THE $L^2$-BOUNDEDNESS OF THE VARIATIONAL CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. In this paper, we verify the $L^2$-boundedness for the jump functions and variations of Calderón-Zygmund singular integral operators with the underlying kernels satisfying
\[
\int_{\varepsilon \leq |x-y| \leq N} K(x,y)dy = \int_{\varepsilon \leq |x-y| \leq N} K(x,y)dx = 0 \quad \forall 0 < \varepsilon \leq N < \infty,
\]
in addition to some proper size and smooth conditions. This result should be the first general criteria for the variational inequalities for kernels not necessarily of convolution type. The $L^2$-boundedness assumption that we verified here is also the starting point of the related results on the (sharp) weighted norm inequalities appeared in many recent papers.

1. INTRODUCTION

A singular integral operator in $\mathbb{R}^n$ is a continuous linear mapping $T$ from test functions $\mathcal{D}(\mathbb{R}^n)$ into distributions $\mathcal{D}'(\mathbb{R}^n)$ associated to a Calderón-Zygmund standard kernel $K(x,y)$ in the sense that
\[
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dydx
\]
whenever $f \in \mathcal{D}(\mathbb{R}^n)$ and $x$ not in the support of $f$. If $T$ admits further a bounded extension on $L^2(\mathbb{R}^n)$, that is $||Tf||_{L^2} \lesssim ||f||_{L^2}$, $\forall f \in C_0^\infty(\mathbb{R}^n)$, then $T$ is called a Calderón-Zygmund operator associated with the standard kernel $K$. The $L^2$ boundedness criteria for singular integrals, commonly known as $T1$ or $Tb$ theorems, which has arisen from harmonic analysis and partial differential equations. For $\varepsilon > 0$,
we define the truncated Calderón-Zygmund operator

\[ T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y) f(y) dy. \]

The family of truncated Calderón-Zygmund operators \( \{T_\varepsilon\}_{\varepsilon>0} \) will be denoted by \( T \) for simplicity.

Let \( 2 < q \leq \infty \), the \( q \)-variation operator for this family \( T \) is defined by

\[ V_q(T f)(x) := V_q(T_\varepsilon f(x) : \varepsilon > 0) = \sup_{\varepsilon_j \searrow 0} \left( \sum_{j=1}^{\infty} |T_{\varepsilon_{j+1}} f(x) - T_{\varepsilon_j} f(x)|^q \right)^{1/q}, \tag{1.2} \]

where the supremum is taken over all sequences \( \{\varepsilon_j\} \) decreasing to zero. Note that when \( q = \infty \), this is just the maximal Calderón-Zygmund operator.

Motivated by its studies in probability theory and ergodic theory [24, 33, 18, 2, 1, 13, 34, 18, 20, 19], the first variational inequalities associated with singular integrals were established in [4] in the case of the Hilbert transform. Later, this result was generalized to higher dimensions for homogeneous singular integrals [5, 21, 28, 6]. The results for singular integrals with kernels of convolution type but without homogeneous properties seemingly first appeared in [30] under the additional assumption

\[ \int_{|x|>\varepsilon \leq N} K(x) dx = 0 \quad \forall 0 < \varepsilon \leq N < \infty. \]

Even though this subject on variational inequalities has attracted a lot of attention from analysts and there appeared many papers [14, 23, 10, 21, 28, 32, 30, 31, 36, 22] ranging from the strengthened versions of Bourgain’s variational estimates [31] to the variational Carleson theorem [32] and the dimension-free variational estimates [3], there are very few complete results on the Calderón-Zygmund operators associated with standard kernels of non-convolution type except the individual works like the one on Calderón’s commutators [7] or the one on Cauchy integrals [28]. Indeed, in these papers [17, 8, 11, 27, 26], the \( L^2 \)-variational inequality

\[ \|V_q(T f)\|_{L^2} \leq \|f\|_{L^2}, \quad \forall f \in L^2(\mathbb{R}^n), \tag{1.3} \]

has a priori been assumed to complete their results on the (sharp) weighted estimates of variational Calderón-Zygmund operators. Note that this assumption is quite natural since the \( L^2 \)-boundedness of \( \infty \)-variational (or the maximal) Calderón-Zygmund operator are known to be true (see e.g. [15, 35]).

\[ \|\sup_{\varepsilon>0} |T_{\varepsilon} f|\|_{L^2} \leq \|f\|_{L^2}. \tag{1.4} \]

However with a moment of thought, the assumption (1.3) for \( q < \infty \) is wrong in the general setting because of the following two reasons. First, it is well-known that the validity of the \( q \)-variational inequalities with \( q < \infty \) would imply immediately the a.e. convergence of truncated Calderón-Zygmund operators without the density argument; Secondly, there exist numerous examples such as the fractional singular integrals with complex-valued powers, which do not admit the a.e. convergence as \( \varepsilon \to 0 \).

On the other hand, with the maximal inequality (1.4) at hand, by some compactness and density arguments, there exists a decreasing subsequence \( \{\varepsilon_j\} \) with \( \varepsilon_j \to 0 \) as \( j \to \infty \) such that

\[ \forall f \in L^2(\mathbb{R}^n), \quad T_{\varepsilon_j} f \text{ converges a.e. as } j \to \infty. \tag{1.5} \]

One can find all the above assertions for instance in [15, 35].

Thus the following conjecture for general Calderón-Zygmund operators arises naturally.
Conjecture. Let $T$ be a Calderón-Zygmund operator with a standard kernel satisfying (1.5) for a fixed sequence $\{e_j\}$; then for $2 < q < \infty$, we have
\[ \|V_q(T_{e_j} f) : j \in \mathbb{N})\|_{L^q} \leq \|f\|_{L^2} \ \forall f \in L^2(\mathbb{R}^n). \]

In the present paper, we give a positive answer to the conjecture for kernels satisfying
\[ \int_{e \leq |x-y| \leq N} K(x,y)dy = \int_{e \leq |x-y| \leq N} K(x,y)dx = 0 \ \forall 0 < e \leq N < \infty. \]
and verify this $L^2$-boundedness assumption (1.3) for a large class of Calderón-Zygmund operators with kernels not necessarily of convolution type. Actually, we will prove a slightly stronger estimate, that is, the jump estimate. Let $\mathcal{F} = \{F_t : t \in \mathbb{R}_+\}$ be a family of Lebesgue measurable functions defined on $\mathbb{R}^n$. Recall that the $\lambda$-jump quantity $N_{\lambda}(\mathcal{F})$ for $\lambda > 0$ is defined as the supremum of $N$ over all increasing sequences $\{t_k \in I \subset \mathbb{R}_+ : 0 \leq k \leq N\}$ such that
\[ \min_{k \in \{1, \ldots, N\}} |F_{t_k}(x) - F_{t_{k-1}}(x)| > \lambda. \]

Recall that a standard Calderón-Zygmund kernel is a complex-valued function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ satisfying
\[ |K(x,y)| \leq \frac{1}{|x-y|^n} \]
and there exists some $\theta \in (0, 1)$ such that for any $h \in \mathbb{R}^n$ with $2|h| \leq |x-y|$, 
\[ |K(x+h,y) - K(x,y)| + |K(x,y+h) - K(x,y)| \leq \frac{|h|^{\theta}}{|x-y|^{n+\theta}}. \]

Motivated by the study of the sharp weighted normed inequalities in papers such as [25, 8, 11], we will consider a weaker condition called the Dini condition
\[ |K(x+h,y) - K(x,y)| + |K(x,y+h) - K(x,y)| \leq \frac{\omega(|h|/|x-y|)}{|x-y|^{n}}, \]
where $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is a function called a modulus of continuity of finite Dini norm
\[ \|\omega\|_{Dini} := \int_0^1 \omega(t)\frac{dt}{t} < \infty. \]
Let us recall some basic properties of a modulus of continuity which will be used implicitly in the paper. A modulus of continuity is sub-additive in the sense that
\[ u \leq t + s \Rightarrow \omega(u) \leq \omega(t) + \omega(s). \]
Substituting $s = 0$ one sees that $\omega(u) \leq \omega(t)$ for all $0 \leq u \leq t$. It is easy to check that for any $c > 0$ the integral in (1.9) can be equivalently replaced by the sum over $2^{-j/c}$ with $j \in \mathbb{N}$ up to a $c$-dependent multiplicative constant. The basic example is $\omega(t) = t^\theta$, which goes back to the Lipschitz condition with regularity index $\theta$. Note that the composition or the sum of two modulus of continuity is again a modulus of continuity. In particular, if $\omega(t)$ is a modulus of continuity and $\theta \in (0, 1)$, then $\omega(t^\theta)$ and $\omega(t^\theta)$ are also moduli of continuity.

Now we present our main result.
Theorem 1.1. Let $K$ be a Calderón-Zygmund kernel satisfying (1.6) and (1.8) with some modulus of continuity $\omega$. If in addition, for any $0 < \varepsilon \leq N$

(1.10) \[
\int_{|x-y| \leq N} K(x, y) dy = \int_{|x-y| \leq N} K(x, y) dx = 0,
\]

then

(1.11) \[\sup_{\lambda > 0} \| \lambda \sqrt{N_{\lambda}(T f)} \|_{L^2} \leq (\|\omega^{1/2}\|_{L^1} + 1) \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n)\]

and for all $2 < q \leq \infty$,

(1.12) \[\|V_q(T f)\|_{L^2} \leq (\|\omega^{1/2}\|_{L^1} + 1) \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n).\]

This result should be the first general criteria for the variational inequalities for kernels not necessarily of convolution type. As mentioned previously, the $L^2$-boundedness assumption that we verified here is also the starting point of the related results on the (sharp) weighted norm inequalities appeared in many recent papers, see e.g. [17, 8, 11, 27, 26] etc.

Regarding the proof of Theorem 1.1 we will show only the jump estimate (1.11) since the variational estimate (1.12) can be proven similarly or can be deduced from the jump estimate by first obtaining all the $L^p$-variational estimates via Calderón-Zygmund theory as in [26, 27] and then using the interpolation techniques as in Lemma 2.1 of [21]. Even though we shall show estimate (1.11) in Theorem 1.1 by checking separately the corresponding inequalities for the dyadic jump and the short variation as in most of the previously-cited papers (especially see Lemma 1.3 in [21]), there appear a lot of difficulties. Indeed, for the dyadic jump estimate we have to establish Theorem 3.1 and two square function estimates (4.1) and (4.7), all of which are results of $T1$ type and the kernels of convolution type fails completely in the present setting since the rapid decay estimates are not available, and our proof is based on again a $T1$ type argument.

The rest of this paper is organized as follows. In Section 2, we will show Lemma 2.1 and 2.3 which are key ingredients in dealing with the dyadic jump estimate. Then we establish Theorem 3.1 in Section 3. In Section 4 and Section 5, we will show Theorem 1.1. More precisely, in Section 4, we will show the following dyadic jump estimate,

(1.13) \[\sup_{\lambda > 0} \| \lambda \sqrt{N_{\lambda}(T_{2\lambda} f)} \|_{L^2} \leq \|f\|_{L^2};\]

while in Section 5, we will establish the short variational estimate,

(1.14) \[\|S_2(T f)\|_{L^2} \leq \|f\|_{L^2}\]

with

\[
\begin{align*}
S_2(T f)(x) &= \left( \sum_{j \in \mathbb{Z}} |V_{2, j}(T f)(x)|^2 \right)^{1/2}; \\
V_{2, j}(T f)(x) &= \left( \sup_{2^{j+1} \leq \tau_0 < \cdots < \tau_{n+1} \leq 2^{j+1}} \sum_{k=0}^{N-1} |T_{\tau_k+1} f(x) - T_{\tau_k} f(x)|^2 \right)^{1/2}.
\end{align*}
\]

Notation. From now on, $p' = p/(p - 1)$ represents the conjugate number of $p \in [1, \infty)$; $X \leq Y$ stands for $X \leq CY$ for a constant $C > 0$ which is independent of the essential variables depending on $X$ & $Y$; and $X \approx Y$ denotes $X \lesssim Y \lesssim X$. 


2. Two Key Lemmas

Let \( Q_s \) be defined by \( Q_s f = \psi_s * f \) where \( \psi_s(x) = s^{-n} \psi(x/s) \) with \( \psi \in C_c^\infty(B(0,1)) \) a radial real-valued function of mean zero. Noting that \( Q_s \) is self-ajoint and

\[
\int_0^\infty Q_s^2 \frac{ds}{s} = I,
\]

where \( I \) is the identity operator on \( L^2(\mathbb{R}^n) \).

**Lemma 2.1.** Let \( \omega \) be a modulus of continuity and \( T_j \) be a bounded linear operator in \( L^2(\mathbb{R}^n) \) for each \( j \in \mathbb{Z} \). Suppose

\[
\|T_j Q_s\|_{L^2 \rightarrow L^2} \leq \min\left\{ \omega\left(\frac{2^j}{s}\right), \omega\left(\frac{s}{2^j}\right) \right\}
\]

and

\[
\|Q_s T_j\|_{L^2 \rightarrow L^2} \leq \min\left\{ \omega\left(\frac{2^j}{s}\right), \omega\left(\frac{s}{2^j}\right) \right\}.
\]

Then we get

\[
\left\| \sum_{j \in \mathbb{Z}} T_j f \right\|_{L^2} \leq \omega^{1/2} \left\| \omega \right\|_{\mathcal{D}_{\text{mod}}} f \|_{L^2}, \quad \forall f \in L^2(\mathbb{R}^n),
\]

where the infinite sum is convergent in \( L^2 \)-norm.

**Proof.** There exists a function \( g \in L^2 \) with \( \|g\|_{L^2} = 1 \) such that

\[
\left\| \sum_{j \in \mathbb{Z}} T_j f \right\|_{L^2} = \sum_{j \in \mathbb{Z}} \int_0^\infty \int_0^\infty \langle Q_s^2 T_j Q_t^2 f, g \rangle \frac{ds}{s} \frac{dt}{t} =: I + II.
\]

By symmetry, it is enough to consider the case \( s \leq t \), so we only estimate \( I \). We split \( I \) into three parts as follows:

\[
I = \sum_{j \in \mathbb{Z}} \left( \int_0^{2^j} \int_0^\infty \int_0^{2^j} \int_0^{2^j} \right) \langle Q_s^2 T_j Q_t^2 f, g \rangle \frac{ds}{s} \frac{dt}{t}
\]

\[
= I_1 + I_2 + I_3.
\]

We now consider the first part appeared in the splitting of (2.4). By the Hölder inequality, we get

\[
I_1 = \sum_{j \in \mathbb{Z}} \int_0^{2^j} \int_0^\infty \langle \omega^{-1}(s/2^j) Q_s T_j Q_t^2 f, \omega^{1/2}(s/2^j) Q_s g \rangle \frac{ds}{s} \frac{dt}{t}
\]

\[
\leq \left( \sum_{j \in \mathbb{Z}} \int_0^{2^j} \int_0^\infty \omega^{-1}(s/2^j) \|Q_s T_j Q_t^2 f\|_{L^2}^2 \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \int_0^{2^j} \int_0^\infty \omega(s/2^j) \|Q_s g\|_{L^2}^2 \frac{ds}{s} \frac{dt}{t} \right)^{1/2}.
\]

Since \( s \leq 2^j \), by \( \|Q_s T_j f\|_{L^2} \leq \omega(s/2^j) \|f\|_{L^2} \), then

\[
I_1 \leq \left( \int_0^\infty \sum_{2^{2j} \geq t} \omega(s/2^j) \frac{ds}{s} \|Q_t^2 f\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \left( \int_0^\infty \sum_{2^{2j} \geq t} \omega(s/2^j) \frac{dt}{t} \|Q_s g\|_{L^2}^2 \frac{ds}{s} \right)^{1/2}.
\]
Since
\[
\sum_{2^{j} \geq t} \int_{0}^{s} \frac{\omega(s/2^j)}{s} ds \leq \left( \sum_{2^{j} \geq t} \omega^{1/2}(t/2^j) \right) \int_{0}^{1} \omega^{1/2}(s) \frac{ds}{s} \approx \|\omega^{1/2}\|_{\text{Dini}}^2,
\]
and
\[
\int_{s}^{\infty} \sum_{2^{j} \geq t} \frac{\omega(s/2^j)}{t} dt \leq \int_{0}^{1} \left( \sum_{2^{j} \geq t} \omega^{1/2}(t/2^j) \right) \omega^{1/2}(t) \frac{dt}{t} \approx \|\omega^{1/2}\|_{\text{Dini}}^2.
\]
Therefore,
\[
I_1 \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \left( \int_{0}^{\infty} \|Q_s^2 f\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \|g\|_{L^2} \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \|f\|_{L^2}.
\]
Similarly, for $I_2$, by $\|Q_s T_{f} f\|_{L^2} \leq \omega(2^{-j} s) \|f\|_{L^2}$, we get
\[
I_2 \leq \left( \int_{0}^{\infty} \int_{0}^{t} \sum_{2^{j} \geq s} \omega(2^{-j} s) \|Q_s^2 f\|_{L^2}^2 \frac{ds dt}{s t} \right)^{1/2} \left( \int_{0}^{\infty} \int_{s}^{\infty} \sum_{2^{j} \geq s} \omega(2^{-j} s) \frac{dt}{t} \|Q_s g\|_{L^2}^2 \frac{ds}{s} \right)^{1/2}.
\]
Note that $s \leq t$, $t \geq 2^{j} \geq s^{1/2} t^{1/2} \geq s$.

\[
\int_{s}^{\infty} \sum_{2^{j} \geq s} \omega(2^{-j} s) \frac{dt}{t} \leq \left( \sum_{2^{j} \geq s} \omega^{1/2}(2^{-j} s) \right) \int_{s}^{\infty} \omega^{1/2}((s/t)^{1/2}) \frac{dt}{t} \approx \|\omega^{1/2}\|_{\text{Dini}}^2,
\]
and
\[
\int_{0}^{t} \sum_{2^{j} \geq s} \omega(2^{-j} s) \frac{ds}{s} \leq \int_{0}^{t} \left( \sum_{2^{j} \geq s} \omega^{1/2}(2^{-j} s) \right) \omega^{1/2}((s/t)^{1/2}) \frac{ds}{s} \approx \|\omega^{1/2}\|_{\text{Dini}}^2,
\]
then we get
\[
I_2 \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \left( \int_{0}^{\infty} \|Q_s f\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \left( \int_{0}^{\infty} \|Q_s g\|_{L^2}^2 \frac{ds}{s} \right)^{1/2} \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \|f\|_{L^2}.
\]
Let us deal with $I_3$. Similarly, by $\|T_{f} Q_s f\|_{L^2} \leq \omega(2^{j} / t) \|f\|_{L^2}$, we get
\[
I_3 \leq \left( \int_{0}^{\infty} \int_{0}^{t} \sum_{2^{j} \geq s} \omega(2^{j} / t) \|Q_s^2 f\|_{L^2}^2 \frac{ds dt}{s t} \right)^{1/2} \left( \int_{0}^{\infty} \int_{0}^{t} \sum_{2^{j} \geq s} \omega(2^{j} / t) \|Q_s^2 g\|_{L^2}^2 \frac{ds dt}{s t} \right)^{1/2}.
\]
Since
\[
\int_{0}^{t} \sum_{2^{j} \geq s} \omega(2^{j} / t) \frac{ds}{s} \leq \left( \sum_{2^{j} \geq s} \omega^{1/2}(2^{j} / t) \right) \int_{0}^{t} \omega^{1/2}((s/t)^{1/2}) \frac{ds}{s} \approx \|\omega^{1/2}\|_{\text{Dini}}^2,
\]
and
\[
\int_{s}^{\infty} \sum_{2^{j} \geq s} \omega(2^{j} / t) \frac{dt}{t} \leq \int_{s}^{\infty} \left( \sum_{2^{j} \geq s} \omega^{1/2}(2^{j} / t) \right) \omega^{1/2}((s/t)^{1/2}) \frac{dt}{t} \approx \|\omega^{1/2}\|_{\text{Dini}}^2,
\]
then we get
\[
I_3 \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \|f\|_{L^2}.
\]
Together with the estimates of $I_1$, $I_2$ and $I_3$, we get
\[
I \leq \|\omega^{1/2}\|_{\text{Dini}}^2 \|f\|_{L^2}.
\]
**Remark 2.2.** Using the same proof, under the same assumptions as in Lemma 2.1 one can show
\[
\left\| \sum_{j \in \mathbb{Z}} \varepsilon_j T_j f \right\|_{L^2} \lesssim \|\omega^{1/2}\|_{\text{Dini}} \|f\|_{L^2}
\]
uniformly for all independent Rademacher sequences \((\varepsilon_j)_j\). Then taking averages, we get the square function estimate
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_j f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \|\omega^{1/2}\|_{\text{Dini}} \|f\|_{L^2}.
\]

However, we observe that the above square function requires less condition as in the following Lemma, which is crucial for the application in the proof of Theorem 1.1.

**Lemma 2.3.** Let \(\omega\) be a modulus of continuity and \(T_j\) be a bounded linear operator in \(L^2(\mathbb{R}^n)\) for each \(j \in \mathbb{Z}\). If
\[
\|T_j Q_s f\|_{L^2 \to L^2} \lesssim \min \left\{ \omega\left(\frac{2^j}{s}\right), \omega\left(\frac{s}{2^j}\right) \right\},
\]
then we get
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_j f|^2 \right)^{1/2} \right\|_{L^2} \lesssim \|\omega\|_{\text{Dini}} \|f\|_{L^2}, \forall f \in L^2(\mathbb{R}^n).
\]

**Remark 2.4.** Note that \(\|\omega\|_{\text{Dini}} \lesssim \|\omega^{1/2}\|_{\text{Dini}}^2\), the square function estimate is better than (2.5).

**Proof.** There exists \(\{g_j\} \in L^2(\ell^2)\) with \(\sum_{j \in \mathbb{Z}} \|g_j\|^2_{L^2} = 1\) such that
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_j f|^2 \right)^{1/2} \right\|_{L^2} = \sum_{j \in \mathbb{Z}} \int_0^{2^j} \langle T_j Q_s f, g_j \rangle \frac{ds}{s} + \sum_{j \in \mathbb{Z}} \int_0^\infty \langle T_j Q_s f, g_j \rangle \frac{ds}{s} =: I + II.
\]
For \(I\), using the Hölder inequality and \(\|T_j Q_s f\|_{L^2} \lesssim \omega\left(\frac{s}{2^j}\right)\|f\|_{L^2}\), we get
\[
I = \sum_{j \in \mathbb{Z}} \int_0^{2^j} \langle \omega^{-\frac{1}{2}}(\frac{s}{2^j}) T_j Q_s f, \omega^{-\frac{1}{2}}(\frac{s}{2^j}) g_j \rangle \frac{ds}{s} \leq \left( \int_0^\infty \left( \sum_{2^j \leq s} \omega\left(\frac{s}{2^j}\right) \|Q_s f\|_{L^2}^2 \frac{ds}{s} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \|g_j\|_{L^2}^2 \right)^{1/2} \right)^2.
\]
Since \(\sum_{2^j \leq s} \omega\left(\frac{s}{2^j}\right) \approx \|\omega\|_{\text{Dini}}\) and \(\int_0^{2^j} \omega\left(\frac{s}{2^j}\right) \frac{ds}{s} = \|\omega\|_{\text{Dini}}\), then we get
\[
I \leq \|\omega\|_{\text{Dini}} \left( \int_0^\infty \|Q_s f\|_{L^2}^2 \frac{ds}{s} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \|g_j\|_{L^2}^2 \right)^{1/2} \leq \|\omega\|_{\text{Dini}} \|f\|_{L^2}.
\]

For \(II\), similarly, using \(\|T_j Q_s f\|_{L^2} \lesssim \omega\left(\frac{2^j}{s}\right)\|f\|_{L^2}\), we get
\[
II \leq \left( \int_0^\infty \sum_{2^j \leq s} \omega\left(\frac{2^j}{s}\right) \|Q_s f\|_{L^2}^2 \frac{ds}{s} \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \int_0^\infty \omega\left(\frac{2^j}{s}\right) \|g_j\|_{L^2}^2 \frac{ds}{s} \right)^{1/2} \leq \|\omega\|_{\text{Dini}} \|f\|_{L^2}.
\]
Together with the estimates of \(I\) and \(II\), we finish the proof of Lemma 2.3. \(\square\)
3. A T1 TYPE THEOREM

In the course of establishing the main result, we show the following T1 type theorem, which might be of independent interest. Given a Calderón-Zygmund kernel $K$, we denote

$$\sigma_j f(x) := \int_{2^j \leq |x-y| < 2^{j+1}} K(x,y) f(y) \, dy, \quad \forall j \in \mathbb{Z}.$$ 

**Theorem 3.1.** Let $K$ be a Calderón-Zygmund kernel satisfying (1.6) and (1.8) with some modulus of continuity $\omega$. If in addition, for any $j \in \mathbb{Z}$,

$$\sum_{j=-N}^{M} \sigma_j$$

then $\sum_{j=-N}^{M} \sigma_j$ converges in the strong operator topology as $M,N \to \infty$. Moreover the limit denoted by $T$ satisfies

$$\|T f\|_{L^2} \leq (\|\omega^{1/2}\|_{Dini}^2 + 1) \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n).$$

When the Dini condition is replaced by the Lipschitz condition, that is, $\omega(t) = t^\theta$ with $\theta \in (0,1)$, Theorem 3.1 has been previously established, see e.g. Proposition 8.5.3 of [15].

**Proof.** We split $K(x,y)$ into $K(x,y) = \sum_{j \in \mathbb{Z}} K_j(x,y)$, where $K_j(x,y) = K(x,y) \chi_{(2^j \leq |x-y| < 2^{j+1})}$. Then $T f = \sum_{j \in \mathbb{Z}} \sigma_j f$, where

$$\sigma_j f(x) := \int_{\mathbb{R}^n} K_j(x,y) f(y) \, dy.$$ 

By (1.6), it is easy to verify that for any fixed $j \in \mathbb{Z}$,

$$|K_j(x,y)| \leq \frac{1}{|x-y|^n} \chi_{(2^j \leq |x-y| < 2^{j+1})},$$

and by (1.8) for $2|h| \leq 2^j$, we get

$$|K_j(x,y + h) - K_j(x,y)| \leq \frac{\omega(|h|/|x-y|)}{|x-y|^n} \chi_{(2^j \leq |x-y-h| < 2^{j+1})} + \frac{1}{|x-y|^n} \chi_{(2^j \leq |x-y-h| < 2^{j+1})}.$$ 

Note that

$$\chi_{(2^j \leq |x-y-h| < 2^{j+1})} - \chi_{(2^j \leq |x-y| < 2^{j+1})} \neq 0,$$

if and only if at least one of the following two statements holds:

(i) $2^{j+1} \leq |x-y-h| < 2^{j+1}$ and $|x-y| < 2^j$; or $|x-y| \geq 2^{j+1}$;

(ii) $2^j \leq |x-y| < 2^{j+1}$ and $|x-y-h| < 2^j$; or $|x-y-h| \geq 2^{j+1}$.

This together with the fact that $2|h| \leq 2^j$ implies the following two cases:

(i) $2^j - |h| \leq |x-y| \leq 2^j + |h|$;

(ii) $2^{j+1} - |h| \leq |x-y| \leq 2^{j+1} + |h|$.

Therefore by (1.6) we get for $2|h| \leq 2^j$

$$|K_j(x,y + h) - K_j(x,y)| \leq \frac{\omega(|h|/|x-y|)}{|x-y|^n} \chi_{(2^j \leq |x-y-h| < 2^{j+1})} + \frac{1}{|x-y|^n} \chi_{(2^j \leq |x-y-h| < 2^{j+1})}.$$
Similarly, we get for $2^j |h| \leq 2^j$,

$$\left| K_j(x + h, y) - K_j(x, y) \right| \leq \frac{\omega(|h|/|x - y|)}{|x - y|^{n}} \chi(2^{j/2} \leq |x - y| \leq 2^{j+2}) + \frac{1}{2^m} \chi(2^{j/2} \leq |x - y| \leq 2^{j+2} |h| |y|^{2^{j+1}} + |y|^{2^{j+1}} + |h|).$$

In the following we would like to use (3.3), (3.4) and $\sigma_j = 0$ (resp. (3.3), (3.5) and $\sigma_j = 0$) for any $j \in \mathbb{Z}$, to prove that

$$\|\sigma_j Q_s f\|_{L^2} \leq \min \{\omega(\frac{2^j}{s}) \chi(2^{j/2} \leq |x - y| \leq 2^{j+2}), \omega(\frac{2^j}{s}) \chi(2^{j/2} \leq |x - y| \leq 2^{j+2}) \} \leq \|Q_s f\|_{L^2} \leq \min \{\omega(\frac{2^j}{s}) \chi(2^{j/2} \leq |x - y| \leq 2^{j+2}), \omega(\frac{2^j}{s}) \chi(2^{j/2} \leq |x - y| \leq 2^{j+2}) \}$$

where $\omega(t) = \omega(t) + t^\theta$, with some $\theta \in (0, 1)$ and $\|\omega_1^{1/2}\|_{\text{Dini}} \leq \|\omega_1^{1/2}\|_{\text{Dini}} + 1$. Then by Lemma 2.1, we get

$$\|T f\|_{L^2} \leq \|\omega_1^{1/2}\|_{\text{Dini}} + 1 \|f\|_{L^2}.$$  

First, we are ready to verify the first estimate in (3.6). First of all, for the case of $2^{j-1} \leq s$, by $\sigma_j = 0$ for any fixed $j \in \mathbb{Z}$, and (3.3)

$$\|\sigma_j Q_s f\|_{L^2} \leq (\frac{2^j}{s})^{1/2} (M^2 f(x) + M f(x))$$

Here $M f(x)$ is the usual Hardy-Littlewood maximal function and $M^2 f = M(M f)$. Then we get

$$\|\sigma_j Q_s f\|_{L^2} \leq \|\sigma_j Q_s f\|_{L^2} \leq (\frac{2^j}{s})^{1/2} \|f\|_{L^2}.$$  

Next, for $s \leq 2^{j-1}$, using $Q_s = 0$ and the estimate (3.4), we get

$$\|\sigma_j Q_s f\|_{L^2} \leq \|\sigma_j Q_s f\|_{L^2} \leq \|\sigma_j Q_s f\|_{L^2} \leq (\frac{s}{2^j})^{1/2} \|M f\|_{L^2}.$$
where $1 < p < 2$ can be chosen arbitrarily. Then we get for some $\theta \in (0, 1)$,

$$\|\sigma_j Q_s f\|_{L^2} \lesssim (\omega(S/2^j)) + (S/2^j)\|f\|_{L^2}.$$ 

This together with (3.8) implies the desired estimate (3.6).

Now we are ready to prove the desired estimate (3.6). First, for the case of $2^{j-1} \leq s$, by $\sigma_j^* 1 = 0$ for any fixed $j \in \mathbb{Z}$, (3.3) and the smoothness of $\psi$, we get

$$|Q_s \sigma_j f(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\psi_s(x-z) - \psi_s(x-y)) K_j(z,y) dz f(y) dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} \int_{|z| \leq s} |\psi_s(x-z) - \psi_s(x-y)||K_j(z,y)| dz |f(y)| dy$$

$$+ \int_{\mathbb{R}^n} \int_{|z| > s} \frac{2^j}{|z-y|^{n+1}} |f(y)| dy |\psi_s(x-z)| dz + \int_{\mathbb{R}^n} \int_{|z| > s} \frac{2^j}{|z-y|^{n+1}} |\psi_s(x-y)||f(y)| dy$$

$$\lesssim \left( \frac{2^j}{s} \right)^{1/2} \frac{1}{s^n} \int_{|x-z| \leq 4s} \int_{|z| \leq s} \frac{2^{j/2}|z-y|^{1/2}}{(2^j + |z-y|)^{n+1}} |f(y)| dy + \frac{2^j}{s} (M^2 f(x) + M f(x))$$

$$\lesssim \left( \frac{2^j}{s} \right)^{1/2} (M f(x) + M^2 f(x)).$$

Then we get

$$\|Q_s \sigma_j f\|_{L^2} \lesssim \left( \frac{2^j}{s} \right)^{1/2} \|f\|_{L^2}. \tag{3.9}$$

Thus it remains to show the case of $s \leq 2^{j-1}$. Since $Q_s 1 = 0$ for any fixed $s$, then by (3.5), as the arguments after (3.8), one gets

$$|Q_s \sigma_j f(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_s(x-z)(K_j(z,y) - K_j(x,y)) dz f(y) dy \right|$$

$$\lesssim \int_{2^{-j-1} \leq |x-y| \leq 2^{-j+1}} \int_{\mathbb{R}^n} \frac{\omega(|x-z|/|x-y|)}{|x-y|^n} |\psi_s(x-z)| dz |f(y)| dy$$

$$+ \frac{1}{2^n} \int_{2^{-j+1} \leq |x-y| \leq 2^{-j-1}} \int_{\mathbb{R}^n} |\psi_s(x-z)| dz |f(y)| dy$$

$$\lesssim \frac{M f(x)}{2^j} + \left( \frac{S}{2^j} \right)^{1/p'} (M (|f|^p)^{1/p}(x),$$

where $1 < p < 2$ can be chosen arbitrarily. Then we get for some $\theta \in (0, 1)$,

$$\|Q_s \sigma_j f\|_{L^2} \lesssim (\omega(S/2^j)) + (S/2^j)\|f\|_{L^2}.$$ 

This together with (3.9) implies the second estimate in (3.6). \qed

4. Proof of Theorem 1.1—The dyadic jump estimate (1.13)

Now choose a radial $\varphi \in C_c^\infty(B(0, 1/2))$ with $\int \varphi = 1$, and let $\varphi_j(x) = 2^{-jn} \varphi(2^{-j}x)$ for $j \in \mathbb{Z}$. Following [12], we write $T_j f(x) = \int_{|x-y| \geq 2^{j+1}} K(x,y) f(y) dy$ and $T_j f(x) = \int_{|x-y| \leq 2^{j+1}} K(x,y) f(y) dy,$ then

$$T_j f(x) = \varphi_j * T f - \varphi_j * T_j f + (\delta - \varphi_j) * T_j f =: I_1 - I_2 + I_3.$$ 

In the following, we will estimate $I_1$, $I_2$ and $I_3$, respectively.
4.1. Proof of $I_1$. To estimate the term $I_1$, we first give a known lemma which is stated as follows.

**Lemma 4.1.** ([21], [9]) Let $\varphi \in C_0^\infty (\mathbb{R}^n)$ with $\int \varphi = 1$. Then for $1 < p < \infty$,

$$\sup_{x > 0} \| \lambda \sqrt{N_\lambda((\varphi_j * f)_{j \in \mathbb{Z}})} \|_{L^p} \leq \| f \|_{L^p} \forall f \in L^p(\mathbb{R}^n).$$

For $I_1$, by Lemma 4.1 and Theorem 3.1, it is easy to get

$$\sup_{x > 0} \| \lambda \sqrt{N_\lambda((\varphi_j * f)_{j \in \mathbb{Z}})} \|_{L^2} \leq \| T f \|_{L^2} \leq (\| \omega^{1/2} \|_{Dini}^2 + 1) \| f \|_{L^2}.$$

4.2. Proof of $I_2$. For $I_2$, by the definition of the jump quantity and Chebychev’s inequality,

$$\sup_{x > 0} \| \lambda \sqrt{N_\lambda((\varphi_j * T^j f)_{j \in \mathbb{Z}})} \|_{L^2} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |\varphi_j * T^j f|^2 \right)^{1/2} \right\|_{L^2}.$$

So to estimate $I_2$, we need to show

$$\left\| \left( \sum_{j \in \mathbb{Z}} |\varphi_j * T^j f|^2 \right)^{1/2} \right\|_{L^2} \leq (\| \omega^{1/2} \|_{Dini}^2 + 1) \| f \|_{L^2}. \hspace{2cm} (4.1)$$

For any $j \in \mathbb{Z}$, set $A_j := \varphi_j * T^j$ and write the kernel of $A_j$ as

$$a_j(x, y) = \int_{\mathbb{R}^n} \varphi_j(x - z) K^j(z, y) dz,$$

where $K^j(z, y) = K(z, y)\chi_{|z - y| < 2^j}$. Observe that $a_j$ is supported on the set $|x - y| \leq 2^j$.

We will apply Lemma 2.3 to handle the inequality (4.1). By Lemma 2.3, we need to verify that $A_j$ satisfies (2.6). First, we deal with the case of $2^j \leq s$. Since \( \int_{\mathbb{R}^n} K^j(z, y) dz = \lim_{s \to 0} \int_{|z - y| > s} K^j(z, y) dz = 0 \), we get

$$a_j(x, y) = \int_{\mathbb{R}^n} (\varphi_j(x - z) - \varphi_j(x - y)) K^j(z, y) dz.$$ A trivial computation gives that

$$|a_j(x, y)| \leq \int_{|z - y| \leq 2^j} \frac{|y - z|}{2^{j(n+1)}} |K(z, y)| dz \leq \frac{1}{2^j} \chi_{|x - y| \leq 2^j}. \hspace{2cm} (4.2)$$

Since $\int_{\mathbb{R}^n} K^j(\eta, z) dz = \lim_{s \to 0} \int_{|z - \eta| > s} K^j(\eta, z) dz = 0$, then $\int_{\mathbb{R}^n} a_j(x, z) dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_j(x - \eta) K^j(\eta, z) d\eta dz = 0$, we get

$$|A_j \mathcal{Q}_s f(x)| = \left\| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a_j(x, z) \psi_s(z - y) - \psi_s(x - y) dz f(y) dy \right\| \leq \int_{|x - y| \leq s} \int_{|z - y| \leq 2^j} \frac{|z - x|}{s^{n+1}} |dz f(y)| dy \leq \frac{2^j}{s} M f(x).$$

Thus,\n
$$\|A_j \mathcal{Q}_s f\|_{L^2} \leq \frac{2^j}{s} \| f \|_{L^2}. \hspace{2cm} (4.3)$$

Next, we consider the case of $s \leq 2^j$. For $|h| \leq 2^j$, since $\int_{\mathbb{R}^n} K^j(z, y) dz = 0$, we get

$$|a_j(x, y + h) - a_j(x, y)| = \left| \int_{\mathbb{R}^n} (\varphi_j(x - z) - \varphi_j(x - y - h)) K^j(z, y + h) dz - \int_{\mathbb{R}^n} (\varphi_j(x - z) - \varphi_j(x - y)) K^j(z, y) dz \right|.$$
Deduced from the support of $\varphi_j$ and $K^j$, one has $a_j(\cdot, \cdot + h) - a_j(\cdot, \cdot)$ is supported on the subset $|x - y| \lesssim 2^j$. We proceed with the proof by separating the domain of integration in (4.4),

$$
|a_j(x, y + h) - a_j(x, y)|
= \left| \int_{|z| < 2^j} (\varphi_j(x - z) - \varphi_j(x - y - h)) K^j(z, y + h) \, dz - \int_{|z| < 2^j} (\varphi_j(x - z) - \varphi_j(x - y)) K^j(z, y) \, dz \right|
+ \left| \int_{|z| \geq 2^j} (\varphi_j(x - z) - \varphi_j(x - y - h)) K^j(z, y + h) \, dz - \int_{|z| \geq 2^j} (\varphi_j(x - z) - \varphi_j(x - y)) K^j(z, y) \, dz \right|
=: A_1 + A_2.
$$

For $A_1$, by (1.6), we get $|K^j(z, y)| \lesssim \frac{1}{|z - y|^{n+1}}$, then for some $\theta \in (0, 1)$,

$$
A_1 \lesssim \int_{|z| \leq 2^j, |z - y| \leq 2^j} \frac{|z - y - h|}{2^j} \frac{1}{|z - y - 2^j|^{n+\theta}} \, dz + \int_{|z| \leq 2^j, |z - y - 2^j| \leq 2^j} \frac{|z - y|}{2^j} \frac{1}{|z - y|^{n+\theta}} \, dz
\lesssim \frac{|h|^{\theta}}{2^j \theta} \chi(|x - y| \leq 2^j).
$$

For $A_2$, we get

$$
A_2 \lesssim \int_{|z| \leq 2^j} |\varphi_j(x - y) - \varphi_j(x - y - h)||K^j(y, z)| \, dz
+ \int_{|z| \leq 2^j} |\varphi_j(x - z) - \varphi_j(x - y - h)||K^j(y, z + h) - K^j(y, z)| \chi(|y - z - h| \leq 2^j) \, dz
+ \int_{|z| \leq 2^j} |\varphi_j(x - z) - \varphi_j(x - y - h)||K^j(z, y) (\chi(|y - z - h| \leq 2^j) - \chi(|y - z| \leq 2^j))| \, dz
=: A_3 + A_4 + A_5.
$$

For $A_3$, by $|K^j(z, y)| \lesssim \frac{1}{|y - h|^{n+1}}$, we get for $|h| \leq 2^j$,

$$
A_3 \lesssim \int_{|y| \leq 2^j, |y - z| \leq 2^j} \frac{|h|}{2^j} \frac{1}{|y - z|^{n+\theta}} \, dz
\lesssim \frac{|h|^{\theta}}{2^j \theta} \chi(|y| \leq 2^j).
$$

For $A_4$, since $|z - y| \geq 2h$, one has $\frac{|y - z - h|}{2} < |y - z| \leq 2|y - z|$. By the sub-additivity of $\omega(t)$, we get for any fixed $k \in \mathbb{N}_+$, $\omega(kt) \leq k \omega(t)$. Then apply (1.3),

$$
A_4 \lesssim \int_{|h| \leq 2^j, |y - z| \geq 2^j} \frac{|z - y - h|}{2^j} \frac{\omega(|h|/|y - z|)}{|y - z|^{n+\theta}} \, dz
\lesssim \frac{\omega^{1/2}(1)}{2^j} \frac{1}{\omega^{1/2}(2^j/2^j)} \int_{2^j < |y - z| \leq 2^j} \frac{1}{|y - z|^{n+\theta}} \, dz
\lesssim \frac{\omega^{1/2}(1)}{2^j} \frac{1}{\omega^{1/2}} \frac{1}{2^j} \frac{1}{2^k} \omega^{1/2}(2^j/2^j)
\lesssim \frac{\omega^{1/2}}{2^j} \frac{1}{\omega^{1/2}} \frac{1}{\omega^{1/2}} \frac{1}{2^j} \chi(|x - y| \leq 2^j).
$$
To estimate $A_5$, note that
\[ \mathcal{X} \{ |y - z - h| \leq 2^{j+1} \} \neq 0 \]
if and only if at least one of the following two statements holds:
(i) $|y - z| \leq 2^{j+1}$ and $|y - z - h| > 2^{j+1}$;
(ii) $|y - z| \leq 2^{j+1}$ and $|y - z - h| > 2^{j+1}$.
This together with the fact that $|h| \leq 2^j$ implies the following two cases:
(i) $2^{j+1} \leq |y - z| \leq 2^{j+1} + |h|$;
(ii) $2^{j+1} - |h| \leq |y - z| \leq 2^{j+1}.$
Since $|y - z| \geq 2|h|$, then $|y - z| = |z - y|$. Thus,
\[ A_5 \leq \frac{1}{2^{j(n+1)}} \int_{|y - z| \leq 2|y|} \frac{1}{|y - z|^{n-1}} dz \leq \frac{|h|}{2^{j(n+1)}} \mathcal{X} \{ |y - z| \leq 2^j \}. \]
Combined with the estimates of $A_1$ and $A_2$, we finally obtain the desired estimate: for $|h| \leq 2^j$ and some $\theta \in (0, 1)$,
\[ |a_j(x, y + h) - a_j(x, y)| \leq \left( \frac{|h|^{\theta}}{2^j} + \|\omega^{1/2}\|_{Dini} \frac{\omega^{1/2}(\frac{|h|}{2^j})}{2^n} \right) \mathcal{X} \{ |y - z| \leq 2^j \}. \]
(4.5)
Thus for $s \leq 2^j$, it follows from (4.5) by noting $Q_s 1 = 0$,
\[ |A_j Q_s f(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (a_j(x, z) - a_j(x, y)) \psi_s(z - y) dz f(y) dy \right| \leq \int_{|x - y| \leq 2^j} \int_{|z - y| \leq s} \frac{|z - y|^{\theta}}{2^j} \psi_s(z - y) dz f(y) dy \]
\[ + \|\omega^{1/2}\|_{Dini} \int_{|x - y| \leq 2^j} \int_{\mathbb{R}^n} \frac{\omega^{1/2}(|z - y|/2^j)}{2^n} \psi_s(z - y) dz f(y) dy \]
\[ \leq \frac{s^{\theta}}{2^{j\theta}} M f(x) + \|\omega^{1/2}\|_{Dini} \|\omega^{1/2}(\frac{s}{2^j}) M f(x)\|. \]
(4.6)
Hence for $s \leq 2^j$, we also get for some $\theta \in (0, 1)$
\[ \|A_j Q_s f\|_{L^2} \leq \left( \frac{s^{\theta}}{2^{j\theta}} + \|\omega^{1/2}\|_{Dini} \frac{s^{\theta}}{2^j} \right) \|f\|_{L^2}. \]
(4.6)
Together (4.3) with (4.6), we verify for $A_j$ the assumption (2.6) in Lemma 2.3 Therefore we establish (4.1).

4.3. **Proof of I3.** For $I_3$, by the definition of the jump quantity and Chebychev’s inequality,
\[ \sup_{\lambda > 0} \left\| \lambda \sqrt{\mathcal{N}_a(\{|(\delta - \varphi_j) * T_j f| \}_j \}} \right\|_{L^2} \leq \left\| \left( \sum_{j \in \mathbb{Z}} \{|(\delta - \varphi_j) * T_j f| \} \right)^{\frac{1}{2}} \right\|_{L^2}. \]
So to estimate $I_3$, it suffices to prove
\[ \left\| \left( \sum_{j \in \mathbb{Z}} \{|(\delta - \varphi_j) * T_j f| \} \right)^{\frac{1}{2}} \right\|_{L^2} \leq (1 + \|\omega^{1/2}\|_{Dini}) \|f\|_{L^2}. \]
(4.7)
We will apply Lemma 2.3 to deal with the inequality (4.7) by verifying the assumption (2.6) for $B_j = (\delta - \varphi_j) * T_j$. Denote the kernel of $B_j$ as
\[ b_j(x, y) = \int_{\mathbb{R}^n} \varphi_j(x - z)(K(x, y)\chi_{|x - y| \leq 2^{j+1}} - K(z, y)\chi_{|z - y| \leq 2^{j+1}}) dz. \]
Observe that \(b_j\) is supported on the set \(|x-y| > 2^j\) and \(\int_{\mathbb{R}^n} b_j(x,y) \, dy = 0\) by \(\int_{\mathbb{R}^n} K(\eta,y)\chi_{|\eta-y|>2^{j+1}} \, dy = \lim_{N \to \infty} \int_{|\eta-y|\leq N} K(\eta,y)\chi_{|\eta-y|>2^{j+1}} \, dy = 0\). First, we deal with the case of \(2^j \leq s\). Since \(|x-z| \leq 2^j \leq \frac{|x-y|}{2}\), then by (1.8) we get

\[
|b_j(x,y)| \leq \int_{\mathbb{R}^n} |\varphi_j(x-z)||K(x,y) - K(z,y)|\chi_{|\eta-y|>2^{j+1}} \, dz
\]

\[
+ \int_{\mathbb{R}^n} |\varphi_j(x-z)||K(x,y)||\chi_{|\eta-y|>2^{j+1}} - \chi_{|\eta-y|>2^{j+1}}| \, dz
\]

\[
\leq \int_{\mathbb{R}^n} |\varphi_j(x-z)| \omega(|x-z|/|y-x|) |\chi_{|\eta-y|>2^j}| \, dz + \frac{1}{|x-y|^n} \chi_{|\eta-z| \leq 2^{j+2}} \int_{\mathbb{R}^n} |\varphi_j(x-z)| \, dz
\]

where we have used that \(|x-z| \leq 2^j/2, \chi_{|\eta-z|>2^{j+1}} - \chi_{|\eta-y|>2^{j+1}}\) is non-zero if \(2^j < |x-y| \leq 2^{j+2}\). Write \(\omega(t) = \omega(t) + t\). Again, by \(\int_{\mathbb{R}^n} b_j(x,z) \, dz = 0\), we get

\[
|B_j Q_s f(x)| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} b_j(x,z)(\psi_s(z-y) - \psi_s(x-y))dzf(y)dy \right|
\]

\[
\leq \int_{|x-y| \leq s} \int_{2^{j} \leq |x-z| \leq s} \omega_1(2^j/|x-z|) |\psi_s(z-y) - \psi_s(x-y)|dzf(y)dy
\]

\[
+ \int_{\mathbb{R}^n} \int_{|x-z| > s} \omega_1(2^j/|x-z|) |\psi_s(z-y)|dzf(y)dy
\]

\[
+ \int_{\mathbb{R}^n} \int_{|x-z| > s} \omega_1(2^j/|x-z|) |\psi_s(x-y)|dzf(y)dy
\]

\( =: B_1 + B_2 + B_3\).

We will estimate \(B_1, B_2\) and \(B_3\) separately. First for \(B_1\),

\[
B_1 \leq \int_{|x-y| \leq s} \int_{2^{j} \leq |x-z| \leq s} \omega_1(2^j/|x-z|) |x-z| |z|^{n+1} |\psi_s(z-y)|dzf(y)dy
\]

\[
\leq s^n \int_{|x-y| \leq s} |f(y)|dy \int_{2^{j} \leq |x-z| \leq s} \omega_1(2^j/|x-z|) |x-z|^{n-1}dz
\]

\[
\leq Mf(x)\omega_1^{1/2}(1) \sum_{k=-\infty}^{0} \frac{\omega_1^{1/2}(2^k/2^j/s)}{|x-z|^n} \int_{2^{k-1}s \leq |x-z| \leq 2^k s} \frac{1}{|x-z|^{n-1}}dz.
\]

By the sub-additivity of \(\omega_1(t)\), we get for any fixed \(k \in \mathbb{N}_+\), \(\omega_1(kt) \leq k \omega_1(t)\). Then we get

\[
B_1 \leq \omega_1^{1/2}(1)\omega_1^{1/2}(2^j/s)Mf(x) \sum_{k=-\infty}^{0} 2^k 2^{-k/2} \leq \|\omega_1^{1/2}\|_{Dini} \omega_1^{1/2}(2^j/s)Mf(x).
\]

For \(B_2\), we get

\[
B_2 \leq \int_{|x-z| > s} \omega_1(2^j/|x-z|) |x-z|^{n} Mf(z)dz
\]

\[
\leq \omega_1^{1/2}(\frac{2^j}{s}) \sum_{k=0}^{\infty} \omega_1^{1/2}(2^{-k}) \int_{2^{k-1}s \leq |x-z| \leq 2^k s} \frac{Mf(z)}{|x-z|^n}dz
\]
For $B_3$, similarly, we get
\[
B_3 \lesssim Mf(x) \int_{|x-z|<s} \frac{\omega_j(2j|x-z|)}{|x-z|^n} \, dz \lesssim \|\omega_{1/2}\|_{\text{Di}n} \omega_{1/2}(s) M^2 f(x).
\]
Together the estimates of $B_1$, $B_2$ and $B_3$, we get
\[
(4.8) \quad \|B_jQ_sf\|_{L^2} \lesssim \|\omega_{1/2}\|_{\text{Di}n} \omega_{1/2}(s) \|f\|_{L^2} \leq (\omega_{1/2}(s/2))\|\omega_{1/2}\|_{\text{Di}n} \left( \frac{s}{2^j} \right)^{1/2} \|f\|_{L^2}.
\]
Secondly, we deal with the case of $s \leq 2^j$. For $|h| \leq 2^j$,
\[
|b_j(x, y + h) - b_j(x, y)| \leq \left| \int_{\mathbb{R}^n} \varphi_j(x-z)(K_j(x, y + h) - K_j(x, y)) \, dz \right| + \left| \int_{\mathbb{R}^n} \varphi_j(x-z)(K_j(z, y + h) - K_j(z, y)) \, dz \right| =: B_4 + B_5.
\]
For $B_4$, we get
\[
B_4 \lesssim \int_{\mathbb{R}^n} |\varphi_j(x-z)||K(x, y + h) - K(x, y)|\chi_{|x-z|>2^{j+1}} \, dz
\]
\[
+ \int_{\mathbb{R}^n} |\varphi_j(x-z)||K(x, y)||\chi_{|x-y-h|>2^{j+1}} - \chi_{|x-y|>2^j}| \, dz
\]
\[
\lesssim \omega(|h|/|x-y|)\chi_{|x-y|>2^j} + \frac{1}{|x-y|^n} \int_{\mathbb{R}^n} |\varphi_j(x-z)||\chi_{|x-y-h|>2^{j+1}} - \chi_{|x-y|>2^j}| \, dz,
\]
where we used the fact $|h| \leq 2^j$ and $|x-y-h| \geq 2^{j+1}$ imply $|x-y| > 2^j$. For the second quantity of the right hand side, by $|h| \leq 2^j$, then $\chi_{|x-y-h|>2^{j+1}} - \chi_{|x-y|>2^{j+1}}$ is nonzero if $2^{j+1} - |h| \leq |x-y| \leq 2^{j+1} + |h|$. Therefore, one has
\[
B_4 \lesssim \omega(|h|/|x-y|)\chi_{|x-y|>2^j} + \frac{1}{|x-y|^n} \chi_{2^{j+1}-|h| \leq |x-y| \leq 2^{j+1}+|h|}.
\]
Similarly, for $B_5$,
\[
B_5 \lesssim \int_{\mathbb{R}^n} |\varphi_j(x-z)||\omega(|h|/|x-y|)\chi_{|x-y|>2^j} \, dz + \int_{\mathbb{R}^n} |\varphi_j(x-z)||\chi_{|z-y|>2^j} - \chi_{2^{j+1}-|h| \leq |z-y| \leq 2^{j+1}+|h|} \, dz.
\]
For $|x-z| \leq 2^{j/2}$ and $|z-y| > 2^j$, then $2|z-y| \geq |x-y| \geq \frac{|z-y|}{2} > 2^{j/2}$. We get for some $\theta \in (0, 1)$ and $\frac{1}{\rho'} = \theta$,
\[
B_5 \lesssim \omega(|h|/|x-y|)\chi_{|x-y| \geq 2^{j/2}} + \frac{1}{|x-y|^n} \chi_{|z-y| \geq 2^{j/2}} \left( \int_{\mathbb{R}^n} |\varphi_j(x-z)| \rho' \, dz \right)^\frac{1}{\theta'}
\]
\[
\times \left( \int_{2^{j+1}-|h| \leq |z-y| \leq 2^{j+1}+|h|} |z-y|^\theta' \, dz \right)^\frac{1}{\theta}
\]
\[
\lesssim \omega(|h|/|x-y|)\chi_{|x-y| \geq 2^{j/2}} + \frac{|h|^\theta}{|x-y|^n} \chi_{|x-y| \geq 2^{j/2}}.
\]
Combined the estimate of $B_4$ and $B_5$, we get for $|h| \leq 2^j$
\[
(4.9) \quad |b_j(x, y+h) - b_j(x, y)| \lesssim \omega(|h|/|x-y|) + \frac{|h|^\theta}{|x-y|^n} \chi_{|x-y| \geq 2^{j/2}} + \frac{1}{|x-y|^n} \chi_{|2^{j+1}-|h| \leq |x-y| \leq 2^{j+1}+|h|}.
\]
Now we apply (4.9) to estimate \( \|B_j Q_s f\|_{L^2} \). Since \( Q, 1 = 0 \), and \( s \leq 2^j \),
\[
|B_j Q_s f(x)| = \left| \int_{\mathbb{R}^n} \int_{|z-y| \leq 2^j} (b_j(x, z) - b_j(x, y)) \psi_s(z-y)dz f(y)dy \right|
\]
\[
\lesssim \int_{\mathbb{R}^n} \int_{|x-y| \leq 2^j} \frac{\omega_z}{|z-y|^n} |\psi_s(z-y)|dz |f(y)|dy + \int_{\mathbb{R}^n} \int_{|x-y| > 2^j} \frac{\omega_z}{|z-y|^n} |\psi_s(z-y)|dz |f(y)|dy
\]
\[
\lesssim \left( \int_{|x-y| \leq 2^j} \frac{\omega_z}{|z-y|^n} |f(y)|dy \right) \left( \int_{|x-y| > 2^j} \frac{1}{|x-y|^{\theta n}} dy \right)^{1/p'} \left( \int |f(y)|^{p'} dy \right)^{1/p}
\]
\[
\lesssim \omega^{1/2}(s/2^j) \left( \int_{|x-y| \leq 2^j} \frac{\omega_z}{|z-y|^n} |f(y)|dy \right) + \left( \frac{s}{2^j} \right)^{1/p'} \left( \int |f(y)|^{p'} dy \right)^{1/p}
\]
where \( 1 < p < 2 \). Therefore we get for \( s \leq 2^j \) and some \( \theta \in (0, 1) \)
\[
(4.10) \quad \|B_j Q_s f\|_{L^2} \lesssim (\omega^{1/2}(s/2^j))^{1/2} + (\frac{s}{2^j})^{1/p'} \|f\|_{L^2}.
\]
Therefore, combined with (4.8), we verify the assumption (2.6) in Lemma 2.3. Therefore we establish (4.7).

5. PROOF OF THEOREM 1.1—THE SHORT VARIATIONAL ESTIMATE (1.14)

Let \( T_{j,r} f(x) = \int_{2^{j-r} \leq |x-y| \leq 2^{j+r}} K(x, y) f(y)dy \), \( r \in [1, 2] \). Then
\[
S_2(\mathcal{T} f)(x) = \left( \sum_{j \in \mathbb{Z}} \sup_{t_1, \ldots, t_N} \sum_{t_i+1}^{N-1} |T_{j,t_i} f(x) - T_{j,t_{i+1}} f(x)|^2 \right)^{1/2}
\]
\[
= \left( \sum_{j \in \mathbb{Z}} \sup_{t_1, \ldots, t_N} \sum_{t_i+1}^{N-1} |T_{j,t_i, t_{i+1}} f(x)|^2 \right)^{1/2},
\]
where the operator \( T_{j,t_i, t_{i+1}} \) is given by
\[
T_{j,t_i, t_{i+1}} f(x) = \int_{2^{j-r} \leq |x-y| \leq 2^{j+r}} K(x, y) f(y)dy, \quad [t_i, t_{i+1}] \subset [1, 2].
\]
Then by the Calderón identity \( \int_0^\infty Q_s^2 \frac{ds}{s} = \mathcal{I} \), one has
\[
\|S_2(\mathcal{T} f)\|_{L^2}^2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left( \sup_{t_1, \ldots, t_N} \sum_{t_i+1}^{N-1} |T_{j,t_i, t_{i+1}} f(x)|^2 \right) dx
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left( \sup_{t_1, \ldots, t_N} \sum_{t_i+1}^{N-1} \int_0^\infty \left| T_{j,t_i, t_{i+1}} Q_s f(x) \right|^2 \frac{ds}{s} \right) dx
\]
\[
+ \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left( \sup_{t_1, \ldots, t_N} \sum_{t_i+1}^{N-1} \int_0^{2^{j-1}} \left| T_{j,t_i, t_{i+1}} Q_s Q_s f(x) \right|^2 \frac{ds}{s} \right) dx
\]
For $I$, denote by $K_{j,l,s}(x,y)$ the kernel of $T_{j,l,t_1}Q_s$. Since $T_{j,l,t_1}1 = 0$ for any $j \in \mathbb{Z}$, then
\[
\sum_{l=1}^{N-1} |K_{j,l,s}(x,y)| = \sum_{l=1}^{N-1} \left| \int_{2^{-l} < |x-y| \leq 2^{-l+1}} K(x,z)(\psi_s(z-y) - \psi_s(x-y))dz \right| \\
\leq \int_{2^{-l} < |x-y| \leq 2^{-l+1}} |K(x,z)||\psi_s(z-y) - \psi_s(x-y)|dz.
\]
First, by (1.6) and $t_1$, $t_N \in [1,2]$, we get $|K(x,z)||\psi_s(z-y) - \psi_s(x-y)| \lesssim \frac{2^{j/2}}{|x-z|^{n+1}}$, then
\[
\sum_{l=1}^{N-1} |T_{j,l,t_1}Q_s f(x)| \leq \int_{|x-y| \leq 4} \int_{|z| \leq s} \frac{2^{j/2}}{|x-z|^{n+1/2}} |\psi_s(z-y) - \psi_s(x-y)|dz|f(y)|dy \\
+ \int_{\mathbb{R}^n} \int_{|x-y| > s} \frac{2^{j/2}}{|x-z|^{n+1/2}} |\psi_s(z-y) - \psi_s(x-y)|dz|f(y)|dy \\
+ \int_{\mathbb{R}^n} \int_{|x-y| > s} \frac{2^{j/2}}{|x-z|^{n+1/2}} |\psi_s(x-y)|dz|f(y)|dy \\
\lesssim \int_{|x-y| \leq 4} \int_{|z| \leq s} \frac{2^{j/2}}{|x-z|^{n+1/2}} dz|f(y)|dy + \left(\frac{2^{j}}{s}\right)^{1/2}(M^2f(x) + Mf(x)) \\
\lesssim \left(\frac{2^{j}}{s}\right)^{1/2}(Mf(x) + M^2f(x)).
\]
Then by the Hölder’s inequality
\[
I \lesssim \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left( \int_{2^{j-1}}^{2^j} \frac{2^{j}}{s} (MQ_s f(x) + M^2Q_s f(x)) ds \right)^2 dx \\
\lesssim \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left( \int_{2^{j-1}}^{2^j} \frac{2^{j}}{s} ds \right)^2 \left( \int_{2^{j-1}}^{2^j} \frac{2^{j}}{s} (MQ_s f(x) + M^2Q_s f(x)) ds \right) dx \\
\lesssim \int_{0}^{\infty} \int_{\mathbb{R}^n} \left( \sum_{2^{j-1} \leq s \leq 2^j} \frac{2^{j}}{s} \right)^2 (MQ_s f(x) + M^2Q_s f(x))^2 dx ds \\
\lesssim \int_{\mathbb{R}^n} \int_{0}^{\infty} |Q_s f(x)|^2 s^{1/2} ds dx \\
\lesssim \|f\|_{L^2}^2.
\]
Next, for $II$, since $Q_s1 = 0$, then
\[
\sum_{l=1}^{N-1} |T_{j,l,t_1}Q_s f(x)|^2 = \sum_{l=1}^{N-1} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (K(x,z)\chi_{[2^{-l} < |x-z| \leq 2^{-l+1}]} - K(x,y)\chi_{[2^{-l} < |x-y| \leq 2^{-l+1}]}) \psi_s(z-y) dz df(y) dy \right|^2 \\
\lesssim \left( \sum_{l=1}^{N-1} \int_{\mathbb{R}^n} \int_{2^{-l} < |x-z| \leq 2^{-l+1}} |K(x,z) - K(x,y)||\psi_s(z-y)| dz|f(y)| dy \right)^2 \\
+ \sum_{l=1}^{N-1} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,z)||\psi_s(z-y)| dz|f(y)| dy \right)^2 \\
=: II_1 + II_2,
\]
For $II_1$, by $s \leq 2^{j-1}$ and (1.8) we get

$$II_1 \leq \left( \int_{\mathbb{R}^n} \int_{2^{j_1} < |x-z| \leq 2^{j_s}} |K(x,z) - K(x,y)||\psi_s(z-y)| dz |f(y)| dy \right)^2$$

$$\leq \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\omega(|y-z|/|x-y|)}{|x-y|^{p'}} \chi_{\{2^{j_1} < |x-z| \leq 2^{j_s}\}} |\psi_s(z-y)| dz |f(y)| dy \right)^2$$

$$\leq \omega^2 \left( \frac{s}{2^j} \right) (Mf(x))^2.$$

For $II_2$. Note that

$$\chi_{\{2^{j_1} < |x-z| \leq 2^{j_s}\}} - \chi_{\{2^{j_1} < |x-y| \leq 2^{j_s}\}} \neq 0,$$

if and only if at least one of the following four statements holds,

(i) $2^{j_1} < |x-z| \leq 2^{j_s}$ and $|x-y| \leq 2^{j_1}$;  
(ii) $2^{j_1} < |x-z| \leq 2^{j_s}$ and $|x-y| > 2^{j_1}$;  
(iii) $2^{j_1} < |x-y| \leq 2^{j_s}$ and $|x-z| \leq 2^{j_1}$;  
(iv) $2^{j_1} < |x-y| \leq 2^{j_s}$ and $|x-z| > 2^{j_1}$.

This together with the fact that $|y-z| \leq s$ implies the following four cases

(i) $2^{j_1} < |x-z| \leq 2^{j_s}$ and $2^{j_1} < |x-y| \leq 2^{j_1} + s$;  
(ii) $2^{j_1} < |x-z| \leq 2^{j_s}$ and $2^{j_1} + s < |x-y| \leq 2^{j_1}$;  
(iii) $2^{j_1} < |x-y| \leq 2^{j_s}$ and $2^{j_1} < |x-z| \leq 2^{j_1} + s$;  
(iv) $2^{j_1} < |x-y| \leq 2^{j_s}$ and $2^{j_1} + s < |x-z| \leq 2^{j_1}$.

Case (i) and Case (ii) can be dealt with similarly. We only consider Case (i). Taking an arbitrary $1 < p < 2$, by (1.6) and the Hölder inequality, we get

$$II_2 \leq \sum_{l=1}^{N-1} \left( \frac{1}{2^m} \int_{2^{j_1} < |x-z| \leq 2^{j_s} \cap 2^{j_1} < |x-y| \leq 2^{j_1}+s} \int_{\mathbb{R}^n} |\psi_s(z-y)||f(y)| dy dz \right)^2$$

$$\leq \sum_{l=1}^{N-1} \left( \frac{1}{2^m} \int_{2^{j_1} < |x-z| \leq 2^{j_s} \cap 2^{j_1} < |x-y| \leq 2^{j_1}+s} Mf(z) dz \right)^2$$

$$\leq \sum_{l=1}^{N-1} \left( \frac{1}{2^m} \int_{2^{j_1} < |x-z| \leq 2^{j_s} \cap 2^{j_1} < |x-y| \leq 2^{j_1}+s} Mf(z) dz \right)^{2/p} \left( \frac{1}{2^m} \int_{2^{j_1} < |x-z| \leq 2^{j_s} \cap 2^{j_1} < |x-y| \leq 2^{j_1}+s} dz \right)^{2/p'}$$

$$\leq \left( \frac{1}{2^m} \int_{2^{j_1} < |x-z| \leq 2^{j_s} \cap 2^{j_1} < |x-y| \leq 2^{j_1}+s} Mf(z) dz \right)^{2/p} \left( \frac{s}{2} \right)^{2/p'}$$

$$\leq (M(Mf)^p)^{2/p} (x) \left( \frac{s}{2} \right)^{2/p'}.$$
To conclude the estimate of $II_1$ and $II_2$ we get for some $\theta \in (0, 1)$, $1 < p < 2$,
\begin{equation}
\sum_{j=1}^{N-1} |T_{j,t_{l+1}} Q_s f(x)|^2 \leq (\omega(\frac{s}{2}) + (\frac{s}{2})^\theta)^2 (M(Mf)^p)^{2/p}(x) \simeq \omega^2(\frac{s}{2})(M(Mf)^p)^{2/p}(x).
\end{equation}

Here $\omega(t) = \omega(t) + t^\theta$. Now by (5.1) we can conclude the estimate of $II$,
\begin{align*}
II & \leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \sup_{t_{l-1} < s \leq t_{l+1}} \int_{t_{l-1} < s \leq t_{l+1}} |\omega^{1/2}(\frac{s}{2})\omega^{-1/2}(\frac{s}{2})||T_{j,t_{l+1}} Q_s f(x)|^2 dx ds \\
& \leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \sup_{t_{l-1} < s \leq t_{l+1}} \left( \int_{0}^{2^{j-1} - 1} |\omega(\frac{s}{2})| ds \right) \left( \int_{0}^{2^{j-1}} |\omega^{-1}(\frac{s}{2})| ds \right) \sum_{l=1}^{N-1} |T_{j,t_{l+1}} Q_s f(x)|^2 dx ds \\
& \leq \|\omega\|_{Dini} \int_{\mathbb{R}^n} \int_{0}^{\infty} \left( \sum_{2^{j-1} \geq s} |\omega(\frac{s}{2})| (M(MQ_s f)^p)^{2/p}(x) dx ds \right) \\
& \leq \|\omega\|_{Dini}^2 \int_{\mathbb{R}^n} \int_{0}^{\infty} |Q_s f(x)|^2 dx ds \\
& \leq (\|\omega\|_{Dini} + 1)^2 \|f\|_{L^2}^2.
\end{align*}

Combining the estimates of $I$ and $II$, we get
\[ |S_2(T f)|_{L^2} \leq (1 + \|\omega\|_{Dini}^2) \|f\|_{L^2}^2. \]
\[ \Box \]

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