Chirally improving Wilson fermions

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It is possible to improve the chiral behaviour and the approach to the continuum limit of correlation functions in lattice QCD with standard and twisted Wilson fermions by taking arithmetic averages of correlators computed in theories regularized with Wilson terms of opposite sign. To avoid the problem of the spurious zero-modes of the Wilson–Dirac operator, twisted-mass lattice QCD should be used for the actual computation of the correlators taking part in the averages. A particularly useful choice for the twisting angle is $\pm \pi/2$ where many physical quantities (e.g. masses and zero-momentum matrix elements) are automatically improved with no need of averaging.

1. INTRODUCTION

In this talk I wish to present a simple strategy\cite{1} which allows to get from simulations employing standard or twisted Wilson fermions lattice data that are free of $O(a)$ discretization errors and have a somewhat smoother and more chiral behaviour near the continuum limit than data obtained in unimproved simulations.

Indeed it can be shown that $O(a)$ discretization effects are absent from the average of correlators (Wilson average, WA) computed with lattice actions having Wilson terms of opposite sign and a common value of the subtracted lattice quark mass $m_q = M_0 - M_{cr}$ ($M_0$ and $M_{cr}$ are bare and critical quark masses). Thus fully $O(a)$ improved lattice data for energy levels (hence hadronic masses), matrix elements and renormalization constants can be obtained, without the need of computing anyone of the usual lattice improvement coefficients.

Absence of $O(a)$ discretization errors in WA’s is proved by referring to the Symanzik expansion (SE) of connected, on-shell lattice correlators in terms of continuum Green functions and exploiting the relations derived by matching the “$R_5$-parity” of lattice correlators under

$$R_5 : \psi \rightarrow \psi' = \gamma_5 \psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = -\bar{\psi} \gamma_5$$

(1)
to the $R_5$-parities of the related continuum Green functions. The latter correspond to the parity of continuum Green functions under the change of sign of the continuum quark mass parameter.

Standard Wilson fermions have the well known problem of being afflicted by the existence of “exceptional configurations”. The problem is even more worrisome in the present approach because one will always have to face the situation in which the relative sign of the coefficients of Wilson and quark mass terms are such that they give opposite contributions to the real part of the eigenvalues of the Wilson–Dirac (WD) operator.

The remedy to this situation is to use tm-LQCD\cite{2}. No one of the previous cancellations and improvements is lost by taking averages of correlators evaluated with tm-LQCD actions having (twisted) Wilson terms of opposite sign.

Moreover, if the special choice $\pm \pi/2$ for the twisting angle is made, many interesting physical quantities (e.g. masses and zero-momentum matrix elements) can be extracted from lattice
2. THE GENERAL ARGUMENT

For illustrative purpose we present the key argument in the simple case of standard Wilson fermions. We are interested in computing 
\[ \langle x \rangle_{(r,M_0)} = \frac{1}{Z_{\text{QCD}}^{\text{sp}}} \int D\mu e^{-[S^R_{\text{YM}} + S^{\text{WF}}]} O(x) \]

with \( O \) a gauge invariant, multi-local and multi-plicatively renormalizable (m.r.) operator. The subscript \( (r,M_0) \) means that the correlator is taken with the specified values of the Wilson parameter, \( r \), and bare quark mass, \( M_0 \).

The key observation of this work is that under the fermionic action \( S^{\text{WF}} \) goes into itself, if at the same time we change sign to the Wilson term (i.e. to \( r \)) and \( M_0 \). In the spirit of spurion analysis, a quick way of studying the situation is to momentarily treat \( r \) and \( M_0 \) as fictitious fields and consider the combined transformation

\[ R_{\text{sp}} \equiv R_5 \times [r \rightarrow -r] \times [M_0 \rightarrow -M_0] \]

as a symmetry of the lattice theory. Thus any (m.r.) operator will be either even or odd under \( R_{\text{sp}} \), because \([R_5]^2 = [R_{\text{sp}}]^2 = 1\). From this argument one deduces the relation

\[ \langle O(x) \rangle_{(r,M_0)} = (-1)^{P_{R_5}[O]} \langle O(x) \rangle_{(-r,-M_0)}, \]

where \((-1)^{P_{R_5}[O]}\) is the \( R_5 \)-parity of the bare counterpart of \( O \). Eq. 4 follows by performing in the functional integral 2 the change of fermionic integration variables induced by 4.

Correspondingly, the relation 4, which expresses the implications of the spurionic symmetry \( R_{\text{sp}} \) on lattice correlators, takes the form

\[ \langle O(x) \rangle_{(r,m_q)} = (-1)^{P_{R_5}[O]} \langle O(x) \rangle_{(-r,-m_q)}. \]

We have now indicated the parameters that specify the fermionic action by using, besides \( r \), \( m_q \) instead of \( M_0 \).

In order to discuss the issue of \( O(a) \) improvement we need to make reference to the SE of lattice correlators in terms of correlators of the continuum theory 3. Schematically up to \( O(a) \) terms, one gets for the lattice expectation value of a m.r. operator \( \langle n_x > 0 \rangle\)

\[ \langle O(x) \rangle_{(r,m_q)} = \left[ \zeta^O (r) + a m_q \zeta^O (r) \right] \langle O(x) \rangle_{m_q}^{\text{cont}} + \]

\[ + a \sum_\ell \left[ n_\ell \right] \langle O(x) \rangle_{\ell, (m_q)}^{\text{cont}} + O(a^2). \]

It is important to stress that, in order for the formal counting of powers of \( a \) yielded by the SE to be meaningful, it is necessary that one is dealing with expectation values of m.r. lattice operators.

At this point one can prove that the Symanzik coefficients \( \zeta \) are even functions of \( r \), while all the others \( (\xi, \eta) \) are odd. This is the consequence of eqs. 4, 4, and the transformation properties of lattice and continuum correlators under the functional change of variables induced by the transformation \( R_5 \times D_d \), where

\[ D_d : \left\{ \begin{array}{l}
U_\mu(x) \rightarrow U^\dagger_\mu(-x - a\hat{\mu}) \\
\psi(x) \rightarrow e^{i\pi/2} \psi(-x) \\
\tilde{\psi}(x) \rightarrow e^{i\pi/2} \tilde{\psi}(-x)
\end{array} \right. \]

It then follows that the arithmetic average \( \langle O(x) \rangle_{m_q}^{\text{WA}} = \frac{1}{2} \left[ \langle O(x) \rangle_{(r,m_q)} + \langle O(x) \rangle_{(-r,-m_q)} \right] \) is free of \( O(a) \) discretization effects, because from the above \( r \)-parity considerations one gets

\[ \langle O(x) \rangle_{m_q}^{\text{WA}} = \zeta^O (r) \langle O(x) \rangle_{m_q}^{\text{cont}} + O(a^2). \]

From this relation the \( O(a) \) improvement of \( W_A \)'s of hadronic masses and on-shell matrix elements can be immediately proved 4.

3. TWISTED-MASS LATTICE QCD

Wilson averaging can be straightforwardly extended to tm-LQCD 4. To this end it is convenient to write the fermionic tm-LQCD action for
an SU(2)\textsubscript{f} mass degenerate doublet in the form
\begin{equation}
S_{\text{f,tm}}^{\text{\(h,n\)}} = a^4 \sum_x \bar{\psi}(x) \left( \frac{1}{2} \sum_{\mu} \gamma_\mu (\nabla_\mu + \nabla_\mu) + \right)
+ (-r/2) \sum_{\mu} \nabla_\mu \nabla_\mu + M_{\text{cr}} e^{-i\omega \gamma_5 \tau_3} + m_q \psi(x),
\end{equation}
for in this quark basis (physical basis) the fermionic mass term is real.

Since the transformation \[1\] is still a spurionic symmetry of \[10\] and it does not affect the twisting angle \(\omega\), the whole line of arguments developed above goes through with correlators and derived quantities only having (at finite \(a\)) an extra dependence upon the label \(\omega\).

While working with \(\omega \neq 0\) solves all the problems related to spurious modes of the WD operator, it implies a breaking of the parity (and isospin) symmetry. However, if the parity operation, \(\mathcal{P}\), is accompanied by a change of sign of \(\omega\), the action \[10\] remains invariant. Consequently, the spurionic symmetry \(\mathcal{P} \times (\omega \rightarrow -\omega)\) can be used to label states and operators with a binary quantum number, which in the continuum limit (where by universality the \(\omega\) dependence of lattice quantities drops out) is to be identified with the physical parity. For instance, the eigenstates of the lattice transfer matrix can always be taken to satisfy the formula
\begin{equation}
\mathcal{P} |h,n,k\rangle^{(\omega)}_{(r,m_q)} = \eta_{h,n} |h,n,-k\rangle^{(-\omega)}_{(r,m_q)}, \quad \eta_{h,n}^2 = 1,
\end{equation}
while the corresponding energy eigenvalues can be proved to obey the relations
\begin{equation}
E_{h,n}(k;\omega,r,m_q) = E_{h,n}(\pm k;\pm \omega,r,m_q).
\end{equation}

3.1. A special case: \(\omega = \pm \pi/2\)

The choice \(\omega = \pm \pi/2\) in the tm-LQCD action \[10\] is especially worth mentioning, because all quantities that are even under \(\omega \rightarrow -\omega\) are \(O(a)\) improved with no need of any averaging. This follows from the fact that for the particular value \(\omega = \pm \pi/2\) a sign inversion of the twisting angle is equivalent \((\text{mod } 2\pi)\) to a shift by \(\pi\). This operation is in turn the same as inverting the sign of \(r\): quantities even in \(\omega\) are hence also even in \(r\). As a result the two terms entering the WA are identical and averaging is unnecessary to get \(O(a)\) improvement. Automatically \(O(a)\) improved quantities are e.g. hadron masses and on-shell matrix elements at zero three-momentum. More examples are discussed in \[1\].

Another remarkable fact about the choice \(\pm \pi/2\) is that it is possible to get an \(O(a)\) improved estimate of \(F_\pi\) which requires neither Wilson averaging nor the computation of any renormalization factor. To this end it is enough to use the 1-point split axial current \((x' \equiv x + a\hat{n})\)
\begin{align}
\bar{A}_{\mu}^1(x)^{1-\text{pt}} & = \frac{1}{2} [\bar{\psi}(x) \tau_5 \gamma_\mu U_\mu(x) \psi(x)]
+ \bar{\psi}(x') \tau_5 \gamma_\mu U_\mu^\dagger(x) \psi(x) + \\
& - \frac{r}{2} [\bar{\psi}(x) \tau_5 U_\mu(x) \psi(x') - \bar{\psi}(x') \tau_5 U_\mu^\dagger(x) \psi(x)],
\end{align}
which is exactly conserved at \(m_q = 0\), and observe that only zero three-momentum lattice correlators need be evaluated to extract \(F_\pi\).

4. CONCLUSIONS AND OUTLOOK

The strategy we have proposed, besides ensuring \(O(a)\) improvement, has the virtue of being very flexible and leaves the freedom of regularizing different flavors with Wilson terms of different chiral phases. One can prove \[4\] that, without loosing \(O(a)\) improvement, this freedom can be exploited to have a real and positive fermionic determinant, while at the same time canceling all finite and infinite contributions due to mixing with operators of “wrong chirality” (i.e. those due to the breaking of chiral symmetry induced by the presence of the Wilson term in the lattice action) in the calculation of the CP-conserving matrix elements of the effective weak Hamiltonian. In particular no power divergent mixings survive in the amplitudes relevant for the \(\Delta I = 1/2\)-rule.

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