Thermalization of entanglement

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We explore the dynamics of the entanglement entropy near equilibrium in highly-entangled pure states of two quantum-chaotic spin chains undergoing unitary time evolution. We examine the relaxation to equilibrium from initial states with either less or more entanglement entropy than the equilibrium value, as well as the dynamics of the spontaneous fluctuations of the entanglement that occur in equilibrium. For the spin chain with a time-independent Hamiltonian and thus an extensive conserved energy, we find slow relaxation of the entanglement entropy near equilibration. Such slow relaxation is absent in a Floquet spin chain with a Hamiltonian that is periodic in time and thus has no local conservation law. Therefore, we argue that slow diffusive energy transport is responsible for the slow relaxation of the entanglement entropy in the Hamiltonian system.

Quantum entanglement has recently been a central topic in theoretical physics. Many aspects of the dynamics of entanglement have been recently studied, such as ballistic spreading of the entanglement in integrable and nonintegrable systems, logarithmic spreading in many-body localized systems, and sub-ballistic spreading due to quantum Griffiths effects. In many of these examples, the entanglement spreads more rapidly than conserved quantities that must be transported by currents.

Much of the previous work on the dynamics of entanglement, however, has emphasized far-from-equilibrium regimes, particularly those following a quantum quench. Here, we instead explore the entanglement dynamics near equilibrium in nonintegrable, thermalizing spin chains of finite length. For example, if we start in a nonentangled initial pure state, the entanglement entropy grows linearly with time at early time due to the “ballistic” spreading of entanglement, but then saturates to its “volume-law” equilibrium value at long time. The lower two data sets in Fig. 1 illustrate this. In the limit of a long spin chain, this isolated system is reservoir that thermalizes all of its subsystems. Then the extensive part of the final equilibrium value of the entanglement entropy is equal to the thermal equilibrium entropy at the corresponding temperature, and that temperature is set by the total energy of the initial state. We call this process, in which the entanglement entropy approaches the thermal equilibrium entropy, the “thermalization of entanglement.”

In this paper we focus on the late time, near equilibrium regime of the entanglement dynamics, as well as the spontaneous fluctuations of entanglement in pure states sampled from the equilibrium density operator. We show that this near-equilibrium entanglement dynamics is slowed down by energy diffusion for a Hamiltonian system, while it is faster in the corresponding Floquet system, whose dynamics are not constrained by energy conservation. We also explore the entanglement dynamics for initial pure states that are generalized Bell states with maximal entanglement and that relax to lower entanglement entropy as they thermalize.

![FIG. 1: (color online) Time evolution of the entanglement entropy for $L = 14$: for product random (PR) initial states under Floquet dynamics (2) with $\tau = 0.8$ (red line with circles) as well as Hamiltonian dynamics (1) (blue line with down triangles); “generalized Bell” initial states made from pairs of random pure (RP) states (green line with up triangles) and from “oppositely paired” (OP) states (purple line with diamonds) both under Hamiltonian dynamics. Each case is averaged over 400 initial pure states. See main text for more details.](image-url)
chain \((i \rightarrow (L + 1 - i))\). This system’s “hydrodynamics” are simply its conserved energy moving diffusively and subject to random local currents due to the system’s quantum-chaotic unitary dynamics. We set the Planck constant \(\hbar\) to unity.

To explore the effects of removing the conservation of total energy, we also study a Floquet system that is a modification of (1). We decompose the Hamiltonian into two parts, \(H_z = \sum_i (\hbar \sigma_i^z + \sigma_i^x \sigma_{i+1}^x)\) and \(H_x = \sum_i g \sigma_i^x\). We periodically drive the system with a time-dependent Hamiltonian that is in turn \(H(t) = 2H_z\) for a time interval of \(\tau/2\) and then \(H(t) = 2H_x\) for the next \(\tau/2\), and repeat. The time-averaged Hamiltonian is thus unchanged, but the periodic switching changes the energy conservation from conservation of the extensive total energy to conservation of energy only modulo \((2\pi/\tau)\). This change removes the diffusive transport of energy as a slow “hydrodynamic” mode. The Floquet operator that produces the unitary time evolution through one full period is

\[
U_F(\tau) = e^{-iH_z \tau} e^{-iH_x \tau}.
\]

We choose \(\tau = 0.8\), which causes rapid decay of the average total energy within a few time steps. The eigenvalues of \(U_F(\tau)\) are complex numbers of magnitude one. Note that time is in a certain sense discrete (integer multiples of \(\tau\)) for this Floquet system.

Throughout this paper, we consider the bipartite entanglement entropy of pure states, quantified by the von Neumann entropy of the reduced density operator of a half chain: \(S = -\text{Tr}\{\rho_L \log_2 \rho_L\} = -\text{Tr}\{\rho_R \log_2 \rho_R\}\). We study chains of even length, and \(\rho_L\) and \(\rho_R\) are the reduced density operators of the left and right half chains, respectively. Note that we measure the entropy in bits.

We first look at the entanglement entropy of the eigenstates of the Hamiltonian (1) and of the Floquet operator (2), compared to random pure states of the full chain. Figure 2 shows the distributions of these entanglement entropies for \(L = 16\).

We can see that the entanglement of the eigenstates of the Floquet operator is close to that of random pure states, first derived by Page

\[
S^R(L) = \frac{L}{2} \ln 2 - \frac{1}{2} \ln 2 - O\left(\frac{1}{L^2}\right).
\]

This is consistent with previous studies which have shown that a Floquet dynamics thermalizes a subsystem to infinite temperature. The eigenstates of the Hamiltonian, on the other hand, all have entanglement entropies that are a fraction of a bit or more less than random pure states. What is the source of this difference? It is because the Hamiltonian eigenstates are eigenstates of the extensive conserved total energy, while the random pure states and the Floquet eigenstates are not constrained by an extensive conserved quantity. This causes the probability distribution of the energy of a half chain to be narrower for the Hamiltonian system, since if one half chain has a high energy (compared to its share of the eigenenergy) then the other half chain has to have an energy that is low by the same amount. This suppresses the volume of the possible space of half-chain states whose energy is either high or low, resulting in a reduced entropy of the half-chain and thus reduced entanglement entropy, even for the Hamiltonian eigenstates at energies that correspond to infinite temperature. This goes along with the recent observation that the finite-size deviations of the eigenstates of the Hamiltonian from the Eigenstate Thermalization Hypothesis are larger than those of the eigenstates of the Floquet operator energy conservation somewhat impedes thermalization.

Now we turn to the dynamics of the entanglement entropy. The dynamics of a linear operator is set by the matrix elements of the operator between energy eigenstates (or eigenstates of the Floquet operator) and the eigenenergies. But the entanglement is not a linear operator, so its dynamics cannot be determined so simply. We explore the near-equilibrium dynamics of the entanglement in two different ways. First we study the relaxation of the entanglement to its equilibrium value from particular initial states with either low or high entanglement. We then explore the dynamics of the spontaneous fluctuations of the entanglement entropy during the unitary time evolution of a random pure state of the full spin chain. From these studies we can clearly show that the entanglement dynamics is slower for the Hamiltonian system, since some of the entanglement entropy is connected to the slow diffusion of energy between the two half-chains. In the Floquet system, on the other hand, near equilibrium the entanglement relaxes to equilibrium with a simple-exponential behavior in time, with a relaxation time that is apparently independent of the system...
First we study dynamics of entanglement starting from a product of random half-chain pure states $|\psi(t = 0)\rangle = |\psi_L\rangle \otimes |\psi_R\rangle$, where $|\psi_L\rangle$ and $|\psi_R\rangle$ are picked from the ensemble of random pure states of the left and the right half chain, respectively. On average the states have energy close to 0, so the system is near infinite temperature. Fig. 1 plots the time-dependent entanglement entropy under Hamiltonian and Floquet dynamics for $L = 14$. The long-time average $S(\infty)$ is estimated by averaging $S(t)$ from $t = 250\tau$ to $t = 2999\tau$.

It is clear from Fig. 1 that the Floquet system has faster relaxation of the entanglement entropy towards its saturation value at long times, even though the initial spreading rate of the entanglement is the same for these two systems. Since the only significant difference between these two unitary dynamics is whether or not energy conservation and thus energy transport is present, Fig. 1 suggests a slower long-time thermalization of the entanglement entropy is associated with diffusive energy transport.

Another class of initial states that we explore are those of maximum entanglement entropy, that we call “generalized Bell states”. These states have Schmidt decomposition

$$|\psi_B\rangle = \frac{1}{\sqrt{2^L/2}} \sum_{i=1}^{2^L/2} |L_i\rangle \otimes |R_i\rangle,$$

where the sets $\{|L_i\rangle\}$ and $\{|R_i\rangle\}$ are respectively complete orthonormal bases for left and right half chains. Since these initial states have higher entanglement entropy than equilibrium, their entropy decreases as it thermalizes. This is an amusing apparent “violation” of the second law of thermodynamics, but it is actually not thermodynamics, since the decrease is by less than one bit (very close to $1/(2\ln 2)$ by Eq. (3)), and thus far from extensive. The random pure (“RP”) Bell states are made by independently choosing a random orthonormal basis for each half-chain. To make initial Bell states that also have very large energy differences between the two half-chains, we make the opposite paired (“OP”) states that can be written as

$$|\psi(t = 0)\rangle = \frac{1}{\sqrt{2^L/2}} \sum_{i=1}^{2^L/2} e^{i\theta_i} |E_i^{h_L}|_{left} \otimes |E_i^{h_R}|_{right},$$

where the $|E_i^{h}\rangle$ are the eigenstates of the half-chain Hamiltonian (Hamiltonian with $L/2$ sites), with their eigenenergies ordered according to $E_i^{h_L} \leq E_i^{h_R}$. The ensemble of OP states that we average over is obtained by choosing random phases $\{\theta_i\}$.

The time evolution of the entanglement entropy for a $L = 14$ spin chain starting from generalized Bell states of pairs of random pure (RP) states as well as generalized Bell states with opposite pairing (OP) under Hamiltonian dynamics are shown in Fig. 1 with the estimated long time average subtracted. For the opposite paired (OP) initial states, the initial large energy differences between the two half-chains in many of the Schmidt pairs make the excess entanglement long-lived, since the relaxation of these energy differences requires diffusion of the energy over the full length of the chain. For the RP initial states, on the other hand, the half-chain states are random so do not show nonequilibrium energy correlations, and the excess entanglement relaxes to equilibrium much more rapidly than it does for the OP states.

Fig. 3 gives a more detailed view of the thermalization of the excess entanglement entropy starting from these generalized Bell initial states. Here RP initial states under Floquet dynamics are also shown; since the Floquet system does not have conserved energy we cannot construct an OP initial state for it. This figure again shows the clear importance of energy transport for entanglement thermalization. The excess entropy of the RP initial states decays away faster for the Floquet system as compared to the Hamiltonian system, since the thermalization of the Floquet system is not constrained by an extensive conserved energy. The strong initial anticorrelation between the energies of the two half-chains greatly slows down the thermalization of the entanglement for the OP initial states under Hamiltonian dynamics.

Next we examine the dynamics of the spontaneous fluctuations of the entanglement entropy at equilibrium at infinite temperature, where all pure states are equally likely. Therefore, we simply pick many random pure
In conclusion, we have investigated the thermalization of the entanglement entropy by comparing state evolution of spin chains under Hamiltonian dynamics and Floquet dynamics, with the two systems having the same time-averaged Hamiltonian. Eigenstates of these two dynamics have quite different distributions of the entanglement entropy. The Floquet eigenstates all have entanglement close to that of random pure states, while the Hamiltonian eigenstates all have significantly less entanglement due to the constraint of total energy conservation. We show that the entanglement entropy relaxes to equilibrium more slowly under Hamiltonian dynamics, both for initial states well away from equilibrium and for the spontaneous fluctuations of the entanglement entropy at equilibrium. The Hamiltonian system has slow diffusive energy transport, while the Floquet system does not. This slow diffusive relaxation of the energy distribution in the Hamiltonian system results in slow relaxation near equilibrium of the entanglement entropy.

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