A NEW SECOND CRITICAL EXPONENT AND LIFE SPAN FOR A QUASILINEAR DEGENERATE PARABOLIC EQUATION WITH WEIGHTED NONLOCAL SOURCES

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Abstract. In this paper, we consider positive solutions of a Cauchy problem for the following quasilinear degenerate parabolic equation with weighted nonlocal sources:

\[ u_t = \Delta_p u + \left( \int_{\mathbb{R}^N} K(x)u^q(x,t)dx \right)^{\frac{r-1}{q}} u^{s+1}, \quad (x,t) \in \mathbb{R}^N \times (0,T), \]

where \( N \geq 1, p > 2, q, r \geq 1, s \geq 0, \) and \( r + s > 1. \) We classify global and non-global solutions of the equation in the coexistence region by finding a new second critical exponent via the slow decay asymptotic behavior of an initial value at spatial infinity, and the life span of non-global solution is studied.

1. Introduction. It is well known that the positive solution of a Cauchy problem for the semilinear parabolic equation

\[ u_t = \Delta u + u^s, \quad (x,t) \in \mathbb{R}^N \times (0,T), \]

possesses the critical Fujita exponent \( s_c = 1 + \frac{2}{N} \) (cf. [3]). Fujita [3] also showed that the positive solution of the Cauchy problem (1) blows up at finite time for any nontrivial initial data, whenever \( 1 < s < s_c; \) while there are global solutions for small initial data and non-global solutions for large initial data, if \( s > s_c. \) Furthermore, Hayakawa [7] and Weissler [14] proved that the critical case \( s = s_c \) belongs to blow-up case. From then on, considerable attention has been paid to the study on the critical Fujita exponent for the parabolic equation with local or nonlocal sources. For a parabolic equation with local source, Galaktionov [5] considered the positive solution of a Cauchy problem for the following quasilinear degenerate parabolic equation:

\[ u_t = \Delta_p u + u^s, \quad (x,t) \in \mathbb{R}^N \times (0,T), \]

where \( p > 2, \) and obtained the critical Fujita exponent \( s_c = p-1 + \frac{2}{N}. \) Afterwards, Qi [13] pointed out that the critical case \( s = s_c \) belongs to blow-up case. Concerning a nonlocal source problem, Galaktionov and Levine [6] investigated the positive

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solution of a Cauchy problem for the following semilinear parabolic equation with weighted nonlocal source:

\[ u_t = \Delta u + \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{r-1}{q}} u^{s+1}, \quad (x,t) \in \mathbb{R}^N \times (0,T), \tag{3} \]

where \( q, r \geq 1, s \geq 0, \) and \( r+s > 1, \) and they derived the critical Fujita exponent by the parameter \( r \) to classify solutions of the equation. When the nonnegative weight function \( K(x) \) belongs to \( L^1(\mathbb{R}^N) \), the critical Fujita exponent \( r_c = 1 + \frac{2}{N} - s; \) while if the nonnegative weight function \( K(x) \) does not belong to \( L^1(\mathbb{R}^N) \) and \( K(x) \sim |x|^{-m} \) for \( |x| \) large enough, the critical Fujita exponent \( r_c = 1 + \frac{2q(1-\frac{m}{q})}{N(q-1)+m} \) for \( \frac{N}{q} < 1, \) which is included in blow-up case. Moreover, they also considered the \( p \)-Laplace equation with weight nonlocal sources when \( K(x) \in L^1(\mathbb{R}^N) \) and found the critical Fujita exponent \( r_c = p - 1 - s + \frac{p-2}{q} + \frac{p}{N} \) for \( s < \frac{p-2}{q} + \frac{p}{N}, \) which is included in blow-up case. Afterwards, Afanas’eva and Tedeev [1] studied the positive solution of a Cauchy problem for the following doubly degenerate parabolic equation with weighted nonlocal sources:

\[ u_t = \nabla \cdot (u^l |\nabla u|^{p-2} \nabla u) + \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{r-1}{q}} u^{s+1}, \quad (x,t) \in \mathbb{R}^N \times (0,T), \]

where \( l + p - 1 > 0, \ l, s \geq 0, \ p, r > 0, \ r+s > 1, \ q \geq 1, \ K(x) = (1+|x|)^{-m}, \) and \(-N(q-1) < m < N \). They obtained the critical Fujita exponent \( s_c = p + l - 1 + \frac{p+1}{N} - (m + N(q-1)) \frac{r-1}{q} \), with respect to the parameter \( s \), but this Fujita exponent does not belong to blow-up case.

Note that for the critical Fujita exponent, the region satisfying \( s > s_c \) or \( r > r_c \) is a coexistence region of global and non-global solutions for the Cauchy problem. In order to identify global and non-global solutions in the coexistence region, Lee and Ni [8] introduced a new second critical exponent \( \alpha^* = \frac{2}{r+1} \) for problem (1) with \( s > s_c = 1 + \frac{2}{q} \) by virtue of the slow decay behavior of the initial data at spatial infinity. More precisely, for problem (1) with initial data \( u_0(x) = \lambda \varphi(x) \) and \( s > s_c = 1 + \frac{2}{q} \), there exist constants \( \mu, \Lambda, \Lambda_0 > 0 \) such that the solution blows up in finite time, whenever \( \liminf_{|x| \to \infty} |x|^\alpha \varphi(x) > \mu > 0 \) and \( \lambda > \Lambda, \) or exists globally, if \( \limsup_{|x| \to \infty} |x|^\alpha \varphi(x) < \infty \) with \( \alpha \geq \alpha^* \) and \( \lambda < \Lambda_0. \) Afterwards, Mu et al. [10] considered problem (2) with \( s > s_c = p - 1 + \frac{2}{q} \), and they derived a new second critical exponent \( \alpha^* = \frac{p}{p+1} \) and a life span of non-global solution. Moreover, concerning the second critical exponent for the Cauchy problem of porous medium equation or doubly degenerate parabolic equation with local sources, one can refer to [9, 11]. On the nonlocal problem, recently, Yang et al. [15] studied problem (3) with \( r > r_c, \ K(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \), and \( K(x) \sim |x|^{-m} \) for \( |x| \) large, and they found a new second critical exponent \( \alpha^* = \frac{2q(r-1)(N-m)\mu}{q(r+s-1)} \).

To the best of our knowledge, much less effort has been devoted to the second critical exponent and life span for the Cauchy problem of a quasilinear degenerate parabolic equation with weighted nonlocal sources. At a glance, our main difficulties lie in the treatment of \( p \)-Laplace operator and weighted nonlocal source, and the selection of test function. Motivated by the above works, we investigate the
parabolic $p$-Laplace equation with weighted nonlocal source

\[ u_t = \Delta_p u + \left( \int_{\mathbb{R}^N} K(x)u^q(x,t)dx \right)^{\frac{q}{q-1}} u^{s+1}, \quad (x,t) \in \mathbb{R}^N \times (0,T), \tag{4} \]

subject to the initial condition

\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{5} \]

where $N \geq 1$, $p > 2$, $q, r \geq 1$, $s \geq 0$, $r + s > 1$, and the initial data $u_0(x)$ is a nonnegative and continuous function. Meanwhile, the nonnegative weight function $K(x) \in L^1(\mathbb{R}^N)$ satisfies $K(x) \sim |x|^{-m}$ for $|x|$ large, where $m$ is a positive constant.

Problem (4)-(5) arises in the theory of quasiregular and quasiconformal mappings, which can describe non-Newtonian flux in the mechanics of fluid, population of biological species, and so on (cf. [12, 4]). The essential purpose of this paper is to seek the effect of the slow decay behavior of initial data at spatial infinity for the positive solution of problem (4)-(5) and to derive a life span of non-global solution.

Firstly, recall that the critical Fujita exponent $r_c$ to problem (4)-(5), given by Galaktionov and Levine [6], is such that $r_c = p - 1 - s + \frac{p-2}{q}$ for $s < \frac{p-2}{q}$.

Throughout the rest of this paper, $\mathcal{C}_b(\mathbb{R}^N)$ denotes the space of all bounded continuous functions in $\mathbb{R}^N$ and let

\[ \Pi_\alpha = \{ \varphi \in \mathcal{C}_b(\mathbb{R}^N) \mid \varphi(x) \geq 0, \liminf_{|x| \to \infty} |x|^\alpha \varphi(x) > 0 \}, \]
\[ \Pi^\alpha = \{ \varphi \in \mathcal{C}_b(\mathbb{R}^N) \mid \varphi(x) \geq 0, \limsup_{|x| \to \infty} |x|^\alpha \varphi(x) < \infty \}. \]

Moreover, $u(x,t)$ denotes the solution of problem (4)-(5) with $r > r_c$ and $u_0(x) = \lambda \varphi(x)$. Then we can establish a new second critical exponent $\alpha^* = \frac{pq + (r-1)(N-m)}{q(r+s+1-p)}$.

The main results of this paper are as follows:

**Theorem 1.1.** Suppose that for $p > \max\left\{ 2, \frac{N[(r-1)q+2]}{N+q} \right\}$ and $r > r_c$, we have $\varphi(x) \in \Pi_\alpha$ and $0 < \alpha < \alpha^*$ or $\alpha \geq \alpha^*$ with $\lambda$ large enough. Then the solution $u(x,t)$ of problem (4)-(5) blows up at finite time.

**Theorem 1.2.** Suppose that for $p > 2$ and $r > r_c$, we have $\varphi(x) \in \Pi^\alpha$ and $\alpha \geq \alpha^*$. Then there exist positive constants $\lambda_0$ and $C$ such that the solution $u(x,t)$ of problem (4)-(5) exists globally, if $\lambda \in (0, \lambda_0)$, and $u(x,t)$ satisfies the inequality

\[ \|u(x,t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\alpha \beta} \quad \text{for all } t > 0, \]

where $\beta = \frac{1}{\alpha(p-2)+2}$.

**Theorem 1.3.** Suppose that $u(x,t)$ is a solution of problem (4)-(5) with initial data $u_0(x) = \lambda \varphi(x)$ which blows up at finite time $T$, and $\|\varphi\|_{L^\infty(\mathbb{R}^N)} = \lim_{|x| \to \infty} \varphi = \varphi_\infty$. Then the life span of $u(x,t)$ satisfies

\[ \frac{c_4}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)} \leq T \leq \frac{c_5}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)}, \]

where $0 < c_4 \leq \min \left\{ \left( \int_{\mathbb{R}^N} K(x)dx \right)^{-\frac{r-1}{r}}, c_5 \right\}$.

The rest of this paper is organized as follows: In Section 2, by virtue of the test function method, we prove Theorem 1.1. Theorem 1.2 is proved in Section 3 by establishing a global supersolution for problem (4)-(5). Finally, we obtain a life span of the non-global solution for problem (4)-(5) in Section 4.
2. Non-global solution. In this section, by using the test function method, we derive a sufficient condition for which the solution of (4)-(5) blows up at finite time. We give a proof of Theorem 1.1 below:

Proof. Since \( \varphi(x) \in \Pi_\alpha \) and \( K(x) \sim |x|^{-m} \) for \( |x| \) large enough, there exist positive constants \( R_0 \), \( c_1 \), and \( c_2 \) such that \( \varphi(x) \geq c_1|x|^{-\alpha} \) and \( K(x) \geq c_2|x|^{-m} \) for \( |x| \geq R_0 \).

Now, we define the following test function:

\[
\phi_\epsilon(x) = A_\epsilon e^{-\epsilon |x| - R_0},
\]

where \( A_\epsilon = \frac{1}{\int_{E_{R_0}} e^{-\epsilon |x| - R_0} dx} \) and \( E_{R_0} = \{ x \in \mathbb{R}^N \mid |x| > R_0 \} \). Then it can be easily seen that

\[
\nabla \phi_\epsilon(x) = -\epsilon \phi_\epsilon \frac{x}{|x|},
\]

and

\[
\int_{E_{R_0}} \phi_\epsilon(x) dx = 1.
\]

Next, we introduce the auxiliary function

\[
\Theta(t) = \frac{1}{\sigma} \int_{E_{R_0}} u^\sigma \phi_\epsilon(x) dx,
\]

where \( 0 < \sigma < \frac{1}{p} \). Firstly, differentiating \( \Theta(t) \), and using (6) and Green’s formula, we get

\[
\Theta'(t) = \int_{E_{R_0}} u^{\sigma-1} \phi_\epsilon(x) \left[ \Delta_p u + \left( \int_{\mathbb{R}^N} K(x) u^2(x,t) dx \right) \frac{1}{p-1} u^{p+1} \right] dx,
\]

\[
= -\int_{E_{R_0}} (\sigma - 1) \phi_\epsilon(x) u^{\sigma-2} \| \nabla u \|_p^p dx + \epsilon \int_{E_{R_0}} u^{\sigma-1} \phi_\epsilon(x) \frac{\| \nabla u \|_p^{p-2}}{|x|} \nabla u \cdot x dx
\]

\[
+ \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{p-1}{q}}
\]

\[
\geq (1 - \sigma) \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma-2} \| \nabla u \|_p^p dx - \epsilon \int_{E_{R_0}} u^{\sigma-1} \phi_\epsilon(x) \| \nabla u \|_p^{p-1} dx
\]

\[
+ \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{p-1}{q}}.
\]

Applying Young’s inequality to the second term on the right-hand side of (8), we have the inequality

\[
\epsilon \int_{E_{R_0}} u^{\sigma-1} \phi_\epsilon(x) \| \nabla u \|_p^{p-1} dx
\]

\[
\leq \frac{p-1}{p} \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma-2} \| \nabla u \|_p^p dx + \frac{\epsilon}{p} \int_{E_{R_0}} \phi_\epsilon(x) u^{p+\sigma-2} dx.
\]

Since \( 0 < \sigma < \frac{1}{p} \), it follows from (8) and (9) that

\[
\Theta'(t) \geq \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{p-1}{q}} - \frac{\epsilon}{p} \int_{E_{R_0}} \phi_\epsilon(x) u^{p+\sigma-2} dx.
\]
Then employing Hölder’s inequality and (7) to the last term on the right-hand side of (10), we obtain the inequality
\[
\int_{E_{R_0}} \phi_\epsilon(x) u^{p+\sigma-2} dx \leq \left( \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} dx \right)^{\frac{p+\sigma-2}{\sigma+s}},
\]
by virtue of \( r > r_c = p - 1 - s + \frac{p-2}{q} + \frac{p}{N} \) and \( p > \max \left\{ 2, \frac{N[(r-1)q+2]}{N+q} \right\} \), where \( \frac{p+\sigma-2}{\sigma+s} \in (0,1) \). Thus, substituting (11) into (10), one can derive the inequality
\[
\Theta'(t) \geq \left( \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} dx \right)^{\frac{p+\sigma-2}{\sigma+s}} \times \left[ \left( \int_{E_{R_0}} \phi_\epsilon(x) u^{\sigma+s} dx \right)^{\frac{\sigma}{\sigma+s}} \left( \int_{\mathbb{R}^N} K(x) u^q(x,t) dx \right)^{\frac{\sigma+p-1}{\sigma+s}} - \frac{\epsilon^p}{p} \right].
\]
Comparing (18) and (21), when \( \alpha > \alpha^* \), we have

\[
\frac{1}{2} c^{p+\|s\|+1} \epsilon^{- \frac{(N-m)_{1.5}}{q}} \|y\|^{-\alpha \sigma} \geq \epsilon^{p}, \quad \text{for all } t \in [0, T).
\] (17)

By virtue of (17) and \( r > r_c \), we are led to

\[
\Theta(t) \geq \left( \frac{2}{c^p} \right)^{\frac{\sigma}{r+s-1}} \epsilon^{- \frac{(N-m)_{1.5}}{q}} \|y\|^{-\alpha \sigma}.
\]

Therefore, if \( \Theta(0) \) satisfies (18), then \( \Theta(t) \) increases and is bounded below by \( c_0 \) for all \( t \in [0, T) \). Integrating (16) over \( [0, t] \), we have the inequality

\[
\Theta(t) \geq \left( (\Theta(0))^{- \frac{r+s-1}{\sigma}} - \frac{(r+s-1) c'}{2 \sigma} t \right)^{- \frac{\sigma}{r+s-1}}.
\] (19)

Hence, one can see that the solution \( u(x,t) \) of problem (4)-(5) blows up in the measure of \( \Theta(t) \) at some finite time \( T \) that satisfies

\[
T \leq \frac{2 \sigma}{(r+s-1)c}(\Theta(0))^{- \frac{r+s-1}{\sigma}}.
\] (20)

Finally, we verify the blow-up condition (18). Since \( \varphi(x) \in \Pi_{\alpha} \), there exist positive constants \( R_0 \) and \( c_1 \) such that \( \varphi(x) \geq c_1 |x|^{-\alpha} \) for \( |x| \geq R_0 \), and hence, we can obtain

\[
\Theta(0) = \frac{1}{\sigma} \int_{E_{R_0}} u_0^2 \varphi_s(x) dx \geq \frac{\lambda^* c^p A_s}{\sigma} \int_{E_{R_0}} e^{-\epsilon |x-R_0|} |x|^{-\alpha \sigma} dx,
\]

\[
\geq \frac{\lambda^* c^p A_s}{\sigma} e^{-N+\alpha \sigma} \int_{E_{R_0}} e^{-\epsilon |x-R_0|} |y|^{-\alpha \sigma} dy.
\] (21)

Comparing (18) and (21), when \( 0 < \alpha < \alpha^* = \frac{pq+(r-1)(N-m)_{1.5}}{q(r+s+1-p)} \), we get

\[
\alpha \sigma < \frac{\sigma p}{r+s-p+1} + \frac{\sigma (r-1)(N-m)_{1.5}}{q(r+s+1-p)},
\]

and so (18) holds for \( \epsilon > 0 \) small enough. If \( \alpha \geq \alpha^* \), there exists \( \lambda > 0 \) for any fixed \( \epsilon > 0 \), such that (18) holds for all \( \lambda > \lambda_c \). The proof is completed. \( \square \)

3. Global existence. In this section, we will prove Theorem 1.2 by constructing a global supersolution for problem (4)-(5).

**Proof.** Firstly, we consider the following Cauchy problem:

\[
U_t = \text{div}(|\nabla U|^{p-2} \nabla U), \quad x \in \mathbb{R}^N, \quad t > 0,
\] (22)

\[
U(x,0) = M |x|^{-\alpha}, \quad x \in \mathbb{R}^N,
\] (23)
where $M$ is a positive constant given in (26). It is known that the existence and
uniqueness of the solution of (22)-(23) have been well established and the radially
symmetric self-similar solution
\[ U(x, t) = t^{-\alpha \beta} f(\eta), \]
was given, see [16], where $\eta = \frac{|x|}{R}$, $\beta = \frac{1}{\alpha(p-2)+p}$, and the positive function $f(\eta)$ is
the solution of the following problem:
\[ \left( |f'(\eta)|^{p-2} f'(\eta) \right)' + \frac{N-1}{\eta} |f'(\eta)|^{p-2} f'(\eta) + \eta \beta f'(\eta) + \alpha \beta f(\eta) = 0, \quad \eta > 0, \quad (25) \]
\[ f(\eta) \geq 0, \quad \eta \geq 0, \quad f'(0) = 0, \quad \text{and } \lim_{\eta \to \infty} \eta^\alpha f(\eta) = M. \quad (26) \]
The solution of problem (25)-(26) is decreasing, we can refer to [2].

Since $\varphi(x) \in \Pi^\alpha$, there exists a positive constant $c_3$ such that $\varphi(x) \leq c_3(1+|x|)^{-\alpha}$
for all $x \in \mathbb{R}^N$. Hence, we can choose $c_3$ such that $\lim_{\eta \to \infty} \eta^\alpha f(\eta) = M > c_3$, and so,
there is a positive constant $R_1$ such that
\[ \eta^\alpha f(\eta) > c_3 \text{ for } \eta \geq R_1. \quad (27) \]

By virtue of (27), it is not difficult to verify that there exists $t_0 \in (0, 1)$ such that
\[ \|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq U(x, t_0). \quad (28) \]

Next, we claim that
\[ \int_{\mathbb{R}^N} K(x) U^q(x, t+t_0) dx \leq c(t+t_0)^{-\beta[\alpha q-(N-m)+]}. \quad (29) \]

Since $K(x) \in L^1(\mathbb{R}^N)$, $K(x) \sim |x|^{-m}$ for $|x|$ large enough, and $f(\eta)$ is decreasing,
when $m > N$, we have the inequalities
\[ \int_{\mathbb{R}^N} K(x) U^q(x, t+t_0) dx \leq c\|U(t+t_0)\|_{L^\infty(\mathbb{R}^N)}^q \leq c(t+t_0)^{-\beta \alpha q}, \quad (30) \]
by (24). When $m < N$, since $\lim_{\eta \to \infty} \eta^\alpha f(\eta) = M$, we get $f(x) \sim |x|^{-\alpha}$ for $|x|$ large
enough, and because $\alpha > \alpha^* = \frac{p+1-(r-1)(N-m)}{q(r+s+1-p)}$, which can be led to $m > N - \alpha q$. We then obtain
\[ \int_{\mathbb{R}^N} K(x) U^q(x, t+t_0) dx \leq c(t+t_0)^{-\beta \alpha q} \int_{\mathbb{R}^N} |x|^{-m} f^q \left( \frac{|x|}{t+t_0} \right) dx, \]
\[ = c(t+t_0)^{-\beta[\alpha q-(N-m)]} \int_{\mathbb{R}^N} |y|^{-m} f^q(|y|) dy, \quad (31) \]
by a simple calculation.

If $m = N$, we can derive the inequalities
\[ \int_{\mathbb{R}^N} K(x) U^q(x, t+t_0) dx \]
\[ \leq \int_{|x| \leq R_0} K(x) U^q(x, t+t_0) dx + c \int_{|x| > R_0} |x|^{-N} U^q(x, t+t_0) dx, \quad (32) \]
\[ \leq c\|U\|_{L^\infty}^q + c(t+t_0)^{-\beta \alpha q} \int_{\mathbb{R}^N} |y|^{-m} f^q(|y|) dy, \]
\[ \leq c(t+t_0)^{-\beta \alpha q}. \]

Thus, combining (30)-(32), it can be seen that (29) holds.
Let $h(t)$ be the solution of the following ordinary differential equation:
\[
\begin{cases}
h'(t) = c\lambda^{r+s-1}(t + t_0)^{-\theta}h^{r+s}(t), & t > 0, \\
h(0) = 1,
\end{cases}
\] (33)
where $\theta > 1$. The local existence and uniqueness of the solution $h(t)$ for (33) follow from the standard theory of initial value problem on ordinary differential equation. Afterwards, we claim that there exists $\lambda_0 > 0$ such that $h(t)$ is bounded in $[0, +\infty)$ for all $\lambda \in [0, \lambda_0)$.

Integrating (33) over $[0, t]$, one can have
\[
1 - h^{1-r-s}(t) = c(r+s-1)\lambda^{r+s-1} \int_0^t (\tau + t_0)^{-\theta}d\tau \leq \frac{c(r+s-1)\lambda^{r+s-1}t_0^{-\theta+1}}{\theta - 1}. 
\] (34)

Let $\lambda_0 > 0$ satisfy $\frac{c(r+s-1)\lambda_0^{r+s-1}t_0^{-\theta+1}}{\theta - 1} = 1$, and define $c_\lambda$ and $h_\lambda$ as
\[
c_\lambda = \frac{c(r+s-1)\lambda^{r+s-1}t_0^{-\theta+1}}{\theta - 1}, \quad h_\lambda = \left( \frac{1}{1 - c_\lambda} \right)^{\frac{1}{r+s}}.
\]

We then have $h(t) \leq h_\lambda$ for all $t \in [0, +\infty)$ and $\lambda \in [0, \lambda_0)$.

Now, we construct the following global solution:
\[
\tilde{u}(x, t) = \lambda h(t)U(x, t + t_0), 
\] (35)
where $U(x, t + t_0)$ is the solution of (22)-(23) and $h(t)$ solves (33) with $\theta = \alpha\beta(r + s - 1) - \frac{\beta(N-m)_+(r-1)}{q}$. Note that $\theta > 1$ for $\alpha > \alpha^* = \frac{pq + (r-1)(N-m)_+(r-1)}{q(r+s+1-p)}$. Since $f(\eta)$ is decreasing, it follows from (29) that
\[
\tilde{u}_t - \Delta_p \tilde{u} - \left( \int_{\mathbb{R}^N} K(x)\tilde{u}^q(x, t)dx \right)^{\frac{r-1}{q}}\tilde{u}^{s+1} 
\]
\[
= \lambda h'(t)U - \lambda^{r+s}h^{r+s}(t) \left( \int_{\mathbb{R}^N} K(x)U^q(x, t)dx \right)^{\frac{r-1}{q}}U^{s+1},
\]
\[
\geq \lambda U \left[ h'(t) - c\lambda^{r+s-1}(t + t_0)^{-\theta}h^{r+s}(t) \right] \leq \lambda U \left[ h'(t) - c\lambda^{r+s-1}(t + t_0)^{-\theta}h^{r+s}(t) \right] = 0,
\] (36)

where
\[
\theta - \alpha\beta s = \alpha\beta(r + s - 1) - \frac{\beta(N-m)_+(r-1)}{q} - \alpha\beta s = \alpha\beta(r - 1) - \frac{\beta(N-m)_+(r-1)}{q} = \beta[\alpha q - (N-m)_+(r-1)] \frac{1}{q}.
\]

Moreover, utilizing (28) to the initial data, we have
\[
\tilde{u}(x, 0) = \lambda U(x, t_0) \geq \lambda \varphi(x) = u_0. 
\] (37)

Hence, combining (36) with (37), one can see that $\tilde{u}(x, t)$ is a global supersolution of problem (4)-(5). Furthermore, we can easily show that the decay estimate of the
solution \( u(x,t) \) for (4)-(5) is

\[
\|u(x,t)\|_{L^\infty(\mathbb{R}^N)} \leq \lambda h\|U(x,t)\|_{L^\infty(\mathbb{R}^N)} \leq ct^{-\alpha\beta},
\]

for all \( t > 0 \). The proof is completed.

4. Life span. In this section, we will give a life span of the non-global solution \( u(x,t) \) for problem (4)-(5) by giving a proof of Theorem 1.3.

Proof. Firstly, we have already obtained an upper bound of the blow-up time for \( u(x,t) \) in the measure of \( \Theta(t) \), given in the proof of Theorem 1.1, and the upper bound is given as

\[
T \leq \frac{2\sigma}{(r+s-1)c'} \left( \frac{\lambda^\sigma}{\sigma} \int_{E_{R_0}} \varphi^\sigma \phi_\epsilon dx \right)^{-\frac{r+s-1}{\sigma}}.
\]

Then, it follows from \( \|\varphi\|_{L^\infty(\mathbb{R}^N)} = \lim_{|x| \to \infty} \varphi = \varphi_\infty \) that there exists \( R_2 > 0 \) such that \( |\varphi - \varphi_\infty| < \varepsilon \) for \( |x| > R_0 \) and any \( \varepsilon > 0 \). Meanwhile, by the definition of test function \( \phi_\epsilon(x) \), we must have

\[
T < \frac{2\sigma}{(r+s-1)c'} \left( \frac{\lambda^\sigma}{\sigma} (\varphi_\infty - \varepsilon)^\sigma \right)^{-\frac{r+s-1}{\sigma}},
\]

for \( R_0 > R_2 \). Thus, from the arbitrariness of \( \varepsilon \), let \( \varepsilon \to 0 \) can yield that

\[
T \leq \frac{c_5}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)},
\]

where \( c_5 = \frac{2\sigma}{c'} \). On the other hand, in order to get a lower bound of the blow-up time for the non-global solution \( u(x,t) \), we construct a supersolution of (4)-(5). Consider the following ordinary differential equation:

\[
\begin{cases}
g'(t) = \frac{1}{c_4} g^{r+s}(t), \quad t > 0 \\
g(0) = \lambda \varphi_\infty.
\end{cases}
\]

By a direct calculation, one can see that the solution \( g(t) \) of (39) is given by

\[
g(t) = \left\{ \left( (\lambda \varphi_\infty)^{-(r+s-1)} - (r + s - 1)c_4^{-1}t \right) \right\}^{-\frac{1}{r+s-1}}.
\]

Now, applying \( c_4 \leq \left( \int_{\mathbb{R}^N} K(x)dx \right)^{-\frac{1}{r+s-1}} \) and the comparison principle, it can be easily shown that \( g(t) \) is a supersolution of problem (4)-(5). We then obtain a lower bound of the blow-up time, i.e.,

\[
T \geq \frac{c_4}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)}.
\]

Therefore, combining (38), (41), and \( c_4 \leq c_5 \), we get the life span of the non-global solution for problem (4)-(5) as follows:

\[
\frac{c_4}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)} \leq T \leq \frac{c_5}{r+s-1} (\lambda \varphi_\infty)^{-(r+s-1)}.
\]

The proof is completed.
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