MINIMAL COUNTEREXAMPLES FOR CONTRACTIBLE GRAPHS
AND RELATED NOTIONS

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ABSTRACT. The notion of a contractible transformation on a graph was introduced by Ivashchenko
as a means to study molecular spaces arising from digital topology and computer image analysis, and
more recently has been applied to topological data analysis. Contractible transformations involve a list
of four elementary moves that can be performed on the vertices and edges of a graph, and it has been
shown by Chen, Yau, and Yeh that these moves preserve the simple homotopy type of the underlying
clique complex. A graph is said to be $I$-contractible if one can reduce it to a single isolated vertex via
a sequence of contractible transformations. Inspired by the notions of collapsible and non-evasive
simplicial complexes, in this paper we study certain subclasses of $I$-contractible graphs where one
can collapse to a vertex using only a subset of these moves. Our main results involve constructions of
minimal examples of graphs for which the resulting classes differ, as well as a minimal counterexample
to an erroneous claim of Ivashchenko from the literature. We also relate these classes of graphs to the
notion of $k$-dismantlable graphs and $k$-collapsible complexes, and discuss some open questions.

1. INTRODUCTION

In many applications of graph theory it is of interest to find combinatorial operations on a graph
that preserve the topology of its clique complex. In [15] Ivashchenko introduced what we will call
$I$-contractible transformations, a collection of four modifications one can make on the vertices and
edges of a graph (see Definition 2 for a precise statement). These local operations on graphs are
used in computer image analysis, the theory of molecular spaces, and digital topology. For instance
to model a digital image $S$ embedded in $n$-dimensional Euclidean space one can divide $\mathbb{R}^n$ into
a set of cubes of a certain scale and consider the molecular space $M_1$ obtained by the set of cubes
intersecting $S$. Changing the scale of these cubes gives rise to another molecular space $M_2$. Under
certain conditions the intersection graphs of these two molecular spaces can be transformed into
each other via a sequence of $I$-contractible transformations.

Allowing for sequences of $I$-contractible transformations defines an equivalence relation on the
set of finite graphs. In particular the set of $I$-contractible graphs, denoted $I$, consists of those graphs
that can be reduced to a single isolated vertex through such a sequence. These notions mimic
constructions in combinatorial topology and simple homotopy theory, where similar operations on
simplicial (or more generally CW-) complexes are used to define various classes within a homotopy
class. In the case of graphs and their clique complexes, these moves can be described in terms
of simple graph theoretic constructions that are more well-suited for computer implementation.
As such they have applications to topological data analysis and in particular the computation of
persistent homology, where the homology of the Rips complex of a graph is of interest.

In [15] Ivashchenko proved that $I$-contractible transformations on a graph $G$ do not change
the homology groups of its clique complex $\Delta(G)$. This implies that the clique complex of an
$I$-contractible graph has vanishing homology. More recently in [7] Chen, Yau, and Yeh proved that
\( I \)-contractible transformations in fact preserve the *simple homotopy type* of the underlying clique complex. In particular if \( G \) is an \( I \)-contractible graph then \( \Delta(G) \) is contractible. For example, any cycle on four or more vertices is not \( I \)-contractible.

There are some important subfamilies of \( I \) obtained by only allowing a subset of the \( I \)-transformations described above. Again these mimic constructions for simplicial complexes where the classes of *collapsible* and *nonevasive* complexes are obtained by allowing only certain kinds of collapsing operations, providing the (strict) hierarchy:

\[
\text{nonevasive} \Rightarrow \text{collapsible} \Rightarrow \text{contractible}.
\]

For graphs, we define the class \( I_S \) of *strong \( I \)-contractible* graphs to be those obtained by only applying the *gluing* type operations (I2 and I4 of Definition 2). In [9] it is proved that if a graph \( G \) is strong \( I \)-contractible then the clique complex of \( G \) is collapsible. We define the class \( I_V \) of *vertex \( I \)-contractible* graphs by restricting to the deleting and gluing transformations that only involve *vertices* (I1 and I2 of Definition 2). Finally, the class \( I_{SV} \) of *strong vertex \( I \)-contractible* are those obtained by when we only allow vertex gluing (I2).

A natural question to ask is whether the definitions of \( I \)-contractible, strong \( I \)-contractible, vertex \( I \)-contractible, and strong vertex \( I \)-contractible graphs actually constitute different classes of graphs; that is if there is any redundancy in the transformations described in Definition 2. In [7] Chen, Yau, and Yeh show that in fact the class of \( I \)-contractible and vertex \( I \)-contractible graphs coincide. In particular they prove that edge deletion (gluing) can be realized by the composition of a vertex gluing (deletion) and a vertex deletion (gluing). The authors also describe a flag triangulation of Bing’s house which provides an example of a graph that is \( I \)-contractible but not strong \( I \)-contractible. In summary we have the following containments of graph classes.

\[
(1) \quad I_{SV} \subseteq I_S \subseteq I = I_V.
\]

The first inequality in Equation 1 will be discussed below. In addition to [7] a number of papers have studied these and related operations on graphs, but unfortunately confusion has arisen surrounding exactly which transformations are needed to define various classes. Indeed, in a follow up paper [16] authored by Ivashchenko himself he claims that any \( I \)-contractible graph can be obtained from an isolated vertex by a series of \( I \)-contractible gluings of vertices. It turns out that this is not the case, and a counterexample was described by the third author in [12].

We will see that the class of strong vertex \( I \)-contractible graphs also have close connections to the class of *k-dismantlable* graphs as introduced by Fieux and Jouve in [11]. This class of graphs generalize the well-studied notion of *dismantlable* graphs (here corresponding to 0-dismantlable), which have seen applications in the study of homomorphism complexes [8], statistical physics and Gibbs measures [6], pursuit-evasion games on graphs [19], iterated clique graphs [13], and chordal graphs [1].

The class of k-dismantlable graphs can also be seen as graph-theoretical analogues of the k-*collapsible* simplicial complexes of Barmak and Minian, who used these concepts to define a notion of *strong homotopy type*. In [3] the authors prove that a simplicial complex is non-evasive if and only if it is k-collapsible for some \( k \), and also provide examples of complexes that are k-collapsible but not \((k+1)\)-collapsible. We see our work as providing flag complex analogues of some of these constructions.
1.1. **Our contributions.** In this paper we seek to further understand the distinction between the classes of graphs discussed above. We construct minimal examples of graphs to demonstrate how these families differ, and how various constructions and statements in the literature relate to each other. Our examples were constructed by first creating a bank of all isomorphism types of connected graphs up to 11 vertices, employing the scripts available at [10]. We then implemented the algorithms describe below to check for the various contractible properties. To create our bank of graphs we developed a novel algorithm to tackle the isomorphism problem employing a notion of canonical labeling [2]. In this paper we are not interested in a deep analysis of this problem, and hence we can simply use our database of connected graphs, which agrees with those listed on McKay’s site [18].

We will also be interested in the subfamily of graphs whose graphs have an acyclic clique complex (such graphs will be called *acylic*). For graphs up to 11 vertices, we determined this family using the software Ripser [4]. Here a graph $G$ is provided as input and the homology of the clique complex of $G$ is quickly determined.

Our first result provides a minimal example of a graph that is strong $I$-contractible but not strong vertex $I$-contractible, demonstrating the inequality in the first containment of Equation 1. In particular edge deletions will be needed in the collapsing of these graphs.

**Theorem 12.** The graphs depicted in Figure 1 are the smallest graphs (in terms of vertices) whose clique complexes have trivial homology but which are not strong vertex $I$-contractible. Furthermore these graphs are strong $I$-contractible.

As a corollary to Theorem 12 (see Corollary 14) we conclude that the family of acyclic, $I$-contractible, strong $I$-contractible, and strong vertex $I$-contractible graphs all coincide for the class of graphs on at most ten vertices.

We next consider graphs that are $I$-contractible but not strong $I$-contractible, addressing the second inequality in Equation 1. As mentioned above, Chen, Yau, and Yeh [7] describe a flag triangulation of Bing’s house that provides such a graph on 21 vertices. We construct an example of such a graph on 15 vertices via a flag triangulation of the Dunce Hat (see Figure 2).

**Proposition 15.** The graph in Figure 2a) is $I$-contractible but is not strong-$I$-contractible.

We have not been able to verify whether this graph is minimal with this property, although in Conjecture 16 we posit that it is indeed the smallest example of a graph that is $I$-contractible but not strong $I$-contractible.

Our next class of minimal counterexamples addresses an ‘axiom’ that Ivashchenko formulates in [16]. Here it is claimed that if $G$ is an $I$-contractible graph $G$ and if $v \in G$ is any non-cone vertex then one can find a vertex $u \in G$ such that $N_G(u, v)$ is $I$-contractible (see Statement 18 in Section 4 for a precise statement).

Ivashchenko claims that the statement can be ‘easily verified on graphs with a small number of vertices’, but does not provide a proof (indeed it is not clear why it is stated as an ‘axiom’). In [12] the third author describes a graph on 13 vertices that in fact contradicts the statement. A smaller graph on 11 vertices was found by Ghosh and Ghosh in [14], who also challenged to reader to find an even smaller counterexample. Our next result is a response to their question, namely a construction of two graphs on eight vertices that also contradict Axiom 3.4. We also verify that they are the smallest graphs with this property.
Theorem 19. The two graphs on 8 vertices depicted in Figure 3 are the smallest graphs (in terms of vertices) that contradict Axiom 3.4 from [16]. There are 133 graphs on 9 vertices that contradict the Axiom.

In [16] Ivashchenko uses Statement 18 to establish many other properties of $\mathcal{I}$-contractible graphs which are also seen to be erroneous, for instance that every $\mathcal{I}$-contractible graph is strong vertex $\mathcal{I}$-contractible [16, Theorem 3.8]. Our Theorem 12 provides a counterexample to this. As we mentioned above, in [7] it is shown that in fact there exist $\mathcal{I}$-contractible graphs which are not even strong $\mathcal{I}$-contractible.

By definition the class of strong vertex $\mathcal{I}$-contractible graphs involves removing vertices whose neighborhoods satisfy certain properties, and a natural question to ask is whether one can apply a greedy algorithm to describe this removal. In other words, if $G$ is strong vertex $\mathcal{I}$-contractible and $v \in G$ is an $\mathcal{I}_{SV}$-contractible vertex is it true that $G - v$ is strong vertex $\mathcal{I}$-contractible? It turns out that this is not always the case, that in fact one can get ‘stuck’ in the process of removing such vertices. Our next result provides the smallest examples of graphs where this happens.

Theorem 21. The graphs depicted in Figure 4 are the smallest graphs $G$ (in terms of vertices) that have the property that

1. $G$ is strong vertex $\mathcal{I}$-contractible;
2. There exists a vertex $v \in G$ such that $N_G(v)$ is strong vertex $\mathcal{I}$-contractible and yet $G - v$ is not strong vertex $\mathcal{I}$-contractible.

Finally we relate our constructions to the notion of k-dismantlability of graphs. As mentioned above this class of graphs was introduced by Fieux and Jouve in [11], and again involves an inductive definition in terms of the removal of vertices satisfying certain properties. One justification for the study of 0-dismantlable graphs is that order does not matter in the removal of these vertices. Namely, if $G$ is 0-dismantlable then a greedy algorithm of removing dismantlable vertices will always result in a single vertex. We provide a minimal example of a graph that is strong vertex $\mathcal{I}$-contractible but not 0-dismantlable.

Theorem 24. The smallest graph (in terms of vertices) that is strong vertex $\mathcal{I}$-contractible but not 0-dismantlable is depicted in Figure 6.

The notion of k-dismantlability also has an analogue for simplicial complexes, in the context of k-collapsibility first introduced by Barmak and Minian in [3]. Here it is shown that a simplicial complex is non-evasive if and only if it is k-collapsible for some $k$. We prove an analogous fact for graphs.

Theorem 27. A graph $G$ is strong vertex $\mathcal{I}$-contractible if and only if $G$ is k-dismantlable for some $k$.

The rest of the paper is organized as follows. In Section 2 we review some concepts from combinatorial topology and define the various classes of graphs that we study. In Section 3 we provide examples of graphs that illustrate the hierarchy of contractible graphs depicted in Equation 1 and establish Theorem 12 and Proposition 15. In Section 4 we discuss minimal counterexamples to the Ivaschenko axiom and prove Theorem 19. In Section 5 we discuss the relevance of vertex orderings for strong vertex $\mathcal{I}$-contractible graphs and prove Theorem 21. In Section 6 we discuss k-dismantlability of graphs, and establish Theorems 24 and 27. In Section 7 we discuss some open problems and future directions. Here we describe a graph that we conjecture is the smallest graph that is $\mathcal{I}$-contractible but not strong $\mathcal{I}$-contractible.
Remark 1 (Colors in the figures). In the figures depicting our various counterexamples we use colors to assist in the visualization, and to signify various properties of the edges. We explain the conventions here. In all cases we start with a triangulation of a disk in the plane using black edges. We then add edges with particular colors according the following rules (in this order):

1. If the graph contains a complete graph $K_4$ (corresponding to the 1-skeleton of a tetrahedron in the clique complex) we color the edges green.
2. If there is an edge that can be deleted to preserve $\mathcal{I}$-contractibility (that is, has property (13) from Definition 2) then we color the edge red.
3. The remaining crossing edges we color orange.

In addition, in the figures that depict the addition of vertices and edges, we use blue to depict the new edges.

Although we have added these colors to aid the reader, we also note that the colors in the figures are not essential to understand the figures, and the reader can perfectly manage without them.

2. Definitions and preliminaries

Here we collect some basic definitions and set some notation. Throughout the paper we use $G = (V, E)$ to denote a finite simple graph, with no loops and no multiple edges. We let $K(n)$ denote the complete graph on $n$ vertices, so that in particular $K(1)$ denotes a single isolated vertex. If $G$ is a graph and $v \in V(G)$ is a vertex we let $N_G(v)$ (resp. $N_G[v]$) denote the open neighborhood (resp. closed neighborhood) of $v$ in $G$, defined by

$$N_G(v) = \{u \in V(G) : (u, v) \in E(G)\},$$

$$N_G[v] = N_G(v) \cup \{v\}.$$ 

If $v$ and $w$ are vertices of $G$, their common neighborhood $N_G(v, w)$ is given by the intersection

$$N_G(v, w) = N_G(v) \cap N_G(w).$$

Given a graph $G$, we will often abuse notation and use $G$ to also denote its set of vertices. Similarly, given a subset of vertices $B \subset G$, we will use $B$ to also denote the subgraph of $G$ induced on the elements of $B$. In particular, we use $N_G(v)$ to denote the set of neighbors of $v$, as well as the subgraph of $G$ induced by these vertices. For a vertex $v \in G$ we let $G - v$ denote the subgraph of $G$ obtained by deleting $v$.

From [15] we have the follow notion.

Definition 2. The class of $\mathcal{I}$-contractible graphs, denoted $\mathcal{I}$, is defined as follows.

1. The graph $K(1)$, consisting of a single isolated vertex, is in $\mathcal{I}$.
2. A graph is in $\mathcal{I}$ if it can be obtained from an $\mathcal{I}$-contractible graph by a sequence of the following operations:
   (I) Deleting a vertex: A vertex $v$ of a graph $G$ can be deleted if $N_G(v) \in \mathcal{I}$.
   (II) Gluing a vertex: If $G'$ is a contractible subgraph of $G$, then a vertex $v \notin G$ can be glued to $G$ to produce a new graph $G''$ with $N_{G''}(v) = G'$.
   (I3) Deleting an edge: An edge $\{v_1, v_2\}$ of $G$ can be deleted if the joint neighborhood satisfies $N_G(v_1, v_2) \in \mathcal{I}$.
   (I4) Gluing an edge: $\{v_1, v_2\}$. For two non-adjacent vertices $v_1$ and $v_2$ of $G$, the edge $\{v_1, v_2\}$ can be glued to $G$ if the joint neighborhood satisfies $N_G(v_1, v_2) \in \mathcal{I}$.

We say that a vertex $v \in G$ (resp. an edge $\{v_1, v_2\}$) is $\mathcal{I}$-contractible if $N_G(v)$ (resp. $N_G(v_1, v_2)$) is in $\mathcal{I}_{SV}$. 
It is clear that all complete graphs are 3-contractible and more generally it is not hard to show that chordal graphs are also in the class 3. Two graphs G and H are said to be 3-homotopy equivalent if one can be obtained from other via a sequence of the transformations (I1)–(I4). One can see that 3-homotopy equivalence defines an equivalence relation on the set of finite graphs.

We next recall how one can obtain a higher-dimensional complex from a graph. First recall that a simplicial complex on a finite set \( V \) is a collection \( \Delta \) of subsets of \( V \) that is closed under taking subsets; that is if \( \sigma \in \Delta \) and \( \tau \subset \sigma \) then \( \tau \in \Delta \). Elements of \( \Delta \) are called faces or sometimes simplices. A simplicial map \( f: \Delta_1 \to \Delta_2 \) is a function on the underlying ground sets that takes faces to faces, so that if \( \sigma \in \Delta_1 \) then \( f(\sigma) \in \Delta_2 \). A simplicial complex is said to be flag if its minimal nonfaces have dimension 2. There is a standard functorial way to associate a topological space \( |\Delta| \) to a simplicial complex \( \Delta \) known as the geometric realization.

Given a graph \( G \) its clique complex \( \Delta(G) \) is by definition the simplicial complex on the vertex set \( V(G) \) whose simplices are the complete subgraphs of \( G \). To save on notation we let \( H_n(G; A) \) denote the \( n \)th homology group of the geometric realization clique complex of \( G \) with coefficients in the abelian group \( A \). We let \( H_n(G) := H_n(G; \mathbb{Z}) \) denote the homology groups of \( G \) with integer coefficients.

Convention. We will often speak about topological properties (e.g. homology groups) of a graph \( G \), by which we mean those of its clique complex \( \Delta(G) \).

One can check that the transformations on a graph \( G \) described in Definition 2 are functorial, in the sense that they induce a continuous map on the realizations of the underlying clique complexes. A natural question to ask is how these transformations affect the (simple) homotopy type of the underlying topological spaces. This was studied by Chen, Yau, and Yeh in [7] where in particular they establish the following.

**Theorem 3.** [7, Corollary 3.6] If \( G \) is an 3-contractible graph then \( \Delta(G) \) is contractible.

As a step in proving Theorem 3, Chen, Yau, and Yeh prove that if \( G \) is any graph and \( v \) is a vertex not in \( G \) then the cone graph \( G \ast v \) is 3-contractible. They also prove that the contractible transformations in Definition 2 preserve the simple homotopy type of the underlying clique complexes. This for instance strengthens the main result from [15].

As far as we know the converse of Theorem 3 is still open; that is, if \( \Delta(G) \) is contractible is not known if \( G \) is necessarily 3-contractible. It might be surprising if the class of 3-contractible graphs happens to coincide with the class of contractible flag simplicial complexes. Indeed, as the barycentric subdivision of any simplicial complex is flag, classifying contractible flag complexes should be as hard as classifying contractible simplicial complexes.

### 2.1. Special subclasses of 3-contractible graphs.

Having defined the class of 3-contractible graphs in Definition 2, a natural question to ask is whether all four transformations are actually necessary to define the class. In [7] Chen, Yau, and Yeh prove that in fact there is a redundancy.

**Lemma 4.** [7, Lemma 3.4] Edge deletion (I3) can be realized by the composition of a vertex gluing (I2) and vertex deletion (I1). Similarly, edge gluing (I4) can be realized as a composition of (I1) and (I2).

It then follows that in our definition of 3-contractible graphs we in fact only needed the transformations (I1) and (I2) to define the class. In this paper we will primarily focus on classes of graphs defined by other subsets of the transformations described in Definition 2. In particular we define the following classes.
Definition 5. The class of strong \( I \)-contractible graphs, denoted \( I_S \), is defined as follows:

1. The trivial graph \( K(1) \) is in \( I_S \).
2. A graph is in \( I_S \) if it can be obtained from a strong \( I \)-contractible graph by applying a sequence of the following transformations:
   (I2) Gluing a vertex \( v \): If \( G' \) is in \( I_S \) then a vertex \( v \) not in \( G \) can be added to produce a graph \( G'' \) whenever \( N_G(v) \in I_S \);
   (I4) Gluing an edge \( \{v_1,v_2\} \). For two non-adjacent vertices \( v_1 \) and \( v_2 \) of \( G \), the edge \( \{v_1,v_2\} \) can be glued to \( G \) whenever \( N_G(v_1,v_2) \in I_S \).

Definition 6. The class of strong vertex \( I \)-contractible graphs, denoted by \( I_{SV} \), is defined as follows:

1. The trivial graph \( K(1) \) is in \( I_{SV} \).
2. A graph is in \( I_{SV} \) if it can be obtained from a strong vertex \( I \)-contractible graph via a sequence of the following operation:
   (I2) Gluing a vertex \( v \): If \( G' \) is in \( I_{SV} \) then a vertex \( v \) not in \( G \) can be added to produce a graph \( G'' \) whenever \( N_G(v) \in I_{SV} \);
   (I5) We will say that a vertex \( v \in G \) is \( I_{SV} \)-contractible if \( N_G(v) \in I_{SV} \).

2.2. Simplicial complexes. To motivate the definitions of these graphs we discuss some analogous constructions in the setting of simplicial complexes. For this we recall the following notions.

Definition 7. Suppose \( \Delta \) is a simplicial complex and \( \sigma \in \Delta \) is a face. Then the link, deletion, and face deletion of \( \sigma \) are the subcomplexes of \( \Delta \) defined as

\[
\text{link}_\Delta(\sigma) = \{ \tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset \},
\]
\[
\text{del}_\Delta(\sigma) = \{ \tau \in \Delta : \sigma \cap \tau = \emptyset \},
\]
\[
\text{fdel}_\Delta(\sigma) = \{ \tau \in \Delta : \sigma \not\subseteq \tau \}.
\]

Note that if \( \sigma = \{v\} \) is a single element (a vertex) then the notion of deletion and face deletion coincide. Also note that if \( \Delta \) is a flag simplicial complex then if \( \sigma \) has \( |\sigma| \geq 3 \) then \( \text{fdel}_\Delta(\sigma) \) is no longer flag in general.

The class of collapsible simplicial complexes is typically defined in terms of free faces and the notion of elementary collapses. We will use an equivalent definition described, for instance, in [17, Def. 3.14].

Definition 8. The class of collapsible simplicial complexes is defined recursively as follows:

- The void complex \( \emptyset \) and any 0-simplex \( \{\emptyset,\{v\}\} \) are collapsible.
- If \( \Delta \) contains a nonempty face \( \sigma \) such that the face deletion \( \text{fdel}_\Delta(\sigma) \) and link of \( \sigma \) are collapsible, then \( \Delta \) is collapsible.

One can see that collapsible complexes are contractible but the converse is not true. If we insist that the face \( \sigma \) in the second condition of Definition 8 is a singleton, we obtain another well known class of simplicial complexes.

Definition 9. The class of nonevasive simplicial complexes is defined recursively as follows:

- The void complex \( \emptyset \) and any 0-simplex \( \{\emptyset,\{v\}\} \) are collapsible.
• If $\Delta$ contains a vertex $v$ such that the face deletion $\text{del}_\Delta(v)$ and $\text{link}_\Delta(v)$ are collapsible, then $\Delta$ is collapsible.

It is clear that nonevasive complexes are collapsible, but the converse is not true. With these definitions we see that the class of strong $J$-contractible graphs can be thought of as a graph-theoretical analogue of collapsible simplicial complexes. Indeed one can check that the clique complex of a $J$-contractible graph is collapsible [9], although it is in an open question whether the reverse implication holds. We recall that if $\Delta$ is a flag simplicial complex then if $\sigma$ has $|\sigma| \geq 3$ then $f\text{del}_\Delta(\sigma)$ is no longer flag, and hence the definitions do not directly translate. In [9] it is conjectured that if $\Delta(G)$ is a flag collapsible complex then $G$ is strong $J$-contractible, and some computational evidence for this conjecture is provided. We refer to Section 7 for more discussion.

Similarly, the class of strong vertex $J$-contractible graphs can be seen as a graph theoretical analogue of nonevasive simplicial complexes. Indeed we show in Section 6 that a flag simplicial complex $\Delta$ is nonevasive if and if it is the clique complex of a strong vertex $J$-contractible graph.

2.3. Dismantlings. For our study we will need some further notions from combinatorial topology that have analogues in graph theory. In [3] Barmak and Minian study the notion of a strong collapse of a simplicial complex and related notions of $k$-collapsibility. In what follows a vertex $v$ of a simplicial complex $\Delta$ is a simplicial cone if there exists a vertex $v' \in \text{link}_\Delta(v)$ such that any simplex $\sigma \in \text{link}_\Delta(v)$ can by written as $(v') \cup \tau$, for some $\tau \in \text{link}_\Delta(v)$. Following the notation in [3] we have the following definitions.

**Definition 10.** A vertex $v$ of a finite simplicial complex $\Delta$ is 0-collapsible if the $\text{link}_\Delta(v)$ is a simplicial cone. A simplicial complex $\Delta$ is 0-collapsible if we can apply a sequence of deleting 0-collapsible vertices that ends in a single vertex.

Inductively we say that a vertex $v$ is $k$-collapsible if the complex $\text{link}_\Delta(v)$ is $(k-1)$-collapsible. We say that $\Delta$ is $k$-collapsible if we can obtain a single vertex via a sequence of deleting $k$-collapsible vertices.

It is not hard to see that a 0-collapsible complex is collapsible (and hence contractible). One motivation for studying 0-collapsible complexes come from the fact that the order of deleting vertices in such a complex does not matter. As is proved in [3], if $\Delta$ is 0-collapsible then a greedy algorithm of removing 0-collapsible vertices will always end with a single vertex. This is in contrast to the class of collapsible complexes, where one can ‘get stuck’ in the process of performing elementary collapses. In particular not all collapsible complexes are 0-collapsible. The notion of $k$-collapsibility also relates to the notions discussed above, and for instance in [3] it is shown that if a simplicial complex $\Delta$ is $k$-collapsible for some $k \geq 0$ then $\Delta$ is nonevasive.

These notions also have graph theoretical analogues first introduced in [11]. Here a graph $H$ is a cone if there exists a vertex $v$ with $N_H(v) = V(H) \setminus v$.

**Definition 11.** For a graph $G$ and vertex $v \in G$, we say that $v$ is 0-dismantlable (or simply dismantlable if the context is clear) if its open neighborhood $N_G(v)$ is a cone. A graph $G$ is 0-dismantlable (or simply dismantlable) if it can be reduced to a single vertex by successive deletions of 0-dismantlable vertices.

Proceeding inductively, a vertex $v \in G$ is $k$-dismantlable if its open neighborhood $N_G(v)$ is $(k-1)$-dismantlable, and a graph $G$ is $k$-dismantlable if it can be reduced to a single vertex by successive deletions of $k$-dismantlable vertices.
3. Separating the hierarchy

In this section we discuss minimal examples of graphs that distinguish the classes depicted in Equation 1. We first consider examples of graphs which are strong $\mathcal{I}$-contractible but not strong vertex $\mathcal{I}$-contractible. In Figure 1 we describe the smallest such graphs, each with 11 vertices. One has 30 edges, six have 31 edges, four have 32 edges and one has 33 edges. We present them in four groups, where an arrow indicates that we can pass from one graph to another by adding an edge.

**Theorem 12.** The smallest graphs (in terms of vertices) that have trivial homology but which are not strong vertex $\mathcal{I}$-contractible are depicted in Figure 1. These graphs are all strong $\mathcal{I}$-contractible.

**Proof.** To determine the graphs in Figure 1, we first obtained a list of all isomorphism types of graphs on at most 11 vertices from McKay’s bank [18]. We then used an adaptation of the Ripser program [4] to determine those graphs whose clique complexes have trivial homology. We then applied Algorithm 1 to determine those graphs that are not in $\mathcal{I}_{SV}$, the result is depicted in Figure 1. Among graphs that have 10 or fewer vertices, all those that have acyclic clique complexes are in fact in the class $\mathcal{I}_S$, which establishes the second claim.

```
Algorithm 1: SIVcontractible.graph

Input: A graph $G$ and the cardinality $n$ of the vertex set.
Output: The logical TRUE if $G \in \mathcal{I}_{SV}$, or FALSE otherwise.

1 if $n = 0$ then
2     return FALSE
3 else
4     if $n = 1$ then
5         return TRUE
6     else
7         for $i \leftarrow 1$ to $n$ do
8             if SIVcontractible.graph($N_G(v_i), k$) = TRUE then
9                 return SIVcontractible.graph($G - v_i, n - 1$)
10             end
11         end
12     end
13     return FALSE
14 end
```

To verify that the graphs in Figure 1 are strong $\mathcal{I}$-contractible, one can check that any of the red edges are $\mathcal{I}$-contractible and after removing such an edge there exists an $\mathcal{I}$-contractible vertex.

In particular for graph 30 the removal of the $\mathcal{I}$-contractible edge 46 leads to the $\mathcal{I}$-contractible vertex 5. For the graphs 313 and 314 the removal of the edge 37 leads to the $\mathcal{I}$-contractible vertex 7. For the graph 315, removing the edge 47 leads to the $\mathcal{I}$-contractible vertex 6. Finally for 316 we delete the edge 49 to obtain the $\mathcal{I}$-contractible vertex 11. With this we obtain the following corollary.

**Corollary 13.** The graphs in Figure 1 are the smallest graphs (in terms of vertices) that are strong $\mathcal{I}$-contractible but not strong vertex $\mathcal{I}$-contractible.
We expect that there exist graphs that have trivial homology but which are not strong $\mathcal{I}$-contractible (for instance the 1-skeleton of a certain flag triangulation of the 2-skeleton of the Poincaré homology sphere) but these will all have many more vertices than the graphs depicted in Figure 1. From our findings we can also observe the following.

**Corollary 14.** The family of acyclic, $\mathcal{I}$-contractible, strong $\mathcal{I}$-contractible, and strong vertex $\mathcal{I}$-contractible graphs all coincide for the class of graphs on at most ten vertices.

![Figure 1. Minimal examples of graphs that are in $\mathcal{I}_S$ but not in $\mathcal{I}_{SV}$. An arrow indicates that we can pass from one graph to the other by adding an $\mathcal{I}$-contractible edge according to rule I4 of Definition 2.](image)

From Theorem 12 it follows that the family of graphs on 11 vertices with trivial homology differs from the family of strong vertex $\mathcal{I}$-contractible graphs on 11 vertices. In particular, any graph in Figure 1 has trivial homology but is not in $\mathcal{I}_{SV}$. On the other hand, we observe that any strong vertex $\mathcal{I}$-contractible graph $G$ on at most 11 vertices has the following greedy property: the removal of any $\mathcal{I}_{SV}$-contractible vertex will lead to a strong vertex $\mathcal{I}$-contractible graph, where again the removal of any $\mathcal{I}_{SV}$-contractible vertex will be in $\mathcal{I}_{SV}$. In other words the choice of contracting vertices does not matter as we collapse to a single vertex.

This follows from the fact that for graphs with at most 10 vertices, the classes of $\mathcal{I}_S$ and $\mathcal{I}_{SV}$ coincide. In particular, if $G \in \mathcal{I}_{SV}$ had 11 vertices and did not satisfy the greedy property described above, then there would exist a graph on 10 or fewer vertices that had trivial homology but was not in $\mathcal{I}_{SV}$, a contradiction.

In addition there exist graphs on 11 vertices that are in $\mathcal{I}_S$ but not $\mathcal{I}_{SV}$, but which become strong vertex $\mathcal{I}$-contractible with this greedy property after the removal of any $\mathcal{I}_S$-contractible edge (see for instance graphs $30, 31_3, 31_4, 31_5,$ and $31_6$ in Figure 1). We will see in Section 5 that the order of vertex removals does matter for strong vertex $\mathcal{I}$-contractible graphs with at least 12 vertices.

We next consider graphs that are $\mathcal{I}$-contractible but not strong $\mathcal{I}$-contractible, addressing the second inequality in Equation 1. In [7] a construction of such a graph on 21 vertices is described, based on ‘the house with 2 rooms’ (or Bing’s house). Recall that Bing’s house is a 2-dimensional simplicial complex that is contractible but not collapsible, and hence is a natural place to look for such an example. We will be interested in graphs that are minimal with this property. By considering a triangulation of the Dunce Hat we obtain a graph on 15 vertices with this property.
Proposition 15. The graph in Figure 2a) is $\mathcal{I}$-contractible (in particular its clique complex is contractible) but is not strong-$\mathcal{I}$-contractible.

Unfortunately our computational power is exhausted on graphs up to eleven vertices. In particular we do not know if there are graphs on $n = 12, 13, 14$ vertices that are $\mathcal{I}$-Contractible but not strong $\mathcal{I}$-contractible, although we conjecture that this is not the case.

Conjecture 16. The smallest graph (in terms of vertices) with trivial homology that it is not strong $\mathcal{I}$-contractible is shown in Figure 2a). In particular this is the smallest graph that is $\mathcal{I}$-contractible but not strong $\mathcal{I}$-contractible.

We note that if we add the edge $\{5, 7\}$ to the graph in Figure 2a) according to rule (14) we obtain the graph in Figure 2b), which can be seen to be strong $\mathcal{I}$-contractible. Hence we only need to ‘go up’ one step in order to then collapse via removing contractible edges and vertices. We furthermore conjecture that the graph in Figure 2b) is a minimal example of this phenomenon in the following sense.

Conjecture 17. The graph depicted in Figure 2b) is the smallest graph $G$ (in terms of vertices) that has the following properties.

1. $G$ is strong $\mathcal{I}$-contractible;
2. There exists an edge $\{v, w\}$ in $G$ such that $N(v, w) \in \mathcal{I}_S$ and such that $G \setminus \{v, w\}$ is not strong $\mathcal{I}$-contractible.

4. SMALL COUNTEREXAMPLES TO THE IVASHCHENKO AXIOM

In this section we describe minimal counterexamples to a claimed result of Ivashchenko from [16]. Here the following ‘axiom’ regarding $\mathcal{I}$-contractible graphs is stated.

Statement 18. [16, Axiom 3.4] Suppose that $G$ is a $\mathcal{I}$-contractible graph, and a vertex $v \in G$ is not adjacent to some vertices of $G$. Then there exists a nonadjacent vertex $u \in G$, such that the subgraph $N_G(v, u)$ is $\mathcal{I}$-contractible.
Ivashchenko uses Statement 18 to establish other erroneous properties of \( I \)-contractible graphs. For instance he makes the claim ([16, Theorem 3.8]) that any \( I \)-contractible graph can be obtained from an isolated vertex by only allowing contractible gluings of vertices, that is that all \( I \)-contractible graphs are strong vertex \( I \)-contractible. As we have seen, a counterexample to this last statement was described in [7].

In [16] Ivashchenko claims that he had verified the axiom for all ‘small graphs’. However, in [12] it is shown that there exists a graph on 13 vertices where Statement 18 does not hold. A counterexample on 11 vertices was found by Ghosh and Ghosh in [14], where the authors also challenged the reader to find yet a smaller construction. We have the following answer to their question.

**Figure 3.**

**Theorem 19.** The two graphs depicted in Figure 3 (both on 8 vertices) contradict Axiom 3.4 from [16], and are the smallest graphs (in terms of vertices) with this property. There are 133 graphs on 9 vertices that contradict the Axiom.

**Proof.** To obtain the graphs in Figure 3, a bank of connected graphs from 1 to 9 vertices was reviewed (see for instance [18]). We first applied Algorithm 2 to determine if each graph belongs to the family \( I_S \) (note that by Corollary 14 this is equivalent to checking containment in \( I \)). We then applied Algorithm 3 to determine for which graphs Ivashchenko’s axiom fails.

To explicitly check that each graph in Figure 3 contradicts the Statement 18, note that in both graphs the vertex 6 is not adjacent to vertex 1, and yet there does not exist \( u \in G \) that is not adjacent to 6 and such that \( N(6, u) \) is contractible. Indeed, in each graph the non-neighbors of vertex 6 are 1 and 4, and yet \( N(6, 1) = \{7, 3, 2, 5\} \), \( N(6, 4) = \{7, 8, 3, 5\} \), which are both disconnected and hence not \( I \)-Contractible.

Using our software we verified that there are no graphs on 7 vertices or less that are in \( I_S \) and which contradict Statement 18, and hence the graphs in Figure 3 are indeed the minimal counterexamples. In addition we found 133 graphs on nine vertices that contradict the statement. The software used to verify these claims, along with the graphs themselves, are available in the repository [10].
Algorithm 2: \textit{SI}contractible\.\textit{graph}

\begin{algorithmic}
\State \textbf{Input :} A graph $G$ and the cardinality $n$ of the vertices set.
\State \textbf{Output:} The logical \texttt{TRUE} if $G \in \mathcal{I}_S$, or \texttt{FALSE} otherwise.
\If {$n = 0$}
\State \textbf{return} \texttt{FALSE}
\ElsIf {$n = 1$}
\State \textbf{return} \texttt{TRUE}
\Else
\State # Delete $\mathcal{I}_S$-contractible vertices
\For {$i \leftarrow 1$ \textbf{to} $n$}
\If {$\text{SI}contractible\.\textit{graph}\left(NG\left(v_i\right)\right) = \texttt{TRUE}$}
\State \textbf{return} $\text{SI}contractible\.\textit{graph}\left(G - v_i, n - 1\right)$
\EndIf
\EndFor
\State # Delete $\mathcal{I}_S$-contractible edges
\For {$i \leftarrow 1$ \textbf{to} $n - 1$}
\For {$j \leftarrow i + 1$ \textbf{to} $n$}
\If {$G_{ij} = 1$ \& \& \text{SI}contractible\.\textit{graph}\left(NG\left(v_i, v_j\right)\right) = \texttt{TRUE}$}
\State \textbf{return} $\text{SI}contractible\.\textit{graph}\left(G - \left(v_i, v_j\right), n\right)$
\EndIf
\EndFor
\EndFor
\State \textbf{return} \texttt{FALSE}
\EndIf
\EndIf
\EndIf
\EndAlgorithmic

5. Order matters for strong vertex $\mathcal{I}$-contractible graphs

Recall that a graph $G$ is strong vertex $\mathcal{I}$-contractible if it can be obtained from a single isolated vertex by a sequence of vertex gluings along $\mathcal{I}_S$-contractible vertices (See Definition 6). Hence $G$ can be reduced to a single vertex by a sequence of vertex deletions, and a natural question to ask is whether the ordering of vertices matters in the deletion process. More specifically, given a strong vertex $\mathcal{I}$-contractible graph $G$ and a contractible vertex $v$ (where $NG\left(v\right)$ is strong vertex $\mathcal{I}$-contractible), is it always true that $G - v$ is strong vertex $\mathcal{I}$-contractible? If this property indeed held, a greedy algorithm could be used to detect whether a graph is strong vertex $\mathcal{I}$-contractible, which is especially relevant for implementation and applications.

In [11] Fieux and Jouve describe a graph on 16 vertices that is strong vertex $\mathcal{I}$-contractible, but where the order of vertex deletion matters. In this section we describe the smallest graphs that have this property. In fact they are obtained from adding one additional vertex to the graphs in the last section. For our results we will need the following observation. Recall that a graph $G$ is a \textit{cone} if there exists a vertex $v \in G$ that is adjacent to all other vertices.

\textbf{Lemma 20.} If a graph $G$ is a cone then $G$ is strong vertex $\mathcal{I}$-contractible.
**Algorithm 3: CheckAxiom**

**Input**: A graph $G \in \mathcal{J}$ and the cardinality $n$ of the vertices set.

**Output**: The logical TRUE if $G$ satisfies the axiom, or FALSE otherwise.

1. $n_{VNC} \leftarrow 0$;
2. for $i \leftarrow 1$ to $n$ do
   3. grade $\leftarrow 0$;
   4. for $j \leftarrow 1$ to $n$ do
      5. if $G_{ij} = 1$ then
         6. grade $\leftarrow$ grade + 1;
      end
   end
   7. if (grade $< n$) then
      8. $n_{VNC} \leftarrow n_{VNC} + 1$;
      9. $VNC_{n_{VNC}} \leftarrow i$;
   end
end
14. Check axiom:
15. for $i \leftarrow 1$ to $n_{VNC} - 1$ do
   16. CanAddEdge $\leftarrow 0$;
   17. for $j \leftarrow i + 1$ to $n_{VNC}$ do
      18. if $G_{VNC_i,VNC_j} = 0$ then
         19. if $SI\text{contractible}\text{.graph}(N(G, n, VNC_i, VNC_j)) = \text{TRUE}$ then
            20. CanAddEdge $\leftarrow 1$;
         end
      end
   end
23. if (CanAddEdge $= 0$) then
   24. return FALSE
end
26. return TRUE

**Proof.** We prove this by induction on $n = |V(G)|$. If $n = 1$ the statement is clear. Next suppose $n > 1$ and let $v \in G$ be a cone point. For any vertex $w \neq v$ we have that both $N_G(w)$ and $G - w$ are cones (with cone point $v$) and hence strong vertex $\mathcal{J}$-contractible by induction. The result follows. 

**Theorem 21.** The graphs depicted in Figure 4 are the smallest graphs $G$ (in terms of vertices) that have the property that

1. $G$ is strong vertex $\mathcal{J}$-contractible;
2. There exists a vertex $v$ such that $N_G(v)$ is strong vertex $\mathcal{J}$-contractible and yet $G - v$ is not strong vertex $\mathcal{J}$-contractible.

Before we prove Theorem 21 we establish an auxiliary lemma that we need regarding the automorphism group of a related graph.
Figure 4. Minimal examples of graphs that are strong vertex $\mathcal{J}$-contractible but where order of vertex removal matters. If one deletes the contractible vertex 12 (in both cases) we obtain the graph $G_{11\nu30\epsilon}$ depicted in Figure 1, and hence one cannot continue with vertex deletions. On the other hand deleting the contractible vertex 5 (for graph a)) and the vertex 1 (for graph b)) allows us to continue deleting contractible vertices until we arrive at a single vertex.

Lemma 22. The graph $G_{11\nu30\epsilon}$ depicted in Figure 5 has automorphism group given by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. We let $G_{11\nu30\epsilon}$ denote the graph depicted in Figure 5. Note that there are four isomorphism classes of open vertex neighborhoods, given by $\{1, 2, 6, 7\}$, $\{4, 5, 8, 9\}$, $\{3, 10\}$ and $\{11\}$. We partition the vertex set of $G_{11\nu30\epsilon}$ accordingly. Any symmetry should fix those sets and should fix rigidly the hexagon $\{1, 2, 3, 6, 7, 10, 11\}$ depicted in Figure 5.b). Therefore the only possible symmetries are the vertical, horizontal and point reflections (rotation by 180 degrees). We then have that

\[
(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11), \quad (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11), \quad (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11), \quad (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)
\]

are the automorphisms of the graph $G_{11\nu30\epsilon}$. One can check that this set forms a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. \qed

Figure 5. The graph $G_{11\nu30\epsilon}$, the smallest graph (in terms of edges) among those depicted in Figure 1. The same graph is redrawn in b) to illustrate the proof of Lemma 22.
Proof of Theorem 21. Let $G$ denote the graph depicted in Figure 4a). First note that $N_H(12)$, the neighborhood of the vertex 12, is a cone and hence is strong vertex $J$-contractible by Lemma 20. If we delete the vertex 12 we obtain the graph $G_{11}v_{30}e$ depicted in Figure 1 (on the bottom left), and redrawn in Figure 5. We have seen in Theorem 12 that this graph is not strong vertex $J$-contractible.

To see that $G$ in Figure 4 is itself strong vertex $J$-contractible, note that $N(5)$ is in $J_{SV}$. If we remove vertex 5, we can continue wiping up the rest of vertices, ($N(4)$ is path, and so on), until we obtain an isolated vertex.

We next verify that $G$ is indeed the smallest example (in terms of number of vertices) with this property. We will also see that it has the least number of edges among counterexamples on this many vertices. First note that any minimal counterexample $H$ would require at least 12 vertices. Indeed assuming that $H$ is in $J_{SV}$, when we delete a vertex $v$ with $N(v) \subseteq J_{SV}$ we obtain a graph whose clique complex has trivial homology. We have seen in Corollary 14 that all such graphs on 10 or fewer vertices with this property are strong vertex $J$-contractible.

Now, suppose $H$ is a graph that satisfies the conditions of Theorem 21. Let $v \in H$ be a vertex such that $N_H(v)$ is in $J_{SV}$ and where $H - v$ is not in $J_{SV}$, so that in particular $H - v$ is one of the graphs depicted in Figure 1. Also since $H$ is in $J_{SV}$ let $w \in H$ be a vertex such that $N_H(w)$ and $H - w$ are both strong vertex $J$-contractible.

The graphs in Figure 1 have the property that each vertex contains an induced cycle in its neighborhood. In particular $N_{H-v}(w)$ has an induced cycle $C$ with at least four vertices. Since $N_H(w)$ is $J$-contractible, this implies that $v$ is adjacent to $w$ in $H$, and also that $v$ is adjacent to every vertex in $C$. In particular $N_{H}(v)$ has at least 5 vertices.

First assume $H - v$ is not isomorphic to $G_{11}v_{30}e$. In this case $H$ will have at least 36 edges (since we’ve removed at least 5 edges from $H$ to obtain a graph with at least 31 edges), and hence not a minimal counterexample. So we can assume that $H - v$ is isomorphic to $G_{11}v_{30}e$. In this case we need $N_H(w)$ to have same number of vertices as the minimum cycle among the neighborhoods of every vertex in $G_{11}v_{30}e$, since we are assuming that $H$ is minimum with respect to edges. Hence the possibilities for $w$ are $\{1, 2, 6, 7\} \cup \{4, 5, 8, 9\}$, since these are exactly the vertices of $G_{11}v_{30}e$ whose neighborhoods contain a 4-cycle. Using the automorphisms of Theorem 22 we observe that there are only two choices up to isomorphism. The resulting graphs are depicted in Figure 4, which completes the proof.

6. $k$-Dismantlable Graphs and $k$-Collapsible Complexes

In this section we discuss examples related to $k$-dismantlable graphs, as defined in Section 2. For the case of 0-dismantlable graphs it is known that the order of removing dismantlable vertices does not matter, and in fact the set of all dismantlings of a graph forms a greedoid. As discussed in Section 1, a similar fact holds for the class of strong collapsible simplicial complexes, see Theorem 2.11 from [3].

In Section 5 we saw that order does in general matter for removing contractible vertices of strong vertex $J$-contractible graphs. We first observe that 0-dismantlable graphs are in $J_{SV}$.

Lemma 23. If $G$ is a 0-dismantlable graph then $G$ is strong vertex $J$-contractible.

Proof. Once again we use induction on $n = |V(G)|$. The statement is clearly true for $n = 1$. For $n > 1$ suppose $v \in V(G)$ is 0-dismantlable. Then by definition $N_G(v)$ is a cone, which is strong
vertex $I$-contractible by Lemma 20. The deletion $G - v$ is by definition $0$-dismantlable, which by induction is also strong vertex $I$-contractible. We conclude that $G$ is in $J_{SV}$, as desired.

Hence it is of interest to find examples of graphs that are in $J_{SV}$ but not $0$-dismantlable. Our next result describes the smallest such graphs.

**Theorem 24.** The smallest graphs (in terms of vertices) that are strong vertex $I$-contractible but not $0$-dismantlable have $8$ vertices. Among these, the graph with the least number of edges is depicted in Figure 6. There are six other graphs on $8$ vertices with this property, two of which are shown in Figure 3.

**Proof.** Again we verify this by explicitly checking all graphs with $8$ or fewer vertices. We first consider all graphs that are acyclic and among those check which are $0$-dismantlable using script available at [10].

![Figure 6](image_url)

**Figure 6.** The smallest graph that is strong vertex $I$-contractible but not $0$-dismantlable.

As described in [3], there exists simplicial complexes that are non-evasive but not $0$-collapsible. Our Theorem 24 provides a minimal example among flag complexes.

Recall from Section 2 that a notion of $k$-collapsible for vertices of a simplicial complex was defined by Barmak and Minian in [3]. The definition is very similar to that of $k$-dismantlable vertices and in fact the two concepts coincide for flag simplicial complexes.

**Lemma 25.** [11, Proposition 4] For a graph $G$, a vertex $v \in G$ is $k$-dismantlable if and only if $v$ is $k$-collapsible in the clique complex $\Delta(G)$.

**Proof.** This follows from the fact that for any vertex $v \in G$ we have $\text{link}_{\Delta(G)}(v) = \Delta(N_G(v))$.

As a Corollary we get the following observation, also established by Fieux and Jouve.

**Proposition 26.** [11, Proposition 4] A graph $G$ is $k$-dismantlable if and only if $\Delta(G)$ is $k$-collapsible.

**Theorem 27.** A graph $G$ is strong vertex $I$-contractible if and only if $G$ is $k$-dismantlable for some $k$.

**Proof.** Suppose $G$ is a $k$-dismantlable graph. We will prove by induction on $k$ that $G \in J_{SV}$. For $k = 0$, we have from Lemma 23 that $G$ is in $J_{SV}$. Suppose the result is true for $k$ and suppose that $G$ is $(k + 1)$-dismantlable. By assumption we can reduce $G$ to a single vertex by deleting
k-dismantlable vertices, that is, vertices \( v \) whose open neighborhood \( N(v) \) is a k-dismantlable graph. Then by induction we have that each \( N(v) \in \mathcal{I}_{SV} \), and thus \( G \in \mathcal{I}_{SV} \).

Now suppose that \( G \in \mathcal{I}_{SV} \). We prove that \( G \) is k-dismantlable, for some k, by induction \( n = |V(G)| \). If \( n = 1 \) the claim is clear so we assume \( n > 1 \). By assumption we have some vertex \( v \in G \) such that \( N_G(v) \in \mathcal{I}_{SV} \). Hence by induction we have that \( v \) is \( \ell \)-dismantlable for some \( \ell \). Also, since \( G - v \) is in \( \mathcal{I}_{SV} \) we have that \( G - v \) is \( \ell' \)-dismantlable for some \( \ell' \). Recall that if \( G \) is m-dismantlable then \( G \) is \( m' \)-dismantlable for all \( m' \geq m \) (see [11, Proposition 2]). Hence if we let \( k = 1 + \max(\ell, \ell') \), we conclude that \( G \) is k-dismantlable. \( \square \)

**Corollary 28.** In Figure 1 we have the smallest graph \( G \) (in terms of vertices) that has the properties:

1. Its clique complex \( \Delta(G) \) has trivial homology;
2. \( G \) is not k-dismantlable for any k.

*Proof.* This follows from Theorem 12 and Theorem 27. \( \square \)

**Corollary 29.** In Figure 4 we have the smallest graph \( G \) (in terms of vertices) that has the properties:

1. \( G \) is j-dismantlable for some j;
2. There exists a vertex \( v \in G \) such that \( N_G(v) \) is k-dismantlable for some k, but where \( G - v \) is not \( \ell \)-dismantlable for any \( \ell \).

*Proof.* This follows from Theorem 21 and again Theorem 27. \( \square \)

7. FURTHER DISCUSSION AND OPEN QUESTIONS

We end with some open questions. Some of these have been mentioned above but we collect them here for convenience. Recall from Section 3 that we are interested in graphs that are \( \mathcal{I} \)-contractible but not strong \( \mathcal{I} \)-contractible. Our conjectures from that section were as follows.

**Conjecture 16.** The smallest graph (in terms of vertices) with trivial homology that it is not strong \( \mathcal{I} \)-contractible is shown in Figure 2.a).

**Conjecture 17.** The graph depicted in Figure 2.b) is the smallest graph \( G \) (in terms of vertices) that has the following properties.

1. \( G \) is strong \( \mathcal{I} \)-contractible;
2. There exists and edge \( \{v, w\} \) in \( G \) such that \( N(v, w) \in \mathcal{I}_S \) and such that \( G\setminus\{v, w\} \) is not strong \( \mathcal{I} \)-contractible.

Our next collection of open questions address the connection between the classes of graphs \( \mathcal{I}_{SV} \), \( \mathcal{I}_S \), and \( \mathcal{I} \) and topological properties of their clique complexes. First recall from Theorem 27 and Proposition 26 that a graph \( G \) is strong vertex \( \mathcal{I} \)-contractible if and only if \( \Delta(G) \) is non-evasive.

As for the other classes of graphs, recall also that if \( G \) is strong \( \mathcal{I} \)-contractible then \( \Delta(G) \) is collapsible [9]. As far as we know the converse is still open. The statement was first formulated as a conjecture by the last three authors in [9], where some computational evidence was also discussed.

**Conjecture 30.** Suppose \( G \) is a graph such that its clique complex \( \Delta(G) \) is collapsible. Then \( G \) is strong \( \mathcal{I} \)-contractible.
Finally recall from Theorem 3 that if $G$ is $I$-contractible then $\Delta(G)$ is contractible. Again the converse is still open; that is, if $\Delta(G)$ is a contractible flag complex is it true that $G$ is $I$-contractible? It might be surprising if the class of $I$-contractible graphs happens to coincide with the class of contractible flag simplicial complexes. Indeed, as the barycentric subdivision of any simplicial complex is flag, classifying contractible flag complexes should be as hard as classifying contractible simplicial complexes. Hence we ask the following.

**Question 31.** Does there exists a graph $G$ that is not $I$-contractible but such that $\Delta(G)$ is contractible.

We have seen that the contractible transformations from Definition 2, when applied to a graph $G$, preserve the homotopy type of the underlying clique complex $\Delta(G)$. Hence if one deletes contractible vertices and edges of $G$ to obtain a single vertex, we have then have a certificate that $\Delta(G)$ is contractible. On the other hand our examples show that there exist graphs $H$ such that $\Delta(H)$ is contractible, yet where a greedy collapsing will not result in a vertex.

Our Algorithm 1 checks if a graph $G$ is strong vertex $I$-contractible. As a greedy algorithm it is quite straight-forward, and we have seen that it is sufficient to check for $I$-contractibility of any graph on less than 11 vertices. For larger graphs one could perhaps modify the algorithm, and for this it would be useful to have other examples of graphs that are in $3_{SV}$ but where the order of vertex deletion matters. Similarly, Algorithm 2 can fail to detect $I$-contractibility of a graph on 15 or more vertices, since in general one must ‘anti-collapse’ to determine if a graph is in $I$.

The notions of contractible transformations on a graph also have potential applications to the computation of persistent homology. In this context one is given a filtration of topological spaces, each of which is the Rips complex of some graph. As the contractible transformations on a graph $G$ are functorial and preserve the homotopy type of $\Delta(G)$ one can speed up the process of computing persistent homology by first applying the collapsing transformations to each graph in the sequence. This approach was utilized for the case of strong collapses (which correspond to 0-collapsible vertices in our context) by Boissonnat and Pritam in [5]. Since 0-collapsible vertices are a special case of contractible vertices, our approach will a priori lead to smaller graphs in the sequence.

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