Semiring identities of the Brandt monoid*

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Abstract

The 6-element Brandt monoid $B^1_2$ admits a unique addition under which it becomes an additively idempotent semiring. We show that this addition is a term operation of $B^1_2$ as an inverse semigroup. As a consequence, we exhibit an easy proof that the semiring identities of $B^1_2$ are not finitely based.

We assume the reader’s acquaintance with basic concepts of universal algebra such as an identity and a variety; see, e.g., [1, Chapter II].

The 6-element Brandt monoid $B^1_2$ can be represented as a semigroup of the following zero-one $2 \times 2$-matrices

\[
\begin{pmatrix}
0 & 0 \\ 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\ 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\ 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\ 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\ 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\ 0 & 1
\end{pmatrix}
\]

under the usual matrix multiplication $\cdot$ or as a monoid with presentation

$$
\langle E_{12}, E_{21} \mid E_{12} E_{21} E_{12} = E_{12}, E_{21} E_{12} E_{21} = E_{21}, E_{12}^2 = E_{21}^2 = 0 \rangle.
$$

Quoting from a recent paper [3], ‘This Brandt monoid is perhaps the most ubiquitous harbinger of complex behaviour in all finite semigroups’. In particular, $(B^1_2, \cdot)$ has no finite basis for its identities (Perkins [13,14]) and is one of the four smallest semigroups with this property (Lee and Zhang [10]).

The monoid $(B^1_2, \cdot)$ has a natural involution that swaps $E_{12}$ and $E_{21}$ and fixes all other elements. In terms of the matrix representation (1) this involution is nothing but the usual matrix transposition; we will, however, use the notation $x \mapsto x^{-1}$ for

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*Supported by the Ministry of Science and Higher Education of the Russian Federation (Ural Mathematical Center project No. 075-02-2020-1537/1)
the involution, emphasizing that $x^{-1}$ is the unique inverse of $x$. Recall that elements $x, y$ of a semigroup $(S, \cdot)$ are said to be inverses of each other if $x y x = x$ and $y x y = y$. A semigroup is called inverse if every its element has a unique inverse; inverse semigroups can therefore be thought of as algebras of type $(2,1)$. Being considered as an inverse semigroup, the monoid $(B_2^1, \cdot, ^{-1})$ retains its complex equational behaviour: $B_2^1$ has no finite basis for its inverse semigroup identities (Kleiman [6]) and is the smallest inverse semigroup with this property (Kleiman [5, 6]).

In the present note we consider equational properties of yet another enhancement of the monoid $(B_2^1, \cdot)$ with an additional operation, this time binary. Recall that an additively idempotent semiring an algebra $(S, +, \cdot)$ of type $(2,2)$ such that the additive reduct $(S, +)$ is a semilattice (that is, a commutative idempotent semigroup), the multiplicative reduct $(S, \cdot)$ is a semigroup, and multiplication distributes over addition on the left and on the right, that is, $(S, + , \cdot)$ satisfies the identities $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$. In papers which motivation comes from semigroup theory, objects of this sort sometimes appear under the name semilattice-ordered semigroups, see, e.g., [8] or [12]. We will stay with the term ‘additively idempotent semiring’, abbreviated to ‘ai-semiring’ in the sequel.

Our key observation is the following:

**Lemma 1.** Let $(S, \cdot, ^{-1})$ be an inverse semigroup satisfying the identity

$$x^n \approx x^{n+1}$$

for some $n$. Define

$$x \oplus y := (xy^{-1})^n x.$$ 

Then $(S, \cdot, \oplus)$ is an ai-semiring.

**Proof.** Let $E(S)$ stand for the set of all idempotents of $S$. The relation

$$\leq := \{(a, b) \in S \times S \mid a = eb \text{ for some } e \in E(S)\}$$

is a partial order on $S$ referred to as the natural partial order; see [15] Section II.1 or [9] pp. 21–23. We need two basic properties of the natural partial order:

1) $\leq$ is compatible with both multiplication and inversion;
2) $a \leq b$ if and only if $a = bf$ for some $f \in E(S)$.

Take any $a, b \in S$ and suppose that $c \leq a$ and $c \leq b$. Then $c^{-1} \leq b^{-1}$ whence by the compatibility with multiplication

$$c = (cc^{-1})^n c \leq (ab^{-1})^n a = a \oplus b.$$ 

In presence of the identity (2), $(ab^{-1})^n = (ab^{-1})^{n+1} = \cdots = (ab^{-1})^{2n}$. Hence

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\[ a \oplus b = (ab^{-1})^n \cdot a \leq a. \]

Further,

\[
\begin{align*}
 a \oplus b &= (ab^{-1})^n a = (ab^{-1})^{n+1} a = \cdots = (ab^{-1})^{2n-1} a = \\
 &\phantom{=} \cdot (ab^{-1})^{n-1} a = (ab^{-1})^n \cdot a(b^{-1}a)^{n-1} = \text{(using } b^{-1} = b^{-1}bb^{-1}) \ \\
 &\phantom{=} \cdot (b^{-1}a)^n \leq b \cdot (b^{-1}a)^n \leq b
\end{align*}
\]

since \((b^{-1}a)^n \in E(S)\). We see that \(a \oplus b\) is nothing but the infimum of \(\{a, b\}\) with respect to the natural partial order. Thus, \((S, \oplus)\) is a semilattice. It is known \cite[Proposition 1.22]{16}, see also \cite[Proposition 19]{9} that if a subset \(H \subseteq S\) possesses an infimum under the natural partial order, then so do the subsets \(sH\) and \(Hs\) for any \(s \in S\), and \(\inf(sH) = s(\inf H)\), \(\inf(Hs) = (\inf H)s\). This implies that multiplication distributes over \(\oplus\) on the left and on the right.

**Remark 1.** The essence of Lemma \(1\) is known. Leech, in the course of his comprehensive study of inverse monoids \((S, \cdot, ^{-1}, 1)\) that are inf-semilattices under the natural partial order, has verified that \((S, \leq)\) is an inf-semilattice whenever \(S\) is a periodic combinatorial\(1\) inverse monoid; see \cite[Example 1.21(d), item (iv)]{11}. Of course, the requirement of \(S\) being a monoid is not essential: if a semigroup \(S\) periodic and combinatorial then so is the monoid \(S^1\) obtained by adjoining a formal identity to \(S\). Clearly, if a semigroup satisfies \(\mathcal{L}\), then it is both periodic and combinatorial whence Leech’s observation applies. We have preferred the above direct proof of Lemma \(1\) because we need a \((\cdot, ^{-1})\)-term for the semilattice operation, and such a term is not explicitly present in \cite{11}.

Obviously, the 6-element Brandt monoid satisfies the identity \(x^2 \approx x^3\). Thus, Lemma \(1\) applies, and \((B_2^1, \oplus, \cdot)\) is an ai-semiring. It is known (and easy to verify) that \(\oplus\) is the only addition on \(B_2^1\) under which \(B_2^1\) becomes an ai-semiring.

Our main result states that, similarly to the plain semigroup \((B_2^1, \cdot)\) and the inverse semigroup \((B_2^1, \cdot, ^{-1})\), the ai-semiring \((B_2^1, \oplus, \cdot)\) admits no finite identity basis. Its proof employs a series of inverse semigroups \(C_n, n = 2, 3, \ldots, \) constructed in \cite{6} as semigroups of partial one-to-one transformations. Here, to align with the matrix representation chosen for the \(B_2^1\), we describe them as semigroups of zero-one matrices.

The set \(R_m\) of all zero-one \(m \times m\)-matrices which have at most one entry equal to 1 in each row and column forms an inverse monoid under usual matrix multipli-

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\(1\)A semigroup \(S\) is **periodic** if all monogenic subsemigroups of \(S\) are finite and **combinatorial** if all subgroups of \(S\) are trivial.
cation \cdot and transposition. The inverse monoid $R_m$ is called the rook monoid\(^2\) as its matrices encode placements of nonattacking rooks on an $m \times m$ chessboard.

Let $m = 2n + 1$ and define $m \times m$-matrices $c_1, \ldots, c_n$ by

$$c_k := E_{k+1k} + E_{n+k\ n+k+1}, \ k = 1, \ldots, n,$$

where, as usual, $E_{ij}$ denotes the $m \times m$-matrix unit with an entry 1 in the $(i, j)$ position and 0’s elsewhere. For instance, if $n = 2$, then $c_1$ and $c_2$ are the following $5 \times 5$-matrices:

$$c_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad c_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Let $C_n$ be the inverse subsemigroup of the rook monoid $R_m$ generated by the matrices $c_1, \ldots, c_n$. As a plain subsemigroup, $C_n$ is generated by $c_1, \ldots, c_n$ and their inverses (i.e., transposes) $c_1^{-1}, \ldots, c_n^{-1}$.

The next lemma collects properties of the semigroups $C_n$ that we need.

**Lemma 2.** (i) The semigroup $(C_n, \cdot)$ does not belong to the semigroup variety generated by the monoid $(B_1 \cdot, \cdot)$.

(ii) The semigroup $(C_n, \cdot)$ satisfies the identity $x^2 \approx x^3$.

(iii) For each $k = 1, \ldots, n$, $M_k(n) := C_n \setminus \{c_k, c_k^{-1}\}$ forms an inverse subsemigroup of the inverse semigroup $(C_n, \cdot, -1)$.

(iv) For each $k = 1, \ldots, n$, the inverse semigroup $(M_k(n), \cdot, -1)$ belongs to the inverse semigroup variety generated by the inverse monoid $(B_2^1, \cdot, -1)$.

**Proof.** (i) This property was established in \cite{7} Lemma 3] by exhibiting, for each $n \geq 2$, a semigroup identity that holds in $(B_2^1, \cdot)$ and fails in $(C_n, \cdot)$.

(ii) This is easy to verify (and also follows from the proof of Lemma 1 in \cite{6}).

(iii) This is clear (and is a part of Lemma 1 in \cite{6}).

(iv) This is Property (C) in \cite{6}. \hfill \square

**Remark 2.** Items (i)–(iii) of Lemma 2 are easy. In contrast, the proof of (iv) in \cite{6} is long and complicated. We mention in passing that now the proof can be radically simplified by using a deep result by Kadourek [4] who provided an effective membership test for the inverse semigroup variety generated by $(B_2^1, \cdot, -1)$.

\(^2\)The rook monoid is nothing but the matrix representation of the symmetric inverse monoid; see \cite{15} Section IV.1 or \cite{9} p. 6. The name ‘rook monoid’ was suggested by Solomon [17].
Theorem 3. The semiring identities of the additively idempotent semiring \((B_2^1, +, \cdot)\) admit no basis involving only finitely many variables, and hence, no finite basis.

Proof. Arguing by contradiction, assume that \((B_2^1, +, \cdot)\) has an identity basis \(\Sigma\) such that each identity \(u \approx v\) in \(\Sigma\) involves less than \(n\) variables. Consider the inverse semigroup \((C_n, \cdot, \cdot^{-1})\). By Lemmas 1 and 2(ii), the addition defined by \(x + y := (xy^{-1})^2 x\) makes \((C_n, +, \cdot)\) an ai-semiring. Consider an arbitrary evaluation \(\varepsilon\) of variables \(x_1, \ldots, x_\ell\) involved in the identity \(u \approx v\) in this ai-semiring. By the pigeonhole principle, there exists an index \(k \in \{1, \ldots, n\}\) such that neither \(c_k\) nor \(c_k^{-1}\) belongs to the set \(\{\varepsilon(x_1), \ldots, \varepsilon(x_\ell)\}\) as this set contains at most \(\ell < n\) elements. Thus, \(\{\varepsilon(x_1), \ldots, \varepsilon(x_\ell)\} \subset M_k(n)\).

Since \(x + y\) expresses as \((-1)^\cdot\)-term, one can rewrite the identity \(u \approx v\) into an identity \(u' \approx v'\) in which \(u'\) and \(v'\) are \((-1)^\cdot\)-terms. Since \(u \approx v\) holds in \((B_2^1, +, \cdot)\), the rewritten identity \(u' \approx v'\) holds in the inverse semigroup \((B_2^1, \cdot, \cdot^{-1})\). By Lemma 2(iv) the latter identity holds also in the inverse semigroup \((M_k(n), \cdot, \cdot^{-1})\), and so \(u'\) and \(v'\) take the same value under every evaluation of the variables \(x_1, \ldots, x_\ell\) in \(M_k(n)\). Hence \(\varepsilon(u) = \varepsilon(u') = \varepsilon(v') = \varepsilon(v)\). We conclude that the identity \(u \approx v\) holds in the ai-semiring \((C_n, +, \cdot)\). Since an arbitrary identity from \(\Sigma\) holds in \((C_n, +, \cdot)\), this ai-semiring belongs to the ai-semiring variety generated by \((B_2^1, +, \cdot)\). This, however, contradicts Lemma 2(i), according to which even the semigroup reduct \((C_n, \cdot)\), does not belongs to semigroup variety generated by \((B_2^1, \cdot)\).

Remark 3. To the best of my knowledge, the result of Theorem 3 has not been published up to now. However, after preparing the present article I have learnt that the result has also been obtained by colleagues in Xi’an and Melbourne but with an entirely unrelated proof.

I mention also a related paper by Dolinka [2] where he introduces a 7-element ai-semiring denoted \(\Sigma_7\) and proves that its identities are not finitely based. The semigroup reduct of \(\Sigma_7\) is just the monoid \(B_2^1\) with an extra zero adjoined so that \((\Sigma_7, \cdot, \cdot^{-1})\) and \((B_2^1, \cdot, \cdot^{-1})\) satisfy the same inverse semigroup identities. However, the addition in \(\Sigma_7\) is not derived from its inverse semigroup structure, and one can easily see that the semiring identities of \((\Sigma_7, +, \cdot)\) and \((B_2^1, +, \cdot)\) are essentially different. It should be also mentioned that in [2] Dolinka actually considers ai-semirings with 0 as algebras of type \((2,2,0)\).

Remark 4. Leech [11] defined an inverse algebra as an algebra \((A, \cdot, \cdot, \cdot^{-1}, 1)\) of type \((2,2,1,0)\) such that the reduct \((A, \cdot, \cdot^{-1}, 1)\) is an inverse monoid, the reduct \((A, \cdot)\) is a meet semilattice, and the natural partial order of the inverse monoid coincides with that of the semilattice. Clearly, \((B_2^1, \cdot, \cdot^{-1}, E)\) constitutes an inverse algebra in Leech’s sense, and the above proof of Theorem 3 can be easily adapted to show that \((B_2^1, \cdot, \cdot^{-1}, E)\) has no finite identity basis also as such algebra.
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