Spiky membranes

Maciej Trzetrzelewski *

Institute of Physics,
Jagiellonian University,
Reymonta 4, 30-059 Kraków,
Poland

October 20, 2009

Abstract

We study spiky configurations of membranes in the $SO(d) \times SU(N)$
invariant matrix models. A class of exact solutions (analogous to
plane-waves) of the corresponding Schrödinger equation for an arbitrary $N$ is discussed. If the large $N$ limit is performed so that the
energy scales like $N^2$, the $N = \infty$ wavefunctions reduce to the ground
state of the $d$-dimensional harmonic oscillator.

*e-mail: 33lewski@th.if.uj.edu.pl
1 Introduction

The understanding of the large $N$ behavior of the $SO(d) \times SU(N)$ invariant matrix models [1] and its supersymmetric version [2] is one of the most important problems in the quantum (super)membrane theory. Ideally, one would like to solve the system for finite $N$ exactly and then make the $N \to \infty$ limit according to the prescription given by Hoppe [1]. In that limit the system is equivalent to the light-cone description of the quantum (super)membrane in $D = d + 2$ dimensional Minkowski space. Unfortunately, after almost three decades from the formulation of the problem, there are no known (normalizable) solutions even for $N = 2$ although there is a number of results concerning the properties of the ground state wavefunction in the supersymmetric case [3]. This should not be a surprise since already at the classical level, membrane equations of motion turn out to be very difficult due to the interacting term appearing in e.g. the Hamiltonian formulation of the theory. There is however a distinguished configuration of a membrane when the theory is in fact tractable. This is the case when membrane develops a spiky configuration i.e. an infinitely thin tube outgoing from the membrane surface [4]. If one focuses on these kind of “stringy” configurations then the theory turns out to be solvable.

The idea that string theory might correspond to a specific sector of the whole membrane theory is not new [5]. In this paper we investigate a related possibility i.e. that the wavefunctions of spikes could correspond to excitations of the open string. To do so we use the matrix formulation of the quantum membrane, concentrate on spiky configurations, solve the corresponding Schrödinger equation and then perform the large $N$ limit of the wave functions. We analyze specific solutions for which the final result is surprisingly simple - if the energy scales like $N^2$ the wave function becomes the Gauss function. Accordingly, this function should correspond to the ground state wavefunction of open string excitations. We also make some remarks how the excited states could appear in this context.

2 Spiky configurations

The Dirac membrane [6] in $D$ dimensional Minkowski spacetime is given by the action functional

$$S = \int d^3 \phi \sqrt{|G|}, \quad G_{\alpha \beta} = \partial_\alpha X^\mu \partial_\beta X_\mu, \quad D = 0, 1, \ldots, D - 1$$
where $\phi^\alpha, \alpha = 0, 1, 2$ and $G$ are the internal coordinates and the determinant of the induced metric of the membrane world-volume respectively. A particularly useful formulation of the theory is obtained in the light-cone coordinates

$$
\tau := \frac{1}{2}(X_0 + X_D), \quad \zeta := X_0 - X_D, \quad X^s = X^s(\tau, \phi^1, \phi^2),
$$

$$
s = 1, \ldots, D - 2
$$

(we use the notation as in \cite{7}) after imposing the light-cone gauge $\tau = \phi^0$. In that gauge one finds that the Hamiltonian of the theory is

$$
\mathcal{H} = \frac{1}{2\eta} \int d\phi^1 d\phi^2 \frac{p_s^2 + g}{\rho} \quad (1)
$$

where $p_s$ are the canonical momenta conjugated to $X^s$, $g$ is the determinant of $g_{ab} := \partial_a X^s \partial_b X^s$ ($a, b = 1, 2$), $\eta$ is a constant and $\rho = \rho(\phi^1, \phi^2)$ is a $\tau$ independent scalar density emerging from the EOM involving $\zeta$ (for details see \cite{7}). Furthermore, using the remaining reparametrization invariance one can fix an additional gauge $G_{0a} = 0$ which implies a consistency condition

$$
\epsilon^{ab} \partial_a (p_s / \rho) \partial_b X^s = 0. \quad (2)
$$

In that setup one can successfully quantize the theory by first expanding $X^s$ and $p_s / \rho$ in terms of modes of the surface of the membrane and second by introducing a cutoff in the mode expansion \cite{1}. At the end the resulting system can be viewed as a the $D - 2$ dimensional quantum-mechanics with additional matrix degrees of freedom given by

$$
H \psi = E \psi, \quad G_A \psi = 0,
$$

where

$$
H = -\frac{1}{2} \partial^2_{sA} + \frac{1}{4} (f^{(N)}_{ABC} x_{Bs} x_{Cs})^2, \quad G_A = i f^{(N)}_{ABC} x_{Bs} \partial_{Cs},
$$

$$
A, B, C = 1, \ldots, N^2 - 1. \quad (3)
$$

Here the indices $A, B, C$ correspond to the adjoint representation of the $su(N)$ algebra with the structure constants $f^{(N)}_{ABC}$. The hamiltonian $H$ and the $SU(N)$ singlet constraint $G_A \psi = 0$ are a regularized counterparts of (1) and (2) respectively. The removal of the cutoff results in the large $N$ limit of the theory which has to be appropriately taken according to the prescription given in \cite{1}.  

3
Looking at the hamiltonian (1) one notices that there is a special region in spacetime where the membrane degenerates i.e. the determinant of the induced metrics $g_{ab}$ vanishes. Since $G_{a0} = 0$ it follows that $G = det G_{ab}$ is also zero and hence the Riemann curvature of the world-volume of the membrane diverges. Geometrically this corresponds to one dimensional lines (spikes, strings) in $\mathbb{R}^{D-1}$ extending from the membrane surface [4].

At quantum level these configurations are realized by the vanishing of the potential term in (3) therefore we deduce that the relevant Schrödinger operator describing spikes (and only them) is the $(D-2)(N^2-1)$ dimensional Laplace operator accompanied by the singlet constraint

$$H = -\frac{1}{2}\partial_{sA}^2, \quad G_A = 0.$$  \hfill (4)

The solutions of the above system and their large $N$ limit is the main goal of this paper.

### 3 Finite $N$ solutions

The singlet constraint can be automatically solved by introducing the matrix variables $X_s = T_A^{(N)} x_{As}$ where $T_A^{(N)}$ are the fundamental representation of $su(N)$ satisfying

$$[T_A^{(N)}, T_B^{(N)}] := i f_{ABC}^{(N)} T_C^{(N)},$$

(we use the normalization $Tr(T_A^{(N)} T_B^{(N)}) = \delta_{AB}$). It follows that any function depending on the traces $Tr(X_1 X_2 \ldots)$ satisfies the constraint. Such functions span the entire set of $su(N)$ invariant functions however they are not independent due to the Cayley-Hamilton theorem. Moreover since the Hamiltonian (4) is separable we find it useful to concentrate on the solutions of the form

$$\Psi(x) = \psi_1(X_1) \ldots \psi_{D-2}(X_{D-2})$$

so that the problem reduces to solving a single $N^2 - 1$ dimensional Laplace equation in terms of trace dependent functions

$$-\partial_A^2 \psi = E\psi,$$

$$\psi = \psi(Tr(X^2), Tr(X^3), \ldots, Tr(X^N)), \quad X = x_A T_A. \hfill (5)$$

4
We now assume that the wave function is regular at the origin implying that \( \psi \) can be expanded as
\[
\psi(x) = \sum_{i_2, \ldots, i_N=0}^{\infty} c_{i_2 \ldots i_N} Tr(X^2)^{i_2} \ldots Tr(X^N)^{i_N}.
\]

It is possible to find a class of solutions of this type in terms of bilinear traces \( Tr(X^2) \). We have [8]
\[
\psi_k(X) = \frac{1}{kr} \sin_{N^2-4}(kr), \quad r = \sqrt{Tr(X^2)}, \quad -\Delta \psi_k(X) = k^2 \psi_k(X),
\]
with
\[
\sin_t(y) := \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{1 \cdot 2(3 + t)4(5 + t) \ldots 2k(2k + 1 + t)}
= \frac{y}{1 + t} \ {}_1F_0 \left( \frac{t + 3}{2}, -\frac{y^2}{4} \right),
\]
where \( \ {}_1F_0 \) is the hypergeometric function which, in this case, can be expressed in terms of the Gamma function and the Bessel function of the first kind
\[
\psi_k(X) = \left( \frac{2}{kr} \right)^{\frac{N^2-3}{2}} \Gamma \left( \frac{N^2 - 1}{2} \right) J_{\frac{N^2-3}{2}}(kr).
\]

The generalization of these solutions is done by introducing homogeneous polynomials \( P_{i_3, \ldots, i_N} (Tr(X^2), \ldots, Tr(X^N)) \) of order \( \sum_{k=3}^{N} ki_k \) such that

\[
\Delta P_{i_3, \ldots, i_N} = 0, \quad P_{i_3, \ldots, i_N} = Tr(X^3)^{i_3} \ldots Tr(X^N)^{i_N} + O(1/N)W(X)
\]
where \( W(X) \) is a polynomial depending on traces \( Tr(X^k)^j \) chosen s.t. \( \Delta P_{i_3, \ldots, i_N} = 0 \) [1]. A special case namely \( P_{0,\ldots,1,\ldots,0} \) was discussed in [8].

Now, the wave functions become
\[
\psi_{i,k}(X) = P_i(X) \frac{1}{kr} \sin_{N^2-4 + 2\sum_{k=2}^{N} ki_k}(kr)
= \left( \frac{2}{kr} \right)^{\frac{N^2-3}{2}} \Gamma \left( \frac{t + 3}{2} \right) J_{\frac{t+1}{2}}(kr), \quad t = N^2 - 4 + 2 \sum_{k=2}^{N} ki_k,
\]
\[
- \Delta \psi_{i,k}(X) = k^2 \psi_{i,k}(X), \quad i = (i_3, \ldots, i_N). \quad (6)
\]

\(^1\)The coefficients of the polynomial \( W(X) \) are at most \( O(1) \). By writing \( O(1/N)W(X) \) we indicate the fact that among all the coefficients in \( P_{i_3, \ldots, i_N} \) the coefficient next to \( Tr(X^3)^{i_3} \ldots Tr(X^N)^{i_N} \) is a leading one in the large \( N \) limit.
3.1 Plane-wave properties

Among solutions (6), the one given by \( \psi_0(k,X) \) plays a distinguished role (we use a shortcut notation: \( 0 = (0, \ldots, 0) \)). To see this let us calculate the scalar product

\[
(\psi_{0,k}, \psi_{0,k'}) := \int [dX] \psi_{0,k}(X) \psi_{0,k'}(X), \quad [dX] := \prod_{A=1}^{N^2-1} \frac{dx_A}{\sqrt{\pi}}
\]

where the measure is such that the Gauss function \( g(X) = \exp(-\frac{1}{2}Tr(X^2)) \) is normalized: \( (g,g) = 1 \). Going into the spherical coordinates in \( \mathbb{R}^{N^2-1} \) we find that

\[
(\psi_{0,k}, \psi_{0,k'}) = \left( \frac{2}{k} \right)^{N^2-2} \Gamma\left( \frac{N^2-1}{2} \right) \delta(k - k')
\]

(where we used the orthogonality of Bessel functions) which implies that the functions

\[
\tilde{\psi}_{0,k}(X) := \sqrt{\frac{1}{\Gamma\left( \frac{N^2-1}{2} \right)}} \left( \frac{k}{2} \right)^{N^2-2} \psi_{0,k}(X)
\]

are normalized to the Dirac delta: \( (\tilde{\psi}_{0,k}, \tilde{\psi}_{0,k'}) = \delta(k - k') \). Moreover these functions satisfy a completeness-like identity

\[
\int dk \tilde{\psi}_{0,k}(X) \tilde{\psi}_{0,k}(X') = \frac{1}{2^{r/N^2-4}} \Gamma\left( \frac{N^2-1}{2} \right) \delta(r - r').
\]

Let us also note that for large \( r \) we have

\[
\psi_0(k,X) = \frac{1}{\sqrt{\pi}} \left( \frac{2}{kr} \right)^{\frac{N^2-2}{2}} \Gamma\left( \frac{N^2-1}{2} \right) \cos\left( kr - \frac{\pi}{4} (N^2 - 4) \right).
\]

Because of above properties, functions \( \tilde{\psi}_0(k,X) \) behave similarly to plane-waves which is what one expects from the solutions of the system (4).

On the contrary the solutions (6) with \( i \neq 0 \) do not possess the anticipated property of normalization to the Dirac delta since they are not bounded. However they might turn out to be important in the \( N \to \infty \) limit (see section 4).
3.2 Some properties of polynomials \( P_i \)

The polynomials \( P_{i_3, \ldots, i_N}(X) \) play a role analogous to Hermite polynomials. To see this let us consider the supersymmetric harmonic oscillator for \( SU(N) \) group

\[
H_{osc.} = a_A^\dagger a_A + f_A^\dagger f_A = \frac{1}{2}(p_A p_A + x_A x_A) - \frac{1}{2}(N^2 - 1) + f_A^\dagger f_A,
\]

where \( a_A^\dagger \) and \( f_A^\dagger \) are bosonic and fermionic creation operators. Since the hamiltonian is simply the operator of the number of quanta, the vacuum of the system is the Fock vacuum \( |0\rangle \). We now consider solutions of the following form

\[
\psi(X) = P_{i_3, \ldots, i_N}(X) \sum_{k=0}^{\infty} c_k Tr(X^2)^k e^{-\frac{1}{2} Tr(X^2)}.
\]

The eigenvalue equation \( H_{osc.} \psi = E\psi \) implies the recurrence equation

\[
c_{n+1} = \frac{2n + \left( \sum_{k=3}^{N} ki_k \right) - E}{(n + 1)(2n + N^2 - 1 + 2 \sum_{k=3}^{N} ki_k)} c_n
\]

(where we used \( \Delta P_i = 0 \) and \( \partial_A P_i = (\sum_k ki_k)P_i \) which can be easily solved. In order to make the wave function \( \psi \) square-integrable we impose the condition \( c_{k>n} = 0 \). This gives the eigenvalues \( E = \sum_{k=2}^{N} ki_k \) corresponding to the wave functions

\[
\psi_{i_2, i_3, \ldots, i_N}(X) = P_{i_3, \ldots, i_N}(X) \mathcal{H}_{i_2}(X) e^{-\frac{1}{2} Tr(X^2)},
\]

\[
\mathcal{H}_{i_2}(X) = \sum_{k=0}^{i_2} c_k Tr(X^2)^k.
\]

Solutions (8) are linearly independent moreover, although we started with the ansatz (7), these eigenfunctions already span the entire space of solutions. To see this we use the Fock space argument.

The entire (bosonic) Fock space is spanned by the states

\[ |i_2, i_3, \ldots, i_N\rangle = Tr(a_{i_2}^\dagger a_{i_3}^\dagger \ldots Tr(a_{i_N}^\dagger)^{i_N} |0\rangle, \quad a_A^\dagger := a_A^\dagger T_A^{(N)} \]

which are all independent eigenstates of \( H_{osc.} \) with the eigenvalue equal \( \sum_i ki_k \). It follows that there are as many eigenstates with given number of quanta \( n_B \) as there are natural solutions to the equation
2i_2 + 3i_3 + \ldots + Ni_N = n_B. Let us call this number \( q(n_B) \). We see that there are exactly \( q(n_B) \) solutions (8) with energy \( E = n_B \). Therefore the polynomials \( P_{i_2, \ldots, i_N}(X)H_{i_2}(X) \) guarantee that the functions (8) form a complete set in the space of square-integrable, \( SU(N) \) invariant functions (which is analogous to the role of Hermite polynomials in the space of normalizable functions). As for the orthogonality relations the polynomials \( P_{i_1, \ldots, i_N}(X)H_{i_2}(X) \) are only partly orthogonal (with weight \( e^{-Tr(X^2)} \)) i.e.

\[
\int [dX] P_{i_1, \ldots, i_N}(X)H_{i_2}(X)P_{j_1, \ldots, j_N}(X)H_{j_2}(X)e^{-Tr(X^2)} = 0
\]

for \( \sum_{k=1}^{N} k_i \neq \sum_{k=1}^{N} k_j \).

(9)

However for \( \sum_{k=2}^{N} k_i = \sum_{k=2}^{N} k_j \), i.e. in the subspace of solutions with equal number of quanta, these polynomials are not orthogonal.

4 The large \( N \) limit

Before going into a detailed computation of the \( N \rightarrow \infty \) limit of solutions (6) let us make a few comments on the peculiarities of the limit itself.

Having found a class of exact solutions at finite \( N \) one is only halfway to perform the \( N \rightarrow \infty \) limit. This is due to the fact that structure constants might be \( N \) dependent and hence a wavefunction depending on structure constants is sensitive to the form of the function \( f_{ABC} = f_{ABC}(N) \). Typical examples of this behavior are membranes with different topology e.g. the \( S^2 \) case [1] and the \( T^2 \) case [9]. In fact there are infinitely many ways in which one could make the large \( N \) limit [10]. Fortunately the solutions \( \psi_0(X) \) do not depend on structure constants therefore their large \( N \) limit should be both simple and relevant for membranes with arbitrary topology.

Another issue that we would like to discuss is the "limiting" Hilbert space (let us denote it by \( \mathcal{H} \)) which emerges out of a sequence of Hilbert spaces \( \mathcal{H}_N \) in the large \( N \) limit and the prescription: how \( \mathcal{H}_N \ni \psi_N \rightarrow \psi \in \mathcal{H} \). Certainly \( \mathcal{H} \) should correspond to a \((D-2)\)-dimensional quantum theory where the matrix degrees of freedom (in our case the color indices \( A, B, C \)) are absent leaving only the spatial
indices. A subsequent question is: what is the finite $N$ prototype of a radial coordinate in such theory. A good candidate is clearly

$$r^{(N)} := \sqrt{\sum_{s=1}^{D-2} Tr(X_s^2)}$$

however there are infinitely many other candidates which are (at this point) equally good e.g.

$$r_{2k}^{(N)} := 2^{k} \sqrt{\sum_{s=1}^{D-2} Tr(X_s^{2k})}, \quad k = 2, 3, \ldots.$$

On the other hand bilinear operators $Tr(X_s X_t)$ are distinguished as they do not depend on the structure constants. Moreover, among square-integrable wave functions in $\mathcal{H}_N$ there is another argument in favor of bilinear operators. Let us illustrate it considering an example with

$$\psi_N(X) = \frac{4}{N^3} \left( Tr(X^4) + \frac{A}{N} Tr(X^2)^2 \right) e^{-\frac{1}{2} Tr(X^2)}, \quad A \neq 0.$$

It turns out that the large $N$ limit of the norm of $\psi_N$ is given by

$$\lim_{N \to \infty} \|\psi_N\|^2 = (A + 2)^2. \quad (10)$$

To see this it is convenient to make the calculation in the occupation number representation by making the identification of the Fock vacuum

$$e^{-\frac{1}{2} Tr(X^2)} \leftrightarrow |0\rangle$$

and in general case

$$W(X)e^{-\frac{1}{2} Tr(X^2)} \leftrightarrow W(\hat{X})|0\rangle$$

where

$$\hat{X} = T_A \hat{x}_A, \quad \hat{x}_A = \frac{1}{\sqrt{2}}(a_A + a_A^\dagger).$$

With this in mind we obtain

$$\langle \psi_N, \psi_N \rangle = \frac{16}{N^6} \left( 0|Tr(X^4)^2|0\rangle + \frac{A^2}{N^2} 0|Tr(X^2)^4|0\rangle + \frac{2A}{N} 0|Tr(X^4)Tr(X^2)^2|0\rangle \right). \quad (11)$$
The three terms on the r.h.s. in (11) can be calculated by replacing \( \hat{X} \) by creation and annihilation operators and then by normal ordering of the appropriate terms. In doing so the SU(3) identities
\[
[T_A]_{ij}[T_A]_{kl} = \delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{il}\delta_{jk}
\]
are useful and we find that for large \( N \)
\[
\langle 0 | Tr(X^4)^2 | 0 \rangle \rightarrow \frac{N^6}{4}, \quad \langle 0 | Tr(X^2)^4 | 0 \rangle \rightarrow \frac{N^8}{16},
\]
\[
\langle 0 | Tr(X^4)Tr(X^2)^2 | 0 \rangle \rightarrow \frac{N^7}{8}
\]
implying (11). Therefore the norm admits a nontrivial contribution from the \( 1/N \) term which is perhaps contrary to what one may have expected. An interesting case is \( A = -2 \) when the norm \( \psi_N(X) \) approaches 0 in the large \( N \) limit.

Even though the scaling dimensions of \( Tr(X^4) \) and \( Tr(X^2)^2 \) are the same, this example shows that, when a square integrable function is considered, in the large \( N \) limit the bilinear operators dominate. The example can be easily generalized to many-matrix case.

Lastly, let us discuss the role of Cayley-Hamilton theorem when doing the \( N \rightarrow \infty \) limit. The theorem implies that the traces \( Tr(X^k) \) with \( k > N \) can be expressed in terms of traces \( Tr(X^k) \) with \( k \leq N \). Therefore for fixed \( N \) the terms with \( Tr(X^n) \), \( n > N \) in the Taylor expansion of the wave function, could be misleading as can be seen on the following example. Consider the function which Taylor expansion is
\[
\psi_N(X) := 1 + aTr(X^2) + bTr^2(X^2) + cTr(X^4) + \ldots \quad X \in su(N).
\]
For \( N = 3 \) using the C-H theorem \( X^3 = \frac{1}{2}XTTr(X^2) + \frac{1}{3}Tr(X^3) \) we find that
\[
\psi_3(X) := 1 + aTr(X^2) + \left( b + \frac{1}{2}c \right)Tr^2(X^2) + \ldots \quad X \in su(3)
\]
and one could get a false impression that the function does not depend on \( Tr(X^k) \) with \( k > 2 \). Therefore the large \( x_A \) behavior of \( \psi_N(x) \) could be different form the large \( x_A \) behavior of \( \psi_M(x) \) when \( M \gg N \), hence in analyzing the large \( N \) limit of \( \psi_N \) one should concentrate mainly on the terms that are not altered by the C-H theorem at all.
4.1 The limit of the wave functions

We now focus on solutions (6) and their large $N$ behavior. Let us first discuss the solution (6) with $i_3 = i_4 = \ldots = i_N = 0$. We have

$$\psi_{k,0}(X) = \left(1 - \frac{k^2 Tr(X^2)}{2(3 + N^2 - 4)} + \frac{k^4 Tr(X^2)^2}{2(3 + N^2 - 4)4(5 + N^2 - 4)} - \cdots \right).$$

If $k$ is independent of $N$ we obtain

$$\psi_{k,0}(X) \xrightarrow{N \to \infty} 1,$$

hence the solution is trivial in the large $N$ limit.

We now observe that it is possible to perform the limit in such a way that the square-integrable solutions will nevertheless appear. To see this we first introduce new momenta $\kappa = k/N$. The differential equation (5) becomes now

$$-\partial_A \partial_A \psi = N^2 \kappa^2 \psi$$

which is in accordance with the standard large $N$ techniques \[11\] i.e. the energy scales like $N^2$. Now the $m$th term of the solution (12) can be written as

$$(-1)^{m-1} \frac{1}{(m-1)!} \left(\frac{\kappa^2 r^2}{2}\right)^{m-1} \prod_{l=1}^{m} \frac{1}{(1 + (2l - 3)/N^2)},$$

therefore the solution (12) converges to

$$\psi_{\kappa,0}(x) \xrightarrow{N \to \infty} \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \left(\frac{\kappa^2 r^2}{2}\right)^m = \exp\left(-\frac{\kappa^2 r^2}{2}\right),$$

so that the solution becomes the Gauss function in variable $r$. Clearly a Gauss function is not a solution of the Laplace equation (5) (that solution is in Eqn. (6)) however what we are finding here is that the $N \to \infty$ limit of (6) converges to the Gauss function provided we let the momentum scale with $N$ as $\kappa = k/N$. Now it is interesting to analyze the $N \to \infty$ limit of the rest of solutions (6) i.e. $\psi_i = P_i \psi_0$ with $i \neq 0$. Due to the damping factor emerging from $\psi_0$, one cannot exclude the possibility that $\psi_i$ have a well defined counterpart in the limit. However the polynomials $P_i$ are troublesome since they depend on other independent variables $r_k := |Tr(X^k)|^{1/k}$. In order to
overcome the problematic role of the variables $r_k$ we consider wavefunctions averaged over $r_{k>2}$ in the following sense

$$
\psi^{\text{averaged}}(r) = \frac{1}{A_{S^{N^2-2}}} \int_{Tr(X^2)=r^2} \psi(X) dA_{S^{N^2-2}}. \quad (13)
$$

therefore we are averaging over a sphere $S^{N^2-2}$ in $\mathbb{R}^{N^2-1}$. The hyperarea of that sphere is

$$
A_{S^{N^2-2}} = \int dA_{S^{N^2-2}} = r^{N^2-2} \int d\Omega_{S^{N^2-2}} = r^{N^2-2} \frac{2\pi^{\frac{N^2-1}{2}}}{\Gamma\left(\frac{N^2-1}{2}\right)}
$$

where $d\Omega_{S^{N^2-2}}$ is the measure corresponding to the angels.

If the function depends only on a radial variable $r$ then $\psi^{\text{averaged}}(r) = \psi(r)$ as it should be. The evaluation of the average in general case may be difficult however the case of homogenous functions: $\psi(kX) = k^\Delta \psi(X)$, $\Delta \in \mathbb{N}$ can be done by making the use of occupation number representation. To see this let us calculate $\langle 0 | \psi(\hat{X}) | 0 \rangle$ in the spherical coordinates, we have

$$
\langle 0 | \psi(\hat{X}) | 0 \rangle = \int [dX] \psi(X)e^{-Tr(X^2)}
$$

$$
= \frac{1}{\pi^{\frac{N^2-1}{2}}} \left( \int_0^\infty dr r^{N^2-2+\Delta} e^{-r^2} \right) \left( \int_{Tr(X^2)=1} d\Omega_{S^{N^2-2}} \psi(X) \right)
$$

where we changed the variables $X \to X/r$ and separated the $r$ dependence which allows us to separate the appropriate integrals. Performing the integral over $r$ we obtain

$$
\int_{Tr(X^2)=1} d\Omega_{S^{N^2-2}} \psi(X) = \frac{2\pi^{\frac{N^2-1}{2}} \langle 0 | \psi(\hat{X}) | 0 \rangle}{\Gamma\left(\frac{N^2-1+\Delta}{2}\right)}
$$

therefore

$$
\psi^{\text{averaged}}(r) = \frac{r^{N^2-2+\Delta}}{A_{S^{N^2-2}}} \int_{Tr(X^2)=1} d\Omega_{S^{N^2-2}} \psi(X) = \frac{r^\Delta \Gamma\left(\frac{N^2-1}{2}\right) \langle 0 | \psi(\hat{X}) | 0 \rangle}{\Gamma\left(\frac{N^2-1+\Delta}{2}\right)}. \quad (14)
$$
We now see that averaging over solutions (6) with \( i \neq 0 \) gives 0 since \( \langle 0 | P_i(X) | 0 \rangle = 0 \) due to the orthogonality relation (9). Therefore according to the definition (13) the solutions (6) with \( i \neq 0 \) are not relevant in the large \( N \) limit.

Generalization of the above discussion to a \( d \)-dimensional case (4) is straightforward. The solutions of (4) are given by products of solutions in (6)

\[
\Psi_{i,k}(X) := \Psi_{i_1,k_1,\ldots,i_d,k_d}(X_1,\ldots,X_d) = \psi_{i_1,k_1}(X_1) \cdots \psi_{i_d,k_d}(X_d)
\]

\[- \sum_s \partial^2_{A_s} \Psi_{i,k}(X) = (k_1^2 + \cdots + k_d^2) \Psi_{i,k}(X) \quad (15)\]

where \( \psi_{i_s,k_s}(X_s) \) is as in (6). The large \( N \) limit of the case \( i_s = 0 \) is

\[
\Psi_s(X) = \exp \left( -\frac{d\kappa^2 r^2}{2} \right), \quad \kappa^2 = \sum_{s=1}^d \kappa_s^2, \quad r = \sqrt{\sum_s \text{Tr}(X_s^2)} \quad (16)
\]

provided that we let \( k_s \) scale as \( k_s = N \kappa_s \). The solutions with \( i_s \neq 0 \) when averaged over \( S^{(N^2-2)-1} \) according to

\[
\Psi^{\text{averaged}}(r) = \frac{1}{A_{S^{(N^2-2)-1}}} \int_{\sum_s \text{Tr}(X_s^2) = r^2} \Psi(X) dA_{S^{(N^2-1)-d-1}}
\]

are zero since the formula (14) can be applied also here and

\[
(0 | P_{i_1}(X_1) \cdots P_{i_d}(X_d) | 0) = 0, \quad \langle X | 0 \rangle = e^{-\frac{1}{2} \sum_s \text{Tr}(X_s^2)}.
\]

5 Discussion

In this paper we discussed special kind of membrane configurations, in \( D \)-dimensional Minkowski spacetime, in which a surface of a membrane develops spikes. In the light-cone description of the theory, spikes are defined by the vanishing of the induced metric hence they correspond to one dimensional, extended objects. Using the regularized formulation of the quantum theory in terms of \( SU(N) \) matrices, these "stringy" configurations turn out to be described by a Schrödinger equation of a free particle in \((N^2-1)(D-2)\) dimensions accompanied by the \( SU(N) \) singlet constraint. Due to the constraint the solutions, instead of being ordinary plane waves, have some unusual properties when one considers the \( N \to \infty \) limit. In particular if we let the momenta scale as \( k \sim N \) (or equivalently the energy as \( E \sim N^2 \) then
the solutions approach the Gauss function (16) which is the main result of the paper. This solution should be taken seriously as it does not depend on the structure constants and therefore is independent of topological aspects of the membrane. We speculate that the solution corresponds to the ground state of open string excitations.

In search for the excited states we focus on a different class of solutions which depend on $r_k = \sqrt{|\text{Tr}(X^k)|}, k > 2$. In order to remove the unwanted variables $r_k$ we introduce the averaging procedure over matrix degrees of freedom and find that the averaged solutions are zero. This indicates that one should either reconsider the averaging procedure or concentrate on yet another solutions.

Quite surprisingly the $N = \infty$ solution is perfectly normalizable although it emerges from a sequence of wave functions that are not square integrable. From this point of view one cannot exclude the possibility that the anticipated ground state of the full (super)membrane theory cloud emerge out of a sequence of non-normalizable wavefunctions of the matrix model.

6 Acknowledgments

I thank J. Hoppe for many important comments regarding the manuscript. I also thank J. Wosiek for discussions. This work was supported by Marie Curie Research Training Network ENIGMA (contract MRNT-CT-2004-5652).

References

[1] J. Hoppe, Quantum Theory of a Massless Relativistic Surface and a two dimensional bound state problem, PhD Thesis MIT, (1982), (scanned version available at http://www.aei-potsdam.mpg.de/~hoppe);
[2] D. de Wit, J. Hoppe, H. Nicolai, On the quantum mechanics of supermembranes, Nucl. Phys. B305 [FS23] (1988) 545-581,
[3] J. Fröhlich, J. Hoppe, On Zero-Mass Ground States in Super-Membrane Matrix Models, arXiv:hep-th/9701119
J. Hoppe, On the Construction of Zero Energy States in Supersymmetric Matrix Models, arXiv:hep-th/9709132
M. B. Halpern, C. Schwartz, *Asymptotic Search for Ground States of SU(2) Matrix Theory*, Int. J. Mod. Phys. A13 (1998) 4367-4408, arXiv:hep-th/9712133

A. Konechny, *On Asymptotic Hamiltonian for SU(N) Matrix Theory*, JHEP 9810 (1998) 018, arXiv:hep-th/9805046

M. Bordemann, J. Hoppe, R. Suter, *Zero Energy States for SU(N): A Simple Exercise in Group Theory?*, arXiv:hep-th/9909191

J. Hoppe, *Asymptotic Zero Energy States for SU(N greater or equal 3)*, arXiv:hep-th/9912163

J. Fröhlich, G. M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, *Asymptotic form of zero energy wave functions in supersymmetric matrix models*, Nucl. Phys. B567 (2000) 231-248, arXiv:hep-th/9904182

J. Hoppe, J. Plefka, *The Asymptotic Groundstate of SU(3) Matrix Theory*, arXiv:hep-th/0002107

D. Hasler, J. Hoppe, *Asymptotic Factorisation of the Ground-State for SU(N)-invariant Supersymmetric Matrix-Models*, arXiv:hep-th/0206043

V. Bach, J. Hoppe, D. Lundholm, *Dynamical Symmetries in Supersymmetric Matrix Models*, arXiv:hep-th/07060355

J. Hoppe, D. Lundholm, *On the Construction of Zero Energy States in Supersymmetric Matrix Models IV*, arXiv:0706.0353

J. Hoppe, D. Lundholm, M. Trzetrzelewski, *Construction of the Zero-Energy State of SU(2)-Matrix Theory: Near the Origin*, Nucl. Phys. B817:155-166, 2009 arXiv:0809.5270

[4] H. Nicolai, R. Helling, *Supermembranes and M(atrix) Theory* arXiv:hep-th/9809103

[5] E. Floratos, J. Iliopoulos, *A note on the classical symmetries of the closed bosonic membranes*, Phys. Lett. B, Vol. 201, p. 237 (1988);
I. Bars, C. N. Pope, E. Sezgin, *Central extensions of area preserving membrane algebras*, Phys. Lett. B, Vol. 210, Issue 1-2, p. 85-91 (1988);
I. Antoniadis, P. Ditsas, E. Floratos, J. Iliopoulos, *New realizations of the Virasoro algebra as membrane symmetries*, Nucl. Phys. B, Vol. 300, p. 549-558 (1988);
M. J. Duff, Paul S. Howe, T. Inami, K. S. Stelle, *Superstrings in D=10 from Supermembranes in D=11*, Phys. Lett. B191:70, (1987),
[6] P. A. Dirac, *An Extensible Model Of The Electron*, Proc. Roy. Soc. Lond. A268:57-67, 1962,

[7] J. Hoppe, *Membranes and Matrix Models*, arXiv:hep-th/0206192

[8] M. Trzetrzelewski, *Large N behavior of two dimensional supersymmetric Yang-Mills quantum mechanics*, J. Math. Phys. 48:012302, (2007),

[9] D. Fairlie, P. Fletcher, C. N. Zachos, *Trigonometric structure constants for new infinite algebras*, Phys. Lett. B218:203,(1989);
E. G. Floratos, Phys. Lett. *The heisenberg-weyl group on the Z(N) × Z(N) discretized torus membrane*, Phys. Lett. B223:37, (1989),

[10] J. Hoppe, P. Schaller, *Infinitely many versions of SU(∞)*, Phys. Lett. B, Volume 237, Issue 3-4, p. 407-410, (1990),

[11] *The Large N Expansion in Quantum Field Theory and Statistical Physics* (eds. E. Brezin and S. R. Wadia), World Scientific Publishing Company, Singapore (1993).