Improved light quark masses from pseudoscalar sum rules

Stephan Narison

*Laboratoire Univers et Particules de Montpellier; CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier, France.

Abstract

Using ratios of the inverse Laplace transform sum rules within stability criteria for the subtraction point $\mu$ in addition to the ones of the usual $\tau$ spectral sum rule variable and continuum threshold $t_c$, we extract the $\pi(1300)$ and $K(1460)$ decay constants to order $\alpha_s^4$ of perturbative QCD by including power corrections up to dimension-six condensates, tachyonic gluon mass, instanton and finite width corrections. With these inputs, we extract, in a model-independent way, the sum of the renormalization group invariant (RGI) quark masses ($\tilde{m}_u + \tilde{m}_d$) : $q \equiv d$, $s$ and the corresponding running masses ($\tilde{m}_u + \tilde{m}_d$) evaluated at 2 GeV. By giving the value of the ratio $m_u/m_d$, we deduce the running quark masses $\tilde{m}_{u,d,s}$ and the scale-independent mass ratios : $2m_s/(m_u + m_d)$ and $m_u/m_d$. Using the positivity of the spectral function, we also deduce new lower bounds on these running masses from the inverse Laplace transform sum rules. Our results are summarized in Table 2.

Keywords: QCD spectral sum rules, meson decay constants, light quark masses, chiral symmetry.

1. Introduction and a short historical overview

Pseudoscalar sum rules have been introduced for the first time in [2] for giving a bound on the sum of running light quark masses defined properly for the first time in the $\overline{MS}$-scheme by [3]. Its Laplace transform version including power corrections introduced by SVZ [4] 1,2 has been applied few months later to the pseudoscalar channel in [10] and extended to the estimate of the SU(3) corrections to kaon PCAC in [11]. Its first application to the scalar channel was in [12]. Later on, the previous analysis has been reconsidered in [13] for extracting e.g. the $\pi(1300)$ and $K(1460)$ decay constants. The first FESR analysis in the pseudoscalar channel has been done in [14, 15] which has been used later on by various authors 3.

However, the light pseudoscalar channel is quite delicate as the PT radiative corrections ([2, 16] for the $\alpha_\tau$, [14, 17] for the $\alpha_\pi^2$, [18] for the $\alpha_\pi^3$ and [19] for the $\alpha_\pi^4$ corrections) are quite large for low values of $Q^2 \approx 1$ GeV$^2$ where the Goldstone pion contribution is expected to dominate the spectral function, while (less controlled) and controversial instanton-like contributions [20–22]4 might break the operator product expansion (OPE) at a such low scale. However, working at higher values of $Q^2$ for avoiding these QCD series convergence problems, one has to face the dominant contribution from radial excited states where a little experimental information is known. Some models have been proposed in the literature for parametrizing the high-energy part of the spectral function. It has been proposed in [13] to extract the $\pi(1300)$ and $K(1460)$ decay constants by combining the pseudoscalar and scalar sum rules which will be used in the Laplace sum rules for extracting the light quark masses. Though interesting, the analysis was quite qualitative (no estimate of the errors) such that it is not competitive for an accurate determination of the quark masses. This estimate has been improved in [7] using a narrow width approximation (NWA). Later on, a much more involved ChPT based parametrization of the pion spectral function has been proposed in [24] where the model dependence mainly appears in the interference between the $\pi(1300)$ and the $\pi(1800)$. Using FESR with some weight functions inspired from $\tau$-decay [26–31], the authors of [32] have extracted the decay constants of the $\pi(1300)$ and the $\pi(1800)$ by assuming that they do not interfere in the spectral function. The results for the spectral function are one of the main ingredient for extracting the light quark masses from pseudoscalar sum rules and it is important to have a good control (and a model-independence) of its value for a more precise and model-independent determination of such light quark masses.

In this paper, our aim is to extract the spectral function or the $\pi(1300)$ and K(1460) decay constants from the ratio of Laplace sum rules known to order $\alpha_s^4$ of perturbation theory (PT) and including power corrections up to dimension six within the SVZ expansion plus those beyond it such as the tachyonic gluon mass and the instanton contributions. With this result, we shall extract the light quark mass values at the same approximation of the QCD series.

2. The pseudoscalar Laplace sum rule

- The form of the sum rules

We shall be concerned with the two-point correlator:

$$\psi(x) = \int d^4x \, e^{iq \cdot x} \langle 0 | T J_5^a(x) J_5^a(0) | 0 \rangle$$

(1)
where $J_5^q(x)$ is the local pseudoscalar current:

$$J_5^q(x) \equiv (m_u + m_d) \bar{u}(i \gamma_5) q, \quad q = d, \, s; \, P = \pi, \, K. \quad (2)$$

The associated Laplace sum rules (LSR) $L^0_q(\tau, \mu)$ and its ratio $R^0_q(\tau, \mu)$ are deduced in \cite{34}:

$$L^0_q(\tau, \mu) = \int_0^\tau dt \, e^{-\tau t} \frac{1}{\pi} \text{Im} \phi^0_q(t, \mu), \quad (3)$$

$$R^0_q(\tau, \mu) = \int_0^\tau dt \, e^{-\tau t} \frac{1}{\pi} \text{Im} \phi^0_q(t, \mu), \quad (4)$$

where $\mu$ is the subtraction point which appears in the approximate QCD series. The ratio of sum rules $R^0_q(\tau, \mu)$ is useful here for extracting the contribution of the radial excitation $P'$ to the spectral function, while the Laplace sum rule $L^0_q(\tau, \mu)$ will be used for determining the sum of light quark masses.

**The QCD expression within the SVZ expansion**

As mentioned earlier, the perturbative expression of the two-point correlator $\phi^0_q(q^2)$ is known up to order $\alpha_s^2$ from successive works \cite{2, 14, 17-19}. For a convenience of the reader, we give below the numerical expression \cite{34}:

$$L^0_q(\tau) = \frac{3}{8\pi^2} \bar{m}_q + m_q^2 \tau^2 \left[ 1 + \sum_{n=1, 2} \delta_n(0) \tau^n \right] + 2m_q^2 \left[ 1 + \sum_{n=1, 2} \delta_n(2) \tau^n \right] + \tau^2 \delta(4) + \tau^3 \delta(6), \quad (5)$$

where $\bar{m}_q$ is the running quark mass evaluated at the scale $\mu$.

From the analytic expression compiled in \cite{34}, we derive the numerical PT corrections:

$$\delta_1(0) = 4.82107 - 2\Delta, \quad \delta_1(2) = 21.976 - 28.0729\Delta, \quad \delta_1(4) = 53.1386 - 677.987\Delta, \quad \delta_1(6) = -31.6283 + 756071\Delta, \quad \delta_1(8) = 321.968\Delta + \frac{7735\Delta^4}{384}$

$$\delta_2(0) = 7.64213 - 4\Delta, \quad \delta_2(2) = 51.0915 - 62.93\Delta + \frac{25}{2}\Delta^2, \quad (6)$$

where $\Delta \equiv \alpha_s/\pi, \, \Delta = -\log(\tau^2\mu^2)$. The non-perturbative corrections are combinations of RGI quantities defined in \cite{27, 35, 36}:

$$\overline{m}_q(\bar{q}q) = m_q(\bar{q}q) + \frac{3}{2\pi^2} m^4_q \left( \frac{1}{a_s} \frac{53}{24} \right)$$

$$\langle \alpha_s G^2 \rangle = \langle \alpha_s G^2 \rangle \left( 1 + \frac{16}{9} a_s \right) - \frac{16}{9} a_s \left( 1 + \frac{91}{24} a_s \right) m_q(\bar{q}q), \quad (7)$$

In terms of these quantities, they read \cite{2, 7, 37, 38}:

$$\delta_q(4) = \frac{4\pi^2}{3} \delta_q(4) + \delta_q(4), \quad (8)$$

with:

$$\delta_q(4) = -2m_q(\bar{u}u) \left[ 1 + a_s \left( 5.821 - 2\Delta \right) \right] + m_q(\bar{q}q) \left[ 1 + a_s \left( 5.266 - 2\Delta \right) \right] - \frac{3}{7\pi^2} m^4_q \left( \frac{1}{a_s} + 2.998 - \frac{15}{4} \Delta \right), \quad (9)$$

The contribution of the $d = 6$ condensates is:

$$\delta(6) = -\frac{4\pi^2}{3} m_q(\bar{u}u) + \frac{32}{27} \pi a_s \left[ (\bar{u}u)^2 + (\bar{q}q)^2 - 9(\bar{u}u)(\bar{q}q) \right], \quad (10)$$

where $\langle \bar{u}u \rangle \equiv \langle (\bar{u}u)/2 \rangle G^\alpha_{\mu\nu} \sigma_{\mu\nu} \equiv M^2_0(\bar{u}u)$ with $M^2_0 = (0.8 \pm 0.2)$ GeV$^2$ \cite{39-41} is the quark-gluon mixed condensate; $\rho = (2.6 \pm 0.6)$ \cite{29, 39, 42} indicates the violation of the vacuum saturation assumption of the four-quark operators.

**The tachyonic gluon mass contribution**

The tachyonic gluon mass $\lambda$ of dimension two has been introduced in \cite{43, 44} for parametrizing the uncalculated higher order terms of the PT series \cite{45}. It appears naturally in most holographic QCD models \cite{46}. Its contribution reads \cite{44}:

$$L^m_q(\tau) = -\frac{3}{2\pi} (\bar{m}_q + m_q^2) a_s \lambda^2 \tau^{-1}, \quad (11)$$

Its value has been estimated from $e^+ e^-$ \cite{29, 47} and $\tau$-decay \cite{30} data:

$$a_s \lambda^2 = -(0.07 \pm 0.03) \text{ GeV}^2. \quad (12)$$

**The instanton contribution**

The inclusion of this contribution into the operator product expansion (OPE) is not clear and controversial \cite{20-22}. In addition, an analogous contribution might lead to some contradiction to the OPE in the scalar channel \cite{23}. Therefore, we shall consider the sum rule including the instanton contribution as an
alternative approach. For our purpose, we parametrize this contribution as in [20, 21], where its corresponding contribution to the Laplace sum rule reads:

\[ L_3^\pi(\tau)_{\rm inst} = \frac{3}{8\pi^2} (m_u + m_d)^2 \tau^3 \rho_c^2 e^{-\tau \rho_c} [K_0(r_c) + K_1(r_c)] , \tag{13} \]

where \( K_n \) is the Bessel-Mac-Donald function; \( r_c \equiv \rho_c^2/(2\tau) \) and \( \rho_c = (1.89 \pm 0.11) \text{ GeV}^{-1} \) [48] is the instanton radius.

Table 1: Input parameters: the value of \( \hat{m}_q \) has been obtained from the running masses evaluated at 2 GeV: \( \overline{m}_u + \overline{m}_d \approx 7.9(3) \text{ MeV} \) [7, 49]. Some other predictions and related references can be found in [1]. The error on \( \Gamma_\pi \) is a guessed estimate.

| Parameters | Values | Ref. |
|-----------|--------|-----|
| \( \Lambda(n_f = 3) \) | (353 \pm 15) MeV | [31, 50] |
| \( \hat{m}_q \) | (0.114 \pm 0.021) GeV | [7, 31, 49, 51] |
| \( \hat{\rho}_q \) | (263 \pm 7) MeV | [7, 49, 51] |
| \( \kappa \equiv (3\hat{s})/(\hat{g}u) \) | (0.74 \pm 0.06) | [52] |
| \( a_s^2 \) | (7 \pm 3) \times 10^{-2} \text{ GeV}^2 | [30, 47] |
| \( \langle \pi, \rho \rangle^2 \) | (7 \pm 2) \times 10^{-2} \text{ GeV}^4 | [29, 30, 33, 42, 47, 48, 53-57] |
| \( M_0^2 \) | (0.8 \pm 0.2) \text{ GeV}^2 | [39-41] |
| \( \rho_\pi \) | (1.89 \pm 0.11) \text{ GeV}^{-1} | [48] |
| \( \Gamma_\pi \) | (0.6 \pm 0.2) \text{ GeV} | [1] |
| \( \Gamma_\pi \) | (0.25 \pm 0.05) \text{ GeV} | [1] |

### The QCD input parameters

We shall work in the analysis with the input parameters given in Table 1. \( \hat{m}_q \) and \( \hat{\rho}_q \) are RGI invariant mass and condensates which are related to the corresponding running parameters as [3]:

\[ \bar{m}_q(\tau) = \frac{\hat{m}_q}{(-\log \sqrt{\Lambda})^{2/\beta_1}} (1 + \rho_m), \]
\[ \langle \bar{q}q \rangle(\tau) = \frac{-\hat{\rho}_q}{(-\log \sqrt{\Lambda})^{2/\beta_1}} (1 + \rho_m), \]

where \( \beta_1 = (-1/2)(11 - 2n_f/3) \) is the first coefficient of the QCD \( \beta \)-function for \( n_f \)-flavours. \( \rho_m \) is the QCD correction which reads to N4LO accuracy for \( n_f = 3 \) [7, 58]:

\[ \rho_m = 0.8951 a_\pi + 1.3715 a_\pi^2 + 0.1478 a_\pi^3, \]

where \( a_\pi = \alpha_s/\pi \) is the QCD running coupling.

3. A Laplace sum rule estimate of the decay constant \( f_\pi \)

#### The spectral function

We shall parametrize the spectral function as:

\[ \frac{1}{\pi} \text{Im}\hat{\Sigma}(t) = \sum_{n, \pi'} 2f_{\pi'}^2 \bar{m}_0^2 \delta(t - m_{\pi'}^2) + \text{"QCD cont."} \theta(t-t_\pi), \tag{16} \]

where the higher states (\( n', \ldots \)) contributions are smeared by the “QCD continuum” coming from the discontinuity of the QCD diagrams and starting from a constant threshold \( t_\pi \). \( f_\pi \) is the well-known decay constant:

\[ \langle 0|J_\pi^\mu(x)|P \rangle = \sqrt{2} m_\pi^2 f_\pi \rho_c, \tag{17} \]

normalized here as: \( f_\pi = (93.2 \pm 0.2) \text{ MeV} \) and \( f_K \approx (1.20 \pm 0.01)f_\pi [59] \). We improve the \( \pi' \equiv \pi(1300) \) contribution by taking into account the finite width correction by replacing the delta function with a Breit-Wigner shape:

\[ \pi_\pi(t - m_{\pi'}^2) \to BW(t) = \frac{m_\pi \Gamma_{\pi'}}{(t - m_{\pi'}^2)^2 + m_{\pi'}^2 \Gamma_{\pi'}^2}. \tag{18} \]

Defined in this way, the \( \pi' \) can be considered as an “effective resonance” parametrizing the higher state contributions not smeared by the QCD continuum and may take into account some possible interference between the \( \pi(1300) \) and \( \pi(1800) \) contributions.

- \( f_\pi \) from the ratio \( R_\pi^\pi \) within the SVZ expansion\(^7\) at arbitrary \( \mu \)

One expect from some chiral symmetry arguments that \( f_\pi \) behaves like \( m_\pi^2 \). Therefore, one may expect that the \( \pi' \) will dominate over the pion contribution in the derivative of the Laplace sum rule:

\[ -\frac{\partial}{\partial \tau} L_3^\pi(\tau, \mu), \tag{19} \]

from which one can extract the decay constant \( f_\pi \) or the \( \pi(1300) \) contribution to the spectral function. In order to eliminate the unknown value of the sum of light quark masses \( (m_u + m_d) \), it is convenient to work with the ratio of Finite Energy Laplace sum rules \( R_\pi^\pi(\tau, \mu) \) defined in Eq. (4). In so doing, we define the quantity:

\[ r_\pi \equiv \frac{M_\pi^2 f_\pi^2}{m_\pi^2 f_{\pi'}^2}, \tag{20} \]

which quantifies the relative weight between the \( \pi' \) and the pion contribution into the spectral function. It is easy to deduce the sum rule:

\[ r_\pi = \frac{R_{\pi|^\pi}_{\mu \equiv \tau}}{BW_{\pi|-\pi_{\mu}|\tau}BW_{\pi^0}} e^{-m_{\pi'}^2 \tau}. \tag{21} \]

\( R_{\pi|^\pi}_{\mu \equiv \tau} \) is the QCD expression of the FESR in Eq. (4) where we have parametrised the spectral function by a step function corresponding to the perturbative expression for massless quarks from the threshold \( t_\pi \). \( BW_{\pi^0} \) is the Breit-Wigner integral:

\[ BW_{\pi^0} \equiv \frac{1}{\pi} \int_{m_\pi^2}^\infty dt t^2 e^{-t} BW(t) = n = 0, 1, \tag{22} \]

where \( BW(t) \) has been defined in Eq. (18). With the set of input parameters in Table 1, we show in Fig. 1a the \( \tau \)-behaviour of \( r_\pi \) at a given value of \( \mu = 1.53 \text{ GeV} \). We extract the optimal result at the value of \( t_\pi = 2 \text{ GeV}^2 \) where both a minimum in the change of \( t_\pi \) is obtained and where a large range of \( \tau \)-stability

\(^7\)Here and in the following we shall denote by SVZ expansion the OPE without the instanton contribution.
is reached. In Fig. 1b, we study the influence of the choice of $\mu$ on this result. Our final optimal result corresponds to the values of $\tau \approx (0.50 - 0.55) \text{ GeV}^{-2}$ and of the inflexion point $\mu = 1.53 \text{ GeV}$ at which we deduce:

$$r_\pi^{SVZ} = 4.63(10)\lambda(1)c(19)_{\mu}(14)cGR(0)_{\mu}(91)(181)r_0$$

$$= 4.63 \pm 1.83,$$  

(23)

where the main error comes from the experimental width of the $\pi(1300)$ which needs to be improved.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure1.png}
\caption{a) $\tau$-behaviour of $r_\pi$ for $\mu = 1.53 \text{ GeV}$ and for different values of $\tau$ within the SVZ expansion. b) $\mu$-behaviour of the optimal value of $r_\pi$ deduced from a).}
\end{figure}

• **Convergence of the QCD series**

- We study in Fig 2a, the convergence of the PT QCD series at the value of the subtraction scale $\mu = 1.53 \text{ GeV}$ for the ratio $R_{\pi}^2(\tau, \mu)$ entering in the estimate of $r_\pi$ (lower family of curves). One can notice that for $\tau \approx (0.5 - 0.6) \text{ GeV}^{-2}$, the $a_1, \Delta_1^2, \Delta_2^2$ and $\alpha_s^2$ effects are respectively -7, -9, -9.5 and -7.2% of the preceding PT series: LO, NLO, N2LO and N3LO. The convergence of the PT series is not quite good but each corrections to $r_\pi$ are reasonably small.

- We show in Fig 2b, the convergence of the power corrections for $r_\pi$ (lower family of curves). We see that the $d = 2, 4, 6$ dimension operator effects are -1.8, -6.8 and -4.6% of the preceding sum of contributions indicating a good convergence and relatively small corrections.

• **Tachyonic gluon mass contribution to $r_\pi$**

The tachyonic gluon mass contribution has been included into the result in Eq. (23). It decreases the value of $r_\pi$ by about 0.1 which is relatively negligible. By duality with the higher order of PT series [45], this small effect of the tachyonic gluon mass confirms the good convergence of the PT series obtained previously.

$$r_\pi^{inst} = 3.10(2)\lambda(1)c(1)_{\mu}(3)cGR(0)_{\mu}(101)_{\mu}(114)r_0$$

$$= 3.10 \pm 1.19,$$  

(24)

which is in better agreement with the one obtained by using the SVZ expansion.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{a) $\tau$-behaviour of the PT series of $\sqrt{R_{\pi}^2(\tau, \mu)}$ (upper group of curves) and of $R_{\pi}^2(\tau, \mu)$ (lower group of curves) appropriately normalised to 1 for $\tau = 0$ and using $\mu = 1.53 \text{ GeV}$; b) the same as a) but for the power corrections within the SVZ expansion.}
\end{figure}

• **$r_\pi$ from instanton sum rule at arbitrary $\mu$**

We include the instanton contribution into the OPE using the expression given in Eq. (13). The variations of $r_\pi$ versus $\tau$ and $t_\pi$ for different values of $\mu$ are similar to the one in Fig. 3a which are also qualitatively similar to the ones discussed without the instanton contribution. The optimal result is obtained for $\tau \approx (0.3 - 0.4) \text{ GeV}^{-2}$ and $t_\pi \approx 2 \text{ GeV}^{-2}$ (minimum in $t_\pi$). If we take $\mu = 1.53 \text{ GeV}$ where the optimal result in Eq. (23) is obtained, one would get $r_\pi^{inst} \approx 1.62$. At this value of $\mu$ and for $\tau \approx 0.3 \text{ GeV}^{-2}$, where the optimal result is obtained, the sum of the $d \leq 6$ condensates is $-6\%$ of the PT contributions, while the one due to the instanton is $+8\%$ of the PT $\oplus d \leq 6$ contributions which are relatively large corrections. One can improve the convergence of the OPE by working with larger value of $\mu$. Then, we study in Fig. 3b the $\mu$-dependence of $r_\pi$ where a minimum in $\tau$ (resp. $t_\pi$) is obtained for various values of $\mu$ at $(0.3-0.4) \text{ GeV}^{-2}$ (resp. 2 GeV$^2$) like in Fig. 3a. A $\mu$-stability is obtained at about (4-4.75) GeV with a minimum at $\mu \approx 4.25 \text{ GeV}$. At this scale, we show in Fig. 4 the convergence of the OPE for the ratio of moments (lowest families of curves). The instanton contribution is about 50% of the PT $\oplus d \leq 6$ contributions which is a more reasonable correction. Then, we deduce the optimal value of $r_\pi$ at this value $\mu=4.25 \text{ GeV}$:

\begin{align*}
\text{(24)}
\end{align*}

The shape of the curves for $\mu = 1.53 \text{ GeV}$ is similar to the one in Fig. 3b.
\[
\tau \left[ \text{GeV}^{-2} \right]
\]

\[
\text{SVZ} \oplus \text{Instanton}
\]

1. \( r_\pi \) from the Laplace sum rule at \( \mu = \tau^{-1/2} \)

We complete your analysis in the case where the subtraction constant \( \mu \) is equal to the sum rule variable \( 1/\sqrt{\tau} \). This case is interesting as it does not possess the Log \( \mu \tau \) terms appearing in the PT series which have large coefficients and which are now absorbed into the running of \( \alpha_s(\tau) \) from the renormalization group equation. This case has been largely used in the literature (for reviews see, e.g.: [6–9]). The analysis is very similar to the previous case. In Fig. 5, we show the \( \tau \)-behaviour of the results in the case of the SVZ expansion and SVZ \( \oplus \) instanton contribution where in both cases a minimum in \( t_c \) is obtained at \( t_c = 2 \text{ GeV}^2 \). We obtain:

\[
r_\pi^{\text{SVZ}} = 5.70(88)\alpha_s(11)\beta_0(86)\beta_\pi(63)\xi^2(2)\alpha_\pi(113)\beta_\pi(90)r_\pi, \]

\[
r_\pi^{\text{inst}} = 3.54(80)\alpha_s(1)\beta_0(6)\beta_\pi(11)\xi^2(0)\alpha_\pi(8)\beta_\pi(119)r_\pi, \]

and:

\[
r_\pi = \frac{5.12}{\pi} + 1.36 \Rightarrow \frac{f_\pi}{f_\pi} = (2.58 \pm 0.35) \times 10^{-2}, \]

\[
r_\pi^{\text{inst}} = 3.27 \pm 0.95 \Rightarrow \frac{f_\pi}{f_\pi} = (2.06 \pm 0.30) \times 10^{-2} \]

where we have separated the determinations from the SVZ and SVZ \( \oplus \) instanton sum rules. In Fig. 6, we compare the above two results, with the existing ones in the current literature: NPT83 [13], SN02 [7], BPR95 [24], KM02 [32] for the quantity:

\[
L_\pi(\tau) \equiv r_\pi \text{BW}_I(0),
\]

involved in the Laplace sum rule estimate of \( (m_u + m_d) \) which we shall discuss in the next sections. Here \( \text{BW}_I(0) \) defined in Eq. (22) is the integrated spectral function entering into the lowest moment Laplace sum rule \( L_\pi(\tau) \). For this comparison, we have used:

\[
r_\pi = (9.5 \pm 2.5) \text{ and consistently the NWA for the results in NPT83 and SN02 from [7] (see also [17]).}
\]

- For KM02, we add coherently the \( \pi(1300) \) and \( \pi(1800) \) contributions which may be an overestimate as they may have a destructive interference like in [24]. We use the decay constants \( f_\pi(1300) = (2.2 \pm 0.57) \text{ MeV} \) and \( f_\pi(1800) = (1.36 \pm 0.21) \text{ MeV} \) obtained in [32] and consistently a Breit-Wigner parametrization of the spectral function. We have not introduced the (large)
error due to the width of the π(1300) because it is not clear if this error is already included in the quoted value of the decay constants. Therefore, we may have underestimated the errors for this comparison.

- For BPR95, we add, into their parametrization, the error due to the π(1300) width which is not included in their original work.

- The CHPT parametrization from [24] without any resonance is also given in Fig. 6.

One can see that there is (within the errors) a complete agreement between the different determinations though the previous results tend to slightly overestimate the value of \( r_\tau \) at smaller values of \( \tau \) which would imply a slightly larger value of \((m_u + m_d)\). The results obtained in [24] and [32] are model-dependent as they depend on the way of treating the \( \pi(1800) \) contribution into the spectral function. In [24], a destructive interference of about 10% suppression in front of the \( \pi(1800) \) propagator is adopted, while, in [32], it is supposed that there is a coherent sum without interference of the \( \pi(1300) \) and \( \pi(1800) \). The \( \pi(1800) \) contribution is relevant in these two approaches because the FESR used there is sensitive to the high-energy tail of the spectral function. This is not the case of the Laplace sum rule discussed here. The exponential weight safely suppresses this less controlled region while the QCD continuum introduced above the \( \pi(1300) \) mass is expected (like in some other many channels) to smear the contributions of higher mass states namely the one of the \( \pi(1800) \).

![Figure 6: Comparison of some other determinations of \( r_\tau \) for a given value of \( \tau = 5 \) GeV\(^2\) which corresponds to the optimal value of \((m_u + m_d)\). The blue continuous line with a circle is the CHPT prediction without a resonance. The results of NPT83 [13] and SN02 [7] are within a narrow width approximation. The errors due to the experimental width of the \( \pi(1300) \) have been introduced in the result of BPR95 [24].](image)

From the previous comparison, we notice that the prediction from the SVZ expansion agrees (within the errors) with all existing predictions shown in Fig. 6, while the one including the instanton tends to underestimate the \( \pi(1300) \) contribution to the spectral function. One can also notice that the tachyonic gluon mass contribution decreases \( r_\pi \) by about 0.1 which is negligible.

4. Estimate of \((m_u + m_d)\) within the SVZ expansion

- **The Laplace sum rule analysis at \( \mu = 1.53 \) GeV**

We find convenient to extract the RGI scale independent mass defined in Eq. (14):

\[
\hat{m}_{ud} \equiv \frac{1}{2}(\tilde{m}_u + \tilde{m}_d) \tag{29}
\]

from the Laplace sum rule \( L_2^\mu(\mu, \tau) \) in Eq. (3). The QCD expression of \( L_2^\mu(\mu, \tau) \) is given in Eq. (5). We shall use into the spectral function, parametrized as in Eq. (16), the value of \( r_\tau \) obtained in Eq. (27). In Fig. 7, we study the \( \tau \)-dependence of \( \hat{m}_{ud} \) for different values of \( \tau \) and for a given value of the subtraction point \( \mu = 1.53 \) GeV, where the optimal value of \( r_\tau \) has been obtained.

One can find from this figure that the largest \( \tau \)-stability range is obtained for \( \tau = (5 \pm 0.5) \) GeV\(^2\). Using the values of the parameters in Table 1, we extract the optimal value of the sum of the RGI \( u, d \) quark masses at the \( \tau \)-minimum of 0.5 GeV\(^2\):

\[
\hat{m}_{ud}^{\text{SVZ}} = 4.47(17)\tau(12),(6)_{\text{SVZ}}(4)_{\text{SVZ}}(0)_{\text{SVZ}}(1)_{\text{SVZ}}(13)_{\text{SVZ}}(39)_\mu \text{ MeV},
\]

\[
= (4.47 \pm 0.47) \text{ MeV}, \tag{30}
\]

where the error due to \( \tau \) corresponds to the localisation of the minimum, while moving \( \tau \) around the minimum leads to a negligible uncertainty.

![Figure 7: \( \tau \)- and \( \tau \)-dependence of \( \hat{m}_{ud} \) from the Laplace sum in Eq. (3) at the subtraction scale \( \mu = 1.53 \) GeV.](image)

- **Convergence of the QCD series**

  - The different contributions of the truncated PT series to \( \sqrt{L_2^\mu(\mu, \tau)} \) for \( \mu = 1.53 \) GeV are given in Fig 2a (upper family of curves).

    One can deduce that for \( \tau = 0.5 \) GeV\(^{-2}\), the \( \alpha_s, \alpha_s^2, \alpha_s^3 \) and \( \alpha_s^4 \) effects are respectively +27, +15.5, +4.6 and +2% of the preceding PT series: LO, NLO, N2LO and N3LO, which indicate a good convergence.

    - We show in Fig 2b, the convergence of the power corrections for \( \sqrt{L_2^\mu(\mu, \tau)} \) for \( \mu = 1.53 \) GeV (upper family of curves).

      We see that the \( d = 2, 4, 6 \) contributions are +3.4, +3.1 and +1.3% of the preceding sum of contributions (PT, PT \( \oplus \) d = 2, PT \( \oplus d = 2 + 4 \)) indicating a good convergence and a relatively small correction.
• Estimate from the Laplace sum rule at $\mu = \tau^{-1/2}$

As mentioned previously, this sum rule has been widely used in the current literature for extracting $m_{ud}$. We shall use it here as another method for determining $m_{ud}$. The analysis is similar to the one for arbitrary $\mu$. We show the $\tau$-dependence for different $t_c$ in Fig. 8. One can observe a $\tau$-stability for $t_c = (5.5 \pm 0.5)$ GeV$^2$ and $\tau = 0.45$ GeV$^{-1/2}$ at which we extract the optimal result:

$$m_{ud} = 4.30(12)_{r_c}(11)_{\lambda}(5)_{t_c}(1)_{\text{inst}}(3)_{G}G(0)_{\mu G}(1)_{\rho}$$

(31)

• Tachyonic gluon mass contribution to $\hat{m}_{ud}$

If we do not include the tachyonic gluon mass into the SVZ expansion, the value of $\hat{m}_{ud}$ obtained in Eq. (30) would increase by 0.14 MeV which is relatively negligible confirming again the good convergence of the PT series if the duality between the tachyonic gluon mass and the non yet calculated higher order PT corrections are used [45].

Figure 8: $\tau_c$ and $t_c$-dependence of $\hat{m}_{ud}$ from the Laplace sum in Eq. (3) at the subtraction scale $\mu = \tau^{-1/2}$

- Final estimate of $\hat{m}_{ud}$ within the SVZ expansion

We consider, as a final estimate of $\hat{m}_{ud}$ within the SVZ expansion, the mean value of the results in Eqs. (30) and (31):

$$(\hat{m}_{ud}^{\text{RGI}}) = (4.38 \pm 0.33) \text{ MeV},$$

(32)

where the error is the mean of the sum of quadratic errors (the overlap of the two determinations leads to about the same value).

5. $m_{ud}$ from the instanton Laplace sum rules

For optimizing the instanton contribution, we work at the same subtraction point $\mu = 4.25$ GeV where $r_{\text{inst}}^{(3)}$ has been obtained. We repeat the previous analysis by taking into account the instanton contribution. Its contribution to $\sqrt{\mathcal{E}}(\mu, \tau)$ compared to the OPE up to $d=6$ condensates is shown in Fig. 4 (upper family of curves) for $\mu = 4.25$ GeV where the estimate of $r_{\tau}$ is optimized in this case (see Fig. 3). The instanton contribution is about $+14\%$ of the perturbative $\oplus d \leq 6$ condensates ones at $\tau = 0.4$ GeV$^{-2}$ where the instanton sum rule is optimized. The

Figure 9: a) $\tau$-behaviour of $m_{ud}$ in the case $\mu = 4.25$ GeV. b) the same as in a) but in the case $\mu = \tau^{-1/2}$.

6. $\hat{m}_{ud}$ and $m_{ud}(2)$ from Laplace sum rules

We consider, as a final estimate of the RGI mass $\hat{m}_{ud}$, the results obtained in Eqs. (32) and (34) from which we deduce the running masses at order $a_s^3$ evaluated at 2 GeV in units of MeV:

$$\hat{m}_{ud}^{\text{svz}} = 3.77(28), \quad \hat{m}_{ud}^{\text{inst}} = 2.82(18).$$

(35)

We have not taken the mean value of the two results taking into account the controversial contribution of the instanton into the pseudoscalar sum rule [20–23]. Using the mean of the range of different results quoted in PDG13 [1] for the ratio:

$$\frac{m_u}{m_d} = 0.50(3),$$

(36)
which (a priori) does not favour the solution \( m_{u} = 0 \), one can deduce the value of the \( u \) and \( d \) running quark masses at 2 GeV in units of MeV:

\[
\begin{align*}
\bar{m}_{u}^{\text{inst}} & = 2.51(21), \quad \bar{m}_{u}^{\text{inst}} = 1.88(14), \\
\bar{m}_{d}^{\text{inst}} & = 5.02(42), \quad \bar{m}_{d}^{\text{inst}} = 3.76(28),
\end{align*}
\]

where one can notice that the sum rule within the SVZ \( \oplus \) instanton contribution tends to give lower values of the light quark masses. We summarize our results in Table 2.

7. Laplace sum rule estimate of \( f_{K} \)

Using the same method as in the case of the \( \pi^\prime \), we shall estimate the \( K' \equiv K(1460) \) decay constant through:

\[
r_{K} \equiv \frac{M_{K}^{2}f_{K}^{2}}{m_{K}^{2}}, \tag{38}
\]

• **Analysis within the SVZ expansion for arbitrary \( \mu \)**

We show the \( \tau \)- and \( t_{c} \)-behaviours of \( r_{K} \) in Fig. 10a for a given value of the subtraction point \( \mu = 4 \) GeV, where we have a \( t_{c} \)-minimum at 2.5 GeV\(^{2} \) and a \( \tau \)-stability around \( \tau = 0.8 \) GeV\(^{-2} \). At this scale, one can inspect in Fig. 11a (lower families of curves) obtained at \( \mu = 4 \) GeV that the PT corrections to \( R_{K}^{\mu}(\tau, \mu) \) are small: the \( \alpha_{r}, \alpha_{\beta}, \alpha_{\gamma} \) and \( \alpha_{\delta} \) effects are respectively -4.2, -4.5, -4.7 and -4.6\% of the preceding PT series including: LO, NLO, N2LO and N3LO contributions. The NP corrections are shown in Fig. 11b. They remain reasonably small: the \( d = 2, 4, 6 \) dimension operators contributions are -2.4, -10 and -11\% of the preceding sum of contributions (\( PT, \ PT \oplus d = 2, PT \oplus d = 2 + 4 \)).

![Figure 10: a) \( \tau \)-behaviour of \( r_{K} \) for a given value \( \mu = 4 \) GeV of subtraction point and for different values of \( t_{c} \). b) \( \mu \)-behaviour of the optimal value of \( r_{K} \) deduced from a). The error corresponds to \( t_{c} = (2.5 \pm 0.25) \) GeV. The dashed region is the mean value obtained from \( \mu = (3 \pm 6) \) GeV, where \( r_{K} \) is almost stable.

In Fig. 10b, we show the \( \mu \)-behaviour of the optimal results obtained from Fig. 10a. One can notice that no inflexion point like in the case of the pion appears here. Instead \( r_{K} \) increases slowly for \( \mu \geq 3 \) GeV. We consider the mean of the results obtained in the range \( \mu = 3 \) to 6 GeV from which we deduce:

\[
r_{K}^{\text{opt}} = 4.79 \pm 0.08, \tag{39}
\]

where the error due to the choice of \( t_{c} = (2.5 \pm 0.25) \) GeV\(^{2} \) is the mean of the quadratic sum of each individual errors. The value of this mean value corresponds to the one at \( \mu = 4 \) GeV.

One can notice that the tachyonic gluon mass contribution included here has decreased the value of \( r_{K} \) by a small amount of 0.10. We evaluate the different errors from some other sources and obtain the final estimate:

\[
r_{K}^{\text{opt}} = 4.79 \pm 0.60 \implies \frac{f_{K}}{f_{K}} = (25.0 \pm 1.5)10^{-2}. \tag{40}
\]

One can remark that \( r_{K} \approx r_{K} \) as might be expected from chiral symmetry arguments.

• **Analysis within the SVZ expansion for \( \mu = \tau^{-1/2} \)**

We show the result of the analysis in Fig. 12 where maxima are not obtained for reasonable values of \( \tau \). Therefore, we shall not consider the result of this sum rule in the following. However, one can notice that, for the value of \( \tau \approx 0.8 \) GeV\(^{-2} \) where the sum rule for arbitrary \( \mu \) is optimized (see Fig. 10), the two sum rules give about the same predictions.

![Figure 11: a) \( \tau \)-behaviour of the PT series of \( \sqrt{\tau}_{K}^{\mu}(\tau, \mu) \) (upper group of curves) and of \( R_{K}^{\mu}(\tau, \mu) \) (lower group of curves) appropriately normalised to 1 for \( \tau = 0 \) and using \( \mu = 4 \) GeV. b) the same as a) but for the power corrections.

8
• \( r_K \) from instanton sum rules for arbitrary \( \mu \)

The analysis for \( \mu = \tau^{-1/2} \) does not also lead to a conclusive result like in the case of the usual SVZ expansion. Then, we shall not also consider it here and study the sum rule for arbitrary value of \( \mu \). We show in Fig. 13a the \( \tau \)-behaviour of \( r_K \) for a

![Graph](image)

Figure 13: a) \( \tau \)-behaviour of \( r_K \) for a given value \( \mu = 4 \text{ GeV} \) of subtraction point and for different values of \( t_c \) from the instanton sum rule. One has a \( \tau \)-minimum for \( \tau \approx 0.3 \text{ GeV}^{-2} \), while a \( t_c \)-minimum is obtained for \( t_c = 2.5 \text{ GeV}^2 \). At this \( \tau \)-scale, the instanton contribution to \( R^K(K, \mu) \) is about 60% of the perturbative +d \( \leq 6 \) condensate contributions.

We show in Fig. 13b the \( \mu \)-behaviour of the optimal value of \( r_K \) deduced from a)

![Graph](image)

Figure 12: \( \tau \)-behaviour of \( r_K \) for \( \mu = \tau^{-1/2} \) and for different values of \( t_c \).

given value \( \mu = 4 \text{ GeV} \) of the subtraction point and for different values of \( t_c \) from the instanton sum rule. One has a \( \tau \)-minimum for \( \tau \approx 0.3 \text{ GeV}^{-2} \), while a \( t_c \)-minimum is obtained for \( t_c = 2.5 \text{ GeV}^2 \). At this \( \tau \)-scale, the instanton contribution to \( R^K(K, \mu) \) is about 60% of the perturbative +d \( \leq 6 \) condensate contributions.

We show in Fig. 13b the \( \mu \)-behaviour of the optimal value of \( r_K \) deduced from Fig. 13a. One can notice like in the previous analysis without instanton that the curve is (almost) stable for \( \mu \geq 3 \text{ GeV} \). We use as a final result the mean obtained for \( \mu \) in the range 3 to 6 GeV:

\[
\tau_K^{\text{inst}} \approx 1.8(1)\mu(3.8)\sqrt{2}(2.4)\mu \Rightarrow \frac{f_K}{f_K^c} = (15.3\pm2.0)10^{-2} \quad (41)
\]

Comparison with some other predictions

We compare in Fig. 14, our results from Eqs. (40) and (41) for:

\[
L_K(t) \equiv r_K BW_t, \quad (42)
\]

with the existing ones in the current literature (NPT83 [13], SN02 [7], KM02 [32] and DPS98 [25]). The results of NPT83 [13] and SN02 [7] are obtained within a narrow width approximation. The ones of KM02 [32] and DPS98 [25] include finite width correction. There are fair agreement between different determinations with the exception of the one from [25] which is relatively high and quoted without any error in the original paper. This high value may be either due to the coherent sum and equal coupling of the \( K(1460) \) and \( K(1800) \) contribution assumed in the amplitude or due to an overall normalization factor\(^9\). We also see in Fig. 14 that the instanton sum rule estimate is relatively small compared with the one from the sum rule within the SVZ expansion and with some other determinations.

![Graph](image)

Figure 14: Comparison of our determination of \( r_K \) from SVZ and instanton sum rules with the ones in the current literature: NPT83 [13], SN02 [7], KM02 [32] and DPS98 [25]). We use \( t_c = 5.5 \text{ GeV}^2 \).

8. Laplace sum rule estimate of \( m_{us} \equiv (m_u + m_s) \)

Defining:

\[
m_{us} = (m_u + m_s), \quad (43)
\]

we now turn to the estimate of the RGI \( \hat{m}_{us} \) and running \( \bar{m}_{us} \) sum of masses.

• \( \hat{m}_{us} \) within the SVZ expansion for arbitrary \( \mu \)

We show in Fig. 15a the \( \tau \)-behaviour of \( \hat{m}_{us} \) for a given value \( \mu = 2.1 \text{ GeV} \) of subtraction point and for different values of \( t_c \) where we have used the value of \( r_K \) in Eq. (40). The largest range of \( \tau \)-stability of about (0.4-0.6) GeV\(^{-2} \) is reached at \( t_c \approx (5.25 - 5.75) \text{ GeV}^2 \). We show in Fig. 15b the \( \mu \)-behaviour of

\(^9\)Notice that a destructive interference has been assumed by [24] in the pion channel, while we do not recover from the expression of the spectral function in [25], the one given in [24] in the chiral limit \( m_\pi = m_K = 0 \).

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the optimal value of $\hat{m}_{us}$ deduced from Fig. 15a. We obtain a stability at $\mu = 2.1$ GeV at which corresponds the value:

$$\hat{m}_{us}^{\text{SVZ}} = 117.3(16)\tau(30)\Lambda(13)\bar{c}(4)\bar{m}(10)\bar{G}(2)\Lambda\bar{G}(4)\rho,$$

$$\hat{m}_{us}^{\text{SVZ}} = (30)\tau\bar{c}(2)\bar{m}(2)\Lambda\bar{G}(4)\rho,$$

$$= (117.3 \pm 4.9) \text{ MeV}. \quad (44)$$

- $\hat{m}_{us}$ within the SVZ expansion for $\mu = \tau^{1/2}$

We redo the previous analysis but for $\mu = \tau^{-1/2}$. This is shown in Fig. 16, where one can see that a $\tau$-stability (inflexion point here) is obtained for $\tau_c \geq 5.75 \text{ GeV}$ and where the result is almost $\tau_c$-stable. We obtain for $\tau_c = (6 \pm 0.25) \text{ GeV}^2$ and $\tau = 0.4 \text{ GeV}^{-1/2}$ :

$$\hat{m}_{us}^{\text{SVZ}} = 110.5(10)\hat{c}(18)\hat{c}(27)\Lambda(5)\hat{c}(3)\bar{h}(7)\Lambda\bar{G}(2)\rho,$$

$$\hat{m}_{us}^{\text{SVZ}} = (30)\tau\bar{c}(9)\bar{c}(1)\bar{m}(1)\Lambda\bar{G}(4)\rho,$$

$$= (110.5 \pm 4.8) \text{ MeV}. \quad (45)$$

Notice that in both cases ($\mu = 2.1$ GeV and $\tau^{-1/2}$), the tachyonic gluon mass decreases by about 3 MeV the value of $\hat{m}_{us}$ which is relatively small but indicates, by duality with the uncalculated higher order terms [45], that the QCD PT series converge quite well.

- $\hat{m}_{us}$ from the instanton sum rule at arbitrary $\mu$

We show in Fig. 17 the $\tau$-behaviour of $\hat{m}_{us}$ from the instanton sum rule at different values of $\mu$, and for a given value $\mu = 2$ GeV. The largest $\tau$-stability is obtained for $\tau_c = (3.5 \pm 0.5) \text{ GeV}^2$. We take the value at the $\tau$-minimum of about 0.6 GeV$^2$. As shown in Fig. 17, the results increase smoothly for $\mu \geq 3$ GeV. Therefore, we take for definiteness, the mean of the results for $\mu = (3 - 6) \text{ GeV}$:

$$\hat{m}_{us}^{\text{inst}} = 105.3(22)\tau(17)\rho(44)\rho,$$

$$\hat{m}_{us}^{\text{inst}} = (105.3 \pm 5.2) \text{ MeV}. \quad (46)$$

where the value corresponds approximately to $\mu = 4.25 \text{ GeV}$.

- Final value of $\hat{m}_{us}$ and $\bar{m}_{us}$

By combining the previous determinations from Eqs. (44) and (45) and from Eqs. (46) and (47), one can deduce the mean final estimate in units of MeV:

$$\hat{m}_{us}^{\text{SVZ}} = 113.8(34) \quad \hat{m}_{us}^{\text{inst}} = 89.2(34), \quad \hat{m}_{us}^{\text{inst}} = 76.8(29). \quad (48)$$

and the corresponding running mass evaluated at 2 GeV:

$$\bar{m}_{us}^{\text{SVZ}} = 98.0(29) \quad \bar{m}_{us}^{\text{inst}} = 76.8(29). \quad (49)$$
Using as input the values of \( \hat{m}_u \) given in Eq. (37), one can deduce:

\[
\hat{m}_{ud}^{\text{SVZ}} = 95.5(29), \quad \hat{m}_{ud}^{\text{inst}} = 74.9(29) .
\]  

(50)

Combining this result with the value of \( \hat{m}_{ud} \) in Eq. (35), one predicts the scale-independent mass ratios:

\[
\left( \frac{m_s}{m_{ud}} \right)^{\text{SVZ}} = 25.3 \pm 1.1, \quad \left( \frac{m_s}{m_{ud}} \right)^{\text{inst}} = 26.6 \pm 0.7 ,
\]

and:

\[
\left( \frac{m_s}{m_d} \right)^{\text{SVZ}} = 19.0 \pm 1.0, \quad \left( \frac{m_s}{m_d} \right)^{\text{inst}} = 19.9 \pm 0.7. 
\]

These results are summarized in Table 2.

9. Lower bounds on \((\bar{m}_u + \bar{m}_d)\) from Laplace sum rule

Lower bounds on quark masses have been first derived in [2, 10] and improved later on in [60] using a finite number of \(Q^2\)-derivatives of the two-point function. In [8], the \( \alpha_s^3 \) corrections to the result of [60] have been included leading to the (improved) lower bounds:

\[
\bar{m}_{ud} \equiv \frac{1}{2}(\bar{m}_u + \bar{m}_d) \geq (3.0 \pm 0.5) \text{ MeV} , \quad \bar{m}_{us} \equiv (\bar{m}_u + \bar{m}_s) \geq (82.7 \pm 13.3) \text{ MeV} .
\]

(53)

In the present work, we shall use the positivity of the spectral function into the Laplace sum rule \( L^\lambda(\tau, \mu) \) defined in Eq. (3) for extracting a lower bound on the sum of light quark RGI masses \((\hat{m}_u + \hat{m}_d)\).

**Bounds from Laplace sum rules at \( \mu = \tau^{-1/2} \)**

We study the lower bounds obtained from sum rules within the SVZ expansion (Fig. 19a) and the ones where the instanton contribution is added into the OPE (Fig. 19b). Similar curves are obtained in the s-quark channel. For \( \mu = \tau^{-1/2} \), we obtain \(^{10}\)

\[
\hat{m}_{ud}^{\text{SVZ}} = 2.84(14) \lambda(4)_{\nu}(4)_{\nu}(8)_{G}^{\nu}(0)_{G}^{\nu}(5)_{P}^{\nu} , \quad \hat{m}_{us}^{\text{SVZ}} = 2.84 \pm 0.18 , \\
\hat{m}_{ud}^{\text{inst}} = 75(3) \lambda(1)_{\nu}(0.7)_{\nu}(1.5)_{G}^{\nu}(0)_{G}^{\nu}(0)_{P}^{\nu}(0)_{G}^{\nu}(0)_{G}^{\nu}(0)_{P}^{\nu} = (75.0 \pm 3.6) , 
\]

and:

\[
\hat{m}_{us}^{\text{inst}} = 2.52(16)_{\nu}(0.3)_{P}^{\nu} , \quad \hat{m}_{us}^{\text{inst}} = 65.2(3.1)_{\nu}(0.4)_{P}^{\nu} .
\]

(54)

One can translate the previous bounds on the RGI masses into the ones for the running masses evaluated at 2 GeV in units of MeV:

\[
\bar{m}_{ud}^{\text{SVZ}} = 2.66(12) , \quad \bar{m}_{us}^{\text{SVZ}} = 69.6(2.4) , \\
\bar{m}_{ud}^{\text{inst}} = 2.29(10) , \quad \bar{m}_{us}^{\text{inst}} = 59.9(2.1) .
\]

(56)

Using the value of the ratio \( m_{ud}/m_{us} \) in Eq. (36), one can deduce from the bound on \( m_{ud} \) in units of MeV:

\[
\bar{m}_{ud}^{\text{SVZ}} \geq 1.77(11) , \quad \bar{m}_{us}^{\text{SVZ}} \geq 1.53(9) , \\
\bar{m}_{ud}^{\text{inst}} \geq 3.54(21) , \quad \bar{m}_{us}^{\text{inst}} \geq 3.06(18) .
\]

(57)

Using the value of \( \bar{m}_u \) in Eq. (37), one can deduce from the bound on \( m_{us} \) in units of MeV:

\[
\bar{m}_{ud}^{\text{inst}} \geq 67.1(24) , \quad \bar{m}_{us}^{\text{inst}} \geq 58.0(21) .
\]

(58)

These bounds are interesting though weaker than the ones in Eq. (53). The results are summarized in Table 2.

**Summary and conclusions**

We have re-estimated the \( \pi(1300) \) and \( K(1460) \) decay constants using pseudoscalar Laplace sum rules which we have compared with some existing ones in the literature. We have

---

\(^{10}\)The sum rules with an arbitrary \( \mu \) do not present a \( \tau \)-stability and will not be considered here.
used these results for improving the determinations of $(m_u + m_d)$: $q \equiv d,s$ from these channels. Our results are summarized in Table 2. The novel features in the present analysis are:

- In addition to the usual sum rule evaluated at $\mu = \tau^{-1/2}$ where $\tau$ is the Laplace sum rule variable, we have used an arbitrary subtraction point $\mu$ which value has been fixed from a $\mu$-stability criterion. Here this $\mu$-stability appears as an inflexion point or an (almost) stable plateau.

- The improved model-independent extraction of the experimentally unknown contribution of the $\pi(1300)$ and $K(1460)$ into the spectral function and the inclusion of finite width corrections. Here, we have used the QCD continuum contribution from a threshold $t_c$ above the $\pi(1300)$ and $K(1460)$ which is expected to smear the effects of higher radial excitations including their interference which are not under good control in the existing literature. The value of $t_c$ which optimizes the duality between the experimental and QCD sides of the sum rule is fixed from the $t_c$-stability criteria which value may differ for each sum rule analyzed.

- An inclusion of the tachyonic gluon mass into the SVZ expansion showing that its effect is relatively small. It decreases $r_\pi$ and $r_K$ by 0.1 and $\hat{m}_{ud}$ (resp. $\hat{m}_u$) by 0.13 (resp. 0.3) MeV. This is reassuring as by duality the tachyonic gluon mass contribution provides an estimate of the uncalculated higher order terms of the QCD PT series [45].

- An explicit study of the Laplace sum rule including instanton contribution which we have considered as an alternative determination of $(m_u + m_d)$ despite the controversial role of the instanton into the pseudoscalar sum rule.

One may consider our results as improvements of the determinations of $(m_u + m_d)$: $q \equiv d,s$ from the pseudoscalar sum rules. One can notice that the sum rules including the instanton contribution into the OPE tend to give lower values of the $\pi(1300)$ and $K(1460)$ contributions to the spectral functions and consequently predict lower values of the sum of light quark masses. We have not taken the mean value of the two different determinations due to the controversial instanton role into the pseudoscalar sum rules. The results using the SVZ expansion without the instanton contribution agree within the errors with our previous determinations from the pseudoscalar and nucleon sum rules [7, 49, 51] and the ones from $e^+e^-$ and $\tau$-decay data [28, 49]. Therefore, this comparison tends to favour the results from the SVZ expansion without instanton. However, both results using the SVZ expansion and the SVZ expansion @ instanton agree within the errors with some other determinations and lattice calculations compiled in [1] and in [61].

### Table 2: Summary of the main results of this work.

| Parameters | SVZ | SVZ @ instanton | Eq. |
|-----------|-----|----------------|-----|
| $f_\pi/f_\rho$ | (2.58 ± 0.35) $10^{-2}$ | (2.06 ± 0.30) $10^{-2}$ | 27 |
| $f_K/f_\rho$ | (25.0 ± 1.5) $10^{-2}$ | (15.3 ± 2.0) $10^{-2}$ | 40,41 |
| $\bar{m}_{ud}$ | 3.77 ± 0.28 | 2.82 ± 0.18 | 35 |
| $\bar{m}_u$ | 2.51 ± 0.21 | 1.88 ± 0.14 | 37 |
| $\bar{m}_d$ | 5.02 ± 0.42 | 3.76 ± 0.28 | 37 |
| $\bar{m}_{sv}$ | 98.0 ± 2.9 | 76.8 ± 2.9 | 49 |
| $\bar{m}_s$ | 95.5 ± 2.9 | 74.9 ± 2.9 | 50 |
| $\bar{m}_{u/m_{ud}}$ | 25.3 ± 1.1 | 26.6 ± 0.7 | 51 |
| $\bar{m}_{d/m_{ud}}$ | 19.0 ± 1.0 | 19.9 ± 0.7 | 52 |

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