RATIONAL CERTIFICATES OF POSITIVITY ON COMPACT
SEMIALGEBRAIC SETS

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Let \( \mathbb{R}[X] \) denote the real polynomial ring \( \mathbb{R}[X_1, \ldots, X_n] \) and write \( \sum \mathbb{R}[X]^2 \) for the set of sums of squares in \( \mathbb{R}[X] \). Given \( g_1, \ldots, g_s \in \mathbb{R}[X] \) such that the semialgebraic set \( \mathcal{K} := \{ x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for all } i \} \) is compact, Schmüdgen’s theorem says that if \( f \in \mathbb{R}[X] \) such that \( f > 0 \) on \( \mathcal{K} \), then \( f \) is in the preordering in \( \mathbb{R}[X] \) generated by the \( g_i \)'s, i.e., \( f \) can be written as a finite sum of elements \( \sigma g_1^{e_1} \cdots g_s^{e_s} \), where \( \sigma \) is a sum of squares in \( \mathbb{R}[X] \) and each \( e_i \in \{0, 1\} \).

Putinar’s theorem says that under a condition on the set of generators \( \{g_1, \ldots, g_s\} \) (which is a stronger condition than the compactness of \( \mathcal{K} \)), any \( f > 0 \) on \( \mathcal{K} \) can be written \( f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \), where \( \sigma_i \in \sum \mathbb{R}[X]^2 \). Both of these theorems can be viewed as statements about the existence of certificates of positivity on compact semialgebraic sets. In this note we show that if the defining polynomials \( g_1, \ldots, g_s \) and polynomial \( f \) have coefficients in \( \mathbb{Q} \), then in Schmüdgen’s theorem we can find a representation in which the \( \sigma \)'s are sums of squares of polynomials over \( \mathbb{Q} \).

We prove a similar result for Putinar’s theorem assuming that the set of generators contains \( N - \sum X_i^2 \) for some \( N \in \mathbb{N} \).

1. Introduction

We write \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{Q} \) for the set of natural, real, and rational numbers. Let \( n \in \mathbb{N} \) be fixed and let \( \mathbb{R}[X] \) denote the polynomial ring \( \mathbb{R}[X_1, \ldots, X_n] \). We denote by \( \sum \mathbb{R}[X]^2 \) the set of sums of squares in \( \mathbb{R}[X] \).

For \( S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[X] \), the basic closed semialgebraic set generated by \( S \), denoted \( \mathcal{K}_S \), is

\[
\{ x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0 \}.
\]

Associated to \( S \) are two algebraic objects: The quadratic module generated by \( S \), denoted \( \mathcal{M}_S \), is the set of \( f \in \mathbb{R}[X] \) which can be written

\[
f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s,
\]
where each $\sigma_i$ lies in $\sum \mathbb{R}[X]^2$, and the preordering generated by $S$, denoted $T_S$, is the quadratic module generated by all products of elements in $S$. In other words, $T_S$ is the set of $f \in \mathbb{R}[X]$ which can be written as a finite sum of elements

$$\sigma g_1^{e_1} \ldots g_s^{e_s}, \quad \text{for } \sigma \in \mathbb{R}[X] \text{ and each } e_i \in \{0, 1\}.$$  

A polynomial $f \in \sum \mathbb{R}[X]^2$ is obviously globally nonnegative in $\mathbb{R}^n$ and writing $f$ explicitly as a sum of squares gives a “certificate of positivity” for the fact that $f$ takes only nonnegative values in $\mathbb{R}^n$. (Note: To avoid having to write “nonnegativity or positivity” we use the term “positivity” to mean either.) More generally, for a basic closed semialgebraic set $K_S$, if $f \in T_S$ or $f \in M_S$, then $f$ is nonnegative on $K_S$ and an explicit representation of $f$ in $M_S$ or $T_S$ gives a certificate of positivity for $f$ on $K_S$.

Schmüdgen [1991] showed that if the semialgebraic set $K_S$ is compact, then any $f \in \mathbb{R}[X]$ which is strictly positive on $K_S$ is in the preordering $T_S$. A preordering or quadratic module is archimedean if it contains $N - \sum X_i^2$ for some $N \in \mathbb{N}$. We note that if $M_S$ is archimedean, then it follows immediately that $K_S$ is compact, however the converse is not true in general. Putinar [1993] showed that if $M_S$ is archimedean then any $f \in \mathbb{R}[X]$ which is strictly positive on $K_S$ is in $M_S$. In other words, these results say that under the given conditions a certificate of positivity for $f$ on $K_S$ exists.

Recently, techniques from semidefinite programming combined with Schmüdgen’s and Putinar’s theorems have been used to give numerical algorithms for applications such as optimization of polynomials on semialgebraic sets. However since these algorithms are numerical they might not produce exact certificates of positivity. With this in mind, Sturmfels asked whether any $f \in \mathbb{Q}[X]$ which is a sum of squares in $\mathbb{R}[X]$ is a sum of squares in $\mathbb{Q}[X]$. Hillar [2009] showed that the answer is yes in the case where $f$ is known to be a sum of squares over a totally real field $K$. The general question remains unsolved.

It is natural to ask a similar question for Schmüdgen’s and Putinar’s theorems: If the polynomials defining the semialgebraic set and the positive polynomial $f$ have rational coefficients, is there a certificate of positivity for $f$ in which the sums of squares have rational coefficients? In this note, we show that in the case of Schmüdgen’s theorem the answer is yes. This follows from an algebraic proof of the theorem, originally due to T. Wörmann [1998]. In the case of Putinar’s theorem, we show that the answer is also yes as long as the generating set contains $N - \sum X_i^2$ for some $N \in \mathbb{N}$. This follows easily from an algorithmic proof of the theorem, due to Schweighofer [2005]. For Lasserre’s method [2001] for optimization of polynomials on compact semialgebraic sets, it is usual in concrete cases to add a polynomial of the type $N - \sum X_i^2$ to the generators in order to insure that Putinar’s theorem holds. Thus our assumption in this case is reasonable.
2. Rational certificates of positivity for Schmüdgen’s theorem

Fix $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[X]$ and define $K_S$ and $T_S$ as above.

**Theorem 1** (Schmüdgen). Suppose that $K_S$ is compact. If $f \in \mathbb{R}[X]$ and $f > 0$ on $K_S$, then $f \in T_S$.

In this section we show that if $f$ and the generating polynomials $g_1, \ldots, g_s$ are in $\mathbb{Q}[X]$, then $f$ has a representation in $T_S$ in which all sums of squares $\sigma_\epsilon$ are in $\sum \mathbb{Q}[X]^2$. This follows from T. Wörmann’s algebraic proof of the theorem using the classical Abstract Positivstellensatz, and a generalization of Wörmann’s crucial lemma due to M. Schweighofer.

**The abstract Positivstellensatz.** We will need a version of the abstract Positivstellensatz, a result traditionally attributed to Kadison and Dubois, but now thought to have been proved earlier by Krivine or Stone. For details on its history, see [Prestel and Delzell 2001, Section 5.6]. The setting is preordered commutative rings, and we state the version we need as Theorem 2 below.

Let $A$ be a commutative ring with $\mathbb{Q} \subseteq A$. A subset $T \subseteq A$ is a preorder if $T + T \subseteq T$, $T \cdot T \subseteq T$, and $-1 \notin T$. For $S = \{a_1, \ldots, a_k\} \subseteq A$, we define the preorder generated by $S$, $T_S$, exactly as for $A = \mathbb{R}[X]$.

An ordering in $A$ is a preorder $P$ such that $P \cup -P = A$ and $P \cap -P$ is a prime ideal. Any $a \in A$ has a unique sign in $\{-1, 0, 1\}$ with respect to a fixed ordering $P$ and we use the notation $a \geq_P 0$ if $a \in P$, $a > P 0$ if $a \in P \setminus (P \cap -P)$, etc.

Fix a preordered ring $(A, T)$ and denote by $\text{Sper} A$ the real spectrum of $(A, T)$, i.e., the set of orderings of $A$ which contain $T$. Then define

$$H(A) = \{a \in A \mid \text{there exists } n \in \mathbb{N} \text{ with } n \pm a \geq_P 0 \text{ for all } P \in \text{Sper} A\},$$

the ring of geometrically bounded elements in $(A, T)$, and

$$H'(A) = \{a \in A \mid \text{there exists } n \in \mathbb{N} \text{ with } n \pm a \in T\},$$

the ring of arithmetically bounded elements in $(A, T)$. Clearly, $H'(A) \subseteq H(A)$.

The preorder $T$ is archimedean if $H'(A) = A$.

**Theorem 2** [Schweighofer 2002, Theorem 1]. Given the preordered ring $(A, T)$ as above and suppose $A = H'(A)$. For any $a \in A$, if $a > P 0$ for all $P \in \text{Sper} A$, then $a \in T$.

Consider the case where $A = \mathbb{R}[X]$ and $T = T_S$ for $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[X]$. Let $K = K_S$, then $K$ embeds densely in Sper $A$ and hence $H(A) = \{f \in \mathbb{R}[X] \mid f \text{ is bounded on } S\}$. If $S$ is compact, this implies $H(A) = A$ and Schmüdgen’s theorem follows from the following result:
Lemma 3 [Berr and Wörmann 2001, Lemma 1]. With \( A, T, \) and \( S \) as above, if \( H(A) = A \) then \( H'(A) = A \).

Our result follows from a generalization of this lemma:

Theorem 4 [Schweighofer 2002, Theorem 4.13]. Let \( F \) be a subfield of \( \mathbb{R} \) and \((A, T)\) a preordered \( F \)-algebra such that \( F \subseteq H'(A) \) and \( A \) has finite transcendence degree over \( F \). Then

\[
A = H(A) \implies A = H'(A).
\]

We can now prove the existence of rational certificates of positivity in Schmüdgen’s theorem. The argument is exactly that of the proof of the general theorem above.

Theorem 5. Given \( S = \{g_1, \ldots, g_s\} \subseteq \mathbb{Q}[X] \) and suppose \( K_S \subseteq \mathbb{R}^n \) is compact. Then for any and \( f \in \mathbb{Q}[X] \) such that \( f > 0 \) on \( K_S \), there is a representation of \( f \) in the preordering \( T_S \),

\[
f = \sum_{e \in \{0,1\}^s} \sigma_e g_1^{e_1} \cdots g_s^{e_s},
\]

with all \( \sigma_e \in \sum \mathbb{Q}[X]^2 \).

Proof. Let \( T \) be the preordering in \( \mathbb{Q}[X] \) generated by \( S \). Since \( K_S \) is compact, every element of \( \mathbb{Q}[X] \) is bounded on \( K_S \). Then \( K_S \) dense in \( \text{Sper } A \) implies that \( H(\mathbb{Q}[X]) = \mathbb{Q}[X] \), hence by Theorem 4 we have \( \mathbb{Q}[X] = H'(A) \). Note that the condition \( F \subseteq H'(A) \) holds in this case since \( \mathbb{Q}^+ = \sum \mathbb{Q}^2 \). The result follows from Theorem 2. \(\square\)

3. Rational certificates for Putinar’s theorem

Given \( S = \{g_1, \ldots, g_s\} \), recall that the quadratic module generated by \( S \), \( M_S \), is the set of elements in the preordering \( K_S \) with a “linear” representation, i.e.,

\[
M_S = \{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_i \in \sum \mathbb{R}[X]^2\}.
\]

In order to guarantee representations of positive polynomials in the quadratic module, we need a condition stronger than compactness of \( K_S \), namely, we need \( M_S \) to be archimedean.

The quadratic module \( M_S \) is archimedean if all elements of \( \mathbb{R}[X] \) are bounded by a positive integer with respect to \( M_S \), i.e., if for every \( f \in \mathbb{R}[X] \) there is some \( N \in \mathbb{N} \) such that \( N - f \in M_S \). It is not too hard to show that \( M_S \) is archimedean if there is some \( N \in \mathbb{N} \) such that \( N - \sum X_i^2 \in M_S \). Clearly, if \( M_S \) is archimedean, then \( K_S \) is compact; the polynomial \( N - \sum X_i^2 \) can be thought of as a “certificate of compactness”. However, the converse is not true; see [Prestel and Delzell 2001, Example 6.3.1]. The key to the algebraic proof of Schmüdgen’s theorem from the
previous section is showing that in the case of the preordering generated by a finite set of elements from \( \mathbb{R}[X] \), the compactness of the semialgebraic set implies that the corresponding preordering is archimedean.

Putinar [1993] showed that if the quadratic module \( M_S \) is archimedean, we can replace the preordering \( T_S \) by the quadratic module \( M_S \).

**Theorem 6** (Putinar). Suppose that the quadratic module \( M_S \) is archimedean. Then for every \( f \in \mathbb{R}[X] \) with \( f > 0 \) on \( K_S \), \( f \in M_S \).

Lasserre’s method [2001] for minimizing a polynomial on a compact semialgebraic set involves defining a sequence of semidefinite programs corresponding to representations of bounded degree in \( M_S \) whose solutions converge to the minimum. In this context, if \( M_S \) is archimedean then Putinar’s theorem implies the convergence of the semidefinite programs. In practice, it is not clear how to decide if \( M_S \) is archimedean for a given set of generators \( S \), however in concrete cases a polynomial \( N - \sum X_i^2 \) can be added to the generators if an appropriate \( N \) is known or can be computed.

Using an algorithmic proof of Putinar’s theorem due to M. Schweighofer [2005] we can show that rational certificates exist for the theorem as long as we have a polynomial \( N - \sum X_i^2 \) as one of our generators:

**Theorem 7.** Suppose \( S = \{g_1, \ldots, g_s\} \subseteq \mathbb{Q}[X] \) and \( N - \sum X_i^2 \in M_S \) for some \( N \in \mathbb{N} \). Then given any \( f \in \mathbb{Q}[X] \) such that \( f > 0 \) on \( K_S \), there exist \( \sigma_0 \cdots \sigma_s, \sigma \in \sum \mathbb{Q}[X]^2 \) so that

\[
f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s + \sigma (N - \sum X_i^2).
\]

**Proof.** The idea of Schweighofer’s proof is to reduce to Pólya’s theorem. We follow the proof, making sure that each step preserves rationality.

Let

\[
\Delta = \left\{ y \in [0, \infty)^{2n} \mid y_1 + \cdots + y_{2n} = 2n(\frac{N + 1}{4}) \right\} \subseteq \mathbb{R}^{2n}
\]

and let \( C \) be the compact subset of \( \mathbb{R}^n \) defined by \( C = l(\Delta) \), where \( l : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \) is defined by

\[
y \mapsto \left( \frac{y_1 - y_{n+1}}{2}, \ldots, \frac{y_n - y_{2n}}{2} \right).
\]

Scaling the \( g_i \)'s by positive elements in \( \mathbb{Q} \), we can assume that \( g_i \leq 1 \) on \( C \) for all \( i \). The key to the proof is the observation that \( \text{there exists } \lambda \in \mathbb{R}^+ \text{ such that } q := f - \lambda \sum (g_i - 1)^{2k} g_i > 0 \text{ on } C \) [Schweighofer 2005, Lemma 2.3]. Since we can always replace \( \lambda \) by a smaller value, we can assume \( \lambda \in \mathbb{Q} \), whence \( q \in \mathbb{Q}[X] \).
Let $d = \deg q$ and let $Q_i$ be the homogeneous part of $q$ of degree $i$, so $q = \sum_{i=1}^d Q_i$. Let $Y = (Y_1, \ldots, Y_{2n})$ and define in $\mathbb{Q}[Y]$

$$F(Y_1, \ldots, Y_{2n}) := \sum_{i=1}^d Q_i \left( \frac{Y_1 - Y_{n+1}}{2}, \ldots, \frac{Y_n - Y_{2n}}{2} \right) \left( \frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})} \right)^{d-i}.$$ 

Then $F$ is homogenous and $F > 0$ on $[0, \infty)^{2n} \setminus \{0\}$. By Pólya’s theorem, there is some $k \in \mathbb{N}$ so that

$$G := \left( \frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})} \right)^k F$$

has nonnegative coefficients as a polynomial in $\mathbb{R}[Y]$. Furthermore, since $F \in \mathbb{Q}[Y_1, \ldots, Y_{2n}]$, it is easy to see that $G \in \mathbb{Q}[Y]$.

Define $\phi : \mathbb{Q}[Y_1, \ldots, Y_{2n}] \to \mathbb{Q}[X]$ by

$$\phi(Y_i) = N + \frac{1}{4} + X_i, \quad \phi(Y_{n+i}) = (N + \frac{1}{4}) - X_i \quad \text{for } i = 1, \ldots, n$$

and note that $\phi(G) = q$ and

$$\phi(Y_i) = (N + \frac{1}{4}) \pm X_i$$

$$= \sum_{j \neq i} (X_j^2 + (X_i \pm \frac{1}{2})^2) + (N - \sum X_j^2) \in \sum \mathbb{Q}[X]^2 + (N - \sum X_j^2).$$

Thus $\phi(G) = q$ implies there is a representation of $q$ of the required type and then, since $f = q + \lambda \sum (g_i - 1)^{2k} g_i$ with $\lambda \in \mathbb{Q}$, we are done. \qed

**Remark 8.** In the preordering case (Schmüdgen’s theorem), as noted above if the semialgebraic set $K_S$ is compact, then it follows that the preordering $T_S$ in $\mathbb{Q}[X]$ is archimedean. However it is more subtle in the quadratic module case since it is not always clear how to decide if $M_S$ is archimedean for a given set of generators $S$. Thus an open question is the following: Suppose $S \subseteq \mathbb{Q}[X]$ is a finite set of polynomials and $M_S$ is archimedean as a quadratic module in $\mathbb{R}[X]$. Is it true that $M_S$ is archimedean as a quadratic module in $\mathbb{Q}[X]$? To put it more concretely, suppose $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{Q}[X]$ and we know that there is some $N \in \mathbb{N}$ such that

$$N - \sum X_i^2 = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s,$$

with $\sigma_i \in \sum \mathbb{R}[X]^2$. Does there exist a representation with $\sigma_i \in \sum \mathbb{Q}[X]^2$? Equivalently, does there exist $N \in \mathbb{N}$ such that for each $i = 1, \ldots, n$ we can write

$$N \pm X_i = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s,$$

with $\sigma_i \in \sum \mathbb{Q}[X]^2$?
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