Complex manifolds with split tangent bundle

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Introduction

The theme of this note is to investigate when the tangent bundle of a compact complex manifold $X$ splits as a direct sum of sub-bundles. This occurs typically when the universal covering space $\tilde{X}$ of $X$ splits as a product $\prod_{i \in I} U_i$ of manifolds on which the group $\pi_1(X)$ acts diagonally (that is, $\pi_1(X)$ acts on each $U_i$ and its action on $\tilde{X} = \prod_{\tilde{U}_i} U_i$ is the diagonal action $g.(u_i) = (gu_i)$): the vector bundles $T_{U_i}$ on $\tilde{X}$ are stable under $\pi_1(X)$, hence the decomposition $T_{\tilde{X}} = \bigoplus_i T_{U_i}$ descends to a direct sum decomposition of $T_X$. For Kähler manifolds, it is tempting to conjecture that the converse is true, namely that any direct sum decomposition of the tangent bundle $T_X$ (perhaps with the additional assumption that the direct summands are integrable) gives rise to a splitting of the universal covering. We will show that this is indeed the case in three different situations:

- $X$ admits a Kähler-Einstein metric;
- $T_X$ is a direct sum of line bundles of negative degree;
- $X$ is a Kähler surface.

Case a) is a direct consequence of the fact that on a compact Kähler-Einstein manifold, any endomorphism of the tangent bundle is parallel (this idea appears for instance in [Y], and in a more implicit form in [K]). Case b) is a slight improvement of a uniformization result of Simpson [S]. To treat case c) we use the classification of surfaces and some simple remarks about connections. The result in this case is actually an easy consequence of the paper [K-O], where the authors classify surfaces with a holomorphic conformal structure – this turns out to be closely related to the question we are studying here. However we found simpler and more enlightening to give an independent proof rather than extracting from [K-O] the pieces of information that we need.

In §2 we give a few examples which show that for non-Kähler manifolds a splitting of the tangent bundle does not necessarily imply a splitting of the universal covering.

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2 Throughout the paper we will abuse notation and write $T_{U_i}$ instead of $pr_i^*T_{U_i}$. 

1. Kähler-Einstein manifolds

**Theorem A.** — Let $X$ be a compact complex manifold admitting a Kähler-Einstein metric. Assume that the tangent bundle of $X$ has a decomposition $T_X = \bigoplus E_i$. Then the universal covering space of $X$ is a product $\prod_{i \in I} U_i$ of complex manifolds, in such a way that the decomposition $T_X = \bigoplus_{i \in I} E_i$ lifts to the decomposition $T_{\prod U_i} = \bigoplus_{i \in I} T_{U_i}$; the group $\pi_1(X)$ acts diagonally on $\prod_{i \in I} U_i$.

The proof follows closely that of thm. 2.1 in [Y] (I am indebted to J. Wahl for pointing out this reference).

**Proof:** (1.1) As a consequence of the Bochner formula, every endomorphism of $T_X$ is parallel [K]. This applies in particular to the projectors associated to the direct sum decomposition of $T_M$; therefore the sub-bundles $E_i$ are preserved by the hermitian connection, hence the holonomy representation of $X$ is the direct sum of a family of representations corresponding to the $E_i$’s. By the De Rham theorem, the universal covering space of $X$ splits as a product $\prod_{i \in I} U_i$, such that the decomposition $T_X = \bigoplus_{i \in I} E_i$ pulls back to the decomposition $T_{\prod U_i} = \bigoplus_{i \in I} T_{U_i}$.

(1.2) The last assertion follows from the following simple observation: if a group $\Gamma$ acting on a product $\prod_{i \in I} U_i$ preserves the decomposition $T_{\prod U_i} = \bigoplus_{i \in I} T_{U_i}$, it acts diagonally. Let indeed $\gamma$ be an automorphism of $\prod_{i \in I} U_i$; for $j \in I$, put $\gamma_j = \text{pr}_j \circ \gamma$. The condition $\gamma^* T_{U_j} = T_{U_j}$ means that the partial derivatives of $\gamma_j$ in the directions of $U_k$ for $k \neq j$ vanish, hence $\gamma_j((u_i)_{i \in \mathbb{I}})$ depends only on $u_j$, which gives our claim. ■

2. Non-Kähler examples

In this section we give examples of manifolds for which the tangent bundle is a direct sum of line bundles, but which do not satisfy the conclusions of Theorem A.

(2.1) **Hopf manifolds**

Let $T = \text{diag}(\alpha_1, \ldots, \alpha_n)$ be a diagonal matrix, with $0 < |\alpha_i| < 1$ for each $i$. The cyclic group $T^\mathbb{Z}$ generated by $T$ acts freely and properly on $\mathbb{C}^n \setminus \{0\}$; the quotient $X$ is a compact complex manifold, called a Hopf manifold. For each non-zero complex number $\theta$, denote by $L_\theta$ the flat line bundle associated to the character of $\pi_1(X) = T^\mathbb{Z}$ mapping $T$ to $\theta$; in other words, $L_\theta$ is the quotient of the trivial line bundle $(\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}$ by the action of the automorphism $(T, \theta)$. By construction we have $T_X = \bigoplus_{i \in \mathbb{I}} L_{\alpha_i}$, but the universal covering space $\mathbb{C}^n \setminus \{0\}$ of $X$ is clearly not a product.

(2.2) **Complex compact nilmanifolds**

These are compact manifolds $X = G/\Gamma$, where $G$ is a nilpotent complex Lie
group and \( \Gamma \) a discrete subgroup of \( G \). We may assume that \( G \) is simply-connected and non-commutative (to exclude the trivial case of complex tori). A well-known example is the Iwasawa manifold \( U(\mathbb{C})/U(\mathbb{Z}[i]) \), where \( U \) is the group of upper-triangular \( 3 \times 3 \) matrices with diagonal entries 1; many examples can be obtained in an analogous way.

The tangent bundle of \( X = G/\Gamma \) is trivial, and its universal covering space \( G \) is isomorphic to \( \mathbb{C}^n \); however we claim that whatever isomorphism \( G \xrightarrow{\sim} \mathbb{C}^n \) we choose, the action of \( \Gamma \) cannot be diagonal. Indeed if \( \Gamma \) acts diagonally, the standard trivialization of \( T_{\mathbb{C}^n} \) deduced from the coordinate system descends to a trivialization of \( T_X \). Any such trivialization lifts to a trivialization of \( T_G \) defined by a basis of right invariant vector fields; therefore the standard trivialization of \( T_{\mathbb{C}^n} \) is \( G \)-equivariant. In view of 1.2 this means that \( G \) itself acts diagonally on \( \mathbb{C}^n \), hence \( G \) embeds into \( \text{Aut}(\mathbb{C})^n \). Now any nilpotent connected subgroup of the affine group \( \text{Aut}(\mathbb{C}) \) is commutative, so we conclude that \( G \) is commutative, contrary to our hypothesis.

3. Simpson’s uniformization result

The following lemma, which is a variation on the Baum-Bott theorem [B-B], will allow us to slightly improve Simpson’s result:

**Lemma 3.1.**— Let \( X \) be a complex manifold, and \( E \) a direct summand of \( T_X \). The Atiyah class at\((E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E)) \) comes from \( H^1(X, E^* \otimes \mathcal{E}nd(E)) \). In particular, any class in \( H^r(X, \Omega^r_X) \) given by a polynomial in the Chern classes of \( E \) vanishes for \( r > \text{rk}(E) \).

**Proof:** Write \( T_X = E \oplus F \); let \( p : T_X \to E \) be the corresponding projection. For any sections \( U \) of \( E \) and \( V \) of \( F \) over some open subset of \( X \), put \( D_V U = p([V, U]) \). This expression is \( \mathcal{O}_X \)-linear in \( V \) and satisfies the Leibnitz rule \( D_V(fU) = fD_V(U) + (V f)U \), so that \( D \) is a \( F \)-connection on \( E \) [B-B]: if we denote by \( D^1(E) \) the sheaf of differential operators \( \Delta : E \to E \), of degree \( \leq 1 \), whose symbol \( \sigma(\Delta) \) is scalar, this means that \( D \) defines an \( \mathcal{O}_X \)-linear map \( F \to D_1(E) \) such that \( \sigma(D_V) = V \) for all local sections \( V \) of \( F \). Thus the exact sequence
\[
0 \to \mathcal{E}nd(E) \longrightarrow D^1(E) \xrightarrow{\sigma} T_X \to 0
\]
splits over the sub-bundle \( F \subset T_X \); therefore its extension class at\((E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}nd(E)) \) vanishes in \( H^1(X, F^* \otimes \mathcal{E}nd(E)) \), hence comes from \( H^1(X, E^* \otimes \mathcal{E}nd(E)) \). The last assertion follows from the definition of the Chern classes in terms of the Atiyah class. ■

We denote as usual by \( \mathbf{H} \) the Poincaré upper half-space.
Theorem B.— Let $X$ be a compact Kähler manifold, with a Kähler class $\omega$. Assume that the tangent bundle $T_X$ is a direct sum of line bundles $L_1, \ldots, L_n$ with $\omega^{n-1} c_1(L_i) < 0$ for each $i$. Then the universal covering space of $X$ is $H^n$, and the decomposition $T_X = \bigoplus L_i$ lifts to the canonical decomposition $T_{H^n} = (T_H)^{\oplus n}$.

Proof: This is Cor. 9.7 of [S], except that Simpson makes the extra hypothesis $\omega^{n-2} (c_1(X)^2 - 2c_2(X)) = 0$ (the assertion about the compatibility of decompositions is not stated in loc. cit., but follows directly from the proof). Now lemma 3.1 gives $c_1(L_i)^2 = 0$ for each $i$, hence $c_1(X)^2 - 2c_2(X) = 0$.

4. The surface case

Theorem C.— Let $X$ be a compact complex surface. The tangent bundle of $X$ splits as a direct sum of two line bundles if and only if one of the following occurs:

a) The universal covering space of $X$ is a product $U \times V$ of two (simply-connected) Riemann surfaces and the group $\pi_1(X)$ acts diagonally on $U \times V$; in that case the given splitting of $T_X$ lifts to the direct sum decomposition $T_{U \times V} = T_U \oplus T_V$.

b) $X$ is a Hopf surface, with universal covering space $\mathbb{C}^2 \setminus \{0\}$. Its fundamental group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$, for some integer $m \geq 1$; it is generated by a diagonal automorphism $(x, y) \mapsto (ax, by)$ with $|a| \leq |b| < 1$, and a diagonal automorphism $(x, y) \mapsto (\lambda x, \mu y)$ where $\lambda$ and $\mu$ are primitive $m$-th roots of 1.

As a corollary, for Kähler surfaces we see that any direct sum decomposition of the tangent bundle gives rise to a splitting of the universal covering, as announced in the introduction.

(4.1) Before starting the proof we will need a few preliminaries. From now on we denote by $X$ a compact complex surface; we assume given a direct sum decomposition $\Omega^1_X \cong L \oplus M$. By lemma 3.1 (or by [B-B]) the Chern class $c_1(L) \in H^1(X, \Omega^1_X)$ belongs to the subspace $H^1(X, L)$, and similarly for $M$. As a consequence, we get:

(4.2) We have $L^2 = M^2 = 0$, and therefore $c_1^2(X) = 2L.M = 2c_2(X)$.

The following consequence is less obvious.

Proposition 4.3.— Let $C$ be a smooth rational curve in $X$. Then $C^2 \geq 0$.

Proof: Put $C^2 = -d$ and assume $d > 0$. Since $H^1(C, \mathcal{O}_C(d + 2)) = 0$, the exact sequence

$$0 \to \mathcal{O}_C(d) \to \Omega^1_{X|C} \to \Omega^1_C \to 0$$

splits, giving an isomorphism $\Omega^1_{X|C} \cong \mathcal{O}_C(d) \oplus \mathcal{O}_C(-2)$. Thus one of the line bundles $L$ or $M$, say $L$, satisfies $L|_C \cong \mathcal{O}_C(d)$. Consider the commutative diagram
\[ \begin{array}{ccc}
H^1(X, L) & \longrightarrow & H^1(X, \Omega^1_X) \\
\downarrow & & \downarrow \\
H^1(C, L|_C) & \longrightarrow & H^1(C, \Omega^1_C) 
\end{array} \]

since \( d > 0 \) we have \( H^1(C, L|_C) = 0 \); thus \( c_1(L) \) goes to 0 in \( H^1(C, \Omega^1_C) \), which means \( d = 0 \), a contradiction. \( \blacksquare \)

(4.4) We shall come across situations where the vector bundle \( \Omega^1_X = L \oplus M \) appears as an extension
\[ 0 \to P \to \Omega^1_X \to Q \to 0 \]
of two line bundles \( P \) and \( Q \). In that case,
- either the restriction of \( p \) to one of the direct summands of \( \Omega^1_X \), say \( M \), is surjective; then the exact sequence splits, \( Q \) is isomorphic to \( M \) and \( P \) to \( L \);
- or the restriction of \( p \) to both \( L \) and \( M \) is not surjective; then there exists effective (non-zero) divisors \( A \) and \( B \), whose supports do not intersect, such that \( L \cong Q(-A) \), \( M \cong Q(-B) \) and \( P \cong Q(-A - B) \); the exact sequence does not split.

In particular, if \( \text{Hom}(P, Q) = 0 \), the exact sequence splits.

(4.5) Finally we will need some classical facts about connections (see [E]). Let \( p : M \to B \) be a smooth holomorphic map between complex manifolds, whose fibres are isomorphic to a fixed variety \( F \). A connection on \( p \) is a splitting of the exact sequence
\[ 0 \to p^*\Omega^1_B \to \Omega^1_M \to \Omega^1_{M/B} \to 0 , \]
that is a sub-bundle \( L \subset \Omega^1_M \) mapping isomorphically onto \( \Omega^1_{M/B} \); the connection is flat (or integrable) if \( dL \subset L \wedge \Omega^1_M \) (this is automatic if \( B \) is a curve). In that case the group \( \pi_1(B) \) acts on \( F \) by complex automorphisms, and \( M \) is the fibre bundle on \( B \) with fibre \( F \) associated to the universal covering \( \tilde{B} \to B \), that is the quotient of \( \tilde{B} \times F \) by the group \( \pi_1(B) \) acting diagonally; the splitting \( \Omega^1_M = p^*\Omega^1_B \oplus L \) pulls back to the decomposition \( \Omega^1_{\tilde{B} \times F} = \Omega^1_{\tilde{B}} \oplus \Omega^1_F \).

5. Proof of theorem C

(5.1) Kodaira dimension 2

If \( \kappa(X) = 2 \), the canonical bundle \( K_X \) is ample by Prop. 4.3. The Aubin-Calabi-Yau theorem implies that \( X \) admits a Kähler-Einstein metric; we can therefore apply Theorem A.
(5.2) Kodaira dimension 1

If \( \kappa(X) = 1 \), \( X \) admits an elliptic fibration \( p : X \to B \). By 4.2 we have \( c_2(X) = 0 \); this implies that the only singular fibres of \( p \) are multiples of smooth elliptic curves (see [B1], VI.4 and VI.5). For \( b \in B \), we write \( p^*[b] = m(b)F_b \), where \( F_b \) is a smooth elliptic curve; we have \( m(b) \geq 1 \) and \( m(b) = 1 \) except for finitely many points. Put \( \Delta = \sum_b (m(b) - 1)F_b \). We have an exact sequence

\[
0 \to p^*\Omega^1_B(\Delta) \to \Omega^1_X \to \omega_{X/B} \to 0,
\]

where \( \omega_{X/B} \) is the relative dualizing line bundle. Since \( \chi(O_X) = 0 \) by Riemann-Roch, we deduce from [B-P-V], V.12.2 and III.18.2, that \( \omega_{X/B} \) is a torsion line bundle. Since \( K_X = p^*\Omega^1_B(\Delta) \otimes \omega_{X/B} \), the hypothesis \( \kappa(X) = 1 \) implies \( \text{Hom}(p^*\Omega^1_B(\Delta), \omega_{X/B}) = 0 \), hence the exact sequence (5.3) splits by 4.4.

Let \( \rho : \tilde{B} \to B \) be the orbifold universal covering of \( (B, m) \): this is a ramified Galois covering, with \( \tilde{B} \) simply-connected, such that the stabilizer of a point \( \tilde{b} \in \tilde{B} \) is a cyclic group of order \( m(\rho(\tilde{b})) \) (see for instance [K-O], lemma 6.1; note that because of the hypothesis \( \kappa(X) = 1 \) and the formula for \( K_X \), there are at least 3 multiple fibers if \( B \) is of genus 0). Let \( \tilde{X} \) be the normalization of \( X \times_B \tilde{B} \). We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow \tilde{p} & & \downarrow p \\
\tilde{B} & \xrightarrow{\rho} & B
\end{array}
\]

where \( \tilde{p} \) is smooth and \( \pi \) is étale ([B1], VI.7'). The exact sequence

\[
0 \to \tilde{p}^*\Omega^1_B \to \Omega^1_X \to \Omega^1_{\tilde{X}/\tilde{B}} \to 0
\]

coincides with the pull back under \( \pi \) of the exact sequence (5.3); therefore \( p \) admits an integrable connection, given by the subbundle \( \pi^*M \) of \( \Omega^1_X \). The result follows from 4.5 and 1.2.

(5.4) Kodaira dimension 0

Assume \( \kappa(X) = 0 \). By 4.2 and the classification of surfaces, \( X \) is either a complex torus, a bielliptic surface, or a Kodaira surface. Complex tori and bielliptic surfaces fall into case a) of the theorem (a bielliptic surface is the quotient of a product \( E \times F \) of elliptic curves by a finite abelian group acting diagonally).

A primary Kodaira surface has trivial canonical bundle and admits a smooth elliptic fibration \( p : X \to B \). Thus the exact sequence (4.2) realizes \( \Omega^1_X \) as an
extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. Since $h^{1,0}(X) = 1$, this extension is non-trivial, and it follows from 4.4 that $\Omega^1_X$ does not split.

A secondary Kodaira surface admits a primary Kodaira surface as a finite étale cover, hence its tangent bundle cannot split either.

**(5.5) Ruled surfaces**

We consider the case when $X$ is algebraic and $\kappa(X) = -\infty$. By 4.2 and 4.3, $X$ is a geometrically ruled surface, that is a projective bundle $p : X \to B$ over a curve. We again consider the exact sequence

$$0 \to p^*\Omega^1_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0 ;$$

since $\Omega^1_{X/B}$ has negative degree on the fibres, we have $\text{Hom}(p^*\Omega^1_B, \Omega^1_{X/B}) = 0$, hence by 4.4 the above exact sequence splits: one of the direct summands of $\Omega^1_X$ defines an integrable connection for $p$. The result follows then from 4.5.

**(5.6) Inoue surfaces**

We now assume that $X$ is not algebraic and $\kappa(X) = -\infty$, so that $X$ is what is usually called a surface of type VII$_0$. These surfaces have $b_1 = h^{0,1} = 1$ and therefore $c_1^2 + c_2 = 12\chi(\mathcal{O}_X) = 0$; in our case this gives $c_1^2 = c_2 = 0$ in view of 4.2, and finally $b_2 = 0$. Moreover we have $\text{H}^0(X, \Omega^1_X \otimes L^{-1}) \neq 0$. The surfaces with these properties have been completely classified by Inoue [I]: they are either Hopf surfaces, or belong to three classes of surfaces constructed by Inoue (loc. cit.).

We first consider the Inoue surfaces. The surfaces $S_{M}$ of the first class are quotients of $H \times \mathbb{C}$ by a group acting diagonally, hence they fall into case a) of the theorem.

The surfaces $S_{N, p, q, r, t}^{(+)}$ of the second class are quotients of $H \times \mathbb{C}$ by a group which does not act diagonally. This action leaves invariant the vector field $\partial/\partial z$ on $\mathbb{C}$, which therefore descends to a non-vanishing vector field $v$ on $X$. This gives rise to an exact sequence

$$0 \to K_X \xrightarrow{i(v)} \Omega^1_X \xrightarrow{i(v)} \mathcal{O}_X \to 0 ,$$

which does not split since $h^{1,0}(X) = 0$. We have $\text{H}^0(X, K_X^{-1}) = 0$, for instance because $X$ contains no curves; we infer from 4.4 that $\Omega^1_X$ does not split.

The surfaces $S_{N, p, q, r}^{(-)}$ of the third class are quotients of certain surfaces of the second class by a fixed point free involution; therefore their tangent bundle does not split either.

**(5.7) Primary Hopf surfaces**

It remains to consider the class of Hopf surfaces, which are by definition the surfaces of class VII$_0$ whose universal covering space is $W := \mathbb{C}^2 \smallsetminus \{0\}$. We
consider first the primary Hopf surfaces, which are quotients of $W$ by the infinite cyclic group generated by an automorphism $T$ of $W$. According to [Ko], §10, there are two cases to consider:

a) $T(x, y) = (\alpha x, \beta y)$ for some complex numbers $\alpha, \beta$ with $0 < |\alpha| \leq |\beta| < 1$;

b) $T(x, y) = (\alpha^m x + \lambda y^m, \alpha y)$ for some positive integer $m$ and non-zero complex numbers $\alpha, \lambda$ with $|\alpha| < 1$.

As in 2.1, we denote by $L_\theta$, for $\theta \in \mathbb{C}$, the flat line bundle associated to the character of $\pi_1(X)$ mapping $T$ to $\theta$. In case $a$) we find $\Omega^1_X = L_{-1}^\alpha \oplus L_{-1}^\beta$, so the tangent bundle splits.

Let us consider case $b$). The form $dy$ on $W$ satisfies $T^*dy = \alpha dy$, hence descends to a form $dy$ in $H^0(X, \Omega^1_X \otimes L_\alpha)$; similarly the function $y$ descends to a non-zero section of $L_\alpha$. We have an exact sequence

$$0 \to L_{-1}^\alpha \to \Omega^1_X \to L_{-m}^\alpha \to 0.$$ 

Since $L_\alpha$ has a non-zero section, the space $\text{Hom}(L_{-1}^\alpha, L_{-m}^\alpha)$ is zero for $m > 1$. Hence if $\Omega^1_X$ splits, we deduce from 4.4 that the exact sequence splits. This means that there exists a form $\overline{\omega} \in H^0(X, \Omega^1_X \otimes L_{-m}^\alpha)$ such that $\overline{\omega} \wedge \overline{dy} \neq 0$. Then $\overline{\omega} \wedge \overline{dy}$ is a generator of the trivial line bundle $K_X \otimes L_{-m}^\alpha$, hence pulls back to $c dx \wedge dy$ on $W$, for some constant $c \neq 0$. Therefore the pull back $\omega$ of $\overline{\omega}$ to $W$ is of the form $c dx + f(x, y)dy$ for some holomorphic function $f$ on $\mathbb{C}^2$. The flat line bundle $L_{-m}^\alpha$ carries a flat holomorphic connection $\nabla$; the 2-form $\nabla \omega$, which is a global section of $K_X \otimes L_{-m}^\alpha \cong L_{-1}^\alpha$, is zero. This implies $d\omega = 0$, so the function $f(x, y)$ is independent of $x$; let us write it $f(y)$. Now the condition $T^* \omega = \alpha^m \omega$ reads $\alpha f(\alpha y) + c\lambda my^{m-1} = \alpha^m f(y)$. Differentiating $m$ times we find $f^{(m)} = 0$, then differentiating $m - 1$ times leads to a contradiction.

(5.8) Secondary Hopf surfaces

A secondary Hopf surface $X$ is the quotient of $W$ by a group $\Gamma$ acting freely, containing a central, finite index subgroup generated by an automorphism $T$ of the above type. We assume that $\Omega^1_X$ splits. The primary Hopf surface $Y = W/\mathbb{Z}$ is a finite étale cover of $X$, so $\Omega^1_Y$ also splits; it follows from 5.7 that $T$ is of type $a$), and that $\Gamma$ does not contain any transformation of type $b$). According to [Ka], §3, this implies that after an appropriate change of coordinates, the group $\Gamma$ acts linearly on $\mathbb{C}^2$.

We claim that $\Gamma$ is contained in a maximal torus of $\text{GL}(2, \mathbb{C})$. This is clear if $\alpha \neq \beta$, because $T$ is central in $\Gamma$. If $\alpha = \beta$, the direct sum decomposition of $\Omega^1_X$ pulls back to a decomposition $\Omega^1_Y = L_{\alpha^{-1}}^\alpha \oplus L_{-\alpha^{-1}}^\alpha$ (5.7), which for an appropriate choice of coordinates comes from the decomposition $\Omega^1_W = \mathcal{O}_W dx \oplus \mathcal{O}_W dy$. Since $\Gamma$ must preserve this decomposition, it is contained in the diagonal torus.
Thus we may identify $\Gamma$ with a subgroup of $(\mathbb{C}^*)^2$; since it acts freely on $W$, the first projection $\Gamma \rightarrow \mathbb{C}^*$ is injective. Therefore the torsion subgroup of $\Gamma$ is cyclic, and we are in case $b)$ of the theorem. ■

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