LONG TIME DECAY FOR 3D-NSE IN GEVREY-SOBOLEV SPACES

JAMEL BENAMEUR AND LOTFI JLALI

ABSTRACT. In this paper we prove, if $u$ is a global solution to Navier-Stokes equations in the Sobolev-Gevrey spaces $H^s_{a,\sigma}(\mathbb{R}^3)$, then $\|u(t)\|_{H^s_{a,\sigma}}$ decays to zero as time goes to infinity. Fourier analysis is used.

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1. Introduction

The 3D incompressible Navier-Stokes equations are given by:

$$
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u &= -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
u u(0, x) &= u^0(x) \quad \text{in } \mathbb{R}^3,
\end{align*}
$$

where, we suppose that the fluid viscosity $\nu = 1$, and $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. $(u, \nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$, and $u^0 = (u^0_1(x), u^0_2(x), u^0_3(x))$ is a given initial velocity. If $u^0$ is quite regular, the divergence free condition determines the pressure $p$.

We define the Sobolev-Gevrey spaces as follows; for $a, s \geq 0$ and $\sigma > 1$,

$$
H^s_{a,\sigma}(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3); \ e^{a|D|^{1/\sigma}} f \in H^s(\mathbb{R}^3) \}.
$$

It is equipped with the norm

$$
\|f\|_{H^s_{a,\sigma}} = \|e^{a|D|^{1/\sigma}} f\|_{H^s}
$$

and its associated inner product

$$
\langle f, g \rangle_{H^s_{a,\sigma}} = \langle e^{a|D|^{1/\sigma}} f, e^{a|D|^{1/\sigma}} g \rangle_{H^s}.
$$

There are several authors who have studied the behavior of the norm of the solution to infinity in the different Banach spaces. For example: Wiegner proved in [9] that the $L^2$ norm of the solutions vanishes for any square integrable initial
data, as times goes to infinity and gave a decay rate that seems to be optimal for a class of initial data. In [8] [10] M.E.Schonber and M.Wieger derived some asymptotic properties of the solution and its higher derivatives under additional assumptions on the initial data. In [5] J.Benameur and R.Selmi proved that if $u$ be a Leray solution of (2d $\Delta$ NSE) then $\lim_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R}^2)} = 0$. In [7] for the critical Sobolev spaces $\dot{H}^s$, I.Gallagher, D.Iftimie and F.Planchon proved that $\|u(t)\|_{\dot{H}^s}$ goes to zero at infinity. In [2] J.Benameur proved if $u \in C([0,\infty), X^{-1}(\mathbb{R}^3))$ be a global solution to 3D Navier-Stokes equation, then $\|u(t)\|_{X^{-1}}$ decay to zero as times goes to infinity.

We state our main result.

**Theorem 1.1.** Let $a > 0$ and $\sigma > 1$. Let $u \in C([0, \infty), H^1_{a,\sigma}(\mathbb{R}^3))$ be a global solution to (NSE) system. Then

$$
\lim_{t \to \infty} \|u(t)\|_{H^1_{a,\sigma}} = 0.
$$

**Remark 1.2.** The existence of local solutions to (NSE) was studied in a recent paper [1].

The paper is organized in the following way: In section 2, we give some notations and important preliminary results. The section 3 is devoted to prove that, if $u \in C(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global solution to (NSE), then $\|u(t)\|_{H^s}$ decays to zero as time goes to infinity. This proof uses the fact that

$$
\lim_{t \to \infty} \|u(t)\|_{H^s} = 0
$$

and the energy estimate

$$
\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2.
$$

In section 4, we generalize the results of Foias-Temam (see [6]) to $\mathbb{R}^3$. In section 5, we prove the main theorem. This proof is based on the obtained results in sections 3 and 4.

2. Notations and Preliminaries Results

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as

$$
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix,\xi)f(x)dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
$$

- The inverse Fourier formula is

$$
\mathcal{F}^{-1}(g)(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \exp(i\xi, x)g(\xi)d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.
$$

- For $s \in \mathbb{R}$, $H^s(\mathbb{R}^3)$ denotes the usual non-homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle ., .\rangle_{H^s}$ denotes the usual scalar product on $H^s(\mathbb{R}^3)$.

- For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^3)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle ., .\rangle_{\dot{H}^s}$ denotes the usual scalar product on $H^s(\mathbb{R}^3)$.

- The convolution product of a suitable pair of functions $f$ and $g$ on $\mathbb{R}^3$ is given by

$$
(f \ast g)(x) := \int_{\mathbb{R}^3} f(y)g(x-y)dy.
$$

- If $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ are two vector fields, we set

$$
f \otimes g := (g_1f, g_2f, g_3f),
$$

and

$$
\text{div} (f \otimes g) := (\text{div} (g_1f), \text{div} (g_2f), \text{div} (g_3f)).
$$
2.2. Preliminary results.

**Lemma 2.1.** (See [1]) Let \((s, t) \in \mathbb{R}^2\), such that \(s < \frac{3}{2}\) and \(s + t > 0\). Then, there exists a constant \(C > 0\), such that

\[
\|uv\|_{H^{s+t} - \frac{3}{2}(\mathbb{R}^3)} \leq C(\|u\|_{H^s(\mathbb{R}^3)} \|v\|_{H^t(\mathbb{R}^3)} + \|u\|_{H^s(\mathbb{R}^3)} \|v\|_{H^t(\mathbb{R}^3)}).
\]

If \(s < \frac{3}{2}\), \(t < \frac{1}{2}\) and \(s + t > 0\), then there exists a constant \(C > 0\), such that

\[
\|uv\|_{H^{s+t} - \frac{3}{2}(\mathbb{R}^3)} \leq C\|u\|_{H^s(\mathbb{R}^3)} \|v\|_{H^t(\mathbb{R}^3)}.
\]

**Lemma 2.2.** Let \(f \in \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)\), where \(s_1 < \frac{3}{2} < s_2\). Then, there is a constant \(c = c(s_1, s_2)\) such that

\[
\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|\hat{f}\|_{L^1(\mathbb{R}^3)} \leq c \|f\|_{\dot{H}^{s_1}(\mathbb{R}^3)} \|f\|_{\dot{H}^{s_2}(\mathbb{R}^3)}.
\]

**Proof.** We have

\[
\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} \\
\leq \int_{\xi} |\hat{f}(\xi)| d\xi \\
\leq \int_{|\xi| < \lambda} |\hat{f}(\xi)| d\xi + \int_{|\xi| > \lambda} |\hat{f}(\xi)| d\xi.
\]

We take

\[
I_1 = \int_{|\xi| < \lambda} |\xi|^{s_1} |\hat{f}(\xi)| d\xi.
\]

Using the Cauchy-Schwarz inequality, we obtain

\[
I_1 \leq \left( \int_{|\xi| < \lambda} \frac{1}{|\xi|^{s_1}} d\xi \right)^{\frac{1}{2}} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^3)} \\
\leq \left( \int_{0}^{\lambda} \frac{1}{r^{2s_1 - 2}} dr \right)^{\frac{1}{2}} \|f\|_{H^{s_1}(\mathbb{R}^3)} \\
\leq c_{s_1} \lambda^{\frac{2}{2} - s_1} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^3)}.
\]

Similarly, take

\[
I_2 = \int_{|\xi| > \lambda} |\xi|^{s_2} |\hat{f}(\xi)| d\xi,
\]

we have

\[
I_2 \leq \left( \int_{|\xi| > \lambda} \frac{1}{|\xi|^{s_2}} d\xi \right)^{\frac{1}{2}} \|f\|_{H^{s_2}} \\
\leq \left( \int_{\lambda}^{\infty} \frac{1}{r^{2s_2 - 2}} dr \right)^{\frac{1}{2}} \|f\|_{H^{s_2}} \\
\leq c_{s_2} \lambda^{\frac{2}{2} - s_2} \|f\|_{H^{s_2}}.
\]

Therefore,

\[
\|f\|_{L^\infty} \leq A \lambda^{\frac{2}{2} - s_1} + B \lambda^{\frac{2}{2} - s_2},
\]

with \(A = c_{s_1} \|f\|_{H^{s_1}}\) and \(B = c_{s_2} \|f\|_{H^{s_2}}\).

Posing

\[
\varphi(\lambda) = A \lambda^{\frac{2}{2} - s_1} + B \lambda^{\frac{2}{2} - s_2},
\]

we have
Then, $\varphi'(\lambda) = 0 \iff \lambda = c(s_1, s_2) \left( \frac{2}{\lambda} \right)^{1/2}$

So,

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq c^{2/s_2-\frac{3}{4}} A^{s_2-\frac{5}{4}} B^{s_2-\frac{3}{4}}. \quad \square$$

**Remark 2.3.** In particular, for $s_1 = 1$ and $s_2 = 2$, where $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^2(\mathbb{R}^3)$, we get

$$\|f\|_{L^\infty} \leq \|f\|_{\dot{H}^1}^{\frac{1}{2}} \|f\|_{\dot{H}^2}^{\frac{1}{2}}.$$ 

### 3. Long time decay of \((NSE)\) system in $H^1(\mathbb{R}^3)$

In this section, we want to prove: If $u \in C([\mathbb{R}^+, H^1(\mathbb{R}^3)])$ is a global solution to \((NSE)\) system, then

$$\lim_{t \to \infty} \sup \|u(t)\|_{H^1} = 0.$$ 

This proof is done in two steps.

- **Step 1:** In this step, we shall prove that

$$\lim_{t \to \infty} \sup \|u(t)\|_{H^1} = 0.$$ 

We have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^{s_2}} + \|u\|^2_{H^{s_1}} \leq c\|u\|_{H^{s_2}} \|u\|^2_{H^{s_2}}.$$ 

From (1.2), let $t_0 > 0$ such that $\|u(t_0)\|_{H^{s_2}} < \frac{1}{2c}$. Then

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^{s_2}} + \frac{1}{2} \|u\|^2_{H^{s_1}} \leq 0, \quad \forall t \geq t_0.$$ 

Integrating with respect to time, we obtain

$$\|u(t)\|^2_{H^{s_2}} + \int_{t_0}^t \|u(\tau)\|^2_{H^{s_2}} \leq \|u(t_0)\|^2_{H^{s_2}}, \quad \forall t \geq t_0.$$ 

Let $s > 0$ and $c = c_s$. There exists $T_0 = T_0(s, \nu, u^0) > 0$, such that

$$\|u(T_0)\|_{H^{s_2}} < \frac{1}{2c_s}.$$ 

Then

$$\|u(t)\|_{H^{s_2}} < c_s, \quad \forall t \geq t_0.$$ 

Now, for $s > 0$ we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^s} + \|u\|^2_{H^{s+1}} \leq \|u \otimes u\|_{H^s} \|u\|_{H^{s+1}} \leq c_s \|u\|_{H^{s_2}} \|u\|^2_{H^{s_2}}.$$ 

Then

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^s} + \|u\|^2_{H^{s+1}} \leq c_s \|u\|_{H^{s}} \|u\|^2_{H^{s+1}}, \quad \forall t \geq T_0.$$ 

Thus

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^s} + \frac{1}{2} \|u(t)\|^2_{H^{s+1}} \leq 0, \quad \forall t \geq T_0.$$ 

So, for $T_0 \leq t' \leq t$,

$$\|u(t)\|^2_{H^s} + \int_{t'}^t \|u(\tau)\|^2_{H^{s+1}} d\tau \leq \|u(t')\|^2_{H^s}.$$ 

In particular, for $s = 1$

$$\|u(t)\|^2_{H^1} + \int_{t'}^t \|u(\tau)\|^2_{H^{2}} d\tau \leq \|u(t')\|^2_{H^1}.$$
Then \( t \to \|u(t)\|_{H^1} \) is decreasing on \([T_0, \infty)\) and \( u \in L^2([0, \infty), \dot{H}^2(\mathbb{R}^3))\).

Now, let \( \varepsilon > 0 \) small enough. The \( L^2 \)-energy estimate
\[
\|u(t)\|_{L^2}^2 + 2 \int_{T_0}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u(T_0)\|_{L^2}^2, \quad \forall t \geq T_0
\]
implies that \( u \in L^2([T_0, \infty), \dot{H}^1(\mathbb{R}^3)) \) and there is a time \( t_\varepsilon \geq T_0 \) such that
\[
\|u(t_\varepsilon)\|_{H^1} < \varepsilon.
\]
As \( t \to \|u(t)\|_{H^1} \) is decreasing on \([T_0, \infty)\), then
\[
\|u(t)\|_{H^1} < \varepsilon, \quad \forall t \geq t_\varepsilon.
\]
Therefore (3.2) is proved. □

• Step 2: In this step, we prove that
\[
\lim_{t \to \infty} \sup_{\varepsilon > 0} \|u(t)\|_{L^2} = 0.
\]

This proof is inspired by [3] and [5]. For \( \delta > 0 \) and a given distribution \( f \), we define the operators \( A_\delta(D) \) and \( B_\delta(D) \), as following:
\[
A_\delta(D)f = \mathcal{F}^{-1}(1_{\{|\xi| < \delta\}}\mathcal{F}(f)), \quad B_\delta(D)f = \mathcal{F}^{-1}(1_{\{|\xi| \geq \delta\}}\mathcal{F}(f)).
\]

It is clear that when applying \( A_\delta(D) \) (respectively, \( B_\delta(D) \)) to any distribution, we are dealing with its low-frequency part (respectively, high-frequency part).

Let \( u \) be a solution to (NSE). Denote by \( \omega_\delta \) and \( v_\delta \), respectively, the low-frequency part and the high-frequency part of \( u \) and so on \( \omega_\delta^0 \) and \( v_\delta^0 \) for the initial data \( u^0 \). Applying the pseudo-differential operators \( A_\delta(D) \) to the (NSE), we get
\[
\partial_t \omega_\delta - \nu \Delta \omega_\delta + A_\delta(D)[\mathbb{P}(\nabla u)] = 0.
\]

Taking the \( L^2(\mathbb{R}^3) \) inner product, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\omega_\delta(t)\|_{L^2}^2 + \|\nabla \omega_\delta(t)\|_{L^2}^2 \leq \langle A_\delta(D)[\mathbb{P}(\nabla u)]/\omega_\delta(t) \rangle_{L^2}.
\]

Lemma 2.1 yields
\[
\frac{1}{2} \frac{d}{dt} \|\omega_\delta(t)\|_{L^2}^2 \leq \|\omega_\delta^0\|_{L^2}^2 + CM \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \|\nabla \omega_\delta(\tau)\|_{L^2}^2 d\tau.
\]

Integrating with respect to time, we obtain
\[
\|\omega_\delta(t)\|_{L^2}^2 \leq \|\omega_\delta^0\|_{L^2}^2 + CM \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \|\nabla \omega_\delta(\tau)\|_{L^2}^2 d\tau.
\]

Hence, we have \( \|\omega_\delta(t)\|_{L^2}^2 \leq M_\delta \) for all \( t \geq 0 \), where
\[
M_\delta = \|\omega_\delta^0\|_{L^2}^2 + CM \int_0^\infty \|\nabla u(\tau)\|_{L^2}^2 \|\nabla \omega_\delta(\tau)\|_{L^2}^2 d\tau.
\]

On the one hand, it is clear that \( \lim_{\delta \to 0} \|\omega_\delta^0\|_{L^2(\mathbb{R}^3)} = 0 \). On the other hand, the Lebesgue-Dominated Convergence Theorem implies that
\[
\lim_{\delta \to 0} \int_0^\infty \|\nabla u(\tau)\|_{L^2}^2 \|\nabla \omega_\delta(\tau)\|_{L^2}^2 d\tau = 0.
\]
Hence, \( \lim_{\delta \to 0} M_\delta = 0 \), and thus
\[
\limsup_{\delta \to 0} \|\omega_\delta(t)\|_{L^2} = 0. \tag{3.5}
\]
At this point, we note that it makes sense to take time equal to \( \infty \) in the integral \((\ref{3.4})\). In fact, by definition of \( \omega_\delta \) we have \( \|\nabla \omega_\delta\|_{L^2} \leq \|\nabla u\|_{L^2} \). It is clear that, \( \lim_{\delta \to 0} \|\nabla \omega_\delta(t)\|_{L^2} = 0 \) almost everywhere. So, the integrand sequence
\[
\|\nabla u(t)\|_{L^2} \|\nabla \omega_\delta(t)\|_{L^2}
\]
converges point-wise to zero. Moreover, using the above computations and \((\ref{1.3})\), we obtain
\[
\|\nabla u(t)\|_{L^2} \|\nabla \omega_\delta(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2}^2 \in L^1(\mathbb{R}^+) \).
Thus, the integral sequence is dominated by an integrable function. Then the limiting function is integrable and one can take the time \( T = \infty \) in \((\ref{3.4})\).
Now, let us investigate the high-frequency part. To do so, one applies the pseudo-differential operators \( B_\delta(D) \) to the \((NSE)\) to get
\[
\partial_t v_\delta - \Delta v_\delta + B_\delta(D) \mathcal{P}(u, \nabla u) = 0.
\]
Taking the Fourier transform with respect to the space variable, we obtain
\[
\partial_t \hat{v}_\delta(t, \xi)^2 + 2|\xi|^2 |\hat{v}_\delta(t, \xi)|^2 \leq 2|\mathcal{F}(B_\delta(D) \mathcal{P}(u, \nabla u))(t, \xi)| |\hat{v}_\delta(t, \xi)| \leq 2|\xi| |\mathcal{F}(B_\delta(D) \mathcal{P}(u \otimes u))(t, \xi)| |\hat{v}_\delta(t, \xi)| \leq 2 |\mathcal{F}(u \otimes u)(t, \xi)| |\nabla \hat{v}_\delta(t, \xi)|.
\]
Multiplying the obtained equation by \( \exp(2\nu|\xi|^2) \) and integrating with respect to time, we get
\[
|\hat{v}_\delta(t, \xi)|^2 \leq e^{-2\nu|\xi|^2} |\hat{v}_\delta(0, \xi)|^2 + 2 \int_0^t e^{-2(\nu-\rho)|\xi|^2} |\mathcal{F}(u \otimes u)(\tau, \xi)| |\nabla \hat{v}_\delta(\tau, \xi)| d\tau.
\]
Since \( |\xi| > \delta \), we have
\[
|\hat{v}_\delta(t, \xi)|^2 \leq e^{-2\nu\delta^2} |\hat{v}_\delta(0, \xi)|^2 + 2 \int_0^t e^{-2(\nu-\rho)\delta^2} |\mathcal{F}(u \otimes u)(\tau, \xi)| |\nabla \hat{v}_\delta(\tau, \xi)| d\tau.
\]
Integrating with respect to the frequency variable \( \xi \) and using Cauchy-Schwartz inequality, we obtain
\[
\|v_\delta(t)\|_{L^2}^2 \leq e^{-2\nu\delta^2} \|v_\delta(0)\|_{L^2}^2 + 2 \int_0^t e^{-2(\nu-\rho)\delta^2} \|u \otimes u\|_{L^2} \|\nabla u\|_{L^2} d\tau.
\]
By the definition of \( v_\delta \), we have
\[
\|v_\delta(t)\|_{L^2}^2 \leq e^{-2\nu\delta^2} \|v_\delta(0)\|_{L^2}^2 + 2 \int_0^t e^{-2(\nu-\rho)\delta^2} \|u \otimes u\|_{L^2} \|\nabla u\|_{L^2} d\tau.
\]
Lemma \((\ref{2.1})\) and inequality \((\ref{1.2})\) yield
\[
\|v_\delta(t)\|_{L^2(\mathbb{R}^2)}^2 \leq e^{-2\nu\delta^2} \|v_\delta(0)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t e^{-2(\nu-\rho)\delta^2} \|u\|_{H^1} \|\nabla u\|_{L^2}^2 d\tau.
\]
Hence,
\[
\|v_\delta(t)\|_{L^2}^2 \leq \|v_\delta(0)\|_{L^2}^2 + CM \int_0^t e^{-2(\nu-\rho)\delta^2} \|\nabla u\|_{L^2}^2 d\tau, \quad (M = \sup_{t \geq 0} \|u\|_{H^1}).
\]
Hence, \( \|v_\delta(t)\|_{L^2}^2 \leq N_\delta(t) \), where
\[
N_\delta(t) = e^{-2\nu\delta^2} \|v_\delta(0)\|_{L^2}^2 + CM \int_0^t e^{-2(\nu-\rho)\delta^2} \|\nabla u\|_{L^2}^2 d\tau.
\]
Using Young inequality and inequality \((\ref{1.3})\), we get \( N_\delta \in L^1(\mathbb{R}^+) \) and
\[
\int_0^\infty N_\delta(t) dt \leq \frac{\|v_\delta(0)\|_{L^2}^2}{2\nu\delta^2} + \frac{CM \|u_0\|_{L^2}^2}{4\delta^2}.
\]
So $t \to \|v_\delta(t)\|_{L^2}^2$ is continuous and belongs to $L^1(\mathbb{R}^+)$. Now, let $\varepsilon > 0$. At first, (3.5) implies that there exist some $\delta_0 > 0$ such that
\[
\|v_{\delta_0}(t)\|_{L^2} \leq \varepsilon/2, \quad \forall t \geq 0.
\]
Let us consider the set $R_{\delta_0}$ defined by $R_{\delta_0} := \{ t \geq 0, \|v_\delta(t)\|_{L^2(\mathbb{R}^3)} > \varepsilon/2 \}$. If we denote by $\lambda_1(R_{\delta_0})$ the Lebesgue measure of $R_{\delta_0}$, we have
\[
\int_0^\infty \|v_\delta(t)\|_{L^2(\mathbb{R}^3)}^2 dt \geq \int_{R_{\delta_0}} \|v_\delta(t)\|_{L^2(\mathbb{R}^3)}^2 dt \geq (\varepsilon/2)^2 \lambda_1(R_{\delta_0}).
\]
By doing this, we can deduce that $\lambda_1(R_{\delta_0}) = T_{\delta_0} < \infty$, and there exists $t_{\delta_0} > T_{\delta_0}$ such that
\[
\|v_\delta(t_{\delta_0})\|_{L^2}^2 \leq (\varepsilon/2)^2.
\]
So, $\|u(t_{\delta_0})\|_{L^2} \leq \varepsilon$ and from (3.3) we have
\[
\|u(t)\|_{L^2} \leq \varepsilon, \quad \forall t \geq t_{\delta_0}.
\]
This completes the proof of (3.3). \hfill \Box}

4. Generalization of Foias-Temam result in $H^1(\mathbb{R}^3)$

In [6] Fioas and Temam proved an analytic property for the Navier-Stokes equations on the torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Here, we give a similar result on whole space $\mathbb{R}^3$.\hfill

**Theorem 4.1.** We assume that $u^0 \in H^1(\mathbb{R}^3)$. Then, there exists a time $T$ that depends only on the $\|u^0\|_{H^1(\mathbb{R}^3)}$, such that:

(NSE) possesses on $(0,T)$ a unique regular solution $u$ such that $(t \to e^{t(D)}u(t))$ is continuous from $[0,T]$ into $H^1(\mathbb{R}^3)$. Moreover if $u \in C(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global solution to (NSE) system, then there are $M \geq 0$ and $t_0 > 0$ such that
\[
\|e^{t_0(D)}u(t)\|_{H^1(\mathbb{R}^3)} \leq M, \quad \forall t \geq t_0.
\]

Before proving this theorem, we need the following lemmas

**Lemma 4.2.** Let $t \mapsto e^{t(D)}u \in H^2(\mathbb{R}^3)$, where $|D| = (\Delta)^{1/2}$. Then
\[
\|e^{t(D)}u.\nabla v\|_{L^2(\mathbb{R}^3)} \leq \|e^{t(D)}u\|_{H^1(\mathbb{R}^3)}\|e^{t(D)}\nabla u\|_{H^1(\mathbb{R}^3)}\|e^{t(D)}\Delta^{1/2}v\|_{L^2(\mathbb{R}^3)}.
\]

**Proof.** We have
\[
\|e^{t(D)}u.\nabla v\|_{L^2} = \int_{\mathbb{R}^3} e^{t|\xi|} |\widehat{u.\nabla v}(\xi)|^2 d\xi
\]
\[
\leq \int_{\mathbb{R}^3} e^{t|\xi|} \left( \int_{\mathbb{R}^3} |\widehat{u}(\xi - \eta)| |\nabla v(\eta)| d\eta \right)^2 d\xi
\]
\[
\leq \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} e^{t|\xi - \eta|} |\widehat{u}(\xi - \eta)| |\nabla v(\eta)| d\eta \right)^2 d\xi
\]
\[
\leq \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} e^{t|\xi - \eta|} |\widehat{u}(\xi - \eta)| |\nabla v(\eta)| d\eta \right)^2 d\xi
\]
\[
\leq \left( \int_{\mathbb{R}^3} e^{t|\xi - \eta|} |\widehat{u}(\xi)| d\xi \right)^2 \|e^{t(D)}\Delta^{1/2}v\|_{L^2}.
\]

Hence, for $f = F^{-1}(e^{t|\xi|} |\widehat{u}(\xi)|) \in H^2(\mathbb{R}^3)$ and $(s_1 = 1; \ s_2 = 2)$, lemma 2.2 gives the desired result.

**Lemma 4.3.** Let $t \mapsto e^{t(D)}u \in H^2(\mathbb{R}^3)$. Then
\[
\left| \langle e^{t(D)}(u, \nabla v) / e^{t(D)}u \rangle_{H^1} \right| \leq \|e^{t(D)}u\|_{H^1}^{1/2} \|e^{t(D)}u\|_{H^1}^{1/2} \|e^{t(D)}\Delta^{1/2}v\|_{L^2} \|e^{t(D)}u\|_{L^2} \|e^{t(D)}w\|_{L^2}.
\]
Therefore

Thus

Hence, the sum of (4.2) and (4.3) yields

\[ \langle \hat{e}^{t[D]} u, \nabla v / \hat{e}^{t[D]} u \rangle_{H^1} = \left| \langle \hat{e}^{t[D]} u, \nabla v / \hat{e}^{t[D]} u \rangle_{L^2} \right| \leq \| \hat{e}^{t[D]} u, \nabla v \|_{L^2} \| \hat{e}^{t[D]} u \|_{L^2} \]

Finally, using lemma 4.2 we obtain the desired result.

**Proof of theorem 4.7.** We have

(4.1) \[ \partial_t u - \Delta u + u.\nabla u = -\nabla p. \]

Applying the fourier transform to the last equation and multiplying by \( \overline{\hat{u}} \), we have

(4.2) \[ \partial_t \hat{u} \overline{\hat{u}} + |\xi|^2 |\hat{u}|^2 = -\langle \hat{u}, \hat{u} \rangle \hat{u}. \]

Again, the fourier (bar) of (4.1) multiplied by \( \hat{u} \) gives

(4.3) \[ \partial_t \hat{u} \hat{u} + |\xi|^2 |\hat{u}|^2 = -\langle \hat{u}, \nabla u \rangle \hat{u}. \]

Hence, the sum of (4.2) and (4.3) yields

\[ \partial_t |\hat{u}|^2 + 2 |\xi|^2 |\hat{u}|^2 = -2 \text{Re} \langle (\hat{u}, \nabla u) \rangle \hat{u}. \]

This implies

\[ \partial_t |\hat{u}|^2 (1 + |\xi|^2 e^{2t|\xi|} + 2(1 + |\xi|^2) |\xi|^2 e^{2t|\xi|} |\hat{u}|^2 = -2 \text{Re} \langle (u, \nabla u) \hat{u} \rangle (1 + |\xi|^2) e^{2t|\xi|}. \]

Then

\[ \int_{\mathbb{R}^3} (1 + |\xi|^2) e^{2t|\xi|} \partial_t |\hat{u}|^2 d\xi + 2 \int_{\mathbb{R}^3} (1 + |\xi|^2) |\xi|^2 e^{2t|\xi|} |\hat{u}|^2 d\xi = -2 \text{Re} \int_{\mathbb{R}^3} ((u, \nabla u) \hat{u}) (1 + |\xi|^2) e^{2t|\xi|} d\xi. \]

Thus

\[ \langle \hat{e}^{t[D]} u / \hat{e}^{t[D]} u \rangle_{H^1} + 2 \| \hat{e}^{t[D]} \nabla u \|_{H^1(\mathbb{R}^3)}^2 = -2 \text{Re} \langle e^{t[D]} (u, \nabla u) / e^{t[D]} u \rangle_{H^1}. \]

At time \( \tau \), we have

(4.4) \[ \langle e^{t[D]} u(\tau) / e^{\tau[D]} u(\tau) \rangle_{H^1} + 2 \| e^{t[D]} \nabla u \|_{H^1}^2 = -2 \text{Re} \langle e^{\tau[D]} (u, \nabla u) / e^{t[D]} u \rangle_{H^1}. \]

Therefore

\[ \langle e^{t[D]} u(t) / e^{t[D]} u(t) \rangle_{H^1} = \langle (e^{t[D]} u(t))' - |D| e^{t[D]} u(t) / e^{t[D]} u(t) \rangle_{H^1} = \frac{1}{2} \frac{d}{dt} \| e^{t[D]} u \|_{H^1}^2 - \langle e^{t[D]} D u(t) / e^{t[D]} u(t) \rangle_{H^1} \]

\[ \geq \frac{1}{2} \frac{d}{dt} \| e^{t[D]} u \|_{H^1}^2 - \| e^{t[D]} u \|_{H^1} \| e^{t[D]} u \|_{H^2}. \]

Using the Young inequality, we obtain

(4.5) \[ \frac{d}{dt} \| e^{t[D]} u \|_{H^1}^2 - 2 \| e^{t[D]} u \|_{H^1}^2 \leq 2 \langle e^{t[D]} u(t) / e^{t[D]} u(t) \rangle_{H^1}. \]
Hence, using the lemma 4.3 and Young inequality the right hand of (4.4) satisfies
\[ | - 2Re(e^{\tau|D|}u \nabla u / e^{\tau|D|}u)_{H^1} | \leq 2\|e^{\tau|D|}u\|_{H^1}^2 \|e^{-\tau|D|}u\|_{H^1}^2 \|e^{\tau|D|}|D|u\|_{L^2} \|e^{\tau|D|}\Delta u\|_{L^2} \]
\[ \leq 2\|e^{\tau|D|}u\|_{H^1}^2 \|e^{-\tau|D|}u\|_{H^1}^2 \]
\[ \leq \frac{3}{4}\|e^{\tau|D|}u\|_{H^1}^2 + \frac{c_1}{2}\|e^{\tau|D|}u\|_{H^1}^6, \]
where \(c_1\) is a positive constant.

Then (4.6) yields
\[ (e^{\tau|D|}u'(t) / e^{\tau|D|}u(t))_{H^1} + 2\|e^{\tau|D|}\nabla u\|_{H^1}^2 \leq \frac{3}{4}\|e^{\tau|D|}u\|_{H^1}^2 + \frac{c_1}{2}\|e^{\tau|D|}u\|_{H^1}^6. \]

Hence, using (4.3) and (5.1), we get
\[ \frac{d}{dt}\|e^{\tau|D|}u\|_{H^1}^2 + 2\|e^{\tau|D|}\nabla u\|_{H^1}^2 \leq 4\|e^{\tau|D|}u\|_{H^1}^2 + c_1\|e^{\tau|D|}u\|_{H^1}^6 \]
\[ \leq c_2 + 2c_1\|e^{\tau|D|}u\|_{H^1}^6, \]
where also \(c_2\) is a positive constant.

Finally, we obtain
\[ y'(t) \leq K_1 y^3(t), \]
where
\[ y(t) = 1 + \|e^{\tau|D|}u(t)\|_{H^1}^2 \quad \text{and} \quad K_1 = 2c_1 + c_2. \]

Then
\[ y(t) \leq y(0) + K_1 \int_0^t y^3(s)ds. \]

Let
\[ T_1 = \frac{2}{K_1 y^2(0)} \]
and \(0 < T \leq T^*\) such that \(T = \sup\{t \in [0, T^*) \mid \sup_{0 \leq s \leq t} y(s) \leq 2y(0)\}\). Hence for \(0 \leq t \leq \min(T_1, T)\), we have
\[ y(t) \leq 2y(0), \forall t \in [0, T_1]. \]

Therefore \(t \mapsto e^{\tau|D|}u(t) \in H^1(\mathbb{R}^3), \forall t \in [0, T_1]\).

In particular
\[ \|e^{T_1|D|}u(T_1)\|_{H^1}^2 \leq 2 + 2\|u_0\|_{H^1}^2. \]

Now, if we know that
\[ \|u(t)\|_{H^1} \leq M_1 \forall t \geq 0. \]

Defining the system
\[
\begin{align*}
\partial_t w - \Delta w + w \cdot \nabla w &= -\nabla p_2 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div } w &= 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\
w(0) &= u(b) \quad \text{in} \quad \mathbb{R}^3,
\end{align*}
\]
where \( w(t) = u(T + t) \).

Using a similar technic, we can prove that there exists \( T_2 = \frac{2}{K_4}(1 + M_1^2)^{-2} \) such that

\[
y(t) = 1 + \| e^{[t]} u(t) \|^2_{H^1} \leq 2(1 + M_2^2), \ \forall t \in [0, T_2].
\]

This implies that \( 1 + \| e^{[t]} u(T + t) \|^2_{H^1} \leq 2(1 + M_2^2) \). Hence, for \( t = T_2 \) we have

\[
\| e^{[t]} u(T + t) \|^2_{H^1} \leq 2(1 + M_2^2).
\]

Since \( t = T + T_2 \geq T_2, \ \forall T \geq 0 \), we obtain

\[
\| e^{[t]} u(t) \|^2_{H^1} \leq 2(1 + M_1^2), \ \forall t \geq T_2.
\]

Then

\[
\| e^{[t]} u(t) \|^2_{H^1} \leq 2(1 + M_1^2), \ \forall t \geq T_2,
\]

where

\[
T_2 = T_2(M_1) = \frac{2}{K_4}(1 + M_1^2)^{-2}.
\]

\[
\square
\]

5. Proof of main result

In this section, we prove the main theorem 1.1. This proof uses the result of sections 3 and 4. Let \( u \in C(\mathbb{R}^+, H^{1,\sigma}_0(\mathbb{R}^3)) \). As \( H^{1,\sigma}_0(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3) \), then \( u \in C(\mathbb{R}^+, H^1(\mathbb{R}^3)) \).

Applying the theorem 1.1 there exist \( t_0 > 0 \) and \( \alpha > 0 \) such that

\[
\| e^{[t]} u(t) \|_{H^1} \leq c_0 = 2 + M_1^2, \ \forall t \geq t_0,
\]

where \( \alpha = \varphi(t_0) \) and \( t_0 = \frac{2}{K_4}(1 + M_1^2)^{-2} \).

Therefore, let \( a > 0, \beta > 0 \). It shows that there exists \( c_3 \geq 0 \) such that

\[
a x \leq c_3 + \beta x, \ \forall x \geq 0.
\]

Indeed: \( \frac{a}{\alpha} + \frac{\sigma - 1}{\sigma} = \frac{1}{p} + \frac{1}{q} = 1 \). Using the Young inequality, we obtain

\[
ax = a\beta \left( \beta x \right)^{\frac{1}{q}} \leq \frac{(a\beta^q)^q}{q} + \frac{(\beta x)^p}{p} \leq c_3 + \frac{\beta x}{\sigma} \leq c_3 + \beta x,
\]

where \( c_3 = \frac{\sigma - 1}{\sigma} a\sigma^\frac{\sigma}{\sigma - 1} \beta^\frac{1}{\sigma - 1} \).

Take \( \beta = \frac{2}{q} \), using (5.1) and the Cauchy Schwarz inequality, we have

\[
\| u(t) \|_{H^{1,\sigma}_0} = \| e^{[t]} u(t) \|_{H^1}
\]

\[
= \int (1 + |\xi|^2)e^{2(\sigma - 1)|\xi|} |\hat{u}(t, \xi)|^2 d\xi
\]

\[
= \int (1 + |\xi|^2)e^{2(\sigma + \beta)|\xi|} |\hat{u}(t, \xi)|^2 d\xi
\]

\[
= \int (1 + |\xi|^2)e^{2\alpha |\xi|} |\hat{u}(t, \xi)|^2 d\xi
\]

\[
\leq e^{2c_3} \left( \int (1 + |\xi|^2)^2 |\hat{u}(t, \xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int (1 + |\xi|^2) e^{2\alpha \xi^2} |\hat{u}(t, \xi)|^2 d\xi \right)^{\frac{1}{2}}
\]

\[
\leq e^{2c_3} \| u \|_{H^1}^{\frac{1}{2}} \| e^{[t]} u(t) \|_{H^1}^{\frac{1}{2}}
\]

\[
\leq c \| u \|_{H^1},
\]
where $c = e^{2c_2}c_1^2$.

Using (3.1), we get

$$\limsup_{t \to \infty} \|e^{\alpha |D|^{1/\sigma}} u(t)\|_{H^1} = 0.$$  \hfill $\square$

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Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Kingdom of Saudi Arabia

E-mail address: jbenateur@ksu.edu.sa

Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Kingdom of Saudi Arabia

E-mail address: ljlali@ksu.edu.sa