ABSTRACT. We present tools and definitions to study abstract tropical manifolds in dimension 2, which we call simply tropical surfaces. This includes explicit descriptions of intersection numbers of 1-cycles, normal bundles to some curves and tropical Chern cycles and numbers. We provide a new method for constructing tropical surfaces, called the tropical sum, similar to the fiber sum of usual manifolds.

We prove a tropical adjunction formula for curves in compact tropical surfaces satisfying a local condition, a partial Castelnuovo-Enriques criterion for contracting $-1$-curves, and also invariance of $(p,q)$-homology and Chow groups under tropical modification. Finally we prove a tropical version of Noether’s formula for compact surfaces constructed from tropical toric surfaces by way of summations and tropical modifications.

1. INTRODUCTION

Tropical geometry is a recent field of mathematics which can be loosely described as geometry over the tropical semi-field. This semi-field is also known as the max-plus semi-field and is $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ equipped with the operations of maximum and addition.

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Tropical spaces are polyhedral in nature and carry an integral affine structure. Sometimes these spaces arise via limits, degenerations, or skeletons of algebraic varieties defined over a field. However, this is not always the case. A popular example comes from the theory of matroids. The tropicalization of a linear space defined over a field depends only on an associated (perhaps, valuated) matroid [Stu02]. This led to the notion of the Bergman fan of a matroid, which gives a representation of any matroid as a tropical linear space. In classical manifold theory, manifolds are locally modeled on affine spaces, therefore we take matroidal fans as the local models of tropical manifolds. Interestingly, whether a matroid is representable over a field or not, the associated tropical linear space has many properties analogous to those of smooth spaces in classical geometry. This eventually lead to properties of tropical manifolds akin to properties of non-singular algebraic varieties.

Abstract tropical manifolds of dimension 1, tropical curves, are simply metric graphs. A Riemann-Roch theorem for divisors on graphs given by Baker and Norine [BN07] and extended to tropical curves [GK08], [MZ08], sparked a study of linear series on graphs with many applications to the study of linear series on classical algebraic curves. These applications come usually by way of specialization theorems, which allow one to deduce inequalities between ranks of divisors and their tropicalizations [Bak08]. For example, there is a proof of Brill-Noether theorem [CDPR12], the study of gonality of curves [AK] and even applications to the arithmetic geometry of curves [KRZB]. On the combinatorial side, the theory of divisors on tropical curves is related to chip-firing games, see [BS13].

Before this, the enumerative geometry of tropical curves in \( \mathbb{R}^2 \) had powerful applications to real and complex enumerative geometry by way of Mikhalkin’s correspondence theorem [Mik05]. Abstract tropical curves have also had applications to enumerative geometry via Hurwitz numbers [CJM10], [BBM11], [MRa], as well as via the study of the intersection theory of their moduli spaces [Mik07a], [GKM09], [MR09].

The goal of this paper is to provide necessary definitions and tools to study higher dimensional tropical manifolds, focusing on dimension 2. The definition of abstract tropical varieties from [Mik06] has been adapted to consider tropical spaces locally modeled on matroidal fans. These spaces will be called tropical manifolds. Rather than adopt the term “non-singular tropical variety”, the name “tropical manifold” is more fitting since there are examples of tropical manifolds which are models of non-algebraic complex manifolds [RS].

Before summarizing the main results, we would like to point out the work of Cartwright on tropical complexes [Carb], [Cara]. A tropical complex is a simplicial complex equipped with some extra numerical data. One should think of the simplicial complex as the dual complex of a degeneration and the additional data encoding intersection numbers. Tropical complexes of dimension 1, do not require this extra numerical information and can be considered simply as discrete graphs appearing in the work of Baker and Norine.

Cartwright proves specialization lemmas for ranks of divisors on tropical complexes. For tropical complexes of dimension 2, there is an analogue of the Hodge Index Theorem on the tropical Neron-Severi group, as well as a tropical version of Noether’s formula. For tropical surfaces studied here, there is also an intersection product on the (1,1)-tropical homology group of a compact tropical surface, see [MZ], [Sha]. The intersection product on the (1,1)-tropical homology group does not necessarily satisfy the analogue of the Hodge Index Theorem [Sha13a]. However, this does not imply that the tropical Hodge Index Theorem fails when the intersection pairing is restricted to the Neron-Severi group of a tropical surface.

We now summarize the outline and results of this paper. We begin by describing the local models of tropical manifolds; fan tropical linear spaces in \( \mathbb{R}^N \) and in tropical affine space. A fan tropical linear space in \( \mathbb{R}^N \) is given by a matroid and a choice of \( \mathbb{Z}^N \)-basis. Here we are only interested in
the support of the fan, also called the coarse structure in \[\text{AK06}\]. In general, it is possible to obtain the same tropical fan plane from non-isomorphic matroids, if we also choose different bases of \(\mathbb{Z}^N\). However, in restricting to dimension 2 we prove that this is not possible.

**Proposition 1.** The underlying matroid of a fan tropical plane in \(\mathbb{R}^N\) is unique.

This however does not imply that the coordinates with respect to which a fan tropical linear space is equal to a Bergman fan of a matroid is unique, just that it is not possible to change the underlying matroid by changing the \(\mathbb{Z}^N\)-basis, see Example 2.7. There are known counter-examples to the above statement for fan tropical linear spaces of dimension greater than 2.

The rest of Section 2 defines tropical modifications of fan tropical linear spaces, 1-cycles in fan tropical planes, and also intersections of 1-cycles in fan tropical planes using the definitions from \[\text{BS15}\]. Tropical intersection theory is, for the most part, defined locally and these definitions will be used when passing to general tropical surfaces.

Section 3 switches from local to global considerations. Section 3.1 gives the general definition of tropical manifolds and Section 3.2 describes their boundaries. In Section 3.4, the definition of Chern cycles of tropical manifolds are extended beyond the first Chern cycles. For tropical surfaces we give a description of \(c_2(X)\) in terms of combinatorial data of tropical fan planes. We also show that the canonical cycle of a tropical manifold is balanced.

**Proposition 2.** The canonical cycle \(K_X\) of a tropical manifold \(X\) is a balanced tropical cycle.

Section 3.5 describes how to pass from local intersection theory from Section 2.4 to intersections of 1-cycles of curves on surfaces. Also we describe the normal bundles to boundary curves, in order to define their self-intersections. This will also be used to define the tropical sum in Section 4. At the boundary of a tropical manifold, intersection products are no longer defined on the cycle level, and must be defined up to equivalence. Here we consider only numerical equivalence for simplicity.

There is also a tropical intersection theory based on Cartier divisors, see \[\text{Mik06}\], \[\text{AR10}\], and the 2 theories agree. Moreover, we show that on any tropical manifold there is an equivalence of Weil and Cartier divisors. This is a property of non-singular spaces in classical algebraic geometry.

**Theorem 3.** Every codimension 1 cycle on a tropical manifold is a Cartier divisor.

Finishing off Section 3 we summarize the definitions of tropical rational equivalence from \[\text{Mik06}\], and also \((p,q)\)-holomology from \[\text{IKMZ}\]. Via a cycle map a 1-cycle \(A\) in a tropical surface produces a \((1,1)\) homology cycle \([A]\). For consistency, we show that the intersection of 1-cycles in a tropical surface is compatible with their intersection as \((1,1)\)-cycles. It follows that rational equivalence respects the numerical intersection pairing.

**Proposition 4.** The intersection of 1-cycles \(A, B\) in a compact tropical surface \(X\) is numerically equivalent to their intersection as \((1,1)\)-cycles.

It is also the case that the intersection product on tropical cycles descends to rational equivalence, however we do not require this here so we do not include the proof. Such statements have been proved in the non-compact setting, using a bounded version of rational equivalence \[\text{AHR}\].

In Section 4 we describe 2 operations to construct tropical surfaces. The first is already familiar in tropical geometry and is known as tropical modification. This operation was introduced by Mikhalkin in \[\text{Mik06}\] and produces from a tropical variety \(X\) and a tropical Cartier divisor, a new tropical variety \(\tilde{X}\). Between the 2 varieties there is a tropical morphism \(\delta : \tilde{X} \to X\). We restrict to so-called locally degree 1 modifications \(\delta : \tilde{X} \to X\) which are along codimension 1-cycles satisfying an extra condition. This additional condition ensures that the modification produces a tropical manifold. The next proposition then follows from Theorem 3.
Proposition 5. Let $X$ be a tropical manifold, and $D \subset X$ a locally degree 1, then there is a tropical manifold $\tilde{X}$ such that $\delta : \tilde{X} \to X$ is the tropical modification along $D$.

In dimension 2, we use the above proposition to prove a tropical version of the adjunction formula. This formula relates the 1st Betti number of the tropical curve $C$ in a surface $X$ to its self-intersection and intersection with the canonical cycle. Recall that the genus of a tropical curve $C$ is $b_1(C)$.

Theorem 6 (Tropical Adjunction Formula). Let $C$ be a locally degree 1 tropical curve in a compact tropical surface $X$, then

$$b_1(C) = \frac{K_X \cdot C + C^2}{2} + 1,$$

where $b_1(C)$ is the 1st Betti number of $C$.

We also show that tropical manifolds $\tilde{X}$ and $X$ related by a locally degree 1 modification have the same tropical Chow groups and also $(p,q)$-homology groups.

Theorem 7. Let $\delta : \tilde{X} \to X$ be a modification of tropical manifolds then the tropical Chow groups are isomorphic for all $k$

$$CH_k(\tilde{X}) \cong CH_k(X).$$

Theorem 8. Let $\delta : \tilde{X} \to X$ be a modification of tropical manifolds then their tropical $(p,q)$-homology groups are isomorphic for all $p,q$

$$H_{p,q}(\tilde{X}) \cong H_{p,q}(X).$$

The second operation described in Section 4 is analogous to the fibre sum of manifolds. Given tropical surfaces $X_1, X_2$ each containing isomorphic boundary curves $C_1, C_2$ respectively, under suitable conditions we may glue the 2 surfaces together to produce a new tropical surface $X_1 \# X_2$. In general, this sum relies on a choice of identification of neighborhoods of the curves $C_1$ in $X_1$ and $C_2$ in $X_2$.

For example, together tropical sums and modifications can be used to contract locally degree 1 tropical rational curves of self-intersection $-1$ (Example 4.17).

Theorem 9 (Partial Castelnuovo-Enriques Criterion). Let $E$ be a locally degree 1 tropical curve in a tropical surface $X$, such that $b_1(E) = 0$ and $E^2 = -1$ then there exists a tropical modification $\delta : \tilde{X} \to X$, another tropical surface $Y$, and a tropical morphism

$$\pi : \tilde{X} \to Y$$

which sends $\tilde{E}$ to a point $y$ and is an isomorphism on $\tilde{X} \setminus \tilde{E} \to Y \setminus y$.

Section 5 treats a tropical analogue of Noether’s formula for classical projective surfaces. Classically, this formula relates the holomorphic Euler characteristic $\sum_{q=0}^{2} (-1)^q h^{0,q}(X)$ with the Todd class of the surface $\frac{1}{12}(K_X^2 + c_2(X))$. This formula is a special case of the Riemann-Roch formula for the trivial line bundle of a surface.

To translate this tropically, firstly following [IKMZ], the ranks of the $(p,q)$-homology groups from [IKMZ] play the role of Hodge numbers of tropical surfaces. When $p = 0$ the tropical $(p,q)$-homology groups correspond to the usual singular homology groups. Therefore, in the tropical version of Noether’s formula, the holomorphic Euler characteristic is replaced with the topological Euler characteristic of the tropical surface. The top tropical Chern class $c_2(X)$ and also the square of the canonical class $K_X^2$ are defined in Section 3.4.

Conjecture 10 (Tropical Noether’s Formula). A compact tropical surface $X$ satisfies

$$\chi(X) = \frac{K_X^2 + c_2(X)}{12},$$
where $\chi(X)$ is the topological Euler characteristic of $X$.

Proving a sequence of lemmas which relate $K^2_X$ and $c_2(X)$ for sums of surfaces and modifications we prove Noether’s formula in special cases.

**Theorem 11.** Tropical Noether’s formula holds for any compact tropical surface obtained by way of successive modifications and summations of tropical toric surfaces.

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### 2. Preliminaries

#### 2.1. The tropical semi-field.

The tropical semi-field consists of the tropical numbers $T = [-\infty, \infty)$ equipped with the operations; tropical multiplication, which is usual addition, and tropical addition, which is the maximum;

$$“xy” = x + y \quad \text{and} \quad “x + y” = \max\{x, y\}.$$

The additive and multiplicative identities are respectively, $-\infty$ and $0$. Tropical division is usual subtraction, whereas additive inverses do not exist due to the idempotency of addition; $“x + x” = x$.

Tropical polynomial functions are convex integer affine functions $f : T^N \to T$ given by finite tropical sums of the form

$$f(x) = “\sum_{I \in \mathbb{Z}_{\geq 0}^N} a_I x^I” = \max_{I \in \mathbb{Z}_{\geq 0}^N} \{a_I + \langle I, x \rangle \}.$$

Notice that 2 different tropical polynomial expressions may produce the same function $T^N \to T$. Tropical rational functions are the difference of 2 tropical polynomial functions $"g/h" = g - h$.

Notice that this difference of functions is always defined on $\mathbb{R}^N$. The extension of such a function to points in $T^N$ with coordinates equal to $-\infty$ may have indeterminacies.

#### 2.2. Standard tropical affine and projective spaces.

Standard tropical affine space is $T^N$ and the tropical torus is $(T^*)^N = \mathbb{R}^N$. We equip $T$ with the topology whose basis of open sets are the open intervals in $\mathbb{R}$ and $(-\infty, a)$.

The space $T^N$ has a stratification coming from the order of sedentarity of points introduced by Losev and Mikhalkin.

**Definition 2.1.** [Mik06] The sedentarity of a point $x \in T^N$ is

$$S(x) = \{i \mid x_i = -\infty\} \subseteq \{1, \ldots, N\},$$

the order of sedentarity of $x$ is denoted by $s(x) = |S(x)|$.

For $I \subset \{1, \ldots, N\}$, let $\mathbb{R}_I^N$ denote the points of $T^N$ of sedentarity $I$. Then $\mathbb{R}_I^N \cong \mathbb{R}^{n - |I|}$ and together these sets define a stratification of $T^N$. We denote the closure in $T^N$ of the stratum $\mathbb{R}_I^N$ by $T_I^N$.

Tropical projective space is defined analogously to classical projective space [Mik06] as

$$\mathbb{P}^N = \frac{T^N+1\setminus(-\infty, \ldots, -\infty)}{(x_0, \ldots, x_N) \sim \lambda(x_0, \ldots, x_N)^N}.$$

This representation provides tropical homogeneous coordinates, denoted by $[x_0 : \cdots : x_N]$. Using these coordinates, the notion of sedentarity extends to $\mathbb{P}^N$. We use the same notation to denote the strata of $\mathbb{P}^N$, namely $\mathbb{R}_I^N$, where now $I \subseteq \{0, \ldots, N\}$. Denote the closure of $\mathbb{R}_I^N$ in $\mathbb{P}^N$ by
$\mathbb{T}P^N_k$, then we have $\mathbb{T}P^N_j \cong \mathbb{T}P^{n-|J|}$. The stratification of $\mathbb{T}P^N$ induced by the sedentarity is the same as the face structure of the $N$-dimensional simplex.

Tropical projective space $\mathbb{T}P^N$ can also be glued from $N+1$ copies of tropical affine space as in the classical situation. Let $U_i$ be the open set of points in $\mathbb{T}P^N$ with homogeneous coordinate $x_i \neq -\infty$ and the map $\phi_i : U_i \to \mathbb{T}^N$ be given by

$$\phi_i([x_0 : \cdots : x_N]) = (x_0 - x_i, \ldots, x_N - x_i) \in \mathbb{T}^N.$$ 

On the overlaps of the charts the coordinate changes

$$\Phi_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

are given explicitly by,

$$(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \mapsto (x_0 - x_j, \ldots, x_{i-1} - x_j, -x_j, x_{i+1} - x_j, \ldots, x_N - x_j).$$

Notice that the functions on the right hand side can be expressed using tropical operations as “$\frac{x_i}{x_j}$.” In the image, the $j^{th}$ coordinate is removed (notice, $x_j - x_j = 0$), so we have a map from $\mathbb{T}^N$ to $\mathbb{T}^N$. Let $e_1, \ldots, e_N$ denote the standard basis vectors of $\mathbb{R}^N$, then $-e_1, \ldots, -e_N, e_0 := \sum_{i=1}^N e_i$ form distinguished directions in $\mathbb{R}^N$ with respect to this compactification to $\mathbb{T}P^N$. By this we mean that the closure in $\mathbb{T}P^N$ of a half ray in $\mathbb{R}^N$ intersects the interior of a boundary hyperplane $\mathbb{T}P^N_{\{i\}}$ if and only if it is in one of these $N+1$ directions.

Given any other $\mathbb{Z}^N$-basis $\Delta = \{u_1, \ldots, u_N\}$, the tropical torus $\mathbb{R}^N$ can be compactified to projective space using $\Delta$ so that the distinguished directions are $u_0, \ldots, u_N$, where $u_0 := -\sum_{i=1}^N u_i$. Denote this compactification by $\mathbb{T}P^N_\Delta$. The notation $\mathbb{T}P^N$ is reserved to denote $\mathbb{T}P^N_{\{-e_1, \ldots, -e_N\}}$.

### 2.3. Tropical fan planes.

The Bergman fan [Ber71] of an algebraic variety $V \subset (\mathbb{C}^*)^N$ is defined as

$$\lim_{t \to \infty} \Log_t(V) \subset \mathbb{R}^N.$$ 

In other words, the Bergman fan is the limit as $t \to \infty$ of amoebas of $V$, from [GKZ94]. For $V \subset (\mathbb{C}^*)^N$ of dimension $k$, the set $\lim_{t \to \infty} \Log_t(V)$ is also $k$ dimensional and can be equipped with a polyhedral fan structure. Moreover, positive integer weights can be assigned to the top dimensional faces so that the weighted polyhedral complex satisfies to the balancing condition well known in tropical geometry, see [MS], [BIMS]. The resulting weighted fan is denoted $\Trop(V) \subset \mathbb{R}^N$.

A $k$-dimensional variety $\mathcal{L} \subset (\mathbb{C}^*)^N$ is a linear space if it is defined by a linear ideal in some monomials $z^{u_1}, \ldots, z^{u_N}$ where the collection $u_1, \ldots, u_N \subset \mathbb{Z}$ forms a $\mathbb{Z}^N$-basis. A linear space $\mathcal{L} \subset (\mathbb{C}^*)^N$ defines a matroid $M$ of rank $k+1$ on $N+1$ elements. The reader is directed to the standard reference on matroid theory [Oxl11] or also the more algebro-geometric [Kat]. A geometric way of obtaining the matroid is to first compactify to $\mathcal{L} = \mathbb{C}P^k \subset \mathbb{C}P^N$ by way of the coordinates $z^{u_1}, \ldots, z^{u_N}$, so that the $\mathcal{L}$ is of degree 1. Intersecting $\mathcal{L}$ with any of the $N+1$ coordinate hyperplanes in $\mathbb{C}P^N$ defines a hyperplane arrangement $\mathcal{H}_0, \ldots, \mathcal{H}_N$ on $\mathbb{C}P^k = \mathcal{L}$. Define the rank function of the matroid on the ground set $\{0, \ldots, N\}$ by

$$\text{rank}(I) = \text{codim}_{\mathbb{C}P^k}(\cap_{i \in I} \mathcal{H}_i).$$

For a linear space $\mathcal{L} \subset (\mathbb{C}^*)^N$, the tropicalization $\Trop(\mathcal{L})$ is a fan equipped with weight 1 on all of its faces. The fan is determined by the underlying matroid of $\mathcal{L}$ and the $\mathbb{Z}^N$-basis $u_1, \ldots, u_N$. This leads to the notion of Bergman fan for any matroid in [Stu02]. In general, the Bergman fan of a loopless matroid $M$ of rank $k+1$ on $N+1$ elements with respect to a $\mathbb{Z}^N$-basis $\Delta = \{u_1, \ldots, u_N\}$ is a $k$-dimensional tropical variety which we denote, $\Trop_{\Delta}(M) \subset \mathbb{R}^N$ where $\Delta$ is a $\mathbb{Z}^N$-basis. The weights on all top dimensional faces of $\Trop_{\Delta}(M)$ are always 1.
A description of the Bergman fan of a matroid \( M \) can be given in terms of its lattice of flats \( \Lambda_M \) \([AK06]\). A flat of a matroid is a subset of the ground set that is closed under the rank function. We review this construction only in the case of rank 3 matroids which yield 2 dimensional fans.

**Construction 2.2.** \([AK06]\) Let \( M \) be a loopless matroid of rank 3 on \( N + 1 \) elements labeled \( 0,\ldots,N \). Let \( \Delta = \{u_1,\ldots,u_N\} \) denote a \( \mathbb{Z}^N \)-basis of \( \mathbb{R}^N \) and set \( u_0 = -\sum_{i=1}^{N} u_i \). To construct \( \text{Trop}_\Delta(M) \), first for every flat \( F_I \) in the lattice of flats \( \Lambda_M \) fix the direction \( u_I = \sum_{i \in I} u_i \). Then for every flag of flats

\[
\mathcal{F} = \{ \emptyset \subsetneq F_{I_1} \subsetneq \cdots \subsetneq F_{I_k} \subsetneq \{0,\ldots,N\} \}
\]

contained in the lattice of flats \( \Lambda_M \) there is a cone in \( \text{Trop}_\Delta(M) \) spanned by the vectors \( u_{I_1},\ldots,u_{I_k} \). For a matroid of rank 3 the maximum length of a chain of flats is 4, including the flats \( \emptyset \) and \( \{0,\ldots,N\} \) for which the corresponding direction \( u_I \) is 0. Thus the maximal cones of \( \text{Trop}_\Delta(M) \) are 2 dimensional. This collection of cones is what is known as the fine fan structure \([AK06]\). Here we are only concerned with the support of this fan \( \mathbb{R}^N \) which we denote \( \text{Trop}_\Delta(M) \). This is known as the coarse structure.

To obtain just the cones necessary for the support \( \text{Trop}_\Delta(M) \) we can delete any rays of the above fan corresponding to flats \( F_I \) with \( |I| = 2 \), and also rays corresponding to single element flats \( F_{\{i\}} \) which are contained in exactly 2 flats of size 2. These 2 types of flats produce exactly the rays which subdivide a cone of dimension 2.

**Lemma 2.3.** Let \( M \) be a rank 3 simple matroid (loopless matroid and without double points) on \( \{0,\ldots,N\} \) and \( \Delta \) a \( \mathbb{Z}^N \)-basis. Suppose that the Bergman fan \( \text{Trop}_\Delta(M) \subset \mathbb{R}^N \) does not contain the ray in the coarse structure in the direction \( u_i \in \Delta \) for some \( i \in \{0,\ldots,N\} \). Then \( M \) is a parallel connection of 2 rank 2 matroids.

**Proof.** Since we assume that \( M \) has no loops or double points the rank 1 flats of \( M \) are precisely the elements of the ground set \( \{0,\ldots,N\} \). If there is no ray in direction \( u_i \) in the coarse fan structure of \( \text{Trop}_\Delta(M) \) then the element \( i \) is contained in precisely 2 flats of \( M \) of rank 2, call them \( K \) and \( L \). Moreover, by the axioms of flats the sets \( K \) and \( L \) partition the ground set \( \{0,\ldots,N\} \setminus i \). Any other rank 2 flat \( J \) is \( \{k,l\} \) where \( k \in K \) and \( l \in L \), since the intersection \( K \cap J \) is also a flat it must be a single element, and similarly for \( L \cap J \). Moreover, \( i \notin J \) if \( J \neq K,L \). Along with \( \emptyset \) and \( \{0,\ldots,N\} \) this determines all of the flats of the rank 3 matroid \( M \). It is easy to check that these are precisely the flats of the parallel connection of \( U_{2,k+1} \) and \( U_{2,l+1} \) where \( |K| = k + 1 \) and \( |J| = j + 1 \).

**Corollary 2.4.** Let \( M \) be a rank 3 simple matroid (loopless matroid and without double points) on \( \{0,\ldots,N\} \) and \( \Delta \) a \( \mathbb{Z}^N \)-basis. Suppose that the Bergman fan \( \text{Trop}_\Delta(M) \subset \mathbb{R}^N \) does not contain the ray in the coarse structure in the direction \( u_i \in \Delta \) for some \( i \in \{0,\ldots,N\} \). Then \( \text{Trop}_\Delta(M) \) is one of the following,

1. \( \text{Trop}_\Delta(M) = \mathbb{R}^2 \), so that \( M = U_{3,3} \);
2. \( \text{Trop}_\Delta(M) = \mathbb{R} \times L \), where \( L \) is a fan tropical line, the underlying matroid is \( M = U_{2,N} \oplus \{e\} \);
3. \( \text{Trop}_\Delta(M) \) is the cone over a complete bipartite graph \( K_{k+1,l+1} \) and \( M \) is the parallel connection of the uniform matroids \( U_{2,k+1} \) and \( U_{2,l+1} \), where \( k + l = N \).

**Proof.** Using \( k,l \) from Lemma 2.3. When both \( k,l = 1 \) then we are in Case (1). If exactly one of \( k,l \) is 1 we are in Case (2), otherwise we are in Case (3). This completes the proof of the lemma.

Notice that the parallel connection of the uniform matroids \( U_{2,k+1} \) and \( U_{2,l+1} \) is realizable over characteristic 0, see Figure 2.3. Case (2) of Corollary 2.4 consists of a \( k \) lines through a point and 1 line not through the point. The line arrangement in case (3), consists of \( k + 1 \) lines containing a point \( p \) and \( l + 1 \) lines containing a distinct point \( q \), and a single line containing only the points
Given a Bergman fan of a matroid \( L = \text{Trop}_\Delta(M) \), using \( \Delta \) we may take the closure of \( L \) in \( T^N_\Delta \) or even compactify it in \( TP^N_\Delta \).

**Definition 2.5.** A \( k \) dimensional fan tropical linear space \( L \subset \mathbb{R}^N \) is a tropical cycle equipped with weight \( 1 \) on all of its facets such that \( L = \text{Trop}_\Delta(M) \) for some \( \mathbb{Z}^N \)-basis \( \Delta \) and rank \( k+1 \) matroid \( M \).

A \( k \) dimensional fan tropical linear space \( L \subset T^N_\Delta \), (respectively \( L \subset TP^N_\Delta \)), of sedentarity 0 is a tropical cycle equipped with weight \( 1 \) on all of its facets such that it is the closure of \( L^0 = \text{Trop}_\Delta(M) \subset \mathbb{R}^N \) (respectively \( TP^N_\Delta \)) where \( \Delta = \{-e_1, \ldots, -e_N\} \) and \( M \) is a rank \( k+1 \) matroid \( M \).

A fan tropical linear space is non-degenerate if it is not contained in a subspace of \( \mathbb{R}^N \). Equivalently a fan tropical linear space is non-degenerate if it is the Bergman fan of a matroid without double points.

Set \( \Delta = \{-e_1, \ldots, -e_N\} \), and let \( \bar{L} \) denote the compactification of \( L = \text{Trop}_\Delta(M) \subset \mathbb{R}^N \) in \( TP^N \). Then \( \bar{L} \) defines an arrangement of \( k-1 \)-dimensional fan tropical linear spaces located at the boundary of \( \bar{L} \) which we call \( A_M \), namely \( H_i = \bar{L} \cap TP^N_{\{i\}} \) for all \( i = 0, \ldots, N \). If \( L = \text{Trop}_\Delta(M) \) then the rank function of the matroid \( M \) coincides with the one defined by

\[
\text{rank}(I) = k - \dim(\cap_{i \notin I}(L \cap TP^N_{\{i\}})).
\]

Because of this geometric realization of a matroid \( M \) as a tropical hyperplane arrangement \( A_M \), flats of rank \( k \) of a rank \( k+1 \) matroid will be referred to as points, and flats of rank \( k-1 \) as lines, and so on, when the matroid is not representable over a field.

Tropical manifolds introduced in Section 3.1 are locally modeled on fan tropical linear spaces and coordinate changes are integer affine maps. Because of this we are interested in integer linear maps which act as automorphisms of fan tropical linear spaces.

**Definition 2.6.** An automorphism of a fan tropical \( k \)-plane \( L \subset \mathbb{R}^N \) is a map \( \phi \in GL_N(\mathbb{Z}) \) which preserves \( L \) as a set in \( \mathbb{R}^N \).

An automorphism is trivial if it corresponds to a permutation of the elements of the basis \( \Delta \).

If a fan tropical plane has a trivial automorphism it implies that the underlying matroid \( M \) has automorphisms itself; these are permutations of the ground set preserving the matroid. For example, the whole symmetric group \( S_n \) acts on the uniform matroid \( U_{k,n} \). A non-trivial automorphism of a fan tropical linear space \( L \subset \mathbb{R}^N \) is equivalent to the existence of another choice of \( \mathbb{Z}^N \)-basis \( \Delta' \) for which \( L = \text{Trop}_{\Delta'}(M') \) for a matroid \( M' \). Here \( M' \) and \( M \) may or may not be distinct. Some examples of tropical planes in \( \mathbb{R}^N \) having non-trivial automorphisms were previously presented in [BSL15, Example 2.23].

**Example 2.7.** Consider the arrangement of 6 lines known as the braid arrangement, drawn on the left of Figure 2.3. Up to automorphism of \( \mathbb{CP}^2 \) there is only one such arrangement. We may suppose that the lines can be given by the linear forms \( z_i = 0 \) for \( i = 0, 1, 2 \) and \( z_i - z_j = 0 \) for \( 0 \leq i < j \leq 2 \). The tropicalization of a linear embedding of the complement \( \mathbb{CP}^2 \setminus A \to (\mathbb{C}^*)^5 \) is a fan tropical plane \( P \subset \mathbb{R}^5 \) which is the cone over the Petersen graph [AK06].

The complement \( \mathbb{CP}^2 \setminus A \) can be identified with the moduli space \( M_{0,5} \) of 5-marked rational curves up to automorphism. Similarly, the fan \( P \) is isomorphic to the moduli space of 5-marked complex rational tropical curves, \( M_{0,5} \), see [AK06], [Mik07b]. There is also an action of \( S_5 \) on \( P \cong M_{0,5} \).
induced by permuting the markings of the 5-marked rational tropical curves. This gives the entire group of non-trivial automorphisms of $P$ which is the entire symmetric group $S_5$, see [RSS].

In the example above, the automorphism group of the fan is non-trivial, but it turns out the underlying matroid $M$ is still determined by the fan $P \subset \mathbb{R}^N$. If a fan tropical linear space $L \subset \mathbb{R}^N$ of dimension $k$ is equal to $\text{Trop}_{\Delta}(M)$, it may be the case that $L = \text{Trop}_{\Delta'}(M')$ for a distinct matroid $M'$. The next proposition shows that, when restricting to rank 3 matroids on the same number of elements, this is not possible. In this case, given a 2 dimensional matroidal fan in $\mathbb{R}^N$ the underlying rank 3 matroid is unique, despite the possible existence of non-trivial automorphisms of the fan.

**Theorem 2.8.** Let $M_1, M_2$ be rank 3 simple (containing no loops or double points) matroids on $N + 1$ elements and suppose that $\text{Trop}_{\Delta_1}(M_1) = \text{Trop}_{\Delta_2}(M_2) \subset \mathbb{R}^N$ for some $\mathbb{Z}^N$-bases, $\Delta_1$ and $\Delta_2$, then $M_1, M_2$ are isomorphic matroids.

**Proof.** Denote the ground sets $E_1 = \{0, \ldots, N\}$ and $E_2 = \{0', \ldots, N'\}$. We will show that there is a bijection $f : E_1 \rightarrow E_2$ inducing an isomorphism of matroids. For this we just need to see that $f$ induces a bijection on the rank 2 flats of $M_1$ and $M_2$, in other words of the points of the tropical line arrangements.

Let $P = \text{Trop}_{\Delta}(M)$ and let the graph $G$ be the link of $P$. Notice that $G$ has no 2 valent vertices. The identification of $P$ as the fan of the matroid $M_1$ gives a partial labelling of the vertices of $G$ by the elements of the ground set $E_1$. Any unlabeled vertices correspond to points of the tropical line arrangement (flats of rank 2). Similarly, there is another partial labelling of $G$ by the ground set of $M_2$. Figure 2 shows the graphs $G$ for the line arrangements from Figure 2.3.

Assume that all elements of $E_i$ label vertices of $G$, otherwise we are in the situation of Lemma 2.3 and the topology of $G$ is unique to the matroids listed in Corollary 2.4. In the case when all elements of $E_i$ label the vertices of $G$, it is important to remark that 2 vertices of $G$ corresponding to points of $M_i$ cannot be adjacent in the graph $G$. This is not the case for the graphs $G$ in Figure 2b) and c) since not every line corresponds to a vertex of $G$.

Let $\tilde{V}$ denote the collection of vertices of $G$ which are labeled by exactly one of $E_1$ or $E_2$ but not both. In other words a vertex is in $\tilde{V}$ if it corresponds to a line in $M_1$ and a point in $M_2$ or vice versa. Suppose, without loss of generality, that a vertex in $V' \setminus \tilde{V}$ is labeled by $m$ in $E_1$ and $m'$ in $E_2$.

We claim that either $|\tilde{V}| = 6$ or that the subgraph $\tilde{G}$ on the vertices of $\tilde{V}$ corresponds to the adjacency graph of a finite projective plane. If $\tilde{V} = 2$, then we are in Case (2) of Corollary 2.4 which was already excluded, so $\tilde{V} > 2$. Suppose $v_1, v_2 \in \tilde{V}$ are labeled by elements $E_1$. Then $v_1, v_2$ must either be connected by a unique edge of $G$ or they are both adjacent to a unique vertex $v$ of

**Figure 1.** The braid arrangement from Example 2.7 and examples of arrangements from case (2) and (3) of Corollary 2.4.
\( G \) corresponding to a point of the tropical line arrangement of \( M_1 \). We claim that only the second case is possible and that moreover, \( v \in \tilde{V} \) as well.

Notice that \( v_1, v_2 \) are points in \( M_2 \), therefore they cannot be connected by an edge, and there must be a unique vertex \( v \) incident to \( v_1, v_2 \) in \( G \). This vertex must be labeled by a line in \( M_2 \). The argument above says that \( v \) is labeled by a point of \( M_1 \), so \( v \in \tilde{V} \).

On the other hand \( 2 \) vertices \( v_1, v_2 \in \tilde{V} \) corresponding to points in \( M_1 \) are labeled by elements of \( E_2 \) and thus they must both be incident to a unique vertex which is labeled by \( E_1 \). We can conclude that the subgraph of \( G \) with vertices of \( \tilde{V} \) is the incidence graph of a finite projective plane, or \( |\tilde{V}| = 6 \) and \( G \) is a hexagon.

If it is a finite projective plane, the integer linear map \( \phi \) sending \( \Delta_1 \) to \( \Delta_2 \) cannot be in \( \text{GL}_N(\mathbb{Z}) \) since it is a composition of a permutation matrix with a linear map given by the incidence matrix of a finite projective plane. The latter has determinant \( > 1 \) and so \( \Delta_1 \) and \( \Delta_2 \) cannot both be \( \mathbb{Z}^N \) bases. Therefore, \( |\tilde{V}| = 6 \) and the subgraph of \( G \) on the vertices of \( \tilde{V} \) is a hexagon.

Denote the points of \( M_1 \) corresponding to the vertices of \( \tilde{V} \) labeled by flats \( I, J, K \), and the vertices of \( \tilde{V} \) corresponding to elements of \( E_1 \) by \( i, j, k \). We claim that \( I \cup J \cup K = E_1 \). Suppose \( m \in E_1 \), \( m \neq i, j, k \) and \( m \not\in I \cup J \cup K \). Then in the graph \( G \) the vertex \( v \), labeled by \( m \) must be connected to each of the vertices \( i, j, k \) either by a single edge or by a path consisting of 2 edges with an intermediate vertex which is a labeled by a point of the line arrangement of \( M_1 \). The second case is not possible, since in the labelling of \( G \) given by \( M_2 \), there would be \( 2 \) vertices labeled by points adjacent in \( G \). If \( v \) is adjacent to the vertices labeled by \( i, j, k \), we also obtain a contradiction, since \( v \) is also labeled by a line \( m' \in E_2 \), and \( m' \) and \( i' \) would be contained in \( 2 \) flats of \( M_2 \), (similarly for \( j' \) and \( k' \)). This contradicts the covering axiom for flats of matroids.

Suppose the vertices of \( \tilde{G} \) labeled by elements of \( E_1 \) are labeled \( i, j, k \). The three other vertices of \( \tilde{G} \) are labeled by elements \( i', j', k' \) of \( E_2 \), so that \( i \) and \( i' \) are opposite vertices of the hexagon and similarly for \( j, j' \) and \( k, k' \). Let \( f : E_1 \rightarrow E_2 \) be given by \( i \mapsto i' \), then \( f \) induces a bijection on the flats of \( M_1 \) to flats of \( M_2 \). So the matroids \( M_1 \) and \( M_2 \) are isomorphic. \( \square \)

From the above proof, a matroidal fan has non-trivial automorphisms in \( \text{GL}_N(\mathbb{Z}) \) if and only if there exists a certain configuration in the underlying tropical line arrangement. Call a saturated triangle of an arrangement \( A_M \) a collection of three lines \( L_i, L_j, L_k \), and three points \( p_I, p_J, p_K \) for \( I, J, K \subset \{0, \ldots, N\} \) such that,

\[
L_k \in I \cap J, L_j \in I \cap K, L_i \in J \cap K \quad \text{and} \quad I \cup J \cup K = \{0, \ldots, N\}.
\]

**Corollary 2.9.** A fan tropical plane in \( P \subset \mathbb{R}^N \) has non-trivial automorphisms if and only if the underlying tropical line arrangement contains a saturated triangle.
When a matroid is representable over a field of characteristic 0, an automorphism of its Bergman fan yields a birational automorphism of $\mathbb{P}^2$ regular on the complement of the line arrangement. In this case the above proposition and corollary can be proved using the structure of the Cremona group in dimension 2.

Remark 2.10. It is known that Theorem 2.8 does not hold for matroidal fans of higher dimensions. The first counter-example is a 3-dimensional matroidal fan in $\mathbb{R}^5$. Topologically the fan is the cone over the graph $K_3 \times K_3$, direct product with $\mathbb{R}$. It can be given by two matroids of rank three on 6 elements, namely $U_{2,3} \oplus U_{2,3}$ and $M \oplus \{e\}$ where $U_{2,3}$ is the uniform matroid of rank 2 on 3 elements and $M$ is the matroid of the arrangement of 5 lines drawn in on the left of Figure 1. There are also examples of pairs of connected matroids exhibiting this property.

2.4. One dimensional fan cycles. We restrict to describing 1 dimensional tropical cycles, since we are for the most part interested in cycles in tropical surfaces. Definitions of general tropical cycles in $\mathbb{R}^N$ can be found in [Mik06], [MS].

Definition 2.11. A fan 1-cycle $C \subset \mathbb{R}^N$ is a 1-dimensional rational fan equipped with integer weights on its edges such that

$$\sum_{e \in \text{Edge}(C)} w_e v_e = 0,$$

where $\text{Edge}(C)$ is the set of edges of the fan, $w_e$ is the integer weight assigned to $e \in \text{Edge}(C)$ and $v_e$ is the primitive integer vector in the direction of $e$.

A fan 1-cycle is called a fan tropical curve if all of the weights $w_e$ are positive integers. Cycles with positive weights are also known as effective.

For $\Delta = \{u_1, \ldots, u_N\}$ a choice of $\mathbb{Z}^N$-basis we recall the definition of the degree of $C$ relative to $\Delta$ from [BS15]. Again let $u_0 = \sum_{i=1}^N u_i$. Firstly, the primitive integer vector $v_e$ of an edge $e$ of $C$ can be uniquely expressed as a positive linear combination

$$v_e = \sum_{i=0}^N r_e(i) u_i,$$

where $r_e(i) \geq 0$ and $r_e(i) > 0$ for at most $N$ of the $u_i$.

Definition 2.12. Let $\Delta = \{u_1, \ldots, u_N\}$ be a $\mathbb{Z}^N$-basis of $\mathbb{R}^N$, and $C \subset \mathbb{R}^N$ a fan tropical cycle. Then the degree of $C$ with respect to $\Delta$ is

$$\deg_{\Delta}(C) = \sum_{e \in \text{Edge}(C)} w_e r_e(i),$$

for any choice of $i \in \{0, \ldots, N\}$.

The fact that the above definition does not depend on the choice of $i$ follows from the balancing condition. The above definition of $\deg_{\Delta}(C)$ is equivalent to the multiplicity of the tropical stable intersection of $C$ with $H_{\Delta} \subset \mathbb{R}^N$, where $H_{\Delta}$ is the standard tropical hyperplane with respect to the basis $\Delta$ [BS15, Lemma 3.5]. In other words, $H_{\Delta} = \text{Trop}_{\Delta}(U_{N,N+1})$.

The degree of a fan 1-cycle is dependent on the choice of $\Delta$. Even when the fan 1-cycle is contained in a fan tropical plane $P$, and we consider only $\mathbb{Z}^N$-bases $\Delta$ for which $P = \text{Trop}_{\Delta}(M)$ [BS15, Examples 3.3 and 3.4]. Define the degree of a 1-cycle in a tropical plane to be the minimal of these degrees.

Definition 2.13. The relative degree of a fan tropical 1-cycle $C$ in a fan tropical plane $P \subset \mathbb{R}^N$ is

$$\deg_P(C) = \min_{\Delta \in \mathcal{B}} \{\deg_{\Delta}(C)\},$$
where $\mathcal{B}$ be the collection of $\mathbb{Z}^N$-bases $\Delta$ such that $P = \text{Trop}_\Delta(M)$.

It is also possible to define the degree of a fan 1-cycle in $\mathbb{R}^N$, using the minimum of $\Delta$ degrees,

$$\deg(C) = \min\{\deg_\Delta(C) \mid \Delta \text{ is a } \mathbb{Z}^N\text{-basis of } \mathbb{R}^N\}.$$

Clearly, $\deg(C) \leq \deg_P(C) \leq \deg_\Delta(C)$.

**Definition 2.14.** A fan tropical curve $C \subset P \subset \mathbb{R}^N$ for which $\deg_P(C) = 1$ is said to have degree 1 in $P$.

Intersection numbers of tropical cycles in fan tropical planes have been defined in different places using various methods [AR10, Sha13b, BS15]. Here we will recall the definition presented in [BS15] for the sake of completeness. Recall from Section 2.3, that for a fan tropical plane $P \subset \mathbb{R}^N$ and a $\mathbb{Z}^N$-basis $\Delta$ such that $\text{Trop}_\Delta(M) = P$ we can take the compactification $\overline{P} \subset \text{TP}_\Delta^N$, and obtain a tropical line arrangement with the same rank function as $M$, we denote this line arrangement $\mathcal{A}_M$. For two fan curves $C_1, C_2 \subset P$ we will define intersection multiplicities of the curves in $\overline{P}$ at points $p_I$ of the arrangement $\mathcal{A}_M$. That is to say at flats of rank 2 of the matroid $M$.

**Definition 2.15.** [BS15 Definition 3.1] Let $P \subset \mathbb{R}^N$ be a fan tropical plane and $\Delta = \{u_1, \ldots, u_N\}$ be a $\mathbb{Z}^N$-basis such that $\text{Trop}_\Delta(M) = P$ for some matroid $M$. Given two fan tropical curves $C_1, C_2 \subset P$, let $\overline{C}_1$ denote their compactifications in $\overline{P} \subset \text{TP}_\Delta^N$. Let $p_I \in \overline{P}$ be a point of $\mathcal{A}_M$ and suppose that $\overline{C}_1$ and $\overline{C}_2$ both have exactly one edge containing the point $p_I$. The intersection multiplicity of $\overline{C}_1$ and $\overline{C}_2$ at the point $p_I$ is defined as follows:

1. If $I = \{i, j\}$ choose an affine chart $U_m$ of $\text{TP}_\Delta^N$ where $m \not\in I$ described in Section 2.2. Let $\pi_{i,j}: U_m \to \mathbb{T}^2$ be the map induced by extending the linear projection $\mathbb{R}^N \to \mathbb{R}^2$ with kernel $u_{m'}$ for $m' \neq i, j$. Suppose the ray of $\pi_{i,j}(\overline{C}_1 \cap U_m) \subset \mathbb{T}^2$ has weight $w_1$ and primitive integer direction $(k_1, k_2)$, and similarly the ray of $\pi_{i,j}(\overline{C}_2 \cap U_m) \subset \mathbb{T}^2$ has weight $w_2$ and primitive integer direction $(l_1, l_2)$ then,

$$\langle \overline{C}_1 \cdot \overline{C}_2 \rangle_{p_I} = w_1 w_2 \min\{k_1 l_2, k_2 l_1\}.$$

2. If $|I| > 2$ choose an affine chart, $U_m \ni p_I$ for $m \not\in I$, and a projection $\pi_{i,j}: U_m \to \mathbb{T}^2$ where $i, j \in I$ such that the rays of $\overline{C}_1$ and $\overline{C}_2$ are contained in the union of the closed faces generated by $u_i, u_j$. Then

$$\langle \overline{C}_1 \cdot \overline{C}_2 \rangle_{p_I} = (\pi_{i,j}(\overline{C}_1 \cap U_m) \cdot \pi_{i,j}(\overline{C}_2 \cap U_m))(-\infty, -\infty).$$

The intersection multiplicity is extended by distributivity in the case when $\overline{C}_1$ and $\overline{C}_2$ have more than one ray containing the point $p_I$.

**Definition 2.16.** Let $P \subset \mathbb{R}^N$ be a non-degenerate plane and $\Delta$ a $\mathbb{Z}^N$-basis such that $P = \text{Trop}_\Delta(M)$ for some matroid $M$. The tropical intersection multiplicity of fan tropical 1-cycles $C_1, C_2$ in $P$ at the vertex of the fan $p \in P$ is

$$\langle C_1 \cdot C_2 \rangle_p = \deg_\Delta(C_1) \deg_\Delta(C_1) - \sum_{p_I \in \mathcal{A}(M)} \langle \overline{C}_1 \cdot \overline{C}_2 \rangle_{p_I}.$$

Although a choice of $\mathbb{Z}^N$-basis $\Delta$ is used in the above definition, the multiplicity of 2 fan curves at the vertex of the fan plane $P$ is independent of the choice as long as $P = \text{Trop}_\Delta(M)$ since the above definition is equivalent to the intersection multiplicities of tropical cycles in matroidal fans given in [Sha13b] and [FRI13].
Notice that by definition, fan 1-cycles $\overline{C}_1, \overline{C}_2$ satisfy Bézout’s theorem in the compactification of $P$ to $\overline{P} \subset \mathbb{T}P^N_\Delta$. That is if we define the total intersection multiplicity of fan 1-cycles in $\overline{P}$ to be,

$$\overline{C}_1 \cdot \overline{C}_2 = \sum_{x \in (\overline{C}_1 \cap \overline{C}_2)(0)} (\overline{C}_1 \cdot \overline{C}_2)_x,$$

then we immediately have the following proposition.

**Proposition 2.17.** Let $C_1, C_2$ be fan 1-cycles in a fan tropical plane $P = \text{Trop}_\Delta(M) \subset \mathbb{R}^N$, then

$$\overline{C}_1 \cdot \overline{C}_2 = \deg(\overline{C}_1) \deg(\overline{C}_2),$$

where $\overline{C}_i$ is the closure of $C_i$ in the compactification of $\mathbb{R}^N$ to $\mathbb{T}P^N_\Delta$.

Moreover, the above definition of local intersection multiplicity of 2 fan tropical curves at the vertex of the fan plane $P$ reflects the complex intersection multiplicities in the case when the curves arise as tropicalizations of complex curves $C_1, C_2 \subset P \subset \mathbb{C}^*$, see [BIMS, Theorem 3.8].

### 2.5. Fan Modifications

General tropical modifications were introduced by Mikhalkin in [Mik06]. Here we recall the definitions of this operation for fans in $\mathbb{T}^N$. Eventually, the restriction will be to so-called degree 1 modifications, which produce a new $k$-dimensional fan tropical linear space $L \subset \mathbb{T}^{N+1}$ from a pair of fan tropical linear spaces $D \subset L \subset \mathbb{T}^N$, of dimensions $k - 1$ and $k$ respectively. A more thorough treatment of degree 1 modifications can be found in [Sha13b, Section 2.4]. A general introduction to tropical modifications, and tropical divisors of regular and rational functions can be found in [BIMS, MRB].

Given a tropical variety $V \subset \mathbb{T}^N$ of dimension $k$ and a tropical regular function $f : \mathbb{T}^N \to \mathbb{T}$ its graph $\Gamma_f(V) \subset \mathbb{T}^{N+1}$ is a rational polyhedral complex of dimension $k$ and it inherits weights from the top dimensional facets of $V$. However, since $f$ is only piecewise linear, $\Gamma_f(V)$ may not be balanced. There is a canonical way to add weighted facets to $\Gamma_f(V)$ to produce a tropical cycle $\overline{V}$. At each codimension 1 face $E$ of $\Gamma_f(V)$ which fails the balancing condition attach the facet,

$$F_E = \{(x - te_{N+1} \mid x \in E \text{ and } t \in [0, \infty]\}. $$

Assign to $F_E$ the unique positive integer weight so that the union $\Gamma_f(V) \cup F_E$ is balanced at $E$. After carrying out this procedure for all unbalanced faces $E$ of $\Gamma_f(E)$ call the resulting polyhedral complex $\overline{V}$.

Let $\delta : \mathbb{T}^{N+1} \to \mathbb{T}^N$ denote the linear projection with kernel $e_{N+1}$. Then the restriction to $\overline{V}$, is $\delta : \overline{V} \to \Gamma_f(V)$ is the tropical modification of $V$ along $f$. If $f(x) \neq -\infty$ for all $x \in \mathbb{T}^N$ then the divisor of $f$, $\text{div}_f(V)$ is a tropical cycle, with support $\{x \in V \mid |\delta^{-1}(x)| > 1\}$. The weight of a top dimensional face $E \subset \text{div}_f(V)$ is assigned the same weight as $F_E$ in $\overline{V}$ where $\delta^{-1}(E) = F_E$. If $f(x) = -\infty$ for some $x \in \overline{V}$ then the divisor of $f$ may have additional components in the boundary strata of $\mathbb{T}^N$, here we ignore this case for simplicity, it is treated in [Sha13b] and [BIMS].

A tropical rational function is $f = \text{“}g/h\text{”}$ where $g, h : \mathbb{T}^N \to \mathbb{T}$ are both tropical polynomial functions. For simplicity, again assume that $g(x) \neq -\infty$ for all points $x \in V \subset \mathbb{T}^N$ and similarly for $h$. Define the divisor of $f$ restricted to $V$ by

$$\text{div}_f(V) = \text{div}_g(V) - \text{div}_h(V),$$

If the divisor of $f$ is an effective cycle, we can once again take the graph of $V$ along $f$ and complete it to a tropical variety $\overline{V}$ as above.

**Definition 2.18.** Let $V \subset \mathbb{T}^N$ be a tropical variety and $f = \text{“}g/h\text{”}$ a tropical rational function such that $g, h$ do not both attain $-\infty$ at any point $x \in V$ and that $\text{div}_f(V)$ is effective. The tropical modification of $V$ along $f$ is $\delta : \overline{V} \to V$ where $\overline{V}$ is described above.
**Definition 2.19.** Suppose \( L \subset \mathbb{T}^N \) is a fan tropical linear space and let \( f \) be a tropical rational function on \( \mathbb{T}^N \) such that \( \text{div}_L(f) \) is also a fan tropical linear space in \( \mathbb{T}^N \). Then the tropical modification \( \delta : \tilde{L} \to L \) along \( f \) is said to be a degree 1 modification of \( L \subset \mathbb{T}^N \).

Given a degree 1 fan tropical modification, \( \delta : \tilde{L} \to L \), the tropical cycle \( \tilde{L} \subset \mathbb{T}^{N+1} \) is also a fan tropical plane. In particular, a degree 1 modification of a fan linear space \( L \subset \mathbb{T}^N \) corresponds to a so-called matroidal extension on the underlying matroids [Sha13b, Proposition 2.25].

We can also define open fan tropical modifications of a fan tropical linear space \( L \subset \mathbb{R}^N \). Given an open modification \( \delta^o : \tilde{L} \to L \) along an effective divisor \( D \subset L \), the space \( \tilde{L} \subset \mathbb{R}^{N+1} \) should be thought of as the complement of \( D \) in \( L \), see Section 2.4 of [Sha13b] for more details.

One difference between open fan modifications of degree 1 and the tropical modifications described in Definition 2.19 is that the \( \mathbb{Z}^N \)-basis \( \Delta \) is no longer fixed in the open case; for a fan tropical linear space \( L \subset \mathbb{R}^N \), an open degree 1 modification may be along any effective divisor \( D \subset L \) as long as there exist a \( \mathbb{Z}^N \)-basis \( \Delta \) and matroids \( M, N \) such that \( \text{Trop}_\Delta(M) = L \) and \( \text{Trop}_\Delta(M) = D \).

**Definition 2.20.** Let \( L = \text{Trop}_\Delta(M) \subset \mathbb{R}^N \) be a fan tropical linear space and let \( f \) be a tropical rational function on \( \mathbb{R}^N \) such that \( \text{div}_L(f) = \text{Trop}_\Delta(N) \) for a matroid \( N \), denote \( \tilde{L} \subset \mathbb{R}^{N+1} \) the tropical cycle obtained by completing the graph of \( L \) along \( f \). Then \( \delta^o : \tilde{L} \to L \) is an open degree 1 tropical modification along \( f \), where \( \delta^o \) is induced by the linear projection \( \mathbb{R}^{N+1} \to \mathbb{R}^N \) with kernel generated by \( e_{N+1} \).

Once again the tropical cycle \( \tilde{L} \) in the above definition is a fan tropical linear space in \( \mathbb{R}^{N+1} \). The \( \mathbb{Z}^N \)-basis \( \Delta \) of \( \mathbb{R}^{N+1} \), for which \( \deg_\Delta(L) = 1 \) is obtained by adding \( -e_{N+1} \) to the \( \mathbb{Z}^N \)-basis \( \Delta \) of \( \mathbb{R}^N \) for which \( \deg_\Delta(D) = \deg_\Delta(L) = 1 \).

The next example shows that the local degree of a 1-cycle in a fan tropical plane from Definition 2.13 is not invariant under open degree 1 tropical modifications.

**Example 2.21.** Let \( P = \text{Trop}_\Delta(U_{3,4}) \subset \mathbb{R}^3 \) where \( \Delta = \{-e_1, -e_2, -e_3\} \) see Figure 3. Let \( C \) be the fan tropical curve with rays of weight 1 in directions 
\[
(-2, -1, 0), \quad (1, 0, 1), \quad (1, 1, -1),
\]
shown in red in Figure 3. Then \( C \subset P \).

There is a unique \( \mathbb{Z}^3 \)-basis \( \Delta \) for which \( P = \text{Trop}_\Delta(M) \) for some matroid, and \( \deg_P(C) = \deg_\Delta(C) = 2 \). The tropical 1-cycle \( L \) with support the affine line in direction \((1, 1, 0)\) equipped with weight 1, is of degree 1 in \( P \). There is a tropical rational function \( f \), such that \( L = \text{div}_P(f) \).
Performing the modification of $P$ along $f$ yields a fan tropical plane $\tilde{P} \subset \mathbb{R}^4$. Applying the same modification to the above fan tropical curve $C$ yields the fan tropical curve $\tilde{C} \subset \tilde{P} \subset \mathbb{R}^4$ with directions,

$$(-2, -1, 0, -1), \quad (1, 0, 1, 1), \quad (1, 1, -1, 0).$$

The fan $\tilde{P}$ is a matroidal fan with respect to the $\mathbb{Z}^4$-bases $\Delta_1, \Delta_2$. The degrees of $\tilde{C}$ with respect to these bases are $\deg_{\Delta_1}(\tilde{C}) = 2$ and $\deg_{\Delta_2}(\tilde{C}) = 1$, therefore $\deg_{\tilde{P}}(C) = 1$. See [BS15] Examples 2.3 and 3.4 for details.

**Remark 2.22.** Let $P \subset (\mathbb{C}^*)^3$ be the plane defined by the equation $z_1 + z_2 + z_3 + 1 = 0$, then $\text{Trop}(P) = P$. This plane defines the arrangement of 4 bold lines on the left side of Figure 3 and the $\mathbb{Z}^4$-bases in the above example give two compactifications of the complex plane $\tilde{P} \subset (\mathbb{C}^*)^4$ to $\mathbb{C}P^2$ from [BS15] Example 2.3. The 2 compactifications are related by performing the Cremona transformation at the 3 points $p, q, r$ in Figure 3. The fan tropical curve $C$ is the tropicalization of the conic $C$ drawn with respect to a line arrangement on the left. The image of $C$ under the Cremona transformation is a line.

Given a plane curve $C \subset \mathbb{C}P^2$, consider a linear embedding $\phi : \mathbb{C}P^2 \to \mathbb{C}P^N$. Let $P' = \phi(\mathbb{C}P^2) \cap (\mathbb{C}^*)^N$ and analogously for $C'$. There may be another compactification of $P'$ to a linear space in $\mathbb{C}P^N$ such that the closure of $C'$ in this compactification may have smaller degree than the original plane curve $\text{Trop}(C)$, as is the case in the example above. We may ask what is the minimal degree of a curve $C \subset \mathbb{C}P^2$ which can be obtained by such a procedure. This minimal degree is bounded below by the Cremona degree of a curve. Rational curves in $\mathbb{C}P^2$ of Cremona degree 1 are called rectifiable. There are known examples of rational curves which are not rectifiable see [CC10].

### 3. Tropical surfaces

#### 3.1. Tropical manifolds.

An integer affine map $f : \mathbb{R}^N \to \mathbb{R}^M$ is a composition of an integer linear map and a translation in $\mathbb{R}^M$. Such a map can be given by $M$ tropical monomials $i.e.$

$$(x_1, \ldots, x_N) \mapsto ("a_1 \cdot x^{a_1}", \ldots, "a_M \cdot x^{a_M}");$$

where $(a_1, \ldots, a_M) \in \mathbb{R}^M$ encodes the translation and together the $a_i \in \mathbb{Z}^N$ form an integer $N \times M$ matrix. An integer affine map $f : T^N \to T^M$, is defined to be the extension of an integer affine map $\mathbb{R}^N \to \mathbb{R}^M$.

Tropical manifolds are instances of abstract tropical varieties from [Mik06], or [MZ] (also called tropical spaces) which are locally modeled on fan tropical linear spaces. Just as for tropical spaces, the coordinate changes are restrictions of, possibly partially defined, integer affine maps $T^N \to T^M$.

**Definition 3.1.** A tropical manifold $X$ of dimension $n$ is a Hausdorff topological space equipped with an atlas of charts $\{U_\alpha, \phi_\alpha\}$, $\phi_\alpha : U_\alpha \to V_\alpha \subset T^{N_\alpha}$, such that the following hold,

1. for every $\alpha$ there is an open embedding $\phi_\alpha : U_\alpha \to V_\alpha \subset T^{N_\alpha}$, where $V_\alpha$ is a non-degenerate fan tropical linear space of dimension $n$;

2. coordinate changes on overlaps $\phi_{\alpha_1} \circ \phi_{\alpha_1}^{-1} : \phi_{\alpha_2}(U_{\alpha_1} \cap U_{\alpha_2}) \to \phi_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2})$ are restrictions of (possibly partially defined) integer affine linear maps $\Phi_{\alpha_1\alpha_2} : T^{N_{\alpha_2}} \to T^{N_{\alpha_1}}$;

3. $X$ is of finite type: there is a finite collection of open sets $\{W_i\}_{i=1}^s$ such that $\bigcup_{i=1}^s W_i = X$ and $W_i \subset U_\alpha$ for some $\alpha$ and $\phi_\alpha(W_i) \subset \phi_\alpha(U_\alpha) \subset T^{N_\alpha}$.

Just as with smooth manifolds, we say 2 atlases $\{U_\alpha, \phi_\alpha\}$, $\{U'_\beta, \phi'_\beta\}$ on $X$ are equivalent if their union is also an atlas. An equivalence class of atlases on $X$ has a unique maximal atlas. We often
work with this atlas even if a tropical manifold is defined with a more manageable collection of charts. See [BIMS, Example 7.2] for a discussion of the finite type condition.

Call a tropical manifold of dimension 2 simply a tropical surface. In this case the charts are $\phi_\alpha : U_\alpha \to P_\alpha \subset T^N$, where $P_\alpha$ is a non-degenerate tropical fan plane.

**Example 3.2** (Integral affine manifolds). An integer affine manifold satisfying the finite type condition is also a tropical manifold. In dimension 2, the diffeomorphism type of a compact integer affine manifold is either $S^1 \times S^1$ or the Klein bottle (see Example 3.32). For the orientable case, a tropical structure on $S^1 \times S^1$ can be given by the quotient $\mathbb{R}^2/\Lambda$ where $\Lambda \subset \mathbb{R}^2$ is a lattice of full rank [MZ08].

**Example 3.3** (Tropical toric surfaces). A tropical toric variety $X$ of dimension $n$ is locally modeled on $T^N$ and so it has an atlas $\{U_\alpha, \phi_\alpha\}$ where $\phi_\alpha : U_\alpha \to T^N$. Tropical projective space appeared in the beginning of Section 2. In general, copies of affine space are glued together along tropical monomial maps, which classically are maps in $\text{GL}_N(\mathbb{Z})$. Just as in the classical case a tropical toric variety can be encoded by a simplicial fan $\Sigma \subset \mathbb{R}^N$ and the resulting space is a tropical manifold if and only if $\Sigma \subset \mathbb{R}^2$ is unimodular fan, see Section 3.2 [MRb]. The tropical variety is compact if and only if $\Sigma$ is complete.

**Example 3.4** (Non-singular tropical hypersurfaces in toric varieties). A tropical hypersurface $X_f \subset \mathbb{R}^N$ is the divisor of a tropical polynomial function $f : \mathbb{R}^3 \to \mathbb{R}$, see [RGST05, Mik06]. It is a weighted polyhedral complex dual to a regular subdivision of the Newton polytope of the defining polynomial. If the dual subdivision is *primitive*, meaning each polytope in the subdivision has normalized volume equal to 1, the hypersurface is called non-singular and produces a tropical manifold.

Examples of tropical surfaces in $\mathbb{R}^3$ defined by tropical polynomial functions of degrees 1 and 2 respectively are shown in Figure 4. When $X_f$ is 2 dimensional and non-singular, then locally, up to the action of $\text{GL}_3(\mathbb{Z})$, $X_f$ is the standard tropical hyperplane in $\mathbb{R}^3$ shown in Figure 4.

**Example 3.5** (Products of curves). A non-singular abstract tropical curve is equivalent to a graph equipped with a complete inner metric, [MZ08, BIMS].

Given 2 tropical curves $C_1$ and $C_2$, their product $X = C_1 \times C_2$ is a tropical surface in the sense of Definition 3.1 above. A point of the 0-skeleton of $X$ which is not on the boundary, arises as the product of 2 non-boundary vertices of $C_1$ and $C_2$. The link of every such vertex of $X$ is a complete...
bipartite graph $K_{l_1, l_2}$ where $l_i$ is valency of the corresponding vertex of $C_i$. The local model of $X$ at such a vertex is a fan tropical plane from part (3) of Corollary 2.4.

Using examples of superabundant tropical curves in $\mathbb{R}^N$ for $N \geq 3$ [Spe], [Mik], it is possible construct examples of tropical surfaces in $\mathbb{R}^M$ for $M \geq 4$ which are products of tropical curves and are not approximable in the sense of Definition 5.16 of [BIMS].

3.2. Boundary arrangements. The sedentarity of a point in $\mathbb{T}^N$ from Definition 2.1 is coordinate dependent and does not translate directly to tropical manifolds. However, the order of sedentarity is still well defined since the local models of tropical manifolds are non-degenerate fan tropical linear space and the coordinate changes come from extensions of integer linear maps over $\mathbb{R}$. For a point $x$ in a surface $X$ choose a chart $\phi_x : U_x \rightarrow \mathbb{T}^{N_0}$ such that $x \in U_x$ and define the order of sedentarity of the point $x$ by $s(x) := s(\phi_x(x))$.

**Definition 3.6.** The boundary of a tropical manifold $X$ is

$$\partial X = \{ x \in X \mid s(x) > 0 \}.$$  

The points of sedentarity order 0, or interior points, of $X$ are denoted

$$X^0 = \{ x \in X \mid s(x) = 0 \} = X \setminus \partial X.$$  

An irreducible boundary divisor $D$ of a tropical manifold $X$ is $D^0$ where $D^0$ is a connected component of the set $\{ x \in X \mid s(x) = 1 \}$.

**Example 3.7.** The points of a tropical surface can be classified into 3 types based on their order of sedentarity. There are interior points ($s(x) = 0$), boundary edge points ($s(x) = 1$), and corner points ($s(x) \geq 2$).

Every irreducible boundary divisor is of codimension 1 in $X$. The irreducible boundary divisors of a tropical manifold $X$ form an arrangement $A_X$. When $X$ is a surface call them boundary curves. There is a chart independent notion of sedentarity for points in a tropical manifold in terms of this boundary arrangement.

**Definition 3.8.** The sedentarity of a point $x$ in a tropical manifold $X$ is

$$S(x) = \{ D \mid x \in D \in A_X \} \subset A_X.$$  

Notice that the set of points of sedentarity $I \subset A(X)$ of a tropical manifold $X$ may not be connected.

**Definition 3.9.** A tropical manifold $X$ of dimension $n$ has simple normal crossing boundary divisors $D_1, \ldots, D_k$ if every connected component of the intersection $\bigcup_{i \in I} D_i$ is of dimension $n - |I|$.

**Example 3.10** (Boundary arrangements of linear spaces of dimension 2 in $\mathbb{T}P^N$). In [HJJS09], a tropical linear space $L$ of dimension 2 in $\mathbb{R}^N$ is described by a so-called tree arrangement. The tropical linear space $L$ can be compactified to $\tilde{L} \subset \mathbb{T}P^N$, just as done for fan tropical linear spaces. The boundary arrangement of $\tilde{L}$ is exactly the tree arrangement from [HJJS09].

The tree arrangements of tropicalizations of del Pezzo surfaces from [RSS] can also be seen as boundary divisor arrangements on compactifications of these tropical surfaces. Each such tree is in correspondence with a $(-1)$-curve on the del Pezzo surface.

**Example 3.11.** A tropical hypersurface $X_f \subset \mathbb{R}^N$ from Example 3.4 can be naturally compactified in the tropical toric variety $X(\Sigma)$ from Example 3.3 where $\Sigma$ is the dual fan of the Newton polytope of $f$. Under the same assumptions for non-singularity as in Example 3.4 the compactification $\overline{X_f} \subset X(\Delta)$ is a tropical manifold and the boundary divisors of $\overline{X_f}$ are in correspondence with the facets of $\Delta$ and have simple normal crossings. A collection of boundary divisors intersect if and only if the corresponding facets of the polytope $\Delta$ intersect.
3.3. Cycles in tropical surfaces.

**Definition 3.12.** A 0-cycle $Z$ in a tropical surface $X$ is a finite formal sum of points in $X$ with integer coefficients,

$$Z = \sum_{i=1}^{N} m_i x_i.$$ 

The degree of a 0-cycle is $\deg(Z) = \sum_{i=1}^{N} m_i$.

**Definition 3.13.** A tropical 1-cycle of sedentarity $\emptyset$ in a tropical surface $X$ is a subset $C \subset X$ such that in every chart $\phi_\alpha : U_\alpha \to P_\alpha \subset \mathbb{T}^{N_\alpha}$ there exists a 1-cycle $C_\alpha \subset P_\alpha$ of sedentarity $\emptyset$ with

$$\phi_\alpha(C \cap U_\alpha) = C_\alpha \cap \phi_\alpha(U_\alpha)$$

and the weights on the edges of $C_\alpha$ are consistent on the intersections $U_\alpha \cap U_\beta$.

**Definition 3.14.** A boundary 1-cycle $C$ in a tropical surface $X$ is a finite linear combination of boundary curves of $X$ with integer coefficients.

Given $k$-cycles $A, B$ in a tropical surface $X$ their sum $A + B$ is the tropical $k$-cycle supported on union of the supports $A \cup B$ (with refinements if necessary) along with addition of weight functions when facets coincide. Two cycles are equivalent if they differ by a tropical cycle of weight 0. Denote the set of $k$-cycles in a surface $X$ up to the above equivalence by $Z_k(X)$. Then $Z_k(X)$ forms a group, see [AR10]. The group of tropical 1-cycles in a surface $X$ splits as a direct sum of sedentarity $\emptyset$ cycles and Z-multiples of $D_i$ for every irreducible boundary curve $D_i \in \mathcal{A}_X$,

$$Z_1(X) = Z_1(X^0) \oplus \bigoplus_{i=1}^{s} \mathbb{Z}D_i$$  \hspace{2cm} (3.1)

As before an effective tropical 1-cycle in $X$ is also called a tropical curve. A tropical curve is irreducible if it cannot be expressed as a sum of effective tropical cycles.

3.4. Chern cycles. The $k$th Chern cycle of a tropical variety $X$ is a cycle supported on its codimension $k$-skeleton [Mik06]. The weights of the top dimensional faces of $c_k$ were defined in the case of the canonical class $K_X = -c_1(X)$.

**Definition 3.15.** [Mik06] Given a tropical manifold $X$ of dimension $n$, its canonical cycle $K_X$ is supported on the codimension 1 skeleton $X^{(n-1)}$ of $X$. The weight of a top dimensional face $E \subset X^{(n-1)}$ is given by $w_E = \text{val}(E) - 2$, where $\text{val}(E)$ is the number of facets in $X$ adjacent to $E$.

For 1 dimensional tropical manifolds (tropical curves) this is the canonical class used in [BN07], [GK08], [MZ08] in relation to the tropical Riemann-Roch theorem. In the case of a tropical surface $X$, the canonical cycle is a 1-cycle supported on the 1-skeleton of $X$, the balancing condition is proved here in Proposition 3.16. By the above definition and the direct sum decomposition of the cycle group in Definition 3.1 the canonical class of a surface $X$ splits into a cycle supported on the boundary of $X$ and a cycle $K_X^0$ supported on the closure of the 1-skeleton of the points of sedentarity 0 of $X$. Therefore,

$$K_X = K_X^0 - \partial X,$$

since an edge $E$ of the surface located at the boundary has valency 1, and is equipped with weight $\partial E$.

**Proposition 3.16.** The canonical cycle $K_X$ of a tropical surface satisfies the balancing condition.

**Proof.** The condition is non-trivial to check only at points $x$ in the 0-skeleton $X^{(0)}$ of $X$ of sedentarity 0. A neighborhood of such a point has a chart to some tropical plane $P_x \subset \mathbb{R}^{N_x}$ where $P_x = \text{Trop}_\Delta(M)$ for a matroid $M$ and $\mathbb{Z}^{N_x}$-basis $\Delta = \{u_1, \ldots, u_{N_x}\}$. Let $w_u$ denote the weight in $K_X$ of
the edge in $X^{(0)}$ in the direction $u$. Construction 2.2 determines the number of neighboring facets of an edge in direction $u_i$, so that $w_{ui} = |\{p_I \mid i \in I\}| - 2$ and $w_{ui} = |I| - 2$. Therefore,

$$w_{ui} + \sum_{i \in I} w_{ui} = (|\{p_I \mid i \in I\}| - 2) + \sum_{p_I \mid i \in I} (|I| - 2) = -2 + \sum_{p_I \mid i \in I} (|I| - 1).$$

Since $P_x$ is non-degenerate, $M$ has no double points, and by the covering axiom of flats of a matroid, $\sum_{p_I \mid i \in I} (|I| - 1) = N_x$, since $N_x + 1$ is the number of elements of the ground set of $M$. Let $\Lambda(M)$ be the lattice of flats of $M$. Then the balancing of $K_P$ at $x$ follows since

$$\sum_{I \in \Lambda(M)} w_{ui} u_I = (N_x - 2) \sum_{i=0}^{N_x} u_i = 0.$$

This proves the proposition. \[ \square \]

It is enough to check that $K_X$ is balanced for a tropical surface $X$ to deduce that $K_Y$ is balanced for any manifold $Y$.

**Corollary 3.17.** The canonical cycle $K_X$ of an $n$-dimensional tropical manifold $X$ is balanced.

**Proof.** The balancing condition for $K_Y$ is a local condition on codimension 2 faces of $Y$ of sedentarity $\emptyset$. Let $x$ be a point in a codimension 2 face $E$ and consider a chart $\phi_x : U_x \to V_x$ such that $V_x \cong \mathbb{R}^{n-2} \times P_x$ where $P_x$ is a fan tropical plane. Then $K_X$ is balanced along $E$ if and only if $K_{P_x}$ is balanced at $x$. \[ \square \]

The proof of Proposition 3.16 also shows that for a non-degenerate fan plane $P \subset \mathbb{R}^N$ the degree of $K_P$ in $P$ is $N - 2$. If $P \subset \mathbb{R}^N$ is a fan tropical plane, the self-intersection of $K_P^2$ in $P$ by Definition 2.16 is,

$$K_P^2 = (N - 2)^2 - \sum_{p_I} (|I| - 2)^2,$$

where $p_I$ are the points of the arrangement of the matroid associated to $P$. The next Proposition gives another expression for $K_P^2$.

**Proposition 3.18.** Let $P \subset \mathbb{R}^N$ be a fan tropical plane and let $\text{Edge}(P)$ and $\text{Face}(P)$ denote the set of 1 dimensional and 2 dimensional faces of $P$ respectively,

1. if $P$ contains a linearity space then $K_P^2 = 0$;
2. if $P$ is a product of tropical lines from part (3) of Corollary 2.4 then
   $$K_P^2 = 8 - 4|\text{Edge}(P)| + 2|\text{Face}(P)|;$$
3. otherwise
   $$K_P^2 = 10 + N - 5|\text{Edge}(P)| + 2|\text{Face}(P)| - \sum_{E \in \text{Face}(P)} \sigma(E),$$

where $\sigma(E)$ is defined by

$$\sigma(E)v_E = - \sum_{E, E' \text{ span a face}} v_{E'},$$

and $v_E$ is the primitive integer vector in the direction of an edge $E \in \text{Edge}(P)$.

**Remark 3.19.** Suppose $P = \text{Trop}_\Delta(M)$, if a ray $E \in \text{Edge}(P)$ corresponds to a point of the associated tropical line arrangement $\mathcal{A}_M$ then $\sigma(E) = -1$, and if it corresponds to a line, $L_i$ then $\sigma(E) = 1 - (n - |\{p_I \mid i \in I\}|)$. 


Proof. Statement (1) is clear. For statement (2), if $P \subset \mathbb{R}^N$ is a product of tropical lines, then its link is a complete bipartite graph $K_{k,l}$ where $N = k + l - 2$ and $k, l \geq 3$. Therefore, $\text{Faces}(P) = kl$ and $\text{Edges}(P) = k + l$. Moreover, there are exactly 2 points $p_I$ of the corresponding tropical line arrangement for which $|I| > 2$. Now Equation $3.2$ becomes,

$$K_{k,l}^3 = (N - 2)^2 - \sum_{p_I} (|I| - 2)^2$$

$$= (k + l - 4)^2 - (k - 2)^2 - (l - 2)^2$$

$$= 2|\text{Face}(P)| - 4|\text{Edge}(P)| + 8.$$

Part (3) uses that $N(N + 1) = \sum_{p_I} |I|(|I| - 1)$ for the arrangements of non-degenerate tropical planes, this follows again from the covering axiom of flats of a matroid. Furthermore, from Remark $3.19$ preceding the proof we have

$$\sum_{E \in \text{Edge}(P)} \sigma(E) = N + 1 - |\{p_I \mid |I| \geq 3\}| - \sum_{p_I \text{ s.t. } |I| \geq 3} |I|.$$

Applying this equality, we arrive at the statement in Part (3) above.

We will now give the weights for the points in the top Chern cycle $c_n(X)$ of an $n$ dimensional tropical manifold, using the local matroidal structure. Firstly, for a matroid $M$, let $\chi_M(t)$ denote its characteristic polynomial. The reduced characteristic polynomial of $M$ is

$$\overline{\chi}_M(t) = \frac{\chi_M(t)}{1 - t}.$$

If $\mathcal{A}$ is a complex hyperplane arrangement in $\mathbb{C}P^k$ then the Euler characteristic of the complement is given by $\overline{\chi}_M(1)$. The absolute value of $\overline{\chi}_M(1)$ is also known as the $\beta$-invariant of a matroid, introduced by Crapo [Cra67]. For real hyperplane arrangements, the $\beta$-invariant gives the number of bounded components of the complement [Zas75].

For a tropical manifold let $X^{(0)}$ denote the 0-skeleton of $X$; that is points of $X$ which are in the 0-skeleton of $V_\alpha$ for some chart $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{T}^{N_\alpha}$.

**Definition 3.20.** Let $X$ be an $n$ dimensional tropical manifold with simple normal crossing boundary divisors. Then its top Chern cycle $c_n(X)$ is a 0-cycle supported on $X^{(0)}$. For $x \in X^{(0)}$ let there be a neighborhood $U_x$ and chart $\phi_x : U_x \to V_x \subset \mathbb{T}^{N_x}$ with sedentarity $S(\phi_x(x)) = I$. Then, the multiplicity of $x$ in $c_n(X)$ is

$$m_{c_n}(x)(x) = \overline{\chi}_{M_x}(1)$$

where $M_x$ is a matroid on $N_x - |I| + 1$ elements such that $\text{Trop}_\Delta(M_x) = V_x \cap \mathbb{R}_1^{N_x}$.

For a matroid $M$, $\overline{\chi}_M(1)$ can also be given in terms of the Orlik-Solomon algebra of $M$. From [Zha13] this algebra can be constructed from just the support of the fan $\text{Trop}_\Delta(M)$, see also [Sha, Theorem 2.2.6] for an alternative proof. Therefore the multiplicity of a point $x \in X^{(0)}$ in $c_n(X)$ is independent of matroid chosen to represent $V_x$. For a tropical surface $X$, the following lemma expresses the weight of a point $x$ in $c_2(X)$ without recalling the underlying matroid.

**Lemma 3.21.** Let $x$ be a point in the 0-skeleton of a tropical surface $X$,

1. if $x \in X^{(0)}$ is a point of sedentarity order 2 then,

$$m_{c_2}(x)(x) = 1;$$

2. if $x \in X^{(0)}$ is a point of sedentarity order 1 then,

$$m_{c_2}(x)(x) = 2 - \text{val}(x),$$

where $\text{val}(x)$ is the valuation of $x$.
where \( \text{val}(x) \) is the number edges of sedentarity order 1 adjacent to \( x \) in a chart \( \phi_x : U_x \to P_x \subset \mathbb{T}^{N_x} \);

(3) if \( x \in X^{(0)} \) is a point of sedentarity order 0, then

\[
m_x(c_2(X)) = 2 - N_x + |\text{Face}(x)| - |\text{Edge}(x)|,
\]

where \( \phi_x : U_x \to P_x \subset \mathbb{T}^{N_x} \) for a non-degenerate fan tropical plane \( P_x \subset \mathbb{T}^{N_x} \) and \( \phi_x(x) \) is the vertex of \( P_x \). The sets \( \text{Edge}(x), \text{Face}(x) \) are the sets of 1 and 2 dimensional faces of \( P_x \).

**Proof.** In the case of points of sedentarity order 1 or 2 the statement concerns rank 1 or 2 matroids and can be checked directly.

For part (3), the coefficients of the reduced characteristic polynomial of a matroid \( M \) are given by the Möbius function on \( \Lambda_{dA_M} \), the lattice of flats of the decone \( dA_M \) of the arrangement \( A_M \) associated to \( \text{Trop}(\Delta^x) \), see [Kat]. The decone, \( dA_M \), is the affine arrangement obtained by declaring 1 of the hyperplanes of \( A_M \) to be the hyperplane at infinity and then removing it. Let \( \mathfrak{p}(dA) \) denote the set of points (flats of rank 2) of \( dA_M \), and \( \mathfrak{p}(A) \) denote the set of points of \( A \).

Then,

\[
\bar{\chi}_{M_x}(t) = \sum_{F \in \Lambda_{dA_M}} \mu(F) t^{rk(F)}
\]

\[
\bar{\chi}_{M_x}(1) = 1 - N_x + \sum_{\hat{p}_I \in \mathfrak{p}(dA)} (|I| - 1)
\]

\[
= 1 - 2N_x + \sum_{p_I \in \mathfrak{p}(A)} (|I| - 1).
\]

If all lines of \( A_{M_x} \) correspond to 1-dimensional rays in the support of the fan \( \text{Trop}(M_x) \), then we have

\[
|\text{Edge}(x)| = N_x + 1 + |\{p_I \in \mathfrak{p}(A) \mid |I| \geq 3\}|
\]

\[
|\text{Face}(x)| = |\{p_I \in \mathfrak{p}(A) \mid |I| = 2\}| + \sum_{p_I \in \mathfrak{p}(A) \text{ s.t. } |I| \geq 3} |I|.
\]

These expressions show that (3.3) is equal to (3.6) above.

If there is a line of \( A_{M_x} \) not corresponding to a 1 dimensional ray of the fan then the possible arrangements are described explicitly in Corollary 2.4 and the statement of this lemma can be checked directly in these cases. This completes the proof.

\[\square\]

3.5. **Intersections of 1-cycles in surfaces.** A convenient feature of tropical intersection theory is the ability to calculate intersection products locally and on the level of cycles in many cases. A first example of this is the stable intersection of tropical cycles in \( \mathbb{R}^N \) [RGST05, Mik06]. It is also the case that there is an intersection product defined on the level of tropical cycles in matroidal fans [Sha13b, FR13], which extends to tropical manifolds without boundary.

Here we describe the intersections of 1-cycles in tropical surfaces. Such intersections are defined on the cycle level with the exception of self-intersections of boundary divisors. By abuse of notation we will use \( A \cdot B \) to sometimes denote the 0-cycle of the intersection and also the total degree of the intersection, which is the integer

\[
\sum_{x \in X} (A \cdot B)_x.
\]
3.5.1. Intersections of 1-cycles of sedentarity 0. Given 1-cycles $A$ and $B$ in a tropical surface $X$ and a point $x \in A \cap B$ we may choose an open set $U_x \ni x$ and chart $\phi_x : U_x \to P_x \subset T^{N_x}$ such that $A_x = \phi_x(A \cap U_x)$ and $B_x = \phi_x(B \cap U_x)$ are fan 1-cycles.

Definition 3.22. Let $A, B$ be tropical 1-cycles of sedentarity $\emptyset$ in a tropical surface $X$, then their intersection is the 0-cycle

$$A \cdot B = \sum_{x \in (A \cap B)^{(0)}} (A \cdot B)_x,$$

where $(A \cdot B)_x$ is the intersection multiplicity from Definition 2.16 of $A_x$ and $B_x$ in $P_x$ at $\phi_x(x)$ in the chart $\phi_x : U_x \to P_x \subset T^{N_x}$.

3.5.2. Intersection of a boundary divisor and 1-cycle of sedentarity 0. An irreducible boundary curve $C$ and a non-boundary cycle $A$ in a tropical surface always intersect in a finite collection of points of sedentarity order 1 or 2. The intersection product of $A$ and $C$ is a well defined 0-cycle supported on $A \cap C$.

Definition 3.23. Let $A$ be a cycle of sedentarity $\emptyset$ in a tropical surface $X$ and $C \subset X$ an irreducible boundary curve. We define,

$$A \cdot C = \sum_{x \in A \cap C} (A \cdot C)_x,$$

where,

1. if $x$ is a point of sedentarity order 1 in $A \cap C$ adjacent to an edge $E$ of $A$ then $(A \cdot C)_x = w_E$, where $w_E$ is the weight of $E$;
2. if $x$ is a point of sedentarity order 2 of $X$, choose a neighborhood $U_x$ of $x$ and take a chart $\phi_x : U_x \to P_x \subset T^{N_x}$, suppose $\phi_x(C \cap U_x) \subset T^{N_x}$ is contained $\{x_i = -\infty\}$. Let $v_1, \ldots, v_s$ denote the primitive integer directions of all edges $E_1, \ldots, E_s$ of $\phi_x(A \cap U_x)$ which contain $\phi_x(x) \in T^{N_x}$ and $w_1, \ldots, w_s$ their respective weights. Then

$$(A \cdot C)_x = \sum_{j=1}^{s} w_j(e_i, v_j),$$

where $e_i$ is $i^{th}$ standard basis vector of $\mathbb{R}^{N_x}$.

When $D$ is a boundary divisor, which is not an irreducible boundary curve then extend the above definition by distributivity.

3.5.3. Intersection of boundary divisors.

Definition 3.24. Given distinct irreducible boundary curves $C_1, C_2$ in a tropical manifold $X$, then

$$C_1 \cdot C_2 = \sum_{x \in C_1 \cap C_2} x.$$

Again, the above definition can be extended by distributivity to reducible boundary divisors $D_1, D_2$ as long as the components of $D_1$ are distinct from the components of $D_2$.

3.5.4. Self-intersections of boundary divisors. The self-intersection of a boundary divisor in a tropical surface is not defined on the cycle level. To define self-intersections in this case, we define normal bundles of irreducible boundary curves and take sections. Tropical line bundles on curves were introduced in [MZ08], and more general vector bundles in [All12].

Given an irreducible boundary curve $C$ of a tropical surface $X$ having simple normal crossings with the other boundary divisors of $X$, then a neighborhood of $C$ in $X$ defines a tropical line bundle.
When $C$ does not have simple normal crossings it is possible to alter a neighborhood of $C$ in $X$ to produce its normal bundle, for simplicity we do not describe the construction in this case.

Let $\{U_\beta\}$ be a covering of $C$ in $X$, such that $U'_\beta = C \cap U_\beta$ is simply connected for all $\beta$. Assume also that the atlas is fine enough so that $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R} \times T$ for all $U_\alpha \cap U_\beta \neq \emptyset$. Then for each pair $U_\alpha \cap U_\beta$, the map $\phi_\alpha \circ \phi_\beta^{-1}$ is induced by an integer affine map $\Phi_{\alpha\beta} : \mathbb{R} \times T \to \mathbb{R} \times T$.

**Definition 3.25.** Let $C \subset X$ be an irreducible boundary curve of a tropical surface $X$ having simple normal crossings with the other boundary curves of $X$, let $\{U_\beta\}$ be a covering of $C$ in $X$, satisfying the conditions above. The normal bundle $\pi : N_X(C) \to C$ is the tropical surface given by the quotient $\bigsqcup (U'_\beta \times T) \setminus \sim$ where the equivalence relation is given by identifying points in $U_\alpha \times T$ and $U_\beta \times T$ via the maps $\Phi_{\alpha\beta} : \mathbb{R} \times T \to \mathbb{R} \times T$ from above.

A section of the normal bundle, $\pi : N_X(C) \to C$, is a continuous function $\sigma : C \to N_X(C)$ such that $\pi \circ \sigma = id$, with the additional requirement that the restriction $\sigma|_{U_\beta} : U_\beta \to U_\beta \times T$ is induced by a piecewise integer affine function $T^{N_\beta} \to \mathbb{T}$ in each chart $\phi_\beta : U_\beta \to T^{N_\beta}$. In other words, $\sigma \circ \phi_\beta^{-1}$ is given by a tropical rational function. By Proposition 4.6 of [MZ08] a section of $N_X(C)$ always exists. We can also make the requirement that the section is bounded from below, so that there exists an $M >> 0$ such that $\sigma \circ \phi_\beta^{-1} > -M$.

The graph of a section $\sigma : C \to N_X(C)$ can be made into a balanced tropical 1-cycle in $N_C(X)$, similar to the process of tropical modification from Section 2.5. At each point $x$ of the graph $\sigma(C) \subset N_X(C)$ which is not balanced, in each chart add an edge in the direction towards $C$, equipped with an integer weight. That is, add the half-line $(x, a)$ for $-\infty \leq a \leq \sigma(x)$ in some trivialisation of the bundle. This edge can be equipped with a unique $\mathbb{Z}$-weight $w_x$ so that the resulting 1 dimensional complex satisfies the balancing condition in each chart. Define the degree of a section to be

$$\deg(\sigma) = \sum_{x \in \sigma(C)(0)} w_x.$$ 

Given two sections $\sigma_1, \sigma_2 : N_X(C) \to C$ of the same bundle their degrees are the same by [All12 Lemma 1.20].

**Definition 3.26.** The self-intersection of an irreducible boundary curve $C$ in a tropical surface $X$ is $\deg(\sigma)$, where $\sigma : C \to N_X(C)$ is a section of $N_X(C)$.

3.5.5. **Tropical Cartier divisors.** A Cartier divisor on a tropical space is a collection of tropical rational functions $\{f_\alpha\}$ defined on $T^{N_\alpha}$ where $\phi : U_\alpha \to V_\alpha \subset T^{N_\alpha}$, such that on the overlaps the functions agree up to an integer affine function, see [AR10] or [Mik06]. Every tropical Cartier divisor produces a codimension 1 tropical cycle. The next proposition says that on a tropical manifold $X$ the converse is also true.

**Proposition 3.27.** Every codimension 1 tropical cycle $D$ in a tropical manifold $X$ is a tropical Cartier divisor.

**Proof.** Choose a collection of charts $U_\alpha$ of $X$ so that $D_\alpha = D \cap U_\alpha$ is a fan cycle in a fan linear space $V_\alpha$. In every chart the fan cycle is the divisor of a tropical rational function $f_\alpha$ restricted to the fan linear space [Sha13b Lemma 2.23]. On the overlap $U_\alpha \cap U_\beta$, the cycles $D_\alpha$ and $D_\beta$ agree, therefore $\tilde{f_\alpha} / f_\beta$ is an invertible tropical function on the overlap. So $f = \{f_\alpha\}$ is a Cartier divisor on $X$ and $\text{div}_X(f) = D$. \qed

The intersections of 1-cycles in a surface can also be given in terms of Cartier divisors [AR10] at least when the divisors are of sedentarity 0. For a Cartier divisor $f = \{f_\alpha\}$ on $X$ such that $\text{div}_X(f) = D$ and a 1-cycle $C$, then

$$C \cdot D = \deg(f|_C).$$
The intersection product in terms of Cartier divisors is compatible with the product described above. However, this approach requires first expressing a 1-cycle \( D \) as a Cartier divisor.

3.6. **Rational equivalence.** Tropical rational equivalence was introduced in [Mik06] by way of families. There is another version of tropical rational equivalence coming from bounded tropical rational functions in [AR10], [AHR]. That equivalence relation is finer than the one defined here, see Remark 3.29 for a comparison.

**Definition 3.28.** Let \( A, B \) be tropical cycles in a tropical manifold \( X \), then \( A \) and \( B \) are rationally equivalent if there exists a tropical cycle \( Z \subset X \times \Tp^1 \) such that

\[
\pi_*((X \times \{\infty\}) \cdot Z - (X \times \{-\infty\}) \cdot Z) = A - B,
\]

where \( \pi \) is the projection \( X \times \Tp^1 \rightarrow X \).

For brevity we refer the reader to [Mik06] and [AR10] for the definition of pushforwards \( \pi_* \) of tropical cycles. Also the intersections of sedentarity 0 cycles with boundary divisors is locally determined as for the intersection with boundary divisors in \( T^1 \) from [Sha13b].

**Remark 3.29.** The above definition of rational equivalence is equivalent to taking the equivalence relation given by

\[
\pi_*((X \times \{t_1\}) \cdot Z - (X \times \{t_2\}) \cdot Z) = A - B,
\]

for a cycle \( Z \) and any points \( t_1, t_2 \in T^1 \) [Sha, Proposition 2.1.23]. The tropical rational equivalence from [AHR] restricts \( t_1, t_2 \) to be finite, i.e. \( t_i \neq \pm \infty \) [AHR, Proposition 3.5]. For example, Theorem 4.7 to be proved in Section 4.1 does not hold for this bounded version of rational equivalence.

As in classical algebraic geometry we can define the tropical Chow groups as the quotients of the cycle groups by rational equivalence. Denote the set of \( k \)-cycles rationally equivalent to 0 by \( R_k(X) \). Then \( R_k(X) \) forms a subgroup of \( Z_k(X) \).

**Definition 3.30.** The \( k^{th} \) Chow group of a tropical manifold \( X \) is

\[
CH_k(X) = \frac{Z_k(X)}{R_k(X)}.
\]

3.7. **Tropical \((p,q)\)-homology.** Tropical \((p,q)\)-homology from [IKMZ] is homology of tropical varieties with respect to a coefficient system denoted by \( F_p \) defined by the local structure of the variety. Other references on the subject include [Sha13b], [MZ] and the more introductory [BIMS].

Here we outline the definitions of this homology theory for tropical manifolds which are polyhedral. A tropical manifold \( X \) comes with a combinatorial stratification, see Section 1.5 of [MZ]. Here we assume that \( X \) has a stratification (perhaps a refinement of the combinatorial stratification) which is polyhedral in the following sense.

**Definition 3.31.** [MZ, Definition 1.10] A tropical manifold \( X \) of dimension \( n \) is polyhedral if there are finitely many closed sets \( F_j \subset X \), called facets, such that

- \( X = \cup F_j \);
- for each \( F_j \) there is a chart \( \phi_\alpha : U_\alpha \rightarrow V_\alpha \subset T^{N_\alpha} \) such that \( F_j \subset U_\alpha \) and \( \phi_\alpha(F_j) \) is a polyhedron of dimension \( n \) in \( T^{N_\alpha} \);
- for each collection \( \{F_i\}_{i \in I} \) of facets of \( X \), the intersection \( \cap_{i \in I} F_i \) must be a face of every \( F_j \).

Tropical \((p,q)\)-homology can still be defined when \( X \) is not polyhedral, see for example [BIMS, Section 7].
For a face $E$ of $X$ let $\text{int}(E)$ denote its relative interior. Let $x \in \text{int}(E)$, and suppose $\phi_\alpha(x)$ has sedentarity $I$ in $\mathbb{T}^{N_\alpha}$. Suppose $U_\alpha$ is a neighborhood of a face $E$ and $\phi_\alpha : U_\alpha \to \mathbb{T}^{N_\alpha}$. The $p^{th}$ integral multi-tangent module at $E$ is,

$$\mathcal{F}^p_\alpha(E) = \{v_1 \wedge \cdots \wedge v_p \mid v_1, \ldots, v_p \in \mathbb{Z}^{N_\alpha}_I \text{ and } v_1, \ldots, v_p \in \sigma \subset T_x(X)\} \subset \Lambda^k(\mathbb{Z}^{N_\alpha}_I),$$

and the $p^{th}$ multi-tangent space is $\mathcal{F}^p_\alpha(E) \cong \mathbb{R}$.

For a pair of faces satisfying $E \subset E'$ there are maps $\iota_p : \mathcal{F}^p_\alpha(E') \to \mathcal{F}^p_\alpha(E)$ which are inclusions when $E$ and $E'$ have the same sedentarity, and compositions of quotients and inclusions otherwise. The fact that $\mathcal{F}^p_\alpha(E)$ and $\iota_p$ are uniquely defined follows from the assumption that $X$ is polyhedral.

A tropical $(p, q)$-cell is a singular $q$-cell, respecting the polyhedral structure of $X$, i.e. $\sigma : \Delta_q \to E \subset X$, such that for each face $\Delta' \subset \Delta_q$, $\sigma(\text{int}(\Delta'))$ is contained in the interior of a single face of $E$. Throughout we will abuse notation and use $\sigma$ to also denote the image of the map in $X$. A singular $q$-chain $\sigma$ is equipped with a coefficient $\beta_\sigma \in \mathcal{F}^p_\alpha(E)$ to produce a $(p, q)$-cell, $\beta_\sigma \sigma$. Singular tropical $(p, q)$-chains are finite formal sums of $(p, q)$-cells. Denote the group of $(p, q)$-chains by $C_{p,q}(X; \mathbb{Z})$. For a $(p, q)$-chain $\gamma$ denote its supporting $q$-chain by $|\gamma|$.

The boundary map is the usual singular boundary map

$$\partial : C_{p,q}(X; \mathbb{Z}) \to C_{p,q-1}(X; \mathbb{Z})$$

composed with given maps on the coefficients

$$\iota_p : \mathcal{F}^p_\alpha(E) \to \mathcal{F}^p_\alpha(E') \quad \text{for} \quad E' \subset E,$$

when the boundary of a supporting $q$-cell $\sigma$ is in a face of $X$ different from $\text{int}(\sigma)$. Just as for singular chains with constant coefficients the boundary operator satisfies $\partial^2 = 0$. The tropical integral $(p, q)$-homology groups are the homology groups of this complex, and are denoted $H_{p,q}(X; \mathbb{Z})$.

The $p^{th}$ multi-tangent spaces over $\mathbb{R}$, denoted $\mathcal{F}_p := \mathcal{F}^p_\alpha \otimes \mathbb{R}$ can also be used as coefficients for tropical homology. Denote these groups by $H_{p,q}(X; \mathbb{R})$.

When $X$ is a polyhedral, the tropical homology groups over $\mathbb{R}$ and the cellular cosheaf homology (see [Cun]) are equal [MZ Proposition 2.2]. The cellular $(p, q)$-chain groups of a tropical manifold have the advantage of being finitely generated since $X$ satisfies the finite type condition.

Example 3.32. Let $K$ be a Klein bottle obtained from the product of intervals $[-1, 1] \times [-1, 1]$ by identifying $[-1, 1] \times \{-1\}$ and $[-1, 1] \times \{1\}$ with the same orientation and $\{-1\} \times [-1, 1]$ with $\{1\} \times [-1, 1]$ with the opposite orientation, as drawn in Figure 5. Then $K$ comes with the structure of an integral affine manifold, and so it is a compact non-singular tropical manifold with charts to $\mathbb{R}^2$. It can also be made polyhedral by choosing an appropriate subdivision of $K$. However, the computations are reduced and the same integral $(p, q)$-homology groups are obtained by using the simplicial subdivision of $K$ from Figure 5 along with the groups $\mathcal{F}^p_\alpha$ and maps $\iota_p$ specified below. In the subdivision of $K$ shown in Figure 5 the edges are labeled by $E_i$, the faces by $F_i$, and the single point by $x$. 

![Figure 5. A Klein bottle $K$ with the cellular structure from Example 3.32](image-url)
For every face $F$ of the subdivision of $K$ in Figure 5, the $p$th multi-tangent space is $\mathcal{F}_p^2(F) = \Lambda^p(\mathbb{Z}^2)$. Let $e_1, e_2$ denote the standard basis vectors of $\mathbb{Z}^2$. The maps $\iota_p : \mathcal{F}_p^2(F_1) \to \mathcal{F}_p^2(E_1)$ are the identity for $j = 1, 2$ and $i = 1, 2$ and all $p$. For $E_3$ we have,

\[
\begin{align*}
\iota_1 : \quad & \mathcal{F}_1^2(F_1) \to \mathcal{F}_1^2(E_3) & \iota_2 : \quad & \mathcal{F}_2^2(F_1) \to \mathcal{F}_2^2(E_3) \\
& e_1 \mapsto e_1 & & e_1 \wedge e_2 \mapsto (-1)^i e_1 \wedge e_2.
\end{align*}
\]

Lastly, $\iota_p : \mathcal{F}_p^2(E_i) \to \mathcal{F}_p^2(x)$ are the identity maps for all $i$ and $p$.

The groups $H_{0,q}(K; \mathbb{Z})$ are just the usual homology groups of $K$ with integral coefficients. The complex of cellular $(2, q)$-chains is,

\[
0 \to C_{2,0}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,1}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,2}^{\text{cell}}(K; \mathbb{Z}) \to 0 \quad \text{is} \quad 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0.
\]

The chain groups coincide with the constant coefficient case since $\mathcal{F}_2 = \Lambda^2(\mathbb{Z}^2) \cong \mathbb{Z}$. However, the homology groups differ due to the maps $\iota_p$. The map $C_{2,2}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,1}^{\text{cell}}(K; \mathbb{Z})$ has kernel

\[
\langle (e_1 \wedge e_2)F_1 + (e_1 \wedge e_2)F_2, \rangle
\]

so $H_{2,2}(K; \mathbb{Z}) \cong \mathbb{Z}$. Also, $C_{2,1}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,0}^{\text{cell}}(K; \mathbb{Z})$ has kernel

\[
\langle (e_1 \wedge e_2)E_3, (e_1 \wedge e_2)E_1 + (e_1 \wedge e_2)E_2 \rangle.
\]

Quotienting by the image

\[
\text{Im}(C_{2,2}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,1}^{\text{cell}}(K; \mathbb{Z})) = \langle (e_1 \wedge e_2)E_1 + (e_1 \wedge e_2)E_2 - (e_1 \wedge e_2)E_3 \rangle,
\]

gives $H_{2,1}(K; \mathbb{Z}) = \mathbb{Z}$. Finally, the image of $C_{2,1}^{\text{cell}}(K; \mathbb{Z}) \to C_{2,0}^{\text{cell}}(K; \mathbb{Z})$ is $\langle 2(e_1 \wedge e_2) \rangle$. Therefore, $H_{2,0}(K; \mathbb{Z}) \cong \mathbb{Z}_2$.

To compute the groups $H_{1,q}(K; \mathbb{Z})$ notice that the cellular chain complex splits as the direct sum of 2 chain complexes, one consisting of cells with coefficients $e_1$ and the other with coefficients $e_2$. The complex with coefficients $e_1$ behaves exactly as the constant coefficient case, whereas the complex with coefficients $e_2$ behaves exactly as the $(2, q)$-complex. Therefore $H_{1,q}(K; \mathbb{Z}) = H_{0,q}(K; \mathbb{Z}) \oplus H_{2,q}(K; \mathbb{Z})$. This gives all the integral tropical $(p, q)$-homology groups of $K$.

The tropical $(p, q)$-homology groups over $\mathbb{Z}$ can be arranged in a Hodge like diamond, where the row at height $n$ has entries $H_{p,q}(X; \mathbb{Z})$ for $n = p + q$ with $p$ increasing from left to right. The diamond for $K$ is

\[
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z}_2 + \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 \\
\mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
\end{array}
\]

The $(1, 1)$-homology group of a tropical surface carries an intersection form \cite{Sha}. More generally there are intersection pairings between $(p, q)$-homology groups of tropical manifolds \cite{MZ}.

A pair of $(1, 1)$-cycles $\gamma_1, \gamma_2$ in a tropical surface $X$ are said to intersect transversally if $|\gamma_1| \cap |\gamma_2|$ consists of a finite number of points in the interior of facets of $X$, and each such point $x \in |\gamma_1| \cap |\gamma_2|$ is in the relative interior of just 2 supporting 1-cells which intersect transversally in the usual sense. Suppose $\gamma_1$ and $\gamma_2$ intersect transversally in $X$, then each point $x \in |\gamma_1| \cap |\gamma_2|$ is contained in a facet $F_x$ of $X$. Suppose that $\beta_1(x)$ and $\beta_2(x)$ are the coefficients of $\gamma_1$ and $\gamma_2$ respectively of
the supporting 1-cells $\sigma_1$ and $\sigma_2$ containing $x$. Then consider the volume form $\Omega_{F_x}$ evaluated at $\beta_1(x) \wedge \beta_2(x)$. Define,

$$\gamma_1 \cdot \gamma_2 = \sum_{x \in \gamma_1 \cap \gamma_2} e\Omega_{F_x}(\beta_1(x) \wedge \beta_2(x)) x \in C_0(X, \mathbb{Z}).$$

The coefficient $\epsilon$ is 1 if the orientations of $\sigma_1, \sigma_2$ at $x$ induce the same orientation on $F$ as the ordered vectors $\beta_1(x), \beta_2(x)$ and $\epsilon = -1$ if the orientations are opposite. Alternatively, for $\gamma_1$ and $\gamma_2$ intersecting transversally in $X$, the contribution to $\gamma_1 \cdot \gamma_2$ at $x$ is

$$\epsilon[\Lambda_F : \Lambda_{\beta_1(x)} \oplus \Lambda_{\beta_2(x)}],$$

where $\Lambda_F$ is the integer lattice parallel to $\phi_i(F)$ for some chart and $\Lambda_{\beta_1(x)} \oplus \Lambda_{\beta_2(x)}$ is the sublattice generated by the vectors $\beta_1(x), \beta_2(x) \in F_1(F)$. Notice that $\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_1$ and also that the sign of the intersection multiplicity does not rely on choosing an orientation of facets of $X$.

When $X$ is a compact tropical surface the above product on transversally intersecting $(1, 1)$-cycles descends to homology,

$$\cdot : H_{1,1}(X; \mathbb{Z}) \times H_{1,1}(X; \mathbb{Z}) \to H_0(X, \mathbb{Z}) \cong \mathbb{Z}.$$

See [Shai Section 3.1.4], or the more general [MZ Theorem 6.14].

**Example 3.33.** Returning to Example 3.32 let $\gamma$ be the cycle in $C_{1,1}(K; \mathbb{Z})$ supported on $\sigma$ in Figure 5 and equipped with coefficient $e_1$. Let $\gamma_1, \gamma_2$ be the $(1, 1)$-cycles supported on the 1-cycle $\tau$ in Figure 5 and equipped with coefficients $e_1$ and $e_2$ respectively. Then $\gamma$ intersects both $\gamma_1$ and $\gamma_2$ transversally in $K$ in a single point, $\gamma_1$ and $\gamma_2$ are disjoint. The intersection product on $H_{1,1}(K; \mathbb{Z})$ is

$$\gamma \cdot \gamma = 0 \quad \gamma_1 \cdot \gamma_1 = 0 \quad \gamma \cdot \gamma_2 = 1 \quad \gamma_1 \cdot \gamma_j = 0.$$

Given a tropical 1-cycle $A$ in a compact tropical surface $X$, there is a cycle map which produces a tropical $(1, 1)$-cycle $[A],$

$$[\ ] : Z_1(X) \to Z_{1,1}(X).$$

Firstly, equip each edge $E$ of $A$ with an orientation to obtain a 1-cell $\sigma_E$. These cells form the underlying 1-chain of $[A]$. The coefficient of $\sigma_E$ in $F_1$ is the integer vector parallel to $\sigma_E$ multiplied by the weight of $E$ in $A$. The fact that $[A]$ is a closed chain follows immediately from the balancing condition on $A$ [MZ, Sha]. If $X$ is not compact, then we can construct a cycle $[A]$ in the Borel-Moore version of tropical $(p, q)$-homology of $X$. As usual the Borel-Moore homology groups have the same definitions as the singular $(p, q)$-tropical homology groups except that chains consist of possibly infinite sums of cells satisfying a locally finite condition [BM60].

Say that a $(1, 1)$-cycle $\gamma \in \mathbb{T}^N$ is parallel if each $(1, 1)$-cell $\beta_\gamma \sigma$ has support contained in an affine line of direction $\beta_\gamma$. A $(1, 1)$-cycle in $X$ is parallel if it is parallel in each chart. For example if $A$ is a tropical cycle then $[A]$ is a parallel integral 1-cycle.

The next proposition shows that the intersection product on 1-cycles in a compact surface $X$ from Section 3.3 is numerically equivalent to their product considered as tropical $(1, 1)$-cycles.

**Proposition 3.34.** Let $A, B$ be 1-cycles in a compact tropical surface $X$, then the total degree of the intersection of 1-cycles $A \cdot B$ is equal to the intersection multiplicity $[A] \cdot [B]$ as $(1, 1)$-cycles.

The next lemma considers a local situation for fan tropical cycles in a fan tropical plane $P \subset \mathbb{T}^N$. Since $P$ is not compact we are using Borel-Moore homology.

**Lemma 3.35.** Let $A, B$ be fan 1-cycles of sedentarity $\emptyset$ in a tropical fan plane $P \subset \mathbb{T}^N$. There exist arbitrarily small neighborhoods $U_A, U_B$ of $A, B$ respectively, and $(1, 1)$ Borel-Moore cycles $a, b$ homologous to $[A], [B]$ in $P$ respectively, such that

1. $|a| \subset U_A$ and $|b| \subset U_B$ and $a$ and $b$ intersect transversally in $P$.
Proof. Since contained in the fan $y$

(2) for each $x_i \in (A \cap B)^{(0)}$ there is a neighborhood $U_{x_i}$ of $x_i$ such that $U_{x_i} \subset U_A \cap U_B$, $|a| \cap |b| \subset \bigcup_i U_{x_i}$, and $m_{x_i}(A \cdot B) = \sum_{x'_i(\cap|b|\cap U_{x_i})} m_{x'_i}(a \cdot b)$.

(3) outside of $\bigcup_{x_i(\cap|A\cap B|)(0)} U_{x_i}$, the support $|a|$ is contained in the union of a collection of affine lines which are in bijection with the rays of $A$, and each line in the collection is parallel to the corresponding ray of $A$. Moreover, outside of $\bigcup_{x_i(\cap|A\cap B|)(0)} U_{x_i}$ the $(1,1)$-cycle $a$ is parallel.

Proof. Since $A$ and $B$ are fan cycles, the only point of sedentarity $\emptyset$ in $(A \cap B)^{(0)}$ is the vertex $x$ of the fan $P \subset \mathbb{T}^N$. We prove the lemma only in the case of the vertex $x$ of the fan $P$, omitting the details for points of sedentarity since this case follows in a similar fashion.

Enumerate the rays of $P$ from $1, \ldots, s$ and let $v_i$ denote the primitive integer vector pointing in the direction of the $i^{th}$ ray. By Construction 2.2 each $v_i$ is in a direction of an indicator vector $u_j$, where $u_i = -e_i$ for $i = 1, \ldots, N$ and $u_0 = e_0$. We assume that in $[A]$ and $[B]$ the supporting 1-cells are oriented outwards from the vertex $x$ of $P$. First we will find cycles $a, b$ homologous to $[A], [B]$ respectively, and satisfying conditions (1) and (3) of the lemma, and then compare the intersection multiplicities.

The first step is to move $[A]$ to a homologous $(1,1)$-cycle supported on the 1-skeleton of $P$ in a neighborhood $U_x$ of the vertex of $P$. Let $U_x$ be the intersection of a convex neighborhood of $x$ with $P$, so that $U_x \subset U_A \cap U_B$. On the $i^{th}$ ray of $P$ choose a point $y_i \subset U_x$ along the ray and do this for each $i$. Denote the face of $P$ spanned by vectors $v_i$ and $v_j$ by $F_{i,j}$. Let $r$ be a ray of $A$ in the interior of $F_{i,j}$ and suppose that $r$ is equipped with weight $w_r \in \mathbb{Z}$ and is in direction $v_r = k_i v_i + k_j v_j$ for $p_i, p_j$ relatively prime positive integers. Let $\gamma_i = w_r k_i v_i \tau_i$ be a $(1,2)$-cell where $\tau_i$ is a 2-simplex contained in $F \cap U$ which is the convex hull of 0, $y_i$ and the intersection point of $A$ with the segment from $y_i$ to $y_j$, see the middle of Figure 6.

Orient the simplex $\tau_i$ appropriately, so that $[A] + \partial(\gamma_1 + \gamma_2)$ no longer has support on $r$ in the neighborhood $U_x$, see right side of Figure 6. Apply this procedure to every ray of $A$ contained in the interior of a face of $P$, to obtain a $(1,1)$-cycle $a'$ homologous to $[A]$ which is supported on the 1-skeleton of $P$ in $U_x$ and is homologous to $[A]$.

If a ray of $P$ is in direction $u_l$ for $|l| > 1$, then for each $i \in I$, the 2 dimensional cone spanned by $u_l, u_i$ is also in $P$. Apply the same procedure as above to construct a cycle $a''$, again homologous to $[A]$, which in $U_x$ is supported on the rays of $P$ in the directions $u_i$. The union of these rays is the standard tropical line $L \subset \mathbb{T}^N$, i.e. $L = \text{Trop}(U_{2,N+1})$. The coefficient in $\mathcal{F}_1$ along the ray spanned by $u_i$ is $c_i u_i$, for some constant $c_i$. The closedness of the cycle $a''$ at the vertex $x$ implies that the $c_i$’s are all equal. Calculating this coefficient for just one $i$ shows that it is $d_1 = \deg_{\Delta}(A)$, where $\Delta = \{-e_1, \ldots, -e_{N+1}\}$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{The steps to construct the cycle $a$ from $[A]$ in Lemma 3.35.}
\end{figure}
Finally, the standard tropical line can be translated in Trop$(M)$ for any $M$. Therefore, we can find a homologous cycle $a \sim a'' \sim [A]$ contained in a neighborhood $U_A$ of $A$ and such that in a neighborhood $U'_x \subset U_x$ it coincides with $[d_1L]$.

The intersection of $a$ and $[B]$ may still be 1-dimensional or consist of points contained on the 1-skeleton of $P$ if $B$ has rays in common with $A$ or with the 1-skeleton of $P$. This can be avoided by choosing another representative $b$ of $[B]$ contained in a neighborhood $U_B$, whose edges are still in bijection with the edges of $B$, so that corresponding edges are parallel.

The intersection points of $a$ and $b$ of sedentarity $\emptyset$ come in 2 types, those supported on the cells of $A$ which are segments from the chosen points $y_i$ to $y_j$ and those supported on the segments of $a$ in directions $u_i$. The sum of the multiplicities of all intersection points contained on the rays of $a$ in directions $u_i$ is exactly $d_1d_2$. This is because the intersection points of $B$ and the tropical line $L$ coincide with the intersection points of $B$ and $a$ and the total intersection multiplicity of the former is $d_1d_2$, where $d_2 = \deg_\Delta(B)$.

The other intersection points of $a$ and $b$ are supported on the segments from $y_i$ and $y_j$. For simplicity, suppose a face $F_{ij}$ of $P$ is generated by rays $u_i$ and $u_j$ and contains exactly 1 ray from each of $A$ and $B$ in its interior. Let the ray of $A$ be in direction $k_iu_i + k_ju_j$ and the ray of $B$ be in direction $l_iu_i + l_ju_j$. Then the intersection point of $a$ and $b$ contained in the segment from $y_i$ to $y_j$ is $-w_Aw_B \min\{p_{i2}q_{i2}, q_{i2}p_{j2}\}$, where $w_A, w_B$ are the weights of the rays of $A$ and $B$ in the face $F_{ij}$ respectively. The right side of Figure 6 shows that the orientation induced by the underlying 1-cells of $a$ and $b$ are not consistent with the orientation of the coefficient vectors. The above calculated contribution to the intersection of $a$ and $b$ is equal to the intersection multiplicity, $-(A,B)_{p_{ij}}$ from Definition 2.15. When the rays of $A$ and $B$ are in a face of $P$ spanned by a pair of vectors $u_i$ and $u_j$ the result follows in a similar fashion. Extending by distributivity over all the rays of $A$ and $B$ and comparing with the local intersection multiplicity of $A$ and $B$ at $x$ from Definition 2.10 completes the proof for points in $A \cdot B$ of sedentarity $\emptyset$.

A similar argument works for points of positive order of sedentarity, which completes the proof. □

**Proof of Proposition 3.34.** If at least one of $A$ or $B$ is a boundary cycle find a rational section as in 3.5.4 so that the 1-cycles intersect transversally in $X$. Then the statement follows immediately by applying the cycle map.

We can choose an open covering $\{U_\alpha\}$ of the union $A \cup B$ and charts, $\phi_\alpha : U_\alpha \to P_\alpha \subset \mathbb{T}^N$ such that in each chart $A \cap U_\alpha$ is contained in a fan cycle $A_\alpha$ and similarly $B \cap U_\alpha$ is contained in a fan cycle $B_\alpha$. Moreover, on the overlaps $U_\alpha \cap U_\beta$, we can insist that the image of $A \cap U_\alpha \cap U_\beta$ is a single affine line, and similarly for $B$.

In each chart $\phi : U_\alpha \to P_\alpha$ there are $(1,1)$-cycles $a_\alpha$ and $b_\alpha$ satisfying the conditions of Lemma 3.35 however, $a_\alpha$ and $a_\beta$ may not coincide on the intersections $U_\alpha \cap U_\beta$. On the overlap $U_\alpha \cap U_\beta$ the

![Figure 7](image-url)
cycle \( A \) is supported on an affine line. By choosing an appropriate covering \( \{ U_\alpha \} \), we can assume that \( a_\alpha \) and \( a_\beta \) are supported on 1-chains parallel to the affine line supporting \( A \) in \( U_\alpha \cap U_\beta \). By Lemma 3.35 the coefficient of \( a_\alpha \) in this intersection is also parallel to \( A \) and similarly for \( a_\beta \). Now \( a_\alpha \) and \( a_\beta \) can be patched together on the overlap \( U_\alpha \cap U_\beta \) by patching together the underlying 1-chains \( a_\alpha \) and \( a_\beta \) with 1-chains in the usual way and equipping them with the same coefficient in \( \mathcal{F}_1 \) as the edge of \( A \) to which it is parallel. This can also be done for \( B \). Denote the resulting cycles in \( X \) by \( a \) and \( b \).

We can assume that \( a \) and \( b \) intersect transversally in \( X \). Moreover, intersection points in the overlaps \( U_\alpha \cap U_\beta \) produced from the patching do not contribute to the intersection multiplicity of \( a \) and \( b \). This is because the coefficients in \( \mathcal{F}_1 \) of the underlying \((1,1)\)-cells at these new intersection points are parallel, hence the multiplicity of intersection is 0. Therefore, the comparison of the intersection multiplicities \( A \cdot B \) with \( a \cdot b \) follows from the local comparison in the previous lemma.

The following proposition is a direct extension of [Zha, Lemma 4].

**Proposition 3.36.** Suppose \( A, A' \) are rationally equivalent 1-cycles in a tropical surface \( X \), then \([A],[A']\) are homologous \((1,1)\)-cycles.

**Corollary 3.37.** Suppose \( A, A', B \) are 1-cycles in \( X \) such that \( A \) and \( A' \) are rationally equivalent, then \( A \cdot B = A' \cdot B \).

**Proof.** If \( A \) and \( A' \) are rationally equivalent then \([A],[A']\) are homologous. By Proposition 3.36 \( A \cdot B = [A] \cdot [B] = [A'] \cdot [B] = A' \cdot B \).

4. Constructing tropical surfaces

In this section we describe 2 operations for constructing tropical surfaces; modifications and summations.

4.1. Tropical Modifications.

**Definition 4.1 ([Mik06]).** Let \( X \) and \( Y \) be tropical manifolds. A map \( f : X \to Y \) is a tropical linear morphism if for every point \( x \in X \) there is a neighborhood \( U_x \) of \( x \) and \( U_y \) of \( y = f(x) \) with charts \( \phi_x : U_x \to V_x \subset \mathbb{T}^{N_x} \), \( \phi_y : U_y \to V_y \subset \mathbb{T}^{N_y} \) such that \( \phi_y \circ f \circ \phi_x^{-1} : V_x \subset \mathbb{T}^{N_x} \to V_y \subset \mathbb{T}^{N_y} \) is an integer affine map \( \mathbb{T}^{N_x} \to \mathbb{T}^{N_y} \).

The following is a restriction of modifications of tropical varieties from [Mik06] to tropical manifolds.

**Definition 4.2.** Let \( \hat{X} \) and \( X \) be a pair of tropical manifolds. A tropical modification is a tropical morphism \( \delta : \hat{X} \to X \) such that, if in borrowing from notation of Definition 4.1, at every point \( x \in \hat{X} \) the composition \( \phi_x \circ \delta \circ \phi_x^{-1} : V_x \subset \mathbb{T}^{N_x} \to V_x \subset \mathbb{T}^{N_x} \) is a degree 1 tropical modification.

**Definition 4.3.** A divisor \( D \subset X \) is locally degree 1 if all of its facets are weight 1 and there exists an atlas \( \{ U_\alpha, \phi_\alpha \} \) with \( \phi_\alpha : U_\alpha \to V_\alpha \) and \( \phi_\alpha : D \cap U_\alpha \to D_\alpha \) is an open embedding such that \( D_\alpha = \text{Trop}_{\Delta_\alpha}(N_\alpha) \) and \( X_\alpha = \text{Trop}_{\Delta_\alpha}(M_\alpha) \) for a pair of matroids \( N_\alpha, M_\alpha \).

Recall from Lemma 3.27 that every codimension 1 tropical cycle in a tropical manifold is a Cartier divisor.

**Proposition 4.4.** If \( D \) is a locally degree 1 divisor in a tropical manifold \( X \), then there exists a tropical manifold \( \hat{X} \) and a modification \( \delta : \hat{X} \to X \) along \( D \).
Proof. Let \( g = \{ g_α \} \) be the tropical Cartier divisor such that \( \text{div}_X(g) = D \). In each chart \( φ_α : U_α \to V_α \subset \mathbb{T}^{N_α} \) there is a degree 1 modification along \( D_α \), given by \( δ_α : \tilde{V}_α \to V_α \) given by the tropical rational function \( g_α \). To construct the surface \( \tilde{X} \) from the atlas \( \{ U_α, φ_α \} \) for \( X \), first set \( \tilde{U}_α = δ_α^{-1}(φ_α(U_α)) \). The sets \( \tilde{U}_α \) come with charts \( \tilde{φ}_α : \tilde{U}_α \to V_α \), and maps \( \tilde{Φ}_{αβ} : ϕ_β(\tilde{U}_α \cap \tilde{U}_β) \to φ_α(\tilde{U}_α \cap \tilde{U}_β) \). The maps \( \tilde{Φ}_{αβ} \) are restrictions of integer affine functions \( \mathbb{T}^{N_β+1} \to \mathbb{T}^{N_α+1} \) given by

\[
(x_1, \ldots, x_{N_α+1}) \mapsto \left( (Φ_{αβ})_1(x_1, \ldots, x_{N_α}), \ldots, (Φ_{αβ})_{N_β}(x_1, \ldots, x_{N_α}), \left[ \frac{x_{N_α+1}g_α}{g_β} \right] \right).
\]

The topological space \( \tilde{X} \) is defined by quotienting the disjoint union \( \sqcup_α \tilde{U}_α \) by the relation \( x \sim y \) if \( \tilde{Φ}_{αβ} \circ φ_α(x) = \tilde{ϕ}_β(y) \). The above relation is indeed an equivalence relation since the functions \( \tilde{Φ}_{αβ} \) satisfy:

\[
\tilde{Φ}_{αβ} \circ ϕ_α = Id \quad \text{and} \quad \tilde{Φ}_{γα} \circ \tilde{Φ}_{αβ} = \tilde{Φ}_{γβ}.
\]

The above relations hold since they hold for \( Φ_{αβ} \) and the functions \( g_α \) form a tropical Cartier divisor. The quotient \( \tilde{X} \) is Hausdorff and is equipped with charts to fan tropical linear spaces. Moreover it satisfies the finite type condition since so does \( X \), thus \( \tilde{X} \) is a tropical manifold.

**Definition 4.5.** Given a tropical modification of manifolds \( δ : \tilde{X} \to X \) there is the pushforward map \( δ_* : Z_*(\tilde{X}) \to Z_*(X) \) and pullback map \( δ* : Z_*(X) \to Z_*(\tilde{X}) \) on the cycle groups.

The definition of these maps in a local chart are given in [Sha, Definition 2.15]. To summarise, the pushforward of a cycle \( δ_A \) is supported on the image \( δ(A) \). The weights of a facet \( F \) of \( δ_A \) are defined by:

\[
w_{δ_A}(F) = \sum_{F_i \subset T, δ(F_i) = F} w_A(F_i) \Lambda_{\tilde{F}_i} \Lambda_F,
\]

where \( \Lambda_{\tilde{F}_i} \) is the image under \( δ \) of the integer lattice parallel to \( F_i \) and \( \Lambda_F \) is the integer lattice parallel to \( F \). If the modification \( δ : \tilde{X} \to X \) is along the Cartier divisor \( g = \{ g_α \} \) then the pullback of a cycle \( A \) is given by restricting the modification along \( g \) to \( A \). As in the local case, \( δ_*δ^* \) is the identity map on \( Z_*(X) \).

**Definition 4.6.** Let \( δ : \tilde{X} \to X \) be a modification of tropical manifolds, for a cycle \( A \subset X \) call \( \tilde{A} \subset \tilde{X} \) a lift of \( A \) to \( \tilde{X} \) if \( δ_*\tilde{A} = A \).

**Theorem 4.7.** Given a tropical modification \( δ : \tilde{X} \to X \) of tropical manifolds the pushforward and pullback maps induce isomorphisms of the Chow groups. In other words,

\[
δ_* : CH_k(\tilde{X}) \to CH_k(X) \quad \text{and} \quad δ^* : CH_k(X) \to CH_k(\tilde{X}),
\]

are isomorphisms.

**Proof.** The modification \( δ : \tilde{X} \to X \) can be extended to a modification on the products \( δ : \tilde{X} \times \mathbb{T}P^1 \to X \times \mathbb{T}P^1 \). For a family \( B \subset X \times \mathbb{T}P^1 \) we may pullback the cycle \( δ^*B \) to obtain a family in \( \tilde{X} \times \mathbb{T}P^1 \). Similarly, the pushforward map can be applied to any family \( B \subset \tilde{X} \times \mathbb{T}P^1 \). Therefore, the pushforward and pullback, \( δ_* \) and \( δ^* \) descend to homomorphisms on the Chow groups.

The composition \( δ_*δ^* \) is already the identity map on the cycle level. To establish the isomorphism we must show that for a cycle \( A \in Z_k(\tilde{X}) \), the cycle \( δ^*δ_*A \) is rationally equivalent to \( A \). In other words, any two lifts of a cycle in \( \tilde{X} \) to \( X \) are rationally equivalent. Notice that the difference \( Δ_A = A - δ^*δ_*A \) is contained in the closure of the faces contracted by the modification \( δ \), so this cycle can be “translated” in \( \tilde{X} \). More precisely, for \( t \geq 0 \), identify \( Δ_A \) with its image in a chart \( φ_α : U_α \to V_α \subset \mathbb{T}^{N_α+1} \) and define for \( t \geq 0 \) a cycle in \( \tilde{X} \) given by:

\[
B_t \cap \tilde{V}_α = \{ x - t e_{N_α+1} \mid x \in Δ_A \cap \tilde{V}_α \}.
\]
To construct a family $B$, first over $[-\infty, \infty]$ take the product $B_0 \times [-\infty, 0]$, and let the fiber over $t \in [0, \infty]$ be $B_t$ from above. Let $\tilde{B}$ be the union of all of these fibers. In other words $\tilde{B}$ is the image of a map $\Delta_A \times \mathbb{T}P^1 \to X \times \mathbb{T}P^1$ which is piecewise linear in each chart and the locus of non-linearity is exactly at $\Delta_A \times \{0\} \subset \Delta_A \times \mathbb{T}P^1$. Therefore, $\tilde{B}$ violates the balancing condition exactly at its codimension 1 faces which are facets of $\Delta_A \times \{0\}$. For every top dimensional face of $\Delta_A$, add to $\tilde{B}$ the set,

$$\{(x + te_{N_{\alpha}+1}, 0) \mid x \in \Delta_A \cap \tilde{V}_{\alpha} \text{ and } t \in [-\infty, 0]\} \subset X \times \{0\}$$

in each chart. Let $B$ denote this union. Equip all facets $B \setminus \tilde{B}$ with weight $-1$ then $B$ is balanced in every chart and is a tropical cycle in $X \times \mathbb{T}P^1$.

By construction, $B(-\infty) = A - \delta^* \delta_* A$ and since $\delta_* (A - \delta^* \delta_* A) = 0$ we have $B(\infty) = 0$. Therefore, $A$ and $\delta^* \delta_* A$ are rationally equivalent and the Chow groups of $\tilde{X}$ and $X$ are isomorphic. \hfill \Box

**Corollary 4.8.** Let $\delta: \tilde{X} \to X$ be a tropical modification of tropical surfaces. Suppose $C_1, C_2$ are 1-cycles in $X$, then numerically we have,

$$C_1 \cdot C_2 = \delta^* C_1 \cdot \delta^* C_2.$$

Furthermore, if $\tilde{C}_1, \tilde{C}_2$ are 1-cycles in $\tilde{X}$ then numerically we have,

$$\tilde{C}_1 \cdot \tilde{C}_2 = \delta_* \tilde{C}_1 \cdot \delta_* \tilde{C}_2.$$

**Proof.** For cycles of sedentarity 0, the first statement follows immediately from Definition 3.6 of [Sha13b] of the local intersection products in matroidal fans. If one or both of $C_1, C_2$ are boundary curves, the statement follows also from the definitions of intersections given in [3.3]. Now given lifts $\tilde{C}_1, \tilde{C}_2$ of cycles $C_1, C_2$ respectively, then $\delta^* C_i$ and $\tilde{C}_i$ are rationally equivalent by the proof of Theorem 4.7. Corollary 3.37 combined with the previous statement gives $\tilde{C}_1 \cdot \tilde{C}_2 = \delta^* C_1 \cdot \delta^* C_2 = C_1 \cdot C_2$. \hfill \Box

**Corollary 4.9.** Let $\delta: \tilde{X} \to X$ be a modification of tropical manifolds, then $\delta_* K_{\tilde{X}} = K_X$, moreover, $K_{\tilde{X}}$ is rationally equivalent to $\delta^* K_X$. If $\tilde{X}$ and $X$ are surfaces we have equality of intersection numbers $K_{\tilde{X}}^2 = K_X^2$.

**Lemma 4.10.** Let $\delta: \tilde{X} \to X$ be a modification of tropical surfaces then, $\delta_* c_2(\tilde{X}) = c_2(X)$, in particular $\deg(c_2(X)) = \deg(c_2(\tilde{X}))$.

**Proof.** Notice that for a point in $X$ there are at most 2 points $x_1, x_2 \in \tilde{X}^{(0)}$ such that $\delta(x_i) = x$. Using Lemma 3.21 it can be directly checked that the multiplicity in $c_2(X)$ of a point $x \in X$ is equal to the sum of the multiplicities $m_{x_1}(c_2(\tilde{X})) + m_{x_2}(c_2(\tilde{X}))$. This proves the lemma. \hfill \Box

Tropical modifications can be used to show that any locally degree 1 curve $C$ in a compact tropical surface satisfies the adjunction formula.

**Theorem 4.11 (Tropical Adjunction Formula).** An locally degree 1 tropical curve $C$ in a compact tropical surface $X$ satisfies

$$b_1(C) = \frac{K_X \cdot C + C^2}{2} + 1,$$

where $b_1(C)$ is the 1st Betti number of $C$.

**Proof.** Suppose $C$ is an irreducible boundary curve of $X$. Then we may write:

$$K_X = K_{\tilde{X}}^2 - C - \sum_{i=1}^d D_i$$
where the irreducible boundary curves of \( X \) are \( C, D_1, \ldots, D_d \). Then (4.1) becomes

\[
b_1(C) = \frac{K_C^0 \cdot C - \sum_{i=1}^d D_i \cdot C}{2} + 1.
\]

This equality is verified by a simple Euler characteristic computation as follows. For every leaf \( l \) (1-valent vertex) of the curve \( C \), there is a collection of boundary curves \( D_i \) which meet \( C \) at this point. We construct a new graph \( G \) from \( C \) by adding to each leaf \( l \) of \( C \) an edge for each boundary curve which meets \( C \) at \( l \). Let \( V(G) \) and \( E(G) \) denote the edge and vertex sets of the graph \( G \), and \( \text{val}_C(x) \) denote the valency of \( x \in V(G) \) in \( G \). Definitions 3.22 and 3.23 give

\[
K_C^0 \cdot C = \sum_{x \in V(C)} \text{val}_C(x) - 2 \quad \text{and} \quad \sum_{i=1}^d D_i \cdot C = |L(G)|.
\]

(4.2)

So that,

\[
K_X^0 \cdot C - \sum_{i=1}^d D_i \cdot C = \sum_{x \in \text{v}(G)} \text{val}_C(x) - 2 = 2b_1(G) - 2,
\]

and \( b_1(C) = b_1(G) \). This proves the statement when \( C \) is an irreducible boundary curve.

When \( C \) is not a boundary curve there is a tropical modification \( \delta : \tilde{X} \to X \) along \( C \) by Proposition 4.4. The tropical surface \( \tilde{X} \) has a boundary curve \( \tilde{C} \) such that \( \delta_1(\tilde{C}) = C \), and moreover \( b_1(C) = b_1(\tilde{C}) \). Since \( \tilde{C} \subset \tilde{X} \) is a boundary curve it satisfies the adjunction formula by the above argument. Corollaries 4.8 and 4.9 give \( \tilde{C}^2 = C^2 \) and \( K_{\tilde{X}} \cdot \tilde{C} = K_X \cdot C \). Thus \( C \subset X \) satisfies the tropical adjunction formula as well. \( \square \)

**Remark 4.12.** The tropical adjunction formula from Theorem 4.11 does not hold for tropical curves in general. In fact, the right hand side of the above formula is neither an upper nor lower bound for the \( 1^{\text{st}} \) Betti number of a tropical curve in a surface. Already for fan tropical curves in a fan tropical plane the the right hand side of the equation from Theorem 4.11 can be greater than or less than 0, see [BS15]. Moreover, the condition that \( C \) be locally degree 1 in \( X \) is not a necessary condition to satisfy the tropical adjunction formula. For example the fan tropical curve \( C \) from Example 2.21 is not locally degree 1, yet it satisfies the tropical adjunction formula in the compactification of the fan plane \( \mathcal{P} \subset \mathbb{R}^4 \) to \( \mathcal{P} \subset \mathbb{T} \mathcal{P}^4 \).

Lastly, given a tropical modification of manifolds, \( \delta : \tilde{X} \to X \), we show that the tropical \((p, q)\)-homology groups of \( \tilde{X} \) and \( X \) are isomorphic. Firstly, there is a map on the chain level

\[
\delta_* : C_{p,q}(\tilde{X}; \mathbb{Z}) \to C_{p,q}(X; \mathbb{Z}).
\]

Suppose that a \((p, q)\)-cell \( \beta_{\sigma} \sigma \) in \( C_{p,q}(\tilde{X}; \mathbb{Z}) \) is such that \( \sigma \) is contained in a single chart \( \tilde{U}_\alpha \to \tilde{V}_\alpha \) of \( \tilde{X} \), and \( \delta \) is given by \( \delta_\alpha : \tilde{V}_\alpha \to V_\alpha \) in that chart. Recall that \( \delta_\alpha : \mathbb{T}^{N_{\alpha+1}} \to \mathbb{T}^{N_{\alpha}} \) is the extension of the linear map \( \mathbb{R}^{N_{\alpha+1}} \to \mathbb{R}^{N_{\alpha}} \) which has kernel generated by \( e_{N_{\alpha+1}} \). Then \( \delta_\alpha(\beta_\sigma \sigma') = \beta'_{\sigma'} \sigma' \) where \( \sigma' = \delta_\alpha(\sigma) \in X \) and \( \beta'_{\sigma'} \) is the image of the coefficient \( \beta_\sigma \) under the map \( F^\mathbb{Z}_{\mathcal{P}}(\tilde{X}) \to F^\mathbb{Z}_{\mathcal{P}}(X) \) induced by the linear projections \( \delta_* \). In fact, in each chart the following diagram commutes:

\[
\begin{array}{ccc}
F^{\mathbb{Z}}_{\mathcal{P}}(E) & \xrightarrow{\iota^p} & F^{\mathbb{Z}}_{\mathcal{P}}(E') \\
\delta_* \downarrow & & \delta_* \downarrow \\
F^{\mathbb{Z}}_{\mathcal{P}}(\delta(E)) & \xrightarrow{\iota^p} & F^{\mathbb{Z}}_{\mathcal{P}}(\delta(E'))
\end{array}
\]
which implies $\delta_s \partial = \partial \delta_s$ and so the map $\delta_s$ of chain groups descends to a homomorphism on the tropical $(p, q)$-homology groups.

For $p = 0$, the map $\delta_s$ gives an isomorphism on homology since the $(0, q)$ groups are just the singular homology and $X$ is a strong deformation retract of $\tilde{X}$. When $p > 0$, showing that $\delta_s$ induces an isomorphism of homology groups, follows the standard argument in basic algebraic topology while keeping track of the coefficients.

**Theorem 4.13.** Let $\delta : \tilde{X} \to X$ be a tropical modification of tropical spaces, then

$$H_{p,q}(\tilde{X}; \mathbb{Z}) \cong H_{p,q}(X; \mathbb{Z})$$

for all $p, q$.

**Proof.** The tropical space $X$ can be identified as a subspace of $\tilde{X}$ by taking $f_\alpha(V_\alpha) \subset \tilde{V}_\alpha$ in each chart. We denote this map by $f : X \to \tilde{X}$. There is a strong deformation retract $F : \tilde{X} \times I \to \tilde{X}$ such that $F : \tilde{X} \times \{0\} \to \tilde{X}$ is the identity and the image of $F : \tilde{X} \times \{1\} \to \tilde{X}$ is in $f(X)$. Here each point in $f(X)$ is fixed by $F$ for all $t \in I$. Moreover the homotopy respects the stratification of $\tilde{X}$ up until the end when in each chart $U_\alpha \to V_\alpha \subset T^{N_\alpha + 1}$, the points of sedentarity $N_\alpha + 1$ are sent to $f(D)$.

For homology with constant coefficients recall the prism operator $P : C_q(\tilde{X}; \mathbb{Z}) \to C_{q+1}(\tilde{X}; \mathbb{Z})$, see Theorem 2.10 [Hat02] for details. For example, in our particular case, if $\sigma$ is a singular $q$-cell contained in $\tilde{X} \setminus f(X)$ and contained in a chart $\tilde{U}_\alpha \to \tilde{V}_\alpha \subset T^{N_\alpha + 1}$, then the image of the prism operator applied to $\sigma$ is

$$P(\sigma) = \{x + te_{N_\alpha + 1} \mid t \in [0, f_\alpha(\delta(x))])\},$$

where $f_\alpha$ is the tropical rational function $f_\alpha : \mathbb{T}^{N_\alpha} \to \mathbb{T}$ defining the modification in the chart $\phi_\alpha : U_\alpha \to V_\alpha \subset T^{N_\alpha}$. In homology with constant coefficients, the prism operator satisfies the relation,

$$\partial P = id - f_* - P \partial. \quad (4.3)$$

To extend this to the situation of $(p, q)$-cells, we need only to assign coefficients in $\mathbb{F}_p$ to the cells in $P(\sigma)$ in a way that maintains the relation (4.3). If $\beta_\sigma \sigma$ is a $(p, q)$-cell in $\tilde{X} \setminus f(X)$, assign to each face of $P(\sigma)$ the coefficient $\beta_\sigma$. This is possible since either $\sigma$ and $P(\sigma)$ are contained in the same face of $\tilde{X}$; or in a chart $\sigma$ is contained in a face $E \subset T^{N_\alpha + 1}$ where $N_\alpha + 1 \in I$, and $P(\sigma)$ is contained in an adjacent face $E' \subset T^{N_\alpha + 1}$ where the $I' = I \setminus \{N_\alpha + 1\}$. In this second case, there is still an injection $\mathbb{F}_p(E) \to \mathbb{F}_p(E')$, so $\beta_\sigma \in \mathbb{F}_p(E')$.

This assignment of coefficients still satisfies the relation in (4.3). So by the usual argument in algebraic topology the map $\delta_s$ induces an isomorphism of the $(p, q)$-homology groups. \qed

### 4.2. Summation of tropical surfaces

Given tropical surfaces $X_1, X_2$ each with boundary curves $C_1, C_2$ satisfying certain conditions we define their tropical sum, which is a new tropical surface obtained by “glueing” together $X_1$ and $X_2$. This construction is analogous to the fibre sum for manifolds or the symplectic sum in the symplectic category [Gom95].

The tropical sum of $X_1$ and $X_2$ depends firstly on a chosen identification of the boundary curves $C_1, C_2$ as well an orientation reversing isomorphism of their normal bundles. This information determines a 1-parameter family of tropical surfaces, similar to the situation for symplectic sums.

Recall, that an abstract tropical curve is a graph with a complete inner metric defined away from the 1-valent vertices, see [BIMS] Section 7.2.

**Definition 4.14.** An isomorphism of tropical curves is an isometry of metric graphs $f : C_1 \to C_2$. 

Proof. Proposition 4.16. The sum spectively, so that the tropical sum is also the quotient of the disjoint union $U$ from open coverings of $X$. This cover comes with charts $\phi_{\alpha}$ and coordinate changes on the overlaps $U_\alpha \cap U_\beta$ either induced by the coordinate changes from $X_1$ or $X_2$ or from $\tilde{g}$. The quotient is a Hausdorff topological space.

To verify the finite type condition, let $\{W_i\}$ be the finite open cover of $X_1$ and $\{Y_j\}$ be a finite open cover of $X_2$. Simply intersecting $W_i \cap U_1$ and $Y_j \cap U_2$ may not provide a finite open cover

![Figure 8. A depiction of the sum of 2 surfaces $X_1, X_2$ along 2 curves $C_1, C_2$ where $f_i : \mathbb{P}^1 \to C_i$ sends $0 \mapsto a_i$ and $\tilde{g}(\sigma_1) = \sigma_2$.](image)
satisfying the finite type condition. For example, if $\phi_\alpha(W_i \cap U_1) \not\subset \phi_\alpha(U_0 \cap U_1) \cap W_i \cap \bar{U}_1 \setminus U_1$. The image of $\phi_\alpha(W_i \cap U_1) \subset T^{N_\alpha}$ can be shrunk by an arbitrarily small amount $\epsilon$ so that it no longer intersects the image of $U_1 \setminus U_1$. Pull this set back to obtain a new open set $W'_i \subset U_0 \cap U_1$. This can be done for each $W_i$ and $Y_j$ by a sufficiently small $\epsilon$ so that we still obtain an open cover of the sum which now satisfies the finite type condition. Therefore the sum operation produces a new tropical surface.

Example 4.17 (Contracting rational $(-1)$-curves). Let $E \subset X$ be an irreducible boundary curve of a surface $X$ with simple normal crossings with the other boundary curves. Suppose that $E$ has $k$-leaves, $E^2 = -1$ and that $b_1(E) = 0$, so that $E$ is a rational tropical curve. Let $E^o$ denote the complement of the points of sedentarity order 2 in $E$. The normal bundle restricted to $E^o$ is

$$N_X(E^o) = L^o \times T \subset \mathbb{R}^{k-1} \times \mathbb{T}.$$  

where $L^o$ is a tropical line in $\mathbb{R}^{k-1}$. The graph of a section $\sigma : E \to N_X(E)$ has $k$ unbounded edges identifying $N_X(E^o) \subset \mathbb{R}^{k-1} \times \mathbb{T}$ then these outgoing rays have primitive integer directions: $-e_1, \ldots, -e_{k-1}$, and $e_1 + \cdots + e_{k-1} - e_k$.

To “contract” $E$, perform a tropical sum with a tropical plane $V \subset TP^k$ which is the cone over a tropical line $L$ isomorphic to $E$ contained in the hyperplane $TP^k_{\{0\}}$ of $TP^k$. Since $V$ is a cone (has no internal vertices) the resulting sum is independent of the choice of section and identification of $E$ with $L$. Denote the sum $Y = X \#_{E,L} V$, and by $y$ the point of $Y$ corresponding to the cone point of $V$. There is a tropical morphism $f : X \to Y$ which is an isomorphism restricted to $X \setminus E \to Y \setminus y$.

Let $\mathbb{T}^k$ denote the tropical toric variety corresponding to the toric blow up of affine space at the origin. The fan $\Sigma$ producing this toric variety has for example, 1-dimensional rays $-e_1, \ldots, -e_N$ and $-e_0 = -\sum_{i=1}^N e_i$. Let $\hat{E}$ denote the boundary divisor of $\mathbb{T}^k$ corresponding to the ray in direction $-e_0$. There is a tropical linear morphism $\hat{f} : \hat{\mathbb{T}}^k \to \mathbb{T}^k$ which is the identity when restricted to $\hat{\mathbb{T}}^k \setminus \hat{E} \to \mathbb{T}^k \setminus -\infty$. A neighborhood $U$ of $E$ in $X$ has an open embedding $\hat{L} \subset \hat{\mathbb{T}}^k$, where $\hat{L} \cap \hat{\mathbb{T}}^k \setminus \hat{E}$ is a tropical linear space in $\hat{\mathbb{T}}^k \setminus -\infty$. The tropical morphism $\hat{f} : X \to Y$ is just the identity outside of a neighborhood of $E$ and the restriction of $\hat{f}$ on a neighborhood of $E$.

If $E \subset X$ is not necessarily a boundary curve but still a locally degree 1 curve in $X$ with $E^2 = -1$ and $b_1(E) = 0$, then perform a non-singular modification, $\delta : \hat{X} \to X$ along $E$. Then $\hat{X}$ has a boundary divisor $\hat{E}$ with $\hat{E}^2 = -1$ and we may take the sum $\hat{X}$ with a tropical plane $V$ along $\hat{E}$ and $L$ as above. Therefore, up to tropical modification, we can contract rational curves of self-intersection $-1$, which are locally degree 1 in $X$.

![Figure 9](attachment:image.jpg)  

**Figure 9.** A neighborhood of a $-1$-fan curve $E$ in a surface $X$, and the result after summing with $\nabla$. 


This gives a partial tropical version of the Castelnuovo-Enriques criterion for blowing down \((-1)\)-curves in classical algebraic geometry. Recall from Example 2.21 that the local degree of a tropical curve may be decreased upon performing modifications. However, there are examples of rational \((-1)\)-curves in \(\mathbb{C}P^2\) which are not rectifiable. The tropicalization of such a \((-1)\)-curve could not be contracted in this way.

**Example 4.18.** The tropical Hirzebruch surface \(X_k\) of degree \(k\) can be constructed following Example 3.3. Then \(X_k\) is a compactification of \(\mathbb{R}^2\) with four boundary curves all of which are isomorphic to \(\mathbb{T}P^1 = [-\infty, \infty]\) and satisfying \(C_1^2 = C_3^2 = 0\) and \(C_2^2 = -C_4^2 = k\). Moreover, \(C_1 \cap C_3 = C_2 \cap C_4 = \emptyset\). These four divisors correspond to the directions in \(\mathbb{R}^2\)

\[
\begin{array}{l}
v_1 = (-1, 0), \quad v_2 = (0, -1), \quad v_3 = (1, k), \quad \text{and} \quad v_4 = (0, 1),
\end{array}
\]

in \(\mathbb{R}^2 \subset X_k\). The tropical self-sum of \(X_k\) can be formed along either pair \(C_1, C_3\) or \(C_2, C_4\).

Fix identifications \(f_1 : \mathbb{T}P^1 \to C_1 \subset X_k\) and \(f_3 : \mathbb{T}P^1 \to C_3 \subset X_k\), so the underlying topological space of the sum is a cylinder. For any choice of \(g\) and \(\sigma\), the self-sum \((X_k)\#f_1, f_3, g, \sigma\) produces a tropical projective line bundle over a tropical elliptic curve \(E\), (which is just a circle equipped with a length). The length of \(E\) is determined by the choice of \(g\) and \(\sigma\). The degree of a section of this bundle is equal to \(k\).

Precomposing the identification \(f_1 : \mathbb{T}P^1 \to C_1\) with the map \(x \mapsto -x\) to obtain \(f_1' : \mathbb{T}P^1 \to C_1\), then the sum \((X_k)\#f_1', f_3, g, \sigma\) is diffeomorphic to a M"obius band with boundary. The result is still a tropical projective line bundle over an elliptic curve, \(E'\) which is a circle now of length twice the length of \(E\). The degree of a section of the bundle is 0.

We may also form the self-sum along the tropical divisors \(C_2\) and \(C_4\) after fixing identifications \(f_2 : \mathbb{T}P^1 \to C_2 \subset X_k\), \(f_4 : \mathbb{T}P^1 \to C_4 \subset X_k\) to obtain a tropical surface \(Y = (X_k)\#f_2, f_4, g, \sigma\). Again the underlying topological space is either a cylinder or M"obius band but depending the choices of \(f_2, f_4, g\) and \(\sigma\) it may or may not be fibered by tropical elliptic curves. These are examples of tropical models of Hopf surfaces, see [RS].

**Example 4.19** (Logarithmic transformations). Consider the tropical surface \(Y\) obtained from quotienting \(\mathbb{T} \times \mathbb{R}\) by the extension of a translation \(\mathbb{R}^2 \to \mathbb{R}^2\). If the translation is by \((a, b)\), where \(a/b\) is rational then \(Y\) is fibered over \(\mathbb{T}\) by tropical elliptic curves.

The elliptic curves over \(\mathbb{R} \subset \mathbb{T}P^1\) are all circles of length \(l\) where \((a, b) = l(n, m)\) for \(n, m\) relatively prime integers. The boundary curve is also a circle but instead it has length \(b\), since the restriction of the translation to the boundary of \(\mathbb{T} \times \mathbb{R}\) is given by \((-\infty, x) \mapsto (-\infty, x + b)\). The boundary curve behaves like a fibre of multiplicity \(m\), where \(b = ml\).
Suppose $X$ is another tropical surface with a boundary curve $C$, that is again a tropical elliptic curve which is a circle of length $l$. Suppose moreover that the normal bundle $N_X(C)$ is trivial (i.e. $N_X(C) = C \times \mathbb{T}$). Denote by $X \#_C Y$ the surface obtained by removing a neighborhood of $C$ in $X$ and gluing in a copy of $Y$ by the identification of fibers, as in the tropical sum. Then $X \#_C Y$ is an example of a tropical logarithmic transformation of $X$ performed at $C$.

The description of a tropical surface as a sum of other surfaces provides a means for calculating some invariants. As a first example, there is a Mayer-Vietoris sequence for tropical $(p, q)$-homology groups which can be applied to the sum of tropical surfaces. In the next section Lemma 5.4 relates the Chern numbers of tropical sums of surfaces.

**Proposition 4.20.** Let $X$ be a tropical manifold and $A, B \subset X$ open subsets covering $X$, then the following sequence of tropical $(p, q)$-homology groups is exact,

$$
\cdots \to H_{p,q}(A \cap B) \to H_{p,q}(A) \oplus H_{p,q}(B) \to H_{p,q}(X) \to H_{p,q-1}(A \cap B) \to \cdots
$$

To apply the above Mayer-Vietoris sequence in the case of a sum of tropical surfaces, take $A = X_1 \setminus D$ and $B = X_1 \setminus D$, so that $A \cap B = N_{X_1}^o(D)$.

5. **Noether’s formula**

Classically, Noether’s formula for a non-singular projective surface $X$ relates its holomorphic Euler characteristic with its 1st and 2nd Chern classes [GH93],

$$
\chi(O_X) = \frac{K_X^2 + c_2(X)}{12}.
$$

Here $\chi(O_X)$ is the holomorphic Euler characteristic of $X$, $K_X$ is its canonical class of $X$ and $c_2(X)$ is its second Chern class. This formula is a special case of the Hirzebruch-Riemann-Roch formula for surfaces, which states that for a line bundle $\mathcal{L}$ on a compact complex surface $X$:

$$
\chi(\mathcal{L}) = \chi(O_X) + \frac{\mathcal{L}.\mathcal{L} - \mathcal{L}.K_X}{2}.
$$

By the relation of tropical $(p, q)$-homology to Hodge numbers from [IKMZ], (see also [BIMS, Theorem 7.34]), we replace the holomorphic Euler characteristic $\chi(X) = \sum_{p=0}^{N} h^{0,n}(X)$ by the topological Euler characteristic of the tropical surface $X$.

**Theorem 5.1.** A compact tropical surface $X$ obtained from tropical toric surfaces via summations and modifications satisfies,

$$
\chi(X) = \frac{K_X^2 + c_2(X)}{12},
$$

where $\chi(X)$ is the topological Euler characteristic of $X$.

To prove the above theorem we require a series of lemmas.

**Lemma 5.2.** Suppose $X$ is a tropical sum of $X_1$ and $X_2$ along the irreducible boundary divisors $C_1 \subset X_1$ and $C_2 \subset X_2$ then,

1. $(K_X^n)^2 = (K_{X_1}^n)^2 + (K_{X_2}^n)^2$;
2. $K_X^n \cdot \partial X = K_{X_1}^n \cdot \partial X_1 + K_{X_2}^n \cdot \partial X_2 - 2 \sum_{v \in V(C_1)} (val_{C_1}(v) - 2)$;
3. $C_1^2 = -C_2^2$.

**Proof.** The intersection $(K_X^n)^2$ is supported on points of $X^{(0)}$ that are of sedentarity $0$ and also points of sedentarity, which are the intersection of more than 2 boundary divisors of $X$. Such a point is in exactly one of the skeletons $X_1^{(0)}$ or $X_2^{(0)}$ by the assumption that $C_1, C_2$ have simple
normal crossings with the other boundary divisors of $X_1$ and $X_2$ respectively. Therefore (1) follows from local intersection multiplicities.

For each of the surfaces there is the equality,

$$K_X^o \cdot \partial X = \sum_{D \in A} \sum_{v \in V(D^o)} (\text{val}_D(v) - 2) + \sum_{x \text{ s.t. } S(x)=2} m_x(K_X^o \cdot \partial X).$$

When $X$ is a tropical sum of $X_1$ and $X_2$ there is a contribution of $\text{val}_{C_i}(v) - 2$ for an internal vertex of $C_i$ to $K_X^o \cdot \partial X_i$ for $i = 1, 2$ which does not appear in $K_X^o \cdot \partial X$, otherwise the expressions are the same. Since $C_1$ and $C_2$ are isomorphic the equality in (2) follows.

Part (3) follows simply from Definition 3.26 and that the sections $\sigma, \sigma'$ of the normal bundles of $C_1, C_2$ respectively are related by $\sigma' = g \circ (\sigma)$ where $g$ is orientation reversing. \hfill \Box

**Lemma 5.3.** Suppose $X$ is a tropical sum of $X_1, X_2$ along boundary curves $C_1 \subset X_1, C_2 \subset X_2$. Denote the boundary arrangement of $X_i$ by $A_i$ for $i = 1, 2$ and the boundary arrangement of $X$ by $A_X$. Then,

$$\left( \sum_{\partial_D \in A_X} \hat{D}_D \right)^2 = \left( \sum_{E_i \in A_1} E_i \right)^2 + \left( \sum_{F_j \in A_2} F_j \right)^2 - 4|l(C)|,$$

where $l(C)$ is the set of leaves of $C \cong C_1$. \hfill \Box

**Proof.** The section $\sigma$ from the construction of the tropical sum is a subset of $X$. For every $\hat{D}_D \in A_X$, we can find a cycle $\hat{D}_D'$ rationally equivalent to $\hat{D}_D$ and contained in a neighborhood of $\hat{D}_D$, by taking a section in the sense of Definition 3.26 and completing it to a tropical cycle. We can also insist that $\hat{D}_D$ intersects $\hat{D}_D'$ in a finite number of points in the interior of 1 dimensional faces of both $\hat{D}_D$ and $\hat{D}_D'$ and in the interior of a top dimensional stratum of $X$. We can also suppose that at each point $x \in \hat{D}_D \cap \hat{D}_D'$ the tropical intersection multiplicity is 1, and similarly for the intersection with $\hat{D}_D'$ and $\sigma$. Denote $D = \sum \hat{D}_D'$.

The two connected components $V_1, V_2$ of $X \setminus \sigma$ are identified with open subsets of $X_1$ and $X_2$ respectively. By intersecting $D$ with $V_1$ and extending any open rays of $D \cap V_1$ to the rest of $X_1$ in the direction of $C_1$ we produce a cycle $E$ rationally equivalent to $\sum_{E_i \in A_1} E_i - C_1$. Moreover, $E \cdot C_1 = |l(C_1)|$ since $C_1$ has simple normal crossings with the rest of the divisors of $X_1$ and the cycles $\hat{D}_D'$ are chosen so that they had intersection multiplicity 1 with $\sigma$. The analogous statements hold in $X_2$ using $F$ to denote the cycle $\sum_{F_j \in A_2} F_j - C_2$.

By the local intersection multiplicities for cycles of sedentarity $0$, we obtain $D^2 = E^2 + F^2$. Now, $(\sum_{E_i \in A_1} E_i)^2 = (E + C_1)^2$ and similarly $(\sum_{F_j \in A_2} F_j)^2 = (F + C_2)^2$. Using that $C_1^2 = -C_2^2$ and $E \cdot C_1 = F \cdot C_2 = |l(C_1)|$, the lemma then follows by equivalence of intersection numbers for rationally equivalent cycles from Proposition 3.37. \hfill \Box

**Lemma 5.4.** Let $X_1, X_2$ be compact tropical surfaces and $C_1 \subset X_1, C_2 \subset X_2$ boundary curves such that the tropical sum along $C_1, C_2$ exists. Suppose $X = X_1 \#_f X_2$, then

1. $c_2(X) = c_2(X_1) + c_2(X_2) + 2K_C$,
2. $K_X^c = K_{X_1}^c + K_{X_2}^c + 4K_C$,

where $K_C$ is the canonical class of the curve $C \cong C_1$. \hfill \Box

**Proof.** A comparison of vertices in $X_1, X_2$ and $X$ gives the first formula for $c_2(X)$

$$c_2(X) = c_2(X_1) + c_2(X_2) - 2 \sum_{v \in V(C)} (2 - \text{val}_C(v))$$

$$= c_2(X_1) + c_2(X_2) + 2K_C,$$

where $V(C)$ is the vertex set of the curve $C$ and $\text{val}_C(v)$ is the valency of a vertex.
Recall that $K_X = K_X^o - \partial X$ and apply Lemmas 5.2 and 5.3 to obtain:

$$K_X^2 - (K_{X_1}^2 + K_{X_2}^2) = -2\left[\sum E_i \cdot C_1 + \sum F_i \cdot C_2 - K_{X_1}^o \cdot C_1 - K_{X_2}^o \cdot C_2\right]$$

$$= 4\left[\sum_{v \in V(C)} \left(\text{val}_C(v) - 2\right)\right]$$

$$= 4K_C.$$

This proves the lemma. □

A similar sequence of lemmas can be proved when $X$ is the result of a self-sum of a surface $X'$ as in Example 4.18. We summarize the statement in a single lemma below. The proof is omitted since it follows the same lines as for the sum of distinct surfaces.

**Lemma 5.5.** Let $X$ be the result of the self-sum of a tropical surface $X'$ along disjoint boundary curves $C_1, C_2$, then

1. $c_2(X) = c_2(X') + 2K_C$;
2. $K_X^2 = K_{X'}^2 + 4K_C$,

where $K_C$ is the canonical class of the curve $C \cong C_i$.

The next corollary immediately follows the above lemma.

**Corollary 5.6.** Let $X_1, X_2$ be compact tropical surfaces and $C_1 \subset X_1, C_2 \subset X_2$ irreducible boundary curves such that the tropical sum $X$ along $C_1, C_2$ exists. Then,

$$K_X^2 - 2c_2(X) = K_{X_1}^2 + K_{X_2}^2 - 2\left[c_2(X_1) + c_2(X_2)\right].$$

Hirzebruch’s signature formula states that for a compact complex surface $X$,

$$3\text{Sign}(X) = K_X^2 - 2c_2(X),$$

where $\text{Sign}(X)$ is the signature of the intersection form on $H^2(X)$. Once again, this formula is a special case of the Hirzebruch-Riemann-Roch formula applied to the exterior algebra bundle of the cotangent bundle of a manifold, see Section 2 of [Hir71]. By Novikov additivity, the signature of 4-manifolds is additive under cobordism [Kir89], and the above corollary shows that tropically the right hand side of Hirzebruch’s formula is also additive under taking tropical sums of surfaces. This leads to the following conjecture.

**Conjecture 5.7.** A compact tropical surface $X$ satisfies

$$3\text{Sign}_{1,1}(X) = K_X^2 - 2c_2(X),$$

where $\text{Sign}_{1,1}(X)$ is the signature of the intersection form on $H_{1,1}(X)$.

**Remark 5.8.** In the above conjecture we could equivalently replace $\text{Sign}_{1,1}(X)$ by $\text{Sign}(X)$ which is the signature of the intersection pairing on $H_{2,0}(X) \oplus H_{1,1}(X) \oplus H_{0,2}(X)$. The intersection pairing on $(0,2)$ and $(2,0)$ classes is defined in [MZ].

**Proof of Theorem 5.1.** For a tropical toric variety we have $\chi(X) = 1$. The translation of Noether’s formula for toric surfaces to properties of polygons is quite simple and can be found in [Ful93, Section 4.3]

If $f : \tilde{X} \to X$ is a tropical modification we have $\chi(\tilde{X}) = \chi(X)$ since $\tilde{X}$ is homotopic to $X$. Combining this with Lemmas 4.9 and 4.10 proves Noether’s formula for $\tilde{X}$. 
Suppose $X_1, X_2$ both satisfy tropical Noether’s formula, and let $X$ denote the sum of $X_1, X_2$ along $C_1, C_2$ for some choices of $f_1, f_2, g, \sigma$. Apply both parts of Lemma 5.4 to obtain:

\[
K^2_X + c_2(X) = K^2_{X_1} + c_2(X_1) + K^2_{X_2} + c_2(X_2) + 6K_C
\]

\[
= 12\chi(X_1) + 12\chi(X_2) + 6K_C
\]

\[
= 12\chi(X_1) + 12\chi(X_2) - 12\chi(C),
\]

where $C \cong C_i$. By the formula for Euler characteristics of non-disjoint unions Noether’s formula holds. When $X$ is a self-sum of $X'$ along disjoint boundary curves applying Lemma 5.5 establishes the tropical Noether’s formula for the self-sum. This proves the theorem. \hfill \Box

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