Broadcasting of entanglement in three-particle GHZ state via quantum copying

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Abstract

We introduce entanglement measures to describe entanglement in a three-particle system and apply it to studying broadcasting of entanglement in three-particle GHZ state. We show that entanglement of three-qubit GHZ state can be partially broadcasted with the help of local or non-local copying processes. It is found that non-local cloning is much more efficient than local cloning for the broadcasting of entanglement.

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Quantum entanglement, first noted by Einstein-Podolsky-Rosen and Schrödinger [1], is one of the essential features of quantum mechanics. It has been well known that quantum entanglement plays a key role in many such applications like quantum teleportation [2], super-dense coding [3], quantum error correction [4], and quantum computational speedups [5]. Recently, maximally entangled states of three particles, i.e., three-particle GHZ states, have been produced experimentally [6]. Usually, it is difficult to obtain ideal entangled multi-particle state. An interesting question is: whether there are ways which can broadcast the entanglement of correlated systems? The answer is ok. Masiak and Knight [7] have shown that copies of entangled pair of qubits can be generated through using a universal quantum cloning machine (UQCM) [8], although the degree of entanglement of the resultant copies is substantially reduced due to a residual entanglement between the copied output and the copying machine. On the other hand, up to now there is not an appropriate measure to describe quantitatively the entanglement of three and more subsystems due to the high complexity of entanglement in multi-particle system [9]. The purpose of this paper is to propose measures of entanglement in three qubit system in terms of the entanglement tensor approach [10], and use these measures to investigate broadcasting of entanglement [11] in three-particle GHZ state.

Consider a system consisting of three qubits. The density operator of the system can be expanded as a sum of tensor products in terms of Pauli matrices,

\[
\hat{\rho} = \frac{1}{8} [\hat{1} \otimes \hat{1} \otimes \hat{1} + \sum_{i=1}^{3} \lambda_i(1)(\hat{\sigma}_i \otimes \hat{1} \otimes \hat{1}) \\
+ \sum_{j=1}^{3} \lambda_j(2)(\hat{1} \otimes \hat{\sigma}_j \otimes \hat{1}) + \sum_{k=1}^{3} \lambda_k(3)(\hat{1} \otimes \hat{1} \otimes \hat{\sigma}_k) \\
+ \sum_{i,k=1}^{3} K_{ik}(1,3)(\hat{\sigma}_i \otimes \hat{1} \otimes \hat{\sigma}_k) + \sum_{i,j=1}^{3} K_{ij}(1,2) \\
\times (\hat{\sigma}_i \otimes \hat{\sigma}_j \otimes \hat{1}) + \sum_{j,k=1}^{3} K_{jk}(2,3)(\hat{1} \otimes \hat{\sigma}_j \otimes \hat{\sigma}_k) \\
+ \sum_{i,j,k=1}^{3} K_{ijk}(1,2,3)(\hat{\sigma}_i \otimes \hat{\sigma}_j \otimes \hat{\sigma}_k)]. \tag{1}
\]
Here \( \{\hat{\sigma}_i, i = 1, 2, 3\} \) are the Pauli matrices. \( \lambda(1), \lambda(2) \) and \( \lambda(3) \) are the three coherence vectors belonging to qubit 1, 2 and 3, respectively, which determine the properties of the individual particles. \( K_{ij}(1, 2), K_{jk}(2, 3), \) and \( K_{ik}(1, 3) \) are the second-rank correlation tensors which describe the correlation between qubit 1 and 2 (qubit 2 and 3, qubit 1 and 3) respectively. \( K_{ijk}(1, 2, 3) \) are the three-qubit correlation tensor.

Making use of properties of Pauli matrices \( \text{tr} \{\hat{\sigma}_i\} = 0, \) and \( \text{tr} \{\hat{\sigma}_i\hat{\sigma}_j\} = 2\delta_{ij}, \) we get the following relations:

\[
\lambda_i(1) = \text{tr}(\hat{\rho} \cdot \hat{\sigma}_i \otimes \hat{1} \otimes \hat{1}), \tag{2}
\]

\[
\lambda_j(2) = \text{tr}(\hat{\rho} \cdot \hat{1} \otimes \hat{\sigma}_j \otimes \hat{1}), \tag{3}
\]

\[
\lambda_k(3) = \text{tr}(\hat{\rho} \cdot \hat{1} \otimes \hat{1} \otimes \hat{\sigma}_k), \tag{4}
\]

\[
K_{ij}(1, 2) = \text{tr}(\hat{\rho} \cdot \hat{\sigma}_i \otimes \hat{\sigma}_j \otimes \hat{1}), \tag{5}
\]

\[
K_{ik}(1, 3) = \text{tr}(\hat{\rho} \cdot \hat{\sigma}_i \otimes \hat{1} \otimes \hat{\sigma}_k), \tag{6}
\]

\[
K_{jk}(2, 3) = \text{tr}(\hat{\rho} \cdot \hat{1} \otimes \hat{\sigma}_j \otimes \hat{\sigma}_k), \tag{7}
\]

\[
K_{ijk}(1, 2, 3) = \text{tr}(\hat{\rho} \cdot \hat{\sigma}_i \otimes \hat{\sigma}_j \otimes \hat{\sigma}_k). \tag{8}
\]

After performing the partial trace over the second qubit and the third qubit, from Eq.(1) we obtain the reduced density operator for the first qubit:

\[
\hat{\rho}^{(1)} = \text{tr}_{2,3}(\hat{\rho}) = \frac{1}{2}(\hat{1} + \sum_{i} \lambda_i(1)\hat{\sigma}_i). \tag{9}
\]

Similarly, we can calculate the reduced density operators \( \hat{\rho}^{(2)}, \hat{\rho}^{(3)} \) for qubit 2 and 3.

Comparing the direct product \( \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \hat{\rho}^{(3)} \) with Eq.(1), we can identify the difference by tensors \( M(m, n)(1 \leq m < n \leq 3), \) \( M(1, 2, 3) \) defined by

\[
M_{ij}(m, n) = K_{ij}(m, n) - \lambda_i(m)\lambda_j(n), \tag{10}
\]

\[
M_{ijk}(1, 2, 3) = K_{ijk}(1, 2, 3) - \lambda_i(1)M_{jk}(2, 3)
- \lambda_j(2)M_{ik}(1, 3) - \lambda_k(3)M_{ij}(1, 2)
- \lambda_i(1)\lambda_j(2)\lambda_k(3), \tag{11}
\]
which indicates that $M(1, 2, 3) = 0$ for any product state of three qubits. From the above, we also find that a three-qubit entanglement state necessarily involves entanglement between any two qubits. Hence entanglement measures in a three-qubit system should involve both an inter-three-qubit entanglement measure and an inter-two-qubit entanglement measure.

Based on $M(1, 2, 3)$, $M(m, n)(1 \leq m < n \leq 3)$, we introduce an inter-three-qubit entanglement measure $E_3$ and an inter-two-entanglement measure $E_2$ in the following form,

$$E_3 = \frac{1}{4} \sum_{i,j,k=1}^{3} M_{ijk}(1, 2, 3)M_{ijk}(1, 2, 3),$$

$$E_2(m, n) = \frac{1}{3} \sum_{i,j=1}^{3} M_{ij}(m, n)M_{ij}(m, n).$$

It is easy to check that $E_2$ and $E_3$ obey all conditions as entanglement measures indicated in Ref.[10]. These measures are invariant under local unitary transformations of the subsystems and varies between 0 (product states) and 1 (maximum entangled states). $E_3$ quantifies the three-qubit entanglement. The larger $E_3$ is, the stronger the three-qubit entanglement is. And $E_2(m, n)$ quantifies the entanglement between any two qubits $m, n$ in the three qubit system. The larger $E_2(m, n)$ is, the stronger the entanglement between two qubit $m, n$ is.

As an example of three-qubit entanglement, let us consider the GHZ state $|\psi\rangle = (|111\rangle + |000\rangle)/\sqrt{2}$. Making use of Eqs.(2)-(9), from Eqs.(10) and (11) we can get the nonzero entanglement tensors, $M_{xxx} = 1, M_{xyy} = M_{yxy} = M_{gxx} = -1, M_{zz}(1, 2) = M_{zz}(2, 3) = M_{zz}(1, 3) = 1$. All other entanglement tensors vanish as well as the coherence vectors. Then from Eqs.(12)and (13) we find that

$$E_3 = 1, \quad E_2(1, 2) = E_2(2, 3) = E_2(1, 3) = \frac{1}{3},$$

which indicate that the GHZ state is the maximally entangled inter-three-qubit state and contains inter-two-qubit entanglement, as expected.

In what follows we investigate broadcasting of entanglement in a three-particle GHZ state via quantum copying. We shall consider the two cases of local cloning and non-local cloning, and assume that the three qubits are prepared in the GHZ state:
Firstly, we consider the case of local cloning. In this case, the three qubits $1_0$, $2_0$ and $3_0$ is copied by the UQCM denoted by the local unitary transformations [8]:

\[
U_1 | 0 \rangle_{a_0} | 0 \rangle_{a_1} | X \rangle_x = \sqrt{\frac{2}{3}} | 00 \rangle_{a_0a_1} | \uparrow \rangle_x \\
+ \sqrt{\frac{1}{3}} | + \rangle_{a_0a_1} | \downarrow \rangle_x,
\]

(16)

\[
U_1 | 1 \rangle_{a_0} | 0 \rangle_{a_1} | X \rangle_x = \sqrt{\frac{2}{3}} | 11 \rangle_{a_0a_1} | \downarrow \rangle_x \\
+ \sqrt{\frac{1}{3}} | + \rangle_{a_0a_1} | \uparrow \rangle_x
\]

(17)

where $| + \rangle_{a_0a_1} = (| 10 \rangle_{a_0a_1} + | 01 \rangle_{a_0a_1})/\sqrt{2}$. The system labelled by $a_0$ is the original (input) qubit, while the other system $a_1$ represents the target qubit onto which the information is copied. The states of the copying machine are labelled by $x$. The state space of the copying machine is two dimensional and we assume that it is always in the same state $| X \rangle_x$ initially.

Suppose that each of the three original qubits $1_0$, $2_0$ and $3_0$ is cloned separately by three distant local cloners $X_1$, $X_2$ and $X_3$. The cloner $X_1$ ($X_2, X_3$) generates out of qubit $1_0$ ($2_0, 3_0$) two $l_0$ and $l_1$ ($2_0$ and $2_1$, $3_0$ and $3_1$). After cloning, we get the following total output state

\[
| \psi \rangle_{\text{total}}^{(\text{out})} = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} \prod_{m=1}^{3} [U_1(m) | i \rangle_{m_0} | 0 \rangle_{m_1} | X_m \rangle_x].
\]

(18)

After performing trace over the three cloners we obtain a six-qubit density operator $\hat{\rho}_{10120303}^{(\text{out})}$ which also describes two nonlocal three-qubit systems $\hat{\rho}_{102030}$ and $\hat{\rho}_{11231}$. The two three-qubit systems are the clones of the original three-qubit GHZ state in Eq.(15) and they are described by the density operators:

\[
\hat{\rho}_{102030}^{(\text{out})} = \frac{7}{24} (| 111 \rangle \langle 111 | + | 000 \rangle \langle 000 |) \\
+ \frac{7}{54} (| 111 \rangle \langle 000 | + | 000 \rangle \langle 111 |) \\
+ \frac{5}{72} (| 110 \rangle \langle 110 | + | 011 \rangle \langle 011 |) \\
+ | 101 \rangle \langle 101 | + | 100 \rangle \langle 100 | \\
+ | 010 \rangle \langle 010 | + | 001 \rangle \langle 001 |).
\]

(19)
Now we check whether entanglement is broadcasted. Making use of Eq.(2)-(8), we can obey nonzero entanglement tensors in the output state \( M_{xxx} = 7/27, M_{xyy} = M_{yyx} = -7/27 \) and \( M_{zz}(1,2) = M_{zz}(1,3) = M_{zz}(2,3) = 4/9 \) while all other entanglement tensors vanish as well as the coherence vectors. The values \( E_3 \) and \( E_2(m,n) \) are then given by

\[
E_3 = \frac{49}{729}, \quad E_2(1,2) = E_2(2,3) = E_2(1,3) = \frac{16}{243}, \tag{20}
\]

which indicate that both the inter-three-qubit entanglement and inter-two-qubit entanglement in a tree-qubit systems characterized by \( E_3 \) and \( E_2(m,m) \), respectively, can be broadcasted via local quantum cloners, although both the inter-three-qubit entanglement and inter-two-qubit entanglement of the copied state are less than those of the original state which are given by Eq.(14).

Secondly, we consider the case of non-local cloning. In this case, the entangled state of the three-qubits is treated as a state in a larger Hilbert space and cloned as a whole. The non-local quantum copying machine [12] is an \( N \) dimensional quantum system, and we shall let \( | X_i \rangle_x \) \((i = 1, \cdots, N)\) be a set of orthonormal basis of the copying machine Hilbert space. This copier is initially prepared in a particular state \(| X \rangle_x \). The action of the cloning transformation can be specified by a unitary transformation acting on the basis vectors of the tensor product space of the original quantum system \(| \phi_i \rangle_{a_0}, \) the copier, and an additional \( N \)-dimensional system which is to become the copy (which is initially prepared in an arbitrary state \(| 0 \rangle_{a_1} \)). The corresponding transformation \( U_2 \) is given by

\[
U_2 \left| \phi_i \right\rangle_{a_0} \left| 0 \right\rangle_{a_1} \left| X \right\rangle_x = c \left| \phi_i \right\rangle_{a_0} \left| \phi_i \right\rangle_{a_1} \left| X_i \right\rangle_x + d \sum_{j \neq i} \left| \phi_j \right\rangle_{a_0} \left| \phi_i \right\rangle_{a_1} \left| X_j \right\rangle_x, \tag{21}
\]

where \( i = 1, \cdots, N, c^2 = 2/(N+1), \) and \( d^2 = 1/2(N+1). \)

For a three-qubit system, we have eight basis vectors \(| \phi_1 \rangle = |000\rangle, | \phi_2 \rangle = |001\rangle, | \phi_3 \rangle = |010\rangle, | \phi_4 \rangle = |011\rangle, | \phi_5 \rangle = |100\rangle, | \phi_6 \rangle = |101\rangle, | \phi_7 \rangle = |110\rangle, \) and \(| \phi_8 \rangle = |111\rangle \). So that the
original three-qubit GHZ state (15) can be simply expressed as \( |\psi\rangle = (|\phi_1\rangle + |\phi_8\rangle)/\sqrt{2} \).

The copying is now performed by the transformation (21) with \( N = 8 \). After cloning, we get the total output state:

\[
|\psi^{(\text{out})}_{\text{total}}\rangle = \frac{1}{\sqrt{2}} U_2[ (|\phi_1\rangle_{a_0} + |\phi_8\rangle_{a_0}) |0\rangle_{a_1} |X\rangle_x].
\]

(22)

Then performing trace over the cloner, from Eq.(22) we obtain a six-qubit density operator \( \hat{\rho}^{(\text{out})}_{102030} \), which also describes two nonlocal three-qubit systems \( \hat{\rho}^{(\text{out})}_{112131} \). These two three-qubit systems are the clones of the original three-qubit GHZ state in Eq.(4.1) and they are described by the same density operator as:

\[
\hat{\rho}^{(\text{out})}_{102030} = \frac{1}{3}(|111\rangle\langle111| + |000\rangle\langle000|) + \frac{5}{18}(|111\rangle\langle000| \\
+ |000\rangle\langle111|) + \frac{1}{18}(|110\rangle\langle110| + |011\rangle\langle011| \\
+ |101\rangle\langle101| + |100\rangle\langle100| + |010\rangle\langle010| \\
+ |001\rangle\langle001|)
\]

(23)

For the output state (23) we find the nonzero entanglement tensors \( M_{xxx} = 5/9 \), \( M_{xyy} = M_{yxy} = M_{yyx} = -5/9 \), and \( M_{zz}(1,2) = M_{zz}(1,3) = M_{zz}(2,3) = 5/9 \). While all other entanglement tensors vanish as well as the coherence vectors. Then the values the entanglement measures \( E_3 \) and \( E_2(m,n) \) are are found to be

\[
E_3 = \frac{25}{81}, \quad E_2(1,2) = E_2(2,3) = E_2(1,3) = \frac{25}{243},
\]

(24)

which implies that entanglements are broadcasted through nonlocal quantum copying, but entanglement of the copied state is less than that of the original state. Comparing Eq.(24) with Eq.(20), we see that non-local cloning is much efficient than the local copying process for broadcasting entanglement.

Finally, it is interesting to compare the fidelity \( F_1 \) of the output density operator after local copying relative to \( |\psi\rangle \) with the fidelity \( F_2 \) of the output density operator after non-local copying relative to \( |\psi\rangle \). The fidelity of a density matrix \( \rho \) relative to \( |\psi\rangle \) is defined by \( F = \langle \psi | \rho | \psi \rangle \). Through calculating, we find that \( F_1 = 91/216 \) and \( F_2 = 11/18 \). Therefore,
we can see that the output state after non-local copying is closer to the original state than
the output state after local copying.

In conclusion, we have proposed entanglement measures for a three-particle system and
applied them to the study of broadcasting of entanglement for a thrr-qubit GHZ state. By
using local and non-local cloning transformations, we have shown that entanglement of the
three-qubit GHZ state can be locally or no-locally copied with the help of local quantum
copiers or non-local quantum copiers, and that the degree of entanglement of the resultant
copies is reduced. And we have found that non-local cloning is much more efficient than
local cloning for the broadcasting of entanglement.

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