A generalized model for two dimensional quantum gravity and dynamics of random surfaces for \( d > 1 \)

M. Martellini

*I.N.F.N., Sezione di Roma "La Sapienza", Roma, Italy*

M. Spreafico

*Dipartimento di Matematica, Università di Milano, Milano and I.N.F.N., Sezione di Milano, Italy*

K. Yoshida

*Dipartimento di Fisica, Università di Roma, Roma and I.N. F. N., Sezione di Roma, Italy*

In memory of Giuseppe Occhialini

ABSTRACT

The possible interpretations of a new continuum model for the two-dimensional quantum gravity for \( d > 1 \) (\( d \)=matter central charge), obtained by carefully treating both diffeomorphism and Weyl symmetries, are discussed. In particular we note that an effective field theory is achieved in low energy (large area) expansion, that may represent smooth self-avoiding random surfaces embedded in a \( d \)-dimensional flat space-time for arbitrary \( d \). Moreover the values of some critical exponents are computed, that are in agreement with some recent numerical results.

* On leave of absence from Dipartimento di Fisica, Università di Milano, Milano, Italy and I.N.F.N., Sezione di Pavia, Italy
1. Introduction

In ref. [1] we have argued that the fixing of the two dimensional diffeomorphisms, by the so-called conformal gauge ($g_{\mu\nu} \rightarrow e^{\alpha \phi_L} \hat{g}_{\mu\nu}$), as well as of the Weyl structure, by the gauge $R(g) = R_0$ ($R(g)$ is the curvature scalar relative to the metric $g$), produces the following two scalar Liouville like action, up to corrections which are negligible for large intrinsic area $A$,

$$S[\phi_L, \phi_W; \hat{g}] = S_0 + \frac{1}{2\lambda} \int d^2 \xi \sqrt{\hat{g}} (R(e^{\alpha \phi_L} \hat{g}) - R_0)^2 + O\left(\frac{1}{A}\right)$$

$$S_0[\phi_L, \phi_W; \hat{g}] = \frac{1}{8\pi} \int d^2 \xi \sqrt{\hat{g}} \left[ -M_{ij} \Delta_g \Phi^i - Q_i \Phi^i R\hat{g} \right]$$

(1)

where

$$\Phi^i \equiv (\phi_L, \phi_W)$$

$$Q_i \equiv (Q_L, Q_W)$$

$$M_{ij} \equiv \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}$$

and we use here the same notations as in ref. [1].

This model, which is a sort of generalization of the DDK one [2], was constructed in [1] by requiring conformal invariance and the "boundary condition" that for $d \rightarrow 1^+$ the action (1) should reproduce the DDK model. Ignoring the last terms in eq. (1), at least to calculate the genus $h$ string susceptibility $\Gamma_h$, we have shown that for $d > 1$ this model gives a totally new physics. Indeed in order to calculate $\Gamma_h$ defined by the scaling relation

$$Z(A) \sim K A^{\Gamma_h - 3} e^{-\mu A}, \quad A \rightarrow \infty$$

(2)

it is sufficient to consider only the large $A$ behaviour. This procedure is consistent with the recent result of Kawai and Nakayama (KN) [3] which show that, for large
A, the $R^2$-term does not influence the leading behaviour. Furthermore notice that the non-analytic scaling breaking term found by KN, namely $\exp\left(-\frac{32\pi^2(1-h)}{A}\right)$, is exactly cancelled in (2) by the contribution coming from the fixed curvature scalar $R_0$ in (1) by using the Gauss-Bonet identity. Therefore we must think of (1) as a sort of "low-energy effective field theory" depending on two free parameters $B$ and $d$.

From now on, our basic strategy is to regard $S_0$ in (1) as the starting point to construct the fixed area partition function

$$Z(A) = \int Dg XD_g \phi_L D_g \phi_W D_g b D_g c d \tau \epsilon_{S_M}[X;\hat{g}]-S_{GH}[b,c;\hat{g}]-S_0[\phi_L,\phi_W;\hat{g}] \delta \left( \int d^2 x \sqrt{\hat{g}} - A \right) e^{-\mu A}$$

(3)

where $S_M$ is the usual Polyakov action associated with the immersion maps $X^a : \Sigma_h \rightarrow R^d, a = 1, ..., d$, implying the matter central charge $c_M = d$. The main result of [1] was that the genus $h$ string susceptibility associated with (3) is

$$\Gamma_h = \frac{2(1-h)B\sqrt{1+Bd} - \sqrt{(1+B^2)(25+(B-1)d)}}{\sqrt{25+(B-1)d} - \sqrt{1+(B-1)d}} + 2$$

(4)

$\Gamma_h$ is real for $d > 1$ if we assume that $B \geq B_c = O(1)$. Notice that for $B = 0$, (4) is the DDK string susceptibility. This fact implies that we may regard (3) as an off-critical string theory for $d \geq 1$, which extends the DDK model (that exists only for $d \leq 1$) and belongs to the same universality class as DDK at $d = 1$.

The first (almost trivial) consequence of (2) and (3) is that, even for $c_M = d > 1$, the mean (intrinsic) area is a linear function of the genus $h$. Indeed by defining

$$Z_h(\mu) = \int dA Z_h(A) \sim \mu^{-(\Gamma_h-2)}$$

we have that

$$<A>_h = -\frac{\partial}{\partial \mu} ln Z_h(\mu) \sim \frac{\Gamma_h - 2}{\mu}$$

$$\frac{\partial \Gamma_h}{\partial h} = -\frac{2}{\sqrt{1+B^2}} \frac{B\sqrt{1+Bd} - \sqrt{(1+B^2)(25+(B-1)d)}}{\sqrt{25+(B-1)d} - \sqrt{1+(B-1)d}}$$

(5)
The linear behaviour (5) is consistent with some recent numerical simulations [4] [5] [6]. We shall see also that the values of $\Gamma_h$ and its slope $\frac{\partial \Gamma_h}{\partial h}$ calculated according to eqs. (4) and (5) agree well with the numerical results obtained in [4] [5] [6].

In this letter we obtain two main results:

- If we consider (3) as a generalized DDK model, we get a closer information on the value of $B$ by studying the dynamics, i.e. the associated tachyon scattering amplitudes. It turns out that $B \sim B_c \equiv 1 - \frac{1}{d}$. Notice that this choice automatically guarantees our boundary condition for $d \to 1^+$. This analysis is developed in section 2.

- As it is well known, in the so-called dynamical triangulated approach of the two-dimensional quantum gravity, there are no obstruction in the construction of a random surface model for $d > 1$ [7]. Therefore, it is of interest to ask if one can relate for $d > 1$ the continuum theory described by the model (3) with some underlying random surface model. We shall show in section 3, that if we regard $B$ as an effective coupling parameter independent from $d$, our model may describe a phase of smooth, self-avoiding random surface embedded in $\mathbb{R}^d$ for $B \sim B_H \equiv 1$, with a positive $\Gamma$.

In the following we shall assume the planar topology, i.e. $h = 0$.

2. Tachyon scattering amplitudes

We proceed to study the tachyon scattering amplitudes of our model coupled to a conformal matter represented by the following action

\[
S_M = \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}}[-X^a \Delta \hat{g} X_a]
\]

Consistently with our interpretation of the area operator, the tachyon vertex operators $V_k = e^{ik_a X^a}$ get a gravitational dressing by the Liouville field $\Phi^1 = \phi_L$.
alone
\[ T_k = e^{ik_a X^a + \beta_1 \Phi^1} \quad (k_a = k_{a'} \forall a, a') \]
where the parameter \( \beta_1 = \beta(k) \) is defined by conformal invariance \( (\Delta[T_k] = 1) \)
\[ \beta_{\pm}(B, k) = -\frac{\sqrt{1 + B^2}}{2\sqrt{3}} \left[ \sqrt{25 + d(B - 1) \mp \sqrt{1 + d(B - 1) + 12d}} \right] \]

As a result we can express the tachyon scattering amplitudes over a fixed background geometry with planar topology as follows
\[
\mathcal{A}_n(k_1, ..., k_n) := < T_{k_1} ... T_{k_n} > = 2 \sqrt{\pi} A^{-1-s} \int Dg \Phi' D\bar{g} X' e^{-I[\Phi'; \bar{g}] - S_M[X'; \bar{g}]} \left[ \int dx^2 \sqrt{g} e^{\alpha_1 \Phi_0(x)} \right]^s T_{k_1} ... T_{k_n} \tag{6}
\]
where \( \Phi' \) and \( X' \) represent the non zero modes of the fields, \( I \) is the quadratic part of the action \( S_0 \) defined in eq. (1). In (6) \( s \) is given by
\[
s = -\frac{Q_1}{\alpha_1} - \frac{1}{\alpha_1} \sum_{i=1}^{n} \beta(k_i) \tag{7}
\]
\[
Q_1 = Q_L \equiv -\frac{1}{\sqrt{3}} \left[ B \sqrt{1 + Bd} - \sqrt{(1 + B^2)(25 + (B - 1)d)} \right]
\]
\[
\alpha_1 = -\frac{\sqrt{1 + B^2}}{2\sqrt{3}} \left[ \sqrt{25 + (B - 1)d} - \sqrt{1 + (B - 1)d} \right]
\]
The charge neutrality of the matter sector requires
\[
\sum_{i=1}^{n} k_i = 0 \tag{8}
\]
In case of integer \( s \) we can reduce (6) to the following multiple integral
\[
\mathcal{A}_n = \frac{2\sqrt{\pi}}{\alpha_1} A^{-1-s} \prod_{i=1}^{n} \prod_{j=1}^{s} \int d^2x_i d^2t_j |x_i - t_j|^{-\frac{2}{1 + B^2 \beta_i \alpha_1}} \prod_{i' < i} |x_{i'} - x_i|^{-\frac{2}{1 + B^2 \beta_i \beta_i + 2k_i k_i}} \prod_{j' < j} |t_{j'} - t_j|^{-\frac{2}{1 + B^2 \alpha_1^2}}
\]
We must now find what are the values of \( B \) for which these correlators have a
“good behaviour”. In ref. [8] we calculated the three-points correlation function and, studying its convergence conditions, we got an indication that $B \geq B_c \equiv 1 - \frac{1}{d}$.

Furthermore $B = B_c$ is the minimum value that ensures the reality of $\Gamma$. We also found that the allowed integer values for $s$ in (6) and (7), were 1 and $s = 2$.

Applying similar technique to the four-points function

$$A_4(k_1, k_2, k_3, k_4) =$$

$$= \frac{2\sqrt{\pi}}{\alpha_1} A^{-1-s} \int d^2x |x|^{2a'} |1 - x|^{2b'} \prod_{i=1}^{s} \int d^2t_i |t_i|^{2a_i} |1 - t_i|^{2b_i} |x - t_i|^{2p} \prod_{i' < i} |t_{i'} - t_i|^{4c}$$

where

$$a = -\frac{1}{1 + B^2} \alpha_1 \beta(k_1); \quad b = -\frac{1}{1 + B^2} \alpha_1 \beta(k_3)$$

$$p = -\frac{1}{1 + B^2} \alpha_1 \beta(k_4)$$

$$a' = k_1 k_4 - \frac{1}{1 + B^2} \beta(k_1) \beta(k_4); \quad b' = k_3 k_4 - \frac{1}{1 + B^2} \beta(k_3) \beta(k_4)$$

$$c = -\frac{1}{2(1 + B^2)} \alpha_1^2$$

we find that it never converges for $s = 2$. Instead it converges for $s = 1$ under the following conditions

$$\text{Re} \ a, b, a', b' > -1$$

$$- \text{Min} \left[ \frac{1}{2}; \text{Re} \frac{a + a'}{2} + 1; \text{Re} \frac{b + b'}{2} + 1 \right] < \text{Re} \frac{p}{2} <$$

$$< - \text{Max} \left[ \frac{1}{2} \text{Re}(a + b + 1); \frac{1}{2} \text{Re}(a' + b' + 1); \frac{1}{2} \text{Re}(a + b + a' + b' + 1) + 1 \right]$$

(9)

Remembering (7) and (8) and the above conditions, we find “experimentally” that for fixed $d$ and for $B \in [1 - \frac{1}{d}, 1 - \frac{1}{d} + \epsilon], \epsilon \sim 0$ there exists a region in the $\{k_1, k_2, k_3, k_4\}$ 4-dimensional momentum-space where $A_4$ converges. Moreover we note that a strong simplification arises in the form of the parameters when
$B = B_c \equiv 1 - \frac{1}{d}$, since in this case they becomes linear functions of (the moduli of) the momenta. In fact substituting $B = B_c$ in the definitions we find that

$$Q_1(d) = 2\sqrt{2}\frac{\sqrt{2d^2 + 1 - 2d}}{d} - \left(1 - \frac{1}{d}\right) \sqrt{\frac{d}{3}}$$

$$\alpha_1(d) = -\sqrt{2}\frac{\sqrt{2d^2 + 1 - 2d}}{d}$$

$$\beta(d, k) = \frac{\sqrt{2d^2 + 1 - 2d}}{d} (\sqrt{d}|k| - \sqrt{2})$$

$$a, b, p = \sqrt{2d}|k_{1,3,4}| - 2$$

$$a', b' = \sqrt{2d}(|k_{1,3}| + |k_4|) - 3|k_{1,3}|k_4| - 2$$

We end this section by outlining the argument that allow us to identify the region ("window") of convergence of $A_4$ (for $s = 1$). We note that, because of (7) and (8), there are only two independent momenta, say $k_1$ and $k_4$. Indeed by choosing the kinematic region $k_1, k_3, k_4 > 0$ and $k_2 < 0$, we get

$$k_2 = -\frac{1}{\sqrt{2d}} \left(\frac{(d - 1)\sqrt{d}}{\sqrt{12d^2 - 12d + 6}} + 3\right)$$

$$k_3 = -k_1 + |k_2| - k_4$$

Using the inequality (9) (with $B = B_c$), we find the condition of convergence in the $(k_1, k_4)$ plane, that is

$$k_1 > \frac{1}{\sqrt{2d}}$$

$$\frac{1}{\sqrt{2d}} < k_4 < \frac{2\sqrt{2d}}{4d - 3}$$

An explicit calculation of the four-points correlation function can be found in [9].

3. Planar random surface models

In this section we analyse which kind of planar random surface model may be
related to the "generalized" two-dimensional quantum gravity theory discussed in
the previous section. We have already remarked that (3), regarded as a low-energy
(large-area) effective field theory, in the planar limit, depends on the "coupling
constant" $B$ and on the dimension $d$ of the target space. Therefore if we understand
the model (3) as a possible "phase" of the two-dimensional quantum gravity for
$d > 1$ (complementary to that described by the DDK model for $d < 1$), $B$ is an
effective parameter and the geometrical complexity of the associated planar random
surfaces changes by varying $B$ in the ranges determined by some "critical" values,
as we shall show below. For this purpose, we consider, together with the string
susceptibility $\Gamma = \Gamma(B, d)$ defined in eq. (4), the value of the Hausdorff dimension
$d_H$ as a function of $B$ and $d$.

We compute $d_H(B, d)$ following the method of ref. [10]. The key formula is
given by the relation between the mean square size of the embedded surface and
the two-points correlation function in momenta space, which in our model is a
particular case of eq. (6). The result for large $A$ reads

$$d_H(B, d) = 2\left(\frac{\beta_+(B, d, k^2) + \beta_-(B, d, k^2)}{\alpha(B, d)}\right) - 2|_{k^2 = 0} = \sqrt{\frac{25 + d(B - 1)}{1 + d(B - 1)}} - 1 \quad (10)$$

In the following we distinguish two classes of planar random surface models
associated to (3).

(a) Off-critical string model for $d \geq 1$

In section 2 the model (3) is seen as an off-critical string with $d \geq 1$, which must
be in the same "universality class" of the DDK model at $d = 1$. This implies in
particular that at $d = 1$ it must have the same string susceptibility and Hausdorff
dimension as the DDK model. Furthermore, analyticity conditions on the $n$-points
correlation functions require \( B = B_c + \epsilon, \epsilon \sim 0 \). Thus we get, for \( d \geq 1 \) (\( \epsilon \sim 0 \))

\[
\Gamma(d) = \Gamma_c(d) + o(\epsilon)
\]

\[
\Gamma_c(d) = \frac{1}{\sqrt{6}} \sqrt{\frac{d}{2d^2 - 2d + 1}(d - 1)}
\]

\[
\frac{\partial \Gamma(h(B_c, d))}{\partial h} = 2 - \Gamma_c(d)
\]  

(11)

and

\[
d_H(d) \sim \sqrt{\frac{24}{d}} \epsilon^{\frac{1}{2}}
\]

The plot of \( \Gamma_c(d) \) is given in figure 1. Note that the values of \( \Gamma_c(d) \) for \( 1 < d \leq 4 \) fit rather well with the recent numerical data obtained in ref.s [4] [5] [6]. In particular, we get \( \Gamma_c(d = 3) = 0.392 \sim 0.4 \) which must be compared with the planar exponents \( b = 2 - \Gamma_c(3) \) and \( \gamma = \Gamma_c(3) \) of [4] and [6] respectively. Indeed, we find \( b \sim 1.6 \), which is consistent (within the large numerical errors) with the upper bound of \( b (b \sim 0.93) \) given by Caselle et al. [4] in the case of the "smaller" lattice. The correspondence appears to improve if one uses the new efficient algorithm for sampling of random surfaces in the Monte Carlo simulations used by Ambjorn et al. [6]. These last named authors analyse the fractal structure of the two-dimensional quantum gravity coupled to conformal matter with central charge \( c_M \), by measuring the distribution of the so-called baby universes and the numerical algorithm based on the cutting and pasting of the baby universes. In [6], one uses several fits to compute the critical exponent \( \gamma \). However, in order to compare \( \gamma(c_M) \) with our \( \Gamma_c(c_M = d) \), one must choose in [6] the fit which gives \( \gamma(c_M) \sim 0 \) as \( c_M \to 0 \). This is the “fit(b)” in ref. [6] and, for which this one, one finds \( \gamma_b(3) = 0.53 \) (11) [6], a value very close to our result \( \Gamma_c(3) \sim 0.4 \). As a final check of eq. (11), note also that \( \frac{\partial \Gamma_h(B_c, d=3)}{\partial h} = 1.608 \sim 1.6 \) and this value agrees well with the slope found in [5]. The authors of ref. [5] find, by using a body centered cubic lattice and the Swendsen-Wang (SW) Monte Carlo algorithm, a slope 1.25 ± 0.1.

As for the geometrical complexity of the random surface associated to the behaviour summarized by eq. (11), we have mentioned in the introduction the
equivalence between our model and the higher curvature ($R^2$) KN model obtained by adding the $R^2$ term to the Gibbs measure of (3). This observation suggests that the surfaces involved avoid the branched-polymer configurations by the KN mechanism. However, the fact that the Hausdorff dimension is very large with respect to the physical dimension $d$, indicates that these surfaces are not self-avoiding.

(b) Self-avoiding planar random surfaces

Alternatively we may think of (3) as an effective field description of the phase of the two-dimensional quantum gravity for $d > 1$. Thus $B$ is an effective parameter and we may ask whether there exists a value $B_H$ of $B$ such that

$$d_H(B_H, d) = k$$

$$\Gamma(B_H, d) > 0$$

for some open interval of $d > 1$, like for example $d \in (1, 24]$, with $k$ a fixed positive finite constant. Then we have self-avoiding (SA) planar surfaces if we choose $k < d$. From eq. (12), we get that

$$B_H = B_H(k, d) = 1 + \frac{(6 + k)(4 - k)}{kd(k + 2)}$$

We have studied the two most relevant cases:

i) $k=1$. Then by construction $d > d_H$ for any $d \in (1, 24]$ (hence SA regime) and by (13) $B_H = 1 + \frac{7}{d}$; the plot of $\Gamma(1 + \frac{7}{d}, d)$ is given in figure 2. Notice that $\Gamma(d = 1) \cong 0.105$. Therefore this SA random surface model is not in the same universality class of DDK, but for say $d = 3$ we have $\Gamma(d = 3) \cong 0.246$. So we conjecture that our model may supply a microscopic field theory description for the interface magnetization domains of the 3D-Ising model [4].

ii) $k=4$. Now by (13) $B_H = 1$ and the plot of $\Gamma(1, d)$ for $d \in (1, 24]$ is given in figure 3. In particular $\Gamma(1, d = 1) = 0$. Therefore, in this limiting case, we get
the same asymptotic partition function for large $A$ as the DDK model, but with a finite Hausdorff dimension. This class of planar random surfaces is SA only for $d > 4$.

Let us conclude with an observation. We have shown above that our model may describe the statistical mechanics of planar random surfaces, in a SA-phase or not, if $B$ is treated as an effective coupling parameter. This is consistent with our approximation of neglecting the potential interaction term in (1) and of regarding (3) as a sort of low-energy effective field theory. Of course we do not know up to now the complete quantum field theory leading to (3), and hence we can not say whether or not the consistency conditions applied to the full theory are brought to fix the value of the parameter $B$. As we have shown in [1] this does not happen at the level of the effective field theory model (3).

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FIGURE CAPTIONS

1) Behaviour of $\Gamma_c(d)$ for arbitrary values of $d$.

2) Behaviour of $\Gamma(1 + \frac{7}{d}, d)$ for arbitrary values of $d$.

3) Behaviour of $\Gamma(1, d)$ for arbitrary values of $d$. 