Chalker-Coddington model described by an S-matrix with odd dimensions

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Abstract

The Chalker-Coddington network model is often used to describe the transport properties of quantum Hall systems. By adding an extra channel to this model, we introduce an asymmetric model with profoundly different transport properties. We present a numerical analysis of these transport properties and consider the relevance for realistic systems.

Key words: quantum Hall effect, Chalker-Coddington network model, quantum transport, nanographene

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1 Introduction

The Chalker-Coddington network model (C-C model) has been accepted as one of the simplest models [1,2] to explain and analyze the quantum Hall effect [4]. In the C-C model, the equipotential lines of the long-ranged disordered potential are regarded as the links of a network, while the saddle points of these equipotential lines are regarded as the nodes of this network. Quantum tunneling at the nodes of the network is described by a $2 \times 2$ scattering matrix $S_2$

$$S_2 = \begin{pmatrix} -r & t \\ t & r \end{pmatrix},$$ (1)

where $r$ and $t$ are reflection and transmission amplitude, respectively. They are real numbers and satisfy $r^2 + t^2 = 1$. 

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In this paper we introduce the asymmetric Chalker-Coddington network model. This is obtained from the standard C-C model by adding an extra channel in such a way that the number of incoming fluxes on one side of the system differs from that on the other side (see Fig. 1). This leads to the creation of a single perfectly transmitting channel. We analyze the conductance distribution for the square geometry (2D), and the size dependence of the average conductance in both 2D and quasi-one dimensional (quasi-1D) geometries. The results are interpreted using random matrix theory and the repulsion of transmission eigenvalues from the perfectly transmitting channel. We emphasize that the asymmetry due to the addition of single channel dramatically changes the transport properties. This asymmetry can be realized in parallel Hall bars where the contact does not touch one edge of a bar. Another possibility is the transport in graphene, where $N$ ($N'$) left (right) going and $N'$ ($N$) right (left) going channels are realized near $K$ ($K'$) point. For long ranged impurities, scattering between states near $K$ and $K'$ is suppressed, resulting in the asymmetric number of left and right going channels [3].

2 Model

From $S_2$, we define a 2 by 2 transfer matrix $T_2$ that relates the currents in the left side of the node to those in the right,

$$T_2 = \begin{pmatrix} 1/t & r/t \\ r/t & 1/t \end{pmatrix},$$

and then construct the full transfer matrix $T$. It consists of two types of transfer matrices. One is the transfer matrix (see the left most column of nodes in Fig. 1),

$$V_1 = \begin{pmatrix} 1/r & t/r & 0_2 & \cdots & 0 \\ 0_2 & \cdots & 0 \end{pmatrix},$$

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where $0_2$ is a $2 \times 2$ zero matrix, and the other is (see the next column in Fig. 1)

$$V_2 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1/r & t/r & \cdots & 0_2 & \vdots \\
0 & t/r & 1/r & \cdots & 0_2 & \vdots \\
\vdots & 0_2 & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 1/r_{\text{ext}} & t_{\text{ext}}/r_{\text{ext}} & \vdots \\
0 & \cdots & \cdots & t_{\text{ext}}/r_{\text{ext}} & 1/r_{\text{ext}} & \end{pmatrix}. \tag{4}
$$

Here the new parameters $r_{\text{ext}}$ and $t_{\text{ext}}$ describe scattering due to the addition of a new link. Between the nodes, we assume that the electron wavefunctions acquire random phases $e^{i\phi}$, with $\phi$ uniform random numbers on $[0, 2\pi)$. When the number of links in a layer is $N$, we need a set of $N$ independent random numbers $\{\phi\}$. The transfer matrix $T$ that relates current fluxes in the left side to that in the right side is

$$T = \prod_i V_D(\{\phi_i\}) V_i V_D(\{\phi_i\}) V_2, \tag{5}$$

where $i$ indicates $i$-th layer, and $V_D(\{\phi_i\}) = \text{diag}[e^{i\phi_1}, \cdots, e^{i\phi_N}]$. The transfer matrix is then related to the reflection and transmission matrices [2].

To focus on the quantum Hall critical point, we set $r = t = 1/\sqrt{2}$. Unless explicitly stated, we assume $r = r_{\text{ext}}$ and $t = t_{\text{ext}}$.

Before reporting our numerical results we describe the structure of the scattering matrix $S$. This matrix relates the incoming and outgoing particle flux amplitudes

$$\begin{pmatrix}
\Psi_L^{\text{out}} \\
\Psi_R^{\text{out}}
\end{pmatrix} = S \begin{pmatrix}
\Psi_L^{\text{in}} \\
\Psi_R^{\text{in}}
\end{pmatrix} = \begin{pmatrix} r & t' \\
t & r'
\end{pmatrix} \begin{pmatrix}
\Psi_L^{\text{in}} \\
\Psi_R^{\text{in}}
\end{pmatrix}. \tag{6}
$$

Here $\Psi_{L(R)}^{\text{in(out)}}$ is the incoming (outgoing) current flux amplitude in the left (right) side of the system. The $t$-matrix and $t'$-matrix are $L \times L$ and $(L + 1) \times (L + 1)$ square matrices, respectively, while $r$ is $(L + 1) \times L$ and $r'$ is $L \times (L + 1)$. As a result, both $S$ and $T$ have odd dimensions $(2L+1) \times (2L+1)$. The asymmetric C-C model is a type of quantum railroad [5,6].

3 Results

The dimensionless conductances $G$ and $G'$

$$G = \text{Tr} tt^\dagger, \quad G' = \text{Tr} t't'^\dagger, \tag{7}$$

3
describe current from left to right and right to left, respectively. Using the relationships obtained from the unitarity of the scattering matrix,

\[ t^\dagger t' + r'^\dagger r' = I_{L+1}, \quad tt^\dagger + r' r'^\dagger = I_L, \quad (8) \]

and performing a singular value decomposition of \( r' \), we obtain

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & & \\
& & \ddots & \\
0 & & & \lambda_L & 0
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
\begin{pmatrix}
U(L, L)
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
= U'(L+1, L+1), \quad (9)
\]

where \( U \) and \( U' \) are unitary matrices and \( 1 \geq \lambda_i \geq 0 \) \( (i = 1, 2, \cdots, L) \). Taking its Hermitian conjugate,

\[
\begin{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \\
& & \ddots & \\
0 & & & \lambda_L
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
\begin{pmatrix}
U(L, L)
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
= U^\dagger(L, L), \quad (10)
\]

we obtain

\[ tt^\dagger = U \text{diag}(1 - \lambda_1^2, 1 - \lambda_2^2, \cdots, 1 - \lambda_L^2) U^\dagger \quad (11) \]

and

\[ t'^\dagger t' = U'^\dagger \text{diag}(1 - \lambda_1^2, 1 - \lambda_2^2, \cdots, 1 - \lambda_L^2, 1) U'. \quad (12) \]

Note that the transmission eigenvalues are the same except for a single extra unit eigenvalue that appears in \( t'^\dagger t' \). As a result, the difference between \( G' \) and \( G \) is always unity,

\[ G' - G = 1. \quad (13) \]

We might expect that the conductance distribution simply shifts by unity but, as we show below, this expectation is too naive.

In the following, we calculate the transmission matrix \( t \) via the transfer matrix technique. The system width \( L \) corresponds to the number of nodes in a column, while the system length \( M \) to that in a row.

### 3.1 Conductance distribution for 2D

The distribution of the conductance \( G' \) in 2D at the critical point is shown in Fig. 2. \( L = M = 32 \) and \( 10^6 \) samples have been realized. The conductance is
always greater than unity as expected. The distribution has a kink at unity, and decreases rapidly with increasing $G'$. This is explained by the fact that the unit eigenvalue repels the other eigenvalues. This behavior is profoundly different from that of the standard C-C model, where a broad distribution is found [7].

3.2 Averaged conductances in 2D and quasi-1D

The averaged conductances $\langle G' \rangle$ for 2D and quasi-1D geometries are shown in Figs. 3 and 4. In the limit $L \to \infty$, the conductance converges to a finite value of, approximately, 1.24 for the 2D geometry (Fig. 3) and to unity for the quasi-1D geometry (Fig. 4). We see that the asymmetric C-C model exhibits metallic behavior even for a long wire. This is again very different to the standard C-C model where the conductance converges to zero. Also, the convergence to the limiting value is faster than that for the standard C-C model. This is consistent with predictions based on random matrix theory of an increase of the Lyapunov exponents associated with localised channels due to the repulsion with the unit eigenvalue [7].

4 Summary and Concluding Remarks

In summary, we have proposed an asymmetric C-C model and analyzed its transport properties. We find that, even for a long wire, the conductance remains finite and the system is metallic. Transmission eigenvalues are repelled by the unit eigenvalue resulting in an unusual form for the conductance distribution.

We have also varied the coupling to the additional link, $r_{\text{ext}}$ and $t_{\text{ext}}$. Preliminary results suggest that conductance in sufficiently large systems is not sensitive to the choice of coupling, indicating that the observed transport phenomena are of bulk origin.

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\[
\begin{align*}
\psi_{\text{out},1} & \quad V_1 & \quad V_2 & \quad \psi_{\text{in},1} \\
\psi_{\text{in},1} \\
\psi_{\text{out},2} & \quad V_1 & \quad \vdots & \quad V_2 & \quad \psi_{\text{in},2} \\
\psi_{\text{in},2} \\
\psi_{\text{out},3} & \quad \vdots & \quad \psi_{\text{in},3} \\
\psi_{\text{in},3} \\
\vdots \\
\psi_{\text{out},L} & \quad \psi_{\text{in},L} \\
\psi_{\text{in},L} \\
\psi_{\text{out},L+1} & \quad \psi_{\text{in},L+1} \\
\end{align*}
\]

Fig. 1. Schematic of model. Note the lowest links that dramatically change the transport properties.

\[
\begin{align*}
P(G') & \quad G' \\
0.0 & \quad 0.5 & \quad 1.0 & \quad 1.5 & \quad 2.0 & \quad 2.5 & \quad 3.0 \\
0.0 & \quad 0.5 & \quad 1.0 & \quad 1.5 & \quad 2.0 & \quad 2.5 & \quad 3.0
\end{align*}
\]

Fig. 2. Conductance distribution at critical point for the asymmetric CC model (×). For comparison, results for the standard CC model (+) is plotted. The conductance is measured in units of $e^2/h$. $L = M = 36. \ 10^5$ different random configurations have been realized.
Fig. 3. Averaged conductance as function of system size $L$ for square geometry. Average over $10^6$ samples has been performed for each size.

Fig. 4. Averaged conductance as function of system length $M$ in the quasi-1D wire. Average over $10^5$ samples has been performed. The width $L = 32$. 

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