Construction of frames for shift-invariant spaces

Stevan Pilipović, Suzana Simić

2000 Mathematics Subject Classification: 42C15, 42C40, 42C99, 46B15, 46B35, 46B20

Key Words and Phrases: $p$-frame; Banach frame; weighted shift-invariant space.

Abstract

We construct a sequence $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, i = 1, \ldots, r\}$ which constitutes a $p$-frame for the weighted shift-invariant space

$$V_p^\mu(\Phi) = \left\{ \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}} c_i(j) \phi_i(\cdot - j) \mid \{c_i(j)\}_{j \in \mathbb{Z}} \in \ell_p^\mu, i = 1, \ldots, r \right\}, \quad p \in [1, \infty],$$

and generates a closed shift-invariant subspace of $L_p^\mu(\mathbb{R})$. The first construction is obtained by choosing functions $\phi_i, i = 1, \ldots, r$, with compactly supported Fourier transforms $\hat{\phi}_i, i = 1, \ldots, r$. The second construction, with compactly supported $\phi_i, i = 1, \ldots, r$, gives the Riesz basis.

1 Introduction and preliminaries

The shift-invariant spaces $V_p^\mu(\Phi), \quad p \in [1, \infty]$, quoted in the abstract, are used in the wavelet analysis, approximation theory, sampling theory, etc. They have been extensively studied in recent years by many authors [2]–[19]. The aim of this paper is to construct $V_p^\mu(\Phi), \quad p \in [1, \infty], \quad \mu$ spaces with specially chosen functions $\phi_i, \quad i = 1, \ldots, r$, which generate its $p$-frame. These results expand and correct the construction obtained in [20]. For the first construction, we take functions $\phi_i, \quad i = 1, \ldots, r$, so that the Fourier transforms are compactly supported smooth functions. Also, we derive the conditions for the collection $\{\phi_i(\cdot - k) \mid k \in \mathbb{Z}, i = 1, \ldots, r\}$ to form a Riesz basis for $V_p^\mu(\Phi)$. We note that the properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in $V_p^\mu(\Phi)$ [22]. We generalize these results for a shift-invariant subspace of $L_p^\mu(\mathbb{R}^d)$. The second construction is obtained by choosing compactly supported functions $\phi_i, \quad i = 1, \ldots, r$. In this way, we obtain the Riesz basis.

This paper is organized as follows. In Section 2 we quote some basic properties of certain subspaces of the weighted $L^p$ and $\ell^p$ spaces. In Section 3 we derive the conditions for the functions of the form $\hat{\phi}_i(\xi) = \theta(\xi + k_i \pi), \quad k_i \in \mathbb{Z},$
We consider the weighted function spaces $L^p$ which constitute a $i$-ω basis by using compactly supported functions $\varphi_i$. In Section 2 we construct a sequence $\{\varphi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 0, \ldots, r\}$, where $r \in 2\mathbb{N}$ or $r \in 3\mathbb{N}$, which constitutes a p-frame for the weighted shift-invariant space $V_p^\mu(\Phi)$. Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust in the sense of [2]. In Section 3 we construct p-Riesz basis by using compactly supported functions $\varphi_i, i = 1, \ldots, r$.

## 2 Basic spaces

Let a function $\omega$ be nonnegative, continuous, symmetric, submultiplicative, i.e., $\omega(x + y) \leq \omega(x)\omega(y), x, y \in \mathbb{R}^d$, and let a function $\mu$ be $\omega$-moderate, i.e., $\mu(x + y) \leq C\omega(x)\mu(y), x, y \in \mathbb{R}^d$. Functions $\mu$ and $\omega$ are called weights. We consider the weighted function spaces $L_p^\mu$ and the weighted sequence spaces $\ell_p^\mu(\mathbb{Z}^d)$ with $\omega$-moderate weights $\mu$ (see [20]). Let $p \in [1, \infty)$. Then (with obvious modification for $p = \infty$)

$$
L_p^\mu = \left\{ f \mid \|f\|_{L_p^\mu} = \left( \int_{[0,1]^d} \left( \sum_{j \in \mathbb{Z}^d} |f(x + j)|\omega(x + j) \right)^p dx \right)^{1/p} < +\infty \right\},
$$

$$
\ell_p^\mu := \left\{ f \mid \|f\|_{\ell_p^\mu} = \left( \sum_{j \in \mathbb{Z}^d} \sup_{x \in [0,1]^d} |f(x + j)|^p\omega(j)^p \right)^{1/p} < +\infty \right\}.
$$

In what follows, we use the notation $\Phi = (\varphi_1, \ldots, \varphi_r)^T$. Define $\|\Phi\|_{\mathcal{H}} = \sum_{i=1}^r \|\varphi_i\|_{\mathcal{H}}$, where $\mathcal{H} = L_p^\mu, L_p^\omega$ or $\ell_p^\mu, p \in [1, \infty]$. With $\mathcal{F}\varphi = \hat{\varphi}$ we denote the Fourier transform of the function $\varphi$, i.e., $\hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x)e^{-i\pi x \cdot \xi} \, dx, \xi \in \mathbb{R}^d$.

The concept of a p-frame is introduced in [2]:

It is said that a collection $\{\varphi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\}$ is a p-frame for $V_p^\mu(\Phi)$ if there exists a positive constant $C$ (dependant upon $\Phi, p$ and $\omega$) such that

$$
C^{-1}\|f\|_{L_p^\mu} \leq \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x)\varphi_i(x - j) \, dx \right\}_{j \in \mathbb{Z}^d} \right\|_{\ell_p^\mu} \leq C\|f\|_{L_p^\mu}, \quad f \in V_p^\mu(\Phi).
$$

Recall [1] that the shift-invariant spaces are defined by

$$
V_p^\mu(\Phi) := \left\{ f \in L_p^\mu \mid f(\cdot) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} c_{ij} \varphi_i(\cdot - j), \quad \{c_{ij}\}_{j \in \mathbb{Z}^d} \in \ell_p^\mu, i = 1, \ldots, r \right\}.
$$
Remark 2.1. [21] Let \( \Phi \in W^1_\omega \) and let \( \mu \) be \( \omega \)-moderate. Then \( V^p_\mu(\Phi) \) is a subspace (not necessarily closed) of \( L^p_\mu \) and \( W^p_\mu \) for any \( p \in [1, \infty] \). Clearly (2.1) implies that \( \ell^p_\mu \) and \( V^p_\mu(\Phi) \) are isomorphic Banach spaces.

Let \( \Phi = (\phi_1, \ldots, \phi_r)^T \). Let
\[
\hat{\Phi}, \hat{\Phi}|(\xi) = \left[ \sum_{k \in \mathbb{Z}^d} \hat{\phi}_i(\xi + 2k\pi)\hat{\phi}_j(\xi + 2k\pi) \right]_{1 \leq i \leq r, 1 \leq j \leq r},
\]
where we assume that \( \hat{\phi}_i(\xi)\hat{\phi}_j(\xi) \) is integrable for any \( 1 \leq i, j \leq r \). Let \( A = [a(j)]_{j \in \mathbb{Z}^d} \) be an \( r \times \infty \) matrix and \( A^T \). Then
\[
\text{rank } A = \text{rank } A^T.
\]
We will recall some results from [2] and [20] which are needed in the sequel.

Lemma 2.1 ([2]). The following statements are equivalent.
\begin{enumerate}
\item \( \text{rank } [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}^d} \) is a constant function on \( \mathbb{R}^d \).
\item \( \text{rank } [\hat{\Phi}, \hat{\Phi}](\xi) \) is a constant function on \( \mathbb{R}^d \).
\item There exists a positive constant \( C \) independent of \( \xi \) such that
\[
C^{-1}[\hat{\Phi}, \hat{\Phi}](\xi) \leq [\hat{\Phi}, \hat{\Phi}](\xi) \leq C[\hat{\Phi}, \hat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d.
\]
\end{enumerate}

The next theorem ([20]) derives necessary and sufficient conditions for an indexed family \( \{\phi_i(· - j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\} \) to constitute a \( p \)-frame for \( V^p_\mu(\Phi) \), which is equivalent with the closedness of this space in \( L^p_\mu \). Thus, it is shown that under appropriate conditions on the frame vectors, there is an equivalence between the concept of \( p \)-frames, Banach frames and the closedness of the space they generate.

Theorem 2.1 ([20]). Let \( \Phi = (\phi_1, \ldots, \phi_r)^T \in (W^1_\omega)^r, \ p_0 \in [1, \infty], \) and let \( \mu \) be \( \omega \)-moderate. The following statements are equivalent.
\begin{enumerate}
\item \( V^{p_0}_\mu(\Phi) \) is closed in \( L^{p_0}_\mu \).
\item \( \{\phi_i(· - j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\} \) is a \( p_0 \)-frame for \( V^{p_0}_\mu(\Phi) \).
\item There exists a positive constant \( C \) such that
\[
C^{-1}[\hat{\Phi}, \hat{\Phi}](\xi) \leq [\hat{\Phi}, \hat{\Phi}](\xi) \leq C[\hat{\Phi}, \hat{\Phi}](\xi), \quad \xi \in [-\pi, \pi]^d.
\]
\item There exist positive constants \( C_1 \) and \( C_2 \) (depend on \( \Phi \) and \( \omega \)) such that
\[
C_1\|f\|_{L^{p_0}_\mu} \leq \inf_{f = \sum_{i=1}^r \phi_i e^i} \sum_{i=1}^r \|c^{j}_{e^i}\|_{\ell^{p_0}_\mu} \leq C_2\|f\|_{L^{p_0}_\mu}, \quad f \in V^{p_0}_\mu(\Phi).
\]
\end{enumerate}

(2.2)
v) There exists $\Psi = (\psi_1, \ldots, \psi_r)^T \in (W_\omega^1)^r$, such that

$$f = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(-j) \rangle \psi_i(-j) = \sum_{i=1}^r \sum_{j \in \mathbb{Z}^d} \langle f, \phi_i(-j) \rangle \psi_i(-j), \ f \in V^p_\mu(\Phi).$$

**Corollary 2.1** (20). Let $\Phi = (\phi_1, \ldots, \phi_r)^T \in (W_\omega^1)^r$, $p_0 \in [1, \infty]$, and let $\mu$ be $\omega$-moderate.

i) If $\{\phi_i(-j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\}$ is a $p_0$-frame for $V^p_\mu(\Phi)$, then the collection $\{\phi_i(-j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\}$ is a $p$-frame for $V^p_\mu(\Phi)$ for any $p \in [1, \infty]$.

ii) If $V^p_\mu(\Phi)$ is closed in $L^p_\mu$ and $W^p_\mu$, then $V^p_\mu(\Phi)$ is closed in $L^p_\mu$ and $W^p_\mu$ for any $p \in [1, \infty]$.

iii) If (2.2) holds for $p_0$, then it holds for any $p \in [1, \infty]$.

## 3 Construction of frames using a compactly supported smooth function

Considering the length of the support of a function $\theta$ and defining a function $\Phi$ in an appropriate way using $\theta$, we have different cases for the rank of the matrix $[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ described in Theorem 3.1.

First, we consider the next claim:

Let $\theta \in C^\infty_0(\mathbb{R})$ be a positive function such that $\theta(x) > 0$, $x \in A$, $A \subset [-\pi, \pi]$, and $\text{supp} \, \theta \subset [-\pi, \pi]$. Moreover, let

$$\hat{\phi}_k(\xi) = \theta(\xi + k\pi), \quad k \in \mathbb{Z},$$

and $\Phi = (\phi_1, \phi_{i+1}, \ldots, \phi_{i+r})^T$, $i \in \mathbb{Z}$, $r \in \mathbb{N}$.

Then the rank of the matrix $[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

As a matter of fact, by the Paley-Wiener theorem, $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_\mu(\mathbb{R})$, $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$, the matrix $[\hat{\phi}_2(\xi + k\pi)](\xi) = \sum_{j \in \mathbb{Z}} |\theta(\xi + k\pi + 2j\pi)|^2$, $\xi \in \mathbb{R}$, has the rank 0 or 1, depending on $\xi$. Moreover, we have $[\hat{\phi}_2(\xi + k\pi)](\xi) > 0$. Because of that, the rank of the matrix $[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

**Theorem 3.1.** Let $\theta \in C^\infty_0(\mathbb{R})$ be a positive function such that $\theta(x) > 0$, $x \in (-\pi - \varepsilon, \pi + \varepsilon)$, and $\text{supp} \, \theta = [-\pi - \varepsilon, \pi + \varepsilon]$, where $0 < \varepsilon < 1/4$. Moreover, let

$$\hat{\phi}_i(\xi) = \theta(\xi + k_i\pi), \quad k_i \in \mathbb{Z}, \ i = 1, 2, \ldots, r, \ r \in \mathbb{N},$$
and \( \Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \).

1) If \(|k_2 - k_1| = 2\) and \(|k_i - k_j| \geq 2\) for different \(i, j \leq r\), then the rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is a constant function on \( \mathbb{R} \) and equals \( r \).

2) If \(|k_2 - k_1| = 2\) and, at least for \(k_{i_1}\) and \(k_{i_2}\), it holds that \(|k_{i_1} - k_{i_2}| = 1\), where \(1 \leq i_1, i_2 \leq r\), then the rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is a non-constant function on \( \mathbb{R} \).

Proof. By the Paley-Wiener theorem, \( \phi_i \in S(\mathbb{R}) \subset W^1(\mathbb{R})\), \(i = 1, \ldots, r\). All supporting cases are described in the following lemmas.

**Lemma 3.1.** Let \( \Phi = (\phi_{k_1}, \phi_{k_2})^T\), \(k_2 - k_1 = 2\), \(k_1, k_2 \in \mathbb{Z}\). The rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is a constant function on \( \mathbb{R} \) and equals 2.

Proof. We have the next two cases.

1° If \(\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi), \ell \in \mathbb{Z}\), for the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) we obtain a \(2 \times \infty\) matrix

\[
\begin{bmatrix}
\cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\
\cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots 
\end{bmatrix},
\]

for some \(0 < a^i, b^i \leq 1, i = 1, 2\). It is obvious that \(\text{rank}[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2\), \(\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi), \ell \in \mathbb{Z}\).

2° For \(\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi], \ell \in \mathbb{Z}\), there are only two non-zero values \(a^1\) and \(a^2\) which are in different columns of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \). Since

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & 0 & a^1 & 0 & \cdots \\
\cdots & 0 & a^2 & 0 & 0 & \cdots 
\end{bmatrix},
\]

it has the rank 2 for all \(\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi], \ell \in \mathbb{Z}\).

We conclude that the rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} , \Phi = (\phi_{k_1}, \phi_{k_2})^T, k_2 - k_1 = 2, k_1, k_2 \in \mathbb{Z}\), is a constant function on \( \mathbb{R} \) and equals 2.

\[\square\]

**Lemma 3.2.** The rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is not a constant function on \( \mathbb{R} \) if \( \Phi = (\phi_{k_1}, \phi_{k_2})^T, k_2 - k_1 = 1, k_1, k_2 \in \mathbb{Z}\).

Proof. In the same way, as in the proof of the Lemma 3.1, we have four different cases for the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \). Without losing generality, let us suppose that \(k_1 \in 2\mathbb{Z}\).

1° If \(\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi), \ell \in \mathbb{Z}\), then we have

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & a^1 & b^1 & 0 & \cdots \\
\cdots & 0 & a^2 & 0 & 0 & \cdots 
\end{bmatrix}, \ 0 < a^1, a^2, b^1 \leq 1,
\]

and \(\text{rank}[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2\), for all \(\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi), \ell \in \mathbb{Z}\).
2° For $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, non-zero values $a^1$ and $a^2$ are in the same column of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$. For any choice of a $2 \times 2$ matrix, we get that the determinant equals 0. So, we obtain

$$\text{rank} \begin{bmatrix} \cdots & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & \cdots \end{bmatrix} = 1,$$

for all $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

3° If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then the matrix

$$\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & b^2 & a^2 & 0 & \cdots \end{bmatrix},$$

for some $0 < a^1, a^2, b^2 \leq 1$, has the rank 2, for all $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

4° For $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, there are two non-zero values $a^1$ and $b^2$ in different columns of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$ and the block with these elements determines the rank 2 for all $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

Considering all cases, we conclude that the rank of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$, $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 1$, $k_1, k_2 \in \mathbb{Z}$, depends on $\xi \in \mathbb{R}$ and equals 1 or 2. This rank is a non-constant function on $\mathbb{R}$.

**Proof of Theorem 3.2**

1) Using Lemma 3.1 and Lemma 3.2, it is obvious that if $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then the position of the first non-zero element in each row of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$ is unique for each row. So, the rank of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals $r$ for all $\xi \in \mathbb{R}$.

2) If $|k_2 - k_1| = 2$ and, at least for $k_{i_1}$ and $k_{i_2}$, it holds that $|k_{i_1} - k_{i_2}| = 1, 1 \leq i_1, i_2 \leq r$, then, in the row with the index $i_2$ (suppose, without losing generality, that $i_2 \in 2\mathbb{Z} + 1$) we will have a new column with a non-zero element for $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, but for $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, the positions of all non-zero elements in that row will already appear in the previous columns. It is obvious that the rank of the matrix $\hat{\Phi}(\xi + 2j\pi)|_{j \in \mathbb{Z}}$ depends on $\xi \in \mathbb{R}$ and is not the same for all $\xi \in \mathbb{R}$.

As a consequence of Theorem 2.1 and Theorem 3.1, we have the following result.

**Theorem 3.2.** Let the functions $\theta$ and $\Phi$ satisfy all the conditions of Theorem 2.1). Then the space $V_p^\mu(\hat{\Phi})$ is closed in $L_p^\mu$ for any $p \in [1, \infty]$ and the family $\{\phi_i(-j) \mid j \in \mathbb{Z}, 1 \leq i \leq r\}$ is a p-Riesz basis for $V_p^\mu(\hat{\Phi})$ for any $p \in [1, \infty]$.

The following theorem is a generalisation of Theorem 3.1 and can be proved in the same way, so we omit the proof.
Proof. Note that ξ rank of the matrix ᾱMoreover, let constant function on \( R \) where \( c, d \) and \( Φ = (\hat{A}, \hat{B}) \) the matrix has a constant rank and equals 1.

Theorem 3.3. Let \( θ ∈ C_0^∞(R) \) be a positive function such that \( θ(x) > 0, x ∈ A, A ⊂ [a, b], b > a \), and supported by \([a, b]\) where \( b − a > 2π \). Moreover, let

\[
\hat{φ}_i(ξ) = θ(ξ + k_iπ), \quad k_i ∈ Z, \quad i = 1, 2, ..., r, \quad r ∈ N,
\]

and \( Φ = (φ_1, φ_2, ..., φ_r)^T \).

1) If \( |k_2 − k_1| = 2 \) and \( |k_i − k_j| ≥ 2 \) for different \( i, j ≤ r \), then the rank of the matrix \( [Φ(ξ + jπ)]_{j ∈ Z} \) is a constant function on \( R \) and equals \( r \).

2) If \( |k_2 − k_1| = 2 \) and, at least for \( k_i \), \( i ≤ r \), it holds that \( |k_i − k_{i+2}| = 1 \), where \( 1 ≤ i_1, i_2 ≤ r \), then the rank of the matrix \( [Φ(ξ + jπ)]_{j ∈ Z} \) is not a constant function on \( R \).

4 Construction of frames using several compactly supported smooth functions

Firstly, we consider two smooth functions with proper compact supports.

Lemma 4.1. Let \( θ ∈ C_0^∞(R), ψ ∈ C_0^∞(R) \) be positive functions such that

\[
θ(x) > 0, x ∈ (-ε, 2π + ε), \quad \text{supp} \ θ = [-ε, 2π + ε],
\]

\[
ψ(x) > 0, x ∈ (ε, 2π − ε), \quad \text{supp} \ ψ = [ε, 2π − ε], \quad 0 < ε < 1/4.
\]

Moreover, let \( \hat{φ}_1(ξ) = θ(ξ), \hat{φ}_2(ξ) = ψ(ξ), ξ ∈ R \), and \( Φ = (φ_1, φ_2)^T \). Then the rank of the matrix \( [Φ(ξ + jπ)]_{j ∈ Z} \) is a constant function on \( R \) and equals 1.

Proof. Note that \( φ_i ∈ S(R) ⊂ W_1^∞(R) \), \( i = 1, 2 \).

We have the following two cases.

1° If \( ξ ∈ (-ε + 2ℓπ, ε + 2ℓπ), ℓ ∈ Z \), then the matrix

\[
[Φ(ξ + jπ)]_{j ∈ Z} = \begin{bmatrix}
... & 0 & a & b & 0 & ... \\
... & 0 & 0 & 0 & 0 & ...
\end{bmatrix}, \quad 0 < a, b ≤ 1,
\]

has a constant rank and equals 1.

2° For \( ξ ∈ (ε + 2ℓπ, 2π − ε + 2ℓπ), ℓ ∈ Z \), the rank of the matrix

\[
[Φ(ξ + jπ)]_{j ∈ Z} = \begin{bmatrix}
... & 0 & c & 0 & ... \\
... & 0 & 0 & d & 0 & ...
\end{bmatrix},
\]

where \( c, d \) are non-zero values, equals 1. The equivalent matrix is obtained for \( ξ = ε + 2ℓπ \) and \( ξ = -ε + 2ℓπ \), so we conclude that \( \text{rank}[Φ(ξ + jπ)]_{j ∈ Z} = 1 \), for \( ξ ∈ [ε + 2ℓπ, 2π − ε + 2ℓπ], ℓ ∈ Z \).

Considering these two cases, the rank of the matrix \( [Φ(ξ + jπ)]_{j ∈ Z} \) is a constant function on \( R \) and \( \text{rank}[Φ(ξ + jπ)]_{j ∈ Z} = 1, ξ ∈ R \).

\[\square\]
Using functions $\theta$ and $\psi$ from Lemma 4.1, in the next lemma we construct the $p$-frame with four appropriate functions.

**Lemma 4.2.** Let the functions $\theta$ and $\psi$ satisfy all the conditions of Lemma 4.1. Moreover, let

$$\hat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \quad \hat{\phi}_{k+2}(\xi) = \psi(\xi + 2k\pi), \quad k = 0, 1,$$

and $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$.

The rank of the matrix $[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals 2.

**Proof.** The proof is similar to the proof of Lemma 4.1.

1° If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then the matrix

$$[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\
\cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix},$$

where $0 < a^i, b^i \leq 1$, $i = 1, 2$, has a constant rank and equals 1.

2° For $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, we have

$$\text{rank}[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & 0 & c^1 & 0 & \cdots \\
\cdots & 0 & 0 & d^1 & 0 & \cdots \\
\cdots & 0 & c^2 & 0 & 0 & \cdots \\
\cdots & 0 & d^2 & 0 & 0 & \cdots
\end{bmatrix} = 2,$$

where $0 < c^i, d^i \leq 1$, $i = 1, 2$.

We conclude that the rank of the matrix $[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals 2.

Lemma 4.1 can be easily generalised for an even number of functions $\phi_i$, $i = 0, \ldots, 2r - 1$, with compactly supported $\hat{\phi}_i$, $i = 0, \ldots, 2r - 1$. The proof of the next theorem is similar to the previous proofs.

**Theorem 4.1.** Let the functions $\theta$ and $\psi$ satisfy all the conditions of Lemma 4.1. Moreover, let

$$\hat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \quad \hat{\phi}_{k+r}(\xi) = \psi(\xi + 2k\pi), \quad k = 0, \ldots, r - 1, \quad r \in \mathbb{N},$$

and $\Phi = (\phi_0, \phi_1, \ldots, \phi_{2r-1})^T$.

The following statements hold.

1° $\text{rank}[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = r$ for all $\xi \in \mathbb{R}$.

2° $V^p_{\mu}(\Phi)$ is closed in $L^p_{\mu}$ for any $p \in [1, \infty]$.

3° $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq i \leq 2r - 1\}$ is a $p$-frame for $V^p_{\mu}(\Phi)$ for any $p \in [1, \infty]$. 

8
Now we consider three functions with compact supports.

**Lemma 4.3.** Let the function \( \theta \) satisfies all the conditions of Lemma 4.1 and let \( \tau \in C^\infty_c(\mathbb{R}) \) and \( \omega \in C^\infty_c(\mathbb{R}) \) be positive functions such that

\[
\tau(x) > 0, \ x \in (\varepsilon, \pi - \varepsilon) \cup (\pi + \varepsilon, 2\pi - \varepsilon), \quad \text{supp } \tau = [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon],
\]

\[
\omega(x) > 0, \ x \in (-3\pi - \varepsilon, -\pi + \varepsilon), \quad \text{supp } \omega = [-3\pi - \varepsilon, -\pi + \varepsilon], \ 0 < \varepsilon < 1/4.
\]

Moreover, let \( \hat{\phi}_1(\xi) = \theta(\xi), \hat{\phi}_2(\xi) = \tau(\xi), \hat{\phi}_3(\xi) = \omega(\xi), \xi \in \mathbb{R}, \) and \( \Phi = (\phi_1, \phi_2, \phi_3)^T. \) Then the rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is a constant function on \( \mathbb{R} \) and equals 2.

**Proof.** We have four different forms for the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) and in all cases the rank of the matrix is 2.

Now we will show all possible cases. Denote with \( a^i, i = 1, 2, 3 \) and \( b^i, i = 1, 2, \) some positive values.

1°

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & a^1 & b^1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & a^2 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}, \ \xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi).
\]

2°

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & b^1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & a^2 & 0 & \cdots \\
\cdots & 0 & a^3 & 0 & \cdots
\end{bmatrix}, \ \xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi].
\]

3°

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & b^1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & a^2 & b^2 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots
\end{bmatrix}, \ \xi \in (\pi - \varepsilon + 2\ell\pi, \pi + \varepsilon + 2\ell\pi).
\]

4°

\[
[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = \begin{bmatrix}
\cdots & 0 & b^1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & b^2 & \cdots \\
\cdots & 0 & a^3 & 0 & 0 & \cdots
\end{bmatrix}, \ \xi \in [\pi + \varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi].
\]

\[\square\]

**Remark 4.1.** In Lemma 4.3 the support of the function \( \omega \) must have an empty intersection with the supports of \( \theta \) and \( \tau. \) In the opposite case, i.e. \( \text{supp } \theta \cap \text{supp } \tau \cap \text{supp } \omega \neq \emptyset, \) the rank of the matrix \( [\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} \) is a non-constant function on \( \mathbb{R}. \)

Lemma 4.3 can be easily generalised for functions \( \phi_i, i = 0, \ldots, 3r - 1, \) with compactly supported \( \hat{\phi}_i, i = 0, \ldots, 3r - 1. \) The proof of the next theorem is similar to the previous proofs.
Theorem 4.2. Let the functions $\theta$, $\tau$ and $\omega$ satisfy all the conditions of Lemma 4.3. Moreover, let

$$\hat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \quad \hat{\phi}_{k+r}(\xi) = \tau(\xi + 2k\pi), \quad \hat{\phi}_{k+2r}(\xi) = \omega(\xi + 2k\pi),$$

$k = 0, \ldots, r - 1, \ r \in \mathbb{N},$ and $\Phi = (\phi_0, \phi_1, \ldots, \phi_{3r-1})^T$.

The following statements hold.

1° rank$[\hat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2r$ for all $\xi \in \mathbb{R}$.

2° $V_p^\mu(\Phi)$ is closed in $L_p^\mu$ for any $p \in [1, \infty]$.

3° $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}, 0 \leq i \leq 3r - 1\}$ is a $p$-frame for $V_p^\mu(\Phi)$ for any $p \in [1, \infty]$.

5 Construction of frames of functions with finite regularities and compact supports; one-dimensional case

Let $H(x), \ x \in \mathbb{R}$, be the characteristic function of the semiaxis $x \geq 0$, i.e. $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$ (Heaviside's function). We construct a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in the following way. Let $\phi_1(x) := (H(x) - H(x - a))/a$, $a > 0$, $\phi_2 := \phi_1 * \phi_1$, $\phi_3 := \phi_1 * \phi_1 * \phi_1$, $\ldots$, i.e.,

$$\phi_n := \underbrace{\phi_1 * \phi_1 * \cdots * \phi_1}_{\text{n-1 times}}, \ n \in \mathbb{N},$$

where $*$ denotes the convolution of the functions.

We obtain

$$\phi_2(x) = \frac{1}{a^2} \left( xH(x) - 2(x-a)H(x-a) + (x - 2a)H(x - 2a) \right),$$

$$\phi_3(x) = \frac{1}{2a^3} \left( x^2H(x) - 3(x-a)^2H(x-a) + 3(x - 2a)^2H(x - 2a) - (x - 3a)^2H(x - 3a) \right),$$

$$\phi_4(x) = \frac{1}{3a^4} \left( x^3H(x) - 4(x-a)^3H(x-a) + 6(x - 2a)^3H(x - 2a) - 4(x - 3a)^3H(x - 3a) + (x - 4a)^3H(x - 4a) \right).$$
Continuing in this manner, for all $n \in \mathbb{N}$, we have

$$
\phi_n(x) = \frac{1}{a^n(n-1)!} \left( \binom{n}{0} x^{n-1} H(x) - \binom{n}{1} (x-a)^{n-1} H(x-a) + \binom{n}{2} (x-2a)^{n-1} H(x-2a) - \binom{n}{3} (x-3a)^{n-1} H(x-3a) + \cdots + (-1)^{n-1} \binom{n}{n-1} (x-(n-1)a)^{n-1} H(x-(n-1)a) + (-1)^n \binom{n}{n} (x-na)^{n-1} H(x-na) \right).
$$

Calculating the Fourier transform of functions $\phi_n$, $n \in \mathbb{N}$, we get

$$
\hat{\phi}_1(\xi) = \frac{-i}{a} \text{v.p.} \left( \frac{1}{\xi} \right) (e^{ia\xi} - 1),
$$

$$
\hat{\phi}_2(\xi) = \frac{(-i)^2}{a^2} \text{v.p.} \left( \frac{1}{\xi^2} \right) (e^{ia\xi} - 1)^2,
$$

$$
\hat{\phi}_3(\xi) = \frac{(-i)^3}{a^3} \text{v.p.} \left( \frac{1}{\xi^3} \right) (e^{ia\xi} - 1)^3.
$$

Continuing in this manner, we obtain $\hat{\phi}_n(\xi) = \frac{(-i)^n}{a^n} \text{v.p.} \left( \frac{1}{\xi^n} \right) (e^{ia\xi} - 1)^n$, $n \in \mathbb{N}$, where v.p. denotes the principal value.

Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$, $r \in \mathbb{N}$. The matrix $[\Phi(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ has for all $\xi \in \mathbb{R}$ the same rank as the matrix

$$
R(\xi) = \begin{bmatrix}
\cdots & \alpha_{-4\pi}^2 \beta_{-4\pi}^2 & \alpha_{-4\pi} \beta_{-4\pi} & \alpha_{-4\pi} & \alpha_{2\pi} \beta_{2\pi} & \alpha_{4\pi} \beta_{4\pi} & \cdots \\
\cdots & \alpha_{4\pi}^2 \beta_{4\pi}^2 & \alpha_{4\pi} \beta_{4\pi} & \alpha_{4\pi} & \alpha_{2\pi} \beta_{2\pi} & \alpha_{4\pi} \beta_{4\pi} & \cdots \\
\cdots & \alpha_{-4\pi}^3 \beta_{-4\pi}^3 & \alpha_{-2\pi}^3 \beta_{-2\pi}^3 & \alpha_{-4\pi}^3 & \alpha_{2\pi}^3 \beta_{2\pi}^3 & \alpha_{4\pi}^3 \beta_{4\pi}^3 & \cdots \\
\cdots & \alpha_{4\pi}^3 \beta_{4\pi}^3 & \alpha_{2\pi}^3 \beta_{2\pi}^3 & \alpha_{4\pi}^3 & \alpha_{2\pi}^3 \beta_{2\pi}^3 & \alpha_{4\pi}^3 \beta_{4\pi}^3 & \cdots \\
\cdots & \alpha_{-4\pi}^4 \beta_{-4\pi}^4 & \alpha_{-2\pi}^4 \beta_{-2\pi}^4 & \alpha_{-4\pi}^4 & \alpha_{2\pi}^4 \beta_{2\pi}^4 & \alpha_{4\pi}^4 \beta_{4\pi}^4 & \cdots \\
\cdots & \alpha_{4\pi}^4 \beta_{4\pi}^4 & \alpha_{2\pi}^4 \beta_{2\pi}^4 & \alpha_{4\pi}^4 & \alpha_{2\pi}^4 \beta_{2\pi}^4 & \alpha_{4\pi}^4 \beta_{4\pi}^4 & \cdots \\
\cdots & \alpha_{-4\pi}^5 \beta_{-4\pi}^5 & \alpha_{-2\pi}^5 \beta_{-2\pi}^5 & \alpha_{-4\pi}^5 & \alpha_{2\pi}^5 \beta_{2\pi}^5 & \alpha_{4\pi}^5 \beta_{4\pi}^5 & \cdots \\
\cdots & \alpha_{4\pi}^5 \beta_{4\pi}^5 & \alpha_{2\pi}^5 \beta_{2\pi}^5 & \alpha_{4\pi}^5 & \alpha_{2\pi}^5 \beta_{2\pi}^5 & \alpha_{4\pi}^5 \beta_{4\pi}^5 & \cdots \\
\cdots & \alpha_{-4\pi}^6 \beta_{-4\pi}^6 & \alpha_{-2\pi}^6 \beta_{-2\pi}^6 & \alpha_{-4\pi}^6 & \alpha_{2\pi}^6 \beta_{2\pi}^6 & \alpha_{4\pi}^6 \beta_{4\pi}^6 & \cdots \\
\cdots & \alpha_{4\pi}^6 \beta_{4\pi}^6 & \alpha_{2\pi}^6 \beta_{2\pi}^6 & \alpha_{4\pi}^6 & \alpha_{2\pi}^6 \beta_{2\pi}^6 & \alpha_{4\pi}^6 \beta_{4\pi}^6 & \cdots 
\end{bmatrix},
$$

where $\alpha_k^m = \text{v.p.} \left( \frac{1}{\xi - k} \right)^m$ and $\beta_k^m = (e^{ia(\xi - k)})^m$. Since the rank of $R(\xi)$ is equal to $r$ for all $\xi \in \mathbb{R}$, we have the next result.

**Theorem 5.1.** Let $\Phi = (\phi_k, \phi_{k+1}, \ldots, \phi_{k+(r-1)})^T$, for $k \in \mathbb{Z}$, $r \in \mathbb{N}$. Then $V^p_\mu(\Phi)$ is closed in $L^p_\mu$ for any $p \in [1, \infty]$ and $\{\phi_{k+s}(:, -j) \mid j \in \mathbb{Z}, 0 \leq s \leq r-1\}$ is a $p$-Riesz basis for $V^p_\mu(\Phi)$ for any $p \in [1, \infty]$.

**Remark 5.1.** (1) We refer to [4] and [22] for the $\gamma$-dense set $X = \{x_j \mid j \in J\}$. Let $\phi_k(x) = F^{-1}(\theta(\cdot - k\pi))(x)$, $x \in \mathbb{R}$. Following the notation of [22], we put $\psi_{x_j} = \phi_{x_j}$ where $\{x_j \mid j \in J\}$ is $\gamma$-dense set determined by $f \in V^2(\phi) = V^2(F^{-1}(\theta))$. Theorems 3.1, 3.2 and 4.1 in [22] give the conditions and explicit form of $C_p > 0$ and $c_p > 0$ such that the inequality
\[ c_p \|f\|_{L_p^\mu} \leq \left( \sum_{j \in J} |\langle f, \psi_x \rangle \mu(x_j)|^p \right)^{1/p} \leq C_p \|f\|_{L_p^\mu} \text{ holds.} \]

This inequality guarantees the feasibility of a stable and continuous reconstruction algorithm in the signal spaces \( V_p^\mu(\Phi) \).

(2) Since the spectrum of the Gram matrix \([\hat{\Phi}, \hat{\Phi}]\)(\(\xi\)), where \(\Phi\) is defined in Theorem 5.1, is bounded and bounded away from zero (see [8]), it follows that the family \(\{\Phi(\cdot - j) \mid j \in \mathbb{Z}\}\) forms a \(p\)-Riesz basis for \( V_p^\mu(\Phi) \).

(3) Frames of the above sections may be useful in applications since they satisfy assumptions of Theorem 3.1 and Theorem 3.2 in [5]. They show that error analysis for sampling and reconstruction can be tolerated, or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions \(\phi_{k_1}, \ldots, \phi_{k_r}\).

Acknowledgment

The authors were supported in part by the Serbian Ministry of Science and Technological Developments (Project #174024).

References

[1] A. Aldroubi, K. Gröchenig, Non-uniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (2001), 585–620.

[2] A. Aldroubi, Q. Sun, W. Tang, \(p\)-frames and shift-invariant subspaces of \(L^p\), J. Fourier Anal. Appl. 7 (2001), 1–21.

[3] A. Aldroubi, Q. Sun, W. Tang, Non-uniform sampling in multiply generated shift-invariant subspaces of \(L^p(\mathbb{R}^d)\). Wavelet analysis and applications (Guangzhou, 1999), AMS/IP Stud. Adv. Math., 25 (2002), 1–8.

[4] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, Appl. Comput. Harmon. Anal. 13 (2002), 151–161.

[5] A. Aldroubi, I. Krishtal, Robustness of sampling and reconstruction and Beurling-Landau type theorems for shift-invariant spaces, Appl. Comput. Harmon. Anal. 20 (2006), 250–260.

[6] A. Aldroubi, M. Unser, Sampling procedure in function spaces and asymptotic equivalence with Shannon’s sampling theory, Numer. Funct. Anal. Optim. 15 (1994), 1–21.

[7] A. Aldroubi, A. Baskakov, I. Krishtal, Slanted matrices, Banach frames, and sampling, J. Funct. Anal. 255 (2008), 1667–1691.

[8] C. de Boor, R. A. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in \(L^2(\mathbb{R}^d)\), J. Funct. Anal. 119 (1994), 37–78.
[9] R. A. DeVore, B. Jawerth, B. J. Lucier, Image compression through wavelet transform coding, IEEE Trans. Inform. Theory 38 (1992), 719–746.

[10] H. G. Feichtinger, Banach convolution algebras of Wiener type, in Functions, Series, Operators, Vols. I, II (Budapest, 1980), North-Holland, Amsterdam, 1983, 509–524.

[11] H. G. Feichtinger, K. Gröchenig, A unified approach to atomic decomposition via integrable group representations, In: Proc. Conf. ”Function Spaces and Applications”, Lecture Notes in Maths, 1302, Berlin-Heidelberg-New York, Springer (1988), 52–73.

[12] H. G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, J. Funct. Anal. 86 (1989), 307–340.

[13] H. G. Feichtinger, Generalized amalgams, with applications to Fourier transform, Canad. J. Math. 42 (1990), 395–409.

[14] H. G. Feichtinger, K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, SIAM J. Math. Anal. 23 (1992), 244–261.

[15] H. G. Feichtinger, K. Gröchenig, Theory and practice of irregular sampling, in Wavelets-Mathematics and Applications. J.J. Benedetto and W. Frazier, eds., CRC, Boca Raton, FL, 1993, 305–363.

[16] H. G. Feichtinger, K. Gröchenig, T. Strohmer, Efficient numerical methods in nonuniform sampling theory, Numer. Math. 69 (1995), 423–440.

[17] H. G. Feichtinger, T. Strohmer, eds., Gabor Analysis and Algorithms, Birkhäuser, Boston, 1998.

[18] K. Gröchenig, Describing functions: Atomic decomposition versus frames, Monatsh. Math 112 (1991), 1–41.

[19] R. Q. Jia, C. A. Micchelli, On linear independence of integer translates of a finite number of functions, Proc. Edinb. Math. Soc. 36 (1992), 69–85.

[20] S. Pilipović, S. Simić, Frames for shift invariant spaces, Mediterr. J. Math., DOI: 10.1007/s00009-011-0155-3, (preprint on Arxiv 2011, arXiv:1105.6105).

[21] S. Simić, Fréchet frames for shift invariant weighted spaces, Novi Sad J. Math. 39 (2009), 119–128.

[22] J. Xian, S. Li, Sampling set conditions in weighted multiply generated shift-invariant spaces and their applications, Appl. Comput. Harmon. Anal. 23 (2007), 171–180.