Three-Neutrino MSW Effect and the Lehmann Mass Matrix

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Abstract. Recent work on analytical solutions to the MSW equations for three neutrino flavours is reviewed, with emphasis on the exponential density. Application to a particular mass matrix, proposed by Lehmann, Newton and Wu, is also discussed. Within this model, the experimental data allow a determination of the three neutrino masses. They are found to be 0.002–0.004, 0.01 and 0.05 eV.

1. Introduction

We here review some recent results on analytical solutions of the Mikheyev–Smirnov–Wolfenstein (MSW) effect [1] for the propagation of three neutrino flavours. Such analytic results have been obtained for both the exponential density [2, 3] and the linear density [4]. In the case of an exponential electron density, the three neutrino wave functions can be expressed in terms of generalized hypergeometric functions, \( _2F_2 \) [2, 3]. For the linear density, the solutions can be expressed in terms of a Fourier transform [4]. In the case of two flavours, these reduce to parabolic cylinder functions or confluent hypergeometric functions, but of different parameters and arguments [5, 6].

A particular neutrino mass matrix, originally proposed by Lehmann, Newton and Wu (LNW) for quarks [7] is also reviewed [8, 9, 10]. Within this model, the current data on atmospheric and solar neutrinos permit a determination of the neutrino masses.

2. Exponential density

In a medium where the electron neutrino (described by \( \phi_1(r) \)) interacts differently from the others, the propagation is given by the equation

\[
\frac{id}{dr} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} D(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2\rho} \begin{bmatrix} M_{11}^2 & M_{12}^2 & M_{13}^2 \\ M_{21}^2 & M_{22}^2 & M_{23}^2 \\ M_{31}^2 & M_{32}^2 & M_{33}^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}.
\]

The mass matrix is real and symmetric, \( M_{ji}^2 = M_{ij}^2 \equiv (M^2)_{ij} \), but otherwise arbitrary. (The LNW mass matrix will be discussed later.) Furthermore, \( D(r) = \sqrt{2}G_F N_e(r) \), with \( G_F \) the Fermi constant and \( N_e(r) \) the solar electron density.

For the sun, the density [11] is well approximated by an exponential,

\[
N_e(r) = N_e(0) e^{-r/r_0}, \quad r_0 \simeq 0.1 \times R_\odot.
\]

Introducing a new radial variable: \( u = r/r_0 + u_0 \), and performing a rotation on the second and third components, Eq. (1) can be written as

\[
\frac{id}{du} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix} = \begin{bmatrix} \omega_1 + e^{-u} & \chi_2 & \chi_3 \\ \chi_2 & \omega_2 & 0 \\ \chi_3 & 0 & \omega_3 \end{bmatrix} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix}.
\]
The eigenvalues of the $3 \times 3$ matrix
\[
\begin{bmatrix}
\omega_1 & \chi_2 & \chi_3 \\
\chi_2 & \omega_2 & 0 \\
\chi_3 & 0 & \omega_3
\end{bmatrix}
\] (4)
are denoted $\mu_1$, $\mu_2$ and $\mu_3$, they are the squares of the neutrino masses multiplied by $r_0/(2p)$. Together with $\omega_1$ and $\omega_2$ they control the evolution of the $\psi_i$.

Introducing now the variable $z = ie^{-u}$, the solutions to Eq. (3) can be expressed in terms of solutions to the third-order ordinary differential equation
\[
\left[ \left( \frac{d}{dz} - i\mu_1 \right) \left( \frac{d}{dz} - i\mu_2 \right) \left( \frac{d}{dz} - i\mu_3 \right) - z \left( \frac{d}{dz} - i\omega_2 \right) \left( \frac{d}{dz} - i\omega_3 \right) \right] \psi = 0,
\] (5)

namely generalized hypergeometric functions $[3, 12]$:
\[
\psi^{(1)} = e^{-i\mu_1 u} \, \, _2F_2 \left[ \begin{array}{c}
- i(\omega_2 - \mu_1), \\
1 - i(\mu_2 - \mu_1), \\
1 - i(\mu_3 - \mu_1)
\end{array} \right] \left( e^{i u} \right)
\]
\[
\psi^{(2)} = e^{-i\mu_2 u} \, \, _2F_2 \left[ \begin{array}{c}
- i(\omega_2 - \mu_2), \\
1 - i(\mu_2 - \mu_2), \\
1 - i(\mu_3 - \mu_2)
\end{array} \right] \left( e^{i u} \right)
\]
\[
\psi^{(3)} = e^{-i\mu_3 u} \, \, _2F_2 \left[ \begin{array}{c}
- i(\omega_2 - \mu_3), \\
1 - i(\mu_2 - \mu_3), \\
1 - i(\mu_2 - \mu_3)
\end{array} \right] \left( e^{i u} \right)
\] (6)
The solutions to Eq. (3) are thus
\[
\psi_i = C_1 \psi^{(1)}_i + C_2 \psi^{(2)}_i + C_3 \psi^{(3)}_i,
\] (7)
where the constants $C_j$ are determined by the boundary conditions: $\psi_1 = 1$, $\psi_2 = \psi_3 = 0$ at the center of the sun, $r = 0$.

The $\, _2F_2$ functions can be defined in terms of the series expansions
\[
\, _2F_2 \left[ \begin{array}{c}
\alpha_1, \\
\alpha_2
\end{array} \right. \left. \frac{\rho_1}{\rho_2}, \, \, z \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\rho_1)_k (\rho_2)_k} \frac{z^k}{k!}
\] (8)
where $(\alpha)_k$ is a Pochhammer symbol, but (somewhat complicated) asymptotic approximations are actually more useful for numerical work $[3, 8, 3]$.

3. The LNW mass matrix

For quarks, it was found $[7]$ that a particular, simple texture for the $d$ ($d$, $s$, $b$) and $u$ ($u$, $c$, $t$) quark mass matrices leads to a good description of the CKM matrix $[13]$. The mass matrix is assumed to have the form
\[
M = \begin{bmatrix}
0 & d & 0 \\
d & c & b \\
0 & b & a
\end{bmatrix}
\] (9)
with $b^2 = 8c^2$. This is known as a two-zero texture, but differs from those normally considered (see, e.g., $[14, 15]$) in the additional relation between $b$ and $c$. The eigenvalues are given by $m_1$, $m_2$, and $m_3$, with $m_1 \leq m_3$. 

Actually, in the case of quarks, since there is CP violation, a complex CKM matrix is obtained by replacing the parameter $d$ in $\mathcal{M}$ by $\pm id$ for $u$ quarks (such that $M$ remains Hermitian). The CKM matrix becomes

$$V_{\text{CKM}} = R^T(u) \text{diag}(-i,1,1)R(d),$$

(10)

The Jarlskog determinant \[14\] thus obtained is $J = 2.6 \times 10^{-5}$, in good agreement with data.

This same mass matrix has been applied to the case of three neutrinos \[8, 9, 10\], and rather good fits to the atmospheric \[17\] and solar \[18, 19, 20, 21, 22, 23, 24, 25\] neutrino data have been obtained.

The matrix $M$ is diagonalized, whereby $M = RM_{\text{diag}}R^T$. For this purpose, the following notation is convenient

$$S_1 \equiv m_3 - m_2 + m_1,$$

$$= a + c$$

$$-S_2 \equiv m_2 m_2 - m_3 m_1 + m_2 m_1,$$

$$= 8c^2 + d^2 - ac$$

$$-S_3 \equiv m_1 m_2 m_3$$

$$= ad^2.$$

(11)

Eliminating $d$ and $c$, the resulting cubic equation for the parameter $a$ can be written as

$$9a^3 - 17S_1 a^2 + (8S_1^2 + S_2)a - S_3 = 0.$$

(12)

A physical solution requires $a$ real and positive. This is equivalent to having three real solutions for $a$. One of these is negative and two are positive. At any point inside the allowed domain in the $(m_1/m_3)-(m_2/m_3)$ plane (See Fig. 1), there are thus two allowed solutions, denoted Solutions 1 and 2.

This diagonalization has to be carried out for both neutrinos and charged leptons, in order to obtain the neutrino mixing matrix \[10\]

$$U = \begin{bmatrix}
  U_{e1} & U_{e2} & U_{e3} \\
  U_{\mu1} & U_{\mu2} & U_{\mu3} \\
  U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{bmatrix},$$

(13)

where ($\epsilon = 1$ or $i$, depending on whether or not there is CP violation)

$$U = (V^\ell_{\text{CKM}})^\dagger = R^T(\ell) \text{diag}(\epsilon,1,1)R(v)$$

(14)

relates the neutrino mass eigenstates to the flavor states of charged-current interactions:

$$|\nu_e\rangle = U_{e1}|\nu_1\rangle + U_{e2}|\nu_2\rangle + U_{e3}|\nu_3\rangle,$$

etc.

(15)

There are four possible solutions to Eq. (12), two for the neutrino sector, and two for the charged lepton sector. Furthermore, the model has two signs (denoted “parities”) to be specified: $b$ and $d$ can each be either positive or negative (unless $d$ is imaginary \[10\]). However, only the product of the “$b$ parities” in the neutrino and charged lepton sector matters, and similarly for the “$d$ parities”, so we may put both “parities” of the charged lepton sector to $+1$. 


4. Fits to data

Let us consider first the atmospheric neutrino data. The Super-Kamiokande results \[17\] give \(\Delta m^2 \simeq (2 - 3) \times 10^{-3}\) eV, with \(\sin^2(2\theta) \simeq 1\). The survival of muon neutrinos is given by

\[
P_{\nu_\mu \rightarrow \nu_\mu}(t) = 1 - 4\left[U_{\mu 1}^2 U_{\mu 2}^2 \sin^2 \left(\frac{\Delta m^2_{\mu \tau} t}{4\rho}\right) + U_{\mu 1}^2 U_{\mu 3}^2 \sin^2 \left(\frac{\Delta m^2_{\mu \tau} t}{4\rho}\right) + U_{\mu 2}^2 U_{\mu 3}^2 \sin^2 \left(\frac{\Delta m^2_{\mu \tau} t}{4\rho}\right)\right],
\]

where \(U\) is the neutrino mixing matrix of Eqs. (13) and (14). In the limit of \(\Delta m^2_{\mu \tau}/4\rho \ll 1\) this simplifies, and invoking further unitarity, one finds

\[
P_{\nu_\mu \rightarrow \nu_\tau}(t) \simeq 4 U_{\mu 3}^2 U_{\tau 3}^2 \sin^2 \left(\frac{\Delta m^2_{\mu \tau} t}{4\rho}\right),
\]

which suggests that one needs \(|U_{\mu 3} U_{\tau 3}| = O(1)|. This can be achieved within the model (for both Solutions 1 and 2), for \(m_1 \ll m_3\), with also \(m_2\) small compared with \(m_3\). Furthermore, the data suggest that the scale \(m_3\) must be such that \(m_3^2 \simeq (2 - 3) \times 10^{-3}\) eV, or \(m_3 = O(0.05)\) eV.

Fits to atmospheric data confirm this qualitative analysis. Invoking also the solar Cl \[20\], Ga \[21, 22\], Super-Kamiokande \[18, 23\] and SNO \[24, 25\] neutrino data, one finds that Solution 2 for the neutrino sector, Solution 1 for the charged lepton sector and \(b\) parity\(^{-1}\) give good fits for \(m_1 \ll m_3\), with \(m_2\) also small as compared with \(m_3\). A \(\chi^2\) determined from these different atmospheric and solar survival probabilities leads to good fits (see Fig. 1) with \(m_3\) about 0.052 eV, \(m_2\) about 0.01 eV, and \(m_1 \sim 0.002-0.004\) eV \[8, 9, 10\]. The \(d\) parity is unimportant.

![Figure 1](image-url)

**Figure 1.** Fits to the atmospheric and solar neutrino data for Solution 2 (neutrino sector) and 1 (charged leptons) and both “parities” negative. Contours are given at \(\chi^2 = 5, 10, 15, 20, 25\). Sum (right panel) also at 30, 35, 40, 45, 50. The best-fit point (right panel, at the lower left), is marked “low”.

The effect of the CHOOZ data \[26\] is to disfavor the region \(m_2 = O(m_3)\), where \(|U_{e 3}|\) is large. But this region is already disfavored by the solar neutrino data, as is seen in the middle panel of Fig. 1. In terms of the more conventional two-flavour analyses for the solar-neutrino sector, these fits roughly correspond to the large-mixing-angle solution.
5. Summary

Analytic results for the solutions to the MSW equations for three neutrino flavours are very valuable for a fast scanning over the parameters of some given model for the mass matrix.

The LNW mass matrix is a very constrained model that in the quark sector describes the CKM matrix, and in the neutrino sector gives the mixing in terms of the mass eigenvalues. Applied to the neutrino data, the model gives a very good fit.

The solar neutrino data has also been studied within the same model, using numerical integration methods (no $F_2$'s) [27]. An additional fit was then found at $m_1 \simeq 2.8 \times 10^{-6}$ eV, corresponding to the small-mixing-angle solutions. However, this point is disfavoured by the atmospheric neutrino data, and by the flat electron recoil spectrum [19].

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