Insufficiency of the Quantum State for Deducing Observational Probabilities

Don N. Page

Theoretical Physics Institute
Department of Physics, University of Alberta
Room 238 CEB, 11322 – 89 Avenue
Edmonton, Alberta, Canada T6G 2G7

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It is usually assumed that the quantum state is sufficient for deducing all probabilities for a system. This may be true when there is a single observer, but it is not true in a universe large enough that there are many copies of an observer. Then the probability of an observation cannot be deduced simply from the quantum state (say as the expectation value of the projection operator for the observation, as in traditional quantum theory). One needs additional rules to get the probabilities. What these rules are is not logically deducible from the quantum state, so the quantum state itself is insufficient for deducing observational probabilities. This is the measure problem of cosmology.

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I. INTRODUCTION

All probabilities for a system are believed to be encoded in its quantum state. This may be true, but there is the question of how to decode the quantum state to give these probabilities. In traditional quantum theory, the probabilities are given by the expectation values of projection operators. Once a possible observation is specified (including the corresponding projection operator), then its probability is given purely by the quantum state as the expectation value the state assigns to the projection operator, a mathematization of the Born rule [1].

This prescription works well in ordinary single laboratory settings, where there are no copies of the observer. Then distinct observations are mutually exclusive, so that different ones cannot both be observed. If one assigns a projection operator to each possible distinct observation in a complete exhaustive set, then these projection operators will be orthonormal, and their (non-negative) expectation values will sum to unity, which are conditions necessary for them to be interpreted as the probabilities of the different possible observations.

However, in cosmology there is the possibility that the universe is so large that there are many copies of each observer, no matter how precisely the observer is defined. This raises the problem [2, 3, 4, 5] that two observations that are seen as distinct for an observer are not mutually exclusive in a global viewpoint; both can occur for different copies of the observer (though neither copy may be aware of that). This would not be a problem for a putative superobserver who can observe all possible sets of observations by all observers over the entire universe, but it is a problem for the assignment of normalized probabilities for the possible observations that are distinct for each copy of the observer. The result [4] is that one cannot get such a set of normalized probabilities as the expectation values of projection operators in the full quantum state of the universe.

One can still postulate that there are rules for getting the probabilities of all possible observations from the quantum state, but then the question arises as to what these rules are. Below we shall give examples of several different possibilities for these rules, showing that they are not uniquely determined and thus that they are logically independent of the question of what the quantum state is. Therefore, the quantum state just by itself is insufficient to determine the probabilities of observations.

The main application of the logical independence of the probability rules is to the measure problem in cosmology (see [2] for many references), the problem of how to make statistical predictions for observations in a universe that may be so large that almost all theoretically possible observations actually occur somewhere. The logical independence implies that the solution to the measure problem is not just the quantum state of the universe but also other independent elements, the rules for getting the probabilities of the observations from the quantum state.

II. OBSERVATIONAL PROBABILITIES WITH MANY COPIES OF THE OBSERVER

A goal of science is to come up with theories $T_i$ that predict the probabilities of results of observations (observational results). Here for simplicity I shall assume that there is a countable set of possible distinct observations $O_j$ out of some exhaustive set of all such observations. This set of possible observations might be, for example, all possible conscious perceptions [6], all possible data sets for one person, all possible contents for an eprint arXiv, or all possible data sets for a human scientific information gathering and utilizing system [2]. If one imagines a continuum for the set of observations (which seems to be logically possible, though not required), in that case I shall assume that they are binned into a countable number of exclusive and exhaustive subsets that each may be...
considered to form one distinct observation $O_j$. Then the goal is to calculate the probability $P_j(i) \equiv P(O_j|T_i)$ for the observation $O_j$, given the theory $T_i$.

One might think that once one has the quantum state, there would be a standard answer to the question of the probabilities for the various possible observations. For example, one might take traditional quantum theory (what I there called standard quantum theory) to give the example [4], one might take traditional quantum theory probabilities for the various possible observations. For there would be a standard answer to the question of the value of each observation operator $O_j$ in the quantum state given by the theory $T_i$, of a projection operator $P_j$ onto the observational result $O_j$.

That is, one might take

$$P_j(i) = \langle P_j \rangle_i,$$  
(2.1)

where $\langle \rangle_i$ denotes the quantum expectation value of whatever is inside the angular brackets in the quantum state $i$ given by the theory $T_i$. This traditional approach works in the case of a single laboratory setting where the projection operators onto different observational results are orthogonal, $P_j P_k = \delta_{jk} P_j$ (no sum over repeated indices).

However, in the case of a sufficiently large universe, one may have observation $O_j$ occurring ‘here’ and observation $O_k$ occurring ‘there’ in a compatible way, so that $P_j$ and $P_k$ are not orthogonal. Then the traditional quantum probabilities given by Eq. (2.1) will not be normalized to obey

$$\sum_j P_j(i) = 1.$$  
(2.2)

Thus one needs a different formula for normalizable probabilities of a mutually exclusive and exhaustive set of possible observations, when distinct observations within the complete set cannot be described by orthogonal projection operators.

Although many other rules are also possible, as I shall illustrate below, the simplest class of modifications of Eq. (2.1) would seem to be to replace the projection operators $P_j$ with some other observation operators $Q_j(i)$ normalized so that $\sum_i (Q_j(i))_i = 1$, giving

$$P_j(i) = \langle Q_j(i) \rangle_i.$$  
(2.3)

Of course, one also wants $P_j(i) \geq 0$ for each $i$ and $j$, so one needs to impose the requirement that the expectation value of each observation operator $Q_j(i)$ in each theory $T_i$ is nonnegative.

The main point is that in cases with more than one copy of the observer, such as in a large enough universe, one cannot simply use the expectation values of projection operators as the probabilities of observations, so that, if Eq. (2.3) is to apply, each theory must assign a set of observation operators $Q_j(i)$, corresponding to the set of possible observations $O_j$, whose expectation values are used instead as the probabilities of the observations. Since these observation operators are not given directly by the formalism of traditional quantum theory, they must be added to that formalism by each particular complete theory. In other words, a complete theory $T_i$ cannot be given merely by the dynamical equations and initial conditions (the quantum state), but it also requires the set of observation operators $Q_j(i)$ whose expectation values are the probabilities of the observations $O_j$ in the complete set of possible observations (or else some other rule for the probabilities, if they are not to be expectation values of operators). The probabilities are not given purely by the quantum state but have their own logical independence in a complete theory.

Let us suppose that we can hypothetically partition spacetime into a countable set of disjoint regions labeled by the index $L$, with each region having its own reference frame and being sufficiently small that for each $L$ separately there is a set of orthogonal projection operators $P^L_j$ whose expectation values give good approximations to the probabilities that the observations $O_j$ occur within the region $L$. Each region has its own algebra of quantum operators, and for simplicity I shall make the somewhat unrealistic assumption that the regions are either space-separated or are so far apart that each operator in one region, such as $P^L_j$, commutes with each observable in a different region, such as $P^M_k$ for $L \neq M$. (However, I am assuming that each observation $O_j$ can in principle occur within any of the regions, so that the content of the observation is not sufficient to distinguish what $L$ is; the observation does not determine where one is in spacetime. One might imagine that an observation determines as much as it is possible to know about some local region, but it does not determine the properties outside, which might go into the specification of the index $L$ that is only known to a hypothetical superobserver that makes the partition.)

Now one might propose that one construct the projection operator

$$P_j = I - \prod_L (I - P^L_j)$$  
(2.4)

(this being the only place where I need the $P^L_j$’s to commute for different $L$) and use it in Eq. (2.1) to get a putative probability of the observation $O_j$ in the quantum state given by the theory $T_i$. Indeed, this is essentially in quantum language [4] what Hartle and Srednicki [2] propose, that the probability of an observation is the probability that it occur at least somewhere. However, because the different $P_j$’s defined this way are not orthogonal, the resulting traditional quantum probabilities given by Eq. (2.1) will not be normalized to obey Eq. (2.2). This lack of normalization is a consequence of the fact that even though it is assumed that two different observations $O_j$ and $O_k$ (with $j \neq k$) cannot both occur within the same region $L$, one can have $O_j$ occurring within one region and $O_k$ occurring within another region. Therefore, the existence of the observation $O_j$ at least somewhere is not incompatible with the existence of the distinct observation $O_k$ somewhere else, so the sum of the existence probabilities is not constrained to be unity.
If one were the hypothetical superobserver who has access to what is going on in all the regions, one could make up a mutually exclusive and exhaustive set of joint observations occurring within all of the regions. However, for us observers who are confined to just one region, the probabilities that such a superobserver might deduce for the various combinations of joint observations are inaccessible for us to test or to use to predict what we might be expected to see. Instead, we would like probabilities for the observations we ourselves can make. I am assuming that each \( O_j \) is an observational result that in principle we could have, but that we do not have access to knowing which region \( L \) we are in. (The only properties of \( L \) that we can know are its local properties that are known in the observation \( O_j \) itself, but that is not sufficient to determine \( L \), which might be determined by properties of the spacetime beyond our local knowledge.)

### III. Examples of Different Observational Probabilities for the Same Quantum State

Let us demonstrate the logical freedom in the rules for the observational probabilities \( P_j(i) \equiv P(O_j | T_i) \) by exhibiting various examples of what they might be. For simplicity, let us restrict attention to theories \( T_i \) that all give the same pure quantum state \(|\psi\rangle\), which can be written as a superposition, with fixed complex coefficients \( a_N \), of component states \(|\psi_N\rangle\) that each have different numbers \( N \) of observational regions:

\[
|\psi\rangle = \sum_{N=0}^{\infty} a_N |\psi_N\rangle, \quad (3.1)
\]

where \( \langle \psi_M | \psi_N \rangle = \delta_{MN} \). (Different values of \( N \) might model different sizes of universes produced by differing amounts of inflation in the cosmological measure problem.) The different theories will then differ only in the prescriptions they give for calculating the observational probabilities \( P_j(i) \) from the single quantum state \( |\psi\rangle \). These differences will illustrate the logical independence of the observational probabilities from the quantum state, the fact that the observational probabilities are not uniquely determined by the state.

In each component state, the index \( L \) can run from 1 to \( N \) (except for the component state \(|\psi_0\rangle\), which has no observational regions at all). As above, let us suppose that \( P_L^j \) is a complete set of orthogonal projection operators for the observation \( O_j \) to occur in the region \( L \). Then if only the region \( L \) existed, and the state were \(|\psi_N\rangle\), then the quantum probability of the observation \( O_j \) would be

\[
p_{NLj} = \langle \psi_N | P_L^j | \psi_N \rangle. \quad (3.2)
\]

However, in reality, even just in the component state \(|\psi_N\rangle\) for \( N > 1 \), there are other regions where the observation could occur, so the total probability \( P_j(i) \) for the observation \( O_j \) in the theory \( T_i \) can be some \( i \)-dependent function of all the \( p_{NLj} \)'s. The freedom of this function is part of the independence of the observational probabilities from the quantum state itself.

Let us define existence probabilities \( p_j(1) \) that might not be normalized to add up to unity when summed over \( j \) for the different possible observational results \( O_j \), and then use \( P_j(i) \) for normalized observational probabilities obeying Eq. \( (2.2) \). The different indices \( i \) will denote different theories, in this case different rules for calculating the probabilities, since for simplicity we are assuming that all the theories have the same quantum state \(|\psi\rangle\).

Next, let us turn to different possible examples.

For theory \( T_1 \), let us suppose that the existence probability \( p_j(1) = 0 \) if there is no region \( L \) in any nonzero component of the quantum state \(|\psi_N\rangle\) with \( a_N \neq 0 \) that has a positive expectation value for \( P_L^j \), so that \( \sum_{N,L} |a_N|^2 p_{NLj} = 0 \), but that \( p_j(1) = 1 \) otherwise, that is if \( \sum_{N,L} |a_N|^2 p_{NLj} > 0 \). This theory is essentially taking the Everett many worlds interpretation to imply that if there is any nonzero amplitude for the observation to occur, it definitely exists somewhere in the many worlds (and hence has existence probability unity).

Now of course these existence probabilities \( p_j(1) \) are not necessarily normalized, since \( M > 1 \) of them can be unity, with the rest zero, so the sum of these existence probabilities is \( M \). However, one could say that so far as the probability goes of making a particular one of the \( M \) actually existing observations, that could be considered equally divided between the \( M \) actually existing possibilities (if \( M > 0 \), so that one has normalized observational probabilities \( P_j(1) = p_j(1)/M \). This would be the theory that every observation that actually does exist is equally probable.

For theory \( T_2 \), define the existence probabilities to be \( p_j(2) = \langle \psi | P_j | \psi \rangle \), the expectation value in the full quantum state \(|\psi\rangle\) of the projection operator \( P_j \) defined by Eq. \( (2.2) \) for the existence of the observation \( O_j \) in at least one region \( L \). This is not the full many-worlds existence probability, which is unity if the observation does occur somewhere, but it might be regarded as the quantum probability for a superobserver to find that at least one instance of the observation \( O_j \) occurs.

Again, these existence probabilities will not in general sum to unity, but one can normalize by dividing by the sum \( M \) (now generically not an integer) to get normalized probabilities \( P_j(2) = p_j(2)/M \). For a universe that is very large (e.g., most of the \( |a_N|^2 \)'s concentrated on very large \( N \)'s), one would expect a large number of \( p_j(2) \)'s to be very near unity (since it would be almost certain that the observation \( O_j \) occurs at least somewhere among the huge number of regions), so that there will be a large number of nearly equal but very small \( P_j(2) \)'s.

For theory \( T_3 \), refrain from defining the existence probabilities \( p_j(3) \) (since I am regarding it sufficient for a theory to prescribe only the observational probabilities of observations in regions within the universe, not for observations by some putative superobserver). Instead,
define unnormalized observational measures

$$\mu_j(3) = \sum_{L} \langle \psi | P_j^L | \psi \rangle = \sum_{N=1}^{\infty} \sum_{L=1}^{N} |a_N|^2 \rho_{NLj}. \quad (3.3)$$

Then normalize these to let the observational probabilities be defined as

$$P_j(3) = \frac{\mu_j(3)}{\sum_k \mu_k(3)}. \quad (3.4)$$

For theory $T_4$, define unnormalized observational measures

$$\mu_j(4) = \sum_{N=1}^{\infty} \frac{1}{N} \sum_{L=1}^{N} |a_N|^2 \rho_{NLj}$$

and then normalized observational probabilities

$$P_j(4) = \frac{\mu_j(4)}{\sum_k \mu_k(4)}. \quad (3.6)$$

Theory $T_3$ in its sum over $L$ effectively weights each component state $|\psi_N\rangle$ by the number of observational regions where the observation $O_j$ can potentially occur. On the other hand, theory $T_4$ has an average over $L$ for each total number $N$ of observational regions, so that component states $|\psi_N\rangle$ do not tend to dominate the probabilities for observations just because of the greater number of observational opportunities within them. Theory $T_3$ is analogous to volume weighting in the cosmological measure, and theory $T_4$ is analogous to volume averaging.

Of these four rules, $T_1$ does not give the probabilities as the expectation values of natural observation operators $Q_j(i)$, but the other three do, with the corresponding observation operators being

$$Q_j(2) = \frac{P_j}{\langle \sum_k P_k \rangle_i},$$

$$Q_j(3) = \frac{\sum_j P_j^L}{\langle \sum_k \sum_L P_k^L \rangle_i},$$

$$Q_j(4) = \frac{\sum_{N=1}^{\infty} \frac{1}{N} \sum_{L=1}^{N} P_N P_j^L P_N}{\langle \sum_k \sum_{N=1}^{\infty} \frac{1}{N} \sum_{L=1}^{N} P_N P_k^L P_N \rangle_i}, \quad (3.7)$$

where $P_N = |\psi_N\rangle\langle \psi_N|$ is the projection operator onto the component state with $N$ observation regions.

These examples show that there is not just one unique rule for getting observational probabilities from the quantum state. It remains to be seen what the correct rule is. Of the four examples given above, I suspect that with a suitable quantum state, theory $T_1$ would have the highest likelihood $P_j(i)$, given our actual observations, since theories $T_1$ and $T_2$ would have the normalized probabilities nearly evenly distributed over a huge number of possible observations, and theory $T_3$ seems to be plagued by the Boltzmann brain problem. One might conjecture that theory $T_4$ can be implemented in quantum cosmology to fit observations better than other alternatives.

Thus we see that in a universe with the possibility of multiple copies of an observer, observational probabilities are not given purely by the quantum state, but also by a rule to get them from the state. There is logical freedom in what this rule is (or in what the observation operators $Q_j(i)$ are if the rule is that the probabilities are the expectation values of these operators). In cosmology, finding the correct rule is the measure problem.

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