Explicit Drinfeld Moduli Schemes and
Abhyankar’s Generalized Iteration Conjecture

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Dedicated to Mira Breuer on the occasion of her 0th birthday

Abstract

Let \( k \) be a field containing \( \mathbb{F}_q \). Let \( \psi \) be a rank \( r \) Drinfeld \( \mathbb{F}_q \left[ t \right] \)-module determined by \( \psi_t(X) = tX + a_1X^q + \cdots + a_{r-1}X^{q^{r-1}} + X^{q^r} \), where \( t, a_1, \ldots, a_{r-1} \) are algebraically independent over \( k \). Let \( n \in \mathbb{F}_q[T] \) be a monic polynomial. We show that the Galois group of \( \psi_n(X) \) over \( k(t, a_1, \ldots, a_{r-1}) \) is isomorphic to \( GL_r(\mathbb{F}_q[t]/n\mathbb{F}_q[t]) \), settling a conjecture of Abhyankar. Along the way we obtain two explicit constructions of Drinfeld moduli schemes of level \( tn \).

Keywords: Drinfeld modules, Drinfeld moduli schemes, Galois groups

1. Introduction

A classical theorem of Weber (see [9, Chapter 6, Corollary 1]) states that, if \( E \) is an elliptic curve over \( K = \mathbb{Q}(j) \) with transcendental \( j \)-invariant \( j \), and \( K_n = K(E[n]) \) denotes the field obtained by adjoining the coordinates of all \( n \)-torsion points of \( E \) to \( K \), then

\[ \text{Gal}(K_n/K) \cong GL_2(\mathbb{Z}/n\mathbb{Z}). \]

The goal of this paper is to prove the analogous statement for Drinfeld modules. Our result was conjectured by S. S. Abhyankar in [1, §19], who didn’t know about Drinfeld modules at the time, and called it the “Generalized Iteration Conjecture”. It was part of his quest to find nice equations for nice groups.

Our approach is based on the observation that a particular morphism of Drinfeld moduli schemes is étale with a suitable Galois group, a result due to V. G. Drinfeld [5].

We start by constructing some suitable rings in §2 and then in §3 we prove our first main result, Theorem 2 which gives an explicit construction of the Drinfeld moduli scheme of level \( tn \). This is a generalization of the level \( t \) construction due to R. Pink [13].

In §4 we prove our second main result, Theorem 5 which shows that this moduli scheme can be obtained from the torsion module of any “sufficiently generic” Drinfeld module, including the one defined by Abhyankar. Both of these results may be of independent interest.

In §5 we then state the third main result, Theorem 6 which settles Abhyankar’s Generalized Iteration Conjecture. This is proved in §§6–9.

Most of this paper will be comprehensible to anybody familiar with the basics of Drinfeld modules over a field, see for example [6, Chapter 4] or [16, Chapter 13]. To state Theorem 2 we need Drinfeld modules over a scheme, for which we recommend the exposition in [11].

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2. Rings generated by torsion points

Let $\mathbb{F}_q$ denote a finite field of $q$ elements, and let $V \cong \mathbb{F}_q^r$ denote an $\mathbb{F}_q$-vector space of dimension $r \geq 1$. We denote by $V' = V \setminus \{0\}$ the set of non-zero vectors in $V$.

Denote by $A = \mathbb{F}_q[t]$ the polynomial ring over $\mathbb{F}_q$, let $n \in A$ be a monic polynomial and set $B := \mathbb{F}_q[t, \frac{1}{tn}] = A[\frac{1}{tn}]$.

We denote by $S_V = B[v \mid v \in V']$ the symmetric algebra of $V$ over $B$, which is isomorphic to a polynomial ring over $B$ in $r$ independent variables; by $K_V$ the quotient field of $S_V$; by $R_V = B[\frac{1}{v} \mid v \in V']$ the $B$-subalgebra of $K_V$ generated by $\frac{1}{v}$ for every $v \in V'$; and finally by $RS_V = B[v, \frac{1}{v} \mid v \in V']$ the $B$-subalgebra of $K_V$ generated by $S_V$ and $R_V$. The rings $S_V$, $R_V$ and $RS_V$ are $\mathbb{Z}$-graded $B$-algebras with respect to the grading $\deg(v) = -1$ and $\deg(\frac{1}{v}) = 1$ for all $v \in V'$. The homogeneous component of $RS_V$ of degree zero is denoted $RS_{V,0}$. We have

$$RS_{V,0} = B\left[\frac{v}{v'} \mid v, v' \in V'\right].$$

These definitions essentially come from [14].

We define a rank $r$ Drinfeld $A$-module $\varphi$ over $K_V$ by setting

$$\varphi_t(X) := tX \prod_{v \in V'} \left(1 - \frac{1}{v}X\right) = tX + g_1X^q + \cdots + g_rX^{q^r} \in R_V[X].$$

The coefficients $g_1, \ldots, g_r$ are algebraically independent over $B$, the highest coefficient $g_r = t \prod_{v \in V'} \frac{1}{v}$ is a unit in $RS_V$, and by construction the $t$-torsion submodule of $\varphi$ is $\varphi[t] = V \subset K_V$.

Next, we want to define similar rings for the $tn$-torsion module of $\varphi$. We construct the ring $RS_W$ via generators and relations, as follows. Choose a basis $v_1, v_2, \ldots, v_r$ of $V$.

Inside the polynomial ring $RS_V[w_1, w_2, \ldots, w_r, z]$, in $r + 1$ independent variables, we consider the ideal $I_W$ generated by the following elements:

$$\varphi_a(w_i) - v_i, \quad i = 1, 2, \ldots, r,$$

$$\left[z \prod_{(a_1, \ldots, a_r) \in (A/tnA)^r \setminus \{0\}} (\varphi_{a_1}(w_1) + \varphi_{a_2}(w_2) + \cdots + \varphi_{a_r}(w_r))\right] - 1.$$

Then we define

$$RS_W := RS_V[w_1, w_2, \ldots, w_r, z]/I_W.$$ By abuse of notation, we have written $\varphi_a(w_i)$ with $a \in A/tnA$ when we actually mean that $a \in A$ represents a certain class in $A/tnA$. Since $\varphi_a(w_i) \equiv \varphi_b(w_i) \pmod{\varphi_n(w_i) - v_i}$ if $a \equiv b \pmod{tn}$, it does not matter which representative we take.

**Proposition 1.** Consider the following subset of $RS_W$:

$$W := \{\varphi_{a_1}(w_1) + \varphi_{a_2}(w_2) + \cdots + \varphi_{a_r}(w_r) \mid a_1, \ldots, a_r \in A/tnA\} \subset RS_W.$$ (i) Then $a \cdot w := \varphi_a(w)$, for $a \in A$, and $w \in W$, turns $W$ into an $A/tnA$-module, which is free of rank $r$.

(ii) We have $V \subset W$ and every element of $W' := W \setminus \{0\}$ is invertible in $RS_W$.

(iii) The following identity holds in $RS_W[X]$:

$$\varphi_{tn}(X) = tnX \prod_{w \in W'} \left(1 - \frac{1}{w}X\right).$$
Proof. Let $J_W$ be the ideal in the polynomial ring $RS_V[w_1, w_2, \ldots, w_r]$ generated by the elements $\varphi_n(w_i) - v_i$, for $i = 1, 2, \ldots, r$, and set $S_W := RS_V[w_1, w_2, \ldots, w_r]/J_W$. Since $\frac{d}{dw_i}(\varphi_n(w_i) - v_i) = n \in RS_V^*$, it follows that the nilradical of $S_W$ is trivial. Every prime ideal of $S_W$ is the kernel of a ring homomorphism $\theta : S_W \to F$, where $F$ is an algebraically closed field. To such a $\theta$ we associate the Drinfeld module $\varphi^{\theta}$ over $F$ by

$$\varphi^\theta(X) := \theta(t)X \prod_{w \in W'} \left(1 - \theta \left(\frac{1}{w}\right)X\right) \in F[X].$$

Its characteristic, $\ker(\theta|A)$, is prime to $tn$, so $\varphi^{\theta}[tn] \cong (A/tnA)^r$ as $A$-modules, and furthermore $\theta$ maps $W$ isomorphically onto $\varphi^{\theta}[tn]$. This completes the proof of (ii).

Now let $\Delta := \prod_{w \in W'} w \in S_W$. We next show that $\Delta$ is not a zero-divisor of $S_W$. Suppose $c \in S_W$ with $c\Delta = 0$. Then

$$\theta(\Delta) = \prod_{u \in \varphi^{\theta}[tn]\smallsetminus\{0\}} u \neq 0 \text{ in } F,$$

and so $\theta(c) = 0$ for each homomorphism $\theta$ as above. It follows that $c$ lies in the intersection of all prime ideals of $S_W$, which is the nilradical of $S_W$, so $c = 0$.

It follows that $RS_W = S_W[z]/(z\Delta - 1)$ is the localization of $S_W$ at $\Delta$, $S_W \subset RS_W$ and each $w \in W'$ is invertible in $RS_W$. Since $v_i = \varphi_n(w_i) \in W$ for each $i$, it follows that $V \subset W$. This proves (ii).

The homomorphisms $\theta : S_W \to F$ above extend to homomorphisms $\theta : RS_W \to F$, and again the nilradical of $RS_W$ is trivial. For each such $\theta$ we have

$$\varphi^{\theta}_{tn}(X) = \theta(tn)X \prod_{w \in W'} \left(1 - \theta \left(\frac{1}{w}\right)X\right) \in F[X],$$

since both polynomials have the same roots and linear term. It follows that each coefficient of

$$\varphi_{tn}(X) - tnX \prod_{w \in W'} \left(1 - \frac{1}{w}X\right) \in RS_W[X]$$

lies in $\cap \ker(\theta) = \{0\}$, which completes the proof of (iii). \qed

We note that $RS_W$ is generated over $RS_V$ by the elements of the form $w$ and $\frac{1}{w}$, for $w \in W'$. At this point it is far from clear that $RS_W$ is integral, but this will be shown later (Theorem 3).

Next, we define a ring generated only by the quotients of torsion points.

Recall that $v_1 \in V'$ is fixed. The Drinfeld module $\varphi' := v_1^{-1}\varphi v_1$, defined by

$$\varphi'_1(X) = v_1^{-1}\varphi_1(v_1X) = tx \prod_{v \in V'} \left(1 - \frac{v}{v_1}X\right) \in RS_{V,0}[X],$$

is isomorphic to $\varphi$ over $K_V$.

Inside the polynomial ring $RS_{V,0}[w'_1, w'_2, \ldots, w'_r, z']$, in $r + 1$ independent variables, we define the ideal $I_{W,0}$ generated by the following elements:

$$\varphi'_i(w'_i) - \frac{v_i}{v_1}, \quad i = 1, 2, \ldots, r,$$

$$\left[z' \prod_{(a_1, \ldots, a_r) \in (A/tnA)^r \smallsetminus \{0\}} \left(\varphi'_{a_1}(w'_1) + \varphi'_{a_2}(w'_2) + \cdots + \varphi'_{a_r}(w'_r)\right)\right] - 1.$$

Then we define

$$RS_{W,0} := RS_{V,0}[w'_1, w'_2, \ldots, w'_r, z']/I_{W,0}.$$
The ring $RS_{W,0}$ embeds into $RS_W$ via $w_i \mapsto w_i/v_1$, and the above relations reflect the fact that $\varphi'_n(w_i/v_1) = w_i/v_1$ for $i = 1, 2, \ldots, r$, as well as ensuring that all elements of the form $w/v_1$ for $w \in W'$ are invertible. Thus we have
\[
RS_{W,0} = RS_{V,0} \left[ \frac{w}{w'} \mid w, w' \in W' \right] = B \left[ \frac{w}{w'} \mid w, w' \in W' \right] \subset RS_W.
\]
Moreover, we see that $RS_{W,0}$ is the degree zero component of $RS_W$ with respect to the $\mathbb{Z}$-grading defined by $\deg(w) = -1$ for all $w \in W'$.

Lastly, it follows from Proposition 1.(iii) that
\[
\varphi'_{tn}(X) = tnX \prod_{w \in W'} \left( 1 - \frac{v_1}{w}X \right) \in RS_{W,0}[X].
\]

3. Explicit Drinfeld moduli schemes

Let $S$ be a scheme over Spec $B$. Then recall (see e.g. [11]) that a Drinfeld $A$-module of rank $r$ over $S$ is a pair $(L, \psi)$, where $L$ is the additive group scheme of a line bundle over $S$, and $\psi : A \rightarrow \text{End}(L), \quad a \mapsto \psi_a,$
is a ring homomorphism that is defined over a trivializing open Spec$(R) \subset S$ by
\[
t \mapsto \psi_t(X) = tX + e_1X^q + \cdots + e_nX^{q^n},
\]
where for each $i = 1, 2, \ldots, n$ we have $e_i \in R$, $e_r \in R^*$ and $e_i$ is nilpotent for all $i > r$. We usually drop the $L$ from our notation and refer to the Drinfeld module as $\psi$. When $e_i = 0$ for all $i > r$, we say that the Drinfeld module $\psi$ is standard.

The $tn$-torsion submodule $\psi[tn]$ of $\psi$ is the closed subscheme of $L$ defined locally over Spec $(R)$ by Spec $(R[X]/\langle \psi_{tn}(X) \rangle)$. When $R = F$ is a field, we identify $\psi[tn]$ with $\{ x \in \bar{F} \mid \psi_{tn}(x) = 0 \}$.

A level-$tn$ structure on $\psi$ is a homomorphism of $A$-modules $\mu : ((tn)^{-1}A/A)^r \rightarrow L(S)$ which induces an equality of divisors
\[
\sum_{\alpha \in (tn)^{-1}A/A} \mu(\alpha) = \psi[tn].
\]

For Drinfeld modules over Spec $R$, where $R$ is a $B$-algebra, this is equivalent to the following more convenient formulation. Fix $A$-module isomorphisms
\[
((tn)^{-1}A/A)^r \sim (A/tnA)^r \sim W \quad \text{and} \quad (t^{-1}A/A)^r \sim (A/tA)^r \sim V
\]
such that the following diagram commutes:
\[
\begin{array}{c}
\begin{array}{ccc}
((tn)^{-1}A/A)^r & \sim & (A/tnA)^r & \sim & W \\
(\downarrow) & & (\downarrow) & & (\downarrow) \\
(t^{-1}A/A)^r & \sim & (A/tA)^r & \sim & V
\end{array}
\end{array}
\]

Then a level-$tn$ structure on a Drinfeld module $\psi$ over Spec $R$ is equivalent to an $A$-module homomorphism $\mu : W \rightarrow R$.
such that $\mu(W') \subset R^*$ and
\[ \psi_{tn}(X) = tnX \prod_{w \in W'} \left( 1 - \frac{X}{\mu(w)} \right) \in R[X]. \]

Here we have made essential use of the fact that the characteristic $\ker(A \to R)$ of $\psi$ is prime to $tn$, since $R$ is a $B$-algebra and $tn \in B^*$.

In particular, by Proposition \[\Box\] $\varphi'$ carries the level-$tn$ structure
\[ \lambda : W \to RS_{W,0}; \quad w \mapsto \frac{w}{v_1}. \]

Our first main result is the fact that $\Spec(RS_{W,0})$ is the fine moduli scheme for rank $r$ Drinfeld $A$-modules with level-$tn$ structure. Denote by $E = \mathbb{G}_{a,RS_{W,0}}$ the additive group scheme over $\Spec(RS_{W,0})$. Then the triple $(E, \varphi', \lambda)$ forms a rank $r$ Drinfeld $A$-module with level $tn$-structure over $\Spec(RS_{W,0})$.

**Theorem 2.** The affine scheme $M^r_{tn,B} := \Spec(RS_{W,0})$, together with the universal family $(E, \varphi', \lambda)$, represents the functor from $B$-Schemes to Sets which sends a scheme $S$ over $\Spec(B)$ to the set of isomorphism classes of triples $(L, \psi, \mu)_S$, where $(L, \psi)$ is a rank $r$ Drinfeld $A$-module, and $\mu : W \to L(S)$ is a level-$tn$ structure.

The special case $M^r_{t,B} \cong \Spec(RS_{V,0})$ is due to Pink [13, §7] and inspired Theorem 2.

**Proof.** Step 1. Let $S$ be a scheme over $\Spec(B)$ and $(L, \psi, \mu)_S$ a triple as above. We must associate to the isomorphism class of $(L, \psi, \mu)_S$ an $S$-valued point $\eta$ on $\Spec(RS_{W,0})$ such that the pullback of the universal family $(E, \varphi', \lambda)$ to $\eta$ is isomorphic to $(L, \psi, \mu)_S$.

First notice that the line bundle $L/S$ must be trivial, since for any $v \in V'$, $\mu(v) \in L(S)$ is a nowhere zero section, as $t$ is prime to the characteristic of $\psi$. Now cover $S$ with open affines $\Spec(R)$; it suffices to prove that the isomorphism class of each pullback $(L, \psi, \mu)_{\Spec(R)}$ corresponds to a $\Spec(R)$-valued point on $\Spec(RS_{W,0})$. Thus we assume that $S = \Spec(R)$ is affine, where $R$ is a $B$-algebra, and that $L = \mathbb{G}_{a,R}$ is the additive group scheme over $\Spec(R)$.

Next, we may replace $\psi$ by an isomorphic Drinfeld module which is standard, i.e. for which
\[ \psi_1(X) = tX + a_1X^q + \cdots + a_rX^{q^r}, \]
where $a_1, \ldots, a_{r-1} \in R$ and $a_r \in R^*$, see [11, §2.2.3, p21].

The level structure $\mu$ is a morphism $\mu : W \to R$ such that $\mu(W') \subset R^*$ and
\[
\psi_1(X) = tX \prod_{v \in V'} \left( 1 - \frac{X}{\mu(v)} \right), \quad \psi_{tn}(X) = tnX \prod_{w \in W'} \left( 1 - \frac{X}{\mu(w)} \right).
\]

Recall that we have fixed $v_1 \in V'$. Consider the Drinfeld module $\psi' := \mu(v_1)^{-1}\psi\mu(v_1)$; it is isomorphic to $\psi$ over $R$, and
\[
\psi'_1(X) = tX \prod_{v \in V'} \left( 1 - \frac{\mu(v_1)}{\mu(v)} \frac{X}{\mu(v)} \right), \quad \psi'_{tn}(X) = tX \prod_{w \in W'} \left( 1 - \frac{\mu(v_1)}{\mu(w)} \frac{X}{\mu(w)} \right).
\]

We now consider the $B$-algebra homomorphism
\[ \theta : RS_{V,0} \to R \]
which is determined by
\[
\frac{v}{v'} \mapsto \frac{\mu(v)}{\mu(v')}, \quad \text{for } v, v' \in V'.
\]
This exists because \( M_{r,B}^t \cong \text{Spec } RS_{V,0} \), by [13, §7], but one can also see this directly: all relations satisfied by the \( v/v' \) in \( RS_{V,0} \) are also satisfied by the \( \mu(v)/\mu(v') \) in \( R \).

We extend \( \theta \) to the \( B \)-algebra homomorphism
\[
\theta : RS_{V,0}[w'_1, w'_2, \ldots, w'_r, z'] \to R
\]
determined by
\[
w'_i \mapsto \frac{\mu(w_i)}{\mu(v_1)},
\]
\[
z' \mapsto \prod_{w \in W_v} \frac{\mu(w)}{\mu(v)}.
\]
It is clear that \( I_{W,0} \subset \ker \theta \), so \( \theta \) extends to a \( B \)-algebra homomorphism \( \theta : RS_{W,0} \to R \). Furthermore, \( \theta \) defines a Spec(\( R \))-valued point \( \eta \) on Spec(\( RS_{W,0} \)), and the triple \((L, \psi, \mu)\) is isomorphic to the pullback \((L, \psi', \mu(v_1)^{-1} \mu)\) of the universal family \((E, \varphi', \lambda)\) to \( \eta \), as required.

**Step 2.** Conversely, suppose we are given an \( S \)-valued point \( \eta \) on Spec(\( RS_{W,0} \)). We may again assume that \( S = \text{Spec}(R) \) is affine, and so we are given a \( B \)-algebra homomorphism \( \theta : RS_{W,0} \to R \).

Define
\[
\mu : W \to R; \quad \mu(w) := \theta \left( \frac{w}{v_1} \right).
\]
Then
\[
\frac{\mu(w)}{\mu(w')} = \theta \left( \frac{w}{w'} \right)
\]
and it is easy to check that \( \mu(W') \subset R^* \), and \( \mu \) is \( \mathbb{F}_q \)-linear and injective.

Now
\[
\psi_t(X) := tX \prod_{v \in V} \left( 1 - \frac{1}{\mu(v)} X \right) = tX \prod_{v \in V} \left( 1 - \theta \left( \frac{v_1}{w} \right) X \right) \in R[X]
\]
defines a rank \( r \) Drinfeld \( A \)-module \( \psi \) over \( R \) with level-\( t \) structure \( \mu|_V \). Its coefficients are the images under \( \theta \) of the coefficients of \( \varphi' \). It remains to show that \( \mu : W \to R \) defines a level-\( tn \) structure. \( \psi_{tn}(X) \) is contracted from \( \psi_t(X) \) in the usual way, hence it is the image under \( \theta \) of \( \varphi'_{tn}(X) \), thus
\[
\psi_{tn}(X) = tnX \prod_{w \in W'} \left( 1 - \theta \left( \frac{v_1}{w} \right) X \right) = tnX \prod_{w \in W'} \left( 1 - \frac{1}{\mu(w)} X \right).
\]
Since \( \mu : W \to \mu(W) \) is already an isomorphism of \( \mathbb{F}_q \)-vector spaces, it suffices to show that \( \mu \) respects the \( A \)-module structure. For this, we compute, for \( w \in W \),
\[
\mu(t \cdot w) = \mu(\varphi_t(w)) = \mu \left( tw \prod_{v \in V} \left( 1 - \frac{w}{v} \right) \right)
\]
\[
= \theta \left( \frac{w}{v_1} \prod_{v \in V} \left( 1 - \frac{w}{v} \right) \right)
\]
\[
= t \theta \left( \frac{w}{v_1} \right) \prod_{v \in V} \left( 1 - \theta \left( \frac{w}{v} \right) \right) = t \mu(w) \prod_{v \in V} \left( 1 - \frac{\mu(w)}{\mu(v)} \right)
\]
\[
= \psi_t(\mu(w)) = t \cdot \mu(w).
\]
We thus obtain a triple \((G_{a,R}, \psi, \mu)\), as desired.

This construction is the inverse of the construction in step 1. This completes the proof. \( \square \)

Next, we collect here the following fundamental results on Drinfeld moduli schemes.
Theorem 3. Let $n \in A = \mathbb{F}_q[t]$ be monic and recall that $B = A[\frac{1}{tn}]$.

(i) The scheme $M_{tn,B}^r$ is smooth of relative dimension $r - 1$ over Spec $B$.

(ii) The group $\text{GL}_r(A/tnA)$ acts on the level structure of the universal Drinfeld module $\varphi'$, and this induces an action of $\text{GL}_r(A/tnA)$ on $M_{tn,B}^r$.

(iii) The canonical morphism $M_{tn,B}^r \to M_{tn,B}^1$ is étale with Galois group

$$G_r(n) := \ker \left( \text{GL}_r(A/tnA) \to \text{GL}_r(A/tA) \right).$$

(iv) There is a morphism, defined over Spec $B$,

$$w_{tn} : M_{tn,B}^r \to M_{tn,B}^1,$$

which is compatible with the action of $\text{GL}_r(A/tnA)$, in the sense that, for every $\sigma \in \text{GL}_r(A/tnA)$, we have $w_{tn} \circ \sigma = \det(\sigma) \circ w_{tn}$.

(v) The scheme $M_{tn,B}^r$ is integral, and the rings $RS_{W,0}$ and $RS_W$ are integral.

Proof. The first three statements are essentially due to Drinfel’d [5], who proved this more generally over Spec $A$ but for level structures divisible by two distinct primes. In our situation, the level $tn \neq 1$ is invertible in $B$. Thus, as in the proof of Theorem [2] if $(L, \psi, \mu)_S$ is a Drinfeld module with level-$tn$ structure over a $B$-scheme $S$ then any $v \in V'$ gives a nowhere vanishing section $\mu(v) \in L(S)$, which trivializes $L/S$. For $S$ over Spec $A$ such a trivialization is only achieved if the level structure is divisible by two distinct primes, see [11] Prop. 2.5.1 and Theorem 3.4.1 for details. Thus in our case, Drinfeld’s proofs give (i), (ii) and (iii) above. See also [19], as well as [7] for a very clear exposition of the situation over the quotient field of $A$.

Alternatively, the interested reader is challenged to deduce (ii) and (iii) directly from Theorem [2]. Smoothness in (i) is probably harder to verify directly, but we will not need it in this paper.

Statement (iv) is essentially due to Anderson [3], see also [19] for details.

To prove (v), let $F = \mathbb{F}_q(t)$. Denote by $J_{W,0}$ the ideal in the polynomial ring $RS_{V,0}[w'_1, w'_2, \ldots, w'_r]$ generated by the elements $\varphi_n' (w'_i) - v_i/v_1$ for $i = 1, 2, \ldots, r$, and set $S_{W,0} = RS_{V,0}[w'_1, w'_2, \ldots, w'_r]/J_{W,0}$.

Then as in the proof of Proposition [1] $RS_{W,0}$ is the localization of $S_{W,0}$ at $\prod_{w \in W'} \frac{w}{v_i}$ and $S_{W,0} \subset RS_{W,0}$. Now $S_{W,0} \otimes_B F \subset RS_{W,0} \otimes_B F$ and

$$S_{W,0} \otimes_B F \cong \left( F \left[ \frac{w}{v'_i} \bigg| v, v' \in V' \right] [w'_1, w'_2, \ldots, w'_r] \right) / J_{W,0}. $$

Since $\frac{d}{dw} (\varphi_n' (w'_i) - v_i/v_1) = n \in F^*$ it follows as in the proof of Proposition [1] that $RS_{W,0} \otimes_B F$ contains no nilpotents, and so $M_{tn,F}^r = M_{tn,B}^r \times_{\text{Spec}(B)} \text{Spec}(F) = \text{Spec}(RS_{W,0} \otimes_B F)$ is reduced. On the other hand, it is shown in [3] Cor. 3.4.5 that the scheme $M_{tn,F}^r$ is $F$-irreducible, so $RS_{W,0} \otimes_B F$ is integral.

Thus also $S_{W,0} \otimes_B F$ is integral. But then also $S_{W,0}$ is integral, for, if any $\varphi'_n(w'_i) - v_i/v_1$ were to factorize in

$$B \left[ \frac{v}{v'} \bigg| v, v' \in V' \right] [w'_1, w'_2, \ldots, w'_r]/\langle \varphi'_n(w'_i) - v_1, \ldots, \varphi'_n(w'_{i-1}) - v_{i-1} \rangle,$$

then it would also factorize in

$$F \left[ \frac{v}{v'} \bigg| v, v' \in V' \right] [w'_1, w'_2, \ldots, w'_r]/\langle \varphi'_n(w'_i) - v_1, \ldots, \varphi'_n(w'_{i-1}) - v_{i-1} \rangle.$$

It follows that the localization $RS_{W,0}$ is also integral, so $M_{tn,B}^r$ is integral.

Lastly, $RS_W = RS_{W,0}[v_1]$, and $v_1$ is transcendental over $RS_{W,0}$, so $RS_W$ is also integral. □
4. **Sufficiently generic Drinfeld modules**

Now let \( \psi \) be a rank \( r \) Drinfeld \( A \)-module over an integral \( B \)-algebra \( R \) defined by

\[
\psi_1(X) = tX + a_1X^q + \cdots + a_rX^{q^r},
\]

where \( a_1, \ldots, a_{r-1} \in R \) and \( a_r \in R^* \). We define the invariants

\[
J_i := \frac{a_i (q^r - 1)/d_i}{a_r (q^r - 1)/d_r}, \quad i = 1, \ldots, r - 1,
\]

where \( d_i := \gcd(q^i - 1, q^r - 1) \). (Actually, we could choose \( d_i \) to be any common divisor of \( q^i - 1 \) and \( q^r - 1 \).) These are isomorphism invariants, although for \( r \geq 3 \) they do not determine the isomorphism class of \( \psi \) completely, see [15].

**Definition 4.** A Drinfeld module \( \psi \) of rank \( r \geq 1 \) is sufficiently generic if \( r = 1 \), or if \( r \geq 2 \) and the invariants \( J_1, \ldots, J_{r-1} \) are algebraically independent over \( \mathbb{F}_q(t) \).

This condition is equivalent to the ring of isomorphism invariants (see [15]) of \( \psi \) having transcendence degree \( r \) over \( \mathbb{F}_q \).

Consider the subfield \( K := \mathbb{F}_q(t, a_1, \ldots, a_r) \) of the quotient field of \( R \), and denote by \( K_{tn} \) the splitting field of \( \psi_{tn}(X) \) over \( K \). We denote by

\[
RS_{tn,0} := B[w/w' \mid w, w' \in \psi[tn], w' \neq 0]
\]

the \( B \)-subalgebra of \( K_{tn} \) generated by the quotients \( w/w' \) with \( w, w' \in \psi[tn], w' \neq 0 \).

Our second main result is the following.

**Theorem 5.** If \( \psi \) is sufficiently generic, then \( RS_{tn,0} \cong RS_{W,0} \). In particular,

\[
M_{F_{tn,B}}^r \cong \text{Spec}(RS_{tn,0}).
\]

**Proof.** When \( r = \text{rank}(\psi) = 1 \) we can show directly that \( RS_{tn,0} \cong RS_{W,0} \). Let \( u_1 \) be a generator of \( \psi[t] \cong A/tnA \), and set \( \psi' := u_1^{-1}\psi u_1 \). Then

\[
\psi'_1(X) = u_1^{-1}\psi_1(u_1X) = tX \prod_{\varepsilon \in \mathbb{F}_q^*} \left( 1 - \frac{u_1X}{\varepsilon u_1} \right) = \varphi'_1(X).
\]

Now

\[
RS_{tn,0} = B[w/w' \mid 0 \neq w \in \psi[tn]] = B[w', 1/w' \mid 0 \neq w' \in \psi'[tn]] = B[w', 1/w' \mid 0 \neq w' \in \varphi'[tn]]
\]

as a subalgebra of \( K_{tn} \). But the last expression is isomorphic to \( B[w/w' \mid w, w' \in \varphi[tn] \setminus \{0\}] \cong RS_{W,0} \).

Now suppose that \( r \geq 2 \). Choose a level-\( tn \) structure \( \mu : W \to \psi[tn] \subset K_{tn} \). Then \( \mu(W') \subset K_{tn}^* \) and similarly to part 1 of the proof of Theorem 2, we construct a \( B \)-algebra homomorphism

\[
\theta : RS_W \to K_{tn}; \quad \theta(w_i) = \mu(w_i), \quad \theta(z) = \prod_{w \in W'} \mu(w)^{-1} \in K_{tn}^*, \quad i = 1, 2, \ldots, r.
\]

We must show that \( \ker \theta \cap RS_{W,0} = \{0\} \), so suppose that \( 0 \neq f \in \ker \theta \cap RS_{W,0} \). By Theorem 3(iii), \( \prod_{\sigma \in G_r(n)} \sigma(f) \in RS_{V,0} \). Multiplying this by a suitable unit \( u \in RS_V^* \), we obtain a homogeneous element

\[
\hat{f} = u \prod_{\sigma \in G_r(n)} \sigma(f) \in \ker \theta \cap R_V.
\]
Now, by \textit{[14, Theorem 3.1]}, $GL_r(F_q)$ acts on $R_V$ and the ring of invariants is $R_V^{GL_r(F_q)} = B[g_1, \ldots, g_r]$, where

$$\varphi_t(X) = tx \prod_{v \in V'} \left(1 - \frac{1}{v} X \right) = tx + g_1 X^q + \cdots + g_r X^{q^r}.$$ 

Thus we obtain

$$\tilde{f} := \prod_{\sigma \in GL_r(F_q)} \sigma(\tilde{f}) \in \ker \theta \cap B[g_1, \ldots, g_r]$$

which is homogeneous of some degree $d$ with respect to the grading $\deg(g_i) = q^i - 1$ for $i = 1, \ldots, r$. Notice that $\tilde{f} \neq 0$ since $RS_W$ is integral, by Theorem \textit{[5]}(v).

Since $a_i = \theta(g_i)$ for $i = 1, \ldots, r$, we see that

$$\tilde{f}(a_1, \ldots, a_r) = 0.$$ 

Now, let $\delta \in K^{sep}$ be such that $\delta^{q-1} = a_r$, and set

$$u_i := \delta^{-q^i} a_i, \quad \text{so} \quad J_i = u_i^{(q^r - 1)/d_i}, \quad i = 1, \ldots, r - 1.$$ 

Then $0 = \delta^d \tilde{f}(u_1, u_2, \ldots, u_{r-1}, 1)$. It follows that $F_q(t, u_1, \ldots, u_{r-1})$ has transcendence degree at most $r - 2$ over $F_q(t)$. Since $F_q(t, J_1, \ldots, J_{r-1}) \subset F_q(t, u_1, \ldots, u_{r-1})$, this contradicts the algebraic independence of $J_1, \ldots, J_{r-1}$ over $F_q(t)$. \hfill \Box

5. Abhyankar’s Generalized Iteration Conjecture

We now come to the heart of this article.

Let $k$ be a field containing $F_q$ such that $t$ is transcendental over $k$, and set $F := k(t)$ and $K := F(a_1, a_2, \ldots, a_{r-1})$, where $a_1, a_2, \ldots, a_{r-1}$ are algebraically independent over $F$. Consider the rank $r$ Drinfeld module $\psi$ over $K$ defined by

$$\psi_t(X) = tx + a_1 X^q + \cdots + a_{r-1} X^{q^{r-1}} + X^{q^r}.$$ 

Notice that here the highest coefficient is $a_r = 1$.

Let $n \in A$ be any monic polynomial and denote by $K_n$ the splitting field of $\psi_n(X)$ over $K$. Our third main result is the following, which was conjectured by S. S. Abhyankar in \textit{[1, §19]}:

**Theorem 6 (Generalized Iteration Conjecture).** $\text{Gal}(K_n/K) \cong GL_r(A/nA)$.

Equivalently, the Galois representation attached to the $n$-torsion of $\psi$ over $K$ is surjective, for all non-zero $n \in A$.

Since $[K_n : K]$ does not depend on $k$, we obtain

**Corollary 7.** $K_n/K$ is a purely geometric extension.

A number of special cases of Theorem \textit{6} are known, see for example \textit{[2, 17]} and further references in \textit{[1, §19]}. A related result is in \textit{[8]}. In particular, the case $r = 1$ follows from the work of Carlitz \textit{[4]}, while the case where $n = t$ dates back to E. H. Moore \textit{[12]}:

**Theorem 8 (Moore).** $\text{Gal}(K_1/K) \cong GL_r(F_q)$.

**Proof.** See \textit{[2, §3]} for a particularly simple proof. \hfill \Box

It is clear that $\text{Gal}(K_n/K)$ is isomorphic to a subgroup of $GL_r(A/nA)$, and this subgroup cannot be enlarged by enlarging $k$, so we may assume that $F_q^r \subset k$.

Denote by $K_{in}$ the splitting field of $\psi_{in}(X)$ over $K$. It will be sufficient to prove the following result.
Proposition 9. Let $n \in A$ be monic. Then

$$\text{Gal}(K_{tn}/K_t) \cong G_r(n) = \ker \left( \text{GL}_r(A/tnA) \to \text{GL}_r(A/tA) \right).$$

Proof of Theorem [8].

By Proposition [9] $\text{Gal}(K_{tn}/K_t) \cong G_r(n)$. Consider the field extensions in the diagram. We have $\text{Gal}(K_t/K) \cong \text{GL}_r(A/tA)$ by Theorem [8] thus $[K_{tn} : K] = \#G_r(n) \cdot \#\text{GL}_r(A/tA) = \#\text{GL}_r(A/tnA)$ and it follows that $\text{Gal}(K_{tn}/K) \cong \text{GL}_r(A/tnA)$. Now from the action of $\text{Gal}(K_{tn}/K)$ on $\psi[n]$ we see that $\text{Gal}(K_{tn}/K_n) \cong \ker \left( \text{GL}_r(A/tnA) \to \text{GL}_r(A/nA) \right)$ and thus $\text{Gal}(K_n/K) \cong \text{GL}_r(A/nA)$.

\[\square\]

It remains to prove Proposition [9]. This will be the goal of the rest of this paper.

6. Moore and Carlitz

We associate to $\psi$ its \textit{determinant Drinfeld module} $\rho$, which is the rank 1 Drinfeld module defined over $F$ by

$$\rho_t(X) = tX - (-1)^r a_r X^q = tX - (-1)^r X^q.$$  

When $r$ is even, then $\rho$ is the original Carlitz module (as studied by Carlitz in the 1930s, [4]), whereas, when $r$ is odd then $\rho$ is the “modern” Carlitz module, as defined in modern texts such as [6, Chapter 3] and [16, Chapter 12].

We denote by $F_t$ and $F_{tn}$ the splitting fields of $\rho_t(X)$ and $\rho_{tn}(X)$ over $F$, respectively.

Proposition 10 (Carlitz). We have $\text{Gal}(F_{tn}/F) \cong (A/tnA)^* \text{ and } \text{Gal}(F_t/F) \cong \mathbb{F}_q^*.$

\textbf{Proof.} When $k = \mathbb{F}_q$ and $F = \mathbb{F}_q(t)$ this is well known, see [3] or [16, Theorem 12.8]. By [16, Corollary to Theorem 12.14], the extension $F_{tn}/F$ is purely geometric, so the proposition remains true for $k$ replaced by any algebraic extension of $\mathbb{F}_q$. Lastly, since $t, a_1, \ldots, a_{r-1}$ are algebraically independent over $k$, the result holds in general.

\[\square\]

The determinant Drinfeld module $\rho$ plays the same role for $\psi$ that the multiplicative group $\mathbb{G}_m$ plays for elliptic curves, and the analogue of the Weil Pairing, developed in [3, 18] in general, has a particularly simple description in the case of $t$-torsion using the Moore determinant. Recall (see [6, §1.3]) that the Moore determinant of a tuple $(x_1, x_2, \ldots, x_r)$ of elements in a field containing $\mathbb{F}_q$ is defined by

$$M(x_1, x_2, \ldots, x_r) := \begin{vmatrix} x_1 & x_2 & \ldots & x_r \\ x_1^q & x_2^q & \ldots & x_r^q \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{q^{r-1}} & x_2^{q^{r-1}} & \ldots & x_r^{q^{r-1}} \end{vmatrix}$$

and has the property that $M(x_1, x_2, \ldots, x_r) \neq 0$ if and only if $x_1, x_2, \ldots, x_r$ are linearly independent over $\mathbb{F}_q$.

Choose a basis $v_1, v_2, \ldots, v_r$ of the vector space $\psi[t] \cong \mathbb{F}_q^*$, then we have

$$\psi_t(X) = M(v_1, v_2, \ldots, v_r, X)/M(v_1, v_2, \ldots, v_r),$$

since both sides equal the unique monic polynomial with set of roots $\mathbb{F}_q v_1 + \mathbb{F}_q v_2 + \cdots + \mathbb{F}_q v_r$. Comparing $X$-coefficients gives $t = (-1)^r M(v_1, v_2, \ldots, v_r)^{q-1}$, so we see that $M(v_1, v_2, \ldots, v_r) \in \rho[t]$. Thus the Moore determinant defines a map (the analogue of the Weil pairing for $t$-torsion):

$$M : (\psi[t])^r \to \rho[t]; \quad (x_1, x_2, \ldots, x_r) \mapsto M(x_1, x_2, \ldots, x_r).$$

The following result is easily verified directly.
Proposition 11. The map \( M \) above is \( \mathbb{F}_q \)-linear, alternating, non-degenerate and surjective. It follows that \( F_t \subset K_t \). \( \square \)

Via the choice of basis \( v_1, v_2, \ldots, v_r \) for \( \psi[t] \) we identify \( \text{Gal}(K_t/K) \) with \( \text{GL}_r(\mathbb{F}_q) \), see Theorem 8. Since \( K/F \) is purely transcendental, we also have

\[
\text{Gal}(K F_t/K) \cong \text{Gal}(F_t/F) \cong G_1(n) = \ker \left( (A/tnA)^* \rightarrow (A/tA)^* \right).
\]

A direct computation shows the following.

Proposition 12. Let \( \sigma \in \text{Gal}(K_t/K) = \text{GL}_r(\mathbb{F}_q) \) and \( (x_1, x_2, \ldots, x_r) \in (\psi[t])^r \). Then

\[
M(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_r)) = \det(\sigma)(M(x_1, x_2, \ldots, x_r)).
\]

In particular, \( K F_t \) is the fixed field of \( \text{SL}_r(\mathbb{F}_q) \) in \( K_t \). \( \square \)

We summarise our progress thus far in the following diagram of field extensions and Galois groups.

7. Function fields of Drinfeld modular varieties

We define the following fields:

\[
K_{tn,0} := F \left( \frac{w}{w'} \mid w, w' \in \psi[tn], w' \neq 0 \right) \subset K_{tn},
K_{t,0} := F \left( \frac{w}{w'} \mid w, w' \in \psi[t], w' \neq 0 \right) \subset K_t, \quad \text{and}
F_{tn,0} := F \left( \frac{w}{w'} \mid w, w' \in \rho[tn], w' \neq 0 \right) \subset F_{tn}.
\]

Notice that \( K_{t,0} = F(\frac{v_1}{v_1'}, \frac{v_2}{v_2'}, \ldots, \frac{v_r}{v_r'}) \), for any basis \( v_1, v_2, \ldots, v_r \) of \( \psi[t] \), so \( K_{t,0}/F \) is a purely transcendental extension of transcendence degree \( r - 1 \).

Proposition 13. We have

1. \( \text{Gal}(K_{tn,0}/K_{t,0}) \cong G_r(n) = \ker \left( \text{GL}_r(A/tnA) \rightarrow \text{GL}_r(A/tA) \right) \).
2. The subfield of \( K_{tn,0} \) fixed by

\[
S_r(n) := \ker \left( \text{SL}_r(A/tnA) \rightarrow \text{SL}_r(A/tA) \right)
\]

is \( K_{t,0} F_{tn,0} \).
Theorem 3.(iii)), which is isomorphic to the quotient $G$ with a little more effort one can show that in fact $\psi$ is sufficiently generic, its function field over $F$ is $K_{t,0}$, by Theorem 3. Similarly, the function fields of $M_{t}^{1, F}$, $M_{t}^{1, F}$ and $M_{t}^{1, F}$ over $F$ are $K_{t,0}$, $F_{t,0}$ and $F_{t,0} = F$, respectively. Now (1) follows from Theorem 3.(iii).

To prove (2), the fixed field contains $K_{t,0}$, by Theorem 3, while Gal($K_{t,0}$/$F_{t,0}$/$K_{t,0}$) is purely transcendental over $F$. Now Gal($F_{t,0}$/$F$) $\cong G_1(n)$ (by Theorem 3.(iii)), which is isomorphic to the quotient $G_r(n)/S_r(n)$. The result follows in this case.

To extend our result to the case for general $k$, recall that $t, a_1, \ldots, a_{r-1}$ are algebraically independent over $k$, so it suffices to show that the relevant field extensions are purely geometric, i.e. that $F_q$ is algebraically closed in the function field of $M_{t, F_q}(t)$ over $F_q(t)$. We achieve this by constructing a field $L$, in which $F_q$ is algebraically closed, and a rank $r$ Drinfeld $F_q[t]$-module $\rho$ over $L$ with $\rho'[tn] \subset L$.

Let $A' = F_q[\sqrt{t}]$ and $K' = F_q(\sqrt{t})$. Consider the Carlitz $A'$-module $\rho'$ defined over $K'$ by $\rho'(\sqrt{t}) = \sqrt{t}X + X^n$.

As before, $L := K'(\rho'[tn])$ is purely geometric over $K'$. On the other hand, $\rho'$ is also a rank $r$ Drinfeld $F_q[t]$-module (with complex multiplication by $A'$), so it, together with a level-$tn$ structure over $L$, defines an $F_q(t)$-algebra homomorphism $RS_{W,0} \otimes_B F_q(t) \rightarrow L$. It follows that $F_q$ is algebraically closed in the function field of $M_{t, F_q}(t)$ over $F_q(t)$.

Next, notice that the leading coefficient of $\psi_t(X) = tX \prod_{v \in \psi[t] \setminus \{0\}} \left(1 - \frac{X}{v}\right)$ is

$$1 = t \prod_{v \in \varphi[t] \setminus \{0\}} \frac{1}{v}.$$ 

Thus

$$\psi'(v_1) - 1 = t \prod_{v \in \varphi[t] \setminus \{0\}} \frac{v_1}{v} \in K_{t,0},$$

and since $K_t = K_{t,0}(v_1)$ and we have assumed that $F_q \subset k \subset K_{t,0}$, we obtain

**Proposition 14.** The extension $K_t/K_{t,0}$ is Galois with $C := \text{Gal}(K_t/K_{t,0})$ cyclic of order dividing $q^r - 1$.

**Remark 15.** With a little more effort one can show that in fact $C$ has order equal to $q^r - 1$, but we will not need this here.

Furthermore, since $K_t$ contains a generator of $\rho[t] \subset \rho[tn]$, we see that $K_tF_{t,0} = K_tF_{t,0}$.

We summarise our progress in the following diagram.

![Diagram](https://via.placeholder.com/150)

Since the order of $C$ is prime to $p$, we see that

**Proposition 16.** We have $v_p([K_tF_{t,0} : K_t]) = v_p([G_1(n)])$, where $v_p$ denotes the $p$-adic valuation.
8. Some Group Theory

Before we continue, we need to recall some results from group theory.

**Lemma 17.** Every proper Abelian quotient of $\text{SL}_r(\mathbb{F}_q)$ has order $p$.

**Proof.** If we use $\text{der}$ to denote the derived (commutator) subgroup, then by [10, chap. XIII Theorems 8.3 and 9.2] we have

$$\text{SL}_r(\mathbb{F}_q)^{\text{der}} = \text{SL}_r(\mathbb{F}_q),$$

with two exceptions. These are:

- If $r = 2$ and $q = 2$, then $\text{SL}_2(\mathbb{F}_2)^{\text{der}} \cong A_3$, which has index 2 in $\text{SL}_2(\mathbb{F}_2) \cong S_3$, and
- If $r = 2$ and $q = 3$, then $\text{SL}_2(\mathbb{F}_3)^{\text{der}} \cong Q$, the 8-element quaternion group, which has index 3 in $\text{SL}_2(\mathbb{F}_3)$.

The result follows. □

**Proposition 18.** Every proper Abelian quotient of $S_r(n)$ is a $p$-group.

**Proof.** Let the prime factorisation of $n$ in $A$ be given by

$$n = \prod_P P^{a_P}.$$

Then

$$S_r(n) = \ker \left( \text{SL}_r(A/tnA) \to \text{SL}_r(A/tA) \right)$$

$$\cong \ker \left( \text{SL}_r(A/t^{a_P+1}A) \to \text{SL}_r(A/tA) \right) \times \prod_{P|n, P \neq t} \text{SL}_r(A/P^{a_P}A).$$

For every prime polynomial $P \in A$, the group $\ker \left( \text{SL}_r(A/P^{a_P}A) \to \text{SL}_r(A/PA) \right)$ is a $p$-group (of order $q^{\deg(P)(a-1)(r^2-1)}$).

Let $C$ be an Abelian quotient of $S_r(n)$. Then $C$ is a product of a $p$-group and Abelian quotients of $\text{SL}_r(A/P^{a_P}A)$, which reduce modulo $P$ to Abelian quotients of $\text{SL}_r(A/PA)$. By Lemma 17 these are also $p$-groups. Therefore $C$ must be a $p$-group. □

9. Completing the proof

We now have all the ingredients we need. Our next step is

**Proposition 19.** $K_t \cap K F_{tn} = K F_t$. In particular,

$$\text{Gal}(K_t F_{tn}/K_t) \cong G_1(n) \quad \text{and} \quad \text{Gal}(K_t F_{tn}/K F_{tn}) \cong \text{SL}_r(\mathbb{F}_q).$$

**Proof.**

![Diagram]

\[ K_t \cap K F_{tn} = K F_t \]

\[ \text{Gal}(K_t F_{tn}/K_t) \cong G_1(n) \quad \text{and} \quad \text{Gal}(K_t F_{tn}/K F_{tn}) \cong \text{SL}_r(\mathbb{F}_q). \]
Let $H := K_t \cap KF_{tn}$. First notice that $KF_{tn}/H$ is an Abelian extension corresponding to a subgroup of $\text{Gal}(KF_{tn}/K_t) \cong G_1(n)$. By Proposition 16 we see that $v_p([G_1(n)]) = v_p([K_t F_{tn} : K_t]) = v_p([KF_{tn} : H])$, and so $p \nmid [H : KF_{tn}]$. Now $\text{Gal}(H/KF_{tn})$ is Abelian of order prime to $p$; it is also a quotient of $\text{Gal}(K_t/KF_{tn}) \cong \text{SL}_r(F_q)$, hence by Lemma 17 it must be trivial. The result follows.

Since we now know that $\text{Gal}(K_t F_{tn,0}/K_t) = \text{Gal}(K_t F_{tn}/K_t) \cong G_1(n)$, we see that $\text{Gal}(K_t F_{tn,0}/K_t, 0)$ is Abelian of order prime to $p$. Proceeding as in the proof of Proposition 19 we see that $\text{Gal}(K_{tn,0} F_{tn,0}/K_t) \cong S_r(n)$ of order prime to $p$, and hence, by Proposition 18 trivial. It follows that $\text{Gal}(K_{tn,0} F_{tn,0}/K_t, 0) \cong \text{Gal}(K_{tn,0}/K_t, 0) \cong S_r(n)$, which completes the proof of Proposition 9, since $K_{tn,0} F_{tn,0} = K_{tn}$.

Remark 20. Given our explicit description for the various fields concerned, it is tempting to search for a direct proof that $[K_{tn} : K_{tn,0}] = [K_t : K_{t,0}]$, which would allow us to cut short much of the above argument and simplify the proof of Proposition 9. Alas, the author was not successful with this.

References

[1] S. S. Abhyankar, Resolution of singularities and modular Galois theory. Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 2, 131–169.
[2] S. S. Abhyankar and G. S. Sundaram, Galois theory of Moore-Carlitz-Drinfeld modules. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 4, 349–353.
[3] G. Anderson, t-motives. Duke Math. J. 53 (1986), no. 2, 457–502.
[4] L. Carlitz, A class of polynomials. Trans. Amer. Math. Soc. 43 (1938), no. 2, 167–182.
[5] V. G. Drinfel’d, Elliptic modules (Russian), Math. Sbornik, 94 (1974), 594-627. Translated in Math. USSR. S., 23 (1974), 561 – 592.
[6] D. Goss, Basic structures in function field arithmetic, Springer-Verlag, 1996.
[7] P. Hubschmid, The André-Oort conjecture for Drinfeld modular varieties, Compos. Math. 149 (2013), no. 4, 507–567.
[8] K. Joshi, A family of étale coverings of the affine line, *J. Number Theory* **59** (1996), 414–418.

[9] S. Lang, Elliptic Functions, 2nd edition, *Graduate Texts in Mathematics* **112**, Springer-Verlag, 1987.

[10] S. Lang, Algebra, 3rd edition, *Graduate Texts in Mathematics* **211**, Springer-Verlag, 2002.

[11] T. Lehmkuhl, Compactification of the Drinfeld Modular Surfaces, *Mem. Amer. Math. Soc.* **197** (2009), no. 921.

[12] E. H. Moore, A two-fold generalization of Fermat’s theorem. *Bull. Amer. Math. Soc.* **2** (1896), no. 7, 189–199.

[13] R. Pink, Compactification of Drinfeld modular varieties and Drinfeld Modular Forms of Arbitrary Rank, *Manuscripta Math.* **140** (2013), no. 3-4, 333–361.

[14] R. Pink, S. Schieder, Compactification of a Drinfeld Period Domain over a Finite Field, *J. Algebraic Geometry* **23** (2014), no. 2, 201–243.

[15] I. Y. Potemine, Minimal terminal $\mathbb{Q}$-factorial models of Drinfeld coarse moduli schemes, *Math. Phys. Anal. Geom.* **1** (1998), 171–191.

[16] M. Rosen, Number Theory in Function Fields, *Graduate Texts in Mathematics* **210**, Springer-Verlag, 2002.

[17] A. Thiery, $F_q$-linear Galois theory, *J. London Math. Soc.* (2) **53** (1996), 441–454.

[18] G.-J. van der Heiden, Weil pairing for Drinfeld modules, *Monatsh. Math.* **143** (2004), 115–143.

[19] G.-J. van der Heiden, Drinfeld modular curves and the Weil pairing, *J. Algebra* **299** (2006), 374–418.