Models and Information Rates for Wiener Phase Noise Channels

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Abstract—A waveform channel is considered where the transmitted signal is corrupted by Wiener phase noise and additive white Gaussian noise. A discrete-time channel model that takes into account the effect of filtering on the phase noise is developed. The model is based on a multi-sample receiver, i.e., an integrate-and-dump filter whose output is sampled at a rate higher than the signaling rate. It is shown that, at high Signal-to-Noise Ratio (SNR), the multi-sample receiver achieves a rate that grows logarithmically with the SNR if the number of samples per symbol (oversampling factor) grows with the cubic root of the SNR. Moreover, the pre-log factor is at least 1/2 and can be achieved by amplitude modulation. For an approximate discrete-time model of the multi-sample receiver, the capacity pre-log at high SNR is shown to be at least 3/4 if the number of samples per symbol grows with the square root of the SNR. The analysis shows that phase modulation achieves a pre-log of at least 1/4 while amplitude modulation still achieves a pre-log of 1/2. This is strictly greater than the capacity pre-log of the (approximate) discrete-time Wiener phase noise channel with only one sample per symbol, which is 1/2. Numerical simulations are used to compute lower bounds on the information rates achieved by the multi-sample receiver. The simulations show that oversampling is beneficial for both strong and weak phase noise at high SNR. In fact, the information rates are sometimes substantially larger than when using commonly-used approximate discrete-time models.

Index Terms—Phase noise, Wiener process, waveform channel, capacity, oversampling.

I. INTRODUCTION

Phase noise arises due to the instability of oscillators [1] in communication systems such as satellite communication [2], microwave links [3] or optical fiber communication [4]. The statistical characterization of the phase noise depends on the application. In systems with phase-locked loops (PLL), the residual phase noise follows a Tikhonov distribution [5]. In Digital Video Broadcasting DVB-S2, an example of a satellite communication system, the phase noise process is modeled by the sum of the outputs of two infinite-impulse response filters driven by the same white Gaussian noise process [2]. In optical fiber communication, the phase noise in laser oscillators is modeled by a Wiener process [4].

A commonly studied discrete-time channel model is

\[ Y_k = X_k e^{j\Theta_k} + Z_k \] (1)

where \( \{Y_k\} \) are the output symbols, \( \{X_k\} \) are the input symbols, \( \{\Theta_k\} \) is the phase noise process and \( \{Z_k\} \) is additive white Gaussian noise (AWGN). For example, Katz and Shamai [6] studied the model (1) when \( \{\Theta_k\} \) is independent and identically distributed (i.i.d.) according to \( p_\Theta(\cdot) \), when \( \Theta \) is uniformly distributed (called a non-coherent AWGN channel) and when \( \Theta \) has a Tikhonov (or von Mises) distribution (called a partially-coherent AWGN channel). The i.i.d. Tikhonov phase noise models the residual phase error in systems with phase-tracking devices, e.g., phase-locked loops (PLL) and ideal interleavers/deinterleavers. Katz and Shamai [6] characterized some properties of the capacity-achieving distribution. Tight lower bounds on the capacities of memoryless non-coherent and partially coherent AWGN channels were computed by solving an optimization problem numerically in [6] and [7], respectively. Lapidoth studied in [8] a phase noise channel [1] at high SNR. He considered both memoryless phase noise and phase noise with memory. He showed that the capacity grows logarithmically with the SNR with a pre-log factor 1/2 if the differential entropy rate of \( \{\Theta_k\} \) is finite. The pre-log 1/2 is achieved by amplitude modulation only, and the phase modulation contributes only a bounded number of bits. Daubels and Loeliger [9] proposed a particle filtering method to compute information rates for discrete-time continuous-state channels with memory and applied the method to [1] for Wiener phase noise and auto-regressive-moving-average (ARMA) phase noise. Barletta, Magarini and Spalvieri [10] computed lower bounds on information rates for [1] with Wiener phase noise by using the auxiliary channel technique proposed in [11] and they computed upper bounds in [12]. They also developed a lower bound based on Kalman filtering in [13]. Barbieri and Colavolpe [14] computed lower bounds with an auxiliary channel slightly different from [10]. Durisi et al. [15, 16] developed capacity bounds for a multi-input multi-output (MIMO) extension of the discrete-time model (1).

A less-studied model since 1990 is the continuous-time model for phase noise channels, also called a waveform phase noise channel, see Fig. [1] The received waveform \( r(t) \) is

\[ r(t) = x(t) e^{j\theta(t)} + n(t), \quad \text{for } t \in \mathbb{R} \] (2)

where \( x(t) \) is the transmitted waveform while \( n(t) \) and \( \theta(t) \) are additive noise and phase noise, respectively. Continuous-
time white phase noise was recently considered in [17] Section IV.C and [18], [19]. We study the waveform channel [2] with Wiener phase noise. A detailed description of the channel is given in Sec. III. This model is reasonable, for example, for optical fiber communication with low to intermediate power and laser phase noise, see [20]–[22]. Since the sampling of a continuous-time Wiener process yields a discrete-time Wiener process (Gaussian random walk), it is tempting to use the model (1) with \( \{ \Theta \} \) as a discrete-time Wiener process. However, this ignores the effect of filtering prior to sampling. It was pointed out in [20] that “even coherent systems relying on amplitude modulation (phase noise is obviously a problem in systems employing phase modulation) will suffer some degradation due to the presence of phase noise”. This is because the filtering converts phase fluctuations to amplitude variations. It is worth mentioning that filtering is necessary before sampling to limit the variance of the noise samples.

The model (1) thus does not fit\(^1\) the channel (2) and it is not obvious whether a pre-log 1/2 is achievable. The model that takes the effect of filtering into account is

\[
Y_k = X_k H_k + N_k \tag{3}
\]

where \( \{ H_k \} \) is a fading process. The model (3) falls in the class of non-coherent fading channels, i.e., the transmitter and receiver have knowledge of the distribution of the fading process \( \{ H_k \} \), but have no knowledge of its realization. For such channels, Lapidoth and Moser showed in [23] that, at high SNR, the capacity grows double-logarithmically with the SNR, where the process \( \{ H_k \} \) is stationary, ergodic and has finite differential entropy rate.

Rather than using a filter and sampling its output at the symbol rate, we use a multi-sample receiver, i.e., a filter whose output is sampled many times per symbol. We show that this receiver achieves a rate that grows logarithmically with the SNR if the number of samples per symbol grows with the cubic root of the SNR. Furthermore, we show that a pre-log of 1/2 is achievable through amplitude modulation. We also show for an approximate multi-sample model, where the approximation is to ignore the filtering effect over a sample, that phase modulation contributes 1/4 to the pre-log factor at high SNR if the oversampling factor grows with the square root of SNR. We corroborate the results of the high-SNR analysis with numerical simulations at finite SNR. This paper collects and extends results presented in [24]–[26].

Benefits of oversampling were reported for other problems. For example, in some digital storage systems, it was shown by numerical simulations that oversampling can increase the information rate [27]. In a low-pass-filter-and-limiter (nonlinear) channel, it was found that oversampling (with a factor of 2) offers a higher information rate [28]. It was demonstrated in [29] that doubling the sampling rate recovers some of the loss in capacity incurred on the bandlimited Gaussian channel with a one-bit output quantizer. For non-coherent block fading channels, it was shown in [30] that by doubling the sampling rate the information rate grows logarithmically at high SNR with a pre-log \( 1 - 1/N \) rather than \( 1 - Q/N \) which is the pre-log achieved by sampling at the symbol rate (\( N \) is the length of the fading block and \( Q \) is the rank of the covariance matrix of the discrete-time channel gains within each block).

This paper is organized as follows. The continuous-time model is described in Sec. III and the discrete-time models of the matched filter receiver and the multi-sample receiver are described in Sec. III. We give a lower bound on the capacity in Sec. IV based on amplitude modulation and show that the pre-log factor is at least 1/2 if the oversampling factor grows with the cubic root of the SNR. We also give a lower bound on the rate achieved by phase modulation for the approximate multi-sample discrete-time model. For such a model, we show that phase modulation contributes at least 1/4 to the pre-log factor, resulting in an overall pre-log factor of 3/4 when both amplitude and phase modulation are used. This is achieved by using an oversampling factor that grows with the square root of the SNR. We develop algorithms to compute tight lower bounds on the information rates of a multi-sample receiver in Sec. V. In Sec. VI, we report the results of numerical simulations which demonstrate the importance of increasing the oversampling factor with the SNR to achieve higher rates. We discuss in Sec. VII our results, related work, open problems and intuitions for our proofs. Finally, we conclude with Sec. VIII.

II. WAVEFORM PHASE NOISE CHANNEL

We use the following notation: \( j = \sqrt{-1} \). \( * \) denotes the complex conjugate, \( \delta_D \) is the Dirac delta function, \( [\cdot] \) is the ceiling operator. We use \( X^k \) as a shorthand for \( \{ X_1, X_2, \ldots, X_k \} \). Suppose the transmit-waveform is \( x(t) \) and the receiver observes

\[
r(t) = x(t) e^{j\theta(t)} + n(t) \tag{4}
\]

where \( n(t) \) is a realization of a white circularly-symmetric complex Gaussian process \( N(t) \) with

\[
\begin{align*}
\mathbb{E} [N(t)] &= 0 \\
\mathbb{E} [N(t_1)N^*(t_2)] &= \sigma_N^2 \delta_D(t_2 - t_1).
\end{align*} \tag{5}
\]

The phase \( \theta(t) \) is a realization of a Wiener process

\[
\Theta(t) = \Theta(0) + \int_0^t W(\tau) d\tau \tag{6}
\]

where \( \Theta(0) \) is uniform on \( [-\pi, \pi] \) and \( W(t) \) is a real Gaussian process with

\[
\begin{align*}
\mathbb{E} [W(t)] &= 0 \\
\mathbb{E} [W(t_1)W(t_2)] &= 2\pi \beta \delta_D(t_2 - t_1) \tag{7}
\end{align*}
\]
where $0 < \beta < \infty$. The processes $N(t)$ and $\Theta(t)$ are independent of each other and independent of the input. $N_0 = 2\sigma_N^2$ is the single-sided power spectral density of the additive noise. We define $U(t) \equiv \exp(j\Theta(t))$. The auto-correlation function of $U(t)$ is

$$R_U(t_1, t_2) = \mathbb{E}[U(t_1)U^*(t_2)] = \exp(-\pi\beta|t_2 - t_1|)$$

and the power spectral density of $U(t)$ is

$$S_U(f) = \int_{-\infty}^{\infty} R_U(t, t + \tau) e^{-j2\pi \tau f} d\tau = \frac{1}{\pi} \frac{\beta/2 + f^2}{(\beta/2)^2 + f^2}.$$  

(9)

The spectrum is said to have a Lorentzian shape. We have $\beta = f_{FWHM} = 2f_{FWHM}$ where $f_{FWHM}$ is the full-width at half-maximum and $f_{FWHM}$ is the half-width at half-maximum. One also refers to $\beta$ as the linewidth of the phase noise.

Let $T$ be the transmission interval, then the transmitted waveforms must satisfy the power constraint

$$\mathbb{E}\left[\frac{1}{T} \int_0^T |X(t)|^2 dt\right] \leq P$$

where $X(t)$ is a random process whose realization is $x(t)$.

### III. DISCRETE-TIME MODELS

#### A. Transmitter

Let $(x_1, x_2, \ldots, x_b)$ be the codeword sent by the transmitter. Suppose the transmitter uses a unit-energy pulse $g(t)$ whose time support is $[0, T_s]$ where $T_s$ is the symbol interval, which is finite and fixed. The waveform sent by the transmitter is

$$x(t) = \sum_{m=1}^b x_m g(t - (m-1)T_s).$$

(11)

We remark that using pulses that are limited in time to one symbol interval simplifies the analysis because it eliminates intersymbol interference at the receiver. We focus on rectangular pulses for the high SNR analysis in Sec. VI. A disadvantage of a rectangular pulses is the slow decay of the sidelobes in frequency domain, which motivates us to also consider cosine-squared pulses for numerical rate computation in Sec. VI. The sidelobes of a cosine-squared pulses, which is equivalent to a (time-domain) raised-cosine pulse with roll-off factor 1, decay much faster than those of a rectangular pulse. We also remark that Martalò et al [31] extended some of our numerical results for bandlimited pulses. However, their model does not include fully the filtering effects of a continuous-time model.

#### B. Matched Filter Receiver

The received signal $r(t)$ is fed to a matched filter and the output $y(t)$ of the filter is sampled at the symbol rate as illustrated in Fig. 2. The $k$-th output sample is

$$y_k = r(t) \ast g^\ast(T_s - t)|_{t = kT_s}$$

$$= x_k \int_{-\infty}^{\infty} e^{j\beta(\tau)} |g(\tau - (k-1)T_s)|^2 d\tau + n_k$$

(12)

where $k = 1, \ldots, b$, $\ast$ denotes convolution and $n_k$ is the part of the output due to the additive noise. Therefore, we have the discrete-time model

$$Y_k = X_k H_k + N_k$$

(13)

where

$$H_k \equiv \int_{-\infty}^{\infty} e^{j\beta(\tau)} |g(\tau - (k-1)T_s)|^2 d\tau.$$  

The transmitter and receiver know the distribution of the fading process $\{H_k\}$ but have no knowledge of its realization. We strongly suspect that $\{H_k\}$ in (14) is stationary, ergodic and has a finite differential entropy rate, i.e., $h(\{H_k\}) > -\infty$. Lapidoth and Moser showed in [23] that, at high SNR, the capacity grows double-logarithmically with the SNR under the aforementioned conditions on $\{H_k\}$.

A commonly-used approximation for $H_k$ is

$$\int_{-\infty}^{\infty} e^{j\beta(\tau)} |g(\tau - (k-1)T_s)|^2 d\tau \approx e^{j\beta(kT_s)} \int_{-\infty}^{\infty} |g(\tau - (k-1)T_s)|^2 d\tau$$

(15)

which yields the approximate discrete-time model

$$Y_k = X_k e^{j\beta(kT_s)} + N_k.$$  

(16)

We refer to (13) and (16) as the symbol-rate model with filtering and without filtering, respectively. We also refer to (16) as the approximate symbol-rate model. We do not claim that this approximation is always accurate. We introduce the approximate model to examine its validity by comparing it with the discrete-time model based on the multi-sample receiver. We remark that the differential entropy of $\{\Theta(kT_s)\}$ is finite because the linewidth $\beta$ of $\Theta(t)$ and the symbol interval $T_s$ are finite and therefore, the pre-log of the model (16) is 1/2.

#### C. Multi-sample Receiver

Consider next a multi-sample receiver (see Fig. 3). Let $L$ be the number of samples per symbol ($L \geq 1$) and define the sample interval as

$$\Delta = \frac{T_s}{L}.$$  

(17)

The received waveform $r(t)$ is filtered using an integrator over a sample interval to give the output signal

$$y(t) = \int_{t-\Delta}^{t} r(\tau) d\tau.$$  

(18)

The signal $y(t)$ is a realization of a random process $Y(t)$ that is sampled at $t = k\Delta$, $k = 1, \ldots, n = bL$. Hence, we have the discrete-time model in which the $k$-th output sample is

$$Y_k = X_{[k/L]} \Delta e^{j\beta_k} F_k + N_k$$

(19)
where \( Y_k \equiv Y(k\Delta) \), \( \Theta_k \equiv \Theta((k-1)\Delta) \),
\[
F_k \equiv \frac{1}{\Delta} \int_{(k-1)\Delta}^{k\Delta} g \left( \tau - \left( \frac{k}{\Delta} \right) T_s \right) e^{i(\Theta(\tau)-\Theta_k)} \, d\tau
\]  
and
\[
N_k \equiv \int_{(k-1)\Delta}^{k\Delta} N(\tau) \, d\tau.
\]
The process \( \{N_k\} \) is an i.i.d. circularly-symmetric complex Gaussian process with
\[
E[N_k] = 0
\]
\[
E[N_kN_k^*] = \sigma_N^2 \delta_k \quad \text{for} \quad k \neq 0
\]
\[
E[N_kN_\ell] = 0
\]
where \( \delta_k[\cdot] \) is the Kronecker delta. The process \( \{\Theta_k\} \) is the discrete-time Wiener process
\[
\Theta_k = \Theta_{k-1} + W_k
\]
for \( k = 2, \ldots, n \), where \( \Theta_1 \) is uniform on \([-\pi, \pi)\) and \( \{W_k\} \) is an i.i.d. real Gaussian process with mean 0 and \( E[|W_k|^2] = 2\pi \beta \Delta \), i.e., the probability density function (pdf) of \( W_k \) is
\[
p_{W_k}(w) = G(w; 0, \sigma_W^2)
\]
and \( \sigma_W^2 = 2\pi \beta \Delta \). The random variable \((W_k \mod 2\pi)\) is a wrapped Gaussian and its pdf is
\[
p_{W}(w; \sigma^2) = \sum_{i=-\infty}^{\infty} G(w - 2\pi i; 0, \sigma^2).
\]
Moreover, \( \{F_k\} \) and \( \{W_k\} \) are independent of \( \{N_k\} \) but not independent of each other. Equations (10) and (11) imply the power constraint
\[
\frac{1}{b} \sum_{m=1}^{b} E[|X_m|^2] \leq P \equiv \mathcal{P}T_s
\]
The signal-to-noise ratio SNR is defined as
\[
\text{SNR} \equiv \frac{P}{\sigma_N^2 T_s} = \frac{P}{\sigma_N^2}.
\]

### D. Information Rates

We use \( Y^k \) as a shorthand for \( (Y_1, Y_2, \ldots, Y_k) \) where
\[
Y_k \equiv (Y((k-1)L+1), Y((k-1)L+2), \ldots, Y((k-1)L+L)).
\]

The capacity of a point-to-point channel is given by
\[
C(\text{SNR}) = \lim_{b \to \infty} \frac{1}{b} \sup I(X^b; Y^b)
\]
where the supremum is over all joint distributions of the input symbols satisfying the power constraint. For a given input distribution, the achievable rate \( R \) is
\[
R(\text{SNR}) = I(X; Y) \equiv \lim_{b \to \infty} \frac{1}{b} I(X^b; Y^b).
\]

### IV. HIGH-SNR INFORMATION RATES

In this section, we study rectangular pulses, i.e., we consider
\[
g(t) \equiv \left\{ \begin{array}{ll}
1/T_s, & 0 \leq t < T_s, \\
0, & \text{otherwise}.
\end{array} \right.
\]
It follows that \( F_k \) in (20) becomes
\[
F_k \equiv \frac{1}{\Delta} \int_{(k-1)\Delta}^{k\Delta} e^{i(\Theta(\tau)-\Theta_k)} \, d\tau.
\]
We define \( X_A \equiv |X| \) and \( \Phi_X = \arg(X) \) where \( \arg(\cdot) \) denotes the phase angle (also called the argument) of a complex number. We decompose the mutual information using the chain rule into two parts:
\[
I(X^b; Y^b) = I(X^b_A; Y^b_A) + I(X^b_A; Y^b | X^b_A)
\]
where \( X^b_A = (|X_1|, \ldots, |X_b|) \) and \( \Phi^b_X = (\arg X_1, \ldots, \arg X_b) \). Define
\[
I(X_A; Y) \equiv \lim_{b \to \infty} \frac{1}{b} I(X^b_A; Y^b)
\]
and
\[
I(\Phi_X; Y | X_A) \equiv \lim_{b \to \infty} \frac{1}{b} I(\Phi^b_X; Y^b | X^b_A).
\]

It follows from (32)-(37) that
\[
I(X; Y) = I(X_A; Y) + I(\Phi_X; Y | X_A).
\]

The first term represents the contribution of the amplitude modulation while the second term represents the contribution of the phase modulation. The results of this section are summarized in Theorem 1 and Theorem 2 which address the contribution of amplitude modulation and phase modulation to the information rate, respectively.

Before we proceed to the theorems, we introduce an important result that we use multiple times in the paper. Let \( X \) and \( Y \) be the input and output of a channel, respectively and let \( g(\cdot) \) be a valid conditional pdf, which represents an “auxiliary channel”. The following auxiliary-channel lower bound holds:
\[
I(X; Y) \geq I(X; Y) = \mathbb{E}[\log g_Y|X(Y|X) - \log g_Y(Y)]
\]
where \( g_Y(\cdot) \) is the output pdf obtained by connecting the true input source to the auxiliary channel. Note that \( \mathbb{E}[\cdot] \) is the expectation according to the true distribution of \( X \) and \( Y \). The lower bound is the rate achieved by a mismatched decoder. See [32, pp. 6-7] for a recent comprehensive review on mismatched decoding.

**Theorem 1 (Amplitude Modulation):** Consider the discrete-time model in (19) with \( F_k \) as defined in (34). If \( L = \sqrt{\beta T_s/\text{SNR}} \) and the process \( \{|X_k|\} \) is i.i.d. such that...
if logarithmically at high SNR with a pre-log factor of at least 1/2 if the oversampling factor \( L \) grows with the cubic root of SNR. More generally, for \( L = \lceil \sqrt{\text{SNR}} \rceil \), we have

\[
\lim_{\text{SNR} \to \infty} \frac{C(\text{SNR})}{\log(\text{SNR})} \geq \frac{3}{4}
\]

which follows from (43) and (44) by using C(SNR) \( \geq I(X; \Psi) = I(X_A; \Psi) + I(\Phi_X; \Psi | X_A) \). This shows that the capacity grows logarithmically at high SNR with a pre-log factor of at least 3/4. We remark that Barletta and Kramer \[33\] extended the result of Theorem 2 to the full model \( [19] \) that includes the filtering effect, i.e., they showed that C(SNR) \( \geq 3/4 \) for \( L = \lceil \sqrt{\text{SNR}} \rceil \). More generally, for \( L = \lceil \text{SNR}^{1/2} \rceil \), they showed for the full model that

\[
\lim_{\text{SNR} \to \infty} \frac{C(\text{SNR})}{\log(\text{SNR})} \geq \left\{ \begin{array}{ll}
2\alpha, & 0 < \alpha \leq 1/3 \\
(1 + \alpha)/2, & 1/3 \leq \alpha \leq 1/2 \\
3/4, & 1/2 \leq \alpha < 1.
\end{array} \right.
\]

V. RATE COMPUTATION AT FINITE SNR
We develop algorithms to compute the information rate numerically. We use \( X_k \) to denote the \( k \)th input sample which is defined by

\[
X_k = X_{[k/L]} g ((k \mod L) \Delta).
\]

The information rate \( I(X; Y) \) in (42) can be expressed as

\[
I(X; Y) = \lim_{b \to \infty} \frac{1}{b} I(X^n; Y^n)
\]

which follows from (47) and (40). One difficulty in evaluating (48) is that the joint distribution of \( \{ F_k \} \) and \( \{ W_k \} \) is not available in closed form. Even the distribution of \( F_k \) is not available in closed form (there is an approximation for small linewidth, see (16) in [21]). However, we can numerically compute tight lower bounds on \( I(X; Y) \) by using the auxiliary-channel technique (cf. [39]). We use the following algorithm [11].

1. Sample a long sequence \( (x^n, y^n) \) according to the true joint distribution of \( X^n \) and \( Y^n \).
2. Compute \( q_{Y^n|X^n}(y^n | x^n) \) and

\[
q_{Y^n}(y^n) = \sum_{x^n} p_{X^n}(x^n) q_{Y^n|X^n}(y^n | x^n)
\]

where \( p_{X^n} \) is the true distribution of \( X^n \).
3. Estimate \( I(X; Y) \) using

\[
I(X; Y) \approx \frac{1}{b} \log \left( \frac{q_{Y^n|X^n}(y^n | x^n)}{q_{Y^n}(y^n)} \right)
\]

Auxiliary Channel I: Consider the auxiliary channel

\[
\Psi_k = X_k \Delta \ e^{j\theta_k} + N_k
\]

where \( \{ \Theta_k \} \) and \( \{ N_k \} \) are defined in Sec. III-C and \( X_k \) is defined by (47). The channel output \( \Psi \) is the same as \( Y \) in (19) except that \( F_k \) is replaced by 1 or more generally with \( g ((k \mod L) \Delta) \). The channel is described by the conditional distribution

\[
p_{\Psi^n|X^n}(\psi^n | x^n) = \int_{\theta^n} p_{\Psi^n|X^n}(\theta^n, y^n | x^n) \ d\theta^n
\]

where

\[
p_{\Psi^n|X^n}(\theta^n, y^n | x^n) = \prod_{k=1}^{n} p_{\Psi_k | \Theta_{k-1}}(\theta_k | \theta_{k-1}) p_{\Psi_k | X^n}(y_k | x_k, \theta_k)
\]

with

\[
p_{\Psi_k|\Theta_{k-1}}(\theta | \theta_{k-1}) = \left\{ \begin{array}{ll}
p_W(\theta - \hat{\theta}; \sigma_N^2), & k \geq 2 \\
1/(2\pi), & k = 1
\end{array} \right.
\]

and

\[
p_{\Psi_k|X^n}(y_k | x, \theta) = \frac{1}{\pi \sigma_N^2 \Delta} \exp \left( -\frac{|y_k - x \Delta e^{j\theta}|^2}{\sigma_N^2 \Delta} \right)
\]

The channel \( p_{\Psi^n|X^n} \) has continuous states \( \theta^n \), which implies that step 2 of the algorithm cannot be computed directly. To compute the integrals, we quantize the states (the phase noise). Finer quantization results in more accurate results.
The conditional probability is a phase-noise tracker which outputs an estimate $\hat{\theta}_{k-1}$ of the phase noise where $\hat{\theta}_{k-1} = \arg(y_{k-1}/\tilde{x}_{m-1})$ where $m = [k/L]$ (cf. (157)). The phase decoder uses the estimate of the phase noise $\hat{\theta}_{k-1}$ and the estimated amplitude $\tilde{x}_{A,m}$ to produce an estimate $\hat{\phi}_{X,m}$ of the phase of $x_m$.

Fig. 5. Two-stage decoding in a multi-sample receiver. The upper branch is part of the double filtering receiver for amplitude decoding. The lower branch is a phase-noise tracker which outputs an estimate $\hat{\phi}_{X,m}$ of the phase noise. When there is no significant difference in the rate estimates, we take that to be a (lower bound) valid estimate of the rate. When there is no significant change in the rate estimates, we take that to be a (lower bound) valid estimate of the rate.

Next, we describe our choice of $S$ and $Q(\cdot)$. We partition $[-\pi, \pi]$ into $S$ intervals with equal lengths and pick the midpoints of these intervals to be the elements of $S$, i.e., we have

$$S = \{\hat{s}_i : i = 1, \ldots, S\} \quad \text{where} \quad \hat{s}_i = \frac{2\pi}{S} - \frac{\pi}{S} - \pi. \quad (60)$$

The state transition probability $Q(\cdot)$ is chosen similar to (10) and (12):

$$Q(s|\hat{s}) = \frac{2\pi}{S} \int_{(\phi, \hat{\phi}) \in \mathcal{R}(s) \times \mathcal{R}(\hat{s})} p_W(\phi - \hat{\phi}; \sigma^2_W) \, d\phi d\hat{\phi} \quad (61)$$

where $\mathcal{R}(s) = [s - \pi/S, s + \pi/S]$, i.e., $\mathcal{R}(s)$ is the interval whose midpoint is $s$. The larger $S$ and $L$ are, the better the auxiliary channel (56) approximates the actual channel (19). We remark that the auxiliary channel gives a valid lower bound on $I(X;Y)$ even for small $S$ and $L$.

**Auxiliary Channel II:** We use the following auxiliary channel for the numerical simulations:

$$\Upsilon_k = X_k \Delta e^{jS_k} + N_k. \quad (56)$$

The conditional probability is

$$p_{T^n|X^n}(y^n|x^n) = \sum_{s^n \in S^n} p_{S^n, T^n|X^n}(s^n, y^n|x^n) \quad (57)$$

where $S$ is a finite set and

$$p_{S^n, T^n|X^n}(s^n, y^n|x^n) = \prod_{k=1}^{n} p_{S_{k}|S_{k-1}}(s_k|s_{k-1}) \cdot p_{T_{k}|S_{k}}(y_k|x_k, s_k) \quad (58)$$

where

$$p_{S_{k}|S_{k-1}}(s|\hat{s}) = \begin{cases} Q(s|\hat{s}), & k \geq 2 \\ 1/|S|, & k = 1. \end{cases} \quad (59)$$

**A. Computing The Conditional Probability**

Suppose the input $X^n$ has the distribution $p_{X^n}$. A Bayesian network for $X^n, S^n, \Upsilon^n$ is shown in Fig. 6. The probability $p_{T^n|X^n}(y^n|x^n)$ can be computed using

$$p_{T^n|X^n}(y^n|x^n) = \sum_{s \in S} p_{\rho_n}(s) \quad (62)$$
where we recursively compute

\[
\rho_k(s) \equiv P_{S_k, Y^k | X^k}(s, y^k | x^n) \\
= \sum_{\tilde{s} \in S} P_{S_k-1, S_k, Y^k | X^k}(\tilde{s}, s, y^k | x^n) \\
= \sum_{\tilde{s} \in S} \rho_{k-1}(\tilde{s}) P_{S_k, Y_k | S_{k-1}, Y^{k-1}, X^n}(s, y_k | \tilde{s}, y^{k-1}, x^n) \\
= \sum_{\tilde{s} \in S} \rho_{k-1}(\tilde{s}) Q(s | \tilde{s}) \psi_{Y_k}(y_k | x_k, s)
\]

(63)

with the initial value \( \rho_0(s) = 1/|S| \). Step (a) is a marginalization, (b) follows from Bayes’ rule and the definition of \( \rho_k \) in (63), while (64) follows from the structure of Fig. 6. We remark that (63) is the same as with independent \( X_1, \ldots, X_n \), e.g., see equation (9) in [34] Sec. IV.

B. Computing The Marginal Probability

Define \( X_m \equiv (X_{(m-1) L+1}, X_{(m-1) L+2}, \ldots, X_{(m-1) L+L}) \). Suppose the input symbols are i.i.d. and \( X_m \in \mathcal{X} \) where \( \mathcal{X} \) is a finite set. Therefore, \( p_X \) has the form

\[
p_X(x^n) = \prod_{m=1}^b p_X(x_m).
\]

(65)

A Bayesian network for \( X^n, S^n, Y^n \) is shown in Fig. 7. The probability \( p_{T^n}(y^n) \) can be computed using

\[
p_{T^n}(y^n) = \sum_{s \in S} \psi_b(s)
\]

(66)

where \( \psi_b(s) \equiv P_{S_m, Y^n | X^n}(s, y^n) \) which can be computed using the recursion:

\[
\psi_m(s) = \sum_{\tilde{s} \in \mathcal{X}_L} p_X(\tilde{x}) \sum_{\tilde{s} \in S} \psi_{m-1}(\tilde{s}) P_{S_m, Y_m | S_{(m-1) L}, X_m}(s, y_m | \tilde{s}, \tilde{x})
\]

(67)

with the initial value \( \psi_0(s) = 1/|S| \). The set \( \mathcal{X}_L \) is

\[
\mathcal{X}_L = \{x \cdot (g(\Delta), g(2\Delta), \ldots, g(L\Delta)) : x \in \mathcal{X}\}
\]

(68)

We remark that \( |\mathcal{X}_L| = |\mathcal{X}|^L \) and not \( |\mathcal{X}|^{L} \). Next, we define

\[
\chi_{m,L}(s, \tilde{s}, \tilde{x}) = P_{S_m, Y_m | S_{(m-1) L}, X_m}(s, y_m | \tilde{s}, \tilde{x})
\]

(69)

for \( s, \tilde{s} \in S \) and \( \tilde{x} \in \mathcal{X}_L \). Computing \( \chi_{m,L}(s, \tilde{s}, \tilde{x}) \) is similar to computing \( \rho_n \) (see (64)). Intuitively, this is because a block of \( L \) samples in Fig. 7 has a structure similar to Fig. 6. More precisely, \( \chi_{m,L}(s, \tilde{s}, \tilde{x}) \) can be computed recursively by using

\[
\chi_{m,L}(s, \tilde{s}, \tilde{x}) = \sum_{s \in S} \chi_{m,L-1}(s, \tilde{s}, \tilde{x}) Q(s | \tilde{s}) \psi_{Y_m}(y_m | x_m, s)
\]

(70)

with the initial value

\[
\chi_{m,0}(s, \tilde{s}, \tilde{x}) = \begin{cases} 1, & s = \tilde{s} \\ 0, & \text{otherwise} \end{cases}
\]

(71)

Therefore, computing \( p_{T^n}(y^n) \) involves two levels of recursion: 1) a recursion over the symbols as described by (67) and 2) a recursion over the samples within a symbol as described by (70).

VI. NUMERICAL SIMULATIONS

We use two pulses with a symbol-interval time support:

- A unit-energy square pulse

\[
g_1(t) = \frac{1}{\sqrt{T_s}} \text{rect} \left( \frac{t}{T_s} \right)
\]

(72)

where

\[
\text{rect}(t) \equiv \begin{cases} 1, & |t| \leq 1/2 \\ 0, & \text{otherwise} \end{cases}
\]

(73)

- A unit-energy cosine-squared pulse

\[
g_2(t) = \frac{1}{\sqrt{T_s/2}} \cos^2 \left( \frac{\pi t}{T_s} \right) \text{rect} \left( \frac{t}{T_s} \right)
\]

(74)

The first step of the algorithm is to sample a long sequence according to the true joint distribution of \( X^n \) and \( Y^n \). To generate samples according to the original channel (19), we must accurately represent digitally the continuous-time waveform (4). We use a simulation oversampling rate \( L_{\text{sim}} = 1024 \) samples/symbol. After the filter (18), the receiver has \( L \) samples/symbol distributed according to (19). Next, to choose a proper sequence length, we follow the approach suggested in (11): for a candidate length, run the algorithm about 10 times (each with a new random seed) and check whether all estimates of the information rate agree up to the desired accuracy. We used \( b = 10^4 \) unless otherwise stated.

For efficient implementation of (64), \( \psi_{Y_k}(.)|\cdot| \cdot \) can be factored out of the summation to yield:

\[
\rho_k(s) = p_{Y_k|X_k}(y_k | x_k, s) \sum_{\tilde{s} \in S} \rho_{k-1}(\tilde{s}) Q(s | \tilde{s})
\]

(75)

Moreover, since \( Q(\cdot | \cdot) \) can be represented by a circulant matrix due to symmetry, \( \rho_k(\cdot) \) can be computed efficiently using the Fast Fourier Transform (FFT). Similarly, the computation of (70) can be done efficiently by factoring out \( p_{Y_k|X_k}(\cdot | \cdot, \cdot) \) and by using the FFT.
A. Very Large Linewidth

Suppose $f_{\text{HWHM}}T_s = 0.125$ and the input symbols are independently and uniformly distributed (i.u.d.) 16-QAM. Fig. 8 shows an estimate of $I(X; Y)$ for a square transmit-pulse, i.e., $g(t) = g_{\text{sq}}(t - T_s/2)$ and an $L$-sample receiver with $L = 4, 8, 16$ and $S = 16, 32, 64$. The curves with $S = 64$ are indistinguishable from the curves with $S = 32$ over the entire SNR range for all values of $L$, and hence it seems that $S = 32$ is adequate up to 25 dB. Even $S = 16$ seems to be adequate up to 20 dB. We remark that the non-monotonicity of some lower bounds is an artifact of the auxiliary channel being “far” from the true channel, specifically because the number $S$ of quantization levels of Auxiliary Channel II is too low for some SNR values. The important trend in Fig. 8 is that higher oversampling rate $L$ is needed at high SNR to extract all the information from the received signal. For example, $L = 4$ suffices up to $\text{SNR} \sim 10$ dB, $L = 8$ suffices up to $\text{SNR} \sim 15$ dB but $L \geq 16$ is needed beyond that. It was pointed out in [11] that the lower bounds can be interpreted as the information rates achieved by mismatched decoding. For example, $I(X; Y)$ for $L = 8$ and $S \geq 32$ in Fig. 8 is essentially the information rate achieved by a multi-sample (8-sample) receiver that uses maximum-likelihood decoding for the simplified channel [51] when it is operated in the original channel [19].

Fig. 9 shows an estimate of $I(X; Y)$ for a cosine-squared transmit-pulse, i.e., $g(t) = g_{\text{csq}}(t - T_s/2)$ and an $L$-sample receiver at $L = 4, 8, 16$ and $S = 16, 32, 64$. We find that $S = 32$ suffices up to $\sim 25$ dB. We see in Fig. 9 the same trend as in Fig. 8 higher $L$ is needed at higher SNR. Furthermore, the square pulse is better than the cosine-squared pulse for the same oversampling rate $L$. It might seem strange the the cosine-squared pulse curves are substantially lower than the square pulse curves despite taking into account the pulse shape in the auxiliary channel model, see [56] and [47]. One obtains insight on this effect with the following Gedankenexperiment. Suppose we use a pulse that is narrow and peaky in time. Then an integrate-and-dump receiver with oversampling puts out essentially only one useful sample per symbol. One may as well, therefore, use only one sample per symbol.

B. Large Linewidth

As the linewidth decreases, the benefit of oversampling at the receiver becomes apparent only at higher SNR. For example, for $f_{\text{HWHM}}T_s = 0.0125$ and i.u.d. 16-PSK input, Fig. 10 shows an estimate of $I(X; Y)$ for a square transmit-pulse and an $L$-sample receiver at $L = 1, 2, 4, 8, 16$ and $S = 64$. We see that $L = 4$ suffices up to $\text{SNR} \sim 19$ dB, $L = 8$ suffices up to $\text{SNR} \sim 24$ dB and only beyond that $L \geq 16$ is necessary.
The symbol-rate model with filtering (13) that are valid up to some SNR that grows with the oversampling rate. We observe from Fig. 11 the following:

1) The rates of ASM and MTR saturate well below the rate achieved by the multi-sample receiver, which implies that symbol-rate sampling results in information loss at high SNR.

2) The multi-sample receiver rates are indistinguishable from the MTR rates at low SNR which implies that matched filtering with symbol-rate sampling incurs practically no information loss, i.e., using (16) to compute rates at low/moderate SNR gives accurate results.

3) MTR is indistinguishable from ASM at low SNR which implies that approximating (13) as (16) can be justified at low SNR, i.e., the filtering effect may be ignored.

We remark that the saturation at high SNR of the rates of the symbol-rate models at a value lower than the entropy of the source should not be surprising. The impact of the additive noise is smaller at high SNR, but the effect of phase noise and/or filtering are not necessarily reduced. For example, consider the memoryless channel $Y = HX + Z$ and a fixed-size constellation (e.g. $M$-QAM). We have the following upper bound (by using a genie-aided argument)

$$I(X;Y) \leq I(X;YZ)$$

$$= H(X) - H(X|HX).$$

The term $H(X|HX)$ represents impact of the ambiguity due to the randomness of $H$. For the special case of $Y = e^{j\theta}X + Z$, we have

$$I(X;Y) \leq H(X) - H(\Phi_X|\Phi_X + \Theta)$$

i.e., one may fully recover the amplitude of $X$ but not the phase of $X$ because of the ambiguity due to $\Theta$. Moreover, for $Y = e^{j\theta}X + Z$, there is a more general upper bound on the contribution of phase modulation given by

$$I(\Phi_X;Y|X_A) \leq I(\Phi_X;Y|X_A, Z)$$

$$= I(\Phi_X;\Phi_X + \Theta|X_A)$$

$$= h(\Phi_X + \Theta|X_A) - h(\Theta)$$

$$\leq \log(2\pi) - h(\Theta)$$

This upper bound is more general than the earlier one in the sense that it is valid for any input constellation (of any size). Unfortunately, we do not have a simple upper bound expression similar to $\log(2\pi) - h(\Theta)$ to predict the cap on the information rate for the multi-sample receiver. Obtaining such an upper bound is beyond the scope of this paper.

VII. DISCUSSION

1) The authors of [21] treated on-off keying in the presence of Wiener phase noise by using a double-filtering receiver, which is composed of an intermediate frequency (IF) filter, followed by an envelope detector (square-law device) and then a post-detection filter. They showed that by optimizing the IF receiver bandwidth the double-filtering receiver outperforms the single-filtering (matched filter) receiver. Furthermore, they showed

![Graph showing comparison of information rates for different models.](image-url)
via simulation that the optimum IF bandwidth increases with the SNR. This is similar to our result in the sense that we require the number of samples per symbol to increase with the SNR in order to achieve a rate that grows logarithmically with the SNR. We remark that Dallal and Shamai [39] (cf. (141)-(143)).

2) The phase modulation pre-log of 1/4 in (44) requires only 2 samples per symbol for which the time resolution 1/Δ grows as the square root of the SNR. In other words, one does not need an analog-to-digital converter (ADC) with sampling every T_s/2 seconds. Instead, one may use two ADCs that sample every T_s seconds and the sampling instants are offset by T_s/2 seconds. For large L, two ADCs with low sampling rate would be less complex than one ADC with a very high sampling rate. It is interesting to contemplate whether another receiver, e.g., a non-coherent receiver, can achieve the maximum amplitude modulation pre-log of 1/2 but requires only 1 sample per symbol. If so, one would need only 3 samples per symbol to achieve a pre-log of 3/4.

3) The paper [24] shows that a pre-log of 1/2 is achievable for the model (19) when L ∝ SNR. Theorem 1 implies that a pre-log of 1/2 can be achieved using L ∝ SNR and this perhaps can be reduced further. But using L ∝ SNR may simplify the receiver design. Specifically, the amplitude decoder designed for the auxiliary channel in [24] can be used for any phase noise linewidth β because the auxiliary channel is (99) with G = 1 which is independent of β. In contrast, using L = 3 SNR requires different amplitude decoders for different linewidths because the auxiliary channel in this case is (99) with G = E[G] which depends on β.

4) For the multi-sample approximate model [42], Barletta and Kramer [38] derived an upper bound on the rate and showed that

\[ I(X; \Gamma_i)_{i=0}^\infty = I(x(t); r(t)) \]  

where \( \Gamma_i = \int_0^{T_s} r(t) f_i^*(t) dt, \quad i = 0, 1, \ldots \) (89)

i.e., \( \Gamma_i_{i=0}^\infty \) are sufficient statistics for X. Therefore, an infinite number of projections per symbol seems to be needed to have sufficient statistics in general. Moreover, we believe that a finite subset of \( \{\Gamma_i\}_{i=0}^\infty \) captures essentially all the information in the received waveform at a given SNR, and that larger subsets would be needed at larger SNRs. The discrete-time model based on the multi-sample receiver in Fig. 3 seems to be easier to analyze than the discrete-time model [40].

5) We briefly discuss sufficient statistics for the continuous-time phase noise channel. One expects intuitively at any SNR that

\[ \lim_{L \to \infty} I(X^h; Y^{hL}) = I(x(t); r(t)). \] 

This is because at infinite sampling rate, one captures essentially the entire received waveform. In general, it seems that an infinite number of projections per symbol are needed to obtain sufficient statistics for all SNRs. For example, consider the Karhunen-Loève expansion of \( U(t) = e^{j\theta(t)} \) on \( t \in [0, T_s] \), for which we have

\[ U(t) \stackrel{m}{=} \sum_{i=0}^{\infty} U_i f_i(t) \]  

where \( \{f_i(t)\} \) are the orthonormal functions [40] Sec. 6.1:

\[ f_i(t) = \sqrt{\frac{2}{T_s}} \cos(\alpha_i (t - T_s/2)), \quad i \text{ odd} \]  

\[ f_i(t) = \sqrt{\frac{2}{T_s}} \sin(\alpha_i (t - T_s/2)), \quad i \text{ even} \]  

where \( \alpha_i \) is the solution to

\[ 2 \arctan(\alpha_i/(\beta \pi)) + \alpha_i T_s = i \pi, \quad i = 0, 1, \ldots \]  

and the \( \{U_i\} \) are uncorrelated random variables with variance \( \lambda_i \) where

\[ \lambda_i = \frac{2 \pi \beta}{\pi^2 \beta^2 + \alpha_i^2}. \]  

Suppose that one symbol X is transmitted and consider a rectangular pulse for simplicity. We have

\[ r(t) = X e^{j\theta(t)} + n(t) = \sum_{i=0}^{\infty} X U_i f_i(t) + n(t). \]  

We conclude, in this simplified case, that

\[ I(X; \{\Gamma_i\}_{i=0}^\infty) = I(x(t); r(t)) \]  

where

\[ \Gamma_i = \int_0^{T_s} r(t) f_i^*(t) dt, \quad i = 0, 1, \ldots \]  

6) Minimum energy per information bit is an important metric at low SNR. For example, Fig. 12 plots the information rates of Fig. 8 against \( E_b/N_0 \), where the energy per information bit is \( E_b = P/R \) and R is the information rate in bits/symbol. We notice a significant impact on the minimum \( E_b \) due to phase noise. Furthermore, oversampling does not affect the rate at low \( E_b/N_0 \), which suggests that using symbol-rate sampling suffices and that several results available in the literature on noncoherent fading channels and discrete-time phase noise channels (with the commonly-used approximation) should remain valid, e.g., see Fig. 2–Fig. 5 in [14].
VIII. Conclusion

We studied a waveform channel impaired by Wiener phase noise and AWGN. A discrete-time channel model based on oversampling was derived taking into account filtering effects on phase noise. At high SNR, the multi-sample receiver achieves rates that grow logarithmically with SNR with at least a 1/2 pre-log factor, if the number of samples per symbol grows with the cubic root of the SNR. In addition, we computed via numerical simulations lower bounds on the overall capacity pre-log is 1/2, if the number of samples per symbol grows with the square root of the SNR. Thus, we demonstrated that multi-sample receivers at f_{\text{HWM}} T_s = 0.125.

Fig. 12. Rates vs $E_b/N_0$ for 16-QAM, square transmit-pulse and multi-sample receiver at f_{\text{HWM}} T_s = 0.125.

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APPENDIX A

AMPLITUDE MODULATION

This Appendix provides a proof of Theorem 1. We set $T_s = 1$ for simplicity. Since $X_A^b$ is i.i.d., we have

$$I(X_A^b; Y^b) \equiv \sum_{k=1}^b I(X_{A,k}; Y^b | X_A^{k-1})$$

$$= \sum_{k=1}^b H(X_{A,k}) - H(X_{A,k} | Y^b, X_A^{k-1})$$

$$\geq \sum_{k=1}^b I(X_{A,k}; Y_k)$$

$$\geq \sum_{k=1}^b I(X_{A,k}; V_k)$$

(90)

where

$$V_k = \sum_{\ell=1}^L |Y_{(k-1)L+\ell}|^2.$$  

(91)

Step (a) follows from the chain rule of mutual information, (b) follows from the independence of $X_{A,1}, X_{A,2}, \ldots, X_{A,b}$, (c) holds because conditioning does not increase entropy, and (d) follows from the data processing inequality. Since $X_A^b$ is identically distributed, we find that $V^b$ is also identically distributed and we have, for $k \geq 2$,

$$I(X_{A,k}; V_k) = I(X_{A,1}; V_1).$$

(92)

In the rest of this section, we consider only one symbol ($k = 1$) and drop the time index. By combining (91) and (19), we have

$$V = \sum_{\ell=1}^L (X_A^2 \Delta^2 |F_{\ell}|^2 + 2 X_A \Delta R[e^{j\Phi_{\ell}} e^{j\Theta_{\ell}} F_{\ell} N_{\ell}^r] + |N_{\ell}|^2)$$

$$= X_A^2 \Delta \Delta G + 2 X_A \Delta Z_1 + Z_0$$

(93)

where $G$, $Z_1$ and $Z_0$ are defined as

$$G \equiv \frac{1}{L} \sum_{\ell=1}^L |F_{\ell}|^2$$

(94)

$$Z_1 \equiv \sum_{\ell=1}^L R[e^{j\Phi_{\ell}} e^{j\Theta_{\ell}} F_{\ell} N_{\ell}^r]$$

(95)

$$Z_0 \equiv \sum_{\ell=1}^L |N_{\ell}|^2.$$  

(96)

The second-order statistics of $Z_1$ and $Z_0$ are

$$E[Z_1] = 0 \quad \text{Var}[Z_1] = E[G] \sigma_N^2 / 2$$

$$E[Z_0] = \sigma_N^2 \quad \text{Var}[Z_0] = \sigma_N^2 \Delta$$

(97)

By using the auxiliary-channel lower bound (cf. (39)), we have

$$I(X_A; V) \geq E[- \log q_V(V)] - E[- \log q_V | X_A(V | X_A)]$$

(98)
where \( q_{V|X_A}(v|x_A) \) is the conditional distribution of the auxiliary channel, which we choose to be
\[
q_{V|X_A}(v|x_A) = \frac{1}{\sqrt{4\pi x_A^2 \Delta^2 \sigma_N^2}} \exp \left( -\frac{(v - x_A^2 \Delta G - \sigma_N^2)^2}{4x_A^2 \Delta^2 \sigma_N^2} \right)
\]
(99) with \( 0 < \Delta \leq 1 \) and
\[
q_V(v) = \int_0^\infty p_{X_A}(x_A)q_{V|X_A}(v|x_A)dx_A
\]
(100) where \( p_{X_A}(\cdot) \) is the (true) pdf of \( X_A \). The intuition behind the choice of (99) is given in Remark 3 below. It follows that
\[
E[-\log(q_{V|X_A}(V|X_A))] = \mathbb{E} \left[ \frac{(V - X_A^2 \Delta \tilde{G} - \sigma_N^2)^2}{4X_A^2 \Delta^2 \sigma_N^2} \right] + \log \Delta + \frac{1}{2} \log(4\pi \sigma_N^2) + \frac{1}{2} \mathbb{E}[\log(X_A^2)]
\]
and by using (93) and (97), we have
\[
\mathbb{E} \left[ \frac{(V - X_A^2 \Delta \tilde{G} - \sigma_N^2)^2}{4X_A^2 \Delta^2 \sigma_N^2} \right] = \frac{1}{4\sigma_N^2} \mathbb{E}[X_A^2] \mathbb{E}[(G - \tilde{G})^2] + \frac{1}{\sigma_N} \mathbb{E}[Z_1^2] + \log \Delta + \frac{1}{2} \log(4\pi \sigma_N^2) + \frac{1}{2} \mathbb{E}[\log(X_A^2)]
\]
(101) where we also used
\[
\mathbb{E}[(G - \tilde{G})Z_1] = 0.
\]
(103) Substituting (102) into (101) and using \( \mathbb{E}[G] \leq 1 \) yield
\[
E[-\log(q_{V|X_A}(V|X_A))]
\leq \log \Delta + \frac{1}{2} \log(4\pi \sigma_N^2) + \frac{1}{2} \mathbb{E}[\log(X_A^2)]
\]
\[
+ \frac{P}{4\sigma_N} \mathbb{E} \left[ (G - \tilde{G})^2 \right] + \frac{1}{2} + \frac{\sigma_N^2}{\Delta} \mathbb{E} \left[ \frac{1}{X_A^2} \right].
\]
(104)

It is convenient to define \( X_P = X_A^2 \). The input distribution of \( X_P \) is
\[
p_{X_P}(x_P) = \begin{cases} \frac{1}{\lambda} \exp \left( -\frac{x_P}{\lambda} \right), & x_P \geq \lambda \\
0, & \text{otherwise} \end{cases}
\]
(105) where \( \lambda = P/2 \). It follows from (100) and (105) that
\[
q_V(v) = \int_\lambda^\infty \frac{1}{\lambda} \exp \left( -\frac{x_P}{\lambda} \right) q_{V|X_P}(v|x_P) \, dx_P
\leq e \, f_V(v)
\]
(106) where
\[
q_{V|X_P}(v|x_P) = q_{V|X_A}(v\sqrt{x_P})
\]
(107) and
\[
f_V(v) = \int_0^\infty \frac{1}{\lambda} \exp \left( -\frac{x_P}{\lambda} \right) q_{V|X_P}(v|x_P) \, dx_P.
\]
(108)

The inequality (106) follows from the non-negativity of the integrand. By combining (99), (107), (108) and making the change of variables \( x = xp\Delta \), we have
\[
f_V(v) = \int_0^\infty e^{-x/(\lambda \Delta)} \frac{1}{\sqrt{4\pi x \Delta \sigma_N^2}} \exp \left( -\frac{(v - x - \sigma_N^2)^2}{4x \Delta^2 \sigma_N^2} \right) \, dx
\leq \frac{1}{\sqrt{\lambda \Delta (\Delta \Delta + 4\Delta \sigma_N^2)}}
\]
\[
\times \exp \left( \frac{2}{4\sigma_N^2} \left[ v - \sigma_N^2 - |v - \sigma_N^2| \sqrt{1 + 4\sigma_N^2 / \lambda \Delta} \right] \right)
\]
(109) where we used equation (140) in Appendix A of [41]:
\[
\int_0^\infty \frac{1}{a} \exp \left( -\frac{x}{a} \right) \frac{1}{\sqrt{\pi bx}} \exp \left( -\frac{(u - x)^2}{bx} \right) \, dx
\leq \frac{1}{\sqrt{a(a + b)}} \exp \left( \frac{2}{b} \left[ u - |u| \sqrt{1 + \frac{b}{a}} \right] \right).
\]
(110)

Therefore, we have
\[
E[-\log(f_V(V))]
\leq \frac{1}{2} \log(\Delta^2 (\lambda^2 + 4\lambda \sigma_N^2))
\]
\[
- \frac{1}{2\Delta \sigma_N^2} \mathbb{E}[V - \sigma_N^2] - \mathbb{E}[|V - \sigma_N^2|] \sqrt{1 + \frac{4\sigma_N^2}{\lambda}}
\]
\[
(a) \geq \log(\Delta \lambda) + \frac{1}{2\sigma_N^2} \mathbb{E}[V - \sigma_N^2] \left[ \sqrt{1 + \frac{4\sigma_N^2}{\lambda}} - 1 \right]
\]
\[
(b) \geq \log(\Delta \lambda)
\]
(111) where (a) holds because the logarithmic function is monotonic and \( \mathbb{E}[|\cdot|] \geq \mathbb{E}[\cdot] \), and (b) holds because
\[
\mathbb{E}[V - \sigma_N^2]
\leq \mathbb{E}[X_A^2] \Delta \mathbb{E}[G] + 2\mathbb{E}[X_A] \Delta \mathbb{E}[Z_1] + \mathbb{E}[Z_0] - \sigma_N^2
\]
\[
= P \mathbb{E}[G] \geq 0.
\]
(112)

The monotonicity of the logarithmic function and (106) yield
\[
E[-\log(f_V(V))] \geq E[-\log(e \cdot f_V(V))] \geq \log \Delta + \log \lambda - 1
\]
(113) where the last inequality follows from (111). It follows from (98), (104) and (113) that
\[
I(X_A; V) \geq \log \lambda - 1 - \frac{1}{2} \log(4\pi \sigma_N^2) - \frac{1}{2} \mathbb{E}[\log(X_A^2)]
\]
\[
- \frac{P}{4\sigma_N^2} \mathbb{E} \left[ (G - \tilde{G})^2 \right] - \frac{1}{2} - \frac{\sigma_N^2}{4\Delta} \mathbb{E} \left[ \frac{1}{X_A^2} \right].
\]
(114)

We have
\[
\mathbb{E} \left[ \frac{1}{X_P} \right] \leq \frac{2}{P}
\]
(115)
and
\[ \mathbb{E} [\log (X_P)] = \int_1^{\infty} \frac{1}{\lambda} e^{-(x-\lambda)/\lambda} \log(x) dx \]
\[ \leq \log \lambda + 1 \tag{116} \]
where (a) follows by the change of variables \( u = x/\lambda \), and (b) holds because \( \log(u) \leq u - 1 \) for all \( u > 0 \). Substituting into (114), we obtain
\[ I(X_A; V) = \frac{1}{2} \log \text{SNR} \geq -2 - \frac{1}{2} \log(8\pi) \]
\[ - \frac{1}{4} \text{SNR} \mathbb{E} [(G - \tilde{G})^2]. \tag{117} \]
By combining (58), (90), (92) and (117), we have
\[ I(X_A; Y) = \frac{1}{2} \log \text{SNR} \geq -2 - \frac{1}{2} \log(8\pi) - \frac{1}{2} \text{SNR}\Delta \]
\[ - \frac{1}{4} \text{SNR} \mathbb{E} [(G - \tilde{G})^2]. \tag{118} \]
Since (118) is valid for any \( \tilde{G} \in (0, 1] \), we choose \( \tilde{G} \) to maximize the right hand side of (118). The optimal \( \tilde{G} \) is
\[ \arg \min \mathbb{E} [(G - \tilde{G})^2] = \mathbb{E} [G]. \tag{119} \]
Since \( L = [\sqrt{2}\beta \text{SNR}] \) and \( \Delta = 1/L \), it follows that
\[ \lim_{\text{SNR} \to \infty} \text{SNR}\Delta = \infty \text{ and } \lim_{\text{SNR} \to \infty} \text{SNR}\Delta^3 = \frac{1}{\beta^2}. \tag{120} \]
Moreover, we have (see Appendix C)
\[ \lim_{\text{SNR} \to \infty} \text{SNR} \text{Var}(G) = \lim_{\Delta \to 0} \frac{\text{SNR}\Delta^3 \cdot \text{Var}(G)}{\Delta^3} \]
\[ = \frac{1}{\beta^2} \cdot \frac{4(\pi\beta)^2}{45} = \frac{4\pi^2}{45}. \tag{121} \]
Hence, for \( \tilde{G} = \mathbb{E} [G] \), we have
\[ \lim_{\text{SNR} \to \infty} I(X_A; Y) = \frac{1}{2} \log \text{SNR} \geq -2 - \frac{1}{2} \log(8\pi) - \frac{\pi^2}{45}. \tag{122} \]

**Remark 3:** We motivate the auxiliary channel in (92). We first express the auxiliary channel as \( \tilde{V} \):
\[ \tilde{V} = X_A^2 \Delta \tilde{G} + 2X_A \Delta \tilde{Z}_1 + \tilde{Z}_0 \tag{123} \]
where \( \tilde{G}, \tilde{Z}_1 \) and \( \tilde{Z}_0 \) are defined as
\[ \tilde{G} \equiv \mathbb{E} [G] \tag{124} \]
\[ \tilde{Z}_1 \equiv \frac{1}{L} \sum_{l=1}^{L} \mathbb{E}[e^{j\phi_l} e^{j\theta_l} N_l^l] \tag{125} \]
\[ \tilde{Z}_0 \equiv \sigma^2_N. \tag{126} \]
We make this choice for \( \tilde{V} \) because \( \tilde{Z}_1 \) and \( \tilde{Z}_0 \) converge (in mean square) to \( Z_1 \) and \( Z_0 \), respectively, as \( L \to \infty \). Moreover, \( \tilde{Z}_1 \) is Gaussian and \( \tilde{Z}_0 \) is a constant, which are easier quantities to handle than \( Z_1 \) and \( Z_0 \).

### APPENDIX B

**Phase Modulation**

This Appendix provides a proof of Theorem 2. We assume that \( T_x = 1 \) and \( \sigma^2_N = 1 \) for simplicity. By using the chain rule, we have
\[ I(\Phi_X; \Psi^n, X^n_A, X^{k-1}_X) \]
\[ \geq \sum_{k=2}^{n} I(\Phi_X; \Psi^n, X^n_A, X^{k-1}_X) \]
\[ \geq \sum_{k=2}^{n} I(\Phi_X; \Psi_k, X^n_A, X^{k-1}_X) \tag{127} \]
where \( \Psi_k \) is a deterministic function of \( (\Psi^n, X^n_A, X^{k-1}_X) \). Inequality (a) follows from the non-negativity of mutual information and (b) follows from the data processing inequality. We choose
\[ \Psi_k = \frac{\Psi(k-1)L_1 \tilde{\Phi}_k}{\sqrt{\Delta}} \]
\[ \tag{128} \]
The intuition behind the choice of (128) is outlined in Remark 3 below. We express \( \tilde{\Phi}_k \) as
\[ \tilde{\Phi}_k = \left| X_k \right| \sqrt{\Delta} e^{j\Phi_X^k} + \tilde{N}_k \left( 1 + \tilde{Z}_k \right) e^{jW(k-1)L_1} \tag{129} \]
where
\[ \tilde{Z}_k = \frac{N_{kL} e^{-j\theta_{kL}}}{X_k^L} \tag{130} \]
\[ \tilde{N}_k = \frac{N_{(k-1)L_1} e^{-j\theta_{(k-1)L_1}}}{\sqrt{\Delta}}. \tag{131} \]
The processes \( \tilde{N}_k \) and \( \tilde{Z}_{k-1} \) are statistically independent and \( \tilde{N}_k \sim \mathcal{N}(0, 1) \)
\[ \tilde{Z}_{k-1} \mid \{|X_{k-1}| = |x_{k-1}|\} \sim \mathcal{N}(0, \frac{1}{|x_{k-1}|^2 \Delta}). \tag{132} \]
The notation (133) means that, conditioned on \( \{|X_{k-1}| = |x_{k-1}|\} \), \( \tilde{Z}_{k-1} \) is a Gaussian random variable with mean 0 and variance \( 1/|x_{k-1}|^2 \Delta \). Moreover, \( W_{(k-1)L_1} \) is statistically independent of \( \tilde{N}_k \) and \( \tilde{Z}_{k-1} \). The choice of \( \tilde{\Phi}_k \) in (128) implies that
\[ I(\Phi_X; \Psi_k, X^n_A, X^{k-1}_X) = I(\Phi_X; \tilde{\Phi}_k, X^n_A, X^{k-1}_X). \tag{134} \]
Define \( \Phi_{\Psi, k} \equiv \mathbb{E}[\tilde{\Phi}_k] \) and
\[ q_{\Phi_{\Psi, k} | X_{A,k}, X_{k-1}} (\phi_y, \phi_x) \equiv \frac{\exp(\alpha \cos(\phi_y - \phi_x))}{2\pi I_0(\alpha)} \tag{135} \]
where \( I_0(\cdot) \) is the zeroth-order modified Bessel function of the first kind and \( \alpha > 0 \). This distribution is known as Tikhonov (or von Mises) distribution [42]. Furthermore, define
\[ q_{\Phi_{\Psi, k} | X_{A,k}, X_{k-1}} (\phi_y, |x_k|, |x_{k-1}|) \equiv \int_{-\pi}^{\pi} p_{\Phi_{\Psi, k} | X_{A,k}, X_{k-1}} (\phi_x, |x_k|, |x_{k-1}|) q_{\Phi_{\Psi, k} | X_{k-1}} (\phi_y, \phi_x) d\phi_x \]
\[ = \frac{1}{2\pi}. \tag{136} \]
The last equality holds because $X_1, \ldots, X_n$ are statistically independent and $\Phi_{X,k}$ is independent of $X_{A,k}$ with a uniform distribution on $[\pi, -\pi)$. We have
\begin{equation}
\begin{aligned}
I(\Phi_{X,k}; \tilde{Ψ}_k|X_{A,k}, X_{-k}) &\geq I(\Phi_{X,k}; \hat{Ψ}_{Φ,k}|X_{A,k}, X_{-k}) \\
&\geq E \left[ \log q_{Φ_{X,k}}(\hat{Ψ}_{Φ,k}|X_{A,k}) \right] \\
&\geq E \left[ \log q_{Φ_{X,k}}(\hat{Ψ}_{Φ,k}|X_{A,k}) \right] - E \left[ \log(2π) \right] \\
&\geq \frac{1}{2} \log\alpha - \frac{\sigma_W^2}{2} - \frac{4α}{SNR\Delta}
\end{aligned}
\end{equation}
where $(a)$ follows from the data processing inequality, $(b)$ follows by extending the result of the auxiliary-channel lower bound (cf. (39)), $(c)$ follows from (135) and (136), $(d)$ follows from (43) Lemma 2
\begin{equation}
I_0(z) \leq \frac{\sqrt{\pi}}{2} \frac{e^z}{\sqrt{\pi}} \leq e^z
\end{equation}
and the last inequality holds because
\begin{equation}
E \left[ \cos(\tilde{Ψ}_{Φ,k} - \Phi_{X,k}) \right] \geq 1 - \frac{\sigma_W^2}{4} \frac{4}{SNR\Delta}
\end{equation}
for $SNR\Delta > 2$, as we will show. Define
\begin{equation}
\begin{aligned}
S_k &\equiv 1 + Z_k \\
\tilde{Ψ}_k &\equiv |X_k|\sqrt{\Delta}e^{jΦ_{X,k}} + \tilde{N}_k
\end{aligned}
\end{equation}
Therefore, we have
\begin{equation}
\begin{aligned}
E \left[ \cos(\tilde{Ψ}_{Φ,k} - \Phi_{X,k}) \right] &\equiv E \left[ \cos(W_{(k-1)L+1} + S_{k} - \Phi_{Φ,k} - \Phi_{X,k}) \right] \\
&\geq E \left[ \cos(W_{(k-1)L+1} + S_{k} - \Phi_{Φ,k} - \Phi_{X,k}) \right] \\
&\geq E \left[ \cos(W_{(k-1)L+1} + S_{k}) \right] E \left[ \cos(\Phi_{Φ,k} - \Phi_{X,k}) \right] \\
&\geq E \left[ \sin(W_{(k-1)L+1} + S_{k}) \right] E \left[ \sin(\Phi_{Φ,k} - \Phi_{X,k}) \right]
\end{aligned}
\end{equation}
where $Φ_{S,k} = \text{arg}(S_k)$ and $Φ_{Ψ,k} = \text{arg}(\tilde{Ψ}_k)$. Step $(a)$ follows from (129), while step $(b)$ follows from the trigonometric relation $\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$ and because $W_{(k-1)L+1} + S_{k}$ is independent of $\Phi_{Φ,k}$ and $\Phi_{X,k}$.

By using Lemma 6 in Appendix D we have
\begin{equation}
\begin{aligned}
E \left[ \cos(\Phi_{Φ,k} - \Phi_{X,k}) \right] &\geq 1 - \frac{1}{|x_k|^2\Delta} \\
E \left[ \sin(\Phi_{Φ,k} - \Phi_{X,k}) \right] &\geq 1 - \frac{1}{|x_k|^2\Delta}
\end{aligned}
\end{equation}
Equations (142) and (144) imply that we do not need to compute $E \left[ \sin(W_{(k-1)L+1} + \Phi_{S,k-1}) \right]$. We have
\begin{equation}
\begin{aligned}
E \left[ \cos(W_{(k-1)L+1} + \Phi_{S,k-1}) \right] &\equiv E \left[ \cos(W_{(k-1)L+1} + \Phi_{S,k-1}) \right] \\
&\geq E \left[ \sin(W_{(k-1)L+1} + \Phi_{S,k-1}) \right] \\
&\equiv E \left[ \sin(W_{(k-1)L+1} + \Phi_{S,k-1}) \right]
\end{aligned}
\end{equation}
because $W_{(k-1)L+1} + \Phi_{S,k-1}$ are independent. Since $W_{(k-1)L+1}$ is Gaussian with mean 0 and variance $\sigma_W^2 = 2πβ\Delta$, the characteristic function of $W_{(k-1)L+1}$ is
\begin{equation}
E \left[ e^{jW_{(k-1)L+1}} \right] = e^{-\sigma_W^2/2}
\end{equation}
and by using the linearity of expectation, we have
\begin{equation}
\begin{aligned}
E \left[ \cos(W_{(k-1)L+1}) \right] &\equiv \Re \left[ E \left[ e^{jW_{(k-1)L+1}} \right] \right] = e^{-\sigma_W^2/2} \\
E \left[ \sin(W_{(k-1)L+1}) \right] &\equiv \Im \left[ E \left[ e^{jW_{(k-1)L+1}} \right] \right] = 0
\end{aligned}
\end{equation}
where $\Re[.]$ and $\Im[.]$ denote the real and imaginary parts, respectively. By using Lemma 6 in Appendix D again, we have
\begin{equation}
\begin{aligned}
E \left[ \cos(\Phi_{S,k-1}) \right] &\geq 1 - \frac{1}{|x_k|^2\Delta}
\end{aligned}
\end{equation}
It follows from (145), (147), (148) and (149) that
\begin{equation}
\begin{aligned}
E \left[ \cos(W_{(k-1)L+1} + \Phi_{S,k-1}) \right] &\geq e^{-\sigma_W^2/2} \left( 1 - E \left[ \frac{1}{|x_k|^2\Delta} \right] \right) \\
&\geq e^{-\sigma_W^2/2} \left( 1 - \frac{1}{2} \right) \\
&\geq e^{-\sigma_W^2/2} \left( 1 - \frac{2}{P\Delta} \right)
\end{aligned}
\end{equation}
By combining (142), (143), (144), (150), we have (for $P\Delta > 2$)
\begin{equation}
\begin{aligned}
E \left[ \cos(\Phi_{Φ,k} - \Phi_{X,k}) \right] &\geq e^{-\sigma_W^2/2} \left( 1 - E \left[ \frac{1}{|x_k|^2\Delta} \right] \right) \\
&\geq e^{-\sigma_W^2/2} \left( 1 - \frac{2}{P\Delta} \right)
\end{aligned}
\end{equation}
where $(a)$ holds because $|X_k|^2 \geq P/2$ and $|X_{k-1}|^2 \geq P/2$, while $(b)$ follows from $e^{-\sigma_W^2/2} \leq 1$ and the non-negativity of $1/(P\Delta)$ and $(c)$ follows from $e^{-x} \geq 1 - x$.
It follows from (127), (134) and (137) that
\begin{equation}
\begin{aligned}
\frac{1}{b} I(\Phi_{X,k}; \Psi|X_A) &\geq \frac{1}{b} - \frac{1}{b} \left[ \frac{1}{2} \log\alpha - \alpha\pi\beta\Delta - \frac{4α}{SNR\Delta} \right]
\end{aligned}
\end{equation}
Hence, we have
\begin{equation}
\begin{aligned}
I(\Phi_{X,k}; \Psi|X_A) &\equiv \lim_{b \to \infty} \frac{1}{b} I(\Phi_{X,k}; \Psi|X_A) \\
&\geq \frac{1}{2} \log\alpha - \alpha\pi\beta\Delta - \frac{4α}{SNR\Delta}
\end{aligned}
\end{equation}
Since the lower bound (153) is valid for any $\alpha > 0$, we have

$$I(\Phi_X; \Psi|X_A) \geq \max_{\alpha > 0} \frac{1}{2} \log \alpha - \alpha \left( \frac{\pi \beta \Delta + \frac{4}{\text{SNR}}}{2} \right)$$

$$= -\frac{1}{2} \log \left( \frac{2\pi \beta \Delta + \frac{8}{\text{SNR}}}{2} \right) - \frac{1}{2}$$

$$= -\frac{1}{2} \log \left( \frac{2\pi \beta L + \frac{8L}{\text{SNR}}}{2} \right) - \frac{1}{2} \tag{154}$$

where we used $\Delta = 1/L$ in the last equation. We maximize the lower bound (154) over $L$ to obtain

$$L = \sqrt{\pi \beta \text{SNR}} / 2. \tag{155}$$

By choosing an oversampling factor $L = \lceil \sqrt{\pi \beta \text{SNR}} / 2 \rceil$, it follows that

$$\lim_{\text{SNR} \to \infty} I(\Phi_X; \Psi|X_A) - \frac{1}{4} \log \text{SNR} \geq -\frac{1}{4} \log(64\pi \beta) - \frac{1}{2}. \tag{156}$$

The proof of (133) is similar to the proof in Appendix A for the full model and is omitted.

Remark 4: We outline the intuition for choosing $\tilde{\Psi}_k$ in (128). The basic idea is to process $(\Psi^n, X^n, \Phi^{k-1})$ in three steps: first estimate the phase noise, then cancel it, and then decode. We outline the motivation for each step:

1) Since the past inputs $X^{k-1}$ and outputs $Y^{k-1}$ are available, one may use the simple estimator

$$e^{j\tilde{\Theta}_{k'}} = \frac{\Psi_k}{X_{[k'/L]}^{k}} \tag{157}$$

to estimate $\tilde{\Theta}_{k'}$ for $k' = 1, \ldots, (k - 1)L$. This is a reasonable estimate as long as the energy per sample is large relative to the noise variance per sample, i.e., when $\text{SNR} \Delta \gg 1$.

2) Next, we use the previous estimates to cancel from $Y_k^n$ the phase noise that accumulated up to sample $(k - 1)L$. More precisely, we use only the most recent estimate $\tilde{\Theta}_{(k - 1)L}$. We make this choice because the discrete-time Wiener process $\{\Theta_k\}$ is a first-order Markov process so, conditioned on $\tilde{\Theta}_{(k - 1)L}$, the random variables $\Theta_{L+1}, \ldots, \Theta_{nL}$ are independent of $\Theta_0, \ldots, \Theta_{k-1}$. This means that if the SNR is high enough so that $\tilde{\Theta}_{k'}$ is an accurate estimate of $\tilde{\Theta}_{k'}$ for $k' = 1, \ldots, (k - 1)L$, then it would suffice to use only $\tilde{\Theta}_{(k - 1)L}$.

3) Since the input is i.i.d. and there is no inter-symbol interference, we discard $\Psi_{k+1}$ and use only $\Psi_k$ (after canceling the phase noise) for decoding $\Phi_{X,k}$. Moreover, we use only the first sample $\tilde{\Psi}_{(k - 1)L+1}$ in $\tilde{\Psi}_k$. This is because, in absence of additive noise, we have

$$I(\Phi_X, \Psi; \Psi) = I(\Phi_X; \Psi|X_A) \geq \max_{\alpha > 0} \frac{1}{2} \log \alpha - \alpha \left( \frac{\pi \beta \Delta + \frac{4}{\text{SNR}}}{2} \right)$$

$$= -\frac{1}{2} \log \left( \frac{2\pi \beta \Delta + \frac{8}{\text{SNR}}}{2} \right) - \frac{1}{2}$$

$$= -\frac{1}{2} \log \left( \frac{2\pi \beta L + \frac{8L}{\text{SNR}}}{2} \right) - \frac{1}{2}$$

which implies that

$$\lim_{\text{SNR} \to \infty} \frac{\text{Var}(F_1^2)}{(\log a)^2} = \frac{4}{45} \tag{164}$$

Finally, (158) follows from (164), (162) and $L = 1/\Delta$ (because $T_s = 1$).

**APPENDIX D**

**Supporting Lemmas for Appendix B**

This appendix contains two lemmas. The main lemma is Lemma 6 which is used in Appendix B. Lemma 7 is a supporting lemma for Lemma 6.

1. Ignoring $L \in \mathbb{Z}$.
2. The inequality (113) is valid for the approximate model except that $G$ is set to 1 because $F_k = 1$ in the approximate model. Hence, by setting $G = 1$, the term $-\pi^2/45$ in (49) does not appear in (113).

3. If the pulse is not rectangular, the process $\{F_k\}$ may not be identically distributed because the phase noise $e^{j\Theta(t)}$ is weighted in a given sample interval according to the pulse shape.
Lemma 6: Let $R$ be a fixed positive real number and let $\Phi \equiv \arg(Y)$ where

$$Y = R + Z$$

(165)

and $Z$ is a circularly-symmetric complex Gaussian random variable with mean 0 and variance 1. Then, we have

$$\mathbb{E}[\cos(\Phi)] \geq 1 - \frac{1}{R^2} \quad (166)$$

$$\mathbb{E}[\sin(\Phi)] = 0 \quad (167)$$

Proof: The probability density function of $\Phi$ is $45$

$$p_{\Phi}(\phi) = \frac{1}{2\pi} e^{-R^2} + \frac{1}{\sqrt{4\pi}} R \cos(\phi) e^{-R^2 \sin^2 \phi} \text{erfc}(-R \cos \phi)$$

(168)

where $\text{erfc}(\cdot)$ is the complementary error function

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt. \quad (169)$$

Therefore, we have

$$\mathbb{E}[\cos(\Phi)]$$

$$= \int_{-\pi}^{\pi} \cos(\phi) p_{\Phi}(\phi) \, d\phi$$

$$(a) \geq 2 \int_{0}^{\pi} \cos(\phi) p_{\Phi}(\phi) \, d\phi$$

$$(b) \geq \int_{0}^{\pi/2} \frac{R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \text{erfc}(-R \cos \phi) \, d\phi$$

$$(c) \geq \int_{0}^{\pi/2} \frac{R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \text{erfc}(-R \cos \phi) \, d\phi$$

$$(d) \geq \int_{0}^{\pi/2} \frac{R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \left(2 - e^{-R^2 \cos^2 \phi}\right) \, d\phi$$

$$= \int_{0}^{\pi/2} \frac{2R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \, d\phi - \int_{0}^{\pi/2} \frac{R}{\sqrt{\pi}} e^{-R^2} \int_{0}^{\pi/2} \cos^2 \phi \, d\phi$$

$$(e) \geq \int_{0}^{\pi/2} \frac{2R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \, d\phi - \int_{0}^{\pi/2} \frac{R}{\sqrt{\pi}} e^{-R^2} \int_{0}^{\pi/2} \cos^2 \phi \, d\phi$$

(170)

where $(a)$ follows because both $p_{\Phi}(\cdot)$ and $\cos(\phi)$ are even functions, $(b)$ follows from $(168)$ and $\int_{0}^{\pi} \cos(\phi) p_{\Phi}(\phi) \, d\phi = 0$, $(c)$ follows because the integrand is non-negative over the interval $\phi \in (\pi/2, \pi]$, $(d)$ holds because $\text{erfc}(-R \cos \phi) \geq 2 - e^{-R^2 \cos^2 \phi}$ (see Lemma 7) and $\cos^2(\phi) e^{-R^2 \sin^2 \phi} \geq 0$ and finally $(e)$ follows by direct integration.

We bound the integral in (170) as follows

$$\int_{0}^{\pi/2} \frac{2R}{\sqrt{\pi}} \cos^2(\phi) e^{-R^2 \sin^2 \phi} \, d\phi$$

$$(a) \geq \frac{2R}{\sqrt{\pi}} \int_{0}^{\pi/2} (1 - \phi^2) e^{-R^2 \phi^2} \, d\phi$$

$$= \frac{2R}{\sqrt{\pi}} \int_{0}^{\pi/2} e^{-R^2 \phi^2} \, d\phi - \frac{2R}{\sqrt{\pi}} \int_{0}^{\pi/2} \phi^2 e^{-R^2 \phi^2} \, d\phi$$

$$(b) \geq \frac{1}{2R^2} \text{erf} \left(\frac{\pi}{2} R\right) - \frac{1}{2R^2} \text{erf} \left(\frac{\pi}{2} R\right)$$

$$(c) \geq \text{erf} \left(\frac{\pi}{2} R\right) - \frac{1}{2R^2}$$

(171)

where $\text{erf}(\cdot)$ is the error function defined as

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt. \quad (172)$$

Step (a) follows from $\cos^2 \phi e^{-R^2 \sin^2 \phi} \geq (1 - \phi^2)e^{-R^2 \phi^2}$

(see Lemma 7). (b) follows by using the definition of $\text{erf}(\cdot)$ and $1.3.321(5)$:

$$\int_{0}^{\pi/2} \sqrt{\pi} e^{-a^2 t^2} \, dt = \frac{1}{2a^2} \left[ \frac{\pi}{2} \text{erf}(ab) - ab \, e^{-a^2 t^2} \right]$$

(173)

and $(c)$ holds because $\text{erf}(\cdot) \leq 1$ and the last term is non-negative. Substituting back in (170) yields

$$\mathbb{E}[\cos(\Phi)] \geq \text{erf} \left(\frac{\pi}{2} R\right) - \frac{1}{2R^2} - \frac{\sqrt{\pi}}{4} R e^{-R^2} \quad (174)$$

$$\geq 1 - e^{-\pi^2 R^2/4} - \frac{1}{2R^2} - \frac{\sqrt{\pi}}{4} R e^{-R^2} \quad (175)$$

where the last inequality follows from the relations $\text{erf}(\pi R/2) = 1 - \text{erfc}(\pi R/2)$ and $\text{erfc}(\pi R/2) \leq e^{-\pi^2 R^2/4}$ (see [47] eq. (5)). Therefore, we have

$$R^2 \mathbb{E}[\cos(\Phi)] \geq R^2 - \frac{4}{\pi^2 e} - \frac{1}{2} - \frac{\sqrt{\pi}}{4} \left(\frac{3}{2e}\right)^{3/2} \quad (176)$$

because

$$\max_x x^m e^{-ax^2} = \left(\frac{m}{2ae}\right)^{m/2} \quad (177)$$

for any $a > 0$ and positive integer $m$, which implies that $R_3 e^{-R^2} \leq (3/2e)^{3/2}$ and $R_2 e^{-\pi^2 R^2/4} \leq 4/\pi^2 e$. We conclude that

$$\mathbb{E}[\cos(\Phi)] \geq 1 - 0.83 \geq 1 - \frac{1}{R^2} \quad (178)$$

Finally, we have

$$\mathbb{E}[\sin(\Phi)] = \int_{-\pi}^{\pi} \sin(\phi) p_{\Phi}(\phi) \, d\phi = 0 \quad (179)$$

because the integrand is odd.

Lemma 7: The following bounds hold.

1. For $z \geq 0$:

$$\text{erfc}(z) \geq 2 - e^{-z^2} \quad (180)$$

2. For $a \geq 0$ and $t \geq 0$:

$$\cos^2 t e^{-a e^{\sin^2 t}} \geq (1 - t^2)e^{-a t^2} \quad (181)$$

Proof:

1. Since for any real number $z$

$$\text{erf}(z) = 1 - \text{erfc}(z) \quad (182)$$

$$\text{erfc}(z) = 1 - \text{erf}(z) \quad (183)$$

we have

$$\text{erfc}(-z) = 1 - \text{erf}(-z) = 1 + \text{erf}(z) = 2 - \text{erfc}(z). \quad (184)$$

To complete the proof, we use [47] eq. (5)

$$\text{erfc}(z) \leq e^{-z^2} \text{ for } z \geq 0. \quad (185)$$
2) The trigonometric identity \( \cos^2 t + \sin^2 t = 1 \) and the bound \( \sin t \leq t \) (for \( t \geq 0 \)) imply that
\[
\cos^2 t \geq 1 - t^2. \tag{186}
\]
Since the exponential function is monotone, we have
\[
e^{-a \sin^2 t} \geq e^{-at^2}. \tag{187}
\]
By combining the two inequalities, we obtain
\[
\cos^2 t \ e^{-a \sin^2 t} \geq (1 - t^2) e^{-at^2} \tag{188}
\]

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