LONG-TIME DYNAMICS OF THE STRONGLY DAMPED SEMILINEAR PLATE EQUATION IN $\mathbb{R}^n$

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ABSTRACT. We investigate the initial-value problem for the semilinear plate equation containing localized strong damping, localized weak damping and nonlocal nonlinearity. We prove that if nonnegative damping coefficients are strictly positive almost everywhere in the exterior of some ball and the sum of these coefficients is positive a.e. in $\mathbb{R}^n$, then the semigroup generated by the considered problem possesses a global attractor in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We also establish boundedness of this attractor in $H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.

1. Introduction

In this paper, our main purpose is to study the long-time dynamics (in terms of attractors) of the plate equation

$$u_{tt} + \gamma \Delta^2 u - \text{div} (\beta(x) \nabla u) + \alpha(x) u_t + \lambda u - f(\| \nabla u(t) \|_{L^2(\mathbb{R}^n)}) \Delta u + g(u) = h(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

with initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,$$

where $\gamma > 0$, $\lambda > 0$, $h \in L^2(\mathbb{R}^n)$ and the functions $\alpha(\cdot)$, $\beta(\cdot)$, $f(\cdot)$ and $g(\cdot)$ satisfy the following conditions:

$$\alpha, \beta \in L^\infty(\mathbb{R}^n), \quad \alpha(\cdot) \geq 0, \quad \beta(\cdot) \geq 0 \text{ a.e. in } \mathbb{R}^n,$$

$$\alpha(\cdot) \geq \alpha_0 > 0 \text{ and } \beta(\cdot) \geq \beta_0 > 0 \text{ a.e. in } \{ x \in \mathbb{R}^n : |x| \geq r_0 \}, \text{ for some } r_0 > 0,$$

$$f \in C^1(\mathbb{R}^+), \quad f(z) \geq 0, \text{ for all } z \in \mathbb{R}^+,$$

$$g \in C^1(\mathbb{R}), \quad |g'(s)| \leq C \left( 1 + |s|^{p-1} \right), \quad p \geq 1, \quad (n-4)p \leq n,$$

The problem (1.1)-(1.2) can be reduced to the following Cauchy problem for the first order abstract differential equation in the space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$:

$$\begin{cases}
\frac{d}{dt} \theta(t) = A \theta(t) + \mathcal{F}(\theta(t)), \\
\theta(0) = \theta_0,
\end{cases}$$

where $\theta(t) = (u(t), u_t(t))$, $\theta_0 = (u_0, u_1)$, $A(u, v) = (v, -\gamma \Delta^2 u + \text{div} (\beta(\cdot) \nabla v) - \alpha(\cdot) v - \lambda u)$, $D(A) = \{(u, v) \in H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n) : \gamma \Delta^2 u - \text{div} (\beta(\cdot) \nabla v) \in L^2(\mathbb{R}^n)\}$ and $\mathcal{F}(u, v) = (0, f(\| \nabla u \|_{L^2(\mathbb{R}^n)}) \Delta u - g(u) + h)$. Defining suitable equivalent norm in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, it is easy to see that the operator $A$, thanks to (1.3), is maximal dissipative in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and consequently, due to Lumer-Phillips Theorem (see [11] Theorem 4.3), it generates a linear continuous semigroup $\{ e^{tA} \}_{t \geq 0}$. Also, by (1.6)-(1.7), we find that the nonlinear operator $\mathcal{F} : H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is Lipschitz continuous on bounded subsets of $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. So, applying semigroup theory (see, for example [2], p. 56-58), and taking advantage of energy estimates, we have the following well-posedness result.
Theorem 1.1. Assume that the conditions (1.3), (1.6), (1.7) and (1.8) hold. Then, for every \((u_0, u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), the problem (1.1)-(1.2) has a unique weak solution \(u \in C\left([0, \infty); H^2(\mathbb{R}^n)\right) \cap C^1\left([0, \infty); L^2(\mathbb{R}^n)\right)\), which depends continuously on the initial data and satisfies the energy equality

\[
E(u(t)) + \int_{\mathbb{R}^n} G(u(t,x)) \, dx + \frac{1}{2} F\left(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2\right) - \int_{\mathbb{R}^n} h(x) u(t,x) \, dx
\]

\[
+ \int_0^t \int_{\mathbb{R}^n} \alpha(x)|u_t(\tau,x)|^2 \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^n} \beta(x)|\nabla u_t(\tau,x)|^2 \, dx \, d\tau
\]

\[
= E(u(s)) + \int_{\mathbb{R}^n} G(u(s,x)) \, dx + \frac{1}{2} F\left(\|\nabla u(s)\|_{L^2(\mathbb{R}^n)}^2\right) - \int_{\mathbb{R}^n} h(x) u(s,x) \, dx, \quad \forall t \geq s \geq 0,
\]

where \(F(z) = \int_0^z f(\sqrt{s}) \, ds\) for all \(z \in \mathbb{R}^n\), \(G(z) = \int_0^z g(s) \, ds\) for all \(z \in \mathbb{R}\) and \(E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^n} \left( |u_t(t,x)|^2 + \gamma |\Delta u(t,x)|^2 + \lambda |u(t,x)|^2 \right) \, dx\). Moreover, if \((u_0, u_1) \in D(A)\), then \(u(t,x)\) is a strong solution satisfying \((u, u_t) \in C\left([0, \infty); D(A)\right) \cap C^1\left([0, \infty); H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\right)\).

Thus, due to Theorem 1.1, the problem (1.1)-(1.2) generates a strongly continuous semigroup \(\{S(t)\}_{t \geq 0}\) in \(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) by the formula \(S(t)(u_0, u_1) = S(t)(u_0, u_1)\), where \(u(t,x)\) is a weak solution of (1.1)-(1.2) with the initial data \((u_0, u_1)\).

Attractors for hyperbolic and hyperbolic like equations in unbounded domains have been extensively studied by many authors over the last few decades. To the best of our knowledge, the first works in this area were done by Feireisl in [3] and [4], for the wave equations with the weak damping (the case \(\gamma = 0, \beta \equiv 0\) and \(f \equiv 1\) in (1.1)). In those articles the author, by using the finite speed propagation property of the wave equations, established the existence of the global attractors in \(H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). The global attractors for the wave equations involving strong damping in the form \(-\Delta u_t\), besides weak damping, were investigated in [5] and [6], where the authors, by using splitting method, proved the existence of the global attractors in \(H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), under different conditions on the nonlinearities. Recently, in [7], the results of [5] and [6] have been improved for the wave equation involving additional nonlocal nonlinear term in the form \(-a + b \|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 \Delta u\) \((a \geq 0, b > 0)\). For the plate equation with only weak damping and local nonlinearity (the case \(\gamma = 1, \beta \equiv 0\) and \(f \equiv 0\) in (1.1)), attractors were investigated in [8] and [9], where the author, inspired by the methods of [10] and [11], proved the existence, regularity and finite dimensionality of the global attractors in \(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). The situation becomes more difficult when the equation contains localized damping terms and nonlocal nonlinearities. Recently, in [12] and [13], the plate equation with localized weak damping (the case \(\beta \equiv 0\) in (1.1)) and involving nonlocal nonlinearities as \(-f(\|\nabla u\|_{L^2(\mathbb{R}^n)}) \Delta u + f(\|u\|_{L^p(\mathbb{R}^n)}) |u|^{p-2} u\) have been considered. In these articles, the existence of global attractors has been proved when the coefficient \(\alpha(\cdot)\) vanishes in a set of positive measure, the existence of the global attractor for (1.1) with \(\beta \equiv 0\) remained as an open question (see [12] Remark 1.2)]. On the other hand, in the case when \(\alpha \equiv 0\) and even \(\beta \equiv 1\), the semigroup \(\{S(t)\}_{t \geq 0}\) generated by (1.1)-(1.2) does not possess a global attractor in \(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). Indeed, if \(\{S(t)\}_{t \geq 0}\) possesses a global attractor, then the linear semigroup \(\{e^{tA}\}_{t \geq 0}\) decay exponentially in the real and consequently, complex space \(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), which, due to Hille-Yosida Theorem (see [11] Remark 5.4), implies necessary condition \(i \mathbb{R} \subset \rho(A)\). This condition is equivalent to the solvability of the equation \((i\mu I - A)(u,v) = (y,z)\) in \(H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), for every \((y,z) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and \(\mu \in \mathbb{R}\). Choosing \(\mu = \sqrt{\lambda}\) and \(y = 0\), we have \(v = i\sqrt{\lambda} au\) and \(\Delta(\Delta u - iu) = \gamma z\). If the last equation for every \(z \in L^2(\mathbb{R}^n)\) has a solution \(u \in H^3(\mathbb{R}^n)\), then denoting \(\varphi = \Delta u - iu\) we can say that the equation \(\Delta \varphi = \gamma z\) has a solution in \(H^3(\mathbb{R}^n)\), for every \(z \in L^2(\mathbb{R}^n)\). However, the last equation, as shown in [6], is not solvable in \(H^3(\mathbb{R}^n)\) for some \(z \in L^2(\mathbb{R}^n)\). Hence, the necessary condition \(i \mathbb{R} \subset \rho(A)\) does not hold. Thus, in the case when \(\alpha \equiv 0\) and \(\beta \equiv 1\), the problem (1.1)-(1.2) does not have a global attractor, and in the case when \(\beta \equiv 0\) and \(\alpha(\cdot)\) vanishes in a set of positive measure, the existence of the global for (1.1)-(1.2) is an open question.
In this paper, we impose conditions (1.3)-(1.5) on damping coefficients \( \alpha(\cdot) \) and \( \beta(\cdot) \), which, unlike the conditions imposed in the previous articles dealing with the wave and plate equations involving strong damping and/or nonlocal nonlinearities, allow both of them to be vanished in the sets of positive measure such that in these sets the strong damping and weak damping complete each other. Thus, our main result is as follows:

**Theorem 1.2.** Under the conditions (1.3)-(1.8) the semigroup \( \{S(t)\}_{t \geq 0} \) generated by the problem (1.1)-(1.2) possesses a global attractor \( \mathcal{A} \) in \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) and \( \mathcal{A} = \mathcal{M}^u(N) \). Here \( \mathcal{M}^u(N) \) is unstable manifold emanating from the set of stationary points \( N \) (for definition, see \([14, 359]\)). Moreover, the global attractor \( \mathcal{A} \) is bounded in \( H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \).

The plan of the paper is as follows: In the next section, after the proof of two auxiliary lemmas, we establish asymptotic compactness of the global attractor manifold emanating from the set of stationary points \( N \). Assume that the condition (1.6) holds. Also, assume that the sequence \( \{v_m\}_{m=1}^{\infty} \) is weakly star convergent in \( L^\infty(0, \infty; H^2(\mathbb{R}^n)) \), the sequence \( \{v_{mt}\}_{m=1}^{\infty} \) is bounded in \( L^\infty(0, \infty; L^2(\mathbb{R}^n)) \) and the sequence \( \left\{\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}\right\}_{m=1}^{\infty} \) is convergent, for all \( t \geq 0 \). Then, for every \( r > 0 \) and \( \phi \in C^1_0(B(0, r)) \)

\[
\lim_{m \to \infty} \limsup_{t \to \infty} \left| \int_0^t \int_{B(0, r)} \tau \left( f(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}) \Delta v_m(\tau, x) - f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \Delta v_l(\tau, x) \right) \right| d\tau dx = 0, \quad \forall t \geq 0,
\]

where \( B(0, r) = \{x \in \mathbb{R}^n : |x| < r\} \).

**Proof.** Firstly, we have

\[
\left| \int_0^t \int_{B(0, r)} \tau f(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}) \Delta v_m(\tau, x) - f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \Delta v_l(\tau, x) \right| d\tau dx
\]

\[
\times \phi(x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) d\tau dx| \leq \frac{1}{2} \int_0^t \tau f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \frac{d}{d\tau} \int \phi(x)|v_m(\tau, x) - v_l(\tau, x)|^2 d\tau dx + K^m,t(t),
\]

where \( K^m,t(t) = \int_0^t \tau \left( f(\|\nabla v_m(\tau)\|_{L^2(\mathbb{R}^n)}) - f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \right) \int \phi(x) \Delta v_m(\tau, x)
\]

\[
\times (v_{mt}(\tau, x) - v_{lt}(\tau, x)) d\tau dx - \int_0^t \int_{B(0, r)} \tau f(\|\nabla v_l(\tau)\|_{L^2(\mathbb{R}^n)}) \nabla \phi(x) \cdot \nabla (v_m(\tau, x) - v_l(\tau, x)) \times (v_{mt}(\tau, x) - v_{lt}(\tau, x)) d\tau dx.
\]

Applying \([14, Corollary 4]\), we have that the sequence \( \{v_m\}_{m=1}^{\infty} \) is relatively compact in \( C([0, T]; H^{2-\varepsilon}(B(0, r))) \), for every \( \varepsilon > 0, T > 0 \) and \( r > 0 \). So,

\[
v_m \to v \text{ strongly in } C([0, T]; H^{2-\varepsilon}(B(0, r))),
\]
for some \( v \in C([0, T]; H^{2-\varepsilon}(B(0, r))) \). Hence, we find
\[
\lim_{m \to \infty} \limsup_{t \to \infty} |K_r^{m,t}(t)| = 0, \quad \forall t \geq 0.
\] (2.3)

Now, denoting \( f_\varepsilon(u) = \begin{cases} f(u), & u \geq \varepsilon \\ f(\varepsilon), & 0 \leq u < \varepsilon \end{cases} \) for \( \varepsilon > 0 \), we get
\[
|f\left(\|\nabla v_\varepsilon(t)\|_{L^2(\mathbb{R}^n)}\right) - f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right)| \leq \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|,
\]
and then, for the first term on the right hand side of (2.1), we obtain
\[
\int_0^t \tau f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \frac{d}{d\tau} \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau
\]
\[
\leq \int_0^t \tau f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \frac{d}{d\tau} \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau
\]
\[
+ c_1 t^2 \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|, \quad \forall t \geq 0.
\] (2.4)

Let us estimate the first term on the right hand side of (2.4). By using integration by parts, we have
\[
\int_0^t \tau f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \frac{d}{d\tau} \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau
\]
\[
= t f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau
\]
\[
- \int_0^t f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau
\]
\[
- \int_0^t \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau.
\] (2.5)

By the conditions of the lemma and the definition of \( f_\varepsilon \), it follows that \( \left\{ f_\varepsilon\left(\|\nabla v_m(\cdot)\|_{L^2(\mathbb{R}^n)}\right)\right\}_{m=1}^\infty \) is bounded in \( W^{1, \infty}(0, \infty) \). Then, considering (2.2) in (2.5), we get
\[
\lim_{m \to \infty} \limsup_{t \to \infty} \int_0^t \tau f_\varepsilon\left(\|\nabla v(t)\|_{L^2(\mathbb{R}^n)}\right) \frac{d}{d\tau} \int_{B(0,r)} \phi(x) |\nabla v_m(\tau, x) - \nabla v_l(\tau, x)|^2 dx d\tau = 0.
\] (2.6)

Taking into account (2.3), (2.4) and (2.6) in (2.1), we obtain
\[
\limsup_{m \to \infty} \limsup_{t \to \infty} \int_0^t \int_{B(0,r)} \tau \left( f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}) \Delta v_m(t, x) - f(\|\nabla v_l(t)\|_{L^2(\mathbb{R}^n)}) \Delta v_l(t, x) \right)
\]
\[
\times \phi(x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \leq c_1 t^2 \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|, \quad \forall t \geq 0,
\]
which yields the claim of the lemma, since \( \varepsilon > 0 \) is arbitrary. \( \square \)
Lemma 2.2. Assume that the condition (1.7) holds. Also, let the sequence \( \{v_m\}_{m=1}^\infty \) be weakly star convergent in \( L^\infty (0, \infty; H^2(\mathbb{R}^n)) \) and the sequence \( \{v_m\}_{m=1}^\infty \) be bounded in \( L^\infty (0, \infty; L^2(\mathbb{R}^n)) \). Then, for every \( r > 0 \) and \( \phi \in L^\infty (B(0, r)) \)

\[
\lim_{m \to \infty} \lim_{l \to \infty} \int_0^t \int_{B(0,r)} \tau (g(v_m(\tau,x)) - g(v_l(\tau,x))) \phi(x) (v_m(\tau,x) - v_l(\tau,x)) \, dx \, d\tau = 0, \quad \forall t \geq 0.
\]

Proof. We have

\[
\int_0^t \int_{B(0,r)} \tau (g(v_m(\tau,x)) - g(v_l(\tau,x))) \phi(x) (v_m(\tau,x) - v_l(\tau,x)) \, dx \, d\tau
\]

\[
= \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_m(\tau,x)) v_m(\tau,x) \, dx \, d\tau + \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_l(\tau,x)) v_l(\tau,x) \, dx \, d\tau
\]

\[
- \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_m(\tau,x)) v_l(\tau,x) \, dx \, d\tau - \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_l(\tau,x)) v_m(\tau,x) \, dx \, d\tau.
\]

(2.7)

Let us estimate the first two terms on the right hand side of (2.7). Applying integration by parts, we get

\[
\int_0^t \int_{B(0,r)} \tau \phi(x) g(v_m(\tau,x)) v_m(\tau,x) \, dx \, d\tau + \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_l(\tau,x)) v_l(\tau,x) \, dx \, d\tau
\]

\[
= t \int_{B(0,r)} \phi(x) G(v_m(t,x)) \, dx + t \int_{B(0,r)} \phi(x) G(v_l(\tau,x)) \, dx
\]

\[
- \int_0^t \int_{B(0,r)} \phi(x) G(v_m(\tau,x)) \, dx \, d\tau - \int_0^t \int_{B(0,r)} \phi(x) G(v_l(\tau,x)) \, dx \, d\tau.
\]

(2.8)

By the conditions of the lemma, we obtain

\[
\begin{align*}
&\left\{ \begin{array}{l}
v_m \to v \text{ weakly star in } L^\infty (0, \infty; H^2(\mathbb{R}^n)), \\
v_m \to v_l \text{ weakly star in } L^\infty (0, \infty; L^2(\mathbb{R}^n)),
\end{array} \right.
\end{align*}
\]

(2.9)

for some \( v \in L^\infty (0, \infty; H^2(\mathbb{R}^n)) \cap W^{1,\infty} (0, \infty; L^2(\mathbb{R}^n)) \). Applying [15 Corollary 4], by (2.9), we have

\[
v_m \to v \text{ strongly in } C \left([0,T]; H^{2-\varepsilon} (B(0,r))\right),
\]

for every \( \varepsilon > 0 \) and \( T > 0 \). Hence, taking into account (1.7), we get

\[
G(v_m) \to G(v) \text{ strongly in } C \left([0,T]; L^1 (B(0,r))\right).
\]

(2.10)

Then, passing to the limit in (2.8) and using (2.10), we obtain

\[
\lim_{m \to \infty} \lim_{l \to \infty} \left( \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_m(\tau,x)) v_m(\tau,x) \, dx \, d\tau + \int_0^t \int_{B(0,r)} \tau \phi(x) g(v_l(\tau,x)) v_l(\tau,x) \, dx \, d\tau \right)
\]

\[
= 2t \int_{B(0,r)} \phi(x) G(v(t,x)) \, dx - 2t \int_0^t \int_{B(0,r)} \phi(x) G(v(\tau,x)) \, dx \, d\tau.
\]

(2.11)
Now, for the last two terms on the right hand side of (2.7), considering (2.9), we get

\[
\lim_{m \to \infty} \lim_{l \to \infty} \left( -\int_0^t \int_{B(0,r)} \tau \phi(x) g \left( v_m(\tau, x) \right) v_t(\tau, x) \, dx \, d\tau - \int_0^t \int_{B(0,r)} \tau \phi(x) g \left( v_l(\tau, x) \right) v_{mt}(\tau, x) \, dx \, d\tau \right) = -2t \int_{B(0,r)} \phi(x) G(v(t, x)) \, dx + 2 \int_0^t \int_{B(0,r)} \phi(x) G(v(\tau, x)) \, dx \, d\tau.
\]

(2.12)

Hence, considering (2.11)-(2.12) and passing to the limit in (2.7), we obtain the claim of the lemma. □

Now, we can prove the asymptotic compactness of \( \{S(t)\}_{t \geq 0} \) in the interior domain.

**Theorem 2.1.** Assume that the conditions (1.3)-(1.8) hold and \( B \) is a bounded subset of \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Then every sequence of the form \( \{S(t) \varphi_k\}_{k=1}^\infty \), where \( \{\varphi_k\}_{k=1}^\infty \subset B \), \( t_k \to \infty \), has a convergent subsequence in \( H^2(B(0, r)) \times L^2(B(0, r)) \), for every \( r > 0 \).

**Proof.** We will use the asymptotic compactness method introduced in [16]. Considering (1.3), (1.6), (1.7) and (1.8) in (1.9), we have

\[
\sup_{t \geq 0} \sup_{\varphi \in B} \|S(t) \varphi\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} < \infty.
\]

(2.13)

Due to the boundedness of the sequence \( \{\varphi_k\}_{k=1}^\infty \) in \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), by (2.13), it follows that the sequence \( \{S(\cdot) \varphi_k\}_{k=1}^\infty \) is bounded in \( L^\infty(0, \infty; H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \). Then for any \( T \geq 1 \) there exists a subsequence \( \{k_m\}_{m=1}^\infty \) such that \( t_{k_m} \geq T \), and

\[
\begin{dcases}
  v_m \to v \text{ weakly star in } L^\infty(0, \infty; H^2(\mathbb{R}^n)), \\
  v_{mt} \to v_t \text{ weakly star in } L^\infty(0, \infty; L^2(\mathbb{R}^n)), \\
  \|\nabla v_m\|_{L^2(\mathbb{R}^n)} \to q \text{ weakly star in } W^{1, \infty}(0, \infty), \\
  v_m \to v \text{ strongly in } C([0, T]; H^2(\mathbb{R}^n), B(0, r)), \quad \varepsilon > 0,
\end{dcases}
\]

(2.14)

for some \( v \in L^\infty(0, \infty; H^2(\mathbb{R}^n)) \cap W^{1, \infty}(0, \infty; L^2(\mathbb{R}^n)) \) and \( q \in W^{1, \infty}(0, \infty) \), where \( (v_m(t), v_{mt}(t)) = S(t + t_{k_m} - T) \varphi_{k_m} \).

Now, taking into account (1.4) in (1.9), we find

\[
\int_0^\infty \|v_{mt}(t)\|^2_{L^2(\mathbb{R}^n \setminus B(0, r_0))} \, dt + \int_0^\infty \|\nabla v_{mt}(t)\|^2_{L^2(\mathbb{R}^n \setminus B(0, r_0))} \, dt \leq c_1.
\]

(2.15)

By (1.1), we have

\[
v_{mtt}(t, x) - \text{div} (\beta (x) \nabla v_{mt}(t, x)) + \gamma \Delta^2 v_m(t, x) + \alpha(x) v_{mt}(t, x) + \lambda v_m(t, x) = f(\|v_m(t)\|_{L^2(\mathbb{R}^n)}) \Delta v_m(t, x) - g(v_m(t, x)) + h(x).
\]

(2.16)

Let \( \eta \in C^\infty(\mathbb{R}^n) \), \( 0 \leq \eta(x) \leq 1 \), \( \eta(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| \geq 2 \end{cases} \) and \( \eta_r(x) = \eta(\frac{x}{r}) \). Multiplying (2.16) with \( \eta_r^2 v_m \) and integrating the obtained equality over \( (0, T) \times \mathbb{R}^n \), we get

\[
\int_0^T \left( \gamma \|\eta_r \Delta v_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\eta_r v_m(t)\|_{L^2(\mathbb{R}^n)}^2 \right) \, dt
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^n} \eta_r^2(x) \beta(x) \left|\nabla v_m(T, x)\right|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} \eta_r^2(x) \beta(x) \left|\nabla v_m(0, x)\right|^2 \, dx
\]
\[-\frac{2}{r} \sum_{i=1}^{n} \int_0^T \beta (x) v_{mtx_i}(t, x) \eta_r \eta_{x_i} \left( \frac{x}{r} \right) v_m(t, x) dx dt \]

\[+ \int_0^T \| \eta_r v_{mt} (t) \|_{L^2(\mathbb{R}^n)}^2 dt - \int \eta_r^2 (x) v_{ml} (T, x) v_m(T, x) dx + \int \eta_r^2 (x) v_{mt} (0, x) v_m(0, x) dx \]

\[= \frac{4}{r} \sum_{i=1}^{n} \int_0^T \eta_r (x) \eta_{x_i} \left( \frac{x}{r} \right) \Delta v_m(t, x) v_{mx_i}(t, x) dx dt - \gamma \int_0^T \int \Delta (\eta_r^2 (x)) \Delta v_m(t, x) v_m(t, x) dx dt \]

\[-\frac{1}{2} \int \eta_r^2 (x) \alpha (x) |v_m(T, x)|^2 dx + \frac{1}{2} \int \eta_r^2 (x) \alpha (x) |v_m(0, x)|^2 dx \]

\[-\int_0^T f(\|v_m(t)\|_{L^2(\mathbb{R}^n)}) \int \eta_r^2 (x) |\nabla v_m(t, x)|^2 dx dt \]

\[-\frac{2}{r} \sum_{i=1}^{n} \int_0^T f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}) \int \eta_r \eta_{x_i} \left( \frac{x}{r} \right) v_{mx_i}(t, x) v_m dx dt \]

\[-\int_0^T \int g (v_m (t, x)) \eta_r^2 (x) v_m (t, x) dx dt \]

\[+ \int_0^T \int h (x) \eta_r^2 (x) v_m (t, x) dx dt. \quad (2.17)\]

Taking into account (1.3), (1.6), (1.8), (1.9), (2.13) and (2.15) in (2.17), we obtain

\[\lim_{m \to \infty} \sup_{T} \int_0^T \left( \gamma \| \Delta v_m(t) \|_{L^2(\mathbb{R}^n \setminus B(0, 2r))}^2 + \lambda \| v_m(t) \|_{L^2(\mathbb{R}^n \setminus B(0, 2r))}^2 \right) dt \]

\[\leq C_2 \left( 1 + \sqrt{T} r + \frac{T}{r} + T \| h \|_{L^2(\mathbb{R}^n \setminus B(0, r))} \right), \quad \forall r \geq r_0. \quad (2.18)\]

Now, by (1.1), we have

\[v_{mtt}(t, x) - v_{tt}(t, x) - \div (\beta (x) \cdot \nabla (v_{mt}(t, x) - v_t(t, x))) + \gamma \Delta^2 (v_m(t, x) - v_t(t, x)) \]

\[+ \alpha (x) (v_{mt}(t, x) - v_t(t, x)) + \lambda (v_m(t, x) - v_t(t, x)) \]

\[= f(\|\nabla v_m(t)\|_{L^2(\mathbb{R}^n)}) \Delta v_m(t, x) - f(\|\nabla v_t(t)\|_{L^2(\mathbb{R}^n)}) \Delta v_t(t, x) - g (v_m) + g (v_t). \quad (2.19)\]

Multiplying (2.19) by \(\sum_{i=1}^{n} x_i (1 - \eta_{4r}) (v_m - v_t) x_i + \frac{1}{2} (n - 1) (1 - \eta_{4r}) (v_m - v_t),\) integrating the obtained equality over \((0, T) \times \mathbb{R}^n\) and taking into account (2.13), we obtain

\[\frac{3\gamma}{2} \int_0^T \| \Delta (v_m(t) - v_t(t)) \|_{L^2(B(0, 4r))}^2 dt + \frac{1}{2} \int_0^T \| v_{mt} (t) - v_{tt} (t) \|_{L^2(B(0, 4r))}^2 dt \]

\[\leq c_3 (1 + T + rT) \| v_m - v_t \|_{C(0, T), H^2(B(0, 8r))} \]

\[+ c_3 \left( \sqrt{T} + r \sqrt{T} \right) \left\| \sqrt{\beta} (v_m - \nabla v_t) \right\|_{L^2((0, T) \times B(0, 8r))} \]

\[+ c_3 \left( \| v_{mt} - v_{tt} \|_{L^2(0, T; L^2(B(0, 8r) \setminus B(0, 4r)))} + \| v_m - v_t \|_{L^2(0, T; H^2(B(0, 8r) \setminus B(0, 4r)))} \right). \quad (2.20)\]
Thus, considering (2.14), (2.15), (2.18) and passing to the limit in (2.20), we get

\[
\limsup_{m \to \infty} \limsup_{t \to \infty} \int_0^T \left[ \| \Delta (v_m(t) - v(t)) \|^2_{L^2(B(0,4r))} + \| v_{mt}(t) - v_{lt}(t) \|^2_{L^2(B(0,4r))} \right] dt \\
\leq c_4 \left( 1 + \frac{\sqrt{T}}{r} + \frac{T}{r} + r\sqrt{T} + T \| h \|_{L^2(\mathbb{R}^n \setminus B(0,2r))} \right), \quad \forall r \geq r_0. \tag{2.21}
\]

Now, multiplying (2.19) by \((1 - \eta_{2r})^4 t \left[ 2 (v_{mt} - v_{lt}) + \alpha_0 \eta_{4}^2 (v_m - v_l) \right] \) and integrating the obtained equality over \((0, T) \times \mathbb{R}^n\), we obtain

\[
\gamma T \| \Delta (v_m(T) - v_l(T)) \|^2_{L^2(B(0,2r))} + T \| v_{mt}(T) - v_{lt}(T) \|^2_{L^2(B(0,2r))} + \\
+ T \lambda \| v_m(T) - v_l(T) \|^2_{L^2(B(0,2r))} \leq \int_0^T \| v_{mt}(t) - v_{lt}(t) \|^2_{L^2(B(0,4r))} dt \\
+ \gamma \int_0^T \| \Delta (v_m(t) - v_l(t)) \|^2_{L^2(B(0,4r))} dt + \lambda \int_0^T \| v_m(t) - v_l(t) \|^2_{L^2(B(0,4r))} dt \\
+ 2 \int_0^T \int_{B(0,4r)} t \left( f(\| \nabla v_m(t) \|_{L^2(\mathbb{R}^n)}) \Delta v_m(t, x) - f(\| \nabla v_l(t) \|_{L^2(\mathbb{R}^n)}) \Delta v_l(t, x) \right) \\
\times (1 - \eta_{2r})^4 \eta_{4}^2(x) (v_m(t, x) - v_l(t, x)) dx dt \\
+ 2 \int_0^T \int_{B(0,4r)} t (g(v_m(t, x)) - g(v_l(t, x))) (1 - \eta_{2r})^4 (v_{mt}(t, x) - v_{lt}(t, x)) dx dt \\
+ \alpha_0 \int_0^T \int_{B(0,4r)} t \left( f(\| \nabla v_m(t) \|_{L^2(\mathbb{R}^n)}) \Delta v_m(t, x) - f(\| \nabla v_l(t) \|_{L^2(\mathbb{R}^n)}) \Delta v_l(t, x) \right) \\
\times (1 - \eta_{2r})^4 \eta_{4}^2(x) (v_m(t, x) - v_l(t, x)) dx dt \\
+ \alpha_0 \int_0^T \int_{B(0,4r)} t (g(v_m(t, x)) - g(v_l(t, x))) (1 - \eta_{2r})^4 \eta_{4}^2(x) (v_m(t, x) - v_l(t, x)) dx dt \\
+ \frac{c_5 T}{r} \int_0^T \int_{B(0,4r) \setminus B(0,r)} \| \nabla (v_{mt}(t, x) - v_{lt}(t, x)) \|^2 dx dt \\
+ \frac{c_5 T}{r} \int_0^T \int_{B(0,4r) \setminus B(0,r)} \| v_{mt}(t, x) - v_{lt}(t, x) \|^2 dx dt \\
+ c_5 T \| v_m - v_l \|^2_{C([0,T]; H^2(B(0,4r)))}, \quad \forall r \geq r_0, \forall T \geq 1. \tag{2.22}
\]

Then, taking into account (2.14), (2.15), (2.21), Lemma 2.1 and Lemma 2.2, and passing to the limit in (2.22), we find

\[
\limsup_{m \to \infty} \limsup_{t \to \infty} \left( \| v_m(T) - v_l(T) \|^2_{H^2(B(0,2r))} + \| v_{mt}(T) - v_{lt}(T) \|^2_{L^2(B(0,2r))} \right) \\
\leq c_6 \left( \frac{1}{T} + \frac{1}{\sqrt{T}} + \frac{1}{r} + \frac{r}{\sqrt{T}} + \| h \|_{L^2(\mathbb{R}^n \setminus B(0,2r))} \right), \quad \forall r \geq r_0, \forall T \geq 1. \tag{2.23}
\]
Thus, by the definition of $v_m$, the inequality (2.23) yields
\[
\limsup_{m \to \infty} \limsup_{l \to \infty} \|S(t_{m,n}) \varphi_{k_m} - S(t_{i}) \varphi_{k_i}\|_{H^2(B(0,r))}^2 \leq c_7 \left( \frac{1}{r} + \frac{1}{\sqrt{r}} + \frac{1}{r} + \frac{r}{\sqrt{T}} + \|h\|_{L^2(B(0,r))} \right), \quad \forall r \geq 2r_0, \quad \forall T \geq 1.
\] (2.24)

Passing to the limit as $T \to \infty$ in (2.24), we obtain
\[
\liminf_{l \to \infty} \liminf_{m \to \infty} \|S(t_{k}) \varphi_{k} - S(t_{m}) \varphi_{m}\|_{H^2(B(0,r))}^2 \leq c_7 \left( \frac{1}{r} + \|h\|_{L^2(B(0,r))} \right), \quad \forall r \geq 2r_0,
\]
which gives
\[
\liminf_{l \to \infty} \liminf_{m \to \infty} \|S(t_{k}) \varphi_{k} - S(t_{m}) \varphi_{m}\|_{H^2(B(0,r))}^2 \leq c_7 \left( \frac{1}{r} + \|h\|_{L^2(B(0,r))} \right), \quad \forall r \geq 2r_0.
\] (2.25)

Consequently, by passing to the limit as $r \to \infty$ in (2.25), we deduce
\[
\liminf_{l \to \infty} \liminf_{m \to \infty} \|S(t_{k}) \varphi_{k} - S(t_{m}) \varphi_{m}\|_{H^2(B(0,r))}^2 = 0, \quad \forall r > 0.
\] (2.26)

Let $r_i \nearrow \infty$ as $i \to \infty$. Taking $r = r_i$ in (2.26) and using the arguments at the end of the proof of [7], Lemma 3.4, we can say that there exist subsequences $\{k^{(i)}_{m}\}$ such that
\[
\left\{k^{(1)}_{m}\right\} \supset \left\{k^{(2)}_{m}\right\} \supset \ldots \supset \left\{k^{(i)}_{m}\right\} \supset \ldots
\]

and
\[
\left\{S(t_{k^{(i)}_{m}}) \varphi_{k^{(i)}_{m}}\right\} \text{ converges in } H^2(B(0,r)) \times L^2(B(0,r)).
\]

Thus, the diagonal subsequence $\left\{S(t_{k^{(m)}_{m}}) \varphi_{k^{(m)}_{m}}\right\}$ converges in $H^2(B(0,r)) \times L^2(B(0,r))$, for every $r > 0$.

To establish the tail estimate, we need the following lemma.

**Lemma 2.3.** Let the conditions (1.3)-(1.6) hold and $B$ be a bounded subset of $H^2(\mathbb{R}^n)$. Then for every $\varepsilon > 0$ there exist a constant $\delta = \delta(\varepsilon) > 0$ and functions $\psi_{\varepsilon} \in L^\infty(\mathbb{R}^n)$, $\varphi_{\varepsilon} \in C^\infty(\mathbb{R}^n)$, such that
\[
0 \leq \psi_{\varepsilon} \leq \min \left\{1, \delta^{-1} \beta \right\} \text{ a.e. in } \mathbb{R}^n, \quad 0 \leq \varphi_{\varepsilon} \leq 1 \text{ in } \mathbb{R}^n, \quad \text{supp}(\varphi_{\varepsilon}) \subset \{x \in \mathbb{R}^n : \alpha(x) \geq \delta \text{ a.e. in } \mathbb{R}^n\}
\]
and
\[
\left| f \left( \left\| \nabla u \right\|_{L^2(\mathbb{R}^n)} \right) - f_\delta \left( \sqrt{\left\| \psi_{\varepsilon} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\varphi_{\varepsilon}} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2} \right) \right| < \varepsilon,
\] (2.27)
for every $u \in B$, where $f_\delta$ is the function defined in the proof of Lemma 2.1.

**Proof.** Let $A_0 = \{x \in B(0,r_0) : \alpha(x) = 0\}$ and $A_k = \{x \in B(0,r_0) : 0 \leq \alpha(x) < \frac{1}{k}\}$. It is easy to see that $A_{k+1} \subset A_k$, and $A_0 = \bigcap_{k > 0} A_k$. Hence, \( \lim_{k \to \infty} mes(A_k) = mes(A_0) \). So, for $\delta > 0$, there exists $k_\delta$ such that
\[
mes(A_{k_\delta} \setminus A_0) \leq \frac{2}{3}.
\] (2.28)

Since $A_{k_\delta}$ is a measurable subset of $B(0,r_0)$, there exists an open set $O^{(1)}_{\delta} \subset B(0,r_0)$ such that $A_{k_\delta} \subset O^{(1)}_{\delta}$ and
\[
mes \left( O^{(1)}_{\delta} \setminus A_{k_\delta} \right) \leq \frac{\delta}{3}.
\] (2.29)

Now, let $\eta_{\delta} \in C_0(\mathbb{R}^n)$ such that $0 \leq \eta_{\delta} \leq 1$, $\eta_{\delta}|_{O^{(1)}_{\delta}} = 1$ and $\text{supp}(\eta_{\delta}) \subset O^{(2)}_{\delta}$, where $O^{(1)}_{\delta} \subset O^{(2)}_{\delta}$ and
\[
mes \left( O^{(2)}_{\delta} \setminus O^{(1)}_{\delta} \right) \leq \frac{\delta}{3}.
\] (2.30)

Then setting $\varphi_{\delta} := 1 - \eta_{\delta}$, we have $\varphi_{\delta} \in C(\mathbb{R}^n)$, $0 \leq \varphi_{\delta} \leq 1$, $\varphi_{\delta}|_{\mathbb{R}^n \setminus O^{(2)}_{\delta}} = 1$ and $\text{supp}(\varphi_{\delta}) \subset \mathbb{R}^n \setminus O^{(1)}_{\delta}$.
By (2.28)-(2.30), we obtain
\[
\left| \int_{O_δ^{(2)}} \varphi_δ |\nabla u(x)|^2 \, dx - \int_{O_δ^{(2)} \setminus A_0} |\nabla u(x)|^2 \, dx \right| = \left| \int_{O_δ^{(2)}} \varphi_δ |\nabla u(x)|^2 \, dx - \int_{O_δ^{(2)} \setminus A_0} |\nabla u(x)|^2 \, dx \right| \\
\leq 2 \int_{O_δ^{(2)} \setminus A_0} |\nabla u(x)|^2 \, dx \leq 2c \|u\|_{H^2(\mathbb{R}^n)}^2 \left( \max (O_δ^{(2)} \setminus A_0) \right)^n \\
< 2c \delta^n \|u\|_{H^2(\mathbb{R}^n)}^2, \tag{2.31}
\]
for every \( u \in H^2(\mathbb{R}^n) \), where \( n^* = \begin{cases} 
1, & n = 1, \\
q, & 0 < q < 1, \\
2, & n = 2, \\
\geq 3, & n \geq 3
\end{cases} \) and \( c > 0 \).

Now, by (1.5), it follows that
\[ \beta > 0 \text{ a.e. in } A_0. \]
Hence, by Lebesgue dominated convergence theorem, there exists \( \lambda_δ > 0 \) such that
\[ \int_{A_0} \frac{\lambda_δ}{\lambda_δ + \beta(x)} \, dx < \delta, \]
which yields
\[ \left| \int_{A_0} \frac{\beta(x)}{\lambda_δ} \left| \nabla u(x) \right|^2 \, dx - \int_{A_0} \frac{\beta(x)}{\lambda_δ + \beta(x)} \left| \nabla u(x) \right|^2 \, dx \right| < 3c \delta^n \|u\|_{H^2(\mathbb{R}^n)}^2, \tag{2.32} \]
Thus, denoting \( \psi_δ = \begin{cases} 
\frac{\beta(x)}{\lambda_δ + \beta(x)}, & x \in A_0, \\
0, & x \in \mathbb{R}^n \setminus A_0,
\end{cases} \) by (2.31) and (2.32), we get
\[ \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 - \left\| \sqrt{\psi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \sqrt{\varphi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 < 3c \delta^n \|u\|_{H^2(\mathbb{R}^n)}^2, \]
and consequently
\[ \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 - \sqrt{\left\| \sqrt{\psi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\varphi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2} \leq \sqrt{\left\| \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \sqrt{\psi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \sqrt{\varphi_δ} \nabla u \right\|_{L^2(\mathbb{R}^n)}^2} < \sqrt{3c \delta^n} \|u\|_{H^2(\mathbb{R}^n)}^2. \]
The last inequality, together with the differentiability of the function \( f \), yields (2.27). \( \square \)

Now, let us proof the following tail estimate.

**Theorem 2.2.** Assume that the conditions (1.3)-(1.8) hold and \( B \) is a bounded subset of \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). Then for any \( \varepsilon > 0 \) there exist \( T = T(B, \varepsilon) \) and \( R = R(B, \varepsilon) \) such that
\[ \|S(t)\varphi\|_{H^2(\mathbb{R}^n \setminus B(0, r)) \times L^2(\mathbb{R}^n \setminus B(0, r))} < \varepsilon, \]
for every \( t \geq T, r \geq R \) and \( \varphi \in B. \)
Proof. Let \((u_0, u_1) \in \mathcal{B}\) and \((u(t), u_t(t)) = S(t)(u_0, u_1)\). Multiplying (1.1) with \(\eta^2 u_t\), integrating the obtained equality over \(\mathbb{R}^n\) and taking into account (2.13), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \|\eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\eta \Delta u(t)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\eta u(t)\|_{L^2(\mathbb{R}^n)}^2 \right)
+ \frac{d}{dt} \left( \int_{\mathbb{R}^n} \eta^2(x) G(u(t, x)) \, dx \right)
+ \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2
+ \|\sqrt{\alpha} \eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2
- \int_{\mathbb{R}^n} f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}) \Delta u\eta_t^2 u_t \, dx
\leq c_2 \left( \frac{1}{r} + \frac{1}{r} \right) \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)} + \|\phi \nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\eta \nabla (u(t))\|_{L^2(\mathbb{R}^n)}^2 \right),
\]}

(2.33)

Now, let us estimate the last term on the left hand side of (2.33). By Lemma 2.3, we have

\[
- f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}) \int_{\mathbb{R}^n} \Delta u\eta_t^2 u_t \, dx
\geq -\varepsilon \|\eta \Delta u(t)\|_{L^2(\mathbb{R}^n)} \|\eta u_t(t)\|_{L^2(\mathbb{R}^n)} - \frac{c_3}{r}
+ \frac{1}{2} f_\delta \left( \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\phi \nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 \right) \frac{d}{dt} \left( \|\eta \nabla (u(t))\|_{L^2(\mathbb{R}^n)}^2 \right).
\]

(2.34)

Moreover, for the last term on the right hand side of (2.34), by using the definition of \(f_\delta\) and the properties of \(\psi_\varepsilon\) and \(\varphi_\varepsilon\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|\eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\eta \Delta u(t)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\eta u(t)\|_{L^2(\mathbb{R}^n)}^2 \right)
+ \frac{d}{dt} \left( \int_{\mathbb{R}^n} \eta^2(x) G(u(t, x)) \, dx \right)
+ \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2
+ \|\sqrt{\alpha} \eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2
- \int_{\mathbb{R}^n} f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}) \Delta u\eta_t^2 u_t \, dx
\leq c_2 \left( \frac{1}{r} + \frac{1}{r} \right) \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)} + \|\phi \nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\eta \nabla (u(t))\|_{L^2(\mathbb{R}^n)}^2 \right),
\]

(2.35)

Considering (2.34) and (2.35) in (2.33), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|\eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2 + \gamma \|\eta \Delta u(t)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\eta u(t)\|_{L^2(\mathbb{R}^n)}^2 \right)
+ \frac{d}{dt} \left( \int_{\mathbb{R}^n} \eta^2(x) G(u(t, x)) \, dx \right)
+ \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)}^2
+ \|\sqrt{\alpha} \eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2
- \int_{\mathbb{R}^n} f(\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}) \Delta u\eta_t^2 u_t \, dx
\leq c_2 \left( \frac{1}{r} + \frac{1}{r} \right) \|\sqrt{\beta} \eta \nabla u_t(t)\|_{L^2(\mathbb{R}^n)} + \|\phi \nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\eta \nabla (u(t))\|_{L^2(\mathbb{R}^n)}^2 \right).
\]

(2.36)

Multiplying (1.1) with \(\mu \eta^2 u\), integrating the obtained equality over \(\mathbb{R}^n\) and taking into account (1.6), (1.8) and (2.13), we get

\[
\mu \gamma \|\eta \Delta u(t)\|_{L^2(\mathbb{R}^n)}^2 + \mu \lambda \|\eta u(t)\|_{L^2(\mathbb{R}^n)}^2 + \mu \|\eta u_t(t)\|_{L^2(\mathbb{R}^n)}^2
+ \frac{\mu}{2} \frac{d}{dt} \left( \|\sqrt{\beta} \eta \nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{\alpha} \eta u(t)\|_{L^2(\mathbb{R}^n)}^2 \right)
\]
We obtain

Moreover, there exist

Then, by Gronwall inequality,

where \(c_i (i = 6, 7)\) are positive constants. By denoting

we get

Moreover, there exist \(\hat{c} \equiv \hat{c} \left( B \right) > 0\) such that

So, considering (2.39) in (2.38), we have

where \(H (t) = c_8 - c_7 \left( \| \sqrt{\beta} \nabla u_\tau \|_{L^2(\mathbb{R}^n)} + \| \sqrt{\alpha} u_\tau \|_{L^2(\mathbb{R}^n)} \right)\) and \(c_8 > 0\). Then, by Gronwall inequality, we obtain

\(2.38\)
Furthermore, applying Young inequality and taking into account (1.9), we have
\[ e^{-\int_{0}^{\tau} H(\sigma) d\sigma} \leq e^{-\frac{4}{3} c_{d}(t-\tau)+c_{3} \int_{0}^{\tau} \left( \| \nabla u_{t}(\tau) \|_{L^{2}(\mathbb{R}^{n})}^{2} + \| \nabla u_{t}(\tau) \|_{L^{2}(\mathbb{R}^{n})}^{2} \right) d\sigma} \]
\[ \leq c_{10} e^{-\frac{4}{3} c_{d}(t-\tau)}, \quad \forall t \geq \tau \geq 0. \]  
(2.41)

Therefore, considering (2.41) in (2.40), we get
\[ \Phi(t) \leq c_{10} e^{-\frac{4}{3} c_{d} t} \Phi(0) + c_{11} \left( \frac{1}{r} + \| h \|_{L^{2}(\mathbb{R}^{n} \setminus B(0, r))} \right), \quad \forall t \geq 0, \]
which completes the proof of the theorem. \( \square \)

Now, we are in a position to prove the existence of the global attractor.

**Theorem 2.3.** Let the conditions (1.3)-(1.8) hold. Then the semigroup \( \{ S(t) \}_{t \geq 0} \) generated by the problem (1.1)-(1.2) possesses a global attractor \( A \) in \( H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n}) \) and \( A = A^{\infty}(\mathcal{N}) \).

**Proof.** By Theorem 2.1 and Theorem 2.2, it follows that every sequence of the form \( \{ S(t_{k}) \varphi_{k} \}_{k=1}^{\infty} \), where \( \{ \varphi_{k} \}_{k=1}^{\infty} \subset \mathcal{B}, \ t_{k} \to \infty, \) and \( \mathcal{B} \) is bounded subset of \( H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n}) \), has a convergent subsequence in \( H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n}) \). Since, by (1.6) and (1.8), the set \( \mathcal{N} \), which is the set of stationary points of \( \{ S(t) \}_{t \geq 0} \) is bounded in \( H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n}) \), to complete the proof, it is enough to show that the pair \( (S(t), H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n})) \) is a gradient system (see [3]).

Now, for \((u(t), u(t)) = S(t)(u_{0}, u_{1})\), let the equality
\[ L(u(t), u(t)) = L(u_{0}, u_{1}), \quad \forall t \geq 0, \]
hold, where \( L(u, v) = \frac{1}{2} \int_{\mathbb{R}^{n}} \left( |v(x)|^{2} + \gamma |xu(x)|^{2} + \lambda |u(x)|^{2} \right) dx + \int G(u(x)) \, dx + \frac{1}{2} F \left( \| \nabla u \|_{L^{2}(\mathbb{R}^{n})}^{2} \right) - \int h(x) u(x) dx \). Then considering (1.3) and (1.9), we have
\[ \alpha u_{t}(t, \cdot) = 0 \quad \text{and} \quad \beta \nabla u_{t}(t, \cdot) = 0 \quad \text{a.e. in } \mathbb{R}^{n}, \]
for \( t \geq 0 \). Taking into account (1.5), from the above equalities, it follows that
\[ u_{t}(t, \cdot) u_{x_{i}}(t, \cdot) = 0 \quad \text{a.e. in } \mathbb{R}^{n}, \]
and consequently
\[ \frac{\partial}{\partial x_{i}} (u_{t}^{2}(t, \cdot)) = 0 \quad \text{a.e. in } \mathbb{R}^{n}, \]
for \( i = 1, \ldots, n \) and \( t \geq 0 \). The last equality means that \( u_{t}^{2}(t, \cdot) \) is independent of variable \( x \), for every \( t \geq 0 \). Hence, by \( u_{t}(t, \cdot) \in L^{2}(\mathbb{R}^{n}) \), we have
\[ u_{t}(t, \cdot) = 0 \quad \text{a.e. in } \mathbb{R}^{n}, \]
for \( t \geq 0 \). So,
\[ (u(t), u_{t}(t)) = (\varphi, 0), \quad \forall t \geq 0, \]
where \((\varphi, 0) \in \mathcal{N}\). Thus, the pair \( (S(t), H^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n})) \) is a gradient system. \( \square \)

### 3. Regularity of the Global Attractor

We start with the following lemma.

**Lemma 3.1.** Let the condition (1.7) hold and \( K \) be a compact subset of \( H^{2}(\mathbb{R}^{n}) \). Then for every \( \varepsilon > 0 \) there exists a constant \( C_{\varepsilon} > 0 \) such that
\[ \| g(u_{1}) - g(u_{2}) \|_{L^{2}(\mathbb{R}^{n})} \leq \varepsilon \| u_{1} - u_{2} \|_{H^{2}(\mathbb{R}^{n})} + C_{\varepsilon} \| u_{1} - u_{2} \|_{L^{2}(\mathbb{R}^{n})}, \]  
(3.1)
for every \( u_{1}, u_{2} \in K \).
Proof. By Mean Value Theorem, Hölder inequality and the embedding $H^2(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have

$$\|g(u) - g(v)\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \int_0^1 g'(\tau u(x) + (1 - \tau)v(x)) \, d\tau \right|^2 |u(x) - v(x)|^2 \, dx$$

$$\leq \int_0^1 \int_{\{x \in \mathbb{R}^n : |\tau u(x) + (1 - \tau)v(x)| > M\}} |g'(\tau u(x) + (1 - \tau)v(x))|^2 |u(x) - v(x)|^2 \, dx \, d\tau$$

$$+ \|g'\|_{C[-M,M]} \| u - v \|_{L^2(\mathbb{R}^n)}^2$$

$$\leq c_1 \int_{\{x \in \mathbb{R}^n : |u(x) + v(x)| > M\}} \left( 1 + |u(x)|^{2(p-1)} + |v(x)|^{2(p-1)} \right) |u(x) - v(x)|^2 \, dx$$

$$+ \|g'\|_{C[-M,M]} \| u - v \|_{L^2(\mathbb{R}^n)}^2$$

$$\leq c_2 \left( \int_{\{x \in \mathbb{R}^n : |u(x) + v(x)| > M\}} \left( 1 + |u(x)|^{2(p-1)} + |v(x)|^{2(p-1)} \right) \, dx \right)^{\frac{1}{2}}$$

where $q = \max\{1, \frac{4}{3}\}$.

Since, by (1.7), $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p-1)q}(\mathbb{R}^n)$, we have that $K$ is compact subset of $L^{2(p-1)q}(\mathbb{R}^n)$. Hence,

$$\lim_{M \to \infty} \sup_{u,v \in K} \int_{\{x \in \mathbb{R}^n : |u(x) + v(x)| > M\}} \left( 1 + |u(x)|^{2(p-1)q} + |v(x)|^{2(p-1)q} \right) \, dx = 0. \tag{3.3}$$

Thus, (3.2) and (3.3) give us (3.1).

\[\Box\]

Theorem 3.1. The global attractor $A$ is bounded in $H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.

Proof. Let $\varphi \in A$. Since $A$ is invariant, there exists an invariant trajectory $\Gamma = \{(u(t), u_\sigma(t)) : t \in \mathbb{R}\}$ $\subset A$ such that $(u(0), u_\sigma(0)) = \varphi$ (see [15] p. 159). Now, let us define

$$v(t,x) := \frac{u(t + \sigma, x) - u(t, x)}{\sigma}, \quad \sigma > 0.$$ 

Then, by (1.1), we get

$$\frac{d}{dt} E(v(t)) + \|\sqrt{\beta} v(t)\|^2_{L^2(\mathbb{R}^n)} + \|\sqrt{\alpha} v(t)\|^2_{L^2(\mathbb{R}^n)}$$

$$\leq -\frac{1}{2} f \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} \right) + \frac{f(\|\nabla u(t + \sigma)\|_{L^2(\mathbb{R}^n)}) - f \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \right)}{\sigma}$$

$$\times \int_{\mathbb{R}^n} \Delta u(t + \sigma, x) v(t, x) \, dx - \frac{1}{\sigma} \int_{\mathbb{R}^n} \left( g(u(t + \sigma, x)) - g(u(t, x)) \right) v(t, x) \, dx$$

$$\leq -\frac{1}{2} f \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} \right)$$

$$+ c_1 \|\nabla v(t)\|_{L^2(\mathbb{R}^n)} v(t) + c_2 \|\nabla v(t)\|_{L^2(\mathbb{R}^n)} v(t) + \frac{1}{\sigma} \|g(u(t + \sigma)) - g(u(t))\|_{L^2(\mathbb{R}^n)} \| v(t)\|_{L^2(\mathbb{R}^n)}.$$
Taking into account Lemma 3.1 in the last inequality, we obtain
\[
\frac{d}{dt} E(v(t)) + \left\| \sqrt{\beta} \nabla v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\alpha} v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2
\leq - \frac{1}{2} f \left( \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \left\| \nabla v(t) \right\|_{L^2(\mathbb{R}^n)} \right)
\]
\[+ c_1 \| \nabla v(t) \|_{L^2(\mathbb{R}^n)} \| v_t(t) \|_{L^2(\mathbb{R}^n)} + \left( \varepsilon \| v(t) \|_{H^2(\mathbb{R}^n)} + C_\varepsilon \| v(t) \|_{L^2(\mathbb{R}^n)} \right) \| v_t(t) \|_{L^2(\mathbb{R}^n)}, \tag{3.5}
\]
for any \( \varepsilon > 0 \). Moreover, by (2.13), we have
\[
\| v(t) \|_{L^2(\mathbb{R}^n)} = \frac{\left\| u(t + \sigma, x) - u(t, x) \right\|_{L^2(\mathbb{R}^n)}}{\sigma} \leq \sup_{0 \leq t < \infty} \| u_t(t) \|_{L^2(\mathbb{R}^n)} < \hat{C}, \ \forall t \in \mathbb{R}. \tag{3.6}
\]
Then, considering (3.6) in (3.5), we get
\[
\frac{d}{dt} E(v(t)) + \left\| \sqrt{\beta} \nabla v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\alpha} v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2
\leq - \frac{1}{2} f \left( \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \left\| \nabla v(t) \right\|_{L^2(\mathbb{R}^n)} \right)
\[+ \left( c_2 \| v(t) \|_{H^2(\mathbb{R}^n)} + \varepsilon \| v(t) \|_{H^2(\mathbb{R}^n)} + \hat{C}_\varepsilon \| v_t(t) \|_{L^2(\mathbb{R}^n)} \right). \tag{3.7}
\]
Now, let us estimate the first term on the right hand side of (3.7). By (2.13) and (3.6), we have
\[
- \frac{1}{2} f \left( \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \left\| \nabla v(t) \right\|_{L^2(\mathbb{R}^n)} \right)
\leq c_3 \max_{0 \leq s_1, s_2 \leq \varepsilon} | f(s_1) - f(s_2) | \| v_t(t) \|_{L^2(\mathbb{R}^n)} \| \Delta u(t) \|_{L^2(\mathbb{R}^n)}
\[+ \frac{1}{2} f_{\varepsilon} \left( \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^n)} \right) \frac{d}{dt} \left( \left\| \nabla v(t) \right\|_{L^2(\mathbb{R}^n)} \right)
\leq c_3 \max_{0 \leq s_1, s_2 \leq \varepsilon} | f(s_1) - f(s_2) | \| v_t(t) \|_{L^2(\mathbb{R}^n)} \| \Delta u(t) \|_{L^2(\mathbb{R}^n)} + c_4 \| v(t) \|_{H^2(\mathbb{R}^n)}
\[+ \left( c_2 \| v(t) \|_{H^2(\mathbb{R}^n)} + \varepsilon \| v(t) \|_{H^2(\mathbb{R}^n)} + \hat{C}_\varepsilon \right) \| v_t(t) \|_{L^2(\mathbb{R}^n)} \right), \tag{3.8}
\]
for any \( \varepsilon > 0 \), where \( f_{\varepsilon} \) is the function defined in the proof of Lemma 2.1. Considering (3.8) in (3.7), we obtain
\[
\frac{d}{dt} \left( E(v(t)) + \frac{1}{2} f_{\varepsilon} \left( \left\| \nabla u(t) \right\|_{L^2(\mathbb{R}^n)} \right) \left\| \nabla v(t) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\beta} \nabla v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \sqrt{\alpha} v_t(t) \right\|_{L^2(\mathbb{R}^n)}^2 \right)
\leq \frac{3}{2} \| \Delta (v(t)) \|_{L^2(B(0,2r_0))}^2 + \frac{1}{2} \| v_t(t) \|_{L^2(B(0,2r_0))}^2
\[+ \frac{d}{dt} \left( \sum_{i=1}^{n} \int_{\mathbb{R}^n} x_i (1 - \eta_{2r_0}(x)) v_{x_i}(t, x) v_t(t, x) dx + \frac{1}{2} (n - 1) \int_{\mathbb{R}^n} (1 - \eta_{2r_0}(x)) v_t(t, x) v(t, x) dx \right)
\leq c_5 \| v_t(t) \|_{L^2(B(0,4r_0))}^2 + c_5 \| \Delta v(t) \|_{L^2(B(0,4r_0))}^2 + c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2
\[+ c_5 \| \sqrt{\beta} v_{x}(t) \|_{L^2(B(0,4r_0))}^2 \cdot \left( \| v(t) \|_{H^2(B(0,4r_0))}^2 + \| v(t) \|_{H^2(B(0,4r_0))}^2 \right)
\leq \frac{3}{2} \| \Delta (v(t)) \|_{L^2(B(0,2r_0))}^2 + \frac{1}{2} \| v_t(t) \|_{L^2(B(0,2r_0))}^2 + c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2
\[+ c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2 \cdot \left( \| v(t) \|_{H^2(B(0,4r_0))}^2 + 1 \right)
\[+ c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2 + c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2 + c_5 \| v(t) \|_{H^2(B(0,4r_0))}^2 + c_5. \tag{3.10}
\]
Multiplying (3.4) by $\eta_{r_0}^2 v$ and integrating over $\mathbb{R}^n$, we find

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^n} \eta_{r_0}^2 v (t, x) v_t (t, x) \, dx + \frac{1}{2} \left\| \sqrt{\alpha} \eta_{r_0} v (t) \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \left\| \sqrt{\beta} \eta_{r_0} \nabla v (t) \right\|_{L^2(\mathbb{R}^n)}^2 \right)
$$

$$
+ \| \Delta v (t) \|_{L^2(\mathbb{R}^n \setminus B(0, r_0))}^2 + \lambda \| v (t) \|_{L^2(\mathbb{R}^n \setminus B(0, r_0))}^2 - \| v_t (t) \|_{L^2(\mathbb{R}^n \setminus B(0, r_0))}^2
$$

$$
\leq c_6 \| v (t) \|_{H^2(\mathbb{R}^n)}^2 + c_6 \left\| \sqrt{\beta} \nabla v_t (t) \right\|_{L^2(\mathbb{R}^n)}^2 \left( 1 + \| v (t) \|_{H^2(B(0, 4r_0))}^2 \right) + c_6. \quad (3.11)
$$

Multiplying (3.10) and (3.11) by $\delta^2$ and $\delta$, respectively, then summing the obtained inequalities with (3.9), choosing $\varepsilon > 0$ and $\delta > 0$ sufficiently small and applying Young inequality, we get

$$
\frac{d}{dt} \Psi (t) + c_7 E (v (t)) \leq c_8, \quad \forall t \in \mathbb{R}, \quad (3.12)
$$

where

$$
\Psi (t) := E(v(t)) + \frac{1}{2} \int_{\mathbb{R}^n} \left\| \nabla u (t) \right\|_{L^2(\mathbb{R}^n)}^2 \| \nabla v (t) \|_{L^2(\mathbb{R}^n)}^2
$$

$$
+ \delta \left( \int_{\mathbb{R}^n} \eta_{r_0}^2 v (t, x) v_t (t, x) \, dx + \frac{1}{2} \left\| \sqrt{\alpha} \eta_{r_0} v (t) \right\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \left\| \sqrt{\beta} \eta_{r_0} \nabla v (t) \right\|_{L^2(\mathbb{R}^n)}^2 \right)
$$

$$
+ \delta^2 \left( \sum_{i=1}^{n} \int_{\mathbb{R}^n} x_i (1 - \eta_{2r_0} (x)) v_{x_i} (t, x) v_t (t, x) \, dx + \frac{1}{2} (n - 1) \int_{\mathbb{R}^n} \left( 1 - \eta_{2r_0} (x) \right) v_t (t, x) v (t, x) \, dx \right),
$$

and the positive constant $c_8$, as the previous $c_i$ $(i = 1, 7)$, is independent of the trajectory $\Gamma$.

Since $\delta > 0$ is sufficiently small, there exist constants $c > 0$, $\bar{c} > 0$ such that

$$
cE (v (t)) \leq \Psi (t) \leq \bar{c} E (v (t)), \quad \forall t \in \mathbb{R}. \quad (3.13)
$$

Taking into account (3.13) in (3.12), we obtain

$$
\frac{d}{dt} \Psi (t) + c_9 \Psi (t) \leq c_8, \quad \forall t \in \mathbb{R},
$$

which yields

$$
\Psi (t) \leq e^{-c_9 (t-s)} \Psi (s) + \frac{c_8}{c_9}, \quad \forall t \geq s.
$$

Passing to the limit as $s \to -\infty$ and considering (3.13), we get

$$
E (v (t)) \leq c_{10}, \quad \forall t \in \mathbb{R}.
$$

By using the definition of $v$, after passing to the limit as $\sigma \to 0$ in the last inequality, we find

$$
E (u_t (t)) \leq c_{10}, \quad \forall t \in \mathbb{R}. \quad (3.14)
$$

Considering (3.14) in (1.1), we obtain

$$
\| u (t) \|_{H^3(\mathbb{R}^n)} \leq c_{11}, \quad \forall t \in \mathbb{R}.
$$

Thus, the last inequality, together with (3.14), yields

$$
\| \varphi \|_{H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)} \leq c_{12}, \quad \forall \varphi \in \mathcal{A},
$$

which completes the proof of the theorem. \qed
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