On the $q$-Laplace transform in the non-extensive statistical physics

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In this paper, $q$-Laplace transforms related to the non-extensive thermodynamics are investigated by using the algebraic operation of the non-extensive calculus. The deformed simple harmonic problem is discussed by using the $q$-Laplace transform.

I. INTRODUCTION

Boltzman-Gibbs statistical mechanics shows how fast microscopic physics with short-range interaction has as effect on much larger space-time scale. The Boltzman-Gibbs entropy is given by

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} \ln \frac{1}{p_i}$$  (1)

where $k$ is a Boltzman constant, $W$ is a total number of microscopic possibilities of the system and $p_i$ is a probability of a given microstate among $W$ different ones satisfying $\sum_{i=1}^{W} p_i = 1$. When $p_1 = \frac{1}{W}$, we have $S_{BG} = k \ln W$.

Boltzman-Gibbs theory is not adequate for various complex, natural, artificial and social system. For instance, this theory does not explain the case that a zero maximal Lyapunov exponent appears. Typically, such situations are governed by power-laws instead of exponential distributions. In order to deal with such systems, the non-extensive statistical mechanics is proposed by C.Tsallis [1,2]. The non-extensive entropy is defined by

$$S_q = k (\sum_{i=1}^{W} p_i^q - 1) / (1 - q)$$  (2)

The non-extensive entropy has attracted much interest among the physicist, chemist and mathematicians who study the thermodynamics of complex system [3]. When the deformation parameter $q$ goes to 1, Tsallis entropy (2) reduces to the ordinary one (1). The non-extensive statistical mechanics has been treated along three lines:

1. Mathematical development [4, 5, 6]
2. Observation of experimental behavior [7]
3. Theoretical physics (or chemistry) development [8]

The basis of the non-extensive statistical mechanics is $q$-deformed exponential and logarithmic function which is different from those of Jackson’s [9]. The $q$-deformed exponential and $q$-logarithm of non-extensive statistical mechanics is defined by [10]

$$\ln_q t = \frac{t^{1-q} - 1}{1 - q}, \quad (t > 0)$$  (3)

$$e_q(t) = (1 + (1 - q)t)^{\frac{1}{1-q}}, \quad (x, q \in R)$$  (4)

where $1 + (1 - q)t > 0$.

From the definition of $q$-exponential and $q$-logarithm, $q$-sum, $q$-difference, $q$-product and $q$-ratio are defined by [5, 6]

$$x \oplus y = x + y + (1 - q)xy$$

$$x \odot y = \frac{x - y}{1 + (1 - q)y}$$

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\[
x \otimes y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}
\]
\[
x \odot y = [x^{1-q} - y^{1-q} + 1]^{\frac{1}{1-q}}
(5)
\]

It can be easily checked that the operation \( \oplus \) and \( \otimes \) satisfy commutativity and associativity. For the operator \( \oplus \), the identity additive is 0, while for the operator \( \otimes \) the identity multiplicative is 1. Indeed, there exist an analogy between this algebraic system and the role of hyperbolic space in metric topology [11]. Two distinct mathematical tools appears in the study of physical phenomena in the complex media which is characterized by singularities in a compact space [12].

For the new algebraic operation, q-exponential and q-logarithm have the following properties:

\[
\ln_q(xy) = \ln_q x \oplus \ln_q y \\
\ln_q(x \otimes y) = \ln_q x + \ln_q y \\
\ln_q(x/y) = \ln_q x \odot \ln_q y \\
\ln_q(x \otimes y) = \ln_q x - \ln_q y
(6)
\]

From the associativity of \( \oplus \) and \( \otimes \), we have the following formula :

\[
t \oplus t \oplus t \oplus \cdots \oplus t = \frac{1}{1-q} \left\{ (1+1-q)t^n - 1 \right\}
(7)
\]

\[
t^{\otimes n} = t \otimes t \otimes t \otimes \cdots \otimes t = [n t^{1-q} - (n-1)]^{\frac{1}{1-q}}
(8)
\]

II. Q-LAPLACE TRANSFORM

In this section, we find the q-Laplace transform related to the non-extensive thermodynamics. From the relation

\[
e_q(x)e_q(y) = e_q(x \oplus y),
(9)
\]

the q-analogue of \( e^{nx} \), \( n \in Z \) is given by

\[
e_q(n \odot t) = [e_q(t)]^n = e_q \left( \frac{1}{1-q} [(1+1-q)t^n - 1] \right) = (1+1-q)t^{\frac{n}{1-q}}
(10)
\]

Then we have

\[
e_q(0 \odot t) = 1
(11)
\]

and the inverse of \( e_q(n \odot t) \) is \( e_q((-n) \odot t) \). For this adoption, q- Laplace kernel is defined by

\[
e_q((-s) \odot t) = [e_q(t)]^{-s} = e_q \left( \frac{1}{1-q} [(1+1-q)t^{-s} - 1] \right) = (1+1-q)t^{-\frac{s}{1-q}}
(12)
\]

Therefore q-Laplace transform is defined by

\[
L_s(F(t)) = \int_0^\infty [e_q(t)]^{-s} F(t) dt, \quad (s > 0)
(13)
\]

Limiting \( q \to 1 \), the eq.(13) reduces to an ordinary Laplace transform. Form now on we assume that \( s \) is sufficiently large.

Since, for two functions \( F(t) \) and \( G(t) \), for which the integral exist

\[
L_s(aF(t) + bG(t)) = aL_s(F(t)) + bL_s(G(t))
(14)
\]

the q-Laplace transform is linear.

For \( F(t) = t^N \), \( N = 0, 1, 2, \cdots \) , we have the following result.
Theorem 1 For sufficiently large $s$, when $q < 1$, the following holds:

$$L_s(t^N) = \frac{N!}{(s; 1-q)_{N+1}}$$  \hspace{1cm} (15)$$

where

$$(a; Q)_n = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^{n} (a-kQ) & (n \geq 1) \end{cases}$$  \hspace{1cm} (16)$$

Proof. Let us assume that the eq.(15) holds for $t^N$. Then,

$$L_s(t^{N+1}) = \int_{0}^{\infty} [e_q(t)]^{-s} t^{N+1} dt$$

$$= \int_{0}^{\infty} \frac{(1 + (1-q)t)^{-\frac{s}{1-q}}}{1-q-s} t^{N+1} dt$$

$$= \left[ \frac{(1 + (1-q)t)^{-\frac{s}{1-q}}}{1-q-s} t^{N+1} \right]_{0}^{\infty} + \frac{N+1}{s-(1-q)} \int_{0}^{\infty} (1 + (1-q)t)^{-\frac{s}{1-q}} t^{N} dt$$

$$= \frac{N+1}{s-(1-q)} L_{s-(1-q)}(t^{N+1})$$

$$= \frac{(N+1)N!}{(N+1)!(s-(1-q))(s-(1-q); 1-q)_{N+1}}$$

$$= \frac{1}{(s; 1-q)_{N+2}}$$  \hspace{1cm} (17)$$

In a similar way, we can obtain the q-Laplace transform for $e_q(a \odot t)$ which is given by

$$e_q(a \odot t) = [e_q(t)]^a$$

Theorem 2 For sufficiently large $s$, when $q < 1$, the following holds:

$$L_s(e_q(a \odot t)) = \frac{1}{s-a-(1-q)}$$  \hspace{1cm} (18)$$

Proof. It is trivial.

In order to obtain q-Laplace transform for the trigonometric function, we need q-analogue of Euler identity. The q-Euler formula is given by

$$e_q(ia \odot t) = C_q(a \odot t) + iS_q(a \odot t),$$  \hspace{1cm} (19)$$

where q-cosine and q-sine functions are defined by

$$C_q(a \odot t) = \cos\left(\frac{a}{1-q} \ln(1 + (1-q)t)\right)$$

$$S_q(a \odot t) = \sin\left(\frac{a}{1-q} \ln(1 + (1-q)t)\right)$$  \hspace{1cm} (20)$$

and we used the following identity.

$$p^i = e^{i \ln p} = \cos \ln p + i \sin \ln p$$  \hspace{1cm} (21)$$
Indeed, q-cosine and q-sine functions can be expressed in terms of q-exponential as follows:

\[
C_q(a \odot t) = \frac{1}{2}(e_q(ia \odot t) + e_q((-ia) \odot t))
\]

\[
S_q(a \odot t) = \frac{1}{2i}(e_q(ia \odot t) - e_q((-ia) \odot t))
\]

(22)

Then we have the q-Laplace transform for q-sine and q-cosine functions:

**Theorem 3** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[
L_s(C_q(a \odot t)) = \frac{s - (1 - q)}{(s - (1 - q))^2 + a^2}
\]

\[
L_s(S_q(a \odot t)) = \frac{a}{(s - (1 - q))^2 + a^2}
\]

(23)

*Proof.* It is trivial.

The eq. (23) can be written in terms of the ordinary sine and cosine functions:

\[
L_s(\cos[\frac{a}{1 - q}\ln(1 + (1 - q)t)]) = \frac{s - (1 - q)}{(s - (1 - q))^2 + a^2}
\]

\[
L_s(\sin[\frac{a}{1 - q}\ln(1 + (1 - q)t)]) = \frac{a}{(s - (1 - q))^2 + a^2}
\]

(24)

The q-sine function and q-cosine function have the following zeros:

\[
S_q(1 \odot t_n) = S_q(t_n) = 0, \quad C_q(1 \odot u_n) = C_q(u_n) = 0,
\]

(25)

where

\[
t_n = \ln_q e^{n\pi}, \quad u_n = \ln_q e^{(n+\frac{1}{2})\pi}, \quad n \in \mathbb{Z}
\]

(26)

From the zeros of the q-sine function and q-cosine function, we have the following Theorem:

**Theorem 4** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[
S_q(t) = t \prod_{j=1}^{\infty} \left( 1 - \frac{t}{\ln_q e^{j\pi}} \right) \left( 1 - \frac{t}{\ln_q e^{-j\pi}} \right)
\]

(27)

\[
C_q(t) = \left( 1 - \frac{t}{\ln_q e^{-\pi/2}} \right) \prod_{j=1}^{\infty} \left( 1 - \frac{t}{\ln_q e^{(j+1/2)\pi}} \right) \left( 1 - \frac{t}{\ln_q e^{-(j+1/2)\pi}} \right)
\]

(28)

*Proof.* From the zeros of the q-sine function, we can set

\[
\frac{S_q(t)}{t} = A \prod_{j=1}^{\infty} \left( 1 - \frac{t}{\ln_q e^{j\pi}} \right) \left( 1 - \frac{t}{\ln_q e^{-j\pi}} \right)
\]
Because \( \lim_{t \to 0} S_q(t) = 1 \), we have \( A = 1 \), which proves the eq.(27). Similarly we can easily prove the eq.(28).

Theorem 4 can be also written as follows:

**Theorem 5** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[
S_q(t) = t \prod_{j=1}^{\infty} \left[ 1 + (1-q)t - \left( \frac{(1-q)t}{2 \sinh \left( \frac{(1-q)t}{2} \right)} \right)^2 \right] 
\]

\[
C_q(t) = \left( 1 - \frac{t}{\ln_q e^{-\pi/2}} \right) \prod_{j=1}^{\infty} \left[ 1 + (1-q)t - \left( \frac{(1-q)t}{2 \sinh \left( \frac{1}{2} (1-q)(j+1/2) \pi \right)} \right)^2 \right] 
\]

**Proof.** It is trivial from the formula \( \cosh x - 1 = 2 \sinh^2 \frac{x}{2} \).

### III. Q-LAPLACE TRANSFORM AND DIFFERENTIAL EQUATION

Now we discuss the q-differential equation. The main purpose of q-Laplace transform is in converting q-differential equation into simpler forms which may be solved more easily. Like the ordinary Laplace transform, we can compute the q-Laplace transformation of derivative by using the definition of the q-Laplace transform, which is given by

\[
L_s(F'(t)) = sL_s + 1 - q(F(t)) - F(0) 
\]

An extension gives

\[
L_s(F''(t)) = s(s + 1 - q)L_{s+2(1-q)}(F(t)) - sF(0) - F'(0) 
\]

Generally, we have following theorem:

**Theorem 6** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[
L_s(F^{(n)}(t)) = [s; 1-q]_n L_{s+n(1-q)}(F(t)) - \sum_{i=0}^{n-1} [s; 1-q]_{n-1-i} F^{(i)}(0), 
\]

where \( F^{(0)}(0) = F(0) \) and

\[
[a; Q]_n = \begin{cases} 
1 & (n = 0) \\
\prod_{k=1}^{n} (a + kQ) & (n \geq 1) 
\end{cases} 
\]

**Proof.** Let us assume that the eq.(34) holds for \( n \). Then,

\[
L_s(F^{(n+1)}(t)) = \int_0^{\infty} (1 + (1-q)t) \frac{d}{dt} F^{(n+1)}(t) dt 
\]

\[
= sL_{s+1-q}(F^{(n)}(t)) - F^{(n)}(0) 
\]

\[
= s\{[s + 1 - q; 1-q]_n L_{s+(n+1)(1-q)}(F(t)) - \sum_{i=0}^{n-1} [s + 1 - q; 1-q]_{n-1-i} F^{(i)}(0)\} - F^{(n)}(0) 
\]

\[
= [s; 1-q]_{n+1} L_{s+(n+1)(1-q)}(F(t)) - \sum_{i=0}^{n} [s; 1-q]_{n-i} F^{(i)}(0) 
\]
We have another formula for the q-Laplace transform of derivative as follows:

\[ L_s(F'(t)) = sL_s\left(\frac{F(t)}{1 + (1 - q)t}\right) - F(0) \quad (36) \]

An extension gives

\[ L_s(F''(t)) = s(s + 1 - q)L_s\left(\frac{F(t)}{(1 + (1 - q)t)^2}\right) - sF(0) - F'(0) \quad (37) \]

Generally, we have the following theorem:

**Theorem 7** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[ L_s(F^{(n)}(t)) = [s; 1 - q]_nL_s\left(\frac{F(t)}{(1 + (1 - q)t)^n}\right) - \sum_{i=0}^{n-1}[s; 1 - q]_{n-1-i}F^{(i)}(0) \quad (38) \]

**Proof.** It is not hard to prove Theorem 7.

Comparing Theorem 6 with Theorem 7, we have the following Lemma:

**Lemma 8** For sufficiently large \( s \), when \( q < 1 \), the following holds:

\[ L_s\left(\frac{F(t)}{(1 + (1 - q)t)^n}\right) = L_{s+n(1-q)}(F(t)) \quad (39) \]

**Proof.** It is trivial.

With the help of q-Laplace transform of the derivative, we can solve some differential equation. It is worth noting that \( e_q(t) \) is not invariant under the ordinary derivative, instead it obeys

\[ \frac{d}{dt}(e_q(t)) = \frac{1}{1 + (1 - q)t}e_q(t) \quad (40) \]

Now consider the following differential equation:

\[ F'(t) = \frac{F(t)}{1 + (1 - q)t}, \quad F(0) = 1 \quad (41) \]

It is evident that \( e_q(t) \) is a solution of the eq.(40).

Let us consider the vertical motion of a body in a resisting medium in which there again exists a retarding force proportional to the velocity. Let us consider that the body is projected downward with zero initial velocity \( v(0) = 0 \) in a uniform gravitational field. The equation of motion is then given by

\[ m\frac{d}{dt}v = mg - kv(t) \quad (42) \]

This equation is not solved by using the q-Laplace transform, instead we solve the following equation:

\[ m\frac{d}{dt}v = mg - k\frac{v(t)}{1 + (1 - q)t} \quad (43) \]

The solution of the eq.(43) is then given by

\[ v(t) = \frac{g}{1 - q + \frac{k}{m}(1 + (1 - q)t - [e_q(t)]^{-\frac{1}{q}})} \quad (44) \]
Similarly, we can modify the harmonic problem whose equation of motion is given by
\[
m \left( \frac{d}{dt} \right)^2 x(t) = -mw^2 \frac{x(t)}{(1+(1-q)t)^2},
\]
where \(x(0) = A, \frac{d}{dt}x(0) = 0\). The solution of the eq.(48) is then given by
\[
x(t) = A(1+(1-q)t)^2 \left\{ C_{w,1} \left( \sqrt{w^2 - \left(\frac{q-1}{2}\right)^2} t \right) + \frac{q-1}{\sqrt{w^2 - \left(\frac{q-1}{2}\right)^2}} S_{w,1} \left( \sqrt{w^2 - \left(\frac{q-1}{2}\right)^2} t \right) \right\}
\]
(46)

The eqs.(43) and (45) seem to be too artificial due to the factor \(1+(1-q)t\). Instead, we can introduce the q-derivative [6] instead of the ordinary time derivative as follows:
\[
D_t F(t) = \lim_{s \to t} \frac{F(t) - F(s)}{t \ominus s} = [1+(1-q)t] \frac{dF}{dt}
\]
The Leibniz rule for q-derivative is as follows:
\[
D_t [F(t)G(t)] = D_t [F(t)]G(t) + F(t)D_t [G(t)]
\]
(47)

Then the eq.(43) is replaced as follows:
\[
m D_t v = mg - kv(t)
\]
(48)
The solution of the eq.(48) is then given by
\[
v(t) = \frac{mg}{k} \left( 1 - [e_q(t)]^{-\frac{1}{m}} \right)
\]
(49)

Similarly, the eq.(45) is replaced as follows:
\[
m D_t^2 x(t) = -mw^2 x(t)
\]
(50)

Using the q-Laplace transform, we get the solution of the eq.(50):
\[
x(t) = AC_q(w \odot t)
\]
(51)

Obtaining these solutions, we used the following theorem:

**Theorem 9** For sufficiently large \(s\), when \(q < 1\), the following holds:
\[
L_s((1+(1-q)t)^n F^{(n)}(t)) = \left[ s - (n+1)(1-q); 1-q \right]_n L_s(F(t)) - \sum_{i=0}^{n-1} \left[ s - (n+1)(1-q); 1-q \right]_{n-i} F^{(i)}(0)
\]
(52)

**Proof.** Let us assume that the eq.(52) holds for \(n\). Then,
\[
L_s((1+(1-q)t)^n F^{(n+1)}(t)) = \int_0^\infty (1+(1-q)t)^{-\frac{n+1}{m}} F^{(n+1)}(t) dt
\]
\[
= (s - (n+1)(1-q))L_s((1+(1-q)t)^n F^{(n)}(t)) - F^{(n)}(0)
\]
\[
= (s - (n+1)(1-q))\left\{ \left[ s - (n+1)(1-q); 1-q \right]_n L_s(F(t)) - \sum_{i=0}^{n-1} \left[ s - (n+1)(1-q); 1-q \right]_{n-i} F^{(i)}(0) \right\} - F^{(n)}(0)
\]
\[
= \left[ s - (n+2)(1-q); 1-q \right]_{n+1} L_s(F(t)) - \sum_{i=0}^{n} \left[ s - (n+2)(1-q); 1-q \right]_{n-i} F^{(i)}(0)
\]
(53)
The eq.(50) is also obtained by using the variational method whose Lagrangian is given by

\[ L = \int dt \left( \frac{1}{2} (D_t x)^2 - U(x) \right) \]  

(54)

The equation of motion is then given by

\[ D_t \left( \frac{\partial L}{\partial (D_t x)} \right) - \frac{\partial L}{\partial x} = 0, \]  

(55)

where the momentum \( p \) is defined by

\[ p = \frac{\partial L}{\partial (D_t x)} \]  

(56)

For the harmonic potential \( U = \frac{1}{2} m w^2 x^2 \), we have the eq.(50). This equation is rewritten by

\[ (1 + (1 - q)t)^2 \ddot{x} + (1 - q)(1 + (1 - q)t) \dot{x} = -w^2 x \]  

(57)

Replacing

\[ \eta = \frac{1}{1 - q} \ln(1 + (1 - q)t), \]  

(58)

the eq.(57) is then as follows:

\[ \frac{\partial^2 x}{\partial \eta^2} = -w^2 x(\eta) \]  

(59)

Solving the eq.(59), we have

\[ x(t) = A \cos \omega \eta = AC_q(\omega \otimes t) \]  

(60)

Here, let us investigate the times \( t_n \) when a body goes back to the initial position. This time is determined by

\[ t_n = \ln_q e^{\frac{\pi n}{w}}, \quad (t = 0, 1, 2, \ldots) \]  

(61)

When \( q < 1 \) and \( w > 0 \), we have the following inequality:

\[ t_{n+1} - t_n > t_n - t_{n-1} \]  

(62)

Thus, the time when a body goes back to the initial position keeps increasing.

\[ \text{IV. CONCLUSION} \]

In this paper, we used the algebraic operation and differential calculus related to the non-extensive thermodynamics to investigate the q-Laplace transform. We used the q-Laplace transform to solve some differential equation such as harmonic oscillator problem.

We think that this work will be applied to some q-differential equation which might appear in the study of the non-extensive statistical mechanics. We hope that these work and their related topics will be clear in the near future.

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