SOME PROPERTIES OF NONCONVEX ORIENTED DISTANCE FUNCTION AND APPLICATIONS TO VECTOR OPTIMIZATION PROBLEMS

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Abstract. In this paper, we study some interesting properties of nonconvex oriented distance function. In particular, we present complete characterizations of monotonicity properties of oriented distance function. Moreover, the Clark subdifferentials of nonconvex oriented distance function are explored in the solid case. As applications, fuzzy necessary optimality conditions for approximate solutions to vector optimization problems are provided.

1. Introduction. Scalarization is a powerful tool for vector optimization as solutions of vector optimization problems can be characterized by the solutions of the corresponding scalar optimization problems. This technique not only allows one to study vector optimization problem but also to solve it by numerical methods applicable for the scalar problem. A traditional way is weighted sum method (i.e., linear scalarization) [36]. However, this approach is only available for convex vector optimization problems. In order to deal with nonconvex vector optimization problems, two types of nonlinear scalarization functions are widely used: one is the so-called Gerstewitz function (also the smallest strictly monotonic function) [14], the other is oriented distance function (also known as signed distance function) [8, 21, 22]. In the past two decades, these two scalarization functions were intensively studied due to their important applications in various aspects of vector optimization, such as characterization of different types of optimal solutions (see [2, 5, 16, 18, 15, 25, 35, 37]), optimality conditions (see [1, 11]), duality (see [3]), well-posedness (see [19, 30]) and stability (see [26]), etc.

In the majority of published papers, one can easily find that Gerstewitz function seems to be more appropriate, because it allows us to consider both solid and
nonsolid optimization (i.e., the ordered sets have nonempty topological interior or empty topological interior) (see [5, 10, 13, 15, 16, 17, 20, 25, 26, 40]), while oriented distance function gives the desired results in the convex or nonsolid case (i.e., the case when the ordered sets are convex or have empty topological interior). For example, Zaffaroni [37] has pointed out that when the ordered set is a solid convex cone, the oriented distance function is sublinear and nondecreasing. These nice properties were used to characterize various types of solutions to a vector optimization problem in [37]. Liu et al. [28] presented some dual representations for oriented distance function when the associated set was convex and solid. Durea et al. [9] provided optimality conditions for Pareto efficient solutions in terms of oriented distance function when the associated set was convex and nonsolid. Jiménez et al. [24] characterized several types of minimal solutions to a set optimization problem, when the considered set was a convex cone. Gao and Yang [12] investigated some properties of oriented distance function when the considered set was convex and co-radiant, and derived Lagrange multiplier rules for approximate solutions of vector optimization problems. However, to the best of our knowledge, in the literatures only a few papers focus on nonconvex oriented distance function in the solid case (i.e., the case when the considered set has nonempty topological interior). Therefore, it is of importance to ask whether there is any useful feature of the nonconvex oriented distance function in the solid case.

This work has two objectives. The first one is to study some elementary properties of oriented distance function in the nonconvex or solid setting. Specifically, we are interested in characterizing some monotonicity properties of the oriented distance function in terms of nonconvex free disposal set. The second one is to state optimality conditions for approximate solutions to vector optimization problems via oriented distance function.

This work is structured as follows. In Sect. 2, some notations are fixed and mathematical tools are recalled. Moreover, some basic properties of oriented distance function are clarified. In Sect. 3, three kinds of monotonicity properties of oriented distance function: \( K^- \) nonincreasce, \( K^- \) decrease and strict \( K^- \) decrease are completely characterized, especially, by virtue of the nonconvex free disposal set. In Sect. 4, some results about the Clarke subdifferentials of nonconvex oriented distance function are discussed in the solid setting. We mainly concentrate on the subdifferential properties at the boundary points of the associated set, and conclude that for nonconvex free disposal set, the formulas for the Clarke subdifferential of oriented distance function at the boundary point behave in a fashion similar to the convex case. As applications, some necessary approximate optimality conditions for approximate solutions are established for vector optimization problem in Sect. 5. Finally, in Sect. 6, we summarize the main conclusions from this work.

2. Preliminaries. Throughout the following, let \( X \) and \( Y \) be normed linear spaces over the real field \( \mathbb{R} \); their dual spaces are denoted by \( X^* \) and \( Y^* \), respectively. Let \( B_X(x, r) \) be the closed ball of \( X \) centered at \( x \) with radius \( r \geq 0 \). \( \mathbb{B} \) and \( \mathbb{B}^* \) denote the closed unit ball in \( Y \) and \( Y^* \), respectively. We refer to the nonnegative orthant and the positive orthant of \( \mathbb{R}^p \) by \( \mathbb{R}_{+}^p \) and \( \mathbb{R}_{++}^p \), respectively, and denote \( \mathbb{R}_{+} := \mathbb{R}_{+}^1 \).

For a nonempty set \( A \subseteq Y \), \( intA, clA, bdA, A^c \) and \( convA \) denote the topological interior, closure, boundary and complement, and the convex hull of \( A \), respectively. The cone generated by a set \( A \) is defined as \( coneA := \bigcup_{\alpha \geq 0} \alpha A \). We say that a subset \( A \) of \( Y \) is nontrivial if \( \emptyset \neq A \neq Y \). The negative dual cone and positive dual cone of \( A \) are denoted by \( A^- \) and \( A^+ \), respectively, i.e.,
The Clarke tangent cone of the set $A$ is defined by

$$T(A; \bar{y}) := \{ v \in X : \forall y_n, c_{clA} \bar{y}, \forall \lambda_n \downarrow 0, \exists v_n \rightarrow v \text{ with } y_n + \lambda_n v_n \in A, \forall n \},$$

where $y_n, c_{clA} \bar{y}$ means $y_n \rightarrow \bar{y}$ with $y_n \in clA$, and $\lambda_n \downarrow 0$ means $\lambda_n \rightarrow 0$ with $\lambda_n \in \mathbb{R}_+$. The Clarke normal cone of the set $A$ at $\bar{y} \in clA$ is defined by the dual correspondence

$$N(A; \bar{y}) := T(A; \bar{y})^-.$$

For a subset $A$ of $Y$, the distance function associated with $A$ is $d_A : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$d_A(y) := \inf_{a \in A} \|y - a\|,$$

with $d_A(y) := +\infty$. As in [21, 22], the oriented distance function associated with $A$ is defined as

$$\Delta_A(y) := d_A(y) - d_{A^c}(y),$$

with $\Delta_A(y) := +\infty$ and $\Delta_{A^c}(y) := -\infty$.

Zaffaroni [37] has discussed some basic properties of oriented distance function, and we list below these properties that will be used in the sequel.

**Lemma 2.1.** [37] Let $A$ be a nontrivial subset of $Y$. The following assertions hold:

(i) $\Delta_A$ is real valued.
(ii) $\Delta_A$ is Lipschitz of rank 1.
(iii) $clA = \{ y : \Delta_A(y) \leq 0 \}$, $intA = \{ y : \Delta_A(y) < 0 \}$, $bdA = \{ y : \Delta_A(y) = 0 \}$, $intA^c = \{ y : \Delta_A(y) > 0 \}$.
(iv) If $A$ is a convex set, then $\Delta_A$ is convex.

In finite-dimensional Euclidean space, some nice properties of oriented distance function were explored by Luo et al. [29]. Also, these properties remain valid in any normed linear space. Taking into account the crucial role of these properties in our main results, we collect some fundamental properties in the following proposition, with more direct proofs.

**Proposition 2.2.** Let $A$ be a nonempty subset of $Y$, and $\lambda \in \mathbb{R}_+$. Then the following properties hold:

(i) $\Delta_{A^c} = -\Delta_A$.
(ii) $\Delta_A(y' + y) = \Delta_{A - y'}(y), \forall y, y' \in Y$.
(iii) $\Delta_A(\lambda y) = \lambda \Delta_A(1)(y), \forall y \in Y, \lambda > 0$.
(iv) If $A \subseteq S \subseteq Y$, then $\Delta_A \geq \Delta_S$. Moreover, if $A$ and $S$ are closed subsets of $Y$, then $A \subseteq S$ if and only if $\Delta_A \geq \Delta_S$.
(v) $clA = cl(intA)$ if and only if $\Delta_A = \Delta_{intA}$.
(vi) $intA = int(clA)$ if and only if $\Delta_A = \Delta_{clA}$.

**Proof.** (i) By definition, it holds obviously.

(ii) Observe that for any subset $C \subseteq Y$, $d_C(y' + y) = d_{C - y'}(y), \forall y, y' \in Y$. Hence (ii) holds.
Remark 1. (i) The first part of (iv) has also been proved using the infimal convolution in [22].

(ii) For any nonempty subset \( A \subseteq Y \), we always have that \( \Delta_A \geq \Delta_{clA} \). However, \( \Delta_A = \Delta_{clA} \) does not hold in general (see [29], Example 3.1).

(iii) Under the assumption that \( A \neq \emptyset \), it can easily be shown that \( intA \neq \emptyset \) in part (v).

(iv) If \( intA = \emptyset \), then the result of part (vi) still holds.

(v) By parts (v) and (vi), we can conclude that for two subsets \( A \) and \( B \),

\[
\Delta_A = \Delta_B \iff clA = clB \text{ and } intA = intB
\]

\[
\iff clA = clB \text{ and } bdA = bdB.
\]

Recall that a set \( A \subseteq Y \) is \textit{nearly convex}, if there is a convex set \( C \subseteq Y \) such that \( C \subseteq A \subseteq clC \). In view of Proposition 2.2 (v)-(vi), we have the following results.

Corollary 2.3. \textit{Let} \( A \subseteq Y \) \textit{be a nonempty nearly convex set with} \( intA \neq \emptyset \). \textit{Then}, \( \Delta_A = \Delta_{clA} = \Delta_{intA} \) \textit{and} \( \Delta_A \) \textit{is convex.}

Proof. Since \( A \subseteq Y \) is a nearly convex set with \( intA \neq \emptyset \), we have \( clA = cl(intA) \) and \( intA = int(clA) \). By Proposition 2.2 (v)-(vi) and Lemma 2.1 (iv), it follows that \( \Delta_A = \Delta_{clA} = \Delta_{intA} \) and \( \Delta_A \) is convex. \( \square \)

Recall that a set \( A \subseteq Y \) is \textit{free disposal} with respect to a convex cone \( K \subseteq Y \), if \( A + K = A \). When \( intK \neq \emptyset \), one has \( intA \neq \emptyset \). In fact, \( A + intK = clA + intK = intA \), which is also given by Zhao and Yang [39]. Moreover, it is easy to verify that \( clA = cl(intA) \) and \( intA = int(clA) \). Due to Proposition 2.2 (v)-(vi), we obtain the following result.

Corollary 2.4. \textit{Let} \( A \subseteq Y \) \textit{be free disposal with respect to a convex cone} \( K \subseteq Y \). \textit{If} \( intK \neq \emptyset \), \textit{then}, \( \Delta_A = \Delta_{clA} = \Delta_{intA} \).

Remark 2. It’s worth noting that the condition \( intK \neq \emptyset \) cannot be replaced by \( intA \neq \emptyset \). Otherwise, the result of Corollary 2.4 cannot hold. See the following example.
Example 2.1. Consider the set
\[ A := \{(a_1, a_2)^T \in \mathbb{R}^2 : a_2 \in [0, 1]\} \cup \{(a_1, a_2)^T \in \mathbb{R}^2 : a_2 \text{ is rational number in the interval } (1, +\infty)\}. \]
Then, \( \text{cl} A = \{(a_1, a_2)^T \in \mathbb{R}^2 : a_2 \geq 0\} \) and \( \text{int} A = \{(a_1, a_2)^T \in \mathbb{R}^2 : a_2 \in (0, 1)\} \).
It is easy to check that \( A \) is free disposal with respect to convex cone \( K := \{(t, 0)^T \in \mathbb{R}^2 : t \geq 0\} \) with \( \text{int} K = \emptyset \). Taking \( \bar{y} = (0, 2)^T \in \text{bd} A \), we have \( \Delta A(\bar{y}) = 0 \), \( \Delta_{\text{cl}A}(\bar{y}) = -2 \) and \( \Delta_{\text{int}A}(\bar{y}) = 1 \), see Fig. 2.1. Hence \( \Delta_A \neq \Delta_{\text{cl}A} \neq \Delta_{\text{int}A} \).

Fig. 2.1 \( \bar{y} \in \text{bd} A \) with \( \Delta_A(\bar{y}) = 0 \), \( \Delta_{\text{cl}A}(\bar{y}) = -2 \) and \( \Delta_{\text{int}A}(\bar{y}) = 1 \) in Example 2.1

3. Some monotonicity properties of oriented distance function. Motivated by the complete characterizations of the monotonicity properties of Gerstewitz function obtained by Gutiérrez et al. [20], in this section, we intend to characterize the monotonicity properties of oriented distance function in the nonconvex case.

Theorem 3.1. Let \( A \) and \( K \) be two nontrivial subsets of \( Y \). Let us consider the following statements:

(a) \( A + K \subseteq A \).
(b) \( \Delta A \) is \( K \)-nonincreasing, i.e.,
\[ y_1 - y_2 \in K \implies \Delta A(y_1) \leq \Delta A(y_2). \]
(c) \( A + K \subseteq \text{cl} A \).
(d) \( \text{cl} A + K \subseteq \text{cl} A \).

The following hold:
\[ (a) \Rightarrow (b) \Rightarrow (c) \iff (d). \]

Proof. \( (a) \Rightarrow (b) \). Take \( y_1, y_2 \in Y \) such that \( y_1 - y_2 \in K \). Then there exists \( k_0 \in K \) such that \( y_2 = y_1 - k_0 \), which implies that \( \Delta A(y_2) = \Delta A(y_1 - k_0) \). Owing to Proposition 2.2(ii), we get \( \Delta A(y_2) = \Delta A + k_0(\bar{y}_1) \). By (a), it follows that \( A + k_0 \subseteq A \). Hence, due to Proposition 2.2(iv), we obtain that \( \Delta A(y_2) = \Delta A + k_0(\bar{y}_1) \geq \Delta A(\bar{y}_1) \), which implies that \( \Delta A \) is \( K \)-nonincreasing.

(b) \( \Rightarrow (c) \). Suppose that \( \Delta A \) is \( K \)-nonincreasing, and fix \( a \in A \) and \( k \in K \). Since \( k = (a + k) - a \in K \), by (b) and Lemma 2.1(iii) we have that
\[ \Delta_A(a + k) \leq \Delta_A(a) \leq 0. \]

Hence, again by Lemma 2.1(iii), we see that \( a + k \in clA \), and the implication is proved.

(c) \( \Leftrightarrow \) (d) is straightforward. \( \Box \)

In case \( A = clA \), we get the following characterization of the monotonicity of \( \Delta_A \), whose proof directly follows from Theorem 3.1.

**Theorem 3.2.** Let \( A \) and \( K \) be two nontrivial subsets of \( Y \). Assume that \( A \) is closed. Then, the following assertions are equivalent:

(a) \( A + K \subseteq A \).
(b) \( \Delta_A \) is \( K \)-nonincreasing, i.e., \( y_1 - y_2 \in K \Rightarrow \Delta_A(y_1) \leq \Delta_A(y_2) \).

According to Theorem 3.1 and Theorem 3.2, we obtain the \( K \)-nonincreasing property of \( \Delta_A \) when the associated set \( A \) is free disposal with respect to a convex cone \( K \).

**Corollary 3.3.** Let \( A \subseteq Y \) be a nontrivial subset, and \( K \subseteq Y \) be a convex cone.

(i) If \( A \) is free disposal with respect to \( K \), then \( \Delta_A \) is \( K \)-nonincreasing, i.e.,
\[ y_1 - y_2 \in K \Rightarrow \Delta_A(y_1) \leq \Delta_A(y_2). \]

(ii) Assume that \( A \) is closed. Then \( A \) is free disposal with respect to \( K \) if and only if \( \Delta_A \) is \( K \)-nonincreasing.

The next result is the analogue to Theorem 3.2 when topological interior is employed.

**Theorem 3.4.** Let \( A \) and \( K \) be two nontrivial subsets of \( Y \). Assume that \( intA \neq \emptyset \). Then, the following statements are equivalent:

(a) \( clA + K \setminus \{0\} \subseteq intA \).
(b) \( \Delta_A \) is \( K \)-decreasing, i.e.,
\[ y_1 - y_2 \in K \setminus \{0\} \Rightarrow \Delta_A(y_1) < \Delta_A(y_2). \]

**Proof.** (a) \( \Rightarrow \) (b). First, we show that when \( C \) is a nonempty closed set and \( C \subseteq intA \), we have
\[ \Delta_C > \Delta_A. \] (1)

Consider the following three cases.

**Case 1:** \( y \notin A \). Take any \( c \in C \), there exists \( b \in (y, c) \cap bdA \) satisfying
\[ \|y - c\| = \|y - b\| + \|b - c\|, \]
which implies that
\[ d_C(y) \geq d_A(y) + d_C(b) > d_A(y), \]
since \( b \notin C \) and \( C \) is closed. This yields that \( \Delta_C(y) > \Delta_A(y) \).

**Case 2:** \( y \in A \setminus intC \). It is easy to see that \( \Delta_C(y) > \Delta_A(y) \).

**Case 3:** \( y \in intC \). As \( C \subseteq intA \), we have that \( cl(A^c) \subseteq C^c \), which implies that \( int(A^c) \subseteq cl(C^c) \). On the lines of Case 1, we have \( d_{C^c}(y) < d_{A^c}(y) \), i.e.,
\[ \Delta_{C^c}(y) > \Delta_A(y). \]

Now, take \( y_1, y_2 \in Y \) such that \( y_1 - y_2 \in K \setminus \{0\} \). Follow immediately on the similar lines of Theorem 3.1, there is \( k_0 \in K \setminus \{0\} \) such that \( \Delta_A(y_2) = \Delta_{A+k_0}(y_1) \geq \Delta_{clA+k_0}(y_1) \). By (a), \( clA + k_0 \subseteq intA \). Consequently, it follows from (1) that
\[ \Delta_A(y_2) \geq \Delta_{clA+k_0}(y_1) > \Delta_A(y_1), \]
which implies that \( \Delta_A \) is \( K \)-decreasing.
Suppose that $\Delta_A$ is $K$-decreasing, and fix $a \in clA$ and $k \in K\setminus\{0\}$. As $k = (a + k) - a \in K\setminus\{0\}$, by (b) and Lemma 2.1(iii) we have that $\Delta_A(a + k) < \Delta_A(a) \leq 0$.

Hence, again by Lemma 2.1(iii) we see that $a + k \in intA$, and the implication is proved.

Similar to Theorem 3.4, we can also establish the strictly $K$-decreasing property of $\Delta_A$.

**Theorem 3.5.** Let $A$ and $K$ be two nontrivial subsets of $Y$. Assume that $intK \neq \emptyset$. Then, the following statements are equivalent:

1. $clA + intK \subseteq intA$.
2. $\Delta_A$ is strictly $K$-decreasing, i.e.,

$$y_1 - y_2 \in intK \implies \Delta_A(y_1) < \Delta_A(y_2).$$

**Remark 3.** Indeed, $A + intK \subseteq intA \iff clA + intK \subseteq intA$ in Theorem 3.5.

Observing that when $A$ is free disposal with respect to a convex cone $K$ with $intK \neq \emptyset$, one always has that $clA + intK = intA$ (see also [39], Theorem 3.1) and $intA = int(clA)$. Then, according to Theorem 3.5, we get the desired results.

**Corollary 3.6.** Let $A \subseteq Y$ be a nontrivial subset, and $K \subseteq Y$ be a convex cone with $intK \neq \emptyset$.

1. If $A$ is free disposal with respect to $K$, then $\Delta_A$ is strictly $K$-decreasing, i.e.,

$$y_1 - y_2 \in intK \implies \Delta_A(y_1) < \Delta_A(y_2).$$

2. Assume that $A$ is closed. Then $A$ is free disposal with respect to $K$ if and only if $\Delta_A$ is strictly $K$-decreasing.

**Remark 4.**

(i) The monotonicity of $\Delta_A$ was also discussed by Li et al. [27] in terms of the recession cone $A^\infty$, where

$$A^\infty := \{u \in Y : A + tu \subseteq A, \forall t \in \mathbb{R}_+\}.$$

In fact, $A$, in essence, is free disposal with respect to $A^\infty$. Here, we obtain complete characterizations of the monotonicity of $\Delta_A$ via free disposal set. Therefore, our results improve and generalize those of [27].

(ii) Gao and Yang [12] have established the monotonicity properties of the oriented distance function under the assumption that the associated set is closed, convex and co-radiant. Observe that convex co-radiant set is free disposal. In addition, we establish the characterizations of the monotonicity of $\Delta_A$ without any convexity assumption. Hence, our results about the monotonicity of $\Delta_A$ improve and generalize Theorem 3.1 (iii) in [12].

(iii) If $A$ is a proper convex cone with $intA \neq \emptyset$, then the results of Theorem 3.1 and Theorem 3.5 reduce to the monotonicity features of $\Delta_A$ in Proposition 3.2 given by Zaffaroni in [37].

4. **The Clarke subdifferential of oriented distance function.** It is well-known that if $A$ is a nontrivial subset of $Y$, then $\Delta_A$ is finite-valued and Lipschitz of rank 1. Accordingly, the Clarke subdifferential $\partial \Delta_A(\cdot)$ is a nonempty weak∗-compact subset of $Y^*$. Due to Corollary 3.3, when $A \subseteq Y$ is nontrivial and free disposal with respect to a convex cone $K \subseteq Y$, the oriented distance function $\Delta_A$ is $K$-nonincreasing.
Then, for each \( y \in Y \) and \( k \in K \), the Clarke generalized directional derivative of \( \Delta_A \) at \( y \) in the direction \( k \)

\[
\Delta_A^*(y; k) := \limsup_{y' \to y} \Delta_A(y' + tk) - \Delta_A(y') \leq 0.
\]

This yields that

\[
\partial \Delta_A(y) \subseteq K^- \cap \mathbb{B}^*, \forall y \in Y. \tag{2}
\]

Taking into account the fact that if two proper functions \( f : Y \to \mathbb{R} \cup \{+\infty\} \) and \( g : Y \to \mathbb{R} \cup \{+\infty\} \) coincide on a neighborhood of some point \( \bar{y} \in \text{dom} f \cap \text{dom} g \), where \( \text{dom} f := \{ y \in Y : f(y) < +\infty \} \) is the domain of \( f \), then \( \partial f(\bar{y}) = \partial g(\bar{y}) \).

Consequently, in view of the Clarke subdifferential of the distance function (see [4]), we have

\[
\partial \Delta_A(\bar{y}) = \begin{cases} 
-\partial d_{A^c}(\bar{y}) \subseteq -N(\partial A^c(\bar{y}); \bar{y}), & \bar{y} \in \text{int} A, \\
\partial d_A(\bar{y}) \subseteq N(\partial A(\bar{y}); \bar{y}), & \bar{y} \notin \text{cl} A,
\end{cases} \tag{3}
\]

where \( A^c(\bar{y}) := \{ y \in Y : d_{A^c}(\bar{y}) \leq d_{A^c}(\bar{y}) \} \) if \( \bar{y} \in \text{int} A \), and \( A(\bar{y}) := \{ y \in Y : d_{A}(\bar{y}) \leq d_{A}(\bar{y}) \} \) if \( \bar{y} \notin \text{cl} A \). But, what about the Clarke subdifferential of \( \Delta_A \) at \( \bar{y} \in \text{bd} A \)? In order to answer the question, we are devoted to studying the Clarke subdifferential of \( \Delta_A \) at the boundary point of \( A \) in the sequence.

We recall the concept of epi-Lipschitz set. Due to Rockafellar [33], a set \( A \) of \( Y \) is said to be **epi-Lipschitz** at \( \bar{y} \in A \) in a direction \( v \in Y \) if there exist a real number \( \gamma > 0 \), and neighborhoods \( U \) and \( V \) in \( Y \) of \( \bar{y} \) and \( v \) respectively such that

\[
U \cap A + (0, \gamma)V \subseteq A.
\]

We simply say that \( A \) is epi-Lipschitz at \( \bar{y} \) if there exists \( v \in Y \) such that \( A \) is epi-Lipschitz at \( \bar{y} \) in the direction \( v \). If \( A \) is epi-Lipschitz at any of its points, or equivalently at any of its boundary points, it is called epi-Lipschitzian.

Recently, Cornet and Czarnecki [7] represented the Clarke normal cone of epi-Lipschitz set at its boundary point and establish the following characterization of epi-Lipschitz set in terms of oriented distance function.

**Proposition 4.1.** [7] Let \( S \) be a nontrivial subset of the normed linear space \( Y \) and \( \bar{y} \in S \cap \text{bd} S \).

(i) If \( S \) is epi-Lipschitz at \( \bar{y} \), then the Clarke normal cone of \( S \) at \( \bar{y} \) can be expressed as

\[
N(S; \bar{y}) = \mathbb{R}_+ \partial \Delta_S(\bar{y}). \tag{4}
\]

(ii) If \( \text{int}(\text{cl} S) \cap U \subseteq S \) for some neighbourhood \( U \) of \( \bar{y} \), then the set \( S \) is epi-Lipschitz at \( \bar{y} \) if and only if \( 0 \notin \partial \Delta_S(\bar{y}) \).

Here we point out the fact that free disposal set is epi-Lipschitz, when the topological interior is employed.

**Theorem 4.2.** Let \( A \subseteq Y \) be a nonempty free disposal set with respect to a convex cone \( K \subseteq Y \) with \( \text{int} K \neq \emptyset \). Then the following assertions hold:

(i) \( A \) is epi-Lipschitz at any \( \bar{y} \in A \) (i.e., \( A \) is epi-Lipschitz).

(ii) \( \text{cl} A \) is epi-Lipschitz at any \( \bar{y} \in \text{cl} A \) (i.e., \( \text{cl} A \) is epi-Lipschitz).

(iii) \( \text{int} A \) is epi-Lipschitz at any \( \bar{y} \in \text{int} A \) (i.e., \( \text{int} A \) is epi-Lipschitz).

(iv) \( A^c \) is epi-Lipschitz at any \( \bar{y} \in A^c \) (i.e., \( A^c \) is epi-Lipschitz).

(v) \( \text{cl}(A^c) \) is epi-Lipschitz at any \( \bar{y} \in \text{cl}(A^c) \) (i.e., \( \text{cl}(A^c) \) is epi-Lipschitz).

(vi) \( \text{int}(A^c) \) is epi-Lipschitz at any \( \bar{y} \in \text{int}(A^c) \) (i.e., \( \text{int}(A^c) \) is epi-Lipschitz).
Proof. Since $A$ is free disposal with respect to $K$ with $intK \neq \emptyset$, it follows that $intA = A + intK \neq \emptyset$ (see also [39], Theorem 3.1).

(i) Fix $\bar{y} \in A$. For any neighborhood $U$ of $\bar{y}$, we always have that
\[ A \cap U + \gamma \cdot (k_0 + intK) \subseteq intA, \forall \gamma > 0, \forall k_0 \in intK, \]
as $A + K = A$. Then by definition, it follows that $A$ is epi-Lipschitz at any $\bar{y} \in A$.

(ii) The proof is similar to the proof of (i), since $clA + K = clA$.

(iii) The proof is similar to the proof of (i), since $intA + K = intA$.

(iv) The proof is similar to the proof of (i), since $A^c - K = A^c$.

(v) The proof is similar to the proof of (i), since $cl(A^c) - K = cl(A^c)$.

As a consequence of Proposition 4.1(ii) and Theorem 4.2, we obtain the following result.

**Theorem 4.3.** Let $A \subseteq Y$ be a nontrivial subset, and $K \subseteq Y$ be a convex cone with $intK \neq \emptyset$. Assume that $A$ is free disposal with respect to $K$. Then $0 \notin \partial \Delta_A(\bar{y})$ at any $\bar{y} \in bdA$.

Proof. Since $A$ is free disposal with respect to $K$ and $intK \neq \emptyset$, it follows from Corollary 2.4 that
\[ \Delta_{clA} = \Delta_A. \] (5)

By Theorem 4.2(ii), $clA$ is epi-Lipschitz at any $\bar{y} \in clA$. Then, applying Proposition 4.1(ii) with $clA$ in place of $S$, we deduce that
\[ 0 \notin \partial \Delta_{clA}(\bar{y}), \forall \bar{y} \in clA \cap bd(clA). \] (6)

Notice that $intA = int(clA)$. It is easy to see that $bdA = bd(clA)$, which means that
\[ clA \cap bd(clA) = bdA. \] (7)

This together with (5) and (6) yields the conclusion.

In what follows, we establish the formula of the Clarke subdifferential of $\Delta_A$ at the boundary point of $A$, when the associate set $A$ is free disposal.

**Theorem 4.4.** Let $A \subseteq Y$ be nontrivial and free disposal with respect to a convex cone $K \subseteq Y$ with $intK \neq \emptyset$. Then the Clarke normal cone of $A$ at $\bar{y} \in bdA$ can be expressed as
\[ N(A; \bar{y}) = \mathbb{R}_+ \partial \Delta_A(\bar{y}). \] (8)

Moreover, the boundary of $A$ can be characterized by
\[ \bar{y} \in bdA \Leftrightarrow N(A; \bar{y}) \neq \{0\}. \] (9)

Proof. We first show (8). Fix $\bar{y} \in bdA$. Since $N(A; \bar{y}) = N(clA; \bar{y})$ is always true (see [25], Theorem 4.2.10), it suffices to show that
\[ N(clA; \bar{y}) = \mathbb{R}_+ \partial \Delta_A(\bar{y}). \] (10)

Following the same lines of the proof of Theorem 4.3, we have that (5) and (7) hold. Then, by using Proposition 4.1(i) with $clA$ in place of $S$, it follows that (10) holds.

Now, we prove (9). By Theorem 4.3, it follows from (8) that
\[ \bar{y} \in bdA \Rightarrow N(A; \bar{y}) \neq \{0\}. \]
On the other hand, when \( \bar{y} \in \text{int} A \), we always have that \( N(A; \bar{y}) = \{0\} \). Accordingly,
\[
N(A; \bar{y}) \neq \{0\} \Rightarrow \bar{y} \in \text{bd} A.
\]
This finishes the proof. \( \square \)

Combining with the conclusion (3) and Theorem 4.4, we get the Clarke subdifferential of \( \Delta_A \) at any \( y \in Y \).

**Theorem 4.5.** Let \( A \subseteq Y \) be nontrivial and free disposal with respect to a convex cone \( K \subseteq Y \). Assume that \( \text{int} K \neq \emptyset \). Then
\[
\partial \Delta_A(y) \subseteq K^- \bigcap B^*, \forall y \in Y.
\]
Moreover,
\[
\partial \Delta_A(\bar{y}) \subseteq \begin{cases} N(A; \bar{y}) \setminus \{0\} \bigcap B^*, & \bar{y} \in \text{bd} A, \\ N(A(\bar{y}); \bar{y}) \bigcap B^*, & \bar{y} \notin \text{cl} A, \\ -N(A^c(\bar{y}); \bar{y}) \bigcap B^*, & \bar{y} \in \text{int} A, \end{cases}
\]
where \( A(\bar{y}) = \{y \in Y : d_A(y) \leq d_A(\bar{y})\} \), \( A^c(\bar{y}) = \{y \in Y : d_{A^c}(y) \leq d_{A^c}(\bar{y})\} \).

**Remark 5.** It must be noted that the condition \( \text{int} K \neq \emptyset \) in Theorem 12 plays an essential role in guaranteeing \( 0 \notin \partial \Delta_A(\bar{y}) \) for \( \bar{y} \in \text{bd} A \). That is to say, if \( \text{int} K = \emptyset \), then we may get that \( 0 \in \partial \Delta_A(\bar{y}) \) for some \( \bar{y} \in \text{bd} A \). The following example is provided to illustrate this issue.

**Example 4.1.** Consider \( A := \{(a_1, a_2)^T \in \mathbb{R}^2 : a_1 \in \mathbb{R}, a_2 \in \mathbb{Q}\} \), where \( \mathbb{Q} \) denotes the set of rational numbers, \( K := \{(k_1, k_2)^T \in \mathbb{R}^2 : k_1 > 0, k_2 = 0\} \) with \( \text{int} K = \emptyset \). Clearly, \( A \) is nontrivial and free disposal with respect to \( K \). Take \( \bar{y} = (0, 0)^T \in \text{bd} A \).

It can be easy to verify that
\[
\Delta^0_A(\bar{y}; k) = \lim_{y \to \bar{y}} \frac{\Delta_A(y + tk) - \Delta_A(y)}{t} \geq 0, \forall k \in \mathbb{R}^2.
\]

This implies that \( 0 \in \partial \Delta_A(\bar{y}) \).

When \( A \) is nearly convex, \( \partial \Delta_A(\cdot) \) coincides with the classical subdifferential in the sense of convex analysis. Furthermore, if \( \text{int} K \neq \emptyset \), then we have more precise description of \( \partial \Delta_A(\cdot) \) at any \( y \in Y \).

**Theorem 4.6.** Let \( A \subseteq Y \) be nontrivial and free disposal with respect to a convex cone \( K \subseteq Y \). Assume that \( A \) is nearly convex, and \( \text{int} K \neq \emptyset \). Then,
\[
\partial \Delta_A(y) \subseteq K^- \setminus \{0\} \bigcap B^*, \forall y \in Y.
\]
More precisely,
\[
\partial \Delta_A(\bar{y}) \subseteq \begin{cases} N(A; \bar{y}) \setminus \{0\} \bigcap B^*, & \text{if } \bar{y} \in \text{bd} A, \\ N(A(\bar{y}); \bar{y}) \bigcap B^*, & \text{if } \bar{y} \notin \text{cl} A, \\ -N(A^c(\bar{y}); \bar{y}) \bigcap B^*, & \text{if } \bar{y} \in \text{int} A, \end{cases}
\]
where \( A(\bar{y}) = \{y \in Y : d_A(y) \leq d_A(\bar{y})\} \) and \( A^c(\bar{y}) = \{y \in Y : d_{A^c}(y) \leq d_{A^c}(\bar{y})\} \).

**Proof.** It can easily be shown that \( A(\bar{y}) \) is free disposal with respect to \( K \) if \( \bar{y} \notin \text{cl} A \), and \( A^c(\bar{y}) \) is free disposal with respect to \( -K \) if \( \bar{y} \in \text{int} A \). Using the definitions of the Clarke tangent cone and the Clarke normal cone, we obtain that \( N(A(\bar{y}); \bar{y}) \subseteq \)
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K− if \( \bar{y} \notin clA \) and \( N(A^c(\bar{y}); \bar{y}) \subseteq K^+ \) if \( \bar{y} \in intA \). Consequently, it suffices to show that

\[
0 \notin \partial \Delta_A(y), \forall y \in Y. \tag{15}
\]

Since \( A \) is nearly convex, it follows from Corollary 2.3 that \( \Delta_A \) is convex. In view of Corollary 3.6(i), we have that \( \Delta_A \) is strictly \( K \)-decreasing, as \( intK \neq \emptyset \). Then, for each \( y \in Y \),

\[
\Delta_A(y) > \Delta_A(y + k_0) \geq \Delta_A(y) + \langle y^*, k_0 \rangle, \forall y^* \in \partial \Delta_A(y), \forall k_0 \in intK.
\]

This implies (15) holds.

\[\square\]

Remark 6. It is necessary to say that the assumption \( intK \neq \emptyset \) in Theorem 4.6 is crucial for guaranteeing that \( 0 \notin \partial \Delta_A(y) \) at any \( y \in Y \). For example, consider \( A = K = \mathbb{R}_+ \times \{0\} \subseteq \mathbb{R}^2 \). Clearly, \( A \) is nearly convex and free disposal with respect to convex cone \( K \) with \( intK = \emptyset \). Take \( \bar{y} = (0,0)^T \in \mathbb{R}^2 \). It can easily be seen that \( 0 \notin \partial \Delta_A(y) \). Also, this example shows that even though \( A \) is a nontrivial closed convex co-radiant subset in \( Y \), but \( intA(0) = \emptyset \) (here \( A(0) = \bigcup_{\alpha > 0} \alpha A \)), we may have \( 0 \in \partial \Delta_A(y) \). Therefore, Theorem 3.4 in [12] can be improved as follows.

Corollary 4.7. Let \( A \subseteq Y \) be a nontrivial convex co-radiant subset in \( Y \). Assume that \( intA \neq \emptyset \). Then, the following assertions hold:

(i)

\[
\partial \Delta_A(y) \subseteq A^c \cap \{0\} \cap B^*, \forall y \in Y. \tag{16}
\]

(ii) If \( 0 \in A \), then for every \( y \in Y \) and for every \( \epsilon \in \mathbb{R}_+ \),

\[\partial \Delta_A(y) = \{y^* \in \partial \Delta_A(0): (y^*, y) = \Delta_A(y)\}, \tag{17}\]

\[\partial \Delta_A(y) = \{y^* \in \partial \Delta_A(0): (y^*, y) \geq \Delta_A(y) - \epsilon\}, \tag{18}\]

\[\partial \Delta_A(0) = \partial \Delta_A(0), \tag{19}\]

where \( \partial \Delta_A(y) := \{y^* \in \mathcal{Y}^* : \Delta_A(y') - \Delta_A(y) \geq \langle y^*, y' - y \rangle - \epsilon, \forall y' \in \mathcal{Y} \} \).

Proof. (i) Since \( A \) is a nontrivial convex co-radiant subset with \( intA \neq \emptyset \), we have that \( A \) is free disposal with respect to \( coneA \) and \( int(coneA) \neq \emptyset \). Then, by Theorem 4.5, it follows that (16) holds, as \( A^c = (coneA)^- \).

(ii) If \( 0 \in A \), then \( A \) is a nontrivial convex cone with \( 0 \in bdA \), and \( \Delta_A(\cdot) \) is sublinear. By Theorem 2.4.14 in [38], it is easy to see that (17), (18) and (19) hold.

\[\square\]

Remark 7. Note that the closure of \( A \) is not needed here, because \( \Delta_{clA} = \Delta_A \) under the assumptions of Corollary 4.7.

5. Optimality conditions of nonconvex vector optimization. Consider the following nonconvex vector optimization problem:

\[
(\text{VP}) \quad \min f(x) \text{ subject to } x \in S,
\]

where \( f : X \to Y \), and the feasible set \( S \subseteq X \) is nonempty.

In what follows, we assume that \( X \) and \( Y \) are Banach spaces, \( E \subseteq Y \) is nontrivial and free disposal with respect to a convex cone \( K \subseteq Y \) with \( intK \neq \emptyset \). Firstly, we recall some definitions of approximate solutions for (VP).

\[\square\]

\[1\]A is called a co-radiant set if \( \lambda A \subseteq A \) for all \( \lambda > 1 \), see [16].
contains a norm convergent subsequence whenever
\[ \text{(VP)}, \text{denoted by } \bar{x} \in \text{WOP}(f; S; E), \text{if} \]
\[ (f(S) - f(\bar{x})) \cap (-E) \subseteq \{0\}. \]

**Definition 5.1.** [17] A point \( \bar{x} \in S \) is said to be \( E \)-efficient solution of (VP), denoted by \( \bar{x} \in OP(f; S; E) \), if
\[ (f(S) - f(\bar{x})) \cap (-E) \subseteq \{0\}. \]

**Definition 5.2.** [40] A point \( \bar{x} \in S \) is said to be weakly \( E \)-efficient solution of (VP), denoted by \( \bar{x} \in \text{WOP}(f; S; E) \), if
\[ (f(S) - f(\bar{x})) \cap (-\text{int}E) = \emptyset. \]

In terms of oriented distance function, the corresponding scalar optimization problem can be written as:
\[ (P_{-E,y_0}) \min_{x \in S} \Delta_E(f(x) - y_0), \]
where \( y_0 \in Y \). The set of \( (P_{-E,y_0}) \) with error \( \varepsilon \geq 0 \) is denoted by \( \text{argmin}_S(\Delta \circ f, y_0, \varepsilon) \) (resp. \( \text{argmin}_S^< (\Delta \circ f, y_0, \varepsilon) \)), i.e.,
\[ \text{argmin}_S(\Delta_E \circ f, y_0, \varepsilon) = \{ \bar{x} \in S : \Delta_E(f(\bar{x}) - y_0) - \varepsilon \leq \Delta_E(f(x) - y_0), \forall x \in S \}, \]
\[ (\text{resp. } \text{argmin}_S^< (\Delta_E \circ f, y_0, \varepsilon) ) \]
\[ = \{ \bar{x} \in S : \Delta_E(f(\bar{x}) - y_0) - \varepsilon < \Delta_E(f(x) - y_0), \forall x \in S \setminus \{\bar{x}\} \}. \]

We denote \( \text{argmin}_S(\Delta_E \circ f, y_0) := \text{argmin}_S(\Delta_E \circ f, y_0, 0) \), i.e., the set of exact minima of \( (P_{-E,y_0}) \).

By definition, we can easily establish the following complete characterizations of approximate solutions to (VP).

**Theorem 5.3.** Assume that \( 0 \notin \text{int}E \). Then the following assertions hold:
(i) \( \bar{x} \in \text{WOP}(f; S; E) \Leftrightarrow \bar{x} \in \text{argmin}_S(\Delta_E \circ f, f(\bar{x}), \Delta_E(0)) \).
(ii) \( \bar{x} \in \text{OP}(f; S; E) \Leftrightarrow \bar{x} \in \text{argmin}_S^< (\Delta_E \circ f, f(\bar{x}), \Delta_E(0)) \). In addition, if \( E \) is closed, then \( \bar{x} \in \text{OP}(f; S; E) \Leftrightarrow \bar{x} \in \text{argmin}_S^< (\Delta_E \circ f, f(\bar{x}), \Delta_E(0)) \).

**Remark 8.** The assumption \( 0 \notin \text{int}E \) in the above theorem guarantees that \( \Delta_E(0) \geq 0 \).

Next, we establish optimality conditions via oriented direction function. We start studying the nonconvex case (i.e., the set \( E \) and the objective function \( f \) of problem (VP) are nonconvex).

**Theorem 5.4.** Let \( S \subseteq X \) be closed and \( 0 \notin \text{int}E \). Denote \( \varepsilon := \Delta_E(0) \). Assume that \( f : X \to Y \) is strictly Lipschitz\(^2\). If \( \bar{x} \in \text{WOP}(f; S; E) \), then for any \( \lambda > 0 \),

\(^2f\) is strictly Lipschitz at \( \bar{x} \) if \( f \) is Lipschitz around \( \bar{x} \), and there is a neighborhood \( V \) of the origin of \( X \) such that the sequence
\[ y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in \mathbb{N}, \]
contains a norm convergent subsequence whenever \( v \in V \), \( x_k \to x \), and \( t_k \downarrow 0 \). If \( f \) is strictly Lipschitz at every point \( x \in X \), it is called strictly Lipschitz on \( X \).

Ralph [32] presents the chain rule for Clarke subdifferentials under the assumption of compactly Lipschitzian mappings. Hence, we use strictly Lipschitz property here.
there exists \( \hat{x} \in S \cap B(\bar{x}; \lambda) \) such that
\[
0 \in \text{weak}^* - \text{cl conv} \quad \bigcup_{y^* \in \partial \Delta_E(f(\hat{x}) - f(\bar{x}))} \partial(y^* \circ f)(\hat{x}) + \frac{\varepsilon}{\lambda} B^* + N(S; \hat{x}).
\] (20)
Furthermore, if \( f : X \to Y \) is continuously Fréchet differentiable, then for any \( \lambda > 0 \), there exist \( \hat{x} \in S \cap B(\bar{x}; \lambda) \) and \( y^* \in K^+ \) such that
\[
0 \in \nabla f(\hat{x})^* y^* + \frac{\varepsilon}{\lambda} B^* + N(S; \hat{x}),
\] (21)
where \( \nabla f(\hat{x})^* \) is the adjoint of the Fréchet derivative of \( f \) at \( \hat{x} \).

**Proof.** Since \( \hat{x} \in \text{WOP}(f, S; E) \), it follows from Theorem 5.3 (i) that
\[
\Delta_E(f(\hat{x}) - f(\bar{x})) - \varepsilon \leq \Delta_E(f(x) - f(\hat{x})), \forall x \in S,
\]
where \( \varepsilon = \Delta_E(0) \geq 0 \), as \( 0 \notin \text{int}E \). From the Ekeland variational principle applied to \( \Delta_E(f(x) - f(\hat{x})) \) on \( S \), we have that for any \( \lambda > 0 \), there exists \( \hat{x} \in S \cap B(\bar{x}, \lambda) \) such that
\[
\Delta_E(f(\hat{x}) - f(\bar{x})) \leq \Delta_E(f(x) - f(\hat{x})) + \frac{\varepsilon}{\lambda} \|x - \hat{x}\|, \forall x \in S,
\]
which means that \( \hat{x} \) is an optimal solution of \( \Delta_E(f(\cdot) - f(\hat{x})) + \frac{\varepsilon}{\lambda} \cdot \|\cdot - \hat{x}\| \) on \( S \).

By calculus rules of the Clarke subdifferential, we get
\[
0 \in \partial \left( \Delta_E(f(\cdot) - f(\hat{x})) + \frac{\varepsilon}{\lambda} \|\cdot - \hat{x}\| \right) (\hat{x}) + N(S; \hat{x})
\leq \partial \Delta_E(f(\cdot) - f(\hat{x}))(\hat{x}) + \frac{\varepsilon}{\lambda} B^* + N(S; \hat{x}).
\] (22)
Owing to the chain rule given by Ralph [32] and Theorem 4.5, it is easy to see that there exists \( y^* \in K^+ \) such that (20) holds.

Furthermore, according to Clarke’s chain rule in [6], if \( f : X \to Y \) is continuously Fréchet differentiable, then
\[
\partial \Delta_E(f(\cdot) - f(\hat{x}))(\hat{x}) \subseteq \partial \Delta_E(f(\hat{x}) - f(\bar{x})) \circ \nabla f(\hat{x}).
\]
By Theorem 4.5, there exists \( y^* \in K^+ \) such that (21) holds. \( \square \)

Since \( \Delta_A \) is Lipschitz on \( Y \), by Corollary 2.25 in [31], we have that the Mordukhovich subdifferential \( \partial_M \Delta_A(\cdot) \neq \emptyset \) if \( Y \) is an Asplund space. Consequently, we can restrict ourselves to the Asplund space setting, and get more sharper optimality conditions for weakly \( E \)-optimal solutions to \((VP)\) in terms of Mordukhovich subdifferential and Mordukhovich normal cone. The proof is similar as before.

**Theorem 5.5.** Let \( X \) and \( Y \) be two Asplund spaces, \( S \) be closed, and \( 0 \notin \text{int}E \). Denote \( \varepsilon := \Delta_E(0) \). Assume that \( f : X \to Y \) is strictly Lipschitz. If \( \hat{x} \in \text{WOP}(f, S; E) \), then for any \( \lambda > 0 \), there exist \( \hat{x} \in S \cap B(\bar{x}; \lambda) \) and \( y^* \in K^+ \) such that
\[
0 \in \partial_M(y^* \circ f)(\hat{x}) + \frac{\varepsilon}{\lambda} B^* + N_M(S; \hat{x}),
\] (23)
where \( \partial_M \) is the Mordukhovich subdifferential and \( N_M(S; \hat{x}) \) is the Mordukhovich normal cone of \( S \) at \( \hat{x} \).

**Remark 9.** (i) Fuzzy necessary optimality conditions of Theorem 5.4 and Theorem 5.5 remain valid for \( E \)-efficient solutions to \((VP)\), if \( E \) is a closed subset of \( Y \).
(ii) We can conclude that there exists \( y^* \in K^+ \setminus \{0\} \) such that (21) and (23) hold when \( E \) is convex.
(iii) Let us consider the case when $E = K$ is a nontrivial convex cone with nonempty topological interior. Obviously, $\varepsilon = \Delta_E(0) = 0$. Then fuzzy necessary optimality conditions of Theorem 5.4 and Theorem 5.5 reduce to necessary optimality conditions of Theorem 3.1 and Theorem 3.2 in [11], respectively. Furthermore, if $S$ is a closed convex subset of $X$, by (21), we still get necessary optimality condition of Theorem 3.1 in [9]. Here we do not need any convexity of $f$ and the solidness of $f(S)$.

(iv) Let $E$ be a nontrivial convex co-radiant set with nonempty topological interior. It is easy to check that $E$ is free disposal with respect to $cone E$. Then, fuzzy necessary optimality condition of Theorem 5.5 reduces to Lagrange multiplier rule in Theorem 4.4(i) given by Gao et al. [12].

6. Conclusion. Oriented distance function is intensively studied due to its important applications in vector optimization. However, in the literatures, most of the results are established in the case that the associated set is convex or nonsolid. A natural question is whether there is any interesting property of oriented distance function when the associated set is nonconvex or solid. This paper concerns, on the one hand, the monotonicity properties of oriented distance function in the nonconvex and solid cases. Subsequently, the Clarke subdifferentials of oriented distance function are discussed by virtue of the nonconvex free disposal set. We conclude that when the associated set is free disposal, the formulas for the Clarke subdifferential of oriented distance function at the boundary point behave in a fashion similar to the convex case. On the other hand, fuzzy necessary approximate optimality conditions for approximate (weakly) efficient solutions to vector optimization problem are expressed in terms of Clarke subdifferentials and obtained on the basis of nonlinear scalarization technique. These necessary conditions we establish are new and can reduce to the ones in [9, 11, 12].

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REFERENCES

[1] Q. H. Ansari, E. Köbis and P. K. Sharma, Characterizations of multiobjective robustness via oriented distance function and image space analysis, J. Optim. Theory Appl., 181 (2019), 817–839.
[2] Y. Araya, Four types of nonlinear scalarizations and some applications in set optimization, Nonlinear Anal., 75 (2012), 3821–3835.
[3] R. I. Bot, S.-M. Grad and G. Wanka, A general approach for studying duality in multiobjective optimization, Math. Methods Oper. Res., 65 (2007), 417–444.
[4] J. V. Burke, M. C. Ferris and M. Qian, On the Clarke subdifferential of the distance function of a closed set, J. Math. Anal. Appl., 166 (1992), 199–213.
[5] G.-Y. Chen, X. Huang and X. Yang, Vector Optimization. Set-Valued and Variational Analysis, Lecture Notes in Economics and Mathematical Systems, 541, Springer-Verlag, Berlin, 2005.
[6] F. Clarke, Functional Analysis, Calculus of Variations and Optimal Control, Graduate Texts in Mathematics, 264, Springer, London, 2013.
[7] M.-O. Czarnecki and L. Thibault, Sublevel representations of epi-Lipschitz sets and other properties, Math. Program., 168 (2018), 555–569.
[8] M. C. Delfour and J.-P. Zolésio, Shape analysis via oriented distance functions, J. Funct. Anal., 123 (1994), 129–201.
[9] M. Durea, J. Dutta and C. Tammer, Lagrange multipliers for ϵ-Pareto solutions in vector optimization with nonsolid cones in Banach spaces, J. Optim. Theory Appl., 145 (2010), 196–211.
[10] M. Durea, R. Strugariu and C. Tammer, Scalarization in geometric and functional vector optimization revisited, J. Optim. Theory Appl., 159 (2013), 635–655.
[11] J. Dutta and C. Tammer, Lagrangian conditions for vector optimization in Banach spaces, Math. Methods Oper. Res., 64 (2006), 521–540.
[12] Y. Gao and X.-M. Yang, Properties of the nonlinear scalar functional and its applications to vector optimization problems, J. Global Optim., 73 (2019), 869–889.
[13] Y. Gao, X. Yang and K. L. Teo, Optimality conditions for approximate solutions of vector optimization problems, J. Ind. Manag. Optim., 7 (2011), 483–496.
[14] C. Gerth and P. Weidner, Nonconvex separation theorems and some applications in vector optimization, J. Optim. Theory Appl., 67 (1990), 297–320.
[15] C. Gutiérrez, B. Jiménez, E. Miglierina and E. Molho, Scalarization in set optimization with solid and nonsolid ordering cones, J. Global Optim., 61 (2015), 552–555.
[16] C. Gutiérrez, B. Jiménez and V. Novo, A unified approach and optimality conditions for approximate solutions of vector optimization problems, SIAM J. Optim., 17 (2006), 688–710.
[17] C. Gutiérrez, B. Jiménez and V. Novo, Improvement sets and vector optimization, European J. Oper. Res., 223 (2012), 304–311.
[18] C. Gutiérrez, B. Jiménez and V. Novo, Nonlinear scalarizations of set optimization problems with set orderings, in Set Optimization and Applications - The State of the Art, Springer Proc. Math. Stat., 151, Springer, Heidelberg, 2015, 43–63.
[19] C. Gutiérrez, E. Miglierina, E. Molho and V. Novo, Pointwise well-posedness in set optimization with cone proper sets, Nonlinear Anal., 75 (2012), 1822–1833.
[20] C. Gutiérrez, V. Novo, J. L. Ródenas-Pedregosa and T. Tanaka, Nonconvex separation functional in linear spaces with applications to vector equilibria, SIAM J. Optim., 26 (2016), 2677–2695.
[21] J.-B. Hiriart-Urruty, Tangent cones, generalized gradients and mathematical programming in Banach spaces, Math. Oper. Res., 4 (1979), 79–97.
[22] J.-B. Hiriart-Urruty, New concepts in nondifferentiable programming. Analyse non convexe, Bull. Soc. Math. France Mém., (1979), 75–85.
[23] B. Jiménez, V. Novo and A. Vilchez, A set scalarization function based on the oriented distance and relations with other set scalarizations, Optimization, 67 (2018), 2091–2116.
[24] B. Jiménez, V. Novo and A. Vilchez, Characterization of set relations through extensions of the oriented distance, Math. Methods Oper. Res., 91 (2020), 89–115.
[25] A. A. Khan, C. Tammer and C. Zălinescu, Set-Valued Optimization. An Introduction with Applications, Vector Optimization, Springer, Heidelberg, 2015.
[26] C. S. Lalitha and P. Chatterjee, Stability and scalarization in vector optimization using improvement sets, J. Optim. Theory Appl., 166 (2015), 825–843.
[27] G. H. Li, S. J. Li and M. X. You, Relationships between the oriented distance functional and a nonlinear separation functional, J. Math. Anal. Appl., 466 (2018), 1109–1117.
[28] C. G. Liu, K. F. Ng and W. H. Yang, Merit functions in vector optimization, Math. Program., 119 (2009), 215–237.
[29] H. Luo, X. Wang and B. Lukens, Variational analysis on the signed distance functions, J. Optim. Theory Appl., 180 (2019), 751–774.
[30] E. Miglierina, E. Molho and M. Rocca, Well-posedness and scalarization in vector optimization, J. Optim. Theory Appl., 126 (2005), 391–409.
[31] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. I. Basic Theory, Fundamental Principles of Mathematical Sciences, 330, Springer-Verlag, Berlin, 2006.
[32] D. Ralph, A chain rule for nonsmooth composite functions via minimisation, Bull. Austral. Math. Soc., 49 (1994), 129–137.
[33] R. T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, Canadian J. Math., 32 (1980), 257–280.
[34] L. Thibault, On compactly Lipschitzian mappings, in Recent Advances in Optimization, Lecture Notes in Econ. and Math. Systems, 452, Springer, Berlin, 1997, 356–364.
[35] H.-Z. Wei, C.-R. Chen and S.-J. Li, Characterizations for optimality conditions of general robust optimization problems, *J. Optim. Theory Appl.*, **177** (2018), 835–856.

[36] L. Zadeh, Optimality and non-scalar-valued performance criteria, *IEEE Trans. Automatic Control*, **8** (1963), 59–60.

[37] A. Zaffaroni, Degrees of efficiency and degrees of minimality, *SIAM J. Control Optim.*, **42** (2003), 1071–1086.

[38] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[39] K. Q. Zhao and X. M. Yang, $E$–Benson proper efficiency in vector optimization, *Optimization*, **64** (2015), 739–752.

[40] K.-Q. Zhao, X.-M. Yang and J.-W. Peng, Weak $E$-optimal solution in vector optimization, *Taiwanese J. Math.*, **17** (2013), 1287–1302.

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