Open strings in the system D5/D9

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Abstract

We construct the six-dimensional Lagrangian for the massless twisted open strings with one end-point ending on a stack of D5 and the other on a stack of D9 branes, interacting with the gauge multiplets living respectively on the D5 and D9 branes. It is first obtained by uplifting to six dimensions the four-dimensional Lagrangian of the $\mathcal{N}=2$ hypermultiplet and manifestly exhibits an $SU(2)$ symmetry. We show by an explicit calculation that it is $\mathcal{N}=1$ supersymmetric in six dimensions and then we check various terms of this Lagrangian by computing string amplitudes on the disk. Finally, starting from this Lagrangian and assuming the presence of non-zero magnetic fluxes along the extra compact dimensions, we determine the spectrum of the Kaluza–Klein states which agrees with the corresponding one obtained from string theory in the field theory limit.

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A very powerful aspect of string theory consists of being able to view different field theories as different limits of the same string model. An example of this is provided by open strings attached to magnetized D9-branes on the background $R^3 \times T^2 \times T^2 \times T^2$. In this case the spectrum of open strings ending on branes with different magnetizations (called in the following twisted open strings) and their interaction depends on the difference of magnetizations $\nu_i$ in the three tori $T^2$. In particular, in the limit where the D branes have the same magnetization, i.e. $\nu_i = 0$ with $i = 1, 2, 3$, the open strings stretched between them are described by the ten-dimensional $\mathcal{N} = 1$ super Yang–Mills (SYM) theory on $R^3 \times T^2 \times T^2 \times T^2$. On the other hand, by taking $\nu_1 = 0$, $\nu_2, \nu_3 = \frac{1}{2}$, one has effectively a system of D5/D9 branes with different kinds of open strings. There are the open strings attached to two D9 branes that are again described by the ten-dimensional $\mathcal{N} = 1$ SYM on $R^3 \times T^2 \times T^2 \times T^2$. Then, there are open strings attached to two D5 branes that are described by a six-dimensional gauge theory obtained by dimensionally reducing the ten-dimensional $\mathcal{N} = 1$ SYM compactified on $R^3 \times T^2$. Finally, there are the open strings stretched between a D9 and a D5 that are described by a six-dimensional theory whose Lagrangian is that of the $\mathcal{N} = 1$ hypermultiplet in six dimensions.

Therefore, in string theory, one has a complete description of the open strings ending on two D branes with different magnetizations [1–6] and it is possible to interpolate between the two field theories discussed above by suitably changing the magnetization. Magnetized branes allow one to construct semirealistic extensions of the Standard Model\(^5\) and therefore it can be very helpful to derive the low-energy effective actions of the open strings attached to them. In this perspective, if one is not interested in string corrections, it is easier to derive such effective actions in a purely field theoretical context rather than extracting them from the string amplitudes. In fact, as shown in [10–12] (see also [13–17]), for the open strings attached to two unmagnetized D9 branes, one can start from $\mathcal{N} = 1$ SYM in ten dimensions,

\(^5\) See for instance [7–9] and references therein.
and introducing background magnetic fields in the extra dimensions, one obtains a description of the twisted open strings in the limit $\alpha' \to 0$. It is then natural to expect that, if one starts instead from the low-energy effective action describing the open strings attached to a D9 and a D5, and introduces background magnetic fields in the extra dimensions, then a description of the twisted D5/D9 strings is obtained in the limit $\alpha' \to 0$. This description is T-dual to the one given in [18, 19]. Naturally, it is expected that the field theory description of magnetized branes has to coincide with the stringy one in the limit $\alpha' \to 0$.

In this paper we show that this is indeed the case also for the D5/D9 twisted open strings. In order to do that, we need to construct the complete six-dimensional action describing the massless D5/D9 twisted open strings attached to unmagnetized D branes interacting with the untwisted ones, corresponding to the gauge multiplets living respectively in the world-volume of the D5 and D9 branes. The new feature of this action, with respect to $\mathcal{N} = 1$ SYM, is that, while the D5/D9 and D5/D5 open strings live in six dimensions, the D9/D9 live instead in the entire ten-dimensional spacetime.

In this paper we construct this six-dimensional action by two different methods. The first one consists in starting from the four-dimensional Lagrangian of the $\mathcal{N} = 2$ hypermultiplet interacting with both gauge multiplets living respectively on the D5 and D9 branes and then uplifting it to six dimensions. In this way, we obtain a six-dimensional theory where not only the fields corresponding to the twisted open strings, but also both gauge multiplets living on the D5 and D9 branes are six dimensional. This approach is straightforward because the four-dimensional Lagrangian for the $\mathcal{N} = 2$ hypermultiplet can be easily constructed using a superfield formalism and also its uplift to six dimensions is straightforward. A nice and new feature of this six-dimensional theory is the fact that the $SU(2)$ R-symmetry of the original four-dimensional theory is still manifest although only an $\mathcal{N} = 1$ supersymmetry survives in six dimensions. During the uplift we may, in general, lose the original supersymmetry and therefore we also check, through a direct calculation, that the six-dimensional action is supersymmetric. In this approach, however, the gauge multiplet living on the D9 brane is still treated as six and not as ten dimensional. Furthermore, the six-dimensional Lagrangian so constructed still contains the auxiliary fields of the gauge multiplet living on the D9 branes and, in order to eliminate them, we should consider it together with the Lagrangian of the D9/D9 open strings. When we eliminate them, by using their algebraic equation of motion, we obtain in the six-dimensional Lagrangian for the open strings D5/D9 quartic terms for the twisted scalars. We stress, however, that all fields, including those that live in the world-volume of the D9 branes, are treated as six-dimensional fields in this field theoretical approach.

In order to have an independent check of this six-dimensional Lagrangian, we construct the vertex operators corresponding to the massless open strings D9/D9, D5/D5 and D5/D9 and we use them to derive some of the terms of the D5/D9 and D5/D5 Lagrangians by computing string amplitudes. In string theory, however, the fields of the gauge multiplet living on the D9 are treated as ten-dimensional fields. This means that in string theory we are not only able to check the previously constructed six-dimensional Lagrangian, but we can also determine the couplings where the fields corresponding to the twisted open strings D5/D9 are six dimensional, while those corresponding to the open strings D9/D9 are ten dimensional. In conclusion, by using these two combined methods, we can construct the complete low-energy six-dimensional Lagrangian for the system D5/D9 and we get various hints on how to extend it to a Lagrangian where the fields of the gauge multiplet living on the D9 branes are treated as ten-dimensional fields.

We face, however, a problem. The uplift of the six-dimensional supersymmetric theory to ten dimensions in general does not ensure that the uplifted theory still remains supersymmetric. This problem may be connected with the fact that, in principle in string theory, we should
treat all fields, including those living on the D5 brane, as ten-dimensional fields and that then
the six-dimensional action is obtained by integrating over their wavefunction in the four extra
dimensions. We do not analyze this problem in this paper and we hope to be able to discuss it
in a future publication.

Finally, starting from the complete D5/D9 Lagrangian where the fields living on the
world-volume of the D5 and D9 branes are respectively six and ten dimensional, and assuming
the presence of non-zero magnetic fluxes along the six extra dimensions, we determine the
spectrum of Kaluza–Klein states and show that it is identical to the one one obtains directly
from string theory in the corresponding field theory limit. This procedure generalizes to the
system D5/D9 which has already been done for the system D9/D9.

In conclusion, the most important and new results of this paper are the construction
of the complete six-dimensional Lagrangian describing the interaction of the massless fields,
corresponding to the twisted D5/D9 open strings, with the gauge multiplets living respectively
on the D5 and D9 branes and the proof that, by introducing in this Lagrangian background
magnetic fields, one obtains the spectrum of Kaluza–Klein states that one obtains from that
of the open strings attached to magnetized D branes in the field theory limit ($\alpha' \to 0$). It is
also new, as far as we know, that the Lagrangian has a manifest $SU(2)$ symmetry that is the
remnant of the $SU(2)$ R-symmetry of the four-dimensional Lagrangian of the hypermultiplet.

The paper is organized as follows. In section 2, we give the spectrum of the twisted open
strings attached to two magnetized D9 branes and we perform two different field theory limits.

In section 3, we construct the six-dimensional Lagrangian describing the interaction of the
massless twisted D5/D9 open strings interacting with the gauge multiplets living respectively
on the D5 and D9 branes, and in section 4, we show that this six-dimensional Lagrangian is
supersymmetric.

Section 5 is devoted to the construction of the vertex operators associated with all massless
open strings of the system D5/D9 and section 6 to compute some string amplitudes checking
some terms of the previously constructed Lagrangians.

Finally, in section 7, we introduce background magnetizations and recover the spectrum
of the Kaluza–Klein excitations that agrees with the one obtained directly from string theory
in the field theory limit.

This paper also contains seven appendices where many technical details are presented. In
appendix A, we discuss our notations for the spinors in four, six and ten dimensions and for the
$\Gamma$-matrices used, while in appendix B, we give some more detail about magnetized D branes.
In appendices C and D, we discuss the uplift from four to six dimensions respectively for the
Lagrangian of the massless open strings 55 and that of the massless open strings 59, and in
appendix E we consider the uplift from six to ten dimensions of the Lagrangian describing
the massless open strings 99. In appendix F, we show in detail that the six-dimensional
Lagrangian for the strings 59 is supersymmetric. Finally, in appendix G, we compute some
 correlators in string theory to have an independent check of the various Lagrangians.

2. Open strings attached to magnetized D branes and the field theory limit

In section 2.1, we give the spectrum of open strings attached to two magnetized D branes
having different magnetizations and we perform two kinds of field theory limits. The first one,
presented in section 2.2, provides the spectrum of open strings attached to two magnetized D9
branes, while the other, presented in section 2.3, gives the spectrum of open strings attached
to a D9 and a D5 brane, both in the limit $\alpha' \to 0$. 

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2.1. Open string spectrum for magnetized D branes

The spectrum of open strings attached to two magnetized D9 branes living on $R^{3,1} \times T^2 \times T^2 \times T^2$ is given by

$$M^2 = \frac{1}{\alpha'} \left[ N_X^X + N_X^\psi + \sum_{r=1}^{3} \left( N_r^Z + N_r^\psi \right) - \frac{x}{2} + \frac{x}{2} \sum_{r=1}^{3} \nu_r \right],$$

(1)

where $x = 1$ (0) for the NS (R) sector, the number operators

$$N_X = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n; \quad N^\psi = \sum_{n=1}^{\infty} n \psi_n^\dagger \cdot \psi_n$$

(2)

contain the harmonic oscillators along the non-compact directions, while the number operators

$$N_Z = \sum_{n=0}^{\infty} \left[ (n + \nu_r)a_{n+\nu_r}^{(r)} a_{n+\nu_r}^{(r)\dagger} + (n + 1 - \nu_r) a_{n+1-\nu_r}^{(r)} a_{n+1-\nu_r}^{(r)\dagger} \right],$$

(3)

$$N^\psi = \sum_{n=\frac{1}{2}}^{\infty} (n + \nu_r) \Psi_{n+\nu_r}^{(r)\dagger} \Psi_{n+\nu_r}^{(r)} + \sum_{n=\frac{1}{2}}^{\infty} (n - \nu_r) \bar{\Psi}_{n-\nu_r}^{(r)\dagger} \bar{\Psi}_{n-\nu_r}^{(r)}$$

(4)

correspond to the contribution of the oscillators along the three tori $T^2$.

Let us denote by $a$ and $b$ the two D9 branes with different magnetizations. Their magnetizations are encoded in the quantities $\nu_r^{(a)}$ and $\nu_r^{(b)}$ ($r = 1, 2, 3$) along the three tori that are given by

$$\tan \pi \nu_r^{(a,b)} = \frac{I_r^{(a,b)}}{n_r^{(a,b)} T_2^{(r)}}, \quad T_2^{(r)} = \frac{V_{T_2^{(r)}}}{(2\pi \sqrt{\alpha'})^2},$$

(5)

where $V_{T_2^{(r)}}$ is the physical volume of the $r$th torus, while $I_r^{(a,b)}$ and $n_r^{(a,b)}$ are respectively the integer magnetic flux and the number of times that the D brane is ‘wrapped’ on the $r$th torus. They are related to the Chern class by

$$\int \frac{\text{Tr}_{\text{NS}} \left( F_{(a,b)}^{(r)} \right)}{2\pi} = I_r^{(a,b)},$$

(6)

The quantity that gives the shift in the frequency of the oscillators is given by

$$\nu_r \equiv \nu_r^{(a)} - \nu_r^{(b)}$$

(7)

corresponding to the difference of the magnetizations of the two D branes along the three tori.

The previous expressions for the number operators in the six compact directions are valid for $0 \leq \nu_r \leq \frac{1}{2}$. They can be extended to the interval $-\frac{1}{2} \leq \nu_r \leq \frac{1}{2}$ that is the natural range for the NS sector. In this range the first equation in (5) shows that $\tan \pi \nu_r$ varies monotonically from $-\infty$ to $+\infty$. When $\nu_r$ is negative the number operators along the compactified directions change their form in terms of the harmonic oscillators. In particular, the number operators for the fermionic coordinate of the NS sector are left unchanged, while in those for the fermionic coordinate in the R sector and for the bosonic coordinate we have to exchange $\Psi$ with $\bar{\Psi}$ and $a$ with $\bar{a}$. This is explained in detail in appendix B. Here we limit ourselves to give the mass spectrum of open strings to include both the case of positive and negative $\nu_r$. One obtains

$$M^2 = \frac{1}{\alpha'} \left[ N_X^X + N_X^\psi + \sum_{r=1}^{3} \left( N_r^Z + N_r^\psi \right) - \frac{x}{2} + \frac{x}{2} \sum_{r=1}^{3} |\nu_r| \right],$$

(8)
where the number operators $N^2_r$ and $N^\Psi_1$ are defined above if $\nu_r$ is positive, while they are given in appendix B when $\nu_r$ is negative.

Having introduced different magnetizations on the two D branes, the previous formula describes both the spectrum of open strings ending on two magnetized D9 branes and the spectrum of the open strings with one end-point on a magnetized D9 and the other end-point attached to a magnetized D5 brane. All this is true in string theory. The situation changes when we take the field theory limit that consists in sending $\alpha' \to 0$, while keeping the physical volume $V_{T^2}$ of the torus $T^2$ fixed. In the following, two subsections we take two different field theory limits corresponding to open string excitations attached to two D9 branes or to a D9 and a D5 brane.

2.2. Field theory limit for D9/D9

In the case of two D9 branes, as it can be seen from equation (5), the field theory limit as defined above corresponds to values of $\nu_r$ proportional to $\alpha'$. The states left in this limit are those for which the factor $\alpha'$ in the denominator of equation (1) is canceled by a similar factor in the numerator. This happens for the following states in the NS sector:

$|\Psi_1(r)\rangle^\dagger_{\pm|\nu_r|}; r = 1, 2, 3$

For the scalar states in the first line of the previous equation the upper (lower) sign is valid if $\nu_r \geq 0$ ($\nu_r \leq 0$). We work in the light-cone gauge and therefore the index $i$ runs over the transverse directions 2 and 3. The mass of the previous eight physical states is given respectively by

$M^2_{r} = \frac{1}{2} \sum_{s=1}^{3} (|\nu_s| \pm |\nu_r|); r = 1, 2, 3$

$M^2_{i} = \frac{1}{2} \sum_{s=1}^{3} |\nu_s|; i = 2, 3.$

They become all massless in the limit of $\nu_r \to 0$ reproducing the eight on-shell components of a massless gauge field. By using equation (5) it is easy to see that their mass remains finite in the field theory limit and is equal to [20]

$M^2_{r} = 2\pi \left[ \sum_{s=1}^{3} \frac{|\tilde{I}^{(r)}_{ab}|}{V_{T^2}} \pm 2 \frac{\tilde{I}^{(r)}_{ab}}{V_{T^2}} \right]; r = 1, 2, 3$

$M^2_{i} = \sum_{s=1}^{3} \frac{\tilde{I}^{(s)}_{ab}}{V_{T^2}}.$

where

$\tilde{I}^{(r)}_{ab} = \frac{I^{(r)}_{ab} n^{(r)}_a - I^{(r)}_{ba} n^{(r)}_b}{n^{(r)}_a n^{(r)}_b}.$

We have a massless scalar if there is a value of $r$ for which one of the previous masses is zero.

They are not, however, the only states that survive in the field theory limit. We can also include an additional bosonic oscillator that we call $a^{(r)}$ and that corresponds to $a^{(r)}_n$ or to
\[ \bar{a}_r \nu r \text{ depending on the sign of } \nu r. \] By including these extra excitations one finally obtains the complete field theoretical result:

\[
M_r^2 = 2\pi \sum_{s=1}^{3} \frac{|\bar{I}_{ab}^{(s)}|}{V_T^2} (2N^{(s)} + 1) \pm 2 \frac{|\bar{I}_{ab}^{(r)}|}{V_T^2} ; \quad r = 1, 2, 3
\]

\[
M_i^2 = 2\pi \sum_{s=1}^{3} \frac{|\bar{I}_{ab}^{(s)}|}{V_T^2} (2N^{(s)} + 1) ; \quad i = 2, 3,
\]

where \( N^{(s)} = a^{(s)} a^{(s)} \) is the oscillator number.

The fermionic mass spectrum of the limiting field theory can be easily obtained from the \( R \) sector proceeding as in the NS sector and it results to be

\[
M^2 = 4\pi \sum_{s=1}^{3} \frac{|\bar{I}_{ab}^{(s)}|}{V_T^2} (N^{(s)} + N^{(s)}_F),
\]

where \( N^{(s)}_F = b^{(s)} b^{(s)} \) is the oscillator number corresponding to the fermionic oscillators \( \Psi_{v_s} \) or \( \bar{\Psi}_{v_s} \) depending on the sign of \( v_s \). The lowest state is the vacuum that, because of the GSO projection, is a four-dimensional chiral spinor.

The mass spectrum obtained in the field theory limit agrees completely, as already noticed in [20], with the spectrum of states found in [10] and [12] starting from the ten-dimensional \( N = 1 \) SYM after the introduction of background magnetic fields in the six extra dimensions.

### 2.3. Field theory limit for D5/D9

In this subsection we perform a different field theory limit that gives the spectrum of open strings attached to a D5 and a D9 in the limit \( \alpha' \to 0 \). If we assume that the world-volume of the D5 branes is along the four non-compact dimensions and along the first torus \( T^2(1) \), then, requiring that \( v_r \) lies in the interval between \(-\frac{1}{2}\) and \( \frac{1}{2}\) implies the following three values of \( \nu r \) in the mass formula:

\[
v_1 \equiv v^{(5)}_1 - v^{(9)}_1
\]

\[
v_{2,3} = \begin{cases} 
\frac{1}{2} - v^{(9)}_{2,3} & \text{if } v^{(9)}_{2,3} > 0 \\
\frac{1}{2} & \text{if } v^{(9)}_{2,3} < 0
\end{cases}
\]

By inserting the previous expressions into equation (8) one obtains

\[
M^2 = \frac{1}{\alpha'} \left[ N^X + N^\Psi + \sum_{r=1}^{3} \left( N_r^Z + N_r^{(i)} \right) \right] .
\]

where \( N^X \) and \( N^\Psi \) are given in equations (2), \( N^Z_1 \) is given in equations (B.3) and (B.4) with analogous expressions for \( N^\Psi_i \) in the Ramond sector, while for the NS sector \( N^{(i)}_1 \) is given in equation (4) that is valid in the entire interval \(-\frac{1}{2} \leq v_1 \leq \frac{1}{2}\). The number operators for the second and third torus are listed in appendix B.

We now want to perform the field theory limit taking into account equation (5). Let us start from the NS sector. In the field theory limit we are left with the following mass formula:

\[
M^2 = \frac{1}{\alpha'} \left[ |v_1| \left( N^{(1)}_1 + \frac{1}{2} \right) + |v^{(9)}_2| \left( N^{(2)}_f - \frac{1}{2} \right) + |v^{(9)}_3| \left( N^{(3)}_f - \frac{1}{2} \right) \right] ,
\]

where \( N^{(i)}_f \) is the number operator for the fermionic oscillators \( \Psi^{(i)}_n \).
The other two states have an even number of fermions: spectrum, in the field theory limit, is given by GSO projection which keeps the states written in equation (21). The mass spectrum in equation (23) corresponds to the one obtained by means of the usual expressions in equations (20) and (22) respectively as follows: Two towers of Kaluza–Klein states are generated by acting on these two fermionic states with the bosonic creation operators: the first excited level is obtained by applying on this vacuum the fermionic creation operator \( a_1 \). The magnetization in the first torus and the Neumann–Dirichlet boundary conditions on the other two tori allow fermionic zero modes only along the four non-compact directions. The first excited level is obtained by applying on this vacuum the fermionic creation operator \( a_1 \). Two towers of Kaluza–Klein states are generated by acting on these two fermionic states with the bosonic creation operators: In the following, for simplifying the notation, we name \( N^{(1)} = a_1^\dagger a_1 \) and \( N_f^{(2,3)} = b_2^\dagger b_2, b_3^\dagger b_3 \). Therefore, the following four states survive in the field theory limit. Two of them have an odd number of fermions:

\[
N^{(1)} = a_1^\dagger a_1 \quad \text{if} \quad v_1 > 0 \\
N^{(1)} = a_1^\dagger a_1 \quad \text{if} \quad v_1 < 0
\]

\[
N_f^{(2,3)} = \Psi^{(2,3)\dagger} \Psi^{(2,3)} \quad \text{if} \quad v_{2,3}^{(9)} > 0 \\
N_f^{(2,3)} = \Psi^{(2,3)\dagger} \Psi^{(2,3)} \quad \text{if} \quad v_{2,3}^{(9)} < 0.
\]

In the following, for simplifying the notation, we name \( N^{(1)} = a_1^\dagger a_1 \) and \( N_f^{(2,3)} = b_2^\dagger b_2, b_3^\dagger b_3 \). Therefore, the following four states survive in the field theory limit. Two of them have an odd number of fermions:

\[
(a_1^\dagger)^m b_1^m (0); \quad (a_1^\dagger)^m b_3^m (0),
\]

with a mass given respectively by

\[
M^2 = \frac{1}{\alpha'} \left[ |v_1| \left( m + \frac{1}{2} \right) \mp \frac{1}{2} (|v_2^{(9)}| - |v_3^{(9)}|) \right].
\]

The other two states have an even number of fermions:

\[
(a_1^\dagger)^m (0); \quad (a_1^\dagger)^m b_2^m b_3^m (0),
\]

with a mass given respectively by

\[
M^2 = \frac{1}{\alpha'} \left[ |v_1| \left( m + \frac{1}{2} \right) \mp \frac{1}{2} (|v_2^{(9)}| + |v_3^{(9)}|) \right].
\]

By performing the field theory limit through equation (5) we can group together the two expressions in equations (20) and (22) respectively as follows:

\[
M_{\text{GSO}}^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'})^2} \left[ \frac{|I_{59}^{(1)}|}{T_2^{(1)}} (2N^{(1)} + 1) \mp \left( \frac{|I_9^{(2)}|}{T_2^{(2)}} n_9^{(2)} - \frac{|I_9^{(3)}|}{T_2^{(3)}} n_9^{(3)} \right) \right]
\]

and

\[
M_{\text{GSO}}^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'})^2} \left[ \frac{|I_{59}^{(1)}|}{T_2^{(1)}} (2N^{(1)} + 1) \mp \left( \frac{|I_9^{(2)}|}{T_2^{(2)}} n_9^{(2)} + \frac{|I_9^{(3)}|}{T_2^{(3)}} n_9^{(3)} \right) \right].
\]

The mass spectrum in equation (23) corresponds to the one obtained by means of the usual GSO projection, while the spectrum in equation (24) corresponds to the one with the opposite GSO projection which keeps the states written in equation (21).

The previous analysis can be easily extended to the R sector. In this case the mass spectrum, in the field theory limit, is given by

\[
M^2 = \frac{1}{\alpha'} \left[ |v_1| a_1^\dagger a_1 + |v_1| b_1^\dagger b_1 \right].
\]

The lowest state, the vacuum, is a four-dimensional massless spinor because the presence of the magnetization in the first torus and the Neumann–Dirichlet boundary conditions on the other two tori allow fermionic zero modes only along the four non-compact directions. The first excited level is obtained by applying on this vacuum the fermionic creation operator \( b_1^\dagger \). Two towers of Kaluza–Klein states are generated by acting on these two fermionic states with the bosonic creation operators:

\[
(a_1^\dagger)^m (0); \quad (a_1^\dagger)^m b_1^m (0); \quad m = 0, 1, \ldots
\]
These two towers differ by the number of fermionic oscillators, having respectively an even and an odd number of such operators and their masses are

\[ M_1^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'})^2} \left| \frac{T_{59}^{(1)}}{T_2^{(1)}} \right|^2 2N^{(1)}; \quad M_2^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'})^2} \left| \frac{T_{59}^{(1)}}{T_2^{(1)}} \right|^2 (2N^{(1)} + 2), \tag{27} \]

where we have used equation (5).

In order to have a consistent string model the GSO projection has to be imposed. The standard GSO projection selects, in the NS sector, states with an odd number of fermionic oscillators, while in the R sector fixes the chirality of the vacuum. In particular, the vacuum becomes a chiral massless state. The remaining states of the two towers have in pair the same mass but opposite chirality. They can be combined together to form a single tower of Kaluza–Klein Dirac fermions. Supersymmetry is achieved if the following condition is imposed:

\[ |v_1| = |v_2^{(9)}| - |v_3^{(9)}|. \tag{28} \]

In this case the states in equation (20) have, respectively, the following masses:

\[ M_2^2 = \frac{1}{\alpha'} |v_1| (m + 1); \quad M_2^2 = \frac{1}{\alpha'} |v_1| m, \tag{29} \]

which coincide with the ones of the R sector in equation (26).

One obtains the same conclusion by imposing, instead of the condition in equation (28), the following one:

\[ |v_1| = |v_3^{(9)}| - |v_2^{(9)}|. \tag{30} \]

The masses of the states in equation (29) are just exchanged.

It is also interesting to note that, considering both in the NS and R-sector the opposite GSO condition, we have Bose–Fermi degeneracy if the following condition is imposed:

\[ |v_1| = |v_3^{(9)}| + |v_2^{(9)}|. \tag{31} \]

In the untwisted sector of the theory the opposite GSO projection is never taken in consideration because, in the absence of magnetic fluxes, it does not project out the tachyon and, therefore, leads to inconsistent string models. In the unmagnetized twisted D5/D9 open string sector, instead, the tachyon is not present in the spectrum and both GSO projections give a supersymmetric spectrum. Therefore, from this point of view both projections are allowed. This is an interesting aspect of the D5/D9 twisted sector and should be further analyzed.

So far we have derived the spectrum of the open strings D5/D9 that one obtains from string theory in the field theory limit. Can the same spectrum be obtained directly from a field theoretical calculation? In the case of the strings D9/D9 it has been shown \[10, 12, 20\] that it follows from the low-energy effective action for the massless open strings D9/D9 that is \( \mathcal{N} = 1 \) SYM. Therefore, in this case, we expect that the spectrum obtained above from string theory can also be obtained from the low-energy effective Lagrangian for the massless D5/D9 strings. That is the reason why we are now going to construct this Lagrangian.

### 3. Lagrangians for the open strings 99, 55 and 59

In this section, we construct the six-dimensional Lagrangian describing the massless open strings attached to a stack of D9 and a stack of D5 branes interacting with the gauge multiplets living respectively on the D5 and D9 branes. As shown in appendix D, we start from the four-dimensional Lagrangian describing the \( \mathcal{N} = 2 \) hypermultiplet interacting with the
gauge multiplets and we uplift it to six dimensions by explicitly keeping the original \( SU(2) \) R-symmetry of the four-dimensional Lagrangian. This may seem strange at the first sight because the six-dimensional Lagrangian has only an \( \mathcal{N} = 1 \) supersymmetry, but string calculations, which will be discussed later, confirm that such a symmetry is indeed present in six dimensions. It turns out that one can keep an \( SU(2) \) symmetry provided that some of the fields as the gaugino that are \( SU(2) \) doublets satisfy some constraints.

In this way we obtain a Lagrangian in which all fields, including those living on the D9 branes, are six dimensional. On the other hand, the Lagrangian so constructed still contains the auxiliary fields of the gauge multiplet living on the D9 branes. In order to eliminate them we have to consider the Lagrangian of the open strings D5/D9 together with that of the open strings D9/D9. The elimination of the auxiliary fields living on the D9 branes generates, in the Lagrangian for the strings 59, a four-scalar interaction involving the twisted scalar of the hypermultiplet and a trilinear term containing two twisted scalar fields and the field strength of the gauge field living on the D9 branes. After eliminating the auxiliary fields, we can finally uplift to ten dimensions that part of the Lagrangian containing the gauge multiplet on the D9 branes.

The six-dimensional Lagrangian of the \( \mathcal{N} = 2 \) gauge multiplet living on a stack of D5 branes is derived in appendix C and is given by

\[
\mathcal{L}_{65} = 2\text{Tr} \left[ -\frac{1}{4} F_{\mu\nu}^6 F_{\mu\nu}^6 + \frac{1}{2} \sum_{a=1}^3 (D_{c+5}^6)^2 - \frac{1}{2} \bar{\Lambda}_i^6 \Gamma^a \Lambda_i^6 \right] \\
+ 2\text{Tr} \left[ - (D_{\mu}^6 (Z^6))^\dagger (D_{\mu}^6 (Z^6))^\dagger + g_5 \bar{Z}_i^6 \sum_{c=1}^3 (D_{c+5}^6 \tau^c)^\dagger (Z^6)^j + i \Psi^{(5)} \Gamma^a D_{\mu}^6 \Psi^{(5)} \\
+ i \sqrt{2} g_5 b \left( [\Psi^{(5)}, Z^{(5)}] \Gamma_{ij}(\Lambda^{(5)})^j - \bar{\Lambda}_i^6 \right) [\Psi^{(5)}, Z^{(5)}] \left( \epsilon^{ij} \right) \right],
\]

where \( b = \pm 1 \), and the indices \( i, j \) label the \( SU(2) \) symmetry and are lowered and raised by means of the \( \epsilon \) tensor as described in appendix A. The trace is over the fundamental matrices of the group \( U(N_3) \). In the previous Lagrangian the six-dimensional gaugino is described by an \( SU(2) \) doublet \( \Lambda^i \) subject to the symplectic Majorana condition \[21\]

\[
B_6 \Lambda^i = ace_\epsilon e_{ij}(\Lambda^j)^*,
\]

where \( a = \pm 1 \) and \( c \) is a phase factor. \( B_6 \) is the operator which relates the six-dimensional Dirac-matrices \( \Gamma_{(6)} \) to their complex conjugates [22]. More details on the properties of such an operator are given in appendix A and its relation with the \( \Gamma_{(6)} \) is given by \( B_6 = c \Gamma_{(6)} \Gamma_{(6)} \). The constraint in equation \[33\] reduces by one half the number of independent fermions giving, as expected, the correct number of degrees of freedom for a gaugino.

Before proceeding further, let us discuss the three factors \( a, c, b \) equal to \( \pm 1 \) that appear in the Lagrangian \[32\] and in the constraint for the gaugino in equation \( \text{(C.24)} \). The six-dimensional fermions \( \Lambda^{(5)} \) and \( \Psi^{(5)} \) come from a Weyl–Majorana ten-dimensional spinor that satisfies equation \[15\] and this implies that they satisfy equations \[19\]. As a consequence, we obtain a factor of \( b \) in the last two terms of equation \[32\]. In the string calculations we have taken the ten-dimensional spinor to be anti-chiral \( (b = 1) \), but here in the field-theoretical formulation we leave \( b \) to be arbitrary. The two other factors \( c \) and \( a \) appear respectively in the definition of \( B_6 \) in equations \( \text{(A.17)} \) and \( \text{(A.18)} \). For the sake of simplicity, we will take them to be equal to \( a = c = 1 \).
The six-dimensional Lagrangian in equation (32) is obtained by dimensional reduction from \( \mathcal{N} = 1 \) SYM in ten dimensions and should be invariant under the R-symmetry group \( SO(4) \equiv SU(2) \times SU(2) \). Since, however, only one of the two \( SU(2) \) is maintained in the Lagrangian for the strings D5/D9, we write it in a form that shows only a manifest invariance under this \( SU(2) \).

In appendix D, we also construct the six-dimensional Lagrangian corresponding to the uplift of the \( \mathcal{N} = 2 \) hypermultiplet from four to six dimensions. It is given in equation (D.23). When we include both the gauge theory living on the D5 and that on the D9 branes we obtain the following six-dimensional Lagrangian:

\[
\mathcal{L}_{69} = \epsilon^{ij} \left( D_\mu \bar{w}_i \right)_a^u \left( D^\mu w_j \right)_u^a - g_s \bar{w}_i^a \sum_{c=1}^{3} \left( \tau^c \right)_j^i \left( \hat{D}_c^{(5)} \right)_a^b w_j^b \\
- i \bar{\mu}^u \Gamma_\mu \bar{\mu}_u^a + \sqrt{2} g_5 i b \left[ \bar{\mu}^u \left( \Lambda^{(5)} \right)_b^a \epsilon_{ij} (w^j)_u^b + \left( \bar{w}_i \right)_u^a \epsilon^{ij} \left( \hat{\Lambda}^{(5)} \right)_b^a \right] \\
+ \frac{g_9}{\sqrt{2}} \bar{w}_i \sum_{c=1}^{3} \left( \tau^c \right)_j^i \left( \hat{D}_c^{(9)} \right)_a^b w_j^b \\
- \sqrt{2} g_9 i b \left[ \bar{\mu}^u \left( \Lambda^{(9)} \right)_b^a \epsilon_{ij} (w^j)_v^b + \left( \bar{w}_i \right)_v^a \epsilon^{ij} \left( \hat{\Lambda}^{(9)} \right)_b^a \right],
\]

(34)

where \( \bar{w}_i = (w^i)_v \) and \( g_9 \) is the gauge coupling constant of the gauge theory living on the D9 branes properly rescaled with the volume factors in order to have the same dimension as \( g_5 \):

\[
\frac{1}{g^2_9} = \frac{1}{g^2_5} \left( 2 \pi \sqrt{a} \right)^2 T_2^{(2)} T_2^{(3)}
\]

(35)

and

\[
\left( D_\mu w^j \right)_u^a = \partial_\mu w^j_{ua} + i g_5 \left( \Lambda^{(5)} \right)_b^a \bar{w}_j^b - i \sqrt{2} g_9 \left( \hat{\Lambda}^{(9)} \right)_b^a \bar{w}_j^b \\
\left( D_\mu \bar{w}_i \right)_a^u = \partial_\mu \bar{w}_i^{ua} - i g_5 \left( \Lambda^{(5)} \right)_b^a \bar{w}_i^b + i \sqrt{2} g_9 \left( \hat{\Lambda}^{(9)} \right)_b^a \bar{w}_i^b \\
\left( D_\mu \bar{\mu}_u \right)_a^b = \partial_\mu \bar{\mu}_u^{ab} + i g_5 \left( \Lambda^{(5)} \right)_b^a \bar{\mu}_u^b - i \sqrt{2} g_9 \left( \hat{\Lambda}^{(9)} \right)_b^a \bar{\mu}_u^b.
\]

(36)

Here, \( T_2^{(2,3)} \) are the imaginary parts of the Kähler moduli of the two tori associated with the directions 6 . . . 9. We have put a hat on the fields living on D9 to remember that they are six dimensional. The previous Lagrangian has a manifest \( SU(2) \) symmetry that is the remnant of the \( SU(2) \) R-symmetry of the Lagrangian of the hypermultiplet in four dimensions. We are able to keep it in six dimensions imposing the constraint in equation (33) that eliminates the redundant components of the gaugino.

The Lagrangian in equation (34) has been obtained by uplifting to six dimensions a four-dimensional \( \mathcal{N} = 2 \) supersymmetric Lagrangian. Although the original Lagrangian was supersymmetric, it is not clear that the uplifted Lagrangian is still supersymmetric as the original one. In section 4, we show, with an explicit calculation, that indeed the uplifted Lagrangian has the same amount of conserved supersymmetry charges as the original four-dimensional one.

Finally, we also have the six-dimensional Lagrangian of the fields living on the D9 branes:

\[
\mathcal{L}_{99} = 2 \text{Tr} \left[ -\frac{1}{4} \hat{F}_{\hat{\mu} \hat{\nu}} \hat{F}^{\hat{\mu} \hat{\nu}} + \frac{1}{2} \sum_{c=1}^{3} \left( \left( \hat{D}_c^{(5)} \right)^2 + i \sqrt{2} g_9 \hat{D}_c^{(5)} \Psi_{(ic+5)mn} \left( \hat{A}^{(9)}_m, \hat{A}^{(9)}_n \right) \right) \\
- \frac{1}{2} \sum_{m=6}^{9} D_{\hat{\mu}} \hat{A}^{(9)}_m D_{\hat{\nu}} \hat{A}^{(9)}_m - \frac{1}{2} \hat{A}^{(9)}_i \Gamma^\hat{\mu} D_{\hat{\nu}} \hat{A}^{(9)}_i - i \hat{\Psi}^{(9)} \Gamma^\hat{\mu} D_{\hat{\mu}} \hat{\Psi}^{(9)} \\
+ i \sqrt{2} g_9 b \left( \hat{\Psi}^{(9)}, \hat{Z}^{(9)}_j \right) \epsilon_{ij} \left( \hat{A}^{(9)}_i \right) - \frac{1}{\Lambda_0} \left( \hat{\Psi}^{(9)} \hat{Z}^{(9)} \right) \epsilon_{ij} \right].
\]

(37)
The trace is over the fundamental matrices of the group $U(N_9)$ and $\eta^{c}_{mn}$ are the 't Hooft symbols defined in equation (C.9) [23].

It is important to stress that each of the three previous Lagrangians is supersymmetric independently from the others. In particular, the Lagrangians of the open strings 55 and 99 are invariant under 16 supercharges, while that of the strings 59 is invariant under 8 supercharges. They still contain the auxiliary fields of the gauge multiplets. In the following we will eliminate them by using their equation of motion obtaining a complete Lagrangian for the strings 59 and a Lagrangian for the strings 99 that can be uplifted from six to ten dimensions.

Let us consider the terms in equations (34) and (37) that contain the auxiliary field $\hat{D}^{(9)}$. There are

$$\mathcal{L}_D = i \tilde{g}_9 \sum_{c=1}^{3} (\hat{D}^{(9)}_{c+5})^u_{v} \eta_{(c+5)mn} ([\hat{A}_m^{(9)}, \hat{A}_n^{(9)}])^v_u + \sum_{c=1}^{3} (\hat{D}^{(9)}_{c+5})^u_{v} (\hat{D}^{(9)}_{c+5})^v_u$$

$$+ \tilde{g}_9 \tilde{u}_{ia}^u \sum_{c=1}^{3} (\tau^c)^j_j (\hat{D}^{(9)}_{c+5})^u_{v} \tilde{u}^{oj}_v .$$

(38)

The equation of motion for $\hat{D}^{(9)}$ is given by

$$(\hat{D}^{(9)}_{c+5})^u_{v} = \frac{1}{2} (\hat{D}^{(9)}_{c+5})^u_{v} - i \tilde{g}_9 \eta_{(c+5)mn} ([\hat{A}_m^{(9)}, \hat{A}_n^{(9)}])^v_u .$$

(39)

which, inserted back in equation (38), yields

$$\mathcal{L}_D = \frac{1}{4} \tilde{g}_9^2 \sum_{c=1}^{3} \eta_{(c+5)mn} \eta_{(c+5)pq} ([\hat{A}_m^{(9)}, \hat{A}_n^{(9)}])^v_u ([\hat{A}_p^{(9)}, \hat{A}_q^{(9)}])^u_v$$

$$- \frac{1}{2} \tilde{g}_9^2 \tilde{u}_{ia}^u \sum_{c=1}^{3} (\tau^c)^j_j \eta_{(c+5)mn} ([\hat{A}_m^{(9)}, \hat{A}_n^{(9)}])^v_u \tilde{u}^{oj}_v$$

$$- \frac{1}{4} \tilde{g}_9^2 \sum_{c=1}^{3} \tilde{u}_{ia}^u (\tau^c)^j_j \tilde{u}^{oj}_v \tilde{u}_{ib}^v (\tau^c)^b_i \tilde{u}^{bk}_u .$$

(40)

The last two terms of the previous equation go into the Lagrangian of the strings 59 that becomes

$$\mathcal{L}_{59} = \epsilon^{ij} (D_\mu \tilde{u}_i^u) (D^5 w)_u^j - g_5 \tilde{u}_{ia}^u \sum_{c=1}^{3} (\tau^c)^j_j (\hat{D}^{(9)}_{c+2})^u_{j} \tilde{u}^{bij}_u$$

$$- i \tilde{u}_{ia}^u \Gamma^\mu (D_\mu \tilde{u}_i^u) + \sqrt{2} g_5 b \tilde{u}_{ia}^u (\tilde{A}_i^{(9)})^u_{j} \tilde{u}^{bij}_u$$

$$- \frac{1}{2} \tilde{g}_9^2 \sum_{c=1}^{3} \tilde{u}_{ia}^u (\tau^c)^j_j \tilde{u}^{oj}_v \tilde{u}_{ib}^v (\tau^{c})^b_i \tilde{u}^{bk}_u$$

$$- \sqrt{2} g_5 b \tilde{u}_{ia}^u (\tilde{A}_i^{(9)})^u_{j} \tilde{u}^{bij}_u + (\tilde{u}_i^u)^j_j (\tilde{A}_j^{(9)})^v_u$$

$$- \frac{1}{2} \tilde{g}_9 \tilde{u}_{ia}^u \sum_{c=1}^{3} (\tau^c)^j_j \eta_{(c+5)mn} ([\hat{A}_m^{(9)}, \hat{A}_n^{(9)}])^v_u \tilde{u}^{oj}_v .$$

(41)

This Lagrangian is six dimensional and also the fields of the gauge multiplet living on the D9 branes are six-dimensional fields. In the Lagrangian for the strings 55 and the one for the strings 59 there is still an auxiliary field living on the world-volume of the D5 branes. By eliminating it one obtains additional quartic terms involving the fields $w$ and $Z$. 


The first term on the right-hand side of equation (40) goes instead together with the others terms in equation (37) to give

\[ L_{99} = 2 \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu}^{(9)} F^{(9)}_{\mu\nu} + \frac{g_s}{4} \sum_{c=1}^{3} \eta_{(c+5)mn} \eta_{(c+5)pq} \left[ \tilde{A}_{m}^{(9)}, \tilde{A}_{n}^{(9)} \right] \left[ \tilde{A}_{p}^{(9)}, \tilde{A}_{q}^{(9)} \right] ight] \\
- \frac{1}{2} \sum_{m=6}^{9} D_{\mu} A_{m}^{(9)} D_{\nu} A_{m}^{(9)} - \frac{i}{2} \tilde{A}_{j}^{(9)} \Gamma^{\mu} D_{\mu} (\tilde{A}_{j}^{(9)})' - i \Psi^{(9)} \Gamma^{\mu} D_{\mu} \Psi^{(9)} \\
+ i \sqrt{2} g_s b \left[ \left[ \Psi^{(9)}, (Z^{(9)})' \right] \epsilon_{ij} (\Lambda^{(9)})' - \tilde{A}_{j}^{(9)} \left[ \Psi^{(9)}, Z_{j}^{(9)} \right] \epsilon_{ij} \right]. \tag{42} \]

This Lagrangian can be uplifted from six to ten dimensions. This is easy to perform for the purely bosonic part of the action, while it requires some attention for the fermionic part. The point is that the six-dimensional gaugino is an SU(2) doublet and in \( N = 1 \) SYM in ten dimensions that symmetry is lacking. The presence of such a doublet is a consequence of the ten-dimensional Majorana–Weyl condition, as is explained in appendix C. In this appendix, it is also shown that the uplifting from six to ten dimensions yields \( N = 1 \) SYM:

\[ L_{99} = 2 \text{Tr} \left[ -\frac{1}{4} F_{MN} F^{MN} - \frac{i}{2} \lambda \Gamma^{M} D_{M} \lambda \right]. \tag{43} \]

In appendices C and D, the Lagrangians (32), (34), and (42) have been obtained by uplifting four-dimensional supersymmetric Lagrangians to six dimensions. While the uplift to ten dimensions for the strings D9/D9 is straightforward and gives rise to a Lagrangian that is supersymmetric in ten dimensions, this is not quite so for the strings D5/D9. The reason is that, while the fields corresponding to the open strings D5/D9 and D5/D5 are six dimensional, those living on the D9 branes are ten dimensional. In order to get an intuition on how this is going to work, in section 5 we will go back to string theory that treats the fields living on the D9 branes as ten dimensional, and we compute various terms of the D5/D9 action, checking, on the one hand, the previously constructed Lagrangian and seeing, on the other hand, how the ten-dimensional fields appear in it.

4. Supersymmetry invariance of the action for the strings D5/D9

The six-dimensional Lagrangian in equation (34), describing the interaction of the massless open strings stretched between the D9 and D5 branes (twisted-matter) with the gauge multiplets living respectively on the D5 and D9 branes, has been obtained from the four-dimensional supersymmetric \( N = 2 \) Lagrangian of the hypermultiplet. It is expected to preserve \( N = 1 \) supersymmetry in six dimensions. The uplifting procedure, however, in general does not generate all the terms of the six-dimensional theory; for example, it does not give terms depending on the derivative with respect to the compact dimensions. The requirement of the gauge invariance allows one to obtain many of the missing terms but it does not ensure, in general, that the uplifted action is complete. The explicit proof of the invariance of the six-dimensional uplifted theory under \( N = 1 \) supersymmetry transformations, which we are now going to discuss, is a strong check that the uplifted Lagrangian is correct.

The six-dimensional Lagrangians for the massless D5/D5 and D9/D9 open strings contain a gauge multiplet and a hypermultiplet, transforming in the adjoint representation of the gauge group, and are invariant under \( N = 2 \) supersymmetry, while that of the twisted matter coupled to the two previous gauge multiplets preserves only half of the previous supersymmetry and is therefore only invariant under \( N = 1 \) supersymmetry. For the sake of simplicity, in the
analysis of this section we neglect the gauge multiplet living on the D9 branes, which, however, can be trivially included. The \( \mathcal{N} = 1 \) supersymmetry transformations of the gauge multiplet can be found from the requirement that they leave invariant the first line of equation (32) corresponding to the Lagrangian of an \( \mathcal{N} = 1 \) gauge multiplet in six dimensions [24] given by

\[
S_\mathcal{E} = 2 \int d^6 x \text{Tr} \left[ -\frac{1}{4} F_{\hat{\mu}\hat{\nu}}^2 + \frac{1}{2} \sum_{c=1}^{3} D_c^2 - \frac{i}{2} \hat{\Lambda}_i \hat{\Gamma}^\mu \hat{D}_\mu \hat{\Lambda}_i \right].
\]  

(44)

It is easy to see that the following supersymmetry transformations leave the previous action invariant:

\[
\delta A^\mu = \frac{i}{2} (\hat{\epsilon}_i \hat{\Gamma}^\mu \Lambda^i - \hat{\Lambda}_i \hat{\Gamma}^\mu \epsilon^i); \quad \delta \Lambda^i = \frac{1}{2} F_{\hat{\mu}\hat{\nu}} \hat{\Gamma}^\mu \epsilon^i + i D^\mu \epsilon^i
\]

\[
\delta D^\mu = \frac{i}{2} (\epsilon^i \hat{\Gamma}^\mu \Lambda^i + \hat{\epsilon}^i \hat{\Gamma}^\mu \Lambda^i),
\]

(45)

where \( \hat{\Gamma}^\mu = \frac{1}{2} [\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] \) and \( D^\mu = D^\nu (\epsilon^\nu) \). The parameters of the supersymmetry transformations \( \epsilon^i \) are two six-dimensional spinors having, for consistency with equations (F.4), the same chirality as the gaugino and, as the gaugino, they have to satisfy equation (A.20). More details about the properties satisfied by the \( \epsilon^i \)'s can be found in appendix F. We are not going to show here explicitly that the action in equation (44) is invariant under the transformations in equation (F.4), but we refer to [24] where the proof has been given. For more details see also section 9 of [25].

In appendix F, we show that the extra terms coming from the presence of the auxiliary fields \( D^\nu \) also give rise to a total derivative that leaves the action invariant.

Having determined the supersymmetry transformations of the fields of the gauge multiplet, we are now going to study the supersymmetry invariance of the Lagrangian for the twisted strings D5/D9 given in equation (34) without including, for the sake of simplicity, the gauge multiplet living on the D9 branes. The Lagrangian is equal to

\[
\mathcal{L}_{S9} = \epsilon^i (D^\nu \bar{w}_{ij}) a (D^\mu w_j)^a - i \bar{\mu}_a \hat{\Gamma}^\mu (D^\mu \mu)^a - g \bar{\mu}_a \sum_{c=1}^{3} (\epsilon^i)^c (\bar{\Lambda}_c)^a \epsilon^i (w_j)^b + \sqrt{2} g b \bar{\mu}_a \left[ \bar{\Lambda}_a \epsilon^i (w_j)^b + \bar{\mu}_a \epsilon^i (w_j)^b \right].
\]

(46)

In appendix F, we show that it transforms as a total derivative under the action of the transformations in equation (F.4) that act on the fields of the gauge multiplet, together with the following supersymmetry transformations acting on the twisted fields:

\[
\delta w^a = -\sqrt{2} b \epsilon^i \bar{w}_{ij} \epsilon^a; \quad \delta \mu^a = -i \sqrt{2} b \epsilon^i (D^\mu w_j)^a.
\]

(47)

These transformations are similar to the ones involving the hypermultiplets in [26] with the main difference that they preserve \( \mathcal{N} = 1 \) in six dimensions instead of \( \mathcal{N} = 2 \) in five dimensions. The doubling of the SUSY parameters, constrained by equation (F.8), is necessary to manifestly keep the \( SU(2) \) invariance of the action. The proof of the supersymmetric invariance of the twisted Lagrangian is long and tedious. The details are again given in appendix F. Here we quote just the final result of the supersymmetry transformations given by

\[
\delta \mathcal{L}_{S9} = \partial_{\bar{\mu}} \left\{ -\sqrt{2} b \bar{w}_{ij} \epsilon^i \bar{\epsilon}^j (\bar{\Gamma}^\mu, \Gamma^\mu) (D^\mu \mu)^a + \sqrt{2} b \bar{\mu}_a \epsilon^i (D^\mu w_j)^a \\
+ g \bar{\mu}_a \epsilon^i (\bar{\Lambda}_a \epsilon^j (\bar{\Lambda}_b \epsilon^i (\bar{\Lambda}_c)^a \epsilon^b w^b) \right\}.
\]

(48)

In conclusion, we have shown that the Lagrangian for the open strings D5/D9 is invariant under the supersymmetry transformations given in equations (45) and (47).
5. Vertex operators for the open strings 99, 55 and 59

In this section, we write the vertex operators of the massless open string states in the system D5/D9. We have three kinds of open strings: those with the two end-points attached to a system of \(N_0\) parallel D9 branes, those with the two end-points attached to a system of \(N_5\) parallel D5 branes and the mixed open strings having one end-point attached to a D9 brane and the other end-point attached to a D5 brane. All the following vertices are in a one-to-one correspondence with those in [27] due to the T-duality between the system D9/D5 and the system D3/D(−1) discussed in that paper. The main difference consists in the momentum dependence in all sectors of our vertices and in the \(SU(2)\) structure that is manifest in the expression of the string vertex operators.

5.1. D9/D9 open strings

The massless fields living on the system of \(N_9\) D9 branes consist of a gauge field \((A_{\mu}^{(9)})_\nu^v(x, y)\) and a ten-dimensional Majorana–Weyl spinor with negative chirality \((\bar{\psi}^{(9)})_\alpha^v(x, y)\). All the fields carry a ten-dimensional momentum with \(k_M k^M = 0\) and their vertex operators are

\[
V_{(9)}^{(1)}(z) = ig_9 \sqrt{2\pi \alpha'} (A_{\mu}^{(9)})_\nu^v(k_M) e^{i\phi(z)} \bar{\psi}^M(z) e^{i\sqrt{2\pi \alpha'} k_M X^M(z)}
\]

for the gauge field in the picture \((-1)\) and

\[
V_{(9)}^{(0)}(z) = ig_9 \sqrt{2\pi \alpha'} (A_{\mu}^{(9)})_\nu^v(k_M) (\partial^M X^M(z)) + i\sqrt{2\pi \alpha'} k_N \bar{\psi}^N(z) \psi^M(z) e^{i\sqrt{2\pi \alpha'} k_M X^M(z)}
\]

in the picture 0. For both of them the transversality condition \(k_M A^M = 0\) holds. Here \(M, N = 0, \ldots, 9; \ A = 1, \ldots, 16\) and \(u, v = 1, \ldots, N_9\) are the indices of the gauge group \(U(N_9)\) living on the set of the \(N_9\) D9 branes.

The vertex for the gaugino is

\[
V_{\overline{\chi}^{(9)}}^{(1/2)}(z) = g_9 (\pi \alpha')^2 (A_{\mu}^{(9)})_\nu^v(k_M) e^{-i\phi(z)} S^A(z) e^{i\sqrt{2\pi \alpha'} k_M X^M(z)}.
\]

The presence of the D5 branes breaks the original ten-dimensional Lorentz group \(SO(1, 9)\) into \(SO(1, 5) \otimes SO(4)\). Hence, it is convenient to split the ten-dimensional Weyl index \(\overline{\chi}\) as follows:

\[
S^A = (S^A, S_\overline{\chi}_0, S_{\overline{\chi}_0}),
\]

where the upper (lower) index \(\overline{\chi}\) is the index of a Weyl spinor in six dimensions with positive (negative) chirality and similarly \(\alpha (\overline{\alpha})\) is the index of a four-dimensional Weyl spinor with positive (negative) chirality. Introducing the decomposition (52) in the vertex in equation (51), one obtains

\[
V_{\overline{\chi}^{(9)}}^{(-1/2)}(z) = \sqrt{2} g_9 (\pi \alpha')^2 (A_{\mu}^{(9)})_\nu^v(k_M) e^{-i\phi(z)} S^A(z) S_\overline{\chi}_0(z) e^{i\sqrt{2\pi \alpha'} k_M X^M(z)}.
\]

The latter expression has an index \(\overline{\alpha}\) that for simplicity, in the previous section, we have denoted by the indices \(i\) and \(j\), and that corresponds to the \(SU(2)\) symmetry previously discussed. In string theory the role of the internal symmetry is thus played by one of the two \(SU(2)\) factors of the \(SO(4)\) group associated with the directions 6...9.

The normalization factor of the vertex operator in equation (53) has been determined by requiring that the three-point function computed in string theory and involving two gauginos and a gauge field agrees with the corresponding coupling in field theory. In particular, the

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6. This is a consequence of the fact that we have chosen the GSO projection in the ten-dimensional IIB string theory so as to keep spinors with negative chirality.
factor $\sqrt{2}$ in equation (53) is a direct consequence of the constraint for the gaugino field in equation (33).

In principle one should also introduce a vertex for the field $\tilde{A}_i$ besides the one for $A_i$, but in this case this is not necessary because the two fields are not independent being related by the symplectic Majorana condition.

The last vertex coming from the decomposition of the ten-dimensional spinor is

$$V_{\psi\phi\alpha\mu}(z) = \sqrt{2}g_5(\pi\alpha')^2 \left( \lambda^{\alpha\mu}_a(k) e^{-i\phi(z)} S_\lambda(z) S_\mu(z) e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)} \right),$$

which again contains an $SU(2)$-doublet corresponding to $SU(2)$ that is a symmetry of the Lagrangian of the open strings D5/D5, but not of the Lagrangian of the open strings D5/D9.

In the six-dimensional Lagrangian of the open strings D5/D5 in equation (32) such a doublet is, however, missing because this $SU(2)$ symmetry is not manifestly realized and we have a field $\Psi$ instead of a doublet. It turns out that the vertex operator corresponding to this field is given in equation (54) where the six-dimensional spinor is taken to be

$$\psi^\lambda_{\alpha a} = \left( \psi^\lambda, 0 \right).$$

Note that the ten-dimensional spinor index is dimensionally reduced, but the four-momentum of the vertices is still ten dimensional.

### 5.2. D5/D5 open strings

The vertex operators corresponding to open strings of the gauge multiplet living on the D5 branes are obtained through a trivial dimensional reduction from those living on a D9 brane. They are a gauge field $(\hat{A}_\mu^{(5)})^a_b(x^\mu)$ (with $\hat{\mu}, \hat{\nu} = 0, \ldots, 5$ and $a, b$ running over the number of D5 branes, namely $a, b = 1, \ldots, N_5$), four real scalars $(\phi_m^{(5)})^a_b(x^\mu)$, a couple of six-dimensional Weyl spinors $(\Lambda^{5,\alpha})^\mu_a(x^\mu)$ with positive six-dimensional chirality and a couple of six-dimensional Weyl spinors $(\Psi^{(5)\beta}_{\delta a})^a_b(x^\mu)$ with negative six-dimensional chirality. The components of the latter spinor are given as in equation (55).

The corresponding vertex operators, carrying a six-dimensional momentum with $k^2 = 0$, are

$$V_{\Lambda^{\alpha}}(z) = i(2\pi\alpha')^2 g_5 \left( \Lambda^{(5)}_\mu_b^a(k) \right) e^{-i\phi(z)} \hat{A}_\mu(z) e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)}$$

for the gauge field in picture $(-1)$,

$$V_{\phi_n}(z) = i(2\pi\alpha')^2 g_5 \left( \phi_n^{(5)}_a(k) \right) [\partial_\alpha X^\alpha(z) + i(2\pi\alpha')^2 k_\beta \psi^\beta(z) \psi^\nu(z)] e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)}$$

for the gauge field in the picture 0, and for both of them the transversality condition $k_\hat{\alpha} A^{(5)\hat{\alpha}} = 0$ holds.

For the scalars in the picture $(-1)$ we have

$$V_{\phi^{(-1)}}(z) = i g_5(2\pi\alpha')^2 \left( \phi_m^{(5)}_a(k) \right) e^{-i\phi(z)} \phi_{\lambda}(z) e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)},$$

while for the scalars in the picture 0 we have

$$V_{\phi^{(0)}}(z) = i g_5(2\pi\alpha')^2 \left( \phi_m^{(5)}_a(k) \right) [\partial_\alpha X^\alpha(z) + i(2\pi\alpha')^2 k_\beta \psi^\beta(z), \psi^\nu(z)] e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)}.$$  

The vertex for the gaugino with negative six-dimensional chirality is

$$V_{\Lambda^{\alpha}}(z) = \sqrt{2}g_5(\pi\alpha')^2 \left( \Lambda^{(5)}_\mu_b^a(k, \beta) \right) e^{-i\phi(z)} S^\lambda(z) S_\mu(z) e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)}$$

and for $\Psi^{(5)}$, which has the same $SU(2)$ structure as given in equation (55), we have

$$V_{\Psi^{\alpha\beta}} = \sqrt{2}g_5(\pi\alpha')^2 \left( \Psi^{(5))}_{\alpha a}(k, \beta) \right) e^{-i\phi(z)} S_\alpha(z) S_\beta(z) e^{i\sqrt{2}\pi\alpha\nu_5 X^\nu(z)}.$$
Finally, we also add the vertex operator of the auxiliary fields whose introduction is convenient for computing only the cubic couplings instead of the more difficult quartic ones:

\[ V_{D,\alpha}^{(5)} = i(\pi \alpha') g_5 \left( \mathcal{D}^{(5)} \right)_{\alpha} \eta_{\alpha \beta} \psi^m \psi^n \varepsilon^{\mu \nu} e^{-\phi(z)} e^{i\sqrt{2\pi \alpha'} k_\mu X^\mu(z)} \]  

(62)

where \( c = 1, 2, 3; m, n = 6, 7, 8, 9 \).

5.3. D9/D5 and D5/D9 open strings

In the system D5/D9, along the four directions of the last two tori, there are Ramond (Neveu–Schwarz) boundary conditions in the NS (R) sector. This means that the massless state of the NS sector is an \( SO(4) \) spinor. Due to the GSO projection it is a chiral spinor, \( w_\alpha \). In the R sector, the massless state is an \( SO(6) \) spinor that, again because of the GSO projection, is a chiral spinor in six dimensions: the four non-compact directions and the two compact ones belonging to the first torus. In conclusion, the massless D9/D5 sector contains a doublet of scalars \( (w_\alpha)^a_{\mu}(x) \) under \( SU(2) \) and a six-dimensional Weyl spinor \( (\mu^\alpha)^a_{\mu}(x) \).

Their vertex operators are [27]

\[ V_{w}^{(-1)}(z) = g_5 (\pi \alpha')^{1/2} (w_\alpha)^a_{\mu}(k) \Delta(z) S^a(z) e^{-\psi(z)} e^{i\sqrt{2\pi \alpha'} k_\mu X^\mu(z)} \]

\[ V_{\mu}^{(-1)}(z) = g_5 (\pi \alpha')^{1/2} (\mu^\alpha)^a_{\mu}(k) \Delta(z) S^a(z) e^{-\phi(z)} e^{i\sqrt{2\pi \alpha'} k_\mu X^\mu(z)} \]  

(63)

where \( \Delta(z) \) is the twist operator with conformal weight 1/4 which changes the boundary conditions of the four compact coordinates from Neumann to Dirichlet. \( k_\mu \) is the six-dimensional momentum. The vertex operators in equation (63) describe the strings D9/D5. The strings D5/D9 are described instead by

\[ V_{w}^{(-1)}(z) = g_5 (\pi \alpha')^{1/2} (\bar{w}_\alpha)^a_{\mu}(k) \bar{\Delta}(z) S^a(z) e^{-\psi(z)} e^{i\sqrt{2\pi \alpha'} \bar{k}_\mu X_i(z)} \]

\[ V_{\mu}^{(-1)}(z) = g_5 (\pi \alpha')^{1/2} (\bar{\mu}^\alpha)^a_{\mu}(k) \bar{\Delta}(z) S^a(z) e^{-\phi(z)} e^{i\sqrt{2\pi \alpha'} \bar{k}_\mu X^\mu(z)} \]  

(64)

with \( \Delta(z) \) replaced by the anti-twist \( \bar{\Delta}(z) \) operator, corresponding to DN (instead of ND) boundary conditions along the directions orthogonal to the world-sheet of the D5 brane. The upper index in brackets of each vertex denotes, as usual, the corresponding picture. All previous states transform according to the bifundamental of the group \( U(N_6) \times U(N_5) \).

6. Computing string amplitudes

Having identified the massless states of the three open string sectors and their vertex operators, we can now perform string disk calculations in order to determine the various terms of their low-energy Lagrangian. This approach is complementary to that presented in section 3, where we have constructed the supersymmetric low-energy Lagrangians in six dimensions by uplifting to six dimensions the four-dimensional ones.

The calculation of various three-point couplings provides an additional check of the six-dimensional Lagrangians already derived in section 3. An example of this is the coupling of two adjoint fermions and a gauge boson living on the D5 branes. This coupling has been computed in equation (G.4) and agrees with the one in equation (32). Other three-point couplings can also be easily computed and one can see that they reproduce the various trilinear couplings of the Lagrangian in equation (32). In particular, one can use the vertex operator of the auxiliary fields to determine also their couplings with the propagating fields. In this way all trilinear couplings can be computed. The quartic terms present in the Lagrangian (32) can be obtained from the cubic one by gauge invariance. In this way one can derive the three six-dimensional Lagrangians for the strings D5/D5, D9/D9 and D5/D9.
On the other hand, string theory treats the fields of the gauge multiplet living on the D9 branes as ten dimensional and therefore one can determine their couplings with the twisted fields living on the D5 branes. In this way one can obtain a six-dimensional Lagrangian for the massless open strings D5/D9 interacting with the gauge multiplet living on the D9 branes, where the latter fields are ten dimensional and not six dimensional as in equation (34).

Examples are provided by the three-point amplitudes involving a twisted scalar, a twisted fermion and a D5/D9 or D5/D5 gaugino. These are computed in appendix G. Particularly interesting is the three-point amplitude involving two twisted scalar fields \( w \) and \( \bar{w} \) and a gauge boson living on the D9 branes, given by

\[
A^{\hat{w}A w} = C_0 \int \frac{d x_1 d x_2 d x_3}{d V} \langle 0 | V_{\hat{w}}^{(-1)}(x_1) V_A^{(0)}(x_2) V_{w}^{(-1)}(x_3) | 0 \rangle, \tag{65}
\]

where \( C_0 \) is the normalization of the disk which is related to the five-dimensional gauge coupling \( g_5 \) through the relation \( C_0 = g_5^{-2} (\pi \alpha')^{-2} \) and \( d V \) is the projective invariant Möbius volume. This amplitude, which can be read from equation (65), is given by

\[
A^{\hat{w}A w} = g_9 \langle \bar{w}_0 \rangle^u_3 \langle p_{\hat{\mu}}, y_0 \rangle \langle q_{\hat{\mu}}, y_0 \rangle \left\{ \epsilon^{\hat{\alpha}\hat{\beta}} \left( A_{\mu}^{(0)} \right)^u_{\hat{\alpha}}(k_{\hat{\mu}}, y_0)(p_{\hat{\mu}} - q_{\hat{\mu}}) - \frac{1}{2} \left( A_m^{(0)} \right)^u_{\hat{\alpha}}(k_{\hat{\mu}}, y_0) k_{\hat{\alpha}} - \left( A_m^{(0)} \right)^u_{\hat{\alpha}}(k_{\hat{\mu}}, y_0) k_{\hat{\beta}} \right\}, \tag{66}
\]

where \( y_0 \) is the position of the D5 branes in the last two tori. The first term of the previous equation reproduces the coupling of the twisted scalars with the D9 gauge fields that can be read from the first term of equation (41). Having computed a three-point coupling implies that the second term contains only the derivative part of the gauge field strength living on the D9 branes. It can be easily made gauge invariant by adding the term with the commutator obtaining:

\[
A^{\hat{w}A w}_{(2)} = - \frac{i}{2} \langle \bar{w}_0 \rangle^u_3 \left\{ (p_{\hat{\mu}}, y_0) \langle w^{\hat{\mu}} q_{\hat{\mu}}, y_0 \rangle (F_{\mu \nu})_v^u(k, y_0) \sum_{c=1}^3 \eta_{c 5}^{mn} (\tau^c)^{ij} \right\}, \tag{67}
\]

where the identity

\[
(\sigma^{mn})^i_j = i \sum_{c=1}^3 \eta_{c 5}^{mn} (\tau^c)^{ij} \tag{68}
\]

has been used and the indices \( (\hat{\alpha}, \hat{\beta}) \equiv (i, j) \) have been redefined. The quantities \( \eta^c \)'s are again the 't Hooft symbols, previously introduced.

The amplitude in equation (67) is the string result of the last term in the Lagrangian (41). It differs from the corresponding one obtained in field theory because it depends on the whole field strength of the internal components of the gauge field instead of only the commutator. Furthermore, the field strength of the D9 gauge field is ten dimensional and is computed at the point \( y_0 \) of the four-dimensional space where the D5 branes are located.

In conclusion, we have shown that, by performing string disk calculations, we can uplift the six-dimensional Lagrangian in (34) to contain the ten-dimensional fields of the gauge multiplet living on the D9 branes (and not just their reduction to six dimensions as obtained in the field theoretical approach). In practice, one obtains the same Lagrangian as in equation (41), where now all the ten-dimensional fields are computed at \( y_0 \) with the last term in equation (41) that becomes

\[
- \frac{1}{2} \langle \bar{w}_0 \rangle^u_3 \langle F_{mn} \rangle^v_u w^{uj}. \tag{69}
\]

The Lagrangian in equation (41), with the modifications discussed above, will be used in the following section to compute the spectrum of the Kaluza–Klein states in the presence of a background magnetization in the six extra dimensions.
7. The Kaluza–Klein reduction

After having determined the six-dimensional Lagrangian describing the interaction of the massless fields corresponding to the open strings D5/D9 with the gauge multiplets living respectively on the D5 and D9 branes, in this section we perform the Kaluza–Klein reduction of the six-dimensional Lagrangian to $R^{11} \times T^2$ with a non-zero magnetization on the D9 branes and we show that the spectrum of states agrees with the one obtained from string theory in the field theory limit given in section 2.3. Since we are interested only in the spectrum of Kaluza–Klein states, we can limit ourselves to the following part of the Lagrangian in equation (41):

$$\mathcal{L}_{59} = \varepsilon^{\alpha \beta} (D^\beta \tilde{u}_a)_{\mu}^{\alpha} (D_\mu w_\beta)^a - i \bar{D}^\alpha \Gamma^{\alpha} (D_\mu \mu)^a_{\mu}$$

$$- \frac{i}{2} (\tilde{u}_a)_{\mu}^{\alpha} (\bar{\sigma}^{m \alpha \gamma}_{\beta}) (F_{m \alpha 
abla}^a (x, y_0)) e^{\beta \gamma} (w_\gamma)^a_{\mu}.$$ (70)

Let us consider the bosonic part of this Lagrangian that, after a partial integration, is equal to

$$\mathcal{L}_{59}^{\text{bos}} = \varepsilon^{\alpha \beta} (D^\beta \tilde{u}_a)_{\mu}^{\alpha} (D_\mu w_\beta)^a - \varepsilon^{\alpha \beta} (\bar{u}_a)_{\mu}^{\alpha} (\bar{D}^\beta \bar{D}_\mu)^a_{\mu}$$

$$- \frac{i}{2} (\tilde{u}_a)_{\mu}^{\alpha} (\bar{\sigma}^{m \alpha \gamma}_{\beta}) (F_{m \alpha 
abla}^a (x, y_0)) e^{\beta \gamma} (w_\gamma)^a_{\mu},$$ (71)

where $\mu = 0, \ldots, 3$, $i = 4, 5$ and $m, n = 6, 7, 8, 9$ and the indices $m, n$ are contracted with the flat metric. Furthermore,

$$(\bar{\sigma}^{m \alpha \gamma}_{\beta}) (F_{m \alpha 
abla}^a (x, y_0)) = \frac{1}{2} (\bar{\sigma}^{m \alpha \gamma}_{\beta}) (\sigma^{m \alpha \gamma}_{\beta}) (x, y_0),$$ (72)

with $\sigma^m \equiv (i \tau^1, \|)$ and $\bar{\sigma}^m \equiv (i \tau^2, \|).$ In the following, we restrict ourselves to a six-dimensional compact manifold that is the product of three tori $T^2$ with a background magnetic field $F \equiv (F_{5 \alpha}, F_{6 \beta}, F_{89})$ turned on only on the D9 branes. The three background magnetic fields must satisfy the constraint that the corresponding Chern class is an integer. For the sake of simplicity, we assume that the magnetic fields lie in the $U(1)$ gauge group. This implies that they have the following form:

$$F_{5 \alpha} = \frac{2\pi I^{(1)}}{(2\pi \sqrt{\alpha'})^2 T_2}; \quad F_{6 \beta} = \frac{2\pi I^{(2)}}{(2\pi \sqrt{\alpha'})^2 T_2}; \quad F_{89} = \frac{2\pi I^{(3)}}{(2\pi \sqrt{\alpha'})^2 T_2},$$ (73)

where $I^{(1)}, I^{(2)}, I^{(3)}$ are three integers. We take the D5 branes unmagnetized.

The extension to the case of magnetized D5 branes is easily obtained by writing $I^{(1)} = I^{(1)}_5 - I^{(1)}_\alpha$, while the extension to a non-Abelian bundle is obtained by the substitution $I \rightarrow \frac{I}{n}$ where $I$ is the first Chern class and $n$ is the wrapping number.

Because of the background magnetizations the quantity appearing in the last term of equation (71) is equal to

$$(\bar{\sigma}^{m \alpha \gamma}_{\beta}) (F_{m \alpha 
abla}^a (x, y_0)) = 2i \left( F_{6 \beta}^{90} + F_{89}^{90} - F_{67}^{90} - F_{89}^{90} \right).$$ (74)

Furthermore, according to the notation of [12], we have

$$- \bar{D}^\alpha D_\alpha \equiv \frac{2\pi |I^{(1)}|}{(2\pi \sqrt{\alpha'})^2 T_2^2} (2a^{(1)} - 1).$$ (75)

By inserting equations (74) and (75) into equation (71) we obtain the following expression for the quadratic term involving the twisted field $w$:

$$\mathcal{L}_{59}^{\text{bos}} = -\varepsilon^{\alpha \beta} \bar{D}^{\alpha} \bar{u}_a D_\alpha w_\beta + \bar{u}_a \left( \frac{4\pi |I^{(1)}|}{(2\pi \sqrt{\alpha'})^2 T_2} N^{(1)} \delta^\alpha_\gamma + M^\alpha_\gamma \right) e^{\beta \gamma} w_\beta.$$ (76)

Footnote:

7 The identity matrix corresponds to the index 9, while the matrix $\bar{\sigma}$ corresponds to the indices 6, 7, 8.
with $N^{(1)} = a^{(1)} a^{(1)}$ and

$$M^a_\gamma = \frac{2\pi}{(2\pi \sqrt{\alpha'}^2)^2} \begin{pmatrix} \frac{|I^{(1)}|}{T_{2}^{(1)}} + \left( \frac{I^{(2)}}{T_{2}^{(2)}} + \frac{I^{(3)}}{T_{2}^{(3)}} \right) & 0 \\ 0 & \frac{|I^{(1)}|}{T_{2}^{(1)}} - \left( \frac{I^{(2)}}{T_{2}^{(2)}} + \frac{I^{(3)}}{T_{2}^{(3)}} \right) \end{pmatrix}. \quad (77)$$

The previous matrix is diagonal with eigenvalues given by

$$\lambda_a = \frac{2\pi}{(2\pi \sqrt{\alpha'}^2)^2} \left[ \frac{|I^{(1)}|}{T_{2}^{(1)}} \pm \left( \frac{I^{(2)}}{T_{2}^{(2)}} + \frac{I^{(3)}}{T_{2}^{(3)}} \right) \right]. \quad (78)$$

The massless state is obtained by choosing $N^{(1)} = 0$ and imposing the constraint $\frac{|I^{(1)}|}{T_{2}^{(1)}} \pm \left( \frac{I^{(2)}}{T_{2}^{(2)}} + \frac{I^{(3)}}{T_{2}^{(3)}} \right) = 0$ where the sign $\pm$ to be chosen depends on the sign of the round bracket. The corresponding wavefunction can be obtained by solving the equation $a^{(1)} w_a = 0$ with suitable boundary conditions. These are given by the identification, up to a gauge transformation, of the fields under a translation on the torus [10]. The solution to this problem is provided in [12]. Therefore, we conclude that the wavefunction is again a theta-function depending just on the moduli of the first torus. The general expression for the mass is then

$$m_n^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'}^2)^2} \left[ \frac{|I^{(1)}|}{T_{2}^{(1)}} (2N^{(1)} + 1) \pm \left( \frac{I^{(2)}}{T_{2}^{(2)}} + \frac{I^{(3)}}{T_{2}^{(3)}} \right) \right]. \quad (79)$$

Let us now show that the spectrum of Kaluza–Klein states in equation (79) agrees with the one obtained from string theory in equations (23) and (24). In equation (79) $I^{(2)}$ and $I^{(3)}$ can be both positive and negative. When they have opposite sign, the previous mass formula becomes

$$m_n^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'}^2)^2} \left[ \frac{|I^{(1)}|}{T_{2}^{(1)}} (2N^{(1)} + 1) \pm \left( \frac{|I^{(2)}|}{T_{2}^{(2)}} + \frac{|I^{(3)}|}{T_{2}^{(3)}} \right) \right], \quad (80)$$

which is equal to equation (23). On the other hand, when they have the same sign, we obtain

$$m_n^2 = \frac{2\pi}{(2\pi \sqrt{\alpha'}^2)^2} \left[ \frac{|I^{(1)}|}{T_{2}^{(1)}} (2N^{(1)} + 1) \pm \left( \frac{|I^{(2)}|}{T_{2}^{(2)}} + \frac{|I^{(3)}|}{T_{2}^{(3)}} \right) \right], \quad (81)$$

which is equal to equation (24). In conclusion, one obtains the string theory spectrum in the field theory limit.

The Kaluza–Klein spectrum of the fermionic states is obtained by expanding the six-dimensional fermionic field in terms of its internal wavefunctions associated with the compact directions of the first torus. These latter are determined by solving the eigenvalue equation of the two-dimensional Dirac equation [10]

$$\Gamma^\mu D_\mu \eta_a(x^l) = m_n \eta_a \quad (82)$$

Squaring the Dirac equation and using the six-dimensional $\Gamma$-matrices given in appendix A, one easily arrives at writing [12]

$$\left( -D_\mu D^\mu + \frac{2\pi |I^{(1)}|}{(2\pi \sqrt{\alpha'})^2 T_{2}^{(1)}} \right) \eta_a = m_n^2 \eta_a. \quad (83)$$

Using equation (75) we see that we have again two towers of Kaluza–Klein fermionic states with masses

$$m_n^2 = \frac{2\pi |I^{(1)}|}{(2\pi \sqrt{\alpha'})^2 T_{2}^{(1)}} (2N^{(1)} + 1) \pm \frac{2\pi |I^{(1)}|}{(2\pi \sqrt{\alpha'})^2 T_{2}^{(1)}}. \quad (84)$$
From the previous expression we see that, once we fix the sign of the first Chern class, the spectrum contains two towers of fermionic Kaluza–Klein states having opposite internal two-dimensional chirality. The six-dimensional spinors being also chiral spinors, we conclude that the two sets of states have also opposite four-dimensional chirality. This is exactly what happens in string theory and the previous expression coincides with the string result given in equation (27). This agreement is another check on the correctness of the interaction term given in equation (69).

8. Conclusions and outlook

In this paper, we have constructed the six-dimensional Lagrangian for the open strings D5/D9 interacting with the gauge multiplets living on the world-volumes of respectively the D5 and D9 branes. We have shown with an explicit calculation that, when we treat the fields of the gauge multiplet living on the D9 branes as six dimensional, this Lagrangian is $\mathcal{N}=1$ supersymmetric. In order to check various couplings of this Lagrangian and to extend it to the case in which the fields of the gauge multiplet living on the D9 branes are treated as ten dimensional, we construct the vertex operators corresponding to the massless states of our system D5/D9 and we use them to compute three-point couplings. In this way we obtain a six-dimensional Lagrangian where the open strings D5/D5 and D5/D9 are treated as six dimensional, while the open strings D9/D9 are treated as ten dimensional. Finally, by introducing background magnetizations in the extra dimensions both on the D5 and D9 branes, we compute the spectrum of the open strings attached with one end-point to a D5 and the other end-point to a D9 brane where the two D branes have different magnetizations. The spectrum of states completely agrees with that obtained in string theory in the field theory limit $(\alpha' \to 0)$.

Actually, string theory requires that all states, including the open strings D5/D9 and D5/D5, should be treated as ten dimensional. We hope to be able to come back to this point in a future publication.

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Appendix A. Notations

We start writing the various Lagrangians in four dimensions in terms of Weyl spinors $\psi^\alpha$ and $\chi^{\dot{\alpha}}$ that are lowered or raised by means of the antisymmetric $\epsilon$ tensor given by

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \epsilon^{\alpha\gamma} \epsilon_{\gamma\beta} = \delta_{\beta}^{\alpha}.$$

(A.1)

Analogous formulas are valid for the dotted indices.

We can define a Dirac spinor and a Majorana spinor in terms of two Weyl spinors as follows:

$$\Psi_D = \begin{pmatrix} \psi^\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix}; \quad \Psi_M = \begin{pmatrix} \psi^\alpha \\ \psi^\alpha \end{pmatrix}.$$

(A.2)
In the case of a Dirac spinor the two Weyl spinors are different, while in the case of a Majorana spinor the two are the complex conjugate of each other.

The four-dimensional \( \gamma \)-matrices are given by

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}; \quad \sigma^\mu = (1, \tau^i); \quad \bar{\sigma}^\mu = (1, -\tau^i) \tag{A.3}
\]

in terms of the two-dimensional Pauli matrices \( \tau^i \) and the identity matrix. The previous notations are the same as those in [28] except that in their case \( \sigma^0 = -1 \), while in our case \( \sigma^0 = 1 \).

In order to go to from four to six dimensions we introduce the following representation of the six-dimensional gamma matrices in terms of the four-dimensional ones:

\[
\Gamma^\mu = \gamma^\mu \otimes 1; \quad \Gamma^4 = \gamma_5 \otimes i\tau^1; \quad \Gamma^5 = \gamma_5 \otimes i\tau^2; \quad \Gamma^7 = \gamma_5 \otimes \tau^3, \tag{A.4}
\]

where

\[
\Gamma_7 \equiv \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5; \quad \gamma_5 \equiv -i\gamma^0 \gamma^1 \gamma^2 \gamma^3; \quad \{\Gamma^0, \Gamma^i\} = -2\gamma^0, \tag{A.5}
\]

where we are using, in any spacetime number of dimensions, the mostly plus metric \((-+, \ldots, +)\). A chiral spinor in six dimensions satisfies the condition

\[
(1 - b\Gamma_7)\zeta = 0, \tag{A.6}
\]

where \( \Gamma_7 \) is defined in equation (A.4) and \( \zeta \) has eight components. The most general solution of the previous equation is given by

\[
\zeta = \begin{pmatrix} \frac{1 + b\gamma_5}{2} \\ a \frac{1 - b\gamma_5}{2} \end{pmatrix} \beta = \zeta_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \zeta_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{A.7}
\]

where

\[
\zeta_1 = \frac{1 + b\gamma_5}{2}; \quad \zeta_2 = a \frac{1 - b\gamma_5}{2}, \tag{A.8}
\]

and

\[
\bar{\xi} \equiv \zeta^\dagger \Gamma^0 = \begin{pmatrix} \zeta_1^\dagger \otimes (1 \ 0) + \zeta_2^\dagger \otimes (0 \ 1) \end{pmatrix} (\gamma^0 \otimes 1) \\
= \zeta_1^\dagger \gamma^0 \otimes (1 \ 0) + \zeta_2^\dagger \gamma^0 \otimes (0 \ 1) = \begin{pmatrix} \beta \frac{1 - b\gamma_5}{2} & a\beta \frac{1 + b\gamma_5}{2} \end{pmatrix}, \tag{A.9}
\]

where \( a = \pm 1 \) and \( b = \pm 1 \). This means that in going from four to six dimensions the six-dimensional Lagrangian contains the parameters \( a \) and \( b \). Using the previous formulas and the six-dimensional \( \Gamma \)-matrices given in equations (A.4) we can compute

\[
\bar{\xi} \Gamma^0 D_\mu \xi = \beta [\gamma^\mu D_\mu + i\gamma_5 D_4 - ab D_3]\beta. \tag{A.10}
\]

In the final part of this appendix, we discuss the embedding of the six-dimensional Dirac matrices in the ten-dimensional ones that will be useful for uplifting the fermions from six to ten dimensions.

It is convenient to use the following representation of the ten-dimensional \( \gamma \)-matrices and the Pauli matrices \( \bar{\gamma} \):}

\[
\begin{align*}
\Gamma^0 &= \gamma_4^0 \otimes I \otimes I \otimes I \otimes I; & \Gamma^5 &= \gamma_5 \otimes \tau^2 \otimes I \otimes I \\
\Gamma^1 &= \gamma_4^1 \otimes I \otimes I \otimes I \otimes I; & \Gamma^6 &= \gamma_5 \otimes \tau^3 \otimes -i\tau^1 \otimes I \\
\Gamma^2 &= \gamma_4^2 \otimes I \otimes I \otimes I \otimes I; & \Gamma^7 &= \gamma_5 \otimes \tau^3 \otimes i\tau^2 \otimes I \\
\Gamma^3 &= \gamma_4^3 \otimes I \otimes I \otimes I \otimes I; & \Gamma^8 &= \gamma_5 \otimes \tau^3 \otimes \tau^3 \otimes i\tau^1 \\
\Gamma^4 &= \gamma_5 \otimes i\tau^1 \otimes I \otimes I; & \Gamma^9 &= \gamma_5 \otimes \tau^3 \otimes \tau^3 \otimes i\tau^2.
\end{align*} \tag{A.11}
\]
They satisfy the ten-dimensional Clifford algebra:
\[
\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}
\]
\[
\eta^{MN} = (-, +, \ldots).
\]  \hspace{1cm} (A.12)

They can also be conveniently expressed in terms of the six-dimensional \( \Gamma \)-matrices introduced in equation (A.4) as follows:
\[
\begin{align*}
\Gamma^0 &= \Gamma^0_{(6)} \otimes I \otimes I \\
\Gamma^1 &= \Gamma^1_{(6)} \otimes I \otimes I \\
\Gamma^2 &= \Gamma^2_{(6)} \otimes I \otimes I \\
\Gamma^3 &= \Gamma^3_{(6)} \otimes I \otimes I \\
\Gamma^4 &= \Gamma^4_{(6)} \otimes I \otimes I \\
\Gamma^5 &= \Gamma^5_{(6)} \otimes I \otimes I \\
\Gamma^6 &= \Gamma^6_{(6)} \otimes -i\tau^1 \otimes I \\
\Gamma^7 &= \Gamma^7_{(6)} \otimes i\tau^2 \otimes I \\
\end{align*}
\]  \hspace{1cm} (A.13)

where we have added an index 6 to the six-dimensional \( \Gamma \)-matrices to distinguish them from the ten-dimensional ones and
\[
\Gamma^{11} = \Gamma^0 \ldots \Gamma^8 = \Gamma^7 \otimes \tau^3 \otimes \tau^3 \\
\Gamma^9 = \Gamma^0_{(6)} \Gamma^1_{(6)} \Gamma^2_{(6)} \Gamma^3_{(6)} \Gamma^4_{(6)} \Gamma^5_{(6)}. 
\]  \hspace{1cm} (A.14)

The spinor \( \lambda \) in ten-dimensional SYM theory is Majorana–Weyl and therefore satisfies the Weyl condition
\[
\left(\frac{1 + b\Gamma^{11}}{2}\right)\lambda = \left(\frac{1 + b\Gamma^7 \otimes \tau^3 \otimes \tau^3}{2}\right)\lambda_6 \otimes \lambda_4 = 0. 
\]  \hspace{1cm} (A.15)

The Majorana condition, instead, is encoded in the constraint
\[
\xi = B^{-1}\xi^*, 
\]  \hspace{1cm} (A.16)

where \( B \) allows one to connect the ten-dimensional Dirac matrices with their complex conjugates [22]:
\[
B \Gamma^M B^{-1} = \Gamma^{M*},
\]
which implies that \( B B^* = I \). In our ten-dimensional representation, \( B \) is given by
\[
B = c \Gamma^2 \Gamma^4 \Gamma^6 \Gamma^8,
\]
where \( c \) is an arbitrary constant such that \( |c| = 1 \). In this paper we take it to be \( \pm 1 \). In the six-dimensional representation of the Dirac matrices, \( B \) becomes
\[
B = c \Gamma^2_{(6)} \Gamma^4_{(6)} \otimes i\tau_2 \otimes \tau_1 \equiv B_6 \otimes i\tau_2 \otimes \tau_1 
\]  \hspace{1cm} (A.17)

with \( B_6 \) being the operator that allows us to connect the six-dimensional Dirac matrices with their conjugates. It satisfies \( B_6 B_6^* = -I \).

We now have all the ingredients to decompose the ten-dimensional Majorana–Weyl spinor in terms of the six-dimensional components. It is given by the following expression:
\[
\lambda = \Psi \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (B_6 \Psi)^* \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \Lambda^2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c^* a \Lambda^1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} 
\]  \hspace{1cm} (A.18)

where \( a = \pm 1 \) to be fixed later. It satisfies the Weyl condition in equation (A.15). if \( \Psi \) and \( \Lambda^i \) satisfy the two conditions,
\[
(1 + b\Gamma^7) \Psi = (1 - b\Gamma^7) \Lambda^i = 0. 
\]  \hspace{1cm} (A.19)

It satisfies the Majorana condition \( \lambda^* = B \lambda \) if the six-dimensional spinor \( \Lambda^i \) satisfies the relation
\[
B_6 \Lambda^i = -ace_{ij} (\Lambda^j)^* \quad \epsilon_{12} = -1. 
\]  \hspace{1cm} (A.20)
The spinor $\lambda$ is then given by
\[
\lambda = \tilde{\Psi} \otimes (1, 0) \otimes (1, 0) - (B_0 \Psi)^T \Gamma_0 \otimes (0, 1) \otimes (0, 1)
+ \tilde{\Lambda}_2 \otimes (1, 0) \otimes (0, 1) + c a \tilde{\Lambda}_1 \otimes (0, 1) \otimes (1, 0),
\]
where, in analogy with two-dimensional spinors (see equation (C.14)), we are following the convention $(\lambda^i)^* = \lambda_i^*$ and lowering and rising indices through the antisymmetric tensor $\epsilon$. The same rule is adopted for the field $\Psi^a$, in detail $(\Psi^a)^* = \Psi^*_a$, which determines
\[
\tilde{\Psi}^a = \left[ \begin{array}{c} 0 \\ -\tilde{\Psi} \end{array} \right].
\]
The ten-dimensional spinors $\lambda$ and $\tilde{\lambda}$ will be used in appendix E for the uplift from six to ten dimensions.

Appendix B. Magnetized D branes

In equation (8) of section 2, we have given the mass spectrum of the open strings attached to two magnetized D branes in terms of the number operators. It is valid in the entire interval $-\frac{1}{2} \leq \nu_r \leq \frac{1}{2}$, but the explicit expression of the number operators is in general different for positive and negative values of $\nu_r$ ($r = 1, 2, 3$). In section 2, we have given their form when $0 \leq \nu_r \leq \frac{1}{2}$. Here we extend it to the case $-\frac{1}{2} \leq \nu_r \leq 0$.

The interval $-\frac{1}{2} \leq \nu_r \leq \frac{1}{2}$ is the natural range for the fermions in the NS sector, and therefore, in this case, the number operator given in equation (4) for $x = 1$ is valid in the entire interval $-\frac{1}{2} \leq \nu_r \leq \frac{1}{2}$. This range is also natural according to the first equation in (5). For the bosonic coordinate and for the R sector we have to change the number operators in equations (3) and (4) for $x = 0$ as follows:
\[
N_r^2 = \sum_{n=0}^{\infty} \left[ (n + 1 + \nu_r) a_n^{(r)} a_{n+\nu_r}^{(r)} + (n - \nu_r) \tilde{a}_{n-\nu_r} a_{n-r}^{(r)} \right] \tag{B.1}
\]
and
\[
N_r^\Psi = \sum_{n=0}^{\infty} \left[ (n + 1 + \nu_r) \Psi_{n+\nu_r}^{(r)} \Psi_{n+\nu_r}^{(r)} + (n - \nu_r) \tilde{\Psi}_{n-\nu_r} \tilde{\Psi}_{n-\nu_r} \right]. \tag{B.2}
\]
We see that, in going from the interval $0 \leq \nu_r \leq \frac{1}{2}$ to the interval $-\frac{1}{2} \leq \nu_r \leq 0$, we exchange the role of the oscillators $a^{(r)}$ and $\tilde{a}^{(r)}$ and $\Psi^{(r)}$ and $\tilde{\Psi}^{(r)}$ in the R sector. This can be seen more directly by writing equations (3) and (B.1) as follows:
\[
N_r^2 = \sum_{n=0}^{\infty} \left[ (n + |\nu_r|) a_{n+|\nu_r|}^{(r)} a_{n+|\nu_r|}^{(r)} + (n + 1 - |\nu_r|) \tilde{a}_{n+1-|\nu_r|}^{(r)} \tilde{a}_{n+1-|\nu_r|}^{(r)} \right] \tag{B.3}
\]
and
\[
N_r^2 = \sum_{n=0}^{\infty} \left[ (n + 1 - |\nu_r|) a_{n+1-|\nu_r|}^{(r)} a_{n+1-|\nu_r|}^{(r)} + (n + |\nu_r|) \tilde{a}_{n+|\nu_r|} \tilde{a}_{n+|\nu_r|} \right]. \tag{B.4}
\]
and analogously in the R sector. In the NS sector, as well as for the four-dimensional non-compact directions, instead nothing changes.

In the second part of this appendix, we give the number operators for the second and third torus for the open strings D5/D9. They are given by
\[
N_{r=2,3}^2 = \sum_{n=0}^{\infty} \left[ (n + 1 - |\nu_r^{(g)}|) a_{n+1-|\nu_r^{(g)}|}^{(r)} a_{n+1-|\nu_r^{(g)}|}^{(r)} + (n + 1 + |\nu_r^{(g)}|) \tilde{a}_{n+1+|\nu_r^{(g)}|}^{(r)} \tilde{a}_{n+1+|\nu_r^{(g)}|}^{(r)} \right]. \tag{B.5}
\]
for the bosonic coordinate,
\[
(N^\Psi_{r=2,3})_R = \sum_{n=0}^{\infty} \left[ \left( n + \frac{1}{2} - |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}|-|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}|-|v_r^{(9)}|} \right]
\]
\[
\left. + \left( n + \frac{1}{2} + |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right] \quad \text{(B.6)}
\]
for the fermionic coordinate in the Ramond sector and
\[
(N^\Psi_{r=2,3})_{NS} = \sum_{n=\frac{1}{2}}^{\infty} \left[ \left( n + \frac{1}{2} - |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}|-|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}|-|v_r^{(9)}|} \right]
\]
\[
\left. + \left( n + \frac{1}{2} - |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right] \quad \text{(B.7)}
\]
for the fermionic coordinate in the NS sector.

The previous expressions (B.5), (B.6) and (B.7) are valid if \( 0 \leq v_{2,3}^{(9)} < \frac{1}{2} \). If instead \(-\frac{1}{2} < v_{2,3}^{(9)} \leq 0 \), we have the following expressions:

\[
N^Z_{r=2,3} = \sum_{n=0}^{\infty} \left[ \left( n + \frac{1}{2} + |v_r^{(9)}| \right) a^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} a^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right] + \left( n + \frac{1}{2} + |v_r^{(9)}| \right) a^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} a^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|}
\]
and for the Ramond sector
\[
(N^\Psi_{r=2,3})_R = \sum_{n=0}^{\infty} \left[ \left( n + \frac{1}{2} + |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right]
\]
\[
\left. + \left( n + \frac{1}{2} + |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right]
\]
while for the NS sector we have
\[
(N^\Psi_{r=2,3})_{NS} = \sum_{n=\frac{1}{2}}^{\infty} \left[ \left( n + \frac{1}{2} + |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right]
\]
\[
\left. + \left( n + \frac{1}{2} + |v_r^{(9)}| \right) \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \Psi^{(r)}_{n+\frac{1}{2}+|v_r^{(9)}|} \right].
\]

**Appendix C. Uplift from four to six dimensions for the strings**

We start from the Lagrangian of \( N = 4 \) SYM in four dimensions given in the \( N = 1 \) superfield formalism by
\[
L_4 = 2 \text{Tr} \left[ \int d^2 \theta \, d^2 \bar{\theta} \sum_{i=1}^{3} \Phi_i \, e^{2g} \Phi_i + i \sqrt{2g} \left( \int d^2 \bar{\theta} \, \Phi_1 [\Phi_2, \Phi_3] + \int d^2 \bar{\theta} \, \Phi_1 [\bar{\Phi_2}, \bar{\Phi_3}] \right) \right]
\]
\[
+ 2 \text{Tr} \left( \frac{1}{4} \int d^2 \bar{\theta} \, W^a W_a + \int d^2 \bar{\theta} \, W_a W^a \right),
\]
where \( V \) is the vector superfield in the Wess–Zumino gauge:
\[
V(x, \theta, \bar{\theta}) = -\theta \sigma^\mu \bar{\theta} A_\mu + i \theta^2 \bar{\theta} \lambda - i \theta^2 \bar{\theta} \lambda + \frac{1}{2} \theta^2 \bar{\theta} D ,
\]
\( W_a \) is its superfield strength:
\[
W_a(x, \theta) = -i \lambda_a + \bar{\delta}_a^\mu D - i (\sigma^\mu)^a_{\nu} F_{\nu} \theta^\mu + \theta^2 \sigma_{\nu a} D_\nu \bar{\lambda}^a
\]

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where we have introduced a triplet of auxiliary fields
\( \Phi_i(x, \theta) = A_i(x) + \sqrt{2} \theta^a \psi_{iA}(x) + \theta^2 F_i(x) \).  

Any of the previous fields that we denote collectively by \( \phi \) is a matrix:
\[ \phi \equiv \phi^A (T^A)_B^{\alpha B} \quad \text{Tr}(T^A T^B) = \frac{1}{2} \delta_{AB} \quad A, B = 1, \ldots, N^2, \]
where \( T^A \) are the matrices of \( U(N) \) in the fundamental representation.

In terms of the component fields the previous Lagrangian is equal to
\[ L_4 = 2 \text{Tr} \left[ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - i \bar{\psi} \sigma^\mu D_\mu \phi + \frac{1}{2} \sum_{i=1}^{3} \left( (D_{c+i})^2 + i g D_{c+i} \eta_{cma} [A^m, A^a] \right) \right. \]
\[ + \sum_{i=1}^{3} \left[ (D^a A_i)^I (D^a A_i) - i \bar{\psi} \sigma^\mu D_\mu \psi + \bar{\Phi}_I F_2 + \bar{\Phi}_3 F_3 - g D_s [A_1, \bar{A}_1] \right. \]
\[ + i \sqrt{2} g (F_2 [A_3, A_1] + F_3 [A_1, A_2] + F_2 [\bar{A}_3, \bar{A}_1] + F_3 [\bar{A}_1, \bar{A}_2]) \]
\[ + i \sqrt{2} g (\psi_I [\psi_{3a}, A_2] + \psi_I [A_3, \psi_{2a}] + \psi_I [\psi_{2a}, A_1]) \]
\[ + \bar{\psi}_I [\bar{\psi}_I, \bar{A}_2] + \bar{\psi}_I [\bar{A}_3, \psi_I^*] + \bar{\psi}_I [\bar{A}_1, \psi_I^*]) \]
\[ + \sqrt{2} i g \sum_{i=1}^{3} \left( \psi_I [\lambda_a, \bar{A}_1] + \bar{\psi}_I [\lambda_a, A_1] \right) \lbrack \right], \]
where we have introduced a triplet of auxiliary fields \( D_{c+i} \) with \( c = 1, 2, 3 \):
\[ D_8 \equiv -D \quad i F_1 = \frac{D_6 - i D_7}{\sqrt{2}}. \]

the real fields \( A_m \) \((m = 6, 7, 8, 9)\) given by
\[ A_2 \equiv \frac{A_6 + i A_7}{\sqrt{2}} \quad A_3 \equiv \frac{A_8 + i A_9}{\sqrt{2}} \]
and the ’t Hooft symbols
\[ \eta_{cma} = \epsilon_{cma} \eta_9 + \delta_{cm} \delta_9 a \quad \bar{\eta}_{cma} = \epsilon_{cma} \eta_9 - \delta_{cm} \delta_9 a, \]
where \( m, n = 6, 7, 8, 9, c = 6, 7, 8 \) and \( \epsilon_{6789} = 1 \).

In the following, we split the previous Lagrangian in a part \( L_1 \), corresponding to \( \mathcal{N} = 2 \) SYM, and in a part \( L_2 \), corresponding to its interaction with an hypermultiplet in the adjoint, we write them in a formalism where the \( SU(2) \) R-invariance is manifest and finally we uplift them to six dimensions.

The Lagrangian corresponding to \( \mathcal{N} = 2 \) SYM is given by
\[ L_1 = 2 \text{Tr} \left[ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - i \bar{\psi} \sigma^\mu D_\mu \phi + \frac{1}{2} \sum_{i=1}^{3} \left( (D_{c+i})^2 + i g D_{c+i} \eta_{cma} [A^m, A^a] \right) \right. \]
\[ + \sum_{i=1}^{3} \left[ (D^a A_i)^I (D^a A_i) - i \bar{\psi} \sigma^\mu D_\mu \psi + \bar{\Phi}_I F_2 + \bar{\Phi}_3 F_3 - g D_s [A_1, \bar{A}_1] \right. \]
\[ + i \sqrt{2} g (F_2 [A_3, A_1] + F_3 [A_1, A_2] + F_2 [\bar{A}_3, \bar{A}_1] + F_3 [\bar{A}_1, \bar{A}_2]) \]
\[ + i \sqrt{2} g (\psi_I [\psi_{3a}, A_2] + \psi_I [A_3, \psi_{2a}] + \psi_I [\psi_{2a}, A_1]) \]
\[ + \bar{\psi}_I [\bar{\psi}_I, \bar{A}_2] + \bar{\psi}_I [\bar{A}_3, \psi_I^*] + \bar{\psi}_I [\bar{A}_1, \psi_I^*]) \]
\[ + \sqrt{2} i g \sum_{i=1}^{3} \left( \psi_I [\lambda_a, \bar{A}_1] + \bar{\psi}_I [\lambda_a, A_1] \right) \lbrack \right], \]
where we have redefined one of the three auxiliary fields:
\[ D_{c+i} = D_{c+i} \quad \text{for} \quad c = 1, 2; \quad D_8 = D_8 - g [\phi, \phi^*]. \]
and we have introduced the following fields \( A_4 \) and \( A_5 \):
\[ A_4 \equiv a \frac{A_4 - i b A_5}{\sqrt{2}} \quad A_5 \equiv a \frac{A_4 + i b A_5}{\sqrt{2}}. \]
where we have written the complex scalar field $A_1$ belonging to the first chiral multiplet in terms of the two real scalar fields $A_4$ and $A_3$ that in the uplift from four to six dimensions will provide the extra two components of the six-dimensional gauge field. The presence of the phases $a = \pm 1$ and $b = \pm 1$ will become clear later on.

The Lagrangian corresponding to the interaction with a hypermultiplet is given by

$$L_2 = 2\text{Tr} \left[ \frac{1}{2} g \sum_{i=1}^{3} D_{\mu} \eta_{\mu\nu} [M^m, A^n] + \sum_{i=2}^{3} \left( -(D^\mu A_i)^{(D_\mu A_i)} - i\bar{\psi}_i \gamma^\mu D_\mu \psi_i + \tilde{F} F_i \right) - \frac{b}{2} g^2 [A_4, A_5] \eta_{mn} [M^m, A^n] 
+ i\sqrt{2} g (F_2 [A_3, A_1] + F_3 [A_1, A_2] + \bar{F}_2 [\bar{A}_3, \bar{A}_1] + \bar{F}_3 [\bar{A}_1, \bar{A}_2]) 
+ \bar{\psi}_{1a} [\bar{\psi}_1^a, \bar{A}_2] + \bar{\psi}_{1a} [\bar{A}_3, \bar{\psi}_2^a] + \bar{\psi}_{2a} [\bar{\psi}_1^a, A_1] 
- \sqrt{2} i g \sum_{i=2}^{3} (\bar{\psi}_i^a [\lambda_{ai}, \bar{A}_1] + \bar{\psi}_{1a} [\bar{\lambda}^a, A_1]) \right].$$

Let us now write the two Lagrangians in a way that the convention:

$$\begin{align*}
A_{1, 2}^\mu \equiv \left( -\bar{\psi}_{1a}^\dagger, \bar{A}_2 \right), & \quad \lambda_{ai} \equiv \left( -\bar{\lambda}_{ai}^\dagger, \bar{A}_1 \right), \\
\bar{A}_{1, 2}^\mu \equiv \left( \bar{\psi}_{1a}^\dagger, A_2 \right), & \quad \bar{\lambda}_{ai} \equiv \left( \bar{\lambda}_{ai}^\dagger, A_1 \right).
\end{align*}$$

In order to show the consistency of the previous equations we have to use the following convention:

$$\begin{align*}
\lambda^{ia} & \equiv \bar{\lambda}_{ai} = \epsilon_{ij} \bar{\lambda}_{aj}, & \bar{\lambda}^{ia} & \equiv \bar{\lambda}_{ai} = \epsilon_{ij} \bar{\lambda}_{aj},
\end{align*}$$

where

$$\begin{align*}
\epsilon^{ij} & = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & \epsilon_{ij} & = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).
\end{align*}$$

It is easy to see that, using the first equation in (14.15), one can show that the last two equations in (14.14) are consistent. The consistency of the first two equations in (14.14) can be shown using the last equation in (14.15).

Using the previous formulas we can rewrite equation (10.10) as follows:

$$L_1 = 2\text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sum_{i=4}^{5} (D^\mu A_i)^{(D_\mu A_i)} + \frac{g^2}{2} [A_4, A_5]^2 
- i\bar{\lambda}_{ai} \gamma^\mu D_\mu \lambda^{ia} + \frac{1}{2} \sum_{\epsilon=1}^{3} D_{\epsilon^5}^2 + i\sqrt{2} g \frac{1}{2} (\lambda_{ai} [\lambda_{ai}, \bar{A}_1] + \bar{\lambda}_{ai} [\bar{\lambda}^a, A_1]) \right],$$

where now the invariance under $SU(2)$ is manifest.

Introducing the following $SU(2)$ doublets:

$$Z_i \equiv (Z^i)^\dagger = (A_3, \bar{A}_2)$$

together with

$$Z_i \equiv (\epsilon_{i2} Z^2 = -A_2, \epsilon_{i1} Z^1 = \bar{A}_3) \quad \tilde{Z}_i \equiv \left( \begin{array}{c} \epsilon_{i2} \bar{Z}_2 = \bar{A}_2 \\ \epsilon_{i1} \bar{Z}_1 = -A_3 \end{array} \right)$$

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and eliminating the auxiliary fields $F_2$ and $F_3$, we can write equation (C.13) as follows:

$$L_2 = 2 \text{Tr} \left[ - (D^\mu Z^j)(D_\mu Z^j) - i \bar{\psi}_3 \delta^{\mu \nu} D_\mu \psi_2 - i \bar{\psi}_3 \delta^{\mu \nu} D_\mu \psi_3 
+ g \bar{Z}_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} Z^j - g Z_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} \bar{Z}^j 
+ g^2 \left( [A_4, \bar{Z}_i][A_4, Z^j] + [A_5, \bar{Z}_i][A_5, Z^j] \right) 
+ i \sqrt{2} g \left( \lambda \alpha [\bar{\psi}_3, \bar{Z}_j] \right) \epsilon_{ij} - \bar{\lambda} \alpha \left[ \bar{\psi}_3, Z_j \right] \epsilon_{ij} 
- \bar{\lambda} \alpha \lbrack \bar{\psi}_2, \bar{Z}_j \rbrack \epsilon_{ij} \right],$$

(C.20)

which is manifestly $SU(2)$ invariant. In deriving the previous equation we have used the following identity:

$$\text{Tr} \left[ \frac{1}{2} g \sum_{c=1}^3 D_{c5} \eta_{c5} \alpha \beta \left[ A^\alpha, A^\beta \right] \right] = \text{Tr} \left[ g \bar{Z}_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} Z^j - g Z_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} \bar{Z}^j \right],$$

that follows from equations (C.8) and (C.18). The next step is to write both equations (C.17) and (C.20) using four-dimensional Dirac fermions:

$$\xi \equiv \left( \begin{array}{c} \psi_{2a} \\
\psi_3 \end{array} \right); \quad \bar{\xi} \equiv \left( \begin{array}{c} \bar{\psi}_3 \\
\bar{\psi}_{2a} \end{array} \right); \quad \eta \equiv \left( \begin{array}{c} \bar{\lambda} \alpha \\
\bar{\lambda} \alpha \end{array} \right); \quad \bar{\eta} \equiv \left( \begin{array}{c} -\bar{\lambda} \alpha \\
\bar{\lambda} \alpha \end{array} \right).$$

(C.21)

We obtain

$$L_1 = 2 \text{Tr} \left[ - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \sum_{c=4}^5 (D_\mu A_c) (D_\mu A_c) + \frac{g^2}{2} [A_4, A_5]^2 
+ \frac{1}{2} \sum_{c=3}^5 D_{c5}^2 - \frac{i}{2} \bar{\eta} \gamma^\mu D_\mu \eta - a \gamma 5 g [A_4, \eta] - a b g [A_5, \eta] \right],$$

(C.22)

and

$$L_2 = 2 \text{Tr} \left[ + g^2 \left( [A_4, \bar{Z}_i][A_4, Z^j] + [A_5, \bar{Z}_i][A_5, Z^j] \right) + g \bar{Z}_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} Z^j 
- g Z_i \sum_{c=1}^3 (D_{c5} \bar{\tau}^c)_{ij} \bar{Z}^j - i \bar{\xi} \left( \gamma^\mu D_\mu \xi + g a b [A_5, \xi] + \gamma 5 g a [A_4, \xi] \right) 
- \sqrt{2} g \left( \bar{\xi} \bar{\xi} \eta \gamma^j \gamma^j + \bar{\eta} \gamma 5 \eta \gamma^j \gamma^j \right) \right].$$

(C.23)

We are now ready to uplift the two previous Lagrangians to six dimensions by introducing the two six-dimensional chiral spinors:

$$(1 - b \gamma_5) \Lambda^i = 0; \quad \Lambda^i = \left( \begin{array}{c} 1 + b \gamma 5 \\
2 - b \gamma 5 \eta \end{array} \right); \quad \bar{\Lambda}_i = \left( \begin{array}{c} \bar{\eta}_i \frac{1 - b \gamma 5}{2} + a \bar{\eta}_i \frac{1 + b \gamma 5}{2} \end{array} \right)$$

(C.24)

and

$$(1 - b \gamma_5) \Psi = 0; \quad \Psi = \left( \begin{array}{c} 1 + b \psi \gamma 5 \\
2 - b \psi \gamma 5 \xi \end{array} \right); \quad \bar{\Psi} = \left( \begin{array}{c} \xi \frac{1 - b \psi \gamma 5}{2} + a \xi \frac{1 + b \psi \gamma 5}{2} \end{array} \right).$$

(C.25)
Appendix D. Uplift from four to six dimensions for the strings 59

This means that the two chiral spinors in six dimensions have opposite chirality.

Equation (C.22) becomes

\[ L_{\mathcal{N}=2}^{\text{SYM}} = 2 \text{Tr} \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \sum_{c=3}^{8} D^2_{c \times 5} - \frac{i}{2} \bar{\Lambda}_c \Gamma^\mu D_\mu \Lambda^c \right] \]  

(C.26)

while equation (C.23) becomes

\[ L_2 = 2 \text{Tr} \left[ -(D_\mu Z^J)^{a(J)} (D_\mu Z^J) + g \bar{Z} \sum_{c=1}^{3} (D_{c \times 5} t^c)^J Z^J - g \bar{Z} \sum_{c=1}^{3} (D_{c \times 5} t^c)^J \bar{Z}^J \right. 

\[ \left. -i \bar{\psi} \Gamma^\mu \psi + i \sqrt{2} g b (\bar{\psi} \Gamma^J Z^J - \bar{\Lambda}_c [\bar{\psi}, \bar{Z}^J] \Gamma^J \right] \]  

(C.27)

where \( \bar{\mu} = (\mu, 4, 5) \). The previous equations have been obtained by imposing that

\[ a_{\psi} = -a; \quad b_{\psi} = -b. \]  

(C.28)

This means that the two chiral spinors in six dimensions have opposite chirality.

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An \( \mathcal{N} = 2 \) hypermultiplet consists of two \( \mathcal{N} = 1 \) chiral superfields \( R_1 \) and \( R_2 \) coupled to the gauge superfield \( V \) and with a very special superpotential. Its Lagrangian is given by

\[ L_{\text{hyper}} = \int d^2 \theta d^2 \bar{\theta} \left[ R_{1a}^* (\bar{\phi}^2 \psi)^a_{\bar{B}} R_{1a}^b + R_{2a} (\bar{\phi}^2 \psi)^a_{\bar{B}} R_{2a}^{ab} \right] 

\[ + \sqrt{2} g \left[ \int d^2 \theta R_{2a} (\phi^* \psi)^a_{\bar{B}} + \int d^2 \bar{\theta} R_{1a}^* (\phi^* \psi)^a_{\bar{B}} \right]. \]  

(D.1)

where

\[ R_{i} = z_{i} + \sqrt{2} \theta \psi_{i} + \theta^2 G_{i}; \quad \Phi = \phi + \sqrt{2} \psi + \theta^2 F. \]  

(D.2)

Here * stands for a complex conjugation and the superfield \( \Phi \) has to be identified with \( \Phi_1 \) of the previous section. In terms of the component fields the first two terms of equation (D.1) are equal to

\[ \int d^2 \theta d^2 \bar{\theta} \left[ R_{1a}^* (\bar{\phi}^2 \psi)^a_{\bar{B}} R_{1a}^b + R_{2a} (\bar{\phi}^2 \psi)^a_{\bar{B}} R_{2a}^{ab} \right] 

\[ = \left[ -(D_\mu z_{1})^a (D_\mu z_{1})^a + G_{1a}^* G_{1a}^* - \psi_{1a} \bar{\psi}_{1a}^a (D_\mu \psi_{1})^b \right] 

\[ + \left[ -(D_\mu z_{2})^a (D_\mu z_{2})^a + G_{2a}^* G_{2a}^* - \psi_{2a} \bar{\psi}_{2a}^a (D_\mu \psi_{2})^b \right] 

\[ + g \left( z_{1a}^* D_{\mu z_{1}}^a - z_{2a} D_{\mu z_{2}}^a \right) + i \sqrt{2} g \left( \bar{\psi}_{1a} \lambda_{a}^\dagger \psi_{1}^b + z_{2a} (\bar{\lambda}_{b}^a \psi_{2}^b - \bar{\psi}_{2a} \lambda_{b}^a \psi_{2}^b) \right. \]  

\[ \left. - \psi_{1a} \bar{\lambda}_{a}^b \psi_{1}^b - \psi_{2a} (\bar{\lambda}_{b}^a \psi_{2}^b) \right]. \]  

(D.3)

where the covariant derivatives are given by

\[ (D_\mu z_{1})^a = \partial_\mu z_{1}^a + ig(A_\mu)^a_{\bar{B}} (A_\mu)^{\bar{B}}_{b} z_{1}^b; \quad (D_\mu z_{2})^a = \partial_\mu z_{2}^a + ig(A_\mu)^a_{\bar{B}} (A_\mu)^{\bar{B}}_{b} z_{2}^b \]  

(D.4)

\[ (D_\mu z_{1})^a = ig(A_\mu)^a_{\bar{B}} (A_\mu)^{\bar{B}}_{b} (z_{1}^a); \quad (D_\mu z_{2})^a = ig(A_\mu)^a_{\bar{B}} (A_\mu)^{\bar{B}}_{b} (z_{2}^a) \]  

(D.5)

\[ (D_\mu z_{2})^a = \partial_\mu z_{2}^a - ig(A_\mu)^a_{\bar{B}} (A_\mu)^{\bar{B}}_{b} (z_{2}^a) \]  

(D.6)

\[ D_\mu \psi_{1}^a = \partial_\mu \psi_{1}^a + ig(A_\mu)^a_{\bar{B}} \psi_{1}^b; \quad D_\mu \psi_{2a} = \partial_\mu \psi_{2b} - ig(A_\mu)^a_{\bar{B}} \psi_{2b}, \]  

with

\[ (A_\mu)^a_{\bar{B}} \equiv (T^A)^a_{\bar{B}} A_\mu^A; \quad (\phi)^a_{\bar{B}} \equiv (T^A)^a_{\bar{B}} \phi^A; \quad (\phi^*)^a_{\bar{B}} \equiv (T^A)^a_{\bar{B}} \phi^A. \]  

(D.8)
We can now write the superpotential in terms of the component fields. We obtain
\[
\sqrt{2} g \left[ \int d^2 \theta R_{a b} (\Phi^\alpha b) R^a_2 + \int d^2 \tilde{\theta} R^*_{a b} (\Phi^\alpha b)^* R^a_2 \right] = \sqrt{2} g \left[ z_{2 a} F_{b c} + z_{1 a} (F^*)_b \right] + G_{a b} \phi^a \zeta^b + \tilde{G}_{a b} \tilde{\psi}^a \tilde{\zeta}^b
\]
\[
+ G_{a b} \phi^a \tilde{\psi}^b + \psi_{2 a} \phi^a \psi_1^b - \psi_{1 a} (\psi^a \psi^b) - \psi_{2 a} (\psi^a \tilde{\psi}^b) - \psi_{1 a} (\tilde{\psi}^a \tilde{\psi}^b) - \psi_{2 a} (\tilde{\psi}^a \psi_1^b) - \psi_{1 a} (\tilde{\psi}^a \tilde{\psi}^b).
\]
(D.9)

We want to rewrite the previous Lagrangian in a way where the SU(2) R-symmetry is manifest. We introduce the following SU(2) doublets [29]:
\[
\tilde{w}_1 \equiv (\tilde{w}_1 \tilde{w}_2) = (-i z_2 \ z_1^*); \quad w' \equiv \left( \frac{w^1}{w^2} \right) = -\left( \frac{iz_2}{z_1} \right).
\]
The index \( i \) is raised or lowered by the antisymmetric tensor \( \epsilon \) given in equation (C.16). The complex variables \( w, \tilde{w} \) are not independent, but satisfy the relation
\[
\epsilon^{ij} \tilde{w}_j = (w_i)^*.
\]
(D.11)
The previous relations imply
\[
\tilde{w}^2 = -\tilde{w}_1 = (w_3)^* = (w_1)^*; \quad -w^1 = \tilde{w}_2 = (w_1)^* = -(w_2)^*.
\]
(D.12)

In terms of the previous doublets for the scalar fields and of those connected with the gauginos given in equation (C.14), the total Lagrangian, that is the sum of equations (D.3) and (D.9), can be written in the following equivalent forms:
\[
L_{\text{hyper}} = (D_\mu \tilde{w})_a (D^\mu w)^a - g \tilde{w}_a \sum_{c=1}^3 (\tau^c)_j (D_{c+5} w^a)^j
\]
\[
- i \tilde{w}_a [\tilde{\sigma}^a (D_\mu \psi_1) + \tilde{\sigma}^a (D_\mu \psi_2)] - \sqrt{2} g \left( \psi_{2 a} \phi^a \psi_1^b + \tilde{\psi}_{1 a} (\phi^a \psi_1^b) \right)
\]
\[
+ \sqrt{2} g \left[ i \tilde{w}_a \lambda_1^* \psi_1 - \tilde{w}_a \lambda_1^* \psi_2 - \psi_1^2 \lambda_1^* w' + i \tilde{\psi}_{1 a} \lambda_1^* w' \right] - \bar{G}_1 G^1
\]
\[
- \sqrt{2} g \left( \bar{G} \psi_1^a \psi_1^b + \bar{\psi}_{1 a} (\phi^b \psi_1^a) \right) w'^a b + \tilde{w}_a \left( \psi_1^a \psi_1^b \right)^a b^i \bar{G}_1 G^1.
\]
(D.13)
The equations of motion for the auxiliary fields \( G \) and \( \bar{G} \) are
\[
G^a + \sqrt{2} g \left( \begin{array}{cc} (\phi^b)^a_j & 0 \\ 0 & (\phi^b)^a_j \end{array} \right) w'^a b = 0
\]
\[
\bar{G}^a + \sqrt{2} g \left( \begin{array}{cc} (\phi^b)^a_i & 0 \\ 0 & (\phi^b)^a_i \end{array} \right) = 0.
\]
(D.14)

Inserting them in equation (D.13) we obtain
\[
L_{\text{hyper}} = \epsilon^{ij} (D_\mu \tilde{w})_a (D_\mu w)^a - g \tilde{w}_a \sum_{c=1}^3 (\tau^c)_j (D_{c+5} w^a)^j
\]
\[
- i \tilde{w}_a [\tilde{\sigma}^a (D_\mu \psi_1) + \tilde{\sigma}^a (D_\mu \psi_2)] - \sqrt{2} g \left( \psi_{2 a} \phi^a \psi_1^b + \tilde{\psi}_{1 a} (\phi^a \psi_1^b) \right)
\]
\[
+ \sqrt{2} g \left[ i \tilde{w}_a \lambda_1^* \psi_1 - \tilde{w}_a \lambda_1^* \psi_2 - \psi_1^2 \lambda_1^* w' + i \tilde{\psi}_{1 a} \lambda_1^* w' \right] + g^2 \tilde{w}_a ([\phi, \phi^a]) w'^a b + g^2 \tilde{w}_a (\tau^c)_j ([\phi, \phi^a]) w'^a b.
\]
(D.15)
Using the redefinition of the auxiliary fields given in equation (C.11) we can rewrite the previous Lagrangian as follows:

\[
\mathcal{L}_{\text{hyper}} = (D_\mu \bar{w})_a (D^\mu w)^a - g \bar{w} w a \sum_{c=1}^{3} (\tau^c)^j_j (D_{c+5} w)^{ab} + g^2 \bar{w} w a (\{\phi, \phi^*\})^a \bar{w} w^b,
\]

\[
= -i \bar{\psi}_{1a} (\bar{\sigma}^\mu)^a \phi (D_\mu \psi) - i \bar{\psi}_{2a} (\bar{\sigma}^\mu)^a \phi (D_\mu \psi) - g \bar{\psi}_a b (\phi^a \phi^* b) \]
\[
- \bar{\psi}_a b \left[ (D_\mu \bar{w})_a (\bar{\phi}^a)^a_b + i(D_\mu w)^a_b \right],
\]

where

\[
(D_\mu \bar{w})^a = \partial_\mu \bar{w}^a + ig (A_\mu)^a_b \bar{w}^b;
\]

\[
(D_\mu w)^a = \partial_\mu w^a - ig (A_\mu)^a_b w^b.
\]

Using the notations given in appendix A we can write the previous four-dimensional Lagrangian in terms of Dirac spinors. One obtains

\[
\mathcal{L}_{\text{hyper}} = \epsilon_{ij} (D_\mu \bar{w})_a (D^\mu w)^a - g \bar{w} w a \sum_{c=1}^{3} (\tau^c)^j_j (D_{c+5} w)^{ab} + g^2 \bar{w} w a (\{\phi, \phi^*\})^a \bar{w} w^b
\]

\[
- \bar{\psi}_a b \left[ (D_\mu \bar{w})_a (\bar{\phi}^a)^a_b + i(D_\mu w)^a_b \right],
\]

where in the first line we have written what is in the first and last line of equation (D.15). The Dirac field \( \psi \) is defined in terms of the Weyl fields by

\[
\psi = \begin{pmatrix} \psi_{1a} \\ i \psi_{2a} \end{pmatrix}
\]

and the covariant derivative is given by

\[
(D_\mu \psi)^a = \partial_\mu \psi^a + ig (A_\mu)^a_b \psi^b.
\]

It remains now to rewrite the last two lines of equation (D.18) in six-dimensional notations.

This can be done by introducing the following six-dimensional Weyl spinor:

\[
(1 - b_\mu \Gamma_7) \mu = 0; \quad \mu = \begin{pmatrix} \psi_{1a} \\ i \psi_{2a} \end{pmatrix}; \quad \bar{\mu} = (\bar{\psi}_{1a} 1 + b_\mu \Gamma_7)
\]

where \( \mu \) and \( \bar{\mu} \) are six-dimensional Weyl spinors with opposite chirality.

If the following conditions are satisfied

\[
\begin{align*}
& a_\mu = -a; \\
& b_\mu = -b,
\end{align*}
\]

we can uplift the previous Lagrangian to the following six-dimensional one:

\[
\mathcal{L}_{\text{hyper}} = \epsilon^{ij} (D_\mu \bar{w})_a (D^\mu w)^a - g \bar{w} w a \sum_{c=1}^{3} (\tau^c)^j_j (D_{c+5} w)^{ab} \]

\[
- i \bar{\mu}_{\alpha} b (D_\mu \mu) + \sqrt{2} g b \left[ \bar{\mu}_a (\Lambda^a)^b_b + \bar{\mu}_a \epsilon_{ij} w^b \right],
\]

where \( \mu, \Lambda, \) and \( \Lambda \) are six-dimensional Weyl spinors with opposite chirality.
Appendix E. Uplift from six to ten dimensions for the strings 99

In this appendix we uplift the Lagrangian in equation (42) to ten dimensions.

The uplift of the purely bosonic part is easy if we observe that the term with the double commutator can be written as follows:

\[
\frac{\tilde{g}_0^2}{4} \sum_{c=1}^{3} \eta_{(c+5)mn} \eta_{(c+5)pq} [\hat{A}^{(9)}_m, \hat{A}^{(9)}_n] [\hat{A}^{(9)}_p, \hat{A}^{(9)}_q] = \frac{\tilde{g}_0^2}{4} \sum_{m,n} [\hat{A}_m, \hat{A}_n]^2.
\] (E.1)

which can be uplifted to ten dimensions to become

\[
\frac{\tilde{g}_0^2}{4} \sum_{m,n} [\hat{A}_m, \hat{A}_n]^2 \rightarrow -\frac{1}{4} \hat{F}^{mn} \hat{F}_{mn}.
\] (E.2)

Then it is easy to see that together with the two other purely bosonic terms it gives rise to the Lagrangian of pure Yang–Mills theory in ten dimensions:

\[
\mathcal{L}_{99} = -\frac{1}{4} \hat{F}_{MN} \hat{F}^{MN} + \text{fermions}.
\] (E.3)

In the following we want to uplift the terms with the fermions present in equation (42).

It is easier to start from the ten-dimensional expression that is given by

\[
- i \text{Tr}(\hat{\lambda} \Gamma^M D_M \lambda)
\] (E.4)

and use in it the representation for the ten-dimensional spinors given in equations (A.18) and (A.21). For \( M = 0, \ldots, 5 \), we obtain

\[
- 2 i \text{Tr}(\hat{\Psi} \Gamma^D D_D \Psi + \frac{1}{2} \hat{\lambda} \Gamma^D D_D \lambda'),
\] (E.5)

while for the other four components we obtain respectively

\[
- i \text{Tr}(\hat{\lambda} \Gamma^6 D_6 \lambda) = -2b[c^* a \hat{\Psi} D_6 \lambda - c a \hat{\lambda} D_6 \Psi]
\]
\[
- i \text{Tr}(\hat{\lambda} \Gamma^7 D_7 \lambda) = -2b[i c a \hat{\Psi} D_7 \lambda^1 + c a \hat{\lambda} D_7 \Psi]
\]
\[
- i \text{Tr}(\hat{\lambda} \Gamma^8 D_8 \lambda) = 2bi\hat{\Psi} D_8 \lambda^2 - \hat{\lambda} D_8 \Psi
\]
\[
- i \text{Tr}(\hat{\lambda} \Gamma^9 D_9 \lambda) = -2ib[\hat{\Psi} D_9 \lambda^3 + \hat{\lambda} D_9 \Psi].
\] (E.6)

In deriving the previous equations we have used the following relations:

\[
B_6^T = -B_6; \quad B_6 \Gamma_6^0 = \Gamma_6^0 B_6; \quad (\Gamma_6^0)^T = \Gamma_6^0.
\]
\[
\Gamma_6^0 = \Gamma_6^1; \quad \Gamma_6^1 = -\Gamma_6^0; \quad \Gamma_6^0 | \Gamma_6^0 = \Gamma_6^0 | \Gamma_6^0.
\] (E.7)

The terms in equation (E.5) reproduce the kinetic terms of the fermions in equation (42), while the four terms in the previous equation reproduce the Yukawa terms in the last line of equation (42) provided that we make the following identifications:

\[
c^* a (A_6 + i A_7) = \sqrt{2} A_2; \quad A_8 + i A_9 = \sqrt{2} A_3
\] (E.8)

with \( ac^* = 1 \). \( A_2 \) and \( A_3 \) are connected to the variables \( \tilde{Z}^i \) and \( \tilde{Z}_i \) through equations (C.18).

In conclusion, if we start from the following ten-dimensional Lagrangian:

\[
L_{10} = 2 \text{Tr} \left( -\frac{1}{4} F_{MN} F^{MN} - \frac{i}{2} \hat{\lambda} \Gamma^M D_M \lambda \right).
\] (E.9)

we reproduce the six-dimensional Lagrangian in equation (42).
Appendix F. $\mathcal{N} = 1$ supersymmetry transformations

In this appendix we prove that the following six-dimensional actions:

\begin{equation}
S_g = 2 \int d^6 x \left[ -\frac{1}{4} F_{\hat{\mu}\hat{\nu}}^2 + \frac{1}{2} \sum_{c=1}^3 \bar{D}_c - \frac{i}{2} \bar{\Lambda}_i \Gamma^{\hat{\mu}} D_{\hat{\mu}} \Lambda^i \right]
\end{equation}

for the gauge sector living on the D5 branes, and

\begin{equation}
S_m = \int d^6 x \left[ \epsilon^{ij} (D_{\hat{\mu}} \bar{w}_i)_{\bar{a}} (D_{\hat{\nu}} w_j)^a - i \bar{\mu}_a \Gamma^{\hat{\mu}} (D_{\hat{\mu}} \mu)^a - g \bar{w}_{ia} \sum_{c=1}^3 (\tau^c)_{ij} (\bar{D}_{c+2})_{\bar{a}} w^{ab} + \sqrt{2} g b (\bar{\mu}_a (\Lambda^0)_{\bar{a}} \epsilon_{ij} (w^j)^b + (\bar{w}_i)_{\bar{a}} \epsilon_{ij} (\bar{\Lambda})_a \mu^b) \right]
\end{equation}

for the twisted matter, are invariant under the following $\mathcal{N} = 1$ supersymmetry transformations:

\begin{align*}
\delta \Lambda^0 &= \frac{i}{2} \left( \bar{\epsilon}_i \Gamma^\lambda \Lambda^i - \bar{\Lambda}_i \Gamma^\lambda \epsilon^i \right); \quad \delta \bar{D}^i = \frac{i}{2} (\epsilon^{ij} \bar{D}^j)_{\bar{a}} \\
\delta \bar{\Lambda}^i &= \frac{i}{2} \bar{\Gamma}^{\hat{\mu}} F_{\hat{\mu}\bar{a}} \epsilon^i + i D_{\hat{a}}^i \epsilon^i; \quad \delta \bar{\Lambda}^i = -\bar{\epsilon}_i F_{\hat{\mu}\bar{a}} \Gamma^{\hat{\mu}} - i \epsilon_j D_{\hat{a}}^i \\
\delta w^a &= -\sqrt{2} b \epsilon_{ij} \epsilon^i \mu^a; \quad \delta \bar{w}_{ia} = -\sqrt{2} b \bar{\mu}_a \epsilon_{ij} \\
\delta \mu^a &= -i \sqrt{2} b \Gamma^\lambda \epsilon_{ij} (D_{\hat{\mu}} \bar{w}^j)^a; \quad \delta \bar{\mu}_a = -i \sqrt{2} b (D_{\hat{\mu}} \bar{w}_i)_{\bar{a}} \epsilon^i \epsilon^j \bar{\Gamma}^{\hat{\mu}}
\end{align*}

where $D^i = D^c (\tau^c)^i_j$, and $\Gamma^{\hat{\mu}\hat{\nu}} = \frac{i}{2} [\Gamma^\mu, \Gamma^\nu]$.

Before showing the supersymmetry of the previous actions let us discuss some properties of the parameter of the supersymmetry transformation $\epsilon^i$. It is a chiral fermion with the same chirality of the gaugino. This property follows from the requirement that, if the gaugino is a chiral fermion, also its supersymmetry variation must be a chiral fermion. This means that

\begin{equation}
0 = \delta \left( \frac{1 - b \Gamma_7}{2} \right) \Lambda^i = \frac{1}{4} F_{\hat{\mu}\bar{a}} \Gamma^{\hat{\mu}} \left( \frac{1 - b \Gamma_7}{2} \right) \epsilon^i + i D^i_j \left( \frac{1 - b \Gamma_7}{2} \right) \epsilon^j,
\end{equation}

which implies

\begin{equation}
\left( \frac{1 - b \Gamma_7}{2} \right) \epsilon^i = 0,
\end{equation}

which is the same chirality condition as imposed on the $\Lambda^i$. For the same reason as before it must also satisfy relation (A.20) as the gaugino. We obtain in fact

\begin{equation}
0 = \delta \left( B_6 \Lambda^i + a \epsilon_{ij} (\Lambda^i)^* \right) = \frac{1}{2} F_{\hat{\mu}\bar{a}} (\Gamma^{\hat{\mu}})^* (\epsilon^i)^* (B_6 (\tau_c)^i_j + a \epsilon_{ij} (\Lambda^i)^*) + i D_{\hat{a}}^i \left( B_6 \epsilon^i + a \epsilon_{ij} (\epsilon^j)^* \right),
\end{equation}

which implies

\begin{equation}
B_6 \epsilon^i = -a \epsilon_{ij} (\epsilon^j)^*.
\end{equation}

In deriving equation (F.7) we have used the transformation properties of $\Lambda^*$:

\begin{equation}
\delta (\Lambda^i)^* = \frac{1}{2} F_{\hat{\mu}\bar{a}} (\Gamma^{\hat{\mu}})^* (\epsilon^i)^* - i D_{\hat{a}}^i (\epsilon^i)^*,
\end{equation}

where we have used that $D^i = D$ which implies $(D^i)^j_j = D^i$, the relation

\begin{equation}
B_6 \Gamma^{\hat{\mu}\hat{\nu}} = (\Gamma^{\hat{\mu}\hat{\nu}})^* B_6,
\end{equation}

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which is valid because in our representation of the Dirac matrices $\Gamma^{2,4}$ are purely imaginary, while the other $\Gamma$-matrices are real, and the identity

$$\left(\tau^i\right)_{ij} = -\epsilon_{ijm} \left(\tau^m\right)^i_j.$$  

Let us now start by analyzing the SUSY invariance of the gauge action in equation (F.1). The invariance under the supersymmetry transformations of the action in equation (F.1) without the auxiliary field has been shown in [24]. Here we consider only the additional contribution given by the presence of the auxiliary fields. We obtain the following extra contributions to the variation of the action in equation (F.1):

$$\delta S_g^D = 2 \int d^6x \text{Tr} \left[ \frac{1}{2} \partial_\mu \left( \bar{A}_i \Gamma^\mu D^i \epsilon^j \right) - \frac{1}{2} \left( D_\mu \bar{A}_i \Gamma^\mu \epsilon^j + \bar{\epsilon}_j \Gamma^\mu D_\mu \Lambda^j \right) D^j + D^c \delta D^c \right]$$

$$= 2 \int d^6x \text{Tr} \left[ \frac{1}{2} \partial_\mu \left( \bar{A}_i \Gamma^\mu D^i \epsilon^j \right) \right],$$

which is a total derivative. This implies that the action in equation (F.1) is invariant under the supersymmetry transformations in equation (F.3). In the last step in equation (F.12) we have used the supersymmetry transformation of the auxiliary field in equation (F.3).

Let us consider now the matter sector that is described by the following Lagrangian:

$$\mathcal{L}_{59} = \epsilon^{ij} (D_\mu \bar{w}_i)_{a} (D^{\mu} w_j)^a - i \bar{\mu}_a \Gamma^{\mu} (D_\mu \mu)^a - g w_{ia} \sum_{c=1}^{3} (\tau^c)^i_j (\bar{D}_{c+2})^a_b w^{jb}$$

$$+ \sqrt{2} g b \left[ \bar{\mu}_a (\bar{A}^{j} \gamma^{a} \epsilon_{ij} (w^{j})^{b} + (\bar{w}_{i})_{a} \epsilon^{ij} (\bar{A}^{j})^{a}_{b} \mu^{b} \right].$$

Let us start evaluating the variation of the various terms present in the previous Lagrangian. The variation of the kinetic term for the scalars is given by

$$\delta \left[ \epsilon^{ij} (D_\mu \bar{w}_i)_{a} (D^{\mu} w_j)^a \right] = -ig \epsilon^{ij} \bar{w}_{ia} \delta (A^{\mu})^a_b (D^{\mu} w_j)^b + \epsilon^{ij} (D_\mu \delta \bar{w}_i)_{a} (D^{\mu} w_j)^a$$

$$+ i g \epsilon^{ij} (D_\mu \bar{w}_i)_{a} \delta (A^{\mu})^a_b w_{jb} + \epsilon^{ij} (D_\mu \bar{w}_i)_{a} (D^{\mu} w_j)^a$$

$$= \frac{g}{2} \bar{w}_{ia} \left( \epsilon_j \gamma^{\mu} A^j - \bar{A}_j \gamma^{\mu} \epsilon^j \right)_a^b (D_\mu w^a)^b$$

$$- \frac{g}{2} (D_\mu \bar{w}_i)_{a} \left( \epsilon_j \gamma^{\mu} A^j - \bar{A}_j \gamma^{\mu} \epsilon^j \right)_a^b w^{jb}$$

$$+ \sqrt{2} b (D_\mu \bar{\mu}_a) \epsilon^j \epsilon_{ij} (D^{\mu} w^a) - \sqrt{2} b (D_\mu \bar{\mu}_a) \epsilon^{ij} \epsilon_{ij} (D^{\mu} \mu)^a$$

while the variation of the kinetic term for the fermions is equal to

$$\delta \left[ -i \bar{\mu}_a \Gamma^{\mu} (D_\mu \mu)^a \right] = -i \delta \bar{\mu}_a \Gamma^{\mu} (D_\mu \mu)^a + g \bar{\mu}_a \Gamma^{\mu} \delta (A^{\mu})^a_b \mu^b - i \bar{\mu}_a \Gamma^{\mu} (D_\mu \delta \mu)^a$$

$$= -\sqrt{2} b (D_\mu \bar{w}_i)_{a} \epsilon^j \epsilon_{ij} \left( \frac{1}{2} \left[ \Gamma^{\mu}, \mu^a \right] + \frac{1}{2} \left[ \mu^a, \Gamma^{\mu} \right] \right) (D_\mu \mu)^a$$

$$- \sqrt{2} b \bar{\mu}_a \left( \frac{1}{2} \left[ \Gamma^{\mu}, \mu^a \right] + \frac{1}{2} \left[ \mu^a, \Gamma^{\mu} \right] \right) \epsilon^j \epsilon_{ij} (D_\mu \bar{w}_i)_{a} (D^{\mu} \mu)^a$$

$$+ \frac{g}{2} \bar{\mu}_a \Gamma_{\mu} \left( \epsilon^j \gamma^{\mu} A^j - \bar{A}_j \gamma^{\mu} \epsilon^j \right)_a^b \mu^b$$

$$= \sqrt{2} b (D_\mu \bar{w}_i)_{a} \epsilon^j \epsilon_{ij} (D^{\mu} \mu)^a + \frac{ig}{\sqrt{2}} w_{ja} \epsilon^j \epsilon_{ij} (D^{\mu} \bar{\mu}_a) \gamma^{\mu} (F_{\mu})^a_b \mu^b$$

$$- \sqrt{2} b (D_{\mu} \bar{\mu}_a) \epsilon^j \epsilon_{ij} (D^{\mu} w^a)$$

8 See also section 9 of [25] for more details.
where \( [D_h, D_e] = ig \mathcal{F}_{\rho \rho} \). Summing the two previous variations we see that the last two terms in equation (F.14) cancel two equal terms in equation (F.15) and we obtain

\[
\delta \left[ \epsilon^{ij}(D_{\rho_\mu})_{a} (D^0 w_j)^a - i \mu_a \Gamma^\beta (\bar{D}_\mu)^a \right] = \frac{g}{2} \bar{\omega}_{ja} \left( \bar{e}_j \Gamma^{\beta} \Lambda^j - \bar{\Lambda}_j \Gamma^{\beta} \bar{e}_j \right)_{\beta} (D_{\rho} w^j)^b \\
- \frac{g}{2} (D_{\rho} \bar{\omega})_{ja} \left( \bar{e}_j \Gamma^{\beta} \Lambda^j - \bar{\Lambda}_j \Gamma^{\beta} \bar{e}_j \right)_{\beta} w^b + \frac{ig}{2} \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b \\
- \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} \epsilon_{ij} (D^0)^a \mu_b + \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} \epsilon_{ij} (D^0)^a \mu_b \\
- \sqrt{2} b \hat{\partial}_{i} \left[ \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b + \sqrt{2} b \hat{\partial}_{i} \left[ \bar{\mu}_a \epsilon^{ij} (D^0)^b \mu_b \right] \right] \\
\text{(F.16)}
\]

The variation of the Yukawa couplings is equal to

\[
\sqrt{2} gb \delta \left[ \bar{\mu}_a \Lambda^\beta (\Lambda^\beta)^a \epsilon_{ij} (w^j)^b + (\bar{\omega}_i)_{a} \epsilon^{ij} (\tilde{\Lambda}_j)^a \right] \\
= 2 g \left( D_{\rho} \bar{\omega} \right)_{ja} \epsilon^{ij} \epsilon_{ij} \epsilon_{ij} \epsilon_{ij} (D^0)^b \mu_b + \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} \epsilon_{ij} (D^0)^a \mu_b \\
- \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b + \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b \\
+ 2 g \bar{\omega}_{ja} \epsilon^{ij} (\tilde{\Lambda}_j)^a \Gamma^{\beta} \epsilon_{ij} \epsilon_{ij} (D^0)^b \mu_b. \\
\text{(F.17)}
\]

The first and the last term of the previous expression can be written as

\[
2 g \left( (D_{\rho} \bar{\omega})_{ja} \right) \epsilon^{ij} \epsilon_{ij} \epsilon_{ij} \epsilon_{ij} (D^0)^b \mu_b \\
= 2 g \left( D_{\rho} \bar{\omega} \right)_{ja} \epsilon^{ij} \epsilon_{ij} \epsilon_{ij} \epsilon_{ij} (D^0)^b \mu_b \\
+ \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b + \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b \\
+ \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b + \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b.
\text{(F.18)}
\]

Using the following identities which are proved at the end of this section:

\[
\bar{e}_j \Gamma^{\beta} \Lambda^j - \bar{\Lambda}_j \Gamma^{\beta} \bar{e}_j = \delta^{b} \frac{1}{2} \left( \bar{e}_j \Gamma^{\beta} \Lambda^j - \bar{\Lambda}_j \Gamma^{\beta} \bar{e}_j \right) = 0, \\
\text{(F.19)}
\]

we can write the variations of the Yukawa couplings as follows:

\[
\sqrt{2} gb \delta \left[ \bar{\mu}_a \Lambda^\beta (\Lambda^\beta)^a \epsilon_{ij} (w^j)^b + (\bar{\omega}_i)_{a} \epsilon^{ij} (\tilde{\Lambda}_j)^a \right] = \sqrt{2} gb \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b \\
- \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b + \sqrt{2} g b \bar{\omega}_{ja} \epsilon^{ij} \epsilon_{ij} (D^0)^b \mu_b \\
+ \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} \epsilon_{ij} (D^0)^a \mu_b \\
+ \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} \epsilon_{ij} (D^0)^a \mu_b \\
+ \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} (F_{\rho \rho})^a \mu_b \\
+ \frac{ig}{2} \bar{\mu}_a \Gamma^{\beta} (F_{\rho \rho})^a \mu_b \\
+ g \hat{\partial}_{i} \left[ \bar{\omega}_{ja} \epsilon^{ij} (\tilde{\Lambda}_j)^a (\Lambda^\beta)^b + \bar{\Lambda}_j \Gamma^{\beta} \bar{e}_j \epsilon_{ij} (w^j)^b \right]. \\
\text{(F.20)}
\]

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Summing equations (F.16) and (F.20), we see that the first four terms of equation (F.16) cancel the corresponding terms in equation (F.20) and we obtain

\[
\delta \left[ \epsilon^{ij} (D_\mu \bar{u}_i)_{a}(D^\mu u)_j - i\bar{u}_i \Gamma^\mu \Lambda^j - \bar{u}_i \Gamma^\mu \Lambda^j \right] = \frac{1}{2} g \mu_a \left( \bar{\epsilon}_i \Gamma^\mu \Lambda^j - \bar{\epsilon}_i \Gamma^\mu \Lambda^j \right) \mu_b - 2ig \mu_a \left( (\Lambda^j)^a \bar{\epsilon}_i - \bar{\epsilon}_i (\Lambda^j)^a \right) \mu_b
\]

\[
- \sqrt{2}gb\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \sqrt{2}gb\bar{u}_i \epsilon^{ij} \epsilon_{jk} (D^\mu \bar{u}_i)_{a}\mu_b
\]

\[
- g\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \bar{\epsilon}_i \Gamma^\mu \Lambda^j \epsilon^{jk} \mu_b
\]

\[
- \sqrt{2}b \delta_{ij} \left[ \bar{u}_i \left( \epsilon^{ij} \right) D^\mu \left( \Lambda^i \right) \right] + \sqrt{2}b \delta_{ij} \left[ \bar{\mu}_a \epsilon^{ij} (D^b w^j)^a \right]
\]

\[
- g\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j k} + \bar{\epsilon}_i \Gamma^\mu \Lambda^j \epsilon^{jk} \mu_b \right].
\]

The last term of the action to consider is the variation of the term with the auxiliary fields given by

\[
- g\delta \left[ \bar{u}_a \left( \epsilon^{ij} \right) D^\mu \left( \Lambda^j \right) \right] = \sqrt{2}gb\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \sqrt{2}gb\bar{u}_i \epsilon^{ij} \epsilon_{jk} (D^\mu \bar{u}_i)_{a}\mu_b
\]

\[
- g\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \bar{\epsilon}_i \Gamma^\mu \Lambda^j \epsilon^{jk} \mu_b
\]

Using the identity in equation (12.21) of [33]

\[
\frac{1}{2} (\epsilon^{ij}) \delta \left[ \epsilon^{ij} \right] = \delta^b \delta^k - \frac{2}{2} \delta^b \delta^j = \left( \delta^b \delta^j - \delta^j \delta^b \right) + \frac{1}{2} \delta^b \delta^j = e^b e^j + \frac{1}{2} \delta^b \delta^j
\]

The last term of equation (F.22) becomes

\[
- g\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \bar{\epsilon}_i \Gamma^\mu \Lambda^j \epsilon^{jk} \mu_b
\]

\[
- g\bar{\mu}_a \epsilon^{ij} \epsilon_{ik} w^{j b} + \bar{\epsilon}_i \Gamma^\mu \Lambda^j \epsilon^{jk} \mu_b
\]

These two terms cancel the terms in the third line of equation (F.21).

In the last part of this appendix we first prove the identities written in equation (F.19) and finally that the second line of equation (F.21) is identically zero.

The starting point to prove the equations (F.19) is the identity

\[
\bar{\epsilon}_i \Gamma^{b_1 ... b_n} \Lambda^j = -\bar{\epsilon}_i B^*_b B^*_a \Gamma^{b_1 ... b_n} \Lambda^j = -(-)^{\sum a_i} \bar{\epsilon}_i B^*_b B^*_a \Gamma^{b_1 ... b_n} \Lambda^j
\]

where we have denoted with \( \Gamma^{b_1 ... b_n} \) the completely antisymmetrized product of the transposed Dirac matrices and used the identity \( B^*_b B^*_a \Gamma^{b_1 ... b_n} = \Gamma^{b_1 ... b_n} \) with

\[
\Gamma^{b_1 ... b_n} = (-)^{b_2 ... b_n + 1} \Gamma^{b_1 ... b_n}
\]

Furthermore, observing that \( \Gamma^0 B^*_a = B^*_a \Gamma^0 \) and using, both for the gaugino and the SUSY parameters, equations (33) and (F.8) we obtain

\[
\bar{\epsilon}_i \Gamma^{b_1 ... b_n} \Lambda^j = (-)^{n-2} \bar{\epsilon}_i \epsilon_{ik} \Lambda^j \Gamma^{b_1 ... b_n} \epsilon^k = (-)^{n-1} \bar{\epsilon}_i \epsilon_{ik} \Lambda^j \Gamma^{b_1 ... b_n} \epsilon^j
\]

\[
\bar{\epsilon}_i \Gamma^{b_1 ... b_n} \Lambda^j = (-)^{n-1} \bar{\epsilon}_i \epsilon_{ik} \Lambda^j \Gamma^{b_1 ... b_n} \epsilon^j
\]

with \( n = 0, \ldots, 4 \). Along the same lines we can prove an analogous relation where the role of the gaugino and SUSY parameters are exchanged. Equations (F.19) can be easily derived by considering the case \( n = 0 \) of the first of the two previous identities.
Let us now consider the following Fierz identity valid for two generic six-dimensional spinors, here denoted with $\chi$ and $\psi$, having the same chirality [30]:

$$\psi^A \bar{\chi}_B = -\frac{1}{2} \left( \bar{\chi} \Gamma^B \psi \right) \Gamma^A_{\mu B} + \frac{1}{38} \left( \bar{\chi} \Gamma^{\mu} \Gamma^B \psi \right) \Gamma^A_{\mu B}. \tag{F.28}$$

This identity allow us to write

$$\bar{\mu}_A \Lambda^A_i \bar{\epsilon}_{i B} \epsilon^B = -\frac{1}{16} (\mu \Gamma^B \mu)(\bar{\epsilon}_i \Lambda^i) \text{Tr}[\Gamma \bar{\mu} \bar{\mu} \Gamma \bar{\mu} \bar{\mu}],$$

where we have used the condition $\text{Tr}[(\Gamma \bar{\mu} \bar{\mu})] = 0$, and

$$\bar{\mu}_A \epsilon^A \bar{\Lambda}_{i B} \epsilon^B = -\frac{1}{16} (\mu \Gamma^B \mu)(\bar{\Lambda}_i \Gamma^i \epsilon^i) \text{Tr}[\Gamma \bar{\mu} \bar{\mu} \Gamma \bar{\mu} \bar{\mu}], \tag{F.29}$$

Subtracting equations (F.29) and (F.30) and using the first identity in equation (F.27) with $n = 0$ and $n = 1$, we obtain

$$\bar{\mu}_A \Lambda^A_i \bar{\epsilon}_{i B} \epsilon^B = -\bar{\epsilon}_i \Lambda^i \bar{\Lambda}_{i B} \epsilon^B = -(\mu \Gamma^B \mu)(\bar{\Lambda}_i \Gamma^i \epsilon^i), \tag{F.31}$$

which prove the vanishing of the second line of equation (F.21).

**Appendix G. Calculation of correlators**

In this appendix, we compute in detail various three-point functions in order to check some of the terms of the Lagrangians written in section 3.

**G.1. Two fermions in the adjoint and a gauge field**

The first amplitude that we consider is the one involving two fermions in the adjoint representation of the gauge group $U(N_3)$ and the gauge field $A_{\mu}^{(5)}$.

$$A^{(\bar{\psi}^{(n)} A^{(5)} \psi^{(n)})} = C_0 \int \prod_{i=1}^3 \frac{d\phi_i}{V_{\bar{\psi}^{(n)} A_b \psi^{(n)}}(x_1, x_2, x_3)}, \tag{G.1}$$

where $C_0$ is defined after equation (65) and

$$dV = \frac{dx_a dx_b dx_c}{x_{ab} x_{ac} x_{bc}}; \quad x_{ab} = x_a - x_b. \tag{G.2}$$

The amplitude (G.1) can be easily computed using for the fields $\Psi^{(5)}$ and $A_{\mu}$ the vertex operators defined in section 5.1 and observing that the vertex for $\Psi^{(5)}$ is given by equation (54) with $\Psi^\alpha$ replaced by the doublet given in equation (A.22). This latter definition follows from the observation that, in our conventions, the complex conjugation of the $SU(2)$ doublets does not change the four-dimensional chirality associated with the compact directions transverse to the D5 brane and, being the ten-dimensional chirality fixed by the GSO, it does not change also the six-dimensional one. It follows that the spin fields associated with the vertex $\Psi^\alpha$ must have the same chirality as the ones associated with the field $\Psi^\alpha$.

The result is

$$A^{(\bar{\psi}^{(n)} A^{(5)} \psi^{(n)})} = i 2 \sqrt{2} s_5 \left( \Psi^{(5)} A_{\mu} \right)_{a}^{c} (p) (A_{\mu}^{(5)})^{b} (k) (\Psi^{(5)} B_{\mu})_{c}^{b} (q) [\chi_{12} \chi_{13} \chi_{23}] \times \left( [S_a (x_1)] \Psi^\beta (x_2) [S_b (x_3)] [S_d (x_1)] S_d (x_3) \right) \left( e^{i \phi (x_1)} e^{i \phi (x_3)} e^{-i \phi (x_2)} \right) \times (d \sqrt{2} s_\alpha p X (x_1) e^{i \sqrt{2} s_\alpha k X (x_2)} e^{i \sqrt{2} s_\alpha q X (x_3)}), \tag{G.3}$$

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Using the following correlators [27, 31]:
\[
\langle S_A(x_1)\psi^\beta(x_2)S_B(x_3) \rangle = -\frac{1}{\sqrt{2}}(\Sigma^\beta)_{AB}x_{12}^{-1/2}x_{13}^{-1/4}x_{23}^{-1/2}
\]
\[
(e^{-i\phi(x_1)}e^{-i\phi(x_2)}) = x_{12}^{-1/2}x_{13}^{-1/4}x_{23}^{-1/2}; \quad \langle S_B(x_1)S_B(x_3) \rangle = -\epsilon_{\alpha\beta}x_{13}^{-1/2}
\]
and taking into account that the correlator involving the bosonic coordinate gives 1 because the momentum is conserved and the three states are massless, we obtain:
\[
A_b^\mu A^\nu_b = -2g_5 \langle \bar{\psi}(5)_a(p)\lambda^\mu_a(k)\psi(5)_b(q)\bar{\psi}(5)_b(k)\psi(5)_a(q) \rangle = 2g_5 \langle \bar{\psi}(p)\lambda^\mu_a(k)\Gamma^\nu_b\psi(5)_a \rangle. \quad (G.4)
\]
where we have introduced the six-dimensional Weyl spinors and Dirac matrices so defined [32]:
\[
\psi^{(5)}_\alpha = \left( \begin{array}{c} \psi^{(5)}_\beta \\ \Gamma^\alpha \end{array} \right) ; \quad \Gamma^\alpha = \left( \begin{array}{c} 0 \\ \Sigma^\alpha \end{array} \right)
\]
being \(\Sigma^\alpha\) and \(\Sigma^\alpha\) \(4 \times 4\) matrices satisfying the anticommutation algebra \(\{\Sigma^\alpha, \Sigma^\beta\} = -2\eta^{\alpha\beta}\) and \(A, B = 1, \ldots, 4\). Note that the representation of Dirac matrices which we are using in this section differs from the one adopted in the field theory calculation of the effective action of our system. However, this difference will not generate any confusion because the final expression of the string amplitudes will always be written in a form which is independent of the chosen representation of the Dirac matrices.

Equation (G.4) reproduces the correct coupling of two adjoint fermions with a gauge field, written for example in equation (32).

G.2. A twisted scalar, a twisted fermion and a D5 gaugino

The next string amplitude which we are going to consider is the one involving the interaction between a twisted scalar, a twisted fermion and the gaugino of the gauge theory living on the D5 brane:
\[
A^\mu A^\nu = C_0 \int \frac{d^3x}{dV} \langle V^{(4)}_{\mu}(x_1)V^{(4)}_{\nu}(x_2)\Delta(0)(x_3) \rangle. \quad (G.5)
\]
We need to compute
\[
A^\mu A^\nu = \sqrt{2}g_5x_{12}x_{13}x_{23} \langle \bar{\psi}(5)_a(k)\psi(5)_b(q)\bar{\psi}(5)_b(k)\psi(5)_a(q) \rangle \langle \Delta(0) \Delta(0) \rangle
\]
\[
\times \langle S^{(5)}(x_1)S^{(5)}(x_2) \rangle \langle S^{(5)}(x_2)S^{(5)}(x_3) \rangle \langle \Delta(0)(x_2) \Delta(0)(x_3) \rangle
\]
\[
\times \langle e^{i\sqrt{2\pi\alpha}p_a x^{(5)}(x_1)}e^{i\sqrt{2\pi\alpha}p_b x^{(5)}(x_2)}e^{i\sqrt{2\pi\alpha}q_a x^{(5)}(x_3)} \rangle. \quad (G.6)
\]
Using the following correlators:
\[
\langle S^{(5)}(x_1)S^{(5)}(x_3) \rangle \sim \frac{1}{(x_{13})^{1/4}}; \quad \langle S^{(5)}(x_1)S^{(5)}(x_2) \rangle \sim \frac{\epsilon^{\alpha\beta}}{(x_{12})^{1/2}}
\]
\[
\langle e^{-i\phi(x_1)}e^{-i\phi(x_2)} \rangle \sim \frac{1}{x_{12}^{1/2}x_{13}^{1/4}x_{23}^{1/2}} \Delta_1^{1/2} \Delta_2 \Delta_3 \Delta_4 \Delta_5 \Delta_6 \Delta_7,
\]
where \(\Delta_1 = \Delta_3 = \frac{3}{2}\) is the conformal dimension of \(e^{-i\phi(x)}\) and \(\Delta_2 = \Delta_4 = \frac{1}{2}\) is the conformal dimension of \(e^{-i\phi(x)}\), we obtain
\[
A^\mu A^\nu = \sqrt{2}g_5 \langle \bar{\psi}(5)_a(k)\lambda^\mu_a(k)\psi(5)_a(w)q_a(q) \rangle.
\]

Here and in the following, we omit writing the delta function of momentum conservation.
G.3. A twisted scalar, a fermion and a D9 gaugino

It is also enlightening to repeat the previous calculation in the case of the gaugino living in the world-volume of the D9 brane. The result is

\[
A^{\hat{\mu}A^{(0)w}} = C_0 \int \prod_{j=1}^{3} \frac{dx_j}{V} \left[ V_{w}^{(5)}(x_1) V_{A^{(0)}}(x_2) V_{\hat{\mu}}^{(9)}(x_3) \right]
\]

\[
= -i\sqrt{2} g_9 e^{i\varphi_0} \mu^\mu_A(p) \left( A^{(9)}(k^\mu, k^\bar{\nu}) \right)_A^{\mu} \epsilon_{ij}(w^j)^{\bar{\nu}}(q),
\]

where \(\varphi_0\) is the position of the D5 in the last two tori, and we have used the correlator

\[
\langle \Delta(x_1) e^{i\lambda x^a A^a_0 \Delta(x_3)} \rangle = -x_{12}^{-\pi\nu_{\lambda}} k_{\lambda}^{\nu} x_{13}^{1/2} \pi\nu_{\lambda} k_{\lambda}^{\nu} x_{23}^{1/2} \pi\nu_{\lambda} k_{\lambda}^{\nu} e^{i\lambda x^a A^a_0},
\]

that is consistent with the general formula for the correlator of three operators \(A, B\) and \(C\) with the conformal dimensions \(\Delta_A, \Delta_B\) and \(\Delta_C\), respectively:

\[
\langle A(x_1) B(x_2) C(x_3) \rangle = \frac{1}{x_{12}^{\Delta_A + \Delta_B - \Delta_C} x_{13}^{\Delta_A + \Delta_C - \Delta_B} x_{23}^{\Delta_B + \Delta_C - \Delta_A}}.
\]

Recall that in our case \(\Delta_A = \Delta_C = \frac{1}{2}\) and \(\Delta_B = \pi\nu_{\lambda} k_{\lambda}^{\nu}\). By performing the Fourier transformations along the compact momenta, according to the relation

\[
A^{(9)}(k^\mu, \varphi_0) = \sum_{k_a} A^{(9)}(k^\mu, k^\bar{\nu}) e^{i\lambda x^a A^a_0},
\]

we obtain

\[
A^{\hat{\mu}A^{(0)w}} = -\sqrt{2} g_9 \mu^\mu_A(p) \left( A^{(9)}(k^\mu, \varphi_0) \right)_A^{\mu} \epsilon_{ij}(w^j)^{\bar{\nu}}(q).
\]

Finally by observing that \(g_9 A^{(0)w}_e = g_9 A^{(9)}_e\), we reproduce, from a string calculus, the interaction term written in the fourth line of equation (41).

G.4. Two twisted scalars and a D9 gauge field

The last amplitude that we consider is the three-point correlator involving the gauge field interaction term written in the fourth line of equation (41).

\[
A^{\hat{\mu}A^{(0)w}} = C_0 \int \prod_{j=1}^{3} \frac{dx_j}{V} \left[ V_{w}^{(5)}(x_1) V_{A^{(0)}}(x_2) V_{\hat{\mu}}^{(9)}(x_3) \right].
\]

Using the vertex operators introduced in section 5, we have to compute the following quantity:

\[
A^{\hat{\mu}A^{(0)w}} = g_9 \left( \frac{2}{\pi \alpha'} \bar{\omega}_{a_0}(p) A^a_{MB}(k) w^b_{\mu}(q) \right) \times \left\{ \left( \tilde{\Delta} S^a e^{-\psi} e^{i\sqrt{2\pi \alpha'} p_a X^a} \right)(x_1) \times \left[ i\varphi X^a + i\sqrt{2\pi \alpha'} k_N \psi^N \psi^M \right] e^{i\sqrt{2\pi \alpha'} \bar{\psi}_q X^q}(x_2) \right\} \times \left( \tilde{\Delta} S^b e^{-\psi} e^{i\sqrt{2\pi \alpha'} q_b X^b} \right)(x_3)
\]

\[
= g_9 \left( \frac{2}{\pi \alpha'} \bar{\omega}_{a_0}(p) A^a_{MB}(k) w^b_{\mu}(q) \right) (x_1, x_2, x_3, x_{23}) \times C^{MB}.\]

We start with the gauge polarization \(M = \hat{\mu}\). In this case, the previous correlator reduces to

\[
C^{\hat{\mu}\hat{\nu}} = \left( \tilde{\Delta} S^a e^{-\psi} e^{i\sqrt{2\pi \alpha'} p_a X^a} \right)(x_1) \times \left( i\varphi X^a + i\sqrt{2\pi \alpha'} k_N \psi^N \psi^M \right) e^{i\sqrt{2\pi \alpha'} \bar{\psi}_q X^q}(x_2) \times \left( \tilde{\Delta} S^b e^{-\psi} e^{i\sqrt{2\pi \alpha'} q_b X^b} \right)(x_3)
\]

\[
= \left( e^{-\varphi(x_1)} e^{-\psi(x_1)} \right) \times \left( \Delta(x_1) e^{i\sqrt{2\pi \alpha'} k_{\lambda}^{\nu} X_{\lambda}^{(3)}}(x_1) \right) \times \left( \Delta(x_2) e^{i\sqrt{2\pi \alpha'} k_{\lambda}^{\nu} X_{\lambda}^{(2)}}(x_2) \right) \times \left( \Delta(x_3) e^{i\sqrt{2\pi \alpha'} k_{\lambda}^{\nu} X_{\lambda}^{(1)}}(x_3) \right)
\]

\[
= x_{12}^{-1} \times x_{13}^{-1} \times x_{23}^{1/2} \frac{2\pi \alpha'}{\lambda} \left[ \frac{p_{\hat{\mu}}}{x_{12}} - \frac{q_{\hat{\nu}}}{x_{23}} \right].
\]
We can rewrite equation (G.14) as follows:

\[
\mathcal{C}_{\alpha \beta} = i \sqrt{2 \pi \alpha'} e^{i \delta_k \phi} e^{\alpha' \beta} \times \left[ \frac{p^\beta}{x_{12}} - \frac{q^\beta}{x_{23}} \right]
\]

where we have used the momentum conservation \( p_\beta + q_\beta + k_\beta = 0 \) that implies, together with mass-shell conditions \( p_\beta p^\beta = q_\beta q^\beta = 0 \), the following conditions: 

\[ 2 \alpha' p_\beta q^\beta = -2 \alpha' p_\beta k_\beta = -2 \alpha' q_\beta k^\beta = \alpha' k_\beta k^\beta. \]

We have also used the mass-shell condition for the vector field: \( k_\beta k^\beta = m^2 \). Note that the piece proportional to \( k_\beta \) does not have the right dependence on the Koba–Nielsen variables and therefore it must be canceled out in the final result.

We now consider the polarization of the vector field to be \( M = m \). In this case, the amplitude written in equation (G.13) is the sum of two pieces. The first one is

\[
\mathcal{C}_{\alpha m}^{\alpha m} = \left( \bar{\Delta}^\alpha \beta e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_1) \bar{\Delta} X^m e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_2) \left( \Delta S^\beta e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_3) \right) \right)
\]

\[
= (e^{-\psi(x_1)} e^{\psi(x_2)}) \left( \bar{\Delta} (x_1) \Delta X^m (x_2) e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_3) \right) \left( \Delta S^\beta e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_3) \right)
\]

\[
\times (e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_1) e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_2) e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_3))
\]

\[
= x_{12}^{-1} \sqrt{2 \pi \alpha'} \phi \beta (x_1 + x_2) \frac{1}{x_{13}} \frac{1}{x_{23}} \left( x_{12}^{1/2} e^{2 \alpha' p_\beta k_\beta} x_{13}^{1/2} e^{2 \alpha' q_\beta k^\beta} x_{23}^{1/2} e^{2 \alpha' k_\beta k^\beta} \right)
\]

\[
= \frac{i \sqrt{2 \pi \alpha'} \phi \beta (x_1 + x_2) e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_3)}{2 \chi_{12} \chi_{13} \chi_{23}}.
\]

The only correlator that needs some explanation is the one containing the twist fields and that is equal to

\[
\langle \bar{\Delta} (x_1) \bar{\Delta} X^m e^{i \sqrt{2 \pi \alpha'} \phi \beta} (x_2) \Delta (x_3) \rangle = \frac{(x_{12} - x_{23}) (i \sqrt{2 \pi \alpha'} \phi \beta (x_1 + x_2))}{\sqrt{2 \pi \alpha'} \phi \beta} (x_1 + x_2) \frac{1}{x_{13}} \frac{1}{x_{23}} \frac{1}{x_{12} \chi_{13} \chi_{23}}
\]

in order to get the proper conformal dependence of the total correlator. There are several reasons why we think that this choice is the correct one. If we assume the correlator in equation (G.8), then the previous one can be obtained from it by taking the derivative with respect to \( x_2 \) of equation (G.8). The operator \( \partial X^m e^{i \sqrt{2 \pi \alpha'} \phi \beta} \) is not a good conformal operator, but the operator \( A_M \bar{X}^M e^{i \sqrt{2 \pi \alpha'} \phi \beta} \) with \( A_M k^M = 0 \) is a good conformal operator and therefore, when used in a correlator, one must obtain the proper dependence on the Koba–Nielsen variables. The choice in equation (G.17) is symmetric with the exchange of 1 and 3.
where we have again used the momentum conservation and the mass-shell conditions.

The final result is the sum of the three contributions in equations (G.15), (G.16) and (G.18) that is equal to

\[
A^{\mu A\nu w} = 2i g_9 \delta_{\mu a}(p) u^b_{\rho a}(q) e^{iV_{10}(x_{12}, x_{13}, x_{23})} \times \left\{ \epsilon^{\alpha\beta} \left[ A^{a}_\mu(k) \left( \frac{p^\mu}{x_{12} x_{13} x_{23}} + \frac{k^\mu}{x_{13} x_{23}} \right) + A^{a}_\alpha(k) k^\mu \left( \frac{x_{12} - x_{23}}{2x_{12} x_{13} x_{23}} \right) \right] - A^{a}_\mu(k) k^\mu \left( \frac{(\partial^{mn})^{\alpha\beta}}{2x_{12} x_{13} x_{23}} \right) \right\}. \tag{G.19}
\]

It can be checked that the previous expression is gauge invariant. In fact, if we make the

\[
A_\hat{\mu} \rightarrow k_\hat{\mu}; \quad A_m \rightarrow k_m, \tag{G.20}
\]

it is easy to check that the last term in equation (G.19) vanishes, while the term between the square brackets becomes

\[
k_\hat{\mu} \left( \frac{p^\hat{\mu}}{x_{12} x_{13} x_{23}} + \frac{k^\hat{\mu}}{x_{13} x_{23}} \right) + k_m k^\mu \left( \frac{x_{12} - x_{23}}{2x_{12} x_{13} x_{23}} \right). \tag{G.21}
\]

Using the mass-shell condition \( k_\hat{\mu} k^\hat{\mu} + k_m k^m = 0 \) and \( k_\hat{\mu} p^\hat{\mu} = -\frac{1}{2} k_\mu k^\hat{\mu} \), we can rewrite the previous equation as follows:

\[
k_\mu k^\mu \left[ -\frac{1}{2x_{12} x_{13} x_{23}} + \frac{1}{x_{13} x_{23}} \right] = k_\mu k^\mu \left[ -\frac{1}{2x_{12} x_{13} x_{23}} + \frac{1}{2x_{13} x_{23}} + \frac{1}{2x_{13} x_{12}} \right] = k_\mu k^\mu \frac{-x_{13} + x_{12} + x_{23}}{2x_{13} x_{23} x_{12}} = 0 \tag{G.22}
\]

that proves the gauge invariance of the correlator.

Using the condition

\[
k^M A_M = k^\mu A_\mu + k^m A_m = 0,
\]

we can rewrite equation (G.19) as follows:

\[
A^{\mu A\nu w} = i \hat{w}_{\mu a}(p) u^b_{\rho a}(q) e^{iV_{10}(x_{12}, x_{13}, x_{23})} \left[ \epsilon^{\alpha\beta} \left( A^{(0)}_{\mu}(k)^{\alpha\beta}(p^\mu - q^\mu) - \left( A^{(0)}_{\mu}\right)_{\beta}(k) k^\mu (\partial^{mn})^{\alpha\beta} \right) \right], \tag{G.23}
\]

that has the correct dependence on the Koba–Nielsen variables. Finally, by performing the Fourier transformation as shown after equation (G.9), we obtain an amplitude which is evaluated at the position of the D5 brane in the last two tori:

\[
A^{\mu A\nu w} = i \hat{w}_{\mu a}(p) u^b_{\rho a}(q) \left[ \epsilon^{\alpha\beta} \left( A^{(0)}_{\mu}(k, \gamma_0)^{\alpha\beta}(p^\mu - q^\mu) - \left( A^{(0)}_{\mu}\right)_{\beta}(k, \gamma_0) k^\mu (\partial^{mn})^{\alpha\beta} \right) \right]. \tag{G.23}
\]

The previous calculation can easily be extended to the interaction of two twisted scalars with the gauge field living in the world-volume of the D5 brane. Since the vertex of the gauge field \( A^{(5)}_{\mu} \) depends only on the six-dimensional momentum, one has that the corresponding amplitude contains only one term which is equal and opposite in sign, due to the different ordering of the twisted fields, to the one in equation (G.15) taken at \( k_m = 0 \). Using the trasversality condition \( k^\mu A^{(5)}_{\mu} = 0 \), we arrive at the following expression:

\[
A^{\mu A\nu w} = -i \hat{w}_{\mu a}(p) u^b_{\rho a}(q) \left( A^{(5)}_{\mu}(k)^{\alpha\beta}(p^\mu - q^\mu) \right), \tag{G.23}
\]

\[
A^{\mu A\nu w} \propto \left( \bar{\alpha}_{\mu A} - \alpha_{\mu A} \right) \propto \left( \bar{\alpha}_{\mu A} - \alpha_{\mu A} \right).
\]
which is in agreement with the one obtained from the kinetic term of the twisted scalar in the field theoretical approach.

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