Revisit to Fritz John’s paper on the blow-up of nonlinear wave equations∗

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Abstract
In Fritz John’s famous paper (1979), he discovered that for the wave equation
$\Box u = |u|^p$, where $1 < p < 1 + \sqrt{2}$ and $\Box$ denoting the d’Alembertian, there
is no global solution for any nontrivial and compactly supported initial data. This paper is intended to simplify his proof by applying a Gronwall’s type
inequality.

Keywords: Blow-up, Wave equations, Gronwall’s type inequality

1. Introduction
In 1979, Fritz John published his pioneering work [1], which was the first
one that discovered the critical power of the blow-up phenomenon for wave
equations. After this article, many people worked on this kind of blow-up
problem. For details and many other related references, see [2, 3, 4, 5, 6,
7, 8, 9, 10, 11, 12, 13]. These work generalize the critical power to other
dimensions, some of them also provide simpler proof by imposing additional
assumptions or by applying different methods.

The paper [1] claims the following well-known Theorem.

Theorem 1.1. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function which satisfies
$\phi(0) = 0$, $\limsup_{s \to 0} \phi(s)/|s| < \infty$.

∗The original paper is ”Blow-up of solutions of nonlinear wave equations in three space
dimensions, Manuscripta Mathematica, 28, 235-268 (1979)”. See [1].
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Moreover, suppose there exists $A > 0$ and $1 < p < 1 + \sqrt{2}$ such that for all $s \in \mathbb{R}, \phi(s) \geq A|s|^p$. Then for any function $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ that solves

\[
\begin{cases}
\Box u(x,t) = \phi(u(x,t)) & \text{for } x \in \mathbb{R}^3, t \in (0, \infty), \\
u(x,0) = f(x) & \text{for } x \in \mathbb{R}^3, \\
u_t(x,0) = g(x) & \text{for } x \in \mathbb{R}^3,
\end{cases}
\]

where $f \in C^3(\mathbb{R}^3), g \in C^2(\mathbb{R}^3)$ and both of them have compact support, we have $u \equiv 0$ in $\mathbb{R}^3 \times [0, \infty)$. Here $\Box$ denotes the d'Alembertian operator:

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}.$$ 

The key step to prove this Theorem is the statement as following.

**Theorem 1.2.** Let $A > 0$, $1 < p < 1 + \sqrt{2}$, let $u$ be a $C^2(\mathbb{R}^3 \times [0, \infty))$ solution of $\Box u \geq A|u|^p$. Moreover, suppose there exists a point $(x^0, t_0) \in \mathbb{R}^4$ such that $u^0(x,t) \geq 0$ for $(x,t) \in \Gamma^+(x^0, t_0)$, then $u$ has compact support and $\text{supp } u \subset \Gamma^-(x^0, t_0)$.

**Remark 1.1.** If $u$ satisfies $\Box u = w$ with initial data $f$ and $g$, then one decomposes $u$ by $u = u^0 + u^1$, where $u^0$ solves

\[
\begin{cases}
\Box u^0(x,t) = 0 & \text{for } x \in \mathbb{R}^3, t \in (0, \infty), \\
u^0(x,0) = f(x) & \text{for } x \in \mathbb{R}^3, \\
u^0_t(x,0) = g(x) & \text{for } x \in \mathbb{R}^3,
\end{cases}
\]

and $u^1$ suffices

\[
\begin{cases}
\Box u^1(x,t) = w(x,t) & \text{for } x \in \mathbb{R}^3, t \in (0, \infty), \\
u^1(x,0) = u^0_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3.
\end{cases}
\]

**Remark 1.2.** For any $(x^0, t_0) \in \mathbb{R}^4$, the forward cone $\Gamma^+(x^0, t_0)$ and the backward cone $\Gamma^-(x^0, t_0)$ are defined as $\Gamma^+(x^0, t_0) = \{(x,t) : |x - x^0| \leq t - t_0, t \geq 0\}$ and $\Gamma^-(x^0, t_0) = \{(x,t) : |x - x^0| \leq t_0 - t, t \geq 0\}$.

In the proof of Theorem 1.2, this paper gives a much more succinct proof which follows a Gronwall’s type inequality.

The organization of this paper is as following: In Section 2 it is shown how Theorem 1.2 implies Theorem 1.1 the argument is from [1]. In addition,
some notations and a basic Lemma are introduced, where the Lemma is the key technique used in Section 3.2. In Section 3, we prove Theorem 1.2. More precisely, Section 3.1, the first part of the proof, follows from [1] with modifications while Section 3.2, the rest part of the proof, comes from our own observation.

2. Preliminaries

2.1. Theorem 1.2 implies Theorem 1.1

Proof. Firstly, one can assume that both $f$ and $g$ have support in $B(0, \rho) \triangleq \{x \in \mathbb{R}^3 : |x| < \rho\}$, then by Huygens’ principle, $u^0 \equiv 0$ in $\Gamma^+(0, \rho)$. It follows from Theorem 1.2 that $\text{supp } u \subset \Gamma^-(0, \rho)$. Secondly, one considers the function $v(x, t) \equiv u(x, \rho - t)$ for $x \in \mathbb{R}^3, 0 \leq t \leq \rho$. Using the assumptions on $\phi$ in Theorem 1.1, one can see $|\Box v| \leq M|v|$ for some fixed $M$ depending on $u$ and $\phi$. Then by energy estimate, $v \equiv 0$ in $\Gamma^-(0, \rho)$. Thus Theorem 1.1 is verified.

2.2. Some Notations

For any function $h : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$, its radial average function (with respect to spatial variable) $\bar{h}$ is defined by

$$\bar{h}(r, t) = \frac{1}{4\pi} \int_{|\xi|=1} h(r\xi, t) dS_\xi, \quad \forall (r, t) \in [0, \infty) \times [0, \infty).$$ (4)

Now let’s calculate the radial average of the solution $u$ to the wave equation $\Box u = w$ with zero initial data. To do so, one defines $v : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $v(r, t) = r \bar{u}(r, t)$, then we will have

$$\left\{ \begin{array}{ll}
\Box v(r, t) = r \bar{w}(r, t) & \text{for } r \in [0, \infty), t \in [0, \infty), \\
v(r, 0) = 0, \quad v_t(r, 0) = 0 & \text{for } r \in [0, \infty).
\end{array} \right.$$

Hence

$$v(r, t) = \frac{1}{2} \int_0^t \int_{|r-t+s|}^{r+s-t} \lambda \bar{w}(\lambda, s) d\lambda ds,$$

which implies

$$\bar{u}(r, t) = \frac{1}{2r} \int_0^t \int_{|r-t+s|}^{r+s-t} \lambda \bar{w}(\lambda, s) d\lambda ds.$$ (5)
For convenience, one defines the operator \( P \) acting on \( \sigma(r,t) \) with domain \([0, \infty) \times [0, \infty)\) by

\[
P\sigma(r,t) = \int \int_{R_{r,t}} \frac{\lambda}{2r} \sigma(\lambda,s) \, d\lambda \, ds, \quad (r,t) \in [0, \infty) \times [0, \infty),
\]

where \( R_{r,t} = \{ (\lambda,s) : 0 \leq s \leq t, |r-t+s| \leq \lambda \leq r+t-s \} \) (See Figure 1).

It is clear that \( P \) is a positive operator and now we can rewrite (5) as

\[
\bar{u}(r,t) = P\bar{w}(r,t), \quad (r,t) \in [0, \infty) \times [0, \infty),
\]

whenever \( u \) solves \( \Box u = w \) with initial data.

2.3. A basic Lemma

Next, we introduce a Lemma which is a generalized Gronwall’s type inequality with weight.

**Lemma 2.1.** Let \( t_0, t_1 \in \mathbb{R} \) with \( t_0 \leq t_1 \), suppose \( H : [t_0, \infty) \to [0, \infty) \) to be continuous and \( H(r) > 0 \) for any \( r > t_1 \). Then there does not exist constants \( C > 0, a > 1, b \geq -1 \) such that

\[
H(r) \geq C \int_{t_1}^{r} H^{a}(\alpha)(\alpha - t_0)^b \, d\alpha, \quad \forall r \geq t_1.
\]

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Proof. Suppose there exist $C > 0, a > 1, b \geq -1$ such that (8) holds, then one defines $J : [t_1 + 1, \infty) \to (0, \infty)$ by

$$J(r) = \int_{t_1}^{r} H^a(\alpha)(\alpha - t_0)^b d\alpha.$$  

Now for any $r \geq t_1 + 1$, it follows from (8) that $0 < J(r) \leq \frac{H(r)}{C}$ and therefore

$$J'(r) = H^a(r)(r - t_0)^b \geq C^a J^a(r)(r - t_0)^b.$$  

As a result, for any $r_0 > t_1 + 1$,

$$\int_{t_1 + 1}^{r_0} \frac{J'(r)}{J^a(r)} dr \geq C^a \int_{t_1 + 1}^{r_0} (r - t_0)^b dr.$$  

Now the left hand side is bounded by

$$\frac{J^{1-a}(t_1 + 1)}{a - 1}$$  

which is a fixed number. However, the Right Hand Side $\to \infty$ when $r_0 \to \infty$, since $b \geq -1$. Thus, the Lemma follows.

Later in Section 3.2, in order to prove the finite time blow-up, the goal is to construct a function $H$, associated with the solution of (1), which satisfies (8) with $a, b$ related to the exponent $p$.

3. Proof of Theorem 1.2

3.1. Set-up steps

To verify Theorem 1.2 first of all, without loss of generality, one can assume $x_0 = 0$ to be the origin in $\mathbb{R}^3$, otherwise just doing a translation. In addition, we can suppose $A = 1$, otherwise just doing a dilation.

Now using proof by contradiction, one assumes supp $u$ is not in $\Gamma^- (0, t_0)$, then there exists $(x^1, t_1) \notin \Gamma^- (0, t_0)$ but $u(x^1, t_1) \neq 0$. Set $t_2 = t_1 + |x^1|$, then $t_2 > t_0$ and therefore $(0, t_2) \in \Gamma^+(0, t_0)$.

Since $u^0 \geq 0$ in $\Gamma^+(0, t_0)$ and $(0, t_2) \in \Gamma^+(0, t_0)$, then $u^0 \geq 0$ in $\Gamma^+(0, t_2)$. As a consequence, for any $0 \leq r \leq t - t_2$,

$$\bar{u}(r, t) = \bar{u}^0(r, t) + \bar{u}^I(r, t) \geq \bar{u}^I(r, t).$$  

(9)
Noticing the fact that $P$ is positive and the assumption $\Box u^1 \geq |u|^p$ in Theorem 1.2, it follows from (7) that

$$\overline{u}(r, t) \geq u^1(r, t) \geq P\left(\frac{1}{2r} |u|^p(r, s) d\lambda ds. \right) \quad (10)$$

Because of the simple fact $|u|^p(\lambda, s) \geq |\overline{u}(\lambda, s)|^p$, one has that

$$\overline{u}(r, t) \geq \int_{R_{r,t}} \frac{\lambda}{2r} |\overline{u}(\lambda, s)|^p d\lambda ds, \quad \forall 0 \leq r \leq t - t_2. \quad (11)$$

From (10), we have

$$\overline{u}(\delta, t_2 + \delta) > 0, \quad \forall \delta > 0, \quad (12)$$

since the point $(|x^1|, t_2)$ lies in $R_{\delta, t_2 + \delta}$ and $u(x^1, t_2) \neq 0$.

Now one fixes $t_2$ and a positive number $\delta$, then considers the regions (See Figure 2)

$$T = \{(\lambda, s) : t_2 + \delta \leq s + \lambda \leq t_2 + 2\delta, s - \lambda \leq t_2, s \geq 0\},$$

$$Q = \{(\lambda, s) : t_2 + 2\delta \leq s + \lambda, t_2 \leq s - \lambda \leq t_2 + \delta\}.$$

Figure 2: $T$ and $Q$
It is easy to check that the fixed region $T \subset R_{r,t}$ for any $(r, t) \in Q$. Then it follows from (11) that for any $(r, t) \in Q$,

$$\bar{u}(r, t) \geq \int \int_{T} \frac{\lambda}{2r} |\bar{u}(\lambda, s)|^p d\lambda ds = \frac{M}{r},$$

(13)

where $M$ is a positive constant due to (12).

Let $\Sigma = \{(r, t) : 0 \leq r \leq t - t^*\}$, where $t^* \triangleq t_2 + 2\delta$. For any $(r, t) \in \Sigma$, one defines the sets (See Figure 3)

$$Q_{r,t} = \{(\lambda, s) : t - r \leq \lambda + s \leq t + r, \ t_2 \leq s - \lambda \leq t_2 + \delta\}$$

$$B_{r,t} = \{(\lambda, s) : t - r \leq \lambda + s \leq t + r, \ t_2 + 2\delta \leq s - \lambda \leq t - r\}.$$ 

Then for any $(r, t) \in \Sigma$, we get

$$Q_{r,t} \subset R_{r,t}, \ B_{r,t} \subset R_{r,t}, \ Q_{r,t} \subset Q, \ B_{r,t} \subset \Sigma.$$ 

(14)
Thus from (11), (13), (14), one obtains

\[ \bar{u}(r,t) \geq \iint_{Q_{r,t}} \frac{\lambda}{2r} |\bar{u}(\lambda,s)|^{p} \, d\lambda \, ds \geq \iint_{Q_{r,t}} \frac{\lambda}{2r} \frac{M^{p}}{\lambda^{p}} \, d\lambda \, ds = \frac{M^{p}}{2r} \iint_{Q_{r,t}} \lambda^{1-p} \, d\lambda \, ds. \]

The area of \( Q_{r,t} \) is \( r\delta \) and for any point \((\lambda, s)\) in it, \( \lambda \leq t + r \). As a result,

\[ \bar{u}(r,t) \geq \frac{M^{p}}{2r} r\delta (t + r)^{1-p} = C_{0}(t + r)^{1-p} , \quad \forall (r,t) \in \Sigma, \quad (15) \]

where \( C_{0} = M^{p}\delta /2 \) is a positive constant.

3.2. Using Gronwall’s type inequality

Until now, the ideas are natural and all the estimates are not difficult to get. But after this step, [11] claims an induction:

\[ \bar{u}(r,t) \geq C_{k}(t + r)^{-1}(t - r - t^{*})^{a_{k}}(t - r)^{-b_{k}}, \quad \forall (r,t) \in \Sigma, \quad \forall k \geq 1, \]

where \( a_{k}, b_{k}, c_{k} \) satisfy complex recurrence formulas. By tedious computations, one finds that \((t - r - t^{*})^{a_{k}}\) is the dominant term and \( a_{k} \to \infty \), then the blow-up follows when taking \((t - r - t^{*})\) to be a fixed large number and \( k \to \infty \).

In the following, we will carry out a much more concise argument by introducing a suitable nonlinear functional and making use of Lemma 2.1.

Firstly, we observe from [11] and [14] that for any \((r,t) \in \Sigma, \)

\[ \bar{u}(r,t) \geq \iint_{B_{r,t}} \frac{\lambda}{2r} |\bar{u}(\lambda,s)|^{p} \, d\lambda \, ds \geq \iint_{B_{r,t}} \frac{\lambda}{2r} |\bar{u}(\lambda,s)|^{p} \, d\lambda \, ds, \quad (16) \]

which implies \( \bar{u} \geq 0 \) on \( \Sigma \) and especially by [14], \( \bar{u} \geq 0 \) on \( B_{r,t} \). So we can remove the absolute value sign in the last term of (16) to get

\[ \bar{u}(r,t) \geq \iint_{B_{r,t}} \frac{\lambda}{2r} \bar{u}^{p}(\lambda,s) \, d\lambda \, ds, \quad \forall (r,t) \in \Sigma. \quad (17) \]
By change of variable: $\alpha = \lambda + s$ and $\beta = s - \lambda$, we obtain

$$\bar{u}(r,t) \geq \frac{1}{8r} \int_{t-r}^{t+r} \int_{t-r}^{t-r} (\alpha - \beta) \bar{u}^p \left( \frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) d\beta d\alpha. \quad (18)$$

Here comes an important observation by considering $F : \Sigma' \to \mathbb{R}$, where $F(\alpha, \beta) \triangleq \bar{u}(\frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2})$ and $\Sigma' = \{(r, t) : t^* \leq t \leq r\}$ corresponding to the definition of $\Sigma$. Now (15) becomes

$$F(r, t) \geq C_0 r^{1-p}, \quad \forall (r, t) \in \Sigma'. \quad (19)$$

Moreover, (18) becomes

$$F(r, t) \geq \frac{1}{4(r - t)} \int_{t}^{r} \int_{t}^{t^*} (\alpha - \beta) F^p(\alpha, \beta) d\beta d\alpha, \quad \forall (r, t) \in \Sigma'. \quad (20)$$

From here, it attempts to employ the Gronwall’s inequality technique, which reduces the blow-up problem to a pure analysis technique. This is our motivation.

However we can not apply it directly since Gronwall’s inequality only deals with single variable and the right hand side being a double integral. In addition, (20) involves some weight functions. To overcome these difficulties, we introduce some new functions and make use of Lemma 2.1 together with (19).

For $q \geq 1$ to be determined later, we define $G : \Sigma' \to \mathbb{R}$ by

$$G(r, t) = (r - t)^q F(r, t).$$

From (20),

$$G(r, t) \geq \frac{1}{4(r - t)} \int_{t}^{r} \int_{t}^{t^*} G^p(\alpha, \beta)(\alpha - \beta)^{1-p} d\beta d\alpha. \quad (21)$$

We define $H : [t^*, \infty) \to \mathbb{R}$ by

$$H(r) = \int_{t^*}^{r} G(r, t) dt$$

and integrate (21) for $t$ from $t^*$ to $r$. Then the left hand side of (21) becomes $H(r)$. In order to exploit Lemma 2.1, the right hand side, hopefully after
changing the order of integration, can become a single integral of $H(r)$. We calculate as follows,

$$
H(r) \geq \frac{1}{4} \int_{t^*}^{r} \int_{t^*}^{r} \int_{t^*}^{t} G^p(\alpha, \beta)(\alpha - \beta)^{1-qp} (r-t)^q d\beta d\alpha dt
$$

$$= \frac{1}{4} \int_{t^*}^{r} \int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-qp} \int_{\beta}^{\alpha} (r-t)^q dt d\beta d\alpha
$$

$$= \frac{1}{4q} \int_{t^*}^{r} \int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-qp} [(r-\beta)^q - (r-\alpha)^q] d\beta d\alpha.
$$

Since $q \geq 1$, then $(r-\beta)^q - (r-\alpha)^q \geq (\alpha-\beta)^q$. As a result,

$$H(r) \geq \frac{1}{4q} \int_{t^*}^{r} \int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-qp+q} d\beta d\alpha. \quad (22)
$$

We choose $q = p/(p-1)$ and denote $C$ to be a constant which is independent of the variable $r$ but may be different from line to line, then

$$H(r) \geq C \int_{t^*}^{r} \int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-p} d\beta d\alpha. \quad (23)
$$

Using Holder’s inequality,

$$H(\alpha) = \int_{t^*}^{\alpha} G(\alpha, \beta) d\beta
$$

$$= \int_{t^*}^{\alpha} G(\alpha, \beta)(\alpha - \beta)^{1-p} (\alpha - \beta)^{\frac{p-1}{p}} d\beta
$$

$$\leq \left( \int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-p} d\beta \right)^{\frac{1}{p}} \left( \int_{t^*}^{\alpha} (\alpha - \beta) d\beta \right)^{\frac{p-1}{p}}.
$$

So

$$\int_{t^*}^{\alpha} G^p(\alpha, \beta)(\alpha - \beta)^{1-p} d\beta \geq H^p(\alpha) \left( \int_{t^*}^{\alpha} (\alpha - \beta) d\beta \right)^{1-p}
$$

$$= CH^p(\alpha) (\alpha - t^*)^{2-2p}.
$$

Plugging in (23) gives

$$H(r) \geq C \int_{t^*}^{r} H^p(\alpha)(\alpha - t^*)^{2-2p} d\alpha, \quad \forall r \geq t^*. \quad (24)$$
Now it is almost done! Only trouble for applying Lemma 2.1 is that $2 - 2p$ may be less than $-1$. In order to raise the power of $\alpha - t^*$, we borrow the idea from [2]. Namely, we write $H^p(\alpha) = H^{1+\varepsilon}(\alpha) H^{p-1-\varepsilon}(\alpha)$ and hope to find $H(\alpha) \geq C(\alpha - t^*)^d$ for some $d$, which can increase the power of $(\alpha - t^*)$ by $(p - 1 - \varepsilon)d$.

From (19) and the definition of $G$ and $H$, we have for any $\alpha \geq 2t^*$,

$$H(\alpha) = \int_{t^*}^{\alpha} G(\alpha, \beta) \, d\beta \geq C_0 \int_{t^*}^{\alpha} (\alpha - \beta)^q \alpha^{1-p} \, d\beta \geq \frac{C_0}{q+1} \alpha^{1-p} (\alpha - t^*)^{q+1} \geq \frac{C_0}{q+1} 2^{1-p} (\alpha - t^*)^{1-p} (\alpha - t^*)^{q+1} = C (\alpha - t^*)^{2-2p+q}. \tag{25}$$

Getting back to (24), for any $r \geq 2t^*$, one gets

$$H(r) \geq C \int_{2t^*}^{r} H^{1+\varepsilon}(\alpha) H^{p-1-\varepsilon}(\alpha) (\alpha - t^*)^{2-2p} \, d\alpha \geq C \int_{2t^*}^{r} H^{1+\varepsilon}(\alpha) (\alpha - t^*)^{s(p,\varepsilon)} \, d\alpha,$$

where

$$s(p, \varepsilon) = (p - 1 - \varepsilon)(2 - p + q) + 2 - 2p = -p^2 + 2p - \varepsilon \left(2 - p + \frac{p}{p-1}\right).$$

Since $1 < p < 1 + \sqrt{2}$, then $-p^2 + 2p > -1$ and therefore it is possible to choose small $\varepsilon$ in $(0, p - 1)$ such that $s(p, \varepsilon) \geq -1$.

Now applying Lemma 2.1 with $t_0 = t^*$, $t_1 = 2t^*$, $a = 1+\varepsilon$, $b = s(p, \varepsilon)$, we get contradiction. Thus we reject the assumption $\text{supp } u \notin \Gamma^-(0, t_0)$, hence Theorem 1.2 follows.

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