Research article

New formulation for discrete dynamical type inequalities via \( h \)-discrete fractional operator pertaining to nonsingular kernel

Maysaa Al Qurashi\(^1\), Saima Rashid\(^{2,*}\), Sobia Sultana\(^3\), Hijaz Ahmad\(^4\) and Khaled A. Gepreel\(^5\)

\(^1\) Department of Mathematics, King Saud University, P.O.Box 22452, Riyadh 11495, Saudi Arabia
\(^2\) Department of Mathematics, Government College University, Faisalabad, Pakistan
\(^3\) Department of Mathematics and Statistics, Imam Muhammad Ibn Saud Islamic University, Riyadh, Saudi Arabia
\(^4\) Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan
\(^5\) Department of Mathematics, Faculty of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

* Correspondence: Email: saimarashid@gcuf.edu.pk.

Abstract: Discrete fractional calculus (DFC) use to analyse nonlocal behaviour of models has acquired great importance in recent years. The aim of this paper is to address the discrete fractional operator underlying discrete Atangana-Baleanu (AB)-fractional operator having \( h \)-discrete generalized Mittag-Leffler kernels in the sense of Riemann type (ABR). In this strategy, we use the \( h \)-discrete AB-fractional sums in order to obtain the Grüss type and certain other related variants having discrete generalized \( h \)-Mittag-Leffler function in the kernel. Meanwhile, several other variants found by means of Young, weighted-arithmetic-geometric mean techniques with a discretization are formulated in the time domain \( h\mathbb{Z} \). At first, the proposed technique is compared to discrete AB-fractional sums that uses classical approach to derive the numerous inequalities, showing how the parameters used in the proposed discrete \( h \)-fractional sums can be estimated. Moreover, the numerical meaning of the suggested study is assessed by two examples. The obtained results show that the proposed technique can be used efficiently to estimate the response of the neural networks and dynamic loads.

Keywords: discrete fractional calculus; Atangana-Baleanu fractional differences and sums; discrete Mittag-Leffler function; Grüss type inequality; Young inequality

1. Introduction

DFC has captivated a lot of consideration across various analysis and engineering disciplines, particularly in modelling [1], neural networks [2] and image encryption [3]. The developing approach
portraying real-world problems have been exhibited to be helpful in numerical devices to analyze, comprehend and predict the nature within humankind live [4–10]. In 1974, Daiz et al. [11] introduced the idea of DFC and composed it with an infinite sum. Later on, in 1988, Gray et al. [12] extended this concept and implemented it on the finite sum. This concept is known as the nabla difference operator in the literature. Atici and Eloe [13] proposed the theory of fractional difference equations, although the practical implementation is presented in [14]. Yilmazer [15] proposed discrete fractional solution of a nonhomogeneous non-Fuchsian differential equations. Yilmazer and Ali [16] derived the discrete fractional solutions of the Hydrogen atom type equations. Many researchers’ focus is directed towards modeling and analysis of various problems in bio-mathematical sciences. This field demonstrates several distinguished kernels depending on discrete power law, discrete exponential-law and discrete Mittag-Leffler law kernels which correspond to the Liouville-Caputo, Caputo-Fabrizio and the Atangana-Baleanu nabla(d) difference operators generalized ℏZ time scale [17–19].

Numerous utilities have been developed via DFC such as the solution of fractional difference equations and discrete boundary value problems are proposed in terms of new mathematical techniques [20–23]. Therefore, the conventional methodology of DFC have some intriguing and less-acknowledged opportunities for modelling. DFC is proposed to depict the customary practice of time scale analysis, with discussing its numerical approximations in ℏZ. Furthermore, we observe that ℏ-discrete fractional calculus is tremendously momentous in applied sciences and can also address the requirements of synchronous operation of various mechanisms, see [24–26].

Among the computational models formulated in fractional calculus, discrete AB-fractional operators, which is a universal operator of fractional calculus that has been traditionally employed to develop modern operators and their characterizations have been proposed in research article [27, 28]. Moreover, DFC has been theoretically presented more by introducing and analyzing discrete forms of these fractional operators [13]. Here, we intend to find the discrete fractional inequalities analogous to fractional operators having ℏ-discrete Mittag-Leffler kernels, encompassing and simplifying these operators in such a manner as to recuperate certain appropriate traits such as discrete inequalities for AB-fractional sums.

Mathematical inequalities [29–38] initially alluded to adjust, harmony, and coordination. Until modern times, refinements of inequalities were characterized as invariance to change [39–43]. Physics comprehends fractional inequalities as predictability, while Psychology accentuates that inequality is the trait of magnificence and art [44].

Numerous investigations have been directed on fractional inequalities in the natural science [45], engineering sciences, see [41, 46–48] and the references cited therein. Landscapes, structures, and mechanical equipment all demonstrate inequalities attributes. Therefore, we intend to find the discrete version of the Grüss type and some further connected modifications by the ℏ-discrete AB-fractional sums depending on ℏ-discrete generalized Mittag-Leffler kernel. This stands as an inspiration for the current paper. The intensively investigated Grüss inequality can be presented as follows:

**Theorem 1.1.** (See [49]) Let \(\mathcal{F}, \mathcal{G} : [c, d] \rightarrow \mathbb{R}\) be two positive functions such that \(\alpha \leq \mathcal{F}(x) \leq \mathcal{A}\) and \(\beta \leq \mathcal{G}(x) \leq \mathcal{B}\) for all \(x \in [c, d]\) and \(\alpha, \beta, \mathcal{A}, \mathcal{B} \in \mathbb{R}\). Then

\[
\left| \frac{1}{d-c} \int_{c}^{d} \mathcal{F}(x)\mathcal{G}(x)dx - \frac{1}{(d-c)^2} \int_{c}^{d} \mathcal{F}(x)dx \int_{c}^{d} \mathcal{G}(x)dx \right|
\] (1.1)
\[ \frac{1}{4}(A - \alpha)(B - \beta), \]

where the constant 1/4 cannot be improved.

The Grüss inequality Eq (1.1) has been broadly and intensely investigated in engineering and applied analysis, and various developed consequences have been acquired so far. Nevertheless, the prevalent existence of Grüss inequality in scientific fields is not in direct proportion to the consideration it has acknowledged. In application viewpoint, practically all mechanical structures are found to have inequality Eq (1.1), and the vast majority of them have the qualities of discrete and continuous fractional operators [50–63].

Inspired by the excellent dynamical properties of \( h \)-discrete AB-fractional sums differences formulation [64], the limitations of fractional calculus can be ameliorated via discrete and continuous state-of-the-art techniques for effective information chaotic map applications, that can be inferred as a generalization of nonlocal/nonsingular type kernels. These investigations promote further sum/difference operators and related inequalities. It is our aim in this investigation to explore the discrete version of the Grüss type and certain other associated variants with some traditional and forthright inequalities in the frame of \( h \)-discrete AB-fractional sums. We also would like to mention that besides these variants, several other intriguing generalizations are derived. The comparison of Grüss type with other discrete fractional calculus frameworks is currently under investigation. Finally, two examples are presented that correlate with some well-known inequalities in the relative literature and with the proposed strategy.

2. Preliminaries on discrete fractional calculus

In this section, we evoke some basic ideas related to fractional operator, discrete generalized Mittag Leffler functions and the time scale calculus, see the detailed information in [13]. For the sake of simplicity, we use the notation, for \( c, d \in \mathbb{R} \) and \( h > 0 \), \( \mathbb{N}_{c,h} = \{ c, c + h, c + 2h, ... \} \) and \( \mathbb{N}_{d,h} = \{ d, d + h, d + 2h, ... \} \).

2.1. Basics on delta and nabla \( h \)-factorials

**Definition 2.1.** ([65]) The backward difference operator of a function \( \mathcal{F} \) on \( h\mathbb{Z} \) is stated as

\[
\tilde{\nabla}_h \mathcal{F}(t) = \frac{\mathcal{F}(t) - \mathcal{F}(\rho_h(t))}{h},
\]

where \( \rho_h(t) = t - h \) denotes the backward jump operator. Also, the forward difference operator of a function \( \mathcal{F} \) on \( h\mathbb{Z} \) is stated as

\[
\tilde{\Delta}_h \mathcal{F}(t) = \frac{\mathcal{F}(\sigma_h(t)) - \mathcal{F}(t)}{h},
\]

where \( \sigma_h(t) = t + h \) denotes the forward jump operator.

**Definition 2.2.** ([65]) (i) For any \( t, \alpha \in \mathbb{R} \) and \( h > 0 \), the delta \( h \)-factorial function is stated as

\[
t_h^{(\alpha)} = h^\alpha \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)},
\]

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where $\Gamma$ denotes the Euler gamma function. For $h = 1$, then $t^{(\alpha)}_h = \frac{\Gamma((\alpha+1))}{\Gamma((\alpha+1) - \alpha)}$. Also, the division by a pole leads to zero.

(ii) For any $t, \alpha \in \mathbb{R}$ and $h > 0$, the nabla $h$-factorial function is stated as

$$t^{(\alpha)}_h = h^\alpha \frac{\Gamma(\frac{\alpha}{h} + \alpha)}{\Gamma(\frac{\alpha}{h})}.$$  

(2.4)

For $h = 1$, we observe that $t^{(\alpha)} = \frac{\Gamma((\alpha+1))}{\Gamma((\alpha+1) - \alpha)}$.

**Lemma 2.3.** ([64]) Let $t \in \mathbb{T} = \mathbb{N}_{c,h}$, then for all $t \in \mathbb{T}$, we obtain

$$\nabla_{t,h} \left( \frac{(x-t)^{\underline{\alpha}}_h}{(\alpha+1)!} \right) = \frac{(x-t)^\alpha_h}{\alpha!}.$$  

(2.5)

**Lemma 2.4.** ([66]) For the time scale $\mathbb{T} = \mathbb{N}_{c,h}$ then the nabla Taylor polynomial

$$\nabla_{t,h}(x,t) = \frac{(x-t)^\alpha_h}{t!}, \quad t \in \mathbb{N}_0.$$  

(2.6)

2.2. Nabla $h$-discrete Mittag-Leffler function

Now we present the concept of nabla $h$-discrete Mittag-Leffler function which is introduced by [6].

**Definition 2.5.** ([6]) Let $\alpha, \varrho, \Omega \in \mathbb{C}$ having $\Re(\alpha) > 0$ such that $\lambda \in \mathbb{R}$ with $|\lambda h^\alpha| < 1$, then the nabla discrete Mittag-leffler function is defined

$$h^{\hat{E}}_{\alpha,\varrho}(\lambda, \Omega) = \sum_{i=0}^{\infty} \lambda^i \frac{\Omega_h^{i\varrho}}{\Gamma(\alpha i + \varrho)} \quad |\lambda h^\alpha| < 1.$$  

(2.7)

For $\varrho = 1$, we have

$$h^{\hat{E}}_{\alpha}(\lambda, y) = h^{\hat{E}}_{\alpha,1}(\lambda, y) = \sum_{i=0}^{\infty} \lambda^i \frac{y_h^{\varrho}}{\Gamma(\alpha i + 1)} \quad |\lambda h^\alpha| < 1.$$  

(2.8)

The following remark illustrates the strengthening properties why $h\mathbb{Z}$ is important.

**Remark 1.** In view of $h\mathbb{Z}$:

I. letting $h = 1$, we attain the nabla discrete Mittag-Leffler function stated in [67, 68].

II. letting $0 < h < 1$, the interval of convergence to which $\lambda$ lies. Observe that, when $h \rightarrow 0$, then $\alpha \in (0, 1)$. Moreover, when $h \rightarrow 1$ guarantee convergence for $\lambda = \frac{\alpha}{1 - \alpha}, \alpha \in (0, \frac{1}{2})$.

For further investigation of the discrete Mittag-Leffler function we refer the reader to [4].

2.3. Left and right delta fractional sums on $h\mathbb{Z}$

**Definition 2.6.** ([26]) For some $\iota \in \mathbb{N}$, $\alpha > 0$ and let $d = c + \iota h$. Assume that a function $\mathcal{F}$ be defined on $\mathbb{T} = \mathbb{N}_{c,h} \cap \mathbb{N}_{d,h}$. Then the delta $h$-fractional sums in the left and right case are defined as follows

$$(\hat{\Delta}^{-\alpha}_h \mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{t = c/h}^{x/h-\alpha} (x - \sigma(\iota h))(\alpha-1) \mathcal{F}(\iota h)h, \quad x \in \{x + \alpha h : x \in \mathbb{T}\}$$
and

\[ (\hat{\Delta}_d^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{i=x/h}^{d/h-\alpha} (ih - \sigma(x))^{(\alpha-1)}_h\mathcal{F}(ih), \quad x \in \{x - \alpha h : x \in \mathbb{T}\}, \]

respectively.

2.4. Left and right nabla fractional sums on \(h\mathbb{Z}\)

**Definition 2.7.** ([6,66]) Assume that \(h > 0\) and the backward jump operator is \(\rho(x) = x - h\). A function \(\mathcal{F} : \mathbb{N}_{c,h} \mapsto \mathbb{R}\) is said to be nabla \(h\)-fractional sum of order \(\alpha\), if

\[ (\hat{\nabla}_h^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{i=x/h}^{s/h-\alpha} (x - \rho(x))^{(\alpha-1)}_h\mathcal{F}(th), \quad x \in \mathbb{N}_{c,h}. \]

Also, the nabla right \(h\)-fractional sum of order \(\alpha > 0\) (ending at \(d\)) for \(\mathcal{F} : \mathbb{N}_{d,h} \mapsto \mathbb{R}\) is described as follows

\[ (\hat{\nabla}_d^{-\alpha}\mathcal{F})(t) = \frac{1}{\Gamma(\alpha)} \sum_{i=x/h}^{d/h-\alpha} (ih - \rho(x))^{(\alpha-1)}_h\mathcal{F}(th). \]

2.5. Nabla \(h\)-fractional differences with \(h\)-discrete Mittag Leffler kernels

Now, we demonstrate the some new concepts which we will be utilized for proving coming results of this paper, see [4]. Also, we use the notation, \(\lambda = -\frac{\alpha}{1-\alpha}\) and \(\rho(x) = x - h\).

**Definition 2.8.** ([64]) For \(\alpha \in [0,1]\), \(h > 0\) with \(|\lambda h^n| < 1\) and let \(\mathcal{F}\) be a function defined on \(\mathbb{N}_{c,h} \cap d,h\mathbb{N}\) with \(c < d\) such that \(c \equiv d(mod\ h)\), then the left nabla \(ABC\)-fractional difference (in the sense of Atangana and Baleanu) is described as

\[ (\hat{\Delta}_c^{ABC}\mathcal{F})(x) = \mathbb{B}(\alpha,h) \frac{1-\alpha + \alpha h}{1-\alpha} \sum_{i=c/h}^{x/h} h\hat{\nabla}_h\mathcal{F}(ih) \hat{E}_{\alpha}(\lambda, x - \rho(\lambda)) \]

(2.9)

and in the left Riemann sense by

\[ (\hat{\Delta}_c^{ABR}\mathcal{F})(x) = \mathbb{B}(\alpha,h) \frac{1-\alpha + \alpha h}{1-\alpha} \hat{\nabla}_h \sum_{i=c/h}^{x/h} h\mathcal{F}(ih) \hat{E}_{\alpha}(\lambda, x - \rho(\lambda)). \]

(2.10)

**Definition 2.9.** ([64]) For \(0 < \alpha < 1\) and let the left \(h\)-fractional sum concern to \((\hat{\Delta}_c^{ABR}\mathcal{F})(x)\) defined on \(\mathbb{N}_{c,h}\) is stated as follows

\[ (\hat{\Delta}_c^{ABR^{-\alpha}}\mathcal{F})(x) = \frac{1-\alpha}{\mathbb{B}(\alpha,h)(1-\alpha + \alpha h)} \mathcal{F}(x) \]
Taking product both sides of Eq. (3.4) by \(1\), therefore,

\[
\frac{\alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)\Gamma(\alpha)} \sum_{i=c/h+1}^{x/h} (x - \rho(\vartheta))^{-1}_h \mathcal{F}(\vartheta)h. 
\]

(2.11)

The right \(h\)-fractional sum is defined on \(d, n\) by

\[
(\hat{\nabla}_d^\alpha \mathcal{F})(x) = \frac{1 - \alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)\Gamma(\alpha)} \mathcal{F}(x) + \frac{\alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)\Gamma(\alpha)} \sum_{i=x/h+1}^{d/h-1} (\vartheta - \rho(\vartheta))^{-1}_h \mathcal{F}(\vartheta)h.
\]

(2.12)

3. Discrete Grüss type inequalities

In this section, we present a different concept of Grüss type inequalities, which consolidates the ideas of \(h\)-discrete AB-fractional sums.

**Theorem 3.1.** Let \(\alpha \in (0, 1)\) and let \(\mathcal{F}\) be a positive function on \(\mathbb{N}_{c+h}\). Suppose that there exist two positive functions \(\phi_1, \phi_2\) on \(\mathbb{N}_{c,h}\) such that

\[
\phi_1(x) \leq \mathcal{F}(x) \leq \phi_2(x), \quad \forall x \in \mathbb{N}_{c,h}. 
\]

(3.1)

Then, for \(x \in \{c, c + h, c + 2h, \ldots\}\), one has

\[
\begin{align*}
\hat{\nabla}_c^\beta \phi_2(x) & \hat{\nabla}_c^\alpha \mathcal{F}(x) + \frac{\alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)\Gamma(\alpha)} \sum_{i=1}^{h} (\vartheta - \rho(\vartheta))^{-1}_h \mathcal{F}(\vartheta)h, \\
\hat{\nabla}_c^\beta \phi_2(x) & \hat{\nabla}_c^\alpha \mathcal{F}(x) \geq \frac{\alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)\Gamma(\alpha)} \sum_{i=1}^{h} (\vartheta - \rho(\vartheta))^{-1}_h \mathcal{F}(\vartheta)h.
\end{align*}
\]

(3.2)

**Proof.** From Eq (3.1), for \(\theta, \lambda \in \mathbb{N}_{c,h}\), we have

\[
(\phi_2(\theta) - \mathcal{F}(\theta))(\mathcal{F}(\lambda) - \phi_1(\lambda)) \geq 0.
\]

(3.3)

Therefore,

\[
\phi_2(\theta)\mathcal{F}(\lambda) + \phi_1(\lambda)\mathcal{F}(\theta) \geq \phi_1(\lambda)\phi_2(\theta) + \mathcal{F}(\theta)\mathcal{F}(\lambda).
\]

(3.4)

Taking product both sides of Eq (3.4) by \(\frac{1-\alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)}\), we get

\[
\frac{(1 - \alpha)\phi_2(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)} + \frac{(1 - \alpha)\phi_1(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)} \geq \frac{(1 - \alpha)\phi_1(\lambda)\phi_2(\theta)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)} + \frac{(1 - \alpha)\mathcal{F}(\theta)\mathcal{F}(\lambda)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)}.
\]

(3.5)

Replacing \(\lambda\) by \(t\) in Eq (3.5) and conducting product both sides by \(\frac{\alpha(x - \rho(t))^{-\alpha}_h}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\), we have

\[
\frac{\alpha(x - \rho(t))^{-\alpha}_h}{\mathbb{B}(\alpha, h)\Gamma(\alpha)} \phi_2(\theta)\mathcal{F}(t) + \frac{\alpha(x - \rho(t))^{-\alpha}_h}{\mathbb{B}(\alpha, h)\Gamma(\alpha)} \phi_1(t)\mathcal{F}(\theta)
\]

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Adding Eqs (3.5) and (3.6), we have

\[
\sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(t))}{h} \phi_1(t)\phi_2(\theta) + \frac{\alpha(x - \rho(t))}{h} \phi_1(\theta)\mathcal{F}(\theta)\mathcal{F}(t).
\]

Summing both sides for \( t \in \{c, c + h, c + 2h, \ldots\} \), we get

\[
\sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \phi_1(\theta)\mathcal{F}(i)h\mathcal{F}(t) + \sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \phi_1(ih)\mathcal{F}(\theta)\mathcal{F}(i)h.
\]

Adding Eqs (3.5) and (3.6), we have

\[
\frac{(1 - \alpha)\phi_2(\theta)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} + \sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \phi_2(\theta)\mathcal{F}(i)h\mathcal{F}(t) + \frac{(1 - \alpha)\phi_1(\lambda)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} + \sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \phi_1(\lambda)\mathcal{F}(\theta)\mathcal{F}(t)
\]

\[
\geq \frac{(1 - \alpha)\phi_1(\lambda)\phi_2(\theta)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} + \sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \phi_1(\theta)\mathcal{F}(i)h\mathcal{F}(t) + \frac{(1 - \alpha)\mathcal{F}(\theta)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} + \sum_{i=c/h+1}^{x/h} \frac{\alpha(x - \rho(ih))}{h} \mathcal{F}(\theta)\mathcal{F}(i)h.
\]

arrives at

\[
\phi_2(\theta) \frac{AB\nabla^{-\alpha}_h}{\mathcal{B}(\alpha, h)} [\mathcal{F}(x)] + \frac{\mathcal{F}(\theta)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} [\phi_1(x)] \geq \phi_2(\theta) \frac{AB\nabla^{-\alpha}_h}{\mathcal{B}(\alpha, h)} [\mathcal{F}(x)] + \frac{\mathcal{F}(\theta)}{\mathcal{B}(\alpha, h)(1 - \alpha + ah)} [\mathcal{F}(x)].
\]

Taking product both sides of Eq (3.7) by \( \frac{1 - \beta}{\beta(\beta)(1 - \beta + \beta h)} \), we have

\[
\frac{(1 - \beta)\phi_2(\theta)}{\mathcal{B}(\beta, h)(1 - \beta + ah)} + \frac{(1 - \beta)\mathcal{F}(\theta)}{\mathcal{B}(\beta, h)(1 - \beta + ah)} [\phi_1(\lambda)] \geq \frac{(1 - \beta)\phi_1(\lambda)\phi_2(\theta)}{\mathcal{B}(\beta, h)(1 - \beta + ah)} + \frac{(1 - \beta)\mathcal{F}(\theta)}{\mathcal{B}(\beta, h)(1 - \beta + ah)} [\phi_1(x)].
\]

Also, replacing \( \theta \) by \( \tilde{\theta} \) in Eq (3.8) and conducting product both sides by \( \frac{\beta(x - \rho(t))}{\beta(\beta)(1 - \beta + \beta h)} \), we have

\[
\frac{\beta(x - \rho(t))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \phi_2(\theta) [\phi_1(x)] + \frac{\beta(x - \rho(t))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \mathcal{F}(\theta) [\phi_1(x)]
\]

\[
\geq \frac{\beta(x - \rho(t))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \phi_2(\theta) [\phi_1(x)] + \frac{\beta(x - \rho(t))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \mathcal{F}(\theta) [\phi_1(x)].
\]

Summing both sides for \( \tilde{\theta} \in \{c, c + h, c + 2h, \ldots\} \), we get

\[
\sum_{j=c/h+1}^{x/h} \frac{\beta(x - \rho(jh))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \phi_2(jh) \frac{AB\nabla^{-\alpha}_h}{\mathcal{B}(\beta, h)(1 - \beta + ah)} [\mathcal{F}(x)] + \sum_{j=c/h+1}^{x/h} \frac{\beta(x - \rho(jh))}{\mathcal{B}(\beta, h)(1 - \beta + ah)} \mathcal{F}(jh) \frac{AB\nabla^{-\alpha}_h}{\mathcal{B}(\beta, h)(1 - \beta + ah)} [\phi_1(x)].
\]


g \geq \sum_{j=c/h+1}^{x/h} \frac{\beta(x - \rho(jh))}{\mathbb{B}(\beta, h)\Gamma(\beta)} \phi_2(jh) h^\alpha \phi_1(x) + \sum_{j=c/h+1}^{x/h} \frac{\beta(x - \rho(jh))}{\mathbb{B}(\beta, h)\Gamma(\beta)} F(jh) h^\alpha \phi_1(x).

(3.9)

Adding Eqs (3.8) and (3.9), then in view of Definition 2.9, yields the inequality Eq (3.11). This completes the proof.

Some special cases which can be derived immediately from Theorem 3.1.
Choosing \( h = 1 \), then we attain a new result for discrete AB-fractional sum.

**Corollary 1.** Let \( \alpha \in (0, 1) \) and let \( F \) be a positive function on \( \mathbb{N}_c \). Suppose that there exist two positive functions \( \phi_1, \phi_2 \) on \( \mathbb{N}_c \) such that

\[
\phi_1(x) \leq F(x) \leq \phi_2(x), \quad \forall x \in \mathbb{N}_c.
\]

(3.10)

Then, for \( x \in \{c, c + 1, c + 2, \ldots\} \), one has

\[
\frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_1(x) + \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_2(x) + \frac{\alpha}{c} \phi_1(x) = \frac{1}{c} \phi_1(x) \frac{1}{c} \phi_2(x) \frac{1}{c} \phi_2(x) + \frac{1}{c} \phi_2(x) = \frac{1}{c} \phi_2(x).
\]

(3.11)

**Theorem 3.2.** Let \( \alpha, \beta \in (0, 1) \) and let \( F \) and \( G \) be two positive functions on \( \mathbb{N}_{c,h} \). Suppose that Eq (3.1) satisfies and also one assumes that there exist two positive functions \( \Omega_1, \Omega_2 \) on \( \mathbb{N}_{c,h} \) such that

\[
\Omega_1(x) \leq G(x) \leq \Omega_2(x), \quad \forall x \in \mathbb{N}_{c,h}.
\]

(3.12)

Then, for \( x \in \{c, c + h, c + 2h, \ldots\} \), one has

\[
\begin{align*}
(M_1) & \quad \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_1(x) + \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_2(x) + \frac{1}{c} \phi_2(x) = \frac{1}{c} \phi_2(x) \frac{1}{c} \phi_2(x) + \frac{1}{c} \phi_2(x), \\
(M_2) & \quad \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_1(x) + \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_2(x) + \frac{1}{c} \phi_2(x) = \frac{1}{c} \phi_2(x) \frac{1}{c} \phi_2(x) + \frac{1}{c} \phi_2(x), \\
(M_3) & \quad \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_1(x) + \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_2(x) + \frac{1}{c} \phi_2(x) = \frac{1}{c} \phi_2(x) \frac{1}{c} \phi_2(x) + \frac{1}{c} \phi_2(x), \\
(M_4) & \quad \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_1(x) + \frac{\alpha}{c} \phi_2(x) \frac{\alpha}{c} \phi_2(x) + \frac{1}{c} \phi_2(x) = \frac{1}{c} \phi_2(x) \frac{1}{c} \phi_2(x) + \frac{1}{c} \phi_2(x).
\end{align*}
\]

(3.13)

**Proof.** To prove Eq (M_1), from Eqs (3.1) and (3.12), we have for \( \lambda, \theta \in \mathbb{N}_{c,h} \) that

\[
(\phi_2(\theta) - F(\theta))(G(\lambda) - \Omega_1(\lambda)) \geq 0.
\]

(3.14)

Therefore,

\[
\phi_2(\theta)G(\lambda) + \Omega_1(\lambda)F(\theta) \geq \Omega_1(\lambda)\phi_2(\theta) + G(\lambda)\Omega_1(\lambda).
\]

(3.15)
Taking product both sides of Eq (3.17) by \( \frac{1 - \alpha}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)} \), we get

\[
(1 - \alpha)\varphi_2(\theta)\mathcal{G}(\lambda) + (1 - \alpha)\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{(1 - \alpha)\varphi_2(\theta)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)} + \frac{(1 - \alpha)\Omega_1(\lambda)\mathcal{F}(\theta)}{\mathbb{B}(\alpha, h)(1 - \alpha + \alpha h)}.
\]

(3.16)

Moreover, replacing \( \lambda \) by \( t \) in Eq (3.17) and conducting product both sides by \( \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)} \), we have

\[
\frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]

(3.17)

Summing both sides for \( t \in [c, c + h, c + 2h, \ldots] \), we get

\[
\sum_{t \in \mathbb{C}/h+1} \frac{x}{h} \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\varphi_2(\theta)\mathcal{G}(\lambda) + \sum_{t \in \mathbb{C}/h+1} \frac{x}{h} \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \sum_{t \in \mathbb{C}/h+1} \frac{x}{h} \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\varphi_2(\theta)\mathcal{G}(\lambda) + \sum_{t \in \mathbb{C}/h+1} \frac{x}{h} \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\alpha, h)\Gamma(\alpha)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]

Then, we have

\[
\frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]

(3.18)

Taking product both sides of Eq (3.18) by \( \frac{1 - \beta}{\mathbb{B}(\beta, h)(1 - \beta + \beta h)} \), we have

\[
\frac{1 - \beta}{\mathbb{B}(\beta, h)(1 - \beta + \beta h)} \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{1 - \beta}{\mathbb{B}(\beta, h)(1 - \beta + \beta h)} \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{1 - \beta}{\mathbb{B}(\beta, h)(1 - \beta + \beta h)} \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{1 - \beta}{\mathbb{B}(\beta, h)(1 - \beta + \beta h)} \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]

(3.19)

Further, replacing \( \theta \) by \( t \) in Eq (3.19) and conducting product both sides by \( \frac{\beta(x - \rho(t))^{\varpi}}{\mathbb{B}(\beta, h)\Gamma(\beta)} \), we have

\[
\frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\alpha(x - \rho(t))^{\varpi}}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]

(3.20)

Summing both sides for \( t \in [c, c + h, c + 2h, \ldots] \), we get

\[
\frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta) \\
\geq \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\varphi_2(\theta)\mathcal{G}(\lambda) + \frac{\vartheta}{\mathbb{B}(\beta, h)\Gamma(\beta)}\Omega_1(\lambda)\mathcal{F}(\theta).
\]
\[
\geq c^{\frac{AB\nabla^\beta[\Omega_1(x)]}{h}} \sum_{j=c/h+1}^{\frac{3/h}{j}} \beta(x - \rho(jh))^\frac{\beta}{h} \epsilon(\beta, h) \Gamma(\beta) \phi_2(jh) h + c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}} \sum_{j=c/h+1}^{\frac{3/h}{j}} \beta(x - \rho(jh))^\frac{\beta}{h} \epsilon(\beta, h) \Gamma(\beta) - \mathcal{F}(j) h.
\]

(3.21)

Adding Eqs (3.19) and (3.21), we conclude the desired inequality Eq (M_1).
To prove Eqs (M_2)–(M_4), we utilize the following inequalities:

\begin{align*}
(M_2) & \quad (\Omega_2(\theta) - \mathcal{G}(\theta))(\mathcal{F}(\lambda) - \phi_1(\lambda)) \geq 0, \\
(M_3) & \quad (\phi_2(\theta) - \mathcal{G}(\theta))(\mathcal{F}(\lambda) - \Omega_2(\lambda)) \leq 0, \\
(M_4) & \quad (\phi_1(\theta) - \mathcal{G}(\theta))(\mathcal{F}(\lambda) - \Omega_1(\lambda)) \leq 0.
\end{align*}

Some special cases which can be derived immediately from Theorem 3.2.
Choosing \( h = 1 \), then we attain a new result for discrete AB-fractional sums.

**Corollary 2.** Let \( \alpha, \beta \in (0, 1) \) and let \( \mathcal{F} \) and \( \mathcal{G} \) be two positive functions on \( \mathbb{N}_c \). Suppose that Eq (3.1) satisfies and also one assumes that there exist two positive functions \( \Omega_1, \Omega_2 \) on \( \mathbb{N}_c \) such that

\[ \Omega_1(x) \leq \mathcal{G}(x) \leq \Omega_2(x), \quad \forall x \in \mathbb{N}_c. \]

Then, for \( x \in \{c, c + 1, c + 2, \ldots\} \), one has

\begin{align*}
(M_5) & \quad c^{\frac{AB\nabla^{-\beta}[\phi_2(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_1(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\phi_2(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_1(x)]}{h}},
\end{align*}

\begin{align*}
(M_6) & \quad c^{\frac{AB\nabla^{-\beta}[\phi_1(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_2(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\phi_1(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_2(x)]}{h}},
\end{align*}

\begin{align*}
(M_7) & \quad c^{\frac{AB\nabla^{-\beta}[\phi_2(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_2(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\phi_2(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_2(x)]}{h}},
\end{align*}

\begin{align*}
(M_8) & \quad c^{\frac{AB\nabla^{-\beta}[\phi_1(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_1(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\phi_1(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\Omega_1(x)]}{h}}.
\end{align*}

**Theorem 3.3.** Let \( \alpha, \beta \in (0, 1) \) and let \( \mathcal{F} \) and \( \mathcal{G} \) be two positive functions on \( \mathbb{N}_{c,h} \) with \( p, q > 0 \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, for \( x \in \{c, c + h, c + 2h, \ldots\} \), one has

\begin{align*}
(M_9) & \quad \frac{1}{p} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{h}},
\end{align*}

\begin{align*}
(M_{10}) & \quad \frac{1}{p} c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}},
\end{align*}

\begin{align*}
(M_{11}) & \quad \frac{1}{p} c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}} \\
& \geq c^{\frac{AB\nabla^{-\beta}[\mathcal{G}(x)]}{c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}} c^{\frac{AB\nabla^{-\beta}[\mathcal{F}(x)]}{h}}.
\end{align*}
Proof. According to the well-known Young’s inequality:

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab, \quad \forall a, b \geq 0, \quad p, q > 0, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

setting $a = \mathcal{F}(\theta)\mathcal{G}(\lambda)$ and $b = \mathcal{F}(\lambda)\mathcal{G}(\theta)$, $\theta, \lambda > 0$, we have

$$\frac{1}{p} (\mathcal{F}(\theta)\mathcal{G}(\lambda))^p + \frac{1}{q} (\mathcal{F}(\lambda)\mathcal{G}(\theta))^q \geq (\mathcal{F}(\theta)\mathcal{G}(\lambda))(\mathcal{F}(\lambda)\mathcal{G}(\theta)).$$

Taking product both sides of Eq (3.24) by $\frac{1 - \alpha}{B(a, h)(1 - \alpha + \alpha h)}$, we have

$$\frac{1}{p} \frac{1}{B(a, h)(1 - \alpha + \alpha h)} \frac{(1 - \alpha)\mathcal{F}^p(\theta)\mathcal{G}^p(\lambda)}{\mathcal{F}(\theta)\mathcal{G}(\lambda)} + \frac{1}{q} \frac{1}{B(a, h)(1 - \alpha + \alpha h)} \frac{(1 - \alpha)\mathcal{F}^q(\lambda)\mathcal{G}^q(\theta)}{\mathcal{F}(\lambda)\mathcal{G}(\theta)} \geq \frac{1 - \alpha}{B(a, h)(1 - \alpha + \alpha h)} \frac{(1 - \alpha)\mathcal{F}(\theta)\mathcal{G}(\lambda)(\mathcal{F}(\lambda)\mathcal{G}(\theta))}{\mathcal{F}(\theta)\mathcal{G}(\lambda)}.$$

Moreover, replacing $\lambda$ by $t$ in Eq (3.25) and conducting product both sides by $\frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)}$, we have

$$\mathcal{F}^p(\theta) \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{G}^p(\lambda) + \mathcal{G}^q(\theta) \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}^q(\lambda) \geq \mathcal{F}(\theta)\mathcal{G}(\lambda)\mathcal{F}(\lambda)\mathcal{G}(\theta).$$

Summing both sides for $t \in \{c, c + h, c + 2h, \ldots\}$, we get

$$\mathcal{F}(\theta)\mathcal{G}(\lambda) \sum_{t = c}^{x/h} \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}(\lambda)\mathcal{G}(\theta) \geq \mathcal{F}(\theta)\mathcal{G}(\lambda) \sum_{t = c}^{x/h} \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}(\lambda)\mathcal{G}(\theta).$$

Adding Eqs (3.24) and (3.27), we get

$$\frac{1}{p} \frac{1}{B(a, h)(1 - \alpha + \alpha h)} \frac{(1 - \alpha)\mathcal{F}^{p+q}(\theta)\mathcal{G}^{p+q}(\lambda)}{\mathcal{F}(\theta)\mathcal{G}(\lambda)} + \mathcal{F}^p(\theta) \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{G}^p(\lambda) + \mathcal{G}^q(\theta) \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}^q(\lambda) \geq \mathcal{F}(\theta)\mathcal{G}(\lambda)\mathcal{F}(\lambda)\mathcal{G}(\theta).$$

In view of Definition 2.9, yields

$$\frac{\mathcal{F}^p(\theta)}{p} \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}^q(\lambda) + \frac{\mathcal{G}^q(\theta)}{q} \frac{\alpha(x - \rho(t))^{\frac{n-1}{n}}}{B(a, h)\Gamma(\alpha)} \mathcal{F}^p(\lambda) \geq \mathcal{F}(\theta)\mathcal{G}(\lambda)\mathcal{F}(\lambda)\mathcal{G}(\theta).$$
Again, taking product both sides of Eq (3.29) by $\frac{1-\beta}{\Xi(\beta, h)(1-\beta + \beta h)}$, we have

$$
\frac{F^p(\theta)}{p} (1 - \beta) \hat{\nabla}_{\eta}^\alpha [G^p(x)] + \frac{G^q(\theta)}{q} (1 - \beta) \hat{\nabla}_{\eta}^\alpha [F^p(x)] \geq (1 - \beta) \hat{\nabla}_{\eta}^\alpha [F(x)G(x)] - \frac{1-\beta}{\Xi(\beta, h)(1-\beta + \beta h)} F(\theta)G(\theta).
$$

(3.30)

Further, replacing $\theta$ by $i$ in Eq (3.29) and conducting product both sides by $\frac{\beta(x-\rho(i))^{\beta}}{\Xi(\beta, h)(\rho)}$, we have

$$
\frac{1}{p} \hat{\nabla}_{\eta}^\alpha [G^p(x)] \beta(x-\rho(i))^{\beta} \hat{\nabla}_{\eta}^\alpha [F^p(i)] + \frac{1}{q} \hat{\nabla}_{\eta}^\alpha [F^p(x)] \beta(x-\rho(i))^{\beta} \hat{\nabla}_{\eta}^\alpha [G^q(i)] \geq \hat{\nabla}_{\eta}^\alpha [F(x)G(x)] \beta(x-\rho(i))^{\beta} \hat{\nabla}_{\eta}^\alpha [G^q(i)].
$$

(3.31)

After summing the above inequality Eq (3.31) both sides for $i \in \{c, c + h, c + 2h, \ldots\}$, yields the desired assertion Eq (M9).

The remaining variants can be derived by adopting the same technique and accompanying the selection of parameters in Young inequality.

(M10) $a = \frac{F(\theta)}{\Phi(\lambda)}$, $b = \frac{G(\theta)}{\Phi(\lambda)}$, $\Phi(\lambda), G(\lambda) \neq 0$,

(M11) $a = \frac{F(\theta)G^\frac{2}{\gamma}(\lambda)}{\Phi(\lambda)}, b = \frac{\Phi^\frac{2}{\gamma}(\lambda)G(\theta)}{\Phi(\lambda)}$,

(M12) $a = \frac{\Phi^\frac{2}{\gamma}(\theta)F(\lambda)}{\Phi(\lambda)}, b = \frac{G^\frac{2}{\gamma}(\theta)G(\lambda)}{\Phi(\lambda)}, \Phi(\lambda), G(\lambda) \neq 0$.

Repeating the foregoing argument, we obtain Eqs (M10)–(M12).

(1) Letting $h = 1$, then we attain a result for discrete AB-fractional sums.

**Corollary 3.** Let $\alpha, \beta \in (0, 1)$ and let $F$ and $G$ be two positive functions on $\mathbb{N}$ with $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $x \in \{c, c + 1, c + 2, \ldots\}$, one has

(M13) $\frac{1}{p} \hat{\nabla}_{\eta}^\alpha [F^p(x)] \hat{\nabla}_{\eta}^\alpha [G^p(x)] + \frac{1}{q} \hat{\nabla}_{\eta}^\alpha [G^q(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)] \geq \hat{\nabla}_{\eta}^\alpha [F(x)G(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)]$,

(M14) $\frac{1}{p} \hat{\nabla}_{\eta}^\alpha [G^q(x)] \hat{\nabla}_{\eta}^\alpha [F^p(x)] + \frac{1}{q} \hat{\nabla}_{\eta}^\alpha [F^q(x)] \hat{\nabla}_{\eta}^\alpha [G^q(x)] \geq \hat{\nabla}_{\eta}^\alpha [F^{-1}F^{-1}(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)]$,

(M15) $\frac{1}{p} \hat{\nabla}_{\eta}^\alpha [F^q(x)] \hat{\nabla}_{\eta}^\alpha [G^q(x)] + \frac{1}{q} \hat{\nabla}_{\eta}^\alpha [F^q(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)] \geq \hat{\nabla}_{\eta}^\alpha [F^{-1}F^{-1}(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)]$,

(M16) $\frac{1}{p} \hat{\nabla}_{\eta}^\alpha [G^q(x)] \hat{\nabla}_{\eta}^\alpha [F^p(x)] + \frac{1}{q} \hat{\nabla}_{\eta}^\alpha [G^q(x)] \hat{\nabla}_{\eta}^\alpha [F^p(x)] \geq \hat{\nabla}_{\eta}^\alpha [F^{-1}G^{-1}(x)] \hat{\nabla}_{\eta}^\alpha [F^q(x)]$.

(3.32)
Example 3.4. Let $\alpha, \beta \in (0, 1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,h}$ with $p, q > 0$ satisfying $p + q = 1$. Then, for $x \in \{c, c + h, c + 2h, \ldots\}$, one has

\[ p^c e^{-\beta c}[\mathcal{F}(x)] e^{-\alpha c}[\mathcal{G}(x)] + q^c e^{-\beta c}[\mathcal{F}(x)] e^{-\alpha c}[\mathcal{G}(x)] \geq e^{-\beta c}[\mathcal{F}(x)] e^{-\alpha c}[\mathcal{F}(x)] e^{-\beta c}[\mathcal{F}(x)], \]

Then, for $x \in \{c, c + h, c + 2h, \ldots\}$, one has

\[ (M_1) \quad a = \mathcal{F}(\theta)\mathcal{G}(\lambda), \quad b = \mathcal{F}(\lambda)\mathcal{G}(\theta). \]

\[ (M_2) \quad a = \mathcal{F}(\lambda), \quad b = \frac{\mathcal{G}(\theta)}{\mathcal{G}(\lambda)}, \quad \mathcal{F}(\theta), \mathcal{G}(\lambda) \neq 0. \]

\[ (M_3) \quad a = \mathcal{F}(\theta)\tilde{\mathcal{G}}(\lambda), \quad b = \tilde{\mathcal{F}}(\lambda)\mathcal{G}(\theta), \quad \mathcal{F}(\theta), \mathcal{G}(\lambda) \neq 0. \]

\[ (M_4) \quad a = \frac{\tilde{\mathcal{F}}(\theta)}{\tilde{\mathcal{G}}(\lambda)}, \quad b = \frac{\tilde{\mathcal{F}}(\lambda)}{\tilde{\mathcal{G}}(\theta)}, \quad \mathcal{F}(\theta), \mathcal{G}(\theta) \neq 0. \]

Example 3.5. Let $\alpha \in (0, 1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be two positive functions on $\mathbb{N}_{c,h}$ with $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Let

\[ \gamma = \min_{\theta \in \mathbb{N}_{c,h}} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \quad \text{and} \quad \Upsilon = \max_{\theta \in \mathbb{N}_{c,h}} \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}. \]

Then, for $x \in \{c, c + h, c + 2h, \ldots\}$, one has

\[ (i) \quad 0 \leq \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \leq \frac{\gamma + \Upsilon}{4\gamma\Upsilon} \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right)^2, \]

\[ (ii) \quad 0 \leq \sqrt{\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}} \sqrt{\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}} - \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right)^2 \leq \frac{\sqrt{\Upsilon} - \sqrt{\gamma}}{2\sqrt{\gamma\Upsilon}} \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right)^2, \]

\[ (iii) \quad 0 \leq \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \leq \frac{\gamma + \Upsilon}{4\gamma\Upsilon} \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right)^2. \]

Proof. From Eq. (3.34) and the inequality

\[ \left( \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} - \gamma \right) \left( \Upsilon - \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \right) \mathcal{G}^2(\theta) \geq 0, \quad \theta \in \mathbb{N}_{c,h} \]
then we can write as,

$$F^2(\theta) + \gamma Y G^2(\theta) \leq (\gamma + T) F(\theta) G(\theta). \quad (3.36)$$

Taking product both sides of Eq (3.36) by \(\frac{1-\alpha}{B(\alpha, h)(1-\alpha + ah)}\), we have

$$\frac{(1-\alpha)F^2(\theta)}{B(\alpha, h)(1-\alpha + ah)} + \frac{(1-\alpha)G^2(\theta)}{B(\alpha, h)(1-\alpha + ah)} \gamma x \leq \frac{1-\alpha}{B(\alpha, h)(1-\alpha + ah)}(\gamma + x)F(\theta) G(\theta). \quad (3.37)$$

Replacing \(\theta\) by \(t\) in Eq (3.36) and conducting product both sides by \(\frac{\alpha(x-\rho(t)_{\widehat{\Gamma}(\alpha)}}{\alpha, h)}{B(\alpha, h)\Gamma(\alpha)}\), we have

$$\frac{\alpha(x-\rho(t))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} F^2(t) + \gamma Y \frac{\alpha(x-\rho(t))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} G^2(t) \leq (\gamma + T) \frac{\alpha(x-\rho(t))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} F(t) G(t). \quad (3.38)$$

Summing both sides for \(t \in \{c, c + h, c + 2h, \ldots\}\, we get

$$\sum_{t=c/h+1}^{s/h} \frac{\alpha(x-\rho(h))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} F^2(ah)h + \gamma Y \sum_{t=c/h+1}^{s/h} \frac{\alpha(x-\rho(h))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} G^2(ah)h \leq (\gamma + T) \sum_{t=c/h+1}^{s/h} \frac{\alpha(x-\rho(h))_{h}^{\gamma \Gamma}}{B(\alpha, h)\Gamma(\alpha)} F(ah)hG(ah)h. \quad (3.39)$$

Adding Eqs (3.37) and (3.39), yields

$$A^c B^h \nabla^{-\alpha} [F^2(x)] + \gamma Y A^c B^h \nabla^{-\alpha} [G^2(x)] \leq (\gamma + T) A^c B^h \nabla^{-\alpha} [F G(x)], \quad (3.40)$$

on the other hand, it follows from \(\gamma Y > 0\) and

$$\left(\sqrt{A^c B^h \nabla^{-\alpha} [F^2(x)]} - \sqrt{\gamma Y A^c B^h \nabla^{-\alpha} [G^2(x)]}\right)^2 \geq 0, \quad (3.41)$$

that

$$2 \sqrt{A^c B^h \nabla^{-\alpha} [F^2(x)]} \sqrt{\gamma Y A^c B^h \nabla^{-\alpha} [G^2(x)]} \leq \sqrt{A^c B^h \nabla^{-\alpha} [F^2(x)]} + \sqrt{\gamma Y A^c B^h \nabla^{-\alpha} [G^2(x)]} \quad (3.42)$$

then from Eqs (3.40) and (3.42), we obtain,

$$4\gamma Y A^c B^h \nabla^{-\alpha} [F^2(x)] A^c B^h \nabla^{-\alpha} [G^2(x)] \leq (\gamma + T)^2 (A^c B^h \nabla^{-\alpha} [F G(x)]). \quad (3.43)$$

Which implies (i). By some change of (i), analogously, we get (ii) and (iii).
4. Conclusions

Unlike some known and established inequalities in the literature, the Grüss type inequalities have been presented via the $h$-discrete AB-fractional sums with different values of parameters on the domain $h\mathbb{Z}$ that can be implemented to solve the qualitative properties of difference equations. Our consequences can be applied to overcome the obstacle of obtaining estimation on the explicit bounds of unknown functions and also to extend and unify continuous inequalities by using the simple technique. Several novel consequences have been derived by the use of discrete $h$-fractional sums. The noted consequences can also be extended to the weighted function case. Certainly, the case $h \mapsto 1$ recaptures the outcomes of the discrete $AB$-fractional sums. For indicating the strength of the offered fallouts, we employ them to investigate numerous initial value problems of fractional difference equations.

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Conflict of interest

The authors declare no conflict of interest.

References

1. F. M. Atici, S. Sengul, Modeling with fractional difference equations, *J. Math. Anal. Appl.*, **369** (2010), 1–9.
2. G. C. Wu, D. Baleanu, Discrete fractional logistic map and its chaos, *Nonlinear Dyn.*, **75** (2014), 283–287.
3. G. C. Wu, Z. G. Deng, D. Baleanu, D. Q. Zeng, New variable-order fractional chaotic systems for fast image encryption, *Chaos*, **29** (2019), 083103.
4. T. Abdeljawad, D. Baleanu, On fractional derivatives with generalized Mittag-Leffler kernels, *Adv. Differ. Equations*, **2018** (2018), 1–15.
5. T. Abdeljawad, S. Banerjee, G. C. Wu, Discrete tempered fractional calculus for new chaotic systems with short memory and image encryption, *Optik*, **218** (2020), 163698.
6. T. Abdeljawad, F. Jarad, J. Alzabut, Fractional proportional differences with memory, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3333–3354.
7. A. Atangana, D. Baleanu, New fractional derivative with non-local and non-singular kernel, *Therm. Sci.*, **20** (2016), 757–763.
8. D. Baleanu, J. A. T. Machado, A. C. J. Luo, *Fractional Dynamics and Control*, Springer Science and Business Media, London, UK, 2012.
9. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, **1** (2015), 73–85.
10. J. Danane, K. Allali, Z. Hammouch, Mathematical analysis of a fractional differential model of HBV infection with antibody immune response, *Chaos, Solitons Fractals*, **136** (2020), 109787.
11. J. B. Daiz, T. J. Osler, Differences of fractional order, *Math. Comput.*, **28** (1974), 185–202.
12. H. L. Gray, N. F. Zhang, On a new definition of the fractional difference, *Math. Compt.*, **50** (1988), 513–529.
13. F. M. Atici, P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equations*, **3** (2009), 1–12.
14. S. Sengul, *Discrete fractional calculus and its applications to tumor growth*, Master Thesis, Western Kentucky University, 2010.
15. R. Yilmazer, Discrete fractional solutions of a non-homogeneous non-Fuchsian differential equations, *Thermal Sci.*, **23** (2019), S121–S127.
16. R. Yilmazer, K. K. Ali, On discrete fractional solutions of the Hydrogen atom type equations, *Thermal Sci.*, **23** (2019), S1935–S1941.
17. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 1–12.
18. T. Abdeljawad, F. M. Atici, On the definitions of nabla fractional operators, *Abstr. Appl. Anal.*, **2012** (2012), 1–13.
19. T. Abdeljawad, Fractional difference operators with discrete generalized Mittag-Leffler kernels, *Chaos Solitons Fractals*, **126** (2019), 315–324.
20. A. Fernandez, T. Abdeljawad, D. Baleanu, Relations between fractional models with three-parameter Mittag-Leffler kernels, *Adv. Differ. Equations*, **2020** (2020), 1–13.
21. G. C. Wu, T. Abdeljawad, J. Liu, D. Baleanu, K. T. Wu, Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique, *Nonlinear Anal. Modell. Control*, **24** (2019), 919–936.
22. L. L. Huang, J. H. Park, G. C. Wu, Variable-order fractional discrete-time recurrent neural networks, *J. Comput. Appl. Math.*, **370** (2019), 112633.
23. X. F. Wang, G. Chen, Synchronization in small-world dynamical networks, *Int. J. Bifurc. Chaos*, **12** (2002), 187–192.
24. T. Abdeljawad, Different type kernel h-fractional differences and their fractional ħ-sums, *Chaos Solitons Fractals*, **116** (2018), 146–156.
25. C. S. Goodrich, Continuity of solutions to discrete fractional initial value problems, *Comput. Math. Appl.*, **59** (2010), 3489–3499.
26. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst.*, **29** (2011), 417–437.
27. M. T. Holm, The Laplace transform in discrete fractional calculus, *Comput. Math. Appl.*, **62** (2011), 1591–1601.
28. A. G. M. Selvam, J. Alzabut, R. Dhineshbabu, S. Rashid, M. Rehman, Discrete fractional order two-point boundary value problem with some relevant physical applications, *J. Inequal. Appl.*, **2020** (2020), 1–19.
29. S. Rashid, Z. Hammouch, R. Ashraf, Y. M. Chu, New computation of unified bounds via a more general fractional operator using generalized Mittag-Leffler function in the kernel, *CMES-Comput. Modell. Eng. Sci.*, **126** (2021), 359–378.

30. Z. Khan, S. Rashid, R. Ashraf, D. Baleanu, Y. M. Chu, Generalized trapezium-type inequalities in the settings of fractal sets for functions having generalized convexity property, *Adv. Differ. Equations*, **2020** (2020), 1–24.

31. S. B. Chen, S. Rashid, Z. Hammouch, M. A. Noor, R. Ashraf, Y. M. Chu, Integral inequalities via Raina’s fractional integrals operator with respect to a monotone function, *Adv. Differ. Equations*, **2020** (2020), 1–20.

32. S. Rashid, R. Ashraf, K. S. Nisar, T. Abdeljawad, Estimation of integral inequalities using the generalized fractional derivative operator in the Hilfer sense, *J. Math.*, **2020** (2020), 1626091.

33. S. Rashid, H. Ahmad, A. Khalid, Y. M. Chu, On discrete fractional integral inequalities for a class of functions, *Complexity*, **2020** (2020), 8845867.

34. T. Abdeljawad, S. Rashid, Z. Hammouch, Y. M. Chu, Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications, *Adv. Differ. Equations*, **2020** (2020), 1–26.

35. T. Abdeljawad, S. Rashid, A. A. AL.Deeb, Z. hammouch, Y. M. Chu, Certain new weighted estimates proposing generalized proportional fractional operator in another sense, *Adv. Differ. Equations*, **2020** (2020), 1–16.

36. S. B. Chen, S. Rashid, M. A. Noor, R. Ashraf, Y. M. Chu, A new approach on fractional calculus and probability density function, *AIMS Math.*, **5** (2020), 7041–7054.

37. H. G. Jile, S. Rashid, M. A. Noor, A. Suhail, Y. M. Chu, Some unified bounds for exponentially tgs-convex functions governed by conformable fractional operators, *AIMS Math.*, **5** (2020), 6108–6123.

38. T. Abdeljawad, S. Rashid, Z. Hammouch, Y. M. Chu, Some new local fractional inequalities associated with generalized \((s, m)\)-convex functions and applications, *Adv. Differ. Equations*, **2020** (2020), 1–27.

39. S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, Y. M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics*, **7** (2019), 1225.

40. G. A. Anastassiou, About discrete fractional calculus with inequalities, in *Intelligent mathematics: computational analysis*, Springer, Berlin, Heidelberg, (2011), 575–585.

41. B. Zheng, Some new discrete fractional inequalities and their applications in fractional difference equations, *J. Math. Inequal.*, **9** (2015), 823–839.

42. M. Bohner, R. A. C. Ferreira, Some discrete fractional inequalities of Chebyshev type, *Afr. Diaspora J. Math.*, **11** (2011), 132–137.

43. F. M. Atici, Y. Yaldiz, Refinements on the discrete Hermite-Hadamard inequality, *Arabian J. Math.*, **7** (2018), 175–182.

44. B. G. Pachpatte, Integral and Finite Difference Inequalities and Applications, in *Mathematics Studies*, Elsevier, (2006).
45. R. L. Magin, *Fractional Calculus in Bioengineering*, Redding: Begell House, 2006.

46. M. K. Wang, H. H. Chu, Y. M. Li, Y. M. Chu, Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind, *Appl. Anal. Discrete Math.*, **14** (2020), 255–271.

47. W. M. Qian, Z. Y. He, Y. M. Chu, Approximation for the complete elliptic integral of the first kind, *RACSAM*, **114** (2020), 1–12.

48. Z. H. Yang, W. M. Qian, W. Zhang, Y. M. Chu, Notes on the complete elliptic integral of the first kind, *Math. Inequal. Appl.*, **23** (2020), 77–93.

49. G. Grüss, Über das Maximum des absoluten Betrages von \( \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \), *Math. Z.*, **39** (1935), 215–226.

50. E. Akin, S. Asliyüce, A. F. Güvenilir, B. Kaymakçalan, Discrete Grüss type inequality on fractional calculus, *J. Inequal. Appl.*, **2015** (2015), 1–7.

51. S. Rashid, F. Jarad, M. A. Noor, K. I. Noor, D. Baleanu, J. B. Liu, On Grüss inequalities within generalized K-fractional integrals, *Adv. Differ. Equations*, **2020** (2020), 1–18.

52. T. Abdeljawad, F. M. Atici, On the definitions of nabla fractional operators, *Abstr. Appl. Anal.*, **2012** (2012), 1–13.

53. C. Goodrich, A. C. Peterson, *Discrete Fractional Calculus*, Springer, Berlin, 2015.

54. G. A. Anastassiou, About discrete fractional calculus with inequalities, in *Intelligent Mathematics: Computational Analysis*, Springer, Berlin, Heidelberg, (2011), 575–585.

55. F. M. Atici, Y. Yal diz, Refinements on the discrete Hermite-Hadamard inequality, *Arabian J. Math.*, **7** (2018), 175–182.

56. M. Bohner, R. A. C. Ferreira, Some discrete fractional inequalities of Chebyshev type, *Afr. Diaspora J. Math.*, **11** (2011), 132–137.

57. S. Rashid, Y. M. Chu, J. Singh, D. Kumar, A unifying computational framework for novel estimates involving discrete fractional calculus approaches, *Alexandria Eng. J.*, **60** (2021), 2677–2685.

58. P. O. Mohammed, T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, *Adv. Differ. Equations*, **2020** (2020), 1–19.

59. T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications, *J. Funct. Spaces*, **2020** (2020), 4352357.

60. P. O. Mohammed, T. Abdeljawad, Opial integral inequalities for generalized fractional operators with nonsingular kernel, *J. Inequal. Appl.*, **2020** (2020), 1–12.

61. A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Methods Appl. Sci.*, (2020), 1–18.

62. L. Xu, Y. M. Chu, S. Rashid, A. A. El. Deeb, K. S. Nisar, On new unified bounds for a family of functions via fractional q-calculus theory, *J. Funct. Spaces*, **2020** (2020), 4984612.

63. T. Abdeljawad, S. Rashid, H. Khan, Y. M. Chu, On new fractional integral inequalities for p-convexity within interval-valued functions, *Adv. Differ. Equations*, **2020** (2020), 1–17.
64. T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, *Chaos, Solitons Fractals*, 116 (2018), 146–156.

65. I. Suwan, T. Abdeljawad, F. Jarad, Monotonicity analysis for nabla h-discrete fractional Atangana-Baleanu differences, *Chaos, Solitons Fractals*, 117 (2018), 50–59.

66. I. Suwan, S. Owies, T. Abdeljawad, Monotonicity results for h-discrete fractional operators and application, *Adv. Differ. Equations*, 2018 (2018), 1–17.

67. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, *Discrete Dyn. Nat. Soc.*, 2013 (2013), 1–12.

68. T. Abdeljawad, Fractional difference operators with discrete generalized Mittag-Leffler kernels, *Chaos, Solitons Fractals*, 126 (2019), 315–324.