Quantum Gravity as Escher’s Dragon

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Abstract

The main obstacle in attempts to construct a consistent quantum gravity is the absence of independent flat time. This can in principle be cured by going out to higher dimensions. The modern paradigm assumes that the fundamental theory of everything is some form of string theory living in space of more than four dimensions. We advocate another possibility that the fundamental theory is a form of $D = 4$ higher-derivative gravity. This class of theories has a nice feature of renormalizability so that perturbative calculations are feasible. There are also finite $\mathcal{N} = 4$ supersymmetric conformal supergravity theories. This possibility is particularly attractive. Einstein’s gravity is obtained in a natural way as an effective low-energy theory.

The $\mathcal{N} = 1$ supersymmetric version of the theory has a natural higher-dimensional interpretation due to Ogievetsky and Sokatchev, which involves embedding of our curved Minkowsky space-time manifold into flat 8-dimensional space. Assuming that a variant of the finite $\mathcal{N} = 4$ theory also admit a similar interpretation, this may eventually allow one to construct consistent quantum theory of gravity.

We argue, however, that even though future gravity theory will probably use higher dimensions as construction scaffolds, its physical content and meaning should refer to 4 dimensions where observer lives.

\footnote{Invited talk at the conference “Modern trends in classical approaches” devoted to the 80-th birthday of K.A. Ter-Martirosyan, Moscow, September 2002.}

\footnote{On leave of absence from ITEP, Moscow, Russia.}
1 Introduction.

Karen Avetovitch belongs to the first generation of the Landau school. A characteristic feature of Landau and his disciples was dislike of “philosophy”. The latter was understood in broad sense as any kind of discussion without explicit formulas or numbers. A scientific paper should involve a derivation of some new formula or new number — this was the main lesson which Landau taught to K.A. and which K.A. taught to his students including myself. In my own scientific activity, I mostly tried to follow this commandment, but human beings are weak and sinful, and cannot really be Good all the time. Sometimes, when the task to derive things scientifically is too hard (as it is the case for quantum gravity), it is very difficult to resist the temptation to think and, which is worse, to talk about these matters. When discussing the foundations of quantum gravity, one has to do a philosophical talk or no talk at all. Today I’ve chosen the first option and can only hope that K.A. will not condemn me too much.

Actually, we do not understand what quantum gravity is. To understand why we do not understand this, let me briefly remind the things that we understand well.

- We understand well Newton’s laws and, generically, the dynamics of any classical system where equations of motion have Cauchy form: you set up the initial conditions at a given time moment and find out how the system will look like at later times. The number of dynamic variables can be finite (this is called classical mechanics) or continuously infinite (this is called classical field theory). Such dynamic systems often enjoy extra symmetries. The symmetries might be global (with Nöther currents, etc) or dynamical (involving the Hamiltonian). The important representative of the latter is Lorentz symmetry. There are also gauge symmetries, which are not symmetries but rather additional constraints imposed in phase space, which are respected during the time evolution of the system prescribed by its Hamiltonian.

- We know how to construct quantum counterparts for all theories mentioned above. You introduce Hilbert space and write the Schrödinger equation for wave functions (in case when the number of degrees of freedom is finite) or wave functionals (in case when the number of degrees of freedom is continuous). To tackle with the continuous number of
dynamic variables in field theories, one should first make it finite (introduce ultraviolet and infrared regularization) and then explore the limit when the corresponding cutoffs are lifted. In some cases (like for QED or for $\lambda\phi^4$ theory or for any field theory with space-time dimension 5 or greater), this leads to a trouble: the continuum limit does not exist. But in many physically important cases ($D = 4$ non–Abelian gauge theories), the continuous limit is well defined.

And this is all that we know for sure. The reader might be surprised why did I not mention classical gravity. A common believe is that though quantum gravity is, indeed, not constructed and not understood yet, the classical theory, Einstein’s gravity, is something which we know well and are sure about. Mostly, this is true, but not quite. The discussion of this nontrivial point is what I would like to begin with.

2 Einstein’s gravity.

The action of the theory is

$$S = m^2_P \int R \sqrt{-g} \, d^4 x + \int L_{\text{matter}} \sqrt{-g} \, d^4 x .$$

The equations of motion are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{m^2_P} T_{\mu\nu} ,$$

where $R$ is scalar curvature, $R_{\mu\nu} -$ Ricci’s tensor, $m_P -$ Planck mass, and $T_{\mu\nu} -$ energy-momentum tensor of the matter fields.

The main problem of this theory is the problem of time (see e.g. Ref.[1] for an extensive discussion). In “normal” systems, time is an independent variable, not a dynamical one. In gravity, time is just one of the coordinates on a $D = 4$ manifold and is intertwined with spatial coordinates, which are related to the dynamic variables. The dependence on time cannot be disentangled from other dependencies. At the classical level, this means that the problem of solving the Einstein’s equations (2) cannot be always reduced to a Cauchy problem.

We hasten to comment that, in all cases representing physical interest, it can. This can be done if the 4-dimensional manifold can be represented as
a set of three–dimensional slices of the same topology (the interval between any two points on such a slice is space-like). In other words, the topology of space–time should be $\Sigma \times \mathbb{R}$. In the physically interesting case, $\Sigma$ is topologically equivalent to $\mathbb{R}^3$ and is asymptotically flat. Choosing some coordinate along the timelike factor $\mathbb{R}$, we may call it time and rewrite Einstein’s equations such that they would express evolution with respect to this time. This procedure is called canonical Arnowitt-Deser-Misner formalism [2].

The trouble strikes back in the following way. Suppose we pose some initial conditions at the spacelike slice $\Sigma$ corresponding to the moment $t = 0$ and are interested what happens at later times. For generic initial conditions, singularities will develop (black holes will be formed). The formation of black holes as such does not lead to inconsistencies. The matter is that the singularity in the center of the hole is normally surrounded by an event horizon (as is the case for the Schwarzschild solution) and is thereby unreachable: if we place the observer far away from the holes, where the metric is nearly flat, he will not get signals from the regions close to singularities and, as far as this observer is concerned, the future evolution of the system is uniquely determined by the Cauchy data in the past.

The conjecture of R. Penrose was that singularities are always surrounded by horizons and a “naked” singularity is never possible (the so-called Cosmic censorship principle) [3]. It was found, however, that this conjecture is not true in its strong form: there are solutions to Einstein’s equations involving naked singularities (see Ref.[4] for recent review). A separate question is whether these solutions are physically realized. The answer to this is probably negative: all such solutions seem to be unstable so that a small fluctuation of initial conditions destroy them. But in principle, naked singularities are not forbidden in general relativity.

The presence of a naked singularity means that a distant observer receives information from regions of arbitrary large curvature where classical theory does not apply. Still, he does not receive in this case information from the singularity proper, and Cauchy interpretation is not spoiled yet on this stage. But there are cases when it is. First of all, the symmetry of the equations with respect to time reversal tells one that, on top of black hole solutions, there are white hole solutions, for which the world lines, matter, and information flow out of the singularity through the horizon to infinity. Again, these solutions are not stable and are not physically realized (at least, at the macroscale), but, at the foundational level, they present a trouble.
Even more this refers to the wormhole solutions with closed time loops [5]. They have roughly the same status as the naked singularity solutions and white hole solutions. The topology of the corresponding 4–manifolds is more complicated than \( \Sigma \times \mathbb{R} \) (so that the ADM canonical formalism does not apply here) and involves a “handle” with two “mouths”. The distance between the mouths in the usual space may be large while the geodesic distance measured through the wormhole may be small. As a result, the particles travelling through the wormhole will effectively move faster than light from the viewpoint of an outer space observer, and this means violation of causality, which is a trouble. In particular, no Cauchy interpretation for the equations of motion is possible in this case [6].

In other words, general relativity describes well observable physical events at macroscale, but it has inherent problems at the foundational level. The same difficulty appears in any gravity theory including general covariance principle. The basic reason for this is the absence of independent flat time.

3 Quantization

If the problems are there at the classical level, they are not going to disappear when we try to quantize the theory. Actually, they become much more severe. If in the classical case non–causality showed up only for rather special solutions, it is an inherent and unavoidable feature of quantum gravity.

I mean here in the first place Hawking’s paradox [7] associated with black hole formation. As was discussed above, in the classical theory, there are “benign” solutions, which describe the formation of black holes prudently surrounded by a horizon. These solutions present no conceptual problems. But in quantum theory, black holes are not completely black, they radiate by Hawking mechanism. This radiation is purely stochastic and does not carry any information on what particular kind of matter fell in the black hole. This information is lost completely. Therefore, our system, having presented a pure quantum state at \( t = 0 \), is necessarily transformed into mixed state after the black hole was formed and radiated a little bit. This means loss of unitarity\(^3\). In a quantum system with well–defined Hilbert space endowed by

\(^3\)In quantum theory, unitarity and causality are related notions, and breaking of unitarity leads usually to breaking of causality (see more detailed discussion at the end of Sect. 5). Causality in quantum gravity is broken also more directly via production of
a norm invariant under time evolution, such a transformation of pure states into mixed states does not happen, and nobody knows how to formulate a quantum theory where the norm in Hilbert space is not conserved.

To be more precise, there were attempts to formulate non–Schrödinger quantum theories. In the framework of the ADM approach, one can naturally derive the so–called Wheeler – de Witt equation [8]. It says

$$\hat{H} \Psi = 0$$

(no term $i\dot{\Psi}$ on the right side). One obtains zero on the right side, because the ADM Hamiltonian, the generator of time translations, represents here one of the gauge constraints: in gravity, the symmetry with respect to coordinate translations is local, not global one. There are comparatively “cosher” quantum systems described by the wave equation of the Wheeler – de Witt type. One of them is a quantum relativistic particle. The Klein-Gordon equation $(\hat{p}^2 - m^2)\Psi = 0$ has exactly the form (3), and this is not accidental. The classical action

$$S = \frac{m}{2} \int \left( \frac{dx_{\mu}}{d\tau} \right)^2 d\tau$$

is invariant with respect to reparametrizations $\tau \rightarrow f(\tau)$ and reminds gravity in this respect. The Klein Gordon operator plays exactly the role of the ADM Hamiltonian. However, this theory can also be formulated in a standard Schrödinger form if choosing $x_0$ as time. The Schrödinger Hamiltonian is then $\hat{H}_{\text{Schrod}} = p_0 = \sqrt{\hat{p}^2 + m^2}$. For the systems with the wave function, which is changed not too rapidly (so that the square root $\sqrt{-\partial_i^2 + m^2}$ is well defined), the equations $\hat{H}_{\text{ADM}} \Psi = 0$ and $\hat{H}_{\text{Schrod}} \Psi = i\dot{\Psi}$ are equivalent.

For gravity, one can in principle also use this trick, but

- Even for the simple system (4), there is still no complete equivalence of the Schrödinger equation and the Wheeler – de Witt one; the restriction for the wave functions not to change too rapidly should be imposed. Moreover, at least in the case when external electromagnetic field is present, the Klein-Gordon equation (as well as the Dirac one) is known to be not internally self–consistent because it does not take into account the creation of particle–antiparticle pairs, which always occurs in strong fields.

virtual wormholes.
• In gravity (in contrast to the relativistic particle), we do not have a unique natural recipe how to choose time. As a result, the system (3) meets very serious, probably insurmountable difficulties in interpretation [1].

4 String Story.

Besides the difficulties discussed above, a standard quantum gravity also has another problem: it is a theory with dimensional constant $m_P$ and, as such, is non-renormalizable. This refers to the quantum version of the standard Einstein’s gravity and also to its supersymmetric versions (though some divergences cancel out in supergravity, even $\mathcal{N} = 8$ supersymmetry is not powerfull enough to get rid of the infinite number of counterterms). To cure this problem, string theory was invented. The latter cures it by the simple reason: a finite size of a string serves as an ultraviolet regulator and the ultraviolet divergences are effectively cut off.

There are two points which I want to emphasize here.

1. Even though perturbative string theory is, indeed, benign in ultraviolet, it is in some sense not constructed until now! We understand it well at the tree level: we can very well calculate tree string amplitudes described by the picture in Fig. 1a (2-sphere with sources) and also at the 1-loop level (torus with sources). But already the calculation of the two-loop graph in Fig. 1c is a tremendously difficult task. It involves integration over moduli space for 2-manifolds of genus 2, and
the latter has a very complicated structure. This moduli space (called Teichmüller space) involves certain singular points corresponding to the cases when the width of one of the handles in Fig. 1c shrinks to zero. The integral for string amplitudes becomes divergent at these singular points, and though these divergences are not ultraviolet, but rather infrared in nature, they are also nasty. Very recently, the solution of the problem for two–loop amplitudes was announced [9] (see also Ref.[10]), but we still do not know how to treat divergences and calculate string amplitudes in the general case.

2. Even if consistent string perturbation theory for an arbitrary number of loops will ever be constructed, it will solve the problem of renormalizability, but will hardly solve real conceptual problems of quantum gravity discussed above: the absence of causality and unitarity.

Let us discuss this point in some more details. String theory has one nice feature compared to simple-minded quantum gravity: if strings are embedded into flat multidimensional target space (usually called bulk), there is a natural definition of time. However, strings are nonlocal objects, and, in the full theory treated nonperturbatively, this should bring about noncausalities at Planck scale (though perturbative string amplitudes are probably causal). Noncausalities in the bulk are bound to lead to noncausalities in effective 4–dimensional theory.

The question whether string theory is unitary has different answers depending on whether we consider it in the bulk (then hopefully it is) or from the viewpoint of a 4–dimensional observer. In the latter case, it is definitely not because the effective 4–dimensional theory is still Einstein’s (super)gravity and Hawking’s paradox is still there.

My personal opinion (I will give more arguments in its favor later) is that string theory (at least the conventional string theory in the framework of mid-eighties paradigm) has little chances to prove to be the fundamental theory of quantum gravity and/or of Everything. Actually, nowadays most string theorists also think that one should look beyond string theory to find a really fundamental one (M–theory?).

My own suggestion, however, is that instead of looking beyond strings, one can try to look in a different direction.
5 Conformal gravity

Going to strings instead of the fields is a rather bold and radical step. The main conceptual problem is impossibility to define Hilbert space and path integral in reasonably rigorous terms.

Of course, mathematicians maintain that the path integral is not defined even in field theory, but for a physicist, there is no problem there. The Euclidean path integral is defined constructively and has been calculated numerically by thousands of people since last 20 years. We believe that the Minkowski path integral can also be calculated and the problem here is purely technical. But for strings we have no idea how to do it. Whereas in field theory, we have an infinite number of dynamic variables marked by spatial points $x$, in string field theory dynamic variables are functionals on loop space, i.e. the argument for the string field variable is a particular embedding of the string in space $\{x(\sigma)\}$. In quantum theory, the basic object would be a complex-valued "hyper-functional" defined on the set of all such functionals. Many people tried to obtain some practical results in this direction, but to no avail. Two loops is the limit of our understanding now.

Bearing this in mind, it is reasonable to explore less revolutionary approaches. String theory makes gravity renormalizable, but is it not possible to make it renormalizable in a conservative field theory framework?

Yes, it is — is the answer. Quantum version of Einstein gravity is non-renormalizable due to the presence of a dimensional constant. It is easy to write a generally covariant Lagrangian where the coupling is dimensionless and the theory is renormalizable. The Einstein–Hilbert action (1) is linear in $R$. Renormalizable gravity is quadratic in $R$. There is a family of such theories with the actions

$$ S = \alpha \int R_{\mu\nu} R^{\mu\nu} \sqrt{-g} \, d^4 x + \beta \int R^2 \sqrt{-g} \, d^4 x . \quad (5) $$

The structure $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ is reduced (at least, in the perturbation theory) to the two structures in Eq. (5) due to the Gauss–Bonnet identity

$$ R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \text{total derivative} . \quad (6) $$

It is known since long time that the theories of the class (5) are renormalizable. Moreover, they are asymptotically free! [11]. We will concentrate on
one particular theory in the family (5) with the action

\[ S = -\frac{1}{\hbar} \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{-g} \, d^4 x, \]  

(7)

where

\[ C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2} [g_{\mu\sigma} R_{\nu\rho} + g_{\nu\rho} R_{\mu\sigma} - g_{\mu\rho} R_{\nu\sigma} - g_{\nu\sigma} R_{\mu\rho}] \]

+ \frac{R}{6} [g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}] \]  

(8)

is the Weyl tensor. A distinguishing feature of the theory (7) is its invariance under local scale transformations,

\[ g_{\mu\nu}(x) \rightarrow \lambda(x) g_{\mu\nu}(x). \]  

(9)

Bearing in mind the relation (6), the action (7) is perturbatively equivalent to (5) with \( \beta = -\alpha/3 = 2/(3\hbar) \).

An immediate objection against the idea that the theory (7) describes the real world could be that it does not have Newtonian limit. Nonrelativistic potential corresponding to the action (7) is not Coulomb-like, but grows \( \propto r \) (this follows from dimensional counting). The objection to this objection is that effective long-distance theory needs not to coincide with the fundamental one. In fact, it can well coincide with Einstein’s gravity!

As was mentioned, conformal gravity is an asymptotically free theory. The explicit 1–loop calculation gives [11]

\[ \frac{1}{\hbar} \bigg|_{\mu} = \frac{1}{\hbar_0} - \frac{199}{30} \frac{1}{16\pi^2} \ln \frac{\Lambda_{UV}}{\mu}, \]

(10)

where \( \Lambda_{UV} \) is the ultraviolet cutoff. Asymptotic freedom makes the physics of conformal gravity rather similar to that of QCD. At large energies, perturbation theory works, but at some scale \( \mu \sim \Lambda_{\text{Conf. Grav}} \), where the effective constant becomes large, nonperturbative effects come into play. The scale \( \Lambda_{CG} \) determines the mass of hadron–like states. This is the standard dimensional transmutation. It is natural to associate the scale \( \Lambda_{CG} \) with the Planck scale.

In QCD, there are distinguished states, the pions, which remain massless in the chiral limit. Thus, the effective theory for massless QCD is the chiral
theory describing pion interactions. The form of the leading-order chiral effective Lagrangian

$$L_{\text{chiral}} = \frac{F^2}{4} \text{Tr}\{\partial_{\mu} U \partial_{\mu} U^\dagger\}$$

is dictated by symmetry considerations.

The effective Lagrangian for conformal gravity is not invariant under local scale transformations (9), but general covariance should still be there. This dictates

$$S_{\text{eff}} = \Lambda \int \sqrt{-g} \, d^4 x + \kappa \int R \sqrt{-g} \, d^4 x,$$

where $\Lambda$ is now cosmological term. *A priori*, $\Lambda \sim m_P^4$ and $\kappa \propto m_P^2$. The estimate $\Lambda \sim m_P^4$ is about 130 orders of magnitude larger than the experimental value of cosmological constant. Thus, the theory (7) is not viable as a realistic fundamental theory of gravity. This refers actually to any nonsupersymmetric theory. But if we start with supersymmetric conformal gravity without cosmological term, the induced cosmological constant vanish. In other respects, the physics of conformal supergravity is similar to that of conformal gravity. In particular, conformal supergravity is asymptotically free and involves dimensional transmutation.

The second term in Eq. (12) is the induced Einstein’s gravity. The idea, by which the Einstein–Hilbert action is not present in the tree action, but is generated spontaneously due to loops of usual matter fields was put forward long time ago by Sakharov [12]. It was mentioned in Ref.[13] that this mechanism works also for conformal (super)gravity and the analogy with the dimensional transmutation mechanism in QCD was emphasized.

At the scale $p_{\text{char}} \sim \Lambda_{\text{CG}} \sim m_P$, nonperturbative effects come into play. In QCD, the nonperturbative effects are not reduced to, but are well represented by instantons, classical solutions to Euclidean field equations. In gravity, there are also such solutions, they are Ricci-flat 4-dimensional manifolds called gravitational instantons. The simplest such solution is the Eguchi–Hanson solution [14] with the metric

$$ds^2 = \frac{dr^2}{1 - a^4} + r^2 \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \left( 1 - \frac{a^4}{r^4} \right) \right],$$

*In real world, supersymmetry is broken and it is not clear, again, why cosmological constant is so small. Nobody can answer now this troublesome question.*
where

\[
\begin{align*}
\sigma_x &= \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi) \\
\sigma_y &= -\frac{1}{2} (\cos \psi d\theta + \sin \theta \sin \psi d\phi) \\
\sigma_z &= \frac{1}{2} (d\psi + \cos \theta d\phi)
\end{align*}
\] (14)

are Cartan–Mauer forms. The metric (13) is locally asymptotically flat. It satisfies the condition \( R_{\mu\nu} = 0 \), which are equations of motion for Einstein’s gravity without matter, but Ricci flatness implies also that the equations of motion for conformal gravity

\[
g_{\mu\nu}(3C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + 2R_{\nu}^{\alpha\nu}) - 4(RR_{\mu\nu} - R_{\mu\nu}) + 12(2R_{\mu}^{\alpha}R_{\nu}\alpha - R_{\mu\nu}^{\alpha\nu} - R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}) = 0
\] (15)

are satisfied.

In contrast to Einstein’s Euclidean action, which is not positive definite and the corresponding path integral is ill-defined, the Weyl action is positive definite. The Weyl action of Eguchi–Hanson instanton is

\[
S_{\text{inst}} = \frac{48\pi^2}{h}
\] (16)

The contribution of the Eguchi-Hanson instanton to the path integral is nonanalytic in \( h \), \( \propto \exp\{-(48\pi^2)/h\} \), which is much similar to what happens in Yang–Mills theory. The EH instanton is analogous to the BPST instanton also in other aspects: (i) The Riemann tensor for the EH instanton is self-dual, as field strength tensor for BPST instanton is; (ii) Like the BPST instanton, the EH instanton can be interpreted as an Euclidean tunneling trajectory interpolating between two topologically distinct vacua [15]. In the Yang–Mills case, different vacua are characterized by different Chern–Simons numbers. In the gravity case, there are two classical vacua with flat \( R^3 \) metric, but with different orientation. Following the EH instanton tunneling trajectory, flat \( R^3 \) space turns inside out and goes over to its mirror image.
Questions and answers.

Not everything is so rosy, however. Conformal gravity has also certain difficulties which we are in a position to discuss now.

First of all, when writing Eq.(10), we tacitly assumed (and this is true) that the one–loop counterterm has the same functional form as the tree action. However, the classical conformal symmetry of the Weyl action is broken by quantum effects. This means that we cannot guarantee that higher–loop counterterms are all proportional to (7). The admixture of the structure $R^2$ cannot been ruled out. Thus, pure Weyl gravity is not renormalizable. Of course, one could consider the theory (5) with two charges. Its physics is roughly the same as for the conformal gravity, but it is much less beautiful and hence much more suspicious. The same concerns the $\mathcal{N} = 1$ supersymmetric version of Weyl theory. It is also asymptotically free, conformal symmetry is anomalous, and nonconformal counterterms are bound to appear at the two loop level and higher.

Aesthetically more appealing are the models where conformal symmetry of the classical action is sustained at quantum level. They are not only renormalizable, but simply finite: $\beta$ function vanishes identically and counterterms of dimension 4 do not appear whatsoever. The most known example of such theory is $\mathcal{N} = 4$ supersymmetric Yang–Mills. Finite theories based on conformal gravity are also known. The minimal variant of $\mathcal{N} = 4$ conformal supergravity happens not to be finite, but the coupling constant ceases to run if including an extra $\mathcal{N} = 4$ SYM multiplet with the gauge group $U(1)^4$ or $SU(2) \times U(1)$ [11].

If $\beta$ function vanishes, we do not have the mechanism of dimensional transmutation at our disposal and the question arises how the effective Einstein action involving a dimensional coupling is generated. The answer is rather transparent: conformal symmetry is not broken explicitly by quantum effects in this case, but it can be broken spontaneously. The point is that $\mathcal{N} = 4$ finite theories involve scalar Higgs fields. For certain nonzero values of the fields, classical potential vanishes. Supersymmetry dictates that the potential is not generated also at quantum level: classical flat directions remain flat in quantum theory. A set of all Higgs values where potential vanishes is called vacuum valley or vacuum moduli space. This is a situation of neutral equilibrium: no particular point on the vacuum moduli space is preferred, and we have a family of theories characterized by particular
Higgs expectation values. This all is very well known for $\mathcal{N} = 4$ finite gauge theories, but it is also true for finite $\mathcal{N} = 4$ conformal supergravities.

Higgs expectation values bring about dimensional constants so that an effective low-energy theory is not conformal anymore. In the case of finite gauge theories, the effective theory is akin to the Standard Model, involving spontaneous breaking of gauge symmetry by Higgs mechanism. The effective theory for the finite conformal supergravity involves Einstein’s term and its superpartners.

Let us discuss another difficulty that conformal supergravity has. The Lagrangian (5) involves four derivatives of the metric. Field theories with higher derivatives are usually considered sick because they are intrinsically noncausal. The latter applies also to conformal gravity. To understand this, consider the theory involving on top of the higher derivative terms also the Einstein term, $\mathcal{L} \sim m_P^2 R + R^2$. The propagator of graviton has then the form

$$D(k^2) \propto \frac{1}{m_P^2 k^2 - k^4} = \frac{1}{m_P^2} \left( \frac{1}{k^2} - \frac{1}{k^2 - m_P^2} \right). \quad (17)$$

In other words, on top of an ordinary massless graviton $G$, a massive particle $G^*$ with negative residue at the pole appears. Production of particles with negative residues would violate unitarity.

However, it is known that unitarity is actually not violated here [16, 17]. What is violated is causality. The point is that, when loop corrections are taken into account, the massive pole is shifted from the real axis, the “particle” $G^*$ ceases to be an asymptotic state and cannot be produced in collision of usual massless gravitons. Indeed, nothing prevents the particle $G^*$ to go into a set of massless gravitons, and this makes the polarization operator $\Pi(m_P^2)$ corresponding to the propagator (17) complex. If $G^*$ were a “normal” particle with positive metric, the resultant propagator

$$\frac{1}{k^2 - m_P^2 - \Pi(m_P^2)}$$

would involve a pole in the lower half-plane of $k_0^2$ ($\text{Im}[\Pi(m_P^2)] < 0$ in this normalization). When the residue is negative, the propagator

$$\frac{1}{-k^2 + m_P^2 - \Pi(m_P^2)}$$

\footnote{We are not worried with numerical factors now.}
has the pole in the upper half-plane of $k^2_0 \Pi(m_P^2)$ is determined by the same graphs as for a usual particle and has the same value. This property prevents making a usual Wick rotation and is not consistent with causality. \footnote{In the papers \cite{16}, higher derivative theories were studied mainly in association with Pauli–Villars regularization procedure. The conclusion was that the regularized Lagrangians lead to unitary amplitudes, but that causality is broken at the regulator scale.}

Currently, it is not clear whether the causality breaking at Planck scale persists in the finite superconformal theories discussed above. In Ref. \cite{11} a careful optimism was expressed that may be it does not. But even if it does, we do not see why it should be considered as a major problem. At nonperturbative level, microcausality is broken in any gravity theories, with string theory not presenting an exception. In conformal supergravity models it is probably also broken perturbatively.

So what?

## 6 Supergravity as a theory of 3–brane: Ogievetsky–Sokatchev approach.

In the previous section, we argued that conformal supergravity (probably, a finite, anomaly–free version thereof) can be considered as a viable candidate for the fundamental gravity theory. It solves the problem of nonrenormalizability of standard gravity even better than string theory does (we say better, because perturbative calculations to any order in coupling constant present no essential technical difficulties there) and the difficulties it has are intrinsic for any gravity theory.

String theory has one attractive feature, however. It is formulated not in curved 4–dimensional space, but in the flat multidimensional bulk. This gives a principle solutions to the problem of time, and brings forward hopes to construct self–consistent quantum theory.

We want to notice here that similar hopes can actually be associated with standard supergravity if describing the latter in the superfield formalism due to Ogievetsky and Sokatchev \cite{18}.

Ogievetsky–Sokatchev approach to supergravity has a lot of advantages compared to the standard Wess–Zumino approach. Unfortunately, the former is not so widely known, and we are in a position to explain briefly its
basic features. In Wess–Zumino approach, the basic superfield is $E_A^M$, a supersymmetric generalization of vierbein. This superfield has a lot of unphysical components; to get rid of them, one has to impose constraints of a rather complicated form.

The Ogievetsky–Sokatchev approach is based on a beautiful geometric construction. Consider a curved $(4 + 4)$-dimensional supermanifold (it has 4 bosonic coordinates $x^m$ and 4 real or 2 complex fermionic coordinates $\theta_\alpha$) embedded into flat $(8 + 4)$-dimensional superspace involving 4 complex (which is equivalent to 8 real) bosonic coordinates $z^m$ and 2 complex fermionic coordinates. Such an embedding is characterized by the superfield $H^m(x^n, \bar{\theta}^\dot{\alpha}, \theta_\alpha)$, where $H^m$ coincides with the imaginary parts of flat coordinates $z^m$ and $x^m$ — with their real parts. The Lagrangian of the standard Einstein supergravity is none other than the supervolume of the associated hypersurface:

$$S_{\text{sugra}} = m_p^2 \int \text{Ber} \parallel E_A^M \parallel d^4x \ d^4\theta , \quad (18)$$

where $E_A^M$ is the induced super–vielbein on the hypersurface and “Ber” stands for the Berezinian (or superdeterminant). Now, $E_A^M$ and $\text{Ber} \parallel E \parallel$ can be expressed in terms of $H^m(x^n, \bar{\theta}^\dot{\alpha}, \theta_\alpha)$ (in a not so simple, but explicit way). One can check that they obey the constraints that are imposed on $E_A^M$ in the Wess–Zumino approach. On the other hand, no constraints on the axial superfield (we are using the Ogievetsky–Sokachev terminology) $H^m$ need be imposed.

The Lagrangian (18) is invariant with respect to general reparametrisations of all bosonic and fermionic coordinates on the hypersurface. This group is too large, however, which is not convenient. In addition, a generic such reparametrisation destroys the simple form

$$\text{Im}(z^m) = H^m \left( \text{Re}(z^n), \bar{\theta}, \theta \right) \quad (19)$$

chosen by us to describe the hypersurface.

The form (19) is preserved by a subgroup of the general reparametrisation group. To describe it, introduce left and right coordinates $x^m_{L,R} = x^m \pm iH^m$ and require them to reduce to the familiar

$$x^m_{L,R} = x^m \pm i\bar{\theta}\sigma^m\theta . \quad (20)$$
in the limit when the embedded hyper–surface represents a hyper–plane. Then the transformations

\[ x_L^m \rightarrow f^m(x_L^n, \theta_\beta) \]
\[ \theta_\alpha \rightarrow \chi_\alpha(x_L^n, \theta_\beta) \]  

(21)
obviously preserve the form (19). To provide for the invariance of the action (18) or, which is the same, to provide for that the transformations (21) represented a reparametrization of the coordinates on the hypersurface, it is sufficient to require that the super-Jacobian of the transformation (21) be equal to 1,

\[
\text{Ber} \left( \frac{\partial (x', \theta')}{\partial (x, \theta)} \right) = \det \left( \frac{\partial x_L^m}{\partial x_L^n} - \frac{\partial x_L^m}{\partial \theta_\alpha} \frac{\partial \theta'_\beta}{\partial x_L^n} \right) \det^{-1} \left( \frac{\partial \theta'_\alpha}{\partial \theta_\beta} \right) = 1.
\]  

(22)
The gauge symmetry (21) allows one to greatly reduce the number of components of \( H^m \). There are all together 64 components. The transformations (21) involve 48 parameters, but the condition (22) fixes 8 of them leaving 40 free parameters. As a result, we obtain 24 (12 bosonic and 12 fermionic) gauge–invariant degrees of freedom. They exactly correspond to component language counting [19]. The Lagrangian involves 38 components (16 for the vierbein \( e_a^m \), 16 for the gravitino \( \psi^m_\alpha \), and 6 for the auxiliary fields \( S, P, A^m \).

There are 4(general coordinate) plus 6(local Lorentz) plus 4(supersymmetry) = 14 gauge parameters. Now, 38 – 14 = 64 – (48 – 8) = 24.

It is convenient to choose the normal gauge (analogous to the Wess–Zumino gauge used in the analysis of supersymmetric gauge theories), in which case

\[ H^m \sim e_a^m \bar{\theta} \sigma^a \theta + \text{other terms}. \]  

(23)
One can then be directly convinced (though the calculation is tedious) that the bosonic part of the action (18) coincides (up to a total derivative !) with \( R \). The other terms in the component Lagrangian are restored by supersymmetry.

Now, \( N = 1 \) conformal supergravity can also be described in these terms: its Lagrangian can be expressed via the unconstrained axial superfield \( H^m \). This Lagrangian (see the papers [18] for explicit formulae) is invariant with respect to the general transformations (21) (not restricted by the requirement of unit super–Jacobian ).
Note in passing that also the variant of supergravity with cosmological term is nicely expressed in the Ogievetsky–Sokatchev formalism. It turns out that the corresponding action represents a total derivative and the problem is reduced to the choice of boundary conditions. Thus, the question why the cosmological term vanishes acquires the same status as the question why the $\theta$ term in QCD vanishes. No comprehensive answer to any of these questions is known, but we are sure at least that, if we start with a supersymmetric theory with vanishing cosmological term, the latter is not generated by quantum effects, by the same token as the $\theta$ term in QCD is not generated.

Our main point is that, once flat space appeared in the formulation of the theory, a natural definition of time exists, which should allow one to present the equations of motion in the Cauchy form. The theory becomes much similar to string theory, only it is in a sense much more complicated: the latter deals with embeddings of 2-surfaces into flat Minkowski bulk, while the former depends on embeddings of 4-surfaces (3–branes in modern terminology) there.

On the other hand, supergravity is still much simpler than the full string field theory. Indeed, in spite of the fact that the action (18) describes multi-dimensional geometry, it is four-dimensional in nature. The basic dynamic variables in such theory are embeddings themselves rather than frightening functionals in the loop space, which we would eventually have to learn to deal with if sticking to the conventional string theory paradigm.

7 Discussion.

Before going further, let us reiterate briefly the main points of our reasoning so far (you may call it party line, bearing in mind that the corresponding party is not numerous and in opposition).

1. We do not know how to construct a consistent gravity theory strictly in a four-dimensional framework. The main problem here is the problem of time, which has not been fully solved even in classical general relativity and becomes a real mayhem when one attempts to quantize it.

2. Quantum version of Einstein’s gravity has another problem: nonrenormalizability. It persists in supersymmetric generalizations.
3. The latter problem is cured in string theory, but a *simpler* and in many respects nicer medicin is provided by conformal gravity. The *effective* low–energy theory for conformal gravity is Einstein’s theory (modulo the problem of cosmological term, which is more tractable for supersymmetric versions of the theory, but is far from being fully resolved). In a nonsupersymmetric or $\mathcal{N} = 1, 2, 3$ supersymmetric versions of the theory involving conformal anomaly, Einstein’s constant is generated due to dimensional transmutation mechanism. We like better $\mathcal{N} = 4$ finite superconformal theories, where Einstein’s constant is generated due to spontaneous breaking of conformal symmetry when a particular point on flat Higgs moduli space is picked up.

4. With all probability, causality is broken in these theories at perturbative level (though this was not explicitly demonstrated) due to the presence of higher derivatives in the Lagrangian and complexification of negative metric poles by Lee and Wick mechanism. But *any* gravity theory is acausal in four dimensions.

5. $\mathcal{N} = 1$ supergravity and conformal supergravity have a nice interpretation due to Ogievetsky and Sokatchev, where the classical field configuration can be thought of as an embedding of a 3–brane into 8–dimensional flat bulk space. This gives one a natural definition of time, and one can hope to construct a unitary quantum theory with well–defined Hilbert space *in the bulk*. The reasons are the same that the reasons why we believe that string theory (we mean string theory in the second quantization framwork, when it is a form of 2–dimensional field theory) is unitary in the bulk.

As the reader has probably already guessed, we *believe* that the future fundamental theory of gravity (and probably of Everything) is a variant of finite superconformal gravity theory. We also believe that this theory can be represented as a theory of 3–brane embedded into a higher–dimensional flat space.

There are still several points which are not clear now. The last one is especially worrysome.

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As we live now in civilized times and the risk of being severely punished (beaten by stones, etc) for a false prophecy is comparatively low, I am allowing myself to make one.

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1. We believe that Ogievetsky–Sokatchev supergravity is unitary and causal in the bulk, but do not know how to prove it. This is going to be much more complicated problem than proving unitarity for string theory (such a proof is also absent now).

2. Nice geometric interpretation discussed above has been found so far only for $\mathcal{N} = 1$ theories. Little is known in this respect about $\mathcal{N} = 4$ theory. An educated guess is that the bulk is this case is 10–dimensional rather than 8–dimensional. One can notice in this respect that, in the problem of embedding of a 4–dimensional manifold into $R^m$, the dimensions 8 and 10 are distinguished. Namely, (i) one can always embed an $n$–dimensional manifold into $R^{2n}$ without self–crossings and (ii) one can always embed an $n$–dimensional manifold into $R^{2n+2}$ without knots (so that all embeddings of a given manifold are topologically equivalent) [20].

3. The finite superconformal gravity theories discussed above do not have realistic matter content. They are based on the gauge group $SU(2) \times U(1)$ or $U(1)^4$, whereas we need the group $SU(3) \times SU(2) \times U(1)$ or larger, three fermion generations, etc. It is not clear, however, that realistic superconformal gravity theories will never be found.

4. There is also a major philosophical problem. The physics of 20-th century is based on positivistic philosophy. We want to formulate theory in terms of physical observables and dismiss as meaningless all attempts to talk about “real” electron trajectories, etc. A real physical observable is by definition something which can be measured by a real physical observer, who is four–dimensional, like we are. But if we treat the theory in a multidimensional bulk, the wave function of the Universe and (in the proposed approach) the $D3$–brane transition amplitudes can be measured only by a “divine” observer living in the bulk. This smells mystics, but I do not know how to get rid of it here.

For any oppositioner, the negative program is usually much stronger than the positive one. That is why I want to finish with some comments on what, I think, the fundamental theory of Everything is not.

I am personally rather skeptical towards the assertions that higher dimensions are really there. The matter is that even ordinary field theory is
Figure 2: Dragon.
ill-defined if the dimension of space–time is 5 or more: the path integral simply does not have continuum limit there (at least, for $D \geq 5$, we are not aware of any example where such a limit existed). I cannot imagine that the string field theory path integral is defined any better. Thus, I do not believe in the ideas (rather popular now) of large extra dimensions, the brane new world, etc.

It may be beneficial and even necessary to think of our physical space as being embedded into a multidimensional flat bulk, but the physical space itself should be four–dimensional. In other words, my attitude towards higher dimensions is close to the standpoint of Catholic Church with respect to heliocentric ideas of Kopernicus and Galileo. No problems as far as they were proposed as a convenient mathematical tool to facilitate calculation of physical observables like planet positions, etc (for people of 16-th century, the physical observer must, of course, dwell on Earth), but the suggestion that Earth really rotates around Sun was unacceptable.  

Close to the end, but not in the very end, I want to present, on top of a historico-philosophical analogy, an artistic one and simultaneously justify the queer title of this paper. “Dragon” is a gravure by Escher. It is reproduced in Fig. 2. As was emphasized in Ref.[21], this dragon seems to be very much three–dimensional, it kind of tries to escape the sheet of paper where it is drawn. But the only “physical dragon” that is at our disposal is the gravure itself, which is two–dimensional. It tries to make us believe that his real dimensionality is more than two, but it is a false claim. Likewise, gravity may be conveniently formulated in higher–dimensional terms, but our physical world has only 4 dimensions.

The last paragraph of the paper is reserved to a physical argument. The idea that an essentially four–dimensional theory can be conveniently described with fictitious higher–dimensional scaffolds is not new. This is exactly the content of Maldacena’s conjecture on AdS/CFT correspondence: the correlators of four–dimensional SYM theory coincide with certain correlators in 10–dimensional supergravity defined on the boundary of some particular background [22]. Many other quantities in 10–dimensional theory can be defined and considered, but they are declared to be meaningless as far as SYM theory is concerned.

My reasons are not religious, however, but simply a desire to be able to define the path integral.
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References

[1] C.J. Isham, gr-qc/9210011, published in the Proceedings of GIFT Int. Seminar on Theor. Physics, Salamanca, 15-27 June, 1992.

[2] R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. 117 (1960) 1595.

[3] R. Penrose, Riv. Nouvo Cim. 1 (1969) 252.

[4] T. Harada, Progr. Theor. Phys. 107 (2002) 449.

[5] K. Gödel, Rev. Mod. Phys. 21 (1949) 447; C.W. Misner and A.H. Taub, Sov. Phys. JETP 28 (1969) 122; M. Visser, Lorentzian wormholes: from Einstein to Hawking (AIP press, New York, 1995).

[6] A. Carlini and I.D. Novikov, Int. J. Phys. D5 (1996) 445.

[7] S. Hawking, Comm. Math. Phys. 87 (1982) 395.

[8] J. Wheeler, in: C. DeWitt and B. DeWitt, eds, Relativity, Groups and Topology (Gordon and Breach, New York and London, 1964), pp. 316-520; B. DeWitt, Phys. Rev. 160 (1967) 1113.

[9] E. D’Hoker and D.H. Phong, Phys. Lett. B529 (2002) 241; Nucl. Phys. B636 (2002) 3,61; B639 (2002) 129.

[10] G.S. Danilov, hep-th/0112022.

[11] E.S. Fradkin and A.A. Tseytlin, Phys. Repts. 119 (1985) 233, and references therein.

[12] A.D. Sakharov, Sov. Phys. Doklady 12 (1968) 1040.

[13] A.V. Smilga: in Proceedings of the Int. Symposium on Group theoretical methods in Physics, Zvenigorod, 1982, v.2, p.73.
[14] T. Eguchi and A.J. Hanson, *Ann. Phys.* **120** (1979) 82.

[15] A.V. Smilga, *Nucl. Phys.* **B234** (1984) 402.

[16] T.D. Lee and G.C. Wick, *Nucl. Phys.* **B9** (1969) 209; *Phys. Rev.* **D6** (1970) 1033.

[17] I. Antoniadis and E.T. Tomboulis, Phys. Rev. **D33** (1986) 2756.

[18] V.I. Ogievetsky and E.S. Sokatchev, *Sov. J. Nucl. Phys.* **31** (1980) 140,424; **32** (1980) 589, 870.

[19] P. van Nieuwenhuizen, *Phys. Repts.* **68** (1981) 189, and references therein.

[20] see e.g. V. Guillemin and A. Pollack, *Differential topology* (Englewood Cliffs, N.J., 1974).

[21] D.R. Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid* (Basic Books, 1979).

[22] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231; S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. **B428** (1998) 105.