When the positivity of the $h$-vector implies the Cohen-Macaulay property

F. Cioffi · R. Di Gennaro

Abstract We study relations between the Cohen-Macaulay property and the positivity of the $h$-vector of a locally Cohen-Macaulay equidimensional closed subscheme $X \subset \mathbb{P}^n_K$, showing that these two conditions are equivalent for those $X$ which are close to a complete intersection $Y$ (of the same codimension) in terms of the difference between the degrees. More precisely, let $X$ be contained in $Y$, either of codimension two with $\text{deg}(Y) - \text{deg}(X) \leq 5$ or of codimension $\geq 3$ with $\text{deg}(Y) - \text{deg}(X) \leq 3$. Over a field $K$ of characteristic 0, we prove that $X$ is arithmetically Cohen-Macaulay if and only if its $h$-vector is positive, improving results of a previous work. If $X$ is a curve, this result holds in every characteristic different from 2. We find also other classes of schemes for which the positivity of the $h$-vector implies the Cohen-Macaulay property and provide several examples.

Keywords Cohen-Macaulayness · $h$-Vector · Liaison · Borel ideal

Mathematics Subject Classification (2000) 14M05 · 14M06 · 14M10

1 Introduction

Let $X \subset \mathbb{P}^n_K$ ($K$ algebraically closed field) be a locally Cohen-Macaulay (LCM, for short) equidimensional closed subscheme, which is contained in a complete
intersection $Y$ of the same codimension $c$. It is well-known that, if $X$ is arithmetically Cohen-Macaulay (aCM, for short), then its $h$-vector is admissibile, and in particular also positive. Anyway, there are subschemes $X$ with positive $h$-vector that are not aCM (see Example 3.4).

We show that the Cohen-Macaulay property and the positivity of the $h$-vector are equivalent for those subschemes $X$ which are close to $Y$ in terms of the difference between the degrees, precisely in the following cases:

- $X$ is 1-dimensional over a field $K$ of characteristic different from two, and either $n = 3$ and $\deg(Y) - \deg(X) \leq 5$ or $n \geq 4$ and $\deg(Y) - \deg(X) \leq 3$ (Theorem 3.3);
- $X$ has dimension $\geq 2$ over a field $K$ of null characteristic, and either $c = 2$ and $\deg(Y) - \deg(X) \leq 5$ or $c \geq 3$ and $\deg(Y) - \deg(X) \leq 3$ (Theorems 5.2 and 6.1).

Nevertheless, there are also other classes of subschemes for which the positivity of the $h$-vector implies the Cohen-Macaulay property (see Propositions 5.3 and 6.3, and [6, Proposition 4.6]). These statements improve results described in [5,6], where admissible $h$-vectors are considered, instead of positive $h$-vectors.

The results for curves cannot be improved, due to suitable examples provided by Davis (see Sect. 3 of [10]) and to some extensions of them (see [6, Examples 4.6 and 4.7 and the Appendix]). Here, we provide a class of non-aCM but ICM equidimensional surfaces $X \subset \mathbb{P}_K^4$ with admissible $h$-vectors, such that $\deg(Y) - \deg(X) \geq 10$, by exploiting the Davis’ technique (see Proposition 5.5). The subschemes that belong to this class are constructed applying an odd number of direct algebraic linkages, instead of sequences of basic double links, that are the tools used in [24] to obtain an analogous construction in even liaison classes.

As it is common in the study of the Cohen-Macaulay property, we use general hyperplane sections and, hence, the properties of 0-dimensional schemes, which are always aCM. Nevertheless, the only knowledge of 0-dimensional schemes is not sufficient to give answers to our question because we deal also with schemes that are not aCM. So, we look also at the features of liaison, at the notion of extremal curves and at properties of Borel ideals. We compute saturated Borel ideals with a given Hilbert polynomial by the applet BORELGENERATOR of P. Lella (see at www.personalweb.unito.it/paolo.lella/HSC/borelGenerator.html) and based on an algorithm described in [7] and further developed in [22] (an analogous algorithm is described in [26]).

Similar problems have been treated by means of the admissibility of the $h$-vector in [6,10–12,24], in different situations, with different approaches, and also in the context of simplicial complexes (see [16] and the references therein).

2 Basic definitions and results

Let $K$ be an algebraically closed field and $S := K[x_0, \ldots, x_n]$ be the polynomial ring over $K$ in $n + 1$ variables. Let $M = \bigoplus_{t \geq 0} M_t$ be a finitely generated standard graded $K$-algebra of Krull dimension $k + 1$. 
Definition 2.1 The Hilbert function of $M$ is the function $H_M(t) := \dim_K M_t$, for every $t \geq 0$.

For every integer $i \geq 0$, the $i$-th difference of $H_M$ is the function $\Delta^i H_M : \mathbb{N} \to \mathbb{N}$ defined letting $\Delta^0 H_M := H_M$ and, for each $i > 0$, $\Delta^i H_M(0) := 1$ and $\Delta^i H_M(t) := \Delta^{i-1} H_M(t) - \Delta^{i-1} H_M(t-1)$ for all $t > 0$.

For every integer $i \geq 0$, the $i$-th sum of $H_M(t)$ is the function $\Sigma^i H_M : \mathbb{N} \to \mathbb{N}$ defined letting $\Sigma^0 H_M := H_M$ and, for each $i > 0$, $\Sigma^i H_M(0) := 1$ and $\Sigma^i H_M(t) := \Sigma^i H_M(t-1) + \Sigma^{i-1} H_M(t)$ for each $t > 0$.

It is well-known that there is a polynomial $P_M(z) \in \mathbb{Q}(z)$ such that $H_M(t) = P_M(t)$, for $t \gg 0$. This polynomial is the Hilbert polynomial of $M$ and has degree $k$. The regularity of the Hilbert function $H_M$ is $\rho_M := \min\{i \mid H_M(t) = P_M(t), \forall t \geq i\}$.

Recall that the Hilbert series $\sum_{t \in \mathbb{N}} H_M(t)z^t$ of $M$ is equal to a rational function of type $h(z)/(1 - z)^{k+1}$, where $k+1$ is the Krull dimension of $M$ and $h(z)$ is a polynomial with integer coefficients.

Definition 2.2 The polynomial $h(z) = h_0 + h_1 z + \cdots + h_s z^s \in \mathbb{Z}[z]$ is the $h$-polynomial of $M$ and $(h_0, h_1, \ldots, h_s)$ is the $h$-vector of $M$. We say that an $h$-vector $(h_0, h_1, \ldots, h_s)$ is positive if the integers $h_i$ are positive, for all $0 \leq i \leq s$.

Remark 2.3 Observe that $(h_0, h_1, \ldots, h_s) = (\Delta^{k+1} H_M(0), \Delta^{k+1} H_M(1), \ldots, \Delta^{k+1} H_M(\rho_M + k))$ and $\Sigma^{k+1}(\Delta^{k+1} H_M) = H_M$.

Given two positive integers $a$ and $t$, $a$ can be written uniquely in the form $a = \binom{k(t)}{t} + \binom{k(t-1)}{t-1} + \cdots + \binom{k(j)}{j}$, where $k(t) > k(t-1) > \cdots > k(j) \geq j \geq 1$. Let

$$a^{(t)} := \binom{k(t) + 1}{t + 1} + \binom{k(t - 1) + 1}{t} + \cdots + \binom{k(j) + 1}{j + 1}.$$

A numerical function $H : \mathbb{N} \to \mathbb{N}$ is admissible if $H(t+1) \leq H(t)^{<t>}$ for all $t \geq 1$ and $H(0) = 1$. A finite sequence of positive integers $h_0, h_1, \ldots, h_s$ is admissible if the corresponding function given by $H(0) = h_0$, $H(1) = h_1$, $\ldots$, $H(s) = h_s$, $H(s + i) = 0$, for every $i > 0$, is admissible.

Remark 2.4 It is well-known that a finite sequence $h_0, h_1, \ldots, h_s$ of (positive) integers is the $h$-vector of a Cohen-Macaulay (standard) graded $K$-algebra if and only if it is admissible [29, Theorem 1.5]. In particular, the $h$-vector of a Cohen-Macaulay graded $K$-algebra is positive. Anyway, there are graded $K$-algebras with positive but non-admissible $h$-vector, as next example shows.

Example 2.5 The function $H(t) : 1 \ 4 \ 10 \ P(t) = 9t - 10$ is the Hilbert function of a standard homogeneous $K$-algebra, with positive but non-admissible $h$-vector $(1, 2, 3, 1, 2)$. By [15, Theorem 3.3], we can construct a reduced $K$-algebra with this property, because the first difference of $H(t)$ is admissible.

Definition 2.6 (i) A finitely generated graded $K$-module $M$ is $m$-regular if the $i$-th syzygy module of $M$ is generated in degree $\leq m + i$, for all $i \geq 0$. The regularity $\text{reg}(M)$ of $M$ is the smallest integer $m$ for which $M$ is $m$-regular.
(ii) With the common notation of ideal sheaf cohomology (we refer to [18,25]), a coherent sheaf $F$ on $\mathbb{P}_k^n$ is $m$-regular if $H^i(F(m-i)) = 0$ for all $i > 0$. The Castelnuovo-Mumford regularity (or regularity) $\text{reg}(F)$ of $F$ is the smallest integer $m$ for which $F$ is $m$-regular.

The saturation of a homogeneous ideal $I \subseteq S$ is $I^{\text{sat}} = \{ f \in S \mid \forall j = 0, \ldots, n, \exists r \in \mathbb{N} : x_j^r f \in I \}$ and the ideal $I$ is saturated if $I^{\text{sat}} = I$.

Let $X \subset \mathbb{P}_K^n$ be a closed subscheme of dimension $k$ and $I_X$ its homogeneous (saturated) defining ideal. Instead of $H_S/I_X$, $P_S/I_X$, $\rho_S/I_X$ we can write $H_X$, $P_X$, $\rho_X$. The $h$-vector of $X$ is the $h$-vector of $S/I_X$. If $P_X := \frac{d^X}{k!} + \ldots$, then $\deg(X) := d$ is the degree of $X$ and

$$\deg(X) = \sum_{t=0}^{\rho_X+k} \Delta^{k+1} H_X(t).$$

The regularity $\text{reg}(X)$ of $X$ is defined as the regularity $\text{reg}(I_X)$, since the regularity of a saturated homogeneous ideal $I$ equals the regularity of its sheafification.

3 Cohen-Macaulayness and positive $h$-vector for curves

With the same notation of Sect. 2, we say that the scheme $X \subset \mathbb{P}_K^n$ has Cohen-Macaulay postulation if there is an aCM closed subscheme $W \subset \mathbb{P}_K^n$ such that $H_W = H_X$. If $X$ is a closed subscheme with Cohen-Macaulay postulation and with odd dimension, then $\text{reg}(X) > \rho_X + 1$ [6, Proposition 2.4]. Now, we see that an analogous result holds more generally for odd-dimensional subschemes with a positive $h$-vector.

**Proposition 3.1** If $X \subset \mathbb{P}_K^n$ is a closed subscheme with odd dimension $k$ and positive $h$-vector, then $\text{reg}(X) > \rho_X + 1$.

**Proof** Let $(h_0, h_1, \ldots, h_s)$ be the $h$-vector of $X$. By Remark 2.3, we have $s = \rho_X + k$ and $h_t = \Delta^{k+1} H_X(t) > 0$, for every $0 \leq t \leq \rho_X + k$, because the $h$-vector is positive. Hence, again by Remark 2.3 and by formula (1), we have $\Delta^k P_X(\rho_X + k) = \deg(X) = \sum_{t=0}^{\rho_X+k} \Delta^{k+1} H_X(t) > \sum_{t=0}^{\rho_X+k-1} \Delta^{k+1} H_X(t) = \Delta^k H_X(\rho_X + k - 1)$.

Observe that $\Delta^{k-i} P_X(\rho_X + k - i) = \Delta^{k-i} H_X(\rho_X + k - i) \neq \Delta^{k-i} H_X(\rho_X + k - i - 1)$, for every $0 \leq i \leq k$. In particular, for $i = 1$ we obtain

$$\Delta^{k-1} P_X(\rho_X + k - 2) < \Delta^{k-1} H_X(\rho_X + k - 2)$$

and, repeating the same argument for every $1 \leq i \leq k$, we find $\Delta^{k-i} P_X(\rho_X + k - i) > \Delta^{k-i} H_X(\rho_X + k - i)$, if $i$ is even, and $\Delta^{k-i} P_X(\rho_X + k - i - 1) < \Delta^{k-i} H_X(\rho_X + k - i - 1)$, if $i$ is odd. Thus, if $i = k$ is odd, we obtain $H_X(\rho_X - 1) > P_X(\rho_X - 1)$ and we can apply [8, Proposition 2.5].

Let $C$ be a curve, i.e. a 1-dimensional closed subscheme of a projective space $\mathbb{P}_K^n$. By Proposition 3.1, if $C$ has Cohen-Macaulay postulation then $\text{reg}(C) > \rho_C + 1$. An
other interesting consequence of the positivity of the h-vector involves the arithmetic genus of a curve.

**Proposition 3.2** If \( C \subset \mathbb{P}_K^n \) is a curve with positive h-vector, then its arithmetic genus \( g \) is non-negative.

**Proof** If \((h_0, h_1, \ldots, h_e)\) is the h-vector of \( C \) and \( H_C \) is the Hilbert function of \( C \), then the first difference \( \Delta H_C = (h_0, h_0 + h_1, \ldots, \sum_{0 \leq i \leq s} h_i, \sum_{0 \leq i \leq s} h_i, \ldots) \) is strictly increasing until it becomes equal to \( \deg(C) = \sum_{0 \leq i \leq s} h_i \), because the h-vector is positive. Hence, by construction we get \( \deg(C)(s-1) + 1 \geq \sum_{0 \leq i \leq s-1} \Delta H_C(i) = H_C(s-1) = P_C(s-1) = \deg(C)(s-1) + 1 - g \), so \( g \geq 0 \).

We are already able to state our results for curves, giving a new version of [6, Theorem 4.2] and [6, Proposition 4.6].

**Theorem 3.3** Let \( ch(K) \neq 2 \) and \( C \subset \mathbb{P}_K^n \) be a lCM curve. If either \( n = 3 \) and \( \deg(Y) - \deg(C) \leq 5 \) or \( n \geq 4 \) and \( \deg(Y) - \deg(C) \leq 3 \), then \( C \) is aCM if and only if the h-vector of \( C \) is positive.

**Proof** It is enough to apply Proposition 3.1 and then either [6, Theorem 4.2] or [6, Proposition 4.6].

The \( K \)-algebra of Example 2.5 defines a space curve with positive but non-admissible h-vector. Anyway, that curve has embedded components, so it is not lCM because a lCM curve is also equidimensional. Now, we describe an example of lCM space curve that is non-aCM and has positive (but non-admissible) h-vector.

**Example 3.4** Let \( C \) be the rational curve given by the rational map \( \Phi : \mathbb{P}^1_K \rightarrow \mathbb{P}^3_K \) such that \( \Phi(u, v) := (u^5 + v^5, u^4v + uv^4, u^3v^2 + v^5, u^2v^3) \). The Hilbert function of \( C \) is \( H_C(i) : 1495s + 1 \). We apply to \( C \) three successive basic double links (we refer to [25] for the definition of a basic double link) of type \((1, 7), (1, 7)\) and \((1, 9)\), respectively, obtaining a curve \( \tilde{C} \) with non-admissible h-vector \((1, 2, 3, 4, 5, 5, 1, 2)\), by [25, Proposition 5.4.5(d)].

To construct the curve \( \tilde{C} \) of Example 3.4, we have used the notion of basic double link, for which we referred to [25]. In next section we will recall the notion of direct algebraic linkage that we will allows us to give an alternative proof of Theorem 3.3 in which the combinatorial nature of the h-vector is exploited (see Remark 4.6).

### 4 Liaison and hyperplane sections

In this section we recall some basic results on liaison theory and, especially, on algebraically linked schemes, referring mainly to [25, Chapter 5]. As in Sect. 2, let \( X \subset \mathbb{P}_K^n \) be a closed subscheme of dimension \( k \) and \( I_X \) its (saturated) defining ideal.

It is always possible to find positive integers \( \beta_1 \leq \cdots \leq \beta_{n-k} \) and a complete intersection \( Y \) of type \((\beta_1, \ldots, \beta_{n-k}) \) containing \( X \) (see Theorem 3.14 of Chapter VI of [21]). Recall that \( \text{reg}(Y) = \sum \beta_i - (n-k) + 1 \).

The h-vector of a complete intersection \( Y \subset \mathbb{P}_K^n \) of type \((\beta_1, \beta_2) \) and of dimension \( k = n-2 \) is:

\[
\begin{array}{c}
\beta_1 & \beta_2 \\
\beta_2 & \beta_3 \\
\vdots & \vdots \\
\beta_{n-k} & \beta_{n-k+1} \\
\end{array}
\]
\[ \Delta^{k+1} H_Y(t) : 1 \ 2 \ \ldots \ \beta_1 \ \ldots \ \beta_1 \ \beta_1 - 1 \ \ldots \ 2 \ 1 \ 0, \]  
\[ (2) \]

where \( \beta_1 \) appears \( \beta_2 - \beta_1 + 1 \) times, and \( \text{reg}(Y) = \beta_1 + \beta_2 - 1 \). A complete intersection \( Y \) is also an arithmetically Gorenstein scheme and an arithmetically Gorenstein scheme of codimension two is always a complete intersection.

If we suppose that \( X \) is also ICM and equidimensional, we can say that \( X \) and an other closed ICM equidimensional subscheme \( X' = \mathbb{P}^n \) are algebraically directly linked (linked, for short) by \( Y \) if \( I_{X'} = (I_Y : I_X) \).

A liaison is an equivalent relation generated by direct algebraic linkage (linkage, for short). Recall that the dimension is preserved under liaison and linkage is preserved by general hypersurface section \([25, \text{Proposition 5.2.17}]\).

**Theorem 4.1** \([9, \text{Theorem 3}]\) Let \( Y \) be a \( k \)-dimensional Gorenstein scheme containing properly a \( k \)-dimensional aCM scheme \( W \) defined by a saturated ideal \( I_W \). Let \( W' \) be the scheme linked to \( W \) by \( Y \). Let \( \alpha \) and \( \alpha' \) be the initial degrees of \( I_W / I_Y \) and of \( I_{W'}/I_Y \), respectively. Then,

(i) \( \text{reg}(W) + \alpha' = \text{reg}(W') + \alpha = \text{reg}(Y) \);

(ii) \( \Delta^{k+1} H_Y(t) = \Delta^{k+1} H_W(t) + \Delta^{k+1} H_{W'}(\text{reg}(Y) - 1 - t), \) for every \( 0 \leq t \leq \text{reg}(Y) - 1 \).

**Notation 4.2** If \( X \subset \mathbb{P}^n_K \) is a subscheme of dimension \( k > 0 \), we denote by \( X_{k-i} \) the subscheme obtained by applying \( i \) successive general hyperplane sections to \( X \), where \( 0 \leq i \leq k \). In particular, if \( k \geq 2 \), \( C := X_1 \) is the curve obtained by applying \( k - 1 \) successive general hyperplane sections to \( X \) and \( Z := X_0 \) is a general hyperplane section of \( C \). Moreover, when \( X \) is equidimensional and ICM, we denote by \( C' \) and \( Z' \), respectively, the curve and the \( 0 \)-dimensional scheme linked to \( C \) and \( Z \) by general hyperplane sections of a complete intersection \( Y \) containing properly \( X \).

**Remark 4.3** (a) If \( k > 0 \), \( h \in S_1 \) is a general linear form which is not a zero-divisor on \( S/I_X \) and \( J := (I_X, h) \), the saturated ideal \( J^{sat}/(h) \) of \( X_1 \) is the curve obtained by applying \( k \) successive general hyperplane sections to \( X \) and \( Z := X_0 \) is a general hyperplane section of \( C \). Moreover, when \( X \) is equidimensional and ICM, we denote by \( C' \) and \( Z' \), respectively, the curve and the \( 0 \)-dimensional scheme linked to \( C \) and \( Z \) by general hyperplane sections of a complete intersection \( Y \) containing properly \( X \).

(b) If \( X \) is aCM, then \( \Delta H_X(t) = H_{X_{k-1}}(t) \) for all \( t \). The converse is false in general, but true for curves. Anyway, an equidimensional and ICM closed subscheme \( X \) of dimension \( k \geq 2 \) is aCM if and only if its general hyperplane section is aCM ([20, \text{Proposition 2.1}] or [25, \text{Theorem 1.3.3}]).

(c) A curve \( C \) without Cohen-Macaulay postulation has a non-admissible \( h \)-vector \((h_0, h_1, \ldots, h_s)\), but the first sum \((h_0, h_0 + h_1, \ldots, \sum_i h_i, \ldots)\), which is equal to \( \Delta H_C \), has to be admissible by (a).

**Lemma 4.4** (i) If \( X \subset \mathbb{P}^n_K \) is a closed ICM equidimensional subscheme, then \( X_{k-i} \) is equidimensional and ICM, for every \( i \leq k \). If moreover \( X \) has codimension 2 and is non-degenerate, then \( X_{k-i} \) is non-degenerate, for every \( i \neq k \).
(ii) Let \( t_0 := \min\{t \in \mathbb{N} : H_X(t) < H_Y(t)\} \) and, for every \( 1 \leq i \leq k \), \( t_i := \min\{t \in \mathbb{N} : H_{X_{k-i}}(t) < \Delta^i H_Y(t)\}. \) Then

\[
t_0 \geq t_1 \geq \cdots \geq t_k = \min\{t \in \mathbb{N} : \Delta H_Z(t) < \Delta^{k+1} H_Y(t)\}. \tag{3}
\]

**Proof** (i) For \( i = k \), the first part of the statement is trivial because \( X_0 = Z \) is a 0-dimensional scheme, so it is aCM.

For \( i < k \), recall that a scheme \( X \) is equidimensional and ICM if and only if its deficiency modules \( \bigoplus_i H^j(I_X(t)), \ j > 0 \), have finite length. This property is preserved by general hyperplane section due to the sequence

\[
0 \rightarrow H^0(I_X(t-1)) \rightarrow H^0(I_X(t)) \rightarrow H^0(I_{X_{k-1}, H}(t)) \rightarrow
\]

\[
H^1(I_X(t-1)) \rightarrow H^1(I_X(t)) \rightarrow H^1(I_{X_{k-1}}(t)) \rightarrow \ldots . \tag{4}
\]

where \( H \) is a general hyperplane. Hence, \( X_{k-i} \) is equidimensional and ICM, for every \( i < k \).

Further, in codimension two, if \( X \) is non-degenerate also \( X_{k-i} \) is non-degenerate, for every \( i < k \) [6, Proposition 1.4].

(ii) By Remark 4.3(a), we have \( \Delta H_Y(t_0) > \Delta H_X(t_0) \geq H_{X_{k-1}}(t_0) \) and we deduce that \( t_1 \leq t_0 \); from \( \Delta^2 H_Y(t_1) > \Delta H_{X_{k-1}}(t_1) \geq H_{X_{k-2}}(t_1) \) we deduce that \( t_2 \leq t_1 \); and so on.

**Lemma 4.5** The arithmetic genus of \( C' \) is

\[
g' = D_t + \deg(C') \left( -t + \sum \beta_i - (c + 1) - 1 \right) + 1, \tag{5}
\]

where \( D_t = \Delta^{k-1} P_Y(t) - P_C(t) \), for every \( t \geq \max\{\rho_C, \reg(Y) - 2\} \).

**Proof** Denoting by \( \bar{g} \) and \( g \) the arithmetic genus respectively of \( Y \) and \( C \), we obtain

\[
\left( \prod_i \beta_i - \deg(C') \right) \cdot t + 1 - g = \prod_i \beta_i \cdot t + 1 - \bar{g} - D_t .
\]

By applying [25, Corollary 5.2.14], for which \( g - g' = \frac{1}{2} \left( \sum_{i=1}^n \beta_i - n - 1 \right) (\deg(C) - \deg(C')) \), and by the shape of the arithmetic genus of a complete intersection curve (see, for example, [25, page 36]), for which we have \( \bar{g} = \frac{1}{2} \sum_i \beta_i \left( \sum_{i=1}^n \beta_i - n - 1 \right) + 1 \), we obtain the thesis. \( \Box \)

**Remark 4.6** By analyzing the admissibility of the \( h \)-vector, we can prove the part of Theorem 3.3 for space curves also in the following geometric way, that is of course more complicated, but that highlights the combinatorial nature of the notion of \( h \)-vector.

We look for all the possible positive \( h \)-vectors of \( C \), taking into account the relations among the integers \( t_0 \) and \( t_1 \) of Lemma 4.4 and the fact that \( \Delta H_Z \) can differ from the \( h \)-vector of \( Y \) only in degrees \( \geq \reg(Y) - \deg(Z') \), by Theorem 4.1.

\( \Box \) Springer
Table 1  Possible positive $h$-vectors for a non-aCM curve $C$ with minimal arithmetic genus, when $\Delta H_{Z'}$: 111111

| $\beta_1$ | $t$ | $\text{reg}(Y) - 5$ | $\text{reg}(Y) - 4$ | $\text{reg}(Y) - 3$ | $\text{reg}(Y) - 2$ | $\text{reg}(Y) - 1$ |
|-----------|-----|--------------------|--------------------|--------------------|--------------------|--------------------|
| 2         | $\Delta^2 H_Y$ | 2                 | 2                  | 2                  | 2                  | 1                  |
|           | $\Delta^2 H_C$ | 2                 | 2                  | 1                  | 0                  | 0                  |
|           | $\Delta H_Z$   | 1                 | 1                  | 1                  | 1                  | 0                  |
| 3         | $\Delta^2 H_Y$ | 3                 | 3                  | 3                  | 2                  | 1                  |
|           | $\Delta^2 H_C$ | 3                 | 3                  | 1                  | 0                  | 0                  |
|           | $\Delta H_Z$   | 2                 | 2                  | 2                  | 1                  | 0                  |
| 4         | $\Delta^2 H_Y$ | 4                 | 4                  | 3                  | 2                  | 1                  |
|           | $\Delta^2 H_C$ | 4                 | 4                  | 1                  | 0                  | 0                  |
|           | $\Delta H_Z$   | 3                 | 3                  | 2                  | 1                  | 0                  |
| $\geq 5$  | $\Delta^2 H_Y$ | 5                 | 4                  | 3                  | 2                  | 1                  |
|           | $\Delta^2 H_C$ | 5                 | 4                  | 1                  | 0                  | 0                  |
|           | $\Delta^2 H_C$ | 4                 | 6                  | 0                  | 0                  | 0                  |
|           | $\Delta H_Z$   | 4                 | 3                  | 2                  | 1                  | 0                  |

If $Z'$ is not degenerate or is degenerate of degree at most 4, then all the possible positive $h$-vectors of $C$ are admissible and we get the thesis by [6, Theorem 4.2]. So, let $Z'$ be degenerate of degree 5, i.e. $\Delta H_{Z'}$: 1111110.

If $\text{ch}(K) = 0$ then $C'$ is a plane curve by [19, Theorem 2.1] and so $C$ is aCM. If $\text{ch}(K) = p > 0$, we find that the arithmetic genus $g'$ of $C'$ is always positive, in contradiction with [19, proof of Theorem 3.3], where Hartshorne studies non-degenerate space curves with degenerate general hyperplane section. Indeed, by looking at all the possible sequences that can be $h$-vectors of $C$ in this case, we obtain that $\rho_C \leq \text{reg}(Y) - 1$. So, by Lemma 4.5 applied with $t = \text{reg}(Y) - 1 = \beta_1 + \beta_2 - 2 \geq \rho_C$, we obtain $g' = D_{\text{reg}(Y) - 1} - 9$, where $D_{\text{reg}(Y) - 1}$ is the difference between the values assumed on $\text{reg}(Y) - 1$ by the Hilbert polynomials of $Y$ and of $C$, respectively. In Table 1 we collect all the sequences that give rise to Hilbert polynomials $P_C(t)$ for $C$ that assume the maximum possible value on $\text{reg}(Y) - 1$, with the consequence that the corresponding value of $D_{\text{reg}(Y) - 1} = 11$ is the minimum possible. Hence, we obtain $g' \geq 11 - 9 > 0$.

5 Cohen-Macaulayness and positive $h$-vector in codimension two

With the notation stated in Sect. 4, let $X \subset \mathbb{P}_K^n$ be a non-degenerate codimension two subscheme that is ICM and equidimensional. Thus, $k = n - 2$ is the dimension of $X$ and $C$ is a space curve. Recall that, for every $i \neq k$, $X_{k-i}$ is non-degenerate, equidimensional and ICM as $X$, by Lemma 4.4.

We will need the following characterization of a space extremal curve in terms of the $h$-vector. We refer to [4, 13] for the definition and geometric descriptions of a space extremal curve. For a large class of explicit examples of extremal space curves see [23] and the references therein.
Table 2  The case $\Delta H_Z(t) : 122$

| $t$ | $0$ | $\ldots$ | $\text{reg}(Y) - 3$ | $\text{reg}(Y) - 2$ | $\text{reg}(Y) - 1$ | $\text{reg}(Y)$ |
|-----|-----|-----------|-----------------|-----------------|-----------------|----------------|
| $\Delta^{k+1} H_Y$ | 1   | $\ldots$ | $a_{\text{reg}(Y) - 3}$ | 2               | 1               | 0              |
| $\Delta^{k+1} H_X$ | 1   | $\ldots$ | $a_{\text{reg}(Y) - 3} - 2$ | 0               | 0               | 0              |
| $\Delta^2 H_C$     | 1   | $\ldots$ | $a_{\text{reg}(Y) - 3} - 2$ | 0               | 0               | 0              |
| $\Delta H_Z$       | 1   | $\ldots$ | $a_{\text{reg}(Y) - 3} - 2$ | 0               | 0               | 0              |

**Proposition 5.1** Let $ch(K) = 0$ and $C \subset \mathbb{P}^3_K$ be a space curve of degree $d \geq 5$ with general hyperplane section $Z$. Then $C$ is an extremal curve if and only if $\Delta H_Z(t) : 121 \ldots 1$.

**Proof** It is enough to apply [13, Theorem 8], because $Z$ has character $(d - 1, 2)$ iff $\Delta H_Z(t) : 121 \ldots 1$, by the definition of character of a plane 0-dimensional scheme (see [14]). One can also use geometric arguments due to [28] (to find a plane curve $C_\pi$ of degree $d - 1$ contained in $C$) and then [4, Theorem 2.1]. □

In the following, we use Notation 4.2.

**Theorem 5.2** Assuming $ch(K) = 0$ and $n \geq 4$, let $X \subset \mathbb{P}^n_K$ be a lCM equidimensional codimension two subscheme contained in a complete intersection $Y$ with $\deg(Y) - \deg(X) \leq 5$. Then $X$ is aCM if and only if $X$ has positive $h$-vector.

**Proof** For the cases $\deg(Y) - \deg(X) \leq 4$ we refer to the proof of [6, Theorem 4.9], because in that proof only the positivity of the $h$-vector has been used.

For the case $\deg(Y) - \deg(X) = 5$, according to the notation of Sect. 1, by Remark 4.3(b) we have that $\Delta H_C(t) \geq H_Z(t)$ for every $t$ and the equality holds for every $t$ iff $C$ is aCM. In the hypothesis that the $h$-vector of $X$ is positive, we consider all possible $\Delta H_Z(t)$ such that $deg(Z') = 5$.

If $\Delta H_{Z'} : 11111$, then $Z'$ is degenerate with $deg(C') = deg(Z') = 5 \geq 3$; so, being $ch(K) = 0$, $C'$ is a plane curve [19, Theorem 2.1], then $C'$ is aCM, thus $C$ is.

If $\Delta H_{Z'}(t) : 122$, according to Table 2 we get $\Delta H_Z(t) = 0$, for every $t \geq \text{reg}(Y) - 2$, by Theorem 4.1. Then, recalling that $k = n - 2$ is the dimension of $X$, we obtain

$$
\sum_t \Delta^{k+1} H_X(t) = \deg(X) = \deg(Z) = \sum_{t \leq \text{reg}(Y) - 3} \Delta H_Z(t),
$$

by formula (1), and $t_i \geq t_k = \text{reg}(Y) - 3$, for every $0 \leq i < k$, by Lemma 4.4. In particular, we have

$$
\Delta^{k+1-i} H_{X_{k-i}}(t) = \Delta^{k+1} H_Y(t), \text{ for every } t < \text{reg}(Y) - 3 \text{ and } 0 \leq i \leq k,
$$

and $\Delta^{k+1} H_X(\text{reg}(Y) - 3) \geq \Delta H_Z(\text{reg}(Y) - 3)$. So, it follows $\Delta^{k+1} H_X(t) = \Delta H_Z(t)$, for every $t \geq \text{reg}(Y) - 3$, otherwise the $h$-vector of $X$ would have some
negative entries, for \( t > \text{reg}(Y) - 3 \). As a consequence, we have also \( \Delta^2 H_C(t) = \Delta H_Z(t) \), for every \( t \geq \text{reg}(Y) - 3 \), because \( t_{k-1} \geq t_k = \text{reg}(Y) - 3 \). Now, we can apply Remark 4.3(b).

If \( \Delta H_{Z'}(t) : 1 \ 2 \ 1 \ 1 \), then the curve \( C' \) is extremal by Proposition 5.1. Thus, \( C' \) is aCM or there is not a lCM surface with \( C' \) as general hyperplane section, by [3, Theorem 1.1 and Corollary 3.6].

By the same arguments applied in the proof of Theorem 5.2 we get the following result.

**Proposition 5.3** The conclusion of Theorem 5.2 holds also for every \( d' = \text{deg}(Y) - \text{deg}(X) \geq 6 \) when

1. \( Z' \) is degenerate; or
2. \( Z' \) has maximal rank, i.e. \( H_{Z'}(t) = \min \{ d', \left( \frac{d' + 2}{2} \right) \} \).

Moreover, in \( \mathbb{P}^n_K \), \( n \geq 4 \), there is not a lCM subscheme \( X' \) of codimension two with \( Z' \) as 0-dimensional hyperplane section such that \( \Delta H_{Z'}(t) : 1 \ 2 \ 1 \ldots \ 1 \).

**Remark 5.4** By Proposition 5.3, the condition \( \text{deg}(Y) - \text{deg}(X) \leq 5 \) does not characterize the equidimensional lCM schemes that are forced to be aCM by the positivity of their \( h \)-vector.

In [6, Remark 4.10], we gave an example of a non-aCM, but lCM equidimensional surface \( X \) in \( \mathbb{P}^4_K \) with Cohen-Macaulay postulation and \( \text{deg}(Y) - \text{deg}(X) = 10 \), by exploiting the technique of Davis to construct examples of non-aCM space curves \( C' \) of degree \( d' \geq 6 \) (see [10] and [6, Appendix]) linked to curves with Cohen-Macaulay postulation. More precisely, we applied an odd number of suitable linkages to the Veronese surface \( V \) in \( \mathbb{P}^4_K \) (e.g. [18, Cap. II, Ex. 7.7]) for which we know that \( h^1(\mathcal{I}_V(t)) = 0 \) if \( t \neq 1 \), \( h^1(\mathcal{I}_V(1)) = 1 \) and \( h^2(\mathcal{I}_V(t)) = 0 \) for every \( t \) [2, Example 3.7] (see also [27] for a study of the Veronese surface and its degenerations in an analogous topic). Starting from this example, now we exhibit a class of non-aCM surfaces in \( \mathbb{P}^4_K \) with Cohen-Macaulay postulation.

As recalled in [6, Appendix], by applying successive suitable linkages to the union of two skew lines in \( \mathbb{P}^3_K \), Davis constructs the following two types of space curves (where \( Z \) is the general hyperplane section):

1. curves \( D \subset \mathbb{P}^3_K \) of type \([a, r]\), with \( a > r > 0 \), such that

   \[ \alpha(I_Z) = \rho_Z - 1 = a; \quad \Delta H_Z(a) = a; \quad \Delta H_Z(a + 1) = r; \quad h^1(\mathcal{I}_D(a)) = 1; \]

2. curves \( D \subset \mathbb{P}^3_K \) of type \([a, r]\), with \( a > r > 1 \), such that

   \[ \alpha(I_Z) = \rho_Z - 1 = a; \quad \Delta H_Z(a) = r; \quad \Delta H_Z(a + 1) = 1; \quad h^1(\mathcal{I}_D(a)) = 1. \]

For every \( d' \geq 6 \), \( d' \neq 7, 8, 12 \), let \( e_{d'} := \max \{ t : \left( \frac{t}{2} \right) \leq d' \} \) and \( f_{d'} := d' - \left( \frac{e_{d'}}{2} \right) \).

A non-aCM curve \( C_{d'} \) with Cohen-Macaulay postulation and contained in a complete intersection \( Y \) with \( \text{deg}(Y) - \text{deg}(C_{d'}) = d' \) is obtained by applying a linkage with
complete intersection of type \((\beta_1, \beta_2) = (e_{d'} + 1, e_{d'} + 1)\) to a curve \(C'_{d'}\), chosen among the curves \(D\) of type \([a, r]\) or \([[a, r]]\) according to the table below:

| \(d'\) | \(e_{d'} - f_{d'}\) | \(C'_{d'}\) |
|---|---|---|
| 1 | \([e_{d'} - 1, f_{d'} - 1]\) | \([[d]'\) |
| 2 | \([e_{d'} - 1, 2]\) | \([[d]'\) |
| 3 | \([e_{d'} - 1, f_{d'}]\) | \([[d]'\) |
| \(\geq 4\) | \([e_{d'} - 2, f_{d'} + 1]\) | \([[d]'\)

For \(d' = 7, 8, 12\), the curves \(C_{d'}\) are constructed in a slightly different way.

**Proposition 5.5** There exists an ICM equidimensional surface \(X \subset \mathbb{P}_k^4\) contained in a complete intersection \(Y\), with \(\deg(Y) - \deg(X) = d'\), such that \(X\) is non-aCM, but has Cohen-Macaulay postulation, for every integer \(d' \in \{10, 14, 15, 19, 20, 21, 22\} \cup \left( \bigcup_{t \geq 8} \left( \left( \frac{t}{2} \right) - 3, \left( \frac{t}{2} \right) - 2, \left( \frac{t}{2} \right) - 1, \left( \frac{t}{2} \right), \left( \frac{t}{2} \right) + 1 \right) \right)\).

**Proof** It is crucial for our purpose that the general hyperplane section of the Veronese surface \(V\) of \(\mathbb{P}_k^4\) is a curve \(C\) that is linked to two skew lines by a linkage of type \((2, 3)\), as one can check by a computation and by [10, Lemma-Definition, page F8]. Indeed, many of the curves of types \([a, r]\) and \([[a, r]]\) are constructed from two skew lines by linkages, the first of which generates \(C\). Thanks to the fact that linkage preserves general hyperplane sections, by applying to \(V\) the same linkages, we obtain surfaces \(X'_{d'}\), whose general hyperplane sections are curves of type \([a, r]\) or \([[a, r]]\). With a further linkage of type \((\beta_1, \beta_2) = (e_{d'} + 1, e_{d'} + 1)\), we obtain a surface \(X\) whose general hyperplane section is the curve \(C_{d'}\) of Davis, which has Cohen-Macaulay postulation. Moreover, if the number of applied linkages is odd, by the Hartshorne-Schenzel Theorem we obtain that \(h^1(I_X(t)) = h^2(I_V(t)) = 0\), for every \(t\), so that \(\Delta H_X(t) = H_{C_{d'}}(t)\) [25, Remark 2.1.3] and also \(X\) has Cohen-Macaulay postulation, because \(X\) shares its \(h\)-vector with \(C_{d'}\).

First, we observe that the described strategy works in the cases considered in the following table: we apply successively linkages of the listed types to the surface of \(\mathbb{P}_k^4\) denoted by \(S\), obtaining the above surface \(X'_{d'}\) of degree \(d'\); after a further linkage, we get the surface \(X\) with Cohen-Macaulay postulation.

| \(d'\) | \(S\) | Linkages | Further linkage | \(h\)-Vector of \(X\) |
|---|---|---|---|---|
| 10 | \(V\) | \((3, 4), (3, 6)\) | \((6, 6)\) | \([1, 2, 3, 4, 5, 6, 5]\) |
| 14 | \(X'_{10}\) | \((4, 6), (4, 7)\) | \((6, 6)\) | \([1, 2, 3, 4, 5, 6, 1]\) |
| 15 | \(V\) | \((3, 3), (3, 4), (4, 5), (4, 7)\) | \((7, 7)\) | \([1, 2, 3, 4, 5, 6, 7, 6]\) |
| 19 | \(X'_{14}\) | \((5, 7), (5, 8)\) | \((7, 7)\) | \([1, 2, 3, 4, 5, 6, 7, 2]\) |
| 20 | \(X'_{15}\) | \((5, 7), (5, 8)\) | \((7, 7)\) | \([1, 2, 3, 4, 5, 6, 7, 1]\) |
| 21 | \(V\) | \((3, 3), (3, 4), (4, 4), (4, 5), (5, 6)(5, 8)\) | \((8, 8)\) | \([1, 2, 3, 4, 5, 6, 7, 8, 7]\) |
| 22 | \(V\) | \((3, 4), (4, 5), (5, 6), (5, 8)\) | \((8, 8)\) | \([1, 2, 3, 4, 5, 6, 7, 8, 6]\) |
In this construction there is a type of recursion useful to prove that our strategy works well also in the remaining cases. Indeed, for $d' \in \mathbb{U}_{t \geq 8}\left\{ \left(\frac{d}{2}\right) - 3, \left(\frac{d}{2}\right) - 2, \left(\frac{d}{2}\right) - 1 \right\}$, by the construction of [10] it is enough to apply successively two linkages of type $(t - 2, t)$ and $(t - 2, t + 1)$ to $X_{\tilde{d}}$, where $\tilde{d} = \left(\frac{d - 1}{2}\right) - 2, \left(\frac{d - 1}{2}\right) - 1, \left(\frac{d - 1}{2}\right)$, respectively. With a further linkage of type $(t, t)$ we obtain the desired surface $X$, after a total odd number of linkages applied to $V$.

For $d' = \left(\frac{d}{2}\right)$, we observe that the curve $C_{\tilde{d}'}$ is of type $[t - 2, 1]$ and is obtained from $C$ by linkages of type $(3, 3), (3, 4), \ldots, (t - 3, t - 3), (t - 3, t - 2), (t - 2, t - 1), (t - 2, t + 1)$. Starting from the Veronese surface $V$, with a further linkage of type $(t + 1, t + 1)$ we obtain the desired surface $X$.

For $d' = \left(\frac{d}{2}\right) + 1$, the curve $C_{\tilde{d}'}$ is of type $[t - 2, 2]$ and is obtained from $C$ by linkages of two types: first $(3, 3), (3, 4), \ldots, (t - 4, t - 3), (t - 3, t - 2), (t - 2, t - 1), (t - 2, t + 1)$. Starting from the Veronese surface $V$, with a further linkage of type $(t + 1, t + 1)$ we obtain the desired surface.

\[ \square \]

6 Cohen-Macaulayness and positive $h$-vector in codimension higher than two

With the same notation of the previous sections, $Y$ is a complete intersection of type $(\beta_1, \ldots, \beta_{n-k})$ containing properly $X$ and $\text{reg}(Y) = \sum \beta_i - c + 1$, where $c = n - k$ is the codimension of $X$ and $Y$.

In this section, we suppose $\text{ch}(K) = 0$ and use the notion of Borel ideal. Recall that a Borel ideal is an ideal fixed under the action of the Borel subgroup of upper-triangular invertible matrices, if $x_0 < x_1 < \cdots < x_n$, or under the action of the lower-triangular invertible matrices, if $x_0 > x_1 > \cdots > x_n$. Here, we consider the latter setting.

In generic coordinates, the initial ideal of an ideal $I$, with respect to a fixed term order $<$, is a constant Borel monomial ideal called the generic initial ideal of $I$. We denote by $\text{gin}(I)$ the generic initial ideal of a homogeneous ideal $I$ with respect to the degree reverse lexicographic term order and we set $\text{gin}(X) := \text{gin}(I(X))$ for any subscheme $X$. For a survey on this subject we refer to [17]. It is well-known that $\text{reg}(X) = \text{reg}(\text{gin}(X))$ ([1, Theorem 4.2] and that, if $Z$ is a general hyperplane section of $X$, then $\text{gin}(Z) = (\text{gin}(X), x_n)^{\text{sat}} / (x_n)$ [17, Proposition 2.9].

**Theorem 6.1** Let $X \subset \mathbb{P}^n_K$ $(n \geq 4)$ be a lcm equidimensional subscheme of codimension $c = n - k \geq 3$ contained in a complete intersection $Y$ with $\text{deg}(Y) - \text{deg}(X) \leq 3$. Then $X$ is aCM if and only if $X$ has positive $h$-vector.

**Proof** We will follow the same approach of the proof of Theorem 5.2 and use Notation 4.2.

If $\Delta H_Z$ is either 1 0 or 1 1 0 or 1 2 0, then $\Delta H_Z(t) = 0$ for every $t \geq \text{deg}(Y) - 1$ by Theorem 4.1, hence $\text{reg}(Y) - 2 \leq t_k$ by the definition of $t_k$ in Lemma 4.4. So, we have $\Delta H_Z(\text{deg}(Y) - 2) \leq \Delta^{k+1} H_X(\text{deg}(Y) - 2) \leq \Delta^{k+1} H_Y(\text{deg}(Y) - 2)$. In particular, $\Delta H_Z(\text{deg}(Y) - 2) = \Delta^{k+1} H_X(\text{deg}(Y) - 2)$, otherwise the $h$-vector of $X$ should be non-positive by the same argument on the degree already applied in the
proof of Theorem 5.2. As a consequence, we get \( \Delta H_Z(t) = \Delta H_C(t) \) for every \( t \) and the thesis follows by Remark 4.3(b).

It remains to analyze the case \( \Delta H_{Z'} : 1 \ 1 \ 1 \), in which \( \text{reg}(Y) - 3 \leq t_k \). Recall that the arithmetic genus of a lCM curve of degree 3 is \( \leq 1 \) and the equality holds if and only if the curve is planar (e.g. [19]).

Let \( c = 3 \) and \( X \) be non-aCM. Then, by Theorem 4.1 and by degree arguments, i.e. \( \sum_{t \geq 0} \Delta^{k+1} H_X(t) = \text{deg}(X) = \text{deg}(Y) - 3 \), we obtain the following situation:

| \( t \) | 0 | \ldots | \text{reg}(Y) - 3 | \text{reg}(Y) - 2 | \text{reg}(Y) - 1 | \text{reg}(Y) |
|-------|---|-------|-----------------|-----------------|-----------------|-----------------|
| \( \Delta^{k+1} H_Y \) | 1 | \ldots | \text{areg}(Y) - 3 | 3 | 1 | 0 |
| \( \Delta^{k+1} H_X \) | 1 | \ldots | \text{areg}(Y) - 3 | 1 | 0 | 0 |
| \( \Delta H_Z \) | 1 | \ldots | \text{areg}(Y) - 3 - 1 | 2 | 0 | 0 |

and \( \Delta^{k+1} H_X(t) = \Delta^{k+1} H_Y(t) = 0 \), for every \( t \geq \text{reg}(Y) = \sum \beta_i - c + 1 \). Applying Lemma 4.5 with \( t = \text{reg}(Y) - 2 = \sum \beta_i - c + 1 \) and computing \( D_t = \Delta^{k-1} P_Y(t) - P_C(t) = 2 \), we obtain \( g' = 0 \), hence \( P_C(t) = 3t + 1 \). Using the already cited applet BorelGenerator of P. Lella, \( gin(C') \) can be one of the following Borel ideals:

\[
J_1 = (x_0, x_1, x_2^4, x_2^3 x_3), \quad J_2 = (x_0, x_1^2, x_1 x_2, x_1 x_3, x_2^3), \quad J_3 = (x_0, x_1^2, x_1 x_2, x_2^3).
\]

Since \( Z' \) is a planar scheme, then \( gin(Z') \) is univocally determined by its \( h \)-vector; so, \( gin(Z') = (x_0, x_1, x_2^3) \) and by [17, Proposition 2.9] we exclude \( J_3 \). We exclude also \( J_1 \) because in this case \( g' = 1 \neq 0 \), as \( C' \) would be a plane curve of degree 3. Finally, we exclude \( J_2 \), because in that case \( C' \) should be a non-degenerate space curve of degree 3, meanwhile \( Z' \) is degenerate in \( \mathbb{P}^2_K \), contrary to [19, Theorem 2.1]. Hence, we obtain the thesis for \( c = 3 \).

Let \( c \geq 4 \) and \( X \) be non-aCM. Then, we obtain several possible situations. The first one generalizes the case \( c = 3 \) and is

| \( t \) | 0 | \ldots | \text{reg}(Y) - 3 | \text{reg}(Y) - 2 | \text{reg}(Y) - 1 | \text{reg}(Y) |
|-------|---|-------|-----------------|-----------------|-----------------|-----------------|
| \( \Delta^{k+1} H_Y \) | 1 | \ldots | \text{areg}(Y) - 3 | \( c \) | 1 | 0 |
| \( \Delta^{k+1} H_X \) | 1 | \ldots | \text{areg}(Y) - 3 | \( c - 2 \) | 0 | 0 |
| \( \Delta H_Z \) | 1 | \ldots | \text{areg}(Y) - 3 - 1 | \( c - 1 \) | 0 | 0 |

the second one is

| \( t \) | 0 | \ldots | \text{reg}(Y) - 3 | \text{reg}(Y) - 2 | \text{reg}(Y) - 1 | \text{reg}(Y) |
|-------|---|-------|-----------------|-----------------|-----------------|-----------------|
| \( \Delta^{k+1} H_Y \) | 1 | \ldots | \text{areg}(Y) - 3 | \( c \) | 1 | 0 |
| \( \Delta^{k+1} H_X \) | 1 | \ldots | \text{areg}(Y) - 3 | \( c - 3 \) | 1 | 0 |
| \( \Delta H_Z \) | 1 | \ldots | \text{areg}(Y) - 3 - 1 | \( c - 1 \) | 0 | 0 |
where $\Delta^{k+1} H_X(t) = \Delta^{k+1} H_Y(t) = 0$, for every $t \geq \text{reg}(Y) = \sum b_i - c + 1$; only for $c > 4$, there are other possible cases, in which we have always $\Delta^{k+1} H_X(\text{reg}(Y) - 2) < c - 3$.

In the first case, as for $c = 3$, we obtain $g' = 0$; in the second case, we apply Lemma 4.5 for $t = \text{reg}(Y) - 2$, computing $D_t = \Delta^{k-1} P_Y(t) - P_C(t) = 3$ and obtaining $g' = 1$. So, we have either $P_{C'}(t) = 3t + 1$ or $P_{C'}(t) = 3t$.

In the first case, as for $c = 3$, we obtain for $C'$ the possible generic initial ideals $J'_1 = (x_0, \ldots, x_{n-3}, x^4_{n-2}x_{n-1}, x^3_{n-2}), J'_2 = (x_0, \ldots, x_{n-4}, x^2_{n-3}, x_{n-3}x_{n-2}, x_{n-3}x_{n-1}, x^3_{n-2})$ and $J'_3 = (x_0, \ldots, x_{n-4}, x^2_{n-3}, x_{n-3}x_{n-2}, x^2_{n-2})$, that we exclude with the same arguments as before. In the second case, we get $\text{gin}(C') = (x_0, \ldots, x_{n-3}, x^2_{n-2})$ and $C'$ would be aCM. In the other cases, applying Lemma 4.5 with $t = \rho_C \geq \text{reg}(Y) - 2$, we obtain $g' > 1$, that is absurd.

**Example 6.2** Just to give an example of the situations that can occur for $c > 4$ in the proof of Theorem 6.1, we consider the following case

| $t$ | $\text{reg}(Y) - 3$ | $\text{reg}(Y) - 2$ | $\text{reg}(Y) - 1$ | $\text{reg}(Y)$ |
|-----|----------------------|----------------------|----------------------|------------------|
| $\Delta^{k+1} H_Y$ | 1 | $a_{\text{reg}(Y) - 3}$ | $c$ | 1 |
| $\Delta^{k+1} H_X$ | 1 | $a_{\text{reg}(Y) - 3}$ | $c - 4$ | 1 |
| $\Delta H_Z$ | 1 | $a_{\text{reg}(Y) - 3} - 1$ | $c - 1$ | 0 |

For $t = \text{reg}(Y) - 1$, we obtain $D_t = \Delta^{k-1} P_Y(t) - P_C(t) = 8$ and $g' = 3$.

**Proposition 6.3** The statement of Theorem 6.1 holds also with $\text{deg}(Y) - \text{deg}(X) \geq 4$ if $H_{Z'}(t) = \min \{d', \left(\frac{t+c}{c}\right)\}$, i.e. $Z'$ has maximal rank in $\mathbb{P}_K^r$.

**Acknowledgments** We thank Silvio Greco who introduced us to the geometry of extremal curves and the referee for useful comments and suggestions. The authors were partially supported by GNSAGA and by the PRIN “Geometria delle Varietà Algebriche”, cofinanced by MIUR (Italy) (cofin 2010–2011).

**References**

1. Bayer, D., Stillman, M.: A criterion for detecting $m$-regularity. Invent. Math. 87(1), 1–11 (1987)
2. Bolondi, G., Miró-Roig, R.M.: Two-codimensional Buchsbaum subschemes of $P^n$ via their hyperplane sections. Commun. Algebra 17(8), 1989–2016 (1989)
3. Chiarli, N., Greco, S., Nagel, U.: Surfaces in $\mathbb{P}^d$ with extremal general hyperplane section. J. Algebra 257(1), 65–87 (2002)
4. Chiarli, N., Greco, S., Nagel, U.: Families of space curves with large cohomology. J. Algebra 307(2), 704–726 (2007)
5. Cioffi, F., Marinari, M.G., Ramella, L.: Regularity bounds by minimal generators and Hilbert function. Collect. Math. 60(1), 89–100 (2009)
6. Cioffi, F., Di Gennaro, R.: Liaison and Cohen-Macaulayness conditions. Collect. Math. 62(2), 173–186 (2011)
7. Cioffi, F., Lella, P., Marinari, M.G., Roggero, M.: Segments and Hilbert schemes of points. Discrete Math. 311, 2238–2252 (2011)
8. Cioffi, F., Lella, P., Marinari, M.G., Roggero, M.: Minimal Castelnuovo-Mumford regularity fixing the Hilbert polynomial. arXiv:1307.2707 (2013)
9. Davis, E.D., Geramita, A.V., Orecchia, F.: Gorenstein algebras and the Cayley-Bacharach theorem. Proc. Am. Math. Soc. 93(4), 593–597 (1985)
When the positivity of the $h$-vector implies the Cohen-Macaulay property

10. Davis, ED.: Curves which are close to complete intersections, The Curves Seminar at Queen’s, vol. VII (Kingston, ON, 1990), Queen’s Papers in Pure and Appl. Math., vol. 85, Queen’s Univ., Kingston, pp. Exp. No. F, 14 (1990)

11. Davis, E.D., Geramita, A.V., Maroscia, P.: Perfect homogeneous ideals: Dubreil’s theorems revisited. Bull. Sci. Math. 108(2), 143–185 (1984)

12. de Quehen, V.E., Roberts, L.G.: Non-Cohen-Macaulay projective monomial curves with positive $h$-vector. Canad. Math. Bull. 48(2), 203–210 (2005)

13. Ellia, P.: On the cohomology of projective space curves. Boll. Un. Mat. Ital. A (7) 9(3), 593–607 (1995)

14. Ellia, P., Peskine, C.: Groupes de points de $P^2$: caractère et position uniforme, Algebraic geometry (L’Aquila, 1988), Lecture Notes in Math., vol. 1417, pp. 111–116. Springer, Berlin (1990)

15. Geramita, A.V., Maroscia, P., Roberts, L.G.: The Hilbert function of a reduced $k$-algebra. J. Lond. Math. Soc. (2) 28(3), 443–452 (1983)

16. Green, M.L.: Generic Initial Ideals, Six Lectures on Commutative Algebra, Mod. Birkhäuser Class, pp. 119–186. Birkhäuser Verlag, Basel (2010)

17. Hartshorne, R.: Algebraic Geometry, Graduate Texts in Mathematics, no. 52. Springer, New York (1977)

18. Hartshorne, R.: The genus of space curves. Ann. Univ. Ferrara Sez. VII (N.S.) 40, 207–223 (1994/1996)

19. Huneke, C., Ulrich, B.: General hyperplane sections of algebraic varieties. J. Algebraic Geom. 2(3), 487–505 (1993)

20. Kunz, E.: Introduction to commutative algebra and algebraic geometry, Birkhäuser Boston Inc., Boston (1985). Translated from the German by Michael Ackerman, With a preface by David Mumford.

21. Lella, P.: An efficient implementation of the algorithm computing the Borel-fixed points of a Hilbert scheme. In: ISSAC 2012-Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation, pp. 242–248. ACM, New York (2012)

22. Martin-Deschamps, M., Perrin, D.: Le schéma de Hilbert des courbes gauches localement Cohen-Macaulay n’est (presque) jamais réduit. Ann. Sci. École Norm. Sup. (4) 29(6), 757–785 (1996)

23. Migliore, JC., Nagel, U.: Numerical macaulification. arXiv:1202.2275 (2012)

24. Migliore, J.C.: Introduction to Liaison Theory and Deficiency Modules, Progress in Mathematics, vol. 165. Birkhäuser Boston Inc., Boston (1998)

25. Moore, D., Nagel, U.: Algorithms for strongly stable ideals. http://arxiv.org/abs/1110.4080 (2011)

26. Roggero, M.: Laudal-type theorems in $P^N$. Indag. Math. (N.S.) 14(2), 249–262 (2003)

27. Strano, R.: Curves and their hyperplane sections. J. Pure Appl. Algebra 152(1–3), 337–341 (2000). Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998)

28. Valla, G.: Problems and results on Hilbert functions of graded algebras, six lectures on commutative algebra (Bellaterra, 1996). Progr. Math. 166, 293–344 (1998)