Non-linear dynamics of reheating

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Abstract

Resonant decay of the inflaton produces highly non-equilibrium states dominated by interacting classical waves. We discuss several topics in the theory of formation and evolution of such states.

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1 Introduction

Inflationary cosmology explains the oldness and the large-scale uniformity of the observable universe. It may also explain the origin of the large-scale structure (galaxies, clusters of galaxies). It assumes the existence of a separate epoch in the cosmological history, when the expansion of the universe was extremely fast, “superluminal”. That epoch is called inflation.

Inflation ends through a process known as reheating, in which the energy is transferred from a homogeneous scalar field, the inflaton, to other fields and particles. In inflationary cosmology, all the matter in our universe has been created as a result of this process.

Reheating can be viewed as amplification of quantum fluctuations of various fields in the time-dependent inflaton background. In slow-roll inflation, that background is an oscillating homogeneous inflaton field. It can be viewed alternatively as a Bose condensate of zero-momentum particles—the inflaton quanta. Then, reheating is the decay of the Bose condensate.

It now appears probable that the postinflationary universe had been a much livelier place than it was thought to be. In many models of inflation, the decay of the inflaton condensate starts out as a rapid, explosive process, called preheating, during which fluctuations of Bose fields coupled to the inflaton grow quasexponentially and achieve large occupation numbers. This growth can be thought of as a result of parametric resonance. At the subsequent stage, called semiclassical thermalization, the resonance smears out, and the fields reach a slowly evolving turbulent state with smooth power spectra.

The highly non-equilibrium states produced in the inflaton decay can support non-thermal symmetry restoration and baryogenesis. Collisions of the classical waves produced in the decay process give rise to a strong, conceivably detectable background of relic gravitational waves.

Below, I discuss several topics in the theory of the resonant decay of the inflaton and in the description of the resulting non-equilibrium states. This discussion is based upon partially published work that I have done in collaboration with I. Tkachev.

2 A model

Consider the class of models in which the inflaton $\phi$ interacts with another scalar field $X$. The Lagrangian of the matter fields is

$$L = \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g^{\mu \nu} \partial_\mu X \partial_\nu X - V(\phi, X)$$

(1)

where

$$V(\phi, X) = V_\phi(\phi) + \frac{1}{2} g^2 \phi^2 X^2 + \frac{1}{2} M_X^2 X^2$$

(2)
and $V_{\phi}$ is a potential for the field $\phi$. The space-time metric $g_{\mu\nu}$ is assumed to be that of a spatially flat Friedmann-Robertson-Walker (FRW) space-time with the scale factor $a(t)$. In our simulations, the scale factor can be determined self-consistently, using the Einstein equations.

For definiteness, let us now consider the so-called massive case, when

$$V_{\phi}(\phi) = \frac{1}{2} m^2 \phi^2$$

Later, we will consider other potentials for $\phi$. If we want our simple model to be a realistic model of inflation, the inflaton mass $m$ should be of order $10^{-6} M_{\text{Pl}}$, to comply with the COBE data.

Both $\phi$ and $X$ are regarded (for a while) as quantum operators, and we are working in the Heisenberg picture: the time dependence of averages is due to the time-dependence of operators, while the quantum state is time-independent. The system is translationally invariant in space. So, the average of the inflaton field $\phi$ in the quantum state is space-independent:

$$\langle \phi(x, t) \rangle = \phi_0(t)$$

We consider the initial data problem formulated as follows.

The inflaton’s zero-momentum (homogeneous) mode $\phi_0$ rolls slowly during inflation, until it reaches a certain value, of order $M_{\text{Pl}}/3$. At that time, the slow roll changes into oscillations of $\phi_0$ about the minimum of $V_{\phi}$. Thus the inflation ends and the reheating starts. The inflaton’s homogeneous mode decays into fluctuations of $X$ and those of its own field $\phi$. The change from the slow-roll to the oscillatory behavior of $\phi_0$ is not instantaneous, and in principle the method described below allows us to follow the evolution of the fluctuations through the inflationary stage and into the reheating. The results, however, are practically unchanged if we adopt, instead, the following simpler formulation.

To begin with, we solve for $\phi_0$ and the scale factor $a$ at the last stages of inflation and during the crossover from the slow roll to the oscillating stage. For this purpose, the fluctuations can be neglected. The crossover solution is insensitive to the values of $\phi_0$ and $d\phi_0/dt$ with which the slow roll starts. It turns out that the solution for $\phi_0$, as a function of the conformal time $\eta$, $d\eta = dt/a(t)$, has an extremum during the crossover, before it makes the first oscillation. We will refer to the extremum as the time when the inflaton oscillations start; we take that time as $\eta = 0$. In the massive model, $\phi_0(0) = 0.28 M_{\text{Pl}}$. The fluctuations at $\eta = 0$ are small, and in our formulation we assume that they are in the conformal vacuum, with respect to annihilation operators defined below. This specifies the (quantum) initial conditions for the fluctuations. (The assumption of the conformal vacuum is not a crucial one, as long as it gives initial fluctuations of the correct order of magnitude.)

It is convenient to rescale to the dimensionless conformal variables:
\[ \tau = m\eta; \quad \varphi = \phi a/\phi_0(0) \]
\[ \xi = m\chi; \quad \chi = Xa/\phi_0(0) \]

We have normalized the scale factor \( a(\tau) \) to 1 at \( \tau = 0 \), when the inflaton oscillations start: \( a(0) = 1 \). By virtue of our rescaling, \( \varphi_0(0) = 1 \). The action for the matter fields in the new variables takes the form

\[ S = \frac{\phi_0^2(0)}{m^2} \int d^3\xi d\tau \left[ \frac{1}{2}(\dot{\varphi} - h\varphi)^2 + \frac{1}{2}(\dot{\chi} - h\chi)^2 - \frac{(\nabla_\xi \varphi)^2}{2} - \frac{(\nabla_\xi \chi)^2}{2} \right. \]

\[ \left. - \frac{1}{2}a^2 \varphi^2 - 2q\varphi^2\chi^2 - \frac{1}{2}a^2\chi^2 \right] \]  

(5)

Hereinafter dots denote derivatives with respect to \( \tau \); \( h = \dot{a}/a \) is the rescaled Hubble parameter.

The parameter \( q \) appearing in (5) is defined as

\[ q = \frac{g^2\phi_0^2(0)}{4m^2} \]  

(6)

and is called the resonance parameter.

In this type of inflationary models, the resonance parameter \( q \) is naturally large. Indeed, with \( \phi_0^2(0) \sim 0.1M_{Pl}^2 \) and \( m^2 \sim 10^{-12}M_{Pl}^2 \), one gets \( q \sim 10^{10}g^2 \), so even for fairly small \( g^2 \) one still has \( q \gg 1 \). A large value of \( q \) is necessary for efficient resonant decay of the inflaton in this model.

The Heisenberg equations of motion following from (5) are

\[ \ddot{\varphi} - \nabla_\xi^2 \varphi - \frac{\ddot{a}}{a} \varphi + a^2 \varphi + 4q\varphi^2\chi = 0 \]  

(7)

\[ \ddot{\chi} - \nabla_\xi^2 \chi - \frac{\ddot{a}}{a} \chi + m_\chi^2a^2\chi + 4q\varphi^2\chi = 0 \]  

(8)

where \( m_\chi \equiv M_X/m \). We can write

\[ \varphi(\xi, \tau) = \varphi_0(\tau) + \delta\varphi(\xi, \tau) \]  

(9)

where \( \delta\varphi(\xi, \tau) \) is the (quantum) fluctuation. Recall that \( \varphi_0(\tau) \) defined in (4) and, consequently, \( \varphi_0(\tau) \) are c-number functions.

### 3 Quantities to compute

What are the interesting quantities to compute? One is the power spectrum of fluctuations. This is defined as follows. Introduce the spatial Fourier components

\[ \chi(\xi, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \chi_k(\tau)e^{ik\xi}, \quad \delta\varphi(\xi, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \varphi_k(\tau)e^{ik\xi} \]  

(10)
Thus \( k \) is a **comoving momentum** (in the units of the mass \( m \)); it does not redshift in an expanding universe. The power spectra are then defined as
\[
P_\chi(k) = \frac{\langle |\chi_k|^2 \rangle}{V}, \quad P_\varphi(k) = \frac{\langle |\varphi_k|^2 \rangle}{V}
\]
(11)
where \( V \) is a normalization spatial volume. Because the system is isotropic, the power spectra are functions only of \( k = |k| \).

The angular brackets so far denote averaging over the quantum state of the system. The actual calculations were done using the classical approximation; we will come to this important point shortly. In the classical approximation, the averages over the quantum state are replaced by averages over realizations of the random initial data.

Another interesting quantity is the variance of a field. With the above definitions, it would be just an integral over \( d^3k \) of the field’s power spectrum, if we did not have to subtract away the high-momentum quantum fluctuations in order to make the variance finite. A natural scheme is to subtract the full initial value of the variance:
\[
\langle \chi^2 \rangle_1(\tau) = \int [P_\chi(k, \tau) - P_\chi(k, 0)]d^3k
\]
(12)
If we introduce a maximal (cutoff) momentum \( k_{\text{max}} \), we can view (12) as the limit of
\[
\langle \chi^2 \rangle_2(\tau) = \int_{k < k_{\text{max}}} P_\chi(k, \tau)d^3k - \int_{k < k_{\text{max}}} P_\chi(k, 0)d^3k
\]
(13)
as \( k_{\text{max}} \) goes to infinity. A maximal momentum is automatically present in our lattice calculations. We have to make sure that it is large enough so that (13) is insensitive to the cutoff or, equivalently, the modes with \( k \sim k_{\text{max}} \) do not give a large contribution to (13).

The variances measure the typical size of the fluctuations. For example, they determine the fluctuations’ symmetry restoring power. Of main interest are the values of the variances at sufficiently large times, when the fluctuations are the largest. For the values of \( k_{\text{max}} \) that we use, the first term on the right-hand side of (13), when resonance is efficient, quickly becomes much larger than the second. So, we will “forget” the second term and use
\[
\langle \chi^2 \rangle(\tau) = \int_{k < k_{\text{max}}} P_\chi(k, \tau)d^3k
\]
(14)
The parameters of our model \((m, M_X, g^2)\) can be viewed as the running parameters of quantum field theory, normalized at the momentum point \( k_{\text{max}} \).

In our actual calculations, the variances are obtained as lattice averages:
\[
\langle \chi^2 \rangle = \frac{1}{N_s} \sum_{i=1}^{N_s} \chi_i^2
\]
(15)
\[
\langle (\delta \varphi)^2 \rangle = \frac{1}{N_s} \sum_{i=1}^{N_s} \varphi_i^2 - \left( \frac{1}{N_s} \sum_{i=1}^{N_s} \varphi_i \right)^2
\]
(16)
where \( i \) labels the lattice sites, and \( N_s \) is their total number;

\[
\frac{1}{N_s} \sum_{i=1}^{N_s} \varphi_i = \varphi_0
\]  

is the inflaton’s zero-momentum mode.

Note that in the classical approximation, the variances (15), (16) and the zero mode (17) need not be averaged over realizations of the initial data. These quantities are self-averaging, as they are defined as averages over a large number lattice sites. Similarly, for the power spectra in the classical approximation, we use, instead of (11),

\[
P_\chi(k) = \frac{1}{V} \int |\chi_k|^2 \frac{d\Omega}{4\pi}, \quad P_\varphi(k) = \frac{1}{V} \int |\varphi_k|^2 \frac{d\Omega}{4\pi}
\]

where \( \Omega \) is the direction of \( k \).

### 4 The linear stage

As seen from (5), the parameter \( m^2/\phi_0^2(0) \sim 10^{-11} \) suppresses the initial magnitude of the fluctuations relative to the scale at which their non-linear interactions become important. Unless the parameter \( q \) is too large, in which case the scale of non-linearity is itself too small, the initial evolution of the fluctuations proceeds in the linear regime.

Expanding the equations (7)–(8) to the first order in \( \delta \varphi \) and \( \chi \), we obtain the following linearized equations

\[
\ddot{\varphi}_0 - (\ddot{a}/a)\varphi_0 + a^2\varphi_0 = 0
\]

\[
\delta \ddot{\varphi} - (\ddot{a}/a)\delta \varphi + a^2\delta \varphi = 0
\]

\[
\ddot{\chi} - \nabla^2 \chi - (\dddot{a}/a)\chi + m^2_{\chi} a^2 \chi + 4q\varphi_0^2 \chi = 0
\]

\( m_{\chi} \equiv M_X/m \). The Fourier expansion (10) turns the equation (21) into

\[
\ddot{\chi}_k + \omega_k^2(\tau)\chi_k = 0
\]

where

\[
\omega_k^2(\tau) = m^2_{\chi} a^2(\tau) + k^2 - \dddot{a}/a + 4q\varphi_0^2(\tau)
\]

Eq. (22) is still a quantum equation, but it is a linear equation and can be solved exactly.

The operator solution to (24) can be written as

\[
\chi_k(\tau) = f_k(\tau)b_k(0) + f^*_k(\tau)b^*_k(0)
\]

The c-number function \( f_k(\tau) \) satisfies the same equation as \( \chi_k \): 

\[
\ddot{f}_k + \omega_k^2 f_k = 0
\]
and the initial conditions
\[ f_k(0) = \frac{m}{\phi(0) \sqrt{2\tilde{\omega}_k(0)}} \] (26)
\[ \dot{f}_k(0) = [-i\tilde{\omega}_k(0) + h(0)] f_k(0) \] (27)
where
\[ \tilde{\omega}_k^2(\tau) = m_X^2 a^2(\tau) + k^2 + 4q\varphi_0^2(\tau) = \omega_k^2(\tau) + \ddot{a}/a \] (28)
Note that \( f_k \) depends only on the absolute value of \( k \). The operators \( b \) and \( b^\dagger \) are time-independent creation and annihilation operators normalized according to
\[ [b_k, b^\dagger_{k'}] = \delta(k - k') \] (29)
The quantum state of the system is assumed to be the vacuum of \( b_k \). The time-dependence of the operators \( \chi_k \) is entirely due to that of the \( c \)-number mode functions \( f_k \).

5 Non-adiabatic amplification
In the linear approximation, the evolution of the inflaton zero-momentum mode is given by (19). In a static universe, i.e. for \( a(\tau) = 1 \), the relevant solution to (19) is
\[ \varphi_0(\tau) = \cos \tau \] (30)
Then, the equation (25) reduces to the well-known Mathieu equation:
\[ \ddot{f}_k + [A + 2q \cos 2\tau] f_k = 0 \] (31)
where \( A = k^2 + m_X^2 + 2q \). The instability zones of the Mathieu equation, in the plane of \( A \) and \( q \), determine momenta \( k \) for which there are exponentially growing solutions. This growth is known as \textit{parametric resonance}. The case of large \( q \) is known in the literature as the broad resonance case.

In the expanding universe, solutions to (19) are not periodic. The crucial point is that, when \( q \) is sufficiently large, there still are growing solutions (although the growth is no longer simply exponential), see Fig. 1. The range of \( q \) for which there is an efficient growth of the variance of \( X \) has been determined numerically. For example, for \( M_X = 0 \), it is \( q \gtrsim 10^3 \). The growing solutions in the expanding universe have been recently explored analytically.

To distinguish the case with a non-periodic \( \varphi_0 \) from the case of a periodic \( \varphi_0 \), we will use, instead of the term parametric resonance, the term \textit{non-adiabatic amplification}. This stresses that the periodicity of \( \varphi_0 \) is not crucial for the existence of the
Figure 1: Variances of the physical fields $X$ and $\phi$, together with the inflaton’s zero-momentum mode, in the massive model with $q = 10^4$, $M_X = 0$ in the expanding universe.

Growing solutions. What is crucial for the efficient amplification is that, at least once in a while, the adiabatic (WKB) condition for Eq. (22)

$$\left| \frac{\dot{\omega}_k(\tau)}{\omega_k^2(\tau)} \right| \ll 1$$

breaks down.

Soon after the inflaton oscillations start, the inflaton zero-momentum mode is well approximated by

$$\varphi_0(\tau) \approx \bar{\varphi}(\tau) \cos(a\tau + \alpha)$$

where $\bar{\varphi}$ is a relatively slowly decreasing amplitude. During the linear stage, its decrease is entirely due to the redshift of the field:

$$\bar{\varphi}(\tau) \sim a^{-1/2}(\tau)$$

The amplitude of the physical field $\phi_0$ is $\bar{\phi}(\tau) = \phi_0(0)\bar{\varphi}(\tau)/a(\tau)$.

It is convenient to introduce the effective resonance parameter:

$$q_{\text{eff}}(\tau) = \frac{g^2\bar{\varphi}^2(\tau)}{4m^2} = \frac{\bar{\varphi}^2(\tau)}{a^2(\tau)}$$

For example, in Fig. at $\tau \sim 10$, we have $\bar{\varphi}^2 \sim 10^{-5}M_{\text{Pl}}^2$, so $q_{\text{eff}} \sim 1$.

The most interesting case is that when the resonance parameter $q$ is large enough for the amplification to fully develop. That means that the growth of the amplified modes is cut-off by non-linear effects, rather than by the expansion of the universe.
This is the case when the fluctuations can reach the largest values. It occurs when at the end of the linear stage the effective resonance parameter is still fairly large: $q_{\text{eff}} \gtrsim \max\{1, (M_X/m)^4\}$, which translates into $q \gtrsim 10^4 \max\{1, (M_X/m)^4\}$.

As long as $q_{\text{eff}} \gg 1$, the adiabatic approximation is good most of the time. The condition (32) breaks down only during small intervals when $\varphi_0$ is close to zero:

$$\cos(a \tau + \alpha) \lesssim q_{\text{eff}}^{-1/4}$$

and, even then, only provided that

$$k^2/a^2 + m^2 \chi \lesssim q_{\text{eff}}^{1/2}$$

These intervals of time correspond to spikes in $\langle X^2 \rangle$, which are seen in Fig. 1. We will distinguish between $\langle \chi^2 \rangle_s$ in spikes and in “valleys” between them by a subscript: “s” or “v”. One can show analytically that during the linear stage $\langle \chi^2 \rangle_s / \langle \chi^2 \rangle_v \sim q_{\text{eff}}^{1/4}$.

The band of momenta specified by (37) is analogous to the strongest (for a given $q$) instability band of the Mathieu equation (31). Notice that $k/a$ is the physical momentum in the units of $m$. For a given $k$, even one that satisfies (37), the mode function $f_k$ will not grow each time the adiabatic condition is broken, so the power spectrum of $\chi$ at the linear stage becomes jagged.

### 6 Quantum-to-classical transition

The modes of $\chi$ with growing $f_k$ achieve large occupation numbers and begin to behave classically. The theory of this phenomenon is parallel to the linear theory describing how large-scale classical density perturbations are generated from the vacuum fluctuations during inflation.

The occupation number in the modes with $|k| = k$, obtained by a Bogoliubov transformation, is

$$n_k(\tau) = \frac{\rho_{\chi}^2(0)}{2m^2 \omega_k(\tau)} \left[ |\hat{f}_k - h \hat{f}_k|^2(\tau) + \omega_k^2(\tau) |\hat{f}_k|^2(\tau) \right] - 1/2$$

One can show that up to $O(1/n_k)$ the quantum averages of the operators in the modes with $|k| = k$ can be approximated by classical averages computed with the help of a certain distribution function. When the state vector is the vacuum of $b_k$ and $b_{-k}$, the distribution function for the pair of modes with momenta $k$ and $-k$ is

$$\mathcal{F}[\chi_k, \dot{\chi}_k; \tau] = \mathcal{N} \exp \left( -\frac{(2\pi)^3 |\chi_k|^2}{V |f_k(\tau)|^2} \right) \delta \left( f_k(\tau) \chi_k - \dot{f}_k(\tau) \chi_k \right)$$

where the delta function is a shorthand for the product of the delta functions for real and imaginary parts; $\mathcal{N}$ is a time-independent normalization. (Note that half of the values of $k$ give all the independent $\chi_k$.)
It is useful to notice that (39) satisfies the classical Liouville equation during the entire linear stage. So, we can evolve it back and use it as an initial condition at \( \tau = 0 \), where \( f_k \) and its derivative are given by (26), (27). This leads to the classical problem with random initial conditions distributed according to

\[
\mathcal{F}_k[\chi_k, \dot{\chi}_k; 0] = \mathcal{N}' \exp \left(-\frac{2\phi^2(0)}{V'm^2} \tilde{\omega}_k(0)|\chi_k|^2 \right) \delta \left( \dot{\chi}_k + [i\tilde{\omega}_k(0) - h(0)]\chi_k \right)
\]

(40)

\( V' \equiv V/(2\pi)^3 \). This classical problem and the original quantum problem “converge” as the modes are amplified.

How good is this classical approximation? For example, if \( q_{\text{eff}} \sim 1 \) at the end of the linear stage, the typical occupation number in the efficiently amplified modes at that time is \( n_{\text{amp}} \sim \tilde{\sigma}^2/m^2 \). The corrections to the classical approximation are then of order \( 1/n_{\text{amp}} \sim 10^{-7} \).

When

\[
|f_k| |\dot{f}_k| \sim m^2 n_k / \phi_0^2(0)
\]

(41)

and \( n_k \gg 1 \), one can derive the classical approximation in the following simple way. The solutions \( f_k(\tau) \) and \( f^*_k(\tau) \) to (23)–(27) satisfy

\[
f_k(\tau) \dot{f}^*_k(\tau) - f^*_k(\tau) \dot{f}_k(\tau) = (m^2 / \phi_0^2(0))i
\]

(42)

Given (11) and \( n_k \gg 1 \), the right-hand side of (12) can be neglected compared to each of the terms on the left-hand side. That means that the phase \( \theta_k \) defined by

\[
f^*_k = f_k \exp(i\theta_k)
\]

(43)

becomes almost time-independent. With the definition (13), the operator solution (24) takes the form

\[
\chi_k(\tau) = f_k(\tau) \left[ b_k + e^{i\theta_k} b^\dagger_k \right] \equiv f_k(\tau) \mathcal{O}_k
\]

(44)

When the phase \( \theta_k \) is almost time-independent, the time-derivative of \( \chi_k \) can be found as

\[
\dot{\chi}_k' \approx \dot{f}_k' \mathcal{O}_k'
\]

(45)

provided \( |\dot{f}_k| \) is large enough, cf. (11). The canonical momentum conjugate to \( \chi_k \), as obtained from the action (3), is proportional to \( (\dot{\chi}_k' - h\chi_k') \). A direct calculation shows that \( \{ \mathcal{O}(k), \mathcal{O}(k') \} = 0 \), for any \( k \) and \( k' \). So, in this approximation, \( \chi_k \) and its canonical momentum commute and can therefore be regarded as random classical variables.

Even if at some time the condition (11) breaks down, i.e. one of the two quantities, \( |f_k| \) or \( |\dot{f}_k| \), happens to be small, the classical approximation will still work, as long as \( n_k \gg 1 \). The reason is that the other of the two quantities is necessarily large, cf. (18), so the quantum “noise” will not be able to disturb the evolution of the modes.
In the vacuum of $b_k$, the variance (12) is

$$\langle \chi^2 \rangle_1(\tau) = \int \frac{d^3k}{(2\pi)^3}|f_k|^2(\tau)$$

As long as $q_{\text{eff}} \gg 1$, $f_k$ oscillates rapidly in the “valleys”. When $\theta_k$ becomes almost time-independent, $|f_k|^2$ also oscillates rapidly there. As a result, the variance acquires a high-frequency “beard”, see Fig. [4].

7 Rescattering

The modes outside the band (37) and fluctuations of the field $\varphi$ do not undergo an efficient non-adiabatic amplification in the massive model. The system thus arrives into the regime when the low-momentum fluctuations of $\chi$ are already large, while its higher-momentum fluctuations and the fluctuations of $\varphi$ are still small.

Fluctuations of $\varphi$ now get amplified via rescattering [5] of the already large fluctuations in the low-momentum modes of $\chi$. Further rescattering of the $\chi$ and $\varphi$ fluctuations leads to amplification of the higher-momentum modes of both $\varphi$ and $\chi$.

The early stages of these processes can be studied using the following approximation. We assume that the fluctuations of $\varphi$ are much smaller than the amplitude of its zero-momentum mode:

$$\langle (\delta \varphi)^2 \rangle \ll \varphi^2$$

but we make no assumptions about the size of the fluctuations of $\chi$. Expanding the full equations (7), (8) in $\delta \varphi$ (but no longer in $\chi$), we obtain, to the first order, the following approximate equations ($p \neq 0$)

$$\ddot{\chi}_k + \omega_k^2(\tau)\chi_k + 8q\varphi_0(\tau) \int d^3p \varphi_0^* \chi_{k+p} = 0$$

$$\ddot{\varphi}_p + \Omega_p^2(\tau)\varphi_p + 4q\varphi_0(\tau) \int d^3k \chi_k^* \chi_{k+p} = 0$$

$$\dot{\varphi}_0 - (\ddot{a}/a)\varphi_0 + a^2(\tau)\varphi_0 + 4q\langle \chi^2 \rangle \varphi_0 = 0$$

where $\Omega_p^2 = p^2 - \ddot{a}/a + a^2$.

The main contribution to the last term on the left-hand side of (49) comes from the low-momentum, classical modes of $\chi$. The fluctuations of $\varphi$ are driven by a classical force and therefore become classical themselves. This quantum-to-classical transition happens in the linear regime with respect to $\delta \varphi$. In this respect, it is similar to the one for the low-momentum modes of $\chi$, even though it is due to the force-driven rather than parametric amplification. We obtain the initial distribution function for $\varphi_p$ and $\dot{\varphi}_p$ analogous to (24). In fact, within the accuracy of the classical approximation, we could have taken the initial values for $\varphi$ and $\dot{\varphi}$ to be zeroes: the classical force would produce fluctuations of $\varphi$ anyway.
The amplified fluctuations of $\varphi$, together with the low-momentum fluctuations of $\chi$ form a classical force for the higher-momentum fluctuations of $\chi$, see (48). In this way, the power spectra propagate to larger momenta.

According to (49), the $\delta \varphi$ subsystem is a resonator of high quality (the universe already expands slowly at this stage). Therefore, the rapid growth of $\delta \varphi$ will continue until $\delta \varphi \sim \bar{\varphi}$, and the approximation (48)–(49), based on (47), breaks down. A more precise condition for the end of the rapid growth of $\delta \varphi$, obtained from the simulations, is

$$\langle (\delta \varphi)^2 \rangle \sim 0.1 \bar{\varphi}^2$$

(51)

The system then enters the fully non-linear stage.

8 The fully non-linear calculation

To include all non-linear effects, we simulate the complete non-linear equations of motion (8), (8) as a classical non-linear problem with random initial conditions for $\chi$ and $\delta \varphi$. The random initial conditions are distributed according to (40) and a similar expression for $\delta \varphi$. The initial conditions for $\varphi_0$ are: $\varphi_0(0) = 1$, $\dot{\varphi}_0(0) = 0$. The simulations were done on $128^3$ lattices, with $m^2 = 10^{-12} M_{Pl}^2$, both for massless and for massive $X$. For massless $X$, we determined the evolution of the scale factor $a(\tau)$ self-consistently, taking into account the effect of the produced fluctuations in the Einstein equations. For massive $X$, we used the matter-dominated form of $a(\tau)$. Simulations were done also for the conformally-invariant model, discussed below.

Additional results of the simulations are given in Figs. 1, 2, 3. The variances and the power spectra are defined according to (15), (16), (18).

We used the classical distribution (40) for all momenta present on the lattice. This is possible because, even at the largest times that we consider, the main contributors to the physical quantities are modes with large occupation numbers. In general, a lattice calculation of the power spectrum can be trusted not up to the maximal momentum $k_{\text{max}}$ present on the lattice, but only up to the Nyquist momentum $k_{\text{Ny}}$; on a cubic lattice $k_{\text{Ny}} = k_{\text{max}}/\sqrt{3}$. For example, in Fig. 3 (obtained for the conformally-invariant model), $k_{\text{max}} \approx 28$ and $k_{\text{Ny}} \approx 16$. For $3 < k < 16$, the latest outputs of the power spectrum are close to a perfect straight line in the log-linear plot:

$$P_\chi(k) \approx P_0 \exp(-k/k_0)$$

(52)

Such exponential fronts are obtained also for the power spectrum of $\varphi$ and the fields in other models. Integrals of modest powers of $k$ computed with the power spectrum (52) will be saturated at $k \sim k_0$, where $P_\chi(k) \sim P_0$, and the occupation numbers are large.

This distinguishes the highly non-equilibrium states emerging after the inflaton’s resonant decay from the equilibrium, thermal states. For a real scalar field with a
Figure 2: Power spectrum of the field $\chi$ in the conformally invariant model, output every half-period at the extrema of $\varphi_0(\tau)$.

time-independent dispersion law $\epsilon(k)$, the occupation numbers are given by the Bose distribution $n_B(k)$, and the power spectrum, at temperature $T$, is

$$P_T(k) = (2\pi)^{-3} \epsilon^{-1}(k)n_B(k) = \frac{(2\pi)^{-3} \epsilon^{-1}(k)}{1 - \exp[\epsilon(k)/T]}$$

The variance subtracted at $T = 0$ is

$$\langle \chi^2 \rangle_T - \langle \chi^2 \rangle_0 = \int d^3k P_T(k)$$

The modes with $\epsilon(k) \ll T$ are essentially classical but the main contributors to (54) are the quantal modes, those with $n_B \sim 1$.

The new states are to a good accuracy classical, precisely because they are highly non-thermal, with the power spectra behaving like (52). For these states, there is no ultraviolet catastrophe (no Rayleigh-Jeans problem). Eventually, the system will thermalize, and will be dominated by quantum modes. Nevertheless, there is a prolonged stage in the evolution when the power spectra are concentrated at small momenta, and the system is essentially classical.

The non-linear phenomena described by the classical equations of motion are creation, scattering, decay, and annihilation of the classical waves arising through the amplification of quantum fluctuations. We refer to these phenomena as rescattering. In the language of particle physics, rescattering can be viewed as stimulated creation, scattering, decay and annihilation of “particles” in states with large occupation numbers. A good analogy would be the scattering of waves from throwing pebbles in a pond, if those waves could be made high enough for their non-linear interactions.
(rather than merely interference) to become important. Such non-linear interactions of waves occur in plasma physics and fluid dynamics.[14]

9 Approximate methods for the non-linear stage

9.1 The Hartree approximation

We will see below that as long as

$$\langle (\delta \varphi)^2 \rangle \ll \varphi^2 / q_{\text{eff}}$$

we can neglect the fluctuations of $\varphi$ altogether. For large $q_{\text{eff}}$, (55) is a stronger condition than (47). In terms of the physical field $\phi$, (55) reads

$$\langle (\delta \phi)^2 \rangle \ll \phi^2(0) / q$$ (56)

We then get a system of two equations

$$\ddot{\chi}_k + \omega_k^2(\tau) \chi_k = 0$$ (57)
$$\ddot{\varphi}_0 - (\ddot{a} / a) \varphi_0 + a^2(\tau) \varphi + 4q\langle \chi^2 \rangle \varphi = 0$$ (58)

These comprise the Hartree approximation for the present model. The classical average in (58) approximates the corresponding quantum average with the accuracy $O(1/n_{\text{amp}})$, where $n_{\text{amp}}$ is the typical occupation number in the amplified modes.

The equation (57) is formally identical to (22), so the operator solution (24) still applies, although the form of $\varphi_0$ and, consequently, of $f_k$ will now be different. The phase $\theta_k$, see (43), becomes almost time-independent under the same conditions (41) and $n_k \gg 1$. So, as long as $q_{\text{eff}} \gg 1$, the variance of $\chi$ in the Hartree approximation has the high-frequency “beard”.

The time $\tau_{\text{sc}}$ when the “beard” disappears on our pictures while $q_{\text{eff}} \gg 1$ (e.g. $\tau \approx 10.5$ in Fig. 3) is the time when the Hartree approximation breaks down, and rescattering becomes a dominant effect.

9.2 Corrections to the Hartree approximation

To derive the condition (55), we need to estimate corrections to the Hartree approximation and see when they become large. While (17) still applies, we can use for that purpose the approximate equations (48)–(49). We will do the estimate for the general case of a field $\chi$ with $N$ real components. The components will be labeled by Greek indices $\alpha, \beta, \ldots$; a repeated index is summed over and is sometimes omitted.

Eq. (49) is solved by ($p \neq 0$)

$$\varphi_p(\tau) = \varphi_p(0) + 4q \int_0^\tau d\tau' G_p(\tau, \tau') \varphi_0(\tau') \int d^3k \chi^*_k \chi_k + p, \alpha(\tau')$$ (59)
where $\varphi_P^{(0)}$ is the solution to the free equation (20), and $G_P(\tau, \tau')$ is the retarded Green function for (20): $G_P(\tau, \tau') \approx -\Omega_P^{-1}(\tau) \sin[\omega_P(\tau - \tau')]$, where $\Omega_P^2 \approx p^2 + a^2$. The magnitude of $\varphi_P^{(0)}$ is of the order of the initial condition for $\varphi_P$ and is very small.

At the stage when the fluctuations of $\chi$ still rapidly grow, the main contribution to the time integral in (59) comes from an interval of duration of order $a^{-1}$ near $\tau' = \tau$, when the $\chi$ fluctuations are the largest. We then estimate (59) as

$$\varphi_P(\tau) \sim \frac{q^2 \varphi^2(\tau)}{a^2(\tau)} \int d^3k \chi_k^* \chi_{k+p,\beta}(\tau)$$

(60)

where we have used the estimate $p \sim a(\tau)$ for the typical momentum of the $\varphi$ fluctuations, so that also $\Omega_P \sim a(\tau)$.

Eq. (48) is solved by

$$\chi_{k\beta}(\tau) = \chi_{k\beta}^{(0)}(\tau) + 8q \int_0^\tau d\tau' F_k(\tau, \tau') \varphi(\tau') \int d^3p \varphi^* P_k k+p,\beta$$

(61)

where $\chi_{k\beta}^{(0)}(\tau)$ solves Eq. (54) of the Hartree problem (57)–(58), and $F_k(\tau, \tau')$ is the retarded Green function for (57); $F_k(\tau = \tau') = 0$, $\dot{F}_k(\tau = \tau') = -1$. The integral over time in (61) is estimated in the same manner as the one in (59)

$$\frac{\varphi(\tau)}{a\omega_k(\tau)} \int d^3p \varphi^* P_k k+p,\beta$$

(62)

where $\omega_k(\tau) \sim \sqrt{q}\varphi(\tau)$.

Using the estimates (60) and (62) in (61) we obtain

$$\chi_{k\beta} - \chi_{k\beta}^{(0)} \sim \frac{q^2 \varphi^2}{a^2 \omega_k} \int d^3k d^3p \chi_{k'\alpha} k'\alpha \chi_{k+p,\beta}$$

(63)
We recall at this point that $\chi$ and $\varphi$ are random variables. By the translational invariance and the $O(N)$ symmetry,

$$\langle \chi_{k\alpha}^* \chi_{k'\beta} \rangle = N^{-1} \langle \chi_{k\alpha}^* \chi_{k\beta} \delta_{\alpha\beta} \delta_{kk'} \rangle \to (2\pi)^3 N^{-1} V^{-1} \langle \chi_{k\alpha}^* \chi_{k\beta} \delta_{\alpha\beta} \delta(k-k') \rangle$$

(64)

So, for instance, the average of $\varphi_p (p \neq 0)$ over realizations of the initial data is zero.

The first correction to the Hartree approximation is obtained when we replace $\chi$ on the right-hand sides of (59), (61) with $\chi(0)$. In the Hartree approximation, different pairs of modes of $\chi$ with the opposite momenta ($k$ and $-k$) are statistically independent. For example, for any $p \neq 0$

$$\langle \chi_{k\alpha}^*(0) \chi_{k'\beta} \rangle = \langle \chi_{k\alpha}^*(0) \chi_{k'\beta} \rangle + \langle \chi_{k\alpha}^*(0) \chi_{k'\beta} \rangle$$

for any $p \neq 0$. Higher-order correlators decompose in a similar way.

Using this approximation in (60), we estimate the mean square of $|\varphi_p|$ as

$$\langle |\varphi_p|^2 \rangle \sim q^2 \bar{\varphi}^2 \int d^3 k \langle |\chi_k|^2 \rangle \langle |\chi_{k+p}|^2 \rangle$$

(66)

and the variance $\langle (\delta \varphi)^2 \rangle = V^{-1} \int d^3 p |\varphi_p|^2$ as

$$\langle (\delta \varphi)^2 \rangle \sim q^2 \varphi^2 \langle \chi^2 \rangle / a^4 N$$

(67)

We can similarly estimate the first correction to $\langle \chi_{k\alpha}^* \chi_{k\beta} \rangle$. Towards the end of the Hartree stage, it is of order of the average square of the right-hand side of (63). (The cross correlation between the right-hand side and $\chi(0)$ is at these times a smaller effect for large $N$.) We find that the first correction becomes of the order of the leading term when

$$\langle \chi^2 \rangle \sim \sqrt{N} \frac{a^3(\tau)}{\bar{\varphi}^{3/2} \bar{\varphi}_c}$$

i.e. $\langle X^2 \rangle \sim \sqrt{N} \frac{\phi_0^2(0)}{q a^{3/2} \phi_{\text{eff}}}$

(68)

According to (57), at that time

$$\langle (\delta \varphi)^2 \rangle \sim \frac{a^2}{q}$$

i.e. $\langle \varphi^2 \rangle \sim \phi_0^2(0) / q$

(69)

cf. the condition (56).

The estimate (68) grows indefinitely with $N$, for a fixed $q$. The physical variance cannot do that. So, for sufficiently large $N$, $N > N_0(q)$, where $N_0$ increases with $q$, the Hartree approximation does not break down down at all during the rapid growth of $\langle \chi^2 \rangle$, i.e. the condition (69) is never reached.

Otherwise, the estimate (69) determines the time $\tau_{sc}$ after which rescattering begins to strongly influence the evolution of $\langle \chi^2 \rangle$. Because in deriving (69) we neglected
various numerical factors, it should be more of a scaling law with respect to $q$ than of an actual numerical estimate. Nevertheless, it turns out to correspond numerically quite well to $\langle (\delta \phi)^2 \rangle$ at the time when the “beard” disappears on our pictures, whenever that happens while $q_{\text{eff}} \gg 1$. For example, in Fig. 3, the beard disappears at $\tau = \tau_{\text{sc}} \approx 10.5$. At that time, $\langle (\delta \phi)^2 \rangle \sim 10^{-7} M_{\text{Pl}}^2$, in agreement with (69). On the other hand, in Fig. 1, $q_{\text{eff}}$ becomes of order 1 already at $\tau \sim 10$, and even the maximal value of $\langle (\delta \phi)^2 \rangle$ is somewhat smaller than the estimate (69).

At $\tau > \tau_{\text{sc}}$, the system exhibits chaotic behavior, characteristic of a non-linear classical system. The stage when

$$\phi_0^2 (0) / q < \langle (\delta \phi)^2 \rangle \ll 0.1 \bar{\phi}^2$$

(70)

has been called the chaotization stage. When the second condition in (70) is broken, the system enters the fully non-linear stage with developed chaotic behavior. After chaotization, the system is in a slowly evolving (quasi-steady) state with smooth power spectra, see e.g. Fig. 2. Such states are naturally called turbulent. It is the chaotic evolution that finally establishes the maximal values attained by the variances of the fields.

The turbulent states emerge in a variety of models of the inflaton’s resonant decay (cf. the conformally invariant case below). The power spectra in these states in general consist of a segment of a power law at the smallest $k$, $k \sim 1$, from where the exponential front (52) takes off. We expect that at later times, the front will slowly move to larger momenta, and the region of $k$ occupied by the power law will grow.

## 10 Decay of the inflaton

An interesting feature of the massive model in the expanding universe is that the inflaton decays completely fairly soon after it starts to oscillate, see Figs. 4, 5. This early decay does not occur in the same model without the expansion or in the conformally invariant case (discussed below).

Because the inflaton decays fast, and the expansion of the universe is already slow, the total energy is almost conserved during the decay. One can obtain a simple, albeit non-rigorous, estimate of the maximal size of $X$ fluctuations from the energy conservation. We have (in terms of the physical fields)

$$m^2 \bar{\phi}^2(\tau_1) \sim g^2 \langle (\delta \phi)^2 \rangle (\tau_2) \langle X^2 \rangle_{\text{max}}$$

(71)

where $\tau_1$ is a moment right before the decay, and $\tau_2$ is the moment when $\langle X^2 \rangle$ is maximal; we have assumed that a significant portion of the inflaton’s energy went into the $X$ fluctuations. Using $\langle (\delta \phi)^2 (\tau_1) \rangle \sim 0.1 \bar{\phi}^2 (\tau_1)$, see (51), and assuming that $\langle (\delta \phi)^2 \rangle$ does not change drastically during the decay, we get

$$\langle (\delta \phi)^2 \rangle (\tau_2) \sim 0.1 \bar{\phi}^2 (\tau_1)$$

(72)
From (71) and (72), $\langle X^2 \rangle_{\text{max}} \sim m^2/g^2 \sim \phi_0^2(0)/q$ (neglecting numerical factors), or
$$
\langle X^2 \rangle_{\text{max}} = \epsilon \phi_0^2(0)/q
$$
(73)
where $\epsilon$ takes into account the fraction of the inflaton’s energy that went to $X$ and other numerical factors.

The $1/q$ estimate (73) agrees with the scaling of our data points and with an estimate obtained from the self-similarity hypothesis. From the data, we find $\epsilon \sim 0.1–1$.

11 Conformally-invariant model

This model has the scalar potential
$$
V(\phi, X) = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g^2 \phi^2 X^2
$$
(74)
The fields $\phi$ and $X$ are massless. For the inflaton self-coupling $\lambda$ we take a realistic value $\lambda = 10^{-13}$.

The inflaton oscillations start at $\phi_0(0) \approx 0.35 M_{\text{Pl}}$, $d\phi_0(0)/d\eta = 0$; $\eta$ is the conformal time: $d\eta = dt/a(t)$. The rescaled conformal variables are
$$
\tau = \sqrt{\lambda} \phi_0(0) \eta \ ; \quad \varphi = \phi a/\phi_0(0)
$$
$$
\xi = \sqrt{\lambda} \phi_0(0) x \ ; \quad \chi = X a/\phi_0(0)
$$
so that $\varphi_0(0) = 1$. The universe is radiation dominated, $a(\tau)/a(0) \approx 0.51 \tau + 1$.

The equations of motion in the conformal variables
$$
\ddot{\varphi} - \nabla^2 \varphi + \varphi^3 + 4q \chi^2 \varphi = 0 ,
$$
$$
\ddot{\chi} - \nabla^2 \chi + 4q \varphi^2 \chi = 0
$$
(75)
are the same as in the flat space-time. The resonance parameter in this case is $q \equiv g^2/4\lambda$.

The power spectrum of $\chi$ for $q = 30$ is shown in Fig. 2. Notice the formation of the exponential tail, cf. (52). The latest outputs in Fig. 2 fall almost exactly onto each other—a sign of a quasi-steady state.

12 Conclusions

Resonant decay of the inflaton leads to formation of highly non-equilibrium states after inflation. These states are dominated by interacting classical waves. As we have shown elsewhere, collisions of these classical waves give rise to a potentially observable background of relic gravitational waves.
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