QUASI-OPTIMAL CONTROL WITH A GENERAL QUADRATIC CRITERION IN A SPECIAL NORM FOR SYSTEMS DESCRIBED BY PARABOLIC-HYPERBOLIC EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. In this work, we consider a dynamical system generated by a parabolic-hyperbolic equation with non-local boundary conditions. The optimal control problem for this system is studied using a notion of quasi-optimal solution. Existence and uniqueness of quasi-optimal control are proved.

1. Introduction. At present, there is a need for generalization of the classical problems of mathematical physics, as well as the formulating qualitatively new problems, arising in the study of different nature objects. The results of research in this direction can be found in such areas as economics, physics, etc [8] [9]. An example of such a problem is mentioned as follows.

Let some medium be filled with gas, and at some instant of time an ionizing radiation, for example it could be X-rays, has an effect on this gas. As the result of sufficient ionization, we obtain a medium with a higher conductivity. Thus, the determination of the electric field strength at the time of changing is related to the solution of the boundary value problem for two equations: parabolic and hyperbolic types [11].

This paper is a continuation of the work on the study of parabolic-hyperbolic equations with non-local boundary conditions and is devoted to the construction of optimality conditions for quasi-optimal distributed control with a general quadratic criterion in a special norm. The elliptic and parabolic case were considered in [5],[6].

The paper is organized as follows. In Section 2, we give a statement of the problem. Section 3 contains the main results, where we give the conditions for finding control and prove the theorems of existence of the solution.

2010 Mathematics Subject Classification. 49J20.

Key words and phrases. Dynamical system, optimal control, parabolic-hyperbolic equations, non-local boundary conditions, quadratic criterion, distributed systems.

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2. Statement of the problem. Let the controlled process \( y(x,t) \in C^1(\bar{D}) \cap C^2(D_-) \cap C^{2,1}(D_+) \) in \( D \) satisfy the equation

\[
Ly(x,t) = \hat{u}(x,t) \tag{1}
\]

with initial condition

\[
y(x,-\alpha) = \varphi(x) \tag{2}
\]

and boundary conditions

\[
y(0,t) = 0, \quad y'(0,t) = y'(1,t), \quad -\alpha \leq t \leq T,
\]

where \( D = \{(x,t) : 0 < x < 1, -\alpha < t \leq T, \alpha > 0 \}, \quad D_- = \{(x,t) : 0 < x < 1, -\alpha < t \leq 0 \}, \quad D_+ = \{(x,t) : 0 < x < 1, 0 < t \leq T \}, \quad L_y = \begin{cases} y_t - y_{xx}, & t > 0, \\ y_{tt} - y_{xx}, & t < 0, \end{cases}
\]

and

\[
\hat{u}(x,t) = \begin{cases} u(x,t), & t \geq 0, \\ v(x,t), & t < 0. \end{cases}
\]

This boundary value problem was solved in [2].

The differential operator \( L_y \) and the non-local conditions (3) generate a biorthogonal in \( L_2(0,1) \) Riesz basis:

\[
W_0 = \{X_j(x), j = 0, 1, \ldots \}, \quad X_{2k-1}(x) = x \cos(2\pi k x), \quad X_{2k}(x) = \sin(2\pi k x),
\]

\( k = 1, 2, \ldots, X_0(x) = 1; \quad R_0 = \{Y_i(x), i = 0, 1, \ldots \}, \quad Y_{2k-1}(x) = 4 \cos(2\pi k x), Y_{2k}(x) = 4(1-x) \sin(2\pi k x), \quad k = 1, 2, \ldots, Y_0(x) = 2. \)

Thus, the functions of the problem are expanded as follows: \( y_i(t) = (y_i(., t), Y_i(., 0, t))_{L_2(0,1)} \), \( \hat{u}_i(t) = (\hat{u}_i(., t), Y_i(., 0, t))_{L_2(0,1)} \), \( \varphi_i = (\varphi(., X_i(., t)))_{L_2(0,1)} \), \( \psi_i = (\psi(., X_i(., t)))_{L_2(0,1)}, \alpha > 0 \).

It is required to find the piecewise continuous by \( t \) control \( \hat{u}^*(x,t) \), for which the following functional takes a minimum value:

\[
I(\hat{u}) = 0.5(\hat{\alpha}||y(., T) - \psi(.)||^2_D + \hat{\beta}_1 \int_{-\alpha}^0 ||y(., t)||^2_D dt + \hat{\beta}_2 \int_{0}^T ||y(., t)||^2_D dt + \\
+ \hat{\gamma}_1 \int_{-\alpha}^0 ||v(., t)||^2_D dt + \hat{\gamma}_2 ||u(., 0)||^2_D + \int_{0}^T ||u(., t)||^2_D dt) = \\
= 0.5 \sum_{i=0}^\infty (\hat{\alpha}(y_i(T) - \psi_i)^2 + \hat{\beta}_1 \int_{-\alpha}^0 y_i^2(t) dt + \hat{\beta}_2 \int_{0}^T y_i^2(t) dt + \\
+ \hat{\gamma}_1 \int_{-\alpha}^0 \hat{v}_i^2(t) dt + \hat{\gamma}_2 (\hat{u}_i^2(0) + \int_{0}^T \hat{u}_i^2(t) dt)), \tag{4}
\]

where \( \psi(x) \) is fixed function, \( \hat{\alpha}, \hat{\beta}_i \geq 0, \hat{\gamma}_i > 0, i = 1,2; \hat{\alpha} + \hat{\beta}_1 + \hat{\beta}_2 > 0. \)
The expression $\hat{\gamma}_2(u^2_1(0) + \int_0^T \hat{u}^2_1(t)dt)$ is used for the functional (4), since the classical form $\hat{\gamma}_2(\int_0^T u^2_1(t)dt)$ leads to the impossibility of finding a continuous solution $y(x,t)$ of the problem.

The origin problem was solved in [3].

Based on the type of functional (4) and solution of (1) - (3) for the problem (1) - (3), (4) we can construct a quasi-optimal control.

**Definition 2.1.** Quasi-optimal control is defined as the control for which the solution for the odd members $\hat{u}_{2k-1}(t)$ of decomposition in the Riesz basis is found firstly, and then for the even ones $\hat{u}_{2k}(t)$.

In this case the problem is formally equivalent to a sequence of finite-dimensional problems [1]:

1) to find control $v^*_0(t) \in C[-\alpha, 0)$, $u^*_0(0) \in R^1$, $\xi^*_0(t) \in L2[0,T]$, which minimizes functional

$$I_0 = 0.5(\hat{\alpha}(y_0(T) - \psi_0)^2 + \hat{\beta}_1 \int_{-\alpha}^0 y^2_0(t)dt + \hat{\beta}_2 \int_0^T y^2_0(t)dt +$$

$$+ \hat{\gamma}_1 \int_{-\alpha}^0 v^2_0(t)dt + \hat{\gamma}_2(u^2_0(0) + \int_0^T \xi^2_0(t)dt),$$

(5)

on the basis of the boundary problem

$$\frac{d^2y_0(t)}{dt^2} = v_0(t), t \in (-\alpha, 0), y_0(-\alpha) = \varphi_0;$$

$$\frac{dy_0(t)}{dt} = u_0(0) + \int_0^t \xi_0(\tau)d\tau, t \in (0, T];$$

$$y_0(0-) = y_0(0+), \dot{y}_0(0-) = y_0(0+) = \psi_0(0);$$

(6)

2) to find control $v^*_2(t) \in C[-\alpha, 0)$, $u^*_2(t) \in R^1$, $\xi^*_2(t) \in L2[0,T]$, which minimizes functional

$$I_{2k-1} = 0.5(\hat{\alpha}(y^2_{2k-1}(T) - \psi_{2k-1})^2 + \hat{\beta}_1 \int_{-\alpha}^0 y^2_{2k-1}(t)dt + \hat{\beta}_2 \int_0^T y^2_{2k-1}(t)dt +$$

$$+ \hat{\gamma}_1 \int_{-\alpha}^0 v^2_{2k-1}(t)dt + \hat{\gamma}_2(u^2_{2k-1}(0) + \int_0^T \xi^2_{2k-1}(t)dt),$$

(7)

on the basis of the boundary problem

$$\frac{dy_{2k-1}(t)}{dt} = -\lambda^2_t y_{2k-1}(t) + u_{2k-1}(0) + \int_0^t \xi_{2k-1}(\tau)d\tau, t > 0;$$

$$\frac{d^2y_{2k-1}(t)}{dt^2} = -\lambda^2_t y_{2k-1}(t) + v_{2k-1}(t), t < 0;$$

$$y_{2k-1}(0-) = y_{2k-1}(0+), y_{2k-1}(-\alpha) = \varphi_{2k-1},$$

$$\dot{y}_{2k-1}(0-) = \dot{y}_{2k-1}(0+) = -\lambda^2_t y_{2k-1}(0+) + u_{2k-1}(0);$$

(8)
3) to find control $v^*_2(t) \in C[-\alpha, 0]$, $u^*_2(0) \in R^1$, $\xi^*_2(t) \in L_2[0, T]$, which minimizes functional

$$I_{2k} = 0.5(\hat{\alpha} \left((y_{2k}(T) - \psi_{2k})^2 + \hat{\beta}_1 \int_{-\alpha}^{0} y_{2k}'(t)dt + \hat{\beta}_2 \int_{0}^{T} y_{2k}''(t)dt\right) + \hat{\gamma}_1 \int_{-\alpha}^{0} v_{2k}^2(t)dt + \hat{\gamma}_2 \int_{0}^{T} \xi_{2k}'(t)dt),$$

(9)

on the basis of the boundary problem

$$\frac{dy_{2k}(t)}{dt} = -\lambda_k^2 y_{2k}(t) - 2\lambda_k y_{2k-1}(t) + u_{2k}(0) + \int_{0}^{t} \xi_{2k}(\tau)d\tau, t > 0,$$

$$\frac{d^2y_{2k}(t)}{dt^2} = -\lambda_k^2 y_{2k}(t) - 2\lambda_k y_{2k-1}(t) + v_{2k}(t), t < 0,$$

$$y_{2k}(-\alpha) = \varphi_{2k}, y_{2k}(0-) = y_{2k}(0+),$$

$$\dot{y}_{2k}(0-) = \dot{y}_{2k}(0+) = -\lambda_k^2 y_{2k}(0+) - 2\lambda_k y_{2k-1}(0+).$$

(10)

Here $\dot{u}_i$ is denoted as $\xi_i$, $\lambda_k = 2\pi k$.

3. Optimality conditions. Optimality conditions for problem 1. The functional (5) is strictly convex by the controls. Therefore the problem (5) - (6) has no more than one minimum point in $C[-\alpha, 0] \times R^1 \times L_2(0, T)$, which is characterized by optimality conditions

$$\hat{\gamma}_1 v_0(t) + \int_{-\alpha}^{0} K_{0,1}^{(1)}(t, \tau)v_0(\tau)d\tau + K_{0,2}^{(1)}(t)u_0(0) + \int_{0}^{T} K_{0,3}^{(1)}(t, \tau)\xi_0(\tau)d\tau =$$

$$= M_{1,1}^{(1)}(t)\varphi_0 + M_{1,2}^{(1)}(t)\psi_0, t \in [-\alpha, 0),$$

$$\hat{\gamma}_2 u_0(0) + \int_{-\alpha}^{0} K_{0,1}^{(2)}(t, \tau)v_0(\tau)d\tau + K_{0,2}^{(2)}(t)u_0(0) + \int_{0}^{T} K_{0,3}^{(2)}(t, \tau)\xi_0(\tau)d\tau =$$

$$= M_{1,1}^{(2)}(t)\varphi_0 + M_{1,2}^{(2)}(t)\psi_0,$$

$$\hat{\gamma}_2 \xi_0(t) + \int_{-\alpha}^{0} K_{0,1}^{(3)}(t, \tau)v_0(\tau)d\tau + K_{0,2}^{(3)}(t)u_0(0) + \int_{0}^{T} K_{0,3}^{(3)}(t, \tau)\xi_0(\tau)d\tau =$$

$$= M_{1,1}^{(3)}(t)\varphi_0 + M_{1,2}^{(3)}(t)\psi_0, t \in (0, T],$$

(11)

where

$$K_{0,1}^{(1)}(t, \tau) = \hat{\alpha}V_{0,1}^0(T, t)V_{0,1}^0(T, \tau) + \hat{\beta}_1 \int_{-\alpha}^{0} V_{0,-}^0(\xi, t)V_{0,-}^0(\xi, \tau)d\xi +$$

$$+ \int_{0}^{T} V_{0,-}^0(\xi, t)V_{0,-}^0(\xi, \tau)d\xi + \int_{t}^{T} V_{0,-}^0(\xi, t)V_{0,-}^0(\xi, \tau)d\xi +$$
$$\begin{aligned}
\mathcal{K}_{0,2}^{(1)}(t) &= \hat{\alpha}(U_{0,+}^{0}(T) + \int_{0}^{T} U_{0,+}^{0}(T,\tau)d\tau)V_{0,+}^{0}(T, t) + \hat{\beta}_{1} \int_{-\infty}^{0} U_{0,-}^{0}(\tau)V_{0,-}^{0}(\tau, t)d\tau + \\
&+ \hat{\beta}_{2} \int_{0}^{T} (U_{0,+}^{0}(\tau) + U_{0,+}^{0}(\tau, \xi)d\xi)V_{0,+}^{0}(\tau, t)d\tau,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{K}_{0,3}^{(1)}(t, \tau) &= \hat{\alpha}V_{0,+}^{0}(T, t) \int_{\tau}^{T} U_{0,+}^{0}(T, \mu)d\mu + \hat{\beta}_{2} \int_{\tau}^{T} U_{0,+}^{0}(\xi, \mu)d\mu V_{0,+}^{0}(\xi, t)d\xi, \\
\mathcal{M}_{0,1}^{(1)}(t) &= -\hat{\alpha}\Phi_{0,+}^{0}(T)V_{0,+}^{0}(T, t) - \hat{\beta}_{1} \int_{-\infty}^{0} \Phi_{0,-}^{0}(\xi)V_{0,-}^{0}(\xi, t)d\xi - \\
&- \hat{\beta}_{2} \int_{0}^{T} \Phi_{0,+}^{0}(\xi)V_{0,+}^{0}(\xi, t)d\xi, \\
\mathcal{M}_{0,2}^{(1)}(t) &= \hat{\alpha}V_{0,+}^{0}(T, t); \mathcal{K}_{0,1}^{(2)}(t) = \mathcal{K}_{0,2}^{(1)}(t), \\
\mathcal{K}_{0,2}^{(2)} &= \hat{\alpha}(U_{0,+}^{0}(T) + \int_{0}^{T} U_{0,+}^{0}(T,\tau)d\tau)^{2} + \hat{\beta}_{1} \int_{-\infty}^{0} (U_{0,-}^{0}(\xi))^{2}d\xi + \\
&+ \hat{\beta}_{2} \int_{0}^{T} (U_{0,+}^{0}(\xi) + U_{0,+}^{0}(\xi, \tau)d\tau)^{2}d\xi, \\
\mathcal{K}_{0,3}^{(2)}(t) &= \hat{\alpha}(U_{0,+}^{0}(T) + \int_{0}^{T} U_{0,+}^{0}(T,\tau)d\tau)\int_{0}^{T} U_{0,+}^{0}(T, \tau)d\tau + \\
&+ \hat{\beta}_{2} \int_{0}^{T} (U_{0,+}^{0}(\xi) + \int_{0}^{T} U_{0,+}^{0}(\xi, \tau)d\tau)\int_{0}^{T} U_{0,+}^{0}(\xi, \mu)d\mu d\xi, \\
\mathcal{M}_{0,1}^{(2)} &= -\hat{\alpha}(U_{0,+}^{0}(T) + \int_{0}^{T} U_{0,+}^{0}(T,\tau)d\tau)\Phi_{0,+}^{0}(T) - \hat{\beta}_{1} \int_{-\infty}^{0} U_{0,-}^{0}(\xi)\Phi_{0,-}^{0}(\xi)d\xi - \\
&- \hat{\beta}_{2} \int_{0}^{T} (U_{0,+}^{0}(\xi) + \int_{0}^{T} U_{0,+}^{0}(\xi, \tau)d\tau)\Phi_{0,+}^{0}(\xi)d\xi, \\
\mathcal{M}_{0,2}^{(2)} &= \hat{\alpha}(U_{0,+}^{0}(T) + \int_{0}^{T} U_{0,+}^{0}(T,\tau)d\tau); \\
\mathcal{K}_{0,1}^{(3)}(t, \tau) &= \mathcal{K}_{0,3}^{(1)}(\tau, t), \mathcal{K}_{0,2}^{(3)}(t) = \mathcal{K}_{0,3}^{(2)}(t), \\
\mathcal{K}_{0,3}^{(3)}(t, \tau) &= \hat{\alpha} \int_{\tau}^{T} U_{0,+}^{0}(T, \mu)d\mu \int_{\tau}^{T} U_{0,+}^{0}(T, \xi)d\xi +
The system (11) has a unique solution in space $L_2(-\alpha,0) \times R^1 \times L_2(0,T)$. We establish the unique solvability of the system (11). Let us define an operator

\[ \tilde{A}_0 \theta(\cdot) = \Gamma_{3 \times 3} \theta(t) + A_0 \theta(\cdot), \]

where $(\theta_0(t))' = (v_0(t),u_0(0),\xi_0(t)) \in L_2(-\alpha,0) \times R^1 \times L_2(0,T)$, $\Gamma_{3 \times 3} = \text{diag}(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$, operator $A_0$ is determined by the remainder terms of the left-hand sides of the equation system (11).

Obviously, $\tilde{A}_0$ operates from $L_2(-\alpha,0) \times R^1 \times L_2(0,T)$ to $L_2(-\alpha,0) \times R^1 \times L_2(0,T)$. It is linear and continuous. Let’s prove the following theorem.

**Theorem 3.1.** The system (11) has a unique solution in space $C(-\alpha,0) \times R^1 \times L_2(0,T)$.

**Proof.** The space $L_2(-\alpha,0) \times R^1 \times L_2(0,T)$ is a Hilbert space with the scalar product

\[ \langle \theta_0, \tilde{\theta}_0 \rangle_3 = \int_{-\alpha}^{0} v_0(t)\tilde{v}_0(t)dt + u_0(0)\tilde{u}_0(0) + \int_{\alpha}^{T} \xi_0(t)\tilde{\xi}_0(t)dt, \]

where $(\theta_0(t))' = (v_0(t),u_0(0),\xi_0(t)), (\tilde{\theta}_0(t))' = (\tilde{v}_0(t),\tilde{u}_0(0),\tilde{\xi}_0(t))$.

We select the quadratic by controls $v_0(t), t \in [-\alpha,0); u_0(0), \xi_0(t) \in [0,T]$ part and subtract from it the value

\[ 0.5(\tilde{\gamma}_1 \int_{-\alpha}^{0} v_0^2(t)dt + \tilde{\gamma}_2 (u_0^2(0) + \int_{0}^{T} \xi_0^2(t)dt)), \]

that is, we consider the functional

\[ \tilde{I}_0 = 0.5[\tilde{\alpha}(\int_{-\alpha}^{0} V_{0,+}(T,\tau)v_0(\tau)d\tau + (U_{0,+}^0(T) + \int_{0}^{T} U_{0,+}(T,\mu)d\mu)u_0(0) + \int_{0}^{T} \int_{-\alpha}^{T} U_{0,+}(T,\mu)d\mu\xi_0(\tau)d\tau)]^2 + \tilde{\beta}_1 \int_{-\alpha}^{0} (\int_{-\alpha}^{0} V_{0,-}(t,\tau)v_0(\tau)d\tau + U_{0,-}^0(t)u_0(0) + \int_{-\alpha}^{T} V_{0,-}(t,\tau)v_0(\tau)d\tau)^2dt + \tilde{\beta}_2 (\int_{0}^{T} V_{0,+}(t,\tau)v_0(\tau)d\tau + (U_{0,+}^0(t) + \int_{0}^{T} U_{0,+}(t,\mu)d\mu\xi_0(\tau)d\tau)^2dt]. \]
It is clear that $\tilde{I}_0 \geq 0$. Now the values of the operator $\mathcal{A}_0\theta_0(\cdot)$ scalar multiply by $\theta_0(t)$, that is, consider a quadratic form

$$\Pi_0 = \langle \mathcal{A}_0\theta_0(\cdot), \theta_0(\cdot) \rangle_3 = \int_0^0 \int_{-\alpha}^{T} K_{0,1}^{(1)}(t, \tau) v_0(\tau)d\tau \ v_0(t)dt + \int_0^0 K_{0,2}^{(1)}(t)v_0(t)dt \times$$

$$\times u_0(0) + \int_0^T K_{0,3}^{(1)}(t, \tau) \xi_0(\tau)d\tau v_0(t)dt + \int_{-\alpha}^{0} K_{0,1}^{(2)}(\tau)v_0(\tau)d\tau u_0(0) +$$

$$+ K_{0,2}^{(2)} u_0^2(0) + \int_0^T K_{0,3}^{(2)}(\tau) \xi_0(\tau)d\tau u_0(0) + \int_0^T K_{0,1}^{(3)}(\tau)v_0(\tau)d\tau \xi_0(\tau)dt +$$

$$+ \int_0^T K_{0,2}^{(3)}(\tau) \xi_0(\tau)d\tau u_0(0) + \int_0^T \int_0^T K_{0,3}^{(3)}(\tau, \tau) \xi_0(\tau)d\tau \xi_0(\tau)dt.$$

Substituting into the quadratic form $\Pi_0$ the explicit form of kernel $K_{0,i,j}^{(j)}$, $i, j = 1, 3$, we obtain equality $\Pi_0 = 2\tilde{I}_0$. This implies the positive definiteness of the operator $\mathcal{A}_0$ and the unique solvability of the system (11) in the space $L_2(-\alpha, 0) \times R^1 \times L_2(0, T)$.

From the first equation of (11) we find the estimation

$$\| \frac{dv_0(\cdot)}{dt} \|_{L_2(-\alpha, 0)} \leq C(\| \frac{dK_{0,1}^{(1)}(\cdot)}{dt} \|_{L_2(-\alpha, 0) \times L_2(-\alpha, 0)} \| v_0 \|_{L_2(-\alpha, 0)} +$$

$$+ \| \frac{dK_{0,2}^{(1)}(\cdot)}{dt} \|_{L_2(-\alpha, 0)} \| u_0(0) \| + \| \frac{dK_{0,3}^{(1)}(\cdot)}{dt} \|_{L_2(-\alpha, 0) \times L_2(0, T)} \| \xi_0 \|_{L_2(0, T)} +$$

$$+ \| \frac{dM_{0,1}^{(1)}(\cdot)}{dt} \|_{L_2(-\alpha, 0)} \| \varphi_0 \| + \| \frac{dM_{0,2}^{(1)}(\cdot)}{dt} \|_{L_2(-\alpha, 0)} \| \psi_0 \| < \infty.$$n

It follows that $v_0(t)$ is absolutely continuous. \hfill \Box

Let’s consider the following case.
Let $v_0(t) = 0$, $t \leq 0$; $\xi_0(t) = 0$, $t > 0$, $\hat{\beta}_1 = \hat{\beta}_2 = 0$. Then

$$u_0(0) = \frac{\mathcal{M}_{0,1}^{(1)} \varphi_0 + \mathcal{M}_{0,2}^{(1)} \psi_0}{\hat{\gamma}_2 + K_{0,2}^{(2)}} = \hat{\alpha}(\alpha + T) \frac{- \varphi_0 + \psi_0}{\hat{\gamma}_2 + \hat{\alpha}(\alpha + T)^2},$$

$$y_0(T) = \varphi_0 + \hat{\alpha}(\alpha + T)^2 \frac{- \varphi_0 + \psi_0}{\hat{\gamma}_2 + \hat{\alpha}(\alpha + T)^2},$$

and the optimal value of the criterion takes the form

$$I_0 = 0.5(\hat{\alpha}(\varphi_0(T) - \psi_0)^2 + \hat{\gamma}_2 u_0^2(0)) = 0.5 \frac{\hat{\gamma}_2 \hat{\alpha}(\varphi_0 - \psi_0)^2}{\hat{\gamma}_2 + \hat{\alpha}(\alpha + T)^2}.$$

Optimality conditions for problem 2). By virtue of the strict convexity of the functional (7) the problem (7) - (8) has no more than one minimum point in $C[-\alpha, 0] \times R^1 \times L_2(0, T)$, which is characterized by optimality conditions

$$\hat{\gamma}_1 v_{2k-1}(t) + \int_{-\alpha}^{0} \hat{K}_{2k-1,1}^{(1)}(t, \tau)v_{2k-1}(\tau)d\tau + \hat{K}_{2k-1,2}^{(1)}u_{2k-1}(0) +$$

$$+ \int_{-\alpha}^{0} \hat{K}_{2k-1,3}^{(1)}(t, \tau) \xi_0(\tau)d\tau = 0,$$
where

$$
\begin{align*}
\hat{\gamma}_2u_{2k-1}(0) + \int_{-\alpha}^{0} \hat{\mathcal{K}}_{2k-1,1}^{(2)}(\tau)u_{2k-1}(\tau)d\tau + \hat{\mathcal{K}}_{2k-1,2}^{(2)}u_{2k-1}(0) + \\
+ \int_{0}^{T} \hat{\mathcal{K}}_{2k-1,3}(\tau)u_{2k-1}(\tau)d\tau = \hat{\mathcal{M}}_{2k-1,1}^{(2)}(\tau)\varphi_{2k-1} + \hat{\mathcal{M}}_{2k-1,2}(\tau)\psi_{2k-1},
\end{align*}
$$

(12)

where

$$
\begin{align*}
\hat{\mathcal{K}}_{2k-1,1}(t, \tau) = \hat{\alpha} V_{2k-1,1}(T, t) V_{2k-1,1}(T, \tau) + \hat{\beta}_1 \left( \int_{-\alpha}^{0} V_{2k-1,1}(\xi, t) V_{2k-1,1}(\xi, \tau)d\xi + \int_{t}^{T} \gamma_{1} V_{2k-1,1}(\xi, t) V_{2k-1,1}(\xi, \tau)d\xi + \right.
\end{align*}
$$

(13)

$$
\begin{align*}
+ \left. \frac{1}{T} \left( \int_{0}^{T} V_{2k-1,1}(\xi, t) V_{2k-1,1}(\xi, \tau)d\xi \right) \right|_{\tau \leq t} + \hat{\beta}_2 \left( \int_{0}^{T} V_{2k-1,1}(\xi, t) V_{2k-1,1}(\xi, \tau)d\xi \right)_{\tau > t},
\end{align*}
$$

(14)

$$
\begin{align*}
\hat{\mathcal{K}}_{2k-1,2}(t) = \hat{\alpha} U_{2k-1,1}(T) + \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(T, \tau)d\tau + \hat{\beta}_1 \int_{-\alpha}^{0} U_{2k-1,1}(\tau) V_{2k-1,1}(\tau, \tau)d\tau + \\
+ \hat{\beta}_2 \left( \int_{0}^{T} (U_{2k-1,1}(\tau) + \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(\tau, \xi)d\xi) V_{2k-1,1}(\tau, \tau)d\tau, 
\end{align*}
$$

(15)

$$
\begin{align*}
\hat{\mathcal{M}}_{2k-1,3}(t, \tau) = \hat{\alpha} U_{2k-1,1}(T, t) \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(T, \mu)d\mu + \hat{\beta}_1 \int_{-\alpha}^{0} U_{2k-1,1}(\tau) V_{2k-1,1}(\tau, \tau)d\tau + \\
+ \hat{\beta}_2 \left( \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(\tau, \xi)d\xi \right) \right|_{\tau \leq t} + \hat{\beta}_2 \left( \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(\tau, \xi)d\xi \right)_{\tau > t},
\end{align*}
$$

(16)

$$
\begin{align*}
\hat{\mathcal{M}}_{2k-1,1}(t) = -\hat{\alpha} \Phi_{2k-1,1}(T) V_{2k-1,1}(T, t) - \hat{\beta}_1 \int_{-\alpha}^{0} U_{2k-1,1}(\tau) V_{2k-1,1}(\tau, \tau)d\tau + \\
+ \hat{\beta}_2 \left( \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(\tau, \xi)d\xi \right) \right|_{\tau \leq t} + \hat{\beta}_2 \left( \int_{0}^{T} \hat{\mathcal{U}}_{2k-1,1}(\tau, \xi)d\xi \right)_{\tau > t},
\end{align*}
$$

(17)
$$\begin{align*}
-\hat{\beta}_1 & \int_{-\alpha}^{0} \Phi_{2k-1,1}^{-1}(\xi)V_{2k-1,1}(\xi,t)d\xi - \hat{\beta}_2 \int_{0}^{T} \Phi_{2k-1,1}^{-1}(\xi)V_{2k-1,1}(\xi,t)d\xi, \\
\tilde{\mathcal{N}}_{2k-1,2}^{(1)}(t) &= \hat{\alpha}V_{2k-1,1}(T,t); \\
\tilde{\mathcal{K}}_{2k-1,1}^{(2)}(t) &= \tilde{\mathcal{K}}_{2k-1,2}^{(2)}(t), \\
\tilde{\mathcal{K}}_{2k-1,2}^{(2)} &= \hat{\alpha}(U_{2k-1,1}^{-1}(T) + \int_{0}^{T} U_{2k-1,1}^{-1}(T,\tau)d\tau)^2 + \hat{\beta}_1 \int_{-\alpha}^{0} (U_{2k-1,1}^{-1}(\xi))^2d\xi + \\
+\hat{\beta}_2 \int_{0}^{T} (U_{2k-1,1}^{-1}(\xi) + \int_{0}^{\xi} U_{2k-1,1}^{-1}(\xi,\tau)d\tau)^2d\xi, \\
\tilde{\mathcal{K}}_{2k-1,3}^{(2)}(t) &= \hat{\alpha}(U_{2k-1,1}^{-1}(T) + \int_{0}^{T} U_{2k-1,1}^{-1}(T,\tau)d\tau) \int_{0}^{T} U_{2k-1,1}^{-1}(T,\tau)d\tau + \\
+\hat{\beta}_2 \int_{0}^{T} (U_{2k-1,1}^{-1}(\xi) + \int_{0}^{\xi} U_{2k-1,1}^{-1}(\xi,\tau)d\tau) \int_{0}^{\xi} U_{2k-1,1}^{-1}(\xi,\mu)d\mu d\xi, \\
\tilde{\mathcal{N}}_{2k-1,1}^{(2)} &= -\hat{\alpha}(U_{2k-1,1}^{-1}(T) + \int_{0}^{T} U_{2k-1,1}^{-1}(T,\tau)d\tau) \Phi_{2k-1,1}^{-1}(T) - \\
-\hat{\beta}_1 \int_{-\alpha}^{0} U_{2k-1,1}^{-1}(\xi) \Phi_{2k-1,1}^{-1}(\xi)d\xi - \\
-\hat{\beta}_2 \int_{0}^{T} (U_{2k-1,1}^{-1}(\xi) + \int_{0}^{\xi} U_{2k-1,1}^{-1}(\xi,\tau)d\tau) \Phi_{2k-1,1}^{-1}(\xi)d\xi, \tilde{\mathcal{N}}_{2k-1,2}^{(2)} \\
= \hat{\alpha}(U_{2k-1,1}^{-1}(T) + \int_{0}^{T} U_{2k-1,1}^{-1}(T,\tau)d\tau); \\
\tilde{\mathcal{K}}_{2k-1,1}^{(3)}(t,\tau) &= \tilde{\mathcal{K}}_{2k-1,3}^{(1)}(\tau,\tau), \tilde{\mathcal{K}}_{2k-1,2}^{(3)}(t) = \tilde{\mathcal{K}}_{2k-1,1}^{(2)}(t), \\
\tilde{\mathcal{K}}_{2k-1,3}^{(3)}(t,\tau) &= \hat{\alpha} \int_{t}^{\tau} U_{2k-1,1}^{-1}(T,\mu)d\mu \int_{t}^{\tau} U_{2k-1,1}^{-1}(T,\xi)d\xi + \\
+\hat{\beta}_2 \left\{ \int_{t}^{\tau} U_{2k-1,1}^{-1}(\xi,\mu)d\mu \int_{t}^{\tau} U_{2k-1,1}^{-1}(\xi,\nu)d\nu d\xi, \tau \leq t, \\
\int_{t}^{\tau} U_{2k-1,1}^{-1}(\xi,\mu)d\mu \int_{t}^{\tau} U_{2k-1,1}^{-1}(\xi,\nu)d\nu d\xi, \tau > t \right\}, \\
\tilde{\mathcal{N}}_{2k-1,1}^{(3)}(t) &= -\hat{\alpha} \Phi_{2k-1,1}^{-1}(T) \int_{t}^{T} U_{2k-1,1}^{-1}(T,\mu)d\mu - \\
-\hat{\beta}_2 \int_{t}^{T} \Phi_{2k-1,1}^{-1}(\tau) \tau \int_{t}^{T} U_{2k-1,1}^{-1}(T,\mu)d\mu d\tau, \tilde{\mathcal{N}}_{2k-1,2}^{(3)} \end{align*}
Optimality conditions for problem 3. By virtue of the strict convexity of the conditions \( C \) and the optimal value of the criterion has the form

\[
\theta = \theta_1 + \theta_2,
\]

with

\[
\theta_1 = \gamma_1 + \gamma_2, \quad \theta_2 = \beta_1 + \beta_2.
\]

Theorem 3.2. The system (12) has a unique solution in space \( C(-\alpha, 0) \times R^1 \times L_2(0, T) \).

Proof. Coincides with the proof of the theorem 3.1, if we replace the system (11) by (12).

Let us consider one of the simple cases of the problem.

Let \( v_{2k-1}(t) = 0, t \leq 0; \xi_{2k-1}(t) = 0, t > 0, \beta_1 = \beta_2 = 0. \) Then

\[
u_{2k-1}(0) = \frac{M_{\alpha, 1}(t)\varphi_{2k-1} + M_{\alpha, 2}(t)\psi_{2k-1}}{\gamma_2 + \bar{\gamma}_2} = \frac{\hat{\lambda}_k^2 \exp(-\lambda_2^2 T) \exp(-\lambda_2^2 T) \varphi_{2k-1} - \delta_k(\alpha)u_{2k-1} \cos \lambda_k \alpha}{\gamma_2 + \bar{\gamma}_2},
\]

and the optimal value of the criterion has the form

\[
I_{2k-1} = 0.5(\hat{\alpha}(y_{2k-1}(T) - \psi_{2k-1})^2 + \gamma_2 u_{2k-1}^2(0)).
\]

Optimality conditions for problem 3). By virtue of the strict convexity of the functional (9) the problem (9) - (10) has no more than one minimum point in \( C[-\alpha, 0) \times R^1 \times L_2(0, T) \), which is characterized by the following optimality conditions

\[
\hat{\gamma}_1 v_{2k}(t) + \int_{-\alpha}^{0} \hat{K}_{2k, 1}(t, \tau) v_{2k}(\tau) d\tau = \hat{M}_{2k, 1}(t) \varphi_{2k} + \hat{M}_{2k, 2}(t) \psi_{2k} + P_1, t \in [-\alpha, 0),
\]

\[
\hat{\gamma}_2 u_{2k}(0) + \int_{-\alpha}^{0} \hat{K}_{2k, 1}(t, \tau) u_{2k}(\tau) d\tau = \hat{M}_{2k, 1}(t) \varphi_{2k} + \hat{M}_{2k, 2}(t) \psi_{2k} + P_2,
\]

\[
\hat{\gamma}_2 \xi_{2k}(0) + \int_{-\alpha}^{0} \hat{K}_{2k, 1}(t, \tau) v_{2k}(\tau) d\tau + \hat{K}_{2k, 2}(t) u_{2k}(0) + \int_{0}^{T} \hat{K}_{2k, 3}(t, \tau) \xi_{2k}(\tau) d\tau = \hat{M}_{2k, 1}(t) \varphi_{2k} + \hat{M}_{2k, 2}(t) \psi_{2k} + P_3, t \in [0, T],
\]

(13)
where

\[
\hat{\mathcal{K}}^{(1)}_{2k,1}(t, \tau) = \hat{\alpha} V_{2k,+}^2(T, t) V_{2k,+}^2(T, \tau) + \hat{\beta}_1 \left( \int_0^T \int_{-\alpha}^\tau V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi + \int_{-\alpha}^\tau V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi \right)
\]

\[
+ \int_\tau^T \int_{-\alpha}^\tau V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi + \int_{-\alpha}^\tau V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi + \left\{ \begin{array}{ll}
\int_0^t V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi,
\tau \leq t,
\int_t^\tau V_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, \tau) d\xi,
\tau > t
\end{array} \right\} + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau V_{2k,+}^2(\xi, t) V_{2k,+}^2(\xi, \tau) d\xi,
\]

\[
\hat{\mathcal{K}}^{(1)}_{2k,2}(t) = \hat{\alpha} \left( \int_0^T U_{2k,+}^2(T, \tau) d\tau \right) V_{2k,+}^2(T, t) + \hat{\beta}_1 \int_{-\alpha}^\tau U_{2k,-}^2(\xi, \tau) d\tau + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau U_{2k,+}^2(\tau, \xi) V_{2k,+}^2(\tau, \xi) d\tau d\xi,
\]

\[
\hat{\mathcal{K}}^{(1)}_{2k,3}(t, \tau) = \hat{\alpha} \left( \int_0^T U_{2k,+}^2(T, \tau) d\tau \right) V_{2k,+}^2(T, t) + \hat{\beta}_1 \int_{-\alpha}^\tau U_{2k,-}^2(\xi, \tau) d\tau + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau U_{2k,+}^2(\tau, \xi) V_{2k,+}^2(\tau, \xi) d\tau d\xi,
\]

\[
\hat{\mathcal{M}}^{(1)}_{2k,1}(t) = -\hat{\alpha} \Phi_{2k,+}^2(T, t) V_{2k,+}^2(T, t) - \hat{\beta}_1 \int_{-\alpha}^\tau \Phi_{2k,-}^2(\xi, t) V_{2k,-}^2(\xi, t) d\xi + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau \Phi_{2k,-}^2(\xi, \tau) V_{2k,-}^2(\xi, \tau) d\xi d\tau,
\]

\[
\hat{\mathcal{M}}^{(1)}_{2k,2}(t) = \hat{\alpha} \Phi_{2k,+}^2(T, t); \\
\hat{\mathcal{K}}^{(2)}_{2k,1}(t) = \hat{\mathcal{K}}^{(1)}_{2k,2}(t),
\]

\[
\hat{\mathcal{K}}^{(2)}_{2k,2} = \hat{\alpha} \left( \int_0^T U_{2k,+}^2(T, \tau) d\tau \right)^2 + \hat{\beta}_1 \left( \int_0^T U_{2k,-}^2(\xi, \tau) d\tau \right)^2 + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau U_{2k,+}^2(\tau, \xi) V_{2k,+}^2(\tau, \xi) d\tau d\xi,
\]

\[
\hat{\mathcal{K}}^{(2)}_{2k,3}(t) = \hat{\alpha} \left( \int_0^T U_{2k,+}^2(T, \tau) d\tau \right) \int_0^T U_{2k,+}^2(T, \tau) d\tau + \hat{\beta}_2 \int_0^T \int_{-\alpha}^\tau U_{2k,+}^2(\tau, \xi) V_{2k,+}^2(\tau, \xi) d\tau d\xi.
\]
\[
\begin{align*}
\tilde{\mathcal{M}}_{2k,1}^{(2)} &= -\hat{\alpha}(U_{2k,+}^{2k}(T) + \int_0^T U_{2k,+}^{2k}(T, \tau) d\tau) \Phi_{2k,+}^{2k}(T) - \hat{\beta}_1 \int_{-\alpha}^0 U_{2k,-}^{2k}(\xi) \Phi_{2k,-}^{2k}(\xi) d\xi - \hat{\beta}_2 \int_0^T (U_{2k,+}^{2k}(\xi) + \int_0^\xi U_{2k,+}^{2k}(\xi, \tau) d\tau) \Phi_{2k,+}^{2k}(\xi) d\xi, \\
\tilde{\mathcal{M}}_{2k,2}^{(2)} &= \hat{\alpha}(U_{2k,+}^{2k}(T) + \int_0^T U_{2k,+}^{2k}(T, \tau) d\tau); \\
\tilde{\mathcal{C}}_{2k,1}^{(3)}(t, \tau) &= \tilde{\mathcal{C}}_{2k,3}^{(1)}(\tau, t), \quad \tilde{\mathcal{C}}_{2k,2}^{(3)}(t) = \tilde{\mathcal{C}}_{2k,3}^{(2)}(t), \\
\tilde{\mathcal{C}}_{2k,3}^{(3)}(t, \tau) &= \hat{\alpha} \int_t^T U_{2k,+}^{2k}(T, \mu) d\mu \int_t^T U_{2k,+}^{2k}(T, \xi) d\xi + \hat{\beta}_2 \int_t^T \int_t^\xi U_{2k,+}^{2k}(\xi, \nu) d\nu d\xi, \quad \tau \leq t, \\
&\quad + \hat{\beta}_2 \int_t^T \int_t^\xi U_{2k,+}^{2k}(\xi, \nu) d\nu d\xi, \quad \tau > t, \\
\tilde{\mathcal{M}}_{2k,1}^{(3)}(t) &= -\hat{\alpha} \Phi_{2k,+}^{2k}(T) \int_t^T U_{2k,+}^{2k}(T, \mu) d\mu - \hat{\beta}_2 \int_t^T \Phi_{2k,+}^{2k}(\tau) \int_t^\tau U_{2k,+}^{2k}(\tau, \mu) d\mu d\tau, \\
\tilde{\mathcal{M}}_{2k,2}^{(3)}(t) &= \hat{\alpha} \int_t^T U_{2k,+}^{2k}(T, \mu) d\mu. 
\end{align*}
\]
\[ \hat{K}^{(1)}_{2k-1,1}(t, \tau) = \hat{\alpha} V^{2k}_{2k,+}(T, t)V^{2k-1}_{2k,+}(T, \tau) + \hat{\beta}_1 \left( \int_0^T V^{2k}_{2k,-}(\xi, t)V^{2k-1}_{2k,-}(\xi, \tau)d\xi + \int_0^T V^{2k}_{2k,-}(\xi, t)V^{2k}_{2k,-}(\xi, \tau)d\xi \right) + \hat{\beta}_2 \left( \int_0^T V^{2k}_{2k,+}(\xi, t)V^{2k-1}_{2k,+}(\xi, \tau)d\xi + \int_0^T V^{2k}_{2k,+}(\xi, t)V^{2k}_{2k,+}(\xi, \tau)d\xi \right) \]

\[ + \hat{\beta}_1 \int_0^T \left( \int_0^T V^{2k}_{2k,-}(\xi, t)V^{2k-1}_{2k,-}(\xi, \tau)d\xi \right)d\tau + \hat{\beta}_2 \int_0^T \left( \int_0^T (U^{2k}_{2k,+}(\tau, \tau)d\tau)V^{2k}_{2k,+}(T, t) + \int_0^T \left( \int_0^T (U^{2k}_{2k,+}(\tau, \tau)d\tau)V^{2k}_{2k,+}(T, t) \right) \right) \]

\[ \hat{K}^{(1)}_{2k-1,2}(t) = \hat{\alpha} (U^{2k-1}_{2k,+}(T) + \int_0^T U^{2k-1}_{2k,+}(T, \tau)d\tau)V^{2k}_{2k,+}(T, t) + \hat{\beta}_1 \int_0^T \left( \int_0^T (U^{2k}_{2k,+}(\tau, \tau)d\tau)V^{2k}_{2k,+}(T, t) \right) \]

\[ \hat{M}^{(1)}_{2k-1,1}(t) = -\hat{\beta}_2 \int_0^T (U^{2k}_{2k,+}(\xi))V^{2k}_{2k,+}(\xi, t)d\xi, \]

\[ \hat{M}^{(1)}_{2k-1,2}(t) = \hat{\alpha} V^{2k}_{2k,+}(T, t); \]

\[ \hat{K}^{(2)}_{2k-1,1}(t) = \hat{K}^{(1)}_{2k-1,1}(t), \]

\[ \hat{K}^{(2)}_{2k-1,2}(t) = \hat{\alpha} (U^{2k}_{2k,+}(T) + \int_0^T U^{2k}_{2k,+}(T, \tau)d\tau))(U^{2k-1}_{2k,+}(T) + \int_0^T U^{2k-1}_{2k,+}(T, \tau)d\tau) + \]

\[ + \hat{\beta}_1 \int_0^T \left( \int_0^T (U^{2k}_{2k,+}(\xi))(U^{2k-1}_{2k,+}(\xi))d\xi \right) + \hat{\beta}_2 \left( \int_0^T (U^{2k}_{2k,+}(\xi))d\xi + \int_0^T (U^{2k}_{2k,+}(\xi, \tau)d\tau)(U^{2k-1}_{2k,+}(\xi) + \int_0^T (U^{2k}_{2k,+}(\xi, \tau)d\tau)(U^{2k}_{2k,+}(\xi, \tau)d\tau) \right) \]

\[ \hat{K}^{(2)}_{2k-1,3}(t) = \hat{\alpha} (U^{2k}_{2k,+}(T) + \int_0^T U^{2k}_{2k,+}(T, \tau)d\tau))(U^{2k-1}_{2k,+}(T) + \int_0^T U^{2k-1}_{2k,+}(T, \tau)d\tau) + \]

\[ + \hat{\beta}_2 \left( \int_0^T (U^{2k}_{2k,+}(\xi))d\xi + \int_0^T (U^{2k}_{2k,+}(\xi, \tau)d\tau)(U^{2k}_{2k,+}(\xi, \tau)d\tau) \right) \int_0^T (U^{2k-1}_{2k,+}(\xi, \mu)d\mu d\xi, \]
\[
\hat{\mathcal{M}}^{(2)}_{2k-1,1} = -\beta_1 (U^{2k}_{2k,+}(T)) + \int_0^T U^{2k}_{2k,+}(T,\tau)d\tau \Phi^{2k-1}_{2k,+}(T) - \\
- \beta_2 \int_0^T U^{2k}_{2k,+}(\xi) + \int_0^\xi U^{2k}_{2k,+}(\xi,\tau)d\tau \Phi^{2k-1}_{2k,+}(\xi)d\xi,
\]

\[
\hat{\mathcal{M}}^{(2)}_{2k-1,2} = \beta_1 (U^{2k}_{2k,+}(T)) + \int_0^T U^{2k}_{2k,+}(T,\tau)d\tau;
\]

\[
\hat{\mathcal{M}}^{(3)}_{2k-1,1}(t,\tau) = \hat{\mathcal{M}}^{(1)}_{2k-1,3}(\tau,\tau, t, \tau) \hat{\mathcal{M}}^{(3)}_{2k-1,2}(t) = \hat{\mathcal{M}}^{(2)}_{2k-1,3}(t),
\]

\[
\hat{\mathcal{M}}^{(3)}_{2k-1,1}(t) = -\beta_1 \Phi^{2k-1}_{2k,+}(T) \int_t^T U^{2k}_{2k,+}(T,\mu)d\mu - \beta_2 \int_t^T \Phi^{2k-1}_{2k,+}(T,\mu)d\mu d\tau,
\]

\[
\hat{\mathcal{M}}^{(3)}_{2k-1,2}(t) = \beta_1 \int_t^T U^{2k}_{2k,+}(T,\mu)d\mu.
\]

To establish the unique solvability of the system (13) we define the operator

\[
\hat{A}_{2k}\theta_{2k-1} = \Gamma_{3 \times 3} \theta_{2k} + A_{2k} \theta_{2k} + \mathcal{P}_2
\]

where \((\theta_{2k}(t))' = (v_{2k}(t), u_{2k}(t), \xi_{2k}(t)) \in L^2(-\alpha, 0) \times R^1 \times L^2(0, T), \Gamma_{3 \times 3} = diag(\gamma_1, \gamma_2, \gamma_2),\) operator \(A_0\) is determined by the remainder terms of the left-hand sides of the equation system (13).

Evidently, \(\hat{A}_{2k}\) operates from \(L^2(-\alpha, 0) \times R^1 \times L^2(0, T)\) to \(L^2(-\alpha, 0) \times R^1 \times L^2(0, T)\). It is linear and continuous. We prove the following theorem.

**Theorem 3.3.** The system (13) has a unique solution in space \(C(-\alpha, 0) \times R^1 \times L^2(0, T)\).

**Proof.** Coincides with the proof of the theorem 3.1, if we replace the system (11) by the system (13).

Let us consider one of the simple cases of the problem.

Let \(v_{2k}(t) = 0, t \leq 0; \xi_{2k}(t) = 0, t > 0, \beta_1 = \beta_2 = 0.\) Then

\[
u_{2k}(t) = \frac{\hat{\mathcal{M}}^{(2)}_{2k,1}\varphi_{2k} + \hat{\mathcal{M}}^{(2)}_{2k,2}\psi_{2k} + \mathcal{P}_2}{\gamma_2 + \hat{\mathcal{M}}^{(2)}_{2k,2}}
\]
is given by the formulas:

\[
-\hat{\alpha} \Phi^{2k}_{2k,+}(T)(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau) \varphi_{2k} + \hat{\gamma}_{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau)^{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau) \psi_{2k} + \hat{\gamma}_{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau)^{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau) \varphi_{2k-1} + \hat{\gamma}_{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau)^{2} + \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau) \varphi_{2k-1} = 0.
\]

\[
\gamma_{2} - \hat{\alpha}(U^{2k}_{2k,+}(T) + \int_{0}^{T} \mathcal{U}^{2k}_{2k,+}(T, \tau) d\tau) \psi_{2k} = 0.
\]

Then the quasi-optimal control of the problem (1) - (3), (4) exists, it is unique and is given by the formulas:

\[
u(x, t) = v_{0}(t)X_{0}(x) + \sum_{k=1}^{\infty} (v_{2k-1}(t)X_{2k-1}(x) + v_{2k}(t)X_{2k}(x)),\]

Theorem 3.4. Let functions \( \varphi(x) \), \( \psi(x) \) in the optimal control problem (1) - (3), (4) satisfy the conditions

\[
\sum_{k=1}^{\infty} \lambda_{k}^{2} (|\varphi_{2k-1}| + |\varphi_{2k}|) < \infty,
\]

\[
\sum_{k=1}^{\infty} \left( \frac{|\psi_{2k-1}| + |\psi_{2k}|}{\lambda_{k}} \right) < \infty, \quad \lambda_{k} = 2\pi k. \quad (14)
\]

Then the quasi-optimal control of the problem (1) - (3), (4) exists, it is unique and is given by the formulas:

\[
u(x, t) = v_{0}(t)X_{0}(x) + \sum_{k=1}^{\infty} (v_{2k-1}(t)X_{2k-1}(x) + v_{2k}(t)X_{2k}(x)),\]
\[ u(x,t) = u(x,0) + \int_0^t \xi(x,\tau) d\tau, \]  

where

\[ u(x,0) = u_0(0)X_0(x) + \sum_{k=1}^{\infty} (u_{2k-1}(0)X_{2k-1}(x) + u_{2k}(0)X_{2k}(x)), \]

\[ \xi(x,t) = \xi_0(t)X_0(x) + \sum_{k=1}^{\infty} (\xi_{2k-1}(t)X_{2k-1}(x) + \xi_{2k}(t)X_{2k}(x)), \]

and the coefficients of these representations are defined as solutions of the systems of equations (11), (12), (13).

By solving the problem (1) - (3), (4) for the procedure given in [4], and also solving the same problem by the method proposed in this article, we establish that the deviation of the values of the corresponding quality criteria will be from 1.5 to 3 percent.

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Received November 2017; revised March 2018.

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