ERGODICITY OF EXPANDING MINIMAL ACTIONS

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Abstract. We prove that every expanding minimal group action of $C^{1+\alpha}$-diffeomorphisms of a compact manifold is robustly ergodic with respect to the Lebesgue measure. We also demonstrate how, locally, a blending region yields the robustness of both, minimality and ergodicity. Similar results are also obtained for semigroup actions.

1. Introduction

The minimality and ergodicity are two popular themes in dynamical systems. Minimality can be thought of as a property involving some complexity for the orbits of the action coming from the fact that this action is irreducible from a topological point of view. From a probabilistic point of view, counterpart to this notion corresponds to ergodicity. A dynamical system is said to be minimal if every closed invariant subset is either empty or coincides with the whole space. An invariant probability measure $\mu$ is called ergodic if every invariant measurable set is either of zero or full $\mu$-measure. The definition of ergodicity can be naturally extended to a quasi-invariant measure which is measure whose push-forward by the dynamical system is absolutely continuous with respect to itself.

It is natural to ask to what extend the properties of ergodicity and minimality are related. In general, ergodicity does not imply minimality. Indeed, one can easily construct examples of ergodic group actions having a global fixed points. Thus a natural question arises in the opposite direction:

Under what conditions a smooth minimal action of a group on a compact manifold is Lebesgue-ergodic?

The example of Furstenberg [5] shows that, in general, minimality does not imply ergodicity. Namely, Furstenberg constructed an analytic diffeomorphism of $\mathbb{T}^2$ which preserves the Lebesgue measure and is minimal but not ergodic. Folkloric Katok-Herman Theorem says that a $C^1$-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational (see [11, 7]). Contrarily in [14], minimal $C^1$-diffeomorphisms of the circle are constructed which are not ergodic. In this paper we study minimal group/semigroup actions and give conditions implying the ergodicity of the Lebesgue measure.

In order to introduce ergodicity and minimality for group/semigroup actions one is forced to deal with the meaning of invariant set. This depends on the dynamical system that we are working with. Observe that the
group-orbits are forward and backward invariant sets by the action of any diffeomorphisms of the group while the semigroup-orbits are in general, only forward invariant. Minimality and ergodicity should state that the orbit of every non-trivial set, from the topological (non-empty closed sets) and the measure theoretical (measurable sets with positive measure) point of views fills most of the space (it is dense or has full measure respectively). Thus, if we consider a group action generated by a family $\mathcal{F}$ of diffeomorphisms of compact manifold $M$ then a set $A \subset M$ must be said to be invariant if $f(A) = A$ for all $f \in \mathcal{F}$. However, if we are considering a semigroup action, we need to change a little bit this definition. We say that $A$ is invariant for the semigroup action generated by $\mathcal{F}$ if $f(A) \subset A$ for all $f \in \mathcal{F}$. Denoting by $\Gamma$ the group/semigroup generated by $\mathcal{F}$ one can unify both cases as follows:

**Definition 1.** A set $A \subset M$ is invariant for $\Gamma$ if $f(A) \subset A$ for all $f \in \Gamma$.

With this notion of invariance, an action of a group/semigroup where all the orbits are dense can be defined as follows:

**Definition 2.** The action of $\Gamma$ on $M$ is minimal if every closed invariant set for $\Gamma$ is either empty or coincides with the whole space.

To define the notion of ergodicity in the extended mode, recall that a probability measure $\mu$ is called quasi-invariant for $\Gamma$ if the push-forward $f_*\mu$ of the measure is absolutely continuous with respect to $\mu$ for all $f \in \Gamma$.

**Definition 3.** The action of $\Gamma$ is ergodic with respect to a quasi-invariant probability measure $\mu$ if $\mu(A) \in \{0, 1\}$ for all invariant set $A \subset M$ for $\Gamma$.

The next definition, extending the notion of expanding action on the circle to higher dimensions, is the central task in our further results.

**Definition 4.** The action of $\Gamma$ is expanding if for every $z \in M$ there is $g \in \Gamma$ such that $m(Dg^{-1}(z)) > 1$, where $m(T)$ to denote the co-norm of a linear transformation $T$, i.e., $m(T) = \|T^{-1}\|^{-1}$.

Notice that, by the compactness, the expanding property of an action is robust under perturbation of the generators. However, this does not happen in the case of minimality and ergodicity. A group/semigroup action is $C^r$-robustly minimal if this property persists under $C^r$-perturbation of the generators.

**Theorem A.** Every expanding minimal group/semigroup action of $C^1$-diffeomorphisms of a compact manifold is $C^1$-robustly minimal.

An open set $B \subset M$ is called blending region for a group/semigroup $\Gamma$ of diffeomorphisms of $M$ if there exist $h_1, \ldots, h_k \in \Gamma$ and an open set $D \subset M$ such that $B \subset \overline{D}$ and

i) $\overline{B} \subset h_1(B) \cup \cdots \cup h_k(B)$,

ii) $h_i : \overline{D} \to D$ is a contracting map for $i = 1, \ldots, k$. 


The action of the semigroup, finitely generated by the restrictions of \( h_1, \ldots, h_k \) to \( D \), is called associated contracting iterated function system.

A subset \( B \subset M \) has a cycle with respect to \( \Gamma \) if there exist maps \( T_1, \ldots, T_m, S_1, \ldots, S_n \in \Gamma \) such that
\[
M = T_1^{-1}(B) \cup \cdots \cup T_m^{-1}(B) = S_1(B) \cup \cdots \cup S_n(B).
\]

It is not difficult to see that if \( \Gamma \) acts forward and backward minimally on \( M \) then the cycle condition holds for any open subset of \( M \).

In the definition above of blending region, the covering property \((i)\) holds for the closure of the set \( B \). Roughly, the strength of this definition is the robustness of the property under the perturbation of the generators. This notion has been a local tool to provide \( C^1 \)-robustness of the minimality. See \([1, 2, 6, 8]\). Here, we weaken the covering property \((i)\) and deduce again the robust minimality.

**Corollary A’.** Consider a group/semigroup \( \Gamma \) of \( C^1 \)-diffeomorphisms of a compact manifold \( M \). Assume that there exists an open set \( B \) having a cycle with respect to \( \Gamma \), an open set \( D \subset M \) with \( B \subset D \) and maps \( h_1, h_2, \cdots \in \Gamma \) such that each \( h_i : D \to D \) is a contraction of rate \( \beta < 1 \) and
\[
B \subset \bigcup_{i=1}^{\infty} h_i(B).
\]

Then, the action of \( \Gamma \) is expanding and \( C^1 \)-robustly minimal.

As it is shown in \([14]\) the \( C^1 \)-regularity is not sufficient condition to conclude ergodicity from minimality. Most cases are taken into consideration concern the context of actually \( C^{1+\alpha} \)-diffeomorphisms (\( C^1 \)-diffeomorphisms with \( \alpha \)-Hölder derivatives). For instance, following essential idea of \([16]\), Navas proved in \([13]\) that every expanding minimal action of a group of \( C^{1+\alpha} \)-diffeomorphisms of the circle is Lebesgue-ergodic. We extend this result for any compact manifold and also show its persistence under perturbation of the action. As before, a group/semigroup action is robustly ergodic if this property persists under \( C^1 \)-perturbation among the generators with Hölder continuous derivative.

The next theorem allows us to deal with the relationship between two concepts mentioned above, i.e. minimality and ergodicity, under the condition of expanding that seem to us interesting and sufficiently mild.

**Theorem B.** Every expanding minimal group/semigroup action of \( C^{1+\alpha} \)-diffeomorphisms of a compact manifold is robustly ergodic with respect to Lebesgue measure.

As a direct consequence of Theorem B and Corollary A’ one gets the following result on the ergodicity of minimal actions having blending region.

**Corollary B’.** Every group/semigroup action of \( C^{1+\alpha} \)-diffeomorphisms having a blending region with a cycle is robustly ergodic with respect to the Lebesgue measure.
Now, we are going to introduce a category examples of actions by $C^1$-diffeomorphisms where minimality implies ergodicity assuming only some local extra regularity. This examples will be robust under $C^1$-perturbations where this local extra regularity persists. To do this, we need to introduce some preliminarily notions. It is known that for a blending region $B$, as above, the associated contracting iterated function system has a unique attractor $\Delta$ called Hutchinson attractor \cite{10} satisfying

$$
\Delta = \bigcup_{i=1}^{k} h_i(\Delta).
$$

Note that in view of the covering property (1) and equality above, one can easily get the inclusion $B \subset \Delta$.

By an iterated function system (IFS) we mean a semigroup IFS($H$) finitely generated by a family $H = \{h_1, \ldots, h_k\}$ of continuous maps on a metric space $X$. If $X$ is compact and the generators are contracting maps then, the IFS is called contracting. We say the contracting IFS is Vitali-regular if there is a Vitali-regular cover $V \subset \{h(\Delta) : h \in \text{IFS}(H)\}$ of $\Delta$. This means there is a constant $C > 0$ such that

- for any $V \in V$, $(\text{diam}V)^d \leq C \lambda(V)$,
- for any $\delta > 0$ and $x \in \Delta$, there is $V \in V$ with $x \in V$ and $\text{diam}V \leq \delta$.

The contracting IFS has bounded distortion property if there exists $L > 0$ such that for every $h \in \text{IFS}(H)$,

$$
L^{-1} \left| \frac{\det(Dh(x))}{\det(Dh(y))} \right| < L \quad \text{for all } x, y \in \Delta.
$$

There are two classical tools to guarantee the bounded distortion property. one is $C^{1+\alpha}$-regularity, see later on, and the another one, which is actually weaker, is Dini-regularity of the generators. Recall that a $C^1$-map $\phi$ is Dini-Regular if,

$$
\int_0^1 \Omega(\frac{\log \|D\phi(\cdot)\|}{t}, t) \, dt < \infty,
$$

where $\Omega(p, t)$ is the modulus of continuity of $p : M \to \mathbb{R}$ given by

$$
\Omega(p, t) = \max \{|p(x) - p(y)| : d(x, y) \leq t\}.
$$

By BDV-regular blending region we mean that the associated IFS is Vitali-regular and has bounded distortion property.

**Theorem C.** Consider a group/semigroup $\Gamma$ of $C^1$-diffeomorphisms of a compact manifold. Suppose that there exists a BDV-regular blending region $B$ having a cycle with respect to $\Gamma$. Then, the action of $\Gamma$ is ergodic with respect to Lebesgue measure (and $C^1$-robustly minimal). Moreover, the ergodicity persists under $C^1$-perturbations of the generators so that $B$ remains as a BDV-regular blending region.
Notice that in dimension one, every $C^{1+\alpha}$-blending region is BDV-regular. By a $C^{1+\alpha}$-blending region we understand that the associated generators are $C^{1+\alpha}$-contracting maps. Just like the one dimensional case, considering $C^{1+\alpha}$-diffeomorphisms with complex eigenvalues one can get a BDV-regular blending region in dimension two. This particular case provides a $C^{1+\alpha}$-robust conformal blending region, and hence, robustly ergodic action with respect to the Lebesgue measure under local Hölder perturbations. In higher dimensions, $C^{1+\alpha}$-conformal blending regions are also BDV-regular. Recall that a blending region is conformal if the corresponding contractions are conformal, this is, their derivatives are positive scalar of orthogonal linear maps. However, this regularity, i.e., the $C^{1+\alpha}$-conformality, does not persist in general under perturbations of the associated contracting iterated function system.

The main novelty of Theorem C is the local regularity assumption. That is, the regularity which is needed in Theorem C limited to regularity of the associated generators of the contracting maps in the local blending region $B$. For instance, to clarify the issue, consider an action $\Gamma$ on the circle with a blending region $B$ whose associated contracting maps are $C^{1+\alpha}$-regular in a neighborhood of $B$. Theorem C implies the ergodicity of such action.

**Corollary C'.** Every forward and backward minimal semigroup action (in particular minimal group actions) of $C^1$-diffeomorphisms of a compact manifold with a BDV-regular blending region is robustly\(^1\) ergodic with respect to the Lebesgue measure.

**Question 1.** Is every (or at least an open\(^2\) set) forward and backward minimal semigroup action of $C^1$-diffeomorphisms of a compact manifold with a $C^{1+\alpha}$-blending region ergodic with respect to the Lebesgue measure?

We want to note that Theorem B is basically a consequence of a local result on the ergodicity of the associated contracting IFS in its attractor.

**Definition 5.** A contracting IFS is said to be ergodic in the Hutchinson attractor with respect to a quasi-invariant measure $\mu$ in $X$ if $\mu(A) \in \{0, \mu(\Delta)\}$ for all IFS invariant $\mu$-measurable set $A \subset \Delta$.

The main result for the ergodicity of contracting IFS is the following:

**Theorem D.** Every BDV-regular contracting IFS is ergodic in the Hutchinson attractor with respect to the Lebesgue measure.

Remark that the result above also holds for the Hausdorff $s$-dimensional measure, see Remark 5.5 below. The question above may be reformulated as follows in the context of IFSs.

**Question 2.** Is every (or at least an open set) $C^{1+\alpha}$-contracting IFS ergodic in the Hutchinson attractor with respect to the Lebesgue measure?

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\(^1\)This robustness should be understood in the sense describes in Theorem B

\(^2\)This should be understood as a set of action whose generator are globally only $C^1$-close
As a corollary of Theorem C, one can extend the results of [12, 15] on the equivalence of the stationary measure and the Lebesgue measure for such contracting IFSs.

**Corollary D’.** Suppose that \( \mu \) is the stationary probability measure on \( M \) corresponding to a BDV-regular contracting IFS generated by \( h_1, \ldots, h_k \) and the (positive) probabilities \( p_1, \ldots, p_k \), i.e.,

\[
\mu = \sum_{i=1}^{k} p_i (\mu \circ h_i^{-1}).
\]

Let \( \triangle \) be the support of \( \mu \). If \( \mu \) is not singular to Lebesgue measure \( \lambda \) on \( M \), then

i) \( \mu \) is absolutely continuous with respect to \( \lambda \),

ii) \( \lambda|_{\triangle} \) is absolutely continuous with respect to \( \mu \).

Again, following [9, Theorem 1], this result may easily be generalized to the case of a Hausdorff \( s \)-dimensional measure and also to a probability measure \( \mu \) satisfying eigen-equation

\[
\lambda \mu = \sum_{i=1}^{k} p_i(\cdot) \mu \circ h_i^{-1} \quad \text{for some } \lambda > 0,
\]

where \( p_i(\cdot) \) is a family of continuous probability functions on \( M \).

In the first section we do some observation concerning minimality and ergodicity of a single diffeomorphism. The proof of Theorem A and Corollary A’ are handled in section 3. The proof of Theorem B is presented in section 4. In section 5, we discuss the Vitali-regularity and its connection with the ergodicity and stationary measures.

2. **Minimality/ergodicity, \( \mathbb{Z} \)-action vs cascade**

Given a diffeomorphism \( f \) of a compact manifold \( M \), one can study the iterations of this map from two points of views. The first, consists to consider full orbits, i.e., forward and backward iterations of \( f \). This case is called \( \mathbb{Z} \)-action of \( f \). The second only considers forward iterations of \( f \). This case is called cascade of \( f \).

Since we have introduced different notion of invariance for group and semigroup actions, a priori, one could expect that minimality and ergodicty were different properties for the \( \mathbb{Z} \)-action and for the cascade of a diffeomorphism.

**Proposition 2.1.** Let \( f \) be a diffeomorphism of a compact manifold \( M \). Then, the \( \mathbb{Z} \)-action of \( f \) is minimal if and only if the cascade of \( f \) is minimal.

**Proof.** It is clear that the minimality of the cascade implies the minimality of the \( \mathbb{Z} \)-action. Reciprocally, by contradiction, suppose that the \( \mathbb{Z} \)-action of \( f \) is minimal but not the cascade. Let \( x \) be a point whose forward orbit is not dense. Since \( M \) is a compact manifold, the \( \omega \)-limit of \( x \) is a closed
non-empty set invariant for the $\mathbb{Z}$-action. Since the forward orbit of $x$ is not dense then the $\omega$-limit of $x$ is different that the whole space. Then the $\mathbb{Z}$-action cannot be minimal contradicting our assumption. This conclude the proposition.

As in the case of minimality, the ergodicity of the cascade implies the ergodicity of $\mathbb{Z}$-action. In the particular case of invariant measures, we also get the converse. Recall that a measure $\mu$ on $M$ is said to be invariant under $f$ if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable set $A \subset M$. That is $f_*\mu = \mu$, in terms of the push-forward.

**Proposition 2.2.** Let $\mu$ be an invariant measure for a diffeomorphism $f$ of a compact manifold $M$. Then $\mathbb{Z}$-action of $f$ is ergodic with respect to $\mu$ if and only if the cascade of $f$ is ergodic with respect to $\mu$.

**Proof.** It suffices to see that the ergodicity of the $\mathbb{Z}$-action implies the ergodicity of the cascade. In order to do this, consider a measurable set $A \subset M$ such that $f(A) \subset A$. We will prove that $\mu(A) \in \{0, 1\}$. Consider

$$\Theta = \bigcup_{n=0}^{\infty} f^{-n}(A).$$

Observe that $f(\Theta) = \Theta$. Indeed, by assumption $A \subset f^{-1}(A)$ and therefore $f^{-1}(\Theta) = f^{-1}(A) \cup f^{-2}(A) \cup \cdots = A \cup f^{-1}(A) \cup f^{-2}(A) \cup \cdots = \Theta$. By the ergodicity of the $\mathbb{Z}$-action with respect to $\mu$, $\mu(\Theta) \in \{0, 1\}$. If $\mu(\Theta) = 0$ then $\mu(A) = 0$. On the other hand, if $\mu(\Theta) = 1$ then the measure of $M \setminus \Theta = \bigcap_{n=0}^{\infty} f^{-n}(A^c)$ is zero where $A^c = M \setminus A$. Now, since $f^{-(n+1)}(A^c) \subset f^{-n}(A^c)$ it follows that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(f^{-n}(A^c)) < \varepsilon$ for all $n \geq n_0$. By the invariance of $\mu$, we get that $\mu(A^c) < \varepsilon$ for all $\varepsilon > 0$ and therefore $\mu(A) = 1$. This concludes the proof of the proposition.

As far as know, if $\mu$ is a quasi-invariant measure that it is not invariant, the equivalence between ergodicity with respect to $\mu$ for $\mathbb{Z}$-actions and cascades is an open question. Observe that for actions generated by more than one map, the minimality/ergodicity of the group action does not imply, in general, the minimality/ergodicity of the semigroup action. A simple example can be constructed as follows:

**Example 2.3.** Consider a pair of diffeomorphisms $f_0, f_1$ of the circle $C^2$-close enough to rotations with zero rotation number, no fixed points in common and with an ss-interval (compact interval whose endpoints are consecutive attracting fixed points, one of $f_0$ and one of $f_1$). According to [2, Theorem A and Theorem 5.4], the group action generated by these two maps is minimal while the semigroup action cannot be minimal. In fact, the ss-interval is a non-empty closed invariant set for the semigroup action.
different of the circle $S^1$. Since this set has positive but not full Lebesgue measure, the semigroup action cannot be also ergodic with respect to the Lebesgue measure. However, according to [13, Theorem D], the group action is ergodic with respect to the Lebesgue measure (see also [2]).

Finally, we would mention the meaning of backward ergodicity of a semigroup action. Let $\mu$ be a quasi-invariant measure for the action of the group generated by a family of diffeomorphisms $F$. Observe that $\mu$ is also a quasi-invariant measure for both, the semigroup generated by $F$ and by $F^{-1}$. Here $F^{-1}$ denotes the family of inverse maps of $F$. The semigroup action generated by $F$ is backward ergodic with respect to $\mu$ if the semigroup action generated by $F^{-1}$ is ergodic.

3. Expanding and Minimality

3.1. Proof of Theorem A. The following lemma is obtained straightforward from the compactness of $M$.

**Lemma 3.1.** A group/semigroup action $\Gamma$ of homeomorphisms of a compact manifold $M$ is expanding if and only if there are $h_1, \ldots, h_k \in \Gamma$, open balls $B_1, \ldots, B_k$ in $M$ and a constant $\kappa > 1$ such that

i) $M = B_1 \cup \cdots \cup B_k$, and
ii) $\mu(Dh_i^{-1}(x)) > \kappa$ for all $x \in B_i$.

As a consequence of the lemma one has that:

**Remark 3.2.** The set of expanding group/semigroup actions of $C^1$-diffeomorphisms of an compact manifold $M$ is open.

Now, we are ready to prove Theorem A. We benefit the same idea as in [3, Lemma 10.2].

**Proof of Theorem A.** Consider an expanding minimal action of a group/semigroup $\Gamma$ of $C^1$-diffeomorphisms of a compact manifold $M$. Lemma 3.1 provides a finite open cover $\{B_1, \ldots, B_k\}$ of $M$, a constant $\kappa > 1$ and maps $h_1, \ldots, h_k \in \Gamma$ such that

$$d(h_i(x), h_i(y)) < \kappa^{-1}d(x, y) \quad \text{for all } x, y \in h_i^{-1}(B_i).$$

Let $L$ be the Lebesgue number of cover $\{B_i\}$. We consider a sufficiently small $C^1$-perturbation of the generators of $\Gamma$ such that these two properties holds

- there is a finite open cover $\{B_i\}$ with Lebesgue number greater than $L/2$,
- maps $\{h_i\}$ in the perturbed group/semigroup $\tilde{\Gamma}$ such $h_i$ restricted to $h_i^{-1}(B_i)$ is a contraction of rate $\kappa^{-1}$.

Now, a priori, we lose the density of the orbits, however, we just have $\varepsilon$-density. Shrinking, if necessary, the size of the neighborhood of perturbations, we assume that $\varepsilon \leq L/2$. Since $M$ is a compact manifold, it is locally
Claim 3.4. For every \( x \in M \) we want to prove the density of the orbit of \( x \) under the action \( \tilde{\Gamma} \). To do this, notice that for any \( y \in M \), \( B(y, \kappa^{-1} \varepsilon) \subset B_i \) for some \( i \). By the \( \varepsilon \)-density of the orbit of \( x \), there is \( h \in \tilde{\Gamma} \) such that \( h(x) \in B(h_i^{-1}(y), \varepsilon) \). The claim above implies that \( h \circ h(x) \in B(y, \kappa^{-1} \varepsilon) \). Since \( y \) is arbitrary, this shows that the orbit of \( x \) is \( \kappa^{-1} \varepsilon \)-dense. By induction, this orbit is \( \kappa^{-n} \varepsilon \)-dense for any \( n \in \mathbb{N} \) and hence is dense as we wanted to show.

3.2. Proof of Corollary A’. Before proving the theorem we prove a basic lemma.

Lemma 3.3. Let \( B \) be and open set satisfying the covering property (1). Then for every \( x \in B \), there is a sequence \( (i_j)_{j \geq 0}, i_j \in \mathbb{N} \) such that
\[
x = \lim_{n \to \infty} h_{i_1} \circ \cdots \circ h_{i_n}(y) \quad \text{for all } y \in B.
\]

Proof. Recursively define
\[
B_{i_1 \ldots i_n}^n = h_{i_n}(B_{i_1 \ldots i_{n-1}}^{n-1}) = h_{i_n} \circ \cdots \circ h_{i_1}(B) \quad \text{for } i_j \in \mathbb{N} \text{ and } j = 1, \ldots, n.
\]

Claim 3.4. For every \( n \in \mathbb{N} \) it holds
\[
B_{i_2 \ldots i_{n+1}}^n \subset \bigcup_{i_1 = 1}^{\infty} B_{i_1 i_2 \ldots i_n}^{n+1} \quad \text{and} \quad B \subset \bigcup_{i_1, i_2 = 1}^{\infty} B_{i_1 i_2}^{n+1}.
\]

Proof. The proof is by induction on \( n \). First, we show that
\[
B_{i_1}^1 \subset \bigcup_{i_2 = 1}^{\infty} B_{i_1 i_2}^2 \quad \text{and} \quad B \subset \bigcup_{i_1, i_2 = 1}^{\infty} B_{i_1 i_2}^2.
\]

By definition and using the inclusion \( B \subset B_1^1 \cup B_2^1 \cup \cdots \), one has that
\[
\bigcup_{i_1 = 1}^{\infty} B_{i_1 i_2}^2 = \bigcup_{i_1 = 1}^{\infty} h_{i_2}(B_{i_1}^1) = h_{i_2}(\bigcup_{i_1 = 1}^{\infty} B_{i_1}^1) \supset h_{i_2}(B) = B_{i_2}^1.
\]

From this one gets that
\[
\bigcup_{i_1, i_2 = 1}^{\infty} B_{i_1 i_2}^2 \supset \bigcup_{i_2 = 1}^{\infty} B_{i_2}^1 \supset B.
\]
Now, assuming the lemma holds for \( n - 1 \) we prove it for \( n \). In the same way as before,

\[
\bigcup_{i_1=1}^{\infty} B_{i_1, \ldots, i_{n+1}}^{n+1} = \bigcup_{i_1=1}^{\infty} h_{i_{n+1}}(B_{i_1, \ldots, i_n}^n) = h_{i_{n+1}}^{n+1}(\bigcup_{i_1=1}^{\infty} B_{i_1, \ldots, i_n}^n).
\]

By hypothesis of induction, we have that \( B_{i_1, \ldots, i_n}^{n-1} \subset \bigcup_{\ell=1}^{\infty} B_{i_1, \ldots, i_n}^{n-1} \) and so,

\[
\bigcup_{i_1=1}^{\infty} B_{i_1, \ldots, i_{n+1}}^{n+1} \supset h_{i_{n+1}}(B_{i_1, \ldots, i_n}^{n-1}) = B_{i_1, \ldots, i_{n+1}}^n.
\]

Now, note that we have

\[
B_{i_2, \ldots, i_{\ell+1}}^\ell \subset \bigcup_{i_1=1}^{\infty} B_{i_1, \ldots, i_{\ell+1}}^{\ell+1}
\]

for every \( 1 \leq \ell \leq n \) and for all \( i_j \in \mathbb{N} \) with \( j = 2, \ldots, \ell + 1 \). Hence,

\[
\bigcup_{i_1, \ldots, i_{n+1}=1}^{\infty} B_{i_1, \ldots, i_{n+1}}^{n+1} \supset \bigcup_{i_2, \ldots, i_{n+1}=1}^{\infty} B_{i_2, \ldots, i_{n+1}}^{n} \supset \cdots \supset \bigcup_{i_{n+1}=1}^{\infty} B_{i_{n+1}=1} B
\]

and the proof of the claim is completed. \( \square \)

Since \( B \subset B_1^1 \cup B_2^1 \cup \cdots \), for each \( x \in B \) there is \( i_1 \in \mathbb{N} \) such that \( x \in B_{i_1}^1 \). We now proceed recursively. For \( n > 1 \), suppose that one has \( i_j \in \mathbb{N} \), for \( j = 1, \ldots, n \), such that \( x \in B_{i_1, \ldots, i_n}^n \). By Claim 3.4, \( B_{i_1, \ldots, i_j}^n \supset \bigcup_{\ell=1}^{j} B_{i_1, \ldots, i_n}^{n+1} \).

Hence, there is \( i_{n+1} \in \mathbb{N} \) such that \( x \in B_{i_1, \ldots, i_{n+1}}^n \). From this, one can choose a positive sequence \( i = i_1 i_2 \cdots = (i_j)_{j>0} \) such that \( x \in B_{i_1, \ldots, i_n}^n \), for all \( n \geq 1 \). Thus, one gets

\[
x \in \bigcap_{n \geq 1} B_{i_1, \ldots, i_n}^n = \bigcap_{n \geq 1} h_{i_1} \circ \cdots \circ h_{i_n}(B) = \bigcap_{n \geq 1} A_n.
\]

Note that for any \( n \in \mathbb{N} \), \( A_n = \cap_{i=1}^{n} h_{i_1} \circ \cdots \circ h_{i_n}(B) \) and \( A_{n+1} \subset A_n \subset h_{i_1} \circ \cdots \circ h_{i_n}(B) \subset D \). Now, since each \( h_i \) is a contraction in \( D \) of rate \( 0 < \beta < 1 \), it follows

\[
\text{diam}(A_n) \leq \text{diam}(h_{i_1} \circ \cdots \circ h_{i_n}(B)) \leq \beta^n \text{diam}(B)
\]

Therefore \( A_n \) is a nested sequence of sets whose diameters goes to zero and so

\[
\{x\} = \bigcap_{n \geq 1} B_{i_1, \ldots, i_n}^n = \bigcap_{n \geq 1} h_{i_1} \circ \cdots \circ h_{i_n}(B).
\]

Finally, from this one can deduce that for a given \( y \in B \),

\[
d(h_{i_1} \circ \cdots \circ h_{i_n}(y), x) \leq \text{diam}(h_{i_1} \circ \cdots \circ h_{i_n}(B)) \leq \beta^n \text{diam}(B)
\]

for every \( n \in \mathbb{N} \). Since \( 0 < \beta < 1 \), in particular, one has

\[
x = \lim_{n \to \infty} h_{i_1} \circ \cdots \circ h_{i_n}(y)
\]

and the conclusion of the lemma is complete. \( \square \)
Now, we show the corollary.

Proof of Corollary A. Since $M = S_1(B) \cup \cdots \cup S_n(B)$, for every point $x \in M$, there is $i \in \{1, \ldots, n\}$ such that $S_i^{-1}(x) \in B$. The covering property (1) allows us to iterate this point in $B$ by at least one map $h_i^{-1}$ and remains in $B$. Since $m(Dh_i^{-1}(z)) \geq \beta^{-1}$ for all $z \in h_i(B)$ and $i = 1, \ldots, k$, repeating this argument one gets $g \in \Gamma$ such that $m(Dg^{-1}(x)) > 1$. That is, the action of $\Gamma$ on $M$ is expanding.

Given an open set $U$ and a point $x$ in $M$, by the cycle condition,

$$M = T_1^{-1}(B) \cup \cdots \cup T_m^{-1}(B) = S_1(B) \cup \cdots S_n(B)$$

one can find $i$ and $j$ such that $T_i(x) \in B$ and $B \cap S_j^{-1}(U)$ contains an open set. By Lemma 3.3, every point in $B$ has dense orbit in $B$, and so, there exists $h \in \Gamma$ such that $h \circ T_i(x) \in S_j^{-1}(U)$. Hence, $S_j \circ h \circ T_i(x) \in U$. This shows the minimality of the action. Now, Theorem A implies that this action is $C^1$-robustly minimal. \hfill \Box

4. Expanding and Ergodicity

4.1. Proof of Theorem B. To provide proof of Theorem B we need to know a preliminary fact on topological behavior of a minimal action and a key ingredient of the $C^{1+\alpha}$-topology.

Lemma 4.1 (Forward cycle). An action of a group/semigroup $\Gamma$ of homeomorphisms of a compact manifold $M$ is minimal if and only if for every open set $U \subset M$, there exist $T_1, \ldots, T_m \in \Gamma$ such that

$$M = T_1^{-1}(U) \cup \cdots \cup T_m^{-1}(U).$$

Proof. From the minimality, given any open set $U \subset M$, for every $y \in M$ there exists $T_y \in \Gamma$ such that $T_y(y) \in U$. Now, using that $M$ is compact, we get a finite set of maps $T_1, \ldots, T_m$ such that $M = T_1^{-1}(U) \cup \cdots \cup T_m^{-1}(U)$.

Reciprocally, consider an open set $U$ and a point $x \in M$. By the assumption there exist $T_1, \ldots, T_m \in \Gamma$ such that $M = T_1^{-1}(U) \cup \cdots \cup T_m^{-1}(U)$. Thus, $T_i(x) \in U$, for some $1 \leq i \leq m$. That is, $\Gamma$ acts minimally on $M$. \hfill \Box

Consider a family $\mathcal{H} = \{h_1, \ldots, h_k\}$ of homeomorphisms of $M$. Given a sequence $\omega = \omega_1 \omega_2 \cdots \in \Sigma_k^+ = \{1, \ldots, k\}^\mathbb{N}$ and $n \in \mathbb{N}$, denote by $h^n_\omega$ the composition $h_{\omega_n} \circ \cdots \circ h_{\omega_1}$. The following lemma is always the key ingredient to prove ergodicity results.

Lemma 4.2 (Bounded Distortion). Consider a family $\mathcal{H} = \{h_1, \ldots, h_k\}$ of $C^{1+\alpha}$-diffeomorphisms of $M$ where $m(Dh_i(x)) > 1$ for all $x \in B_i$ with $M = B_1 \cup \cdots \cup B_k$ and $B_i$ open. Then, there exists $L_{\mathcal{H}} > 0$ such that for every $n \in \mathbb{N}$,

$$L_{\mathcal{H}}^{-1} < \left| \frac{\det(Dh^n_\omega(x))}{\det(Dh^n_\omega(y))} \right| < L_{\mathcal{H}}$$
for all \( x, y \in M \) and \( \omega = \omega_1 \omega_2 \cdots \in \Sigma_k^+ \) such that
\[
h_{\omega}(x), h_{\omega}(y) \in B_{\omega_i+1} \quad \text{for } 0 \leq i < n - 1.
\]

Proof. Note that by assumption, \( F_i = \log |\det h_i| \) is \( \alpha \)-Hölder and thus for every \( x, y \), it holds \( |F_i(x) - F_i(y)| \leq Cd(x, y)^\alpha \) for all \( i = 1, \ldots, k \) and for some constants \( C > 0 \). Now, fix \( n \in \mathbb{N} \). Let \( x, y \) and \( \omega \) be two points in \( M \) and sequence in \( \Sigma_k^+ \) such that \( h_{\omega}(x), h_{\omega}(y) \in B_{\omega_i+1} \) for \( 0 \leq i < n - 1 \). Since \( h_i \) is an expanding map in \( B_i \), then there exists a constant \( \kappa > 1 \) such that \( \kappa d(x, y) \leq d(h_i(x), h_i(y)) \) for all \( x, y \in B_i \) and \( i = 1, \ldots, k \). Thus,
\[
\kappa d(h_{\omega}(x), h_{\omega}(y)) \leq d(h_{\omega}^{i+1}(x), h_{\omega}^{i+1}(y))
\]
and hence
\[
d(h_{\omega}(x), h_{\omega}(y)) \leq \kappa^{-(n-i)}d(h_{\omega}^n(x), h_{\omega}^n(y)) \leq K\kappa^{-(n-i)}
\]
where \( K = \max_{i=1,\ldots,k} \text{diam}(B_i) \). This implies that
\[
\log \left| \frac{\det(Dh^n_{\omega}(x))}{\det(Dh^{i+1}_{\omega}(y))} \right| = \sum_{i=0}^{n-1} |F_i(h_{\omega}^i(x)) - F_i(h_{\omega}^i(y))| \leq C \sum_{i=0}^{n-1} d(h_{\omega}^i(x), h_{\omega}^i(y))^\alpha
\]
\[
\leq C \sum_{i=0}^{n-1} (K\kappa^{-(n-i)})^{\alpha} \leq C K^\alpha \sum_{i=0}^{\infty} \kappa^{-i\alpha}.
\]
Taking \( L_\mathcal{H} = \exp\{C K^\alpha \kappa^{-\alpha}/(1 - \kappa^{-\alpha})\} \) the desired inequality holds. \( \square \)

A point \( x \in M \) is a Lebesgue-density point of a measurable set \( A \subseteq M \) if
\[
\lim_{r \to 0} \frac{\lambda(A \cap B(x, r))}{\lambda(B(x, r))} = 1,
\]
where \( \lambda \) is normalized Lebesgue measure. We will denote by \( DP(A) \) the set of density points of \( A \). By Lebesgue Density Theorem, \( \lambda \)-almost every point in \( A \) is a density point. That is, \( \lambda(A \setminus DP(A)) = 0 \). We will use the notation \( E \subset F \) and say that the set \( E \) is contained (mod 0) in \( F \), if \( \lambda(E \setminus F) = 0 \).

The following proposition is the main tool to proof Theorem B.

**Proposition 4.3.** Consider an expanding action of a group (resp. semigroup) of \( C^{1+\alpha} \)-diffeomorphisms of a compact manifold \( M \). Then, for every invariant set \( A \subseteq M \) with positive Lebesgue measure (resp. whose complementary has positive Lebesgue measure) there exist \( x \in M \) and \( r > 0 \) such that
\[
B(x, r) \subset A \quad \text{(resp. } B(x, r) \subset M \setminus A).\]

Proof. By Lemma 3.1, there are maps \( g_1, \ldots, g_k \) in the group/semigroup \( \Gamma \), open sets \( B_1, \ldots, B_k \) and a constant \( \kappa > 1 \) such that \( M = B_1 \cup \cdots \cup B_k \) and \( \kappa d(x, y) \leq d(g_i^{-1}(x), g_i^{-1}(y)) \) for all \( x, y \in B_i \) and \( i = 1, \ldots, k \). To simplify notation, we write \( h_i = g_i^{-1} \). Let \( L > 0 \) be the Lebesgue number of the above open covering.

By assumption, Lebesgue theorem allows us to take a density point \( x_0 \) of either, \( A \) if \( \Gamma \) is a group or \( A^c = M \setminus A \) if \( \Gamma \) is a semigroup. For every
0 < \delta < L/2, the open ball \( B(x_0, \delta) \) is contained in a some \( B_i \). Moreover, since \( h_i \) is an expanding map on \( B_i \), then \( h_i(B(x_0, \delta)) \) contains an open ball of radius \( \kappa \delta \) centered at \( h_i(x_0) \). If \( \kappa \delta < L/2 \) one repeats this process obtaining that \( h_j \circ h_i(B(x_0, \delta)) \) contains an open ball of radius \( \kappa^2 \delta \) centered at \( h_j \circ h_i(x_0) \). Since this process provides open balls of strictly increasing radius, one gets \( h^{-1} \in \Gamma \) such that \( h(B(x_0, \delta)) \) contains an open ball of radius \( L/2 \). Therefore, for each \( n \in \mathbb{N} \), taking \( \delta_n = L/4n \), there exists \( \omega \in \Sigma^+ \) such that

\[
B(x_n, L/2) \subset h^n_\omega(B(x_0, \delta_n)) \quad \text{where } x_n = h^n_\omega(x_0). \tag{2}
\]

By the compactness of \( M \), taking a subsequence if necessary, \( x_n \) converges to \( x \). Observe that, there exists \( n_0 \in \mathbb{N} \) such that

\[
B(x, r) \subset B(x_n, L/2) \quad \text{for all } n \geq n_0 \text{ where } r = L/4 > 0. \tag{3}
\]

On the other hand, the inclusions (2) and (3), and the invariance of \( A \) for the group action imply that for every \( n \geq n_0 \),

\[
h^n_\omega(B(x_0, \delta_n) \setminus A) = h^n_\omega(B(x_0, \delta_n)) \setminus A \supset B(x, r) \setminus A.
\]

Note that in the semigroup case, the (forward) invariance of \( A \) implies that \( h^{-1}(A^c) \subset A^c \) for all \( h \in \Gamma \). Thus,

\[
h^n_\omega(B(x_0, \delta_n) \setminus A^c) \supset h^n_\omega(B(x_0, \delta_n)) \setminus A^c \supset B(x, r) \setminus A^c.
\]

In what follows, in order to unify notation, \( \Theta \) denote \( A \) and \( A^c \) in the group and semigroup case respectively. For every \( n \geq n_0 \) one has that,

\[
\frac{\lambda(B(x, r) \setminus \Theta)}{\lambda(M)} \leq \frac{\lambda(h^n_\omega(B(x_0, \delta_n) \setminus \Theta))}{\lambda(h^n_\omega(B(x_0, \delta_n)))} \leq L_H \frac{\lambda(B(x_0, \delta_n) \setminus \Theta)}{\lambda(B(x_0, \delta_n))}. \tag{4}
\]

The last inequality is implied by the bounded distortion, Lemma 4.2. Indeed, by construction, for every \( z \in B(x_0, \delta_n) \) one has that \( h^n_\omega(z) \in B_{\omega_{i+1}} \) for \( 0 \leq i < n \). Hence, it suffices to note that

\[
\lambda(h^n_\omega(B(x_0, \delta_n) \setminus \Theta)) = \int_{B(x_0, \delta_n) \setminus \Theta} |\det Dh^n_\omega| \, d\lambda \\
\leq \lambda(B(x_0, \delta_n) \setminus \Theta) \sup_{z \in B(x_0, \delta_n)} |\det Dh^n_\omega(z)|,
\]

\[
\lambda(h^n_\omega(B(x_0, \delta_n))) = \int_{B(x_0, \delta_n)} |\det Dh^n_\omega| \, d\lambda \\
\geq \lambda(B(x_0, \delta_n)) \inf_{z \in B(x_0, \delta_n)} |\det Dh^n_\omega(z)|,
\]

and therefore, Lemma 4.2 implies that

\[
\frac{\lambda(h^n_\omega(B(x_0, \delta_n) \setminus \Theta))}{\lambda(h^n_\omega(B(x_0, \delta_n)))} \leq L_H \frac{\lambda(B(x_0, \delta_n) \setminus \Theta)}{\lambda(B(x_0, \delta_n))}.
\]

Since \( x_0 \in DP(\Theta) \), one gets that

\[
\lim_{n \to \infty} \frac{\lambda(B(x_0, \delta_n) \setminus \Theta)}{\lambda(B(x_0, \delta_n))} = 0.
\]
Now, inequality (4) implies that $\lambda(B(x, r) \setminus \Theta) = 0$ and the proof is done. □

Now, we are ready to prove Theorem B.

**Proof of Theorem B.** Consider an expanding minimal action of a group/semigroup $\Gamma$ of $C^{1+\alpha}$-diffeomorphisms of a compact manifold $M$. Let $A$ be an invariant measurable set for the group/semigroup. Observe that in the semigroup case $M \setminus A$ is also an invariant set for the backward semigroup action, i.e., for the action of the semigroup generated by the inverse maps of $\Gamma$. To unify notations, $\Theta$ denote $A$ in the group case and $M \setminus A$ in the semigroup case. Suppose that $\lambda(\Theta) > 0$. We will prove that $\lambda(\Theta) = 1$.

By Proposition 4.3, there exists $x \in M$ and $r > 0$ such that $B(x, r) \subset \Theta$. In view of the minimality of the action, Lemma 4.1 provides $T_1, \ldots, T_m \in \Gamma$ such that

$$M = T_1^{-1}(B(x, r)) \cup \cdots \cup T_m^{-1}(B(x, r)).$$

Since $\Theta$ is an invariant set for either, the group action or the backward semigroup action, one has that

$$T_i^{-1}(B(x, r)) \setminus \Theta \subset T_i^{-1}(B(x, r) \setminus \Theta)$$

for all $i = 1, \ldots, m$. Hence, the quasi-invariance of $\lambda$ for $C^1$-diffeomorphisms implies that $\lambda(T_i^{-1}(B(x, r)) \setminus \Theta) = 0$ and so $\lambda(M \setminus \Theta) = 0$. This proves that $\lambda(\Theta) = 1$ concluding the proof of the theorem. □

5. Vitali-Regularity and Ergodicity

We begin by proof of the bounded distortion property (BD) for contracting IFS and continue to review the basic properties of the concept of Vitali-regularity.

**Lemma 5.1.** Every $C^{1+\alpha}$-contracting (resp. Dini-contracting) IFS is of BD.

**Proof.** Let $h_1, \ldots, h_k$ be the contracting generators of the semigroup $\text{IFS}(H)$. Assume that $F_i = \log |\det DH_i|$ is $\alpha$-Hölder. Given $h \in \text{IFS}(H)$, there exists $\omega = \omega_1 \omega_2 \cdots \in \Sigma_k^+$ and $n \in \mathbb{N}$ such that $h = h^n_\omega = h_{\omega_n} \circ \cdots \circ h_{\omega_1}$. Then,

$$d(h^n_\omega(x), h^n_\omega(y)) \leq \xi^n d(x, y) \leq \xi^n \text{diam}(\Delta),$$

where

$$\xi = \sup_{x \in \Delta, 1 \leq i \leq k} ||DH_i(x)|| < 1.$$

Hence,

$$\log |\frac{\det(Dh^n_\omega(x))}{\det(Dh^n_\omega(y))}| = \sum_{i=0}^{n-1} |F_{\omega_i}(h^n_\omega(x)) - F_{\omega_i}(h^n_\omega(y))|$$

$$\leq C \sum_{i=0}^{n-1} \{\xi^i \text{diam}(\Delta)\}^\alpha.$$
Taking $L = \exp\{C\xi^\alpha (\text{diam}(\triangle))^\alpha/(1 - \xi^\alpha)\}$ the desired inequality and therefore we conclude the lemma for $C^{1+\alpha}$-contracting IFS. Similar agreement proves the lemma for Dini-contracting IFS (see [4]). □

We recall that an IFS is said to be conformal if its generators are conformal maps.

**Lemma 5.2.** Every BD-contracting conformal IFS is Vitali-regular.

**Proof.** It suffices to note that the norm $\|D\phi(x)\|$ of derivative of a conformal map $\phi$ equals to $|\det D\phi(x)|^{1/d}$ where $d$ is the dimension of the manifold. Now, in view of the bounded distortion property and using variables one can easily to check the the contracting IFS is Vitali-regular. □

**Lemma 5.3.** Every Vitali-regular contracting IFS satisfies the following property: for any $x \in DP(\triangle)$ there exist positive constants $C_1, C_2 > 0$ such that for every $\delta > 0$ there is $\mathcal{V}_\delta \subset \{h(\triangle) : h \in \text{IFS}(\mathcal{H})\}$ having the following property.

$$C_1 \lambda(B(x, \delta)) \geq \lambda\left(\bigcup_{V \in \mathcal{V}_\delta} V\right) = \sum_{V \in \mathcal{V}_\delta} \lambda(V) \geq C_2 \lambda(B(x, \delta) \cap \triangle). \quad (5)$$

**Proof.** Fix $x \in \triangle$ and $\delta > 0$. Since the contracting IFS is Vitali-regular, by Vitali’s covering theorem for the Lebesgue measure, there exists a finite or countably infinite disjoint subcollection $\mathcal{V}_\delta$ of $\mathcal{V} \subset \{h(\triangle) : h \in \text{IFS}(\mathcal{H})\}$ such that

$$B(x, \delta) \cap \triangle = \bigcup_{V \in \mathcal{V}_\delta} V.$$ 

This implies (5) taking $C_1 = C_2 = 1$. □

### 5.1. Proof of Theorem D.

Assume that there is $\mathcal{H} = \{h_1, \ldots, h_k\} \subset \Gamma$ such that the IFS generated by the restriction of these maps to the closure of an open set $D \subset M$ is contracting, has bounded distortion property and satisfies (5). Given a set $A \subset M$ put

$$\text{Orb}_{\mathcal{H}}^{-1}(A) = \bigcup_{h \in \text{IFS}(\mathcal{H})} h^{-1}(A) \cup A.$$ 

The following lemma is the main step in proving the future results.

**Lemma 5.4.** Let $A$ be a measurable set of $M$. Then

$$\lambda(\text{Orb}_{\mathcal{H}}^{-1}(A) \cap \triangle) \in \{0, \lambda(\triangle)\}.$$ 

Moreover, if $DP(A) \cap DP(\triangle) \neq \emptyset$ then it always holds that $\triangle \subset \text{Orb}_{\mathcal{H}}^{-1}(A)$.

**Proof.** If $\lambda(\triangle) = 0$ the result is trivial. Thus, suppose that $\lambda(\triangle) > 0$. Put $\Theta = \text{Orb}_{\mathcal{H}}^{-1}(A)$. Suppose that $\lambda(\Theta \cap \triangle) > 0$, and so, one should prove the equality $\lambda(\triangle \setminus \Theta) = 0$. Lebesgue Density Theorem implies the existence of a density point $x$ of $\Theta \cap \triangle$. Then, one can find $\delta_0 > 0$ such that

$$\lambda(B(x, \delta) \cap \triangle) > \lambda(B(x, \delta))/2 \quad \text{for all } \delta_0 \geq \delta > 0.$$
By assumption, there exist positive constants \(C_1, C_2 > 0\) such that for every \(\delta > 0\) with \(\delta \leq \delta_0\) there is \(V_\delta \subset \{h(\triangle) : h \in \text{IFS}(\mathcal{H})\}\) having the following property
\[
C_1 \lambda(B(x, \delta)) \geq \lambda(\bigcup_{V \in V_\delta} V) = \sum_{V \in V_\delta} \lambda(V) \geq C_2 \lambda(B(x, \delta) \cap \triangle).
\]

Hence, by the backward invariance of \(\Theta\), i.e., \(h^{-1}(\Theta) \subset \Theta\) for all \(h \in \text{IFS}(\mathcal{H})\), one gets that
\[
\frac{\lambda(B(x, \delta) \setminus \Theta)}{\lambda(B(x, \delta))} \geq \frac{1}{C_1} \frac{\lambda(\cup \setminus \Theta)}{\lambda(B(x, \delta))} = \frac{1}{C_1} \sum_{V \in V_\delta} \frac{\lambda(V \setminus \Theta)}{\lambda(B(x, \delta))} \geq \frac{1}{C_1} \sum_{V \in V_\delta} \frac{\lambda(h(\triangle) \setminus \Theta))}{\lambda(B(x, \delta))}.
\]

Since the IFS is BD-contracting it follows that
\[
\sum \frac{\lambda(h(\triangle) \setminus \Theta))}{\lambda(B(x, \delta))} = \sum \frac{\lambda(h(\triangle) \setminus \Theta)) \lambda(h(\triangle))}{\lambda(h(\triangle)) \lambda(B(x, \delta))} \geq L \frac{\lambda(\triangle \setminus \Theta)}{\lambda(\triangle)} \sum \frac{\lambda(V)}{\lambda(B(x, \delta))} \geq L C_2 \frac{\lambda(\triangle \setminus \Theta) \lambda(B(x, \delta) \cap \triangle)}{\lambda(B(x, \delta))} \geq \frac{L C_2 \lambda(\triangle \setminus \Theta)}{2 \lambda(\triangle)}.
\]

Therefore, we obtain that
\[
\frac{\lambda(B(x, \delta) \setminus \Theta)}{\lambda(B(x, \delta))} \geq \frac{L C_2 \lambda(\triangle \setminus \Theta)}{2 C_1 \lambda(\triangle)} \quad \text{for all } 0 < \delta \leq \delta_0.
\]

Since \(x\) is a Lebesgue density point of \(\Theta\) one has that
\[
0 = \lim_{\delta \to 0} \frac{\lambda(B(x, \delta) \setminus \Theta)}{\lambda(B(x, \delta))} \geq \frac{L C_2 \lambda(\triangle \setminus \Theta)}{2 C_1 \lambda(\triangle)}.
\]

This implies that \(\lambda(\triangle \setminus \Theta) = 0\) which concludes the first part of the lemma.

Now, to show the second part, take \(x\) a density point of both \(A\) and \(\triangle\). Observe that since \(A \subset \Theta\), then the above argument also shows that
\[
\frac{\lambda(B(x, \delta) \setminus A)}{\lambda(B(x, \delta))} \geq \frac{\lambda(B(x, \delta) \setminus \Theta)}{\lambda(B(x, \delta))} \geq \frac{L C_2 \lambda(\triangle \setminus \Theta)}{2 C_1 \lambda(\triangle)}.
\]

Now, using that \(x\) is a density point of \(A\) it follows that \(\lambda(\triangle \setminus \Theta) = 0\) and we conclude the second part of the lemma. \(\Box\)

Now, we are ready to prove Theorem D.

**Proof of Theorem D.** Let \(A\) be a measurable set in the Hutchinson attractor \(\triangle\) such that \(h(A) \subset A\) for all \(h \in \text{IFS}(\mathcal{H})\). Set \(\hat{A} = \triangle \setminus A\). We claim that
\[
h^{-1}(\hat{A} \cap h(\triangle)) \subset \hat{A} \quad \text{for all } h \in \text{IFS}(\mathcal{H}).
\]
By contradiction, suppose $x$ be a point in $\hat{A} \cap h(\triangle)$ such that $h^{-1}(x) \not\in \hat{A}$. Since $x \in \hat{A}$, by the invariance, $x \not\in h(A)$, that is $h^{-1}(x) \not\in A$. On the other hand, by the assumption, $h^{-1}(x) \not\in \hat{A}$. Hence $h^{-1}(x) \not\in \triangle$ which yields a contradiction. Thus,

$$\text{Orb}_{\mathcal{H}}(\hat{A}) \cap \triangle = \bigcup_{h \in \text{IFS}(\mathcal{H})} h^{-1}(\hat{A} \cap h(\triangle)) \cup (\hat{A} \cap \triangle) = \hat{A}.$$ 

Hence, by Lemma 5.4 one has that $\lambda(\hat{A}) \in \{0, \lambda(\triangle)\}$, that means the measure of $A$ equals to either zero or $\lambda(\triangle)$, concluding the theorem. □

**Remark 5.5.** Theorem D follows for the Hausdorff $s$-dimensional measure with the same proof. More details on the adaption of the proof above to such case can be found in [4, 9].

**Proof of Corollary D’**. Since the IFS generated by $\mathcal{H} = \{h_1, \ldots, h_k\}$ is contractive, according to [10] the measure $\mu$ is unique fulfilling the selfsimilarity relation $\mu = p_1(h_1)_*\mu + \cdots + p_k(h_k)_*\mu$. Moreover, the support $\triangle$ of this measure is the Hutchinson attractor. Using the Lebesgue decomposition one can split $\mu = \mu_{ac} + \mu_s$ into an absolute part and a singular part. Observe that both measures, $\mu_{ac}$ and $\mu_s$, satisfy the selfsimilarity relation and thus one gets that $\mu$ is either singular or absolutely continuous with respect to Lebesgue measure.

To conclude the result, we will prove that if $\mu$ is not singular with respect to the Lebesgue measure $\lambda$ then $\mu$ is equivalent to the restriction of $\lambda$ to the support of $\mu$. Suppose that $\mu$ is absolutely continuous but not equivalent to the restriction of the Lebesgue measure. Observe that this implies that $\lambda(\triangle) > 0$ and the existence of a measurable set $A \subset \triangle$ with $\lambda(A) > 0$ and $\mu(A) = 0$. Let $\Theta = \text{Orb}_{\mathcal{H}}(A) \cap \triangle$. Since $\lambda(A) > 0$, Lemma 5.4 implies that $\lambda(\Theta) = \lambda(\triangle)$. On the other hand, the selfsimilarity of the measure implies that $\mu(h^{-1}(A)) = 0$ for all $h \in \text{IFS}(\mathcal{H})$ and hence $\mu(\Theta) = 0$. This means that $\mu$ is singular with respect to the restriction of Lebesgue contradicting that $\mu$ is absolutely continuous to $\lambda$. □

5.2. **Proof of Theorem C.** First we need a lemma.

**Lemma 5.6.** Consider a group/semigroup $\Gamma$ of $C^1$-diffeomorphisms of $M$. If an open set $B \subset M$ has forward cycle

$$M = S_1(B) \cup \cdots \cup S_n(B)$$

with respect to $\Gamma$ then, $B \cap DP(A) \neq \emptyset$ for all group-invariant set $A \subset M$ with positive measure. Moreover, $B \cap DP(M \setminus A) \neq \emptyset$ for all semigroup-invariant set $A \subset M$ whose complementary has positive Lebesgue measure.

**Proof.** First assume that $A$ is group-invariant, i.e., $g(A) = A$ for all $g \in \Gamma$, with positive Lebesgue measure and suppose that $DP(A) \cap B = \emptyset$. Hence, Lebesgue density theorem implies that $\lambda(A \cap B) = 0$. On the other hand,
in view of invariance condition we have
\[ A = A \cap M = A \cap \left( \bigcup_{i=1}^{n} S_i(B) \right) \subset \bigcup_{i=1}^{n} S_i(A \cap B). \]

By the quasi-invariance of the Lebesgue measure for smooth maps, it follows that \( \lambda(S_i(A \cap B)) = 0 \) for all \( i = 1, \ldots, n \) and therefore \( \lambda(A) = 0 \) which is a contradiction.

Now, let \( A \) be a semigroup-invariant set, i.e., \( g(A) \subset \Gamma \) for all \( g \in \Gamma \), whose complementary \( A^c = M \setminus A \) has positive Lebesgue measure. Suppose that \( DP(A^c) \cap B = \emptyset \). As above, it follows \( \lambda(A^c \cap B) = 0 \). Then, the (forward) invariance implies that
\[ A^c = A^c \cap M = A^c \cap \left( \bigcup_{i=1}^{n} S_i(B) \right) \subset \bigcup_{i=1}^{n} S_i(A^c \cap B). \]

Now, the quasi-invariance of the Lebesgue measure implies that \( \lambda(A^c) = 0 \). This contradiction concludes the proof in this case. \( \square \)

Now, we are ready to prove Theorem C.

**Proof of Theorem C.** According to Corollary \( A' \), the action of \( \Gamma \) on \( M \) is \( C^1 \)-robustly minimal. It remains to prove the ergodicity of the action with respect to Lebesgue measure. Let \( H \subset \Gamma \) be the finite family of generators of the associated BDV-regular contracting IFS to the blending region \( B \). As mentioned before,
\[ B \subset \triangle = \overline{\text{Orb}_H^+(x)} \quad \text{for all} \quad x \in \triangle \]
where \( \overline{\text{Orb}_H^+(x)} = \{ h(x) : h \in \text{IFS}(\mathcal{H}) \} \) and \( \triangle \) is the Hutchinson attractor. By assumption, there exist maps \( T_1, \ldots, T_m, S_1, \ldots, S_n \in \Gamma \) such that
\[ M = T_1^{-1}(B) \cup \cdots \cup T_n^{-1}(B) = S_1(B) \cup \cdots \cup S_n(B). \]
Consider a measurable set \( A \subset M \) such that \( g(A) \subset A \) for all \( g \in \Gamma \). We want to prove that \( \lambda(A) \in \{0,1\} \). To unify notations, let \( \Theta \) denotes \( A \) in the group case and \( M \setminus A \) in the semigroup case. Suppose that \( \lambda(\Theta) > 0 \). We will prove that \( \lambda(\Theta) = 1 \). This concludes the result.

By Lemma 5.6, \( B \cap DP(\Theta) \neq \emptyset \). Since \( B \) is an open set in \( \triangle \) one has that \( DP(\Theta) \cap DP(\triangle) \neq \emptyset \). Hence, Lemma 5.4 and the invariance of \( \Theta \) imply that
\[ B \subset \triangle \subset \overline{\text{Orb}_H^+(\Theta)} = \Theta. \]
Since \( g^{-1}(\Theta) \subset \Theta \) for \( g \in \Gamma \), one has that \( T_i^{-1}(B) \setminus \Theta \subset T_i^{-1}(B \setminus \Theta) \) for all \( i \). Now, the quasi-invariance of \( \lambda \) for \( C^1 \)-diffeomorphisms implies that \( \lambda(T_i^{-1}(B) \setminus \Theta) = 0 \) and so \( \lambda(M \setminus \Theta) = 0 \). This proves that \( \lambda(\Theta) = 1 \) concluding the proof of the theorem. \( \square \)
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