Weak decays of H–like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions

A. N. Ivanov$^a$, M. Faber$^a$, R. Reda$^c$, P. Kienle$^{b,c}$

February 10, 2008

$^a$Atominstitut der Österreichischen Universitäten, Technische Universität Wien, Wiedner Hauptstrasse 8-10, A-1040 Wien, Österreich,

$^b$Physik Department, Technische Universität München, D–85748 Garching, Germany,

$^c$Stefan Meyer Institut für subatomare Physik, Österreichische Akademie der Wissenschaften, Boltzmanngasse 3, A-1090, Wien, Österreich

Abstract

The nuclear K–shell electron–capture ($EC$) and positron ($\beta^+$) decay constants, $\lambda_{EC}$ and $\lambda_{\beta^+}$ of H–like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions, measured recently in the ESR ion storage ring at GSI, were calculated using standard weak interaction theory. The calculated ratios $R = \lambda_{EC}/\lambda_{\beta^+}$ of the decay constants agree with the experimental values within an accuracy better than 3%.

PACS: 12.15.Ff, 13.15.+g, 23.40.Bw, 26.65.+t
1 Introduction

The neutral $^{140}\text{Pr}^{0+}$ atoms decay with a 99.4\% branch to the ground state of $^{140}\text{Ce}^{0+}$ via a pure $1^+ \rightarrow 0^+$ Gamow–Teller transition [1]. The K–shell $EC$ to $\beta^+$ ratio was measured as $R_{EC/\beta^+} = 0.74(3)$ by Biryukov and Shimanskaya [2] and $R_{EC/\beta^+} = 0.90(8)$ by Evans et al. [3]. Because of the discrepancy of these results the experiment was repeated by Campbell et al., who found $R_{EC/\beta^+} = 0.73(3)$ and claimed a disagreement of about 15\% with the theoretical value by Bambynek et al. [5]. This problem came up again, when Litvinov et al. succeeded in measuring the $\beta^+$ and orbital electron–capture decay rates in fully ionised, H–like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions in the Experimental Storage Ring (ESR) at GSI in Darmstadt [6]. For the He–like $^{140}\text{Pr}^{57+}$ ion a ratio $R_{EC/\beta^+} = 0.96(8)$ agrees well with $R_{EC/\beta^+} = 0.90(8)$ by Evans et al. [3], but deviates by about 2.5 standard deviations from the values of [2] and [4]. In addition Litvinov et al. [6] found that the H–like $^{140}\text{Pr}^{58+}$ ion with one electron in the K–shell decays 1.49(8) times faster than the He–like one with two electrons in the K–shell. In this work we show that using standard weak interaction theory one can reproduce the experimental values of the ratios of the $EC$ and $\beta^+$ decay constants of the H–like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions with an accuracy better than 3\%.

The paper is organised as follows. In Section 2 we give the results of the calculation of the weak decay constants of the H-like $^{140}\text{Pr}^{58+}$ and the He–like $^{140}\text{Pr}^{57+}$ ions and their ratios. We compare the theoretical results with the experimental data. In the Conclusion we discuss the obtained results. In Appendices A and B we aduce the detailed calculations of the weak decay constants of the H-like $^{140}\text{Pr}^{58+}$ and the He–like $^{140}\text{Pr}^{57+}$ ions.

2 Weak decay constants

For the calculation of the weak decay constants of H-like and He–like ions we use the Hamiltonian of the weak interaction taken in the standard form [7]

\[
H_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} [\bar{\psi}_n(x)\gamma^\mu(1 - g_A\gamma^5)\psi_p(x)] [\bar{\psi}_\nu_e(x)\gamma_\mu(1 - \gamma^5)\psi_e(x)] + \text{h.c.},
\]

where $G_F = 1.166 \times 10^{-11}$ MeV$^{-2}$ is the Fermi constant, $V_{ud} = 0.9738 \pm 0.0005$ is the CKM matrix element, $g_A = 1.2695 \pm 0.0058$ is the axial coupling constant [8], $\psi_n(x)$, $\psi_p(x)$, $\psi_{\nu_e}(x)$ and $\psi_e(x)$ are the operators of the neutron, proton, neutrino and electron/positron fields, respectively.

The detailed calculations of the weak decay constants of the H–like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions are given in Appendices A and B. Since the $EC$–decay of the H–like $^{140}\text{Pr}^{58+}$ ion from the hyperfine state $^{140}\text{Pr}^{58+}_{F=\frac{3}{2}}$ with $F = \frac{3}{2}$ is suppressed (see Appendix B and [6, 9]), we take into account that the H–like $^{140}\text{Pr}^{58+}$ ion decays from the hyperfine ground state $^{140}\text{Pr}^{58+}_{F=\frac{1}{2}}$ with $F = \frac{1}{2}$.

Using the Hamiltonian of the weak interaction [11] and following [5] for the $EC$–decay constants of the H–like $^{140}\text{Pr}^{58+}_{F=\frac{1}{2}}$ and the He–like $^{140}\text{Pr}^{57+}$ ions in the ground states we
obtain the following expressions (see Appendix A)

\[
\begin{align*}
\lambda_{EC}^{(H)} &= \frac{1}{2F+1} \frac{3}{2} |M_{\text{GT}}|^2 |\langle \psi_{1s}^{(Z)} \rangle|^2 \frac{Q_H^2}{\pi}, \\
\lambda_{EC}^{(He)} &= \frac{1}{2F+1} \frac{3}{2} |M_{\text{GT}}|^2 |\langle \psi_{1s}^{(Z-1)} \psi_{(1s)^2}^{(Z)} \rangle|^2 \frac{Q_{\text{He}}^2}{\pi},
\end{align*}
\]

where \( F = 1/2 \) and \( I = 1 \) are the total angular momenta of the H–like \(^{140}\text{Pr}^{58+}\) and the He–like \(^{140}\text{Pr}^{57+}\) ions, respectively, \( Q_H = (3348 \pm 6) \) keV and \( Q_{\text{He}} = (3351 \pm 6) \) keV are the \( Q \)-values of the decays \(^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + \nu_e\) and \(^{140}\text{Pr}^{57+} \rightarrow ^{140}\text{Ce}^{57+} + \nu_e\), \( M_{\text{GT}} \) is the nuclear matrix element of the Gamow–Teller transition \[^{10}\] , \( \psi_{1s}^{(Z)} \) and \( \psi_{1s}^{(Z-1)} \) are the wave functions of the H–like ions \(^{140}\text{Pr}^{58+}\) and \(^{140}\text{Ce}^{57+}\), respectively, \( Z = 59 \) is the electric charge of the mother nucleus \(^{140}\text{Pr}^{59+}\) and \( \psi_{(1s)^2}^{(Z)} \) is the wave function of the He–like ion \(^{140}\text{Pr}^{57+}\), \( \langle \psi_{1s}^{(Z)} \rangle \) and \( \langle \psi_{1s}^{(Z-1)} \psi_{(1s)^2}^{(Z)} \rangle \) are defined by

\[
\begin{align*}
\langle \psi_{1s}^{(Z)} \rangle &= \frac{\int d^3x \psi_{1s}^{(Z)}(\vec{r}) \rho(\vec{r})}{\int d^3x \rho(\vec{r})}, \\
\langle \psi_{1s}^{(Z-1)} \psi_{(1s)^2}^{(Z)} \rangle &= \frac{\int d^3x_1 d^3x_2 \psi_{1s}^{(Z-1)}(\vec{r}_1) \psi_{(1s)^2}^{(Z)}(\vec{r}_1, \vec{r}_2) \rho(\vec{r}_2)}{\int d^3x \rho(\vec{r}_2)},
\end{align*}
\]

where \( \rho(\vec{r}) \) is the nuclear density \[^{10}\] . Using the Woods–Saxon shape for the nuclear density \( \rho(r) \) and the Dirac wave functions for the H–like ion and the He–like ion \[^{11, 12}\] one gets \( \langle \psi_{1s}^{(Z)} \rangle = \langle \psi_{1s}^{(Z-1)} \psi_{(1s)^2}^{(Z)} \rangle = 1.66/\sqrt{\pi a_B^2} \), where \( a_B = 1/Z \alpha m_e = 897 \) fm is the Bohr radius of the H–like ion \(^{140}\text{Pr}^{58+}\); \( m_e = 0.511 \) MeV is electron mass and \( \alpha = 1/137.036 \) is the fine–structure constant \[^{3}\] .

The \( \beta^+ \)–decay constants of the H–like \(^{140}\text{Pr}^{58+}\) and the He–like \(^{140}\text{Pr}^{57+}\) ions are defined by (see Appendix A)

\[
\begin{align*}
\lambda_{\beta^+}^{(H)} &= \frac{2}{2F+1} \frac{1}{4\pi^3} |M_{\text{GT}}|^2 f(Q_{\beta^+}^H, Z - 1), \\
\lambda_{\beta^+}^{(He)} &= \frac{3}{2I+1} \frac{1}{4\pi^3} |M_{\text{GT}}|^2 f(Q_{\beta^+}^{He}, Z - 1).
\end{align*}
\]

Since the \( Q \)-values of the decays \(^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + e^+ + \nu_e\) and \(^{140}\text{Pr}^{57+} \rightarrow ^{140}\text{Ce}^{57+} + e^+ + \nu_e\) are equal, \( Q_{\beta^+}^H = Q_{\beta^+}^{He} = Q_{\beta^+} = (3396 \pm 6) \) keV, the Fermi integral \( f(Q_{\beta^+}, Z - 1) = (2.21 \pm 0.03) \) MeV\(^5\) is defined by the phase volume of the final states of the decays and the Fermi function, describing the Coulomb repulsion between the positron and the nucleus \(^{140}\text{Ce}^{58+}\) for \( Z = 59 \) \[^{7}\].

The theoretical values of the weak decay constants are defined up to the unknown nuclear matrix element \( M_{\text{GT}} \) \[^{10}\] of the Gamow–Teller transition, which cancels in the ratios. The theoretical ratios of the weak decay constants are given by

\[
R_{EC/\beta^+}^{(H), \text{th}} = \frac{3\pi^2 Q_{\beta^+}^2 |\langle \psi_{1s}^{(Z)} \rangle|^2}{f(Q_{\beta^+}, Z - 1)} = 1.40(4),
\]

\(^1\)In the ratio \( \frac{4}{3} \) of the EC–decay constant \( \lambda_{EC}^{(H)} \), the factors 3 and \( \frac{4}{3} \) are caused by the hyperfine structure of the ground state of the H–like \(^{140}\text{Pr}^{58+}\) ion and the phase volume of the final state of the decay, respectively (see Appendix A).
\[ R_{EC/\beta^+}^{(\text{He}, \text{th})} = \frac{2\pi^2 Q_{\text{He}}^2 |\langle \psi_{1s}^{(Z-1)} \psi_{1s}^{(Z)} \rangle|^2}{f(Q_{\beta^+}, Z - 1)} = 0.94(3), \]
\[ R_{EC/\beta^+}^{(\text{H/He}, \text{th})} = \frac{2I + 1}{2F + 1} \frac{|\langle \psi_{1s}^{(Z)} \rangle|^2}{|\langle \psi_{1s}^{(Z-1)} \psi_{1s}^{(Z)} \rangle|^2} \frac{Q_{\text{He}}^2}{Q_{\text{H}}^2} = 1.50(4), \]

(5)

calculated for \( F = \frac{1}{2} \) and \( I = 1 \). The experimental data on the ratios of the weak decay constants are \[ R_{EC/\beta^+}^{(\text{H})} = 1.36(9) , \]
\[ R_{EC/\beta^+}^{(\text{He})} = 0.96(8) , \]
\[ R_{EC/\beta^+}^{(\text{H/He})} = 1.49(8) . \]

(6)

The theoretical values agree well with the experimental data \[ ]\text{[6]}. \]

3 Conclusion

We have calculated the \( EC \) and \( \beta^+ \) decay constants of the H–like and He-like ions \( ^{140}\text{Pr}^{58+} \) and \( ^{140}\text{Pr}^{57+} \), respectively. Following the standard theory of weak decays of heavy nuclei \[ ]\text{[5] we have expressed the decay constants in terms of the nuclear matrix element } M_{\text{GT}} \text{ of the Gamow–Teller transition. We have shown that the complete set of experimental data on the ratios of weak decay constants of the ions } ^{140}\text{Pr}^{58+} \text{ and } ^{140}\text{Pr}^{57+} \text{ can be explained within the standard theory of weak interactions of heavy ions.} \]

Our theoretical values of the ratios of the decay constants for H–like \( ^{140}\text{Pr}^{58+} \) and He-like \( ^{140}\text{Pr}^{57+} \) ions agree with the experimental data within an accuracy better than 3%. This high precision of the theoretical analysis of the weak decays of the H–like \( ^{140}\text{Pr}^{58+} \) and He–like \( ^{140}\text{Pr}^{57+} \) ions is due to the small number of electrons, the behaviour of which can be described by the solution of the Dirac equation. The dependence of the ratio \( R_{EC/\beta^+} \) on the electron structure of the decaying system is confirmed for both the experimental data and our theoretical analysis of the He–like ion \( ^{140}\text{Pr}^{57+} \). Indeed, for the He–like \( ^{140}\text{Pr}^{57+} \) ion the ratio \( R_{EC/\beta^+}^{(\text{He})} = 0.96(8) \) is smaller than for the H–like \( ^{140}\text{Pr}^{58+} \) ion \( R_{EC/\beta^+}^{(\text{H})} = 1.36(9) \). Of course, such a regularity should be confirmed experimentally by the measurements of \( EC \) and \( \beta^+ \) decays of the Li–like \( ^{140}\text{Pr}^{56+} \) ions.

According to the hyperfine structure of the H–like ion \( ^{140}\text{Pr}^{58+} \) \[ ]\text{[13], the bound electron can be in two states with a total angular momentum } F = \frac{1}{2} \text{ and } F = \frac{3}{2} \text{ with the energy splitting equal to} \]

\[ \Delta E = E_{1s F = \frac{1}{2}} - E_{1s F = \frac{3}{2}} = -2\alpha(\alpha Z)^3 \frac{\mu}{\mu_N} m_e^2 \frac{\mu}{m_p} \left\{ \frac{(1 - \delta)(1 - \varepsilon)}{(1 + \gamma)(1 + 2\gamma)} + x_{\text{rad}} \right\}, \]

(7)

where \( \gamma = \sqrt{1 - (\alpha Z)^2} - 1, \mu = +2.5\mu_N \) is the magnetic moment of the nucleus \( ^{140}\text{Pr}^{59+} \) \[ ]\text{[3], } \mu_N = e/2m_p \text{ is the nuclear magneton, } m_p = 938.27 \text{ MeV/c}^2 \text{ is the proton mass, } \delta \text{ is the nuclear charge distribution correction, } \varepsilon \text{ is the nuclear magnetisation distribution correction (the Bohr–Weisskopf correction } \text{[14] \text{), } x_{\text{rad}} \text{ denotes the radiative correction, calculated to lowest order in } \alpha \text{ and } \alpha Z \text{[15].} \] Numerical estimates, carried our for different
ions by Shabaev et al. [13], show that for the calculation of \( \Delta E \) with an accuracy better than 1\% one can drop the contributions of the corrections \( \delta, \varepsilon \) and \( x_{\text{rad}} \) and get \( \Delta E = -1, 12, \text{eV} \). The lifetime \( \tau_{F=3/2} \) of the hyperfine state of the H–like \( ^{140}\text{Pr}^{58+} \) ion is defined by the radiative transition \( ^{140}\text{Pr}^{58+}_{F=3/2} \rightarrow ^{140}\text{Pr}^{58+}_{F=5/2} + \gamma \) only. It is equal to \( \tau_{F=3/2} = 8.5 \times 10^{-3} \text{sec} \) [6]. Since the lifetime \( \tau_{F=3/2} = 8.5 \times 10^{-3} \text{sec} \) is much shorter than the cooling time of about 2s [6], all H–like \( ^{140}\text{Pr}^{58+}_{F=3/2} \) ions decay into the hyperfine ground states \( ^{140}\text{Pr}^{58+} \).

In our calculation of the \( EC \)–decay of the H–like \( ^{140}\text{Pr}^{58+} \) ion from the hyperfine ground state \( ^{140}\text{Pr}^{58+}_{F=3/2} \), the hyperfine structure is taken into account in the spinorial wave function of the bound electron (see Appendices A and B).

The influence of the hyperfine structure on the probabilities of weak decays of heavy ions at finite temperature has been investigated by Folan and Tsirfinovich [9]. As has been mentioned by Folan and Tsirfinovich [9], the \( EC \)–decay of the H–like ion \( ^{140}\text{Pr}^{58+} \) from the hyperfine state with \( F = 3/2 \) is forbidden by a conservation of angular momentum. This agrees with our analysis (see Appendix B).

Our result for the ratio \( R_{EC/EC}^{(\text{H/He}),\text{th}} = 3/2 \) agrees quantitatively well with that obtained by Patyk et al. [16], who also accounted for the hyperfine structure of the H–like \( ^{140}\text{Pr}^{58+} \) ion, and the estimate carried out by Litvinov et al. [6]. According to Patyk et al. [16] and Litvinov et al. [6], the ratio \( R_{EC/EC}^{(\text{H/He}),\text{th}} = 3/2 \) is fully caused by the statistical factors

\[
R_{EC/EC}^{(\text{H/He})} = \frac{2I + 1}{2F + 1} = \frac{3}{2},
\]

calculated for \( I = 1 \) and \( F = 1/2 \). Unfortunately, this assertion is not completely correct. The result \( R_{EC/EC}^{(\text{H/He}),\text{th}} = 3/2 \) appears only at the neglect of the electron screening of the electric charge of the nucleus \( Z \) in the He–like \( ^{140}\text{Pr}^{57+} \) ion. Having neglected the electron screening of the electric charge of the nucleus \( Z \) one can represent the wave function of the bound \((1s)^2 \) state of the He–like ion in the form of the product of the one–electron Dirac wave functions. In this case \( \langle \psi(Z) \rangle = \langle \psi(Z^{-1}) \psi((1s)^2) \rangle \) and the ratio \( R_{EC/EC}^{(\text{H/He}),\text{th}} \), given by Eq.(5), reduces to Eq.(8). Unlike, our analysis of the weak decays of the H–like and He–like ions Patyk et al. [16] as well as Litvinov et al. [6] did not analyse the contributions of the Coulomb wave functions of the bound electrons averaged over the nuclear density. Nevertheless, such contributions are important for the correct calculation of both the ratio \( R_{EC/EC}^{(\text{H/He})} \) for ions with small electric charges \( Z \) and the ratios \( R_{EC/\beta^+}^{(\text{H})} \) and \( R_{EC/\beta^+}^{(\text{He})} \) (see Eq.(5)), which were calculated by neither Patyk et al. [16] nor Litvinov et al. [6].

**Acknowledgement**

We acknowledge many fruitful discussions with M. Kleber, F. Bosch and Yu. A. Litvinov in the course of this work. One of the authors (A. Ivanov) is grateful to N. I. Troitskaya and V. M. Shabaev for the discussions of properties of heavy ions.
Appendix A: Calculation of weak decay constants of H-like $^{140}\text{Pr}^{58+}$ and He–like $^{140}\text{Pr}^{57+}$ ions

In this Appendix we adduce the detailed calculations of the $EC$ and $\beta^+$ decay constants of the H–like $^{140}\text{Pr}^{58+}$ ion in the hyperfine ground state $^{140}\text{Pr}^{58+}_{F=\frac{1}{2}}$ and the He–like $^{140}\text{Pr}^{57+}_{I=1}$ ion in the ground $(1s)^2$ state. For the calculation of the weak decay constants of H-like and He–like ions we follow [5] and use the Hamiltonian Eq.(A).

$EC$–decay of the H–like $^{140}\text{Pr}^{58+}$ ion

The K–shell electron capture decay (the $EC$–decay) $^{140}\text{Pr}^{58+}_{F=\frac{1}{2}} \rightarrow ^{140}\text{Ce}^{58+}_{I=0} + \nu_e$ describes a transition of the H–like mother ion $^{140}\text{Pr}^{58+}$ from the hyperfine ground state $|F, M_F\rangle$ with $F = 1/2$ and $M_F = \pm 1/2$ into the daughter ion $^{140}\text{Ce}^{58+}$ in the ground state $|I, I_z\rangle = |0, 0\rangle$. The $EC$–decay constant of the H–like $^{140}\text{Pr}^{58+}$ ion is defined by

$$\lambda_{EC}^{(H)} = \frac{1}{2M_m} \frac{1}{2F+1} \sum_{M_F} \int |M_{EC,F,M_F}^{^{140}\text{Pr}^{58+}}|^2 (2\pi)^4 \delta^{(4)}(p_d + k - p_m) \frac{d^3p_d}{(2\pi)^3 2E_d} \frac{d^3k}{(2\pi)^3 2E_\nu},$$

(A-1)

where $p_m$, $p_d$ and $k$ are 4–momenta of the mother ion, the daughter ion and the neutrino, respectively. According to [5], the amplitudes $M_{EC,F,M_F}^{^{140}\text{Pr}^{58+}}$ of the $EC$–decay are equal to

$$M_{EC}^{^{140}\text{Pr}^{58+}_{F=\frac{1}{2}}} = \sqrt{2M_m2E_dE_\nu} \mathcal{M}_{GT}^{EC} \left\{ \sqrt{\frac{2}{3}} [\varphi_{n_{-\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{+\frac{1}{2}}}] \cdot [\varphi_{\nu_{-\frac{1}{2}}}^{\dagger} (1 - \vec{n} \cdot \vec{\sigma}) \varphi_{e_{-\frac{1}{2}}}] \right\} - \sqrt{\frac{1}{6}} [\varphi_{n_{+\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{+\frac{1}{2}}} - \varphi_{n_{-\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{-\frac{1}{2}}}] \cdot [\varphi_{\nu_{-\frac{1}{2}}}^{\dagger} (1 - \vec{n} \cdot \vec{\sigma}) \varphi_{e_{+\frac{1}{2}}}] \} ,$$

$$M_{EC}^{^{140}\text{Pr}^{58+}_{F=-\frac{1}{2}}} = \sqrt{2M_m2E_dE_\nu} \mathcal{M}_{GT}^{EC} \left\{ \sqrt{\frac{2}{3}} [\varphi_{n_{+\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{-\frac{1}{2}}}] \cdot [\varphi_{\nu_{-\frac{1}{2}}}^{\dagger} (1 - \vec{n} \cdot \vec{\sigma}) \varphi_{e_{-\frac{1}{2}}}] \right\} - \sqrt{\frac{1}{6}} [\varphi_{n_{+\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{+\frac{1}{2}}} - \varphi_{n_{-\frac{1}{2}}}^{\dagger} \vec{\sigma} \varphi_{p_{-\frac{1}{2}}}] \cdot [\varphi_{\nu_{-\frac{1}{2}}}^{\dagger} (1 - \vec{n} \cdot \vec{\sigma}) \varphi_{e_{+\frac{1}{2}}}] \} ,$$

(A-2)

where we have used the non–relativistic approximation for nucleons [5], $\sqrt{2/3}$ and $\sqrt{1/6}$ are the Clebsch–Gordan coefficients of the spinorial wave function of the bound electron in the hyperfine state with $F = \frac{1}{2}$, caused by the spinorial wave functions of the nucleus $^{140}\text{Pr}^{59+}$ with spin $I = 1$ and the electron with spin $s = \frac{1}{2}$, $\vec{n} = \vec{k}/E_\nu$ is a unit vector alone the 3–momentum of the neutrino and $\mathcal{M}_{GT}^{EC}$ is the matrix element of the Gamow–Teller transition defined by

$$\mathcal{M}_{GT}^{EC} = -g_A \frac{G_F}{\sqrt{2}} V_{ud} \int d^3x \Psi_d^*(\vec{r})\Psi_m(\vec{r}) \psi_{1s}^{(Z)}(r),$$

(A-3)

where $\Psi_d^*(\vec{r})$ and $\Psi_m(\vec{r})$ are the wave functions of the daughter $^{140}\text{Ce}^{58+}$ and mother $^{140}\text{Pr}^{59+}$ nuclei, respectively, and $\psi^{(Z)}_{1s}(r)$ is the radial wave function of the bound electron in the hyperfine ground state of the H–like $^{140}\text{Pr}^{58+}_{F=\frac{1}{2}}$ ion with electric charge $Z$. It is equal to [11]

$$\psi^{(Z)}(r) = \sqrt{\frac{1}{\pi a^2_B}} \frac{2 + \gamma}{\Gamma(3 + 2\gamma)} \left(\frac{2r}{a_B}\right)^\gamma e^{-r/a_B},$$

(A-4)
where \( \gamma = \sqrt{1 - \alpha^2 Z^2} - 1 \) \([11]\). For numerical calculations we set \( \Psi_d^*(\vec{r}) \Psi_m(\vec{r}) \sim \rho(r) \), where \( \rho(r) \) has the Woods–Saxon shape \([10]\). For the subsequent analysis it is convenient to rewrite the matrix element Eq.(A-3) as follows

\[
\mathcal{M}^{EC}_{GT} = \frac{1}{2\sqrt{2}} \mathcal{M}_{GT} \langle \psi^{(Z)}_{1s} \rangle,
\]  

(A-5)

where we have denoted

\[
\mathcal{M}_{GT} = -2 g_A G_F \int d^3x \rho(r) \langle \psi^{(Z)}_{1s} \rangle \frac{\int d^3x \rho(r) \langle \psi^{(Z)}_{1s} \rangle}{\int d^3x \rho(r)}.
\]  

(A-6)

Since a neutrino is polarised anti–parallel to its spin, i.e. the spinorial wave function is the eigenfunction of the operator \( \vec{n} \cdot \vec{\sigma} \) with the eigenvalue \(-1\) (see \([7, 17]\))

\[
\vec{n} \cdot \vec{\sigma} \varphi_{\nu, -\frac{1}{2}} = -\varphi_{\nu, -\frac{1}{2}},
\]  

(A-7)

a non–zero contribution to the \( EC \)–decay constant comes from the amplitude \( M^{EC}_{\frac{1}{2}, -\frac{1}{2}} \) only

\[
M^{EC}_{\frac{1}{2}, -\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{2M_m 2E_d E_v} \mathcal{M}_{GT} \langle \psi^{(Z)}_{1s} \rangle \left\{ \sqrt{\frac{2}{3}} \left[ \varphi^\dagger_{n,+\frac{1}{2}} \vec{\sigma} \varphi_{p,-\frac{1}{2}} \right] \cdot \left[ \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{e,+\frac{1}{2}} \right] \right\} - \sqrt{\frac{1}{6}} \left[ \varphi^\dagger_{n,+\frac{1}{2}} \vec{\sigma} \varphi_{p,-\frac{1}{2}} - \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{p,-\frac{1}{2}} \right] \cdot \left[ \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{e,-\frac{1}{2}} \right],
\]  

(A-8)

where we have taken into account Eq.(A-7). Since the spinorial matrix elements are equal to

\[
\left[ \varphi^\dagger_{n,+\frac{1}{2}} \vec{\sigma} \varphi_{p,-\frac{1}{2}} \right] \cdot \left[ \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{e,+\frac{1}{2}} \right] = +2,
\]

\[
\left[ \varphi^\dagger_{n,+\frac{1}{2}} \vec{\sigma} \varphi_{p,+\frac{1}{2}} - \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{p,-\frac{1}{2}} \right] \cdot \left[ \varphi^\dagger_{n,-\frac{1}{2}} \vec{\sigma} \varphi_{e,-\frac{1}{2}} \right] = -2,
\]  

(A-9)

the amplitude \( M^{EC}_{\frac{1}{2}, -\frac{1}{2}} \) is

\[
M^{EC}_{\frac{1}{2}, -\frac{1}{2}} = \sqrt{3 \sqrt{2M_m 2E_d E_v} \mathcal{M}_{GT}^{EC} \langle \psi^{(Z)} \rangle}.
\]  

(A-10)

Taking the squared absolute value of the amplitude Eq.(A-10) and substituting into Eq.(A-1) we get

\[
\lambda^{(H)}_{EC} = \frac{3}{2F + 1} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s} \rangle|^2 \int \delta^{(4)}(p_d + k - p_m) \frac{d^3p_d d^3k}{8\pi^2} = \frac{1}{2F + 1} \frac{3}{2} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s} \rangle|^2 \frac{Q_H^2}{\pi},
\]  

(A-11)

where in the ratio \( \frac{3}{2} \) the factors 3 and \( \frac{1}{2} \) are caused by the hyperfine structure of the ground state of the H–like \(^{140}\text{Pr}^{58+}\) ion and the phase volume of the final state of the decay, respectively.

Thus, the \( EC \)–decay constant of the H–like \(^{140}\text{Pr}^{58+}\) ion is

\[
\lambda^{(H)}_{EC} = \frac{1}{2F + 1} \frac{3}{2} |\mathcal{M}_{GT}|^2 |\langle \psi^{(Z)}_{1s} \rangle|^2 \frac{Q_H^2}{\pi},
\]  

(A-12)

where \( Q_H = (3348 \pm 6) \) keV is the \( Q \)–value of the \( EC \)–decay of the H–like \(^{140}\text{Pr}^{58+}\) ion.
$\beta^+\text{–decay of H–like }^{140}\text{Pr}^{58+}$ ion

The $\beta^+\text{–decay }^{140}\text{Pr}^{58+}$ in the ground state $|F, M_F\rangle$ with $F = 1/2$ and $M_F = \pm 1/2$ into the H–like daughter ion $^{140}\text{Ce}^{57+}$ in the ground state $|F', M'_F\rangle$ with $F' = 1/2$ and $M'_F = \pm 1/2$. The $\beta^+\text{–decay constant of the H–like }^{140}\text{Pr}^{58+}$ ion is defined by

$$
\lambda_{\beta^+}^{(H)} = \frac{1}{2M_m} \frac{1}{2F + 1} \sum_{M_{F':M_F'=\mp\pm}} \int |M^\beta_{F,M_F=F',M'_F}|^2 F(Z - 1, E_+) \times (2\pi)^4 \delta^{(4)}(p_d + k + p_+ - p_m) \frac{d^3p_d}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3p_+}{(2\pi)^3} F^p_{58+} \delta_{M'_F, M_F}, \quad (A-13)
$$

where $p_m, p_d, k$ and $p_+$ are 4–momenta of the mother ion, the daughter ion, the positron, respectively, $F(Z - 1, E_+) = 2E_d/(2\pi)^3$ is the Fermi function [7].

The amplitudes of the $\beta^+\text{–decay }^{140}\text{Pr}^{58+}$ in the ground state $|F, M_F\rangle$ are equal to

$$
M^{\beta+}_{F = 1/2, M_F = \pm 1/2} = \sqrt{2M_m 2E_d} \mathcal{M}^{\beta+}_{GT} \left\{ \frac{2}{3} \left[ \varphi_{n_+, \pm \frac{1}{2} \sigma} \varphi_{p_+, \pm \frac{1}{2}} \right] \cdot \left[ \bar{u}_\nu(k, -\frac{1}{2}) \gamma (1 - \gamma^5) v_e + (\bar{p}_+, \sigma_+) \right] \times \delta_{M'_F = \pm 1/2, M_F}, \quad (A-14)
$$

where $M'_F = \pm 1/2$ defines a polarisation of the electron in the final state. The matrix element of the Gamow–Teller transition $\mathcal{M}^{\beta+}_{GT}$ is defined by

$$
\mathcal{M}^{\beta+}_{GT} = -g_A \frac{G_F}{\sqrt{2}} V_{ud} \langle \psi^{(Z-1)}_{1s} | \psi^{(Z)}_{1s} \rangle \int d^3x \rho(r) = \frac{1}{2\sqrt{2}} \mathcal{M}^{\beta+}_{GT} \langle \psi^{(Z-1)}_{1s} | \psi^{(Z)}_{1s} \rangle, \quad (A-15)
$$

where $\langle \psi^{(Z-1)}_{1s} | \psi^{(Z)}_{1s} \rangle$ is the matrix element of the transition of the bound electron from the initial state with the wave function $\psi^{(Z)}_{1s}$ into the final state with the wave function $\psi^{(Z-1)}_{1s}$. Since for $Z = 59$ this matrix element is equal to unity with a good accuracy, we set $\langle \psi^{(Z-1)}_{1s} | \psi^{(Z)}_{1s} \rangle = 1$ and $\mathcal{M}^{\beta+}_{GT} = \mathcal{M}^{\beta+}_{GT}/2\sqrt{2}$, where $\mathcal{M}^{\beta+}_{GT}$ is defined by Eq. (A-6). Having calculated the proton–neutron matrix elements

$$
\begin{align}
[\varphi_{n_+, \pm \frac{1}{2} \sigma} \sigma \varphi_{p_+, \pm \frac{1}{2}}] &= \delta_{1j} - i \delta_{2j}, \\
[\varphi_{n_+, \pm \frac{1}{2} \sigma} \sigma \varphi_{p_+, \pm \frac{1}{2}}] &= \delta_{1j} + i \delta_{2j}, \\
[\varphi_{n_+, \pm \frac{1}{2} \sigma} \sigma \varphi_{p_+, \pm \frac{1}{2}} - \varphi_{n_+, \pm \frac{1}{2} \sigma} \sigma \varphi_{p_+, \pm \frac{1}{2}}] &= 2 \delta_{3j}
\end{align}
$$

(A-16)
we get

\[ M^\beta_{\frac{1}{2} + \frac{1}{2}} = \sqrt{2M_m2E_d} \frac{M_{\text{GT}}}{2\sqrt{2}} \sqrt{\frac{2}{3}} \left\{ [\bar{u}_\nu(k, -\frac{1}{2})/(\gamma_1 - i\gamma_2)(1 - \gamma_5) v_+ + (\vec{p}_+, \sigma_+)] \delta_{M_F, -\frac{1}{2}} - [\bar{u}_\nu(k, -\frac{1}{2})/(\gamma_1 - i\gamma_2)(1 - \gamma_5) v_+ + (\vec{p}_+, \sigma_+)] \delta_{M_{F'}, \frac{1}{2}} \right\}, \]

\[ M^\beta_{\frac{1}{2} - \frac{1}{2}} = \sqrt{2M_m2E_d} \frac{M_{\text{GT}}}{2\sqrt{2}} \sqrt{\frac{2}{3}} \left\{ [\bar{u}_\nu(k, -\frac{1}{2})/(\gamma_1 + i\gamma_2)(1 - \gamma_5) v_+ + (\vec{p}_+, \sigma_+)] \delta_{M_F, \frac{1}{2}} - [\bar{u}_\nu(k, -\frac{1}{2})/(\gamma_1 + i\gamma_2)(1 - \gamma_5) v_+ + (\vec{p}_+, \sigma_+)] \delta_{M_{F'}, -\frac{1}{2}} \right\}. \]

The squared absolute values of the amplitudes, summed over the polarisations of the final state are equal to

\[
\sum_{M_{F'} = \pm 1/2} \left| M^\beta_{\frac{1}{2} + \frac{1}{2}} \right|^2 = 2M_m2E_d \frac{|M_{\text{GT}}|^2}{8} \frac{2}{3} \left\{ \text{tr}\{\hat{k}\gamma_3(1 - \gamma_5) (\hat{p}_+ - m_e)(1 - \gamma_5) \gamma_3(1 - \gamma_5) \} \right\} + \text{tr}\{\hat{k}(\gamma_1 + i\gamma_2)(1 - \gamma_5)(\hat{p}_+ - m_e)(\gamma_1 - i\gamma_2)(1 - \gamma_5) \} \}
\]

\[
\sum_{M_{F'} = \pm 1/2} \left| M^\beta_{\frac{1}{2} - \frac{1}{2}} \right|^2 = 2M_m2E_d \frac{|M_{\text{GT}}|^2}{8} \frac{2}{3} \left\{ \text{tr}\{\hat{k}\gamma_3(1 - \gamma_5) (\hat{p}_+ - m_e)(1 - \gamma_5) \gamma_3(1 - \gamma_5) \} \right\} + \text{tr}\{\hat{k}(\gamma_1 - i\gamma_2)(1 - \gamma_5)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma_5) \}. \]

The results of the calculation of the traces over Dirac matrices are given by

\[
\text{tr}\{\hat{k}(\gamma_1 - i\gamma_2)(1 - \gamma_5)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma_5) \} = 2 \text{tr}\{\hat{k}(\gamma_1 - i\gamma_2)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma_5) \} = 2 \text{tr}\{\hat{k}(\gamma_1 - i\gamma_2)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma_5) \} = 8 \left( 2k_xp_{+x} + k \cdot p_+ \right) + 8 \left( 2k_yp_{+y} + k \cdot p_+ \right) + 8i \left( k_xp_{+y} + k_yp_{+x} \right) - 8i \left( k_yp_{+x} + k_xp_{+y} \right) = 16 \left( E_\nu E_+ - k_zp_{+z} \right),
\]

\[
\text{tr}\{\hat{k}(\gamma_1 + i\gamma_2)(1 - \gamma_5)(\hat{p}_+ - m_e)(\gamma_1 - i\gamma_2)(1 - \gamma_5) \} = 8 \left( 2k_xp_{+x} + k \cdot p_+ \right) + 8 \left( 2k_yp_{+y} + k \cdot p_+ \right) + 8i \left( k_xp_{+y} + k_yp_{+x} \right) - 8i \left( k_yp_{+x} + k_xp_{+y} \right) = 16 \left( E_\nu E_+ - k_zp_{+z} \right),
\]

\[
\text{tr}\{\hat{k}\gamma_3(1 - \gamma_5)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma_5) \} = 8 \left( E_\nu E_+ - k_zp_{+z} \right).
\]

Using Eq. (A-19), for the sum of the squared absolute values of the amplitudes Eq. (A-18) we obtain the following expression

\[
\sum_{M_F, M_{F'} = \pm 1/2} \left| M^\beta_{F, M_{F'} \rightarrow F', M_{F'}} \right|^2 = 2M_m2E_d \left| M_{\text{GT}} \right|^2 4 \left( E_\nu E_+ - \frac{1}{3} k_zp_{+z} \right). \]  

(A-20)

Substituting Eq. (A-20) into Eq. (A-13) and integrating over the phase volume of the final state we get

\[
\lambda^{(H)}_{\beta_+} = \frac{1}{2F + 1} \left| M_{\text{GT}} \right|^2 \int F(Z - 1, E_+) \delta^{(4)}(p_d + k + p_+ - p_m) \frac{d^3p_d d^3kd^3p_+}{32\pi^5} = \frac{1}{2F + 1} \left| M_{\text{GT}} \right|^2 \frac{f(Q_{\beta_+}, Z - 1)}{4\pi^3}, \]

(A-21)
where \( f(Q_{\beta^+}, Z - 1) \) is defined by [7]

\[
f(Q_{\beta^+}, Z - 1) = \int_{m_e}^{Q_{\beta^+} - m_e} (Q_{\beta^+} - m_e - E_+)^2 \sqrt{E_+^2 - m_e^2} F(Z - 1, E_+) \, E_+ \, dE_+.
\] (A-22)

Thus, the \( \beta^+ \)-decay constant of the H–like \(^{140}\text{Pr}^{58+}\) ion is equal to

\[
\lambda^{(\text{H})}_{\beta^+} = \frac{2}{2F + 1} \frac{|M_{\text{GT}}|^2}{4\pi^3} f(Q_{\beta^+}, Z - 1)
\] (A-23)

and \( Q_{\beta^+} = (3396 \pm 6) \) keV is the \( Q \)-value of the \( \beta^+ \)-decay of the H–like \(^{140}\text{Pr}^{58+}\) ion.

**EC–decay of He–like \(^{140}\text{Pr}^{57+}\) ion**

The K–shell electron capture decay (the EC–decay) \(^{140}\text{Pr}^{57+}_{I=1} \rightarrow ^{140}\text{Ce}^{57+}_{F=\frac{1}{2}} + \nu_e\) describes a transition of the He–like mother ion \(^{140}\text{Pr}^{57+}\) from the ground \((1s)\) state \(|I, I_z\rangle\) with \( I = 1 \) and \( I_z = 0, \pm 1 \) into the H–like daughter ion \(^{140}\text{Ce}^{57+}\) in the ground state \(|F, M_F\rangle\) with \( F = 1/2 \) and \( M_F = \pm 1/2 \). The EC–decay constant is defined by

\[
\lambda^{(\text{He})}_{EC} = \frac{1}{2M_m} \frac{1}{2I + 1} \sum_{I_z, M_F} |M^{EC}_{I, I_z = -F, M_F}|^2 (2\pi)^4 \delta(4) (p_d + k - p_m) \frac{d^3p_d}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}.\]
(A-24)

The amplitudes of the EC–decay \(M^{EC}_{I, I_z = -F, M_F}\) are equal to

\[
M^{EC}_{1, + 1 \rightarrow \pm 1, + \frac{1}{2}} = \sqrt{2M_m 2E_d E\nu} \tilde{\mathcal{N}}_{\text{GT}}^{(57+)} \frac{1}{2} \left[ \varphi_{\nu, -\frac{1}{2}}^\dagger (1 - \vec{n} \cdot \vec{\sigma}) \vec{\sigma} \varphi_{\nu, -\frac{1}{2}} - \varphi_{\nu, -\frac{1}{2}}^\dagger \varphi_{\nu, -\frac{1}{2}} \right],
\]

\[
M^{EC}_{1, + 1 \rightarrow -1, - \frac{1}{2}} = \sqrt{2M_m 2E_d E\nu} \tilde{\mathcal{N}}_{\text{GT}}^{(57+)} \frac{1}{2} \left[ \varphi_{\nu, -\frac{1}{2}}^\dagger (1 - \vec{n} \cdot \vec{\sigma}) \vec{\sigma} \varphi_{\nu, -\frac{1}{2}} - \varphi_{\nu, -\frac{1}{2}}^\dagger \varphi_{\nu, -\frac{1}{2}} \right],
\]

\[
M^{EC}_{1, -1 \rightarrow \pm 1, + \frac{1}{2}} = \sqrt{2M_m 2E_d E\nu} \tilde{\mathcal{N}}_{\text{GT}}^{(57+)} \frac{1}{2} \left[ \varphi_{\nu, -\frac{1}{2}}^\dagger (1 - \vec{n} \cdot \vec{\sigma}) \vec{\sigma} \varphi_{\nu, -\frac{1}{2}} - \varphi_{\nu, -\frac{1}{2}}^\dagger \varphi_{\nu, -\frac{1}{2}} \right],
\]

\[
M^{EC}_{1, -1 \rightarrow -1, - \frac{1}{2}} = \sqrt{2M_m 2E_d E\nu} \tilde{\mathcal{N}}_{\text{GT}}^{(57+)} \frac{1}{2} \left[ \varphi_{\nu, -\frac{1}{2}}^\dagger (1 - \vec{n} \cdot \vec{\sigma}) \vec{\sigma} \varphi_{\nu, -\frac{1}{2}} - \varphi_{\nu, -\frac{1}{2}}^\dagger \varphi_{\nu, -\frac{1}{2}} \right],
\]

(A-25)

where \( \vec{n} = \vec{k}/E\nu \) is a unit vector alone the 3–momentum of the neutrino and we have denoted

\[
\tilde{\mathcal{N}}_{\text{GT}}^{(57+)} = \frac{M_{\text{GT}}}{2\sqrt{2}} \langle \psi_{1s}^{(Z-1)} \psi_{1s}^{(Z)} \rangle,
\]

\[
\langle \psi_{1s}^{(Z-1)} \psi_{1s}^{(Z)} \rangle = \frac{\int d^3x_1 d^3x_2 \psi_{1s}^{(Z-1)}(\vec{r}_1) \psi_{1s}^{(Z)}(\vec{r}_1, \vec{r}_2) \rho(r_2)}{\int d^3x_2 \rho(r_2)},\]

(A-26)
where $\psi^{(Z)}_{(1s)^2}(\vec{r}_1, \vec{r}_2)$ is the wave function of the ground $(1s)^2$ state of the He–like $^{140}\text{Pr}^{57+}$ ion and $\mathcal{M}_{\text{GT}}$ is the matrix element of the Gamow–Teller transition defined by Eq. (A-6).

For the amplitudes of the $EC$–decay of the He–like $^{140}\text{Pr}^{57+}$ ion we get

$$
M_{1,1-\frac{1}{2},-\frac{1}{2}}^{EC} = M_{1,0-\frac{1}{2},-\frac{1}{2}}^{EC} = M_{1,-1-\frac{1}{2},\frac{1}{2}}^{EC} = 0,
$$

$$
M_{1,0-\frac{1}{2},-\frac{1}{2}}^{EC} = \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} \frac{1}{\sqrt{2}} \left[ \varphi_{n,\frac{1}{2}}^{+} \tilde{\sigma} \varphi_{p,-\frac{1}{2}} - \varphi_{n,-\frac{1}{2}}^{+} \tilde{\sigma} \varphi_{p,\frac{1}{2}} \right]
= \left[ \varphi_{\nu,-\frac{1}{2}}^{+} \left( 1 - \tilde{n} \cdot \tilde{\sigma} \right) \tilde{\sigma} \varphi_{e,-\frac{1}{2}} \right],
$$

$$
M_{1,-1-\frac{1}{2},\frac{1}{2}}^{EC} = \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} \left[ \varphi_{n,\frac{1}{2}}^{+} \tilde{\sigma} \varphi_{p,-\frac{1}{2}} \right] \left[ \varphi_{\nu,-\frac{1}{2}}^{+} \left( 1 - \tilde{n} \cdot \tilde{\sigma} \right) \tilde{\sigma} \varphi_{e,+\frac{1}{2}} \right].
$$

Having calculated the spinorial matrix elements we obtain

$$
M_{1,1-\frac{1}{2},-\frac{1}{2}}^{EC} = M_{1,0-\frac{1}{2},-\frac{1}{2}}^{EC} = M_{1,-1-\frac{1}{2},\frac{1}{2}}^{EC} = 0,
$$

$$
M_{1,0-\frac{1}{2},-\frac{1}{2}}^{EC} = -2 \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} = -\sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} \left( \psi^{(Z-1)}_{1s} \psi^{(Z)}_{(1s)^2} \right),
$$

$$
M_{1,-1-\frac{1}{2},\frac{1}{2}}^{EC} = 4 \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} = \sqrt{2} \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}}^{EC} \left( \psi^{(Z-1)}_{1s} \psi^{(Z)}_{(1s)^2} \right),
$$

where we have used Eq. (A-26). As a result the $EC$–decay constant of the He–like $^{140}\text{Pr}^{57+}$ ion is defined by

$$
\lambda_{EC}^{(\text{He})} = \frac{1}{2 E_m} \int \left( |M_{1,0-\frac{1}{2},-\frac{1}{2}}^{EC}|^2 + |M_{1,-1-\frac{1}{2},\frac{1}{2}}^{EC}|^2 \right) (2\pi)^4 \delta^{(4)}(p_d + k - p_m)
\times \frac{d^3 p_d}{(2\pi)^3 2E_d} \frac{d^3 k}{(2\pi)^3 2E_{\nu}} = \frac{1}{2 I + 1} \frac{3}{2} |\mathcal{M}_{\text{GT}}|^2 |\psi^{(Z-1)}_{1s} \psi^{(Z)}_{(1s)^2}|^2 \frac{Q_{\text{He}}^2}{\pi}.
$$

Thus, the $EC$–decay constant of the He–like $^{140}\text{Pr}^{57+}$ ion is equal to

$$
\lambda_{EC}^{(\text{He})} = \frac{1}{2 I + 1} \frac{3}{2} |\mathcal{M}_{\text{GT}}|^2 |\psi^{(Z-1)}_{1s} \psi^{(Z)}_{(1s)^2}|^2 \frac{Q_{\text{He}}^2}{\pi},
$$

where $Q_{\text{He}} = (3352 \pm 6)$ keV is the $Q$–value of the $EC$–decay of the He–like $^{140}\text{Pr}^{57+}$ ion.

**$\beta^+$–decay of the He–like $^{140}\text{Pr}^{57+}$ ion**

The $\beta^+$–decay $^{140}\text{Pr}^{57+}_{I=1} \rightarrow ^{140}\text{Ce}^{56+}_{I'_{1s}} + e^+ + \nu_e$ describes a transition of the He–like mother ion $^{140}\text{Pr}^{57+}_{I=1}$ from the ground $(1s)^2$ state $|I, I_z\rangle$ with $I = 1$ and $I_z = 0, \pm 1$ into the He–like daughter ion $^{140}\text{Ce}^{56+}_{I'_{1s}}$ in the ground $(1s)^2$ state $|I', I'_z\rangle$ with $I' = 0$ and $I'_z = 0$. The $\beta^+$–decay constant of the He–like $^{140}\text{Pr}^{57+}$ ion is defined by

$$
\lambda_{\beta^+}^{(\text{He})} = \frac{1}{2 E_m} \frac{1}{2 I + 1} \sum_{I_z = 0, \pm 1} \int |M_{I, I_z = 0, \pm 1}^{\beta^+}|^2 (2\pi)^4 \delta^{(4)}(p_d + k + p_+ - p_m) F(Z - 1, E_+)
\times \frac{d^3 p_d}{(2\pi)^3 2E_d} \frac{d^3 k}{(2\pi)^3 2E_{\nu}} \frac{d^3 p_+}{(2\pi)^3 2E_+}.
$$

Using the results obtained above and following [5, 7, 17], for the amplitudes $M_{I, I_z = 0, \pm 1}^{\beta^+}$ of the $\beta^+$–decay $^{140}\text{Pr}^{57+}_{I=1} \rightarrow ^{140}\text{Ce}^{56+}_{I'_{1s}} + e^+ + \nu_e$ we obtain the following expressions

$$
M_{I, I_z = 0, \pm 1}^{\beta^+} = \sqrt{2} M_{2E_d E_{\nu}} \mathcal{M}_{\text{GT}} \frac{1}{2 \sqrt{2}} \left[ \varphi_{n,\frac{1}{2}}^{+} \tilde{\sigma} \varphi_{p,-\frac{1}{2}} \right] \left[ \tilde{u}_\nu(k, -\frac{1}{2}) \gamma(1 - \gamma^5)\nu_e \left( \vec{p}_+, \sigma_+ \right) \right],
$$

11
Substituting Eq. (A-35) into Eq. (A-31) and integrating over the phase volume of the final state we get

\[
M_{1,0-00}^{\beta^+} = \sqrt{2M_mE_d} \frac{M_{GT}}{2\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \phi_{n,+}^\dagger \bar{\phi}_{p,+} - \phi_{n,-}^\dagger \bar{\phi}_{p,-} \right] \\
\cdot \left[ \bar{u}_\nu(\vec{k}, -\frac{1}{2}) \gamma_1 (1 - \gamma^5) v_{e+}(\vec{p}_+,\sigma_+) \right],
\]

\[
M_{1,-1-00}^{\beta^+} = \sqrt{2M_mE_d} \frac{M_{GT}}{2\sqrt{2}} \frac{1}{\sqrt{2}} \left[ \phi_{n,+}^\dagger \bar{\phi}_{p,-} \right] \\
\cdot \left[ \bar{u}_\nu(\vec{k}, -\frac{1}{2}) \gamma_1 (1 - \gamma^5) v_{e+}(\vec{p}_+,\sigma_+) \right].
\]

(A-32)

Using Eq. (A-16) for the calculation of the proton–neutron matrix elements we get

\[
M_{1,+1-00}^{\beta^+} = \sqrt{2M_mE_d} \frac{M_{GT}}{2\sqrt{2}} \left[ \bar{u}_\nu(\vec{k}, -\frac{1}{2}) (\gamma_1 - i\gamma_2) (1 - \gamma^5) v_{e+}(\vec{p}_+,\sigma_+) \right],
\]

\[
M_{1,0-00}^{\beta^+} = \sqrt{2M_mE_d} \frac{M_{GT}}{2\sqrt{2}} \left[ \bar{u}_\nu(\vec{k}, -\frac{1}{2}) \gamma_3 (1 - \gamma^5) v_{e+}(\vec{p}_+,\sigma_+) \right],
\]

\[
M_{1,-1-00}^{\beta^+} = \sqrt{2M_mE_d} \frac{M_{GT}}{2\sqrt{2}} \left[ \bar{u}_\nu(\vec{k}, -\frac{1}{2}) (\gamma_1 + i\gamma_2) (1 - \gamma^5) v_{e+}(\vec{p}_+,\sigma_+) \right].
\]

(A-33)

The squared absolute values of the amplitudes \( M_{1,+1-00}^{\beta^+}, M_{1,0-00}^{\beta^+} \) and \( M_{1,-1-00}^{\beta^+} \) are

\[
|M_{1,+1-00}^{\beta^+}|^2 = 2M_mE_d \frac{|M_{GT}|^2}{8} \text{tr}\{\hat{k}(1 + i\gamma_2)(1 - \gamma^5)(\hat{p}_+ - m_e)(\gamma_1 - i\gamma_2)(1 - \gamma^5)\} =
\]

\[
= 2M_mE_d |M_{GT}|^2 \left( E_\nu E_+ - k_z p_{+z} \right),
\]

\[
|M_{1,0-00}^{\beta^+}|^2 = 2M_mE_d \frac{|M_{GT}|^2}{8} \left[ 2 \text{tr}\{\hat{k}\gamma_3(1 - \gamma^5)(\hat{p}_+ - m_e)\gamma_3(1 - \gamma^5)\} \right] =
\]

\[
= 2M_mE_d |M_{GT}|^2 \left( E_\nu E_+ + k_z p_{+z} \right),
\]

\[
|M_{1,-1-00}^{\beta^+}|^2 = 2M_mE_d \frac{|M_{GT}|^2}{8} \text{tr}\{\hat{k}(1 - i\gamma_2)(1 - \gamma^5)(\hat{p}_+ - m_e)(\gamma_1 + i\gamma_2)(1 - \gamma^5)\} =
\]

\[
= 2M_mE_d |M_{GT}|^2 \left( E_\nu E_+ - k_z p_{+z} \right),
\]

(A-34)

where we have used Eq. (A-19). The sum of the squared absolute values of the amplitudes Eq. (A-34) is

\[
\sum_{l_z} |M_{l_z}^{\beta^+}|^2 = 2M_mE_d |M_{GT}|^2 6 \left( E_\nu E_+ - \frac{1}{3} k_z p_{+z} \right).
\]

(A-35)

Substituting Eq. (A-35) into Eq. (A-31) and integrating over the phase volume of the final state we get

\[
\lambda_{\beta^+}^{(He)} = \frac{3}{2I + 1} |M_{GT}|^2 \int \delta^{(4)}(p_d + k + p_+ - p_m) F(Z - 1, E_+) \frac{d^3 p_d d^3 k d^3 p_+}{64\pi^5} =
\]

\[
= \frac{3}{2I + 1} |M_{GT}|^2 \frac{f(Q_{\beta^+}, Z - 1)}{4\pi^3},
\]

(A-36)

where \( f(Q_{\beta^+}, Z - 1) \) is defined by Eq. (A-22). Thus, the \( \beta^+ \)–decay constant of the He–like \( ^{140}\text{P}^{57+} \) ion is equal to

\[
\lambda_{\beta^+}^{(He)} = \frac{3}{2I + 1} \frac{|M_{GT}|^2}{4\pi^3} f(Q_{\beta^+}, Z - 1),
\]

(A-37)

where \( Q_{\beta^+} = (3396 \pm 6) \text{ keV} \) is the \( Q \)–value of the \( \beta^+ \)–decay of the He–like \( ^{140}\text{P}^{57+} \) ion.
Appendix B: Calculation of the EC–decay constant of the H-like $^{140}$Pr$^{58+}$ ion from the hyperfine state $^{140}$Pr$^{58+}_{F=3/2}$

The calculation of the EC–decay constant of the H–like $^{140}$Pr$^{58+}$ ion from the hyperfine state $^{140}$Pr$^{58+}_{F=3/2}$ with $F = \frac{3}{2}$ is similar to that for the EC–decay from the hyperfine ground state $^{140}$Pr$^{58+}_{F=1/2}$ with $F = \frac{1}{2}$. The EC–decay constant is defined by

$$\lambda_{EC}^{(H)} = \frac{1}{2M_n} \frac{1}{2F+1} \sum_{M_p=\pm \frac{1}{2}, \frac{3}{2}} |M_{EC,F,M_p}^{(EC)}|^2 (2\pi)^4 \delta^{(4)} (p_d + k - p_m) \frac{d^3p_d}{(2\pi)^3 2E_d} \frac{d^3k}{(2\pi)^3 2E_\nu},$$

(B-1)

where $p_m, p_d$ and $k$ are 4–momenta of the mother ion, the daughter ion and the neutrino, respectively. The amplitudes $M_{EC,F,M_p}^{(EC)}$ of the EC–decay are given by

$$M_{\frac{3}{2}+, \frac{1}{2}}^{EC} = 2 \sqrt{2M_n2E_dE_\nu} M_{GT}^{EC} \{\varphi_{n, \frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \}
+ \sqrt{\frac{1}{3}} \left[ \varphi_{n, +\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}} \varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \]

M_{\frac{3}{2}-, \frac{1}{2}}^{EC} = 2 \sqrt{2M_n2E_dE_\nu} M_{GT}^{EC} \{\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \}
+ \sqrt{\frac{1}{3}} \left[ \varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}} \varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \]

M_{\frac{3}{2}, \frac{1}{2}-}^{EC} = 2 \sqrt{2M_n2E_dE_\nu} M_{GT}^{EC} \{\varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, +\frac{3}{2} \sigma \varphi_{e, -\frac{1}{2}}} \}
+ \sqrt{\frac{1}{3}} \left[ \varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}} \varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, +\frac{3}{2} \sigma \varphi_{e, -\frac{1}{2}}} \]

M_{\frac{3}{2}, \frac{1}{2}}^{EC} = 2 \sqrt{2M_n2E_dE_\nu} M_{GT}^{EC} \{\varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \}
+ \sqrt{\frac{1}{3}} \left[ \varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}} \varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \]

(B-2)

Since the spinorial matrix elements vanish

$$[\varphi_{n, -\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

$$[\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

$$[\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}} \varphi_{n, -\frac{1}{2} \sigma \varphi_{p, -\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

$$[\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

$$[\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, +\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

$$[\varphi_{n, +\frac{1}{2} \sigma \varphi_{p, +\frac{1}{2}}} \cdot [\varphi_{\nu, -\frac{3}{2} \sigma \varphi_{e, +\frac{1}{2}}} \] = 0,$$

(B-3)

the amplitudes Eq.(B-2) of the EC–decay from the hyperfine state $^{140}$Pr$^{58+}_{F=3/2}$ are equal to zero

$$M_{\frac{3}{2}+, \frac{1}{2}}^{EC} = M_{\frac{3}{2}+, \frac{3}{2}}^{EC} = M_{\frac{3}{2}, \frac{1}{2}+}^{EC} = M_{\frac{3}{2}, \frac{1}{2}+}^{EC} = 0.$$

(B-4)

This confirms the suppression of the EC–decay of the H–like $^{140}$Pr$^{58+}$ ion from the hyperfine state $^{140}$Pr$^{58+}_{F=3/2}$ pointed out in [6, 9].
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[10] In our approach the matrix element of the Gamow–Teller transition is defined by

\[ \mathcal{M}_{\text{GT}} = -2g_A G_F V_{ud} \int d^3x \Psi_d^*(r) \Psi_m(r), \]

where \( \Psi_d^*(r) \) and \( \Psi_m(r) \) are the wave functions of the daughter and mother nuclei, respectively. We set \( \Psi_d^*(r) \Psi_m(r) \sim \rho(r) \), where \( \rho(r) \) has a Woods–Saxon shape with \( R = 1.1 \ A^{1/3} \) fm and slope parameter \( a = 0.50 \) fm taken from W. N. Cottingham and D. A. Greenwood, in An introduction to nuclear physics, Cambridge University Press, Second Edition, 2001.

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