Applying Renormalization Group to Quantum Walks

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Abstract. We provide a brief introduction to quantum walks. In particular, we describe how to administer and interpret the real-space renormalization group for such walks, a subtle task due to replacing the stochastic propagator in classical random walks by a unitary operator.

1. Introduction
For random walks, the probability density $\rho(\vec{x}, t)$ to detect a walk at time $t$ at site $\vec{x}$, a distance $x = |\vec{x}|$ from its origin, obeys the scaling collapse with the scaling variable $x/t^{1/d_w}$,

$$\rho(\vec{x}, t) \sim t^{-d_w/2} f\left(\frac{x/t}{d_w}\right),$$

where $d_w$ is the walk-dimension and $d_f$ is the fractal dimension of the network [1]. On a translationally invariant lattice of any spatial dimension $d = d_f$, it is easy to show that the walk is always purely “diffusive”, $d_w = 2$, with a Gaussian scaling function $f$, which is the content of many classic textbooks on random walks and diffusion [2]. The scaling in Eq. (1) still holds when translational invariance is broken or the network is fractal (i.e., $d_f$ is non-integer). Such “anomalous” diffusion with $d_w \neq 2$ may arise in many transport processes [1, 3]. For quantum walks, the only known value for a finite walk dimension is that for ordinary lattices [4], where Eq. (1) generically holds with $d_w = 1$, indicating a “ballistic” spreading of the quantum walk from its origin. This value has been obtained for various versions of one- and higher-dimensional quantum walks, for instance, with so-called weak-limit theorems [4, 5].

The renormalization group (RG) [6] expands the analytic tools to understand quantum walks [7, 8, 9, 10], since it works for networks that lack translational symmetries. We have obtained exact results for complex networks, with the scaling exponents shown in Tab. 1. We conjecture that the quantum walk dimension $d_w$ with a Grover coin always is half of that for the corresponding random walk [9],

$$d_w^{QW} = \frac{1}{2} d_w^{RW}.$$  

(2)

This ability to explore a given geometry much faster than diffusion is essential for the effectiveness of quantum search algorithms [11, 12].

2. Evolution equation for a walk
Our walks are governed by the discrete-time “master” equation [3] for the state of the system,

$$|\Psi(t + 1)\rangle = \mathcal{U} |\Psi(t)\rangle.$$ 

(3)
with propagator $U$. It is a stochastic operator for a classical, dissipative random walk, but in the quantum case $U$ is unitary and, thus, reversible [yet, we will still refer to Eq. (3) as a master equation]. Then, in the discrete $N$-dimensional site-basis $|x\rangle$ of some network, the PDF is given by $\rho(x, t) = |\psi_{x,t}\rangle = \langle x|\Psi(t)\rangle$ for random walks, or by $\rho(x, t) = |\tilde{\psi}_{x,t}\rangle^2$ for quantum walks.

Assuming that we possess the eigensolution for the propagator, $U\phi_j = u_j\phi_j$ with eigenvalues $u_j$ and an orthonormal set of eigenvectors $\phi_j(x)$, then the formal solution of Eq. (3) becomes $\psi_{x,t} = \sum_j a_j u_j^* \phi_j(x)$. For a stochastic $U$, aside from the unique $(+1)$-eigenvalue of the stationary state, the remaining eigenvalues have $|u_j| < 1$, thus, according to Eq. (3), the dynamics is determined by $\rho(\tilde{x}, t) \sim e^{-t/\tau}$ for large times $t$ with $\tau = -1/\ln \max_j \{ |u_j| < 1 \}$. In turn, for unitary $U$ all eigenvalues are uni-modular, $|u_j| = 1$ for all $j$, such that $u_j = e^{i\theta}$ with real $\theta$. A discrete Laplace-transform [3] of the site amplitudes $\tilde{\psi}_x(z) = \sum_{t=0}^{\infty} \tilde{\psi}_{x,t} z^t$ has its poles – hence those for $\overline{\rho}(x, z)$ – located on the unit-circle in the complex-$z$ plane [10],

$$
\overline{\rho}(x, z) \propto \prod_{j,l} \left[ 1 - z e^{(i\theta_j - \theta_l)} \right]^{-1}.
$$

**Walks on the 1d-line:** To appreciate the promise of the RG, we review the time evolution of random and quantum walks on the 1d-line, although these are well-understood and quantum search cannot improve on the classical result here. The propagator in Eq. (3) is

$$
U = \sum_x \{ A \ |x + 1\rangle \langle x| + B \ |x - 1\rangle \langle x| + M \ |x\rangle \langle x| \}
$$

for nearest-neighbor transitions. For quantum walks the norm of $\rho$ demands unitary propagation, $I = U^*U$. The rules then impose the conditions $I_r = A^1A + B^1B + M^1M$ and $0 = A^1M + M^1B = A^1B$, consistent with $A + B + M$ being unitary. This algebra requires at least $r = 2$-dimensional matrices, and it is customary [14] to use the most general unitary coin-matrix,

$$
C = \begin{pmatrix}
\sin \eta & e^{i\chi} \cos \eta \\
e^{-i\chi} \cos \eta & -e^{i(\chi+\theta)} \sin \eta
\end{pmatrix},
$$

to define $U = S (C \otimes I_N)$ with shift $S$ using matrices $S^{(A,B,M)}$ for transfer in a direction either out of a site or back to itself, i.e., $A = S^A C$, $B = S^B C$, and $M = S^M C$ with $S^A + S^B + S^M = I_r$. The

### Table 1. Walk dimensions for (hypercubic) lattices and some complex networks, as determined with the RG. The classical value for $d_{rw}^{\text{QW}}$ is known for DSG [1] and HN3 [13], or easily derived for MK [9]. The values for $d_{rw}^{\text{QW}}$ were first conjectured [8, 9], then analytically verified (except HN3) [10] for a reflective Grover coin. In each case, these $d_{rw}$ satisfies Eq. (2).

| Network | $d_{rw}^{\text{QW}}$ | $d_{rw}^{\text{QW}}$ (Grover) |
|----------|----------------------|-----------------------------|
| Lattice  |                      |                             |
| MK3      | $\log_4(21) \approx 2.196$ | 1.098079...                |
| MK4      | $\log_4(\frac{21}{7}) \approx 2.571$ | 1.285253...               |
| HN3      | $\log_2(24 - 8\sqrt{5}) \approx 2.612$ | 1.305758...               |
| DSG      | $\log_2(5) \approx 2.322$ | 1.160964...                |

**Figure 1.** Plot of Laplace-poles in the complex-$z$ plane at RG-steps $k = 2, 3$ for site-amplitude $\tilde{\psi}_0^{(k)}(z)$ (in blue, with $\circ$ for $k = 2$ and $\times$ for $k = 3$) and for the hopping parameters $\tilde{a}_k(z)$ (in red, with $\Box$ for $k = 2$ and $+$ for $k = 3$) for quantum walk on the 1d-line and DSG with a Grover coin. All poles for the site-amplitude are on the unit-circle, while all poles for the hopping parameters are outside. Either type of poles impinge increasingly on the real-$z$ axis (arrows).
quantum-coin entangles all \( r \) components of \( \psi_{x,t} \) and the shift-matrices facilitate the subsequent transitions to neighboring sites. For \( r = 2 \), the degree of each site, there are no self-loops \( (S^A = 0, M = 0) \) and we shift upper (lower) components of each \( \psi_{x,t} \) to the right (left) using projectors \( S^A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( S^B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). Alternatively, these coin-degrees of freedom could be replaced by more intricate next-nearest-neighbor hoppings without coin [15, 16], for which schemes equivalent to the following RG can be developed.

**Renormalization of walks:** To obtain the RG-recursions, a Laplace-transform is used such that the master equation (3) with \( \mathcal{U} \) in Eq. (5) becomes \( \overline{\psi}_x = z M \overline{\psi}_x + \overline{z A \psi}_{x-1} + \overline{z B \psi}_{x+1} \). We first consider the walk problem here with initial conditions (IC) localized at the origin, \( \psi_{x,t=0} = \delta_{x,0} \psi IC \). We recursively eliminating \( \overline{\psi}_x \) for all sites for which \( x \) is an odd number, then set \( x \to x/2 \) and repeat, step-by-step for \( k = 0, 1, 2, \ldots \). Starting at \( k = 0 \) with the “raw” hopping coefficients \( A_0 = z A, B_0 = z B, \) and \( M_0 = z M, \) after each step, the master equation becomes self-similar in form when redefining the renormalized hopping coefficients \( A_k, B_k, M_k \).

E.g., for sites near any even site \( x \) at step \( k \) we have [7]:

\[
\begin{align*}
\overline{\psi}_{x-1} &= M_k \overline{\psi}_{x-1} + A_k \overline{\psi}_{x-2} + B_k \overline{\psi}_x, \\
\overline{\psi}_x &= M_k \overline{\psi}_x + A_k \overline{\psi}_{x-1} + B_k \overline{\psi}_{x+1} + \delta_{x,0} \psi IC, \\
\overline{\psi}_{x+1} &= M_k \overline{\psi}_{x+1} + A_k \overline{\psi}_{x+2} + B_k \overline{\psi}_{x+1}.
\end{align*}
\]

Solving this linear system at \( x \) yields \( \overline{\psi}_x = M_{k+1} \overline{\psi}_{x-2} + B_{k+1} \overline{\psi}_{x+2} \) with RG “flow”

\[
\begin{align*}
A_{k+1} &= A_k \left( \mathbb{I} - M_k \right)^{-1} A_k, & B_{k+1} &= B_k \left( \mathbb{I} - M_k \right)^{-1} B_k, \\
M_{k+1} &= M_k + A_k \left( \mathbb{I} - M_k \right)^{-1} B_k + B_k \left( \mathbb{I} - M_k \right)^{-1} A_k.
\end{align*}
\] (8)

For the quantum walk, we evolve Eqs. (8) from its unrenormalized values [7]. Already after two iterations, a recurring pattern emerges that suggest the Ansatz \( A_k = a_k S^A C, B_k = -a_k S^B C, \) and \( M_k = m_k \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C \). For \( a_k \) and \( m_k \), the RG-flow (8) closes for

\[
\begin{align*}
a_{k+1} &= \frac{a_k^2 \sin \eta}{1 - 2 m_k \cos \eta + m_k^2}, & m_{k+1} &= m_k + \frac{(m_k - \cos \eta) a_k^2}{1 - 2 m_k \cos \eta + m_k^2}
\end{align*}
\] (9)

for \( 0 < \eta < \pi/2 \) and \( \chi = \vartheta = 0 \) in Eq. (6). These recursions have a non-trivial fixed point at \( (a_{\infty}, m_{\infty}) = (\sin \eta, \cos \eta) \), yet, its Jacobian at \( k \to \infty \) is, in fact, \( \eta \)-independent and has a degenerate eigenvalue \( \lambda_{1,2} = 2 \), suggesting \( \delta_w = \log_2 \lambda_1 = 1 \). Basically, the same procedure as in Sec. 2, although algebraically more laborious, can be applied to obtain closed RG-recursions for a number of fractal networks [8, 9]. These have provided the first exact weak-limit results for non-trivial networks beyond regular lattices, as listed in Tab. 1.

### 3. Interpreting the RG

Although the above analysis of the fixed points of the RG-flow appear to follow the conventional approach [3] for both, the classical and the quantum walk, puzzling discrepancies emerge when such a naive approach is extended beyond 1d. When applied to the dual Sierpinski gasket (DSG) [8], or a number of other hierarchical networks [9], the largest Jacobian eigenvalue corresponds merely to their fractal dimension, i.e., \( \log_2 \lambda_1 = d_f \). It is an accident that \( d_f = d_{\text{SW}}^0 = 1 \) on the 1d-line! Hence, we had to reconsider the basis of the Tauberian theorems [2, 3] and obtained the following interpretation [10]: The poles closest to \( z_\infty = 1 \) in the
Laplace-transforms $\overline{\psi}_x(z)$ determine the large-$t$ behavior of the walk dynamics. For increasing length (i.e., $k \rightarrow k + 1$ in the RG-flow), these poles move towards $z_\infty$ in a characteristic way [3]. In the classical case, for a stochastic propagator in Eq. (3), these poles move radially along the real-$z$ axis with $z_k > 1$ and only the pole with smallest to $|z_k/z_\infty| - 1$ determines the scaling between length and time in Eq. (1). In the quantum case, with a unitary operator, all poles of $\overline{\psi}_x(z)$ are unimodular, i.e., $|z_k/z_\infty| \equiv 1$ for all $k$, see Eq. (4). Instead, they approach $z_\infty$ tangentially along the complex unit-circle in a characteristic manner that provides the correct scaling. Yet, this behavior is somewhat obscured within the RG-flow: The poles of hopping parameters, such as $a_k(z)$ or $m_k(z)$ in Eq. (9), for example, move radially as well as tangentially, see Fig. 1. As in the classical case, it is the radial motion of poles that is captured by the largest Jacobian eigenvalue $\lambda_1$, while we have shown [10] that the tangential motion is described by the geometric mean of the two largest eigenvalues $\lambda_{1,2}(>1)$, so that $d_{QW}^\text{d} = \log_2 \sqrt{\lambda_1 \lambda_2}$ indeed reproduces the correct quantum scaling consistent with the numerical results (to within some 6-8 digits accuracy!) [8, 9]. For example, the RG for quantum walk on DSG yields $\lambda_1 = 3$, $\lambda_2 = \frac{5}{3}$, such that $\sqrt{\lambda_1 \lambda_2} = \sqrt{5}$ reproduces the numerically predicted scaling, $d_{QW}^\text{d} = \log_2 \sqrt{5}$. Since classically $\lambda_{1,2}^{\text{RW}} = 5 (= \lambda_1 \lambda_2)$, it verifies the conjecture in Eq. (2). The same pattern holds for two Migdal-Kadanoff networks [9]: For the 3-regular network called MK3, the Jacobian eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 3$ such that $\lambda_1 \lambda_2 = \lambda_{1,2}^{\text{RW}} = 21$, and for the 4-regular network called MK4 they are $\lambda_1 = 13$ and $\lambda_2 = \frac{10}{7}$, such that $\lambda_1 \lambda_2 = \lambda_{1,2}^{\text{RW}} = \frac{240}{49}$.

4. Conclusions
Some remarks are in order: First, it is unfortunate that the quantum walk on the 1d-line, the only independently solvable model to compare to the RG, provides no insights because $\lambda_1 = \lambda_2 = \sqrt{\lambda_1 \lambda_2} = 2$ can not distinguish between radial and tangential motion of poles. Second, our explanation of the quantum case here is somewhat simplified because the poles of observables like $\rho(x,t) = |\psi_{x,t}|^2$ depend on $|\psi_{x,t}|^2$ and thus those poles result from all mutual ratios $\exp \{i (\theta_n - \theta_m)\}$ of unimodular eigenvalues $\exp \{i \theta_j\}$ of the unitary propagator $U$, see Eq. (4). We argue [10] that, if all such eigenvalues scale with $k$ as $\theta_j^{(k)} \sim (\lambda_1 \lambda_2)^{-\frac{k}{d}}$ or as a constant, then so does any difference $\theta_n - \theta_m$, and thus it is sufficient to follow any convenient pole of $a_k(z)$ such as that nearest to $z_\infty = 1$. Unfortunately, this assumption appears to fail in some cases, like the network HN3 we considered before [9], see Tab. 1.

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