Composition of choreography automata

(Technical Report)

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Abstract. Choreography automata are an automata-based model of choreographies, that we show to be a compositional one. Choreography automata represent global views of choreographies (and rely on the well-known model of communicating finite-state machines to model local behaviours). The projections of well-formed global views are live as well as lock- and deadlock-free. In the class of choreography automata we define an internal operation of composition, which connects two global views via roles acting as interfaces. We show that under mild conditions the composition of well-formed choreography automata is well-formed. The composition operation enables for a flexible modular mechanism at the design level.

1 Introduction

Choreographic approaches to the modelling, analysis, and programming of message-passing applications abound. Several models based on behavioural types have been proposed to analyse properties such as liveness or deadlock-freedom (e.g., [33,18,12] and the survey [20] to mention but few) while other approaches have considered syntax-free models [35]. Abstract models have also been advocated to verify and debug choreographic specifications [23,2] using modelling languages such as BPMN [30]. At a programming level, choreographic programming has been explored in [25,28]. The ICT industrial sector is also starting to acknowledge the potential of choreographies [8,36].

A distinguished feature of choreographies is the coexistence of two distinct yet related views of a distributed system: the global and the local views. The former is an abstraction that yields a description of the system from a holistic point of view. A global view indeed describes the coordination necessary among the various components of the system “altogether”. In contrast, the local view of a choreography specifies the behaviour of the single components in “isolation”. In the so called top-down approach to choreographies, the local view can be derived by projecting the global one on each single component. This yields results that typically guarantee that the execution of the local components reflects the specification of the global one without spoiling communication soundness (e.g., deadlock freedom, liveness, etc.). These results do not hold in general and require to restrict to well-formed global views.
The composition of local views is simple. The local components just execute in parallel on a specific communication infrastructure. The situation is more complex for global views and, despite some attempts (cf. Section 6.1), the problem of composing global views of choreographies is still open.

This paper, after recalling the choreographic model proposed in [6], describes a proposal for the global views composition problem via a composition operation over interfaces. The modelling language of global views of [6] are choreography automata (c-automata for short), which are basically finite-state automata whose transitions are labelled with interactions expressing communications between some participants (also called roles). More precisely, an interaction \( A \rightarrow B : m \) represents the fact that participant \( A \) sends message \( m \) to participant \( B \) and participant \( B \) receives it. For instance, consider the following c-automata

\[
\begin{align*}
C_1 & \rightarrow A: \text{tick} \\
& \rightarrow A \rightarrow B : m \\
& \rightarrow A \rightarrow H : \text{tock} \\
\text{and} \\
C_2 & \rightarrow C \rightarrow K : \text{tick} \\
& \rightarrow K \rightarrow C \rightarrow \text{tock}
\end{align*}
\]

where the left-most c-automaton \( C_1 \) represents a system where \( A \) sends \( B \) message \( m \) once it receives a \text{tick} from an external “cron” service \( H \). Participant \( A \) acknowledges the completion of the exchange with \( B \) by sending \( H \) message \( \text{tock} \). The right-most c-automaton \( C_2 \) specifies that the cron service \( C \) repeatedly sends message \text{tick} and receives message \text{tock} from \( K \).

Our model of local views are a variant of communicating systems [9] in which we adopt the synchronous semantics defined in [6].

For us, an interface is a designated role of a c-automaton representing behaviour “scattered in the environment”, namely provided by other c-automata. For instance, \( H \) is an interface of \( C_1 \) in (1) whereby we “delegate” the cron service; this outsourced behaviour is provided by another system in the execution environment of \( C_1 \) such as the c-automaton \( C_2 \).

The composition of c-automata, say \( CA \) and \( CB \), over their interfaces, say \( H \) for \( CA \) and \( K \) for \( CB \), is the result of the combination of two operations:

(i) the first operation is simply the product \( CA \times CB \) of the two automata and
(ii) the second operation is (interface) blending, namely the removal of the interfaces transforming participants \( H \) and \( K \) into “forwarders”.

More precisely, if the first transition \( t \) of two consecutive transitions in the product \( CA \times CB \) has an interface role as the receiver of a message \( m \) and the second, say \( t' \), the other interface role as sender of \( m \), then \( t \) and \( t' \) are replaced with a transition where the sender of \( t \) and the receiver of \( t' \) exchange \( m \). Any transition involving \( H \) or \( K \) that is not replaced is then removed.
Let us illustrate how to compose the c-automata in (1). The product of $C_1$ and $C_2$ and their blending on $H$ and $K$ are respectively the c-automata

![Diagram]

where the dotted and the dashed transitions in the product automaton on the left form the consecutive transitions to blend because one interface participant receives a message and the other interface participant sends it.

As shown in this simple example, blending enables modular design of choreographies. We note that modularity can be attained also with other mechanisms. For instance, in the approach in [29] one can specify partial designs and then check that their combination enjoys properties of interest, without explicitly building a composed choreography. Another example is the use of refinement of global views as suggested in [14]. Choreography automata instead are equipped with an internal operation of composition that given two c-automata yields another one. Composition preserves the interesting behaviours of the components and guarantees communication soundness (liveness and deadlock freedom) of the resulting c-automaton under mild conditions. We argue that blending-based composition is more flexible an approach to modularity than other proposals; for instance, the result of the composition of two c-automata is another c-automaton that provides a global specification of the composition. Also, the blending operation allows one to “self-compose” a c-automaton and this corresponds to a sort of refinement operation where blending two roles $H$ and $K$ of a c-automaton amounts to consider them as “forwarders” that are then made implicit and removed. We will further demonstrate the flexibility of blending-based composition on a non-trivial working example where more systems can be composed together.

Structure & Contributions of the paper The basic notions used through the paper are surveyed in Section 2. Section 3 recalls c-automata and their projections; the former model is essentially borrowed from [6] and the latter are basically the communicating systems of [9]. Choreography automata yield a simple yet very expressive language for global views. Sections 4 and 5 report our main results.

Well-formedness The notion of well-formedness and its relations with the projection operation [6] are recalled in Section 4. The projection is correct (cf. Theorem 4.12), and guarantees liveness as well as lock- and deadlock-freedom (cf. Propositions 4.13, 4.15 and 4.17).
Composition The composition of c-automata is defined in Section 5 as a refinement of the product of c-automata through the blending operation. This operation is inspired by one of the composition approaches in [4,5]. The composition of arbitrary c-automata may not preserve well-formedness (and hence the behaviour at the local level).

Properties We identify a class of c-automata that, imposing mild conditions (cf. Definition 5.15) on interfaces, guarantees that our notion of composition preserves well-formedness (cf. Corollary 5.26). Moreover, interface blending ensures that the composition does not add new behaviours to the roles not in the interface (cf. Proposition 5.7) for all c-automata, while for reflective c-automata we can also prove that behaviours are preserved (cf. Theorem 5.12).

Compatibility Finally, the notion of compatibility of interfaces (cf. Definition 5.27, borrowed from [4]) is sufficient to establish reflectiveness (Theorem 5.29).

Besides being based on the well-known theory of formal languages, c-automata are also technically convenient as they allow us to reuse the rich theory of automata. Section 6 discusses these points more extensively together with other concluding remarks, related work, and future lines of research.

2 Preliminaries

We recall some basic notions of finite-state automata (FSAs). A finite-state automaton is a tuple $A = \langle S, L, \rightarrow, s_0 \rangle$ where

- $S$ is a finite set of states (ranged over by $s, q, \ldots$),
- $L$ is a finite set of labels (ranged over by $l, \lambda, \ldots$) and $\epsilon \notin L$,
- $\rightarrow \subseteq S \times (L \cup \{ \epsilon \}) \times S$ is a set of transitions (where $\epsilon$ represents the empty word), and
- $s_0 \in S$ is the initial state.

We use the usual notation $\rightarrow^*$ for the reflexive and transitive closure of $\rightarrow$. The set of reachable states of $A$ is $R(A) = \{ s \mid s_0 \rightarrow^* s \}$. For $t = (s_1, \lambda, s_2)$ we write $s_1 \xrightarrow{\lambda} s_2$ and occasionally $t \in A$ when $t \in \rightarrow$.

Remark 2.1. Our definition of FSA omits the set of accepting states since we consider only FSAs where each state is accepting. We discuss this point further at the end of the paper.

We define on FSAs traces, trace equivalence, and bisimilarity.

Definition 2.2 (Trace equivalence). A run of an FSA $A = \langle S, L, \rightarrow, s_0 \rangle$ is a (possibly empty or infinite) sequence of consecutive transitions starting at $s_0$. The trace (or word) $w$ of a run $(s_{i-1} \xrightarrow{\lambda_{i-1}} s_i)_{1 \leq i \leq n}$ is obtained by concatenating

Assume $n = \infty$ if the run is infinite.
the labels of the run, namely \( w = \lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_n \); if the run is empty then \( w = \varepsilon \).

The language of \( A \), written \( \mathcal{L}(A) \), is the set of traces of \( A \). Two FSAs \( A \) and \( B \) are trace equivalent if \( \mathcal{L}(A) = \mathcal{L}(B) \). We say that \( A \) accepts \( w \) from \( s \) if \( w \in \mathcal{L}(\langle S, \mathcal{L}, \rightarrow, s \rangle) \).

Since we have to deal with possibly infinite traces, it is natural to use coinduction as the main logical tool. We recall the notion of bisimulation.

**Definition 2.3 (Bisimilarity).** A bisimulation on two FSAs \( A = \langle S, \mathcal{L}, \rightarrow, s_0 \rangle \) and \( A' = \langle S', \mathcal{L}', \rightarrow', s'_0 \rangle \) is a relation \( R \subseteq S \times S' \) such that if \( (s, s') \in R \) then

- if \( s \xrightarrow{\lambda} s_1 \) then \( s' \xrightarrow{\lambda} s'_1 \) and \( (s_1, s'_1) \in R \),
- if \( s' \xrightarrow{\lambda} s'_1 \) then \( s \xrightarrow{\lambda} s_1 \) and \( (s_1, s'_1) \in R \).

Bisimilarity is the largest bisimulation.

Recall that trace equivalence and bisimulation equivalence coincide on deterministic labelled transition systems (see e.g., [16, Theorem 2.3.12]).

In this paper we adopt communicating finite-state machines (CFSMs) [9] to model the local behaviour of distributed components. The following definitions are borrowed from [9] and adapted to our context.

**Definition 2.4 (Communicating system).** A communicating finite-state machine (CFSM) is an FSA \( M \) on the set \( \mathcal{L}_{\text{act}} = \{ AB!m, AB?m \mid A, B \in P, m \in M \} \) of actions. The subject of an output (resp. input) action \( AB!m \) (resp. \( AB?m \)) is \( A \) (resp. \( B \)). Machine \( M \) is \( A \)-local if all its transitions have subject \( A \).

A (communicating) system is a map \( S = \langle M_A \rangle_{A \in P} \) assigning a \( A \)-local CFSM \( M_A \) to each participant \( A \in P \).

We consider the synchronous semantics for communicating systems as a transition system labelled in a set of interactions

\[ \mathcal{L}_{\text{int}} = \{ A \rightarrow B \; m \mid A \neq B \in P, m \in M \} \]

defined as follows.

**Definition 2.5 (Synchronous semantics).** Let \( S = \langle M_A \rangle_{A \in P} \) be a communicating system where \( M_A = \langle S_A, \mathcal{L}_{\text{act}}, \rightarrow, q_0A \rangle \) for each participant \( A \in P \). A (synchronous) configuration for \( S \) is a map \( \vec{q} = (q_A)_{A \in P} \) assigning a local state \( q_A \in S_A \) to each participant \( A \in P \). We denote \( q_A \) with \( \vec{q}(A) \).

The (synchronous) semantics of \( S \) is the transition system \( \llbracket S \rrbracket = \langle S, \mathcal{L}_{\text{int}}, \rightarrow, \vec{q}_0 \rangle \) defined as follows

- \( S \) is the set of synchronous configurations and \( \vec{q}_0 = (q_0A)_{A \in P} \in S \) is the initial configuration
- \( \vec{q}_1 A \rightarrow B \; m \rightarrow \vec{q}_2 \in \rightarrow \) if
• \( q_1^A \xrightarrow{A \rightarrow B; m} q_2^A \) and \( q_1^B \xrightarrow{A \rightarrow B; m} q_2^B \), and
• for all \( C \neq A, B \), \( q_1^C = q_2^C \).

In this case, we say that \( q_1 \) and \( q_2 \) are component transitions of \( q_1 \xrightarrow{A \rightarrow B; m} q_2 \).

If \( q_1 \xrightarrow{\alpha} q_2 \), we say that \( q_2 \) is (synchronously) reachable from the configuration \( q_1 \) by firing the transition \( \alpha \).

3 Choreography Automata

We introduce choreography automata (c-automata) as an expressive and flexible model of global specifications, following the styles of choreographies [10, 21, 30], global graphs [35] and multiparty session types [17, 19]. As customary in choreographic frameworks, we show how to project c-automata on local specifications.

As anticipated, our projections yield CFSMs formalising the local behaviour of the participants of a choreography.

C-automata (ranged over by \( CA, CB, \) etc.) are FSAs with labels in \( L_{\text{int}} \).

**Definition 3.1 (Choreography automata).** A choreography automaton (c-automaton) is an \( \epsilon \)-free FSAs on the alphabet \( L_{\text{int}} \). Elements of \( L_{\text{int}} \) are choreography words, subsets of \( L_{\text{int}}^* \) are choreography languages.

Admitting \( \epsilon \)-transitions in c-automata would force us to deal with specifications where participants may take inconsistent decisions. For instance, consider the following patterns

![Diagram](image)

At state \( q \), \( A \) and \( B \) should locally decide which message to exchange; for instance, \( A \) may decide for message \( m \) while \( B \) may choose for message \( n \) leading to miscommunications. Note that we still admit non-determinism in c-automata.

**Example 3.2 (Working Example).** We consider a publication system \( S_P \) and a validation system \( S_V \) in combination with a cron system \( S_C \) (alike the one in Section 1). The c-automata \( CC, CP, \) and \( CV \) in Fig. 1 formalise the overall behaviour of \( S_C, S_P, \) and \( S_V \), respectively.

**System** \( S_C \) is composed of participants \( R, D, S, \) and \( F \); it repeatedly ticks \( D \) to execute some tasks; before starting, \( D \) informs \( S \) on the number of tasks, and \( F \) informs \( R \) with message tock that the tasks are completed.

**System** \( S_V \) is composed of participants \( C, Q, H, \) and \( I \) as follows.

1. Participant \( H \) repeatedly checks if texts received from \( Q \) can be accepted or not, accordingly informing \( Q \) by means of messages ack or nack;
— refusal (nack): Q sends the text to participant I which modifies it and sends it back to Q for resubmission to H;
— acceptance (ack): Q informs I that her help is not needed.

2. Each cycle of the above protocol can start only when Q does receive a tick message from participant C.

System $S_p$ is composed of participants A, B, K, and E as follows.

1. Participant K repeatedly sends text messages to, alternately, participants A and B (starting with A), who can accept or refuse the texts, sending back a message ack or nack, respectively:
   — in case of acceptance (ack): the participant who received the text sends the message go to the other receiver in order to inform her that it is her turn to get a text message from K;
   — in case of refusal (nack): the participant who received the text, say A, sends an alt to the other receiver B in order to inform her that it is not her turn yet to get a text message from K, since A will keep on receiving texts from K until one is accepted;

2. Participants A and B inform participant E, by means of a message tock, that one of their cycles has been completed.

Given a c-automaton, our projection operation builds the corresponding communicating system consisting of the set of projections of the c-automaton on each participant, which yields a CFSM.
Definition 3.3 (Projection). The projection on $A$ of a transition $t = s \xrightarrow{\alpha} s'$ of a c-automaton, written $t_{\downarrow A}$, is defined by:

$$
t_{\downarrow A} = \begin{cases} 
  s \xrightarrow{ACm} s' & \text{if } \alpha = B \rightarrow C: m \land B = A \\
  s \xrightarrow{BA\land m} s' & \text{if } \alpha = B \rightarrow C: m \land C = A \\
  s \xrightarrow{\epsilon} s' & \text{if } \alpha = B \rightarrow C: m \land B, C \neq A \\
  s \xrightarrow{\epsilon} s' & \text{if } \alpha = \epsilon
\end{cases}
$$

The projection of a c-automaton $CA = \langle S, L_{\text{int}}, \rightarrow, s_0 \rangle$ on a participant $A \in P$, denoted $CA_{\downarrow A}$, is obtained by minimising up-to-language equivalence the intermediate automaton

$$A_A = \langle S, L_{\text{act}}, \{ s \xrightarrow{q_A} s' \mid s \xrightarrow{\epsilon} s' \in \rightarrow \}, s_0 \rangle$$

The projection of $CA$, written $CA_{\downarrow}$, is the communicating system $(CA_{\downarrow A})_{A \in P}$.

Remark 3.4. The projection with respect to $A$ of a c-automaton is $A$-local. $\diamondsuit$

Notably, minimisation also removes $\epsilon$-transitions. For technical reasons, we assume the process of minimisation to be performed through the partition refinement algorithm [31]. In fact, to establish the correspondence between runs of a c-automaton and the ones of its blending (e.g., c.f Theorem 5.12) we need to map the states of the former on the states of the latter. This correspondence is immediate in the partition refinement algorithm (a state of the c-automaton corresponds to the equivalence class containing it in the minimised automaton).

Our results do not depend on the use of the partition refinement algorithm, however, adopting a different algorithm might require to explicitly track this correspondence. Therefore, given $CA = \langle S, L_{\text{int}}, \rightarrow, s_0 \rangle$ and $A \in P$, the states of $CA_{\downarrow A}$, as well as the components of $[CA_{\downarrow}]$, are subsets of $S$.

Besides, by construction, we have

Fact 1. Let $CA = \langle S, L_{\rightarrow}, s_0 \rangle$ be a c-automaton.

i) Given $s \in R(CA)$, there is a unique configuration $\bar{q} \in R([CA_{\downarrow}])$ such that $s \in \bar{q}(A)$ for each $A \in P$.

ii) For any configuration $\bar{q} \in R([CA_{\downarrow}])$, there exists $s \in R(CA)$ such that for each $A \in P$ we have $s \in \bar{q}(A)$.

Example 3.5 (Working example: projections). The projections of $C$ are

Note that the states are subsets of states of $CC$ due to minimisation and that all projections do not have $\epsilon$-transitions. $\diamondsuit$
The projection operation is well-behaved with respect to trace equivalence.

**Proposition 3.6.** If $CA$ and $CA'$ are trace-equivalent c-automata then $CA_{\downarrow A} = CA'_{\downarrow A}$ for each participant $A \in \mathcal{P}$.

**Proof.** Let us consider the intermediate automata $A_A$ and $A_{A'}$ in the projection of $CA$ and $CA'$, respectively. If $A_A$ and $A_{A'}$ are trace-equivalent then their minimal automaton up to trace equivalence is of course the same. We have to show that $A_A$ and $A_{A'}$ are trace equivalent. Take a word $w_A$ in the language of $A_A$. It is obtained from some computation that accepts a word $w_{CA}$ in $CA$. By trace equivalence, the same word is accepted in $CA'$. Executing the computation generating it in $A_{A'}$ will produce $w_A$. This proves the thesis. $\Box$

Thanks to the result above, without loss of generality, we can assume that c-automata are minimal up-to language equivalence (in particular, they do not have $\epsilon$-transitions and they are deterministic).

The following proposition relates the language of the projection with the language of the original automaton.

**Proposition 3.7.** For all c-automata $CA$ and $A \in \mathcal{P}$, $L(CA_{\downarrow A}) = L(CA)_{\downarrow A}$.

**Proof.** Let $A_A$ be the intermediate automaton built in the first step in the projection of $CA$ on $A$. We have

$$L(CA_{\downarrow A}) = L(A_A) = L(CA)_{\downarrow A}$$

where the first equality follows since $CA_{\downarrow A}$ and $A_A$ are trace-equivalent by definition, and the second equality follows since $A_A$ is obtained transition per transition by projecting the labels of $CA$ on $A$. $\Box$

### 4 Well-formed Choreography Automata

In order to ensure that the projection of a c-automaton on a communicating system is well-behaved we require some conditions on the c-automaton. We first define the notion of concurrent transitions.

**Definition 4.1 (Concurrent transitions).** Let $CA = \langle S, \mathcal{L}, \rightarrow, s_0 \rangle$. Two transitions $s \xrightarrow{1} s_1$ and $s \xrightarrow{2} s_2$ are concurrent iff there is a state $s_3 \in S$ and transitions $s_1 \xrightarrow{2} s_3$ and $s_2 \xrightarrow{1} s_3$.

We can now define the key notion of well-branchedness, which intuitively states that each participant is aware of “what it needs to do in the current state”. In other words, each participant is aware of choices made by others when its behaviour depends on those choices. The awareness of choice is checked on spans, namely runs that may constitute alternative branches of choices. Spans are formalised as follows.
Definition 4.2 (s-span). Given a state $s$ of a c-automaton $\mathcal{CA}$, a pair of runs $(\pi_1, \pi_2)$ is an s-span if $\pi_1$ and $\pi_2$ are coinitial at $s$, transition-disjoint, acyclic, and either $\pi_1$ and $\pi_2$ are cofinal otherwise they are both maximal in $\mathcal{CA}$ and share only the state $s$.

Example 4.3 (Working example: spans). In the c-automaton $\mathcal{CV}$ the only state with spans is state 2. The unique 2-span is $(\pi, \pi')$ where

\[
\pi = \begin{array}{c}
2 \xrightarrow{H: \text{ack}} 6 \\
3 \xrightarrow{Q: \text{ack}} 5
\end{array} \quad \pi' = \begin{array}{c}
2 \xrightarrow{H: \text{nack}} 6 \\
4 \xrightarrow{Q: \text{text}} 5
\end{array}
\]

Indeed, $\pi$ and $\pi'$ are coinitial, cofinal, acyclic, and transition disjoint. Note that any other pair of paths from 2 in not a 2-span since the transition from 0 to 1 of $\mathcal{CV}$ would make the paths non-cofinal and either make the paths non transition-disjoint or have the state 0 on both paths. ◇

Intuitively, a choice is well-branched when it has a determined selector, and the other participants either behave uniformly in each branch, or can ascertain which branch has been chosen from the messages they receive.

Definition 4.4 (Well-branchedness). A c-automaton $\mathcal{CA}$ is well-branched if for each state $s$ in $\mathcal{CA}$ and participant $B \in \mathcal{P}$ sender in a transition from $s$:

1. all transitions from $s$ involving $B$, have sender $B$ and pairwise distinct labels;
2. for each transition $t$ from $s$ whose sender is not $B$ and each transition $t'$ from $s$ whose sender is $B$, $t$ and $t'$ are concurrent
3. for each participant $A \neq B \in \mathcal{P}$ and s-span $(\pi_1, \pi_2)$, the first pair of different actions in $\pi_1 \downarrow_A$ and $\pi_2 \downarrow_A$ (if any) is of the form $(\mathcal{CA}^?m, \mathcal{DA}^?n)$ with $C \neq D$ or $m \neq n$.

We dub $B$ a selector at $s$.

Example 4.5 (Working example: well-branchedness). In $\mathcal{CV}$ all the states satisfy the conditions of Definition 4.4; the only non-trivial case is state 2. Condition (1) holds for $H$, which is the selector of the choice at 2; condition (2) holds vacuously, and condition (3) holds for both $Q$ and $I$ in the unique 2-span $(\pi, \pi')$, described in Example 4.3. Indeed, the first actions of $Q$ on $\pi$ and $\pi'$ are the input from $H$ which have different messages; likewise, the first actions of $I$ on $\pi$ and $\pi'$ are the inputs from $Q$, which have different messages as well. ◇

Condition (2), vacuously true in the example, is needed when multiple participants act as sender in the same state: this ensures that the only possibility is that actions of different participants are concurrent so that possible choices at a state are not affected by independent behaviour.

We also rule out c-automata having consecutive transitions involving disjoint participants and not independent of each other.

Definition 4.6 (Well-sequencedness). A c-automaton $\mathcal{CA}$ is well-sequenced if each pair of consecutive transitions $s \xrightarrow{A:B:m} s' \xrightarrow{C:D:n} s''$ either
– share a participant, that is \( \{A, B\} \cap \{C, D\} \neq \emptyset \), or
– are part of a diamond, i.e. there is \( s'' \) such that \( s \xrightarrow{c \to D; n} s''' \xrightarrow{A \to B; m} s'' \).

**Notation.** For the sake of readability, a well-sequenced c-automaton \( CA \) can be represented by omitting, for each diamond, two of its consecutive transitions. We call such representation compact. Notice that, given a compact representation, it is always possible to recover the original c-automaton. So far and hereafter we assume that all c-automata are compactly represented.

**Example 4.7 (Working example).** Both \( CV \) and \( CP \) are well-sequenced because they enjoy the first condition of Definition 4.6. Instead, the participants on the transitions from 1 to 2 and from 2 to 0 in \( CC \) are disjoint; in fact, we assume \( CC \) to be compactly represented.

Well-sequencedness is necessary to establish a precise correspondence between the language of a c-automaton and its corresponding communicating system (cf. Theorem 4.12) as well as to ensure the applicability of the blending operation (and hence systems composition to work).

**Definition 4.8 (Well-formedness).** A c-automaton is well-formed if it is both well-branched and well-sequenced.

**Remark 4.9.** Not any c-automaton can be “completed” to a well-sequenced automaton, as shown by the following example.

Let us consider the following c-automaton

![Diagram](attachment:image.png)

The above c-automaton is not well sequenced because of the transitions 0 – 1 and 1 – 2. So we complete the diamond.

![Completed Diagram](attachment:image2.png)

The resulting automaton is still not well-sequenced, because of the transitions 3 – 2 and 2 – 0. So we complete the diamond.
The resulting automaton is still not well-sequenced, because of the transitions 4 - 0 and 0 - 3. So we complete the diamond.

The resulting automaton is still not well-sequenced, because of the transitions 5 - 3 and 3 - 4. So we complete the diamond.... And we can go on forever.

The impossibility to complete the initial c-automaton depends on the fact that the intended completed automaton should generate a non regular language (since we should generate strings with a number of $C \rightarrow D: c$ interaction which is, roughly, double of the number of $A \rightarrow B: a$ interactions. It would hence be interesting to know whether, in case the expected completed interaction language of a c-automaton is regular and prefix-closed, it is possible to generate it also by means of a well-formed c-automaton. It would be also interesting to know whether a condition can be given on cycles such that the completion of a c-automaton is possible.

4.1 Languages and well-formedness

We show a closure property of the language of well-sequenced c-automata.

**Definition 4.10 (Commutation of independent interactions).** Two choreography words $w$ and $w'$ are equal up-to-commutation of independent interactions, written $w \sim w'$, if one can be obtained from the other by repeatedly swapping consecutive interactions with disjoint sets of participants.
Given a choreography language $L$

$$
\text{close}(L) = \{ w \in L_{\text{out}} \mid \exists w' \in L. w \sim w' \}
$$
is the closure under commutation of independent interactions of $L$.

**Proposition 4.11.** Let $CA$ be a well-sequenced $c$-automaton. Then $\mathcal{L}(CA)$ is closed under commutation of independent actions, i.e. $\mathcal{L}(CA) = \text{close}(\mathcal{L}(CA))$.

**Proof.** Let us assume $\mathcal{L}(CA) \neq \text{close}(\mathcal{L}(CA))$. Clearly, $\mathcal{L}(CA) \subseteq \text{close}(\mathcal{L}(CA))$. Then there exist $w' \in \text{close}(\mathcal{L}(CA))$ such that $w' \notin \mathcal{L}(CA)$. By definition of closure, there is $w \in \mathcal{L}(CA)$ such that $w \sim w'$. By definition of $\sim$, there exist two choreographic words $w_1$ and $w_2$ such that $w \sim w_1 \cdot (A \to B: b) \cdot (C \to D: c) \cdot w_2 \in \mathcal{L}(CA)$ and $w_1 \cdot (C \to D: c) \cdot (A \to B: b) \cdot w_2 \notin \mathcal{L}(CA)$, where $\{A, B\} \cap \{C, D\} = \emptyset$. This implies that $CA$ is not well-sequenced. In fact, if that were not the case, let $s$ and $s'$ be the states of $CA$ from which the subwords $(A \to B: b) \cdot (C \to D: c)$ and $w_2$ are, respectively, generated. By well-sequencedness, there is also a path from $s$ to $s'$ generating $(C \to D: c) \cdot (A \to B: b)$, so implying that $w_1 \cdot (C \to D: c) \cdot (A \to B: b) \cdot w_2 \in \mathcal{L}(CA)$, which is a contradiction. \(\Box\)

Notice that the converse of the above proposition does not hold without the assumption of implicit diamonds. In fact, consider the following $c$-automaton

![Diagram of a c-automaton]

we can check that $\mathcal{L}(CA) = \text{close}(\mathcal{L}(CA))$ but $CA$ is not well-sequenced.

The next result establishes that the language of a well-formed $c$-automaton coincides with the language of the communicating system made of its projection. This provides a correctness criterion for our projection operation. The proof requires to reason coinductively to account for possibly infinite runs. In particular, we adopt the coinduction style advocated in [22] which, without any loss of formal rigour, promotes readability and conciseness.

**Theorem 4.12.** If $CA$ is a well-formed $c$-automaton then $\mathcal{L}(\llbracket CA \rrbracket) = \mathcal{L}(CA)$.

**Proof.** We first show that $\mathcal{L}(CA) \subseteq \mathcal{L}(\llbracket CA \rrbracket)$.

Let us take a trace $\sigma \in \mathcal{L}(CA)$. We need to show that $\sigma \in \mathcal{L}(\llbracket CA \rrbracket)$. We shall actually prove a stronger thesis:

If $CA$ accepts $\sigma$ starting from a state $s$, then $\llbracket CA \rrbracket$ accepts $\sigma$ starting from the unique (by Fact[13]) state $\vec{q} = (q_A)_{A \in \mathcal{P}}$ such that for each $A \in \mathcal{P}$ we have $s \in q(A)$.

The proof is by coinduction. If $\sigma$ is empty, there is nothing to prove. Otherwise, let $\sigma$ be $(B \to C: m) \cdot \sigma'$, where $B \to C: m$ is accepted through a transition $s \xrightarrow{B \to C: m} s'$ and where $\sigma'$ is accepted by $CA$ starting from $s'$. 

Consider now the intermediate automata built in the first step of the projection of \( CA \) (Definition 3.3). By construction, \( s \xrightarrow{BC^m} s' \) in \( A_B \), \( s \xrightarrow{BC^m} s' \) in \( A_C \) and \( s \xrightarrow{m} s' \) in \( A_B \) for each \( D \neq B, C \). By coinduction, \( \sigma' \) is accepted by \( [CA] \) starting from the unique state \( q' = (q_A)_{A \in P} \) such that for each \( A \in P \) we have \( s' \in q'(A) \). Hence, by definition of minimization and by definition of \( [CA] \), we get that \( [CA] \) accepts \( \sigma \) starting from the state \( q = (q_A)_{A \in P} \) such that for each \( A \in P \) we have \( s \in q(A) \).

We now show that \( L([CA]) \subseteq L(CA) \). As before, we consider a stronger thesis:

- if \( [CA] \) accepts \( \sigma \) starting from a state \( q = (q_A)_{A \in P} \), then \( CA \) accepts \( \sigma \) starting from any state \( s \) such that for each \( A \in P \) we have \( s \in q(A) \).

The thesis will follow since by definition of synchronous semantics \( s_0 \in q_0(A) \) for each \( A \in P \).

The proof is by coinduction. If \( \sigma \) is empty, there is nothing to prove. Otherwise, let \( \sigma = (B \rightarrow C: m) \cdot \sigma' \), where \( B \rightarrow C: m \) is accepted by a transition \( q \xrightarrow{B \rightarrow C: m} q' \). Hence, by definition of synchronous semantics, there are transitions \( q(B) \xrightarrow{BC} q'(B) \) and \( q(C) \xrightarrow{BC} q'(C) \). Moreover, \( q(D) = q'(D) \) for each \( D \neq B, C \).

Let now \( s \) be any state such that for each \( A \in P \) we have \( s \in q(A) \) (by Fact 1.10). Hence, in the intermediate automata built in the first step of the projection of \( CA \) (Definition 3.3), by definition of minimization up-to language equivalence, we have

\[
s \xrightarrow{1} \cdots \xrightarrow{1} \tilde{s}_B \xrightarrow{BC} s'_B \text{ in } A_B \text{ and } s \xrightarrow{1} \cdots \xrightarrow{1} \tilde{s}_C \xrightarrow{BC} s'_C \text{ in } A_C. \tag{2}
\]

The two transitions \( \tilde{s}_B \xrightarrow{BC} s'_B \) and \( \tilde{s}_C \xrightarrow{BC} s'_C \) may be both obtained from the projection of the same transition of \( CA \) or not. Let us consider the two cases separately.

- \( \tilde{s}_B \xrightarrow{BC} s'_B \) and \( \tilde{s}_C \xrightarrow{BC} s'_C \) are both obtained from the projection of some transition \( t : \tilde{s} \xrightarrow{B \rightarrow C: m} s' \) of \( CA \), with \( \tilde{s} = \tilde{s}_B = \tilde{s}_C \) and \( s' = s'_B = s'_C \).

In a such a case we will show that \( t \) is enabled in \( s \), that is \( t \) is of the form \( s \xrightarrow{B \rightarrow C: m} s' \), with \( s' = s'_B = s'_C \). By contradiction, let us assume that not to be the case, and let \( \pi_1 \) be one of the runs with minimal length \( n > 0 \) from \( s \) to a state \( \tilde{s} \) such that \( \tilde{s} \xrightarrow{B \rightarrow C: m} s' \). Let us call \( t' \) the last transition in \( \pi_1 \). By well-sequencedness (Definition 4.10) \( t' \) involves either \( B \) or \( C \), or it is concurrent to \( t \).

- If \( t' \) is concurrent with \( t \), then there is a transition with label \( B \rightarrow C: m \) coinital with \( t' \). This implies the existence of a run shorter than \( \pi_1 \), which was assumed to be minimal, and hence a contradiction.
- If \( t' \) is not concurrent with \( t \), then \( t' \) should involve either \( B \) or \( C \). Let us assume it to involve \( B \) but not \( C \), the other cases are analogous. By 2,
we know that there is a run \( s \overset{\epsilon}{\rightarrow} \cdots \overset{\epsilon}{\rightarrow} \hat{s}_B \) in \( A_B \) and a run \( s \overset{\epsilon}{\rightarrow} \cdots \overset{\epsilon}{\rightarrow} \hat{s}_C \) in \( A_C \) both with all labels \( \epsilon \) and such that \( \hat{s}(\hat{s}_B = \hat{s}_C) \) enables \( t \) in \( CA \).

Because of the projection on \( B \), there exists a run \( \pi_2 \), in \( CA \), not involving \( B \). We can assume that the last transition of \( \pi_2 \) is not concurrent to \( t \), as above. Thus, thanks to well-sequencedness, it necessarily ends with a transition involving \( C \). We can also assume \( \pi_2 \) to have minimal length.

Note that \( \pi_1 \) and \( \pi_2 \) are coinitial, cofinal, and different since one contains transitions involving \( B \) but not \( C \) and the other one contains transitions involving \( C \) but not \( B \). Since we assumed that they are minimal they are also acyclic. By removing common transitions and selecting any of the remaining run fragments \( \pi'_1 \) of \( \pi_1 \) involving \( B \) and the corresponding fragment \( \pi'_2 \) of \( \pi_2 \) (which does not involve \( B \)) we obtain runs which are coinital, cofinal, acyclic and transition disjoint. By construction, \( \pi'_1 \downarrow_B \) and \( \pi'_2 \downarrow_B \) accept different words (since the second is empty while the first is not) and the first pair of different characters is not an input for \( B \) in both the cases (since in \( \pi'_2 \) no such character exists). However, this contradicts well-branchedness hence this case can never happen.

Thus, there is a transition \( s \overset{B \rightarrow C: m}{\rightarrow} s' \) in \( CA \). The state \( s' \) is such that \( s' \in \bar{q}'(A) \) for each \( A \in \mathcal{P} \), thanks to the properties of \( s \) and of the definition of synchronous semantics. By coinduction, this state accepts \( \sigma' \), and the thesis follows.

\( \hat{s}_B \overset{B \rightarrow C: m}{\rightarrow} \hat{s}'_B \) and \( \hat{s}_C \overset{B \rightarrow C: m}{\rightarrow} \hat{s}'_C \) are obtained by projecting two different transitions \( t_1 \) and \( t_2 \) in \( CA \) with the same label \( B \rightarrow C: m \).

Since \( CA \)s are deterministic (Definition 3.1), there are two different runs \( \pi_1 \) and \( \pi_2 \) in \( CA \), the former ending with \( t_1 \) and the latter ending with \( t_2 \). We consider now two possible subcases.

- If \( \pi_1 \downarrow_B = \pi_2 \downarrow_B \) and \( \pi_1 \downarrow_C = \pi_2 \downarrow_C \),
  then the two runs project to the same run in \( CA \downarrow_B \) and \( CA \downarrow_C \). As a result, any of the two transitions can be used to answer the challenge, and the thesis follows by coinduction as above.

- If instead the projections on either \( B \) or \( C \) are different,
  then we can remove common transitions and cycles and select any of the pairs of run fragments (which are coinitial, transition disjoint and acyclic) highlighting the difference in the projection. If they are not the final fragments, then they are also cofinal, otherwise we can extend them till they become cofinal or both maximal. By well-branchedness the first action involving the role on which the projections differ should be an input on both the runs, and the two inputs should be different. If the projection is different on \( B \), since we assumed that \( B \rightarrow C: m \) is the first action involving \( B \), then we have a contradiction (since it is an output). If the projection is different on \( C \), then \( B \rightarrow C: m \) is the first action involving \( C \), and it is the same on both runs, against the hypothesis that the two projections on \( C \) were different.

Anyway, this case cannot happen. \( \square \)
4.2 Communication soundness and well-formedness

In this section we show that the projection of well-formed choreography automata enjoys relevant correctness properties.

The projection of a well-formed c-automaton is live \cite{33}: if there is a participant willing to take a transition, there is a computation executing such transition.

Proposition 4.13 (Liveness). Let $CA$ be a well-formed c-automaton. For each $\vec{q} \in R([CA]_\downarrow)$, if there is $A \in P$ with an outgoing transition $t$ from $\vec{q}(A)$ in $CA_{\downarrow A}$, then there exists a run of $[CA]_\downarrow$ including a transition which has $t$ as component transition.

Proof. Since $\vec{q}$ is reachable, there is a run to $\vec{q}$ producing some word $w$. Consider the unique state $q_s$ of $CA$ such that $q_s \in \vec{q}(A)$ for each $A \in P$. The c-automaton $CA_s$ obtained from $CA$ by setting $q_s$ as initial state is well-formed. Now, assume there exists $A$ such that $\vec{q}(A)$ has an outgoing transition $t$. Thanks to the definition of projection, there is a transition $t'$ in $CA_s$ involving $A$ and another participant, say $B$. Such a transition is on a run with word $w'$. A run with word $w'$ exists also in $[CA]_\downarrow$ thanks to Theorem 4.12, it goes through $\vec{q}$, since the state reached is fully determined by the label (since participants are deterministic), and it executes $t$ as desired. This proves the thesis. $\square$

Proposition 4.15 below shows that the projection of a well-formed c-automaton is lock-free \cite{19}, namely that enabled transitions of a participant are sooner or later fired. We formalise locks first. (Recall that the transitions of projected CFSMs are minimal up-to language equivalence and hence deterministic.)

Definition 4.14 (Lock). Let $CA$ be a c-automaton. A state $\vec{q} \in R([CA]_\downarrow)$ is a lock if there is $A \in P$ with an outgoing transition $t$ from $\vec{q}(A)$ in $CA_{\downarrow A}$, yet in all runs of $[CA]_\downarrow$ starting from $\vec{q}$ there is no transition $t'$ involving $A$.

Proposition 4.15 (Lock freedom). Let $CA$ be a well-formed c-automaton. For each $\vec{q} \in R([CA]_\downarrow)$, $\vec{q}$ is not a lock.

Proof. Since $\vec{q}$ is reachable, there is a run to $\vec{q}$ producing some word $w$. Consider the unique state $q_s$ of $CA$ such that $q_s \in \vec{q}(A)$ for each $A \in P$. The c-automaton $CA_s$ obtained from $CA$ by setting $q_s$ as initial state is well-formed. Now, assume there exists $A$ such that $\vec{q}(A)$ has an outgoing transition $t$. Thanks to the definition of projection, there is a transition $t'$ in $CA_s$ involving $A$ and another participant, say $B$. Transition $t'$ is on a run with word $w'$ in $CA_s$, and thanks to Theorem 4.12 also in $[CA_s]_\downarrow$. We have to show that each run in $[CA_s]_\downarrow$ has a transition involving $A$. This will imply the thesis.

Assume towards a contradiction that there is a run not involving $A$. Such a run and the one including $t'$ are coinitial. By removing common transitions they can be made transition disjoint, and by removing cycles they can also be taken cycle free. Thanks to well-branchedness, since one of the two paths contains a transition with $A$, the other one also need to include a transition with $A$, against the hypothesis. This proves the thesis. $\square$
Interestingly, Proposition 4.15 does not require any notion of fairness like in frameworks that syntactically constrain loops (e.g., in behavioural types loops do not have continuations). Instead, look-freedom requires some notion of fairness in frameworks admitting continuations after loops (e.g., [35]). In our case, fairness is not required since well-branchdness imposes that each participant involved in a continuation after a loop should also be involved within the loop.

The next proposition shows that the projections of a well-formed c-automaton form a deadlock-free communicating system. We first define deadlock configurations as those of communicating systems with no outgoing transitions, yet with at least one component willing to take a transition. Definition 4.16 adapts the ones in [26, 35] to a synchronous setting.

**Definition 4.16 (Deadlock).** A state \( \vec{q} \in R(S) \) reachable in a communicating system \( S = (M_A)_{A \in \mathcal{P}} \) is a deadlock if \( \vec{q} \) has no outgoing transitions, yet there exists \( A \in \mathcal{P} \) such that \( \vec{q}(A) \) has an outgoing transition in the CFSM of \( S(A) = M_A \).

**Proposition 4.17 (Deadlock freedom).** If \( CA \) is a well-formed c-automaton and \( \vec{q} \in R([\{CA\}]) \), \( \vec{q} \) is not a deadlock.

**Proof.** We prove the contrapositive. If participant \( A \in \mathcal{P} \) has an outgoing transition \( t \) from \( \vec{q}(A) \) in \( CA \downarrow A \), then (Proposition 4.15) there is a run involving \( \vec{q} \) where a transition \( \vec{q}(A) \) is fired. \( \square \)

### 5 Composition of Choreography Automata

In this section we study how to compose two or more (well-formed) c-automata into a single c-automaton. Composition is obtained by means of two operations:

1. the product of c-automata, building a c-automaton corresponding to the concurrent execution of the two original c-automata;
2. a blending operation that, given two roles of a c-automaton, removes them and adjusts the c-automaton as if they became two “coupled” forwarders.

For instance, blending \( H \) and \( K \) removes participants \( H \) and \( K \) and sends each message \( m \) originally sent to \( H \) to whoever \( K \) used to send \( m \), and vice versa.

#### 5.1 The blending algorithm

We present and discuss the blending operation, which will then be used for defining c-automata composition. We start by giving an informal algorithmic presentation of the blending operation on two roles \( H \) and \( K \) in a given c-automaton. (A more formal presentation of the algorithm is in Appendix A.2.)

**The (informal) BLENDING algorithm**

**Input:** a c-automaton and two roles, \( H \) and \( K \).

**Output:** a c-automaton.

**begin**
I) each transition with \( p \xrightarrow{A \rightarrow H: m} q \) (resp. \( p \xrightarrow{A \rightarrow K: m} q \)) is removed, and for each transition \( q \xrightarrow{K \rightarrow B: m} r \) (resp. \( q \xrightarrow{H \rightarrow B: m} r \)) a transition \( p \xrightarrow{A \rightarrow B: m} r \) (resp. \( p \xrightarrow{A \rightarrow B: m} r \)) is added provided that \( A \neq B \); if \( A = B \) the blending is not defined;
II) transitions involving neither \( H \) nor \( K \) are preserved, whereas all other transitions are removed;
III) states and transitions unreachable from the initial state are removed.

end

Notice how the blending operation can be interpreted as a sort of “self-composition” of a single system. It is worth remarking that self-composition is a relevant operation per-se; it can be used, for instance, to get rid of two participants \( H \) and \( K \) by transforming them into a “hidden” bidirectional forwarder, letting their partners in the interaction to communicate directly. Of course, blending \( H \) and \( K \) could cause the overall behaviour of the system to drastically change, unless some conditions are satisfied (see the notion of reflectiveness in Definition 5.10 and Theorem 5.12). Conditions essentially require that if a message is forwarded from \( H \) to \( K \) then \( K \) expects it, and vice versa.

Remark 5.1. The blending operation can be performed in time \( O(n^2) \) where \( n \) is the number of transitions in the c-automaton, see Proposition A.1 in Appendix A.2.

Before proceeding with the formalisation of the blending operation, it is convenient to introduce some notations. Let \( \hat{CA} \) denote the c-automaton obtained by removing the states unreachable from the initial state of the c-automaton \( CA \). The set of transitions of \( CA \) involving participants in \( P \subseteq \mathcal{P} \) is

\[
CA_{\alpha \delta} = \{ p \xrightarrow{A \rightarrow B: m} q \in CA \mid \{A, B\} \cap P \neq \emptyset \}
\]

Given two participants \( H \) and \( K \),

\[
CA_{(H,K)} = \{ p \xrightarrow{A \rightarrow B: m} q \mid \exists r \in S : p \xrightarrow{A \rightarrow H: m} r, r \xrightarrow{K \rightarrow B: m} q \in CA \}
\]

is the set of blended transitions of \( CA \) with respect to \( H \) and \( K \).

Definition 5.2 (Blending). Given two participants \( H \) and \( K \), the blending in a c-automaton \( CA = \langle S, \mathcal{L}_{int}, \rightarrow, s_0 \rangle \) of \( H \) and \( K \), written \( CA^{H\bowtie K} \), is the c-automaton

\[
CA^{H\bowtie K} = \begin{cases} 
\langle S, \mathcal{L}_{int}, \rightarrow_{\bowtie}, s_0 \rangle, & \text{if } \forall p \xrightarrow{A \rightarrow B: m} q \in \rightarrow_{\bowtie} : A \neq B \\
\bot, & \text{otherwise}
\end{cases}
\]

where \( \rightarrow_{\bowtie} = \rightarrow \cup CA_{(H,K)} \cup CA_{(K,H)} \setminus CA_{\alpha \delta} \).

The decoration \((H,K)\) is pleonastic; it is a convenience for Definition 5.3 and some proofs.
Example 5.3. In Section 1, the construction in Definition 5.2 is used to perform the blending on H and K of the product automaton on the left, producing the automaton on the right.

The blending operation enjoys two different forms of commutativity.

Lemma 5.4. Given a c-automaton CA and participants H, K, I, and J, we have

1. $CA^{H \bowtie K} = CA^{K \bowtie H}$
2. $(CA^{H \bowtie K})^{b\sigma J} = (CA^{K \bowtie J})^{b\sigma H}$

Proof. The proof of (1) straightforwardly follows from Definition 5.2 since

$\rightarrow_{\bowtie} = (\rightarrow \cup CA_{(H,K)} \cup CA_{(K,H)}) \setminus CA_{(H,K)}$

To prove (2) we show that the c-automata on the two sides of the equality have the same set of transitions. By Definition 5.2 the transitions of $CA^{H \bowtie K}$ are

$\rightarrow_{1} = (\rightarrow \cup CA_{(H,K)} \cup CA_{(K,H)}) \setminus CA_{(H,K)}$

where $\rightarrow$ are the transitions of CA. Hence, again by Definition 5.2,

$\rightarrow_{2} = (\rightarrow_{1} \cup CA^{H \bowtie K}_{(i,j)} \cup CA^{K \bowtie J}_{(j,i)}) \setminus CA_{(i,i)}$

are the transitions of $(CA^{H \bowtie K})^{b\sigma J}$. Similarly, we have that

$\rightarrow'_{2} = (\rightarrow_{1} \cup CA^{K \bowtie K}_{(H,K)} \cup CA^{H \bowtie J}_{(K,H)}) \setminus CA_{(i,i)}$

are the transitions of $(CA^{K \bowtie j})^{b\sigma H}$ provided that $\rightarrow'_{1}$ are the transitions of $CA^{b\sigma J}$.

We prove $\rightarrow_{2} = \rightarrow'_{2}$. Let $t \in \rightarrow_{2}$ be a transition with label $A \rightarrow B : m$. It is a simple observation that $\{A, B\} \cap \{I, J\} = \emptyset$ (by construction any transition where participants are those in the interface are removed). Hence we have three cases:

- If $t \in \rightarrow_{1}$
- If $t \in CA^{H \bowtie K}_{(i,j)}$ then we also have $\{A, B\} \cap \{H, K\} = \emptyset$. And, assuming $t = p \xrightarrow{A \rightarrow B : m} q$, there is $r$ such that $p \xrightarrow{A \rightarrow I : m} r \xrightarrow{J \rightarrow B : n} q$ are in $\rightarrow_{1}$. We proceed by case analysis:
  - Both $p \xrightarrow{A \rightarrow I : m} r$ and $r \xrightarrow{J \rightarrow B : m} q$ are in CA. In this case, $p \xrightarrow{A \rightarrow I : m}$ $q$ is in $CA^{b\sigma J}$ and hence in $\rightarrow'_{2}$ since $\{A, B\} \cap \{I, J\} = \emptyset$. This implies that $p \xrightarrow{A \rightarrow B : m}$ $q$ is in $(CA^{b\sigma J})^{b\sigma H}$ since $\{A, B\} \cap \{H, K\} = \emptyset$.
  - Only one between $p \xrightarrow{A \rightarrow I : m} r$ and $r \xrightarrow{J \rightarrow B : m} q$ is in CA. With no loss of generality, assume that only $r \xrightarrow{J \rightarrow B : m} q$ is in CA. Then $p \xrightarrow{A \rightarrow I : m} r$ is in $CA^{H \bowtie K}$ and therefore there is $r'$ such that $p \xrightarrow{A \rightarrow H : m}$ $r'$ $\xrightarrow{K \rightarrow I : m}$ $q$ is in CA (the case $p \xrightarrow{A \rightarrow K : m}$ $r'$ $\xrightarrow{H \rightarrow I : m}$ $r$ is CA is similar). Therefore, $r' \xrightarrow{K \rightarrow B : m} q$ in $CA^{b\sigma J}$ which in turn implies that $t \in (CA^{b\sigma J})^{b\sigma H}$.

• None of $p \xrightarrow{A} m \rightarrow r$ and $r \xrightarrow{J \rightarrow B} m \rightarrow q$ is in CA. Applying twice the argument in the previous item we get the thesis.

- If $t \in CA_{(H,K)}^{(H \bowtie K)}$ we proceed as in the previous case.

The inclusion $\rightarrow_2' \subseteq \rightarrow_2$ is proved similarly.

Whereas it is quite natural to expect property (1) to hold – due to the perfect symmetry between $H$ and $K$ in the definition of blending – property (2) sounds less natural. In fact, one could imagine that blending on $H$ and $K$ could affect the behaviour of the other participants of CA, including $I$ and $J$. Remarkably, blending is such that property (2) holds. Such result, besides being interesting on its own, will be used also to prove associativity of the composition. This is quite relevant if one takes into account that the development of systems in a modular way is one of the main goals of our framework for choreographies composition.

In fact modularity naturally requires the composition operation to be insensitive on the order of application.

We provide a condition ensuring that blending does not essentially affect the behaviour of participants, namely that the only difference between CA and $CA_{(H,K)}^{(H \bowtie K)}$ is that some messages sent/received by $H$ or $K$ in CA are sent/received by other participants in $CA_{(H,K)}^{(H \bowtie K)}$. The following definition is instrumental to formally establish the informal property above; it casts the projection operation on c-automata where some interfaces have been blended.

**Definition 5.5 (Up-to-interface projection).** The projection on participant $A$ up-to-interface $(H,K)$ of a c-automaton $CA = \langle S, L_{\text{int}}, \rightarrow, s_0 \rangle$ is the automaton $CA_{\downarrow A}^{(H \bowtie K)} = \langle S, L_{\text{act}}, \rightarrow', s_0 \rangle$ where

\[
\rightarrow' = \{ t \downarrow A \}_{t \rightarrow} \cup \{ p \xrightarrow{A,X,m} q \mid p \xrightarrow{H \bowtie K} q \in \rightarrow' \} \cup \{ p \xrightarrow{Y \bowtie m} q \mid p \xrightarrow{A} m \xrightarrow{X,Y} q \in \rightarrow' \}
\]

**Lemma 5.6.** If $CA_{(H,K)}^{(H \bowtie K)}$ is defined, its states are included in the states of CA.

**Proof.** By construction, since the first two steps do not change the set of states, and the last one may only remove states. \( \square \)

**Proposition 5.7 (Simulation).** If $CA_{(H,K)}^{(H \bowtie K)}$ is defined then for each participant $A \neq H,K$ we have that $CA_{\downarrow A}$ simulates $CA_{(H \bowtie K)}^{(H \bowtie K)}$.

**Proof.** Recall that states of projection are equivalence classes of states of the projected automata up to trace equivalence. Let us consider the relation

\[ \mathcal{R} = \{ (S', S) \mid S \cap S' \neq \emptyset, S \in CA_{\downarrow A}, S' \in CA_{(H \bowtie K)}^{(H \bowtie K)} \} \]

The proof is by coinduction. Take any transition $S' \xrightarrow{\alpha} S'_1$ in $CA_{(H \bowtie K)}^{(H \bowtie K)}$. By definition of projection there is a transition $t' = s' \xrightarrow{\alpha} s'_1$ in $CA_{(H \bowtie K)}^{(H \bowtie K)}$ with $s' \in S'$ whose projection up-to-interface on $A$ yields label $\alpha$. By definition of blending, transition $t'$ is derived by transitions of CA. There are a few cases:
– if \( t' \) involves neither \( H \) nor \( K \) then \( s' \xrightarrow{\alpha} s'_1 \) is in \( CA \), and \( [s'] \xrightarrow{\alpha'} \ [s'_1] \) is in \( CA \downarrow_A \) as desired since the two target states are \( R \)-related;
– if \( t' = s' \xrightarrow{A \rightarrow H : m} H K \) then \( CA \) has transitions

\[
s' \xrightarrow{A \rightarrow H : m} s'_2 \quad \text{and} \quad s'_2 \xrightarrow{K \rightarrow B : m} s'_1
\]

hence in \( CA \downarrow_A \) we have \( [s'] \xrightarrow{HA} \ [s'_2] = [s'_1] \) as desired.

The case where \( A \) is the receiver is analogous to the last case above. \( \square \)

One would hope the operation of blending to affect only the participants that are blended; however, this is not the case in general as shown by the following example.

Example 5.8. Consider the following well-formed c-automaton and its (empty) blending on \( H \) and \( K \):

```
CA  \xrightarrow{B \rightarrow m} K
   \xrightarrow{D \rightarrow n} B
```

Trivially, \( D \) behaves differently in \( CA \) than in \( CA^{H \bowtie K} \).

Remark 5.9. The above example highlights an undesirable feature of blending. We therefore identify a condition on c-automata that avoids that blending modifies the behaviour on non-interface participants. \( \diamond \)

Definition 5.10 (Reflective c-automata). Let \( H, K \) be two participants. A c-automaton \( CA \) is \( H \)-to-\( K \) reflective if

(1) for each input transition \( S \xrightarrow{AH} S' \) in \( CA \downarrow H \) we have:
   (a) there is \( S'' \xrightarrow{K} S''' \) in \( CA \downarrow K \) with \( S' \cap S'' \neq \emptyset \);
   (b) for each \( S'' \xrightarrow{K} S''' \) in \( CA \downarrow K \), there are states \( s \in S, s' \in S', S'' \subseteq S''' \) such that \( s \xrightarrow{A} \ x \rightarrow H \rightarrow \ x \rightarrow K \rightarrow B \rightarrow m \rightarrow s' \) in \( CA \).

(2) for each output transition \( S \xrightarrow{HA} S' \) in \( CA \downarrow H \) we have:
   (a) there is \( S'' \xrightarrow{B} S''' \) in \( CA \downarrow K \) with \( S'' \cap S \neq \emptyset \);
   (b) for each \( S'' \xrightarrow{K} S''' \) in \( CA \downarrow K \), there are states \( s'' \in S'', s''' \in S''' \) such that \( s'' \xrightarrow{X} s''' \xrightarrow{H} A \rightarrow m \rightarrow s' \) in \( CA \).

If \( CA \) is \( H \)-to-\( K \) reflective and \( K \)-to-\( H \) reflective then we say that \( CA \) is reflective on \( H \) and \( K \).

A non-trivial example of reflective c-automaton is given below.
Example 5.11. The \( c \)-automaton below is reflective on \( H \) and \( K \).

![Diagram of the c-automaton]

For instance, the three inner states on the dashed path would collapse in the same equivalence class when projecting on \( H \).

Instead, the \( c \)-automaton \( CA \) in Example 5.8 is not reflective on \( H \) and \( K \) (in particular it is not \( K \)-to-\( H \) reflective). In fact, on the projections

\[
\begin{align*}
CA_{\downarrow H} &\ {\{1, 2, 3\}} & \quad \text{CA}_{\downarrow K} &\ {\{1\}} & \quad \text{KB} \ ? \ {\{2, 3\}}
\end{align*}
\]

condition (1a) of Definition 5.10 does not hold.

We now formally show that reflectiveness on \( H \) and \( K \) does guarantee that the behaviour of any participant \( A \neq H, K \) is not “affected” by the blending \( H \) and \( K \).

Theorem 5.12 (Bisimulation). If \( CA \) is reflective on \( H \) and \( K \) then \( CA_{\downarrow A} \) and \( CA_{\downarrow H} \bowtie \downarrow K \) are bisimilar, for each participant \( A \neq H, K \).

Proof. Fix the participant \( A \in \mathcal{P} \). Let us consider the relation \( R \) defined as follows: \( R = \{(S', S)| S \cap S' \neq \emptyset, S \in CA_{\downarrow A}, S' \in CA_{\downarrow H} \bowtie \downarrow K \} \).

The proof is by coinduction. One direction follows from Proposition [a].

Let us consider the other direction, namely that \( CA_{\downarrow H} \bowtie \downarrow K \) simulates \( CA_{\downarrow A} \).

Take any transition \( S \xrightarrow{\alpha} S' \) in \( CA_{\downarrow A} \).

There are a few cases depending on the transition:

- If the transition does not involve \( H \) nor \( K \) then by definition of projection in \( CA \) there is \( s \in S \) such that \( s \xrightarrow{\alpha'} s' \) for some \( s' \in S' \) such that the projection of \( \alpha' \) on \( A \) is \( \alpha \). Also, in \( CA_{\downarrow H} \bowtie \downarrow K \) \( s \xrightarrow{\alpha'} s' \), and in \( CA_{\downarrow H} \bowtie \downarrow K \) \( [s] \xrightarrow{\alpha'} [s'] \) as desired. The two target states are in the relation hence we are done.

- If the transition is \( S \xrightarrow{A H \ x m} S' \) then by (2a) in Definition 5.10 there is a transition \( S'' \xrightarrow{K B \ x m} S''' \) in \( CA_{\downarrow K} \) with \( S' \cap S'' \neq \emptyset \). By the (2b) in Definition 5.10 there are states \( s \in S, s' \in S' \cap S'', s''' \in S''' \) such that \( s \xrightarrow{A H \ x m} s' \) and \( s'' \xrightarrow{K B \ x m} s''' \) in \( CA \). As a consequence in \( CA_{\downarrow H} \bowtie \downarrow K \) we have \( s \xrightarrow{A H \ x m} s' \). Thus, in \( CA_{\downarrow A} \bowtie \downarrow H \) we have \( [s] \xrightarrow{A H \ x m} [s'] \). The thesis follows since \( [s'''] = [s'] \) given that in the projection on \( A \) they are connected by an \( \epsilon \)-transition and \( [s''] \cap [s'] \neq \emptyset \).
– If the transition is \( S^H \xrightarrow{a} S' \) then by \((\text{1a})\) in Definition 5.10 there is \( S'' \xrightarrow{B} S''' \) in \( CA\downarrow_k \) with \( S'' \cap S \neq \emptyset \). Also, by \((\text{1b})\) in Definition 5.10 there are states \( s'' \in S'', s''' \in S''' \cap S, s' \in S' \) such that \( s'' \xrightarrow{B} s''' \xrightarrow{H} s' \) in \( CA \). As a consequence in \( CA \downarrow_k \) we have \( s'' \xrightarrow{B} s''' \xrightarrow{H} s' \). Thus, in \( CA \downarrow_k \) we have \( [s'] \xrightarrow{H} [s'] \) as desired.

\( \square \)

Remark 5.13. The blending operation is sensitive with respect to the representation of c-automata, as shown in the following example. If we apply the blending operation on participants \( H \) and \( K \) directly on the canonical automaton below

\[
\begin{array}{cccc}
CA & \xrightarrow{a} & A & \xrightarrow{a} \circ & C & \xrightarrow{b} \circ & B & \xrightarrow{a} \circ & CA \downarrow_k \xrightarrow{b} \circ \\
\end{array}
\]

then we get the empty automaton, whereas a “correct” blending should give

\[
\begin{array}{cccc}
CA \downarrow_k & \xrightarrow{b} & C & \xrightarrow{b} \circ & A & \xrightarrow{b} \circ & B & \xrightarrow{a} \circ & CA \downarrow_k \xrightarrow{b} \circ \\
\end{array}
\]

since the interaction \( C \xrightarrow{b} B \) is independent from both \( A \xrightarrow{a} H \) and \( K \xrightarrow{a} B \).

Let us consider now a well-sequenced c-automaton producing the language \( close(L(CA)) \), i.e. where the independent interactions were made “visible” by the presence of “diamonds”, and precisely

\[
\begin{array}{cccc}
CA' & \xrightarrow{a} & A & \xrightarrow{a} \circ & C & \xrightarrow{b} \circ & B & \xrightarrow{a} \circ & CA' \downarrow_k \xrightarrow{b} \circ \\
\end{array}
\]

The blending of \( H \) and \( K \) in \( CA' \) yields exactly the expected c-automaton \( [3] \).

\( \diamond \)

We show that blending preserves well-formedness. This means that c-automata obtained by blending can be correctly projected. We begin with well-sequencedness.

**Proposition 5.14 (Blending preserves well-sequencedness).** If \( CA \) is a well-sequenced c-automaton then so is \( CA \downarrow_k \) (if defined).
Proof. By contradiction, let us assume $CA^{H\sqcup K}$ not to be well-sequenced. So there exist in it a pair of transitions of the following form

![Diagram](image)

such that

- $\{C, D\} \cap \{A, B\} = \emptyset$; and there is no state $q'$ such that

![Diagram](image)

Moreover by definition of Blending,

- the transitions correspond, in $CA$, to

![Diagram](image)

- $H \not\in \{C, D\}$ (otherwise the $p$-to-$q$ transition would have been dropped by the blending procedure)

By well-sequencedness of $CA$, it follows that there exist diamonds in it enabling to go from $p$ to $q$ following all possible combinations of transitions with labels in $\{C \rightarrow D : c, A \rightarrow H : a, K \rightarrow B : a\}$. This implies that, by the blending procedure, in $CA^{H\sqcup K}$ we have actually

![Diagram](image)

Contradiction. $\square$

Well-branchedness, unfortunately, is not generally preserved by blending, as shown by the following counterexample. Consider

![Diagram](image)

which can be checked to be well-branched. By blending $H$ and $K$ in $CA$ we get

![Diagram](image)

which is not well-branched. We therefore restrict to the class of $A$-univocal $c$-automata.
Definition 5.15 (A-univocity). A c-automaton CA is A-univocal if for any transitions \( \frac{X \rightarrow A : m}{\rightarrow q} \) and \( \frac{Y \rightarrow C : m}{\rightarrow q'} \) in CA, \( C = A \) and for any transitions \( \frac{A \rightarrow X : m}{\rightarrow q} \) and \( \frac{C \rightarrow Y : m}{\rightarrow q'} \) in CA, \( C = A \).

It is immediate to check that CA in (4) above is not \( H \)-univocal.

Lemma 5.16. If \( CA^{H \bowtie K} \) is defined then for each run \( \pi \) in \( CA^{H \bowtie K} \) there is a run \( \hat{\pi} \) in CA such that if \( t \) precedes \( t' \) in \( \pi \) and both occur in CA then \( t \) precedes \( t' \) in \( \hat{\pi} \).

Proof. It suffices to observe that for any transition \( t'' = p \frac{A \rightarrow B : m}{H K} q \) in \( \pi \) which is not in CA there is a state \( r \) such that transitions \( p \frac{A \rightarrow H : m}{H K} r \frac{K \rightarrow B : m}{H K} q \) are in CA (and likewise for any \( t'' = p \frac{A \rightarrow B : m}{H K} q \) in \( \pi \) which is not in CA). Hence, \( \pi' \) is obtained by replacing \( t'' \) with such transitions. \( \square \)

Corollary 5.17. If \( CA^{H \bowtie K} \) is defined then for each acyclic run \( \pi \) in \( CA^{H \bowtie K} \) there is an acyclic run \( \hat{\pi} \) in CA such that if \( t \) precedes \( t' \) in \( \pi \) and \( t \) and \( t' \) occur in CA then \( t \) precedes \( t' \) in \( \hat{\pi} \).

Proof. Observe that the construction in the proof of Lemma 5.16 never uses a same transition to replace different transitions on \( \pi \). \( \square \)

The next results establish that blending preserves well-branchedness on univocal and reflective c-automata.

Proposition 5.18. Fix two participants \( H \) and \( K \) and a well-branched c-automaton CA such that \( CA^{H \bowtie K} \) is defined. If CA is \( H \)- and \( K \)-univocal then conditions (1) and (2) of Definition 4.4 hold on \( CA^{H \bowtie K} \). If moreover CA is \( H \)- and \( K \)-reflective then condition (3) of Definition 4.4 holds on \( CA^{H \bowtie K} \).

Proof. By reductio ad absurdum. Let \( p \) be a state in \( CA^{H \bowtie K} \) with no selector. Then one of the following must hold for each participant \( B \) of \( CA^{H \bowtie K} \) (note that \( B \neq H \) and \( B \neq K \)):

1. there are transitions \( t = p \frac{B \rightarrow A : m}{\rightarrow q} \) and \( t' = p \frac{B \rightarrow A : m}{\rightarrow q'} \) with \( q \neq q' \); or
2. there are transitions \( t = p \frac{B \rightarrow A : m}{\rightarrow q} \) and \( t' = p \frac{X \rightarrow Y : n}{\rightarrow q'} \) such that \( X \neq B \) and \( t \) and \( t' \) are not concurrent, or else
3. there are a participant \( X \neq B \) and a p-span \( (\pi_1, \pi_2) \) in \( CA^{H \bowtie K} \) such that (i) \( \pi_1 \downarrow X \neq \pi_2 \downarrow X \) and (ii) the first pair of characters where \( \pi_1 \downarrow X \) and \( \pi_2 \downarrow X \) differ are not both inputs.

We derive a contradiction in each of the cases above.
Case (1). Not both $t$ and $t'$ can be transitions of $CA$ otherwise $CA$ would not be well-branched. With no loss of generality, suppose that $t$ is not in $CA$. There must be $r$ such that

$$p \xrightarrow{B\rightarrow H} m \xrightarrow{r \rightarrow K} A \xrightarrow{m} q \quad \text{or} \quad p \xrightarrow{B\rightarrow K} m \xrightarrow{H\rightarrow A} r \xrightarrow{m} q$$

in $CA$; \hfill (5)

Then $t' \notin CA$ otherwise $CA$ would not be $H$- or $K$-univocal. Hence, there is $r'$ s.t.

$$p \xrightarrow{B\rightarrow H} r' \xrightarrow{K\rightarrow A} m \xrightarrow{r'} q'$$

or

$$p \xrightarrow{B\rightarrow K} r' \xrightarrow{H\rightarrow A} m \xrightarrow{r'} q'$$

in $CA$;

If $p \xrightarrow{B\rightarrow H} m \xrightarrow{r} K \xrightarrow{A} m \xrightarrow{r} q$ and $p \xrightarrow{B\rightarrow K} m \xrightarrow{H} A \xrightarrow{m} q'$ are in $CA$ then $p$ would have two transitions with the same label violating the well-branchedness of $CA$. If $p \xrightarrow{B\rightarrow H} m \xrightarrow{r} K \xrightarrow{A} m \xrightarrow{r} q'$ are in $CA$ then, contrary to our hypothesis, $CA$ would not be $H$- and $K$-univocal. Swapping $H$ and $K$ would yield similar contradictions in the other cases.

Case (2). Not both $t$ and $t'$ can be transitions of $CA$ otherwise $CA$ would not be well-branched.

If $t \notin CA$, there is $r$ such that (5) holds. By the well-branchedness of $CA$, there is $r'$ such that

for a $q''$ where the diamond $r-q-r'-q''$ exists by the well-branchedness of $CA$. Therefore, in $CA^{h\rightarrow k}$ there is a diamond $p-q'-q''$ with labels $B\rightarrow H$: $m$ and $X\rightarrow Y$: $n$, contrary to our hypothesis.

If $p \xrightarrow{B\rightarrow K} m \xrightarrow{H} A \xrightarrow{m} q$ the proof is similar.

If $t' \notin CA$, there is $r$ such that

where the diamond $r'-r-q'-q''$ exists by the well-branchedness of $CA$ and the thesis follows as before.
Case (b) The assumption \( \pi_1 \downarrow x \neq \pi_2 \downarrow x \) implies that

(a) \( X \) is the sender in a transition of \( \pi_1 \) or of \( \pi_2 \) (otherwise both projections

would be the automaton accepting the empty word), or

(b) \( X \) is the receiver in a transition of \( \pi_1 \) or of \( \pi_2 \), but not both (otherwise either

the inputs should be different by well-branchedness of \( CA \) contrary to our hypothesis).

With no loss of generality, suppose that \( X \) is the sender or the receiver of a

transition in \( \pi_1 \).

Case (a) Let \( \pi_1' \) be the longest prefix of \( \pi_1 \) whose projection on \( X \) is also the

projection of a prefix of \( \pi_2 \), and \( t \) the first transition in \( \pi_1 \) after \( \pi_1' \). Then,

for some \( \pi_1'', \pi_2', \) and \( \pi_2'' \) we have

\[
\pi_1 = \pi_1', t, \pi_1'', \quad \pi_2 = \pi_2', \pi_2'', \quad \pi_1 \downarrow x = \pi_2' \downarrow x, \quad \text{and} \quad (t, \pi_1'') \downarrow x \neq \pi_2'' \downarrow x
\]

Using the construction of the proof of Lemma 5.16 we have two co-initial

runs \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) in \( CA \) such that, for \( i \in \{1, 2\} \), \( \hat{\pi}_i \) has the transitions of \( CA \)

occurring in \( \pi_i \) preserving the order they have in \( \pi_i \). Note that the construction

guarantees that \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are transition disjoint and Corollary 5.17 ensures

that they are acyclic. Moreover, if \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are not co-final, then they can be

extended to form a span adding transition from the maximal states of \( \pi_1 \) and \( \pi_2 \). Let \( \hat{\pi}_1' \) and \( \hat{\pi}_2' \) be the part of \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) which include the transitions

of \( \pi_1' \) and \( \pi_2' \); then \( X \) cannot behave uniformly \( \hat{\pi}_1' \) and \( \hat{\pi}_2' \) (otherwise the first

action of \( X \) distinguishing its behaviour on the two runs would be the output

at \( t \) and this is not possible by the well-branchedness of \( CA \)). This implies

that the first actions distinguishing the behaviour of \( X \) on \( \hat{\pi}_1' \) and \( \hat{\pi}_2' \) should

be labelled with \( H \rightarrow X : m \) and \( K \rightarrow X : m \) (obtained through the construction

of Lemma 5.16 which should have replaced a transition in \( \pi_1' \) and in \( \pi_2' \) with receiver \( X \)). Then the univocity of \( CA \) is violated.

Case (b) As in the previous case, we can use the construction of Lemma 5.16

to extend runs \( \pi_1 \) and \( \pi_2 \) to two coinitial runs \( \pi_1' \) and \( \pi_2' \) in \( CA \). Notice that

this construction cannot yield a transition with receiver \( X \) in \( \pi_2' \) since it at

most adds transitions where receivers are either \( H \) or \( K \). Then, the span

\((\pi_1, \pi_2)\) cannot be cofinal, otherwise we would violate the well-branchedness

of \( CA \). Let \( q \) and \( q' \) the states reached by \( \pi_1 \) and \( \pi_2 \) (and hence the states

reached by \( \pi_1' \) and \( \pi_2' \)). By the well-branchedness of \( CA \), \( \pi_2' \) cannot be maximal,

hence there should be a transition from \( q' \) involving an interface which the

blending operation cuts away (otherwise \( q' \) would not be the last state on

\( \pi_2 \)) because it cannot be blended with a forwarding transition involving the

other interface. Call such transition \( t \) and assume that its arrival state is \( q'' \)

and, with no loss of generality, that it involves \( H \).

If \( H \) is the receiver of \( t \) then, by reflectiveness (condition \( \mathbb{I} \) in Definition 5.10),

we can find a transition \( q'' \xrightarrow{H \rightarrow X : m} q''' \) which violates the maximal-

ity of \( \pi_2 \) in \( CA^{H=K} \). Likewise, if \( H \) is the receiver of \( t \) then, by reflectiveness
(condition (2) in Definition 5.10), we can find a transition $q'' \xrightarrow{H \rightarrow Y; m} q'''$ which violates the maximality of $\pi_2$ in $\text{CA}^{H \bowtie K}$.

In all the cases we obtain a contradiction, which gives the thesis.    

The following result is immediate from Proposition 5.18.

**Corollary 5.19.** Given a well-branched c-automaton $\text{CA}$ and participants $H$ and $K$ such that $\text{CA}$ is $H$- and $K$-univocal and reflective on $H$ and $K$, then $\text{CA}^{H \bowtie K}$, if defined, is well-branched.

## 5.2 Modular development of choreographies

In the previous section we studied blending in isolation. Now, as informally described at the beginning of Section 5, we show how to combine blending with product to provide a composition operator, which is the basis of our modular development of choreographies.

We first define the product of c-automata. The product will be defined on c-automata having disjoint sets of participants. We wish in fact the systems we compose to interact only through the interfaces.

**Definition 5.20 (Product of c-automata).** The product of two c-automata is the standard product of FSA if the automata have disjoint sets of participants and it is undefined otherwise.

We can now define the composition operation.

**Definition 5.21 (Composition).** Let $\text{CA}$ and $\text{CB}$ be two c-automata, $H$ and $K$ be participants respectively in $\text{CA}$ and in $\text{CB}$. The composition of $\text{CA}$ and $\text{CB}$ via $H$ and $K$ is

$$\text{CA}^{H \bowtie K} \text{CB} = (\text{CA} \times \text{CB})^{H \bowtie K}$$

provided that $\text{CA}$ and $\text{CB}$ have disjoint participants.

A reasonable modular development of systems must rely on composition operations that do not “break” the relevant properties that modules separately possess. The relevant properties are those considered in Section 4.2, that are guaranteed by projecting well-formed c-automata. Thus, we will show below that composition preserves well-formedness.

We begin by showing that composition does preserve well-branchedness and in order to do that the following definition will be handy.

**Notation.** We shall often denote by $A \in \text{CA}$ the fact that $p$ is a participant in the c-automaton $\text{CA}$ and denote the set of its participants by $\mathcal{P}(\text{CA})$

**Definition 5.22.** Let $\pi$ be a run in $\text{CA} \times \text{CB}^{H \bowtie K}$. We define the restriction of $\pi$ to $\text{CA}$, dubbed $\pi|_{\text{CA}}$, as follows

$$\varepsilon|_{\text{CA}} = \varepsilon \quad \text{(where } \varepsilon \text{ is the empty string)}$$

$$\begin{align*}
(t \pi')|_{\text{CA}} &= \left\{ \begin{array}{ll}
\pi'|_{\text{CA}} & \text{if } t \in \text{CA} \\
\pi'|_{\text{CB}} & \text{if } t \in \text{CB} \\
p \xrightarrow{A \rightarrow H; m} \xrightarrow{r \pi'|_{\text{CA}}} & \text{if } t = p \xrightarrow{A \rightarrow B; m} q \\
\end{array} \right.
\end{align*}$$

where (*)
(*) $r$ is such that $p \xrightarrow{A \rightarrow K}{\text{H:}}m r \xrightarrow{\text{H:}}{\text{K:}}q$ has been obtained by blending $p \xrightarrow{A \rightarrow H}{\text{m:}}r$ and $r \xrightarrow{\text{K:}}{\text{A:}}m q$ in $\text{CA} \times \text{CB}$.

Similarly for the definition of $\pi_{|\text{CB}}$.

**Lemma 5.23.** Let $\pi$ be a run in $\text{CA} \times \text{CB}^{\text{H:K}}$. Then $\pi_{|\text{CA}}$ (resp. $\pi_{|\text{CB}}$) is a run in $\text{CA}$ (resp. $\text{CB}$).

**Proof.** Easy, by definition of product, blending and restriction (·)|(·).  

**Proposition 5.24 (Composition preserves well-branchedness).** Given two participants $\text{H} \in \text{CA}$ and $\text{K} \in \text{CB}$, where $\text{CA}$ and $\text{CB}$ are well-branched, $(\text{CA} \times \text{CB})^{\text{H:K}}$ is well-branched.

**Proof.** By reductio ad absurdum. Let $p$ be a state in $(\text{CA} \times \text{CB})^{\text{H:K}}$ with no selector. Then one of the following must hold for each participant $\text{B}$ of $(\text{CA} \times \text{CB})^{\text{H:K}}$ (note that $\text{B} \neq \text{H}$ and $\text{B} \neq \text{K}$):

1. there are transitions $t = p \xrightarrow{\text{B} \rightarrow \text{A:}}m q$ and $t' = p \xrightarrow{\text{B} \rightarrow \text{A:}}m q'$ with $q \neq q'$, or
2. there are transitions $t = p \xrightarrow{\text{B} \rightarrow \text{A:}}m q$ and $t' = p \xrightarrow{\text{X} \rightarrow \text{Y:}}n q'$ such that $X \neq B$ and $t$ and $t'$ are not concurrent, or else
3. there are a participant $X \neq B$ and a $p$-span $(\pi_1, \pi_2)$ in $(\text{CA} \times \text{CB})^{\text{H:K}}$ such that (i) $\pi_1 \downarrow X \neq \pi_2 \downarrow X$ and (ii) the first pair of labels where $\pi_1 \downarrow X$ and $\pi_2 \downarrow X$ differ are not of the form $(\text{C:}m, \text{D:}n)$ with $\text{C} \neq \text{D}$ or $m \neq n$

We derive a contradiction in each of the cases above. In the following we can assume, without any loss of generality, that $\text{B} \in \text{CA}$.

**Case [1]** It is impossible that $\text{A} \in \text{CA}$, otherwise $\text{CA}$ would not be well-branched. So, necessarily there must be $r$ and $r'$ such that $r \neq r'$ and

\[ p \xrightarrow{\text{B} \rightarrow \text{H:}}m r \xrightarrow{\text{K} \rightarrow \text{A:}}m q \quad \text{and} \quad p \xrightarrow{\text{B} \rightarrow \text{H:}}m r' \xrightarrow{\text{K} \rightarrow \text{A:}}m q' \]  

(6)

then $p$ would have two transitions in $\text{CA}$ with the same label violating the well-branchedness of $\text{CA}$.

**Case [2]** Not both $t$ and $t'$ can be transitions of $\text{CA}$, otherwise $\text{CA}$ would not be well-branched.

If $t \notin \text{CA}$ and $t' \in \text{CB}$, there is $r$ such that the first conjunct of [1] holds. Moreover, by definition of product we can infer that there is $r'$ and $q''$ such that
are in $CA \times CB$, and hence

\[
\begin{array}{c}
B \rightarrow A : m \\
X \rightarrow Y : n \\
q' \\
q \\
q'' \\
B \rightarrow A : m
\end{array}
\begin{array}{c}
p \\
X \rightarrow Y : n \\
q \\
q' \\
q'' \\
B \rightarrow A : m
\end{array}
\]

(7)

is in $(CA \times CB)^{H \circ K}$. Contradiction.

If $t \notin CA$ and $t' \in CA$, then we would have

\[
\begin{array}{c}
B \rightarrow H : m \\
X \rightarrow Y : n \\
q' \\
q \\
q'' \\
B \rightarrow H : m
\end{array}
\begin{array}{c}
p \\
X \rightarrow Y : n \\
q \\
q' \\
q'' \\
B \rightarrow H : m
\end{array}
\]

both in $CA$. Now, since we assumed that $X \neq B$, by well-branchedness of $CA$ we would have

\[
\begin{array}{c}
B \rightarrow H : m \\
K \rightarrow A : m \\
q' \\
q \\
q'' \\
K \rightarrow A : m
\end{array}
\begin{array}{c}
p \\
X \rightarrow Y : n \\
q \\
q' \\
q'' \\
B \rightarrow H : m
\end{array}
\]

and hence

\[
\begin{array}{c}
B \rightarrow H : m \\
K \rightarrow A : m \\
q' \\
q \\
q'' \\
K \rightarrow A : m
\end{array}
\begin{array}{c}
p \\
X \rightarrow Y : n \\
q \\
q' \\
q'' \\
B \rightarrow H : m
\end{array}
\]

are in $CA \times CB$, where $r-q$ is in $CB$ and hence $q-q''$ and $r'-q''$ exist by definition of product. So, we can obtain that (7) is in $(CA \times CB)^{H \circ K}$ and we get a contradiction since we assumed $t$ and $t'$ not to be concurrent.

If $t \notin CA$ and $X \in CB$ and $t' \notin CB$, then, by reasoning similarly to the case $(t \notin CA$ and $t' \in CB)$ we would arrive to conclude (7) to be in $(CA \times CB)^{H \circ K}$. Contradiction.

If $t \in CA$ and $X \in CA$ and $t' \notin CA$, then, we would have

\[
\begin{array}{c}
B \rightarrow A : m \\
X \rightarrow Y : n \\
q \\
q' \\
B \rightarrow A : m
\end{array}
\begin{array}{c}
p \\
X \rightarrow Y : n \\
q \\
q' \\
B \rightarrow A : m
\end{array}
\]
both in \( CA \). Now, since we assumed that \( X \neq B \), by the well-branchedness of \( CA \) there exists also the rest of the diamond and we can reason similarly to what done for the case \( t \notin CA \) and \( t' \in CA \), getting to a contradiction with \( t \) and \( t' \) not concurrent.

If \( t \in CA \) and \( t' \in CB \), then, we would infer the existence of \( t \) by definition of product, getting a contradiction with \( t \) and \( t' \) not concurrent.

\textbf{Case (3)} Without loss of generality, we can assume \( X \in CA \) (recall that we have also that \( X \neq H \)).

The assumptions for the present case implies that the \( p \)-span \( (\pi_1, \pi_2) \) is such that there exist \( t_1, t_2 \notin CB \) such that

\begin{itemize}
  \item [a)] either \( t_1, t_2 \in CA \) or \( [t_1 \in CA \) and \( t_2 \notin CA] \) or \( [t_2 \in CA \) and \( t_1 \notin CA] \);
  \item [b)] \( \pi_1 = \pi_1', \pi_1'' \) and \( \pi_2 = \pi_2' t_2 \pi_2'' \), for some \( \pi_1', \pi_2', \) and \( \pi_2'' \);
  \item [c)] \( \pi_1' \downarrow_X = \pi_2' \downarrow_X \)
  \item [d)] \( (t_1 \downarrow_X, t_2 \downarrow_X) \neq (CX?, m, DX?n) \) with \( C \neq D \) or \( m \neq n \);
  \item [e)] \( \pi_1' \) and \( \pi_2' \) with the above properties have maximal length.
\end{itemize}

Without loss of generality we can assume \( t_1 \in CA \). Let now \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) be the runs of \( CA \) defined out of \( \pi_1 \) and \( \pi_2 \) as in Lemma 5.23. It is not difficult to check that \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are co-initial. Note that the construction guarantees that \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are transition disjoint and acyclic. Moreover, if \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) are not co-final, then \( \pi_1 \) and \( \pi_2 \) can be extended (by adding transition from the maximal states of \( \pi_1 \) and \( \pi_2 \)) in such a way

\[ \hat{\pi}_1 = \pi_1' (t_1(\downarrow_X)) \hat{\pi}_1'' \quad \text{and} \quad \hat{\pi}_2 = \pi_2' (t_2(\downarrow_X)) \hat{\pi}_2'' \]

So that \( (\hat{\pi}_1, \hat{\pi}_2) \) forms a \( p \)-span of \( CA \). Besides, by (3) above, it is possible to infer that

\[ \hat{\pi}_1' = \hat{\pi}_2' \quad (8) \]

We proceed now by considering the two possible cases:

\textbf{\( t_1, t_2 \in CA \)}

By definition of restriction, we get \( t_i|CA = t_i \) for \( i = 1, 2 \). Hence the above properties (3) and (4) of the \( p \)-span \( (\hat{\pi}_1, \hat{\pi}_2) \) immediately contradict the well-branchedness of \( CA \).

\textbf{\( t_1 \in CA \) and \( t_2 \notin CA \)}

We consider two further possibilities

\textbf{\( t_2 \) has been obtained by blending \( p' \) and \( q \) in \( CA \times CB \).}

Since \( t_1 \in CA \), by definition of restriction we have \( t_1|CA = t_1 \). Moreover, by definition, \( (t_2)\downarrow_X = X!m \). Then the \( p \)-span \( (\hat{\pi}_1, \hat{\pi}_2) \) contradicts the well-branchedness of \( CA \) both in case \( t_1 \downarrow_X \) is an input and in case it is an output.

\textbf{\( t_2 \) has been obtained by blending \( p' \) and \( q \) in \( CA \times CB \).}

In case \( t_1 \downarrow_X \) is an output, we get a contradiction with the well-branchedness of \( CA \).

In case, instead, \( t_1 \downarrow_X \) is an input, we have necessarily that \( t_1 \) is of the
form \( q' \xrightarrow{\xi} q'' \) with \( C \in CA \). Since we have also that, necessarily, \( Y \in CB \), it follows that \((t_1 \downarrow_X, t_2 \downarrow_X) = (CX^n, YX^m)\) with \( Y \neq C \). This, however, is impossible. In fact in the present main case we are assuming condition \( C \) – stated at the beginning of the proof – to hold, according to which the first pair of labels where \( \pi_1 \downarrow_X \) and \( \pi_2 \downarrow_X \) differ are not of the form \((CX^m, DX^n)\) with \( C \neq D \) or \( m \neq n \). \( \square \)

It is worth noticing that composition can be proven to preserve well-branchedness without any univocity assumptions, which has been instead necessary to prove well-branchedness preservation by blending in Proposition 5.18. This is due to the fact that Proposition 5.18 deals with blending on general c-automata, not those obtained from product.

For what concerns well-sequencedness, since we already know that blending preserve well-sequencedness (see Proposition 5.14), it is enough to show well-sequencedness to be preserved by product.

**Lemma 5.25 (Product preserves well-sequencedness).** Let \( CA \) and \( CB \) be two well-sequenced c-automata. Then \( CA \times CB \) is well-sequenced (if defined).

**Proof.** Let us have two consecutive transitions in \( CA \times CB \) with disjoint sets of participants. If the two transitions belong both to \( CA \) or both to \( CB \), we have a diamond by well-sequencedness of \( CA \) and \( CB \). Otherwise we have a diamond by definition of product. \( \square \)

Well-formedness preservation by composition can hence be obtained as a corollary.

**Corollary 5.26 (Composition preserves well-formedness).** Given two participants \( H \in CA \) and \( K \in CB \), where \( CA \) and \( CB \) are well-formed c-automata, \( CA \sqcirc K \) is well-formed.

**Proof.** Well sequencedness of \( CA \sqcirc K \) is a consequence of Lemma 5.25 and Proposition 5.14, while its well branchedness is proved in Proposition 5.24. \( \square \)

In order to show that composition does not lose behaviours (that is, that Theorem 5.12 holds) we also need to show that \( CA \times CB \) is reflective on \( H \) and \( K \).

In order to define compatibility, a few simple definitions are handy. Let \( \mathcal{L}_{i/o} = \{ ?m, !m \mid m \in \mathcal{M} \} \) and define the functions

\[
(\cdot)^P : \mathcal{L}_{act} \cup \{ \varepsilon \} \rightarrow \mathcal{L}_{i/o} \cup \{ \varepsilon \}
\]

by the following clauses

\[
(A B ? m)^P = ?m \quad (A B ! m)^P = !m \quad \varepsilon^P = \varepsilon
\]

\[
\overline{?m} = !m \quad \overline{!m} = ?m \quad \tau = \varepsilon
\]
which extend to CFSMs in the obvious way; given a CFSM \( M = (S, \mathcal{L}_{act}, \rightarrow, q_0) \), we define \( M_1^p = (S, \mathcal{L}_{i/o}, \rightarrow^p, q_0) \) where \( \rightarrow^p = \{ p \xrightarrow{\alpha} q \mid p \xrightarrow{\alpha} q \in \rightarrow \} \); and likewise for \( M_2^p \).

**Definition 5.27 (Compatibility).** Two CFSMs \( M_1 \) and \( M_2 \) are compatible if \( M_1^p \) is bisimilar to \( M_2^p \).

Given CA and CB, participants \( H \) of CA and \( K \) of CB are compatible if \( CA \downarrow_H \) and \( CB \downarrow_K \) are compatible.

In general, reflectiveness implies compatibility, but, in the special case of product automaton, compatibility of the components implies reflectiveness of the product.

**Proposition 5.28 (Reflectiveness implies compatibility).** If a c-automaton CA is reflective on \( H \) and \( K \), then \( CA \downarrow_H \) and \( CA \downarrow_K \) are compatible.

**Proof.** We have to show that \( CA \downarrow_H \) and \( CA \downarrow_K \) are compatible, namely that \( CA \downarrow_H^p \) is bisimilar to \( CA \downarrow_K^p \).

Remember that the states of the projections are sets of states of the original c-automata.

Let us consider the relation \( R \subseteq \mathcal{R}(CA \downarrow_H) \times \mathcal{R}(CA \downarrow_K) \) defined as:

\[
R = \{ (S_H, S_K) \mid S_H \in \mathcal{R}(CA \downarrow_H), S_K \in \mathcal{R}(CA \downarrow_K), S_H \cap S_K \neq \emptyset \}
\]

By definition it relates the two initial states. Also, consider a challenge from \( CA \downarrow_H \) (the other case is analogous), that is a transition \( S_H \xrightarrow{\alpha} S'_H \). Let us consider the case \( \alpha = ?m \) (the case \( \alpha = !m \) is analogous), that is we have a transition \( S_H \xrightarrow{AH,m} S'_H \) in \( CA \downarrow_H \).

By reflectiveness we have a transition \( S''_K \xrightarrow{KB,m} S'''_K \) in \( CA \downarrow_K \) with \( S''_K \cap S'''_K \neq \emptyset \).

Also, in \( CA \) we have transitions \( s \xrightarrow{AH,H} s' \) and \( s' \xrightarrow{KB,K} s'' \). By projecting to \( H \) we get \( s'' \in S''_H \) (since the second transition projects to \( \epsilon \) as well as \( s'' \in S'''_K \)) and \( S_K = S''_K \) (since the first transition projects to \( \epsilon \)).

As a result, \( K \) can answer the challenge with \( S_K \xrightarrow{m} S''_K \) with \( S''_H \cap S''_K \supseteq \{ s'' \} \neq \emptyset \) as desired. \( \square \)

**Theorem 5.29 (Compatibility implies reflectiveness of products).** If \( CA = CB \times CC \) and \( CB \downarrow_B \) and \( CC \downarrow_K \) are compatible then \( CA \) is reflective on \( H \) and \( K \).

**Proof.** For each \( S \xrightarrow{AH,m} S' \) in \( CA \downarrow_H \) we have that \( S = X \times Q_C \) and \( S' = X' \times Q_c \) with \( X, X' \subseteq Q_B \). Hence, \( X \xrightarrow{AH,m} X' \) in \( CB \downarrow_B \). Thanks to the bisimilarity condition there is \( S_1 \xrightarrow{KB,m} S'_1 \) in \( CC \downarrow_K \). Hence, \( Q_B \times S_1 \xrightarrow{KB,m} Q_B \times S'_1 \) in \( CA \downarrow_K \).

Since \( X' \times Q_c \cap Q_B \times S_1 \neq \emptyset \) the first condition follows. The proofs of the other conditions are similar. \( \square \)
As a corollary, if \( H \) in \( CA \) and \( K \) in \( CB \) are compatible, then roles different from \( H \) and \( K \) behave the same in the original c-automata and in the composition.

**Corollary 5.30.** If \( H \) in \( CA \) and \( K \) in \( CB \) are compatible then \( CA \downarrow_A \) and \( CA_{\bowtie A} CB_{\bowtie A} \) are bisimilar, for each participant \( A \neq H, K \).

**Proof.** From Theorem 5.12, where the condition of reflectiveness on \( CA \times CB \) holds thanks to Theorem 5.29.

We show now, step by step, how composition works in a simple example.

**Example 5.31 (Composition at work).** Let us compose the c-automata below on the interfaces \( H \) and \( K \).

Note that \( CA \) and \( CB \) are well-formed, \( CA \) is \( H \)-univocal and \( CB \) is \( K \)-univocal. As discussed, we shall first compute \( CA \times CB \) and then blending the two roles \( H \) and \( K \) in \( CA \times CB \). The resulting c-automaton is well-formed thanks to Corollary 5.26.

To save space, we represent these steps in the following decorated c-automaton.

where the decorations have the following interpretation:
- the transitions of \( CA \times CB \) are all the transitions except the dotted ones, that from state \((0, 0)\) go to state \((1, 1)\) and to state \((2, 2)\), which are added by blending of the dashed transitions and the doubly-lined ones, respectively;
- the grey transitions are the ones removed either because of blending (like the dashed and doubly-lined ones) or removal of unreachable states (like those involving greyed states \((2, 1)\) and \((1, 2)\)).
– the non-grey transitions (regardless if they are solid or dotted) form the final result of the composition.

We now apply step by step the blending algorithm on $CA \times CB$ and $H$ and $K$.

**step I**: Each transition $p \xrightarrow{A\rightarrow H; m} q$ is removed, and for each transition $q \xrightarrow{K\rightarrow B; m} r$ a transition $p \xrightarrow{A\rightarrow B; m} r$ is added provided that $A \neq B$; and similarly, by swapping $H$ and $K$.

In this step, the dashed and the doubly-lined transitions are replaced by the blended transitions, represented by the dotted arrows.

**step II**: Transitions involving neither $H$ nor $K$ are preserved, whereas all other transitions are removed.

This step gets rid of grey transitions, but for the ones outgoing from the greyed states, which are removed in the next step.

**step III**: States and transitions unreachable from state $(0, 0)$ are removed.

This last step hence removes the greyed states and their outgoing transitions, obtaining the overall composition of $CA$ and $CB$ through $H$ and $K$.

It is worth pointing out that the composition operation can be performed on any pair of roles, that is any role can be chosen as an interface. However, if the conditions discussed above are not verified, then the composition may not be well-behaved. In particular, it may happen that the behaviour of some participants in the original systems (different from $H$ and $K$) will change, in particular some transitions may be disabled.

This can be seen on the simple example below.

Their composition $CA_{H\rightarrow CB}$ yields the empty automaton, which is well-formed (thanks to Corollary 5.26). However, participants $A$ and $D$ lose their transitions. This is due to the fact that $H$ in $CA$ and $K$ in $CB$ are not compatible, hence $CA \times CB$ is not reflective.

**Remark 5.32.** Compatibility does not ensure reflectiveness when taking into account c-automata which are not products, as the following example shows.

It is easy to check that the c-automata $CA$ is well-formed and that $CA_{\downarrow H}$ and $CA_{\downarrow K}$ are compatible. Nonetheless $CA$ is not $H$-$K$-reflective; in particular requirement (1b) of Definition 5.10 does not hold.
Our composition operator is associative and commutative up-to isomorphism. We first show that our composition operation commutes with the product of c-automata.

**Lemma 5.33.** If CA, CB and CC are c-automata with pairwise disjoint sets of participants then \((CA \times CB) \times CC = ((CA \times CB) \times CC)^{H \bowtie K}\).

**Proof.** We show that each run in \(((CA \times CB)^{H \bowtie K}) \times CC\) is also a run in \(((CA \times CB) \times CC)^{H \bowtie K}\) and vice versa.

For each blended transition \(t = ((p, q), r) \xrightarrow{A \to B : m} ((p', q'), r')\) on a run \(\pi\) in \(((CA \times CB)^{H \bowtie K}) \times CC\) there is a state \(((p'', q''), r'')\) in \((CA \times CB) \times CC\) such that

\[
((p, q), r) \xrightarrow{A \to B : m} ((p'', q''), r'') \xrightarrow{H \bowtie K} ((p', q'), r') \text{ in } (CA \times CA) \times CC \quad (9)
\]

Note that any of transitions above may be in CC by hypothesis. By replacing each \(t\) in \(\pi\) with the corresponding transitions in (9) we obtain a run in \((CA \times CB) \times CC\) and therefore \(\pi\) is also a run in \(((CA \times CB) \times CC)^{H \bowtie K}\). The other direction is similar.

The following result establishes that our operation of composition commutative and associative, modulo the structure of the states of the c-automaton resulting from composition.

**Corollary 5.34.** If CA, CB and CC are c-automata with pairwise disjoint sets of participants then

1. \(CA \times CB\) is isomorphic to \(CB \times CA\) (commutativity up-to isomorphism)
2. \((CA \times CB) \times CC\) is isomorphic to \(CA \times (CB \times CC)\) (associativity up-to isomorphism).

**Proof.** For commutativity, it is a trivial observation that there is a label-preserving bijection between the runs in \(CA \times CB\) and those in \(CB \times CA\). Since the blending operation does not depend on the structure of the states of a c-automaton then we have thesis.

For associativity we have

\[
(CA \times CB) \times CC = ((CA \times CB) \times CC)^{H \bowtie K} = ((CA \times (CB \times CC))^{H \bowtie K})^{H \bowtie K} \equiv ((CA \times (CB \times CC))^{H \bowtie K})^{H \bowtie K} = (CA \times (CB \times CC))^{H \bowtie K} = CA \times (CB \times CC)
\]

where \(\equiv\) is the up-to isomorphism equality on FSAs.

By Corollary 5.34, the operation of composition basically induces a sort of commutative monoidal algebraic structure on the class of c-automata. The term-algebra of this structure are trees that represent compositions of automata. We can picture those terms as a binary tree, dubbed composition tree, where leaves
are c-automata and edges are labelled with participants; a node $n$ with children $n_1$ and $n_2$ such that the edge from $n$ to $n_1$ is labelled $H$ and the edge from $n$ to $n_1$ is labelled $K$ represents $n_1H\bowtie\triangleright\leftarrow K n_2$. Corollary 5.34 is relevant for the use of our choreographic framework for a modular development of systems as it guarantees that the order of composition is immaterial.

Composition trees composition cannot produce “circular” compositions. However, combining composition and blending allows us to produce arbitrary graph structures. This can be attained through the following alternative representation. A composition graph is a labelled undirected graph where nodes are c-automata and edges between $CA$ and $CB$ labelled with $(H, K)$ iff there is a blending between role $H$ of $CA$ and role $K$ of $CB$. For instance, Example 5.35 shows how we can produce working example to produce a “triangle-shaped” circular composition.

Example 5.35 (Working example cont’d.). A compact representation of the composition of $CP$ and $CV$ via $H$ and $K$ is in Fig. 2. (The full automaton $CP_{H\bowtie\triangleright\leftarrow K CV$ is in Appendix A.1.)

The above c-automaton can be composed with $CC$ via $C$ and $D$. Then, in the resulting automaton, $E$ and $F$, originally belonging to different c-automata, can be blended. A compact representation of $(CP_{H\bowtie\triangleright\leftarrow K CV\bowtie\triangleright\leftarrow D CC)$ is in Fig. 3. In this way we connect systems $S_P$, $S_V$ and $S_C$ in a triangle shaped structure.

In case we intended to compose a number of c-automata, it would be helpful that the compatibility checks of the various interfaces we would like connect could actually were all made at the very beginning. This fact, relevant for the modular composition of system is guaranteed by the following result.

**Notation.** Below we denote by $A \in CA$ the fact that $p$ is a participant in the c-automaton $CA$. 

---

Fig. 2. A compact representation of the composition of $CP$ and $CV$ via $H$ and $K$. 

Fig. 3. A compact representation of $(CP_{H\bowtie\triangleright\leftarrow K CV\bowtie\triangleright\leftarrow D CC)$.
Proposition 5.36. Let CA, CB and CC be c-automata with pairwise disjoint sets of participants and such that H ∈ CA, K, I ∈ CB and J ∈ CC. If CA ↓ H is compatible with CB ↓ K, and CB ↓ I is compatible with CC ↓ J, then (CA ↓ H, CB) ↓ I is compatible with CC ↓ J.

Proof (Sketch). By Theorem 5.12 (CA ↓ H, CB) ↓ I is bisimilar with CC ↓ J. The thesis follows by checking that compatibility is a subrelation of bisimilarity.

Remark 5.37. In general, compatibility is not a sufficient condition to guarantee that several choreographies can be composed in a circular structure as we managed to do in Example 5.35. In fact, the possibility of obtaining “sound” systems out of a circular composition is equivalent to the following property:

If CA is a c-automaton with participants containing \{I, J, H, K\}, then compatibility of I and J is preserved by blending H and K.

This however, does not hold. The intuitive reason is that a “circular” sequence of compositions is “sound” if they do not introduce deadlocks; the absence of these particular form of deadlocks cannot be guaranteed simply by compliance. The following counterexample make this intuition more evident. Let us consider the following c-automata with disjoint sets of participants.

We can immediately check that CA ↓ J and CB ↓ I are compatible as well as CA ↓ H and CB ↓ K. By Theorem 5.29 compatibility of CA ↓ H and CB ↓ K guarantees that
the composition of CA and CB through H and K preserves the behaviour outside the interfaces. Namely, the construction of the blended automaton \((CA \times CB)^{Hb\cdot K}\) does not affect the behaviour of the participants other than H and K. In fact,

where the product CA \times CB consists of all the transitions above but the dotted ones and the result of the composition is made of the solid and dotted transitions.

Now we notice that the following c-automata

\[
(CA \times CB)^{Hb\cdot K} \downarrow I \quad J \downarrow C\!
\]

are trivially compatible (as they were before the composition). Nonetheless \((CA \times CB)^{Hb\cdot K}\) is not reflective on I and J. We have in fact that \(((CA \times CB)^{Hb\cdot K}) \downarrow I \downarrow J\) is the empty c-automaton.

So, in order to safely compose choreography in a circular way, reflectiveness must be checked on the participants whose blending “completes the circle”, namely I and J.

It is worth remarking that providing a condition for sound circular connections is a complex problem. This has been already recognised in the literature. Tree-like composition is the most adopted safe form of composition. For instance, [24] discusses the problem (on page 2, paragraph 3) and provides a generalisation of acyclic architectures by considering so-called “disjoint circular wait free component systems”. In the context of session types, a similar problem arises when processes work on more than one session at the same time as in [12], where an “interaction type system” is used to control the dependencies among roles.

6 Related and Future Work

We introduced a compositional model of choreographies and demonstrated its suitability to cope with modular descriptions of global specifications of message-passing applications. The operation that we devise is basically the composition of
classical automata product and of the blending operation introduced here. Our notion of composition preserves well-formedness under mild conditions that, in practice, do not hinder its applicability.

The adoption of an automata-based model brings in two main benefits. Firstly, the constructions that we provided are based on basic notions and mainly syntax-independent. In fact, it is often the case that syntax-driven models (such as behavioural type systems) introduce complex constructions and constraints to define well-formedness that restrict their applicability. Secondly, we can re-use well-known results of the theory of automata (e.g., we used notions of bisimulation, product and minimization) and related tools.

6.1 Related work

The present paper includes two main contributions: (i) the definition of a choreography model based on FSAs and (ii) the definition of a notion of composition of choreographies. We comment related work according to these contributions.

Composing choreographies While choreographies have been deeply explored in the literature, the problem of their composition has not. Indeed, only a few works consider the issue of composition.

The closest to our approach are [3] and [4], which inspired the present work. The former introduces the idea of connecting choreographies via forwarders, but did not provide a way to compute the corresponding choreography, that is composition was performed only at the level of systems. The latter provides a composition at the choreography level, but has stronger well-formedness conditions than ours (e.g., in choices all interactions should have the same pair of participants), and did not allow for self-connections. As a result, only tree structures could be created.

Modularity of choreographies has been considered in [29]. There, one can specify partial choreographies where some roles are not supposed to be fully specified. Partial choreographies give some support to modular development, although with some limitations. For instance, complete choreographies cannot be composed, while in our case any choreography can be composed by selecting one of its roles as an interface. In fact, a type system can check if partial choreographies are consistent with respect to global types (using a “dually incomplete” typing principle), but no composition of partial choreographies is actually defined in [29].

Approaches to choreography composition in the setting of adaptive systems has been discussed in [32] and in [11]. In [32], choreographies form a programming language and are executable, and composition is obtained by replacing a choreography component $CC_1$ inside a choreography context $CA$ with a new choreography component $CC_2$ from outside the system. Notably, the only allowed interactions between the component and the context are auxiliary interactions introduced by the projection and needed to ensure well-formedness conditions at the border between the component and the context. In [11] instead, adaptiveness
is handled at the type level rather than in a choreographic programming language. There the main limitation is that adaptiveness only allows one to switch inside a number of pre-defined behaviours (in a kind of context-driven choice), and new behaviours cannot be introduced.

**Expressiveness of c-automata** First, we note that the use of automata-based models for specifying the local behaviour of distributed components is commonplace in the literature [9,13], less so for the global specifications of choreographies. To the best of our knowledge, the automata model used in the line of work in [7] (and references therein) is the only one in the literature. The main difference between models like in [7] and c-automata (besides some constraints on the usage of messages in interactions that c-automata do not have), is that in this context the realisability of choreographies has been studied (using e.g., model checking) while compositionality is not considered in this line of research. Moreover, as shown in [15] some of the main decidability results were flawed.

While the main aim of c-automata is to provide a choreography model based on FSAs, we remark here that it is rather expressive and complements existing models of choreographies or multiparty session types.

In particular, the expressive power of c-automata is not comparable with the one of the multiparty session types in [33], which subsumes most systems in the literature. More precisely, the c-automaton

\[
\begin{align*}
A \rightarrow B : p & \quad A \rightarrow B : n \\
0 & \rightarrow 1 & 1 & \rightarrow 2 & 2 & \rightarrow 3
\end{align*}
\]

(10)

cannot be syntactically written in [33] due to the two entangled loops. The example (10) cannot be expressed in global graphs [35] either, again due to the intersecting loops. We note that the infinite unfolding of the c-automaton (10) is regular and therefore it fits in the session type system considered in [34]. However, this type system has not been conceived for choreographies (it is binary session type system) and does not allow non-determinism.

On the other side, examples such as [33, Ex. 2, Fig. 4] cannot be written in our model (since we expect the same roles to occur in branches which are coinitial, branches inside loops require that all participants in a loop are notified when the loop ends). We conjecture that a refinement of well-branchness is possible to address this limitation. An advantage of global graphs is that they feature parallel composition of global specifications, which c-automata lack. We note however that the product of c-automata allows one to model parallel composition in the case where the two branches have disjoint sets of participants (as typically assumed in multiparty session types with parallel composition). Mapping global graphs without parallel composition into c-automata is trivial. The same considerations apply to choreography languages where possible behaviours are defined by a suitable process algebra with parallel composition such as [25,10].
Future work We leave as future work the extension of our theory to asynchronous communications. While the general approach should apply, well-formedness and the conditions ensuring a safe composition should be refined. Indeed, our blending operation does not work for asynchronous semantics as it is, as the following example shows. Consider

$$\begin{align*}
\text{CA} & \rightarrow A \xrightarrow{a} K \xrightarrow{b} \text{and} \\
\text{CB} & \rightarrow B \xrightarrow{h} J \xrightarrow{a}
\end{align*}$$

whose produce is the automaton in Example 5.11. The interfaces $H$ and $K$ are compatible, so let us compute $$(\text{CA} \times \text{CB})^{H \bowtie K}$$

If we now compute $$((\text{CA} \times \text{CB})^{H \bowtie K})^{I \bowtie J}$$, we obtain

which is precisely the expected choreography under the synchronous semantics.

If, instead, the asynchronous semantics was considered, we should have obtained

An interesting future development is to adopt Büchi automata as c-automata. This extension is technically straightforward (we just add a set of final states in the definition of c-automata and define $\omega$-languages accordingly), but it probably impacts greatly on our underlying theory. For instance, it would be interesting, yet not trivial, to devise well-formedness conditions on this generalised class of c-automata that guarantee a precise correspondence with the $\omega$-languages of the projections. In this new setting, one can then explore milder forms of liveness where, for instance, it is not the case that all participants have to terminate (as the termination awareness condition of \[35\]).

The interplay between FSAs and formal languages could lead to a theory of projection and composition of choreographies based on languages instead of automata. For instance, one could try to characterise the languages accepted by well-formed c-automata, similarly to what done in \[127135\]. In those approaches global specifications are rendered as partial orders and the distributed realisability is characterised in terms of some closure properties of languages instead of using well-formedness conditions.

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A On blending

A.1 Composition in the working example

The composition of $CP$ and $CV$ on $H$ and $K$ of Example 3.2 is in Fig. 4.

Fig. 4. The c-automaton $CP_{H \Rightarrow K} CV$
A.2 Blending algorithm

The blending algorithm is given by the following pseudo-code.

Listing 1. The blending algorithm

```python
input: CA c-automaton, roles H and K
output: CA^H\bowtie^K (cf. Definition 5.2)
def forwarding(CX, U, V):
    Caux = CX
    for each t = p \rightarrow_{U, m} q in CX:
        Caux = Caux \setminus \{t\}
    for each q \rightarrow_{V, B, m} r in CX:
        if A \neq B:
            Caux = Caux \cup \{p \rightarrow_{A, m} B\}
        else:
            throw("Undefined")
    return Caux
Ctmp = forwarding(forwarding(CA, H, K), K, H)
return Ctmp \setminus \{state and transitions in Ctmp not reachable from the initial state\}
```

Lines 6-14 correspond to Step I of the informal presentation of the blending algorithm in Section 5.1, line 15 corresponds to step II, and line 16 corresponds to step III.

Proposition A.1 (Complexity of blending). The complexity of Algorithm 1 is $O(n^2)$ where $n$ is the number of transitions of input c-automaton CA.

Proof. We assume that the c-automaton, seen as a graph, is stored as an array of nodes (the states), each one with an unordered list of outgoing edges (the transitions). Step I has complexity $O(n^2)$. After the step, the number of nodes is unchanged and the number of edges is at most $O(n^2)$. Step II has linear cost on the new size. Step III requires to visit nodes and edges at most once, but since in the worst case the number of edges is larger, it also has linear complexity. The thesis follows. □