\[ U_{\text{PMNS}} = U_\ell^\dagger U_\nu \]

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Abstract

We consider corrections to vanishing \( U_{e3} \) and maximal atmospheric neutrino mixing originating from the relation \( U = U_\ell^\dagger U_\nu \), where \( U \) is the PMNS mixing matrix and \( U_\ell \) (\( U_\nu \)) is associated with the diagonalization of the charged lepton (neutrino) mass matrix. We assume that in the limit of \( U_\ell \) or \( U_\nu \) being the unit matrix, one has \( U_{e3} = 0 \) and \( \theta_{23} = \pi/4 \), while the solar neutrino mixing angle is a free parameter. Well-known special cases of the indicated scenario are the bimaximal and tri-bimaximal mixing schemes. If \( U_{e3} \neq 0 \) and \( \theta_{23} \neq \pi/4 \) due to corrections from the charged leptons, \( |U_{e3}| \) can be sizable (close to the existing upper limit) and we find that the value of the solar neutrino mixing angle is linked to the magnitude of CP violation in neutrino oscillations. In the alternative case of the neutrino sector correcting \( U_{e3} = 0 \) and \( \theta_{23} = \pi/4 \), we obtain a generically smaller \( |U_{e3}| \) than in the first case. Now the magnitude of CP violation in neutrino oscillations is connected to the value of the atmospheric neutrino mixing angle \( \theta_{23} \). We find that both cases are in agreement with present observations. We also introduce parametrization independent “sum-rules” for the oscillation parameters.

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1 Introduction

The low energy neutrino mixing implied by the neutrino oscillation data can be described by the Lagrangian (see, e.g., [1])

\[ \mathcal{L} = -\frac{g}{\sqrt{2}} \bar{\ell}_L \gamma^\mu \nu_L W_\mu - \frac{1}{2} \bar{\nu}_R m_\nu \nu_L - \bar{\ell}_R m_\ell \ell_L + h.c. \]  

(1)

which includes charged lepton and Majorana neutrino mass terms. When diagonalizing the neutrino and charged lepton mass matrices via

\[ m_\nu = U_\nu^\dagger m_\nu^{\text{diag}} U_\nu \]  

and

\[ m_\ell = V_\ell m_\ell^{\text{diag}} V_\ell^\dagger, \]

we obtain the lepton mixing (PMNS) matrix in the weak charged lepton current

\[ U = U_\ell^T U_\nu. \]

(2)

From the analyzes of the currently existing neutrino oscillation data it was found [2] that the present best-fit values of the CHOOZ and atmospheric neutrino mixing angles, \( \theta_{13} \) and \( \theta_{23} \), correspond to \( |U_{e3}| = \sin \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \), i.e., to \( |U_{\mu 3}| = |U_{\tau 3}| \). Accordingly, the “best-fit” PMNS matrix is given by

\[ U = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \frac{\theta_{12}}{\sqrt{2}} & \cos \frac{\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\sin \frac{\theta_{12}}{\sqrt{2}} & \cos \frac{\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \]

(3)

where we have not written the Majorana phases [3, 4] and used \( \theta_{23} = -\pi/4 \) in the usual Particle Data Group (PDG) parametrization of the PMNS matrix. One well-known possibility to construct this “phenomenological” mixing matrix is to require a \( \mu - \tau \) exchange symmetry for the neutrino mass matrix in the basis in which the charged lepton mass matrix is diagonal [5]. Well-known examples of neutrino mixing with \( \mu - \tau \) symmetry are the bimaximal [6] and tri-bimaximal [7] mixing matrices

\[ U_{\text{bi}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \]  

and

\[ U_{\text{tri}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \]

(4)

A common feature of these two mixing matrices is \( \theta_{23} = \pm \pi/4 \) and \( \theta_{13} = 0 \), which is perfectly compatible with the current data. However, they differ in their prediction for the value of the solar neutrino mixing angle, namely, \( \sin^2 \theta_{12} = 1/2 \) and \( 1/3 \), respectively. The best-fit value of \( \sin^2 \theta_{12} \) determined from the neutrino oscillation data is \( \sin^2 \theta_{12} = 0.30 \). Actually, \( \sin^2 \theta_{12} = 1/2 \) is ruled out by the data at more than 6 \( \sigma \) [8].

A natural possibility to obtain a phenomenologically viable PMNS neutrino mixing matrix, and to generate non-zero \( |U_{e3}| \) and non-maximal \( \theta_{23} \), is to assume that one of the two
matrices in $U = U_\ell^\dagger U_\nu$ corresponds to Eq. (3) or (4), and is “perturbed” by the second matrix leading to the required PMNS matrix. Following this assumption, corrections to bimaximal [9, 10, 11, 12, 13] and tri-bimaximal [14, 15, 13] mixing have previously been analyzed. For instance, scenarios in which the CKM quark mixing matrix corrects the bimaximal mixing pattern are important for models incorporating Quark-Lepton Complementarity (QLC) [16, 17, 18] (for earlier reference see [19]). Corrections to mixing scenarios with $\theta_{12} = \pi/4$ and $\theta_{13} = 0$ were considered in [20] (motivated by the $L_e - L_\mu - L_\tau$ flavor symmetry [21]) and in [12]. The case with $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ has been investigated in Refs. [22, 23, 13, 24].

Up to now in most analyzes it has been assumed that $U_\nu$ possesses a form which leads to $\sin^2 \theta_{23} = 1/2$ and $\theta_{13} = 0$. However, the alternative possibility of $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ originating from $U_\ell$ is phenomenologically equally viable. We are aware of only few papers in which that option is discussed [11, 12, 25, 26]. A detailed study is still lacking in the literature. In the present article we perform, in particular, a comprehensive analysis of this possibility. We also revisit the case of $U_{e3} \neq 0$ and $\theta_{23} \neq \pi/4$ due to corrections from $U_\ell^\dagger$ and derive parametrization independent sum-rules for the relevant oscillation parameters. We point out certain “subtleties” in the identification of the relevant phases governing CP violation in neutrino oscillations with the Dirac phase of the standard parametrization of the PMNS matrix.

Our paper is organized as follows: Section 2 briefly summarizes the formalism and the relevant matrices from which the neutrino mixing observables can be reconstructed. We analyze the possibility of $U_\nu$ leading to $\sin^2 \theta_{23} = 1/2$ and $\theta_{13} = 0$ and being corrected by a non-trivial $U_\ell$ in Sec. 3. In Sec. 4 the alternative case of $U_\ell$ causing $\sin^2 \theta_{23} = 1/2$ and $\theta_{13} = 0$ and being modified by a non-trivial $U_\nu$ is discussed. Section 5 contains our conclusions.

2 Formalism and Definitions

We will use the following parametrization of the PMNS matrix:

$$U = V \text{diag}(1, e^{i\alpha}, e^{i(\beta+\delta)}) = O_{23}(\theta_{23}) U_{13}(\theta_{13}, \delta) O_{12}(\theta_{12}) \text{diag}(1, e^{i\alpha}, e^{i(\beta+\delta)})$$

$$= \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix} \text{diag}(1, e^{i\alpha}, e^{i(\beta+\delta)}),$$

(5)
where $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ and $O_{ij}(\theta_{ij})$ is a $3 \times 3$ orthogonal matrix of rotations on angle $\theta_{ij}$ in the $ij$-plane. We have also defined

$$U_{13}(\theta_{13}, \delta) = \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix}.$$  \hspace{1cm} (6)

Hereby we have included the Dirac CP violating phase $\delta$ and the two Majorana CP violating phases $\alpha$ and $\beta$ \cite{[3] [4]}. In general, all phases and mixing angles of $U$ are functions of the parameters characterizing $U_\nu$ and $U_\ell$. It can be shown that \cite{[27] [10]} after eliminating the unphysical phases, $U$ can be written as $U = \tilde{U}_\ell^\dagger U_\nu$, where in the most general case $U_\nu$ and $\tilde{U}_\ell$ are given by

$$U_\nu = P \tilde{U}_\nu Q = \text{diag}(1, e^{i\phi}, e^{i\omega}) \tilde{U}_\nu \text{diag}(1, e^{i\sigma}, e^{i\tau})$$

$$= P O_{23}(\theta_{23}^\nu) U_{13}(\theta_{13}^\nu, \xi) O_{12}(\theta_{12}^\nu) Q$$

$$= P \begin{pmatrix} c_{12}^\nu c_{13}^\nu & s_{12}^\nu c_{13}^\nu & s_{13}^\nu e^{-i\xi} \\ -s_{12}^\nu c_{23}^\nu - c_{12}^\nu s_{23}^\nu s_{13}^\nu e^{i\xi} & c_{12}^\nu c_{23}^\nu - s_{12}^\nu s_{23}^\nu s_{13}^\nu e^{i\xi} & s_{23}^\nu c_{13}^\nu \\ s_{12}^\nu s_{23}^\nu - c_{12}^\nu c_{23}^\nu s_{13}^\nu e^{i\xi} & -c_{12}^\nu s_{23}^\nu - s_{12}^\nu c_{23}^\nu s_{13}^\nu e^{i\xi} & c_{23}^\nu c_{13}^\nu \end{pmatrix} Q \hspace{1cm} \text{(7)}$$

where $P = \text{diag}(1, e^{i\phi}, e^{i\omega})$, $Q = \text{diag}(1, e^{i\sigma}, e^{i\tau})$ are rather important for the results to be obtained, and

$$\tilde{U}_\ell = O_{23}(\theta_{23}^\ell) U_{13}(\theta_{13}^\ell, \psi) O_{12}(\theta_{12}^\ell)$$

$$= \begin{pmatrix} c_{12}^\ell c_{13}^\ell & s_{12}^\ell c_{13}^\ell & s_{13}^\ell e^{-i\psi} \\ -s_{12}^\ell c_{23}^\ell - c_{12}^\ell s_{23}^\ell s_{13}^\ell e^{i\psi} & c_{12}^\ell c_{23}^\ell - s_{12}^\ell s_{23}^\ell s_{13}^\ell e^{i\psi} & s_{23}^\ell c_{13}^\ell \\ s_{12}^\ell s_{23}^\ell - c_{12}^\ell c_{23}^\ell s_{13}^\ell e^{i\psi} & -c_{12}^\ell s_{23}^\ell - s_{12}^\ell c_{23}^\ell s_{13}^\ell e^{i\psi} & c_{23}^\ell c_{13}^\ell \end{pmatrix}. \hspace{1cm} \text{(8)}$$

Here we have defined $c_{ij}^{\ell,\nu} = \cos \theta_{ij}^{\ell,\nu}$ and $s_{ij}^{\ell,\nu} = \sin \theta_{ij}^{\ell,\nu}$. Thus, $\tilde{U}_\nu$ and $\tilde{U}_\ell$ contain one physical CP violating phase each $\mathbb{I}$\cite{[1]}. The remaining four phases are located in the diagonal matrices $P$ and $Q$. Note that $Q$ is “Majorana-like” $\mathbb{II}$, i.e., the phases $\sigma$ and $\tau$ contribute only to the low energy observables related to the Majorana nature of the neutrinos with definite mass. Typically that are specific observables associated with $|\Delta L| = 2$ processes, like neutrinoless double beta decay $(A, Z) \rightarrow (A, Z + 2) + e^- + e^-$ (see, e.g., $\mathbb{III}[28][29]$). In the following we will be interested in models and the phenomenological consequences that result if $\tilde{U}_\nu$ corresponds to Eq. (3), while $\tilde{U}_\ell$ contains comparatively small angles, and vice versa. It proves convenient to introduce the abbreviations $\sin \theta_{ij}^{\ell,\nu} = \lambda_{ij} > 0$ for the small quantities we will use as expansion parameters in our further analysis.

\textsuperscript{1}In Section 4 it will be convenient to define instead of $U_\ell$ its transposed matrix as $U_\ell^T = O_{23}(\theta_{23}^\ell) U_{13}(\theta_{13}^\ell, \psi) O_{12}(\theta_{12}^\ell)$. In addition, $U_\nu^\dagger = P \tilde{U}_\nu Q$ will be used there.

4
Turning to the observables, the sines of the three mixing angles of the PMNS matrix $U$ are given by

$$\sin^2 \theta_{13} = |U_{e3}|^2, \quad \sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, \quad \sin^2 \theta_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2}.$$  (9)

The expressions quoted above are in terms of the absolute values of the elements of $U$, which emphasizes the independence of parametrization. In the case of 3-$\nu$ mixing under discussion there are, in principle, three independent CP violation rephasing invariants, associated with the three CP violating phases of the PMNS matrix. The invariant related to the Dirac phase $\delta$ is given as

$$J_{CP} = \text{Im}\left\{U^*_{\tau1} U^*_{\mu2} U_{e3} U_{\mu1}\right\},$$

which controls the magnitude of CP violation effects in neutrino oscillations and is a directly observable quantity [30]. It is analogous to the rephasing invariant associated with the Dirac phase in the Cabibbo-Kobayashi-Maskawa quark mixing matrix, introduced in Ref. [31]. In addition to $J_{CP}$, there are two rephasing invariants associated with the two Majorana phases in the PMNS matrix, which can be chosen as $S_1, S_2$ (see also [29]):

$$S_1 = \text{Im}\left\{U_{e1} U^*_{e3}\right\}, \quad S_2 = \text{Im}\left\{U_{e2} U^*_{e3}\right\}.$$  (11)

The rephasing invariants associated with the Majorana phases are not uniquely determined. Instead of $S_1$ defined above we could also have chosen $S'_1 = \text{Im}\{U^*_{\tau1} U_{\tau2}\}$ or $S''_1 = \text{Im}\{U_{\mu1} U^*_{\mu2}\}$, while instead of $S_2$ we could have used $S'_2 = \text{Im}\{U^*_{\tau2} U_{\tau3}\}$ or $S''_2 = \text{Im}\{U_{\mu2} U^*_{\mu3}\}$. The Majorana phases $\alpha$ and $\beta$, or $\beta$ and $(\beta - \alpha)$, can be expressed in terms of the rephasing invariants in this way introduced [29], for instance via $\cos\beta = 1 - S_1^2/|U_{e1} U_{e3}|^2$. The expression for, e.g., $\cos\alpha$ in terms of $S_1'$ is somewhat more cumbersome (it involves also $J_{CP}$) and we will not give it here. Note that CP violation due to the Majorana phase $\beta$ requires that both $S_1 = \text{Im}\{U_{e1} U^*_{e3}\} \neq 0$ and $\text{Re}\{U_{e1} U^*_{e3}\} \neq 0$. Similarly, $S_2 = \text{Im}\{U^*_{e2} U_{e3}\} \neq 0$ would imply violation of the CP symmetry only if in addition $\text{Re}\{U^*_{e2} U_{e3}\} \neq 0$.

Finally, let us quote the current data on the neutrino mixing angles [2, 8]:

$$\sin^2 \theta_{12} = 0.30^{+0.02, 0.10}_{-0.03, 0.06},$$
$$\sin^2 \theta_{23} = 0.50^{+0.08, 0.18}_{-0.07, 0.16},$$
$$|U_{e3}|^2 = 0^{+0.012, 0.041}_{-0.000},$$

where we have given the best-fit values as well as the $1\sigma$ and $3\sigma$ allowed ranges.

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2The expressions for the invariants $S_{1,2}$ that we give here and will use further in the discussion correspond to Majorana conditions for the fields of neutrinos with definite mass $\nu_j$ that do not contain phase factors, see, e.g., [29].
3 Maximal Atmospheric Neutrino Mixing and $U_{e3} = 0$
from the Neutrino Mass Matrix

In this Section we assume that maximal atmospheric neutrino mixing and vanishing $|U_{e3}|$ are realized in the limiting case, where $U_\ell$ corresponds to the unit matrix. We can obtain $\theta_{23}' = -\pi/4$ and $\theta_{13}' = 0$ by requiring $\mu-\tau$ exchange symmetry \[5, 23\] of the neutrino mass matrix in the basis in which the charged lepton mass matrix is diagonal. Under this condition we have

$$m_\nu = \begin{pmatrix}
A_\nu & B_\nu & B_\nu \\
\cdot & D_\nu + E_\nu & E_\nu - D_\nu \\
\cdot & \cdot & D_\nu + E_\nu
\end{pmatrix},$$

with $A_\nu \equiv m_1 c_{12}^2 + e^{-2i\alpha} m_2 s_{12}^2$, $B_\nu \equiv (e^{-2i\alpha} m_2 - m_1) c_{12} s_{12}/\sqrt{2}$, $D_\nu \equiv e^{-2i\beta} m_3/2$, $E_\nu \equiv \frac{1}{2} (e^{-2i\alpha} m_2 c_{12}^2 + m_1 s_{12}^2)$,\[12\]

where $m_{1,2,3}$ are the neutrino masses. The indicated symmetry is assumed to hold in the charged lepton mass basis, although the charged lepton masses are obviously not $\mu-\tau$ symmetric. However, such a scenario can, for example, be easily realized in models with different Higgs doublets generating the up- and down-like particle masses.

For the sines of the “small” angles in the matrix $U_\ell$ we introduce the convenient notation $\sin \theta_{ij}^e = \lambda_{ij} > 0$ with $ij = 12, 13, 23$. We obtain the following expressions for the observables relevant for neutrino oscillation in the case under consideration:

$$\sin^2 \theta_{12} \simeq \sin^2 \theta_{12}' - \frac{1}{\sqrt{2}} \sin 2\theta_{12}' (\lambda_{12} \cos \phi + \lambda_{13} \cos(\omega - \psi)),$$

$$|U_{e3}| \simeq \frac{1}{\sqrt{2}} \left| \lambda_{12} e^{i\phi} - \lambda_{13} e^{i(\omega - \psi)} \right|,$$

$$\sin^2 \theta_{23} \simeq \frac{1}{2} + \lambda_{23} \cos(\omega - \phi) - \frac{1}{4} (\lambda_{12}^2 - \lambda_{13}^2) + \frac{1}{2} \cos(\omega - \phi - \psi) \lambda_{12} \lambda_{13},$$

$$J_{CP} \simeq \frac{1}{4\sqrt{2}} \sin 2\theta_{12}' (\lambda_{12} \sin \phi - \lambda_{13} \sin(\omega - \psi)).$$

Setting in these equations $\theta_{ij}'$ to $\pi/4$ (to $\sin^{-1} \sqrt{1/3}$) reproduces the formulas from [10] (also [15]).

A comment on the CP phases is in order. The relevant Dirac CP violating phase(s) can be identified from the expression for the rephasing invariant $J_{CP}$: these are $\phi$ or $(\omega - \psi)$, depending on the relative magnitude of $\lambda_{12}$ and $\lambda_{13}$. However, within the approach we are employing, a Dirac CP violating phase appearing in $J_{CP}$ does not necessarily coincide with the Dirac phase in the standard parametrization of the PMNS matrix. For illustration it is sufficient to consider the simple case of $\lambda_{12} \neq 0$ and $\lambda_{13} = \lambda_{23} = 0$. Working to leading order in $\lambda_{12}$, it is easy to find that in this case the PMNS matrix can be written as

$$U \simeq \tilde{P} \begin{pmatrix}
\frac{c_{12}^\nu e^{-i\phi} + \lambda_{12} s_{12}^\nu}{\sqrt{2}} e^{i\theta_{12}} & \frac{1}{\sqrt{2}} \frac{\lambda_{13}}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \frac{-s_{12}^\nu}{\sqrt{2}} & \frac{c_{12}^\nu e^{-i\phi} - \lambda_{12} s_{12}^\nu}{\sqrt{2}}
\end{pmatrix} Q,$$

\[14\]
where \( \tilde{P} = \text{diag}(e^{i\phi}, e^{i\psi}, e^{i\omega}) \) and \( \tilde{Q} = \text{diag}(1, e^{i\sigma}, e^{i\tau}) \). The phase matrix \( \tilde{P} \) can be eliminated from \( U \) by a redefinition of the phases of the charged lepton fields. The Majorana phases \( \alpha \) and \( \beta' \equiv (\beta + \delta) \) can be directly identified (modulo \( 2\pi \)) with \( \sigma \) and \( \tau \). It is clear from the expressions (1) and (14) for \( U \), however, that the phase \( \phi \) does not coincide with the Dirac phase \( \delta \) of the standard parametrization of \( U \). Actually, the phase \( \phi \) could be directly identified with the Dirac CP violating phase of a different parametrization of the PMNS matrix, namely, the parametrization in which \( \tilde{U} \) in Eq. (14) is given by

\[
\tilde{U} = O_{12}(\theta_{12}) \text{diag}(e^{-i\delta'}, 1, 1) O_{23}(\theta_{23}) O_{12}(\theta'_{12})
\]

\[
= \left( \begin{array}{ccc}
\tilde{c}_{12} \tilde{c}_{12} e^{-i\delta'} - \tilde{c}_{23} s'_{12} \tilde{s}_{12} & \tilde{c}_{12} s'_{12} e^{-i\delta'} + \tilde{c}_{12} \tilde{c}_{23} \tilde{s}_{12} & \tilde{s}_{12} \tilde{s}_{23} \\
-\tilde{c}_{12} \tilde{c}_{23} s'_{12} - \tilde{c}_{12} \tilde{s}_{12} e^{-i\delta'} & \tilde{c}_{12} \tilde{c}_{23} \tilde{s}_{12} - s'_{12} \tilde{s}_{23} & \tilde{s}_{12} \tilde{s}_{23} \\
\tilde{s}_{12} s'_{23} & -\tilde{c}_{12} \tilde{s}_{23} & \tilde{c}_{23}
\end{array} \right).
\]

From this parametrization it would follow (using \( |U_{e3}| = \sin \theta_{23} \sin \theta_{12} \) and \( |U_{\mu3}/U_{\tau3}|^2 = \cos^2 \theta_{12} \tan^2 \theta_{23} \)) that \( \theta_{12} \) should be small and that atmospheric neutrino mixing was governed in leading order by \( \theta_{23} \). In the limit of \( \theta_{23} = \pm \pi/4 \) and \( \theta_{12} = 0 \) one would have \( |U_{e2}/U_{e1}|^2 = \tan^2 \theta'_{12} \). Hence, to leading order the solar neutrino mixing would be governed by \( \theta'_{12} \) and leptonic CP violation in neutrino oscillations would be described by 

\[ J_{\text{CP}} = -\frac{1}{8} \sin 2\theta'_{12} \sin 2\theta_{12} \sin 2\theta_{23} \sin \delta' \]. We would recover Eq. (14) from Eq. (15) if we identified \( \theta_{23} = -\pi/4 \), \( s_{12} = -\lambda_{12}, c_{12} = c'_{12}, s'_{12} = s_{12}, \) and \( \delta' = \phi \).

We are not going to use the parametrization (15) in the following. Instead, the three neutrino mixing angles \( \theta_{13}, \theta_{12} \) and \( \theta_{23} \) will be determined using the absolute values of the elements of the PMNS matrix, Eq. (9). Concerning the issue of CP violation in neutrino oscillations, we will work only with the CP violating rephasing invariant \( J_{\text{CP}} \). However, it is still useful to keep in mind that, as the example discussed above illustrates, in the approach we are following the resulting Dirac CP violating phase, which is the source of CP violation in neutrino oscillations, cannot always be directly identified with the Dirac CP violating phase of the standard parametrization (5) of the neutrino mixing matrix (3).

Returning to Eq. (13), we note that both \( |U_{e3}| \) and \( \sin^2 \theta_{23} \) do not depend on the mixing angle \( \theta'_{12} \). The quantities \( \lambda_{12} \) and \( \lambda_{13} \) are crucial for the magnitudes of \( |U_{e3}| \), \( \sin^2 \theta_{12} \) and \( J_{\text{CP}} \), whereas they enter into the expression for \( \sin^2 \theta_{23} \) only quadratically. In fact, \( \sin^2 \theta_{23} \) receives first order corrections only from \( \lambda_{23} \), which in turn contributes to the other observables only via terms proportional to \( \lambda^2_{23} \). Unless there are accidental cancellations, \( |U_{e3}| \) is lifted from its zero value due to non-zero \( \lambda_{12} \) and/or \( \lambda_{13} \). Atmospheric neutrino oscillations

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3 The same conclusion is valid, e.g., for the Dirac phase in the relation given in Eqs. (1) of the third and fourth articles quoted in Ref. (13).

4 The matrix \( V \) in the parametrization (5) of the PMNS matrix, and the matrix \( \tilde{U} \) in the parametrization (15) are connected by a unitary matrix: \( V = WV \). The latter reduces to the unit matrix (or to a diagonal phase matrix) only when the Dirac CP violating phases \( \delta \) and \( \delta' \), present in \( V \) and \( \tilde{U} \), take CP conserving values: \( \delta = k\pi, \delta' = k'\pi, k, k' = 0, 1, 2, \ldots \). In this case we can write \( V = \tilde{U} \) and can express the angles of \( V \) in terms of the angles of \( \tilde{U} \), and vice versa.
mixing can be maximal, or very close to maximal, for instance if $\omega - \phi = \pi/2$. Note that $\lambda_{12}$ and $\lambda_{13}$ in the expressions for $\sin^2 \theta_{12}$, $|U_{e3}|$ and $J_{CP}$ are multiplied by cosines and/or sines of the same phases $\phi$ and $(\omega - \psi)$, respectively. This means that if the terms proportional to $\lambda_{12}$ (to $\lambda_{13}$) dominate over the terms proportional to $\lambda_{13}$ (to $\lambda_{12}$) – we will refer to this possibility as $\lambda_{12}(\lambda_{13})$-dominance$^5$ – we have $\sin \theta_{12} = \sin \theta_{12}^\nu - 2 \theta_{12}^\nu |U_{e3}| \cos \gamma$.

\[
\sin^2 \theta_{12} = \sin^2 \theta_{12}^\nu - \sin 2 \theta_{12}^\nu |U_{e3}| \cos \gamma , \tag{16}
\]

where $\gamma = \phi$ or $(\psi - \omega)$ is the CP violating phase (combination) appearing in the expression for $J_{CP}$, $J_{CP} \propto \sin \gamma$. The relation (16) implies a correlation of the initial 12-mixing in $U_\nu$ with $|U_{e3}|$ and the observable CP violation in neutrino oscillations. If $\tilde{U}_\nu$ is a bimaximal mixing matrix, we have $\sin^2 \theta_{12}^\nu = 1/2$ and $\cos \gamma$ has to take a value close to one (while $|U_{e3}|$ has to be relatively large) in order to obtain sufficiently non-maximal solar neutrino mixing. Consequently, in the case of $\lambda_{12}(\lambda_{13})$-dominance, CP violation would be suppressed even though $|U_{e3}|$ can be sizable. On the other hand, if $\tilde{U}_\nu$ is a tri-bimaximal mixing matrix, we have $\sin^2 \theta_{12}^\nu = 1/3$ which already is in good agreement with the present data. Hence, $|U_{e3}| \cos \gamma$ has to be relatively small. Consequently, CP violation can be sizable if $|U_{e3}|$ has a value close to the existing upper limit. This interesting feature has first been noticed in Ref. [15]. Generally, in the case of $\lambda_{12}(\lambda_{13})$-dominance we get from Eq. (13):

\[
\sin^2 \theta_{12} = \sin^2 \theta_{12}^\nu - 4 J_{CP} \cot \gamma . \tag{17}
\]

where $\gamma = \phi$ ($\gamma = \psi - \omega$) for $\lambda_{12}$-dominance ($\lambda_{13}$-dominance). The following “sum-rule” holds as well:

\[
\sin^2 \theta_{12} = \sin^2 \theta_{12}^\nu \pm \sqrt{|U_{e3}|^2 \sin^2 2 \theta_{12}^\nu - 16 J_{CP}^2} , \tag{18}
\]

where the minus (plus) sign represents a positive (negative) cosine of the relevant Dirac CP violating phase. The sign ambiguity is unavoidable because the CP conserving quantity $\sin^2 \theta_{12}$ can only depend on the cosine of a CP violating phase, whereas any CP violating quantity like $J_{CP}$ can only depend on the sine of this phase. Knowing the cosine of a phase will never tell us the sign of the sine.$^6$ Note that since all parameters in Eq. (18) are rephasing invariant quantities, it can be applied to any parametrization of the PMNS matrix $U$ and of the matrix $\tilde{U}_\nu$. If $\tilde{U}_\nu$ is a bimaximal (tri-bimaximal) mixing matrix, we get

\[
\sin^2 \theta_{12} = \frac{1}{2} \pm \sqrt{|U_{e3}|^2 - 16 J_{CP}^2} \quad \text{and} \quad \sin^2 \theta_{12} = \frac{1}{3} \left( 1 \pm 2 \sqrt{2} \sqrt{|U_{e3}|^2 - 6 J_{CP}^2} \right) , \tag{19}
\]

respectively. The first relation has been obtained also in Ref. [18]. Obviously, one has to choose here the negative sign.

In Fig. [1] we show the allowed parameter space for the exact equations in the cases of $\sin^2 \theta_{12}^\nu = 1/2$ (bimaximal mixing), $1/3$ (tri-bimaximal mixing) and 0.2. We have chosen the

$^5$More concretely, the conditions for, e.g., $\lambda_{12}$-dominance are: $|\lambda_{12} \cos \phi| \gg |\lambda_{13} \cos(\omega - \psi)|$ and $|\lambda_{12} \sin \phi| \gg |\lambda_{13} \sin(\omega - \psi)|$.

$^6$The same ambiguity will show up if one identifies the phase $\phi$ with the phase $\delta$ of a given parametrization of the PMNS matrix, as done, e.g., in Ref. [13]. See also the comments given after Eq. (13).
Figure 1: Correlations resulting from $U = U_\ell^\dagger U_\nu$ if $U_\ell$ is CKM-like and $U_\nu$ has maximal $\theta_{23}^\nu$, vanishing $\theta_{13}^\nu$, but free $\theta_{12}^\nu$, for three representative values of $\theta_{12}^\nu$ (see text for details). The currently allowed 1σ and 3σ ranges of the observables are also indicated.
Figure 2: The sum-rule from Eq. (18).

\[ \lambda_{ij} \] to obey a CKM-like hierarchy: \( 0.1 \leq \lambda_{12} \leq 0.3, \ 0.02 \leq \lambda_{23} \leq 0.08 \) and \( 0 \leq \lambda_{13} \leq 0.01 \). As \( |U_{e3}| \) and \( \sin^2 \theta_{23} \) are independent of \( \theta_{12}' \) we have plotted these observables only once. The chosen ranges of the \( \lambda_{ij} \) lead from Eq. (13) to a lower limit of \( |U_{e3}| > \sim 0.09/\sqrt{2} \simeq 0.06 \), as is seen in the figure. Improved future limits on the range of \( \sin^2 \theta_{12} \) and, in particular, on the magnitude of \( |U_{e3}| \) can give us valuable information on the structure of \( U_e \). The allowed parameter space of \( \sin^2 \theta_{23} \) is roughly half of its allowed 3 \( \sigma \) range. The interplay of \( \theta_{12}' \) and leptonic CP violation in neutrino oscillations mentioned above results in the “falling donut” structure when \( J_{CP} \) is plotted against \( \sin^2 \theta_{12} \). We can also directly plot the sum-rule from Eq. (18), which is shown in Fig. 2. As a consequence of varying the observables in Eq. (18) we can extend the parameter space to smaller values of \( |U_{e3}| \). In fact, if \( U_\nu \) corresponds to tri-bimaximal mixing, \( U_{e3} \) is allowed to vanish. Equation (13) can be used to understand the results in Fig. 2 if, for instance, we have \( \sin^2 \theta_{12}' = 1/2 \), the experimental upper limit of \( (\sin^2 \theta_{12})_{\text{max}} = 0.4 \) implies that \( |U_{e3}| \geq 1/2 - (\sin^2 \theta_{12})_{\text{max}} \approx 0.1 \). On the other hand, for \( \sin^2 \theta_{12}' = 0.2 \), and therefore \( \sin 2\theta_{12}' = 0.8 \), we have with \( (\sin^2 \theta_{12})_{\text{min}} = 0.24 \) that \( |U_{e3}| \geq ((\sin^2 \theta_{12})_{\text{min}} - 0.20)/0.8 \approx 0.05 \), which is in agreement with the figure. A more stringent limit on, or a value of, \( |U_{e3}|^2 \leq 0.01 \) would strongly disfavor (or rule out) the simple bimaximal mixing scenario.

The equations given up to this point are also valid if neutrinos are Dirac particles. We will discuss now briefly the observables describing the CP violation associated with the
Majorana nature of the massive neutrinos. We find that in the case under discussion

\begin{equation}
S_1 \simeq -\frac{1}{\sqrt{2}} \cos \theta_{12}' \left( \lambda_{12} \sin(\phi + \tau) - \lambda_{13} \sin(\omega - \psi + \tau) \right), \\
S_2 \simeq \frac{1}{\sqrt{2}} \sin \theta_{12}' \left( \lambda_{12} \sin(\sigma - \phi - \tau) - \lambda_{13} \sin(\sigma - (\omega - \psi + \tau)) \right).
\end{equation}

(20)

According to the parameterization of Eq. (5), we have

\begin{align*}
S_1 & = -c_{12} c_{13} s_{13} \sin \beta \\
S_2 & = s_{12} c_{13} s_{13} \sin(\alpha - \beta).
\end{align*}

Hence, we find that in the case of \(\lambda_{12}\)-dominance, \(\beta\) is associated with \(\phi + \tau\), while if the terms proportional to \(\lambda_{13}\) dominate over the terms proportional to \(\lambda_{12}\), the phase \(\beta\) is associated with \(\psi - \omega - \tau\). In both cases \(\alpha\) is associated with \(\sigma\). Obviously, if \(\sigma = 0\) we get in the case of \(\lambda_{12}\)- or \(\lambda_{13}\)-dominance that

\begin{equation}
S_1 \simeq \frac{1}{\sqrt{2}} \cos \theta_{12}' \left( \lambda_{12} \sin(\phi + \tau) - \lambda_{13} \sin(\sigma - \phi - \tau) \right), \\
S_2 \simeq \frac{1}{\sqrt{2}} \sin \theta_{12}' \left( \lambda_{12} \sin(\sigma - \phi - \tau) - \lambda_{13} \sin(\sigma - (\omega - \psi + \tau)) \right).
\end{equation}

(21)

4 Maximal Atmospheric Mixing and \(\nu_3 = 0\) from the Charged Lepton Mass Matrix

Now we study the equally interesting possibility that maximal \(\theta_{23}\) and vanishing \(|U_{e3}|\) are realized in the limiting case, where \(U_{\nu}\) is equivalent to the unit matrix. In this scenario we have

\begin{equation}
U_{\nu}^\dagger = U_{\nu}^T = \begin{pmatrix}
c_{12} & s_{12} & 0 \\
\frac{-s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{s_{12}}{\sqrt{2}} & \frac{-c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\end{equation}

(23)

\footnote{Actually such an identification is always valid modulo \(2\pi\). For simplicity, we will omit stating this explicitly from here on.}
where we have to define \( U_c^T = O_{23}(\theta_{23}^\ell) U_{13}(\theta_{13}^\ell, \psi) O_{12}(\theta_{12}^\ell) \) in order to have the rotations in the correct order, cf. Eq. (23). Note that \( U_c \) is real and therefore \( m_\ell^\dagger m_\ell = U_c (m_\ell^\text{diag})^2 U_c^\dagger \) is symmetric. Reconstructing this matrix gives

\[
\begin{pmatrix}
    m_e^2 (c_{12}^\ell)^2 + \frac{1}{2} (s_{12}^\ell)^2 (m_\mu^2 + m_\tau^2) & c_{12}^\ell s_{12}^\ell (m_e^2 - \frac{1}{2} m_\mu^2 - \frac{1}{2} m_\tau^2) & \frac{1}{2} s_{12}^\ell (m_\mu^2 - m_\tau^2) \\
    \cdot & m_e^2 (s_{12}^\ell)^2 + \frac{1}{2} (c_{12}^\ell)^2 (m_\mu^2 + m_\tau^2) & \frac{1}{2} c_{12}^\ell (m_\mu^2 - m_\tau^2) \\
    \cdot & \cdot & \frac{1}{2} (m_\mu^2 + m_\tau^2)
\end{pmatrix}
\]

(24)

which does not obey a simple exchange symmetry as the neutrino mass matrix in Eq. (12). However, there are relations between the entries: for instance, if we additionally assume \( \theta_{12}^\ell = -\pi/4 \), we find

\[
m_\ell^\dagger m_\ell = \begin{pmatrix}
    A_\ell + D_\ell & A_\ell - D_\ell & B_\ell \\
    \cdot & A_\ell + D_\ell & B_\ell \\
    \cdot & \cdot & 2 A_\ell
\end{pmatrix},
\]

(25)

with \( A_\ell \equiv \frac{1}{4} (m_\mu^2 + m_\tau^2) \), \( B_\ell \equiv (m_\mu^2 - m_\tau^2) / \sqrt{8} \), \( D_\ell \equiv \frac{1}{2} m_e^2 \).

Discrete symmetries might be capable of generating such a texture. Another hint towards a possible origin of such a matrix can be obtained by noting that due to \( m_\tau^2 \gg m_\mu^2 \gg m_e^2 \) the entries are all of similar magnitude \([11]\), and therefore \( m_\ell^\dagger m_\ell \) resembles the mass matrices of the “flavor democratic” type.

We have to multiply \( U_c^\dagger = U_c^T \) from Eq. (23) with the matrix \( U_\nu \) to obtain the PMNS matrix. Let us first assume that \( U_\nu \) is given by the hermitian adjoint of Eq. (7): \( U_\nu^\dagger = P O_{23}(\theta_{23}^\nu) U_{13}(\theta_{13}^\nu, \xi) O_{12}(\theta_{12}^\nu) Q \). This will bring the 12-rotations of \( U_\ell \) and \( U_\nu \) directly together and, in absence of phases, would lead to \( \theta_{12} = \theta_{12}^\ell - \theta_{12}^\nu \), a feature which makes this possibility interesting for Quark-Lepton Complementarity scenarios \([16, 17, 18]\). For the neutrino oscillation observables we get

\[
\sin^2 \theta_{12} \simeq \sin^2 \theta_{12}^\ell - \lambda_{12} \sin 2 \theta_{12}^\ell \cos \sigma + \frac{1}{4} \left( \lambda_{13}^2 - \lambda_{23}^2 \right) \sin^2 2 \theta_{12}^\ell + \lambda_{12}^2 \cos 2 \theta_{12}^\ell ,
\]

\[
|U_{e3}| \simeq \left| \lambda_{23} \sin \theta_{12}^\ell + \lambda_{13} \cos \theta_{12}^\ell e^{i(\xi - \sigma)} \right| ,
\]

\[
\sin^2 \theta_{23} \simeq \frac{1}{2} + \lambda_{23} \cos \theta_{12}^\ell \cos(\xi - \sigma + \tau) - \lambda_{13} \sin \theta_{12}^\ell \cos \tau ,
\]

\[
J_{CP} \simeq -\frac{1}{4} \sin 2 \theta_{12}^\ell \left( \lambda_{23} \sin \theta_{12}^\ell \sin(\xi - \sigma + \tau) + \lambda_{13} \cos \theta_{12}^\ell \sin \tau \right).
\]

(26)

The parameter \( \lambda_{12} \) is crucial for obtaining a sufficiently non-maximal angle \( \theta_{12} \) in the case of a bimaximal \( U_\nu^\dagger \). However, \( \lambda_{12} \) appears only in terms proportional to \( \lambda_{12}^3 \) in \( |U_{e3}|, \sin^2 \theta_{23} \) and \( J_{CP} \). In these latter observables \( \lambda_{13} \) and \( \lambda_{23} \) are multiplied by the sines or cosines of the same phases. As a consequence, we can write down a correlation analogous to the one given in Eq. (17). Namely, if the terms proportional to \( \lambda_{23} \) dominate over the terms
proportional to $\lambda_{13}$ ("$\lambda_{23}$-dominance"), we have

$$\sin^2 \theta_{23} \simeq \frac{1}{2} - 2J_{\text{CP}} \frac{\cot(\xi - \sigma + \tau)}{\sin^2 \theta_{12}}.$$  \hspace{1cm} (27)$$

The analogue of the sum-rule in Eq. (18) is

$$\sin^2 \theta_{23} \simeq \frac{1}{2} \pm \frac{1}{\sin^2 \theta_{12}} \sqrt{|U_{e3}|^2 \cos^2 \theta_{12} \sin^2 \theta_{12} - 4J_{\text{CP}}^2},$$  \hspace{1cm} (28)$$

where the plus (minus) sign corresponds to $\cos(\xi - \sigma + \tau) > 0$ ($\cos(\xi - \sigma + \tau) < 0$). In this scenario the value of the atmospheric neutrino mixing angle is correlated with the magnitude of CP violation effects in neutrino oscillations. In the case of $\sin^2 \theta_{12} = 1/2$ or $1/3$ we find

$$\sin^2 \theta_{23} - \frac{1}{2} = \pm \sqrt{|U_{e3}|^2 - 16J_{\text{CP}}^2} \quad \text{or} \quad \sin^2 \theta_{23} - \frac{1}{2} = \pm \sqrt{2 \sqrt{|U_{e3}|^2 - 18J_{\text{CP}}^2}},$$  \hspace{1cm} (29)$$

respectively. The first relation has been obtained also in Ref. [18]. A high precision measurement of $\sin^2 \theta_{23}$, combined with a sufficiently stringent limit on, or a relatively small measured value of, $|U_{e3}|^2$ might allow to discriminate between the simple bimaximal and tri-bimaximal mixing scenarios we are considering.

The corresponding relations in the case of $\lambda_{13}$-dominance are

$$\sin^2 \theta_{23} \simeq \frac{1}{2} + 2J_{\text{CP}} \frac{\cot \tau}{\cos^2 \theta_{12}},$$  \hspace{1cm} (30)$$

and

$$\sin^2 \theta_{23} \simeq \frac{1}{2} \mp \frac{1}{\cos^2 \theta_{12}} \sqrt{|U_{e3}|^2 \cos^2 \theta_{12} \sin^2 \theta_{12} - 4J_{\text{CP}}^2},$$  \hspace{1cm} (31)$$

where the minus (plus) sign corresponds to $\cos \tau > 0$ ($\cos \tau < 0$). The results for $\sin^2 \theta_{12} = 1/2$ or $1/3$ can be easily obtained as

$$\sin^2 \theta_{23} - \frac{1}{2} = \mp \sqrt{|U_{e3}|^2 - 16J_{\text{CP}}^2} \quad \text{or} \quad \sin^2 \theta_{23} - \frac{1}{2} = \mp \frac{1}{\sqrt{2}} \sqrt{|U_{e3}|^2 - 18J_{\text{CP}}^2}.$$  \hspace{1cm} (32)$$

In Fig. 3 we show the allowed parameter space for the exact equations in the cases of $\sin^2 \theta_{12} = 1/2$ (bimaximal), $1/3$ (tri-bimaximal) and $0.2$. We have chosen again the $\lambda_{ij}$ to follow a CKM-like hierarchy with $0.1 \leq \lambda_{12} \leq 0.3$, $0.02 \leq \lambda_{23} \leq 0.08$ and $0 \leq \lambda_{13} \leq 0.01$. Note that – in contrast to the first scenario – $|U_{e3}|$ is much smaller and can even vanish exactly not only when $\sin^2 \theta_{12} = 1/3$, but also for $\sin^2 \theta_{12} = 1/2$ or $0.2$. Moreover, the range of the $\lambda_{ij}$ and the dependence of $\sin^2 \theta_{23}$ on them lead to the absence of a characteristic donut-like structure as seen in Fig. 1. For a CKM-like $U_{\nu}$, the importance of $\sin^2 \theta_{12}$ for $\sin^2 \theta_{23}$ and $|U_{e3}|$ is not as strong as it is the first scenario considered in Sec. 3. As mentioned above, the value of $\sin^2 \theta_{12}$ is important mainly for the required magnitude of $\lambda_{12}$ which is responsible only for subleading contributions to the other parameters. As in
Figure 3: Correlations resulting from $U = U_\nu^\dagger U_\nu$ if $U_\nu$ is CKM-like and $U_\ell^\dagger$ has maximal $\theta_{23}^\ell$ and vanishing $\theta_{13}^\ell$, but free $\theta_{12}^\ell$. The results shown correspond to three representative values of $\theta_{12}^\ell$. The currently allowed 1 $\sigma$ and 3 $\sigma$ ranges of the observables are also indicated.
the first scenario, atmospheric neutrino mixing can be maximal. If $|U_{e3}|$ will be observed to be close to its current limit, scenarios in which a CKM-like $U_{\nu}$ corrects $U_\ell$ corresponding to $|U_{e3}| = 0$ and $\theta_{23} = \pi/4$ will be ruled out.

The rephasing invariants associated with the Majorana CP violation are given by

$$S_1 \simeq - \cos \theta_{12}^\ell \left( \cos \theta_{12}^\ell \sin(\omega + \xi) \lambda_{13} + \sin \theta_{12}^\ell \sin(\omega + \sigma) \lambda_{23} \right),$$

$$S_2 \simeq - \sin \theta_{12}^\ell \left( \cos \theta_{12}^\ell \sin(\omega - \phi + \xi - \sigma) \lambda_{13} + \sin \theta_{12}^\ell \sin(\omega - \phi) \lambda_{23} \right).$$

(33)

In the case of $\lambda_{23}$-dominance ($\lambda_{13}$-dominance) we find that $\beta$ is associated with $\omega + \sigma$ ($\omega + \xi$). In both cases $\alpha$ is associated with $\phi + \sigma$.

Finally, we give the formulas for the case of a CKM-like $U_{\nu}$, i.e., $\lambda_{23} = A \lambda_{12}^2$ and $\lambda_{13} = B \lambda_{12}^3$ with $A$ and $B$ of order one:

$$\sin^2 \theta_{12} \simeq \sin^2 \theta_{12}^\ell - \cos \sigma \sin 2\theta_{12}^\ell \lambda_{12} + \cos 2\theta_{12}^\ell \lambda_{12}^2,$$

$$|U_{e3}| \simeq B \sin \theta_{12}^\ell \lambda_{12}^2,$$

$$\sin^2 \theta_{23} \simeq \frac{1}{2} + B \cos \theta_{12}^\ell \cos(\xi - \sigma + \tau) \lambda_{12}^2,$$

$$J_{CP} \simeq -\frac{1}{4} B \sin 2\theta_{12}^\ell \sin \theta_{12}^\ell \sin(\xi - \sigma + \tau) \lambda_{12}^2.$$  

(34)

We note that for an identical in magnitude correction, $|U_{e3}|$ is smaller by one order in $\lambda_{12}$, i.e., $|U_{e3}| \propto \lambda_{12}^2$ if the correction comes from $U_\nu$ in contrast to $|U_{e3}| \propto \lambda_{12}$ if the correction comes from $U_\ell$.

Consider next the case of $U_\nu$ (and not $\nu^\dagger$ as before) given by Eq. (7). For the neutrino oscillation observables we obtain

$$\sin^2 \theta_{12} \simeq \sin^2 \theta_{12}^\ell + \lambda_{12} \sin 2\theta_{12}^\ell \cos \phi + \frac{1}{4} (\lambda_{13}^2 - \lambda_{23}^2) \sin^2 2\theta_{12}^\ell + \lambda_{12}^2 \cos 2\theta_{12}^\ell,$$

$$|U_{e3}| \simeq |\lambda_{23} \sin \theta_{12}^\ell + \lambda_{13} \cos \theta_{12}^\ell e^{i(\phi + \xi)}|,$$

$$\sin^2 \theta_{23} \simeq \frac{1}{2} - \lambda_{12} \cos \theta_{12}^\ell \cos(\omega - \phi) + \lambda_{13} \sin \theta_{12}^\ell \cos(\omega + \xi),$$

$$J_{CP} \simeq -\frac{1}{4} \sin 2\theta_{12}^\ell \left( \lambda_{23} \sin \theta_{12}^\ell \sin(\omega - \phi) + \lambda_{13} \cos \theta_{12}^\ell \sin(\omega + \xi) \right).$$

(35)

The resulting formulas are very similar to those derived earlier: they can be obtained formally from Eq. (26) by simple changes of phases. Since in addition $\lambda_{13}$ and $\lambda_{23}$ in the expressions for $\sin^2 \theta_{23}$ and $J_{CP}$ in Eq. (35) are multiplied by the sines or cosines of the same phases, both the sum-rule corresponding to $\lambda_{23}$-dominance, Eq. (28), and the sum-rule associated with $\lambda_{13}$-dominance, Eq. (31), are valid in this case as well.

5 Summary

The results from various neutrino oscillation experiments indicate that $\theta_{23}$ is very close to $\pi/4$ and $\theta_{13}$ is very close to zero. It is natural to assume that at leading order these mixing
angles take the quoted extreme values and some form of perturbation leads to non-zero \( \theta_{13} \) and non-maximal \( \theta_{23} \). It is hoped that this perturbation is imprinted in correlations between various observables. Future precision experiments can tell us whether there are such correlations, which can then be used to identify the perturbation and to obtain thereby valuable hints on the flavor structure of the underlying theory. In this paper we have studied one interesting class of perturbations: because the observable lepton mixing matrix is a product of the diagonalization matrices of the charged lepton and neutrino mass matrices, \( U = U^\dagger \mathcal{M}_L U \), we assumed that in the limit of one of these matrices being the unit matrix, maximal \( \theta_{23} \) and zero \( \theta_{13} \) would result. When the second matrix deviates from being the unit matrix, i.e., has a CKM-like form, we investigated the effects on the CP conserving and CP violating observables. Free parameters are the small angles of the “correction matrix”, the 12-mixing angle of the leading matrix, and various phases. Scenarios like bimaximal mixing, tri-bimaximal mixing or Quark-Lepton Complementarity are special cases of our analysis. We consistently worked only with rephasing invariants in order to avoid the subtleties of identifying CP phases within different parameterizations. We should stress here also that our analysis is independent of the neutrino mass ordering and hierarchy.

In the first scenario we have considered, the neutrino sector alone is responsible for zero \( \theta_{13} \) and maximal \( \theta_{23} \). Requiring the neutrino mass matrix to obey a \( \mu-\tau \) symmetry can generate such a mixing pattern. Figures 1 and 2 illustrate the results. We find that \( |U_{e3}| \) will typically be non-zero, proportional to the sine of the largest angle in \( U_\ell \), and in most of the cases will be well within reach of up-coming experiments. If \( U_\nu \) is bimaximal, \( |U_{e3}| \) should satisfy \( |U_{e3}| \gtrsim 0.1 \) in order for \( \sin^2 \theta_{12} \) to be within the 3 \( \sigma \) interval allowed by the current data. There is no similar constraint on \( |U_{e3}| \) in the case of tri-bimaximal \( U_\nu \); even a vanishing value of \( |U_{e3}| \) is allowed. Atmospheric neutrino mixing can be maximal. There is a correlation between the solar neutrino mixing, the magnitude of \( |U_{e3}| \) and CP violation in neutrino oscillations, given by \( \sin^2 \theta_{12} = \sin^2 \theta_{12}' \pm \sqrt{|U_{e3}|^2 \sin^2 \theta_{12}' - 16 J_{\text{CP}}^2} \), where \( \theta_{12}' \) is the 12-rotation angle in \( U_\nu \). The magnitude of leptonic CP violation is rather sensitive to \( \theta_{12}' \). We have shown as well that in the approach we are following the resulting Dirac CP violating phase, which is the source of CP violation in neutrino oscillations, cannot always be directly identified with the Dirac CP violating phase of the standard PDG parametrization of the PMNS matrix. The identification of the Majorana CP violating phases is typically rather straightforward.

The alternative possibility corresponds to the charged lepton sector alone being responsible for zero \( \theta_{13} \) and maximal \( \theta_{23} \). We have identified the required texture of the charged lepton mass matrix in Eq. 24 and plot the observables in Fig. 3. Typically, \( |U_{e3}| \) is smaller than in the first scenario, being proportional to the sine of the second largest angle in \( U_\nu \). Another important difference with the first case is that now there exists a correlation between atmospheric neutrino mixing, the magnitude of \( |U_{e3}| \) and CP violation in neutrino oscillations: with \( \theta_{12}' \), being the 12-rotation angle in \( U_\ell \) we find that \( \sin^2 \theta_{23} \simeq \frac{1}{2} \pm \frac{\sqrt{|U_{e3}|^2 \sin^2 \theta_{12}' \cos^2 \theta_{12}' - 4 J_{\text{CP}}^2}}{\sin^2 \theta_{12}'} \), or \( \sin^2 \theta_{23} \simeq \frac{1}{2} \mp \frac{1}{\cos^2 \theta_{12}'} \sqrt{|U_{e3}|^2 \sin^2 \theta_{12}' \cos^2 \theta_{12}' - 4 J_{\text{CP}}^2} \), depending on whether the 23- or 13-rotation angle in \( U_\ell \) dominates.
We find that both scenarios are in agreement with the existing neutrino oscillation data, have interesting phenomenology and testable differences. Future higher precision determinations of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$, and more stringent constraints on, or a measurement of, $|U_{e3}|$ can provide crucial tests of these simplest scenarios, shedding more light on whether any of the two scenarios is realized in Nature.

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