TOPOLOGICAL LIE BIALGEBRA STRUCTURES AND THEIR CLASSIFICATION OVER $\mathfrak{g}[x]$.

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Abstract. This paper is devoted to a classification of topological Lie bialgebra structures on the Lie algebra $\mathfrak{g}[x]$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over an algebraically closed field $F$ of characteristic 0.

We introduce the notion of a topological Manin pair $(L, \mathfrak{g}[x])$ and present their classification by relating them to trace extensions of $F[x]$. Then we recall the classification of topological doubles of Lie bialgebra structures on $\mathfrak{g}[x]$ and view the latter as a special case of the classification of Manin pairs.

The classification of topological doubles states that up to some notion of equivalence there are only three non-trivial doubles. It is proven that topological Lie bialgebra structures on $\mathfrak{g}[x]$ are in bijection with certain Lagrangian Lie subalgebras of the corresponding doubles. We then attach algebro-geometric data to such Lagrangian subalgebras and, in this way, obtain a classification of all topological Lie bialgebra structures with non-trivial doubles. When $F = \mathbb{C}$ the classification becomes explicit. Furthermore, this result enables us to classify formal solutions of the classical Yang-Baxter equation.

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1. Introduction

Let \( g \) be a finite-dimensional simple Lie algebra over an algebraically closed field \( F \) of characteristic 0. Topological Lie bialgebra structures on \( g[[x]] \) arise naturally as completions of several important classical Lie bialgebras, e.g. any Lie bialgebra structure on \( g[x] \) or \( g[x, x^{-1}] \) extends to \( g[[x]] \). Moreover, these structures are closely related to Hopf-algebra deformations of the universal enveloping algebra of \( g \); see [11, 12]. This makes them important in the theory of integrable systems and quantum groups. This paper is dedicated to the classification of topological Lie bialgebra structures on \( g[[x]] \).

Let us describe one possible approach to the classification of Lie bialgebra structures. Recall that a Lie bialgebra over \( F \) is a vector space \( L \) over \( F \) (not necessarily finite-dimensional) which is equipped with a Lie algebra structure \([·, ·]: L \otimes L \to L\) and a Lie coalgebra structure \( δ: L \to L \otimes L \) subject to the compatibility condition

\[
[δ(x, y)] = [x \otimes 1 + 1 \otimes x, δ(y)] - [y \otimes 1 + 1 \otimes y, δ(x)] \quad ∀x, y ∈ L. \tag{1.1}
\]

Each Lie bialgebra structure \( δ \) on \( L \) is associated with the object \( \mathcal{D}(L, δ) \) called the classical double of \( δ \). It is the vector space \( L + L^\vee \) equipped with the bilinear form

\[
B(x + f, y + g) := f(y) + g(x) \quad ∀x, y ∈ L, ∀f, g ∈ L^\vee, \tag{1.2}
\]

and the commutator map

\[
[x, f] := -f \circ \text{ad}_x + (f \otimes 1)(δ(x)) \quad ∀x ∈ L, ∀f ∈ L^\vee. \tag{1.3}
\]

Having a Lie bialgebra structure \( δ \) on \( L \) we can obtain new Lie bialgebra structures by a procedure called twisting. More explicitly, for any skew-symmetric tensor \( s \in L \otimes L \) such that

\[
\text{CYB}(s) = \text{Alt}((δ \otimes 1)s), \tag{1.4}
\]

where \( \text{CYB}(s) := [s^{12}, s^{13}] + [s^{12}, s^{23}] + [s^{13}, s^{23}] \) and \( \text{Alt}(a \otimes b \otimes c) := a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b \), the linear map \( δ_s := δ + ds \) defines another Lie bialgebra structure on \( L \). Skew-symmetric tensors satisfying \( \text{Eq. (1.4)} \) are called twists of \( δ \). One important property of the double \( \mathcal{D}(L, δ) \) is its invariance under twisting, i.e.

\[
\mathcal{D}(L, δ) = \mathcal{D}(L, δ + ds).
\]
for any twist $s$ of $\delta$. For this reason doubles are sometimes called twisting classes. This observation leads to the following simple classification scheme for Lie bialgebra structures on $L$:

1. Classify all possible classical doubles $\mathcal{D}(L, \delta) = L + L'$;

2. Choose a representative $\delta$ inside each twisting class and describe all its twists. By [10, Theorem 2.4] this is equivalent to describing Lagrangian Lie subalgebras $W \subset \mathcal{D}(L, \delta)$ such that

$$\mathcal{D}(L, \delta) = L + W \quad \text{and} \quad \dim(W + L')/(W \cap L') < \infty.$$  

The second step of the scheme, in the most interesting cases, is equivalent to the classification of certain $r$-matrices, i.e. solutions to the classical Yang-Baxter equation (CYBE)

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = 0. \quad (1.5)$$

These were thoroughly studied by Belavin and Drinfeld [5, 6] and by Stolin [22, 23].

**Example 1.1.** It was shown in [22] that the classical double $\mathcal{D} = \mathcal{D}(g, \delta)$ of a Lie bialgebra structure $\delta$ on $g$ is of the form $g \otimes A$ for a two-dimensional unital associative commutative $F$-algebra $A$. Consequently,

$$\mathcal{D} \cong g \times g \quad \text{or} \quad \mathcal{D} \cong g[x]/x^2g[x]$$

as a Lie algebra and in both cases there is essentially only one possibility for the bilinear form. Moreover, Whitehead’s Lemma implies that in both cases $\delta = dr$ for a constant solution $r \in g \otimes g$ of the CYBE [Eq. (1.5)]. The isomorphism $\mathcal{D} \cong g[x]/x^2g[x]$ holds if and only if $r$ is skew-symmetric. Therefore, the classification of Lie bialgebra structures with $\mathcal{D} \cong g \times g$ (resp. $\mathcal{D} \cong g[x]/x^2g[x]$) is equivalent to the classification of constant non-skew-symmetric (resp. skew-symmetric) $r$-matrices. These classifications can be found in e.g. [10]. ◊

It was proven in [19] that any Lie bialgebra structure $\delta$ on $g[x]$ satisfies

$$\delta(x^n g[x]) \subseteq (x, y)^{n-1} (g \otimes g)[x, y]. \quad (1.6)$$

In particular, if we equip $g[x]$ with the $(x)$-adic topology and $g[x] \otimes g[y] \cong (g \otimes g)[x, y]$ with the projective tensor product topology, both the Lie bracket and the Lie cobracket become continuous. Therefore, they possess unique continuous extensions

$$\hat{\delta}(g[x]) \nsubseteq g[x] \otimes g[y] \quad \text{and} \quad \delta : g[x] \to (g \otimes g)[x, y]. \quad (1.7)$$

In general $\delta(g[x]) \nsubseteq g[x] \otimes g[y]$ and hence it is not a Lie bialgebra in the sense above.

A topological Lie bialgebra structure on $g[x]$ is a linear map $g[x] \to (g \otimes g)[x, y]$ having properties similar to the ones of $\hat{\delta}$ above. As in the classical case, a topological Lie bialgebra structure on $g[x]$ gives rise to a topological double – topological analogue of a classical double. Let $g[x]' := \{f : g[x] \to F \mid f(x^n g[x]) = 0 \quad \text{for some} \quad n \in \mathbb{Z}_+\}$. be the space of continuous functionals on $g[x]$. Then the topological double of a topological Lie bialgebra structure $\delta$ on $g[x]$ is the vector space $\mathcal{D} = g[x] + g[x]'$ with the continuous commutator map defined similar to [Eq. (1.3)] and separately continuous bilinear form [Eq. (1.2)].

The classification of topological Lie bialgebra structures on $g[x]$ can be done using a similar scheme:

1. Classify all possible topological doubles $\mathcal{D}(g[x], \delta) = g[x] + g[x]'$;

2. Choose a representative $\delta$ inside each twisting class and describe all its topological twists. By [Theorem 4.3] this problem is equivalent to classifying Lagrangian Lie subalgebras $W \subset \mathcal{D}(g[x], \delta)$ such that $\mathcal{D}(g[x], \delta) = g[x] + W$.

The first step of this classification is done in [19]: it is shown that up to some notion of equivalence there are only three non-trivial topological doubles $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. More explicitly,
\[ D_i := g((x)) \times g[x]/x^{i-1}g[x] \text{ for } i \in \{1, 2, 3\} \] with the bilinear form

\[ B_i((f_1, [f_2]), (g_1, [g_2])) = \text{coeff}_{i-2}\{\kappa(f_1(x), g_1(x))\} - \text{coeff}_{i-2}\{\kappa(f_2(x), g_2(x))\}, \]

(1.8)

where \( f_1, g_1 \in g((x)), f_2, g_2 \in g[x] \) and \( \kappa: g((x)) \times g((x)) \to F((x)) \) is the \( F((x)) \)-bilinear extension of the Killing form on \( g \).

The main result of this paper is the completion of step (2) in the above-mentioned classification scheme. We achieve this result by using the connection between topological Lie bialgebra structures on \( g[[x]] \) and formal \( r \)-matrices, i.e. series

\[ r(x, y) = \frac{s(y)\Omega}{x - y} + g(x, y) \in (g \otimes g)((x))[y], \]

(1.9)

where \( s \in F[y], g \in (g \otimes g)[x, y] \) and \( \Omega \in g \otimes g \) is the quadratic Casimir element, solving Eq. (1.5).

More precisely, we prove that the assignment \( r \mapsto dr \), where

\[ dr(f) := [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)], \]

(1.10)

establishes a bijection between these objects. The reflection of the classification of topological doubles on the side of formal \( r \)-matrices gives an unexpected restriction on the series \( s(y) \). To be exact, there are no formal \( r \)-matrices of the form Eq. (1.9) with a non-zero \( s \in y^3F[y] \).

Following [1], to each Lagrangian Lie subalgebra of \( D_i \) complementary to \( g[[x]] \) we assign an algebra-geometric datum. Then, using general methods of algebraic geometry, we obtain significant restrictions on formal \( r \)-matrices associated to Lagrangian Lie subalgebras of \( D_i \):  

\( i = 1 \): The \( r \)-matrices are subject to a generalization of the Belavin-Drinfeld trichotomy: they are either of elliptic, trigonometric or rational type;  
\( i = 2 \): The corresponding \( r \)-matrices turn out to be quasi-trigonometric;  
\( i = 3 \): The associated formal \( r \)-matrices are of quasi-rational type.

**Figure 1. Restrictions on formal \( r \)-matrices**

When \( F = \mathbb{C} \), the above-mentioned five classes of formal \( r \)-matrices are completely classified in the available literature: Elliptic and trigonometric \( r \)-matrices are classified in [5]; Rational \( r \)-matrices are classified in [22, 23]; Quasi-trigonometric \( r \)-matrices are described in [21] and independently in [2]; Finally, quasi-rational \( r \)-matrices are classified in [24]. Note that most of these
descriptions are valid for an arbitrary algebraically closed field of characteristic 0 without any adjustments.

1.1. Structure of the paper. Section 2 starts with preliminary results on linearly topologized vector spaces and their completions as well as on linear tensor product topologies. Using these results we then introduce the notion of a topological Lie bialgebra structure on an arbitrary linearly topologized Lie algebra \( L \) with continuous Lie bracket.

Section 3 starts with the classification of Manin pairs \((L, g[x])\) via trace extensions of \( F[x] \). Then we review the classification of topological doubles over \( g[x] \) from [19] in the setting of Manin pairs. In the process we generalize one of the pivotal steps in the classification [19] and obtain the following result.

**Theorem A.** Let \( L \) be a Lie algebra with a non-degenerate invariant symmetric bilinear form and \( A \) be a unital associative commutative reduced \( F \)-algebra. If \( L \) contains a coisotropic subalgebra of the form \( g \otimes A \), then \( L \cong g \otimes \tilde{A} \) for some associative commutative \( F \)-algebra extension \( \tilde{A} \supseteq A \).

In Sections 4 and 5 we introduce topological twists – certain elements in \((g \otimes g)[x, y]\) that allow us to obtain new topological Lie bialgebra structures on \( g[x] \) from a given one with the same topological double. We relate these objects to Lagrangian Lie subalgebras of \( D_i \), \( i \in \{1, 2, 3\} \) and formal \( r \)-matrices. The main results can be summarized as follows.

**Theorem B.** There is a bijection between

1. Topological Lie bialgebra structures on \( g[x] \) with the topological double \( D_i \);
2. Formal \( r \)-matrices of the form \( y^{-1} \Omega/(x - y) + g(x, y) \) and
3. Lagrangian Lie subalgebras of \( D_i \) complementary to \( g[x] \).

Moreover, these bijections preserve the equivalences.

Viewing the classification of topological doubles [10] through the prism of the above-mentioned connections, we obtain an interesting result on the side of formal \( r \)-matrices.

**Corollary C.** If the series

\[ s(y)\Omega \frac{1}{x - y} + g(x, y) \in (g \otimes g)((x))[y], \]

with \( s \in F[y] \) and \( g \in (g \otimes g)[x, y] \), solves CYBE, then \( s = 0 \) or \( s \) has a zero in \( y = 0 \) of multiplicity either 0, 1 or 2. Furthermore, after an appropriate equivalence transformation \( s \in \{0, 1, y, y^2\} \). The existence of such a transformation in case of multiplicity 2 is equivalent to the fact that the coefficient of \( y^3 \) in \( s(y) \) vanishes.

Section 6 is about a natural subclass of topological twists, called commensurable twists. These are topological twists inside \((g \otimes g)[x, y]\). Commensurable twists turn out to be in bijection with rational, quasi-trigonometric and quasi-rational \( r \)-matrices. The classifications of these \( r \)-matrices are already known and presented in this section.

In Section 7 we state the main classification results for topological Lie bialgebra structures on \( g[x] \). More precisely, we show that up to a certain notion of equivalence there are six classes of formal \( r \)-matrices: degenerate (i.e. \( s(y) = 0 \)), elliptic, trigonometric, rational, quasi-trigonometric and quasi-rational. In the case \( F = \mathbb{C} \) the descriptions of these classes become explicit.

**Theorem D.** Let \( r \) be a formal \( r \)-matrix corresponding to a Lie bialgebra structure with the double \( D_i \), \( i \in \{1, 2, 3\} \). Then

\( i = 1 \): \( r \) is equivalent to a Taylor series expansion at \( y = 0 \) of either an elliptic or a trigonometric or a rational \( r \)-matrix;
\( i = 2 \): \( r \) is equivalent to a quasi-trigonometric \( r \)-matrix;
\( i = 3 \): \( r \) is equivalent to a quasi-rational \( r \)-matrix.
The algebra-geometric proof of the classification is given in Section 8. The case $i = 1$ was already thoroughly studied in [1]. We present a sketch of its proof given there. The cases $i = 2$ and $i = 3$ are new and described in full detail.

We use many different notions of equivalence throughout the paper. Appendix A collects all of these notions and can be used as a cheat sheet while reading the work.

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2. Topological Lie bialgebras

We start with a brief review of linearly topologized vector spaces and then introduce the notion of a topological Lie bialgebra. For a detailed exposition on linearly topologized vector spaces we refer to [17].

Let $\mathbb{F}$ be a field of characteristic 0 equipped with the discrete topology. All vector spaces are considered over that field.

2.1. Linearly topologized vector spaces. A vector space $V$ is said to be linearly topologized if it is endowed with a topology $\mathcal{T}$ satisfying the following conditions

1. Vector addition $V \times V \to V$ and scalar multiplication $\mathbb{F} \times V \to V$ are (jointly) continuous;
2. $(V, \mathcal{T})$ is a Hausdorff topological space;
3. $\mathcal{T}$ is translation invariant and $0 \in V$ admits a fundamental system of neighbourhoods consisting of subspaces of $V$.

Such a topology $\mathcal{T}$ will be referred to as linear topology.

Example 2.1. Any vector space equipped with the discrete topology is a linearly topologized vector space. The converse is true for finite-dimensional vector spaces: any finite-dimensional linearly topologized vector space is necessarily discrete. ♦

We refer to complete linearly topologized vector spaces simply as linearly complete. Any linearly topologized vector space $(V, \mathcal{T})$ can be embedded into a smallest linearly complete vector space $(\hat{V}, \hat{\mathcal{T}})$. This space is unique up to topological isomorphism and it is called the completion of $V$. It is common to represent the completion $\hat{V}$ as the following limit

$$\hat{V} = \lim_{i \in I} V/N_i, \quad (2.1)$$

where $\{N_i\}_{i \in I} \subseteq \mathcal{T}$ is a fundamental system of neighbourhoods of $0 \in V$ and each quotient $V/N_i$ is equipped with the discrete topology. The set $\{\text{Cl}_{\hat{V}}(N_i)\}_{i \in I}$ forms a fundamental system of neighbourhoods of $0 \in \hat{V}$. Here $\text{Cl}_{\hat{V}}(N_i)$ denotes the closure of $N_i$ in $\hat{V}$.

Example 2.2. Any discrete vector space $V$ is automatically complete. Let us equip the algebraic dual $V^\vee$ with the weak topology, i.e. the linear topology with $\{U^\perp \mid U \subseteq V \text{ is finite-dimensional}\}$ as a fundamental system of neighbourhoods of $0 \in V^\vee$, where $U^\perp = \{f \in V^\vee \mid f|_U = 0\}$. Then $V^\vee$ is also a linearly complete vector space. In particular, this implies that both $F[x]$ and $F[x] = F[x]^\vee$ are linearly complete spaces. The weak topology on $F[x]$ is precisely the $(x)$-adic topology. ♦

2.2. Linear tensor product topologies. Let $V$ and $W$ be linearly topologized vector spaces. Their usual tensor product $V \otimes W$ has no canonical linear topology on it.

In this work we are interested in the finest linear topology on $V \otimes W$ making the tensor product map $\otimes : V \times W \to V \otimes W$ continuous. More explicitly, we declare a subspace $U \subseteq V \otimes W$ open if it satisfies the following three conditions

- There are open subspaces $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $V_0 \otimes W_0 \subseteq U$;
- For every $v \in V$ there exists an open subspace $W_0 \subseteq W$ such that $v \otimes W_0 \subseteq U$;
For every \( w \in W \) there exists an open subspace \( V_0 \subseteq V \) such that \( V_0 \otimes w \subseteq U \).

The corresponding topology is referred to as projective tensor product topology or the \( \underline{\otimes} \)-topology. The completion of \( V \otimes W \) with respect to it is denoted by \( V \overline{\otimes} W \). The universal property of the \( \underline{\otimes} \)-topology immediately implies that the unique factorization \( f : V \otimes W \to X \) of any continuous bilinear map \( f : V \times W \to X \) is again continuous.

Even though we have an explicit description of open sets in the \( \underline{\otimes} \)-topology, it is not easy to work with them explicitly. For that reason we introduce another auxiliary tensor product topology. It is the finest linear topology on \( V \otimes W \) making the tensor product map \( \otimes : V \times W \to V \otimes W \) uniformly continuous. We call it the \( \overline{\otimes} \)-topology.

Remark 2.3. It is clear that the first topology is finer than the second one. However, when both \( V \) and \( W \) are discrete we get the homeomorphisms
\[
V \overline{\otimes} W \cong V \underline{\otimes} W \quad \text{and} \quad V^\vee \overline{\otimes} W^\vee \cong V^\vee \underline{\otimes} W^\vee \cong (V \otimes W)^\vee,
\]
where the algebraic duals are equipped with the weak topology; see e.g. [8, Lemma 24.17 and Corollary 24.25]. Later we work primarily with the \( \underline{\otimes} \)-topology and the isomorphisms \( \text{Eq. (2.4)} \) allow us to use the explicit description \( \text{Eq. (2.2)} \) of 0-neighbourhoods.

2.3. Topological Lie bialgebras. From now on all tensor products of linearly topologized vector spaces are taken by default with the \( \underline{\otimes} \)-topology.

A topological Lie coalgebra is a pair \( (L, \delta) \), where \( L \) is a linearly topologized vector space and \( \delta : L \to L \overline{\otimes} L \) is a continuous linear map satisfying the conditions
\[
\delta(x) + \tau \delta(x) = 0 \quad \text{and} \quad \overline{\text{Alt}}((\delta \overline{\otimes} 1) \delta(x)) = 0,
\]
where
\[
\tau : L \overline{\otimes} L \to L \overline{\otimes} L, \quad \overline{\text{Alt}} : L \overline{\otimes} L \overline{\otimes} L \to L \overline{\otimes} L \overline{\otimes} L, \quad \delta \overline{\otimes} 1 : L \overline{\otimes} L \to L \overline{\otimes} L \overline{\otimes} L
\]
stand for the unique continuous extensions of the continuous linear maps
\[
\tau : L \otimes L \to L \otimes L, \quad x \otimes y \mapsto y \otimes x, \quad \overline{\text{Alt}} : L \otimes L \otimes L \to L \otimes L \otimes L, \quad x \otimes y \otimes z \mapsto x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y, \quad \delta \otimes 1 : L \otimes L \to (L \overline{\otimes} L) \otimes L, \quad x \otimes y \mapsto \delta(x) \otimes y.
\]

We call a continuous \( F \)-linear map \( \varphi : L_1 \to L_2 \) a morphism of topological Lie coalgebras if
\[
(\varphi \overline{\otimes} \varphi) \delta_1 = \delta_2 \varphi.
\]

In a similar vein we define a topological Lie algebra as a linearly topologized vector space \( L \) together with a continuous Lie bracket \( [\cdot, \cdot] : L \times L \to L \). A morphism between topological Lie algebras is a continuous Lie algebra morphism.

Remark 2.4. Note that by definition of the \( \underline{\otimes} \)-topology we have an isomorphism between the space of continuous bilinear maps \( L \times L \to L \) and the space of continuous linear maps \( L \otimes L \to L \). Moreover, when \( L \) is complete these spaces are also isomorphic to the space of continuous linear maps \( L \overline{\otimes} L \to L \).
A topological Lie bialgebra consists of the following datum
- A topological Lie algebra \((L, [\cdot, \cdot])\);
- A topological Lie coalgebra \((L, \delta)\);
- A linear topology on the space \(L'\) of continuous linear functionals \(L \to F\);

And this datum is subject to the following conditions
(1) The restriction of the dual map \(\delta' : (L \otimes L)' \to L'\) to the subset \(L' \otimes L' \subseteq (L \otimes L)' = (L \otimes L)'\) is again continuous;
(2) The following compatibility condition holds
\[
\delta([x, y]) = (\text{ad}_x 1 + 1 \otimes \text{ad}_x)\delta(y) - (\text{ad}_y 1 + 1 \otimes \text{ad}_y)\delta(x)
\]
for all \(x, y \in L\).

When there is no ambiguity we simply write \((L, \delta)\) to denote a topological Lie bialgebra. A map \(\varphi\) between two topological Lie bialgebras is a topological Lie bialgebra morphism if it is a morphism of both topological Lie algebra and topological Lie coalgebra structures such that \(\varphi'\) is continuous.

**Remark 2.5.** In the following we use the notation \([x \otimes 1, f] := (\text{ad}_x 1) t\) and \([1 \otimes x, f] := (1 \otimes \text{ad}_x) t\) for all \(x \in L\) and \(t \in L \otimes L\). For instance, the compatibility condition in (2) then reads
\[
\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)]
\]
for all \(x, y \in L\).

**Remark 2.6.** If \(\delta : L \to L \otimes L\) is a topological Lie bialgebra structure on \(L\), then so is \(\xi \delta\) for any \(\xi \in F\). It is natural to not distinguish structures that differ only by a non-zero scalar multiple. Therefore, we call two topological Lie bialgebras \((L_1, \delta_1)\) and \((L_2, \delta_2)\) equivalent if there is a constant \(\xi \in F^\times\) such that \((L_1, \delta_1)\) is isomorphic to \((L_2, \xi \delta_2)\). We adopt the notation
\[
(L_1, \delta_1) \sim (L_2, \delta_2)
\]
to denote equivalent topological Lie bialgebras.

### 2.4. Topological Manin Triples and Doubles.

A triple \((L, L_+, L_-)\) is called a topological Manin triple if \(L\) is a topological Lie algebra equipped with a separately continuous invariant non-degenerate symmetric bilinear form \(B : L \times L \to F\) such that \(L_\pm\) are isotropic topological Lie subalgebras of \(L\) and \(L = L_+ + L_-\).

We say that two topological Manin triples \((L_1, L_1, L_-)\) and \((M, M_+, M_-)\) are isomorphic if there is an isomorphism of topological Lie algebras \(\varphi : L \to M\) such that
\[
\varphi(L_\pm) = M_\pm \quad \text{and} \quad B_L(x, y) = B_M(\varphi(x), \varphi(y)) \quad \forall x, y \in L.
\]

Let \((L, \delta)\) be a topological Lie bialgebra. Consider the triple \((L + L', L, L')\) with the form
\[
B(x + f, y + g) := f(y) + g(x) \quad \forall x, y \in L, \forall f, g \in L',
\]
and the bracket
\[
[x, f] := -f \circ \text{ad}_x + (f \otimes 1)(\delta(x)) \quad \forall x \in L, \forall f \in L'.
\]

In the classical (i.e. when all vector spaces are discrete) finite-dimensional framework the construction above gives a bijection between Lie bialgebras and Manin triples. If we allow \(L\) to be infinite-dimensional, the triple \((L + L', L, L')\) is still a Manin triple, but the converse direction is no longer true: not every Manin triple is of this form. Passing further from the classical framework to the topological one we entirely lose this connection: the triple \((L + L', L, L')\) is not a topological Manin triple in general. The first subtlety is that the bracket [Eq. (2.11)] may not be well-defined if \(L\) is not complete. The second problem is that it may not be continuous. However, in the most important cases for us the triple \((L + L', L, L')\) is indeed a topological Manin triple. The space
$L \oplus L'$ with the topological Lie algebra structure and the form mentioned above is then denoted by $\mathcal{O}(L, \delta)$ or simply by $\mathcal{O}$ and called the topological double of $(L, \delta)$.

Suppose $(L, L_{+}, L_{-})$ is a topological Manin triple and we have a topology on $L_{+}$ such that

1. The dual map $\delta' : L_{+} \rightarrow (L_{-} \otimes L_{-})^{\ast}$ restricts to a continuous map $\delta : L_{+} \rightarrow L_{+} \otimes L_{+}$, where $L_{+}$ is viewed as a subset of $L_{+}'$ through the bilinear form $B$ on $L$;

2. The restriction of $\delta'$ to $L_{+}' \otimes L_{+}'$ is continuous, then $(L_{+}, \delta)$ is a topological Lie bialgebra defined by the triple $(L, L_{+}, L_{-})$.

Remark 2.7. As in the classical case the first condition is equivalent to the following: there exists a continuous linear map $\delta : L_{+} \rightarrow L_{+} \otimes L_{+}$ such that $B \otimes 2(\delta(x), y \otimes z) = B(x, [y, z])$ for all $x \in L_{+}$ and $y, z \in L_{-}$. Here $B \otimes 2$ is the unique continuous extension $L_{+} \otimes L_{+} \rightarrow F$ of the continuous map $B(\cdot, y \otimes z) = B(\cdot, y)B(\cdot, z) : L_{+} \otimes L_{+} \rightarrow F$.

Remark 2.8. Suppose $(L, L_{+}, L_{-})$ and $(M, M_{+}, M_{-})$ are two isomorphic topological Manin triples. If $(L, L_{+}, L_{-})$ together with a topology on $L_{+}'$ defines a topological Lie bialgebra $(L_{+}, \delta_{L})$, then $(M, M_{+}, M_{-})$ together with the induced topology on $M_{+}'$ defines a topological Lie bialgebra $(M_{+}, \delta_{M})$. Moreover, if $\varphi : L \rightarrow M$ is the isomorphism between the two Manin triples, then its restriction $\varphi|_{L_{+}} : L_{+} \rightarrow M_{+}$ is an isomorphism of topological Lie bialgebras.

Remark 2.9. Note that if a Manin triple $(L, L_{+}, L_{-})$ with a form $B$ defines a Lie bialgebra structure $(L_{+}, \delta)$, then the same triple $(L, L_{+}, L_{-})$ with the form $\xi B$, $\xi \in F^{\times}$, defines the Lie bialgebra structure $(L_{+}, \xi^{-1} \delta)$. Since we have identified Lie bialgebras that differ by a non-zero scalar multiple (see Remark 2.6 it is natural to identify Manin triples whose forms differ by a non-zero scalar multiple as well. Taking that into account, we call two Manin triples $(L, L_{+}, L_{-})$ and $(M, M_{+}, M_{-})$ with forms $B_{L}$ and $B_{M}$ respectively equivalent if there is $\xi \in F^{\times}$ such that $(L, L_{+}, L_{-})$ is isomorphic to $(M, M_{+}, M_{-})$ with the form $\xi B_{M}$. Such an equivalence of Manin triples is denoted by $(L, L_{+}, L_{-}) \sim (M, M_{+}, M_{-})$.

3. Classification of topological doubles on $\mathfrak{g}[x]$.

In this section we review the classification of topological doubles on $\mathfrak{g}[x]$ achieved in [15] within our framework of topological Lie bialgebras. One of the pivotal steps in the derivation of this result is the fact that these doubles are of the form $\mathfrak{g} \otimes A$ for a so-called trace extension $A$ of $F[x]$. However, the proof presented in [15] is not applicable for $\mathfrak{g} = \mathfrak{g}(n, F)$ with $n \geq 3$. In this section we refine the proof to include all simple Lie algebras as well as show that the result is still valid when $F[x]$ is replaced by any associative commutative reduced unital $F$-algebra.

3.1. Extension of scalars of simple Lie algebras. Throughout this section $F$ denotes an arbitrary field of characteristic 0. Let us recall that a finite-dimensional (not necessarily associative or Lie) algebra $\mathfrak{a}$ over $F$ is called central if its centroid

$$
\Gamma_{F}(\mathfrak{a}) := \{ f \in \text{End}_{F}(\mathfrak{a}) \mid a f(b) = f(a b) = f(a) b, \text{ for all } a, b \in \mathfrak{a} \}
$$

(3.1)

coincides with scalar multiples of the identity map. If $\mathfrak{a}$ is simple, the centroid is a finite field extension of $\mathfrak{a}$. In particular, any finite-dimensional simple $F$-algebra is central if $F$ is algebraically closed.

Lemma 3.1. Let $\mathfrak{g}$ be a finite-dimensional central simple Lie algebra over $F$. Define

$$
\text{Ann}(\mathfrak{g}) := \{ x \in U(\mathfrak{g}) \mid x \cdot \mathfrak{g} = 0 \},
$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. If $V$ is a finite-dimensional irreducible $\mathfrak{g}$-module and $\text{Ann}(\mathfrak{g}) V = 0$, then $V \cong \mathfrak{g}$ as $\mathfrak{g}$-modules.
Therefore, \( P \) is simple we have \( O_{U(g)}(I_a) = g \). The kernel of the surjective \( (U(g)) \)-module homomorphism

\[
U(g) \longrightarrow O_{U(g)}(I_a) = g
\]

\[
x \longmapsto x \cdot I_a
\]

(3.2)
is exactly \( \text{Ann}(I_a) := \{ x \in U(g) \mid x \cdot I_a = 0 \} \). Therefore, \( U(g)/\text{Ann}(I_a) \cong g \) and hence \( \text{Ann}(I_a) \) is a maximal left ideal of \( U(g) \). A similar argument shows that if \( \{ v_i \}_{i=1}^\ell \) is a basis of \( V \), then \( \text{Ann}(v_i) \) are maximal left ideals of \( U(g) \) and \( U(g)/\text{Ann}(v_i) \cong V \).

Consider the \( (U(g)) \)-module homomorphism

\[
U(g) \longrightarrow \text{End}_F(g)
\]

\[
x \longmapsto x \cdot (-).
\]

(3.3)
By Schur’s lemma, Jacobson density theorem and the centrality of \( g \) this map is surjective with kernel \( \text{Ann}(g) \). This means that we have an isomorphism of \( (U(g)) \)-modules \( U(g)/\text{Ann}(g) \cong \text{End}_F(g) \) and, consequently, \( \text{Ann}(g) \) is a maximal two-sided ideal. Moreover, \( \text{Ann}(V) \) is also a two-sided ideal and by our assumption we have the following inclusions

\[
\text{Ann}(g) \subseteq \text{Ann}(V) \subseteq U(g)
\]

(3.4)
that imply the equality \( \text{Ann}(g) = \text{Ann}(V) \).

Combining all previous observations we obtain the following relations between \( g \)-modules

\[
\prod_{\alpha=1}^d g \cong \prod_{\alpha=1}^d U(g)/\text{Ann}(I_a) \cong U(g)/\text{Ann}(g) = U(g)/\text{Ann}(V) \subseteq \prod_{i=1}^\ell U(g)/\text{Ann}(v_i) \cong \prod_{i=1}^\ell V,
\]

which now imply the desired isomorphism \( V \cong g \). \( \blacksquare \)

**Theorem 3.2.** Let \( g \) be a central simple Lie algebra over \( F \) and \( A \) be a reduced unital associative commutative \( F \)-algebra. Equip the tensor product \( g \otimes A \) with the Lie algebra bracket

\[
[a \otimes f, b \otimes g] = [a, b] \otimes fg.
\]

Let \( L \) be a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form \( B \) such that \( g \otimes A \subseteq L \) is a coisotropic subalgebra. Then, as a \( g \)-module, \( L \cong g \otimes \hat{A} \) for a unital associative commutative algebra extension \( \hat{A} \supseteq A \).

**Proof.** The action of \( g \cong g \otimes 1 \subseteq g \otimes A \) on \( L \) extends uniquely to an action of \( U(g) \) on \( L \). Define

\[
P := \text{Ann}(g) = \{ x \in U(g) \mid x \cdot g = 0 \}.
\]

Let

\[
i: U(g) \longrightarrow U(g)
\]

\[
x_1 \ldots x_n \longmapsto (-1)^n x_n \ldots x_1
\]

(3.5)
be the antipode map and \( \kappa \) be the Killing form on \( g \). By invariance of Killing form we have \( \kappa(i(P) \cdot g, g) = \kappa(g, P \cdot g) = 0 \), implying that \( i(P) \subseteq P \). Similarly, by the invariance of \( B \), we obtain

\[
B(P \cdot L, g \otimes A) = B(L, i(P) \cdot (g \otimes A)) = 0.
\]

(3.6)
Therefore, \( P \cdot L \subseteq (g \otimes A)^\perp \subseteq g \otimes A \) and \( P^2 \cdot L = 0 \). Since \( U(g) \) is Noetherian we can write

\[
P = U(g)p_1 + \cdots + U(g)p_m.
\]

(3.7)
The ideal \( P \) can be viewed as the kernel of \( U(g) \)-module homomorphism \( U(g) \rightarrow \text{End}_F(g) \) giving \( \dim(U(g)/P) = \ell < \infty \). Write \( U(g) = P + \text{span}_F\{u_1, \ldots, u_\ell\} \) as a vector space. Then for any \( u \in U(g) \) exists \( x_1, x_{ij} \in U(g) \) and \( \mu_1, \lambda_k \in F \) such that

\[
u = \sum_{i=1}^m x_i p_i + \sum_{k=1}^\ell \lambda_k u_k \quad \text{and} \quad x_i = \sum_{j=1}^\ell x_{ij} p_j + \sum_{t=1}^\ell \mu_t u_t.
\]
Thus, for all $d \in L$ we have

$$u \cdot d = \left( \sum_{i=1}^{m} x_{ip_{i}} + \sum_{k=1}^{t} \lambda_{k} u_{k} \right) \cdot d$$

$$= \sum_{i=1}^{m} \left( \sum_{j=1}^{m} x_{ij} p_{j} + \sum_{t=1}^{m} \mu_{t} u_{t} \right) p_{i} + \sum_{k=1}^{t} \lambda_{k} u_{k} \cdot d$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} (x_{ij} p_{j}) \cdot d + \sum_{i=1}^{m} \sum_{t=1}^{m} \mu_{t} (u_{t} p_{i}) \cdot d + \sum_{k=1}^{t} \lambda_{k} u_{k} \cdot d. \quad (3.8)$$

Taking other $u \in U(\mathfrak{g})$ amounts to changing constants $\mu_{i}, \lambda_{k} \in F$ in the very last line of Eq. (3.8).

For that reason we have the inclusion

$$U(\mathfrak{g}) \cdot d \subseteq \text{span}_{F} \{ u_{k} \cdot d, (u_{k} p_{i}) \cdot d \mid k \in \{1, \ldots, t\}, i \in \{1, \ldots, m\} \} \quad (3.9)$$

for any $d \in L$. In particular, $U(\mathfrak{g}) \cdot d$ is a finite-dimensional $U(\mathfrak{g})$-module and thus semi-simple. If we now consider $I = \{ d \in L \mid U(\mathfrak{g}) \cdot d \text{ is simple} \}$ we can write

$$L = \sum_{d \in I} U(\mathfrak{g}) \cdot d. \quad (3.10)$$

Since $P \cdot (U(\mathfrak{g}) \cdot d)$ is a $U(\mathfrak{g})$-submodule of $U(\mathfrak{g}) \cdot d$, it is either 0 or $U(\mathfrak{g}) \cdot d$ itself for all $d \in I$. The latter equality is impossible, because otherwise $0 = P^{2} \cdot (U(\mathfrak{g}) \cdot d) = U(\mathfrak{g}) \cdot d \neq 0$. Thus $P \cdot (U(\mathfrak{g}) \cdot d) = 0$ and Lemma 3.1 gives isomorphisms of $\mathfrak{g}$-modules $U(\mathfrak{g}) \cdot d \cong \mathfrak{g}$ for all $d \in I$. We call two indices $d, d' \in I$ equivalent $d \sim d'$ if $U(\mathfrak{g}) \cdot d = U(\mathfrak{g}) \cdot d'$. Removing duplicates from Eq. (3.10) we get

$$L = \bigoplus_{[d] \in I/\sim} U(\mathfrak{g}) \cdot d \cong \bigoplus_{[d] \in I/\sim} \mathfrak{g} \cong \mathfrak{g} \otimes \tilde{A}, \quad (3.11)$$

for some vector space $\tilde{A}$. Moreover, by choosing appropriate representatives in each equivalence class we can, without loss of generality, assume $\tilde{A} \subseteq \tilde{A}$.

The next step is to equip $\tilde{A}$ with a unital commutative associative algebra structure such that $A$ becomes a subalgebra of $\tilde{A}$. When $\mathfrak{g}$ is not of type $\mathfrak{sl}(n, F), n \geq 3$, we have $\text{Hom}_{\mathfrak{g}, \text{Mod}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) = \text{span}_{F} \{ [\cdot, \cdot], \cdot \}$. Repeating the proof of [12, Proposition 2.2] we obtain the desired algebra structure. Assume $\mathfrak{g} = \mathfrak{sl}(n, F), n \geq 3$. The space $\text{Hom}_{\mathfrak{g}, \text{Mod}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ is now generated by two $\mathfrak{g}$-module homomorphisms, namely the Lie bracket $[\cdot, \cdot]$ and the map

$$a \otimes b \mapsto a \circ b := ab + ba - \frac{\text{tr}(ab + ba)}{n} I_{n}.$$  

The Lie algebra structure on $L$ induces a unital algebra structure on $\tilde{A}$: the multiplication is now given by

$$[a \otimes f, b \otimes g] = [a, b] \otimes \frac{1}{2}(fg + gf) + (a \circ b) \otimes \frac{1}{2}(fg - gf). \quad (3.12)$$

Furthermore, the algebra $\tilde{A}$ is alternative for $n = 3$ and associative for $n > 3$; for details see e.g. [12]. The following requirement ensures only the alternativity property and hence works in both cases.

Let us denote the induced bilinear form on $\mathfrak{g} \otimes \tilde{A}$ with the same letter $B$. Define the linear functional $t: \tilde{A} \to F$ by

$$t(f) := B(e_{12} \otimes 1, e_{21} \otimes f),$$

where $e_{ij}$ is the $(n \times n)$-matrix having 1 in position $(i, j)$ as its only non-zero entry. Observe that $[e_{ij}, e_{jk}] = e_{ik} = e_{ij} \circ e_{jk}$ for all pairwise different $1 \leq i, j, k \leq n$. By the invariance and the symmetry of $B$ we obtain

$$t(fg) = B(e_{ij} \otimes f, e_{ji} \otimes g) = t(gf).$$
for all $1 \leq i \neq j \leq n$ and $f, g \in \tilde{A}$. We put
\begin{equation}
A^{1+} := \{ f \in \tilde{A} \mid t(Af) = 0 \}.
\end{equation}
(3.13)
Since $g \otimes A$ is a cos isotropic subalgebra of $L$ we have $A^{1+} \subseteq A$. Again, using the invariance and the symmetry of $B$ we get the following chain of identities for any $p, q \in A$ and $f \in \tilde{A}$
\begin{equation}
t(p(qf)) = t((pq)f) = t((qp)f) = t((fq)p) = t(pfq).
\end{equation}
(3.14)
In particular, this means $t(p[q, f]) = 0$ for all $p, q \in A$ and $f \in \tilde{A}$. Therefore, $[A, \tilde{A}] \subseteq A^{1+} \subseteq A$.

Now we proceed by showing that $A$ lies in the center of $\tilde{A}$. For that observe that for any two elements $q \in A$ and $f \in \tilde{A}$ we have
\begin{equation}
0 = [q, [q, f^2]]
= [q, qf^2 - f^2q + f(qf - f(qf))]
= [q, q]f + f[q, f]
= 2[q, f]^2.
\end{equation}
(3.15)
Here we implicitly used Artin’s theorem stating that any subalgebra of an alternative algebra generated by two elements is associative. By assumption $A$ has no non-trivial nilpotent elements implying that $[A, \tilde{A}] = 0$.

Finally, since $\text{char}(F) = 0$ the commutative center of $\tilde{A}$ lies in the associative center of $\tilde{A}$. Therefore, for all $p \in A$ and $f, g \in \tilde{A}$ we have $t(p[f, g]) = t([pf, g]) = 0$ showing that $[A, \tilde{A}] \subseteq A \subseteq Z(\tilde{A})$. Commutativity of $\tilde{A}$ now follows from the equality
\begin{equation}
[f, g]^2 = [[f, g], f, g] = [[f, gf], g] = 0.
\end{equation}
(3.16)
The proof is now complete because any unital commutative alternative $F$-algebra with $\text{char}(F) \neq 3$ is automatically associative.

To classify topological Lie bialgebras or doubles it is important to understand how isomorphisms of a Lie algebra of the form $g \otimes A$ may look like. The following theorem gives us control over such maps.

**Theorem 3.3.** Let $a$ be a finite-dimensional central simple (not necessarily associative or Lie) $F$-algebra and $A$ be a unital associative commutative $F$-algebra. Then any $F$-algebra automorphism $\varphi$ of $a \otimes A$ is a composition of an $F$-algebra automorphism of $A$ and an $A$-algebra automorphism of $a \otimes A$. More precisely, we have $\text{Aut}_{F,\text{-Alg}}(a \otimes A) = \text{Aut}_{F,\text{-Alg}}(A) \times \text{Aut}_{A,\text{-Alg}}(a \otimes A)$.

**Proof.** For any element $a \in A$ we define $\gamma_a \in \text{End}_{A,\text{-Mod}}(a \otimes A)$ by $\gamma_a(x \otimes b) = x \otimes ab$. The set of all such endomorphism $\tilde{A} = \{ \gamma_a \mid a \in A \}$ is an $F$-algebra isomorphic to $A$. We now show that for any $\varphi \in \text{Aut}_{F,\text{-Alg}}(a \otimes A)$ we have
\begin{equation}
\varphi^{-1}(\tilde{A}) = \tilde{A}.
\end{equation}
(3.17)
Let $n := \text{dim}_F(a)$, chose a basis of $a$ so that we can identify $\text{End}_{F,\text{-Mod}}(a)$ with the space of $(n \times n)$-matrices $M_n(F)$ and let $M_F(a)$ be the multiplication algebra of $a$, i.e. the subalgebra of $M_n(F)$ generated by left and right multiplications in $a$. Since $a$ is central, the centroid $\Gamma_F(a)$ of $a$ is precisely the field $F$ itself. Then by [13, Chapter X, Theorem 4] we have $M_F(a) = M_n(F)$ and, consequently, $M_A(a \otimes A) = M_n(A)$. Let $c_n(X_1, X_2, \ldots, X_n)$ be a multilinear central polynomial for $M_n(F)$. Then $c_n(M_n(F), M_n(F), \ldots, M_n(F)) = F \cdot I_{n \times n}$ and
\begin{equation}
c_n(M_A(a \otimes A), M_A(a \otimes A), \ldots, M_A(a \otimes A)) = c_n(M_n(A), M_n(A), \ldots, M_n(A))
= A \cdot I_{n \times n} = \tilde{A}.
\end{equation}
(3.18)
The left-hand side of Eq. (3.18) is stable under conjugation by $\varphi \in \text{Aut}_{F,\text{-Alg}}(a \otimes A)$ implying the identity Eq. (3.17).
Let us now fix some $\varphi \in \text{Aut}_{F,-\text{Alg}}(a \otimes A)$ and take $\psi \in \text{Aut}_{F,-\text{Alg}}(A)$ such that
\[ \varphi^{-1} \gamma_{\alpha} \varphi = \gamma_{\psi(a)} \]  
(3.19)
for all $a \in A$. Such an automorphism exists because of [Eq. (3.17)]. Define $f \in \text{Aut}_{F,-\text{Alg}}(a \otimes A)$ by $f(x \otimes a) = x \otimes \psi(a)$. Then $\varphi = (\varphi \circ f) \circ f^{-1}$, where $\varphi \circ f \in \text{Aut}_{A,-\text{Alg}}(a \otimes A)$, which completes the proof. ■

3.2. Trace extensions of $F[x]$ and topological Manin pairs. Henceforth, $F$ is an algebraically closed field of characteristic 0 and $g$ is a finite-dimensional simple Lie algebra over $F$. Consider the Lie algebra $g[x] := g \otimes F[x]$ with the bracket $[a \otimes f, b \otimes g] := [a, b] \otimes f g$. Endowing $g[x]$ with the $(x)$-adic topology or, equivalently, with the weak topology it becomes a topological Lie algebra.

Let $L$ be another Lie algebra over $F$. We call $(L, g[x])$ a topological Manin pair if

1. $L$ is a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form $B$;
2. $g[x] \subseteq L$ is a Lagrangian subalgebra with respect to $B$;
3. for any continuous functional $T: g[x] \to F$ exists an element $f \in L$ such that $T = B(f, -)$.

This notion is closely related to the notion of trace extensions of $F[x]$ introduced and classified in [13]. Let us recall its definition. Endow $F[x]$ with the $(x)$-adic topology. A unital commutative associative $F$-algebra extension $A \supseteq F[x]$, equipped with a continuous linear map $t: A \to F$, is called a trace extension of $F[x]$ if

1. the bilinear form $(f, g) \mapsto t(fg)$ is non-degenerate;
2. $F[x]_{x} := \{ f \in A \mid t(F[x]f) = 0 \} = F[x]$;
3. for any continuous linear functional $T: F[x] \to F$ there exists an $f \in A$ such that $T(p) = t(pf)$ for all $p \in F[x]$.

The above-mentioned relation is then presented in the following lemma.

**Lemma 3.4.** Let $(L, g[x])$ be a topological Manin pair and $B$ be the bilinear form of $L$. Then there exists a trace extension $(A, t)$ of $F[x]$ such that
\[ L \cong g \otimes A \quad \text{and} \quad B(a \otimes f, b \otimes g) = \kappa(a, b) t(fg), \]  
(3.20)
where $\kappa$ is the Killing form on $g$.

The proof is straightforward and obtained by combining [Theorem 3.2] with [10, Lemma 2.3].

**Example 3.5.** Let $n \geq 1$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ be an arbitrary sequence. Consider the algebra
\[ A(n, \alpha) := F((x)) \oplus F[x]/(x^n), \]
and the functional $t: A(n, \alpha) \to F$, given by
\[ t(x^{n-1}) := 1, \quad t(x^i) := \alpha_i, \quad \text{for } i \leq n - 2, \]
\[ t([x]^{n-1}) := -1, \quad t([x]^i) := -\alpha_i, \quad \text{for } 0 \leq i \leq n - 2 \text{ when } n \geq 2. \]

Then $(A(n, \alpha), t)$ is a trace extension of $F[x]$, where the latter is identified with the image of the inclusion $F[x] \to A(n, \alpha)$, $f \mapsto ([f, f])$. Here $[f]$ is the equivalence class of $f$ in $F[x]/(x^n) = F[x]/(x^n)$.

**Example 3.6.** Let $\alpha = (\alpha_i \mid -\infty < i \leq -2)$ be an arbitrary sequence in $F$. Then the algebra $F((x))$ with functional $t$, defined by $t(x^{-1}) := 1$ and $t(x^i) := \alpha_i$ for $i \leq -2$, is a trace extension of $F[x]$ denoted by $A(0, \alpha)$. Later we implicitly identify $F((x))$ with $F((x)) \oplus \{0\}$ in order to write $A(n, \alpha) = F((x)) \oplus F[x]/(x^n)$ for all $n \geq 0$. ■
Example 3.7. The algebra $A(\infty) := \sum_{i \geq 0} F a_i + F[x]$ with multiplication

$$a_i a_j := 0, \quad a_i x^j := a_{i-j} \quad \text{for} \quad i \geq j \quad \text{and} \quad a_i x^j := 0 \quad \text{otherwise},$$

and the functional $t: A \to F$, defined by $t(a_0) := 1, t(a_i) := 0, \ i \geq 1$ and $t(F[x]) := 0$, is a trace extension called the trivial extension of $F[x]$.

Two trace extensions $(A, t)$ and $(A', t')$ are called equivalent if there exists an algebra isomorphism $T: A \to A'$ identical on $F[x]$ and a non-zero scalar $\xi \in F$ such that $t'(T(a)) = \xi t(a)$ for any $a \in A$. In this case we write $(A, t) \sim (A', t')$.

Proposition 3.8 (Proposition 2.9, [19]). Let $(A, t)$ be a trace extension of $F[x]$. Then either $(A, t) \sim A(\infty)$ or $(A, t) \sim A(n, \alpha)$ for some $n \geq 0$ and some sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$.

Remark 3.9. The equivalence $T: A \to A'$ of two trace extensions $(A, t)$ and $(A', t')$ does not extend in general to an isomorphism of Lie algebras $A \otimes A$ and $A' \otimes A'$ with forms $B(a \otimes f, b \otimes g) := \kappa(a, b) t(fg)$ and $B'(a \otimes f', b \otimes g') := \kappa(a, b) t'(fg')$ respectively. Indeed, by definition of equivalence for trace extensions we have

$$B'(a \otimes T(f), b \otimes T(g)) = \xi B(a \otimes f, v \otimes g).$$

In other words, $T$ does not intertwine the corresponding bilinear forms, but it gives an isomorphism between $(g \otimes A, \xi B)$ and $(g \otimes A', B')$.

Corollary 3.10. Let $(L, g[x])$ be a topological Manin pair with the bilinear form $B$. Then there exists a non-zero $\xi \in F$ such that either $(L, \xi B) \cong g \otimes A(\infty)$ or $(L, \xi B) \cong g \otimes A(n, \alpha)$ for some $n \geq 0$ and some sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$.

Let $(A, t)$ be a trace extension of $F[x]$. One can produce new trace extensions from the given one in the following way: let $\varphi$ be an algebra automorphism of $A$ such that $\varphi(F[x]) = F[x]$, then $(A, t)_{(\varphi)} := (A, t \circ \varphi)$ is another trace extension of $F[x]$.

We call automorphisms of $F[x]$ given by $x \mapsto a_1 x + a_2 x^2 + \ldots, a_i \in F, a_1 \neq 0$, coordinate transformations. Note that coordinate transformations $x \mapsto x + a_2 x^2 + \ldots$ form a group under substitution which we denote by $Aut_0 F[x]$.

Lemma 3.11. A coordinate transformation $\varphi \in Aut_0 F[x]$ induces an automorphism of

$$A(n, \alpha) = F((x)) \otimes F[x]/(x^n)$$

by $f/g \mapsto \varphi(f)/\varphi(g)$ and $[x] \mapsto [\varphi(x)]$.

Moreover, there exists another sequence $\beta := (\beta_i \in F \mid -\infty < i \leq n - 2)$ such that $A(n, \alpha)^{(\varphi)} = A(n, \beta)$.

Proof. Let $\varphi = x + a_2 x^2 + a_3 x^3 + \cdots \in Aut_0 F[x]$. The first part of the statement is clear. Denote the induced automorphism with the same letter $\varphi$. Consider the trace extension $t := t_0 \circ \varphi$, where $t_0$ is the linear functional given by the sequence $\alpha$. Since $t_0(x^k) = 0$ for $k \geq n$ we have $t(x^{n-1}) = 1$ and a well defined sequence $\beta$ given by

$$\beta_i := t(x^i), \ i \leq n - 2.$$

Moreover, when $n \geq 1$ we see that $t([x]^{n-1}) = t_0([\varphi(x)]^{n-1}) = t_0([x]^{n-1}) = -1$ and for $0 \leq i \leq n - 2$ we compute $t([x]^i) = t_0([\varphi(x)]^i) = -t_0(\varphi(x)^i) = -\beta_i$. Therefore $t$ is given by the sequence $\beta$ and $A(n, \alpha)^{(\varphi)} = A(n, \beta)$ as we wanted. \hfill \blacksquare

It was mentioned in [19, Section 2] that by applying $\varphi \in Aut_0 F[x]$ to a trace extension $A(n, \alpha), 0 \leq n \leq 2$, we get a full control over sequence $\alpha$. Namely, we can make it into a zero sequence with $\alpha_0 \neq 0$. The following proposition extends these facts to $n \geq 3$. 

Proposition 3.12. Let $n \geq 0$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)$ be a sequence. There exists a $\varphi \in \text{Aut}_0 F[x]$ such that $A(n, \alpha) \cong A(n, \beta)^{(\varphi)}$, where $\beta$ is the sequence satisfying $\beta_i = 0$ for all $i \neq 0$ and $\beta_0 = \alpha_0$.

Proof. The cases $n \in \{0, 1, 2\}$ are considered in [19]. Assume $n > 2$. Denote by $t$ the trace form corresponding to $A(n, \alpha)$. We want to find a $u = x(1 + u_1 x + \ldots) \in F[x]$ such that $t(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0, n - 1\}$. We do that by defining the coefficients $u_1, u_2, \ldots$ inductively.

We start by considering the positive powers of $u$. Since $t(x^k) = 0$ for $k \geq n$, the equality $t(u^k) = 0$ is automatic for $k \geq n$. Next, we observe that

\[
t(u^{n-2}) = t(x^{n-2} + (n-2)u_1 x^{n-1}) = \alpha_2 + (n-2)u_1
\]  

(3.21)

and defining $u_1 := -\alpha_{n-2}/(n-2)$ we get the desired $t(u^{n-2}) = 0$. For $n = 3$ this concludes the positive powers of $u$. Assume now that for $n > 3$ we have fixed $u_1, \ldots, u_{k-1}$, $2 \leq k \leq n - 2$, such that $t(u^{n-2}) = \ldots = t(u^{n-k}) = 0$. Using the convention $u_0 := 1$ we can write

\[
\sum_{j=0}^\infty u_1 \cdots u_{n-k-1} x^j.
\]  

(3.22)

Therefore, letting $\alpha_{n-1} := 1$ we get

\[
t(u^{n-k-1}) = \sum_{j=0}^k \sum_{\ell_1, \ldots, \ell_{n-k-1} \geq 0} u_{\ell_1} \cdots u_{\ell_{n-k-1}} \alpha_{j+n-k-1}.
\]  

(3.23)

Defining

\[
u_k := -\frac{1}{n-k-1} \sum_{j=0}^k \sum_{\ell_1, \ldots, \ell_{n-k-1} \geq 0} u_{\ell_1} \cdots u_{\ell_{n-k-1}} \alpha_{j+n-k-1}.
\]  

(3.24)

we obtain the identity $t(u^{n-k-1}) = 0$ completing the induction step for positive powers of $u$.

Now we proceed with negative powers of $u$. For the base case, we notice that

\[
t(u^{-1}) = t(x^{-1} - u_1 - (u_2 - u_1^2)x - \cdots - (u_n - p(u_1, \ldots, u_{n-1}))x^{-1})
\]  

\[
= \alpha_1 - u_1 \alpha_0 - (u_2 - u_1^2) \alpha_1 - \cdots - (u_n - p(u_1, \ldots, u_{n-1})),
\]  

(3.25)

where $p$ is a polynomial depending only on $u_1, \ldots, u_{n-1}$. In other words, if we fix $u_1, \ldots, u_{n-2}$ as above and choose an arbitrary $u_{n-1} \in F$, then we can put

\[
u_n := \alpha_1 - u_1 \alpha_0 - (u_2 - u_1^2) \alpha_1 - \cdots - p(u_1, \ldots, u_{n-1})
\]

and obtain $t(u^k) = 0$ for $-1 \leq k \leq n - 2$ and $k \neq 0$. For the inductive step, write $v := u^{-1} := x^{-1}(1 + v_1 x + \ldots) \in x^{-1} F[x]$. The identity $1 = uv$ implies $\sum_{j=0}^k v_j v_{j-k} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Therefore, $u_1, \ldots, u_k$ uniquely determine $v_1, \ldots, v_k$ and vice versa. We have

\[
v^k = x^{-k} \sum_{j=0}^\infty \sum_{\ell_1, \ldots, \ell_k = j} v_{\ell_1} \cdots v_{\ell_k} x^j
\]  

(3.26)

and, as a consequence,

\[
t(u^{-k}) = \sum_{j=0}^{n-1+k} \sum_{\ell_1, \ldots, \ell_k = j} v_{\ell_1} \cdots v_{\ell_k} \alpha_{j-k}.
\]  

(3.27)
In particular, if parameters \(u_1, \ldots, u_{n-2}, \ldots, u_{n+k}, k \geq 0\) are fixed in such a way, that \(t(u^m) = 0\) for \(-k - 1 \leq m \leq n - 2, m \neq 0\) then by defining
\[
v_{n+k+1} := -\frac{1}{k+2} \sum_{j=0}^{n+k+1} \sum_{0 \leq \ell_1 + \cdots + \ell_{k+2} = j} v_{\ell_1} \cdots v_{\ell_{k+2}} \alpha_{j-k-2}, \tag{3.28}
\]
we get the identities \(t(u^m) = 0\) for all \(-k - 2 \leq m \leq n - 2, m \neq 0\). This concludes the proof. \(\blacksquare\)

Combining Corollary 3.10 and Proposition 3.8 results in the following statement.

**Corollary 3.13.** Let \((L, g[x])\) be a topological Manin pair and \(B\) be the bilinear form on \(L\). Then there exists a \(\varphi \in \text{Aut}_\mathbb{C}F[x]\) such that either \((L, B) \cong g \otimes A(\infty)\varphi)\) or \((L, B) \cong g \otimes A(n, \alpha)\varphi)\) for some \(n \geq 0\) and some sequence \(\alpha = (\alpha_i \in F \mid -\infty < i \leq n - 2)\) satisfying \(\alpha_i = 0\) for all \(i \neq 0\).

**Remark 3.14.** Let us explicitly describe the bilinear forms on
\[
g \otimes A(i - 1, 0) = g((x)) \times g((x)) / x^{i-1} g[x], \tag{3.29}
\]
where \(i \in \mathbb{Z}_{>0}\). We start with the bilinear form \(\mathcal{K}_i : g((x)) \times g((x)) \to F\) defined by
\[
\mathcal{K}_i(f, g) := \text{res}_{x=0} \{x^{i-1} \kappa(f(x), g(x))\} = \text{coeff}_{x^{-2}} \{\kappa(f(x), g(x))\}, \tag{3.30}
\]
where \(\kappa\) stands for the Killing form of \(g((x))\) over \(F((x))\). Observe that \(\mathcal{K}_i(x^{i-1}g[x], x^{i-1}g[x]) = 0\). Therefore,
\[
\mathcal{K}_i([f], [g]) := \mathcal{K}_i(f, g) \tag{3.31}
\]
gives a well-defined bilinear form on \(g[x] / x^{i-1} g[x] = g[x] / x^{i-1} g[x]\). The bilinear form \(B_i\) on \(g \otimes A(i - 1, 0)\) is now defined by
\[
B_i((f_1, [f_2]), (g_1, [g_2])) := \mathcal{K}_i(f_1, g_1) - \mathcal{K}_i(f_2, g_2) \tag{3.32}
\]
for all \(f_1, g_1 \in g((x))\) and \(f_2, g_2 \in g[x]\]. \(\diamondsuit\)

**Remark 3.15.** Manin pairs are related to so-called quasi-Poisson structures in a similar fashion as Manin triples are related to Poisson-Lie groups via their connection to Lie bialgebra structures. These quasi-Poisson structures are of independent interest, as they capture interesting phenomena which go beyond the realm of usual Poisson structures; see e.g. [13]. Interestingly, contrary to Poisson case, it is open if the quasi-Poisson structures associated to Manin pairs can be appropriately quantized; see e.g. [13, Remark in Section 16.2.2]. \(\diamondsuit\)

### 3.3. Topological Lie bialgebra structures on \(g[x]\)

As before, we equip the Lie algebra \(g[x]\) with the \((\cdot)\)-adic topology. The continuous dual
\[
g[x]' = \{\text{linear maps } f : g[x] \to F \mid f(x^n g[x]) = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}
\]
of \(g[x]\) is isomorphic as a vector space to the space of polynomials with coefficients in \(g\). We endow it with the discrete topology and, to avoid any confusion, denote it by \(g[x]\). According to our definition in Section 2 a topological Lie bialgebra structure on \(g[x]\) is a continuous linear map
\[
\delta : g[x] \to g[x] \otimes g[y] \cong (g \otimes g)[x, y]\]
such that
1. the dual map \(\delta' : g[x] \otimes g[y] = ((g \otimes g)[x, y])' \to g[x]\) is a Lie bracket on \(g[x]\) and
2. the compatibility condition \(\delta([f, g]) = [f \otimes 1 + 1 \otimes f, \delta(g)] - [g \otimes 1 + 1 \otimes g, \delta(f)]\) holds for all \(f, g \in g[x]\).
Remark 3.16. We omitted the continuity condition on $\delta^i$ because $g \{x \}$ and hence $g \{x \} \otimes g \{y \}$ are discrete. Moreover, repeating the proof of [19, Lemma 4.1] we can easily show that the compatibility condition guarantees the inclusion

$$\delta(x^n g[x]) \subseteq (x, y)^{n-1}(g \otimes g)[x,y]$$

for any positive integer $n$ and thus automatically implies the continuity of $\delta$. Therefore, our definition of a topological Lie bialgebra structure on $g[x]$ coincides with the one given in [19].

The following result shows that, as in the classical case, topological Lie bialgebra structures on $g[x]$ produce topological Manin triples.

Lemma 3.17. Let $(g[x], \delta)$ be a topological Lie bialgebra. The corresponding triple

$$(g[x] + g[x], g[x], g[x])$$

described in Section 2.4 is a topological Manin triple, i.e. the Lie bracket on $g[x] + g[x]$ and the form $B$, defined in Eqs. (2.10) and (2.11), are continuous and separately continuously respectively.

Proof. Fix some arbitrary $X,Y \in g[x]$ and $f,g \in g \{x \}$. The separate continuity follows from the identity

$$B(X + f, (Y + x^{\deg(f)+1} g[x]) + g) = B(X + f, Y + g).$$

To prove the continuity of the Lie bracket it is enough to show that for 0-neighbourhood $x^n g[x] \subseteq g[x] + g \{x \}$ there is a 0-neighbourhood $x^n g[x] \subseteq g[x]$ such that

$$[X + x^n g[x], g] \subseteq [X, g] + x^n g[x].$$

Let $m := \max\{2 \deg(g) + 2, 2n\}$, then

$$g([x^m g[x], \cdot]) + (g \circ 1)\delta(x^m g[x]) \subseteq x^n g[x],$$

providing the continuity of the bracket. ■

Remark 3.18. Using the homeomorphisms Eq. (2.4) it is not hard to show that we have the following converse to Lemma 3.17. Let $M = (L, L^+, L^-)$ be a topological Manin triple with form $B$ such that

(1) $L_-$ is equipped with the discrete topology,

(2) As a topological Lie algebra $L_+ \cong g[x]$ and finally

(3) $B$ identifies $L^+_\alpha$ with $L_-\alpha$.

Then $M$ defines a unique topological Lie bialgebra structure $\delta$ on $g[x]$ and $M$ is isomorphic to $(\Lambda(g[x], \delta), g[x], g\{x\})$ as a topological Manin triple. Consequently, the classification of topological Lie bialgebras on $g[x]$ coincides with the classification of topological Manin triples with the above-mentioned properties. Note that each such Manin triple also gives rise to a topological Manin pair $(L, L_\wedge)$ in the sense of Section 3.2. ■

3.4. Classification of topological doubles. In this section we recall the classification of topological doubles of $g[x]$ from [19], which can be understood as a refinement of Corollary 3.13. Indeed, a topological Manin pair $(L, g[x])$ is a topological double of $g[x]$ if and only if there exists a Lagrangian subalgebra $W \subseteq L$ complementary to $g[x]$. If we choose a representation $L \cong g \otimes A(n, \alpha)$ according to Corollary 3.13, the existence of such a $W$ implies that $0 \leq n \leq 2$ and $\alpha_0 = 0$. Thus, Corollary 3.13 takes the following form in this setting.

Theorem 3.19 (Proposition 2.10 [19]). Let $D = g \otimes A(n, \alpha)$ for some $n \geq 0$ and a sequence $\alpha = (\alpha_i, \alpha_0)$. Then
(1) $D$ contains a Lagrangian Lie subalgebra $W$ such that $D = g \otimes F[x] + W$ if and only if $0 \leq n \leq 2$
and
(2) in this case there is a unique $\varphi \in \text{Aut}_0 F[x]$ such that $A(n, \alpha) \sim A(n, 0)^{(c)}$.

Let us note that there is no ambiguity in the topologies of the respective objects according to the following result.

**Lemma 3.20.** Let $\delta_1$ and $\delta_2$ be two topological Lie bialgebra structures on $g[x]$. If there is a Lie algebra isomorphism $\varphi$ between the corresponding topological doubles $g[x] + g\{x\}_1$ and $g[x] + g\{x\}_2$ that is identical on $g[x]$ and intertwines the forms, then $\varphi$ is a homeomorphism.

**Proof.** Let $\mathcal{D}$ be the topological double of $\delta_1$. As a Lie algebra it has two decompositions into Lagrangian Lie subalgebras coming from $\delta_1$ and $\delta_2$
\[ \mathcal{D} = g[x] + g\{x\}_1 = g[x] + \varphi(g\{x\}_2). \]

These decompositions give rise to two different product topologies generated by
\[ \mathcal{T}_1 := \{ U \times \{ f \} \mid U \text{ open in } g[x] \text{ and } f \in g\{x\}_1 \}, \]
\[ \mathcal{T}_2 := \{ U \times \{ g \} \mid U \text{ open in } g[x] \text{ and } g \in \varphi(g\{x\}_2) \}. \]

For any $U \times \{ f \} \in \mathcal{T}_1$ we can find $n \in \mathbb{N}^+$, $u, u' \in g[x]$ and $g' \in \varphi(g\{x\}_2)$ such that $u + x^n g(x) \subseteq U$ and $f = u' + g'$. Then $(u + u' + x^n g(x)) \times \{ g' \}$ is an element in $\mathcal{T}_2$ that is contained in $U \times \{ f \}$. Similarly, for any element $V$ in $\mathcal{T}_2$ there is an element in $\mathcal{T}_1$ contained in $V$. Therefore, the topologies generated by $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal. $\blacksquare$

**Remark 3.21.** Combining **Lemma 3.20** and **Theorem 3.19** we see that for any topological Lie bialgebra $(g[x], \delta)$ with the topological double $\mathcal{D}(g[x], \delta)$ there exist a scalar $\xi \in F^\times$ and either
- An $F[x]$-linear isomorphism of Lie algebras $\mathcal{D}(g[x], \xi \delta) \cong g \otimes A(\infty)$ identical on $g[x]$ and intertwining the corresponding forms or
- An integer $0 \leq n \leq 2$, a series (change of variable) $\varphi \in \text{Aut}_0 F[x]$ and an $F[x]$-linear isomorphism of Lie algebras $\mathcal{D}(g[x], \xi \delta) \cong g \otimes A(n, 0)^{(c)}$ identical on $g[x]$ and intertwining the corresponding forms.

Moreover, there are unique (independent of $\delta$ and $\xi$) topologies on $g \otimes A(\infty)$ and $g \otimes A(n, 0)^{(c)}$ making the above-mentioned isomorphisms into homeomorphisms. We will see in **Section 5.3** that all these Lie algebras with forms can indeed be realized as topological doubles of some topological Lie bialgebra structures on $g[x]$. $\diamond$

### 4. Topological Twists of $(g[x], \delta)$

Given a classical Lie bialgebra structure $\delta : L \to L \otimes L$ on a Lie algebra $L$ we can obtain new Lie bialgebra structures by means of so called twisting. More precisely, any skew-symmetric tensor $s \in L \otimes L$, satisfying
\[ \text{CYB}(s) = \text{Alt}(\delta \otimes 1)s, \]
gives rise to a twisted Lie bialgebra structure $\delta_s := \delta + ds$ on $L$. Here $ds(a) := [a \otimes 1 + 1 \otimes a, s]$, $\text{Alt}$ was defined in **Eq. (2.7)** and
\[ \text{CYB}(s) := [s^{12}, s^{13}] + [s^{12}, s^{23}] + [s^{13}, s^{23}], \]
is the classical Yang-Baxter equation, where $e.g. \ [(a \otimes b)^{13}, (c \otimes d)^{23}] = a \otimes c \otimes [b, d]$. Since the classical doubles for $\delta_s$ and $\delta$ are equal this procedure can be used for classification of Lie bialgebra structures within a fixed double $\mathcal{D}$.

We now adopt this procedure to our topological setting by introducing topological twists for a topological Lie bialgebra structure on $g[x]$. These are just certain elements in $(g \otimes g)[x, y]$ satisfying a condition similar to **Eq. (4.1)**. In subsequent sections, by classifying topological twists,
we obtain a full description of all topological Lie bialgebra structures $\delta$ on $g[x]$ with doubles $D(g[x], \delta) \cong g \otimes A(i - 1, 0)$, where $i \in \{1, 2, 3\}$.

4.1. Topological twists. We say that an element $s \in g[x] \otimes g[y] = g[x] \otimes g[y] = (g \otimes g)[x, y]$ is a topological twist of a topological Lie bialgebra $(g[x], \delta)$ if

1. $s + \pi(s) = 0$;
2. CYB$(s) = \text{Alt}(\delta \otimes 1)s$.

Remark 4.1. For a topological Lie algebra $L$ the function CYB is the unique continuous extension of the continuous composition

$$L \otimes L \xrightarrow{\text{CYB}} L \otimes L \otimes L \leftarrow \rightarrow L \otimes L \otimes L.$$ 

To see that CYB is continuous, note that e.g. $s \mapsto [s^{12}, s^{13}]$ is the composition of the diagonal map $L \otimes L \to (L \otimes L) \times (L \otimes L)$, the canonical map $(L \otimes L) \times (L \otimes L) \to L^{\otimes 4}$, the endomorphism of $L^{\otimes 4}$, which switches the second and third factors, and $[\cdot, \cdot] \otimes 1 \otimes 1 : L^{\otimes 4} \to L^{\otimes 3}$. All of these maps are continuous, so $s \mapsto [s^{12}, s^{13}]$ is too. Similar arguments apply to $s \mapsto [s^{12}, s^{23}]$ and $s \mapsto [s^{13}, s^{23}]$.

Let $s \in (g \otimes g)[x, y]$ be a topological twist of $\delta$. Define the linear map $ds : g[x] \to (g \otimes g)[x, y]$ by $f \mapsto [f \otimes 1 + 1 \otimes f, s]$. As in the classical case the linear functional $\delta_s := \delta + ds$ is again a topological Lie bialgebra structure on $g[x]$.

We say that two topological twists $s_1$ and $s_2$ of $\delta$ are formally isomorphic, if there is an $F[x]$-linear isomorphism $\phi$ of topological Lie bialgebras $(g[x], \delta + ds_1)$ and $(g[x], \delta + ds_2)$.

Remark 4.2. Observe, that by Theorem 3.3 any $F$-linear automorphism of $g[x]$ is a composition of an $F[x]$-linear automorphism of $g$ and an $F$-linear automorphism of $F[x]$. It is not hard to prove that any $F$-linear automorphism of $F[x]$ is given by

$$x \mapsto a_1 x + a_2 x^2 + \ldots$$ (4.3)

for some non-zero $a \in F$. These observations immediately imply that

1. Any automorphism of $g[x]$ is automatically continuous with respect to the $(x)$-adic topology as a composition of continuous automorphisms and
2. Applying coordinate transformations [Eq. (4.3)] and scaling to the isomorphism classes of topological twists, we obtain all possible Lie bialgebra structures on $g[x]$.

Therefore, we have reduced the classification of topological Lie bialgebras to the classification of topological twists.

4.2. Lagrangian subalgebras. Topological twists of $\delta$ are closely related to Lagrangian Lie subalgebras of $D(g[x], \delta)$. The following theorem describes this relation.

Theorem 4.3. Let $(g[x], \delta)$ be a topological Lie bialgebra with the classical double $D$. Then there are the following one-to-one correspondences:

...
Topological twists of $\delta$, i.e. skew-symmetric tensors $s \in \mathfrak{g}[x] \otimes \mathfrak{g}[y]$ satisfying $\text{CYB}(s) = \text{Alt}((\delta \otimes 1)s)$

Lagrangian
Lie subalgebras $W \subseteq \mathfrak{D}$ complementary to $\mathfrak{g}[x]$

Linear maps $T: \mathfrak{g}[x] \to \mathfrak{g}[x]$ such that for all $p_1, p_2, p_3 \in \mathfrak{g}[x]$
$B(Tp_1, p_2) + B(p_1, Tp_2) = 0$ and $B([Tp_1 - Tp_2 - p_2], Tp_3 - p_3) = 0$

Proof. The proof is done by repeating the arguments in [2, Theorem 2.4] and [16, Theorem 7] within our topological setting. The only step that deserves a clarification is the construction of a basis for $\text{im}(T)$ from a linear map $T: \mathfrak{g}[x] \to \mathfrak{g}[x]$.

Let $\{Tp_i\}_{i \geq 1}$ be a basis for $\text{im}(T)$. We write $\text{mindeg}(Tp_i)$ for the minimal exponent of $x$ contained in $Tp_i$ and $(Tp_i)_m$ for the $m$-th degree part of $Tp_i$, i.e. $(\sum_{i \geq 0} a_i \otimes x^i)_m = a_m \otimes x^m$. Since a basis of $\text{im}(T)$ is at most countable, we can without loss of generality assume that for any non-negative integer $m$ there is a non-negative integer $N_m$ such that
$$
\begin{cases}
\text{mindeg}(Tp_i) \leq m & i \leq N_m; \\
\text{mindeg}(Tp_i) > m & i > N_m; \\
\{ (Tp_i)_m \mid N_{m-1} < i \leq N_m \} & \text{is a linearly independent set.}
\end{cases}
$$

Now let us construct a dual set $\{v_i\} \subseteq \mathfrak{g}[x]$. For each $m \geq 0$ and $N_{m-1} < i \leq N_m$ define $v'_i := a_i x^m$, where $a_i \in \mathfrak{g}$ are chosen in such a way that
$$
B(Tp_j, v'_i) = \begin{cases} 
1 & N_{m-1} < i = j \leq N_m; \\
0 & N_{m-1} < i \neq j \leq N_m.
\end{cases}
$$
For each $m \geq 0$ and $N_{m-1} < i \leq N_m$ we recursively define
$$
v_i := v'_i - \sum_{k=1}^{N_{m-1}} B(Tp_k, v'_i)v_k.
$$
The corresponding twist is then defined as $s := -\sum Tp_i \otimes Tv_i$. Since for any $m \in \mathbb{Z}_{\geq 0}$ there is at most $N_m$ elements $Tp_i$ containing $x^m$ this is a well-defined element of $(\mathfrak{g} \otimes \mathfrak{g})[x,y]$. Moreover, we have
$$
Tp_k = \sum_i B(v_i, Tp_k)Tp_i = -\sum_i B(Tv_i, p_k)Tp_i
$$
for all $k$. Because $T$ is completely defined by its action on $\{p_i\}$ we have the equality
$$
T = -\sum_i B(Tv_i, \cdot)Tp_i.
\boxdot$

Remark 4.4. Let $s = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} s_{k,\alpha} \otimes I_{\alpha} y^{k} \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]$ be a classical twist of $\delta$, for some basis $\{I_{\alpha}\}_i^m$ of $\mathfrak{g}$. The corresponding $T: \mathfrak{g}[x] \to \mathfrak{g}[x]$ is defined by
$$
Tw := \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} B(w, I_{\alpha} x^{k})s_{k,\alpha},
$$
(4.4)
where $B$ is the bilinear form of $\mathcal{D}(\mathfrak{g}[x], \delta)$. The Lagrangian subalgebra associated to $s$ is given by

$$W := \{ Tw - w \mid w \in \mathfrak{g}[x] \}. \quad (4.5)$$

We say that two topological Lie bialgebras $(\mathfrak{g}[x], \delta)$ and $(\mathfrak{g}[x], \tilde{\delta})$ are in the same twisting class if there is an $F[x]$-linear isomorphism of topological Lie algebras $\varphi: \mathcal{D}(\mathfrak{g}[x], \delta) \to \mathcal{D}(\mathfrak{g}[x], \tilde{\delta})$ intertwining the corresponding forms and such that the diagram

$$\begin{array}{ccc}
\mathfrak{g}[x] & \xrightarrow{\varphi} & \mathfrak{g}[x], \tilde{\delta} \\
\mathcal{D}(\mathfrak{g}[x], \delta) \downarrow & & \downarrow \mathcal{D}(\mathfrak{g}[x], \tilde{\delta}) \\
\end{array}$$

commutes. It is clear that if $(\mathfrak{g}[x], \delta)$ and $(\mathfrak{g}[x], \tilde{\delta})$ are in the same twisting class, then $\delta$ is completely determined by a Lagrangian topological Lie subalgebra $W := \varphi(\mathfrak{g}[x]) \subseteq \mathcal{D}(\mathfrak{g}[x], \tilde{\delta})$. Combining Theorem 4.3 and Remark 2.7 we get the following statement.

**Lemma 4.5.** If $(\mathfrak{g}[x], \delta)$ and $(\mathfrak{g}[x], \tilde{\delta})$ are in the same twisting class, then there exists a topological twist $s \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ of $\delta$ such that $\tilde{\delta} = \delta + ds$. Conversely, if $\tilde{\delta} = \delta + ds$, for a topological twist $s$, then $\mathcal{D}(\mathfrak{g}[x], \delta) = \mathcal{D}(\mathfrak{g}[x], \tilde{\delta})$.

Let $\phi$ be an $F[x]$-linear automorphism of $\mathfrak{g}[x]$, then $(\phi \times [\phi])$ is a Lie algebra automorphism of $\mathfrak{g} \otimes A(n, 0) = \mathfrak{g}(x) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x], n \geq 0$. Here we use the same notation $\phi$ for the unique extension of $\phi$ to an automorphism of $\mathfrak{g}(x)$ and $[\phi]$ for the induced map on $\mathfrak{g}[x]/x^n \mathfrak{g}[x]$ given by $[a \otimes x^k] \mapsto [\phi(a) \otimes x^k]$. We call two Lagrangian Lie subalgebras $W$ and $\tilde{W} \subset \mathcal{D}(\mathfrak{g}[x], \delta)$, $i \in \{1, 2, 3\}$, formally isomorphic if there is an $F[x]$-linear automorphism $\phi$ of $\mathfrak{g}[x]$ such that

$$\tilde{W} = (\phi \times [\phi])(W). \quad (4.6)$$

**5. Formal $r$-matrices**

In this section we show that there is a deep connection between topological Lie bialgebra structures on $\mathfrak{g}[x]$ and formal $r$-matrices leading to important observations for both structures. For instance, we use formal $r$-matrices to show that all topological Lie algebras with forms described in Remark 3.21 are obtainable as topological doubles of certain topological Lie bialgebra structures on $\mathfrak{g}[x]$. Furthermore, Lagrangian Lie subalgebras determined by topological twists have a very useful description in terms of formal $r$-matrices, which is used in the study of equivalences. On the other hand, the classification of topological doubles, results in unexpected restrictions on the form of formal $r$-matrices.

**5.1. Formal $r$-matrices and topological Lie bialgebra structures on $\mathfrak{g}[x]$**. Let $\{I_n\}_n$ be an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form $\kappa$ on it. Recall that the quadratic Casimir element is defined as $\Omega := \sum_{\alpha=1}^n I_\alpha \otimes I_\alpha \in \mathfrak{g} \otimes \mathfrak{g}$. Consider the formal Yang’s $r$-matrix

$$r_{\text{Yang}}(x, y) := \frac{\Omega}{x - y} = \sum_{k, l=0}^{\infty} x^{-k-1} I_\alpha \otimes y^k I_\alpha \in (\mathfrak{g} \otimes \mathfrak{g})(x)[y]. \quad (5.1)$$

A series $r \in (\mathfrak{g} \otimes \mathfrak{g})(x)[y]$ of the form

$$r(x, y) = s(y)r_{\text{Yang}}(x, y) + g(x, y), \quad (5.2)$$

where $s \in F[y]$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ is called a formal $r$-matrix if it solves the formal classical Yang-Baxter equation (CYBE)

$$\text{CYB}(r) := \{r^{12}(x_1, x_2), r^{13}(x_1, x_3)\} + \{r^{12}(x_1, x_2), r^{23}(x_2, x_3)\} + \{r^{13}(x_1, x_3), r^{23}(x_2, x_3)\} = 0.$$
Here for instance
\[
[a^{12}(x_1, x_3), b^{23}(x_2, x_3)] := \sum_{j,k=0}^{n} a_{j,\alpha}(x_1) \otimes [x_{2j}^j I_\alpha, b_{k,\beta}(x_2)] \otimes I_\beta x_3^k \tag{5.3}
\]
for all \(a = \sum_{j=0}^{\infty} \sum_{\alpha=1}^{n} a_{j,\alpha}(x) \otimes y^j I_\alpha\) and \(b = \sum_{j=0}^{\infty} \sum_{\alpha=1}^{n} b_{j,\alpha}(x) \otimes y^j I_\alpha\) in \((\mathfrak{g} \otimes \mathfrak{g})(x)[y]\). The other two commutators are defined in a similar way. As the notation suggests, CYB coincides with the map defined in Section 4 for \(r \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]\).

**Remark 5.1.** We can equivalently define a formal r-matrix as a series
\[
r(x, y) = s(x, y) r_{\text{Yang}}(x, y) + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})(x)[y],
\]
where \(s \in F[x, y]\) and \(g \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]\). The equivalence of two definitions follows from the equality
\[
r(x, y) = s(y, y) r_{\text{Yang}}(x, y) + \sum_{x \neq y} \frac{s(x, y) - s(y, y)}{x - y} \Omega + g(x, y). \tag{5.4}
\]

The connection between formal r-matrices and topological Lie bialgebra structures is based on the following observation.

**Lemma 5.2.** For any formal r-matrix \(r \in (\mathfrak{g} \otimes \mathfrak{g})(x)[y]\) the formula
\[
dr(f) := [f \otimes 1 + 1 \otimes f, r] \tag{5.5}
\]
defines a topological Lie bialgebra structure \(dr\) on \(\mathfrak{g}[x]\).

**Proof.** Consider a formal r-matrix \(r(x, y) = s(y, y) r_{\text{Yang}}(x, y) + g(x, y)\), for some \(s \in F[y]\) and \(g \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]\). First of all, observe that the invariance of \(\Omega\) implies that for all \(f = \sum_{j=0}^{\infty} f_j x^j \in \mathfrak{g}[x]\)
\[
[f \otimes 1 + 1 \otimes f, r_{\text{Yang}}] = \sum_{j=0}^{\infty} \left( x^j \frac{y^j}{x - y} [f_j \otimes 1, \Omega] + \frac{y^j}{x - y} [f_j \otimes 1 + 1 \otimes f_j, \Omega] \right) \tag{5.5}
\]
is an element of \((\mathfrak{g} \otimes \mathfrak{g})[x, y]\). Therefore, \(dr : \mathfrak{g}[x] \to (\mathfrak{g} \otimes \mathfrak{g})[x, y]\) is well-defined.

Let us now prove that \(dr\) is skew-symmetric. For that it is enough to prove the skew-symmetry of \(r\). The latter result is obtained by repeating the arguments from the proof of [5, Proposition 4.1]. Namely, by definition \(\text{CYB}(r) \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})(x_1)(x_2)[x_3]\). Swapping variables \(x_1\) and \(x_2\) and applying the \(F(x_1)(x_2)[x_3]\)-linear extension of \(\tau \otimes 1\) to \(\text{CYB}(r)\) results in
\[
- [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] - [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{23}(x_2, x_3), r^{13}(x_1, x_3)] = 0, \tag{5.6}
\]
where \(\tau(x, y) := s(x) r_{\text{Yang}}(x, y) - \tau g(y, x)\). Adding this to \(\text{CYB}(r)\) yields
\[
[r^{12}(x_1, x_2) - \tau^{12}(x_1, x_2), r^{13}(x_1, x_3) + r^{23}(x_2, x_3)] = 0. \tag{5.7}
\]
Multiplying with \((x_1 - x_3)\) and putting \(x_1 = x_3\) results in
\[
[r^{12}(x_1, x_2) - \tau^{12}(x_1, x_2), \Omega^{13}] = 0. \tag{5.8}
\]
Using the fact that \(\sum_{\alpha=1}^{n} \text{ad}(I_\alpha)^2 = 1\), we obtain \(r = \tau\) by applying the map \(a \otimes b \otimes c \mapsto [a, c] \otimes b\) coefficientwise.

It remains to prove the co-Jacobi identity for \(\delta\). For any \(f \in \mathfrak{g}[x]\) and \(a \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})(x_1)(x_2)[x_3]\) we define
\[
f \cdot a := [f \otimes 1 \otimes 1 + 1 \otimes f \otimes 1 + 1 \otimes 1 \otimes f, a]. \tag{5.9}
\]
Then
\[
 f \cdot [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] = \sum_{j,k=0}^{\infty} \sum_{\alpha, \beta = 1}^{n} \left( [f(x_1), [r_{j,\alpha}(x_1), r_{k,\beta}(x_1)]] \otimes x_j^2 I_\alpha \otimes x_k^2 I_\beta + [r_{j,\alpha}(x_1), r_{k,\beta}(x_1)] \otimes [f(x_2), x_j^2 I_\alpha] \otimes x_k^2 I_\beta + [r_{j,\alpha}(x_1), r_{k,\beta}(x_1)] \otimes x_j^2 I_\alpha \otimes [f(x_3), x_k^2 I_\beta] \right).
\] (5.10)

Using the identity
\[
 [f(x_1), [r_{j,\alpha}(x_1), r_{k,\beta}(x_1)]] = [[f(x_1), r_{j,\alpha}(x_1), r_{k,\beta}(x_1)] + [r_{j,\alpha}(x_1), [f(x_1), r_{k,\beta}(x_1)]]
\]
we can rewrite Eq. (5.10) in the form
\[
 f \cdot [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] = [dr(f)^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), dr(f)^{13}(x_1, x_3)].
\] (5.11)

Similarly, we have
\[
 f : [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] = [dr(f)^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{12}(x_1, x_2), dr(f)^{23}(x_2, x_3)],
\]
\[
 f : [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = [dr(f)^{13}(x_1, x_3), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), dr(f)^{23}(x_2, x_3)].
\] (5.12)

Summing up left and right sides of Eqs. (5.11) and (5.12) respectively and using relations of the form
\[
 [r^{12}(x_1, x_2), dr(f)^{13}(x_1, x_3)] + dr(f)^{23}(x_2, x_3) = - (dr \otimes 1) dr(f) (x_1, x_2, x_3)
\] (5.13)
we obtain the desired equality
\[
 0 = f \cdot [\text{CYB}(r) = - \text{CYB}(dr \otimes 1) dr(f)].
\]

5.2. Formal $r$-matrices and Lagrangian Lie subalgebras. Consider a formal $r$-matrix of the form
\[
 r(x, y) = \frac{m \Omega}{x - y} + \tilde{g}(x, y) = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} x^{-k-1} I_\alpha \otimes y^{k+m} I_\alpha + \tilde{g}(x, y),
\] (5.14)
for some $m \in \mathbb{Z}_{\geq 0}$ and $\tilde{g} \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$. According to Lemma 5.2 it defines a Lie bialgebra structure $dr$ on $\mathfrak{g}[x]$. The goal of this section is two-fold: First, we want to prove that $\mathfrak{D}(\mathfrak{g}[x], dr) \sim \mathfrak{g} \otimes \mathfrak{A}(m, 0)$. More precisely, $\mathfrak{D}(\mathfrak{g}[x], dr) \cong \mathfrak{g} \otimes \mathfrak{A}(m, 0)$ as Lie algebras and the bilinear form on $\mathfrak{D}$ corresponds to $-B_{m+1}$ on $\mathfrak{g} \otimes \mathfrak{A}(m, 0)$; Secondly, we want to construct the Lagrangian Lie subalgebra $W_r \subset \mathfrak{g} \otimes \mathfrak{A}(m, 0)$ corresponding to $dr$.

Construction of $W_r$. Using the given $r$-matrix $r$ we now construct a Lagrangian Lie subalgebra $W_r \subset \mathfrak{g} \otimes \mathfrak{A}(m, 0) = \mathfrak{g}(x) \times \mathfrak{g}[x] / x^m \mathfrak{g}[x]$ such that $\mathfrak{g} \otimes \mathfrak{A}(m, 0) = \mathfrak{g}[x] + W_r$. We recall that $\mathfrak{g} \otimes \mathfrak{A}(m, 0)$ is equipped with the bilinear form $B_{m+1}$ defined in Remark 3.1

We start by writing
\[
 g(x, y) = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} p^{\alpha}_k(x) \otimes I_\alpha y^k.
\] (5.15)
Substituting this into Eq. (5.14) we obtain
\[
 r(x, y) = \sum_{k=0}^{m-1} \sum_{\alpha=1}^{n} p^{\alpha}_k(x) \otimes I_\alpha y^k + \sum_{k=m}^{\infty} \sum_{\alpha=1}^{n} (I_\alpha x^{m-k-1} + p^{\alpha}_k(x)) \otimes I_\alpha y^k.
\] (5.16)

Similarly, representing $(x - y)^{-1}$ as series $- \sum_{k=0}^{\infty} x^k y^{-k-1}$ we get another formal $r$-matrix in $(\mathfrak{g} \otimes \mathfrak{g})([[y]])[x]$: \[
 r(x, y) = - \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} I_\alpha x^{m+k} \otimes I_\alpha y^{-k-1} - \sum_{k=0}^{m-1} \sum_{\alpha=1}^{n} (I_\alpha x^{m-k-1} + p^{\alpha}_k(x)) \otimes I_\alpha y^k + \sum_{k=m}^{\infty} \sum_{\alpha=1}^{n} p^{\alpha}_k(x) \otimes I_\alpha y^k.
\] (5.17)
We define \( W_r \) by combining the coefficients in front of \( I_\alpha y^k, k \geq 0 \) in Eqs. (5.16) and (5.17) namely

\[
W_r := \text{span}_F \{ w_{k,\alpha} \mid k \geq 0, \ 1 \leq \alpha \leq n \} \subset g(\langle x \rangle) \times g[x]/x^m g[x],
\]

where

\[
w_{k,\alpha} = \begin{cases} \left( p^\alpha_k(x), -I_\alpha[x]^{m-k-1} + p^\beta_k([x]) \right) & 0 \leq k \leq m-1, \ 1 \leq \alpha \leq n, \\ (I_\alpha x^{-k+m-1} + p^\alpha_k(x), p^\beta_k([x])) & k \geq m, \ 1 \leq \alpha \leq n; \end{cases}
\]

Here \([x]\) denotes the equivalence class of \( x \) in \( F[x]/(x^m) \). It is clear that \( W_r \) is complementary to the diagonal embedding \( \Delta := \{(f, [f]) \mid f \in g[x]\} \) of \( g[x] \) into \( g(\langle x \rangle) \times g[x]/x^m g[x] \). Let \( \{ b_{k,\alpha} := (I_\alpha x^k, I_\alpha [x]^k) \mid 1 \leq \alpha \leq n \text{ and } k \geq 0 \} \) be a topological basis for \( \Delta \). Then by direct calculation we see that

\[
B_{m+1}(w_{j,\alpha}, b_{k,\beta}) = \delta_{jk} \delta_{\alpha \beta}
\]

for all \( j, k \geq 0 \) and \( 1 \leq \alpha, \beta \leq n \).

**Lagrangian property of \( W_r \).** Let us expand each series \( p^\alpha_k(x) \) in Eq. (5.15) as

\[
p^\alpha_k(x) = \sum_{s=0}^{\infty} \sum_{\beta=1}^{n} c^\alpha_{k,s} I_\alpha x^s.
\]

The skew-symmetry of \( r(x, y) \) gives restrictions on the coefficients \( c^\alpha_{k,s} \). More precisely,

\[
g(x, y) + \tau(g(y, x)) = \sum_{i,j=0}^{\infty} \sum_{\alpha, \beta=1}^{n} (c^\alpha_{i,j} + c^\beta_{j,i}) I_\alpha x^i \otimes I_\beta y^j
\]

\[
= \frac{(x^m - y^m) \Omega}{x - y}
\]

\[
= \sum_{k=0}^{m-1} \sum_{\alpha=1}^{n} I_\alpha x^{m-k-1} \otimes I_\alpha y^k.
\]

Therefore, we have

\[
c^\alpha_{i,j} + c^\beta_{j,i} = \delta_{\alpha, \beta} \delta_{i+j, m-1}.
\]

Using these identities we can show that \( W_r \) is isotropic by proving \( B_{m+1}(w_{j,\alpha}, w_{k,\beta}) = 0 \) for all \( k, j \geq 0 \) and \( 1 \leq \alpha, \beta \leq n \). For example, when \( j, k \leq m - 1 \) we get

\[
B_{m+1}\left(p^\alpha_k(x), -I_\alpha[x]^{m-i-1} + p^\beta_i([x]), p^\beta_j(x), -I_\beta[x]^{m-j-1} + p^\alpha_j([x])\right)
\]

\[
= \sum_{i,j=0}^{\infty} \sum_{\alpha, \beta=1}^{n} (c^\alpha_{i,j} + c^\beta_{j,i}) \kappa(I_\alpha, I_\beta) \text{coeff}_{m-1}(x^{m-i-1} + m-j-1)
\]

\[
= 0.
\]

The isotropy of \( W_r \) combined with \( \Delta + W_r \) implies that \( W_r \) is Lagrangian. **An \( r \)-matrix in \((g \otimes g)[x, y]/(x^m, y^m)\)**. We denote the first sum in Eq. (5.17) by \( r_- \) and the other two by \( r_+ \), i.e.

\[
r(x, y) = \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} I_\alpha x^{m+k} \otimes I_\alpha y^{-k-1}
\]

\[
- \sum_{k=0}^{m-1} \sum_{\alpha=1}^{n} (I_\alpha x^{m-k-1} + p^\alpha_k(x)) \otimes I_\alpha y^k + \sum_{k=m}^{\infty} \sum_{\alpha=1}^{n} p^\alpha_k(x) \otimes I_\alpha y^k.
\]
The Classical Yang-Baxter equation for \( r \) then writes as
\[
0 = \text{CYB}(r) = \text{CYB}(r_+) + \text{CYB}(r_-) + [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}].
\]
It can be considered as an element in \((\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})(y)\) of \( \mathfrak{g} \) and then taking the quotient by \((x^m, y^m, z^m)\) we obtain the equality \( \text{CYB}(r) = 0 \). In particular, \( r_-([x, [y]]) \) is a formal \( r \)-matrix in \((\mathfrak{g} \otimes \mathfrak{g})(x, y)/(x^m, y^m)\).

**Subalgebra property of \( W_r \).** The previous result implies that
\[
R := \sum_{j=0}^{\infty} \sum_{\alpha=1}^{n} w_{j,\alpha} \otimes b_{j,\alpha} \in (W \otimes \mathfrak{g})(y, [y])
\]
is an \( r \)-matrix, i.e. it formally solves the CYBE because both right and left components of \( R \) do that. The fact that \( W_r \) is a subalgebra now follows from the identity \([R_{12}, R_{13}] = -[R_{12} + R_{13}, R_{23}]\) that can be rewritten as
\[
\sum_{j, k=0}^{\infty} \sum_{\alpha, \beta} [w_{j,\alpha}, w_{k,\beta}] \otimes b_{j,\alpha} \otimes b_{k,\beta} = -\sum_{j=0}^{\infty} \sum_{\alpha=1}^{n} w_{j,\alpha} \otimes dR(b_{j,\alpha}) \in (W \otimes \mathfrak{g} \otimes \mathfrak{g})(x_2, [x_2]), (x_3, [x_3])
\]
(5.24)
Here \( dR: \Delta \rightarrow \Delta \otimes \Delta \) is defined by \( dR(f) = [f \otimes 1 + 1 \otimes f, R] \) for all \( f \in \Delta \). Observe that the diagonal embedding \( \mathfrak{g}[x] \rightarrow \Delta \) defines an isomorphism \((\mathfrak{g}[x], dR) \cong (\Delta, dR)\) of topological Lie bialgebras.

**The topological double of \( dr \).** Applying
\[
B_{m+1}^3(b_{j_1,\alpha_1} \otimes w_{j_2,\alpha_2} \otimes w_{j_3,\alpha_3}, -)
\]
to Eq. (5.24) and using Eq. (5.20) results in
\[
B_{m+1}^2(dR(b_{j_1,\alpha_1}), w_{j_2,\alpha_2} \otimes w_{j_3,\alpha_3}) = -B_{m+1}(b_{j_1,\alpha_1}, [w_{j_2,\alpha_2}, w_{j_3,\alpha_3}])
\]
for all \( j_1, j_2, j_3 \geq 0 \) and \( 1 \leq \alpha_1, \alpha_2, \alpha_3 \leq n \). This implies that
\[
B_{m+1}^2(dR(f), v \otimes w) = -B_{m+1}(f, [v, w])
\]
for all \( f \in \Delta \) and \( v, w \in W_r \). It is easy to see that \( \mathfrak{g}[x]' \cong \Delta' \cong W_r \) using \( B_{m+1} \). Remark 3.18 now states that
\[
(\mathfrak{D}(\mathfrak{g}[x], -dr), \mathfrak{g}[x], \mathfrak{g}[x]') \cong (\mathfrak{g}(x) \times \mathfrak{g}[x]/x^m \mathfrak{g}[x], \Delta, W_r)
\]
(5.26)

5.3. **Realization of all topological doubles.** It is well-known that the following four series
\[
\begin{align*}
\gamma_0(x, y) &:= 0, & \gamma_1(x, y) &:= \frac{\Omega}{x - y}, & \gamma_2(x, y) &:= \frac{y \Omega}{x - y} + r_{D1}, & \gamma_3(x, y) &:= \frac{x \Omega}{x - y} = \frac{y \Omega}{x - y} + y \Omega
\end{align*}
\]
are formal \( r \)-matrices. Here \( r_{D1} \in \mathfrak{g} \otimes \mathfrak{g} \) is the Drinfeld-Jimbo classical \( r \)-matrix with respect to a triangular decomposition \( \mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_- \). Therefore, the discussion in Section 5.2 implies that \( \delta_0 := -dr_0 = 0 \) and \( \delta_{m+1} := -dr_{m+1} \) define topological Lie bialgebra structures on \( \mathfrak{g}[x] \) with topological doubles
\[
\mathfrak{D}(\mathfrak{g}[x], \delta_0) \cong \mathfrak{g} \otimes A(\infty) \quad \text{and} \quad \mathfrak{D}(\mathfrak{g}[x], \delta_{m+1}) \cong \mathfrak{g} \otimes A(m, 0) = \mathfrak{g}(x) \times \mathfrak{g}[x]/x^m \mathfrak{g}[x]
\]
(5.28)
for \( m \in \{0, 1, 2\} \). These isomorphisms are identical on the images of \( \mathfrak{g}[x] \) under the respective natural embeddings.

Using the construction of \( W_r \) presented in Section 5.2 we can explicitly describe the Lagrangian Lie subalgebras \( W_i \) defining \( \delta_i, i \in \{0, 1, 2, 3\} \):
Lagrangian Lie subalgebras $W$

The assignment

Corollary 5.5.

with

conditions on $s$

5.4. Formal $r$-matrices and topological twists. The following statements describe how the conditions on $s \in (g \otimes g)[x, y]$ for being a topological twist look like on the side of formal $r$-matrices.

Lemma 5.3. Let $r$ be a formal $r$-matrix and $\delta := dr$ be the corresponding Lie bialgebra structure on $g[x]$. Then $s \in (g \otimes g)[x, y]$ is a topological twist of $\delta$ if and only if $r_s := r + s$ is a formal $r$-matrix. Moreover, if this is the case, then $\delta_s = dr_s$.

Proof. The first part of the statement follows from the formal equality in $\langle g \otimes g \rangle \langle x \rangle \langle y \rangle = \mathbb{Z}$

$$\text{CYB}(r + s) = \text{CYB}(r) + \text{CYB}(s) - \text{Alt}((\delta \circ 1)s).$$

(5.29)

Indeed, $\text{CYB}(r) = 0$ implies that $\text{CYB}(r + s) = 0$ is equivalent to $\text{CYB}(s) = \text{Alt}((\delta \circ 1)s)$.

The second part follows by definition of $ds$, namely

$$(\delta_s)(f)(x, y) = \delta(f)(x, y) + ds(f)(x, y)$$

$$= [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] + [f(x) \otimes 1 + 1 \otimes f(y), s(x, y)]$$

$$= [f(x) \otimes 1 + 1 \otimes f(y), r_s(x, y)].$$

(5.30)

Remark 5.4. Let $s$ be a topological twist of $\delta$. By straightforward calculations one can prove that the Lagrangian Lie subalgebra $W_s$ associated to $s$ in Theorem 4.3 (see Remark 4.4) coincides with $W_{r+s}$ presented in Section 5.2. Therefore, when $i$ is fixed we use the notations $W_s$ and $W_{r+s}$ interchangeably.

Corollary 5.5. The assignment $r \mapsto W_s$ presented in Section 5.3 defines a bijection between Lagrangian Lie subalgebras $W \subset \mathcal{D}(g[x], \delta_i), i \in \{0, 1, 2, 3\}$ complementary to $g[x]$, and formal $r$-matrices of the form $r = r_i + s$.

5.5. Reinterpretation of the classification of doubles. The classification of classical doubles has interesting consequences for the form of formal $r$-matrices.

Theorem 5.6. Let $r$ be a formal $r$-matrix of the form

$$r(x, y) = \frac{s(y)\Omega}{x-y} + g(x, y)$$

(5.31)

for some $s \in x^n F[x]^\times$ and $g \in (g \otimes g)[x, y]$. Then $m \in \{0, 1, 2\}$ and there exists $\psi \in x F[x]^\times, \xi \in F^\times$ and $\tilde{g} \in (g \otimes g)[x, y]$ such that

$$\xi r(\psi(x), \psi(y)) = \frac{ym\Omega}{x-y} + \tilde{g}(x, y).$$

(5.32)
Proof. [Lemma 5.2] implies that \( r \) defines a topological Lie bialgebra structure \( dr \) on \( g[x] \). By [Remark 3.21] there is a constant \( \xi \in F^* \), an integer \( 0 \leq m < 2 \), a coordinate transformation \( \varphi \in \text{Aut}_F F[x] \) and an \( F[x] \)-linear isomorphism \( \mathcal{D}(g[x], \xi \delta) \cong g \otimes A(m, 0)(\varphi) \) identical on \( g[x] \). This means that \( \xi(\varphi^{-1} \otimes \varphi^{-1})dr = \delta_{m+1} + dt \) for some topological twist \( t \). Put \( \psi := \varphi^{-1} \). Since \( \delta_{m+1} + dt \) is given by the \( r \)-matrix \( -r_{m+1} + t \) we have

\[
[f(x) \otimes 1 + 1 \otimes f(y), \xi r(\psi(x), \psi(y))] = \xi(\psi \otimes \psi)dr \varphi(f) = (\delta_{m+1} + dt)(f) = [f(x) \otimes 1 + 1 \otimes f(y), -r_{m+1}(x, y) + t(x, y)]
\]

Putting \( f = a \in g \) implies that

\[
\xi r(\psi(x), \psi(y)) + r_{m+1}(x, y) - t(x, y) = h(x, y)\Omega
\]

for some \( h \in F[[x]][y] \), since the \( g \)-invariant elements of \( g \otimes g \) are spanned by \( \Omega \). Letting \( f = ax \in g[x] \) we see that

\[
0 = [ax \otimes 1 + 1 \otimes ay, h(x, y)\Omega] = (x - y)h(x, y)[a \otimes 1, \Omega],
\]

which implies that \( h = 0 \) since \( [a \otimes 1, \Omega] \neq 0 \) for \( a \neq 0 \). In particular,

\[
\xi r(\psi(x), \psi(y)) = -r_{m+1}(x, y) + t(x, y).
\]

We can always write

\[
\frac{s(\psi(y))}{\psi(x) - \psi(y)} = \frac{s(\psi(y))}{\psi(y)(x - y)} + u(x, y)
\]

for some \( u(x, y) \in (g \otimes g)[x, y] \). Combining this with Eq. (5.36) we obtain

\[
\xi \frac{s(\psi(y))}{\psi(y)} = -y^m.
\]

By taking \(-\xi\) instead of \(\xi\) we conclude the proof. \( \blacksquare \)

Remark 5.7. Let

\[
s(y)\Omega \frac{x - y}{x - y} + g(x, y),
\]

where \( s(y) = y^m + a_{m+1}y^{m+1} + \ldots, m \in \{0, 1, 2\}, \) be a formal \( r \)-matrix. One can directly see from the proof of Theorem 5.6 that the equivalence \( \psi \in \text{Aut}_a F[[x]] \), used to transform \( s(y) \) into 1, \( y \) or \( y^2 \), is a solution to the differential equation

\[
y^m \psi'(y) = s(\psi(y)).
\]

Note, that for \( m = 0, 1 \) Eq. (5.40) always has a solution, whereas for \( m = 2 \) a solution exists if and only if \( a_3 = 0 \).

Conversely, if \( s(y) = y^m + a_{m+1}y^{m+1} + \ldots, m \in \{0, 1, 2\} \) and Eq. (5.40) has a solution, then we can find \( r \)-matrix of the form Eq. (5.39). \( \diamond \)

Remark 5.8. The methods in the proof of Theorem 5.6 combined with the previous discussion of formal \( r \)-matrices imply that the assignment \( r \mapsto dr \) defines a bijection between the set of formal \( r \)-matrices and the set of topological Lie bialgebra structures on \( g[x] \). \( \diamond \)

5.6. Equivalences. Two formal \( r \)-matrices \( r \) and \( \tilde{r} \) are called \emph{formally gauge equivalent} if there is an \( F[x] \)-linear automorphism \( \phi \) of \( g[x] \) such that

\[
\tilde{r} = (\phi \otimes \phi)r.
\]

At this stage we have defined three notions of formal equivalence: between topological twists, Lagrangian Lie subalgebras and formal \( r \)-matrices. Now we turn to describing the relation between these notions.

Theorem 5.9. Let \( \phi \) be an \( F[x] \)-linear automorphism of \( g[x] \), \( s \) and \( \tilde{s} \) be topological twists of the topological Lie bialgebra structure \( \delta_i, i \in \{1, 2, 3\} \), on \( g[x] \). The following are equivalent:
Proof. 1 $\iff$ 3: Using $F[x]$-linearity of $\phi$ and Lemma 5.3 condition 1 reads as

\[
(\phi \otimes \phi)(\delta_i + ds) = (\delta_i + ds)\phi;
\]

(2) \((\phi \times [\phi])(W_s) = W_s;\)

(3) \((\phi \otimes \phi)(-r_i + s) = -r_i + \delta_s.
\]

This is equivalent to $[f \otimes 1 + 1 \otimes f, (\phi \otimes \phi)(-r_i + s)] = 0$ for all $f \in g[x]$, which means that $(r_i + \delta_s) - (\phi \otimes \phi)(r_i + s) = h\Omega$, for some $h \in F((x))[[y]]$. Hence,

\[
0 = h(x,y)[f(x) \otimes 1 + 1 \otimes f(y), \Omega] = h(x,y) [(f(x) - f(y)) \otimes 1, \Omega]
\]

for all $f \in g[x]$. Since $[a \otimes 1, \Omega] \neq 0$ for any non-zero $a \in g$ we must have $h = 0$ implying the desired equality $(\phi \otimes \phi)(-r_i + s) = -r_i + \delta_s$. The converse is clear.

2 $\iff$ 3: Write two series expansions of $-r_i + s$ at $y = 0$ and $x = 0$ respectively as follows

\[
\sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} r_{i}^\alpha(x) \otimes I_\alpha y^k,
\]

\[
\sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} I_\alpha x^{k+i-1} \otimes I_\alpha y^{k-1} + \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \tilde{r}_k^\alpha(x) \otimes I_\alpha y^k.
\]

Let $\phi(I_\alpha) = \sum_{j=0}^{\infty} \sum_{\gamma=1}^{c_j \gamma} I_\gamma x^j$. Applying $\phi \otimes \phi$ to Eq. (5.43) we get the series

\[
\sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \sum_{j=0}^{\infty} \sum_{\gamma=1}^{c_j \gamma} \phi(r_{i}^\alpha(x)) \otimes I_\alpha y^{k+j},
\]

\[
\sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \sum_{j=0}^{\infty} \sum_{\gamma=1}^{c_j \gamma} \phi(I_\alpha x^{k+i-1} \otimes I_\alpha y^{k-1} + \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{c_j \gamma} \phi(\tilde{r}_k^\alpha(x)) \otimes I_\alpha y^{k+j}.
\]

Put $t := k + j$. This gives

\[
\sum_{l=0}^{\infty} \sum_{\gamma=1}^{c_l \gamma} \left( \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \phi(r_{i}^\alpha(x)) \right) \otimes I_\gamma y^t,
\]

\[
\sum_{l=0}^{\infty} \sum_{\gamma=1}^{c_l \gamma} \left( \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \phi(I_\alpha x^{k+i-1} \otimes I_\alpha y^{k-1} + \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \phi(\tilde{r}_k^\alpha(x)) \right) \otimes I_\gamma y^t,
\]

where we let $c_{l,\alpha} := 0$ for $l < 0$. Therefore, the Lagrangian Lie subalgebra in $g \otimes A(i-1,0)$ corresponding to $(\phi \otimes \phi)(-r_i + s) = -r_i + \delta_s$ is

\[
W_s := \operatorname{span}_F \left\{ \sum_{k=0}^{\infty} \sum_{\alpha=1}^{n} \phi(r_{i}^\alpha(x)), [\phi]([\tilde{r}_k^\alpha(x)]) \right\} \right|_{0 \leq t, 1 \leq \gamma \leq n}.
\]

It is now clear that $W_s \subseteq (\phi \times [\phi])W_s$. Since $g[x] + W_s = g[x] + (\phi \times [\phi])W_s = g \otimes A(i-1,0)$ we have the equality

\[
W_{(\phi \otimes \phi)(-r_i + s)} = (\phi \times [\phi])W_{-r_i + s}.
\]

Together with bijection Corollary 5.5 we get the desired statement. $\blacksquare$
Therefore, we have shown that isomorphism classes of topological twists of \( \delta_i, i \in \{1, 2, 3\} \), are in one-to-one correspondence with isomorphism classes of the corresponding Lagrangian Lie subalgebras and gauge equivalence classes of formal \( r \)-matrices.

6. Commensurable twists and their classification

One natural subclass of topological twists of \( \delta_i \) for \( i \in \{0, 1, 2, 3\} \) that admits a straightforward classification is the class of topological twists in \( \mathfrak{g}[x] \otimes \mathfrak{g}[y] \subset (\mathfrak{g} \otimes \mathfrak{g})[x, y] \). We say that a subalgebra \( W \subseteq \mathcal{D}(\mathfrak{g}[x], \delta_i) \) is commensurable with \( W_i \subseteq \mathcal{D} \) if
\[
\dim(W_i + W)/(W_i \cap W) < \infty.
\]

Adjusting slightly the argument in Theorem 4.3 we get the following correspondence.

**Lemma 6.1.** Let \( (\mathfrak{g}[x], \delta_i), \; i \in \{0, 1, 2, 3\} \), be the topological Lie bialgebra structures defined in Section 5.3. There are the following one-to-one correspondences:

**Lagrangian Lie subalgebras**

- **W \subseteq D**
- **complementary to \( \mathfrak{g}[x] \)**
- **and commensurable with \( W_i \)**

**Topological twists \( s \) of \( \delta_i \)**

- **lying in \( \mathfrak{g}[x] \otimes \mathfrak{g}[y] \)**

**Linear maps**

- **\( T : W_i \rightarrow \mathfrak{g}[x] \)**
- **such that**
  - \( \dim(\text{im}(T)) < \infty \)
  - **and for all** \( p_1, p_2, p_3 \in W_i \)
  - \( B_i(T p_1, p_2) + B_i(p_1, T p_2) = 0 \)
  - \( B_i([T p_1 - p_1, T p_2 - p_2], T p_3 - p_3) = 0 \)

**Proof.** We start with the identification
\[
\mathfrak{g}[x] + \mathfrak{g}\{x\} \cong \begin{cases} \mathfrak{g} \otimes A(\infty) & \text{if } i = 0, \\ \mathfrak{g} \otimes A(i - 1, 0) & \text{if } i \in \{1, 2, 3\} \end{cases} \quad (6.1)
\]
under which \( \mathfrak{g}\{x\} \cong W_i \) as Lie algebras. It is enough to prove that any linear map \( T : W_i \rightarrow \mathfrak{g}[x] \) satisfying the assumptions above gives rise to a polynomial twist \( s \in \mathfrak{g}[x] \otimes \mathfrak{g}[y] \). As in the proof of Theorem 4.3 let \( \{T p_j\}_{j=1}^n \) be a basis for \( \text{im}(T) \) and \( \{v_j\}_{j=1}^n \subseteq \mathfrak{g}\{x\} \) be its dual basis. The vector space \( \mathfrak{g}\{x\} \) decomposes as
\[
\text{span}\{p_j \mid 1 \leq j \leq n\} + \ker(T).
\]
Put \( m := \max_j \{\deg(p_j)\} \), then it is clear that \( x^{m+1} \mathfrak{g}\{x\} \subseteq \ker(T) \). Here we recall that \( \mathfrak{g}\{x\} = \mathfrak{g}[x]' \) can be identified with \( \mathfrak{g}\{x\} \) as a vector space. Now we observe that for any \( 1 \leq j \leq n \) we have the equality
\[
0 = B(v_j, T(\ker(T))) = -B(T v_j, \ker(T)) = -B(T v_j, x^{m+1} \mathfrak{g}\{x\}),
\]
which means that \( T v_j \) is actually a polynomial of degree at most \( m \). In other words, the tensor \( s := -\sum_{j=1}^n T p_j \otimes T v_j \) lies in \( \mathfrak{g}[x] \otimes \mathfrak{g}[y] \). By skew-symmetry we get \( s \in \mathfrak{g}[x] \otimes \mathfrak{g}[y] \) as we wanted.

With this result in mind we call topological twists in \( \mathfrak{g}[x] \otimes \mathfrak{g}[y] \) *commensurable twists*. It is clear that formal isomorphisms of twists do not necessarily preserve their commensurability. Therefore, we will classify commensurable twists within \( \mathcal{D}(\mathfrak{g}[x], \delta_i) \), \( 1 \leq i \leq 3 \) up to polynomial isomorphisms, i.e. \( F[x] \)-linear automorphisms of \( \mathfrak{g}[x] \) extended to \( \mathfrak{g}[x] \).
6.1. Commensurable twists in $g \otimes A(0,0)$. According to the results of Section 3, isomorphism classes of commensurable twists of $\delta_1$ are in bijection with gauge equivalence classes of formal rational $r$-matrices

$$r_p = r_{\text{Yang}} + p \in (g \otimes g)((x))[y],$$  

(6.2)

where $p \in (g \otimes g)[x, y]$ is a skew-symmetric polynomial.

Let us for a moment assume $F = \mathbb{C}$. Then $r_p$ can be considered as a Taylor series expansion at $y = 0$ of the non-degenerate classical $r$-matrix

$$r_p(x, y) = \frac{\Omega}{x - y} + p(x, y).$$  

(6.3)

By [1, Proposition 4.8] classical non-degenerate $r$-matrices of the form (Eq. (6.3)) are holomorphically gauge equivalent if and only if their Taylor series expansions at $y = 0$ are formally gauge equivalent. Furthermore, the statement of [2, Theorem A.3] tells us that two classical $r$-matrices of the form (Eq. (6.3)) are homomorphically gauge equivalent if and only if they are polynomially gauge equivalent. Combining these results we get the following statement.

Lemma 6.2. Put $F = \mathbb{C}$. Then formal $r$-matrices $r_{p_1}$ and $r_{p_2}$ are formally gauge equivalent if and only if they are polynomially gauge equivalent. More precisely, $r_{p_2} = (\phi \otimes \phi)r_{p_1}$ for a $\phi \in \text{Aut}_{C[x]}\text{-LieAlg}(g[x])$ is equivalent to $r_{p_2} = (\psi \otimes \psi)r_{p_1}$ for some $\psi \in \text{Aut}_{C[x]}\text{-LieAlg}(g[x]).$

Therefore, for $F = \mathbb{C}$ the isomorphism classes of commensurable twists (up to polynomial isomorphisms) are in bijection with gauge equivalence classes of formal $r$-matrices of the form (Eq. (6.2)). Using [22, Theorem 1] we reduce the classification of commensurable twists of $\delta_1$ to the classification of particular Lagrangian Lie subalgebras in $\mathfrak{g}(x^{-1})$.

Theorem 6.3. Formal $r$-matrices of the form $r_p$ are in bijection with Lagrangian Lie subalgebras $W \subset \mathfrak{g}(x^{-1})$ such that

1. $g[x] + W = \mathfrak{g}(x^{-1})$;
2. $x^{-N}g[x^{-1}] \subseteq W$ for some integer $N > 0$.

Moreover, $r_{p_2} = (\varphi \otimes \varphi)r_{p_1}$ for some $\varphi \in \text{Aut}_{C[x]}\text{-LieAlg}(g[x])$ if and only if $W_2 = \varphi(W_1)$, where $W_1$ and $W_2$ are the corresponding Lagrangian Lie subalgebras.

The second condition on $W$ means that $W$ is an order in $\mathfrak{g}(x^{-1})$. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of $\mathfrak{g}$ associated to some triangular decomposition of $\mathfrak{g}$ and $\alpha_0$ be the corresponding maximal root. Let $k_i$ be the unique positive integers such that $\alpha_0 = \sum_{i=1}^{n} k_i \alpha_i$ holds. It was shown in [23] that an order $W$ can be embedded by an appropriate polynomial gauge transformation into a so-called maximal order $W' \subset \mathfrak{g}(x^{-1})$ with the property $W' + g[x] = \mathfrak{g}(x^{-1})$ and that such maximal orders are in one-to-one correspondence with roots $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. The classification of orders within maximal orders corresponding to roots $-\alpha_0$ and $\alpha_i$ with $k_i = 1$ reduces to the classification of pairs $(L, B)$, where $L$ is a subalgebra of $g$ such that $L + P_k = g$ for the parabolic subalgebra $P_k$ corresponding to the simple root $\alpha_k$ and $B$ is a 2-cocycle on $L$ non-degenerate on the intersection $L \cap P_k$. In particular, when $g = \mathfrak{sl}(n, \mathbb{C})$ all coefficients $k_i = 1$ and the classification of orders reduces completely to the classification of such pairs $(L, B)$; for more details see [22].

Remark 6.4. In order to obtain the above-mentioned description of commensurable twists we made an assumption that $F = \mathbb{C}$. This was done in order to apply verbatim the statement of [2, Theorem A.3]. However, the geometric methods used in [2] to prove the theorem are applicable directly to formal $r$-matrices over an arbitrary algebraically closed field $F$ of characteristic 0.

Therefore, the description of commensurable twists in terms of orders $W \subset \mathfrak{g}(x^{-1})$ and pairs $(L, B)$ is true for any algebraically closed field $F$ of characteristic 0. \(\diamond\)
6.2. Commensurable twists in \( g \otimes A(1,0) \). As in the previous case, isomorphism classes of commensurable twists of \( \delta_2 \) are in one-to-one correspondence with gauge equivalence classes of quasi-trigonometric \( r \)-matrices, i.e. formal \( r \)-matrices of the form
\[
r_2(x, y) + p(x, y) = \frac{yQ}{x} + p(x, y),
\]
where \( p \in g[x] \otimes g[y] \).

In case \( F = \mathbb{C} \) quasi-trigonometric \( r \)-matrices were classified up to polynomial gauge equivalence in \cite{[21]} and independently in \cite{[2]}. More precisely, any quasi-trigonometric \( r \)-matrix \( r_2 + p \) is polynomially gauge equivalent to \( r_2 + t_Q \), where \( t_Q \) is a twist defined by a BD quadruple \( Q = (\Gamma_1, \Gamma_2, \gamma, \tau) \) such that \( \Gamma_3 \) does not contain the minimal root \( -\alpha_0 \) of \( g \). The explicit construction of \( t_Q \) is presented in \cite[Section 4]{[2]}.  

6.3. Commensurable twists in \( g \otimes A(2,0) \). In \cite{[24]} \( r \)-matrices of the form \( r_4 + p \) for some polynomial \( p \in g[x] \otimes g[y] \) were called quasi-rational \( r \)-matrices. It was shown there that these are in bijection with Lagrangian Lie subalgebras \( W \) of \( g((x^{-1})) \times g[x]/x^{2}g[x] \) such that
\begin{enumerate}
\item \( W \cap P = 0 \);
\item \( W \oplus P = g \otimes A(2,0) \);
\item \( x^{-N}g[x^{-1}] \times \{0\} \subseteq W \) for some \( N > 0 \),
\end{enumerate}
where \( P = \{(f, [f]) \mid f \in g[x]\} \). It was shown in \cite{[19]} that the classification of such Lagrangian Lie subalgebras up to polynomial gauge equivalence coincides with the classification of orders within maximal orders in \( g((x^{-1})) \) corresponding to \( -\alpha_0 \) and \( \alpha_1 \) with coefficient 1 in the linear decomposition \( \alpha_0 = \sum_{i=1}^{n} k_i \alpha_i \) of \( \alpha_0 \) into the sum of simple roots. As in Section 6.1, this reduces to the classification of pairs \((L, B)\), where \( L \) is a subalgebra of \( g \) such that \( L + P_k = g \) for the parabolic subalgebra \( P_k \) corresponding to the simple root \( \alpha_k \) and B is a 2-cocycle on \( L \) non-degenerate on the intersection \( L \cap P_k \). In particular, for \( g = gl(n, F) \), the classification of commensurable twists of \( \delta_1 \) coincides with the classification of commensurable twists of \( \delta_3 \).

6.4. Commensurable twists in \( g \otimes A(\infty) \). By Lemma 6.1 commensurable twists of \( \delta_0 = 0 \) are precisely the elements \( s \in g[x] \otimes g[y] \) such that \( CYB(s) = 0 \). Such \( r \)-matrices are in one-to-one correspondence with finite-dimensional quasi-Frobenius Lie subalgebras of \( g[x] \). This problem is known to be "wild" and hence we do not expect to see any classification in this case.

7. Classification of topological twists

We now give a classification of topological twists of \( \delta_i, i \in \{1, 2, 3\} \). Its proof is presented in the following section.

7.1. Twisting class \( g \otimes A(0,0) \). In this section we assume that \( F = \mathbb{C} \) in order to present the most explicit classification. However, we establish similar results over arbitrary algebraically closed fields of characteristic 0 in a series of remarks.

Theorem 5.9 states that isomorphism classes of topological twists of \( \delta_1 \) are in bijection with gauge equivalence classes of formal \( r \)-matrices \( r_1 + s \), where \( s \in (g \otimes g)[x, y] \). Following \cite{[3]}, when \( F = \mathbb{C} \) these formal series are actually Taylor series of appropriate two-parametric meromorphic \( r \)-matrices and are subject to an adjusted form of the well-known trichotomy for one-parametric \( r \)-matrices achieved by Belavin and Drinfeld \cite[Theorem 1.1]{[5]}.  

Theorem 7.1 (Theorem 4.7, \cite{[1]}). Let \( r \) be a formal \( r \)-matrix of the form \( r_1 + s \) for some topological twist \( s \in (g \otimes g)[x, y] \). Then it is gauge equivalent to the Taylor series expansion of a classical \( r \)-matrix \( \rho : \mathbb{C}^2 \rightarrow g \otimes g \) with respect to the second variable in point 0, where \( \rho \) is either

Elliptic: In this case there is a lattice \( \Lambda \subset \mathbb{C}^2 \) of rank 2, such that for all \( \lambda_1, \lambda_2 \in \Lambda \)
\[
\rho(x + \lambda_1, y + \lambda_2) = \rho(x, y);
\]
**Trigonometric:**

\[
\rho(x, y) = \frac{1}{\exp(x - y) - 1} \sum_{k=0}^{[\sigma] \cdot 1} \exp \left( \frac{k(x - y)}{\sigma} \right) \Omega_k + t \left( \exp \left( \frac{x}{\sigma} \right), \exp \left( \frac{y}{\sigma} \right) \right),
\]

for some finite-order automorphism \(\sigma \in \text{Aut}_{\mathbb{C}(\text{LieAlg}(g))}(g)\) and an element \(t \in \mathcal{L}(g, \sigma) \otimes \mathcal{L}(g, \sigma)\). Here \(\mathcal{L}(g, \sigma)\) stands for the loop algebra of \(g\) twisted by \(\sigma\);

**Rational:**

\[
\rho(x, y) = \rho_p(x, y) = \frac{\Omega}{x - y} + p(x, y),
\]

where \(p \in (g \otimes g)[x, y]\) is skew-symmetric polynomial.

**Remark 7.2.** We will see in Section 8 that it is possible to assign an irreducible cubic plane curve \(X\) to a formal \(r\)-matrix \(r = r_1 + s\) over any algebraically closed field of characteristic 0. It is then well-known that \(X\) is either an elliptic curve or has a unique singular point, which is either nodal or cuspidal. The type of \(X\) is thereby an invariant of the equivalence class of \(r\). Therefore, this procedure splits (equivalence classes of) formal \(r\)-matrices of the form \(r_1 + s\) into three classes. It is shown in [1] that for \(F = \mathbb{C}\), \(r\) is elliptic (resp. trigonometric, resp. rational) if \(X\) is elliptic (resp. nodal, resp. cuspidal). Therefore, this categorization of \(r\) by its associated curve is the generalization of Theorem 7.1 to arbitrary algebraically closed fields of characteristic 0. In particular, we also call a formal \(r\)-matrix of the form \(r_1 + s\) elliptic (resp. trigonometric, resp. rational) if \(X\) is elliptic (resp. nodal, resp. cuspidal) even if \(F\) is not the field of complex numbers.

By [1, Proposition 4.10] two classical \(r\)-matrices of one of the forms above are holomorphically gauge equivalent if and only if the corresponding Taylor series expansions of these \(r\)-matrices are formally gauge equivalent. This reduces the classification of topological twists of \(\delta_1\) up to formal isomorphisms to the classification of classical \(r\)-matrices of the forms given in Theorem 7.1 up to holomorphic equivalence. The latter classification, in its turn, reduces to the works by Belavin and Drinfeld [3, 4] in the elliptic and trigonometric cases and to works by Stolin [22, 23] in the rational case.

**Elliptic.** It is known from [3] that elliptic \(r\)-matrices exist only for \(g = \mathfrak{sl}(n, \mathbb{C})\). Moreover, their explicit construction involves a choice of a primitive \(n\)-th root of unity. Therefore, all classical elliptic \(r\)-matrices are parametrized by triples \((\lambda_1, \lambda_2, d)\), where \(\lambda_1, \lambda_2 \in \mathbb{C}\) span a two-dimensional lattice and \(n > d > 0\) is an integer coprime with \(n\) that describes our choice a primitive root of unity.

Extending the notion of equivalence by allowing scaling and coordinate transformations we can without loss of generality assume \(\lambda_1 = 1\) and \(\Im(\lambda_2) > 0\). Furthermore, by [1, Lemma 3.3], two classical elliptic \(r\)-matrices corresponding to \((1, \lambda, d)\) and \((1, \lambda', d)\) are equivalent if only if the elliptic curves \(\mathbb{C}/\langle 1, \lambda \rangle\) and \(\mathbb{C}/\langle 1, \lambda' \rangle\) are isomorphic. This gives the following statement.

**Theorem 7.3.** Let \(g = \mathfrak{sl}(n, \mathbb{C})\). Topological twists of \(\delta_1\) of elliptic type are parametrized by doubles \((\lambda, d)\), where \(\lambda \in \mathbb{H}/\text{SL}(2, \mathbb{Z})\) and \(0 < d < n\) is an integer coprime to \(n\).

**Remark 7.4.** It can be shown that two elliptic \(r\)-matrices corresponding to triples \((\lambda, d_1)\) and \((\lambda, d_2)\) are (gauge) equivalent if and only if \(d_2 = n - d_1\). Combining this result with Theorem 7.3 we see that equivalence classes of topological twists of \(\delta_1\) of elliptic type are in bijection with

\[
\mathbb{H}/\text{SL}(2, \mathbb{Z}) \times \{(n, d) \in \mathbb{Z}_+ \mid \gcd(n, d) = 1 \text{ and } 0 < d < n/2\}.
\]

**Remark 7.5.** For a general algebraically closed field \(F\) of characteristic 0, the methods from [1] prove that elliptic formal \(r\)-matrices are parametrized by pairs \((X, \mathcal{A})\) consisting of an elliptic curve
X over F and an acyclic locally free sheaf A of Lie algebras on X with fiber g at every point of X.

**Trigonometric.** It was shown in [2, Theorem 3.4] that any classical r-matrix of the form \( \text{Eq. (7.1)} \) is globally holomorphically equivalent to a classical r-matrix \( X(u - v) \) depending on the difference of its parameters and such that the set of poles of \( X(z) \) is \( 2\pi i \mathbb{Z} \). The last fact means that \( X(z) \) is a trigonometric solution in the sense of [5]. The well-known classification [5, Theorem 6.1] of trigonometric solutions tells us that \( X(z) \) is holomorphically equivalent to another trigonometric solution \( X^*_{g}(z) \), described by a finite-order automorphism \( \sigma \) of \( g \) and a Belavin-Drinfeld (BD) quadruple \( \mathcal{Q} = (\Gamma_1, \Gamma_2, \gamma, t) \).

Conversely, any trigonometric solution \( X^*_{g}(z) \) is automatically of the form described in [Theorem 7.1] and hence its Taylor series expansion at \( y = 0 \) gives a normalized formal r-matrix, which in its turn produces a topological twist.

Therefore, topological twists of \( \delta_1 \) of trigonometric type are fully described by finite-order automorphisms of \( g \) and BD quadruples \( \mathcal{Q} \); for more details see [2, 5].

**Remark 7.6.** For an arbitrary algebraically closed field \( F \) of characteristic 0, the formula \( \text{Eq. (7.1)} \) is well-defined and a formal r-matrix is trigonometric if and only if it is equivalent to a series of \( \text{Eq. (7.1)} \). The classification of trigonometric r-matrices by Belavin-Drinfeld quadruples relies completely on the structure theory of twisted loop algebras and thus remains valid if \( \mathbb{C} \) is replaced by \( F \).

**Rational.** Classification of twists in this case coincides with the classification of commensurable twists inside \( g \otimes A(0,0) \). The latter is presented in Section 6.1.

**Remark 7.7.** For an arbitrary algebraically closed field \( F \) of characteristic 0, the formula \( \text{Eq. (7.2)} \) is well-defined and a formal r-matrix is rational if and only if it is equivalent to a series of this form. As already mentioned in Section 6.1 the results from [22, 23] also apply to this general setting.

7.2. **Twisting classes \( g \otimes A(1,0) \) and \( g \otimes A(2,0) \).** Let \( i \in \{2, 3\} \). Lagrangian Lie subalgebras \( W \subseteq g((x)) \otimes g[x]/x^i-1g[x] \) complementary to the image of \( g[x] \) in \( g((x)) \otimes g[x]/x^i-1g[x] \) are in one-to-one correspondence with formal r-matrices of the form \( r_1 + s \) for some \( s \in (g \otimes g)[x,y] \), defined in Section 5.3.

**Theorem 7.8.** Up to formal isomorphism, Lagrangian Lie subalgebras of \( g \otimes A(1,0) \) and \( g \otimes A(2,0) \) are commensurable. This implies that all topological twists of \( \delta_2 \) and \( \delta_3 \), up to formal isomorphism, are commensurable and hence their classification coincides with the one presented in Section 6.3 and Section 6.4 respectively. In particular, all r-matrices of the form \( r_2 + s \) and \( r_3 + s \), where \( s \in (g \otimes g)[x,y] \), are formally gauge equivalent to quasi-trigonometric and quasi-rational r-matrices respectively.

8. **Algebro-geometric proof of the classification of topological twists**

Finally, we present an algebro-geometric proof of the classification stated in Theorem 7.8.

8.1. **Preliminary results.** Let \( i \in \{1, 2, 3\} \), consider the canonical injection

\[
\iota: g[x] \to g((x)) \otimes g[x]/x^i-1g[x]
\]

\[
f \mapsto (f, [f])
\]

and let \( W \subseteq g \otimes A(i - 1, 0) = g((x)) \oplus g[x]/x^i-1g[x] \) be a Lagrangian subalgebra satisfying

\[
\iota(g[x]) + W = g((x)) \times g[x]/x^i-1g[x].
\]
Furthermore, let $W_+$ and $W_-$ be the projections of $W$ onto $g(\{x\})$ and $g[x]/x^{i-1}g[x]$ respectively. The following results are true:

1. $W_+^\perp \subseteq W_+$ with respect to the bilinear form $\mathcal{K}_i$ defined in Remark 3.14.
2. $g(\{x\}) = g[x] + W_+$ and $\dim(g[x]\cap W_+) < \infty$;
3. $W_+/W_+^\perp \times W_-/W_-^\perp = (\iota(g[x])\cap(W_+ \times W_-)) + W/(W_+^\perp \times W_-^\perp)$ is a Manin triple satisfying $\dim(W_+/W_+^\perp) = \dim(W_-/W_-^\perp) < \infty$;
4. When $i \in \{2, 3\}$ we have $g[x] \cap W_+ \neq \{0\}$.

Proof. Part (1) follows from the inclusions

$$W_+^\perp \times W_-^\perp = (W_+ \times W_-)^\perp \subseteq W^\perp = W \subseteq W_+ \times W_-.$$  \hspace{1cm} (8.3)

The equality $g(\{x\}) = g[x] + W_+$ follows easily from $\iota(g[x]) + W = g(\{x\}) \times g[x]/x^{i-1}g[x]$. For the second part of (2) we have $0 = (g[x] + W_+) = x^{i-1}g[x] \cap W_+^\perp$ which implies that $g[x] \cap W_+^\perp$ can be embedded into $g[x]/x^{i-1}g[x]$ and is therefore finite-dimensional. Consequently, the dimension of $g[x] \cap W_+$ is finite if the quotient $(g[x] \cap W_+)/g[x] \cap W_+^\perp$ is finite-dimensional. The latter space can be identified with a subspace of $W_+/W_+^\perp$. Therefore, the second part of (2) follows from (3).

For (3) observe that the kernel $K$ of the projection $W \rightarrow W_+$ contains $\{0\} \times W_+^\perp$ by virtue of Eq. (8.3). On the other hand, any element of $K$ is of the form $(0, a)$ for some $a \in W_-$, so for all $(w_+, w_-) \in W$

$$0 = B_i((0, a), (w_+, w_-)) = -K_i(a, w_-)$$  \hspace{1cm} (8.4)

holds, implying $a \in W_-^\perp$ and hence $K = \{0\} \times W_-^\perp$. This provides us with an isomorphism $W/(W_+^\perp \times W_-^\perp) \rightarrow W_+/W_+^\perp$ and, similarly, $W/(W_+^\perp \times W_-^\perp) \rightarrow W_-/W_-^\perp$. Composing them we obtain an isomorphism $W_+/W_+^\perp \rightarrow W_-/W_-^\perp$. In particular, $\dim(W_+/W_+^\perp) = \dim(W_-/W_-^\perp) < \infty$. Since $W \subseteq W_+ \times W_-$, the identity $\iota(g[x]) + W = g(\{x\}) \times g[x]/x^{i-1}g[x]$ is equivalent to

$$W_+ \times W_- = (\iota(g[x]) \cap(W_+ \times W_-)) + W.$$  \hspace{1cm} (8.5)

Quoting out $W_+^\perp \times W_-^\perp$ concludes (3).

Let us turn now to (4). Assume that $i \in \{2, 3\}$ and $g[x] \cap W_+ = \{0\}$. Then

$$\iota(g[x]) \cap(W_+ \times W_-) = \{0\},$$  \hspace{1cm} (8.6)

and Eq. (8.5) imply $W = W_+ \times W_- = W_+^\perp \times W_-^\perp$. Since $g(\{x\}) \times g[x]/x^{i-1}g[x] = \iota(g[x]) + (W_+ \times W_-)$ and $g[x] \cap W_+ = \{0\}$ we obtain $W_- = g[x]/x^{i-1}g[x]$, which contradicts $W_+^\perp = W_-$. $\blacksquare$

8.2. Algebro-geometric notation and convention.

In the following, an algebraic variety is an integral scheme of finite type over the algebraically closed field $F$ of characteristic 0. In this setting, a projective variety $X$ is an algebraic variety isomorphic to the projective spectrum $\text{Proj}(R)$ of a finitely-generated graded $F$-algebra $R$ without zero divisors; see e.g. [14, Section II.2] for the definition of the projective spectrum. In particular, a projective curve is such a projective variety of dimension one.

For an algebraic variety $X$, the sheaf of rings of regular functions of $X$ is denoted by $\mathcal{O}_X$ and for any point $p \in X$, $\mathcal{O}_X,p$ is the maximal ideal of the local ring $\mathcal{O}_X,p$.

For an $\mathcal{O}_X$-module $\mathcal{F}$, we write $\mathcal{F}_p$ (resp. $\mathcal{F}_p = \mathcal{F}_p/\mathfrak{m}_p \mathcal{F}_p$) for the stalk (resp. fiber) of $\mathcal{F}$ in some point $p \in X$. Furthermore, $\Gamma(U, \mathcal{F})$ denotes the space of sections of $\mathcal{F}$ on an open subset $U \subseteq X$, $H^k(F)$ denotes the $k$-th global cohomology class of $\mathcal{F}$, and $h^k(F) := \dim(H^k(F))$. In particular, $H^0(F) = \Gamma(X, \mathcal{F})$.

We call $\mathcal{F}$ a sheaf of Lie algebras on $X$ if it is equipped with a bilinear morphism $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ which equips $\mathcal{F}_p$ with the structure of a Lie algebra for all $p \in X$. If $\mathcal{F}$ is also locally free of finite rank, we call the unique bilinear morphism $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{O}_X$ whose stalk at any point $p \in X$...
corresponds to the Killing form of the free $\mathcal{O}_{X,p}$-Lie algebra $\mathcal{F}_p$, Killing form of $\mathcal{F}$ (see e.g. [1] Definition 2.4 & Lemma 2.5)).

8.3. Geometrization of lattices. The methods to prove Theorem 7.1 and Theorem 7.8 are based on the following geometrization scheme of $\mathfrak{g}$-lattices from [1] Section 2.3. Recall that a $\mathfrak{g}$-lattice $L \subseteq \mathfrak{g}([x])$ is a subalgebra

\[
dim(L \cap \mathfrak{g}[x]) < \infty \quad \text{and} \quad \dim(\mathfrak{g}([x])/(\mathfrak{g}[x] + L)) < \infty.
\]  

(8.7)

Then for any unital subalgebra $O \subseteq M_L := \{ f \in F([x]) \mid fL \subseteq L \}$ of finite codimension, the graded $F$-algebra

\[
gr(O) := \bigoplus_{j=0}^{\infty} t^j (O \cap x^{-j}F[x]) \subseteq O[t]
\]

(8.8)

defines a projective curve $X := \text{Proj}(\mathfrak{g}(O))$ satisfying

\[
h^1(O_X) = \dim(F([x])/(F[x] + O)).
\]  

(8.9)

The smooth point $p = (t) \in X$ satisfies $D_p(t) = X \setminus \{ p \}$ and is equipped with an isomorphism $c: \hat{\mathcal{O}}_{X,p} \to F[x]$ such that the extension $c: (\hat{\mathcal{O}}_{X,p} \setminus \{ 0 \})^{-1} \hat{\mathcal{O}}_{X,p} \to F([x])$ has the property

\[
c(\Gamma(X \setminus \{ p \}, O_X)) = O \subseteq F([x]).
\]  

(8.10)

Now the graded $\mathfrak{g}(O)$-Lie algebra

\[
gr(L) := \bigoplus_{j \in \mathbb{Z}} t^j (L \cap x^{-j} \mathfrak{g}[x]) \subseteq L[t, t^{-1}]
\]

(8.11)

defines a coherent sheaf of Lie algebras $\mathcal{L}$ on $X$ satisfying

\[
h^0(\mathcal{L}) = \dim(L \cap \mathfrak{g}[x]) \quad \text{and} \quad h^1(\mathcal{L}) = \dim(\mathfrak{g}([x])/(\mathfrak{g}[x] + L)).
\]  

(8.12)

As $\mathcal{O}_X$-module, $\mathcal{L}$ is simply the the sheaf associated to a graded module on a projective scheme in e.g. [14] Section II.5. There is a natural isomorphism $\zeta: \hat{\mathcal{L}}_p \to \mathfrak{g}[x]$ such that the extension $(\hat{\mathcal{O}}_{X,p} \setminus \{ 0 \})^{-1} \hat{\mathcal{L}}_p \to \mathfrak{g}(x)$ has the property $\zeta(\Gamma(X \setminus \{ p \}, \mathcal{L})) = L$. Let us write

\[
\mathcal{G}(O, L) := ((X, \mathcal{L}), (p, c, \zeta))
\]  

(8.13)

for the geometric datum constructed above.

8.4. Some results on multipliers. Take an $i \in \{ 1, 2, 3 \}$ and consider a Lagrangian subalgebra

\[
W \subseteq \mathfrak{g} \otimes A(i - 1, 0) = \mathfrak{g}([x]) \times \mathfrak{g}[x]/x^{i-1}[x]
\]

(8.14)

complementary to $i(\mathfrak{g}[x])$ and let $W_+ \subseteq \mathfrak{g}([x])$ be the projection of $W$ on $\mathfrak{g}([x])$.

(1) The integral closure $N$ of $M := M_{W_+} := \{ f \in F([x]) \mid fW_+ \subseteq W_+ \}$ satisfies

\[
g := \dim(F([x])/(F[x] + N)) \in \{ 0, 1 \};
\]  

(8.15)

(2) When $g = 1$ we have $M = N$ and $i = 1$;

(3) If $g = 0$ and $i = 1$ then $F[s', s'] \subseteq N$ for some $s \in x^{-1}F[x]^\times$. Here $s'$ stands for the formal derivative of $s$;

(4) For $i \in \{ 2, 3 \}$ we have $N = M = F[x^{-1}]$.

Proof. The statements from Section 8.3 imply that $M$ has Krull dimension one and $W_+$ is finitely-generated over $M$. The canonical inclusion $M \to N$ is finite (see e.g. [14] Chapter I, Theorem 3.9A1) and $(M \setminus \{ 0 \})^{-1}M = (M \setminus \{ 0 \})^{-1}N$, so $N$ is a finitely-generated torsion-free $M$-module of rank one. Therefore, $N/M$ is a torsion module over $M$ and $N/M$ is finite-dimensional since $M$ has Krull dimension one. The subalgebra $L := NW_+ \subseteq \mathfrak{g}([x])$ is now also a finitely-generated $M$-module of the same rank as $W_+$. Therefore, using the same argument $L/W_+$ is finite-dimensional. By (2) of Section 8.1 the subalgebra $W_+$ is a $\mathfrak{g}$-lattice and hence so is $L := NW_+ \subseteq \mathfrak{g}([x])$. Let
\( \mathbb{G}(N, L) := ((X, \mathcal{L}), (p, c, \zeta)) \) be the geometrization described in Section 8.3. Combining Eq. (8.12) with \( g(x) = g[x] + W_+ \subseteq \mathbb{g}[x] + L \) we get \( H^1(\mathcal{L}) = 0 \).

The Killing form \( K : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) induces a morphism \( K^* : \mathcal{L} \to \mathcal{L}^* \) (where \( \mathcal{L}^* \) is the dual as \( \mathcal{O}_X \)-module). The fiber \( K^*[p] \) is an isomorphism because \( K^*|_p \) can be identified with the Killing form of \( \mathcal{L}|_p \cong \mathbb{g} \). Therefore, \( \text{Ker}(K^*) \) and \( \text{Cok}(K^*) \) are torsion sheaves. In particular, \( H^1(\text{Cok}(K^*)) = 0 \) and \( \text{Ker}(K^*) \) vanishes, since it is a torsion subsheaf of the locally free sheaf \( \mathcal{L} \). This results in the short exact sequence

\[
0 \to \mathcal{L} \xrightarrow{K^*} \mathcal{L}^* \to \text{Cok}(K^*) \to 0.
\]

The associated long exact sequence in cohomology reads

\[
0 \to H^0(\mathcal{L}) \to H^0(\mathcal{L}^*) \to H^0(\text{Cok}(K^*)) \to H^1(\mathcal{L}) \to H^1(\mathcal{L}^*) \to H^1(\text{Cok}(K^*)) \to 0.
\]

The identities \( H^1(\mathcal{L}) = 0 = H^1(\text{Cok}(K^*)) \) imply that \( H^1(\mathcal{L}^*) = 0 \).

The Riemann-Roch theorem for \( \mathcal{L} \) and \( \mathcal{L}^* \) (e.g. in the version of [18, Chapter 7, Exercise 3.3]) combined with the fact that \( h^1(\mathcal{O}_X) = g \) implies

\[
0 \leq h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\text{det}(\mathcal{L})) + (1 - g)\text{rank}(\mathcal{L}),
\]

\[
0 \leq h^0(\mathcal{L}^*) - h^1(\mathcal{L}^*) = -\deg(\text{det}(\mathcal{L})) + (1 - g)\text{rank}(\mathcal{L}),
\]

where we used that \( \text{det}(\mathcal{L}^*) = \text{det}(\mathcal{L})^* \) implies \( \deg(\text{det}(\mathcal{L}^*)) = -\deg(\text{det}(\mathcal{L})) \). We conclude \( g \leq 1 \).

Assume \( g = 1 \), then \( X \) is an elliptic curve. Let \( \Omega_X^1 \) be the sheaf of regular 1-forms on \( X \). We have \( H^1(\Omega_X^1) = F\eta \) for some global 1-form \( \eta \) on \( X \) and this choice defines an isomorphism \( \Omega_X^1 \cong \mathcal{O}_X \). Serre duality (see e.g. [14, Chapter III, Corollary 7.7]) provides \( 0 = h^1(\mathcal{L}^*) = h^0(\mathcal{L}) \).

In particular, by Eq. (8.12)

\[
W_+ \cap g[x] \subseteq L \cap g[x] = \{0\},
\]

so Section 8.1.(4) implies \( i = 1 \). Moreover,

\[
W_+ + g[x] = g((x)) = L + g[x] \quad \text{and} \quad W_+ \subseteq L \implies L = W_+, \quad \text{so} \quad M = N.
\]

Assume \( g = 0 \), i.e. \( F((x)) = F[x] + N \). Since \( N \cap F[x] = H^0(\mathcal{O}_X) = F \), we can see that \( N = F[s] \) for the unique \( s \in (x^{-1} + xF[x]) \cap N \neq \{0\} \). The Killing form \( \kappa \) of \( g((x)) \) as \( F((x)) \)-Lie algebra restricts to the Killing form \( L \times L \to N \) of the locally free \( N \)-Lie algebra \( L \). Let \( N^\perp \) be the orthogonal complement of \( N \) with respect to the bilinear form \( R : F((x)) \times F((x)) \to F \) defined by

\[
(f, g) \mapsto \text{res}_{x=0}\{x^{-i}f g\} = \text{coeff}_{i-2}\{f g\}.
\]

For all \( f \in N^\perp \) and \( a, b \in W_+ \) we have

\[
K_1(fa, b) = \text{res}_{x=0}\{x^{-i}f(x)\kappa(a(x), b(x))\} = R(f(x), \kappa(a(x), b(x))) = 0.
\]

In particular, \( fa \in W_+^\perp \subseteq W_+ \). Therefore, \( N^\perp \subseteq \{f \in F((x)) \mid fW_+ \subseteq W_+\} = M \subseteq N \).

From

\[
R(x^{-1} s', s^k) = \text{res}_{x=0}\{s'(x)s^k(x)\} = \frac{1}{k+1} \text{res}_{x=0}\{s^{k+1}(x)\}' = 0
\]

for all \( k \in \mathbb{Z}_{\geq 0} \), we can deduce that \( x^{-1} s' \in N^\perp \subseteq M \).

**Case** \( i = 1 \). Since \( R \) is associative and \( s' \in N^\perp \) we have the inclusion \( s'N \subseteq N^\perp \). Furthermore, since \( s' \in N^\perp \subseteq N = F[s] \), we obtain \( F[s', s'] \subseteq F + s'N \). Combining these results we get

\[
F[s', s'] \subseteq F + s'N \subseteq F + N^\perp \subseteq \{f \in F((z)) \mid fW_+ \subseteq W_+\} = M.
\]

**Case** \( i = 2 \). In this case we have \( xs' \in N^\perp \subseteq F[s] \). Since we assumed that \( s \) has no constant part the following equality must hold \( xs' = -s \). By coefficient comparison we obtain \( s = x^{-1} \). Therefore, \( N = F[x^{-1}] \) and \( x^{-1} F[x^{-1}] = N^\perp \subseteq M \) implying \( M = N = F[x^{-1}] \).

**Case** \( i = 3 \). The fact that \( x^2s' \in N^\perp \cap F[x] \subseteq N \cap F[x] = F \) implies that \( s' = -x^{-2} \). Consequently, \( N = F[x^{-1}] = N^\perp \subseteq M \) concluding the proof. □
8.5. Some results on sheaves of Lie algebras.

(1) Let $\mathcal{A}$ be a locally free sheaf of Lie algebras on an algebraic variety $X$ and $\mathcal{A}|_p$ be finite-dimensional and semi-simple for some closed point $p \in X$. Then $\mathcal{A}|_q \cong \mathcal{A}|_p$ for all closed points $q$ in some open neighbourhood of $p$.

(2) Let $\mathcal{A}$ be a sheaf of Lie algebras on an irreducible cubic plane curve $X$ such that:

(a) $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$;
(b) $\mathcal{A}|_p \cong \mathfrak{g}$ for some smooth point $p \in X$;
(c) There is a bilinear form $K: \mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ which extends the Killing form of $\mathcal{A}|_C$, where $C \subseteq X$ is the set of smooth points on $X$.

Then $\mathcal{A}|_p \cong \mathfrak{g}$ for all smooth closed points $p \in X$.

(3) Let $A$ be a $F[x]$-Lie algebra such that $A/(x-a)A \cong \mathfrak{g}$ for all $a \in F$. Then $A \cong \mathfrak{g}[x]$ as $F[x]$-Lie algebras.

Proof. We can assume that $X$ is an affine variety and that $\mathcal{A}$ is free of rank $n$. Let $X(F) \subseteq X$ be the affine algebraic set of closed points (which coincide with the set of $F$-rational points, since $F$ is algebraically closed) and $B \subseteq \text{Hom}_{F}(F^n \otimes F^n, F^n)$ be the affine algebraic subset of all possible Lie brackets on $F^n$. Then $\Gamma(X, \mathcal{A})$ can be identified with the $\Gamma(X, \mathcal{O}_X)$-module of all regular maps $X(F) \to F^n$ equipped with the Lie bracket $\mu_{A}$ defined by a regular map $\theta: X(F) \to B$ via $\mu_{A}(a \otimes b)(q) = \theta(q)(a(q) \otimes b(q))$ for all $a, b: X(F) \to F^n$ regular and $q \in X(F)$. The group $G = \text{GL}(n, F)$ acts on $B$ via

$$\left(L \cdot \vartheta\right)(v \otimes w) = L^{-1} \vartheta(Lv \otimes Lw) \quad \forall L \in G, \vartheta \in M, v, w \in F^n. \tag{8.22}$$

The orbit $G \cdot \theta(p)$ coincides with the set of Lie brackets on $F^n$ determining Lie algebra structures isomorphic to $\mathcal{A}|_p = (F^n, \theta(p))$. Combining [22, Theorem 7.2] with Whitehead’s Lemma and the fact that $\mathcal{A}|_p$ is semi-simple, we see that $G \cdot \theta(p) \subseteq B$ is open and as a consequence $U := \theta^{-1}(G \cdot \theta(p)) \subseteq X(F)$ is an open neighbourhood of $p$. The observation $\mathcal{A}|_q \cong (F^n, \theta(p)) = \mathcal{A}|_p$ for all $q \in U$ concludes the proof of Part (1).

Let us turn to Part (2). It is well-known that the dualizing sheaf of any irreducible cubic plane curve is trivial. Therefore, Serre duality implies that $H^0(\mathcal{A}^*) = H^1(\mathcal{A}) = 0$. Similar to the proof of Lemma 8.4, the condition (b) implies that there is an exact sequence

$$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{A}^*) \to H^0(\text{Cok}(K^n)) \to H^1(\mathcal{A}) \to H^1(\mathcal{A}^*) \to H^1(\text{Cok}(K^n)) \to 0. \tag{8.23}$$

where $K^n: \mathcal{A} \to \mathcal{A}^*$ is the morphism induced by $K$ and $\text{Cok}(K^n)$ is torsion. Consequently, Serre duality and condition (a) gives $0 = H^1(\mathcal{A}) = H^0(\mathcal{A}^*)$, so $H^0(\text{Cok}(K^n)) = 0$. Since $\text{Cok}(K^n)$ is a torsion sheaf, we see that $K^n: \mathcal{A} \to \mathcal{A}^*$ is an isomorphism. Therefore, the fiber of $K$ at any smooth point $q \in X$, which coincides with the Killing form of $\mathcal{A}|_q$ by assumption, is non-degenerate. By virtue of Cartan’s criterion $\mathcal{A}|_q$ is semi-simple. Using (1) and the fact that the set of smooth points $C$ is connected, we are left with $\mathcal{A}|_q \cong \mathcal{A}|_p \cong \mathfrak{g}$ for all closed points $q \in C$.

Part (3) is a simple reformulation of [1, Theorem 4.12.(2)].

8.6. Twisting class $\mathfrak{g} \otimes A(0, 0)$. Let $W = W_{+} \subseteq \mathfrak{g} \otimes A(0, 0) = \mathfrak{g}((x))$ be a Lagrangian subalgebra complementary to $\mathfrak{g}[x]$. The proof of [Theorem 7.1] given in [1] proceeds in the following steps:

(1) According to (2) and (3) of Section 8.4 there is a unital subalgebra $O \subseteq M_W$ such that

$$h^1(\mathcal{O}_X) = \dim(F((x))/(F[x] + O)) = 1, \tag{8.24}$$

where $X$ is taken from the geometrization $G(O, W) = ([X, \mathcal{A}], (p, c, \zeta))$. It is easy to see that $X$ is an irreducible cubic plane curve (see [1, Remark 3.9]), i.e. it is defined by an equation of the form $v^2 = u^3 + au + b$ for some $a, b \in F$. Moreover, $X$ is smooth if $4a^3 + 27b^2 \neq 0$, in
which case its an elliptic curve, and has a unique singularity \( s \) otherwise. This singularity is nodal if \( 0 \neq 4a^3 = -27b^2 \) and cuspidal otherwise;

(2) The fact that \( W \subseteq \mathfrak{g}(x) \) is a Lagrangian subalgebra with respect to \( B_1 = K_1 \) satisfying \( g[x] + W = \mathfrak{g}(x) \) implies that \( A \) satisfies the conditions (a)-(c) in Section 8.5(2) (see [1], Subsection 4.2). Therefore, \( A_q \cong \mathfrak{g} \) for all smooth closed points \( q \in X \). This can be used to describe \( A|_C \) explicitly in all three cases, where \( C \subseteq X \) is the set of smooth points; see [1], Subsection 4.2. For instance, if \( X \) is cuspidal, \( C \) is isomorphic to the affine line \( \mathbb{A}^1 = \text{Spec}(F[x]) \) and then \( \Gamma(C,A) \cong \mathfrak{g}[x] \) according to Section 8.5(3). The explicit description in the elliptic case was thereby only achieved for \( F = \mathbb{C} \), that is why the base field is restricted to the field of complex numbers;

(3) In [2], the authors construct a distinguished section \( \rho \in \Gamma(C \times C \setminus \Delta, \mathcal{A} \boxtimes \mathcal{A}) \) called geometric \( r \)-matrix. This section has the initial formal \( r \)-matrix \( r \) as Taylor series; see [1], Theorem 3.17. Combined with the explicit description of \( A|_C \) in (2), this implies that \( r \) is elliptic (resp. trigonometric, resp. rational) in the sense of Theorem 7.1 if and only if \( X \) is smooth (resp. nodal, resp. cuspidal); see [1], Subsection 4.3 for details.

8.7. Twisting class \( \mathfrak{g} \otimes A(1,0) \). Fix a Lagrangian Lie subalgebra

\[
W \subseteq \mathfrak{g} \otimes A(1,0) = \mathfrak{g}(x) \times \mathfrak{g},
\]

such that \( \iota(\mathfrak{g}[x]) + W = \mathfrak{g}(x) \times \mathfrak{g} \) and let \( W_{\pm} \) be its projection on the first and second components respectively. Then

\[
((Y,W),(p,c,\zeta)) := \mathcal{G}(M_{W_{\pm}},W_{\pm})
\]

satisfies \( Y = P^1 \) because of Section 8.4.4. Put \( s := p \) and \( s_+ \in P^1 \) be the point corresponding to the ideal \( (x^{-1}) \subseteq F[x^{-1}] \) via \( c(\Gamma(P^1 \setminus \{s_-,\mathcal{O}_X\}) = F[x^{-1}] \).

**Lemma 8.1.** Let \( \mathcal{W} \) be defined as above. Let the sheaf of Lie algebras \( \mathcal{V} \) be defined by the pull-back diagram

\[
\begin{array}{ccc}
\mathcal{V} & \rightarrow & W_{-} \\
\downarrow & & \downarrow \\
\mathcal{W} & \rightarrow & \mathcal{W}|_{s_{-}} \cong \mathfrak{g}
\end{array}
\]

where \( \mathfrak{g} \), \( W_{-} \), and \( \mathcal{W}|_{s_{-}} \) are understood as skyscraper sheaves at \( s_{-} \). Then the following is true:

1. \( \mathcal{V} \) can be identified with a subsheaf of \( \mathcal{W} \);
2. \( H^0(\mathcal{V}) \cong \iota(\mathfrak{g}[x]) \cap (W_{+} \times W_{-}) \), \( H^0(\mathcal{V}) = 0 \) and \( \mathcal{V}|_{P^1 \setminus \{s_{-}\}} = \mathcal{W}|_{P^1 \setminus \{s_{-}\}} \);
3. Let \( \tilde{K} : V \times \mathcal{O}_{P^1} \rightarrow \mathcal{O}_{P^1} \) be the restriction of the Killing form of \( W \) to \( V \). There exist canonical surjective morphisms \( \mathcal{V}|_{s_{\pm}} \rightarrow W_{\pm}/W_{\pm}^{\perp} \) intertwining the corresponding forms if \( \mathcal{V}|_{s_{-}} \) is equipped with the bilinear form \( \tilde{K}|_{s_{\pm}} \).

**Proof.** By definition, \( \mathcal{V} \) fits into the short exact sequence

\[
0 \rightarrow \mathcal{V} \rightarrow \mathcal{W} \oplus W_{-} \rightarrow \mathfrak{g} \rightarrow 0.
\]

The morphism \( W_{-} \rightarrow \mathfrak{g} \) is injective, so \( \mathcal{V} \rightarrow \mathcal{W} \) is also injective and we can identify \( \mathcal{V} \) with a subsheaf of \( \mathcal{W} \), proving (1).

Restricting Eq. (8.28) to \( \mathcal{W}|_{P^1 \setminus \{s_{-}\}} = \mathcal{W}|_{P^1 \setminus \{s_{-}\}} \) yields \( \mathcal{V}|_{P^1 \setminus \{s_{-}\}} = \mathcal{W}|_{P^1 \setminus \{s_{-}\}} \). Since the first cohomology group of torsion sheaves vanishes \( g[x] + W_{+} = \mathfrak{g}(x) \) implies \( H^1(\mathcal{W}) = 0 \), the long exact sequence of Eq. (8.28) in cohomology reads

\[
0 \rightarrow H^0(\mathcal{V}) \rightarrow H^0(\mathcal{W}) \oplus W_{-} \rightarrow \mathfrak{g} \rightarrow H^1(\mathcal{V}) \rightarrow 0.
\]

The fact that \( H^0(\mathcal{W}) \cong \iota(\mathfrak{g}[x]) \cap (W_{+} \times W_{-}) \) gives

\[
H^0(\mathcal{V}) \cong \iota(\mathfrak{g}[x]) \cap (W_{+} \times W_{-}).
\]
The image \( \nabla^+ \) of \( g[x] \cap W_+ \) under the evaluation \( x = 0 \) has the property \( \nabla^+ + W_- = g \). Therefore, the map \( H^0(V) \oplus W_- \rightarrow g \) in Eq. (8.29) is surjective and hence \( H^1(V) = 0 \). This concludes the proof of (2).

Let us turn to the proof of the last statement. Section 8.3 (4) implies that \( W_+ \) is a free Lie algebra over \( F[x^{-1}] \), so \( \kappa(a, b) \in F[x^{-1}] \) for all \( a, b \in W_+ \subseteq g((x)) \). This implies that
\[
B_2(x^{-1}a, b) = \text{res}_{x \to 0} \{ x^{-2}\kappa(a, b) \} = 0
\]
for all \( a, b \in W_+ \). Therefore, \( x^{-1}W_+ \subseteq W_+^\perp \) and we have a surjective morphism

\[
\theta_+ : V|_{s_+} \cong W_+/x^{-1}W_+ \longrightarrow W_+/W_+^\perp
\]
twisting the corresponding forms. By construction of \( V \) we have a canonical morphism \( V \rightarrow W_- \) which is surjective since \( W \rightarrow W|_{s_-} \) is surjective. This morphism factors through a surjective morphism \( V|_{s_-} \rightarrow W_- \) which intertwines the forms. The latter induces the desired morphism \( \theta_- \).

Lemma 8.2. Let \( X \) be an irreducible cubic plane curve with nodal singularity \( s \) and chose the normalization \( \nu : \mathbb{P}^1 \rightarrow X \) in such a way that \( \nu^{-1}(s) = \{ s_+, s_- \} \). Let \( A \) be defined by the pull-back diagram

\[
\begin{array}{ccc}
A & \longrightarrow & W/(W_+^\perp \times W_-^\perp) \\
\downarrow & & \downarrow \\
\nu_* V & \longrightarrow & W_+/W_+^\perp \times W_-/W_-^\perp
\end{array}
\]

where \( W/(W_+^\perp \times W_-^\perp) \) and \( W_+/W_+^\perp \times W_-/W_-^\perp \) are viewed as skyscraper sheaves at \( s \in X \) and \( \theta \) is the direct image under \( \nu \) of the morphism

\[
V \longrightarrow V|_{s_+} \cong W_+/x^{-1}W_+ \longrightarrow W_+/W_+^\perp
\]
for \( \theta_\pm \) from Lemma 8.3. Then \( A|_q \cong g \) for all smooth closed points \( q \in X \).

Proof. It suffices to show that \( A \) satisfies the conditions (a)-(c) of Section 8.5 (2).

- \( A \) satisfies condition (a) of Section 8.5 (2): The long exact sequence in cohomology of

\[
0 \longrightarrow A \longrightarrow \nu_* V \oplus (W/(W_+^\perp \times W_-^\perp)) \longrightarrow W_+/W_+^\perp \times W_-/W_-^\perp \longrightarrow 0,
\]

which is the short exact sequence that defines \( A \), is given by

\[
0 \longrightarrow H^0(A) \longrightarrow H^0(V) \oplus (W/(W_+^\perp \times W_-^\perp)) \longrightarrow W_+/W_+^\perp \times W_-/W_-^\perp \longrightarrow H^1(A) \longrightarrow 0.
\]

Here, we used that the first cohomology group of torsion sheaves vanishes and \( H^1(V) = 0 \); see Lemma 8.1 (2). The canonical map \( H^0(V) \rightarrow W_+/W_+^\perp \times W_-/W_-^\perp \) thereby coincides with the inclusion

\[
\iota|_g \cap (W_+ \times W_-) \longrightarrow W_+/W_+^\perp \times W_-/W_-^\perp
\]
under the identification \( H^0(V) \cong \iota|_g \cap (W_+ \times W_-) \). Therefore, Section 8.1 (3) implies that there exists a closed point \( p \in \mathbb{P}^1 \setminus \{ s_+, s_- \} \) such that \( g \cong W|_p \cong A|_{\nu(p)} \).

- \( A \) satisfies condition (b) of Section 8.5 (2): Section 8.5 (1) and \( W|_{s_-} \cong g \) imply that there exists a closed point \( p \in \mathbb{P}^1 \setminus \{ s_+, s_- \} \) such that \( g \cong W|_p \cong A|_{\nu(p)} \).

- \( A \) satisfies condition (c) of Section 8.5 (2): Let us identify \( A \) with a subsheaf of \( \nu_* V \) and let \( K : A \times A \rightarrow \nu_* O_p \) be the restriction of \( \nu_* \bar{K} \) to \( A \), where we recall that \( \bar{K} : V \times V \rightarrow O_p \) is the restriction of the Killing form of \( V \) to \( V \). We want to see that \( K \) actually takes values in \( O_X \subseteq \nu_* O_p \). Let \( a, b \in A|_{s_-} \) and \( a_\pm, b_\pm \in W_\pm \) be representatives of the images of \( a, b \) under the canonical maps \( A|_{s_-} \rightarrow V|_{s_-} \rightarrow W_+/W_+^\perp \). Then

\[
K|_{s_-}(a, b) = (K_2(a_+, b_+), K_2(a_-, b_-)) \in F \times F \cong \nu_* O_p|_{s_-}
\]
holds, since $\theta_\pm$ is the isomorphism $\psi|_{W_\pm} \to W_\pm/W_\pm^\perp$ intertwine the forms. The definition of $A$ implies that

$$(a_+, a_-), (b_+, b_-) \in W,$$  

(8.36)

and the Lagrangian property of $W$ gives

$$0 = B_2(a_+, a_-), (b_+, b_-) = K_2(a_+, b_+) - K_2(a_-, b_-).$$  

(8.37)

We obtain $K|_{\{a, b\}} \in \{\lambda, \lambda | \lambda \in F\}$. This implies that $K$ takes values in $O_X$. The restriction of $K$ to $A \times A \to O_X$ to the set $C := X \setminus \{s\}$ of smooth points of $X$ coincides with the Killing form of $A$ since $A|C = \nu_*(\mathcal{V}|_{\mathcal{V}|\{s, s\}_s}) = \nu_*(\mathcal{W}|_{\mathcal{W}|\{s, s\}_s})$.

Lemma 8.3. There exists an $F[x]$-linear Lie algebra automorphism $\varphi: \mathfrak{g}[x] \to \mathfrak{g}[x]$ such that $\varphi(W_+) \subseteq \mathfrak{g}[x, x^{-1}]$. Furthermore, $(\varphi \times [\varphi])(W) \subseteq \mathfrak{g}((x)) \times \mathfrak{g}$ is commensurable with $W_2$.

Proof. Lemma 8.2 implies $W|_2 \cong A|\{q\} \cong \mathfrak{g}$ for all $q \in \mathbb{P}^1 \setminus \{s, s\}_s$. Combined with $W|_{s_\pm} \cong \mathfrak{g}$, this implies that $A := \mathfrak{g}(\mathbb{P}^1 \setminus \{s, s\}_s, W) \subseteq \mathfrak{g}[x]$ is a free $F[x] = c(\mathbb{P}^1 \setminus \{s\}, \mathcal{O}_X)$-Lie algebra satisfying $A/(x - a)A \cong \mathfrak{g}$ for all $a \in F$. Therefore, Section 8.3(3) provides an isomorphism $A \cong \mathfrak{g}[x]$. Completing said automorphism in the $(x)$-adic topology yields $\varphi \in Aut_{F[x]}(\mathfrak{g}[x])$ with the property $\varphi(A) = \mathfrak{g}[x]$. Since $W$ is a sheaf, we have

$$\varphi(W_+) = \varphi(\mathfrak{g}(\mathbb{P}^1 \setminus \{s, s\}_s, W)) \subseteq \varphi(\mathfrak{g}(\mathbb{P}^1 \setminus \{s, s\}_s, W)) = \varphi(A)[x^{-1}] = \mathfrak{g}[x, x^{-1}].$$  

(8.38)

Combining $\dim(\varphi(W_+) \cap \mathfrak{g}[x]) < \infty$ with $\varphi(W_+) \subseteq \mathfrak{g}[x, x^{-1}]$ results in

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq \varphi(W_+) \subseteq x^N\mathfrak{g}[x^{-1}]$$  

(8.39)

for a sufficiently large integer $N$. This implies that $(\varphi \times [\varphi])(W) \subseteq \mathfrak{g}((x)) \times \mathfrak{g}$ is commensurable with $W_2$ defined in Section 5.3.

8.8. Twisting class $\mathfrak{g} \otimes A(2, 0)$. As in the previous case we fix a Lagrangian Lie subalgebra

$$W \subseteq \mathfrak{g} \otimes A(2, 0) = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^2\mathfrak{g}[x]$$  

(8.40)

complementary to $\iota(\mathfrak{g}[x])$ with projections $W_\pm$ on the components $\mathfrak{g}((x))$ and $\mathfrak{g}[x]/x^2\mathfrak{g}[x]$ respectively.

Lemma 8.4. The following facts are true.

1. $W = W_+ \times W_-$;
2. $W_+ \cap x^2\mathfrak{g}[x] = \{0\}$, so $W_+ \cap \mathfrak{g}[x]$ can be identified with a subalgebra of $\mathfrak{g}[x]/x^2\mathfrak{g}[x]$;
3. $(W_+ \cap \mathfrak{g}[x]) + W_- = \mathfrak{g}[x]/x^2\mathfrak{g}[x]$.

Proof. For (1) observe that $\kappa(a, b) \in F[x^{-1}]$ for all $a, b \in W_+$, since $W_+$ is a free Lie algebra over $F[x^{-1}]$ by virtue of Section 8.4(4). Therefore, $x^{-2}\kappa(a, b) \in x^{-2}F[x^{-1}]$ implies

$$B_2(a, b) = \text{res}_{x=0}(x^{-2}\kappa(a, b), b(x)) = 0$$  

(8.41)

and hence $W_+ \subseteq W_+^\perp$. Together with $W_+^\perp \subseteq W_+$ we arrive at $W_+ = W_+^\perp$. Section 8.1(3) implies $W_+ = W_-^\perp$, so $W_+^\perp \times W_-^\perp \subseteq W \subseteq W_+^\perp \times W_-$ concludes the proof of (1).

The identities $\{0\} = (\mathfrak{g}[x] + W_+^\perp) = x^2\mathfrak{g}[x] \cap W_+^\perp = x^2\mathfrak{g}[x] \cap W_+$ imply (2). Part (3) now follows from (2) and $\iota(\mathfrak{g}[x]) + (W_+ \cap \mathfrak{g}[x]) = \mathfrak{g}((x)) \times x^2\mathfrak{g}[x]$.

Consider $((Y, W), (p, c, \zeta)) := G(M_{W_+}, W_+)$. As in the last section we have $Y = \mathbb{P}^1$. Let us write $0 := p$ and $\infty \in \mathbb{P}^1$ for the point corresponding to the ideal $(x^{-1})$ in $F[x^{-1}] = c(\mathbb{P}^1 \setminus \{0\}, \mathcal{O}_X)$.

Lemma 8.5. Let $X$ be an irreducible plane cubic curve with cuspidal singularity $s$ and chose the normalization $\nu: \mathbb{P}^1 \to X$ in such a way that $\nu(s) = 0$.

1. The isomorphism $\zeta: \mathcal{W}_p \to \mathfrak{g}[x]$ induces a surjective morphism $\nu_*W \to \mathfrak{g}[x]/x^2\mathfrak{g}[x]$;
(2) Let the sheaf of Lie algebras \( \mathcal{A} \) be defined by the pull-back of

\[
\begin{align*}
\mathcal{A} & \twoheadrightarrow W_- \\
\nu_* W & \twoheadrightarrow \mathfrak{g}[x]/x^2 \mathfrak{g}[x]
\end{align*}
\] (8.42)

where \( \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) and \( W_- \) are understood as skyscraper sheaves at \( s \). Then \( \mathcal{A}|_q \cong \mathfrak{g} \) for all smooth points \( q \in X \).

**Proof.** The isomorphism \( \zeta: \hat{W}_0 \to \mathfrak{g}[x] \) implies that \( \nu_* W|_s \cong \zeta(\hat{W}_0)/x^2 \zeta(\hat{W}_0) = \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \). This yields a surjective morphism \( \nu_* W \to \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \). In order to prove the second part of the statement, it suffices to show that \( \mathcal{A} \) satisfies the conditions (a)-(c) of **Section 8.3** (2).

- **\( \mathcal{A} \) satisfies condition (a) of **Section 8.3** (2):** Globally, the morphism \( \nu_* W \to \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) coincides with the canonical morphism \( \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \to \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) after identifying \( H^0(W) \) with \( \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \). Therefore, the middle arrow in the long exact sequence in cohomology

\[
0 \to H^0(\mathcal{A}) \to H^0(W) \oplus W_- \to \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \to H^1(\mathcal{A}) \to 0,
\] (8.43)

of the short exact sequence

\[
0 \to \mathcal{A} \to \nu_* W \oplus W_- \to \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \to 0,
\] (8.44)

which defines \( \mathcal{A} \), is an isomorphism by virtue of **Lemma 8.4** (3). Here we used again that the first cohomology group of torsion sheaves vanishes and that \( H^1(W) = 0 \) by virtue of **Section 8.1** (2). Consequently, \( H^0(\mathcal{A}) = 0 = H^1(\mathcal{A}) \);

- **\( \mathcal{A} \) satisfies condition (b) of **Section 8.3** (2):** Part (1) in **Section 8.5** and \( W|_0 \cong \mathfrak{g} \) imply that there exists \( p \in \mathbb{P}^1 \setminus \{0\} \) such that \( \mathfrak{g} \cong W|_p \cong \mathcal{A}|_{\nu(p)} \);

- **\( \mathcal{A} \) satisfies condition (c) of **Section 8.3** (2):** Let \( \hat{K} \) be the Killing form of \( W \). Identify \( \mathcal{A} \) with a subsheaf of \( \nu_* W \) and let \( K: \mathcal{A} \times \mathcal{A} \to \nu_* \mathcal{O}_X \) be the the restriction of \( \nu_* \hat{K} \) to \( \mathcal{A} \). For any \( a, b \in \mathcal{A}|_s \) we have

\[
K|_s(a, b) = \kappa(a_1, b_1) + [x](\kappa(a_1, b_2) + \kappa(a_2, b_1)) \in F[x]/(x^2),
\] (8.45)

where \( a_1 + [x]a_2 \) and \( b_1 + [x]b_2 \in \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) are the images of \( a \) and \( b \) respectively under \( \mathcal{A}|_s \to \nu_* W|_s \cong \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \).

By definition of \( \mathcal{A} \), \( a_1 + [x]a_2, b_1 + [x]b_2 \in W_- \) and \( \kappa(a_1, b_2) + \kappa(a_2, b_1) = 0 \) since \( W_- \subseteq \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) is Lagrangian. Therefore, \( K|_s(a, b) = \kappa(a_1, b_1) \in F \), implying that \( K \) takes values \( \mathcal{O}_X \subseteq \nu_* \mathcal{O}_X \).

The restriction of the bilinear form \( K: \mathcal{A} \times \mathcal{A} \to \mathcal{O}_X \) to the set \( C := X \setminus \{s\} \) of smooth points of \( X \) coincides with the Killing form of \( \mathcal{A}|_C \) since \( \mathcal{A}|_C = \nu_* (W|_{\mathbb{P}^1 \setminus \{0\}}) \).

The proof of the following statement is completely analogous to the proof of **Lemma 8.3**

**Lemma 8.6.** There exists an \( F[x] \)-linear Lie algebra automorphism \( \varphi: \mathfrak{g}[x] \to \mathfrak{g}[x] \) such that \( \varphi(W_+) \subseteq \mathfrak{g}[x, x^{-1}] \). Furthermore, \( (\varphi \times [\varphi])|_W \subseteq \mathfrak{g}(x) \times \mathfrak{g}[x]/x^2 \mathfrak{g}[x] \) is commensurable with \( W_3 \).
Appendix A. Different types of equivalence

Now we give a brief description of different equivalence notions used in this paper. We denote by \( F \) an algebraically closed field of characteristic 0.

| Equivalence | Notation | Description |
|-------------|----------|-------------|
| Isomorphism of two topological Lie bialgebra structures | \((L, \delta_1) \cong (L, \delta_2)\) | There exists \( \varphi \in \text{Aut}_{F-LieAlg}(L) \) such that \( \varphi \) and its dual \( \varphi' \) are homeomorphisms and \((\varphi \otimes \varphi')\delta_1 = \delta_2\delta'\). |
| Equivalence of two topological Lie bialgebra structures | \((L, \delta_1) \sim (L, \delta_2)\) | There exists a constant \( \xi \in F^\times \) such that \((L, \xi\delta_1) \cong (L, \delta_2)\). |
| Isomorphism of two topological Manin triples | \((L, L_+, L_-) \cong (M_+, M_-)\) | There exists a Lie algebra isomorphism \( \varphi : L \to M \) such that \( \varphi \) is a homeomorphism, \( \varphi(L_\pm) = M_\pm \) and it intertwines the corresponding forms. |
| Equivalence of two topological Manin triples | \((L, L_+, L_-) \sim (M_+, M_-)\) | There exists a constant \( \xi \in F^\times \) such that the Manin triple \((L_\pm, L_-) \) with form \( \xi B_L \) is isomorphic to \((M_\pm, M_-) \) with form \( B_M \) and it intertwines the corresponding forms. |
| Equivalence of two trace extensions of \( F[x] \) | \((A_1, t_1) \sim (A_2, t_2)\) | There exists an algebra isomorphism \( T : A_1 \to A_2 \), identical on \( F[x] \), and a constant \( \xi \in F^\times \) such that \( t_2 \circ T = \xi t_1 \). |

Lie bialgebra isomorphisms of \( g[x] \), given by \( x \mapsto a_1 x + a_2 x^2 + \ldots \) with \( a_i \in F \) and \( a_1 \neq 0 \), are called coordinate transformations. We show in [Theorem 3.3] that any Lie algebra automorphism of \( g[x] \) decomposes uniquely into a coordinate transformation and an \( F[x] \)-linear Lie algebra automorphism of \( g[x] \).

In general, scaling and coordinate transformations do not preserve the topological double of a topological Lie bialgebra: they change the corresponding form. These equivalences are used to place each topological Lie bialgebra structure into a particular topological double. After that we work primarily with formal isomorphisms, i.e. \( F[x] \)-linear Lie algebra automorphisms of \( g[x] \) because they leave topological doubles invariant.

Topological twists – topological analogue of classical twists – are introduced in [Section 3]. This notion allows to reduce the classification of topological Lie bialgebra structures to the classification of twists within certain topological doubles. We call two topological twists \( s_1, s_2 \in (g \otimes g)[x, y] \) of \( \delta \) formally isomorphic if the corresponding Lie bialgebra structures \( \delta + ds_1 \) and \( \delta + ds_2 \) are formally isomorphic.

In [Section 3.4] we explain that there are only three non-trivial topological doubles of topological Lie bialgebra structures on \( g[x] \). They are denoted by \( \mathcal{D}(g[x], \delta_i), i \in \{1, 2, 3\} \). Each topological twist of \( \delta_i \) is completely determined by a certain Lagrangian Lie subalgebra of \( \mathcal{D}(g[x], \delta_i) \). Conversely, every Lagrangian Lie subalgebra of \( \mathcal{D}(g[x], \delta_i) \), complementary to \( g[x] \), defines a topological twist of \( \delta_i \). Two Lagrangian Lie subalgebras \( W_1, W_2 \subseteq \mathcal{D}(g[x], \delta_i) \) are said to be formally isomorphic if there is an \( F[x] \)-linear automorphism \( \phi \) of \( g[x] \) such that

\[
W_2 = (\phi \times [\phi])(W_1).
\]

Another important notion tightly related to topological twists is the notion of a formal \( r \)-matrix; see [Section 3]. Again, each topological twist of \( \delta_i \) gives rise to a formal \( r \)-matrix \( -r_i + t \) and conversely, any formal \( r \)-matrix of the form \( -r_i + t \) defines a topological twist of \( \delta_i \). Two
formal $r$-matrices $r_1$ and $r_2$ are called \textit{formally} (resp. \textit{polynomially}) \textit{gauge equivalent} if there is an element $\varphi$ in $\text{Aut}_{F[x]}\text{-LieAlg}(\mathfrak{g}[x])$ (resp. in $\text{Aut}_{F[x]}\text{-LieAlg}(\mathfrak{g}[x])$) such that

$$(\varphi(x) \otimes \varphi(y))r_1(x, y) = r_2(x, y).$$

Similarly, classical $r$-matrices $r_1$ and $r_2$ are called \textit{holomorphically gauge equivalent} if there is a holomorphic function $\varphi: U \subseteq \mathbb{C} \to \text{Aut}(\mathfrak{g})$ such that $(\varphi(x) \otimes \varphi(y))r_1(x, y) = r_2(x, y)$.

The statement of \textbf{Theorem 5.9} tells us that the relations between the three objects mentioned above preserve equivalences.
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