PICK’S THEOREM AND SUMS OF LATTICE POINTS

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Abstract. Pick’s theorem is used to prove that if $P$ is a lattice polygon (that is, the convex hull of a finite set of lattice points in the plane), then every lattice point in the $h$-fold sumset $hP$ is the sum of $h$ lattice points in $P$.

For sets $X$ and $Y$ in $\mathbb{R}^n$, we define the $\textit{sumset}$
$$X + Y = \{x + y : x \in X \text{ and } y \in Y\}.$$  
For every positive integer $h$, we have the $\textit{h-fold sumset}$
$$hX = \{x_1 + \cdots + x_h : x_i \in X \text{ for } i = 1, \ldots, h\}$$
and the $\textit{dilation}$
$$h \ast X = \{hx : x \in X\}.$$  

Lemma 1. For every convex subset of $\mathbb{R}^n$ and every positive integer $h$, the sumset equals the dilation, that is,
$$hX = h \ast X.$$  

Proof. If $x \in X$, then
$$hx = \underbrace{x + \cdots + x}_{h \text{ summands}} \in hX$$
and so $h \ast X \subseteq hX$.

If the set $X$ is convex and if $x_1, \ldots, x_h \in X$, then $X$ contains the convex combination
$$(1/h)x_1 + \cdots + (1/h)x_h$$
and so
$$x_1 + \cdots + x_h = h ((1/h)x_1 + \cdots + (1/h)x_h) \in h \ast X.$$  
Thus, if $X$ is convex, then $hX \subseteq h \ast X$. This completes the proof. □

A $\textit{lattice polytope}$ in $\mathbb{R}^n$ is the convex hull of a nonempty finite set of lattice points in $\mathbb{Z}^n$. A $\textit{lattice polygon}$ in $\mathbb{R}^2$ is the convex hull of a nonempty finite subset of $\mathbb{Z}^2$.

Let $n \geq 2$ and $h \geq 2$. If $P, P_1, \ldots, P_h$ are lattice polytopes in $\mathbb{R}^n$, then
(1) $$h(P \cap \mathbb{Z}^n) \subseteq (hP) \cap \mathbb{Z}^n$$
and
(2) $$(P_1 \cap \mathbb{Z}^n) + \cdots + (P_h \cap \mathbb{Z}^n) \subseteq (P_1 + \cdots + P_h) \cap \mathbb{Z}^n.$$  
These set inclusions can be strict. For example, if $P_1$ is the triangle in $\mathbb{R}^2$ whose vertices are $\{(0,0), (1,0), (1,-1)\}$, and if $P_2$ is the triangle whose vertices are $\{(0,0), (1,2), (2,3)\}$, then $P_1 + P_2$ is the hexagon with vertices $\{(0,0), (1,-1), (3,2), (3,3), (2,3), (1,2)\}$. 



We have
\[(1, 1) = (1/2, 0) + (1/2, 1) \in (P_1 + P_2) \cap \mathbb{Z}^2\]
but
\[(1, 1) \notin (P_1 \cap \mathbb{Z}^2) + (P_2 \cap \mathbb{Z}^2).
\]
Therefore,
\[(P_1 \cap \mathbb{Z}^2) + (P_2 \cap \mathbb{Z}^2) \neq (P_1 + P_2) \cap \mathbb{Z}^2.
\]

In \(\mathbb{R}^3\), let \(P\) be the tetrahedron with vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)\)
We have
\[(1, 1, 1) = (1/2, 1/2, 0) + (1/2, 1/2, 1) \in (2P) \cap \mathbb{Z}^3\]
but
\[(1, 1, 1) \notin 2(P \cap \mathbb{Z}^3).
\]
Therefore,
\[2(P \cap \mathbb{Z}^3) \neq (2P) \cap \mathbb{Z}^3.
\]

It is an only partially solved problem to determine the lattice polytopes \(P, P_1, \ldots, P_h\) in \(\mathbb{R}^n\) for which we have equalities and not inclusions in (1) and (2). This is usually discussed in the language of toric geometry [3, 5, 9]. It is known that in the plane we have
\[h(P \cap \mathbb{Z}^2) = (hP) \cap \mathbb{Z}^2\]
for every lattice polygon \(P\) and every positive integer \(h\) (Koelman [7, 6]). In this note we apply Pick’s theorem (Pick [10], Beck and Robins [1, pp. 38–40]) to obtain a simple proof of this result.

Pick’s theorem, proofs of which appear frequently in the Monthly (e.g. [4, 11, 2, 8]), states that if \(P\) is a lattice polygon with area \(A\) and with \(I\) lattice points in its interior and \(B\) lattice points on its boundary, then
\[A = I + \frac{B}{2} - 1.
\]
A triangle is the convex hull of three non-collinear points. A primitive lattice triangle is a lattice triangle whose only lattice points are its three vertices. By Pick’s theorem, a lattice triangle in \(\mathbb{R}^2\) is primitive if and only if its area is 1/2.

**Lemma 2.** Let \(T\) be a primitive lattice triangle with vertex set \(\{0, u, v\}\), and let
\[W = \{iu + jv : i = 0, 1, \ldots, h\ \text{and} \ j = 0, 1, 2, \ldots, h - i\}.
\]
Then
\[W = hT \cap \mathbb{Z}^2.
\]

**Proof.** Because \(u, v \in \mathbb{Z}^2\) and because the set \(T\) is convex and
\[hT = h \ast T = \{\alpha u + \beta v : \alpha \geq 0, \beta \geq 0, \ \text{and} \ \alpha + \beta \leq h\}\]
it follows that \(W \subseteq hT \cap \mathbb{Z}^2\). The linear independence of the vectors \(u\) and \(v\) implies that
\[|W| = \sum_{i=0}^{h} (h - i + 1) = \frac{(h + 1)(h + 2)}{2}.
\]
Let \( A(h) \) denote the area of the dilated triangle \( hT \), and let \( I(h) \) and \( B(h) \) denote, respectively, the number of interior lattice points and boundary lattice points of \( hT \). Because \( T \) is primitive, we have \( A(1) = \frac{1}{2} \) and \( B(1) = 3 \). It follows that
\[
A(h) = A(1)h^2 = \frac{h^2}{2}
\]
and
\[
B(h) = B(1)h = 3h.
\]
By Pick’s theorem,
\[
A(h) = I(h) + \frac{B(h)}{2} - 1
\]
and so the number of lattice points in \( hT \) is
\[
| hT \cap \mathbb{Z}^2 | = I(h) + B(h) = A(h) + \frac{B(h)}{2} + 1 = \frac{(h+1)(h+2)}{2}.
\]
Because \( W \) and \( hT \cap \mathbb{Z}^2 \) are finite sets with \( W \subseteq hT \cap \mathbb{Z}^2 \) and \( |W| = |hT \cap \mathbb{Z}^2| \), it follows that \( W = hT \cap \mathbb{Z}^2 \). This completes the proof. \( \square \)

**Theorem 1.** Let \( P \) be a lattice polygon. If \( w \) is a lattice point in the sumset \( hP \), then there exist lattice points \( a, b, c \) in \( P \) and nonnegative integers \( i, j, k \) such that
\[
h = i + j + k
\]
and
\[
w = ia + jb + kc \in h(P \cap \mathbb{Z}^2).
\]
In particular,
\[
h(P \cap \mathbb{Z}^2) = (hP) \cap \mathbb{Z}^2.
\]
**Proof.** Every lattice polygon \( P \) can be triangulated into primitive lattice triangles. If \( w \) is a lattice point in \( hP = h * P \), then \( w/h \in P \) and so there is a primitive lattice triangle \( T' \) contained in \( P \) with \( w/h \in T' \). Let \( \{a, b, c\} \) be the set of vertices of \( T' \), and let \( T = T' - c \) be the primitive lattice triangle with vertices \( 0, u = a - c, \) and \( v = b - c \). We have \( w/h - c \in T \) and so \( w - hc \in hT \). By Lemma\( ^2 \) there are nonnegative integers \( i \) and \( j \) such that \( i + j = h - k \leq h \) and
\[
w - hc = iu + jv.
\]
This implies that
\[
w = iu + jv + hc = ia + jb + kc \in 3(T' \cap \mathbb{Z}^2) \subseteq 3(P \cap \mathbb{Z}^2).
\]
This completes the proof. \( \square \)

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