Violation of the KSS Bound in Holographic Bjorken-Flow

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Abstract

In a system with a Bjorken-flow (BF), considering the inside of a BF as a subsystem for the reason explained in the body text, we compute entanglement entropy appearing in our side holographically. BF consists of many concentrically scattering quark-gluon plasmas created by colliding two masses of hadrons. Our BF is defined in the boundary space in an asymptotically five-dimensional Anti de-Sitter space with a growing Schwartzshild black hole.

In our analysis we can get a parameter prescribing our bulk space-time in terms of the boundary QCD’s variables. Using this, we find that the $\eta/s$ dips from $1/4\pi$ just for a moment before asymptoting to $1/4\pi$ at infinitely large time. Since $1/4\pi$ is the KSS bound, this result is intriguing as an example of violation of the KSS bound.
1 Introduction

Holographic principle is very important in terms of the degree of freedom of the gravity. Its concrete example is AdS/CFT correspondence (AdS/CFT) [1, 2, 3], which is very important in the understanding of the strongly coupled gauge theories and the open questions in the gravity. One of the key quantities in AdS/CFT is Ryu-Takayanagi formula (RT formula) [4, 5]. One of its important applications is the analysis in the time-dependent system.

We consider a Bjorken-Flow (BF) [6]; BF is a model for the time-evolution of scattering quark-gluon plasma (QGP) created by colliding two masses of hadrons, which is filled with the QGP behaving as a fluid at the late-time regime of the evolution (See Fig.1 in [7, 8]). It is known that its viscosity is least from observation, which means BF is extremely strongly coupled†.

Construction of the gravity dual for a BF based on AdS/CFT (hBF) is important in the context of the holographic QGP. The ground we can say we succeeded in this is we can obtain the stress tensors in the gravity which can agree with those of BF. [8, 10, 11, 12, 13, 14, 15, 16] are important papers in the context of the construction of hBF. However those have two problems: 1) Presence of an event horizon cannot be checked for the reason that the coordinate system in those is the Fefferrman-Graham in which the inside of event horizon is not covered (see Sec.2 in [17]), 2) presence of a logarithmic singularity. (for more detail, see [18]). However a hBF in which these two problem are fixed have been obtained in [17, 18].

We would have no way to know what is happening in the inside of BF except that it is very hot. Therefore we could regard the inside of BF as a thermal bass. We can regard a thermal bass as a subsystem in general, therefore we could regard the inside of BF as a subsystem, which we explain in Sec.4. If this is right, the hEE of hBF is interesting as a new example of hEE in the time-evolution process.

Therefore we in this paper analyze the hEE of hBF in the space-time of [17, 18] by regarding the inside of BF as a subsystem.

Although we evaluate with the RT formula, if the bulk space is time-dependent, the minimal area surface can be considered infinite ways. Therefore we comment that the minimal area we evaluate is right even in the covariant formalism of [27] in Sec.4.

Then, considering the stress tensors in the second-order hydrodynamics we can read off the viscosity $\eta$, from which we can compute the quantity $\eta/s$. Then, we can find that up to a parameter prescribing our space-time it can dip from the $\hbar/(4\pi k_B)$ (we denote it as 1/4$\pi$ in what follows) just for a moment before asymptoting to 1/4$\pi$ at infinitely large time. 1/4$\pi$ is the lowest bound of $\eta/s$ in the theoretical calculation in the ideal fluid, called KSS (Kovtun-Son-Starinets) bound‡.

In this paper, we can determine that parameter. As a result, we can show that $\eta/s$ in hBF violates the KSS bound as in Sec.6. This is intriguing as an example of†.

‡ Since [shear viscosity] $\sim$ [mass density] $\times$ [mean velocity] $\times$ [mean free path], small viscosities is caused of short mean free paths. This means the system is strongly coupled.

† As some related papers, we list [19, 20, 21, 22, 23, 24, 25, 26].
violation of the KSS bound.

We mention the organization of this paper. In Sec. 2, the space-time in this study is given. In Sec. 3, how hBF can correspond to BF is reviewed. In Sec. 4, the equation we compute to evaluate hEE is given. Further we comment that the minimal area we evaluate is right in the covariant formalism of [27]. In Sec. 5, the equation given in Sec. 4 is computed. In Sec. 6, the violation of the KSS bound is shown.

In Appendix A, we review that stress tensors in the gravity dual can generally satisfy the hydrodynamic equation and traceless condition on the boundary space. In Appendix B, the equation of hEE given in Sec. 4 is shown in the form that the arguments in the integrand are written explicitly.

2 Gravity dual to Bjorken-flow in this study

We in this section introduce the space-time we consider in this paper. It is a five-dimensional asymptotic AdS space with a Schwartzshild black hole described by Eddington-Finkelstein type coordinate (EF coordinate):

\[ ds^2 = \frac{1}{R^2} \left( -r^2 dt^2 + 2R^2 dt dr + e^{2(b-c)}(R^2 + r^2) dy^2 + r^2 e^c y^2 d^2 \Omega \right), \]

where \( R \) is the AdS radius. For this form with \( R \), see [28] for example. We can normalize the coordinates as \( \tau/R \) and \( r/R \). At this time \( y \) is not included in those. This is because \( y \) means the rapidity, which is dimensionless (but we interpret \( y \) as velocities as mentioned in the last of this subsection).

Since we in this paper consider a spherical coordinate system in the four-dimensional boundary space, \( d\Omega^2_2 \) is \( d\theta^2 + \sin^2 \theta d\phi^2 \), and the factor \( y^2 \tau^2 \) comes from the fact that in this EF coordinate, the radial direction in the four-dimensional boundary space is specified by \( y\tau \). This is because \( y \) and \( \tau \) respectively play the role of the scattering velocities and the proper time of QGP, as mentioned in the last of this subsection.

The \( a, b \) and \( c \) are given in the late-time expansion, which is technically an expansion around large \( \tau \) (with \( r \) treated as \( u\tau^{-1/3} \)), which corresponds to the late-time regime in the time-evolution of a Bjorken-Flow (BF) where QGP behave as fluid. \( a, b \) and \( c \) are concretely obtained to its sub-subleading order in [17] as

\[
\begin{align*}
a(u, \tau) &= a_0(u) + a_1(u)\tau^{-2/3} + a_2(u)\tau^{-4/3} + \mathcal{O}(\tau^{-8/3}) \\
b(u, \tau) &= b_0(u) + b_1(u)\tau^{-2/3} + b_2(u)\tau^{-4/3} + \mathcal{O}(\tau^{-8/3}) \\
c(u, \tau) &= c_0(u) + c_1(u)\tau^{-2/3} + c_2(u)\tau^{-4/3} + \mathcal{O}(\tau^{-8/3})
\end{align*}
\]

where

\[
u \equiv r\tau^{1/3},
\]

and

\[
a_0(u) = 1 - w^4 u^{-4}, \quad b_0(u) = c_0(u) = 0,
\]

\[\text{2}\] Gravity dual to Bjorken-flow in this study

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\end{align*}
\]

where

\[
u \equiv r\tau^{1/3},
\]

and

\[
a_0(u) = 1 - w^4 u^{-4}, \quad b_0(u) = c_0(u) = 0,
\]
\begin{align*}
a_1(u) &= -\frac{2}{3u^5} \{(1 + \xi_1)u^4 + \xi_1 w^4 - 3\xi_1 uw^4\} \\
b_1(u) &= -\frac{\xi_1}{u} \\
c_1(u) &= \frac{1}{3w} \left\{\arctan \frac{u}{w} - \frac{\pi}{2} + \frac{1}{2} \ln \left(\frac{u-w}{u+w}\right)\right\} - \frac{\xi_1}{2} \ln \left(1 - \frac{w^4}{u^4}\right) - \frac{2\xi_1}{3u}, \\
a_2(u) &= \frac{\xi_2^2}{9w^6} \left(u^4 - 3w^4\right) - \frac{4\xi_2}{9w^5} \left(u^3 - 3\xi_1 w^4\right) - \frac{2\xi_2}{3w^5} \left(u^4 + w^4\right) \\
&\quad - \frac{1}{18w^9} \left[4 \left(u^3 + 3\lambda uw^4\right) + 3\xi_1 w^4 \left\{3\xi_1 u \left(12 \ln u + 5\right) + 4\right\}\right] \\
&\quad + \frac{u^4 + w^4}{6w^5w} \left(9\xi_2^2 w^2 + 1\right) \arctan \left(\frac{u}{w}\right) + \frac{9\xi_2^2 w^4 + w^2}{6w^4} \left(\ln \left(u^2 + w^2\right)\right) \\
&\quad + \frac{9\xi_2^2 w^2 - 1}{12u^5w} \left\{(u^4 + 2uw^3 + w^4) \ln(u + w) - (u^4 - 2uw^3 + w^4) \ln(u - w)\right\}, \\
b_2(u) &= \frac{1}{2w^3} - \frac{\xi_2^2}{6w^2} - \frac{\xi_2}{w} + \frac{\xi_1}{4} \left(-24\xi_1 \ln u - \frac{4}{u} + \frac{\pi}{w}\right) \\
&\quad + \frac{9\xi_2^2 - 2\xi_1 u + w^2 + 1}{4uw} \arctan \left(\frac{u}{w}\right) + \frac{18\xi_2^2 + w^2}{12} \left(\ln \left(u^2 + w^2\right)\right) \\
&\quad + \frac{3\xi_1 w - 1}{24uw^2} \left\{3\xi_1 w(4u - 3w) + 2u - 3w\right\} \ln(u - w) \\
&\quad + \frac{3\xi_1 w + 1}{24uw^2} \left\{3\xi_1 w(4u + 3w) - 2u - 3w\right\} \ln(u + w), \\
c_2'(u) &= \frac{\xi_1}{9 \left(w^5 - uw^4\right)} \left\{6 \left(u^4 - 5u^4\right) \xi_1 u^4 + 4u^3 \left(u^4 + w^4\right)\right\} + \frac{2\xi_2^2}{9w^3} + \frac{2\xi_2}{3u^2} \\
&\quad + \frac{\xi_1 w^5}{3 \left(w^5 - uw^4\right)^2} \left\{12w\xi_1 u^5 - 6wu^4 + \pi \left(u^4 - w^4\right) u + 2w^5\right\} \\
&\quad + \frac{4\xi_1 u^2}{3 \left(w^5 - uw^4\right)} \ln u - \frac{3\xi_1 u^3 + w^2}{9w^5 - 9uw^4} \ln \left(u^2 + w^2\right) - \frac{\pi u^3 - 3w \left(4\xi_2 w^4 + u^2\right)}{9 \left(w^5 - uw^4\right) w} \\
&\quad \times \left\{(3w\xi_1 - 1) \left\{(u + w) \left(u^2 - 2wu + 3w^2\right) - 9w(u - w)(u^2 + w^2)\right\} \xi_1\right\} \ln(u - w) \\
&\quad - \left\{(3w\xi_1 + 1) \left\{(u - w) \left(u^2 + 2wu + 3w^2\right) + 9w(u + w)(u^2 + w^2)\right\} \xi_1\right\} \ln(u + w) \\
&\quad + \left\{\frac{w^4 + 3w^4 - 3w^2\xi_1 \left\{4uw^2 + 9 \left(u^4 - w^4\right)\right\} \xi_1}{18u^2 \left(w^4 - w^4\right)} \arctan \left(\frac{u}{w}\right)\right\}. \\
\end{align*}

According to [17, 18], \(\xi_{1,2}, \xi_{1,2}\) and \(w\) are integral constants. \(\xi_{1,2}\) can be absorbed arbitrarily by exploiting the gauge freedom of coordinate system. \(\xi_1\) and \(w\) can be related each other for the demand for no singularities in the space-time. \(\xi_2\) can be
determined for the demand for no singularities. The values assigned and determined for those in [17, 18] are
\[ \xi_1 = -1, \quad \zeta_1 = \frac{R^2}{3w}, \quad \text{and} \quad \zeta_2 = \frac{1 + 2 \ln 2}{18w^2}, \tag{13} \]
where \( \xi_2 \) is not specified in [17, 18]. \( R \) is involved as above if we write, which can be known from (26), (37) and the fact \([w] = -4/3\), where \([w] = -4/3\) (see (61)).

\( c_2(u) \) is given with a derivative of \( u \). We can integrate it however since its expression is so long, if we involve it into our computation our calculation becomes unfeasible (e.g., \( r_* \) in [53] becomes unobtainable.). For this reason, we perform our computation to the subleading order and disregard the sub-subleading order of \( a_2(u), b_2(u) \) and \( c_2(u) \).

The Hawking temperature of our black hole is given as [17]
\[ T_H = \frac{1}{\pi R^2} \frac{w}{\tau^{1/3}}, \tag{14} \]
where \( w \) is an integral constant in our geometry [2]-[4]. In [17, 18], \( w \) is not fixed concretely. In this study, we can relate \( w \) to the variables in the boundary QCD as in (61) from our analysis, which leads to the violation of the KSS bound in Sec 6. How we derive (14) is given in (32). Since \([w] = -4/3\) (see (61)), \( R \) appears as above.

The coordinate (1) asymptotes to AdS space when \( r \) in \( u \) in (5) grows as
\[ ds^2 \bigg|_{r \to \infty} = \lim_{r \to \infty} \{ r^2 \left( -d\tau^2 + \tau^2 dy^2 + y^2 \tau^2 d\Omega_2^2 \right) + 2d\tau dr + \cdots \} \]
\[ \sim -d\tau^2 + \tau^2 dy^2 + y^2 \tau^2 d\Omega_2^2. \tag{15} \]

- The four-dimensional part in (15) can be obtained from \( ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \) with the coordinate transformation: \((x^0, x^1) = (\tau \cosh y, \tau \sinh y)\), where \((dx^0)^2 + (dx^3)^2\) is changed to the polar coordinate, \( y^2 \tau^2 d\Omega_2^2 \). In Fig.1, we show the \((\tau, y)\)-frame in the \((x^0, x^1)\)-coordinate.

- Then we can see from Fig.1 that the diagonal lines with \( y = 0 \) and \( y = 1 \) correspond to the world-lines of static particles and a particles moving with the speed of light, respectively. From this we can identify the world-lines of the scattering QGP with the diagonal lines.

At this time, the ranges of \( \tau \) and \( y \) are \( 0 \leq \tau \) and \( 0 \leq y \leq 1 \), and \( \tau \) and \( y \) can correspond to the scattering QGP’s proper time and scattering velocities in the concentric fashion. Therefore, the locations of the scattering QGP in the radial direction can be specified as \( y \tau \).

- \((\tau, y)\) coordinate system is named the local rest frame (LRF).

### 3 How gravity dual corresponds to Bjorken-flow

In this section, we explain how the BF is described theoretically and what sense the holographic BF (hBF) in this paper corresponds to the QCD side.
Figure 1: We show the $(\tau, y)$-frame in the $(x^0, x^1)$-coordinate. The diagonal lines, one of which a dashed blue line hovers, are the $y$-constant lines, and the curved lines, one of which a dotted red line hovers, are the $\tau$-constant lines. The diagonal lines with $y = 0$ and $y = 1$ correspond to the world-lines of static particles and particles moving with the speed of light, respectively. From this, we can identify the world-lines of the scattering QGP with the diagonal lines. The ranges of $\tau$ and $y$ are $0 \leq \tau$ and $0 \leq y \leq 1$.

We can interpret $\tau$ as the scattering QGP’s proper time and $y$ as the scattering QGP’s velocity in the concentric fashion. Therefore the locations of the scattering QGP in the radial direction can be specified as $y\tau$.

When we construct the gravity dual of hBF, we perform the following arrangements to the dual geometry: 1) Since the coordinate system in QCD side is LRF, the coordinate system in the gravity side is taken so as to be LRF in its asymptotic region, 2) since the temperature of the BF in QCD side decreases as time lapses, the Hawking temperature is arranged to diminish as the proper time in LRF grows.

Here, the description of the BF in QCD side is given by the stress tensors satisfying the hydrodynamic equation in the late-time expansion in the LRF. If the conformal symmetry is supposed, the traceless condition is also imposed. hBF is also given by the stress tensors obtained in the gravity.

Therefore, once we have constituted the gravity dual, we obtain the stress tensors, which are given with the parameters remained to be fixed. For the arrangements above, the stress tensors in the gravity dual have been already equipped with the necessary conditions as the BF, and the rest work is to fix those parameters. In our case, we can fix the parameters as in [13] from the comparison of only $\tau\tau$-component of the stress tensors between the gravity dual and QCD side [18].

As such, in this section we first review the stress tensors in the second-order hydrodynamics of BF in the QCD side. Next, we review the constitution of the geometry equipped with the two arrangements above. Then we review that we can fix the pa-
rameters in our geometry from a comparison of the $\tau \tau$-components in the stress tensors between the gravity and the QCD sides. By this we can learn what sense hBF corresponds to the QCD side. This section is the review based on [17, 18, 28, 29].

If the stress tensors we have obtained are ones in well-defined theories, it is obvious that those satisfy the hydrodynamic equation. Furthermore, if there is conformal symmetry, it is obvious that those are traceless. However, if those are obtained in the bulk gravities, it would not be obvious whether those can satisfy those relations in the lower dimensional boundary theories. Therefore, we also would like to review the logic in [17] that the stress tensors in the gravity dual generally given in [9] can satisfy the hydrodynamic equation and the traceless condition in not the five-dimensional bulk theories but four-dimensional boundary theories. However, if we can show what sense hBF in this paper corresponds to the QCD side, it is be enough as the issue we address in this section. Therefore we review those in Appendix A.

3.1 Stress tensors of Bjorken-flow in QCD in second-order hydrodynamics with conformal symmetry

Since the form of the late-time regime in the BF is a fluid, the BF we treat in this study can be described by the hydrodynamics.

In general, the description of the hydrodynamics is given by the stress tensors for the flows of long-distance and low-frequency modes after the higher and faster modes are integrated out, which we denote $T^{\mu \nu}$. The equations to prescribe the stress tensors is $\nabla_{\mu} T^{\mu \nu} = 0$ (Hydrodynamic equation). The effects of the integrated higher and faster modes will appear in the coefficients in the expression of $T^{\mu \nu}$, which we call transport coefficients. As such we will focus on the stress tensors.

Then it is the general manner in the hydrodynamics to describe the stress tensors using the velocity fields $u_i(x_\mu)$ (velocities of the fluidities at each point in the LRF and $i$ refers to the spacial directions), energy density $\varepsilon(x_\mu)$ (note the difference of the notations in this paper between $\varepsilon$ (holographic UV cutoff) and $\varepsilon$ (energy density) as mentioned again in Sec.6) and the derivative expansion (it can become a good approximation when the variations in short-distance are small).

The stress tensors in $\nabla_{\mu} T^{\mu \nu} = 0$ up to the first-order of the derivative expansion is given as

$$T^{\mu \nu} = \varepsilon u^\mu u^\nu + T_{\perp}^{\mu \nu},$$

(16)

where

$$T_{\perp}^{\mu \nu} = P \Delta^{\mu \nu} - \eta \sigma^{\mu \nu} - \zeta \delta^{\mu \nu} \nabla u + O(\nabla^2),$$

$$\Delta^{\mu \nu} = g^{\mu \nu} + u^\mu u^\nu, \quad \sigma^{\mu \nu} = 2(\nabla^\mu u^\nu).$$

(17)

In the one above, $\mu, \nu = 0, \cdots, D - 1$. $P = P(\varepsilon)$, $\eta = \eta(\varepsilon)$ and $\zeta = \zeta(\varepsilon)$ are the transport coefficients which can be identified as the pressure, the shear and bulk viscosities, respectively. $T_{\perp}^{\mu \nu}$ satisfies the condition $u_\mu T_{\perp}^{\mu \nu} = 0$ (Transverse condition).

We now assume existence of the conformal invariance in the hydrodynamics. Conformal invariance leads to the traceless condition in the stress tensors. With these
traceless condition and the transverse condition, let us proceed to the second-order of the derivative expansion.

First of all, it is pointed in [31] that the following eight terms:

\[
\nabla^\mu \ln T \nabla^\nu \ln T, \quad \nabla^\mu \nabla^\nu \ln T, \quad \sigma^\mu{}^\nu \nabla u, \quad \sigma^{(\mu}{}^{\nu)\lambda}, \quad \sigma^{(\mu}{}^{\nu}{}^{\lambda}{}^{\omega}, \quad \Omega^{(\mu}{}^{\nu}{}^{\lambda}{}^{\omega}, \quad u_\alpha R^{\alpha}{}^{(\mu}{}^{\nu)}{}^{\beta} u_\beta \text{ and } R^{(\mu\nu)},
\]

(18)
can be the possible contributions under the traceless and the transverse conditions at the second-order, where \(T\) means temperature, which has the relation as \(\varepsilon = CT^D\) (\(\varepsilon\) and \(C\) are the energy density and some constant, respectively.) in the theories with conformal symmetry [31], and \(\Omega^{\mu\nu} \equiv 1/2 \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)\). From (18) we constitute the following quantities:

\[
\begin{align*}
\mathcal{O}_1^{\mu\nu} & = R^{\mu\nu} - (D - 2) \left( \nabla^{(\mu} \nabla^{\nu)} \ln T - \nabla^{(\mu} \ln T \nabla^{\nu)} \ln T \right), \\
\mathcal{O}_2^{\mu\nu} & = R^{\mu\nu} - (D - 2) u_\alpha R^{\alpha(\mu\nu)}{}^{\beta} u_\beta, \\
\mathcal{O}_3^{\mu\nu} & = \sigma^{(\mu}{}^{\nu)}{}^{\lambda}, \\
\mathcal{O}_4^{\mu\nu} & = \sigma^{(\mu}{}^{\nu}{}^{\lambda}{}^{\omega}, \quad \text{and} \quad \mathcal{O}_5^{\mu\nu} \equiv \sigma^{(\mu}{}^{\nu)}{}^{\lambda}{}^{\omega}.
\end{align*}
\]

(19)

However as for \(\mathcal{O}_1^{\mu\nu}\) it is pointed in [31] that we should use the following one denoted \(\mathcal{O}'_1^{\mu\nu}\) instead of \(\mathcal{O}_1^{\mu\nu}\):

\[
\mathcal{O}'_1^{\mu\nu} \equiv \langle D\sigma^{\mu\nu} \rangle + \frac{\sigma^{\mu\nu} \nabla u}{D-1}.
\]

(20)

We can obtain the stress tensors to the second-order derivative expansion under the conformal invariance and the transverse condition from some combination of the terms in (19), which we can write down as

\[
T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} + P \Delta^{\mu\nu} + \Pi^{\mu\nu} + \mathcal{O}(\nabla^3),
\]

(21)

where \(\Pi^{\mu\nu}\) is the part usually referred to as the dissipative part given as

\[
\Pi^{\mu\nu} = - \eta \sigma^{\mu\nu} + \eta \tau_1 \mathcal{O}'_1^{\mu\nu} + \kappa \mathcal{O}_2^{\mu\nu} + \sum_{i=1}^{3} \lambda_i \mathcal{O}_i^{\mu\nu}.
\]

(22)

In the one above, \(\tau_1, \kappa \text{ and } \lambda_{1,2,3}\) are the transport coefficients appearing in the second-order derivative expansion. We can interpret \(\eta\) and \(\tau_1\) as the shear viscosity and relaxation time but physical meanings of the rest four, \(\kappa\) and \(\lambda_{1,2,3}\), are unclear as those are not specified in [31].

According to [31], the \(\kappa\) term vanishes in flat space and the \(\lambda_{1,2,3}\) terms are effective when the target physics is nonlinear. Therefore, if we look at small perturbations, only linear orders would be effective, therefore we could ignore \(\lambda_{1,2,3}\). However since the solutions of BF is known to be non-linear, we would need to involve \(\lambda_{1,2,3}\). However it is known that \(\lambda_{2,3}\) are not needed for irrotational flows. Finally we need to involve only \(\lambda_1\) if we look at BF.
Now let us consider the LRF given in (15) in order to consider the situation of BF specifically. The velocity fields \( u^\alpha \) in the LRF are given as
\[
 u^\alpha = (u^\tau, u^y, \vec{u}^\perp) = (1, 0, 0, 0), \tag{23}
\]
where \( \perp \) means the angular velocities in the \( S^2 \) part denoted as \( \Omega_2 \) in (15).

In the LRF, \( \eta, \tau_\Pi \) and \( \lambda_1 \) in the late-time regime of the BF with the conformal invariance can be determined in [31] as
\[
 \eta = \varepsilon_0 \eta_0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^{3/4}, \quad \tau_\Pi = \tau_0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^{-1/4}, \quad \lambda_1 = \varepsilon_0 \lambda_1^0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^{1/2}, \tag{24}
\]
where
\[
 \frac{\varepsilon}{\varepsilon_0} = \tau^{-4/3} - 2\eta_0 \tau^{-2} + \varepsilon_0^{(2)} \tau^{-8/3} + \ldots, \tag{25}
\]
\[
 \varepsilon_0^{(2)} = \frac{9\eta_0^2 + 4\chi}{6}, \quad \chi = \lambda_0^1 - \eta_0 \tau_0^0. \tag{26}
\]
In the one above, \( \varepsilon_0, \eta_0 \) and \( \tau_0^0 \) are integral constants, and the abbreviated part represents higher order terms in the late-time expansion.

Here let us turn to the dimensions of \( \varepsilon_0, \eta_0 \) and \( \tau_0^0 \). We count the dimensions in the mass dimension in the natural unit. Then based on the fact that \( [\varepsilon] = 4, [\tau] = -1, [\eta] = 3 \) and \( [\tau_\Pi] = -1 \), we can obtain as \( [\varepsilon_0] = 8/3, [\eta_0] = [\tau_0^0] = [\lambda_1^0]/2 = -23/3 \).

Also, the \( \tau\tau \)-components in the stress tensors can be finally obtained as
\[
 T_{\tau\tau} = \varepsilon_0 \left( \tau^{-4/3} - 2\eta_0 \tau^{-2} + \varepsilon_0^{(2)} \tau^{-8/3} + \ldots \right), \tag{27}
\]
where other components \( T_{yy} \) and \( T_{\perp\perp} \) (\( \perp = \theta, \phi \) in the \( S^2 \) direction) can be expressed in terms of \( T_{\tau\tau} \) using the traceless condition and the Hydrodynamic equation [18].

The three transport coefficients in (24) can be evaluated if it comes to the \( \mathcal{N} = 4 \) SYM by using AdS/CFT correspondence as [28]
\[
 \eta = \frac{\pi}{8} N_c^2 T^3 \left( 1 + \frac{135 \zeta(3)}{8} \lambda^{-3/2} + \ldots \right), \tag{28}
\]
\[
 \tau_\Pi = \frac{2 - \ln 2}{2\pi T} + \frac{375 \zeta(3)}{32\pi T} \lambda^{-3/2} + \ldots, \tag{29}
\]
\[
 \lambda_1 = \frac{N_c^2 T^2}{16} \left( 1 + \frac{175 \zeta(3)}{4} \lambda^{-3/2} + \ldots \right). \tag{30}
\]

### 3.2 Stress tensors in gravity dual, and how gravity links to QCD and holographic Bjorken-flow holds

In the previous subsection, we have given the \( \tau\tau \)-components in the stress tensors of the BF in QCD side as in (27). In this section, we first review how to constitute the geometry in such a way that it can be the gravity dual for the BF. Then we see that it will reach (1) with (2)-(4). At this stage, several unfixed parameters remain as mentioned under (13). We show the \( \tau\tau \)-component of the stress tensors in this geometry, which
will be given with those unfixed parameters. Then, we review that we can match that \(\tau\tau\)-component with the QCD side completely by taking those parameters properly.

By this, the unfixed parameters are fixed, and the geometry is fixed. Therefore, the dual geometry for BF in QCD can be fixed. By this, we can learn what sense hBF in this paper corresponds to the QCD side.

We start with a problem in the coordinate system in the dual gravities. In the various studies of hBF, either of GF or EF coordinate is employed. However GF coordinate does not cover the inside of the event horizon [17]. Therefore, it is impossible to confirm the presence of the event horizon in the GF coordinate. On the other hand, EF coordinate covers the trapped and untrapped regions. Therefore, it is possible to confirm the existence of apparent horizon which guarantees the presence of the event horizon in the EF coordinate. In this sense, EF coordinate is appropriate rather than GF coordinate. Finally the following prototype of the EF coordinate is considered:

\[
ds^2 = -r^2 \left\{ 1 - \left( \frac{r_0}{r} \right)^4 \right\} dt_+^2 + 2dt_+dr + r^2d\vec{x}^2,
\]

where \(t_+\) is some time-like coordinate and \(r\) is the coordinate for the five-dimensional radial direction.

Then we can know that the Hawking temperature in (31) is given as \(T_H = r_0/\pi\). Here, the temperature of BF in QCD is known to have a dependence on the proper time as \(1/\tau^{1/3}\). Analogically considering, we are led to replace \(r_0\) in \(T_H\) with \(w\tau^{-1/3}\), where \(w\) is some constant. If we write this deduction in a rough manner,

\[
T_H = r_0/\pi, \quad T_{QCD} \propto 1/\tau^{1/3} \quad \implies \quad r_0 = w/\tau^{1/3} \quad \implies \quad T_H = w/(\pi \tau^{1/3}).
\]

In addition, corresponding to the fact that BF in QCD is described in the LRF, we arrange the boundary part in (31) to the LRF. Those arrangements lead (31) to

\[
ds^2 = -r^2 \left\{ 1 - \left( \frac{w}{r^{1/3}} \right)^4 \right\} d\tau^2 + 2d\tau dr + r^2(\tau^2dy^2 + dx^2_\perp),
\]

where \(\tau\) is the time-like coordinate which will play the role of the proper time in the LRF in the four-dimensional boundary space.

Now we can see that (33) does not become a pure AdS geometry even when \(w\) goes to zero. We will improve this point. Finally, we reach the following one:

\[
ds^2 = -r^2 \left\{ 1 - \left( \frac{w}{r^{1/3}} \right)^4 \right\} d\tau^2 + 2d\tau dr + r^2 \left( 1 + \frac{1}{r^{1/3}} \right)^2 dy^2 + dx^2_\perp.
\]

We have reached (34) by the arrangements and improvements by hand. Therefore it does not satisfy the Einstein equation as it is. Therefore let us consider to arrange it such that it can be a solution.

We first remember that the BF we consider is given in the late-time regime. Therefore we address the problem by performing an expansion with the late-time expansion. At this time we should fix \(u \equiv r\tau^{1/3}\) [17]. Therefore let us rewrite (34) with \(u\) as

\[
(34) \implies -r^2 \left( 1 - \frac{w^4}{u^4} \right) d\tau^2 + 2d\tau dr + r^2 \left( 1 + \frac{1}{u^{2/3}} \right)^2 dy^2 + dx^2_\perp.
\]
Based on (35) we propose (1) with (2)-(4) as the form we aim to get. We obtain $a$, $b$ and $c$ in the range of the late-time expansion order-by-order by solving the equations of motion: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 6g_{\mu\nu} = 0$. The results are (6)-(12).

Once we have obtained an AdS space as a solution, we move to the stage to obtain the stress tensors. If we restrict those in the $r$-constant surface, it will work as the stress tensors in the four-dimensional field theories which have the bulk gravity as its gravity dual. In [18], its $\tau\tau$-component in our gravity duals (1) is shown explicitly as

$$T_{\tau\tau} = \frac{3N^2_\tau w^4}{8\pi^2 R^8} \left( \tau^{4/3} - 2\zeta_1 \tau^{-2} + \zeta_2 \tau^{-8/3} + \cdots \right),$$

(36)

where $G_5 = G_{10}/(\pi^3 R^5) = \pi R^3/(2N^2_c)$ in AdS$_5$/CFT$_4$. We have written $R$ explicitly.

Let us compare (36) and (27). Since we have equipped the arrangements (32) and (33), the stress tensors in the gravity side have the same structure with the QCD side in the boundary space, and those can agree completely if we take in the following way:

$$\frac{3N^2_\tau}{8\pi^2 R^8} w^4 = \varepsilon_0, \quad \zeta_1 = \eta_0 \quad \text{and} \quad \zeta_2 = \varepsilon_0^{(2)}.$$

(37)

By (37) all the parameters remained to be fixed in our geometry (1) are fixed.

Lastly note just the two points: 1) By putting $R$ as in (36), the dimensions in both sides of (37) can agree if $[w] = -4/3$ (see (61)). 2) We can determine $\chi$ in (26) as $\chi = (-1 + \ln 2)/(6w^2)$ from (37) with (13) and (26).

4 Our holographic entanglement entropy

In this section, we first define the equation for the hEE we evaluate based on RT formula. However, in the case the bulk space depends on time, the minimal surface can be considered infinite ways. Therefore, we show that the equation we should evaluate can result in that based on the covariant formulation of hEE [27].

4.1 Equation we evaluate

The BF we will consider is an expanding three dimensional ball composed of the layers of QGP scattering in a concentric fashion with the velocities $y$. We can regard the inside of the BF as a subsystem for us; This is because it is very hot, so we would have no way to know the inside of the BF, therefore we could regard BF as a thermal-bath, and the inside of thermal-bath can be generally considered as a subsystem. Therefore, we can consider the EE in the case that the inside of the BF is a subsystem. We further explain this in what follows.

Based on the fact that we have no way to know the inside of the BF except that it is a very hot, we presume that we can write the whole density matrix as

$$\rho_{\text{whole}} = e^{-\beta_0 H_0} \cdot e^{-\beta_{BF} F_{BF}} = \rho_0 e^{-\beta_{BF} F_{BF}}.$$

(38)

In the one above, $\rho_0 \equiv e^{-\beta_0 H_0}$, which is the density matrix for the space except for the BF’s space, where $\beta_0$ is inverse of some normal temperature around us. Next, $\beta_{BF}$
is the inverse of some BF’s very high temperature, and $F_{\text{BF}}$ is the free energy for the BF’s space given as

$$e^{-\beta_{\text{BF}}F_{\text{BF}}} = \text{tr}_{\text{BF}} \rho_{\text{BF}}.$$  \hspace{2cm} (39) 

$\rho_{\text{BF}}$ is some density matrix for the BF’s space, and $\text{tr}_{\text{BF}}$ is the trace acting on the BF’s part. Then, we can write $\rho_{\text{whole}}$ as

$$\rho_{\text{whole}} = \text{tr}_{\text{BF}} (\rho_0 \rho_{\text{BF}}).$$  \hspace{2cm} (40) 

Using this we can consider the EE written as $S_{\text{EE}} = -\text{tr} (\rho_{\text{whole}} \ln \rho_{\text{whole}})$. This EE is the one defined by regarding the inside of a BF as a subsystem.

We evaluate the EE above holographically (hEE) based on the Rye-Ta kayanagi (RT) formula. To this purpose we consider an area of the surface overhanging in the five-dimensional space as

$$S_{\text{EE}} = \frac{1}{4G_5} \int_{\gamma} \left( \sqrt{g_{\tau'\tau'} (d\tau')^2 + g_{\tau'\tau} d\tau' dr} \cdot \int_{S^2} d\theta d\phi \sin \theta g_{\Omega_2\Omega_2} \right) \bigg|_{y=1}$$

$$= \frac{\pi \tau_0^2}{G_5} \int_{\tau_0 - 1/\delta}^{0} d\tau' \ r^2 e^{\sqrt{r^2a - \dot{r}}}$$

$$= \frac{\pi \tau_0^2}{G_5} \int_{1/\epsilon}^{r_*} dr \r^2 e^{\sqrt{r^2a - \dot{r}}}.$$  \hspace{2cm} (41) (42) 

where $\dot{r} \equiv dr/d\tau'$. In the one above, we have omitted the descriptions concerning the arguments in the integrands and so on. We give the description with those explicitly in Appendix.B Then let us look at each line in the one above.

In the first line,  

- $g_{\tau'\tau'}$, $g_{\tau'\tau}$ and $g_{\Omega_2\Omega_2}$ can be read off from the metric (1). We have put an “absolute value” in the square-root in order to avoid that its content becomes negative when $\dot{r} = 0$ at the turning point of the area.

- We regard the inside of the BF as a subspace for the reason mentioned above, and the most external layer of the BF is expanding with the velocity $y = 1$.

- $\gamma$ means the surface to be integrated, which overhangs from the surface on the boundary of the subsystem in the four-dimensional boundary space to the five-dimensional bulk space.

- $\int_{S^2} d\theta d\phi \sin \theta g_{\Omega_2\Omega_2}$ is independent of the $\gamma$-integral, which gives $4\pi g_{\Omega_2\Omega_2}$.

In the second line,  

- $1/\delta$ is the parameter for the holographic UV cutoff, $1/\epsilon$.

- The $\tau_0 - 1/\delta$ is the lower limit of the $\tau'$-integral, which corresponds to the position of the boundary space in the bulk space, which we represent as $1/\epsilon$ in the third line.
– The “0” is the upper limit of the \( \tau' \)-integral, which corresponds to the turning point of the surface, which we represent as \( r_* \) in the third line.

- All the \( r \) in (41), including the \( r \) in the arguments of the \( a \) and \( c \), are the functions of the \( \tau' \).

- All the \( \tau \) in the \( g_{\Omega_2 \Omega_2} \) in the \( S^2 \)-integral is \( \tau_0 \). Therefore, “\( r^2 c^\prime \)” is given like the third line in Appendix (note the arguments). On the other hand, all the \( \tau \) appearing from the non-trivial part of the area-computation given by a square root are \( \tau' \). Here,
  - \( \tau' \) is the integral variable, which plays the role of the parameter to specify the \( r \)-coordinate of the surface via \( r(\tau') \).
  - \( \tau_0 \) is the proper time at when we will measure EE, which is distinguished from \( \tau' \), and behaves as a constant in the \( \tau' \)-integral.

In the third line,

- The \( \tau' \)-integral in the second line is changed to the \( r \)-integral. Corresponding to this, the fundamental variable in the third line is \( r \). Therefore all the \( \tau' \) are treated as some functions of \( r \) as \( \tau' = \tau'(r) \). As a result,
  - \( a \) and \( \dot{r} \) are some functions of \( \tau' \) in the second line. Therefore, these are some functions of \( r \) as \( a = a(r) \) and \( \dot{r} = \dot{r}(r) \) in the third line. We obtain these concrete expression in the next section.
  - Since \( c \) is independent of \( \tau' \) as \( c = c(r, \tau_0) \), we can use the expression of \( c \) in (9) as it is.

In our computation we use the information at \( \tau' = 0 \) despite the fact that our space-time is given as the late-time expansion. We comment on this in the summary.

In this paper, the AdS radius \( R \) is taken to 1, but if we write it explicitly, we can see that our \( S_{\text{EE}} \) is proportional to \( R^3/G_5 \). Therefore, using the relation of \( G_5 \) given under (36), we can confirm that our \( S_{\text{EE}} \) is proportional to \( N_c^2 \) as well as other \( h\text{EE} \).

### 4.2 Comment in terms of the covariant formulation

When the bulk geometry is time-independent, we can consider the one and only time-constant slice in the bulk space. Consequently the minimum area can be fixed and we can use RT formula. On the other hand if the bulk geometry is time-dependent, we can consider the time-constant slice in infinite ways.

However, the algorithm to prescribe the minimal area in the time-dependent geometries is provided in [27] in the covariant manner. It is based on various geometric knowledge of the space-times and the Bousso bound [32, 33, 34], which we cannot state here in brief. For the details, we would like to refer the reader to those references.

Since we evaluate the minimal area in the frame (11) despite it is time-dependent, we in this subsection comment the area we should evaluate can be given by that based on [27].
The area we evaluate is \((\text{41})\), which is given by a product of the \((\tau, r)\) and the boundary parts. Since the \((\tau, r)\) and the boundary parts represent the contributions from the bulk and boundary spaces respectively, the \((\tau, r)\) part is the problem.

The \((\tau, r)\) part is determined from \(ds^2 = \frac{1}{R^2}(-r^2 d\tau^2 + 2R^2 d\tau dr)\) in \((\text{1})\), which we can represent as

\[
ds^2 = -f(r, \tau) d\tau^2 + 2 d\tau dr, \quad \text{where} \quad f(r, \tau) = r^2 a. \tag{43}
\]

This is nothing but (6.3) in \([27]\) with \(r^2 dx^2\) removed, \(v\) replaced with \(\tau\) and \(f(r, v)\) replaced with the one in \((\text{43})\).

In \([27]\), starting with (6.3), it is shown that the minimal area (6.9) in the frame (6.3) can be the minimal area in the covariant formalism based on the fact written under (6.16). Their discussion can be applied to our case just with the replacements mentioned under \((\text{43})\) (In \([27]\) \(f\) is treated generally, and we should just ignore the differentiated quantities regarding \(x\) because of no \(r^2 dx^2\) in \((\text{43})\)). Therefore we can consider \((\text{41})\) as the minimal area in the covariant formalism.

5 Computation of our holographic entangle entropy

In the previous chapter, we have given the equation we will compute to evaluate \(h_{\text{EE}}\). We evaluate it actually in this section.

5.1 \(a\) and \(\dot{r}\) as functions of \(r\) such that area-computation becomes feasible

We evaluate \(S_{\text{EE}}\) based on \((\text{12})\) with \(\dot{r}\) and \(a\) corresponding to the most minimal surface configuration. We obtain such \(\dot{r}\) and \(a\) from \((\text{11})\) in the manner explained below.

In order to fix \(\dot{r}\) and \(a\) to the most minimal surface configuration, we utilize the general fact that Hamiltonian is constant when configuration is on an extremal.

To be concrete, regarding the integrand in \((\text{11})\) as a Lagrangian density (“density” is abbreviated in what follows), we obtain a Hamiltonian from Legendre transformation. Then supposing it as a constant, we obtain the expressions of \(\dot{r}\) and \(a\).

In general, when Hamiltonian is a constant the system is on the extreme, therefore our system is on an extreme. We suppose it is the most minimal, therefore the \(\dot{r}\) and \(a\) correspond to the most minimal surface configuration. We evaluate \(S_{\text{EE}}\) in \((\text{12})\) with those \(\dot{r}\) and \(a\).

The parameters describing the surface are \(\tau'\) or \(r\). \(S_{\text{EE}}\) in \((\text{12})\) is given by an \(r\)-integral, and \(\dot{r}\) and \(a\) are the quantities determined by the surface configuration. Therefore we obtain \(\dot{r}\) and \(a\) as functions of \(r\).

We describe the procedure mentioned above more specifically below:

- Regarding the integrand in \((\text{11})\) as a Lagrangian, we obtain a Hamiltonian via Legendre transformation. Here, \((r, \dot{r})\) play the role of the conjugate variables and \(\tau'\) plays the role of the time parameter.
Supposing that the Hamiltonian is a constant, we denote it as $H_0$. Then we write as $H_0 = H_0(a, c, r, \dot{r})$, where l.h.s. is the constant, and r.h.s. is the Hamiltonian obtained above.

Solving the equation $H_0 = H_0(a, c, r, \dot{r})$ in terms of $a$, we can obtain $a$ as $a = a(c, r, \dot{r}, H_0)$. However it is insufficient because we need to obtain $\dot{r}$ and $a$ separately.

Substituting $a = a(c, r, \dot{r}, H_0)$ into the $a$ in the first Lagrangian $L(a, c, r, \dot{r})$ we obtain the Hamiltonian as $H_0 = H_0(c, r, \dot{r}, H_0)$ again, then solving it in terms of $\dot{r}$, we can obtain $\dot{r}$ as $\dot{r} = \dot{r}(c, r, H_0)$.

Substituting these $\dot{r}$ and $a$ into the integrand in (41) we can obtain the integrand in (42) written by $r$ and other variables without $\tau'$.

Let us perform what we have described above in what follows, concretely.

To begin with, we regard the integrand in (41) as a Lagrangian:

$$L \equiv e^c r^2 \sqrt{ar^2 - \dot{r}}.$$  (44)

Then, we can obtain the corresponding Hamiltonian as

$$H = \dot{r} \frac{\partial L}{\partial \dot{r}} - L = -\frac{r^2 e^c (2ar^2 - \dot{r})}{2\sqrt{ar^2 - \dot{r}}}.$$  (45)

Now, assuming that $H$ is a constant, let us formally denote it as $H_0$, we write (45) as

$$H_0 = \text{r.h.s. of (45)}.$$  (46)

Solving this with regard to $a$ we can obtain an expression of $a$ as

$$a = \frac{1}{2r^6} \left\{ r^4 \dot{r} + e^{-2c} H_0 \left( H_0 \pm \sqrt{(H_0)^2 - 2e^{2c} r^4 \dot{r}} \right) \right\}.$$  (47)

The system is on an extreme if the Hamiltonian is constant. Therefore this $a$ corresponds to an extremal surface configuration. We now suppose that it is the most minimal, which means this $a$ corresponds to the most minimal surface configuration.

As for the option of “±” in (47), we can see later that there is no changes whichever we choose in the solution of $\dot{r}$ at (49). However, it turns out later that $H_0$ becomes zero trivially in the evaluation at (54) if we choose “+”. We therefore perform our computation in what follows with “−”.

We substitute $a$ in (47) into the $a$ in the r.h.s. of (44). Then we can obtain the Lagrangian again. Toward this Lagrangian we perform Legendre transformation again. As a result we obtain $H_0$ again as

$$H_0 = \frac{1}{2^{3/2}} \left( \frac{H_0}{\sqrt{(H_0)^2 - 2e^{2c} r^4 \dot{r}}} - 1 \right) \sqrt{-e^{2c} r^4 \dot{r} + H_0 \left( H_0 - \sqrt{(H_0)^2 - 2e^{2c} r^4 \dot{r}} \right)}.$$  (48)
Solving the one above with regard to $\dot{r}$, we can obtain the three expressions of $\dot{r}$ written without $\tau'$. One of them is just zero, and the rest two are

$$
\dot{r} = \frac{2(-4 \pm 3\sqrt{2}) (\mathcal{H}_0)^2}{e^{2c} r^4}.
$$

(49)

Since the surface overhangs into the side where the black hole exists, $r$ goes in the opposite direction of the $r$-axis as $\tau'$ decreases. According to this $\dot{r}$ should be positive, therefore we should choose the one with “+”.

We here explain how to parametrize the surface, which is also the explanation about the integral range in the calculation of the area of the surface.

The surface is parametraized by either of $r$ or $\tau'$. We take the range of those as

$$
\begin{array}{ccc}
\tau' & \epsilon^{-1} \rightarrow & r_* \\
\tau & \tau_0 \rightarrow & 0
\end{array}
$$

We explain the correspondence relation between $(\epsilon^{-1}, r_*) \leftrightarrow (\tau_0, 0)$. $\epsilon^{-1}$ and $r_*$ are the locations of the boundary space and the turning point of the overhanging surface, respectively. By considering the situation that we project $\epsilon^{-1}$ and $r_*$ onto the boundary space, $\epsilon^{-1}$ and $r_*$ correspond to the outermost of the BF on the boundary space, and the center of the BF respectively. Therefore, if we parametrize the surface by $\tau'$, it would be natural to take the $\tau'$ corresponding to $r = r_*$ as 0 and to take the $\tau'$ corresponding to $r = 1/\epsilon$ as $\tau_0$.

The point at $\tau' = 0$ can be interpreted as the moment that the BF comes into existence and $\tau_0$ can be interpreted as the lapse time from that. since the $\tau$ parameterizing the surface has a physically different meaning from the $\tau$ in the space-time coordinate, we denote it as $\tau'$ and distinguish it from the space-time coordinate $\tau$.

Substituting (49) into the $a$ in (47), we can obtain $a$ written without $\dot{r}$ as

$$
a = \frac{2(-1 + \sqrt{2}) (\mathcal{H}_0)^2}{e^{2c} r^6}.
$$

(50)

If we had chosen “-” in (49), the “+” in the front of $\sqrt{2}$ would change to “-” and this $a$ would become negative, which is inconsistent with the fact that $a|_{u \to \infty} \to 1$ (our space-time is AdS at the asymptotic region as in [15]).

Using (49) and (50) we can write down $S_{EE}$ in (42) with $r$ and $\tau$ and without $d\tau$ as

$$
S_{EE} = \frac{\pi \tau_0^2 (1 + \sqrt{2})}{2 G_5 \mathcal{H}_0} \int_1^{r_*} dr e^{2c} r^4.
$$

(51)

Before evaluating (51), let us evaluate $r_*$ and $\mathcal{H}_0$ in the next subsection.
5.2 $r_*$ and $\mathcal{H}_0$

We first obtain $r_*$ in terms of $\tau_0$. Since $r_*$ corresponds to the turning point of the surface, the following equation can be held:

$$\dot{r} \bigg|_{r=r_*} = 0.$$  \hspace{1cm} (52)

This equation cannot be solved analytically in terms of $r$ if we treat $\dot{r}$ in (49) as it is. However since we are employing the late-time expansion, we expand with regard to $\tau_0$ to the $\tau_0^{-2}$ order. If we expand further, solving in terms of $r$ becomes unable.

Solving such an expanded (52) with regard to $r$, we can obtain

$$r_* = \frac{w^{3/4}}{3^{1/3}} \sqrt{\frac{R}{\tau_0}}.$$  \hspace{1cm} (53)

Since the dimension of $w$ turns out to be $-4/3$ in (61), $R$ would appear as above.

We evaluate $\mathcal{H}_0$. It can be evaluated with (45) at the turning point, namely $\dot{r} = 0$ at $r = r_*$. As a result, we can write $\mathcal{H}_0$ as

$$\mathcal{H}_0 = -\sqrt{a} \ e^{c} \ r_*^{3} \quad \text{with} \quad a = a(r, \tau') \bigg|_{r=r_*, \tau'=0} \quad \text{and} \quad c = c(r, \tau_0) \bigg|_{r=r_*}.$$  \hspace{1cm} (54)

This Hamiltonian is negative, however it is not problem. This is because our Hamiltonian has been introduced formally to solve the problem of the minimal configuration.

It may appear that we have performed some unreasonable analysis because we use $r_*$ despite that our space-time is given by the late-time expansion, where $r(\tau') \big|_{\tau'=0} = r_*$ ($r_*$ is the value corresponding to $\tau' = 0$). However, this is not problems because $r_*$ is just referring a point on the well-defined area. However $a(r, \tau') \big|_{r=r_*, \tau'=0}$ is a problem (referring $\tau'$ at 0 is problem), which we comment in the next section.

5.3 $r_*$ and $w$ in terms of variables in QCD

Now we can determine $w$ in (13). From $a$ in (50) and $\mathcal{H}_0$ in (54) we can obtain

$$a(r, \tau') = 2(\sqrt{2} - 1) a(r_*, 0) \frac{e^{2c(r, \tau_0)} r_*^{6}}{e^{2c(r, \tau_0)} r_*^{6}}.$$  \hspace{1cm} (55)

Since our space-time is the asymptotic AdS as in (15), the relations:

$$a \big|_{u \to \infty} \to 1 \quad \text{and} \quad c \big|_{u \to \infty} \to 0 \quad \text{for} \quad r \to \epsilon^{-1} \quad \text{with} \quad \epsilon \to 0,$$  \hspace{1cm} (56)

hold. Therefore, r.h.s. of (55) = 1 hold in the neighborhood of the boundary space at $r = \epsilon^{-1}$. From this we can obtain

$$r_* = \frac{\epsilon^{-1}}{2(\sqrt{2} - 1) a(r_*, 0) e^{2c(r, \tau_0)} r_*^{6}}.$$  \hspace{1cm} (57)

\[ \text{Another relation obtained as } dz/dx = f(z, x) \Rightarrow \int dx = \int f^{-1}(z, x), \text{ is employed to obtain the turning point in [4].} \]
Equating this with $r_*$ in (53), the values of $w$ depending on the location of the boundary space $\epsilon^{-1}$ can be determined as

$$w = \left( \frac{27}{4} \left( 3 + 2\sqrt{2} \right) \frac{\tau_0^6}{(a(r_*, 0) e^{2c(r_*, \tau_0)})^2} \right)^{1/9} \epsilon^{-4/3}. \quad (58)$$

Note that $\epsilon^{-1}$ is not a variable but a parameter for the location of the boundary space in the radial direction. Therefore the dependence on $\epsilon^{-1}$ does not mean the $r$-dependence, and the fact that our space-time is a solution is not changed.

$a(r_*, 0)$ in (57) and (58) is the quantity at $\tau' = 0$, which is the moment that two masses of QGP’s collide and a BF comes into existence. In the dual gravity side, it is the moment that a black hole comparable with the plank scale comes into existence in an empty AdS space. At that time, the terms truncated in (2)-(4) for the late-time expansion become necessary, and the late-time expansion cannot work. However, since the asymptotic region would be classical and asymptotes to an AdS space even at such $\tau' = 0$, we suppose that the region around $r_*$ is included in such an asymptotic region. Based on this consideration, we suppose we can treat $a(r_*, 0)$ as

$$a(r_*, 0) = 1. \quad (59)$$

What we will do with (57) and (58) is to get $r_*$ and $w$. Since $r_*$ and $w$ are included in those through $e^{2c(r_*, \tau_0)}$ ($c$ is given in (4)), we have to solve those simultaneously. However, it turns out that we cannot solve those with regard to $r_*$ and $w$ even if we simplify those by the late-time expansion to the first order $\tau_0^{-2/3}$ as

$$r_* = \frac{\epsilon^{-1}}{(2(\sqrt{2} - 1))^{1/6}} \left( 1 - \frac{R^2 c_1}{3} \tau_0^{-2/3} + \mathcal{O}(\tau_0^{-4/3}) \right), \quad (60)$$

$$w = \left( \frac{27}{4} \left( 3 + 2\sqrt{2} \right) \epsilon^{-12} \tau_0^6 \right)^{1/9} \left( 1 - \frac{2R^2 c_1}{9} \tau_0^{-2/3} + \mathcal{O}(\tau_0^{-4/3}) \right), \quad (61)$$

($r_*$ and $w$ are involved through $c_1$ ($c_1$ is given in (9)), where the ones above can be obtained by expanding (57) and (58) with (59). Therefore we consider the leading part only in the r.h.s.; Namely the $r_*$ and $w$ we will consider are the r.h.s. of (60) and (61) to the constant part and we disregard the corrections under the $\tau_0^{-2/3}$ order.

Let us turn to the dimension of $w$. Since the expansion part in (61) is dimensionless, we can find $[w] = -4/3$ counting by the mass dimension in the natural unit. Further, we can see $[c_1 \tau_0^{-2/3}] = 2$, where $c_1$ is given in (2). Therefore, we can determine how $R$ is involved in (60) and (61).

Finally we comment that it seems impossible to determine the dimension of $[w]$ only from the information in [17, 18]. Since we have succeeded in obtaining a relation (61), we have been able to determine the dimension of $w$. As a result, we can know how $R$ is involved in our analysis. To make clear how $R$ is involved would be important in terms of the dimensions.
5.4 Evaluation for our holographic entanglement entropy

Let us evaluate our EE based on (51), where $c(r, \tau_0)$ is given in (41) with $c_0$ and $c_1$ up to subleading order (13) for the reason written under (13), $\dot{r}$ and $a$ are given in (49) and (50), $\mathcal{H}_0$ are given in (54), and $w$ and $r_s$ are given by the ones up to the constant part of (60) and (61) for the reason written under (61).

Since it is difficult to perform the integral as it is, considering that our space-time is given by the late-time expansion to the subleading order, we expand the integrand by the late-time expansion to the subleading order. Its result is given as

$$S_{EE} = \frac{\pi \tau_0^2 (1 + \sqrt{2})}{G_5 \mathcal{H}_1} \int_{1/\epsilon}^{r_s} dr \left[ \frac{r^4 \epsilon^3}{2} + \frac{2r^3 \epsilon^3}{3 \tau_0} \right.$$

$$- \frac{1}{36w_1} \left( 6 \pi + 24 \ln \left( w_1/r \right) - 32 \log \epsilon + 8 \log \tau_0 \right) \frac{r^4 \epsilon^{13/3}}{\tau_0^{4/3}} + \frac{4r^2 \epsilon^3}{9\tau_0^2} + \cdots \left. \right]$$

$$= \frac{\pi \tau_0^2 (1 + \sqrt{2})}{G_5 \mathcal{H}_1} \int_{1/\epsilon}^{r_s} dr \left[ \frac{r^4 \epsilon^3}{2} + \frac{2u_0^2 \epsilon^3}{3 \tau_0} \right.$$

$$+ \left\{ - \frac{1}{36w_1} \left( 6 \pi + 24 \ln \left( w_1^{1/3}/u_0 \right) - 32 \log \epsilon + 8 \log \tau_0 \right) u_0^4 \epsilon^{13/3} + \frac{4u_0^2 \epsilon^3}{9} \right\} \frac{1}{\tau_0^{8/3}}$$

$$+ \mathcal{O} \left( \tau_0^{-10/3} \right) \right],$$

where $u_0 \equiv \epsilon^{-1} r_0^{1/3}$, and we have written as $\mathcal{H}_0 = - \mathcal{H}_1 \epsilon^3$, $w_0 = w_1 \epsilon^{-4/3} r_0^{2/3}$.

In the first line, two expansions around $r = \infty$ and $\epsilon = 0$ are performed. (We can confirm the result does not change if we change the order of the expansions.) In the second line, the integrand has been rewritten into the form of the late-time expansion with rewriting to $u$ (see (4)).

The terms in the first line are the all that will be effective in the late-time expansion to the subleading, and the second line are the one that the first line is summarized with rewriting to $u$, which is in the form of the late-time expansion to the subleading order. Therefore it is enough to perform the $r$-integral to the terms in the first line. Writing the result in the expansion of $\epsilon^{-1}$,

$$S_{EE} = \frac{\pi \tau_0^2 (1 + \sqrt{2})}{G_5 \mathcal{H}_1} \left[ \frac{1}{10 \epsilon^2} \left( \frac{1}{(2 \sqrt{2} - 1)^{5/6}} - 1 \right) + \frac{1}{6 \tau_0 \epsilon} \left( \frac{1}{(2 \sqrt{2} - 1)^{2/3}} - 1 \right) \right.$$

$$- \left( 3 \log \left( \frac{w_0^8}{r_0^6} \right) + 8 \log(\tau) - 32 \log \epsilon + 6 \pi \right) \frac{1}{180w_1 \tau_0^{4/3} \epsilon^{2/3}} \left( \frac{1}{(2 \sqrt{2} - 1)^{5/6}} - 1 \right)$$

$$+ \frac{4}{27 \tau_0^2} \left( \frac{1}{2 \sqrt{2} - 1} - 1 \right) + \mathcal{O} \left( \epsilon \right) \right].$$

(62)

The terms in the one above are $\sim \epsilon^{-2}$, $\sim \epsilon^{-1}$, $\sim \epsilon^{-2/3}$ and $\sim 1$, which are diverged at $\epsilon \to 0$ except for $\sim 1$. However the most hardest order is $\sim (-2)$, which is generally
known in holographic EE\(^1\), and those can be removed by some counter terms. As a result the constant part survives and our final result is given as

\[
S_{\text{EE}} = \frac{4\pi(1 + \sqrt{2})}{27G_5H_1} \left( \frac{1}{\sqrt{2} (\sqrt{2} - 1)} - 1 \right). \tag{64}
\]

The result is independent of \(\tau_0\) (and positive). This may appear against anticipation that our EE would depend on \(\tau_0\) since the BF expands as \(\tau_0\) proceeds. Note that our BH is growing\(^\parallel\), which has an effect to make the area for our EE small. Therefore our result can be interpreted in the way that the effects of black hole’s growing and the BF’s expanding cancel each other.

### 6 Violation of KSS bound

The transport coefficients in the second-order hydrodynamics with conformal symmetry can be generally written as in \([24]\). We write those again:

\[
\eta = \varepsilon_0 \eta_0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^{3/4}, \quad \tau_\Pi = \tau_\Pi^0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^{-1/4}. \tag{65}
\]

with

\[
\frac{\varepsilon}{\varepsilon_0} = \tau_0^{-4/3} - 2\eta_\Pi \tau_0^{-2} + \varepsilon_0^{(2)} \tau_0^{-8/3} + \cdots, \tag{66}
\]

\[
\varepsilon_0^{(2)} = \frac{9\eta_0^2 + 4(\lambda_1^0 - \eta_0 \tau_0^0)}{6}. \tag{67}
\]

The transport coefficients in the BF in the four-dimensional boundary space are also given by these. Note the difference in our notations between \(\varepsilon^{-1}\) (holographic UV cutoff) and \(\varepsilon\) (energy density). We have presented \((66)\) using \(\tau_0\) because the time here is considered as the lapse time measured in the proper time from the moment that BF gets into existence, where \(\tau_0\) is defined under \((49)\). The dimensions of the parameters presented above are given under \((26)\).

We can write \(\varepsilon_0\), \(\tau_\Pi^0\) and \(\varepsilon_0^{(2)}\) in terms of the gravity dual as

\[
\varepsilon_0 = \frac{3N_c^2 w^4}{8\pi^2 R^8}, \quad \tau_\Pi^0 = \frac{2 - \ln 2 R^2}{2 R w}, \quad \varepsilon_0^{(2)} = \frac{1 + 2 \ln 2 R^4}{18 w^2}. \tag{68}
\]

We can obtain \(\varepsilon_0\) from \((37)\). We can obtain \(\tau_\Pi^0\) from the fact that it can be determined as \((2 - \ln 2)/(2\pi\tau_0^{1/3}T_H)\) from the fact that the leading of \((65)\) should agree with that of \((29)\), then using \(T_H\) in \((1)\). We can obtain \(\varepsilon_0^{(2)}\) from \((26)\) with \(\chi\) at the last of Sec 3.2

\(^\dagger\) It is known that the most hardest leading contribution in hEE will be \(S_{\text{EE}} \sim \frac{R^n}{4G_{n+2}} \cdot \frac{1}{\varepsilon^{n-1}}\) for AdS\(_{D=n+2}\) with the AdS radius \(R\)\(^\parallel\).

\(^\parallel\) Since our Hawking temperature diminishes with time, our black hole grows with time. However we have no idea about its source at present.
Indeed there is one more transport coefficient $\lambda_1$ as in (24), however since its physical meaning is unclear as mentioned under (22) we will not consider it. Since $\eta$ is the quantity concerning the universal quantity $\eta/s$ and in this study we can obtain an intriguing result in $\eta/s$, we focus on only $\eta$ in what follows.

$\eta_0$ in (65) and $w$ are can be linked each other by the relation (13) with (37): $\eta_0 \leftrightarrow \zeta_1 \leftrightarrow w$. $\epsilon_0$ and $w$ are linked each other by the relation (37). $w$ and $\epsilon^{-1}$, $\tau_0$ can be linked as in (61): $w \leftrightarrow (\epsilon^{-1}, \tau_0)$. Therefore we can represent $\eta$ in (65) in terms of $\epsilon^{-1}$ or $\tau_0$. Also from the relation (14) it is possible to represent $\eta$ in (65) in terms of the Hawking temperature.

As for the relation (13), (14) and (61) we referred above, (13) and (14) are given in [17, 18], but (61) is newly obtained in our study by putting conditions (46), (52) and (56) to compute $h_{EE}$: (46) is the condition for referring to the most minimal surface configuration, (52) is the condition to refer to the location of the turning point of the surface, and (56) is the condition for our space-time to asymptote to AdS. Therefore it would be interesting to check how $\eta$ in (65) will be determined by the (61).

First, let us expand $\eta$ in (65) around large $\tau_0$ to the second order by substituting (66). Then,

$$\eta = \frac{\epsilon_0 \eta_0}{\tau_0} \left( 1 - \frac{3\eta_0}{2} \tau_0^{-2/3} + \frac{3}{4} \left( \frac{\epsilon_0^{(2)}}{2} - \frac{\eta_0^2}{2} \right) \tau_0^{-4/3} + \cdots \right). \quad (69)$$

Using $T_H$ in (14) and $\varepsilon_0$ in (37), we can write $s = \pi^2 N_c^2 T^3 / 2$ and $\varepsilon_0 \eta_0$ as $N_c^2 w^3 / (2\pi) \cdot \tau_0^{-1}$ and $N_c^2 w^3 / (8\pi^2)$, respectively. Therefore, with (13) (and (37)), we can write $\eta/s$ as

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - \frac{1}{2} \left( \frac{1}{w \tau_0^{2/3}} \right)^2 + \frac{\ln 2}{12} \left( \frac{1}{w \tau_0^{2/3}} \right)^2 + \cdots \right). \quad (70)$$

Since $w = w(\epsilon^{-1}, \tau_0)$ as in (61), if we specify $\epsilon^{-1}$ and $\tau_0$ we can quantify $w$. Then if $w$ is positive, we can see that $\eta/s$ in the BF dips from $1/4\pi$ just for a moment before asymptotes to $1/4\pi$ at infinitely large time. We can see from (61) that $w$ is positive. Therefore, we can predict that $\eta/s$ in BF will behave to dip from $1/4\pi$ just for a moment before asymptotes to $1/4\pi$ at infinitely large time, which is our result.

The $1/4\pi$ is called the KSS bound: It is generally considered as the lowest bound for the values $\eta/s$ can take in the ideal fluid in the theoretical calculation. Therefore our result is theoretically intriguing.

In addition, although the gauge theory in this study is $N = 4$ large-$N_c$ strongly coupled SYM, since the results in hydrodynamics are considered to be independent of each theory, our result is also intriguing as a target observation in the experiments.

Note that not just $1/4\pi$ but some values at the same order with $1/4\pi$ are observed in the experiments. Therefore some values less than $1/4\pi$ might be observed. However, we could observe the characteristic behavior: once dipping, asymptote to some constants around $1/4\pi$, if our result is right. However, since the actual BF undergoes the QCD phase transition since time lapsing it cools down, we may not be able to check the characteristic behavior we predict appears or not.
7 Summary

We have computed hEE in the hBF. Since the BF is expanding as time its result is expected to grow as time grows, however the result has been time-independent. We can understand this point by the way that the effect of growing of hEE arisen from the volume factor is canceled with the effect of growing effect of the black hole.

As the point in this study, in the analysis of the hEE we have been able to link a parameter in the space-time solution to the variables in QCD. Using that we have been able to write the $\eta/s$ in terms of the variables in QCD. Then we have found that it dips from $1/4\pi$ just for a moment before asymptoting to $1/4\pi$ at infinitely large time. This result is intriguing as an example of violation of the KSS bound.

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A On what the stress tensors in the dual gravities satisfy not five- but four-dimensional hydrodynamic equation and the traceless condition

In Sec 3 we have reviewed how our geometry can link to the BF in QCD; The central quantity has been the stress tensors. In this Appendix, we review a logic given in [17] based on the formalism given in [30] that the stress tensors in the gravity dual given generally in [9] can satisfy the hydrodynamic equation and the traceless condition in not the five-dimensional bulk theories but four-dimensional boundary theories. Actually, if what we have obtained are stress tensors in well-defined theories, it is obvious that those satisfy the hydrodynamic equation. Furthermore, if there is conformal symmetry, those would be traceless. However it is not trivial whether those can satisfy those in the lower dimensional boundary theories.

When a space-time asymptoting to AdS space is given, what the stress tensors we can obtain in the asymptotic region has been investigated in [9]. Then assuming that we have obtained the stress tensors, if we look at the indeces of those except that the indeces take the five-dimensional direction (in short exclude the case $\mu = r$), we can regard those as the stress tensors in some four-dimensional field theory on the four-dimensional $r$-constant boundary space.

The stress tensor in the AdS$_5$ given in [9] is

$$T_{\mu \nu} = \frac{N^2}{4\pi^2} r^2 \left. T_{\mu \nu} \right|_{r \to \infty},$$

(71)
where
\[
T_{\mu\nu} = K_{\mu\nu} - K\gamma_{\mu\nu} - 3\gamma_{\mu\nu} + \frac{1}{2}G_{\mu\nu},
\]
(72)
\[
K_{\mu\nu} = -\frac{1}{2}(\nabla_{\mu}n_{\nu} - \nabla_{\nu}n_{\mu}), \quad \gamma_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}.
\]
(73)

Here, \(\gamma_{\mu\nu}\) \((\mu, \nu = 0, \cdots, 4)\) is an induced metric on the four-dimensional \(r\)-constant boundary space [30], and \(K_{\mu\nu}\) is the extrinsic curvature associated with the four-dimensional boundary space. \(\vec{n}\) is the normal vector for the \(r\)-constant boundary space. It is normalized as \(g_{\mu\nu}n^\mu n^\nu = 1\). To be specific, \(n^\mu = (0, 1/\sqrt{g_{rr}}, 0, 0, 0)\) \((n^2\) indicates the \(r\)-component). \(G_{\mu\nu} = R_{\mu\nu} - \gamma_{\mu\nu}/2\cdot R\), where \(R_{\mu\nu}\) is constituted of \(\gamma_{\mu\nu}\).

We here perform a coordinate transformation: \(\tau \rightarrow \tau + 1/r\) to move to the system with the diagonalized metric. Then \(ds^2 = (15) \rightarrow -r^2d\tau^2 + r^2(\tau^2dy^2 + dx_+^2) + dr^2/r^2\), and \(n^\mu = (0, r, 0, 0, 0)\).

In this subsection we review that
- (71) is traceless as the four-dimensional quantity: \(\sum_{\mu=0}^3 T^{\mu\mu} = 0\).
- (71) is conserved as the four-dimensional quantity: \(\sum_{\mu=0}^3 \nabla_\mu T^{\mu\nu} = 0\), where \(\nabla_\nu\) is the covariant derivative constituted of the four-dimensional \(\gamma_{\mu\nu}\).

For this purpose, let us rewrite \(T_{\mu\nu}\) well in what follows.

First let us focus on \(G_{\mu\nu}\) in (72). When the Einstein equation is given as \(R_{\mu\nu} - 1/2Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 0\) \((\Lambda = 6\) in the present case), according to [30], we can write as
\[
G_{\mu\nu} = \frac{2\Lambda}{3}\left\{g_{\rho\sigma}\gamma^{\rho\sigma} + \left(g_{\rho\sigma}n^\rho n^\sigma - \frac{1}{4}g^{\rho\sigma}\right)\gamma_{\mu\nu}\right\} + KK_{\mu\nu} - K^\mu_\sigma K_{\nu\sigma}
\]
\[- \frac{1}{2}\gamma_{\mu\nu} \left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) - C^\alpha_{\mu\nu\beta}n^\alpha n^\beta,
\]
(74)
where \(C^\alpha_{\mu\nu\beta}\) is the Weyl tensor constituted of the five-dimensional bulk space metric, \(g_{\mu\nu}\). If it comes to AdS5, calculating the part \(\{\cdots\}\) in (74) using the relation obtained from (73): \(g_{\mu\nu} = \gamma_{\mu\nu} + n_{\mu}n_{\nu}\),
\[
G_{\mu\nu} = 3\gamma_{\mu\nu} + KK_{\mu\nu} - K^\mu_\sigma K_{\nu\sigma} - \frac{1}{2}\gamma_{\mu\nu} \left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) - C^\alpha_{\mu\nu\beta}n^\alpha n^\beta
\]
(75)
at the asymptotic region. Then let us look at (71) with (73). Considering a quantity, \(\tilde{K}_{\mu\nu} \equiv K_{\mu\nu} + \gamma_{\mu\nu}\), we can rewrite \(T_{\mu\nu}\) in (71) as
\[
2T_{\mu\nu} = \tilde{K}_{\mu\nu}\tilde{K} - \tilde{K}_{\mu\sigma}K^\sigma_{\nu} + \frac{1}{2}\left(\tilde{K}_{\alpha\beta}\tilde{K}^{\alpha\beta} - \tilde{K}^2\right)\gamma_{\mu\nu} - C_{\mu\nu\beta}n^\alpha n^\beta,
\]
(76)
where the point in newly considering \(\tilde{K}_{\mu\nu}\) is that it can vanish at \(r \rightarrow \infty\). Therefore,
\[
2T_{\mu\nu}\big|_{r\rightarrow\infty} = -C_{\mu\nu\beta}n^\alpha n^\beta.
\]
(77)
As a result of (77), we can see readily that \( \sum_{\mu=0}^{4} T_{\mu}^{\mu} \bigg|_{r \to \infty} = 0 \) as the nature of Weyl tensors. Here, \( C_{\alpha \mu \beta \gamma} n^\alpha n^\beta \) vanish trivially when \( \mu = r \). Therefore, we can see

\[
\sum_{\mu=0}^{3} T_{\mu}^{\mu} \bigg|_{r \to \infty} = 0.
\]

(78)

Here we have written the range of the summation explicitly because at present it is crucial to make a distinction if it is the trace in four-dimensional boundary space or not. By (78) we have shown that the traceless condition in the to-do list under (71) can hold. Let us try to show the rest one,

\[ \nabla_\nu T^{\mu \nu} = 0. \]

In order to show that, looking at \( T^{\mu \nu} \) given in (72) we try to act \( \nabla_\nu \) on it. Then covariant derivatives of metrices are zero as the basic formula (in short, \( \nabla_\nu \gamma^{\mu \nu} = 0 \)). Next, covariant derivatives of Einstein tensors are also zero as the Bianchi identity. Therefore, if we act \( \nabla_\nu \) on \( T^{\nu \mu} \) in (72), using the Codazzi equation [17, 30],

\[
\nabla_\nu T^{\nu \mu} = \nabla_\nu (K^{\nu \mu} - K \gamma^{\nu \mu}) = R_{\alpha \beta} \gamma^{\alpha \mu} n^\beta \bigg|_{r \to \infty} - 4g_{\alpha r} \gamma^{\alpha \mu} n^r \bigg|_{r \to \infty} \simeq 0,
\]

(79)

where we have used the facts that \( R_{\alpha \beta} = -4g_{\alpha \beta} \) in the AdS\(_5\) space and the non-zero component of \( n^\mu \) is only \( n^r \) as we wrote under (74). Finally, the non-zero component in \( g_{\alpha r} \) is only \( g_{rr} = 1/r^2 \) at \( r \to \infty \) when \( \alpha \) runs with \( r \) fixed. Therefore, we can confirm \( \nabla_\nu T^{\mu \nu} = 0 \), and have shown the rest one in the to-do list under (71).

**B Expression of holographic entanglement entropy**

We compute with explicitly written arguments

We give the expression of (41) and (42) with the arguments written explicitly.

\[
S_{EE} = \frac{1}{4G_5} \int \gamma \left( \sqrt{g_{\tau \tau'} (d\tau')^2 + g_{\tau \tau'} d\tau' dr} \right) \cdot \int_{S^2} d\theta d\phi \sin \theta g_{\Omega_2 \Omega_2} \bigg|_{y=1}
\]

\[
= \frac{\pi r_0^2}{G_5} \int_{r_{-1/\delta}}^{r_0} d\tau' \left( r(\tau') \right)^2 e^{c(r(\tau'), \tau_0)} \sqrt{r(\tau')^2 a(r(\tau'), \tau) - \frac{dr(\tau')}{d\tau'}}
\]

\[
= \frac{\pi r_0^2}{G_5} \int_{r_{-1/\delta}}^{r_*} d\tau' \left( r(\tau') \right)^2 e^{c(r(\tau'), \tau_0)} \sqrt{\left( r(\tau') \right)^2 a(r(\tau'), \tau) - \frac{dr(\tau')}{d\tau'}}
\]

\[
= \frac{\pi r_0^2}{G_5} \int_{r_{1/\delta}}^{r_*} d\tau' \left( \frac{r}{r'} \right)^2 e^{c(r(\tau'), \tau_0)} \sqrt{\left( \frac{r}{r'} \right)^2 a(r, \tau') - \frac{dr}{d\tau'}}
\]

\[
= \frac{\pi r_0^2}{G_5} \int_{r_{1/\delta}}^{r_*} d\tau' \left( \frac{r}{r'} \right)^2 e^{c(r(\tau'), \tau_0)} \sqrt{\frac{r^2 a(r, \tau'(r))}{r^2} - \frac{dr}{d\tau'}}.
\]

The third line is expression that the arguments in the second line are written explicitly, and the fifth line is the fourth line written explicitly.
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