Extension of lower and upper solutions approach for generalized nonlinear fractional boundary value problems

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ABSTRACT
Our main concern in this study is to present the generalized results to investigate the existence of solutions to nonlinear fractional boundary value problems (FBVPs) with generalized nonlinear boundary conditions. The framework of the presented results relies on the lower and upper solutions approach which allows us to ensure the existence of solutions in a sector defined by well-ordered coupled lower and upper solutions. It is worth mentioning that the presented results unify the existence criteria of certain problems which were treated on a case-by-case basis in the literature. Two examples are supplied to support the results.

1. Introduction
The idea of fractional calculus was conceived on September 30, 1695, in a series of letters. L’Hôpital asked Leibniz about the notion of the non-integer order of differentiation. Leibniz responded (Leibniz, 1965):

“It appears that one day useful consequences will be drawn from these paradoxes.”

However, the topic which was predicted by Leibniz as a paradox has nowadays evolved to contribute in the various realm of science-related disciplines. For a comprehensive study of derivatives and integrals of arbitrary real orders, their method of solution, and applications in various fields the reader may refer (Podlubny, 1999).

Currently, many researchers are focusing on the analytical and numerical study of fractional differential equations (FDEs) due to their potential to model complicated processes in a more efficient manner. For example in (Baleanu & Agarwal, 2021), a tumour-immune surveillance (TIS) mathematical model has been investigated with Caputo-type fractional operators containing different types of kernels. The growth of the naive tumour cell population has been derived in (Baleanu & Agarwal, 2021) by using the Caputo fractional TIS and ABC fractional TIS models. Moreover in (Baleanu & Agarwal, 2021), the effectiveness of the fractional models has also been analyzed by comparing the numerically simulated results with the experimental data. The fractional modeling of the epidemiological phenomenon is well suited to explain and understand the outbreaks of infectious diseases. In this realm (Atangana, 2021) presented the applicability and usefulness of the fractional operators defined with nonsingular kernels in explaining the spread of infectious diseases. In (Kumar, Ghosh, Samet, & Goufo, 2020), an analysis for heat equations has been presented by using the new Yang-Abdel-Aty-Cattani fractional operator. In (Atangana & Baleanu, 2016), a generalized Mittag–Leffler function has been used to build a new fractional derivative with a non-local and non-singular kernel which is then applied to solve the fractional heat transfer model. Thermodynamics,
polymers, biophysics, and other areas of physics are only a few of the many fields the author has explored in detail in (Hilfer, 2000), hence demonstrating the usefulness of fractional calculus. Some latest results on the applications and usefulness of fractional operators are reported in (Agarwal, Ahmad, & Alsaedi, 2017; Agarwal, Ahmad, et al., 2017; Butt, Agarwal, Yousaf, & Guirao, 2022; Goyal, Agarwal, Parmentier, & Cesano, 2021; Khalil, Khalil, Hashim, & Agarwal, 2021).

An extensive study has also been reported on solving the FDEs numerically. For instance, in (Baleanu, Jleli, Kumar, & Samet, 2020), a numerical algorithm based on the Picard iteration technique has been derived to solve FDEs modeled using the fractional derivative involving two singular kernels. In (Jafari & Tajadodi, 2018), a new numerical method based on B-spline polynomials has been derived to solve partial FDEs. A comparative study of the Caputo fractional Lotka–Volterra population model has been provided in (Kumar, Ghosh, et al., 2020) by using the Haar wavelet and Adams–Bashforth Moulton methods. In (Kumar, Ghosh, et al., 2020), a numerical study of the nonlinear Caputo fractional predator-prey model of two species has been presented by employing the Bernstein wavelet and Euler methods. In (Shah, El-Zahar, Aljoufi, & Chung, 2021), the hybrid technique based on the homotopy perturbation Elzaki transform method has been developed to numerically solve Helmholtz FDEs. In (Jafari, Tajadodi, & Ganji, 2019), the operational matrices of Bernstein polynomials have been derived to find the approximate solution of variable order FDEs. In (Ghanbari, Kumar, & Kumar, 2020), the analytical and numerical study of the fractional immunogenetic tumour model has been presented by using the fixed point approach and the Adams-Bashforth-Moulton approach.

The concept of existence is fundamental to the field of mathematical analysis. Knowing whether or not a certain FDE has a solution can greatly aid in the process of solving it. This intuitive inquiry motivates the researchers toward the development of the existence results for FDEs. For example in a recent paper (Baleanu, Jleli, et al., 2020), the authors provided the new existence results for strong-singular FBVPs under some conditions. In (Baleanu, Etemad, Mohammad, & Rezapour, 2021), the authors derived some new existence results for novel modeling of fractional multi-term boundary value problems by considering the Glucose graph.

The lower and upper solutions (LUSs) approach is one of the strongest methodologies available in the literature for investigating the existence of solutions of nonlinear FBVPs. For an exhaustive analysis of this approach in a more general context, the reader may consult (Asif, Talib, & Tunc, 2015; De Coster & Habets, 2006; Franco & O’Regan, 2003; Talib, Asif, & Tunc, 2015). The reason for using this approach is its compatibility with the monotonicity assumptions that are used to define coupled LUSs which allows us to develop the generalized existence results for which periodic and antiperiodic FBVPs are the special cases. Secondly by using this approach not only the existence of solutions is guaranteed, but also the position of solutions could be determined.

Problems with periodic boundary conditions (PBCs) and antiperiodic boundary conditions (APBCs) have been studied extensively as PBCs and APBCs are widely exercised to model numerous physical phenomena. For instance, in the case of infinitely large systems, PBCs are preferably used for the replication of the systems in a small domain. Since macroscopic systems are massive and molecular simulations are computationally expensive in these scenarios, therefore while performing computer simulations and developing mathematical models, scientists prefer to use periodic or antiperiodic boundary conditions (BCs), see (Aubertin, Henneron, Piriou, Guerin, & Mipo, 2010; Garoz, Gilabert, Sevenois, Spronk, & Van Paepegem, 2019; Jacques, De Baere, & Van Paepegem, 2014; Wu, Owino, Al-Ostaz, & Cai, 2014). For more applications of APBCs, see (Delvos & Knoche, 1999), where the authors studied the interpolation problems by utilizing antiperiodic polynomials. For the reader’s interest, we refer to the articles (Chen, Nieto, & Ó Regan, 2007) and (Shao, 2008) to highlight some applications of APBCs in physics.

The extensive use of PBCs and APBCs motivated the mathematical community toward the development of the existence results for FDEs including these conditions. For instance, in (Qiao & Zhou, 2017), the authors employed the Leray-Schauder degree theory together with the Banach contraction theorem to develop the existence results for FDEs with APBCs. In (Ahmad & Nieto, 2010), the authors proposed the existence results of FDEs with APBCs by using the Leray-Schauder degree theory. Very recently, the authors Ahmed, Kumam, Abubakar, Borisut, & Sithithawornkiet, 2020) developed the existence and uniqueness results for impulsive pantograph FDEs with generalized APBCs by employing Banach and Krasnosel’skii fixed point theorems. Moreover in (Lin, Liu, & Fang, 2012), the authors studied the existence and uniqueness results for FDEs with PBCs by applying LUSs approach and monotone iterative algorithm. The comprehensive survey on FDEs modeled with APBCs has been presented in (Agarwal, Ahmad, et al., 2017).

Recently, qualitative analyses of Caputo fractional delay integro-differential equations, Volterra integral equations, non-linear differential systems of second order and continuous and discrete integro-
differential equations have been investigated and some novel results were obtained (Bohner, Tunç & Tunç, 2021; Chauhan, Singh, Tunç & Tunç, 2022; Tunç and Tunç 2017; Graef, Tunç & Sevli, 2021; Tunç, Atan, Tunç & Yao, 2021; Tunç, Atan, Tunç & Yao, 2021). The concept of generalizations is central to the study of mathematics and occurs frequently in applications. Mathematicians always look for an abstract structure that can be studied independently and that expands to several specific cases. Motivated by the results proposed in (Franco & O'Regan, 2003), we extend the study to the fractional domain. Since our proposed results rely on the theory of LUSs approach, so the implementation of this approach necessitates having information on the Caputo fractional differential operator’s behavior at its extreme points. For this purpose, we consult (Al-Refai, 2012) which provides the information at extreme points for Caputo fractional differential operator. Based on the foregoing studies of periodic and antiperiodic FBVPs, as well as the paramount significance of generalized processes and applications of FDEs in various areas of sciences, we propose unified existence results for nonlinear generalized FBVPs, of which periodic and antiperiodic FBVPs are particular cases.

We consider the problem

\[ D_\gamma^c v(z) = h(z,v(z)), \quad z \in [0,1], \]

(1)

corresponding to generalized nonlinear BCs

\[
\begin{aligned}
&f(v(0),v'(1),v'(0)) = 0, \\
&v(1) + g(v(0)) = 0,
\end{aligned}
\]

(2)

where \( h : [0,1] \times \mathbb{R} \to \mathbb{R}, \) \( f : \mathbb{R}^3 \to \mathbb{R}, \) and \( g : \mathbb{R} \to \mathbb{R} \)

are continuous functions, and \( D_\gamma^c \) is the Caputo fractional derivative (Caputo, 1967; 1969; Podlubny, 1999), which is for \( 1 < \gamma < 2 \) defined as

\[
D_\gamma^c v(z) = \frac{1}{\Gamma(2 - \gamma)} \int_0^z (z - s)^{-\gamma} v'(s) ds,
\]

(3)

where

\[
\frac{1}{\Gamma(2 - \gamma)} \int_0^z (z - s)^{-\gamma} v'(s) ds = \mathcal{P} \int_0^z J^{1-\gamma} v(s)
\]

(4)

is the fractional integral operator of order \( 2 - \gamma \) in Riemann–Liouville sense (Podlubny, 1999).

The problem (1)–(2) generalizes some certain FBVPs, for example if \( f(x_1, x_2, x_3) = x_2 - x_3 \) and \( g(x_1) = -x_1, \)

then (1)–(2) is the periodic FBVPs, with

\[
v'(0) = v'(1), \quad v(0) = v(1).
\]

(5)

If \( f(x_1, x_2, x_3) = x_2 + x_3, \) and \( g(x_1) = x_1, \) then (1)–(2) is the antiperiodic FBVPs, with

\[
v'(0) = -v'(1), \quad v(0) = -v(1).
\]

(6)

The novel aspects of our presented study are the development of the generalized existence results for the problem (1) with generalized nonlinear BCs (2).

The proposed results also unify the existence criteria of the problem (1) with PBCs (5) and APBCs (6) that have been studied separately in (Ahmad & Nieto, 2010; Ahmed et al., 2020; Lin et al., 2012; Qiao & Zhou, 2017) by using various approaches. Moreover, in comparison with the existing content, the problem (1) with BCs (2) has not been studied so far by employing LUSs approach.

The paper is organized as follows. Some essential definitions can be found in section 2. The Extremum principle for Caputo fractional derivative is stated and proved in Section 3, which is necessary to apply LUSs approach. A generalized existence result is stated and proved in Section 4. In Section 5, two examples have been included to demonstrate the applicability of the developed theoretical results. Conclusion is given in Section 6.

2. Preliminary results

In this section, some definitions are recalled that are essential to implement the LUSs approach.

**Definition 1.** A function \( \zeta \in C^2[0,1] \) is a lower solution of (1), if it satisfies

\[
D_\gamma^c \zeta(z) \geq h(z, \zeta(z)), \quad z \in [0,1].
\]

(7)

Similarly, a function \( \eta \in C^2[0,1] \) is an upper solution of (1), if it satisfies

\[
D_\gamma^c \eta(z) \leq h(z, \eta(z)), \quad z \in [0,1].
\]

(8)

Following the above, the assumption is

\[
\zeta(z) \leq \eta(z), \quad z \in [0,1].
\]

(9)

For \( v_1, v_2 \in C^1[0,1] \) with \( v_1(z) \leq v_2(z) \) for all \( z \in [0,1] \), the following set can be defined as

\[
[v_1, v_2] = \{ u \in C^1[0,1] : v_1(z) \leq u(z) \leq v_2(z), \quad \text{for all } z \in [0,1] \}.
\]

**Definition 2.** The well ordered functions, \( \zeta, \eta \in C^2[0,1] \) are said to be coupled LUSs for the problem (1)–(2), if they satisfy the inequalities (7)–(9) along with the following set of inequalities:

\[
\begin{aligned}
&\min \{ f(\zeta(0), \zeta'(0), \zeta'(1)), f(\zeta(0), \zeta'(0), \eta'(1)) \} \geq 0, \\
&\max \{ f(\eta(0), \eta'(0), \eta'(1)), f(\eta(0), \eta'(0), \zeta'(1)) \} \leq 0,
\end{aligned}
\]

(10)

and

\[
\begin{aligned}
&\zeta(1) + \max \{ g(\eta(0)), g(\zeta(0)) \} = 0, \\
&\eta(1) + \min \{ g(\zeta(0)), g(\eta(0)) \} = 0.
\end{aligned}
\]

(11)

3. Extremum result in Caputo sense

The extremum results for Riemann–Liouville and Caputo fractional-order derivatives have been
presented by some researchers. These results are very useful for the researchers who are interested in extending the LUSs approach for FBVPs. The first notable results regarding the fractional derivative at extreme points in Caputo sense for \(0 < \gamma < 1\) were presented in (Luchko, 2009) and in Riemann–Liouville sense for \(0 < \gamma < 1\) were presented in (Al-Refaie, 2012).

These results were further extended in (Al-Refaie, 2012; Shi & Zhang, 2009) for Caputo fractional derivative, when \(1 < \gamma < 2\).

By following the results studied in (Al-Refaie, 2012), we present the following extremum result to extend the LUSs approach for the problem (1) with BCs.

**Theorem 3.1.** Let \(v \in C^2[0,1]\) attains its maximum at \(z_0 \in (0,1)\), then

\[
D^\gamma_{v}(z_0) \leq \frac{z_0^{\gamma}}{\Gamma(2-\gamma)} \left[\left(\gamma - 1\right) \left(v(0) - v(z_0)\right) - z_0v'(0)\right],
\]

for all \(1 < \gamma < 2\).

**Proof.** We define an auxiliary function \(r(z) = v(z_0) - v(z), z \in [0,1]\). The followings are true for \(r(z)\) in \([0,1]\)

\[
\begin{align*}
    r(z) & \geq 0, \\
    r(z_0) = r'(z_0) = 0, \\
    r''(z_0) \leq 0, \\
    D^\gamma_{v}(z) = -D^\gamma_{v}(z).
\end{align*}
\]

Since

\[
D^\gamma_{v}(z_0) = \frac{1}{\Gamma(2-\gamma)} \left[\frac{z_0}{0} (z_0 - s)^{1-\gamma} r''(s) ds. \right. \tag{12}
\]

Integrating (12), and using \(D^\gamma_{v}(z) = -D^\gamma_{v}(z), z \in [0,1]\), we have

\[
\Gamma(2-\gamma)D^\gamma_{v}(z_0) = -(z_0 - s)^{1-\gamma} r'(s) ds \bigg|_{s}^{z_0} + \left(\gamma - 1\right) \left(z_0 - s\right)^{-\gamma} r'(s) ds.
\]

Since \(r'(z_0) = 0\), and \(r''(z_0)\) is bounded. So, there exists \(v_1(z) \in C[0,1]\), such that, \(r'(z) = (z_0 - z)v_1(z)\). So for \(1 < \gamma < 2\), we have

\[
\lim_{z \to z_0} \frac{r'(z)}{(z_0 - z)^{-\gamma} - 1} = \lim_{z \to z_0} \frac{(z_0 - z)v_1(z)}{(z_0 - z)^{2-\gamma}} = \lim_{z \to z_0} (z_0 - z)^{2-\gamma} v_1(z) = 0. \tag{13}
\]

Hence,

\[
\Gamma(2-\gamma)D^\gamma_{v}(z_0) = \left(z_0 - z\right)^{-\gamma} r'(0) + \left(\gamma - 1\right) \left(z_0 - s\right)^{-\gamma} r'(s) ds. \tag{14}
\]

Since \(r(z_0) = r'(z_0) = 0\) and \(r''(z_0)\) is bounded, there exists \(v_2(z) \in C[0,1]\), such that, \(r(z) = (z_0 - z)^2 v_2(z)\). Therefore

\[
\int_{0}^{z_0} \left(z_0 - s\right)^{-\gamma - 1} r(s) ds = \int_{0}^{z_0} \left(z_0 - s\right)^{-\gamma + 1} v_2(z) ds, \tag{15}
\]

is bounded, and

\[
\lim_{z \to z_0} \frac{r(z)}{(z_0 - z)^{\gamma}} = \lim_{z \to z_0} \frac{(z_0 - z)^2 v_2(z)}{(z_0 - z)^{\gamma}} = \lim_{z \to z_0} (z_0 - z)^{2-\gamma} v_2(z) = 0. \tag{16}
\]

Now integrating and using (15) and (16) together with \(r(z) \geq 0\) on \([0,1]\), we have

\[
\Gamma(2-\gamma)D^\gamma_{v}(z_0) = (z_0)^{-\gamma} r'(0) - (\gamma - 1)(z_0)^{-\gamma} r(0) - \gamma(\gamma - 1) \left(z_0 - s\right)^{-\gamma} r'(s) ds
\]

\[
\leq (z_0)^{-\gamma} r'(0) - (\gamma - 1)(z_0)^{-\gamma} r(0)
\]

\[
= \frac{z_0^{\gamma - 1}}{\Gamma(2-\gamma)} \left[(\gamma - 1) \left(v(0) - v(z_0)\right) - z_0v'(0)\right].
\]

Therefore, the proof is completed. \(\square\)

**Corollary 3.2.** Let \(v \in C^2[0,1]\) attains its maximum at \(z_0 \in (0,1)\), and \(v'(0) \geq 0\). Then \(D^\gamma_{v}(z_0) \leq 0\), for all \(1 < \gamma < 2\).

**Proof.** In the light of Theorem 3.1

\[
D^\gamma_{v}(z_0) \leq \frac{z_0^{\gamma - 1}}{\Gamma(2-\gamma)} \left[(\gamma - 1) \left(v(0) - v(z_0)\right) - z_0v'(0)\right],
\]

holds. Since \(v(0) \geq v(z)_0\) for \(z_0 > 0\), implies \(v(0) - v(z_0) < 0\), and \(v'(0) \geq 0\), that further implies that \([v(0) - v(z_0)] - z_0v'(0) < 0\), so

\[
D^\gamma_{v}(z_0) \leq \frac{z_0^{\gamma - 1}}{\Gamma(2-\gamma)} \left[(\gamma - 1) \left(v(0) - v(z_0)\right) - z_0v'(0)\right] \leq 0.
\]

Consequently, \(D^\gamma_{v}(z_0) \leq 0\) for all \(1 < \gamma < 2\). \(\square\)

4. Generalized existence result

In this section, we develop the generalized results to ensure the existence of solutions of the problem (1) with BCs (2, 5) and (6).

**Theorem 4.1.** Suppose that \(ζ, η \in C^2[0,1]\) are the coupled LUSs of the problem (1)-(2). Assume that boundary function \(f\) is monotone nondecreasing in the second variable. Furthermore suppose that the function \(g\) is
continuity of constant functions and that the following functions have got the same kind of monotonicity as $g$.

$$f_1(z) := f(\zeta(0), \zeta'(0), z), f_2(z) := f(\eta(0), \eta'(0), z), z \in \mathbb{R},$$

Then the problem (1)-(2) has at least one solution, such that

$$\zeta(z) \leq \nu(z) \leq \eta(z), \quad z \in [0, 1].$$

**Proof.** We introduce the following function in order to define an appropriate modified problem

$$b(z, x_1) := \max\{\zeta(z), \min(x_1, \eta(z))\} \tag{17},$$

then the modified problem is

$$\begin{cases}
D_\zeta \nu(z) = \lambda \nu(z) = H^r(z, \nu(z)), \quad z \in [0, 1], \quad \lambda > 0,

\nu(0) = f^*(\nu(0), \nu'(0), \nu'(1)),

\nu(1) + g^*(\nu(0)) = 0,
\end{cases} \tag{18}$$

where

$$H^r(z, \nu(z)) = \begin{cases}
h(z, \eta(z)) - \lambda \eta(z), \quad \text{if } \eta(z) < \nu(z),

h(z, \nu(z)) - \lambda \nu(z), \quad \text{if } \zeta(z) \leq \nu(z) \leq \eta(z),

h(z, \zeta(z)) - \lambda \zeta(z), \quad \text{if } \nu(z) < \zeta(z),
\end{cases} \tag{19}$$

and

$$\begin{cases}
f^*(x_1, x_2, x_3) = b(0, x_1 + f(x_1, x_2, x_3)),

g^*(x_1) = g(b(0, x_1)). \tag{20}
\end{cases}$$

Since (18) is the modified problem of (1)-(2), so the solution of (18) leads to the solution of (1)-(2) between $\zeta$ and $\eta$. We will divide the proof in steps for the sake of convenience.

**Step 1:** The solution of (18) is equivalent to find the fixed points of the operator, $M^{-1}P: C^1[0, 1] \to C^1[0, 1]$ that is defined by the composition of the following mappings:

$$\begin{cases}
M : C^1[0, 1] \to C[0, 1] \times \mathbb{R} \times \mathbb{R},

\text{and}

P : C^1[0, 1] \to C[0, 1] \times \mathbb{R} \times \mathbb{R},
\end{cases}$$

defined as

$$\begin{cases}
[M \nu](z) = (\nu'(z) - \nu'(0) - (\lambda \nu(s), \nu(0), \nu(1)),

[P \nu](z) = (I_{z} \nu(0), \nu(z), \nu'(0), \nu'(1)), - g^*(\nu(0))). \tag{21}
\end{cases}$$

Since $h$ is bounded and continuous on $[0, 1] \times \mathbb{R}$ which implies the uniform continuity of $H^r$. Likewise the continuity of $[P \nu]$ on $[0, 1]$ is ensured due to the continuity of constant functions $f^*$ and $g^*$ and Riemann-Liouville fractional integral. Now according to Arzelâ(role)-Ascoli theorem, the class $\{P \nu : \nu \in C[0, 1]\}$ is relatively compact being equicontinuous and uniformly bounded. Furthermore $M^{-1}$ exists and is continuous therefore $M^{-1}P$ is continuous and compact, so Schauder fixed point theorem guarantees the existence of at least one fixed point. As a result, the solution of the problem (18) is the fixed point of the operator $M^{-1}P$.

**Step 2:** Now we will show that, if $\nu(z) \in C^1[0, 1]$ is a solution of (18), then it must lies in a sector defined by the well-ordered coupled LUSs, such that $\zeta(z) \leq \nu(z) \leq \eta(z), \quad z \in [0, 1]$. By the definition of $f^*$, we notice that $\nu(0) \in [\zeta(0), \eta(0)]$. So if $g$ is nondecreasing we have by the condition (11)

$$\zeta(1) = -g(\eta(0)) \leq -g(\nu(0)) = \nu(1) \leq -g(\zeta(0)) = \eta(1),$$

and if $g$ is nonincreasing

$$\zeta(1) = -g(\zeta(0)) \leq -g(\nu(0)) = \nu(1) \leq -g(\eta(0)) = \eta(1),$$

which shows that $\nu(1) \in [\zeta(1), \eta(1)].$

Now Our claim is, $\nu(z) \leq \eta(z)$ for all $z \in [0, 1]$. On contrary, we suppose that $\nu(z) \notin \eta(z)$, then $\nu - \eta$ attains a positive maximum at some $z_0 \in [0, 1]$ (by definitions of $f^*$ and $g^*$, we get that $z_0 \in (0, 1)$). Then $\nu - \eta(z_0) = 0$. Hence, $D_\zeta(\nu - \eta)(z_0) \leq 0$ in the light of Corollary 3.2. However, it provides a contradiction

$$0 \geq D_\zeta(\nu - \eta)(z_0) \geq H^r(z_0, \nu(z_0)) + \lambda \nu(z_0) - h(z_0, \eta(z_0)) = h(z_0, \eta(z_0)) - \lambda \eta(z_0) + \lambda \nu(z_0) - h(z_0, \eta(z_0)) = \lambda(\nu(z_0) - \eta(z_0)) > 0.$$

Consequently, $\nu(z) \leq \eta(z)$ for all $z \in [0, 1]$. Similarly, it can be shown that $\zeta \leq \nu$ on $[0, 1]$.

**Step 3:** If $\nu$ is a solution of the modified problem then it must satisfy the boundary conditions (2). From step 2 and by using the definition of $g^*$, it is sufficient to show that,

$$\zeta(0) \leq \nu(0) + f(\nu(0), \nu'(0), \nu'(1)) \leq \eta(0). \tag{22}$$

Since

$$\nu(0) = f^*(\nu(0), \nu'(0), \nu'(1)) = \nu(0) + f(\nu(0), \nu'(0), \nu'(1)).$$

On contrary, assume that

$$\nu(0) + f(\nu(0), \nu'(0), \nu'(1)) < \zeta(0), \tag{23}$$

then using (17), we have

$$\nu(0) = f^*(\nu(0), \nu'(0), \nu'(1)) = \zeta(0). \tag{24}$$

If $g$ is nondecreasing (by the given hypothesis, $f_1$ is also nondecreasing), so we have

$$\nu(1) = -g(\nu(0)) = -g(\zeta(0)) = \eta(1). \tag{25}$$

Using (24, 25) and step 2, we get

$$\nu'(0) \geq \zeta'(0), \quad \nu'(1) \geq \eta'(1),$$

it provides a contradiction as
\[ v(0) + f(v(0), v'(0), v'(1)) = \zeta(0) + f(\zeta(0), v'(0), v'(1)) \]
\[ \geq \zeta(0) + f(\zeta(0), \zeta'(0), v'(1)) = \zeta(0) + f(\zeta(0), v'(1)) \]
\[ \geq \zeta(0) + f(\zeta'(1)) = \zeta(0) + f(\zeta(0), \zeta'(0), \eta'(1)) \geq \zeta(0). \]

On the other hand, if \( g \) is nonincreasing, we have
\[ v(1) = -g(v(0)) = -g(\zeta(0)) = \zeta(1), \]
and this along with second step and \((24)\) implies
\[ v'(0) \geq \zeta'(0), \zeta'(1) \geq v'(1), \]
and this along with second step and \((24)\) implies
\[ v'(0) \leq \zeta'(0), \zeta'(1) \geq v'(1), \]
and we would use the boundary function \( f_1 \) to prove that \( v(0) + f(v(0), v'(0), v'(1)) \leq \eta(0). \)

Consequently, the problem \((1)-(2)\) has at least one solution, such that
\[ \zeta(z) \leq v(z) \leq \eta(z), \quad z \in [0, 1]. \]
This completes the proof. \( \square \)

5. Example

In this section, two examples are included to support the theoretical results.

Example 1. Consider the nonlinear FBVPs
\[ D^3_C v(z) = v^3(z) - \cos^2(\pi z), \quad z \in [0, 1], \]
with nonlinear BCs
\( f(\nu(0), \nu'(0), \nu'(1)) = (\nu'(1)\nu'(0))^2 - \nu(0), \)
\[ v(1) + g(v(0)) = v(1) \sin(\pi) - v(0) \tan(\pi). \]
The functions given by
\[ \zeta(z) = -2z, \text{ and } \eta(z) = z^2 + 2, \]
are the lower and upper solutions of \((30),\) satisfying \((7)\) and \((8),\)
as
\[ D^3_C \zeta(z) = 0 \geq h(z, \zeta(z)) = -8z^3 - \cos^2(\pi z), \quad z \in [0, 1], \]
\[ D^3_C \eta(z) = 2.256578334z^3 \leq h(z, \eta(z)) = (z^2 + 2) - \cos^2(\pi z), \quad z \in [0, 1]. \]
Moreover, the functions \(-2z\) and \(z^2 + 2\) are coupled LUSs of \((30)-(31),\) satisfying \((10)-(11)\) because if the boundary function \( f \) is monotone nondecreasing in the third variable then the following set of inequalities are satisfied.

\[ f(\eta(0), \eta'(0), \eta'(1)) \leq 0, \]
\[ f(\zeta(0), \zeta'(0), \zeta'(1)) \geq 0. \]

Similarly, if the boundary function \( g \) is monotone nonincreasing then the following equalities are satisfied.
\[ f(\zeta(0), \zeta'(0), \zeta'(1)) = 0, \]
\[ \eta(1) + g(\eta(0)) = 0. \]

Also if boundary function \( f \) is monotone nonincreasing in the third variable then the following set of inequalities are satisfied.
\[ f(\zeta(0), \zeta'(0), \zeta'(1)) \leq 0, \]
\[ f(\zeta(0), \zeta'(0), \zeta'(1)) \geq 0. \]

Also the functions
\[ f_1(z) := f(\zeta(0), \zeta'(0), \zeta'(1)), \]
\[ f_\eta(z) := f(\eta(0), \eta'(0), \eta'(1)), \]
are monotone on \([\zeta(0), \eta(0)].\) As all the assumptions of the Theorem 4.1 are satisfied. So, the problem \((30)-(31)\) has at least one solution, such that \( \zeta(z) \leq v(z) \leq \eta(z), \) for all \( z \in [0, 1]. \)

Example 2. Consider the nonlinear FBVPs
\[ D^3_C v(z) = 5v(z) + v^3(z) - \sin^2(\pi z), \quad z \in [0, 1], \]
with nonlinear BCs
\( f(\nu(0), \nu'(0), \nu'(1)) = (\nu'(1))^2 - (\nu'(0))^2, \)
\[ v(1) + g(v(0)) = (\nu(1))^2 - (\nu(0))^2. \]
The functions given by
\[ \zeta(z) = -5\lambda, \text{ and } \eta(z) = 5\lambda, \quad \lambda > 0, \]
are the lower and upper solutions of the problem \((33),\) satisfying \((7)\) and \((8),\)
as
\[ D^3_C \zeta(z) = 0 \geq h(z, \zeta(z)) = -25\lambda - 125(\lambda)^3 - \sin^2(\pi z), \quad z \in [0, 1], \]
\[ D^3_C \eta(z) = 0 \leq h(z, \eta(z)) = 25\lambda + 125(\lambda)^3 - \sin^2(\pi z), \quad z \in [0, 1]. \]
Moreover, the functions \(-5\lambda\) and \(5\lambda\) are coupled LUSs of \((33)-(34),\) satisfying \((10)-(11)\) because if the boundary function \( f \) is monotone nondecreasing in the third variable then the following set of inequalities are satisfied. 

\[ D^3_C \zeta(z) = 0 \geq h(z, \zeta(z)) = -25\lambda - 125(\lambda)^3 - \sin^2(\pi z), \quad z \in [0, 1], \]
\[ D^3_C \eta(z) = 0 \leq h(z, \eta(z)) = 25\lambda + 125(\lambda)^3 - \sin^2(\pi z), \quad z \in [0, 1]. \]
that the Theorem 4.1 are verified. Consequently, of inequalities are satisfied.

\[
\begin{align*}
& f(\eta(0), \eta'(0), \eta'(1)) \leq 0, \\
& f(\zeta(0), \zeta'(0), \zeta'(1)) \geq 0.
\end{align*}
\]

Similarly, if the boundary function \(g\) is monotone nondecreasing then the following set of inequalities are satisfied.

\[
\begin{align*}
& f(\eta(0), \eta'(0), \eta'(1)) \leq 0, \\
& f(\zeta(0), \zeta'(0), \zeta'(1)) \geq 0.
\end{align*}
\]

Also if boundary function \(f\) is monotone nonincreasing in the third variable then the following set of inequalities are satisfied.

\[
\begin{align*}
& \zeta(1) + g(\zeta(0)) = 0, \\
& \eta(1) + g(\eta(0)) = 0.
\end{align*}
\]

Also the functions

\[
f_1(z) := f(\zeta(0), \zeta'(0), z), \\
f_2(z) := f(\eta(0), \eta'(0), z)
\]

are monotone on \([\zeta(0), \eta(0)]\). Hence all the assumptions of the Theorem 4.1 are verified. Consequently, the problem (33)–(34) has at least one solution, such that \(\zeta(z) \leq \nu(z) \leq \eta(z)\), for all \(z \in [0, 1]\).

6. Conclusion

In the present study, the applicability of LUSs approach is extended to generalized nonlinear FBVPs. We proposed the generalized results that unify the existence criteria of certain FBVPs that have been studied separately in the literature. For instance, the periodic and antiperiodic FBVPs are special cases. We found LUSs approach more effective due to the localization of the solution in a sector defined by the well-ordered functions named as lower and upper solutions. The developed theoretical results have been verified by providing two examples. In the future our aim is to extend the proposed study for nonlinear FBVPs involving Hilfer, ABC, and Caputo generalized fractional differential operators studied Talib & Bohner, 2022. In addition, we sought numerical solutions to Caputo FBVPs involving the generalized BCs (2) by applying the spectral methods that are based on the operational matrices of orthogonal polynomials and approximate the solution as the basis of orthogonal polynomials.

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Data availability statement

Our manuscript has no associated data.

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