Periodic orbit theory revisited in the anisotropic Kepler problem

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Gutzwiller’s trace formula for the anisotropic Kepler problem (AKP) is Fourier transformed with a convenient variable $u = 1/\sqrt{-2E}$, which takes care of the scaling property of the AKP action $S(E)$. The proper symmetrization procedure (Gutzwiller’s prescription) is used by the introduction of half orbits that close under symmetry transformations, so that the 2D semiclassical formulas correctly match the quantum subsectors $m^+ = 0^+$ and $m^− = 0^-$. Response functions constructed from half orbits in the periodic orbit theory (POT) side are explicitly given. In particular, the response function $g_X$ from the $X$-symmetric half orbit has an amplitude where the root of the monodromy determinant is inverse hyperbolic. The resultant weighted densities of periodic orbits $D_{m^\pm = 0}(\phi)$ from both quantum subsectors show peaks at the actions of the periodic orbits with correct peak heights and widths corresponding to their Lyapunov exponents. The formulation takes care of the cut-off of the energy levels, and the agreement between the $D(\phi)$s of the quantum mechanical (QM) and POT sides is observed to be independent of the choice of cut-off. The systematics appearing in the densities of the periodic orbits is explained in terms of features of the periodic orbits. It is shown that, from quantum energy levels, one can extract information on AKP periodic orbits, even the Lyapunov exponents—the success of inverse quantum chaology in AKP.

Subject Index \ A32, A33

1. Introduction

1.1. Quantum–classical correspondence and quantum chaos

There is a salient distinction in quantum–classical correspondence between the case of classical theory being integrable and the case of it being chaotic. In the former, the number ($M$) of constants of motion is equal to the number ($N$) of the degrees of freedom (i.e. $M = N$) and the classical system trajectory is confined on an $N$-dimensional torus. Then, the Einstein–Brillouin–Keller (EBK) condition gives a semiclassical quantization scheme \cite{1–3}. In the latter case ($M < N$), on the other hand, the trajectories are exponentially sensitive to the initial conditions. A chaotic trajectory covers the constant-energy surface in phase space uniformly and the phase-space average of some quantity will be equal to the time-average along any one of the trajectories, if the system is entirely in the ergodic regime. Because of the absence of a torus, the EBK quantization is no longer applicable\footnote{Einstein gives a coordinate-independent topological condition $\int_{\Gamma} p \cdot dq = n_i h$ ($i = 1, \cdots, N$). Keller’s improvement accounts for the leakage of the wave function over the turning points and gives a condition on}. Quantum
chaos (QC) is a research field that aims at the elucidation of the quantum–classical correspondence for this latter case. In this paper, we call a quantum system whose classical counterpart is chaotic a QC system for short.

One of the theoretical approaches to QC is the periodic orbit theory (POT) pioneered by Gutzwiller, which is a semiclassical quantization scheme, applicable not only to integrable but also to non-integrable cases [5–10]. As a testing ground for the POT, the anisotropic Kepler problem (AKP) was chosen, and it has successfully produced important results, which are now the basis for the study of quantum chaos [5,7–9,11,12]. In this paper, we revisit this AKP and study the POT in a reverse way, armed with newly obtained quantum data; we show that these data yield accurate information on the classical periodic orbits.

For a QC system in two dimensions, Gutzwiller’s trace formula [5,8] for the response function $g(E)$ is

$$
g(E) = \sum_n \frac{1}{E - E_n + i\epsilon} \approx \frac{i}{\hbar} \sum_{\Gamma_{\text{PO}}} \frac{T_{\Gamma_{\text{PO}}}(E)}{2 \sinh \left( \frac{\lambda_{\Gamma_{\text{PO}}}}{2} \right)} e^{i S_{\Gamma_{\text{PO}}}(E)/\hbar - i \pi v_{\Gamma_{\text{PO}}}/2}. \tag{1}
$$

Here the sum runs over all periodic orbits (primitive ones and their repetitions) and $T_{\Gamma_{\text{PO}}}$, $S_{\Gamma_{\text{PO}}}$, $v_{\Gamma_{\text{PO}}}$, and $\lambda_{\Gamma_{\text{PO}}}$ are respectively the period, action, number of turning points, and Lyapunov exponent of each periodic orbit $\Gamma_{\text{PO}}$. The AKP is a 3D problem, but, for the quantum sector with azimuthal quantum number $m = 0$, this formula is sufficient to work with. The density of states (d.o.s.) is directly read off from the response function by $\rho(E) \equiv \sum_n \delta(E - E_n) = -\frac{1}{\pi} \Im(g(E))$.  

Let us call the first expression of the trace formula (1) the quantum mechanical (QM) side and the second expression, the semiclassical approximation of it, the periodic orbit theory (POT) side.

An important test of (1) is whether the POT side correctly produces the quantum energy levels, when a sufficiently large number of periodic orbits are accounted for. This direction (from POT to QM) has been extensively pursued by Gutzwiller.

It is possible to organize the POT test in the reverse direction (from QM levels to POT). Soon after Gutzwiller’s trace formula, a mathematical framework in this direction was presented by Chazarain [13,14], which is an extension of Poisson’s sum formula, and later on this direction is called inverse quantum chaology and quantum recurrence spectroscopy. One remarkable test in this direction was given by Wintgen [16] in the diamagnetic Kepler problem (DKP)—a hydrogen atom in a uniform magnetic field. He showed that the long-range correlation in the DKP energy spectrum produces, the wave number $k$:

$$
k \oint_{C_i} \nabla \cdot dq = 2\pi n_i + \pi/2 m_i \ (i = 1, \ldots, N),
$$

where $m_i$ is the number of turning points. This may be deemed as a higher-dimensional extension of the Wentzel–Kramers–Brillouin (WKB) approximation. It is interesting to note that Einstein had already noticed that his quantization condition could not be applied to the classically chaotic case as early as 1917 [1]. For mixed dynamics, the EBK condition can only be used for the integrable part.

2 See the discussion at the beginning of Sect. 2.

3 Explicitly writing out the double summation for the POs:

$$
\rho(E) \approx \rho(E) + \sum_{\Gamma_{\text{PO}}} \sum_{n=1}^{\infty} \frac{n T_{\Gamma_{\text{PO}}}(E)}{2 \pi \hbar \sinh \left( n \lambda_{\Gamma_{\text{PO}}}/2 \right)} \cos n \left( S_{\Gamma_{\text{PO}}}(E)/\hbar - \pi v_{\Gamma_{\text{PO}}}/2 \right), \tag{2}
$$

where the first term denotes the contribution of zero-length orbit.

4 We have learned these terms from Ref. [15].
through the trace formula in the reverse direction, peaks due to POs at the right positions on an appropriately scaled action axis. An important step taken in Ref. [16] is the symmetrization technique given by Gutzwiller [8], with which the POT side can accommodate the correct symmetry properties of the QM sector with which the QM side is calculated. It is the inverse quantum chaology in AKP, which we aim at in this paper.

1.2. Anisotropic Kepler problem

The AKP is a system of an electron, with an anisotropic mass tensor, bound in a spherically symmetric Coulomb field, which is realized by an electron moving around a donor impurity of a semiconductor. For the AKP experiment, see Refs. [17,18].

The Hamiltonian (in dimensionless variables) is

\[ H = \frac{1}{2\mu} u^2 + \frac{1}{2\nu} (v^2 + w^2) - \frac{1}{\sqrt{x^2 + y^2 + z^2}} (\mu \geq v, \mu \nu = 1), \]

where the \( x \)-axis is the heavy axis [8]. There remains rotational symmetry around the heavy axis due to the crystal symmetry.

For the limit \( \gamma \equiv v/\mu = 1 \), (3) is the hydrogen atom (the Kepler problem), one of the most well studied integrable quantum models along with the harmonic oscillators. With the decrease of the mass anisotropy parameter \( \gamma \), the classical system increases in randomness. Because of the Coulombic potential, AKP is not a Kolmogorov–Arnold–Moser (KAM) system [19,20]. With a decrease in \( \gamma \) from 1, the phase space does not become the admixture of integrable regions (tori) and ergodic regions, but rather it passes through (around \( \gamma \approx 0.75–0.85 \)) an intermediate regime where, due to the abrupt collapse of KAM tori, the phase space is filled by Cantori [19,20] (stochastic web of chaos; see e.g. Ref. [21]), and finally reaches the completely ergodic regime (\( \gamma \lesssim 0.3 \)). There, the quantum level statistics at the ergodic limit is Gaussian Orthogonal Ensemble (GOE), because, for AKP, the time-reversal symmetry is respected with \( T^2 = 1 \) (rather than with \( T^2 = -1 \)) and rotational symmetry is also respected. On the classical mechanics side, all the periodic orbits are isolated and unstable.\(^5\)

There is a particularly nice feature of AKP. Consider any one of 2D AKP orbits and code it by a binary code \((\ldots, a_{-1}, a_0, a_1, \ldots, a_i, \ldots)\) where \( a_i = 1 \) \((0)\) if the sign of the \( x \) at the \( i \)-th intersection of the orbit with the \( x \)-axis is positive (negative). Then, it is proved that there exists, for \( \gamma < 8/9 \) (or equivalently \( \mu = 1/\nu > \sqrt{9/8} \)), one orbit for one given binary code [11,24]; namely, AKP at high anisotropy is endowed with such great variety. Then, a periodic orbit is described by \( \{a_i\} \) \((1 \leq i \leq 2N)\) with \( N = 1, 2, \ldots \). Let us call \( 2N \) the length of the periodic orbit. It is conjectured that the correspondence between the code of a periodic orbit and the initial value is one to one [8,12]\(^6\). Thus, AKP is the best testing ground for the quantum–classical correspondence—by varying one parameter \( \gamma \), one can observe how it varies, while keeping track of the important ingredients in the classical side, the periodic orbits, by binary codes.

\(^5\) These estimates of anisotropy ranges come from level statistics analysis [19,20,22,23].

\(^6\) This conjecture is drawn based on shooting for POs with lengths \( 2N \leq 10 \) under careful consideration of the orbit symmetries. We are now able to find POs with \( 2N \leq 20 \) and are in the process of making an exhaustive list of them. Furthermore, we have found a plausible explanation of the conjecture based on the devil’s staircase over the 2D plane of initial values \((X_0, U_0)\) with a height function \( \xi(X_0, U_0) \) calculated from the corresponding binary codes to \((X_0, U_0)\). (For a preliminary report, see Ref. [25].) The physics of the devil’s staircase surface, and an accurate test of Gutzwiller’s action approximation based on the statistical Ising spin system (with an exponentially decaying interaction) by POs up to \( 2N = 20 \), will be reported elsewhere. See also footnote 9.) For the use of \((X_0, U_0)\), see Ref. [11].
In his seminal paper in 1971 [5], Gutzwiller presented the trace formula (1) for the QC system and applied it to AKP. The fundamental periodic orbit (FPO; the distorted Kepler ellipse by mass anisotropy) was found by solving Hill's equation iteratively and the prediction of energy levels was shown to be in rough agreement with Faulkner's then-available results by perturbative calculation (the first six levels) [26] for Si and Ge. Then the endeavor of summing up the contributions of all of the periodic orbits was accomplished in Refs. [7,8,12]. Here an amazing formula was utilized, which gives a very good approximation of the action of a periodic orbit as a function of its binary code. In addition, a method was given that calculates the energy levels of particular quantum subsectors in terms of POT by the inclusion of periodic orbits that close under the relevant symmetry transformations [8]. The real part of the pole positions of the response functions turned out to be in better agreement with the energy levels [26] and, most importantly, the imaginary part turned out to be very small (particularly for the parity even sector), so that it is proved that the interferences between periodic orbits do indeed act to produce sharp peaks in (1). Tanner et al. employed the cycle expansion method combined with the functional equation and even obtained very high-lying energy levels of AKP only from $2N \leq 8$ POs [27]. The usual direction of the POT trace formula in AKP has thus been extensively studied.

1.3. Inverse quantum chaos in AKP

In this note, we revisit the anisotropic Kepler problem (AKP), and show that the inverse use of the trace formula works perfectly. Our method is essentially the same as that of Wintgen [16]. We also take full advantage of Gutzwiller's method of symmetrizing the POT side, and we use a naive but very efficient Fourier transformation from the energy to action space, which undoes the interference and predicts classical periodic orbits from quantum energy levels one by one. Our main formula is (31) (and its practical version (33)). The virtue of AKP is its simplicity. For one thing, the action of the AKP orbit has a simple scaling property $S_{\Gamma_{\text{PO}}} (E) = \Phi_{\Gamma_{\text{PO}}} / \sqrt{-2E}$. This makes the experimental inverse test in AKP quite transparent, while in DKP the experimental test is somewhat involved because one has to simultaneously consider the energy levels at different magnetic field strengths [28]. In DKP, we also have to take into account the specific feature that the Lyapunov exponent has a distinct dependence on the energy and field strength depending on whether the PO is longitudinal or transverse to the magnetic field, while in AKP with the azimuthal quantum number $m = 0$, the POT is 2D and there is only one Lyapunov exponent to deal with. Finally, one can be sure in AKP with the choice $\gamma = 0.2$ (the classic value where Gutzwiller's heroic endeavor was performed) that one is perfectly in the ergodic regime.

For the QM side, we use our data on quantum energy levels extending up to $2 \times 10^4$ for both even and odd parity sectors. These data are calculated by the method of Wintgen, Marxer, and Briggs (WMB) [29]. The calculation is performed after a careful test on the necessary choice of the scaling parameter involved in the formulation (see Sect. 3.1, footnote 10 there, and Refs. [22,23]).

For the POT side, we are now armed with periodic orbit data up to rank $2N = 12$ obtained by an extensive shooting calculation (see Sect. 3.1 and footnote 6).

We pay special attention in order to perform the test quantitatively. We check not only the predicted position of the POs but compare the QM and POT sides of the inverse trace formula (33) quantitatively with respect to both real and imaginary parts at various choices of the maximum energy levels.

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7 For silicon $\gamma = 0.208$ and for germanium $\gamma = 0.0513$. 


We show that they agree independently of $n_{\text{max}}$, as they should as long as the semiclassical approximation is correct. As far as we know, this is the first quantitative inverse quantum chaos in AKP. Furthermore, we show that we can extract not only the action but also the Lyapunov exponent of AKP periodic orbits from quantum energy levels.

The organization of the rest of this paper is as follows. In Sect. 2, we present two important ingredients to facilitate the inverse quantum chaos in AKP. One is Gutzwiller’s technique to adapt the POT so that it can match the symmetry properties of the quantum sector under investigation. We give explicit formulas for the POT response functions that have proper symmetries. The other is the Fourier transformation for the inverse quantum chaos using a convenient variable $u = 1/\sqrt{-2E}$, which takes care of the scaling property of the AKP action fully. Our main formula is (31) (and its practical version (33)). The degeneracy factor $\sigma$ in (35), important in performing the quantitative test, is discussed. In Sect. 3, we show our results for inverse quantum chaos in AKP. The systematics of the distribution of AKP orbits on the action space is explained. The characteristic feature of Gutzwiller’s amazing action as a function of the AKP orbit binary is also discussed. Furthermore, we extract periodic orbits in AKP from quantum data. We conclude in Sect. 4.

2. Formulation of trace formula in AKP in the inverse direction

2.1. POT response functions with proper symmetries

The Hamiltonian (3) of AKP is symmetric with respect to the rotation around the $x$-axis. Therefore, the angular momentum $L_x$ around the $x$-axis is conserved. As pointed out by Gutzwiller, the case $L_x = 0$ is most important since, in this case, hard chaos is realized in the classical mechanics. When $L_x = 0$, the orbit of the electron is restricted in a plane that includes the $x$-axis (we choose the $xy$-plane as this plane). The corresponding QM theory is under the restriction that the azimuthal quantum number $m = 0$. In the POT side, it is sufficient to work with the 2D trace formula (1) with the sum over the 2D periodic orbits and lift it to 3D by correctly choosing $\nu/G\Omega_1$, while in the QM side the restricted Hilbert space with $m = 0$ must be calculated with full 3D formulation. Moreover, we can restrict the 3D quantum theory by the parity $\pi$. By considering the quantum subsectors $m^\pi = 0^+$ and $m^\pi = 0^-$ separately, and testing the quantum–classical correspondence sector by sector, we can enhance the accuracy of the test.

Below, we first explain Gutzwiller’s idea [8] of how to formulate the 2D POT so that it can represent the quantum subsectors $m^\pi = 0^+$ and $m^\pi = 0^-$. The strategy is as follows. The 3D Hamiltonian (3) is mirror symmetric under the transformation with respect to the $yz$-plane. Let us call this an $X$-transformation. The quantum Green function for the parity even (odd) sector can be constructed by symmetrizing (antisymmetrizing) the Green function under the $X$-transformation with respect to the end point of the paths. Similarly, (3) is also symmetric under the rotation around the $x$-axis. We call this $Y$-transformation (in the 2D view). The quantum Green function with $m = 0$ can be constructed by symmetrizing the Green function under the $Y$-transformation with respect to the end point of the paths. The procedure of symmetrization (antisymmetrization) can be realized in terms of POT by introducing the POs that close under the respective transformations. Then we give concrete formulas for the response functions that have the correct symmetry properties.

2.1.1. $X$-transformation. The $X$-transformation is given by

$$Xq \equiv X(x, y, z) \quad Xp \equiv X(u, v, w)$$

$$= (-x, y, z), \quad = (-u, v, w). \quad (4)$$
Firstly let us consider the QM side. Because $X^2 = 1$, the eigenfunctions of the Hamiltonian (3) are classified into even or odd functions under $X$-transformation, i.e.

$$\phi(Xq) = \phi(q), \quad \psi(Xq) = -\psi(q).$$

(5)

Accordingly, the sum for the Green function in the QM side can be divided into two distinct sums:

$$G(q'', q', E) = \sum_j \frac{\phi_j(q'')\phi_j^*(q')}{E - E_j + i\epsilon} + \sum_k \frac{\psi_k(q'')\psi_k^*(q')}{E - E_k + i\epsilon}.$$  

(6)

Here $j$ and $k$ label, respectively, even and odd parity quantum states, and the response function is the path-cum-trace

$$g(E) = \int dq' G(q', q', E).$$  

(7)

By replacing $q''$ by $Xq''$ in (6) and using (5), we obtain

$$G(Xq'', q', E) = \sum_j \frac{\phi_j(Xq'')\phi_j^*(q')}{E - E_j + i\epsilon} + \sum_k \frac{\psi_k(Xq'')\psi_k^*(q')}{E - E_k + i\epsilon} = \sum_j \frac{\phi_j(q'')\phi_j^*(q')}{E - E_j + i\epsilon} - \sum_k \frac{\psi_k(q'')\psi_k^*(q')}{E - E_k + i\epsilon},$$

and the corresponding response function is

$$g_X(E) = \int dq' G(Xq', q', E).$$  

(9)

The Green functions for even and odd parity quantum sectors are respectively given by

$$G_e(q'', q', E) = \sum_j \frac{\phi_j(q'')\phi_j^*(q')}{E - E_j + i\epsilon} = \frac{1}{2} \left[ G(q'', q', E) + G(Xq'', q', E) \right],$$

(10)

$$G_o(q'', q', E) = \sum_k \frac{\psi_k(q'')\psi_k^*(q')}{E - E_k + i\epsilon} = \frac{1}{2} \left[ G(q'', q', E) - G(Xq'', q', E) \right].$$

(11)

and the desired response functions in the QM side are finally

$$g_e(E) = \sum_j \frac{1}{E - E_j + i\epsilon} = \frac{1}{2} \left[ g(E) + g_X(E) \right]$$

(12)

$$g_o(E) = \sum_k \frac{1}{E - E_k + i\epsilon} = \frac{1}{2} \left[ g(E) - g_X(E) \right].$$

(13)

Now we derive a semiclassical expression for $g(E)$ and $g_X(E)$, from which $g_e(E)$ and $g_o(E)$ follow. The derivation is almost parallel but one must carefully track down how the difference comes out.

The Green function is a Fourier transform of Feynman's propagator and by a stationary phase approximation it is given by

$$G(q'', q', E) = \left( q'' \right| \frac{1}{E + i\epsilon - \hat{H}} \left| q' \right) \approx \sum_{\Gamma_{cl}} A_{\Gamma} \exp \left( \frac{i}{\hbar} S_{\Gamma}(q'', q', E) - \frac{i\pi\nu_{\Gamma}}{2} \right).$$

(14)

The sum is over all the classical trajectory $\Gamma$ and $S_{\Gamma}(q'', q', E) = \int_{q'}^{q''} pdq$ is its action. The path-cum-trace of the right-hand side calculated by the stationary phase approximation gives a semiclassical approximation for the response function $g(E)$ and $g_X(E)$. Here the difference comes out in
both the linear and quadratic parts of the expansion of $S_\Gamma(q'', q', E)$. Also we must specify the $v_{\Gamma, PO}$ separately.

(i) The linear term:

$g(E)$: The stationary phase condition is

$$
\left. \left( \frac{\partial S_\Gamma(q'', q', E)}{\partial q''} \delta q'' + \frac{\partial S_\Gamma(q'', q', E)}{\partial q'} \delta q' \right) \right|_{q''=q'=0} = (p'' - p') \delta q' = 0. \tag{15}
$$

Thus the dominant trajectories for the response function are periodic orbits.

$g_X(E)$: Condition $q'' = Xq'$ implies $\delta q'' = X\delta q'$; the stationary phase condition becomes

$$
\left. \left( \frac{\partial S_\Gamma(q'', q', E)}{\partial q''} \delta q'' + \frac{\partial S_\Gamma(q'', q', E)}{\partial q'} \delta q' \right) \right|_{q''=Xq'} = (Xp'' - p') \delta q' = 0. \tag{16}
$$

Thus, not only $q'' = Xq'$ but also $p'' = Xp'$ holds for the orbits that contribute to $g_X(E)$. We denote this type of a trajectory as $\Gamma_X$ and call it an $X$-periodic orbit. $\Gamma_X$ is apparently half of a periodic orbit symmetric with respect to the y-axis; see Fig. 1. The action, period, Lyapunov exponent, and the number of conjugate points are all half of the corresponding full orbit. An alternative way of looking at $\Gamma_X$ is that it is a closed periodic orbit in an orbifold space $\mathcal{R}^2/X$ [30].

(ii) The quadratic term:

$g(E)$: For the full periodic orbit the action can be approximated as

$$
S_\Gamma(q', q', E) \approx S_\Gamma(\tilde{q}, \tilde{q}, E) + \frac{1}{2} \left( \frac{\partial^2 S_\Gamma}{\partial q'^2} + 2 \frac{\partial^2 S_\Gamma}{\partial q'_\perp \partial q'_\perp} + \frac{\partial^2 S_\Gamma}{\partial q'^2} \right) (\delta q'_\perp)^2. \tag{17}
$$

for the point $q'$ in the transverse neighborhood of the point $\tilde{q}$ on the periodic orbit ($q' = \tilde{q} + \delta q'_\perp$).

$g_X(E)$: The condition $q'' = Xq'$ implies $\delta q'' = X\delta q'$ and

$$
S_\Gamma(Xq', q', E) \approx S_\Gamma(X\tilde{q}, \tilde{q}, E) + \frac{1}{2} \left( \frac{\partial^2 S_\Gamma}{\partial q'^2} - 2 \frac{\partial^2 S_\Gamma}{\partial q'_\perp \partial q'_\perp} + \frac{\partial^2 S_\Gamma}{\partial q'^2} \right) (\delta q'_\perp)^2. \tag{18}
$$

The negative sign in the middle second derivative term requires the replacement of the hyperbolic sine function by the hyperbolic cosine function in the amplitude factor in the trace formula (1). This important issue will be tested quantitatively in Sect. 3.

(iii) Maslov:

$v_{\Gamma, PO}$: For the AKP periodic orbit, $v_\Gamma$ is twice the length of the orbit ($2N$). Hence $\exp(-i\pi v_\Gamma/2) = 1$.

$v_{\Gamma, X}$: The parent of $\Gamma_X$ is an $X$-symmetric full orbit, for which $N$ must be even, i.e. its length $2N$ is $4n$. Thus, $v_{\Gamma, X} = (1/2)(2 \times (4n)) = 4n$ and $\exp(-i\pi v_{\Gamma, X}/2) = 1^8$.

From this point on, the derivation proceeds perfectly in parallel. The Gaussian integral over $\delta q'_\perp$ yields a determinant factor and the ratio of it to $A_\Gamma$ in (14) merges into the determinant of the monodromy matrix of the periodic orbit. It is solely determined by the Lyapunov exponent of the PO and independent of the choice of $\tilde{q}$. The zeroth-order term in (17) is also (by definition) independent of

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8 If the orbit is symmetric with respect to both the $x$- and $y$-axes, $N$ must be an odd integer. This exception includes, for instance, the fundamental periodic orbit (FPO).
Fig. 1. The $X$-periodic orbit (solid curve) that closes in $\mathcal{R}^2 / X$. This connects $q'$ and $q'' = Xq'$ with $p'' = Xp'$, and its variation (illustrated by a dashed line) with $\delta q'' = X(\delta q')$. Note that $\delta q''_\perp = -\delta q'_\perp$ in the depicted coordinate frames.

$\bar{q} : S_\Gamma(\bar{q}, \bar{q}, E) \equiv S_\Gamma(E)$. Therefore, the longitudinal integral merely produces the period of the orbit as a multiplicative factor, and we obtain an explicit semiclassical formula for the POT side of $g(E)$ and $g_X(E)$:

$$g(E) = \sum_n \frac{1}{E - E_n + i\epsilon} \approx -i \sum_{\Gamma_{PO}} \frac{T_{\Gamma_{PO}}(E)}{2 \sinh(\lambda_{\Gamma_{PO}}/2)} e^{iS_{\Gamma_{PO}}(E)}, \quad (19)$$

$$g_X(E) \approx -i \sum_{\Gamma_X} \frac{T_{\Gamma_X}(E)}{2 \cosh(\lambda_{\Gamma_X}/2)} e^{iS_{\Gamma_X}(E)}. \quad (20)$$

Both are in the dimensionless variables used in the Hamiltonian (3).

2.1.2. $Y$-transformation. In order to manufacture the 3D rotational symmetry around the $x$-axis in terms of the 2D trace formula, we have to select the orbits $\Gamma_Y$ that are closed in the orbifold space $\mathcal{R}^2 / Y$. We call this type of half orbit a $Y$-periodic orbit.

For the $Y$-periodic orbit, $\delta q''_\perp = \delta q'_\perp$, the change of sign seen in (18) does not occur, and there is no need to interchange the hyperbolic functions in the amplitude factor. For the length of the full periodic orbit whose half is $\Gamma_Y$, $N$ is an odd natural number. The number of conjugate points $v_{\Gamma_Y}$ is $2N$ (half of $4N$) with odd $N$. In total we obtain an explicit semiclassical formula for $g_Y(E)$:

$$g_Y(E) \approx i \sum_{\Gamma_Y} \frac{T_{\Gamma_Y}(E)}{2 \sinh(\lambda_{\Gamma_Y}/2)} e^{iS_{\Gamma_Y}(E)}. \quad (21)$$

2.1.3. Combination of $X$- and $Y$-transformations. By considering the $X$- and $Y$-transformations at the same time, we can calculate individually the even and odd response functions under $X$-transformation, both for $m = 0$ subspace as

$$g_e^{m=0}(E) = \frac{1}{4} [g(E) + g_Y(E) + g_X(E) + g_{XY}(E)]$$

$$g_o^{m=0}(E) = \frac{1}{4} [g(E) + g_Y(E) - g_X(E) - g_{XY}(E)]. \quad (22)$$

The new term

$$g_{XY}(E) = \int dq' G(XYq', q', E) \quad (23)$$
in the POT side can be handled in the same manner as above, and we obtain
\[ g_{XY}(E) \approx i \sum_{\Gamma_{XY}} \frac{T_{\Gamma_{XY}}(E)}{2 \cosh(\lambda_{\Gamma_{XY}}/2)} e^{i S_{\Gamma_{XY}}(E)}. \] (24)

Here the orbits \( \Gamma_{XY} \) are closed in the orbifold space \( \mathcal{R}^2/(X \cdot Y) \) and we call these half orbits \( XY \)-periodic orbits. For the length \( 4N \) of the full periodic orbit whose half is \( \Gamma_{XY} \), \( N \) is an odd number. The number of conjugate points \( \nu_{\Gamma_{XY}} \) is \( 2N \) (half of \( 4N \)) with odd \( N \).

2.2. Inverse quantum chaology: use of a variable \( u = 1/\sqrt{-2E} \)

If we follow the trace formula in the direction from POT to QM, the density of state (d.o.s.) will be produced (within the semiclassical approximation) but the interference between the PO contributions necessarily forbids investigation of the contributions of periodic orbits one by one. Instead, in the inverse direction (from QM to POT), we are going to extract information on the periodic orbits from the d.o.s. one by one. For this purpose, we focus on the scaling property of the AKP action. Because the Hamiltonian (3) is homogeneous with respect to coordinates and momenta, the action of AKP has a remarkable scaling property
\[ S_{\Gamma_{PO}}(E) = \frac{1}{\sqrt{-2E}} \Phi_{\Gamma_{PO}}. \] (25)

Here \( \Phi_{\Gamma_{PO}} \) is the value of the action of \( \Gamma_{PO} \) at \( E = -1/2 \) and it is an energy-independent constant distinguishing the periodic orbits. Apart from this factor, all the periodic orbits have the same energy dependence \( \propto 1/\sqrt{-2E} \). From this observation, we introduce a convenient variable
\[ u \equiv \frac{1}{\sqrt{-2E}}. \] (26)

Now we change the variable from \( E \) to \( u \) in both sides of (19). In the QM side, we have simply
\[ \frac{1}{E - E_n + i\epsilon} = \frac{du}{dE} \frac{1}{u - u_n + i\epsilon}. \] (27)

In the POT side, the period is given by \( T(E) = dS(E)/dE \) and we have
\[ T_{\Gamma_{PO}}(E) = \frac{du}{dE} \frac{d}{du} S_{\Gamma_{PO}} = \frac{du}{dE} \Phi_{\Gamma_{PO}}. \] (28)

Because the factor \( du/dE \) in (27) and (28) cancels out, we obtain a concise formula for the response function in terms of \( u \);
\[ g(u) \equiv \sum_n \frac{1}{u - u_n + i\epsilon} \approx -i \sum_{\Gamma_{PO}} \frac{\Phi_{\Gamma_{PO}}}{2 \sinh(\lambda_{\Gamma_{PO}}/2)} e^{iu\Phi_{\Gamma_{PO}}}. \] (29)

Let us consider the Fourier–Laplace transformation from \( u \) to the conjugate variable \( \phi \):
\[ D(\phi) \equiv \int_0^\infty g(u) e^{-i\phi u} du. \] (30)

Then we obtain from (29)
\[ D_{\text{QM}}(\phi) \equiv \sum_{n=1}^\infty e^{-iu_n\phi} \approx D_{\text{POT}}(\phi) \equiv \sum_{\Gamma_{PO}} \frac{\Phi_{\Gamma_{PO}}}{2 \sinh(\lambda_{\Gamma_{PO}}/2)} \delta(\phi - \Phi_{\Gamma_{PO}}). \] (31)

In the POT side, \( D_{\text{POT}}(\phi) \) has delta peaks localized at \( \phi = \Phi_{\Gamma_{PO}} \). Hence there is no interference between contributions from periodic orbits. It is, so to speak, the (weighted) density of the periodic
orbits. On the other hand, in $D_{QM}(\phi)$, the pseudo-energy levels $e^{-iu_n\phi}$ distributed on the unit circle on the complex plane with fluctuations interfere with each other. Equation (31) predicts that $D_{QM}(\phi)$ should have peaks at $\phi = \Phi_{\Gamma_{PO}}$ by this interference. While in (1) it is the POT side that produces peaks by the interference between periodic orbits, in (31) it is the QM side that produces peaks by the interference between pseudo-energy levels. The roles of the left- and right-hand sides are interchanged between the two trace formulas (1) and (31).

In practice, the available energy levels are limited to $n_{\text{max}}$, and we have to introduce a corresponding cut-off $u_{\text{max}} \equiv u^n_{\text{max}}$. Thus we consider

$$D_{\text{cut}}(\phi) = \int_0^\infty \theta(u_{\text{max}} - u) g(u)e^{-iu\phi} du.$$  

(32)

We then obtain

$$D_{QM}^{\text{cut}}(\phi) \equiv \sum_{n=1}^{n_{\text{max}}} e^{-iu_n\phi} \approx D_{POT}^{\text{cut}}(\phi) \equiv \sum_{\Gamma_{PO}} \frac{\Phi_{\Gamma_{PO}}}{2 \sinh(\lambda_{\Gamma_{PO}}/2)} \tilde{\delta}(\Phi_{\Gamma_{PO}} - \phi),$$  

(33)

with a smeared delta function

$$\tilde{\delta}(w) = \frac{1}{2\pi} \frac{e^{iu_{\text{max}}} - 1}{i w}.$$  

(34)

with a peak at $w = 0$ in the real part. The formula (34) provides us with a new test by comparing $D_{QM}^{\text{cut}}(\phi)$ and $D_{POT}^{\text{cut}}(\phi)$ with a freely chosen $n_{\text{max}}$.

If we choose a sufficiently large $n_{\text{max}}$, the interference between peaks in (33) is strongly suppressed. In this situation, the POT side predicts that the peak in $D_{QM}^{\text{cut}}(\phi)$ should appear at $\phi = \Phi_{\Gamma_{PO}}$ for every periodic orbit, and the predicted ($u_{\text{max}}$-dependent) height of the peak of the real part is

$$\frac{\sigma \Phi_{\Gamma_{PO}}}{2 \sinh(\lambda_{\Gamma_{PO}}/2)} u_{\text{max}}^2.$$  

(35)

Here $\sigma$ is the number of periodic orbits that are degenerate at the same $\Phi_{\Gamma_{PO}}$ due to the symmetries of the Hamiltonian. This degeneracy factor $\sigma$ is listed in Table 1 in Ref. [12]. We reproduce it in the upper row of our Table 1. The sequence of the above calculation is

$$g(E) \text{ in (19) } \rightarrow g(u) \text{ in (29) } \xrightarrow{\text{Fourier Tr. in (30)}} D_{POT}(\phi) \text{ in the inverse trace formula (31), and}$$

$$g(E) \rightarrow g(u) \xrightarrow{\text{Fourier Tr. with cut in (32)}} D_{POT}^{\text{cut}}(\phi) \text{ in the inverse trace formula under cut (33).}$$

For the inverse quantum chaology from the quantum subsector $m^\pi = 0^+$ and $m^\pi = 0^-$, we also need $D_X$, $D_Y$, and $D_{XY}$ in POT (see Eqs. (37) and (22)). These can be calculated in just the same scheme starting from $g_X(E)$, $g_Y(E)$, and $g_{XY}(E)$ in (20), (21), and (24), respectively.

As the degeneracy factor $\sigma$ plays a crucial role in the quantitative test of the inverse trace formula (31) (or its practical version (33)) let us briefly discuss it before concluding this subsection. We need to consider three symmetry transformations (acting on the periodic orbits) in two dimensions: $X$-transformation, $Y$-transformation, and time-reversal $T$.

Firstly, let us consider the $X$- and $Y$-transformations. Either of them can produce from a given periodic orbit a partner periodic orbit with the same period and a respectable solution of the Hamilton equations of motion. Now, the key is whether the partner is distinct or not. This depends on the symmetry of the periodic orbit. There are four types of POs:

(i) those which have neither $X$- nor $Y$-symmetry,
This table is prepared for two purposes. The upper two rows show types of periodic orbits and corresponding degeneracy factor $\sigma$ ([12]). The bottom row classifies $2N = 12$ POs (altogether 122 POs) into symmetry types. For instance, there are 57 non-self-retracing periodic orbits that are neither $X$- nor $Y$-symmetric (each has $\sigma = 2^3 = 8$).

|      | Non-self-retracing | Self-retracing |
|------|--------------------|----------------|
| (i)  |                    |                |
| No sym. | 8          | 2  |
| (ii) |                    |                |
| X-sym. | 4           | 4  |
| (iii) |                    |                |
| Y-sym. | 4           | 2  |
| (iv) |                    |                |
| $X,Y$-sym. | 2          | 2  |

(ii) (iii) those which have either $X$-symmetry or $Y$-symmetry but not both, (iv) those which have both $X$- and $Y$-symmetries.

If a periodic orbit is type (i), it is associated with $2 \times 2 = 4$ distinct partners of the same period and we must give a factor four multiplicatively in $\sigma$. Similarly, for types (ii, iii) and (iv) we must give two and one in $\sigma$ respectively.

As for the time-reversal $T$, we should note that there are self-retracing periodic orbits, which retrace themselves after passing the turning point (see, for instance, the orbit in the third inset of Fig. 2). The time-reversal transformation produces from a given periodic orbit its partner rotating in the reverse direction and for this pairing we should, except for the self-retracing type, count a factor of two multiplicatively in $\sigma$. If the orbit is self-retracing, the pairing is immaterial since it only amounts to another choice of the starting point and the multiplicative factor is one.

As a concrete example, let us consider the FPO, which is a variation of the Kepler ellipse under mass anisotropy. It is both $X$- and $Y$-symmetric and hence it is a singlet with respect to these symmetries. As for the time-reversal, it is not self-retracing and hence we must count two for it. Therefore, $\sigma = 1 \times 2$ for the FPO.

As another concrete example, let us consider the maximally degenerate case, $\sigma_{\text{max}} = 2^3 = 8$. This is the case when the orbit has neither $X$, $Y$-symmetry nor the self-retracing property.

3. Results

3.1. The set-up of the inverse quantum chaology in AKP

We choose for the anisotropy $\mu = \sqrt{5}$ (or equivalently $\gamma = 0.2$ in the notation of Refs. [19,29]). This anisotropy is realized by the electron in a silicon semiconductor. At this anisotropy, the electron classical orbit is perfectly chaotic and the unstable periodic orbits are isolated from each other in the classical phase space. The classical theory of AKP at $\mu = \sqrt{5}$ was extensively examined by Gutzwiller and both the action and Lyapunov exponent of periodic orbits with length $2N$ up to ten are tabulated in Ref. [12]. At $2N = 2, 4, 6, 8, 10$ there are respectively $2, 4, 8, 18, 44$ independent periodic orbits; each has its own $\sigma$ according to its symmetries, as discussed in Sect. 2.2.

In order to achieve our test of quantum–classical correspondence, we have extended the Gutzwiller table to include altogether 122 new independent periodic orbits at $2N = 12$. This is sufficient to

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9 The procedure is essentially shooting but with special attention paid to the appearance of flat directions in the chi-squared function, expressing the amount of misfit in the shooting. This flat direction corresponds to the section of the surface of the devil’s staircase over $(X_0, U_0)$ at the height of the binary code of the periodic orbit. See also footnote 6 and Ref. [25].
support the analysis for the region $\phi = 0$–15 (see the discussion in Sect. 3.2 below and Fig. 2). The numbers of periodic orbits with $2N = 12$ subject to the symmetry types (i), (ii), (iii), and (iv) are tabulated in Table 1 along with the degeneracy factor $\sigma$ for each type of PO.

As for the energy levels in the QM side, we have worked out up to around $10^4$ energy levels from the ground state at $\mu = \sqrt[5]{2}$ ($\gamma = 0.2$) using the matrix diagonalization method with Sturmian basis developed by Wintgen, Marxer, and Briggs [29]. This is not a perturbation calculation but rather an exact calculation in the sense that the approximation only comes from the practical limitation of the size $N$ of the (suitably deformed) Hamiltonian matrix. We derived in our earlier work two useful formulas [22,23]; the most optimized value of the scaling parameter $\varepsilon$ involved in the WMB formulation is $\varepsilon \approx -\gamma/4$ and, with this $\varepsilon$,

$$R_{\text{eff}} \equiv \frac{\text{number of reliable energy levels}}{\text{matrix size } N} \approx \sqrt[5]{\gamma} \approx \frac{1}{\mu}. \quad (36)$$

Equation (36) implies that the calculation of higher $\mu$ (lower $\gamma$) becomes increasingly hard. For instance, a large calculation of $10^4$ energy levels, the same number of levels as us, was previously performed at $\gamma = 0.8$ by WMB. But, according to (36), $R_{\text{eff}} = 0.9$ and 0.46 at $\gamma = 0.8$ and 0.2 respectively. Thus we had to use the Hamiltonian matrix with $N$ being roughly twice that of WMB and the necessary CPU memory is about four times their value.$^{10}$

3.2. Quantum–classical correspondence in $D_{\text{QM}}(\phi)$ and $D_{\text{POT}}(\phi)$ and periodic orbits from quantum levels

In Fig. 2 we compare $|D_{\text{QM}}(\phi)|^2$ and $|D_{\text{POT}}(\phi)|^2$ (the latter is flipped vertically for the sake of comparison) for the quantum subsector $m^\pi = 0^+$. (Hereafter we drop the superscript “cut” for simplicity). The three panels show the results at $j_{\text{max}} = 10^3$, $5 \times 10^3$, and $10^4$ respectively. We find that, with the increase of $j_{\text{max}}$, the peaks become higher and the widths become narrower. This trend is common to the QM and POT sides and at every $j_{\text{max}}$ the agreement between $|D_{\text{QM}}(\phi)|^2$ and $|D_{\text{POT}}(\phi)|^2$ is remarkable. This is just as it should be (see Sect. 2.2).

Note that the graphs of $|D_{\text{QM}}(\phi)|^2$ are the result of the sum of the pseudo-energy terms $\exp(-iu_j \phi)$ as given in the QM side (see Eq. (33)). Here, it is checked that the energy levels $E_j$ (recall $u_j \equiv 1/\sqrt{-2E_j}$) are subject to the GOE at this high anisotropy. The statistics, for instance, $\Sigma_2$ and $\Delta_3$, are well described by the GOE [19]. Even knowing that Gutzwiller’s trace formula should give the correct semiclassical description, it is still amazing that the sum of the random $\exp(-iu_j \phi)$s on the unit circle on the complex plane produces by interference the peaks just at the very locations in $\phi$ as the heights and widths just prescribed by the POT side. (For instance, if we plot the distribution of $\exp(-iu_j \phi)$ as a histogram on the circle, we can see no structure in it, and the mock levels created by the Thomas–Fermi formula show no peaks at all on the $\phi$-axis. The fluctuation of the levels creates peaks at the right positions with the correct widths, as expected by the actions and Lyapunov exponents of the periodic orbits.)

Gutzwiller’s trace formula works surprisingly well—even magically. This point may also be stated in a reverse way; it may be said that the QM levels amazingly know information on periodic orbits.

$^{10}$ We have also formulated WMB diagonalization in terms of a tensored 2D harmonic-oscillator function basis in order to study quantum–classical correspondence in the phase space, because the harmonic oscillator basis makes the Husimi function calculation much easier. The comparison between the calculations in two independent bases has in turn given us a guarantee of the accuracy of both calculations. In the new basis, $\varepsilon = \gamma$ and $R_{\text{eff}} = \sqrt[5]{\gamma}/2$ [22,23].
Fig. 2. Comparison of the QM and POT sides of (33) at anisotropy $\mu = \sqrt{5}$ and for the quantum sector $m = 0$, parity even. In each of the three vertical panels with distinct $j_{\text{max}}$, $|D_{\text{QM}}(\phi)|^2$ (upper) and $|D_{\text{POT}}(\phi)|^2$ (lower) are compared after subtracting the Thomas–Fermi contribution to avoid an extraneous peak around $\phi = 0$. Clustering of peaks ($R$) and peak deserts ($D_{1,2,3}$) are observed. The POT side is calculated from 2-dim. orbits ($2N \leq 12$): six full POs (including repetition of FPO) and ten half POs. Peaks due to the former (latter) are connected by solid (dashed) lines perpendicular to the panels. Insets show full POs in the order of increasing action/Phi1/Gamma1, respectively [12]. There are two peaks (indicated by dashed arrows without insets), each of which is produced by two periodic orbits with almost degenerate actions.

Now let us examine Fig. 2 in more detail. The distribution of the peaks in the region $\phi = 0–15$ shown in Fig. 2 has three interesting features:

(i) Peaks usually appear isolated from each other.
(ii) However, a strong accumulation is observed around $\phi \in (8.5, 9.5)$.
(iii) There are three remarkable gaps where no peaks appear.

We explain these features in terms of the POT as follows.

(i) Each of the isolated peaks can be attributed to a corresponding periodic orbit with the same period. In the region $\phi = 0–15$ depicted in the figure, there are altogether 16 isolated peaks. Among these, six peaks are attributed to the full orbits, which are respectively displayed in the insets. The other 10 peaks are attributed to half periodic orbits $\Gamma_X, \Gamma_Y, \Gamma_{XY}$. The agreement between the quantum and classical sides is not only in the positions of the peaks but also in the heights and widths of them. Encouraged by this observation, let us compare the prediction about the actions and Lyapunov exponents given by the quantum levels with the classical data respectively in Figs. 3 and 4.

We observe that the correspondence is extremely accurate for the actions and good for the exponents. This success comes from the fact that we have calculated the weight factor in the
trace formula under a proper formulation of symmetrized Green functions [8]. To our knowledge, this is the first challenge for the determination of Lyapunov exponents from quantum levels.

(ii) Now let us discuss the accumulation of peaks in the region $R \equiv \{ \phi \in (8.5, 9.5) \}$ produced by the energy level data in the QM side. With increasing $j_{\text{max}}$, more peaks are resolved in this
accumulation. This accumulation (the clustering of peaks) produced in the QM side can be succinctly explained from the POT side as follows. In AKP there is a special class of periodic orbits—inﬁnitely many periodic orbits, each evolving along the heavy x-axis with a small-amplitude oscillation in the transverse y-direction. All of the periodic orbits in this class or their half orbits tend to have nearly the same action $\Phi_{\text{PO}}$. The upper bound of these $\Phi_{\text{PO}}$ can be estimated by considering the 1D orbit evolving precisely on the x-axis connecting the two turning points at the kinematical boundary ($x = \pm 2$), i.e., the limit of no oscillation in the y-direction. The action can be calculated analytically and $\Phi_{\text{1dim}} = 2\pi \sqrt{\mu} \simeq 9.396$. Therefore, the accumulation of peaks produced in $\mathcal{R}$ by the QM energy level data is just as predicted from the POT side consideration. Equivalently, we may say that the QM side amazingly knows the accumulation of peaks in $\mathcal{R}$.

(iii) As for the deserts $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ with no peaks at all in the QM side, we have ﬁrst checked explicitly that, consistently, there exist no periodic orbits with the action in these ranges in our collections of periodic orbits up to $2N = 12$. Do any of the periodic orbits with higher lengths come with their actions in these ranges? This is a hard question since it is not tractable to exhaustively search for periodic orbits and examine their actions. But we have two supporting pieces of evidence for the non-existence of such periodic orbits. For one thing, there is a plausible tendency that higher-length orbits come with larger action (i.e. period), and it seems that $2N = 12$ is already the highest length that can give actions in the range $\phi \leq 15$. Better evidence is given by an amazing formula constructed by Gutzwiller for AKP, which approximately gives $\Phi_{\text{PO}}$ from the Bernoulli sequence $(a_i, i = 1, \ldots, 2N)$ of $\Gamma_{\text{PO}}$ [7,8,12]. If this formula, derived from periodic orbits up to $2N = 10$, can also be used for higher-length orbits, we can draw the conclusion that there are no higher-length periodic orbits in the desert regions. In short, the QM side knows not only the accumulation of peaks in $\mathcal{R}$ but also the deserts $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$.

3.3. Verification that the POT side correctly accommodates the symmetries of the QM side
Up to now we have compared $D_{\text{QM}}(\phi)$ and $D_{\text{POT}}(\phi)$ after taking the absolute value squared. Now let us compare in Fig. 5 the real parts of $D_{\text{QM}}(\phi)$ with $D_{\text{POT}}(\phi)$ for the even and odd parity states separately. The latter $D_{\text{POT}}(\phi)$ is calculated by Fourier–Laplace transformation from the symmetrized response functions $g^{m=0}_x(E)$ and $g^{m=0}_y(E)$ in (22). Therefore, for this comparison, the rule of sign factors in the POT side is essential and we can test whether the POT side under the symmetrization in terms of the full and half orbits correctly accommodate the symmetries ($m = 0$ and parity $\pm$) of the QM side. The rule consists of three steps. (We call peaks with positive height “$+$ peaks” and those with negative heights “− peaks”.)

(i) It is easy to see from (20), (21), (24), and (19) that the relative sign of the imaginary part of $g_X(E)$, $g_Y(E)$, and $g_{XY}(E)$ to $g(E)$ is $(+, -, -)$ respectively.

(ii) We see that the $D_{\text{POT}}$ in (33) derived from $g(E)$ by Fourier–Laplace transformation (32) has a positive peak at every $\phi = \Phi_{\text{PO}}$. Combining this with (i), the sign of every peak in $D(\phi)$, $D_X(\phi)$, $D_Y(\phi)$, $D_{XY}(\phi)$ derived from $g_X(E)$, $g_Y(E)$, and $g_{XY}(E)$ is $(+, +, -, -)$ respectively.

11 At least, we have checked that it nicely approximates our new $2N = 12$ data, except for the special class of periodic orbits responsible for the accumulation of peaks in $\mathcal{R}$, discussed in feature (ii) above.
Fig. 5. a) Parity even and $m=0$ sector with $j_{\text{max}} = 10^4$, b) parity odd and $m=0$ sector with $k_{\text{max}} = 10^4$. Real parts of $D_{\text{QM}}(\phi)$ (upper) and $D_{\text{POT}}(\phi)$ (lower) are compared for each sector. The peaks in $\text{Re}D_{\text{POT}}(\phi)$ of full, $X$-, $Y$-, and $XY$-symmetric POs are signified by symbols—circle, X, Y, XY respectively; they appear as, according to trace formulas, $(+, +, -, -)$ for parity even and $(+, -, -, +)$ for parity odd. The peaks in $\text{Re}D_{\text{QM}}(\phi)$ follow perfectly this sign rule and all the heights agree with the POT prediction. The symmetries in the QM side are properly accommodated in the POT orbifolding formulation. For the accumulation region $R[\phi \in (8.5, 9.5)]$, inclusion of POs with $2N \geq 14$ will improve the POT prediction. However, we have been unable to do this, because shooting for these is difficult and also because Gutzwiller’s approximation formula [7, 8, 12] becomes inaccurate for the actions of particular POs (those evolving along the heavy $x$-axis) contributing to $R$.

(iii) Finally, $D_{e}^{m=0}(\phi)$ and $D_{o}^{m=0}(\phi)$ are respectively given by

\[
D_{e}^{m=0}(\phi) = \frac{1}{4} [D(\phi) + D_Y(\phi) + D_X(\phi) + D_{XY}(\phi)]
\]

\[
D_{o}^{m=0}(\phi) = \frac{1}{4} [D(\phi) + D_Y(\phi) - D_X(\phi) - D_{XY}(\phi)]
\]

(37)
corresponding to (22).
Therefore, combining with (ii), we see that the full-, $X$-, $Y$-, and $XY$-symmetric orbits respectively predict $(+, +, -, -)$ peaks for the parity even sector, while they predict $(+, -, -, +)$ peaks for the parity odd sector.

The comparison in Fig. 5 shows that the peaks (and the background) produced by the sum of $\exp(-iu_j\phi)$ in the QM side perfectly agree with those predicted by the POT side with respect to the heights and widths as well as the signs, and also for both the parity even and odd sectors. This succinctly shows that the POT side successfully accommodates the symmetry of the sectors in the QM side.

### 4. Conclusion

In this paper we have shown how accurately Gutzwiller’s trace formula catches the quantum physics of AKP in the completely ergodic region $\mu = 1/\nu = \sqrt{5}$ (or equivalently $\gamma = 0.2$). This in turn has led us to the observation that the information on classical chaotic POs is amazingly embedded in the QM side of AKP; we can extract it by our formulation, in which the classical periodic orbits make peaks with minimal interference in the POT side.

We have called the left- and right-hand sides of the trace formula the QM and POT sides respectively. We followed Gutzwiller’s idea \[8\] and gave trace formulas for the quantum subspaces with definite symmetries.

Restriction to the parity even (odd) sector in 3D AKP quantum physics can be accounted for by the introduction of half periodic orbits $\Gamma_X$ that close by $X$-transformation (periodic in $R^2/X$; see Fig. 1) and taking the symmetrization (antisymmetrization) according to (12) and (13). We remarked there that the response function $g(E)$ (in the POT approximation) has $\sinh(\lambda_{r\text{PO}}/2)$ in the denominator from the contribution of each full periodic orbit $\Gamma_{\text{PO}}$, while $g_X(E)$ has $\cosh(\lambda_{\text{PO}}/2)$ from the contribution of each $\Gamma_X$; see Eqs. (19) and (20). Furthermore, the restriction to the quantum sector with azimuthal quantum number $m = 0$ is realized in the POT side by the introduction of $g_Y(E)$.

The trace formula is usually tested by predicting the energy levels (or d.o.s.) from the POT side. Here the prediction is given under the interference of amplitudes of periodic orbits and hence, in this direction, it is impossible to separately investigate periodic orbits one by one.

Our inverse trace formulas ((31) and its symmetry partners) are basically a Fourier transformation of the usual ones but with a one-shot idea of using the scaling property of the AKP action by the introduction of a convenient variable $u = 1/\sqrt{-2E}$. They extract the pieces of classical periodic orbit information one by one from the quantum level data.

The new test of quantum–classical correspondence is done in Figs. 2 and 5 based on (31), and we have shown that the information on periodic orbits, including the Lyapunov exponents, can indeed be extracted from the quantum energy levels in Figs. 3 and 4.

The QM side of (33) is the sum of the sequence $\exp(-iu_j\phi)$ where $u_j$ is determined from the energy levels—the eigenvalues of the Schrödinger equation. We have worked out the energy levels with sweat, but, by looking at them, it is hardly ever understood how the interference between $\exp(-iu_j\phi)$ makes up the peaks at the prescribed position by the periodic orbits. In particular, the fluctuation of $E_j$ is essential for this make up; if we substitute the Thomas–Fermi approximation for the levels (the averaged $E_j$), the peaks in $D_{\text{QM}}(\phi)$ disappear instantly. It is remarkable that the intricate fluctuation can produce, via the inverse trace formula, periodic orbits that are the solution of a classical Hamilton equation of motion with Coulomb singularity.
One interesting challenge in inverse quantum chaology may be the test whether, in the diamagnetic Kepler problem, it extracts from quantum energy levels the salient distinction between the Lyapunov exponents of the POs that are longitudinal and transverse to the field direction [31].

At the anisotropy $\mu = 1/\nu = \sqrt{5}$ (or $\gamma = 0.2$), the classical phase space is filled by isolated unstable periodic orbits. We have undertaken this work as the first step toward the most ambitious study of the intermediate region around $(\mu \simeq 1.1, \gamma \simeq 0.8)$ between chaos and regularity, where the classical phase space is deemed to be dominated by the web of chaos [20,32]. We have already calculated the $10^4$ energy levels at every $\gamma$ in the interval $\gamma \in (0.2, 0.85)$ and increment 0.005 for the QM side. On the other hand, we have observed that POs gradually disappear with the increase of $\gamma$. We are going to investigate how the trace formula (31) reflects this situation.

The intermediate region of AKP is a fascinating region in that it is conjectured that the wave functions are multifractal under a deep connection to the Anderson localization [20,22,23,32,33]. We will challenge this region in a forthcoming paper.

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