Generalized Poincaré algebras, Hopf algebras and ς-Minkowski spacetime

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Abstract

We propose a generalized description for the ς-Poincaré-Hopf algebra as a symmetry quantum group of underlying ς-Minkowski spacetime. We investigate all the possible implementations of (deformed) Lorentz algebras which are compatible with the given choice of ς-Minkowski algebra realization. For the given realization of ς-Minkowski spacetime there is a unique ς-Poincaré-Hopf algebra with undeformed Lorentz algebra. We have constructed a three-parameter family of deformed Lorentz generators with ς-Poincaré algebras which are related to ς-Poincaré-Hopf algebra with undeformed Lorentz algebra. Known bases of ς-Poincaré-Hopf algebra are obtained as special cases. Also deformation of $\mathfrak{gl}(4)$ Hopf algebra compatible with the ς-Minkowski spacetime is presented. Some physical applications are briefly discussed.

1. Introduction

The Poincaré symmetry is the full symmetry of special relativity and it includes translations, rotations and boosts. Algebraically it is described by the
Poincaré-Lie algebra (Lorentz generators $M_{\mu\nu}$ and momenta $P_\mu$), usually denoted as $\mathfrak{iso}(1,3)$, and defined by the following commutation relations

$$[M_{\mu\nu}, M_{\lambda\rho}] = i \left( M_{\mu\lambda} \eta_{\nu\rho} - M_{\mu\rho} \eta_{\nu\lambda} - M_{\nu\lambda} \eta_{\mu\rho} + M_{\nu\rho} \eta_{\mu\lambda} \right),$$

$$[M_{\mu\nu}, P_\lambda] = i \left( P_\nu \eta_{\mu\lambda} - P_\mu \eta_{\nu\lambda} \right),$$

$$[P_\mu, P_\nu] = 0,$$

where $\eta_{\mu\nu}$ is the metric tensor with Lorentzian signature. As it is known, any Lie algebra provides an example of undeformed Hopf algebra via its universal enveloping algebra. Therefore, the universal enveloping algebra of the Poincaré-Lie algebra $\mathcal{U}_{\mathfrak{iso}(1,3)}$ can be equipped with comultiplication, a counit and antipode

$$\Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \quad \text{and} \quad \Delta_0 P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu$$

$$S(M_{\mu\nu}) = -M_{\mu\nu}; \quad S(P_\mu) = -P_\mu; \quad \epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = 0$$

defined on the generators and then extended to the whole $\mathcal{U}_{\mathfrak{iso}(1,3)}$, which constitutes the undeformed Poincaré-Hopf algebra. Such an algebra can be deformed within the Hopf algebraic framework. First deformations of Poincaré symmetry appeared in the early 90’s. The so-called $\kappa$-deformation \cite{1,2}, was obtained by contraction procedure from $\text{q-deformed } SO_q(3,2)$. It gained much attention since such deformed relativistic symmetries can be interpreted as a description of Planck scale world and the deformation parameter $\kappa$ (of mass dimension) is usually interpreted as the Planck Mass $M_P$ or the quantum gravity scale $M_{QG}$. The $\kappa$-Poincaré-Hopf algebra is the symmetry algebra (quantum group) of underlying quantum space which should replace the undeformed one at this level of energy. It is believed that below the Planck scale a more general spacetime structure appears, e.g. noncommutative one, where (as with quantum mechanics phase space) uncertainty relations naturally arise \cite{3}. The $\kappa$-Minkowski spacetime, as one of the examples of noncommutative spacetime is a (Hopf) module algebra over the $\kappa$-Poincaré-Hopf algebra and it stays invariant under the quantum group of transformations in analogy to the classical case. The $\kappa$-deformed Poincaré algebra as deformed symmetry of the $\kappa$-Minkowski spacetime inspired many authors to, e.g. construct quantum field theories (see e.g., \cite{4,5,6,7}), electrodynamics on $\kappa$-Minkowski spacetime \cite{8,9}, or modify particle statistics (see e.g., \cite{11,10}).

One of the main ideas is to discuss Planck scale (quantum gravity) effects, since it naturally includes $\kappa$ (Planck scale, quantum gravity) corrections.

In this Letter we are interested in the generalization of the description for the $\kappa$-deformed Poincaré symmetry group, especially the Lorentz sector. A deformation of the phase space, compatible with $\kappa$-Minkowski spacetime can be characterized by a set of functions $h_{\mu\nu}(p)$ \cite{12,13,14,15}. For a given choice of $h_{\mu\nu}$ there are
infinitely many ways of implementing the Lorentz algebra, with different Hopf algebra structures generally within \( \mathfrak{isl}(4) \). Among these algebras, for a given choice of \( h_{\mu\nu} \), there exists a unique \( \kappa \)-Poincaré-Hopf algebra. Such examples were considered in \([2, 12, 13, 14, 15]\). The first example of deformed Lorentz algebra with the \( \kappa \)-Poincaré-Hopf algebra was proposed in \([1]\). Undeformed Lorentz algebras that cannot be equipped with a Hopf algebra structure which would be closed in the Poincaré algebra were considered in \([5, 11, 16, 17, 18, 19]\). Our aim is to introduce the most general form of deformation of the Lorentz algebra, compatible with the \( \kappa \)-Minkowski spacetime, which includes the above examples as special cases. Especially, we have constructed an infinite three-parameter class of deformed Lorentz algebras with the \( \kappa \)-Poincaré-Hopf algebra structure which are related to the \( \kappa \)-Poincaré-Hopf algebra with undeformed Lorentz algebra, of which the standard basis \([1]\) is one example.

The plan of the Letter is as follows. In Section 2 we introduce the deformed Heisenberg algebra, including noncommutative \( \kappa \)-Minkowski spacetime coordinates. Later, in Section 3 we focus on the generalization of \( \kappa \)-Poincaré algebra which includes a certain ansatz for Lorentz boost generators. This section constitutes the generalized description of the deformed Poincaré algebra within this framework. In Section 4 we focus on the undeformed Lorentz algebra with a deformed coalgebra which depends on the choice of \( \kappa \)-Minkowski realization. Here the bicrossproduct basis \([2]\) for the \( \kappa \)-Poincaré-Hopf algebra is included as a special case. Section 5 includes the case in which the Lorentz algebra is deformed and here we are able to obtain the standard basis \([1]\) of \( \kappa \)-Poincaré as a special case. Section 6 concerns the twist deformation of \( \mathfrak{isl}(4) \). Finally, in Section 7 conclusions are presented.

2. Deformed Heisenberg algebra

Before going into deformed relativistic symmetries let us start with the deformation of a phase space (Heisenberg algebra) including \( \kappa \)-Minkowski spacetime. In the undeformed case Heisenberg algebra \( H \) can be defined as a free algebra of \( n \) coordinate generators and \( n \) generators of momenta, satisfying the following relations:

\[
[x_\mu, x_\nu] = 0; \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu} \cdot 1; \quad [p_\mu, p_\nu] = 0; \quad (6)
\]

where \( \eta_{\mu\nu} = (-, +, +, +) \) is diagonal metric tensor with Lorentzian signature.

A possible deformation of Heisenberg algebra is to introduce noncommutative coordinates. In the case of \( \kappa \)-Minkowski spacetime algebra we have:

\[
\begin{align*}
[\hat{x}_\mu, \hat{x}_\nu] &= i \left( a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu \right); & [p_\mu, \hat{x}_\nu] &= -ih_{\mu\nu}(p); & [p_\mu, p_\nu] &= 0. \quad (7)
\end{align*}
\]
For simplicity we can choose \( a_\mu = (a_0, 0, 0, 0) \). We shall denote the algebra described by these commutation relations as \( \hat{H} \). The choice of function \( h_{\mu\nu}(p) \) is such that the Jacobi identities for \( \hat{x}_\mu, p_\nu \) are satisfied which implies that \( h_{\mu\nu}(p) \) obeys a system of partial differential equations ([12, 13], see also [14, 15, 20, 21, 22]). In the undeformed limit \( a_\mu \to 0 \) we obtain algebra \( H \), i.e. \( \lim_{a_0 \to 0} h_{\mu\nu} = \eta_{\mu\nu} \).

For the purpose of deformation we shall introduce h-adic extension of \( \hat{H} \) and denote it by \( \hat{H}[[a]] \), which is an algebra of formal power series [22]. In fact, algebra \( \hat{H}[[a]] \) can be obtained from \( H \) by non-linear change of generators (mapping), as example see realization (9) [22].

Let us emphasize that for the given \( \hat{H} \), there is a unique choice of functions \( h_{\mu\nu} \) and the corresponding realization, \( \hat{x}_\mu = x_\alpha h_{\alpha\mu}(p) \). One family of realizations can be described by:

\[
\hat{x}_0 = x_0 \psi(A) - a_0 x_k p_k \gamma(A), \quad \hat{x}_i = x_i \varphi(A) \tag{8}
\]

where \( A = -a_\alpha p^\alpha = a_0 p_0 \). The functions \( \varphi(A), \psi(A) \) are arbitrary real-analytic functions such that \( \varphi(0) = 1; \varphi'(0) = 1 \) and \( \gamma(A) \) obeys: \( \gamma = \frac{e^A}{\varphi} + 1 \), in order to obtain \( \kappa \)-Minkowski commutation relations for noncommutative coordinates [12, 21]. Functions \( h_{\mu\nu} \) can be easily obtained from relations (7) and (8).

As a special case of Eq. (8) one can choose: \( \psi = 1; \varphi = Z^{-\lambda} \). Then we have

\[
\hat{x}_0 = x_0 - a_0 (1 - \lambda) x_k p_k, \quad \hat{x}_i = x_i Z^{-\lambda} \tag{9}
\]

where we introduced the so-called shift operator \( Z \in \hat{H}[[a]] \) defined by commutation relations: \( [Z, \hat{x}_\mu] = i a_\mu Z; [Z, p_\mu] = 0 \).

Note that the above realizations are not Hermitian, however one can construct the Hermitian realizations by simple formula:

\[
\hat{x}^h_\mu = \frac{\hat{x}_\mu + \hat{x}_\mu^\dagger}{2}. \tag{10}
\]

### 3. Generalized Poincaré algebras

We are interested in the most general form for deformation of Poincaré algebra which would be compatible with the above deformed Heisenberg algebra \( \hat{H} \). Let us define deformed Lorentz generators \( \hat{M}_{\mu\nu} \) satisfying general commutation relations

\[
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\lambda}] = i \left( \hat{M}_{\mu\lambda} g_{\nu\rho}(p) - \hat{M}_{\mu\rho} g_{\nu\lambda}(p) - \hat{M}_{\nu\lambda} g_{\mu\rho}(p) + \hat{M}_{\nu\rho} g_{\mu\lambda}(p) \right) \tag{11}
\]

where \( g_{\mu\nu}(p) \) is real, nondegenerate metric in momentum space \( g_{\mu\nu}(p) = g_{\nu\mu}(p) \) and \( \hat{M}_{\mu\nu} \) satisfy \( \hat{M}_{\mu\nu} = -\hat{M}_{\nu\mu} \). This point of view could be related with the recent
idea of relative locality [25] which includes the description of $\kappa$-Poincaré-inspired momentum space geometry. However we will not focus on this here.

We write $\hat{M}_{\mu \nu}$ in order to emphasize the deformation. The generalized Poincaré algebra is defined by $[p_\mu, p_\nu] = 0$ and

$$[\hat{M}_{\mu \nu}, p_\lambda] = iG_{\mu \nu \lambda}(p)$$

for real functions $G_{\mu \nu \lambda}(p)$. We require that all Jacobi identities are satisfied which set restrictions on tensors $g_{\mu \nu}$, $G_{\mu \nu \lambda}$. It is enough to calculate Jacobi identities using Eq. (11) for three $\hat{M}$s and using Eqs. (11) and (12) for two $\hat{M}$s and for one $p$. In order to complete the set of commutation relations, we include

$$[\hat{M}_{\mu \nu}, \hat{x}_\lambda] = -i\hat{x}_\alpha K^\alpha_{\mu \nu \lambda}(p)$$

for real functions $K^\alpha_{\mu \nu \lambda}(p)$. Since $\hat{x}_\mu = x^\mu h_{\alpha \mu}(p)$ satisfy $\kappa$-Minkowski commutation relations (7) and $\hat{M}_{\mu \nu} = x^\alpha G_{\mu \nu \alpha}(p)$ (satisfying Eqs. (11) and (12)), Eq. (13) produces expressions for functions $K^\alpha_{\mu \nu \lambda}$ in terms of $h_{\alpha \mu}$ and $G_{\mu \nu \alpha}$. We also require the smooth limit to the Poincaré algebra when deformation parameter $\alpha$ goes to 0.

The deformed Heisenberg algebras are introduced as subalgebras of the generalized Poincaré algebra extended by $\kappa$-Minkowski noncommutative coordinates. Therefore, introduced realizations for Lorentz algebras are compatible with deformed Heisenberg algebras as well.

One can consider the following ansatz for generators in which the deformation parameter $\alpha$ has the form $(a_0, 0, 0, 0)$ and the rotation subalgebra so(3) is undeformed:

$$\hat{M}_{0 \alpha} = x_\alpha p_0 F_1 (A, b) - x_0 p_\alpha F_2 (A, b) + a_0 (x_\alpha p_k) p_l F_3 (A, b) + a_0 x_\alpha p_0 F_4 (A, b)$$

$$\hat{M}_{ij} = M_{ij} = x_i p_j - x_j p_i,$$

expressed in terms of undeformed Heisenberg algebra, and $A = a_0 p_0; b = a_0 p_0^2$. When $a_0$ goes to 0, $F_1$ and $F_2$ go to 1, $F_3$ and $F_4$ are finite and $g_{00}$ goes to $\eta_{00}$.

The generalized Poincaré algebra is described by the following set of commutation relations:

$$[\hat{M}_{0 \alpha}, p_\beta] = i\delta_{\beta \alpha} \hat{M}_{0 \beta}; \quad [\hat{M}_{0 \alpha}, p_\beta] = i\delta_{\beta \alpha} \left( p_\alpha F_1 + a_0 \hat{F}_1 \right) + i\epsilon_{\beta \alpha \mu} p_\beta F_3 \quad (16)$$

$$[M_{ij}, \hat{M}_{0 \beta}] = i \left( \hat{M}_{0 \beta} \eta_{ij} - \hat{M}_{ij} \eta_{0 \beta} \right)$$

$$[\hat{M}_{0 \alpha}, \hat{M}_{0 \beta}] = i \hat{M}_{ij} (-A F_1 F_2 + A F_1 F_3 - 2AF_1 F_4 - b F_3 F_4 - 2b F_4^2 - A \frac{\partial F_1}{\partial A} F_2 - b F_2 \frac{\partial F_4}{\partial A} (18)$$

$$-2A^2 F_1 \frac{\partial F_1}{\partial b} - 2A b \frac{\partial F_1}{\partial b} F_3 - 2A b \frac{\partial F_1}{\partial b} F_4 - 2A b F_1 \frac{\partial F_4}{\partial b} - 2b^2 F_3 \frac{\partial F_4}{\partial b} - 2b^2 F_4 \frac{\partial F_4}{\partial b} ),$$

$$[\hat{M}_{0 \alpha}, \hat{M}_{0 \beta}] = i \hat{M}_{ij} (-A F_1 F_2 + A F_1 F_3 - 2AF_1 F_4 - b F_3 F_4 - 2b F_4^2 - A \frac{\partial F_1}{\partial A} F_2 - b F_2 \frac{\partial F_4}{\partial A} (18)$$

$$-2A^2 F_1 \frac{\partial F_1}{\partial b} - 2A b \frac{\partial F_1}{\partial b} F_3 - 2A b \frac{\partial F_1}{\partial b} F_4 - 2A b F_1 \frac{\partial F_4}{\partial b} - 2b^2 F_3 \frac{\partial F_4}{\partial b} - 2b^2 F_4 \frac{\partial F_4}{\partial b} ),$$

$$[\hat{M}_{0 \alpha}, \hat{M}_{0 \beta}] = i \hat{M}_{ij} (-A F_1 F_2 + A F_1 F_3 - 2AF_1 F_4 - b F_3 F_4 - 2b F_4^2 - A \frac{\partial F_1}{\partial A} F_2 - b F_2 \frac{\partial F_4}{\partial A} (18)$$

$$-2A^2 F_1 \frac{\partial F_1}{\partial b} - 2A b \frac{\partial F_1}{\partial b} F_3 - 2A b \frac{\partial F_1}{\partial b} F_4 - 2A b F_1 \frac{\partial F_4}{\partial b} - 2b^2 F_3 \frac{\partial F_4}{\partial b} - 2b^2 F_4 \frac{\partial F_4}{\partial b} ),$$
(the rest of commutation relations for Poincaré algebra stays undeformed). Note that Eqs. (14)-(18) also include the case of the undeformed Lorentz algebra.

In the classical limit \( a_0 \to 0 \) from generalized Poincaré algebra we recover the well-known relations of the undeformed (standard) Poincaré algebra \( U_{\text{iso}(1,3)} \):

\[
g_{\mu\nu} \to \eta_{\mu\nu} \quad \text{and} \quad G_{\mu\nu\lambda\rho}(p) \to i \left( p_{\nu} \eta_{\mu\lambda} - p_{\mu} \eta_{\nu\lambda} \right).
\]

One can easily recover the form of the metric \( g_{\mu\nu}(p) \) for which the set of commutation relations (16)-(18) is satisfied. It leads to the following:

\[
g_{00}(p) = -F_1 F_2 + A F_1 F_3 - 2 A F_1 F_4 - b F_3 F_4 - 2 b F_4^2 - A \frac{\partial F_1}{\partial A} F_2 - b F_2 \frac{\partial F_1}{\partial b} - 2 A^2 F_1 \frac{\partial F_1}{\partial b} - 2 A b \frac{\partial F_1}{\partial b} - 2 b^2 \frac{\partial F_1}{\partial b} - 2 A b \frac{\partial F_1}{\partial b} (19)
\]

and \( g_{ij} = \eta_{ij}, \ g_{i0} = 0 \).

Remark: It is possible to define Hermitian realization of Lorentz generators as well:

\[
\hat{M}^h_{i0} = \frac{\hat{M}_{i0} + \hat{M}^\dagger_{i0}}{2}. \quad (20)
\]

The deformed symmetry algebra can be equipped in Hopf algebra structure as well, generally \( \text{igl}(4) \) one. As quantum Hopf algebra the generalized Poincaré algebra will possess an algebraic sector (commutators in the form of (16)-(18)) and the co-algebraic sector (coproduct, counit and antipode), generally in \( \text{igl}(4) \). The quantum (deformed) Hopf algebras will be presented in the next sections. Let us mention that when \( a_0 \) goes to 0 it reduces to undeformed Hopf algebra, (1)-(5).

4. Lorentz algebra and \( \kappa \)-Poincaré-Hopf algebra

In this section we consider the Lie algebra defined by undeformed Lorentz generators \( M_{\mu\nu} \) and \( \hat{x}_\mu \) (7), (see [15, 14, 12, 13, 21, 20, 22, 23]) satisfying the following commutation relation

\[
[M_{\mu\nu}, \hat{x}_\lambda] = i \left( \hat{x}_\nu \eta_{\mu\lambda} - \hat{x}_\mu \eta_{\nu\lambda} + a_{\nu} M_{\mu\lambda} - a_{\mu} M_{\nu\lambda} \right). \quad (21)
\]

In this Lie algebra we use \( M_{\mu\nu} \) instead of \( \hat{M}_{\mu\nu} \). Let us mention that the relation (21) is the unique way to obtain a Lie algebra generated by \( M_{\mu\nu} \) and \( \hat{x}_\mu \) (see [20]).

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5 Within the idea of relative locality [25] the metric dependence on momenta is linear, here we have more general form, however also consistent with the \( \kappa \)-Poincaré algebra which will be shown in Section 5.

6 Note that if \( \hat{M}_{\mu\nu} \) and \( p_\mu \) are Hermitian then the right hand side of (11) should be anti-Hermitian. Also \( g_{00}(A,b) \) commutes with \( M_{ij} \) in (17).
For a class of Heisenberg algebras defined in Section 2 by \( h_{\mu\nu}(p) \), we get the corresponding class of \( \kappa \)-deformed Poincaré algebras defined by

\[
[M_{\mu\nu}, p_\lambda] = iG_{\mu\nu\lambda}(p).
\]  
(22)

Jacobi identities imply that \( G_{\mu\nu\lambda} \) is uniquely determined by \( h_{\mu\nu} \) [13].

One example is particularly interesting: the natural realization (or classical bases) [15, 20, 23, 12, 13]. For the natural realization,

\[
\hat{\chi}_\mu = X_\mu Z^{-1} + a_0 X_0 P_\mu
\]
\[
M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \quad (F_1 = F_2 = 1, F_3 = F_4 = 0),
\]  
(23)

where \( Z^{-1} = -a_0 P_0 + \sqrt{1 - a_0^2 p^2} \). In order to emphasize the natural realization, we write capital letters (to distinguish from generic \( x_\mu \) and \( p_\mu \)). Then the commutator of \( M_{\mu\nu} \) and \( P_\lambda \) has the form

\[
[M_{\mu\nu}, P_\lambda] = i \left( P_\nu \eta_{\mu\lambda} - P_\mu \eta_{\nu\lambda} \right).
\]  
(24)

They generate the undeformed Poincaré algebra. The coalgebra structure is given by

\[
\Delta P_\mu = P_\mu \otimes Z^{-1} + 1 \otimes P_\mu - a_\mu p^L_\alpha Z \otimes P^\alpha
\]
\[
\Delta M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - a_\mu (p^L)^\alpha Z \otimes M_{\alpha\nu} + a_\nu (p^L)^\alpha Z \otimes M_{\alpha\mu}
\]  
(25)

where

\[
p^L_\alpha = P_\alpha - \frac{a_\alpha}{2} \Box
\]  
(26)

and \( \Box = \frac{2}{a^2} \left( 1 - \sqrt{1 + a^2 P^2} \right) \). Here, the deformation is described by the general four-vector \( a_\mu \). Counits are trivial i.e. \( \epsilon(M_{\mu\nu}) = 0 \) and \( \epsilon(P_\mu) = 0 \). The antipodes are given by

\[
S(P_\mu) = (-P_\mu - a_\mu p^L) Z
\]
\[
S(M_{\mu\nu}) = -M_{\mu\nu} - a_\mu p^L_\alpha M_{\alpha\nu} + a_\nu p^L_\alpha M_{\alpha\mu}.
\]  
(27)

It is important to emphasize that using similarity transformations

\[
P_\mu = \mathcal{E} p_\mu \mathcal{E}^{-1} = P_\mu(p)
\]
\[
X_\mu = \mathcal{E} x_\mu \mathcal{E}^{-1} = x^\alpha \Psi_{\alpha\mu}(p)
\]  
(28)

where \( \mathcal{E} = \exp\{x^\alpha \Sigma_\alpha(p)\} \) and functions \( \Sigma_\alpha(p) \) satisfy the boundary condition

\[
\lim_{\alpha \to 0} \Sigma_\alpha = 0,
\]  
(30)
one can obtain formulae for the coproduct and antipode of $p_\mu$ using (28) (and its inverse relation) \[13\].

Let us consider the class of realizations given by (8). The corresponding similarity transformations for $P_\mu$ (28) are given by

$$P_0 = \frac{1 - Z^{-1}}{a_0} + \frac{a_0}{2} \square$$  \hspace{1cm} (31)

$$P_i = P_i Z^{-1} \varphi(A),$$  \hspace{1cm} (32)

where

$$\square = -\vec{p}^2 Z^{-1} \varphi^2(A) + \frac{1}{a_0^2} (Z + Z^{-1} - 2).$$  \hspace{1cm} (33)

It is easy to obtain formulae for $X_0$ and $X_i$ \[12\].

Functions $F_i(A, b)$ are given by

$$F_1(A, b) = \frac{\varphi(A)(Z^2 - 1)}{2A}, \quad F_2(A, b) = \frac{\psi(A)}{\varphi(A)}, \quad F_3(A, b) = \frac{\gamma(A)}{\varphi(A)}, \quad F_4(A, b) = -\frac{1}{2\varphi(A)}.$$  \hspace{1cm} (34)

Let $\Psi(A) = \int_0^A \frac{dt}{\psi(t)}$. The coalgebra structure is given by

$$\Delta p_0 = \frac{1}{a_0} \Psi^{-1}(\ln(Z \otimes Z))$$  \hspace{1cm} (35)

where $\ln(Z \otimes Z) = \ln(Z) \otimes 1 + 1 \otimes \ln(Z)$. Also

$$\Delta p_i = \varphi(a_0 \Delta p_0) \left( \frac{p_i}{\varphi(a_0 p_0)} \otimes 1 + Z \otimes \frac{p_i}{\varphi(a_0 p_0)} \right).$$  \hspace{1cm} (36)

The coproducts of the Lorentz generators are given by

$$\Delta M_{i0} = M_{i0} \otimes 1 + Z \otimes M_{i0} - a_0 \varphi(a_0 p_0) p_j \otimes M_{ij}$$  \hspace{1cm} (37)

and

$$\Delta M_{ij} = M_{ij} \otimes 1 + 1 \otimes M_{ij}.$$  \hspace{1cm} (38)

The counits of all generators are undeformed and the antipodes are given by

$$S(p_0) = \frac{1}{a_0} \Psi^{-1}(\ln(Z^{-1})),$$  \hspace{1cm} (39)

$$S(p_i) = -p_i \frac{\varphi(S(a_0 p_0))}{\varphi(a_0 p_0)} Z^{-1},$$  \hspace{1cm} (40)

$$S(M_{i0}) = -Z^{-1} \left( M_{i0} + a_0 \varphi(a_0 p_0) p_j M_{ij} \right),$$  \hspace{1cm} (41)

$$S(M_{ij}) = -M_{ij}.$$  \hspace{1cm} (42)
For the choice $\psi = 1$ and $\varphi = Z^{-1}$ in (3) the Hopf algebra structure can be written as:
\[
\Delta p_0 = p_0 \otimes 1 + 1 \otimes p_0; \quad \Delta p_i = p_i \otimes Z^{-1} + Z^{1-i} \otimes p_j; \\
\Delta M_{i0} = M_{i0} \otimes 1 + Z \otimes M_{i0} - a_0 Z^{1} p_j \otimes M_{ij} \\
S(p_0) = -p_0; \quad S(p_i) = -Z^{2-i} p_i; \quad S(M_{i0}) = -Z^{-1} \left( M_{i0} + a_0 Z^{1} p_j M_{ij} \right).
\]

Let us consider two choices of $\lambda$:
1) The case $\lambda = 0$. The form of functions $F_i$ is the following: $F_1 = \frac{Z^2 - 1}{2A}$, $F_2 = 1$, $F_3 = 1$, $F_4 = -\frac{1}{2}$. The formulae for the coproduct and antipode of momentum and Lorentz generators are written in formulae (43), (44) and (45). The Hopf algebra obtained in this way is identical to the bicrossproduct basis from [2].
2) The case $\lambda = \frac{1}{2}$. The form of functions $F_i$ is the following: $F_1 = \frac{\sinh A Z^{1/2}}{A}$, $F_2 = Z^{1/2}$, $F_3 = Z^{1/2}$, $F_4 = -Z^{1/2}$. Again, the formulae for the coproduct and antipode of momentum and Lorentz generators are written in formulae (43), (44) and (45). Coproducts and antipodes for momenta, $p_0, p_i$ are identical as in the standard basis [1]. Since the Lorentz algebra is undeformed, the coproducts of $M_{i0}$ are not the same as in the standard basis [1].

Similar examples could be presented for $\psi = 1$ and $\varphi = 1 - A$ (left covariant realization) and $\psi = 1 + A$ and $\varphi = 1$ (right covariant realization). There are some other interesting examples in [13].

The generalized form of the Poincaré algebra does not contain dilatation generator, however the ansatz for Lorentz boosts includes them (see formula (14)). Nevertheless provided examples show that Poincaré algebra does not necessarily need to be extended, even in the case of $F_3 \neq 0$.

We point out that for the given choice of $h_{\mu\nu}$, there is a unique expression for $G_{\mu\nu\lambda}$ (Eq. (22)) and $F_i$ (14) such that the Lorentz algebra is undeformed and relation (21) is satisfied. Note that the reverse statement is not true. Examples with undeformed Poincaré algebras are given by a family of $h_{\mu\nu}$, see Subsection 4.2. in [13].

5. $\kappa$-Deformed Lorentz algebras and $\kappa$-Poincaré-Hopf algebra

Let us consider the given choice of $h_{\mu\nu}$ (Eq. (7)) and a family of $\kappa$-deformed Lorentz algebras defined by functions $F_{d1}^1, F_{d2}^2, F_{d3}^3$ and $F_{d4}^4$ (see (14) and (15)). For a given choice of $h_{\mu\nu}$, there is a unique choice of functions $F_i$ (denoted without index $d$), such that the Lorentz algebra is undeformed and Lorentz generators $M_{\mu\nu}$ and
The inverse relation is given by
\[ \Delta \to \] where \( G_1(A, b) \) are arbitrary functions such that \( G_1 \) goes to 1 and \( G_2 \) and \( G_3 \) are finite as the deformation parameter \( a_0 \) goes to 0. We emphasize that (46) is a subclass of (14). For simplicity we consider the case \( G_2 = 0 \) and (46) transforms to
\[ \hat{\Delta}_{(0)} = \Delta_{(0)} \to G_1 + a_0 M_{ij} p_j G_3. \] (47)
The inverse relation is given by
\[ M_{(0)} = \hat{\Delta}_{(0)} G_1 - a_0 M_{ij} p_j G_3. \] (48)
Relations connecting \( F^d_i \) and \( F_i \) are given by
\[ F^d_1 = F_1 G_1, \quad F^d_2 = F_2 G_1, \quad F^d_3 = F_3 G_1 - G_3, \quad F^d_4 = F_4 G_1 + G_3. \] (49)
It is easy to obtain inverse relations. Let us mention that generally \( g_{00} \neq \eta_{00} \) (11) and (18), implying that the Lorentz algebra is deformed. If \( G_1 = 1 \) and \( G_3 = 0 \), then \( g_{00} = \eta_{00} \).

The coproduct \( \hat{\Delta}_{(0)} \) can be obtained from Eqs. (47) and (48):
\[ \hat{\Delta}_{(0)} = \Delta(M_{(0)}) \Delta(G_1) + a_0 \Delta(M_{ij}) \Delta(p_j) \Delta(G_3). \] (50)
Coproducts \( \Delta p_i \) and \( \Delta M_{ij} \) are calculated in the previous section. One has to express \( M_{(0)} \) in terms of \( \hat{\Delta}_{(0)} \) and \( M_{ij} \) (48) in the Eq. (50).

For the class of examples when \( \psi = 1 \) and \( \varphi = Z^{-1} \) the relation (50) transforms to
\[ \hat{\Delta}_{(0)} = \left( M_{(0)} \otimes 1 + Z \otimes M_{(0)} - a_0 p_\lambda \Delta M_{ij} \right) \Delta G_1 + a_0 \left( M_{ij} \otimes 1 \otimes M_{ij} \right) \left( p_j \otimes Z^{-1} \otimes Z^{1-1} \otimes p_j \right) \Delta G_3.
\]
\[ = \left( \hat{\Delta}_{(0)} \frac{1}{G_1} \otimes 1 + Z \otimes \hat{\Delta}_{(0)} \frac{1}{G_1} - a_0 \left( M_{ij} \otimes \frac{G_3}{G_1} \otimes 1 + Z \otimes M_{ij} \otimes \frac{G_3}{G_1} + p_j \Delta G_3 \right) \right) \Delta G_1 + a_0 \left( M_{ij} \otimes 1 \otimes M_{ij} \right) \left( p_j \otimes Z^{-1} \otimes Z^{1-1} \otimes p_j \right) \Delta G_3. \] (51)

Let us consider the special case \( \lambda = \frac{1}{2} \), \( F_1 = \frac{\sinh A}{A} \), \( F_2 = 1 \), \( F_3 = F_4 = 0 \), \( G_1 = Z^{-\frac{1}{2}} \) and \( G_3 = \frac{1}{2} \):
\[ \hat{\Delta}_{(0)} = M_{(0)} Z^{-\frac{1}{2}} + \frac{1}{2} a_0 M_{ij} p_j \] (52)
and
\[
\Delta \hat{M}_0 = \hat{M}_0 \otimes Z^{-\frac{1}{2}} + Z^{\frac{1}{2}} \otimes \hat{M}_0 + \frac{a_0}{2} \left( M_{ij} Z^{\frac{1}{2}} \otimes p_j - p_j \otimes M_{ij} Z^{-\frac{1}{2}} \right). \tag{53}
\]

Let us mention that \( g_{00} = \eta_{00} \cosh A \). This example for \( \hat{M}_0 \) and \( \Delta \hat{M}_0 \) coincides with the corresponding results in the standard basis in [1] (\( a_0 = -\frac{1}{\kappa} \)).

Eq. (46) produces a three-parameter \((G_1, G_2, G_3)\) family of generalized Poincaré algebras, in which the Lorentz algebra is generally deformed. This family corresponds to different bases of the \( \kappa \)-Poincaré-Hopf algebra. We point out that for all the other deformed Lorentz generators \( \hat{M}_0 \), which are not related to \( M_{\mu \nu} \) via relation (46), \( \hat{M}_0 \) can be related to some undeformed Poincaré algebra, but not to the Poincaré algebra given by the natural realization (23). As a consequence, the coproduct of such \( \hat{M}_0 \) cannot be written in terms of (deformed) Poincaré generators only, the Hopf algebra structure should be extended to \( \kappa \)-deformed igl(4) (or Poincaré-Weyl) Hopf algebra. Note that undeformed Lorentz generators \( \tilde{M}_{\mu \nu} = x_{\mu} p_{\nu} - x_{\nu} p_{\mu} \) generally cannot be written in terms of \( M_{\mu \nu} \) by the Eq. (46). For example

\[
M_0 = X_0 P_0 - X_i P_i = \left( x_i p_0 \frac{\sinh A}{A} - x_0 p_i + \frac{a_0}{2} (x_k p_k) p_i - \frac{a_0}{2} x_i \vec{p}^2 \right) Z^{\frac{1}{2}}, \tag{54}
\]

for the case \( \lambda = \frac{1}{2} \) in Section 4. Inserting generators \( \tilde{M} \) and \( M \) into relation (46), we find that there is no solution for \( G_1, G_2, G_3 \). As a consequence, \( \Delta \tilde{M}_0 \) is not closed in \( U_{iso(1,3)} \), which we demonstrate in Section 6.

6. \( \kappa \)-Minkowski spacetime and twisting igl(4)

To each realization (8), there is a corresponding twist. In [11], Abelian twists for realizations (8) where \( \psi = 1 \) were constructed. In [18], Jordanian twists for one subfamily of realizations (8) with \( \psi \neq 1 \) were constructed.

Let us consider the (undeformed) algebra igl(4) generated by \( L_{\mu \nu} = x_{\mu} p_{\nu} \) and \( p_{\mu} \). The commutation relations are given by

\[
[L_{\mu \nu}, L_{\lambda \rho}] = -i \left( \eta_{\mu \lambda} L_{\nu \rho} - \eta_{\nu \lambda} L_{\mu \rho} \right); \quad [L_{\mu \nu}, p_{\lambda}] = i \eta_{\mu \lambda} p_{\nu}. \tag{55}
\]

The commutation relations \([L_{\mu \nu}, x_i]\) can be easily calculated using (8) and (55).

The coalgebra structure can be obtained by the twist operator \( \mathcal{F} \). For the given twist operator, \( \Delta p_{\mu} = \mathcal{F} \Delta_0 p_{\mu} \mathcal{F}^{-1} \) and \( \Delta L_{\mu \nu} = \mathcal{F} \Delta_0 L_{\mu \nu} \mathcal{F}^{-1} \) (\( \Delta_0 p_{\mu} \) and \( \Delta_0 L_{\mu \nu} \) are primitive coproducts), while the counit is trivial.

For the realizations (8) given by \( \psi = 1 \) and \( \varphi = Z^{-\lambda} \), the corresponding Abelian twists [11, 18] are

\[
\mathcal{F} = \exp \left( i x_k p_k \otimes A - (1 - \lambda) A \otimes i x_k p_k \right). \tag{56}
\]
Coprodacts of $p_{\mu}$, obtained by twist (56) coincide with Eq. (43).

Generators $M_{\mu\nu} = L_{\mu\nu} - L_{\nu\mu}$ generate the subalgebra $\mathfrak{so}(1,3)$ of $\mathfrak{gl}(4)$. It can be shown that

$$
\Delta \tilde{M}_{i0} = L_{i0} \otimes Z^1 + Z^{1-\lambda} \otimes L_{i0} - L_{0i} \otimes Z^{-1} - Z^{1-\lambda} \otimes L_{0i} - (1 - \lambda) a_0 p_i \otimes \sum_{k=1}^{3} L_{ik} Z^{-1} + \lambda \sum_{k=1}^{3} L_{ik} Z^{1-\lambda} \otimes a_0 p_i.
$$

(57)

Hence, $\Delta \tilde{M}_{i0}$ cannot be expressed in terms of Poincaré generators $\tilde{M}_{\mu\nu}$ and $p_{\mu}$. It shows that it is not possible to put the coalgebra structure on the subalgebra $\mathfrak{so}(1,3)$, and we get $\kappa$-deformed $\mathfrak{gl}(4)$ Hopf algebra compatible with $\kappa$-Minkowski spacetime.

Similarly, one can obtain $\Delta L_{\mu\nu}$ and $\Delta p_{\mu}$ using the family of Jordanian twists [18]. Particularly, for left covariant ($\psi = \varphi = 1 - A$), and right covariant ($\psi = 1 + A, \varphi = 1$) realizations, corresponding Jordanian twists lead to the Poincaré-Weyl algebra which includes dilatations [19].

We point out that for the given $h_{\mu\nu}$, there are infinitely many ways of implementing $\kappa$-deformed $\mathfrak{gl}(4)$ or $\kappa$-deformed Lorentz algebras defined by Eqs. (14) and (15).

7. Conclusions

The structure of spacetime, at the scale where quantum gravity effects take place, is one of the most important questions in fundamental physics. One of the examples of noncommutative structure is $\kappa$-Minkowski spacetime. Below the quantum gravity scale the symmetry of spacetime should also be deformed. In this Letter we have generalized the description of such deformations, which include the well-known forms of $\kappa$-Poincaré-Hopf algebra in different basis as special cases.

For the given phase space $\hat{H}$, defined by $h_{\mu\nu}$ (7), there exists a unique undeformed Lorentz algebra such that $M_{\mu\nu}$ and $\hat{x}_\lambda$ generate the Lie algebra (21). Then, the realization of Lorentz generators is fixed. For example, they are given by (14), (15) and (34), corresponding to the family of realizations (8). The relations between $M_{\lambda0}$ and $p_{\mu}$ are given by (16) and (34). The Hopf algebra structure is fixed and it corresponds to the $\kappa$-Poincaré-Hopf algebra with undeformed Lorentz algebra [13].

For the given $h_{\mu\nu}$ we have constructed a three-parameter family of deformed Lorentz generators $\hat{M}_{\lambda0}$, described by $G_1$, $G_2$ and $G_3$ (46). The generators $\hat{M}_{\lambda0}$ are given by (14) and (15). The Hopf algebra structure $\Delta \hat{M}_{\lambda0}$ depends on $G_1$ and $G_3$ (50). The $\kappa$-Poincaré-Hopf algebra with deformed Lorentz algebra is related to
the $\kappa$-Poincaré-Hopf algebra with undeformed Lorentz algebra. A special example where $\lambda = \frac{1}{2}$, $G_1 = Z^{-\frac{1}{2}}$ and $G_3 = \frac{1}{2}$ coincides with the standard basis given in [1].

For the same choice of $h_{\mu\nu}$ there is an infinite family of deformed Lorentz generators $\hat{M}_0$ given by $F_i$ such that the corresponding Hopf algebra structure is not $\kappa$-Poincaré, but lies in $i \mathfrak{gl}(4)$. For example, $\hat{M}_\mu = x_\mu y - x_\nu y_\mu$, ($F_1 = F_2 = 1$, $F_3 = F_4 = 0$) does have the coproduct in $i \mathfrak{gl}(4) \otimes i \mathfrak{gl}(4)$ (for generic $h_{\mu\nu}$ see Section [6]). Note that for the natural realization (23) generators $M_{\mu\nu}$ have the $\kappa$-Poincaré-Hopf algebra structure (see Section [4]).

We point out that for the given phase space $\hat{H}$ defined by $h_{\mu\nu}$, there are infinitely many implementations of $\kappa$-deformed Poincaré (and $i \mathfrak{gl}(4)$) algebras and corresponding Hopf algebras. Our description contains them both ($\kappa$-deformed Poincaré and $i \mathfrak{gl}(4)$) therefore we call it generalized. There is no physical principle which would distinguish some choice in this class. What are good and bad physical consequences of some choice (which generally leads to Lorentz symmetry violation) is still an open question.

Nevertheless such a general form for the $\kappa$-Poincaré-Hopf algebra might be useful in the view for applications in the Planck scale or Quantum Gravity physics. Certain realizations of quantum spacetime influence the form of the symmetry algebra and deform the Casimir operators, which in turn lead to different dispersion relations. As it was already shown (see, e.g., [24]) such dispersion relations result in the so-called time delays for photons, which could be connected with the similar effect measured for high energy photons coming from gamma ray bursts (GRB’s).

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