New Perfect Nonlinear Functions and Their Semifields

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Abstract: In this paper, two new classes of perfect nonlinear functions over $\mathbb{F}_{p^{2m}}$ are proposed, where $p$ is an odd prime. Furthermore, we investigate the nucleus of the corresponding semifields of these functions and show that the semifields are not isotopic to all the known semifields. Particularly, the new perfect nonlinear functions are CCZ-inequivalent to other classes in general.

Keywords: Perfect nonlinear functions, Planar functions, Commutative semifields, Isotopism, CCZ equivalence

1 Introduction

Block ciphers use substitution boxes (S-boxes) whose aim is to create confusion into the cryptosystems. Functions used as S-boxes are required to have low differential uniformity \cite{6}, high nonlinearity \cite{29} and high algebraic degree \cite{23}. A function $f$ from the finite field $\mathbb{F}_{p^n}$ to itself is called differentially $\delta$-uniform if for any nonzero $a$ and $b \in \mathbb{F}_{p^n}$, the equation $f(x+a)-f(x)=b$ has at most $\delta$ solutions in $\mathbb{F}_{p^n}$ \cite{32}. It is well known that for odd prime $p$, the lowest differential uniformity of a function defined on $\mathbb{F}_{p^n}$ is 1 and such functions are called perfect nonlinear (PN). It is worth noting that PN functions have also been studied under the alias of planar functions, which are functions such that $f(x+a)-f(x)$ is a permutation polynomial over $\mathbb{F}_{p^n}$ for any nonzero $a \in \mathbb{F}_{p^n}$. Planar functions can be used to describe projective planes with certain properties \cite{15}. Note that a PN function yields either a skew Hadamard difference set or a Paley type partial difference set \cite{20,38}. On the other hand, PN functions can be applied to describe finite commutative semifields of odd prime power order \cite{11,12}. As the functions “fares” from

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linear functions, PN functions have various applications in many fields [18][19][37]. Therefore, it is an interesting topic to find new PN functions.

An algebraic structure satisfying all the axioms of skewfield except for (possibly) associativity of multiplication is called a **semifield**. Any finite field is a semifield. A **presemifield** is similar to semifield, only (possibly) without the multiplicative identity. Any presemifield can derive semifields. Indeed, if \( S = (\mathbb{F}_{p^n},+,\ast) \) is a commutative presemifield with no identity, then for any \( a \in \mathbb{F}_{p^n} \), define

\[
(x \ast a) \circ (a \ast y) = x \ast y
\]

for all \( x, y \in S \). It is routine to check that \( S' = (\mathbb{F}_{p^n},+,\circ) \) is a commutative semifield with identity \( a \ast a \). Hence from one presemifield, there are many associated semifields, but they are isotopic (see Definition 2).

A polynomial \( F \in \mathbb{F}_{p^n}[x] \) is a **Dembowski-Öststrom polynomial** (DO-polynomial) if it can be expressed as

\[
F(x) = \sum_{i,j=0}^{n-1} a_{ij} x^{p^i+p^j}, \quad a_{ij} \in \mathbb{F}_{p^n}.
\]

Many constructions of PN functions are based on the relationship between quadratic PN functions and commutative semifields. It is known that any DO type PN function is equivalent to a commutative presemifield in odd characteristic (see [11]). Precisely, if \( \ast \) is the multiplication in the presemifield, then the corresponding PN function is \( f(x) = \frac{1}{2}(x \ast x) \). Conversely, if the PN function \( f \) is given, then the multiplication \( \ast \) in the corresponding presemifield can be defined as

\[
x \ast y = f(x + y) - f(x) - f(y).
\]

As a matter of fact, there are a handful classes of PN functions presented in the literature [3–5, 7, 10, 12, 16, 20, 22, 24, 26, 28, 33–36, 39–42]. In the following, we list all the known families of PN functions and/or the corresponding commutative semifields. For any odd prime \( p \), there are:

**I.**

\[ x^2 \]

over \( \mathbb{F}_{p^n} \), whose corresponding commutative semifield is the finite field \( \mathbb{F}_{p^n} \);

**II.**

\[ x^{p^s+1} \]

over \( \mathbb{F}_{p^n} \) with \( s \leq \frac{n}{2} \) and \( n/(n,s) \) odd, whose corresponding commutative semifields are Albert’s commutative twisted fields [2][14][15].

2
III.

\[ x^{p^s+1} - u^{p^k-1}x^{p^k+p^{2k+s}} \]

over \( \mathbb{F}_{p^{3k}} \), where \( \gcd(3, k) = 1 \), \( k - s \equiv 0 \pmod{3} \), \( s \neq k \), \( k/(s, k) \) is odd and \( u \) is a primitive element of \( \mathbb{F}_{p^{3k}} \), whose corresponding commutative semifields are Zha-Kyureghyan-Wang (ZKW) semifields \[5][10];

IV.

\[ x^{p^s+1} - u^{p^k-1}x^{p^{3k}+p^{k+s}} \]

over \( \mathbb{F}_{p^{4k}} \), where \( p^k \equiv p^s \equiv 1 \pmod{4} \), \( v_2(s) > v_2(k) \) and \( u \) is a primitive element of \( \mathbb{F}_{p^{4k}} \), whose corresponding commutative semifields are Bierbrauer semifields \[5];

V.

\[ \omega \beta x^{p^s+1} + \omega \beta p^k x^{p^k(p^s+1)} + x^{p^k+1} \]

over \( \mathbb{F}_{p^{2k}} \), where

\[ \omega + \omega p^k = 0, \quad \beta \equiv \frac{p^k}{\gcd(p^k+1, p^s+1)} \neq 1 \]

and \( v_2(s) \neq v_2(k) \), whose corresponding commutative semifields are Budaghyan-Helleseth-Bierbrauer (BHB) semifields \[4][8][41];

VI.

\[ x^2 + x^{2q^m} + G(x^{q^2+1}) \]

over \( \mathbb{F}_{q^{2m}} \), where \( q \) is a power of odd prime \( p \), \( m = 2k + 1 \) and \( G(x) = h(x - x^{q^m}) \) with

\[ h(x) = \sum_{i=0}^{k} (-1)^i x^{q^i} + \sum_{j=0}^{k-1} (-1)^{k+j} x^{q^{2j+1}}, \]

whose corresponding commutative semifields are Lunardon-Marino-Polverino-Trombetti-Bierbrauer (LMPTB) semifields \[1][28];

VII. The Dickson semifields

\( (a, b) \star (c, d) = (ac + ab^\sigma d, ad + bc) \)

over \( \mathbb{F}_{p^{2k}} \), where \( q \) is a power of odd prime \( p \) and \( \alpha \) is a nonsquare element in \( \mathbb{F}_{q^k} \) \[17];

VIII. The Zhou-Pott (ZP) semifields

\( (a, b) \star (c, d) = (a \circ_t c + \alpha (b \circ_t d)^\sigma, ad + bc) \)

over \( \mathbb{F}_{p^{2k}} \), where \( v_2(t) \geq v_2(k) \), \( x \circ_t y = x^{p^t} y + xy^{p^t}, \sigma \in \text{Aut}(\mathbb{F}_{p^{k}}) \) and \( \alpha \) is a nonsquare element in \( \mathbb{F}_{p^k} \) \[42].

For \( p = 3 \), there are some extra semifields and/or PN functions as follows:
• The Coulter-Matthews-Ding-Yuan semifields over $\mathbb{F}_{3^n}$ with $n$ odd \[14, 20\];
• The Cohen-Ganley semifields over $\mathbb{F}_{3^n}$ \[10\];
• The Ganley’s semifields over $\mathbb{F}_{3^{2k}}$ with $k$ odd \[35\];
• The Coulter-Henderson-Kosick semifields over $\mathbb{F}_{3^8}$ \[12\];
• The Penttila-Williams semifields over $\mathbb{F}_{3^{10}}$ \[33\];
• The PN functions $x^2 + x^{90}, x^2 - x^{84} + x^{108} + x^{162}$ over $\mathbb{F}_{3^5}$ and $3x^6 + x^{50}$ over $\mathbb{F}_{5^5}$ \[3, 13, 39\].

Up till now, all the known PN functions are DO polynomials, except for the family $x^{3^{k+1}/2}$ over $\mathbb{F}_{3^n}$, where $k \geq 3$ is odd and $(k, n) = 1$. This function corresponds to Coulter-Matthews planes \[14, 22\]. In addition, a switching construction of PN functions is presented in \[35\] and a character theoretic approach to prove the planarity of a function is given in \[36\]. Some new PN functions as the products of two linearized polynomials are introduced in \[24\]. In this paper, we will construct two new classes of PN functions over $\mathbb{F}_{p^{2m}}$ for any odd prime $p$. Moreover, we can prove that the corresponding semifields of the new functions are not isotopic to all the semifields of the previously known PN functions in general.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary knowledge and auxiliary results. Two new classes of PN functions will be constructed in Section 3. In Section 4, we shall analyze the non-isotopism of the new PN functions.

2 Preliminaries

For odd prime $p$ and positive integer $n$, let $\mathbb{F}_{p^n}$ be the finite field with $p^n$ elements and $\mathbb{F}_{p^n}^*$ its multiplicative group. Denote by $v_2(m)$ the largest power of 2 dividing $m$, i.e., $m = 2^{v_2(m)}t$ with $t$ odd.

Recall that any function $F$ from the finite field $\mathbb{F}_{p^n}$ to itself can be represented as a polynomial of degree less than $p^n$. The graph $G_F$ of the function $F$ is defined as the set $G_F = \{ (x, F(x)) \mid x \in \mathbb{F}_{p^n} \} \subset \mathbb{F}_{p^n}^2$.

A polynomial $L \in \mathbb{F}_{p^n}[x]$ with the form

$$L(x) = \sum_{i=0}^{n-1} a_i x^{p^i}, \quad a_i \in \mathbb{F}_{p^n}$$

is called linearized polynomial or $p$-polynomial over $\mathbb{F}_{p^n}$ \[27\]. A polynomial over $\mathbb{F}_{p^n}$ is affine if it is the sum of a linearized polynomial and a constant in $\mathbb{F}_{p^n}$.
**Definition 1.** Two functions $F, F': \mathbb{F}_p^n \to \mathbb{F}_p^n$ are called:

- extended affine equivalent (EA-equivalent) if

$$F' = A_1 \circ F \circ A_2 + A$$

for some affine permutations $A_1, A_2$ and affine function $A$.

- Carlet-Charpin-Zinoviev equivalent (CCZ-equivalent) if there exists some affine permutation $L: \mathbb{F}_p^{2^n} \to \mathbb{F}_p^{2^n}$ such that

$$L(G_F) = G_{F'}.$$  

Note that CCZ-equivalent functions have equal differential uniformity and EA-equivalent is a particular case of CCZ-equivalence [9]. However, CCZ-equivalence coincides with EA-equivalence for PN functions [8,25]. Furthermore, two DO type PN functions are CCZ-equivalent if and only if the corresponding presemifields are strongly isotopic [11]. It was shown by Albert that two semifields coordinatize isomorphic planes if and only if they are isotopic [1]. In what follows, we introduce the notation of isotopism and strong isotopism.

**Definition 2.** Let $S_1 = (\mathbb{F}_p^n, +, \circ)$ and $S_2 = (\mathbb{F}_p^n, +, \star)$ be two presemifields. If there exist three linearized permutations $L, M, N$ over $\mathbb{F}_p^n$ such that

$$L(x \circ y) = M(x) \star N(y)$$

for any $x, y \in \mathbb{F}_p^n$, then $S_1$ and $S_2$ are called isotopic. The triple $(M, N, L)$ is called the isotopism between $S_1$ and $S_2$. If $M = N$, then $S_1$ and $S_2$ are called strongly isotopic.

Let $(\mathbb{S}, +, \star)$ be a semifield. Then the subsets

$$N_l(\mathbb{S}) = \{a \in \mathbb{S} | a \star (x \star y) = (a \star x) \star y, \forall x, y \in \mathbb{S}\},$$

$$N_m(\mathbb{S}) = \{a \in \mathbb{S} | x \star (a \star y) = (x \star a) \star y, \forall x, y \in \mathbb{S}\},$$

and

$$N_r(\mathbb{S}) = \{a \in \mathbb{S} | x \star (y \star a) = (x \star y) \star a, \forall x, y \in \mathbb{S}\}$$

are called the left, middle and right nucleus of $\mathbb{S}$, respectively. The intersection

$$N(\mathbb{S}) = N_l(\mathbb{S}) \cap N_m(\mathbb{S}) \cap N_r(\mathbb{S})$$

is called the nucleus of $\mathbb{S}$. If $\mathbb{S}$ is commutative, then it can be verified that

$$N(\mathbb{S}) = N_l(\mathbb{S}) = N_r(\mathbb{S}) \subseteq N_m(\mathbb{S}).$$
Note that the order of the respective nucleus is invariant under isotopism. This allows us to investigate the isotopism of corresponding semifields among the new PN functions and the known PN functions by comparing the order of their nucleus.

In order to calculate the middle nucleus of the corresponding semifields of the new PN functions, we recall the following result from [21].

**Lemma 1.** ([21]) Let \( f(x) = x^{p^l+1} + x^{(p^l+1)p^r} + x^{p^k+1} - x^{(p^k+1)p^r} \) be perfect nonlinear over \( \mathbb{F}_{p^{2r}} \). If \( x^{p^k+1} \) is perfect nonlinear over \( \mathbb{F}_{p^{2r}} \), then the middle nucleus of a commutative semifield associated with \( f \) has order equal to \( p^{\gcd(2l,k,2r)} \).

The following lemma characterizes the distance between isotopism and strong isotopism. It will be used to show the non-isotopism of the corresponding semifields among the new PN functions and the known PN functions in Section 4.

**Lemma 2.** ([11]) Let \( S_1 = (\mathbb{F}_{p^n},+,*) \) and \( S_2 = (\mathbb{F}_{p^n},+,o) \) be isotopic commutative semifields. Then there exists an isotopism \((M,N,L)\) between \( S_1 \) and \( S_2 \) such that either

(i) \( M = N \) or

(ii) \( M(X) \equiv \alpha \star N(X) \mod (X^{p^n} - X) \) where \( \alpha \in N_m(S_1) \setminus N(S_1) \).

The isotopism between the corresponding semifields of functions \( V \) and \( VI \) is shown in [30] as follows.

**Lemma 3.** ([30]) The BHB semifields and LMPTB semifields, which are the corresponding commutative presemifields of functions \( V \) and \( VI \), are isotopic.

### 3 New Classes of Perfect Nonlinear Functions

In this section, based on the block matrix interpretation, for any odd prime \( p \), we construct two new classes of PN functions over \( \mathbb{F}_{p^n} \) by counting the number of solutions of certain equations.

The first class of PN functions over \( \mathbb{F}_{p^n} \) with \( n = 2m \) for any odd prime \( p \) is proposed as follows.

**Theorem 1.** Let \( p \) be an odd prime. Suppose \( s \) and \( m \) are positive integers. Let also \( n = 2m \), \( d = \gcd(m,s) \), \( m = ld \) and \( s = hd \). Then, for any non-negative integer \( r \),

\[
f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + \left(x^{p^{m-s}+1} + x^{p^m(p^{m-s}+1)}\right)^{p^r}
\]

is PN over \( \mathbb{F}_{p^n} \) if \( p^m \equiv 1 \pmod{4} \) and \( h, l \) are odd.
Proof. Since \( f \) is a DO polynomial, in order to prove that \( f \) is PN, it is sufficient to show that for any \( a \in \mathbb{F}_p^m \), the equation

\[
\Delta_{f,a}(x) := f(x + a) - f(x) - f(a) = 0
\]

has exactly one solution: \( x = 0 \). It can be verified that

\[
\Delta_{f,a}(x) = (ax^{p^s} + a^{p^s}x) + \left( a^{p^n}x^{p^{m+s}} + a^{p^{m+s}}x^{p^n} \right)
+ \left( ax^{p^{m-s}} + a^{p^{m-s}}x \right)^{p^r} - \left( a^{p^n}x^{p^{2m-s}} + a^{p^{2m-s}}x^{p^n} \right)^{p^r}.
\]

By taking \( \Delta_1 = ax^{p^s} + a^{p^s}x \) and \( \Delta_2 = a^{p^{m-s}} + a^{p^{m-s}}x \), this equation is equivalent to

\[
\Delta_{f,a}(x) = \Delta_1 + \Delta_1^{p^m} + \left( \Delta_2 - \Delta_2^{p^m} \right)^{p^r}.
\]

If \( x \) is a solution of \( \Delta_{f,a}(x) = 0 \), then

\[
\Delta_1 + \Delta_1^{p^m} + \left( \Delta_2 - \Delta_2^{p^m} \right)^{p^r} = 0. \tag{1}
\]

Raising both sides of (1) to \( p^m \)-th power, one obtains

\[
\Delta_1 + \Delta_1^{p^m} + \left( \Delta_2 - \Delta_2^{p^m} \right)^{p^r} = 0. \tag{2}
\]

It follows from (1) and (2) that

\[
\Delta_1 + \Delta_1^{p^m} = 0 \tag{3}
\]

and

\[
\Delta_2 = \Delta_2^{p^m}. \tag{4}
\]

Let \( g \) be a primitive element of \( \mathbb{F}_{p^d} \) and \( \omega = g^{p^d+1} \). It can be verified that \( \omega^{p^m} = -\omega \) and \( \omega^{p^s} = -\omega \) as \( h, l \) are odd. Denote by \( \alpha = \omega^2 \). Then \( \alpha \) is a non-square in \( \mathbb{F}_{p^m} \) and \( \{ 1, \omega \} \) is a \( \mathbb{F}_{p^m} \)-basis of \( \mathbb{F}_{p^n} \). If one sets \( x = x_1 + x_2 \omega \) and \( a = a_1 + a_2 \omega \) with \( x_1, x_2, a_1, a_2 \in \mathbb{F}_{p^m} \), then

\[
\Delta_1 = a_1x_1^{p^s} + a_1^{p^s}x_1 - \left( a_2x_2^{p^s} + a_2^{p^s}x_2 \right) \alpha + \left( a_1^{p^s}x_2 - a_1x_2^{p^s} + a_2x_1^{p^s} - a_2^{p^s}x_1 \right) \omega
\]

and

\[
\Delta_2 = a_1x_1^{p^s} + a_1^{p^s}x_1 + \left( a_2x_2^{p^s} + a_2^{p^s}x_2 \right) \alpha + \left( a_1^{p^s}x_2 + a_1x_2^{p^s} + a_2x_1^{p^s} + a_2^{p^s}x_1 \right) \omega.
\]

It follows from (3) that

\[
a_1x_1^{p^s} + a_1^{p^s}x_1 - \left( a_2x_2^{p^s} + a_2^{p^s}x_2 \right) \alpha = 0 \tag{5}
\]

and (4) indicates

\[
a_1x_2^{p^s} + a_1^{p^s}x_2 + a_2x_1^{p^s} + a_2^{p^s}x_1 = 0. \tag{6}
\]
Then

By \( p^m \equiv 1 \pmod{4} \), one can obtain that \(-1\) is a square in \( \mathbb{F}_{p^m} \) and \(-1 = \delta^2 \) for some \( \delta \in \mathbb{F}_{p^m} \). Raising both sides of (5) and (6) to \( p^s \)-th, \( p^{2s} \)-th, \( \cdots \), \( p^{(l-1)s} \)-th powers, respectively, one has

\[
\begin{pmatrix} A & -\alpha B \\ B & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where

\[
A = \begin{pmatrix} a_1^{p^s} & a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_1^{p^s} & a_1^{p^{2s}} & \cdots & 0 & 0 \\ 0 & 0 & a_1^{p^{3s}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1^{p^{(l-1)s}} & a_1^{p^{(l-2)s}} \\ a_1^{p^{(l-1)s}} & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}_{l \times l},
\]

\[
B = \begin{pmatrix} a_2^{p^s} & a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2^{p^s} & a_2^{p^{2s}} & \cdots & 0 & 0 \\ 0 & 0 & a_2^{p^{3s}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_2^{p^{(l-1)s}} & a_2^{p^{(l-2)s}} \\ a_2^{p^{(l-1)s}} & 0 & 0 & \cdots & 0 & a_2 \end{pmatrix}_{l \times l},
\]

\[
X_1 = \begin{pmatrix} x_1 \\ x_1^{p^s} \\ \vdots \\ x_1^{p^{(l-1)s}} \end{pmatrix}_{l \times 1}, \quad X_2 = \begin{pmatrix} x_2 \\ x_2^{p^s} \\ \vdots \\ x_2^{p^{(l-1)s}} \end{pmatrix}_{l \times 1}
\]

and \( 0 \) is the \( l \times 1 \) zero matrix. It can be checked that

\[
\begin{pmatrix} E_l & \delta \omega E_l \\ O_l & E_l \end{pmatrix} \begin{pmatrix} A & -\alpha B \\ B & A \end{pmatrix} \begin{pmatrix} E_l & -\delta \omega E_l \\ O_l & E_l \end{pmatrix} = \begin{pmatrix} A + \delta \omega B & O_l \\ B & A - \delta \omega B \end{pmatrix},
\]

where \( E_l \) is the \( l \times l \) identity matrix and \( O_l \) is the \( l \times l \) zero matrix. Denote by

\[
D = \begin{pmatrix} A & -\alpha B \\ B & A \end{pmatrix}.
\]

Then

\[
\det (D) = \det (A + \delta \omega B) \det (A - \delta \omega B).
\]

It follows from

\[
\det (A \pm \delta \omega B) = 2 (a_1 \pm a_2 \delta \omega) (a_1^{p^s} \pm a_2^{p^s} \delta \omega) \cdots (a_1^{p^{(l-1)s}} \pm a_2^{p^{(l-1)s}} \delta \omega)
\]

8
that \( \text{det}(D) = 0 \) if and only if \( a_1 = a_2 = 0 \) since \( a_1, a_2 \in \mathbb{F}_{p^m} \) and \( \delta \omega \notin \mathbb{F}_{p^m} \). This shows that \( \text{det}(D) \neq 0 \) holds for any \( a \in \mathbb{F}_{p^m}^* \). Note that (1) is equivalent to (7). It follows that (1) has only one solution \( x = 0 \). Consequently, \( f(x) \) is PN if \( p^m \equiv 1 \pmod{4} \) and \( h, l \) are odd. This proves the desired conclusion. \( \square \)

**Remark 1.** The function in Theorem 1

\[
f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + \left(x^{p^{-s}+1} - x^{p^m(p^{-s}+1)}\right)^{pr}
\]

is not PN if \( p^m \equiv 3 \pmod{4} \). Indeed, for \( p^m \equiv 3 \pmod{4} \), we can choose \( \alpha = -1 \) in the proof of Theorem 1. It follows from (5) and (6) that

\[
\begin{align*}
\begin{cases}
    a_1 x_1^{p^s} + a_1^{p^s} x_1 + a_2 x_2^{p^s} + a_2^{p^s} x_2 = 0, \\
    a_1 x_2^{p^s} + a_1^{p^s} x_2 + a_2 x_1^{p^s} + a_2^{p^s} x_1 = 0,
\end{cases}
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\begin{cases}
    (a_1 + a_2)(x_1 + x_2)^{p^s} + (a_1 + a_2)^{p^s}(x_1 + x_2) = 0, \\
    (a_1 - a_2)(x_1 - x_2)^{p^s} + (a_1 - a_2)^{p^s}(x_1 - x_2) = 0.
\end{cases}
\end{align*}
\]

(8)

It can be verified that (8) has nonzero solutions \( x_1 = x_2 \neq 0 \) when \( a_1 = -a_2 \neq 0 \).

In [21], the authors show several sporadic examples of PN functions in the form of

\[
x^{p^k+1} + x^{p^3k(p^k+1)} + x^{p^{2k}+1} - x^{p^{3k}(p^{2k}+1)},
\]

which are contained in Theorem 1 when \( s = k, m = 3k \) and \( r = 0 \).

In the sequel, for any finite field \( \mathbb{F}_{p^m} \) with \( n = 2m \), another new class of PN functions can be obtained using a similar method as in Theorem 1.

**Theorem 2.** Let \( p \) be an odd prime, \( m \) positive integer. Let also \( r \) and \( s \) be non-negative integers, \( n = 2m, d = \gcd(m, s), m = ld \) and \( s = hd \). Then the function

\[
f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + \left(x^{p^{-s}+1} - x^{p^m(p^{-s}+1)}\right)^{pr}
\]

is PN over \( \mathbb{F}_{p^m} \) if \( h \) is even and \( l \) is odd.

**Proof.** Since \( f \) is a DO polynomial, to prove that \( f \) is PN, it suffices to show that for any \( a \in \mathbb{F}_{p^m}^* \), the equation

\[
\Delta_{f,a}(x) := f(x + a) - f(x) - f(a) = 0
\]
has exactly one solution: \( x = 0 \). Note that

\[
\Delta_{f,a}(x) = (ax^{p^s} + a^{p^s}x) + \left(a^{p^m}x^{p^{m+s}} + a^{p^{m+s}}x^m\right) + \left(a_2x^{p^{n-s}} + a^{p^{n-s}}x\right)^{p^r} - \left(a^{p^m}x^{p^{n-s}} + a^{p^{n-s}}x^m\right)^{p^r}.
\]

Let \( \Delta_1 = ax^{p^s} + a^{p^s}x \) and \( \Delta_2 = ax^{p^{n-s}} + a^{p^{n-s}}x \). Then the previous equation can be rewritten as

\[
\Delta_{f,a}(x) = \Delta_1 + \Delta_1^{p^m} + \left(\Delta_2 - \Delta_2^{p^m}\right)^{p^r}.
\]

If \( x \) is a solution of \( \Delta_{f,a}(x) = 0 \), one has

\[
\Delta_1 + \Delta_1^{p^m} + \left(\Delta_2 - \Delta_2^{p^m}\right)^{p^r} = 0. \tag{9}
\]

Raising both sides of (9) to \( p^m \)-th power, one can obtain

\[
\Delta_1 + \Delta_1^{p^m} + \left(\Delta_2 - \Delta_2^{p^m}\right)^{p^r} = 0. \tag{10}
\]

Then (9) and (10) imply

\[
\Delta_1 + \Delta_1^{p^m} = 0 \tag{11}
\]

and

\[
\Delta_2 = \Delta_2^{p^m}. \tag{12}
\]

Let \( g \) be a primitive element of \( \mathbb{F}_{p^{m+1}} \) and \( \omega = g^{p^r} \). Since \( h \) is even and \( l \) is odd, one has \( \omega^{p^m} = -\omega \) and \( \omega^{p^s} = \omega \). It follows that \( \omega \notin \mathbb{F}_{p^m} \) and \( \alpha := \omega^2 \in \mathbb{F}_{p^m} \). Thus \( \{1, \omega\} \) is a \( \mathbb{F}_{p^m} \)-basis of \( \mathbb{F}_{p^n} \). Let \( x = x_1 + x_2\omega \) and \( a = a_1 + a_2\omega \) with \( x_1, x_2, a_1, a_2 \in \mathbb{F}_{p^m} \). Then

\[
\Delta_1 = a_1x_1^{p^r} + a_1^{p^r}x_1 + \left(a_2x_2^{p^r} + a_2^{p^r}x_2\right)\alpha + \left(a_2x_1^{p^r} + a_2^{p^r}x_1 + a_1x_2^{p^r} + a_1^{p^r}x_2\right)\omega
\]

and

\[
\Delta_2 = a_1x_1^{p^r} + a_1^{p^r}x_1 + \left(a_2x_2^{p^r} + a_2^{p^r}x_2\right)\alpha + \left(a_2x_1^{p^r} + a_2^{p^r}x_1 + a_1x_2^{p^r} + a_1^{p^r}x_2\right)\omega.
\]

Hence (11) implies

\[
a_1x_1^{p^r} + a_1^{p^r}x_1 + \left(a_2x_2^{p^r} + a_2^{p^r}x_2\right)\alpha = 0 \tag{13}
\]

and (12) indicates

\[
a_2x_1^{p^r} + a_2^{p^r}x_1 + a_1x_2^{p^r} + a_1^{p^r}x_2 = 0. \tag{14}
\]

Raising both sides of (13) and (14) to \( p^s \)-th, \( p^{2s} \)-th, \( \ldots \), \( p^{(l-1)s} \)-th power, respectively, one obtains

\[
\begin{pmatrix}
A & \alpha B
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
=
\begin{pmatrix}
0
\end{pmatrix}.
\tag{15}
\]
implies that (9) has only one solution: $x$ and $a$
completes the proof.

Note that $E_0$ where

$$E_0 = \begin{pmatrix} 0 \end{pmatrix},$$

is the $l \times 1$ zero matrix. It follows that

$$\det (O) = 0.$$ 

and 0 is the $l \times 1$ zero matrix. It follows that

$$\begin{pmatrix} E_l & \omega E_l \\ O_l & E_l \end{pmatrix} \begin{pmatrix} A & \alpha B \\ B & A \end{pmatrix} \begin{pmatrix} E_l & -\omega E_l \\ O_l & E_l \end{pmatrix} = \begin{pmatrix} A + \omega B & O_l \\ B & A - \omega B \end{pmatrix},$$

where $E_l$ is the $l \times l$ identity matrix and $O_l$ is the $l \times l$ zero matrix. Therefore,

$$\det \begin{pmatrix} A & \alpha B \\ B & A \end{pmatrix} = \det (A + \omega B) \det (A - \omega B).$$

Note that

$$\det (A + \omega B) = 2 (a_1 \pm a_2 \omega) (a_1^{p^s} \pm a_2^{p^s} \omega) \cdots (a_1^{p^{(l-1)s}} \pm a_2^{p^{(l-1)s}} \omega)$$

and $a_1, a_2 \in \mathbb{F}_{p^m}$ and $\omega \notin \mathbb{F}_{p^m}$. It follows that $\det (D) = 0$ if and only if $a_1 = a_2 = 0$. This implies that (2) has only one solution: $x = 0$. Hence $f(x)$ is PN if $h$ is even and $l$ is odd. This completes the proof.

4 Non-isotopism of New Perfect Nonlinear Functions

In this section, we shall analyze the non-isotopism of the corresponding semifields among the new PN functions in Theorems 1 and 2 and all the known PN functions mentioned in Section 1.
Note that the new PN functions are DO polynomials and exist for any odd prime \( p \), it suffices to consider the functions I-VIII.

### 4.1 Nuclei of New Perfect Nonlinear Functions

For brevity, we only consider the case \( r = 0 \). The left and middle nucleus of the corresponding semifields of the new PN functions in Theorem 1 is calculated as follows.

**Proposition 1.** Let \( p \) be an odd prime, \( n = 2m \), \( p^m \equiv 1 \pmod{4} \), \( d = \gcd(m, s) \), \( m = ld \), \( s = hd \) with \( h, l \) odd. Let \( S = (F_{p^n}, +, \cdot) \) be the corresponding semifield of the function
\[
f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^{m-s}+1} - x^{p^m(p^{m-s}+1)}.
\]
Then the left nucleus of \( S \) is equal to the middle nucleus and has order \( p^{2d} \).

**Proof.** Observe that \( x^{p^{m-s}+1} \) is PN over \( F_{p^n} \) as \( h \) and \( l \) are odd. It follows from Lemma 1 that the order of the middle nucleus of \( S \) is
\[
p^{\gcd(2s, m-s, n)} = p^{2d}.
\]
Since the left nucleus of \( S \) is a subfield of the middle nucleus, it suffices to prove that the left nucleus of \( S \) contains \( F_{p^{2d}} \).

Note that the presemifields of \( f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^{m-s}+1} - x^{p^m(p^{m-s}+1)} \) can be expressed by \( S = (F_{p^n}, +, \cdot) \) with
\[
x \cdot y = xy^{p^s} + x^{p^m} y + x^{p^{m-s}} y^{p^m + s} + x^{p^m+y^{p^m+s} + x^{p^m-s} y - x^m y^{2m-s} - x^{2m-s} y^m} \]
and \( x^{p^m} = x^{p^s} \) holds for any \( x \in F_{p^n} \). Let \( L(x) = 1 * x = 2x + x^p + x^{p(l-1)s} + x^{p(l+1)s} - x^{p(2l)s} \). Then it can be verified that
\[
L^{-1}(x) = \frac{1}{4} x + \frac{1}{4} \sum_{i=1}^{(l-1)/2} \left( x^{p^{(2i+1)s}} - x^{p^{(2i-1)s}} \right).
\]
It follows that
\[
L^{-1}(x \cdot y) = \frac{1}{4} \left( x^{p^s} + x^{p(l-1)s} + x^{p(l+1)s} + x^{p(2l-1)s} \right) y
\]
\[
+ \frac{1}{4} \sum_{i=1}^{l-1} \left( (-1)^{l-1} x^{p(i-1)s} + (-1)^i x^{p(i+1)s} - x^{p(i-1-1)s} + x^{p(i+1+1)s} \right) y^{p^s}
\]
\[
- \frac{1}{4} \left( x^{p^s} - x^{p(l-1)s} - x^{p(l+1)s} + x^{p(2l-1)s} \right) y^{p^s}
\]
\[
+ \frac{1}{4} \sum_{i=l+1}^{2l-1} \left( (-1)^l x^{p(i-l)s} + (-1)^{l+1} x^{p(i+1)s} + x^{p(i-1-1)s} - x^{p(i+1+1)s} \right) y^{p^s}.
\]
For any \( z \in \mathbb{F}_p^n, a \in \mathbb{F}_{p^{2d}} \), it can be checked that

\[
z \ast a = \left( z + z^{p^s} + z^{p^{(l-1)s}} + z^{p^s} \right) a + \left( z - z^{p^s} + z^{p^{(l+1)s}} - z^{p^{(2l-1)s}} \right) a^{p^d}
\]

since

\[
a^{kd} = \begin{cases} 
  a^{p^d}, & k \text{ is odd}, \\
  a, & k \text{ is even}.
\end{cases}
\]

Substituting \( z \) by \( L^{-1}(x \ast y) \), one yields

\[
z + z^{p^s} + z^{p^{(l-1)s}} + z^{p^s} = \frac{1}{2} \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y + \frac{1}{2} \left( x + x^{p^s} \right) y^{p^s}
\]

\[
+ \frac{1}{2} \left( x + x^{p^s} \right) y^{p^{(l-1)s}} + \frac{1}{2} \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y^{p^s}
\]

\[
- \frac{1}{2} \left( x - x^{p^s} \right) y^{p^{(l+1)s}} + \frac{1}{2} \left( x - x^{p^s} \right) y^{p^{(2l-1)s}},
\]

and

\[
z - z^{p^s} + z^{p^{(l+1)s}} - z^{p^{(2l-1)s}} = \frac{1}{2} \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y + \frac{1}{2} \left( x - x^{p^s} \right) y^{p^s}
\]

\[
+ \frac{1}{2} \left( x - x^{p^s} \right) y^{p^{(l-1)s}} - \frac{1}{2} \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y^{p^s}
\]

\[
+ \frac{1}{2} \left( x + x^{p^s} \right) y^{p^{(l+1)s}} - \frac{1}{2} \left( x + x^{p^s} \right) y^{p^{(2l-1)s}}.
\]

It follows that

\[
L^{-1}(x \ast y) \ast a = \frac{1}{2} \left( x^{p^s} + x^{p^{(l-1)s}} - x^{p^{(l+1)s}} + x^{p^{(2l-1)s}} \right) y + \frac{1}{2} a \left( x + x^{p^s} \right) y^{p^s}
\]

\[
+ \frac{1}{2} a \left( x + x^{p^s} \right) y^{p^{(l-1)s}} + \frac{1}{2} a \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y^{p^s}
\]

\[
- \frac{1}{2} a \left( x - x^{p^s} \right) y^{p^{(l+1)s}} + \frac{1}{2} a \left( x - x^{p^s} \right) y^{p^{(2l-1)s}}
\]

\[
+ \frac{1}{2} a^{p^d} \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y + \frac{1}{2} a^{p^d} \left( x - x^{p^s} \right) y^{p^s}
\]

\[
+ \frac{1}{2} a^{p^d} \left( x - x^{p^s} \right) y^{p^{(l+1)s}} - \frac{1}{2} a^{p^d} \left( x + x^{p^s} \right) y^{p^{(2l-1)s}}.
\]
On the other hand, due to (16), one obtains

\[
L^{-1}(x \ast a) = \frac{1}{4} a \sum_{i=1}^{(l-1)/2} \left( (-1)^{2i-1} x^{p^{(2i-1)s}} + (-1)^{2i} x^{p^{(2i+1)s}} - x^{p^{(l+2i-1)s}} + x^{p^{(l+2i+1)s}} \right) \\
+ \frac{1}{4} a \sum_{i=(l+1)/2}^{l-1} \left( (-1)^{2i} x^{p^{(2i-1)s}} + (-1)^{2i+1} x^{p^{(2i+1)s}} + x^{p^{(l+2i-1)s}} - x^{p^{(l+2i+1)s}} \right) \\
+ \frac{1}{4} \sum_{i=1}^{(l-1)/2} \left( (-1)^{2i-2} x^{p^{(2i-2)s}} + (-1)^{2i-1} x^{p^{(2i)s}} - x^{p^{(l+2i-2)s}} + x^{p^{(l+2i)s}} \right) \\
+ \frac{1}{4} \sum_{i=(l+1)/2}^{l-1} \left( (-1)^{2i+1} x^{p^{(2i)s}} + (-1)^{2i+2} x^{p^{(2i+2)s}} + x^{p^{(l+2i)s}} - x^{p^{(l+2i+2)s}} \right) \\
- \frac{1}{4} \left( x^{p^s} - x^{p^{l-1}s} - x^{p^{l+1}s} + x^{p^{2l-1}s} \right)
\]

\[
= \frac{1}{2} \left( x + x^{p^s} \right) a + \frac{1}{2} \left( x - x^{p^s} \right) a^{p^d}.
\]

It can be verified that

\[
L^{-1}(x \ast a) \ast y = \left( \frac{1}{2} \left( x + x^{p^s} \right) a + \frac{1}{2} \left( x - x^{p^s} \right) a^{p^d} \right) \ast y \\
= \frac{1}{2} a \left( x^{p^s} + x^{p^{l-1}s} - x^{p^{l+1}s} + x^{p^{2l-1}s} \right) y + \frac{1}{2} a \left( x + x^{p^s} \right) y^{p^s} \\
+ \frac{1}{2} a \left( x + x^{p^s} \right) y^{p^{(l-1)s}} + \frac{1}{2} a \left( x^{p^s} + x^{p^{(l-1)s}} + x^{p^{(l+1)s}} - x^{p^{(2l-1)s}} \right) y^{p^d} \\
- \frac{1}{2} a \left( x - x^{p^s} \right) y^{p^{(l+1)s}} + \frac{1}{2} a \left( x - x^{p^s} \right) y^{p^{(2l-1)s}} \\
+ \frac{1}{2} a^{p^d} \left( x^{p^s} + x^{p^{l-1}s} + x^{p^{l+1}s} - x^{p^{2l-1}s} \right) y + \frac{1}{2} a^{p^d} \left( x + x^{p^s} \right) y^{p^s} \\
+ \frac{1}{2} a^{p^d} \left( x - x^{p^s} \right) y^{p^{(l-1)s}} - \frac{1}{2} a^{p^d} \left( x^{p^s} + x^{p^{(l-1)s}} - x^{p^{(l+1)s}} + x^{p^{(2l-1)s}} \right) y^{p^s} \\
+ \frac{1}{2} a^{p^d} \left( x + x^{p^s} \right) y^{p^{(l+1)s}} - \frac{1}{2} a^{p^d} \left( x + x^{p^s} \right) y^{p^{(2l-1)s}}.
\]

Therefore,

\[
L^{-1}(x \ast a) \ast y = L^{-1}(x \ast y) \ast a
\]

holds for any \( a \in \mathbb{F}_{p^{2d}} \), which yields that the left nucleus of \( S \) is \( \mathbb{F}_{p^{2d}} \). This proves the desired conclusion. \( \square \)

The following proposition derives the left and middle nucleus of the corresponding semifields of the new PN functions in Theorem 2 for the special case of \( r = 0, n = 6k \) and \( s = 2k \).
Proposition 2. Let $S = (\mathbb{F}_{p^{2k}}, +, *)$ be the corresponding semifield of the function
\[ f(x) = x^{p^{2k}+1} + x^{p^3k(p^{2k}+1)} + x^{p^{4k}+1} - x^{p^{3k}(p^{4k}+1)}. \]

Then the left nucleus of $S$ is equal to the middle nucleus and has order $p^{2k}$.

Proof. Since $x^{p^{3k}+1}$ is PN over $\mathbb{F}_{p^{2k}}$, according to Lemma 1, the order of the middle nucleus of $S$ is
\[ p^{\gcd(2k,4k,6k)} = p^{2k}. \]

Note that the left nucleus of $S$ is a subfield of the middle nucleus. It only needs to show that the left nucleus of $S$ contains $\mathbb{F}_{p^{2k}}$.

It is worth noting that the presemifields of $f(x) = x^{p^{2k}+1} + x^{p^3k(p^{2k}+1)} + x^{p^{4k}+1} - x^{p^{3k}(p^{4k}+1)}$ can be represented by $S = (\mathbb{F}_{p^{2k}}, +, *)$ with
\[ x \ast y = xy^{p^{2k}} + x^{p^{2k}} y + x^{p^3k} y^{p^{2k}} + x^{p^5k} y^{p^{3k}} + xy^{p^{4k}} + x^{p^{4k}} y - x^{p^{2k}} y^{p^{4k}} - x^{p^{3k}} y^{p^{2k}}. \]

Let $L(x) = x \ast 1 = 2x - x^{p^k} + x^{p^{2k}} + x^{p^{4k}} + x^{p^{5k}}$. Then one has
\[ L^{-1}(x) = \frac{1}{4} x + \frac{1}{4} x^{p^k} - \frac{1}{4} x^{p^{5k}}. \]

Hence
\[ L^{-1}(x \ast y) = \frac{1}{2} \left( xy^{p^{2k}} + x^{p^{2k}} y + xy^{p^{4k}} + x^{p^{4k}} y - x^{p^{2k}} y^{p^{4k}} - x^{p^{3k}} y^{p^{2k}} \right). \]

For any $z \in \mathbb{F}_{p^{2k}}, a \in \mathbb{F}_{p^{2k}}$, one can obtain
\[ z \ast a = \left( 2z + z^{p^{2k}} + z^{p^{4k}} \right) a + \left( z^{p^{5k}} - z^{p^{k}} \right) a^{p^k}. \]

Substituting $z$ by $L^{-1}(x \ast y)$, one yields
\[ 2z + z^{p^{2k}} + z^{p^{4k}} = xy^{p^{2k}} + x^{p^{2k}} y + x^{p^{4k}} y + xy^{p^{4k}} \]
and
\[ z^{p^{5k}} - z^{p^{k}} = x^{p^{3k}} y^{p^{5k}} + x^{p^{5k}} y^{p^{3k}} - x^{p^{3k}} y^{p^{k}} - x^{p^{k}} y^{p^{3k}}. \]

It follows that
\[ L^{-1}(x \ast y) \ast a = \left( xy^{p^{2k}} + x^{p^{2k}} y + x^{p^{4k}} y + xy^{p^{4k}} \right) a \]
\[ + \left( x^{p^{3k}} y^{p^{5k}} + x^{p^{5k}} y^{p^{3k}} - x^{p^{3k}} y^{p^{k}} - x^{p^{k}} y^{p^{3k}} \right) a^{p^k}. \]

15
On the other hand,

\[ L^{-1}(x*a)*y = \frac{1}{2} \left( ax^{p^2k} + a^p x + ax^{p^4k} + a^{p^2} x - a^{p^4k} x^{p^2k} - a^{p^2k} x^{p^4k} \right) * y \
\]

\[ = (ax) * y \
\]

\[ = a^{p^2k} x^{p^2k} y + a x y^{p^2k} + a^{p^4k} x^{p^4k} y^{p^2k} + a^{p^2} x y^{p^4k} + a^{p^4} x^{p^4k} y + a x y^{p^4k} \
\]

\[ - a^{p^2k} x^{p^2k} y^{p^2k} - a^{p^4k} x^{p^4k} y^{p^2k} \
\]

\[ + (x y^{p^2k} + x y^{p^4k}) a \
\]

\[ + (x^{p^2k} y^{p^2k} + x^{p^4k} y^{p^2k} - x^{p^4k} y^{p^4k} - x^{p^2k} y^{p^4k}) a^{p^2k}. \
\]

It is straightforward that for any \( a \in \mathbb{F}_{p^{2k}} \),

\[ L^{-1}(x*a)*y = L^{-1}(x*y) * a. \]

This shows that the left nucleus of \( S \) is \( \mathbb{F}_{p^{2k}} \). The desired conclusion follows. \( \square \)

**Remark 2.** It is conjectured that both the left nucleus and middle nucleus of the associated semifield of \( f(x) \) in Theorem 2 have order \( p^{2d} \). But here we can not prove it since we have no idea on how to find an explicit expression of \( L^{-1}(x) \).

### 4.2 Non-isotopism of New Perfect Nonlinear Functions

Note that Propositions 1 and 2 show that for some special cases, the corresponding semifields of the new PN functions in Theorems 1 and 2 have the same left nucleus and middle nucleus, respectively. It follows from Lemma 2 that the isotopism of the corresponding semifields is equivalent to the strong isotopism of them.

**Proposition 3.** Let \( p \) be an odd prime, \( n = 2m \), \( 0 \leq t < n \), \( v_2(t) > v_2(m) \), \( p^m \equiv 1 \pmod{4} \), \( 0 < s < n \), \( s \neq m \), \( d = \gcd(m, s) \), \( m = ld \) and \( s = hd \) with \( h, l \) odd. Then the associated semifield of

\[ f(x) = x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^m-s+1} - x^{p^m} (p^m-s+1) \]

is not isotopic to the associated semifield of

\[ g(x) = x^{p^s+1}. \]

In particular, \( f(x) \) is CCZ-inequivalent to \( g(x) \) in general.

**Proof.** Note that

isotopism of the semifields \( \iff \) by Lemma 2 and Prop 1 strong isotopism of the semifields
\[ \iff CCZ-equivalence of the associated PN functions \\iff EA-equivalence of the associated PN functions. \]

Therefore it is sufficient to show that \( f(x) \) and \( g(x) \) are EA-inequivalent. Otherwise, assume \( f(x) \) is EA-equivalent to \( g(x) \). Precisely, there exist linear permutations

\[ L_1(x) = \sum_{i=0}^{n-1} d_i x^{p^i} \]

and

\[ L_2(x) = \sum_{i=0}^{n-1} e_i x^{p^i} \]

such that

\[ L_1(f(x)) = g(L_2(x)). \]  \tag{17} \]

In the following, all the subscripts of \( d_i \) and \( e_i \) are regarded as module \( n \). For instance,

\[ d_{-i} = d_{n-i}, \quad e_{-i} = e_{n-i}. \]

If \( t = 0 \), then \( \text{(17)} \) can be simplified to

\[ \sum_{i=0}^{n-1} d_i \left( x^{p^i+1} + x^{p^m(p^i+1)} + x^{p^m-s+1} - x^{p^m(p^m-s+1)} \right)^{p^i} = \sum_{j,l=0}^{n-1} e_j e_l x^{p^j+p^l}. \]  \tag{18} \]

Comparing the coefficients of \( x^{2p^i} \) with \( 0 \leq i \leq n - 1 \) on both sides of \( \text{(18)} \), we have

\[ 0 = e_i^2. \]  \tag{19} \]

It follows that \( e_i = 0 \) holds for any \( 0 \leq i \leq n - 1 \), which yields

\[ L_2(x) = 0. \]

This is impossible since \( L_2(x) = 0 \) cannot be a permutation.

For \( 1 \leq t \leq n - 1 \), we discuss the following two cases.

**Case I:** If \( t \neq m - s, m + s \), then \( \text{(17)} \) can be expanded as

\[ \sum_{i=0}^{n-1} d_i \left( x^{p^i+1} + x^{p^m(p^i+1)} + x^{p^m-s+1} - x^{p^m(p^m-s+1)} \right)^{p^i} = \sum_{j,l=0}^{n-1} e_j e_l x^{p^j+p^l+p^t}. \]  \tag{20} \]

Comparing the coefficients of \( x^{2p^i} \) and \( x^{p^j(p^i+1)} \) with \( 0 \leq i \leq n - 1 \) on both sides of \( \text{(20)} \), respectively, one has

\[ 0 = e_i e_{i-t}. \]  \tag{21} \]

17
Assume that there exists some \( j \) such that \( e_j \neq 0 \). Then, (21) indicates \( e_{j-t} = 0 \). It follows from (22) that \( e_j = 0 \), a contradiction. This indicates \( e_i = 0 \) for any \( 0 \leq i \leq n - 1 \). Thus \( L_2(x) = 0 \), a contradiction.

**Case II:** If \( t = m - s \) or \( t = m + s \), we only consider the case \( t = m - s \) since \( x^{p_{m-s}+1} \) and \( x^{p_{m+s}+1} \) are EA-equivalent. Note that (17) can be expressed as

\[
0 = e_i x + e_{i+m} x^{p^i + i + m} + e_{i+2s} x^{p^i + i + 2s} + e_{i+m+2s} x^{p^i + i + m + 2s},
\]

where \( e_i \neq 0 \). Thus (23) can be simplified as

\[
\sum_{j=0}^{n-1} d_j \left( x^{p^j + 1} + x^{p^m(p^s + 1)} + x^{p^m-s+1} - x^{p^m(p^m-s+1)} \right)^{p^j} = \sum_{i,j=0}^{n-1} e_i e_j x^{p^i + p^j + m - s}.
\]

For any \( j \neq i \), \( i + m \), \( i + 2s \), \( i + m + 2s \), comparing the coefficients of \( x^{2p^i} \) and \( x^{p^i + p^j} \) on both sides of (23), respectively, one obtains

\[
0 = e_i^{m-s} e_i,
\]

and

\[
0 = e_{i+m+s} e_j + e_{i+m+2s} e_i.
\]

Assume that there exists some \( e_i \neq 0 \) with \( 0 \leq i \leq n - 1 \). Then it follows from (24) that \( e_{i+m+s} = 0 \). According to (25), one yields \( e_j = 0 \) for any \( j \neq i \), \( i + m \), \( i + 2s \), \( i + m + 2s \). This indicates

\[
L_2(x) = e_i x + e_{i+m} x^{p^i + i + m} + e_{i+2s} x^{p^i + i + 2s} + e_{i+m+2s} x^{p^i + i + m + 2s},
\]

and

\[
0 = e_i^{m-s} e_{i+2s}.
\]

Since \( m \neq s \), \( m/d \) and \( s/d \) are odd, one obtains \( m - 3s \neq \pm s \), \( m \pm s \) and \( 3s \neq \pm s \), \( m \pm s \). Then observing the coefficients of \( x^{p^i + i + m + 2s} \) and \( x^{p^i + i + m + 2s} \) on both sides of (25), one has

\[
0 = e_i^{m-s} e_{i+2s}.
\]
and
\[ 0 = e^i p^m s e^{i+m+2s}. \] (28)

This shows that \( e^{i+2s} = e^{i+m+2s} = 0 \) as \( e_i \neq 0 \). Consequently, (25) can be transformed into
\[
\sum_{j=0}^{n-1} d_j \left( x^{p^j+1} + x^{p^m(p^j+1)} + x^{p^{m-s}+1} - x^{p^m(p^{m-s}+1)} \right) p^j
\] (29)

\[ = e^i p^{m-s+1} x^{p^i(p^{m-s}+1)} + e_i p^{m-s} e_{i+m} x^{p^i+m} + e_i e_{i+m}^s x^{p^i-s} + e_i e_{i+m}^s x^{p^i+m}. \]

Comparing the coefficients of \( x^{p^i(p^{m-s}+1)} \), \( x^{p^i+m} \), \( x^{p^i-s} \), \( x^{p^i+m} \) on both sides of (29), respectively, one yields
\[ d_i - d_{i+m} = e_i p^{m-s+1}, \] (30)
\[ d_{i+m} - d_i = e_{i+m} \] (31)
\[ d_{i+m-s} + d_{i-s} = e_i p^{m-s}, \] (32)

and
\[ d_{i-s} + d_{i+m-s} = e_i e_{i+m}. \] (33)

It follows from (30)-(33) that
\[ e_{i+m}^{p^{m-s+1}} + e_i^{p^{m-s+1}} = 0 \quad \text{and} \quad e_i^{p^{m-s}} e_{i+m} = e_{i+m} e_i \]
which yields
\[ e_i^2 + e_{i+m}^2 = 0. \] (34)

Due to \( p^m \equiv 1 \pmod{4} \), by (34) it is routine to check that \( L_2(x) = e_i x^{p^i} + e_{i+m} x^{p^i+m} \) is not a permutation, a contradiction. This completes the proof.

Similarly, the following proposition demonstrates that the corresponding semifields of the new PN functions in Theorems 2 with \( r = 0 \) are not strongly isotopic to the semifields of functions I-II.

**Proposition 4.** Let \( p \) be an odd prime, \( n = 2m \), \( 0 \leq t < n \), \( v_2(t) > v_2(m) \), \( 0 < s < n \), \( d = \gcd(m,s) \), \( m = ld \) and \( s = hd \) where \( h \) is even and \( l \) is odd. Then the corresponding semifield of
\[ f(x) = x^{p^t+1} + x^{p^m(p^t+1)} + x^{p^{n-s}+1} - x^{p^m(p^{n-s}+1)} \]
is not strongly isotopic to the semifield of
\[ g(x) = x^{p^t+1}. \]

In particular, if \( s = 2k \) and \( m = 3k \), the corresponding semifields are not isotopic.
Proof. Similar as in the proof of Prop. 3, it suffices to show that \( f(x) \) and \( g(x) \) are EA-inequivalent. On the contrary, assume there exist linear permutations

\[
L_1(x) = \sum_{i=0}^{n-1} d_i x^{p^i}
\]

and

\[
L_2(x) = \sum_{i=0}^{n-1} e_i x^{p^i}
\]

such that

\[
L_1(f(x)) = g(L_2(x)). \tag{35}
\]

If \( t = 0 \), then (35) can be expressed as

\[
\sum_{i=0}^{n-1} d_i \left(x^{p^i+1} + x^{p^m(p^+1)} + x^{p^n-s+1} - x^{p^m(p^n-s+1)}\right) = \sum_{j,l=0}^{n-1} e_j e_l x^{p^j+p^l}. \tag{36}
\]

Comparing the coefficients of \( x^{2p^i} \) with \( 0 \leq i \leq n - 1 \) on both sides of (36), one has

\[
0 = e_i^2. \tag{37}
\]

It follows that \( e_i = 0 \) holds for any \( 0 \leq i \leq n - 1 \), which yields

\[
L_2(x) = 0.
\]

This is absurd since \( L_2(x) = 0 \) cannot be a permutation.

The case \( 1 \leq t \leq n - 1 \) is divided into two subcases.

**Subcase I:** If \( t \neq s, n - s \), then (35) can be simplified as

\[
\sum_{i=0}^{n-1} d_i \left(x^{p^i+1} + x^{p^m(p^+1)} + x^{p^n-s+1} - x^{p^m(p^n-s+1)}\right) = \sum_{j,l=0}^{n-1} e_j e_l x^{p^j+p^l+t}. \tag{38}
\]

Comparing the coefficients of \( x^{2p^i} \) and \( x^{p^j(p^l+1)} \) with \( 0 \leq i \leq n - 1 \) on both sides of (38), respectively, one obtains

\[
0 = e_{i-t} e_i \tag{39}
\]

and

\[
0 = e_{i+t}^p + e_{i-t} e_{i+t}. \tag{40}
\]

Assume that there exists some \( j \) such that \( e_j \neq 0 \). Then, (39) indicates \( e_{j-t} = 0 \). It follows from (40) that \( e_j = 0 \), a contradiction. Therefore, \( e_i = 0 \) for any \( 0 \leq i \leq n - 1 \). Thus \( L_2(x) = 0 \), a contradiction.
**Subcase II:** If \( t = s \) or \( t = n - s \), it suffices to consider the case \( t = s \) since \( x^{p^s+1} \) and \( x^{p^{n-s}+1} \) are EA-equivalent. Then (35) can be expanded as

\[
\sum_{j=0}^{n-1} d_j \left( x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^{n-s}+1} - x^{p^m(p^{n-s}+1)} \right)^{p^j} = \sum_{i,j=0}^{n-1} e_i e_j x^{p^j + p^{j+s}}. \tag{41}
\]

Observing the coefficients of \( x^{2p^i} \) with \( 0 \leq i \leq n-1 \) on both sides of (41), one has

\[
0 = e_{i-s} e_i. \tag{42}
\]

For any \( j \neq i, i - 2s \), comparing the coefficients of \( x^{p^i+p^{j+s}} \) on both sides of (41), one can obtain

\[
0 = e_{i-s} e_{j+s} + e_j x^{p^i}. \tag{43}
\]

Assume that there exists some \( e_i \neq 0 \). By (42) and (43), we deduce that

\[ e_{i-s} = 0 \]

and \( e_j = 0 \) for any \( j \neq i, i - 2s \). As a consequence,

\[ L_2(x) = e_i x^{p^i} + e_{i-2s} x^{p^{i-2s}} \]

and (41) is transformed to

\[
\sum_{j=0}^{n-1} d_j \left( x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^{n-s}+1} - x^{p^m(p^{n-s}+1)} \right)^{p^j} = e_i x^{p^i} + e_{i-2s} x^{p^{i-2s}} + e_i e_{i-2s} x^{p^{i-2s}+p^i} + e_{i-2s} x^{p^{i-2s}+p^i}. \tag{44}
\]

The fact \( m/d \) is odd and \( s/d \) is even imply \( 3s \equiv \pm s \) (mod \( n \)). By comparing the coefficients of \( x^{p^{i+s}+p^{j-2s}} \) on both sides of (44), one yields

\[
0 = e_i x^{p^i} e_{i-2s},
\]

which yields

\[ e_{i-2s} = 0 \]

as \( e_i \neq 0 \). Then (41) can be simplified as

\[
\sum_{j=0}^{n-1} d_j \left( x^{p^s+1} + x^{p^m(p^s+1)} + x^{p^{n-s}+1} - x^{p^m(p^{n-s}+1)} \right)^{p^j} = e_i x^{p^i} x^{(p^s+1)}. \tag{45}
\]

Observing the coefficients of \( x^{p^i(p^s+1)} \) and \( x^{p^{i+m}(p^s+1)} \) on both sides of (45), one obtains

\[ d_i + d_{i+m} = e_i \]

21
and
\[ d_i + d_{i+m} = 0, \]
which implies that \( e_i = 0 \), a contradiction.

As a result, \( f(x) \) is EA-inequivalent to \( g(x) \) implies that they are not CCZ-equivalent, i.e., their associated semifields are not strongly isotopic. In particular, if \( s = 2k \) and \( m = 3k \), the middle nucleus and the left nucleus are the same which yields that the associated semifields are not isotopic by Lemma 2.

Now, we are ready to investigate the isotopism of the corresponding semifields among the new PN functions in Theorems 1 and 2 and all the known PN functions mentioned in Section 1. The main result in this section is given by the following theorem.

**Theorem 3.** Let \( p \) be an odd prime. Then the corresponding semifields of the new PN functions in Theorems 1 and 2 are not isotopic to any semifields of the known PN functions I-VIII listed in Section 1 in general.

**Proof.** The proofs for Theorems 1 and 2 are similar. We only present the proof for the non-isotopism of the corresponding semifields among the new PN functions in Theorem 1 and the known PN functions I-VIII, respectively. For brevity, we just consider the case \( r = 0 \).

For functions I and II, it follows from Proposition 1, Proposition 3 and Lemma 2 that the corresponding semifields of the new PN functions in Theorem 1 are not isotopic to the semifields of functions I and II.

For functions III and IV, note that the functions III and IV cannot cover all the new PN functions in Theorem 1. It follows that the corresponding semifields of the new PN functions in Theorem 1 are not isotopic to the semifields of functions III and IV.

For functions V and VI, it follows from [31], Theorem 4.1 that the presemifields of the functions V have middle nucleus of order \( p^{2\gcd(m,s)} \) and left nucleus of order \( p^{\gcd(m,s)} \). Then, by Proposition 1, the corresponding semifields of the new PN functions in Theorem 1 are not isotopic to the semifields of functions V. Moreover, due to Lemma 3, one can obtain that the corresponding semifields of the new PN functions in Theorem 1 are also not isotopic to the semifields of functions VI.

For functions VII, it is shown in [16] that the order of the middle nucleus of Dickson semifields is \( p^m \). Hence, Proposition 1 indicates that the corresponding semifields of the new PN functions in Theorem 1 are not isotopic to the semifields in VII.

For functions VIII, note that [12], Theorem 2 shows that the order of the middle nucleus
of ZP semifields is \( p^{2 \gcd(m,t)} \) when \( \sigma = 1 \) and \( p^{\gcd(m,t)} \) when \( \sigma \neq 1 \), while the left nucleus is \( p^{\gcd(m,t,l)} \) for \( \sigma(x) = x^{p^l} \).

- For \( \sigma = 1 \), the order of the nucleus of ZP semifields is
  \[ p^{\gcd(m,t,0)} = p^{\gcd(m,t)}, \]
  which implies that the left nucleus of ZP semifields is a proper subfield of the middle nucleus. Thus, in this case the corresponding semifields of the new PN functions in Theorem 1 are not isotopic to ZP semifields.

- For \( \sigma \neq 1 \), the middle nucleus of ZP semifields is the same with the left nucleus if and only if \( \gcd(m,t) \mid l \). In this case the left nucleus of ZP semifields has size \( p^{\gcd(m,t)} \) with \( m/\gcd(m,t) \) odd, while the left nucleus in Theorem 1 has size \( p^{2\gcd(m,s)} \) with \( m/\gcd(m,s) \) odd. Since \( v_2(\gcd(m,t)) = v_2(m) < v_2(2\gcd(m,s)) \), then their left nucleus have different size which implies that they are not isotopic.

As a result, the associated semifields of the new PN functions in Theorems 1 and 2 are not isotopic to the semifields of all the known PN functions I-VIII listed in Section 1 in general.

**Remark 3.** It is expected that semifields derived from Theorem 1 is not isotopic to those derived from Theorem 2. One clue to make us believe they are not isotopic is that \( s/\gcd(s,m) \) is odd in Theorem 1 while \( s/\gcd(s,m) \) is even in Theorem 2. Indeed, their non-isotopism can be shown in a similar way as in the proof of Prop. 3 and Prop. 4. Here we omit the details.

### 5 Conclusions

In this paper, two new families of PN functions are proposed. Moreover, we derive the nucleus of the corresponding semifields of these functions and deduce that the corresponding semifields of the new PN functions are not isotopic to all the semifields of known PN functions in general.

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