1 Introduction

The algorithmic principle of the bootstrap method is quite simple: reiterate the mechanism that produces an estimator on pseudo-samples. But when it comes to estimators that are numerically complicated to obtain, the bootstrap is less attractive to use due to the numerical burden. If one estimator is hard to find, reiterating compounds this issue. Paraphrasing Emile in the French comedy La Cité de la Peur: we can implement the bootstrap when the estimator is simple to obtain or we can compute a numerically complex point estimator, but it is too computationally cumbersome to do both.

Although this limitation is purely practical and tends to be reduced by the ever increasing computational power at our disposal, everyone would agree that it is nonetheless attractive to have a method that frees the user from the computational burden, or at least provides an answer within a reasonable time. In this chapter, we explore a special case of the efficient method of moments ([1]) that encompasses both the computation of numerically complex estimators and of a “bootstrap distribution” at a reduced cost. The idea deviates from the algorithmic principle of the bootstrap: the proposed method no longer attempts at reproducing the sample mechanism that lead to an estimator, but instead, tries to find every estimators that may have produced the observed sample, or more often, some statistics on the sample. The idea is not new though, several methods follow this pattern. The indirect inference method ([2, 3]) similarly attempts at finding the point estimate that lead to statistics obtained from simulated samples as close as possible to the same statistics on the observed sample. Mostly used in econometric and financial contexts, indirect inference has been successfully applied to the estimation of stable distribution ([4]), stochastic volatility models ([5, 6]), financial contingent claims ([7]), dynamic panel models ([8]), dynamic stochastic equilibrium models ([9]), continuous time models ([10]), diffusion processes ([11]); but it has also been used in queueing theory ([12]), robust estimation of generalized linear latent variable models ([13]), robust income distribution ([14]), high dimensional generalized linear model and penalized regression ([15]). Often presented as the Bayesian counterpart of the indirect inference, the approximate Bayesian computation ([16, 17]) aims at finding the values that match the statistics computed on simulated samples and the statistics on the observed sample, with a certain degree approximation. The method has however grown in a different context of applications. For example, it has been successfully employed in population genetics ([18]), in ecology ([19]), in evolutionary biology ([20, 21]). Less popular, R.A. Fisher’s fiducial inference (see for instance [22, 23, 24, 25, 26]) and related methods such as the generalized fiducial inference ([27, 28, 29]), D.A.S. Fraser’s structural inference ([30], see also [31]), Dempster-Shafer theory ([32, 33]) and inferential models ([34, 35]).
A possible counter-example is the following:

We consider a sequence of random variables \( \{ X_n \} \) where it is different than parametric bootstrap estimators, except in the case of a location parameter. This section with the approach we endorse in this chapter. By letting the statistics be the solution of an estimating function of the same dimension as the quantity of interest, we demonstrate that it is possible to bypass the computation of the same statistics on simulated sample by directly estimating the quantity of interest within the estimating function, thereby in a potential significant gain in computational time. In Section 3 we demonstrate in finite sample that under some weak conditions the estimators resulting from our approach is equivalent to the estimators one would have obtained using certain forms of indirect inference, approximate Bayesian computation or fiducial inference approaches, whereas it is different than parametric bootstrap estimators, except in the case of a location parameter. This section innovates on two aspects. First, it implicates that our approach can be employed in practice to solve problems that relate to indirect inference, approximated Bayesian computation and fiducial inference in a computationally efficient manner. Second, it proves or disproves formally the link between the aforementioned methods, and this in the most general situation as the results remain true for any sample size.

Constructing tests or confidence regions that controls over the error rates in the long-run is probably one of the most important problem in statistics ever since at least Neyman-Pearson famous article \(^{36}\). Yet, the theoretical justification for most methods in statistics is asymptotic. The bootstrap for example, despite its simplicity and its widespread usage is an asymptotic method \(^{38}\); for the other methods, see for example \(^{39}\) for approximate Bayesian computation, \(^2\) for indirect inference and \(^{29}\) for generalized fiducial inference. There are in general no claim about the exactness of the inferential procedures in finite sample (see \(^{36}\) for one of the exceptions). In Section 4 we study theoretically the frequentist error rates of confidence regions constructed on the distribution issued from our proposed approach. In particular, we demonstrate under some strong, but frequently encountered, conditions that the confidence regions have exact coverage probabilities in finite sample. Asymptotic justification is nonetheless provided in Section 5. In addition, we bear the comparison with the asymptotic properties of indirect inference method to conclude that, surprisingly, both approaches reach the same conclusion but under distinct conditions. Some leads are evoked, but we lack to elucidate the fundamental reason behind such discrepancy.

Although the proposed method is first and foremost computational, surprisingly in some situations explicit closed-form solutions may be found. We gather a non-exhaustive number of such examples, some important, in Section 6. The numerical study in Section 7 ends this chapter. We study via Monte Carlo simulations the coverage probabilities obtained from our approach and compare with others on a variety of problems. We conclude that in most situations, exact coverage probability computed within a reasonable computational time can be claimed with our method.

### 2 Setup

Let \( \mathbb{N} (\mathbb{N}^+) \) be the sets of all positive integers including (excluding) 0. For any positive integer \( n \), let \( \mathbb{N}_n \) be the set whose elements are the integers 0, 1, 2, \ldots, \( n \); similarly \( \mathbb{N}_n^+ = \{1, 2, \ldots, n\} \).

We consider a sequence of random variables \( \{ X_i : i \in \mathbb{N}_n^+ \} \), possibly multivariate, to follow an assumely known distribution \( F_{\theta} \), indexed by a vector of parameters \( \theta \in \Theta \subset \mathbb{R}^p \). We suppose that it is easy to generate artificial samples \( x^* \) from \( F_{\theta} \). Specifically, we generate the random variable \( x \) with a known algorithm that associates \( \theta \) and a random variable \( u \). We denote the generating mechanism as follows:

\[ x = g(\theta, u). \]

The random variable \( u \) follows a known model \( F_u \) that does not depend on \( \theta \). Using this notation, the observed sample is \( x_0 = g(\theta_0, u_0) \) and the artificial sample is \( x^* = g(\theta, u^*) \), where \( u_0 \) and \( u^* \) are realizations of \( u \).

**Example 1 (Normal).** Suppose \( x \sim N(\theta, 1) \), then four examples of possible generating mechanism are:

1. \( g(\theta, u) = \theta + u \) where \( u \sim \mathcal{U}(0, 1) \).
2. \( g(\theta, u) = \theta + \sqrt{2 \text{erf}^{-1}(2u - 1)} \) where \( u \sim \mathcal{U}(0, 1) \) and \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \) is the error function.
3. \( g(\theta, u) = \theta + \sqrt{-2 \ln(u_1) \cos(2\pi u_2)} \) where \( u = (u_1, u_2)^T \), \( u_1 \sim \mathcal{U}(0, 1) \) and \( u_2 \sim \mathcal{U}(0, 1) \).
4. \( g(\theta, u) = \theta + u_2 \sqrt{-\frac{2 \ln(u_3)}{u_3}} \) where \( u = (u_1, u_2, u_3) \), \( u_3 = u_1 + u_2 \), \( u_1 \sim \mathcal{U}(0, 1) \), \( u_2 \sim \mathcal{U}(0, 1) \).

A possible counter-example is the following: \( g(\theta, u) = u - \theta \) where \( u \sim N(2\theta, 1) \). Clearly \( x = g(\theta, u) \), but this \( g \) is not adequate because the distribution of \( u \) depends on \( \theta \).
We now define the estimators we wish to study.

**Definition 2 (SwiZs).** We consider the following sequence of estimators:

\[ \hat{\pi}_n \in \Pi_n = \arg\min_{\pi \in \Pi} \frac{1}{n} \sum_{i=1}^{n} \psi \left( g \left( \theta_0, u_{0i} \right), \pi \right) = \arg\min_{\pi \in \Pi} \Psi_n \left( \theta_0, u_0, \pi \right), \]

\[ \hat{\theta}_n^{(s)} \in \Theta_n^{(s)} = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \psi \left( g \left( \theta, u_{si} \right), \hat{\pi}_n \right) = \arg\min_{\theta \in \Theta} \Psi_n \left( \theta, u_s, \hat{\pi}_n \right), \]

where \( \psi \) is an estimating function and \( s \in \mathbb{N}_S^+ \). The estimators \( \hat{\pi}_n \) are referred as the auxiliary estimators. Any sequence of estimators \( \{\hat{\theta}_n^{(s)} : s \in \mathbb{N}_S^+\} \) is called Switched Z-estimators, or in short, SwiZs. The collection of the solutions is \( \Theta_n = \cup_{s \in \mathbb{N}_S^+} \Theta_n^{(s)} \).

**Remark 1.** The SwiZs in the Definition 2 may arguably be viewed as a special case of the Efficient Method of Moment (EMM) estimator proposed by [1]. Indeed, to have an EMM estimator the only modification to the Definition 2 is

\[ \hat{\theta}^{(s)}_{EMM,n} \in \Theta^{(s)}_{EMM,n} = \arg\min_{\theta \in \Theta} \frac{1}{H} \sum_{h=1}^{H} \Psi_n \left( \theta, u_{sh}, \hat{\pi}_n \right), \]

where \( H \in \mathbb{N}^+ \). Ergo, the SwiZs and EMM coincide whenever \( H = 1 \). Note that in general the EMM is defined with \( H \) large and \( S = 1 \).

### 3 Equivalent methods

As already remarked, the SwiZs does not appear to be a new estimator. The SwiZs in fact offers a new point of view to different existing methods as it federates several techniques under the same hat. In this Section, we show the equivalence or disequivalence of the SwiZs to other existing methods, for any sample size \( n \), to conclude that the distribution obtained by the SwiZs is (approximatively) a Bayesian posterior, and thereby that it is valid for the purpose of inference.

The EMM and the indirect inference estimator of \([3,2]\) are known to have the same asymptotic distribution when \( \dim(\pi) = \dim(\theta) \) (see Proposition 4.1 in [20]). In the next result, we demonstrate that the SwiZs and a certain form of indirect inference estimator are equivalent for any \( n \).

**Definition 3 (indirect inference estimators).** Let \( \hat{\pi}_n \) and \( \{u_j : j \in \mathbb{N}\} \) be defined as in the Definition 2. We consider the following sequence of estimators, for \( s \in \mathbb{N}_S^+ \):

\[ \hat{\pi}^{(s)}_{II,n}(\theta) \in \Pi^{(s)}_{II,n} = \arg\min_{\pi \in \Pi} \Psi_n \left( \theta, u_{si}^*, \pi \right), \quad \theta \in \Theta, \]

\[ \hat{\theta}^{(s)}_{II,n} \in \Theta^{(s)}_{II,n} = \arg\min_{\theta \in \Theta} d \left( \hat{\pi}_n, \hat{\pi}^{(s)}_{II,n}(\theta) \right), \quad \hat{\pi}_n \in \Pi_n, \quad \hat{\pi}^{(s)}_{II,n} \in \Pi^{(s)}_n, \]

where \( d \) is a metric. We call \( \{\hat{\theta}^{(s)}_{II,n} : s \in \mathbb{N}_S^+\} \) the indirect inference estimators. The collections of solutions are denoted \( \Pi_{II,n} = \cup_{s \in \mathbb{N}_S^+} \Pi^{(s)}_{II,n} \) and \( \Theta_{II,n} = \cup_{s \in \mathbb{N}_S^+} \Theta^{(s)}_{II,n} \).

**Remark 2.** In Definition 3 we are implicitly assuming that \( \Theta \) contains at least one of, possibly many zeros, of the distance between the auxiliary estimators on the sample and the pseudo-sample. Therefore, the theory is the same for any measure of distance that we denote generically by \( d \).

**Remark 3.** The indirect inference estimators in Definition 3 is a special case of the more general form

\[ \hat{\theta}^{(s)}_{II,B,m} \in \Theta^{(s)}_{II,B,m} = \arg\min_{\theta \in \Theta} d \left( \hat{\pi}_n, \frac{1}{B} \sum_{b=1}^{B} \hat{\pi}^{(s)}_{II,b,m}(\theta) \right), \]

\( B \in \mathbb{N}^+, m \geq n \). In Definition 3 we fixed \( B = 1 \) and \( m = n \). [2] considered two cases: first, \( B \) large, \( m = n \) and \( S = 1 \), second, \( B = 1 \), \( m \) large and \( S = 1 \). For both cases, the \( \ell_2 \)-norm was used as the measure of distance (see the preceding remark).

**Assumption 4 (uniqueness).** For all \((\theta, s) \in \Theta \times \mathbb{N}_S^+\), \( \arg\min_{\pi \in \Pi} \Psi_n(\theta, u_s, \pi) \) has a unique solution.
Theorem 5 (Equivalence SwiZs/indirect inference). If Assumption 4 is satisfied, then the following holds for any $s \in \mathbb{N}_S$:

$$\Theta_n^{(s)} = \Theta_{l,n}^{(s)}.$$  

Theorem 5 is striking because it concludes that a certain form of EMM, the SwiZs, and indirect inference estimators (as in Definition 4) are actually the very same estimators, not only asymptotically, but for any sample size, and under a very mild condition. Indeed, Assumption 4 requires the roots of the estimating function to be well separated so there exists a unique solution. This requirement is restrictive and it is typically satisfied. One may even wonder what would be the purpose of an estimating function for which Assumption 4 would not hold. In this spirit, Assumption 4 may be qualified as the “minimum criterion” for choosing an estimating function.

Even if the optimizer is perfect, Theorem 5 does not imply that the exact same values are found using the SwiZs or the indirect inference estimators, but that they belong to the same set of solutions, and thereby that they share the same statistical properties. Hence, Theorem 5 offers us two different ways of computing the same estimators. Simple calculations however show that the SwiZs is computationally more attractive. Indeed, if we let $k$ denotes the cost evaluation of $\Psi_n$, $l$ the numbers of evaluations of $\Psi_n$ for obtaining an auxiliary estimator or the final estimator, then the SwiZs has a total cost of roughly $\Theta(2kl)$ whereas it is $\Theta(kl + kl^2)$ for the indirect inference estimator, so a reduction in order of $\Theta(kl^2)$. This computational efficiency of the SwiZs accounts for the fact that it is not necessary to compute $\hat{\pi}_n$, and thus avoids the numerical problem of the indirect inference estimator of having an optimization nested within an optimization. This discrepancy is also, quite surprisingly, reflected in the theory we develop in Section 4 for the finite sample properties and in Section 5 for the asymptotic properties.

At first glance, the SwiZs may appear similar to the parametric bootstrap (see the Definition 6 below). If we strengthen our assumptions and think of the auxiliary estimator as an unbiased estimator of $\theta$, it is natural to think of the SwiZs and the parametric bootstrap as being equivalent. In any cases, both methods use the exact same ingredients, so we may wonder whether actually they are the same. The next result demonstrates that in fact, they will be seldom equivalent.

Definition 6 (parametric bootstrap). Let $\hat{\pi}_n$ and $\{u_j : j \in \mathbb{N}\}$ be defined as in Definition 2. We consider the following sequence of estimators:

$$\hat{\theta}_{\text{Boot},n}^{(s)} \in \Theta_{\text{Boot},n}^{(s)} = \arg\min_{\theta \in \Theta} \Psi_n(\hat{\pi}_n, u_s, \theta), \quad s \in \mathbb{N}_S.$$  

The collection of the solutions is $\Theta_{\text{Boot},n} = \bigcup_{s \in \mathbb{N}_S} \Theta_{\text{Boot},n}^{(s)}$.

Remark 4. For the solutions $\Theta_{\text{Boot},n}^{(s)}$ in Definition 6 to be nonempty, the parametric bootstrap requires that $\Pi_n \subset \Theta$. The SwiZs has not such requirement.

Assumption 7. The zeros of the estimating functions are symmetric on $(\theta, \pi)$, that is

$$\Psi_n(\theta, u_s, \pi) = \Psi_n(\pi, u_s, \theta) = 0.$$  

Theorem 8 (equivalence SwiZs/parametric bootstrap). If and only if Assumption 7 is satisfied, then it holds that

$$\Theta_n^{(s)} = \Theta_{\text{Boot},n}^{(s)}.$$  

Assumption 7 is very restrictive, so Theorem 8 suggests that in general the SwiZs and the parametric bootstrap are not equivalent. This may appear as a surprise as only the argument $\theta$ and $\pi$ are interchanged in the estimating function. Then, if they are different, the question of which one should be preferred naturally arises. We do not attempt at answering this question, but we rather prefer to stimulate debates by giving motivations for using the SwiZs. Popularized by [41], the bootstrap has been a long-standing technique for (frequentist) statistician, it is relatively straightforward to implement and has a well-established theory (see for instance [38]). On the other hand, although the idea of the SwiZs has been arguably around for decades (see the comparison with the fiducial inference at the end of this section), we lack evidence of its widespread usage, at least not under the form presented here. When facing situations where $\hat{\pi}_n$ is an unbiased estimator of $\theta_0$, compared to the parametric bootstrap, the SwiZs is more demanding for the implementation and is generally less numerically efficient (see Section 4) suggesting that solving $\Psi_n(\theta, \pi)$ in $\theta$ is computationally more involved than in $\pi$. However, in all the other situations where for example $\hat{\pi}_n$ may be an asymptotically biased estimator of $\theta_0$, a sample statistic or a consistent estimator of a different model, the parametric bootstrap cannot be invoked directly, at least not with the same form as in Definition 8. Indeed, the parametric bootstrap requires $\hat{\pi}_n$ to be a consistent estimator of $\theta_0$. Therefore, when considering complex model for which a consistent estimator is not readily available at a reasonable cost, the SwiZs may be computationally more attractive. The rest of this section aims at demonstrating that the distribution of the SwiZs is valid for the purpose of inference, whereas the following section theorizes the inferential properties of the SwiZs in finite sample for which
Sections 6 and 7 gather evidences. But before, having emphasized their differences, we would like to share a rather common problem on which the parametric bootstrap and the SwiZs are equivalent.

The condition under which the SwiZs and the parametric bootstrap are equivalent (Assumption 7) is very strong and generally not met. There is one situation however where this condition holds, if the inferential problem is on the parameter of a location family as formalized in the next Proposition.

**Proposition 9** (equivalence SwiZs/parametric bootstrap in location family problems). Suppose that $x$ is a univariate random variable identically and independently distributed according to a location family, that is $x = \theta + y$, where $\theta \in \mathbb{R}$ is the location parameter. If the auxiliary parameter is estimated by the sample average and $x$ is symmetric around 0, that is $x \overset{d}{=} -x$, then

$$\Theta_n(s) = \Theta_{\text{Boot}, n}^s.$$  

The conditions which satisfies Proposition 9 are restrictive. Indeed, they are satisfied for location families for which the centered random variable is symmetric. Proposition 9 holds for example with a Gaussian, a Student, a Cauchy and a Laplace random variables (variance and degrees of freedom known), but not, for example, for a generalized extreme value, a skewed Laplace and a skewed $t$ random variables (even with non-location parameters being fixed). The proof uses an average as the auxiliary estimator, but it should be easily extended to other estimator of location such as the trimmed mean. Proposition 9 is illustrated with a Cauchy random variable in Example 40 of Section 6.

Although the parametric bootstrap and the SwiZs will lead rarely to the same estimators, in spite of the similitude of their forms, the next result demonstrates that the distribution of the SwiZs corresponds in fact to (some sort of) a Bayesian posterior. Likewise the indirect inference, the approximate Bayesian computation (ABC) techniques were proposed to respond to complex problems. The two techniques are often presented to be respectively the frequentist and the Bayesian approaches to a same problem and have even been mixed sometimes (see [16, 17]). We now show under what conditions the SwiZs and the ABC are equivalent, but before, we need to give more precision on what type of posterior distribution. If $\hat{\pi}_n$ and $\{u_j : j \in \mathbb{N}\}$ be defined as in Definition 2, we consider the following algorithm. For a given $\varepsilon \geq 0$, for a given infinite sequence $\{u_s : s \in \mathbb{N}^+_2\}$, for a given infinite sequence of empty sets $\{\Theta_{\text{ABC}, n}^s(\varepsilon) : s \in \mathbb{N}^+_2\}$, for a given prior distribution $\mathcal{D}$ of $\Theta$, repeat (indeﬁnitely) the following steps:

1. Generate $\theta^* \sim \mathcal{D}$.
2. Compute $\hat{\pi}_n^s(\theta^*)$.
3. If the following criterion is satisfied

$$d\left(\hat{\pi}_n, \hat{\pi}_n^s(\theta^*)\right) \leq \varepsilon,$$

add $\theta^*$ to the set $\Theta_{\text{ABC}, n}^s(\varepsilon)$, i.e. $\Theta_{\text{ABC}, n}^s(\varepsilon) = \Theta_{\text{ABC}, n}^s(\varepsilon) \cup \{\theta^*\}$.

For a given $s \in \mathbb{N}^+_2$, we denote by $\tilde{\Theta}_{\text{ABC}, n}^s(\varepsilon)$ an element of $\Theta_{\text{ABC}, n}^s(\varepsilon)$. The collection of the solutions is denoted $\Theta_{\text{ABC}, n}(\varepsilon) = \cup_{s \in \mathbb{N}^+_2} \tilde{\Theta}_{\text{ABC}, n}^s(\varepsilon)$.

**Remark 5.** The ABC algorithm presented in Definition 10 is a speciﬁc version of the simple accept/reject algorithm proposed by [16, 17], where the auxiliary estimators are the solution of an estimating function and the dimensions of $\pi$ and $\theta$ are the same.

**Definition 11** (posterior distribution). The distribution of the infinite sequence $\{\tilde{\Theta}_{\text{ABC}, n}(\varepsilon) : s \in \mathbb{N}^+_2\}$ issued from Definition 10 is referred to as the $(\varepsilon, \hat{\pi}_n)$-approximate posterior distribution. If $\varepsilon = 0$, we have the $\hat{\pi}_n$-approximate posterior distribution. If $\hat{\pi}_n$ is a suﬃcient statistic, we have the $\varepsilon$-approximate posterior distribution. If both $\varepsilon = 0$ and $\hat{\pi}_n$ is suﬃcient, then we simply refer to the posterior distribution.

**Remark 6.** In Definition 11 we mention two sources of approximation to the posterior distribution, $\varepsilon$ and $\hat{\pi}_n$. There is actually a third source of approximation stemming from the number of simulations $S$, if indeed $S < \infty$. Since it is common to every methods presented, it is left implicit.
Assumption 12 (existence of a prior). For every $s \in \mathbb{N}^+_0$ and for all $n$, there exists a prior distribution $\mathcal{P}$ such that

$$
\lim_{\varepsilon \downarrow 0} \Pr \left( d \left( \hat{\pi}_n, \hat{\pi}_{n,\mathcal{P}_n}^{(s)} (\theta^*) \right) \leq \varepsilon \right) = 1, \quad \theta^* \sim \mathcal{P}.
$$

Theorem 13 (Equivalence SwiZs/ABC). If Assumptions 4 and 12 are satisfied, then the following holds:

$$
\Theta_n^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{ABC,n}^{(s)} (\varepsilon).
$$

From Theorem 13 and Definition 11, we have clearly established that the distribution obtained by the SwiZs is a $\hat{\pi}_n$-approximate posterior distribution. Yet, the conclusion reached by Theorem 13 is surprising at two different levels: first, Theorem 13 implies the possibility of obtaining an $\hat{\pi}_n$-approximate posterior distribution without specifying explicitly a prior distribution by using the SwiZs, second, whereas, for each $s \in \mathbb{N}^+_0$, it would in general require a very large number of sampled $\theta^*$ for the ABC to approach an $\hat{\pi}_n$-approximate posterior distribution ($\varepsilon = 0$), it is obtainable by the SwiZs at a much reduced cost. Indeed, for a given $s \in \mathbb{N}^+_0$, it demands in general a considerable number of attempts to sample a $\theta^*$ that satisfies the matching criterion with an error of $\varepsilon \approx 0$, whereas it is replaced by one optimization for the SwiZs, so it may be more computationally efficient to use the SwiZs. Note also that in the situation where one has a prior knowledge on $\theta$, the SwiZs may be modified, for example, by including an importance sampling weight, in the same fashion that the ABC would be modified when the prior distribution is improper (see e.g. [51]). However, for some problems, the optimizations to obtain the SwiZs distribution may be numerically cumbersome and the ABC may prove itself a facilitating alternative (for example [52] argued in this direction for some of their examples when comparing the indirect inference and the ABC).

Switching between the SwiZS and the ABC algorithms for estimating a posterior poses the fundamental and practical question of which prior distribution to use. Assumption 12 stating that a prior distribution exists is very reasonable and widely accepted (although a frequentist fundamentalist may argue differently), but the result of Theorem 13 brings at least three questions: which prior distribution satisfies both the SwiZs and the ABC at the same time, whether the prior distribution under which Theorem 13 holds is unique and whether there is an “optimal” prior in the numerical sense (that would produce $\theta^*$ satisfying “rapidly” the matching criteria as defined at the point 3 of Definition 10). We do not answer these questions because, firstly, the numerical problems we face in Section 7 are achievable quite efficiently by the SwiZs, secondly, they would deserve much more attention than what we are able to conduct in the present. Thus, we content ourselves by mentioning only briefly studies made on this direction. In order to approach this topic, we first need to present an ultimate technique.

The possibility of obtaining an (approximate) Bayesian posterior without specifying explicitly a prior distribution on the parameters of interest inescapably links the SwiZs to R.A. Fisher’s controversial fiducial inference (see for instance [22, 23, 24, 25, 26]). Here we keep the SwiZs neutral and do not aim at reanimating any debate. It is delicate to give an unequivocal definition of the fiducial inference as it has changed on many occasion over time (see [53] for a comprehensive historical review) and we rather give the presentation with the generalized fiducial inference proposed by [27] (see also [28, 29]) which includes R.A. Fisher’s fiducial inference. Other efforts to generalize R.A. Fisher’s fiducial inference include Fraser’s structural inference ( [30], see also [31]), the Dempster-Shafer theory ( [32, 33], see also [54]) refined later with the concept of inferential models ( [34, 35]). As argued by [27], Fraser’s structural inference may be viewed as a special case of the generalized fiducial inference where the generating function $g$ has a specific structure. The concept of inferential models is similar to the generalized fiducial inference in appearance but they differ in their respective theory. The departure point of the inferential models is to conduct inference with the conditional distribution of the pivotal quantity $u$ given $x_i$ after the sample has been observed. It is argued that keeping $u \sim F_u$ after the sample has been observed makes the whole procedure subjective ( [35]), but the idea is essentially a gain in efficiency of the estimators. Also this idea is sound (see Lemma 12 in the next section), we do not see how it can be applied for the practical examples we use in Section 7 and more fundamentally, we do not understand how such conditional distribution may be built without some form of prior (and arguably subjective) knowledge on $u_i$. We therefore leave such consideration for further research and limit the equivalence to the generalized fiducial inference given in the next definition.

Definition 14 (Generalized fiducial inference). The generalized fiducial distribution is given by

$$
\hat{\theta}_{GFD,n}^{(s)} \in \Theta_{GFD,n}^{(s)} = \arg \min_{\theta \in \Theta} d (x, g (\theta, u_i^*)).
$$

Remark 7. The generalized fiducial distribution in Definition 14 is slightly more specific than usually defined in the literature. In Definition 1 in [29], it is given by

$$
\lim_{\varepsilon \downarrow 0} \left[ \arg \min_{\theta \in \Theta} \| x - g (\theta, u_i^*) \| \cdot \min_{\theta} \| x - g (\theta, u_i^*) \| \leq \varepsilon \right],
$$

for any norm. Here, in addition, we assume that $\Theta$ contains at least one of, possibly many, zeros.
If we let the sample size equals the dimension of the parameter of interest, \( n = p \), then it is obvious from their definitions that the generalized fiducial distribution and the indirect inference estimators are equivalent. We formalize this finding for the sake of the presentation.

**Assumption 15.** The followings hold:

i. \( \pi_n = x; \)

ii. \( \pi_{II,n}(\theta) = g(\theta, u). \)

**Proposition 16.** If Assumption 15 is satisfied, then the following holds:

\[
\Theta_{II,n}^{(s)} = \Theta_{GFD,n}^{(s)}.
\]

Also the link between the indirect inference and the generalized fiducial inference seems self-evident, it was, at the best of our knowledge, never mentioned in the literature. It may be explained by the two different goals that each of these methods target, that may respectively be loosely summarized as finding a point-estimate of a complex problem and making Bayesian inference without using a prior distribution. Having established this equivalence, the connection with the SwiZs is direct from Theorem 5 and formalize in the next proposition.

**Proposition 17.** If Assumptions 4 and 15 are satisfied, then the following holds:

\[
\Theta_{II}^{(s)} = \Theta_{GFD,n}^{(s)}.
\]

In the light of Proposition 17, the SwiZs may appear equivalent to the generalized fiducial inference under a very restrictive condition. Indeed, the only possibility for Assumption 15 to hold is that the sample size must equal the dimension of the problem. But we would be willing to concede that this apparent rigidity is thinner as one may propose to use sufficient statistics with minimal reduction on the sample, thereby leaving \( n > p \), and Proposition 16 would still hold. Such situation however is confined to problems dealing with exponential families as demonstrated by the Pitman-Koopman-Darmois theorem, so in general, when \( n > p \) and the problem at hand is outside of the exponential family, the SwiZs and the generalized fiducial inference are not equivalent.

Although the link between the generalized fiducial inference and the indirect inference has remained silent, the connection with the former to the ABC has been much more emphasized. Indeed, the algorithms proposed to solve the generalized fiducial inference problem.

The generalized fiducial inference is also linked by 29 to what may be called “non-informative” prior approaches (see 56 for a broad discussion of this concept). More specifically, it appears that some distribution resulting from the generalized fiducial inference corresponds to the posterior distribution obtained by 57 based on a data-dependent prior proportional to the likelihood function in the absence of information. This result enlarges the previous vision brought by 58 that concluded that R.A. Fisher’s fiducial inference is “Bayes inconsistent” (in the sense that the Bayes’ theorem cannot be invoked) apart from problems on the Gaussian and the gamma distributions. 58’s results relied on a narrower definition of fiducial inference than brought by the generalized fiducial inference, so whether the generalized fiducial inference has become Bayes consistent for broader problems nor 57 approach with an uninformative prior is Bayes inconsistent remains an open question. But most importantly, the strong link between the generalized fiducial inference and this non-informative prior approach reveals the common goal towards which these approaches tends, which might be stated as tackling the individual subjectivism in the Bayesian inference that has been one of the major subject of criticism ever since at least 22.

Last but not least, we complete the loop by the following Corollary which is a consequence of Theorems 5, 8 and 13, and Propositions 16 and 17.

**Corollary 18.** We have the followings:

i. If Assumptions 4 and 12 are satisfied, then \( \Theta_{II,n}^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{ABC,n}^{(s)}(\varepsilon); \)

ii. If Assumptions 4, 12 and 7 are satisfied, then \( \Theta_{Boot,n}^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{ABC,n}^{(s)}(\varepsilon); \)

iii. If Assumptions 4 and 7 are satisfied, then \( \Theta_{II,n}^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{Boot,n}^{(s)}(\varepsilon); \)

iv. If Assumptions 4, 12 and 7 are satisfied, then \( \Theta_{Boot,n}^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{GFD,n}^{(s)}(\varepsilon); \)

v. If Assumptions 4, 12 and 7 are satisfied, then \( \Theta_{ABC,n}^{(s)} = \lim_{\varepsilon \downarrow 0} \Theta_{GFD,n}^{(s)}(\varepsilon). \)
4 Exact frequentist inference in finite sample

Having demonstrated that the distribution of the SwiZs sequence, for a single experiment, is approximatively a Bayesian posterior, we now turn our interest to the long-run statistical properties of the SwiZs. Our point of view here is frequentist, that is we suppose that we have an indefinite number of independent trials with fixed sample size \( n \) and fixed \( \theta_0 \in \Theta \). For each experiment we compute an exact \( \alpha \)-credible set, as given in the Definition 20 below, using the SwiZs independently: the knowledge acquired on an experiment is not used as a prior to compute the SwiZs on another experiment. The goal of this Section is to demonstrate under what conditions the SwiZs leads to exact frequentist inference when the sample size is fixed.

**Definition 19** (sets of quantiles). Let \( F_{\theta_n|\pi_n} \) be a \( \hat{\pi}_n \)-approximate posterior cumulative distribution function. We define the following sets of quantiles:

1. Let \( \underline{Q}_\alpha = \{ \theta_n \in \Theta_n, \alpha \in (0,1) : F_{\theta_n|\pi_n}(\theta_n) \leq \alpha \} \) be the set of all \( \theta_n \) for which \( F_{\theta_n|\pi_n} \) is below the threshold \( \alpha \).

2. Let \( \overline{Q}_\alpha = \{ \theta_n \in \Theta_n, \alpha \in (0,1) : F_{\theta_n|\pi_n}(\theta_n) \geq 1 - \alpha \} \) be the set of all \( \theta_n \) for which \( F_{\theta_n|\pi_n} \) is above the threshold \( 1 - \alpha \).

**Definition 20** (credible set). Let \( F_{\theta_n|\pi_n} \) be a \( \hat{\pi}_n \)-approximate posterior cumulative distribution function. A set \( C_{\pi_n} \) is said to be an \( \alpha \)-credible set if

\[
\Pr\left( \theta_n \in C_{\pi_n} | \pi_n \right) \geq 1 - \alpha, \quad \alpha \in (0,1),
\]

(1)

where

\[
C_{\pi_n} = \Theta_n \setminus \left\{ \underline{Q}_\alpha \cup \overline{Q}_\alpha \right\}, \quad \alpha_1 + \alpha_2 = \alpha.
\]

If we replace “\( \geq \)” by the equal sign in (1), we say that the coverage probability of \( C_{\pi_n} \) is exact.

Definition 20 is standard in the Bayesian literature (see e.g. [59]). Note that an \( \alpha \)-credible set can have an exact coverage only if the random variable is absolutely continuous. Such credible set is referred to as an “exact \( \alpha \)-credible set”.

The next result gives a mean to verify the exactness of frequentist coverage of an exact \( \alpha \)-credible set.

**Proposition 21** (Exact frequentist coverage). If a \( \hat{\pi}_n \)-approximate posterior distribution evaluated at \( \theta_0 \in \Theta_n \) is a realization from a standard uniform variate identically and independently distributed, \( F_{\theta_n|\pi_n}(\theta_0) = u, u \sim U(0,1) \), then every exact \( \alpha \)-credible set built from the quantiles of \( F_{\theta_n|\pi_n} \) leads to exact frequentist coverage probability in the sense that \( \Pr\left( C_{\pi_n} \ni \theta_0 \right) = 1 - \alpha \) (unconditionally).

Proposition 21 states that if the cumulative distribution function (cdf), obtained from the SwiZs, varies (across independent trials!) uniformly around \( \theta_0 \) (fixed!), so does any quantities computed from the percentiles of this cdf, leading to exact coverage in the long-run. The proof relies on Borel’s strong law of large number. Although this result may be qualified of unorthodox by mixing both Bayesian posterior and frequentist properties, it arises very naturally. Replacing \( \hat{\pi}_n \)-approximate posterior distribution by any conditional distribution on \( \pi_n \) in Proposition 21 leads to the same result. This proposition is similar in form to the concept of confidence distribution formulated by [60] and later refined by [61, 62, 63]. The confidence distribution is however a concept entirely frequentist and could not be directly exploited here. The general theoretical studies on the finite sample frequentist properties are quite rare in the literature, we should eventually mention the study of [36], although the theory developed is around inferential models and different than ours, the author uses the same criterion of uniformly distributed quantity to demonstrate the frequentist properties.

**Remark 8.** In Proposition 21 we use a standard uniform variable as a mean to verify the frequentist properties. With the current statement of the proposition, other distributions with support in \([0,1]\) may be candidates to verify the exactness of the frequentist coverage. However, if we restrain the frequentist exactness to be \( \Pr(C_{\pi_n} \ni \theta_0) = 1 - \alpha \), \( \Pr(Q_{\alpha_2} \ni \theta_0) = \alpha_2 \) and \( \Pr(Q_{\alpha_1} \ni \theta_0) = \alpha_1 \), for \( \alpha = \alpha_1 + \alpha_2 \), then the uniform distribution would be the only candidate.

In the light of Proposition 21 we now give the conditions under which the distribution of the sequence \( \{\hat{\theta}_n^{(s)} : s \in \mathbb{N}^+\} \), \( F_{\theta_n|\pi_n} \), leads to exact frequentist coverage probabilities. We begin with a lemma which is essential in the construction of our argument.
Lemma 22. If the mapping $\pi \mapsto \Psi_n$ has unique zero in $\Pi$ and the mapping $\theta \mapsto \Psi_n$ has unique zero in $\Theta$, then the following holds

$$\theta_0 = \hat{\theta}_n = \arg\min_{\theta \in \Theta} \Psi_n(\theta, u_0, \hat{\pi}_n).$$

The idea behind Lemma 22 is that if one knew the true pivotal quantity $u_0$ that generated the data, then one could directly recover the true quantity of interest $\theta_0$ from the sample. Of course, both $u_0$ and $\theta_0$ are unknown (otherwise statisticians would be extinct species!), but here we are exploiting the idea that, for a sufficiently large number of simulations $S$, at some point we will generate $u_n$ “close enough” to $u_0$. This idea is reflected in the following assumption.

Assumption 23. Let $\Theta_n \subseteq \Theta$ be the set of the solutions of the SwiZs in the Definition 2. We have the following:

$$\theta_0 \in \Theta_n.$$ 

The following functions are essential for convenient data reduction.

Assumption 24 (data reduction). We have:

1. There exists a Borel measurable surjection such that $b(u)$ has the same dimension as $x$.
2. There exists a Borel measurable surjection such that $h \circ b(u)$ has the same dimension as $\theta$.

Remark 9. The function $b$ allows to work with a random variable of the same dimension as the observed variable. Indeed we have

$$x = g(\theta, u) = g \circ (\text{id}_\Theta \times b)(\theta, u) = g(\theta, v),$$

where $v = b(u)$ has the same dimension as $x$ and $\text{id}_\Theta$ is the identity function on the set $\Theta$. On the other hand, the function $h$ allows us to deal with random variables of the same dimension as $\theta$, and thus $\pi$.

Remark 10. In Assumption 24 by saying the functions $h$ and $b$ are Borel measurable, we want to emphasis thereby that after applying these functions we still work with random variables, which is essential here.

To fix ideas, we consider the following example:

Example 25 (Explicit form for $h$ and $b$). As in Example 2 we suppose that $x = x_1, \cdots, x_n$ is identically and independently distributed according to $\Pi(\theta, \sigma^2)$, where $\sigma^2$ is known, and consider the generating function $g \in G$ where

$$g(\theta, u, \sigma^2) = \theta + \sigma \sqrt{-2 \ln(u_1)} \cos(2\pi u_2),$$

where $u_{1i}, u_{2i}, i = 1, \cdots, n$, are identically and independently distributed according to $\mathcal{U}(0, 1)$. Letting $v = b(u)$ be uniform in $[0, 1]$, we clearly have that $v \sim \mathcal{U}(0, 1)$ is a random variable of the same dimension as $x$. Next, if we consider $h$ as the function that averages its argument, we have

$$w = h \circ b(u) = \frac{1}{n} \sum_{i=1}^n v_i,$$

so by properties of Gaussian random variable we have that $w$ has a Gaussian distribution with mean 0 and variance $\frac{1}{n}$. Since $w$ is a scalar, it has the same dimensions as $\theta$.

Example 25 shows explicit forms for functions in Assumption 24. It is however not requested to have an explicit form as we will see. Indeed, under Assumption 24 we can construct the following estimating function:

$$\Psi_n(\theta, u^*, \pi) = \varphi_p(\theta, w, \pi),$$

where $w = h \circ b(u^*)$ is a $p$-dimensional random variable. The index $p$ in the estimating function $\varphi_p$, aims at emphasizing that $w$ has the same dimensions as $\theta$ and $\pi$, which is essential in our argument. Since the sample size $n$ and dimension $p$ are fixed here, it is disturbing. For some fixed $\theta_1 \in \Theta$ and $\pi_1 \in \Pi$, it clearly holds that:

$$\hat{\pi}_n = \arg\min_{\pi \in \Pi} \Psi_n(\theta_1, u^*, \pi) = \varphi_p(\theta_1, w, \pi_1),$$

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} \Psi_n(\theta, u^*, \pi_1) = \varphi_p(\theta, w, \pi_1).$$

Assumption 26 (characterization of $\varphi_p$). Let $\Theta_n \subseteq \Theta$ and $W_n$ be open subsets of $\mathbb{R}^p$. Let $\hat{\pi}_n$ be the unique solution of $\Psi_n(\theta_0, u_0, \pi)$. Let $\varphi_{\pi_n}(\theta, w) \equiv \varphi_p(\theta, w, \pi_n)$ be the map where $\pi_n$ is fixed. We have the following:

1. $\varphi_{\pi_n} \in C^1(\Theta_n \times W_n, \mathbb{R}^p)$ is once continuously differentiable on $(\Theta_n \times W_n) \setminus K_n$, where $K_n \subset \Theta_n \times W_n$ is at most countable,
2. $\det(D_\theta \varphi_{\pi_n}(\theta, w)) \neq 0$, $\det(D_w \varphi_{\pi_n}(\theta, w)) \neq 0$ for every $(\theta, w) \in (\Theta_n \times W_n) \setminus K_n$.

9
We have the followings:

Theorem 28.

Assumption 27

Theorem 28 is very powerful as it concludes that the SwiZs (Assumptions 24, 23 and 26) and the indirect inference Assumption 24. We do not need to know explicitly Assumption 26 (\( \theta \)), see also [65, 35] for the construction of \( \theta \). The estimators \( \hat{\theta} \) have exact frequentist coverage probabilities in finite sample. Our argument

seems not possible to reach the conclusion of Theorem 28 with a local implicit function theorem (usually encountered by conditioning). Yet, it remains unclear if this condition holds in the situations when the likelihood function does not exist. The indirect inference and ABC literatures are overflowing with existence of a maximum likelihood, it is simply impractical to obtain one. Second, Assumption 23 states that the true situation of the estimating functions of, respectively, the SwiZs and the indirect inference estimators. Although we fixed, it is not necessary to explore the whole set \( \Theta \) (that would require \( S \) to be extremely large), but an area sufficiently large of \( \Theta \) such that it includes \( \theta_0 \). Third, Assumptions 26 and 27 are more technical and concerns the finite sample behavior of the estimating functions of, respectively, the SwiZs and the indirect inference estimators. Although we cannot conclude that Assumption 26 is weaker than Assumption 27 it seems easier to deal with the former.

Assumption 26 (i) requires the estimating function to be once continuously differentiable in \( \theta \) and \( w \) almost everywhere. The estimators \( \hat{\theta}_n \) and \( \hat{\pi}_n \) are not known in an explicit form, but they can be characterized by their derivatives

\[
\begin{align*}
iii. \lim_{n\to\infty} \left\| \varphi_{\pi_n}(\theta, w) \right\| &= \infty,

\end{align*}
\]

Assumption 27 (characterization of \( \varphi_p \)). Let \( \Theta_n \subseteq \Theta \), \( W_n \) and \( \Pi_n \subseteq \Pi \) be open subsets of \( \mathbb{R}^p \). Let \( \varphi_{\theta}(w, \pi) = \varphi_p(\theta_1, w, \pi) \) be the map where \( \theta_1 \in \Theta \) is fixed. Let \( \varphi_{\pi_1}(\theta, \pi) = \varphi_p(\theta_1, \pi) \) be the map where \( \pi_1 \in \Pi_n \) is fixed. We have the followings:

i. \( \varphi_{\theta} \in C^1 (W_n \times \Pi_n, \mathbb{R}^p) \) is once continuously differentiable on \( (W_n \times \Pi_n) \setminus K_{1n} \), where \( K_{1n} \subseteq W_n \times \Pi_n \) is at most countable,

ii. \( \varphi_{\pi} \in C^1 (\Theta_n \times \Pi_n, \mathbb{R}^p) \) is once continuously differentiable on \( (\Theta_n \times \Pi_n) \setminus K_{2n} \), where \( K_{2n} \subseteq \Theta_n \times \Pi_n \) is at most countable,

iii. \( \det(D_{\pi} \varphi_{\theta}(w, \pi)) \neq 0, \det(D_{\pi} \varphi_{\pi}(w, \pi)) \neq 0 \) for every \( (w, \pi) \in (W_n \times \Pi_n) \setminus K_{1n} \),

iv. \( \det(D_{\pi} \varphi_{\pi}(w, \pi)) \neq 0 \) for every \( (\theta, \pi) \in (\Theta_n \times \Pi_n) \setminus K_{1n} \),

v. \( \lim_{(w, \pi)\to(\infty, \infty)} \left\| \varphi_{\theta}(w, \pi) \right\| = \infty,

vi. \( \lim_{(w, \pi)\to(\infty, \infty)} \left\| \varphi_{\pi}(\theta, \pi) \right\| = \infty.

Theorem 28. If Assumptions 24 and 27 and one of Assumptions 26 or 27 are satisfied, then the followings hold:

1. There is a C1-diffeomorphism map \( a : W_n \to \Theta_n \) such that the distribution function of \( \hat{\theta}_n \) given \( \hat{\pi}_n \) is

\[
\int_{\Theta_n} f_{\hat{\theta}_n} \left( \hat{\theta}_n | \hat{\pi}_n \right) \, d\theta = \int_{W_n} f(\hat{\theta}(w) | \hat{\pi}_n) \, |J(w | \hat{\pi}_n)| \, dw,
\]

where

\[
J(w | \hat{\pi}_n) = \frac{\det(D_{\theta} \varphi_{\pi}(\theta(w), w))}{\det(D_{\pi} \varphi_{\pi}(\theta(w), w))}.
\]

2. For all \( \alpha \in (0, 1) \), every exact \( \alpha \)-credible set built from the percentiles of the distribution function have exact frequentist coverage probabilities.
using an implicit function theorem argument. Since \( \theta \) and \( w \) appears in the generating function \( g \), this assumption may typically be verified with the example at hand using a chain rule argument: the estimating function must be once continuously differentiable in the observations represented by \( g \), and \( g \) must be once continuously differentiable in both its arguments. Discrete random variables are automatically ruled out by this last requirement, but this should not appear as a surprise as exactness of the coverage cannot be claimed in general for discrete distribution (see e.g. \([66]\)).

The smoothness requirement on the estimating function excludes for example estimators based on order statistics. In general, relying on non-smooth estimating function leads to less efficient estimators and less stable numerical solutions, but they may be an easier estimating function to choose in situations where it is not clear which one to select. Although, non-smooth estimating functions and discrete random variables are dismissed, the condition may nearly be satisfied when considering a \( n \) large enough. Assumption \([27](i, ii)\) requires in addition the estimating equation to be once continuously differentiable in \( \pi \).

Assumption \([26]\)(ii), as well as Assumption \([27]\)(iii, iv), essentially necessitate the estimating function to be “not too flat” globally. It is one of the weakest condition to have invertibility of the Jacobian matrices. Usually only one of the Jacobian has such requirement for an implicit function theorem, but since we are targeting a \( C^1 \)-diffeomorphism, we strengthen the assumption on both Jacobians. Once verified the first derivative of the estimating function as explained in the preceding paragraph, the non-nullity of determinant may be appreciated, it typically depends on the model and the chosen estimating function. An example for which this condition is not globally satisfied is when considering robust estimators as the estimating function is constant on an uncountable set once exceeding some threshold. This consideration gives raise to the question on whether this condition may be relaxed to hold only locally, condition which would be satisfied by the robust estimators, but Example \([50]\) with the robust Lomax distribution in the Section \([7]\) seems to indicate the opposite direction.

Assumption \([26]\)(iii), as well as Assumption \([27]\)(v, vi), is a necessary and sufficient condition to invoke Palais’ global inversion theorem \([\{67\}]\) which is a key component of the global implicit function theorem of \([68]\) we use. It can be verified in two steps by, first, letting \( g \) diverges in the estimating function, and then letting \( \theta \) and \( w \) diverges in \( g \) once continuously differentiable on \( \pi \). Occasionally, robust estimators do not fulfill this requirement as their estimating functions do not diverge with rather stay constant.

Once again, robust estimators do not fulfill this requirement for an implicit function theorem, but since we are targeting a particular, the distribution function is:

\[
\pi_n = h(x_0) \quad \text{where} \quad h \text{ is a known (surjective) function of the observations (see Assumption} \ [24].\]

We can define a (new) indirect inference estimator as follows:

\[
\hat{\theta}_{II,n} = \arg \min_{\theta \in \Theta} d[h(x_0), g(\theta, w)].
\]

**Remark 11.** The estimator defined in Equation \([2]\) is a special case of the indirect inference estimators as expressed in Definition \([3]\) and thus of the SwiZs by Theorem \([5]\) where the auxiliary estimators \( \hat{\pi}_n \) and \( \pi_{II,n} \) are known in an explicit form.

**Assumption 29** (characterization of \( g \)). Let \( \Theta_n \subseteq \Theta \), \( W_n \) be subsets of \( \mathbb{R}^p \) and \( K_n \subset \Theta_n \times W_n \) be at most countable. The followings hold:

1. \( g \in C^1(\Theta_n \times W_n, \mathbb{R}^p) \) is once continuously differentiable on \( (\Theta_n \times W_n) \setminus K_n \).
2. \( \det(D_\theta g(\theta, w)) \neq 0 \) and \( \det(D_w g(\theta, w)) \neq 0 \) for every \( (\theta, w) \in (\Theta_n \times W_n) \setminus K_n \).
3. \( \lim_{\|w\| \to \infty} \|g(\theta, w)\| = \infty \).

**Proposition 30.** If Assumptions \([24, 23, 29]\) are satisfied, then the conclusions (1) and (2) of Theorem \([28]\) hold. In particular, the distribution function is:

\[
\int_{\Theta_n} f_{\theta_n|\pi_n}(\theta|h(x_0)) \, d\theta = \int_{W_n} f(w|h(x_0)) \left| J(w|h(x_0)) \right| \, dw,
\]

where

\[
J(w|h(x_0)) = \frac{\det(D_\theta g(a(w), w))}{\det(D_w g(a(w), w))}.
\]

The message of Proposition \([30]\) is fascinating: once the auxiliary estimator is known in an explicit form, the conditions to reach the conclusion of Theorem \([28]\) simplify accounting for the fact that the implicit function theorem is no longer necessary. The discussion we have after Theorem \([28]\) still holds, but the verification process of the conditions is reduced to inspecting the generating function.
5 Asymptotic properties

When \( n \to \infty \), different assumptions than in Section 4 may be considered to derive the distribution of the SwiZs. By Theorem 34, the SwiZs in Definition 2 and the indirect inference estimators in Definition 3 are equivalent for any \( n \). Yet, due to their different forms, the conditions to derive their asymptotic properties differ, at least in appearance. We treat both the asymptotic properties of the SwiZs and the indirect inference estimators in an unified fashion and highlight their differences. We do not attempt at giving the weakest conditions possible as our goal is primarily to demonstrate in what theoretical aspect the SwiZs is different from the indirect inference estimators. The asymptotic properties of the indirect inference estimators were already derived by several authors in the literature, and we refer to [40], Chapter 4, for the comparison.

The following conditions are sufficient to prove the consistency of any estimator \( \hat{\theta}_n^{(s)} \) in Definitions 2 and 3. When it is clear from the context, we simply drop the suffix and denote \( \hat{\theta}_n \) for any of these estimators.

**Assumption 31.** The followings hold:

i. The sets \( \Theta, \Pi \) are compact,

ii. For every \( \pi_1, \pi_2 \in \Pi, \theta \in \Theta \) and \( u \sim F_u \), there exists a random value \( A_n = O_p(1) \) such that, for a sufficiently large \( n \),

\[
\| \Psi_n(\theta, u, \pi_1) - \Psi_n(\theta, u, \pi_2) \| \leq A_n \| \pi_1 - \pi_2 \|,
\]

iii. For every \( \theta \in \Theta, \pi \in \Pi \), the estimating function \( \Psi_n(\theta, u, \pi) \) converges pointwise to \( \Psi(\theta, \pi) \).

iv. For every \( \theta \in \Theta, \pi_1, \pi_2 \in \Pi \), we have

\[
\Psi(\theta, \pi_1) = \Psi(\theta, \pi_2),
\]

if and only if \( \pi_1 = \pi_2 \).

**Assumption 32 (SwiZs).** The followings hold:

i. For every \( \theta_1, \theta_2 \in \Theta, \pi \in \Pi \) and \( u \sim F_u \), there exists a random value \( B_n = O_p(1) \) such that, for a sufficiently large \( n \),

\[
\| \Psi_n(\theta_1, u, \pi) - \Psi_n(\theta_2, u, \pi) \| \leq B_n \| \theta_1 - \theta_2 \|,
\]

ii. For every \( \theta_1, \theta_2 \in \Theta, \pi \in \Pi \), we have

\[
\Psi(\theta_1, \pi) = \Psi(\theta_2, \pi),
\]

if and only if \( \theta_1 = \theta_2 \).

**Assumption 33 (IIIE).** The followings hold:

i. For every \( \theta_1, \theta_2 \in \Theta \), there exists a random value \( C_n = O_p(1) \) such that, for sufficiently large \( n \),

\[
\| \hat{\pi}_{H,n}(\theta_1) - \hat{\pi}_{H,n}(\theta_2) \| \leq C_n \| \theta_1 - \theta_2 \|;
\]

ii. Let \( \pi(\theta) \) denotes the mapping towards which \( \hat{\pi}_{H,n}(\theta) \) converges pointwise for every \( \theta \in \Theta \). For every \( \theta_1, \theta_2 \in \Theta \), we have

\[
\pi(\theta_1) = \pi(\theta_2),
\]

if and only if \( \theta_1 = \theta_2 \).

**Theorem 34 (consistency).** Let \( \{ \hat{\pi}_n \} \) be a sequence of estimators of \( \{ \Psi_n(\pi) \} \). For any fix \( \theta \in \Theta \), let \( \{ \hat{\pi}_{H,n}(\theta) \} \) be the sequence of estimators of \( \{ \Psi_n(\theta, \pi) \} \). Let \( \{ \hat{\theta}_n \} \) be a sequence of estimators of \( \{ \Psi_n(\theta) \} \). We have the following:

1. If Assumption 31 holds, then any sequence \( \{ \hat{\pi}_n \} \) converges in probability to \( \pi_0 \) and any sequence \( \{ \hat{\pi}_{H,n}(\theta) \} \) converges in probability to \( \pi(\theta) \);

2. Moreover, if one of Assumptions 32 or 33 holds, then any sequence \( \{ \hat{\theta}_n \} \) converges in probability to \( \theta_0 \).

Theorem 34 demonstrates the consistency of \( \hat{\theta}_n \) under two sets of conditions. Assumptions 31 and 33 or the conditions that are implied by these Assumptions, are regular in the literature of the indirect inference estimators (see [40], Chapter 4). More specifically, the mapping \( \theta \mapsto \pi \), usually referred to as the “binding” function (see e.g. [2]) or the “bridge relationship” (see [69]), is central in the argument and is required to have a one-to-one relationship.
We now turn our interest to the asymptotic distribution of an estimator \( \hat{\theta}_n \). Surprisingly, in Theorem 34 such requirement may be substituted by the bijectivity of the deterministic estimating function with respect to \( \theta \) (Assumption 32 (ii)). Whereas the bijectivity of \( \pi(\theta) \) can typically only be assumed (if \( \theta \mapsto \pi \) was known explicitly, then one would not need to use the indirect inference estimator unless of course one would be willing to lose statistical efficiency and numerical stability for no gain), there is more hope for Assumption 32 (ii) to be verifiable. Since both Assumptions 32 and 33 leads to the same conclusion, one would expect some strong connections between them. Since \( \pi(\theta) \) may be interpreted as the implicit solution of \( \Psi(\theta, \pi(\theta)) = 0 \), it seems possible to link both Assumptions with the help of an implicit function theorem, but it typically requires further conditions on the derivatives of \( \Psi \) that are not necessary for obtaining the consistency results, and we thus leave such considerations for further research.

Proving the consistency of an estimator relies on two major conditions: the uniform convergence of the stochastic objective function and the bijectivity of the deterministic objective function (Assumption 31 (iv), Assumption 32 (ii), Assumption 33 (ii)). This second condition is referred to as the identifiability condition. It can sometimes be verified, or sometimes it is only assumed to hold, but it is typically appreciated in accordance with the chosen probabilistic model. Discrepancy among approaches mainly occurs on the demonstration of the uniform convergence. Here we rely on a stochastic version of the classical Arzelà-Ascoli theorem, see [70] for alternative approaches based on the theory of empirical processes. To satisfy this theorem, we require the parameter sets to be compact (Assumption 31 (ii)), the stochastic objective function to converge pointwise (Assumption 31 (iii)) and the stochastic objective function to be Lipschitz (Assumption 31 (ii), Assumption 32 (i), Assumption 33 (i)). Note that the last requirement is in fact for the objective function to be stochastically equicontinuous, requirement verified by the Lipschitz condition, see also [71] for a broad discussion on this condition and alternatives. Some authors proposed to relax the compactness condition, see for example [72], but this is generally not a sensitive issue in practice. The pointwise convergence of the stochastic objective function may be appreciated up to further details depending on the context. For identically and independently distributed observations, typically the weak law of large numbers may be employed, thus requiring the stochastic objective function to have the same finite expected value across the observations. Other law of large numbers results may be used for serially dependent processes (see the Chapter 7 of [73]) and for non-identically distributed processes (see [74]), each results having its own conditions to satisfy.

We now turn our interest to the asymptotic distribution of an estimator \( \hat{\theta}_n \). Likewise the consistency result, the following sufficient conditions, are separated to outline the difference between the SwiZs and the indirect inference estimators.

**Assumption 35.** The followings hold:

i. Let \( \Theta^0, \Pi^0 \), the interior sets of \( \Theta, \Pi \), be open and convex subsets of \( \mathbb{R}^p \).

ii. \( \theta_0 \in \Theta^0 \) and \( \pi_0 \in \Pi^0 \).

iii. \( \Psi_n \in C^1(\Theta^0 \times \Pi^0, \mathbb{R}^p \times \mathbb{R}^p) \) when \( n \) is sufficiently large,

iv. For every \( \theta \in \Theta^0, \pi \in \Pi^0 \), \( D_\theta \Psi_n(\theta, \pi) \) converge pointwise to \( D_\theta \Psi(\theta, \pi) \equiv K(\theta, \pi), D_\pi \Psi_n(\theta, \pi) \) converge pointwise to \( D_\pi \Psi(\theta, \pi) \equiv J(\theta, \pi) \).

v. \( K \equiv K(\theta_0, \pi_0), \) \( J \equiv J(\theta_0, \pi_0) \) are nonsingular;

vi. \( n^{1/2} \Psi_n(\theta_0, \pi) \sim N(0, Q), \|Q\|_\infty < \infty \).

**Assumption 36** (SwiZs II). For every \( \pi_1, \pi_2 \in \Pi^0, \theta \in \Theta^0 \) and \( u \sim \mathcal{N}(0, \Sigma) \), there exists a random value \( E_n = \Theta_{\odot}(1) \) such that, for sufficiently large \( n \)

\[
\|D_\theta \Psi_n(\theta, \pi_1) - D_\theta \Psi_n(\theta, \pi_2)\| \leq E_n \|\pi_1 - \pi_2\| .
\]

**Assumption 37** (IE II). The followings hold:

i. \( \pi_{n_0} \in C^1(\Theta^0, \mathbb{R}^p) \) for sufficiently large \( n \);

ii. For every \( \theta \in \Theta^0, D_\theta \hat{\pi}_{n_0}(\theta) \) converges pointwise to \( D_\theta \pi(\theta) \).

**Theorem 38** (asymptotic normality). If the conditions of Theorem 34 are satisfied, we have the following additional results:

1. If Assumption 35 holds, then

\[
n^{1/2} (\hat{\pi}_n - \pi_0) \sim N(0, K^{-1}QK^{-T}) ,
\]

and

\[
n^{1/2} (\hat{\pi}_{n_0}(\theta) - \pi(\theta)) \sim N(0, K^{-1}QK^{-T}) ;
\]
2. Moreover, if Assumption 36 or 37 holds, then
\[ n^{1/2} \left( \hat{\theta}_n - \theta_0 \right) \rightsquigarrow N \left( 0, 2J^{-1}QJ^{-T} \right). \]

Theorem 38 gives the asymptotic distribution of both the auxiliary estimator and the estimator of interest. The conditions to derive the asymptotic distribution of the auxiliary estimator as expressed in Assumption 35 is regular for most estimators in the statistical literature. The proof of the first statement relies on the possibility to apply a delta method, which requires the estimating function to be once continuously differentiable (Assumption 35 (i), (ii) and (iii)). The case where this condition is not met is typically when \( \theta_0 \) is a boundary point of \( \Theta \). Not devoid of interest, this case is atypical and deserves to be treated on its own, this situation is therefore excluded by Assumption 35 (ii). In contrast, relaxing the smoothness requirement on the estimating function has received a much larger attention in the literature (see \( [72, 75, 70] \) among others). Here we content ourselves with the stronger smooth condition on the estimating function (Assumption 35 (iii)), maybe because it is largely admitted, but also maybe because the smoothness of the estimating function is already required when \( n \) is finite by Theorem 28 to demonstrate the exact coverage probabilities, a situation that encourages us to consider smooth estimating function in the practical examples. The conditions for the Jacobian matrices to exist (Assumption 35 (iv)) and to be invertible (Assumption 35 (v)) are regular ones. The last condition is that a central limit theorem is applicable on the estimation equation (Assumption 35 (vi)). This statement is very general and its validity depends upon the context. For identically and independently distributed observations, one typically needs to verify Lindeberg’s conditions (72)), which essentially requires that the two first moments exist and are finite. The requirements are similar if the observations are non-identically observed (see e.g. (77)). The conditions are also similar for stationary processes (see e.g. \( [78] \), for a review). Note eventually that, also as minor as it might be, the delta method (which is essentially a mean value theorem) largely in use in the statistical literature has recently been shown to be wrongly used for many by vector-valued function (79), this flaw has been taken into account in the present.

The proof of the second statement of Theorem 38 on the asymptotic distribution of the estimator of interest is more specific to the indirect inference literature. Compared to the proof of the first statement, it requires in addition that, for \( n \) large enough, the binding function to be asymptotically differentiable with respect to \( \theta \) for the indirect inference estimator (Assumption 37) or the derivative of the estimating function with respect to \( \theta \) to be stochastically Lipschitz for the SwiZs (Assumption 39). For the same arguments we presented after the consistency Theorem 34, it may be more practical to verify Assumption 36 as the verification of Assumption 37 is impossible, at least directly, as the binding function is unknown. This is actually not entirely true as one may express the derivative of the binding function by invoking an implicit function theorem, the condition then may be verified on the resulting explicit derivative. The proof we use under Assumption 37 uses this mechanism, the derivative of the binding function is thus given by

\[ D_\theta \hat{\pi}(\theta) = -K^{-1}J, \]

for every \( \theta \) in a neighborhood of \( \theta_0 \) (see the proof in Appendix for more details). It is only by using this implicit function theorem argument that the exact same explicit distribution for both the SwiZs and the indirect inference estimators may be obtained. The same idea may be used then to find the derivative of \( \hat{\pi}_{n,s}(\theta) \) and verify Assumption 36.

Note eventually that \( [40] \) have an extra condition not required here (but that would as well be required) because they include a stochastic covariate with their indirect inference estimator.

Having demonstrated the asymptotic properties of one of the SwiZs estimators, \( \hat{\theta}^{(s)}_n, s \in N^+_S \), we finish this section by giving the property of the average of the SwiZs sequence. The mean is an interesting estimator on its own and it is often considered as a point estimate in a Bayesian context.

**Proposition 39.** Let \( \bar{\theta}_n \) be the average of \( \{ \hat{\theta}^{(s)}_n : s \in N^+_S \} \). If the conditions of Theorem 38 are satisfied, then it holds that

\[ n^{1/2} \left( \bar{\theta}_n - \theta_0 \right) \rightsquigarrow N \left( 0, \gamma J^{-1}QJ^{-T} \right), \]

where the factor \( \gamma = 1 + 1/S \).

The discussion of the proof and the condition to obtain Theorem 38 are also valid for Proposition 39. The only point that deserves further explanations is on the factor \( \gamma \). This factor accounts for the numerical approximation of the \( \pi_n \)-approximate posterior when \( S \) is finite. It is not surprising though for someone familiar with the indirect inference literature. What may appear unclear is how this factor pass from 2 for one the SwiZs estimate in Theorem 38 to \( \gamma < 2 \) for the mean in Proposition 38. If the \( \{ \hat{\theta}^{(s)}_n : s \in N^+_S \} \) are independent, then it is well-known from the properties of the convolution of independent Gaussian random variables that \( \gamma \) should equal 2. In fact, the pivotal quantities \( \{ u_s : s \in N^+_S \} \) are indeed independent, but each of the \( \{ \hat{\theta}^{(s)}_n : s \in N^+_S \} \) shares a “common factor”, namely \( \pi_n \), and thus this common variability may be reduced by increasing \( S \). Note eventually that the average estimator in Proposition 39 has the same asymptotic distribution as the two indirect inference estimators considered by \( [2] \) (given that the dimension of \( \Theta \) and \( \pi \) matches and that our implicit function theorem argument is used).
6 Examples

In this section, we illustrate the finite sample results of the Section 4 with some examples for which explicit solutions exist. Indeed, for all the examples, we are able to demonstrate analytically that the SwiZs’ \( \hat{\pi}_n \)-approximate posterior distribution follows a uniform distribution when evaluated at the true value \( \theta_0 \), and thus concluding by Proposition 21 that any confidence regions built from the percentiles of this posterior have exact coverage probabilities in the long-run. In addition, and maybe more surprisingly, for most examples we are able to derive the explicit posterior distribution that the SwiZs targets. This message is formidable, one may not even need computations to characterize the distribution of \( \hat{\theta}_n \) given \( \hat{\pi}_n \), but as one may foresee, these favorable situations are limited in numbers. Lastly, we illustrate Proposition 9 on the equivalence between the SwiZs and the parametric bootstrap with a Cauchy random variable in Example 40 to conclude that they are indeed the same. Since the SwiZs and the parametric bootstrap are seldom equivalent (see the discussion after Theorem 8), we also demonstrate the nonequivalence of the two methods in the case of uniform random variable with unknown upper bound (Example 41) and a gamma random variable with unknown rate (Example 43). The considerations of this section are not only theoretical but also practical as we treat the linear regression (Example 45) and the geometric Brownian motion when observed irregularly (Example 48), two models widely use.

Example 40 (Cauchy with unknown location). Let \( x_i \sim \text{Cauchy}(\theta, \sigma) \), \( \sigma > 0 \) known, \( i = 1, \ldots, n \), be identically and independently distributed. Consider the generating function \( g(\theta, u) = \theta + u \) where \( u \sim \text{Cauchy}(0, \sigma) \) and the average as the (explicit) auxiliary estimator, \( \hat{\pi}_n = \bar{x} \). We have

\[
\hat{\pi}_{II,n}(\theta) = \frac{1}{n} \sum_{i=1}^{n} g(\theta, u_i) = \theta + w,
\]

where \( w = \frac{1}{n} \sum_{i=1}^{n} u_i \). By the properties of the Cauchy distribution, we have that \( w \sim \text{Cauchy}(0, \sigma) \), that is the average of independent Cauchy variables has the same distribution of one of its components. Let \( \hat{\theta}_n \) be the solution of \( d(\hat{\pi}_n, \hat{\theta}_n + w) = 0 \), hence we have the explicit solution \( \hat{\theta}_n = \hat{\pi}_n - w \). Note that by symmetry of \( w \) around \( 0 \) we have \( w \overset{d}{=} -w \), so \( \hat{\theta}_n = \hat{\pi}_n + w \). We therefore have that

\[
\Pr(\hat{\theta}_n \leq \theta_0 | \hat{\pi}_n) = \Pr(\hat{\pi}_n + w \leq \theta_0 | \hat{\pi}_n) = \Pr(\theta_0 - w_0 + w \leq \theta_0 | \theta_0, w_0) = \Pr(w \leq w_0) \sim U(0, 1),
\]

and by Proposition 21 the coverage obtained on the percentiles of the distribution of \( \hat{\theta}_n | \hat{\pi}_n \) are exact in the long-run (frequentist).

The distribution of \( \hat{\theta}_n | \hat{\pi}_n \) can be known in an explicit form. From the solution of \( \hat{\theta}_n \), we let \( w = a(\theta) = \hat{\pi}_n + \theta \). Following Proposition 20 we have

\[
f_{\hat{\theta}_n}(\theta | \hat{\pi}_n) = f_w(a(\theta) | \hat{\pi}_n) \frac{\partial}{\partial \theta} g(\theta, w)\bigg|_{\theta = \theta_0}.
\]

Since \( g(\theta, w) = \theta + w \), the scaling factor is 1 and \( \hat{\theta}_n | \hat{\pi}_n \sim \text{Cauchy}(\hat{\pi}_n, \sigma) \).

Eventually, we illustrate Theorem 8 more specifically Proposition 9 by showing that the parametric bootstrap is equivalent. The bootstrap estimators is \( \hat{\theta}_{\text{Boot},n} = \frac{1}{n} \sum_{i=1}^{n} g(\hat{\pi}_n, u_i) = \hat{\pi}_n + w \). It follows immediately that \( \hat{\theta}_n = \hat{\theta}_{\text{Boot},n} \) and both estimators are equivalently distributed.

Example 41 (uniform with unknown upper bound). Let \( x_i \sim U(0, \theta) \), \( i = 1, \ldots, n \), be identically and independently distributed. Consider the generating function \( g(\theta, u) = u \theta \) where \( u \sim U(0, 1) \) and the (explicit) auxiliary estimator \( \max_i x_i \). Clearly, \( \max_i x_i = \theta \max_i u_i \). Denote \( w = \max_i u_i \) so the auxiliary estimator on the sample is \( \hat{\pi}_n = w_0 \theta_0 \). Now define the estimator \( \hat{\theta}_n \) to be the solution such that \( d(\hat{\pi}_n, \hat{\theta}w) = 0 \). An explicit solution exists and is given by \( \hat{\theta}_n = \frac{\theta_0 w_0}{w} \). We therefore have that

\[
\Pr(\hat{\theta}_n \leq \theta_0 | \hat{\pi}_n) = \Pr\left(\frac{\theta_0 w_0}{w} \leq \theta_0 | \theta_0, w_0\right) = \Pr\left(w^{-1} \leq w_0^{-1}\right) \sim U(0, 1),
\]

and by Proposition 21 the coverage obtained on the percentiles of the distribution of \( \hat{\theta}_n \) are exact in the frequentist sense.
We can even go further by expliciting the distribution of $\hat{\theta}_n$ given $\hat{n}$. Let define the mapping $\alpha(\theta) = \frac{n\bar{\theta}}{\theta}$. By the change-of-variable formula we obtain:

$$f_{\hat{\theta}_n}(\theta|\hat{n}) = f_w(\alpha(\theta)|\hat{n}) \left| \frac{\partial}{\partial \theta} \alpha(\theta) \right|.$$ 

The maximum of $n$ standard uniform random variables has the density $f_w(w) = n w^{n-1}$. The derivative is given by $\partial \alpha(\theta)/\partial \theta = -\theta_0 w_0/\theta^2$. Note that by Proposition 27 we equivalently have

$$\frac{\partial}{\partial w} g(\theta, w) \bigg|_{w = \alpha(\theta)} = \frac{w}{\theta} \bigg|_{w = \theta_0 w_0/\theta} = \frac{\theta_0 w_0}{\theta^2}.$$ 

Hence, we eventually obtain:

$$f_{\hat{\theta}_n}(\theta|\hat{n}) = \frac{n\pi_n^{\theta_0}}{\theta_0^{n+1}} \cdot \pi_n = \theta_0 w_0.$$ 

Note that $\hat{n}$ is a sufficient statistic. Therefore we have obtained that the posterior distribution of $\hat{\theta}_n$ given $\hat{n}$ is a Pareto distribution parametrized by $\hat{n}$, the minimum value of the support, and the sample size $n$, as the shape parameter.

In view of the preceding display, it is not difficult to develop a similar result for the parametric bootstrap (see the Definition 28). The bootstrap estimator solution is simple, it is given by $\hat{\theta}_{\text{Bootstrap}} = \max_i u_i \hat{n} = \theta_0 w_0 w$. We thus obtain

$$\Pr \left( \hat{\theta}_{\text{Bootstrap}} \leq \theta_0 | \hat{n} \right) = \Pr \left( \theta_0 w_0 w \leq \theta_0 | \theta_0, w_0 \right) = \Pr \left( w \leq w_0^{-1} \right),$$

so it cannot be concluded that $F_{\theta_{\text{Bootstrap}} | \hat{n}} (\theta_0)$ follows a uniform distribution and we cannot invoke Proposition 27. Note that however we cannot exclude that the parametric bootstrap leads to exact coverage probability in virtue of Proposition 27 (see Remark 28). The parametric bootstrap is well-known to be inadequate in such problem. This fact may be made more explicit as we give now the distribution of the parametric bootstrap estimators. Let define the mapping $w = b(\hat{\theta}) = \frac{\hat{\theta}}{\theta_0 w_0}$. Note that $b(\theta_0) = 1/w_0 \neq w_0$. We obtain by the change-of-variable formula

$$f_{\theta_{\text{Bootstrap}}}(\theta | \hat{n}) = f_w \left( b(\hat{\theta}) | \hat{n} \right) \left| \frac{\partial}{\partial \theta} b(\hat{\theta}) \right| = \frac{n\hat{n}^{n-1}}{\pi_n^{\theta_0}}.$$ 

This distribution is known to be the power-function distribution, a special case of the Pearson Type I distribution (see [53]). More interestingly, we have the following relationship between the parametric bootstrap and the SwiZs estimates:

$$\hat{\theta}_{\text{Bootstrap}} \sim \frac{1}{\theta_0}.$$ 

Ultimately, note that the support of the distribution of $\hat{\theta}_{\text{Bootstrap}}$ is $(0, \hat{n})$ whereas it is $(\hat{n}, +\infty)$ for the SwiZs, so both distributions never cross! Since $\hat{n}$ is systematically bias downward the true value $\theta_0$, the coverage of the parametric bootstrap is always null. We illustrate this fact in the next figure.

**Example 42** (exponential with unknown rate parameter). Let $x_1 \sim \mathcal{E}(\theta), i = 1, \ldots, n$, be identically and independently distributed. Consider the generating function $g(\theta, u) = \frac{u}{\theta}$, where $u \sim \Gamma(1, 1)$, and the inverse of the average as auxiliary estimator, denoted $\bar{x}^{-1}$. Clearly we have $\bar{x}^{-1} = \theta/w$, where $w = \sum_{i=1}^n u_i/n$, so $\hat{n} = \theta_0/w_0$. The solution of $d(\hat{n}, \theta/w) = 0$ in $\theta$ is given by $\hat{\theta}_n = \theta_0 w/w_0 = w \hat{n}$. We therefore have

$$\Pr \left( \hat{\theta}_n \leq \theta_0 | \hat{n} \right) = \Pr \left( w \leq w_0 \right) \sim U(0, 1).$$

It results from Proposition 27 that any intervals built from the percentiles of the distribution of $\hat{\theta}_n$ has exact frequentist coverage. The distribution can be found in explicit form. We have by the additive property of the Gamma distribution that $w \sim \Gamma(n, 1/n)$ (shape-rate parametrization). It immediately results from the change-of-variable formula that

$$\hat{\theta}_n | \hat{n} \sim \Gamma \left( n, \frac{1}{\sum_{i=1}^n x_i} \right).$$

Note that $\hat{n}$ is a sufficient statistic so the obtained distribution is a posterior distribution.

This last example on an exponential variate can be (slightly) generalized to a gamma random variable as follows.
We also have that any intervals built from the percentiles of the posterior have exact frequentist coverage probabilities. Therefore, by Proposition 21, any region built from the percentiles of the posterior distribution of \( \hat{\theta} \) has exact frequentist coverage. This posterior distribution has a closed form.

Example 43 (gamma with unknown rate parameter). Consider the exact same setup as in Example 42 with the exception that \( x_i \sim \Gamma(\alpha, \theta) \) and \( u \sim \Gamma(\alpha, 1) \), where \( \alpha > 0 \) is a known shape parameter. Following the same steps as in Example 42, we find the following posterior distribution:

\[
\hat{\theta}_n | \tilde{\pi}_n \sim \Gamma \left( \alpha n, \sum_{i=1}^{n} x_i \right).
\]

We also have that any intervals built from the percentiles of the posterior have exact frequentist coverage probabilities. In view of this display and Example 42, we can derive the distribution of the parametric bootstrap. The estimator is obtained as follows:

\[
\hat{\theta}_{\text{Boot}, n} = \frac{\sum_{i=1}^{n} u_i}{n} \sim \tilde{\pi}_n.
\]

where \( w \sim \Gamma(n\alpha, 1/n) \). It follows by the inverse of gamma variate and the change-of-variable formula that

\[
\hat{\theta}_{\text{Boot}, n} \sim \Gamma^{-1} \left( n\alpha, \sum_{i=1}^{n} x_i \right),
\]

so \( \hat{\theta}_{\text{Boot}, n} \equiv 1/\hat{\theta}_n \). Since \( \tilde{\pi}_n = \hat{\theta}_0 / u_0 \), we can also conclude that the parametric bootstrap is not uniformly distributed:

\[
\Pr \left( \hat{\theta}_{\text{Boot}, n} \leq \hat{\theta}_0 | \tilde{\pi}_n \right) = \Pr \left( \frac{\theta_0}{u_0 w} \leq \hat{\theta}_n | \theta_0, w \right) = \Pr \left( \frac{1}{w} \leq w_0 \right).
\]

The posterior distribution we obtained for the SwiZs in the last example coincides with the fiducial distribution [see Table 1 [81], Example 21.2 [82]]. This correspondence is not surprising in view of the discussion held after Proposition 17. Indeed the gamma distribution is a member of the exponential family and we use a sufficient statistic as the auxiliary estimator, so the SwiZs and the generalized fiducial distribution are equivalent.

We now turn our attention to more general examples where \( \theta \) is not a scalar.

Example 44 (normal with unknown mean and unknown variance). Let \( x_i \sim \mathcal{N}(\mu, \sigma^2) \) be identically and independently distributed and consider \( g(\mu, \sigma^2, u) = \mu + \sigma^2 u \) where \( u \sim \Gamma(0, 1) \). Take the following auxiliary estimator:

\[
\hat{\pi}_n = (\bar{x}, k s^2)^T = h(x), \quad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2, \quad k \in \mathbb{R}
\]

is any constant. Note for example that \( k < 0 \), so the auxiliary estimator of the variance may be negative. Indeed the SwiZs accepts situation for which \( \Pi \cap \Theta = \emptyset \), it is clearly not the case of the parametric bootstrap for example (see Remark 4). We have that

\[
\tilde{\pi}_n = (\bar{x}, s^2)^T = h(u) = \left( \frac{1}{n} \sum_{i=1}^{n} u_i, \left( \sum_{i=1}^{n} u_i - \frac{1}{n} \sum_{j=1}^{n} u_j \right)^2 \right).
\]

An explicit solution exists for \( d(\tilde{\pi}_n, g(\mu, \sigma^2, w)) = 0 \) in \( (\mu, \sigma^2) \) and is given by

\[
\hat{\theta}_n = \left( \frac{\bar{x}}{\sigma^2} = a(w) \right).
\]

Note that \( \bar{x}_0 = \mu_0 + \sigma_0 w_0,1 \) and \( s^2 = \sigma_0^2 w_0,2 \). We obtain the following

\[
\Pr \left( \hat{\theta}_n \leq \theta_0 \right) = \Pr \left( \left( \frac{\mu_0 + \sigma_0 w_0,1 - \sigma_0 w_1}{\sigma_0^2 w_0,2} \right) \leq \left( \frac{\mu_0}{\sigma_0^2} \right) \right) \leq \mathcal{U}(0, 1).
\]

Therefore, by Proposition 21, any region built from the percentiles of the posterior distribution of \( \hat{\theta}_n \) has exact frequentist coverage. This posterior distribution has a closed form.

Note that \( w_1 \sim \mathcal{N}(0, 1/n) \). Once realized that \( u_i - \frac{1}{n} \sum_{j=1}^{n} u_j \sim \mathcal{N}(0, (n-1)/n) \), it is not difficult to obtain that \( w_2 \sim \Gamma(n/2, n/2(n-1)) \), a gamma random variable (shape-rate parametrization). It is straightforward to remark that

\[
\hat{\mu} | (\sigma^2, \tilde{\pi}_n) \sim \mathcal{N} \left( \frac{\bar{x}_0}{n}, \frac{\sigma^2_0}{n} \right), \quad \sigma^2 \sim \Gamma^{-1} \left( \frac{n}{2} \right, \frac{\sigma_0^2}{2(n-1)} \right).
\]
where \( \Gamma^{-1} \) represents the inverse gamma distribution. The joint distribution is known in the Bayesian literature as the normal-inverse-gamma distribution (see [83]). We thus have the following joint distribution

\[
\hat{\theta}_n | \hat{\pi}_n \sim \mathcal{N}(\mathbf{x}_0, n, \frac{\mathbf{1}_n \mathbf{1}_n}{2(n-1)})
\]

The distribution of \( \hat{\mu} \) unconditionally on \( \hat{\sigma}^2 \) is a non-standardized \( t \)-distribution with \( n \) degrees of freedom,

\[
\hat{\mu} | \hat{\pi}_n \sim t \left( \bar{x}_0, \frac{s_0^2}{n-1}, n \right).
\]

The results on the normal distribution (Example 44) can be generalized to the linear regression.

**Example 45 (linear regression).** Consider the linear regression model \( y = X\beta + \epsilon \) where \( \epsilon \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_n) \) and \( \dim(\beta) = p \). Suppose the matrix \( X^T X \) is of full rank. A natural generating function is \( g(\beta, \sigma^2, X) = X\beta + \sigma \mathbf{u} \) where \( \mathbf{u} \sim \mathcal{N}(0, \mathbf{I}_n) \) (see Example 43 for other suggestions). Take the ordinary least squares as the auxiliary estimator so we have the following explicit form:

\[
\hat{\pi}_n = \begin{pmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{X}^T \mathbf{X}^{-1} \mathbf{X}^T y_0 \\ k y_0^T \mathbf{P} y_0 \end{pmatrix},
\]

where \( \mathbf{P} = \mathbf{I}_n - \mathbf{H} \) is the projection matrix, \( \mathbf{H} = \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \) is the hat matrix, \( y_0 \) denotes the observed responses and \( k \) is any constant. Note that \( \mathbf{P} \) and \( \mathbf{H} \) are symmetric idempotent matrices and that \( \mathbf{P} \mathbf{X} = 0 \). An explicit solution exists for \( \hat{\theta}_n = \left( \hat{\beta}^T \hat{\sigma}^2 \right)^T \). To find it, we use the indirect inference estimator, which by Theorem 5 is the equivalent to the SwiZs estimator. Using \( y \overset{d}{=} X\beta + \sigma \mathbf{u} \), we have

\[
\hat{\pi}_{\beta, n}(\theta) = \begin{pmatrix} \hat{\pi}_1(\theta) \\ \hat{\pi}_2(\theta) \end{pmatrix} = \begin{pmatrix} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{X} \beta + \sigma \mathbf{u}) \\ k \sigma^2 \mathbf{u}^T \mathbf{P} \mathbf{u} \end{pmatrix}.
\]

Since \( \hat{\pi}_2(\theta) \) depends only on \( \sigma^2 \), solving \( d(\hat{\pi}_2, \hat{\pi}_2(\theta)) = 0 \) in \( \sigma^2 \) leads to

\[
\hat{\sigma}^2 = \frac{y_0^T \mathbf{P} y_0}{\mathbf{u}^T \mathbf{P} \mathbf{u}}.
\]

On the other hand, solving \( d(\hat{\pi}_1, \hat{\pi}_1(\theta)) = 0 \) in \( \beta \) leads to

\[
\hat{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (y_0 + \hat{\sigma} \mathbf{u}).
\]

Since \( y_0 = X\beta_0 + \sigma_0 \mathbf{u}_0 \), we obtain the following:

\[
\Pr(\hat{\theta}_n \leq \theta_0) = \Pr\left( \hat{\beta} \leq \beta_0, \hat{\sigma}^2 \leq \sigma_0^2 \right)
\]

\[
= \Pr\left( \mathbf{X}^T \mathbf{X}^{-1} \mathbf{X}^T (X\beta_0 + \sigma_0 \mathbf{u}_0 + \hat{\sigma} \mathbf{u}) \leq \beta_0, (X\beta_0 + \sigma_0 \mathbf{u}_0)^T \mathbf{P} (X\beta_0 + \sigma_0 \mathbf{u}_0) \leq \sigma_0^2 \right)
\]

\[
= \Pr\left( \mathbf{X}^T \mathbf{X}^{-1} \mathbf{X}^T (\sigma_0 \mathbf{u}_0 - \hat{\sigma} \mathbf{u}) \leq 0, \sigma_0^2 \mathbf{u}_0^T \mathbf{P} \mathbf{u}_0 \leq \sigma_0^2 \right)
\]

\[
= \Pr\left( \frac{\mathbf{X}^T \mathbf{u}}{\mathbf{u}^T \mathbf{P} \mathbf{u}} \leq \frac{\mathbf{X}^T \mathbf{u}_0}{\mathbf{u}_0^T \mathbf{P} \mathbf{u}_0}, \frac{\mathbf{1}^T \mathbf{P} \mathbf{u}_0}{\mathbf{u}_0^T \mathbf{P} \mathbf{u}_0} \leq \frac{\mathbf{1}^T \mathbf{P} \mathbf{u}_0}{\mathbf{u}_0^T \mathbf{P} \mathbf{u}_0} \right) \sim \mathcal{U}(0, 1).
\]

Note that at the third equality we use the fact that \( \mathbf{u} \overset{d}{=} -\mathbf{u} \) since \( \mathbf{u} \) is symmetric around \( 0 \). The last development, together with Proposition 27 demonstrates that any region built on the percentiles of the distribution of \( \hat{\theta}_n \) leads to exact frequentist coverage probabilities. The distribution of \( \hat{\theta}_n \) can be obtained in an explicit form.

Since \( \mathbf{P} \) is symmetric and idempotent, it is well known that \( \mathbf{u}^T \mathbf{P} \mathbf{u} \sim \chi^2_{n-p} \) [see Theorem 5.1.1 [84]]. Hence we obtain that

\[
\hat{\beta} | (\hat{\sigma}^2, \hat{\pi}_n) \sim \mathcal{N}(\hat{\pi}_1, \hat{\sigma}^2 (\mathbf{X}^T \mathbf{X})^{-1}), \quad \hat{\sigma}^2 | \hat{\pi}_n \sim \Gamma^{-1}\left(\frac{n-p}{2}, \frac{y_0^T \mathbf{P} y_0}{2}\right).
\]

As shown in Example 43 it follows that the joint distribution of \( \hat{\theta}_n \) conditionally on \( \hat{\pi}_n \) is a normal-inverse-gamma distribution

\[
\hat{\theta}_n | \hat{\pi}_n \sim \mathcal{N}(\mathbf{x}_0, \frac{1}{k y_0^T \mathbf{P} y_0} \mathbf{X}^T \mathbf{X}^{-1} \mathbf{X}^T y_0, \mathbf{X}^T \mathbf{X}^{-1}, \mathbf{I}_p, \frac{n-p}{2}, \frac{y_0^T \mathbf{P} y_0}{2})
\]
and the distribution of \( \hat{\beta} \), unconditionally on \( \hat{\sigma}^2 \), is a multivariate non-standardized t distribution with \( n - p \) degrees of freedom

\[
\hat{\beta} | \hat{\pi}_n \sim t \left( (X^T X)^{-1} X^T y_0, \frac{y_0^T P y_0}{n - p} (X^T X)^{-1}, n - p \right).
\]

In this last example on the linear regression, we employed the OLS as the auxiliary estimator, which is known to be an unbiased estimator. In fact, it is not a necessity to have unbiased auxiliary estimator. The next example illustrate this point.

**Example 46** (ridge regression). Consider the same setup as in Example 45 \( y = X \beta + \epsilon, \epsilon \sim N(0, \sigma^2 I_n) \) and \( \text{rank}(X^T X) = p \). Take the ridge estimator as the auxiliary estimator, so for the regression coefficients we have

\[
\hat{\pi}_1^R (\theta) = \left( \hat{\pi}_1^R (\theta), \hat{\pi}_2^R (\theta) \right) = \left( (X^T X + \lambda I_p)^{-1} X^T (X \beta + \sigma u), \frac{y_0^T P y_0}{k} (X \beta + \sigma u)^T P \lambda P \lambda (X \beta + \sigma u) \right).
\]

Let \( \hat{\beta} \) denotes the solution of \( d(\hat{\pi}_1^R, \hat{\pi}_1^R (\theta)) = 0 \) in \( \beta \). We have the explicit solution given by

\[
\hat{\beta} = (X^T X)^{-1} X^T (y_0 - \sigma u).
\]

Using \( \hat{\beta} \) in \( \hat{\pi}_1^R (\theta) \) leads to

\[
\hat{\pi}_2^R (\theta) = k(H y_0 - \sigma Pu)^T P \lambda P \lambda (Hy_0 - \sigma Pu),
\]

where \( H = X (X^T X)^{-1} X \) and \( P = I_n - H \). We have the followings: \( HH \lambda = H \lambda, PP \lambda = P \) and \( PH = 0 \). Finding \( \hat{\sigma}^2 \) such that \( d(\hat{\pi}_2^R, \hat{\pi}_2^R (\theta)) = 0 \) gives

\[
\hat{\sigma}^2 = \frac{y_0^T P y_0}{u^T P u}.
\]

Therefore, \( \hat{\sigma}^2 \) is the same as \( \hat{\sigma}^2 \) we found in Example 45 and we directly have that \( \hat{\beta} = \hat{\beta} \). As a consequence, the distribution of \( \hat{\theta} \) is exactly the same as \( \hat{\theta}_n \) in Example 45 and the frequentist coverage probabilities are exact.

From Example 44 on the normal distribution, the derivation to closely related distribution is straightforward, as we see now with the log-normal distribution.

**Example 47** (log-normal with unknown mean and unknown variance). Let \( x_i \sim \log N(\mu, \sigma^2) \) be identically and independently distributed and consider \( g(\mu, \sigma^2, u) = e^{\mu} e^{\sigma u} \) where \( u \sim N(0, 1) \). If we take the maximum likelihood estimator as the auxiliary estimator, we have

\[
\hat{\pi}_n = \left( \hat{\pi}_1, \hat{\pi}_2 \right) = \left( \frac{1}{n} \sum_{i=1}^{n} \ln(x_i), \left( \ln(x_i) - \frac{1}{n} \sum_{j=1}^{n} \ln(x_j) \right)^2 \right)
\]

The solution is the following

\[
\hat{\theta}_n = \left( \hat{\mu}, \hat{\sigma}^2 \right) = \left( \frac{\hat{\pi}_1 - \hat{\sigma} w_1}{w_2}, \frac{\hat{\pi}_2}{w_2} \right)
\]

where \( w_1 = \frac{1}{n} \sum_{i=1}^{n} u_i \) and \( w_2 = \sum_{i=1}^{n} u_i - \frac{1}{n} \sum_{j=1}^{n} u_j \). It is the same solution as Example 44 hence the posterior distribution of \( \hat{\theta}_n \) is normal-inverse-gamma and any alpha-credible region built on this posterior have exact frequentist coverage.
We consider the following auxiliary estimators:

\[ y_t = y_0 \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right]. \]

Suppose we observe the process at \( n \) points in time: \( t_1 < t_2 < \ldots < t_n, \forall t_i \in \mathbb{R}^+ \). Define the difference in time by \( \Delta_i = t_i - t_{i-1} \), so we have \( n-1 \) time differences. Note that all the time differences are positive, \( \Delta_i > 0 \), and we allow the process to be irregularly observed, \( \Delta_i \neq \Delta_j, i \neq j \). Instead of working directly with the process \( \{y_{t_i} : i \geq 1\} \), it is more convenient to work with the following transformation of the process \( \{x_{t_i} = \ln(y_{t_i}/y_{t_{i-1}}) : i \geq 2\} \). Indeed, we have

\[ x_{t_i} = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta_i + \sigma (W_{t_i} - W_{t_{i-1}}). \]

By the properties of the Wiener process, we have \( W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, \Delta_i) \) and \( W_{t_i} - W_{t_{i-1}} \) is independent from \( W_{t_j} - W_{t_{j-1}} \) for \( i \neq j \). Hence the vector \( x = (x_{t_2} \ldots x_{t_n})^T \) is independently but non-identically distributed according to the joint normal distribution

\[ x \sim \mathcal{N} \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta, \sigma^2 \Sigma \right), \]

where \( \Delta = (\Delta_2 \ldots \Delta_n)^T \) and \( \Sigma = \text{diag}(\Delta) \). Note that \( \Delta = \Sigma 1_{n-1}, \) where \( 1_{n-1} \) is a vector of \( n-1 \) ones, and \( \Delta^T 1_{n-1} = \Delta^{T/2} \Delta^{1/2} \) since all the \( \Delta \) are positives.

We consider the following auxiliary estimators:

\[ \hat{\pi}_n = \left( \hat{\pi}_1 \hat{\pi}_2 \right) = \left( \frac{x_0^T 1_{n-1}}{x_0^T \Sigma^{-1} x_0} \right). \]

Since \( x \overset{d}{=} (\mu - \sigma^2/2) \Delta + \sigma \Sigma^{1/2} z, \) where \( z \sim \mathcal{N}(0, \Sigma_{n-1}), \) we obtain the following indirect inference estimators (or equivalently SwiZs),

\[ \hat{\pi}_1(\theta) = \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta + \sigma \Sigma^{1/2} z \right] \Sigma^{-1} 1_{n-1} = \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta^{T/2} \Delta^{1/2} + \sigma z^T \Delta^{1/2}, \]

and

\[ \hat{\pi}_2(\theta) = \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta + \sigma \Sigma^{1/2} z \right] \Sigma^{-1} \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta + \sigma \Sigma^{1/2} z \right] = \left( \mu - \frac{1}{2} \sigma^2 \right)^2 \Delta^{T/2} \Delta^{1/2} + 2 \sigma \left( \mu - \frac{1}{2} \sigma^2 \right) z^T \Delta^{1/2} + \sigma^2 z^T z. \]

Solving \( d(\hat{\pi}_1, \hat{\pi}_1(\hat{\theta})) = 0 \) in \( \hat{\mu} \) gives

\[ \hat{\mu} = \frac{1}{2} \sigma^2 - \hat{\theta} z^T \Delta^{1/2} \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1} x_0^T 1_{n-1} \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1}. \]

Now solving \( d(\hat{\pi}_2, \hat{\pi}_2(\hat{\theta})) = 0 \) in \( \hat{\sigma}^2 \) and substituting \( \hat{\mu} \) by the above expression in \( (1) \) leads to

\[ \hat{\sigma}^2 = x_0^T Q x_0 \]

where \( Q = I_{n-1} - \Delta^{1/2} \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1} \Delta^{T/2} \) is symmetric and idempotent, and \( Q = \Sigma^{-1} - 1_{n-1} \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1} 1_{n-1}^T \). By the properties of the rank of a matrix, we have rank(\( P \)) = trace(\( P \)) = \( n - 2 \).
Note that by independence \( z^T \Delta^{1/2} \overset{d}{=} z(\Delta^{T/2} \Delta^{1/2}) \), where \( z \) is a single standard normal random variable. Similarly to the example on the linear regression (Example 45), we obtain the explicit distributions

\[
\hat{\mu} | (\hat{\pi}_n, \hat{\sigma}^2) \sim N \left( \frac{1}{2} \hat{\sigma}^2 + x_0^T 1_{n-1} \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1}, \hat{\sigma}^2 \left( \Delta^{T/2} \Delta^{1/2} \right)^{-1} \right), \\
\hat{\sigma}^2 | \hat{\pi}_n \sim \Gamma^{-1} \left( \frac{n-2}{2}, \frac{x_0^T Q x_0}{2} \right).
\]

As with Example 45, this findings suggest that \( \theta_n | \pi_n \) is jointly distributed according to a normal-inverse-gamma distribution. However, \( \hat{\sigma}^2 \) appears in the mean of \( \hat{\mu}(\pi_n, \hat{\sigma}^2) \) so such conclusion is not straightforward. We leave the derivation of the joint distribution and the distribution of \( \hat{\mu} \) unconditioned on \( \hat{\sigma}^2 \) for further research.

We now demonstrate that the \( \pi_n \)-approximate posterior distribution of \( \hat{\theta}_n \) leads to exact frequentist coverage probabilities. Once realized that \( \Sigma^{-1} = \Sigma^{-1/2} \Sigma^{1/2} \Sigma^{1/2} 1_{n-1} = \Delta^{1/2}, \) and \( \Delta^{T} \Sigma^{-1} = 1_{n-1}, \) it is not difficult to show that \( \Delta^T Q \Delta = 0, \Delta^T Q \Sigma^{1/2} = 0 \) and \( \Sigma^{1/2} Q^{1/2} \Sigma^{1/2} = P. \) Since \( x_0 = (\mu_0, \sigma_0^2/2) \Delta + \sigma_0 \Sigma^{1/2} z_0, \) we obtain

\[
\hat{\sigma}^2 = \sigma_0^2 \frac{z_0^T P z_0}{z^T P z} = \sigma_0^2 \frac{w_0}{w}, \\
\hat{\mu} = \sigma_0^2 \frac{w_0}{w} - \sigma_0 \sqrt{\frac{w_0}{w}} z + \mu_0 - \frac{1}{2} \sigma_0^2 + \sigma_0 z_0.
\]

Therefore,

\[
\Pr \left( \hat{\mu} \leq \mu_0, \hat{\sigma}^2 \leq \sigma_0^2 \right) = \Pr \left( \frac{\sigma_0^2}{2} \frac{w_0}{w} - \sigma_0 \sqrt{\frac{w_0}{w}} z \leq \frac{-1}{2} \hat{\sigma}^2 + \sigma_0 z_0 \leq 0, \frac{w_0}{w} \leq 1 \right) = \Pr \left( \frac{k_0}{w} - \frac{z}{\sqrt{w}} \leq \frac{k_0}{w} - \frac{z_0}{\sqrt{w}} \leq \frac{w_0}{w} \right) \sim U(0, 1),
\]

where \( k_0 = \sigma_0 \sqrt{w_0}/2. \) Thus, any region on the joint distribution of \( \hat{\theta}_n \) leads to exact frequentist coverage by Proposition 27.

### 7 Simulation study

The main goal of this section is threefold. First, we illustrate the results of the Section 4 on the frequentist properties in finite sample of the SwiZs in the general case where no solutions are known in explicit forms, as opposed to the Section 6 and thus requiring numerical solutions. In order to achieve this point, we measure at different levels the empirical coverage probabilities of the intervals built from the percentiles of the \( \pi_n \)-approximate posterior obtained by the SwiZs. Note that for \( \dim(\theta) > 1, \) we only considered marginal intervals to avoid a supplementary layer of numerical nuisance, the coverage probabilities are not concerned by this choice, only the length of the intervals. Second, we elaborate on the verification of the conditions of Theorem 28 with the examples at hand. As already motivated, the emphasis is on the estimation function. It seems easier to verify Assumption 26 than Assumption 27 since only one of them is necessary to satisfy Theorem 28, we concentrate our efforts on the former. We also brighten the study up to situations where Assumption 26 does not entirely hold or cannot be verified to measure its consequences empirically. Third, we give the general idea on how to implement the SwiZs. Indeed, anyone familiar with the numerical problem of solving a point estimator such as the maximum likelihood estimator has a very good idea on how to obtain the auxiliary estimator \( \pi_n. \) Solving the estimating function for the parameters of interest is very similar, it requires the exact same tools but has the inconvenient of needing further analytical derivations and implementations details. As already remarked, the parametric bootstrap does not possess such inconvenient. The counterpart is that the SwiZs may lead to exact coverage probabilities. The motto “no pain, no gain” is particularly relevant here. For this purpose, the parametric bootstrap is proposed as the point of comparison for all the examples of this section. We measure the computational time as experienced by the user in order to appreciate the numerical burden. In case both the SwiZs and the parametric bootstrap have very similar coverage probabilities, we also quantify the length of the intervals as a mean of comparison.

As a subsidiary goal of this section, we study the point estimates of the SwiZs. Indeed, the indirect inference is also a method for reducing the small sample bias of an initial (auxiliary) estimator, even in situations where it may be “unnatural” to call such method, as for example, when a maximum likelihood estimator may be easily obtained (see [14]). Since the SwiZs is a special case of indirect inference, it would be interesting to gauge the ability of the SwiZs to correct the bias. We explore the properties of the mean and the median of the SwiZs. This choice is arbitrary but largely admitted.
The determinant here is

Assumption 26 (i) is satisfied. As a consequence, given that

We select five different scenarios. First, we start with a toy example by considering a standard Student’s $t$-distribution with unknown degrees of freedom (Example 49). Although the Student distribution is ubiquitous in statistics since at least Gosset’s Biometrika paper (85), there are no simple tractable way to construct an interval of uncertainty around the degrees of freedom. In addition, the degrees of freedom is a parameter that gauges the tail of the distribution and is not particularly easy to handle. The existence of the moments of this distribution depends upon the values that this parameter takes. We take a particular interest in small values of this parameter for which, for example the variance or the kurtosis are infinite.

**Example 49** (standard $t$-distribution with unknown degrees of freedom). Let $x_i \sim t(\theta), i = 1, \ldots, n$, be identically and independently distributed with density

$$f(x_i, \theta) = \frac{1 + x_i^2/\pi}{\sqrt{\theta B(1/2, 1/2)}};$$  \hspace{1cm} (4)

where $\theta$ represents the degrees of freedom and $B$ is the beta function. We consider the likelihood score function as the estimating function and we take the MLE as the auxiliary estimator. In this situation, $\Theta$ and $\Pi$ are equivalent, and thus, there are no reasons to disqualify the parametric bootstrap. Substituting $\theta$ by $\pi$ in the Equation 2 taking then the derivative with respect to $\pi$ of the log-density leads to the following

$$\Phi_n(\theta, u, \pi) = \psi\left(\frac{\pi + 1}{2}\right) - \psi\left(\frac{\pi}{2}\right) - \frac{1}{n} \sum_{i=1}^{n} \ln \left(\frac{g(\theta, u_i)^2 + 1}{\pi}\right) + \frac{1}{n} \sum_{i=1}^{n} \frac{g(\theta, u_i)^2 - 1}{g(\theta, u_i)^2 + \pi},$$

where $\psi$ is the digamma function. We now verify Assumption 26 so Theorem 28 can be invoked. Suppose Assumption 24 holds so we can write the following scalar-valued function

$$\varphi_{\hat{\pi}_n}(\theta, w) = \frac{1}{2} \psi\left(\hat{\pi}_n + 1\right) - \frac{1}{2} \psi\left(\hat{\pi}_n\right) - \frac{1}{2} \ln \left(\frac{g(\theta, w)^2 + 1}{\hat{\pi}_n}\right) + \frac{1}{2} \frac{g(\theta, w)^2 - 1}{g(\theta, w)^2 + \hat{\pi}_n},$$

where $\hat{\pi}_n$ is fixed. The first derivative with respect to $\theta$ is given by

$$\frac{\partial}{\partial \theta} \varphi_{\hat{\pi}_n}(\theta, w) = g(\theta, w) \frac{\partial}{\partial \theta} g(\theta, w) \left[\frac{\hat{\pi}_n - 1}{\left(g(\theta, w)^2 + \hat{\pi}_n\right)^2} - \frac{1}{g(\theta, w)^2 + 1}\right].$$  \hspace{1cm} (5)

Substituting $(\partial/\partial \theta) g$ by $(\partial/\partial u) g$ gives the first derivative with respect to $w$. The derivative exists everywhere so $K_n = \emptyset$. Therefore, if the generating function $g(\theta, w)$ is once continuously differentiable in both its arguments then Assumption 26 (i) is satisfied.

The determinant here is $|\frac{\partial}{\partial \theta} \varphi_{\hat{\pi}_n}(\theta, w)|$. It will be zero on a countable set of points: if $g(\theta, w) = 0$, if $(\partial/\partial \theta) g(\theta, w) = 0$ or if the rightest term of the Equation 5 is 0. Substituting $(\partial/\partial \theta) g$ by $(\partial/\partial u) g$ gives the same analysis. Hence, the determinant of the derivatives of the estimating function is almost everywhere non-null and Assumption 26 (ii) is satisfied.

Eventually, we clearly have that

$$\lim_{|g| \to \infty} |\varphi_{\hat{\pi}_n}(\theta, w)| = +\infty.$$

As a consequence, given that $\lim_{||(\theta, w)|| \to \infty} |g(\theta, w)| = \infty$, Assumption 26 (iii) is satisfied.

In the light of these findings, the choice of generating function is crucial and there are many candidates [see e.g. 89]. The inverse cumulative distribution function is a natural choice, but a numerically complicated one in this case. Indeed, it can be obtained by

$$g_1(\theta, u_1) = \text{sign} \left(u_1 - \frac{1}{2}\right) \left(\frac{\theta (1 - z)}{z}\right)^{1/2},$$

where $z = \sqrt{\theta B(1/2, 1/2)}$. The inverse cumulative distribution function is a natural choice, but a numerically complicated one in this case.
where \( u_1 \sim U(0, 1) \) and \( z \) is equal to the incomplete beta function inverse parametrized by \( \theta \) and depending on \( u_1 \). An alternative choice, numerically and analytically simpler, is to consider Bailey’s polar algorithm \( [87] \), which is given by

\[
g_2(\theta, u_2) = u_{2,1} \sqrt{\frac{\theta}{u_{2,2}} \left( \frac{u_{2,2} - 1}{u_{2,2}} \right)},
\]

where \( u_{2,2} = u_{2,1}^2 + u_{2,3}^2 \) if \( u_{2,2} \leq 1 \) and \( u_{2,1}, u_{2,3} \sim U(-1, 1) \). Clearly \( g_2(\theta, u_2) \) is once continuously differentiable in each of its arguments and the limit is \( \lim_{(\theta, u_{2,1}, u_{2,2}) \to (\infty, 1, 1)} |g_2(\theta, u_{2,1}, u_{2,2})| = \infty \). Hence, even if \( \omega \) is unknown, these results strongly suggest that the conditions of Theorem 28 hold, and as a conclusion, any intervals built on the percentiles of the distribution of \( \theta \), given \( \bar{s}_n \), have exact frequentist coverage.

The coverage probabilities in the Table 1 below are computed for three different values of \( \theta_0 = \{1.5, 3.5, 6\} \) and a sample size of \( n = 50 \). When \( \theta_0 = 1.5 \), the variance of a Student’s random variable is infinite and the skewness and kurtosis of the distribution are undefined. When \( \theta_0 = 3.5 \), the variance is finite and the kurtosis is infinite. When \( \theta_0 = 6 \), the first five moments exist.

| \( \theta_0 \) | \( \alpha \) | \( \bar{c} \) | \( \bar{s} \) | \( \bar{c} \) | \( \bar{s} \) |
|---|---|---|---|---|---|
| 1.5 | 50% | 50.66% | 0.5129 | 0.1622 | 49.13% | 0.0358 | 47.69% | 0.0333 |
| | 75% | 75.39% | 0.8839 | 1.0504 | 71.64% | 0.8607 |
| | 90% | 90.15% | 1.2861 | 1.6734 | 86.64% | 1.2815 |
| | 95% | 94.68% | 1.5540 | 2.1935 | 91.82% | 1.5800 |
| | 99% | 98.84% | 2.1052 | 3.8820 | 97.13% | 2.2714 |
| 3.5 | 50% | 50.08% | 1.7594 | 0.2010 | 47.65% | 0.0349 | 44.94% | 0.0322 |
| | 75% | 74.62% | 3.2780 | 2.8832 | 89.63% | 2.1935 | 96.83% | 2.1935 |
| | 90% | 90.39% | 5.2129 | 5.2066 | 84.36% | 5.2066 |
| | 95% | 94.85% | 8.4161 | 9.4161 | 80.83% | 6.9584 |
| | 99% | 98.73% | 10.788 | 10.788 | 96.05% | 29.011 |
| 6 | 50% | 48.61% | 4.2027 | 0.2093 | 46.54% | 0.0342 | 44.29% | 0.0305 |
| | 75% | 74.39% | 8.3688 | 4.2027 | 87.45% | 4.2027 | 69.99% | 12.45 |
| | 90% | 89.56% | 16.087 | 8.3688 | 11.46% | 12.45 |
| | 95% | 94.61% | 26.250 | 16.087 | 87.45% | 41.335 |
| | 99% | 98.90% | 361.28 | 26.250 | 90.39% | 29.011 |

Table 1: \( \bar{c} \): estimated coverage probabilities, \( \bar{I} \): median interval length, \( \bar{s} \): average time in seconds to compute the intervals for one trial.

The SwiZs is accurate at all the confidence levels with a maximum discrepancy of 1.39% in absolute value. This is very reasonable considering the numerical task we perform. In comparison, the parametric bootstrap has a minimum discrepancy of 0.87% for an average of 4.44%. The SwiZs is also more efficient, it dominates the parametric bootstrap with a median interval length systematically smaller. The parametric bootstrap is however about six times faster than the SwiZs to compute the intervals. The comparison is not totally fair in disfavor of the SwiZs as we were able here to use directly the log-likelihood for the parametric bootstrap, which is numerically simpler to evaluate than the estimating functions. We also bear the comparison with the bias-corrected and accelerated (BCa) resampling bootstrap of \( [88] \). Performances of this bootstrap scheme are comparable to the parametric bootstrap. Finally, when considered in absolute value, 0.2 second do not seem to be a hard effort for obtaining interval which is nearly exact and shorter.

Second, we consider a more practical case with the two-parameters Lomax distribution \( [89] \) (Example 50), also known as the Pareto II distribution. This distribution has been used to characterise wealth and income distributions as well as business and actuarial losses (see \( [90] \) and the references therein). Because of this close relationship to the application, we also measure the coverage probabilities of the Gini index, the value-at-risk and the expected shortfall, quantities that may be of interest for the practitioner. The maximum likelihood estimator has been shown in \( [91] \) to suffer from small sample bias when \( n \) is relatively small and the parameters are close to the boundary of the parameter space. We add their proposal for bias adjustment to the basket of comparative methods. To keep the comparison fair, we use a similar simulation scenario to the ones they proposed, which were also motivated by their closeness to situations encountered in practice. Situations where the Lomax distribution is employed has been shown to suffer from
influential outliers ever since at least [92], we therefore consider, in a second time, the weighted maximum likelihood ([93]) as the auxiliary estimator to gain robustness. Interestingly, the weighted maximum likelihood estimator is generally not a consistent estimator (see [94, 13]) so the parametric bootstrap cannot be invoked directly, whereas, on the contrary, the SwiZs may be employed without any particular care.

**Example 50** (two-parameters Lomax distribution). Let \( x_i \sim \text{Lomax}(\theta) \), \( i = 1, \ldots, n \), \( \theta = (b, q) \), be identically and independently distributed with density

\[
f(x_i, \theta) = \frac{q}{b} \left(1 + \frac{x_i}{b}\right)^{-q-1}, \quad x_i > 0,
\]

where \( b, q > 0 \) are shape parameters. We consider the likelihood score function as the estimating function and we take the MLE as the auxiliary estimator. The parameter sets \( \Theta \) and \( \Pi \) are equivalent with this setup, and thus, the parametric bootstrap may be employed. Substituting \( \theta \) by \( \pi \) in the Equation 6 taking then the derivative with respect to \( \pi \) of the log-density leads to the following

\[
\psi_n(\theta, u, \pi) = \left( \frac{1}{\pi^2} - \sum_{i=1}^n \log \left(1 + \frac{g(\theta, u_i)}{\pi}\right) \right).
\]

We now verify Assumption 24 so Theorem 28 can be invoked. Suppose Assumption 24 on the existence of a random variable with the same dimensions as \( \theta \) holds, and let denote it by \( w = (w_1, w_2)^T \). Now assume that we can re-express the estimating function as follows

\[
\varphi_n(\theta, w) = \left( \frac{1}{\pi^2} - \log \left(1 + \frac{g(\theta, w_1)}{\pi}\right) \right).
\]

where \( \pi \) is fixed. The Jacobian matrix with respect to \( \theta \) is given by

\[
D_\theta \varphi_n(\theta, w) = \begin{pmatrix}
\kappa_1(\theta) D_\theta g(\theta, w_1) \\
\kappa_2(\theta) D_\theta g(\theta, w_2)
\end{pmatrix},
\]

where

\[
\kappa_1(\theta) = \frac{-1}{\pi + g(\theta, w_1)}
\]

\[
\kappa_2(\theta) = \frac{\pi^2}{\pi + g(\theta, w_2)}.
\]

Note that \( \pi \) and \( g(\theta, w) \) are strictly positive, so \( \kappa_1(\theta) < 0 \) and \( \kappa_2(\theta) > 0 \). Substituting \( D_\theta g \) by \( D_\theta g \) leads to the Jacobian matrix with respect to \( w \), given by

\[
D_w \varphi_n(\theta, w) = \begin{pmatrix}
\kappa_1(\theta) \frac{\partial}{\partial w_1} g(\theta, w_1) & 0 \\
0 & \kappa_2(\theta) \frac{\partial}{\partial w_2} g(\theta, w_2)
\end{pmatrix}.
\]

We see by inspection that the derivatives are defined everywhere and \( K_n = \{\emptyset\} \). If \( D_\theta g \) and \( D_w g \) exist and are continuous, then Assumption 26(i) is satisfied.

The determinants are given by

\[
det(D_\theta \varphi_n(\theta, w)) = \kappa(\theta, w) \left[ \frac{\partial}{\partial \theta} g(\theta, w_1) \frac{\partial}{\partial \theta} g(\theta, w_2) - \frac{\partial}{\partial \alpha} g(\theta, w_2) \frac{\partial}{\partial \alpha} g(\theta, w_1) \right],
\]

\[
det(D_w \varphi_n(\theta, w)) = \kappa(\theta, w) \left[ \frac{\partial}{\partial w_1} g(\theta, w_1) \frac{\partial}{\partial w_2} g(\theta, w_2) \right],
\]

where \( \kappa(\theta, w) = \kappa_1(\theta) \kappa_2(\theta) \) and \( \kappa(\theta, w) < 0 \). The only scenario where these determinants are zero are whether all the partial derivatives are zero, or if \( (\partial/\partial \alpha) g(\theta, w_1) (\partial/\partial \theta) g(\theta, w_2) = (\partial/\partial \alpha) g(\theta, w_2) (\partial/\partial \theta) g(\theta, w_1) \). Since the Lomax random variables are absolutely continuous, it is impossible for the generating function to be flat on \( \theta \) and on \( w \), except maybe in extreme cases. Therefore, situations where the determinants are zero are countable, and Assumption 26(ii) is satisfied.
Suppose the generating function satisfies the following property:

$$\lim_{||\theta, w|| \to \infty} g(\theta, w_1) = \infty.$$  

Since the limit of the natural logarithm tends to infinity when its argument diverges, we clearly have that

$$\lim_{||\theta, w|| \to \infty} ||\varphi_{\pi_n}(\theta, w)|| = +\infty,$$

and as a consequence, Assumption 26(iii) is satisfied.

It remains to demonstrate that a generating function satisfies the above properties. A natural and computationally easy choice for the generating function is the inverse cdf, it is given by

$$g(\theta, u) = b + bu^{-1/q}, \quad u \sim U(0, 1).$$

Clearly the generating function is once continuously differentiable in each $(b, q, u)$. The only possibilities for the partial derivatives of $g$ to be zero are whether $q = \{+\infty\}$ or $u = \{0\}$. The generating function tends to infinity when $b$ diverges whereas it remains constant when $q$ or $u$ diverges. All these findings strongly suggest that Theorem 28 is applicable here, and as a conclusion that any intervals built on the percentiles of the SwiZs distribution lead to exact frequentist coverage probabilities.

However, the situation is less optimistic with the weighted maximum likelihood. Indeed, the estimating function is typically modified as follows:

$$\Psi_n(\theta, u, \pi) = w(\theta, u, \pi, k)\Psi_n(\theta, u, \pi),$$

where $w(\theta, u, \pi, k)$ is some weight function typically taking values in $[0, 1]$ that depends upon a tuning constant $k$. Usual weight functions are Huber’s type ($\frac{25}{5}$) and Tukey’s biweight function ($\frac{26}{3}$); see [27] for a textbook on robust statistics. For an estimating function to be robust, the weight function either decreases to 0 or remains constant for large values of $x$. As a consequence, at least two out of the three hypothesis of Assumption 26 do not hold. Indeed, the determinants will be zero on an uncountable set and $\lim_{||\theta, w|| \to \infty} \Psi_n < \infty$.

For the simulations, we set $\theta_0 = (2, 2, 3)^T$ and use $n = \{35, 50, 100, 150, 250, 500\}$ as sample sizes. As already mentioned, this setup is close to the ones proposed in [27], and we thus add their proposal for correcting the bias of the maximum likelihood estimator to the basket of the compared methods. The bias-adjustment estimator is given by

$$\theta_{BA,n} = \hat{\pi}_n - B(\hat{\pi}_n) A(\hat{\pi}_n) \operatorname{vec}(B(\hat{\pi}_n)),$$

where

$$A(\pi) = n \begin{pmatrix}
\frac{2\pi_2}{\pi_1(\pi_2+2)} & \frac{\pi_3}{\pi_1(\pi_2+2)} & -1
\frac{\pi_3(\pi_2+2)}{\pi_1(\pi_2+2)^2} & \frac{\pi_2}{\pi_1^2(\pi_2+1)^2} & \frac{-1}{\pi_2^2}
-1 & \frac{-1}{\pi_2} & \frac{1}{\pi_2}
\end{pmatrix},$$

and

$$B^{-1}(\pi) = n \begin{pmatrix}
\frac{\pi_2}{\pi_1(\pi_2+2)} & \frac{\pi_3}{\pi_1(\pi_2+2)} & \frac{-1}{\pi_2^2}
\frac{-1}{\pi_1^2(\pi_2+1)^2} & \frac{-1}{\pi_2} & \frac{1}{\pi_2}
\end{pmatrix}.$$
Figure 1: Coverage probabilities of the SwiZs, the parametric bootstrap (Boot) and the bias-adjustment (BA) proposal of [91] for different sample sizes. On the left panel is the coverage for the first estimator, and the second is on the right. The gray horizontal dotted-lines indicate the perfect coverage probabilities. The closer to these lines is the better.

Figure 2: On the left panel: representation of the median interval lengths for a confidence level of 95% for the SwiZs, the parametric bootstrap (Boot) and the bias-adjustment (BA) proposal of [91] for three different sample sizes. The ellipses are just a representation and do not reflect the real shapes of the confidence regions. All the ellipses are on the same scale. The centre of the ellipses is chosen for aesthetical reason and have no special meaning. The $y$-axis corresponds to the median interval length of the first parameter, the $x$-axis the one of the second parameter. The smaller the ellipse is, the better it is. On the right panel: the average computational time in seconds of the SwiZs and the Boot for the different sample sizes. Note that the computational time of the the BA (not on the figure) is quasi-identical to the Boot. The lower is the better.
Figure 3: *On the left panel:* the sum of absolute value of the median bias for the two estimators divided by their respective true values for the mean of SwiZs distribution, the median of the SwiZs distribution and the bias-adjustment (BA) proposal of [91] evaluated on the different sample sizes. *On the right panel:* likewise the left panel, but for a different measure: the average of the median absolute deviation for the two estimators divided by their respective true values. The lower is the better.

Figure 4: Coverage probabilities for different sample sizes of the SwiZs (RSwiZs) and the parametric bootstrap (RBoot) when taking the weighted maximum likelihood as auxiliary estimator. *On the left panel* is the coverage for the first estimator, and the second is on the right. The gray horizontal dotted-lines indicate the perfect coverage probabilities. The closer to these lines is the better.
Figure 5: On the left panel: the sum of absolute value of the median bias for the two estimators divided by their respective true values for different sample sizes for the mean of SwiZs distribution (RSwiZs: mean), the median of the SwiZs distribution (RSwiZs: median) when considering the weighted maximum likelihood (WMLE) as the auxiliary estimator. On the right panel: likewise the left panel, but for a different measure: the average of the median absolute deviation for the two estimators divided by their respective true values. The lower is the better.

| Sample Size (n) | Cumulative Relative Median Bias | Mean Relative Median Absolute Deviation |
|-----------------|-------------------------------|----------------------------------------|
| 35              | 35.78%                        | 30.78%                                 |
| 50              | 21.94%                        | 20.00%                                 |
| 100             | 3.02%                         | 3.02%                                  |
| 150             | 0.40%                         | 0.40%                                  |

Table 2: Empirical proportion of times the bias-adjusted maximum likelihood estimator is jointly out of the parameter space \( \Theta \).

Third, we investigate a linear mixed-model. These models are very common in statistics as they incorporate both parameters associated with an entire population and parameters associated with individual experimental units facilitating thereby the study of, for example, longitudinal data, multilevel data and repeated measure data. Although being widespread, the inference on the parameters remain a formidable task. We study a rather simple model, namely the random intercept and random slope model when data is balanced.

**Example 51** (random intercept and random slope linear mixed model). Consider the following balanced Gaussian mixed linear model expressed for the \( i \)th individual as

\[
y_i = (\beta_0 + \alpha_i)1_m + (\beta_1 + \gamma_i)x_i + \epsilon_i, \quad i = 1, \ldots, n,
\]

where \( \epsilon_i, \alpha_i \) and \( \gamma_i \) are identically and independently distributed according to centered Gaussian distributions with respective variances \( \sigma^2_\epsilon1_m, \sigma^2_\alpha \) and \( \sigma^2_\gamma \). \( m \) being the number of replicates, the same for each individual, and \( 1_m \) is a vector of \( m \) ones. The vector of parameters of interest is \( \theta = (\beta_0, \beta_1, \sigma^2_\epsilon, \sigma^2_\alpha, \sigma^2_\gamma)^T \). Let \( \pi = (\pi_0, \ldots, \pi_4)^T \) be the corresponding vector of auxiliary parameters. We take the MLE as the auxiliary estimator and thus consider the likelihood score function as the estimating function. With this setup, the parameter spaces \( \Theta \) and \( \Pi \) are equivalent, and the parametric bootstrap may be employed. Denote by \( N = nm \) the total sample size. The negative log-likelihood may be expressed as

\[
\ell(y, \theta) = k + \frac{1}{2N} \sum_{i=1}^{n} \log \left( \det \left( \Omega_i(\theta) \right) \right) + (y_i - \beta_01_m - \beta_1x_i)^T \Omega_i^{-1}(\theta)(y_i - \beta_01_m - \beta_1x_i),
\]
We now motivate the possibility to employ Theorem 28 by verifying Assumption 26. First, we suppose that a random variable \( w \) do not depend on parameters, let denotes \((0 0 0 \cdots 1)^\text{T} \), then substituing \( \pi \) of the same dimension as \( \theta \), then substituing \( \pi \), then substituing \( \pi \), then substituing \( \pi \), then substituing \( \pi \). Then, we assume that the estimating function may be re-expressed as

\[
\Psi_N(\theta, u, \pi) = \begin{pmatrix}
\frac{-1}{N} \sum_{i=1}^{n} z^T(\theta, u_i, \pi) \Omega_i^{-1}(\pi) 1_m \\
\frac{-1}{N} \sum_{i=1}^{n} z^T(\theta, u_i, \pi) \Omega_i^{-1}(\pi)x_i \\
\frac{1}{N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi) \frac{\partial}{\partial \pi_j} \Omega_i(\pi) \right) \\
\frac{-1}{N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi) \frac{\partial}{\partial \pi_j} \Omega_i(\pi) \right) z_i(\theta, u_i, \pi), j = 2, 3, 4
\end{pmatrix},
\]

where \( z(\theta, u_i, \pi) = g(\theta, u_i) - \pi_0 1_m - \pi_1 x_i \) (see also [98] for more details on these derivations). The derivatives of \( \Omega_i(\pi) \) are easily obtained: \((\partial/\partial \pi_2) \Omega_i(\pi) = 1_m\), \((\partial/\partial \pi_3) \Omega_i(\pi) = 1_m 1_m^T\) and \((\partial/\partial \pi_4) \Omega_i(\pi) = x_i x_i^T\). Since they do not depend on parameters, let denotes \((\partial/\partial \pi_j) \Omega_i(\pi) \equiv D_{ij}^\text{N}\).

We now motivate the possibility to employ Theorem 28 by verifying Assumption 26. First, we suppose that a random variable \( w \) of the same dimension as \( \theta \) exists. Then, we assume that the estimating function may be re-expressed as follows:

\[
\varphi_{N}(\theta, w) = \begin{pmatrix}
\frac{-1}{N} \sum_{i=1}^{n} z_i^T(\theta, w_0, \pi_N) \Omega_i^{-1}(\pi_N) 1_m \\
\frac{-1}{N} \sum_{i=1}^{n} z_i^T(\theta, w_1, \pi_N) \Omega_i^{-1}(\pi_N)x_i \\
\frac{1}{2N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi_N) D_{ij} \right) \\
\frac{-1}{2N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi_N) D_{ij} \right) z_i(\theta, w_j, \pi_N), j = 2, 3, 4
\end{pmatrix},
\]

for some constant \( k \) and where \( \Omega_i(\theta) = \sigma^2 1_m + \sigma^2 1_m 1_m^T + \sigma^2 x_i x_i^T \) is clearly a symmetric positive definite matrix.

Taking the derivatives with respect to \( \theta \), then substituting \( \theta \) by \( \pi \) and \( y_i \) by \( g(\theta, u_i) \) leads to

\[
\Psi_N(\theta, u, \pi) = \begin{pmatrix}
\frac{-1}{N} \sum_{i=1}^{n} z^T(\theta, u_i, \pi) \Omega_i^{-1}(\pi) 1_m \\
\frac{-1}{N} \sum_{i=1}^{n} z^T(\theta, u_i, \pi) \Omega_i^{-1}(\pi)x_i \\
\frac{1}{N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi) \frac{\partial}{\partial \pi_j} \Omega_i(\pi) \right) \\
\frac{-1}{N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi) \frac{\partial}{\partial \pi_j} \Omega_i(\pi) \right) z_i(\theta, u_i, \pi), j = 2, 3, 4
\end{pmatrix},
\]

where \( z(\theta, u_i, \pi) = g(\theta, u_i) - \pi_0 1_m - \pi_1 x_i \) (see also [98] for more details on these derivations). The derivatives of \( \Omega_i(\pi) \) are easily obtained: \((\partial/\partial \pi_2) \Omega_i(\pi) = 1_m\), \((\partial/\partial \pi_3) \Omega_i(\pi) = 1_m 1_m^T\) and \((\partial/\partial \pi_4) \Omega_i(\pi) = x_i x_i^T\). Since they do not depend on parameters, let denotes \((\partial/\partial \pi_j) \Omega_i(\pi) \equiv D_{ij}^\text{N}\).

We now motivate the possibility to employ Theorem 28 by verifying Assumption 26. First, we suppose that a random variable \( w \) of the same dimension as \( \theta \) exists. Then, we assume that the estimating function may be re-expressed as follows:

\[
\varphi_{N}(\theta, w) = \begin{pmatrix}
\frac{-1}{N} \sum_{i=1}^{n} z_i^T(\theta, w_0, \pi_N) \Omega_i^{-1}(\pi_N) 1_m \\
\frac{-1}{N} \sum_{i=1}^{n} z_i^T(\theta, w_1, \pi_N) \Omega_i^{-1}(\pi_N)x_i \\
\frac{1}{2N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi_N) D_{ij} \right) \\
\frac{-1}{2N} \sum_{i=1}^{n} \text{trace} \left( \Omega_i^{-1}(\pi_N) D_{ij} \right) z_i(\theta, w_j, \pi_N), j = 2, 3, 4
\end{pmatrix},
\]
where \( z_i(\theta, w_j, \pi_N) = g(\theta, w_j) - \pi_0 1_m - \pi_1 x_i, \) \( j = 0, 1, 2, 3, 4 \), and \( \pi_N \) is fixed. The Jacobian matrix with respect to \( \theta \) is given by
\[
D_\theta \varphi_{\pi_N}(\theta, w) = \begin{pmatrix}
\sum_{i=1}^n \frac{D_\theta g^T(\theta, w_i) \Omega_i^{-1}(\pi_N) 1_m}{N} \\
\sum_{i=1}^n \frac{D_\theta g^T(\theta, w_i) \Omega_i^{-1}(\pi_N) x_i}{N} \\
\sum_{i=1}^n \frac{D_\theta g^T(\theta, w_i) \Omega_i^{-1}(\pi_N) D_{ij}}{N} \times \Omega_i^{-1}(\pi_N) g(\theta, w_j), \quad j = 2, 3, 4
\end{pmatrix}.
\]

Substituting \( D_\theta g^T \) by \( D_w g^T \) in the above delivers immediately the Jacobian matrix with respect to \( w \). Note that this second Jacobian is a diagonal matrix. Clearly, the differentiability and continuity of \( \varphi_{\pi_N} \) depends exclusively upon the differentiability and continuity of \( g \). Ergo, if \( D_\theta g \) and \( D_w g \) exist and are continuous, then Assumption 26(i) holds.

These Jacobian matrices may have a null determinant under two circumstances: whether the generating function \( g \) is flat on \( \theta \) and/or \( w \), and/or whether they are linearly dependent. Since the Normal distribution is absolutely continuous, \( g \) may be flat only on extreme cases. The Jacobian of \( D_w \varphi_{\pi_N} \) is a diagonal matrix, so its determinant is null if and only if one of its diagonal element is null. Since both the design and \( \pi_N \) are fixed, situations where \( D_\theta \varphi_{\pi_N} \) is linearly dependent may occur if the vectors \((\partial/\partial \theta_j) g(\theta, w) = k(\partial/\partial \theta_{j'}) g(\theta, w), j \neq j', \) for some constant \( k \in \mathbb{R} \). But because \( w \) is random, this situation is unlikely to occur, and, depending on \( g \), Assumption 26(ii) is plausible.

Eventually, it clearly holds that
\[
\lim_{\|\theta, w\| \to \infty} \| \varphi_{\pi_N}(\theta, w) \| = \infty
\]
if \( \| g(\theta, w) \| \to \infty \) as \( \| (\theta, w) \| \to \infty \), so Assumption 26(iii) is satisfied given that \( g \) fulfills the requirement.

Once again, the plausibility of Assumption 26 is up to the choice of the generating function. A popular choice is the following:
\[
g(\theta, u_i) = \beta_0 1_m + \beta_1 x_i + C_i(\theta) u_i, \quad u_i \sim N(0, I_m),
\]
where \( C_i(\theta) \) is the lower triangular Cholesky factor such that \( C_i(\theta) C_i^T(\theta) = \Omega_i(\theta) \). It is straightforward to remark that \( g \) is once continuously differentiable in \( \beta_0, \beta_1 \) and \( u_i \). For the variances components, the partial derivatives of the Cholesky factor is given by Theorem A.1 in [99]:
\[
\frac{\partial}{\partial \theta_j} C_i(\theta) = C_i(\theta) L \left( C_i^{-1}(\theta) \frac{\partial}{\partial \theta_j} \Omega_i(\theta) C_i^{-T}(\theta) \right), \quad j = 2, 3, 4,
\]
where the function \( L \) returns the lower triangular and half of the diagonal elements of the inputed matrix, that is:
\[
L_{ij}(A) = \begin{cases}
A_{ij}, & i > j, \\
\frac{1}{2} A_{ij}, & i = j, \\
0, & i < j.
\end{cases}
\]

The partial derivatives of the covariance matrix are given by: \((\partial/\partial \sigma^2) \Omega_i(\theta) = I_m \), \((\partial/\partial \sigma^2) \Omega_i(\theta) = I_m \) and \((\partial/\partial \sigma^2) \Omega_i(\theta) = x_i x_i^T \). Hence, \( C_i(\theta) \) is once differentiable. For the continuity of the partial derivative of \( C_i(\theta) \), note that \( C_i(\theta) \) and \( C_i^{-1}(\theta) \) are once differentiable and thus continuous. Indeed, \((\partial/\partial \theta_j) C_i^{-1}(\theta) = -C_i^{-1}(\theta) ([\partial/\partial \theta_j] C_i(\theta)) C_i^{-1}(\theta) \). Eventually, \((\partial/\partial \theta_j) \Omega_i(\theta) \) is constant in \( \theta \), and therefore continuous. Since matrix product preserves the continuity, the Cholesky factor is once continuously differentiable. The partial derivatives of \( g \) may be zero if the design is null or if the pivotal quantity is zero, two extreme situations unlikely encountered. It is straightforward to remark that the estimating function diverges as \( \theta \) and \( u_i \) tends to infinity. All these findings make usage of Theorem 23 highly plausible.

Let us turn our attention to simulations. We set \( \theta_0 = (1, 0.5, 0.5^2, 0.5^2, 0.2^2)^T \) and considered \( n = m = \{5, 10, 20, 40\} \) such that \( N = nm = \{25, 100, 400, 1,600\} \). The detailed results of simulations may be found in the tables of Appendix 2. In Figure 7 we can observe the outstanding performances of the SwiZs in terms of coverage probabilities, which supports our analysis and the possibility of using Theorem 23. The parametric bootstrap meets the performance of the SwiZs as the sample size increases, however, when the sample size is small, it is off the ideal level for the variance components. The length of the marginal intervals of uncertainty are comparable between the two methods, except for the smallest sample size considered where it is anyway harder to interpret the size of the interval of the parametric bootstrap since it is off the confidence level. We also bear the comparison with profile likelihood confidence intervals which are based on likelihood ratio test. The coverage probabilities are almost undistinguishable from the SwiZs whereas interval lengths for variance components are the shortest. We interpret such good performances as follows: first, as shown in Example [25] on linear regression, asymptotic and finite sample distributions coincides in theory, coincidance that may be still hold in the present case with balanced linear mixed
Figure 7: On the left panel: Representation of the coverage probabilities for different sample sizes of the SwiZs, the parametric bootstrap (Boot) and the confidence intervals based on the likelihood ratio test (Asymptotic) for the five estimators. The gray line represents the ideal level of 95% coverage probability. On the right panel: median length of the marginal intervals of uncertainty at a level of 95%. For graphical reason, the lengths corresponding to $\hat{\sigma}_0^2$ and $\hat{\sigma}_2^2$ on the right is downsized by a factor of 5 compared to the lengths corresponding to the other estimators.

Figure 8: On the left panel: the sum of absolute value of the median bias for the five estimators divided by their respective true values for different sample sizes for the mean of SwiZs distribution (SwiZs: mean), the median of the SwiZs distribution (SwiZs: median) and the maximum likelihood estimator (MLE). On the right panel: likewise the left panel, but for a different measure: the average of root mean square error for the five estimators. For both panels, the lower is the better.
model; second, larger intervals accounts for the fact that no simulations are needed. A good surprise appears in Figure 8 where the median of the SwiZs shows good performances in terms of relative median bias.

Fourth, we study inference in queuing theory models (see [100] for a monograph). In particular, we re-investigate the M/G/1 model studied by [12, 101, 52]. Although the underlying process is relatively simple, there is no known closed-form for the likelihood function and inference is not easy to conduct.

**Example 52 (M/G/1-queueing model).** Consider the following stochastic process

\[ x_i = \begin{cases} 
  v_i, & \text{if } \sigma_i^2 \leq \sigma_{i-1}^2, \\
  v_i + \sigma_i^2 - \sigma_{i-1}^2, & \text{if } \sigma_i^2 > \sigma_{i-1}^2,
\end{cases} \]

for \( i = 1, \ldots, n \), where \( \sigma_i^2 = \sum_{j=1}^{i} \varepsilon_j, \sigma_1^2 = \sum_{j=1}^{i} x_j \), \( v_i \) is identically and independently distributed according to a uniform distribution \( U(\theta_1, \theta_2) \), \( 0 \leq \theta_1 < \theta_2 < \infty \) and \( \varepsilon_i \) is identically and independently distributed according to an exponential distribution \( \delta(\theta_3), \theta_3 > 0 \). In queuing theory, random variables have special meaning, for the \( i \)th customer: \( x_i \) represents interdeparture time, \( v_i \) is service time and \( \varepsilon_i \) corresponds to interarrival time. Only the interdeparture times \( x_i \) are observed, \( v_i \) and \( \varepsilon_i \) are latent. All past information influence the current observation and therefore this process is not Markovian. Finding an “appropriate” auxiliary estimator is challenging as we now discuss.

In this context, semi-automatic ABC approaches by [101] and [52] use several quantiles as summary statistics for the auxiliary estimator. This method cannot be employed here for the SwiZs because, first, the restriction that \( \dim(\theta) = \dim(\pi) \) would be violated, and second, the quantiles are non-differentiables with respect to \( g \) and consequently, as already discussed, Assumptions [26] and [27] would not hold. However, [12] present different choices and motivate a particular auxiliary model with the following closed-form:

\[ f(x_i, \pi) = \begin{cases} 
  0, & \text{if } x_i \leq \pi_1, \\
  (\pi_2 - \pi_1)^{-1} \left[ 1 - \alpha \exp \left( -\pi_3^{-1}(x_i - \pi_1) \right) \right], & \text{if } \pi_1 < x_i \leq \pi_2, \\
  \frac{\alpha}{\pi_2 - \pi_1} \left[ \exp \left( -\pi_3^{-1}(x_i - \pi_2) \right) - \exp \left( -\pi_3^{-1}(x_i - \pi_1) \right) \right], & \text{if } x_i > \pi_2,
\end{cases} \]

where \(-1 \leq \alpha \leq 1\) is some constant. motivations for this auxiliary model are based on a graphical analysis of the sensitivity of \( \hat{\pi}_n(\theta) \) with respect to \( \theta \) and the root mean squared errors performances of \( \hat{\theta}_n \) on simulations. Unfortunately, Assumption [26] is not satisfied with this choice. Indeed, by taking the likelihood scores of the auxiliary model as the estimating equation, one can realize that the score relative to \( \pi_2 \) is

\[ \Phi_n(\theta, u, \pi) = \begin{cases} 
  0, & \text{if } g(\theta, u) < \pi_1, \\
  \frac{1}{\pi_2 - \pi_1}, & \text{if } \pi_1 \leq g(\theta, u) < \pi_2, \\
  \frac{1}{\pi_2 - \pi_1} - \frac{\pi_2 - \pi_1}{\pi_2 - \pi_1}, & \text{if } g(\theta, u) \geq \pi_2,
\end{cases} \]

hence, it does not depend on \( \theta \)! This result implies directly that all the partial derivatives with respect to \( \theta \) and \( w \) are null and \( \det(\varphi_{\pi_n}) = 0 \) for all \((\theta, w) \in (\Theta_n \times W_n)\). Assumption [27] is also violated and Theorem [26] cannot be invoked. Worse, the behaviour of this score does not depend on \( n \) and the identifiability condition in Assumption [22] (ii) does not hold since \( \Phi_2(\theta_1, \pi) = \Phi_2(\theta_2, \pi) \) for all \((\theta_1, \theta_2) \in \Theta\), so using this auxiliary model does not lead to a consistent estimator. It is however not clear whether Assumption [28] the alternative to Assumption [22] holds or not because the quantities to verify are unknown. Note however that in view of the equivalence theorem between the SwiZs and the indirect inference estimator (Theorem 5), it would appear as a contradiction for Assumption [22] not to hold but Assumption [28] to be satisfied.

[12] idea is to select an auxiliary model where \( \hat{\pi}_n(\theta) \) is both sensitive to \( \theta \) and efficient for a given \( \theta \). Since they justify their choice on a graphical analysis with simulated samples, one may wonder whether the authors were unlucky or misled by the graphics on this particular example. In fact, although \( \hat{\pi}_n(\theta) \) is unknown in an explicit form, its Jacobian may be derived explicitly by mean of an implicit function theorem, so for a given \( \theta_1 \in \Theta \) we have:

\[ D\theta \hat{\pi}_n(\theta_1) = -\left[ D_\pi \Psi_n(\theta_1, u, \pi) \right]^{-1}_{\pi = \hat{\pi}_n(\theta_1)} D_\theta \Psi_n(\theta_1, u, \hat{\pi}_n(\theta_1)) \]

The Jacobian \( D_\pi \Psi_n \) is non zero. Yet, as already discussed, the second partial derivative of \( \Psi_n \) with respect to \( \theta \) is null. Because only the second row of \( D_\theta \Psi_n \) has zero entries, there is no reason to believe that \( D_\theta \hat{\pi}_n(\theta) \) has zero entries. Consequently, the authors were not misled by the graphics or unlucky, it is the criterion itself that is misleading.

We now face ourselves to the delicate task of choosing an auxiliary model which non-only respects the constraint \( \dim(\theta) = \dim(\pi) \), but also makes Assumption [26] plausible. In view of this particular M/G/1 stochastic process, using
the convolution between a gamma with shape parameter \( n \) and unknown rate parameter and a uniform distributions may be a “natural” choice, yet, terms computationally complicated to evaluate readily appear. We propose instead of using Fréchet’s three parameters extreme value distribution, whose density is given, for \( i = 1, \ldots, n \), by:

\[
f(x_i; \pi) = \frac{\pi_1}{\pi_2} \left( \frac{x_i - \pi_3}{\pi_2} \right)^{-1-\pi_1} \exp \left\{ - \left( \frac{x_i - \pi_3}{\pi_2} \right)^{-\pi_1} \right\}, \quad \text{if } x_i > \pi_3,
\]

where \( \pi_1 > 0 \) is a shape parameter, \( \pi_2 > 0 \) is a scale parameter and \( \pi_3 \in \mathbb{R} \) is a parameter representing the location of the minimum. The relationship between \( \pi_3 \) and \( \theta_1 \) as the minimum of the distribution seems natural and we thus further constrain here \( \pi_3 \) to be non-negative, so \( \pi > 0 \). However, the existence of a potential link between \( (\theta_2, \theta_3)^T \) and \( (\pi_1, \pi_2)^T \) is not self-evident, but certainly that the shape \( (\pi_1) \) and scale \( (\pi_2) \) parameters offer enough flexibility to “encompass” the distribution of the M/G/1 stochastic process as illustrated in Figure 9. Note that the “closeness” between M/G/1 and Fréchet models is also dependent on the parametrization. It remains to advocate this choice in the

M/G/1 empirical distribution with Fréchet density

![Histogram of a simulated M/G/1 stochastic process of size \( n = 10^4 \) on which the density (solid line) of Fréchet distribution has been added. The true parameter is \( \theta_0 = [0.3 \ 0.9 \ 1]^T \), the auxiliary estimator we obtain here is approximately \( \hat{\pi}_n = [0.02 \ 0.60 \ 2.05]^T \).](image)

**Figure 9:** Histogram of a simulated M/G/1 stochastic process of size \( n = 10^4 \) on which the density (solid line) of Fréchet distribution has been added. The true parameter is \( \theta_0 = [0.3 \ 0.9 \ 1]^T \), the auxiliary estimator we obtain here is approximately \( \hat{\pi}_n = [0.02 \ 0.60 \ 2.05]^T \).

*light of Assumption 26* We take the maximum likelihood estimator of Fréchet’s distribution as the auxiliary estimator and thus the likelihood score as the estimating function, which is given by:

\[
\Psi_n(\theta, \mathbf{u}, \pi) = \begin{pmatrix}
\frac{1}{\pi_1} + \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{g(\theta, u_i) - \pi_3}{\pi_2} \right) \left[ 1 - \left( \frac{g(\theta, u_i) - \pi_3}{\pi_2} \right)^{-\pi_1} \right] \\
- \frac{\pi_1}{\pi_2 \ n} \sum_{i=1}^{n} 1 - \left( \frac{g(\theta, u_i) - \pi_3}{\pi_2} \right)^{-\pi_1} \\
- \frac{1}{n} \sum_{i=1}^{n} \frac{1}{g(\theta, u_i) - \pi_3} + \frac{\pi_1}{\pi_2 \ n} \sum_{i=1}^{n} \left( \frac{g(\theta, u_i) - \pi_3}{\pi_2} \right)^{-\pi_1-1}
\end{pmatrix}.
\]
Let us assume that a random variable \( w \) with the same dimension as \( \theta \) exists such that the estimating function may be expressed as follows:

\[
\varphi_{\hat{\pi}_n}(\theta, w) = \begin{pmatrix}
\frac{1}{\pi_1} + \log(z_1) \left[ 1 - z_1^{-\hat{\pi}_1} \right] \\
-\frac{1}{\pi_2} \left[ 1 - z_2^{-\hat{\pi}_2} \right] \\
-\frac{1}{\pi_3} \left[ 1 - z_3^{-\hat{\pi}_3} \right]
\end{pmatrix},
\]

where \( \hat{\pi}_n \) is fixed and \( z_i = \frac{E(\theta, w_i)^{1-\hat{\pi}_3}}{\pi_2}, i = 1, 2, 3 \). The Jacobian matrix with respect to \( \theta \) is given by:

\[
D_\theta \varphi_{\hat{\pi}_n}(\theta, w) = \begin{pmatrix}
D_{\theta^T} g(\theta, w_1) \left[ \frac{1}{\pi_2} \left( 1 - z_1^{-\hat{\pi}_1} \right) + \frac{1}{\pi_2} \log(z_1) z_1^{-\hat{\pi}_1-1} \right] \\
D_{\theta^T} g(\theta, w_2) \left[ -\frac{1}{\pi_2} z_2^{-\hat{\pi}_2-1} \right] \\
D_{\theta^T} g(\theta, w_3) \left[ \frac{1}{\pi_2} z_3^{-\hat{\pi}_3-1} - \frac{1}{\pi_3} + \frac{1}{\pi_2} z_2^{-\hat{\pi}_2-1} \right]
\end{pmatrix}.
\]

Substituting \( D_\theta g^T \) by \( D_w g^T \) in the above equation gives the Jacobian matrix with respect to \( w \), a matrix which is diagonal. It is straightforward to remark that the differentiability and continuity depends exclusively on the smoothness of \( g \). Thus, if \( g \) is once continuously differentiable in both \( \theta \) and \( w \), then Assumption 26(i) holds.

Concerning the determinant of these Jacobian matrices, they may be null only on unlikely situations: first, if \( g \) equals \( \hat{\pi}_3 \) then \( z_i \) is zero for \( i = 1, 2, 3 \), second, if \( D_\theta g \) or \( D_w g \) are zero. The choice of \( g \) may be guided by this restriction so typically the determinants may be null, but only on a countable set, and Assumption 26(ii) is verified. For Assumption 26(iii), it is straightforward to remark that

\[
\lim_{\|\theta, w\| \to \infty} \|\varphi_{\hat{\pi}_n}(\theta, w)\| = \infty,
\]

as long as \( \lim_{\|\theta, w\| \to \infty} \|g(\theta, w)\| = \infty \), since \( \log(z_1) \) would diverge. Depending on \( g \), Assumption 26(iii) is satisfied.

Therefore, the plausibility of Assumption 26 is up to the choice of the generating equation \( g \). Here, the choice is quasi immediate as it is driven by the form of the process:

\[
g(\theta, u) = \begin{cases}
v_1(\theta), & \text{if } \sigma^1_1(\theta) \leq \sigma^i_{i-1}(\theta), \\
v_2(\theta) + \sigma^i_i(\theta) - \sigma^i_{i-1}(\theta), & \text{if } \sigma^i_i(\theta) > \sigma^i_{i-1}(\theta),
\end{cases}
\]

where \( u_i = (u_{1i}, u_{2i})^T, u_{ji} \sim U(0,1), j = 1, 2, u_{1i} \) and \( u_{2i} \) are independent, \( v_i(\theta) \equiv \theta_1 + (\theta_2 - \theta_1)u_{1i}, \sigma^i_1(\theta) = \sum_{j=1}^i \varepsilon_j(\theta), \varepsilon_j(\theta) = -\theta_3^{-1} \log(u_{2j}) \) and \( \sigma^i_2 = \sum_{j=1}^i g(\theta, u_j) \). Let \( E_i \) correspond to the event \( \{ \sigma^i_1(\theta) \leq \sigma^i_{i-1}(\theta) \} \) and \( \bar{E}_i \), be the contrary. The partial derivatives may be found recursively as follows:

\[
\frac{\partial}{\partial \theta_1} g(\theta, u_i) = \begin{cases}
1 - u_{1i}, & \text{if } i = 1, \\
1 - u_{1i}, & \text{if } i > 1 \text{ and } E_i, \\
1 - u_{1i} - \sum_{j=1}^{i-1} \frac{\partial}{\partial \theta_1} g(\theta, u_j), & \text{if } i > 1 \text{ and } \bar{E}_i.
\end{cases}
\]

\[
\frac{\partial}{\partial \theta_2} g(\theta, u_i) = \begin{cases}
u_{1i}, & \text{if } i = 1, \\
u_{1i}, & \text{if } i > 1 \text{ and } E_i, \\
u_{1i} - \sum_{j=1}^{i-1} \frac{\partial}{\partial \theta_2} g(\theta, u_j), & \text{if } i > 1 \text{ and } \bar{E}_i.
\end{cases}
\]

\[
\frac{\partial}{\partial \theta_3} g(\theta, u_i) = \begin{cases}0, & \text{if } i = 1, \\
-\frac{1}{\varepsilon_3} \sum_{j=1}^i \log(u_{2j}) - \sum_{j=1}^{i-1} \frac{\partial}{\partial \theta_3} g(\theta, u_j), & \text{if } i > 1 \text{ and } E_i.
\end{cases}
\]

\[
\frac{\partial}{\partial u_{1i}} g(\theta, u_i) = \begin{cases}\theta_2 - \theta_1, & \text{if } i = 1, \\
\theta_2 - \theta_1, & \text{if } i > 1 \text{ and } E_i, \\
\theta_2 - \theta_1 - \sum_{j=1}^{i-1} \frac{\partial}{\partial u_{1j}} g(\theta, u_j), & \text{if } i > 1 \text{ and } \bar{E}_i.
\end{cases}
\]

\[
\frac{\partial}{\partial u_{2i}} g(\theta, u_i) = \begin{cases}0, & \text{if } i = 1, \\
-\theta_3^{-1} \sum_{j=1}^i \frac{1}{u_{2j}} - \sum_{j=1}^{i-1} \frac{\partial}{\partial u_{2j}} g(\theta, u_j), & \text{if } i > 1 \text{ and } E_i.
\end{cases}
\]

Clearly \( g \) is once continuously differentiable in both its arguments with non-zero derivatives. Eventually, we have that \( v_i(\theta) \) goes to \( \infty \) when \( \theta_1 \to \infty, \theta_2 \to \infty \) and \( u_{1i} \to 1 \), whereas \( \varepsilon_i(\theta) \) tends to zero whenever \( \theta_3 \to \infty \) and \( u_{2i} \to 1 \). It is not clear whether \( v_i(\theta) + \sigma^i_1(\theta) - \sigma^i_2(\theta) \) diverges or converges to 0 when \( \|\theta, w(u)\| \to \infty \), but in
any case \( \|q(\theta, u_i)\| \) tends to \( \infty \) since \( v_i(\theta) \) diverges. As a consequence, Assumption 26 is highly plausible and thus Theorem 28 seems invokable.

For the simulation, we set \( \theta_0 = [0.3 \ 0.9 \ 1]^T \) and \( n = 100 \) as in [12]. We compare the SwiZs with indirect inference in Definition 3 and the parametric bootstrap using the indirect inference with \( B = 1 \) as the initial consistent estimator (see Definition 6). By Theorem 5 the SwiZs and the indirect inference are equivalent, but as argued, the price for obtaining the indirect inference is higher so here we seek empirical evidence, and Table 3 speaks for itself, the difference is indeed monstrous. The parametric bootstrap is even worse in terms of computational time. It is maybe good to remind the reader that the comparison is fair: all three methods benefit from the same level of implementation and uses the very same technology. The complete results may be found in Appendix D.3 In Figure 10 we can realize that the SwiZs do not offer an exact coverage in this case, it is even far from ideal for \( \hat{\theta}_2 \). It is nonetheless better than the parametric bootstrap. Especially the coverage of \( \hat{\theta}_1 \) and \( \hat{\theta}_3 \) are close to the ideal level. Considering the context of this simulation: moderate sample size, no closed-form for the likelihood, the results are very encouraging. A good surprise appears from Figure 11, where the SwiZs demonstrates better performances of its point estimates (mean and median) compared to indirect inference approaches in terms of absolute median bias and mean absolute deviation.

It is however not clear which one, if not both, we should blame for failure of missing exact coverage probability between our analysis on the applicability of Theorem 28 to this case or the numerical optimization procedure. The previous examples seem to indicate for the latter. To this end, we re-run the same experiment only for the SwiZs (for pure operational reason) by changing the starting values to be the true parameter \( \theta_0 \) to measure the implication. Indeed, starting values are a sensitive matter for quasi-Newton routine and since \( \hat{\pi}_n \) is not a consistent estimator of \( \theta_0 \), using it as a starting value might have a persistent influence on the sequence \( \{\hat{\theta}_n^{(s)} : s \in \mathbb{N}_S^+\} \). Results are reported.

### Table 3: Average time in seconds to estimate a conditional distribution on \( S = 10,000 \) points and total time in hours for the \( M = 10,000 \) independent trials.

|                         | SwiZs  | indirect inference | parametric bootstrap |
|-------------------------|--------|--------------------|----------------------|
| Average time [seconds]  | 0.97   | 134.18             | 197.15               |
| Total time [hours]      | 2.7    | 372.5              | 547.4                |
in Table in Appendix D.3. The coverage probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$ becomes nearly perfect, which shows that indeed good starting values may reduce the numerical error in the coverage probabilities. However, coverage probability for $\hat{\theta}_2$ persistently shows result off the desired levels, which seems rather to indicate a problem related to the applicability of Theorem 28. Increasing the sample size to $n = 1,000$ (see Table 19) makes the coverage of all three parameters nearly perfect.

Fifth and last, we consider logistic regression. This is certainly one of the most widely used statistical model in practice. This case is challenging at least on two aspects. First, the random variable is discrete and the finite sample theory in Section 4 does not hold. Second, the generating function is non-differentiable with respect to $\theta$, therefore gradient-based optimization routines cannot be employed. In what follows, we circumvent this inconvenient by smoothing the generating function. To this end, we start by introducing the continuous latent representation of the logistic regression.

**Example 53.** Suppose we have the model

$$ y = X\theta + \epsilon, $$

where $\epsilon = (\epsilon_1, \cdots, \epsilon_n)^T$ and $\epsilon_i, i = 1, \cdots, n,$ are identically and independently distributed according to a logistic distribution with mean 0 and unity variance. This distribution belongs to symmetric location-scale families. It is similar to the Gaussian distribution with heavier tails. The unknown parameters $\Theta$ of this model could be easily estimated by the ordinary least squares:

$$ \hat{\pi}_n = (X^TX)^{-1}X^Ty. $$

The corresponding estimating function is:

$$ \Psi_n(\theta, u, \pi) = X^TX\pi - X^Tg(\theta, u). $$

A straightforward generating function is $g(\theta, u) = X\theta + u$ where $u_i \sim \text{Logistic}(0,1)$. Evaluating this function at $\pi = \hat{\pi}_n$ leads to

$$ \Psi_n(\theta, u, \hat{\pi}_n) = X^Ty - X^TX\theta - X^Tu. $$

Solving the root of this function in $\theta$ gives the following explicit solution:

$$ \hat{\theta}_n = (X^TX)^{-1}X^T(y - u). \quad (7) $$

Following Example 45 on linear regression, it is easy to show that inference based on the distribution of this estimator leads to exact frequentist coverage probabilities.
Let us turn our attention to logistic regression. In this case, \(y\) is not observed. Instead, we observe a binary random variable \(y\), whose elements are:

\[
y_i = \begin{cases} 
1, & X_i \theta + \epsilon_i \geq 0, \\
0, & X_i \theta + \epsilon_i < 0,
\end{cases}
\]

where \(X_i\) is the \(i\)th row of \(X\). Saying it differently, this consideration implies that the generating function is modified to the following indicator function:

\[
g(\theta, u_i) = 1 \{X_i \theta + u_i \geq 0\}.
\]

Clearly, this change implies that \(\Psi_n\) has a flat Jacobian matrix and Assumptions\(^{26}\) and \(^{27}\) do not hold. Moreover, this problem becomes numerically more involved, especially if we want to pursue with a gradient-based optimization routine. As mentioned, in practice we seek the solution of the following problem:

\[
\arg\min_{\theta \in \Theta} \|X^T y - X^T g(\theta, u)\|_2^2 \equiv \arg\min_{\theta \in \Theta} f(\theta).
\]  

Note that \(X^T y\) is the sufficient statistic for a logistic regression (see Chapter 2 in \(^{102}\)). The gradient of \(f(\theta)\) is

\[
-D_{\theta g}(\theta, u) X [X^T y - X^T g(\theta, u)].
\]

However, the Jacobian \(D_{\theta g}(\theta, u)\) is 0 almost everywhere and alternatives are necessary for using gradient-based methods. A possibility is to smooth \(g(\theta, u)\) by using for example a sigmoid function:

\[
g(\theta, u) = \lim_{t \to \infty} \frac{1}{1 + \exp (-\langle x, \theta + u_i \rangle / \epsilon)}.
\]

The value of \(t\) tunes the approximation and the value of the gradient. However, from our experience, large values of \(t\), say \(t > 0.1\), leads to poor results and small values, say \(t < 0.1\), leads to numerical instability. We thus prefer to use a different strategy by taking \(-f(\theta)\) as the gradient. This strategy corresponds to the iterative bootstrap procedure (\(^{17}\)). In Figure \(12\) we illustrate the difference between these two approximations and the “ideal” distribution we would have obtained by observing the continuous underlying latent process. Clearly, the loss of information induced from the possibility of only observing a binary outcome results in an increase of variability. Nonetheless, the difference is not enormous. Both approximations lead to similar distributions in terms of shapes. We can notice a little difference in their modes. Since the iterative bootstrap approximation is numerically advantageous, we use it in the next study.

For simulation, we setup \(\theta_0 = (0, 5, 5, -7, -7, 0, \ldots, 0)^T\) and sample size \(n = 200\). We compare coverage probabilities of 95% confidence intervals obtained by the SwiZs and by asymptotic theory. We report results in Table \(4\). We can clearly see that the SwiZs have the most precise confidence intervals for all coefficients with coverage close to the target level of 95%.

References

[1] A Ronald Gallant and George Tauchen. Which moments to match? Econometric Theory, 12(4):657–681, 1996.

[2] Christian Gourieroux, Alain Monfort, and Eric Renault. Indirect inference. Journal of applied econometrics, 8(51), 1993.

[3] Anthony A Smith. Estimating nonlinear time-series models using simulated vector autoregressions. Journal of Applied Econometrics, 8(51), 1993.

[4] René Garcia, Eric Renault, and David Veredas. Estimation of stable distributions by indirect inference. Journal of Econometrics, 161(2):325–337, 2011.

[5] Chiara Monfardini. Estimating stochastic volatility models through indirect inference. The Econometrics Journal, 1(1):113–128, 1998.

[6] Marco J Lombardi and Giorgio Calzolari. Indirect estimation of \(\alpha\)-stable stochastic volatility models. Computational Statistics & Data Analysis, 53(6):2298–2308, 2009.

[7] Peter CB Phillips and Jun Yu. Simulation-based estimation of contingent-claims prices. The Review of Financial Studies, 22(9):3669–3705, 2009.

[8] Christian Gouriéroux, Peter CB Phillips, and Jun Yu. Indirect inference for dynamic panel models. Journal of Econometrics, 157(1):68–77, 2010.
Figure 12: Simulated SwisZ distribution of a single logistic regression with coefficient $\theta = 2$ and sample size of 10. “Ideal” is (7). “Smoothing” approximates the gradient with a sigmoid function and $t = 0.01$. “Iterative bootstrap” uses $-f(\theta)$ as the gradient.

[9] Ramdan Dridi, Alain Guay, and Eric Renault. Indirect inference and calibration of dynamic stochastic general equilibrium models. *Journal of Econometrics*, 136(2):397–430, 2007.

[10] A Ronald Gallant and George Tauchen. Simulated score methods and indirect inference for continuous-time models. *Handbook of financial econometrics*, 1:427–477, 2010.

[11] Laurence Broze, Olivier Scaillet, and Jean-Michel Zakoian. Quasi-indirect inference for diffusion processes. *Econometric Theory*, 14(2):161–186, 1998.

[12] Knut Heggland and Arnoldo Frigessi. Estimating functions in indirect inference. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 66(2):447–462, 2004.

[13] Irini Moustaki and Maria-Pia Victoria-Feser. Bounded-influence robust estimation in generalized linear latent variable models. *Journal of the American Statistical Association*, 101(474):644–653, 2006.

[14] Stéphane Guerrier, Elise Dupuis-Lozeron, Yanyuan Ma, and Maria-Pia Victoria-Feser. Simulation-based bias correction methods for complex models. *Journal of the American Statistical Association*, pages 1–12, 2018.

[15] Stéphane Guerrier, Mucyo Karemera, Samuel Orso, and Maria-Pia Victoria-Feser. On the properties of simulation-based estimators in high dimensions. *arXiv preprint arXiv:1810.04443*, 2018.

[16] Simon Tavaré, David J Balding, Robert C Griffiths, and Peter Donnelly. Inferring coalescence times from dna sequence data. *Genetics*, 145(2):505–518, 1997.
|   | SwiZs | asymptotic |
|---|-------|------------|
| $\theta_1$ | 0.9442 | 0.9187 |
| $\theta_2$ | 0.9398 | 0.8115 |
| $\theta_3$ | 0.9382 | 0.8121 |
| $\theta_4$ | 0.9432 | 0.7688 |
| $\theta_5$ | 0.9450 | 0.7737 |
| $\theta_6$ | 0.9397 | 0.9233 |
| $\theta_7$ | 0.9357 | 0.9170 |
| $\theta_8$ | 0.9398 | 0.9237 |
| $\theta_9$ | 0.9391 | 0.9218 |
| $\theta_{10}$ | 0.9400 | 0.9208 |
| $\theta_{11}$ | 0.9424 | 0.9208 |
| $\theta_{12}$ | 0.9375 | 0.9214 |
| $\theta_{13}$ | 0.9368 | 0.9204 |
| $\theta_{14}$ | 0.9389 | 0.9210 |
| $\theta_{15}$ | 0.9400 | 0.9207 |
| $\theta_{16}$ | 0.9400 | 0.9183 |
| $\theta_{17}$ | 0.9361 | 0.9183 |
| $\theta_{18}$ | 0.9449 | 0.9241 |
| $\theta_{19}$ | 0.9412 | 0.9218 |
| $\theta_{20}$ | 0.9427 | 0.9240 |

Table 4: 95% coverage probabilities of confidence intervals from the SwiZs and asymptotic theory.

[17] Jonathan K Pritchard, Mark T Seielstad, Anna Perez-Lezaun, and Marcus W Feldman. Population growth of human Y chromosomes: a study of Y chromosome microsatellites. *Molecular biology and evolution*, 16(12):1791–1798, 1999.

[18] Mark A Beaumont, Wenyang Zhang, and David J Balding. Approximate bayesian computation in population genetics. *Genetics*, 162(4):2025–2035, 2002.

[19] Mark A Beaumont. Approximate bayesian computation in evolution and ecology. *Annual review of ecology, evolution, and systematics*, 41:379–406, 2010.

[20] Jean-Marie Cornuet, Filipe Santos, Mark A Beaumont, Christian P Robert, Jean-Michel Marin, David J Balding, Thomas Guillemaud, and Arnaud Estoup. Inferring population history with diy abc: a user-friendly approach to approximate bayesian computation. *Bioinformatics*, 24(23):2713–2719, 2008.

[21] Richard D Wilkinson, Michael E Steiper, Christophe Soligo, Robert D Martin, Ziheng Yang, and Simon Tavaré. Dating primate divergences through an integrated analysis of palaeontological and molecular data. *Systematic Biology*, 60(1):16–31, 2010.

[22] R.A. Fisher. On the mathematical foundations of theoretical statistics. *Phil. Trans. R. Soc. Lond. A*, 222(594-604):309–368, 1922.

[23] R.A. Fisher. Inverse probability. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 26, pages 528–535. Cambridge University Press, 1930.

[24] R.A. Fisher. The concepts of inverse probability and fiducial probability referring to unknown parameters. *Proc. R. Soc. Lond. A*, 139(838):343–348, 1933.

[25] R.A. Fisher. The fiducial argument in statistical inference. *Annals of eugenics*, 6(4):391–398, 1935.

[26] R.A. Fisher. *Statistical methods and scientific inference*. Oxford, England: Hafner Publishing Co., 1956.

[27] Jan Hannig. On generalized fiducial inference. *Statistica Sinica*, pages 491–544, 2009.

[28] Jan Hannig. Generalized fiducial inference via discretization. *Statistica Sinica*, pages 489–514, 2013.

[29] Jan Hannig, Hari Iyer, Randy CS Lai, and Thomas CM Lee. Generalized fiducial inference: A review and new results. *Journal of the American Statistical Association*, 111(515):1346–1361, 2016.
[30] D.A.S. Fraser. *The structure of inference*. Wiley, New York, 1968.

[31] A Philip Dawid, Mervyn Stone, and James V Zidek. Marginalization paradoxes in bayesian and structural inference. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 189–233, 1973.

[32] Glenn Shafer. *A mathematical theory of evidence*, volume 42. Princeton university press, 1976.

[33] Arthur P Dempster. The dempster-shafer calculus for statisticians. *International Journal of approximate reasoning*, 48(2):365–377, 2008.

[34] Ryan Martin and Chuanhai Liu. Inferential models: A framework for prior-free posterior probabilistic inference. *Journal of the American Statistical Association*, 108(501):301–313, 2013.

[35] Ryan Martin and Chuanhai Liu. Conditional inferential models: combining information for prior-free probabilistic inference. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 77(1):195–217, 2015.

[36] Ryan Martin. Plausibility functions and exact frequentist inference. *Journal of the American Statistical Association*, 110(512):1552–1561, 2015.

[37] Jerzy Neyman and Egon S Pearson. Ix. on the problem of the most efficient tests of statistical hypotheses. *Phil. Trans. R. Soc. Lond. A*, 231(694-706):289–337, 1933.

[38] Peter Hall. *The bootstrap and edgeworth expansion*. Springer-Verlag, New York, 1992.

[39] David T Frazier, Gael M Martin, Christian P Robert, and Judith Rousseau. Asymptotic properties of approximate bayesian computation. *Biometrika*, 105(3):593–607, 2018.

[40] Christian Gourieroux and Alain Monfort. *Simulation-based econometric methods*. Oxford university press, 1996.

[41] B. Efron. Bootstrap methods: Another look at the jackknife. *The Annals of Statistics*, 7(1):1–26, 1979.

[42] Christopher C Drovandi, Anthony N Pettitt, and Anthony Lee. Bayesian indirect inference using a parametric auxiliary model. *Statistical Science*, 30(1):72–95, 2015.

[43] Peter J Diggle and Richard J Gratton. Monte carlo methods of inference for implicit statistical models. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 193–227, 1984.

[44] Paul Marjoram, John Molitor, Vincent Plagnol, and Simon Tavaré. Markov chain monte carlo without likelihoods. *Proceedings of the National Academy of Sciences*, 100(26):15324–15328, 2003.

[45] Paola Bortot, Stuart G Coles, and Scott A Sisson. Inference for stereological extremes. *Journal of the American Statistical Association*, 102(477):84–92, 2007.

[46] Scott A Sisson, Yanan Fan, and Mark M Tanaka. Sequential monte carlo without likelihoods. *Proceedings of the National Academy of Sciences*, 104(6):1760–1765, 2007.

[47] Mark A Beaumont, Jean-Marie Cornuet, Jean-Michel Marin, and Christian P Robert. Adaptive approximate bayesian computation. *Biometrika*, 96(4):983–990, 2009.

[48] Tina Toni, David Welch, Natalja Strelkowa, Andreas Ipsen, and Michael PH Stumpf. Approximate bayesian computation scheme for parameter inference and model selection in dynamical systems. *Journal of the Royal Society Interface*, 6(31):187–202, 2009.

[49] Jean-Michel Marin, Pierre Pudlo, Christian P Robert, and Robin J Ryder. Approximate bayesian computational methods. *Statistics and Computing*, 22(6):1167–1180, 2012.

[50] SA Sisson, GW Peters, M Briers, and Y Fan. A note on target distribution ambiguity of likelihood-free samplers. *arXiv preprint arXiv:1005.5201*, 2010.

[51] Pierre Del Moral, Arnaud Doucet, and Ajay Jasra. Sequential monte carlo samplers. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(3):411–436, 2006.
[52] Paul Fearnhead and Dennis Prangle. Constructing summary statistics for approximate bayesian computation: semi-automatic approximate bayesian computation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 74(3):419–474, 2012.

[53] Sandy L Zabell. Raisher and fiducial argument. *Statistical Science*, 7(3):369–387, 1992.

[54] Jianchun Zhang and Chuanhai Liu. Dempster-shafer inference with weak beliefs. *Statistica Sinica*, pages 475–494, 2011.

[55] Jan Hannig, Randy CS Lai, and Thomas CM Lee. Computational issues of generalized fiducial inference. *Computational Statistics & Data Analysis*, 71:849–858, 2014.

[56] Robert E Kass and Larry Wasserman. Formal rules for selecting prior distributions: A review and annotated bibliography. *Journal of the American Statistical Association*, 1994.

[57] DAS Fraser, N Reid, E Marras, and GY Yi. Default priors for bayesian and frequentist inference. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 72(5):631–654, 2010.

[58] Dennis V Lindley. Fiducial distributions and bayes’ theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 102–107, 1958.

[59] Christian Robert. *The Bayesian choice: from decision-theoretic foundations to computational implementation*. Springer Science & Business Media, 2007.

[60] Tore Schweder and Nils Lid Hjort. Confidence and likelihood. *Scandinavian Journal of Statistics*, 29(2):309–332, 2002.

[61] Kesar Singh, Minge Xie, William E Strawderman, et al. Combining information from independent sources through confidence distributions. *The Annals of Statistics*, 33(1):159–183, 2005.

[62] Minge Xie, Kesar Singh, and William E Strawderman. Confidence distributions and a unifying framework for meta-analysis. *Journal of the American Statistical Association*, 106(493):320–333, 2011.

[63] Min-ge Xie and Kesar Singh. Confidence distribution, the frequentist distribution estimator of a parameter: A review. *International Statistical Review*, 81(1):3–39, 2013.

[64] Robert V Hogg, Joseph McKeans, and Allen T Craig. *Introduction to mathematical statistics*. Pearson Education, 6 edition, 2005.

[65] Ailana M Fraser, Donald AS Fraser, and Ana-Maria Staicu. Second order ancillary: A differential view from continuity. *Bernoulli*, pages 1208–1223, 2010.

[66] T Tony Cai. One-sided confidence intervals in discrete distributions. *Journal of Statistical planning and inference*, 131(1):63–88, 2005.

[67] Richard S Palais. Natural operations on differential forms. *Transactions of the American Mathematical Society*, 92(1):125–141, 1959.

[68] Mihai Cristea. On global implicit function theorem. *Journal of Mathematical Analysis and Applications*, 456(2):1290–1302, 2017.

[69] Wenxin Jiang and Bruce Turnbull. The indirect method: inference based on intermediate statistics—a synthesis and examples. *Statistical Science*, 19(2):239–263, 2004.

[70] Aad W Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 1998.

[71] Benedikt M Pötscher and Ingmar R Prucha. Generic uniform convergence and equicontinuity concepts for random functions: An exploration of the basic structure. *Journal of Econometrics*, 60(1-2):23–63, 1994.

[72] Peter J Huber. The behavior of maximum likelihood estimates under nonstandard conditions. In *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, volume 1, pages 221–233. University of California Press, 1967.

[73] James Douglas Hamilton. *Time series analysis*, volume 2. Princeton university press Princeton, NJ, 1994.
[74] Donald WK Andrews. Laws of large numbers for dependent non-identically distributed random variables. *Econometric theory*, 4(3):458–467, 1988.

[75] Whitney K Newey and Daniel McFadden. Large sample estimation and hypothesis testing. *Handbook of econometrics*, 4:2111–2245, 1994.

[76] Jarl Waldemar Lindeberg. Eine neue herleitung des exponentialgesetzes in der wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, 15(1):211–225, 1922.

[77] Patrick Billingsley. *Probability and Measure*, volume 939. John Wiley & Sons, 2012.

[78] Wei Biao Wu. Asymptotic theory for stationary processes. *Statistics and its Interface*, 4(2):207–226, 2011.

[79] Changyong Feng, Hongyue Wang, Yu Han, Yinglin Xia, and Xin M Tu. The mean value theorem and taylor’s expansion in statistics. *The American Statistician*, 67(4):245–248, 2013.

[80] N.L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous univariate distributions*, volume 1. John Wiley & Sons, Inc., 2nd edition, 1994.

[81] Piero Veronese and Eugenio Melilli. Fiducial and confidence distributions for real exponential families. *Scandinavian Journal of Statistics*, 42(2):471–484, 2015.

[82] Maurice Kendall and Alan Stuart. *The advanced theory of statistics*, volume 2nd: Inference and relationship. Charles Griffin & Company Limited, 3rd edition, 1961.

[83] Karl-Rudolf Koch. *Introduction to Bayesian statistics*. Springer Science & Business Media, 2007.

[84] Arakaparampill M Mathai and Serge B Provost. *Quadratic forms in random variables: theory and applications*. Dekker, 1992.

[85] Student. The probable error of a mean. *Biometrika*, pages 1–25, 1908.

[86] Luc Devroye. *Non-uniform random variate generation*. Springer-Verlag, New York, 1986.

[87] Ralph W Bailey. Polar generation of random variates with the t-distribution. *Mathematics of Computation*, 62(206):779–781, 1994.

[88] Bradley Efron and Robert J Tibshirani. *An introduction to the bootstrap*. CRC press, 1994.

[89] KS Lomax. Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, 49(268):847–852, 1954.

[90] Christian Kleiber and Samuel Kotz. *Statistical size distributions in economics and actuarial sciences*, volume 470. John Wiley & Sons, 2003.

[91] David E Giles, Hui Feng, and Ryan T Godwin. On the bias of the maximum likelihood estimator for the two-parameter lomax distribution. *Communications in Statistics-Theory and Methods*, 42(11):1934–1950, 2013.

[92] Maria-Pia Victoria-Feser and Elvezio Ronchetti. Robust methods for personal-income distribution models. *Canadian Journal of Statistics*, 22(2):247–258, 1994.

[93] C Field and B Smith. Robust estimation: A weighted maximum likelihood approach. *International Statistical Review/Revue Internationale de Statistique*, pages 405–424, 1994.

[94] Debbie J Dupuis and Stephan Morgenthaler. Robust weighted likelihood estimators with an application to bivariate extreme value problems. *Canadian Journal of Statistics*, 30(1):17–36, 2002.

[95] Peter J Huber et al. Robust estimation of a location parameter. *The annals of mathematical statistics*, 35(1):73–101, 1964.

[96] Albert E Beaton and John W Tukey. The fitting of power series, meaning polynomials, illustrated on band-spectroscopic data. *Technometrics*, 16(2):147–185, 1974.

[97] Frank R Hampel, Elvezio M Ronchetti, Peter J Rousseeuw, and Werner A Stahel. *Robust statistics: the approach based on influence functions*, volume 196. John Wiley & Sons, 2011.
A Technical results

Lemma 54. Let \( X \) and \( Y \) be open subsets of \( \mathbb{R}^n \). If \( f : X \to Y \) is a \( C^1 \)-diffeomorphism, then the Jacobian matrices of the maps \( x \mapsto f \) and \( y \mapsto f^{-1} \) are invertible, and the derivatives at the points \( a \in X \) and \( b \in Y \), are given by:
\[
D_x f(a) = [D_y f^{-1}(y = f(a))]^{-1}, \quad D_y f(b) = [D_x f(x = f^{-1}(b))]^{-1}.
\]

Proof. By assumption, \( f \) is invertible, once continuously differentiable and \( f^{-1} \) is once continuously differentiable.

We have \( f^{-1} \circ f = \text{id}_X \), where \( \text{id}_X \) is the identity function on the set \( X \). Fix \( a \in X \). By the chain rule, the derivative at \( a \) is the following:
\[
D_y f^{-1}(f(a)) \cdot D_x f(a) = I_n,
\]
where \( I_n \) is the identity matrix. Since \( D_y f^{-1} \) and \( D_x f \) are square matrices, we have:
\[
\det (D_y f^{-1}(f(a))) \cdot \det (D_x f(a)) = 1.
\]
The determinants cannot be 0, there are either 1 or -1 for both matrices, ergo, the Jacobian is invertible and we can write
\[
D_x f(a) = [D_y f^{-1}(f(a))]^{-1}.
\]
The proof for \( f \circ f^{-1} = \text{id}_Y \) follows by symmetry.

Lemma 55. Let \( \Theta \) and \( W \) be open subsets of \( \mathbb{R}^p \). If there exists a \( C^1 \)-diffeomorphic mapping \( a : W \to \Theta \), that is, \( w \mapsto a \) is continuously once differentialbe in \( \Theta \times W \) and the inverse map \( \theta \mapsto a^{-1}(\theta) \) is continuously once differentiable in \( \Theta \times W \), then the cumulative distribution function of \( \{\theta_n^{(s)} : s \in \mathbb{N}\} \) is given by:
\[
\int_{\Theta_n} f_{\theta_n}(\theta | \pi_n) \, d\theta = \int_W f_w(a(w) | \pi_n) \frac{1}{|\det (D_w a(w))|} \, dw,
\]
provided that \( f \) is a nonnegative Borel function and \( \text{Pr}(\pi_n \neq \emptyset) = 1 \).

Proof of Lemma 55. By assumption, \( w \mapsto a \) is a \( C^1 \)-diffeomorphism so by Lemma 54, the Jacobian of \( a \) and \( a^{-1} \) are invertible. All the conditions of the change-of-variable formula for multidimensional Lebesgue integral in [77, Theorem 17.2, p.239] are satisfied, so we obtain
\[
\int_{\Theta_n} f_{\theta_n}(\theta | \pi_n) \, d\theta = \int_{\Theta_n} f_{a^{-1}(\theta)}(\theta | \pi_n) \det (D_{\theta} a^{-1}(\theta)) \, d\theta
\]

By Lemma 54 we have that \( D_{\theta} a^{-1} = [D_w a]^{-1} \). Taking the determinant ends the proof. \( \square \)
B Finite sample

"Proof of Theorem" We proceed by showing first that $\Theta_{\Pi,n}^{(s)} \subset \Theta_n^{(s)}\), and second that $\Theta_{\Pi,n}^{(s)} \supset \Theta_n^{(s)}\). It follows from Assumption that $\pi_n$ is the unique solution of argzero$_{\pi \in \Pi} \Psi_n(\theta_0, u_0, \pi)\), ergo $\Pi_n$ in the Definition is a singleton.

(1) Fix $\theta_1 \in \Theta_{\Pi,n}^{(s)}\). By Definition it holds that

$$\pi_n = \pi_{\Pi,n}^{(s)}(\theta_1)\), \quad \Psi_n \left(\theta_1, u_s, \pi_{\Pi,n}^{(s)}(\theta_1)\right) = 0,$$

where $\pi_{\Pi,n}^{(s)}$ is the unique solution of argzero$_{\pi \in \Pi} \Psi_n(\theta_1, u_s, \pi)\). Ergo, it holds as well that

$$\Psi_n \left(\theta_1, u_s, \pi_n\right) = 0,$$

implying that $\theta_1 \in \Theta_n^{(s)}\) by Definition Thus $\Theta_{\Pi,n}^{(s)} \subset \Theta_n^{(s)}\).

(2) Fix $\theta_2 \in \Theta_n\). By Definition we have

$$\Psi_n \left(\theta_2, u_s, \pi_n\right) = 0.$$

By Definition we also have

$$\Psi_n \left(\theta_2, u_s, \pi_{\Pi,n}^{(s)}(\theta_2)\right) = 0,$$

where $\pi_{\Pi,n}^{(s)}(\theta_2)$ is the unique solution of argzero$_{\pi \in \Pi} \Psi_n(\theta_2, u_s, \pi)\). It follows that $\pi_n = \pi_{\Pi,n}^{(s)}(\theta_2)$ uniquely, implying that $\theta_2 \in \Theta_{\Pi,n}^{(s)}$ by Definition Thus $\Theta_{\Pi,n}^{(s)} \supset \Theta_n^{(s)}\), which concludes the proof.

"Proof of Theorem" We proceed by showing first that (A) $\Theta_n^{(s)} \subset \Theta_{\text{Boot},n}^{(s)}\) implies (B) $\Psi_n(\theta, u_s, \pi) = \Psi_n(\pi, u_s, \theta) = 0$, then that (B) implies (A).

1. Suppose (A) holds. Fix $\theta_1 \in \Theta_n^{(s)}$ and $\pi_n \in \Pi_n\). We have by the Definition

$$\Psi_n \left(\theta_1, u_s, \pi_n\right) = 0.$$

By (A), we also have that $\theta_1 \in \Theta_{\text{Boot},n}^{(s)}$ so by the Definition

$$\Psi_n \left(\pi_n, u_s, \theta_1\right) = 0.$$

Since both estimating equations equal zero, we have

$$\Psi_n \left(\pi_n, u_s, \theta_1\right) = \Psi_n \left(\theta_1, u_s, \pi_n\right) = 0.$$

Hence (A) implies (B).

2. Suppose now that (B) holds. Fix $\theta_1 \in \Theta_n^{(s)}$ and $\pi_n \in \Pi_n$ so $\Psi_n(\theta_1, u_s, \pi_n) = 0$. By (B), we have

$$\Psi_n \left(\theta_1, u_s, \pi_n\right) = \Psi_n \left(\pi_n, u_s, \theta_1\right) = 0,$$

so $\theta_1 \in \Theta_{\text{Boot},n}^{(s)}$ and thus $\Theta_n^{(s)} \subset \Theta_{\text{Boot},n}^{(s)}\). The same argument shows that $\Theta_n^{(s)} \supset \Theta_{\text{Boot},n}^{(s)}$ which ends the proof.

"Proof of Proposition" Since $\hat{x}_n = \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i\), the sample average, we can write the following estimating equation

$$\hat{x}_n = \text{argzero}_{\pi \in \Pi} (\bar{x} - \pi) = \text{argzero}_{\pi \in \Pi} \Phi_n(\theta_0, u_0, \pi),$$

where $x \overset{d}{=} g(\theta_0, u_0)$ Since $x$ follows a location family, we have that $x \overset{d}{=} \theta_0 + g(0, u_0) \overset{d}{=} \theta_0 + y$. The SwiZs is defined as

$$\hat{\theta}_n^{(s)} = \text{argzero}_{\theta \in \Theta} \Phi_n \left(\theta, u_s, \pi_n\right).$$

On the other hand, the parametric bootstrap estimator is

$$\hat{\theta}_{\text{Boot},n}^{(s)} = \text{argzero}_{\theta \in \Theta} \Phi_n \left(\hat{x}_n, u_s, \theta\right).$$
Eventually, we obtain that
\[ \Phi_n \left( \hat{\theta}^{(s)}_n, u_s, \hat{\pi}_n \right) = \hat{\theta}^{(s)}_n + \hat{y} - \hat{\pi}_n = 0, \]
\[ \Phi_n \left( \hat{\pi}_n, u_s, \hat{\theta}^{(s)}_n \right) = \hat{\pi}_n + \hat{y} - \hat{\theta}_\text{Boot,n}^{(s)} = -\hat{\pi}_n + \hat{y} + \hat{\theta}_\text{Boot,n}^{(s)} = 0. \]
where we use the fact that \( \hat{y} \overset{d}{=} -\bar{y} \). Therefore, \( \hat{\theta}^{(s)}_n = \hat{\theta}_\text{Boot,n}^{(s)} \), or equivalently \( \Phi_n (\theta, u_s, \pi) = \Phi_n (\pi, u_s, \theta) = 0 \), which ends the proof.

**Proof of Theorem 13** Fix \( \varepsilon = 0 \). The Theorem 5 is satisfied so \( \Theta^{(s)}_n = \Theta^{(s)}_{\Pi,n} \) for any \( s \). It is sufficient then to prove \( \Theta^{(s)}_{\text{ABC},n}(0) = \Theta^{(s)}_{\Pi,n} \) for any \( s \in \mathbb{N}_S^+ \). We proceed by verifying that first \( \Theta^{(s)}_{\text{ABC},n}(0) \subset \Theta^{(s)}_{\Pi,n} \), and second that \( \Theta^{(s)}_{\text{ABC},n}(0) \supset \Theta^{(s)}_{\Pi,n} \).

(1). Fix \( \theta_1 \in \Theta^{(s)}_{\text{ABC},n}(0) \). By the Assumption 12 \( \theta_1 \) is also a realization from the prior distribution \( \mathcal{P} \). By Definition 10 we have
\[ d \left( \hat{\pi}_n, \hat{\pi}^{(s)}_{\Pi,n} (\theta_1) \right) = 0. \]
By Definition 8 \( \theta_1 \in \Theta^{(s)}_{\Pi,n} \), thus \( \Theta^{(s)}_{\text{ABC},n}(0) \subset \Theta^{(s)}_{\Pi,n} \).

(2). Fix \( \theta_2 \in \Theta^{(s)}_{\Pi,n} \). By Definition 5 we have
\[ d \left( \hat{\pi}_n, \hat{\pi}^{(s)}_{\Pi,n} (\theta_2) \right) = 0. \]
By Assumption 12 and Definition 10 \( \theta_2 \in \Theta^{(s)}_{\text{ABC},n}(0) \), ergo \( \Theta^{(s)}_{\text{ABC},n}(0) \supset \Theta^{(s)}_{\Pi,n} \), which ends the proof.

**Proof of Proposition 27** Fix \( \alpha_1, \alpha_2 > 0 \) such that \( \alpha_1 + \alpha_2 = \alpha \in (0, 1) \). Since we consider an exact \( \alpha \)-credible set \( C_{\hat{\pi}_n} \), we have
\[ 1 - \alpha = \Pr \left( \theta \in C_{\hat{\pi}_n} | \hat{\pi}_n \right) \]
\[ = \Pr \left( \theta \in \Theta_n \setminus \left( Q_{\alpha_1} \cup Q_{\alpha_2} \right) \right) \]
\[ = \Pr \left( F_{\hat{\theta}_n | \hat{\pi}_n} (\theta) \in (\alpha_1, 1 - \alpha_2) \right). \]
Consider the event \( E = \{ u \in (\alpha_1, 1 - \alpha_2) \} \) taking value one with probability \( p \) if \( u \) is inside the interval and 0 otherwise. Let \( u = F_{\hat{\theta}_n | \hat{\pi}_n} (\theta_0) \) so at each trial there is one such event. Now consider indefinitely many trials, so we have \( \{ E_i : i \in \mathbb{N}^+ \} \) where \( \mathbb{E}(E_i) = \Pr (E_i = 1) = p_i \). Denote by \( N \) the number of trials. The frequentist coverage probability is given by
\[ \lim_{N \to \infty} \frac{\sum_{i=1}^{N} E_i}{N}. \]
By assumption, \( u \) is an independent standard uniform variable, so the events are independent and \( p_i = 1 - \alpha \) for all \( i \geq 1 \) and for every \( \alpha \in (0, 1) \). It follows that \( \{ E_i : i \in \mathbb{N}^+ \} \) are identically and independently distributed Bernoulli random variables. The proof follows by Borel’s strong law of large numbers (see 103).

**Proof of Lemma 22** Fix \( u_0 \). Fix \( \theta_1 \in \Theta \). By definition we have
\[ \hat{\pi}_n = \text{argzero} \right \}_{\pi \in \Pi} \Psi_n (\theta_1, u_0, \pi). \]
By assumption, the following equation
\[ \Psi_n (\theta_1, u_0, \hat{\pi}_n) = 0 \]
is uniquely defined. Now fix \( \pi_1 \in \Pi \). By definition we have
\[ \hat{\theta}_n = \text{argzero} \right \}_{\theta \in \Theta} \Psi_n (\theta, u_0, \pi_1), \]

45
and by assumption
\[ \Psi_n \left( \hat{\theta}_n, u_0, \pi_1 \right) = 0 \]
is uniquely defined. It follows that \( \theta_1 = \hat{\theta}_n \) if and only if \( \pi_1 = \hat{\pi}_n \).

**Proof of Theorem 28** We gives the demonstration under the Assumptions 26 and 27 separately.

1. We proceed by showing that we have a \( C^1 \)-diffeomorphism which is unique so Lemma 55 and Lemma 22 apply. We then demonstrate that the obtained cumulative distribution function evaluated at \( \hat{\theta}_n \in \Theta \) is a realization from a standard uniform random variable. The conclusion is eventually reached by the Proposition 21.

Let \( \pi_1 : \Theta_n \times W_n \rightarrow \Pi_n \) and \( \pi_2 : \Theta_n \times W_n \rightarrow \Pi_n \) be the projections defined by \( \pi_1(\theta, w) = \theta \) and \( \pi_2(\theta, w) = w \) if \( (\theta, w) \in \Theta_n \times W_n \). By Assumption 26 the conditions of the global implicit function theorem of [68, Theorem 1] are satisfied, so it holds that there exists a unique (global) continuous implicit function \( a : W_n \rightarrow \Theta_n \) such that \( a(w_0) = \theta_0 \) and \( \varphi_{\pi_1}(w, a(w)) = 0 \) for every \( w \in W \). In addition, the mapping is continuously differentiable on \( W_n \setminus \pi_2(K_n) \) with derivative given by

\[
D_w a = -\left[D_{\theta_1} \psi_{\theta_1}(\theta = a(w))\right]^{-1} D_w \varphi_{\theta_1}
\]

for every \( w \in W_n \setminus \pi_2(K_n) \). Clearly the map \( a \) is invertible with a continuous inverse. Since the derivative \( D_w \varphi_{\theta_1} \) is continuous and invertible for \( (\theta, w) \in \Theta_n \times W_n \), we immediately have that \( a \) is a \( C^1 \)-diffeomorphism with derivative of the inverse given by

\[
D_{\theta_1} a^{-1} = -\left[D_w \varphi_{\theta_1}(w = a^{-1}(\theta))\right]^{-1} D_{\theta} \varphi_{\theta_1}
\]

for \( \theta \in \Theta_n \setminus \pi_1(K_n) \). The conditions of Lemma 55 are satisfied and we obtain the cumulative distribution function

\[
F_{\theta_n \mid \pi_n}(\theta_0) = \int_{W_n} f_w (\theta_1 | \pi_1) \frac{\det (D_{\theta_1} \varphi_{\theta_1})}{\det (D_w \varphi_{\theta_1})} \, dw = F_{w \mid \pi_n}(w_0) = u \sim U(0, 1),
\]

proving point (i). Since \( \pi_n \) is the unique zero of \( \Psi_n(\theta_0, u_0, \pi) \), and hence of \( \varphi_{\pi_n}(w, a(w), \pi) \), and \( \theta = a(w) \) is the unique zero of \( \varphi_{\pi_n}(\theta, w, \pi_n) \), we have by Lemma 22 that \( \theta_0 = a(w_0) \), and therefore that \( w_0 = a^{-1}(\theta_0) \). In consequence, evaluating the above distribution at \( \theta_0 \) leads to

\[
F_{\theta_n \mid \pi_n}(\theta_0) = F_{w \mid \pi_n}(w_0) = u \sim U(0, 1),
\]

that is, the distribution evaluated at \( \theta_0 \) is a realization from a standard uniform random variable. The conclusion follows by the Proposition 21.

2. Fix \( \theta_0 \in \Theta_n \) and \( w_0 \in W_n \). Fix \( \pi_n \in \Pi_n \), the point such that \( \varphi_{\pi_n}(\theta_0, w_0, \pi_n) = 0 \). Let \( \pi_1 : W_n \times \Pi_n \rightarrow W_n \) and \( \pi_2 : W_n \times \Pi_n \rightarrow \Pi_n \) be the projections such that \( \pi_1(w, \pi) = w \) and \( \pi_2(w, \pi) = \pi \) if \( (w, \pi) \in W_n \times \Pi_n \). By Assumption 27(i), (ii), (iii), (iv), the Theorem 1 in [68] is satisfied, as a consequence it holds that \( \varphi_{\theta_0} \) admits a unique global implicit function \( \pi_{\theta_0} : W_n \rightarrow \Pi_n \) such that \( \varphi_{\pi_1}(w, \pi_{\theta_0}(w)) = 0 \) for every \( w \in W_n \), \( \pi_{\theta_0}(w_0) = \pi_n \), and \( \pi_{\theta_0} \) is once continuously differentiable on \( W_n \setminus \pi_2(K_{1n}) \) with derivative given by

\[
D_w \pi_{\theta_0} = -[D_{\pi_1} \varphi_{\theta_0}]^{-1} D_w \varphi_{\theta_0}.
\]

Clearly \( w \mapsto \pi_{\theta_0} \) is a homeomorphism. Since \( D_w \varphi_{\theta_0} \) is continuous and invertible on \( W_n \times \Pi \setminus K_{1n} \), we have that \( \pi_{\theta_0} \) is a \( C^1 \)-diffeomorphism with differentiable inverse function on \( \Pi \setminus \pi_2(K_{1n}) \) given by Lemma 54.

\[
D_{\theta_1} \pi_{\theta_0} = -[D_{w} \varphi_{\theta_0}]^{-1} D_{\theta_1} \varphi_{\theta_0}.
\]

Let \( \pi_3 : \Theta_n \times \Pi_n \rightarrow \Theta_n \) and \( \pi_4 : \Theta_n \times \Pi_n \rightarrow \Pi_n \) denote the projections such that \( \pi_3(\theta, \pi) = \theta \) and \( \pi_4(\theta, \pi) = \pi \). By using the same argument presented above, the Assumption 27(iii), (iv), (vi) permits us to have an implicit \( C^1 \)-diffeomorphism \( \pi_{w_0} : \Theta_n \rightarrow \Pi_n \) with the following continuous derivatives:

\[
D_{\theta} \pi_{w_0} = -[D_{\pi_1} \varphi_{w_0}]^{-1} D_{\theta_1} \varphi_{w_0}, \quad \theta \in \Theta \setminus \pi_3(K_{2n}),
\]

\[
D_{\pi} \pi_{w_0} = -[D_{w} \varphi_{w_0}]^{-1} D_{\pi_1} \varphi_{w_0}, \quad \pi \in \Pi \setminus \pi_4(K_{2n}).
\]

Now define the function \( \xi(\theta) = \pi_{\theta_0}^{-1} \circ \pi_{w_0}(\theta) \). It is trivial to show that this mapping \( \theta \mapsto \xi \) is a \( C^1 \)-diffeomorphism. We have from the preceding results and the chain rule that

\[
D_{\theta} \xi = [D_{w_0} \varphi_{\theta_0}]^{-1} D_{\theta_1} \varphi_{\theta_0} [D_{\pi_1} \varphi_{w_0}]^{-1} D_{\pi_1} \varphi_{w_0}.
\]
We make the following remarks. First, note that all these derivatives are square matrices of dimension $p \times p$. Second, we have that $D_{\varphi} \theta_0 (w_0, \pi_n) = D_{\varphi_0} (\theta_0, w_0, \pi_n) = D_{\varphi} \theta_0 (\theta_0, w_0, \pi_n)$ so $D_{\varphi} \theta_0 [D_{\varphi} \theta_0]^{-1} = I_p$. Third, it holds that $D_{w \varphi} \theta_0 (w_0, \pi_n) = D_{w \varphi_0} (\theta_0, w_0)$ and $D_{\varphi_0} \theta_0 (\theta_0, \pi_n) = D_{\varphi_0} (\theta_0, \pi_n)$. As a consequence, we obtain that

$$\det (D_\theta \xi) = \frac{\det (D_{\varphi_0} (\theta_0, \pi_n))}{\det (D_{w \varphi_0} (\theta_0, \pi_n))}.$$ 

Using Lemma 55 ends the proof of point (i) in Theorem 28. From the above display, we have that the relation $\pi_0 (w_0) = \pi_n = \pi_{w_0} (\theta_0)$ is uniquely defined, so $\xi (\theta_0) = \pi_{\theta_0}^{\pi_0} (\pi_n) = w_0$. Since $\xi$ is a diffeomorphism, then $\xi^{-1} (w_0) = \theta_0$, which finishes the proof.

**Proof of Proposition 30** This is a special case of the Theorem 28. Let define $\varphi_\pi_n (w, \theta) = h(x_0) - g (\theta, w)$, where $h(x_0) = \pi_n$ is fixed. Following the proof of Theorem 28 we have by assumption that $a : W_n \rightarrow \Theta_n$ is a $C^1$-diffeomorphism with derivatives

$$D_{w} a = -\left[ D_{\theta} g \big|_{\theta = a(w)} \right]^{-1} D_{w} g, \quad w \in W_n \setminus \pi_2 (K_n),$$

$$D_{\theta} a = -\left[ D_{w} g \big|_{w = a^{-1} (\theta)} \right]^{-1} D_{\theta} g, \quad \theta \in \Theta_n \setminus \pi_1 (K_n).$$

The rest of the proof is identical to the proof of Theorem 28.

**C Asymptotics**

**Proof of Theorem 34** We start by showing the claim 1: the pointwise convergence of $\hat{\pi}_n$. Then we demonstrate the claim 2 with two different approaches corresponding respectively to the Assumptions 32 and 33.

1. Fix $\pi_0 \in \Pi$. Since $\{ \Psi_n (\theta, u, \pi) \}$ is stochastically Lipschitz in $\pi$, it is stochastically equicontinuous by the Lemma 59. In addition, $\Pi$ is compact and $\{ \Psi_n \}$ is pointwise convergent by assumption, so by the Lemma 58 $\{ \Psi_n \}$ converges uniformly and the limit $\Psi$ is uniformly continuous. By $\Pi$ compact and the continuity of the norm, the infimum of the norm of $\Psi$ exists. The infimum of $\Psi$ is well-separated by the bijectivity of the function. Therefore, all the conditions of Lemma 56 are satisfied and $\{ \pi_n \}$ converges pointwise to $\pi_0$.

2 (i). For this proof, we consider $\theta$ and $\pi$ jointly. Let $\mathcal{K} = \Theta \cap \Pi$ be the set for both $\theta$ and $\pi$. Fix $(\theta_0, \pi_0) \in \mathcal{K}$. Since $\Pi \subset \mathbb{R}^p$ and $\Theta \subset \mathbb{R}^p$ are compact subsets of a metric space, they are closed (see the Theorem 2.34 in [104]), and $\mathcal{K}$ is compact (see the Corollary to the Theorem 2.35 in [104]) and nonempty (Theorem 2.36 in [104]). Having $\mathcal{K}$ compact, it is now sufficient to show that $\{ \Psi_n \}$ is jointly stochastically Lipschitz as the rest of the proof follows exactly the same steps as the claim 1.

For every $(\theta_1, \pi_1), (\theta_2, \pi_2) \in \mathcal{K}$, $n$, and $u \sim F_u$, we have by the triangle inequality that

$$\| \Psi_n (\theta_1, u, \pi_1) - \Psi_n (\theta_2, u, \pi_2) \| = \| \Psi_n (\theta_1, u, \pi_1) - \Psi_n (\theta_1, u, \pi_2) \| + \| \Psi_n (\theta_1, u, \pi_1) - \Psi_n (\theta_2, u, \pi_2) \|$$

$$\leq \| \Psi_n (\theta_1, u, \pi_1) - \Psi_n (\theta_1, u, \pi_2) \| + \| \Psi_n (\theta_1, u, \pi_2) - \Psi_n (\theta_2, u, \pi_2) \|$$

$$\leq D_n (\| \pi_1 - \pi_2 \| + \| \theta_1 - \theta_2 \|),$$

where for the last inequality we make use of the marginal stochastic Lipschitz assumptions and $D_n = \max (A_n, B_n)$. Let $a = \| \theta_1 - \theta_2 \|$ and $b = \| \pi_1 - \pi_2 \|$. Now remark that for the $\ell_2$-norm we have

$$\left\| \begin{pmatrix} \theta_1 \\ \pi_1 \end{pmatrix} - \begin{pmatrix} \theta_2 \\ \pi_2 \end{pmatrix} \right\| = \sqrt{a^2 + b^2}.$$ 

Since $a, b$ are positive real numbers, a direct application of the inequality of arithmetic and geometric means gives

$$\sqrt{2 \sqrt{a^2 + b^2}} \geq a + b.$$ 

Therefore, we have that

$$D_n (\| \pi_1 - \pi_2 \| + \| \theta_1 - \theta_2 \|) \leq D_n^* \left\| \begin{pmatrix} \theta_1 \\ \pi_1 \end{pmatrix} - \begin{pmatrix} \theta_2 \\ \pi_2 \end{pmatrix} \right\|,$$

where $D_n^* = \sqrt{2} D_n$. Consequently, $\{ \Psi_n \}$ is jointly stochastically Lipschitz, and following the proof of claim 1 we have that $\theta_n \overset{p}{\rightarrow} \theta_0$. More precisely, we even have that $(\theta_n, \pi_n) \overset{p}{\rightarrow} (\theta_0, \pi_0)$. 

47
2 (ii). This proof is different from 2 (i) since \( \hat{\pi}_{\Pi,n} \) is considered as a function of \( \theta \). Fix \( \pi_0 \in \Pi \). Since \( \{\hat{\pi}_{\Pi,n}\} \) is stochastically Lipschitz in \( \theta \), it is stochastically equicontinuous by the Lemma 59. In addition, \( \Theta \) is compact and \( \{\hat{\pi}_{\Pi,n}\} \) is pointwise convergent by the claim 1, so by the Lemma 58 \( \{\hat{\pi}_{\Pi,n}\} \) converges uniformly and the limit \( \pi \) is uniformly continuous in \( \theta \). Let the stochastic and deterministic objective functions be \( Q_n(\theta) = ||\hat{\pi}_n - \hat{\pi}_{\Pi,n}(\theta)|| \) and \( Q(\theta) = ||\pi_0 - \pi(\theta)|| \), for any norms. Now, we have by using successively the reverse and the regular triangle inequalities

\[
|Q_n(\theta) - Q(\theta)| = ||\hat{\pi}_n - \hat{\pi}_{\Pi,n}(\theta)|| - ||\pi_0 - \pi(\theta)|| \\
\leq ||\hat{\pi}_n - \hat{\pi}_{\Pi,n}(\theta) - \pi_0 + \pi(\theta)|| \\
\leq ||\pi_n - \pi_0|| + ||\pi(\theta) - \pi_{\Pi,n}(\theta)||.
\]

By the convergence of \( \{\pi_n\} \) and the uniform convergence of \( \{\hat{\pi}_{\Pi,n}\} \), we have

\[
\lim_{n \to \infty} \Pr \left( \sup_{\theta \in \Theta} |Q_n(\theta) - Q(\theta)| \right) = o_p(1).
\]

By \( \Pi \) compact and the continuity of the norm, the infimum of the norm of \( \Psi \) exists. The infimum of \( \Psi \) is well-separated by the bijectivity of the function. Therefore, all the conditions of Lemma 56 are satisfied and \( \{\pi_n\} \) converges pointwise to \( \pi_0 \).

**Proof of Theorem 38** We first demonstrate the asymptotic distribution of the auxiliary estimator, then separately shows the result for \( \hat{\theta}_n \) using independently the Assumption 36 and 37.

1. The result on \( \hat{\pi}_n \) is a special case of \( \hat{\pi}_{\Pi,n}(\theta) \). Fix \( \theta_0 \in \Theta^o \) and denote \( \pi(\theta_0) \equiv \pi_0 \). By assumptions, the conditions for the delta method in Lemma 63 are satisfied so we have

\[
\Psi_n(\theta_0, u_s, \hat{\pi}_{\Pi,n}(\theta_0)) - \Psi_n(\theta_0, u_s, \pi_0) = D_\pi \Psi_n(\theta_0, u_s, \pi_0) \cdot (\hat{\pi}_{\Pi,n}(\theta_0) - \pi_0) + o_p(||\hat{\pi}_{\Pi,n}(\theta_0) - \pi_0||). \tag{9}
\]

By the Definition 3, we have \( \Psi_n(\theta_0, u_s, \hat{\pi}_{\Pi,n}(\theta_0)) = 0 \). By the Theorem 34, \( o_p(||\hat{\pi}_{\Pi,n}(\theta_0) - \pi_0||) = o_p(1) \). By assumptions, \( D_\pi \Psi_n(\theta_0, u_s, \pi_0) \) is nonsingular. Multiplying by square-root \( n \), the proof results from the central limit theorem assumption on \( \Psi_n \) and the Slutsky’s lemma.

2 (i). From the delta method in Lemma 63 we obtain

\[
\Psi_n(\hat{\theta}_n, u_s, \hat{\pi}_n) - \Psi_n(\theta_0, u_s, \hat{\pi}_n) = D_\theta \Psi_n(\theta_0, u_s, \hat{\pi}_n) \cdot (\hat{\theta}_n - \theta_0) + o_p(||\hat{\theta}_n - \theta_0||).
\]

By definition we have \( \Psi_n(\hat{\theta}_n, u_s, \hat{\pi}_n) = 0 \). Using again the delta method on the non-zero left-hand side element, we obtain from (9)

\[
0 = [\Psi_n(\theta_0, u_s, \pi_0) + D_\pi \Psi_n(\theta_0, u_s, \pi_0) \cdot (\hat{\pi}_n - \pi_0) + o_p(||\hat{\pi}_n - \pi_0||)]
\]

\[
= D_\theta \Psi_n(\theta_0, u_s, \hat{\pi}_n) \cdot (\hat{\theta}_n - \theta_0) + o_p(||\hat{\theta}_n - \theta_0||).
\]

Since \( \{D_\theta \Psi_n(\theta_0, u_s, \pi)\} \) is stochastically Lipschitz in \( \pi \), it is stochastically equicontinuous by the Lemma 59. In addition, \( \Pi \) is compact and \( \{\psi_n\} \) is pointwise convergent by assumption, so by the Lemma 58 \( \{D_\theta \Psi_n\} \) converges uniformly and the limit \( J \) is uniformly continuous in \( \pi \).

Next, we obtain the following

\[
||D_\theta \Psi_n(\hat{\pi}_n) - J(\pi_0)|| \leq \sup_{\pi \in \Pi} ||D_\theta \Psi_n(\pi) - J(\pi)|| + ||J(\hat{\pi}_n) - J(\pi_0)||
\]

\[
\leq \sup_{\pi \in \Pi} ||D_\theta \Psi_n(\pi) - J(\pi)|| + ||J(\hat{\pi}_n) - J(\pi_0)||.
\]

By uniform convergence \( \sup_{\pi \in \Pi} ||D_\theta \Psi_n(\pi) - J(\pi)|| = o_p(1) \) and by the continuous mapping theorem \( ||J(\hat{\pi}_n) - J(\pi_0)|| = o_p(1) \).

The central limit theorem is satisfied for the estimating equation thus \( n^{1/2} \Psi_n \to c N(0, Q) \). Let \( y \) be a random variable identically and independently distributed according to \( N(0, Q) \). Therefore, by multiplying by square-root \( n \) we obtain

\[
-\frac{y - K^{1/2} (\pi_n - \pi_0)}{n^{1/2}} = J n^{1/2} (\hat{\theta}_n - \theta_0) + o_p(||\hat{\theta}_n - \theta_0||).
\]

By the Theorem 34 we have \( o_p(||\pi_n - \pi_0||) = o_p(1) \) and \( o_p(||\hat{\theta}_n - \theta_0||) = o_p(1) \). By the result of the claim 1 and the nonsingularity of \( J \), we have

\[
n^{1/2} (\hat{\theta}_n - \theta_0) = -J^{-1} (y + K \cdot K^{-1} y + o_p(1)) + o_p(1).
\]
Slutsky’s lemma ends the proof.

2 (ii). Let $g_n(\theta) = \hat{\pi}_n - \pi_{I,n}(\theta)$. The conditions for the delta method in Lemma 63 are satisfied by assumption so we have

$$g_n(\hat{\theta}_n) - g_n(\theta_0) = D_\theta g_n(\theta_0) \cdot (\hat{\theta}_n - \theta_0) + o_p\left(\|\theta_n - \theta_0\|\right).$$

(10)

Since $\hat{\theta}_n = \arg\min_{\theta} \ell(\theta_n, \hat{\theta}_{I,n}(\theta))$, we have $\hat{\theta}_n - \hat{\theta}_{I,n}(\theta_n) = 0$ and thus $g_n(\hat{\theta}_n) = 0$. By the Theorem 34 we have $o_p\left(\|\theta_n - \theta_0\|\right) = o_p(1)$. We have $D_\theta g_n(\theta_0) = -D_\theta \pi_{I,n}(\theta_0)$ which, by assumption converges pointwise to $D_\theta \pi(\theta_0)$. By the claim 1, we have $n^{1/2}(\hat{\pi}_n - \pi_0) \overset{d}{=} n^{1/2}(\pi_{I,n}(\theta) - \pi_0) \overset{d}{=} K^{-1}y$ as $n \to \infty$. Hence, multiplying the Equation (10) by square-root $n$, gives the following

$$K^{-1}(y - y) = D_\theta \pi(\theta_0) \cdot n^{1/2}(\hat{\theta}_n - \theta_0) + o_p(1),$$

for sufficiently large $n$. Remark that the mapping $\theta \mapsto \pi$ is implicitly defined by

$$\Psi(\theta, \pi(\theta)) = 0.$$

Since $\Psi$ is once continuously differentiable in $(\theta, \pi)$ and the partial derivatives are invertible, the conditions for invoking an implicit function theorem are satisfied (see for example the Theorem 9.28 in [104]) and one of the conclusion is that

$$D_\theta \pi(\theta_0) = -K^{-1}J.$$

Since $J$ is invertible, the conclusion follows by Slutsky’s lemma.

Proof of Proposition 32. The proof follows essentially the same steps as the proof of Theorem 38. From the proof of Theorem 38 the following holds: $n^{1/2}(\hat{\pi}_n - \pi_0) \overset{d}{=} K^{-1}y_0$ and $n^{1/2}(\pi_{I,n}(\theta_0, u_s, \pi_0) \overset{d}{=} y_s$ as $n \to \infty$ where $y_j \sim N(0, Q), j \in \mathbb{N}^+$. $D_\pi \Psi_n(\theta_0, u_s, \pi_0)$ converges in probability to $K$ and $D_\theta \Psi_n(\theta_0, u_s, \pi)$ converges uniformly in probability to $J$. The $\{u_j : j \in \mathbb{N}\}$ are assumed independent and so are $\{y_j : j \in \mathbb{N}\}$.

From the delta method in Lemma 63 we obtain

$$\frac{1}{S} \sum_{s \in N_S^\pi} \Psi_n(\hat{\theta}_n(s), u_s, \pi_n) - \frac{1}{S} \sum_{s \in N_S^\pi} \Psi_n(\theta_0, u_s, \pi_n) = \frac{1}{S} \sum_{s \in N_S^\pi} D_\theta \Psi_n(\theta_0, u_s, \pi_n) \cdot (\hat{\theta}_n(s) - \theta_0) + o_p(1).$$

By definition $\frac{1}{S} \sum_{s \in N_S^\pi} \Psi_n(\hat{\theta}_n(s), u_s, \pi_n) = 0$. Using the delta method on $\frac{1}{S} \sum_{s \in N_S^\pi} \Psi_n(\theta_0, u_s, \pi_n)$, multiplying by square-root $n$, we obtain from the results of Theorem 38

$$\frac{1}{S} \sum_{s \in N_S^\pi} y_s - KK^{-1}y_0 - o_p(1) = Jn^{1/2}(\hat{\theta}_n - \theta_0) + o_p(1).$$

Clearly $\frac{1}{S} \sum_{s \in N_S^\pi} y_s \sim N(0, \frac{1}{S}Q)$. The conclusion follows from Slutsky’s lemma.
D Additional simulation results

D.1 Lomax distribution

|       | SwiZs | Boot  | AB   | RSwiZs | RBoot |
|-------|-------|-------|------|--------|-------|
| $n = 35$ | 0.1430 | 0.0222 | 0.0197 | 0.5613 | 0.0998 |
| $n = 50$  | 0.2002 | 0.0293 | 0.0268 | 0.7889 | 0.1320 |
| $n = 100$ | 0.3826 | 0.0526 | 0.0504 | 1.3520 | 0.2314 |
| $n = 150$ | 0.5580 | 0.0753 | 0.0736 | 1.7792 | 0.3291 |
| $n = 250$ | 0.8998 | 0.1228 | 0.1211 | 2.3141 | 0.5174 |
| $n = 500$ | 1.7763 | 0.2364 | 0.2398 | 3.2132 | 0.9848 |

Table 5: Average computationnal time in seconds to approximate a distribution on $S = 10,000$ points.
| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 49.48 | 50.07 | 43.10 | 44.26 | 0.00 | 0.00 | 42.73 | 44.07 | 36.72 | 36.84 |
| 75% | 74.49 | 75.14 | 65.82 | 65.39 | 0.00 | 0.00 | 65.84 | 66.59 | 55.00 | 55.06 |
| 90% | 89.31 | 89.39 | 80.64 | 78.74 | 0.00 | 0.00 | 81.41 | 81.97 | 64.47 | 64.26 |
| 95% | 94.27 | 94.34 | 86.71 | 84.28 | 0.03 | 0.00 | 87.58 | 87.41 | 67.33 | 67.13 |
| 99% | 98.26 | 98.43 | 91.23 | 91.07 | 0.75 | 0.00 | 93.84 | 93.53 | 69.64 | 70.39 |

$n = 50$

| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 49.59 | 49.88 | 44.48 | 45.30 | 0.01 | 0.00 | 45.70 | 46.93 | 37.37 | 37.64 |
| 75% | 74.73 | 76.67 | 68.43 | 67.84 | 0.08 | 0.00 | 67.40 | 68.21 | 57.44 | 56.73 |
| 90% | 89.89 | 90.62 | 83.15 | 81.57 | 0.76 | 0.00 | 82.51 | 82.75 | 69.52 | 68.81 |
| 95% | 94.67 | 94.94 | 89.26 | 87.11 | 1.92 | 0.00 | 88.47 | 88.35 | 73.01 | 72.49 |
| 99% | 98.40 | 98.46 | 95.19 | 93.69 | 10.86 | 0.00 | 94.79 | 94.80 | 75.97 | 76.43 |

$n = 100$

| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 50.12 | 49.80 | 48.36 | 48.58 | 47.05 | 47.78 | 49.80 | 49.82 | 33.94 | 33.00 |
| 75% | 75.37 | 75.88 | 72.00 | 71.59 | 44.13 | 57.82 | 73.07 | 74.32 | 57.01 | 55.61 |
| 90% | 90.20 | 90.42 | 86.69 | 85.86 | 69.68 | 81.85 | 86.54 | 86.83 | 73.68 | 71.96 |
| 95% | 95.41 | 95.67 | 92.06 | 90.96 | 81.89 | 91.13 | 91.69 | 91.52 | 80.75 | 79.17 |
| 99% | 99.08 | 99.10 | 97.92 | 97.43 | 97.34 | 98.72 | 99.28 | 97.81 | 97.69 | 90.07 | 88.56 |

$n = 150$

| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 49.46 | 49.84 | 48.60 | 49.01 | 47.61 | 47.09 | 49.55 | 49.90 | 29.16 | 28.45 |
| 75% | 75.02 | 74.49 | 73.59 | 72.75 | 72.09 | 72.63 | 74.83 | 74.80 | 49.94 | 47.56 |
| 90% | 95.08 | 95.35 | 93.03 | 92.11 | 93.26 | 94.89 | 93.60 | 93.74 | 80.17 | 78.15 |
| 99% | 99.08 | 99.10 | 97.92 | 97.43 | 99.18 | 99.50 | 98.61 | 98.70 | 89.37 | 87.24 |

$n = 250$

| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 50.08 | 49.89 | 49.29 | 49.81 | 48.76 | 48.67 | 50.26 | 49.64 | 20.51 | 18.95 |
| 75% | 74.73 | 74.36 | 73.90 | 73.64 | 73.68 | 73.85 | 74.55 | 74.68 | 37.76 | 34.96 |
| 90% | 89.53 | 89.75 | 88.86 | 88.69 | 89.03 | 89.22 | 89.45 | 89.80 | 56.15 | 52.68 |
| 95% | 94.92 | 94.86 | 94.11 | 94.22 | 94.33 | 94.77 | 94.92 | 94.80 | 66.89 | 63.51 |
| 99% | 98.97 | 98.99 | 98.62 | 98.40 | 99.01 | 99.07 | 98.94 | 99.03 | 83.63 | 80.06 |

Table 6: Estimated coverage probabilities.
| $\alpha$ | SwiZs | Boot | BA | RSwiZs | RBoot |
|---|---|---|---|---|---|
| $n = 35$ | | | | | |
| 50% | 50.22 | 44.26 | 0.02 | 44.27 | 36.84 |
| 75% | 76.03 | 65.44 | 0.72 | 67.12 | 55.06 |
| 90% | 91.07 | 78.96 | 68.11 | 83.07 | 64.36 |
| 95% | 96.76 | 84.35 | 100.00 | 89.43 | 67.19 |
| 99% | 98.84 | 91.10 | 100.00 | 93.88 | 70.41 |
| $n = 50$ | | | | | |
| 50% | 49.89 | 45.30 | 0.00 | 46.94 | 37.64 |
| 75% | 76.86 | 67.84 | 0.00 | 68.26 | 56.73 |
| 90% | 90.83 | 81.58 | 41.20 | 82.68 | 68.82 |
| 95% | 95.17 | 87.16 | 71.42 | 88.40 | 72.49 |
| 99% | 98.92 | 93.76 | 99.82 | 95.14 | 76.45 |
| $n = 100$ | | | | | |
| 50% | 49.95 | 48.04 | 32.96 | 49.80 | 35.48 |
| 75% | 75.88 | 71.59 | 59.90 | 74.32 | 55.61 |
| 90% | 90.42 | 85.86 | 82.63 | 86.83 | 71.96 |
| 95% | 95.74 | 90.98 | 91.44 | 91.64 | 79.19 |
| 99% | 98.85 | 96.46 | 98.73 | 96.83 | 86.43 |
| $n = 150$ | | | | | |
| 50% | 49.80 | 48.58 | 46.30 | 49.82 | 33.00 |
| 75% | 75.32 | 72.63 | 72.68 | 74.69 | 53.45 |
| 90% | 90.32 | 86.85 | 89.18 | 89.22 | 70.01 |
| 95% | 95.35 | 92.12 | 94.87 | 93.73 | 78.15 |
| 99% | 99.06 | 97.47 | 99.27 | 97.71 | 88.60 |
| $n = 250$ | | | | | |
| 50% | 49.84 | 49.01 | 46.99 | 49.90 | 28.45 |
| 75% | 74.49 | 72.75 | 72.41 | 74.80 | 47.56 |
| 90% | 89.81 | 88.11 | 88.95 | 89.58 | 65.25 |
| 95% | 94.81 | 93.34 | 94.99 | 94.69 | 74.43 |
| 99% | 99.04 | 97.93 | 99.48 | 98.68 | 87.34 |
| $n = 500$ | | | | | |
| 50% | 49.89 | 49.81 | 48.67 | 49.64 | 18.95 |
| 75% | 74.36 | 73.64 | 73.85 | 74.68 | 34.96 |
| 90% | 89.75 | 88.69 | 89.22 | 89.80 | 52.68 |
| 95% | 94.86 | 94.22 | 94.77 | 94.79 | 63.57 |
| 99% | 98.98 | 98.41 | 99.03 | 99.02 | 80.28 |

Table 7: Estimated coverage probabilities of Gini index.
| α   | SwiZs | Boot | BA | RSwiZs | RBoot | 95% value-at-risk |
|-----|-------|------|----|--------|--------|------------------|
|     |       |      |    |        |        |                  |
|     |   n = 35 |      |    |        |        |                  |
| 50% | 47.30 | 46.08 | 20.92 | 45.34 | 41.13  |                  |
| 75% | 73.76 | 67.53 | 55.77 | 70.38 | 61.00  |                  |
| 90% | 90.05 | 80.35 | 93.73 | 88.08 | 73.92  |                  |
| 95% | 95.67 | 85.36 | 98.92 | 94.80 | 79.41  |                  |
| 99% | 99.17 | 91.63 | 99.97 | 99.25 | 87.26  |                  |
|     |   n = 50 |      |    |        |        |                  |
| 50% | 48.14 | 47.23 | 31.76 | 46.40 | 41.27  |                  |
| 75% | 73.39 | 69.40 | 63.30 | 70.22 | 61.47  |                  |
| 90% | 89.63 | 82.24 | 91.60 | 87.07 | 74.72  |                  |
| 95% | 94.89 | 87.41 | 97.72 | 93.60 | 80.20  |                  |
| 99% | 99.23 | 93.17 | 99.90 | 99.27 | 87.87  |                  |
|     |   n = 100 |     |   |        |        |                  |
| 50% | 49.75 | 48.90 | 48.33 | 49.18 | 39.94  |                  |
| 75% | 74.68 | 72.61 | 75.68 | 72.93 | 61.39  |                  |
| 90% | 89.48 | 86.38 | 91.97 | 87.16 | 75.97  |                  |
| 95% | 95.07 | 91.17 | 96.79 | 94.17 | 82.45  |                  |
| 99% | 99.23 | 96.31 | 99.75 | 99.11 | 90.45  |                  |
|     |   n = 150 |     |   |        |        |                  |
| 50% | 50.10 | 49.19 | 49.47 | 49.91 | 37.43  |                  |
| 75% | 74.13 | 73.17 | 75.42 | 73.57 | 59.31  |                  |
| 90% | 89.77 | 87.25 | 91.21 | 88.49 | 75.26  |                  |
| 95% | 94.76 | 92.57 | 96.18 | 93.31 | 81.76  |                  |
| 99% | 98.89 | 97.34 | 99.61 | 98.46 | 91.00  |                  |
|     |   n = 250 |     |   |        |        |                  |
| 50% | 50.28 | 49.52 | 50.02 | 50.24 | 34.09  |                  |
| 75% | 75.29 | 74.25 | 74.87 | 74.75 | 55.55  |                  |
| 90% | 89.43 | 88.10 | 90.27 | 89.13 | 72.28  |                  |
| 95% | 94.66 | 93.26 | 95.15 | 94.14 | 80.35  |                  |
| 99% | 98.89 | 97.85 | 99.10 | 98.67 | 90.11  |                  |
|     |   n = 500 |     |   |        |        |                  |
| 50% | 49.15 | 48.63 | 49.00 | 49.22 | 27.45  |                  |
| 75% | 74.88 | 74.01 | 74.63 | 74.53 | 45.61  |                  |
| 90% | 90.02 | 89.46 | 90.37 | 89.93 | 62.84  |                  |
| 95% | 94.47 | 94.45 | 95.18 | 94.85 | 72.65  |                  |
| 99% | 98.92 | 98.32 | 98.87 | 98.96 | 86.63  |                  |

Table 8: Estimated coverage probabilities of value-at-risk at 95%.
| $\alpha$ | SwiZs | Boot | BA | RSwiZs | RBoot |
|----------|-------|------|----|--------|-------|
| 50%      | 50.33 | 48.55 | 0.02 | 50.08  | 47.38 |
| 75%      | 74.97 | 72.60 | 0.72 | 74.70  | 71.28 |
| 90%      | 89.61 | 87.63 | 68.11 | 89.24  | 86.35 |
| 95%      | 94.65 | 92.87 | 100.00 | 94.37  | 92.23 |
| 99%      | 98.80 | 97.97 | 100.00 | 98.72  | 97.48 |
| $n = 50$ |       |      |     |        |       |
| 50%      | 49.48 | 48.24 | 0.00 | 49.28  | 47.06 |
| 75%      | 74.81 | 72.74 | 0.00 | 74.45  | 71.28 |
| 90%      | 89.76 | 88.07 | 41.20 | 89.25  | 86.85 |
| 95%      | 94.74 | 93.32 | 71.42 | 94.48  | 92.16 |
| 99%      | 98.89 | 97.92 | 99.82 | 98.62  | 97.48 |
| $n = 100$|       |      |     |        |       |
| 50%      | 49.94 | 49.16 | 32.96 | 49.64  | 47.22 |
| 75%      | 74.47 | 74.12 | 59.90 | 74.37  | 72.21 |
| 90%      | 90.13 | 89.15 | 82.63 | 89.99  | 87.57 |
| 95%      | 95.10 | 94.23 | 91.44 | 95.00  | 93.13 |
| 99%      | 98.98 | 98.55 | 98.73 | 98.91  | 98.10 |
| $n = 150$|       |      |     |        |       |
| 50%      | 49.91 | 49.49 | 46.30 | 49.81  | 48.13 |
| 75%      | 74.57 | 74.25 | 72.68 | 74.95  | 72.45 |
| 90%      | 90.13 | 89.31 | 89.18 | 89.74  | 87.76 |
| 95%      | 95.05 | 94.37 | 94.87 | 94.98  | 93.15 |
| 99%      | 98.91 | 98.62 | 99.27 | 98.86  | 98.14 |
| $n = 250$|       |      |     |        |       |
| 50%      | 50.53 | 50.64 | 46.99 | 50.44  | 47.94 |
| 75%      | 75.01 | 74.97 | 72.41 | 74.91  | 72.31 |
| 90%      | 89.96 | 89.72 | 88.95 | 89.98  | 87.75 |
| 95%      | 95.11 | 94.58 | 94.99 | 95.13  | 93.16 |
| 99%      | 99.04 | 98.70 | 99.48 | 99.06  | 98.14 |
| $n = 500$|       |      |     |        |       |
| 50%      | 49.25 | 49.34 | 48.67 | 49.48  | 46.61 |
| 75%      | 74.50 | 74.29 | 73.85 | 74.28  | 70.91 |
| 90%      | 90.02 | 89.56 | 89.22 | 89.99  | 86.47 |
| 95%      | 95.05 | 94.77 | 94.77 | 95.13  | 92.52 |
| 99%      | 99.01 | 99.01 | 99.03 | 99.04  | 98.23 |

Table 9: Estimated coverage probabilities of expected shortfall at 95%.
| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50% | 2.19 | 1.85 | 7.52 | 6.04 | 0.26 | 0.34 | 1.89 | 1.64 | 7.97 | 6.73 |
| 75% | 4.79 | 3.92 | 179.86 | 150.90 | 0.46 | 0.54 | 3.84 | 3.37 | 27.20 | 23.62 |
| 90% | 11.18 | 8.56 | 8673.53 | 7420.31 | 1.31 | 1.09 | 6.96 | 5.97 | 86.42 | 75.85 |
| 95% | 24.30 | 18.00 | $2.55 \times 10^4$ | $2.18 \times 10^4$ | 8.99 | 8.89 | 9.89 | 9.72 | 161.13 | 142.10 |
| 99% | 2488.08 | 1849.66 | $1.19 \times 10^5$ | $1.05 \times 10^5$ | $3.18 \times 10^9$ | $3.30 \times 10^9$ | 22.62 | 17.10 | 435.99 | 401.28 |
| $n = 35$ | | | | | | | | | | | | |
| 50% | 1.78 | 1.51 | 3.61 | 2.98 | 0.39 | 0.42 | 1.56 | 1.34 | 5.04 | 4.20 |
| 75% | 3.60 | 2.97 | 10.55 | 8.78 | 0.66 | 0.68 | 3.11 | 2.65 | 14.89 | 12.37 |
| 90% | 6.78 | 5.41 | 551.95 | 494.65 | 1.22 | 0.94 | 5.57 | 4.83 | 44.67 | 38.37 |
| 95% | 10.78 | 8.38 | $7.40 \times 10^3$ | $6.31 \times 10^3$ | 6.13 | 5.27 | 7.80 | 6.70 | 84.42 | 73.24 |
| 99% | 54.20 | 39.06 | $5.57 \times 10^4$ | $4.82 \times 10^4$ | $1.09 \times 10^7$ | $1.04 \times 10^7$ | 15.60 | 12.65 | 231.61 | 202.96 |
| $n = 50$ | | | | | | | | | | | | |
| 50% | 1.26 | 1.06 | 1.69 | 1.39 | 0.64 | 0.60 | 1.19 | 1.01 | 2.73 | 2.27 |
| 75% | 2.32 | 1.92 | 3.32 | 2.74 | 1.08 | 1.02 | 2.23 | 1.87 | 6.01 | 5.01 |
| 90% | 3.74 | 3.04 | 6.28 | 5.20 | 1.55 | 1.36 | 3.67 | 3.03 | 13.00 | 10.88 |
| 95% | 4.92 | 3.94 | 10.30 | 8.58 | 1.93 | 1.54 | 4.89 | 4.00 | 22.10 | 18.69 |
| 99% | 8.58 | 6.63 | 181.34 | 153.63 | 20.11 | 16.79 | 8.41 | 6.95 | 64.18 | 55.35 |
| $n = 100$ | | | | | | | | | | | | |
| 50% | 1.02 | 0.86 | 1.21 | 1.01 | 0.71 | 0.62 | 1.00 | 0.85 | 2.02 | 1.68 |
| 75% | 1.82 | 1.52 | 2.24 | 1.88 | 1.23 | 1.08 | 1.80 | 1.52 | 4.00 | 3.35 |
| 90% | 2.78 | 2.30 | 3.71 | 3.11 | 1.78 | 1.59 | 2.80 | 2.32 | 7.50 | 6.28 |
| 95% | 3.52 | 2.89 | 5.05 | 4.26 | 2.12 | 1.90 | 3.58 | 2.95 | 11.12 | 9.34 |
| 99% | 5.38 | 4.35 | 10.39 | 8.97 | 2.86 | 2.27 | 5.62 | 4.52 | 26.58 | 22.47 |
| $n = 150$ | | | | | | | | | | | | |
| 50% | 0.78 | 0.66 | 0.85 | 0.72 | 0.64 | 0.55 | 0.79 | 0.66 | 1.45 | 1.21 |
| 75% | 1.36 | 1.15 | 1.52 | 1.29 | 1.13 | 0.96 | 1.38 | 1.16 | 2.68 | 2.24 |
| 90% | 2.01 | 1.69 | 2.34 | 1.99 | 1.68 | 1.44 | 2.07 | 1.72 | 4.41 | 3.68 |
| 95% | 2.48 | 2.08 | 2.97 | 2.52 | 2.07 | 1.78 | 2.56 | 2.12 | 5.94 | 4.97 |
| 99% | 3.56 | 2.92 | 4.72 | 4.01 | 2.96 | 2.55 | 3.69 | 3.01 | 10.84 | 9.10 |
| $n = 250$ | | | | | | | | | | | | |
| 50% | 0.55 | 0.46 | 0.57 | 0.48 | 0.50 | 0.42 | 0.56 | 0.47 | 0.97 | 0.81 |
| 75% | 0.94 | 0.80 | 0.99 | 0.84 | 0.87 | 0.74 | 0.96 | 0.81 | 1.71 | 1.43 |
| 90% | 1.37 | 1.16 | 1.47 | 1.25 | 1.27 | 1.08 | 1.41 | 1.18 | 2.63 | 2.20 |
| 95% | 1.66 | 1.40 | 1.80 | 1.53 | 1.54 | 1.32 | 1.71 | 1.43 | 3.31 | 2.78 |
| 99% | 2.27 | 1.90 | 2.55 | 2.16 | 2.16 | 1.83 | 2.35 | 1.95 | 5.05 | 4.22 |

Table 10: Estimated median interval length.
| n   | SwiZs: mean | SwiZs: median | MLE | AB | RSwiZs: mean | RSwiZs: median | WMLE |
|-----|-------------|---------------|-----|----|-------------|---------------|------|
|     | $\theta_1$ | $\theta_2$    | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
| 35  | 2511.13     | 2226.09       | 2504.27 | 2230.19 | 2492.15    | 2241.82       | -1.38x10^{-12} | -1.34x10^{-12} | 13.33      | 11.50      | 13.38      | 11.53      | 13.78      | 12.10      |
| 50  | 832.02      | 739.28        | 829.87  | 739.77  | 827.45     | 742.50        | -1.54x10^{-11} | -1.55x10^{-11} | 5.99       | 5.19       | 6.07       | 5.22       | 6.52       | 5.70       |
| 100 | 45.96       | 37.47         | 45.71   | 37.28   | 45.81      | 37.48         | -6.65x10^{-8}  | -5.22x10^{-8}  | 1.20       | 1.03       | 1.26       | 1.05       | 1.72       | 1.47       |
| 150 | 1.03        | 0.91          | 0.96    | 0.82    | 1.06       | 0.92          | -1.60x10^{-4}  | -1.48x10^{-4}  | 0.48       | 0.42       | 0.52       | 0.43       | 0.96       | 0.82       |
| 500 | 0.08        | 0.07          | 0.07    | 0.06    | 0.10       | 0.08          | 0.00           | 0.00          | 0.08       | 0.08       | 0.08       | 0.06       | 0.45       | 0.39       |

| n   | Median bias |     |     |     | Median bias |     |     |     |     | Median bias |     |     |     |     |     |     |     |
|-----|-------------|-----|-----|-----|-------------|-----|-----|-----|-----|-------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 35  | 0.4583      | 0.4894 | 0.0538 | 0.0276 | 0.5885      | 0.4654       | -1.5551       | -1.2966      | 0.2523     | 0.3257     | 0.0561     | 0.0309     | 0.9571     | 0.7846     |
| 50  | 0.2083      | 0.2374 | 0.0250 | 0.0197 | 0.3684      | 0.3008       | -1.1319       | -0.9168      | 0.1691     | 0.2039     | 0.0335     | 0.0213     | 0.7112     | 0.5986     |
| 100 | 0.0801      | 0.0824 | 0.0191 | 0.0135 | 0.1770      | 0.1389       | -0.4093       | -0.3267      | 0.0813     | 0.0905     | 0.0228     | 0.0195     | 0.5025     | 0.4289     |
| 150 | 0.0358      | 0.0434 | 0.0051 | 0.0021 | 0.1011      | 0.0851       | -0.2259       | -0.1848      | 0.0385     | 0.0470     | 0.0063     | 0.0041     | 0.4140     | 0.3623     |
| 250 | 0.0151      | 0.0265 | -0.0022 | 0.0028 | 0.0541      | 0.0521       | -0.1255       | -0.1011      | 0.0184     | 0.0268     | -0.0017    | 0.0029     | 0.3686     | 0.3268     |
| 500 | 0.0129      | 0.0150 | 0.0050 | 0.0046 | 0.0331      | 0.0275       | -0.0560       | -0.0473      | 0.0145     | 0.0163     | 0.0049     | 0.0034     | 0.3449     | 0.3056     |

| n   | Root mean squared error |     |     |     | Mean absolute deviation |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-------------------------|-----|-----|-----|-------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 35  | 17263.26                | 15552.08 | 17233.4 | 15587.83 | 17137.54 | 15667.69 | 2.97x10^{-13} | 2.95x10^{-13} | 59.16      | 50.35      | 59.00      | 50.44      | 58.45      | 50.95      |
| 50  | 7996.07                 | 7382.94 | 7982.45 | 7395.00 | 7957.28 | 7418.68 | 5.15x10^{-12} | 5.62x10^{-12} | 27.55      | 24.08      | 27.52      | 24.13      | 27.32      | 24.35      |
| 100 | 1331.57                 | 1059.16 | 1330.24 | 1056.18 | 1328.51 | 1057.59 | 4.41x10^{-10} | 3.36x10^{-10} | 6.15       | 5.21       | 6.22       | 5.27       | 6.26       | 5.37       |
| 150 | 36.30                   | 32.42   | 36.27   | 32.44   | 36.24   | 32.48   | 1.11x10^{-6} | 1.06x10^{-6} | 2.46       | 2.13       | 2.56       | 2.20       | 2.70       | 2.34       |
| 250 | 0.77                    | 0.66    | 0.75    | 0.63    | 0.78    | 0.66    | 0.58        | 0.49         | 0.92       | 0.79       | 1.01       | 0.85       | 1.26       | 1.07       |
| 500 | 0.46                    | 0.39    | 0.46    | 0.38    | 0.47    | 0.40    | 0.35        | 0.49         | 0.41       | 0.50       | 0.42       | 0.77       | 0.66       | 0.66       |

Table 11: Performances of point estimators.
D.2 Random intercept and random slope linear mixed model

Table 12: Average computational time in seconds to approximate a distribution on $S = 10,000$ points.

| $N$  | $\text{SwiZs}$ | $\text{Parametric bootstrap}$ |
|------|----------------|-------------------------------|
| 25   | 1.87           | 0.20                          |
| 100  | 6.49           | 0.73                          |
| 400  | 35.60          | 4.58                          |
| 1,600| 245.59         | 37.80                         |

| $n$  | $m$  | $\alpha$ | $\beta_0$ | $\beta_1$ | $\sigma_r^2$ | $\sigma_a^2$ | $\sigma^2_\gamma$ | $\beta_0$ | $\beta_1$ | $\sigma_r^2$ | $\sigma_a^2$ | $\sigma^2_\gamma$ |
|------|------|----------|-----------|-----------|--------------|--------------|----------------|-----------|-----------|--------------|--------------|----------------|
| 5    | 5    | 50%      | 51.78     | 53.87     | 48.54        | 54.18        | 70.38          | 42.37     | 43.61     | 44.60        | 32.27        | 28.10        |
|      |      | 75%      | 76.89     | 78.87     | 73.58        | 81.67        | 89.09          | 64.17     | 66.19     | 66.20        | 48.35        | 41.80        |
|      |      | 90%      | 91.87     | 92.93     | 88.89        | 94.10        | 98.80          | 78.38     | 81.94     | 81.07        | 61.72        | 46.87        |
|      |      | 95%      | 96.45     | 97.04     | 94.32        | 97.83        | 99.98          | 84.58     | 88.45     | 86.61        | 68.68        | 47.30        |
|      |      | 99%      | 99.54     | 99.71     | 98.73        | 99.87        | 100.00         | 91.93     | 95.40     | 93.54        | 79.03        | 47.61        |
| 10   | 10   | 50%      | 50.10     | 51.20     | 50.70        | 50.65        | 62.48          | 46.25     | 45.37     | 50.05        | 40.01        | 39.84        |
|      |      | 75%      | 75.16     | 77.08     | 74.92        | 75.64        | 85.74          | 69.81     | 68.68     | 74.48        | 60.54        | 59.68        |
|      |      | 90%      | 90.38     | 92.03     | 90.20        | 90.61        | 95.49          | 84.81     | 84.32     | 88.65        | 75.01        | 73.29        |
|      |      | 95%      | 95.23     | 96.40     | 95.23        | 94.96        | 97.86          | 90.71     | 90.32     | 93.95        | 81.30        | 79.29        |
|      |      | 99%      | 99.16     | 99.54     | 99.25        | 99.09        | 99.64          | 96.45     | 96.76     | 98.41        | 89.37        | 84.71        |
| 20   | 20   | 50%      | 50.78     | 49.10     | 49.97        | 49.74        | 49.85          | 49.03     | 47.58     | 49.63        | 45.40        | 45.75        |
|      |      | 75%      | 75.28     | 74.45     | 75.24        | 74.89        | 75.88          | 73.08     | 71.87     | 75.06        | 67.66        | 66.98        |
|      |      | 90%      | 90.06     | 89.79     | 89.95        | 90.28        | 90.75          | 87.59     | 87.02     | 89.73        | 81.76        | 81.83        |
|      |      | 95%      | 95.05     | 94.83     | 94.79        | 95.06        | 95.97          | 93.10     | 92.69     | 94.59        | 87.48        | 87.52        |
|      |      | 99%      | 98.96     | 98.97     | 98.93        | 98.90        | 99.50          | 97.77     | 97.82     | 98.75        | 94.20        | 94.15        |
| 40   | 40   | 50%      | 49.52     | 48.48     | 49.80        | 52.42        | 53.19          | 49.41     | 48.92     | 49.94        | 47.47        | 47.95        |
|      |      | 75%      | 74.70     | 72.86     | 75.27        | 77.89        | 78.39          | 74.22     | 73.34     | 75.63        | 70.93        | 71.46        |
|      |      | 90%      | 90.07     | 88.10     | 89.69        | 91.81        | 92.46          | 89.30     | 87.99     | 89.70        | 85.62        | 86.34        |
|      |      | 95%      | 95.15     | 94.09     | 94.71        | 96.27        | 96.59          | 94.37     | 93.65     | 94.82        | 91.29        | 91.82        |
|      |      | 99%      | 99.01     | 98.62     | 98.99        | 99.37        | 99.43          | 98.56     | 98.39     | 98.90        | 96.80        | 96.67        |

Table 13: Estimated coverage probabilities.
|     | α    | β₀   | β₁   | σₓ²   | σₓ²   | β₀   | β₁   | σₓ²   | σₓ²   |
|-----|------|------|------|-------|-------|------|------|-------|-------|
| n = 5 m = 5 |
| 50% | 0.3303 | 0.2243 | 0.4976 | 1.2050 | 0.1755 | 0.2712 | 0.1728 | 0.4453 | 1.5575 |
| 75% | 0.5940 | 0.3882 | 0.8552 | 2.0974 | 0.4491 | 0.4606 | 0.2947 | 0.7607 | 3.5624 |
| 90% | 0.9314 | 0.5682 | 1.2436 | 3.1286 | 1.1761 | 0.6577 | 0.4217 | 1.0909 | 12.9753 |
| 95% | 1.1956 | 0.6934 | 1.5222 | 3.7094 | 0.7845 | 0.5031 | 1.3051 | 13.9626 | 0.0036 |
| 99% | 1.8698 | 1.0031 | 2.3468 | 8.6739 | 1.0290 | 0.6623 | 1.7335 | 15.3409 | 0.0070 |
| n = 10 m = 10 |
| 50% | 0.2230 | 0.1198 | 0.2136 | 0.7311 | 1.0080 | 0.2038 | 0.1069 | 0.2099 | 0.7676 |
| 75% | 0.3902 | 0.2068 | 0.3638 | 1.2540 | 1.8614 | 0.3471 | 0.1818 | 0.3594 | 1.3370 |
| 90% | 0.5817 | 0.3008 | 0.5210 | 1.8131 | 2.9290 | 0.4953 | 0.2601 | 0.5144 | 1.9844 |
| 95% | 0.7162 | 0.3658 | 0.6218 | 2.1764 | 3.9196 | 0.5887 | 0.3097 | 0.6140 | 2.4462 |
| 99% | 1.0284 | 0.5130 | 0.8177 | 2.8992 | 7.9667 | 0.7745 | 0.4075 | 0.8055 | 3.6688 |
| n = 20 m = 20 |
| 50% | 0.1547 | 0.0699 | 0.1006 | 0.4750 | 0.5665 | 0.1482 | 0.0674 | 0.0998 | 0.4733 |
| 75% | 0.2672 | 0.1205 | 0.1718 | 0.8065 | 0.9934 | 0.2530 | 0.1149 | 0.1708 | 0.8102 |
| 90% | 0.3900 | 0.1752 | 0.2455 | 1.1499 | 1.4857 | 0.3622 | 0.1643 | 0.2447 | 1.1655 |
| 95% | 0.4718 | 0.2117 | 0.2926 | 1.3701 | 1.8096 | 0.4311 | 0.1957 | 0.2918 | 1.3964 |
| 99% | 0.6436 | 0.2894 | 0.3833 | 1.8121 | 2.4686 | 0.5645 | 0.2569 | 0.3825 | 1.8686 |
| n = 40 m = 40 |
| 50% | 0.1056 | 0.0452 | 0.0490 | 0.2816 | 0.1124 | 0.1056 | 0.0451 | 0.0493 | 0.3194 |
| 75% | 0.1810 | 0.0772 | 0.0834 | 0.4466 | 0.3469 | 0.1804 | 0.0770 | 0.0839 | 0.5429 |
| 90% | 0.2596 | 0.1107 | 0.1191 | 0.6923 | 0.6031 | 0.2576 | 0.1102 | 0.1197 | 0.7759 |
| 95% | 0.3100 | 0.1323 | 0.1420 | 0.8523 | 0.7672 | 0.3070 | 0.1313 | 0.1423 | 0.9257 |
| 99% | 0.4094 | 0.1747 | 0.1870 | 1.1467 | 1.1309 | 0.4020 | 0.1724 | 0.1864 | 1.2163 |

Table 14: Estimated median interval length.
|        | SwiZs: mean      | SwiZs: median   | Maximum likelihood |
|--------|------------------|-----------------|--------------------|
|        | $\beta_0$ | $\beta_1$ | $\sigma_2^2$ | $\sigma_\gamma^2$ | $\beta_0$ | $\beta_1$ | $\sigma_2^2$ | $\sigma_\gamma^2$ | $\beta_0$ | $\beta_1$ | $\sigma_2^2$ | $\sigma_\gamma^2$ |
| $N = 25$ | -0.0647 | -0.3827 | -3.1193 | 1.8554 | 1.5502 | -0.0761 | -0.3732 | -2.4630 | 6.4149 | 3.3175 | -0.0708 | -0.4203 | -1.2985 | -5.8224 | -0.3807 |
| $N = 100$ | 0.2843 | -0.0320 | -0.2911 | 2.4583 | 0.6119 | 1.6374 | -0.1452 | 0.7182 | -1.8475 | 1.8127 | 0.0685 | 0.0314 | -0.0166 | -2.8806 | -0.6425 |
| $N = 400$ | 0.0163 | 0.0374 | 0.0739 | 1.2927 | 0.0944 | 0.0149 | 0.0386 | 0.0514 | 0.9056 | 0.1565 | 0.0245 | 0.0417 | 0.0133 | -1.3425 | -0.2785 |
| $N = 1,600$ | 0.0010 | 0.0385 | 0.0183 | -0.9811 | -0.2965 | -0.0011 | 0.0394 | 0.0120 | -1.1600 | -0.2121 | 0.0130 | 0.0343 | -0.0021 | -0.6265 | -0.1253 |

|        | Median bias $\times 100$ | Root mean squared error $\times 100$ | Mean absolute deviation $\times 100$ |
|--------|--------------------------|-----------------------------------|-----------------------------------|
| $N = 25$ | -0.0341 | -0.2171 | -3.8669 | -6.5130 | -0.0876 | -0.0018 | -0.2114 | -3.3736 | -0.8483 | 0.0121 | 0.0327 | -0.2932 | -2.1012 | -10.1138 | -3.9990 |
| $N = 100$ | 0.4345 | 0.0289 | -0.4759 | 0.1208 | -0.0951 | 5.3959 | -1.4459 | 0.5589 | -0.7598 | 0.0354 | 0.1838 | 0.0069 | -0.1815 | -4.8730 | -1.2975 |
| $N = 400$ | 0.0020 | -0.0378 | 0.0422 | 0.4196 | -0.1116 | 0.0149 | -0.0286 | 0.0211 | -0.0405 | -0.0068 | -0.0140 | -0.0261 | -0.0220 | -2.1176 | -0.4517 |
| $N = 1,600$ | 0.0032 | 0.0500 | 0.0082 | -1.0639 | -0.1813 | -0.0060 | 0.0543 | 0.0041 | -0.0818 | -0.0021 | -0.0098 | 0.0480 | -0.0098 | -1.1378 | -0.1833 |

|        | Root mean squared error $\times 100$ | Mean absolute deviation $\times 100$ |
|--------|-----------------------------------|-----------------------------------|
| $N = 25$ | 24.6914 | 16.0625 | 9.2357 | 27.0499 | 6.2389 | 24.7198 | 16.0766 | 8.6916 | 24.2014 | 8.3432 | 24.7291 | 16.0853 | 8.1605 | 18.5249 | 6.8108 |
| $N = 100$ | 16.4663 | 8.8542 | 3.9374 | 14.7976 | 3.5251 | 14.3449 | 7.7017 | 4.1388 | 11.5080 | 3.3680 | 16.5630 | 8.7967 | 3.8703 | 12.0714 | 3.1774 |
| $N = 400$ | 11.4174 | 5.2549 | 1.8779 | 9.1330 | 1.8623 | 11.4174 | 5.2550 | 1.8752 | 8.9859 | 1.7515 | 11.4182 | 5.2554 | 1.8689 | 8.2404 | 1.7092 |
| $N = 1,600$ | 7.8721 | 3.4528 | 0.9119 | 4.7681 | 0.6698 | 7.9083 | 3.4524 | 0.9117 | 4.4706 | 0.5759 | 7.9891 | 3.4532 | 0.9110 | 5.7583 | 1.0216 |

Table 15: Performances of point estimators
| Coverage probability | Median interval length |
|----------------------|-----------------------|
| \( \alpha \) | \( \beta_0 \) | \( \beta_1 \) | \( \sigma^2_r \) | \( \sigma^2_\alpha \) | \( \sigma^2_\gamma \) | \( \beta_0 \) | \( \beta_1 \) | \( \sigma^2_r \) | \( \sigma^2_\alpha \) | \( \sigma^2_\gamma \) |
|----------------------|-----------------------|
| 50%                  | 43.16                 | 44.94                 | 48.66                 | 40.42                 | 36.49                 | 0.2770                 | 0.1791                 | 0.1043                 | 0.1868                 | 0.0375 |
|                      | 46.38                 | 45.98                 | 50.75                 | 45.86                 | 44.91                 | 0.2060                 | 0.1082                 | 0.0525                 | 0.1422                 | 0.0383 |
| 75%                  | 67.51                 | 69.17                 | 73.83                 | 64.17                 | 70.73                 | 0.4942                 | 0.3180                 | 0.1836                 | 0.3625                 | 0.0945 |
|                      | 70.85                 | 71.03                 | 75.36                 | 70.65                 | 68.84                 | 0.3591                 | 0.1888                 | 0.0901                 | 0.2583                 | 0.0690 |
| 90%                  | 83.68                 | 86.75                 | 88.79                 | 81.88                 | 96.33                 | 0.7612                 | 0.4897                 | 0.2764                 | 0.6358                 | 0.2010 |
|                      | 87.23                 | 87.08                 | 90.04                 | 86.58                 | 85.82                 | 0.5321                 | 0.2806                 | 0.1304                 | 0.4088                 | 0.1078 |
| 95%                  | 90.37                 | 93.23                 | 93.83                 | 88.93                 | 98.88                 | 0.9671                 | 0.6226                 | 0.3431                 | 0.8982                 | 0.3095 |
|                      | 93.20                 | 93.27                 | 95.12                 | 92.37                 | 93.09                 | 0.6534                 | 0.3449                 | 0.1569                 | 0.5299                 | 0.1392 |
| 99%                  | 97.04                 | 98.93                 | 98.54                 | 96.95                 | 99.75                 | 1.4991                 | 0.9746                 | 0.4982                 | 1.8138                 | 0.7069 |
|                      | 98.41                 | 98.53                 | 98.95                 | 98.02                 | 99.59                 | 0.9264                 | 0.4903                 | 0.2111                 | 0.8593                 | 0.2265 |
| \( n = 10 \) \( m = 10 \) |                      |                      |                      |                      |                      |                      |                      |                      |                      |          |
| 50%                  | 46.38                 | 45.98                 | 50.75                 | 45.86                 | 44.91                 | 0.2060                 | 0.1082                 | 0.0525                 | 0.1422                 | 0.0383 |
|                      | 70.85                 | 71.03                 | 75.36                 | 70.65                 | 68.84                 | 0.3591                 | 0.1888                 | 0.0901                 | 0.2583                 | 0.0690 |
| 90%                  | 87.23                 | 87.08                 | 90.04                 | 86.58                 | 85.82                 | 0.5321                 | 0.2806                 | 0.1304                 | 0.4088                 | 0.1078 |
|                      | 94.09                 | 94.02                 | 94.81                 | 93.80                 | 93.72                 | 0.4524                 | 0.2055                 | 0.0735                 | 0.3445                 | 0.0712 |
| 99%                  | 98.56                 | 98.61                 | 98.94                 | 98.40                 | 98.59                 | 0.6167                 | 0.2801                 | 0.0972                 | 0.5019                 | 0.1038 |
| \( n = 20 \) \( m = 20 \) |                      |                      |                      |                      |                      |                      |                      |                      |                      |          |
| 50%                  | 49.20                 | 47.62                 | 49.92                 | 48.00                 | 47.31                 | 0.1491                 | 0.0677                 | 0.0251                 | 0.1048                 | 0.0216 |
|                      | 73.66                 | 72.54                 | 75.09                 | 72.49                 | 72.86                 | 0.2571                 | 0.1168                 | 0.0429                 | 0.1845                 | 0.0381 |
| 90%                  | 88.70                 | 88.34                 | 89.97                 | 88.33                 | 88.10                 | 0.3742                 | 0.1700                 | 0.0616                 | 0.2774                 | 0.0573 |
|                      | 94.09                 | 94.02                 | 94.81                 | 93.80                 | 93.72                 | 0.4524                 | 0.2055                 | 0.0735                 | 0.3445                 | 0.0712 |
| 99%                  | 98.56                 | 98.61                 | 98.94                 | 98.40                 | 98.59                 | 0.6167                 | 0.2801                 | 0.0972                 | 0.5019                 | 0.1038 |
| \( n = 40 \) \( m = 40 \) |                      |                      |                      |                      |                      |                      |                      |                      |                      |          |
| 50%                  | 49.46                 | 49.32                 | 49.79                 | 48.67                 | 49.01                 | 0.1060                 | 0.0452                 | 0.0122                 | 0.0748                 | 0.0136 |
|                      | 74.46                 | 73.78                 | 75.28                 | 73.52                 | 74.77                 | 0.1819                 | 0.0776                 | 0.0209                 | 0.1295                 | 0.0236 |
| 90%                  | 89.88                 | 88.76                 | 89.70                 | 88.89                 | 89.83                 | 0.2623                 | 0.1119                 | 0.0299                 | 0.1899                 | 0.0346 |
|                      | 94.95                 | 94.28                 | 94.85                 | 94.22                 | 94.71                 | 0.3148                 | 0.1343                 | 0.0356                 | 0.2310                 | 0.0420 |
| 99%                  | 98.98                 | 98.86                 | 98.99                 | 98.77                 | 98.82                 | 0.4212                 | 0.1797                 | 0.0468                 | 0.3194                 | 0.0582 |

Table 16: Asymptotic results
### D.3 M/G/1 queueing model

|                  | SwiZs | Indirect Inference | Parametric Bootstrap |
|------------------|-------|--------------------|----------------------|
|                  | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
| 50%              | 46.92  | 38.68              | 56.73               | 40.59          | 9.95          | 54.59          | 18.31          | 10.23          | 20.96          |
| 75%              | 71.56  | 55.41              | 81.80               | 68.01          | 34.11          | 84.50          | 32.70          | 20.96          | 37.38          |
| 90%              | 87.55  | 67.77              | 94.47               | 87.62          | 57.13          | 96.04          | 48.62          | 35.24          | 53.71          |
| 95%              | 93.16  | 74.78              | 97.97               | 94.66          | 70.22          | 98.75          | 57.05          | 46.03          | 63.21          |
| 99%              | 98.17  | 90.06              | 99.90               | 98.84          | 94.89          | 99.94          | 71.99          | 65.43          | 77.64          |

Table 17: Estimated coverage probabilities.

|                  | SwiZs | Indirect Inference | Parametric Bootstrap |
|------------------|-------|--------------------|----------------------|
|                  | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
| 50%              | 0.0235 | 0.0805             | 0.1379              | 0.0382        | 0.0468        | 0.1368        | 0.0263        | 0.0420        | 0.1134        |
| 75%              | 0.0404 | 0.1467             | 0.2357              | 0.0911        | 0.0978        | 0.2389        | 0.0460        | 0.0757        | 0.2051        |
| 90%              | 0.0585 | 0.2207             | 0.3378              | 0.1563        | 0.1914        | 0.3835        | 0.0708        | 0.1185        | 0.3131        |
| 95%              | 0.0705 | 0.2733             | 0.4032              | 0.2225        | 0.2952        | 0.5432        | 0.0895        | 0.1533        | 0.3855        |
| 99%              | 0.0952 | 0.3934             | 0.5407              | 0.5331        | 0.7152        | 1.6084        | 0.1327        | 0.2514        | 0.5562        |

Table 18: Estimated median interval length.

|                  | SwiZs: starting value is $\theta_0$ | SwiZs: sample size is $n = 1,000$. |
|------------------|--------------------------------------|-------------------------------------|
|                  | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
| 50%              | 50.22       | 58.64       | 49.98       | 50.07       | 46.06       | 49.37       |
| 75%              | 75.24       | 91.25       | 74.24       | 75.24       | 71.82       | 74.77       |
| 90%              | 90.52       | 99.82       | 89.55       | 89.73       | 89.84       | 89.49       |
| 95%              | 95.37       | 100.00      | 94.87       | 94.81       | 95.41       | 94.69       |
| 99%              | 99.09       | 100.00      | 99.02       | 98.95       | 99.28       | 99.10       |

Table 19: Estimated coverage probabilities under different conditions than Table 17.
|                | SwiZs: mean | SwiZs: median | Indirect inference | Indirect inference: mean | Indirect inference: median |
|----------------|-------------|---------------|--------------------|--------------------------|---------------------------|
|                | $\theta_1$  | $\theta_2$   | $\theta_3$        | $\theta_1$              | $\theta_2$               | $\theta_3$              |
| Mean bias      | 0.0037      | -0.0149       | 0.0006             | 0.0057                   | -0.0096                   | 0.0002                   |
| Median bias    | 0.0026      | -0.0219       | -0.0044            | 0.0046                   | -0.0157                   | -0.0041                  |
| RMSE           | 0.0197      | 0.0764        | 0.0890             | 0.0200                   | 0.0762                    | 0.0888                   |
| MAD            | 0.0192      | 0.0705        | 0.0884             | 0.0190                   | 0.0718                    | 0.0882                   |

Table 20: Performances of point estimator.
E  Generic results

This chapter assembles some generic theoretical results useful for the other Chapters.

We generically denote \( \{g_n : n \geq 1\} \) a sequence of a random vector-valued function and \( \theta \in \Theta \) a vector of parameters.

The next Lemma is Theorem 5.9 in [70]. The proof is given for the sake of completeness.

**Lemma 56** *(weak consistency).* Let \( \{g_n(\theta)\} \) be sequence of a random vector-valued function of vector parameter \( \theta \) with a deterministic limit \( g(\theta) \). If \( \Theta \) is compact, if the random function sequence converges uniformly as \( n \to \infty \)

\[
\sup_{\theta \in \Theta} \|g_n(\theta) - g(\theta)\| \xrightarrow{p} 0, \tag{11}
\]

and if there exist \( \delta > 0 \) such that

\[
\inf_{\theta \notin B(\theta_0, \delta)} \|g(\theta)\| > 0 = \|g(\theta_0)\|, \tag{12}
\]

then any sequence of estimators \( \{\hat{\theta}_n\} \) converges weakly in probability to \( \theta_0 \).

**Proof.** Choose \( \hat{\theta}_n \) that nearly minimises \( \|g_n(\theta)\| \) so that

\[
\|g_n(\hat{\theta}_n)\| \leq \inf_{\theta \in \Theta} \|g_n(\theta)\| + o_p(1)
\]

Clearly we have \( \inf_{\theta} \|g_n(\theta)\| \leq \|g_n(\theta_0)\| \), and by (11) \( \|g_n(\theta_0)\| \xrightarrow{p} \|g(\theta_0)\| \) so that

\[
\|g_n(\hat{\theta}_n)\| \leq \|g(\theta_0)\| + o_p(1)
\]

Now, substracting both sides by \( \|g(\hat{\theta}_n)\| \), we have by the reverse triangle inequality

\[
-\|g_n(\hat{\theta}_n) - g(\hat{\theta}_n)\| \leq \|g(\theta_0)\| - \|g(\hat{\theta}_n)\| + o_p(1)
\]

The left-hand side is bounded by the negative supremum, thus

\[
\|g(\theta_0)\| - \|g(\hat{\theta}_n)\| \geq -\sup_{\theta \in \Theta} \|g_n(\theta) - g(\theta)\| - o_p(1)
\]

It follows from (11) that the limit in probability of the right-hand side tends to 0. Let \( \varepsilon > 0 \) and choose a \( \delta > 0 \) as in (12) so that

\[
\|g(\theta)\| > \|g(\theta_0)\| - \varepsilon
\]

for every \( \theta \notin B(\theta_0, \delta) \). If \( \hat{\theta}_n \notin B(\theta_0, \delta) \), we have

\[
\|g(\theta_0)\| - \|g(\hat{\theta}_n)\| < \varepsilon
\]

The probability of this event converges to 0 as \( n \to \infty \). \Halmos

The next definition is taken from [105] (see also [106, Chapter 7.1])

**Definition 57.** \( \{g_n(\theta)\} \) is stochastically uniformly equicontinuous on \( \Theta \) if for every \( \varepsilon > 0 \) there exist a real \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta} \Pr \left( \sup_{\theta \in \Theta} \sup_{\theta \in B(\theta, \delta)} \|g_n(\theta') - g_n(\theta)\| > \varepsilon \right) < \varepsilon \tag{13}
\]

**Lemma 58** *(uniform consistency).* If \( \Theta \) is compact, if the sequence of random vector-valued function \( \{g_n(\theta)\} \) is pointwise convergent for all \( \theta \in \Theta \) and is stochastically uniformly equicontinuous on \( \Theta \), then

i. \( \{g_n(\theta)\} \) converges uniformly,

ii. \( g \) is uniformly continuous.
Then, by the triangle inequality we have
\[ \limsup_{n \to \infty} \Pr \left( \| g_n(\theta_i) - g(\theta_i) \| > \varepsilon \right) < \varepsilon, \]
whenever \( 1 \leq l \leq k \). If \( \theta \in \Theta \), so \( \theta \in \mathcal{B}(\theta_i, \delta) \) for some \( l \), so that
\[ \limsup_{n \to \infty} \Pr \left( \| g_n(\theta_i) - g_n(\theta) \| > \varepsilon \right) < \varepsilon \]
Then, by the triangle inequality we have
\[ \limsup_{n \to \infty} \Pr \left( \sup_{\theta \in \Theta} \| g_n(\theta) - g(\theta) \| > \varepsilon \right) \]
\[ \leq \limsup_{n \to \infty} \Pr \left( \sup_{\theta \in \Theta} \sup_{\theta' \in \mathcal{B}(\theta, \delta)} \| g_n(\theta) - g_n(\theta') \| > \varepsilon \right) \]
\[ + \limsup_{n \to \infty} \Pr \left( \| g_n(\theta') - g(\theta') \| > \varepsilon \right) + \Pr \left( \sup_{\theta \in \Theta} \sup_{\theta' \in \mathcal{B}(\theta, \delta)} \| g(\theta) - g(\theta') \| > \varepsilon \right) < 3\varepsilon \]

(ii). The proof follows the same steps.

The next Lemma is similar to [105, Lemma 1]. The result of [105] is on the difference between a random and a nonrandom functions and requires the extra assumption of absolute continuity of the nonrandom function. The proof provided here is also different.

**Lemma 59.** If for all \( \theta, \theta' \in \Theta \), \( \| g_n(\theta) - g_n(\theta') \| \leq B_n d(\theta, \theta') \) with \( B_n = \Theta_p(1) \), then \( \{ g_n(\theta) \} \) is stochastically uniformly equicontinuous.

**Proof.** By \( B_n = \Theta_p(1) \), there is \( M > 0 \) such that for all \( n, \Pr(\| B_n \| > M) < \varepsilon \). Let \( \varepsilon > 0 \) and choose a sufficiently small \( \delta > 0 \) such that for all \( \theta', \theta \in \Theta \), \( d(\theta, \theta') < \varepsilon/M = \tau, \delta \leq \tau \). Let \( \mathcal{B}(\theta, \delta) = \{ \theta' \in \Theta : d(\theta, \theta') < \delta \} \). Then, we have
\[ \limsup_{n \to \infty} \Pr \left( \sup_{\theta \in \Theta} \sup_{\theta' \in \mathcal{B}(\theta, \delta)} \| g_n(\theta) - g_n(\theta') \| > \varepsilon \right) \]
\[ \leq \limsup_{n \to \infty} \Pr \left( B_n \sup_{\theta \in \Theta} \sup_{\theta' \in \mathcal{B}(\theta, \delta)} d(\theta, \theta') > \varepsilon \right) \]
\[ \leq \limsup_{n \to \infty} \Pr (B_n > \varepsilon) \leq \limsup_{n \to \infty} \Pr (\| B_n \| > M) < \varepsilon \]

The next Lemma is a special case of [107, Corollary 3.1].

**Lemma 60.** Let \( \{ x_i : i \geq 1 \} \) be an i.i.d. sequence of random variable and let \( g_n(\theta) = n^{-1} \sum_{i=1}^{n} g(x_i, \theta) \). If for all \( i = 1, \ldots, n \) and \( \theta, \theta' \in \Theta \), \( \| g(x_i, \theta) - g(x_i, \theta') \| \leq b_n(x_i) d(\theta, \theta') \) with \( \mathbb{E}[b_n(x_i)] = \mu_n = \Theta(1) \), then \( \{ g_n(\theta) \} \) is stochastically uniformly equicontinuous.

**Proof.** Let \( B_n = n^{-1} \sum_{i=1}^{n} b_n(x_i) \), so \( \mathbb{E}[B_n] = \Theta(1) \). We have by triangle inequality
\[ \| g_n(\theta) - g_n(\theta') \| \leq \frac{1}{n} \sum_{i=1}^{n} \| g(x_i, \theta) - g(x_i, \theta') \| \leq B_n d(\theta, \theta') \]

The rest of the proof follows from Lemma 59.

**Lemma 61** (uniform weak law of large number). If, in addition to Lemma 60, for each \( \theta \in \Theta \), \( g_n(\theta) \) is pointwise convergent, then \( \{ g_n(\theta) \} \) converges uniformly.
The supremum of the norm exists because the affine line \( \lambda t \) is in the closed ball. Hence, we have \( \lambda t \rightarrow \lambda \) as \( t \rightarrow 0 \).

**Lemma 62 (mean value inequality).** Let \( U \) be a convex open set in \( \Theta \). Let \( \theta_1 \in U \) and \( \theta_2 \in \Theta \). If \( g : U \rightarrow F \) is a \( C^1 \)-mapping, then

\[ g(\theta_1 + \theta_2) - g(\theta_1) = \int_0^1 Dg(\theta_1 + t\theta_2) dt \cdot \theta_2 \]

**Proof.** (i) Fix \( \theta_1 \in U, \theta_2 \in \Theta \). Let \( \lambda = (1-t)\theta_1 + t\theta_2 \) for \( t \in [0,1] \). We have by the convexity of \( U \) that \( \lambda \in U \), and so \( \theta_1 + t\theta_2 \) is in \( U \) as well. Put \( h(t) = g(\theta_1 + t\theta_2) \), so \( Dh(t) = Dg(\theta_1 + t\theta_2) \cdot \theta_2 \). By the fundamental theorem of calculus we have that

\[ \int_0^1 Dh(t) dt = h(1) - h(0) \]

Since \( h(1) = g(\theta_1 + \theta_2), h(0) = g(\theta_1) \), and \( \theta_2 \) is allowed to be pulled out of the integral, part (i) is proven.

(ii). We have that

\[ \|g(\theta_1 + \theta_2) - g(\theta_1)\| \leq \left\| \int_0^1 Dg(\theta_1 + t\theta_2) dt \right\| \cdot \|\theta_2\|, \]

\[ \leq |1| \sup_{0 \leq t \leq 1} \|Dg(\theta_1 + t\theta_2)\| \cdot \|\theta_2\|, \]

where we use the Cauchy-Schwarz inequality for the first inequality, and the upper bound of integral for the second. The supremum of the norm exists because the affine line \( \theta_1 + t\theta_2 \) is compact and the Jacobian is continuous.

**Lemma 63 (delta method).** If conditions of Lemma 62 holds, then

\[ g(\theta_1 + \theta_2) - g(\theta_1) = Dg(\theta_1) \cdot \theta_2 + o(\|\theta_2\|) \]

**Proof.** Fix \( \theta_1 \in U \) and \( \theta_2 \in \Theta \). By Lemma 62 we have

\[ \left\| \int_0^1 Dg(\theta_1 + t\theta_2) dt \right\| \leq \sup_{0 \leq t \leq 1} \|Dg(\theta_1 + t\theta_2)\| \]

Let \( \lambda = \theta_1 + \theta_2 \) so \( \lambda = (1-t)\theta_1 + t\theta_2 \), \( t \in [0,1] \), is in \( U \) and \( \theta_1 + t\theta_2 \) as well. Let \( B(\theta_1, 2\|\theta_2\|) \) = \( \{ \|\theta_1 - \theta\| \leq 2\|\theta_2\| \} \). We have

\[ \|t\theta_1 + (1-t)\theta_3 - \theta\| \leq t \|\theta_1 - \theta\| + (1-t) \|\theta_3 - \theta\| \]

\[ \leq t \|\theta_2\| + (1-t) \|\theta_2\| = \|\theta_2\|, \]

so the line segment \( \lambda \) is in the closed ball. Hence, we have

\[ \left\| \int_0^1 Dg(\theta_1 + t\theta_2) dt \right\| \leq \sup_{\theta \in B(\theta_1, \|\theta_2\|)} \|\theta\| \] \[ \sup_{\theta \in B(\theta_1, \|\theta_2\|)} \|Dg(\theta) - Dg(\theta_1)\| \rightarrow 0 \]

as \( \|\theta_2\| \rightarrow 0 \).

**Lemma 64 (asymptotic normality).** Let \( U \) be a convex open set in \( \Theta \). Let \( \{\hat{\theta}_n\} \) be a sequence of estimator (roots of) the mapping \( g_n : U \rightarrow F \). If

i. \( \hat{\theta}_n \) converges in probability to \( \theta_0 \in U \).
\( \{ g_n \} \) is a \( C^1 \)-mapping,

\( n^{1/2} g_n(\theta_0) \to n(0, V) \).

iv. \( Dg_n(\theta_0) \) converges in probability to \( M \).

v. \( Dg_n(\theta_0) \) is nonsingular, then

\[ n^{1/2}(\hat{\theta}_n - \theta_0) \to n(0, \Sigma), \]

where \( \Sigma = M^{-1} V M^{-T} \).

**Proof.** Fix \( \theta_1 = \theta_0 \) and \( \theta_2 = \hat{\theta}_n - \theta_0 \), from Lemma 62 and Lemma 63 we have

\[ g_n(\hat{\theta}_n) = g_n(\theta_0) + Dg_n(\theta_0) \cdot (\hat{\theta}_n - \theta_0) + o_p(\|\hat{\theta}_n - \theta_0\|) \]

By definition \( g_n(\hat{\theta}_n) = 0 \). Multiplying by square-root \( n \) leads to

\[ n^{1/2}(\hat{\theta}_n - \theta_0) = -[Dg_n(\theta_0)]^{-1} n^{1/2} g_n(\theta_0) - n^{1/2} [Dg_n(\theta_0)]^{-1} o_p(\|\hat{\theta}_n - \theta_0\|) \]

By the continuity of the matrix inversion \( [Dg_n(\theta_0)]^{-1} \xrightarrow{\text{P}} M^{-1} \). Since the central limit theorem holds for \( n^{1/2} g_n(\theta_0) \), the proof results from Slutsky’s lemma. \( \square \)

The next Lemma is Theorem 9.4 in [109] and is given without proof.

**Lemma 65** (implicit function theorem). Let \( \Xi \times \Theta \) be an open subset of \( \mathbb{R}^m \times \mathbb{R}^p \). Let \( g : \Xi \times \Theta \to \mathbb{R}^p \) be a function of the form \( g(\xi, \theta) = k \). Let the solution at the points \( (\xi_0, \theta_0) \in \Xi \times \Theta \) and \( k_0 \in \mathbb{R}^p \) be

\[ g(\xi_0, \theta_0) = k_0 \]

If

i. \( g \) is differentiable in \( \Xi \times \Theta \),

ii. The partial derivative \( D_\xi g \) is continuous in \( \Xi \times \Theta \),

iii. The partial derivative \( D_\theta g \) is invertible at the points \( (\xi_0, \theta_0) \in \Xi \times \Theta \),

then, there are neighborhoods \( X \subset \Xi \) and \( O \subset \Theta \) of \( \xi_0 \) and \( \theta_0 \) on which the function \( \hat{\theta} : O \to X \) is uniquely defined, and such that:

1. \( g(\xi, \hat{\theta}(\xi)) = k_0 \) for all \( \xi \in X \).

2. For each \( \xi \in X \), \( \hat{\theta}(\xi) \) is the unique solution lying in \( O \) such that \( \hat{\theta}(\xi_0) = \theta_0 \).

3. \( \hat{\theta} \) is differentiable on \( X \) and

\[ D_\xi \hat{\theta} = -[D_\theta g]^{-1} D_\xi g \]