Disorder driven collapse of the mobility gap and transition to an insulator in fractional quantum Hall effect

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We study the \( \nu = 1/3 \) quantum Hall state in the presence of random disorder. We calculate the topologically invariant Chern number, which is the only quantity known at present to unambiguously distinguish between insulating and current carrying states in an interacting system. The mobility gap can be determined numerically this way, which is found to agree with experimental value semiquantitatively. As the disorder strength increases towards a critical value, both the mobility gap and plateau width narrow continuously and ultimately collapse leading to an insulating phase.

The situation is very different in the case of FQHE. Due to the topological and many-body nature of the problem, in the FQHE system at filling factor \( \nu = n \hbar / eB = p/q \) (\( B \) is the magnetic field and \( n \) the areal density), there exists a manifold of \( q \) nearly degenerate low energy states on the torus, whose energy differences disappear in the thermodynamic limit (Ref. 9 and see below). The quantization of the Hall conductance can not be tied, in a physically meaningful way, to any particular one of the \( q \) ground states. For example, in the absence of disorder (in which case the degeneracy is exact), when an external flux quanta is inserted adiabatically in a region inaccessible to the electrons, the states within a given manifold evolve into each other. In the presence of disorder, the Chern number of the individual states fluctuates while the sum of the Chern number of all the states turns out to be \( p \) and robust. As a consequence, we may regard the total Chern number \( p \) to be shared by the \( q \) degenerate states, which results in fractionally quantized Hall conductance \( \sigma_H = pe^2/\hbar q \) for the system.

Based on this picture, we have developed a numerical method for studying the topological Chern number of the ground state and low energy excited states for an interacting system. We will show that the presence of disorder leads to several interesting and important results for 1/3 FQHE: (i) A weak random disorder lifts the degeneracy of the ground state for a finite number of electrons. However, the level spacings between the lowest three states decrease monotonically with the increase of electron number, indicating the recovery of the degeneracy in the thermodynamic limit. (ii) The mobility gap, which separates the higher energy extended excitations from the low energy FQHE states, can be determined from the distribution of the Chern number of the many-body states. This is the only way to unambiguously distinguish between insulating and current carrying states in an interacting system. (iii) In general the mobility gap, determined this way, will be different from the spec-
trum gap (which separates the lowest three states from other higher states). It is the mobility gap that should be compared with the experimentally obtained activation energies. (iv) There exists a critical disorder strength $W = W_c \approx 0.2 e^2/\epsilon$ for Gaussian white noise potential, which marks a transition from the FQHE to insulator. (v) The physics of the destruction of the FQHE can be described as the continuous collapse of the mobility gap; the closing of the mobility gap and quantization of the Hall conductance occur at the same time.

We consider a two-dimensional interacting electron system in an $L_1 \times L_2$ square cell with twisted boundary conditions: $T(L_j)\Psi(r) = e^{i\theta_j}\Psi(r)$, where $T(L_j)$ is the magnetic translation operator and $j = 1, 2$ represents the $x$ and $y$ directions, respectively. Calculating the Hall conductance $\sigma_H$ directly from a Kubo formula would require a knowledge of all the many body eigenstates for a fixed number of electrons. This proves impractical for systems with 3 electrons or larger. As first realized by Thouless and co-workers, a topological property of the wavefunctions, known as the first Chern number, can be used to calculate the boundary condition averaged $\sigma_H$. The importance of Chern numbers, however, goes beyond obtaining $\sigma_H$. It appears to be the only quantity that distinguishes between insulating and current carrying states in an interacting system. Because we are dealing with many-body wavefunctions, other simpler numerical methods (such as inverse participation ratio and Thouless numbers) used for determining the localization of the single-particle wavefunctions have no obvious extension here. After a unitary transformation $\Psi_k = exp[-i\sum_{j=1}^{N_e}(\frac{\phi_{j1}}{e^2}x_1 + \frac{\phi_{j2}}{e^2}y_1)]\Phi_k$, the boundary-phase averaged Hall conductance for the $k$-th many-body eigenstate is $\sigma_H^k = C(k)e^2/h$, with

$$C(k) = \frac{i}{4\pi} \int d\theta_j \{\langle \psi_k | \frac{\partial \psi_k}{\partial \theta_j} \rangle - \langle \frac{\partial \psi_k}{\partial \theta_j} | \psi_k \rangle\},$$

where the closed path integral is along the boundary of a unit cell $0 \leq \theta_1, \theta_2 \leq 2\pi$ and summation over $j$ is implied. $C(k)$ is exactly the Berry phase (in units of $2\pi$) accumulated for such a state when the boundary phase evolves along the closed path. To determine the Chern number uniquely, we separate the boundary phase space into approximately 36 – 100 mesh points and get the sum of the Berry phases from each mesh. In cases where there are near-level-crossings, the integration contour has to be chosen reasonably close to these points. This determines the size of the mesh, at least locally. For the mesh sizes we chose, we found the Chern numbers had converged and did not change by further reducing the size of the mesh.

In the presence of a strong magnetic field, one can project the Hamiltonian onto the partially-filled, lowest Landau level. The projected Hamiltonian in the presence of both Coulomb interaction and disorder can be written as:

$$H = \sum_{i < j} \sum_q e^{-q^2/2}V(q)e^{iq}(\mathbf{R}_i - \mathbf{R}_j) + \sum_i \sum_q e^{-q^2/4}U_q e^{iq}\mathbf{R}_i,$$

where $\mathbf{R}_i$ is the guiding center coordinate of the $i$-th electron, $V(q) = 2\pi e^2/eq$ is the Coulomb potential, and $U_q$ is the impurity potential with the wave vector $q$. We set the magnetic length $\ell = 1$ and $e^2/\ell e = 1$ for convenience. The Gaussian white noise potential we use is generated according to the correlation relation in $q$-space $\langle U_q U_{q'} \rangle = (W^2/A)\delta_{q,-q'}$, which corresponds to $\langle U(r) U(r') \rangle = W^2\delta(r-r')$ in real space, where $W$ is the strength of the disorder and $A$ is the area of the system. We consider the case of $\nu = N_e/N_0 = 1/3$, where $N_e$ and $N_0$ are the number of electrons and flux quanta. We obtain the exact low energy eigenvalues and eigenstates using the Lanczos method for systems up to $N_e = 8$ electrons in 24 flux quanta, spanning a Hilbert space of size $N_{\text{basis}} = 735, 471$. We then calculate the Chern numbers using Eq. (1).

Independent of boundary conditions, we find that, for weak disorder, low energy states are separated into groups. The lowest group has three closely spaced states. This property comes from the three-fold center of mass degeneracy of the pure system at $\nu = 1/3$ and is shared by the entire spectrum. For finite sizes, there is a finite level spacing between the lowest three states, which is much smaller than the energy difference between the 3rd and the 4th state. The latter is the low energy spectrum gap denoted as $E_s$. At $W = 0.06$, we have $E_s = 0.04 \pm 0.002$ (averaged over 100-2000 disorder configurations), which is size independent for $N_e = 5 - 8$. It remains finite for weak W until $W$ is further increased to $W \approx 0.15$, at which point $E_s$ becomes too small and its value in the thermodynamic limit cannot be extrapolated from the sizes accessible to our approach.

On the other hand, the level spacings between the lowest three states depend strongly on $N_e$. They are expected to vanish exponentially as the linear dimension of the system. This can be seen clearly in Fig. 1, where we show a semilog plot of the “bandwidth” $E_b$ vs $\sqrt{N_e}$ for three different disorder strengths. It is apparent that $E_b$ drops to zero for large $N_e$, a direct consequence of the topological order in the ground state. This is in contrast to the usual effect of disorder in the IQHE, where the degeneracy of all the states in a Landau level, not being a topological property, will be lifted by the perturbation of arbitrarily weak disorder.

In the presence of weak disorder ($W < 0.12$), we find that the total Chern number carried by the lowest level is always 1, for $N_e = 3 - 8$ and thousands of disorder configurations. Note that the vanishing $E_b$ and the existence of a finite spectrum gap $E_s$, at relatively weak...
and 7, respectively. As seen in Fig. 2, \( P(1) = 1 \) for the first group \((N_g = 1)\), and \( P(C) = 0 \) for \( C \neq 1 \). This means all disorder configurations have \( C = 1 \), which corresponds to the 1/3 FQHE because each state carries a definite average Hall conductance of \( e^2/3h \). For \( N_g = 2 \), both \( P(0) \) and \( P(2) \) are nonzero, indicating that a small number of configurations have \( C = 0 \) or 2. As a result, \( P(1) \) is reduced to 0.9, 0.91, and 0.92 for \( N_e = 5, 6, \) and 7. The increase of \( P(1) \) with \( N_e \) indicates that \( P(1) \) may recover to 1 at large \( N_e \). For the \( N_g = 3 \) case, \( P(1) \) is significantly reduced to about 0.7; it behaves nonmonotonically as a function of \( N_e \). This results from the coexistence of three different Chern numbers in the thermodynamic limit, which characterizes the delocalization of quasi-particle excitations. Namely, these excitations carry nonzero Chern numbers which are extended in real space. For \( N_g > 3 \), \( P(1) \) is further reduced and seems to saturate at a value near 0.5.

We may regard the fluctuation of the Chern number to be an indication of the degree of delocalization. In analogy to the physics of noninteracting systems\(^8\), we define \( P_{\text{ext}} = 1 - P(1) \) to be the likelihood of the breakdown of the quantization of the Hall conductance and thus a measure of the delocalization of the charged excitations\(^\text{12}\). In the FQHE plateau regime, \( P_{\text{ext}} \) goes to zero as a result of the localization or nondissipative nature of the state. Here nonzero \( P_{\text{ext}} \) occurs as we go to higher energy states \((N_g \geq 3)\). The energy that separates these two kinds of states is called the mobility edge, where \( P_{\text{ext}} \) has a large increase, which probably becomes a finite jump in the thermodynamic limit. For \( W = 0.06 \), we find that, from \( N_g = 2 \) to 3, \( P_{\text{ext}} \) has the largest increase, which puts the \( N_g = 3 \) group at the mobility edge. Measuring the energy at the mobility edge relative to the energy of the lowest level, we get the mobility gap \( E_m \) for \( N_e = 4 - 8 \), which is \( N_e \)-dependent. For \( W > 0.06 \), we determine the mobility gap by extrapolating the finite size data to the thermodynamic limit. For weaker disorder, the sizes that can be treated are not sufficient to produce a meaningful extrapolation to \( N \to \infty \).

The extrapolated \( E_m \) vs. \( W \) is shown in Fig. 3a. In the inset we plot \( E_m \) vs. \( 1/N_e \) for \( W = 0.17 \), which can be best fit to \( E_m = 0.10/N_e + 0.005 \); thus we obtain \( E_m = 0.005 \pm 0.003 \) in the thermodynamic limit. We see that at \( W \approx 0.17 \) such a gap is strongly reduced, consistent with the drop of the spectrum gap for similar \( W \), signaling the FQHE is on the verge of being destroyed by disorder.

The energy gap \( \Delta \) in the excitation spectrum of the correlated many-body ground state can be extracted experimentally from the temperature dependence of the magnetoresistivity, \( \rho_{xx} \propto \exp(-\Delta/2k_BT) \), where \( \Delta/2 \) is the activation energy and \( k_B \) the Boltzmann’s constant\(^\text{11,12}\). Boebinger et al.\(^\text{14}\) systematically studied the activation energy for \( \nu = 1/3, 2/3, 4/3, \) and \( 5/3 \) and its dependence on sample mobility \( \mu \) (an indication of dis-
order) in GaAs-Al_{x}Ga_{1-x}As. For a class of high-mobility samples, they found that $\Delta(\mu) \approx C_0(\mu)(e^2/\ell) - \Gamma(\mu)$, consistent with a simple phenomenological model that assumes a disorder-broadened excitation energy level with half-width $\Gamma(\mu)$. Figure 3 compares this empirical formula of $\Delta(\mu)$ with fitting parameters ($C_0 = 0.049$ and $\Gamma = 6K$) for high-mobility samples with the mobility gap we obtained in our calculation. To do this, we assume that both the (zero field) mobility and (high field) mobility gap are dominated by short-range scatterers (appropriate for high-mobility samples) and, in the Born approximation, $\mu = e\hbar^3/(n^2\sigma W^2)$. We also use an empirical relation $\mu \propto n^{1.5}$ between $\mu$ and electron density $n$ (both for zero gate bias) extracted from Fig. 1 of Ref. \textsuperscript{14}. Here, we do not include the effects of layer thickness and Landau level mixing which, nevertheless, exist in experimental samples and are known to reduce the gap by as much as a factor of 2 \textsuperscript{15,17} (for high-field mobility) for zero or very weak disorder.

As we further increase $W$, the FQHE becomes unstable. This can be discerned by following the evolution of the Chern number of the lowest level and $\sigma_H$ averaged over the lowest three states. For example, for $N_e = 6$, at $W \leq 0.14$, we have $\sigma_H = e^2/(3\hbar)$; it drops to 0.924$e^2/3\hbar$ at $W = 0.17$. At larger $W$, we find a very strong enhancement of the fluctuation in the Chern number and, correspondingly, a rapid reduction of $\sigma_H$. Similar results are obtained for $N_e = 8$. As shown in Fig. 4, $-\Delta\sigma_H/\Delta W$ has its largest value near $W_c = 0.22 \pm 0.025$ for all $N_e = 5-8$, which determines the critical disorder for the $\nu = 1/3$ state plateau to insulator transition.

To illustrate the nature of the phase transition, we change the filling to be slightly off $1/3$. At weak $W$, the $\nu = 1/3$ FQHE plateau has a finite width due to the nonzero mobility gap that survives to fillings slightly below and above $1/3$. But at $W = 0.17$, which is slightly below the critical $W_c$, we find the mobility gap $E_m$ at $N_e = N_e/3 + 1$ is already reduced to zero. This suggests that the plateau for disorder close to $W_c$ is still pinned at filling $\nu = 1/3$, but with zero width. Therefore, the disappearance of the FQHE at $W_c$ is caused by the collapse of the mobility gap.

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\begin{figure}[h]
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\includegraphics[width=\textwidth]{figure3.png}
\caption{(a) The extrapolated mobility gap $E_m$ as a function of $W$. Inset shows $E_m$ at $W = 0.17$ for $N_e = 4-8$ electrons. The dashed line indicates $E_m$ can be extrapolated to 0.005 at $1/N_e \to 0$. The dot at $W = 0$ is the creation energy for a quasiparticle-quasihole pair at infinite separation, extrapolated from pure systems with up to $N_e = 10$. (b) Dependence of $E_m$ on mobility $\mu$. The dashed line is converted from a fit to experimental data (taken from Ref. \textsuperscript{13}). Here, we use an empirical mobility-density relation as well as a mobility-disorder relation in the Born approximation (see text for detail).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The relative decrease of the Hall conductance of the lowest level over the change of disorder strength $\Delta W$ is plotted as a function of disorder strength $W$ for $N_e = 5, 6, 7$ and $8$.}
\end{figure}

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