(\(p, q\))-Generalization of Szász–Mirakjan operators and their approximation properties

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Abstract
We introduce a new modification of \((p, q)\)-analogue of Szász–Mirakjan operators. Firstly, we give a recurrence relation for the moments of \((p, q)\)-analogue of Szász–Mirakjan operators and present some explicit formulae for the moments and central moments up to order 4. Next, we obtain quantitative estimates for the convergence in the polynomial weighted spaces. In addition, we give the Voronovskaya theorem for the new \((p, q)\)-Szász–Mirakjan operators.

Keywords: \((p, q)\)-Integers; \((p, q)\)-Szász–Mirakjan operators; Weighted approximation

1 Introduction
In the last two decades, quantum calculus plays a significant role in the approximation of functions by a positive linear operator. In 1987, Lupaş [1] introduced the Bernstein (rational) polynomials based on the \(q\)-integers. In 1996, Phillips [2] introduced another generalization of Bernstein polynomials based on \(q\)-integers. In [3–23], in the case of \(0 < q < 1\), many operators have been introduced and examined. Among the most important operators there are \(q\)-Szász operators. In [18–21] the authors constructed and studied different \(q\)-generalizations of Szász–Mirakjan operators in the case \(0 < q < 1\). In 2012, Mahmudov [24] introduced the \(q\)-Szász operator in the case \(q > 1\) and studied quantitative estimates of convergence in polynomial weighted spaces and gave the Voronovskaya theorem.

In recent years, the rapid rise of \((p, q)\)-calculus has led to the discovery of new generalizations of Bernstein polynomials containing \((p, q)\)-integers. In 2015, Mursaleen [25] introduced \((p, q)\)-Bernstein operators and studied approximation properties based on a Korovkin-type approximation theorem of \((p, q)\)-Bernstein operators. Also, in 2017, Khan and Lobiyal [26] constructed a \((p, q)\)-analogue of Lupaş–Bernstein functions. In [27–39], the authors constructed many operators by using \((p, q)\)-integers and studied their approximation properties. Acar [40] introduced \((p, q)\)-Szász–Mirakjan operators. In addition, Acar gave a recurrence relation for the moments of these operators. In the same year, H. Sharma and C. Gupta [41] introduced the generalization of the \((p, q)\)-Szász–Mirakjan Kantorovich operators and examined their approximation properties. In 2017, Mursaleen, AAH Al-Abied, and Alotaibi [42] constructed new Szász–Mirakjan operators based on \((p, q)\)-calculus and studied weighted approximation and a Voronovskaya-type...
theorem. Also, \((p, q)\)-analogues of Szász–Mirakjan–Baskakov operators \([18]\) and Stancu-type Szász–Mirakjan–Baskakov operators \([43]\) were defined, and their approximation properties were investigated. Acar, Agrawal, and Kumar \([44]\) introduced a sequence of \((p, q)\)-Szász–Mirakjan operators, and their weighted approximation properties were investigated.

2 Construction of \(K_{l, p, q}\) and moment estimations

We give some basic notations and definitions of the \((p, q)\)-calculus.

The \((p, q)\)-integer and \((p, q)\)-factorial are defined by

\[
[l]_{p, q} := \begin{cases} \frac{p^l - q^l}{p - q} & \text{if } 0 < q < p \leq 1, \\ l & \text{if } p = q = 1 \end{cases} \quad \text{for } l \in \mathbb{N}, \quad [0]_{p, q} = 0, \\
[l]_{p, q}! := [1]_{p, q}[2]_{p, q} \cdots [l]_{p, q} \quad \text{for } l \in \mathbb{N}, \quad [0]_{p, q}! = 1.
\]

For integers \(0 \leq k \leq l\), the \((p, q)\)-binomial is defined by

\[
\binom{l}{k}_{p, q} := \frac{[l]_{p, q}!}{[k]_{p, q}![l-k]_{p, q}!}.
\]

The \((p, q)\)-derivative \(D_{p, q}g\) of a function \(g(z)\) is defined by

\[
(D_{p, q}g)(z) := \frac{g(pz) - g(qz)}{(p - q)z}, \quad z \neq 0, \quad (D_{p, q}g)(0) = g'(0)
\]

The product and quotient formulae for the \((p, q)\)-derivative are as follows:

\[
\begin{align*}
D_{p, q} \left( f(z)g(z) \right) &= g(pz)D_{p, q}f(z) + f(qz)D_{p, q}g(z), \\
D_{p, q} \left( \frac{f(z)}{g(z)} \right) &= \frac{g(pz)D_{p, q}f(z) - f(pz)D_{p, q}g(z)}{g(pz)g(qz)}.
\end{align*}
\]

It is known that

\[
D_{p, q} z^l = [l]_{p, q} z^{l-1}.
\]

The \((p, q)\)-anallogues of an exponential function, denoted by \(e_{p, q}(z)\) and \(E_{p, q}(z)\), are defined by

\[
e_{p, q}(z) := \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}} z^k}{[k]_{p, q}!}, \quad E_{p, q}(z) := \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{[k]_{p, q}!},
\]

and the \((p, q)\)-derivatives of \(e_{p, q}(az)\) and \(E_{p, q}(az)\) are

\[
D_{p, q}e_{p, q}(az) = ae_{p, q}(apz), \quad D_{p, q}E_{p, q}(az) = ae_{p, q}(aqz).
\]

Further, the \((p, q)\)-power is defined by

\[
(z - y)^{l}_{p, q} = (z - y)(pz - qy)(p^2z - q^2y) \cdots (p^{l-1}z - q^{l-1}y).
\]
For any integer \( l \),

\[
D_{p,q}(z-y)_p^l = \frac{[l]_{p,q}}{[l]_{p,q}} (pz-y)_p^{l+1},
\]

and \( D_{p,q}(z-y)_p^0 = 0 \).

The formula of the \( k \)th \((p,q)\)-derivative of the polynomial \((z-y)_p^l\) is

\[
D_{p,q}^k (z-y)_p^l = \frac{[l]_{p,q}^k}{[k]_{p,q}!} (p^k z - y)_p^{l-k},
\]

where \( l \in \mathbb{Z}_+ \) and \( 0 \leq k \leq l \).

The \((p,q)\)-analogue of the Taylor formulas for any function \( g(z) \) is defined by

\[
g(z) = \sum_{k=0}^{\infty} (-1)^k q^{-\frac{k}{2}} \frac{D_{p,q}^k g(z) (z-t)^k}{[k]_{p,q}!}.
\]

Let \( C_{p,q} \) denote the set of all real-valued continuous functions \( g \) on \([0, \infty)\) such that \( w_{p,q} g \) is bounded and uniformly continuous on \([0, \infty)\) endowed with the norm

\[
\|g\|_{m} := \sup_{z \in [0, \infty)} w_{p,q} |g(z)|,
\]

where \( w_{q}(z) = 1 \), and \( w_{p,q}(z) = \frac{1}{1 + z} \) for \( \beta \in \mathbb{N} \).

The corresponding Lipschitz class is given for \( 0 < \alpha \leq 2 \) by

\[
\Delta_{p,q}^\alpha g(z) := g(z+2\beta) - 2g(z+\beta) + g(z),
\]

\[
w_{p,q}^{\alpha}(g;\gamma) := \sup_{0 \leq \gamma \leq \gamma} \|\Delta_{p,q}^\alpha g\|, \quad \text{Lip}_{p,q}^{\alpha} := \{ g \in C_{p,q}: w_{p,q}^{\alpha}(g;\gamma) = 0(\gamma^\alpha), \gamma \rightarrow 0^{+} \}.
\]

Now we introduce the \((p,q)\)-Szász–Mirakjan operator.

**Definition 1** Let \( 0 < q < p \leq 1 \) and \( l \in \mathbb{N} \). For \( g : [0, \infty) \rightarrow R \), we define the \((p,q)\)-Szász–Mirakjan operator as

\[
K_{l,p,q}(g; z) := \sum_{k=0}^{\infty} g \left( p^{l-k} [k]_{p,q} \right) \frac{p^k q^{l-k}}{k^{l-1/2} [k]_{p,q}!} e_{p,q} (-[l]_{p,q} q^{k-1} z^{k-1}).
\]

It is clear that the operator \( K_{l,p,q} \) is linear and positive. It is known that the moments \( K_{l,p,q}(t^m; z) \) play a fundamental role in the approximation theory of positive operators.

**Lemma 2** Let \( 0 < q < p \leq 1 \) and \( m \in \mathbb{N} \). We have the following recurrence formula:

\[
K_{l,p,q}(t^{m+1}; z) = \sum_{j=0}^{\infty} \binom{m}{j} \frac{z^q p^{m(l-1)-j}}{[l]_{p,q}^j} K_{l,p,q}(t^j; pq^{-1} z).
\]
Proof  According to the definition of $K_{l,p,q}$ (8), we have

$$K_{l,p,q}(t^{m+1}; z)$$

$$= \frac{\sum_{k=0}^{\infty} p^{(l-k)(m+1)} [k]_{p,q}^{m+1} p^{k-l} \frac{[l]^{k}}{[l]_{p,q}} \sum_{j=0}^{\infty} \frac{q^{(k-l)j}}{[k]_{p,q}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{j}z)}{\sum_{k=0}^{\infty} \frac{p^{(l-k)(m+1)} [k]_{p,q}^{m} p^{k-l} \frac{[l]^{k}}{[l]_{p,q}} \sum_{j=0}^{\infty} \frac{q^{(k-l)j}}{[k]_{p,q}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{j}z)}$$

Next, we use the identity $q[k]_{p,q} + p^k = [k+1]_{p,q}$ to get the desired formula

$$K_{l,p,q}(t^{m+1}; z)$$

$$= \frac{\sum_{k=0}^{\infty} p^{(l-k)(m+1)} [k]_{p,q}^{m+1} p^{k-l} \frac{[l]^{k}}{[l]_{p,q}} \sum_{j=0}^{\infty} \frac{q^{(k-l)j}}{[k]_{p,q}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{j}z)}{\sum_{k=0}^{\infty} \frac{p^{(l-k)(m+1)} [k]_{p,q}^{m} p^{k-l} \frac{[l]^{k}}{[l]_{p,q}} \sum_{j=0}^{\infty} \frac{q^{(k-l)j}}{[k]_{p,q}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{j}z)}$$

Lemma 3 Let $0 < q < p \leq 1$, $z \in [0, \infty)$, $l \in I$, and $k \geq 0$. We have the following identities related to the $(p,q)$-derivative:

$$zD_{p,q}s_{k}(p,q;z) = [l]_{p,q} \left( p^{k} [k]_{p,q}^{l} - zp^{-(l-1)} \right) s_{k}(p,q;pz),$$

$$K_{l,p,q}(t^{m+1}; z) = \frac{z}{[l]_{p,q}^{l-1}} D_{p,q}K_{l,p,q}(t^{m}; z) + zK_{l,p,q}(t^{m}; z), \quad (10)$$

where $s_{k}(p,q;z) = \frac{p^{k-l} [k]_{p,q}^{l}}{[l]_{p,q}^{l-1}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{k}z)$.

Proof  We take the $(p,q)$-derivative of $s_{k}(p,q;z)$:

$$D_{p,q}s_{k}(p,q;z) = \frac{p^{k-l} [k]_{p,q}^{l}}{[l]_{p,q}^{l-1}} \left( z^{k} e_{p,q}(-[l]_{p,q}p^{k-l}q^{k}z) \right)$$

$$= \frac{p^{k-l} [k]_{p,q}^{l}}{[l]_{p,q}^{l-1}} e_{p,q}(-[l]_{p,q}p^{k-l}q^{k}pz)$$

$$= \frac{p^{k-l} [k]_{p,q}^{l}}{[l]_{p,q}^{l-1}} (qz)^{k} e_{p,q}(-[l]_{p,q}p^{k-l}q^{k}pz).$$
Then
\[
zD_{p,q} s_k(p, q; z) = p^{-k} \left[ k \right]_{p,q} \frac{p^k}{q^k} \frac{p}{q} e_{p,q} \left( -\left[ l \right]_{p,q} p^{k-l+1} q^{-k} (pz) \right)
- z \left[ l \right]_{p,q} p^{-(l-1)} \frac{p^k}{q^k} \frac{p}{q} e_{p,q} \left( -\left[ l \right]_{p,q} p^{k-l+1} q^{-k} (pz) \right)
= p^{-k} \left[ k \right]_{p,q} s_k(p, q; pz) - z \left[ l \right]_{p,q} p^{-(l-1)} s_k(p, q; pz)
= \left[ l \right]_{p,q} \left( p^k \left[ k \right]_{p,q} -zp^{-(l-1)} \right) s_k(p, q; pz).
\]

Using the obtained formula and the definition of the operator $K_{l,p,q}$, we get the second desired formula:
\[
zD_{p,q} K_{l,p,q}(t^m; z) = \left[ l \right] \sum_{k=0}^{\infty} \left( \frac{p^{l-k}[k]_{p,q}}{[l]_{p,q}} \right)^m p^{-(l-1)} \left( p^{-k}[k]_{p,q} - z \right) s_k(p, q; pz)
= \left[ l \right] \sum_{k=0}^{\infty} \left( \frac{[k]_{p,q}}{[l]_{p,q}} \right)^m \frac{p^{l-k}(m+1)}{p} s_k(p, q; pz)
- \left[ l \right] p^{-(l-1)} \sum_{k=0}^{\infty} \left( \frac{[k]_{p,q}}{[l]_{p,q}} \right)^m p^{l-k} s_k(p, q; pz)
= p^{-l} \left[ l \right] p^{-(l-1)} z K_{l,p,q}(t^{m+1}; pz) - \left[ l \right] p^{-(l-1)} z K_{l,p,q}(t^m; pz).
\]

Lemma 4 For $0 < q < p \leq 1$ and $l \in \mathbb{N}$, we have
\[
K_{l,p,q}(1; z) = 1, \quad K_{l,p,q}(t; z) = z, \quad K_{l,p,q}(t^2; z) = z^2 + \frac{1}{p^{-l-1}[l]_p} z^2,
K_{l,p,q}(t^3; z) = z^3 + \frac{2p + q}{p^{-2l-1}[l]_p} z^2 + \frac{1}{p^{-3l-1}[l]_p} z,
K_{l,p,q}(t^4; z) = z^4 + \left( \frac{3p^2 + 2pq + q^2}{p^2} \right) \frac{z^3}{p^{-2l-1}[l]_p}
+ \left( \frac{3p^2 + 3pq + q^2}{p^2} \right) \frac{z^2}{p^{-3l-1}[l]_p} + \frac{1}{p^{-4l-1}[l]_p} z.
\]

Proof From the $(p, q)$-Taylor theorem [45] we have
\[
\psi_l(t) = \sum_{k=0}^{\infty} (-1)^k q^{-\left( \frac{k}{2} \right)} [k]_{p,q}^{-1} \left( D^k_{p,q} \psi_l \right) (zq^{-k})(z \otimes t)_p^k.
\]

For $t = 0$, having in mind the equalities
\[
(z)_p^k = z^k p^{k(k-1)/2} \left( D^k_{p,q} \psi_l \right) (-l)_p P^{-(l-1)} z
= (-1)^k p^{k(k-1)/2} \left[ l \right]_p e_\psi \left( -l \right) P^{-(l-1)} p^k z
\]
for $\psi(z) = e_{pq}(-[l]_{p,q}p^{-(l-1)}z)$, we get the formula

$$1 = \psi(0) = \sum_{k=0}^{\infty} \frac{(-1)^k q^k \psi_k(z)_{p,q}}{[k]_{p,q}!} (D^k_{p,q}[q]) (zq^{-k})$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^k \psi_k(z)_{p,q}}{[k]_{p,q}!} (-1)^k [l]_{p,q} e_{pq} (-[l]_{p,q}p^{-(l-1-k)}q^{-k}z)$$

$$= \sum_{k=0}^{\infty} \frac{[l]_{p,q}^k z^k}{[k]_{p,q}!} p^{k(k-1)/2} \left( \frac{p}{q} \right)^k p^{-(l-1-k)} e_{pq} (-[l]_{p,q}p^{-(l-1)} \left( \frac{p}{q} \right)^k z),$$

that is, $K_{l,p,q}(1;z) = 1$.

For $i = 2, 3, 4$, recurrence formula (10) gives us the following results:

$$K_{l,p,q}(t^2;z) = \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ D_{p,q} K_{l,p,q} \left( \frac{t}{p} ; z \right) + \frac{[l]_{p,q}}{p^{-(l-1)}} K_{l,p,q}(t; z) \right\}$$

$$= \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ 1 + \frac{[l]_{p,q}}{p^{-(l-1)}} \right\} = \frac{z}{p^{-(l-1)}[l]_{p,q}} + z^2,$$

$$K_{l,p,q}(t^3;z) = \frac{z}{p^{-(l-1)}[l]_{p,q}} D_{p,q} K_{l,p,q} \left( \frac{t^2}{z} ; \frac{z}{p} \right) + z K_{l,p,q}(t^2; z)$$

$$= \frac{z}{p^{-(l-1)}[l]_{p,q}} \frac{1}{[l]_{p,q}p^{-(l-1)}} + \frac{[2]_{p,q} z}{p} + z \left\{ z^2 + \frac{z}{p^{-(l-1)}} \right\}$$

$$= z^3 + \frac{(2p + q) z}{p^{-(l-2)}[l]_{p,q}} + \frac{z}{[l]_{p,q}p^{2(l-1)}},$$

$$K_{l,p,q}(t^4;z) = \frac{z}{p^{-(l-1)}[l]_{p,q}} D_{p,q} K_{l,p,q} \left( \frac{t^3}{z} ; \frac{z}{p} \right) + z K_{l,p,q}(t^3; z)$$

$$= \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ \frac{[3]_{p,q} z^2}{p^2} + \frac{[2]_{p,q} (2p + q)}{pp^{-(l-2)}[l]_{p,q}} z + \frac{1}{[l]_{p,q}p^{2(l-1)}} \right\}$$

$$+ z \left\{ z^3 + \frac{(2p + q) z}{p^{-(l-2)}[l]_{p,q}} + \frac{z}{[l]_{p,q}p^{2(l-1)}} \right\}$$

$$= z^4 + \left( \frac{3p^2 + 2qp + q^2}{p^2} \right) \frac{z^3}{p^{-(l-1)}[l]_{p,q}} + \frac{1}{p^{-(3l-1)}[l]_{p,q}}\frac{z^2}{z^3}.$$

\[ \square \]

**Lemma 5** For every $z \in [0, \infty)$, we have

$$K_{l,p,q}((t-z); z) = 0,$$

$$K_{l,p,q}((t-z)^2; z) = \frac{z}{p^{-(l-1)}[l]_{p,q}},$$

$$K_{l,p,q}((t-z)^3; z) = \frac{1}{[l]_{p,q}p^{2(l-1)}z} + \frac{z^2}{p^{-(l-1)}[l]_{p,q}} \left( \frac{q}{p} - 1 \right),$$

$$K_{l,p,q}((t-z)^4; z) = \frac{1}{[l]_{p,q}p^{3(l-1)}z} + \left( \frac{3pq + q^2 - p^2}{p^2} \right) \frac{z^2}{[l]_{p,q}p^{2(l-1)}} + \frac{(p-q)^2 z^3}{p^2p^{-(l-1)}[l]_{p,q}}. \quad \tag{12}$$
Proof In fact, we may easily calculate third- and fourth-order central moments as follows:

\[ K_{l,p,q}(t^3; z) = K_{l,p,q}(t^3; z) - 3zK_{l,p,q}(t^2; z) + 3z^2K_{l,p,q}(t; z) - z^3 \]
\[ = \left( z^3 + \frac{2p + q}{[l]_{p,q}p^{l-1}} z^2 + \frac{1}{[l]_{p,q}p^{2[l-1]}} z \right) \]
\[ - 3z \left( z^2 + \frac{z}{[l]_{p,q}p^{l-1}} \right) + 3z^3 - z^3 \]
\[ = \frac{1}{[l]_{p,q}^2p^{2[l-1]}} z + z^2(q - p) \]
\[ K_{l,p,q}(t^4; z) \]
\[ = K_{l,p,q}(t^4; z) - 4zK_{l,p,q}(t^3; z) + 6z^2K_{l,p,q}(t^2; z) - 4z^3K_{l,p,q}(t; z) + z^4 \]
\[ = z^4 + \left( \frac{3p^2 + 2pq + q^2}{p^2} \right) \frac{z^3}{[l]_{p,q}p^{l-1}} + \left( \frac{3p^2 + 3pq + q^2}{p^2} \right) \frac{z^2}{[l]_{p,q}p^{2[l-1]}} + \frac{1}{p^{3[l-1]}[l]_{p,q}^3} z^2 \]
\[ - 4z \left( z^3 + \frac{(2p + q)}{p^{l-2}[l]_{p,q}} z^2 + \frac{z}{[l]_{p,q}^2p^{2[l-1]}} \right) + 6z^2 \left( \frac{z}{p^{l-1}[l]_{p,q}} + z^2 \right) - 4z^4 + z^4 \]
\[ = \frac{1}{[l]_{p,q}^3p^{3[l-1]}} z + \left( \frac{3pq + q^2 - p^2}{p^2} \right) \frac{z^2}{[l]_{p,q}^2p^{2[l-1]}} + \frac{(p - q)z^3}{p^2p^{3[l-1]}[l]_{p,q}}. \]

Remark 6 For \( 0 < q < p \leq 1 \),

\[ \lim_{l \to \infty} [l]_{p,q} = 0 \quad \text{or} \quad \frac{1}{p - q}, \]

In our study, we assume that \( q = q_l \in (0, 1) \) and \( p = p_l \in (q, 1) \) are such that

\[ \lim_{l \to \infty} q_l = 1, \quad \lim_{l \to \infty} p_l = 1, \]

and

\[ \lim_{l \to \infty} q_l' = 1, \quad \lim_{l \to \infty} p_l' = 1. \]

Therefore

\[ \lim_{l \to \infty} [l]_{p_l,q_l} = \infty. \]

For all \( 0 < q < p \leq 1 \) and \( j \geq 0 \), the \((p, q)\)-difference operators are defined as

\[ \Delta_{p,q}^0 g(z_l) = 0, \quad \Delta_{p,q}^j g(z_l) = \Delta_{p,q} g(z_l), \]

and

\[ \Delta_{p,q}^{k+1} g(z_l) = p^k \Delta_{p,q}^k g(z_{l+1}) - q^k \Delta_{p,q}^k g(z_{l+1}), \]

where \( z_l = \frac{p^{j+l+l} [l]_{p,q}}{[l]_{p,q}} \). Using this definition, we can prove the following lemmas.
Lemma 7  For all $0 < q < p \leq 1$ and $j, k \in \mathbb{N} \cup \{0\}$, we have
\[
g[z_j, z_{j+1}, \ldots, z_{j+k}] = \frac{p^{-[l-1]j} [l]_p^k p^{k(k-1)/2} \Delta^k_{p,q} g(z_j)}{p^{-k(2j-k+1)/2} q^{k(k-1)/2} [k]_p q^k}.
\]

Lemma 8  For all $0 < q < p \leq 1$, we have
\[
\Delta^k_{p,q} g(z_0) = g \left[ 0, \frac{1}{p^{-[(l-1)]} [l]_p}, \ldots, \frac{[k]_p}{p^{[(l-1)]} [l]_p} \right] q^{k(k-1)/2} [k]_p q^k.
\]

Lemma 9  We have
\[
\Delta^k_{p,q} g(z_j) = \sum_{i=0}^{k} \frac{(-1)^i q^{-[l-1]i} p^{-i(k-1)}}{p^{[l-1]i}} \binom{k}{i} g(z_{j+i}).
\]

Lemma 10  The $(p,q)$-Szász–Mirakjan operator can be represented as
\[
K_{p,q}(g; z) = \sum_{k=0}^{\infty} \left[ 0, \frac{1}{p^{-[(l-1)]} [l]_p}, \ldots, \frac{[k]_p}{p^{[(l-1)]} [l]_p} \right] z^k.
\]

Proof  Indeed,
\[
K_{p,q}(g; z) = \sum_{k=0}^{\infty} g \left( \frac{p^{[(l-1)]k} [l]_p}{[l]_p q^k} \right) p^{k(k-1)/2} [k]_p q^k \sum_{j=0}^{\infty} \frac{(-1)^j [l]_p j p^{k-j} q^j z^j}{p^{[(l-1)j]} [j]_p q!}.
\]

The result gives an explicit formula for the moments $K_{p,q}(t^m; z)$ in terms of Stirling numbers, which is a $(p,q)$-analogue of Becker’s formula; see [46].

Lemma 11  For $0 < q < p \leq 1$ and $m \in \mathbb{N}$, we have
\[
K_{p,q}(t^m; z) = \sum_{k=0}^{m} S_{p,q}(m,k) \frac{z^k}{p^{-[(l-1)m]} [l]_p q^m}.
\]

where
\[
S_{p,q}(m,k) = \frac{1}{q \frac{[l-1]k}{p^{-[(l-1)k]} [k]_p q^k}} \sum_{j=0}^{k} \frac{(-1)^j q^{-[l-1]j}}{p^{-[(l-1)(j-1)]} j [j]_p q^j} \binom{k}{j} p^{[-(l-1)m]} [k-j]_p q^m.
\]
are the second-type Stirling polynomials satisfying the equalities
\[
S_{p,q}(m+1,j) = p^{-(j-1)}[j]_{p,q}S_{p,q}(m,j) + S_{p,q}(m,j-1), \quad m \geq 0, j \geq 1, \\
S_{p,q}(0,0) = 1, \quad S_{p,q}(m,0) = 0, \quad m > 0, \quad S_{p,q}(m,j) = 0, \quad m < j. \tag{14}
\]

Clearly, \(K_{p,q}(t^m; z)\) are polynomials of degree \(m\) without a constant term.

**Proof** Because of \(K_{p,q}(t; z) = z\) and \(K_{p,q}(t^2; z) = z^2 + \frac{z}{p^{-(2-1)}[2]_{p,q}}\), representation (13) holds for \(m = 1, 2\) with \(S_{p,q}(2, 1) = 1, S_{p,q}(1, 1) = 1\).

Using mathematical induction, assume (13) to be valued for \(m\). Then from Lemma 3 we get
\[
K_{p,q}(t^m; z) = \sum_{k=1}^{m} S_{p,q}(m, k) \frac{z^k}{p^{-(l-1)(m-k)}[l]_{p,q}^{m-k}},
\]
\[
K_{p,q}(t^{m+1}; pz)
= \frac{zp^j}{[j]_{p,q}} D_{p,q} K_{p,q}(t^m; z) + zp K_{p,q}(t^m; pz)
= \frac{zp^j}{[j]_{p,q}} \sum_{j=1}^{m} [j]_{p,q} S_{p,q}(m, j) \frac{z^{j-1}}{p^{-(l-1)(m-j)}[l]_{p,q}^{m-j-1}} + zp \sum_{j=1}^{m} S_{p,q}(m, j) \frac{(pz)^j}{p^{-(l-1)(m-j)}[l]_{p,q}^{m-j}}
= \frac{1}{p^{-(l-1)}[l]_{p,q}} \sum_{j=1}^{m} [j]_{p,q} S_{p,q}(m, j) \frac{z^j}{p^{-(l-1)(m-j)}[l]_{p,q}^{m-j+1}} + \sum_{j=1}^{m} S_{p,q}(m, j) \frac{(pz)^j}{p^{-(l-1)(m-j)}[l]_{p,q}^{m-j}}
= \frac{zp}{p^{-(l-1)m}[l]_{p,q}^{m+1}} S_{p,q}(m, 1) + \sum_{j=2}^{m} \frac{(zp)^j}{p^{-(l-1)(m-j+1)}[l]_{p,q}^{m-j+1}} + \sum_{j=2}^{m} \frac{(zp)^j}{p^{-(l-1)(m-j)}[l]_{p,q}^{m-j}}.
\]

**Remark 12** For \(p = q = 1\), formulae (14) become recurrence formulas satisfied by the second-type Stirling numbers from [8].

3 (\(p, q\))-Szász–Mirakjan operators in a polynomial weighted space

**Lemma 13** For given any fixed \(\beta \in \mathbb{N} \cup \{0\}\) and \(0 < q < p \leq 1\), we have
\[
\|K_{p,q}(1/w^\beta; z)\|_\beta \leq K_1(p, q, \beta), \quad l \in l, \tag{15}
\]
where \(K_1(p, q, \beta)\) are positive constants. Moreover, for every \(g \in C_{\beta}\), we have
\[
\|K_{p,q}(g)\|_\beta \leq K_1(p, q, \beta)\|g\|_\beta, \quad l \in l. \tag{16}
\]

Thus \(K_{p,q}\) is a linear positive operator from \(C_{\beta}\) into \(C_{\beta}\).
Proof. Inequality (15) is obvious for $\beta = 0$. Let $\beta \geq 1$. Then by (13) we have

$$w_\beta(z)M_{lq}(1/w_\beta; z) = w_\beta(z)M_{lq}(1 + z^\beta; z)$$

$$= w_\beta(z)M_{lq}(1; z) + w_\beta(z)M_{lq}(z^\beta; z)$$

$$= w_\beta(z) + w_\beta(z) \sum_{j=1}^{\beta} S_{pq}(\beta,j) \frac{z^j}{p^{-(\beta+1)/[l]_{pq}^j}} \leq K_1(p,q,\beta),$$

where $K_1(p,q,\beta) > 0$ is a constant depending on $\beta$, $p$, and $q$. From this (15) follows. Moreover, for every $g \in C_\beta$,

$$\|K_{l,p,q}(g)\|_\beta \leq \|g\|_\beta \|K_{l,p,q}(1/w_\beta)\|_\beta.$$

By applying (15) we obtain

$$\|K_{l,p,q}(g)\|_\beta \leq K_1(p,q,\beta)\|g\|.$$

□

Lemma 14. For given any fixed $\beta \in \mathbb{N} \cup \{0\}$ and $0 < q < p \leq 1$, we have

$$\left\|K_{l,p,q}\left(\frac{(t-z)^2}{w_\beta(t)}; z\right)\right\|_\beta \leq \frac{K_2(p,q,\beta)}{p^{\beta/\beta_1}[l]_{pq}^{\beta_1}}, \quad l \in l,$$

where $K_2(p,q,\beta)$ are positive constants.

Proof. Formula (11) imply (17) for $\beta = 0$. We have

$$K_{l,p,q}\left(\frac{(t-z)^2}{w_\beta(t)}; z\right) = K_{l,p,q}(t-z)^2; z) + K_{l,p,q}(\beta z^2; z)$$

for $\beta, l \in \mathbb{N}$. If $\beta = 1$, then we get

$$K_{l,p,q}(t-z)^2(1 + t; z) = K_{l,p,q}(t-z)^2; z) + K_{l,p,q}(t-z)^2; z) = K_{l,p,q}(t-z)^2; z) + (1 + z)K_{l,p,q}(t-z)^2; z),$$

which by Lemma 5 yields (17) for $\beta = 1$.

Let $\beta \geq 2$. By applying (13) we get

$$w_\beta(z)K_{l,p,q}(t-z)^2; z)$$

$$= w_\beta(z)K_{l,p,q}(t^\beta; z) - 2zK_{l,p,q}(t^{\beta+1}; z) + 2 K_{l,p,q}(t^\beta; z))$$

$$= w_\beta(z) \left( z^{\beta+2} + \sum_{j=1}^{\beta+1} S_{pq}(\beta, 2, j) \frac{z^j}{p^{-(\beta+1)/[l]_{pq}^j}} \right)$$

$$- 2z^{\beta+2} - 2 \sum_{j=1}^{\beta} S_{pq}(\beta+1, j) \frac{z^{\beta+1}}{p^{-(\beta+1)/[l]_{pq}^j}}$$

$$+ z^{\beta+2} + \sum_{j=1}^{\beta} S_{pq}(\beta, j) \frac{z^{\beta+1}}{p^{-(\beta+1)/[l]_{pq}^j}}$$
By the Taylor formula, we obtain that

\[
K_p(z) = \left( \sum_{j=2}^{\beta} S_{p,q}(\beta + 2,j) - 2S_{p,q}(\beta + 1,j) + S_{p,q}(\beta,j - 1) \right) \frac{z^{j+1}}{p^{-(j+1)\beta+1-\beta}} \\
+ S_{p,q}(\beta + 2,1) \frac{z}{p^{-(j+1)\beta+1-\beta}} + (S_{p,q}(\beta + 2,2) - 2S_{p,q}(p + 2,1)) \frac{z^2}{p^{-(j+1)\beta+1-\beta}}
\]

\[
= w_p(z) \frac{z}{p^{-(j+1)\beta+1-\beta}} \varphi_p(p,q;z),
\]

where \( \varphi_p(p,q;z) \) is a polynomial of degree \( \beta \). Therefore we have

\[
w_p(z)K_{l,p,q}(t) \leq K_2(p,q,\beta) \frac{z}{p^{-(j+1)\beta+1-\beta}}.
\]

In the next theorem, we give an approximation property of \( K_{l,p,q} \).

**Theorem 15** Let \( g \in C_p^2 \), \( 0 < q < p \leq 1 \), and \( z \in [0, \infty) \). There exist positive constants \( K_3(p,q,\beta) > 0 \) such that

\[
w_p(z)K_{l,p,q}(g;z) - g(z) \leq K_3(p,q,\beta) \left\| g'' \right\|_{p} \frac{z}{p^{-(j+1)\beta+1-\beta}}.
\]

**Proof** By the Taylor formula

\[
g(t) = g(z) + g'(z)(t-z) + \int_z^t \int_z^s g''(u) du ds, \quad g \in C_p^2,
\]

we obtain that

\[
w_p(z)K_{l,p,q}(g;z) - g(z)
\]

\[
= w_p(z)K_{l,p,q} \left( \int_z^t \int_z^s g''(u) du ds; z \right)
\]

\[
\leq w_p(z)K_{l,p,q} \left( \left\| g'' \right\|_{p} \int_z^t \int_z^s (1 + u'''(s)) du ds; z \right)
\]

\[
\leq w_p(z) \frac{1}{2} \left\| g'' \right\|_{p} K_{l,p,q} \left( (t-z)^2 \left( 1/w_p(z) + 1/w_p(t) \right); z \right)
\]

\[
\leq w_p(z) \frac{1}{2} \left\| g'' \right\|_{p} K_{l,p,q} \left( (t-z)^2 \left( 1/w_p(z) + 1/w_p(t) \right); z \right)
\]

\[
\leq K_3(p,q,\beta) \left\| g'' \right\|_{p} \frac{z}{p^{-(j+1)\beta+1-\beta}}.
\]

We consider the modified Steklov means

\[
g_h(z) := \frac{4}{h^2} \int_0^h \int_0^h \left[ 2g(z + s + t) - g(z + 2(s + t)) \right] ds dt,
\]
which have the following properties:
\[
 g(z) - g_h(z) = \frac{4}{h^2} \int_0^\frac{1}{h} \int_0^h \Delta^2 v g(z) ds dt, \\
 g_h''(z) = h^{-2} \left( 8 \Delta^2 v g(z) - \Delta^2 g(z) \right),
\]
and therefore
\[
\|g - g_h\|_\beta \leq o_{\beta}^2(g; h), \\
\|g_h''\|_\beta \leq \frac{1}{9h^2} o_{\beta}^2(g; h).
\]

We may prove the following so-called direct approximation theorem.

**Theorem 16** For given any \( \beta \in \mathbb{N} \cup \{0\}, g \in C_\beta, z \in [0, \infty) \), and \( 0 < q < p \leq 1 \), we have
\[
 w_p(z)\|K_{l,p,q}(g; z) - g(z)\| \leq M_\beta o_{\beta}(g; \sqrt{\frac{z}{p^{-1} - 1}/[l]_{p,q}}) = M_\beta o_{\beta}^2 \left( g; \sqrt{\frac{p^{l-1}(q - p)z}{(q - p)}} \right).
\]
Particularly, if \( \text{Lip}^2 \alpha \) for some \( \alpha \in (0, 2] \), then
\[
 w_p(z)\|K_{l,p,q}(g; z) - g(z)\| \leq M_\beta \left( \frac{z}{p^{-1} - 1}/[l]_{p,q} \right)^{\frac{1}{2}}.
\]

**Proof** For \( g \in C_\beta \) and \( h > 0 \),
\[
\|K_{l,p,q}(g; z) - g(z)\| \leq \|K_{l,p,q}(g - g_h; z) - (g - g_h)(z)\| + \|K_{l,p,q}(g_h; z) - g_h(z)\|,
\]
and therefore
\[
w_p(z)\|K_{l,p,q}(g; z) - g(z)\| \leq \|g - g_h\|_\beta \left( w_p(z)K_{l,p,q} \left( \frac{1}{w_p(t)^{l-1}} \right) + 1 \right) + K_\beta(p, q, \beta)\|g_h''\|_\beta \frac{z}{p^{-1} - 1}/[l]_{p,q}.
\]
Since \( w_p(z)K_{l,p,q}(\frac{1}{w_p(t)^{l-1}}) \leq K_1(p, q, \beta) \), we get that
\[
w_p(z)\|K_{l,p,q}(g; z) - g(z)\| \leq l(p, q, \beta)w_p^2(g, h) \left( 1 + \frac{z}{h^2 p^{-1} - 1}/[l]_{p,q} \right).
\]
Thus choosing \( h = \sqrt{\frac{z}{p^{-1} - 1}/[l]_{p,q}} \), we complete the proof. \(\square\)

**Corollary 17** If \( \beta \in \mathbb{N} \cup \{0\}, g \in C_\beta, 0 < q < p \leq 1 \), and \( z \in [0, \infty) \), then
\[
\lim_{l \to \infty} K_{l,p,q}(g; z) = g(z)
\]
uniformly on every \([c, d], 0 \leq c < d\).
4 Convergence of \((p, q)\)-Szász–Mirakjan operators

In [47, Theorem 1] and [48, Theorem 1], Totik and de la Cal investigated the class problem of all continuous functions \(g\) such that \(K_{l,p,q}(g)\) converges to \(g\) uniformly on the whole interval \([0, \infty)\) as \(l \to \infty\). The following theorem is a \((p, q)\)-analogue of Theorem 1 in [48].

**Theorem 18** Assume that \(g : [0, \infty) \to \mathbb{R}\) is either bounded or uniformly continuous. Let

\[
g^*(z) = g(z^2), \quad z \in [0, \infty).
\]

Then, for all \(t > 0\) and \(z \geq 0\),

\[
|K_{l,p,q}(g; z) - g(z)| \leq 2 \omega \left( g^*; \sqrt[2-l]{\frac{[k]_{p,q}}{[l]_{p,q}}} \right),
\]

(18)

Therefore \(K_{l,p,q}(g; z)\) converges to \(g\) uniformly on \([0, \infty)\) as \(l \to \infty\) whenever \(g^*\) is uniformly continuous.

**Proof** By the definition of \(g^*\) we have

\[
K_{l,p,q}(g; z) = K_{l,p,q}(g^*(\sqrt[2-l]{(\cdot)}); z).
\]

Thus we can write

\[
|K_{l,p,q}(g; z) - g(z)| = |K_{l,p,q}(g^*(\sqrt[2-l]{(\cdot)}); z) - g^*(\sqrt[2-l]{(\cdot)})|
\]

\[
= \left| \sum_{k=0}^{\infty} \left( g^* \left( \sqrt[2-l]{\frac{[k]_{p,q}}{[l]_{p,q}}} \right) - g^*(\sqrt[2-l]{(\cdot)}) \right) s_{l,k}(p, q; z) \right|
\]

\[
\leq \sum_{k=0}^{\infty} \left( \left| g^* \left( \sqrt[2-l]{\frac{[k]_{p,q}}{[l]_{p,q}}} \right) - g^*(\sqrt[2-l]{(\cdot)}) \right| s_{l,k}(p, q; z) \right)
\]

\[
\leq \sum_{k=0}^{\infty} \omega \left( g^*; \sqrt[2-l]{\frac{[k]_{p,q}}{[l]_{p,q}}} \right) s_{l,k}(q; z)
\]

Finally, from the inequality

\[
\omega(g^*; \alpha \delta) \leq (1 + \alpha) \omega(g^*; \delta), \quad \alpha, \delta \geq 0,
\]

we obtain

\[
|K_{l,p,q}(g; z) - g(z)| \leq \omega(g^*; K_{l,p,q}(\sqrt[2-l]{(\cdot)}; z)) \sum_{k=0}^{\infty} \left( 1 + \sqrt[2-l]{\frac{[k]_{p,q}}{[l]_{p,q}}} - \sqrt[2-l]{(\cdot)} \right) s_{l,k}(p, q; z)
\]

\[
= 2 \omega(g^*; K_{l,p,q}(\sqrt[2-l]{(\cdot)}; z)).
\]
To complete the proof, we need to show that for all \( t > 0 \) and \( z > 0 \), we have

\[
M_{l,q}(\sqrt{r} - \sqrt{z};z) \leq \frac{1}{p^{(l-1)}[l]}.
\]

Indeed, from the Cauchy–Schwarz inequality it follows that

\[
K_{l,p,q}(\sqrt{r} - \sqrt{z};z)
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{|k|_{p,q}}{p^{(l-1)}[l]_{p,q}} - \sqrt{z} \right] s_{l,k}(p,q;z)
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{|k|_{p,q}}{p^{(l-1)}[l]_{p,q}} - \sqrt{z} \right] s_{l,k}(p,q;z) \leq \frac{1}{\sqrt{z}} \sum_{k=0}^{\infty} \left| \frac{|k|_{p,q}}{p^{(l-1)}[l]_{p,q}} - z \right| s_{l,k}(p,q;z)
\]

\[
\leq \frac{1}{\sqrt{z}} \sum_{k=0}^{\infty} \left| \frac{|k|_{p,q}}{p^{(l-1)}[l]_{p,q}} - z \right| s_{l,k}(q_1;z) = \frac{1}{\sqrt{z}} \sqrt{K_{l,p,q}(\cdot - z)_2;z}
\]

\[
= \frac{1}{\sqrt{z}} \sqrt{\frac{1}{p^{(l-1)}[l]_{p,q}} - z} = \sqrt{\frac{1}{p^{(l-1)}[l]_{p,q}}}. \quad \Box
\]

Our next results is a Voronovskaya-type theorem for \((p,q)\)-Szász–Mirakjan operators.

**Theorem 19** Let \( 0 < q < p \leq 1 \). For any \( g \in C_\beta^2(0, \infty) \), we have the equality

\[
\lim_{l \to \infty} |l|_q (K_{l,p,q}(g;z) - g(z)) = \frac{z}{2} g''(z)
\]

for every \( z \in [0, \infty) \).

**Proof** Let \( z \in [0, \infty) \) be fixed. By the Taylor formula we may write

\[
g(t) = g(z) + g'(z)(t-z) + \frac{1}{2} g''(z)(t-z)^2 + r(t;z)(t-z)^2,
\]

where \( r(t;z) \) is the Peano form of the remainder, \( r(\cdot ; z) \in C_\beta \), and \( \lim_{t \to z} r(t;z) = 0 \). Applying \( K_{l,p,q} \) to (19), we obtain

\[
p_q(K_{l,p,q}(g;z) - g(z)) = g'(z) |l|_{p,q} K_{l,p,q}(t-z;z)
\]

\[
+ \frac{1}{2} g''(z) |l|_{p,q} K_{l,p,q}(t-z)_2;z) + |l|_{p,q} K_{l,p,q}(r(t;z)(t-z)_2;z).
\]

Applying the Cauchy–Schwarz inequality, we have

\[
K_{l,p,q}(r(t;z)(t-z)_2;z) \leq \sqrt{K_{l,p,q}(r^2(t;z)z)} \sqrt{K_{l,p,q}(t-z)_4;z).
\]

Obviously, \( r^2(z;z) = 0 \). Then it follows from Corollary 17 that

\[
\lim_{l \to \infty} K_{l,p,q}(r^2(t;z)z) = r^2(z;z) = 0.
\]
Now from (20), (21), and Lemma 5 we immediately get
\[
\lim_{l \to \infty} \left[ l \right]_{p,q} K_{l,p,q} \left( r(t;z)(t - z)^2; z \right) = 0.
\]

The proof is completed. \(\square\)

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