SPATIAL RACK DRIVES PITCH CONFIGURATIONS: ESSENCE AND CONTENT

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ABSTRACT: The practical realization of all types of mechanical motions converters is preceded by solving the task of their kinematic synthesis. In this way, the determination of the optimal values of the constant geometrical parameters of the chosen structure of the created mechanical system is achieved. The searched result is a guarantee of the preliminary defined kinematic characteristics of the synthesized transmission and in the first place, to guarantee the law of motions transformation. The kinematic synthesis of mechanical transmissions is based on adequate mathematical modelling of the process of motions transformation and on the object, realizing this transformation. Basic primitives of the mathematical models for synthesis upon a pitch contact point are geometric and kinematic pitch configurations. Their dimensions and mutual position in space are the input parameters for the processes of design and elaboration of the synthesized mechanical device. The study presented here is a brief review of the theory of pitch configurations. It is an independent scientific branch of the spatial gearing theory (theory of hyperboloid gears). On this basis, the essence and content of the corresponding primitives, applicable to the synthesis of spatial rack drives, are defined.

KEY WORDS: Hyperboloid gears, spatial rack mechanisms/drives, mathematical modelling, kinematic synthesis, pitch configurations.

1. INTRODUCTION

The design of multibody system mechanics, realizing a preliminary defined law of motions transformation, is a complex task, which solution generally involves the following three stages [1]:

(a) construction of an adequate kinematic scheme of the mechanism;
(b) elaboration of the design documentation of the mechanical system;
(c) technological preparation of the processes related to the manufacture.

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The first stage of the design is known in the *Applied Mechanics* and in the *Theory of Mechanisms*, under the name *Synthesis of Mechanisms*, which in the most common case includes the following two main tasks:

- Choice of the structure of the designed mechanism by carrying out a structural synthesis.

- Design of the chosen kinematic scheme of the mechanism. This task is known as a *kinematic synthesis of the mechanism*. Through its solution it is achieved the determination of the constant geometric parameters of the chosen structure of the mechanism. They are such parameters that satisfy its preliminary defined kinematic characteristics and above all - the function of the law of motions transformation.

In the case of synthesis of spatial rack drives, as well as the synthesis of hyperboloid gear mechanisms, it is not necessary to solve the problem related to the determination of the structure of the mechanisms. As a rule, both type mechanisms realize a transformation of motions by a frame and two movable links, which configure high kinematic joints. For the gear drives, an object of this study, always one of the movable links performs rotation and the other – a rectilinear translation. Between these two motions a predefined functional dependence exists. Therefore, the synthesis of spatial rack mechanisms is reduced to finding out a solution of the *kinematic synthesis task*.

For all types gear mechanisms, including rack drives, the solution of the task of their kinematic synthesis should give in general answers to two sets of questions regarding: (a) the geometry and dimensions of the movable links, having high kinematic joints; (b) the geometry and dimensions of the tooth surfaces, representing the geometric elements of the kinematic joints (mated rack and gear).

Answers to the first group of questions are contained in the synthesis of so-called *pitch configurations* of the rack transmissions. The formulation of the second group questions and their adequate answers, form in the theory of gearing the task for the synthesis of conjugated rack - gear mating (conjugate rack meshing).

The kinematic theory of the spatial motions transformation of the type \((R \leftrightarrow T)\), offered in [2,3], is applied in defining the geometric and kinematic nature of the basic elements, that built the mathematical models, used for the synthesis of spatial rack drives. Known as *pitch configurations* [1], they treat one actual and contemporary direction in the theory of gearing.

In the theory of spatial gearing, the terms *primary surfaces* [4,5] and *pitch surfaces* [6-8,12-14] have been used at the same time. In most cases, these terms (*primary and pitch surfaces*) have been used for one and the same surfaces. To the
maximum extent, in the cited literature with those two terms – *primary and pitch surfaces* are defined the same concepts.

The well-known Russian scientist Prof. F. Litvin, in the beginning of the seventies of the last century, gave one of the first interpretations in [4] about *primary surfaces*. Later on, at the end of 20th century, Professor F. Litvin, already a professor of mechanical engineering at the University of Illinois in Chicago, USA, and an active researcher in the field of gear transmissions, keeps almost unchanged the concept of the *primary surfaces*. In [7], the primary surfaces are called *operating pitch surfaces* in relation to their practical realization in the design of spatial gear mechanisms with crossed axes, unlike the axodes of the movable links:

“The operating pitch surfaces represents: (i) two cylinders for a worm-gear and helical gears with crossed axes and (ii) two cones for a hypoid gear drives. The chosen surfaces that are called in the technical literature “operating pitch surfaces” must satisfy the following requirements:

(i) The axes of cylinders (cones) have to form the same crossed angle and be at the same shortest distance as for the designed gears.

(ii) The cylinders (cones) must be in tangency at the middle point of contact of the surfaces of the gears to be designed.

(iii) The relative sliding velocity \( \vec{V}_{12} \) at point \( P \) of tangency of the cylinders (cones) must lie in the plane that is tangent to the cylinders (cones) and \( \vec{V}_{12} \) must be directed along the common tangent to the helices of the gears to be designed.

The term “helix” is a conventional one. Actually, we have to consider a spatial curve that belongs to the operating cylinders (cones) and represents the line of intersection of the gear tooth surfaces with the operating cylinders (cones). For the case of a helical gear, a cylinder worm, this line of intersection is indeed a helix. For the case of spatial bevel gears and hypoid gears, the line of intersection is a spatial curve that differs from a helix and might be represented with complicated equations.

(iv) The tangent point \( P \) of operating pitch cylinders (cones) will be simultaneously the point of tangency of gear tooth surfaces if the surfaces have a common normal \( n - n \) at \( P \), and \( n - n \) is perpendicular to \( \vec{V}_{12} \) ...

The version of the pitch surfaces’ idea, mentioned above, is oriented to the synthesis and design of worm, helical and hypoid gear transmissions. The defined in [5,9] conditions for existence of the primary surfaces possess similar meaning. In these studies, algorithms for calculation of the geometric parameters of the primary surfaces of the spatial gear set with external mating of the tooth surfaces, are offered.
It should be noted that the algorithm, suggested there, is oriented for the synthesis of hyperboloid gear sets with arbitrary crossed axes.

In publication [8], W. Nelson, chief engineer at Illinois Tool Works, Chicago, treated the pitch surfaces for one concrete type of spatial gears - the Spiroid\textsuperscript{2} ones. There, he used the terms \textit{primary pitch cone} (the coaxial cone limiting the tips of the Spiroid pinion threads) and \textit{pitch surface} (an envelope of the primary pitch cone in its relative motion with respect to the axis of the second movable link of the Spiroid gear). On the common line of contact of both pitch surfaces, he searched for that pitch contact point which determines the most suitable spatial curve, used as a longitudinal line of the synthesized tooth surfaces of the Spiroid pinion. This is an approach that does not differ from those approaches, which have been already considered.

In relation to the treated theme in [1], scientific studies are presented. They are oriented to illustrate the essence and the content of the two basic concepts, which form the common name \textit{pitch configurations}: \textit{pitch circles} and \textit{pitch surfaces}. This is realized in the context of treating the spatial rack mechanisms as a special case of hyperboloid gear sets [2].

2. \textbf{SYNTHESIS OF GEOMETRIC PITCH CONFIGURATIONS}

2.1. 2.1. \textbf{STRUCTURAL, GEOMETRIC AND KINEMATIC SIMILARITY BETWEEN THE PROCESSES OF SPATIAL MOTIONS TRANSFORMATION “ROTATION INTO ROTATION (R ↔ R)” AND “ROTATION INTO TRANSLATION (R ↔ T)”}

The wide spread in mechanics devices for motions transformation are three-link mechanisms, which movable links are set of discreet conjugate surfaces, that form high kinematic joints, when the motions transformation is realized [3-5]. Depending on the quality characteristics of the realized spatial motions transformation, they are divided into two basic types: gear mechanisms with crossed axes (hyperboloid gear mechanisms), which realize transformation of type \((R ↔ R)\) (rotation into rotation); spatial rack drives, accomplishing motions transformation of type \((R ↔ T)\) (rotation into translation and vice versa) [6].

Here briefly, the kinematic, structure and geometric similarity of the spatial gear transmissions, which ensure the motions transformation \((R ↔ R)\) and \((R ↔ T)\), will be illustrated. \textit{For a start, the basic kinematic parameters of the spatial transformations “rotation into rotation \((R ↔ R)\)” and “rotation into translation \((R ↔ T)\)” will be defined.}

The transformation of rotation motion, around fixed in space axis \(1–1\), which is realized with angular velocity \(\dot{\omega}_1\), into rotation motion with angular velocity \(\dot{\omega}_2\) around another fixed in space axis \(2–2\) (which is crossed with the axis \(2–2\)) is called

\textsuperscript{2}Spiroid and Helicon are trademarks registered by the Illinois Tool Works, Chicago, Ill.
transformation of type “rotation into rotation \((R \leftrightarrow R)\)”. It is realized by means of high kinematic joints \((\Sigma_1 : \Sigma_2)\).

The kinematic scheme of such type transformations is illustrated in Fig. 1a, and it is characterized with the following kinematic conditions [1,2]:

\[
\begin{align*}
\omega_1 &= \text{constant}, \\
\omega_2 &= \text{constant}, \\
\frac{\omega_1}{\omega_2} = i_{12} = \frac{1}{\omega_2 \omega_1} = \frac{1}{i_{21}} = \text{constant},
\end{align*}
\]

where \(\omega_1\) is the value of the angular velocity vector \(\bar{\omega}_1\); \(\omega_2\) – the value of the angular velocity vector \(\bar{\omega}_2\); \(i_{12}, i_{21}\) – law of transformation \((R \leftrightarrow R)\) (velocity ratio). The distance between the axes of rotations \(1-1\) and \(2-2\) is \(a_\omega = \text{constant}\), and \(\angle(\bar{\omega}_1, \bar{\omega}_2) = \delta = \text{constant}\) is the crossed angle of the rotations axes.

Fig. 1. Structural transition from hyperboloid gear transmission into spatial rack drive:
(a) kinematic scheme of gear mechanism, realizing transformation of type \((R \leftrightarrow R)\); (b) kinematic scheme of gear mechanism, realizing transformation of type \((R \leftrightarrow T)\).
The transformation of rotation motion around fixed in space axis 1–1, that is realized with angular velocity \( \bar{\omega}_1 \) into translation motion with velocity \( \bar{V}_2 \) along the direction 2t–2t, (placed in the space against axis 1–1 under an angle \( \angle(\bar{\omega}_1, \bar{V}_2) = \delta_r = \text{constant} \)), by means of the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) is called transformation of type “\((R \leftrightarrow T)\)”.

The kinematic scheme of this type transformation is shown in Fig. 1b. It is characterized with the following kinematic conditions [2]:

\[
\omega_1 = \text{constant}, \quad V_2 = \text{constant}, \quad \frac{\omega_1}{V_2} = j_{12} = \frac{1}{V_2/\omega_1} = \frac{1}{j_{21}} = \text{constant},
\]

where \( V_2 \) – value of the translation velocity vector \( \bar{V}_2 \); \( j_{12}, j_{21} \) – law of transformation of type “rotation into translation \((R \leftrightarrow T)\)”.

The spatial transformation of type \((R \leftrightarrow T)\) can be considered as a special case of the spatial transformation \((R \leftrightarrow R)\). In other words, the spatial rack mechanisms can be treated as a border case of hyperboloid gear set, i.e. it is obtained from a three-link spatial gear transmission with crossed axes under the following structural and kinematic changes (according to Figs. 1a and b):

- the number of the teeth connected with one of the rotating links is increased to infinity; for example, the number of the geometric elements \( \Sigma_2 \) of the high kinematic joint (\( \Sigma_1 : \Sigma_2 \)) is increased to infinity;
- the number of mutually meshed tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \), i.e. the number of the high kinematic joints (\( \Sigma_1 : \Sigma_2 \)) remains finite;
- the axis of rotation 2–2 of the surfaces \( \Sigma_2 \) is moved into the infinity;
- rotation motion of \( \Sigma_2 \) around 2–2 with an angular velocity \( \bar{\omega}_2 \) is transformed into a translation one with velocity \( \bar{V}_2 \).

2.2. Common characteristics of geometric pitch configurations

While adhering to the presented in Section 1, short analysis, based on the studies in [1], some specific characteristics, of the primary surfaces and geometric pitch circles for the hyperboloid gear mechanisms will be presented briefly (see Fig. 2). When a spatial rotations transformation with a constant velocity ratio \( i_{12} = \text{constant} \) is realized, arbitrary rotating surfaces \( H_1 \) and \( H_2 \) can be used as primary surfaces for
the synthesis of hyperboloid gear mechanisms, if the following basic conditions are fulfilled:

- the rotations axes of $H_1$ and $H_2$ have to coincide with the axes 1–1 and 2–2 of the movable links of the three-links gear mechanism;
- surfaces $H_1$ and $H_2$ have to obtain a common tangent plane $T_m$, passing through the common (for $H_1$ and $H_2$) contact point $P$. The plane $T_m$ contains the relative velocity vector $\vec{V}_{12}$ and circumferential velocities vectors $\vec{V}_i$ ($i = 1, 2$) of the point $P$.

An illustration of these conditions (in the case of hyperboloid gears with external gearing) is shown in Fig. 2. From the analysis of what has been said so far, as well as from the illustration in the figure, the following conclusions are derived in [1]:

![Fig. 2. Geometric pitch configuration: $H_i^G$ ($i = 1, 2$) - geometric pitch circles; $H_i$ ($i = 1, 2$) – geometric pitch surfaces (cones); $B_i$ ($i = 1, 2$) – primary surfaces; $P$ – pitch contact point (pole of meshing); $T_m$ – geometric pitch plane; $m$–$m$ – pitch normal.](image-url)

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If the law of rotations transformation \( i_{12} = \omega_1 / \omega_2 = \text{constant} \) between fixed and crossed axes 1–1 and 2–2 (with an offset \( \omega_\omega = \text{constant} \) and a shaft angle \( \delta = \angle(\bar{\omega}_1, \bar{\omega}_2) = \text{constant} \)) is given, and the place of the point \( P \) (treated as a contact point of the conjugate tooth surfaces \( \Sigma_i \) \( (i = 1, 2) \)) in the fixed space is defined, then the diameters and mutual positions of the circles \( H^G_i \) \( (i = 1, 2) \) are completely determined. The circumferential velocities \( \bar{V}_i \) of the common point \( P \), and correspondingly, the relative velocity \( \bar{V}_{12} \) at the same point are defined uniquely. The plane \( T_m \) (where the vectors \( \bar{V}_i \) \( (i = 1, 2) \) and \( \bar{V}_{12} \) lie) and the normal \( m-m \) to \( T_m \) at the point \( P \), are defined.

The primary surfaces \( H_1 \) and \( H_2 \) (discussed above) are not defined in a unique way, because they are all surfaces of revolution, that include the pair of circles \( H^G_i \) \( (i = 1, 2) \). In [4,5], the conical surfaces of revolution contacting at only one point \( P \) are discussed as a primary surfaces. The algorithms written there, for their synthesis practically define the diameters and the mutual position of the circles \( H^G_i \) passing through their common point \( P \).

From all said above, one can conclude that it is sufficient to define in advance the mutual position of the rotation axes 1–1 and 2–2, and the position of the point \( P \) (as a common point of the tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \)), so that the pair of pitch circles \( H^G_i \) \( (i = 1, 2) \) (as dimensions and parameters of mutual position) to be determined uniquely. The plane \( T_m \), containing the tangent lines to the circles \( H^G_i \) \( (i = 1, 2) \) at point \( P \), and also the normal \( m-m \) to \( T_m \) at \( P \) are determined uniquely [1]. The parameters mentioned above are geometric once, because the circles \( H^G_i \) \( (i = 1, 2) \), which serve for their determining, are not “brought into rotation” in accordance with the defined law of motions transformation \( i_{12} = \omega_1 / \omega_2 \). But, if the law is being active, the geometric parameters together with the kinematic parameters of the mechanical system serve for defining the longitude and profile orientations of the conjugate tooth surfaces \( \Sigma_i \) \( (i = 1, 2) \), their pitches and the module of the teeth, corresponding.

The said above serves as a reason the mentioned pair of circles to be called geometric pitch circles. Depending on their location with respect to the plane \( T_m \), it is distinguished: external contacting pitch circles – when \( H^G_i \) \( (i = 1, 2) \) are located in different half-spaces with respect to \( T_m \) and internal contacting pitch circles – when \( H^G_i \) \( (i = 1, 2) \) are unilaterally placed to \( T_m \). In summary, their choice is based on the following conditions [1]:

- Centers \( C_i \) \( (i = 1, 2) \) of \( H^G_1 \) and \( H^G_2 \) lie on the axes of rotation 1–1 and 2–2. Moreover, the planes of the circles are perpendicular to the corresponding axes of rotation 1–1 and 2–2.
• $H_G^1$ and $H_G^2$ have a common point $P$, which is called a pitch contact point. This point is also common point for the conjugated tooth surfaces $\Sigma_i \ (i = 1, 2)$.

• The circumferential velocities $\bar{V}_1$ and $\bar{V}_2$, as well the sliding velocity $\bar{V}_{12}$ between the conjugated tooth surfaces at point $P$, lie in the plane $T_m$, called a pitch plane.

Treatment of one point $P$ of conjugated tooth surfaces $\Sigma_1$ and $\Sigma_2$ as a common point of the geometric pitch circles $H_G^1$ and $H_G^2$, makes possible to understand easily the approaches to the constructing of mathematical models for the synthesis of hyperboloid gear mechanisms with external and internal mating. In this context, it is important to be noted once again, that geometric pitch circles are used to define the longitudinal orientation of the contacted in point $P$ tooth surfaces $\Sigma_i \ (i = 1, 2)$. Also, the geometric pitch circles can be an element of the co-axial (reference) or the same-type co-axial (top or bottom) surfaces of revolution.

The conical surfaces, containing geometric pitch circles and having a taper angle at the apex equal to the angle $\delta_i \ (i = 1, 2)$, which the planes containing those circles conclude with the pitch normal $m$–$m$, are called geometric pitch surfaces (geometric pitch cones) or just pitch surfaces/cones. The pairs of geometric pitch circles and pairs of geometric pitch surfaces form the set of geometric pitch configurations [1]. It will be noted specially, that a common practice in the synthesis of hyperboloid transmissions is to choose the reference surfaces of the gears as pitch surfaces [1,5]. There is also a possibility [8], one of the geometric pitch surfaces to coincide with the addendum surface of one gear of the gear mechanism. In this case, the other geometric pitch surface is the envelope of the first once, which contains the second geometric pitch circle.

2.3. **Geometric Pitch Configurations at Conjugate Rack Meshing**

We will define the geometric pitch configurations of the spatial rack drive in the context of the ideology, that this mechanism can be considered as a special case of hyperboloid gear set. If the axis of the rotation of one link moves into infinity (for example, the axis of the gear having an index $i = 2$), then the pitch circle $H_G^2$ transforms into a pitch straight line, passing through the pitch contact point $P$ and coinciding with the tangent to the circle $H_G^2$, at point $P$. Hence, into the geometric pitch configurations of the spatial rack mechanisms, two elements are included. The following conditions are fulfilled for them [1]:

• The pitch circle $H_G^1$ with radius $r_1$, which center $C_1$ is placed on the axis of rotation $1–1$, lies in the plane, perpendicular to the same axis.
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- Geometric pitch line \( H^G_2 \) lies in the pitch plane \( T_m \) and its placement on \( T_m \) is defined from the direction of the translation velocity vector \( \dot{V}_2 \) of the link \( i = 2 \).
- The geometric pitch circle \( H^G_1 \) and pitch straight line \( H^G_2 \) have for a common point – the pitch contact point \( P \). \( P \) is also a common point of the conjugated tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \).
- In the most common case, the angle \( \delta_1 \) between the plane, containing \( H^G_1 \), and the normal \( m-m \) to the pitch plane \( T_m \), is \( \delta_1 \neq 0 \). The angle between the axis of rotation \( 1-1 \) of \( H^G_1 \) and the direction of translation \( 2t-2t \) of \( H^G_2 \) is \( \delta_r = \angle(\bar{\omega}_1, \dot{V}_2) \).

The analytical study of the geometric pitch circle and the line is realized by usage of Fig. 3. The mutual placement of these pitch elements is completely defined by the angles \( \delta_r, \delta_1 \) and \( \theta_1 \), by matrices \( m \) and \( h \) of the unit vectors \( \bar{m} \) and \( \bar{h} \), for which the condition \( \bar{m} \cdot \bar{h} = 0 \) is fulfilled:

\[
m = \begin{bmatrix} \cos \delta_1 \cos \theta_1 \\ \cos \delta_1 \sin \theta_1 \\ - \sin \theta_1 \end{bmatrix},
\]

\[
h = \begin{bmatrix} - \cos \alpha \sin \delta_1 \cos \theta_1 + \sin \alpha \cos \theta_1 \\ - \cos \alpha \sin \delta_1 \sin \theta_1 - \sin \alpha \cos \theta_1 \\ - \cos \alpha \cos \delta_1 \end{bmatrix},
\]

where

\[
\cos \alpha = - \frac{\cos \delta_r}{\cos \delta_1}, \quad \sin \alpha = \pm \sqrt{\cos^2 \delta_1 - \cos^2 \delta_r}.
\]

It is evident from (5), that for the cases of pitch configuration, for which \( \delta_1 \in [0, \pi/2) \), the following inequality is realized:

\[
|\cos \delta_r| \leq \cos \delta_1.
\]

2.4. **Helix angles** \( \beta_i (i = 1, 2) \) **of the longitudinal lines of** \( L_1 \) **and** \( L_2 \) **of the conjugated tooth surfaces** \( \Sigma_i (i = 1, 2) \) **in the pitch contact point** \( P \)

Let us define the longitudinal orientation of the active tooth surfaces \( \Sigma_i \), of the movable links of spatial rack drive, when its synthesis is based on the application of the geometric pitch configurations. In this case, the longitudinal orientation of the active surfaces \( \Sigma_i (i = 1, 2) \), is determined through the angles of their inclination \( \beta_i \)
(i = 1, 2). In accordance with [10], this angle of the rotating link is the acute angle between the crossing at this point longitudinal line (the tangent line to the longitudinal line in a concrete particular point) and the intersection of the geometric pitch surface (pitch cone or pitch cylinder) with the plane of the axial section (the plane containing the rotation axis 1–1), determined by the corresponding point. The same angle (in relation to the active tooth surfaces of the rack gear) is placed between the tangent to the longitudinal line of the corresponding active tooth surfaces (obtained as a section between their crossing with the pitch plane) and the normal to the intersection of the last one with its normal plane containing the trajectory of its rectilinear translation 2r–2r. Hence, in the mentioned points of the longitudinal lines of the teeth of both movable links of the rack drive, the angles $\beta_i$ ($i = 1, 2$) are those angles, which will form $90^\circ$ with the angles between the vectors $\vec{V}_i$ ($i = 1, 2$) of the circumferential motion in these points and the relative velocity $\vec{V}_{12}$ (these angles according to [10] are called spiral angles (ascent angles) of the longitudinal lines). Then, for these angles by definition it can be written [4, 11]:

$$\tan \beta_i = \cot(\vec{V}_{12}, \vec{V}_i) = \frac{\vec{V}_{12} \cdot \vec{V}_i}{|\vec{V}_{12} \times \vec{V}_i|}.$$  \hfill (7)

For the circumferential velocities $\vec{V}_i$ ($i = 1, 2$) and relative velocity $\vec{V}_{12}$ in point $P$, in accordance with Fig. 3, it can be written:

$$\vec{V}_1 = \vec{\omega}_1 \times \vec{p}_1 = y\vec{i} + (-x)\vec{j},$$  \hfill (8)

$$\vec{V}_2 = j_{21} \sin \delta_r \vec{j} - j_{21} \cos \delta_r \vec{k},$$  \hfill (9)

$$\vec{V}_{12} = y\vec{i} + [-(x + j_{21} \sin \delta_r)]\vec{j} + j_{21} \cos \delta_r \vec{k},$$  \hfill (10)

where $\vec{\omega}_1$ is angular velocity vector of the rotating link, which magnitude is accepted to be $\omega_1 = 1$ rad/s; $\vec{p}_1 = \vec{p}_1(x, y, z)$ – radius-vector of the pitch contact point $P$ in the fixed co-ordinate system $S(O, x, y, z)$; $j_{21} = V_2$ – velocity ratio of the rack drive.

The helix angles (inclination angles) $\beta_i$ ($i = 1, 2$) are defined by using (7). Then, for the study case, they are

$$\tan \beta_1 = \frac{x^2 + y^2 + j_{21} \sin \delta_r x}{j_{21} \sqrt{\cos^2 \delta_r x^2 + y^2}},$$  \hfill (11)

$$\tan \beta_2 = -\frac{j_{21} + \sin \delta_r x}{\sqrt{\cos^2 \delta_r x^2 + y^2}}.$$

In accordance with Fig. 3, the coordinates of point $P$, can be obtained as follows:

$$x = -r_1 \cos \theta_1, \quad y = -r_1 \sin \theta_1, \quad z = -r_1 \cot \delta_1.$$  \hfill (12)
Since the geometric pitch line $H^G_2$ is a directrix of the circumferential velocity $\bar{V}_2$ in the pitch contact point, then for it the following condition is fulfilled:

$$\bar{m} \cdot \bar{V}_2 = 0 .$$

Substituting (3) and (9) into (13), the next expression is obtained:

$$\sin \theta_1 = -\tan \delta_1 \cot \delta_r .$$

Formula (14) defines the following condition for existence of the geometric pitch configurations:

$$|\tan \delta_1 \cot \delta_r| \leq 1 .$$

The symbols in Fig. 3 are as follows: $l$–$l$ – axis of rotation of the link $i = 1$; $2t$–$2t$ – direction of translation of the link $i = 2$; $H^G_1$ – geometric pitch circle with radius $r_1$; $H^G_2$ – geometric pitch line; $h$ – unit vector of $H^G_2$; $H_1$ – geometric pitch

![Fig. 3. Geometric pitch configurations of spatial rack.](image-url)
cone; \( H_2 \) – geometric pitch plane; \( T_m \) – pitch plane; \( m-m \) – pitch normal; \( \vec{m} \) – unit vector; \( \vec{V}_i \) – circumferential velocity of the pitch contact point \( P \); \( \vec{V}_{12} \) – sliding velocity; \( L_i (i = 1, 2) \) – longitudinal lines of the active tooth surfaces \( \Sigma_i (i = 1, 2) \); \( t-t \) – tangent line to \( L_i \) in point \( P \); \( m-m \) – normal line to \( L_i \) in point \( P \); \( \delta_1 \) – angle, that determines the orientation of the plane containing \( H_G^1 \) against \( m-m \) (angle of the geometric pitch cone \( H_1 \)). The following equalities, which define the coordinates of the pitch contact point \( P \), are obtained from (12) and (14):

\[
\begin{align*}
x &= -r_1 \sqrt{1 - \tan^2 \delta_1 \cot^2 \delta_r}, \\
y &= +r_1 \tan \delta_1 \cot \delta_r, \\
z &= -r_1 \cot \delta_1.
\end{align*}
\]

After substituting (16) in (11), for \( \beta_i (i = 1, 2) \) is received

\[
\begin{align*}
\tan \beta_1 &= \frac{\left( \frac{r_1}{j_{21}} \right) \cos \delta_1 + \sqrt{\sin(\delta_r + \delta_1) \sin(\delta_r - \delta_1)}}{\cos \delta_r}, \\
\tan \beta_2 &= \frac{\left( \frac{r_1}{j_{21}} \right)^{-1} \cos \delta_1 - \sqrt{\sin(\delta_r + \delta_1) \sin(\delta_r - \delta_1)}}{\cos \delta_r}.
\end{align*}
\]

For the studied case, shown in Fig. 3, the geometric pitch circle \( H_G^1 \) of the rotating link of the spatial rack drive corresponds to the geometric pitch surface, which represents straight circular cone \( H_1 \) with angle at the top \( \delta_1 \) (pitch cone). The plane \( T_m \), where the geometric pitch plane \( H_G^2 \) lies, coincides with the geometric pitch plane \( H_2 \) of the rack mechanism. The geometric pitch circle \( H_G^2 \) and straight line \( H_G^2 \) and the geometric pitch cone \( H_1 \) and plane \( H_2 \) are the elements of so-called geometric pitch configurations of the spatial rack drive.

If \( \delta_1 = 0 \), then the geometric pitch cone \( H_1 \) of the rack drive is transformed into geometric pitch cylinder \( H_1 \). In this case, the angles, which determine the direction of the longitudinal lines \( L_1 \) and \( L_2 \), passing through the pitch contact point, are defined with the expressions:

\[
\begin{align*}
\tan \beta_1 &= \frac{\left( \frac{r_1}{j_{21}} \right) \sin \delta_r}{\cos \delta_r}, \quad \tan \beta_2 = \frac{\left( \frac{r_1}{j_{21}} \right)^{-1} \sin \delta_r}{\cos \delta_r}.
\end{align*}
\]

3. Synthesis of Kinematic Pitch Configurations

3.1. Essence and Content of the Kinematic Pitch Configurations

Kinematic pitch surfaces, are surfaces obtained as envelopes of a family of normalized relative helices of the vector function \( \vec{V}_{12} = \vec{V}_{12}(x_0, y_0, z_0) \) in their relative
motion against the axes, or motions transformation of type \((R \leftrightarrow R)\) and \((R \leftrightarrow T)\). Normalized relative helices are placed on the specifically defined kinematic cylinders of level [1,2]. In other words, the kinematic pitch surfaces, for the study case, should be considered as envelopes of the corresponding kinematic pitch cylinders for the above mentioned relative motions. These surfaces are fixed to the co-ordinate systems \(S_i (i = 1,2)\). In relation to this and based on Fig. 4, the kinematic cylinders of level, in case of motions transformation of type \((R \leftrightarrow T)\), are defined with the following:

\[
\begin{align*}
x_0 &= R_0 \sin t, & y_0 &= R_0 \cos t, \\
z_0 &= H, & H &\in (-\infty, +\infty), & t &= \arcsin \frac{R_0}{x_0}.
\end{align*}
\]

Fig. 4. Geometric-kinematic scheme of motions transformation of type \((R \leftrightarrow T)\) and vice versa.
In Fig. 4, the following symbols are shown: \( S(O, x, y, z) \) and \( S_0(0, x_0, y_0, z_0) \) – fixed coordinate systems; \( S_i(O_i, x_i, y_i, z_i) \) – coordinate systems connected with the movable links \( i = 1, 2 \); \( \Sigma_i \) \((i = 1, 2)\) – geometric elements of the high kinematic joint \((\Sigma_1 : \Sigma_2)\); \( D_{12} \) – instantaneous contact line between \( \Sigma_1 \) and \( \Sigma_2 \); \( P \) – instantaneous contact point between \( \Sigma_1 \) and \( \Sigma_2 \); \( I-I \) – rotation axis; \( 2t-2t \) – direction of rectilinear translation; \( \vec{\omega}_1 \) – angular velocity vector of the rotating link \( i = 1 \); \( \vec{V}_2 \) – translation velocity vector of the link \( i = 2 \); \( \vec{p}_{i/s/s_0} \) – radius-vector of the contact point \( P \), as a point from \( \Sigma_1 \) and \( \Sigma_2 \), written in the coordinate systems \( S \) and \( S_0 \); \( \delta_r \) – crossed angle between vectors, characterizing motions transformation; \( \Sigma_r \) – inter-axes angle of the rack drive; \( \varphi_1 \) – meshing parameter.

Let the canonic form (19) of the kinematic cylinders of level is written in the coordinate systems \( S_i \) \((i = 1, 2)\), in accordance with the symbols in Fig. 4. In order to do this, the following matrix equalities are used:

\[
\begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1 \\
  t_1
\end{bmatrix}^T = M_{S_1S_0} \begin{bmatrix}
  x_0 \\
  y_0 \\
  z_0 \\
  t_0
\end{bmatrix}^T,
\]

where

\[
M_{S_1S_0} = \begin{bmatrix}
  \cos \varphi_1 & \sin \varphi_1 & 0 & -j_{21} \sin \Sigma_r \cos \varphi_1 \\
  -\sin \varphi_1 & \cos \varphi_1 & 0 & +j_{21} \sin \Sigma_r \sin \varphi_1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2 \\
  t_2
\end{bmatrix}^T = M_{S_2S_0} \begin{bmatrix}
  x_0 \\
  y_0 \\
  z_0 \\
  t_0
\end{bmatrix}^T,
\]

where

\[
M_{S_2S_0} = \begin{bmatrix}
  1 & 0 & 0 & -j_{21} \sin \Sigma_r \\
  0 & 1 & 0 & j_{21} \sin \Sigma_r \varphi_1 \\
  0 & 0 & 1 & j_{21} \cos \Sigma_r \varphi_1 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

When solving (20) and (21), it is obtained the corresponding:

\[
x_1 = R_0 \sin(t + \varphi_1) - j_{21} \sin \Sigma_r \cos \varphi_1,
\]

\[
y_1 = R_0 \cos(t + \varphi_1) + j_{21} \sin \Sigma_r \sin \varphi_1,
\]

\[
z_1 = H, \quad H \in (-\infty, +\infty)
\]

and

\[
x_2 = R_0 \sin t - j_{21} \sin \Sigma_r,
\]

\[
y_2 = R_0 \cos t + j_{21} \sin \Sigma_r \varphi_1,
\]

\[
z_2 = H + j_{21} \cos \Sigma_r \varphi_1, \quad H \in (-\infty, +\infty).
\]
The equations systems (22) and (23) describe families of the enveloped kinematic surfaces of level. Their envelopes, which are treated as locus of the contacting lines of the enveloped and enveloping surfaces, written in coordinate systems $S_i$ ($i = 1, 2$), are obtained when, the following kinematic condition is added to (22) and (23):

$$\bar{e}_{0,s_i} \cdot \bar{V}_{i,s_i} = 0 \quad (i = 1, 2),$$

where: $\bar{e}_0$ is an unit vector of an arbitrary kinematic cylinder of level in its arbitrary point $P_0(x_0, y_0, z_0)$; $\bar{V}_i$ – relative velocities ($i = 1$ – rotation link and $i = 2$ – translation one) of arbitrary point $P_0$ of the arbitrary kinematic cylinder of level; $S_i$, ($i = 1, 2$) – coordinate systems, connected with the movable link $i = 1$, which realizes a rotation motion with an angular velocity $\omega_1$, and $i = 2$, realizing a translation with velocity $\bar{V}_2$. Then, the studied envelopes are of the following analytical type:

$$x_1^2 + y_1^2 = (R_0 \mp j_{21} \sin \Sigma_r)^2, \quad z_1 = H, \quad H \in (-\infty, +\infty),$$

and

$$x_2 = \pm R_0 - j_{21} \sin \Sigma_r, \quad \cot \Sigma_r y_2 - z_2 + H = 0, \quad H \in (-\infty, +\infty).$$

Therefore, the kinematic pitch cylinders of the spatial rack mechanisms, are two rotational cylinders (25) and contacting with them corresponding planes (26):

$$x_1^2 + y_1^2 = (R_0 - j_{21} \sin \Sigma_r)^2, \quad z_1 = H,$$

$$x_2 = R_0 - j_{21} \sin \Sigma_r, \quad \cot \Sigma_r y_2 - z_2 + H = 0, \quad H \in (-\infty, +\infty),$$

$$x_1^2 + y_1^2 = (R_0 + j_{21} \sin \Sigma_r)^2, \quad z_1 = H,$$

$$x_2 = -(R_0 + j_{21} \sin \Sigma_r), \quad \cot \Sigma_r y_2 - z_2 + H = 0, \quad H \in (-\infty, +\infty).$$

From (27) and (28) it is obvious, that when $R_0 = 0$, the axodes of the rack drive are obtained. They are cylinder and a plane, which are tangent to the zero kinematic cylinder $O_0z_0$ (see Fig. 7).

The kinematic pitch cylinders are visualized, as a result of the created computer program. They are determined for the concrete case of motions transformation, when the equations from the system (27) (see Fig. 5) and (28) (see Fig. 6) are used.

The axodes – cylinder and plane of the spatial rack transmission are shown in Fig. 7.

From the study realized and illustrated here, it is obviously the following [2]:
The kinematic pitch cylinders are visualized, as a result of the created computer program. They are determined for the concrete case of motions transformation, when the equations from the system (27) (see Fig. 5) and (28) (see Fig. 6) are used.

- Every common point of the axodes (cylinder and plane) and of the pitch surfaces (cylinder and plane) coincides with the point of normalized kinematic relative helix;

- The axis of the relative helices $O_{0z}$, which coincide with zero kinematic relative helix, is parallel to the geometric axis $1–1$ of the rotating link $i = 1$;

- Helix angle of the family of the normalized relative helices, which are placed on a concrete kinematic cylinder at every common point of the pitch surfaces (including the axodes), is a constant one.
The axodes -cylinder and plane of the spatial rack transmission are shown in Fig. 7.

From the study realized and illustrated here, it is obviously the following [2]:

- Every common point of the axodes (cylinder and plane) and of the pitch surfaces (cylinder and plane) coincides with the point of normalized kinematic relative helix;
- The axis of the relative helices \( O \), which coincide with zero kinematic relative helix, is parallel to the geometric axis of the rotating link \( 1i \);
- Helix angle of the family of the normalized relative helices, which are placed on a concrete kinematic cylinder at every common point of the pitch surfaces (including the axodes), is a constant one.

The defined features related to the kinematic pitch surfaces allow to determine simply the kinematic pitch circle and the straight line. They are the cross section of the kinematic pitch cylinder, which axis is the axis of rotation and the straight line, which lies in the kinematic pitch plane, and it is parallel to the direction of the translation. They have a common point, which lies on the normalized relative helix. The corresponding pairs of kinematic cylinder and the plane and the pair of pitch kinematic circles and the line, form the kinematic pitch configurations are defined.

3.2. Orientation of the longitudinal lines of the conjugated tooth surfaces

Let’s the longitudinal orientation of the active tooth surfaces \( \Sigma_i \) of the movable links of spatial rack drive is defined, when its synthesis is based on the application of kinematic pitch configurations. The study, realized in [2], shows that from a theoretical view point, (which is based on the kinematic synthesis of this class of mechanisms), the rotating link should be chosen as a cylindrical gear with helical teeth \( \Sigma_1 \), meshed with theoretically conjugated teeth \( \Sigma_2 \) of the second movable link, that performs translation – gear rack. In this case, the longitudinal orientation of the active tooth surfaces \( \Sigma_i \) \((i = 1, 2)\), is determined through the angles of their inclination \( \beta_i \) \((i = 1, 2)\) (helix angles). The following can be written, on the base of the Section 2.4 and symbols given in Fig. 4:

\[
\tan \beta_1 = \frac{x^2 + y^2 + j_{21} \sin \Sigma_r x}{j_{21} \sqrt{\cos^2 \Sigma_r x^2 + y^2}},
\]
\[
\tan \beta_2 = -\frac{\sin \Sigma_r, x + j_{21}}{\sqrt{\cos^2 \Sigma_r, x^2 + y^2}}.
\]

For the studied case, when the angles of inclination of the active tooth surfaces at points of the pitch surfaces, are defined, then the following equalities are true:

\[
x = \pm R_p, \quad y = 0,
\]

where \(R_p\) is radius of the pitch cylinder.

Substituting (31) in (29) and (30), it is obtained:

\[
\tan \beta_1 = \frac{R_p}{j_{21}} \pm \frac{\sin \Sigma_r,}{\cos \Sigma_r} ,
\]

\[
\tan \beta_2 = -\left(\frac{R_p}{j_{21}}\right)^{-1} \pm \frac{\sin \Sigma_r,}{\cos \Sigma_r}.
\]

Considering that:

\[
R_0 = R_p \mp j_{21} \sin \Sigma_r ,
\]

then (32) is presented as follows:

\[
\tan \beta_1 = \frac{R_p}{j_{21}} \pm \frac{j_{21} \sin \Sigma_r,}{\cos \Sigma_r} = \frac{R_0}{p_{z_0}} = \tan \beta_0 ,
\]

where \(\beta_0\) is an angle of inclination of the normalized relative helix.

The equalities \(\beta_1 = \beta_0\), (that is resulted from (35)), is obvious due to the fact that the zero relative helix \(O_0z_0\) is parallel to the rotation axis \(l-l\) \((O_1z_1)\) [2].

4. Conclusion

On the basis of realized review and analysis of the scientific concepts of primary and pitch circles and surfaces applied in the synthesis of hyperboloid gears, the term geometric pitch configurations for the mechanical motions convertor of the type spatial rack drives is introduced, terminologically and as content as well. An adequate mathematical model applied for defining the conditions of existence of geometric pitch configurations and as well the inclination angles of the conjugate tooth surfaces at the pitch contact point (determined by the pitch configurations), is elaborated. When the structural, geometric and kinematic similarity between hyperboloid transmissions and spatial rack drives are used, then the kinematic pitch configurations and the corresponding inclination angles of the conjugate tooth surfaces are defined for the case of spatial rack mechanism.
REFERENCES

[1] ABADJIEV, V. Gearing Theory and Technical Applications of Hyperboloid Mechanisms, Sc. D. Thesis, Institute of Mechanics, Bulgarian Academy of Sciences, Sofia, 2007, 309 (In Bulgarian).

[2] ABADJIEVA, E. Mathematical Models of the Kinematic Processes in Spatial Rack Mechanisms and their Application, Ph. D Thesis, Institute of Mechanics- BAS, Sofia, Bulgaria, 2010, 165 (in Bulgarian).

[3] ABADJIEVA, E. Spatial Rack Drives. Mathematical Modelling for Synthesis, VDM Verlag Dr. Müller e.K., 2011, 72.

[4] LITVIN, F. Theory of Gearing, Moskow, Nauka, 1968, 584 (in Russian).

[5] MINKOV, K. S. Mechano- Mathematical Modeling of Hyperboloid Gear Sets, Sc. D. Thesis, Institute of Mechanics, Bulgarian Academy of Sciences, Sofia, 1986, 303.

[6] LITVIN, F. Theory of Gearing. NASA Reference Publication 1212, AVSCOM Technical Report 88-C-035, Washington, US Government Printing Office, 1989, 470.

[7] LITVIN, F. Gearing Geometry and Applied Theory, New Jersey, PTR Prentice Hall, A Paramount Communication Company, Englewood Eliffs, 07632, 1994, 724.

[8] NELSON, W. Spiroid Gearing Part 1 – Basic Design Practices. J. Machine Design, 1961, 136-144.

[9] PETROV, K. Synthesis Basics of the Spatial Gears with Arbitrary Crossed Axes Primary Surfaces. Journal of Theoretical and Applied Mechanics, 3 (1975), BAS, Sofia, 61-74.

[10] Bulgarian State Standard 8540 – 84 Gear Mechanisms. Common Terminology. Definitions and Notations, Sofia, State Committee of Science, 1985, 172.

[11] KOVATCHEV, G., V. ABADJIEV. On the Synthesis of Spatial Rack Mechanisms, Proc. 6-th National Congress of Theoretial and Applied Mechanics, Varna, 1, 1990, 35-38, (in Russian).

[12] LITVIN, F., A. FUENTES. A Gearing Geometry and Applied Theory, Second Edition, New York, Melbourne, Madrid, cape Town, Singapore, Sao Paulo, Delhi, Tokyo, Mexico City, Cambridge University Press, Cambridge, 2004, 795.

[13] BOONE, D. A. Kinematic Geometry of Gearing. Second Edition, United Kingdom, 2012, A John Wiley & Sons, Ltd., Publicatio, ISBN 978-1-119-95094-3 (hardback), 496.

[14] ABADJIEV, V., E. ABADJIEVA. Geometric Pitch Configurations – Basic Primitives of the Mathematical Models for the Synthesis of Hyperboloid Gear Drives, Advanced Gear Engineering, Mechanism and Machine Science 51, Springer International Publishing Switzerland, 2018, 91-117.