Relations between $\Delta M_{s,d}$ and $B_{s,d} \to \mu \bar{\mu}$ in Models with Minimal Flavour Violation

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Abstract

The predictions for the $B_{s,d} - \bar{B}_{s,d}$ mixing mass differences $\Delta M_{s,d}$ and the branching ratios $Br(B_{q} \to \mu \bar{\mu})$ within the Standard Model (SM) and its extensions suffer from considerable hadronic uncertainties present in the $B_{s,d}$-meson decay constants $F_{B_{s,d}}$ that enter these quantities quadratically. We point out that in the restricted class of models with minimal flavour violation (MFV) in which only the SM low energy operators are relevant, the ratios $Br(B_{q} \to \mu \bar{\mu})/\Delta M_{q}$ ($q = s, d$) do not depend on $F_{B_{q}}$ and the CKM matrix elements. They involve in addition to the short distance functions and $B$ meson lifetimes only the non-perturbative parameters $\hat{B}_{s,d}$. The latter are under much better control than $F_{B_{s,d}}$. Consequently in these models the predictions for $Br(B_{q} \to \mu \bar{\mu})$ have only small hadronic uncertainties once $\Delta M_{q}$ are experimentally known. Of particular interest is also the relation

$$\frac{Br(B_{s} \to \mu \bar{\mu})}{Br(B_{d} \to \mu \bar{\mu})} = \frac{\hat{B}_{d}}{\hat{B}_{s}} \frac{\tau(B_{s})}{\tau(B_{d})} \frac{\Delta M_{s}}{\Delta M_{d}}$$

that is practically free of theoretical uncertainties as $\hat{B}_{s}/\hat{B}_{d} = 1$ up to small $SU(3)$ breaking corrections. Using these ideas within the SM we find much more accurate predictions than those found in the literature: $Br(B_{s} \to \mu \bar{\mu}) = (3.4 \pm 0.5) \cdot 10^{-9}$ and $Br(B_{d} \to \mu \bar{\mu}) = (1.00 \pm 0.14) \cdot 10^{-10}$ were in the first case we assumed as an example $\Delta M_{s} = (18.0 \pm 0.5)/$ps.
1. Among the possible extentions of the Standard Model (SM), of particular interest are the models with minimal flavour violation (MFV), where the only source for flavour mixing is still given by the CKM matrix (see, for instance [1, 2, 3]). In the restricted class of these models [1], in which only the SM low energy operators are relevant, it is possible to derive relations between various observables that are independent of the parameters specific to a given MFV model [1, 4]. Violation of these relations would indicate the relevance of new low energy operators and/or the presence of new sources of flavour violation encountered for instance in general supersymmetric models [5, 6, 7, 8].

In this letter we would like to point out the existence of simple relations between the $B_{s,d} - \bar{B}_{s,d}$ mixing mass differences $\Delta M_{s,d}$ and the branching ratios for the rare decays $B_{s,d} \to \mu\bar{\mu}$ that are valid in models with minimal flavour violation (MFV) as defined in [1]. These relations should be of interest for the Run II at Tevatron and later for the LHC and BTeV experiments where $\Delta M_s$ and $Br(B_{s,d} \to \mu\bar{\mu})$ should be measured. Moreover, they allow one to make much more accurate predictions for $Br(B_{s,d} \to \mu\bar{\mu})$ once $\Delta M_{s,d}$ are precisely known. To our knowledge the relations in question have not been discussed so far in the literature except for a short comment made by us in [9].

2. Within the MFV models $\Delta M_{s,d}$ and $Br(B_{s,d} \to \mu\bar{\mu})$ are given as follows ($q = d, s$) [10]

$$\Delta M_q = \frac{G_F^2}{6\pi^2} \eta_B m_{B_q} (\hat{B}_q F_{B_q}^2) M_W^2 |V_{td}^* V_{tq}|^2 S(x_t, x_{new}),$$

$$Br(B_q \to \mu\bar{\mu}) = \tau(B_q) \frac{G_F^2}{\pi} \eta_Y^2 \left( \frac{\alpha}{4\pi\sin^2\theta_W} \right)^2 F_{B_q}^2 m_{\mu}^2 m_{B_q} |V_{td}^* V_{tq}|^2 Y^2(x_t, \bar{x}_{new}),$$

where $F_{B_q}$ is the $B_q$-meson decay constant and $\hat{B}_q$ the renormalization group invariant parameter related to the hadronic matrix element of the operator $Q(\Delta B = 2)$. See [10] for details. $\eta_B = 0.55 \pm 0.01$ [11, 12] and $\eta_Y = 1.012$ [13] are the short distance QCD corrections evaluated using $m_t \equiv m_t(m_t)$. In writing [2] we have neglected the terms $O(m_{\mu}^2/m_{B_q}^2)$ in the phase space factor. The short distance functions $S(x_t, x_{new})$ and $Y(x_t, \bar{x}_{new})$ result from the relevant box and penguin diagrams specific to a given MFV model. They depend on the top quark mass ($x_t = m_t^2/M_W^2$) and new parameters like the masses of new particles that we denoted collectively by $x_{new}$ and $\bar{x}_{new}$. Explicit expressions for these functions in the MSSM at low tan $\beta$ and in the ACD model [14] in five dimensions can be found in [15] and [16], respectively.

The main theoretical uncertainties in [10] and [2] originate in the values of $\hat{B}_q F_{B_q}^2$ and $F_{B_q}^2$ for which the most recent published values obtained by lattice simulations read [17]

$$F_{B_d} \sqrt{\hat{B}_d} = (235 \pm 33^{+0}_{-24}) \text{ MeV}, \quad F_{B_s} \sqrt{\hat{B}_s} = (276 \pm 38) \text{ MeV},$$

$$F_{B_d} = (203 \pm 27^{+0}_{-20}) \text{ MeV}, \quad F_{B_s} = (238 \pm 31) \text{ MeV} .$$
Similar results are obtained by means of QCD sum rules \[18\]. Consequently the hadronic uncertainties in $\Delta M_{s,d}$ and $\text{Br}(B_{s,d} \to \mu \bar{\mu})$ are in the ballpark of $\pm 30\%$ which is clearly disturbing. The uncertainties in the $B_q$-meson lifetimes are substantially smaller \[19\]:

$$\tau(B_s) = (1.461 \pm 0.057) \text{ ps}, \quad \tau(B_d) = (1.540 \pm 0.014) \text{ ps}, \quad \frac{\tau(B_s)}{\tau(B_d)} = 0.949 \pm 0.038 \ .$$

As noticed by many authors in the past, the uncertainties in $\Delta M_{s,d}$ and $\text{Br}(B_{s,d} \to \mu \bar{\mu})$ can be considerably reduced by considering the ratios

$$\frac{\Delta M_d}{\Delta M_s} = \frac{m_{B_d}}{m_{B_s}} \frac{\hat{B}_d}{\hat{B}_s} \frac{F_{B_d}^2 |V_{td}|^2}{F_{B_s}^2 |V_{ts}|^2}$$

(6)

$$\frac{\text{Br}(B_d \to \mu^+ \mu^-)}{\text{Br}(B_s \to \mu^+ \mu^-)} = \frac{\tau(B_d) m_{B_d} F_{B_d}^2 |V_{td}|^2}{\tau(B_s) m_{B_s} F_{B_s}^2 |V_{ts}|^2}$$

(7)

that can be used to determine $|V_{td}/V_{ts}|$ without the pollution from new physics \[1\]. In particular the relation (3) will offer after the measurement of $\Delta M_s$ a powerful determination of the length of one side of the unitarity triangle, denoted usually by $R_t$. As \[17\]

$$\xi = \sqrt{\frac{\hat{B}_s F_{B_s}}{\hat{B}_d F_{B_d}}} \approx \frac{F_{B_s}}{F_{B_d}} = 1.18 \pm 0.04^{+0.12}_{-0.04} \ ,$$

(8)

(see also $\xi = 1.22 \pm 0.07$ \[20\]) the uncertainties in the relations (6) and (7) are in the ballpark of $\pm 15\%$ and thus by roughly a factor of two smaller than in (1) and (2).

3. Here we would like to point out three useful relations that do not involve the decay constants $F_{B_d}$ and consequently contain substantially smaller hadronic uncertainties than the formulae considered so far. These relations follow directly from (11) and (2) and read

$$\frac{\text{Br}(B_s \to \mu \bar{\mu})}{\text{Br}(B_d \to \mu \bar{\mu})} = \frac{\hat{B}_d \tau(B_d)}{\hat{B}_s \tau(B_s)} \frac{\Delta M_s}{\Delta M_d},$$

(9)

$$\text{Br}(B_q \to \mu \bar{\mu}) = C \frac{\tau(B_q) Y^2(x_t, x_{\text{new}})}{S(x_t, x_{\text{new}})} \Delta M_q, \quad (q = s, d)$$

(10)

with

$$C = \frac{\eta_2^2 \beta_2}{\eta_B} \left( \frac{\alpha}{4\pi \sin^2 \theta_W} \right)^2 \frac{m^2_\mu}{M^2_W} = 4.36 \cdot 10^{-10}$$

(11)

where we have used $\alpha = 1/129$, $\sin^2 \theta_W = 0.23$ and $M_W = 80.423 \text{ GeV}$ \[21\].

The relevant parameters obtained from lattice simulations are \[17\]

$$\frac{\hat{B}_s}{\hat{B}_d} = 1.00 \pm 0.03, \quad \hat{B}_d = 1.34 \pm 0.12, \quad \hat{B}_s = 1.34 \pm 0.12 \ .$$

(12)

The simple relation between $\Delta M_s/\Delta M_d$ and $\text{Br}(B_s \to \mu \bar{\mu})/\text{Br}(B_d \to \mu \bar{\mu})$ in (9) involves only measurable quantities except for the ratio $\hat{B}_s/\hat{B}_d$ that has to be calculated by non-perturbative
methods. As \( \hat{B}_s/\hat{B}_d = 1 \) in the SU(3) flavour symmetry limit, only SU(3) breaking corrections have to be calculated. Now, in contrast to \( F_{B_s}/F_{B_d} \) and \( F_{B_d} \) that suffer from chiral logarithms and quenching \( \text{[22, 17, 20]} \), the chiral extrapolation in the case of \( \hat{B}_q \) is well controlled and very little variation is observed between quenched and \( N_f = 2 \) results. Consequently the error in \( \hat{B}_s/\hat{B}_d = 1 \) is very small and also the separate values for \( \hat{B}_s \) and \( \hat{B}_d \) given in \( \text{[12]} \) are rather accurate \( \text{[22, 17, 20, 23]} \). These results should be further improved in the future. Consequently \( \text{[9]} \) is one of the cleanest relations in B physics but also \( \text{[10]} \) is rather clean theoretically.

We note that once \( \Delta M_s/\Delta M_d \) has been precisely measured, the relation \( \text{[9]} \) will allow one to predict \( Br(B_s \to \mu \bar{\mu})/Br(B_d \to \mu \bar{\mu}) \) without essentially any hadronic uncertainties. On the other hand the relations in \( \text{[10]} \) allow to predict \( Br(B_{s,d} \to \mu \bar{\mu}) \) in a given MFV model with substantially smaller hadronic uncertainties than found by using directly the formulae in \( \text{[2]} \). In particular using the known formulae for the functions \( Y \) and \( S \) in the SM model \( \text{[10]} \), we find

\[
Br(B_s \to \mu \bar{\mu}) = 3.42 \cdot 10^{-9} \left[ \frac{\tau(B_s)}{1.46 \text{ ps}} \right] \left[ \frac{1.34}{\hat{B}_s} \right] \left[ \frac{\overline{m}_t(m_t)}{167 \text{ GeV}} \right]^{1.6} \left[ \frac{\Delta M_s}{18.0/\text{ps}} \right], \tag{13}
\]

\[
Br(B_d \to \mu \bar{\mu}) = 1.00 \cdot 10^{-10} \left[ \frac{\tau(B_d)}{1.54 \text{ ps}} \right] \left[ \frac{1.34}{\hat{B}_d} \right] \left[ \frac{\overline{m}_t(m_t)}{167 \text{ GeV}} \right]^{1.6} \left[ \frac{\Delta M_d}{0.50/\text{ps}} \right]. \tag{14}
\]

Using \( \overline{m}_t(m_t) = (167 \pm 5) \text{ GeV} \), the lifetimes in \( \text{[5]} \), \( \hat{B}_q \) in \( \text{[12]} \), \( \Delta M_d = (0.503 \pm 0.006)/\text{ps} \) \( \text{[24]} \) and taking as an example \( \Delta M_s = (18.0 \pm 0.5)/\text{ps} \) we find the predictions for the branching ratios in question

\[
Br(B_s \to \mu \bar{\mu}) = (3.42 \pm 0.54) \cdot 10^{-9}, \quad Br(B_d \to \mu \bar{\mu}) = (1.00 \pm 0.14) \cdot 10^{-10}. \tag{15}
\]

These results are substantially more accurate than the ones found in the literature (see for instance \( \text{[2, 3, 25, 26]} \)) where the errors are in the ballpark of \( \pm (30 - 50)\% \).

In calculating the errors in \( \text{[15]} \) we have added first the experimental errors in \( \tau(B_q) \), \( \overline{m}_t(m_t) \) and \( \Delta M_q \) in quadrature to find \( \pm 6.8\% \) and \( \pm 4.9\% \) for \( \text{[13]} \) and \( \text{[14]} \), respectively. We have then added linearly the error of \( \pm 9\% \) from \( \hat{B}_{B_s} \). Consequently the total uncertainties in \( Br(B_s \to \mu \bar{\mu}) \) and \( Br(B_d \to \mu \bar{\mu}) \) are found to be \( \pm 15.8\% \) and \( \pm 13.9\% \), respectively. If all errors are added in quadrature we find \( \pm 11.3\% \) and \( \pm 10.2\% \), respectively. As the errors in \( \tau(B_s) \), \( \overline{m}_t(m_t) \) and \( \Delta M_s \) will be decreased considerably in the coming years, the only significant errors in \( \text{[13]} \) and \( \text{[14]} \) will be then due to the uncertainties in \( \hat{B}_{B_q} \). Future lattice calculations should be able to reduce these errors as well, so that predictions for \( Br(B_{s,d} \to \mu \bar{\mu}) \) in the SM will become very accurate and in other MFV their accuracy will mainly depend on the knowledge of the short distance functions \( S \) and \( Y \).

4. The dependence on new physics in \( \text{[10]} \) is given entirely by the ratio \( Y^2/S \). As hadronic uncertainties in \( \text{[10]} \) are substantially smaller than in \( \text{[11]} \) and \( \text{[2]} \), the differences between various
MFV models can be easier seen. For instance in the ACD model with five dimensions the ratio $Y^2/S$ with $m_t(m_t) = 167$ GeV equals 0.58, 0.53, 0.49 and 0.46 for the compactifications scales $1/R = 200$, 250, 300, 400 GeV, respectively. In the SM one has $Y^2/S = 0.40$ and the effects of the Kaluza-Klein modes could in principle be seen when (10) is used, whereas it is very difficult by means of (2).

The relation that is satisfied in any model with MFV violation as defined in [1], is not satisfied in more complicated models in which other operators are relevant. As an example in the MSSM with MFV but large $\tan\beta$ the contributions of new LR scalar operators originating in neutral Higgs exchanges modify $Br(B_{s,d} \to \mu\bar{\mu})$ by orders of magnitude, change $\Delta M_s$ typically by $10\%$--$30\%$ [28], leaving $\Delta M_d$ essentially unchanged with respect to the SM estimates. While a correlation between new physics effects in $Br(B_{s,d} \to \mu\bar{\mu})$ and $\Delta M_s$ exists [28], the absence of relevant new contributions to $\Delta M_d$ results in the violation of (9). Using the formulae of [28] we find

$$\frac{Br(B_s \to \mu\bar{\mu})}{Br(B_d \to \mu\bar{\mu})} = \frac{\hat{B}_d \tau(B_s) \Delta M_s}{\hat{B}_s \tau(B_d) \Delta M_d} \left[ \frac{m_{B_s}}{m_{B_d}} \right]^4 \frac{1}{1 + f_s}$$

(16)

with $f_s$ being a complicated function of supersymmetric parameters that enters $\Delta M_s$ in this model. The dependence on $m_{B_{d}}$ in (16) originates in the LR scalar operators that dominate $Br(B_{s,d} \to \mu\bar{\mu})$ at large $\tan\beta$. Similarly, in a scenario considered in [26] in which new physics effects are assumed to be important in $\Delta M_d$ but negligible in $Br(B_{s,d} \to \mu\bar{\mu})$ and $\Delta M_s$, the relation (9) is violated. Finally, we expect that it is generally violated in models with non-minimal flavour violation [5, 6, 7, 8].

5. In summary we have presented stringent relations between $Br(B_{s,d} \to \mu\bar{\mu})$ and $\Delta M_{s,d}$ that are valid in the MFV models as defined in [1]. The virtue of these relations is their theoretical cleanness that allows to obtain improved predictions for $Br(B_{s,d} \to \mu\bar{\mu})$ as demonstrated above. Other useful relations in the MFV models can be found in [4]. It will be interesting to follow the developments at Tevatron, LHC, BTeV, BaBar, Belle and K physics dedicated experiments to see whether these relations are satisfied. While the present experimental upper bounds are still rather weak

$$Br(B_s \to \mu\bar{\mu}) < 2.6 \times 10^{-6} \quad (95\% \ C.L. \ [29]),$$

(17)

$$Br(B_d \to \mu\bar{\mu}) < 2.0 \times 10^{-7} \quad (95\% \ C.L. \ [30]),$$

(18)

considerable progress is expected in the coming years.

Needless to say the improvement on the accuracy of $F_{B_{d}}$ is very important as $F_{B_{s}}/F_{B_{d}}$ is crucial for the determination of the CKM element $|V_{td}|$ as seen in [6] and [7]. Moreover the measurements of $\Delta M_s$ and $Br(B_s \to \mu\bar{\mu})$ in conjunction with accurate values of $\sqrt{\hat{B}_s F_{B_s}}$
and $F_B$, will determine the function $S$ and $Y$, respectively. This information will allow one to distinguish between various MFV models.

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