Symmetry-allowed phase transitions realized by the two-dimensional fully frustrated XY class

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A 2D Fully Frustrated XY (FFXY)-class of models is shown to contain a new groundstate in addition to the checkerboard groundstates of the standard 2D FFXY model. The spin configuration of this additional groundstate is obtained. Associated with this groundstate there are additional phase transitions. An order parameter accounting for these new transitions is proposed. The transitions associated with the new order parameter are suggested to be similar to a 2D liquid-gas transition which implies $\mathbb{Z}_2$-Ising like transitions. This suggests that the class of 2D FFXY models belongs within a $U(1) \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$-designation of possible transitions, which implies that there are seven different possible single and combined transitions. MC-simulations for the generalized fully frustrated XY (GFFXY) model on a square lattice are used to investigate which of these possibilities can be realized in practice: five of the seven are encountered. Four critical points are deduced from the MC-simulations, three consistent with central charge $c = 3/2$ and one with $c = 1$.

The implications for the standard 2D FFXY-model are discussed in particular with respect to the long standing controversy concerning the characteristics of its phase transitions.

I. INTRODUCTION

The two-dimensional (2D) fully frustrated XY (FFXY)-model describes a 2D Josephson junction array in a perpendicular magnetic field with the strength of the magnetic field corresponding to one magnetic flux quanta for every second plaquette of the array. The phase transitions of this model on a square lattice have been the subject of a long controversy\textsuperscript{1,2}. The emerging canonical picture is that the model has two relevant phase ordering symmetries: an angular $U(1)$-symmetry and a $\mathbb{Z}_2$-chirality symmetry\textsuperscript{3,9,10,11}. As a consequence, the model has often been assumed to belong within the designation $U(1) \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$\textsuperscript{2,3,4,12}. The controversial questions have been: Does the model undergo a single combined transition or two separate transitions and, if the latter, in which order do the transitions occur? The emerging consensus is two separate transitions: as the temperature is increased first a Kosterlitz-Thouless (KT) transition associated with the angular $U(1)$-symmetry and then at a slightly higher temperature a $\mathbb{Z}_2$-chirality transition\textsuperscript{1,2}. The cause of the controversy can, retrospectively, be attributed to the fact that the two transitions are extremely close in temperature.

We here generalize the 2D FFXY model into a wider 2D FFXY-class of models by changing the nearest neighbor interaction in such a way as to keep all symmetries. This generalized 2D FFXY-class is shown to contain an additional groundstate. The existence of this additional groundstate leads to a phase diagram containing four sectors\textsuperscript{13}. We here show that it has seven different phase transitions lines and four multicritical points. We use Monte Carlo simulations to establish the characters of the transitions of this phase diagram. Our simulations suggest that three of the critical points are consistent with the central charge $c = 3/2$ and one with $c = 1$.

In section 2 we define the 2D FFXY-model and in section 3 we describe the structure of the new groundstate. In section 4 we propose an order parameter associated with the transition into this new groundstate. In section 5 we give the results for the various phase transitions obtained from Monte Carlo simulations and determine the character of the four multicritical points by invoking a relation between the central charge $c$ and the bulk critical indices. In section 6 we discuss the original 2D FFXY model in view of our results. We also comment on related models not contained within the class of fully frustrated XY model discussed in the present investigation. Finally, some concluding remarks are given in section 7.

II. GENERALIZED FULLY FRUSTRATED XY MODEL

The Hamiltonian which defines the 2D fully frustrated XY-class models on an $L \times L$ square lattice is given by

$$H = \sum_{\langle ij \rangle} U (\phi_{ij} \equiv \theta_i - \theta_j - A_{ij}),$$

with $\phi_{ij} \in [-\pi, \pi]$, where the sum is over nearest neighbor pairs. The phase angle $\theta_i$ for the $i$th site at the lattice point $(x_i, y_i)$ satisfies the periodicity $\theta_{i+Lx} = \theta_{i-Lx} = \theta_i$. The magnetic bond angle $A_{ij}$ is defined as the line integral along the link from $i$ to $j$, i.e., $A_{ij} \equiv (2\pi/\Phi_0) \int_{\gamma} A \cdot dl$ with the magnetic vector potential $A$ for the uniform magnetic field $B = B_0\hat{z}$ in the $z$ direction. With the Landau gauge taken, $A_{ij} = 2\pi f x_i$ for the vertical link and $A_{ij} = 0$ for the horizontal one, where the frustration parameter $f$ measures the average number of flux quanta per plaquette. The fully frustrated case corresponds to $f = 1/2$ with a half flux quantum per plaquette on the average. The Boltzmann factor, which determines
the thermodynamic properties, is given by $\exp(-H/T)$ where $T$ is the temperature. The interaction potential $U(\phi) = U(\phi + 2\pi)$ is periodic in $2\pi$ and is quadratic to lowest order in $\phi$ so that $U(\phi) \sim \phi^2$. These conditions for the interaction potential defines the class: the members of this class are distinguished by the explicit form of the interaction potential $U(\phi)$. If the relevant symmetry class is $U(1) \otimes Z_2$, then in principle three transitions are possible: separate $U(1)$ and $Z_2$ transitions or a merged $U(1) \otimes Z_2$ transition. However, the number of allowed phase transitions for the FFXY-class is much larger.

The implication is that by just changing the specific form of $U(\phi)$ within the FFXY-class one could encounter a plethora of phase transitions. In order to verify this, we choose a parametrization of $U(\phi)$ and find the phase transitions corresponding to this parametrization using Monte Carlo simulations techniques. This strategy was employed earlier in Ref.\textsuperscript{13}. The parametrization is of the form $U(\phi)$ where\textsuperscript{13,14,15}

$$U(\phi) = \frac{2}{p^2}[1 - \cos^2p\left(\frac{\phi}{2}\right)] \tag{2}$$

and $p = 1$ corresponds to the standard FFXY since $2[1 - \cos^2(\phi/2)] = 1 - \cos(\phi)$. The members of the FFXY class, which belong to this parametrization, was in Ref.\textsuperscript{13} termed the Generalized Frustrated XY (GFFXY) model. Figure 1a shows a sequence of interaction potentials $U(\phi)$.

To sum up: The 2D FFXY class which we discuss here is obtained from the standard 2D FFXY by generalizing the interaction potential within the allowed conditions: $U(\phi)$ is a monotonously increasing function in the interval $\phi \in [0, \pi]$, $U(\phi) = U(\phi + 2\pi)$ is periodic in $2\pi$ and is quadratic to lowest order in $\phi$ so that $U(\phi) \sim \phi^2$. The GFFXY model is by construction contained within this class. The Villain interaction is also contained in this class.\textsuperscript{8} In Fig. 1b the interaction potential for the standard XY model $U(\phi) = 1 - \cos(\phi)$ is compared to the one for the Villain model at the KT-transition ($T = 0.45$) $U(\phi) = -T \ln(\sum_{n=-\infty}^{\infty} \exp(-\phi - 2\pi n^2)/2T)$. The 2D FFXY model with the Villain interaction has the same phase transition scenario as the usual 2D FFXY model i.e. a $U(1)$ KT-transition followed by a $Z_2$ transition (still extremely close together but a little less close than for the standard 2D FFXY model).\textsuperscript{3} Is this true for all models within the FFXY class? The answer is no.\textsuperscript{13} The reason is, according to us, connected to the appearance of a new groundstate.

### III. GROUNDSTATE

Let us first consider the groundstate for the standard 2D FFXY model on a square lattice: The spin configuration corresponding to the groundstate checkerboard is given in Fig. 2a.\textsuperscript{15} A square with (without) a flux quanta is denoted by + (−). The arrows give the spin directions and the thick (thin) links are the links with (without) magnetic bond angles $\pi$ (0) modulo $2\pi$. In this configuration all the links contribute the same energy $U(\frac{\pi}{2})$ to the groundstate. Thus the energy for the four links constituting an elementary square is in this configuration $4U(\frac{\pi}{2})$. The broken symmetry of the free energy is for $T = 0$ directly related to the fact that in order to change + to − squares in Fig. 2a by continuously turning the spin directions from the one groundstate to the other, an increase of the energy is required by a finite amount for a number of links. This required number of links goes to infinity with the size of the system: the two groundstates are separated by an infinite energy barrier in the thermodynamic limit.

The crucial point in the present context is that the groundstate shown in Fig. 2a does not remain the groundstate for all values $p$. As $p$ is increased, the maximum link energy $U(\pi)$ decreases and at a particular value $p_\pi > 1$ the groundstate switches to the spin configuration shown in Fig. 2b. The energy for the links around a square is for this configuration given by $U(\pi) + SU(0)$.
The critical value $p_c$ is hence given by the condition $U(\pi) + 3U(0) = 4U(\frac{\pi}{4})$ leading to the determination

$$p_c = \sqrt{\frac{\ln(3/4)}{2\ln(\cos(\pi/8))}} = 1.3479.$$  

The groundstate for $p > p_c$ shown in Fig. 2b has the property that an infinitesimal change of the middle spin is enough to flip between the two checkerboard patterns (switching between + and − in Fig. 2b). Thus there is no barrier between these two checkerboard patterns for $p > p_c$. This means that the broken symmetry of the free energy associated with the two possible checkerboard patterns states is restored. However, there is a new infinite barrier between the two degenerate groundstates on opposite sides of $p_c$: continuously turning the spins to change from the spin-configuration in Fig. 2a to the spin-configuration in Fig. 2b requires an infinite energy.

### IV. ORDER PARAMETERS

In order to characterize the phase transition properties of the 2D GFFXY model one needs to identify a set of order parameters with which all possible transitions can be characterized:

The checkerboard pattern is usually associated with a $Z_2$ chirality symmetry. For $T = 0$ this symmetry is reflected in the existence of two degenerate groundstates (the two checkerboards) separated by an infinite energy barrier. The corresponding order parameter is related to the staggered magnetization $m$ defined as $\phi_{ij}$.

$$m = \left\langle \frac{1}{L^2} \sum_{l=1}^{L^2} (-1)^{x_l+y_l} s_l \right\rangle,$$  

where $\langle \cdots \rangle$ is the ensemble average and the vorticity for the $l$th elementary plaquette at $(x_l, y_l)$ is computed from $s_l \equiv (1/\pi) \sum_{(ij) \in \phi_{ij}} = \pm 1$ with the sum taken anti-clockwise around the given plaquette. The corresponding broken symmetry is reflected in the following way: for any finite system the quantity $\frac{1}{L^2} \sum_{l=1}^{L^2} (-1)^{x_l+y_l} s_l$ can with finite probability acquire any value in the range $[-1, 1]$ allowed by the model. However, in the thermodynamic limit $L = \infty$ only values in either the range $[−1, 0]$ or the range $[0, 1]$ can be acquired. This means that the order parameter $O = (\frac{1}{L^2} \sum_{l=1}^{L^2} (-1)^{x_l+y_l} s_l)$ in the thermodynamic limit can only take on the two values $O = \pm m$. The probability for the two values are equal but they are separated by an infinite free-energy barrier. This is equivalent to saying that the order parameter $O$ has a $Z_2$ symmetry which is broken. In the broken symmetry region $m \neq 0$ whereas when the symmetry is unbroken $m = 0$. Figure 3 shows the phase diagram in the $(T, p)$-plane. As seen $m \neq 0$ corresponds to a finite region of this plane.

For $T = 0$ and $p = p_c$, the two groundstates in Fig. 2a and b are degenerate and separated by an infinite energy barrier. For $T > 0$ this should instead take the form of an infinite free energy barrier in the thermodynamic limit, separating values which a local order parameter can acquire. To this end one needs to identify an appropriate local order parameter. Such a possible order parameter is the defect density $n_k$ defined by

$$n_k = \left\langle \frac{4}{L^2} \sum_{l=1}^{L^2} |s_l| \right\rangle,$$

where the square lattice has been divided into $L^2$ squares numerated by $t$ where each consists of four elementary plaquettes. Here $s_l$ is the sum of the phase difference around a four-plaquette $s_l \equiv (1/\pi) \sum_{(ij) \in \phi_{ij}} = \pm 1$ which means that $|s_l|$ can be 0, 1 or 2. Thus the defect density can be described in the following way: Think of the elementary plaquettes as being either black ($s = 1$) or white ($s = -1$). There are always equally many black and white squares. The defect density measures the average difference in the number of white and black squares contained in a four-plaquette. Obviously the checkerboard groundstate corresponds to a zero defect density $n_k = 0$. However, for a finite temperature the checkerboard groundstate may contain a kink. This situation is illustrated in Fig. 4: start from a checkerboard pattern. The thick dotted line is a boundary between the two possible checkerboard patterns. The 90 degree turn of this line is associated with a four-plaquette with $s_l = 1$. 

![FIG. 3: Phase diagram of the 2D GFFXY model in the $(p, T)$ plane. The staggered magnetization $m$ and the helicity modulus $Y$ give us all four combinations, all of which are realized in the phase diagram. The horizontal dotted line at $p = 1$ corresponds to the standard FFXY model which has two distinct, extremely close transitions.](attachment:phase_diagram.png)
which is denoted as thick solid line surrounding four plaquettes in Fig. 4. Thus a kink corresponds to a defect with \( |s| = 1 \) according to our definition. The defect density defined here can be regarded as a generalization of the kink concept, since it does not rest on the possibility of uniquely identifying domain boundaries. Thus the defect density remains a well defined concept even when the checkerboard symmetry is completely restored and \( m = 0 \). The groundstate shown in Fig. 2b is an example of a situation when \( m = 0 \), because switching between + and - in Fig. 2b does not involve passing any energy barrier. Thus the defect density remains finite as \( T \) is lowered towards zero for any \( p > p_c \). Consequently, the groundstate in Fig. 2b corresponds to a finite defect density \( n_k > 0 \). It is also obvious that the defect density is monotonously increasing with \( T \).

A phase transition associated with this order parameter is signaled by either a discontinuous or a non-analytical behavior of \( n_k \) as a function of \( T \) and \( p \). The defect density makes a discontinuous jump from zero to a finite value at \( p_c \) in the limit of small temperatures and these two values are separated by an infinite energy barrier: the point \((p,T) = (p_c, 0)\) is the starting point of a phase transition line (see Fig.3). On this phase transition line the order parameter \( n_k \) can only take on two values. These two values are equally probable but are separated by an infinite free energy barrier. Thus the order parameter \( n_k \) on this phase transition line possesses a \( Z_2 \) symmetry which is broken.

One should note that in case of \( n_k \) the infinite free energy barrier between two different but equally probable values of \( n_k \) only resides on well defined lines in the \( (p,T) \)-plane, whereas the infinite barrier for the chirality transition resides on an area of the \( (p,T) \)-plane (see Fig.3). Thus the phase transition associated with the defect density \( n_k \) is more akin to a liquid-gas transition in the pressure temperature plane: the order parameter is the density difference on the two sides of the transition line and the infinite free energy barrier only exists precisely on the transition line.

The \( U(1) \)-symmetry is in 2D at most “quasi” broken because of the Mermin-Wagner Theorem\(^{16} \). As a consequence the corresponding phase transitions cannot be described by a local order parameter. Instead the phase transitions can be monitored by the increase of the free energy caused by a uniform twist \( \delta \) of the spin angles across the system. Expanding the free energy \( F(\delta) \) for small values of \( \delta \) to lowest orders gives

\[
F(\delta) = \frac{Y_4 \delta^4}{4!}.
\]

Here, \( Y \) is the helicity modulus. It is finite in the low-temperature phase and zero in the high-temperature phase.\(^{17} \) \( Y_4 \) is the fourth order modulus and can be used to verify that the helicity modulus \( Y \) makes a discontinuous jump to zero at the transition.\(^{18} \) This discontinuous jump is a key characteristics of the KT-transition.\(^{19,20} \)

### V. Phase Diagram and Phase Transitions

In Ref.\(^{33} \) the phase transitions associated with the \( U(1) \)-symmetry and the \( Z_2 \)-chirality symmetry were investigated. The corresponding phase diagram is reproduced in Fig. 3. This phase diagram has four sectors corresponding to all four possible combinations of transitions for a combined symmetry \( U(1) \otimes Z_2 \): The four sectors are characterized by the four possible combinations \((\Upsilon, m) = (0, 0), (0, \neq 0), (\neq 0, 0), (\neq 0, \neq 0)\). The dashed horizontal line at \( p = 1 \) in Fig. 3 corresponds to the usual FFXY model. In this case the phase \((\Upsilon \neq 0, m = 0)\) is not realized.\(^{33} \)

In the present paper we use all the three order parameters described in the previous section together with Monte Carlo simulations in order to deduce the nature of the various phase boundaries.

Fig. 5 gives a sketch of the resulting “horizontal” phase boundary in Fig. 3. In this blown up scale one finds that it has one maximum and one minimum, as well as three multicritical points ending three distinct phase lines. The critical points are denoted by \( A, B, \) and \( C \). A fourth multicritical point is found along the ”vertical” phase line at much higher \( p \) and lower \( T \) (see Figs.3 and 5). Let us first consider the phase boundary from \( T = 0 \) to the critical point \( A \). Across this first section of the phase boundary the phase transition associated with the defect density \( n_k \) is first order. Fig. 6a illustrates the discontinuous change in the defect density \( n_k \). The defect-density histogram along this phase line has two distinct values of equal probability which remain distinct in the large \( L \)-limit. An example is given in Fig. 6b: For a given temperature \( T \), the lower value corresponds to the low-\( p \) phase and the
higher to the high-\(p\) phase. As pointed out above, this is analogous to the density for a liquid-gas transition. Note that for \(T = 0.1\) the \(p\)-value for the first order line is lower than \(p_c(0)\). However, as \(T\) is further increased, the \(p\)-value for the first order line increases. Finally, at a critical temperature \(T_c(A)\) the density difference vanishes with increasing system size. This is the signature of the critical point \(A\) which is hence the critical point ending the first order transition line for the defect-density. Thus the critical point \(A\) is analogous to the critical point ending the first order line for a gas-liquid transition. Fig. 6c shows the defect-density histogram close to the critical point: at the critical point the free energy barrier between the two phases is \(L\)-independent. To good approximation this means that the ratio between the maximum and minimum in the kink-density histogram should be size-independent, whereas it increases (decreases) for lower (higher) temperatures\(^{21}\). This condition is fulfilled to good approximation for the \(T\)-value in Fig. 6c. At the critical point the defect-density difference \(\Delta n_k\) (the difference between the two maxima in Fig. 6c) should vanish with size as \(\Delta n_k \sim L^{-\beta/\nu}2\). The \(\Delta n_k\) size scaling is shown in Fig. 6d and is consistent with an exponent \(\beta/\nu = 0.25\). One can express this exponent in terms of the central charge \(c\) as \(\beta/\nu = c/4\).\(^{22,23}\) The central charge \(c\) is coupled to the symmetry of the order parameter. The defect density, the staggered magnetization and the magnetization for the 2D Ising model can all acquire precisely two distinct values with equal probability separated by an infinite energy barrier. The broken symmetry reflected by these order parameters does hence have a \(Z_2\)-character and the phase transitions are Ising-like. The central charge is \(c = 1/2\) for 2D Ising like transitions. If the order parameter on the other hand is a 2D vector then the symmetry is \(U(1)\) (which means that the order parameter with equal probability have the same magnitude and any direction, but that all these possibilities are separated by an infinite energy barrier) then the central charge is \(c = 1\). Provided that our three order parameters covers all possibilities, then a phase transition can \textit{a priori} be any combination of single and joint transitions involving these order parameters and is hence contained within the designation \(U(1) \otimes Z_2 \otimes Z_2\). These imply that the central charge can have the four values \(c = 1/2, 1, 3/2,\) and \(2\). Here a \(Z_2\)-transition corresponds to \(c = 1/2\), an individual KT-transition or a combined \(Z_2 \otimes Z_2\) transition corresponds to \(c = 1\), the two possible combined \(U(1) \otimes Z_2\)-transitions correspond to \(c = 3/2\), and a combined \(U(1) \otimes Z_2 \otimes Z_2\) corresponds to \(c = 2\). These possibilities are tested in Fig. 6d and singles out \(c = 1\) or equivalently \(\beta/\nu = 1/4\). This means that of the four possible values only \(c = 1\) is consistent with the data. As will be explained below, the helicity modulus remains non-zero in this part of the phase diagram (compare Fig.3) and consequently this suggests that the critical point \(A\) reflects a combined \(Z_2 \otimes Z_2\) defect-density and chirality transition. The defect-density transition ends at the critical point \(A\) as \(T\) is increased the free energy barrier vanishes in the large \(L\)-limit. However, there is a second defect-density transition line for higher \(p\)-values associated with a \(U(1) \otimes Z_2\) combined KT and defect-density transition, as illustrated in Fig. 6e and f. This transition is first order for higher \(p\) and ends at a critical point \(D\) : for a \(T\) higher than the critical point \(D\) there is no defect-density transition, just as for the case of the critical point \(A\).

Fig. 7a illustrates the chirality transition along the same phase boundary. Up to the critical point \(A\) (see Fig. 5) the transition is first order (see Fig. 7a). The chirality transition cannot cease at the critical point \(A\) because for a fixed \(T\) the free energy barrier between the \(O = \pm |m|\) always vanishes for a large enough \(p\). There are then two possibilities: it can continue alone as a \(Z_2\)-transition or it can combine with the KT-transition into a joint \(U(1) \otimes Z_2\)-transition. To deduce which possibility is the correct one, we calculate the size scaling of \(m \sim L^{-\beta/\nu}\) and decide which of the two possible symmetry allowed values \(\beta/\nu = c/4 = 1/8\) or \(3/8\) is consistent with the data. Here we use standard size scaling and calculate \(m(T,p)\) for a fixed \(T\) for a sequence of \(p\) which crosses the phase line. As seen in Fig. 7b, a unique crossing point is to good approximation obtained for \(\beta/\nu = 1/8\). From this we conclude that the chirality transition continues alone from the critical point \(A\) as a \(Z_2\)-transition. However as we increase the temperature further the character of the chirality transition changes: using the same procedure we instead find that the value \(\beta/\nu = 3/8\) is consistent with the data (see Fig. 7c). This is consistent with a joint \(U(1) \otimes Z_2\) KT-chirality transition. As we increase \(T\) further we come to the critical point \(C\) where the KT and chirality splits up into two separate transitions\(^{13}\). At this point it is possible to instead calculate the size scaling for a fixed \(p\). The advan-
The transition is a combined KT-transition. We argued that between the critical point \(T_{cA} \) and the critical point \(D \), the chirality transition merges with the KT-transition. This means that the helicity modulus must now vanish at the transition. This is illustrated in Fig. 8b, which shows that the \(Y \)-minimum now vanishes in the large \(L\)-limit (compare inset in Fig. 8b). Fig. 8c shows the same construction close to the critical point \(C \). The fact that \(Y \) vanishes as a power law can be verified for the critical point \(C \) by instead varying \(T \) for fixed \(p \). In these variables the critical point \(C \) obeys a standard scaling relation \(T = T_{cA} - a(T - T_{cC})L^{1/\nu} \) which confirms the power law decay of \(Y \), as opposed to the KT-universal jump signaling the isolated \(U(1)\)-transition for the XY-model (see Fig. 8d). We also note that the obtained critical index \(\nu \approx 0.77 \) is consistent with the data for \(m \) in Fig. 7d. It is also possible to use the fourth order helicity modulus \(\Upsilon_4 \) to determine the character of the \(U(1)\)-transition. In Ref. it was found from the \(\Upsilon_4 \)-data that, in the interval \(1.346 \leq p \leq 1.35 \), the character of the \(U(1)\)-transition was consistent with a transition without a discontinuous jump in the helicity modulus \(\Upsilon \). This is consistent with a combined \(U(1) \otimes Z_2\)-transition between the multicritical points \(B \) to \(C \).

The following picture emerges: \(A \) and \(D \) end two first order phase lines. \(A \) is associated with a \(Z_2 \otimes Z_2\)-transition with central charge \(c = 1 \) and \(D \) with a \(U(1) \otimes Z_2\)-transition with \(c = 3/2 \). \(B \) and \(C \) are both associated with \(U(1) \otimes Z_2\)-transitions and \(c = 3/2 \) but are not end-points of first order lines.

FIG. 7: a) The staggered magnetization \(|m| \) at \(T = 0.1 \). b) Size scaling \(|m| \sim L^{-\beta/\nu}\) for \(T = 0.118 \) (just above point \(A \), consistent with \(\beta/\nu = 1/8 \). c) Size scaling between point \(B \) and \(C \), consistent with \(\beta/\nu = 3/8 \). d) Size scaling \(m = L^{-\beta/\nu}f((T - T_{cC})L^{1/\nu}) \) for the critical point \(C \); Good scaling collapse obtained for \(T_{cC} \approx 0.17 \) with \(\nu \approx 0.77 \) (from Fig. 4d) and \(\beta/\nu = 3/8 \).

FIG. 6: a) First order transition of \(n_k \) at \(T = 0.1 \). b) Two valued probability distribution \(P(n_k) \) at the abrupt change in a). c) \(P(n_k) \) at the critical point \(A \). The inset shows that the ratio \(P_{\text{max}}/P_{\text{min}} \) is good approximation finite and independent of \(L \). d) Finite size scaling \(\Delta n_k \sim L^{-\beta/\nu} \), is consistent with \(\beta/\nu = 1/4 \). e) \(P(n_k) \) for a large \(p\)-value above the critical point \(D \). The figure illustrates that \(P_{\text{max}}/P_{\text{min}} \to \infty \) with increasing system size. f) Helicity modulus transition at the same \(T \) and \(p \) as in e) indicating a joint first order transition.
VI. STANDARD 2D FFXY MODEL

The usual 2D FFXY model corresponds to the $p = 1$-line in Fig. 3. The critical point $C$ for the 2D FFXY class is the closest multicritical point to the actual phase transitions of the usual 2D FFXY model (compare Fig. 5). The critical point $C$ is characterized by the critical index $\nu \approx 0.77$ and the central charge $c = 1.5$. A single $Z_2$ transition is characterized by $\nu = 1$ and $c = 0.5$. All the earlier papers, in which it was putatively concluded that the 2D FFXY model has a joint transition, the apparent value of $\nu$ was in the interval $0.77 < \nu < 1$ (see table 1 in Ref. 12). In particular in Ref. 12 the values of $\nu$ and $c$ were independently determined and given by $\nu = 0.80(4)$ and $c = 1.61(3)$. Thus the apparent multicritical point for the usual FFXY model appeared to have critical properties inconsistent with a single $Z_2$-transition and with critical $\nu$-values in between a single $Z_2$-transition and the real $\mathcal{U}(1) \otimes Z_2$ multicritical point $C$ for the 2D FFXY class. Furthermore, the closeness of the $\nu$-values and $c$-values ($\nu \approx 0.77$ and $c = 1.5$) for $C$, respectively, $\nu = 0.80(4)$ and $c = 1.61(3)$ obtained for the usual FFXY model in Ref. 12 suggests that the putative multicritical point found for the 2D FFXY model is an artifact of the closeness to the real critical point $C$ for the 2D FFXY class.

The present consensus is that the 2D FFXY model undergoes two separate transition, a KT transition at $T_{KT}$ followed by a $Z_2$-transition at $T_{Z2}$ with $T_{KT} < T_{Z2}$. In particular Korshunov in Ref. 13 has given a general argument which purportedly states that $T_{KT} < T_{Z2}$ should be true not only for the 2D FFXY model, but also for the 2D FFXY class studied in the present work, provided that the interaction is such that its groundstate is the broken symmetry checkerboard state. This is in contradiction with the existence of the multicritical point $C$ at $p < p_c$ (compare Fig. 5) which does correspond to an interaction potential with a checkerboard groundstate. We suggest that the reason for this fallacy of the argument is connected to the closeness to the $(m,n) = (0,0)$-phase.

The most striking feature of the phase transition for 2D FFXY model is the closeness between $T_{KT}$ and $T_{Z2}$. The phase diagram in Fig. 5 gives a scenario for which this feature becomes less surprising: The point is that the chirality transition and the KT-transition merge and cross as a function of $p$ for the 2D GFFXY model. It then becomes more natural that, for some values of $p$, the transitions can be extremely close. The value $p = 1$, which corresponds to the usual FFXY model happens to be such a value.

There are many other $\mathcal{U}(1) \otimes Z_2$-models related to the 2D FFXY model. Although, our results only pertain to the 2D FFXY-class defined in this paper, we note that, to our knowledge, none of the phase diagrams for related models contain a crossing of the KT and an Ising-like transition. In a vast majority, the KT-transition is always at lower temperature than the Ising-like transition or possibly merged. However, in the model in Ref. 12 the situation is reversed with the Ising-like transition below or merging with the KT-transition. Also in this case a crossing is lacking. Because there is no crossing it is notoriously difficult assert whether a merging takes place or whether the two transitions are only extremely close. For example the Ising-XY model was in Ref. 12 found to contain such a line of merged transitions. However, more careful MC simulations in fact suggest that the transitions are extremely close but never merge along this line. The point to note is that for our 2D GFFXY-model the transitions cross from which directly follows that a real merging exists in this case. We believe that this crossing is intimately related to the appearance of the additional groundstate.

VII. FINAL REMARKS

To sum up, we have found that the description of the phase diagram for the 2D FFXY-class of models requires at least three distinct order parameters consistent with the proposed designation $\mathcal{U}(1) \otimes Z_2 \otimes Z_2$: In addition to the usual KT $\mathcal{U}(1)$ transition and the chirality $Z_2$-transition, there is also a defect-density transition with Ising like $Z_2$-character. Within our simple parametrization of the interaction $U(\phi)$, we have found that all combinations of transitions can be realized except two: the single $Z_2$-defect transition and the fully combined $\mathcal{U}(1) \otimes Z_2$-transition. All the others are realized, i.e., the single $Z_2$-chirality transition, the single $\mathcal{U}(1)$-KT transition, the combined $Z_2$-defect and $Z_2$-chirality tran-
sition, the combined $Z_2$-chirality and $U(1)$-KT, and the combined $Z_2$-defect and the $U(1)$-KT transition. Since the GFFXY-model is a subclass of the 2D FFXY-class this means that at least five of the symmetry allowed transitions can be realized. What about the remaining two? Here we speculate that a single $Z_2$-density transition will hardly be realized because it couples too strongly to the other transitions. However, one might imagine that there exists a potential $U(\phi)$ for which the two nearby critical points $A$ and $B$ are merged. This critical point would then correspond to a merged $U(1) \otimes Z_2 \otimes Z_2$-transition with central charge $c = 2$.

We also note that Cristofano et al in Ref.25, argued from general symmetry considerations that the full symmetry of the FFXY-model allows for $U(1) \otimes Z_2 \otimes Z_2$. The present results for the phase diagram of the 2D GFFXY model supports this designation.

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