Monotonicity formulas in potential theory

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Received: 13 July 2019 / Accepted: 31 October 2019 / Published online: 27 November 2019
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Abstract
Using the electrostatic potential $u$ due to a uniformly charged body $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we introduce a family of monotone quantities associated with the level set flow of $u$. The derived monotonicity formulas are exploited to deduce a new quantitative version of the classical Willmore inequality.

Mathematics Subject Classification 35B06 · 53C21 · 35N25

1 Introduction
1.1 Setting of the problem and statement of the main result

We consider the electrostatic potential due to a charged body, modelled by a bounded domain $\Omega$ with $C^2$-boundary, for some $0 < \alpha < 1$. The potential is defined as the unique solution $u$ of the following problem in the exterior domain

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\
u &= 1 \quad \text{on } \partial \Omega, \\
u(x) &\to 0 \quad \text{as } |x| \to \infty.
\end{aligned}
$$

(1.1)

It follows from the strong maximum principle and the Hopf’s Lemma that $|Du| > 0$ on the boundary, so that $\partial \Omega$ is a regular level set of $u$. It is also worth pointing out that for every $0 < t \leq 1$ the level set $\{u = t\}$ is compact, due to the properness of $u$. Moreover, we have that for every $t > 0$ sufficiently small the level set $\{u = t\}$ is diffeomorphic to a $(n - 1)$-dimensional sphere, and thus connected. These properties can be deduced from expansion (1.5) below.
A natural quantity associated with a solution to problem (1.1) is the electrostatic capacity of \( \Omega \), which is defined as

\[
\text{Cap}(\Omega) := \inf \left\{ \frac{1}{(n - 2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |Dw|^2 \, d\mu : w \in C_c^\infty(\mathbb{R}^n), \; w \equiv 1 \text{ in } \Omega \right\}.
\]

We recall that the capacity of \( \Omega \) can be computed in terms of the electrostatic potential \( u \) as

\[
\text{Cap}(\Omega)(n - 2)|\mathbb{S}^{n-1}| = \int_{\partial\Omega} |Du| \, d\sigma = \int_{\{u = 1\}} |Du| \, d\sigma,
\]

where in the second equality we have used the equation \( \Delta u = 0 \) and the Divergence Theorem, together with the fact that each level set of \( u \) has finite \( \mathcal{H}^{n-1} \)-measure (an easy consequence of the analyticity of \( u \), see [25, Theorem 3.4.8]) and that the unit normal is well defined and given by \( -Du/|Du| \mathcal{H}^{n-1} \)-a.e. on each level set of \( u \) (see [12,33,34]). This last fact can be rephrased by saying that the function \( F_0 : [1, \infty) \rightarrow \mathbb{R} \), given by

\[
\tau \mapsto F_0(\tau) = \int_{\{u = 1/\tau\}} |Du| \, d\sigma,
\]

is constant and \( F_0(\tau) \equiv \text{Cap}(\Omega)(n - 2)|\mathbb{S}^{n-1}| \), for every \( \tau \geq 1 \). In analogy with (1.3), we introduce for \( \beta \geq 0 \) the functions \( F_\beta : [1, \infty) \rightarrow \mathbb{R} \) given by

\[
\tau \mapsto F_\beta(\tau) = \tau^{\beta(\frac{n-1}{n-2})} \int_{\{u = 1/\tau\}} |Du|^\beta \, d\sigma.
\]

To describe a first property of the functions \( F_\beta \), observe that, using the following classical asymptotic expansions at infinity (see for instance [39])

\[
\begin{align*}
\quad u & = \text{Cap}(\Omega)|x|^{2-n} + o(|x|^{2-n}), \\
\quad D_\mu u & = -(n - 2)\text{Cap}(\Omega)|x|^{-n}x_\mu + o(|x|^{1-n}), \\
\quad D_\mu D_v u & = (n - 2)\text{Cap}(\Omega)|x|^{-n-2} \left( nx_\mu x_v - |x|^2 \delta_{\mu v}\right) + o(|x|^{-n}),
\end{align*}
\]

one can easily compute the limit

\[
\lim_{\tau \to +\infty} F_\beta(\tau) = \left[ \text{Cap}(\Omega) \right]^{\frac{n-2-\beta}{n-2}} (n - 2)^{\beta + 1} |\mathbb{S}^{n-1}|.
\]

Before proceeding with the statement of the main result, it is worth describing some other features of the functions \( \tau \mapsto F_\beta(\tau) \). First of all, we notice that such functions are well defined, since the integrands are globally bounded and the level sets of \( u \) have finite hypersurface area. In fact, since \( u \) is harmonic, the level sets of \( u \) have locally finite \( (n - 1) \)-dimensional Hausdorff measure \( \mathcal{H}^{n-1} \) (see for instance [34] and the references therein). Moreover, by the properness of \( u \), such level sets are compact and thus with finite hypersurface area. To describe another important feature of the \( F_\beta \)'s, we recall that in the case where \( \Omega \) is a ball the explicit solution to problem (1.1) is given by (a multiple of) the Green function. Hence, the expansions (1.5) deprived of the reminder terms yield in this case explicit formulae for \( u, Du \) and \( DDu \) on the whole \( \mathbb{R}^n \setminus \overline{\Omega} \). Tacking advantage of this observation, one easily realises that the quantities
For every \( \mathbb{R}^n \setminus \Omega \ni x \mapsto \frac{|Du|}{u^{n-2}}(x) \) and \( [1, +\infty) \ni \tau \mapsto \int_{\{u=1/\tau\}} u^{\frac{n-1}{n-2}} d\sigma \) \((1.7)\)

are constant on rotationally symmetric solutions. Also notice en passant that the square of the first quantity is known in the literature as the \( P \)-function naturally associated with problem \((1.1)\), see for instance \([21,23,27,28,30,42,43]\). In Sect. 1.3 we will shed some light on the geometric nature of such a function. We also note that, for every \( \beta \geq 0 \), the functions \( \tau \mapsto F_\beta(\tau) \) can be rewritten in terms of the quantities appearing in \((1.7)\) as

\[
F_\beta(\tau) = \int_{\{u=1/\tau\}} \left( \frac{|Du|}{u^{\frac{n-1}{n-2}}} \right)^{\beta+1} u^{\frac{n-1}{n-2}} d\sigma.
\] \((1.8)\)

Therefore they are constant on rotationally symmetric solutions. In contrast with this, our main result states that the functions \( \tau \mapsto F_\beta(\tau) \) are in general monotonically nonincreasing and that the monotonicity is strict unless \( u \) is rotationally symmetric.

**Theorem 1.1** (Monotonicity–Rigidity Theorem) *Let \( u \) be a solution to problem \((1.1)\) and let \( F_\beta : [1, +\infty) \to \mathbb{R} \) be the function defined in \((1.4)\). Then, the following properties hold true.*

(i) **Continuity.** For every \( \beta \geq 0 \), the function \( F_\beta \) is continuous.

(ii) **Differentiability, Monotonicity and Rigidity.** For every \( \beta \geq (n-2)/(n-1) \), the function \( F_\beta \) is continuously differentiable and the derivative admits for every \( \tau \geq 1 \) the integral representation

\[
F'_\beta(\tau) = -\beta \int_{\{u<1/\tau\}} u^{-2\beta(\frac{n-2}{n-1})}|Du|^{\beta-2} \left\{ \left[ |D^2u|^2 - \left( \frac{n}{n-1} \right) |D|Du|^2 \right] + \left( \beta - \frac{n}{n-2} \right) |D^T|Du|^2 \right. 
+ \left. \left( \beta - \frac{n-2}{n-1} \right) |Du|^2 \left[ H - \left( \frac{n-1}{n-2} \right) |D \log u| \right]^2 \right\} d\mu,
\] \((1.9)\)

where \( H(x) \) is the mean curvature of the level set \( \{u \} \) passing through \( x \), computed with respect to the unit normal vector field \( v = -Du/|Du| \), and \( D^T \) denotes the tangential part of the gradient. In particular, the derivative is always nonpositive. Furthermore, the sign of the derivative is strict for all \( \tau \in [1, +\infty) \), unless the function \( F_\beta \) is constant and \( u \) is rotationally symmetric.

(iii) **Convexity and Rigidity.** For every \( \beta \geq (n-2)/(n-1) \), the function \( F_\beta \) is convex and the convexity is strict, unless the function is constant and \( u \) is rotationally symmetric. Moreover, for every \( \tau \in [1, +\infty) \) where \( F_\beta \) is twice differentiable, the second derivative obeys the formula

\[
F''_\beta(\tau) = \beta \tau^{\beta(\frac{n-2}{n-1})-4} \int_{\{u=1/\tau\}} |Du|^{\beta-1} \left\{ |h|^2 - \left( \frac{1}{n-1} \right) H^2 \right. 
+ \beta \left| \frac{D^T|Du|}{|Du|} \right|^2 
+ \left. \left( \beta - \frac{n-2}{n-1} \right) \left[ H - \left( \frac{n-1}{n-2} \right) |D \log u| \right]^2 \right\} d\sigma,
\] \((1.10)\)

where \( h \) is the second fundamental form of the level set \( \{u = 1/\tau\} \), computed with respect to the unit normal vector field \( v = -Du/|Du| \), and \( D^T \) denotes the tangential part of the...
gradient. In particular, the second derivative is always nonnegative, wherever defined. Furthermore, the sign is strict, unless the function $F_{\beta}$ is constant and $u$ is rotationally symmetric.

It is worth mentioning that monotonicity formulas are nowadays known to play a fundamental role in geometric analysis (dropping any attempt of being complete, we mention [36,37,44] and also the more recent [11]), as well as in the study of geometric properties of harmonic functions on manifolds subject to suitable curvature lower bounds [18–20].

Before stating in Sect. 1.2 the main geometric implications of the above theorem, let us list some remarks, in which some technical details concerning our monotonicity formulas are taken into account.

**Remark 1** Let us observe that if $1/\tau$ is a regular value of $u$, then the function $F_{\beta}$ is differentiable at $\tau$ for every $\beta \geq 0$, and a direct computation gives

$$F'_{\beta}(\tau) = -\beta \tau^{(n-1)/2-2} \int_{\{u=1/\tau\}} |Du|^\beta \left[ H - \left(\frac{n-1}{n-2}\right) |D\log u| \right] d\sigma,$$

where $H$ is the mean curvature of the level set $\{u = 1/\tau\}$ computed with respect to the unit normal vector field $\nu = -Du/|Du|$. In Sect. 4 we will combine the above formula (1.11) with the sign coming from the Monotonicity Formula (1.9) for $\beta \geq (n-2)/(n-1)$, in order to draw several geometric conclusions. For example, such combination implies at once the non existence of minimal level sets of $u$, and in turn the non existence of smooth minimal compact hypersurfaces in $\mathbb{R}^n$.

**Remark 2** Notice that for $\beta \geq 1$ formula (1.11) is well-posed also in the case where $\{u = 1/\tau\}$ is not a regular level set of $u$. Indeed, as already observed, it is well-known that, since $u$ is harmonic and proper, the $\mathcal{H}^{n-1}$-measure of each of its level sets is finite. Moreover, by the results in [33] (see also [12,15]), the Hausdorff dimension of its critical set is bounded above by $(n-2)$. In particular, the unit normal is well defined $\mathcal{H}^{n-1}$-almost everywhere on each level set and so does the mean curvature $H$. In turn, the integrand in (1.11) is well defined $\mathcal{H}^{n-1}$-almost everywhere. Finally, we observe that where $|Du| \neq 0$ it holds

$$|Du|^\beta H = |Du|^\beta - 3 Du(\nabla u)^2 (Du, Du).$$

Since $|D^2 u|$ is uniformly bounded in $\mathbb{R}^n \setminus \Omega_1$, this shows that the integrand in (1.11) is essentially bounded and thus summable on every level set of $u$, provided $\beta \geq 1$. In the case where $(n-2)/(n-1) \leq \beta < 1$, it is no longer possible to infer that the function $|Du|^\beta H$ is essentially bounded on the critical level sets of $u$. However, we will prove in Corollary 3.5—that is at the core of Theorem 1.1(ii)—that the right hand side of (1.11) admits a unique continuous extension to the critical values of $u$. In this sense, the function $|Du|^\beta H$ can be understood as an integrable function also for small admissible values of $\beta$.

**Remark 3** The same facts and arguments recalled in the previous remark can be employed to deduce that for every $\beta \geq (n-2)/(n-1)$ the integrand in the right hand side of formula (1.9) is well defined $\mathcal{H}^{n-1}$-almost everywhere. The fact that such a function is summable represents a significant step in the proof of Theorem 1.1(ii) and it is somehow encoded in the statement that $F'_{\beta}$ is continuous and hence finite-valued.

**Remark 4** Formula (1.10) for the second derivative of $F_{\beta}$ follows at once from (1.9), through the Coarea Formula and the pointwise identity.
\[
|\text{DD}u|^2 - \left(\frac{n}{n-1}\right)|\text{D}Du|^2 - \left(\frac{n-2}{n-1}\right)|\text{DT}Du|^2 = |Du|^2 \left[|h|^2 - \left(\frac{1}{n-1}\right)H^2\right].
\]

However, at a regular value of \(u\), it can also be deduced from (1.11) via a direct computation.

### 1.2 Geometric implications

In this subsection, we describe the main geometric conclusions that can be drawn from the Monotonicity-Rigidity Theorem 1.1, deferring the proofs to Sect. 4.

A nowadays classical statement about the geometry of smooth closed surfaces in the Euclidean three-dimensional space is the so called Willmore Inequality, namely

\[
16\pi \leq \int_{\partial\Omega} H^2 d\sigma,
\]

where the equality is achieved if and only if the domain \(\Omega\) is a round ball. Its validity, together with its extension to higher dimensions, has been established by the joint efforts of several authors (see [16, Theorem 3], [17], and also [47], together with the references therein). Applying the Monotonicity-Rigidity Theorem with \(\beta = n-2\), one gets immediately that

\[
F_{n-2}(1) \geq \lim_{\tau \to +\infty} F_{n-2}(\tau),
\]

where the limit equals \((n-2)^{n-1}|S^{n-1}|\), in view of formula (1.6). On the other hand, it is not hard to use the sign of (1.11) in combination with the Hölder inequality in order to obtain an upper bound for \(F_{n-1}(1)\) in terms of the \(L^{n-1}\)-norm of the mean curvature of \(\partial\Omega\). As a consequence, one recovers a new proof of the \((n-1)\)-dimensional Willmore Inequality

\[
|S^{n-1}| \leq \int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} d\sigma,
\]

together with its corresponding rigidity statement, which characterises the round balls. Using the optimal threshold for the exponent \(\beta\) in our Monotonicity Formulas, we also deduce a novel sharp quantitative version of the above inequality.

**Theorem 1.2** (Quantitative Willmore-type inequality) Let \(\Omega \subseteq \mathbb{R}^n\), \(n \geq 3\), be a bounded domain with smooth boundary. Then, the inequality

\[
\left|F'_{n-2}(1)\right| \leq A(n) \left[\text{Cap}(\Omega)\right]^{(n-2)/(n-1)}
\times \left[\left(\int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} d\sigma\right)^{1/(n-1)} - |S^{n-1}|^{1/(n-1)}\right]
\]

holds true, where \(H\) is the mean curvature of \(\partial\Omega\) and the positive constant \(A(n)\) is explicitly given by

\[
A(n) = (n-2)^{(2n-3)/(n-1)} |S^{n-1}|^{(n-2)/(n-1)}.
\]

Moreover, the deficit on the left hand side is optimal in the sense that if it vanishes, then the right hand side also vanishes and \(\Omega\) is a round ball.

**Remark 5** The optimal deficit on the left hand side of (1.14) can be written more explicitly with the help of (1.9), so that the Quantitative Willmore-type Inequality rewrites as...
\[ \int_{\mathbb{R}^n} u \left[ \frac{|Du|^2 - \left(\frac{n}{n-1}\right)|D|Du|^{\frac{2}{n-1}}}{|Du|^{n/(n-1)}} \right] d\mu \]

\[ \leq \hat{A}(n) \left[ \text{Cap}(\Omega) \right]^{\frac{n-2}{n-1}} \left\{ \left( \int_{\partial \Omega} \frac{H}{n-1} \right) \left[ \int_{\partial \Omega} \frac{H}{n-1} \right] \left[ \frac{1}{n-1} \right] - \frac{1}{n-1} \right\} \]

where \( u \) is the capacitary potential of \( \Omega \) and the positive constant \( \hat{A}(n) \) is explicitly given by

\[ \hat{A}(n) = (n - 1) \left[ (n - 2) \left[ \frac{n-1}{n-1} \right] \right]^{(n-2)/(n-1)}. \]

In [2] we extend the validity of the Willmore-type Inequality to the case of manifolds with non-negative Ricci curvature and Euclidean volume growth, obtaining among the consequences some new characterizations of the Asymptotic Volume Ratio. That context is also the most natural for drawing a detailed comparison with the beautiful papers [18–20] by Colding and Minicozzi, where they obtain monotonicity formulas along the level set of Green’s functions. Here we just observe that in the present setting their formulas do not yield any new information, due to the rotational symmetry of the Euclidean Green’s function. The sake of comparison with the above cited papers by Colding and Minicozzi has also inspired the notation of comparison with the above cited papers by Colding and Minicozzi has also inspired the information, due to the rotational symmetry of the Euclidean Green’s function. The sake

Another important instrument in the study of the geometry of the hypersurfaces in \( \mathbb{R}^n \) is the so called Minkowski inequality. It says that if \( \Omega \) is convex, then

\[ |\partial \Omega|^{(n-2)/(n-1)} \left| \mathbb{S}^{n-1} \right|^{1/(n-1)} \leq \int_{\partial \Omega} \frac{H}{n-1} \, d\sigma. \tag{1.16} \]

A new proof of this classical result is provided in [26], using the level set flow of \( p \)-capacitary potentials. Taking advantage of the beautiful results of [31,46] about the long time behaviour of the Inverse Mean Curvature Flow for domains that are mean convex and starshaped, it is possible to extend the validity of inequality (1.16) to this class of domains (see [32]). Another route, still based on the Inverse Mean Curvature Flow, has been suggested by Huisken [35] and essentially carried out in [29]. It allows to prove the Minkowski inequality for outward minimizing and strictly mean convex domains (which are not necessarily diffeomorphic to spheres). Furthermore, it is observed in [13,14] that the Minkowski inequality for mean convex domains with a non optimal constant follows directly from the Michael-Simon-Sobolev inequality. An open question is whether the mean convexity alone is sufficient to imply the validity of (1.16) or not. We will address this problem in the forthcoming work [1]. Here, we provide a weighted version of inequality (1.16) that has the advantage of not requiring any geometric restriction on \( \Omega \).

**Theorem 1.3 (Weighted Minkowski Inequality)** Let \( \Omega \subseteq \mathbb{R}^n, n \geq 3 \), be a bounded domain with smooth boundary. Then, the inequality

\[ \left| F'_{\frac{n-2}{n-1}}(1) \right| \leq A(n) \left[ \frac{\text{Cap}(\Omega)}{|\partial \Omega|} \right]^{\frac{n-2}{n-1}} \left\{ \int_{\partial \Omega} \frac{H}{n-1} \, d\sigma - |\partial \Omega|^{(n-2)/(n-1)} \left| \mathbb{S}^{n-1} \right|^{1/(n-1)} \right\} \tag{1.17} \]

holds true, where \( H \) is the mean curvature of \( \partial \Omega \), the measure element \( d\sigma \) is defined as

\[ d\bar{\sigma} = \left( \frac{|Du|}{\int_{\partial \Omega} |Du| \, d\sigma} \right)^{(n-2)/(n-1)} \, d\sigma, \]
and the positive constant $A(n)$ is given by formula (1.15). Moreover, the deficit on the left hand side is optimal in the sense that if it vanishes, then the right hand side also vanishes and $\Omega$ is a round ball.

Concerning the weighted measure that appears in the above statement, it is intriguing to observe that by the reverse Jensen’s inequality, one has that

$$\int_{\partial\Omega} d\sigma \leq |\partial\Omega|,$$

so that the mass of $\partial\Omega$ with respect to $\sigma$ is less than or equal to the usual one.

**Remark 6** Let us underline that inequalities (1.14) and (1.17) are obtained exploiting the full power of Theorem 1.1, up to the threshold value $\beta = (n - 2)/(n - 1)$. This is possible only by virtue of the optimal treatment of the critical set that is proposed in the present paper, in sharp contrast with the effective but non optimal analysis contained in [5].

### 1.3 Strategy of the proof

In this subsection, we present the main ideas underlying the proof of the Monotonicity-Rigidity Theorem. To do this, we focus for simplicity on the case $\beta = 2$. Our strategy consists of two main steps. The first step is the construction of a cylindrical ansatz, that is a metric $g$ conformally equivalent to the Euclidean metric $g_{\mathbb{R}^n}$ through the conformal factor $u^2/n - 2$, namely

$$g = u^2/n - 2 g_{\mathbb{R}^n}.$$

The reason for the name is that when $u$ is rotationally symmetric, then $g$ is the cylindrical metric. Before proceeding, we recall that the same strategy described here is at the basis of the results of [5] and of [7,8], where static metrics and the associated static potentials are considered in place of the Euclidean metric and the corresponding electrostatic potential. The cylindrical ansatz leads to a reformulation of problem (1.1) where the new metric $g$ and the $g$-harmonic function $\varphi = -\log u$ fulfill the quasi-Einstein type equation

$$\text{Ric}_g - \nabla\nabla\varphi + \frac{d\varphi \otimes d\varphi}{n - 2} = \frac{|\nabla\varphi|^2_g}{n - 2} g, \quad \text{in } M. \quad (1.18)$$

Here, $M = \mathbb{R}^n \setminus \Omega$ and $\nabla$ is the Levi–Civita connection of $g$. Before proceeding, it is worth pointing out that taking the trace of the above equation gives

$$\frac{R_g}{n - 1} = \frac{|\nabla\varphi|^2_g}{n - 2},$$

where $R_g$ is the scalar curvature of the conformal metric $g$. Hence, if $(M, g)$ is isometric to a round cylinder, then $|\nabla\varphi|^2_g$ has to be constant (observe that the converse is also true in view of Eq. (1.20) below, the consequent isometric splitting, and Remark 7 below). Noticing that

$$|\nabla\varphi|_g = \frac{|Du|}{u^{n-2}}$$

and recalling the little discussion after formula (1.7), we obtain a clear geometric interpretation of the constancy of the $P$-function $x \mapsto P(x) = \left(|Du|^2 / u^{2(n-1)/(n-2)}\right)(x)$, which is naturally associated with problem (1.1).
The second step of our strategy consists in proving via a splitting principle that the metric $g$ has indeed a product structure, provided the hypothesis of the rigidity statement is satisfied (splitting techniques have been successfully employed in the context of partial differential equations for example in [3,24]). More in general, we use the above conformal reformulation of the original system combined with the Bochner identity to deduce the equation

$$
\Delta_g |\nabla \varphi|^2_g - \langle \nabla |\nabla \varphi|^2_g, \nabla \varphi \rangle_g = 2|\nabla \varphi|^2_g.
$$

(1.20)

Observing that the drifted Laplacian appearing on the left hand side is formally self-adjoint with respect to the weighted measure $e^{-\varphi}d\mu_g$, we integrate by parts and obtain, for every $s \geq 0$, the integral identity

$$
\int_{\varphi=s} |\nabla \varphi|^2_g H_g d\sigma_g = e^s \int_{\varphi>s} \frac{|\nabla \nabla \varphi|^2_g}{e^\varphi} d\mu_g,
$$

(1.21)

where $H_g$ is the mean curvature of the level set $\{\varphi = s\}$ inside the ambient $(M,g)$, computed with respect to the normal vector field $\nabla \varphi/|\nabla \varphi|_g$. To give an effective interpretation of the above identity, it is now convenient to consider, for every $\beta \geq 0$, the function $\Phi_{1\beta}$:

$$
\Phi_{1\beta}(s) = \int_{\varphi=s} |\nabla \varphi|^\beta_g d\sigma_g.
$$

A direct computations shows that, for $\beta = 2$, one has

$$
\Phi'_2(s) = -2 \int_{\varphi=s} |\nabla \varphi|^2_g H_g d\sigma_g = -2e^s \int_{\varphi>s} \frac{|\nabla \nabla \varphi|^2_g}{e^\varphi} d\mu_g \leq 0.
$$

(1.22)

so that the function $s \mapsto \Phi_2(s)$ is monotone. Also, under the hypothesis of the rigidity statement, the left hand side of the above identity vanishes at some point and thus the Hessian of $\varphi$ must be zero in an open region of $M$. In turn, by analyticity, it vanishes everywhere. On the other hand, the asymptotic behavior of $u$ implies that $\varphi(x) \to +\infty$ when $x \to \infty$. In particular, $\nabla \varphi$ is a nontrivial parallel vector field. Hence, it provides a natural splitting direction for the metric $g$, which can then be proved to have a product structure. Finally, using the fact that $g_{\mathbb{R}^n}$ is flat and thus $g$ is conformally flat by construction, we can argue as in [3] that the cross sections of the Riemannian product $(M,g)$ are indeed metric spheres and that in turn $(M,g)$ is isometric to a round cylinder.

For an arbitrary $\beta \geq (n - 2)/(n - 1)$, we obtain in place of (1.21) and (1.22) the more general sequel of identities

$$
\Phi'_\beta(s) = -\beta \int_{\varphi=s} |\nabla \varphi|^\beta_g H_g d\sigma_g
$$

$$
= -\beta e^s \int_{\varphi>s} \frac{|\nabla \nabla \varphi|^\beta-2_g}{e^\varphi} \left(|\nabla \nabla \varphi|^2_g + (\beta - 2)|\nabla |\nabla \varphi|^2_g\right) d\mu_g \leq 0.
$$

(1.23)

As for the case $\beta = 2$, the monotonicity and rigidity results are obtained thanks to the nonnegativity of the rightmost hand side of the above identity, which is ensured by the standard Kato inequality when $\beta \geq 1$ and by the refined Kato inequality for harmonic...
functions when \((n - 2)/(n - 1) \leq \beta < 1\). The monotonicity formulas claimed in the statement of Theorem 1.1 are finally obtained via the identities

\[ F_\beta(\tau) = \Phi_\beta(\log \tau) \quad \text{and} \quad \tau F'_\beta(\tau) = \Phi'_\beta(\log \tau). \]

Some comments are in order about the convenience of working in a conformally related setting rather than in the original one. As the reader may realise comparing formula (1.23) with the corresponding (1.9), a first practical advantage comes from the fact that the computations are much simpler when performed on the cylindrical ansatz. A second reason, somehow less intuitive, is that the quasi-Einstein type Eq. (1.18) is more suitable than its flat counterpart (i.e., \(\text{Ric} = 0\)) to investigate the geometry of the level sets of \(\varphi\) (or equivalently of \(u\)) and eventually detect the presence of splitting phenomena. To illustrate this fact, one might observe for example that \(\text{Ric}_g(\nabla \varphi, \nabla \varphi) = 0\) at some point if and only if \(H_g = 0\) at the same point. Ultimately, this comes from the fact that—unlike in the original setting, where one would have to handle the two equations \(\Delta u = 0\) and \(\text{Ric} = 0\) separately—the conformally reformulated system (2.3) is non decoupled, so that the geometric properties of \(g\) are in a more explicit relationship with the analytic features of the harmonic function \(\varphi\).

### 1.4 Concluding remarks and further directions

We conclude this Introduction with some comments and suggestions about the possible implications of our monotonicity formulas. An interesting feature is that they seem to indicate, at least in some specific contexts, that the level set flow of a suitably chosen harmonic function can be employed as a valid substitute of the Inverse Mean Curvature Flow to obtain sharp geometric inequalities. Evidences of this phenomenon have already been exploited in [5], where a new proof of the Riemannian Penrose Inequality is obtained in every dimension for asymptotically flat static metrics, as well as in [2], where the same monotonicity-rigidity theory is carried out in the context of complete metrics with nonnegative Ricci curvature and maximal volume growth.

The advantages in considering the first flow instead of the latter one are quite evident. In fact, all the issues concerning the long time existence of the Inverse Mean Curvature Flow and its prolongation beyond the singular times, are somehow instantaneously ruled out, due to the existence theory for harmonic functions in exterior domains. Also concerning the short time existence the first approach reveals a better flexibility, since the mean convexity—which is necessary to start running the Inverse Mean Curvature Flow—is definitely not needed.

Finally, although the nowadays well understood structure of the nonregular level sets of harmonic functions could possibly be employed to extend the validity of the monotonicity formulas beyond the singular times (i.e., beyond the critical values, in the level set framework), it must be noticed that the analysis carried out in Sect. 3.2 is surprisingly independent of any refined a priori knowledge about the size of the critical set. As such, it can be adapted to derive effective monotonicity formulas even in the case of the level set flow of \(p\)-harmonic functions, where in principle the critical set is allowed to have full measure. This is the content of [1], where, letting \(p \to 1^+\) and taking advantage of a well-known approximation scheme due to Moser [41], we also draw some conclusions about the qualitative behaviour of the Inverse Mean Curvature Flow. More in general, it would be interesting in our opinion to investigate how far the technique based on the level set flow of harmonic and \(p\)-harmonic functions could be employed to re-discover some of the beautiful achievements of the Inverse Mean Curvature Flow theory, possibly simplifying the analysis.
Another beautiful challenge that is somehow suggested by the conformal splitting technique described in Sect. 1.3 concerns the possibility of obtaining enhanced versions of inequality (1.12), under topological constraints for the surface. This would eventually yield a different approach towards the study of the well-known Willmore Conjecture, recently solved in the affirmative by the outstanding work of Marques and Neves [40]. In this direction we just mention that the exterior boundary value problem that is at the basis of our construction should be chosen differently from (1.1). However, looking at the model situation, it is not difficult to figure out some natural candidates. We plan to explore this path in future work.

1.5 Plan of the paper

In Sect. 2, we reformulate problem (1.1) according to the cylindrical ansatz described in Sect. 1.3. This leads to the new problem (2.3), as well as to the conformal version of Theorem 1.1, namely to Theorem 2.2 below. It is then shown how Theorem 1.1 can be deduced after Theorem 2.2. The remaining part of the section is devoted to the proof of some preliminary results for the analysis of system (2.3), such as the gradient estimate of Proposition 2.3 and the subsequent upper bound for the quantities \( \Phi_1 \)'s, which is the content of Lemma 2.4. Section 3 contains the core of our analysis, which consists in the proof of two integral identities. The First Integral identity is proven in Proposition 3.1 and subsequently used in Corollary 3.3 to deduce the continuity of the functions \( \Phi_1 \), according to the statement of Theorem 2.2(i). The Second Integral Identity is proven in Lemma 3.4 and then used in Corollary 3.5 to deduce the differentiability and the monotonicity of the \( \Phi_1 \)'s, which are stated in Theorem 2.2(ii). The proof of Theorem 2.2 is finally completed with the rigidity statement deduced in Corollary 3.6. Section 4 is devoted to the consequences of Theorem 1.1. These are divided into ‘Consequences at the boundary’ (Sect. 4.1) and ‘Global geometric consequences’ (Sect. 4.2). The first include some new geometric upper bounds for the Capacity in terms of the \( L^p \)-norm of the mean curvature of the boundary (see Corollary 4.4), whereas the latter include the geometric inequalities described in Theorems 1.2 and 1.3.

2 A conformally equivalent setting

2.1 A conformal change of metric

Proceeding as in [3, Section 2], we perform a conformal change of the Euclidean metric to obtain an equivalent formulation of problem (1.1). Consider a solution \( u \) to problem (1.1) and note that \( 0 < u < 1 \), by the maximum principle. To set up the notation, we let \( M = \mathbb{R}^n \setminus \Omega \), denote by \( g_{\mathbb{R}^n} \) the flat Euclidean metric of \( \mathbb{R}^n \), and consider the conformally equivalent metric given by

\[
g = u^\frac{2}{n-2} g_{\mathbb{R}^n}.
\]

(2.1)

To reformulate our problem it is also convenient to set

\[
\varphi = - \log u,
\]

(2.2)

so that the metric \( g \) can be equivalently written as \( g = e^{-\frac{2\varphi}{n-2}} g_{\mathbb{R}^n} \). In what follows we denote by \( \langle \cdot | \cdot \rangle \) and \( (\cdot | \cdot)_g \) the scalar products and the covariant derivatives of the metrics \( g_{\mathbb{R}^n} \) and \( g \), respectively. The symbols \( \text{DD}, \nabla \nabla \) and \( \Delta, \Delta_g \) stand for the corresponding

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Hessian and Laplacian operators. \textit{In the sequel, for the sake of simplicity, we will refer to a function in the kernel of }\Delta_g\text{ as to a }g\text{-harmonic function.} The same computation as in [3, Section 2] show that problem (1.1) is equivalent to

\[
\begin{aligned}
\Delta_g \varphi &= 0 & \text{in } M, \\
\text{Ric}_g - \nabla \nabla \varphi + \frac{d\varphi \otimes d\varphi}{n-2} - \frac{|\nabla \varphi|^2_g}{n-2} g &= 0 & \text{in } M, \\
\varphi &= 0 & \text{on } \partial M, \\
\varphi(x) &\to +\infty & \text{as } x \to \infty.
\end{aligned}
\]

(2.3)

Notice that the second equation corresponds to the understatement \text{Ric}_g R_n = 0, which is implicit in the fact that the background metric in problem (1.1) is the flat one.

In the remaining part of Sect. 2 as well as in Sect. 3, we are going to analyse system (2.3), eventually proving Theorem 2.2, which can be thought as the conformal version of Theorem 1.1. We emphasise that our analysis here is independent of the fact that \( g \) and \( \varphi \) are related to \( g_R^n \) and \( u \) through (2.1) and (2.2), respectively. Nonetheless, for a mere solution \((M, g, \varphi)\) to (2.3) the following regularity result holds.

\textbf{Lemma 2.1} \textit{Let }\( (M, g, \varphi)\text{ be a solution to (2.3) such that }\partial M\text{ is a regular level set of }\varphi\text{. Then, both the metric }g\text{ and the function }\varphi\text{ are real analytic (in either harmonic coordinates or in geodesic normal coordinates). In particular, the set of the critical points of }\varphi\text{ has finite }\mathcal{H}^{n-1}\text{-measure.}}

\textbf{Proof} \textit{Setting }\( g_0 = e^{\frac{2\varphi}{n-2}} g\)\text{ and }\( u_0 = e^{-\varphi}\), it is straightforward to realise that \text{Ric}_{g_0} = 0\text{, and that }\Delta_{g_0} u_0 = 0\text{. As a consequence of }g_0\text{ being an Einstein metric, we have that its coefficients are real analytic functions, either harmonic coordinates or in geodesic normal coordinates, due to [22, Theorem 5.2]. With the same choice of coordinates, one can then use equation }\Delta_{g_0} u_0 = 0\text{ to show that }u_0\text{ is real analytic as well. In turn, }g\text{ and }\varphi\text{ are real analytic. The last statement is a consequence of [25, Theorem 3.4.8] and of [45].} \quad \Box

\textbf{2.2 The extrinsic curvature of the level sets}

In the forthcoming analysis it will be important to study the geometry of the level sets of \( \varphi \), which coincide with the level sets of \( u \) by definition. To this end, we denote by \( \text{Crit}(\varphi)\) the set of the critical points of \( \varphi \) and we fix on \( M\setminus\text{Crit}(\varphi)\) the \( g_{\mathbb{R}^n}\)-unit vector field \( v = - Du/|Du| = D\varphi/|D\varphi|\) and the \( g\)-unit vector field \( v_g = - \nabla u/|\nabla u|_g = \nabla \varphi/|\nabla \varphi|_g\). Consequently, the second fundamental forms of the regular level sets of \( u \) or \( \varphi \) with respect to the flat ambient metric and the conformally-related ambient metric \( g \), are given by

\[
\begin{aligned}
h(X, Y) &= - \frac{\text{DD}u(X, Y)}{|Du|} = \frac{\text{DD}\varphi(X, Y)}{|D\varphi|}, \\
h_g(X, Y) &= - \frac{\nabla \nabla u(X, Y)}{|\nabla u|_g} = \frac{\nabla \nabla \varphi(X, Y)}{|\nabla \varphi|_g},
\end{aligned}
\]

respectively, where \( X \) and \( Y \) are vector fields tangent to the level sets. Taking the traces of the above expressions with respect to the induced metrics and using the fact that \( u \) is harmonic and \( \varphi \) is \( g\)-harmonic, we obtain the following expressions for the mean curvatures in the two ambients

\[
\begin{aligned}
H &= \frac{\text{DD}u(Du, Du)}{|Du|^3}, & H_g &= - \frac{\nabla \nabla \varphi(\nabla \varphi, \nabla \varphi)}{|\nabla \varphi|^3_g}.
\end{aligned}
\]

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By a direct computation one can show that the second fundamental forms and the mean curvatures are related by the following formulæ

\[ h_g(X, Y) = u \frac{1}{n-2} \left[ h(X, Y) - \left( \frac{1}{n-2} \right) \frac{|Du|}{u} \langle X \mid Y \rangle \right], \]

(2.5)

\[ H_g = u \frac{1}{n-2} \left[ H - \left( \frac{n-1}{n-2} \right) \frac{|Du|}{u} \right], \]

(2.6)

where, as before, \( X \) and \( Y \) are vector fields tangent to the level sets. For the sake of completeness, we also report the reverse formulæ

\[ h(X, Y) = e^{\psi} \frac{1}{n-2} \left[ h_g(X, Y) + \left( \frac{1}{n-2} \right) |\nabla \varphi|^g \langle X \mid Y \rangle \right], \]

\[ H = e^{-\psi} \frac{1}{n-2} \left[ H_g + \left( \frac{n-1}{n-2} \right) |\nabla \varphi|^g \right]. \]

Concerning the nonregular level sets of \( \varphi \), we just observe that, by the same arguments as in Remark 2, the second fundamental form and mean curvature also make sense \( H_{n-1} \)-almost everywhere on a singular level set \( \{ \varphi = s_0 \} \), namely on the relatively open set \( \{ \varphi = s_0 \} \setminus \text{Crit}(\varphi) \).

### 2.3 A conformally equivalent version of the Monotonicity-Rigidity Theorem

In order to take advantage of our cylindrical ansatz, it is convenient to reformulate the statement of Theorem 1.1 in the new conformally related setting. To do that, we introduce, for every \( \beta \geq 0 \), the function \( \Phi_\beta : [0, +\infty) \rightarrow \mathbb{R} \), setting

\[ \Phi_\beta(s) := \int_{\{ \varphi = s \}} |\nabla \varphi|^{\beta+1} \, d\sigma_g, \]

(2.7)

so that if \( (u, g_{\mathbb{R}^n}) \) and \( (\varphi, g) \) are related by formulæ (2.1) and (2.2), then the following relationships are also in force

\[ F_\beta(\tau) = \Phi_\beta(\log \tau) \quad \text{and} \quad \tau F'_\beta(\tau) = \Phi'_\beta(\log \tau). \]

Having this in mind, we can now re-state the first two points in Theorem 1.1 in the following way.

**Theorem 2.2 (Monotonicity-Rigidity Theorem—Conformal Version)** Let \( (M, g, \varphi) \) be a solution to problem (2.3) such that \( \partial M \) is a regular level set of \( \varphi \), and assume that the following growth condition

\[ |\nabla \varphi|^2_g(x) = o(e^\varphi) \quad \text{as} \quad x \rightarrow \infty \]

is satisfied. Let \( \Phi_\beta : [0, +\infty) \rightarrow \mathbb{R} \) be the function defined in (2.7). Then, the following properties hold true.

(i) **Continuity.** For every \( \beta \geq 0 \), the function \( \Phi_\beta \) is continuous and admits for every \( s \geq 0 \) the integral representation

\[ \Phi_\beta(s) = e^s \int_{\{ \varphi > s \}} \frac{|\nabla \varphi|^{\beta-2}_g \left( |\nabla \varphi|^4_g - \beta \nabla \nabla \varphi(\nabla \varphi, \nabla \varphi) \right)}{e^\varphi} d\mu_g. \]
(ii) Differentiability, Monotonicity and Rigidity. For every $\beta \geq (n-2)/(n-1)$, the function $\Phi_\beta$ is continuously differentiable and the derivative $\Phi'_\beta$ admits for every $s \geq 0$ the integral representation

$$\Phi'_\beta(s) = -\beta e^s \int_{\{\varphi > s\}} |\nabla \varphi|_g^{\beta-2} \left( |\nabla \nabla \varphi|_g^2 + (\beta - 2) |\nabla \varphi|_g^2 \right) e^\varphi d\mu_g \leq 0. \quad (2.9)$$

Moreover, if there exists $s_0 \geq 0$ such that $\Phi'_\beta(s_0) = 0$ for some $\beta \geq (n-2)/(n-1)$, then the manifold $(M, g)$ is isometric to the half cylinder $([0, +\infty) \times \partial M, d\varrho \otimes d\varrho + g_{|\partial M})$, where $\varrho$ is the distance to $\partial M$, and $\varphi$ is an affine function of $\varrho$. The cross sections of the cylinder are locally isometric to round spheres for $n = 3, 4$ and to compact Einstein manifolds for $n \geq 5$.

The above statement will be proven in Sect. 3. More precisely, Theorem 2.2(ii) will be deduced in Sect. 3.1 (Corollaries 3.2 and 3.3) as a consequence of the First Integral Identity (3.4), whereas Theorem 2.2(ii) will be proven in Sect. 3.2 (Corollaries 3.5 and 3.6) with the help of the Second Integral Identity (3.6).

**Remark 7** Observe that if a solution $(M, g, \varphi)$ to problem (2.3) comes from a solution $u$ to problem (1.1) through (2.1) and (2.2), then the classical asymptotic expansions (1.5) imply through formula (1.19) that $|\nabla \varphi|_g^2 = O(1)$, as $x \to \infty$. Thus, the growth condition (2.8) is largely fulfilled in this case. Also the conclusions of the rigidity statement in Theorem 2.2(ii) are much stronger in this situation. In fact, one can take once again advantage of the asymptotic expansions (1.5) to argue that the cross sections of the cylindrical ansatz are asymptotic to round spheres, by construction. In turn, this implies that the cylinders of the rigidity statement are necessarily round (i.e., cylinders with spherical cross sections) in any dimension.

In complete analogy with Remark 1, it is worth stating the following

**Remark 8** We observe that if $s$ is a regular value of $\varphi$, then the function $\Phi_\beta$ is differentiable at $s$ for every $\beta \geq 0$, and a direct computation gives

$$\Phi'_\beta(s) = -\beta \int_{\{\varphi = s\}} |\nabla \varphi|_g^\beta H_g d\sigma_g, \quad (2.10)$$

where $H_g$ is the mean curvature of the level set $\{\varphi = s\}$ computed with respect to the unit normal vector field $v_g = \nabla \varphi / |\nabla \varphi|_g$.

The above observation will be employed in the discussion of the borderline case $\beta = (n - 2)/(n - 1)$ in the proof of the rigidity statement in Corollary 3.6 below.

We conclude this subsection showing how our main Theorem 1.1 can be deduced from Theorem 2.2.

**Proof of Theorem 1.1, after Theorem 2.2** First of all, we notice that the classical asymptotic expansions (1.5) imply through formula (1.19) that $|\nabla \varphi|_g^2 = O(1)$, as $x \to \infty$ and thus the growth condition (2.8) is fulfilled. Now, we observe that the continuity statement in Theorem 1.1(i) is a straightforward consequence of the analogous statement in Theorem 2.2(i), once the relationship $F_\beta(\tau) = \Phi_\beta(\log \tau)$ is taken into account.

The differentiability and the monotonicity of the $F_\beta$’s for $\beta \geq (n-2)/(n-1)$ in Theorem 1.1(ii), follow from the analogous statements for the $\Phi_\beta$’s in Theorem 2.2(ii), using the relationship $\tau F'_\beta(\tau) = \Phi'_\beta(\log \tau)$. In particular, the Monotonicity Formula (1.9) is
the reformulation of (2.9) in terms of \( u \) and of the Euclidean metric, via the—messy but straightforward—identities

\[
|\nabla \nabla \varphi|_g^2 = u^{-\frac{4}{n-2}} \left\{ |DDu|_g^2 + \frac{n(n-1)}{(n-2)^2} |Du|_g^4 - \left( \frac{2n}{n-2} \right) \frac{|Du|_g^3}{H} \right\}
\]

\[
|\nabla \nabla \varphi|_g^2 = u^{-\frac{4}{n-2}} \left\{ |D|Du|_g^2 + \left( \frac{n-1}{n-2} \right)^2 \frac{|Du|_g^4}{H} - 2 \left( \frac{n-1}{n-2} \right) \frac{|Du|_g^3}{H} \right\},
\]

which yield, by adding and subtracting the term \( u^{-4/(n-2)} |D|Du|/u^2 \),

\[
|\nabla \nabla \varphi|_g^2 + (\beta - 2) |\nabla \nabla \varphi|_g^2 = u^{-\frac{4}{n-2}} \left\{ |DDu|_g^2 - \left( \frac{n}{n-1} \right) |D|Du|_g^2 \right\}
\]

\[
+ (\beta - \frac{n-2}{n-1}) |D|Du|_g^2 \left[ H - \left( \frac{n-1}{n-2} \right) \frac{|Du|_g}{H} \right]^2 \right\}.
\]

The rigidity statement follows from Remark 7.

The convexity of the functions \( F_\beta \) in the statement of Theorem 1.1(iii) is a consequence of the fact that the assignment \( \tau \mapsto F_\beta'(\tau) \) is nondecreasing. Indeed, for any given \( 1 \leq \tau_1 \leq \tau_2 < +\infty \), one has that

\[
F_\beta'(\tau_2) - F_\beta'(\tau_1) = \frac{\Phi_\beta'(\log \tau_2)}{\tau_2} - \frac{\Phi_\beta'(\log \tau_1)}{\tau_1}
\]

\[
= \beta \int_{\{\log \tau_1 < \varphi < \log \tau_2\}} \left( |\nabla \nabla \varphi|_g^{\beta-2} (|\nabla \nabla \varphi|_g^2 + (\beta - 2) |\nabla \nabla \varphi|_g^2) \right) \frac{|D|Du|_g^2}{H} \frac{d\mu_g}{e^\varphi},
\]

where the last equality corresponds to identity (3.10) in Corollary 3.5, but can also be deduced at once from (2.9) in Theorem 2.2(ii). Finally, as already observed in Remark 4, the representation formula (1.10) for the second derivatives of the functions \( F_\beta \) is a straightforward consequence of the Coarea Formula.

\[ \square \]

### 2.4 Some pointwise identities and a sharp gradient estimate

This subsection is devoted to some preliminary computations and results, that will be readily applied in the forthcoming Sect. 3 to obtain a couple of integral identities and then the complete proof of Theorem 2.2.

We start by noticing that for a solution \( (M, g, \varphi) \) of problem (2.3), the Bochner formula reduces to the identity

\[
\Delta_g |\nabla \varphi|_g^2 - \langle \nabla |\nabla \varphi|_g^2, \nabla \varphi \rangle_g = 2 |\nabla \nabla \varphi|_g^2.
\]

(2.11)

Now, observe that, wherever the following expressions are well defined, it holds

\[
\nabla |\nabla \varphi|_g^\beta = \left( \frac{\beta}{2} \right) |\nabla \varphi|_g^{\beta-2} \nabla |\nabla \varphi|_g^2,
\]

\[
\Delta_g |\nabla \varphi|_g^\beta = \left( \frac{\beta}{2} \right) |\nabla \varphi|_g^{\beta-2} \Delta_g |\nabla \varphi|_g^2 + \beta(\beta - 2) |\nabla \varphi|_g^{\beta-2} |\nabla \nabla \varphi|_g^2.
\]
Combining these two facts together with identity (2.11), we arrive at
\[ \Delta_g |\nabla \varphi|^2_g - (\nabla |\nabla \varphi|^2_g) = \beta |\nabla \varphi|^2_g - 2 |\nabla \varphi|^2_g \left[ |\nabla \varphi|^2_g + (\beta - 2) |\nabla \varphi|^2_g \right]. \] (2.12)

As we are going to see in the next section, the above relationship is at the core of our fundamental integral identities (3.4) and (3.6), and thus also of the Monotonicity-Rigidity Theorem 2.2, that from these identities is deduced.

As a first effective application of the above computations, we are going to prove the following sharp gradient estimate (in the spirit of [18]), that readily translates into Theorem 4.2, when \((u, g_{\mathbb{R}^n})\) and \((\varphi, g)\) are related by formulæ (2.1) and (2.2).

**Proposition 2.3** Let \((M, g, \varphi)\) be a solution to problem (2.3) and assume that the growth condition
\[ |\nabla \varphi|^2_g = o(e^\varphi), \quad \text{as} \quad x \to \infty, \]
is satisfied. Then, for every \(x \in M\) it holds
\[ |\nabla \varphi|^2_g(x) \leq \max_{\partial M} |\nabla \varphi|^2_g. \] (2.13)
Moreover, the equality is achieved at some interior point of \(M\) if and only if \((M, g)\) is isometric to one half cylinder. The cross sections of the cylinder are locally isometric to round spheres for \(n = 3, 4\) and to compact Einstein manifolds for \(n \geq 5\).

**Proof** Setting \(K = \max_{\partial M} |\nabla \varphi|_g\) and
\[ w = \frac{K^2 - |\nabla \varphi|^2_g}{e^\varphi}, \]
it is immediate to check that
\[ \frac{\nabla |\nabla \varphi|^2_g}{e^\varphi} = -\nabla w - w \nabla \varphi \]
\[ \Delta_g |\nabla \varphi|^2_g = -\Delta_g w - 2 \langle \nabla w |\nabla \varphi\rangle_g - w |\nabla \varphi|^2_g. \]
Hence, with the help of (2.11), one gets
\[ \Delta_g w + \langle \nabla w |\nabla \varphi\rangle_g = -2 e^{-\varphi} |\nabla \varphi|^2_g \leq 0. \] (2.14)
Now, observe that the growth condition (2.8) ensures that \(w(x) \to 0\) as \(x \to \infty\). On the other hand, it follows from the definition of \(K\) that \(w \geq 0\) on \(\partial M\). The desired gradient bound (2.13) is now an easy consequence of the Maximum Principle.

Finally, the rigidity statement follows from the Strong Maximum Principle, which in turn implies the vanishing of \(\nabla \nabla \varphi\) through either Eq. (2.14) or Eq. (2.11). This proves that \((M, g)\) is isometric to one half cylinder, as one can readily check, writing \(g\) in adapted coordinates \(\{\varphi, \theta^1, \ldots, \theta^{n-1}\}\)
\[ g = \frac{d\varphi \otimes d\varphi}{|\nabla \varphi|^2_g} + g_{ij} d\theta^i \otimes d\theta^j, \]
and using the fact that
\[ h_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \varphi} = \frac{\nabla_i \nabla_j \varphi}{|\nabla \varphi|_g} = 0. \]
To check the last assertion of the statement, namely that the cross sections are locally isometric to round spheres for \( n = 3, 4 \) and to Einstein manifolds for \( n \geq 5 \), we first observe that the components \( R_{i\varphi j\varphi} \) of the Riemann tensor are identically zero, so that the (once) traced Gauss Equation reduces to

\[
R_{ij} = \overline{R}_{ij}, \tag{2.15}
\]

where the \( \overline{R}_{ij} \)'s are the components of the Ricci tensor of the metric induced on the cross sections. Moreover, again from the vanishing of \( \nabla \nabla \varphi \), we have that the \((ij)\)-th component of the second equation of problem (2.3) reduces to \( R_{ij} = (|\nabla \varphi|_g^2/(n - 2))\overline{g}_{ij} = (|\nabla \varphi|_g^2/(n - 2))\overline{g}_{ij} \). This fact, coupled with (2.15), gives that \( \overline{R}_{ij} \) is a constant multiple of \( \overline{g}_{ij} \) and thus the metric induced on the cross sections is Einstein. To obtain the enhanced conclusion in dimension \( n = 3 \) or \( n = 4 \) it is sufficient to invoke the classical Uniformization Theorem.

\[\square\]

In the next lemma we are going to show that the gradient estimate just obtained forces the functions \( \Phi_\beta \) defined in (2.7) to be bounded. We then conclude this subsection with an outline of the proof, under favourable assumptions, of the monotonicity claimed in Theorem 2.2. This argument is inspired by the corresponding monotonicity results for the pseudoarea and pseudovolume functionals in [18].

**Lemma 2.4** Let \((M, g, \varphi)\) be a solution to problem (2.3) and assume that the growth condition \( |\nabla \varphi|_g^2 = o(e^\varphi) \) is satisfied, as \( x \to \infty \). Then, for every \( \beta \geq 0 \), there exists a constant \( C_\beta > 0 \) such that

\[
\Phi_\beta(s) \leq C_\beta,
\]

for every \( s \geq 0 \).

**Proof** We have that

\[
\Phi_\beta(s) = \int_{\{\varphi = s\}} |\nabla \varphi|_g^\beta |\nabla \varphi|_g d\sigma_g \leq \left( \max_{\partial M} |\nabla \varphi|_g^\beta \right) \int_{\{\varphi = s\}} |\nabla \varphi|_g d\sigma_g = \left( \max_{\partial M} |\nabla \varphi|_g^\beta \right) \int_{\partial M} |\nabla \varphi|_g d\sigma_g,
\]

where the last equality follows from the Divergence Theorem and from the fact \( \varphi \) is \( g \)-harmonic. \[\square\]

**Proof** (Outline of the proof of the Monotonicity in Theorem 2.2(ii)) Suppose we have a solution \((M, g, \varphi)\) to problem (2.3) satisfying the requirement (2.8), so that the above lemma is in force. Assume in addition that the following integral identity holds true for every \( \beta \geq (n - 2)/(n - 1) \) and every \( 0 \leq S_0 < S \)

\[
\int_{\{\varphi = S_0\}} |\nabla \varphi|_g^\beta H_g \frac{d\sigma_g}{e^{S_0}} - \int_{\{\varphi = S\}} |\nabla \varphi|_g^\beta H_g \frac{d\sigma_g}{e^S} = \int_{\{S_0 < \varphi < S\}} \frac{|\nabla \varphi|_g^{\beta - 2} (|\nabla \nabla \varphi|_g^2 + (\beta - 2)|\nabla \varphi|_g^2)^2}{e^\varphi} d\mu_g.
\]
where $H_g$ is the mean curvature of the level set computed with respect to the unit normal vector field $\nabla \varphi/|\nabla \varphi|_g$. In the forthcoming Lemma 3.4 this identity will be completely justified whenever $S_0$ and $S$ are regular values, and then it will be used in Corollary 3.5 to prove that the function $\Phi_\beta$ is differentiable and satisfies

$$\Phi'_\beta(S)e^{-S} - \Phi'_\beta(S_0)e^{-S_0} = \beta \int_{\{S_0 < \varphi < S\}} \frac{|\nabla \varphi|_g^{\beta-2}\left(|\nabla \nabla \varphi|_g^2 + (\beta - 2)|\nabla \varphi|_g^2\right)}{e^\varphi} \, d\mu_g. \quad (2.16)$$

In particular, since the right-hand side of the above identity is nonnegative in view of the refined Kato inequality for harmonic functions, we have that

$$\Phi'_\beta(S) \geq \Phi'_\beta(S_0)e^{S-S_0}.$$ 

Now, recall that our claim in Theorem 2.2 is that the derivative of $\Phi_\beta$ is everywhere nonpositive. Suppose then by contradiction that $\Phi'_\beta(S_0) > 0$ for some $S_0 \geq 0$. Integrating the above differential inequality gives

$$\Phi_\beta(S) \geq \Phi_\beta(S_0) + \Phi'_\beta(S_0) \frac{e^S - e^{S_0}}{e^{S_0}},$$

for every $0 \leq S_0 < S$. In turn, we would get $\Phi_\beta(S) \to +\infty$ as $S \to +\infty$, contradicting Lemma 2.4. This provides the desired monotonicity.

The crucial role played by integral identities in this very simple argument, motivates the analysis of the following sections.

### 3 Integral identities and proof of Theorem 2.2

In this section we derive a couple of integral identities that will then be used to prove a conformally equivalent version of the Monotonicity-Rigidity Theorem 2.2. They are essentially deduced using identity (2.12)

$$\Delta_g |\nabla \varphi|_g^{\beta} - |\nabla \nabla \varphi|_g^{\beta} |\nabla \varphi|_g = \beta |\nabla \varphi|_g^{\beta-2}\left(|\nabla \nabla \varphi|_g^2 + (\beta - 2)|\nabla \varphi|_g^2\right)$$

and applying the Divergence Theorem to the vector fields

$$X = \frac{|\nabla \varphi|_g^{\beta} \nabla \varphi}{e^\varphi} \quad \text{and} \quad Y = \frac{|\nabla \varphi|_g^{\beta}}{e^\varphi}.$$ 

on sets of the form

$$E^S_s := \{s < \varphi < S\}. \quad (3.1)$$

Care will be needed to justify the integration by parts when critical points of $\varphi$ are present in $E^S_s$. To this aim, it will be convenient to consider a suitable family of neighbourhoods of the set of critical points $\text{Crit}(\varphi)$. Whenever $\text{Crit}(\varphi)$ is nonempty, we set for $\varepsilon > 0$

$$U_\varepsilon := \{|\nabla \varphi|_g^2 < \varepsilon\} \supseteq \text{Crit}(\varphi). \quad (3.2)$$
We recall that since $|\nabla \varphi|^2_g$ is smooth, Sard’s Theorem implies that for almost every $\varepsilon > 0$ the boundary $\partial U_\varepsilon$ is a regular level set of $|\nabla \varphi|^2_g$. In particular, $\partial U_\varepsilon$ is a smooth $(n-1)$-dimensional hypersurface with
\[
\frac{\nabla |\nabla \varphi|^2_g}{\nabla |\nabla \varphi|^2_g}_g
\]
as a unit normal vector field. To go a little further, recall that, by Lemma 2.1, $\varphi$ and $g$ are real analytic, so that $|\nabla \varphi|^2_g$ is real analytic as well. Hence, the results of [45] guarantee that all the properties of $\partial U_\varepsilon$ described above hold true except for a discrete—and thus locally finite—set of values of the parameter $\varepsilon$.

### 3.1 First integral identity and proof of Theorem 2.2(i)

Having fixed the set-up, we are now ready to prove our integral identities. The first integral identity will be directly employed to show that the assignment $s \mapsto \Phi_\beta(s)$ is continuous for every $\beta \geq 0$.

**Proposition 3.1** (First Integral Identity) Let $(M, g, \varphi)$ be a solution to problem (2.3) and assume that the growth condition $|\nabla \varphi|^2_g = o(e^\varphi)$ is satisfied, as $x \to \infty$. Then, for every $\beta \geq 0$ and every $0 \leq s < S$, we have
\[
\hat{\int}_{\{\varphi = S\}} \frac{|\nabla \varphi|^{\beta+1}_g}{e^S} d\sigma_g - \int_{\{|\varphi = x\}|} \frac{|\nabla \varphi|^{\beta+1}_g}{e^x} d\sigma_g = \int_{\{s < \varphi < S\}} \frac{|\nabla \varphi|^\beta_g \left( \beta \nabla \nabla \varphi(\nabla \varphi, \nabla \varphi) - |\nabla \varphi|^4_g \right)}{e^\varphi} d\mu_g, \tag{3.4}
\]

**Proof** To simplify the notation, we drop the subscript $g$ throughout the proof. We consider the vector field
\[
X = \frac{|\nabla \varphi|^{\beta}_g \nabla \varphi}{e^\varphi},
\]
and compute, wherever $|\nabla \varphi| > 0$,
\[
\text{div} X = \frac{|\nabla \varphi|^{\beta-2}_g \left( \beta \nabla \nabla \varphi(\nabla \varphi, \nabla \varphi) - |\nabla \varphi|^4_g \right)}{e^\varphi},
\]
using the fact that $\varphi$ is harmonic. Notice that the harmonicity of $\varphi$ also implies that $\text{div} X$ is locally bounded on $M$, for every $\beta \geq 0$. Now, for $0 \leq s < S$, we consider the set $E^S_s$ defined in (3.1) and observe that if the closure of $E^S_s$ does not contain any critical points of $\varphi$, then the thesis follows directly from the Divergence Theorem. On the other hand, we recall from Lemma 2.1 and [45] that there exists at most a finite number of critical values for $\varphi$ in between $s$ and $S$. Notice that the level sets $\{\varphi = s\}$ and $\{\varphi = S\}$ might be critical as well. For small enough values of the parameter $\varepsilon$, let $\{U_\varepsilon\}_\varepsilon$ be the (nondecreasing) family of tubular neighbourhoods of $\text{Crit}(\varphi)$ defined in (3.2). The Divergence Theorem applied to the vector field $X$ on $E^S_s \setminus U_\varepsilon$ gives
\[
\int_{E^S_s \setminus U_\varepsilon} \text{div} X d\mu = \int_{\partial(E^S_s \setminus U_\varepsilon)} \langle X | n \rangle d\sigma.
\]
where $n$ is the outer unit normal to the boundary of $E^S\setminus U_\varepsilon$. In particular, $n$ is well defined almost everywhere on the boundary and it is given by formula (3.3) on $\partial U_\varepsilon \cap E^S$, whereas it coincides with $\pm \nabla \phi/|\nabla \phi|$ on $\partial E^S\setminus U_\varepsilon$. To prove (3.4), we first observe that, since $\text{div}X$ is locally bounded for every $\beta \geq 0$, the Dominated Convergence Theorem gives

$$\lim_{\varepsilon \downarrow 0} \int_{E^S\setminus U_\varepsilon} \text{div}X \, d\mu = \int_{E^S} |\nabla \phi|^{\beta-2} \left( \beta \nabla \nabla \phi(\nabla \phi, \nabla \phi) - |\nabla \phi|^4 \right) \frac{e^\phi}{\phi} \, d\mu.$$ 

Concerning the boundary integral, it is convenient to split it into several pieces, writing

$$\int_{\partial(E^S\setminus U_\varepsilon)} |X| n \, d\sigma = \int_{\{|s\}=0\}\ U_\varepsilon} |\nabla \phi|^{\beta+1} \left( \frac{\nabla \phi}{|\nabla \phi|} \left| \frac{\nabla \phi}{|\nabla \phi|} \right| \right) \, d\sigma.$$ 

Using the Dominated Convergence Theorem it is not hard to argue that the first two terms of the right hand side converge to the left hand side of (3.4), as $\varepsilon \to 0$. To treat the last term, we observe that

$$\left| \int_{\partial U_\varepsilon \cap E^S} |\nabla \phi|^{\beta+1} \left( \frac{\nabla \phi}{|\nabla \phi|} \left| \frac{\nabla \phi}{|\nabla \phi|} \right| \right) \, d\sigma \right| \leq e^{(\beta+1)/2} \int_{\partial U_\varepsilon \cap E^S} e^{-\phi} \, d\sigma.$$ 

Hence, it tends to 0, as $\varepsilon \to 0$, providing the desired identity. \qed

Remark 9 It is interesting to observe that the above proof does not use the fact that the Hausdorff dimension of $\text{Crit}(\phi)$ is bounded above by $(n-2)$. In fact, if $\bar{s}$ is a critical value for $\phi$, one has that

$$\int_{\{|s\}=\bar{s}\}\ U_\varepsilon} |\nabla \phi|^{\beta+1} \, d\sigma = \int_{\{|s\}=\bar{s}\} \text{Crit}(\phi)} |\nabla \phi|^{\beta+1} \, d\sigma, \quad \text{as } \varepsilon \to 0,$$

since by definition $|\nabla \phi|_{\bar{s}} = 0$ on the critical set.

As an immediate consequence of the first integral identity, we obtain a representation formula for the functions $\Phi_{\beta}$, as well as the continuity of $s \mapsto \Phi_{\beta}(s)$, for every $\beta \geq 0$. This is stated in the following corollaries.

Corollary 3.2 (Theorem 2.2(i)—Representation Formula for $\Phi_{\beta}$) Let $(M, g, \phi)$ be a solution to problem (2.3) and assume that the growth condition $|\nabla \phi|^2_g = o(e^\phi)$ is satisfied, as $x \to \infty$. Then, for every $\beta \geq 0$ and every $s \geq 0$, we have

$$\Phi_{\beta}(s) = e^s \int_{\{|s\}>s\} |\nabla \phi|^{\beta-2} \left( |\nabla \phi|^4_g - \beta \nabla \phi(\nabla \phi, \nabla \phi) \right) \frac{e^\phi}{\phi} \, d\mu_{\bar{s}}. \quad (3.5)$$

Proof It is sufficient to apply the integral identity (3.4) and observe that

$$\lim_{s \to +\infty} e^{-S} \int_{\{|s\}=S\} |\nabla \phi|^{\beta+1} \, d\sigma = \lim_{s \to +\infty} e^{-S} \Phi_{\beta}(S) = 0.$$
by virtue of Lemma 2.4.

**Corollary 3.3 (Theorem 2.2(ii)—Continuity)** Let \((M, g, \varphi)\) be a solution to problem (2.3) and assume that the growth condition \(|\nabla \varphi|^2_g = o(e^\varphi)\) is satisfied, as \(x \to \infty\). Then, for every \(\beta \geq 0\) and every \(s \geq 0\), we have that the function \(s \mapsto \Phi_\beta(s)\) defined in (2.7) is continuous.

**Proof** As already observed, the quantity

\[
|\nabla \varphi|^\beta - 2 \left( \beta |\nabla \varphi|^2 \nabla \varphi \cdot \nabla \varphi - |\nabla \varphi|^4 \right) e^{\varphi}
\]

is locally bounded for every \(\beta \geq 0\). In particular it is locally summable. Hence, by the absolute continuity of the integral, we have that the right hand side of (3.4) tends to 0 when either \(s \to S\) or \(S \to s\). This implies the continuity of the assignment \(s \mapsto e^{-s} \Phi_\beta(s)\). The continuity of \(s \mapsto \Phi_\beta(s)\) follows at once.

\[\square\]

### 3.2 Second integral identity and proof of Theorem 2.2(ii)

We are now ready to prove the second of our integral identities. This formula represents the core of our analysis as it will be used to prove the differentiability of the functions \(\Phi_\beta\) together with the monotonicity and the rigidity statement claimed in Theorem 2.2(ii). For the sake of clearness, we present a version of the second integral identity for regular values of \(\varphi\). In fact, according to Remarks 1, 2 and 8, if \(s\) is a critical value of \(\varphi\), some attention must be payed to the precise definition of the term

\[\hat{\{\varphi = s\}} |\nabla \varphi|^\beta_g H_g d\sigma_g\]

at least when \((n - 2)/(n - 1) \leq \beta < 1\) and thus the integrand is not necessarily bounded a priori.

**Lemma 3.4 (Second Integral Identity for Regular Values)** Let \((M, g, \varphi)\) be a solution to problem (2.3) and assume that the growth condition \(|\nabla \varphi|^2_g = o(e^\varphi)\) is satisfied, as \(x \to \infty\). Then, for every \(\beta \geq (n - 2)/(n - 1)\) and every couple of regular values \(0 \leq s < S\) of the function \(\varphi\), we have

\[
\int_{\{\varphi = s\}} |\nabla \varphi|^\beta_g H_g \frac{d\sigma_g}{e^s} - \int_{\{\varphi = S\}} |\nabla \varphi|^\beta_g H_g \frac{d\sigma_g}{e^S} = \int_{\{s < \varphi < S\}} \frac{|\nabla \varphi|^\beta_g - 2 \left( |\nabla \varphi|^2_g (\beta - 2) |\nabla |\nabla \varphi|^2_g \right)}{e^{\varphi}} \frac{d\mu_g}{\mu_g} \geq 0. (3.6)
\]

Before proceeding with the proof, it is worth noticing that the sign of the right hand side of (3.6) is guaranteed for every \(\beta\) greater that the threshold value \((n - 2)/(n - 1)\) by the fact that \(\varphi\) is harmonic, and thus the refined Kato inequality

\[
|\nabla \nabla \varphi|^2_g - \left( \frac{n}{n - 1} \right) |\nabla |\nabla \varphi|^2_g \geq 0
\]

(see for example [26, Corollary 4.6] with \(p = 2\) for a proof) is in force almost everywhere on \(\{s < \varphi < S\}\).
Proof As in the proof of Proposition 3.1, we drop the subscript $g$ to simplify the notation. For every $\beta \geq (n - 2)/(n - 1)$, we consider the vector field $Y$ defined as

$$ Y = \frac{\nabla|\nabla \varphi|^{\beta}}{e^{\varphi}}. \quad (3.7) $$

Wherever $|\nabla \varphi| > 0$, we compute the divergence of $Y$ with the help of Eq. (2.12), obtaining

$$ \text{div} Y = \beta |\nabla \varphi|^{\beta - 2}\left( |\nabla \varphi|^{2} + (\beta - 2) |\nabla|\nabla \varphi|^{2}\right) \geq 0. \quad (3.8) $$

For $0 \leq s < S$, we set $E^S_s = \{s < \varphi < S\}$ as in (3.1), and observe that if the closure of $E^S_s$ does not contain any critical point of $\varphi$, then the validity of (3.6) follows directly from the Divergence Theorem. As in the proof of Proposition 3.1, we now suppose without loss of generality that there is only a finite number of critical values of $\varphi$ in $(s, S)$. To deal with the presence of critical points inside $E^S_s$, we consider for every $\varepsilon > 0$ sufficiently small a smooth nondecreasing cut-off function $\chi_\varepsilon : [0, +\infty) \rightarrow [0, 1]$ such that

- $\chi_\varepsilon(t) = 0$, for $t \leq \frac{1}{2}\varepsilon$;
- $0 < \dot{\chi}_\varepsilon(t) < 2\varepsilon^{-1}$, for $\frac{1}{2}\varepsilon < t < \frac{3}{2}\varepsilon$;
- $\chi_\varepsilon(t) = 1$, for $\frac{3}{2}\varepsilon \leq t$.

Using $\chi_\varepsilon$, we define the smooth function $\Xi_\varepsilon : M \rightarrow [0, 1]$ as

$$ \Xi_\varepsilon(x) = (\chi_\varepsilon \circ |\nabla \varphi|^2)(x). $$

In particular, with the notation introduced in (3.2), we have that $\text{supp}(\nabla \Xi_\varepsilon) \subset \overline{U_{3\varepsilon/2}\setminus U_{\varepsilon/2}}$, since by the chain rule it holds

$$ \nabla \Xi_\varepsilon(x) = (\dot{\chi}_\varepsilon \circ |\nabla \varphi|^2)(x) \nabla |\nabla \varphi|^2(x). $$

Taking advantage of our cut-off functions, we now apply the classical Divergence Theorem to the smooth vector field $\Xi_\varepsilon Y$, obtaining

$$ \int_{\{\varphi = s\}} |\nabla \varphi|^\beta H \frac{e^s}{e^{\varphi}} d\sigma - \int_{\{\varphi = S\}} |\nabla \varphi|^\beta H \frac{e^S}{e^{\varphi}} d\sigma = \int_{E^S_s} \Xi_\varepsilon \text{div} Y \frac{\beta}{e^{\varphi}} d\mu + \int_{E^S_s} \langle \nabla \Xi_\varepsilon Y \rangle \frac{\beta}{e^{\varphi}} d\mu 

= \int_{E^S_s} \Xi_\varepsilon |\nabla \varphi|^{\beta - 2}\left( |\nabla \varphi|^{2} + (\beta - 2) |\nabla|\nabla \varphi|^{2}\right) \frac{e^s}{e^{\varphi}} d\mu 

+ \int_{U_{3/2}\setminus U_{\varepsilon/2}} \dot{\chi}_\varepsilon (|\nabla \varphi|^2) \frac{|\nabla \varphi|^{\beta - 2} |\nabla|\nabla \varphi|^{2}}{2e^s} d\mu. $$

Now, it is clear that when $\varepsilon \to 0$ the first term in the last row tends to the right hand side of (3.6), by the Monotone Convergence Theorem. We claim that for every $\beta > (n - 2)/(n - 1)$ the last term in the second row tends to 0, as $\varepsilon \to 0$, or equivalently

$$ \lim_{\varepsilon \to 0} \int_{E^S_s} \frac{\langle \nabla \Xi_\varepsilon Y \rangle}{\beta} d\mu = 0. \quad (3.9) $$
To prove such a claim, we use the Coarea Formula to write
\[
\int_{U_r \cup U_{r/2}} \dot{\chi}_e(|\nabla \varphi|^2) \frac{|\nabla \varphi|^{\beta-2} |\nabla |\nabla \varphi|^2|^2}{2 e^\varphi} \, d\mu = \frac{1}{2} \int_{\varepsilon/2}^{3\varepsilon/2} \dot{s}(s) s^{(\beta-2)/2} \, ds \int \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma,
\]
where the last estimate follows by the structural properties of \( \chi_e \). Since the integrand in the last term is a continuous function of \( s \), the Mean Value Theorem for integrals insures the existence of a real number \( r \in (\varepsilon/2, 3\varepsilon/2) \) such that
\[
\frac{1}{\varepsilon} \int_{\varepsilon/2}^{3\varepsilon/2} s^{(\beta-2)/2} \, ds \int \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma = r^{(\beta-2)/2} \int \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma.
\]
If we set
\[
F(r) = \int_{\{|\nabla \varphi|^2 = r\}} \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma,
\]
and its sufficient to prove that \( \lim_{r \to 0} r^{(\beta-2)/2} F(r) = 0 \) in order to obtain claim (3.9) for every \( \beta > (n-2)/(n-1) \). To accomplish this program, we first observe that, if \( n \) is the outer unit normal vector field to the set \( U_r \) defined in (3.2), one can write
\[
F(r) = \int_{\partial U_r} \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma = \int_{U_r} \text{div} \left( \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \right) \, d\mu = 2 \int_{U_r} |\nabla |\nabla \varphi|^2| e^\varphi \, d\mu
\]
\[
= 2 \int_0^r \int_{\{|\nabla \varphi|^2 = s\}} \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \frac{1}{|\nabla |\nabla \varphi|^2|} \, d\sigma,
\]
where in the last two identities we have used formula (2.11) and the Coarea Formula, respectively. Differentiating the above expression with the help of the Fundamental Theorem of Calculus, and using the refined Kato inequality, we get the differential inequality
\[
F'(r) = 2 \int_{\{|\nabla \varphi|^2 = r\}} \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma \geq 2 \left( \frac{n}{n-1} \right) \int_{\{|\nabla \varphi|^2 = r\}} \frac{|\nabla |\nabla \varphi|^2|}{e^\varphi} \, d\sigma = \frac{1}{2} \left( \frac{n}{n-1} \right) \frac{F(r)}{r^{(\beta-2)/2}}
\]
Integrating this inequality between \( r \) and fixed value \( R > r \), one gets
\[
\frac{F(r)}{r^{(\beta-2)/2}} \leq \frac{F(R)}{R^{(\beta-2)/2}}.
\]
In particular, it follows that for every \( \beta > (n-2)/(n-1) \)
\[
0 \leq r^{(\beta-2)/2} F(r) \leq \frac{F(R)}{R^{(\beta-2)/2}}, \quad \text{as } r \to 0.
\]
This completes the proof of claim (3.9), and in turn of the identity (3.6) for every $\beta$ which is strictly above the optimal threshold. To obtain the desired identity also for $\beta = (n-2)/(n-1)$, it is now sufficient to pass to the limit in $\beta$, using the Dominated Convergence Theorem on the left hand side and the Monotone Convergence Theorem on the right hand side. \hfill $\square$

As a direct application of the Second Integral Identity, we are going to show in the next Corollary that for every $\beta \geq (n-2)/(n-1)$ the function $\Phi_\beta$ is differentiable.

**Corollary 3.5** (Theorem 2.2(ii)—Differentiability and Monotonicity) Let $(M, g, \varphi)$ be a solution to problem (2.3) and assume that the growth condition $|\nabla \varphi|^2_g = o(e^{\psi})$ is satisfied, as $x \to \infty$. Then, for every $\beta \geq (n-2)/(n-1)$ and every $s \geq 0$, we have that the function $s \mapsto \Phi_\beta(s)$ defined in (2.7) is continuously differentiable. Moreover, the derivative $\Phi'_\beta$ is everywhere nonpositive and satisfies

$$
\Phi'_\beta(s)e^{-s} - \Phi'_\beta(s)e^{-s} = \beta \int_{\{s < \varphi < S\}} \frac{|\nabla \varphi|^{\beta - 2} \left(|\nabla \varphi|^2_g + (\beta - 2)|\nabla \varphi|^2\right)}{e^{\varphi}} \, d\mu_g,
$$

(3.10)

for every $0 \leq s < S$.

**Proof** We start showing that for every $\beta \geq (n-2)/(n-1)$ the function

$$
[0, +\infty) \setminus \varphi(\text{Crit}(\varphi)) \ni s \mapsto \Psi_\beta(s) = \int_{\{\varphi=s\}} \frac{|\nabla \varphi|^{\beta} H}{e^{s}} \, d\sigma
$$

admits a (unique) continuous extension to the whole range of $\varphi$. Such an extension will be necessarily monotone, as it will satisfy the identity

$$
\Psi_\beta(S_0) - \Psi_\beta(S) = \int_{\{S_0 < \varphi < S\}} \frac{|\nabla \varphi|^{\beta - 2} \left(|\nabla \varphi|^2_g + (\beta - 2)|\nabla \varphi|^2\right)}{e^{\varphi}} \, d\mu \geq 0,
$$

for every $0 \leq S_0 < S$, due to Lemma 3.4. What we need to prove is that if $\overline{\varphi}$ is a critical value of $\varphi$, then the formula

$$
\Psi_\beta(\overline{\varphi}) = \lim_{s \to \overline{\varphi}} \Psi_\beta(s)
$$

yields a good definition for $\Psi_\beta(\overline{\varphi})$. In other words, we have to show that the above limit exists and is finite. Since the singular values of $\varphi$ are discrete (from Lemma 2.1 and [45]), we let $\eta > 0$ be such that the only regular value in $[\overline{\varphi} - \eta, \overline{\varphi} + \eta]$ is given by $\overline{\varphi}$. Using identity (3.6), it is easy to see that the assignment

$$
[\overline{\varphi} - \eta, \overline{\varphi}] \ni s \mapsto \Psi_\beta(s) = \Psi_\beta(\overline{\varphi} - \eta) - \int_{\{\overline{\varphi} - \eta < \varphi < \overline{\varphi}\}} \frac{|\nabla \varphi|^{\beta - 2} \left(|\nabla \varphi|^2_g + (\beta - 2)|\nabla \varphi|^2\right)}{e^{\varphi}} \, d\mu
$$

is nonincreasing for every $\beta \geq (n-2)/(n-1)$. Moreover, by the same identity, it is immediate to deduce that it is bounded from below. In fact, one has that

$$
\Psi_\beta(s) \geq \Psi_\beta(\overline{\varphi} - \eta) - \int_{[\overline{\varphi} - \eta < \varphi < \overline{\varphi} + \eta]} \frac{|\nabla \varphi|^{\beta - 2} \left(|\nabla \varphi|^2_g + (\beta - 2)|\nabla \varphi|^2\right)}{e^{\varphi}} \, d\mu = \Psi_\beta(\overline{\varphi} + \eta).
$$
This proves that $\lim_{t \to +\infty} \Psi_{\beta}(s)$ exists and is finite. Reasoning in the same way, one can prove that also $\lim_{s \to +\infty} \Psi_{\beta}(s)$ exists and is finite. Hence, it remains to show that the two limits coincide, but this follows directly from the absolute continuity of the integral. In fact, an immediate consequence of Lemma 3.4 is that

$$\frac{|\nabla \varphi|^{\beta-2} \left( |\nabla \nabla \varphi|^2 + (\beta - 2) |\nabla |\nabla \varphi|^2 \right)}{\text{e}^\varphi} \in L^1((\overline{s} - \eta < \varphi < \overline{s} + \eta), \mu).$$

Hence, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $E$ is a measurable set with $\mu(E) \leq \delta$, then

$$\int_E \frac{|\nabla \varphi|^{\beta-2} \left( |\nabla \nabla \varphi|^2 + (\beta - 2) |\nabla |\nabla \varphi|^2 \right)}{\text{e}^\varphi} \, d\mu \leq \varepsilon.$$

To conclude it is thus enough to observe that for sufficiently small $t > 0$, one has that $\mu((\overline{s} - t < \varphi < \overline{s} + t)) < \delta$.

We now pass to discuss, for every $\beta \geq (n - 2)/(n - 1)$, the differentiability of the auxiliary function $\Upsilon_{\beta} : [0, +\infty) \longrightarrow \mathbb{R}$, given by $\Upsilon_{\beta}(s) = \text{e}^{-s} \Phi_{\beta}(s)$. By the First Integral Identity (3.4) we have that

$$\frac{\Upsilon_{\beta}(s + h) - \Upsilon_{\beta}(s)}{h} = \frac{1}{h} \int_{\{s < \varphi < s+h\}} \frac{|\nabla \varphi|^{\beta-2} \left( \beta \nabla \nabla \varphi(\nabla \varphi, \nabla \varphi) - |\nabla \varphi|^4 \right)}{\text{e}^\varphi} \, d\mu$$

$$= -\frac{1}{h} \int_s^{s+h} dt \int_{\{\varphi = t\}} \frac{|\nabla \varphi|^{\beta+1}}{\text{e}^\varphi} \, d\sigma - \frac{\beta}{h} \int_s^{s+h} dt \int_{\{\varphi = t\}} \frac{|\nabla \varphi|^{\beta+1}}{\text{e}^\varphi} \, d\sigma$$

$$= -\frac{1}{h} \int_s^{s+h} \Upsilon_{\beta}(t) \, dt - \frac{\beta}{h} \int_s^{s+h} \Upsilon_{\beta}(t) \, dt,$$

Using the continuity of both $\Upsilon_{\beta}$ and $\Psi_{\beta}$ and invoking the Mean Value Theorem for integrals, we arrive at

$$\frac{\Upsilon_{\beta}(s + h) - \Upsilon_{\beta}(s)}{h} = -\Upsilon_{\beta}(\xi_h) - \beta \Psi_{\beta}(\xi_h),$$

for some $\xi_h$ lying between $s$ and $s + h$. Letting $h \to 0$, we finally deduce that $s \mapsto \Upsilon_{\beta}(s)$ is a $C^1$ function and that

$$\Upsilon_{\beta}' = -\Upsilon_{\beta} - \beta \Psi_{\beta}.$$  

This implies in turn that for every $\beta \geq (n - 2)/(n - 1)$ the function $s \mapsto \Phi_{\beta}(s) = \text{e}^{-s} \Upsilon_{\beta}(s)$ is differentiable. Moreover, the derivative coincides with $-\beta \text{e}^{-s} \Psi_{\beta}$, so that it is continuous and satisfies the identity

$$\Phi_{\beta}'(S) \text{e}^{-S} - \Phi_{\beta}'(S_0) \text{e}^{-S_0} = \beta \int_{\{S_0 < \varphi < S\}} \frac{|\nabla \varphi|^{\beta-2} \left( |\nabla \nabla \varphi|^2 + (\beta - 2) |\nabla |\nabla \varphi|^2 \right)}{\text{e}^\varphi} \, d\mu.$$

for every $0 \leq S_0 < S$. Combining the latter identity with the upper bound for $\Phi_{\beta}$ obtained in Lemma 2.4 one easily deduces the monotonicity of $\Phi_{\beta}$ by the same argument outlined at the very end of Sect. 2.  

$\square$
As a consequence of formula (3.10) in the above corollary, we obtain a representation formula for the derivatives of the functions $\Phi_\beta$, together with the rigidity statement claimed in Theorem 2.2(ii).

**Corollary 3.6 (Theorem 2.2(ii)—Representation Formula for $\Phi'_\beta$ and Rigidity statement)** Let $(M, g, \varphi)$ be a solution to problem (2.3) and assume that the growth condition $|\nabla \varphi|^2 = o(e^{\varphi})$ is satisfied, as $x \to \infty$. Then, for every $\beta \geq (n - 2)/(n - 1)$ and every $s \geq 0$, we have

$$\Phi'_\beta(s) = -\beta e^s \int_{\{\varphi > s\}} \frac{|\nabla \varphi|_g^{\beta-2} (|\nabla \nabla \varphi|^2_g + (\beta - 2)|\nabla \varphi||_g^2)}{e^\varphi} \, d\mu_g. \quad (3.11)$$

Moreover, if there exists $s_0 \geq 0$ such that $\Phi'_\beta(s_0) = 0$ for some $\beta \geq (n - 2)/(n - 1)$, then the manifold $(M, g)$ is isometric to the half cylinder $\{(0, +\infty) \times \partial M, d\varrho \otimes d\varrho + g|_{\partial M}\}$, where $\varrho$ is the distance to $\partial M$, and $\varphi$ is an affine function of $\varrho$. The cross sections of the cylinder are locally isometric to round spheres for $n = 3, 4$ and to compact Einstein manifolds for $n \geq 5$.

**Proof** By virtue of identity (3.10), to prove that the representation formula (3.5) is also satisfied, it is sufficient to find a sequence of positive real numbers $(s_n)_{n \in \mathbb{N}}$ tending to $+\infty$ such that $\Phi'_\beta(s_n) \to 0$, as $n \to +\infty$. As a consequence of Corollary 3.5 we have that, for every $\beta$ in the admissible range, the function $s \mapsto \Phi_\beta(s)$ is nonincreasing. On the other hand, such function is also bounded from below and so admits a limit as $s \to +\infty$. In particular, for every $n \in \mathbb{N}$, there exists $k = k(n) \in \mathbb{N}$ such that $0 \leq \Phi_\beta(k) - \Phi_\beta(k + 1) \leq 1/n$. By Lagrange’s Theorem, there exists $s_n \in [k, k + 1)$, such that $|\Phi'_\beta(s_n)| \leq 1/n$.

To prove the rigidity statement, observe that if $\Phi'_\beta(s_0) = 0$ for some $s_0 \geq 0$ and some $\beta \geq (n - 2)/(n - 1)$, then

$$|\nabla \varphi|^2 + (\beta - 2)|\nabla \varphi|| \equiv 0,$$

in $\{\varphi \geq s_0\}$, due to the refined Kato inequality for harmonic functions. Let us now distinguish two cases, depending if either $\beta > (n - 2)/(n - 1)$ or else $\beta = (n - 2)/(n - 1)$. In the former case it is immediate to conclude that $|\nabla \nabla \varphi| \equiv 0$ in $\{\varphi \geq s_0\}$. In turn, $|\nabla \varphi|$ is a nonzero constant in that region, due to the last two conditions in (2.3). Therefore, $\nabla \varphi$ is a nontrivial parallel vector field and by [3, Theorem 4.1-(i)] we deduce that the Riemannian manifold $(\{\varphi \geq s_0\}, g)$ is isometric to the manifold $\{\varphi = s_0\} \times [s_0, +\infty)$ endowed with the product metric $d\varrho \otimes d\varrho + g|_{\{\varphi = s_0\}}$, where $\varrho$ represents the distance to $\{\varphi = s_0\}$. Moreover, from the proof of Theorem 4.1-(i) in the mentioned paper, one gets that the function $\varphi$ can be expressed as an affine function of $\varrho$ in $\{\varphi \geq s_0\}$, i.e. $\varphi = s_0 + \varrho|\nabla \varrho|$, where $|\nabla \varphi||$ is a positive constant. The fact that the product structure holds up to the boundary is now an easy consequence of Lemma 2.1, whereas the roundness of the cross sections in dimension 3 and 4 follows by the same arguments as in the proof of Proposition 2.3. Let us now consider the case where $\beta = (n - 2)/(n - 1)$, which gives

$$|\nabla \nabla \varphi|^2 = \frac{n}{n - 1} |\nabla \varphi||^2, \quad (3.12)$$

in $\{\varphi \geq s_0\}$. Following the proof of [9, Proposition 5.1], it is possible to deduce that $|\nabla \varphi|$ is a (necessarily nonzero) constant along the level sets of $\varphi$ and thus that the metric $g$ has a warped product structure in this region, namely

$$g = d\varrho \otimes d\varrho + \eta^2(\varrho) g|_{\{\varphi = s_0\}}, \quad (3.13)$$
for some positive warping function $\eta = \eta(\varphi)$. Moreover $\varphi$, $\rho$ and the warping factor $\eta$ satisfy the relationship

$$\varphi(p) = s_0 + \kappa \int_0^{\varphi(p)} \frac{d\tau}{\eta(\tau)^{n-1}}$$

for every point $p \in \{ \varphi \geq s_0 \}$ and some $\kappa \geq 0$. In particular, $\varphi$ and $\varphi$ share the same level sets and, by formula (3.13), these are totally umbilic. In fact one has

$$h_{ij}^{(s)} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \varphi} = \frac{d \log \eta}{d \varphi} g_{ij}.$$  

As a consequence, the mean curvature is constant along each level set $\{ \varphi = s \}$ with $s \geq s_0$. Applying formula (2.10) in Remark 8, with $\beta = (n-2)/(n-1)$, to each of these level sets, one gets

$$H \int_{\{\varphi=s\}} |\nabla \varphi|^\frac{n-2}{n-1} d\sigma = 0,$$

since by virtue of (3.12) the right hand side of (3.5) is always zero. This implies in turn that all the level sets $\{ \varphi = s \}$ with $s \geq s_0$ are minimal and thus totally geodesic. From $H \equiv 0$ one can also deduce that $\langle \nabla |\nabla \varphi|^2 \rangle \equiv 0$ in $\{ \varphi \geq s_0 \}$. Hence, $|\nabla \varphi|$ is constant in $\{ \varphi \geq s_0 \}$, and thus the conclusion follows arguing as in the case where $\beta > (n-2)/(n-1)$.

\[\square\]

### 4 Consequences of the Monotonicity-Rigidity Theorem

In this section we discuss some of the analytic and geometric consequences of our Monotonicity-Rigidity Theorem 1.1. The first group of results will be deduced only using the fact that

$$0 \leq -F_\beta'(1) = \beta \int_{\partial \Omega} |Du|^\beta \left[ H - \frac{n-1}{n-2} |Du| \right] d\sigma,$$  

for every $\beta \geq (n-2)/(n-1)$, whereas the geometric inequalities of the second group (Theorems 1.2 and 1.3) will make a substantial use of the fact that

$$\left[ \text{Cap}(\Omega) \right]^{\frac{n-2-\beta}{n-2}} (n-2)^{\beta+1} |S^{n-1}| = \lim_{\tau \to +\infty} F_\beta(\tau) \leq F_\beta'(1) = \int_{\partial \Omega} |Du|^\beta+1 d\sigma,$$  

where the limit has already been discussed in (1.5)–(1.6).

#### 4.1 Consequences at the boundary

We begin with the following sharp inequality, which says that the $L^p$-norm of the normal derivative at $\partial \Omega$ is always bounded above by the $L^p$-norm of the mean curvature, provided $p \geq 2 - 1/(n-1)$.
Theorem 4.1  Let $u$ be a solution to problem (1.1). Then, for every $p \geq 2 - 1/(n - 1)$ the inequality
\[
\left\| \frac{\partial u}{\partial \nu} \right\|_{L^p(\partial \Omega)} \leq \left( \frac{n - 2}{n - 1} \right) \| H \|_{L^p(\partial \Omega)}
\]  
holds true, where $H$ is the mean curvature of $\partial \Omega$ and $\nu$ is the unit normal vector of $\partial \Omega$ pointing toward the interior of $\mathbb{R}^n \cup \Omega$. Moreover, the equality is fulfilled for some $p \geq 2 - 1/(n - 1)$ if and only if $u$ is rotationally symmetric.

Proof  Setting $p = \beta + 1$ in (4.1) and rearranging the terms, one gets
\[
\int_{\partial \Omega} |Du|^p d\sigma \leq \left( \frac{n - 2}{n - 1} \right) \int_{\partial \Omega} |Du|^{p-1} H d\sigma.
\]  
The thesis follows from the Hölder inequality. The rigidity statement is a consequence of the rigidity of the equality case in inequality (4.1), which follows in turns from the rigidity statement in Theorem 1.1(ii).

To proceed, we observe that letting $p \to +\infty$ in the previous inequality, one also obtains that
\[
\max_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right| \leq \left( \frac{n - 2}{n - 1} \right) \max_{\partial \Omega} |H|.
\]  
Unfortunately the corresponding rigidity statement does not survive the passage to the limit. However, one can recover the validity of such a statement either invoking the results in [6], or using the following sharp gradient estimate together with the subsequent corollary

Theorem 4.2  (Sharp gradient estimate à la Colding) Let $u$ be a solution to problem (1.1). Then, for every $x \in \mathbb{R}^n \cup \Omega$, the inequality
\[
|Du|(x) \leq \left( \max_{\partial \Omega} |Du| \right) u^{\frac{n-2}{n-1}}(x)
\]  
holds true. Moreover, the equality is achieved at some point in $\mathbb{R}^n \cup \Omega$ if and only if $u$ is rotationally symmetric.

Proof  Using the expansion (1.5) in combination with (2.1) and (2.2), it is immediate to deduce that $|\nabla \varphi|^2_{\varphi} = O(1)$, as $x \to \infty$. Hence, the hypothesis of Proposition 2.3 are largely satisfied. The thesis can be now easily deduced from (2.13), with the help of (1.19).

From inequality (4.4) in Corollary 4.1 and from Theorem 4.2 above, it is immediate to deduce the following Corollary.

Corollary 4.3  Let $u$ be a solution to problem (1.1). Then, for every $x \in \mathbb{R}^n \cup \Omega$, the inequality
\[
|Du|(x) \leq \left( \frac{n - 2}{n - 1} \right) \left( \max_{\partial \Omega} |H| \right) u^{\frac{n-1}{n-2}}(x)
\]  
holds true. Moreover, the equality is achieved at some point in $\mathbb{R}^n \cup \Omega$ if and only if $u$ is rotationally symmetric.
In particular, the equality case in (4.4) is characterized by a geometric rigidity, as desired. Recalling that the electrostatic capacity of a charged body $\Omega$ can be computed in terms of the exterior normal derivative as

$$\text{Cap}(\Omega) = -\frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial \Omega} \frac{\partial u}{\partial \nu} d\sigma,$$

and using the Hölder inequality, it is not hard to deduce from (4.3) and (4.4) the following geometric upper bounds for the capacity. Observe that $\frac{\partial u}{\partial \nu} = -|Du| < 0$ on $\partial \Omega$, due to the definition of $\nu$. Other geometric upper and lower bounds for the capacity are obtained for example in [10, 29, 38, 48].

**Corollary 4.4** Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be a bounded domain with smooth boundary. Then, for every $p \geq 2 - 1/(n - 1)$, the inequality

$$\text{Cap}(\Omega) \leq \frac{\partial \Omega}{|\mathbb{S}^{n-1}|} \left( \int_{\partial \Omega} \left| \frac{H}{n-1} \right|^p d\sigma \right)^{1/p}$$

holds true, where $H$ is the mean curvature of $\partial \Omega$. Moreover, the equality is fulfilled for some $p \geq 2 - 1/(n - 1)$ if and only if $\Omega$ is a round ball. Finally, letting $p \to +\infty$ in the previous inequality, one has that

$$\text{Cap}(\Omega) \leq \frac{\partial \Omega}{|\mathbb{S}^{n-1}|} \max_{\partial \Omega} \left| \frac{H}{n-1} \right|.$$  

Moreover, the equality is fulfilled if and only if $\Omega$ is a round ball.

### 4.2 Global geometric consequences

So far we have used the local feature of the monotonicity, namely inequality (4.1), to prove a first group of corollaries of Theorem 1.1. To deduce further consequences of our main theorem, we now exploit the global feature of the monotonicity, comparing the behaviour of our quantities at the boundary and at large level sets of $u$, with the help of (4.2). We start with the proof of the Weighted Minkowski Inequality.

**Proof of Theorem 1.3** Multiplying inequality (4.2) by $(n-1)/(n-2)$, we obtain

$$[\text{Cap}(\Omega)]^{n-2-\beta \over n-2} (n-1)(n-2)^\beta |\mathbb{S}^{n-1}| \leq \int_{\partial \Omega} |Du|^{\beta} H d\sigma - \int_{\partial \Omega} |Du|^{\beta} \left[ H - \left( \frac{n-1}{n-2} \right) |Du| \right] d\sigma,$$

where we have added and subtracted the quantity $\int_{\partial \Omega} |Du|^{\beta} H d\sigma$ to the right hand side. A simple rearrangement of the terms leads to

$$\int_{\partial \Omega} \left| \frac{Du}{n-2} \right|^{\beta} \left[ \frac{H}{n-1} - \frac{Du}{n-2} \right] d\sigma \leq \int_{\partial \Omega} \left| \frac{Du}{n-2} \right|^{\beta} \frac{H}{n-1} d\sigma - [\text{Cap}(\Omega)]^{n-2-\beta \over n-2} |\mathbb{S}^{n-1}|.$$  

Notice that the left hand side is nonnegative in view of (4.1). Setting $\beta = (n-2)/(n-1)$, the last summand on the right hand side can be rewritten as

$$[\text{Cap}(\Omega)]^{n-1 \over n-2} |\mathbb{S}^{n-1}| = \left( \int_{\partial \Omega} \left| \frac{Du}{n-2} \right| d\sigma \right)^{n-2 \over n-1} |\partial \Omega|^{1 \over n-1} |\mathbb{S}^{n-1}|^{1 \over n-1}.$$
Substituting this expression in the last inequality, we arrive with some algebraic manipulations at

\[
\int_{\partial \Omega} \left[ \frac{H}{n-1} - \frac{|Du|}{n-2} \right] \left( \int_{\partial \Omega} |Du| \, d\sigma \right)^{\frac{n-2}{n-1}} \, d\sigma 
\leq \int_{\partial \Omega} \frac{H}{n-1} \left( \int_{\partial \Omega} |Du| \, d\sigma \right)^{\frac{n-2}{n-1}} \, d\sigma - |\partial \Omega| \frac{|S^{n-1}|}{n-1} \frac{1}{n-1}
\]

Recalling the definition of the measure \( \tilde{\sigma} \), it is then easy to realise that the last inequality coincides with (1.17). The rigidity follows at once by the rigidity statement in Theorem 1.1(ii).

We are now ready to deduce the Quantitative Willmore-type Inequality.

**Proof of Theorem 1.2** Multiplying the last inequality in the proof of Theorem 1.3 by the factor

\[
\left( \int_{\partial \Omega} |Du| \, d\sigma \right)^{\frac{n-2}{n-1}}
\]

and using (1.2), we obtain

\[
\int_{\partial \Omega} \left[ \frac{H}{n-1} - \frac{|Du|}{n-2} \right] |Du|^{\frac{n-1}{n-2}} \, d\sigma \leq \int_{\partial \Omega} \frac{H}{n-1} |Du|^{\frac{n-1}{n-2}} \, d\sigma - |S^{n-1}|^{1/(n-1)} \left( \int_{\partial \Omega} |Du| \, d\sigma \right)^{\frac{n-2}{n-1}}
\]

\[
\leq \left[ \left( \int_{\partial \Omega} \frac{H}{n-1} \, d\sigma \right)^{1/(n-1)} - |S^{n-1}|^{1/(n-1)} \right] \left( \int_{\partial \Omega} |Du| \, d\sigma \right)^{\frac{n-2}{n-1}},
\]

where in the last inequality we used the Hölder Inequality. The thesis (1.14) follows now from simple algebraic manipulations, noticing that the leftmost hand side is nonnegative by Theorem 1.1(ii). The rigidity also follows at once by the rigidity statement in Theorem 1.1(ii).

**4.3 Quantitative Willmore Inequality in 3-D**

We conclude with the observation that for \( n = 3 \) one can provide a different quantitative version of the Willmore Inequality, beside the one that follows from the general statement of Theorem 1.2. For the ease of reference, we recall that setting \( n = 3 \) in (1.14) one gets

\[
\int_{\partial \Omega} \sqrt{\frac{|Du|}{4\pi \text{Cap}(\Omega)}} (H - 2|Du|) \, d\sigma \leq \left( \int_{\partial \Omega} H^2 \, d\sigma \right)^{1/2} - \sqrt{16\pi},
\]

where the deficit on the left hand side is nonnegative and optimal, due to Theorem 1.1(ii). Here optimal means that as soon as it is zero, the right hand side is also zero.

A slightly different path leads to the following statement, in which the deficit is still optimal, although different.

**Corollary 4.5** Let \( \Omega \subseteq \mathbb{R}^3 \) be a bounded domain with smooth boundary. Then, the inequality

\[
\int_{\partial \Omega} (H - 2|Du|)^2 \, d\sigma \leq \int_{\partial \Omega} H^2 \, d\sigma - 16\pi,
\]

where the deficit on the left hand side is nonnegative and optimal, due to Theorem 1.1(ii). Here optimal means that as soon as it is zero, the right hand side is also zero.
holds true, where $H$ is the mean curvature of $\partial \Omega$ and $u$ is the capacitary potential due to the uniformly charged body $\Omega$. Moreover, the deficit on the left hand side is optimal in the sense that if it vanishes, then the right hand side also vanishes and $\Omega$ is a round ball.

**Proof** Notice that for $n = 3$ and $\beta = 1$ the inequalities (4.1) and (4.2) give

$$4\pi \leq \int_{\partial \Omega} |Du|^2 \, d\sigma \quad \text{and} \quad 0 \leq \int_{\partial \Omega} |Du| (H - 2|Du|) \, d\sigma.$$

Starting from these results, it is possible to deduce at once the following chain of inequalities

$$16\pi \leq 4 \int_{\partial \Omega} |Du|^2 \, d\sigma \leq 2 \int_{\partial \Omega} |Du|Hd\sigma + 2 \int_{\partial \Omega} |Du| (H - 2|Du|) \, d\sigma$$

$$= 4 \int_{\partial \Omega} |Du| (H - |Du|) \, d\sigma = \int_{\partial \Omega} H^2 d\sigma - \int_{\partial \Omega} (H - 2|Du|)^2 \, d\sigma.$$

The thesis follows by a simple rearrangement. If the deficit vanishes, then $H \equiv 2|Du|$ on $\partial \Omega$, and this triggers the rigidity statement in Theorem 1.1(ii). \hfill \Box

**Acknowledgements** The author are grateful to G. Crasta, A. Farina, I. Fragalà, C. Mantegazza, J. Metzger, M. Novaga, and D. Peralta-Salas for useful comments and discussions during the preparation of the paper. The authors are members of Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), which is part of the Istituto Nazionale di Alta Matematica (INdAM), and partially funded by the GNAMPA project “Principi di fattorizzazione, formule di monotonia e disuguaglianze geometriche”. The authors would like to thank the anonymous referee for the careful reading of the manuscript as well as for his/her valuable suggestions.

**References**

1. Agostiniani, V., Fogagnolo, M., Mazzieri, L.: Minkowski inequalities via nonlinear potential theory. arXiv:1906.00322
2. Agostiniani, V., Fogagnolo, M., Mazzieri, L.: Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. arXiv:1812.05022
3. Agostiniani, V., Mazzieri, L.: Riemannian aspects of potential theory. J. Math. Pures Appl. 104(3), 561–586 (2015)
4. Agostiniani, V., Mazzieri, L.: Comparing monotonicity formulas for electrostatic potentials and static metrics. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 28(1), 7–20 (2017)
5. Agostiniani, V., Mazzieri, L.: On the geometry of the level sets of bounded static potentials. Commun. Math. Phys. 355(1), 261–301 (2017)
6. Borghini, S., Mascellani, G., Mazzieri, L.: Some sphere theorems in linear potential theory. Trans. Am. Math. Soc. 371(11), 7757–7790 (2019)
7. Borghini, S., Mazzieri, L.: On the mass of static metrics with positive cosmological constant: II. arXiv:1711.07024
8. Borghini, S., Mazzieri, L.: On the mass of static metrics with positive cosmological constant: I. Class. Quantum Gravity 35(12), 125001 (2018)
9. Bour, V., Carron, G.: Optimal integral pinching results. Ann. Sci. Éc. Norm. Supér. (4) 48(1), 41–70 (2015)
10. Bray, H.L., Miao, P.: On the capacity of surfaces in manifolds with nonnegative scalar curvature. Invent. Math. 172(3), 459–475 (2008)
11. Brendle, S., Hung, P.-K., Wang, M.-T.: A Minkowski inequality for hypersurfaces in the Anti-de Sitter–Schwarzschild manifold. Commun. Pure Appl. Math. 69(1), 124–144 (2016)
12. Caffarelli, L.A., Friedman, A.: Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations. J. Differ. Equ. 60(3), 420–433 (1985)
13. Chang, S.-Y.A., Wang, Y.: On Aleksandrov–Fenchel inequalities for k-convex domains. Milan J. Math. 79(1), 13 (2011)
14. Chang, S.-Y.A., Wang, Y.: Inequalities for quermassintegrals on k-convex domains. Adv. Math. 248, 335–377 (2013)
15. Cheeger, J., Naber, A., Valtorta, D.: Critical sets of elliptic equations. arXiv:1207.4236v3
16. Chen, B.-Y.: On a theorem of Fenchel–Borsuk–Willmore–Chern–Lashof. Math. Ann. 194(1), 19–26 (1971)
17. Chen, B.-Y.: On the total curvature of immersed manifolds, I: an inequality of Fenchel–Borsuk–Willmore. Ann. J. Math. 93(1), 148–162 (1971)
18. Colding, T.H.: New monotonicity formulas for Ricci curvature and applications. I. Acta Math. 209(2), 229–263 (2012)
19. Colding, T.H., Minicozzi, W.P.: Monotonicity and its analytic and geometric implications. Proc. Natl. Acad. Sci. 110(48), 19233–19236 (2013)
20. Colding, T.H., Minicozzi, W.P.: Ricci curvature and monotonicity for harmonic functions. Calc. Var. Partial. Differ. Equ. 49(3), 1045–1059 (2014)
21. Crasta, G., Fragaład, L., Gazzola, F.: On a long-standing conjecture by Pólya–Szegö and related topics. Z. Angew. Math. Phys. 56(5), 763–782 (2005)
22. DeTurck, D., Kazdan, J.L.: Some regularity theorems in Riemannian geometry. Ann. Scie. l’École Norm. Supérieure Ser. 4 14(3), 249–260 (1981)
23. Enciso, A., Peralta-Salas, D.: Symmetry for an overdetermined boundary problem in a punctured domain. Nonlinear Anal. 70(2), 1080–1086 (2009)
24. Farina, A., Mari, L., Valdinoci, E.: Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds. Commun. Partial Differ. Equ. 38(10), 1818–1862 (2013)
25. Federer, H.: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, vol. 153. Springer, New York (1969)
26. Fogagnolo, M., Mazzieri, L., Pinamonti, A.: Geometric aspects of p-capacitary potentials. Ann. Inst. H. Poincaré Anal. Non Linéaire 36(4), 1151–1179 (2019)
27. Fragala, I., Gazzola, F.: Partially overdetermined elliptic boundary value problems. J. Differ. Equ. 245(5), 1299–1322 (2008)
28. Fragala, I., Gazzola, F., Kawohl, B.: Overdetermined problems with possibly degenerate ellipticity, a geometric approach. Math. Z. 254(1), 117–132 (2006)
29. Freire, A., Schwartz, F.: Mass-capacity inequalities for conformally flat manifolds with boundary. Commun. Partial Differ. Equ. 39(1), 98–119 (2014)
30. Garofalo, N., Sartori, E.: Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates. Adv. Differ. Equ. 4(2), 137–161 (1999)
31. Gerhardt, C.: Flow of nonconvex hypersurfaces into spheres. J. Differ. Geom. 32(1), 299–314 (1990)
32. Guan, P., Li, J.: The quermassintegral inequalities for k-convex starshaped domains. Adv. Math. 221(5), 1725–1732 (2009)
33. Hardt, R., Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., Nadirashvili, N.: Critical sets of solutions to elliptic equations. J. Differ. Geom. 51(2), 359–373 (1999)
34. Hardt, R., Simon, L.: Nodal sets for solutions of elliptic equations. J. Differ. Geom. 30(2), 505–522 (1989)
35. Huiskens, G.: An isoperimetric concept for the mass in general relativity. Video. https://video.ias.edu/node/234. Accessed Mar 2009
36. Huiskens, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. 31(1), 285–299 (1990)
37. Huiskens, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differ. Geom. 59(3), 353–437 (2001)
38. Hurtado, A., Palmer, V., Ritoré, M.: Comparison results for capacity. Indiana Univ. Math. J. 61(2), 539–555 (2012)
39. Kellogg, O.D.: Foundations of Potential Theory. Springer, Berlin (1967)
40. Marques, F.C., Neves, A.: Min–Max theory and the Willmore conjecture. Ann. Math. 179(2), 683–782 (2014)
41. Moser, R.: The inverse mean curvature flow and p-harmonic functions. J. Eur. Math. Soc. 9(1), 77–83 (2007)
42. Payne, L.E., Philippin, G.A.: On some maximum principles involving harmonic functions and their derivatives. SIAM J. Math. Anal. 10(1), 96–104 (1979)
43. Payne, L.E., Philippin, G.A.: Some overdetermined boundary value problems for harmonic functions. Z. Angew. Math. Phys. 42(6), 864–873 (1991)
44. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159
45. Souček, J., Souček, V.: Morse–Sard theorem for real-analytic functions. Comment. Math. Univ. Carol. 13(1), 45–51 (1972)
46. Urbas, J.I.E.: On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. 205(3), 355–372 (1990)
47. Willmore, T. J.: Mean curvature of immersed surfaces. Ann. Şti. Univ. “All. I. Cuza” Iaşi Secţ. I a Mat. (N.S.) 14, 99–103 (1968)
48. Xiao, J.: P-capacity vs surface-area. Adv. Math. 308, 1318–1336 (2017)

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