On the variational principle for dust shells in General Relativity

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(November 12, 2018)

Abstract

The variational principle for a thin dust shell in General Relativity is constructed. The principle is compatible with the boundary-value problem of the corresponding Euler-Lagrange equations, and leads to “natural boundary conditions” on the shell. These conditions and the gravitational field equations which follow from an initial variational principle, are used for elimination of the gravitational degrees of freedom. The transformation of the variational formula for spherically-symmetric systems leads to two natural variants of the effective action. One of these variants describes the shell from a stationary interior observer’s point of view, another from the exterior one. The conditions of isometry of the exterior and interior faces of the shell lead to the momentum and Hamiltonian constraints. The canonical equivalence of the mentioned systems is shown in the extended phase space. Some particular cases are considered.

PACS numbers: 04.20.-q, 04.20.Fy, 04.40.-b

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I. INTRODUCTION

A thin spherically-symmetric dust shell is among the simplest popular models of collapsing gravitating configurations. The equations of motion of these objects are obtained in [1], [2]. The construction of a variational principle for such systems was discussed from different points of view in [3]-[7]. There are a number of problems here, most basic of which is the dependence on the choice of the evolution parameter (internal, external, proper). The choice of time coordinate, in turn, affects the choice of a particular quantization scheme, leading, in general, to quantum theories which are not unitarily equivalent.

In most of these papers the variational principle for shells is usually constructed in a comoving frame of reference, or in one of variants of freely falling frames of reference. However, use of such frames of reference frequently leads to effects unrelated to the object under consideration. The essential physics involves a picture of a gravitational collapse from the point of view of an infinitely remote stationary observer. In quantum theory this point of view enables us to treat bound states in terms of asymptotic quantities and to build the relevant scattering theory correctly. On the other hand, to treat primordial black holes in the theory of self-gravitating shells it is convenient to take the viewpoint of a central stationary observer. In the approach related to proper time of the shell reduction of the system leads to complicated Lagrangians and Hamiltonians which creates difficulties on quantization. In particular it leads to theories with higher derivatives or to finite difference equations.

In our opinion, the choice of the exterior or interior stationary observers is most natural and corresponds to the real physics. To provide the necessary properties of invariance, specification of the canonical transformations in an extended phase space which translate the corresponding dynamical systems into one another enough. In addition the action for a shell should satisfy some natural requirements. In the absence of self-forces it should pass into the action for a geodesic motion. Further, according to the correspondence principle, at small velocities and masses of the shell, and also in the absence of other sources of the gravitational field, we should obtain the action for a self-gravitating Newtonian shell (see Appendix C).

The natural Hamiltonian formulation of a self-gravitating shell was considered in works [8], [9]. However this formulation was not obtained by a variational procedure from some initial action containing the standard Einstein-Hilbert term. The action for such a self-gravitating spherical shell of mass $m$ can be introduced with the help of a naive “relativization” of the Newtonian action. It is carried out by simple replacement of the kinetic energy $mv^2/2$ by the relativistic expression $-mc\sqrt{1-v^2/c^2}$ (see below Lagrangians (C17) and (4.3)). If there is an exterior gravitational field then the kinetic and potential energy of the shell have as their general relativistic analog the geodesic Lagrangian $-mc^{(2)}ds_\pm/dt_\pm$. The subsigns “$\pm$” correspond to exterior and interior observers. The gravitational self-action of the shell is the same for all cases, and its sign depends on whether stationary observer can exist inside and outside the shell.

The above Hamiltonian formulation for the shell, as well as the procedure of “relativization” follows from the Lagrange formalism of dust shells constructed in the present paper. We view the system as a compound configuration consisting of two vacuum regions with a spatially-closed boundary surface formed by the shell. The initial action we take as the sum of actions of York type for either region and the action for dust matter. For the complete
action introduced in this way the variational principle is compatible with the boundary-value problem of the corresponding Euler-Lagrange equations for either region of the configuration, and leads to “natural boundary conditions” on the shell. The missing boundary conditions are obtained by consideration of the variations with respect to normal displacements of the shell. The obtained conditions coincide with the known Israel matching conditions at singular hypersurfaces and are considered as constraints. Together with the equations of the gravitational field they are used to eliminate of the gravitational degrees of freedom. The tangential variations of thus-obtained action with constraints lead to the known equations of motion of the Israel [1].

The problem of the complete reduction of the action is solved for spherically-symmetric systems. By transforming the variational formula and using the constraints the obtained action is reduced to two variants of the effective action. One of these variants describes the shell from an interior stationary observer’s point of view, and the other from the exterior one. Then we go over from the Lagrangian to the Hamiltonian description. The conditions of isometry of the exterior and interior sides of the shell lead to the momentum and Hamiltonian constraints. The canonical equivalence of these two variants of the description of the shells in the extended phase space indicates the existence of a “discrete gauge” transformation associated with the transition from the interior observer to the exterior one.

The paper is organized as follows. In Sec.II the full action is constructed for a compound, piecewise smooth Lorentz manifold with a four-dimensional spatially-closed boundary surface between two vacuum regions, corresponding to the world sheet of the shell. From here the Einstein equations for regions outside the shell and surface equations follow. Further, the action for the shell and the equations of motion are constructed.

In Sec.III spherically-symmetric relativistic dust shells are considered. The Lagrangians and Hamiltonians describing the shell from the point of view of the interior or exterior observer are obtained. Then momentum and Hamiltonian constraints are found. They emerge from independent consideration of the interior and exterior faces of the shell using the conditions of isometry of its two faces. In Sec.IV special cases of dust shells and configurations of several shells are briefly considered.

In Appendix A it is shown that the surface equations, obtained in Sec.II, reduce to the known equations for jumps of the extrinsic curvature tensor of the shell. In Appendix B we show the canonical equivalence of the actions for the dust spherically-symmetric shell written relative to the interior and exterior observers. This equivalence is thought of as operating in the extended phase space of the corresponding dynamical system. In Appendix C the action for an arbitrary nonrelativistic gravitating dust shell is constructed. The Lagrangian for the spherical gravitating nonrelativistic dust shell is found. It was deemed worthwhile to consider the nonrelativistic case because it clarifies the interpretation of the results and allows comparisons with the general relativistic approach.

In this work we consider both relativistic and non-relativistic systems. In this connection, we shall keep all the dimensional constants. Here c is the velocity of light, γ is the gravitational constant, \( \chi = \frac{8\pi\gamma}{c^2} \), \( \hbar \) is Planck’s constant. The metric tensor \( g_{\mu\nu} \) (\( \mu, \nu = 0, 1, 2, 3 \)) has the signature (+ − − −).
II. THE VARIATIONAL PRINCIPLE AND EQUATIONS OF MOTION FOR RELATIVISTIC DUST SHELLS

Consider a time-like spatially-closed hypersurface $\Sigma_t^{(3)}$ into some region $D^{(4)}$ of the spacetime $V^{(4)}$. Let it be the world sheet of the infinitely thin dust shell with the surface density of dust $\sigma$. This shell divides the region $D^{(4)}$ into the interior and exterior ones, $D_{-}^{(4)}$ and $D_{+}^{(4)}$. Introduce the general coordinate map $x^\mu$ on our compound manifold $D^{(4)} = D_{-}^{(4)} \cup \Sigma_t^{(3)} \cup D_{+}^{(4)}$ and the metrics $g_{\mu\nu}$ on $D^{(4)}_{\pm}$, so that $g_{\mu\nu} |_{\Sigma_t^{(3)}} = g_{\mu\nu}^{-} |_{\Sigma_t^{(3)}}$.

One defines the elements of the four-volume $d^4\Omega$ on $D^{(4)}_{\pm}$ and three-volume $d^3\Omega$ on $\Sigma_t^{(3)}$ according to the formulas

$$d^4\Omega = \sqrt{-g} dx^4 = \sqrt{-g} dx^a \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

$$d^3\Omega = -\sqrt{-g n^\mu} d\Sigma^\mu = \sqrt{-g} d\Sigma,$$

where $n^\mu$ is the unit normal to $\Sigma_t^{(3)}$, directed from $D_{-}^{(4)}$ to $D_{+}^{(4)}$ ($n_\mu n^\mu = -1$, $u_\mu n^\mu = 0$), $g = \det |g_{\mu\nu}|$. Three-forms $d\Sigma^\mu$ and $d\Sigma$ are determined by the relations

$$dx^\mu \wedge d\Sigma^\nu = \delta^\nu_\mu dx^4, \quad \eta \wedge d\Sigma = d^4x \quad (d\Sigma^\mu = n_\mu d\Sigma),$$

where “$\wedge$” denotes the exterior product, and $\eta = n_\mu dx^\mu$ is a normal covector.

Now let us fix coordinate system $x^\mu$ so that the coordinates $x^a$ ($a = 2, 3$) be Lagrange coordinates of particles on the shell $\Sigma_t^{(3)}$. Then $u^\mu x^a_{,\mu} = n^a x^a_{,a} = 0$, where “$\,_{\mu}$” is derivative with respect to the coordinate $x^\mu$. Hence it follows $u^a = n^a = 0$. The equations $x^a = \text{const}$ determine the world line $\gamma$ of some particle of dust on $\Sigma_t^{(3)}$. The set $\{\gamma\} = \{x^a, \; x^a + dx^a\}$ of the world lines forms the elementary stream tube of dust. On the shell $\Sigma_t^{(3)}$ we shall introduce the basis of one-forms

$$\{e^0 \equiv \omega = u_\mu dx^\mu, \; e^a = dx^a\} \quad (a, b = 2, 3)$$

and the dual vector basis

$$\{e_0 \equiv u = u^\mu \partial_\mu, \; e_a\}, \quad e^i(e_k) = \delta^i_k \quad (i, k = 0, 2, 3).$$

In the basis $\{e^i\} = \{\omega, \; dx^a\}$ the metric tensor and three-form of volume on $\Sigma_t^{(3)}$ are

$$(^3g = \omega \otimes \omega - q_{ab} dx^a \otimes dx^b,$$

$$d^3\Omega = -\omega \wedge d^2\Omega, \quad d^2\Omega = \sqrt{q} dx^2 \wedge dx^3.$$  

Here “$\otimes$” is the sign of a tensor product, $d^2\Omega$ is the surface element of the area for the section which is orthogonal to the elementary stream tube of dust, $q_{ab}$ is the metric on these sections, $q = \det |q_{ab}|$. In the neighbourhood of the hypersurface $\Sigma_t^{(3)}$ the metric tensor $^4g$ and four-form of volume $d^4\Omega$ can be expressed in the form

$$(^4g = (^3g - \eta \otimes \eta,$$

$$d^4\Omega = \eta \wedge d^3\Omega = \omega \wedge \eta \wedge d^2\Omega.$$  

We introduce the two-form of mass on $\Sigma_t^{(3)}$ by the formula $d^2m = \sigma d^2\Omega$, then $\sigma d^3\Omega = \omega \wedge d^2m$.  

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Now we take the full action of the compound configuration in the form

\[ I_{\text{tot}}^{(g)} = I_{EH} - c \int_{\Sigma_{t}^{(3)}} \left( \sigma n^{\mu} + \frac{1}{2\chi}[\omega^{\mu}] \right) \sqrt{-g} d\Sigma_{\mu} + I_{\partial D^{(4)}} + I_{0} . \]  

(2.10)

It is the functional of the metric \( g_{\mu\nu} \), density of the dust \( \sigma \) and hypersurface \( \Sigma_{t}^{(3)} \): \( I_{\text{tot}}^{(g)} \equiv I_{\text{tot}}^{(g)}(g_{\mu\nu}, \sigma, \Sigma_{t}^{(3)}) \). The first term in the right side of \( (2.10) \)

\[ I_{EH} = -c \frac{1}{2\chi} \int_{D^{(4)} \cup \partial D^{(4)}} (4) R d^{4}\Omega \]  

(2.11)

is the Einstein-Hilbert action for the regions \( D^{(4)} \), where \( (4) R \) is the curvature scalar.

The second term in the right side \( (2.10) \) contains the matter term \( c \sigma d^{3}\Omega \) and matching term. The symbol \([\omega^{\mu}] = \omega^{\mu}|_{+} - \omega^{\mu}|_{-}\) denotes the jump of the quantity

\[ \omega^{\mu} = g^{\sigma\rho} \Gamma_{\sigma\rho}^{\mu} - g^{\mu\rho} \Gamma_{\sigma\rho}^{\sigma} , \]  

(2.12)

\[ \Gamma_{\sigma\rho}^{\mu} = \frac{1}{2} g^{\mu\nu} (g_{\nu\sigma,\rho} + g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu}) , \]  

(2.13)

on \( \Sigma_{t} \). The sign “\( |_{+} \)” or “\( |_{-} \)” indicates the marked values to be calculated as a limiting magnitude when approaching the boundary \( \Sigma_{t} \) from outside or inside respectively. In Appendix A it will be shown, that the relation

\[ [\omega^{\mu}] n_{\mu} = 2[K] \]  

(2.14)

takes place. Here \( K = g^{\mu\nu} K_{\mu\nu} \) is the trace of the extrinsic curvature tensor

\[ K_{\mu\nu} = -n_{\mu,\rho} h^{\rho}_{\nu} \quad (h^{\rho}_{\nu} = \delta^{\rho}_{\nu} + n^{\rho} n_{\nu}) , \]  

(2.15)

where \( ;_{\rho} \) is covariant derivative with respect to the coordinate \( x^{\mu} \). The third term

\[ I_{\partial D^{(4)}} = \frac{c}{2\chi} \int_{\partial D^{(4)}} \omega^{\mu} \sqrt{-g} d\Sigma_{\mu} \]  

(2.16)

contains the surface terms which are introduced to fix the metric on the boundary \( \partial D^{(4)} \) of the region \( D^{(4)} \). Note, that the boundary \( \partial D^{(4)} \) consists of the pieces of time-like as well as space-like hypersurfaces. The last term \( I_{0} \) in \( (2.10) \) contains the boundary terms on the time-like infinitely remote hypersurfaces, necessary for normalization of the action.

The relation

\[ \sqrt{-g} (4) R = \sqrt{-g} G + (\sqrt{-g} \omega^{\mu}),_{\mu} \]  

(2.17)

takes place, where

\[ G = g^{\mu\nu} \left( \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\mu\rho}^{\sigma} \Gamma_{\nu\sigma}^{\rho} \right) \]  

(2.18)

contains only the first derivatives of the metric. Therefore the action \( (2.10) \) can be rewritten in a more compact form

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\[ I_{\text{tot}}^{(g)} = I_g + I_m + I_0, \tag{2.19} \]

where

\[ I_g = -\frac{c}{2\chi} \int_{D^{(4)} \cup D^{(4)}_+} \sqrt{-g} G d^4x = \int_{D^{(4)} \cup D^{(4)}_+} L_g d^4x \tag{2.20} \]

is the gravitational action of the first order, and

\[ I_m = c \int_{\Sigma^{(3)}_t} \sigma d^3\Omega = -c \int_{S^{(2)}_t} d^2m \int_\gamma \omega \tag{2.21} \]

is the action for the dust.

The first and the penultimate terms in (2.10) form the action which can be ascribed to that of the York’s type \( I_Y = I_{EH} + I_{\partial D^{(4)}} \). It is used in variational problems with the fixed metric on the boundary \( \partial D^{(4)} \) of the region \( D^{(4)} \). It can also be used in variational problems with the general relativistic version of “natural boundary conditions” for “free edge” [11]. In this case the metric on the boundary is arbitrary and the corresponding momenta vanishes. Together with \( I_0 \) it forms the York-Gibbons-Hawking action \( I_{YGH} = I_Y + I_0 \) for a free gravitational field.

In our case of the compound configuration we also fix the metric on boundary \( \partial D^{(4)} \), as it was done in variational problem for action \( I_Y \). In addition, inside the system there is the boundary surface \( \Sigma^{(3)}_t \), with singular distribution of matter on it. One can interpret this configuration as the two vacuum regions \( D^{(4)}_{\pm} \) with a common “loaded edge” (or with a “massive edge”). The sum of the actions of type \( I_Y \) for these regions and of the action for matter \( I_m \) and normalizing term \( I_0 \) do leads to the action \( I_{\text{tot}}^{(g)} \).

If there is no dust, \( \sigma = 0 \), the common boundary is not “loaded”. Then, the requirement \( \delta I_{\text{tot}}^{(g)} = 0 \), at arbitrary, everywhere continuous variations of the metric, gives generalization of the above “natural boundary conditions” for free hypersurface \( \Sigma^{(3)}_t \). They coincide with the condition of continuity for the extrinsic curvature on \( \Sigma^{(3)}_t \), i.e., with ordinary matching conditions. If the edge, being matched, is “loaded” by some surface distribution of matter, then we obtain the corresponding surface equation or the boundary conditions for \( D^{(4)}_{\pm} \). They are the analog of the generalized “natural boundary conditions” for “loaded edges”. The initial action is chosen so, that the surface equations on \( \Sigma^{(3)}_t \) following from the requirement \( \delta I_{\text{tot}}^{(g)} = 0 \), coincide with the matching conditions on singular hypersurfaces [1]. In this case, the variational principle for the action \( I_{\text{tot}}^{(g)} \) will be compatible with the boundary-value problem of the corresponding Euler-Lagrange equations [13], [14].

Note, that, as a rule, the boundary terms are formulated in terms of the extrinsic curvature of the corresponding hypersurfaces. For the configuration which contains the boundary hypersurface dividing the domain \( D^{(4)} \) into parts and the whole boundary consisting of several pieces of edge, initial, and eventual hypersurfaces, it is more convenient to use the covariant approach. In order to calculate \( \delta I_{\text{tot}}^{(g)} \) we use the complete action in the form (2.10). According to [13] we have

\[ \delta \left( \sqrt{-g} \, (4) R \right) = -\sqrt{-g} \, (4) G^{\mu\nu} \delta g_{\mu\nu} + \left( \sqrt{-g} \Omega^{\mu} \right)_{,\mu}, \tag{2.22} \]
where
\[ \Omega^\mu = g^{\sigma\rho} \delta \Gamma^\mu_{\sigma\rho} - g^{\mu\rho} \delta \Gamma^\sigma_{\sigma\rho}, \]  
(2.23)

and \( G^{\mu\nu} = (4)R^{\mu\nu} - \frac{1}{2} (4)R g^{\mu\nu} \) is the Einstein tensor. In addition, we shall use the following conditions: the boundary of the configuration \( \partial D^{(4)} \), the metric on it, and the normal vector are fixed. Then \( \delta d\Sigma_{\mu|\partial D^{(4)}} = 0 \), \( \delta g_{\mu\nu}|_{\partial D^{(4)}} = 0 \), \( \delta n_{\mu}|_{\partial D^{(4)}} = 0 \). The hypersurface \( \Sigma_t^{(3)} \) is fixed, and the metric and its variations are continuous on \( \Sigma_t^{(3)} \): \( [g_{\mu\nu}]_{\Sigma_t^{(3)}} = 0 \), \( [\delta g_{\mu\nu}]_{\Sigma_t^{(3)}} = 0 \), \( [n_{\mu}]_{\Sigma_t^{(3)}} = 0 \), \( [\delta n_{\mu}]_{\Sigma_t^{(3)}} = 0 \).

For the variation \( \delta I_m \) according to the formula (2.21) we have
\[ \delta I_m = -c \int d^2 m \int \delta \omega = -c \int d^2 m \int \delta \omega_\gamma. \]
(2.24)

If all these conditions are satisfied, then from the requirement \( \delta I_{m}^{(g)} = 0 \) one obtains the vacuum Einstein equations
\[ (4)G^{\mu\nu} = 0, \quad \forall D^{(4)}_\pm \]  
(2.25)

and the surface equations on \( \Sigma_t^{(3)} \)
\[ Q_{\mu\nu} - \frac{1}{2} Q g_{\mu\nu} = -\chi \sigma u_\mu u_\nu, \]  
(2.26)

where \( Q = g^{\mu\nu} Q_{\mu\nu} \), and
\[ Q_{\sigma\rho} = n_\mu [\Gamma^\mu_{\sigma\rho}] - \frac{1}{2} (n_\sigma [\Gamma^\mu_{\mu\rho}] + n_\rho [\Gamma^\mu_{\mu\sigma}]). \]  
(2.27)

It is shown in Appendix A that the surface equations (2.20) reduce to the known equations for the jump discontinuity of the extrinsic curvature tensor of the hypersurface \( \Sigma_t^{(3)} \) \[ [K_{\mu\nu}] - [K] h_{\mu\nu} = -\chi \sigma u_\mu u_\nu, \]  
(2.28)

where \( h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \) is the metric on \( \Sigma_t^{(3)} \). From the relations (2.28) it follows
\[ [K_{\mu\nu}] u^\mu u^\nu = -\frac{\chi}{2} \sigma, \]  
(2.29)

The missing equation for the average tensor of the extrinsic curvature
\[ \bar{K}_\nu^\mu = \frac{1}{2} (K^\mu_{\nu|+} + K^\mu_{\nu|-}) \]  
(2.30)

can be obtained by considering the variations of \( I_{tot}^{(g)} \) with respect to normal displacements of the hypersurface \( \Sigma_t^{(3)} \). For this purpose we define some one-parameter family of time-like hypersurfaces in a neighbourhood of \( \Sigma_t^{(3)} \) so that \( \Sigma_t^{(3)} \) is included in this family. The family
induces (3+1)-decomposition of the objects in the neighbourhood of $\Sigma^{(3)}_i$. Thus for the four-curvature scalar one has

$$(^{(4)}R) = (^{(3)}R + K^\mu _\nu K_\mu ^\nu - K^2 + \frac{2}{\sqrt{-g}} \left\{ \sqrt{-g} (K n^\mu - a^\mu) \right\}_\mu ,$$

(2.31)

where $a^\mu = n^\mu _\nu n^\nu$ and $^{(3)}R$ is the curvature scalar of hypersurfaces of the family. Substituting (2.31) into (2.11) and taking into account the relations $a^\mu n^\nu = 0$ and (2.14), one obtains the action (2.10) in the form

$$I^{(g)}_{tot} = \hat{I}_g + I_m + \hat{I}_{\partial D^{(4)}} + I_0 ,$$

(2.32)

where

$$\hat{I}_g = \int \limits _{D^{(4)}_+ \cup D^{(4)}_-} \hat{L}_g d^4x = \frac{c}{2\chi} \int \limits _{D^{(4)}_+ \cup D^{(4)}_-} \left( ^{(3)} R + K^\mu _\nu K_\mu ^\nu - K^2 \right) \sqrt{-g} d^4x$$

(2.33)

is the gravitational action, containing normal derivatives up to the first order, and $\hat{I}_{\partial D^{(4)}}$ and $I_0$ contain the boundary terms, which are unessential here.

Now let every point $p \in \Sigma^{(3)}_i$ be translated at a coordinate distance $\delta x^\mu (p) = n^\mu \delta \lambda (p)$ in the normal direction. As a result of the displacement one gets a new hypersurface $\tilde{\Sigma}^{(3)}_i$. The initial and eventual positions of a shell are fixed, therefore $\delta \lambda (p) = 0$, $\forall p \in \Sigma^{(3)}_i \cap \partial D^{(4)} = \tilde{\Sigma}^{(3)}_i \cap \partial D^{(4)}$. In addition, we fix the metric $g_{\mu \nu}$ and all the quantities on $\Sigma^{(3)}_i$, so that $\delta I_m = 0$.

As a result of the displacement of the hypersurface $\Sigma^{(3)}_i$, the initial regions $D^{(4)}_+$ and $D^{(4)}_-$ are transformed into new ones $\bar{D}^{(4)}_+$ and $\bar{D}^{(4)}_-$, so that, $\bar{D}^{(4)}_+ \cup \tilde{\Sigma}^{(3)}_i \cup \bar{D}^{(4)}_- = D^{(4)}_+ \cup \Sigma^{(3)}_i \cup D^{(4)}_- = D^{(4)}$. Then, for example, the variation of the region $D^{(4)}_-$ can be expressed in the form $\delta D^{(4)}_+ = \bar{D}^{(4)}_+ \\bar{D}^{(4)}_- = D^{(4)}_+ \\bar{D}^{(4)}_-$. The variation of the action (2.33), under the above conditions, proves to be equal

$$\delta I^{(g)}_{tot} = \delta \hat{I}_g = \int \limits _{D^{(4)}_+ \cup D^{(4)}_-} \hat{L}_g d^4x - \int \limits _{D^{(4)}_+ \cup D^{(4)}_-} \hat{L}_g d^4x \cong - \int \limits _{\delta D^{(4)}_-} (\hat{L}^{+}_g - \hat{L}^{-}_g) d^4x .$$

(2.34)

Here $\hat{L}^{+}_g$ and $\hat{L}^{-}_g$ are Lagrangians defined by the relation (2.33) and calculated as a limiting magnitude when approaching the hypersurface $\Sigma^{(3)}_i$ from outside or inside respectively. Under the infinitesimal normal displacement of the hypersurface $\Sigma^{(3)}_i$, the full action is variated by the formula

$$\delta I^{(g)}_{tot} = \int \limits _{\Sigma^{(3)}_i} (\hat{L}^{+}_g - \hat{L}^{-}_g) \delta x^\mu d\Sigma_\mu = \int \limits _{\Sigma^{(3)}_i} [\hat{L}_g] \delta \lambda d\Sigma .$$

(2.35)

Hence, from arbitrariness of $\delta \lambda (p)$ and the requirement $\delta I^{(g)}_{tot} = 0$, one finds

$$[\hat{L}_g] = \hat{L}^{+}_g - \hat{L}^{-}_g = [K^\mu _\nu K_\mu ^\nu - K^2] = 2K^\nu _\nu ([K^\mu _\mu ] - [K]\delta^\nu _\mu ) = 0 .$$

(2.36)

Here we considered that $^{(3)}R = 0$ on $\Sigma^{(3)}_i$. Then, using (2.28), from (2.36) we obtain
The relations (2.28) and (2.37) form the necessary complete set of algebraic conditions or constraints for the extrinsic curvature tensor $K_{\nu\pm}^\mu$ of the hypersurface $\Sigma_t^{(3)}$.

Now we can eliminate gravitational degrees of freedom in the action $I_{\text{tot}}^{(g)}$ and construct the action for the shell. For this purpose it is necessary to calculate $I_{\text{tot}}^{(g)}$ on the solutions of the vacuum Einstein equations (2.25) taking into account the constraints (2.28) and (2.37). Note, first, that on this stage we use explicitly only the following results of these equations:

$$R = 0, \quad [\omega^\mu]_{\nu\mu} = 2[K] = \chi \sigma. \quad (2.38)$$

Substituting these relations for the corresponding terms in (2.10) one finds

$$I_{\text{tot}}^{(g)} \big|\{(\text{equations } 2.38)\} = I_{sh} + I_{\partial D^{(4)}} + I_0, \quad (2.39)$$

where

$$I_{sh} = \frac{1}{2} \int_{\Sigma_t^{(3)}} c \sigma d^3\Omega = -\frac{c}{2} \int_{\Sigma_t^{(3)}} d^2m_\gamma \int \omega \quad (2.40)$$

is the reduced action for the dust shell. This action must be considered together with constraints (2.28) and (2.37). The action $I_{sh}^{(g)}$ is quite certain if the gravitational fields in the neighbourhood of $\Sigma_t^{(3)}$ are determined as the solutions of the vacuum Einstein equations (2.25) which satisfy the boundary conditions (2.28) and (2.37). That is the finding of these fields that completes the construction of the action for the shell. At this stage all the equations (2.25) and constraints (2.28), (2.37) are already used.

Note, that one usually comes to the action for the shell in the other form. In our approach the action can be obtained at the partial reduction of initial action $I_{\text{tot}}^{(g)}$, when the constraint in (2.38) is not taken into account. As a result we come to the action of the type

$$\tilde{I}_{sh} = -c \int_{\Sigma_t^{(3)}} \left( \sigma - \frac{1}{\chi} [K] \right) \omega \wedge d^3\Omega. \quad (2.41)$$

or to some its modification. In the spherically-symmetric case from here follows the Lagrangian of the shell in a frame of reference of the comoving observer. However quantity $[K]$ contains second derivatives with respect to proper time of the shell. When eliminating them, through the integration by parts, one comes to rather complicated Lagrangians and Hamiltonians.

To find the equations of motion for particles of the shell from action $I_{sh}$ (2.40) one should introduce the independent coordinates $x^\mu_\pm$ in each of the regions $D^{(4)}_\pm$, and the interior coordinates $y^i (i, k = 0, 2, 3)$ on $\Sigma_t^{(3)}$. Let the equations of embedding of $\Sigma_t^{(3)}$ into $D^{(4)}_\pm$ have the form $x^\mu_\pm = x^\mu_\pm (y^i)$. Then we can write the relations

$$\begin{align*}
(4) ds^2_\pm &= g^\mu_\pm dx^\mu_\pm dx^\nu_\pm, \\
(3) ds^2 &= g^\mu_{\pm\pm} x^\mu_\pm x^\nu_\pm dy^k dy^k = h_{ik} dy^i dy^j, \\
\omega &= \omega_\pm = u^\mu_\pm dx^\mu_\pm, \\
\omega_{\gamma\gamma} &= ds_\pm, \\
(3) \omega_{\gamma\gamma} &= (3) ds, \quad u^i = dy^i / (3) ds.
\end{align*} \quad (2.42)$$

$$\begin{align*}
\omega_{\pm\gamma} &= dx^\mu_\pm / ds_\pm, \\
(3) \omega_{\gamma\gamma} &= (3) ds.
\end{align*} \quad (2.43)$$
Non-gravitational interaction between particles of the dust is absent. Therefore we consider quantity $d^2m$ to be unchanged when a flow line is varied.

First, consider variations $I_{sh}$ with respect to the internal coordinates $y^i$. In this case $\int_\gamma \omega = \int_\gamma (3) \omega |_\gamma = \int_\gamma (3) ds$. Then the metric $h_{ik}(y^i)$ is given on $\Sigma^{(3)}_t$ and the variation of $I_{sh}$ leads to the equations of three-dimensional geodesic on the hypersurface $\Sigma^{(3)}_t$

$$u^i_{;k}u^k = 0. \quad (2.45)$$

Here “;” denotes the covariant derivative with respect to the coordinate $y^k$ calculated with the help of the metric $h_{ik}$.

The consideration of the variational principle $\delta I_{sh}^{(g)} = 0$ with respect to the exterior coordinates $x^\mu$ is more interesting treatment. In this case $\int_\gamma \omega = \int_\gamma (4) \omega |_\gamma = \int_\gamma (4) ds$. Then the metrics $g^{\pm}_{\mu\nu}(x^\rho)$ are given in a neighbourhood of the shell. Since the normal variations of the shell are already used, it is possible to consider the variations of dynamical quantities, generated only by the tangential to $\Sigma^{(3)}_t$ variations of the coordinates $x^\mu$. These variations of the values will be denoted by the sign $\tilde{\delta}$. Thus, omitting for simplicity signs “±”, we have

$$\tilde{\delta} x^\mu = \delta x^\mu + n^\mu n_\nu \delta x^\nu \equiv h^\mu_\nu \delta x^\nu \quad (n_\mu \tilde{\delta} x^\mu = 0, \quad h^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu), \quad (2.46)$$

where $\delta x^\mu$ are arbitrary values. Then we find

$$\tilde{\delta} \omega |_\gamma = \tilde{\delta} (4) ds = \tilde{\delta} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = -u_\mu u^\nu h^\mu_\rho \delta x^\rho (4) ds + d(u_\mu \delta x^\mu). \quad (2.47)$$

Supposing that $\delta x^\mu = 0$ on $\Sigma^{(3)}_t \cap \partial D^{(4)}$, from the requirement $\tilde{\delta} I_{sh}^{(g)} = 0$ we obtain the three-dimensional geodesic equations on $\Sigma^{(3)}_t$, but, here, in the four-dimensional form

$$u_{\mu;\nu} u^\nu h^\mu_\rho = 0. \quad (2.48)$$

This equation cab be rewritten as

$$u_{\rho;\nu} u^\nu = -u_{\mu;\nu} u^\nu n^\mu n_\rho. \quad (2.49)$$

Hence, using the definition of $K_{\mu\nu}$ (2.13) one obtains

$$u_{\rho;\nu} u^\nu = n_\rho n_{\mu;\nu} u^\mu u^\nu = -n_\rho K_{\mu\nu} u^\mu u^\nu. \quad (2.50)$$

Here we again introduce signs “±” and use the relations $K_{\mu\nu}|_{\pm} = \tilde{K}_{\mu\nu} \pm \frac{1}{2}[K_{\mu\nu}]$. Then, taking into account constraints (2.29) and (2.37), we come to the equations of motion for the shell’s particles with respect to the exterior coordinates

$$(u_{\mu;\nu} u^\nu) |_{\pm} = \mp \frac{\chi}{4} \sigma n_\mu. \quad (2.51)$$

For completeness one should add the unused constraints

$$[K_{\mu\nu}] u^\mu e^\nu_i = 0, \quad [K_{\mu\nu}] e^\mu_a e^\nu_b = \frac{\chi \sigma}{2} h_{ab}, \quad (2.52)$$

where $e^\mu_a = \partial x^\mu / \partial y^a$.

From (2.51) it follows the well-known Israel equations [1]
where \( Du_\mu = u_\mu,\nu dx^\nu \) is the covariant differential.

The equations of motion of the dust shell (2.51) can immediately be found from the action \( I_{sh} \). Indeed, acting in the same manner as when deducing the equations of motion (2.51), the variational formula (2.47) can be transformed to the form

\[
\tilde{\delta}^{(4)} ds|_{\pm} = -u_{\mu,\nu} u^\nu \delta x^\mu_{\pm} + \frac{1}{2} [K_{\mu
u}] u^\mu u^\nu n_\rho \delta x^\rho ds|_{\pm} + d(u_\mu \delta x^\mu)|_{\pm}.
\]

or

\[
\tilde{\delta} \omega^{\pm}_{\gamma} = \tilde{\delta}^{(4)} ds|_{\pm} = \left\{ -u_{\mu,\nu} u^\nu \pm \frac{\chi \sigma}{4} n_\mu \right\} \delta x^\mu_{(4)} ds + d(u_\mu \delta x^\mu)|_{\pm}.
\]

From here, under the above conditions, the equations of motion follow.

The proposed variational deducing of the equations of motion makes the problem of construction of the effective action for the dust shell free from constrains (2.28) and (2.37).

It turns out that it is possible for some special class of the configurations. To show it, we shall choose such interior coordinates \( y^i \), which at \( i = a = 2, 3 \) are the Lagrange coordinates of particles on the shell \( \Sigma^{(3)}_t \). In addition, we introduce the coordinates \( x^\mu_{\pm} \) in the regions \( D^{(4)}_\pm \) so that, when \( \mu = a = 2, 3 \) the equalities \( x^a_{\pm}|_{\Sigma^{(3)}_t} = x^a|_{\Sigma^{(3)}_t} = y^a \) are satisfied. These coordinates are arbitrary in any other respect. Then the formulas of embedding of \( \Sigma^{(3)}_t \) into \( D^{(4)}_\pm \) have the form \( x^a_{\pm} = x^a(y^0) \) (for \( n = 0, 1 \)) or \( f_{\pm}(x^0, x^1) = 0 \). Therefore we have \( u^\mu_{\pm} = \{ u^0_{\pm}, u^1_{\pm}, 0, 0 \} \) and \( n_\mu = \{ n_0^\pm, n_1^\pm, 0, 0 \} \). Using the conditions \( (u_\mu u^\mu)|_{\pm} = (n_\mu n^\mu)|_{\pm} = 1 \) and \( (u_\mu n^\mu)|_{\pm} = 0 \) one finds \( n_0^\pm = u^0_{\pm}, \ n_1^\pm = -u^1_{\pm} \). Hence it follows

\[
n_\mu \delta x^\mu ds|_{\pm} = (u^1 \delta x^0 - u^0 \delta x^1) ds|_{\pm} = (dx^1 \delta x^0 - dx^0 \delta x^1)|_{\pm}.
\]

Therefore the variational formula (2.56) has the form

\[
\tilde{\delta} \omega^{\pm}_{\gamma} = \tilde{\delta}^{(4)} ds|_{\pm} = \left\{ \delta^{(4)} ds \pm \frac{1}{4} \chi \sigma (dx^1 \delta x^0 - dx^0 \delta x^1) + d(u_\mu \delta x^\mu) \right\}|_{\pm}.
\]

Now we introduce the vector potential \( U_n = U_n(x^0, x^1) \) by the relation

\[
d \wedge (U_n dx^n) \equiv G_{01} dx^0 \wedge dx^1 = -\frac{1}{4} \chi \sigma dx^0 \wedge dx^1,
\]

where \( G_{nm} \equiv U_{m,n} - U_{n,m} \) (\( n, m = 0, 1 \)). Hence it follows that the configurations, being considered, admit such motions of matter for which \( \sigma = \sigma(x^0, x^1) \).

Using the definition (2.59) and the relation

\[
\delta(U_n dx^n) - d(U_n \delta x^n) = G_{10}(dx^0 \delta x^1 - dx^1 \delta x^0),
\]

the variational formula (2.58) can be rewritten in the following form
\[ \delta \omega^-_{/\gamma} = \tilde{\delta} (4) ds|_{\pm} = \{ \delta (ds \mp U_n dx^n) + d[(u_n \pm U_n) \delta x^n + u_a \delta y^a] \}|_{\pm}. \] (2.61)

Returning to action for the shell (2.40), we conclude, that in the case under consideration we have

\[ \delta I_{sh} = \delta I_{sh}^\pm - \frac{c}{2} \int d^2 m \left\{ (u_n \pm U_n) x^0_0 \delta y^0 + u_a \delta y^a \right\} \bigg|_{A}^{B}, \] (2.62)

where

\[ I_{sh}^\pm = -\frac{c}{2} \int d^2 m \int (ds \mp U_n dx^n)|_{\pm}, \quad (n = 0, 1) \] (2.63)

is the effective action for the shell written in terms of the exterior coordinates. Indices \( A \) and \( B \) indicate that the corresponding quantities are taken in initial and final positions of the shell. Since at fixed initial and final positions of particles \( \delta y^j|_{A,B} = 0 \), then it follows \( \delta I_{sh} = \delta I_{sh}^\pm \).

In such a way, under the above conditions, the action of the shell (2.40) with the constraints (2.28), (2.37) and the action (2.62) without these constraints are equivalent. The actions \( I_{sh}^+ \) and \( I_{sh}^- \) are equivalent in the same sense. Let us note, that in the considered above independent treatment of the interior and exterior faces of the shell there are new constraints following from isometry conditions of these faces.

**III. EFFECTIVE ACTION FOR THE SPHERICAL DUST SHELL**

Let us consider spherically-symmetric compound region \( D^{(4)} = D^{(4)}_- \cup \Sigma^{(3)}_t \cup D^{(4)}_+ \subset V^{(4)} \) into the spherically-symmetric space-time \( V^{(4)} \), where \( D^{(4)}_\pm \) are exterior and interior regions separated from each other by spherically-symmetric time-like hypersurface \( \Sigma^{(3)}_t \). By using the curvature coordinates we can choose common in \( D^{(4)}_\pm \), spatial, spherical coordinates \( \{ r, \theta, \alpha \} \), and individual time coordinates \( t_\pm \) for \( D^{(4)}_\pm \) respectively. Then the world sheet for the shell \( \Sigma^{(3)}_t \) respectively the interior and exterior coordinates is determined by the equations \( r = R_-(t_-) \) and \( r = R_+(t_+) \). Under appropriate choice of \( t_\pm \) we have \( R_-(t_-) = R_+(t_+) \). Thus, the interior and exterior regions are determined by the relations

\[ D^{(4)}_- = \{ t_- , r, \theta, \alpha : r_0 < r < R_-(t_-) \}, \quad D^{(4)}_+ = \{ t_+ , r, \theta, \alpha : R_+(t_+) < r < \infty \} \]

for all \( \{ \theta, \alpha \} \in \{ 0 \leq \theta \leq \pi, \ 0 \leq \alpha < 2\pi \} \) and for all admissible \( t_\pm \). The particles of the shell are described by one collective dynamical coordinate \( R = R_+(t_+) \) and by the two fixed individual (Lagrange) angular coordinates \( \theta \) and \( \alpha \). The minimal value of \( r_0 \) is limited by the domain of definition of the curvature coordinates.

The gravitational fields into the regions \( D^{(4)}_\pm \) are given by the metrics

\[ (4) ds^2_{\pm} = f_{\pm}^{-1} c^2 dt_{\pm}^2 - f_{\pm} \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\alpha^2), \] (3.1)

where
\[ f_\pm = 1 - \frac{2\gamma M_\pm}{c^2 r}, \]  

(3.2)

and \( M_\pm \) are the Schwarzschild masses \((M_+ > M_-)\).

Owing to the spherical symmetry \( \sigma = \sigma(t_\pm, R) \). Therefore the conditions of applicability of the modified action (2.62) are satisfied. In this case we have

\[ d^2 m = \sigma d^2 \Omega = \sigma R^2 \sin \theta d\theta d\alpha, \]  

(3.3)

\[ U_n dx^n = c\varphi(t_\pm, R) dt_\pm + U_R(t_\pm, R) dR. \]  

(3.4)

Using the gauge condition \( U_R(t_\pm, r) = 0 \), the action (2.62) can be written in the form

\[ I_{sh}^\pm = -\frac{c}{2} \int s^{(2)}_t \sigma R^2 \sin \theta d\theta d\alpha \int (^{(2)} ds \mp c\varphi dt)|_\pm. \]  

(3.5)

Since the particles move only radially \((\theta = \text{const}, \varphi = \text{const})\) we shall use the truncate interval

\[ (^{(2)} ds_\pm^2 = f_\pm c^2 dt_\pm^2 - f_\pm^{-1} dR^2. \]  

(3.6)

Further, from the formula (2.59) it follows

\[ \frac{1}{4} \chi_\sigma = \frac{\gamma m}{2c^2 R^2} = \frac{\partial \varphi}{\partial R}, \]  

(3.7)

where \( m = 4\pi \sigma R^2 \) is the rest mass of the shell. Hence, up to an additive constant, one finds

\[ \varphi = -\frac{\gamma m}{2c^2 R}. \]  

(3.8)

Finally, integrating in (3.3) over the angles \( \theta \) and \( \alpha \) and making use of (3.6) and (3.8), the effective action of the shell can be expressed in the form

\[ I_{sh}^\pm = \frac{1}{2} \int L_{sh}^\pm dt|_\pm = -\frac{1}{2} \int \left( mc^{(2)} ds \pm \frac{\gamma m^2}{2R} dt \right)|_\pm, \]  

(3.9)

where

\[ L_{sh}^\pm = -mc^2 \sqrt{f_\pm - f_\pm^{-1} R^2/c^2} \pm U \]  

(3.10)

are the Lagrangians of the dust shell with respect stationary observes into the regions \( D^{(4)}_\pm \), \((R_{t_\pm} = dR/dt_\pm)\), and

\[ U^{(G)} = -\frac{\gamma m^2}{2R} \]  

(3.11)

is the effective potential energy of the gravitational self-action of the shell. It is important that the self-action (3.11) has the same form as that in the Newtonian theory (formula (C12) in Appendix C). The Lagrangians (3.10) themselves can be obtained from the corresponding Newtonian analogs (see Appendix C, formulas (C13) and (C14)) by the formal replacement.
of the first and second terms describing the kinetic and potential energies of the shell into the external field by their general relativistic analog, the geodesic Lagrangian $-mc \frac{d^2s}{dt^2}$. It is natural that the Lagrangians (3.13), (3.14), up to an additive constant, are the Newtonian limits of the relativistic Lagrangians (3.10).

It is easy to see that the actions (3.9) transform each into other under the discrete gauge transformation

$$M_\pm \to M_\mp \quad (f_\pm \to f_\mp), \quad U^{(G)} \to - U^{(G)}, \quad t_\pm \to t_\mp.$$ 

This transformation generalizes the corresponding transformation of the Newtonian theory of shells (see Appendix C) and reduces to the transformation from the interior observer to the exterior one and otherwise.

Note, that despite the equivalence of the actions $I_{sh}^\pm$, similar to Newtonian case (3.13), (3.14), they can be considered quite independently. We also can consider the regions $D_\pm$ together with the corresponding gravitational fields (3.1) separately and independently, as manifolds with the edge $\Sigma_{t_\pm}$. The edges $\Sigma_{t_\pm}$ acquire the physical meaning of different faces of the shell with the world sheet $\Sigma_t$, provided the regions $D_{\pm}$ are joined along these edges $\Sigma_{t_\pm}$. This can be performed only if the conditions of isometry of the edges $\Sigma_{t_\pm}$ are satisfied

$$f_+ c^2 dt_+^2 - f_+^{-1} dR^2 = f_- c^2 dt_-^2 - f_-^{-1} dR^2 = c^2 d\tau^2, \quad (3.12)$$

where $\tau$ is the proper time of the shell. In addition we have $\Sigma_{t_+} = \Sigma_{t_-} = \Sigma_t$, $\gamma_+(t_+) = \gamma_-(t_+) = \gamma$.

Now we study some results following from the isometry conditions of the edges. First, we obtain the relations between the velocities

$$c^2 \frac{f_+}{R^2_+} - \frac{1}{f_+} = c^2 \frac{f_-}{R^2_-} - \frac{1}{f_-}, \quad (3.13)$$

$$R^2_\tau \equiv \left( \frac{dR}{d\tau} \right)^2 = \frac{c^2 R^2_\mp}{c^2 f_\pm - f_\mp^{-1} R^2_\mp}, \quad R^2_{t_\pm} \equiv \left( \frac{dR}{dt_\pm} \right)^2 = \frac{c^2 f_\pm^2 R^2_\tau}{c^2 f_\pm + R^2_\tau}. \quad (3.14)$$

Then from the Lagrangians $L_{sh}^\pm$ (3.10) one finds the momenta and Hamiltonians of the shell

$$P_\pm = \frac{\partial L_{sh}^\pm}{\partial R_{t_\pm}} = \frac{m R_{t_\pm}}{f_\pm \sqrt{f_\pm - f_\mp^{-1} R^2_\mp / c^2}} = \frac{m}{f_\pm} R_{\tau}, \quad (3.15)$$

$$H_{sh}^\pm = \frac{mc^2 f_\pm}{\sqrt{f_\pm - f_\mp^{-1} R^2_\mp / c^2}} \mp U = mc^2 f_\pm \frac{dt_\pm}{d\tau} \mp U \quad (3.16)$$

or

$$H_{sh}^\pm = c \sqrt{f_\pm (m^2 c^2 + f_\pm P^2_\pm)} \mp U = mc^2 \sqrt{f_\pm + R^2_\tau / c^2} \mp U = E_\pm, \quad (3.17)$$

where $E_\pm$ are the energies of the shell which are conjugated to the time $t_\pm$ respectively and conserve with respect to the corresponding interior or exterior stationary observers’ point.
of view. After elimination of velocity $R_\tau$ from (3.15) and (3.17), the isometry conditions of the edges can be expressed in the form

\begin{equation}
 f_+ P_+ = f_- P_-, \quad (E_- - U)^2 - m^2 c^4 f_- = (E_+ + U)^2 - m^2 c^4 f_+.
\end{equation}

Substituting $U$ and $f_\pm$ from (3.2) and (3.11) for those in the last relation and equating the coefficients at the same power of $R$ we obtain the relations between the Hamiltonian $H_\pm$ and the Schwarzschild masses $M_\pm$

\begin{equation}
 H_+ = H_- = (M_+ - M_-)c^2 = E.
\end{equation}

Here $E = E_\pm$ is the full energy of the shell. This energy is conjugated to the coordinate time $t_+$ and $t_-$ as well, and does not depend on the position of the stationary observer (inside or outside the shell). We shall interpret the relations (3.18) and (3.20) following from the above independent consideration of the shell faces, as momentum and Hamiltonian constraints.

The Lagrangians $L_\pm$ (3.10), as well as the relations (3.13) - (3.20), are valid only in a limited domain, since the used curvature coordinates are valid outside the event horizon only. Therefore $L_-^\pm$ can be used when $R > 2\gamma M_-/c^2$, and $L_+^\pm$ when $R > 2\gamma M_+/c^2$ ($M_+ > M_-$).

As is known, the complete description of the shells can be performed in the Kruskal-Szekeres coordinates. With respect to these coordinates the full Schwarzschild geometry consists of the four regions $R^+, T^-, R^-, T^+$, detached by the event horizons. Our above consideration concerned with the $R^+$ region only.

Supposing $r$ to be a time coordinate, we can formally use the action in the form (3.9) under the horizon. However, here we encounter the ambiguity when choosing the sign before $(2) ds$. It is usually ascribed to ambiguity of the radial component direction of the unit normal to $\Sigma_t^{(3)}$. The point is that in the curvature coordinates the regions $T^-$ and $T^+$ coincide. Hence the time singularity $r = 0$ contains the two singularities: past singularity and future singularity. Therefore, for instance, the movement of a test particle with the energy $E = 0$ consists of the two stages. At the first stage the particle begins to move into the expanding region $T^+$ from the past singularity $r = 0$ and reaches the horizon at a moment when $r$ reaches $2\gamma M/c^2$. Then it goes over into the contracting $T^-$ region and moves from the horizon to the future singularity $r = 0$. In the coordinates $\{r, t\}$, where $r$ is the time coordinate, the latter stage looks like the movement directed backwards in time.

Similarly, in the curvature coordinates the regions $R^-$ and $R^+$ of the Kruskal-Szekeres diagram coincide and ordinary movement of particles into the future of the $R^-$-region looks as the movement directed backwards in time which corresponds to the change $ds \rightarrow -ds$. It means that ordinary particles moving into the $R^-$-region are mapped into the $R^+$-region as antiparticles (remember the Feynman’s interpretation of antiparticles as ordinary particles moving backwards in time). Such trajectories can be taken into account by the change of the sign before $mc (2) ds_\pm$ in the expression for the action (3.9) of the shell.

In order to use simplicity and convenience of the curvature coordinates and, at the same time, to keep information about shells into the $R^-$-region we introduce an auxiliary discrete variable $\varepsilon = \pm 1$ and make a change $(2) ds_\pm \rightarrow \varepsilon_\pm (2) ds_\pm$ in $I_{sh}^\pm$ (3.9). Herewith $\varepsilon_\pm = 1$ correspond to the shell into the $R^+$-region, and $\varepsilon_- = -1$ to the shell into the $R^-$-region. Then, we introduce the quantities $\mu_\pm = \varepsilon_\pm m$. As a result the extended action has the form
\[ I_{sh}^\pm(\mu_\pm) = \frac{1}{2} \int L_{sh}^\pm(\mu_\pm) dt_{\pm} = -\frac{1}{2} \int \left( \mu c^{(2)} ds \mp U^G dt \right)_{\pm}, \quad (3.21) \]

where

\[ L_{sh}^\pm(\mu_\pm) = -\mu_\pm c^2 \sqrt{f_\pm - f_\pm^{-1} R_{t_\pm}^2/c^2} \pm U \quad (3.22) \]

are the generalized Lagrangians describing the shell inside any of the \( R^\pm \)-regions with respect to the curvature coordinates of the interior \( \{t_-, R\} \) or exterior \( \{t_+, R\} \) regions. The event horizons \( R_g = 2\gamma M_\pm/c^2 \) are, still, singular points of the dynamical systems \((3.21)\) and must be excluded from consideration.

For the extended system \((3.21)\) the Hamiltonian has the form

\[ H_{sh}^\pm(\mu_\pm) = c \varepsilon_\pm \sqrt{f_\pm(m^2 c^2 + f_\pm P_{t_\pm}^2)} \mp U = \mu_\pm c^2 \sqrt{f_\pm + R_{t_\pm}^2/c^2} \mp U. \quad (3.23) \]

Hence, taking into account the Hamiltonian constraints \((3.20)\) one finds the standard relations of the theory of dust spherical shells \([1]\). We shall rewrite them in terms of new designations

\[
\begin{align*}
\mu_- \sqrt{f_- + R_{t_-}^2/c^2} - \mu_+ \sqrt{f_+ + R_{t_+}^2/c^2} &= \frac{\gamma \mu^2}{R c^2}, \\
\mu_- \sqrt{f_- + R_{t_-}^2/c^2} + \mu_+ \sqrt{f_+ + R_{t_+}^2/c^2} &= 2(M_+ - M_-). 
\end{align*} \quad (3.24, 3.25) 
\]

In the end of the section we write out the Hamilton-Jacobi equations corresponding to the Hamiltonians \((3.23)\) and to the constraints \((3.18), (3.20)\) for truncated actions \( S_0^\pm = S_0^\pm(R) \)

\[
\frac{1}{f_\pm} \left( M_+ - M_- \mp U/c^2 \right)^2 - \frac{f_\pm}{c^2} \left( \frac{dS_0^\pm}{dR} \right)^2 = m^2. \quad (3.26) 
\]

\[ f_+ \, dS_0^+ = f_- \, dS_0^- . \quad (3.27) \]

Then, the complete actions are determined by the formula \( S^\pm = -c^2(M_+ - M_-) t_\pm + S_0^\pm \).

**IV. PARTICULAR CASES OF SPHERICAL DUST CONFIGURATIONS**

- **Self-gravitating dust shell.** In this case \( M_- = 0 \). Denote \( M_+ = M \) and consider the shell moving into the \( R_+ \)-region. Then with respect to the exterior coordinates, the Lagrangian and the Hamiltonian of the shell have the form

\[
\begin{align*}
L_{sh}^+ &= -mc^2 \sqrt{1 - \frac{2\gamma M}{c^2 R} - \left( 1 - \frac{2\gamma M}{c^2 R} \right)^{-1} \frac{R_{t_+}^2}{c^2} - \frac{\gamma m^2}{2R}}, \\
H_{sh}^+ &= c \sqrt{1 - \frac{2\gamma M}{c^2 R}} \sqrt{m^2 c^2 + \left( 1 - \frac{2\gamma M}{c^2 R} \right) P_{t_+}^2} + \frac{\gamma m^2}{2R}. \quad (4.1, 4.2) 
\end{align*} \]
The same shell with respect to the interior coordinates is described by the Lagrangian and the Hamiltonian

\[ L_{sh} = -mc^2 \sqrt{1 - \frac{R_t^2}{c^2}} + \frac{\gamma m^2}{2R}, \quad (4.3) \]

\[ H_{sh} = c \sqrt{m^2c^2 + P^2} - \frac{\gamma m^2}{2R}. \quad (4.4) \]

This Hamiltonian was considered in the works [8], [9]. The dynamical systems with \( L_{sh}^\pm \) obey momentum and Hamiltonian constraints

\[ P_- = f_+P_+, \quad H_+ = H_- = M c^2 \]

and they are canonically equivalent (see Appendix B).

- **The dust shell with vanishing full energy.** Now we consider the shell for which the binding energy \( E_b = (m + M_+ - M_-)c^2 \) coincides with the rest energy \( mc^2 \). Denote \( M_+ = M_- \equiv M, \ f_+ = f_- \equiv f = 1 - 2\gamma M/c^2 R, \ t_+ = t_- \equiv t \). Then for the full energy we have \( E = 0 \). This is possible, as it follows from (3.24), (3.25), only when \( \mu_+ = -\mu_- < 0 \), i.e. for the wormhole. Such a shell can be considered as a classical model for “zeroth oscillations” of dust matter with bare mass \( m \) under the gravitational field with \( f = 1 - 2\gamma M/c^2 R \).

In terms \( \{t, R\} \) the trajectories of “zeroth oscillations” are determined by the equation

\[ \frac{dR}{dt} = \frac{2c^3}{\gamma m} \left( 1 - \frac{2\gamma M}{c^2 R} \right) \sqrt{\frac{\gamma^2 m^2}{4c^4} + \frac{2\gamma M}{c^2} R - R^2}. \quad (4.5) \]

Hence for the turning radius we have

\[ R_m = \frac{\gamma}{c^2} \left( M + \sqrt{M + \frac{m^2}{4}} \right). \quad (4.6) \]

In the case of the flat space when \( M = 0 \), from (4.5) and (4.6) we find

\[ \frac{dR}{dt} = c \sqrt{1 - \frac{R^2}{R_{m0}^2}}, \quad R_{m0} = \frac{\gamma m}{2c^2}. \quad (4.7) \]

The equations of motion of such “zeroth” shells coincide with those for the oscillator

\[ \frac{d^2R}{dt^2} + \omega^2 R = 0. \quad (4.8) \]

Its oscillations \( R(t) = R_{m0} \cos \omega(t - t_0) \) occur with the amplitude \( R_{m0} \) and frequency \( \omega = c/R_{m0} = 2c^3/\gamma m \). Hence we find the time of life of these shells into the flat space-time as a half-period of the oscillation

\[ T = \frac{\pi}{\omega} = \frac{\pi \gamma m}{2c^3} = \frac{\pi}{c} R_{m0}. \quad (4.9) \]

For the shell with mass equal to the mass of the Earth we have \( R_g = 2\gamma M/c^2 \approx 4cm, \ R_{m0} = R_g/4 \approx 1cm, \ T \approx 10^{-10}c \). For the shells with Planck’s mass \( m = m_{pl} = \sqrt{\hbar c}/\gamma \) the time of life equals \( T = \pi T_{pl}/2, \) where \( T_{pl} = \sqrt{\hbar \gamma/c^5} \) is Planck’s time. We underline, that
the “zero” shells are characterized by that their gravitational binding energy completely compensate proper energy, leaving their total energy to be equal to zero. These shells can be thought of as a classical prototype of the Wheeler’s space-time foam.

- **The set of concentric dust shells.** Now, consider briefly configurations consisting from the set of \( N \) concentric dust shells. Let \( R_a, m_a, \tau_a \) be the radius, bare mass and proper time of an \( a \)-th shell, respectively \((a = 1, 2, ..., N)\). For simplicity we suppose that \( R_a > R_b \) if \( a > b \). Then let \( M_a \) be the Schwarzschild mass determining the gravitational field \( f_a = 1 - 2\gamma M_a/c^2 r \) on the right side of an \( a \)-th shell, into the region \( R_a < r < R_{a+1} \). Suppose \( f_a^- = 1 - 2\gamma M_{a-1}/c^2 R_a \) and \( f_a^+ = 1 - 2\gamma M_a/c^2 R_a \). Let \( P_a^\pm = m_a dR_a/f_a^\pm d\tau_a \) be momenta of \( a \)-th shell, and \( U_a^{(G)} = -\gamma m_a^2/2R_a \) be its potential energy of the self-action. Then

\[
H_a^\pm = c e_a^\pm \sqrt{f_a^\pm (m_a^2 c^2 + f_a^\pm (P_a^\pm)^2)} \mp U_a
\]  

(4.10)

is the Hamiltonians of an \( a \)-th shell. They, similarly to momenta \( P_a^\pm \), are considered from the stationary observers’ points of view, into the interior \( R_{a-1} < r < R_a \) and exterior \( R_a < r < R_{a+1} \), regions respectively. They satisfy the momentum and Hamiltonian constraints

\[
f_a^+ P_a^+ = f_a^- P_a^-, \quad H_a^+ = H_a^- = (M_a - M_{a-1})c^2.
\]  

(4.11)

Now we are ready to determine the full Hamiltonian of this configuration

\[
H_N = \sum_{a=1}^{N} H_a^\pm.
\]  

(4.12)

For the self-gravitating configuration \( M_0 = 0 \). Then \( H_1^\pm = M_1 c^2 \) and the full Hamiltonian of the configuration satisfies the constrain

\[
H_N = Mc^2.
\]  

(4.13)

Here \( M = M_N \) is the Schwarzschild mass of the configuration. The system admits the discrete gauge transformations

\[
M_a \leftrightarrow M_{a-1}, \quad U_a \leftrightarrow -U_a, \quad t_a \leftrightarrow t_{a-1} \quad (a = 1, 2, ..., N),
\]

where \( t_a \) is coordinate time determined on the right from an \( a \)-th shell. The choice of sides (left or right) of the shells is not fixed beforehand and can be made by the reason of convenience.

**ACKNOWLEDGMENTS**

I would like to acknowledge M.Korkina and S.Stepanov for helpful discussions of problems, touched in this paper.
APPENDIX A: TRANSFORMATIONS OF THE SURFACE EQUATIONS

We show that the surface equations (2.26) reduce to the known equations for the jumps of the extrinsic curvature tensor on the shell [1]. First, we shall calculate \( n_\mu \omega^\mu \).

We suppose that the following conditions are satisfied on the hypersurface \( \Sigma \)

\[
[n_\mu] = 0, \quad [n_\mu,\nu] h_\sigma^\nu = 0, \quad [n_\mu] h_\sigma^\nu = 0, \quad [g_{\mu\nu,\rho}] h_\sigma^\rho = 0.
\]

(A1)

Hence it follows

\[
[\Gamma^\mu_{\rho\sigma}] h_\mu^\nu h_\sigma^\rho h_\alpha^\sigma = 0.
\]

(A2)

Then from the definition (2.15) one finds

\[
[\Gamma^\sigma_{\alpha\nu}] n_\sigma h_\nu^\beta - \frac{1}{2} Q h_\sigma^\beta = -[K]_{\alpha\beta},
\]

(A3)

\[
[\Gamma^\sigma_{\alpha\nu}] n_\sigma h_\nu^\nu = [-K]^\sigma_{\alpha
\}
\]

(A4)

\[
[\Gamma^\sigma_{\alpha\nu}] n_\sigma n_\nu h_\beta^\sigma = 0.
\]

(A5)

According to (C12) we find

\[
n_\mu \omega_\mu = n_\mu g^{\sigma\rho} \Gamma^\mu_{\sigma\rho} - n_\rho \Gamma^\rho_{\sigma\rho} = n_\mu h^{\sigma\rho} \Gamma^\mu_{\sigma\rho} - n_\mu h^{\sigma\rho} \Gamma^\rho_{\sigma\rho}.
\]

Therefore, making use of Eq. (A4) and (A5) we obtain the sought relation (2.14).

Then, projecting the equation (2.26) into the hypersurface \( \Sigma \) and into the normal \( n^\rho \) one finds

\[
Q_{\sigma\rho} h_\sigma^\rho = -n_\mu \omega_\mu = n_\mu g^{\sigma\rho} \Gamma^\mu_{\sigma\rho} - n_\rho \Gamma^\rho_{\sigma\rho} = n_\mu h^{\sigma\rho} \Gamma^\mu_{\sigma\rho} - n_\mu h^{\sigma\rho} \Gamma^\rho_{\sigma\rho}.
\]

(A6)

\[
Q_{\sigma\rho} h_\sigma^\rho = -\frac{1}{2} Q n_\sigma = 0.
\]

(A7)

Using the definitions (2.27) and Eqs. (A2)–(A3) we obtain

\[
Q = g_{\mu\nu} Q^{\mu\nu} = n_\mu [\omega^\mu] = 2[K],
\]

(A8)

\[
Q_{\sigma\rho} h_\sigma^\rho = n_\mu n_\rho [\Gamma^\mu_{\rho\sigma} - \frac{1}{2} n_\sigma n_\rho [\Gamma^\mu_{\rho\mu} + \frac{1}{2} \Gamma^\mu_{\sigma\mu}]] =
\]

\[
- n_\sigma (n_\mu [\Gamma^\mu_{\alpha\beta}] n_\alpha n_\beta + [\Gamma^\mu_{\mu\alpha}] n_\alpha) = -n_\sigma n_\rho [\Gamma^\mu_{\rho\mu}] h_\mu^\nu = [K] n_\sigma,
\]

(A9)

\[
Q_{\sigma\rho} h_\sigma^\rho = n_\mu [\Gamma^\mu_{\sigma\rho}] h_\sigma^\rho h_\alpha^\sigma = [K]_{\alpha\beta}.
\]

(A10)

Thus the equation (A6) is satisfied identically, and the equations (A7) yield the sought relations (2.28).

APPENDIX B: ON THE CANONICAL EQUIVALENCE OF THE ACTIONS \( I^\pm_{sh} \) FOR THE DUST SPHERICAL SHELL

In order to show the canonical equivalence of the actions \( I^\pm_{sh} \) in the extended phase space we write the variational principle (3.9) in the form
\[ \delta I_{sh}^\pm = \delta \int (P_\pm dR - H_\pm dt_\pm) = 0, \quad (B1) \]

where
\[ P_\pm = \frac{1}{c f_\pm} \sqrt{(H_\pm \pm U)^2 - m^2 c^4 f_\pm}. \quad (B2) \]

The dynamical systems with the actions \( I_{sh}^\pm \) are restricted by momentum and Hamiltonian constraints (3.18) and (3.20), which follow from the independent consideration of the faces of the shell.

The systems \( I_{sh}^\pm \) will be canonically equivalent in the extended phase space of variables \( \{P_\pm, H_\pm, R, t_\pm\} \), if
\[ dI_{sh}^+ = dI_{sh}^- + dF, \quad (B3) \]
or
\[ P_\pm dR - H_\pm dt_\pm = P_- dR - H_- dt_- + dF, \quad (B4) \]
where \( F = F(R, t_+, t_-) \) is the generating function of the canonical transformation \( \{P_+ = P_+(P_-, t_-, R), \ t_+ = t_+(P_-, t_-, R)\} \). From (B4) we find
\[ H_+ = -\frac{\partial F}{\partial t_+}, \quad H_- = \frac{\partial F}{\partial t_-}, \quad P_+ = P_- - \frac{\partial F}{\partial R}. \quad (B5) \]

Using these relations the constraints (3.18) and (3.20) can be rewritten in the following way
\[ -\frac{\partial F}{\partial t_+} = \frac{\partial F}{\partial t_-} = E, \quad \frac{\partial F}{\partial R} = P_\left(1 - \frac{f_-}{f_+}\right). \quad (B6) \]

From here we find
\[ F = E(t_- - t_+) + \sigma(R, E), \quad (B7) \]
where
\[ \sigma(R, E) = \frac{1}{c} \int \left(\frac{1}{f_-} - \frac{1}{f_+}\right) \sqrt{(E \pm U)^2 - m^2 c^4 f_\pm} \ dR. \quad (B8) \]

The expression under the radical is invariant with respect to the replacement \( f_\pm \rightarrow f_\mp, \ U \rightarrow -U \).

Then differentiating the expression (B7) over \( E \) one finds the relation between \( t_+ \) and \( t_- \)
\[ \frac{\partial F}{\partial E} = t_- - t_+ + \frac{\partial \sigma}{\partial E} = \alpha, \quad (B9) \]
where the constant \( \alpha \) cab be omitted. Thus, the transformations
\[ \begin{cases} t_+ = t_- + \frac{\partial \sigma(R,E)}{\partial E}, \\ P_+ = P_- + \frac{\partial \sigma(R,E)}{\partial R}, \end{cases} \quad (B10) \]
are the sought canonical transformations of the extended phase space \( \{ P_{\pm}, H_{\pm}, R, t_{\pm} \} \) of the system. Herewith, the corresponding presymplectic form
\[
dP_+ \wedge dR - dE \wedge dt_+ = dP_- \wedge dR - dE \wedge dt_-
\]
is invariant under these transformations.

The difference of the shell actions, which are considered from the point of view of the exterior and interior observers, up to an additive constant is
\[
I_{sh}^+(R, t_+) - I_{sh}^-(R, t_-) = F(R, t_+, t_-),
\]
where \( F = F(R, t_+, t_-) \) is the generating function, is calculated by the elimination of the energy \( E \) from the relations (B7) and (B9).

**APPENDIX C: NON-RELATIVISTIC DUST SHELL**

Let us consider an infinitely thin dust layer in the Euclidean space \( \mathbb{R}^3 \) in the form of the closed surface \( \Sigma_t \) moving in its own Newtonian gravitational field \( \varphi = \varphi(\vec{r}) \). The full action for this configuration has the form
\[
I_{tot}^{(N)} = \int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} \left( \frac{1}{2} \sigma \vec{v}^2 - \sigma \varphi \right) d^2 f - \frac{1}{8 \pi \gamma} \int_{D_- \cup D_+} (\nabla \varphi)^2 dV \right\}.
\]
Here \( \Sigma_t (t_1 \leq t \leq t_2) \) is a one-parameter family of the closed surfaces, \( D_- \) and \( D_+ \) are interior and exterior regions of the shell \( \Sigma_t \) at a moment \( t \), \( d^2 f \) is the surface element on \( \Sigma_t \), \( dV \) is the volume element in \( \mathbb{R}^3 \), \( \vec{v} \) is the velocity of particles of the shell, \( \nabla \) is the nabla operator, \( \sigma \) is surface mass density of dust on \( \Sigma_t \). By virtue of the mass conservation law the value \( dm \equiv \sigma d^2 f \) conserves along the stream tube and in case of dust can be considered as a stationary value under arbitrary variations. Note also, that, we require that the potential \( \varphi \) be continuous, and, together with all its derivatives, vanish at infinity.

The requirement of extremity of the action \( \delta I_{tot}^{(N)} = 0 \) with respect to everywhere continuous variations \( \delta \varphi \), vanishing on infinity, leads to the Laplace equation
\[
\Delta \varphi(\vec{r}) = 0, \quad \vec{r} \in D_- \cup D_+
\]
with the boundary conditions for normal derivatives of \( \varphi \) on \( \Sigma_t \). The latter, for completeness, will be written out together with the continuity conditions for \( \varphi \) on \( \Sigma_t \):
\[
[\varphi] \equiv \varphi_+ - \varphi_- = 0, \quad \left[ \frac{\partial \varphi}{\partial \eta} \right] \equiv \left. \frac{\partial \varphi}{\partial \eta} \right|_+ - \left. \frac{\partial \varphi}{\partial \eta} \right|_- = 4 \pi \gamma \sigma, \quad \vec{r} \in \Sigma_t (t = \text{const}).
\]
Here \( \partial/\partial \eta = (\vec{n} \cdot \nabla) \) is the derivative with respect to the exterior normal \( \vec{n} \) to \( \Sigma_t \), vector \( \vec{n} \) (\( \vec{n}^2 = 1 \)) is directed from \( D_- \) to \( D_+ \). The solution of the equations (C2), (C3) is the potential of a “simple layer”
\[
\varphi_\sigma(\vec{r}) = -\gamma \int_{\Sigma_t} \frac{\sigma(\vec{r'}) d^2 f'}{|\vec{r} - \vec{r'}|}.
\]

Now we can calculate \( I_{tot}^{(N)} \) on the solutions of the equations (C2), (C3). Thereby, the potential \( \varphi \) is excluded from the full action (C1). Note that owing to (C2) we have
\((\nabla \varphi)^2 = \nabla (\varphi \nabla \varphi)\). This allows to transform the volume integral of (C4) into the surface one on the boundaries of the regions \(D_\pm\). Taking into account the boundary conditions (C3) and an asymptotic behaviour of \(\varphi\), we find the reduced action \(I^{(N)}_{\text{tot}}\), as the value of the initial action on the solution (C4) of the equations (C2) and (C3) \(I^{(N)}_{\text{tot}} \mid \{\text{solutions eq. (C2), (C3)}\} = I^{(N)}_{\text{sh}} + I^{(N)}_{0}\). (C5)

Here \(I^{(N)}_{0}\) contains the surface term, which is unessential for further consideration, and

\[
I^{(N)}_{\text{sh}} = \int_{t_1}^{t_2} L^{(N)}_{\text{sh}} \, dt \quad \text{(C6)}
\]

is the effective action for the shell with the Lagrangian

\[
L^{(N)}_{\text{sh}} = \frac{1}{2} \int_{\Sigma_t} \sigma \bar{v}^2 df - U, \quad \text{(C7)}
\]

where

\[
U = \frac{1}{2} \int_{\Sigma_t} \sigma \varphi_\sigma \, df = -\frac{\gamma}{2} \int_{\Sigma_t} \int_{\Sigma_t} \frac{\sigma(r) \sigma(r')}{|r - r'|} \, df \, df', \quad \text{(C8)}
\]

is the functional of the potential energy of the gravitational self-action of the shell.

The Lagrangian of the shell in an external gravitational field \(\varphi_0 = \varphi_0(r)\) has the form

\[
L^{(N)}_{\text{sh}} = \int_{\Sigma_t} \left( \frac{1}{2} \sigma \bar{v}^2 - \sigma \varphi_0 \right) df - U. \quad \text{(C9)}
\]

Now consider the spherical non-relativistic dust shell. Let \(R = R(t)\) be the radius of the spherical shell at a moment \(t\). With respect to the spherical coordinates \(\{r, \theta, \alpha\}\), we have \(\sigma = \sigma(r)\), \(d^2 f = R(t) \sin \theta d \theta d \alpha\), \(\bar{v}^2 = \dot{R}^2 = (dR/dt)^2\). The mass of the shell is \(m = 4\pi \sigma R^2 = \text{const}\). Potential of the external field \(\varphi_0\) on the shell has the value

\[
\varphi_0 \equiv \varphi_- = -\frac{\gamma m}{R(t)} \quad \text{(C10)}
\]

where \(m_-\) is the total mass of the interior source. The potential \(\varphi_\sigma\) and the self-action energy for the shell \(U\) prove to be equal

\[
\varphi_\sigma(r) = \begin{cases} -\gamma m/r, & r \geq R(t) \\ -\gamma m/R(t), & r < R(t) \end{cases} \quad \text{(C11)}
\]

\[
U = \frac{1}{2} m \dot{R}^2 - \frac{\gamma m^2}{2R(t)} \quad \text{(C12)}
\]

Replacing the corresponding terms in (C9) by those of (C10) - (C12), we obtain the Lagrangian of the spherically-symmetric dust shell in Newtonian theory of gravity

\[
L^{(N)}_{\text{sh} -} = \frac{1}{2} m \dot{R}^2 + \frac{\gamma m m_-}{R} - U. \quad \text{(C13)}
\]
A distinctive feature of spherical shell is that the two-valued description of the shell dynamic becomes possible with respect to the observer’s position. From an interior observer’s point of view, except for the force of self-action \(-\partial U/\partial R\), the shell is effected by the external force \(F = -md\varphi /dr\), which determines an interior gravitational field. This situation corresponds to the Lagrangian \(L_{\text{sh} -}^{(N)}\), therefore the latter can be interpreted as the Lagrangian describing the non-relativistic shell from an interior observer’s point of view.

An exterior observer \((r > R(t))\) determines the field, judging by the force \(F = -md\varphi /dr\) acting on the shell in the field \(\varphi = \varphi_+ + \varphi_\sigma = -\gamma m_+/R(t)\). This field is generated by the total mass of the system \(m_+ = m_- + m\). If, by making use of this relation, we eliminate \(m_-\) from \(L_{\text{sh} -}^{(N)}\) we obtain the following Lagrangian
\[
L_{\text{sh} +}^{(N)} = \frac{1}{2} m \dot{R}^2 + \frac{\gamma mm_+}{R}
\]
(C14)
It can be interpreted as the Lagrangian describing the Newtonian shell from an exterior observer’s point of view.

In such a way, transformation from an exterior observer to an interior one stipulates the discrete transformation \(m_+ \rightarrow m_\sigma = m_\pm \mp m\). Its can be interpreted as both the gravitational potential transformation \(\varphi_+ \rightarrow \varphi_+ = \varphi_\pm \pm \gamma m/R\) and the change of sign of the self-action potential \(U^{(N)} \rightarrow -U^{(N)}\). The above two-valued description of spherical shell in the Newtonian theory has a formal character. This ambiguity, arising when describing spherically-symmetric shell, is a matter of principle in General Relativity.

Note other feature of the shell, which has non-trivial meaning in General Relativity. The Lagrangians \(L_{\text{sh} \pm}^{(N)}\) completely and closely determine the motion of boundaries of regions \(D_\pm\). That is why they can be thought of as independent systems with their momenta and Hamiltonians
\[
P_\pm = m \dot{R}_\pm, \quad H_\pm = \frac{P^2_\pm}{2m} - \frac{\gamma mm_\pm}{R_\pm} \mp U_\pm = E_\pm
\]
(C15)
Here \(E_\pm\) are the total energies of these boundaries, \(U^{(N)}_\pm = -\gamma m^2/2R_\pm\) are their potential energies of self-action, and \(R_\pm = R_\pm(t)\) are the radiuses of the regions’ boundary \(D_\pm\). The systems \((C15)\) describe the same shell provided the regions \(D_\pm\) have a common boundary \(R_+(t) = R_-(t) \equiv R(t)\) for all moments \(t\). In this case, eliminating the momentum \(P \equiv P_+ = P_-\) from the equations \(H_\pm = E_\pm\) one has
\[
E_+ - E_- = \frac{\gamma m}{R} (m_+ - m - m_-)
\]
(C16)
Hence it follows an ordinary equality of energies and the additivity of masses \(E_+ = E_-\), \(m_+ = m - m_-\). In General Relativity, a similar but not trivial procedure follows from the isometry conditions for the boundaries of the corresponding four-dimensional regions.

Finally we shall write the corresponding relations for a self-gravitating shell, when \(m_- = 0\):
\[
L_{\text{sh} -}^{(N)} = \frac{1}{2} m \dot{R}^2 + \frac{\gamma m^2}{2R}
\]
(C17)
\[
P = m \dot{R}, \quad H = \frac{P^2}{2m} - \frac{\gamma m^2}{2R} = E
\]
(C18)
REFERENCES

[1] W. Israel, Nuovo Cimento 44B, 1 (1966); 48B, 463(E) (1967).
[2] K. Kuchař, Czech. J. Phys. B 18, 435 (1968).
[3] M. Visser, Phys. Rev. D 43, 402 (1991).
[4] P. Kraus and F. Wilczek, Nucl. Phys. B 433, 403 (1995).
[5] A. Ansoldi, A. Aurilia, R. Balbinot, and E. Spallucci, Phys. Essays 9, 556 (1996); Class. Quantum Grav. 14, 1 (1997).
[6] P. Hájíček and J. Bičák, Phys. Rev. D 56, 4706 (1997); P. Hájíček and J. Kijowski, Phys. Rev. D 57, 914 (1998); P. Hájíček, Phys. Rev. D 57, 936 (1998).
[7] V. A. Berezin, A. M. Boyarsky, and A. Yu. Neronov, Phys. Rev. D 57, 1118 (1998).
[8] P. Hájíček, B. S. Kay, and K. Kuchař, Phys. Rev. D 46, 5439 (1992).
[9] A. D. Dolgov, I. B. Khriplovich, Phys. Lett. B 400, 12 (1997).
[10] J. A. Wheeler, Ann. Phys. (NY) 2, 604 (1957).
[11] G. Hayward and J. Louko, Phys. Rev. D 42, 4032 (1990).
[12] J. W. York Jr., Phys. Rev. Lett. 28, 1082 (1972).
[13] R. Courant, D. Hilbert, Methoden der mathematischen Physik (Springer, v.I, 1931, v.II, 1937).
[14] V. N. Ponomarev, A. O. Barvinsky, Yu. N. Obuhov, Geometrical geometrodynamical methods and the gauge approach to the theory of gravitational interactions (Moscow: Energoatomizdat, 1985) (in Russian).
[15] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon, Oxford, 1971).

[16] In the four-dimensional case, for configurations with the dust density $\rho$, we have $\rho d^4 \Omega = \omega \wedge d^3 m$, where $d^3 m = \rho \sqrt{-g} \rho u^\mu d\Sigma_\mu$ is the 3-form of the element of the mass for some hypersurface. The requirement of the closure $d \wedge d^3 m = 0$ gives a conservation law $(\sqrt{-g} \rho u^\mu)_\mu = 0$. Therefore, in the Lagrange formulation of an perfect fluid, quantity $\sqrt{-g} \rho u^\mu$ does not vary when varying of the metric (see S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, 1973)). Therefore 3-form $d^3 m$ does not vary too. Similarly to this, in a three-dimensional case the 2-form $d^2 m$ is closed on $\Sigma_i^{(3)}$ and, by virtue of a conservation law, does not vary when varying of the metric.