1 Introduction

A global action is an algebraic analogue of a topological space. It consists of group actions $G_\alpha \curvearrowright X_\alpha$, ($\alpha \in \Phi$), which fulfill a certain compatibility condition (see Definition 2.1). In Section 2, immediately below we define three kinds of morphisms between global actions namely a general notion of morphism, then regular morphisms and finally normal morphisms. It turns out that every regular morphism is normal. Further we define the product of two global actions and show how one can put a global action structure on the set $\text{Mor}(X, Y)$ of all morphisms from a global action $X$ to a global action $Y$. The main theorem of Section 2 is the Exponential Law 2.31, which ensures that one can develop a good homotopy theory for global actions.

In Section 3 we define the notion of path in a global action $X$, introduce the notion of homotopy of morphisms and define the fundamental group $\pi_1(X_\ast)$ of a global action $X$ at a fixed point $\ast$. Section 4 introduces the notion of covering morphism. We show important lifting properties and as the main result of this section Theorem 4.8, which establishes a Galois type correspondence between connected coverings of a given connected global action and subgroups of the fundamental group of that action.

2 Basic Definitions

Recall that if $X$ denotes a set and $G$ a group then a (left) group action or just (left) action of $G$ on $X$ is a rule which associates to each pair $(g, x)$ of elements $g \in G$ and $x \in X$ an element $gx \in X$ such that if $g_1, g_2 \in G$ then $g_2(g_1x) = (g_2g_1)x$ and such that if 1 denotes the identity element in $G$ then $1x = x$. An action of $G$ on $X$ is often denoted by $G \curvearrowright X$. A morphism $G \curvearrowright X \rightarrow G' \curvearrowright X'$ of left actions is a pair $(h, s)$ consisting of a group homomorphism $h : G \rightarrow G'$ and a function $s : X \rightarrow X'$ such that for any $g \in G$ and $x \in X$ $s(gx) = h(g)s(x)$.

The notion of a right action of $G$ on $X$ and a morphism of right actions is defined similarly.

**Definition 2.1** (global action) Let $X$ denote a set. A (left) global action on $X$ consists of a set $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ of group actions $G_\alpha \curvearrowright X_\alpha$ where each $X_\alpha \subseteq X$ and $\Phi$ is an indexing set equipped with a reflexive relation $\leq$ such that the following holds:

(a) If $\alpha \leq \beta$ then $X_\alpha \cap X_\beta$ is stable under the action of $G_\alpha$, i.e. if $g \in G_\alpha$ and $x \in X_\alpha \cap X_\beta$ then $gx \in X_\alpha \cap X_\beta$. Thus $G_\alpha \curvearrowright (X_\alpha \cap X_\beta)$.

(b) Suppose $\alpha \leq \beta \in \Phi$. Then as part of the structure of a global action there is a group homomorphism $G_{\alpha \leq \beta} : G_\alpha \rightarrow G_\beta$ such that the pair $(G_{\alpha \leq \beta}, \iota_{X_\alpha \cap X_\beta, X_\beta})$ is a morphism $G_\alpha \curvearrowright (X_\alpha \cap X_\beta) \rightarrow G_\beta \curvearrowright X_\beta$ of group actions where $\iota_{X_\alpha \cap X_\beta, X_\beta} : X_\alpha \cap X_\beta \rightarrow X_\beta$ denotes the canonical inclusion. Such a homomorphism $G_{\alpha \leq \beta}$ is called a structure homomorphism. It is assumed that $G_{\alpha \leq \alpha} : G_\alpha \rightarrow G_\alpha$ is the identity map.
The set $X$ equipped with a global action is called a *global action space* or simply a *global action*. The sets $X_\alpha$ are called *local sets*, the groups $G_\alpha$ *local groups*, and the actions $G_\alpha \acts X_\alpha$ *local actions*.

**Remark** (a) In this thesis it will additionally be assumed that $X = \bigcup_{\alpha \in \Phi} X_\alpha$.

(b) Since it is not assumed that the relation on $\Phi$ is transitive, it follows that if $\alpha \leq \beta$ and $\beta \leq \gamma$ then it does not follow automatically that $\alpha \leq \gamma$, but it is also not forbidden that $\alpha \leq \gamma$. Suppose $\alpha \leq \gamma$. Then $G_{\alpha \leq \beta}$, $G_{\beta \leq \gamma}$ and $G_{\alpha \leq \gamma}$ are structure homomorphisms but it is not assumed that $G_{\beta \leq \gamma} G_{\alpha \leq \beta} = G_{\alpha \leq \gamma}$, although this equality is also not forbidden. However, in many situations which arise naturally, the relation on $\Phi$ is transitive and the composition rule $G_{\beta \leq \gamma} G_{\alpha \leq \beta} = G_{\alpha \leq \gamma}$ holds.

Below are two important examples of global actions.

**Example 2.2** Let $G$ denote a group and let $\Phi$ denote a set which is indexing a set \{ $G_\alpha | \alpha \in \Phi$ \} of subgroups $G_\alpha$ of $G$, i.e. if $G_\alpha = G_{\alpha'}$ then $\alpha = \alpha'$, which is closed under taking intersections, i.e. if $\alpha, \beta \in \Phi$, $\exists \gamma \in \Phi$ s.t. $G_\alpha \cap G_\beta = G_\gamma$. Equip $\Phi$ with the reflexive, transitive relation defined by $\alpha \leq \beta \Leftrightarrow G_\alpha \subseteq G_\beta$. If $\alpha \leq \beta$ let $G_{\alpha \leq \beta}$ denote the canonical inclusion homomorphism $G_\alpha \rightarrow G_\beta$. Let $X = G$ and for each $\alpha \in \Phi$ let $X_\alpha = X$. Let $G_\alpha$ act on $X_\alpha (\sim X)$ by left (or right) multiplication. Then \{ $G_\alpha \acts X_\alpha | \alpha \in \Phi$ \} is a global action on $X$. This example is called a *standard single domain* global action.

**Example 2.3** (line action) Let $X = \mathbb{Z}$, $\Phi = \mathbb{Z} \cup \{ * \}$ and $X_n = \{ n, n+1 \}$ if $n \in \mathbb{Z}$ and $X_* = \mathbb{Z}$. Let $G_n \cong \mathbb{Z}/2\mathbb{Z}$ if $n \in \mathbb{Z}$, $G_* = 1$ and let $G_n \acts \{ n, n+1 \}$ be the group action such that the non-trivial element of $G_{\{ n, n+1 \}}$ exchanges the elements $n$ and $n+1$. Let the only relations in $\Phi$ be $* \leq n$ for all $n \in \mathbb{Z}$. Let $G_{* \leq n} : \{ 1 \} \rightarrow G_{\{ n, n+1 \}}$ denote the unique group homomorphism. This example is called the *line action*.

We prepare now for the notion of a morphism of global actions.

**Definition 2.4** (local frame) Let \{ $G_\alpha \acts X_\alpha | \alpha \in \Phi$ \} denote a global action on the set $X$ and let $\alpha \in \Phi$. A *local frame* at $\alpha$ or simply an $\alpha$-frame is a finite set \{ $x_0, ..., x_p$ \} $\subseteq X_\alpha$ such that $G_\alpha$ acts transitively on \{ $x_0, ..., x_p$ \}, i.e. for each $i$ ($1 \leq i \leq p$), there is an element $g_i \in G_\alpha$ s.t. $g_i x_0 = x_i$.

**Definition 2.5** (morphism of global actions) Let \{ $G_\alpha \acts X_\alpha | \alpha \in \Phi$ \} and \{ $H_\beta \acts Y_\beta | \beta \in \Psi$ \} be global actions on $X$ and $Y$, respectively. Then a *morphism* $f : X \rightarrow Y$ of global actions is a function $f$ which preserves local frames, i.e. if $\{ x_0, ..., x_p \}$ is a local frame in $X$ then $\{ f x_0, ..., f x_p \}$ is a local frame in $Y$.

**Remark** If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are morphism of global actions then obviously their composition $gf : Y \rightarrow Z$ is a morphism of global actions.

**Example 2.6** Let $A$ be a global action on a space $X$, $B = \{ H_\beta \acts Y_\beta | \beta \in \Psi \}$ be a global action on a space $Y$ such that $\exists \beta \in \Psi$ and an element $y \in Y_\beta$ such that
$H_\beta y = Y_\beta$, i.e. all elements of $Y_\beta$ are in the orbit of some element $y \in Y_\beta$. Then any function $f : X \to Y$ whose image lies in $Y_\beta$ is a morphism of global actions.

**Example 2.7** This example is a special case of Example 2.6. Define the global action $B$ as follows: There is only one group $G_1 = \mathbb{Z}$, which acts on itself by left addition (of course $G_{1 \leq 1}$ is the identity map $id_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$). Then the function $id_\mathbb{Z}$ is a morphism from the global action space $\mathbb{Z}$ equipped with the line action defined in Example 2.3 to $\mathbb{Z}$ equipped with the global action $B$.

**Definition 2.8** (regular morphism) Let $\{G_\alpha \rhd X_\alpha | \alpha \in \Phi\}$ denote a global action on $X$ and $\{H_\beta \rhd Y_\beta | \beta \in \Psi\}$ a global action on $Y$. A regular morphism $\eta : X \to Y$ is a triple $(\iota, \kappa, \lambda)$ where

(a) $\iota : \Phi \to \Psi$ is a function such that $\iota(\alpha) \leq \iota(\beta)$ whenever $\alpha \leq \beta$, i.e. $\iota$ is relation preserving.

(b) $\kappa$ is a rule which assigns to each $\alpha \in \Phi$ a group homomorphism $\kappa_\alpha : G_\alpha \to H_{\iota(\alpha)}$ such that if $\alpha \leq \beta$ then the diagram

\[
\begin{array}{ccc}
G_\alpha & \xrightarrow{\kappa_\alpha} & H_{\iota(\alpha)} \\
\downarrow{G_{\alpha \leq \beta}} & & \downarrow{H_{\iota(\alpha) \leq \iota(\beta)}} \\
G_\beta & \xrightarrow{\kappa_\beta} & H_{\iota(b)}
\end{array}
\]

commutes. (Thus, if the relations on $\Phi$ and $\Psi$ are transitive (and therefore $\Phi$ and $\Psi$ are categories in the obvious way) and the composition rules $G_{\beta \leq \gamma}^{-1} G_{\alpha \leq \beta} = G_{\alpha \leq \gamma}$ and $H_{\beta \leq \gamma}^{-1} H_{\alpha \leq \beta} = H_{\alpha \leq \gamma}$ hold then $G$ and $H$ are functors on respectively $\Phi$ and $\Psi$, with values in groups, $\iota$ is a functor $\Phi \to \Psi$, and $\kappa$ defines a natural transformation $G(\_') \to H(\_')$ of functors defined on $\Phi$. ) The rule $\kappa$ will be called a natural transformation $\kappa : G(\_') \to H(\_')$ from $G(\_')$ to $H(\_')$ even though $\Phi$ and $\Psi$ might not even be categories.

(c) $\lambda : X \to Y$ is a function such that $\lambda(X_\alpha) \subseteq Y_{\iota(\alpha)} \forall \alpha \in \Phi$

(d) For all $\alpha \in \Phi$ the pair $(\kappa_\alpha, \lambda|_{X_\alpha}) : G_\alpha \rhd X_\alpha \to H_{\iota(\alpha)} \rhd Y_{\iota(\alpha)}$ is a morphism of group actions.

A regular morphism $\eta = (\iota, \kappa, \lambda) : X \to Y$ is called an extension of a morphism $f : X \to Y$ if $\lambda = f$. A regular morphism $\eta = (\iota, \kappa, \lambda) : X \to Y$ is called a regular isomorphism if there is a regular morphism $\eta' = (\iota', \kappa', \lambda') : Y \to X$ such that $\iota'$ is inverse to $\iota$, $\lambda'$ is inverse to $\lambda$ and $\kappa'_{\iota(\alpha)}$ is inverse to $\kappa_{\alpha}$. Such a regular morphism $\eta'$ is called the inverse of $\eta$. A regular morphism $\eta = (\iota, \kappa, \lambda) : X \to Y$ is called a weak regular isomorphism if there is a regular morphism $\eta' = (\iota', \kappa', \lambda') : Y \to X$ such that $\lambda'$ is inverse to $\lambda$.

**Remark** Condition 2.8(d) implies that a regular morphism is a morphism of global actions (to be more precise if $(\iota, \kappa, \lambda)$ is a regular morphism then $\lambda$ is a morphism of global actions).
To develop a homotopy theory for morphisms of global actions it is necessary to put the structure of a global action on the set $\text{Mor}(X,Y)$ of all morphisms from a global action $X$ to a global action $Y$.

**Definition 2.9** Let $\{G_\alpha \twoheadrightarrow X_\alpha|\alpha \in \Phi\}$ denote a global action on $X$ and $\{H_\beta \twoheadrightarrow Y_\beta|\beta \in \Psi\}$ a global action on $Y$. Define a global action on $\text{Mor}(X,Y)$ as follows. The index set $\Theta$ for the global action to be put on $\text{Mor}(X,Y)$ is $\Theta = \{\beta: X \rightarrow \Psi|\beta \text{ an arbitrary function}\}$. The relation on $\Theta$ is defined by $\beta \leq \beta' \iff \beta(x) \leq \beta'(x) \forall x \in X$. If $\beta \in \Theta$ then the local set $\text{Mor}(X,Y)_\beta$ is defined by $\text{Mor}(X,Y)_\beta = \{f \in \text{Mor}(X,Y)\}$.

(a) $f(x) \in Y_{\beta(x)} \forall x \in X$

(b) if $\{x_0, ..., x_p\} \subseteq X$ is a local frame in $X$ then $\{f(x_0), ..., f(x_p)\}$ is a local b-frame in $Y$ for some $b \in \Psi$ such that $b \geq \beta(x_i) (0 \leq i \leq p)$.

If $\beta \in \Theta$ then the local group $J_\beta$ at $\beta$ is defined by $J_\beta = \prod_{x \in X} H_{\beta(x)}$ and the action of $J_\beta$ on $\text{Mor}(X,Y)_\beta$ is given by

$$J_\beta \times \text{Mor}(X,Y)_\beta \rightarrow \text{Mor}(X,Y)_\beta$$

$$(\sigma, f) \mapsto \sigma f$$

where

$$\sigma f: X \rightarrow Y$$

$$x \mapsto \sigma_x(f(x))$$

and

$\sigma_x$ is the x-coordinate of $\sigma \in \prod_{x \in X} H_\beta(x)$.

One has to check that the function $\sigma f$ defined above is an element of $\text{Mor}(X,Y)_\beta$, but the proof is straightforward. If $\beta, \beta' \in \Theta$ and $\beta \leq \beta'$ then the structure morphism $J_{\beta \leq \beta'}$ is defined by $J_{\beta \leq \beta'} = \prod_{x \in X} H_{\beta(x)}$ where $J_{\beta} = \prod_{x \in X} H_{\beta(x)} \rightarrow J_{\beta'} = \prod_{x \in X} H_{\beta'(x)}$.

One checks straightforward, but with a little effort that $(\Theta, J_{(-)}, \text{Mor}(X,Y)_{(-)})$ defines a global action structure on the set $\text{Mor}(X,Y)$.

The following lemma is easy to check.

**Lemma 2.10** Let $X$ and $Y$ be global actions. If $g: Z \rightarrow X$ is a morphism of global actions then

$$\text{Mor}(g, 1_Y): \text{Mor}(X,Y) \rightarrow \text{Mor}(Z,Y)$$

$f \mapsto fg$
is a morphism of global actions.

Unfortunately it is not necessarily the case that if $Z$ and $X$ are global actions and $g : X \to Y$ a morphism of global actions then $\text{Mor}(1_z,g)$ defines a morphism
\[
\text{Mor}(Z,X) \to \text{Mor}(Z,Y)
\]
\[f \mapsto gf\]
of global actions. If $\text{Mor}(1_z,g)$ is a morphism of global actions then we say that $g$ is $Z$-normal. We make this now a definition.

**Definition 2.11** (Z-normal morphism) Let $Z$ be a global action. A morphism $g : X \to Y$ of global actions is called $Z$-normal, if for any local frame $\{f_0, f_1, \ldots, f_p\}$ at $\beta \in \Theta(Z,X)$ there is a $\gamma \in \Theta(Z,Y)$ such that $\{gf_0, gf_1, \ldots, gf_p\}$ is a local frame at $\gamma$.

**Lemma 2.12** Let $Z$ be a global action and $g : X \to Y$ be a morphism of global actions. Then the function
\[
\text{Mor}(1_z,g) : \text{Mor}(Z,X) \to \text{Mor}(Z,Y)
\]
\[f \mapsto gf\]
is a morphism of global actions if and only if $g$ is $Z$-normal.

**Proof** $\text{Mor}(1_z,g)$ is a morphism of global actions $\Leftrightarrow$ for all local frames $\{f_0, f_1, \ldots, f_p\}$ in $\text{Mor}(Z,X)$ $\{\text{Mor}(1_z,g)f_0, \text{Mor}(1_z,g)f_1, \ldots, \text{Mor}(1_z,g)f_p\} = \{gf_0, gf_1, \ldots, gf_p\}$ is a local frame in $\text{Mor}(Z,Y)$ $\Leftrightarrow$ $g$ is $Z$-normal.

**Definition 2.13** (normal morphism, $\infty$-normal morphism) Let $g : X \to Y$ be a morphism of global actions. $g$ is called normal, if it is $Z$-normal for all global actions $Z$. $g$ is called $\infty$-normal, if for any finite set of global actions $Z_1, \ldots, Z_n$ the map
\[
\text{Mor}(1_{Z_n}, \text{Mor}(1_{Z_{n-1}}, \ldots, \text{Mor}(1_{Z_1},g))) : \text{Mor}(Z_n, \text{Mor}(Z_{n-1}, \ldots, \text{Mor}(Z_1,X))) \to \text{Mor}(Z_n, \text{Mor}(Z_{n-1}, \ldots, \text{Mor}(Z_1,Y)))
\]
is a morphism of global actions.

The following lemma is obvious and useful.

**Lemma 2.14** If $g : X \to Y$ is an $\infty$-normal morphism of global actions then for any global action $Z$ $\text{Mor}(1_z,g) : \text{Mor}(Z,X) \to \text{Mor}(Z,Y)$ is $\infty$-normal.

**Definition 2.15** A morphism $g : X \to Y$ of global actions is called a $Z$-normal (resp. normal, $\infty$-normal) isomorphism, if the map $g$ is bijective and $g^{-1}$ is $Z$-normal(resp. normal, $\infty$-normal).

**Definition 2.16** A global action $X$ is called $Z$-normal (resp. normal, $\infty$-normal), if every morphism whose domain is $X$ is $Z$-normal (resp. normal, $\infty$-normal). A global action $Y$ is called $Z$-conormal (resp. conormal, $\infty$-conormal), if every morphism whose codomain is $Y$ is $Z$-normal (resp. normal, $\infty$-normal).
Lemma 2.17 If \((\iota, \kappa, \lambda) : X \to Y\) is a regular morphism then for any global action \(Z\), the function \(\text{Mor}(1_Z, \lambda) : \text{Mor}(Z, X) \to \text{Mor}(Z, Y)\) extends to a regular morphism.

Proof We shall define a relation preserving map \(\iota' : \Theta(Z, X) \to \Theta(Z, Y)\) and a natural transformation \(\kappa' : \text{J}(Z, X) \to \text{J}(Z, Y)\) and show that the triple \((\iota', \kappa', \text{Mor}(1_Z, g))\) is a regular morphism \(\text{Mor}(Z, X) \to \text{Mor}(Z, Y)\) of global actions. Define \(\iota'\) by \(\iota'(\beta) = \iota\beta \quad \forall \beta \in \Theta(Z, X)\) and \(\kappa'_\beta = \prod_{z \in Z} \kappa_{\beta(z)} \quad \forall \beta \in \Theta(Z, X)\). We show now that \((\iota', \kappa', \text{Mor}(1_Z, g))\) is a regular morphism.

(a) Let \(\beta, \gamma \in \Theta(X, Y), \beta \leq \gamma\) i.e. \(\beta(z) \leq \gamma(z)\) for all \(z\) in \(Z\). Then \((\iota'(\beta))(z) = (\iota\beta)(z) \leq \iota(\gamma)(z) = (\iota'(\gamma))(z)\) for all \(z \in Z\). Hence \(\iota'(\beta) \leq \iota'(\gamma)\).

(b) We show that for all \(\beta, \gamma \in \Theta(Z, X), \beta \leq \gamma\), the diagram

\[
\begin{array}{ccc}
(J(Z,X))_\beta & \xrightarrow{\kappa'_\beta} & (J(Z,Y))_{\iota'(\beta)} \\
(J(Z,X))_{\beta \leq \gamma} \downarrow & & \downarrow (J(Z,Y))_{\iota(\beta) \leq \iota'(\gamma)} \\
(J(Z,X))_\gamma & \xrightarrow{\kappa'_\gamma} & (J(Z,Y))_{\iota'(\gamma)}
\end{array}
\]

commutes. Let \(\beta, \gamma \in \Theta(Z, X), \beta \leq \gamma\), then

\[
\kappa'_\gamma(J(Z,X))_{\beta \leq \gamma} = (\prod_{z \in Z} \kappa_{\beta(z)}) (\prod_{z \in Z} G_{\beta(z) \leq \gamma(z)})
\]

\[
= (\prod_{z \in Z} \kappa_{\beta(z)} G_{\beta(z) \leq \gamma(z)})
\]

\[
= (\prod_{z \in Z} G_{\iota(\beta(z)) \leq \iota(\gamma(z))} \kappa_{\beta(z)})
\]

\[
= (\prod_{z \in Z} G_{\iota(\beta(z)) \leq \iota(\gamma(z))} \prod_{z \in Z} \kappa_{\beta(z)})
\]

\[
= (J(Z,Y))_{\iota(\beta) \leq \iota(\gamma)} \kappa'_\beta
\]

\[
= (J(Z,Y))_{\iota'(\beta) \leq \iota'(\gamma)} \kappa'_\beta
\]

(*) If \(\sigma \in (J(Z,X))_\beta\), then

\[
\prod_{z \in Z} \kappa_{\gamma(z)} (\prod_{z \in Z} G_{\beta(z) \leq \gamma(z)} (\sigma_z))_{z \in Z}
\]

\[
= \prod_{z \in Z} \kappa_{\gamma(z)} (G_{\beta(z) \leq \gamma(z)} (\sigma_z))_{z \in Z}
\]

\[
= (\kappa_{\gamma(z)}) (G_{\beta(z) \leq \gamma(z)} (\sigma_z))_{z \in Z}
\]

\[
= G_{\beta(z) \leq \gamma(z)} (\sigma_z)_{z \in Z}
\]

\[
= \prod_{z \in Z} \kappa_{\beta(z)} (G_{\beta(z) \leq \gamma(z)} (\sigma_z))_{z \in Z}
\]
(**) The diagram

\[ \begin{array}{c}
G_{\beta(z)} \xrightarrow{\kappa_{\beta(z)}} H_{i(\beta(z))} \\
G_{\beta(z) \leq \gamma(z)} \downarrow \quad \downarrow H_{i(\beta(z)) \leq \gamma(z)} \\
G_{\gamma(z)} \xrightarrow{\kappa_{\gamma(z)}} H_{i(\gamma(z))}
\end{array} \]

commutes for all \( z \in Z \), since \((\iota, \kappa, \lambda)\) is a regular morphism.

(c) We show that \( \text{Mor}(1_{Z}, \lambda)(\text{Mor}(Z, X)_{\beta}) \subseteq \text{Mor}(Z, X)_{\iota'(<\beta)} \ \forall \beta \in \Theta(\mathcal{Z}, \mathcal{X}) \). Let \( \beta \in \Theta(\mathcal{Z}, \mathcal{X}) \), \( f \in \text{Mor}(Z, X)_{\beta} \). By definition,

(1) \( f(z) \in X_{\beta(z)} \ \forall \ z \in Z \), and

(2) if \( \{z_{0}, ..., z_{p}\} \subseteq Z \) is a local frame then \( \{f(z_{0}), ..., f(z_{p})\} \) is a local b-frame in \( X \) for some \( b \in \Phi \) such that \( b \geq \beta(z_{i}) \) (\( 0 \leq i \leq p \)), i.e. \( \exists g_{1}, ..., g_{p} \in G_{b} \) such that \( g_{i}f(z_{0}) = f(z_{i}) \) (\( 1 \leq i \leq p \)).

We must show that \( \lambda f \in (\text{Mor}(Z, Y))_{\iota'(<\beta)} \), i.e.

(1') \( (\lambda f)(z) \in X_{\iota'(<\beta)}(z) \ \forall \ z \in Z \), and

(2') if \( \{z_{0}, ..., z_{p}\} \) is a local frame in \( Z \) then \( \{\lambda(f(z_{0})), ..., \lambda(f(z_{p}))\} \) is a local c-frame for some \( c \in \Psi \) such that \( c \geq \iota(\beta(z_{i})) \) (\( 0 \leq i \leq p \)), i.e. \( \exists h_{1}, ..., h_{p} \in H_{c} \) such that \( h_{i}\lambda(f(z_{0})) = \lambda(f(z_{i})) \).

But (1) clearly implies (1'). Let now \( \{z_{0}, ..., z_{p}\} \) be a local frame in \( Z \). Set \( c = \iota(b) \) and \( h_{i} = \kappa_{\iota}(g_{i})(1 \leq i \leq p) \). Then \( h_{i}\lambda(f(z_{0})) = \kappa_{b}(g_{i})\lambda(f(z_{0})) = \lambda(g_{i}f(z_{0})) = \lambda(f(z_{i}))(1 \leq i \leq p) \) whence \( \{z_{0}, ..., z_{p}\} \) is a local c-frame in \( Y \) and obviously \( c \geq \iota(\beta(z_{i})) \) (\( 0 \leq i \leq p \)), since \( \iota \) is relation preserving.

(d) We show that if \( \beta \in \Theta(\mathcal{Z}, \mathcal{X}) \) then \( (\kappa'_{\beta}, \text{Mor}(1_{Z}, \lambda)|_{(\text{Mor}(Z, X))_{\beta}}) : (J_{(Z, X)}\beta) \cap (\text{Mor}(Z, X))_{\beta} \rightarrow (J_{(Z, Y)}\iota'(<\beta)) \cap (\text{Mor}(Z, Y))_{\iota'(<\beta)} \) is a morphism of group actions, i.e. \( (\text{Mor}(1_{Z}, \lambda)(\sigma f)) = \kappa'_{\beta}(\sigma)\text{Mor}(1_{Z}, \lambda)(f) \ \forall \sigma \in (J_{(Z, X)}\beta), f \in (\text{Mor}(Z, X))_{\beta} \).

Let \( \beta \in \Theta(\mathcal{Z}, \mathcal{X}) \), \( \sigma \in (J_{(Z, X)}\beta), f \in (\text{Mor}(Z, X))_{\beta} \), \( z \in Z \). Then

\[
(\text{Mor}(1_{Z}, \lambda)(\sigma f)(z)) = (\lambda(\sigma f))(z) = \lambda(\sigma(z)) = \lambda(\sigma_{z}f(z)) = \kappa_{\beta(z)}(\sigma_{z})\lambda(f(z)) = (\kappa'_{\beta}(\sigma)(\lambda f))(z)
\]

Hence \( \text{Mor}(1_{Z}, \lambda)(\sigma f) = \kappa'_{\beta}(\sigma)\text{Mor}(1_{Z}, \lambda)(f) \). \qquad \square

**Corollary 2.18** A regular morphism is \( \infty \)-normal (more precisely if \((\iota, \kappa, \lambda)\) is a regular morphism then \( \lambda \) is \( \infty \)-normal).

**Proof** Let \((\iota, \kappa, \lambda) : X \rightarrow Y\) be a regular morphism. By applying the lemma above repeatedly one can show that if \( Z_{1}, ..., Z_{n} \) is a sequence of global actions then \( \text{Mor}(1_{Z_{n}}, ..., \text{Mor}(1_{Z_{1}}, \lambda)) ... : \text{Mor}(Z_{n}, \text{Mor}(Z_{n-1}, ..., \text{Mor}(Z_{1}, X)) ... \rightarrow \text{Mor}(Z_{n}, \text{Mor}(Z_{n-1}, ..., \text{Mor}(Z_{1}, Y)) ...) \) extends to a regular morphism and hence is a morphism of global actions (a formal proof would be one by induction). Thus \( \lambda \) is \( \infty \)-normal. \qquad \square
Definition 2.19 (product of global actions) Let \( \{ G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi \} \) denote a global action on \( X \) and \( \{ H_\beta \curvearrowright Y_\beta | \beta \in \Psi \} \) a global action on \( Y \). The global action on \( X \times Y \) defined by \( \Theta = \Phi \times \Psi \), \( (\alpha, \beta) \leq (\alpha', \beta') \iff (\alpha \leq \alpha') \land (\beta \leq \beta') \), \( J_{(\alpha, \beta)} = G_\alpha \times H_\beta \), \( J_{(\alpha, \beta)} \leq (\alpha', \beta') \) = \((G_{\alpha \leq \alpha'}, H_{\beta \leq \beta'})\), \( (X \times Y)_{(\alpha, \beta)} = X_\alpha \times X_\beta \), and \( J_{(\alpha, \beta)} \) acts on \((X \times Y)_{(\alpha, \beta)}\) by \((g, h)(x, y) = (gx, hy)\) is called the product of \( X \) and \( Y \).

Our next goal is to show that the exponential law holds for many global action spaces. This will be needed for developing a good homotopy theory of global actions.

The following notation will be used in the next definition. If \( A \) and \( B \) are sets, let \( \text{Mor}(A,B) = \text{Mor}(\text{sets})(A,B) \). If \( C \) is also a set then the exponential law for sets states that the function \( E_{\text{set}} : (A, (B, C)) \rightarrow (A \times B, C) \) \((2.18.1)\)

\[
f \mapsto E_{\text{set}}f
\]

where \( E_{\text{set}}f(a, b) = f(a)(b) \) is an isomorphism of sets.

Definition 2.20 (exponential map) Let \( \{ G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi \} \) denote a global action on \( X \), \( \{ H_\beta \curvearrowright Y_\beta | \beta \in \Psi \} \) a global action on \( Y \) and \( \{ J_\gamma \curvearrowright Z_\gamma | \gamma \in \Theta \} \) denote a global action on \( Z \). Define a regular morphism \( E = (\iota, \kappa, \lambda) : \text{Mor}(X, \text{Mor}(Y, Z)) \rightarrow \text{Mor}(X \times Y, Z) \) as follows. Define

\[
\iota : \Phi_{(X, (Y, Z))} \longrightarrow \Phi_{(X \times Y, Z)}
\]

\[
\kappa_\alpha : (J_{(X, (Y, Z))})_\alpha \longrightarrow (J_{(X \times Y, Z)})_{\iota(\alpha)}
\]

\[
\prod_{x \in X} \prod_{y \in Y} (J_{\alpha(x)}(y)) \prod_{(x, y) \in X \times Y} J_{\iota(\alpha))(x, y)
\]

\[
\prod_{(x, y) \in X \times Y} J_{\alpha(x)}(y)
\]

in the obvious way. Define \( \lambda \) to be the composite mapping \( \text{Mor}(X, \text{Mor}(Y, Z)) \rightarrow (X, (Y, Z)) \) \( E_{\text{set}} \rightarrow (X \times Y, Z) \) where \( E_{\text{set}} \) is the exponential morphism \((2.18.1)\) (one can show that the image of the mapping defined above lies in \( \text{Mor}(X \times Y, Z) \)).

We want to provide conditions when the map \( \text{Mor}(X, \text{Mor}(Y, Z)) \rightarrow \text{Mor}(X \times Y, Z) \) above is an isomorphism of global actions. This will be answered in Theorem 2.31 below.
Definition 2.21 (∞-exponential) A global action $Z$ is called $\infty$-exponential, if for any global actions $X$ and $Y$ the regular morphism

$$E : \text{Mor}(X, \text{Mor}(Y, Z)) \to \text{Mor}(X \times Y, Z)$$

above is an $\infty$-normal isomorphism. (This means its inverse is $\infty$-normal). A global action $Z$ is called regularly $\infty$-exponential, if for any global actions $X$ and $Y$, the regular morphism $E$ above is a weak regular isomorphism. (This means its inverse is a regular morphism). It follows from Corollary 2.17 that $Z$ is $\infty$-exponential also.

Suppose $Z$ is $\infty$-exponential. Let $X_1, ..., X_n$ be any $n$ global actions where $n \geq 2$. We construct now by induction on $n$ the canonical $\infty$-normal isomorphism

$$e_n : \text{Mor}(X_n, \text{Mor}(X_{n-1}, ..., \text{Mor}(X_1, Z))...) \to \text{Mor}(X_n \times ... \times X_1, Z).$$

Let $e_2 : \text{Mor}(X_2, \text{Mor}(X_1, Z) \to \text{Mor}(X_2 \times X_1, Z)$ be the canonical (regular) morphism defined in 2.20. Since $Z$ is $\infty$-exponential $e_2$ is by definition an $\infty$-normal isomorphism. Suppose $e_{n-1} : \text{Mor}(X_{n-1}, ..., \text{Mor}(X_1, Z))...) \to \text{Mor}(X_{n-1} \times ... \times X_1, Z)$ has been constructed and is an $\infty$-normal isomorphism. Define $e_n$ as the composition of the morphisms $(1_{X_n}, e_{n-1}) : \text{Mor}(X_n, \text{Mor}(X_{n-1}, ..., \text{Mor}(X_1, Z))...) \to \text{Mor}(X_n, \text{Mor}(X_{n-1} \times ... \times X_1, Z))$, which is an $\infty$-normal isomorphism by the induction assumption for $n-1$ and Lemma 2.14, and the canonical morphism $\text{Mor}(X_n, \text{Mor}(X_{n-1} \times ... \times X_1, Z)) \to \text{Mor}(X_n \times ... \times X_1, Z)$, which is an $\infty$-normal isomorphism because $Z$ is $\infty$-exponential. As a composition of two $\infty$-normal isomorphisms, $e_n$ is an $\infty$-normal isomorphism. □

The construction of $e_n$ just given yields the following lemma.

Lemma 2.22 If $Z$ is $\infty$-exponential then the canonical morphism

$$e_n : \text{Mor}(X_n, \text{Mor}(X_{n-1}, ..., \text{Mor}(X_1, Z))...) \to \text{Mor}(X_n \times ... \times X_1, Z)$$

is an $\infty$-normal isomorphism.

The pattern of the proof of Lemma 2.22 can be taken over to prove the following lemma.

Lemma 2.23 If $Z$ is regularly $\infty$-exponential then the canonical isomorphism $e_n$ defined in Lemma 2.22 extends to a weakly regular isomorphism.

Definition 2.24 (strong infimum condition) Let $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ denote a global action on $X$. Let $\Delta \subseteq \Phi$ denote a finite subset and let $\Phi_{\geq \Delta} = \{\alpha \in \Phi | \alpha \geq \beta \forall \beta \in \Delta\}$. Let $U \subseteq X$ be a finite and nonempty subset such that $U \cap X_\beta \neq \emptyset$ for all $\beta \in \Delta$. The strong infimum condition for $X$ says that for any $\Delta$ and $U$ as above, the set $\{\alpha \in \Phi_{\geq \Delta} | U \text{ is an } \alpha\text{-frame}\}$ is either empty or contains an initial element. If $X$ satisfies the condition for $\Delta = \emptyset$ it is called an infimum action.
Lemma 2.25 Let \( \{G_\alpha \cap X_\alpha | \alpha \in \Phi\} \) denote a global action on \( X \). If \( X \) satisfies the conditions (a) and (b) below then it is a strong infimum action.

(a) Let \( \alpha, \beta \in \Phi \). Then \( \alpha \leq \beta \iff \exists x \in X_\alpha \cap X_\beta \) such that \( G_\alpha(x) \subseteq G_\beta(x) \).

(b) Let \( \Psi \subseteq \Phi \). Then for any \( x \in \bigcap_{\alpha \in \Psi} X_\alpha \) there is a \( \beta \in \Phi \) such that \( \bigcap_{\alpha \in \Psi} (G_\alpha(x)) = G_\beta(x) \).

**Proof** Let \( \Delta \in \Phi \) denote a finite subset and \( U \subseteq X \) denote finite and nonempty subset such that \( U \cap X_\gamma \neq \emptyset \ \forall \gamma \in \Delta \). We must show that \( \{\alpha \in \Phi_{\geq \Delta} | U \text{ is an} \ \alpha \text{-frame}\} =: \Psi \) is either empty or contains an initial element. If \( u \in U \) then clearly \( u \in \bigcap_{\alpha \in \Psi} X_\alpha \) and hence, by 2.25(b), there is a \( \beta \in \Phi \) such that \( \bigcap_{\alpha \in \Psi} (G_\alpha(u)) = G_\beta(u) \). Since \( U \subseteq \bigcap_{\alpha \in \Psi} (G_\alpha(u)) = G_\beta(u) \) it is obvious that \( U \) is a \( \beta \)-frame. Since \( G_\beta(u) \subseteq G_\alpha(u) \ \forall \alpha \in \Psi \), it follows by 2.25(a) that \( \beta \leq \alpha \ \forall \alpha \in \Psi \). It remains to show that \( \gamma \leq \beta \ \forall \gamma \in \Delta \).

Let \( \gamma \in \Delta \). Since \( U \cap X_\gamma \neq \emptyset \ \forall \gamma \in \Delta \) there is an \( x \in U \cap X_\gamma \). Since \( \gamma \leq \alpha \ \forall \alpha \in \Psi \) and \( U \subseteq X_\alpha \ \forall \alpha \in \Psi \) it follows from 2.1(a) that \( G_\gamma(x) \subseteq \bigcap_{\alpha \in \Psi} (G_\alpha(x)) = G_\beta(x) \).

Hence, by 2.25(b), \( \gamma \leq \beta \). \( \square \)

**Corollary 2.26** A standard single domain global action is a strong infimum action

**Proof** We show that a standard single domain global action satisfies 2.25(a) and (b).

(a) Let \( \alpha, \beta \in \Phi \).

\[ \Rightarrow: \] Suppose \( \alpha \leq \beta \). We have to show that \( \exists x \in X_\alpha \cap X_\beta = G \) such that \( G_\alpha(x) \subseteq G_\beta(x) \). \( \alpha \leq \beta \) implies \( G_\alpha \subseteq G_\beta \). Hence \( G_\alpha(1) = G_\alpha \subseteq G_\beta = G_\beta(1) \) where 1 denotes the identity element in \( G \).

\[ \Leftarrow: \] Suppose \( \exists x \in X_\alpha \cap X_\beta = G \) such that \( G_\alpha(x) \subseteq G_\beta(x) \). Let \( g \in G_\alpha \). Then \( gx \in G_\alpha(x) \). Since \( G_\alpha(x) \subseteq G_\beta(x) \), \( \exists g' \in G_\beta \) such that \( gx = g'x \). Since \( G_\alpha \) and \( G_\beta \) act on \( G \) by left or right multiplication, it follows that \( g = g' \). Thus \( G_\alpha \subseteq G_\beta \) and hence \( \alpha \leq \beta \).

(b) Let \( \Psi \subseteq \Phi \) be a nonempty subset and \( x \in G \). We have to show that there is a \( \beta \in \Phi \) such that \( \bigcap_{\alpha \in \Psi} (G_\alpha(x)) = G_\beta(x) \). Since \( \{G_\gamma | \gamma \in \Phi\} \) is closed under taking intersections, \( \exists \beta \in \Phi \) such that \( \bigcap_{\alpha \in \Psi} G_\alpha = G_\beta \). Thus \( \bigcap_{\alpha \in \Psi} (G_\alpha(x)) = G_\beta(x) \). We show now that \( \bigcap_{\alpha \in \Psi} (G_\alpha(x)) = \bigcap_{\alpha \in \Psi} (G_\alpha(x)) \).

\[ \subseteq: \] Let \( y \in \bigcap_{\alpha \in \Psi} (G_\alpha(x)) \). Then \( \exists g \in \bigcap_{\alpha \in \Psi} G_\alpha \) such that \( y = gx \). \( g \in \bigcap_{\alpha \in \Psi} G_\alpha \) implies \( g \in G_\alpha \ \forall \alpha \in \Psi \). It follows that \( gx \in G_\alpha x \ \forall \alpha \in \Psi \) and hence \( y = gx \in \bigcap_{\alpha \in \Psi} (G_\alpha(x)) \).

\[ \supseteq: \] Let \( z \in \bigcap_{\alpha \in \Psi} (G_\alpha(x)) \). Then \( z \in G_\alpha(x) \ \forall \alpha \in \Psi \). Hence \( \forall \alpha \in \Psi \ \exists g_\alpha \in G_\alpha \) such that \( y = g_\alpha x \). If \( \alpha, \beta \in \Psi \) then \( g_\alpha x = y = g_\beta x \). But this implies
$g_\alpha = g_\beta$. Hence if $\beta \in \Psi$ then $g_\beta \in G_\alpha \ \forall \alpha \in \Psi$. Thus $g_\beta \in \bigcap_{\alpha \in \Psi} G_\beta$ and hence $z = g_\beta x \in \left( \bigcap_{\alpha \in \Psi} G_\alpha \right)(x)$.

The four lemmas below will be used in the proof of Theorem 2.31. Their proofs will be given after that of Theorem 2.31.

**Lemma 2.27** (Local-Global Lemma) Let $X, Y$ denote global actions and $f_0 \in \text{Mor}(X, Y)$. Then $\{f_0, \ldots, f_p\}$ is a $\beta$-frame in $\text{Mor}(X, Y)$ if and only if $\{f_0(x), f_1(x), \ldots, f_p(x)\}$ is a $\beta(x)$-frame in $Y \ \forall x \in X$.

**Lemma 2.28** Let $X$ and $Y$ be global actions and let $\{f_0, \ldots, f_p\}$ be a $\beta$-frame in $\text{Mor}(X, Y)$. If $\{x_0, \ldots, x_q\}$ is a local frame in $X$ then $\{f_i(x_j) \ | 0 \leq i \leq p, 0 \leq j \leq q\}$ is a $b$-frame in $Y$ for some $b$ such that $b \geq \beta(x_0), \ldots, \beta(x_q)$.

**Lemma 2.29** An infimum action is conormal.

**Lemma 2.30** If $Z$ is an infimum action then for any global action $X$, $\text{Mor}(X, Z)$ is an infimum action. If $Z$ is an strong infimum action and the relation on $\Theta$ is transitive then for any global action $X$, $\text{Mor}(X, Z)$ is a strong infimum action and the relation on $\Theta_{(X,Z)}$ is transitive.

**Theorem 2.31** (exponential law) An infimum action is $\infty$-conormal and $\infty$-exponential. A strong infimum action is $\infty$-conormal and regularly $\infty$-exponential.

**Proof** Let $Z$ be an infimum action.

$Z$ is $\infty$-conormal: We have to show that if $Y$ is a global action and $g : Y \to Z$ is a morphism then $g$ is $\infty$-normal, i.e. if $X_n, \ldots, X_1$ are global actions then the map

$$\text{Mor}(1_{X_0}, \text{Mor}(1_{X_{n-1}}, \ldots, \text{Mor}(1_{X_1}, g))\ldots) : \text{Mor}(X_n, \text{Mor}(X_{n-1}, \ldots, \text{Mor}(X_1, Y))\ldots) \to \text{Mor}(X_n, \text{Mor}(X_{n-1}, \ldots, \text{Mor}(X_1, Z))\ldots)$$

is a morphism of global actions. We prove this by induction on $n$.

$n=1$: Let $X$ be a global action. By Lemma 2.29, $Z$ is conormal and hence the map $\text{Mor}(1_X, g) : \text{Mor}(X, Y) \to \text{Mor}(X, Z)$ is a morphism of global actions.

$n-1 \to n$: Let $n \geq 2$. Suppose by induction that for any global actions $X'_{n-1}, \ldots, X'_1$, the map

$$\text{Mor}(1_{X'_{n-1}}, \text{Mor}(1_{X'_{n-2}}, \ldots, \text{Mor}(1_{X'_1}, g))\ldots) : \text{Mor}(X'_{n-1}, \text{Mor}(X'_{n-2}, \ldots, \text{Mor}(X'_1, Y))\ldots) \to \text{Mor}(X'_{n-1}, \text{Mor}(X'_{n-2}, \ldots, \text{Mor}(X'_1, Z))\ldots)$$

is a morphism of global actions. Let $X_n, \ldots, X_1$ be global actions. By Lemmas 2.29 and 2.30, $\text{Mor}(X_{n-1}, \ldots, \text{Mor}(X_1, Z))\ldots$ is conormal and since by the induction assumption $\text{Mor}(1_{X_{n-1}}, \text{Mor}(1_{X_{n-2}}, \ldots, \text{Mor}(1_{X'_1}, g))\ldots)$ is a morphism of global actions, it follows that

$$\text{Mor}(1_{X_n}, \text{Mor}(1_{X_{n-1}}, \ldots, \text{Mor}(1_{X_1}, g))\ldots) : \text{Mor}(X_n, \text{Mor}(X_{n-1}, \ldots, \text{Mor}(X_1, Y))\ldots) \to \text{Mor}(X_n, \text{Mor}(X_{n-1}, \ldots, \text{Mor}(X_1, Z))\ldots)$$

is a morphism of global actions. Hence $\text{Mor}(1_{X_n}, \ldots, \text{Mor}(1_{X_1}, g))\ldots$ is $\infty$-conormal.
is a morphism of global actions.  

Z is 8∞-exponential: We have to show that if X and Y are global actions then the morphism $E : \text{Mor}(X, \text{Mor}(Y, Z)) \to \text{Mor}(X \times Y, Z)$ is an $\infty$-normal isomorphism. Since E is regular, it is $\infty$-normal by Corollary 2.18. Therefore it suffices to show that E has an $\infty$-normal inverse. But $\text{Mor}(X, \text{Mor}(Y, Z))$ is an infimum action by 2 applications of Lemma 2.30 and therefore $\infty$-conormal. Thus any inverse for E is an $\infty$-normal inverse.

Let X and Y be global actions. Define

$$E' : \text{Mor}(X \times Y, Z) \to \text{Mor}(X, \text{Mor}(Y, Z))$$

$$h \mapsto E'(h)$$

where

$$E'(h) : X \to \text{Mor}(Y, Z)$$

$$x \mapsto E'(h)(x)$$

and

$$E'(h)(x) : Y \to Z$$

$$y \mapsto h(x, y)$$

Then $E'$ is a set theoretical inverse of E. We want to show that $E'$ is a morphism of global actions. There are a number of things to verify.

- $E'(h)(x) : Y \to Z$ is a morphism of global actions $\forall h \in \text{Mor}(X \times Y, Z)$ and $\forall x \in X$.

Let $h \in \text{Mor}(X \times Y, Z)$ and $x \in X$. Let $\{y_0, ..., y_p\}$ be a local frame in Y. We have to show that $\{E'(h)(x)(y_0), ..., E'(h)(x)(y_p)\}$ is a local frame in Z. Since $X = \bigcup_{\alpha \in \Phi} X_{\alpha}$ there is an $\alpha \in \Phi$ such that $x \in X_{\alpha}$. Since $\{y_0, ..., y_p\}$ is a local frame there is an $\beta \in \Psi$ such that $\{y_0, ..., y_p\} \subseteq Y_{\beta}$ and is a $\beta$-frame. Clearly $\{(x, y_0), ..., (x, y_p)\} \subseteq X_{\alpha} \times Y_{\beta} = (X \times Y)_{\alpha, \beta}$. Since $\{y_0, ..., y_p\}$ is a $\beta$-frame there are $h_1, ..., h_p \in H_{\beta}$ such that $h_i y_0 = y_i$ ($1 \leq i \leq p$). If 1 denotes the identity element in the local group $G_{\alpha}$ then $(1, h_i)(x, y_0) = (1, h_i y_0) = (x, y_i)$ ($1 \leq i \leq p$). Thus $\{(x, y_0), ..., (x, y_p)\}$ is a local frame and since $h : X \times Y \to Z$ is a morphism of global actions $\{h(x, y_0), ..., h(x, y_p)\} = \{E'(h)(x)(y_0), ..., E'(h)(x)(y_p)\}$ is a local frame in Z. Thus $E'(h)(x)$ is a morphism of global actions.

- $E'(h) : X \to \text{Mor}(Y, Z)$ is a morphism of global actions $\forall h \in \text{Mor}(X \times Y, Z)$.

Let $h \in \text{Mor}(X \times Y, Z)$ and $\{x_0, ..., x_p\}$ be a local frame in X. We must show that $\{E'(h)(x_0), ..., E'(h)(x_p)\}$ is a local frame in $\text{Mor}(Y, Z)$. As above $\{(x_0, y), ..., (x_p, y)\}$ is a local frame in $X \times Y$ for any $y \in Y$. It follows that $\{h(x_0, y), ..., h(x_p, y)\} = \{E'(h)(x_0)(y), ..., E'(h)(x_p)(y)\}$ is a local frame in Z for any $y \in Y$ because $h : X \times Y \to Z$ is a morphism of global actions. Since Z is an infimum action, the set
\{c \in \Theta \mid \{E'(h)(x_0)(y), ..., E'(h)(x_p)(y)\} \text{ is a c-frame}\} \text{ has an initial element } c_y \text{ for any } y \in Y \text{. Define }

\gamma : Y \rightarrow \Theta
\begin{align*}
y \mapsto c_y
\end{align*}

We shall show that \(E'(h)(x_0) \in (\text{Mor}(Y, Z))_\gamma\), i.e.

(a) \(E'(h)(x_0)(y)) \in Z_{\gamma(y)} \forall y \in Y\), and

(b) if \(\{y_0, ..., y_q\}\) is a local frame in Y then \(\{E'(h)(x_0)(y_0), ..., E'(h)(x_0)(y_p)\}\) is a c-frame in Z for some \(c \geq \gamma(y_i) (0 \leq i \leq q)\).

Clearly if \(y \in Y\) then \(\{E'(h)(x_0)(y), ..., E'(h)(x_p)(y)\} \text{ is a c}_y\text{-frame. Thus } E'(h)(x_0)(y) \in Z_{c_y} = Z_{\gamma(y)} \forall y \in Y\). (b) holds because of the following. Let \(\{y_0, ..., y_q\}\) be a local frame in Y. Then \(\{(x_i, y_j) \mid 0 \leq i \leq p, 0 \leq j \leq q\}\) is a local frame in \(X \times Y\) and hence \(\{h(x_i, y_j) \mid 0 \leq i \leq p, 0 \leq j \leq q\} = \{E'(h)(x_i)(y_j) \mid 0 \leq i \leq p, 0 \leq j \leq q\}\) is a c-frame for some \(c \in \Theta\). On the one hand this implies that \(\{E'(h)(x_0)(y_0), ..., E'(h)(x_0)(y_p)\}\) is a c-frame and on the other hand that \(\{E'(h)(x_0)(y_0), ..., E'(h)(x_0)(y_j)\}\) is a c-frame \(\forall j \in \{0, ..., q\}\) and hence \(c \geq c_{y_j} = \gamma(y_j) \forall j \in \{0, ..., q\}\). Thus \(E'(h)(x_0) \in (\text{Mor}(Y, Z))_\gamma\). By the Local-Global Lemma 2.27 it follows that \(\{E'(h)(x_0), ..., E'(h)(x_p)\}\) is a local frame.

\(\bullet\) \(E' : \text{Mor}(X \times Y, Z) \rightarrow \text{Mor}(X, \text{Mor}(Y, Z))\) is a morphism of global actions.
Let \(\{h_0, ..., h_p\}\) be a local frame in \(\text{Mor}(X \times Y, Z)\). We have to show that \(\{E'(h_0), ..., E'(h_p)\}\) is a local frame in \(\text{Mor}(X, \text{Mor}(Y, Z))\). By Lemma 2.27, \(\{h_0(x, y), ..., h_p(x, y)\} = \{E'(h_0)(x)(y), ..., E'(h_p)(x)(y)\}\) is a local frame in Z \(\forall (x, y) \in X \times Y\).

Since Z is an infimum action, the set \(\{c \in \Theta \mid \{h_0(x, y), ..., h_p(x, y)\}\} \text{ is a c-frame}\} \text{ has an initial element } c_{(x, y)}\). Define

\(\gamma : X \rightarrow (Y, \Theta)\)
\(x \mapsto c_{(x, -)}\).

We will now show that \(E'(h_0)(x) \in \text{Mor}(Y, Z)_{\gamma(x)}\) for any \(x \in X\), i.e.

(a) \(E'(h_0)(x)(y) = h_0(x, y) \in Z_{\gamma(x)(y)} = c_{(x, y)} \forall y \in Y\), and

(b) if \(\{y_0, ..., y_r\}\) is a local frame in Y then \(\{E'(h_0)(x)(y_0), ..., E'(h_0)(x)(y_r)\} = \{h_0(x, y_0), ..., h_0(x, y_r)\}\) is a d-frame in Z for some \(d \geq \gamma(x)(y_j) = c(x, y_j) (0 \leq j \leq r)\).

Clearly \(\{h_0(x, y), ..., h_p(x, y)\}\) is a \(c(x, y)\)-frame \(\forall (x, y) \in X \times Y\). Thus \(h_0(x, y) \in Z_{c(x, y)} \forall (x, y) \in X \times Y\). (b) holds because of the following. Let \(\{y_0, ..., y_r\}\) be a local frame in Y. Then \(\{(x, y_0), ..., (x, y_r)\}\) is a local frame in \(X \times Y\). By Lemma 2.28, \(\{h_i(x, y_j) \mid 0 \leq i \leq p, 0 \leq j \leq r\}\) is a d-frame in Z. But on the one hand this implies that \(\{h_0(x, y_0), ..., h_0(x, y_r)\}\) is a d-frame and on the other hand that \(\{h_0(x, y_j), ..., h_p(x, y_j)\}\) is a d-frame for any \(j \in \{0, ..., r\}\) and hence \(e \geq c(x, y_j) (0 \leq j \leq r)\). Thus (b) holds and hence \(E'(h_0)(x) \in \text{Mor}(Y, Z)_{\gamma(x)}\) for
any \( x \in X \). Since \( \{ E'(h_0)(x)(y), \ldots, E'(h_p)(x)(y) \} \) is a \( \gamma(x)(y) \)-frame \( \forall x \in X \) and \( \forall y \in Y \) it follows from Lemma 2.27 that \( \{ E'(h_0)(x), \ldots, E'(h_p)(x) \} \) is a \( \gamma(x) \)-frame in \( \text{Mor}(Y, Z) \) \( \forall x \in X \).

We show now that that \( E'(h_0) \in (\text{Mor}(X, \text{Mor}(Y, Z)))_\gamma \), i.e.

1. \( E'(h_0)(x) \in \text{Mor}(Y, Z)_\gamma(x) \) \( \forall x \in X \), and

2. if \( \{ x_0, \ldots, x_q \} \) is a local frame in \( X \) then \( \{ E'(h_0)(x_0), \ldots, E'(h_0)(x_q) \} \) is a c-frame in \( \text{Mor}(Y, Z) \) for some \( c \geq \gamma(x_i) \) \( (0 \leq i \leq q) \).

We have shown (1) above. Let \( \{ x_0, \ldots, x_q \} \) be a local frame in \( X \). Then \( \{ (x_0, y), \ldots, (x_q, y) \} \) is a local frame in \( X \times Y \) for any \( y \in Y \). It follows from Lemma 2.28 that \( \{ h_i(x_j, y)|0 \leq i \leq p, 0 \leq j \leq q \} \) is a local frame in \( Z \) for any \( y \in Y \). Since \( Z \) is an infimum action, for any \( y \in Y \) the set \( \{ d \in \Theta|\{ h_i(x_j, y)|0 \leq i \leq p, 0 \leq j \leq q \} \) is a local frame} has an initial element \( d_y \). Define

\[
\delta : Y \to \Theta
\]

\[
y \mapsto d_y.
\]

Since \( \{ h_i(x_j, y)|0 \leq i \leq p, 0 \leq j \leq q \} \) is a \( \delta(y) \)-frame for any \( y \in Y \), it follows that \( \{ h_0(x_j, y)|0 \leq j \leq q \} = \{ E'(h_0)(x_j)(y)|0 \leq j \leq q \} \) is a \( \delta(y) \)-frame for any \( y \in Y \). We show now that \( E'(h_0(x_0)) \in \text{Mor}(Y, Z)_\delta \), i.e.

\[
(1.1) \quad E'(h_0)(x_0)(y) = h_0(x_0, y) \in Z_{\delta(y)=d_y} \forall y \in Y,
\]

(1.2) if \( \{ y_0, \ldots, y_r \} \) is an local frame in \( Y \) then \( \{ E'(h_0)(x_0)(y_0), \ldots, E'(h_0)(x_0)(y_r) \} = \{ h_0(x_0, y_0), \ldots, h_0(x_0, y_r) \} \) is an e-frame in \( Z \) for some \( e \geq \delta(y_j) = d_{y_j} \) \( (0 \leq j \leq r) \).

Clearly if \( y \in Y \) then \( \{ h_0(x_j, y)|0 \leq j \leq q \} \) is a \( \delta(y) \)-frame. Thus \( h_0(x_0, y) \in Z_{\gamma(x)} \forall y \in Y \). We show that (1.2) holds. Let \( \{ y_0, \ldots, y_r \} \) be a local frame in \( Y \). Since \( \{ x_j, y_k|0 \leq j \leq q, 0 \leq k \leq r \} \) is a local frame in \( X \times Y \), it follows from Lemma 2.28 that \( \{ h_i(x_j, y_k)|0 \leq i \leq p, 0 \leq j \leq q, 0 \leq k \leq r \} \) is an e-frame in \( Z \). On the one hand this implies that \( \{ h_0(x_0, y_0), \ldots, h_0(x_0, y_r) \} \) is an e-frame and on the other hand \( \{ h_i(x_j, y_k)|0 \leq i \leq p, 0 \leq j \leq q \} \) is an e-frame for any \( k \in \{ 0, \ldots, r \} \) and hence \( e \geq \delta(y_j) = \delta(y_j) \) \( (0 \leq k \leq r) \). Thus \( E'(h_0(x_0)) \in \text{Mor}(Y, Z)_\delta \). By Lemma 2.27, \( \{ E'(h_0)(x_j)|0 \leq j \leq q \} \) is a \( \delta \)-frame. Obviously if \( y \in Y \) then \( d_y \leq c(x_j, y) \) \( (0 \leq j \leq q) \) and hence \( \delta \geq \gamma(x_j) \) \( (0 \leq j \leq q) \). Thus (2) holds and hence \( E'(h_0) \in (\text{Mor}(X, \text{Mor}(Y, Z)))_\gamma \). Since we have shown that \( \{ E'(h_0)(x), \ldots, E'(h_p)(x) \} \) is a \( \gamma(x) \)-frame in \( \text{Mor}(Y, Z) \) \( \forall x \in X \) it follows from Lemma 2.27 that \( \{ E'(h_0), \ldots, E'(h_p) \} \) is a \( \gamma \)-frame.

Let \( Z \) now be a strong infimum action. We must show that \( Z \) is regularly infinite-exponential, i.e. if \( X \) and \( Y \) are global actions then \( E = (\iota, \kappa, \lambda) : \text{Mor}(X, (\text{Mor}(Y, Z)) \to \text{Mor}(X \times Y, Z) \) is a weak regular isomorphism, i.e. there is a regular morphism \( (\iota', \kappa', \lambda') : \text{Mor}(X \times Y, Z) \to \text{Mor}(X, (\text{Mor}(Y, Z)) \) such that \( \lambda' \) is inverse
to \( \lambda \). Let \( X \) and \( Y \) be global actions. Define \( \iota' \) as the set theoretic inverse of \( \iota \). Define \( \lambda' = E' \) where \( E' \) is the morphism of global actions we constructed above. Define

\[
\kappa'_\alpha : \ (J_{(X \times Y, Z)})_{\alpha} \xrightarrow{\iota'_\alpha} (J_{(X, Mor(Y, Z))})_{\iota'(\alpha)}
\]

\[
\prod_{(x,y) \in X \times Y} J_{\alpha(x,y)} \xrightarrow{\prod_{x \in X} \prod_{y \in Y} (J_{\iota'(\alpha)(x,y)})} \prod_{x \in X} \prod_{y \in Y} (J_{\alpha(x,y)})
\]

in the obvious way. Since \( \lambda' = E' \) is inverse to \( \lambda \) it only remains to show that \( (\iota', \kappa', \lambda') \) is a regular morphism, i.e.

(a) \( \iota'(\alpha) \leq \iota'(\beta) \) whenever \( \alpha \leq \beta \),

(b) if \( \alpha \leq \beta \) then the diagram

\[
\begin{array}{ccc}
(J_{(X \times Y, Z)})_{\alpha} & \xrightarrow{\iota'_\alpha} & (J_{(X, Mor(Y, Z))})_{\iota'(\alpha)} \\
\downarrow_{(J_{(X \times Y, Z)})_{\alpha \leq \beta}} & & \downarrow_{(J_{(X, Mor(Y, Z))})_{\iota'(\alpha) \leq \iota'(\beta)}} \\
(J_{(X \times Y, Z)})_{\beta} & \xrightarrow{\iota'_\beta} & (J_{(X, Mor(Y, Z))})_{\iota'(\beta)}
\end{array}
\]

commutes,

(c) \( \lambda'((Mor(X \times Y, Z))_{\alpha}) \subseteq (Mor(X, Mor(Y, Z)))_{\iota'(\alpha)} \forall \alpha \in \Theta_{(Mor(X \times Y, Z))} \),

(d) for all \( \alpha \in \Theta_{(Mor(X \times Y, Z))} \) the pair \( (\kappa'_\alpha, \lambda')((Mor(X \times Y, Z))_{\alpha}) : (J_{(X \times Y, Z)})_{\alpha} \to (J_{(Mor(X, Mor(Y, Z)))})_{\iota'(\alpha)} \) is a morphism of group actions.

We will now verify (a)-(d).

(a) Let \( \alpha, \beta \in \Theta_{(X \times Y, Z)} = (X \times Y, \Theta) \) such that \( \alpha \leq \beta \). We have to show that \( \iota'(\alpha) \leq \iota'(\beta) \). But

\[
\iota'(\alpha) \leq \iota'(\beta) \\
\iff \iota'(\alpha)(x) \leq \iota'(\beta)(x) \forall x \in X \\
\iff \iota'(\alpha)(x)(y) \leq \iota'(\beta)(x)(y) \forall x \in X, y \in Y \\
\iff \alpha(x, y) \leq \beta(x, y).
\]

(b) Let \( \alpha, \beta \in \Theta_{(X \times Y, Z)} = (X \times Y, \Theta) \) such that \( \alpha \leq \beta \) and \( \sigma = (\sigma_{(x,y)})_{(x,y)} \in \Theta_{(Mor(X \times Y, Z))} \).
(J_{X\times Y,Z})_{\alpha}. Then

$$
\kappa'_\beta((J_{X\times Y,Z})_{\alpha\leq\beta}(\sigma)) \\
= \kappa'_\beta\left( \prod_{(x,y)\in X\times Y} J_{\alpha(x,y)\leq\beta(x,y)}(\sigma) \right) \\
= \kappa'_\beta((J_{\alpha(x,y)\leq\beta(x,y)}(\sigma(x,y)))(x,y)) \\
= ((J_{\alpha(x,y)\leq\beta(x,y)}(\sigma(x,y)))_y)_x \\
= ((J_{\iota'(\alpha)(x)(y)\leq\iota'(\beta)(x)(y)}(\sigma(x,y)))(x,y))_x \\
= \left( \prod_{x\in X} \left( \prod_{y\in Y} J_{\iota'(\alpha)(x)(y)\leq\iota'(\beta)(x)(y)}((\sigma(x,y))_y)_x \right) \right) \\
= \left( J_{(X,Y,Z)}\iota'(\alpha)(x)\leq\iota'(\beta)(x)\right)((\sigma(x,y))_y)_x \\
= \left( J_{(X,Y,Z)}\iota'(\alpha)\leq\iota'(\beta)(\kappa'_\alpha(\sigma)) \right),
$$

i.e. \(\kappa'_\beta \circ (J_{X\times Y,Z})_{\alpha\leq\beta} = (J_{X,Y,Z})\iota'(\alpha)\leq\iota'(\beta) \circ \kappa'_\alpha.\)

(c) Let \(\alpha \in \Theta_{(X\times Y,Z)}\) and \(f \in (Mor(X \times Y, Z))_{\alpha}\), i.e.

1. \(f(x,y) \in Z_{\alpha(x,y)} \forall (x,y) \in X \times Y\), and
2. if \(\{(x_0, y_0), \ldots, (x_p, y_p)\}\) is a local frame in \(X \times Y\) then \(\{f(x_0, y_0), \ldots, f(x_p, y_p)\}\)
   is a c-frame for some \(c \in \Theta\) such that \(c \geq \alpha(x_i, y_i)\) \((0 \leq i \leq p)\).

We have to show that \(\lambda'(f) \in (Mor(X, Mor(Y, Z)))_{\iota'(\alpha)}\), i.e.

1') \(\lambda'(f)(x) \in ((Mor(Y, Z))_{\iota'(\alpha)(x)} \forall x \in X\), and
2') if \(\{x_0, \ldots, x_p\}\) is a local frame in \(X\) then \(\{\lambda'(f)(x_0), \ldots, \lambda'(f)(x_p)\}\)
   is a d-frame for some \(d \in \Theta_{(Y,Z)}\) such that \(d \geq \iota'(\alpha)(x_i)\) \((0 \leq i \leq p)\).

First we show (1'), i.e. if \(x \in X\) then

1'') \(\lambda'(f)(x)(y) = f(x,y) \in Z_{\iota'(\alpha)(x)(y)\leq\alpha(x,y)} \forall y \in Y\), and
2'') if \(\{y_0, \ldots, y_q\}\) is a local frame in \(Y\) then \(\{\lambda'(f)(x)(y_0), \ldots, \lambda'(f)(x)(y_q)\}\) = \(\{f(x,y_0), \ldots, f(x,y_q)\}\)
   is an e-frame for some \(e \in \Theta\) such that \(e \geq \iota'(\alpha)(x)(y_j) = \alpha(x,y_j)\) \((0 \leq j \leq q)\).

(1'') follows from (1). Let \(\{y_0, \ldots, y_q\}\) be a local frame in \(Y\). Then \(\{(x, y_0), \ldots, (x, y_q)\}\)
is a local frame in \(X \times Y\) and hence it follows from (2) that \(\{f(x, y_0), \ldots, f(x, y_q)\}\)
is an e-frame for some \(e \in \Theta\) such that \(e \geq \alpha(x, y_q)\) \((0 \leq j \leq q)\). Thus (2'') holds
and hence (1') holds.

We show now (2'). Let \(\{x_0, \ldots, x_p\}\) be a local frame in \(X\). Then for any \(y \in Y\)
\(\{(x_0, y), \ldots, (x_p, y)\}\)
is a local frame in \(X \times Y\) and since \(f : X \times Y \to Z\) is a morphism of global actions \(\{f(x_0, y), \ldots, f(x_p, y)\}\)
is a local frame in \(Z\) for any \(y \in Y\).
Since $Z$ is a strong infimum action the set \( \{ c \in Z \mid \{ f(x_0, y),..., f(x_p, y) \} \} \) is a \( \sigma \)-frame, \( c \geq \nu'(\alpha)(x_i)(y) \) \( (0 \leq i \leq p) \) has an initial element \( c_y \). Define
\[
\gamma : Y \rightarrow \Theta \\
y \mapsto c_y.
\]
Since \( \gamma(y) = c_y \geq \nu'(\alpha)(x_i)(y) \) \( \forall y \in Y, i \in \{0, ..., p\} \) it follows that \( \gamma \geq \nu'(\alpha)(x_i) \) \( (0 \leq i \leq p) \). We show now that \( \lambda'(f)(x_0) \in Mor(Y, Z)_\gamma \), i.e. \( \lambda'(f)(x_0) = f(x_0, y) \in Z_{\gamma(y)=c_y} \forall y \in Y \), and \( \lambda'(f)(x_0) = f(x_0, y) \in Z_{\gamma(y)=c_y} \forall y \in Y \), and
\[
\lambda'(f)(x_0) = f(x_0, y) \in Z_{\gamma(y)=c_y} \forall y \in Y.
\]
Clearly \( \{ f(x_0, y),..., f(x_p, y) \} \) is a \( \sigma \)-frame for any \( y \in Y \). Thus \( f(x_0, y) \in Z_{c_y} \forall y \in Y \), \( \lambda'(f)(x_0) \) holds because of the following. Let \( \{ y_0, ..., y_q \} \) be a local frame in \( Y \). Then \( \{ (x_i, y_j) \mid 0 \leq i \leq p, 0 \leq j \leq q \} \) is a local frame in \( X \times Y \) and hence it follows from (2) that \( \{ f(x_i, y_j) \mid 0 \leq i \leq p, 0 \leq j \leq q \} \) is a \( \sigma \)-frame for any \( y \in Y \). Thus \( \lambda'(f)(x_0) \) holds and hence \( \lambda'(f)(x_0) \in Mor(Y, Z)_\gamma \). Since \( \{ \lambda'(f)(x_0)(y),..., \lambda'(f)(x_0)(y) \} = \{ f(x_0, y),..., f(x_0, y) \} \) is a \( \sigma \)-frame for any \( y \in Y \), it follows from Lemma 2.27 that \( \lambda'(f)(x_0),..., \lambda'(f)(x_0) \) is a \( \gamma \)-frame.

(d) Let \( \alpha \in \Theta_{(X \times Y, Z)} \), \( \sigma \in (J_{X \times Y, Z})_{\alpha} \) and \( f \in Mor(X \times Y, Z)_\alpha \). We must show that \( \lambda'(\sigma f) = \kappa'_\alpha(\sigma) \lambda'(f) \). Let \( x \in X \), \( y \in Y \). Then
\[
\lambda'(\sigma f)(x)(y) \\
= \sigma_{(x,y)} f(x, y) \\
= \sigma_{(x,y)} \lambda'(f)(x)(y) \\
= ((\sigma_{(x,y)})_y) \lambda'(f) \\
= \kappa'_\alpha(\sigma) \lambda'(f)
\]
and hence \( \lambda'(\sigma f) = \kappa'_\alpha(\sigma) \lambda'(f) \).

\[\square\]

Proof of Lemma 2.27 Let \( X, Y \) denote global actions and \( f_0 \in Mor(X, Y)_\beta \).

\[\Rightarrow:\] Let \( \{ f_0, ..., f_p \} \) be a \( \beta \)-frame in \( Mor(X, Y) \), i.e. \( \{ f_0, ..., f_p \} \subseteq Mor(X, Y)_\beta \) and \( \exists \sigma_1, ..., \sigma_p \in J_{\beta} \) such that \( \sigma_i f_0 = f_i \) \( (1 \leq i \leq p) \). We must show that \( \{ f_0(x),..., f_p(x) \} \) is a \( \beta \)-frame \( \forall x \in X \), i.e. if \( x \in X \) then \( \{ f_0(x),..., f_p(x) \} \subseteq Y_{\beta(x)} \) and \( \exists h_1, ..., h_p \in H_{\beta(x)} \) such that \( h_i f_0(x) = f_i(x) \) \( (1 \leq i \leq p) \). But if \( i \in \{0, ..., p\} \) then \( \sigma_i f_0 = f_i \Leftrightarrow (\sigma_i f_0)(x) = f_i(x) \forall x \in X \Leftrightarrow (\sigma_i)_x f_0(x) = f_i(x) \forall x \in X \) and from \( \{ f_0, ..., f_p \} \subseteq Mor(X, Y)_\beta \) it follows that \( \{ f_0(x),..., f_p(x) \} \subseteq Y_{\beta(x)} \forall x \in X \).

\[\Leftarrow:\] Let \( \{ f_0(x),..., f_p(x) \} \) be a \( \beta \)-frame in \( Y \forall x \in X \), i.e. if \( x \in X \) then \( \{ f_0(x),..., f_p(x) \} \subseteq Y_{\beta(x)} \) and \( \exists h_1(x), ..., h_p(x) \in H_{\beta(x)} \) such that \( h_i(x) f_0(x) = f_i(x) \) \( (1 \leq i \leq p) \). We have to show that \( \{ f_0, ..., f_p \} \) is a \( \beta \)-frame, i.e. \( \{ f_0, ..., f_p \} \subseteq
\[ \text{Mor}(X,Y)_\beta \text{ and } \exists \sigma_1, \ldots, \sigma_p \in J_\beta = \prod_{x \in X} H_{\beta(x)} \text{ such that } \sigma_i f_0 = f_i \ (1 \leq i \leq p). \]

Define \( \sigma_i = (h_i(x))_{x \in X}. \) Then \( (\sigma_i f_0)(x) = (\sigma_i)_x f_0(x) = h_i(x) f_0(x) = f_i(x) \) for any \( x \in X \) and hence \( \sigma_i f_0 = f_i. \) Since \( f_0 \in \text{Mor}(X,Y)_\beta \) it follows that \( f_i \in \text{Mor}(X,Y)_\beta \) (1 \( \leq i \leq p \)) and hence \( \{f_0, \ldots, f_p\} \subseteq \text{Mor}(X,Y)_\beta. \)

**Proof of Lemma 2.28** Let \( \{x_0, \ldots, x_q\} \) be a local frame in \( X. \) Since \( f_0 \in \text{Mor}(X,Y)_\beta \) it follows that \( \{f_0(x_0), \ldots, f_0(x_q)\} \) is a \( b \)-frame in \( Y \) such that \( b \geq \beta(x_0), \ldots, \beta(x_q). \) Hence \( \exists \sigma_1, \ldots, \sigma_q \in H_b \) such that \( \sigma_j f_0(x_0) = f_0(x_j) \) (1 \( \leq j \leq q). \) Since \( \{f_0, \ldots, f_p\} \) is a \( \beta \)-frame in \( \text{Mor}(X,Y) \) \( \exists \tau_1, \ldots, \tau_p \in J_\beta \) such that \( \tau_i f_0 = f_i \) (1 \( \leq i \leq p). \) Let \( i \in \{0, \ldots, p\}, j \in \{0, \ldots, q\}. \) Then (notice that \( f_0(x_j) \in Y_{\beta(x_j) \cap Y_b} \))

\[
(H_{\beta(x_j)} \leq b((\tau_i)_x) \sigma_j) f_0(x_0) \\
= H_{\beta(x_j)} \leq b((\tau_i)_x) (\sigma_j f_0(x_0)) \\
= H_{\beta(x_j)} \leq b((\tau_i)_x) f_0(x_j) \\
= (\tau_i)_x f_0(x_j) \\
= f_i(x_j),
\]

i.e \( H_b \) acts transitively on \( \{f_i(x_j)\}_{0 \leq i \leq p, 0 \leq j \leq q} \) and hence \( \{f_i(x_j)\}_{0 \leq i \leq p, 0 \leq j \leq q} \) is a \( b \)-frame. \( \square \)

**Proof of Lemma 2.29** Let \( Y \) be an infimum action, \( X \) be a global action and \( g : X \rightarrow Y \) a morphism of global actions. We have to show that \( g \) is \( Z \)-normal for any global actions \( Z, \) i.e., if \( Z \) is a global action and \( \{f_0, \ldots, f_p\} \) is a local frame in \( \text{Mor}(Z,X) \) then \( \{gf_0, \ldots, gf_p\} \) is a local frame in \( \text{Mor}(X,Y). \) Let \( Z \) be a global action and \( \{f_0, \ldots, f_p\} \) be a local frame in \( \text{Mor}(Z,X). \) From Lemma 2.27 it follows that \( \{f_0(z), \ldots, f_p(z)\} \) is a local frame in \( X \forall z \in Z. \) Since \( g : X \rightarrow Y \) is a morphism of global actions it follows that \( \{(gf_0)(z), \ldots, (gf_p)(z)\} \) is a local frame in \( Y \forall z \in Z. \) For any \( z \in Z \) the set \( \{b \in \Psi | \{(gf_0)(z), \ldots, (gf_p)(z)\} \text{ is a } b \text{-frame} \} \) has an initial element \( b_z, \) since \( Z \) is an infimum action. Define

\[
\beta : Z \rightarrow \Psi \\
z \mapsto b_z.
\]

We show now that \( gf_0 \in \text{Mor}(Z,Y)_\beta, \) i.e.

(a) \( (gf_0)(z) \in Y_{\beta(z)} = Y_{b_z} \ \forall z \in Z,\)

(b) if \( \{z_0, \ldots, z_q\} \) is a local frame in \( z \) then \( \{(gf_0)(z_0), \ldots, (gf_0)(z_q)\} \) is a \( c \)-frame for some \( c \in \Psi \) such that \( c \geq \beta(z_i) = b_{z_i} \ (0 \leq i \leq q). \)

Clearly \( \{(gf_0)(z), \ldots, (gf_p)(z)\} \) is a \( b_z \)-frame for any \( z \in Z. \) Thus \( (gf_0)(z) \in Y_{b_z} \ \forall z \in Z. \) We show that (b) holds. Let \( \{z_0, \ldots, z_q\} \) be a local frame in \( Z. \) By Lemma 2.28 \( \{f_i(z_j)\}_{0 \leq i \leq p, 0 \leq j \leq q} \) is a \( b_z \)-frame for some \( c \in \Psi. \) This implies \( \{(gf_0)(z_0), \ldots, (gf_0)(z_q)\} \) is a \( c \)-frame and \( \{(gf_0)(z_j), \ldots, (gf_p)(z_j)\} \) is a \( c \)-frame \( \forall j \in \{0, \ldots, q\}, \) whence \( c \geq b_{z_j} \ (0 \leq j \leq q). \) Thus (b) holds and hence \( gf_0 \in \text{Mor}(Z,Y)_\beta. \) Since \( \{(gf_0)(z), \ldots, (gf_p)(z)\} \) is a \( \beta(z) = b_z \)-frame \( \forall z \in Z, \) it follows from Lemma 2.27 that \( \{gf_0, \ldots, gf_p\} \) is a \( \beta \)-frame. \( \square \)

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Proof of Lemma 2.30  First we show that if Z is an infimum action then for any global action X, \( Mor(X, Z) \) is an infimum action. Let Z be an infimum action, i.e. if \( U \subseteq Z \) is a finite and nonempty subset then the set \( \{ c \in \Theta : U \) is a c-frame \} is either empty or contains an initial element. Let X be a global action. We must show that \( Mor(X, Z) \) is an infimum action, i.e. if \( V \subseteq Mor(X, Z) \) is a finite and nonempty subset then the set \( \{ \gamma \in \Theta_{(X,Z)} : V \) is a \( \gamma \)-frame \} is either empty or contains an initial element. Let \( V = \{ f_0, \ldots, f_p \} \subseteq Mor(X, Z) \) be a finite and nonempty subset. Suppose \( \{ \gamma \in \Theta_{(X,Z)} : V \) is a \( \gamma \)-frame \} is nonempty. Let \( \delta \in \{ \gamma \in \Theta_{(X,Z)} : V \) is a \( \gamma \)-frame \}.

Then \( V \) is a \( \delta \)-frame and hence it follows from Lemma 2.27 that \( \{ f_0(x), \ldots, f_p(x) \} \) is a \( \delta(x) \)-frame \( \forall x \in X \). Since \( Z \) is an infimum action, \( \{ c \in \Theta : \{ f_0(x), \ldots, f_p(x) \} \) is a c-frame \} contains an initial element \( c_x \) for any \( x \in X \). Define

\[
\delta' : X \to \Theta \\
x \mapsto c_x.
\]

Clearly \( \delta(x) \geq c_x = \delta'(x) \forall x \in X \iff \delta \geq \delta' \). We will show now that \( f_0 \in Mor(X, Z)_{\delta'} \), i.e.

(a) \( f_0(x) \in Z_{\delta'(x)} \forall x \in X \), and

(b) if \( \{ x_0, \ldots, x_q \} \) is a local frame in X then \( \{ f_0(x_0), \ldots, f_0(x_q) \} \) is a d-frame for some \( d \in \Theta \) such that \( d \geq \delta'(x_i) \) (0 \( i \leq q \)).

Clearly \( \{ f_0(x_0), \ldots, f_0(x_q) \} \) is a \( c_x = \delta'(x) \)-frame \( \forall x \in X \). Thus \( f_0(x) \in Z_{\delta'(x)} \forall x \in X \). (b) holds because of the following. Let \( \{ x_0, \ldots, x_q \} \) be a local frame in X. Since \( \{ f_0, \ldots, f_p \} \) is a \( \delta \)-frame in \( Mor(X, Z) \), it follows from Lemma 2.28 that \( \{ f_i(x_j) : 0 \leq i \leq \gamma, 0 \leq j \leq q \} \) is a d-frame for some \( d \in \Theta \). This implies that \( \{ f_0(x_0), \ldots, f_0(x_q) \} \) is a d-frame and that \( \{ f_0(x_j), \ldots, f_p(x_j) \} \) is a local frame \( \forall j \in \{ 0, \ldots, q \} \), whence \( d \geq c_{x_j} = \delta'(x_j) \) (0 \( i \leq j \leq q \)). Thus (b) holds and hence \( f_0 \in Mor(X, Z)_{\delta'} \). Since \( \{ f_0, \ldots, f_p \} \) is a \( c_x = \delta'(x) \)-frame \( \forall x \in X \), it follows from Lemma 2.27 that \( \{ f_0, \ldots, f_p \} \) is a \( \delta' \)-frame. Thus \( \delta' \in \{ \gamma \in \Theta_{(X,Z)} : V \) is a \( \gamma \)-frame \}.

We show now that if \( Z \) is a strong infimum action and the relation on \( \Theta \) is transitive then for any global action X, \( Mor(X, Z) \) is a strong infimum action and the relation on \( \Theta_{(X,Z)} \) is transitive. Let \( Z \) be a strong infimum action, i.e. if \( \Delta \subseteq \Theta \) denotes a finite subset and \( U \subseteq Z \) denotes a finite and nonempty subset such that \( U \cap Z_{\gamma} \neq \emptyset \ \forall \gamma \in \Delta \) then the set \( \{ c \in \Theta_{\geq \Delta} : U \) is a c-frame \} is either empty or contains an initial element. Let the relation on \( \Theta \) be transitive and let X be a global action. We must show that \( Mor(X, Z) \) is a strong infimum action, i.e. if \( \Gamma \subseteq \Theta_{(X,Z)} \) is a finite subset and \( V \subseteq Mor(X, Z) \) is a finite and nonempty subset such that \( V \cap Mor(X, Z)_{\delta} \neq \emptyset \ \forall \delta \in \Gamma \) then the set \( \{ d \in \Theta_{(X,Z)}_{\geq \Gamma} : V \) is a d-frame \} is either empty or contains an initial element, and that the relation on \( Mor(X, Z) \) is transitive. Let \( \alpha, \beta, \gamma \in \Theta_{(X,Z)} \) such that \( \alpha \leq \beta, \) i.e. \( x \leq \beta(x) \ \forall x \in X \), and \( \beta \leq \gamma, \) i.e. \( \beta(x) \leq \gamma(x) \ \forall x \in X \). Since the relation on \( \Theta \) is transitive it follows that \( \alpha(x) \leq \gamma(x) \ \forall x \in X \) which is equivalent to \( \alpha \leq \gamma \). Thus the relation on \( \Theta_{(X,Z)} \) is transitive. Let \( \Gamma \subseteq \Theta_{(X,Z)} \) be a finite subset and \( V = \{ f_0, \ldots, f_p \} \subseteq Mor(X, Z) \) be
a finite and nonempty subset such that $V \cap Mor(X,Z)_\delta \neq \emptyset \forall \delta \in \Gamma$. Suppose \( \{d \in (\Theta(X,Z))_{\geq 0}| V \text{ is a } d\text{-frame}\} \) is nonempty. Let \( e \in \{d \in (\Theta(X,Z))_{\geq 0}| V \text{ is a } d\text{-frame}\} \). Then $V$ is an $e$-frame and hence it follows from Lemma 2.27 that \( \{f_0(x), ..., f_p(x)\} \) is an $e(x)$-frame $\forall x \in X$. Let $\Delta(x) = \{\delta(x)| \delta \in \Gamma\}$. Let $x \in X$ and $e \in \Delta(x)$. Then $\epsilon = \delta(x)$ for some $\delta \in \Gamma$. Since $V \cap Mor(X,Z)_{\delta'} \neq \emptyset \forall \delta' \in \Gamma$, if $i \in \{0, ..., p\}$ such that $f_i \in Mor(X,Z)_{\delta}$. It follows that $f_i(x) \in Z_{\delta(x)} = Z_{\epsilon}$. Hence if $x \in X$ then \( \{f_0(x), ..., f_p(x)\} \) $\cap Z_{\epsilon} \neq \emptyset \forall \epsilon \in \Delta(x)$. Since $Z$ is a strong infimum action, \( \{c \in \Theta_{\geq \Delta(x)}| \{f_0(x), ..., f_p(x)\} \text{ is a } c\text{-frame}\} \) contains an initial element $c_x$ for any $x \in X$. Define

\[
\epsilon' : X \to \Theta
\]

\[
x \mapsto c_x,
\]

Since $e \geq \delta \forall \delta \in \Gamma$ it follows that $e(x) \geq \delta(x) \forall \delta \in \Gamma \forall x \in X$. Hence $e(x) \in \Theta_{\geq \Delta(x)} \forall x \in X$. Since \( \{f_0(x), ..., f_p(x)\} \) is an $e(x)$-frame for any $x \in X$, it follows $e(x) \in \{c \in \Theta_{\geq \Delta(x)}| \{f_0(x), ..., f_p(x)\} \text{ is a } c\text{-frame}\} \forall x \in X$. Thus $e(x) \geq c_x = \epsilon'(x) \forall x \in X \Leftrightarrow e \geq \epsilon'$. We will show now that $f_0 \in Mor(X,Z)_{\epsilon'}$, i.e.

(a) $f_0(x) \in Z_{\epsilon'(x)} \forall x \in X$, and

(b) if $\{x_0, ..., x_q\}$ is a local frame in $X$ then \( \{f_0(x_0), ..., f_0(x_q)\} \) is an $f$-frame for some $f \in \Theta$ such that $f \geq e'(x_i) (0 \leq i \leq q)$.

Clearly \( \{f_0(x), ..., f_p(x)\} \) is a $c_x = \epsilon'(x)$-frame $\forall x \in X$. Thus $f_0(x) \in Z_{\epsilon'(x)} \forall x \in X$. (b) holds because of the following. Let \( \{x_0, ..., x_q\} \) be a local frame in $X$. Since \( \{f_0, ..., f_p\} \) is an $e$-frame in $Mor(X,Z)$, it follows from Lemma 2.28 that \( \{f_i(x_j)| 0 \leq i \leq p, 0 \leq j \leq q\} \) is an $f$-frame for some $f \in \Theta$ such that $f \geq e(x_0), ..., e(x_p)$. This implies that \( \{f_0(x_0), ..., f_0(x_q)\} \) is an $f$-frame and that \( \{f_0(x_j), ..., f_p(x_j)\} \) is a local frame $\forall j \in \{0, ..., q\}$. Since $\epsilon'(x_i) \geq \delta(x_i) \forall \delta \in \Gamma \forall i \in \{0, ..., q\}$ it follows, by the transitivity of the relation on $\Theta$, that $f \geq \delta(x_i) \forall \delta \in \Gamma \forall i \in \{0, ..., q\}$. Thus $f \in \Theta_{\geq \Delta(x_i)} (0 \leq i \leq q)$. It follows that $f \in \{c \in \Theta_{\geq \Delta(x)}| \{f_0(x_i), ..., f_p(x_i)\} \text{ is a } c\text{-frame}\} (0 \leq i \leq q)$ and hence $f \geq c_{x_i} = \gamma'(x_i) (0 \leq j \leq q)$. Thus (b) holds and hence $f_0 \in Mor(X,Z)_{\epsilon'}$. Since \( \{f_0(x), ..., f_p(x)\} \) is a $c_x = \epsilon'(x)$-frame $\forall x \in X$, it follows from Lemma 2.27 that \( \{f_0, ..., f_p\} \) is a $\epsilon'(x)$-frame. Thus $\epsilon' \in \{d \in \Theta(X,Z)| V \text{ is a } d\text{-frame}\}$. It remains to show that $\epsilon' \geq \delta \forall \delta \in \Gamma$. Let $\delta \in \Gamma$. Since $\epsilon'(x) = c_x \in \{c \in \Theta_{\geq \Delta(x)}| \{f_0(x), ..., f_p(x)\} \text{ is a } c\text{-frame}\} \forall x \in X$, it follows that $\epsilon'(x) \geq \delta(x) \forall x \in X \Leftrightarrow \epsilon' \geq \delta$. Thus $\epsilon' \in \{d \in (\Theta(X,Z))_{\geq 0}| V \text{ is a } d\text{-frame}\}$. \( \Box \)

## 3 Homotopy

**Definition 3.1** Let $X$ and $Y$ denote global actions and let $f, g : X \to Y$ denote morphisms. Let $L$ denote the line action defined in 2.3. We say that $f$ is homotopic to $g$ if there is a morphism $H : X \times L \to Y$ and elements $n_-, n_+ \in \mathbb{Z}, n_- \leq n_+$ such that $H_{n_-} = f \forall n \leq n_-$ and $H_{n_+} = g \forall n \geq n_+$ where \( \circ_i : X \to X \times L \)

\[
x \mapsto (x, n).
\]
Definition 3.2 (homotopy) Let $X$, $Y$ denote global actions. A morphism $H : X \times L \to Y$ is called a homotopy if $\exists$ elements $n_\leq n_+ \in \mathbb{Z}$ such that $H_{H_{n}^{n_{+}}}^{n_{+}} \forall n \geq n_+$ and $H_{H_{n}^{n_{-}}}^{n_{-}} \forall n \leq n_-$.

Definition 3.3 Let $X$ denote a global action. A morphism $f : L \to X$ stabilizes on the left, if $\exists n_\in \mathbb{Z}$ such that $f(n) = f(n_-) \forall n \leq n_-$. If $f$ stabilizes on the right, if $\exists n_+ \in \mathbb{Z}$ such that $f(n) = f(n_+) \forall n \geq n_+$. A pair $(n_+, n_-)$ as above is called a stabilisation pair for $f$.

Definition 3.4 (path) Let $X$ denote a global action. A morphism $f : L \to X$ which stabilizes on the left and on the right is called a path in $X$.

Definition 3.5 Let $X$ denote a global action and $\omega : L \to X$ a path. If $\omega$ is not constant then there is a smallest integer $\text{lus}(\omega)$ called the least upper stabilization of $\omega$ such that $\omega(n) = \omega(\text{lus}(\omega)) \forall n \geq \text{lus}(\omega)$ and a largest integer $\text{gls}(\omega)$ called the greatest lower stabilization of $\omega$ such that $\omega(n) = \omega(\text{gls}(\omega)) \forall n \leq \text{gls}(\omega)$. Suppose $\omega$ is constant then we define $\text{lus}(\omega) = 0$ and $\text{gls}(\omega) = 0$. $\omega(\text{lus}(\omega))$ is called the terminal point of $\omega$ (term(\omega)) and $\omega(\text{gls}(\omega))$ is called the initial point of $\omega$ (init(\omega)).

Definition 3.6 (loop) A loop in $X$ is a path $\omega$ in $X$ whose initial point and terminal point are equal, i.e. if $n_+ = \text{lus}(\omega)$ and $n_- = \text{gls}(\omega)$ then $\omega(n_-) = \omega(n_+)$. If $x = \text{init}(\omega)$ then $\omega$ is called a loop at $x$. The set of all loops at $x$ is denoted $\Omega(x)$.

Definition 3.7 Let $\omega$ and $\omega'$ denote loops in a global action $X$. We say that $\omega$ is loop homotopic to $\omega'$

(a) if $\omega$ is homotopic to $\omega'$ in the usual sense, i.e. there is a morphism $H : L \times L \to X$ and integers $n_-, n_+ \in \mathbb{Z}, n_- \leq n_+$, such that $H_{H_{n}^{n_{+}}}^{n_{+}} = \omega \forall n \leq n_-$ and $H_{H_{n}^{n_{-}}}^{n_{-}} = \omega' \forall n \geq n_+$, and

(b) $H_{H_{n}^{n_{+}}}^{n_{+}}$ is a loop for any $n \in \mathbb{Z}$.

Definition 3.8 (composition of paths) Let $\omega$ and $\omega'$ denote paths in a global action $X$ such that term(\omega) = init(\omega'). If $\omega$ is not constant then their composition $\omega' \omega$ is defined as follows.

$\omega' \omega(n) = \begin{cases} 
\omega(n), & \text{if } n \leq \text{lus}(\omega), \\
\omega'(n - \text{lus}(\omega) + \text{gls}(\omega')), & \text{if } n \geq \text{lus}(\omega).
\end{cases}$

The composition is well defined, since $\omega(\text{lus}(\omega)) = \text{term}(\omega) = \text{init}(\omega') = \omega'(\text{gls}(\omega')) = \omega'((\text{lus}(\omega) - \text{lus}(\omega) + \text{gls}(\omega'))$. Clearly, if $\omega'$ is constant then $\omega' \omega = \omega$. If $\omega$ is constant then define $\omega' \omega = \omega'$.

Remark 3.9 Clearly loops in $\Omega(x)$ compose and their composition is a loop at $x$. The definition of composition in 3.8 is concocted so that the composition law induced on $\Omega(x)$ is a monoid with identity element the constant function $x$. 

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Definition 3.10 (inverse of a path) Let $\omega$ be a path in a global action $X$. The map $\omega^{-1} : I \to X$ defined by

$$\omega^{-1}(n) = \begin{cases} \text{term}(\omega), & \text{if } n < \text{gls}(\omega), \\ \omega(\text{lus}(\omega) - n + \text{gls}(\omega)), & \text{if } \text{gls}(\omega) \leq n \leq \text{lus}(\omega), \\ \text{init}(\omega), & \text{if } n > \text{lus}(\omega), \end{cases}$$

is called the inverse of $\omega$. It is clearly a path.

Remark Let $\omega$ and $\omega'$ be paths in a global action $X$ such that $\text{term}(\omega) = \text{init}(\omega')$. Then $\omega \omega'$ and $\omega^{-1}$ also are paths in $X$.

Definition 3.11 (path connected) Let $\{G_{\alpha} \subset X_{\alpha} | \alpha \in \Phi\}$ denote a global action on $X$. Two points $x, y \in X$ are called path connected if there is a path $\omega$ in $X$ such that $\omega(0) = x$ and $\omega(1) = y$. If $x$ and $y$ are path connected we write $x \sim y$. $X$ is called (path) connected if any two points $x, y \in X$ are path connected.

Definition-Lemma 3.12 Let $X$ be a global action. Then $\sim$ is an equivalence relation on $\bigcup_{\alpha \in \Phi} X_{\alpha}$. By definition $\pi_0(X)$ is the set of all equivalence classes of $\sim$ on $\bigcup_{\alpha \in \Phi} X_{\alpha}$. An equivalence class of $\sim$ is called a path component of $X$.

Proof We must check that $\sim$ is reflexive, symmetric and transitive.

- $\sim$ is reflexive.
  Let $x \in X$. Let $\omega$ be the constant path in $X$ at $x$. Then $\text{init}(\omega) = \text{term}(\omega) = x$ and hence $x \sim x$.

- $\sim$ is symmetric.
  Let $x, y \in X$ such that $x \sim y$. Then there is a path $\omega$ in $X$ such that $\omega(0) = x$ and $\omega(1) = y$. Since $\text{init}(\omega^{-1}) = \text{term}(\omega) = y$ and $\text{term}(\omega^{-1}) = \text{init}(\omega) = x$, it follows that $y \sim x$.

- $\sim$ is transitive.
  Let $x, y, z \in X$ such that $x \sim y$, $y \sim z$. Then there are paths $\omega, \omega'$ in $X$ such that $\text{init}(\omega) = x$, $\text{term}(\omega) = y$, $\text{init}(\omega') = y$ and $\text{term}(\omega) = z$. Clearly $\omega' \omega$ is a path in $X$ such that $\text{init}(\omega' \omega) = x$ and $\text{term}(\omega' \omega) = z$. Hence $x \sim z$. \hfill $\Box$

Definition 3.13 Let $X, Y$ denote global actions, $H : X \times L \to Y$ denote a homotopy. Suppose $\exists m, n \in \mathbb{Z}$, $m \neq n$, such that $H_{\alpha}^o_m \neq H_{\alpha}^o_n$. Then there is a largest $n_-(H) \in \mathbb{Z}$ such that $H_{\alpha}^o_n = H_{\alpha}^o_{n-(H)} \forall n \leq n_-(H)$ and a smallest $n_+(H) \in \mathbb{Z}$ such that $H_{\alpha}^o_n = H_{\alpha}^o_{n+(H)} \forall n \geq n_+(H)$. If $H_{\alpha}^o_n = H_{\alpha}^o_m \forall m, n \in \mathbb{Z}$ then we define $n_-(H) = n_+(H) = 0$. $n_-(H)$ is called the greatest lower stabilization of $H$ (gls($H$)) and $n_+(H)$ is called the least upper stabilization of $H$ (lus($H$)). $H_{\alpha}^o_{n-(H)}$ is called the initial value of $H$ (init($H$)) and $H_{\alpha}^o_{n+(H)}$ is called the terminal value of $H$ (term($H$)).

Definition 3.14 (composition of homotopies) Let $H_1, H_2 : X \times L \to Y$ be homotopies such that $\text{term}(H_1) = \text{init}(H_2)$. Then their composition $H_2 H_1$ is defined as
follows.

\[
H_2H_1((x, n)) = \begin{cases} 
H_1((x, n)), & \text{if } n \leq \text{lus}(H_1), \\
H_2((x, n - \text{lus}(H_1) + \text{gls}(H_2))), & \text{if } n \geq \text{lus}(H_1).
\end{cases}
\]

**Definition 3.15** (inverse of a homotopy) Let \( H : X \times L \to Y \) denote a homotopy. The map \( H^{-1} : X \times L \to Y \) defined by

\[
H^{-1}(x, n) = \begin{cases} 
H(x, \text{gls}(H)), & \text{if } n < \text{gls}(H), \\
H(x, \text{lus}(H) - n + \text{gls}(H)), & \text{if } \text{gls}(\omega) \leq n \leq \text{lus}(\omega), \\
H(x, \text{lus}(H)), & \text{if } n > \text{lus}(H),
\end{cases}
\]

is called the inverse of \( H \). It is clearly a homotopy.

**Definition 3.16** (base point) Let \( X \) denote a global action. Giving \( X \) a base point means fixing some \(* \in X\). If \(*\) is the base point of \( X \) we write sometimes \( X_\ast \) instead of \( X \).

**Definition 3.17** (end point stable homotopy) Let \( X \) denote a global action with base point \(*\) and \( H : L \times L \to X \) a homotopy. If for any \( n \in \mathbb{Z} \), \( H^\ast_{n} \) is a loop whose initial point (=terminal point) is \(*\), then \( H \) is called an end point stable homotopy.

**Definition-Lemma 3.18** Let \( X \) denote a global action with base point \(*\). Define a relation \( \sim \) on \( \Omega(*) \) by \( \omega \sim \omega' \iff \omega \text{ and } \omega' \text{ are end point stable homotopic, i.e. there is a homotopy } H : L \times L \to X \text{ such that } \forall n \in \mathbb{Z} \text{ } H^\ast_{n} \text{ is a loop at } \ast \text{ and for all } n \text{ sufficiently small (resp. large) } H^\ast_{n} = \omega \text{ (resp. } H^\ast_{n} = \omega') \). Then \( \sim \) is an equivalence relation on \( \Omega(*) \) and, by definition, \( \pi_1(X_\ast) \) is the set of all equivalence classes of \( \sim \). If \( X \) is pathwise connected and \( |\pi_1(X_\ast)| = 1 \), \( X \) is called simply connected.

**Proof** We have to check that \( \sim \) is reflexive, symmetric and transitive.

- \( \sim \) is reflexive.
  Let \( \omega \in \Omega(*) \). Define a map \( H : L \times L \to X \) by \( h(m,n) = \omega(m) \) \( \forall m,n \in \mathbb{Z} \).
  One checks easily that \( H \) is an end point stable homotopy such that \( \text{init}(H) = \text{term}(H) = \omega \). Hence \( \omega \sim \omega \).

- \( \sim \) is symmetric.
  Let \( \omega, \omega' \in \Omega(*) \) such that \( \omega \sim \omega' \). Then there is an end point stable homotopy \( H : L \times L \to X \) such that \( \text{init}(H) = \omega \text{ and } \text{term}(H) = \omega' \). Since \( \text{init}(H^{-1}) = \text{term}(H) = \omega' \text{ and } \text{term}(H^{-1}) = \text{init}(H) = \omega \), it follows that \( \omega' \sim \omega \).

- \( \sim \) is transitive.
  Let \( \omega, \omega', \omega'' \in X \) such that \( \omega \sim \omega' \), \( \omega' \sim \omega'' \). Then there are end point stable homotopies \( H, H' : L \times L \to X \) such that \( \text{init}(H) = \omega \text{, } \text{term}(H) = \omega' \text{, } \text{init}(H') = \omega' \text{ and } \text{term}(H') = \omega'' \). Clearly \( H'H : L \times L \to X \) is an end point stable homotopy such that \( \text{init}(H'H) = \omega \text{ and } \text{term}(H'H) = \omega'' \). Hence \( \omega \sim \omega'' \). \( \square \)
Lemma 3.19 Let $X$ denote a global action with basepoint $\ast$. If $\omega \in \Omega(x)$, let $[\omega]$ denote the equivalence class of loops $\omega'$ which are end point stable homotopic to $\omega$. Define

$$\circ : \pi_1(X, x) \times \pi_1(Y, y) \to \pi_1(Y, y)$$

$$([\omega], [\tau]) \mapsto [\tau \omega] .$$

Then $\circ$ is well defined and $(\pi_1(X, x), \circ)$ is a group.

Proof First we show that $\circ$ is well defined. Let $A, B \in \pi_1(X, x)$, $\omega, \omega' \in A$ and $\tau, \tau' \in B$. We must show that $[\omega \tau] = [\omega' \tau']$, i.e. $\omega \tau \sim \omega' \tau'$. Since $\omega \sim \omega'$ and $\tau \sim \tau'$, there are end point stable homotopies $H_1, H_2 : \operatorname{Id} \times L \to X$ such that $\operatorname{init}(H_1) = \omega$, $\operatorname{term}(H_1) = \omega'$, $\operatorname{init}(H_2) = \tau$ and $\operatorname{term}(H_2) = \tau'$. Define a map $H : \operatorname{Id} \times L \to X$ by $H = H_1 \circ_n H_2 \circ_n (n \in \mathbb{Z})$. Obviously $H$ is an end point stable homotopy such that $\operatorname{init}(H) = \omega \tau$ and $\operatorname{term}(H) = \omega' \tau'$ and hence $\omega \tau \sim \omega' \tau'$. We show now that $(\pi_1(X, x), \circ)$ is a group. Since $\Omega(\ast)$ is a monoid by Remark 3.9, it follows that $(\pi_1(X, x), \circ)$ is a monoid. Hence we only have to show the existence of inverse elements. Let $A = [\omega] \in \pi_1(X, x)$. Define a function $f_k : \operatorname{Id} \times L \to X$ by

$$f_k(n) = \begin{cases} 
\omega^{-1} \omega(n), & \text{if } n < \operatorname{lus}(\omega) - k, \\
\omega^{-1} \omega(\operatorname{lus}(\omega) - k), & \text{if } \operatorname{lus}(\omega) - k \leq n \leq \operatorname{lus}(\omega) + k, \\
\omega^{-1} \omega(n), & \text{if } n > \operatorname{lus}(\omega) + k. 
\end{cases}$$

$(0 \leq k \leq \operatorname{lus}(\omega) - \operatorname{gls}(\omega))$. One checks easily that $f_{k-1}$ is end point stable homotopic to $f_k$ by a homotopy $H_k (0 \leq k \leq \operatorname{lus}(\omega) - \operatorname{gls}(\omega) - 1)$. Clearly $f_0 = \omega^{-1} \omega$ is end point stable homotopic to $f_{\operatorname{lus}(\omega) - \operatorname{gls}(\omega)} = \text{constant loop at } \ast$ by $H_{\operatorname{lus}(\omega) - \operatorname{gls}(\omega)}... \operatorname{Id} \times L \times \operatorname{Id}$. Hence $[\omega] \circ [\omega^{-1}] = [\omega^{-1} \omega] = [\tau]$. □

4 Coverings

Definition 4.1 Let $p : Y \to X$ denote a morphism of global actions. $p$ is called a covering morphism if and only if for any local frame $\{x_0, \ldots, x_n\}$ in $X$ and any $y_0 \in p^{-1}(x_0)$ there is a unique local frame $\{y_0, y_1, \ldots, y_n\}$ in $Y$ such that $p(y_i) = x_i \forall i \in \{1, \ldots, n\}$.

Definition 4.2 Let $X$ denote a global action and $x \in X$. Let $\text{star}(x) = \bigcup_{x \in X} G_{\alpha} x$. We give $\text{star}(x)$ the structure of a global action as follows. Let $\Phi_{\text{star}(x)} = \{ \alpha \in \Phi \mid x \in X_{\alpha} \}$, $(G_{\text{star}(x)})_{\alpha} = G_{\alpha}$ and $\text{star}(x)_{\alpha} = G_{\alpha} x$. We let $(G_{\text{star}(x)})_{\alpha}$ act on $\text{star}(x)$ in the obvious way. Note: Let $x, x_1, \ldots, x_p$ be a finite set of elements in $X$. Then $x, x_1, \ldots, x_p$ is a local frame in $X$ if and only if $x, x_1, \ldots, x_p \in \text{star}(x)$ and is a local frame there also.

Lemma 4.3 Let $X, Y$ be global actions and let $p : Y \to X$ be a map. Then $p$ is a covering morphism if and only if $y \in Y$, $p_{\text{star}(y)} : \text{star}(y) \to \text{star}(p(y))$ is an isomorphism of global actions.
Proof Suppose $p : Y \to X$ is a covering morphism. We have to check several things.

- $p|_{\text{star}(y)}(y') \in \text{star}(p(y)) \forall y' \in \text{star}(y)$. Let $y' \in \text{star}(y)$. Since $y' \in \text{star}(y)$, there is a $\beta \in \Psi_{\text{star}(y)}$ and an $h \in H_{\beta}$ such that $hy = y'$. Obviously $\{y, y'\}$ is a local frame in $Y$. Since $p$ is a morphism of global actions, it follows that $\{p(y), p(y')\}$ is a local frame in $X$. Thus $p(y') \in \text{star}(x)$.

- $p|_{\text{star}(y)}$ is a morphism of global actions. Let $\{y_0, ..., y_q\}$ be a local frame in $\text{star}(y)$. It follows that $\{y_0, ..., y_q\}$ is a local frame in $Y$. Since $p$ is a morphism of global actions $\{p|_{\text{star}(y)}(y_0), ..., p|_{\text{star}(y)}(y_q)\} = \{p(y_0), ..., p(y_q)\}$ is a local frame in $X$. Hence $\{p|_{\text{star}(y)}(y_0), ..., p|_{\text{star}(y)}(y_q)\}$ is a local frame in $\text{star}(p(y))$.

- $p|_{\text{star}(y)}$ is injective. Let $y_1, y_2 \in \text{star}(y)$ such that $p(y_1) = p(y_2)$. Since $\{p(y), p(y_1)\}$ is a local frame in $X$, there is a unique local frame $\{y, y'\}$ in $Y$ such that $p(y') = p(y_1)$. Since $\{y, y_1\}$ and $\{y, y_2\}$ are both liftings of the same local frame in $X$, it follows that they are equal and hence $y_1 = y_2$.

- $p|_{\text{star}(y)}$ is surjective. Let $x \in \text{star}(p(y))$, i.e. $\{p(y), x\}$ is a local frame in $X$. Since $p$ is a covering morphism, there is a local frame $\{y, y'\}$ in $Y$ such that $p(y') = x$. Since $\{y, y'\}$ is a local frame, $y' \in \text{star}(y)$. Thus $p|_{\text{star}(y)}$ is surjective.

- Since $p|_{\text{star}(y)}$ is bijective, it suffices to show that $(p|_{\text{star}(y)})^{-1} : \text{star}(p(y)) \to \text{star}(y)$ is a morphism of global actions. Let $\{x_0, ..., x_q\}$ be a local frame in $\text{star}(p(y))$. Then $\{x_0, ..., x_q\} \in G_\alpha p(y)$ for some $\alpha \in \Phi_{\text{star}(p(y))}$. Thus $\{p(y), x_0, ..., x_q\}$ is a local frame in $X$. Hence there is a (unique) local frame $\{y, y_0, ..., y_q\}$ such that $p(y_i) = x_i (0 \leq i \leq q)$. Obviously $y_0, ..., y_q \in \text{star}(y)$. Thus $(p|_{\text{star}(y)})^{-1}$ is a morphism.

Suppose now $p|_{\text{star}(y)} : \text{star}(y) \to \text{star}(p(y))$ is an isomorphism of global actions $\forall y \in Y$. We show first that $p$ is a morphism. Let $\{y_0, ..., y_q\}$ be a local frame in $Y$. Then $\{y_0, ..., y_q\}$ is a local frame in $\text{star}(y_0)$. Since $p|_{\text{star}(y_0)} : \text{star}(y_0) \to \text{star}(p(y_0))$ is a morphism, $\{p(y_0), ..., p(y_q)\}$ is a local frame in $\text{star}(p(y_0))$ and thus also in $X$. We show now that $p$ satisfies the unique local frame lifting property. Let $\{x_0, ..., x_q\}$ be a local frame in $X$ and let $y_0 \in p^{-1}(x_0)$. We must show that there is a unique local frame $\{y_0, y_1, ..., y_q\}$ in $Y$ such that $p(y_i) = x_i \forall i \in \{1, ..., n\}$.

- existense
  Obviously $\{x_0, ..., x_q\} \subseteq \text{star}(x_0)$. Since $p|_{\text{star}(y)} : \text{star}(y_0) \to \text{star}(x_0)$ is an isomorphism of global actions, $(p|_{\text{star}(y_0)})^{-1} : \text{star}(x_0) \to \text{star}(y_0)$ is an isomorphism of global actions. Hence $\{(p|_{\text{star}(y_0)})^{-1}(x_0), (p|_{\text{star}(y_0)})^{-1}(x_1), ..., (p|_{\text{star}(y_0)})^{-1}(x_q)\} = \{y_0, (p|_{\text{star}(y_0)})^{-1}(x_1), ..., (p|_{\text{star}(y_0)})^{-1}(x_q)\}$ is a local frame in $Y$.

- uniqueness
Suppose \( \{y_0, y_1, \ldots, y_q\} \) is a local frame in \( Y \) such that \( p(y_i) = x_i \) \( \forall i \in \{1, \ldots, n\} \). Then \( \{y_0, y_1, \ldots, y_q\} \subseteq \text{star}(y_0) \). Since \( p|_{\text{star}(y_0)} : \text{star}(y_0) \to \text{star}(x_0) \) is injective, it follows that \( y_i = (p|_{\text{star}(y_0)})^{-1}(x_i) \) (1 \( \leq i \leq q \)).

Lemma 4.4 (Unique Path Lifting Property (UPLP)) Let \( p : Y \to X \) be a covering morphism of global actions, \( \omega \) a path in \( X \) and \((n_-, n_+)\) be a stabilisation pair for \( \omega \). Let \( y \in p^{-1}(\text{init}(\omega)) \). Then there is a unique path \( \tilde{\omega} \) in \( Y \) such that \( \text{init}(\tilde{\omega}) = y \). \((n_-, n_+)\) is also a stabilisation pair for \( \tilde{\omega} \).

Proof See [2], p.154, proof of Lemma 10.4. \( \square \)

Definition 4.5 (fixed end point homotopy) Let \( X \) denote a global action, \( x, y \in X \) and \( \omega, \omega' \) be paths in \( X \) such that \( \text{init}(\omega) = \text{init}(\omega') = a \) and \( \text{term}(\omega) = \text{term}(\omega') = b \). A morphism \( H : L \times L \to X \) such that \( \exists n_-, n_+ \in \mathbb{Z}, n_- \leq n_+ \) such that \( \mathcal{O}_n = \omega \forall n \leq n_- \), \( \mathcal{O}_n = \omega' \forall n \geq n_+ \), \( \mathcal{O}_n(m) = a \forall m \leq n \forall n \in \mathbb{Z} \) and \( \mathcal{O}_n(m) = b \forall m \geq n_+ \forall n \in \mathbb{Z} \).

Lemma 4.6 Let \( p : Y \to X \) be a covering of global actions. Let \( \omega, \omega' \) be paths in \( X \). Let \( H : L \times L \to X \) be a fixed end point homotopy from \( \omega \) to \( \omega' \). Define \( \tilde{\omega} \) and \( \tilde{\omega}' \) such that \( \text{init}(\tilde{\omega}) = \text{init}(\tilde{\omega}') = y \) are homotopic by a fixed end point homotopy \( \tilde{H} \) (this implies term(\( \tilde{\omega} \)) = term(\( \tilde{\omega}' \))).

Proof See [2], p.154 f., proof of Lemma 10.5. \( \square \)

Theorem 4.7 (Lifting Criterion) Let \((X, x_0), (Y, y_0)\) and \((Z, z_0)\) denote global actions with base points. Let \( p : Y \to X \) denote a covering morphism such that \( p(y_0) = x_0 \) and \( f : Z \to X \) a morphism such that \( f(z_0) = x_0 \). Let \( p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0) \) and \( f_* : \pi_1(Z, z_0) \to \pi_1(X, x_0) \) be the induced maps. Suppose that \( Z \) is path connected. Then \( f \) lifts to a morphism \( \tilde{f} : Z \to Y \) such that \( \tilde{f}(z_0) = y_0 \) if and only if \( f_* \circ (\pi_1(Z, z_0)) \subseteq p_* \circ \pi_1(Y, y_0)) \).

Proof \( \Rightarrow \): Suppose \( f \) lifts to a morphism \( \tilde{f} : Z \to Y \) such that \( \tilde{f}(z_0) = y_0 \), i.e. there is a morphism \( \tilde{f} : Z \to Y \) such that \( \tilde{f}(z_0) = y_0 \) and \( p \tilde{f} = f \). Then

\[
f_* \circ (\pi_1(Z, z_0)) = (p \tilde{f})_* \circ (\pi_1(Z, z_0)) = p_* \circ (\tilde{f}_* \circ (\pi_1(Z, z_0))) \subseteq p_* \circ \pi_1(Y, y_0)).
\]

\( \Leftarrow \): Suppose \( f_* \circ (\pi_1(Z, z_0)) \subseteq p_* \circ \pi_1(Y, y_0)) \). Define a map \( \tilde{f} : Z \to Y \) as follows. Let \( z \in Z \). Since \( Z \) is connected there is a path \( \omega \) in \( Z \) such that \( \text{init}(\omega) = z_0 \) and \( \text{term}(\omega) = z \). Clearly \( y_0 \in p^{-1}(f(\text{gls}(\omega)))). \) Define \( \tilde{f} \) by the UPLP \( f \omega \) lifts uniquely to a path \( \tilde{\omega} \) in \( Y \) such that \( \text{init}(\tilde{\omega}) = y_0 \). Define \( \tilde{f}(z) = \text{term}(\tilde{\omega}) \). There are several things to check.

- \( \tilde{f} \) is well defined.
- Let \( \omega \) and \( \omega' \) be paths in \( Z \) such that \( \text{init}(\omega) = \text{init}(\omega') = z_0 \) and \( \text{term}(\omega) = \text{term}(\omega') = z \). Then \( (f \omega)^{-1}(f \omega') \) is a loop in \( X \) at \( x_0 \), since \( \text{init}((f \omega)^{-1}(f \omega')) = \text{init}(f \omega') = f \text{init}(\omega')) = f(z_0) = x_0 \) and \( \text{term}((f \omega)^{-1}(f \omega')) = \text{term}(f \omega') = f \text{init}(\omega') = f(z_0) = x_0 \). Clearly \( [(f \omega)^{-1}(f \omega')] \in f_* \circ \pi_1(Z, z_0) \subseteq p_* \circ \pi_1(Y, y_0)) \). Hence there is a loop \( \tau \) in \( Y \) at \( y_0 \) such that \( p_* \circ (\tau) = ([(f \omega)^{-1}(f \omega')] \Rightarrow [p \tau] = [(f \omega)^{-1}(f \omega')] \), i.e. \( p \tau \) and \( (f \omega)^{-1}(f \omega') \) are homotopic by a homotopy
Let $H : L \times L \to X$. By the ULP $(f\omega)^{-1}(f\omega')$ lifts uniquely to a path $\lambda$ in $Y$ with the stabilisation pair $(gls(\omega'),lus(\omega') + lus(\omega) - gls(\omega))$ such that $init(\lambda) = y_0$. By Lemma 4.6 $term(\lambda) = term(\tau) = y_0$. Set $n_0 = lus(\omega') + gls(\omega) - lus(\omega)$. For all $n \in \mathbb{Z}$ define

$$
\mu(n) = \begin{cases} 
\lambda(n), & \text{if } n \leq lus(\omega'), \\
\lambda(lus(\omega')), & \text{if } n \geq lus(\omega')
\end{cases}
$$

and

$$
\nu(n) = \begin{cases} 
y_0, & \text{if } n \leq gls(\omega'), \\
\lambda(lus(\omega')) + lus(\omega) - gls(\omega) - n + lus(\omega'), & \text{if } gls(\omega') \leq n \leq n_0, \\
\lambda(lus(\omega')), & \text{if } n \geq n_0.
\end{cases}
$$

Obviously $\mu = \tilde{\omega}'$ and $\nu = \tilde{\omega}$. Hence $term(\tilde{\omega}) = term(\nu) = \lambda(lus(\omega')) = term(\mu) = term(\tilde{\omega}')$.

- $p\tilde{f} = f$.

Let $z \in Z$. Let $\omega$ be a path in $Z$ such that $init(\omega) = z_0$ and $term(\omega) = z$. Let $\tilde{\omega}$ be the unique lift of $f\omega$ such that $init(\tilde{\omega}) = y_0$. It follows that

$$(p\tilde{f})(z) = p(\tilde{f}(z)) = p(term(\tilde{\omega})) = p(\tilde{\omega}(lus(\omega))) = p(f\omega(lus(\omega))) = f(z).$$

- $\tilde{f}$ is a morphism of global actions.

Let $\{z_1, ..., z_q\}$ be a local frame in $Z$. We have to show that $\{\tilde{f}(z_1), ..., \tilde{f}(z_q)\}$ is a local frame in $Y$. Let $\omega_1$ be an arbitrary path in $Z$ such that $init(\omega_1) = z_0$ and $term(\omega_1) = z_1$. For any $i \in \{2, ..., q\}$, $n \in \mathbb{Z}$ define

$$
\omega_i(n) = \begin{cases} 
\omega_1(n), & \text{if } n \leq lus(\omega_1), \\
z_i, & \text{if } n > lus(\omega_1)
\end{cases}
$$

Then

$$
\tilde{\omega}_i(n) = \begin{cases} 
\tilde{\omega}_1(n), & \text{if } n \leq lus(\omega_1), \\
\tilde{f}(z_i), & \text{if } n > lus(\omega_1)
\end{cases}
$$

for any $i \in \{2, ..., q\}$, $n \in \mathbb{Z}$. Hence $\tilde{f}(z_2), ..., \tilde{f}(z_q) \in \text{star}(\tilde{f}(z_1))$. Since $f$ is a morphism of global actions $\{f(z_1), ..., f(z_q)\}$ a local frame in $X$. Since $p$ is a covering morphism there is a local frame $\{\tilde{f}(z_1), y_2, ..., y_q\}$ in $Y$ such that $p(y_i) = f(z_i)$ $(2 \leq i \leq q)$. But since $p|_{\text{star}(f(z_1))}$ is bijective by Lemma 4.3, $y_i = \tilde{f}(z_i)$ $(2 \leq i \leq q)$.

- $\tilde{f}(z_0) = y_0$.

Let $\omega$ be the constant path at $z_0$. Let $\tilde{\omega}$ be the unique lifting of $\omega$ such that $init(\tilde{\omega}) = y_0$. By definition of $\tilde{f}$, $\tilde{f}(z_0) = term(\tilde{\omega})$. Since $(gls(\omega), lus(\omega)) = (0,0)$ is a stabilisation pair for $\tilde{\omega}$, $\tilde{\omega}$ is the constant path at $y_0$ and hence $term(\tilde{\omega}) = y_0$. □

**Theorem 4.8** Let $(X, x_0)$ denote a connected global action with base point and let $H$ be a subgroup of $\pi_1(X, x_0)$. Then there is a connected global action $X_H$, an element $\tilde{x}_0$ and a covering morphism $p_H : X_H \to X$ such that $p_H(\pi_1(X_H, \tilde{x}_0)) = H$. 

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Proof Let \( \{X_\alpha \sim G_\alpha | \alpha \in \Phi \} \) denote a connected global action with base point \( x_0 \) and let \( \Gamma X \) denote the set of all paths in \( X \) starting at \( x_0 \). Define a relation \( \sim_H \) on \( \Gamma X \) by \( \omega \sim_H \omega' \iff \text{term}(\omega) = \text{term}(\omega') \) and \([\omega']^{-1}\omega \in H \). One checks easily that \( \sim_H \) is an equivalence relation. Let \( X_H \) denote the set of all equivalence classes of \( \sim_H \). Define a function \( p_H : X_H \to X \) by \( p_H(\omega) = \text{term}(\omega) \). We will now show \( X_H \) is a covering morphism such that \( p_H(\pi_1(X_H, \tilde{x}_0)) = H \), where \( \tilde{x}_0 \) is the equivalence class of the constant path at \( x_0 \).

- \( p_H : X_H \to X \) is a morphism.
- \( p_H : X_H \to X \) is a covering morphism.

\( X_H \) is connected.

Let \( [\omega] \in X_H \). By the UPLP \( \omega \) lifts (uniquely) to a path \( \tau \) in \( X_H \) starting at \( \tilde{x}_0 \). Clearly, if \( n \in \{1, \ldots, \ellus(\omega)\} \), then \( \tau(n) = [\rho_n] \), where

\[
\rho_n(m) = \begin{cases} 
\omega(m), & \text{if } m \leq n, \\
\omega(n), & \text{if } m > \ellus(\omega).
\end{cases}
\]

Since \( \text{term}(\rho_{\ellus(\omega)}) = \text{term}(\omega) \) and \([\rho_{\ellus(\omega)}]^{-1}\omega = [\omega^{-1}] \omega = [\text{constant path at } x_0] \in H \), it follows that \( \rho_{\ellus(\omega)} \sim_H \omega \) and hence \( \text{term}(\tau) = [\rho_{\ellus(\omega)}] = [\omega] \). Thus \( \tilde{x}_0 \) and \([\omega] \)
are path connected and since \( \omega \) was an arbitrary element of \( X_H \), \( X_H \) is connected.

\( p_{H,*}(\pi_1(X_H, \tilde{x}_0)) = H \).

Let \([l] \in \pi_1(X_H, \tilde{x}_0)\). Clearly \( p_Hl \) is a loop in \( X \) at \( x_0 \) and \( l \) is its unique lift starting at \( \tilde{x}_0 \). It follows that \( [c] = \tilde{x}_0 = \text{term}(l) = [p_Hl] \) (see above), where \( c \) is the constant path in \( X \) at \( x_0 \). Hence \( [(p_Hl)c^{-1}] \in H \). Since \( \text{term}((p_Hl)c^{-1}) = \text{term}(p_Hl) \) and \( [((p_Hl)c^{-1})^{-1}(p_Hl)] = [(p_Hl)^{-1}(p_Hl)] = [c] \in H \), it follows that \( p_{H,*}([l]) = [p_Hl] = [(p_Hl)c^{-1}] \in H \). \( \square \)

**Corollary 4.9** A connected global action has a simply connected covering space.

**Proof** Let \((X, x_0)\) denote a connected global action. First we show that \( p_{H,*} : \pi_1(X_H, \tilde{x}_0) \to H \) is injective. Let \([l], [l'] \in \pi_1(X_H, \tilde{x}_0)\) such that \( p_{H,*}([l]) = p_{H,*}([l']) \).

By Lemma 4.6, \([l] = [l']\) and hence \( p_{H,*} \) is injective. Let \( H = [c] \), where \( c \) is the constant path in \( X \) at \( x_0 \). It follows that \( |\pi_1(X_H, \tilde{x}_0)| \leq 1 \), i.e. \( X_H \) is simply connected. \( \square \)
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