On the joint distribution of cyclic valleys and excedances over conjugacy classes of $\mathfrak{S}_n$

M. Crossan Cooper William S. Jones Yan Zhuang
Department of Mathematics and Computer Science
Davidson College
{crcooper, wijones, yazhuang}@davidson.edu
January 17, 2020

Abstract

We derive a formula expressing the joint distribution of the cyclic valley number and excedance number statistics over a fixed conjugacy class of the symmetric group in terms of Eulerian polynomials. Our proof uses a slight extension of Sun and Wang’s cyclic valley-hopping action as well as a formula of Brenti. Along the way, we give a new proof for the $\gamma$-positivity of the excedance number distribution over any fixed conjugacy class along with a combinatorial interpretation of the $\gamma$-coefficients.

Keywords: permutation statistics, excedances, cyclic valleys, Eulerian polynomials, $\gamma$-positivity, modified Foata–Strehl action

1. Introduction

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ be a permutation in the symmetric group $\mathfrak{S}_n$. We say that $i \in [n-1]$ is a descent of $\pi$ if $\pi(i) > \pi(i+1)$ and that $i \in [n]$ is an excedance of $\pi$ if $i < \pi(i)$.\footnote{The set $[n]$ is defined by $[n] := \{1, 2, \ldots, n\}$.}

We let des($\pi$) denote the number of descents of $\pi$ and exc($\pi$) the number of excedances of $\pi$. For example, if $\pi = 371896542$, then the descents of $\pi$ are 2, 5, 6, 7, and 8 whereas the excedances of $\pi$ are 1, 2, 4, and 5; thus des($\pi$) = 5 and exc($\pi$) = 4. It is well known that the descent number des and the excedance number exc have the same distribution over $\mathfrak{S}_n$, that is, the number of permutations in $\mathfrak{S}_n$ with exactly $k$ descents is equal to the number of permutations in $\mathfrak{S}_n$ with exactly $k$ excedances.

Given a polynomial $f$ in the variable $t$, we say that $f$ is $\gamma$-positive with center of symmetry $n/2$ if we can write

$$f(t) = \sum_{i=0}^{[n/2]} \gamma_i t^i (1 + t)^{n-2i}$$

2010 Mathematics Subject Classification. Primary 05A15; Secondary 05A05, 05E18.
for some non-negative integers $\gamma_i$ (called the $\gamma$-coefficients of $f$). If a polynomial is $\gamma$-positive, then its sequence of coefficients is symmetric and unimodal. The prototypical example of a family of $\gamma$-positive polynomials are the Eulerian polynomials $\{A_n(t)\}_{n \geq 0}$ defined by

$$A_n(t) := \sum_{\pi \in S_n} t^{\text{des}(\pi)+1} = \sum_{\pi \in S_n} t^{\text{exc}(\pi)+1}$$

for $n \geq 1$ and by $A_0(t) := 1$. The $n$th Eulerian polynomial encodes the distribution of the descent number (equivalently, the excedance number) over $S_n$. The $\gamma$-positivity of Eulerian polynomials was proven by Foata and Schützenberger [4] in 1970, long before the term "$\gamma$-positivity" was coined, but the general notion of $\gamma$-positivity has emerged as a powerful way to prove unimodality results and has connections to many facets of enumerative, algebraic, and geometric combinatorics. See [1] for a comprehensive survey on this topic.

The cycle type of a permutation $\pi$ is a partition of $n$ encoding the number of cycles of $\pi$ of each size. Continuing with the earlier example, the permutation $\pi = 371896542$ in one-line notation can be written as $\pi \in S_9$ ([3, 4, 5, 7, 8, 9], 2, 4) in cycle notation, which has cycle type $\lambda = (1, 2, 4, 2)$.

Conjugacy classes of the symmetric group $S_n$ correspond to sets of permutations with a fixed cycle type, and one may investigate distributions of permutation statistics over conjugacy classes. Perhaps the most famous result in this domain is by Gessel and Reutenauer [6], who proved that the number of permutations in a prescribed conjugacy class with a prescribed descent set is equal to the scalar product of a ribbon Schur function and a Lyndon symmetric function (equivalently, the scalar product of the characters of a Foulkes representation and a Lie representation).

We say that $i \in [n]$ is a cyclic valley of $\pi \in S_n$ if $\pi^{-1}(i) > i < \pi(i)$, and we let $\text{cval}(\pi)$ denote the number of cyclic valleys of $\pi$. In this paper, we will study the polynomials

$$E_\lambda(t) := \sum_{\pi \in S_n(\lambda)} t^{\text{exc}(\pi)}, \quad E_\lambda^{\text{val}}(t) := \sum_{\pi \in S_n(\lambda)} t^{\text{cval}(\pi)},$$

and

$$E_\lambda^{(\text{val}, \text{exc})}(s, t) := \sum_{\pi \in S_n(\lambda)} s^{\text{val}(\pi)} t^{\text{exc}(\pi)},$$

where $S_n(\lambda)$ is the conjugacy class of $S_n$ consisting of all permutations with cycle type $\lambda$.

While the polynomials $E_\lambda^{\text{val}}(t)$ and $E_\lambda^{(\text{val}, \text{exc})}(s, t)$ appear to be new, the $E_\lambda(t)$ were studied earlier by Brenti. For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$ of $n$—that is, a partition with $m_i$ parts of size $i$ for each $i$—Brenti [3, Theorem 3.1] proved the formula

$$E_\lambda(t) = \frac{n!}{z_\lambda} \prod_{i \geq 1} \left[ \frac{A_{i-1}(t)}{(i-1)!} \right]^{m_i}$$

(1)

where the constant $z_\lambda$ is defined by $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$. Since Eulerian polynomials are $\gamma$-positive and products of $\gamma$-positive polynomials are $\gamma$-positive [7, Observation 4.1], Brenti’s formula implies that the polynomials $E_\lambda(t)$ are $\gamma$-positive as well.

Our main result is the following formula:
Theorem 1. Let \( \lambda = (1^{m_1}2^{m_2} \cdots) \) be a partition of \( n \). Then

\[
E^{(\text{cval, exc})}_\lambda(s, t) = \frac{n!}{\tilde{z}_\lambda} \left( \frac{1 + u}{1 + uv} \right)^{n-m_1} \prod_{i \geq 1} \left[ \frac{A_{i-1}(v)}{(i-1)!} \right]^{m_i}
\]

where \( u = \frac{1 + t^2 - 2st - (1-t)\sqrt{(1+t)^2 - 4st}}{2(1-st)} \) and \( v = \frac{(1+t)^2 - 2st - (1+t)\sqrt{(1+t)^2 - 4st}}{2st} \).

This formula allows one to compute the joint distribution of the statistics cval and exc over any fixed conjugacy class directly from Eulerian polynomials. For example, take \( \lambda = (1, 5, 5) \).

Then

\[
E^{(\text{cval, exc})}_\lambda(s, t) = 11! \left( \frac{1 + u}{1 + uv} \right)^{10} \frac{A_0(v)A_4(v)^2}{0!4!^2}
\]

\[
= 1386 \left( \frac{1 + u}{1 + uv} \right)^{10} (v + 11v^2 + 11v^3 + v^4)^2
\]

\[
= (1386t^2 + 8316t^3 + 20790t^4 + 27720t^5 + 20790t^6 + 8316t^7 + 1386t^8)s^2
\]

\[
+ (22176t^3 + 88704t^4 + 133056t^5 + 88704t^6 + 22176t^7)s^3
\]

\[
+ (88704t^4 + 177408t^5 + 88704t^6)s^4.
\]

(The last equality can be verified using a computer algebra system such as Maple.)

Remark 2. Notice that, in the above example, the coefficient of \( s^i \) for each \( i \) is a unimodal and symmetric polynomial in \( t \); we will see that this is true for all \( E^{(\text{cval, exc})}_\lambda(s, t) \).

To prove Theorem 1, we will first extend a group action of Sun and Wang \[10\] defined on derangements (permutations without fixed points) to all permutations; we call this action “cyclic valley-hopping". We will then prove a technical lemma (Lemma 5) using cyclic valley-hopping which is in turn utilized to derive a formula expressing the polynomial \( E^{(\text{cval, exc})}_\lambda(s, t) \) in terms of the polynomial \( E_\lambda(t) \). We then combine this formula with Brenti’s formula \[1\] to yield Theorem 1 and derive a similar result (Theorem 13) expressing the polynomial \( E^{(\text{cval})}_\lambda(t) \) in terms of Eulerian polynomials. Along the way, we use Lemma 5 to obtain an alternative proof for the \( \gamma \)-positivity of the polynomials \( E_\lambda(t) \) in a way which yields a combinatorial interpretation for their \( \gamma \)-coefficients and which will explain our observation in Remark 2.

2. Preliminaries

2.1. Permutation statistics

We begin with a brief discussion of several more permutation statistics which will arise in this paper. Given a permutation \( \pi = \pi(1)\pi(2) \cdots \pi(n) \) in \( S_n \), we say that \( \pi(i) \) is:

- a **valley** of \( \pi \) if \( \pi(i-1) > \pi(i) < \pi(i+1) \);
- a **peak** of \( \pi \) if \( \pi(i-1) < \pi(i) > \pi(i+1) \);
- a **double ascent** of \( \pi \) if \( \pi(i-1) < \pi(i) < \pi(i+1) \);

3
• a double descent of $\pi$ if $\pi(i-1) > \pi(i) > \pi(i+1)$.

In our work, we will be more concerned with cyclic analogues of these notions which have also been well-studied, e.g., in [9, 10, 11, 12]. We have already defined excedances and cyclic valleys of $\pi$. We say that $i \in [n]$ is:

• a cyclic peak of $\pi$ if $\pi^{-1}(i) < i > \pi(i)$;
• a cyclic double ascent of $\pi$ if $\pi^{-1}(i) < i < \pi(i)$;
• a cyclic double descent of $\pi$ if $\pi^{-1}(i) > i > \pi(i)$;
• a fixed point of $\pi$ if $\pi(i) = i$.

It is clear that every cycle of size one is a fixed point, and that in any cycle of size at least two, the first letter is a cyclic peak and the last letter is either a cyclic valley or a cyclic double ascent.

Define $\text{Exc}(\pi)$, $\text{Cval}(\pi)$, $\text{Cpk}(\pi)$, $\text{Cdasc}(\pi)$, $\text{Cddes}(\pi)$, and $\text{Fix}(\pi)$ to be the set of excedances, cyclic valleys, cyclic peaks, cyclic double ascents, cyclic double descents, and fixed points, respectively. Moreover, let $\text{cpk}(\pi) := |\text{Cpk}(\pi)|$, $\text{cdasc}(\pi) := |\text{Cdasc}(\pi)|$, $\text{cddes}(\pi) := |\text{Cddes}(\pi)|$, and $\text{fix}(\pi) := |\text{Fix}(\pi)|$.

As an example, take $\pi = (5, 2, 1)(6)(8)(11, 9, 10, 4, 3, 7)$. Here $\text{Exc}(\pi) = \{1, 3, 7, 9\}$, $\text{Cval}(\pi) = \{1, 3, 9\}$, $\text{Cpk}(\pi) = \{5, 10, 11\}$, $\text{Cdasc}(\pi) = \{7\}$, $\text{Cddes}(\pi) = \{2, 4\}$, and $\text{Fix}(\pi) = \{6, 8\}$. Thus $\text{exc}(\pi) = 4$, $\text{cval}(\pi) = 3$, $\text{cpk}(\pi) = 3$, $\text{cdasc}(\pi) = 1$, $\text{cddes}(\pi) = 2$, and $\text{fix}(\pi) = 2$.

It is clear from the definitions that every letter of a permutation is either a cyclic valley, cyclic peak, cyclic double ascent, cyclic double descent, or fixed point. Thus, we have

$$\text{Cval}(\pi) \cup \text{Cpk}(\pi) \cup \text{Cdasc}(\pi) \cup \text{Cddes}(\pi) \cup \text{Fix}(\pi) = [n]$$

and

$$\text{cval}(\pi) + \text{cpk}(\pi) + \text{cdasc}(\pi) + \text{cddes}(\pi) + \text{fix}(\pi) = n$$

(2)

for any $\pi \in \mathcal{S}_n$. It is also clear that the excedances of a permutation are precisely its cyclic valleys and cyclic double ascents, that is,

$$\text{Cval}(\pi) \cup \text{Cdasc}(\pi) = \text{Exc}(\pi)$$

(3)

and

$$\text{cval}(\pi) + \text{cdasc}(\pi) = \text{exc}(\pi)$$

(4)

for all $\pi \in \mathcal{S}_n$. Finally, it is not difficult to see that, for any $\pi \in \mathcal{S}_n$, the sets $\text{Cpk}(\pi)$ and $\text{Cval}(\pi)$ are in bijection. Hence, we have

$$\text{cpk}(\pi) = \text{cval}(\pi).$$

(5)

Before continuing, we give a couple remarks on cycle notation. When writing permutations in cycle notation, we adopt the convention of writing each cycle with its largest letter in the first position, and writing the cycles from left-to-right in increasing order of their largest
letters. (This convention is sometimes called canonical cycle representation.) For example, the permutation \( \pi = 649237185 \) in one-line notation is written as \( \pi = (42)(716)(8)(953) \) in cycle notation.

We will make use of a map called Foata’s “transformation fondamentale”; this map \( o : \mathfrak{S}_n \rightarrow \mathfrak{S}_n \) is defined by taking as input a permutation in canonical cycle representation and the output is the permutation in one-line notation obtained by erasing the parentheses. Continuing the example with \( \pi = (42)(716)(8)(953) \), we have \( o(\pi) = 427168953 \). It is easy to see that this map is a bijection; we can recover the cycles of \( o^{-1}(\pi) \) from a permutation \( \pi \) by noting the left-to-right maxima of \( \pi \): letters \( \pi(i) \) for which \( \pi(j) < \pi(i) \) for all \( 1 \leq j < i \).

2.2. Cyclic valley-hopping

Our remaining goal in this preliminary section is to define a group action on \( \mathfrak{S}_n \) induced by involutions which toggle between cyclic double ascents and cyclic double descents. Before we define this group action, it will be convenient to first define two related group actions. Fix a permutation \( \pi \in \mathfrak{S}_n \) and a letter \( x \in [n] \). We may write \( \pi = w_1 w_2 x w_4 w_5 \) where \( w_2 \) is the maximal consecutive subword immediately to the left of \( x \) whose letters are all smaller than \( x \), and \( w_4 \) is the maximal consecutive subword immediately to the right of \( x \) whose letters are all smaller than \( x \); this decomposition is called the \( x \)-factorization of \( \pi \). For example, if \( \pi = 834279156 \) and \( x = 7 \), then \( \pi \) is the concatenation of \( w_1 = 8 \), \( w_2 = 342 \), \( x = 7 \), the empty word \( w_4 \), and \( w_5 = 9156 \).

Define \( \varphi_x : \mathfrak{S}_n \rightarrow \mathfrak{S}_n \) by

\[
\varphi_x(\pi) := \begin{cases} 
  w_1 w_4 x w_2 w_5, & \text{if } x \text{ is a double ascent or double descent of } \pi, \\
  \pi, & \text{if } x \text{ is a peak or valley of } \pi.
\end{cases}
\]

(Here, we are using the conventions \( \pi(0) = \pi(n+1) = \infty \).) Equivalently, \( \varphi_x(\pi) = w_1 w_4 x w_2 w_5 \) if exactly one of \( w_2 \) and \( w_4 \) is nonempty, and \( \varphi_x(\pi) = \pi \) otherwise. It is easy to see that \( \varphi_x \) is an involution, and that for all \( x, y \in [n] \), the involutions \( \varphi_x \) and \( \varphi_y \) commute with each other. Given a subset \( S \subseteq [n] \), we define \( \varphi_S: \mathfrak{S}_n \rightarrow \mathfrak{S}_n \) by \( \varphi_S := \prod_{x \in S} \varphi_x \). For example, given \( \pi = 834279156 \) and \( S = \{ 6, 7, 8 \} \), we have \( \varphi_S(\pi) = 734289615 \); see Figure 1. The involutions \( \{ \varphi_S \}_{S \subseteq [n]} \) define a \( \mathbb{Z}_2^n \)-action on \( \mathfrak{S}_n \) which is commonly known as the modified Foata–Strehl action or valley-hopping. This action is based on a classical group action of Foata and Strehl [5], was introduced by Shapiro, Woan, and Getu [8], and later rediscovered by Brändén [2].

Next, we define a group action due to Sun and Wang [10] which is an analogue of valley-hopping for derangements in cycle notation. Let \( \mathcal{D}_n \) be the set of derangements of length \( n \). Define \( \theta_x : \mathcal{D}_n \rightarrow \mathcal{D}_n \) by \( \theta_x(\pi) := o^{-1}(\varphi_x(o(\pi))) \), where we treat the 0th letter of \( o(\pi) \) as 0 and the \( (n+1) \)th letter as \( \infty \). As with the functions \( \varphi_x \), the functions \( \theta_x \) are involutions that commute with each other. Similarly, for a subset \( S \subseteq [n] \), define \( \theta_S : \mathcal{D}_n \rightarrow \mathcal{D}_n \) by \( \theta_S := \prod_{x \in S} \theta_x \). Then Sun and Wang’s cyclic modified Foata–Strehl action is the \( \mathbb{Z}_2^n \)-action defined by the involutions \( \theta_S \).

Sun and Wang’s action can easily be extended to all permutations; simply define \( \psi_x : \mathfrak{S}_n \rightarrow \mathfrak{S}_n \) by

\[
\psi_x(\pi) := \begin{cases} 
  o^{-1}(\varphi_x(o(\pi))), & \text{if } x \text{ is not a fixed point of } \pi, \\
  \pi, & \text{if } x \text{ is a fixed point of } \pi,
\end{cases}
\]
Figure 1: Valley-hopping on $\pi = 834279156$ with $S = \{6, 7, 8\}$ yields $\varphi_S(\pi) = 734289615$

Figure 2: Cyclic valley-hopping on $\pi = (523)(8)(97641)$ with $S = \{3, 7\}$ yields $\psi_S(\pi) = (532)(8)(96417)$

where, as before, we treat the 0th letter of $o(\pi)$ as 0 and the $(n+1)$th letter as $\infty$. Given a subset $S \subseteq [n]$, define $\psi_S: \mathfrak{S}_n \to \mathfrak{S}_n$ by $\psi_S := \prod_{x \in S} \psi_x$. For example, given $\pi = (523)(8)(97641)$ and $S = \{3, 7\}$, we have $\psi_S(\pi) = (532)(8)(96417)$; see Figure 2. In what follows, we will call the $\mathbb{Z}_2^n$-action defined by the involutions $\{\psi_S\}_{S \subseteq [n]}$ cyclic valley-hopping.

We omit the proof of the next proposition, which describes the cyclic valleys, cyclic peaks, cyclic double ascents, cyclic double descents, and fixed points of $\psi_S(\pi)$ in terms of those of $\pi$.

The takeaway is that cyclic valley-hopping does not affect cyclic valleys, cyclic peaks, and fixed points, but toggles between cyclic double ascents and cyclic double descents.

**Proposition 3.** For any $S \subseteq [n]$ and $\pi \in \mathfrak{S}_n$, we have:

(a) $\text{Cval}(\psi_S(\pi)) = \text{Cval}(\pi)$;

(b) $\text{Cpk}(\psi_S(\pi)) = \text{Cpk}(\pi)$;

(c) $\text{Cdasc}(\psi_S(\pi)) = (\text{Cdasc}(\pi) \setminus S) \cup (S \cap \text{Cddes}(\pi))$;
(d) \( \text{Cddes}(\psi_S(\pi)) = (\text{Cddes}(\pi)) \setminus S \cup (S \cap \text{Cdasc}(\pi)) \);

(e) \( \text{Fix}(\psi_S(\pi)) = \text{Fix}(\pi) \).

We say that a subset \( \Pi \subseteq \mathfrak{S}_n \) is \textit{invariant under cyclic valley-hopping} if for every \( S \subseteq [n] \) and permutation \( \pi \in \Pi \), we have \( \psi_S(\pi) \in \Pi \) (equivalently, if \( \Pi \) is a disjoint union of orbits of the cyclic valley-hopping action).

\textbf{Proposition 4.} Any conjugacy class \( \mathfrak{S}_n(\lambda) \) is invariant under cyclic valley-hopping.

\textbf{Proof.} It suffices to show that, for any permutation \( \pi \in \mathfrak{S}_n \) and \( x \in [n] \), the permutation \( \psi_x(\pi) \) has the same cycle type as \( \pi \). Because \( \psi_x(\pi) = \pi \) whenever \( x \) is a fixed point, cyclic valley, or cyclic peak, we only need to consider the cases when \( x \) is a cyclic double ascent or cyclic double descent.

Fix a cyclic double ascent or cyclic double descent \( x \) of \( \pi \). Let us write \( \pi \) as a product of cycles \( \pi = C_1C_2 \cdots C_i \cdots C_k \), and let \( C_i \) be the cycle containing \( x \). First suppose that \( i < k \), i.e., \( C_i \) is not the last cycle of \( \pi \). Let \( c \) denote the first letter of \( C_i \), and let \( d \) denote the first letter of the next cycle \( C_{i+1} \). Then both \( c \) and \( d \) are larger than every other element in \( C_i \). Consider the \( x \)-factorization \( w_1w_2xw_4w_5 \) of \( o(\pi) \). We observe that the letter \( c \) is in \( w_1 \) and the letter \( d \) is in \( w_5 \), that all of the letters in \( w_2xw_4 \) are from the cycle \( C_i \), and that each of the left-to-right maxima of \( o(\pi) \) is in \( w_1 \) or \( w_5 \). Thus, the letter \( x \) is still between the letters \( c \) and \( d \) in \( \varphi_x(o(\pi)) = w_1w_4xw_2w_5 \), and the left-to-right maxima of \( o(\pi) \) and their positions are unchanged after applying \( \varphi_x \). It follows that the number of cycles and the number of elements in each cycle of \( \pi \) are unchanged after applying \( \psi_x \), so \( \psi_x(\pi) \) has the same cycle type as \( \pi \). If \( C_i \) is the last cycle of \( \pi \), then a similar argument works with \( d = \infty \). \( \square \)

\textbf{3. Results}

For a set of permutations \( \Pi \subseteq \mathfrak{S}_n \), let

\[ E(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{exc}(\pi)}. \]

Also let \( \mathfrak{S}_n,k \) be the set of permutations in \( \mathfrak{S}_n \) with exactly \( k \) fixed points, and given \( \sigma \in \mathfrak{S}_n \), we let \( \text{Orb}(\sigma) = \{ \psi_S(\sigma) \mid S \subseteq [n] \} \) denote the orbit of \( \sigma \) under cyclic valley-hopping. We begin by proving a preliminary lemma on the exceedance number distribution over a single orbit.

\textbf{Lemma 5.} Let \( \sigma \in \mathfrak{S}_{n,k} \). Then

\[ \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} = \sum_{\pi \in \text{Orb}(\sigma)} \frac{(s + t)^{\text{exc}(\pi) - \text{cval}(\pi)(1 + st)^{n-k-\text{cval}(\pi)-\text{exc}(\pi)} t^{\text{cval}(\pi)}}{(1 + s)^{n-k-2\text{cval}(\pi)}}. \]

\textbf{Proof.} Given a fixed permutation \( \sigma \in \mathfrak{S}_{n,k} \), first we wish to prove the identity

\[ \left( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} \right)(1 + s)^{\text{cdasc}(\sigma) + \text{cddes}(\sigma)} = \sum_{\pi \in \text{Orb}(\sigma)} (s + t)^{\text{cdasc}(\pi)(1 + st)^{\text{cddes}(\pi)}} t^{\text{cval}(\pi)}, \quad (6) \]
which we do combinatorially by showing that the two sides of the equation encode the same objects.

We begin with the left-hand side. Each summand in the factor \( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} \) corresponds to a permutation in the orbit of \( \sigma \) weighted by its excedance number. Each summand in the factor \((1 + s)^{\text{cdasc}(\sigma) + \text{cddes}(\sigma)} \) corresponds to marking a subset of cyclic double ascents and cyclic double descents of \( \sigma \). Thus, the left-hand side counts permutations in \( \text{Orb}(\sigma) \) where \( t \) is weighting the excedance number and \( s \) is weighting the number of marked letters.

Now, let us examine the right-hand side of Equation (6). Each term on the right-hand side of Equation (6) corresponds to taking a permutation \( \pi \in \text{Orb}(\sigma) \), choosing a subset \( S \) of cyclic double ascents and cyclic double descents of \( \pi \), applying \( \psi_S \) to \( \pi \), marking the letters of \( S \) in \( \psi_S(\pi) \) (which are all cyclic double ascents or cyclic double descents of \( \psi_S(\pi) \)), and weighting the marked letters by \( s \) and the excedances of \( \psi_S(\pi) \) by \( t \). The \((s + t)^{\text{cdasc}(\pi)} \) factor corresponds to selecting the cyclic double ascents, and the \((1 + st)^{\text{cddes}(\pi)} \) factor corresponds to selecting the cyclic double descents.

At this point, we have accounted for all excedances of \( \psi_S(\pi) \) which are cyclic double ascents of \( \psi_S(\pi) \). By Equation (3), the only remaining excedances of \( \psi_S(\pi) \) are the cyclic valleys of \( \psi_S(\pi) \), which are precisely the cyclic valleys of \( \pi \) by Proposition (3) (a); this contributes the factor of \( t^{\text{cval}(\pi)} \). In summary, both sides of Equation (6) count permutations in \( \text{Orb}(\sigma) \) with a marked subset \( S \) of letters by the same weights, but on the right-hand side, we are applying the involution \( \psi_S \) to each \( \pi \in \text{Orb}(\sigma) \) before doing the counting.

Next, observe the following:

- By Equation (4), we have
  \[
  \text{cdasc}(\pi) = \text{exc}(\pi) - \text{cval}(\pi).
  \]

- By Equations (2) and (5), we have
  \[
  \text{cddes}(\pi) = n - (\text{cval}(\pi) + \text{cpk}(\pi) + \text{cdasc}(\pi) + k)
  = n - k - \text{cval}(\pi) - \text{exc}(\pi).
  \]

- By the above two equations, we have
  \[
  \text{cdasc}(\sigma) + \text{cddes}(\sigma) = n - k - 2 \text{cval}(\sigma).
  \]

Therefore, from (6) we have the equation
\[
\left( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} \right) (1 + s)^{n - k - 2 \text{cval}(\sigma)} = \sum_{\pi \in \text{Orb}(\sigma)} (s + t)^{\text{exc}(\pi) - \text{cval}(\pi)} (1 + st)^{n - k - \text{cval}(\pi) - \text{exc}(\pi)} t^{\text{cval}(\pi)},
\]
and dividing both sides by \((1 + s)^{n - k - 2 \text{cval}(\sigma)} = (1 + s)^{n - k - 2 \text{cval}(\sigma)} \) gives us the desired formula.

\[\square\]

\footnote{More precisely, we are partitioning \( \text{Cdasc}(\pi) \) into two sets \( S \cap \text{Cdasc}(\pi) \) and \( \text{Cdasc}(\pi) \setminus S \). By Proposition (3) (d), the letters in \( S \cap \text{Cdasc}(\pi) \) are cyclic double descents of \( \psi_S(\pi) \) and thus non-excedances of \( \psi_S(\pi) \), so they are not given a weight of \( t \) but are given a weight of \( s \) because they belong to \( S \). On the other hand, by Proposition (3) (c), the letters in \( \text{Cdasc}(\pi) \setminus S \) are cyclic double descents of \( \psi_S(\pi) \) and thus excedances of \( \psi_S(\pi) \), so they are given a weight of \( t \) but not a weight of \( s \) because they do not belong to \( S \).}

\footnote{This is by similar reasoning as in the previous footnote.}
3.1. Gamma-positivity results

Before proving our main result (Theorem 1), we use Lemma 5 to prove a \( \gamma \)-positivity result for excedance number distributions over subsets invariant under cyclic valley-hopping and containing permutations with the same number of fixed points.

**Theorem 6.** Let \( \Pi \subseteq S_{n,k} \) be invariant under cyclic valley-hopping. Then

\[
E(\Pi; t) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \gamma_i t^i (1 + t)^{n-k-2i}
\]

where

\[
\gamma_i = \frac{1}{2^{n-k-2i}} \left| \left\{ \pi \in \Pi : \text{cval}(\pi) = i \text{ and } \text{cdasc}(\pi) = 0 \right\} \right|
\]

\[
= \frac{1}{2^{n-k-2i}} \left| \left\{ \pi \in \Pi : \text{cval}(\pi) = i \right\} \right|.
\]

**Proof.** Taking Lemma 5 and setting \( s = 1 \) yields

\[
\sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} = \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{cval}(\pi)} \frac{(1 + t)^{n-k-2\text{cval}(\sigma)}}{2^{n-k-2\text{cval}(\sigma)}},
\]

and noting that \( \text{cval}(\pi) = \text{cval}(\sigma) \) for all \( \pi \in \text{Orb}(\sigma) \)—a consequence of Proposition 3(a)—yields

\[
\sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} = \left( \sum_{\pi \in \text{Orb}(\sigma)} \frac{1}{2^{n-k-2\text{cval}(\sigma)}} \right) t^{\text{cval}(\sigma)} (1 + t)^{n-k-2\text{cval}(\sigma)}.
\]

Since \( n-k-2\text{cval}(\sigma) = \text{cdasc}(\sigma) + \text{cddes}(\sigma) \), it follows that \( |\text{Orb}(\sigma)| = 2^{n-k-2\text{cval}(\sigma)} \). Thus

\[
\sum_{\pi \in \text{Orb}(\sigma)} t^{\text{exc}(\pi)} = t^{\text{cval}(\sigma)} (1 + t)^{n-k-2\text{cval}(\sigma)}.
\]

Summing this equation over all orbits contributing to \( \Pi \) yields

\[
E(\Pi; t) = \sum_{\pi \in \Pi} t^{\text{exc}(\pi)} = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \gamma_i t^i (1 + t)^{n-k-2i}
\]

where \( \gamma_i \) is the number of orbits contributing to \( \Pi \) containing permutations with exactly \( i \) cyclic valleys.

Since \( |\text{Orb}(\sigma)| = 2^{n-k-2\text{cval}(\sigma)} \), we have

\[
\gamma_i = \frac{1}{2^{n-k-2i}} \left| \left\{ \pi \in \Pi : \text{cval}(\pi) = i \right\} \right|.
\]

Moreover, in each cyclic valley-hopping orbit there is a unique permutation with no cyclic double ascents—this is \( \psi_S(\sigma) \) for \( S = \text{Cdasc}(\sigma) \)—so alternatively we have

\[
\gamma_i = \left| \left\{ \pi \in \Pi : \text{cval}(\pi) = i \text{ and } \text{cdasc}(\pi) = 0 \right\} \right|.
\]
We now give several interesting consequences of Theorem 6.

**Corollary 7.** Let \( \lambda \) be a partition of \( n \) with \( k \) parts of size 1. Then

\[
E_\lambda(t) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \gamma_i t^i (1 + t)^{n-k-2i}
\]

where \( \gamma_i = 2^{-n+k+2i} |\{ \pi \in \mathfrak{S}_n(\lambda) : \text{cval}(\pi) = i \}| \).

**Proof.** We know from Proposition 4 that \( \mathfrak{S}_n(\lambda) \) is invariant under cyclic valley-hopping for any partition \( \lambda \) of \( n \). Thus the result follows from Theorem 6 by taking \( \Pi = \mathfrak{S}_n(\lambda) \). \( \Box \)

As mentioned in the introduction, the \( \gamma \)-positivity of \( E_\lambda(t) \) follows from Brenti’s formula (1), but our approach yields a combinatorial interpretation for the \( \gamma \)-coefficients.

The next corollary explains our observation in Remark 2 that the coefficient of each \( s^i \) in \( E_\lambda(\text{cval,exc}) \) is a \( \gamma \)-positive polynomial in \( t \).

**Corollary 8.** Let \( \lambda \) be a partition of \( n \). Then, for any integer \( i \geq 0 \), the coefficient of \( s^i \) in \( E_\lambda(\text{cval,exc})(s,t) \) is a \( \gamma \)-positive polynomial in \( t \).

**Proof.** Let \( \mathfrak{S}_{n,i}(\lambda) \) denote the set of permutations in \( \mathfrak{S}_n(\lambda) \) with exactly \( i \) cyclic valleys. We know that the number of cyclic valleys is constant over any cyclic valley-hopping orbit. Thus, by the same reasoning as in the proof of Lemma 4, the set \( \mathfrak{S}_{n,i}(\lambda) \) is invariant under cyclic valley-hopping, and so \( E(\mathfrak{S}_{n,i}(\lambda); t) \) is \( \gamma \)-positive by Theorem 6. Since

\[
E_\lambda(\text{cval,exc})(s,t) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} E(\mathfrak{S}_{n,i}(\lambda); t)s^i
\]

(where \( k \) is the number of parts of size 1 in \( \lambda \)), the result follows. \( \Box \)

Let us now define \( \mathfrak{S}_{n,k,i} \) to be the set of permutations of length \( n \) with exactly \( k \) fixed points and \( i \) cyclic valleys.

**Corollary 9.** For any \( 0 \leq k \leq n \), we have

\[
E(\mathfrak{S}_{n,k}; t) = \sum_{i=0}^{\lfloor (n-k)/2 \rfloor} \frac{|\mathfrak{S}_{n,k,i}|}{2^{n-k-2i}} t^i (1 + t)^{n-k-2i}.
\]

**Proof.** Since \( \mathfrak{S}_{n,k} \) is the union of all conjugacy classes \( \mathfrak{S}_n(\lambda) \) containing permutations with exactly \( k \) fixed points, each of which is invariant under cyclic valley-hopping, it follows that \( \mathfrak{S}_{n,k} \) is also invariant under cyclic valley-hopping. Thus the result follows from Theorem 6. \( \Box \)

The case \( k = 0 \) of Corollary 9 agrees with the known result for derangements [9, 10]. The numbers \( |\mathfrak{S}_{n,k,i}| \) can be obtained via the generating function

\[
1 + \sum_{n=1}^{\infty} |\mathfrak{S}_{n,k,i}| u^k t^i \frac{x^n}{n!} = \frac{\sqrt{1 - te^{(u-1)x}}}{\sqrt{1 - t} \cosh(x\sqrt{1 - t}) - \sinh(x\sqrt{1 - t})}
\]

(see [11] Section 4.1)).
Corollary 10. For any $0 \leq k \leq n$ and $0 \leq i \leq \lfloor (n-k)/2 \rfloor$, we have

$$E(\mathcal{S}_{n,k,i}; t) = \frac{|\mathcal{S}_{n,k,i}|}{2^{n-k-2i}} t^i (1 + t)^{n-k-2i}.$$ 

Proof. In the proof of Corollary 8, we saw that the set $\mathcal{S}_{n,i}(\lambda)$ consisting of all permutations in the conjugacy class $\mathcal{S}_n(\lambda)$ with exactly $i$ cyclic valleys is invariant under cyclic valley-hopping. Since $\mathcal{S}_{n,k,i}$ is the union of all the sets $\mathcal{S}_{n,i}(\lambda)$ over all partitions $\lambda$ of $n$ with exactly $k$ parts of size 1, it follows that $\mathcal{S}_{n,k,i}$ is also invariant under cyclic valley-hopping. Thus the result follows from Theorem 6. \qed

We note that if $\Pi \subseteq \mathcal{S}_n$ is invariant under cyclic valley-hopping but contains permutations with different numbers of fixed points, then $E(\Pi; t)$ is not necessarily $\gamma$-positive but is a sum of $\gamma$-positive polynomials with different centers of symmetry.

3.2. Proof of Theorem 1 and related results

For a set of permutations $\Pi \subseteq \mathcal{S}_n$, let

$$E^{(\text{cval, exc})}(\Pi; s, t) := \sum_{\pi \in \Pi} s^{\text{cval}(\pi)} t^{\text{exc}(\pi)}.$$

The following theorem allows us to relate the polynomials $E(\Pi; t)$ and $E^{(\text{cval, exc})}(\Pi; s, t)$ whenever the subset $\Pi \subseteq \mathcal{S}_{n,k}$ is invariant under cyclic valley-hopping. Theorem 11 will follow as a corollary.

**Theorem 11.** Let $\Pi \subseteq \mathcal{S}_{n,k}$ be invariant under cyclic valley-hopping. Then

$$E(\Pi; t) = \left(\frac{1 + st}{1 + s}\right)^{n-k} E^{(\text{cval, exc})}(\Pi; \frac{(1 + s)^2 t}{(s + t)(1 + st)}, \frac{s + t}{1 + st}). \quad (7)$$

Equivalently,

$$E^{(\text{cval, exc})}(\Pi; s, t) = \left(\frac{1 + u}{1 + uv}\right)^{n-k} E(\Pi; v) \quad (8)$$

where $u = \frac{1 + t^2 - 2st - (1-t)\sqrt{(1+t)^2 - 4st}}{2(1-s)t}$ and $v = \frac{(1+t)^2 - 2st - (1+t)\sqrt{(1+t)^2 - 4st}}{2st}$.

This theorem is a cyclic analogue of a previous result by the third author [13, Theorem 5.1] which relates the distribution of des and the joint distribution of pk (the number of peaks) and des over sets of permutations invariant under (ordinary) valley-hopping.

**Proof.** Taking Lemma 5 and summing over all orbits contributing to $\Pi$ yields

$$\sum_{\pi \in \Pi} t^{\text{exc}(\pi)} = \sum_{\pi \in \Pi} \frac{(s + t)^{\text{exc}(\pi) - \text{cval}(\pi)}(1 + st)^{n-k - \text{cval}(\pi) - \text{exc}(\pi)} t^{\text{cval}(\pi)}}{(1 + s)^{n-k-2\text{cval}(\pi)}}$$

$$= \left(\frac{1 + st}{1 + s}\right)^{n-k} \sum_{\pi \in \Pi} \left(\frac{(1 + s)^2 t}{(s + t)(1 + st)}\right)^{\text{cval}(\pi)} \left(\frac{s + t}{1 + st}\right)^{\text{exc}(\pi)}$$
and thus Equation (7) follows.

We obtain Equation (8) by setting \( u = \frac{(1+s)^2 t}{(s+t)(1+st)} \) and \( v = \frac{s+t}{1+st} \), solving for \( s \) and \( t \) (which can be done using a computer algebra system such as Maple), and simplifying.\(^4\)

Taking \( \Pi = \mathfrak{S}_n(\lambda) \) where \( \lambda = (1^{m_1}2^{m_2}\cdots) \), we obtain from Equation (8) the formula

\[
E_{\lambda}^{(\text{val,exc})}(s,t) = \left( 1 + \frac{u}{1+uv} \right)^{n-m_1} E_{\lambda}(v)
\]

where \( u \) and \( v \) are defined as in Theorem 11. Combining this formula with Brenti’s formula proves Theorem 11.

Finally, we derive a formula analogous to Theorem 11 which allows one to compute the polynomials \( E_{\lambda}^{(\text{val})}(t) \) using Eulerian polynomials. Given a set of permutations \( \Pi \subseteq \mathfrak{S}_n \), let

\[
E_{\text{val}}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{val}(\pi)}.
\]

**Theorem 12.** Let \( \Pi \subseteq \mathfrak{S}_{n,k} \) be invariant under cyclic valley-hopping. Then

\[
E(\Pi; t) = \left( \frac{1 + t}{2} \right)^{n-k} E_{\text{val}}(\Pi; \frac{4t}{(1+t)^2}) \tag{9}
\]

Equivalently,

\[
E_{\text{val}}(\Pi; t) = (1 + \sqrt{1-t})^{n-k} E(\Pi; w) \tag{10}
\]

where \( w = 2t^{-1}(1 - \sqrt{1-t}) - 1 \).

**Proof.** Equation (9) is obtained by taking Equation (7) and setting \( s = 1 \). Inverting Equation (9) and simplifying the result yields Equation (10). \( \square \)

**Theorem 13.** Let \( \lambda = (1^{m_1}2^{m_2}\cdots) \) be a partition of \( n \). Then

\[
E_{\lambda}^{(\text{val})}(t) = \frac{n!}{z_{\lambda}}(1 + \sqrt{1-t})^{n-m_1} \prod_{i \geq 1} \left[ \frac{A_{i-1}(w)}{(i-1)!} \right]^{m_i} \tag{11}
\]

where \( w = 2t^{-1}(1 - \sqrt{1-t}) - 1 \).

**Proof.** This is proven by taking Equation (10) with \( \Pi = \mathfrak{S}_n(\lambda) \) and combining the result with Brenti’s formula (11). \( \square \)

**Acknowledgements.** We thank Kyle Petersen for his helpful feedback and an anonymous referee for pointing out a couple mistakes on an earlier version of this manuscript.

\(^4\)We exchanged \( u \) and \( v \) with \( s \) and \( t \), respectively, in the statement of Equation (8) in this theorem, so that the \((\text{val,exc})\) polynomial would have variables \( s \) and \( t \) as in its definition.
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