Dynamics of invariant solutions of mKdV-ZK arising in a homogeneous magnetised plasma

Raj Kumar
Veer Bahadur Singh Purvanchal University

Ravi Shankar Verma (✉ rsverma747@gmail.com)
Veer Bahadur Singh Purvanchal University https://orcid.org/0000-0001-9294-6511

Research Article

Keywords: Modified Korteweg de Vries-Zakharov-Kuznetsov, Solitons, Lie symmetry reduction

Posted Date: March 10th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1411278/v1

License: ☀️ This work is licensed under a Creative Commons Attribution 4.0 International License. Read Full License

Version of Record: A version of this preprint was published at Nonlinear Dynamics on April 4th, 2022. See the published version at https://doi.org/10.1007/s11071-022-07389-4.
Dynamics of invariant solutions of mKdV-ZK arising in a homogeneous magnetised plasma

Raj Kumar · Ravi Shankar Verma

Abstract  Classical Lie symmetry analysis is proposed to get similarity reductions of a nonlinear (3+1)-modified Korteweg de Vries-Zakharov-Kuznetsov equation. The equation is often used to control the type of weakly nonlinear ion-acoustic waves in a homogeneous magnetised electron-positron plasma. In a magnetised plasma including some hot and cold ions, such waves exist. Some more general types of analytical solutions to the equation are derived in this article. Through their animation profiles, the dynamics of electrostatic potential indicate elastic single soliton to multi solitons, elastic multi solitons, kink to stationary, bell-shaped to asymptotic. The solutions are confirmed by comparing the existing results.

Keywords  Modified Korteweg de Vries-Zakharov-Kuznetsov · Solitons · Lie symmetry reduction

1 Introduction:

Fully ionised gases with particles of equal and opposite charge and mass are referred to as pair plasmas. Such paired (Electron-positron) plasmas play a significant role in the cooling process and creation of elements of the early universe [1]. The Big Bang, active galactic nuclei, gamma ray bursts, pulsar magnetospheres, and the Solar atmosphere all contribute to the early universe’s slow cooling. High-energy particles are accelerated along the pulsar magnetic field and release curvature photons, which generate new electron-positron couples in the case of pulsars.

It has been also a key concern of many researchers to get the exact solutions of physical systems inherently nonlinear in nature, which are usually governed by nonlinear partial differential equations (NPDEs). It has become a matter of deep attention since these exact solutions are helpful broadly to explain the inner mechanism/behaviour of such complex phenomena governed by these NPDEs arising in numerous fields, like viscoelasticity, solid state, kinematics, magnetized plasma in space, study of coastal waves in ocean solid mechanics, optical fibers, fluid ions, signal processing, astrophysics, electromagnetic waves, biomedicine, fluid mechanics, etc. [2-37]. Pluralistic development of nonlinear science has resulted in the last five decades due to the use of many effective methods and techniques for getting the analytical solutions of NPDEs, taking into account different issues, approximations, assumptions, merits, and demerits of the methods. There is no unique technique to apply to all NPDEs. Therefore, it is important to choose appropriate and established techniques for a problem.

In the ongoing research, we investigate the modified Korteweg de Vries-Zakharov-Kuznetsov (mKdV-ZK) (1) equation to get its analytical solutions. The mKdV-ZK controls the nature of weakly nonlinear ion-acoustic waves in magnetized electron-positron plasma, consisting of the some hot and cold ions [2]. A complete overview of the historical background of mKdV-ZK (1) is presented in Abdullah et al. [3], and Lazarus et al. [4]. The mKdV-ZK can be calculated using the following approximations [5] or conditions [4] in a four-component paired (electron-positron) plasma that is homogeneously magnetised and consists of cold electrons and positrons with identical temperatures and equilibrium densities.

(i) The density of hot electrons (positrons) can be represented in the form of the electrostatic potential since hot isothermal species have a Boltzmann distribution.

(ii) The dynamics of the cooler adiabatic species, follows the continuity equations, equations of motion, and adiabatic pressure equations.

(iii) Such system is closed by the Poisson’s equation.

(iv) Linearization of (i)-(iii) yields some dispersion relations for paired plasmas.

(v) Using nonlinear modes by introducing stretched coordinates and then approximating fluid velocity, density, pressure, and electrical potential with a small parameter, e.g. \( \epsilon \), one can get Poisson’s equation to order \( \epsilon^{\frac{5}{2}} \) and the continuity, momentum, and pressure equations to order \( \epsilon^{2} \), which leads to the following mKdV-
ZK equation:

\[ u_t + a u^2 u_x + b u_{xxx} + c u_{xyy} + c u_{zzz} = 0, \quad (1) \]

where \( u \) is the electrostatic potential or voltage in an \( x, y, z \) frame at time \( t \), \( a \) is a dispersion coefficient, \( b \) and \( c \) are real constants that appear due to the finite-amplitude effects of first-order disturbances. The values of these depend upon initial fluid velocity along \( x \)-direction, thermal velocity of cool component, common mass of electron/positron, charge at hot species, Debye lengths, phase velocity, gyro-frequency, plasma frequency and phase velocity [4]. The mKdV-ZK is renamed as extended KdV-ZK [6] when second term is considered as \( a u u_x \). In a two-component electron-ion plasma, the ion-acoustic wave, which is an ion time-scale event, has been explored, as well as the associated linear [7] and nonlinear behaviour [8, 9].

The mKdV-ZK (1) is obtained by applying the singular perturbation method to a mathematical model whose governing equations explain the dynamics of cooler adiabatic species [10]. This model represents the wave propagation in a three-dimensional homogeneous magnetised, electron-positron plasma, containing equal amount of cool and hot ions. Abdullah et al. [3] employed modified extended mapping method to the mKdV-ZK equation (1), and obtained periodic, kink and antikink, bright, dark soliton solutions. Solitary wave solutions such as electrostatic field potentials, electric fields, magnetic fields, and quantum statistical pressures were obtained by them [3].

Some of the two-dimensional properties of solitary waves arising in the Earth’s auroral area at high altitude were explained by Mace and Hellberg [11]. An aurora is a natural light that can be seen in the sky and is most commonly seen in the Arctic and Antarctic regions of the Earth. Auroras produce beautiful light patterns that seem like curtains, spirals, or dynamic flickers that cover the entire sky. Auroras are created by solar wind-induced instabilities in the magnetosphere. Charged particles, primarily electrons and protons, whose trajectories are altered by these disturbances, precipitate in the thermosphere and exosphere, i.e., in the magnetospheric plasma. Ionization and excitation of atmospheric components provide a wide range of colour and complexity in light. The amount of acceleration imparted to the precipitating particles determines the shape of the aurora, which occurs in bands around both polar regions. Auroras can be found on most of the planets in the Solar System, as well as some natural satellites, brown dwarfs, and even comets. Many of the observed waveforms for KdV-ZK can be attributed to magnetised [11] and unmagnetized planar and ellipsoidal soliton with high time resolution electric and magnetic field measurements.

An interesting literature overview related to the solutions of mKdV-ZK is presented to know more about it like this equation is a mutual combination of the mKdV and ZK equations. For \( a = \alpha \), and \( b = c = 1 \) in (1), the equation governs the oblique propagation of electrostatic modes in magnetized plasmas and has been re-derived in a fully systematic way for general mixtures of hot isothermal, warm adiabatic fluid, and cold immobile background species [5, 12].

Khaliq and Adeyemo [13] explored the Lie symmetry reduction method to find a non-topological soliton solution by solving generalized KdV-ZK equation of the form

\[ u_t + a u^2 u_x + b u_{xxx} + c u_{xyy} + c u_{zzz} = 0. \quad (2) \]

Bibi et al. [14] employed the \( \frac{G'}{G} \)-expansion method to the following form of (3+1)-time fractional KdV-ZK equations

\[ D_t^\alpha u + \epsilon u D_x^\beta u + \eta D_x^2 u + \nu D_x^3 u = 0, \]

\[ t > 0, 0 \leq \alpha, \beta \leq 1, \]

where \( D_t^\alpha \) is used as a modified Riemann-Liouville derivative [14], and \( \epsilon, \eta, \nu \) are constants. They [14] have attained trigonometric and rational type solutions, while Jin et al. [15] attained the dark solitons for the following space-time fractional mKdV-ZK equation by using the fractional complex transform with undetermined coefficients.

\[ D_t^\alpha u + d u^2 u_x + e u_{xxx} + f u_{xyy} + g u_{zzz} = 0, \]

\[ t > 0, 0 < \alpha \leq 1, \]

where \( d, e, f, g \) are constants. They [15] have taken Riemann-Liouville fractional order derivatives. For \( \alpha = 1, f = g \) it becomes the same mKdV-ZK (1).

Moreover, Ali Akbar et al. [16] used rational \( \frac{G'}{G} \)-expansion method to obtain hyperbolic, trigonometric, and rational form solutions of the following form of space-time fractional KdV-ZK equations.

\[ D_t^\alpha u + \epsilon u^2 D_x^2 u + D_x^{2\alpha} u + D_x^{2\alpha} D_y^{2\alpha} u + D_x^{2\alpha} D_z^{2\alpha} u = 0, \]

\[ \epsilon \neq 0, 0 < \alpha \leq 1, \]

where fractional derivatives are considered as Riemann-Liouville’s fractional order [16] and in the meaning of conformable derivative. Abdelrahman [17] investigated a similar form (5) of fractional KdV-ZK and used the Riccati-Bernoulli Sub-ODE approach to find its trigonometric solutions.

Islam et al. [18] used the enhanced \( \frac{G'}{G} \)-expansion method, and Lu et al. [2] used the extended \( \frac{G'}{G} \)-expansion method to solve mKdV-ZK (1) and derived its trivial solutions, while for \( a = \alpha \), and \( b = c = 1 \) in Eq. (1), the mKdV-ZK is investigated by Zhang [12] employing Jacobi elliptic function expansion method and derived travelling wave solutions.

Lie group symmetry method is employed by Saahoo et al. [19] to Eq. (1) treating \( b = c = 1 \) therein, and obtained the trivial form of the solutions without considering the effect of all variables \( x, y, z \) and \( t \) because they [19] have solved to Eq. (1) taking directly the six infinitesimal generators. The modified extended
direct algebraic method by Lu et al. [6] and the modified simple equation method is employed by Khan and Ali Akbar [20] for the same form of the mKdV-ZK and claimed to get some new exact solutions, while the auxiliary equation method is used to get traveling wave solutions of (1) by Tariq and Seadawy [21]. Khalique and Adeyemo [13] solved extended form of KdV-ZK via Lie symmetry reductions and Kudryashov’s method and non–topological soliton, cnoidal and snoidal periodic solutions were found to be possible.

In this paper, the authors investigate the Lie symmetry analysis to derive new solutions to mKdV-ZK (1). Some of the results existing in [6, 13, 18–20] can be derived from this research. More over, the solutions in the existing results [2, 6, 12, 13, 18–21] are different from the derived in this research.

The rest of the framework of this article is arranged as follows: In section 2, the authors recall some basic steps to generate Lie symmetries. The analytical solutions are derived by using the Lie symmetry in sections 3. Comparison with existing analytical solutions are performed in section 4, while in section 5, the solutions are depicted physically via their animation profiles. Conclusions with future scope of the present research appear in section 6.

2 Lie symmetries

This section depicts the steps to generate Lie symmetries of the (3 + 1)-dimensional mKdV-ZK equation (1). The first step of this process is to construct the invariant condition. Secondly, the invariance reduces the number of independent variables in the existing mKdV-ZK. Repeated use of such a process finally gives an ODE. Authors have obtained the solution to the mKdV-ZK equation after obtaining its solution. These solutions are appearing in terms of explaining physical nature and depicting its behaviour with respect to changes in space and time. Due to its nonlinear nature, the mKdV-ZK equation is not easily solvable to get analytical solutions. Thus, the generation of similarity forms via Lie symmetries leads to the reduction of an equivalent PDE with one fewer number of independent variables. With repeated use of Lie symmetry reduction, the (3 + 1)-mKdV-ZK reduces to an ODE, which is much easier to solve. The brief description and applications of the method can be studied from some text books [22, 23] and references therein [18–37].

Assuming one-parameter Lie group of the following transformations

\[ x^* = x + \epsilon \xi^{(1)}(\chi, t) + O(\epsilon^2), \]

\[ y^* = y + \epsilon \xi^{(2)}(\chi, t) + O(\epsilon^2), \]

\[ z^* = z + \epsilon \xi^{(3)}(\chi, t) + O(\epsilon^2), \]

\[ t^* = t + \epsilon \tau(\chi, t) + O(\epsilon^2), \]

\[ u^* = u + \epsilon \eta_u(\chi, t) + O(\epsilon^2), \]

where the notation \((\chi, t)\) denotes \((x, y, z, u, t)\) and \(\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \tau\) and \(\eta_u\) are the infinitesimals among which the each one is a function of \((\chi, t)\).

The generator \(V\) associated with the one-parameter transformations can be explored as

\[ V = \xi^{(1)} \frac{\partial}{\partial x} + \xi^{(2)} \frac{\partial}{\partial y} + \xi^{(3)} \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial t} + \eta_u \frac{\partial}{\partial u}. \]

(7)

Lie symmetry keeps the Eq. (1) invariant satisfying the condition:

\[ P_{r}^{(3)} V [u_t + a u^2 u_x + b u_{xxx} + c u_{yy} + c u_{zzz}] = 0, \]

(8)

where the third prolongation \(P_{r}^{(3)}\) (termed in [23]), can be expressed as

\[ P_{r}^{(3)} V = V + [\eta_y] \frac{\partial}{\partial (u_t)} + [\eta_x] \frac{\partial}{\partial (u_x)} + [\eta_{xxx}] \frac{\partial}{\partial (u_{xxx})} + [\eta_{yy}] \frac{\partial}{\partial (u_{yy})} + [\eta_{zzz}] \frac{\partial}{\partial (u_{zzz})}. \]

After using Eq. (1) into (8), and eliminating one of the partial derivative, from the invariance condition (8), it turns into a PDE which provides the following determining system of the equations:

\[ \xi^{(1)}_y = \xi^{(1)}_t = \xi^{(1)}_u = \xi^{(2)}_u = \xi^{(2)}_t = 0, \]

\[ \xi^{(3)}_x = \xi^{(3)}_u = \xi^{(3)}_y = \tau_x = \tau_y = \tau_z = \tau_u = \tau_t = 0, \]

\[ 3 \xi^{(1)}_x = 3 \xi^{(2)}_x = 3 \xi^{(3)}_u = \tau_x, \xi^{(2)}_z = -\xi^{(3)}_u, \]

\[ 3 \eta_u + \eta_t = 0. \]

(9)

After solving them, the authors obtained the following infinitesimals:

\[ \xi^{(1)} = a_{1} x + a_{2}, \xi^{(2)} = ay + a_{3} z + a_{4}, \]

\[ \xi^{(3)} = -a_{3} y + a_{1} z + a_{5}, \tau = 3 a_{1} t + a_{6}, \]

\[ \eta_u = -a_{1} u. \]

(10)

The constants \(a_i\) (i = 1, 2, 3, 4, 5, 6) are arbitrary. The infinitesimals derived here are same as calculated in Saloo et al. [19] but they did not get even a single solution to Eq. (1) involving the effect of all variables \(x, y, z, t\) on \(u\).

In the following section, the authors have taken care of it and used classical Lie symmetry approach considering with more general choices of arbitrary constants \(a_i\) to get more variety of analytical solutions.

Now, the six dimensional Lie-algebra \(L^6\) can be generated by

\[ X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + 3 \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, X_2 = \frac{\partial}{\partial x}, \]

\[ X_3 = \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, X_4 = \frac{\partial}{\partial y}, X_5 = \frac{\partial}{\partial z}, X_6 = \frac{\partial}{\partial t}. \]

(11)

The commutator table for the Lie-symmetry algebra \(L^6\) is given in table 1.
The generators derived here are same as calculated in Sahoo et al. [19]. They [19] have solved to Eq. (1) taking directly using the six infinitesimals to start with Lagrange’s characteristic equations. Consequently, they obtained the very particular form of the solutions.

3 Invariant solutions

Authors can utilize the characteristic equation with the following Lagrange system for the proposed Eq. (1)

\[
\frac{dx}{a_1 x + a_2} = \frac{dy}{a_1 y + a_3 z + a_4} = \frac{dz}{a_3 y + a_1 z + a_5} = \frac{dt}{3a_1 t + a_6} = \frac{du}{a_1 u - a_6}.
\]

Now, to proceed further, authors can consider the following cases:

**Case (I)** For most general case, one should take \(a_1 \neq 0\) in Eq. (12) to reduce it via Lie symmetry, then the corresponding Lagrange’s characteristic equation recasts as

\[
\frac{dx}{x + a_1} = \frac{dy}{y + a_2 z + a_3} = \frac{dz}{-a_2 y + z + a_4} = \frac{dt}{3t + a_5} = \frac{du}{u - a_6}.
\]

For the similarity variables \(X = (3t + a_5)^{-\frac{1}{3}} x, Y = (3t + a_5)^{-\frac{1}{3}} (y^2 + z^2)^{\frac{1}{2}}, Z = (3t + a_5)^{-\frac{1}{3}} (x^2 + y^2 + z^2)^{\frac{1}{2}},\) it gives the similarity function as \(u = (3t + a_5)^{\frac{1}{3}} F(X, Y, Z),\) and \(a_1 = a_2, a_2 = a_3, a_3 = a_4, a_4 = a_5, a_5 = a_6, a_6.

Then, the first similarity reduction for mKdV-ZK (1) yields

\[
byZ^3 F_{XXX} + 3bXY Z^3 F_{XZZ} + cY Z^3 F_{YY} + 2cY^2 Z^5 F_{YX} + cX Y^3 F_{ZZ} + (cY^2 Z^3 + 3bX^2 Y^3) F_{XZ} + (bX Y^3 Z^2 + cX Y^3 Z^2) F_{XZ} + 3(cX Y^2 Z^3 + X^2 Z^5) F_{XZ} + 2(bX^3 Z^2 + aX Y^3 Z^2 + cX Y^3 Z^2) F_{XZ} = 0.
\]

Employing Lie symmetry to (13), all infinitesimals are zero, and a trivial solution \(F = 0\) is obtained for it, and hence \(u = u_1 = 0\) is not beneficially. The authors are intended to consider another case as:

**Case (II)** Taking \(a_1 = a_3 = 0, a_6 \neq 0\) in Eq. (12), one can get

\[
\frac{dx}{A_1} = \frac{dy}{A_2} = \frac{dz}{A_3} = \frac{dt}{dt}.
\]

It gives the following similarity form

\[
X = x - A_1 t, Y = y - A_2 t, Z = z - A_3 t, u = F(X, Y, Z),
\]

and \(A_1 = \frac{a_2}{a_6}, A_2 = \frac{a_4}{a_6}, A_3 = \frac{a_5}{a_6}.

Therefore, similarity reduction of the system (1) yields

\[
bF_{XXX} + cF_{YY} + cF_{ZZ} + aF^2 - A_1 F - A_2 F_Y - A_3 F_Z = 0.
\]

Again, applying the similarity reduction, the infinitesimals are as follows:

\[
\xi_X = a_7, \quad \xi_Y = a_8, \quad \xi_Z = a_9, \quad \eta_F = 0.
\]

The corresponding characteristic equation for (15) is

\[
\frac{dX}{a_7} = \frac{dY}{a_8} = \frac{dZ}{a_9} = \frac{dF}{0}.
\]

If \(a_9 \neq 0,\) then \(X_1 = X - A_1 Z, Y_1 = Y - A_2 Z\) are similarity variables for the similarity function \(F = G(X_1, Y_1),\) where \(B_1 = \frac{a_2}{a_9}, B_2 = \frac{a_4}{a_9}.

Therefore, further similarity reduction of Eq. (14) provides

\[
(b + B_7^2) G_{X_1 X_1} + 2bB_4 B_2 G_{X_1 Y_1} + c(1 + B_7^2) G_{X_1 Y_1} + (aG^2 - A_1 + A_3 B_1) G_{X_1} + (A_3 B_2 - A_2) G_{Y_1} = 0.
\]

Again, applying the similarity which yields

\[
\xi_{X_1} = a_{10}, \quad \xi_{Y_1} = a_{11}, \quad \eta_{G} = 0.
\]

Thus, Lagrange’s characteristic equation for (17) is

\[
\frac{dX_1}{a_{10}} = \frac{dY_1}{a_{11}} = \frac{dG}{0}.
\]

**Case (IIa)**: For \(a_{11} \neq 0\) in Eq. (18), then

\[
X_2 = X_1 - E_1 Y_1, G = H(X_2),\text{ where } E_1 = \frac{a_{10}}{a_{11}}\text{ gives Lie similarity reduction of the system (16) as}
\]

\[
B_3 H'' + B_4 H' + aH^2 = 0,
\]

where \(B_3 = b + cE_1^2 + cB_7^2 + cB_2 E_1 - 2bB_1 E_1, B_4 = aE_1 - A_1 + A_3 B_1 - A_3 B_2 E_1\) and Integrating

\[
B_3 H'' + B_4 H + \frac{a}{3} H^3 = C_1.
\]

Its integration gives

\[
(H')^2 + \frac{a}{6B_3} H^4 + \frac{B_4}{B_3} H^2 = C_3 H + C_4.
\]
Case (IIa) : For $B_3 = -a^5$, $B_4 = \sqrt{\frac{a}{3}}$, $C_3 = 0$, $C_4 = 1$, the solutions for mKdV-ZK (1) are

$$u_2 = 6^{1/4} a \tan \left[ \frac{x - E_{1} y + (B_{2} E_{1} - B_{1}) z + (\sqrt{3} a^{3}/\sqrt{3}) t}{6^{1/4} a} \right] + C_5,$$

and

$$u_3 = 6^{1/4} a \cot \left[ \frac{x - E_{1} y + (B_{2} E_{1} - B_{1}) z + (\sqrt{3} a^{3}/\sqrt{3}) t}{6^{1/4} a} \right] + C_6. \quad (23)$$

Case (IIa2) : If $B_4 = C_3 = C_4 = 0$, then solution for Eq. (1) is

$$u_4 = -6 \left[ \frac{x - E_{1} y + (B_{2} E_{1} - B_{1}) z}{(6 a B_3 - 6 C_7)} \right] \sqrt{\frac{a}{6}}, \quad (24)$$

Case (IIa3) : Setting $B_3 = -1$, $B_4 = -\frac{a}{6}$, $C_3 = C_4 = 0$, the solutions can be expressed as

$$u_5 = \sec \left[ \frac{x - E_{1} y + (B_{2} E_{1} - B_{1}) z - (a/6) t}{\sqrt{\frac{a}{6}}} \right] + C_8,$$

and

$$u_6 = \csc \left[ \frac{x - E_{1} y + (B_{2} E_{1} - B_{1}) z - (a/6) t}{\sqrt{\frac{a}{6}}} \right] + C_9. \quad (25)$$

Case (IIa4) : For $B_3 = -1$, $B_4 = -\frac{a^3}{3}$, $C_3 = 0$, $C_4 = \frac{a}{6}$, the solutions are

$$u_7 = \tanh \left[ C_{10} \pm \left( x - E_{1} y + (B_{2} E_{1} - B_{1}) z \right) - (a^3/3) t \right] \sqrt{\frac{a}{6}}, \quad (27)$$

$$u_8 = \coth \left[ C_{11} \pm \left( x - E_{1} y + (B_{2} E_{1} - B_{1}) z \right) - (a^3/3) t \right] \sqrt{\frac{a}{6}}, \quad (28)$$

Table 2: Adjoint table for (1)

| Ad  | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |
|-----|-------|-------|-------|-------|-------|-------|
| $X_1$ | $X_1$ | $e^{+} X_2$ | $X_3$ | $e^{+} X_4$ | $e^{+} X_5$ | $e^{+} X_6$ |
| $X_2$ | $e^{-} X_2$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |
| $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_4 - e X_5 - \frac{a}{3} X_4 + O(e^2)$ | $X_5 + e X_4 - \frac{a}{3} X_5 + O(e^2)$ | $X_6$ |
| $X_4$ | $X_1 - e X_4$ | $X_2$ | $X_3 + e X_5$ | $X_4$ | $X_5$ | $X_6$ |
| $X_5$ | $X_1 - e X_5$ | $X_2$ | $X_3 - e X_4$ | $X_4$ | $X_5$ | $X_6$ |
| $X_6$ | $X_1 - 3e X_6$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |

where $C_3 = \frac{2C_1}{B_3}$ and $C_4 = \frac{2C_2}{B_3}$ are integration constants.

To solve it further, it is imperative to split the cases as:

Case (IIa5) : Taking $B_3 = 1$, $B_4 = -\frac{a}{6}$, $C_3 = C_4 = 0$, then solution is

$$u_{10} = \text{sech} \left[ \left( x - E_{1} y + (B_{2} E_{1} - B_{1}) z - (a/6) t \right) \sqrt{\frac{a}{6}} \right] + C_{13}. \quad (30)$$

Case (IIa6) : For $B_3 = -1$, $B_4 = -\frac{a}{6}$, $C_3 = C_4 = 0$, the solution is given by

$$u_{11} = \text{csch} \left[ \left( x - E_{1} y + (B_{2} E_{1} - B_{1}) z + (a/6) t \right) \sqrt{\frac{a}{6}} \right] + C_{14}. \quad (31)$$

Case (IIb) : $a_{10} \neq 0$ in Eq. (18) recasts $Y_2 = Y_1 - E_2 X_1 + C = P(Y_2)$, where $E_2 = \frac{a_{11}}{a_{10}}$.

Therefore, similarity reduction of the system (16) provides

$$B_5 P'' + B_6 P' - a_2 E_2^2 P' = 0, \quad (32)$$

where $B_5 = cE_2 - bE_3^2 - cB_1^2 E_2^3 - cB_2 E_2^2 + 2cB_1 B_2 E_2^2$, $B_6 = A_1 E_2 - A_2 - A_1 B_1 E_2 + A_3 B_2$. Its direct integration results

$$B_5 P'' + B_6 P - \frac{a_2 E_2^2}{3} P^3 = C_{15}, \quad (33)$$

Integrating

$$(P')^2 - \frac{a_2 E_2^2}{6B_5} P^4 + \frac{B_6}{B_5} P^2 = C_{17} P + C_{18}, \quad (34)$$

where $C_{17} = \frac{2C_{15}}{B_5}$ and $C_{18} = \frac{2C_{16}}{B_5}$ are integration constants.

Case (IIb1) : For $B_6 = -\frac{a}{3}$, $C_{17} = 0$, $C_{18} = \frac{a}{6B_5}$, $E_2 = 1$, the solutions for mKdV-ZK (1) are

$$u_{12} = \tan \left[ \left( -x + y + (B_1 - B_2) z - (a/3) t \right) \sqrt{\frac{a}{6B_5}} \right] + C_{19}, \quad (35)$$
\[ u_{13} = \cot \left[ \left( -x + y + (B_1 - B_2)z - (a/3)t \right) \sqrt{\frac{a}{6}B_5} + C_{20} \right]. \] (38)

**Case (IIb):** Assuming \( B_0 = C_{17} = C_{18} = 0, \) the solution is
\[ u_{14} = - \frac{6}{\left( -E_2x + y + (B_1E_2 - B_2)z \right) \sqrt{\frac{aE_2}{6B_5}} - 6C_{21}}. \] (37)

**Case (IIb):** Taking \( B_6 = \frac{aE_2}{6}, \) \( C_{17} = C_{18} = 0, \) the solutions for Eq. (1) are
\[ u_{15} = \sec \left[ \left( -E_2x + y + (B_1E_2 - B_2)z \right) + (aE_2/6)t \right] \sqrt{\frac{aE_2}{6B_5}} + C_{22}, \] (38)
\[ u_{16} = \csc \left[ \left( -E_2x + y + (B_1E_2 - B_2)z \right) + (aE_2/6)t \right] \sqrt{\frac{aE_2}{6B_5}} + C_{23}. \] (39)

**Case (IIb):** For \( B_5 = 1, \) \( B_6 = 2a, \) \( C_{17} = 0, \) \( C_{18} = a, \) \( E_2 = 6, \) the solution for the mKdV-ZK Eq. (1) can be obtained as
\[ u_{17} = \tanh \left[ \left( -6x + y + (6B_1 - B_2)z + 2at \right) \sqrt{a} + C_{24} \right]. \] (40)
\[ u_{18} = \coth \left[ \left( -6x + y + (6B_1 - B_2)z + 2at \right) \sqrt{a} + C_{25} \right], \] (41)
\[ u_{19} = \frac{2}{1 - \exp \left( 2 \sqrt{a}(-6x + y + (6B_1 - B_2)z + 2at + C_{26}) \right)} \] (42).

**Case (IIb):** By setting \( B_6 = -\frac{a}{6}, \) \( E_2 = -1, \) \( C_{17} = C_{18} = 0, \) the solution can be expressed as
\[ u_{20} = \sech \left[ \left( \frac{1}{6}(-E_2x + y + (B_1E_2 - B_2)z) \right) - (a/6)t \right] \sqrt{\frac{a}{B_5}} + C_{27}. \] (43)

**Case (IIb):** For \( B_0 = -\frac{a}{6}, \) \( E_2 = 1, \) \( C_{17} = C_{18} = 0, \) the solution for mKdV-ZK Eq. (1) is
\[ u_{21} = \csch \left[ \left( \frac{1}{6}(-E_2x + y + (B_1E_2 - B_2)z) \right) - (a/6)t \right] \sqrt{\frac{a}{B_5}} + C_{28}. \] (44)

**Case (IIc):** If \( a_{10} = 0 \) is taken in Eq. (18), then similarity reduction of the system (16) with \( G = R(X_1) \) yields
\[ B_8R'' + aR^2R' + B_7R' = 0, \] (45)
where \( B_7 = A_3B_1 - A_1, \) \( B_8 = b + B_1^2c \) and integrating
\[ B_8R'' + \frac{a}{3}R^3 + B_7R = C_{29}. \] (46)

Integrating
\[ (R')^2 + \frac{a}{6B_8}R^4 + \frac{B_7}{B_8}R^2 = C_{31}R + C_{32}. \] (47)
where \( C_{31} = \frac{2C_{29}}{B_8} \) and \( C_{32} = \frac{2C_{30}}{B_8} \) are integration constants.

To solve Eq. (47), some particular solutions are explored among the following cases:

**Case (IIc1):** By setting \( B_7 = \frac{a}{3}, \) \( B_8 = -1, \) \( C_{31} = 0, \) \( C_{32} = \frac{a}{6}, \) the solutions for the Eq. (1) are
\[ u_{22} = \tan \left[ \left( x - B_1z + (a/3)t \right) \sqrt{\frac{a}{6}} + C_{33} \right], \] (48)
\[ u_{23} = \cot \left[ \left( x - B_1z + (a/3)t \right) \sqrt{\frac{a}{6}} + C_{34} \right]. \] (49)

**Case (IIc2):** For \( B_8 = -1, \) \( B_7 = C_{31} = C_{32} = 0, \) the solution for mKdV-ZK (1) is
\[ u_{24} = - \frac{6}{\sqrt{6a(x - B_1z)} - 6C_{35}}. \] (50)

**Case (IIc3):** Assuming \( B_7 = -\frac{a}{6}, \) \( B_8 = -1, \) \( C_{31} = C_{32} = 0, \) the solutions are
\[ u_{25} = \sec \left[ \left( x - B_1z - (a/6)t \right) \sqrt{\frac{a}{6}} + C_{36} \right], \] (51)
\[ u_{26} = \csc \left[ \left( x - B_1z - (a/6)t \right) \sqrt{\frac{a}{6}} + C_{37} \right]. \] (52)

**Case (IIc4):** For \( B_7 = -\frac{2a}{6}, \) \( B_8 = -1, \) \( C_{31} = 0, \) \( C_{32} = \frac{a}{6}, \) the solutions can be obtained as
\[ u_{27} = \tanh \left[ \left( x - B_1z - (a/3)t \right) \sqrt{\frac{a}{6}} + C_{38} \right], \] (53)
\[ u_{28} = \coth \left[ \left( x - B_1z - (a/3)t \right) \sqrt{\frac{a}{6}} + C_{39} \right], \] (54)
\[ u_{29} = \frac{2}{1 - \exp \left( 2(x - B_1z - (a/3)t + C_{40}) \sqrt{\frac{a}{6}} \right)} - 1. \] (55)

**Case (IIc5):** By taking \( B_7 = -\frac{a}{6}, \) \( C_{31} = C_{32} = 0, \) the solution can be found as
\[ u_{30} = \sech \left[ \left( \frac{1}{6}(x - B_1z - (a/6)t) \right) \sqrt{\frac{a}{B_8}} + C_{41} \right]. \] (56)

**Case (IIc6):** For \( B_7 = \frac{a}{6}, \) \( B_8 = -1, \) \( C_{31} = C_{32} = 0, \) the solution for mKdV-ZK (1) is
\[ u_{31} = \csch \left[ \left( x - B_1z + (a/6)t \right) \sqrt{\frac{a}{B_8}} + C_{42} \right]. \] (57)
Table 3: Comparison with existing solutions [6, 13, 18–20].

| Conditions used | Existing results | Solution in this article |
|-----------------|------------------|--------------------------|
| \( C_9 = 0, C_{14} = 0, C_8 = 0 \) | Eqs. (42), (43), (44) in Lu et al. [6] | \( u_6, u_{11}, u_5 \) respectively |
| \( w(k + s) = \frac{a}{5}, C = C_{13}, \alpha = a, \beta = -\frac{1}{2}, \gamma = 0 \) | Eq. (18) in Khalique and Adeyemo [13] | \( u_{10} \) |
| \( C_{38} = 0 \) | Solutions of family 2 for \( \mu < 0 \) in Islam et al. [18] | \( u_7 \), and \( u_8 \) are more general |
| \( C_{38} = 0 \) | Eq. (22) of Sahoo et al. [19] | \( u_{27} \) |
| \( C_{11} = 0, C_{10} = 0, C_{25} = 0, C_{24} = 0 \) | Eqs. (3.53), (3.54), (3.57), (3.58) of Khan and Akbar [20] | \( u_8, u_7, u_{18}, u_{17} \) respectively |

4 Comparison with existing analytical solutions

Table 3 shows that some of the known results [6, 13, 18–20] can be deduced from the solutions found in this study, demonstrating the originality of the findings. The same problem has been discussed in the literature [2, 3, 5, 6, 10, 12, 18–21], but Sahoo et al. [19] used the same approach to derive a very particular form of the solutions from a very specific form of mKdV-ZK (1) when \( b = c = 1 \). Eq. (22) is missing \( y \), whereas Eq. (27) is missing \( z \), and Eq. (43) is missing both \( y \) and \( z \). They [19] did not solve completely to Eq. (31). Other solutions are diametrically opposed to previous findings.

5 Analysis and discussions

The physical behaviour of analytical solutions represented by Eqs. Eqs. (22)-(31), (35)-(44), (48)-(57) is described in this section. Trigonometric, hyperbolic, rational, and exponential forms are investigated in solutions. The animation is produced using symbolic computations in MATLAB to provide a solution profile. With respect to space and time, there is a variation in \( u \) (electrostatic potential). As a result, the authors have captured the dominating behaviour of a frame of animation and displayed it in Figs. 1–4. Because analytical solutions contain arbitrary constants, the authors have given the arbitrary constants an appropriate value in order to depict the dynamics of profiles. Authors took account the space range \(-20 \leq x \leq 20 \) and \(-20 \leq y \leq 20 \), as well as arbitrary constants and parameters such as \( a = 0.9706, B_1 = 0.9572, B_2 = 0.4854, E_1 = 0.8003, \) kept \( z = 0.1576 \) (fix) for all profiles plotted therein. Similar profiles are not repeated.

**Figure 1:** The behavior of electrostatic potential for \( u_2 \) varies from single soliton to multi solitons during \( 0 \leq t \leq 40 \). The arbitrary constant \( C_5 = 0.4218 \) is used to simulate the animation of \( u_2 \). The physical behaviours of others \( u_3, u_5, u_6, u_{12}, u_{13}, u_{15}, u_{16}, u_2, u_{23}, u_{25}, \) and \( u_{26} \), are analogues to \( u_2 \).

**Figure 2:** Elastic multi solitons can be seen in Fig. 2, for the solution \( u_4 \). As the space range for \( x \) and \( y \) expands, the potential function \( u_4 \) decreases. \( C_7 = 0.6324 \) is held constant in the simulation. Potential behaviour remains the same when the time range is extended. The profiles for \( u_6, u_9, u_{11}, u_{14}, u_{18}, u_{19}, u_{21}, u_{24}, u_{28}, u_{29}, \) and \( u_{31} \) are similar to \( u_4 \).

**Figure 3:** The shape of the profile changes from kink to stationary at time \( 0 \leq t \leq 282 \) and with constant \( C_{10} = 0.9706 \), which is shown in figure 3 and the variation in potential \( u_7 \). The nature of the other profiles, \( u_{17} \) and \( u_{27} \), is the same as \( u_7 \).

**Figure 4:** The shape of the figure changes from bell-shaped to the asymptotic profile of the solutions \( u_{10} \). The potential increases at \( 0 \leq t \leq 182 \) with \( C_{13} = 0.9722 \). Other profiles, \( u_{20} \) and \( u_{30} \), are of the same nature as \( u_{10} \).
Fig. 1. Elastic single soliton to multisolitons variation during $0 \leq t \leq 40$ in the electrostatic potential $u_2$.

Fig. 2. Elastic multi solitons profile of potential $u_4$.

(a) Kink at $t = 0$.

(b) Stationary at $t = 282$.

Fig. 3. Kink to stationary behaviour of the $u_7$ at $t = 282$.

(a) Bell-shaped at $t = 0$.

(b) Asymptotic at $t = 182$.

Fig. 4. Bell-shaped to asymptotic at $t = 182$ for $u_{10}$.
Dynamics of invariant solutions of mKdV-ZK arising in a homogeneous magnetised plasma

Compliance with ethical standards

The authors followed all ethical standard as per journal’s guidelines.

Data availability

Authors used MATLAB to trace animation of profiles. No data is taken from any outside source.

Conflicts of interest

Authors have no known conflicts of interest associated with this publication.

6 Conclusions

In this paper, authors have obtained successfully more invariant solutions of (3+1)-modified Korteweg de Vries-Zakharov-Kuznetsov equation by using classical Lie symmetry analysis. Solutions are represented mathematically by Eqs. (22)-(31), (35)-(44), and (48)-(57), which are analyzed physically. Consequently, their profiles confirm single soliton to multi solitons, elastic multi solitons, kink to stationary, bell-shaped to asymptotic nature. The equation can be used to control the structure of weakly nonlinear ion-acoustic waves in a homogeneous magnetised electron-positron plasma. In a magnetised plasma including some hot and cold ions, such waves exist. If equations (13) and (21) can be solved in a more general form than this study, they can expand the scope of future research.

References

1. Rees, M.J.: The Very Early Universe (ed. G. W. Gibbons, S. W. Hawking and S. Siklas). Cambridge: Cambridge University Press 1983
2. Lu, D., Seadawy, A., Yaro, D.: Analytical wave solutions for the nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov and two-dimensional Kadomtsev-Petviashvili-Burgers equations. Results Phys. 12, 2164-2168 (2019). https://doi.org/10.1016/j.rinp.2019.02.049
3. Abdullah, Seadawy, A.R., Wang, J.: Modified KdV-Zakharov-Kuznetsov dynamical equation in a homogeneous magnetised electron-positron-ion plasma and its dispersive solitary wave solutions. Pramana-J. Phys. 91, 26-39 (2018). https://doi.org/10.1007/s12043-018-1595-0
4. Lazarus, I.J., Bharuthram, R., Hellberg, M.A.: Modified Korteweg-de Vries-Zakharov-Kuznetsov solitons in symmetric two-temperature electron-positron plasmas. J. Plasma Physics. 74(4), 519-529 (2008)
5. Verheest, F., Mace, R.L., Pillay, S.R., Hellberg, M.A.: Unified derivation of Korteweg-de Vries-Zakharov-Kuznetsov equations in multi species plasmas. J. Phys. A: Math. Gen. 35, 795-806 (2002)
6. Lu, D., Seadawy, A.R., Arshad, M., Wang, J.: New solitary wave solutions of (3+1)-dimensional nonlinear extended Zakharov-Kuznetsov and modified KdV-Zakharov-Kuznetsov equations and their applications. Results Phys. 4, 1–11 (2017). https://doi.org/10.1016/j.rinp.2017.02.002
7. Ichimaru, S.: Basic Principles of Plasma Physics. A Statistical Approach Benjamin, New York 1973
8. Davidson, R.C.: Methods in Nonlinear Plasma Theory. Academic, New York, 1972
9. Truemann, R.A., Baumjohann, W.: Advanced Space Plasma Physics. Imperial College, London 1997
10. Seadawy, A.R.: Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in a magnetized electron-positron plasma. Physica A. 455, 44–51 (2016). https://doi.org/10.1016/j.physa.2016.02.061
11. Mace, R.L., Hellberg, M.A.: The Korteweg-de Vries-Zakharov-Kuznetsov equation for electron-acoustic waves. Phys. Plasmas. 8, 2649-2658 (2001). https://doi.org/10.1063/1.1363665
12. Zhang, Z.-Y.: Jacobi elliptic function expansion method for the modified Korteweg-de Vries Zakharov-Kuznetsov and the Hirota equations. Rom. Journ. Phys. 60, 1384–1394 (2015)
13. Khalique, C.M., Adeyemo, O.D.: A study of (3+1)-dimensional generalized Korteweg-de Vries-Zakharov-Kuznetsov equation via Lie symmetry approach. Results Phys. 18, 103197–103206 (2020)
14. Bibi, S., Mohyud-Din, S.T., Ullah, R., Ahmed, N., Khan, U.: Exact solutions for STO and (3+1)-dimensional KdV-ZK equations using G'/G2-expansion method. Results Phys. 7, 4434–4439 (2017)
15. Jin, Q., Xia, T., Wang, J.: The exact solution of the space-time fractional modified KdV-Zakharov-Kuznetsov equation. J. Appl. Math. Phys. 5, 844–852 (2017)
16. Ali Akbar, M., Mohd. Ali, N.H., Tarikul Islam, M.: Multiple closed form solutions to some fractional order nonlinear evolution equations in physics and plasma physics. AIMS Math. 4(3), 397–411 (2019). https://doi.org/10.3934/math.2019.3.397
17. Abdelrahman, M.A.E.: A note on Riccati-Bernoulli Sub-ODE method combined with complex transform method applied to fractional differential equations. Nonlinear Eng. 7(4), 279–285 (2018). https://doi.org/10.1515/neng-2017-0145
18. Islam Md, H., Khan, K., Ali Akbar, M., Salam, Md. A.: Exact traveling wave solutions of modified KdV-Zakharov-Kuznetsov equation and vis-
cous Burgers equation. SpringerPlus. 3, 105–114 (2014). https://doi.org/10.1186/2193-1801-3-105

19. Sahoo, S., Garai, G., Saha Ray, S.: Lie symmetry analysis for similarity reduction and exact solutions of modified KdV-Zakharov-Kuznetsov equation. Nonlinear Dyn. 87(3), 1995–2000 (2016). https://doi.org/10.1007/s11071-016-3169-3

20. Khan, K., Akbar, M.A.: Exact and solitary wave solutions for the Tzitzeica–Dodd–Bullough and the modified KdV-Zakharov-Kuznetsov equations using the modified simple equation method. Ain Shams Eng. J. 4, 903–909 (2013). https://doi.org/10.1016/j.asej.2013.01.010

21. Tariq, K.U.-H., Seadawy, A.R.: Soliton solutions of (3+1)-Dimensional Korteweg-deVries Benjamin-Bona-Mahony, Kadomtsev-Petviashvili Benjamin-Bona-Mahony and modified Korteweg deVries-Zakharov-Kuznetsov equations and their applications in water waves. J. King Saud Univ. Sci. 31, 8–13 (2017)

22. Bluman, G.W., Cole, J.D.: Similarity Methods for Differential Equations. Springer, New York 1974

23. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York 1993

24. Ito, M.: An extension of nonlinear evolution equation of the K-dV (mK-dV) type to higher orders. J. Phys. Soc. Japan 49(2), 771–778 (1980)

25. Kumar, R., Kumar, M., Tiwari, A.K.: Dynamics of some more invariant solutions of (3+1)-burgers system. Int. J. Comput. Methods Eng. 22(3), 225–234 (2021)

26. Kumar, M., Tiwari, A.K.: Some group-invariant solutions of potential Kadomtsev-Petviashvili equation by using Lie symmetry approach. Nonlinear Dyn. 92(2), 781–792 (2018)

27. Kumar, M., Tiwari, A.K.: On group-invariant solutions of Konopelchenko-Dubrovsky equation by using Lie symmetry approach. Nonlinear Dyn. 94 (1), 475–487 (2018)

28. Tanwar, D.A., Kumar, M.: Lie symmetries, exact solutions and conservation laws of the Date-Jimbo-Kashiwara-Miwa equation, Nonlinear Dyn. 106(4), 3453–3468 (2021)

29. Kumar, S., Kumar, A.: Lie symmetry reductions and group invariant solutions of (2+1)-dimensional modified Veronese web equation. Nonlinear Dyn. 98(3), 1891–1903 (2019)

30. Jadaun, V., Kumar, S.: Lie symmetry analysis and invariant solutions of (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation. Nonlinear Dyn. 93(2), 349–360 (2018)

31. Kumar, A., Kumar, M., Kumar, R.: Some more invariant solutions of (2+1)-water waves. Int. J. Appl. Comput. Math. 7(18), 1–17 (2021). https://doi.org/10.1007/s40819-020-00945-9.

32. Kumar, R., Kumar, A.: Dynamical behavior of similarity solutions of CKOEs with conservation law. Appl. Math. Comput. 422, 126976(1-18) (2022)

33. Kumar, R., Verma, R.S., Tiwari A.K.: On similarity solutions to (2+1)-dispersive long-wave equations. J. Ocean Eng. Sci. (2021). https://doi.org/10.1016/j.joes.2021.12.005.

34. Demontis, F.: Exact solutions of the modified Korteweg-De Vries equation. Theor. Math. Phys. 168(1), 866–897 (2011)

35. Aslan, I.: Exact solutions of a Fractional-type differential-difference equation related to discrete MKdV equation. Commun. Theor. Phys. 61, 595–599 (2014)

36. Devanandhan, S., Singh, S. V., Lakhina, G. S., Bharuthram, R.: Small amplitude electron acoustic solitary waves in a magnetized super thermal plasma. Commun Nonlinear Sci Numer Simul. 22(3), 1322–1345 (2014)

37. Sahoo, S., Ray, S.S.: Improved fractional sub-equation method for (3+1)-dimensional generalized fractional KdV-Zakharov-Kuznetsov equations. Comput. Math. with Appl. 70, 158–166 (2015)