We revise recent results on the classification of the generalized three-dimensional Hamiltonian Ermakov system. We show that a statement published recently is incorrect, while the solution for the classification problem was incomplete. We present the correct classification for the three-dimensional system by using results which related the background space with the dynamics. Finally, we extend our results for the generalized $n$-dimensional Hamiltonian Ermakov system.

**KEYWORDS**
Ermakov system, integrability, lie symmetries, Noether symmetries

**MSC CLASSIFICATION**
37C80, 37J15, 70H33

1 INTRODUCTION

At the end of the 19th century, Ermakov\(^1\) solved the problem for the derivation of first integral for the time-dependent linear equation

$$\ddot{x} + \omega^2(t) x = 0,$$

where an overdot means total derivative with respect to the time variable, $t$, that is, $\dot{x} = \frac{dx}{dt}$.

By introducing the new variable $\rho(t)$ which satisfies the second-order differential equation

$$\ddot{\rho} + \omega^2(t) \rho = \rho^{-3}, \quad (2)$$

the time-dependent term in (1) can be eliminated and the following function $J(\rho, \dot{\rho}, x, \dot{x})$ can be constructed

$$J(\rho, \dot{\rho}, x, \dot{x}) = \frac{1}{2} \left[ (\rho \dot{x} - \dot{\rho} x)^2 + (x/\rho)^2 \right], \quad (3)$$

which is a conservation law, that is, $\dot{J} = 0$. The second-order ordinary differential equation (2) was solved by Pinney.\(^2\) The analytic solution is

$$\rho^2(t) = c_1(\rho_1(t))^2 + c_2(\rho_2(t))^2 + 2c_3\rho_1(t)\rho_2(t). \quad (4)$$
where \( \rho_1(t) \) and \( \rho_2(t) \) are solutions of Equation (1) and \( c_1, c_2, \) and \( c_3 \) are constants, of which only two are independent. Hence, Equation (2) is known as the Ermakov–Pinney equation. Furthermore, Lewis\(^3\) investigated the integral properties for the time-dependent harmonic oscillator of Equation (1) in classical and quantum systems. Lewis rediscovered the Ermakov invariant (3), and function \( J(\rho, \dot{\rho}, x, \dot{x}) \) nowadays is called Ermakov–Lewis invariant. The oscillator appears in many physical problems and other areas of applied mathematics. Consequently, the Ermakov–Lewis invariant and the Ermakov–Pinney equation have appeared in many studies, from quantum mechanics,\(^3\)–\(^5\) to gravitational theory\(^6,7\) and others.\(^8\)–\(^11\) Recently, in Padmanabhan,\(^12\) the physical interpretation of the Ermakov–Lewis invariant was discussed.

The application of the Lie theory on the Ermakov–Pinney equation provides that Equation (2) admits three Lie point symmetries which form the \( SL(2, R) \) algebra.\(^13\) The representation of the Lie algebra depends upon the term \( \omega^2(t) \). Thus, there exists a point transformation in which the term \( \omega^2(t) \) can be eliminated and Equation (2) to be reduced in the simple form:

\[
\dot{\rho} = \rho^{-3}.
\]

There are many extensions and generalization of the Ermakov–Pinney equation in the literature. The two-dimensional system,

\[
\ddot{x} + \omega^2(t)x = \frac{f\left(\frac{y}{x}\right)}{x^3},
\]

\[
\ddot{y} + \omega^2(t)y = \frac{g\left(\frac{y}{x}\right)}{y^3},
\]

is known as an Ermakov system.\(^14\) Systems (6) and (7) admit the conservation law\(^15\)

\[
I(x, \dot{x}, y, \dot{y}) = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int_{y/x}^{y/x} (uf(u) - u^{-3}g(u)) \, du,
\]

which is a generalization of the Ermakov–Lewis invariant. We remark that there exist a point transformation such that systems (6) and (7) can be written in an equivalent form without the \( \omega^2(t) \) term. Moreover, the two-dimensional Ermakov system admits three point symmetries which form the \( SL(2, R) \) algebra. In the special case for which \( f\left(\frac{y}{x}\right) = V(x, y) \) and \( g\left(\frac{y}{x}\right) = V(x, y) \), the dynamical system follows from a variational principle and the Lie symmetries are also Noether symmetries.

By using the latter property for the Ermakov system, that is, the admitted Lie symmetries for a given dynamical system to form the \( SL(2, R) \) algebra, there have been various generalizations of the Ermakov system in higher dimensions,\(^13\) in curved geometries,\(^16\) or many others (see for instance previous works\(^18\)–\(^21\) and references therein). The three-dimensional Ermakov system was introduced in Moyo and Leach,\(^13\) for which it was found that in the case that the Ermakov system is Hamiltonian, the Ermakov–Lewis invariant is a function of the variational conservation laws which are determined by the applications of Noether’s second theorem for the elements of the \( SL(2, R) \) algebra; see also the discussion in Haas and Goedert.\(^17\) However, such a property was found also in the case of curved geometries. Moreover, for the Hamiltonian Ermakov system, the Ermakov–Lewis invariant can be constructed by Noether’s second theorem for contact symmetries. Indeed, there was found to be an exact relation for the existence of the Killing tensor which provides the Ermakov–Lewis invariant and of the main generator vector for the background space of the \( SL(2, R) \) Lie algebra.\(^16\) The \( SL(2, R) \) Lie algebra plays an important role in the study of the integrability of various Hamiltonian systems; we refer the reader in some recent studies.\(^22\)–\(^28\)

A complete classification of variational symmetries for the Hamiltonian three-dimensional Ermakov\(^13\) system was performed in Naz and Mahomed.\(^29\) The authors determined the functional form for the unknown potential functions for the generalized Ermakov system to admit additional variational symmetries which produce conservation laws except from the elements of the \( SL(2, R) \) algebra. The authors determined eight different potential functions of which they claim that four of these functions are new in the literature and new exact solutions were determined. However, this claim is false. As we shall see with a simple construction method, the potential functions Naz and Mahomed\(^29\) determined are well known in the literature. Nevertheless, inspired by the problem which Naz and Mahomed\(^29\) study, we extend our analysis and we find a general result for the generalized Hamiltonian Ermakov system in \( n \)-dimensional Euclidian geometry. The plan of the paper is as follow.
In Section 2, we present basic properties and definitions for the theory of symmetries of differential equations. We assume a system of second-order ordinary differential equations of the form

\[ \ddot{x} = \Omega(t, x', \dot{x}') , \]  

(9)

in which \( x' \) are the dependent variables and \( t \) is the independent variable. Furthermore, consider the one-parameter point transformation \( \Phi(t, x; \varepsilon) \) with infinitesimal generator \( \xi(t, x') \partial_t + \eta(t, x') \partial_x \) defined in the augmented space \( \{t, x'\} \). Thus, we say that the vector field \( X \) is a Lie point symmetry for the system (9) if the set of differential equations remains invariant under the action of the one-parameter point transformation \( \Phi(t, x; \varepsilon) \), equivalently when the following condition is true\(^{30}\)

\[ X^{[2]} \left( \ddot{x} - \Omega(t, x', \dot{x}') \right) = 0 , \]  

(10)

where \( X^{[2]} \) is the second prolongation of \( X \) defined by the formula \( X^{[2]} = X + (\dot{\eta} - \dot{x'} \dot{\xi}) \partial_{\dot{x'}} + (\dot{\eta} - \dot{x'} \dot{\xi} - 2\ddot{x} \dot{\xi}) \partial_{\xi} \). An equivalent way to express the symmetry condition (10) is\(^{10}\)

\[ [X^{[1]}, A] = \lambda(x') A , \]  

(11)

in which now \( X^{[1]} \) is the first prolongation of \( X \) and \( A \) is the Hamiltonian vector field \( A = \partial_t + \dot{x'} \partial_x + \Omega(t, x', \dot{x}') \partial_{\xi} \).

If the system of differential equations results from a variational principle, that is, there exists a Lagrangian function \( L = L(t, x', \dot{x}') \), then the infinitesimal generator \( X \) for the one-parameter point transformation is characterized as variational symmetry or as Noether symmetry for the given Lagrangian function, if there exists a function \( G \) such that the condition of Noether’s first theorem is satisfied\(^{31}\)

\[ X^{[1]} L + L \frac{d\xi}{dt} = \frac{dG}{dt} . \]  

(12)

Function \( G \) is boundary term introduced to allow for the infinitesimal changes in the value of the action integral produced by the infinitesimal change in the boundary of the domain caused by the infinitesimal transformation of the variables in the action integral.

The importance of the determination of Noether symmetries for a given system is that there exist a unique relation between Noether symmetries and conservation laws. Indeed, if \( X \) is a Noether symmetry vector with corresponding boundary term \( G \), then according to Noether’s second theorem, the following function

\[ I(t, x', \dot{x}') = \xi \left( \dot{x'} \frac{\partial L}{\partial \dot{x}'} - L \right) - \frac{\partial L}{\partial \dot{x}'} \eta' + G \]  

(13)

is a conservation law for the dynamical system, that is \( I(t, x', \dot{x}') = 0 \). For a recent review on Noether’s theorems, we refer the reader to Halder et al.\(^{32}\)

### 3 | THE ERMAKOV SYSTEM

In the following, we consider the Ermakov system and without loss of generality we can omit the linear term with the time-dependent component \( \omega^2(t) \).

As we mentioned before, the Ermakov–Pinney equation (5) admits three Lie point symmetries, the vector fields

\[ X^1 = \partial_t, \quad X^2 = 2t \partial_t + \rho \partial_{\rho} , \quad X^3 = t^2 \partial_t + t \rho \partial_{\rho} . \]  

(14)
Equation (5) admits the Lagrangian function

\[ L(\rho, \dot{\rho}) = \frac{1}{2} (\dot{\rho}^2 - \rho^{-2}). \]  

(15)

Thus, by replacing the vector fields \( \{X^1, X^2, X^3\} \) in Noether’s condition, we find that the three vector fields are Noether symmetries with corresponding boundary terms \( G(X^1) = G_0, G(X^2) = 0 \) and \( G(X^3) = \frac{1}{2} \rho^2 \), respectively.

Hence, from expression (13), we define the conservation laws

\[ I_1(X^1) = \frac{1}{2} (\dot{\rho}^2 + \rho^{-2}) \equiv \mathcal{H}, \]  

(16)

\[ I_2(X^2) = 2t\mathcal{H} - \rho \dot{\rho}, \]  

(17)

\[ I_3(X^3) = t^2\mathcal{H} - t\rho \dot{\rho} + \frac{1}{2} \rho^2, \]  

(18)

where \( \mathcal{H} \) is the Hamiltonian function.

Consider now the two-dimensional Hamiltonian Ermakov system

\[ \ddot{\mathcal{H}} = \left( \frac{V}{J} \right)_x, \quad \ddot{\mathcal{H}} = \left( \frac{V}{J} \right)_y, \]  

(19)

or in polar coordinates \((r, \theta)\),

\[ \dot{\rho} - \rho \dot{\theta}^2 = \frac{V(\theta)}{\rho^3}, \]  

(20)

\[ \ddot{\theta} + 2\rho \dot{\theta} = \frac{V(\theta)}{2\rho^4}. \]  

(21)

with Lagrangian function

\[ L(\rho, \dot{\rho}, \theta, \dot{\theta}) = \frac{1}{2} \left( \dot{\rho}^2 + \rho^2 \dot{\theta}^2 - \frac{V(\theta)}{\rho^2} \right). \]  

(22)

For arbitrary potential function \( V(\theta) \), the dynamical systems (20) and (21) admit as Lie point symmetries the vector fields \( \{X^1, X^2, X^3\} \) which are also Noether symmetries for the Lagrangian function (22). The corresponding conservation laws are the \( I_1(X^1), I_2(X^2) \) and \( I_3(X^3) \) where now

\[ \mathcal{H} = \frac{1}{2} \left( \dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \frac{V(\theta)}{\rho^2} \right). \]  

(23)

By using these three functions, we can define the following expression \(^{13}\)

\[ J(\rho, \dot{\rho}) = 4I_3(X^1)^2 - (I_2(X^2))^2, \]  

(24)

which is nothing else than the Ermakov–Lewis invariant, that is,

\[ J(\rho, \dot{\rho}) = \frac{1}{2} (\dot{\rho}^2 + \rho^{-2}). \]  

(25)

However, for specific forms of \( V(\theta) \), additional Noether symmetries can exist. Indeed when \( V(\theta) = V_0 \), Equation (21) reads \( (\mathcal{L}) = 0 \), where \( \mathcal{L} = \rho^2 \dot{\theta} \) is the angular momentum. The additional Noether symmetry in this case is the vector field \( X^4 = \partial_\phi \). The functional forms of \( V(\theta) \) for which additional variational symmetries exist are presented in Tsamparlis and Paliathanasis.\(^{33}\) We omit the presentation of the symmetry vector and we present the potential functions.

The potentials are \( V_A(\theta) = V_0, V_B(\theta) = V_0(\cos \theta)^{-2}, V_C(\theta) = V_0(\sin \theta)^{-2}, V_D(\theta) = V_1(\cos \theta)^{-2} + V_2(\sin \theta)^{-2}, \) and \( V_E(\theta) = V_0(V_1 \sin \theta - \cos \theta)^{-2}. \) We remark that potentials \( V_B(\theta) \) and \( V_C(\theta) \) are the same potential function, because
\[ \sin \theta = \cos \left( \theta - \frac{\pi}{2} \right). \] Thus, there are only four potentials. As we discussed, potential \( V_A(\theta) \) provides the conservation law of angular momentum.

For potential \( V_B(\theta) \), if we go into Cartesian coordinates, the Lagrangian function for the Ermakov system (19) is

\[ L_B (x, \dot{x}, y, \dot{y}) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 - \frac{V_0}{x^2} \right), \quad (26) \]

which provides the second-order differential equations \( \ddot{x} = \frac{V_0}{x^3}, \) \( \dot{y} = 0. \) The time-independent conservation laws are \( J (x, \dot{x}) = \frac{1}{2} \left( \dot{x}^2 + V_0 x^{-2} \right) \) and \( I = \dot{y}^2. \) The potential \( V_C(\theta) \) gives the same conservation laws with \( (x, y) \to (y, x). \)

For potential \( V_D(\theta) \) in Cartesian coordinates, the Lagrangian function becomes

\[ L_D (x, \dot{x}, y, \dot{y}) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 - \frac{V_1}{y^2} \right), \quad (27) \]

with equations of motion \( \ddot{x} = \frac{V_1}{y^3}, \) \( \dot{y} = \frac{\dot{x}}{y}, \) for which the conservation laws are \( J_1 (x, \dot{x}) = \frac{1}{2} \left( \dot{x}^2 + V_1 x^{-2} \right) \) and \( J_2 (y, \dot{y}) = \frac{1}{2} \left( \dot{y}^2 + V_1 y^{-2} \right). \)

Finally, for the potential function \( V_E(\theta) \), we have the Lagrangian

\[ L_E (x, \dot{x}, y, \dot{y}) = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 - \frac{V_0}{(ay-x)^2} \right), \quad (28) \]

By doing the second change of variables \( x = X + ay, \) it follows that \( L_E (X, \dot{X}, y, \dot{y}) = \frac{1}{2} \left( \dot{X}^2 + 2a\dot{X} \dot{Y} + (1 + a^2) \dot{y}^2 - \frac{V_0}{x^2} \right) \) for which the equations of motion are \( \ddot{X} = \frac{V_0(1+a^2)}{X^3}, \) \( \ddot{Y} = -\frac{aV_0}{X^2}. \) The time-independent conservation laws are \( J (X, \dot{X}) = \frac{1}{2} \left( \dot{X}^2 - V_0 x^{-2} \right) \) and \( I (X, \dot{X}, \dot{Y}) = (2a\dot{X} \dot{Y} + (1 + a^2) \dot{y}^2). \)

We have seen that in all cases for which additional point symmetries exist the Hamiltonian Ermakov system, when it is written in Cartesian coordinates, the potential function \( \frac{V(\theta)}{\rho^2} \) becomes \( V(x, y) = \frac{V_1}{x^2} + \frac{V_2}{y^2} \) and \( V(x, y) = \frac{V_0}{(ay-x)^2}. \) In these cases, the additional point symmetries follow from the the gradient Killing symmetries of the Euclidian space.

### 4 | THE 3D GENERALIZED ERMAKOV SYSTEM

Consider now the three-dimensional Ermakov system with Lagrangian function

\[ L (\rho, \dot{\rho}, \theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{1}{2} \left( \dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \rho^2 \sin^2 \theta \dot{\phi}^2 - \frac{V(\theta, \phi)}{\rho^2} \right), \quad (29) \]

and equations of motion

\[ \ddot{\rho} - \rho \dot{\phi}^2 - \rho \sin^2 \theta \dot{\phi}^2 - \frac{V(\theta, \phi)}{R^2} = 0, \quad (30) \]

\[ \ddot{\theta} + \frac{2}{\rho} \dot{\rho} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \frac{V(\theta, \phi) \theta}{2R^4} = 0, \quad (31) \]

\[ \ddot{\phi} + \frac{2}{\rho} \dot{\rho} \dot{\phi} + \cot \theta \dot{\theta} \dot{\phi} + \frac{V(\theta, \phi) \phi}{2R^4 \sin^2 \theta} = 0. \quad (32) \]

At this point, we discuss the analysis presented in Naz and Mahomed.\(^{29}\) The authors investigated the functional forms of \( V(\theta, \phi) \) for the dynamical system to admit additional conservation laws based upon point symmetries and contact symmetries. The authors determined the following functional forms for the potential \( V(\theta, \phi), \)

\[ V_A (\theta, \phi) = V_0, \quad V_B (\theta, \phi) = V_0 (\cos \theta)^2, \quad V_C (\theta, \phi) = V_0 (\sin \theta)^2, \quad (33) \]
\[
V_D (\theta, \phi) = V_0 (\cos \theta \sin \phi)^{-2}, \quad V_E (\theta, \phi) = V_0 (\cos \theta \cos \phi)^{-2}. \tag{34}
\]
\[
V_F (\theta, \phi) = \frac{8}{\cos (2\phi - 2\theta) + \cos (2\phi + 2\theta) + 2 \cos (2\theta) - 2 \cos (2\phi) + 6}. \tag{35}
\]
\[
V_G (\theta, \phi) = \frac{8}{2 \cos (2\phi) + 2 \cos (2\theta) - \cos (2\phi - 2\theta) - \cos (2\phi + 2\theta) + 6}. \tag{36}
\]

If we simplify potential functions \(V_F\) and \(V_G\) by using trigonometric identities, we find
\[
V_F (\theta, \phi) = (\cos^2 \phi \sin^2 \theta - 1)^{-1}, \tag{37}
\]
\[
V_G (\theta, \phi) = (1 - \sin^2 \phi \sin^2 \theta)^{-1}. \tag{38}
\]

As we discussed above, the potentials are related; for instance, \(V_F\) and \(V_C\) are the same potential function, as are also potentials \(V_D, V_E\) and \(V_F, V_G\). In order to understand that we should write the dynamical systems in Cartesians coordinates, we assume the transformation
\[
(x, y, z) = (r \sin \theta, r \cos \theta \sin \phi, r \cos \theta \cos \phi). \tag{39}
\]

Then
\[
\rho^{-2} V_C (\theta, \phi) = x^{-2}, \quad \rho^{-2} V_D (\theta, \phi) = y^{-2} \tag{40}
\]
and
\[
\rho^{-2} V_F (\theta, \phi) = (x^2 + z^2)^{-1}. \tag{41}
\]

while in a similar way, the remaining potentials reduce to similar potentials in Cartesians coordinates, by doing a change on the variables \(x, y, z\). These potentials are not new and have been found before and widely studied in Tsamparlis et al.\textsuperscript{34} In addition, the classification scheme in Naz and Mahomed\textsuperscript{29} is incomplete, for which see below.

We want to omit any calculation and use results from the previous section as also important properties of the symmetries for the Euclidian space \(\mathbb{R}^3\). There is a direct relation of the Noether symmetries with the elements of the homothetic algebra of the Euclidian space; for more details, we refer the reader in Tsamparlis et al.\textsuperscript{33} The proper homothetic vector of the \(\mathbb{R}^3\) is the generator of the two vector fields of the \(SL(2, \mathbb{R})\) algebra. However, the \(\mathbb{R}^3\) space admits six isometries. Three rotations which form the \(O(3)\) Lie algebra and three translations which form the \(T(3)\) Lie algebra. From the nature of the \(O(3)\), if an element of \(O(3)\) is a symmetry vector, then we can change the coordinates \(\phi \rightarrow (\phi, \bar{\phi})\), the symmetry vector to be always the \(\phi\) which provides the conservation law for the angular momentum \(\mathcal{L} = \rho^2 \sin^2 \theta \dot{x}^2\). If we assume that two elements of \(O(3)\) to be Noether symmetries, then it follows that \(V (\theta, \phi) = V_0\).

Therefore, when \(V (\theta, \phi) = V_A (\theta, \phi)\) or \(V (\theta, \phi) = V_A^2\), the additional Noether symmetries follow from the elements of the \(O(3)\) Lie algebra. On the other hand, if we assume that the symmetry is generated by an element of the \(T(3)\) Lie algebra, it follows that in the Cartesians coordinates, the system is similar to that found above for the two-dimensional system.

Indeed, we perform the symmetry classification
\[
V_A (x, y, z) = \frac{V_1}{x^2} + \frac{V_2}{y^2} + \frac{V_3}{z^2}, \tag{42}
\]
\[
V_B (x, y, z) = \frac{V_0}{x^2 + y^2} + \frac{V_1}{z^2}, \tag{43}
\]
\[
V_E = \frac{V_0}{(ax - \beta y - \gamma z)^2}, \quad V_E = \frac{V_0}{(ax - \beta y)^2} + \frac{V_1}{z^2}, \tag{44}
\]
where \(V(x, y, z) = \rho^{-2} V (\theta, \phi)\).
It is obvious that we have found new potentials which have not been presented before in Naz and Mahomed. Furthermore, if we apply the Killing tensors to determine new potential functions, from the algorithm described in Karpathopoulos et al, it is clear that there are no additional potential functions.

5 | CONCLUSIONS

In this work, we revised previous results on the classification problem of the three-dimensional Hamiltonian Ermakov system by using point and contact symmetries. We show that the classification problem solved in Naz and Mahomed is not correct and incomplete. We were able to show that with a constructive approach by using previous results which relate properties of the background space with the dynamical system.

We show that there is a natural extension on the classification scheme from the two-dimensional Ermakov system to the three-dimensional problem. Hence, we can use the analysis to elevate the solution in higher dimensions. We summarize our analysis in the following proposition.

**Proposition:** Consider the n-dimensional Hamiltonian Ermakov system generated by the Lagrangian function

\[
L(\rho, \dot{\rho}, \theta_1, \dot{\theta}_1, \ldots, \theta_{n-1}, \dot{\theta}_{n-1}) = \frac{1}{2} \left( \rho^2 + \rho^2 \left( \dot{\theta}_1^2 + \sin^2 \theta_1 \left( \dot{\theta}_2^2 + \sin^2(\ldots) \right) \right) - \frac{V(\theta_1, \theta_2, \ldots, \theta_{n-1})}{\rho^2} \right),
\]

which admits the $SL(2, \mathbb{R})$ as Noether symmetries. The functional forms for the potential $V(\theta_1, \theta_2, \ldots, \theta_{n-1})$ in which additional conservation laws linear or quadratic in the momentum are

\[
V(\theta_1, \theta_2, \ldots, \theta_{n-1}) = V_0,
\]

\[
V(x_1, x_2, \ldots, x_n) = \sum_i V_i (x_i)^{-2},
\]

\[
V(x_1, x_2, \ldots, x_n) = \sum_i V_i \frac{x_i}{x_i^2} + \sum_i \frac{V_i}{(x_i^2)^2}, \kappa \neq \lambda,
\]

\[
V(x_1, x_2, \ldots, x_n) = \sum_i V_i \frac{\alpha_i x_i}{(x_i^2)^2} + \sum_i \alpha_i V_i \frac{x_i}{(x_i^2)^2}, \kappa \neq \lambda,
\]

where $(x_1, x_2, \ldots, x_n)$ are the Cartesians coordinates for the n-dimensional Euclidian space and $\rho^{-2}V(\theta_1, \theta_2, \ldots, \theta_{n-1}) = V(x_1, x_2, \ldots, x_n)$.

In a future work, we plan to extend this procedure to other attempts on the generalization of the Ermakov system.

ACKNOWLEDGEMENTS

GL was funded by Comisión Nacional de Investigación Científica y Tecnológica (CONICYT) through FONDECYT Iniciación 11180126. GL thanks the Department of Mathematics and the Vicerrectoría de Investigación y Desarrollo Tecnológico at Universidad Católica del Norte for financial support.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

*Andronikos Paliathanasis* [https://orcid.org/0000-0002-9966-5517](https://orcid.org/0000-0002-9966-5517)

*Genly Leon* [https://orcid.org/0000-0002-1152-6548](https://orcid.org/0000-0002-1152-6548)

REFERENCES

1. Ermakov V. Univ. Isz. Kiev Series III 9, (1880) (translated by Harin A.O.)
2. Pinney E. The nonlinear differential equation $y + p(x)y + cy - 3 = 0$. *Proc Am Math Soc.* 1950;1:681-681.
3. Lewis HR. Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians. *Phys Rev Lett.* 1967;18:510.

4. Lewis HR, Riesenfeld WB. An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field. *J Math Phys.* 1969;10:1458-1473.

5. Paliathanasis A, Leach PGL. Symmetries of differential equations in cosmology. *Symmetry.* 2018;10:233.

6. Mitsopoulos A, Tsamparlis M. The generalized Ermakov conservative system: a discussion. *Eur Phys J Plus.* 2021;136:933.

7. Govinder KS, Leach PGL. Ermakov systems: a group theoretic approach. *Phys Lett A.* 1974;186:391-395.

8. Roberts J, Shkoller S, Sideris TC. Affine motion of 2D incompressible fluids surrounded by vacuum and flows in $SL(2, \mathbb{R})$-symmetries. *Comm Math Phys.* 2020;375:1003-1040.

9. Gu X, Ma WX, Zhang WY. Two integrable Hamiltonian hierarchies in $sl(2, \mathbb{R})$ and $so(3, \mathbb{R})$ with three potentials. *J Math Phys.* 2017;58:053512.

10. de la Cruz M, Gaspar N, Jimenez-Lara L, Linares R. Classification of the classical $SL(2, \mathbb{R})$ gauge transformations in the rigid body. *Ann Phys.* 2017;379:112-130.

11. Naz R, Mahomed FM. Hamiltonian symmetry classification, integrals, and exact solutions of a generalized Ermakov system. *Math Meth Appl Sci.* 2021;44:4467-4478.

12. Stephani H, Stephani H. *Differential Equations: Their Solutions Using Symmetry.* Cambridge: Cambridge University Press; 1989.

13. Noether E. Nachr. v.d. Ges. d. Wiss zu Gottingen, 235; 1918.

14. Halder A, Paliathanasis A, Leach PGL. Noether’s theorem and symmetr. *Symmetry.* 2018;10:744.

15. Tsamparlis M, Paliathanasis A. Two-dimensional dynamical systems which admit Lie and Noether symmetries. *J Phys A Math Theor.* 2011;44:175202.

16. Tsamparlis M, Paliathanasis A, Karpathopoulos L. Autonomous three-dimensional Newtonian systems which admit Lie and Noether point symmetries. *J Phys A Math Theor.* 2012;45:275201.

17. Karpathopoulos L, Tsamparlis M, Paliathanasis A. Quadratic conservation laws and collineations: a discussion. *J Geom Phys.* 2018;133:279-286.

**How to cite this article:** Paliathanasis A, Leon G, Leach PGL. Symmetries and conservation laws for the generalized $n$-dimensional Ermakov system. *Math Meth Appl Sci.* 2022;45(17):10710-10717. doi:10.1002/mma.8413