Arrangements of translates of a curve

József Solymosi *

Endre Szabó †

Abstract

We show that there are four types of planar curves such that arrangements of its translates are combinatorially equivalent to an arrangement of lines. These curves can be used to define norms giving constructions with many unit distances among points in the plane.

1 Introduction

One of the oldest and best known problems in combinatorial geometry is Paul Erdős’ unit distances problem [4]. What is the maximum number of unit distances among \( n \) points on the plane? Erdős conjectured that the maximum number of unit distances is \( n^{1+o(1)} \). (Through the paper we are going to use the Big-O, Little-o, and Omega notations. A real function \( f(x) \) is \( o(T(x)) \) if \( f(x)/T(x) \to 0 \) as \( x \to \infty \). It is \( O(T(x)) \) if there is a \( c > 0 \) such that \( f(x)/T(x) \leq c \) as \( x \to \infty \), and it is \( \Omega(T(x)) \) if there is a \( c > 0 \) such that \( f(x)/T(x) \geq c \) as \( x \to \infty \).)

The conjecture is still open, the best known upper bound is \( O(n^{4/3}) \). This bound was proved by Spencer, Szemerédi, and Trotter [12]. It seems that the exponent \( 4/3 \) is the limit of the known combinatorial methods, even to prove \( o(n^{4/3}) \) is out of range of the known techniques.

One reason behind this barrier is that there are simple norms where one can find \( n \)-element point-sets with \( \Omega(n^{4/3}) \) unit distances. Probably the oldest construction providing such a norm can be derived from Jarnik’s construction [5]. Jarnik defined a sequence of centrally symmetric convex polygons containing \( \Omega(n^{2/3}) \) lattice points of the \( n \times n \) integer grid. Setting such a convex polygon as the unit disk, there are \( \Omega(n^{8/3}) \) unit distances among the \( (2n)^2 \) points of the \( 2n \times 2n \) integer lattice. In this example the norm changes with \( n \). A nice construction,
with a uniform norm was given by Valtr [15] using translates of a parabola and the $n \times n^2$ integer grid. (see the description of the construction on page 194 in [1]) On the other hand it was proved by Matousek that most norms, in the sense of Baire category, determine $O(n \log n \log \log n)$ unit distances among $n$ points [6].

For any strictly convex norm, among $n$ points in the plane there are $O(n^{4/3})$ unit distances. This claim can be proved using the crossing number inequality in the same way as proving the Szemerédi-Trotter theorem. It gives a sharp upper bound on the number of incidences, $I$, between $N$ points and $M$ lines on the real plane, $\mathbb{R}^2$.

**Theorem 1.1** (Szemerédi-Trotter Theorem [11]).

$$I(N, M) = O\left(N^{2/3} M^{2/3} + N + M\right).$$

An elegant proof of the above theorem was given by Székely, who also showed how to use his proof method to give the $O(n^{4/3})$ bound for unit distances [14].

There are known arrangements of $n$ lines and $n$ points such that

$$I(n, n) = \Omega\left(n^{4/3}\right).$$

Such arrangements were found by Erdős, Elekes [3], Sheffer and Silier [13] using lattice points, i.e. a Cartesian product structure. Recently Guth and Silier gave sharp examples not based on a rectangular lattice [10].

If there were maps where the images of lines are translates of a single curve, $C$, then one can map point-line arrangements to point-curve arrangements. Using part of the curve as (part of) the unit circle, with this norm we have $O(n^{4/3})$ unit distances. Such map exists, the map

$$(x, y) \rightarrow (x, y + x^2)$$

sends the $(t, at + b)$ line to $(t, t^2 + at + b) = \left(t, (t + a/2)^2 - a^2/4 + b\right)$ curve, which is a translate of the $y = x^2$ parabola. This map was used over finite fields by Pudlák in [8]. Pudlák noticed that this map gives a one-to-one correspondence between point-parabola incidences and point-line incidences. He used it to define a coloring of the complete bipartite graph without large monochromatic complete subgraphs. If we apply the map in (2) to Elekes’ point-line arrangement we get a construction very similar to Valtr’s.

Based on the above observations it is a natural problem characterising maps of the plane sending lines into translates of a single curve. As we will see there are exactly four more such maps in addition to Pudlák’s map. Three of them are easy to describe, we list them here. For one curve we didn’t find a nice explicit form.
1. \( M_1 : (x, y) \rightarrow (x, \ln(y)) \).

Every line, \( y = ax + b \) with \( a > 0 \), the \( x > -\frac{b}{a} \) part maps to a translate of a log curve, \( y = \ln(x) \). Its image is \( (x, \ln \left(x + \frac{b}{a}\right) + \ln(a)) \).

2. \( M_2 : (x, y) \rightarrow \left(\ln(x), \ln \left(\frac{y}{x}\right)\right) \).

If we use the notation \( \ln(x) = X \), then the image of the \( y = ax + b \) line is the 
\[
\left(X, \ln \left(1 + \frac{b/a}{e^X}\right) + \ln(a)\right) = \left(X, \ln \left(1 + \frac{e^{-X} \ln(b/a)}{e^X}\right) + \ln(a)\right)
\]
curve if \( a > 0 \) and \( b > 0 \). This is the translate of the \( y = \ln(1 + e^{-x}) \) curve.

3. \( M_3 : (x, y) \rightarrow \left(\ln(x), \frac{y}{x}\right) \).

As in the previous case, if we use the notation \( \ln(x) = X \), then the image of the \( y = ax + b \) line is the 
\[
\left(X, e^{-X} \ln(b) + a\right)
\]
curve if \( a > 0 \) and \( b > 0 \). This is the translate of the \( y = e^{-x} \) curve, so it is similar to the first case with map \( M_1 \).

2 Preliminaries

Notation 2.1. The group of invertable \( 3 \times 3 \) matrices and the Lie algebra of all \( 3 \times 3 \) matrices denoted by \( \text{GL}(3, \mathbb{R}) \) and \( \text{gl}(3, \mathbb{R}) \) resp. The quotient group of \( \text{GL}(3, \mathbb{R}) \) by the normal subgroup of scalar matrices is denoted by \( \text{PGL}(3, \mathbb{R}) \), it is the group of projective linear transformations of the projective plane. The Lie algebra of \( \text{PGL}(3, \mathbb{R}) \) is the quotient of \( \text{gl}(3, \mathbb{R}) \) by the ideal of scalar matrices, we denote it with \( \text{pgl}(3, \mathbb{R}) \). It is naturally isomorphic to \( \text{sl}(3, \mathbb{R}) \), the Lie algebra of \( 3 \times 3 \) matrices of trace 0.

Notation 2.2. We denote by \( \text{Aff}(2, \mathbb{R}) \leq \text{GL}(3, \mathbb{R}) \) the subgroup of all inevitable matrices of the form
\[
\begin{pmatrix}
L & v \\
0 & 1
\end{pmatrix}
\]
where \( L \) a \( 2 \times 2 \) invertable matrix (linear transformation), and \( v \) is a \( 2 \)-dimensional column vector (translation). We denote by \( \text{aff}(2, \mathbb{R}) \leq \text{gl}(3, \mathbb{R}) \) the Lie subalgebra of matrices of the form
\[
\begin{pmatrix}
\Lambda & v \\
0 & 0
\end{pmatrix}
\]
where $\Lambda$ is a $2 \times 2$ matrix, and $v$ is a 2-dimensional column vector. The quotient homomorphism $GL(3, \mathbb{R}) \to PGL(3, \mathbb{R})$ maps $\text{Aff}(2, \mathbb{R})$ isomorphically onto its image, and similarly, the quotient homomorphism $\mathfrak{gl}(3, \mathbb{R}) \to \mathfrak{pgl}(3, \mathbb{R})$ maps $\mathfrak{aff}(2, \mathbb{R})$ isomorphically onto its image. We shall often identify $\text{Aff}(2, \mathbb{R})$ and $\mathfrak{aff}(2, \mathbb{R})$ with these images. With this identification $\text{Aff}(2, \mathbb{R})$ becomes the group of affine transformations of $\mathbb{R}^2$, and $\mathfrak{aff}(2, \mathbb{R})$ becomes its Lie algebra.

The following is well-known.

**Fact 2.3.** Let $\psi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism which maps all lines into lines. Then $\psi_0 \in \text{Aff}(2, \mathbb{R})$.

We need the following local version of this.

**Lemma 2.4.** Let $\psi : U \to V$ be a diffeomorphism between connected open subsets $U, V \subseteq \mathbb{R}^2$ which maps all line segments in $U$ into line segments in $V$. Then $\psi$ can be uniquely extended into a projective linear map $\psi \in PGL(3, \mathbb{R})$.

**Proof.** The standard proof of Fact 2.3 works here, if one is careful enough. We recall it for the sake of completeness.

Let $S \subset U$ be a line segment. We define the following set of real numbers.

$$C_S = \left\{ \lambda \in \mathbb{R} \mid \text{If } A, B, C, D \in S \text{ with cross-ratio } (A, B : C, D) = \lambda \text{ then } (\psi(A), \psi(B); \psi(C), \psi(D)) = \lambda \right\}$$

We make several observations.

1. If $\lambda \in C_S$ then $1 - \lambda \in C_S$.
   Indeed, $(A, B : C, D) = 1 - (A, C : B, D)$.

2. If $\lambda < 0 < \mu$ and $\lambda, \mu \in C_S$ then $\frac{\mu}{\lambda} \in C_S$.
   Indeed, let $(A, B; C, D) = \frac{\mu}{\lambda}$. Since this is negative, one of $C$ and $D$ lies inside $\overline{AB}$, the other lies outside. If $C \in \overline{AB}$ then we relabel $A, B, C, D$ to $B, A, D, C$, this does not change their cross-ratio. So we can assume that $C \notin \overline{AB}$. Then there is a unique $E \in \overline{AB}$ with $(A, B; E, C) = \lambda$, and an easy calculation shows that $(A, B; E, D) = \mu$. This implies that $(\psi(A), \psi(B); \psi(E), \psi(C)) = \lambda$ and $(\psi(A), \psi(B); \psi(E), \psi(D)) = \mu$, hence $(\psi(A), \psi(B); \psi(C), \psi(D)) = \frac{\mu}{\lambda}$.

3. $0, 1 \in C_S$.
   Indeed, if $(A, B; C, D) = 0$ then either $A = C$ or $D = B$. This implies that either $\psi(A) = \psi(C)$ or $\psi(D) = \psi(B)$, hence $(\psi(A), \psi(B); \psi(C), \psi(D)) = 0$. Therefore $0 \in C_S$, and then (1) implies $1 \in C_S$.

4. $-1 \in C_S$.
   Indeed, let $W \subseteq U$ be a convex neighborhood of $S$. If $(A, B : C, D) = -1$ then there a complete quadrangle in $W$ which justifies this, i.e. $A$ and $C$ are the intersection points of the opposite sides, and the diagonals intersect the $AB$ line at $C$ and $D$. Then $\psi$ maps this quadrangle to a quadrangle in $V$ justifying that $(\psi(A), \psi(B); \psi(C), \psi(D)) = \lambda$. 

4
5. \( \mathbb{Z} \subseteq \mathcal{C}_S \).
   Indeed, starting with \(-1 \in \mathcal{C}_S\), and applying (1) and (2) alternately, we
   obtain that \(2,-2,3,-3,4,-4, \cdots \in \mathcal{C}_S\). By (1), 1 and 2 are also in \(\mathcal{C}_S\).

6. \( \mathbb{Q} \subseteq \mathcal{C}_S \).
   Indeed, negative rational numbers are in \(\mathcal{C}_S\) by (5) and (2). Then by (1)
   the nonnegative rational numbers also belong to \(\mathcal{C}_S\).

7. \( \mathcal{C}_S = \mathbb{R} \).
   Indeed, by the continuity of the cross-ratio, \(\mathcal{C}_S\) is a closed set.

Now let \(W \subseteq U\) be a convex open subset. The above observations imply that
\(\psi\) preserves all cross-ratios in \(W\), hence there is a unique projective linear map
\(\overline{\psi}_W \in \text{PGL}(3, \mathbb{R})\) which agrees with \(\psi\) on \(W\). For overlapping convex open sets
the corresponding projective linear maps must be equal. Since \(U\) is connected,
all these \(\overline{\psi}_W\) must be equal. This proves the lemma.

\[ \Box \]

Lemma 2.5. Let \(A, B \in \text{gl}(3, \mathbb{R})\) be matrices whose images \(\overline{A}, \overline{B} \in \text{pgl}(3, \mathbb{R})\)
commute. Then, after a suitable base change, \(\overline{A}, \overline{B} \in \text{aff}(2, \mathbb{R})\).

Proof. Commutators have trace zero. So \([A, B]\) is a scalar matrix with trace 0,
hence \(A\) and \(B\) commute in \(\text{gl}(3, \mathbb{R})\). We distinguish three cases.

- If \(A\) has one real, and two conjugate complex eigenvalues, then let \(V\) be
  the real part of the linear span of the complex eigenspaces corresponding
to the non-real eigenvalues of \(A\).

- If \(A\) has three different real eigenvalues then let \(V\) be the linear span of
  any two of the corresponding eigenspaces.

- If \(A\) has only two different real eigenvalues, then let \(V\) be the eigenspace
  corresponding to the eigenvalue with multiplicity two.

- Otherwise \(A\) has a single real eigenvalue of multiplicity three, hence \(A\) is a
  scalar matrix. In this case we switch the role of \(A\) and \(B\), and go through
  this list again. If \(B\) is not a scalar matrix then we obtain our \(V\).

- Finally, if both \(A\) and \(B\) are scalar matrices then let \(V\) be an arbitrary
  plane in \(\mathbb{R}^3\).

In all cases, \(V\) is a plane in \(\mathbb{R}^3\) invariant under both \(A\) and \(B\). After an appro-
priate base change \(V\) will be the hyperplane of vectors whose last coordinate is
zero. Matrices that map this \(V\) into itself are of the form
\[
\begin{pmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
0 & 0 & \ast
\end{pmatrix}
\]
where elements marked with \(\ast\) are arbitrary, and the quotient homomorphism
\(\text{gl}(3, \mathbb{R}) \to \text{pgl}(3, \mathbb{R})\) maps such matrices into \(\text{aff}(2, \mathbb{R})\). Hence, after this base
change, \(\overline{A}, \overline{B} \in \text{aff}(2, \mathbb{R})\). \(\Box\)
3 Affine Structures

Notation 3.1. For a plane curve \( C \subset \mathbb{R}^2 \) and a vector \( t \in \mathbb{R}^2 \) we denote by \( t + C \) the translate of \( C \) with \( t \).

Problem 3.2. Characterise diffeomorphisms \( \phi_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and continuous curves \( C \subset \mathbb{R}^2 \) such that \( \phi_0(t + C) \) is a line for all \( t \in \mathbb{R}^2 \).

Problem 3.3. Characterise continuous curves \( C \subset \mathbb{R}^2 \) and diffeomorphisms \( \phi : U \rightarrow V \) between connected open subsets \( U, V \subseteq \mathbb{R}^2 \) such that \( \phi((t + C) \cap U) \) is contained in a line for all \( t \in \mathbb{R}^2 \).

Let \( T \cong \mathbb{R}^2 \) denote the group of translations of \( \mathbb{R}^2 \). If we conjugate \( T \) with \( \phi_0 \) then by Fact 2.3 we arrive at a subgroup of \( \text{Aff}(2, \mathbb{R}) \). Since \( T \) is connected, these subgroups are uniquely determined by their Lie algebras, which are 2-dimensional commutative subalgebras of \( \text{aff}(2, \mathbb{R}) \).

In the more general setup, if we conjugate a small neighborhood of the identity in \( T \) with \( \phi \) then by Lemma 2.4 we arrive at a small neighborhood of the identity in a connected subgroup of \( \text{PGL}(2, \mathbb{R}) \). Again, these subgroups are uniquely determined by their Lie algebras, which are, in this case, 2-dimensional commutative subalgebras of \( \text{pgl}(2, \mathbb{R}) \). By Lemma 2.5 these subalgebras, after a base change, become subalgebras of \( \text{aff}(2, \mathbb{R}) \).

So in both problems, we need to classify 2-dimensional commutative subalgebras of \( \text{aff}(2, \mathbb{R}) \), and analyze, whether the corresponding subgroups are isomorphic to \( T \) or not.

Affine structures on the two-dimensional abelian Lie algebra were analysed in the work of Rem and Goze [9] where they proved that there are six affinely non-equivalent affine structures. They listed the affine structures on the two-dimensional Lie algebra and the corresponding action. Based on their list, there are six actions we have to check for a potential \( \phi \) or \( \phi_0 \).

3.1 The six affine actions

In what follows we are checking the affine actions listed in [9] whether they are generated by the translation of a vector \((s, t)\). For every affine action, given by the matrix \( A(s, t) \), we are looking for a diffeomorphism \( \phi : U \rightarrow V \) between connected open neighbourhoods \( U, V \) of the origin in \( \mathbb{R}^2 \) such that

\[
\phi^{-1}(\phi(x, y) + (s, t)) = A^*(s, t) \cdot [x, y, 1]^T,
\]

where \( A^*(s, t) \) is the \( 2 \times 3 \) submatrix of \( A(s, t) \) containing the first two rows. We will not specify \( U \) and \( V \), only their existence is important for us. However, studying the formulas we have for \( \Phi \) the reader can easily find appropriate \( U \) and \( V \).

Let us denote the translation by \((s, t)\) as \( T(s, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and let \( a(s, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denote the affine transformation as defined above with the \( A(s, t) \) matrix. With this notation equation (3) becomes

\[
\phi^{-1} \circ T(s, t) \circ \phi = a(s, t)
\]
where ◦ denotes the function composition. After rearranging it we have
\[ \phi^{-1} \circ T(s,t) = a(s,t) \circ \phi^{-1}. \]
Substituting the (0,0) point we can recover \( \phi \) from \( a(s,t) \) (from the rightmost column of \( A(s,t) \)). In the first two cases our \( \phi \) is actually an \( \mathbb{R}^2 \to \mathbb{R}^2 \) diffeomorphism, in the last four cases we get local maps giving solutions in the selected range.

1. Identity (case \( A_5 \) in \cite{9})
\[
A(s,t) = \begin{pmatrix}
1 & 0 & s \\
0 & 1 & t \\
0 & 0 & 1
\end{pmatrix}
\]
Here \( \phi \) is the identity map, it won’t define translates of a single curve.

2. Parabola (case \( A_4 \) in \cite{9})
\[
A(s,t) = \begin{pmatrix}
1 & 0 & s \\
s & 1 & t + \frac{s^2}{2} \\
0 & 0 & 1
\end{pmatrix}
\]
Here \( \phi : (x,y) \to \left( x, y + \frac{x^2}{2} \right) \). Every line, \( (x, ax + b) \) maps to a translate of a parabola \( y = \frac{x^2}{2} \). Its image is \( \left( x, \frac{(x+a)^2}{2} + \frac{a^2}{2} + b \right) \). This is Pudlak’s map we mentioned in the introduction.

![Figure 1: Unit circles defined by parabola and log arcs.](image)

3. Log curve (case \( A_6 \) in \cite{9})
\[
A(s,t) = \begin{pmatrix}
e^s & 0 & e^s - 1 \\
0 & 1 & t \\
0 & 0 & 1
\end{pmatrix}
\]
Here $\phi : (x, y) \rightarrow (\ln(x + 1), y)$. It is more convenient to work with the map $\phi' : (x, y) \rightarrow (x, \ln(y))$, giving the same family of curves. Every line, $y = ax + b$ with $a > 0$, the $x > \frac{b}{a}$ part maps to a translate of a log curve, $y = \ln(x)$. Its image is $(x, \ln \left(\frac{x}{a} + \frac{b}{a}\right) + \ln(a))$.

4. (case $A_1$ in [9])

$$A(s, t) = \begin{pmatrix} e^s & 0 & e^s - 1 \\ e^s(e^t - 1) & e^s & e^s(e^t - 1) \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \rightarrow \left(\ln(x + 1), \ln\left(1 + \frac{y}{x+1}\right)\right)$. The image of the $x > \max\left(-\frac{b+1}{a+1}, -1\right)$ part of the $(x, ax + b)$ line is a curve, however different lines map into different types of curves. Lines of the form $y = c(x + 1)$ map to a horizontal line $y = \ln(1 + c)$. This is still an interesting map. Let us consider a line $y = ax + b$ where $a \neq -1$ and $\frac{b-a}{1+a} > 0$. If we set $\ln(x + 1) = X$, then the image curve is

$$(X, \ln(1 + a + (b-a)e^{-X})) = \left(X, \ln\left(1 + e^{-\ln\left(\frac{b-a}{1+a}\right)}\right) + \ln(1 + a)\right),$$

which is a translate of the $y = \ln(1 + e^{-z})$ curve.

5. (case $A_2$ in [9])

$$A(s, t) = \begin{pmatrix} e^s & 0 & e^s - 1 \\ e^{st} & e^s & e^{st} \\ 0 & 0 & 1 \end{pmatrix}$$

Here $\phi : (x, y) \rightarrow \left(\ln(x + 1), \frac{y}{x+1}\right)$. As in the previous case, lines of the form $y = c(x + 1)$ map to a horizontal line. But if we consider lines $ax + b$, where $b > a \neq 0$, and set $\ln(x + 1) = X$, then the image curve is

$$(X, a + \frac{b - a}{e^X}) = \left(X, e^{-(X - \ln(b-a))} + a\right),$$

which is a translate of the $y = e^{-x}$ curve.

6. Rotations (case $A_3$ in [9])

$$A(s, t) = \begin{pmatrix} e^s \cos t & -e^s \sin t & 1 - e^s \cos t \\ e^s \sin t & e^s \cos t & e^s \sin t \\ 0 & 0 & 1 \end{pmatrix}$$

Like in the previous case, images of different lines could be the same. Images of translations with $(s, t + k2\pi)$ would give the same line for any
integer $k$. Also, lines of the form $y = c(x + 1)$ map to a horizontal line. In this last case the map is given by

$$\phi : (x, y) \mapsto \left( \frac{\ln((1 - x)^2 + y^2)}{2}, \arctan\left(\frac{y}{1 - x}\right) \right).$$

We didn’t find a nice explicit form for this curve (fig. 6), however we can still provide a parametric equation. If a line is given by the $x = t, y = at + b$ equations then its image after the map is

$$x(t) = \ln(t) + \ln \sqrt{1 + \left(\frac{at + b}{1 - t}\right)^2}, \quad y(t) = \arctan\left(\frac{at + b}{1 - t}\right).$$

![Figure 2: Part of the curve from Case 6](image)

From the six affine actions we found four different curves such that arrangements of their translates are (locally) combinatorically equivalent to arrangements of lines. Three of them have a nice description as we summarised in the introduction.

4 Concluding remarks and open problems

- One can ask the same problem in higher dimensions. What are the surfaces such that any finite arrangement of hyperplanes is combinatorially equivalent to translates of the surface? Remm and Goze collected the 15 invariant affinely non-equivalent affine structures on the three-dimensional abelian Lie algebra in [9], so based on their work one can characterise such surfaces in three-space. One of these (an extension of Pudlák’s map) was used by Zahl in [16] to show a norm determining $\Omega(n^{3/2})$ unit distances
among \( n \) points. In the same paper Zahl was able to break the \( n^{3/2} \) barrier showing that for the Euclidean norm the number of unit distances determined by \( n \) points is \( O(n^{3/2-c}) \) for some \( c > 0 \). In dimension four and higher there are \( n \)-element pointsets with \( \Omega(n^2) \) unit distances.

- It follows from Székely’s proof of the Szemerédi-Trotter theorem that \( m \) translates of a convex curve and \( n \) points determine \( O(n^{2/3}m^{2/3} + n + m) \) incidences (See also in [2]) In this paper we listed five curves where the above incidence bound is sharp, i.e. for each curve there are arrangements of \( m \) translates of the curve and \( n \) points with \( \Omega(n^{2/3}m^{2/3} + n + m) \) incidences. Are there such planar curves significantly different from the ones listed above?

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References

[1] P. Brass, W. O. J. Moser, and J. Pach, Research Problems in Discrete Geometry, Springer (2005) 1st. edition, XII, 500 pp.
[2] Gy. Elekes, M. B Nathanson, I. Z Ruzsa, Convexity and Sumsets, Journal of Number Theory, Volume 83, Issue 2, 2000, 194–201.
[3] Gy. Elekes, On linear combinatorics i. concurrency – an algebraic approach. Combinatorica, 17(4):447–458, 1997.
[4] P. Erdős, On sets of distances of \( n \) points. American Mathematical Monthly (1946) 53: 248–250.
[5] Jarník, V. V. Über die Gitterpunkte auf konvexen Kurven. Mathematische Zeitschrift 24 (1926): 500–518.
[6] J. Matoušek, (2011). The number of unit distances is almost linear for most norms. Advances in Mathematics. 226. 2618–2628.
[7] J. Pach and P.K. Agarwal, Combinatorial geometry. Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995. xiv+354 pp. ISBN: 0-471-58890-3
[8] P. Pudlák, On explicit Ramsey graphs and estimates of the number of sums and products. in: Topics in Discrete Mathematics, eds. Klazar, Kratochvíl, Loebl, Matousek, Thomas and Valtr. Springer 2006, 169–175.

[9] Elisabeth Remm, Michel Goze, Affine structures on abelian Lie groups, Linear Algebra and its Applications, Volume 360, 2003, Pages 215–230.

[10] L. Guth and O. Silier, Sharp Szemerédi-Trotter Constructions in the Plane, arXiv:2112.00306 [math.CO]

[11] E. Szemerédi and W. Trotter, Extremal problems in discrete geometry. Combinatorica 3 (1983) 381–392.

[12] J. Spencer, E. Szemerédi, and W. Trotter, Unit distances in the Euclidean plane. Graph theory and combinatorics (Cambridge, 1983), Academic Press, London, (1984) 293–303.

[13] A. Sheffer and O. Silier, A structural Szemerédi-Trotter Theorem for Cartesian Products, arXiv:2110.09692 [math.CO]

[14] L. Székely, (1997). Crossing Numbers and Hard Erdős Problems in Discrete Geometry. Combinatorics, Probability and Computing, 6(3), 353–358.

[15] P. Valtr, Strictly convex norms allowing many unit distances and related touching questions, manuscript.

[16] J. Zahl, Breaking the 3/2 barrier for unit distances in three dimensions, International Mathematics Research Notices, Volume 2019, Issue 20, 2019, 6235–6284.