COSMOLOGICAL POST-NEWTONIAN APPROXIMATION COMPARED WITH PERTURBATION THEORY

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ABSTRACT

We compare the cosmological first-order post-Newtonian (1PN) approximation with the relativistic cosmological linear perturbation theory in a zero-pressure medium with the cosmological constant. We compare equations and solutions in several different gauge conditions available in both methods. In the PN method we have perturbation equations for density, velocity, and gravitational potential independently of the gauge condition to 1PN order. However, correspondences with these 1PN equations are available only in certain gauge conditions in the perturbation theory. Equations of perturbed velocity and the perturbed gravitational potential in the zero-shear gauge exactly coincide with the Newtonian equations, which remain valid even to 1PN order (the same is true for perturbed velocity identified in the comoving gauge), and equations of perturbed density in the zero-shear gauge and the uniform-expansion gauge coincide to 1PN order. We identify other correspondences available in different gauge conditions of the perturbation theory.

Key words: cosmology: theory – large-scale structure of universe

1. INTRODUCTION

Einstein’s gravity is generally accepted as the gravity to handle astronomical phenomena. The theory holds a remarkable track record in the solar-system test based on vacuum Schwarzschild solution and the parameterized post-Newtonian (PN) approximation, where the gravitational fields are supposed to be weak. It is true that Einstein’s theory has not failed in any experimental test based on modern scientific and technological development up until today, but it is also true that there has been no experimental test of the theory in the strong gravitational field and in large scale, even including the galactic scale. Einstein’s gravity is generally accepted in cosmology mainly based on its successes in other astronomical and Earth-bound tests and the theory’s own prestige associated with Einstein’s fame and historical legacy.

A self-consistent treatment of a cosmological world model is possible in Einstein’s gravity. Without a lead by Einstein’s gravity, however, the spatially homogeneous and isotropic cosmological world model based on Newton’s gravity is known to be incomplete and indeterminate (Layzer 1954; Lemons 1988). Despite such troubles in the background world model, evolution of perturbations in Newton’s theory is known to be quite successful in reproducing the corresponding results in Einstein’s gravity (Lifshitz 1946; Bonnor 1957; Noh & Hwang 2004; Jeong et al. 2011). Considering the action-at-a-distance nature of Newton’s gravity, such a coincidence is a non-trivial result. Differences between the two theories, however, appear as the scale approaches the horizon. We will address this issue in this work.

If we accept Einstein’s theory in analyzing the large-scale cosmic structure in the current era, we have two methods available. One well-known method is the perturbation theory, where all dimensionless deviations in the metric and the energy–momentum tensor from the background world model are assumed to be small. If we accept only linear order deviations, we have the linear perturbation theory. The perturbation theory assumes small deviation but is fully relativistic, generally applicable in all scales including the super-horizon scale and to particles with relativistic velocities (Lifshitz 1946; Harrison 1967; Nariai 1969; Bardeen 1980; Peebles 1980; Kodama & Sasaki 1984; Bardeen 1988; Mukhanov et al. 1992; Ma & Bertschinger 1995).

The other less known method is the cosmological PN approximation, where all dimensionless deviations are assumed to be weakly relativistic with \( GM/Rc^2 \ll 1 \) (\( M \) and \( R \) are characteristic mass and length scales) and for a virialized system \( v^2/c^2 \ll 1 \) (\( v \) is the characteristic velocity involved). The first-order PN (1PN) approximation makes expansion up to \( GM/Rc^2 \sim v^2/c^2 \) order. The PN approximation assumes small relativistic effects and is applicable only in sub-horizon scale. But the equations derived are applicable to fully nonlinear situations (Chandrasekhar 1965; Futamase 1988, 1989, 1993a, 1993b; Tomita 1988, 1991; Shibata & Asada 1995; Asada & Futamase 1997; Takada & Futamase 1999; Hwang et al. 2008, PN2008 hereafter). Therefore, the two methods are complementary to each other.

If we encounter cosmological situations where both non-linearity and relativistic effects are important, we may need full-blown numerical relativity implemented in cosmology. Currently such a general relativistic numerical simulation in cosmology is not available. The nonlinear perturbation analysis, being based on the perturbative approach, is not sufficient to handle the genuine nonlinear aspects of structure formation accompanied with self-organization and spontaneous formation of structures. In order to handle the relativistic nonlinear process in cosmology, we believe that the PN approach is currently practically relevant to implement in numerical simulation.

We can find cosmological situations where the cosmological PN approach, being weakly relativistic but fully nonlinear, might have important applications. Especially, the current cosmological paradigm favors a model where the large-scale structures (requiring the relativistic treatment) are in the linear stage, whereas small-scale structures are apparently in the fully nonlinear stage. The often adopted strategy is to assume that the small-scale nonlinear structures are fully under control by the Newtonian gravity. In the galactic and cluster scales, we have the general relativistic measure \( GM/Rc^2 \sim v^2/c^2 \sim 10^{-6} \) to \( 10^{-4} \), thus small but nonvanishing, and indeed the 1PN (weakly relativistic) assumption is quite sufficiently valid. Thus, we
believe that the 1PN approach would be quite relevant to estimate the general relativistic effects in the nonlinear clustering processes of the galaxy-cluster-scale and large-scale structures.

In this work, we will compare the two relativistic methods in the matter-dominated era: the 1PN method versus linear perturbation theory. The 1PN approximation is based on previous studies in Chandrasekhar (1965) and PN2008, whereas the linear perturbation theory is based on previous studies in Bardeen (1988), Hwang (1994), and Hwang & Noh (1999). We will compare the equations and solutions derived in the two methods. In both methods we have gauge degrees of freedom that need to be fixed by the gauge conditions. The Newtonian perturbation theory appears as the zeroth-order PN (0PN) approximation. Thus, we naturally also have Newtonian theory for the comparison. To 1PN order we will show that the equations for density, velocity, and gravitational potential do not depend on the gauge conditions with each variable gauge invariant to 1PN order. In the perturbation theory, however, the perturbation variables for density, velocity, potential, curvature, and other kinematic variables (like expansion and shear) do depend on the gauge conditions adopted. Our emphasis in this work is on the correspondences between the PN variables and perturbation theory variables based on different gauge conditions.

For the velocity and gravitational potential the Newtonian (0PN) equations are valid even to 1PN order, whereas the 0PN equations are valid even to 1PN order, whereas gravitational potential vanishes to the linear or-

Thus, we naturally also have Newtonian theory for the comparison. To 1PN order our metric convention is (Chandrasekhar 1965; Chandrasekhar & Nutku 1969; PN2008) and is valid in the presence of the cosmological constant.

To perturbed order, we have

\[
\sum_{\lambda} U_{\lambda} = -\frac{2}{a^2} \left[ 2 \Delta \Phi - 2 U \Delta U + \left( a P_{\lambda ij} \right)^\lambda \right] + 3 \dot{U} + 9 \frac{\dot{a}}{a} U + 6 \ddot{a} U + 8 \pi G \rho v^2 \left( \rho - \rho_b \right),
\]

where a vertical bar indicates spatial covariant derivative based on \( \gamma_{ij} \) as the metric. These follow from the mass-conservation, momentum-conservation, Raychaudhury (\( G_0^0 - G_0^i \)), and momentum-constraint (\( G_i^j \)) equations, respectively; for the general case, see Equations (114), (115), (119), and (120) in PN2008. Terms on the left-hand side provide the Newtonian (0PN) limit, and the ones on the right-hand side are 1PN contributions. Note that the 1PN terms include up to fourth-order perturbations. To perturbed order, we have \( \rho = \rho_b + \delta \rho \), where we will ignore the subindex \( b \) indicating the background quantity unless necessary.

Our cosmological PN approach assumes near flat background but is valid in the presence of the cosmological constant. Equations for the background,

\[
\begin{align*}
\ddot{a} & = -\frac{4 \pi G}{3} \rho + \frac{\Lambda c^2}{3}, \\
\frac{\dot{a}}{a} & = 8 \pi G \rho + \frac{\Lambda c^2}{3}, \\
\dot{\rho} + 3 \frac{\dot{a}}{a} \rho & = 0,
\end{align*}
\]

were subtracted in deriving the PN equations; see Section 3.2 in PN2008.

### 2.2. Newtonian Equations as the 0PN Limit

The 0PN approximation gives the Newtonian limit. The 0PN metric is

\[
ds^2 = -\left( 1 - \frac{1}{c^2} 2 U \right) c^2 dt^2 + \delta_{ij} dx^i dx^j.
\]
To 0PN limit Equations (4)–(6) give

\[ \delta = -\frac{1}{a} \nabla \cdot [(1 + \delta) \mathbf{v}], \]  

(10)

\[ \dot{\mathbf{v}} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{a} \nabla U, \]  

(11)

\[ -\frac{\Delta}{a^2} U = 4\pi G \Phi \delta, \]  

(12)

where \( \delta \equiv \delta \rho / \rho \), \( \mathbf{v} \equiv \mathbf{v}' \), and \( \Phi \equiv -U \) are the relative density contrast, velocity perturbation, and perturbed Newtonian gravitational potential, respectively.

In the Newtonian context, assuming the presence of spatially homogeneous and isotropic background, the above equations follow from the mass conservation, the momentum conservation, and the Poisson’s equation, respectively (Peebles 1980); for weakness of Newton’s theory in handling the background world model, however, see Layzer (1954) and Lemons (1988). In fact, the complete nonlinear Newtonian hydrodynamic equations are already built in as the 0PN approximation of Einstein’s gravity (Chandrasekhar 1965; Berwicher & Hamilton 1994; Kofman & Pogosyan 1995; see Section 4.2 in PN2008). Up to this point is a review of the cosmological 1PN approach.

To the linear order, from Equations (10)–(12) we can derive

\[ \ddot{\delta} + 2H \dot{\delta} - 4\pi G \Phi \delta = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{\dot{\delta}}{H} \right)^2 \right] = 0, \]  

(13)

\[ (a \nabla \cdot \mathbf{v})' + H (a \nabla \cdot \mathbf{v})' - 4\pi G \Phi (a \nabla \cdot \mathbf{v}) = 0, \]  

(14)

\[ (aU)' + 2H (aU)' - 4\pi G \Phi (aU) = 0, \]  

(15)

where \( H \equiv \dot{a} / a \). Assuming \( K = 0 = \Lambda \), the growing mode solutions are

\[ \delta = -\frac{2}{5} c^2 \Delta C, \quad \frac{1}{a} \nabla \cdot \mathbf{v} = \frac{2}{5} c^2 \Delta C, \quad \frac{1}{c^2} U = \frac{3}{5} C, \]  

(16)

where \( C(x) \) is an integration constant; the normalization of the constant \( C \) will become clear later (see below Equation (75)). Note that as the Newtonian (0PN) equations are not aware of the presence of horizon, Newtonian equations are supposed to be valid in all scales. The Newtonian or the 0PN order equations are not affected by the gauge transformation.

### 2.3. Linear Limit of 1PN Equations

By keeping only linear order terms, Equations (4)–(7) give

\[ \frac{1}{a^2} (a^3 \delta \rho)' + \frac{1}{a} \delta \rho_{ij} = -\frac{3}{c^2} a \dot{U}, \]  

(17)

\[ \frac{1}{a} \left( a v_i \right)' - \frac{1}{a} U_i = \frac{1}{c^2 a} [2 \Phi + (a P_i)], \]  

(18)

\[ \frac{\Delta}{a^2} U + 4\pi G \Phi \delta = -\frac{1}{c^2} \left\{ \frac{1}{a^2} \left[ 2 \Delta \Phi + (a P_i) \right]' \right\} + 3\dot{U} + 9\frac{\dot{a}}{a} U + 6\frac{\ddot{a}}{a} U, \]  

(19)

By taking a divergence, Equation (20) gives

\[ 4\pi G a \dot{v}_i = \Delta U + H U, \]  

(21)

which follows from the 0PN order of Equations (17) and (19). By introducing a combination

\[ U_i = U_i + \frac{1}{c^2} [2 \Phi + (a P_i)], \]  

(22)

Equations (18) and (19) become

\[ \frac{1}{a} (a v_i)' - \frac{1}{a} U_i = 0, \]  

(23)

\[ \frac{\Delta}{a^2} U + 4\pi G \Phi \delta = \frac{3}{c^2} \left( \frac{a}{a} \dot{U} + 2\frac{\ddot{a}}{a} \right). \]  

(24)

From Equations (17)–(19) we can derive (Takada & Futamase 1999)

\[ \ddot{\delta} + 2H \dot{\delta} - 4\pi G \Phi \delta = -\frac{12}{c^2} \left[ \frac{a}{a} \dot{U} + (2H + H^2) \delta \right]. \]  

(26)

Since terms on the right-hand side are already 1PN order, using Equation (19) to the Newtonian order, we have

\[ (a v_i)' + 2H (a v_i)' - 4\pi G \Phi (a v_i) = 0, \]  

(27)

\[ (aU)' + 2H (aU)' - 4\pi G \Phi (aU) = 0, \]  

(28)

where we used Equation (21) to appropriate order. The terms on the right-hand side of Equation (26) are linear contributions from 1PN correction terms. This apparently shows that the cosmological 1PN approximation is valid only far inside the horizon with \( c^2 \Delta \gg 12\pi G a^2 \sim H^2 \sim H \); as we approach the horizon scale, higher order PN corrections, with \( a^2 H^2 / c^2 \Delta \sim (GM/Rc)^6 \) order correction factors for \( n \)PN order, become important.

It is remarkable that for the perturbed velocity and perturbed gravitational potential in Equations (27) and (28) we have no PN correction terms to the 1PN order. We have not imposed the temporal gauge condition to derive Equations (25)–(27), and the variables \( \delta, v, U, \) and \( U \) are gauge invariant; this is the issue we discuss below.

In our 1PN metric convention in Equation (1), we already have taken the spatial gauge condition by setting \( g_{ij} = a^2 (1 + c^2 2V)g_{ij} \); for a thorough examination of the gauge issue in the PN approximation see Section 6 in PN2008. Under the remaining gauge transformation \( x^0 = x^0 + \xi(x^\perp) \) with

\[ \xi^0 = -\frac{1}{c^2} \xi^{(2)} + \frac{1}{c} \xi^{(4)}, \]  

(29)

we can set \( \xi^{(2)} = 0 \) without losing any generality; see Equation (173) in PN2008. Then, \( U \) does not depend on the gauge and we have (see Equations (175) and (176) in PN2008)

\[ \tilde{P}_i = P_i - \frac{1}{a} \xi^{(4)}_i, \quad \tilde{\Phi} = \Phi + \frac{1}{2} \xi^{(4)}. \]  

(30)
Thus, in the PN approach we have freedom to impose the temporal gauge (hypersurface condition) on $P^i_j$ or $\Phi$. A combination $2\Phi_j + (aP_j)$ is gauge-invariant, and therefore $\mathcal{U}$ is also gauge-invariant. The fluid variables $\delta\varrho$ and $\mathbf{v}$ in our case of vanishing internal energy and the flux are gauge invariant; see Section 6 in PN2008.

In PN2008 we have introduced several temporal gauge conditions as summarized in Table 1. In this table, the propagation speed indicates the speed of propagation of the gravitational potential $U$ in Equation (19) or Equation (24). For example, the harmonic gauge condition replaces the Laplacian operator $\Delta$ in the 0PN limit by a d’Alembertian operator $\Box$ by the 1PN correction terms, thus making the Poisson equation in the 0PN limit a wave equation with the propagation speed $c$ by the 1PN correction. Similarly, the uniform-expansion gauge (and the Chandrasekhar gauge) leaves the action-at-a-distance nature of the Poisson equation, and the transverse-shear gauge makes Equation (19) no longer a wave equation. Apparently, the propagation speed of the gravitational potential depends on the gauge choice. However, the propagation speed of the covariantly gauge-invariant Weyl tensor naturally does not depend on the gauge choice and is always $c$; see Section 7 of PN2008.

**Assuming $\Lambda = 0 = K$, for the growing mode we have $\dot{U} = 0$, and**

$$\delta = \delta^{(N)} + \frac{2}{c^2} U, \quad \frac{1}{a} \nabla \cdot \mathbf{v} = -\dot{\delta}, \quad \mathcal{U} = -\frac{4\pi G\varrho a^2}{\Delta} \delta^{(N)},$$

(31)

where $\delta^{(N)}$ is the Newtonian solution with $\delta^{(N)} \propto \tau^{2/3}, \tau^{-1}$; the solution for $\delta$ was presented in Takada & Futamase (1999). This implies that $2\Phi_j + (aP_j) = 0$ and thus $\mathcal{U} = U$. The growing mode solutions valid to 1PN order are

$$\begin{align*}
\delta &= -\frac{2}{5} \left( \frac{c^2 \Delta}{a^2 H^2} - 3 \right) C, \\
\frac{1}{a} \nabla \cdot \mathbf{v} &= \frac{2}{5} \frac{c^2 \Delta}{a^2 H} C, \\
\frac{1}{c^2} \mathcal{U} &= \frac{3}{5} C.
\end{align*}$$

(32)

This should be compared with the Newtonian solutions in Equation (16). Note that only $\delta$ has 1PN correction terms, whereas $\mathbf{v}$ and $U$ have no PN correction to 1PN order.

3. LINEAR PERTURBATION THEORY

We consider scalar-type perturbation in a flat Robertson–Walker background. Our metric convention is (Bardeen 1988)

$$ds^2 = -(1 + 2\alpha) c^2 dt^2 - 2\alpha \beta_j c dt dx^j + a^2 \left[ (1 + 2\varphi) \delta_{ij} + 2\gamma_{ij} \right] dx^i dx^j. \quad (33)$$

We consider a flat background with vanishing pressure and anisotropic stress. The energy–momentum tensor is

$$T_{0}^0 = -(\varrho + \delta \varrho) c^2, \quad T_i^0 = -a \varrho \mathbf{v} \cdot \mathbf{e}_i, \quad T_{ij} = 0, \quad (34)$$

where $x^0 \equiv c dt$. To the linear order, the basic perturbation equations are (Bardeen 1988; Hwang 1991)

$$\kappa \equiv 3H\alpha - 3\varphi - \frac{\Delta}{a^2} \chi, \quad (35)$$

$$4\pi G\delta \varrho + H\kappa + c^2 \frac{\Delta}{a^2} \varphi = 0, \quad (36)$$

$$\kappa + \frac{\Delta}{a^2} \chi - \frac{12\pi G}{c^2} \varrho a \alpha = 0, \quad (37)$$

$$\dot{\kappa} + 2H \kappa - 4\pi G \delta \varrho + \left( 3H + c^2 \frac{\Delta}{a^2} \right) \alpha = 0, \quad (39)$$

$$\dot{\delta} + 3H \delta \varrho - \varrho \left( 3H\alpha + \frac{\Delta}{a} \chi \right) = 0, \quad (40)$$

$$\frac{1}{a} \left( a \alpha \gamma - \frac{c^2}{a} a \alpha \right) = 0, \quad (41)$$

where $\chi \equiv a\beta + a^2 \gamma/c; \kappa$ is the perturbed part of the trace of extrinsic curvature (minus the perturbed expansion scalar of the normal-frame vector field). These equations are presented without taking the gauge conditions.

### Table 1

**Comparison of Often Used Gauge Conditions in the PN Approximation and in the Perturbation Theory (PT)**

| PN Gauge               | PN Definition | Propagation Speed | PT Gauge | PT Definition                  |
|------------------------|---------------|-------------------|----------|--------------------------------|
| General gauge          | $\frac{1}{2} P^i_j + nU + mHU = 0$ | $c/\sqrt{n - 3}$ | $\kappa + (n - 3)\frac{\Delta}{a^2} \chi + 3(n - m)H\varphi \equiv 0$ |
| Chandrasekhar’s gauge  | $n = 3, m =$ arbitrary | $\infty$          | UEG      | $\kappa = 0$                  |
| Uniform-expansion gauge| $n = 3 = m$   | $\infty$          | ZSG      | $\chi \equiv 0$               |
| Transverse-shear gauge | $n = 0 = m$  | $-ic/\sqrt{3}$   |          |                                |
| Harmonic gauge         | $n = 4, m =$ arbitrary | $c$              |          | $\kappa + \frac{1}{2} \frac{\Delta}{a^2} \chi + \frac{1}{2} (4 - m)H\varphi \equiv 0$ |

**Notes.** The ZSG and UEG stand for the zero-shear (or conformal-Newtonian, or longitudinal) gauge and the uniform-expansion (or uniform-Hubble) gauge, respectively; see Sections 4.1 and 4.2. The harmonic gauge in the PT is discussed in Section 4.7. The gauge transformation properties and the gauge issue in the PN approach was thoroughly discussed in Section 6 of PN2008. The “general gauge” condition covering most of the gauge conditions introduced in the PN approach was identified in Equation (210) of PN2008; here, $n$ and $m$ are arbitrary real numbers. Notice the dependence of the propagation speed of the potential on the gauge choice of $n$; see Equation (213) of PN2008. The propagation speed issue (the gauge dependence of the propagation speed of the potential and the real propagation speed of gravity) is resolved based on the Weyl tensor in Section 7 of PN2008.
Under the gauge transformation $\hat{x}^a = x^a + \xi^a(x^i)$, we have (Bardeen 1988; Hwang 1991)

$$\hat{a} = a - \frac{1}{c^2} \xi^0, \quad \hat{\delta} = \delta + \frac{3}{c} H \xi^0, \quad \hat{v} = v - \frac{c}{a} \xi^0,$$

$$\hat{\chi} = \chi - \xi^0, \quad \hat{\kappa} = \kappa + \left(\frac{3 H}{c} + \frac{\Delta}{a^2}\right) \xi^0,$$

$$\hat{\phi} = \phi - \frac{1}{c} H \xi^0. \quad (42)$$

The gauge conditions include the hypersurface or slicing (temporal gauge) condition and congruence (spatial gauge) condition. Our equations are arranged using only spatially gauge-invariant variables; for example, $\chi$ is a spatially gauge-invariant (being independent of $\xi^i$) combination that is the same as $\chi$ in the $\gamma = 0$ spatial gauge condition (Bardeen 1988). Up to this point is a review of the linear perturbation theory.

Comparing Equations (33) and (34) with Equations (1) and (3), we have

$$\alpha = - \frac{1}{c^2} U - \frac{1}{c^4} 2 \Phi, \quad \varphi = \frac{1}{c^2} V, \quad \chi, = \frac{1}{c^6} a P_i,$$

$$\kappa = - \frac{1}{c^4} \left(\frac{1}{a} P_{\mu}^\mu + 3 \hat{V} + 3 H U\right),$$

$$\delta_{\text{PT}} = \delta_{\text{PN}}, \quad v_j = -v_i + \frac{1}{c^2} P_i. \quad (43)$$

where the left- and right-hand sides are perturbation variables in perturbation theory (PT) and in the PN approximation (PN), respectively. Note that we have

$$v_i = -v_{\chi,i}, \quad U = -c^2 \alpha_{\chi}, \quad (44)$$

where

$$v_{\chi} \equiv v - \frac{c}{a} \chi, \quad \alpha_{\chi} \equiv \alpha - \frac{1}{c} \chi \quad (45)$$

are gauge-invariant combinations; $v_{\chi}$ is our notation of gauge-invariant combination made of $v$ and $\chi$, which is the same as $v$ in the zero-shear gauge $\chi \equiv 0$ (Hwang 1991). To 1PN order, $\delta$, and $v_i$ are gauge invariant because we can set $\xi^{120} = 0$ in the PN approximation. Note that gauge transformation properties of the variables in the PN approximation differ from the ones in the perturbation theory.

In terms of the ADM (Arnowitt–Deser–Misner) and the covariant notations we have (Ehlers 1961; Arnowitt et al. 1962; Ellis 1971, 1973; Bertschinger 1996)

$$N = 1 - \alpha, \quad a_i = \alpha_{,i}, \quad R^{ab} = -4 \frac{\Delta}{a^2} \varphi,$$

$$K'_{ij} = \frac{1}{c} (-3 H + \kappa), \quad \overline{K}_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \Delta\right) \chi, \quad (46)$$

where $N$, $a_i$, $R^{ab}$, $K'_{ij}$, and $\overline{K}_{ij}$ are the lapse function, acceleration vector of the normal four-vector ($a_i \equiv n_{,i} n^b$ with $n_i \equiv 0$), scalar curvature of the normal hypersurface, trace of extrinsic curvature (or expansion scalar of the normal four-vector $\theta \equiv n^b_{,b}$ with a minus sign), and trace-free part of extrinsic curvature (or the shear-tensor $\sigma_{ab}$ of the normal four-vector), respectively. Thus, the perturbed variables $\alpha$, $\varphi$, $\kappa$, and $\chi$ can be interpreted as the perturbed part of the lapse function (or the acceleration), perturbed three-space curvature, perturbed part of the trace of extrinsic curvature (or the perturbed expansion of the normal four-vector with a minus sign), and the shear. Thus, we naturally have the following names for the temporal gauge conditions: the synchronous gauge $\alpha \equiv 0$, the uniform-curvature gauge $\varphi \equiv 0$, the uniform-expansion gauge $\kappa \equiv 0$, the zero-shear gauge $\chi \equiv 0$, and the comoving gauge $\alpha \equiv 0$.

With $\kappa$ defined in Equation (35), we can show that Equations (40), (41), and (39) exactly reproduce Equations (17)–(19), respectively. Equation (37) gives

$$4 \pi G G v_i = (\dot{U} + H U)_j, \quad (47)$$

which gives Equation (21).

From Equations (39)–(41) and Equation (41), respectively, we have

$$\ddot{\delta} + 2 H \dot{\delta} - 4 \pi G G \phi \delta = -3 \left(\frac{\dot{a}}{a} + 2 \frac{\dot{a}}{a}\right), \quad (48)$$

$$\frac{1}{a} (a v_i)' = \frac{c^2}{a} \alpha_i. \quad (49)$$

Note that these forms of equations are valid without taking the gauge conditions. Although Equation (48) is in the same form as Equation (25) with an identification $\alpha \equiv -U/c^2$, we will show that only in certain hypersurface conditions specified in the perturbation theory do the two equations become identical.

4. 1PN LIMITS OF LINEAR PERTURBATIONS

4.1. Zero-shear Hypersurface

The zero-shear hypersurface condition takes $\chi \equiv 0$. From Equation (48), Equations (35), (36), and (38), and Equation (41), respectively, we have

$$\ddot{\delta} + 2 H \dot{\delta} - 4 \pi G G \phi \delta = -3 \left(H \alpha_i + 2 \frac{\dot{a}}{a} \alpha_i\right), \quad (50)$$

$$(c^2 \frac{\Delta}{a^2} - 3 H^2) \alpha_i - 3 H \alpha_i = 4 \pi G G \phi \delta, \quad (51)$$

$$\frac{1}{a} (a v_i)' = \frac{c^2}{a} \alpha_i. \quad (52)$$

By identifying

$$\delta = \delta_{\chi}, \quad U = -\frac{1}{c^2} \alpha_{\chi}, \quad (53)$$

Equation (50) becomes Equation (25).

We can derive closed-form equations for $\delta_{\chi}$, $v_{\chi}$, and $\alpha_{\chi}$. From Equations (35)–(41) we have (Hwang & Noh 1999)

$$\ddot{\delta} + 2 H \dot{\delta} - 8 \pi G G \phi \delta = \frac{3 H^2 + 6 H + c^2 \Delta}{3 H^2 - c^2 \Delta \left(1 + \frac{c^2}{12 \pi G G a^3}\right)} \left[H \dot{\delta} + \left(4 \pi G G + c^2 \frac{\Delta}{3 a^2}\right) \delta_{\chi}\right], \quad (54)$$

$$(a v_i)', + H (a v_i)' - 4 \pi G G (a v_i) = 0, \quad (55)$$

$$(a \alpha_i)', + 2 H (a \alpha_i)' - 4 \pi G G (a \alpha_i) = 0. \quad (56)$$
From Equation (38) we have $\varphi_x = -\alpha_x$. In the sub-horizon limit, Equation (54) properly reproduces Equation (26) to the 1PN limit. More remarkable is the case of Equations (55) and (56), which exactly reproduce Newtonian equations in Equations (14) and (15), which are valid to 1PN order; see Equations (27) and (28). Equations (54)–(56) are valid in the fully relativistic situation and in the general scale. The 1PN approximation corresponds to the sub-horizon limit with $c^2\Delta/a^2 H^2 \gg 1$.

The growing mode exact solutions for $\Lambda = 0 = K$ are (Hwang 1994)

$$\delta_x = \frac{2}{5} \left( 3 - \frac{c^2\Delta}{a^2 H^2} \right) C, \quad v_x = -\frac{2}{5} \frac{c^2}{a H} C,$$

$$\varphi_x = -\alpha_x = \frac{3}{5} C. \quad (57)$$

A complete set of exact solutions in the presence of $\Lambda$ and $\delta$ is presented in Table 1 of Hwang (1994). Comparing with Equations (16) and (32), in the small-scale limit we have

$$\delta_x = \delta^{(1PN)}, \quad -\nabla v_x = \varphi^{(1PN)}, \quad \alpha_x = -\varphi_x = -\frac{1}{c^2} U. \quad (58)$$

### 4.2. Uniform-expansion Hypersurface

The uniform-expansion hypersurface condition takes $\kappa \equiv 0$. From Equations (48), (39), and (41), respectively, we have

$$\delta_x + 2H\delta_x - 4\pi G\varrho\delta_x = -3 \left( H\alpha_x + \frac{\ddot{a}}{a} \alpha_x \right), \quad (59)$$

$$\left( \frac{c^2}{a^2} + 3\dot{H} \right) \alpha_x = 4\pi G\varrho\delta_x,$$  

$$\frac{1}{a} (a v_x)' = \frac{c^2}{a} \alpha_x. \quad (60)$$

By identifying

$$\delta = \delta_x, \quad U \equiv -\frac{1}{c^2} \alpha_x, \quad (62)$$

Equation (59) becomes Equation (25).

We can derive closed-form equations for $\delta_x$, $v_x$, and $\alpha_x$. From Equations (35)–(41) we have (Hwang & Noh 1999)

$$\delta_x + 2H\delta_x - 4\pi G\varrho\delta_x = \frac{a H}{\dot{H}} \left( \frac{3H^2/a}{3H + c^2 \Delta/a^2} \delta_x \right), \quad (63)$$

$$(v_x')' + H (av_x') - 4\pi G\varrho (av_x) = \frac{3}{3H + c^2 \Delta/a^2} \left[ -(\dot{H} + 3H\dot{H})(av_x') + H^2 (av_x) \right], \quad (64)$$

$$(\varphi_x')' + 2H (a\varphi_x') - 4\pi G\varrho (a\varphi_x) = \frac{1}{a^2 \dot{H}} \left( \frac{3\dot{H}^2 + 2H}{3H + c^2 \Delta/a^2} \varphi_x \right), \quad (65)$$

$$\alpha_x = -\frac{c^2 \Delta}{3H + c^2 \Delta/a^2} \varphi_x. \quad (66)$$

In the sub-horizon limit, Equation (63) properly reproduces Equation (26) to the 1PN limit, whereas Equations (64) and (65) produce only the Newtonian equation. Equations (63)–(66), however, are valid in the fully relativistic situation and in the general scale. From Equations (36) and (39), respectively, we have

$$c^2 \Delta \varphi_x = -4\pi G\varrho \delta_x,$$

$$\left( \frac{c^2}{a^2} + 3\dot{H} \right) \alpha_x = 4\pi G\varrho \delta_x, \quad (67)$$

either of which can be regarded as the Poisson equation.

The growing mode exact solutions for $\Lambda = 0 = K$ are (Hwang 1994)

$$\delta_x = \frac{c^2}{c^2 - 12\pi G\varrho a^2} \left( 3 - \frac{c^2\Delta}{5a^2 H^2} \right) C,$$

$$v_x = -\frac{2}{5} \frac{c^2}{a H} \left( c^2 - 12\pi G\varrho a^2 \right), \quad (68)$$

$$\varphi_x = -\frac{4\pi G\varrho a^2}{c^2 - 12\pi G\varrho a^2} \delta_x, \quad \alpha_x = -\frac{c^2\Delta}{c^2 - 12\pi G\varrho a^2} \varphi_x. \quad (69)$$

In the large-scale limit, we have $\varphi_x = C$. Comparing with Equations (16) and (32), in the sub-horizon limit we have

$$\delta_x = \delta^{(1PN)}, \quad (69)$$

and

$$-\nabla v_x = \varphi^{(N)}, \quad \alpha_x = -\varphi_x = -\frac{1}{c^2} U, \quad (70)$$

only to Newtonian order.

### 4.3. Comoving Hypersurface

The comoving hypersurface condition takes $v \equiv 0$. From Equation (41), this implies $\alpha = 0$, which is the synchronous hypersurface condition. From Equations (39) and (40) we have

$$\delta_v + 2H\delta_v - 4\pi G\varrho\delta_v = 0, \quad (71)$$

$$(a^2 \kappa_v)' + H(a^2 \kappa_v) - 4\pi G\varrho(a^2 \kappa_v) = 0. \quad (72)$$

Thus, in this hypersurface condition the equation for $\delta_v$ is exactly the same as the Newtonian order equation with no PN correction. With an identification

$$\kappa_v \equiv -\nabla \cdot \mathbf{v}, \quad (73)$$

the equation for $\kappa_v$ is exactly the same as the Newtonian one in Equation (14) with no PN correction. Equations (71) and (72) are valid in the fully relativistic situation and in the general scale. From the gauge transformation properties in Equation (42) we have

$$\kappa_v \equiv \kappa + \left( \frac{3H}{c^2} + \frac{\Delta}{a^2} \right) a v \equiv \left( \frac{3H}{c^2} + \frac{\Delta}{a^2} \right) a v_x. \quad (74)$$

Equations (35), (37), and (41) give

$$\varphi_v = 0. \quad (75)$$

The coefficient of the growing solution $C$ is introduced based on the powerful conservation property of $\varphi_v$ as $\varphi_v = C$. 

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From the gauge transformation properties in Equation (42) we have
\[ \delta_v \equiv \delta + \frac{3aH}{c^2} v = \delta_x + \frac{3aH}{c^2} v_x = \delta_v + 3aH v \varepsilon \]
\[ = \delta_v + \frac{3aH}{c^2} v \varepsilon. \]  
(76)

Using this relation, we can show that Equations (50), (59), and (80) are consistent with Equation (71). In other words, by taking a combination \( \delta_v + (3aH/c^2)v \varepsilon \) and using Equations (50) and (55), we have the equation for \( \delta_v \) in Equation (71) and similarly for \( \delta_x \) and \( \delta_v \) in Equations (59) and (80).

The growing mode exact solutions for \( \Lambda = 0 = K \) are
\[ \delta_v = - \frac{2 c^2 \Delta}{5 a^2 H^2} C, \quad \kappa_v = - \frac{2 c^2 \Delta}{5 a^2 H} C, \quad \varphi_v = C. \]  
(77)

Comparing with Equations (16) and (32), in the sub-horizon limit we have
\[ \delta_v = \delta^{(NP)}, \quad -\kappa_v = \frac{1}{a} \varphi \cdot \nu^{(1PN)}, \]  
(78)
to 1PN order, and
\[ \alpha_v = 0, \quad \varphi_v \neq - \frac{1}{c^2} U, \]  
(79)
even to Newtonian order.

Note that \( \delta_v \) and \( \kappa_v \) exactly reproduce Newtonian density and velocity perturbations without PN correction; this is true even to the second-order perturbations (Noh & Hwang 2004). Despite such remarkable identifications, by lacking proper gravitational potential \( (\varphi_v = C) \), the correspondence with PN approximation and lacking proper physical density perturbation is not lost and is hidden in other combinations of variables in this gauge like \( \delta_v \equiv \delta - 3(aH/c^2)v \equiv -3(aH/c^2)v_\delta \).

4.5. Uniform-density Hypersurface

The uniform-density hypersurface condition takes \( \delta_{\|} \equiv 0 \). This is the perfectly allowed hypersurface condition in a time-varying background. Although the density perturbation in this hypersurface is null by definition, physical information like physical density perturbation is not lost and is hidden in other combinations of variables in this gauge like \( \delta_v \equiv \delta - 3(aH/c^2)v \equiv -3(aH/c^2)v_\delta \).

4.6. Synchronous Hypersurface

The synchronous hypersurface condition takes \( \alpha \equiv 0 \). In the pressureless case, Equation (41) gives \( v \propto a^{-1} \), which is the pure gauge mode: in Equation (42) \( \tilde{\alpha} = 0 = \alpha \) leaves \( \tilde{\xi} = \xi^{0}(x) \) and gives \( v_{\text{gauge}} \propto \xi^{0}/a \propto a^{-1} \). By setting the gauge mode to \( v = 0 \), it becomes the same as taking the comoving gauge.

4.7. Harmonic Hypersurface

The harmonic or de Donder hypersurface condition takes the perturbed part of \( g^{bc} \Pi_{ab} \equiv 0 \). The temporal component gives \( \alpha = H + a + \kappa = 0 \), which is quite a bad choice, leaving two remnant gauge modes; see the Appendix in Hwang (1993). In the PN approximation this becomes \( (1/a)P_{ab} + 4U + 4H = 0 \) with the well-known propagation speed \( c \); however, we note that, as presented in Table 1, the propagation speed of the potential depends on the PN gauge choice. In Table 1, we presented a combination of perturbation variables that reproduces this PN definition of the harmonic gauge with a general \( HU \) term.

5. POST-NEWTONIAN/PERTURBATION THEORY CORRESPONDENCE

We summarize correspondences between the PN approximation and the perturbation theory in Table 2.

As the 1PN correction terms have \( a^2H^2/c^2 \Delta \) order factor smaller than 0PN terms, those become important as the scale approaches the horizon. Thus, the PN approximation is applicable only in the sub-horizon scale. However, Newton’s theory is free from the presence of the horizon scale. This is why we have divided the 0PN case into the exact (all scales) and the sub-horizon cases in Table 2.

Our comparison of PN approximation and perturbation theory shows that the zero-shear gauge is distinguished by the fact that all perturbation variables of the density, velocity, and gravitational potential properly correspond to 1PN (thus, valid in the sub-horizon scale) order variables. Thus,
\[ \delta_x = \delta^{(1PN)}, \quad -\nabla \varphi_x = \nu^{(1PN)} = \nu^{(N)}, \]
\[ \alpha_x = - \varphi_x = - \frac{1}{c^2} U = - \varphi_v = - \frac{1}{c^2} U. \]  
(88)

The growing mode exact solutions for \( \Lambda = 0 = K \) are
\[ \delta_{\|} = \left( 3 - \frac{2}{5} \frac{c^2 \Delta}{a^2 H^2} \right) \frac{C}{2}, \quad \nu_{\|} = - \frac{c^2}{aH} C, \quad \alpha_{\|} = - \frac{3}{2} C. \]  
(85)

Comparing with Equations (16) and (32), in the small-scale limit we have
\[ \delta_{\|} = \delta^{(NP)}, \]  
(86)
even to Newtonian order.
Table 2

| Variable | PN Order and Scale | ZSG | UEG | CG | UCG |
|----------|--------------------|-----|-----|----|-----|
| δ        | 0PN exact          | δχ  | δv  | δv | δv  |
|          | 0PN sub-horizon    | δχ  | δv  | δv | δv  |
| v        | 0PN exact          | vχ  | κv  | κv | κv  |
|          | 0PN sub-horizon    | vχ  | κv  | κv | κv  |
| U        | 0PN exact          | αx  | δκ  | δκ | δκ  |
|          | 0PN sub-horizon    | αx  | δκ  | δκ | δκ  |

Notes. The CG and UCG stand for the comoving gauge and the uniform-curvature (or flat) gauge, respectively; see Sections 4.3 and 4.4. A notation [2nd] means that the Newtonian correspondence is valid to second-order perturbation in the perturbation theory (Hwang et al. 2012).

Note that for the perturbed velocity and gravitational potential the Newtonian (0PN) equations remain valid to 1PN order.

In the uniform-expansion gauge, only the density perturbation has 1PN correspondence, whereas the velocity and gravitational potential perturbations have 0PN correspondences only in the sub-horizon limit.

Newtonian correspondences of all perturbation variables (density, velocity, and gravitational potential) on the sub-horizon scale in the zero-shear gauge and the uniform expansion gauge were known to the linear order (Peebles 1980; Hwang & Noh 1999); the same correspondences are shown to be valid even to the second-order perturbations (Hwang et al. 2012).

Newtonian correspondences of linear density perturbations in various gauge conditions in the sub-horizon scale were pointed out in Bardeen (1980); the same correspondences are shown to be valid even to the second-order perturbations (Hwang et al. 2012). Only in the comoving gauge (to the linear order the same as the synchronous gauge without the gauge mode), the density perturbation has an exact correspondence to the Newtonian one (Lifshitz 1946; Bonnor 1957; Narai 1969; Bardeen 1980).

In the case of the comoving gauge, the relativistic/Newtonian correspondence is quite distinguishing: up to the second order in perturbations, the perturbed density and velocity show exact (thus, valid in all scales) correspondences with the Newtonian ones (Noh & Hwang 2004; Hwang et al. 2012); these correspondences are possible only in the comoving gauge. However, the comoving gauge implies vanishing αv and thus vanishing gravitational potential to the linear order. Without the proper gravitational potential, despite the striking correspondences of density and velocity, whether proper Newtonian interpretation will be available in this gauge condition is unclear; for our suggestion, see below.

For the growing modes, Equations (32), (57), and (77) show that (Takada & Futamase 1999)

\[ \delta_x = \delta_v + \frac{2}{c^2} U, \quad \delta_x = \delta^{(1PN)}, \quad \delta_v = \delta^{(N)} . \]

From Equations (35)–(41) we can derive

\[ \dot{\delta}_v = \frac{\Delta}{a} v_x, \]  
\[ \dot{v}_x + H v_x = \frac{c^2}{a} \alpha_x, \]  
\[ c^2 \frac{\Delta}{d^2} \alpha_x = 4\pi G \varrho \delta_v. \]

These equations can be compared with Equations (10)–(12) in the Newtonian context. Equation (92) is the well-known Poisson equation relating gauge-invariant combinations based on different gauges: density perturbation based on the comoving gauge and gravitational potential based on the zero-shear gauge. Equations (90)–(92) were presented in Equations (4.3)–(4.5) and (4.8) of Bardeen (1980). Thus, Bardeen has suggested identifications

\[ \delta_v \equiv \delta, \quad -\nabla v_x \equiv v, \quad \alpha_x \equiv -\frac{1}{c^2} U. \]

Instead of Equations (90) and (91), from Equations (35)–(41) we can derive

\[ \dot{\delta}_v = \kappa_v, \]  
\[ \dot{\kappa}_v + 2H \kappa_v = 4\pi G \varrho \delta_v, \]

and together with Equation (92) we may also identify

\[ \delta_v \equiv \delta, \quad \kappa_v \equiv -\frac{1}{a} \nabla \cdot v, \quad \alpha_x \equiv -\frac{1}{c^2} U. \]

It is likely that the proper Newtonian correspondence is available based on such a mixed use of the gauges in constructing gauge-invariant combinations. Bardeen (1980) has identified v_x as the velocity perturbation variable; as summarized in Table 2, both v_x and κ_v show distinguished 0PN and 1PN correspondences, but κ_v is better in having the exact Newtonian correspondence even to the second order in all scales.

6. DISCUSSION

In this work we have compared the cosmological 1PN approximation with the relativistic linear perturbation theory
in the case of a zero-pressure fluid. The 1PN equations are applicable in the weakly relativistic situation within the horizon, but fully nonlinear. The 1PN equations contain up to fourth order in perturbation variables; see Equations (4)–(7). In order to handle the 1PN equations, in general we have to choose the temporal gauge condition presented in Table 1. Together with Equations (4) and (5) to the 0PN order, Equations (6) and (7) determine $U$ to the 0PN order and $P_\nu$. Then, we can solve Equations (4)–(6) to determine $\rho$ and $v^\nu$ to the 1PN order and $\Phi$.

Here, we would like to highlight some of the consequences and implications of this work.

1. To the linear order the 1PN equations can be arranged independently of the PN gauge choice.

2. To the linear order the closed-form 1PN equations for density perturbation have the 1PN correction terms, whereas for the velocity and the potential the 0PN equations remain valid; see Equations (25), (27), and (28).

3. In the linear perturbation theory, however, the equations depend sensitively on the gauge choice.

4. Comparison of the metric and the energy–momentum tensor in the two approaches shows that the velocity and the potential to the 1PN order (which are the same as the ones to the 0PN order) correspond to the ones in the zero-shear gauge; see Equation (44).

5. In Section 4, closed-form equations for density, velocity, and potential in several fundamental gauge conditions are compared with the 1PN equations, and the results are summarized in Table 2.

6. Table 2 shows that the zero-shear gauge is distinguished compared with the 1PN approximation for density, velocity, and potential perturbations.

7. However, for the density and the velocity perturbations the comoving gauge is also distinguished by having exact correspondence with the 0PN equation even to the second-order perturbation.

8. Studies of the relativistic/Newtonian and the relativistic/1PN correspondences lead us to a conclusion that it is not a single gauge condition that is suitable for the correspondences. As Bardeen (1980) has suggested, the gauge-invariant combinations that correspond to perturbation variables in different gauges are suitable for the correspondence; see Equation (93). Our suggestion for such a mixed gauge-invariant combination is made in Equation (96), which coincides with Bardeen’s to the linear order perturbation.