THE SYMBOLIC DEFECT OF AN IDEAL

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Abstract. Let \( I \) be a homogeneous ideal of \( k[x_0, \ldots, x_n] \). To compare \( I^{(m)} \), the \( m \)-th symbolic power of \( I \), with \( I^m \), the regular \( m \)-th power, we introduce the \( m \)-th symbolic defect of \( I \), denoted \( \text{sdefect}(I, m) \). Precisely, \( \text{sdefect}(I, m) \) is the minimal number of generators of the \( R \)-module \( I^{(m)} / I^m \), or equivalently, the minimal number of generators one must add to \( I^m \) to make \( I^{(m)} \). In this paper, we take the first step towards understanding the symbolic defect by considering the case that \( I \) is either the defining ideal of a star configuration or the ideal associated to a finite set of points in \( \mathbb{P}^2 \). We are specifically interested in identifying ideals \( I \) with \( \text{sdefect}(I, 2) = 1 \).

1. Introduction

Let \( I \) be a homogeneous ideal of \( R = k[x_0, \ldots, x_n] \). For any positive integer \( m \), let \( I^{(m)} \) denote the \( m \)-th symbolic power of \( I \). In general, we have \( I^m \subseteq I^{(m)} \), but equality may fail. During the last decade, there has been interest in the so-called “ideal containment problem,” that is, for a fixed integer \( m \), find the smallest integer \( r \) such that \( I^{(r)} \subseteq I^m \). The papers [7, 8, 13, 14, 23, 27] are a small subset of the articles on this problem.

In this note, we are also interested in comparing regular and symbolic powers of ideals, but we wish to investigate a relatively unexplored direction by measuring the “difference” between the two ideals \( I^m \) and \( I^{(m)} \). More precisely, because \( I^m \subseteq I^{(m)} \), the quotient \( I^{(m)}/I^m \) is a finitely generated graded \( R \)-module. For any \( R \)-module \( M \), let \( \mu(M) \) denote the number of minimal generators of \( M \). We then define the \( m \)-th symbolic defect of \( I \) to be the invariant

\[
\text{sdefect}(I, m) := \mu(I^{(m)}/I^m),
\]

that is, the minimal number of generators of \( I^{(m)}/I^m \). We will call the sequence

\[
\{\text{sdefect}(I, m)\}_{m \in \mathbb{N}}
\]

the symbolic defect sequence. Note that \( \text{sdefect}(I, m) \) counts the number of generators we need to add to \( I^m \) to make \( I^{(m)} \); this invariant can be viewed as a measure of the failure of \( I^m \) to equal \( I^{(m)} \). For example, \( \text{sdefect}(I, m) = 0 \) if and only if \( I^m = I^{(m)} \).

We know of only a few papers that have studied the module \( I^{(m)}/I^m \). This list includes: Arsie and Vatne’s paper [3] which considers the Hilbert function of \( I^{(m)}/I^m \); Huneke’s work [28] which considers \( P^{(2)}/P^2 \) when \( P \) is a height two prime ideal in a local ring of

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dimension three; Herzog’s paper [25] which studies the same family of ideals as Huneke using tools from homological algebra; Herzog and Ulrich’s paper [26] and Vasconcelos’s paper [36] which also consider a similar situation to Huneke, but with the assumption that $P$ is generated by three elements; and Schenzel’s work [34] which describes some families of prime ideals $P$ of monomial curves with the property that $P^{(2)}/P^2$ is cyclic (see the comment after [34, Theorem 2]).

The introduction of the symbolic defect sequence raises a number of interesting questions. For example, how large can $s\text{defect}(I,m)$ be? how does $s\text{defect}(I,m)$ compare to $s\text{defect}(I,m+1)$? and so on. In some sense, these questions are difficult since one needs to know both $I^{(m)}$ and $I^m$. To gain some initial insight into the behavior of the symbolic defect sequence, in this paper we focus on two cases: (1) $I$ is the defining ideal of a star configuration, and (2) $I$ is the homogeneous ideal associated to a set of points in $\mathbb{P}^2$. In both cases, we can tap into the larger body of knowledge about these ideals.

To provide some additional focus to our paper, we consider the following question:

**Question 1.1.** What homogeneous ideals $I$ of $k[x_0, \ldots, x_n]$ have $s\text{defect}(I,2) = 1$?

Because one always has $s\text{defect}(I,1) = 0$, Question 1.1 is in some sense the first non-trivial case to consider. Note that when $s\text{defect}(I,2) = 1$, then from an algebraic point of view, the ideal $I^2$ is almost equal to $I^{(2)}$ except that it is missing exactly one generator.

We now give an outline of the results of this paper. In Section 2, we provide the relevant background, and recall some useful tools about powers of ideals and their symbolic powers.

In Sections 3 through 5, we study $s\text{defect}(I,m)$ when $I$ defines a star configuration. Note that in this paper, when we refer to star configurations, the forms that define the star configurations are forms of any degree, not necessarily linear, which is required in other papers. Our main strategy to compute $s\text{defect}(I,m)$ is to find an ideal $J$ such that $I^{(m)} = J + I^m$, and then to show that all the minimal generators are $J$ are required. The recent techniques using matroid ideals developed by Geramita, Harbourne, Migliore, and Nagel [18] will play a key role in our proofs. Our results will imply a similar decomposition found by Lampa-Baczyńska and Malara [31] which considers only star configurations defined using monomial ideals.

In Section 3 we also compute some values of $s\text{defect}(I,m)$ with $m \geq 3$ for some special families of star configurations. Section 4 complements Section 3 by showing that under some extra hypotheses, $s\text{defect}(I,2) = 1$ can force a geometric condition. Specifically, we show that if $X$ is a set of points in $\mathbb{P}^2$ with a linear graded resolution, and if $s\text{defect}(I,2) = 1$, then $I$ must be the ideal of a linear star configuration of points in $\mathbb{P}^2$. In Section 5 we apply our results of Section 3 to compute the graded minimal free resolution of $I^{(2)}$ when $I$ defines a star configuration of codimension two in $\mathbb{P}^n$. This result gives a partial generalization of a result of Geramita, Harbourne, and Migliore [17] (see Remark 5.4).

In Section 6, we turn our attention to general sets of points in $\mathbb{P}^2$. Our main result is a classification of the general sets of points whose defining ideals $I_X$ satisfy $s\text{defect}(I_X,2) = 1$.

**Theorem** (Theorem 6.3). Let $X$ be a set of $s$ general points in $\mathbb{P}^2$ with defining ideal $I_X$. Then

(i) $s\text{defect}(I_X,2) = 0$ if and only if $s = 1, 2$ or 4.
(ii) \( \text{sdefect}(I_X, 2) = 1 \) if and only if \( s = 3, 5, 7, \text{ or } 8 \).

(iii) \( \text{sdefect}(I_X, 2) > 1 \) if and only if \( s = 6 \) or \( s \geq 9 \).

Our proof relies on a deep result of Alexander-Hirschowitz [2] on the Hilbert functions of general double points, and some results of Catalisano [10], Harbourne [22], and Ida [30] on the graded minimal free resolutions of double points. We end this paper with an example to show that the symbolic defect sequence is not monotonic by computing some values of \( \text{sdefect}(I_X, m) \) when \( X \) is eight general points in \( \mathbb{P}^2 \) (see Example 6.5).

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2. Background results

We review the required background. We continue to use the notation of the introduction. Let \( I \) be a homogeneous ideal of \( R = \mathbb{k}[x_0, \ldots, x_n] \). The \textit{m-th symbolic power of } \( I \), denoted \( I^{(m)} \), is defined to be

\[
I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R)
\]

where \( \text{Ass}(I) \) denotes the set of associated primes of \( I \) and \( R_P \) is the ring \( R \) localized at the prime ideal \( P \).

Remark 2.1. There is some ambiguity in the literature concerning the notion of symbolic powers. The intersection in the definition is sometimes taken over all associated primes and sometimes just over the minimal primes of \( I \). In general, these two possible definitions yield different results. However, they agree in the case of radical ideals.

In general, \( I^m \subseteq I^{(m)} \), but the reverse containment may fail. If \( \text{sdefect}(I, m) = s \), then there exists \( s \) homogeneous forms \( F_1, \ldots, F_s \) of \( R \) such that

\[
I^{(m)} / I^m = \langle F_1 + I^m, \ldots, F_s + I^m \rangle \subseteq R/I^m.
\]

It follows that \( I^{(m)} = \langle F_1, \ldots, F_s \rangle + I^m \). Note that the ideal \( \langle F_1, \ldots, F_s \rangle \) is not unique. Indeed, if \( G_1, \ldots, G_s \) is another set of coset representatives such that \( I^{(m)} / I^m = \langle G_1 +
\[ I^m_1, \ldots, G^m_s \), we still have \( I^{(m)}_1 = \langle G_1, \ldots, G_s \rangle + I^m_1 \), but \( \langle F_1, \ldots, F_s_\rangle \) and \( \langle G_1, \ldots, G_s \rangle \) may be different ideals.

We state some simple facts about \( s\text{defect}(I, m) \).

**Lemma 2.2.** Let \( I \) be a homogeneous radical ideal of \( R \).

(i) \( s\text{defect}(I, 1) = 0 \).

(ii) If \( I \) is a complete intersection, then \( s\text{defect}(I, m) = 0 \) for all \( m \geq 1 \).

**Proof.** (i) This fact is trivial. (ii) This result follows from Zariski-Samuel \[38, \text{Appendix 6, Lemma 5}\].

Recall that \( I \) is a generic complete intersection if the localization of \( I \) at any minimal associated prime of \( I \) is a complete intersection. A result of \[11, 35, 37\] will prove useful:

**Theorem 2.3** \([11, \text{Corollary 2.6}], [35], [37]\). Let \( I \) be a homogeneous ideal of \( k[x_0, \ldots, x_n] \) that is perfect, codimension two, and a generic complete intersection. If

\[
0 \longrightarrow F \longrightarrow G \longrightarrow I \longrightarrow 0
\]

is a graded minimal free resolution of \( I \), then

\[
0 \longrightarrow 2 \bigwedge F \longrightarrow F \otimes \text{Sym}^1 G \rightarrow \text{Sym}^2 G \longrightarrow I^2 \longrightarrow 0
\]

is a graded minimal free resolution of \( I^2 \).

**Remark 2.4.** Weyman’s paper \[37\] gives the resolution of \( \text{Sym}^2(I) \). As shown in \[11, 35\], the hypotheses on \( I \) imply that \( \text{Sym}^2(I) \cong I^2 \).

Many of our arguments make use of Hilbert functions. The Hilbert function of \( R/I \), denoted \( H_{R/I} \), is the numerical function \( H_{R/I} : \mathbb{N} \to \mathbb{N} \) defined by

\[
H_{R/I}(i) := \dim_k R_i - \dim_k I_i
\]

where \( R_i \), respectively \( I_i \), denotes the \( i \)-th graded component of \( R \), respectively \( I \).

Our primary focus is to understand \( s\text{defect}(I, m) \) when \( I \) defines either a star configuration or a set of points in \( \mathbb{P}^2 \). In the next section, we introduce star configurations in more detail. For now, we review the relevant background about sets of points in \( \mathbb{P}^2 \).

Let \( \mathbb{X} = \{P_1, \ldots, P_s\} \) be a set of distinct points in \( \mathbb{P}^2 \). If \( I_{P_i} \) is the ideal associated to \( P_i \) in \( R = k[x_0, x_1, x_2] \), then the homogeneous ideal associated to \( \mathbb{X} \) is the ideal \( I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s} \). The next lemma allows us to describe \( I^{(m)}_{\mathbb{X}} \); although this result is well-known, we have included a proof for completeness.

**Lemma 2.5.** Let \( \mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2 \) be a set of \( s \) distinct points with associated ideal \( I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s} \). Then for all \( m \geq 1 \), \( I^{(m)}_{\mathbb{X}} = I^{m}_{P_1} \cap \cdots \cap I^{m}_{P_s} \).

**Proof.** The associated primes of \( I_{\mathbb{X}} \) are the ideals \( I_{P_i} \) with \( i = 1, \ldots, s \). Because localization commutes with products, we have

\[
I^{m}_{\mathbb{X}} R_{P_i} = (I_{\mathbb{X}} R_{P_i})^m = (I_{P_i} R_{P_i})^m = I^{m}_{P_i} R_{P_i}.
\]
Note that the second equality follows from the fact that $I_p$ is the only associated prime of $I_X$ contained in $I_p$. Since $I_p^mR_p \cap R = I_p^m$, the result follows. □

For sets of points in $\mathbb{P}^2$, the symbolic defect sequence will either be all zeroes, or all values of the sequence, except the first, will be nonzero. Moreover, we can completely classify when the symbolic defect sequence is all zeroes.

**Theorem 2.6.** Let $X \subseteq \mathbb{P}^2$ be any set of points. Then the following are equivalent:

(i) $I_X$ is a complete intersection.
(ii) $\text{sdefect}(I_X, m) = 0$ for all $m \geq 1$.
(iii) $\text{sdefect}(I_X, m) = 0$ for some $m \geq 2$.

**Proof.** Lemma 2.2 shows (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is immediate. For (iii) $\Rightarrow$ (i), it was noted in [11, Remark 2.12(i)] that when $X$ is not a complete intersection of points in $\mathbb{P}^2$, then $I_X^m \neq I_X^{(m)}$ for all $m \geq 2$. This also follows from [29, Theorem 2.8] or [28, Corollary 2.5]. □

### 3. Symbolic squares of star configurations

In this section, we will consider $\text{sdefect}(I, 2)$ when $I$ defines a star configuration. In fact, we prove a stronger result by finding an ideal $J$ such that $I^{(2)} = J + I^2$. It is interesting to note that the ideal $J$ will also be a star configuration. For completeness, we begin with the relevant background on star configurations.

**Definition 3.1.** Let $n$, $c$ and $s$ be positive integers with $1 \leq c \leq \min\{n, s\}$. Let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of forms in $\mathcal{R} = \mathbb{k}[x_0, x_1, \ldots, x_n]$ with the property that all subsets of $\mathcal{F}$ of cardinality $c + 1$ are regular sequences in $\mathcal{R}$. Define an ideal of $\mathcal{R}$ by setting

$$I_{c, \mathcal{F}} = \bigcap_{1 \leq i_1 < \ldots < i_c \leq s} \langle F_{i_1}, \ldots, F_{i_c} \rangle.$$

The vanishing locus of $I_{c, \mathcal{F}}$ in $\mathbb{P}^n$ is called a star configuration.

When the forms $F_1, \ldots, F_s$ are all linear, we will write $\mathcal{L} = \{L_1, \ldots, L_s\}$ instead of $\mathcal{F} = \{F_1, \ldots, F_s\}$, and we will call the vanishing locus of $I_{c, \mathcal{L}}$ a linear star configuration.

**Remark 3.2.** A.V. Geramita is attributed with first coining the term star configuration to describe the variety defined by $I_{c, \mathcal{F}}$. The name is inspired by the fact that when $n = c = 2$, and $s = 5$, the placement of the five lines $\mathcal{L} = \{L_1, \ldots, L_5\}$ that define a linear star configuration resembles a star. In this case, the locus of $I_{c, \mathcal{L}}$ is a set of 10 points corresponding to the intersections between these lines. It should be noted that linear star configurations were classically called $l$-laterals (e.g. see [12]). On the other hand, our more general definition follows [18], where the geometric objects are called hypersurface configurations. This more general definition of star configurations evolved through a series of papers (see [11, 33, 18]); in particular, the codimension 2 case was studied before the general case. Star configurations have been shown to have many nice algebraic properties, but at the same time, can be used to exhibit extremal properties. The references [7, 8, 9, 17, 20] form a small sample of papers that have studied the ideals $I_{c, \mathcal{F}}$. 
Remark 3.3. Geometrically, the vanishing locus in $\mathbb{P}^n$ of the ideal $\langle F_{i_1}, \ldots, F_{i_c} \rangle$ is a complete intersection of codimension $c$ obtained by intersecting the hypersurfaces defined by the forms $F_{i_1}, \ldots, F_{i_c}$. A star configuration is then a union of such complete intersections.

Remark 3.4. While the definition of a star configuration makes sense for $s < n + 1$, such cases are less interesting (cf. [17, Remark 2.2]). Therefore we will always assume that $s \geq n + 1$.

Theorem 3.5. Let $I_{c,F}$ be the defining ideal of a star configuration in $\mathbb{P}^n$, with $F = \{F_1, \ldots, F_s\}$. Then

$$\{F_{i_1} \cdots F_{i_{s-c+1}} \mid 1 \leq i_1 < \ldots < i_{s-c+1} \leq s\}$$

is a minimal generating set of $I_{c,F}$.

Proof. See [33, Theorem 2.3] for generation (see also [18, Proposition 2.3 (4)]) and [33, Corollary 3.5] for minimality. 

We will make use of the following decomposition of the $m$-th symbolic power; this follows from [18, Theorem 3.6 (i)].

Theorem 3.6. Let $I_{c,F}$ be the defining ideal of a star configuration in $\mathbb{P}^n$, with $F = \{F_1, \ldots, F_s\}$. For all $m \geq 1$, we have

$$I_{c,F}^{(m)} = \bigcap_{1 \leq i_1 < \ldots < i_c \leq s} \langle F_{i_1}, \ldots, F_{i_c} \rangle^m.$$

We will first consider the case of a linear star configuration $I_{c,L}$ in $\mathbb{P}^n$, with $L = \{L_1, \ldots, L_s\}$ when $|L| = n + 1$. In this context, we can reduce to the case of monomial ideals. Then, following [18], we will apply our results to obtain corresponding statements for arbitrary star configurations.

3.1. The monomial case. Let $I_{c,L}$ be the defining ideal of a linear star configuration in $\mathbb{P}^n$, with $L = \{L_1, \ldots, L_s\}$. Suppose that $|L| = n + 1$. Then, up to a change of variables, we may assume that the hyperplanes forming the star configuration are defined by the coordinate functions $x_0, x_1, \ldots, x_n$. By Theorem 3.6, we have

$$I_{c,L}^{(m)} = \bigcap_{0 \leq i_1 < \ldots < i_c \leq n} \langle x_{i_1}, \ldots, x_{i_c} \rangle^m.$$

Clearly, $I_{c,L}$ and its symbolic powers are monomial ideals. A monomial $p = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ belongs to $I_{c,L}^{(m)}$ if and only if it satisfies the condition

$$\forall 0 \leq i_1 < \ldots < i_c \leq n, \; a_{i_1} + a_{i_2} + \ldots + a_{i_c} \geq m.$$

Let $\text{Supp}(p)$ denote the support of $p$, i.e., $\text{Supp}(p) = \{x_i \mid x_i \text{ divides } p\}$.

We are now able to describe an ideal $M$ with the property that $I_{c,L}^{(m)} = I_{c,L}^m + M$.

Theorem 3.7. Let $L = \{x_0, \ldots, x_n\}$. Then $I_{c,L}^{(m)} = I_{c,L}^m + M$, where $M$ is the ideal generated by all monomials satisfying equation (3.1) whose support has cardinality at least $n - c + 3$. 

Proof. Clearly $I_{c,L}^{(m)} \supseteq I_{c,L}^m + M$. To show the other containment, consider a monomial $p = x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} \in I_{c,L}^{(m)}$. Since $p \in I_{c,L}^{(m)}$, we have $p \in I_{c,L}$. Then $\lvert \text{Supp}(p) \rvert \geq n - c + 2$ by Theorem 3.5.

If $\lvert \text{Supp}(p) \rvert = n - c + 2$, then the complement of $\text{Supp}(p)$ in $\{x_0, x_1, \ldots, x_n\}$ has cardinality $c - 1$. Therefore we can write

$$\{x_0, x_1, \ldots, x_n\} \setminus \text{Supp}(p) = \{x_{j_1}, \ldots, x_{j_{c-1}}\}.$$  

For each $x_i \in \text{Supp}(p)$, equation (3.1) implies that

$$a_i = a_i + a_{j_1} + \ldots + a_{j_{c-1}} \geq m.$$  

Thus $p$ is a multiple of

$$\prod_{x_i \in \text{Supp}(p)} x_i^m = \left( \prod_{x_i \in \text{Supp}(p)} x_i \right)^m$$

which is the $m$-th power of a generator of $I_{c,L}$ by Theorem 3.5. Therefore $p \in I_{c,L}^m$.

On the other hand, if $\lvert \text{Supp}(p) \rvert \geq n - c + 3$, then $p \in M$ by definition. \(\square\)

For $m = 2$ and $m = 3$, we can improve upon the statement of Theorem 3.7. \(\square\)

**Corollary 3.8.** Let $L = \{x_0, \ldots, x_n\}$. We have $I_{c,L}^{(2)} = I_{c-1,L} + I_{c,L}^2$.

Proof. By [17, Lemma 2.13], we have $I_{c-1,L} \subseteq I_{c,L}^{(2)}$, which implies the containment $I_{c,L}^{(2)} \supseteq I_{c-1,L} + I_{c,L}^2$ (these containments hold for any linear star configuration ideal, not just a monomial star configuration ideal). To prove the other containment, we use the fact that our ideals are monomial ideals.

Consider a monomial $p = x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n} \in I_{c,L}^{(2)}$. As observed in the proof of Theorem 3.7, $\lvert \text{Supp}(p) \rvert \geq n - c + 2$ and, in the case of equality, $p \in I_{c,L}^2$. Assume $\lvert \text{Supp}(p) \rvert \geq n - c + 3$. Then $p$ is divisible by one of the generators of $I_{c-1,L}$ described in Theorem 3.5. Therefore $p \in I_{c-1,L}$. \(\square\)

**Remark 3.9.** The above result was first proved in [31, Corollary 3.7, Corollary 4.5] in the special cases that $n = c = 2$, and $n = c = 3$. The above statement is also mentioned in [31, Remark 4.6], but no proof is given.

**Corollary 3.10.** Let $L = \{x_0, \ldots, x_n\}$. If $c \geq 3$, we have $I_{c,L}^{(3)} = I_{c-2,L} + I_{c-1,L} I_{c,L} + I_{c,L}^3$.

Proof. We require $c \geq 3$ so that the ideals on the right hand side are defined. We first show that $I_{c-2,L} \subseteq I_{c,L}^{(3)}$. Recall that

$$I_{c-2,L} = \langle x_1, \ldots, x_{i_{n-4}} \mid 0 \leq i_1 < \cdots < i_{n-c+4} \leq n \rangle.$$  

Consider any subset $A = \{x_{i_1}, \ldots, x_{i_c}\}$ of $\{x_0, x_1, \ldots, x_n\}$ with $\lvert A \rvert = c$, and consider any generator $m = x_{i_1} \cdots x_{i_{n-c+4}}$ of $I_{c-2,L}$. Then at least three of the variables of $A$, say $x_{i_1}, x_{i_j},$ and $x_{i_k}$, appear in $\text{Supp}(m) = \{x_{i_1}, \ldots, x_{i_{n-c+4}}\}$. Because $x_{i_1} x_{i_j} x_{i_k} \in \langle x_{i_1}, \ldots, x_{i_c} \rangle^3$, this means that $m \in \langle x_{i_1}, \ldots, x_{i_c} \rangle^3$. But this implies that every generator $m$ of $I_{c-2,L}$ satisfies

$$m \in \left( \bigcap_{0 \leq i_1 < \cdots < i_c \leq n} \langle x_{i_1}, \ldots, x_{i_c} \rangle^3 \right) = I_{c,L}^{(3)}.$$
In other words, \( I_{c-2,L} \subseteq I_{c,L}^{(3)} \).

By [17 Lemma 2.13], we have \( I_{c-1,L} \subseteq I_{c,L}^{(2)} \). This result allows us to conclude that
\[
I_{c-1,L}I_{c,L} \subseteq I_{c,L}^{(2)} I_{c,L} \subseteq I_{c,L}^{(3)}.
\]

Therefore we have the containment \( I_{c,L}^{(3)} \supseteq I_{c-2,L} + I_{c-1,L}I_{c,L} + I_{c,L}^3 \). To prove the other containment, we again exploit the fact that our ideals are all monomial.

Consider a monomial \( p = x_0a_0x_1^{a_1} \cdots x_n^{a_n} \in I_{c,L}^{(3)} \). By Theorem 3.7, \( |\text{Supp}(p)| \geq n - c + 2 \) and, in the case of equality, \( p \in I_{c,L}^2 \). Let \( |\text{Supp}(p)| = n - c + 3 \). In this case, the complement of \( \text{Supp}(p) \) in \( \{x_0, x_1, \ldots, x_n\} \) has cardinality \( c - 2 \), so we can write
\[
\{x_0, x_1, \ldots, x_n\} \setminus \text{Supp}(p) = \{x_j, \ldots, x_{j_{c-2}}\}.
\]
For each pair \( x_{i_1}, x_{i_2} \in \text{Supp}(p) \), equation (3.1) implies that
\[
a_{i_1} + a_{i_2} = a_{i_1} + a_{i_2} + a_{j_1} + \ldots + a_{j_{c-2}} \geq 3.
\]
Thus either \( a_{i_1} \geq 2 \) or \( a_{i_2} \geq 2 \). Repeating the same argument for all pairs \( x_{i_1}, x_{i_2} \) in \( \text{Supp}(p) \), it follows that there are \( n - c + 2 \) elements \( x_h \in \text{Supp}(p) \) such that \( x_h^2 \mid p \). Hence \( p \) is divisible by a monomial of the form
\[
x_{k_0}x_{k_1}^2 \cdots x_{k_{n-c+2}}^2 = (x_{k_0}x_{k_1} \cdots x_{k_{n-c+2}})(x_{k_1} \cdots x_{k_{n-c+2}}),
\]
and therefore \( p \in I_{c-1,L}I_{c,L} \) by Theorem 3.5. As in the previous proof, if \( |\text{Supp}(p)| \geq n - c + 4 \), then \( p \) is divisible by a generator of \( I_{c-2,L} \), which completes the proof. \( \square \)

**Theorem 3.11.** Let \( L = \{x_0, \ldots, x_n\} \). We have \( \text{sdefect}(I_{c,L}, m) = 1 \) if and only if \( c = m = 2 \).

**Proof.** Let \( c = m = 2 \). By Theorem 3.5, \( I_{c-1,L} = I_{1,L} = \langle x_0x_1 \cdots x_n \rangle \) is a principal ideal generated in degree \( n + 1 \). In contrast, \( I_{c,L}^2 \) is generated in degree \( n^2 \). Therefore, the equality \( I_{c,L}^{(2)} = I_{c-1,L} + I_{c,L}^2 \) of Corollary 3.8 implies that \( I_{c,L}^{(2)}/I_{c,L}^2 \) has a single minimal generator. Thus \( \text{sdefect}(I_{c,L}, m) = 1 \).

Conversely, assume \( \text{sdefect}(I_{c,L}, m) = 1 \). By Theorem 3.7, \( I_{c,L}^{(m)} = I_{c,L}^m + M \), where \( M \) is the monomial ideal generated by all monomials satisfying equation (3.1) whose support has cardinality at least \( n - c + 3 \). Since \( \text{sdefect}(I_{c,L}, m) = 1 \), we deduce \( M \neq 0 \). Given any monomial \( p \in M \), we must have
\[
n + 1 \geq |\text{Supp}(p)| \geq n - c + 3.
\]
This implies \( c \geq 2 \). For any choice of indices \( 0 \leq i_1 < \cdots < i_{n-c+3} \leq n \), the monomial
\[
p = x_{i_1}x_{i_2}^{m-1}x_{i_3}^{m-1} \cdots x_{i_{n-c+3}}^{m-1}
\]
satisfies the condition in equation (3.1), and therefore \( p \in M \). We claim that \( p \) is a minimal generator of \( M \). If it was not, then we could divide \( p \) by a variable in its support and obtain a new monomial still in \( M \). However, if we divide \( p \) by any variable in its support, we either obtain a monomial whose support has cardinality less than \( n - c + 3 \) or one that violates equation (3.1). Thus the claim holds. Note also that the degree of \( p \) is \( (m-1)(n-c+2) + 1 \), and this is strictly smaller than the degree of a minimal generator of \( I_{c,L}^m \), i.e., \( m(n-c+2) \). It follows that the residue class of \( p \) can be taken as a
minimal generator of $I_{c,L}^{(m)} / I_{c,L}^m$. Hence each monomial of the same form as $p$ contributes 1 to $\text{sdefect}(I_{c,L}, m)$. Now, if $c > 2$ or $m > 2$, the freedom in the choice of the indices $i_1, \ldots, i_{n-c+3}$ implies that $\text{sdefect}(I_{c,L}, m) > 1$. \hfill $\Box$

3.2. The general case. To extend the results of the monomial case to arbitrary star configurations, we recall a powerful theorem of Geramita, Harbourne, Migliore, and Nagel \cite[Theorem 3.6 (i)]{GeramitaHarbourneMiglioreNagel1995}.

**Theorem 3.12.** Let $I_{c,F}$ be the defining ideal of a star configuration in $\mathbb{P}^n$, with $F = \{F_1, \ldots, F_s\} \subseteq R = k[x_0, x_1, \ldots, x_n]$. Let $S = k[y_1, \ldots, y_s]$ and define a ring homomorphism $\varphi: S \to R$ by setting $\varphi(y_i) = F_i$ for $1 \leq i \leq s$. If $I$ is an ideal of $S$, then we write $\varphi_*(I)$ to denote the ideal of $R$ generated by $\varphi(I)$. Let $L = \{y_1, \ldots, y_s\}$. Then, for each positive integer $m$, we have

$$I_{c,F}^{(m)} = \varphi_*(I_{c,L})^{(m)} = \varphi_*(I_{c,L}^{(m)}).$$

Since the operator $\varphi_*$ commutes with ideal sums and products, Theorem 3.12 applied to our results from the previous section gives the following more general statements.

**Theorem 3.13.** Let $I_{c,F}$ be the defining ideal of a star configuration in $\mathbb{P}^n$, with $F = \{F_1, \ldots, F_s\}$. Then $I_{c,F}^{(m)} = I_{c,F}^m + M$, where $M$ is the ideal generated by all products $F_1^{a_1} \cdots F_s^{a_s}$ such that:

1. $\{|i | a_i > 0| \geq s - c + 2$;
2. $\forall 0 \leq i_1 < \ldots < i_c \leq n$, $a_{i_1} + a_{i_2} + \ldots + a_{i_c} \geq m$.

**Corollary 3.14.** We have $I_{c,F}^{(2)} = I_{c-1,F} + I_{c,F}^2$.

**Corollary 3.15.** We have $\text{sdefect}(I_{c,F}, 2) \leq (\binom{s}{c-2})$. Furthermore, if $F = L = \{L_1, \ldots, L_s\}$, that is, if $I_{c,L}$, is a linear star configuration, then $\text{sdefect}(I_{c,L}, 2) = (\binom{s}{c-2})$.

**Proof.** By Corollary 3.14 $I_{c,F}^{(2)} = I_{c-1,F} + I_{c,F}^2$. By Theorem 3.5 the ideal $I_{c-1,L}$ is generated by $(\binom{s}{c-2})$ minimal generators, so we need to add at most $(\binom{s}{c-2})$ generators to $I_{c,F}^2$ to generate $I_{c,L}^{(2)}$.

If $F = L$, by Theorem 3.5 $I_{c,L}^2$ is generated by forms of degree $2(s - c + 1)$. On the other hand, again by Theorem 3.5 the ideal $I_{c-1,L}$ is generated by generators of degree $s - c + 2$. Since $s - c + 2 < 2(s - c + 1)$, all the generators of $I_{c-1,L}$ need to be added to $I_{c,L}^2$ to generate $I_{c,L}^{(2)}$, i.e., none of them are redundant. \hfill $\Box$

**Remark 3.16.** In the above proof, we appealed to the degrees of the elements of $L$ to justify why all the generators of $I_{c-1,L}$ are required. In the general case, it may happen that some of the minimal generators of $I_{c-1,F}$ have degree larger than a minimal generator of $I_{c,F}^2$, thus preventing us from generalizing this argument.

The following are also immediate consequences of results from the previous section.

**Corollary 3.17.** We have $I_{c,F}^{(3)} = I_{c-2,F} + I_{c-1,F} I_{c,F} + I_{c,F}^3$. In particular,

$$\text{sdefect}(I_{c,F}, 3) \leq (\binom{s}{c-3}) + \binom{s}{c-2} \binom{s}{c-1}.$$
Theorem 3.18. We have $\text{sdefect}(I_{c,F}, m) = 1$ if and only if $c = m = 2$.

3.3. Powers of codimension two linear star configurations. We round out this section by considering the higher $m$-th symbolic powers of the linear star configuration $I_{2,L}$ in $\mathbb{P}^2$. Note that in this case the linear star configuration defines a collection of points in $\mathbb{P}^2$. By applying a result of Huneke and Harbourne [23] Corollary 3.9] (and see also [11] Example 3.9] for additional details), we have the following relationship between the regular and symbolic powers of $I_{2,L}$ in $\mathbb{P}^2$.

Theorem 3.19. Suppose that $I_{2,L}$ defines a linear star configuration in $\mathbb{P}^2$. Then
\[ I_{2,L}^{(2m)} = (I_{2,L}^{(2)})^m \text{ for all } m \geq 1. \]

We can then derive bounds on some of the values of the symbolic defect sequence.

Theorem 3.20. Suppose that $I_{2,L}$ defines a linear star configuration in $\mathbb{P}^2$. Then
\[ \text{sdefect}(I_{2,L}, 2m) \leq 1 + |L|(m - 1) \text{ for all } m \geq 1. \]

Proof. Suppose that $L = \{L_1, \ldots, L_s\}$. By Corollary 3.14 we have
\[ I_{2,L}^{(2)} = \langle L_1 \cdots L_s \rangle + I_{2,L}^2 \]

since $I_{1,L} = \langle L_1 \cdots L_s \rangle$. Let $L = L_1 \cdots L_s$. It then follows by Theorem 3.19 that
\[
I_{2,L}^{(2m)} = \left( \langle L \rangle + I_{2,L}^2 \right)^m = \langle L \rangle^m + \langle L \rangle^{m-1}I_{2,L}^2 + \langle L \rangle^{m-2}I_{2,L}^4 + \cdots + \langle L \rangle I_{2,L}^{2m-2} + I_{2,L}^{2m}.
\]

Since $I_{2,L}$ is generated by forms of degree $(s - 1)$, we can use a degree argument to show that none of the generators of $\langle L \rangle^m + \langle L \rangle^{m-1}I_{2,L}^2 + \langle L \rangle^{m-2}I_{2,L}^4 + \cdots + \langle L \rangle I_{2,L}^{2m-2}$ belong to $I_{2,L}^{2m}$.

Define $J_{2a} = \langle \frac{L_{2a}}{L_1} | i = 1, \ldots, s \rangle$ for $a = 1, \ldots, m - 1$. We claim that for $1 \leq a \leq m - 1$,
\[
\langle L \rangle^m + \langle L \rangle^{m-1}I_{2,L}^2 + \cdots + \langle L \rangle I_{2,L}^{2a} = \langle L \rangle^m + \langle L \rangle^{m-1}J_2 + \cdots + \langle L \rangle I_{2,L}^{2a} + \langle L \rangle^a J_{2a}.
\]

Indeed, the ideal on the right is contained in the ideal on the left because each generator of $J_{2a}$ is a generator of $I_{2,L}^{2a}$.

For the reverse containment, we do induction on $a$. It is straightforward to check that $\langle L \rangle^m + \langle L \rangle^{m-1}I_{2,L}^2 = \langle L \rangle^m + \langle L \rangle^{m-1}J_2$ for the base case. Assume now that $2 \leq a \leq m - 1$. By induction on $a$,
\[
\langle L \rangle^m + \langle L \rangle^{m-1}I_{2,L}^2 + \cdots + \langle L \rangle I_{2,L}^{2a-1} = \langle L \rangle^m + \langle L \rangle^{m-1}J_2 + \cdots + \langle L \rangle I_{2,L}^{2a-1} + \langle L \rangle^{m-a} J_{2a}.
\]

To finish the proof of the claim, we need to show that
\[ \langle L \rangle^{m-a} I_{2,L}^{2a} \subseteq \langle L \rangle^m + \langle L \rangle^{m-1}J_2 + \cdots + \langle L \rangle^{m-a}J_{2(a-1)} + \langle L \rangle^{m-a} J_{2a}. \]

Because of Theorem 3.5, $I_{2,L}$ is generated by elements of the form $F_i = L/L_i$ for some $i = 1, \ldots, s$. So, a generator of $I_{2,L}^{2a}$ has the form $F_{i_1}F_{i_2} \cdots F_{i_{2a}}$ where $i_1, \ldots, i_{2a}$ need not be distinct. If $i_1 = \cdots = i_{2a} = i$, then the generator $F_{i_1}F_{i_2} \cdots F_{i_{2a}} = \frac{L_{2a}}{L_i}$ of $I_{2,L}^{2a}$ is also
a generator of $J_{2a}$, so $L^{m-a}F_1F_2 \cdots F_{2a} \in \langle L \rangle^{m-a}J_{2a}$. If at least two of $i_1, \ldots, i_{2a}$ are distinct, say $i_1 \neq i_2$, then

$$F_1F_2 \cdots F_{2a} = \frac{F_{i_1}}{L_{i_1}} \frac{F_{i_2}}{L_{i_1}} F_3 \cdots F_{2a} = \frac{F_{i_2}}{L_{i_1}} F_1 \cdots F_{i_2a}.$$ But then

$$L^{m-a}F_1F_2 \cdots F_{2a} = L^{m-a+1}F_1 \frac{F_{i_2}}{L_{i_1}} F_3 \cdots F_{2a} \in \langle L \rangle^{m-(a-1)}J_{2(a-1)}.$$ By induction, we then have

$$L^{m-a}F_1F_2 \cdots F_{2a} \in \langle L \rangle^m + \langle L \rangle^{m-1}J_2 + \cdots + \langle L \rangle^{m-a+1}J_{2(a-1)} + \langle L \rangle^{m-a}J_{2a}.$$ This now verifies the claim.

To complete the proof, note that to form $I_{c,L}^{(2m)}$, we can add all of the generators of $\langle L \rangle^m + \langle L \rangle^{m-1}J_2 + \cdots + \langle L \rangle^{m-a}J_{2(a-1)}$ to $I_{2,L}^{2m}$. This ideal has at most $1 + s(m-1)$ minimal generators (our generating set may not be minimal) since each ideal $J_{2a}$ has $s$ generators, so $\text{sdefect}(I_{2,L}, 2m) \leq 1 + s(m-1).$

4. A GEOMETRIC CONSEQUENCE

By Theorem 3.18 if $I_{c,L}$ is a linear star configuration in $\mathbb{P}^m$ of codimension two, then $\text{sdefect}(I_{c,L}, 2) = 1$ since $c = 2$. If $n = 2$, then the linear star configuration defined by $I_{c,L}$ is a collection of points in $\mathbb{P}^2$, and thus, there exist sets of points $\mathbb{X}$ in $\mathbb{P}^2$ with $\text{sdefect}(I_{\mathbb{X}, 2}) = 1$. In general, it would be interesting to classify all the ideals $I_\mathbb{X}$ of sets of points $\mathbb{X}$ in $\mathbb{P}^2$ with $\text{sdefect}(I_{\mathbb{X}, 2}) = 1$. In this section, we show under some additional hypotheses, that if $\mathbb{X}$ is a set of points in $\mathbb{P}^2$ with $\text{sdefect}(I_{\mathbb{X}, 2}) = 1$, then $\mathbb{X}$ must be a linear star configuration.

We first recall some facts about the defining ideals of points in $\mathbb{P}^2$; many of these results are probably known to the experts, but for completeness, we include their proofs. Recall that for any homogeneous ideal $I \subseteq R$, we let $\alpha(I) = \min\{i \mid I_i \neq 0\}$. Note that for any $m \geq 1$, $\alpha(I^m) = m\alpha(I)$.

**Lemma 4.1.** Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a finite set of points. If $\alpha(I_{\mathbb{X}}) = \alpha$, then $I_{\mathbb{X}}$ has at most $\alpha + 1$ minimal generators of degree $\alpha$.

**Proof.** Because $\alpha(I_{\mathbb{X}}) = \alpha$, the Hilbert function of $\mathbb{X}$ at $\alpha-1$ is $H_{R/I_{\mathbb{X}}} (\alpha-1) = \dim_k R_{\alpha-1} = \binom{\alpha+1}{2}$. If $I_{\mathbb{X}}$ has $d > \alpha + 1$ generators of degree $\alpha$, then $H_{R/I_{\mathbb{X}}} (\alpha) = \binom{\alpha+2}{2} - d < \binom{\alpha+1}{2} - (\alpha + 1) = \binom{\alpha+1}{2}$. In other words, $H_{R/I_{\mathbb{X}}} (\alpha-1) > H_{R/I_{\mathbb{X}}} (\alpha)$, contradicting the fact that the Hilbert functions of sets of points must be non-decreasing functions [19, cf. proof of Proposition 1.1 (2)].

The next lemma is a classification of those sets of points which have exactly $\alpha + 1$ minimal generators of degree $\alpha$.

**Lemma 4.2.** Let $\mathbb{X}$ be a set of points of $\mathbb{P}^2$. Then the following are equivalent:

(i) The ideal $I_{\mathbb{X}}$ has $\alpha + 1$ minimal generators of degree $\alpha(I_{\mathbb{X}}) = \alpha$;
(ii) The set $\mathbb{X}$ is a set of $\binom{\alpha+1}{2}$ points in $\mathbb{P}^2$ having generic Hilbert function, i.e.,
$$H_{R/I_\mathbb{X}}(i) = \min\{\dim_k R_i, |\mathbb{X}|\} \text{ for all } i \geq 0; \text{ and}$$

(iii) The ideal $I_\mathbb{X}$ has a graded linear resolution.

Proof. (i) $\Rightarrow$ (ii) If $I_\mathbb{X}$ has $\alpha + 1$ minimal generators of degree $\alpha$, it follows that
$$\binom{\alpha + 1}{2} = H_{R/I_\mathbb{X}}(\alpha - 1) = H_{R/I_\mathbb{X}}(\alpha) = \binom{\alpha + 2}{2} - \binom{\alpha + 1}{1}.$$ Because the Hilbert function of a set of points in $\mathbb{P}^2$ is a strictly increasing function until it reaches $|\mathbb{X}|$, we have $|\mathbb{X}| = \binom{\alpha + 1}{2}$, and the Hilbert function of $R/I_\mathbb{X}$ is given by
$$H_{R/I_\mathbb{X}}(t) = \min\left\{\dim_k R_t, \binom{\alpha + 1}{2}\right\} \text{ for all } t \geq 0.$$

(ii) $\Rightarrow$ (iii) If $R/I_\mathbb{X}$ has the generic Hilbert function, one can use Section 3 of [32] to deduce that the resolution is
$$0 \to R^\alpha(-(\alpha + 1)) \to R^{\alpha + 1}(-\alpha) \to R \to R/I_\mathbb{X} \to 0,$$ i.e., the graded resolution is linear.

(iii) $\Rightarrow$ (i) Assume that $I_\mathbb{X}$ has a linear graded free resolution
$$0 \to R^{\beta - 1}(-(\alpha + 1)) \to R^{\beta}(-\alpha) \to R \to R/I_\mathbb{X} \to 0.$$ Since $H_{R/I_\mathbb{X}}(t) = H_{R/I_\mathbb{X}}(t + 1)$ for $t \geq 0$, we get that
$$\dim_k R_t - \beta \dim_k R_{t-\alpha} + (\beta - 1) \dim_k R_{t-(\alpha + 1)} = \dim_k R_{t+1} - \beta \dim_k R_{(t+1)-\alpha} + (\beta - 1) \dim_k R_{(t+1)-(\alpha + 1)}.$$ This proves that $\beta = \alpha + 1$, i.e., $I_\mathbb{X}$ has $\alpha + 1$ minimal generators of degree $\alpha$. \hfill $\square$

Lemma 4.3. Let $\mathbb{X}$ be a set of points of $\mathbb{P}^2$, and suppose that any of the three equivalent conditions of Lemma 4.2 holds. If $I^{(2)}_\mathbb{X} = I^2_\mathbb{X} + \langle F_1, \ldots, F_r \rangle$, then $\deg(F_i) < 2\alpha(I^2_\mathbb{X})$ for all $i = 1, \ldots, r$.

Proof. We first observe that because $I_\mathbb{X}$ is an ideal of points, then the saturation of $I^2_\mathbb{X}$ is $I^{(2)}_\mathbb{X}$. If $d$ is the saturation degree of $I^2_\mathbb{X}$, i.e., the smallest integer $d$ such that $(I^{(2)}_\mathbb{X})_t = (I^2_\mathbb{X})_t$ for all $t \geq d$, then it is known that $\text{reg}(I^2_\mathbb{X}) \geq d$ (see, for example, the introduction of [11]).

Again, because $I_\mathbb{X}$ is an ideal of points, we have
$$2\text{reg}(I_\mathbb{X}) \geq \text{reg}(I^2_\mathbb{X}) \geq \alpha(I^2_\mathbb{X}) = 2\alpha(I_\mathbb{X}) = 2\text{reg}(I_\mathbb{X}),$$ where the first inequality follows from [15, Theorem 1.1], and the last equality holds from the fact that $I_\mathbb{X}$ has a linear resolution. Thus, we get that $\text{reg}(I^2_\mathbb{X}) = 2\alpha(I_\mathbb{X})$, or in other words, $I^2_\mathbb{X}$ and $I^{(2)}_\mathbb{X}$ agree in degrees $\geq \text{reg}(I^2_\mathbb{X}) = 2\alpha(I_\mathbb{X})$. Therefore, any minimal generators of $I^{(2)}_\mathbb{X}$ have degrees less than $2\alpha(I_\mathbb{X}) = 2\alpha$, as we wished. \hfill $\square$
When $I_X$ is the homogeneous ideal of a finite set of points $X$ in $\mathbb{P}^2$, it is well known that $I_X$ is both perfect and has codimension two. In addition, $I_X$ is a generic complete intersection because $I_X$ is a radical ideal in a regular ring and the minimal associated primes of $I_X$ are simply the ideals of the points $P \in X$, and when we localize $I_X$ at $I_P$, we get the maximal ideal of $k[x_0, x_1, x_2]$ localized at $I_P$, which is a complete intersection. We can thus apply Theorem 2.3 to any homogeneous ideal of a finite set of points in $\mathbb{P}^2$. In particular, we record this fact as a lemma.

**Lemma 4.4.** Let $X \subseteq \mathbb{P}^2$ be a finite set of points. Suppose that $I_X$ has $d$ minimal generators of degree $\alpha = \alpha(I_X)$. Then $I_X^2$ has $\binom{d+1}{2}$ minimal generators of degree $\alpha(I_X^2) = 2\alpha$. In particular,

$$H_{R/I_X^2}(2\alpha) = \binom{2\alpha + 2}{2} - \binom{d + 1}{2}.$$ 

**Proof.** If $F_1, \ldots, F_d$ are the $d$ minimal generators of degree $\alpha = \alpha(I_X)$, then by Theorem 2.3 the elements of $\{ F_i F_j : 1 \leq i \leq j \leq d \}$ will all be minimal generators of $I_X^2$. Each generator will have degree $\alpha(I_X^2) = 2\alpha$ and there are $\binom{d+1}{2}$ such generators. For the last statement, since $I_X^2$ has no generators of degree $< 2\alpha$, we have $\dim_k(I_X^2)_{2\alpha} = \binom{d+1}{2}$. \hfill \Box

We also require a result of Bocci and Chiantini. Statement (i) can be found in the introduction of [6], while (ii) is [6] Theorem 3.3.

**Theorem 4.5.** Let $X \subseteq \mathbb{P}^2$ be a set of points.

(i) If $\alpha(I_X) = \alpha$, then $\alpha(I_X^{(2)}) \geq \alpha + 1$.

(ii) If $\alpha(I_X^{(2)}) = \alpha(I_X) + 1$, then $X$ is a linear star configuration of points or a set of colinear points.

We now come to the main result of this section.

**Theorem 4.6.** Let $X$ be a set of $\binom{\alpha + 1}{2}$ points of $\mathbb{P}^2$ with the generic Hilbert function. If sdefect$(I_X, 2) = 1$, then $X$ is a linear star configuration.

**Proof.** Since sdefect$(I_X, 2) = 1$, there exits a form $F$ such that $I_X^{(2)} = \langle F \rangle + I_X^2$. By Lemma 4.3, $\deg F < 2\alpha$.

We now show that we must, in fact, have $\deg F \leq \alpha + 1$. By Lemma 4.2, $I_X$ has $\alpha + 1$ generators of degree $\alpha$. By Lemma 4.4, the ideal $I_X^2$ will have $\binom{\alpha + 2}{2}$ minimal generators of degree $2\alpha$. Because $I_X^2 \subseteq I_X^{(2)}$, this means

$$H_{R/I_X^{(2)}}(2\alpha) \leq H_{R/I_X^2}(2\alpha) = \binom{2\alpha + 2}{2} - \binom{\alpha + 2}{2} = \frac{(2\alpha + 2)(2\alpha + 1) - (\alpha + 2)(\alpha + 1)}{2} = \frac{3\alpha^2 + 3\alpha}{2}.$$

Suppose that $\deg F > \alpha + 1$. Because $I_X^2$ is generated by forms of degree $2\alpha$ or larger, we have

$$\langle I_X^{(2)} \rangle_{2\alpha - 1} = \langle \langle F \rangle + I_X^2 \rangle_{2\alpha - 1} = \langle \langle F \rangle \rangle_{2\alpha - 1},$$
and consequently,

$$H_{R/I_g^{(2)}}(2\alpha - 1) = H_{R/(F)}(2\alpha - 1) = \binom{2\alpha + 1}{2} - \dim_k \langle F \rangle_{2\alpha - 1}.$$ 

If $\deg F = d$, then $\langle F \rangle \cong R(-d)$ as graded $R$-modules, so $\dim_k \langle F \rangle_{2\alpha - 1} = \dim_k R_{2\alpha - 1 - d} = \binom{2\alpha - d + 1}{2}$. Because $d \geq \alpha + 2$, we have

$$H_{R/I_g^{(2)}}(2\alpha - 1) = \left(\frac{2\alpha + 1}{2}\right) - \left(\frac{2\alpha - d + 1}{2}\right)$$

$$\geq \left(\frac{2\alpha + 1}{2}\right) - \left(\frac{2\alpha - (\alpha + 2) + 1}{2}\right)$$

$$= \frac{(2\alpha + 1)(2\alpha) - (\alpha - 1)(\alpha - 2)}{2}$$

$$= \frac{4\alpha^2 + 2\alpha - (\alpha^2 - 3\alpha + 2)}{2} = \frac{3\alpha^2 + 5\alpha - 2}{2}.$$ 

Since sdefect$(I_X, 2) \neq 0$, $X$ is not a complete intersection, and thus $X$ cannot be a set of points on a line. Consequently, $\alpha \geq 2$. But then we must have

$$H_{R/I_g^{(2)}}(2\alpha - 1) \geq \frac{3\alpha^2 + 5\alpha - 2}{2} > \frac{3\alpha^2 + 3\alpha}{2} \geq H_{R/I_g^{(2)}}(2\alpha).$$ 

This is a contradiction, so $\deg F \leq \alpha + 1$ as claimed.

Because $\deg F > \alpha$ by Theorem 4.5, we must have $\deg F = \alpha + 1$. Hence $\alpha(I_X^{(2)}) = \alpha(I_X) + 1$. Theorem 4.5 then implies that $X$ is a either a linear star configuration or a set of colinear points. If $X$ was a set of colinear points, then Theorem 2.6 would imply that sdefect$(I_X, 2) = 0$ because colinear points are a complete intersection. Thus $X$ must be a linear star configuration in $\mathbb{P}^2$. \hfill $\Box$

**Remark 4.7.** As we will see in Section 6, there exist sets of points $X$ in $\mathbb{P}^2$ with sdefect$(I_X, 2) = 1$, but $X$ is not a linear star configuration.

**Remark 4.8.** It is natural to ask if a similar type of result holds for points in $\mathbb{P}^n$ with n $\geq 3$, i.e., if sdefect$(I_X, 2) = 1$, along with some suitable hypotheses on $X$, implies that $X$ must be a linear star configuration. However, this cannot happen. Indeed, if such a set of points $X$ existed, then $X = V(I_{n,L})$ for some $n \geq 3$ and set of linear forms $L$, because $X$ is a zero-dimensional scheme. But then we would have sdefect$(I_{n,L}, 2) = 1$, contradicting Theorem 3.18.

5. **Application: Resolutions of squares of star configurations in $\mathbb{P}^n$**

In this section, we use Corollary 3.14 to describe a minimal free resolution of the symbolic square of the defining ideal $I_{2,F}$ of a codimension two star configuration in $\mathbb{P}^n$.

**Lemma 5.1.** Let $I_{2,F}$ be the defining ideal of a star configuration of codimension two in $\mathbb{P}^n$. Assume $F = \{F_1, \ldots, F_s\}$, where $F_1, \ldots, F_s$ are forms of degrees $1 \leq d_1 \leq \cdots \leq d_s$,
and let \( d = d_1 + \cdots + d_s \). Then a graded minimal free resolution of \( I_{2,F}^2 \) has the form
\[
0 \to R^{(s-1)}(-2d) \to \bigoplus_{1 \leq i \leq s} R^{s-1}(-(2d - d_i)) \to \bigoplus_{1 \leq i,j \leq s} R(-(2d - (d_i + d_j))) \to I_{2,F}^2 \to 0.
\]

**Proof.** By [33, Theorem 3.4], the ideal \( I_{2,F} \) has a graded minimal free resolution of the form
\[
0 \to R^{s-1}(-d) \to \bigoplus_{1 \leq i \leq s} R(-(d - d_i)) \to I_{2,F} \to 0.
\]

Recall that
\[
I_{2,F} = \bigcap_{1 \leq i < j \leq s} \langle F_i, F_j \rangle.
\]

Let \( P \) be a minimal prime of \( I_{2,F} \) in \( R \). Then \( P \) has height 2.

**Claim.** There exists a unique pair \((i, j)\) such that \( \langle F_i, F_j \rangle \subseteq P \).

**Proof of Claim.** The existence of the pair follows from [4, Prop. 1.11]. Assume \( \langle F_\alpha, F_\beta \rangle \subseteq P \) for some indices \( \alpha, \beta \) with \( \{\alpha, \beta\} \neq \{i, j\} \). Without loss of generality, we may assume that \( \alpha \neq i, j \). Then \( F_i, F_j, F_\alpha \in P \), which is a contradiction, since \( F_i, F_j, F_\alpha \) form a regular sequence of length 3 but \( P \) has height 2. \( \square \)

It follows from the claim that
\[
(I_{2,F})_P = \bigcap_{1 \leq k < l \leq s} \langle F_k, F_l \rangle_P = \langle F_i, F_j \rangle_P = \langle F_1, F_1 \rangle_P.
\]

Since localization preserves regular sequences, \( (I_{2,F})_P \) is a complete intersection ideal in \( R_P \). We deduce that \( I_{2,F} \) is a generic complete intersection ideal. Since \( I_{2,F} \) is also a perfect codimension two ideal, we can apply Theorem 2.3 to derive the stated graded minimal free resolution of \( I_{2,F}^2 \). \( \square \)

**Lemma 5.2.** Let \( I_{2,F} \) be the defining ideal of a star configuration of codimension two in \( \mathbb{P}^n \). Assume \( \mathcal{F} = \{F_1, \ldots, F_s\} \), and set \( F = F_1 \cdots F_s \). Then

(i) \( I_{2,F}^2 : F \) = \( \langle F_{i_1} \cdots F_{i_{s-2}} | 1 \leq i_1 < \cdots < i_{s-2} \leq s \rangle = I_{3,F} ; \)

(ii) \( I_{2,F}^2 \cap \langle F \rangle = F[I_{2,F}^2 : F] \).

**Proof.** (i) First, recall that
\[
I_{2,F}^2 = \langle F_i F_j | 1 \leq i \leq j \leq s \rangle.
\]

Given indices \( 1 \leq i_1 < \cdots < i_{s-2} \leq s \), let \( \{i_{s-1}, i_s\} \) be the complement of \( \{i_1, \ldots, i_{s-2}\} \) in \( \{1, \ldots, s\} \). Then we have
\[
(F_{i_1} \cdots F_{i_{s-2}})F = \frac{F^2}{F_{i_{s-1}} F_{i_s}} \in I_{2,F}^2,
\]
and so \( F_{i_1} \cdots F_{i_{s-2}} \in [I_{2,F}^2 : F] \).
Conversely, let \( G \in [I_{2,F}^2 : F] \). Since \( GF \in I_{2,F}^2 \), we have that

\[
GF = \sum_{1 \leq i \leq s} A_i \frac{F^2}{F_i^2} + \sum_{1 \leq i < j \leq s} B_{i,j} \frac{F^2}{F_i F_j}
\]

(5.1)

for some \( A_i, B_{i,j} \in R \).

**Claim.** For every \( 1 \leq i \leq s \), \( F_i \) divides \( A_i \).

**Proof of Claim.** For \( i = 1 \),

\[
GF = A_1 \frac{F^2}{F_1^2} + \sum_{2 \leq i \leq s} A_i \frac{F^2}{F_i^2} + \sum_{1 \leq i < j \leq s} B_{i,j} \frac{F^2}{F_i F_j}
\]

Hence

\[
GF - \sum_{1 \leq i < j \leq s} B_{i,j} \frac{F^2}{F_i F_j} - \sum_{2 \leq i \leq s} A_i \frac{F^2}{F_i^2} = A_1 \frac{F^2}{F_1^2}
\]

For all \( h \neq 1 \), \( F_1, F_h \) is, by assumption, a regular sequence. This implies that \( F_1 \) and \( F_h \) are coprime. Therefore \( F_1 \) must divide \( A_1 \) because \( F_1 \) divides every term on the left hand side. Similarly, one can show that \( F_i \) divides \( A_i \) for all \( 1 \leq i \leq s \). \( \square \)

Let \( A_i = F_i A'_i \) for some \( A'_i \in R \). We can rewrite equation (5.1) as

\[
GF = \sum_{1 \leq i \leq s} A'_i \frac{F^2}{F_i} + \sum_{1 \leq i < j \leq s} B_{i,j} \frac{F^2}{F_i F_j}
\]

Dividing both sides by \( F \), we obtain

\[
G = \sum_{1 \leq i \leq s} A'_i \frac{F}{F_i} + \sum_{1 \leq i < j \leq s} B_{i,j} \frac{F}{F_i F_j}
\]

proving that \( G \) is in \( \langle F_1, \ldots, F_{i-2} | 1 \leq i_1 < \cdots < i_{s-2} \leq s \rangle \).

\[\text{[iii] If } G \in I_{2,F}^2 \cap \langle F \rangle, \text{ then } G = G'F \in I_{2,F}^2. \text{ So } G' \in [I_{2,F}^2 : F], \text{ and thus } G = FG' \in F[I_{2,F}^2 : F]. \text{ Conversely, if } H \in F[I_{2,F}^2 : F], \text{ we have } H = FH' \text{ with } H' \in [I_{2,F}^2 : F]. \text{ It is then immediate that } H \in I_{2,F}^2 \cap \langle F \rangle, \text{ which completes the proof of this lemma.} \ \square \]

**Theorem 5.3.** Let \( I_{2,F} \) be the defining ideal of a star configuration of codimension two in \( \mathbb{P}^n \). Assume \( F = \{F_1, \ldots, F_s\} \), where \( F_1, \ldots, F_s \) are forms of degrees \( 1 \leq d_1 \leq \cdots \leq d_s \), and let \( d = d_1 + \cdots + d_s \). Then a graded minimal free resolution of \( I_{2,F}^{(2)} \) has the form

\[
0 \to \bigoplus_{1 \leq i \leq s} R(-(2d - d_i)) \to \left( \bigoplus_{1 \leq i \leq s} R(-(2d - 2d_i)) \right) \oplus R(-d) \to I_{2,F}^{(2)} \to 0.
\]

**Proof.** Let \( F = F_1 \cdots F_s \). Thanks to Corollary 3.14 there is a short exact sequence

\[
0 \to I_{2,F}^2 \cap \langle F \rangle \to I_{2,F}^2 \oplus \langle F \rangle \to I_{2,F}^{(2)} \to 0.
\]
We proceed to describe a minimal free resolution of the left term. By Lemma 5.2, \([I_2, \mathcal{F}] : F] = I_{3, \mathcal{F}}\). By Theorem 3.4, a minimal free resolution of \(I_{3, \mathcal{F}}\) has the form

$$0 \to R^{(s-1)}(-d) \to \bigoplus_{1 \leq i \leq s} R^{s-2}(-(d - d_i)) \to \bigoplus_{1 \leq i < j \leq s} R(-(d - (d_i + d_j))).$$

By Lemma 5.2, we have \(I_2, \mathcal{F} \cap \langle F \rangle = F[I_2, \mathcal{F}] : F = FI_{3, \mathcal{F}}\). Since \(F\) has degree \(d\), to obtain a minimal free resolution of \(FI_{3, \mathcal{F}}\) it is enough to add \(d\) to the degrees of the generators of the free modules in the resolution above. More explicitly, a minimal free resolution of \(I_2, \mathcal{F} \cap \langle F \rangle\) has the form

$$0 \to R^{(s-1)}(-2d) \to \bigoplus_{1 \leq i \leq s} R^{s-2}(-(2d - d_i)) \to \bigoplus_{1 \leq i < j \leq s} R(-(2d - (d_i + d_j))).$$

Next we describe a minimal free resolution of the middle term. We found a minimal free resolution for \(I_2, \mathcal{F}\) in Lemma 5.1. Since \(0 \to R(-d) \to \langle F \rangle \to 0\) is a minimal free resolution of \(\langle F \rangle\), we can take a direct sum of the resolutions of the two ideals to obtain the complex

$$0 \to R^{(s-1)}(-2d) \to \bigoplus_{1 \leq i \leq s} R^{s-2}(-(2d - d_i)) \to \left( \bigoplus_{1 \leq i < j \leq s} R(-(2d - (d_i + d_j))) \right) \oplus R(-d),$$

which is a minimal free resolution of \(I_{2, \mathcal{F}} \oplus \langle F \rangle\).

Our goal is to describe a minimal free resolution of the right term in the short exact sequence. Using a mapping cone construction, we obtain a free resolution of \(I_{2, \mathcal{F}}^{(2)}\) of the form

$$0 \to R^{(s-1)}(-2d) \to \left( \bigoplus_{1 \leq i \leq s} R^{s-2}(-(2d - d_i)) \right) \oplus R^{(s-1)}(-2d) \to \left( \bigoplus_{1 \leq i < j \leq s} R(-(2d - (d_i + d_j))) \right) \oplus \left( \bigoplus_{1 \leq i < j \leq s} R^{s-1}(-(2d - d_i)) \right) \to \left( \bigoplus_{1 \leq i < j \leq s} R(-(2d - (d_i + d_j))) \right) \oplus R(-d).$$

The ideal \(I_{2, \mathcal{F}}^{(2)}\) is Cohen-Macaulay by Corollary 3.7. In particular, a graded minimal free resolution of \(I_{2, \mathcal{F}}^{(2)}\) has length 1. Hence the \(R^{(s-1)}(-2d)\) at the end of the resolution must cancel out the \(R^{(s-1)}(-2d)\) in the penultimate module. In addition,
\[ \bigoplus_{1 \leq i \leq s} R^{s-2}(-(2d - d_i)) \] must cancel with part of \[ \bigoplus_{1 \leq i \leq s} R^{s-1}(-(2d - d_i)) \] in homological degree two. After these cancellations, we are left with the smaller resolution

\[
0 \to \left( \bigoplus_{1 \leq i < j \leq s} R(-(2d - (d_i + d_j))) \right) \oplus \left( \bigoplus_{1 \leq i \leq s} R(-(2d - d_i)) \right) \to \\
\to \left( \bigoplus_{1 \leq i, j \leq s} R(-(2d - (d_i + d_j))) \right) \oplus R(-d)
\]

of \( I_{2,F}^{(2)} \). By Corollary 5.14, \( I_{2,F}^{(2)} \) has exactly one generator of degree \( d \), namely \( F \), and the rest of the generators have degrees \( 2d - (d_i + d_j) \). The generators of degree \( 2d - (d_i + d_j) \) with \( d_i \neq d_j \) are redundant since they are multiples of \( F \). Each redundant generator gives rise to a relation of degree \( 2d - (d_i + d_j) \) that expresses the redundant generator in terms of the minimal ones. As such, we can remove these redundant generators along with the corresponding relations. We are then left with

\[
0 \to \bigoplus_{1 \leq i \leq s} R(-(2d - d_i)) \to \left( \bigoplus_{1 \leq i \leq s} R(-(2d - 2d_i)) \right) \oplus R(-d) \to I_{2,F}^{(2)} \to 0.
\]

This resolution must now be minimal since no further cancellation is possible. \( \square \)

**Remark 5.4.** If \( I_{2,C} \) defines a linear star configuration in \( \mathbb{P}^n \), our formula agrees with the formula of [17] Theorem 3.2] with \( c = 2 \). In particular, the graded minimal free resolution of \( I_{2,C}^{(2)} \) is given by

\[
0 \longrightarrow R^s(-(2s - 1)) \longrightarrow R(-s) \oplus R^s(-(2s - 2)) \longrightarrow I_{2,C}^{(2)} \longrightarrow 0.
\]

Theorem 5.3 can thus be viewed as a generalization [17] Theorem 3.2] when \( c = 2 \) in the sense that the star configuration need not be linear.

### 6. General sets of points

In this section, we study general sets \( X \) of points in \( \mathbb{P}^2 \). Specifically, we characterize when \( \text{sdefect}(I_X, 2) = 1 \).

Recall that a set of \( s \) points \( X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2 \) is set of general points if a property holds for some non-empty open dense subset of points \( (P_1, \ldots, P_s) \in (\mathbb{P}^2)^s \). If \( X \subseteq \mathbb{P}^2 \) is a general set of points, then \( X \) has the generic Hilbert function, that is,

\[ H_{R/I_X}(i) = \min \{ \dim_k R_i, |X| \} \text{ for all } i \geq 0. \]

The key ingredient that we require is the following famous result of Alexander and Hirschowitz which computes the Hilbert function of \( R/I_X^{(2)} \) when \( X \) is a set of general points in \( \mathbb{P}^n \) (we have specialized their result to \( \mathbb{P}^2 \)). Roughly speaking, except if \( s = 2 \) or 5, the Hilbert function of \( R/I_X^{(2)} \) is the generic Hilbert function of \( 3|X| \) points.

**Theorem 6.1 (\[2\] Theorem 2).** Let \( X \) be a set of \( s \) general points in \( \mathbb{P}^2 \). If \( s \neq 2, 5 \), then

\[ H_{R/I_X^{(2)}}(i) = \min \{ \dim_k R_i, 3s \} \text{ for all } i \geq 0. \]
If \( s = 5 \), then
\[
\mathbf{H}_{R/I_X^{(2)}}(i) = \begin{cases} 
\dim_k R_i, & i \neq 4 \\
3s, & i = 4.
\end{cases}
\]

In fact, the graded minimal free resolution of \( I_X \) and \( I_X^{(2)} \) for \( s \) general points in \( \mathbb{P}^2 \) is known. The resolution of \( I_X \) and \( I_X^{(2)} \) is the culmination of the work of many people. For simple sets of points, the minimal resolution was worked out by Geramita and Maroscia [19], Geramita, Gregory, and Roberts [16], and Lorenzini [32]. For \( I_X^{(2)} \), Catalisano’s work [10] implies the resolution of \( I_X^{(2)} \) for \( s \leq 5 \), Harbourne [22] proposed a conjecture for all \( s \) (and showed the conjecture was true for \( s \leq 9 \)), while Idà [30] proved the conjecture in complete generality. We record only the consequences we need.

**Lemma 6.2.** Let \( X \) be a set of \( s \) general points in \( \mathbb{P}^2 \).

(i) If \( s = 5 \), then the graded minimal free resolution of \( I_X \), respectively \( I_X^{(2)} \), is
\[
0 \to R^2(-4) \to R(-2) \oplus R^2(-3) \to I_X \to 0, \text{ respectively}
\]
\[
0 \to R^3(-6) \oplus R(-7) \to R(-4) \oplus R^3(-5) \to I_X^{(2)} \to 0.
\]

(ii) If \( s = 7 \), then the graded minimal free resolution of \( I_X \), respectively \( I_X^{(2)} \), is
\[
0 \to R(-4) \oplus R(-5) \to R^3(-3) \to I_X \to 0, \text{ respectively}
\]
\[
0 \to R^6(-7) \to R^7(-6) \to I_X^{(2)} \to 0.
\]

(iii) If \( s = 8 \), then the graded minimal free resolution of \( I_X \), respectively \( I_X^{(2)} \), is
\[
0 \to R^2(-5) \to R^2(-3) \oplus R(-4) \to I_X \to 0, \text{ respectively}
\]
\[
0 \to R^3(-8) \to R^4(-6) \to I_X^{(2)} \to 0.
\]

(iv) If \( s = 9 \), then the graded minimal free resolution of \( I_X \), respectively \( I_X^{(2)} \), is
\[
0 \to R^3(-5) \to R(-3) \oplus R^3(-4) \to I_X \to 0, \text{ respectively}
\]
\[
0 \to R^6(-8) \to R(-6) \oplus R^6(-7) \to I_X^{(2)} \to 0.
\]

We now present the main result of this section.

**Theorem 6.3.** Let \( X \) be a set of \( s \) general points in \( \mathbb{P}^2 \). Then

(i) \( s \text{defect}(I_X, 2) = 0 \) if and only if \( s = 1, 2 \) or 4.

(ii) \( s \text{defect}(I_X, 2) = 1 \) if and only if \( s = 3, 5, 7, \) or 8.

(iii) \( s \text{defect}(I_X, 2) > 1 \) if and only if \( s = 6 \) or \( s \geq 9 \).

**Proof.** By Theorem 2.6, \( s \text{defect}(I_X, 2) = 0 \) if and only if \( X \) is a complete intersection. But a set of \( s \) general points is a complete intersection if and only if \( s = 1, 2, \) or 4 (e.g., [24, Exercise 11.9]). This proves (i).

We next consider the special cases of \( s = 3, 5, 6, 7, 8, 9 \).

If \( s = 3 \), then \( X \) is also a linear star configuration. Indeed, for each pair of points \( P_i, P_j \) with \( i \neq j \), take the unique line \( L_{i,j} \) through those two points. Then \( I_X = I_{2,L} \) where \( L = \{L_{1,2}, L_{1,3}, L_{2,3}\} \). Then \( s \text{defect}(I_X, 2) = 1 \) by Theorem 3.18.
If \( s = 5 \), by Theorem 2.3 and Lemma 6.2 (i), the graded minimal free resolution of \( I_X^2 \) begins

\[
\cdots \longrightarrow R(-4) \oplus R^2(-5) \oplus R^3(-6) \longrightarrow I_X^2 \longrightarrow 0.
\]

In particular, we have \( \dim_k(I_X^2)_{t_5} = 5 \). However, by Theorem 6.1, \( \dim_k(I_X^{(2)})_5 = \dim_k R_5 - H_{R/I_X^{(2)}}(5) = 6 \). So, there exists an \( F \in (I_X^{(2)})_5 \setminus (I_X^2)_5 \). We claim that \( I_X^{(2)} = \langle F \rangle + (I_X^2)_5 \).

Indeed, we see that for all \( t \leq 5 \), we have \( (I_X^{(2)})_t = \langle (F) + I_X^2 \rangle_t \). But Lemma 6.2 (i) implies that \( I_X^{(2)} \) is generated by forms of degrees 5 or less, so the two ideals are equal, and consequently, \( \text{sdefect}(I_X, 2) = 1 \).

If \( s = 6 \), then \( \alpha(I_X) = 3 \), and thus \( \alpha(I_X^2) = 6 \). On the other hand, by Theorem 6.1, we can deduce that \( \alpha(I_X^{(2)}) = 5 \), and furthermore, \( I_X^{(2)} \) has \( \dim_k R_5 - H_{R/I_X^{(2)}}(5) = 3 \) minimal generators of degree five. But then \( \dim_k(I_X^{(2)})_5/(I_X^2)_5 = 3 \) and so \( \text{sdefect}(I_X, 2) \geq 3 \).

If \( s = 7 \), then Lemma 6.2 (ii) implies that \( I_X \) has three minimal generators of degree 3. So, \( I_X^2 \) has six minimal generators of degree 6 by Lemma 4.4. However, by Lemma 6.2 (ii), \( I_X^{(2)} \) has seven minimal generators of degree 6. Hence, there is an \( F \in (I_X^{(2)})_6 \setminus (I_X^2)_6 \) such that \( (I_X^{(2)})_6 = \langle (F) + I_X^2 \rangle_6 \). Again by Lemma 6.2, \( I_X^{(2)} \) is generated by seven elements of degree six, so we actually have \( I_X^{(2)} = \langle F \rangle + I_X^2 \), and thus \( \text{sdefect}(I_X, 2) = 1 \).

If \( s = 8 \), then Lemma 6.2 (iii) implies that \( \alpha(I_X) = 3 \) and \( I_X \) has two minimal generators of degree 3. So, \( I_X^2 \) has three minimal generators of degree 6. By Lemma 6.2 (iii), \( \alpha(I_X^{(2)}) = 6 \) and \( I_X^{(2)} \) has four minimal generators of degree 6. So, there exists a form \( F \in (I_X^{(2)})_6 \setminus (I_X^2)_6 \). But since \( I_X^{(2)} \) has generated by these four generators of degree 6, \( I_X^{(2)} = \langle F \rangle + I_X^2 \), that is, \( \text{sdefect}(I_X, 2) = 1 \).

If \( s = 9 \), Lemma 6.2 (iv) and Theorem 2.3 imply that the resolution of \( I_X^2 \) is given by

\[
0 \longrightarrow R(-10)^3 \longrightarrow R(-8)^3 \oplus R(-9)^9 \longrightarrow R(-6) \oplus R(-7)^3 \oplus R(-8)^6 \longrightarrow I_X^2 \longrightarrow 0.
\]

From the resolution, \( \dim_k(I_X^2)_{t_7} = 6 \). On the other hand, by Lemma 6.2 (iv), we know that \( I_X \) also has a minimal generator of degree 6, and since \( (I_X^2)_6 \subseteq (I_X^{(2)})_6 \), the minimal generator of degree 6 must be the same in both ideals (up to constant). By Theorem 6.1, we deduce that \( \dim_k(I_X^{(2)})_7 = 9 \). So, \( \dim_k(I_X^{(2)})_7/(I_X^2)_7 = 3 \), and since \( (I_X^{(2)})_i = (I_X^2)_i \), for all \( i \leq 6 \), we must have \( \text{sdefect}(I_X, 2) \geq 3 \).

Going forward, we now assume that \( s \geq 10 \). Our goal is to show that \( \text{sdefect}(I_X, 2) > 1 \). To do this, we first will show that if \( \text{sdefect}(I_X, 2) = 1 \) and \( F \) is any homogeneous form such that \( I_X^{(2)} = \langle F \rangle + I_X^2 \), then the degree of \( F \) is restricted. Below, \( \alpha = \alpha(I_X) \).

**Claim.** If \( s \geq 10 \) and \( I_X^{(2)} = \langle F \rangle + I_X^2 \), then \( \deg F \geq 2\alpha - 1 \).

**Proof of Claim.** Suppose that \( d = \deg F \leq 2\alpha - 2 \). Because \( s \geq 10 \),

\[
H_{R/I_X^{(2)}}(d) = H_{R/I_X^{(2)}}(d + 1) = 3|X| \text{ by Theorem 6.1.}
\]

On the other hand, since \( I_X^{(2)} = \langle F \rangle + I_X^2 \) and \( \deg F \leq 2\alpha - 2 \), we have

\[
\dim_k(I_X^{(2)})_d = \dim_k(\langle F \rangle)_d = 1 \quad \text{and} \quad \dim_k(I_X^{(2)})_{d+1} = \dim_k(\langle F \rangle)_{d+1} = 3.
\]
But this then means that
\[
\binom{d+2}{2} - 1 = H_{R/I_X^{(2)}}(d) = 3|X| = H_{R/I_X^{(2)}}(d + 1) = \binom{d+3}{2} - 3.
\]
So \(d\), the degree of \(F\), would have to satisfy
\[
\binom{d+2}{2} - \binom{d+3}{2} + 2 = 0 \iff d = 0.
\]
But \(\deg F > 0\). So, \(\deg F \geq 2\alpha - 1\).

Now suppose that \(s \geq 10\) and \(s\text{defect}(I_X, 2) = 1\). Consequently, there is a homogeneous form \(F\) such that \(I_X^{(2)} = \langle F \rangle + I_X^2\), and furthermore, by the above claim, \(\deg F \geq 2\alpha - 1\) where \(\alpha = \alpha(I_X)\). We now consider the cases \(\deg F = 2\alpha - 1\), and \(\deg F \geq 2\alpha\) separately.

**Case 1.** \(I_X^{(2)} = \langle F \rangle + I_X^2\) with \(\deg F = 2\alpha - 1\).

If \(\deg F = 2\alpha - 1\), then we first claim that \(\alpha \leq 11\). Indeed, by Theorem \(6.1\)
\[
H_{R/I_X^{(2)}}(2\alpha - 2) = \binom{2\alpha}{2} \leq 3|X| = H_{R/I_X^{(2)}}(2\alpha - 1) = \binom{2\alpha + 1}{2} - 1
\]
where the last equality follows from the fact that \(I_X^{(2)}\) has exactly one generator of degree \(2\alpha - 1\). On the other hand, we know that \(|X| < \binom{\alpha + 2}{2}\) since \(s\) general points have the generic Hilbert function, so \(\alpha\) is by definition the smallest number \(i\) such that \(\binom{i+2}{2} > s = |X|\). Combining these inequalities, we have
\[
\binom{2\alpha}{2} \leq 3|X| < 3\binom{\alpha + 2}{2},
\]
or equivalently, \(\alpha\) must satisfy
\[
\frac{2(2\alpha - 1)}{2} - \frac{3(\alpha + 2)(\alpha + 1)}{2} < 0 \iff \alpha^2 - 11\alpha - 6 < 0.
\]
But the last inequality only holds if \(\alpha \leq 11\). Since we are also assuming that \(|X| \geq 10\), we have \(4 \leq \alpha \leq 11\).

As we noted above, if \(\deg F = 2\alpha - 1\), then \(3|X| = \binom{2\alpha + 1}{2} - 1\) must be also be satisfied. From Table \(1\) we see that \(\binom{2\alpha + 1}{2} - 1\) is divisible by 3 with \(4 \leq \alpha \leq 11\) if and only if \(\alpha = 5, 8, 11\).

So, if \(3|X| = \binom{2\alpha + 1}{2} - 1\) with \(4 \leq \alpha \leq 11\), then we have \(|X| = 54/3 = 18\), or \(|X| = 135/3 = 45\), or \(|X| = 252/3 = 84\). In all other cases, we cannot have \(I_X^{(2)} = \langle F \rangle + I_X^2\).

However, if \(|X| = 45\), then \(\alpha(I_X) = 9\), not 8. Also, if \(|X| = 84\), then \(\alpha(I_X) = 12\), not 11. If \(|X| = 18\), then \(\alpha(I_X) = 5\). So we need a separate argument to show that \(I_X^{(2)} \neq \langle F \rangle + I_X^2\).

So, let \(s = 18\) with \(\alpha = 5\) and suppose that \(I_X^{(2)} = \langle F \rangle + I_X^2\) with \(\deg F = 9\). Then \(I_X^{(2)}_{10} = \langle F + I_X^2 \rangle_{10}\). Now by Theorem \(6.1\), \(\dim_k(I_X^{(2)}_{10}) = 12\). On the other hand, \(I_X\) has three generators of degree \(\alpha = 5\), so by Lemma \(4.3\), \(I_X^2\) has six generators of degree \(2\alpha = 10\), and no smaller generators. So \(\dim_k(\langle F \rangle + I_X^2)_{10} \leq \dim_k(\langle F \rangle)_{10} + \dim_k(I_X^2)_{10} = 3 + 6 = 9\). So, by a dimension count, we cannot have \(I_X^{(2)} = \langle F \rangle + I_X^2\).
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\[ \alpha = \alpha(I_X) \]

\[ \binom{2\alpha+1}{2} - 1 \]

\begin{array}{|c|c|}
\hline
\alpha & \binom{2\alpha+1}{2} - 1 \\
\hline
4 & 35 \\
5 & 54 \\
6 & 77 \\
7 & 104 \\
8 & 135 \\
9 & 170 \\
10 & 209 \\
11 & 252 \\
\hline
\end{array}

Table 1. Computing \( \binom{2\alpha+1}{2} - 1 \)

To summarize this case, if \( s \geq 10 \), there is no set of \( s \) general points with \( I_X^{(2)} = \langle F \rangle + I_X^2 \) with \( \deg F = 2\alpha - 1 \).

Case 2. \( I_X^{(2)} = \langle F \rangle + I_X^2 \) with \( \deg F \geq 2\alpha \).

If \( \deg F \geq 2\alpha \), then we claim that \( \alpha \leq 7 \). Indeed, since \( I_X^{(2)} \) will be generated by forms of degree \( 2\alpha \) or larger, we have

\[ \mathbf{H}_{R/I_X^{(2)}}(2\alpha - 1) = \binom{2\alpha + 1}{2} \leq 3|X|. \]

On the other hand, \( |X| < \binom{\alpha+2}{2} \). Combining these two inequalities gives

\[ \binom{2\alpha + 1}{2} \leq 3|X| < 3\binom{\alpha + 2}{2}. \]

So, \( \alpha \) must satisfy

\[ (2\alpha + 1)(2\alpha) < 3(\alpha + 2)(\alpha + 1) \Leftrightarrow \alpha^2 - 7\alpha - 6 < 0 \Leftrightarrow \alpha \leq 7. \]

Moreover, because \( s \geq 10 \), we have \( 4 \leq \alpha \leq 7 \), or equivalently, \( 10 \leq s = |X| \leq 35 \).

Let \( d = \binom{\alpha+2}{2} - |X| \), that is, \( d \) is the number of minimal generators of \( I_X \) of degree \( \alpha \).

If \( \deg F = 2\alpha \) and \( I_X^{(2)} = \langle F \rangle + I_X^2 \), then \( I_X^{(2)} \) has \( \binom{d+1}{2} + 1 \) minimal generators of degree \( 2\alpha \). If \( \deg F > 2\alpha \) and \( I_X^{(2)} = \langle F \rangle + I_X^2 \), then \( I_X^{(2)} \) has \( \binom{d+1}{2} \) minimal generators of degree \( 2\alpha \). So, we will have

\[ \mathbf{H}_{R/I_X^{(2)}}(2\alpha) = 3|X| = \begin{cases} 
\binom{2\alpha + 2}{2} - \binom{d + 1}{2} - 1 & \text{if } \deg F = 2\alpha, \\
\binom{2\alpha + 2}{2} - \binom{d + 1}{2} & \text{if } \deg F > 2\alpha.
\end{cases} \]

Thus, to summarize, if \( I_X^{(2)} = \langle F \rangle + I_X^2 \) with \( \deg F \geq 2\alpha \), then

(a) \( 10 \leq |X| \leq 35 \),
(b) \( \binom{2\alpha+1}{2} \leq 3|X| < \binom{2\alpha+2}{2} \), and
(c) either \(3|X| = \binom{2\alpha+2}{2} - \binom{d+1}{2} - 1\) or \(3|X| = \binom{2\alpha+2}{2} - \binom{d+1}{2}\) must hold with \(d = \binom{\alpha+2}{2} - |X|\).

A direct computation for each value \(10 \leq |X| \leq 35\) shows that no value of \(|X|\) satisfies both of (b) and (c). Table 2 explicitly verifies this statement; note that in the table, (T) denotes true and (F) denotes false.

| \(|X|\) | \(\alpha = \alpha(I_X)\) | \(\binom{2\alpha+1}{2} \leq 3|X| < \binom{2\alpha+2}{2}\) | \(d\) | \(\binom{2\alpha+2}{2} - \binom{d+1}{2} = 3|X|\) | \(\binom{2\alpha+2}{2} - \binom{d+1}{2} = 3|X|\) |
|---|---|---|---|---|---|
| 10 | 4 | 36 \leq 30 < 45 (F) | | | |
| 11 | 4 | 36 \leq 33 < 45 (F) | | | |
| 12 | 4 | 36 \leq 36 < 45 (T) | 3 | 39 = 36 (F) | 38 = 36 (F) |
| 13 | 4 | 36 \leq 39 < 45 (T) | 2 | 41 = 39 (F) | 42 = 39 (F) |
| 14 | 4 | 36 \leq 42 < 45 (T) | 1 | 43 = 42 (F) | 44 = 42 (F) |
| 15 | 5 | 55 \leq 45 < 66 (F) | | | |
| 16 | 5 | 55 \leq 48 < 66 (F) | | | |
| 17 | 5 | 55 \leq 51 < 66 (F) | | | |
| 18 | 5 | 55 \leq 54 < 66 (F) | | | |
| 19 | 5 | 55 \leq 57 < 66 (T) | 2 | 62 = 57 (F) | 63 = 57 (F) |
| 20 | 5 | 55 \leq 60 < 66 (T) | 1 | 64 = 60 (F) | 65 = 60 (F) |
| 21 | 6 | 78 \leq 63 < 91 (F) | | | |
| 22 | 6 | 78 \leq 66 < 91 (F) | | | |
| 23 | 6 | 78 \leq 69 < 91 (F) | | | |
| 24 | 6 | 78 \leq 72 < 91 (F) | | | |
| 25 | 6 | 78 \leq 75 < 91 (F) | | | |
| 26 | 6 | 78 \leq 78 < 91 (T) | 2 | 87 = 78 (F) | 88 = 78 (F) |
| 27 | 6 | 78 \leq 81 < 91 (T) | 1 | 89 = 81 (F) | 90 = 81 (F) |
| 28 | 7 | 105 \leq 84 < 120 (F) | | | |
| 29 | 7 | 105 \leq 87 < 120 (F) | | | |
| 30 | 7 | 105 \leq 90 < 120 (F) | | | |
| 31 | 7 | 105 \leq 93 < 120 (F) | | | |
| 32 | 7 | 105 \leq 96 < 120 (F) | | | |
| 33 | 7 | 105 \leq 99 < 120 (F) | | | |
| 34 | 7 | 105 \leq 102 < 120 (F) | | | |
| 35 | 7 | 105 \leq 105 < 120 (T) | 1 | 118 = 105 (F) | 119 = 105 (F) |

Table 2. Comparing inequalities and equalities with \(\text{deg} F \geq 2\alpha\)

To summarize this case, if \(s \geq 10\), there is no set of \(s\) general points with \(I_X^{(2)} = \langle F \rangle + I_X^2\) with \(\text{deg} F \geq 2\alpha\). Thus combining this case with the previous case, we see that if \(s \geq 10\), then \(s \text{defect}(I_X, 2) > 1\), thus completing the proof.

\(\square\)

**Remark 6.4.** The special case \(s = 6\) in the Theorem 6.3 can also be explained by appealing to Theorem 4.6. The ideal \(I_X\) of six general points in \(\mathbb{P}^2\) has a linear resolution. So, if \(s \text{defect}(I_X, 2) = 1\), then the six points must be a linear star configuration by Theorem
and in particular, three of the six points must be on the same line. But six general points is not a star configuration since three of the six points cannot lie on a line.

**Example 6.5.** As mentioned in the introduction, there are many questions one can ask about the symbolic defect sequence. We end this section with an example to show that the symbolic defect sequence need not be a non-decreasing sequence. Consider the ideal $I_X$ when $X$ is eight general points in $\mathbb{P}^2$. Using Macaulay2 [21], we found that the symbolic defect sequence \{sdefect$(I_X, m)$\}$_{m=0}^{\infty}$ begins

$$0, 1, 3, 6, 10, 9, 7$$

and thus, the symbolic defect sequence can decrease. Understanding the long term behavior of this sequence would be of interest.

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