A CONJECTURAL GENERALIZATION OF $n!$ RESULT TO ARBITRARY GROUPS

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Dedicated to Prof. M.S. Raghunathan on his sixtieth birthday

Abstract. The aim of this paper is to formulate a conjecture for an arbitrary simple Lie algebra $g$ in terms of the geometry of principal nilpotent pairs. When $g$ is specialized to $sl_n$, this conjecture readily implies the $n!$ result and it is very likely that, in fact, it is equivalent to the $n!$ result in this case. In addition, this conjecture can be thought of as generalizing an old result of Kostant. In another direction, we show that to prove the validity of the $n!$ result for an arbitrary $n$ and an arbitrary partition of $n$, it suffices to show its validity only for the staircase partitions.

1. Introduction

To any partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of a positive integer $n$, one associates an element $\Delta_\sigma$ of the complex polynomial ring $R_n := \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ in $2n$-variables by defining it (up to a sign) to be the determinant of the matrix $(X_i^sY_j^t)_{1 \leq s, t \leq n}$ for some enumeration $(i_1, j_1), \ldots, (i_n, j_n)$ of the elements in $D_\sigma := \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i < \sigma_j\}$. (We could think of $\Delta_\sigma$ as a generalization of the Vandermonde determinant.) Let $A_\sigma$ be the quotient of the ring of constant coefficient differential operators $\text{Diff}_{2n}$ on $R_n$ by the ideal $I_\sigma$ consisting of the elements mapping $\Delta_\sigma$ to zero.

A. Garsia and M. Haiman conjectured that $A_\sigma$ should have complex dimension $n!$. This conjecture later became known as the $n!$-conjecture. The interest in the $n!$-conjecture came from its connection with the positivity conjecture by Macdonald, stating that the Kostka-Macdonald coefficients $K_{\lambda,\mu}(q,t)$ should be polynomials in $q$ and $t$ with nonnegative integer coefficients. Relating the coefficients of $K_{\lambda,\mu}(q,t)$ to the dimension of certain bigraded parts of $A_\sigma$, Garsia and Haiman found that the $n!$-conjecture would imply the above positivity conjecture.

In spite of its simple appearance, the $n!$-conjecture turned out to be much harder to prove than first anticipated, and only recently M. Haiman was able to give the first, and until now only, proof of the conjecture using geometry of the Hilbert scheme of $n$-points in $\mathbb{C}^2$ (following some suggestions of C. Procesi). Unfortunately, his proof is quite complicated which makes it less accessible.
The aim of this paper is to relate the $n!$-conjecture with the geometry of principal nilpotent pairs $e$. More specifically, let $g$ be a simple Lie algebra (corresponding to a simple adjoint group $G$) with a Cartan subalgebra $h$ and let $e \in g^d$ be a principal nilpotent pair in $g^d := g \oplus g$ (a notion defined recently by Ginzburg, cf. Definition 1). We associate to $e$ a subgroup $G_e$ of finite index of the full automorphism group $G$ of $g$ and consider the scheme theoretic intersection of the $G_e$-orbit closure $\mathcal{O}_e$ of $e$ in $g^d$ with $h^d$, where $G$ acts on $g^d$ diagonally via its adjoint action. Then one of the main conjectures of this paper asserts that the scheme $\mathcal{O}_e \cap h^d$ is Gorenstein. In addition, we conjecture that the affine coordinate ring $\mathcal{O}_e := C[\mathcal{O}_e \cap h^d]$ supports the regular representation of $W$; in particular, it is of dimension $|W|$ (cf. Conjecture 2). It is shown (cf. Proposition 3) that the above conjecture for $g = sl_n$ readily implies the $n!$ result by using a result of de Concini and Procesi on the cohomology of Springer fibres and a simple characterization of the algebra $A_\sigma$ in terms of the cohomology of Springer fibres (cf. Theorems 4). In fact, by using the deformations of unstable orbits to semistable orbits (cf. Section 4), we show in Proposition 3 that the (conjectural) Gorenstein property of the ring $\mathcal{O}_e$ in the case of $g = sl_n$ already implies the $n!$ result provided we use a certain variant of a conjecture by Ginzburg (cf. Conjecture 1). Thus our conjecture provides a generalization of the $n!$ result to an arbitrary simple $g$ ($n!$ result being viewed as a result for the case of the Lie algebra $sl_n$).

In another direction, using the geometry of Hilbert schemes and Borel’s fixed point theorem, we show that the validity of $n!$ result for an arbitrary $n$ and an arbitrary partition of $n$ follows from its validity for just the staircase partitions $\sigma_m : m \geq m - 1 \geq \cdots \geq 1 > 0$ for any positive integer $m$ (cf. Theorem 4).

Let $D^k_n(\Delta_n)$ be the subspace of the polynomial ring $P_n := C[X_1, \ldots, X_n]$ in $n$ variables over $C$ obtained by applying all the constant coefficient differential operators to the Vandermonde determinant $\Delta_n$. Then it is a well known result that $D^k_n(\Delta_n)$ is of dimension $n!$. Now let $p$ be a prime such that $n \leq p^2$ and let $k$ be a field of char. $p$. Then we conjecture that the space $D^k_n(\Delta_n)$ over $k$ obtained by applying all the constant coefficient divided differential operators to $\Delta_n$ is again of dimension $n!$ (cf. Conjecture 3). This conjecture immediately implies the validity of $n!$ result for $n = p^2$ and the box partition $\mathcal{b}_p : p \geq \cdots \geq p$ of $n$.

For $n = tm + r$ with $t > r \geq 0, t > 1$ and $m \geq 1$, consider the box plus one row partition $\sigma = \sigma(t, m, r) : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of $n$ with $\sigma_i = t$ for $0 \leq i \leq m - 1$ and $\sigma_m = r$. Finally, in Section 4, we realize the algebra $A_\sigma$ for any such partition as the associated graded algebra of the cohomology algebra with complex coefficients $H^\ast(SL_n/B, C)$ with respect to a certain specific filtration depending solely on $t$ (cf. Corollary 2), where $B$ is a Borel subgroup of $SL_n$ consisting of the upper triangular matrices of determinant one. The proof of this corollary relies on a description of the cohomology of Springer fibres due to de Concini and Procesi which we have included in
the appendix. In addition, we have included a different description of the cohomology of Springer fibres due to Tanisaki in the appendix, which has been used in the proof of Theorem 1.

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2. Preliminaries

Let \(n\) be a positive integer and let \(\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0\) be a partition of \(n\). The diagram of \(\sigma\) is the array of lattice points:

\[ D_\sigma := \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i < \sigma_j\}, \]

where \(\mathbb{Z}_+ := \{0, 1, \cdots \}\) is the set of nonnegative integers. We order the elements of \(D_\sigma\) in any manner as \(\{1, \cdots, n\}\) and let \(s \in \{1, \cdots, n\}\) correspond to the lattice point \((i_s, j_s)\). Define a generalization of the Vandermonde determinant as the polynomial in \(2n\) variables \(X_1, \cdots, X_n, Y_1, \cdots, Y_n\) with integer coefficients:

\[ \Delta_\sigma := \det [X_i^s Y_j^t]_{1 \leq s, t \leq n}. \]

Clearly a different choice of the ordering of \(D_\sigma\) will give rise to the same \(\Delta_\sigma\) up to a sign.

Let \(\text{Diff}_{2n}\) be the algebra \(\mathbb{C}[\partial X_1, \cdots, \partial X_n, \partial Y_1, \cdots, \partial Y_n]\) of all the constant coefficient differential operators in \(2n\) variables with coefficients in the field of complex numbers \(\mathbb{C}\), where \(\partial x_i := \partial/\partial X_i, \partial y_i := \partial/\partial Y_i\). Let \(\text{Diff}_{2n}(\Delta_\sigma)\) be the space \(\{D(\Delta_\sigma) : D \in \text{Diff}_{2n}\}\). The following celebrated conjecture was made by Garsia and Haiman \[12\] and proved recently by Haiman \[16\].

We will refer to the following result as \(n!\) result for \(\sigma\).

**Theorem 1.** (\(n!\) result for \(\sigma\)) For any positive integer \(n\) and partition \(\sigma\) of \(n\),

\[ \dim_{\mathbb{C}} \text{Diff}_{2n}(\Delta_\sigma) = n!. \]

2.1. Hilbert schemes. The Hilbert scheme \(\text{Hilb}^n(\mathbb{C}^2)\) of \(n\) points in \(\mathbb{C}^2\) parameterizes the set of ideals \(I\) in \(\mathbb{C}[X, Y]\) whose associated quotient ring has complex dimension \(n\). By a result of Fogarty \[11\], the Hilbert scheme \(\text{Hilb}^n(\mathbb{C}^2)\) is a smooth irreducible variety of dimension \(2n\). Any partition \(\sigma\) of \(n\) defines a point \(I_\sigma \in \text{Hilb}^n(\mathbb{C}^2)\) by letting \(I_\sigma\) denote the ideal spanned by the monomials \(\{X^i Y^j : (i, j) \notin D_\sigma\}\). Thanks to Procesi and Haiman, the \(n!\) result can be reformulated in geometric terms using \(\text{Hilb}^n(\mathbb{C}^2)\) as follows.
There is a ‘tautological’ vector bundle $\mathcal{F}$ with the scheme $\text{Hilb}^n(\mathbb{C}^2)$ as base, where the fibre over any $I \in \text{Hilb}^n(\mathbb{C}^2)$ is the finite dimensional $\mathbb{C}$-vector space (in fact, a $\mathbb{C}$-algebra) $\mathbb{C}[X,Y]/I$. Now consider the map of sheaves $\mathcal{F}^\otimes n \to \wedge^n(\mathcal{F})$, defined by $(a_1 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_n) \mapsto (a_1 b_1) \wedge \cdots \wedge (a_n b_n)$ for any $a_i, b_i \in \mathbb{C}[X,Y]/I$ (which is the fibre over $I$). This map clearly induces the map

$$\beta : \mathcal{F}^\otimes n \to (\mathcal{F}^\otimes n)^* \otimes \wedge^n(\mathcal{F}).$$

Identifying the polynomial ring $R_n := \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ with the ring $\mathbb{C}[X,Y]^\otimes n$, we can view the fibre $(\mathbb{C}[X,Y]/I)^\otimes n$ of the vector bundle $\mathcal{F}^\otimes n$ over $I$ as a quotient of $R_n$. Let $K_\sigma$ be the kernel of $\beta$ over $I_\sigma$ and let $K_\sigma$ denote the associated preimage in $R_n$. An easy calculation then tells us (see [16]):

$$K_\sigma = \{ f \in R_n : f(\partial_{X_1}, \ldots, \partial_{X_n}, \partial_{Y_1}, \ldots, \partial_{Y_n})(\triangle_\sigma) = 0 \}.$$ 

In particular, defining the algebra

$$A_\sigma := R_n/K_\sigma,$$

we find that $A_\sigma \simeq \text{Diff}_{2n}(\triangle_\sigma)$ as vector spaces. Hence the $n!$ result is equivalent to the statement that the map $\beta$ restricted to the fibre over $I_\sigma$ has rank $n!$ for all the partitions $\sigma$ of $n$.

It is not difficult to see that the rank of $\beta$ over the fibre of any $I \in \text{Hilb}^n_{\text{reg}}(\mathbb{C}^2)$ is equal to $n!$, where $\text{Hilb}^n_{\text{reg}}(\mathbb{C}^2) \subset \text{Hilb}^n(\mathbb{C}^2)$ is the open dense subset consisting of those ideals whose zero set consists of $n$ distinct points. Using that the rank of $\beta$ along fibres is a lower semi-continuous function on $\text{Hilb}^n(\mathbb{C}^2)$, we conclude that the rank of $\beta$ along any fibre is at most $n!$. Furthermore, any point in $\text{Hilb}^n(\mathbb{C}^2)$, by using the natural action of the torus $(\mathbb{C}^*)^2$ on $\text{Hilb}^n(\mathbb{C}^2)$, can be deformed into a point of the form $I_\sigma$ for some partition $\sigma$ of $n$. Applying semi-continuity once more this provides us with the following reformulation of the $n!$ result:

**Theorem 2.** For any positive integer $n$ and partition $\sigma$ of $n$, the rank of $\beta$ over the fibre of $I_\sigma$ is equal to $n!$. Thus, $\dim A_\sigma = n!$.

Further, the validity of $n!$ result for all the partitions of $n$ is equivalent to the assertion that the rank of $\beta$ over any fibre of $\text{Hilb}^n(\mathbb{C}^2)$ is constant (which is automatically equal to $n!$).

The algebra $A_\sigma$ will play a fundamental role in the paper. Observe that the algebra $A_\sigma$ is a bigraded algebra, where the two gradings come respectively from the total degree in the variables $X_1, \ldots, X_n$ and the total degree in the variables $Y_1, \ldots, Y_n$. Moreover, there is a $S_n$-action on $A_\sigma$ preserving the bidegrees coming from the (diagonal) action of $S_n$ on the polynomial ring $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, given by: $\theta(X_i) = X_{\theta(i)}$ and $\theta(Y_i) = Y_{\theta(i)}$, for any $\theta \in S_n$.

We say that a point $I \in \text{Hilb}^n(\mathbb{C}^2)$ (or alternatively the algebra $\mathbb{C}[X,Y]/I$; or alternatively the scheme $\text{Spec} \mathbb{C}[X,Y]/I$) satisfies the **maximal rank condition** if the rank of $\beta$ over $I$ is equal to $n!$ (which is the maximum possible...
rank over $\text{Hilb}^n(\mathbb{C}^2))$. Thus the set of points $I \in \text{Hilb}^n(\mathbb{C}^2)$ satisfying the maximal rank condition is open in $\text{Hilb}^n(\mathbb{C}^2)$.

3. A characterization of the algebra $A_\sigma$

Fix a positive integer $n$. Let $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ be a partition of $n$ and let $\sigma^\vee$ be the dual partition. Let $G = SL_n(\mathbb{C})$ and let $X_\sigma$ be the Springer fibre (over complex numbers) corresponding to a fixed nilpotent matrix $M_\sigma$ (of size $n \times n$) with Jordan blocks of sizes $\sigma_0, \sigma_1, \ldots, \sigma_m$. Recall that $X_\sigma$ consists of those Borel subalgebras $b$ of $\mathfrak{sl}_n(\mathbb{C})$ such that $M_\sigma \in b$. Cohomology of a space $X$ with coefficients in $\mathbb{C}$ will be denoted by $H^*(X, \mathbb{C})$ or simply by $H^*(X)$. In this paper, we only encounter spaces $X$ such that $H^*(X)$ is concentrated in even degrees. We will consider them as graded algebras under rescaled grading by assigning degree $i$ to the elements of $H^{2i}(X)$.

For the definition of Gorenstein rings and their general properties, the reader is referred to, e.g., [1].

**Theorem 3.** For any partition $\sigma$, there exists an algebra $T_\sigma$ satisfying the following:

1. $T_\sigma$ is a $S_n$-equivariant graded quotient of $H^*(X_\sigma \times X_\sigma^\vee)$, under the diagonal Springer action of $S_n$ on $H^*(X_\sigma \times X_\sigma^\vee)$.

2. $T^{d_\sigma}_\sigma$ is one dimensional and, moreover, it is the sign representation of $S_n$, where $T^m_\sigma$ denotes the graded component of degree $m$ of $T_\sigma$ and $d_\sigma$ is the complex dimension of the variety $X_\sigma \times X_\sigma^\vee$.

3. $T_\sigma$ is a Gorenstein algebra. (Since $T_\sigma$ is a finite dimensional graded algebra and its top graded component is one dimensional, the condition that $T_\sigma$ is Gorenstein is equivalent to the condition that the pairing

$$
T^m_\sigma \times T^{d_\sigma - m}_\sigma = T^{d_\sigma}_\sigma,
$$

induced by the multiplication map is perfect for all $m \geq 0$.)

Further, such a $T_\sigma$ (satisfying the above three properties) is unique up to an isomorphism. More specifically, if $\varphi_\sigma : H^*(X_\sigma \times X_\sigma^\vee) \to T_\sigma$ and $\varphi'_\sigma : H^*(X_\sigma \times X_\sigma^\vee) \to T'_\sigma$ are two quotients satisfying the above properties, then their kernels are, in fact, equal.

**Proof.** By the Springer correspondence (see, e.g., [1], 13.3 or [2], §4.4), the top cohomologies of $X_\sigma$ and $X_\sigma^\vee$ are irreducible representations of $S_n$ which differ by the sign representation $\epsilon$. Hence, there is a unique copy of $\epsilon$ in $S^{d_\sigma}_\sigma$, where we abbreviate $H^*(X_\sigma \times X_\sigma^\vee)$ by $S_\sigma$. Let $(S^{d_\sigma}_\sigma)^\epsilon$ be the corresponding isotypical component, which is a one dimensional space. Now, consider the symmetric bilinear form $\gamma$ obtained as the composition

$$
S_\sigma \times S_\sigma \to S^{d_\sigma}_\sigma \to (S^{d_\sigma}_\sigma)^\epsilon \simeq \mathbb{C},
$$

where the first map is induced by the multiplication and the second map is induced by the $S_n$ equivariant projection. Now define

$$
T_\sigma := S_\sigma / \text{rad } \gamma,
$$
where $\text{rad} \gamma$ is the radical of $\gamma$. It is easy to see that $T_\sigma$ satisfies all the three properties listed above.

To prove the uniqueness, let $J := \ker \varphi_\sigma$ and $J' := \ker \varphi'_\sigma$. Then if $J'$ is not a subset of $J$, take a nonzero homogeneous element $x \in J + J'/J \subset T_\sigma$. Then, since $T_\sigma$ is Gorenstein, there exists an element $y \in T_\sigma$ such that $xy \neq 0$ in $T_\sigma^{d_\sigma}$. We can take a homogeneous preimage $\tilde{x}$ of $x$ in $J' \subset S_\sigma$. But $J'$ being an ideal, $\tilde{x}y \in J'$ for any $\tilde{y} \in S_\sigma$. Since $(J' \cap S_\sigma^d)' = 0$, this leads to a contradiction, proving that $J' \subset J$. Similarly, $J \subset J'$, and hence $J = J'$.

**Remark 1.** (a) It is easy to see that the Künneth decomposition of $H^*(X_\sigma \times X_\sigma^\vee)$ gives rise to a bigrading on $T_\sigma$ and this bigrading respects the $S_n$-action.

(b) It is well known that $H^*(X_\sigma)$ is concentrated in even degrees (cf. [8], Theorem 3.9). Moreover, it can be easily shown that $T_\sigma^d = H^2(X_\sigma \times X_\sigma^\vee)$.

(c) It is natural to ask if there exists a compact topological manifold $M_\sigma$ of (real) dimension $2d_\sigma$ admitting a $S_n$-action on $H^*(M_\sigma)$ and a continuous map $\phi : M_\sigma \to X_\sigma \times X_\sigma^\vee$ such that the induced map $\phi^\ast$ in cohomology is $S_n$-equivariant and surjective. If so, $H^*(M_\sigma) \simeq T_\sigma$.

We give such a construction for the partition $\sigma : 2 \geq 1$ of $n = 3$.

In this case $\sigma = \sigma^\vee$ and $X_\sigma$ is nothing but two copies of $\mathbb{P}^1$ identified at $\infty$. Thus $X_\sigma \times X_\sigma^\vee$ can be identified with the subset $(\mathbb{P}^1 \times \infty \times \mathbb{P}^1 \times \infty) \cup (\infty \times \mathbb{P}^1 \times \infty \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \infty \times \infty \times \mathbb{P}^1) \cup (\infty \times \mathbb{P}^1 \times \mathbb{P}^1 \times \infty)$ of $(\mathbb{P}^1)^4$. Now take the union of the last two subsets which is join of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ along a point. Finally, take $M_\sigma$ to be the manifold-join of these two (i.e., remove a small disc from these two separately and join them along the boundary). Then $H^*(M_\sigma)$ is a surjective image of that of $H^*(X_\sigma \times X_\sigma^\vee)$. The Weyl group $S_n$ acts on $H^*(M_\sigma)$ making the above map $S_n$-equivariant (but $S_n$ does not act in any natural manner on $M_\sigma$ itself).

(d) It is well known (see, e.g., [21], Section 5) that $d_\sigma = \sum_{(i,j) \in D_\sigma} (i+j)$, where $D_\sigma$ is the diagram of $\sigma$ as in Section 4.

**Theorem 4.** For any partition $\sigma$ of $n$, there exists a bigraded $S_n$-equivariant isomorphism of algebras:

$$T_\sigma \simeq A_\sigma,$$

where $A_\sigma$ is defined in Section 4.

As a preparation to prove the above theorem, we prove the following two lemmas.

Let $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ be a partition of $n$ and let $\sigma^\vee : \sigma'_0 \geq \cdots \geq \sigma'_{m'} > 0$ be the dual partition. For any $m' < i \leq n$, set $\sigma'_i = 0$, and

\footnote{R. MacPherson has mentioned to us that he also constructed such a (possibly different) $M_\sigma$ in this case.}
define the integer (for any $1 \leq k \leq n$):

$$d_k(\sigma) := n - \sum_{s=0}^{n-k-1} \sigma'_s.$$  

Let $S$ be a subset of the diagram $D_\sigma$ of cardinality $k$, together with a total ordering on $S$ written as $S = \{z_1, \ldots, z_k\}$. Define $i_s, j_s$ by $z_s = (i_s, j_s)$. Now define the monomial

$$P(S) = \prod_{s=1}^{k} X_s^{i_s} Y_s^{j_s} \in \mathbb{C}[X_1, \ldots, X_k, Y_1, \ldots, Y_k].$$

For each integer $1 \leq t \leq k$, let $S_t$ denote the subset $\{z_s : s > t\}$ of $S$.

**Lemma 1.** Let $S$ be as above and let $r$ be a positive integer satisfying $k - d_k(\sigma) < r \leq k$. For any sequence $s : 1 \leq s_1 < s_2 < \cdots < s_r \leq k$, define

$$P(S)_s = (\partial_{Y_{s_1}} \partial_{Y_{s_2}} \cdots \partial_{Y_{s_r}}) P(S).$$

Then there exist integers $1 \leq t' < t \leq k$ such that $P(S)_s$ is symmetric in the variables $(X_t, Y_t)$ and $(X_{t'}, Y_{t'})$ (i.e., it is invariant under the involution of $\mathbb{C}[X_1, \ldots, X_k, Y_1, \ldots, Y_k]$ given by $X_t \mapsto X_{t'}, Y_t \mapsto Y_{t'}$).

**Proof.** Notice first of all that if $n - k \geq \sigma_0$ then $d_k(\sigma) = 0$ and hence there is no $r$ satisfying $k < r \leq k$. In the following we therefore assume that $n - k < \sigma_0$. By symmetry, we may assume $s_i = i$, $i = 1, \ldots, r$. Assume that the statement is not true.

If $j_s = 0$ for an integer $s \leq r$ then clearly $P(S)_s = 0$ and the lemma follows in this case. Hence, assume that $j_s > 0$ for all $s \leq r$. Consider $(i_s, j_s) \in S_r$ and assume $(i_s, j_s + 1) \in S$. Let $s'$ denote the integer such that $(i_s, j_s + 1) = (i_{s'}, j_{s'})$. If $s' \leq r$ then the statement follows with $t' = s'$ and $t = s$. Hence we may assume that, for each $(i_s, j_s) \in S_r$, either $(i_s, j_s + 1) \notin S$ or else $(i_s, j_s + 1) \in S_r$.

Consider the set

$$M = \{0 \leq i < \sigma_0 : (i, j) \in S \text{ for all } j = 0, 1, \ldots, \sigma'_i - 1\}.$$  

For each $i \in M$ the element $(i, 0)$ is then contained in $S$ and therefore also in $S_r$. Hence, $(i, 1) \in S_r$ if $(i, 1) \in S$, and then $(i, 2) \in S_r$ if $(i, 2) \in S$ etc. This way we find that $(i, j) \in S_r$ for $i \in M$ and any $0 \leq j < \sigma'_i$. In particular, the number of elements $r$ in $S \setminus S_r$ is bounded by

$$r = |S \setminus S_r| \leq \sum_{s \notin M, s < \sigma_0} (\sigma'_s - 1) \leq \sum_{s=0}^{\sigma_0 - |M| - 1} (\sigma'_s - 1).$$

As the number $n - k$ of elements in $D_\sigma \setminus S$ is bounded below by

$$n - k = |D_\sigma \setminus S| \geq \sigma_0 - |M|,$$

we conclude

$$r \leq \sum_{s=0}^{n-k-1} (\sigma'_s - 1) = k - d_k(\sigma),$$
Lemma 2. Let $\sigma$ be any partition of $n$. Let $k \leq n$ and $r$ be positive integers such that $k - d_k(\sigma) < r \leq k$. Then, for any sequence $s : 1 \leq s_1 < s_2 < \cdots < s_k \leq n$,
\[ e_r(\partial_{Y_{s_1}}, \ldots, \partial_{Y_{s_k}})(\Delta_\sigma) = 0, \]
where $e_r$ denotes the $r$-th elementary symmetric polynomial.

Proof. By symmetry, we may assume $s_i = i$ for all $1 \leq i \leq k$. Expanding $\Delta_\sigma$ as a sum of monomials, we can write (for some index set $A$)
\[ \Delta_\sigma = \sum_{\alpha \in A} f_\alpha(X_{k+1}, \ldots, X_n, Y_{k+1}, \ldots, Y_n)P(S_\alpha), \]
where $P(S_\alpha)$ are monomials in the variables $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ of the form described earlier for some subsets $S_\alpha \subseteq D_\sigma$ of cardinality $k$ together with some total ordering on them. Now, by Lemma 1,
\[ e_r(\partial_{Y_1}, \ldots, \partial_{Y_k})P(S_\alpha) \]
is a sum of monomials each of which is symmetric in two distinct variables $(X_t, Y_t)$ and $(X_{t'}, Y_{t'})$ for $1 \leq t < t' \leq k$. In particular, the polynomial
\[ P := e_r(\partial_{Y_1}, \ldots, \partial_{Y_k})(\Delta_\sigma) \]
can be written as a sum of monomials in the variables $X_1, Y_1, \ldots, X_n, Y_n$ each of which is symmetric in the variables $(X_t, Y_t)$ and $(X_{t'}, Y_{t'})$ for some $1 \leq t < t' \leq k$. The pair $t < t'$ could be different for different monomials occurring in $P$. If $P \neq 0$, fix a monomial $M$ occurring in $P$ and a pair $t < t'$ such that $M$ is symmetric in $(X_t, Y_t)$ and $(X_{t'}, Y_{t'})$. Let $\pi \in S_n$ be the permutation $(t, t')$. As $\Delta_\sigma$ is $S_n$-antisymmetric, $\pi P = -P$. But $\pi(M) = M$, which is a contradiction. Thus $P = 0$.\]

Now we are ready to prove Theorem 4.

Proof. By virtue of Lemma 2 and Theorem 9 (of the appendix), we get a $S_n$-equivariant morphism
\[ \phi_\sigma : H^*(X_\sigma) \to A_\sigma \]
induced from the natural map $k[Y_1, \ldots, Y_n] \to A_\sigma$.

From the definition of $\Delta_\sigma$, it is easy to see that
\[ \tau \Delta_\sigma = \pm \Delta_{\sigma^\vee}, \]
where $\tau$ is the algebra automorphism of $R_n$ taking $X_i \mapsto Y_i, Y_i \mapsto X_i$. Thus, using the identification
\[ H^*(X_{\sigma^\vee}) \simeq k[X_1, \ldots, X_n]/J(\sigma^\vee), \]
where $J(\sigma^\vee)$ is the ideal defined in Theorem 9, we get a morphism
\[ \phi_{\sigma^\vee} : H^*(X_{\sigma^\vee}) \to A_{\sigma} \]
induced from the natural map $k[X_1, \ldots, X_n] \to A_\sigma$. Combining the two, we get a surjective graded $S_n$-equivariant morphism
\[
\phi : H^*(X_\sigma) \otimes H^*(X_\sigma^\vee) \to A_\sigma,
\]
where $\phi(y \otimes x) := \phi_\sigma(y) \cdot \phi_{\sigma^\vee}(x)$.

By Remark 1 (d), $A_{d}^{\sigma} = \mathbb{C}\Delta_\sigma$. Thus, $A_{d}^{\sigma}$ transforms via the sign character of $S_n$. We next show that $A_\sigma$ is Gorenstein: For any $0 \leq m \leq d_\sigma$, take a nonzero $x \in A_m^{\sigma}$ and take a homogeneous lift $\hat{x} \in R_n$. Viewing $\hat{x}$ as a differential operator $D(\hat{x})$ (under the identification $\text{Diff}_n \simeq R_n$, cf. Section 2), we get $D(\hat{y})\Delta_\sigma \neq 0$. Now take $\hat{y} \in R_n^{d_\sigma - m}$ such that $D(\hat{y})(D(\hat{x})\Delta_\sigma) = 1$.

This shows that the pairing $A_m^{\sigma} \times A_{d_\sigma - m}^{\sigma} \to A_{d_\sigma}^{\sigma}$ (obtained by the multiplication) is perfect. Thus, $A_\sigma$ satisfies the hypotheses of Theorem 3 and hence the isomorphism $A_\sigma \simeq T_\sigma$ follows.

Remark 2. As pointed out by M. Haiman, the map $\phi_\sigma$ (and similarly $\phi_{\sigma^\vee}$) defined above is injective. Their existence (and injectivity) can also be obtained from the results in [2] and [13] as explained in [12], Section 3.1. However, for the sake of completeness, we have included a complete proof.

4. Principal nilpotent pairs and deformation

Let $g$ be a semisimple Lie algebra over $\mathbb{C}$, and let $G = G_{\text{ad}}$ be the corresponding semisimple linear algebraic group of adjoint type. Then, as is well known, $G$ can be identified with the identity component of the group $\text{Aut}(g)$ of Lie algebra automorphisms of $g$. We denote the rank of $g$ by $\text{rk}(g)$.

We let $g^d$ denote the direct sum $g \oplus g$ and think of $G$ as acting diagonally on $g^d$ by the adjoint action. The following definition is due to Ginzburg.

**Definition 1.** ([14], Definition 1.1) A pair $e = (e_1, e_2) \in g^d$ is called a principal nilpotent pair if the following three conditions hold:

1. $[e_1, e_2] = 0$.
2. The common centralizer of $e_1$ and $e_2$ in $g$ is of dimension $\text{rk}(g)$.
3. For any pair $t_1, t_2 \in \mathbb{C}^*$, there exists $g \in G$ such that
   \[
   \text{Ad}(g)e = (t_1e_1, t_2e_2).
   \]

(Observe that by (3), $e_1$ and $e_2$ are ad-nilpotent elements.)

We need the following result due to Ginzburg.

**Theorem 5.** ([14], Theorem 1.2) Let $e = (e_1, e_2)$ be a principal nilpotent pair. Then there exists a pair $h = (h_1, h_2) \in g^d$ satisfying:

1. $h_1$ and $h_2$ are semisimple.
2. $[h_1, h_2] = 0$.
3. $[h_i, e_j] = \delta_{ij}e_j$.
4. The common centralizer of $h_1$ and $h_2$ is a Cartan subalgebra in $g$.
5. All the eigenvalues of $\text{ad}_{h_i} : g \to g$ are integral for $i = 1, 2$. 
Such a pair \( h \), called an \textit{associated semisimple pair}, is determined by the principal nilpotent pair \( e \) uniquely up to a \( G \)-conjugacy.

Let \( e = (e_1, e_2) \) be a principal nilpotent pair. As \( e_1 \) and \( e_2 \) are ad-nilpotent and commute, the sum \( e_1 + e_2 \) is also ad-nilpotent. For any \( t \in \mathbb{C} \) we can form the following automorphism of \( g \):

\[
g(t) = \exp(t \, \text{ad}(e_1 + e_2))
\]

or, alternatively, consider \( g(t) \) as an element of \( G \). The action of \( g(t) \) on the associated semisimple pair \( h = (h_1, h_2) \) is then described by

\[
g(t)h_i = h_i - te_i.
\]

From this one immediately concludes:

**Lemma 3.** All the elements of the form \( h(t) = (h_1 - te_1, h_2 - te_2) \) lie in the \( G \)-orbit \( O_h \) of \( h \).

The \textit{induced cone} \( \mathcal{C}O_h \) in \( g^d \) is, by definition, equal to \( \cup_{t \in \mathbb{C}} t \cdot O_h \subseteq g^d \). By the above result, we know that \( (th_1 + e_1, th_2 + e_2) \) is contained in \( \mathcal{C}O_h \) when \( t \neq 0 \). Hence, \( (e_1, e_2) \) is contained in the closure \( \overline{\mathcal{C}O_h} \).

### 4.1. Deformation

For a complex vector space \( V \), let \( \mathbb{C}[V] \) denote the affine coordinate ring of \( V \). This acquires a natural grading coming from the identification \( \mathbb{C}[V] = S(V^*) \), where \( S(V^*) \) denotes the symmetric algebra of the vector space dual \( V^* \) of \( V \). If \( X \) is any subset of \( V \), we denote by \( I(X) \) the ideal of functions in \( \mathbb{C}[V] \) vanishing identically on \( X \). The vector space spanned by the leading terms (i.e., the highest degree components) of all the elements in \( I(X) \) forms an ideal itself. This ideal is denoted by \( \text{gr}(I(X)) \). For any ideal \( I \subseteq \mathbb{C}[V] \), we denote by \( V(I) := \{ x \in V : f(x) = 0 \text{ for all } f \in I \} \) the corresponding closed subvariety of \( V \).

In the following, \( H \) will denote a reductive group over \( \mathbb{C} \). We also consider a finite dimensional complex representation \( V \) of \( H \). For a \( H \)-stable vector subspace \( M \subseteq \mathbb{C}[V] \), \( \text{gr}(M) \) is again \( H \)-stable, where \( \text{gr}(M) \) denotes the vector space spanned by the leading terms of the elements in \( M \). Moreover,

\[
\text{gr}(M) \simeq M, \text{ as } H\text{-modules.}
\]

**Definition 2.** An \( H \)-orbit \( Hv \) in \( V \) is called \textit{semistable} if 0 is not contained in its closure. If \( Hv \) is not semistable it is said to be \textit{unstable}.

For any subset \( U \subseteq V \), we define the \textit{induced cone} (as earlier) \( \mathcal{C}U := \cup_{t \in \mathbb{C}} t \cdot U \), and the \textit{associated cone} \( \mathcal{K}(U) \) as the closed (reduced) subvariety \( V(\text{gr}(I(U))) \) of \( V \).

Let \( Hv \) be a semistable orbit in \( V \). It is said that an unstable orbit \( Hu \) can be deformed into the semistable orbit \( Hv \), if \( Hu \) is contained in the associated cone \( \mathcal{K}(Hv) \) of \( Hv \) and, moreover, \( Hu \) has the same dimension as \( Hv \).

With this notation, we recall the following result from [4], Satz (3.4).
Proposition 1. Let $V$ be a finite dimensional representation of $H$ and let $Hv$ be a semistable orbit. Then 

$$K(Hv) = (\overline{CU} \setminus CU) \cup \{0\},$$

where $U := Hv$.

Moreover, if $Hu$ is an unstable $H$-orbit such that there exists a deformation of $Hu$ into the (semistable) orbit $Hv$, then 

$$I(Hu) \supset \text{gr}(I(Hv)).$$

4.2. Deforming principal nilpotent pairs. We now turn to certain deformations inside the finite dimensional representation $g^d$ of $G$.

Lemma 4. Let $e = (e_1, e_2)$ be a principal nilpotent pair. Then the $G$-orbit $O_e \subset g^d$ is unstable.

Proof. By Definition 1, each element of the form $(te_1, te_2)$, $t \in \mathbb{C}^*$, is contained in $O_e$. Hence 0 must be contained in the closure.

Lemma 5. Let $h = (h_1, h_2)$ be a semisimple pair associated to a principal nilpotent pair $e$. Then the $G$-orbit $O_h$ is closed and hence semistable.

Proof. As $h_1$ and $h_2$ commute, one can find a Cartan subalgebra of $g$ containing both $h_1$ and $h_2$. Hence, the common centralizer of $h_1$ and $h_2$ in $G$ contains a maximal torus. This implies that the orbit is closed.

Lemma 6. The orbit $O_e$ can be deformed into $O_h$.

Proof. As observed earlier, the element $e$ lies in $\overline{CO_h}$ and hence, by Proposition 1, also lies in the associated cone $K(O_h)$ of $O_h$ (since all the elements of $CO_h$ are semisimple pairs). It remains to show that the dimensions of $O_e$ and $O_h$ are the same. For this it is enough to show that the dimension $d_e$ of the common centralizer of $e_1, e_2$ in $g$ is the same as the dimension $d_h$ of the common centralizer of $h_1, h_2$ in $g$. But, by definition, $d_e = d_h = \text{rk}(g)$.

4.3. Orbits of the automorphism group of $g$. For the rest of this section we assume, for simplicity, that $g$ is simple (and, as earlier, $G$ is the associated adjoint group). We fix a maximal torus $T$ of $G$ and a set of simple roots $\Delta$. The full automorphism group $\mathcal{G}$ of the Lie algebra $g$ is then isomorphic to the semidirect product of $G$ with the group $\Gamma$ of the diagram automorphisms of $g$ (for the Dynkin diagram of $g$ with respect to $\Delta$).

When $e$ is a nilpotent element in $g$, by $\mathcal{G}^e$ we mean the subgroup of those elements in $\mathcal{G}$ leaving the $G$-orbit of $e$ invariant. In particular, $G \subseteq \mathcal{G}^e \subseteq \mathcal{G}$. The following lemma gives a complete description of $\mathcal{G}^e$.

Lemma 7. When $g$ is of type $D_l$ with even $l > 4$, and $e$ corresponds to a very even partition (with the usual classification of nilpotent classes [7]), then $\mathcal{G}^e$ coincides with $G$. When $g$ is of type $D_4$ and $e$ corresponds to a very even partition or one of the partitions $(1^35)$ or $(1^53)$, then $\mathcal{G}^e$ is a subgroup of $\mathcal{G}$ of index 3. In all the remaining cases $\mathcal{G}^e$ coincides with $\mathcal{G}$. 
Proof. Up to $G$-conjugacy we can embed $e$ into a standard $sl_2$-triple $(h, e, f)$ with $h$ contained in the Lie algebra $\mathfrak{h}$ of $T$ and such that $\alpha(h)$ is positive for all $\alpha \in \Delta$. The weighted Dynkin diagram of $e$ is then the Dynkin diagram of $g$ with respect to $\Delta$, with values $\alpha(h)$, $\alpha \in \Delta$, attached to the vertices.

Fix a diagram automorphism $g$ and consider the induced $sl_2$-triple $(g \cdot h, g \cdot e, g \cdot f)$. Then $\alpha(g \cdot h)$ is positive for all $\alpha \in \Delta$, and hence the associated weighted Dynkin diagram of $g \cdot e$ is got by applying the diagram automorphism $g$ to the weighted Dynkin diagram of $e$. As the weighted Dynkin diagram of $e$ determines the $G$-conjugacy class of $e$, we see that $g \in G \cdot e$ if and only if the weighted Dynkin diagram of $e$ is invariant under $g$.

We thus conclude that $G \cdot e / G$ is the subgroup of $\Gamma$ consisting of those diagram automorphisms of the Dynkin diagram of $g$ which leave the weighted Dynkin diagram of $e$ invariant. Now, the result follows by the inspection of the possible weighted Dynkin diagrams as found in [7].

Spaltenstein introduced an involution, called the Spaltenstein involution, on the class of ‘special’ nilpotent orbits in $g$ (cf. [20]).

Lemma 8. Let $e \in g$ be a nilpotent element such that the orbit $O_e := G \cdot e$ is special, and let $e'$ denote a nilpotent element in the special nilpotent orbit dual to $G \cdot e$ under the Spaltenstein involution. Then, for any $\gamma \in G$, $\gamma(O_e)$ is special and, moreover, $\gamma(O_{e'})$ is dual to the orbit $\gamma(O_e)$.

In particular,

$$G^e = G^{e'}.$$  

Proof. The first part is known. However we did not find a precise reference, so we give a brief outline: The Spaltenstein involution is the restriction of a map $d$ defined on the set of nilpotent orbits ([20], Chapter III). When $g$ is of classical type, the uniqueness statement Theorem 1.5, Chap. III in [20] implies that $d$ commutes with any automorphism of $g$. Combining this with the fact that the range of $d$ coincides with the special nilpotent orbits, the result follows for the classical types. When $g$ is of type $E_6$ it follows from the diagrams in [7], p. 441, that there exists at most one order reversing involution on the set of special nilpotent orbits. In particular, the Spaltenstein involution must commute with any automorphism of $g$, as it is known to be order reversing, and the result follows. In the remaining cases $G$ coincides with $G$ and the statement is trivial.

Alternatively, as pointed out by G. Lusztig, the first part follows from the fact that the Spaltenstein involution corresponds to tensoring the associated Springer representation by the sign representation.

To prove the ‘In particular’ statement, take $\gamma \in G^e$. Then $\gamma(O_{e'})$ is dual to $\gamma(O_e) = O_e$, and hence $\gamma(O_{e'}) = O_{e'}$. This proves the lemma.

By [14], Lemma 4.13, for a principal nilpotent pair $e = (e_1, e_2)$ in $g$, the $G$-orbits of $e_1$ and $e_2$ are Spaltenstein duals of each other. In particular, by Lemma 8, the groups $G^{e_1}$ and $G^{e_2}$ coincide. In the following, we use the notation $G^e$ for this common group.
Now we conjecture the following variant of a conjecture by Ginzburg [14], Conjecture (2.5). The original conjecture of Ginzburg is false, as seen in Proposition 4.

**Conjecture 1.** Let \( e = (e_1, e_2) \in g^d \) be a principal nilpotent pair. Then there is a \( G \)-module isomorphism:
\[
\mathbb{C}[\mathcal{O}_e] \simeq \mathbb{C}[G/T],
\]
where \( T \subset G \) is a maximal torus and \( \mathcal{O}_e \) denotes the \( G^e \) orbit of \( e \) in \( g^d \).

**Lemma 9.** Let \( e = (e_1, e_2) \) be a principal nilpotent pair with an associated semisimple pair \( h = (h_1, h_2) \). Then the ideal \( gr(I(G \cdot h)) \) is invariant under the action of \( G^e \).

**Proof.** When \( G^e \) coincides with \( G \) there is nothing to prove. By Lemma 3, this takes care of any \( g \) of type \( B_l, C_l, E_7, E_8, F_4 \) or \( A_1 \) and also the cases when \( e_1 \) (and hence also \( e_2 \)) corresponds to a very even partition in type \( D_l \), even \( l > 4 \).

Suppose now that \( g \) is of type \( A_l \) (\( l > 1 \)), \( E_6 \) or \( D_l \) (\( l \) odd). Then \( G^e/G \) coincides with \( G/G \) which is the group of order 2. By Planche I-IX in [5] it follows that the group \( G/G \) is generated by an automorphism of \( g \), which acts on a chosen Cartan subalgebra \( h \) of \( g \) via multiplication by \(-1\). Choosing \( h \) to contain \( h_1 \) and \( h_2 \), since the ideal \( gr(I(G \cdot h)) \) is homogeneous, the result follows in these cases as well.

We are left with the situation when \( g \) is of type \( D_l \), with \( l \) even. Notice that if \( g \in G^e \) leaves the \( G \)-orbit of \( e \) invariant, then by [14], Theorem 1.2(iv), the \( G \)-orbit of \( h \) is also left invariant by \( g \). We may hence concentrate on the situation when \( g \in G^e \) does not leave the \( G \)-orbit of \( e \) invariant.

Now referring to the classification of principal nilpotent pairs as in [10], Proposition 5 and Theorem 2, it follows that any element \( g \in G^e \) leaves the \( G \)-orbit of \( e \) invariant unless \( e \) correspond to a non-integral rectangular skew-graph (see [10] for the notation), which is the same as saying that \( e_1 \) and \( e_2 \) correspond to very even partitions. Thus, when \( l > 4 \), this situation is already handled above. This leaves us with type \( D_4 \) and principal nilpotent pairs \( e = (e_1, e_2) \) consisting of elements \( e_1 \) and \( e_2 \) corresponding to very even partitions. In this situation there exists an automorphism of \( g \) carrying \( e_1 \) and \( e_2 \) to the nilpotent elements corresponding to the partitions \((1^35)\) and \((1^53)\). As the partitions \((1^35)\) and \((1^53)\) are not very even, this reduces the problem to a situation which has already been handled.

**Theorem 6.** Let \( h \) be a semisimple pair associated to a principal nilpotent pair \( e \). Assume the validity of the above Conjecture 1 for \( e \). Then
\[
I(\mathcal{O}_e) = gr(I(\mathcal{O}_h)),
\]
where \( \mathcal{O}_e \) denotes the \( G^e \)-orbit of \( e \) and \( \mathcal{O}_h \) denotes the \( G \)-orbit of \( h \) (which is closed by Lemma 3).
Proof. By Proposition 1, Lemma 6 and Lemma 9,
\[ I(\mathcal{O}_e) \supset \text{gr}(I(O_h)), \]
which induces a surjective map of \( G \)-modules
\[ \mathbb{C}[\mathfrak{g}^d]/(\text{gr}(I(O_h))) \to \mathbb{C}[\mathcal{O}_e] \simeq \mathbb{C}[G/T], \]
where the last isomorphism follows from Conjecture 1. Moreover, \( \text{gr}(I(O_h)) \simeq I(O_h) \), as \( G \)-modules, so that the left side of (2) is isomorphic with \( \mathbb{C}[O_h] \) (as \( G \)-modules). But the \( G \)-isotropy of \( h \) can easily be seen to be equal to a maximal torus \( T \) and hence \( \mathbb{C}[O_h] \simeq \mathbb{C}[G/T] \). This forces the surjective map (2) to be an isomorphism, thus the theorem follows.

Remark 3. Using the classification of principal nilpotent pairs by Elashvili and Panyushev (see [10] and Appendix to [14]), one can work out the following description of the irreducible components of \( \mathcal{O}_e \) for a principal nilpotent pair \( e \).

\( A_i \): \( \mathcal{O}_e \) is the union of the \( G \)-orbit closures of \( e \) and its transpose \( e' := (e_1', e_2') \). These 2 components coincide precisely when \( e = e_\sigma \) for a box partition \( \sigma \) (see the proof of Proposition 3 for the notation \( e_\sigma \)).

\( B_i, C_i, E_i, G_i, F_i, D_i \): \( (l \text{ even}) \): \( \mathcal{O}_e \) coincides with the \( G \)-orbit closure of \( e \) (cf. Remark on p. 560 in [14]).

\( D_i \): \( (l \text{ odd}) \): If \( e \) is defined by a single skew-diagram (see [10], Prop. 5(i)), then \( \mathcal{O}_e \) consists of two distinct \( G \)-orbit closures of principal nilpotent pairs (those defined by the same skew-graph). Otherwise, \( \mathcal{O}_e \) coincides with the \( G \)-orbit closure of \( e \).

5. A CONJECTURAL GENERALIZATION OF n! RESULT TO ARBITRARY GROUPS

In this section \( \mathfrak{g} \) will denote an arbitrary simple Lie algebra over \( \mathbb{C} \) and \( \mathfrak{h} \subset \mathfrak{g} \) a Cartan subalgebra. Let \( G \) be the associated complex adjoint group with maximal torus \( T \) such that \( \text{Lie } T = \mathfrak{h} \). As in the last section, \( \mathfrak{g}^d \) denotes the direct sum \( \mathfrak{g} \oplus \mathfrak{g} \) and also \( \mathfrak{h}^d := \mathfrak{h} \oplus \mathfrak{h} \). Let \( W := N/T \) be the Weyl group of \( G \); \( N \) being the normalizer of \( T \) in \( G \). For a principal nilpotent pair \( e \), recall the definition of \( \mathcal{G}^e \) from the last section.

Proposition 2. Let \( e = (e_1, e_2) \) be a principal nilpotent pair in \( \mathfrak{g} \) and let \( \mathcal{O}_e \) be the diagonal \( \mathcal{G}^e \)-orbit of \( e \) in \( \mathfrak{g}^d \). Consider the affine coordinate ring \( \mathcal{D}_e := \mathbb{C}[\mathcal{O}_e \cap \mathfrak{h}^d] \), where \( \mathcal{O}_e \cap \mathfrak{h}^d \) denotes the scheme theoretic intersection of \( \mathcal{O}_e \) with \( \mathfrak{h}^d \) inside \( \mathfrak{g}^d \). Then \( \mathcal{D}_e \) is a bigraded finite dimensional (commutative) algebra, which admits a natural \( W \)-module structure compatible with the bigrading.

Proof. Consider the action of the two dimensional torus \( \mathbb{C}^* \times \mathbb{C}^* \) on \( \mathfrak{g}^d \) by \( (z, t)(h_1, h_2) = (zh_1, th_2) \), for \( z, t \in \mathbb{C}^* \) and \( h_1, h_2 \in \mathfrak{g} \). By the definition of principal nilpotent pairs, \( \mathbb{C}^* \times \mathbb{C}^* \) keeps \( \mathcal{O}_e \) (and thus \( \mathcal{O}_e \)) stable. Of course, \( \mathbb{C}^* \times \mathbb{C}^* \) also keeps \( \mathfrak{h}^d \) stable. This gives the desired bigrading on
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Since the only element of $g^d$ is of the form $e$, we call an element of $e$.

Then the scheme $\bar{D}$ on $W$ is even a complete intersection in $\text{dim} W$.

Remark 4. (a) When one of the entries $e_1$ or $e_2$ of $e = (e_1, e_2)$ is zero, then $\bar{O}_e$ coincides with the $G$-orbit of $e$. Hence, in this case, Conjecture 2 reduces to a well known result of Kostant.

(b) An old preprint of Bezrukavnikov-Ginzburg contained a variant of Conjecture 2 (cf. [3], Section 8.3). In addition, M. Haiman has informed us that he also conjectured around 1993 (though unpublished) a variant of Conjecture 2 in the special case of $g = sl_n$.

(c) In an earlier version of our paper, Conjecture 2 was formulated with $\bar{O}_e$ replaced by the $G$-orbit closure $\bar{O}_e$ of $e$. This turns out to be false in general (as pointed out by S. Strømme), since Ginzburg’s original conjecture [4], Conjecture 2.5 is false (cf. Proposition 4). In the same version we had conjectured that $\bar{O}_e \cap \bar{b}^d$ is even a complete intersection in $\bar{b}^d$, but it is false in general (as pointed out by Haiman).

(d) By computer calculations, using SINGULAR [15], we have checked the above conjecture for $sl_n, n \leq 4$.

Proposition 3. The validity of the above Conjecture 2 for $g = sl_n$ (and every principal nilpotent pair $e$) implies the $n!$ result for any partition of $n$.

Alternatively, assume the validity of Conjecture 2 and assume, in addition, that $\bar{D}_e$ is Gorenstein. Then the $n!$ result follows for any partition of $n$.

Proof. Take a partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of $n$. Let $\{v_1, \ldots, v_n\}$ be the standard basis of $\mathbb{C}^n$. Define a nilpotent transformation $e' = e'_\sigma \in sl_n$ which takes $v_1 \mapsto v_2 \mapsto v_3 \mapsto \cdots \mapsto v_{\sigma_0} \mapsto 0, v_{\sigma_0+1} \mapsto v_{\sigma_0+2} \mapsto \cdots \mapsto v_{\sigma_0+\sigma_1} \mapsto 0, v_{\sigma_0+\sigma_1+1} \mapsto v_{\sigma_0+\sigma_1+2} \mapsto \cdots \mapsto v_{\sigma_0+\sigma_1+2} \mapsto 0, \cdots$. Similarly, define $e'' = e''_{\sigma} \in sl_n$ which takes $v_1 \mapsto v_{\sigma_0+1} \mapsto v_{\sigma_0+\sigma_1+1} \mapsto \cdots \mapsto v_{\sigma_0+\cdots+\sigma_{m-1}+1} \mapsto 0, v_2 \mapsto v_{\sigma_0+2} \mapsto \cdots \mapsto v_{\sigma_0+\cdots+\sigma_{m-1}+2} \mapsto 0, \cdots$. Then, by [14], $e_{\sigma} := (e', e'')$ is a principal nilpotent pair (cf. also [19]).
Consider the maps
\[ H^*(X_\sigma^+) \otimes H^*(X_\sigma) \cong \mathbb{C}[G: e' \cap \mathfrak{h}] \otimes \mathbb{C}[G: e'' \cap \mathfrak{h}] \]
\[ \cong \mathbb{C}[G'': e'' \cap \mathfrak{h}] \otimes \mathbb{C}[G'': e'' \cap \mathfrak{h}] \]
\[ \cong \mathbb{C}[(G^{e_\sigma} \times G^{e_\sigma}) \cdot e_\sigma \cap \mathfrak{h}^d] \]
\[ \to \mathcal{D}_{e_\sigma}, \]
where the first isomorphism is due to de Concini and Procesi \[9\] and is bigraded and \(W \times W\)-equivariant and the last map is surjective (bigraded and \(W\)-equivariant), being induced by the closed immersion
\[ \overline{O}_{e_\sigma} \cap \mathfrak{h}^d \subset ((G^{e_\sigma} \times G^{e_\sigma}) \cdot e_\sigma \cap \mathfrak{h}^d). \]

By our Conjecture 2, \(\mathcal{D}_{e_\sigma}\) is Gorenstein; in particular, its top graded component is one dimensional (since its zeroth graded component is one dimensional). Moreover, since it supports the regular representation of \(W\) (by Conjecture 2), the top graded component will have to be the sign representation (being the only representation of dimension one apart from the trivial representation and the trivial representation occurs in degree 0). But since \(\mathcal{D}_{e_\sigma}\) is a quotient of \(H^*(X_\sigma^+) \otimes H^*(X_\sigma)\), and the latter has a unique copy of the sign representation which occurs in the top degree \(d_\sigma\) (cf. \[18\], Section 4.4), we conclude that \(\mathcal{D}_{e_\sigma}^d\) is nonzero (and is the sign representation of \(W\)). Thus, by Theorems 3, 4, we get that
\[ \mathcal{D}_{e_\sigma} \cong A_\sigma. \]

But by Conjecture 3 again, \(\dim \mathcal{D}_{e_\sigma} = n\) and thus the \(n!\) result follows for the partition \(\sigma\) (cf. Theorem 3).

Now we prove the ‘Alternatively’ part of the proposition. Let \(\mathfrak{h}_\sigma\) be a semisimple pair associated to \(e_\sigma\). Then, by Theorem 6, \(I(\overline{O}_{e_\sigma}) = \text{gr}(I(O_{\mathfrak{h}_\sigma}))\). Thus
\[ I(\overline{O}_{e_\sigma}) + I(\mathfrak{h}^d) \subset \text{gr}(I(O_{\mathfrak{h}_\sigma}) + I(\mathfrak{h}^d)), \]
and hence the \(S_n\)-module \(\mathbb{C}[O_{\mathfrak{h}_\sigma} \cap \mathfrak{h}^d]\) is a quotient of the \(S_n\)-module \(\mathcal{D}_{e_\sigma}\). But the scheme \(O_{\mathfrak{h}_\sigma} \cap \mathfrak{h}^d\) is reduced and is isomorphic with the \(S_n\)-orbit \(S_n \cdot \mathfrak{h}_\sigma\). It is easy to see that \(S_n\) acts freely on \(\mathfrak{h}_\sigma\), thus \(\mathbb{C}[O_{\mathfrak{h}_\sigma} \cap \mathfrak{h}^d]\) is the regular representation of \(S_n\). In particular, the sign representation occurs in \(\mathbb{C}[O_{\mathfrak{h}_\sigma} \cap \mathfrak{h}^d]\) and hence also in \(\mathcal{D}_{e_\sigma}\). But then, by the argument given above, \(\mathcal{D}_{e_\sigma}\) has a unique copy of the sign representation which occurs in the top degree \(d_\sigma\). Moreover, \(\mathcal{D}_{e_\sigma}\) being Gorenstein (by assumption), \(\mathcal{D}_{e_\sigma}^d\) is one dimensional. Thus, by Theorems 3 and 4, we again get that \(\mathcal{D}_{e_\sigma} \cong A_\sigma\), and hence \(A_\sigma\) has dimension at least \(n!\). Finally, since \(\dim A_\sigma \leq n!\) (from the semicontinuity argument as in Section 2), we get that \(\dim A_\sigma = n!\). \(\square\)

The following was pointed out to us independently by D. Panyushev and S. Strømme for the principal nilpotent pair \(e_\sigma\) in \(sl_3\) associated to the partition \(\sigma : 2 \geq 1\).
Proposition 4. *Ginzburg’s conjecture* [14], Conjecture 2.5, is false in general.

Proof. Let $e = e_\sigma$ denote the principal nilpotent pair in $\mathfrak{sl}_n$ corresponding to a partition $\sigma$ of $n$ (as in the proof of Proposition 3) and let $h$ be an associated semisimple pair. By choosing a suitable basis, one can assume that $-h$ is an associated semisimple pair of the transposed principal nilpotent pair $e^t$. But if Ginzburg’s conjecture were true for both of $e = e_\sigma$ and $e^t$, then, by a variant of Theorem 6, we would get that $I(G \cdot e) = I(G \cdot e^t)$. This would imply that $G \cdot e = G \cdot e^t$. This is true however only for the principal nilpotent pairs $e_\sigma$ defined by box partitions $\sigma$. Thus, for any partition $\sigma$ of $n$ which is not a box partition, Ginzburg’s conjecture is false.

6. Reduction to the staircase partition

Let $T \subset \text{GL}_2(\mathbb{C})$ be the maximal torus consisting of the diagonal matrices (of nonzero determinant) and let $B$ be the Borel subgroup consisting of the upper triangular matrices (of nonzero determinant). Let $v_1 = (1,0)$ and $v_2 = (0,1)$ be the standard basis of $\mathbb{C}^2$. The algebra of regular functions on $\mathbb{C}^2$ is identified with $\mathbb{C}[X,Y]$, where $X$ and $Y$ are the linear maps determined by $X(v_1) = 1$, $X(v_2) = 0$, $Y(v_1) = 0$, $Y(v_2) = 1$.

Consider the standard representation $\mathbb{C}^2$ of $\text{GL}_2(\mathbb{C})$. This induces a natural representation of $\text{GL}_2(\mathbb{C})$ on $\mathbb{C}[X,Y]$. The space of homogeneous polynomials $\mathbb{C}[X,Y]_d$ of degree $d$ in $\mathbb{C}[X,Y]$ forms an irreducible $\text{GL}_2(\mathbb{C})$-submodule of dimension $d + 1$. The following lemma is well known.

Lemma 10. Let $V \subseteq \mathbb{C}[X,Y]_d$ be a nonzero $B$-stable subspace. Then there exists an integer $i \geq 0$ such that

$$V = \text{span}_\mathbb{C}\{Y^d, Y^{d-1}X, \ldots, Y^d \cdot X^i\}.$$

Conversely, for any $i \geq 0$, the vector space

$$\text{span}_\mathbb{C}\{Y^d, Y^{d-1}X, \ldots, Y^d \cdot X^i\}$$

is $B$-stable.

The action of $\text{GL}_2(\mathbb{C})$ on $\mathbb{C}[X,Y]$ clearly gives rise to its natural action on $\text{Hilb}^n(\mathbb{C}^2)$ leaving the punctual Hilbert scheme $\text{Hilb}^n_0(\mathbb{C}^2) \subset \text{Hilb}^n(\mathbb{C}^2)$ invariant, where $\text{Hilb}^n_0(\mathbb{C}^2)$ is the projective subvariety consisting of ideals containing some power of the maximal ideal $(X,Y)$.

Recall from Section 2 that any partition $\sigma$ of $n$ gives rise to the element $I_\sigma \in \text{Hilb}^n(\mathbb{C}^2)$. Furthermore, it is clear from the definition that $I_\sigma$ lies in the punctual Hilbert scheme $\text{Hilb}^n_0(\mathbb{C}^2)$. Moreover, the $T$-stable points of $\text{Hilb}^n(\mathbb{C}^2)$ are precisely the points of the form $I_\sigma$, for some partition $\sigma$ of $n$. Since $\text{Hilb}^n_0(\mathbb{C}^2)$ is a $\text{GL}_2(\mathbb{C})$-stable projective subvariety, the following result follows readily from the Borel’s fixed point Theorem.
Lemma 11. The closure $\overline{B\lambda}$ of the $B$-orbit of any $I \in \text{Hilb}^n_0(\mathbb{C}^2)$ inside $\text{Hilb}^n(\mathbb{C}^2)$ contains a $B$-fixed point (of the form $I_\sigma$).

Lemma 12. For a partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of $n$, the corresponding element $I_\sigma$ is a $B$-invariant point of $\text{Hilb}^n(\mathbb{C}^2)$ if and only if $\sigma_i > \sigma_{i+1}$ for all $i = 0, 1, \cdots, m-1$.

Thus, for a partition $\sigma$ as above such that $I_\sigma$ is $B$-invariant, we have either $\sigma_0 > m + 1$ or else $m = \sigma_0 - 1$ and $\sigma$ is the staircase partition $s_{m+1} : m + 1 > m > \cdots > 1$ (i.e., $\sigma_i = m + 1 - i$ for $i = 0, 1, \cdots, m$).

Proof. By Lemma 10, $I_\sigma$ is $B$-invariant if and only if $X^iY^j \in I_\sigma$, $i > 0 \Rightarrow X^{i-1}Y^{j+1} \in I_\sigma$.

By the definition of $I_\sigma$ this is equivalent to the condition:

$$i \geq \sigma_j, \ i > 0 \Rightarrow i - 1 \geq \sigma_{j+1}.$$ 

This is clearly equivalent to $\sigma_i > \sigma_{i+1}$ for all $i = 0, 1, \cdots, m-1$. This proves the lemma.

For $f \in \mathbb{C}[X,Y]$ we let $\text{Hilb}^n(\mathbb{C}^2)_f$ denote the subset of $\text{Hilb}^n(\mathbb{C}^2)$ defined by

$$\text{Hilb}^n(\mathbb{C}^2)_f = \{ I \in \text{Hilb}^n(\mathbb{C}^2) : f \notin I \}.$$ 

We prove

Lemma 13. $\text{Hilb}^n(\mathbb{C}^2)_f$ is an open subset of $\text{Hilb}^n(\mathbb{C}^2)$.

Proof. Consider the tautological vector bundle $\mathcal{F}$ over the base $\text{Hilb}^n(\mathbb{C}^2)$ of rank $n$ as in Section 2, where the fibre over any $I \in \text{Hilb}^n(\mathbb{C}^2)$ is the $\mathbb{C}$-algebra $\mathbb{C}[X,Y]/I$. Then the multiplication by $f$ on each fibre induces a map of vector bundles $m_f : \mathcal{F} \to \mathcal{F}$. The subset of $\text{Hilb}^n(\mathbb{C}^2)$ where $m_f$ has nonzero rank is then an open subset coinciding with $\text{Hilb}^n(\mathbb{C}^2)_f$.

For a partition $\sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0$ of $n$, consider the ideal

$$I_\sigma(\Lambda) = (Y^{m+1}(Y-\Lambda), Y^{m+1}X) + (X^iY^j : i \geq \sigma_j, \ j \leq m)$$

inside the polynomial ring $\mathbb{C}[X,Y,\Lambda]$. The quotient $\mathbb{C}[X,Y,\Lambda]/I_\sigma(\Lambda)$ is then a free $\mathbb{C}[\Lambda]$-module of rank $n + 1$ with basis $\{X^iY^j : i < \sigma_j \} \cup \{Y^{m+1}\}$. In particular, this defines a flat family

$$\psi : \text{Spec}(\mathbb{C}[X,Y,\Lambda]/I_\sigma(\Lambda)) \to \text{Spec}(\mathbb{C}[\Lambda]) \simeq \mathbb{A}^1,$$

of closed subschemes of $\mathbb{C}^2$ of length $n + 1$. Notice that the fibre of $\psi$ over $\lambda \neq 0$ is $\text{Spec}(\mathbb{C}[X,Y]/I_{\sigma'} \times \mathbb{C}[X,Y]/(Y - \lambda, X))$, while the fibre over $\lambda = 0$ is $\text{Spec}(\mathbb{C}[X,Y]/I_{\sigma'})$ where $\sigma' : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m \geq \sigma_{m+1} > 0$, with $\sigma_{m+1} = 1$, is a partition of $n + 1$.

We conclude

Lemma 14. The validity of the $n!$-result (rather the $n + 1!$-result) for the partition $\sigma'$ implies its validity for the partition $\sigma$, where $\sigma'$ is as above.
Proof. Since the set of points of $\text{Hilb}^{n+1}(\mathbb{C}^2)$ where the maximal rank condition is satisfied is open (cf. Section 3), we get that it is satisfied in an open subset of $\text{Hilb}^{n+1}(\mathbb{C}^2)$ containing $I_{\sigma}$. In particular, from the above family, we get that the maximal rank condition is satisfied for $\psi^{-1}(\lambda)$ for some $\lambda \neq 0$. But $\psi^{-1}(\lambda) \cong \text{Spec}(\mathbb{C}[X,Y]/I_{\sigma} \times \mathbb{C})$. Finally, observe that the maximal rank condition is satisfied for the product of rings $A \times \mathbb{C}$ if and only if it is satisfied for $A$. This proves the lemma.

**Theorem 7.** To prove the validity of the $n!$-result for all $n \geq 1$ and all the partitions of $n$, it suffices to prove its validity for only the staircase partitions $\mathfrak{s}_m : m \geq m - 1 \geq \cdots \geq 1$ (for all $m \geq 1$).

**Proof.** For any $I \in \text{Hilb}^n_0(\mathbb{C}^2)$, we first define $d = d(I)$ as the smallest integer such that $\mathbb{C}[X,Y]_d \subseteq I$. Then clearly $n \leq d(d + 1)/2$ with equality if and only if $I = I_{\mathfrak{s}_d}$ for the staircase partition $\mathfrak{s}_d$. Now define $N(I) = d(d + 1)/2 - n$. Thus $N(I)$ measures, in some sense, how far $I$ is from $I_{\mathfrak{s}_d}$. The assumption of the theorem means that the maximal rank condition is valid for any $I$ with $N(I) = 0$. We prove the theorem by induction on $N(I)$.

So assume $N(I_{\sigma}) > 0$ and that the theorem has been proved for $I_{\mu}$ with $N(I_{\mu}) < N(I_{\sigma})$.

Consider the orbit closure $\overline{B I_{\sigma}}$ inside $\text{Hilb}^n(\mathbb{C}^2)$. By Lemma 1, there exists a $B$-fixed point $I_{\mu} \in \overline{B I_{\sigma}}$ for some partition $\mu$. Since, for any $d$, $\mathbb{C}[X,Y]_d$ is stable under $B$ (in fact, under $\text{GL}_2(\mathbb{C})$), by Lemma 12, $\mathbb{C}[X,Y]_{d(I_{\sigma})}$ is contained in $I_{\mu}$. In particular, $d(I_{\mu}) \leq d(I_{\sigma})$ and thus $N(I_{\mu}) \leq N(I_{\sigma})$. Since the set of points in $\text{Hilb}^n(\mathbb{C}^2)$ satisfying the maximal rank condition is open, we may now assume $\sigma = \mu$ or, in other words, we may assume that $I_{\mu}$ is $B$-stable. Then $\sigma : \sigma_0 > \sigma_1 > \cdots > \sigma_m > 0$ from which we conclude that $d(I_{\sigma}) \geq m + 1$ with equality if and only if $\sigma$ is a staircase partition (use Lemma 13). We therefore assume $d(I_{\sigma}) > m + 1$.

Now, the partition $\sigma'$ of $n + 1$, as defined earlier, also satisfies that $\mathbb{C}[X,Y]_{d(I_{\sigma})}$ is contained in $I_{\sigma'}$. But $N(I_{\sigma'}) < N(I_{\sigma})$. Thus, by induction, the maximal rank condition follows for $I_{\sigma'}$. Finally, by Lemma 14, the maximal rank condition follows for $I_{\sigma}$.

**Remark 5.** Observe that $\text{GL}_2(\mathbb{C})$ has an invariant in $\text{Hilb}^n(\mathbb{C}^2)$ if and only if $n$ is of the form $d(d + 1)/2$ and in this case $I_{\mathfrak{s}_d}$, for the staircase partition $\mathfrak{s}_d$, is the only invariant.

For the staircase partition $\mathfrak{s}_m : m \geq m - 1 \geq \cdots \geq 1$ of $n := m(m + 1)/2$, we have the following (stronger) variant of Theorem 3.

**Proposition 5.** Let $m$ be any positive integer and let $B_m$ be a finite dimensional graded $S_m$-equivariant quotient of the polynomial ring $R_m := \mathbb{C}[X_1, \cdots, X_n, Y_1, \cdots, Y_n]$ such that $B_m^d = 0$ for $d > d_m$ and $B_m^{d_m}$ is the
one dimensional sign representation of $S_n$, where $n := m(m+1)/2$. Assume further that $B_m$ is a Gorenstein algebra. Then

$$B_m \simeq A_{s_m},$$

as graded $S_n$-algebras.

(Observed that $d_{s_m} = m(m-1)(m+1)/3$.)

Proof. The proof of this proposition follows by a similar argument as that of Theorem 3 in view of the following Lemma 15. \qed

Definition 3. Let $n$ be a positive integer. Write $n$ in the form

$$n = 1 + 2 + 3 + \cdots + m + s, \quad 0 \leq s < m + 1.$$

The degree of $n$ is defined as

$$\deg(n) = 1 \cdot 0 + 2 \cdot 1 + \cdots + m \cdot (m-1) + s \cdot m.$$

The integer $s$ is called the remainder of $n$.

Lemma 15. The lowest degree in $R^d_n$ in which the sign representation appears is exactly $\deg(n)$. If the remainder of $n$ is 0 then (and only then) there is a unique copy of the sign representation in degree $\deg(n)$ in $R^d_n$.

Proof. Assume that $L \subseteq R^d_n$ is a copy of the sign representation. Let $l \in L$ be a nonzero element. Write:

$$l = \sum_{\mathbf{i}, \mathbf{j}} \alpha_{\mathbf{i}, \mathbf{j}} X^{\mathbf{i}} Y^{\mathbf{j}},$$

where $\mathbf{i} = (i_1, \ldots, i_n), \mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$, $X^{\mathbf{i}} := X_{i_1}^{i_1} \cdots X_{i_n}^{i_n}$ and $Y^{\mathbf{j}}$ has a similar meaning. Take a nonzero $\alpha_{\mathbf{i}', \mathbf{j}'}$. If there exist $s \neq t$ such that $(\mathbf{i}'_s, \mathbf{j}'_s) = (\mathbf{i}'_t, \mathbf{j}'_t)$ then the permutation $(s, t) \in S_n$ acts on $l$ leaving the coefficient of $X^{\mathbf{i}'} Y^{\mathbf{j}'}$ invariant. As $L$ is a copy of the sign representation this is impossible. Hence, $(\mathbf{i}'_s, \mathbf{j}'_s) \neq (\mathbf{i}'_t, \mathbf{j}'_t)$ whenever $s \neq t$, i.e., $(\mathbf{i}'_1, \mathbf{j}'_1), (\mathbf{i}'_2, \mathbf{j}'_2), \ldots, (\mathbf{i}'_n, \mathbf{j}'_n)$ are distinct elements in $\mathbb{Z}_+^2$.

Consider the function $\phi : \mathbb{Z}_+^2 \rightarrow \mathbb{Z}_+$ defined by $\phi(x, y) = x + y$. Call $\phi(x, y)$ the norm of $(x, y)$. Note that there are exactly $s$ elements in $\mathbb{Z}_+^2$ of norm $s - 1$. Notice also that the degree of $X^{\mathbf{i}'} Y^{\mathbf{j}'}$ is $d = \sum_t \phi(i'_t, j'_t)$. As the pairs $(i'_t, j'_t)$ are distinct, we get that $d \geq \deg(n)$.

Now we construct a copy of the sign representation inside $R^d_n$ : Choose $n$ distinct pairs $(i_t, j_t)$ such that the sum of the norms is $\deg(n)$ (start with pairs of norm 0, then take those of norm 1, \ldots). Notice that there is a unique way of doing this exactly when the remainder of $n$ is 0. Put

$$L = \mathbb{C} \left( \sum_{\sigma \in S_n} \epsilon(\sigma) \sigma(X^{\mathbf{i}'} Y^{\mathbf{j}'}) \right).$$

Then $L$ is a copy of the sign representation sitting in the degree equal to $\deg(n)$. \qed
7. Divided differential operators and the box partition

Let $n$ be a positive integer and let $R^Z_n := \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ be the polynomial ring over $\mathbb{Z}$. For any $m \geq 0$ and any $1 \leq s \leq n$, the divided differential operators $\partial^{(m)}_{X_s} := \frac{1}{m!} \left( \frac{\partial}{\partial X_s} \right)^m$ and $\partial^{(m)}_{Y_s} := \frac{1}{m!} \left( \frac{\partial}{\partial Y_s} \right)^m$ keep $R^Z_n$ stable. In particular, for any $\mathbb{Z}$-module $M$, we obtain the operators (again denoted by) $\partial^{(m)}_{X_s}$ and $\partial^{(m)}_{Y_s}$ on $R^n_M := R^Z_n \otimes_{\mathbb{Z}} M$.

Now let $p$ be a prime and $k$ a field of char. $p$. We will particularly be interested in the operators $\partial_{X_s}, \partial^{(p)}_{X_s}, \partial_{Y_s}$ and $\partial^{(p)}_{Y_s}$ acting on $R^k_n$. Similarly, we have the operators $\partial_{X_s}, \partial^{(p)}_{X_s}$ on $P^k_n := k[X_1, \ldots, X_n]$.

Define a $k$-linear map (in fact, a $k$-algebra homomorphism) $\phi : R^k_n \rightarrow P^k_n$, by $X^i Y^j \mapsto X_i^{1+p^k j}$, for any $i = (i_1, \ldots, i_n)$ and $j = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n$.

**Lemma 16.** For any $1 \leq s \leq n$ and $Q \in R^k_n$,

(a) $\phi(\partial_{X_s}) = \partial_{X_s} \phi$,

(b) $\phi(\partial_{Y_s} Q) = \partial^{(p)}_{X_s} \phi(Q)$, if deg $Q$ in $X_s$ is $< p$.

**Proof.** (a) Let $\delta_s := (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}_+^n$, where 1 is placed in the $s$-th place.

$$\phi(\partial_{X_s} X^i Y^j) = \phi(i_s(X^{\delta_s} Y^j)) = i_s X^{i-\delta_s+p^k j}.$$ 

Also,

$$\partial_{X_s} \phi(X^i Y^j) = \partial_{X_s} X^{i+p^k j} = i_s X^{i+p^k j-\delta_s}.$$ 

This proves (a).

(b) Let $Q = X^i Y^j$. Then

$$\phi(\partial_{Y_s} Q) = \phi(j_s X^{\delta_s} Y^{\delta_s}) = j_s X^p (i-\delta_s).$$

Also,

$$\partial^{(p)}_{X_s} \phi(Q) = \partial^{(p)}_{X_s} X^{i+p^k j} = \left( i_s + p^k j_s \right) X^{i+p^k j-\delta_s} = j_s X^{i+p^k j-\delta_s}, \text{ since } i_s < p.$$ 

This proves (b). \[\square\]

Now take $n = p^2$ (where $p$ is a prime) and consider the box partition $b_p : p \geq p \geq \cdots \geq p$ of $n$. Let $V^k_n := k[\partial_{X_1}, \ldots, \partial_{X_n}, \partial_{Y_1}, \ldots, \partial_{Y_n}](\Delta_{b_p}) \subset R^k_n$.

Note that $\Delta_{b_p}$ has degree $< p$ in each variables $X_s$ and $Y_s$. The following lemma is clear.

**Lemma 17.** $\phi(\Delta_{b_p}) = \pm \Delta_n$ where $\Delta_n \in P^k_n$ is the standard Vandermonde determinant $\prod(X_s - X_t)$ (product being taken over $1 \leq s < t \leq n$) and $n := p^2$. 

Let $D^k_n$ be the algebra of all the divided differential operators in $n$ variables, i.e., the algebra over $k$ generated by $\{\partial^{(m)}_{X_s}; 1 \leq s \leq n, m \geq 0\}$ inside the algebra of all the linear operators of $P^k_n$. Since $\partial^{(p^2)}_{X_s}$ kills $\Delta_n$, by Lemma 16(b), we get the following.

**Proposition 6.**

$$\phi(V^k_n) = k[\partial X_1, \ldots, \partial X_n, \partial^{(p)} X_1, \ldots, \partial^{(p)} X_n] \cdot \Delta_n.$$  

where $D^k_n(\Delta_n)$ denotes the space $\{D(\Delta_n) : D \in D^k_n\}$ inside $P^k_n$.

**Corollary 1.** $\dim_k V^k_n \geq \dim_k D^k_n(\Delta_n)$.

**Conjecture 3.** Let $n$ be any positive integer and let $p$ be a prime such that $p^2 \geq n$. Then, for any field $k$ of char. $p$,

$$\dim_k D^k_n(\Delta_n) = \dim_{\mathbb{C}}\text{Diff}_n(\Delta_n),$$

where $\text{Diff}_n(\Delta_n) := \mathbb{C}[\partial X_1, \ldots, \partial X_n] \cdot \Delta_n$.

The latter of course is well known to have dimension $n!$.

**Remark 6.** (a) It is easy to see that

$$\dim_k D^k_n(\Delta_n) \leq \dim_{\mathbb{C}}\text{Diff}_n(\Delta_n).$$

(b) It is possible that the above conjecture is true for any prime $p$ (not necessarily under the restriction $p^2 \geq n$), as some explicit calculations indicate, e.g., we have verified the conjecture for $n = 5, p = 2$.

(c) The above conjecture is false, in general, for the other polynomials which transform under the action of the symmetric group $S_n$ via the sign character. Take, e.g., $n = 2$ and $f = (X_1 - X_2)(X_1 + X_2)$. Then, for $p = 2$, $\dim_k D^k_n(f) = 2$, whereas $\dim_{\mathbb{C}}\text{Diff}_n(f) = 4$.

Assuming the above conjecture, we get the following theorem.

**Theorem 8.** Assume that $n = p^2$ and $k$ is a field of char. $p$. If the above Conjecture 3 is true, then

$$\dim_k V^k_n = n!.$$  

In particular,

$$\dim_{\mathbb{C}}\text{Diff}_{2n}(\Delta_{b_p}) = n!.$$  

Thus the validity of Conjecture 3 will imply the $n!$ result for the box partition $b_p$ of $n = p^2$.

**Proof.** By combining Corollary 1 and Conjecture 3, we get that $\dim_k V^k_n \geq n!$. But, $\dim_k V^k_n \leq \dim_{\mathbb{C}}\text{Diff}_{2n}(\Delta_{b_p}) \leq n!$. Thus, both the identities of the theorem follow.  

\[\square\]
8. Gr Description of $A_\sigma$ (box plus one row case)

Let $n = pq + r$ with any integers $p > r \geq 0$ and $p > 1, q \geq 1$, and consider the box plus one row partition $\sigma = \sigma(p, q, r)$: $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_q > 0$ of $n$ with $\sigma_i = p$ for $0 \leq i \leq q - 1$ and $\sigma_q = r$. In this section we describe the algebra $A_\sigma$ as the associated graded ring of a specific filtration, depending only on $p$, of the cohomology ring $H^*(SL_n/B, \mathbb{C})$. Here $B$ is the Borel subgroup of $SL_n$ consisting of the upper triangular matrices of determinant one. By a classical result we may identify $H^*(SL_n/B, \mathbb{C})$ with the ring $P_n/J$, where $P_n := \mathbb{C}[X_1, \ldots, X_n]$ and $J \subset P_n$ denotes the ideal generated by the $S_n$-invariant elements of positive degree.

8.1. Preliminary results. Recall the definitions of $S_{h,t,k} \in \mathbb{C}[Z_1, \ldots, Z_t]$ and $n_k(\sigma)$ from the appendix. In this section we will use the notation $n_k$ and $n_k^\lor$ to denote $n_k(\sigma)$ and $n_k(\sigma^\lor)$ respectively. Then, for the partition $\sigma = \sigma(p, q, r)$,

$$n_k = \begin{cases} q(p-k) + (r-k) & \text{if } 0 \leq k \leq r, \\ q(p-k) & \text{if } r \leq k < p, \\ 0 & \text{if } k \geq p. \end{cases}$$

while

$$n_k^\lor = \begin{cases} p(q-k) + r & \text{if } 0 \leq k \leq q, \\ 0 & \text{if } k > q. \end{cases}$$

Let $J_p$ denote the ideal in $P_n$ generated by $J$ and the elements $X_1^p, \ldots, X_n^p$. Then

**Lemma 18.** Let $h$, $k$ and $1 \leq t \leq n$ be nonnegative integers. If $h + t \geq n_k + 1$, then

$$S = S_{h,t,k}(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) \in J_p,$$

whenever $1 \leq i_1 < i_2 < \cdots < i_t \leq n$. In particular, by Theorem 10, $J(\sigma) \subset J_p$ where $\sigma = \sigma(p, q, r)$.

**Proof.** Observe first of all that the statement is clear when $k \geq p$. In the following we hence assume that $k < p$. By symmetry it suffices to consider the case $i_j = j$, $j = 1, \ldots, t$. Assume first that $t \leq q$. Let $M = x_1^{s_1}x_2^{s_2}\cdots x_t^{s_t}$ be a monomial of degree $h$. If $s_j \leq p - k - 1$ for every $1 \leq j \leq t$, then

$$h = \deg(M) \leq t(p-k) - 1 \leq q(p-k) - t \leq n_k - t \leq h - 1,$$

which is a contradiction. Hence, there exists a $j$ such that $s_j \geq p - k$. But then each term in $S$ is divisible by some $X_j^p$ and hence $S \in J_p$.

Assume now that $t \geq q + 1$ and that the result, by induction, is true for smaller values of $t$. If $k = 0$ then by Lemma 1.2 in [9] we find $S \in J \subseteq J_p$. So assume that $k \geq 1$ and that the statement, by induction, is correct for smaller values of $k$. As

$$(h+t) + t \geq h + t + q + 1 \geq n_k + q + 2 \geq n_{k-1} + 1,$$
we know by induction that
\[ S_{h+t,t,k-1}(X_1, \ldots, X_t) \in J_p. \]
Rewriting \( S_{h+t,t,k-1}(X_1, \ldots, X_t) \) as
\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq t} (X_1 \cdots X_t)^{k-1}(X_{i_1}X_{i_2} \cdots X_{i_l})S_{h+t-t-l}(X_{i_1}, \cdots, X_{i_l})
\]
and noticing, by induction, that all the terms with \( l < t \) of the last expression belongs to \( J_p \), we conclude that \( S_{h,t,k}(X_1, \cdots, X_t) \in J_p \).

Let \( J_q^\lor \) denote the ideal in \( P_n \) generated by the elements of the form
\[
(X_{i_1} \cdots X_{i_{r+1}})^q, 1 \leq i_1 < \cdots < i_{r+1} \leq n,
\]
and together with the ideal \( J_{q+1} \) (defined similar to \( J_p \)).

**Lemma 19.** Let \( h, k \) and \( 1 \leq t \leq n \) be nonnegative integers. If \( h + t \geq n_k + 1 \) then
\[
S = S_{h,t,k}(X_{i_1}, X_{i_2}, \ldots, X_{i_t}) \in J_q^\lor,
\]
whenever \( 1 \leq i_1 < i_2 < \cdots < i_t \leq n \). In particular, by Theorem 10, \( \hat{J}(\sigma^\lor) \subset J_q^\lor \), where \( \sigma = \sigma(p, q, r) \).

**Proof.** Notice first of all that the statement is clear when \( k > q \). Hence, in the following, we assume that \( k \leq q \). By symmetry it suffices to consider the case \( i_j = j, j = 1, \ldots, t \). Assume first that \( t \leq p \). Assume, if possible, that \( S \not\in J_q^\lor \). Then, by the definition of \( J_q^\lor \), there exists a monomial \( M = X_1^{s_1}X_2^{s_2} \cdots X_t^{s_t} \) of degree \( h + ket \) in the decomposition of \( S \) such that at most \( r \) of the indices \( s_j \) are \( \geq q \) and none of the indices are \( \geq q + 1 \). Hence
\[
h + tk = \deg(M) \leq rq + (t - r)(q - 1) \leq n_k^q + tk - t \leq h + tk - 1,
\]
which is a contradiction.

Thus we are left with the case \( t > p \). Assume by induction that the statement is correct for smaller values of \( t \). If \( k = 0 \) then by Lemma 1.2 in [4] we find \( S \in J \subseteq J_q^\lor \). So assume, by another induction, that \( k \geq 1 \) and that the statement is correct for smaller values of \( k \). As
\[
(h + t) + t \geq h + t + p + 1 \geq n_k^q + p + 2 \geq n_k^{q-1} + 1,
\]
we know by induction that
\[
S_{h+t,t,k-1}(X_1, \ldots, X_t) \in J_q^\lor.
\]
Using the identity (6) for \( S_{h+t,t,k-1}(X_1, \ldots, X_t) \) and noticing, by induction, that all the terms with \( l < t \) belong to \( J_q^\lor \), we conclude that \( S_{h,t,k}(X_1, \cdots, X_t) \in J_q^\lor \). \( \square \)

**Remark 7.** By the definition of \( \hat{J}(\sigma) \) and \( \hat{J}(\sigma^\lor) \) and Lemmas 18, 19, it follows easily that, in fact, \( J_p = \hat{J}(\sigma) \) and \( J_q^\lor = \hat{J}(\sigma^\lor) \).
8.2. The Gr description. Let $x_i$ denote $X_i + J \in P_n/J$ and consider the filtration of $P_n/J$:

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots ,$$

where $F_0 = C$, $F_1 = \text{span}_C \{1, x_1, x_2, \ldots, x_n, x_1^p, x_2^p, \ldots, x_n^p\}$ and $F_s$ is defined to be $F_1^s$ (for any $s \geq 1$). Let $\text{Gr}(F)$ denote the associated graded algebra

$$\text{Gr}(F) := \bigoplus_{s \geq 0} F_s/F_{s-1},$$

where $F_{-1} := (0)$. Under the standard action of $S_n$ on $P_n$, each $F_s$ is $S_n$-stable and hence $\text{Gr}(F)$ is canonically a $S_n$-module.

Consider now the algebra homomorphism $\pi^\vee : P_n \to \text{Gr}(F)$ defined by $\pi^\vee(X_i) = x_i + F_0 \in F_1/F_0$. By Lemma 18, $J(\sigma) \subseteq J_\pi \subseteq \ker(\pi^\vee)$ and thus, by Theorem 19, we get an induced map of graded algebras

$$\overline{\pi}^\vee : H^*(X_{\sigma^\vee}) \to \text{Gr}(F).$$

Consider also the algebra homomorphism $\pi : P_n \to \text{Gr}(F)$ defined by $\pi(X_i) = x_i^p + F_0 \in F_1/F_0$. As $p(q + 1) \geq n$, we know that $X_i^{p(q+1)} \in J$ and hence $\pi(X_i^{q+1}) = 0$. Also, by [12], Cor. 3.2.2,

$$(X_i^1 \cdots X_{i+1}^p)^p \in J.$$ 

Thus, from Lemma 19, we conclude that $J(\sigma^\vee) \subseteq J_\pi \subseteq \ker(\pi)$. This gives us an induced map of graded algebras

$$\overline{\pi} : H^*(X_{\sigma}) \to \text{Gr}(F).$$

Moreover, both $\overline{\pi}$ and $\overline{\pi}^\vee$ are $S_n$-equivariant.

**Proposition 7.** The map $\phi = \overline{\pi} \otimes \overline{\pi}^\vee : H^*(X_{\sigma}) \otimes H^*(X_{\sigma^\vee}) \to \text{Gr}(F)$ taking $a \otimes b \mapsto \overline{\pi}(a) \cdot \overline{\pi}^\vee(b)$ is an $S_n$-equivariant surjective graded algebra homomorphism. Furthermore, the top graded component of $\text{Gr}(F)$ is of degree $d_{\sigma}$ and is 1-dimensional supporting the sign representation of $S_n$.

**Proof.** As $\text{Im}(\phi)$, by definition, contains $F_1/F_0$ and $F_0$ and as $\text{Gr}(F)$ (by definition) is generated by degree one elements, we get that $\phi$ is surjective. As the top graded component in $H^*(X_{\sigma}) \otimes H^*(X_{\sigma^\vee})$ is of degree $d_{\sigma}$, the top graded component in $\text{Gr}(F)$ is of degree at most $d_{\sigma}$. As a representation of $S_n$, the ring $\text{Gr}(F)$ is of course isomorphic to $P_n/J$ and thus $\text{Gr}(F)$ is isomorphic to the regular representation of $S_n$. In particular, $\text{Gr}(F)$ contains a unique copy of the sign representation of $S_n$. But $H^*(X_{\sigma}) \otimes H^*(X_{\sigma^\vee})$ contains a unique copy of the sign representation which occurs in the top degree $d_{\sigma}$ (cf. the proof of Proposition 3). Therefore, the copy of the sign representation in $\text{Gr}(F)$ must also be placed in degree $d_{\sigma}$. So the proposition follows from the following. 

**Proposition 8.** The top graded component of $\text{Gr}(F)$ is 1-dimensional.
Proof. By Remark 3.4.(ii) in [4], we know that the top degrees of \( H^*(X_\sigma) \) and \( H^*(X_{\sigma^\vee}) \) are respectively
\[
d = \frac{2qr + p(q - 1)q}{2},
\]
\[
d^\vee = \frac{qp(p - 1) + r(r - 1)}{2}.
\]
The image of the top degree in \( H^*(X_\sigma) \otimes H^*(X_{\sigma^\vee}) \) under \( \phi \), can therefore be represented by elements in \( P_n \) of degree
\[
pd + d^\vee = \frac{n(n - 1)}{2}.
\]
But modulo \( J \) there is only one element (up to a constant) in \( P_n \) of this degree.

Corollary 2. Let \( n = pq + r \) be as in the beginning of this section. Then there exists a \( S_n \)-equivariant isomorphism of graded algebras:
\[
\text{Gr}(F) \simeq A_\sigma,
\]
where \( \sigma = \sigma(p, q, r) \).

Proof. Define the ideal \( I \subset \text{Gr}(F) \) by
\[
I = \{ a \in \text{Gr}(F) : \text{the top degree component of } ab \text{ equals 0 for all } b \in \text{Gr}(F) \}.
\]
Clearly \( I \) is a \( S_n \)-stable graded ideal of \( \text{Gr}(F) \). Then \( \text{Gr}(F)/I \) satisfies all the characterizing properties of Theorem 3 (in view of Proposition 7). Thus \( \text{Gr}(F)/I \simeq T_\sigma \). But, by Theorem 4 and the \( n! \) result, we know that \( T_\sigma \) has complex dimension \( n! = \dim \mathbb{C} \text{Gr}(F) \). This forces \( I = 0 \) and hence \( \text{Gr}(F) \simeq T_\sigma \simeq A_\sigma \).

Remark 8. (a) In the above proof we have used the \( n! \) result for the partition \( \sigma = \sigma(p, q, r) \). Conversely, the validity of the above corollary clearly implies the \( n! \) result for \( \sigma \).

(b) In [12], Section 6, a statement equivalent to Corollary 2 for 2-row shapes (i.e., \( q = 1 \)) appears without proof. In the same paper it is noted that computational experiments suggest that Corollary 2 should be true. Recently, M. Haiman informed us that he had an unpublished proof of Corollary 2 (assuming the validity of the \( n! \)-conjecture for \( \sigma(p, q, r) \)) prior to our work.

9. Appendix: Cohomology of Springer fibres

We recall some well known results on the cohomology of Springer fibres in the case of \( G = SL_n(\mathbb{C}) \) (which we have used in the paper).

Let \( \sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0 \) be a partition of \( n \) and let \( \sigma^\vee : \sigma'_0 \geq \cdots \geq \sigma'_{m'} > 0 \) be the dual partition. For any \( m' < i \leq n \), set \( \sigma'_i = 0 \). For any
$1 \leq k \leq n$, recall the definition of the integer $d_k(\sigma)$ from Section 3, equation (1). For any $k \geq 0$, let $n_k = n_k(\sigma)$ denote the integer

$$n_k := \sigma'_k + \sigma'_{k+1} + \cdots.$$ 

Let $G = SL_n(\mathbb{C})$ and let $X_\sigma$ be the Springer fibre (over complex numbers) corresponding to a fixed nilpotent matrix $M_\sigma$ (of size $n \times n$) with Jordan blocks of sizes $\sigma_0, \sigma_1, \ldots, \sigma_m$. Let $H^*(X_\sigma)$ denote the cohomology ring of $X_\sigma$ with coefficients in $\mathbb{C}$.

The following description of $H^*(X_\sigma)$ is due to Tanisaki [22], though in its present form we have taken it from [13], p. 84-85. The notation $e_r$ denotes the $r$-th elementary symmetric polynomial. As earlier, $P_n := \mathbb{C}[X_1, \ldots, X_n]$.

**Theorem 9.** As a graded algebra, we have an $S_n$-equivariant isomorphism

$$H^*(X_\sigma) \simeq P_n/J(\sigma),$$

where $J(\sigma)$ is the ideal generated by all polynomials of the form

$$e_r(X_{s_1}, X_{s_2}, \ldots, X_{s_k})$$

subject to the conditions $1 \leq s_1 < \cdots < s_k \leq n$ and all $k, r \geq 1$ satisfying $k - d_k(\sigma) < r \leq k$.

**Definition 4.** Let $t \geq 1$ and $h, k \geq 0$ be integers. Let $S_h^t \in \mathbb{C}[Z_1, \ldots, Z_t]$ denote the sum of all monomials of degree $h$ in the variables $Z_1, \ldots, Z_t$. We then define

$$S_{h,t,k} = (Z_1 \cdots Z_t)^k S_h^t \in \mathbb{C}[Z_1, \ldots, Z_t].$$

The following description of the cohomology of $X_\sigma$ is taken from [9], Theorem 2.2.

**Theorem 10.** Let $\hat{J}(\sigma)$ denote the ideal in $P_n$ generated by the polynomials $S_{h,t,k}(X_{i_1}, X_{i_2}, \ldots, X_{i_t})$ for nonnegative integers $h, k$ and $1 \leq t \leq n$ subject to the condition $h + t = n_k + 1$, and $1 \leq i_1 < i_2 < \cdots < i_t \leq n$. Then there exists an $S_n$-equivariant isomorphism

$$H^*(X_\sigma^\vee) \simeq P_n/\hat{J}(\sigma).$$

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