frac{1}{2} (F_{12})^2, \]

then, the Lagrangian is written:

\[ L(x, A, dA) = \frac{1}{2} (\partial_1 A_2 - \partial_2 A_1) (\partial_1 A_2 - \partial_2 A_1) = \frac{1}{2} ((\partial_1 A_2)^2 + (\partial_2 A_1)^2) - (\partial_1 A_2) (\partial_2 A_1). \]

Now we construct a non degenerate Legendre correspondence in the 2D-case via the following Poincaré-Cartan form\footnote{We use here the following notation: \( \mathbf{\theta}^{(2)}_{(q,p)} \) means we specify the canonical Poincaré-Cartan form for Maxwell theory in the 2 dimensional case and taking into account forms that involves 2 fields. (namely forms of the type \( \zeta dA_1 \wedge dA_2 \)) Following this logic we write the previous canonical form as \( \mathbf{\theta}^{(2)}_{(q,p)} := \mathbf{\theta}^{(2)}_{(q,p)} \)}\footnote{5.1} \( \mathbf{\theta}^{\text{Lepage-Dedecker}}_{(q,p)} = \mathbf{\theta}^{\text{Poincaré-Cartan}}_{(q,p)} = \mathbf{\theta}^{\text{Poincaré-Cartan}}_{(q,p)} : \)

\[ \mathbf{\theta}^{(2)}_{(q,p)} := \xi d\eta + \pi^{A_\mu A_\nu} dA_\mu \wedge dA_\nu + \zeta dA_1 \wedge dA_2. \]  

(5.1)
and the related Multisymplectic 3 form:

$$\Omega^{[2]}_{(q,p)} := d\epsilon \land d\eta + d\pi^{A\mu\nu} \land dA_{\mu} \land d\eta_\nu + d\zeta \land dA_1 \land dA_2.$$  (5.2)

Then, we concentrate on the expression of \( \langle p, v \rangle \),

$$\langle p, v \rangle = \theta_{(q,p)}^{[2]}(Z) = \epsilon d\eta(Z) + \pi^{A\mu\nu} dA_{\mu} \land d\eta_\nu(Z) + \zeta dA_1 \land dA_2(Z).$$  (5.3)

We demonstrate by direct calculation that:

$$\langle p, v \rangle = \theta_{(q,p)}^{[2]}(Z) = \pi^{A\mu\nu} \partial_\nu A_\mu + 2\zeta(Z_{11}Z_{22} - Z_{12}Z_{21}).$$

**Proof.** Since \( Z_\nu = \frac{\partial}{\partial x^\nu} + Z_{\mu_1} \frac{\partial}{\partial A_{\mu_1}} \), we have:

\( Z_1 = \partial_1 + Z_{1\mu_1} \frac{\partial}{\partial A_{\mu_1}} \) and \( Z_2 = \partial_2 + Z_{2\mu_2} \frac{\partial}{\partial A_{\mu_2}} \) so that we compute \( Z = Z_1 \land Z_2 \):

\[
Z = \sum_{\mu_1 < \mu_2} Z_{1\mu_1}^{\mu_2} \frac{\partial}{\partial q^{\mu_1}} \land \frac{\partial}{\partial q^{\mu_2}} = \sum_{\mu_1 < \mu_2} \left| \begin{array}{c}
Z_{1\mu_1}^{\mu_2} \\
Z_{2\mu_1}^{\mu_2}
\end{array} \right| \frac{\partial}{\partial q^{\mu_1}} \land \frac{\partial}{\partial q^{\mu_2}}
\]

\[
= Z_{12}^{\mu_2} \partial_1 \land \partial_2 + Z_{12}^{\mu_2} \partial_1 \land \frac{\partial}{\partial A_{\mu_2}} + Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}} + Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}}.
\]

With the different terms:

\[
Z_{12}^{\mu_2} = 1 \quad Z_{12}^{\mu_2} = \left| \begin{array}{c}
0 \\
1
\end{array} \right| Z_{12}^{\mu_2} = \left| \begin{array}{c}
1 \\
0
\end{array} \right| Z_{12}^{\mu_2} = \left| \begin{array}{c}
Z_{12}^{\mu_1} \\
Z_{22}^{\mu_1}
\end{array} \right| = \left| \begin{array}{c}
Z_{12}^{\mu_1} \\
Z_{22}^{\mu_1}
\end{array} \right| = \left( Z_{12}^{\mu_1} Z_{22}^{\mu_2} - Z_{12}^{\mu_1} Z_{22}^{\mu_2} \right)
\]

We make the following calculation:

\[
\langle p, v \rangle = \epsilon + \pi^{A\mu\nu} dA_\mu \land d\eta_\nu(Z_{12}^{\mu_2} \partial_1 \land \frac{\partial}{\partial A_{\mu_2}} + Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}}) + \zeta dA_1 \land dA_2(Z_{12}^{\mu_1 \mu_2} \frac{\partial}{\partial A_{\mu_1}} \land \frac{\partial}{\partial A_{\mu_2}})
\]

The first term in the last equation is given by:

\[
[I] = \sum_{\mu, \nu} \pi^{A\mu\nu} dA_\mu \land d\eta_\nu(Z_{12}^{\mu_2} \partial_1 \land \frac{\partial}{\partial A_{\mu_2}} + Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}}) + \pi^{A21} dA_2 \land d\eta_2(Z_{12}^{\mu_2} \partial_1 \land \frac{\partial}{\partial A_{\mu_2}}) + \pi^{A22} dA_2 \land d\eta_2(Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}})
\]

\[
= \pi^{A11} dA_1 \land dx^2(Z_{12}^{\mu_2} \partial_2 \land \frac{\partial}{\partial A_{\mu_2}}) - \pi^{A12} dA_1 \land dx^1(Z_{12}^{\mu_2} \partial_1 \land \frac{\partial}{\partial A_{\mu_2}})
\]

\[
= \pi^{A11} Z_{12} + \pi^{A12} Z_{21} + \pi^{A21} Z_{12} + \pi^{A22} Z_{22}
\]

\[
= A^{A\mu\nu} \partial_\nu A_\mu.
\]

Whereas the second term is given by:

\[
[II] = \zeta dA_1 \land dA_2(Z_{12}^{\mu_1 \mu_2} \frac{\partial}{\partial A_{\mu_1}} \land \frac{\partial}{\partial A_{\mu_2}})
\]

\[
= \zeta dA_1 \land dA_2(Z_{12}^{\mu_1 \mu_2} - Z_{12}^{\mu_2 \mu_1} \frac{\partial}{\partial A_{\mu_1}} \land \frac{\partial}{\partial A_{\mu_2}})
\]

\[
= \zeta(Z_{11} Z_{22} - Z_{12} Z_{21}).
\]
Let us denote:

\[ 2\varepsilon^{\mu
u}Z_{[\mu}Z_{\nu]} = \varepsilon^{\mu
u}Z_{\mu}Z_{\nu} - Z_{\mu}Z_{\nu} = \varepsilon(Z_{11}Z_{22} - Z_{12}Z_{21}) - \varepsilon(Z_{12}Z_{21} - Z_{11}Z_{22}) = 2\varepsilon(Z_{11}Z_{22} - Z_{12}Z_{21}), \]

we write the term \([\Pi] = \varepsilon^{\mu
u}Z_{[\mu}Z_{\nu]}\).

Then we have the expression of \(<p, v>:\)

\[ <p, v> = \pi^{A_{11}}Z_{11} + \pi^{A_{12}}Z_{21} + \pi^{A_{21}}Z_{12} + \pi^{A_{22}}Z_{22} + \varepsilon(Z_{11}Z_{22} - Z_{12}Z_{21}) \]  \hspace{1cm} (5.5)

We can equivalently write in more contracted notation: \(<p, v> = \theta^{[2]}_{(q,p)}(Z) = \pi^{A_{\mu}\nu}\partial_v A_{\mu} + \varepsilon^{\mu\nu}Z_{[\mu}Z_{\nu]}\). With the notation \(Z_{\nu\mu} = \partial_v A_{\mu}\), we write:

\[ <p, v> = \theta^{[2]}_{(q,p)}(Z) = \pi^{A_{\mu}\nu}\partial_v A_{\mu} + \varepsilon^{\mu\nu}\partial_1 A_{[\mu}\partial_2 A_{\nu]} \]

Let us denote:

\[ \kappa_{\mu\nu} = \kappa^{[2]}_{\mu\nu} = \frac{\partial<p, v>}{\partial(p_\mu A_\nu)} = \frac{\partial}{\partial(p_\mu A_\nu)}\theta^{[2]}_{(q,p)}(Z), \]

we work in coordinate expression so that we use the expression (5.5):

\[ \theta^{[2]}_{(q,p)}(Z) = \pi^{A_{11}}\partial_1 A_1 + \pi^{A_{12}}\partial_2 A_1 + \pi^{A_{21}}\partial_1 A_2 + \pi^{A_{22}}\partial_2 A_2 + \varepsilon(\partial_1 A_1\partial_2 A_2 - \partial_1 A_2\partial_2 A_1) \]

Hence, we find the relations (5.7). (i).

\[
\begin{array}{c|c|c}
\kappa_{\mu\nu}|_{\mu=1,\nu=1} = & \pi^{A_{11}} + \varepsilon\partial_2 A_2 & \lambda_{\mu\nu}|_{\mu=1,\nu=1} = 0 \\
\kappa_{\mu\nu}|_{\mu=1,\nu=2} = & \pi^{A_{12}} - \varepsilon\partial_1 A_2 & \lambda_{\mu\nu}|_{\mu=1,\nu=2} = \partial_1 A_2 - \partial_2 A_1 \\
\kappa_{\mu\nu}|_{\mu=2,\nu=1} = & \pi^{A_{21}} - \varepsilon\partial_1 A_2 & \lambda_{\mu\nu}|_{\mu=2,\nu=1} = \partial_2 A_1 - \partial_1 A_2 \\
\kappa_{\mu\nu}|_{\mu=2,\nu=2} = & \pi^{A_{22}} + \varepsilon\partial_1 A_1 & \lambda_{\mu\nu}|_{\mu=2,\nu=2} = 0 \\
\end{array}
\]

On the other side, we denote \(\partial L/\partial(\partial_\mu A_\nu) = \lambda_{\mu\nu}\). We use the coordinate expression of \(L(x, A, dA)\) and we obtain (5.7) (ii). The condition for the Legendre transform is:

\[ \frac{\partial L}{\partial(\partial_\mu A_\nu)} = \frac{\partial<p, v>}{\partial(p_\mu A_\nu)}. \]

We obtain (5.8) (i) and, choosing to work in the case \(\varepsilon = 1\), we then obtain the relations (5.8) (ii).

\[
\begin{array}{c|c|c}
0 = & \pi^{A_{11}} + \varepsilon\partial_2 A_2 & \pi^{A_{11}} = -\partial_2 A_2 \\
\partial_1 A_2 - \partial_2 A_1 = & \pi^{A_{21}} - \varepsilon\partial_1 A_2 & \pi^{A_{21}} = \partial_1 A_2 \\
\partial_2 A_1 - \partial_1 A_2 = & \pi^{A_{22}} - \varepsilon\partial_1 A_1 & \pi^{A_{22}} = -\partial_1 A_1 \\
0 = & \pi^{A_{22}} + \varepsilon\partial_1 A_1 & \pi^{A_{22}} = -\partial_1 A_1 \\
\end{array}
\]

The generalized Legendre correspondence is non degenerate. It is always possible to invert the multimomenta from multivelocities. Now, we give the expression of the Hamiltonian function. From (5.8) (i),

\[
\begin{align*}
\partial_2 A_2 & = -\varepsilon^{-1}\pi^{A_{11}} \\
\partial_2 A_1 & = (\varepsilon(2 - \varepsilon))^{-1}(\pi^{A_{12}} + (1 - \varepsilon)\pi^{A_{21}}) \\
\partial_1 A_2 & = (\varepsilon(2 - \varepsilon))^{-1}(\pi^{A_{22}} + (1 - \varepsilon)\pi^{A_{21}}) \\
\partial_1 A_1 & = -\varepsilon^{-1}\pi^{A_{22}} \\
\end{align*}
\]  \hspace{1cm} (5.9)

\[ \textbf{Proof} \text{ Let us explicite the second line in (5.5), from the second line in (5.5) we find:} \]

\[ \partial_1 A_2 = \pi^{A_{21}} + (1 - \varepsilon)\partial_2 A_1. \]  \hspace{1cm} (5.10)
The third line of (5.8) writes:
\[
\partial_2 A_1 - \partial_1 A_2 = \pi^{A_1 1} - \varsigma \partial_1 A_2 \quad \Rightarrow \quad \partial_2 A_1 = \pi^{A_1 1^2} + (1 - \varsigma) \partial_1 A_2.
\] (5.11)

We insert (5.10) in (5.11) so that:
\[
\partial_2 A_1 = \pi^{A_1 1^2} + (1 - \varsigma)(\pi^{A_1 2} + (1 - \varsigma)\partial_2 A_1)
\]
\[
\partial_2 A_1(1 - (1 - \varsigma)^2) = \pi^{A_1 1^2} + (1 - \varsigma)\pi^{A_1 2} \Longleftrightarrow \partial_2 A_1(2\varsigma - \varsigma^2) = \pi^{A_1 1^2} + (1 - \varsigma)\pi^{A_1 2}
\]
\[
\partial_2 A_1\varsigma(2 - \varsigma) = \pi^{A_1 1^2} + (1 - \varsigma)\pi^{A_1 2}. \text{ An analogous process holds also for the other relation.}
\]

### 5.2 Calculation of the Hamiltonian

We are interested in the expression of the Hamiltonian:
\[
\mathcal{H} = \theta^{[2]}_{(q,p)}(Z) - L,
\] (5.12)
where, \(\theta^{[2]}_{(q,p)}(Z) = k_1 + \cdots + k_6\) and \(-L = k_7 + k_8 + k_9\), with
\[
\begin{align*}
k_1 &= \pi^{A_1 1} \partial_1 A_1 \\
k_2 &= \pi^{A_1 2} \partial_2 A_1 \\
k_3 &= \pi^{A_2 1} \partial_1 A_2 \\
k_4 &= \pi^{A_2 2} \partial_2 A_2 \\
k_5 &= \varsigma \partial_1 A_1 \partial_2 A_2 \\
k_6 &= -\varsigma \partial_1 A_2 \partial_2 A_1
\end{align*}
\]

Let us examine each of these terms. We denote by \(\varsigma = 1/(2 - \varsigma)\). The terms \(k_1, k_4\) correspond to the terms \(\langle p, v \rangle = \theta^{(DDW)}_{(q,p)}(Z)\):
\[
\begin{align*}
k_1 &= -\varsigma^{-1}\pi^{A_1 1} \pi^{A_2 2} \\
k_2 &= \varsigma \pi^{A_1 2}[\pi^{A_1 2} + (1 - \varsigma)\pi^{A_2 1}] \\
k_3 &= \varsigma \pi^{A_2 1}[\pi^{A_2 1} + (1 - \varsigma)\pi^{A_1 2}] \\
k_4 &= -\varsigma^{-1}\pi^{A_2 2} \pi^{A_1 1}
\end{align*}
\] (5.13)

Also the two terms which are related to the Lepage-Dedecker part:
\[
\begin{align*}
k_5 &= (\varsigma^{-1})\pi^{A_2 2} \pi^{A_1 1} \\
k_6 &= -\varsigma^{2}(\pi^{A_2 1} + (1 - \varsigma)\pi^{A_1 2})[\pi^{A_1 2} + (1 - \varsigma)\pi^{A_2 1}]
\end{align*}
\] (5.14)

And finally, the three terms which come from the Lagrangian density:
\[
\begin{align*}
k_7 &= -(1/2)\varsigma^2[\pi^{A_2 1} + (1 - \varsigma)\pi^{A_1 2}][\pi^{A_1 2} + (1 - \varsigma)\pi^{A_2 1}] \\
k_8 &= -(1/2)\varsigma^2[\pi^{A_2 2} + (1 - \varsigma)\pi^{A_1 1}][\pi^{A_1 1} + (1 - \varsigma)\pi^{A_2 2}] \\
k_9 &= \varsigma^2[\pi^{A_2 1} + (1 - \varsigma)\pi^{A_1 2}][\pi^{A_1 2} + (1 - \varsigma)\pi^{A_2 1}]
\end{align*}
\] (5.15)

Let us consider the equations \(k_1, k_4\) and \(k_5\) in (5.13). We denote by \((i) = k_1 + k_4 + k_5\) so that
\[
(i) = -\varsigma^{-1}\pi^{A_2 2} \pi^{A_1 1}.
\] (5.16)

We denote \((ii) = k_2 + k_3\), so that :
\[
(ii) = \varsigma[\pi^{A_1 2} \pi^{A_2 1} + \pi^{A_2 1} (1 - \varsigma) \pi^{A_2 1} + \pi^{A_2 1} \pi^{A_1 2} + \pi^{A_2 1} (1 - \varsigma) \pi^{A_1 2}].
\] (5.17)

It remains the following equations \(k_6\)-\(k_9\). We have respectively :
\[
\begin{align*}
k_6 &= -\varsigma^2[\pi^{A_2 1} \pi^{A_2 1} + \pi^{A_2 1} (1 - \varsigma) \pi^{A_2 1} + \pi^{A_2 1} \pi^{A_1 2} + \pi^{A_2 1} (1 - \varsigma) \pi^{A_1 2}] \\
&= -\varsigma^2[(1 + (1 - \varsigma)^2) \pi^{A_2 1} \pi^{A_2 1} + \pi^{A_2 1} (1 - \varsigma) + (1 - \varsigma) \pi^{A_1 2} (1 - \varsigma) \pi^{A_1 2}] \\
&= -\varsigma^2[(2(1 - \varsigma) + \varsigma^2) \pi^{A_2 1} \pi^{A_2 1} + \pi^{A_2 1} (1 - \varsigma) + (1 - \varsigma) \pi^{A_1 2} (1 - \varsigma) \pi^{A_1 2}]
\end{align*}
\] (5.18)
The second and the third give

\[ k_\tau = \left[ -(1/2)\varsigma^2 \left( \pi A_1 \right)^2 + (1 - \varsigma) \left( \pi A_1 \right)^2 + 2 \pi^2 A_1 \right] \left( 1 - \varsigma \right) \pi A_1 \]

\[ k_\varsigma = \left[ -(1/2)\varsigma^2 \left( \pi A_1 \right)^2 + (1 - \varsigma) \left( \pi A_1 \right)^2 + 2 \pi^2 A_1 \right] \left( 1 - \varsigma \right) \pi A_1 \]  

(5.19)

Now, we denote (iii) = \( k_\tau + k_\varsigma \), so that :

(iii) = \( (1 - \varsigma)\varsigma^2 \left( 2\left( 1 - \varsigma \right) + \varsigma^2 \right) \pi A_1^2 \pi A_1^2 + \left[ \pi^2 A_1 \right]^2 \left( 1 - \varsigma \right) \left( 1 - \varsigma \right) \left[ \pi A_1^2 \right]^2 \]  

(5.20)

and finally we denote (iv) = \( k_\tau + k_\varsigma \), then :

(iv) = \(- \frac{1}{2}\varsigma^2 \left( \pi A_1 \right)^2 \left( 1 - \varsigma \right) \right\} \pi A_1^2 + \left( 1 - \varsigma \right) \pi A_1^2 + \left[ \pi A_1^2 \right]^2 + \left( 1 - \varsigma \right) \pi A_1^2 \]  

(5.21)

Finally, we compute (ii) + (iii) + (iv). We introduce the following notations :

\[ \pi^\circ = \pi A_1 \pi A_1, \quad \pi^c = \pi A_1 \pi A_1, \quad \pi^p = \pi A_1, \pi A_1, \quad \pi^p = \pi A_1 \pi A_1 = \left[ \pi A_1 \right]^2, \]  

(5.22)

So that the equations (5.17), (5.20) and (5.21) are written (5.23) (ii)-(iv) :

(ii) = \( \varsigma \left( \pi^\circ + 2\left( \pi^c + \pi^p \right) \right) \left( \pi^c - \varsigma \right) \]  

(5.23)

where we have denoted \( \varsigma = \left( \pi^c - \varsigma \right)^{-1} \). So that (5.23)-(ii) is written :

(ii) = \( \varsigma \left( \pi^\circ + 2\left( \pi^c + \pi^p \right) \right) \left( \pi^c - \varsigma \right) \]  

(5.24)

If we denote \( \Phi = \left( 2\varsigma^2 \left( \pi^c - \varsigma \right)^2 \right)^{-1} \), we obtain :

(ii) = \( 2\Phi \left( \pi^\circ + 2\left( \pi^c + \pi^p \right) \right) \left( \pi^c - \varsigma \right) \]  

The equation (5.23)-(iii) is written :

(iii) = \( 2\Phi \left( 1 - \varsigma \right) \left( \pi^\circ + 2\left( \pi^c + \pi^p \right) \right) \left( \pi^c - \varsigma \right) \]  

(5.23)

and finally (5.23)-(iv) is written :

(iv) = \(- \Phi \left( \pi^c + \left( \pi^c + \pi^p \right) \right) \left( 1 - \varsigma \right) \]  

(5.23)

We now writes \( H = (i) + (ii) + (iii) + (iv) \). We finally obtain the expression of the Hamiltonian :

\[ H = -\frac{1}{\varsigma} \pi^c + \frac{1}{2\varsigma} \left( \pi^\circ + \pi^p \right) \]  

(5.24)

If we use the transform with \( \varsigma = 1 \) then (5.24(i)) gives the following Hamiltonian :

\[ H = -\pi^c + \frac{1}{2} \left( \pi^\circ + \pi^p \right). \]  

(5.25)
Proof. We compute in coordinate the straightforward calculation:

\[
\mathcal{H} = \pi^{\mu\nu}\partial_{\nu}A_{\mu} + \kappa[Z_{11}Z_{22} - Z_{12}Z_{21}] - \frac{1}{2}(\partial_{1}A_{2})^{2} - \frac{1}{2}(\partial_{2}A_{1})^{2} + (\partial_{1}A_{2})(\partial_{2}A_{1})
\]

\[
= (\pi^{A_{1}A_{2}} + (\pi^{A_{1}})^{2} - 2\pi^{A_{1}}\pi^{A_{2}} + \pi^{A_{1}}\pi^{A_{2}} - \pi^{A_{1}}\pi^{A_{2}} - \frac{1}{2}(\pi^{A_{1}})^{2} - \frac{1}{2}(\pi^{A_{2}})^{2} + \pi^{A_{1}}\pi^{A_{2}})
\]

And the Hamiltonian [5.25] agrees with the general case [5.24].

5.3 Equations of movement

Now let us derive the generalized Hamilton equations. The general form of a vector field is given by:

\[
X_{\alpha} = \frac{\partial}{\partial x^{\alpha}} + \Theta_{\mu\nu}\frac{\partial}{\partial A_{\mu}} + \Upsilon_{\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} + \Upsilon_{\alpha\nu}\frac{\partial}{\partial X^{A_{\mu}\nu}}
\]

so that \(X = X_{1} \wedge X_{2}\) is written:

\[
X = \partial_{1} \wedge \partial_{2} + \partial_{1} \wedge \Theta_{2\mu}\frac{\partial}{\partial A_{\mu}} + \partial_{1} \wedge \Upsilon_{2\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \partial_{1} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \Upsilon_{2\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} + \Upsilon_{1\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} \wedge \partial_{2} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \Theta_{2\nu}\frac{\partial}{\partial A_{\mu}} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \Upsilon_{2\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} + \Upsilon_{1\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} \wedge \partial_{2} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \Theta_{2\nu}\frac{\partial}{\partial A_{\mu}} + \Theta_{1\mu}\frac{\partial}{\partial A_{\mu}} \wedge \Upsilon_{2\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} + \Upsilon_{1\alpha}\frac{\partial}{\partial \pi^{A_{\mu}\nu}} \wedge \partial_{2}.
\]

We compute the first part of generalized Hamilton equations, namely \(X \wedge \Omega^{[2]}\):

\[
X \wedge \Omega^{[2]} = X \left(\frac{d\epsilon}{d\eta} + d\pi^{A_{\mu}\nu} \wedge dA_{\mu} \wedge d\eta_{\nu} + d\varsigma \wedge dA_{1} \wedge dA_{2}\right)
\]

\[
= \frac{d\epsilon}{d\eta} - (d\varsigma \wedge d\epsilon)(X)dx^{\mu} + (dA_{\mu} \wedge d\eta_{\nu})(X)d\pi^{A_{\mu}\nu} - (dA_{\mu} \wedge d\eta_{\nu})(X)dA_{\mu}
\]

\[
+ (d\pi^{A_{\mu}\nu} \wedge dA_{\mu} \wedge d\eta_{\mu})(X)dx^{\mu} - (d\varsigma \wedge dA_{2})(X)dA_{1} + (d\varsigma \wedge dA_{1})(X)dA_{2}.
\]

Since \(d\varsigma = 0\) the multisymplectic form is written \(\Omega^{[2]}|_{\varsigma = 1} = \frac{d\epsilon}{d\eta} + d\pi^{A_{\mu}\nu} \wedge dA_{\mu} \wedge d\eta_{\nu}\). So that \(X \wedge \Omega^{[2]}|_{\varsigma = 1}\) is given by:

\[
X \wedge \Omega^{[2]}|_{\varsigma = 1} = \frac{d\epsilon}{d\eta} - (d\varsigma \wedge d\epsilon)(X)dx^{\mu} + (dA_{\mu} \wedge d\eta_{\nu})(X)d\pi^{A_{\mu}\nu} - (dA_{\mu} \wedge d\eta_{\nu})(X)dA_{\mu}
\]

\[
+ (d\pi^{A_{\mu}\nu} \wedge dA_{\mu} \wedge d\eta_{\mu})(X)dx^{\mu} - (d\varsigma \wedge dA_{2})(X)dA_{1} + (d\varsigma \wedge dA_{1})(X)dA_{2}.
\]

we only keep the interesting part on the decompositions along \(d\pi^{A_{\mu}\nu}\) and \(dA_{\mu}\).

\[
\begin{align*}
|\Theta_{\nu\mu}d\pi^{A_{\mu}\nu}| &= d\mathcal{H} & (i) \quad \Theta_{1\mu}d\pi^{A_{\mu}\nu} + \Theta_{2\mu}d\pi^{A_{\mu}\nu} &= d\mathcal{H} \\
-\Upsilon_{\nu} &= 0 & (ii) \quad -\Upsilon_{\nu} &= 0
\end{align*}
\]
With \( \mathrm{d}\mathcal{H}\big|_{s=1} = -\pi^{\ast\ast} + 1/2(\pi^{\infty} + \pi^{\ast\ast}) = -\pi^{A_1} d\pi^{A_2} - \pi^{A_2} d\pi^{A_1} + \pi^{A_1} d\pi^{A_2} + \pi^{A_1} d\pi^{A_1} + \pi^{A_2} d\pi^{A_2} \).

Finally we obtain from \((5.26)\) (i) the Legendre transform given by:

\[
\begin{align*}
\partial_1 A_1 &= -\pi^{A_2} \\
\partial_2 A_2 &= -\pi^{A_1} \\
\partial_1 A_2 &= \pi^{A_2} \\
\partial_2 A_1 &= \pi^{A_1}
\end{align*}
\]

(5.27)

Whereas from \((5.26)\) (ii) we obtain the Maxwell’s equations:

\[
\partial_\mu \pi^{A_\mu} = 0.
\]

(5.28)

### 5.4 Grassmannian viewpoint and pseudofibers

**Enlarged pseudofibers - Pseudofibers.** We introduce the following fundamental objects: the *enlarged pseudofiber* and the *pseudofiber*. Following [25][26][27] the enlarged pseudofiber is defined to be:

\[
P_q(z) = \left\{ p \in \Lambda^n T^*_q 3 / \frac{\partial W}{\partial z}(q, z, p) = 0 \right\}
\]

(5.29)

The enlarged pseudofiber is understood as the space of \(n\)-forms \(P_q(z) \subset \Lambda^n T^*_q 3\) such that the generalized Legendre correspondence is satisfied: \((q, z) \equiv (q, p)\). We refer to [24][25][26] for further details. The key point is that \(P_q(z)\) is an affine subspace of \(\Lambda^n T^*_q 3\) with \(\dim(P_q(z)) = (n + k)! - nk\). Finally, for a given \((q, z) \in D^{\infty} 3\), we can find *at the same time* an element \(p \in P_q(z)\) and choose the value of \(\mathcal{H}(q, p)\). Therefore, we find the definition of the pseudofiber to be the space defined by \((5.30)\):

\[
P^h_q(z) = \left\{ p \in P_q(z) / \mathcal{H}(q, p) = h \right\}.
\]

(5.30)

Notice that \(\dim(P^h_q(z)) = \dim(P_q(z)) - 1\) and that \(P_q(z)\) and \(P^h_q(z)\) are affine subspaces parallel to \([T_z D_q^{\infty} 3]^\perp\) and \([T_z D_q^{\infty} 3]^\perp\) where the spaces \([T_z D_q^{\infty} 3]^\perp, [T_z D_q^{\infty} 3]^\perp \subset \Lambda^n T^*_q 3\) are respectively defined by \((5.31)\):

\[
[T_z D_q^{\infty} 3]^\perp = \left\{ p \in \Lambda^n T^*_q 3 / \forall \xi \in T_z D_q^{\infty} 3, p(\xi) = 0 \right\}
\]

(5.31)

\[
[T_z D_q^{\infty} 3]^\perp = \left\{ p \in \Lambda^n T^*_q 3 / \forall \xi \in T_z D_q^{\infty} 3, p(\xi) = 0 \right\}
\]

In the general case we have the following dimension for the involved spaces: \(\text{Gr}^{\infty} 3, \Lambda^n T^*_q 3, \Lambda^n T^*_q 3, D_q^{\infty} 3\) and of the enlarged pseudofiber \(P_q(z)\) and the pseudofiber \(P^h_q(z)\).

\[
\begin{align*}
\dim(\text{Gr}^{\infty} 3) &= n + k + nk \\
\dim(\Lambda^n T^*_q 3) &= n + k + \frac{(n + k)!}{n!k!} \\
\dim(\Lambda^n T^*_q 3) &= \frac{(n + k)!}{n!k!} \\
\dim(P_q(z)) &= \frac{(n + k)!}{n!k!} - nk \\
\dim(P^h_q(z)) &= \dim(P_q(z)) - 1
\end{align*}
\]
Grassmannian viewpoint for 2D-Maxwell theory. In this section we follow the method developed by F. Hélein and J. Kouneiher [21] [25] [26] [27] for the general study of variational problem on maps. In the case of Maxwell theory, we understand the problem via the study of the map \( A : T^*\mathcal{X} \to \mathbb{R} \). Here the multisymplectic manifold is \( \mathcal{M} = \Lambda^2T^*(T^*\mathcal{X}) = \Lambda^2T^*3 \). We picture a map by its graph \( G \) a 2-dimensional submanifold of \( T^*\mathcal{X} \). Now we consider a point \( (x, A) \in \mathcal{G} \subset T^*\mathcal{X} \). At \( (x, A) \), we consider the tangent plane to the graph \( G \), which is described by vectors \( X_1 = \frac{\partial}{\partial x^1} + v_1\frac{\partial}{\partial A}, X_2 = \frac{\partial}{\partial x^2} + v_2\frac{\partial}{\partial A} \).

The set of local coordinates on \( \text{Gr}^2(T^*\mathcal{X}) \) is described by \( (x^\mu, A_\nu, v_{\mu
u}) \), whereas the set of local coordinates on \( 3 = T^*\mathcal{X} \) is \( (x^\mu, A_\mu) \). A basis \( B = B[\Lambda^2T^*_3] \) of the 6-dimensional space \( \Lambda^2T^*_3 \) is:

\[
B = \left\{ dx^1 \wedge dx^2, dA_\mu \wedge dx^\nu, dA_1 \wedge dA_2 \right\}_{1 \leq \mu, \nu \leq 2}
\]

\[
= \left\{ dx^1 \wedge dx^2, dA_1 \wedge dx^1, dA_1 \wedge dx^2, dA_2 \wedge dx^1, dA_2 \wedge dx^2, dA_1 \wedge dA_2 \right\}.
\]

For the Maxwell D2-theory we have the following dimensions for the key spaces involved in the Grassmannian construction:

\[
\begin{align*}
\dim[\text{Gr}^n3] & = 8 & \dim[D_q^n3] & = 6 & \dim(P_q(z)) & = 2 \\
\dim[\Lambda^nT^*3] & = 10 & \dim[\Lambda^nT^*_3] & = 4 & \dim(P^h_q(z)) & = 1
\end{align*}
\]

Any form \( p \in \Lambda^2T^*_3 \) can be identified with the coordinates \( (\epsilon, \pi^{A_\mu}, \varsigma) \) such that:

\[
p = \epsilon d\eta + \epsilon_{\rho\nu}\pi^{A_\mu}\rho dA_\mu \wedge dx^\nu + \varsigma dA_1 \wedge dA_2.
\]

Since, \( \epsilon_{\rho\nu}dx^\nu = d\eta_\rho \), we observe \( \epsilon_{\rho\nu}\pi^{A_\mu}\rho dA_\mu \wedge dx^\nu = -\epsilon_{\rho\nu}\pi^{A_\mu}\rho dA_\mu \wedge dA_\mu = \pi^{A_\mu}\rho dA_\mu \wedge d\eta_\rho \).

A tangent space is identified with coordinates on the Grassman bundle \( t \equiv (v_{\mu\nu}) \in \text{Gr}^2(x, A)3 \).

We describe the pairing \( \langle t, p \rangle = p(X_1, X_2) \). Let notice that \( \{X_\mu\}_{1 \leq \mu \leq 2} \) describes a basis of the tangent space \( t \). We have:

\[
\langle t, p \rangle = \epsilon dx^1 \wedge dx^2(X_1, X_2) + \epsilon_{\rho\nu}\pi^{A_\mu}\rho dA_\mu \wedge dx^\nu(X_1, X_2) + \varsigma dA_1 \wedge dA_2(X_1, X_2)
\]

\[
= \epsilon + \pi^{A_\mu}\nu v_{\mu\nu} + \varsigma(v_{11}v_{22} - v_{12}v_{21}).
\]

and we define the function:

\[
W(x, A, t, p) = \langle t, p \rangle - L(x, A, t).
\]

Notice that the Lagrangian density \( L(x, A, t) \) is identified with a function on \( \text{Gr}^2(x, A)3 \). The tangent space \( t \) is in correspondence with \( p \) - denoted \( t \sqcap p \) - if and only if

\[
\partial W/\partial t(x, A, t, p) = 0
\]

(5.33)

Now we are looking for the enlarged pseudofiber \( P_q(z) \) and the pseudofiber \( P^h_q(z) \). A parametrization of \( \{z \in D^2(x, A)(T^*\mathcal{X}) / d\eta(z) > 0\} \) is described via coordinates \( (t, v_{\mu\nu}) \) with:

\[
z = t^2\partial_1 \wedge \partial_2 + t\epsilon^{\mu\nu}v_{\mu\rho}\frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + (v_{11}v_{22} - v_{12}v_{21})\frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2}.
\]
Elements \( \delta z \in T_z \mathcal{D}_q^n(T^*\mathcal{X}) \) are described by coordinates \( \delta t \) and \( \delta v_{\mu\rho} \):

\[
\delta z = \delta t (2t \partial_1 \wedge \partial_2 + \varepsilon^{\mu\nu} v_{\mu\rho} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu}) + \delta v_{\mu\rho} \left( t \varepsilon^{\mu\nu} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma} \frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2} \right)
\]

We also consider the parametrization \( \{ z^{\eta} \in \mathcal{D}_q^n(x, A) / \mathcal{V}(z) = 1 \} \) in such a context, \( z \) is written:

\[
z^{\eta} = \partial_1 \wedge \partial_2 + \varepsilon^{\mu\nu} v_{\mu\rho} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + (v_{11} v_{22} - v_{12} v_{21}) \frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2}.
\]

Now we consider \( \delta z^{\eta} \in T_z \mathcal{D}_q^n(T^*\mathcal{X}) \):

\[
\delta z^{\eta} = \delta v_{\mu\rho} \left( \varepsilon^{\mu\nu} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma} \frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2} \right)
\]

Then, we describe the space \( [T_z \mathcal{D}_q^n(3)]^\perp = \{ p \in \Lambda^n T_q^* \mathcal{X} / \forall \delta z \in T_z \mathcal{D}_q^n(3), \ p(\delta z) = 0 \} \). So that : \( p \in (T_z \mathcal{D}_q^n(3))^\perp \) is equivalent to \( \forall \delta z \in T_z \mathcal{D}_q^n(T^*\mathcal{X}) , \ p(\delta z) = 0 \). Notice that,

\[
\langle \delta z, p \rangle = \left\langle \delta t (2t \partial_1 \wedge \partial_2 + \varepsilon^{\mu\nu} v_{\mu\rho} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu}), p \right\rangle 
\]

\[
+ \left\langle \delta v_{\mu\rho} \left( t \varepsilon^{\mu\nu} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma} \frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2} \right), p \right\rangle 
\]

\[
\quad = \delta t (2t + v_{\mu\nu} \pi^{A,\mu\nu}) + \delta v_{\mu\rho} (\pi^{A,\mu\rho} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma}). \tag{5.34}
\]

On the other side,

\[
[T_z \mathcal{D}_q^n(3)]^\perp = \left[ T_z \mathcal{D}_q^n(T^*\mathcal{X}) \right]^\perp = \{ p \in \Lambda^n T_q^* \mathcal{X} / \forall \delta z^{\eta} \in T_z \mathcal{D}_q^n(T^*\mathcal{X}), \ p(\delta z^{\eta}) = 0 \};
\]

gives

\[
\langle \delta z, p \rangle = \left\langle \delta v_{\mu\rho} \left( t \varepsilon^{\mu\nu} \frac{\partial}{\partial A_\rho} \wedge \frac{\partial}{\partial x^\nu} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma} \frac{\partial}{\partial A_1} \wedge \frac{\partial}{\partial A_2} \right), p \right\rangle 
\]

\[
\quad = \delta v_{\mu\rho} (\pi^{A,\mu\rho} + \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} v_{\nu\sigma}). \tag{5.35}
\]

The Legendre correspondence related to the system \((5.8)(i)\) is characterized by the functional determinant \( \Delta = \left| \Delta^{\mu\nu}_{\rho\sigma} \right| = \left| \frac{\partial^{A,\mu\nu}}{\partial \rho A_\sigma} \right| \), see \(36 \ 26 \ 27 \). We have the following terms:

\[
\begin{array}{cccc}
\frac{\partial^{A,11}}{\partial (A, A_1)} & = 0 & \frac{\partial^{A,12}}{\partial (A, A_1)} & = 0 \\
\frac{\partial^{A,11}}{\partial (A, A_2)} & = 0 & \frac{\partial^{A,12}}{\partial (A, A_2)} & = 1 \\
\frac{\partial^{A,11}}{\partial (A_1, A)} & = 0 & \frac{\partial^{A,12}}{\partial (A_1, A)} & = - (1 - \varsigma) \\
\frac{\partial^{A,11}}{\partial (A_2, A)} & = - \varsigma & \frac{\partial^{A,12}}{\partial (A_2, A)} & = 0 \\
\frac{\partial^{A,22}}{\partial (A, A_1)} & = - \varsigma & \frac{\partial^{A,22}}{\partial (A, A_1)} & = 0 \\
\frac{\partial^{A,22}}{\partial (A_1, A)} & = 0 & \frac{\partial^{A,22}}{\partial (A_1, A)} & = (1 - \varsigma) \\
\frac{\partial^{A,22}}{\partial (A_2, A)} & = 0 & \frac{\partial^{A,22}}{\partial (A_2, A)} & = 1 \\
\frac{\partial^{A,22}}{\partial (A_2, A)} & = 0 & \frac{\partial^{A,22}}{\partial (A_2, A)} & = 0
\end{array}
\]
Then, we have:

\[
\Delta = \begin{vmatrix}
0 & 0 & 0 & -\varsigma \\
0 & 1 & -(1 - \varsigma) & 0 \\
0 & -(1 - \varsigma) & 1 & 0 \\
-\varsigma & 0 & 0 & 0 \\
\end{vmatrix}
\]

\[
= \varsigma \begin{vmatrix}
0 & 1 & -(1 - \varsigma) \\
0 & -(1 - \varsigma) & 1 \\
-\varsigma & 0 & 0 \\
\end{vmatrix} = \varsigma^2 \begin{vmatrix}
1 & -(1 - \varsigma) \\
-(1 - \varsigma) & 1 \\
\end{vmatrix}
\]

\[
= -\varsigma^2 (1 - (1 - \varsigma)^2) = \varsigma^2 (1 - (1 + \varsigma^2 - 2\varsigma)) = -\varsigma^4 + 2\varsigma^3
\]

Finally, \(\Delta = |\Delta_{\mu \nu}^\rho| \neq 0\) if and only if \(\{\varsigma \neq 0, 2\}\). If we generally denote \(\forall q \in \mathbb{Z}\) the object \(\mathcal{P}_q = \bigcup_{z \in D_q} \mathcal{P}_q(z)\), we now find for the Maxwell 2D theory.

\[
\mathcal{P}_q = \left\{ (e, \pi^{A_{\mu \nu}}, \varsigma) \in \Lambda^2 T^*_{(x, A)} / \varsigma \neq 0, 2 \right\}
\]

\[
\bigcup \left\{ (e, \pi^{A_{\mu \nu}}, 0) \in \Lambda^2 T^*_{(x, A)} / \pi^{A_{11}} = \pi^{A_{22}} = \pi^{A_{12}} + \pi^{A_{21}} = 0 \right\}
\]

\[
\bigcup \left\{ (e, \pi^{A_{\mu \nu}}, 2) \in \Lambda^2 T^*_{(x, A)} / \pi^{A_{12}} - \pi^{A_{21}} = 0 \right\}.
\]

### 6 Conclusion

In this note we have described the general method developed by F. Hélein and J. Kouneiher in the series of papers \[21\] \[22\] \[23\] \[24\] \[25\] \[26\] \[27\]. The main focus is on the determination of algebraic observables, dynamical observables and their related observable functionals. We have also detailed some calculation on the copolarization process for the Maxwell theory. The main result is the Poisson bracket (4.9) between canonical forms \((A, \pi)\). A strong result which merges the physical needs and the mathematical construction of copolarization is that we can not include \(C^1 T^* \text{M}_{\text{Maxwell}}\) in observable 1-forms. There is obviously two directions for further studies. The first is the issue of quantization whereas the second is a more detailed treatment of higher Lepagean equivalent theories. The quantization theory for \(n\)-plectic geometry is still at its infancy. We refer to some recent works : F. Hélein \[21\] \[23\], R.D. Harrivel \[19\] \[20\] and also the precanonical quantization developed by I.V. Kanatchikov \[33\] \[34\] \[35\]. Higher Lepagean equivalent theories for the Maxwell variational problem is currently in preparation \[52\].

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