A graphical method in quantum optics

Stefan Ataman

Extreme Light Infrastructure—Nuclear Physics (ELI-NP), IFIN-HH, 30 Reactorului Street, 077125 Măgurele, Romania
E-mail: stefan.ataman@eli-np.ro

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Abstract
In this paper we describe in detail a graphical method allowing the computation of field operator transformations in quantum optics (QO). Its applications include beam splitters (BS), Mach-Zehnder interferometers (MZI), optical resonators (Fabry–Perot etc) as well as non-linear crystals featuring the process of spontaneous parametric down-conversion (SPDC). Its main advantage compared to the traditional computation step-by-step method is its visual and intuitive approach, somehow similar to Feynman’s diagrammatic approach in Quantum Field Theory. It also seems adapted to computer-based implementations since calculations mainly consist on complex additions and multiplications, not matrix operations.

1. Introduction

The quantum optical description [1–3] of optical devices and systems allows one to compute the transformation of classical and non-classical states of light.

Linear passive optical devices feature, among others, the beam splitter (BS). In the quantum-optical description [2–5], field operators replace classical fields. Beam splitters have been extensively discussed [6–9], sometimes in a rather mathematical fashion [6, 7]. A Mach-Zehnder interferometer (MZI) is a device composed of two beam splitters and two mirrors [2–4]. Its popularity has led to its use in countless important experiments [10–13].

Sometimes optical devices contain resonators, the most widely used being the Fabry–Perot cavity [14, 15]. It can be described classically or quantum mechanically. Although resonators can change very abruptly their input-output characteristics, they are linear optical devices.

Following Burnham and Weinberg’s [16] introduction of non-linear crystals featuring SPDC, Mandel’s group set a string of experiments that revolutionized the field of quantum optics [17–19]. Ou, Wang, Zou and Mandel (OWZM) proposed [20] an experimental setup with two non-linear crystals. What they suggested, and then experimentally confirmed [18], was a ‘phase memory’ if two crystals are pumped by the same laser.

Zou, Wang and Mandel (ZWM) [19] used the same two-crystal setup but made the idler mode of the first crystal pass through the second one, making them indistinguishable. A ‘mind boggling’ experiment resulted, that puzzled many physicists. Two decades later, the same ZWM experimental principle triggered a new field of research: quantum imaging [21–24].

The computation of input-output operator relations for optical devices comprising linear optical components is done through a cascade of successive operator (i.e. matrix) transformations. The complexity of these operations rapidly grows with the number of inputs/outputs as well as the complexity of system, obscuring the physics at work behind the used models. For example a single Mach-Zehnder interferometer poses no particular calculation problems, however a double Mach-Zehnder [25] or two nested Mach-Zehnder configurations [26, 27] demand longer and tedious calculations.

The situation gets worse for two or more non-linear crystals. Using a complete Hamiltonian approach rapidly evolves into painfully long calculations [20]. Authors typically use the monochromatic approximation and some arguments to cut down on the computational complexity.
Feynman diagrams [28] are a graphical way to depict various processes in the Quantum Field Theory. By following Stückeberg’s suggestion that positrons can be seen as positive electrons moving backwards in time, Feynman was able to depict graphically all important processes in QED (Quantum Electro-Dynamics). Schwinger is famously quoted for saying ‘the Feynman diagram was bringing computation to the masses’ [29].

The goal of this paper is more modest. It aims to describe a graphical method in quantum optics that can bring computations—not to the masses—but mainly to graduate students and researchers in this field.

Reference [30] introduced a graphical method for linear optical devices (beam splitters and interferometers). Both the monochromatic and the non-monochromatic cases were considered. The same method was used in [31] in order to make a quantum optical characterization of a Fabry–Perot cavity and to predict a HOM antibunching effect. Recently, Alsing et al [32], employed the graphical method to characterize losses in ring resonators.

Further extensions of the graphical method to non-linear optics was done in reference [33] where it was used to describe in a more visual manner three experiments involving non-linear crystals. The same method was employed in [23] in order describe at an acceptable level of computational complexity an experimental setup with three non-linear crystals.

This article puts together and refines the rules of the graphical method in a single paper with emphasis on a pedagogic and clear style. Some clarifications are made. Focus is put mainly on the monochromatic case, with a single section briefly discussing the non-monochromatic case.

This paper is organized as follows. The considered problem is theoretically stated in section 2. The rules needed to draw the graphs for optical devices are described in section 3. Applications of this method are discussed in section 4. The case of non-monochromatic light is briefly discussed in section 5. The paper closes with a concluding discussion in section 6.

2. Field operator transformations in quantum optics

Many interesting devices in QO (see figure 1) have two input ports (Mach–Zehnder interferometer, Fabry–Perot cavity etc). We shall label them with the indexes 0 and 1. (Extension to more input ports is straightforward.) Unitarity also implies the device has two output ports, labelled N and N + 1. We assume linear and lossless optical systems. Lossy optical devices can be modelled with additional beam splitters [1] connected to fictitious input/output modes that have to be traced out in the final calculation.

If the input is in a pure state, and moreover, if we assume monochromatic light quanta, we can write the input state vector of our system as

\[ |\psi_{in}\rangle = f(\hat{a}_0^\dagger, \hat{a}_1^\dagger)|0\rangle \] (1)

where \( f \) is an operator function to be determined and \(|0\rangle\) denotes the vacuum state. For example, if we have the input Fock state \(|\psi_{in}\rangle = |0_0 k_1\rangle = \hat{a}_1^\dagger|0\rangle\) we immediately find

\[ f(\hat{a}_0^\dagger, \hat{a}_1^\dagger) = \hat{a}_1^\dagger. \] (2)

Throughout this paper, \( \hat{a}_k \) (\( \hat{a}_k^\dagger \)) denotes the annihilation (creation) operator for the mode \( k \) and \( |M_0 P_1\rangle \) denotes a Fock state having \( M \) photons in mode 0 and \( P \) photons in mode 1. A coherent state \(|\psi_{in}\rangle = |\alpha_0 \alpha_1\rangle = \hat{D}_1(\alpha_1)\hat{D}_0(\alpha_0)|0\rangle\) where \( \hat{D}_1(\alpha) = e^{\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1} \) [35] acting on input port 1 can be normally ordered yielding

\[ |\psi_{in}\rangle = f(\hat{a}_0^\dagger, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_0)|0\rangle \] because of the normally ordering theorem [34]. Since the annihilation operator at any power (except zero) acting on the vacuum state will simply vanish, we end up with equation (1).

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1 We are not compelled to write \(|\psi_{in}\rangle = f(\hat{a}_0^\dagger, \hat{a}_0, \hat{a}_1^\dagger, \hat{a}_0)|0\rangle\) because of the normally ordering theorem [34]. Since the annihilation operator at any power (except zero) acting on the vacuum state will simply vanish, we end up with equation (1).
If one wishes to find the output state, the fact that an input vacuum state transforms into an output vacuum state (i.e. $|0_{in}\rangle \rightarrow |0_{out}\rangle$) can always be used, no matter how complicated the system is. Therefore, if we find the operator functions $g_0$ and $g_1$ so that

$$\hat{a}_0 = g_0(\hat{a}_N, \hat{a}_{N+1}^\dagger),$$

$$\hat{a}_1^\dagger = g_1(\hat{a}_N^\dagger, \hat{a}_{N+1}^\dagger),$$

then, at least formally, the output state can be written as

$$|\psi_{\text{out}}\rangle = f(g_0(\hat{a}_N, \hat{a}_{N+1}^\dagger), g_1(\hat{a}_N^\dagger, \hat{a}_{N+1}^\dagger))|0\rangle$$

To summarize, the computation of the output state vector is done following the steps:

1. obtain the function $f(\hat{a}_0^\dagger, \hat{a}_1^\dagger)$ by expressing the input state vector in the form given by equation (1)
2. by using the graphical method (detailed in the following) draw the graph corresponding to the optical system under consideration
3. obtain the functions $g_0$ and $g_1$ defined in equation (4) from each output port to the considered input port
4. use equation (5) to obtain the output state vector

The central focus of this paper is point (3) from the above list of tasks.

### 3. Rules for the graphical method

#### 3.1. Rules for beam splitters and interferometers

Input-output relations for beam splitters have been discussed by many authors [36–39]. For a lossless beam splitter, the output annihilation operators ($\hat{a}_0$ and $\hat{a}_1$) can be written in respect with the input field operators [4] as

$$\begin{cases}
\hat{a}_2 = R\hat{a}_0 + T\hat{a}_1 \\
\hat{a}_3 = T\hat{a}_0 + R\hat{a}_1 
\end{cases}$$

where $R$ ($T$) represent the reflection (transmission) coefficients (see figure 2) of the BS. The input field operators obey the usual commutation relations $[\hat{a}_i, \hat{a}_k] = [\hat{a}_i^\dagger, \hat{a}_k^\dagger] = 0$ and $[\hat{a}_i, \hat{a}_k^\dagger] = \delta_{ik}$ where $\delta_{ik}$ is the Kronecker delta and $i, k = 0, 1, 3$. Imposing the same commutation relations to the output field operators yields the constraints

$$|T|^2 + |R|^2 = 1$$

and

$$RT^* + TR^* = 0$$

We discussed here the so-called ‘symmetrical’ beam splitter [3, 4] and it is easy to see that at least $R$ has to be complex in order to satisfy equation (8). From equation (6) one can also obtain the ‘inverse’ relations involving

2. The other popular choice is given by the ‘cubes’ beam splitter [2, 3] where we have

$$\begin{cases}
\hat{a}_2 = R\hat{a}_1 + T\hat{a}_0 \\
\hat{a}_3 = T\hat{a}_1 - R\hat{a}_0 
\end{cases}$$

and this time both $T$ and $R$ can be real.
creation operators,

\[
\begin{align*}
\hat{a}_0^+ &= T\hat{a}_0^0 + R\hat{a}_1^0 \\
\hat{a}_1^+ &= R\hat{a}_1^0 + T\hat{a}_0^0
\end{align*}
\] (10)

From equations (10) it becomes obvious that the coefficients \( T \) and \( R \) represent amplitudes that connect the input to output ports. This suggests the graphical representation from figure 2.

Another building block needed for linear optics is the delay line (see figure 3, left drawing). Since a global phase factor is irrelevant, we can model this delay as a single phase shift \( e^{i\varphi} \) (see figure 3, right graph) where \( \varphi \) is the phase difference caused by the two paths, i.e. \( \varphi = \varphi_3 - \varphi_2' - \varphi_1 - \varphi_2 \). Since we are moving ‘backwards in time’, the factor we have in the graphical representation is \( e^{-i\varphi} \), as depicted in figure 3.

3.2. Rules for optical resonators

Optical resonators imply one or more loops in at least one of the input-output paths. This is depicted figure 4, together with the (generally complex) amplitudes for each \( ++ \) (\( A_L \) the amplitude on the left side of the loop etc).

The path from, say, port \( N \) to port 0, containing at least one vertex of the loop can thus bring—in a first approximation—a contribution \( A_L A_{EB} A_R \) (see figure 4). However, there is a second contribution by making a complete loop \( B \rightarrow C \rightarrow D \rightarrow E \rightarrow B \), yielding a supplementary factor of \( A_{loop} = A_{EB} A_{BC} A_{CD} A_{DE} \). We could make a new contribution by taking the loop \( B \rightarrow C \rightarrow D \rightarrow E \rightarrow B \) twice bringing in the extra factor \( A_{loop}^2 \) and so on. Summing over all possible number of complete loops, one ends up with a factor \( A_L A_{EB} (1 + A_{loop} + A_{loop}^2 + \ldots) A_R \). This is a trivial geometric series, therefore the field operator transformation is

\[
\hat{a}_0^+ = \frac{A_{N-1}}{1 - A_{loop}} \hat{a}_N^+ + \frac{A_{N+1}}{1 - A_{loop}} \hat{a}_{N+1}^+
\] (11)

where in our particular case (see figure 4) the amplitude from ports \( N \rightarrow (N + 1) \rightarrow 0 \) is \( A_{N-1} = A_R A_{EB} A_L \) (\( A_{N+1} = A_{EB} A_{DE} A_{EB} A_L \)).

In section 4.2, a Fabry–Perot (FP) cavity (also called Fabry–Perot resonator) is worked out in detail using the graphical method.

3 Of course, from a mathematical point of view, equation (11) can be ill-defined. In passive linear optical circuits this is never the case since \( |A_{loop}| < 1 \). If one finds \( |A_{loop}| > 1 \) this implies an active element, i.e. there must be an amplifier inside the loop.
3.3. Rules for non-linear optics

The non-linear process of spontaneous parametric down-conversion (SPDC) is undoubtedly the workhorse in generating non-classical states of light for Quantum Optics and Quantum Information Processing (QIP).

Mainly two types of SPDC processes [40–43] are actually used in practice. We shall focus here on what is called Type I SPDC, where photons emerging from the non-linear crystal have both the same polarization, a degree of freedom we shall ignore in the following. Extensions of the graphical method to encompass the polarization degree of freedom, too, are straightforward.

The SPDC (see figure 5) is a non-linear process that produces two photons, typically called signal (s) and idler (i) from one pump photon (p). Energy (E_n = hω_n) and momentum (p_n = hk_n) conservation relations (n ∈ {p, s, i}) impose ω_p = ω_i + ω_s and k_p = k_i + k_s (also called 'phase matching' conditions). In this paper, a very simplified model of the SPDC process will be employed, where this process converts a monochromatic input photon from the pump field into two (equally monochromatic) photons in the signal, and, respectively, idler outputs. From the QO perspective, the operator transformation performed during SPDC is

\[ \hat{a}_p^\dagger \rightarrow \gamma \hat{a}_s^\dagger \hat{a}_i^\dagger \]  \hspace{1cm} (12)

where \( \gamma \) (typically \( \gamma \sim 10^{-6} \)) is a parameter connected to the \( \chi^{(2)} \) non-linearity of the optical medium and to the pumping power of the crystal [44–46]. In scenarios where identical crystals are pumped at equal powers the value of \( \gamma \) is not important.

From a practical point of view, the input state is a strong pump laser and the signal and idler modes are in the vacuum state. In appendix B we give a justification for the bold simplifications we intend to use. The main point is: since we post-select only events where a down-conversion actually takes place, we can write the input state as

\[ |\psi_{in}\rangle = |1_p\rangle = \hat{a}_p^\dagger |0\rangle \]  \hspace{1cm} (13)

therefore, indeed, in our case \( f(\hat{a}_p^\dagger) = \hat{a}_s^\dagger \), a situation that will ease calculations for two reasons: first, there are no powers of \( \hat{a}_p^\dagger \) and second, there is a single input port. Ignoring for a moment the parameter \( \gamma \), we have the function \( g = \hat{a}_s^\dagger \hat{a}_i^\dagger \) and the output state vector after the non-linear crystal is

\[ |\psi_{out}\rangle = \hat{a}_s^\dagger \hat{a}_i^\dagger |0\rangle = |1_s,1_i\rangle \]  \hspace{1cm} (14)

Quite often, in practical scenarios, two non-linear crystals are pumped together and the output ports are typically called s, s′ (for the signal part) and i, i′ (for the idler part). Therefore, the transformation function \( g \) will be

\[ \hat{a}_p^\dagger = g(\hat{a}_s^\dagger, \hat{a}_s'^\dagger, \hat{a}_i^\dagger, \hat{a}_i'^\dagger) \]  \hspace{1cm} (15)

Since the input state is given by equation (13), at least formally, the output state can be written as

\[ |\psi_{out}\rangle = g(\hat{a}_s^\dagger, \hat{a}_s'^\dagger, \hat{a}_i^\dagger, \hat{a}_i'^\dagger)|0\rangle \]  \hspace{1cm} (16)

There is one more important point worth to be mentioned about SPDC As we shall discuss in section 4.3, although SPDC is a spontaneous process, much like the spontaneous emission of an excited atom, coherence can be induced among non-linear crystals. This phenomenon is typically attributed to the vacuum fields [18, 47].

4. Applications of the graphical method

4.1. Mach-Zehnder interferometers

A Mach-Zehnder interferometer is depicted in figure 6. We denote the input (output) creation field operators by \( \hat{a}_p^\dagger \) and \( \hat{a}_s^\dagger \) (\( \hat{a}_i^\dagger \) and \( \hat{a}_i'^\dagger \)).

We illustrate the graphical method by describing this device. Replacing the beam splitters with the butterfly-like graphs from figure 2 and the delay according to figure 3 we end up with the graph depicted in figure 7.
There are two paths connecting $\hat{a}^\dagger_4$ to $\hat{a}^\dagger_0$. Highlighted in red in figure 7 is the first one, from $\hat{a}^\dagger_4$, via $\hat{a}^\dagger_2$ and $\hat{a}^\dagger_2$, to $\hat{a}^\dagger_0$ with an amplitude $T^2$ and no phase shift. In figure 8 the second path is highlighted in red, connecting $\hat{a}^\dagger_4$ to $\hat{a}^\dagger_0$ via $\hat{a}^\dagger_3$ and $\hat{a}^\dagger_3$ with an amplitude $R^2$ and a phase shift $\phi e^{i\varphi}$.

Another two paths connect $\hat{a}^\dagger_5$ to $\hat{a}^\dagger_0$: from $\hat{a}^\dagger_5$, via $\hat{a}^\dagger_2$ and $\hat{a}^\dagger_2$, to $\hat{a}^\dagger_0$ with an amplitude $TR$ and no phase shift and from $\hat{a}^\dagger_5$ via $\hat{a}^\dagger_3$ and $\hat{a}^\dagger_3$ to $\hat{a}^\dagger_0$ with an amplitude $TR$ and phase shift $\phi e^{i\varphi}$. Summing up all these amplitudes takes us to

$$\hat{a}^\dagger_0 = (T^2 + R^2 e^{i\phi}) \hat{a}^\dagger_4 + TR(1 + e^{i\varphi}) \hat{a}^\dagger_5$$  \hspace{0.5cm} (17)$$

We obtained straight away a result that is normally deduced after a product of three matrix multiplications. Moreover, we obtained it by the simple inspection of the graph from figure 7. Similarly, writing down the amplitudes and phase shifts of the paths connecting $\hat{a}^\dagger_1$ to $\hat{a}^\dagger_0$ and $\hat{a}^\dagger_5$ yields

$$\hat{a}^\dagger_1 = TR(1 + e^{i\varphi}) \hat{a}^\dagger_4 + (T^2 e^{i\phi} + R^2) \hat{a}^\dagger_5$$ \hspace{0.5cm} (18)$$

equations (17) and (18) yield the $g_0$ and $g_1$ operator functions from equation (4). One can compute now an output state vector for the MZI given we know the input state vector.

As an example consider the single-photon Fock state $|\psi_{in}\rangle = |0, 1\rangle = \hat{a}^\dagger_1 |0\rangle$. We have the function $f = \hat{a}^\dagger_1$ and $g_1$ is given by equation (18). According to equation (5) the output state vector is
The probability to detect a photon at the output port 4 can be easily computed yielding

\[ P_4 = 2|TR|^2(1 + \cos \varphi) \]  

and if both beam splitters are balanced we get the classical result \( P_4 = \cos^2(\varphi/2) \).

A more complicated scenario where the graphical method shows its clear advantage is discussed in appendix A.

4.2. Fabry–Perot cavities

A Fabry–Perot cavity (or resonator) is composed of two high-reflectivity mirrors facing each other at a fixed distance \( L \) (see figure 9). Each mirror is assumed to have a (low) transmission coefficient \( T \) and a (high) reflectivity coefficient \( R \). The angles are exaggerated, one has to assume that the input light is almost perpendicular to the cavity. One has two input ports (\( \mathcal{I} \) and \( \mathcal{V} \)) and two output ports (\( \mathcal{T} \) and \( \mathcal{R} \)).

If we consider the input mode \( \mathcal{I} \) (with its corresponding creation operator \( \hat{a}_I^\dagger \)) then a photon entering the cavity can reflect directly to port \( \mathcal{R} \). Or, it could enter the cavity, bounce back from the second mirror and exit at the port \( \mathcal{R} \) (see the graphical depiction from figure 10). Each crossing of a highly-reflecting mirror can be modelled by a beam splitter and each passage inside the cavity yields an extra factor of \( j e^{i\varphi} \) where \( j = kL \) and \( k \) is the wavenumber of the (monochromatic) light that illuminates the interferometer. We can immediately obtain the transformation operators by applying the rules from section 3.2. The loop \( BCDE \) (highlighted in dark red) has an amplitude \( A_{loop} = R^2 e^{i2\varphi} \). Therefore, every path having a common vertex with this loop will take an extra factor of \( 1/(1 - A_{loop}) \).

\[ |\psi_{out}\rangle = TR(1 + e^{i\varphi})|0_1 0_3\rangle + (T^2 e^{i\varphi} + R^2)|0_1 1_5\rangle \]  

Adding the paths from \( \hat{a}_R^\dagger \) to \( \hat{a}_T^\dagger \) takes us to the final operator transformation

\[ \hat{a}_I^\dagger |\tau_{\mathcal{I}}\rangle = \frac{T^2 e^{i\varphi}}{1 - R^2 e^{i2\varphi}} \hat{a}_T^\dagger \]  

Adding the paths from \( \hat{a}_R^\dagger \) to \( \hat{a}_T^\dagger \) yields

\[ \hat{a}_T^\dagger = \frac{T^2 e^{i\varphi}}{1 - R^2 e^{i2\varphi}} \hat{a}_T^\dagger + R \left( 1 + \frac{T^2 e^{i2\varphi}}{1 - R^2 e^{i2\varphi}} \right) \hat{a}_R^\dagger \]  

We now need to write \( \hat{a}_I^\dagger \) in respect with \( \hat{a}_T^\dagger \) and \( \hat{a}_R^\dagger \). One notices that from ports \( \mathcal{T} \) to \( \mathcal{V} \) one can directly arrive with a factor \( R \). There is a second path, though, involving the loop: through nodes \( E, C, D \) and finally port \( \mathcal{V} \).
We can define the Fabry–Perot transmission

\[
T_\text{fp} = \frac{T^2 e^{i\phi}}{1 - R^2 e^{i\phi}}
\]  

and reflection

\[
R_\text{fp} = R \left( 1 + \frac{T^2 e^{i\phi}}{1 - R^2 e^{i\phi}} \right)
\]

coefficients and by analyzing again equations (22) and (23) we remark that the FP cavity can be seen as a beam splitter with highly dependent transmittance/reflectivity coefficients:

\[
\begin{align*}
\hat{a}_T^+ &= T_\text{fp} \hat{a}_T^+ + R_\text{fp} \hat{a}_R^+ \\
\hat{a}_R^+ &= R_\text{fp} \hat{a}_T^+ + T_\text{fp} \hat{a}_R^+
\end{align*}
\]

In reference [31], the graphical method is employed to discuss a Fabry–Perot cavity. By applying two simultaneously impinging single photons at the inputs \(I\) and \(V\) (i.e. the input state vector is \(|\psi\rangle = |1_I 1_V\rangle\)) a HOM effect is obtained if the frequencies of the photons are carefully chosen.

4.3. Experiments with non-linear crystals

The graphical method can yield exceptionally fast answers for these configurations, as discussed in section 3.3. We depict in figure 11 the experiment performed by Ou, Wang, Zou and Mandel (OWZM) [18]. The pump laser is divided by a balanced beam splitter (BS_{p}) between two non-linear crystals \(NL_1\) and \(NL_2\) (assumed identical). Next, the signal and, respectively, idler beams from the two crystals are brought together in beam splitters (BS_{s} and, respectively, BS_{i}). The resulting signal (\(s_1, s_2\) and, respectively, idler (\(i_1, i_2\)) modes are sent to the photodetectors (\(D_s\) and, respectively, \(D_i\)). By slightly displacing BS_{p}, the two-photon coincident detection rates were measured (see figure 3 from reference [18]) and the authors report a sinusoidal variation.

Their calculations being rather involved, we try to find a simplified model using the graphical method. The building blocks are beam splitters (BS_{p}, BS_{s} and BS_{i}—assumed identical), a delay line and SPDC crystals (\(NL_1\) and \(NL_2\)). Making use of figures 2, 3 and 5 we arrive at the graph from figure 12. Since the two non-linear crystals are assumed identical and are pumped at the same power, we simply set both \(\gamma\) coefficients to 1.
What we have implicitly assumed in building this graph is that the two SPDC processes have an induced coherence\(^4\), or ‘phase memory’ as the authors dubbed it in [18]. As discussed in section 3.3, the first simplification we take is to consider the input state as
\[ |\psi_{in}\rangle = |1_p\rangle = \hat{a}_p^\dagger |0\rangle \]  
justified by the post-selection of events where a down-conversion actually takes place.

One faces now the task now to express \( \hat{a}_j^\dagger \) as a function of \( \hat{a}_i^\dagger, \hat{a}_i^\dagger, \hat{a}_j^\dagger \) and \( \hat{a}_j^\dagger \) i.e. to find the operator function from equation (15). We visually analyse figure 12 and find: the signal (red) path from NL\(_1\) has a factor \( T \) to \( \hat{a}_j^\dagger \) and a factor \( R \) to \( \hat{a}_i^\dagger \). The idler (green) path from NL\(_1\) to \( \hat{a}_i^\dagger \) takes a factor of \( T \) and to \( \hat{a}_j^\dagger \) a factor of \( R \). Taking into account that from the pump to the crystal NL\(_1\) we have an extra factor of \( 1/\sqrt{2} e^{i\phi} \) brings us to
\[ \hat{a}_j^\dagger_{\text{NL}1} = \frac{e^{i\phi}}{\sqrt{2}} (T\hat{a}_i^\dagger + R\hat{a}_j^\dagger)(\tilde{T}\hat{a}_j^\dagger + \tilde{R}\hat{a}_i^\dagger) \]  

Still by visual inspection we find the contribution from NL\(_2\) to be \( i/\sqrt{2} (T\hat{a}_i^\dagger + \tilde{T}\hat{a}_j^\dagger)(\tilde{R}\hat{a}_j^\dagger + \tilde{T}\hat{a}_i^\dagger) \). Therefore, the final result for the field operator transformation is
\[ \hat{a}_j^\dagger = \frac{1}{\sqrt{2}} (T e^{i\phi} + i R^2)\hat{a}_i^\dagger + \frac{TR}{\sqrt{2}}(e^{i\phi} + i)\hat{a}_i^\dagger \hat{a}_j^\dagger + TR e^{i\phi} + iT^2)\hat{a}_j^\dagger \hat{a}_i^\dagger \]  

In the OWZM experiment all beam splitters were balanced, therefore we take \( T = 1/\sqrt{2} \) and \( R = i/\sqrt{2} \) yielding the output state vector
\[ |\psi_{out}\rangle = \frac{1}{\sqrt{2}} (\sin(\varphi')|1,1\rangle + \cos(\varphi')|1,1\rangle) \]  

where we denoted \( \varphi' = \varphi_p/2 - \pi/4 \) and ignored a common phase factor. As discussed in reference [33], we can reproduce all reported results by OWZM [18]. For example, the coincidence counts at detectors \( D_i \) and \( D_j \) yield
\[ P_{i\!i} = \frac{\sin^2(\varphi')}{2} \]
\[ P_{i\!j} = \frac{\sin^2(\varphi') + \cos^2(\varphi')}{2} = \frac{1}{2} \]  

There is an interesting point about this experiment (applicable also to the other experiments discussed in this section), worth to be stressed. In the OWZM experiment, we have a single pair of photons at any given moment in the whole setup. And, as remarked earlier, since a photon pair creation in SPDC is ‘spontaneous’, some kind of coherence must have been induced.

An experiment that puzzled physicists and launched the field of quantum imaging [21–23] was performed by Zou, Wang and Mandel (ZWM) [19].

The bizarre and mind-bending feature of the ZWM experiment was the fact that seemingly distant photons from the idler mode appear to influence the interference of the photons from the signal modes.

The ZWM experiment is depicted in figure 13. Similar to the OWZM experiment, a pump laser, through a beam splitter, coherently pumps two non-linear crystals. The signal beams (denoted as before \( s_1 \) and \( s_2 \)) are brought together in the beam splitter BS. A photo-detector \( D_1 \) is placed at its output \( s \) (the other output, \( s' \) being unused).

The idler beam from the first crystal (denoted \( i_1 \)) passes through the second one and is superposed on the idler beam \( i_2 \) (see figure 13). A detector \( D_2 \) is placed at the output \( j \) (the output \( j' \) being ignored).

ZWM observed an interference pattern on monitoring the singles detections \( P_i \) at the detector \( D_2 \) while varying the path length difference by slightly moving the beam splitter BS. This should be already surprising since in the OWZM experiment (figure 11) this single rate, as proven by equation (32) was flat. The real surprise comes when one blocks the path of \( i_1 \) before the crystal NL\(_2\). Then, as noted by the experimenters, all single rate interference at \( D_2 \) disappears.

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\(^4\) If this were not the case (e.g. if one pumps the two non-linear crystals with two different, non-synchronized lasers), we should not have the same field operator \( \hat{a}_j^\dagger \) ‘split’ between the two crystals. We should have rather used different field operators, \( \hat{a}_j^\dagger \) and \( \hat{a}_j^\dagger \) and the input would have been a density matrix \( \rho_{in} \sim |T|^2|1_p\rangle \langle 1_p| + |R|^2|1_p\rangle \langle 1_p| \) where \( |1_p\rangle = \hat{a}_p^\dagger |0\rangle \) with \( j = 1, 2 \).
In Figure 14 we use the graphical method to describe the ZWM experiment. As before, we built it in a straightforward manner by using the building blocks from Figures 2, 3 and 5. The effect of moving the beam splitter BS is modelled by the phase shift $e^{i\phi}$.

By visually inspecting the graph from Figure 14, we can express the contribution of $NL_1$ to the field operation transformation

$$\hat{a}_p^{NL_1} = \frac{1}{\sqrt{2}} (T\hat{a}_s^\dagger + R\hat{a}_i^\dagger)\hat{a}_i^\dagger$$

Adding the contribution of $NL_2$ takes us to the final result

$$\hat{a}_p^{NL_1} = \frac{1}{\sqrt{2}} (T + ie^{i\phi}R)\hat{a}_s^\dagger \hat{a}_i^\dagger + \frac{1}{\sqrt{2}} (R + ie^{i\phi}T)\hat{a}_i^\dagger \hat{a}_i^\dagger$$

and for the case of balanced beam splitters ($T = 1/\sqrt{2}$ and $R = i/\sqrt{2}$) we end up with the output state

$$|\psi_{out}\rangle = -\sin\left(\frac{\phi}{2}\right)|1_s,1_i\rangle + \cos\left(\frac{\phi}{2}\right)|1_c,1_i\rangle$$

where a common phase factor was ignored. As discussed in reference [33], all ZWM experimental results can be reproduced by equation (35), including the single detection rate

$$r = \text{Tr} \{\hat{a}_s^\dagger \hat{a}_s \hat{\rho}_{\text{out}}\} = \sin^2\left(\frac{\phi}{2}\right)$$

where $\hat{\rho}_{\text{out}} = |\psi_{out}\rangle \langle \psi_{out}|$ and $\hat{\rho} = \text{Tr}_{s,s',i} (\hat{\rho}_{\text{out}})$. The authors conclude ‘[…] once the $i_1, i_2$ connection is broken it becomes feasible, in principle, to determine from the counts […] $D_j$ whether the detected signal photon comes from $NL_1$ or $NL_2$, and this destroys the interference’ [19]. As already remarked [3, 33], if this affirmation were complete, the singles detection rate in the OWZM experiment should also have shown interference.

The central point in the output state vector from equation (35) is entanglement. Indeed, the state (35) is acutally separable from the signal-idler point of view:

$$|\psi_{out}\rangle = -\sin\left(\frac{\phi}{2}\right)|1_s\rangle + \cos\left(\frac{\phi}{2}\right)|1_c\rangle \otimes |1_i\rangle$$

and looking through these glasses, it is no wonder that the signal part shows interference even when the idler part is discarded.

The fact that the transmittance of an object inserted in the idler beam between the crystals influences the visibility of the signal’s beam triggered an experiment [21] where undetected photons seem to tell a story. The
practical advantage of what is now called ‘quantum imaging’ lies in the fact that one can illuminate an object with a given wavelength (see figure 13) where it is transparent and retrieve an image at a different wavelength, where the object may be opaque and detectors are efficient (see also figure 1 in [21]).

Figure 15 depicts the graphical representation of this experiment. The idler beam \( i \) is now connected through a beam splitter to \( i' \). We also are compelled to include the fictitious modes \( \hat{a}_0' \) (input) and \( \hat{a}_{0'}^\dagger \) (output) for unitarity reasons. For a perfectly transparent (opaque) object we have \( T_o = 1 \) (\( T_o = 0 \)).

Once again, from a merely visual inspection of the graph from figure 15 one can write down the field operator transformation yielding

\[
\hat{a}_p = \frac{1}{\sqrt{2}} (T \hat{a}_i^\dagger + R \hat{a}_i^\dagger) e^{i \phi_o} + R_o \hat{a}_i^\dagger + \frac{i}{\sqrt{2}} (T \hat{a}_i^\dagger + R \hat{a}_i^\dagger) \hat{a}_i^\dagger
\]

Assuming balanced beam splitters (which was the case in [21]) takes us to the output state vector

\[
|\psi_{\text{out}}\rangle = \frac{T e^{i \phi_o}}{2} |1_i, 1_i\rangle + \frac{R e^{i \phi_o}}{2} |1_i, 1_v\rangle + \frac{T e^{i \phi_o}}{2} |1_v, 1_i\rangle + \frac{R e^{i \phi_o}}{2} |1_v, 1_v\rangle
\]

Using the standard procedure, we write down the output density matrix \( \hat{\rho}_{\text{out}} = |\psi_{\text{out}}\rangle \langle \psi_{\text{out}}| \) and tracing out the unused output ports \( (i', i, v) \) takes us to the reduced density matrix \( \hat{\rho}_i = \text{Tr}_{i', i, v} \{ \hat{\rho}_{\text{out}} \} \). The quantity of interest, the probability of singles detection at \( D_i \) is

\[
P_i = \text{Tr} \{ \hat{a}_i^\dagger \hat{a}_i \hat{\rho}_i \} = 1 - \frac{T_o \cos(\phi_o)}{2}
\]

where we used the fact that \( |T_o| = 1 \) and assumed, without loss of generality, \( T_o \) real.

From equation (40) it becomes clear that both the transmittance and the phase shift introduced by the object play a role in the singles rate \( P_i \). A transparent object implies

\[
P_i = 1 - \frac{\cos(\phi_o)}{2}
\]

and phase shifts of \( \pi + 2n\pi \) (with \( n \in \mathbb{Z} \)) radians can be clearly imaged, while a \( 2n\pi \) phase shift cannot. This is consistent with the experimental findings of the authors (see figure 5 in [21]).

As a last example for SPDC, we use the graphical method to describe an experiment reported in reference [47] featuring three non-linear crystals. We show that we can reproduce all results from the original paper however with a reduced computational burden.

Heuer, Menzel and Milonni (HMM) [47] proposed and performed the experiment depicted in figure 16. Three identical non-linear crystals are pumped by a common laser that can distribute variable amounts of power to each crystal. What they observed was an interference pattern limited by the amount of which-path information that becomes available due to the presence of the third crystal.

The graphical description\(^5\) of the HMM experiment is given in figure 17. This time, since the pumping powers are varied, the \( \gamma \) coefficients from the down-conversion process cannot be ignored. Without loss of generality we can assume that the three non-linear crystals are pumped in phase, therefore the input state vector can be written as

\(^5\) Once again, we did not invoke ‘phase memory’, ‘vacuum fluctuations’ or ‘induced coherence’, however we implicitly assumed a coherence in the SPDC process among the three non-linear crystals.
where \(\mathcal{N}\) is the normalization factor. Therefore, we need the field operator transformations \(\hat{a}_j^\dagger = g_j(\hat{a}_1^\dagger + \hat{a}_2^\dagger + \hat{a}_3^\dagger)\) \(j = 1, 2, 3\) in order to obtain the output state vector. This is where the graphical method can speed up computations. From figure 17 we immediately find the field operator transformations:

\[
\begin{align*}
\hat{a}_1^\dagger &= \gamma_1(Te^{i\phi_1} + Re^{i\phi_2})(Te^{i\phi_3} + Re^{i\phi_4}) + \gamma_2(Te^{i\phi_1} + Re^{i\phi_2})(Te^{i\phi_3} + Re^{i\phi_4}) \\
\hat{a}_2^\dagger &= \gamma_2(Te^{i\phi_1} + Re^{i\phi_2})(Te^{i\phi_3} + Re^{i\phi_4}) \\
\hat{a}_3^\dagger &= \gamma_3(Te^{i\phi_1} + Re^{i\phi_2})(Te^{i\phi_3} + Re^{i\phi_4})
\end{align*}
\]  

(44)

Using equations (42) and (44), it is straightforward to obtain the output state vector

\[
|\psi_{out}\rangle \sim (\gamma_1 T^2 e^{i\phi_1 + \phi_2} + \gamma_2 T^2 e^{i\phi_1 + \phi_3} + \gamma_3 T^2 e^{i\phi_1 + \phi_4})|l_A l_C\rangle + (\gamma_1 T e^{i\phi_1 + \phi_2} + \gamma_2 T e^{i\phi_1 + \phi_3} + \gamma_3 T e^{i\phi_1 + \phi_4})|l_A l_D\rangle + (\gamma_1 R^2 e^{i\phi_1 + \phi_2} + \gamma_2 R^2 e^{i\phi_1 + \phi_3} + \gamma_3 R^2 e^{i\phi_1 + \phi_4})|l_B l_D\rangle
\]

(45)

For example, the coincidence count rate at detectors \(A\) and \(D\) is easily found to be

\[
R_{AD} \sim |\gamma_1 T e^{i\phi_1 + \phi_2} + \gamma_2 T e^{i\phi_1 + \phi_3} + \gamma_3 T e^{i\phi_1 + \phi_4}|^2
\]

(46)

and if we take the beam splitters balanced \((T = 1/\sqrt{2}, R = i/\sqrt{2})\) we confirm the result from equation (4) of reference [47]. The single detection rate at detector \(A\) can also be computed and yields

\[
R_A \sim |\gamma_1 e^{i\phi_1} + \gamma_3|^2 + \gamma_3^2
\]

(47)

confirming equation (3) from reference [47].

5. A brief discussion of the non-monochromatic case

Up until now light was assumed monochromatic. For the linear optics case this implied that a single frequency was present from the generation and until the detection of the photons. We could have emphasized this e.g. in equation (17) by clearly stating the frequency \(\omega\),
\[ \hat{a}^\dagger_\omega(\omega) = (T^2 + R^2e^{-i\omega\tau})\hat{a}^\dagger_\omega(\omega) + TR(1 + e^{-i\omega\tau})\hat{a}_\omega(\omega) \] (48)

and now we wrote \( \omega = \omega\tau, \tau = \Delta L/c \) where \( \Delta L \) denotes the optical path length difference inside the interferometer and \( \tau \) the speed of light in vacuum. However, in real life, quantum states of light have a frequency spread and sometimes this spread cannot be ignored.

Since the operators are now time/ frequency-dependent, we need to impose the commutationrule
\[ [\hat{a}_\omega(\omega), \hat{a}^\dagger_\omega'\omega)] = \delta_\omega\delta(\omega - \omega') \]
where \( \delta_\omega \) is the Kronecker delta and \( \delta(\omega) \) denotes Dirac’s delta function [48].

In section 2 we stated our needs and concluded that expressing the input creation field operator(s) in respect with the output ones is enough to formally solve the problem. In the non-monochromatic case, we typically want the positive frequency electric field operator \( \hat{E}_j^{(+)}(t) \) for the output port(s) \( j \in \{N, N + 1\} \). Therefore, we are interested by the inverse problem

\[ \hat{a}_N(\omega) = g_N(\hat{a}_1(\omega), \hat{a}_4(\omega)) \]
\[ \hat{a}_{N+1}(\omega) = g_{N+1}(\hat{a}_1(\omega), \hat{a}_4(\omega)) \] (49)

because e.g.

\[ \hat{E}_j^{(+)}(t) = E_0 \int d\omega e^{-i\omega t}\hat{a}_j(\omega) \] (50)

with \( j \in \{N, N + 1\} \) and \( E_0 \) is simply a normalization constant here since we make the narrowband approximation. The most general input state vector in the Heisenberg picture (field operators evolve, states remain fixed) is

\[ |\psi_\text{in}\rangle = \int d\omega d\omega' f(\hat{a}^\dagger_\omega \hat{a}^\dagger_\omega')(\hat{a}_\omega, \hat{a}_\omega') |0\rangle \] (51)

where \( f \) is an operator function. For example, if we input a single-photon wavepacket at input 1 we have

\[ f = \xi_1(\omega')\hat{a}^\dagger_\omega(\omega) \text{ and } \int d\omega|\xi_1(\omega)|^2 = 1. \]

If we wish to return to the monochromatic case (of frequency \( \omega_0 \)) we simply take the limit \( \xi_1(\omega') \to \delta(\omega' - \omega_0) \). According to Glauber [35], the single-photon detection at the output \( N \) is given by \( \langle \hat{E}_N^{(-)}(t)\hat{E}_N^{(+)}(t) \rangle \) and considering the formalism discussed here, this takes us to the probability to have a photo-detection at detector \( N \) and at time \( t_0 \)

\[ P_N(t_0) = \langle \psi_\text{in}|\hat{E}_N^{(-)}(t_0)\hat{E}_N^{(+)}(t_0)|\psi_\text{in}\rangle = ||\hat{E}_N^{(+)}(t_0)|\psi_\text{in}\rangle||^2 \] (52)

The coincidence count probability at detectors \( N \) (time \( t_0 \)) and \( N + 1 \) (time \( t_0 + \Delta t \)) is given by

\[ P_{N,N+1}(t_0, t_0 + \Delta t) = ||\hat{E}_N^{(+)}(t_0)\hat{E}_{N+1}^{(+)}(t_0 + \Delta t)|\psi_\text{in}\rangle||^2 \] (53)

A lengthier discussion of the non-monochromatic case can be found in reference [30]. In the following we give a simple example for illustration purposes.

Consider again the MZI from section 6. We can construct the graph from figure 18. It is now straightforward to obtain the output-input (annihilation) field operator transformations

\[ \begin{cases} \hat{a}_4(\omega) = (T^2 + e^{-i\omega\tau}R^2)\hat{a}_0(\omega) + TR(1 + e^{-i\omega\tau})\hat{a}_1(\omega) \\
\hat{a}_3(\omega) = TR(1 + e^{-i\omega\tau})\hat{a}_0(\omega) + (T^2e^{-i\omega\tau} + R^2)\hat{a}_1(\omega) \end{cases} \] (54)

6 We typically have \( E_0 = \sqrt{\varepsilon_0/2\omega_0 V} \) [2] where \( \varepsilon_0 \) is the vacuum electric permittivity and \( V \) is the normalization volume. But since in almost all practical situations we have min(\( \omega \)) \( = \) max(\( \omega \)) \( \ll \omega_0 \) where \( \omega_0 \) is the central frequency in the optical bandwidth of interest we can approximate \( E_0 \approx \sqrt{\varepsilon_0/2\omega_0 V} \).

7 We assume that the beam splitter is ‘flat’ over the considered spectral region. If this is not the case, one has to write in equation (54) \( T(\omega) \) and \( R(\omega) \) instead of \( T \) and \( R \).
Since quite often we are interested in $E_t$ and since, according to equation (50) $\hat{E}_t$ is the Fourier transform of $\hat{w}$, we conclude that a phase delay $e^{-i\omega t}$ translates into a delayed operator function i.e. $\hat{E}_t(t - \tau)$. This allows us to construct the graph from figure 19, where frequency-domain annihilation operators have been replaced with time-dependent positive frequency electric field operators. From this graph we can now write the output-input positive frequency field operator transformations

$$\begin{align*}
E_4^{(+)}(t) &= T^2E_0^{(+)}(t) + R^2E_0^{(+)}(t - \tau) + T\hat{R}E_1^{(+)}(t) + \hat{T}\hat{R}E_1^{(+)}(t) \\
E_5^{(+)}(t) &= \hat{T}\hat{R}E_0^{(+)}(t) + T\hat{R}E_0^{(+)}(t - \tau) + T^2E_1^{(+)}(t - \tau) + R^2E_1^{(+)}(t)
\end{align*}$$

Practical uses of these operator transformations can be found in reference [30], section 3.

6. Discussion and conclusions

In this paper we discussed a graphical method for calculating in an intuitive way field operator transformations in quantum optics. This method has several advantages:

- it describes the system, irrespective of the wavevector one intends to apply to it. This corresponds to what happens in real experiments: e.g. a Mach–Zehnder has the same field operator transformations irrespective of what state of light is applied at its input
- for more complicated systems the graphical method is faster and more intuitive than the traditional step-by-step calculation method (see e.g. appendix A)
- extensions to encompass the polarization degree of freedom are straightforward (PBS, waveplates, Type II SPDC etc)
- in computer-based implementations, calculations consist mainly of complex multiplications and additions of scalars. Operators (matrices) are needed only for the input and output ports
- all linear-optics based systems can be described with this method, both for the monochromatic and non-monochromatic case
- resonators and cavities are easily handled for the monochromatic case (see section 3.2)
- configurations comprising mainly non-linear crystals can be easily calculated in the monochromatic approximation

The fact that the graphical method actually sums amplitudes of allowed input–output paths resonates with Feynman’s path integral method. Further extensions might be possible since it is based on the fundamental principle that in quantum mechanics the amplitudes of all allowed paths have to be taken into account.

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Appendix A. Using the graphical method to describe two nested Mach-Zehnder interferometers

As stated earlier, the advantage of the graphical method is more obvious when the optical setup becomes more complicated. Consider the two nested Mach-Zehnder interferometers, as depicted in figure 20. We assume all beam splitters identical (as before, with the transmittance/reflectivity coefficients $T/R$) except BS$_5$ (transmittance/reflectivity coefficient $T_w/R_w$). Moreover, we consider the outer MZI with equal arm lengths. The phase shift $\varphi$ models the path length difference in the inner MZI. As before, we want to express the output state vector while having the input one, therefore, we are looking for the field operator transformations

$$
\begin{align*}
\hat{a}_0^\dagger &= g_0(\hat{a}_4^\dagger, \hat{a}_3^\dagger, \hat{a}_6^\dagger, \hat{a}_7^\dagger) \\
\hat{a}_1^\dagger &= g_1(\hat{a}_4^\dagger, \hat{a}_3^\dagger, \hat{a}_6^\dagger, \hat{a}_7^\dagger)
\end{align*}
$$

(A1)

The same setup can be described with the graphical method in a very straightforward manner, as done in figure 21. Dashed lines denote connections to unused input ports. Using the rules of the graphical method, one can directly write down the field operator transformations

$$
\hat{a}_0^\dagger = R(T^2 + R^2 e^{i\varphi})\hat{a}_4^\dagger + TR^2(1 + e^{i\varphi})\hat{a}_6^\dagger + T(TT_w + RRT_w)\hat{a}_6^\dagger + T(TR_w + RTT_w)\hat{a}_7^\dagger
$$

(A2)

and

$$
\hat{a}_1^\dagger = TR^2(1 + e^{i\varphi})\hat{a}_4^\dagger + R(T^2 e^{i\varphi} + R^2)\hat{a}_6^\dagger + T(TR_w + RTT_w)\hat{a}_6^\dagger + T(TT_w + RRT_w)\hat{a}_7^\dagger
$$

(A3)

by just visual inspection of the graph from figure 21. These field operator transformations were used in reference [27] where a two-photon state was applied at the input of the nested MZI setup. A slightly modified form of the field operator transformations (A2)–(A3) was also used in reference [26].
Appendix B. Validity of the graphical method for SPDC

In the following we give an argument why, under certain circumstances, our treatment of SPDC yields the correct results. We did the following things:

- consider the field operator transformation given by equation (12)
- consider the input state as a single-photon Fock state |\psi_i\rangle
- consider that there is a ‘phase memory’, i.e. the pump phase carries on after the down-conversion (see also the discussion in reference [45], Chapter 15 about this topic)

The process of spontaneous parametric down-conversion is characterized by the Hamiltonian [2, 45, 46]

\[
H(t) = H_0 + \hbar \left( \frac{\eta}{2} \hat{a}_p^\dagger \hat{a}_p \right) - \hbar \left( \frac{\eta}{2} \hat{a}_s \hat{a}_s \right)
\]

(\text{B1})

where \( H_0 = \sum_{j} \hbar \omega_j \hat{a}_j^\dagger \hat{a}_j \) with \( j \in \{ p, s, i \} \) and without loss of generality we considered \( \eta \in \mathbb{R} \). The input state vector is composed of a strong coherent pump (p) and two vacuum modes (s and i):

\[
|\psi_{in}\rangle = |\alpha_p, 0, 0\rangle
\]

(\text{B2})

The interaction Hamiltonian (\text{B1}) is assumed to evolve the input state \(|\psi_{in}\rangle\) to

\[
|\psi_{int}\rangle = \hat{U}(t_0) |\psi_{in}\rangle
\]

(\text{B3})

where \( t_0 = \hbar / c (l_0) \) is the length of the non-linear crystal and \( \hat{U}(t) = e^{-iH_0 t / \hbar} \). Further simplifications can be obtained by making the rather realistic assumption that the pump is undepleted [2]. Therefore we can replace \( \hat{a}_p \rightarrow \alpha_p e^{-i\varphi_p} e^{-i\omega_p t} \) and \( \hat{a}_p^\dagger \rightarrow \alpha_p^* e^{i\varphi_p} e^{i\omega_p t} \) (we intentionally isolated the pump phase, \( \varphi_p \) yielding the Hamiltonian’ (see also equation (1) in [49])

\[
H(t) = H_0 + \frac{\eta}{2} \left( e^{-i\omega_s t} \alpha_p \hat{a}_s \hat{a}_p^\dagger + e^{i\omega_s t} \alpha_p^* \hat{a}_s^\dagger \hat{a}_p \right) + \frac{\eta}{2} \left( e^{i\omega_s t} \alpha_p \hat{a}_i \hat{a}_p^\dagger + e^{-i\omega_s t} \alpha_p^* \hat{a}_i^\dagger \hat{a}_p \right)
\]

(\text{B4})

From a signal and idler point of view, the Hamiltonian (\text{B4}) applies to a vacuum state \(|\psi_{in}\rangle = |0, 0\rangle\), therefore, using the well-known expansion of squeezed states into Fock states (see equation (1) from reference [50]) we have

\[
|\psi_{int}\rangle \approx e^{i\omega_s t} |\psi_{in}\rangle |0, 0\rangle \approx |0, 0\rangle + \frac{\kappa e^{-i\omega_s t}}{2} |1, 1\rangle + \ldots
\]

(\text{B5})

where \( \kappa = \tanh \eta \omega_p / 2 \). If we post-select only events where a down-conversion takes place (higher order processes can be typically ignored), we can conclude all three assumptions we took are justified.

ORCID iDs

Stefan Ataman  
https://orcid.org/0000-0003-2423-578X

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8 The \( e^{-i\omega_p t} \) factor was ignored because actually \( \hat{a}_i^\dagger \) and \( \hat{a}_i \) also ‘rotate’ with frequencies \( \omega_i \) and, respectively, \( \omega_s \) and, as discussed in section 3.3, energy conservation imposes \( \omega_p = \omega_i + \omega_s \).
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