A construction of $\mathcal{C}^*$-algebras from $\mathcal{C}^*$-correspondences

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Abstract. We introduce a method to define $\mathcal{C}^*$-algebras from $\mathcal{C}^*$-correspondences. Our construction generalizes Cuntz-Pimsner algebras, crossed products by Hilbert $\mathcal{C}^*$-modules, and graph algebras.

0. Introduction

In this short article, we introduce a new way to define $\mathcal{C}^*$-algebras from $\mathcal{C}^*$-correspondences, which generalizes many known constructions in various situations. By a $\mathcal{C}^*$-correspondence, we mean a (right) Hilbert $\mathcal{C}^*$-module together with a left action (Definition 1.3). In some articles, this is called a Hilbert $\mathcal{C}^*$-bimodule. In this paper, the term Hilbert $\mathcal{C}^*$-bimodule is reserved to denote a right Hilbert $\mathcal{C}^*$-module which is simultaneously a left Hilbert $\mathcal{C}^*$-module (Definition 3.1). Namely, a Hilbert $\mathcal{C}^*$-bimodule has a left inner product, but a $\mathcal{C}^*$-correspondence may not. See Section 3.3 in this paper for the relation of $\mathcal{C}^*$-correspondences and Hilbert $\mathcal{C}^*$-bimodules.

In [P], Pimsner introduced a class of $\mathcal{C}^*$-algebras now called Cuntz-Pimsner algebras. He constructed $\mathcal{C}^*$-algebras from $\mathcal{C}^*$-correspondences. In [P], left actions of $\mathcal{C}^*$-correspondences are assumed to be injective, because it is very possible that Cuntz-Pimsner algebras become zero if left actions are not injective [P, Remark 1.2 (1)]. The class of Cuntz-Pimsner algebras is so wide that it includes crossed products by automorphisms, Cuntz algebras and Cuntz-Krieger algebras, for example. However it does not contain some interesting examples, such as graph algebras of graphs with sinks [FLR] and crossed products by partial automorphisms [E]. (It was claimed in [P, Example (4)] that crossed products by partial automorphisms can be obtained by the construction in [P]. However, it seems that one needs the assumption that the ideal I is essential.) In [AEE], Abadie, Eilers and Exel gave a method to define $\mathcal{C}^*$-algebras from Hilbert $\mathcal{C}^*$-bimodules, which generalizes crossed products by partial automorphisms. Since Hilbert $\mathcal{C}^*$-bimodules are particular examples of $\mathcal{C}^*$-correspondences, we can apply the construction in [P] for Hilbert $\mathcal{C}^*$-bimodules. This gives the same $\mathcal{C}^*$-algebras as in [AEE] only when the left actions of the Hilbert $\mathcal{C}^*$-bimodules are injective. For Hilbert $\mathcal{C}^*$-bimodules with

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non-injective left actions, the method in [AEE] is different from the one in [P]. In the introduction of [AEE], they mentioned that it seems reasonable to expect that a common generalization of the constructions in [AEE] and in [P] could be found. Our construction gives this common generalization.

We construct $C^*$-algebras from arbitrary $C^*$-correspondences. We assume no conditions on the $C^*$-correspondences. In particular, Hilbert $C^*$-modules need not be full, and left actions need be neither injective nor non-degenerate. When the left action is injective, our construction is same as the one of Cuntz-Pimsner algebras. When the $C^*$-correspondence comes from a Hilbert $C^*$-bimodule, our construction is the same as in [AEE]. Graph algebras of arbitrary graphs can be naturally obtained by our procedure. More generally, $C^*$-algebras of topological graphs [D, K1] are in our class of $C^*$-algebras. Thus one of the advantages of our construction is that it generalizes many known methods. Another advantage is that our $C^*$-algebras are closely related to the $C^*$-correspondences. Namely, the map from a $C^*$-correspondence to its $C^*$-algebra is always injective, and so we can recover all the information about a $C^*$-correspondence from its $C^*$-algebra. This is not the case for Cuntz-Pimsner algebras when the left action of a $C^*$-correspondence is not injective. Finally, we emphasize that even if one starts with a $C^*$-correspondence with an injective and non-degenerate left action, one needs to work with $C^*$-correspondences with non-injective or degenerate left actions, in order to know about the ideals and quotients of its $C^*$-algebra (see [FMR, K2]).

After preliminaries on Hilbert $C^*$-modules and $C^*$-correspondences in Section 1, we give the definition of our $C^*$-algebras constructed from $C^*$-correspondences in Section 2. Section 3 is devoted to the study of examples.

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1. $C^*$-correspondences and their representations.

**Definition 1.1.** Let $A$ be a $C^*$-algebra. A (right) Hilbert $A$-module $X$ is a Banach space with a right action of the $C^*$-algebra $A$ and an $A$-valued inner product $\langle \cdot, \cdot \rangle_X$ satisfying

(i) $\langle \xi, \eta a \rangle_X = \langle \xi, \eta \rangle_X a$,

(ii) $\langle \xi, \eta \rangle_X = (\langle \eta, \xi \rangle_X)^*$,

(iii) $\langle \xi, \xi \rangle_X \geq 0$ and $\|\xi\| = \|\langle \xi, \xi \rangle_X\|^{1/2}$,

for $\xi, \eta \in X$ and $a \in A$.

**Definition 1.2.** For a Hilbert $A$-module $X$, we denote by $\mathcal{L}(X)$ the $C^*$-algebra of all adjointable operators on $X$. For $\xi, \eta \in X$, the operator $\theta_{\xi, \eta} \in \mathcal{L}(X)$ is defined by $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle_X$ for $\zeta \in X$. We define $\mathcal{K}(X) \subset \mathcal{L}(X)$ by

$$\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi, \eta} \mid \xi, \eta \in X\},$$

where $\overline{\text{span}}\{\cdots\}$ means the closure of linear span of $\{\cdots\}$. 
The set $\mathcal{K}(X)$ is an ideal of $\mathcal{L}(X)$, where an ideal of a $C^*$-algebra always means a two-sided closed ideal, which is automatically $*$-invariant.

**Definition 1.3.** For a $C^*$-algebra $A$, we say that $X$ is a $C^*$-correspondence over $A$ when $X$ is a Hilbert $A$-module and a $*$-homomorphism $\varphi_X : A \to \mathcal{L}(X)$ is given.

We refer to $\varphi_X$ as the left action of a $C^*$-correspondence $X$.

**Definition 1.4.** A $C^*$-correspondence $X$ over a $C^*$-algebra $A$ is called faithful if $\varphi_X$ is injective, non-degenerate if $\operatorname{span}\{\varphi_X(a)\xi \in X \mid a \in A, \xi \in X\} = X$, and full if $\operatorname{span}\{(\xi, \eta)X \in A \mid \xi, \eta \in X\} = A$.

**Definition 1.5.** A representation of a $C^*$-correspondence $X$ over $A$ on a $C^*$-algebra $B$ is a pair $(\pi, t)$ consisting of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ satisfying

(i) $t(\xi^*t(\eta) = \pi((\xi, \eta)x)$ for $\xi, \eta \in X$,
(ii) $\pi(a)t(\xi) = t(\varphi_X(a)\xi)$ for $a \in A, \xi \in X$.

A representation of a $C^*$-correspondence is called an isometric covariant representation in [MS]. For a representation $(\pi, t)$ on $B$, we denote by $C^*(\pi, t)$ the $C^*$-algebra generated by the images of $\pi$ and $t$ in $B$. Note that for a representation $(\pi, t)$ of $X$, we have $t(\xi)\pi(a) = t(\xi a)$ automatically because of the condition (i) above, combined with the fact that $\pi$ is a $*$-homomorphism, implies

$$\|t(\xi)\pi(a) - t(\xi a)\|^2 = \|t(\xi)(\pi(a) - t(\xi a))\|^2 = \|t(\xi)\|^2 = 0.$$  

Note also that for $\xi \in X$, we have $\|t(\xi)\| \leq \|\xi\|$ because

$$\|t(\xi)\|^2 = \|t(\xi^*t(\xi))\| = \|\pi(\langle \xi, \xi \rangle x)\| \leq \|\xi, \xi\rangle \times \|\xi\|^2.$$

**Definition 1.6.** A representation $(\pi, t)$ is said to be injective when the $*$-homomorphism $\pi$ is injective.

By the above computation, we see that $t$ is isometric for an injective representation $(\pi, t)$.

**Definition 1.7.** For a representation $(\pi, t)$ of a $C^*$-correspondence $X$ on $B$, we define a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B$ by $\psi_t(\theta_{\xi, \eta}) = t(\xi)t(\eta)^* \in B$ for $\xi, \eta \in X$.

For the well-definedness of the $*$-homomorphism $\psi_t$, see [KPW, Lemma 2.2], for example. Note that $\psi_t(\mathcal{K}(X)) \subset C^*(\pi, t)$. Note also that $\psi_t$ is injective for an injective representation $(\pi, t)$.

### 2. $C^*$-algebras associated with $C^*$-correspondences

**Definition 2.1.** For an ideal $I$ of a $C^*$-algebra $A$, we define $I^\perp \subset A$ by

$$I^\perp = \{a \in A \mid ab = 0 \text{ for all } b \in I\}.$$  

Recall that an ideal $I$ is called essential if $I^\perp = 0$.

**Lemma 2.2.** For an ideal $I$ of a $C^*$-algebra $A$, $I^\perp$ is an ideal of $A$, and for an ideal $J$ of $A$, we have $J \subset I^\perp$ if and only if $J \cap I = \{0\}$.

**Proof.** It is clear that $I^\perp$ is an ideal. It is also clear that $I^\perp \cap I = \{0\}$. Hence if an ideal $J$ satisfies $J \subset I^\perp$, then we have $J \cap I = \{0\}$. Conversely, suppose that an ideal $J$ satisfies $J \cap I = \{0\}$. Since $ab \in J \cap I = \{0\}$ for any $a \in J$ and $b \in I$, we have $J \subset I^\perp$. This completes the proof. $\Box$
Lemma 2.2 means that $I^\perp$ is the maximal ideal satisfying $I^\perp \cap I = \{0\}$.

**Definition 2.3.** For a $C^*$-correspondence $X$ over $A$, we define an ideal $J_X$ of $A$ by

$$J_X = \varphi_X^* (\mathcal{K}(X)) \cap (\ker \varphi_X)^\perp.$$

Note that $J_X = \varphi_X^* (\mathcal{K}(X))$ when $\varphi_X$ is injective. By Lemma 2.2, we see that the ideal $J_X$ is the maximal ideal on which the restriction of $\varphi_X$ is an injection into $\mathcal{K}(X)$. From this fact, we easily get the following.

**Lemma 2.4.** If there exists an ideal $J$ of $A$ such that the restriction of $\varphi_X$ to $J$ is an isomorphism onto $\mathcal{K}(X)$, then $J = J_X$.

**Definition 2.5.** A representation $(\pi, t)$ is said to satisfy Condition $(\ast)$ if we have $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J_X$.

**Definition 2.6.** For a $C^*$-correspondence $X$ over a $C^*$-algebra $A$, the $C^*$-algebra $O_X$ is defined by $O_X = C^*(\pi_X, t_X)$ where $(\pi_X, t_X)$ is the universal representation of $X$ satisfying Condition $(\ast)$.

One can easily show the existence of the universal representation of $X$ satisfying Condition $(\ast)$ (see, for example, [B]). One can also define $O_X$ in a more concrete way using a Fock space as in [P] or [MS]. We have that $O_X$ is isomorphic to the relative Cuntz-Pimsner $C^*$-algebra $O(J_X, X)$ defined in [MS, Definition 2.18] (see Example 3.2 in this paper). Since $J_X \cap \ker \varphi_X = \{0\}$, [MS, Proposition 2.21] gives us the following.

**Proposition 2.7.** The universal representation $(\pi_X, t_X)$ of $X$ on $O_X$ is injective.

In [K2], we show that the $C^*$-algebra $O_X$ is the smallest $C^*$-algebra among $C^*$-algebras $C^*(\pi, t)$ generated by representations $(\pi, t)$ of $X$ which are injective and admit certain actions of $T$ called gauge actions. This fact gives another definition of $O_X$ without using Condition $(\ast)$ or $J_X$.

3. Examples

3.1. Cuntz-Pimsner algebras. In [P], Pimsner defined two kinds of $C^*$-algebras from a $C^*$-correspondence $X$, which are now called the Toeplitz algebra and the Cuntz-Pimsner algebra of $X$, respectively. He also defined the augmented ones. When the $C^*$-correspondence $X$ is faithful, the $C^*$-algebra $O_X$ defined in this paper is the same as the augmented Cuntz-Pimsner algebra of $X$. When the $C^*$-correspondence $X$ is full, the augmented Cuntz-Pimsner algebra of $X$ coincides with its Cuntz-Pimsner algebra. Hence our $C^*$-algebra $O_X$ is the same as the Cuntz-Pimsner algebra of $X$ when the $C^*$-correspondence $X$ is both faithful and full. For a general $C^*$-correspondence $X$, we have a surjection from our $C^*$-algebra $O_X$ onto the augmented Cuntz-Pimsner algebra of $X$. This surjection is never injective unless the $C^*$-correspondence $X$ is faithful.

3.2. Relative Cuntz-Pimsner algebras. Let $X$ be a $C^*$-correspondence over $A$. Let $J_0$ be an ideal of $A$ such that $\varphi_X(J_0) \subset \mathcal{K}(X)$. The relative Cuntz-Pimsner algebra $O(J_0, X)$ is the universal $C^*$-algebra generated by representations $(\pi, t)$ of $X$ satisfying $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J_0$. Actually Muhly and Solel defined relative Cuntz-Pimsner algebras in more concrete way using Fock spaces.
in [MS, Definition 2.18], and then proved that they have the universal property in [MS, Theorem 2.19]. Note that when \( J_0 = 0 \), \( \mathcal{O}(J_0, X) \) is the augmented Toeplitz algebra of \( X \), and when \( J_0 = \varphi_X^{-1}(K(X)) \), \( \mathcal{O}(J_0, X) \) is the augmented Cuntz-Pimsner algebra. Our \( C^* \)-algebra \( \mathcal{O}_X \) is isomorphic to \( \mathcal{O}(J_X, X) \). In [K2], we prove that the relative Cuntz-Pimsner algebra \( \mathcal{O}(J_0, X) \) is isomorphic to \( \mathcal{O}_X \) for a certain \( C^* \)-correspondence \( X' \). This observation helps us to study relative Cuntz-Pimsner algebras.

### 3.3. Crossed products by Hilbert \( C^* \)-bimodules.

**Definition 3.1.** Let \( A \) be a \( C^* \)-algebra. A *Hilbert \( A \)-bimodule* \( X \) is a Hilbert \( A \)-module together with a left action \( \varphi_X : A \rightarrow \mathcal{L}(X) \) and a left inner product \( X(\cdot, \cdot) : X \times X \rightarrow A \), which satisfy

(i) \( x(\varphi_X(a)\xi, \eta) = a \cdot x(\xi, \eta) \),
(ii) \( x(\xi, \eta) = x(\eta, \xi)^* \),
(iii) \( x(\xi, \xi) \geq 0 \),

for \( \xi, \eta \in X, a \in A \) and \( \xi, \eta, \zeta \in X \).

(3.1) \[ \varphi_X(x(\xi, \eta))\zeta = \xi(\eta, \zeta)_X, \]

for \( \xi, \eta, \zeta \in X \).

In the original definition ([BMS, Definition 1.8]), it was not assumed that \( \varphi_X(a) : X \rightarrow X \) is adjointable and that its adjoint is \( \varphi_X(a^*) \). However this fact follows from (3.1) (see [BMS]). Following [BMS], we define an ideal \( I_X \subset A \) by

\[ I_X = \overline{\text{span}}\{ x(\xi, \eta) \in A \mid \xi, \eta \in X \}. \]

**Lemma 3.2.** Let \( X \) be a *Hilbert \( A \)-bimodule*. For \( a \in A \), we have \( \varphi_X(a) = 0 \) if and only if \( a \in I_X^2 \).

**Proof.** If \( \varphi_X(a) = 0 \) then we have \( a \cdot x(\xi, \eta) = x(\varphi_X(a)\xi, \eta) = 0 \) for all \( \xi, \eta \in X \). Hence \( a \in I_X^2 \). If \( a \in I_X^2 \) then we have

\[ x(\varphi_X(a)\xi, \varphi_X(a)\zeta) = a \cdot x(\xi, \varphi_X(a)\zeta) = 0 \]

for all \( \xi \in X \). Hence \( \varphi_X(a) = 0 \). \( \square \)

The equation (3.1) means that

\[ \varphi_X(x(\xi, \eta)) = \theta_{\xi, \eta}, \]

for \( \xi, \eta \in X \). Hence the restriction of \( \varphi_X \) to the ideal \( I_X \) is a surjection onto \( K(X) \). By Lemma 3.2, this restriction is injective. Thus the restriction of \( \varphi_X \) to the ideal \( I_X \) is an isomorphism onto \( K(X) \) ([BMS, Proposition 1.10]). Hence Lemma 2.4 gives us the following.

**Lemma 3.3.** Let \( A \) be a \( C^* \)-algebra, and \( X \) be a *Hilbert \( A \)-bimodule*. If we consider \( X \) as a \( C^* \)-correspondence over \( A \), then we have \( J_X = I_X \).

By Lemma 3.3, the left inner product \( X(\cdot, \cdot) \) of a Hilbert \( A \)-bimodule \( X \) is completely determined by the structure of \( X \) as a \( C^* \)-correspondence. Namely, for \( \xi, \eta \in X \), \( X(\xi, \eta) \in A \) is the unique element \( a \in J_X \) with \( \varphi_X(a) = \theta_{\xi, \eta} \). We can reverse this procedure.
Lemma 3.4. Let $A$ be a $C^*$-algebra, and $X$ be a $C^*$-correspondence over $A$. If
$\varphi_X(J_X) = \mathcal{K}(X)$, then $X$ is a Hilbert $A$-bimodule with the left inner product $\langle \cdot, \cdot \rangle$ given by
$$\langle \xi, \eta \rangle = (\varphi_X|_{J_X})^{-1}(\theta_{\xi,\eta}) \in J_X \subset A$$
for $\xi, \eta \in X$.

Proof. Straightforward. \qed

By the above argument, we can say that Hilbert $A$-bimodules are nothing but $C^*$-correspondences $X$ over $A$ satisfying $\varphi_X(J_X) = \mathcal{K}(X)$, and left inner products of Hilbert $A$-bimodules are uniquely determined by the structures as $C^*$-correspondences. A $C^*$-correspondence arising from a Hilbert $C^*$-bimodule is always non-degenerate, and it is faithful if and only if $J_X$ is an essential ideal in $A$ by Lemma 3.2 and Lemma 3.3. In [AEE], Abadie, Eilers and Exel defined the crossed product $A \rtimes_X Z$ by a Hilbert $A$-bimodule $X$ as follows.

Definition 3.5 ([AEE, Definition 2.1]). Let $X$ be a Hilbert $A$-bimodule. A covariant representation of $X$ on a $C^*$-algebra $B$ is a pair $(\pi, t)$ consisting of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ satisfying

(i) $t(\varphi_X(a)\xi) = \pi(a)t(\xi),$  
(ii) $t(\xi a) = t(\xi)\pi(a),$  
(iii) $\pi(\langle \xi, \eta \rangle) = t(\xi)t(\eta)^*,$  
(iv) $\pi(\langle \xi, \eta \rangle)_X = t(\xi)^*t(\eta),$  

for $a \in A$ and $\xi, \eta \in X$.

In [AEE], they considered only the case that $B = B(H)$ for some Hilbert space $H$.

Definition 3.6 ([AEE, Definition 2.4]). Let $A$ be a $C^*$-algebra, and $X$ be a Hilbert $A$-bimodule. The crossed product $A \rtimes_X Z$ of $A$ by $X$ is the universal $C^*$-algebra generated by the images of covariant representations of $X$.

The conditions (i) and (iv) in Definition 3.5 are the same as the conditions of representations of $X$ as a $C^*$-correspondence. The condition (ii) follows from (iv) (and also the condition (i) follows from (iii), but we do not use this fact). Since $J_X = \text{span}\{\langle \xi, \eta \rangle \in A \mid \xi, \eta \in X\}$ and $\varphi_X(\langle \xi, \eta \rangle) = \theta_{\xi,\eta}$ for $\xi, \eta \in X$, we see that the condition (iii) is equivalent to Condition ($\ast$). Thus we get the following.

Proposition 3.7. Let $A$ be a $C^*$-algebra, and $X$ be a Hilbert $A$-bimodule. The crossed product $A \rtimes_X Z$ of $A$ by $X$ is canonically isomorphic to $O_X$ where $X$ is considered as a $C^*$-correspondence over $A$.

3.4. Graph algebras.

Definition 3.8. A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0, E^1$ which are sets of vertices and edges respectively, and two maps $r, s : E^1 \to E^0$ which indicate the range $r(e)$ and the source $s(e)$ of a directed edge $e \in E^1$.

In [FLR], Fowler, Laca, and Raeburn defined graph algebras for arbitrary graphs, generalizing the definitions of [KPRR] and [KPR].
Definition 3.9. The graph algebra $C^*(E)$ of a graph $E$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_e\}_{e \in E^0}$ and partial isometries $\{s_e\}_{e \in E^1}$ with orthogonal ranges, such that $s^*_e s_e = p_{r(e)}$, $s_e s^*_e \leq p_{s(e)}$ for $e \in E^1$ and
\[
(3.2) \quad p_v = \sum_{e \in s^{-1}(v)} s_e s^*_e \quad \text{if } 0 < |s^{-1}(v)| < \infty.
\]

Take a graph $E = (E^0, E^1, r, s)$. Set $A = C_0(E^0)$. The linear space $C_c(E^1)$ is a pre-Hilbert $A$-module under the following operations
\[
\langle \xi, \eta \rangle_{\mathcal{X}}(v) = \sum_{e \in r^{-1}(v)} \overline{\xi(e)} \eta(e) \in \mathbb{C} \quad \text{for } v \in E^0,
\]
\[
(\xi f)(e) = \xi(e) f(r(e)) \in \mathbb{C} \quad \text{for } e \in E^1,
\]
for $\xi, \eta \in C_c(E^1)$ and $f \in A$. The completion $X = X(E)$ of $C_c(E^1)$ in the norm defined by $\|\| = \|\langle \xi, \xi \rangle_{\mathcal{X}}\|^{1/2}$ is a Hilbert $A$-module. Define $\varphi_X : A \to \mathcal{L}(X)$ by
\[
\varphi_X(f)\xi(e) = f(s(e))\xi(e) \quad \text{for } e \in E^1,
\]
for $f \in A$ and $\xi \in C_c(E^1) \subset X$. We have a $C^*$-correspondence $X$ over $A$. We see that
\[
\varphi_X^{-1}(K(X)) = C_0(\{v \in E^0 \mid |s^{-1}(v)| < \infty\})
\]
\[
\ker \varphi_X = C_0(\{v \in E^0 \mid |s^{-1}(v)| = 0\}).
\]
Hence we get $J_X = C_0(\{v \in E^0 \mid 0 < |s^{-1}(v)| < \infty\})$.

If we have mutually orthogonal projections $\{p_e\}_{e \in E^0}$ and partial isometries $\{s_e\}_{e \in E^1}$ with orthogonal ranges, then we can define a $*$-homomorphism $\pi$ and a linear map $t$ by
\[
\pi(f) = \sum_{v \in E^0} f(v) p_v \quad \text{for } f \in C_0(E^0) = A,
\]
\[
t(\xi) = \sum_{e \in E^1} \xi(e) s_e \quad \text{for } \xi \in C_c(E^1) \subset X(E).
\]
This pair $(\pi, t)$ is a representation of $X(E)$ if and only if two conditions $s^*_e s_e = p_{r(e)}$, $s_e s^*_e \leq p_{s(e)}$ are satisfied for $e \in E^1$. This representation $(\pi, t)$ satisfies Condition (s) whenever (3.2) is fulfilled. Hence we get the following.

Proposition 3.10. The graph algebra $C^*(E)$ of a graph $E$ is isomorphic to the $C^*$-algebra $\mathcal{O}_{X(E)}$ of the $C^*$-correspondence $X(E)$.

A vertex $v \in E^0$ is called a sink if $s^{-1}(v) = \emptyset$, and a source if $r^{-1}(v) = \emptyset$. The $C^*$-correspondence $X$ defined by a graph $E$ is faithful if and only if $E$ has no sinks, and full if and only if $E$ has no sources. In particular, for a graph $E$ with no sinks, the graph algebra $C^*(E)$ is isomorphic to the augmented Cuntz-Pimsner algebra $\mathcal{O}_X$ of the $C^*$-correspondence $X$ defined by the graph $E$ ([FLR, Proposition 12]).

3.5. $C^*$-algebras arising from topological graphs. In [K1], we generalize the construction of graph algebras to topological graphs (see also [D]).

Definition 3.11 ([K1, Definition 2.1]). A topological graph is a quadruple $E = (E^0, E^1, d, r)$ where $E^0$, $E^1$ are locally compact spaces, $d : E^1 \to E^0$ is a local homeomorphism and $r : E^1 \to E^0$ is a continuous map.
The maps \( d, r : E^1 \to E^0 \) indicate the domain and source map, respectively. For a topological graph \( E = (E^0, E^1, d, r) \), the triple \( (E^1, d, r) \) is called a topological correspondence over \( E^0 \) in [K1]. From a topological graph \( E = (E^0, E^1, d, r) \), we can define a \( C^* \)-correspondence \( C_d(E^1) \) over \( C_0(E^0) \) by

\[
C_d(E^1) = \{ \xi \in C(E^1) \mid \langle \xi, \xi \rangle \in C_0(E^0) \},
\]

where the inner product, the right action, and the left action are defined in the same way as in Example 3.4, but we use the domain map \( d \) to define a Hilbert \( C_0(E^0) \)-module, and the range map \( r \) to define a left action. This is opposite from the case of graph algebras. The author believes that this convention is more natural than the one used in the theory of graph algebras, and this convention behaves well when we consider topological graphs as a kind of dynamical system (see Example 3.6). We can show that \( C_c(E^1) \) is dense in \( C_d(E^1) \). We have \( J_{C_d(E^1)} = C_0(E_{rg}) \) where

\[
E_{rg} = \{ v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \text{ such that } r^{-1}(V) \subset E^1 \text{ is compact, and } r(r^{-1}(V)) = V \}.
\]

We define the \( C^* \)-algebra \( \mathcal{O}(E) \) of a topological graph \( E \) in the same way as we define the \( C^* \)-algebra \( \mathcal{O}_X \) in this paper for the \( C^* \)-correspondence \( X = C_d(E^1) \) [K1, Definition 2.10]. The \( C^* \)-correspondence \( C_d(E^1) \) over \( C_0(E^0) \) defined from a topological graph \( E \) is always non-degenerate. It is full if and only if \( d \) is surjective, and faithful if and only if the image of \( r \) is dense in \( E^0 \).

Note that we can define \( C^* \)-correspondences from so-called continuous measured graphs which are a generalization of topological graphs (see [Sc1, Example 1.7]).

3.6. Crossed products by partial morphisms. Let \( X, Y \) be locally compact spaces, and let \( \sigma : X \to Y \) be a continuous map. The map \( \sigma \) determines a \(*\)-homomorphism \( \varphi : C_0(Y) \to C_0(X) \) by \( \varphi(f)(x) = f(\sigma(x)) \). The \( C^* \)-algebra \( C_0(X) \) is isomorphic to the multiplier algebra of \( C_0(X) \), and the map \( \varphi \) is non-degenerate in the sense that \( \operatorname{span}\{f(g)g \in C_0(X) \mid f \in C_0(Y), g \in C_0(X)\} = C_0(X) \). Every non-degenerate \(*\)-homomorphism from \( C_0(Y) \) to \( C_0(X) \) is obtained by this procedure. Thus the non-commutative analogues of continuous maps between locally compact spaces are non-degenerate \(*\)-homomorphisms from one \( C^* \)-algebra \( A \) to the multiplier algebra \( \mathcal{M}(B) \) of another \( C^* \)-algebra \( B \). Such a map is called a morphism from \( A \) to \( B \) in [L]. On the other hand, a \(*\)-homomorphism \( \varphi : C_0(Y) \to C_0(X) \) corresponds to a proper continuous map from some open subset of \( X \) to \( Y \). This open subset is determined by the hereditary subalgebra in \( C_0(X) \) generated by the image of \( \varphi \). Now it is natural to consider a continuous map from some open subset of \( X \) to \( Y \) and its non-commutative analogue.

**Definition 3.12** (Cf. [Sc1, Example 1.2]). Let \( A, B \) be \( C^* \)-algebras. A partial morphism from \( A \) to \( B \) is a non-degenerate \(*\)-homomorphism \( \varphi \) from \( A \) to the multiplier algebra \( \mathcal{M}(B_0) \) of some hereditary subalgebra \( B_0 \) of \( B \).

Partial morphisms between commutative \( C^* \)-algebras correspond bijectively to partially defined continuous maps between their spectra. Any \(*\)-homomorphism from \( A \) to \( B \) determines a partial morphism from \( A \) to \( B \). More generally, we have the following.
Let $A, B$ be $C^*$-algebras, let $I$ be an ideal of $A$, and let $B'$ be a $C^*$-subalgebra of $B$. Let $\varphi : I \to \mathcal{M}(B')$ be a $*$-homomorphism. Define a hereditary subalgebra $B_0$ of $B$ by
\[ B_0 = \text{span} \{ \varphi'(x_1)b'_1b'_2\varphi'(x_2) \mid x_1, x_2 \in I, b'_1, b'_2 \in B', b \in B \}. \]
We define a $*$-homomorphism $\varphi : A \to \mathcal{M}(B_0)$ by
\[ \varphi(a)b_0 = \varphi'(ax_1)b'_1b'_2\varphi'(x_2), \quad b_0 \varphi(a) = \varphi'(x_1)b'_1b'_2\varphi'(x_2)a \in B_0, \]
for $a \in A$ and $b_0 = \varphi'(x_1)b'_1b'_2\varphi'(x_2) \in B_0$. It is standard to see that this gives a well-defined non-degenerate $*$-homomorphism $\varphi : A \to \mathcal{M}(B_0)$. We call this partial morphism $\varphi : A \to \mathcal{M}(B_0)$ the partial morphism determined by $\varphi' : I \to \mathcal{M}(B')$.

**Definition 3.13** (see [Sc1]). A partial automorphism of a $C^*$-algebra $A$ is a triple $(\theta, I, J)$ where $I$ and $J$ are ideals of $A$ and $\theta : I \to J$ is an isomorphism.

As we saw above, a partial automorphism $(\theta, I, J)$ defines a partial morphism $\varphi : A \to \mathcal{M}(J)$ such that $\varphi(a)b = \theta(a\theta^{-1}(b))$ for $a \in A$ and $b \in J$. If a partial morphism $\varphi : A \to \mathcal{M}(B_0)$ from $A$ to $B$ satisfies $\varphi(A) \cap B_0 = B_0$, then $\varphi$ is the partial morphism determined by $\varphi|_I : I \to B_0$ where $I = \varphi^{-1}(B)$. Partial morphisms defined by partial automorphisms are this kind of partial morphisms.

Following [Sc1], we construct $C^*$-correspondences from partial morphisms in the case that $A = B$. Let $\varphi : A \to \mathcal{M}(A_0)$ be a partial morphism from $A$ to itself. The closed right ideal $X = A_0A$ is a Hilbert $A$-module with the inner product defined by $(x, y)_X = x^*y \in A$ and the right action defined by the multiplication. We have $\mathcal{K}(X) \cong A_0$ and $\mathcal{L}(X) \cong \mathcal{M}(A_0)$ naturally. We define $\varphi_X : A \to \mathcal{L}(X)$ to be the composition of $\varphi$ and the natural isomorphism $\mathcal{M}(A_0) \to \mathcal{L}(X)$. Thus we get a $C^*$-correspondence $X$ over $A$ which is denoted by $X(\varphi)$. By definition, the $C^*$-correspondence $X(\varphi)$ over $A$ is non-degenerate. It is faithful if and only if $\varphi$ is injective, and full if and only if the ideal generated by $A_0$ is $A$. It may be reasonable to make the following definition.

**Definition 3.15.** Let $\varphi : A \to \mathcal{M}(A_0)$ be a partial morphism from $A$ to itself.

We define the **crossed product** $A \rtimes_\varphi \mathbb{N}$ of $A$ by $\varphi$ to be the $C^*$-algebra $\mathcal{O}_{X(\varphi)}$ of the $C^*$-correspondence $X(\varphi)$.

This definition generalizes crossed products by automorphisms (see [P, Example (3)]). More generally for an injective endomorphism $\varphi$ of $A$, its crossed product defined here is isomorphic to the crossed product by the endomorphism defined in [St] because both of them are isomorphic to the augmented Cuntz-Pimsner algebra of $\mathcal{C}^*(\varphi)$.

When the partial morphism $\varphi$ of $A$ is defined by a partial automorphism $(\theta, I, J)$, then the $C^*$-correspondence $X(\varphi)$ over $A$ defined by $\varphi$ satisfies $J_X(\varphi) = I$ and $\varphi_X(\varphi)(J_X(\varphi)) = \mathcal{K}(X(\varphi))$. Hence the $C^*$-correspondence $X(\varphi)$ is a Hilbert $A$-bimodule. Therefore the crossed product $A \rtimes_{\varphi} \mathbb{N}$ of $A$ by the partial morphism $\varphi$ defined by a partial automorphism $(\theta, I, J)$ is isomorphic to the crossed product $A \rtimes_{\theta} \mathbb{N}$ of the partial automorphism $(\theta, I, J)$ defined in [E, Definition 3.7] because both of them are isomorphic to the crossed product by the Hilbert $C^*$-bimodule $X(\varphi)$ (see Example 3.2 and [AEE, Example 3.2]).

Note that a partially defined continuous map $\sigma$ from a locally compact space $X$ to itself determines a topological graph $E_\sigma = (X, U, \iota, \sigma)$ where $U$ is the domain of the map $\sigma$ and $\iota : U \to X$ is the embedding. The Hilbert $C^*$-bimodule $X(\varphi)$ of
the partial morphism $\varphi$ defined from $\sigma$ coincides with the one obtained from the
topological graph $E_\sigma$ as in Example 3.5. Conversely every topological graph $E = (E^0, E^1, d, r)$
with an injective domain map $d$ arises in this manner. As explained in
[K1], a topological graph can be considered as a (kind of) multi-valued continuous
map between locally compact spaces. Its natural non-commutative analogue is a
non-degenerate $C^*$-correspondence. Hence we can say that a non-degenerate $C^*$-
correspondence $X$ over $A$ is a multi-valued morphism from $A$ to itself, and that the
$C^*$-algebra $\mathcal{O}_X$ is the crossed product by this multi-valued morphism.

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