Ferromagnetism, antiferromagnetism, and the curious nematic phase of $S = 1$ quantum spin systems

Daniel Ueltschi

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, United Kingdom

We investigate the phase diagram of $S = 1$ quantum spin systems with SU(2)-invariant interactions, at low temperatures and in three spatial dimensions. Symmetry breaking and the nature of pure states can be studied using random loop representations. The latter confirm the occurrence of ferro- and antiferromagnetic transitions and the breaking of SU(3) invariance. And they reveal the peculiar nature of the nematic pure states which minimize $\sum_x (S_x^1)^2$.

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I. INTRODUCTION

A fascinating aspect of phase transitions is the notion of symmetry breaking. The set of equilibrium states (infinite-volume Gibbs states) of the system possesses the same symmetries as the Hamiltonian. There is typically a unique Gibbs state at high temperatures which is symmetric. But there may be many different Gibbs states at low temperatures, that are not symmetric. An arbitrary state has a unique decomposition in pure states, where pure states are characterised by their clustering properties (decay of truncated correlation functions) and by the fact that they cannot be decomposed in other states.

The decomposition in pure states is well described in models such as Ising, where the spin-flip symmetry is broken at low temperatures in dimensions two and higher. Periodic equilibrium states can be written as the convex combination of exactly two periodic pure states. A similar behavior is expected in models with continuous symmetry, such as the Heisenberg and the XY models. In the case of the classical XY model, there are rigorous results due to Pfister and Fröhlich. In this article, we study spin 1 quantum systems with SU(2)-invariant interactions. Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice, and let $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^3$ be the Hilbert space of the system. The Hamiltonian is

$$H_\Lambda = - \sum_{\langle x,y \rangle} (J_1 S_x^1 \cdot S_y^1 + J_2 (S_x^2 \cdot S_y^2)^2),$$  \hspace{1cm} (1)

where the sum is over nearest-neighbors $x, y \in \Lambda$; the spin operators $S_x^i, x \in \Lambda, i = 1, 2, 3$ satisfy $[S_x^1, S_x^2] = i \delta_{x,y} S_x^3$ (and further cyclic relations) and $(S_x^1)^2 + (S_x^2)^2 + (S_x^3)^2 = 2$. The low temperature phase diagram for $d \geq 3$ has been investigated by several authors and it is expected to split in four regions with ferromagnetic; spin nematic; antiferromagnetic; and staggered nematic phases. They are separated by lines where the model has the larger SU(3) symmetry. See Fig. 1 for an illustration.

There exist mathematically rigorous results that provide a few solid pillars of understanding. The presence of Néel order has been proved for $J_1 < 0, J_2 = 0$ and $d \geq 3$ by Dyson, Lieb, and Simon using the method of infrared bounds and reflection positivity proposed by Fröhlich, Simon, and Spencer. This result can be straightforwardly extended to small $J_2 > 0$. Quadrupolar order is proved in the spin nematic region for $0 \leq J_1 \leq \frac{1}{2} J_2$ and $d \geq 5$. For $J_1 \lesssim \frac{1}{2} J_2$, the result also holds for $d \geq 3$. The system has actually Néel order in the direction $J_1 = 0, J_2 > 0$. Occurrence of quadrupolar order has recently been proved for $J_1 \lesssim 0, J_2 > 0$ in $d \geq 6$. The system should have the stronger Néel order, though.

The goal of this article is to write down the pure equilibrium states explicitly, and to provide evidence that all (periodic) pure states have been identified. We restrict ourselves to $J_2 \geq 0$ where the model possesses probabilistic representations; quantum spin correlations are given by certain random loop correlations. In dimensions three and higher, it has been recently understood that the joint distribution of the lengths of long loops takes a universally simple form, the Poisson-Dirichlet distribution, and this allows to calculate the expectation of certain

FIG. 1: Phase diagram of the spin 1 quantum model with SU(2)-invariant interactions. There are four phases, separated by lines where the model has the higher SU(3) symmetry.
long-range two-point functions. They can also be calculated using identities from symmetry breaking. These two methods are quite independent and since they must give the same result, they provide a non-trivial test of our understanding.

This strategy has been successfully employed for spin $\frac{1}{2}$ quantum Heisenberg and XY models and in loop representations of $O(n)$ models. There is in fact little doubt regarding the nature of pure states and of symmetry breaking in these models. A very interesting aspect of the study of Nahum, Chalker, et al. is to give expressions for the moments of long loops and therefore to confirm the Poisson-Dirichlet distribution.

The results of the present article are explained in Section III and the heuristics behind the joint distribution of the lengths of long loops can be found in Section IV. It is found that the nature of ferromagnetic and antiferromagnetic phases is as expected. The nature of SU(3) symmetry breaking is less direct, but in retrospect it is also quite expected. The quantum nematic phase, on the other hand, turns out to be surprising.

II. NATURE OF PURE STATES

Let $\langle \cdot \rangle_{H_A}$ denote the finite-volume Gibbs state with Hamiltonian $H_A$, where the expectation of the observable $A$ is

$$\langle A \rangle_{H_A} = \frac{1}{Z_A} \text{Tr} A e^{-\beta H_A},$$

with $Z_A = \text{Tr} e^{-\beta H_A}$ the partition function. We let $\langle \cdot \rangle$ denote its infinite-volume limit. Our goal is to identify the pure states $\langle \cdot \rangle_\vec{a}$ such that

$$\langle \cdot \rangle = \int \langle \cdot \rangle_\vec{a} \, d\nu(\vec{a}),$$

where $d\nu$ is a probability measure on the set of possible values for $\vec{a}$. We describe separately the ferromagnetic, broken SU(3), spin nematic, broken staggered SU(3), and antiferromagnetic phases. We do not discuss the staggered nematic phase which remains largely mysterious.

A. The ferromagnetic phase

This is the simplest situation, valid when parameters satisfy $J_1 > 0$ and $J_2 < J_1$ (see Fig. 1). To each unit vector $\vec{a} \in \mathbb{R}^3$, there corresponds the ferromagnetic pure state

$$\langle \cdot \rangle_\vec{a}^{\text{ferro}} = \lim_{h \to 0^+} \lim_{\Lambda \to \infty} \langle \cdot \rangle_{H_A - h \sum_x \vec{a} \cdot \vec{S}_x}. \quad (4)$$

Let $S^2$ denote the unit sphere in $\mathbb{R}^3$ and let $d\nu$ be the uniform probability measure. The infinite-volume state $\langle \cdot \rangle$ should satisfy the identity

$$\langle \cdot \rangle = \int_{S^2} \langle \cdot \rangle_\vec{a}^{\text{ferro}} \, d\nu(\vec{a}). \quad (5)$$

We can apply this decomposition to the special case of the two-point function $\langle S^3_x S^3_y \rangle$ with $x, y$ far apart. We have

$$\langle S^3_x S^3_y \rangle = \frac{1}{3} \langle S^3_x \cdot S^3_y \rangle \overset{(a)}{=} \frac{1}{4} \int_{S^2} \langle S^3_x \cdot S^3_y \rangle_\vec{a}^{\text{ferro}} \, d\nu(\vec{a})$$

$$= \frac{1}{3} \langle S^3_x \cdot S^3_y \rangle \overset{(b)}{=} \frac{1}{3} \sum_{i=1}^3 \langle S^3_i \rangle^{\text{ferro}} \langle S^3_i \rangle^{\text{ferro}} \overset{(c)}{=} \frac{1}{3} \langle \langle S^3_i \rangle^{\text{ferro}} \rangle^2 \overset{(d)}{=} \frac{1}{3} \langle \langle S^3_i \rangle^{\text{ferro}} \rangle^2.$$  

We have used

(a) the conjecture Eq. 5;

(b) $\vec{S}_x \cdot \vec{S}_y$ is SU(2)-invariant;

(c) pure states are clustering, hence factorization when $\|x - y\| \gg 1$;

(d) translation invariance and $\langle S^3_i \rangle^{\text{ferro}} = 0$ for $i = 1, 2$; the latter follows from rotation invariance of the pure state around $e_3$.

We provide independent evidence for the identity using the random loop representation and the conjecture about the joint distribution of the lengths of long loops, see Subsection IV C.

B. Breaking SU(3) invariance

When $J_1 = J_2$ the Hamiltonian has the larger SU(3) symmetry. Indeed, it can be checked that

$$\vec{S}_x \cdot \vec{S}_y + \langle \vec{S}_x \cdot \vec{S}_y \rangle^2 = T_{xy} + 1, \quad (7)$$

where $T_{xy}$ is the transposition operator in $\mathbb{C}^3 \otimes \mathbb{C}^3$, that is, $T_{xy}[a, b] = [b, a], a, b = -1, 0, 1$. Given any unitary $U$ in $\mathbb{C}^3$, the tensor product $U_A = \otimes_x U$ commutes with all $T_{xy}$’s.

The group SU(3) has eight dimensions and it is generated e.g. by the Gell-Mann matrices $\lambda^1, \ldots, \lambda^8$. All matrices have empty diagonal except $\lambda^3$ and $\lambda^8$:

$$\lambda^3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (8)$$

Another property that is relevant for our purpose is the identity

$$\sum_{i=1}^8 \lambda^i_x \lambda^i_y = 2T_{xy} - \frac{2}{3}, \quad (9)$$
which shows that $\sum_{x} \lambda^i_x \lambda^i_y$ is SU(3)-invariant. The pure states are given by

$$
\langle \cdot \rangle_{\tilde{a}} = \lim_{\nu \to 0+} \lim_{\Lambda \to 2^n} \langle \cdot \rangle_{H_{\Lambda} + h \sum_x \tilde{a} \tilde{\lambda}_x}.
$$

(10)

Here, $\tilde{a}$ is a vector in $\mathbb{S}^7$, the unit sphere in $\mathbb{R}^8$. Let $d\nu$ be the uniform probability measure on $\mathbb{S}^7$. The decomposition of the SU(3)-invariant Gibbs state is

$$
\langle \cdot \rangle = \int_{\mathbb{S}^7} \langle \cdot \rangle_{\tilde{a}} d\nu(\tilde{a}).
$$

(11)

The two-point function can be calculated as in the ferromagnetic case, Eqs (6):

$$
\langle S_x^3 S_y^3 \rangle = \frac{1}{8} \left( \sum_{i=1}^{8} \lambda^i_x \lambda^i_y \right) = \frac{1}{8} \int_{\mathbb{S}^7} \left( \sum_{i=1}^{8} \lambda^i_x \lambda^i_y \right) d\nu(\tilde{a})
$$

$$
= \frac{1}{8} \left( \sum_{i=1}^{8} \lambda^i_x \lambda^i_y \right)_{\tilde{e}_3} = \frac{1}{8} \left( \sum_{i=1}^{8} \left( \lambda^i_x \right)^2 \right)_{\tilde{e}_3}
$$

$$
= \frac{1}{8} \left( \langle \lambda^3_x \rangle_{\tilde{e}_3} \right)^2 + \frac{1}{8} \left( \langle \lambda^8_x \rangle_{\tilde{e}_3} \right)^2.
$$

(12)

We used the clustering property of pure states when $\|x - y\| \to \infty$. We have $\langle \lambda^i_x \rangle_{\tilde{e}_3} = 0$ if $i \neq 3, 8$, and $\langle \lambda^3_x \rangle_{\tilde{e}_3} = \frac{1}{\sqrt{3}} \langle \lambda^3_x \rangle_{\tilde{e}_3}$. We obtain

$$
\langle S_x^3 S_y^3 \rangle = \frac{1}{6} \left( \langle \lambda^3_x \rangle_{\tilde{e}_3} \right)^2.
$$

(13)

We check this identity using the random loop representation in Subsection IV C.

The matrix $\lambda^8$ is peculiar, as it is not unitarily equivalent to the other seven matrices. It is worth checking that the corresponding pure state gives the same result. The second line of Eq. (12) becomes

$$
\ldots = \frac{1}{8} \left( \sum_{i=1}^{8} \lambda^i_x \lambda^i_y \right)_{\tilde{e}_3} = \frac{1}{8} \left( \sum_{i=1}^{8} \left( \lambda^i_x \right)^2 \right)_{\tilde{e}_3}
$$

$$
= \frac{1}{8} \left( \langle \lambda^3_x \rangle_{\tilde{e}_3} \right)^2 + \frac{1}{8} \left( \langle \lambda^8_x \rangle_{\tilde{e}_3} \right)^2.
$$

(14)

It turns out that $\langle \lambda^8_x \rangle_{\tilde{e}_3} = \frac{1}{\sqrt{3}} \langle \lambda^1_x \rangle_{\tilde{e}_3} = \frac{1}{\sqrt{3}} \langle \lambda^3_x \rangle_{\tilde{e}_3}$. This can indeed be seen using the random loop representation. We recover the identity (13).

C. The curious spin nematic phase

In the classical model with $J_1 = 0, J_2 > 0$, there is a spin nematic phase where the classical spins are aligned or anti-aligned. This suggests that pure states can be defined using an external field of the form $-\sum_x (\tilde{a} \cdot \tilde{S}_x^2)$. The quantum system is very different for two main reasons. First, the direction $J_1 = 0, J_2 > 0$ is not nematic but it has Néel order its broken staggered SU(3) invariance is explained in the next subsection. The second reason is that for $J_2 > J_1 > 0$, pure states are obtained using the external field $+ \sum_x (\tilde{a} \cdot \tilde{S}_x^2)$ with a “+” sign. Rather than projecting onto the eigenstates of $S_x^2$, with eigenvalues $\pm 1$, we project onto the eigensubspace with value $0$. This is rather surprising and this signals a purely quantum phenomenon.

Let $\mathbb{S}^2_+$ denote the unit hemisphere in $\mathbb{R}^3$ with nonnegative third coordinates. Given $\tilde{a} \in \mathbb{S}^2_+$, the corresponding pure state is

$$
\langle \cdot \rangle_{\tilde{a}} = \lim_{\nu \to 0+} \lim_{\Lambda \to 2^n} \langle \cdot \rangle_{H_{\Lambda} + h \sum_x (\tilde{a} \cdot \tilde{S}_x)^2}.
$$

(15)

With $d\nu$ the uniform probability measure on $\mathbb{S}^2_+$, the nematic decomposition reads

$$
\langle \cdot \rangle = \int_{\mathbb{S}^2_+} \langle \cdot \rangle_{\tilde{a}} d\nu(\tilde{a}).
$$

(16)

The spin-spin correlation function is not a suitable order parameter for the nematic phase. Indeed, it certainly has exponential decay, as suggested by its random loop representation; see Eq. (6) below. So we rather pick the observable for quadrupolar order:

$$
A_2 = \langle S_x^3 S_y^3 \rangle - \frac{2}{3}
$$

(17)

Notice that $\langle A_x \rangle = 0$. The calculation of the quadrupolar two-point function can be done in the nematic phase as follows.

$$
\langle A_x A_y \rangle = \int_{\mathbb{S}^2_+} \langle A_x A_y \rangle_{\tilde{a}} d\nu(\tilde{a})
$$

$$
= \int_{\mathbb{S}^2_+} \left( \langle S_x^3 S_y^3 - \frac{2}{3} \rangle_{\tilde{a}} \right)^2 d\nu(\tilde{a})
$$

(18)

The second identity is due to clustering when $\|x - y\| \gg 1$ and the last identity follows by rotating the observable rather than the state. Next, we have

$$
\langle (\tilde{a} \cdot \tilde{S}_x^2)^2 - \frac{2}{3} \rangle_{\tilde{e}_3} = \sum_{i,j=1}^{3} a_i a_j \langle S_x^i S_x^j \rangle_{\tilde{e}_3} - \frac{2}{3}
$$

$$
= \sum_{i=1}^{3} a_i^2 \langle (S_x^3)^2 - \frac{2}{3} \rangle_{\tilde{e}_3}.
$$

(19)

We now use the following identities:

$$
\langle (S_x^3)^2 - \frac{2}{3} \rangle_{\tilde{e}_3} = -2 \langle (S_x^1)^2 - \frac{2}{3} \rangle_{\tilde{e}_3} = -2 \langle (S_x^2)^2 - \frac{2}{3} \rangle_{\tilde{e}_3}.
$$

(20)

The second identity is obvious by symmetry, but the first identity does not seem immediate. It can be obtained using the random loop representation. We obtain

$$
\langle A_x A_y \rangle = \frac{4}{5} \left( \langle S_x^3 \rangle^2 - \frac{2}{3} \right)_{\tilde{e}_3}^2.
$$

(21)
This identity is verified in Subsection IV C using the loop representation.

It is natural to wonder about the states \( \langle i \rangle_{\tilde{a}} \) obtained with external field \(-\sum_{x} (\tilde{a} \cdot \vec{S}_x)^2\). If these were the pure states, one should expect the decomposition \( \langle \cdot \rangle = \int_{S^2} \langle \cdot \rangle_{\tilde{a}} d\nu(\tilde{a}). \) However, the calculations of \( \langle A_x A_y \rangle \) gives a result that is incompatible with the random loop calculations, so this has to be discarded. If \( \langle i \rangle_{\tilde{a}} \) are not pure states, then they should be linear combinations of the nematic states. And indeed, it appears that

\[
\langle i \rangle_{\tilde{a}} = \frac{1}{2\pi} \int_{S^2} \langle \cdot \rangle_{\tilde{b}}^{\text{nem}} d\tilde{b}. \tag{22}
\]

In the case of one-site observables this can be checked using the following identity, that holds for all matrices \( A \in \mathbb{C}^3 \):

\[
\frac{1}{2\pi} \int_{S^2} \langle 0 | U_{\tilde{b}} A U_{\tilde{b}}^\dagger | 0 \rangle d\tilde{b} = \frac{1}{2} \text{Tr} \left( S^3 \right)^2 A. \tag{23}
\]

Here, we set \( U_{\tilde{b}} = e^{i \frac{\pi}{4} (\tilde{b} \times \vec{c}_3) \cdot \vec{S}} \). The identity (22) can certainly be verified for more general many-site observables.

### D. Breaking staggered SU(3) invariance

In the case \( J_1 = 0, J_2 > 0 \), the interaction is equivalent to the projector onto the spin singlet, that is, onto the one-dimensional eigensubspace of \( ||S_x^2 + S_y^2|| \) with eigenvalue 0. Let \( P_{\text{singlet}}^{xy} \) denote the projector in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) whose matrix elements are the Clebsch-Gordon coefficients

\[
\langle a, b | P_{xy}^{\text{singlet}} | c, d \rangle = \frac{1}{3} (-1)^{a-c} \delta_{a,-b} \delta_{c,-d}. \tag{24}
\]

where \( a, b, c, d \in \{-1, 0, 1\} \). Then

\[
(S_x, S_y)^2 = 3 P_{xy}^{\text{singlet}} + 1. \tag{25}
\]

Given any unitary matrix \( U \in \mathbb{C}^3 \), let \( \overline{U} \) denote the matrix with elements

\[
\overline{U}_{a,b} = (-1)^{a-b} U_{a,-b}. \tag{26}
\]

Here, \( U_{-,a,-b} \) is the complex conjugate of \( U_{a,b} \). Notice that \( \overline{\tilde{U}} = U \). One can check that

\[
P_{xy}^{\text{singlet}} U \otimes \overline{U} = U \otimes \overline{U} P_{xy}^{\text{singlet}} = P_{xy}^{\text{singlet}}. \tag{27}
\]

As a consequence, the Hamiltonian \( H_A \) has SU(3) invariance whenever the lattice \( A \) is bipartite. More precisely, if \( A_A, A_B \) are the two sublattices, the unitary operator on \( \otimes_{x \in A} \mathbb{C}^3 \) is

\[
U_A = \otimes_{x \in A_A} U \otimes_{x \in A_B} \overline{U}. \tag{28}
\]

On non-bipartite lattices, the Hamiltonian has only SU(2) invariance and the low temperature phase is the spin nematic phase discussed in the previous subsection.

In the rest of this subsection we restrict to the bipartite lattice \( Z^d, d \geq 3 \). The pure states can be defined using staggered fields. Given a unit vector \( \tilde{a} \in \mathbb{R}^8 \), let

\[
Q_{\tilde{a}}^A = \sum_{x \in A_A} \tilde{a} \cdot \vec{\lambda}_x + \sum_{x \in A_B} \tilde{a} \cdot \vec{\lambda}_x. \tag{29}
\]

As in Subsection II B, \( \vec{\lambda}_x = (\lambda_1^x, ..., \lambda_8^x) \) is the vector of Gell-Man matrices acting on the site \( x \). We also defined \( \vec{\lambda}_x = (\vec{\lambda}_1^x, ..., \vec{\lambda}_8^x) \). Then we set

\[
\langle \cdot \rangle_{\tilde{a}} = \lim_{h \rightarrow 0+} \lim_{A \rightarrow Z^d} \langle \cdot \rangle_{H_A - h Q^\tilde{a}_A}. \tag{30}
\]

The decomposition of the symmetric Gibbs state reads

\[
\langle \cdot \rangle = \int_{S^7} \langle \cdot \rangle_{\tilde{a}} d\nu(\tilde{a}). \tag{31}
\]

The justification of this formula mirrors that of Eq. (11) in the non-staggered SU(3) situation. Without entering details, let us note that

\[
3 P_{xy}^{\text{singlet}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} = \frac{1}{2} \sum_{i=1}^{8} \lambda^i \otimes \bar{\lambda}^i + \frac{1}{2}. \tag{32}
\]

It follows that \( \vec{\lambda}_x \cdot \vec{\lambda}_{-y} \) is left invariant by \( U_A \) whenever \( x \in A_A, y \in A_B \). From Eq. (17), \( \vec{\lambda}_x \cdot \vec{\lambda}_{-y} \) is also left invariant whenever \( x, y \) belong to the same sublattice. The loop representation differs from the one for \( J_1 = J_2 \) but the joint distribution of macroscopic loops is the same PD(3).

### E. The antiferromagnetic phase

The pure states are indexed by unit vectors \( \tilde{a} \in \mathbb{R}^3 \). If the lattice is bipartite they can be defined using the usual staggered fields, namely

\[
\langle \cdot \rangle_{\tilde{a}}^{\text{AF}} = \lim_{h \rightarrow 0+} \lim_{A \rightarrow Z^d} \langle \cdot \rangle_{H_A - h \sum_{x \in A} (\sum_{y} (-1)^{x \cdot y} \tilde{a} \cdot \vec{S}_y)}. \tag{33}
\]

The decomposition reads

\[
\langle \cdot \rangle = \int_{S^2} \langle \cdot \rangle_{\tilde{a}}^{\text{AF}} d\nu(\tilde{a}). \tag{34}
\]

The justification for this formula is quite identical to that of (5) in the ferromagnetic regime. Loops are different but the joint distribution of macroscopic loops is the same PD(2).

Notice that the random loop representation is only valid when the lattice is bipartite. There are problematic signs otherwise. The study of frustrated systems is notoriously difficult.
III. RANDOM LOOP REPRESENTATIONS

The partition function of quantum lattice models can be expanded using the Trotter product formula, or the Duhamel formula, yielding a sort of classical model in one more dimensions. For some models the expansion takes the form of random loops. This turns out to be important for our purpose.

Random loop models were used in studies of the spin $\frac{1}{2}$ Heisenberg ferromagnet\cite{14,15} and of the antiferromagnet\cite{19} Nachtergaele,\cite{19,20,21} and independently Harada and Kawashima\cite{22} have proposed an extension of these representations for the spin 1 model in the upper half of the phase diagram, $J_2 \geq 0$. We describe these representations in details.

A. Transition operators

Consider a system where each site $x \in \Lambda$ hosts two spins $\frac{1}{2}$. The Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Let $\mathcal{S}$ the projector onto the triplet subspace in $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$ S|a, b\rangle = \frac{1}{2}|a, b\rangle + \frac{1}{2}|b, a\rangle, $$

and let

$$ S_x = \mathcal{S} \otimes \mathbb{1}_{\Lambda \setminus \{x\}}, \quad \mathcal{S}_\Lambda = \otimes_{x \in \Lambda} \mathcal{S}. $$

Let $V : \mathbb{C}^3 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ be the operator such that

$$ V^* V = \mathbb{1}_{\mathbb{C}^3}, \quad V V^* = \mathcal{S}. $$

Define the operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by

$$ T^i = V S^i V^*, \quad i = 1, 2, 3, $$

where the $S^i$s are the spin operators in $\mathbb{C}^3$. One can check that the $T^i$s satisfy the following relations:

$$ [T^1, T^2] = i T^3, \quad \text{(and cyclic relations)} $$

$$ (T^1)^2 + (T^2)^2 + (T^3)^2 = 2 \mathcal{S}. $$

These operators can be written with the help of Pauli matrices. Namely,

$$ T^i = (\sigma^i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^i) \mathcal{S} $$

$$ = 2 \mathcal{S} (\sigma^i \otimes \mathbb{1}) \mathcal{S}. $$

Notice that $\sigma^i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^i$ commutes with $\mathcal{S}$.

Using the operators above, we define the following Hamiltonian on $\mathcal{H}_\Lambda'$:

$$ H_\Lambda' = - \sum_{\langle x, y \rangle} (J_1 \vec{T}_x \cdot \vec{T}_y + J_2 (\vec{T}_x \cdot \vec{T}_y)^2). $$

The partition function and the Gibbs states are defined by

$$ Z'_\Lambda = \text{Tr} \mathcal{H}'_\Lambda \mathcal{S} e^{-\beta H'_\Lambda}, $$

$$ \langle A' \rangle_{H'_\Lambda} = \frac{1}{Z'_\Lambda} \text{Tr} A' \mathcal{S} e^{-\beta H'_\Lambda}. $$

The correspondence between the spin 1 model $H_\Lambda$ and the new model $H'_\Lambda$ is then

$$ \langle A \rangle_{H_\Lambda} = \langle V_\Lambda A V_\Lambda^* \rangle_{H'_\Lambda}. $$

We need to rewrite the Hamiltonian $H'_\Lambda$ using suitable operators in order to get the loop representations. We use symbols that are suggestive of the two sites of two spins each, and of the links due to transitions. We also use the notation $|a, b\rangle$ for an element in the one-site Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $|a, b\rangle \otimes |c, d\rangle$ for an element in the two-site Hilbert space. With these conventions in mind, let

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = |c, b\rangle \otimes |a, d\rangle, $$

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = |c, d\rangle \otimes |a, b\rangle. $$

These are transposition operators for one or two spins. We actually need the symmetrized operators

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = |c, b\rangle \otimes |a, d\rangle, $$

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = |c, d\rangle \otimes |a, b\rangle. $$

They can be written in terms of spin operators as follows.

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = \frac{1}{2} \vec{T}_x \cdot \vec{T}_y + \frac{1}{2} \mathcal{S}_{x,y}, $$

$$ \langle \big[ \big[ a, b \otimes |c, d\rangle \big] \big] \rangle = \vec{T}_x \cdot \vec{T}_y + (\vec{T}_x \cdot \vec{T}_y)^2 - \mathcal{S}_{x,y}. $$

Next, let us consider the operators that are related to spin singlets of spins at distinct sites.

$$ \langle a' , b' \otimes c', d' \big[ \big[ |a, b\rangle \otimes |c, d\rangle \big] \big] \rangle = (-1)^{a' - a} (-1)^{b' - b} \delta_{a,-a} \delta_{b,-c} \delta_{b',-d} \delta_{c',-a}, $$

$$ \langle a' , b' \otimes c', d' \big[ \big[ |a, b\rangle \otimes |c, d\rangle \big] \big] \rangle = (-1)^{b' - b} \delta_{a,a'} \delta_{d,d'} \delta_{b,-c} \delta_{c',-a}. $$

We again need the symmetrized operators

$$ \mathcal{S}_{x,y} = \mathcal{S}_{x,y}, \quad \mathcal{S}_{x,y} = \mathcal{S}_{x,y}. $$

In terms of spin operators, they are given by

$$ \mathcal{S}_{x,y} = (\vec{T}_x \cdot \vec{T}_y)^2 - \mathcal{S}_{x,y}, $$

$$ \mathcal{S}_{x,y} = -\frac{1}{2} \vec{T}_x \cdot \vec{T}_y + \frac{1}{2} \mathcal{S}_{x,y}. $$

We can also consider the operator

$$ \langle a' , b' \otimes c', d' \big[ \big[ |a, b\rangle \otimes |c, d\rangle \big] \big] \rangle = (-1)^{b' - b} \delta_{a,a'} \delta_{d,d'} \delta_{b,-c} \delta_{b',-c'}. $$
Its symmetrized version satisfies
\[
\mathcal{L}_\Lambda = -\frac{1}{2} \mathbf{T}_x \cdot \mathbf{T}_y - (\mathbf{T}_x \cdot \mathbf{T}_y)^2 + \frac{3}{2} \mathcal{S}_{x,y}.
\] (51)

The correspondence between the transition operators and the parameters \(J_1, J_2\) is illustrated in Fig. 2. The Hamiltonian \(H'_\Lambda\) can be written as linear combinations of the transition operators. In order to avoid future sign problems, we choose the following linear combinations. For \(0 \leq J_2 \leq J_1\),
\[
H'_\Lambda = -\sum_{(x,y)} \left(2(J_1 - J_2) \mathbf{T}_x \cdot \mathbf{T}_y + J_2 \mathcal{S}_{x,y} - (J_1 + 2J_2) \mathcal{S}_{x,y}\right);
\] (52)
for \(0 \leq J_1 \leq J_2\),
\[
H'_\Lambda = -\sum_{(x,y)} \left((J_2 - J_1) \mathbf{T}_x \cdot \mathbf{T}_y + J_1 \mathcal{S}_{x,y} + J_2 \mathcal{S}_{x,y}\right);
\] (53)
and for \(J_1 < 0, J_2 \geq 0\),
\[
H'_\Lambda = -\sum_{(x,y)} \left(-2J_1 \mathbf{T}_x \cdot \mathbf{T}_y + J_2 \mathcal{S}_{x,y} + (J_1 + 2J_2) \mathcal{S}_{x,y}\right).
\] (54)

**B. Nachtergaele’s loop representations**

Recall that a Poisson point process on the interval \([0, 1]\) describes the occurrence of independent events at random times. Let \(u \geq 0\) the intensity of the process. The probability that an event occurs in the infinitesimal interval \([t, t + dt]\) is \(u dt\); disjoint intervals are independent. Poisson point processes are relevant to us because of the following expansion of the exponential of matrices. Namely,
\[
\exp\left\{ \sum_{i=1}^{k} (A_i - 1) \right\} = \int \rho(d\omega) \prod_{(i,t) \in \omega} A_i,
\] (55)
where \(\rho\) is a Poisson point process on \(\{1, \ldots, k\} \times [0, 1]\) with intensity \(u\), and the product is over the events of the realization \(\omega\) in increasing times. We actually consider an extension where the time intervals are labeled by the edges of the lattice, and where two kinds of events occur with respective intensities \(u\) and \(v\). Then
\[
\exp\left\{ -\sum_{(x,y)} (uA^{(1)}_{xy} + vA^{(2)}_{xy} - u - v) \right\} = \int \rho(d\omega) \prod_{(x,y,i,t) \in \omega} A^{(i)}_{xy},
\] (56)
Let \(\sigma = \otimes_{x \in \Lambda} |\sigma_x^{(1)}, \sigma_x^{(2)}\rangle\), \(\sigma_x^{(i)} = \pm \frac{1}{\sqrt{2}}\), denote the elements of the basis of \(\mathcal{H}_\Lambda\) where all \((\sigma^1 \otimes \mathbb{I})_x\) and \((\mathbb{I} \otimes \sigma^3)_x\) are diagonal. Then
\[
\text{Tr} e^{-\sum_{(x,y)} (uA^{(1)}_{xy} + vA^{(2)}_{xy} - u - v)} = \int \rho(d\omega) \sum_{|\sigma_1, \ldots, |\sigma_k\rangle} \langle \sigma_k | A^{(i_k-1)}_{xy_k-1} \ldots | \sigma_2 | A^{(i_1)}_{xy_1} | \sigma_1\rangle.
\] (57)
Here, \((x_1, y_1, i_1), \ldots, (x_k, y_k, i_k)\) are the events of the realization \(\omega\) in increasing times. Their number \(k\) is random.

Let us examine the case of the Hamiltonian \(H'_\Lambda\) for \(0 \leq J_2 \leq J_1\), given in Eq. (52). We can neglect the term \((-J_1 + 2S_x)\mathcal{S}_{xy}\) and add \(2J_1 + J_2\) times the identity so that we can use the Poisson expansion formula (56).

Indeed, this may change the partition function, but it does not change the Gibbs states. A realization can then be represented as in Fig. 3. There are two vertical lines on top of each site and neighboring lines are sometimes connected by the transitions. The little boxes represent the action of the symmetrization operator \(\mathcal{S}_x\) in Eq. (55), namely to average over an identity and a transposition. Graphically,
\[
\mathcal{S}_x = 1/2 \parallel + 1/2 \\|
\]
Let \(\mathcal{S}_x\) denote the realization of the Poisson point process. The lattice is one-dimensional here but the representation applies to arbitrary dimensions.
notation \( \rho \) for the corresponding measure. Loop configurations are obtained by connecting the vertical lines according to the transition symbols. The sum over configurations \( |\sigma_1\rangle, \ldots, |\sigma_k\rangle \) amounts to a choice over assignments of a spin value \( \pm \frac{1}{2} \) to each loop. The partition function is then given by

\[
Z'_{\Lambda} = \int \rho(\omega) 2^{L(\omega)},
\]

(58)

where \( |L(\omega)| \) is the number of loops in the realization \( \omega \). A similar expansion can be performed for the correlation functions. We find

\[
\text{Tr}(\sigma^3 \otimes \mathbb{1}_x)(\sigma^3 \otimes \mathbb{1}_y) e^{-\beta H'_N} = \int \rho(\omega) \sum_{\sigma_x(1)} \sigma_y(1). \label{eq:59}
\]

If the realization \( \omega \) does not connect the first lines of the sites \( x \) and \( y \), the sum gives zero. If the first lines of \( x, y \) belong to the same loop, the sum gives \( \frac{1}{2} 2^{|L(\omega)|} \).

Recalling Eqs (40) and (43), we get for \( 0 \leq J_2 \leq J_1 \):

\[
\langle s^3_x s^3_y \rangle H_N = \langle T^3_x T^3_y \rangle H'_N = \mathbb{P}((x, 1) \leftrightarrow (y, 1)), \label{eq:60}
\]

where the right side is the probability, with measure \( \frac{1}{2^{|L|}} d\rho \), that the first lines of \( x \) and \( y \) belong to the same loop.

**FIG. 4:** The realization \( \omega \) and its loop configuration. Here, \( |L(\omega)| = 7 \).

The situation with transition operators \( \mathbb{1} \) and \( \mathbb{1} \) is similar but there are two important differences. Indeed, because of the matrix elements of Eqs (47), we have that

- the spin value in the loop changes sign at those transitions where the vertical direction changes;

- there are extra factors, namely \( e^{i \pi a} \) for transitions of the form \( \mathbb{1} \) and \( e^{-i \pi a} \) for transitions of the form \( \mathbb{1} \), where \( a = \pm \frac{1}{2} \) is the value of the spin at sublattice A.

The factors give 1 when the lattice is bipartite and (i) the transitions \( \mathbb{1} \) and \( \mathbb{1} \) are between distinct sublattices only; (ii) the transpositions are between sites in the same sublattices. Thus it is possible to combine \( \mathbb{1} \) and \( \mathbb{1} \) together, but these cannot be combined with \( \mathbb{1} \) and \( \mathbb{1} \). Actually, it is possible to combine \( \mathbb{1} \) and \( \mathbb{1} \) for the nematic phase. But in this case there exists another, simpler, loop representation where each site has a single line. It is described in the next subsection.

The partition function for the Hamiltonian \( H'_N \) in (54) is again \( Z'_{\Lambda} = \int \rho(\omega) 2^{|L(\omega)|} \). As for correlation functions, (59) holds with the new notion of loops. The sum over \( \sigma : \omega \) gives (i) 0 if the first lines of \( x, y \) do not belong to the same loop; (ii) \( \frac{1}{2} 2^{|L(\omega)|} \) if they belong to the same loop and \( x, y \) belong to the same sublattice; (iii) \( -\frac{1}{2} 2^{|L(\omega)|} \) if they belong to the same loop and \( x, y \) belong to different sublattices. Then we find for \( J_1 \leq 0, J_2 \geq 0 \):

\[
\langle s^3_x s^3_y \rangle H_N = \langle T^3_x T^3_y \rangle H'_N = (-1)^{x-y} \mathbb{P}((x, 1) \leftrightarrow (y, 1)). \label{eq:61}
\]

Let us conclude this subsection by giving two identities. A similar calculation as above gives \( \langle (T^3_x)^2 \rangle H'_N = \frac{1}{4} + \frac{1}{2} \mathbb{P}((x, 1) \leftrightarrow (x, 2)) \). Since the first term is equal to \( \frac{2}{3} \), we find that

\[
\mathbb{P}((x, 1) \leftrightarrow (x, 2)) = \frac{1}{3} \label{eq:62}
\]

for all values of \( \beta \) and of interaction parameters. This should be understood as an effect of the symmetrization operators, which connect the two points with probability 1/3 (if initially disconnected), or disconnect them with probability 2/3 (if initially connected). The factor \( 2^{|L(\omega)|} \) is important. Next, we compute the expression for the quadrupolar correlation function. Recall the operator \( A_x = (S^3_x)^2 - \frac{1}{2} \). We find

\[
\langle A_x A_y \rangle H_N = \frac{1}{36} + \frac{1}{4} \mathbb{P} \left( \begin{array}{c} (x, 1) \hline (y, 1) \end{array} \right) + \frac{1}{4} \mathbb{P} \left( \begin{array}{c} (x, 1) \hline (x, 2) \end{array} \right) + \frac{1}{36} \mathbb{P} \left( \begin{array}{c} (y, 1) \hline (y, 2) \end{array} \right) \label{eq:63}
\]

It seems quite equivalent to the spin-spin correlation function (60), and it would be interesting to work out precise comparison. Notice that Eqs (62) and (63) are valid in the whole region \( J_2 \geq 0 \).

**C. A simpler loop representation for the nematic and SU(3) regions**

For parameters \( 0 \leq J_1 \leq J_2 \), that is, for the nematic phase and for its SU(3) boundaries, there exists a simpler loop model. There is a single vertical line on top of each vertices, and we do not need the symmetrization
operators $\mathcal{S}_x$. Let $d\rho$ denote the measure for a Poisson point process for each edge of $\Lambda$, where “crosses” occur with intensity $u$ and “double bars” occur with intensity $1-u$. Loops are defined by moving upwards, say, and by jumping to a neighbor when a transition occurs; if it is a cross, one continues in the same vertical direction; if it is a double bar, one continues in the opposite direction. The vertical direction has periodic boundary conditions. See Fig. 5 for illustrations.

![Figure 5](image)

**FIG. 5**: The simpler random loop model for parameters $0 \leq J_1 \leq J_2$. Both realizations have $|L(\omega)| = 2$ loops.

This loop model gives a representation for the following Hamiltonian:

$$H_\Lambda = -\sum_{(x,y)} (u\vec{S}_x \cdot \vec{S}_y + (\vec{S}_x \cdot \vec{S}_y)^2 - 2).$$

Namely, the partition function is

$$Z_\Lambda = \text{Tr}_{H_\Lambda} e^{-\beta H_\Lambda} = \int 3^{|L(\omega)|} d\omega.$$

Let $P$ denote the probability with respect to the measure $\frac{1}{Z_\Lambda} 3^{|L(\omega)|} d\omega$. The spin-spin correlation function is given by

$$\langle S_x^i S_y^j \rangle_{H_\Lambda} = \frac{2}{3} \left[ P(x \cap y) - P(x \cap \overline{y}) \right].$$

The first event refers to loops that connect the top of $x$ to the bottom of $y$; the second event refers to loops that connect the top of $x$ with the top of $y$. This expression is actually very interesting and suggests that the spin-spin correlation function has exponential decay in the nematic phase, for all temperatures. The quadrupolar correlation function is given by

$$\langle A_x A_y \rangle_{H_\Lambda} = \frac{2}{9} P(x \leftrightarrow y).$$

### IV. Joint Distribution of the Lengths of Long Loops

#### A. Macroscopic loops and random partitions

Let $L^{(1)}, L^{(2)}, \ldots$ denote the lengths of the loops in decreasing order. Here, the “length” of the loop is by definition the sum of lengths of its vertical components; thus $0 \leq L^{(i)} \leq \beta|\Lambda|$ and $\sum_i L^{(i)} = \beta|\Lambda|$. We expect the largest ones to be of order of the volume, and the smaller ones to be of order of unity. We actually expect that the following limit exists for almost all realizations of loop configurations:

$$\lim_{k \to \infty} \lim_{\Lambda \to \mathbb{Z}^d} \frac{k}{|\Lambda|} L^{(i)} = m(\beta).$$

Further, we expect that the limit $m = m(\beta)$ takes a fixed value. The order of limits is of course important, the result is trivially 1 if the order is reversed. Next, in the limits $\Lambda \to \mathbb{Z}^d$ then $k \to \infty$, the sequence $(L^{(1)}/|\Lambda|, L^{(2)}/|\Lambda|, \ldots)$ is a random partition of $[0,1]$. It turns out that its probability measure can be described explicitly; it is a Poisson-Dirichlet distribution $PD(\theta)$ for a suitable parameter $\theta > 0$.

Aldous conjectured that $PD(1)$ occurs in the random interchange model on the complete graph; this was proved by Schramm, who showed that the time evolution of the loop lengths is described by an effective split-merge process (or “coagulation-fragmentation”). The random interchange model is equivalent to the random loop model of Subsection III C without the factor $3^{|L(\omega)|}$. It was later understood that the Poisson-Dirichlet distribution is also present in three-dimensional systems. This behavior is actually fairly general and concerns many systems where one-dimensional objects have macroscopic size. The distribution $PD(1)$ has been confirmed numerically in a model of lattice permutations and analytically in the related annealed model. More general $PD(\theta)$ have been confirmed numerically in $O(n)$ loop models.

There are several definitions of Poisson-Dirichlet distributions. The simplest one is through the Griffiths-Engen-McCloskey (GEM) distribution. Let $X_1, X_2, \ldots$ be independent identically distributed Beta($\theta$) random variables. That is, each $X_i$ takes value in $[0,1]$ and $P(X > s) = (1-s)^\theta$ for $0 \leq s \leq 1$. The following is a random partition of $[0,1]$:

$$\left( X_1, (1-X_1)X_2, (1-X_1)(1-X_2)X_3, \ldots \right).$$

The corresponding distribution is called GEM($\theta$). After ordering the numbers in decreasing order, the resulting random partition has $PD(\theta)$ distribution.

The probability that two random numbers, chosen uniformly in $[0,1]$, belong to the same element in $PD(\theta)$, can be calculated with GEM($\theta$). They both belong to the $k$th element with probability

$$E((1-X_1)^2 \ldots (1-X_{k-1})^2 X_k^2) = E((1-X_1)^2)^{k-1} E(X_k^2).$$

We have $E((1-X_1)^2) = \theta/((\theta + 2)$ and $E(X_k^2) = 2/(\theta + 1)/(\theta + 2)$. Summing over $k$, we find that the probability that two random numbers belong to the same element is equal to $1/(\theta + 1)$. 

The strategy is to view the random loop measure as the stationary measure of a suitable stochastic process. The effective process on random partitions is a split-merge process with suitable rates. Then its invariant measure is a $PD(\theta)$ distribution with $\theta$ that depends on the rates.

We consider the simpler loop model of Subsection III C and explain later that modifications are minimal for the other loop models. Let $R(\omega, \omega')$ be the transition matrix of the stochastic process, that gives the rate at which $\omega'$ occurs when the configuration is $\omega$. With $C(\omega)$ the number of crosses and $B(\omega)$ the number of double bars, the detailed balance property is

$$\theta^{L(\omega)}(udt)^{C(\omega)}((1-u)dt)^{B(\omega)}R(\omega, \omega') = \theta^{L(\omega')}(udt)^{C(\omega')}(1-u)dt)^{B(\omega')}R(\omega', \omega). \quad (70)$$

We have discretized the interval $[0, \beta]$ with mesh $d\beta$. The following process satisfies the detailed balance property:

- A new cross appears in the interval $\{x, y\} \times [t, dt]$ at rate $u\theta^{1/2}dt$ if its appearance causes a loop to split; at rate $u\theta^{-1/2}dt$ if its appearance causes two loops to merge; and at rate $\theta dt$ if its appearance does not modify the number of loops.
- Same with bars, but with rate $1-u$ instead of $u$.
- An existing cross and double bar is removed at rate $\theta^{1/2}$ if its removal causes a loop to split; at rate $\theta^{-1/2}$ if its removal causes two loops to merge; and at rate $1$ if the number of loops remains constant.

Notice that any new cross or double bar between two loops causes them to merge. When $u=1$, any new cross within a loop causes it to split.

Let $\gamma, \gamma'$ be two macroscopic loops of respective lengths $L, L'$. They are spread all over $\Lambda$ and they interact between one another, and among themselves, in an essentially mean-field fashion. Thus a new cross or double bar that causes $\gamma$ to split appears at rate $\frac{1}{2}c_1\theta^{1/2} L^2/|\Lambda|$; a new cross or double bar that causes $\gamma$ and $\gamma'$ to merge appears at rate $c_1\theta^{-1/2} \frac{LL'}{|\Lambda|}$. The rate for an existing cross or double bar to disappear is $\frac{1}{2}c_2\theta^{1/2} L^2/|\Lambda|$ if $\gamma$ is split, and $c_2\theta^{-1/2} \frac{LL'}{|\Lambda|}$ if $\gamma$ and $\gamma'$ are merged. Consequently, $\gamma$ splits at rate

$$\frac{1}{2}(c_1 + c_2)\theta^{1/2} \frac{L^2}{|\Lambda|} = \frac{1}{2}r_s L^2 \quad (71)$$

and $\gamma, \gamma'$ are merged at rate

$$(c_1 + c_2)\theta^{-1/2} \frac{LL'}{|\Lambda|} = r_m LL'. \quad (72)$$

Because of effective averaging over the whole domain, the constants $c_1$ and $c_2$ are the same for all loops and for both the split and merge events. It follows that the lengths of macroscopic loops satisfy an effective split-merge process, and the invariant distribution is Poisson-Dirichlet with parameter $\theta = r_s/r_m$.

The case $u \in (0, 1)$ is different because loops split with only half the rate above. Indeed, the appearance of a new transition within the loop may just rearrange it: topologically, this is like $0 \leftrightarrow 8$. This yields $PD(\frac{\theta}{2})$. Notice that this cannot happen when $u = 1$, or when $u = 0$ on a bipartite graph.

It follows from these considerations that the macroscopic loops of the model described in Subsection III C have the following Poisson-Dirichlet distributions:

- $PD(3)$ when $J_1 = J_2 > 0$ and when $J_1 = 0, J_2 > 0$;
- $PD(\frac{3}{2})$ when $0 < J_1 < J_2$.

The heuristics for the more complicated loop models of Subsection III B is similar. It helps to reformulate the setting, by considering transitions that connect any vertical lines; this allows to neglect the symmetrization boxes (except at time 0). The discussion for transitions $\frac{\gamma}{3}$ or $\frac{\gamma}{5}$ is then as before and the distribution is $PD(2)$.

The transitions $\frac{\gamma}{3}$ and $\frac{\gamma}{5}$ have multiple connections and the argument is a bit more difficult. But it is possible to obtain an effective split-merge process where two changes occur in succession, with the appropriate rates. Hence $PD(2)$ again. It is essential for the heuristics that, if two vertical lines at the same site belong to macroscopic loops, they are uncorrelated. This seems to fail because of (62). But this holds since symmetrization boxes are discarded.

This heuristic does not apply if the model has only transitions $\frac{\gamma}{4}$ (or $\frac{\gamma}{6}$). Indeed, there are non-trivial correlations between lines at the same site. It should be possible to modify the argument so as to identify the correct Poisson-Dirichlet. But this is not necessary since we can use the simpler loop model.

C. Long-range correlation in models of random loops

At low temperatures, our large system contains many loops, that are either macroscopic or microscopic. (Mesoscopic loops have vanishing density.) While the lengths of macroscopic loops varies, their total density is essentially fixed and is denoted $m(\beta)$. The probability that two distant points belong to the same loop is then equal to the probability that they both belong to macroscopic loops $m(\beta)^2$, times the probability that they belong to the same partition element. If the random partition has Poisson-Dirichlet distribution with parameter $\theta$, this probability is $1/(\theta + 1)$ and we find therefore

$$P(x \leftrightarrow y) = \frac{m(\beta)^2}{\theta + 1}. \quad (73)$$
Let us insist that this equation is meant as an identity in the limits $\Lambda \to Z^d$ and $\|x - y\| \to \infty$. We can now calculate the long-distance correlations in the various phases.

In the ferromagnetic phase, we have $\theta = 2$. Combining Eqs (60) and (73), we get that the left side of (6) is equal to \( \frac{1}{6} m(\beta)^2 \); this confirms the SU(3) decomposition (11).

Next is the spin nematic phase, for $0 \leq J_1 \leq J_2$. We again use the simpler loop representation; the correct distribution is now PD(3). The quadrupolar correlation function in the symmetric Gibbs state gives

\[
\langle A_y A_y \rangle_H = \frac{3}{4} \mathbb{P}(x \leftrightarrow y) \lim_{\Lambda \to Z^d, \|x - y\| \to \infty} \frac{1}{6} m(\beta)^2.
\]

This verifies the staggered SU(3) decomposition (8). This confirms that the spin nematic states are also true of the ferromagnetic decomposition.

To summarize, we have given a precise characterization of the symmetry breaking that occurs in a spin 1 quantum model at low temperatures and in dimensions three and higher. We used random loop representations to confirm the decompositions into pure Gibbs states. We have only considered the case $J_2 \geq 0$; the next challenge is to clarify the phase diagram for $J_2 < 0$, and in particular the staggered spin nematic phase.

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