CONVERGENCE OF MULTIPLE ERGODIC AVERAGES

BERNARD HOST

Abstract. These notes are based on a course for a general audience given at the Centro de Modelamiento Matemático of the University of Chile, in December 2004.

We study the mean convergence of multiple ergodic averages, that is, averages of a product of functions taken at different times. We also describe the relations between this area of ergodic theory and some classical and some recent results in additive number theory.

In this paper we present some recent theorems of convergence of multiple ergodic averages. While the classical Ergodic Theorem describes the limit behavior of the time averages of a function, these theorems deal with averages of a product of functions taken at different times. We essentially focus on the case that the \( k \) functions are taken at times \( n, 2n, \ldots, kn \) (Theorem 1.2) but we also consider the case of polynomial times.

These convergence results belong to the field initiated by Furstenberg, exploring recurrence properties in ergodic theory and their relations with combinatorial number theory. The Correspondence Principle (see Section 1) provides a bridge between these two domains by allowing one to deduce combinatorial properties of sets of integers with positive upper density from recurrence theorems. For example, Szemerédi’s Theorem on the existence of arithmetic progressions in sets of positive upper density corresponds to Furstenberg’s Theorem about multiple recurrence along arithmetic progressions. The original proof of Szemerédi’s Theorem is purely combinatorial and some of its generalizations also have combinatorial proofs, apparently completely different from the ergodic ones. But recent progress in both fields leads to the intuition that there exists a hidden relation between the objects and methods of the two areas. Understanding this relation more precisely could be an interesting challenge. For this reason we do not completely ignore the combinatorial point of view in this paper.

As often happens, the convergence theorems of multiple ergodic averages will probably receive short self-contained proofs sometime in the future. But this is not the case at the present time and it would be
completely impossible to give complete proofs within the framework of this paper. Our more limited goal is to present most of the necessary tools and to summarize the main steps. Some partial results are given complete or partial proofs. These are neither the most difficult nor the “most important”, but they are chosen for the enlightenment they bring to some of the main ingredients and to their uses.

We hope that the majority of this text is accessible to a reader with a minimal knowledge in ergodic theory. For easier reading some notes have been postponed to the end of the sections and further comments are included in the Appendix. This material is not used in the main text and is intended as a supplemental material. The paper uses a lot of notation and some of it is not classical. We tried to keep notation as similar as possible to the original papers referred to. For an easier reading, we introduce the notation only when it is needed. However it should be noted that throughout, we use the symbol $\mathbb{N}$ with its European meaning: $\mathbb{N} = \{0, 1, 2, \ldots \}$.

1. Context and Results

1.1. Szemerédi’s Theorem. Our starting point is a celebrated theorem by Szemerédi. We begin with some definitions.

**Definition.** The upper density of a subset $E$ of $\mathbb{N}$ is:

$$d^*(E) = \limsup_{N \to \infty} \frac{1}{N} \text{Card}(E \cap \{0, 1, 2, \ldots, N - 1\}).$$

An arithmetic progression of length $k$ is a set of integers of the form

$$\{a, a + d, \ldots, a + (k - 1)d\}$$

where $a, d$ are integers and $d > 0$.

**Szemerédi’s Theorem (Sz2).** A subset of integers with positive upper density contains arbitrarily long arithmetic progressions.

The result was first conjectured by Erdős and Turán [ET] in 1936 and was solved by Roth [Ro] in 1953 for progressions of length 3 and by Szemerédi [Sz1] in 1969 for progressions of length 4. While Roth’s proof belongs to harmonic analysis, Szemerédi’s method is combinatorial and relies on graph theory.

This theorem can be reformulated in terms of finite sets:

**Theorem 1.1.** For every integer $k \geq 2$ and every real $\delta > 0$ there exists an integer $N(k, \delta)$ such that:

For every $N > N(k, \delta)$, every subset $E$ of $\{1, 2, \ldots, N\}$ with $\text{Card}(E) > \delta N$ contains an arithmetic progression of length $k$. 
Szemerédi’s Theorem follows immediately from its finite version and the converse implication uses a simple compactness argument. It is worth noting that this method cannot provide an explicit value for the constants $N(k,\delta)$. The original proof of Szemerédi did not give (usable) constants either and many in the combinatorial community competed to find the best constants for progressions of length 3 and 4 (the winner for length 3 was Bourgain [Bo2]) until a few years ago when Gowers [G1] gave a new proof with explicit constants for the general case. He proved the theorem in a more analytical form (see 1.4.1), already used by Roth and Bourgain:

**Theorem.** Let $N \geq 2$ be an integer and let $\mathbb{Z}/N\mathbb{Z}$ be endowed with its normalized Haar measure $m$. For every integer $k \geq 2$ and every real $\delta > 0$ there exists a constant $c(k,\delta) > 0$, not depending on $N$, such that:

For every function $f$ on $\mathbb{Z}/N\mathbb{Z}$ with $0 \leq f \leq 1$ and $\int f(x) \, dm(x) \geq \delta$, (1.1)

$$\int \int f(x) f(x+y) f(x+2y) \ldots f(x+(k-1)y) \, dm(x) \, dm(y) \geq c(k,\delta).$$

Gowers’ proof uses methods of both Fourier Analysis (the circle method) and combinatorial number theory, in particular Freiman’s results ([Fr1], [Fr2]) on the sums of sets of integers.

Although it does not seem related to our topic it would be difficult not to mention the spectacular new result by Green and Tao which answered a very old question.

**Theorem** (Green & Tao [GrT]). The set of primes contains arbitrarily long arithmetic progressions.

(See also Note [1.4.2]) There exists a relation between the proof of Green and Tao and ergodic theory, even though this relation is not completely understood at this time (see Appendix A.2).

1.2. **Furstenberg’s Theorem and its generalizations.** Before stating the result we fix some notation and some conventions.

In general, we write $(X, \mu)$ for a probability space, omitting the $\sigma$-algebra; when needed it is denoted by by the corresponding calligraphic letter $\mathcal{X}$. We always assume that $\mathcal{X}$ is countably generated and that $\mathcal{X}$ is the Borel $\sigma$-algebra whenever $X$ is endowed with a (Polish) topology. Throughout these notes, every subset of $X$ is implicitly assumed to be measurable and the term “bounded function on $X$” means bounded and measurable.

By a system, we mean a probability space $(X, \mu)$ endowed with an invertible, bi-measurable, measure preserving transformation $T : X \to X$.
and we write the system as \((X, \mu, T)\). In the main results the hypothesis of invertibility of \(T\) can be removed by passing to the natural extension.

In 1977 Furstenberg proved a very beautiful result about multiple recurrence in ergodic theory:

**Furstenberg’s Theorem** ([Fu2]; [FuKO] is easier to read). Let \((X, \mu, T)\) be a system, let \(A \subset X\) be a set with \(\mu(A) > 0\) and let \(k \geq 1\) an integer.

\[
\lim \inf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(k-1)n} A) > 0.
\]

In particular there exists \(n \geq 1\) such that \(\mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{(k-1)n} A) > 0\). Then Furstenberg deduced Szemerédi’s Theorem by using the following Correspondence Principle (with \(m_j = (j-1)n\) for \(j = 1, 2, \ldots, k\)):

**Furstenberg’s Correspondence Principle** ([Fu3]). Let \(E\) be a set of integers with positive upper density. There exist a system \((X, \mu, T)\) and a subset \(A\) of \(X\) with \(\mu(A) = d^*(E)\) such that

\[
d^*((E + m_1) \cap \cdots \cap (E + m_k)) \geq \mu(A \cap T^{m_1} A \cap \cdots \cap T^{m_k} A)
\]

for all integers \(k \geq 1\) and all \(m_1, \ldots, m_k \in \mathbb{N}\).

This indirect proof cannot provide explicit bounds in the finite version of Szemerédi’s Theorem (Theorem 1.1), but it is conceptually much simpler than the original proof. Moreover the direction initiated by Furstenberg’s Theorem led to several generalizations, each of them inducing its combinatorial counterpart by the Correspondence Principle (or by some variation of it). For some of these generalizations, there is still no known proof other than the ergodic theoretic proof. Below we only discuss two of these generalizations.

Recently Tao [T] gave a new proof of Szemerédi’s Theorem that can be viewed as a cross between combinatorial and ergodic methods: the proof is purely combinatorial in the sense that it deals only with finite sets (subsets of \(\mathbb{Z}/N\mathbb{Z}\)). But the vocabulary and the “philosophy” of the paper are much closer to ergodic theory. The proof uses in particular an induction that mimics the inductive construction of a sequence of extensions used in Furstenberg’s paper. It would be interesting to compare the way Tao uses Gowers’ norms with the way we use the ergodic seminorms in [HK2] (see Section 2 and Appendix A.2).

1.2.1. **The Polynomial Szemerédi Theorem.** In the next theorem proved by Bergelson and Leibman the exponents \(n, 2n, \ldots\), appearing in Furstenberg’s Theorem are replaced by integer polynomials \(p_1(n), p_2(n), \ldots\),
... (an integer polynomial is a polynomial taking integer values on the integers).

**Polynomial Multiple Recurrence Theorem** (Bergelson & Leibman [BL96]). Let \((X, \mu, T)\) be system, let \(\ell \geq 1\) be an integer and let \(p_1(n), p_2(n), \ldots, p_{\ell-1}(n)\) be integer polynomials with \(p_j(0) = 0\) for \(j = 1, 2, \ldots, \ell - 1\). For every \(A \subset X\) with \(\mu(A) > 0\) we have

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu\left( A \cap T^{p_1(n)} A \cap T^{p_2(n)} \cap \cdots \cap T^{p_{\ell-1}(n)} A \right) > 0 .
\]

The Correspondence Principle immediately gives:

**Polynomial Szemerédi’s Theorem** ([BL96]). Let \(E\) be a set of integers with positive upper density and \(p_1(n), p_2(n), \ldots, p_{\ell-1}(n)\) integer polynomials with \(p_j(0) = 0\) for \(j = 1, 2, \ldots, \ell - 1\). Then there exist integers \(a\) and \(d > 0\) such that

\[
\{a, a + p_1(d), a + p_2(d), \ldots, a + p_{\ell-1}(d)\} \subset E .
\]

Until now, the ergodic proof is the only known and in particular, there is no known combinatorial proof.

1.2.2. *The Multidimensional Szemerédi Theorem.* The following theorem of Furstenberg and Katznelson generalizes Furstenberg’s Theorem for several commuting transformations.

**Multidimensional Recurrence Theorem** (Furstenberg & Katznelson [FuK1]). Let \(k \geq 1\) be an integer and \(T_1, T_2, \ldots, T_k\) commuting measure preserving transformations of the probability space \((X, \mu)\). Then for any subset \(A\) of \(X\) with \(\mu(A) > 0\) we have

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T_1^n A \cap T_2^n A \cap \cdots \cap T_k^n A) > 0 .
\]

The original theorem corresponds to the case that \(T_j = T^{j-1}\) for \(j = 1, \ldots, k\).

The upper density of a subset of \(\mathbb{N}^k\) is defined analogously as for a subset of \(\mathbb{N}\). The combinatorial counterpart (see Note 1.4.3) of the theorem above is:

**Multidimensional Szemerédi’s Theorem** ([FuK1]). Let \(E \subset \mathbb{N}^k\) be a set of positive upper density. Then for any finite subset \(F\) of \(\mathbb{N}^k\) there exists \(a \in \mathbb{N}^k\) and an integer \(d > 0\) such that \(a + d F \subset E\).

Here, \(a + d F = \{a + d x : x \in F\}\). When \(F = \{0, 1, \ldots, \ell - 1\}^k\) the set \(a + d F\) can be called an arithmetic progression of dimension \(k\) and
length $\ell$, hence the name of the theorem. The first combinatorial proof of this result was given by Gowers \[G2\].

There exist several other generalizations of Furstenberg’s Theorem and for each of them a generalization of Szemerédi’s Theorem (see for example \[BM\]). The deepest result in this class is the Density Hales-Jewett Theorem \[FuK2\] of Furstenberg and Katznelson.

1.3. **Convergence results.** It is a natural question to ask whether the lim inf in Furstenberg’s Theorem and in its generalizations are actually limits. Our main goal in these lectures is to give an idea of the proof of the following results, that we refer to as *convergence theorems for multiple ergodic averages*. The first theorem shows the convergence for arithmetic progressions:

**Theorem 1.2** (Host & Kra \[HK2\]). Let $(X, \mu, T)$ be a system, $k \geq 1$ be an integer and $f_1, f_2, \ldots, f_k$ be bounded functions on $X$. Then the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_n^m x) f_2(T_n^{2m} x) \cdots f_k(T_n^{km} x)$$

converge in $L^2(\mu)$.

Taking $f_1 = f_2 = \cdots = f_k = 1_A$ and integrating (1.2) over $A$ we get that the lim inf in Furstenberg’s Theorem is a limit.

The convergence in $L^2(\mu)$ of these averages for $k = 3$ with the added hypothesis that the system is totally ergodic was shown by Conze and Lesigne in a series of papers (\[CL1\], \[CL3\], \[CL2\], see also \[Le5\]) and by Host and Kra \[HK1\] in the general case (see also \[FuW\]). Furstenberg \[Fu2\] proved the convergence for every $k$ under the assumption that the system is weakly mixing.

A similar convergence result also holds for polynomials:

**Theorem 1.3** (Host & Kra \[HK3\]; Leibman \[L3\]). Let $(X, \mu, T)$ be a system, $k \geq 1$ an integer, $p_1(n), p_2(n), \ldots, p_k(n)$ integer polynomials and $f_1, f_2, \ldots, f_k$ bounded functions on $X$. Then the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{p_1(n)} x) f_2(T^{p_2(n)} x) \cdots f_k(T^{p_k(n)} x)$$

converge in $L^2(\mu)$.

It follows that the averages appearing in the Polynomial Multiple Recurrence Theorem converge.

The result of Theorem 1.3 was proved by Bergelson \[B1\] for weakly mixing systems. Furstenberg and Weiss \[FuW\] showed the convergence
for two particular cases when \( k = 2 \): for \( p_1(n) = n \), \( p_2(n) = n^2 \) and for \( p_1(n) = n^2, p_2(n) = n^2 + n \). The paper [HK6] contains a proof for the general case, except when the system is not totally ergodic and at least one polynomial is of degree 1 and some other is of degree \( > 1 \). This restriction was lifted by Leibman [L3]. Frantzikinakis and Kra have shown that if the system is totally ergodic and the polynomials \( p_i \) are linearly independent, then the limit of the averages (1.3) is constant and equal to the product of the integrals of the functions \( f_i \).

The paper [HK2] also contains the proof of convergence of another type of multiple ergodic averages, the cubic averages: see Appendix A.1.

In the above results the averages on \([0, N - 1)\) can be replaced by averages on any sequence of intervals whose lengths tend to infinity. It is sufficient to prove these two theorems for ergodic systems, as the general case follows by ergodic decomposition. So we henceforth assume ergodicity.

The case of several commuting transformations remains essentially open. The problem can be formulated as follows:

**Question.** Let \( k \geq 1 \) be an integer and \( T_1, T_2, \ldots, T_k \) commuting measure preserving transformations of the probability space \((X, \mu)\). Is it true that for all bounded functions \( f_1, f_2, \ldots, f_k \) on \( X \) the averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x) \cdots f_k(T_k^n x)
\]

converge in \( L^2(\mu) \)?

The answer was shown to be positive for two transformations by Conze and Lesigne [CL1]. Frantzikinakis and Kra [FK2] proved the convergence for an arbitrary number of transformations under the additional hypothesis that all the transformations \( T_i \) and all the transformations \( T_i T_j^{-1} i \neq j \), are ergodic. These are very strong hypotheses: it can be assumed without loss that the transformations are jointly ergodic but not that any individual transformation is ergodic. The tools developed below for one transformation do not generalize to the case of several transformations.

The convergence almost everywhere of the different averages considered here is an open and probably very difficult problem. The unique result in this direction is due to Bourgain [Bo1]: the convergence a.e. of the averages (1.2) for \( k = 2 \).

1.4. Notes on Section 1
1.4.1. The analytic form. The finite version of Szemerédi’s Theorem (Theorem 1.1) follows easily from the analytic form, but the converse implication is more tricky. The following result is apparently stronger than Szemerédi’s Theorem but can be deduced from it (this is a non-trivial exercise):

**Theorem.** For every integer \( \ell \geq 2 \) and every real \( \delta > 0 \) there exists a constant \( c(\ell, \delta) > 0 \) such that:

For every integer \( N \geq 1 \), every subset \( E \) of \( \{1, 2, \ldots, N\} \) with more than \( \delta N \) elements contains at least \( c(\ell, \delta) N^2 - 1 \) arithmetic progression of length \( \ell \).

(The \(-1\) in the formula is simply a way to eliminate trivialities for small \( N \).)

This immediately implies the analytic form in the particular case that the function \( f \) takes its values in \( \{0, 1\} \). The general case follows by standard methods.

1.4.2. About the theorem of Green and Tao. The set \( P \) of primes satisfies

\[
\sum_{p \in P} \frac{1}{p} = +\infty
\]

and it is natural to ask whether every subset of \( \mathbb{N}^* \) with a divergent series of inverses contains arbitrarily long arithmetic progressions. This question was asked by Erdős and Turán [ET] in 1936 and is still open.

1.4.3. From the Multidimensional Recurrence Theorem to the Multidimensional Szemerédi Theorem. This implication uses a generalization of the Correspondence Principle that we state here. Let \( T_1, T_2, \ldots, T_k \) be commuting measure preserving transformations of the probability space \((X, \mu)\). For \( \mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k \) we write \( T^n = T_1^{n_1} T_2^{n_2} \ldots T_k^{n_k} \).

**Proposition.** Let \( E \subset \mathbb{N}^k \) be a set of positive upper density. There exist \( k \) measure preserving transformations \( T_1, T_2, \ldots, T_k \) of a probability space \((X, \mu)\) and a subset \( A \) of \( X \) with \( \mu(A) = d^*(E) \) and

\[
d^*(\bigcap_{\mathbf{n} \in F} (E + \mathbf{n})) \geq \mu\left(\bigcap_{\mathbf{n} \in F} T^n A\right)
\]

for every finite subset \( F \) of \( \mathbb{N}^k \).

Then apply the Recurrence Theorem with the commuting measure preserving transformations \( T^n, \mathbf{n} \in F \).
2. Nilmanifolds and nilsystems

We present here a class of systems for which the convergence results (Theorems 1.2 and 1.3) can be proven in an easier way. In the rest of these lectures we explain without details how the general case can be deduced from this particular one.

2.1. Definitions and fundamental properties. Let $G$ be a group. For $g, h \in G$ we write $[g, h] = g^{-1}h^{-1}gh$. The lower central series

$G = G_1 \supset G_2 \supset \cdots \supset G_j \supset G_{j+1} \supset \cdots$

of $G$ is defined by $G_1 = G$ and, for $j \geq 1$,

$G_{j+1}$ is the subgroup of $G$ spanned by $\{[g, h] : g \in G, h \in G_j\}$.

Let $k \geq 1$ be an integer. We say that $G$ is $k$-step nilpotent if $G_{k+1} = \{1\}$.

Let $G$ be a $k$-step nilpotent Lie group and let $\Lambda$ be a discrete cocompact subgroup. The compact manifold $X = G/\Lambda$ is called a $k$-step nilmanifold. The fundamental properties of nilmanifolds were established by Malcev [M]. We recall here only the properties we need.

The group $G$ naturally acts on $X$ by left translation and we write $(g, x) \mapsto g \cdot x$ for this action. There exists a unique Borel probability measure $\mu$ on $X$ invariant under this action, called the Haar measure of $X$. We make use of the following property which appears in [M] for connected groups and is proved in Leibman [L2] in a similar way for the general case:

- For every integer $j \geq 1$, the subgroups $G_j$ and $\Lambda G_j$ are closed in $G$. It follows that the group $\Lambda_j = \Lambda \cap G_j$ is cocompact in $G_j$.

Let $t$ be a fixed element of $G$ and let $T : X \to X$ be the transformation $x \mapsto t \cdot x$. Then $(X, T)$ is called a $k$-step topological nilsystem and $(X, \mu, T)$ is called a $k$-step nilsystem. All the notation introduced above is used freely throughout the rest of this section.

Fundamental properties of nilsystems were established by Auslander, Green and Hahn [AGH] and by Parry [P1]. Further ergodic properties were proven by Parry [P2] and Lesigne [Le4] when the group $G$ is connected, and generalized by Leibman (L2, L3). We state the results we need in the next two theorems.

**Theorem 2.1.** The following properties are equivalent:

- $(X, \mu, T)$ is ergodic.
- $(X, T)$ is uniquely ergodic, meaning that $\mu$ is the unique $T$-invariant probability measure on $X$.
- $(X, T)$ is minimal, meaning that every orbit under $T$ is dense.
• \((X, T)\) is transitive, meaning that there exists at least one dense orbit under \(T\).

The second theorem can be viewed as a particular case of general results of Ratner ([Ra]) and Shah ([Sh]).

**Theorem 2.2.** Let \(x \in X\) and let \(Y\) be the closure of the orbit of \(x\) under \(T\). Then \((Y, T)\) can be given the structure of a topological nilsystem, that is \(Y = H/\Gamma\) where \(H\) is a closed Lie subgroup of \(G\) containing \(t\) and \(\Gamma\) is a closed cocompact subgroup of \(H\).

By Theorem 2.1, \((Y, T)\) is uniquely ergodic. We immediately deduce:

**Corollary 2.3.** For every continuous function \(f\) on \(X\) the averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)
\]

converge for every \(x \in X\) as \(N \to +\infty\).

2.2. **Two examples.** We review the simplest examples of 2-step nilsystems.

2.2.1. Let \(G = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}\), with multiplication given by

\[
(k, x, y) \ast (k', x', y') = (k + k', x + x', y + y' + 2kx') .
\]

Then \(G\) is a Lie group. Its commutator subgroup is \(\{0\} \times \{0\} \times \mathbb{T}\) and \(G\) is 2-step nilpotent. The subgroup \(\Lambda = \mathbb{Z} \times \{0\} \times \{0\}\) is discrete and cocompact. Let \(X\) denote the nilmanifold \(G/\Lambda\) and we maintain the notation of the preceding section.

Fix \(\alpha \in \mathbb{T}\) and define \(t = (1, \alpha, \alpha) \in G\) and let \(T : X \to X\) be the translation by \(t\). Then \((X, \mu, T)\) is a 2-step nilsystem. It can be shown that it is ergodic if and only if \(\alpha\) is irrational.

We give an alternate description of this system. The map \((k, x, y) \mapsto (x, y)\) from \(G\) to \(\mathbb{T}^2\) induces a homeomorphism of \(X\) onto \(\mathbb{T}^2\). Identifying \(X\) with \(\mathbb{T}^2\) via this homeomorphism, the measure \(\mu\) becomes equal to \(m_\mathbb{T} \times m_\mathbb{T}\) where \(m_\mathbb{T}\) is the Haar measure of \(\mathbb{T}\) and the transformation \(T\) of \(X\) is given for \((x, y) \in \mathbb{T}^2 = X\) by

\[
T(x, y) = (x + \alpha, y + 2x + \alpha) .
\]

2.2.2. Let \(G\) be the Heisenberg group \(\mathbb{R} \times \mathbb{R} \times \mathbb{R}\), with multiplication given by

\[
(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + xy') .
\]

Then \(G\) is a 2-step nilpotent Lie group. The subgroup \(\Lambda = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) is discrete and cocompact. Let \(X = G/\Lambda\) and let \(T\) be the translation
by some \( t = (t_1, t_2, t_3) \in G \). We have that \((G/\Lambda, T)\) is a nilsystem. It can be showed that it is ergodic if and only if \( t_1 \) and \( t_2 \) are independent over \( \mathbb{Q} \).

2.3. **Convergence of multiple ergodic averages for nilsystems.**

Let \((X, \mu, T)\) be a nilsystem. We use here the notation of Subsection 2.1.

Let \( k \geq 2 \) be an integer. Then \( X^k \) can be given the structure of a nilmanifold, quotient of the Lie group \( G^k \) by its discrete cocompact subgroup \( \Lambda^k \). Set \( s = (t, t^2, \ldots, t^k) \in G^k \) and let \( S \) be the translation by \( s \) on \( X^k \). Then \((X^k, S)\) is a topological nilsystem.

Let \( f_1, f_2, \ldots, f_k \) be continuous functions on \( X \). By applying Theorem 2.2 to the nilsystem \((X^k, S)\) and to the continuous function \((x_1, x_2, \ldots, x_k) \mapsto f_1(x_1)f_2(x_2) \cdots f_k(x_k)\) at the point \((x, x, \ldots, x) \in X^k\) we get that the average

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{nx})f_2(T^{2nx}) \cdots f_k(T^{knx})
\]

converges for every \( x \in X \). By a standard density argument we have:

**Corollary 2.4.** *Theorem 1.2 holds for nilsystems.*

An explicit expression of the limit was given by Ziegler [Z1] (a shorter proof can be found in [BHK]).

A similar result holds for the polynomial averages:

**Corollary 2.5 ([L2]).** *Theorem 1.3 holds for nilsystems.*

In place of Corollary 2.3 the proof uses an extension due to Leibman of this result for polynomial sequences in a nilmanifold.

3. **Some seminorms**

In this Section and the next ones \((X, \mu, T)\) is an ergodic system. We introduce a sequence of seminorms of \( L^\infty(\mu) \) that we use to bound the different averages under consideration. These seminorms should be compared with the norms introduced by Gowers in a completely different context: see Appendix A.2.

3.1. **Notation.** We need some notation to be used throughout the remainder of these notes.

We write \( C: \mathbb{C} \to \mathbb{C} \) for the conjugacy map \( z \mapsto \bar{z} \). Let \( k \geq 1 \) be an integer. We write \( \varepsilon = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_k \) with \( \varepsilon_i \in \{0, 1\} \) for a point of \( \{0, 1\}^k \), without commas or parentheses and \( |\varepsilon| = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k \). \( 0 \) denotes the point \( 00 \ldots 0 \in \{0, 1\}^k \).
For every integer \( k \geq 0 \) we write \( X^k = X^{2^k} \) and for \( k \geq 1 \) the points of \( X^k \) are written \( x = (x_\varepsilon : \varepsilon \in \{0,1\}^k) \). We write \( T^k \) for the transformation \( T \times T \times \cdots \times T \) (\( 2^k \) times) of this space. We often identify \( X^{k+1} \) with \( X^k \times X^k \) writing \( x = (x', x'') \) for a point of \( X^{k+1} \) where \( x', x'' \in X^k \) are defined by

\[
\text{for every } \varepsilon \in \{0,1\}^k, \ x'_\varepsilon = x_{\varepsilon 0} \text{ and } x''_{\varepsilon} = x_{\varepsilon 1}.
\]

### 3.2. Construction of some measures and some seminorms

We define by induction a \( T^k \)-invariant measure \( \mu^k \) on \( X^k \) for every integer \( k \geq 0 \). Set \( \mu^0 = \mu \). Assume that \( \mu^k \) is defined for some \( k \geq 0 \). Let \( \mathcal{I}^k \) denote the \( T^k \)-invariant \( \sigma \)-algebra of \( (X^k, \mu^k, T^k) \). Identifying \( X^{k+1} \) with \( X^k \times X^k \) as explained above, we define the system \( (X^{k+1}, \mu^{k+1}, T^{k+1}) \) to be the relatively independent joining of two copies of \( (X^k, \mu^k, T^k) \) over \( \mathcal{I}^k \).

This means that \( \mu^{k+1} \) is the measure on \( X^{k+1} = X^k \times X^k \) characterized by:

when \( F' \) and \( F'' \) are bounded functions on \( X^k \),

\[
\int_{X^{k+1}} F'(x') F''(x'') d\mu^{k+1}(x) = \int_{X^k} \mathbb{E}(F' \mid \mathcal{I}^k) \mathbb{E}(F'' \mid \mathcal{I}^k) d\mu^k.
\]

This measure is invariant under \( T^{k+1} = T^k \times T^k \) and each of its two natural projections on \( X^k \) is equal to \( \mu^k \). Note that when \( F \) is a function on \( X^k \), measurable with respect to \( \mathcal{I}^k \) that is invariant under \( T^k \), we have

\[
F(x') = F(x'') \text{ for } \mu^{k+1} \text{-almost every } x = (x', x'') \in X^{k+1}.
\]

By induction, each of the \( 2^k \) natural projections of \( \mu^k \) on \( X \) is equal to \( \mu \). It follows immediately from this definition that for every bounded function \( f \) on \( X \) the integral

\[
\int_{X^k} \prod_{\varepsilon \in \{0,1\}^k} C^{[\varepsilon]} f(x_\varepsilon) d\mu^k(x)
\]

is real and nonnegative and we can define

\[
\|f\|_k = \left( \int_{X^k} \prod_{\varepsilon \in \{0,1\}^k} C^{[\varepsilon]} f(x_\varepsilon) d\mu^k(x) \right)^{1/2^k}.
\]

As \( X \) is assumed to be ergodic, the \( \sigma \)-algebra \( \mathcal{I}^0 \) is trivial and \( \mu^1 = \mu \times \mu \). We therefore have

\[
\|f\|_1 = \left( \int_{X \times X} f(x_0) f(x_1) d\mu \times \mu(x_0, x_1) \right)^{1/2} = \left| \int f(x) d\mu(x) \right|.
\]
From the inductive definition (3.1) of the measures we get that
\[ \|f\|_{k+1} \geq \|f\|_k \] for every \( f \) and every \( k \).

The following Lemma follows immediately from the definition of the measures and the Ergodic Theorem.

**Lemma 3.1.** For every integer \( k \geq 0 \) and every bounded function \( f \) on \( X \),
\[ \|f\|_{k+1} = \left( \lim_{N \to \infty} \sum_{n=0}^{N-1} \|f \cdot T^n f\|_k^2 \right)^{1/2k+1}. \]

3.3. The Kronecker factor and the measure \( \mu^{[2]} \).

3.3.1. The notion of a factor. As usual in ergodic theory, we use the term factor with two different but equivalent meanings. First, a factor of the system \((X, \mu, T)\) is a \( T \)-invariant sub-\( \sigma \)-algebra \( \mathcal{Y} \) of \( X \).

On the other hand, if \((Y, \nu, S)\) is a system and \( \pi: X \to Y \) is a measurable map mapping \( \mu \) to \( \nu \) and such that \( S \circ \pi = \pi \circ T \) (\( \mu \)-a.e.), then \( \pi \) is called a factor map and \( Y \) is also called a factor of \( X \). In this situation we always identify the \( \sigma \)-algebra \( \mathcal{Y} \) of \( Y \) with its inverse image \( \pi^{-1}(\mathcal{Y}) \), which is an invariant sub-\( \sigma \)-algebra of \( X \), that is a factor of \( X \) under the first definition. It is thus natural to denote the transformation on \( Y \) by the same letter as the transformation on \( X \), meaning by \( T \) in our case.

It can be shown that every invariant sub-\( \sigma \)-algebra of \( X \) can be associated to a factor map in this way and thus the two definitions are functionally equivalent. We pass freely from one definition to the other.

Let \( f \) be an integrable function on \( X \). We consider \( \mathbb{E}(f \mid \mathcal{Y}) \) as a function defined on \( X \) and we write \( \mathbb{E}(f \mid Y) \) for the function on \( Y \) defined by \( \mathbb{E}(f \mid Y) \circ \pi = \mathbb{E}(f \mid \mathcal{Y}) \). It is characterized by:
\[ \forall g \in L^\infty(\nu), \int_Y \mathbb{E}(f \mid Y)(y)g(y)\,d\nu(y) = \int_X f(x)g(\pi(x))\,d\mu(x). \]

3.3.2. The Kronecker factor. The Kronecker factor of the system \( X \) is written \( Z_1(X) \) or \( Z_1 \) when the system under consideration is clear from the context. We recall here the definition and some classical properties.

Viewed as a \( \sigma \)-algebra, the Kronecker factor \( Z_1 \) of \( X \) is defined to be the sub-\( \sigma \)-algebra of \( X \) generated by the eigenfunctions of this system; is also the smallest sub-\( \sigma \)-algebra of \( X \) such that all the invariant functions of the system \((X \times X, \mu \times \mu, T \times T)\) are measurable with respect to \( Z_1 \otimes Z_1 \).
When considered as a system, the Kronecker factor \((Z_1, m, T)\) is a rotation; this means that \(Z_1\) is a compact abelian group with Haar measure \(m\) and that the transformation \(T\) has the form \(z \mapsto z + \alpha\) where \(\alpha\) is a fixed element of \(Z_1\).

We write \(\pi_1: \times X \to Z_1\) for the factor map. Then every eigenfunction of \(X\) has the form \(f(x) = c \cdot \chi(\pi_1(x))\) where \(c\) is a constant and \(\chi\) is a character of \(Z_1\), that is a continuous group homomorphism from \(Z_1\) to the circle group \(S^1\). Every \(T \times T\)-invariant function on \(X \times X\) can be written \(f(x, y) = g(\pi_1(x) - \pi_2(y))\) where \(g\) is a function on \(Z_1\). Therefore, when \(f_0\) and \(f_1\) are bounded functions on \(X\),

\[
\mathbb{E}(f_0 \otimes f_1 | \mathcal{I}^{[1]})(x_0, x_1) = \int_{Z_1} \mathbb{E}(f_0 | Z_1)(z) \mathbb{E}(f_1 | Z_1)(z + \pi(x_1) - \pi(x_0)) \, dm(z) .
\]

**3.3.3. The measure \(\mu^{[2]}\).** We deduce a more explicit expression for the measure \(\mu^{[2]}\): when \(f_{\varepsilon}, \varepsilon \in \{0, 1\}^2\) are four measurable functions on \(X\),

\[
\begin{align*}
(3.5) \quad & \int_{X^{[2]}} f_{00}(x_{00}) f_{01}(x_{01}) f_{10}(x_{10}) f_{11}(x_{11}) \, d\mu^{[2]}(x) \\
= & \int_{Z_1 \times Z_1 \times Z_1} \tilde{f}_{00}(z) \tilde{f}_{01}(z+s) \tilde{f}_{10}(z+t) \tilde{f}_{11}(z+s+t) \, dm(z) \, dm(s) \, dm(t)
\end{align*}
\]

where we write \(\tilde{f}_{\varepsilon}\) for \(\mathbb{E}(f_{\varepsilon} | Z_1)\). This implies that \(\|f\|_2\) is the \(\ell^4\)-norm of the Fourier Transform of \(\mathbb{E}(f | Z_1)\). In particular:

**Lemma 3.2.** For every bounded function \(f\) on \(X\), \(\|f\|_2 = 0\) if and only if \(\mathbb{E}(f | Z_1) = 0\).

We give one more formula for the measure \(\mu^{[2]}\). For every \(s \in Z_1\) we define a probability measure \(\mu_s\) on \(X \times X\) by

\[
(3.6) \quad \int_{X \times X} f_0(x_0) f_1(x_1) \, d\mu_s(x_0, x_1) = \int_{Z_1} \mathbb{E}(f_0 | Z_1)(z) \mathbb{E}(f_1 | Z_1)(z+s) \, dm(z) .
\]

This measure is invariant under \(T \times T\) and we have

\[
(3.7) \quad \mu \times \mu = \int_{Z_1} \mu_s \, dm(s) .
\]

From the remarks above it follows that this formula is the ergodic decomposition of \(\mu \times \mu\) under \(T \times T\); in particular the system \((X \times X, \mu_s, T \times T)\) is ergodic for \(m\)-almost every \(s \in Z_1\). We have:

\[
\mu^{[2]} = \int_{Z_1} \mu_s \times \mu_s \, dm(s)
\]
(notice that $\mu_s \times \mu_s$ is a measure on $(X \times X) \times (X \times X) = X^2$).

### 3.4. Seminorms and arithmetic progressions.

Later we prove that for every $k \geq 0$ the map $f \mapsto \|f\|_k$ is a seminorm on $L^\infty(\mu)$. Admitting this for the moment we show now how these seminorms arise in the questions of convergence we are studying.

**Proposition 3.3.** Let $f_1, f_2, \ldots, f_k$ be bounded functions on $X$ with $\|f_\ell\|_\infty \leq 1$ for $\ell = 1, 2, \ldots, k$. Then

$$
\limsup_{N \to +\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^nx)f_2(T^{2n}x) \cdots f_k(T^{kn}x) \right\|_2 \leq \min_{1 \leq \ell \leq k} (\ell.\|f_\ell\|_k)
$$

The proof relies on an iterated use of a Hilbert space variant of the van der Corput Lemma.

**Van der Corput Lemma** ([B1]). Let $\{\xi_n\}$ be a sequence in a Hilbert space $\mathcal{H}$, with $\|\xi_n\| \leq 1$ for every $n$. Then

$$
\limsup_{N \to +\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \xi_n \right\|^2 \leq \limsup_{N \to +\infty} \frac{1}{H} \sum_{h=0}^{H-1} \limsup_{N \to +\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \langle \xi_n \mid \xi_{n+h} \rangle \right\|
$$

**Proof of Proposition 3.3.** We proceed by induction. For $k = 1$ the bound is given by the Ergodic Theorem and the definition of $\| \cdot \|$. Let $k \geq 2$ and assume that the bound holds for $k - 1$. Let $f_1, \ldots, f_k$ be as in the theorem, and choose $\ell \in \{2, \ldots, k\}$ (the case $\ell = 1$ is similar).

Write

$$
\xi_n(x) = f_1(T^nx)f_2(T^{2n}x) \cdots f_k(T^{kn}x)
$$

For every $h \geq 0$, by using the Cauchy-Schwarz Inequality and the invariance of $\mu$ under $T^n$ we get

$$
\left| \frac{1}{N} \sum_{n=0}^{N-1} \langle \xi_n \mid \xi_{n+h} \rangle \right| \leq \|f_1 \cdot f_1 \circ T^h\|_2 \cdot \left\| \sum_{n=0}^{N-1} \left( \prod_{i=2}^k (f_i \cdot f_i \circ T^h) \circ T^{(i-1)n} \right) \right\|_2.
$$

By the inductive assumption,

$$
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \langle \xi_n \mid \xi_{n+h} \rangle \right| \leq \ell.\|f_\ell \cdot f_\ell \circ T^h\|_{k-1}.
$$
By the van der Corput Lemma,

$$\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \xi_n \right|^2 \leq \ell_0 \limsup_{H \to +\infty} \frac{1}{H} \sum_{h=0}^{H-1} \| f_\ell \cdot T^{eh} \|_{k-1}$$

$$\leq \ell_0 \limsup_{H \to +\infty} \frac{1}{H} \sum_{h=0}^{H-1} \| f_\ell \cdot T^h \|_{k-1}$$

$$\leq \ell_0 \limsup_{H \to +\infty} \left( \frac{1}{H} \sum_{h=0}^{H-1} \| f_\ell \cdot T^h \|_{k-1}^{2k-1} \right)^{1/2k-1}$$

and this last term is equal to $\ell_0 \| f_\ell \|_k^2$ by Lemma 3.1. □

3.5. Seminorms and polynomial averages. A similar bound holds for the averages (1.3) considered in Theorem 1.3.

**Proposition 3.4.** Let $k \geq 1$ be an integer and $p_1, p_2, \ldots, p_k$ be integer nonconstant polynomials such that for every $1 \leq i \neq j \leq k$ the polynomial $p_i - p_j$ is not constant. There exists an integer $\ell \geq 1$ such that for any bounded functions $f_1, f_2, \ldots, f_k$ on $X$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{p_1(n)}x)f_2(T^{p_2(n)}x)\ldots f_k(T^{p_k(n)}x)$$

converges to zero in $L^2(\mu)$ whenever $\| f_i \|_\ell = 0$ for at least one value of $i \in \{1, 2, \ldots, k\}$.

The proof uses induction on the polynomial family $(p_1, p_2, \ldots, p_k)$, by using the van der Corput Lemma and the Cauchy-Schwarz Inequality at each step. However this induction is much more intricate than for arithmetic progressions and we do not present it here.

3.6. Invariance properties. Let $\{0, 1\}^k$ be identified with the set of vertices of the unit cube $[0, 1]^k$.

We call the group of the $k$-cube the group of isometries of the Euclidean cube $[0, 1]^k$. We consider this group as acting on $X^k$ in the following way. Each element $\sigma$ of the group induces a permutation, written $\sigma$ also, of the set $\{0, 1\}^k$ of vertices and this permutation in turn induces a transformation $\sigma_*$ of $X^k$ by:

$$\text{for every } \varepsilon \in \{0, 1\}^k, \ (\sigma_* x)_\varepsilon = x_{\sigma(\varepsilon)} .$$

**Lemma 3.5.** For every integer $k$ the measure $\mu^k$ is invariant under the action of the group of the $k$-cube.

An inequality similar to the classical Cauchy-Schwarz inequality can be proven inductively by using this symmetry property:
Lemma 3.6. If $f_\varepsilon, \varepsilon \in \{0, 1\}^k$, are $2^k$ bounded functions on $X$ then
\begin{equation}
\left| \int \prod_{\varepsilon \in \{0, 1\}^k} f_\varepsilon(x_\varepsilon) \, d\mu^{[k]}(x) \right| \leq \prod_{\varepsilon \in \{0, 1\}^k} \|f_\varepsilon\|_k .
\end{equation}

It follows that the map $f \mapsto \|f\|_k$ is subadditive:

**Corollary.** For every $k \geq 1$, $\| \cdot \|_k$ is a seminorm on $L^\infty(\mu)$.

We use the geometric vocabulary for subsets of $\{0, 1\}^k$; for example, a side is a subset of $\{0, 1\}^k$ of the form $\{\varepsilon: \varepsilon_i = 0\}$ or $\{\varepsilon: \varepsilon_i = 1\}$ for some $i \in \{1, \ldots, k\}$. When $\alpha$ is a side of $\{0, 1\}^k$ we define the side transformation $T^{[k]}_\alpha$ of $X^{[k]}$ by:
\begin{equation}
(T^{[k]}_\alpha x)_\varepsilon = \begin{cases} \ x_\varepsilon & \text{if } \varepsilon \in \alpha; \\ \ T x_\varepsilon & \text{otherwise.} \end{cases}
\end{equation}

Notice that the product of two side transformations corresponding to opposite sides is equal to $T^{[k]}$.

When $\alpha$ is the side $\{0, 1\}^{k-1} \times \{1\}$, $T^{[k]}_\alpha$ is equal to $\text{Id}^{[k-1]} \times T^{[k-1]}$ under the identification of $X^{[k]}$ with $X^{[k-1]} \times X^{[k-1]}$. By definition of the measure $\mu^{[k]}$, it is invariant under this transformation. As the group of the $k$-cubes leaves the measure $\mu^{[k]}$ invariant and acts transitively on the sides, we get:

**Lemma 3.7.** For every integer $k$ the measure $\mu^{[k]}$ is invariant under all side transformations.

By induction, it can be checked that:

**Lemma 3.8.** The measure $\mu^{[k]}$ is ergodic under the joint action of the side transformations.

3.7. Notes on Section 3. The measures $\mu^{[k]}$ can be described explicitly in the case that the given ergodic system $(X, \mu, T)$ is a $k$-step nilsystem. We put $X = G/\Lambda$ and use the notation of Section 2.

Let $k \geq 1$ be an integer. For every $g \in G$ and every side $\alpha$ of $\{0, 1\}^k$ we define an element $g^{[k]}_\alpha$ of $G^{[k]}$ by
\[ (g^{[k]}_\alpha)_\varepsilon = \begin{cases} g & \text{if } \varepsilon \in \alpha; \\ 1 & \text{otherwise.} \end{cases} \]

We define the side group of dimension $k$ to be the subgroup $G^{[k]}_{k-1}$ of $G^{[k]}$ spanned by all these elements. It can be checked that $G^{[k]}_{k-1}$ is a closed Lie subgroup of $G^{[k]}$ and that the subgroup $\Lambda^{[k]}_{k-1}$ of $\Lambda^{[k]}$ defined in the same way is equal to $\Lambda^{[k]} \cap G^{[k]}_{k-1}$ and is discrete and cocompact.
in \( G_{k-1}^{[k]} \). The nilmanifold \( X_k = G_{k-1}^{[k]}/\Lambda_{k-1}^{[k]} \) is naturally imbedded in \( X^{[k]} \) and the measure \( \mu^{[k]} \) is the Haar measure of this nilmanifold.

4. Building factors

In this Section we continue to assume that \((X, \mu, T)\) is an ergodic system. Lemma 3.2 gives a simple relation between the seminorm \( \| \cdot \|_2 \) and the Kronecker factor \( Z_1 \) introduced in Subsection 3.3.2. For every \( k \geq 2 \) we define here a factor \( Z_k \) of \( X \) with a similar relation to the seminorm \( \| \cdot \|_{k+1} \).

4.1. The definition of the factors \( Z_k \). Let \( k \geq 1 \) be an integer. In the following construction the coordinate of \( X^{[k]} \) indexed by \( 0 \) plays a particular role. Due to the symmetry of the measure \( \mu^{[k]} \), any other choice is possible with the same results, up to obvious changes in notation.

We note \( X^{[k]}_* = X^{2^k-1} \) and write each point \( x \in X^{[k]} \) as \( x = (x_0, \tilde{x}) \),

where

\[
\tilde{x} = (x_\varepsilon : \varepsilon \in \{0, 1\}^k \setminus \{0\}) \in X^{[k]}_*
\]

and thus we identify \( X^{[k]} \) with \( X \times X^{[k]}_* \).

Recall that the projection of \( \mu^{[k]} \) on \( X \) is equal to \( \mu \). We write \( \mu^{[k]}_* \) for the projection of \( \mu^{[k]} \) on \( X^{[k]}_* \). This measure is invariant under \( T^{[k]}_* = T \times T \times \cdots \times T \) (2\( k \) – 1 times). We say that the system \((X^{[k]}, \mu^{[k]}, T^{[k]})\) is a joining of the systems \((X, \mu, T)\) and \((X^{[k]}_*, \mu^{[k]}_*, T^{[k]}_*)\).

Let \( T_1^{[k]}, T_2^{[k]}, \ldots, T_k^{[k]} \) be the side transformations of \( X^{[k]} \) corresponding to the \( k \) sides of \( \{0, 1\}^k \) not containing \( 0 = (0, 0, \ldots, 0) \): For \( 1 \leq i \leq k \), \( x \in X^{[k]} \) and \( \varepsilon \in \{0, 1\}^k \),

\[
(T_i^{[k]} x)_\varepsilon = \begin{cases} x_\varepsilon & \text{if } \varepsilon_i = 0 ; \\ T x_\varepsilon & \text{if } \varepsilon_i = 1 . \end{cases}
\]

For \( 1 \leq i \leq k \) the transformation \( T_i^{[k]} \) leaves the coordinate indexed by \( 0 \) invariant and thus it can be written \( \text{Id} \times T_i^{[k]}_* \) for some measure preserving transformation \( T_i^{[k]}_* \) of \( X^{[k]}_* \). Let \( \mathcal{J}^{[k]}_* \) be the \( \sigma \)-algebra of subsets of \( X^{[k]}_* \) which are invariant under the transformations \( T_i^{[k]}_* \), \( 1 \leq i \leq k \). An induction using relation (3.2) gives:

Lemma 4.1. Let \( B \) be a subset of \( X^{[k]}_* \). Then \( B \) belongs to \( \mathcal{J}^{[k]}_* \) if and only if there exists a subset \( A \) of \( X \) with

\[
1_A(x_0) = 1_B(\tilde{x}) \text{ for } \mu^{[k]} \text{-almost every } x \in X^{[k]}.
\]

This relation between \( B \) and \( A \) defines a bijection (up to null sets) between the \( \sigma \)-algebra \( \mathcal{J}^{[k]}_* \) and some \( \sigma \)-algebra of \( X \). We define:
Definition. \( Z_{k-1}(X) \) is the sigma-algebra of subsets \( A \) of \( X \) such that equality \((4.1)\) holds for some subset \( B \) of \( X^{[k]} \).

\( Z_1(X) \) was already defined to be the Kronecker factor of \( X \). Below (Corollary 4.3) we show that the two definitions coincide. We write \( Z_{k-1} \) instead of \( Z_{k-1}(X) \) whenever it is possible without ambiguity. This \( \sigma \)-algebra is clearly invariant under \( T \) and so it is a factor map. Let the associated factor system be denoted by \((Z_{k-1}, \mu_{k-1}, T)\) or by \((Z_{k-1}, \mu_{k-1}, T)\) and let \( \pi_{k-1} : X \to Z_{k-1} \) be the factor map.

4.2. Elementary properties.

Proposition 4.2.

i) Consider the \( \sigma \)-algebra \( Z_{k-1} \) on \( X \) and the \( \sigma \)-algebra \( J^{[k]} \) on \( X^{[k]} \) are identified by relation \((4.1)\). Then \( (X^{[k]}, \mu^{[k]}) \) is the relatively independent joining of \( (X, \mu) \) and \( (X^{[k]}, \mu^{[k]}) \) over their common \( \sigma \)-algebra \( Z_{k-1} = J^{[k]} \).

ii) For a bounded function \( f \) on \( X \), \( \|f\|_k = 0 \) if and only if \( \mathbb{E}(f \mid Z_{k-1}) = 0 \).

iii) The measure \( \mu^{[k]} \) is relatively independent over its projection on \( Z_{k-1}^{[k]} \); this means that when \( f_{\varepsilon}, \varepsilon \in \{0, 1\}^k \), are bounded functions on \( X \) then

\[
\int_{X^{[k]}} \prod_{\varepsilon \in \{0, 1\}^k} f_{\varepsilon}(x_{\varepsilon}) \, d\mu^{[k]}(x) = \int_{X^{[k]}} \prod_{\varepsilon \in \{0, 1\}^k} \mathbb{E}(f_{\varepsilon} \mid Z_{k-1})(x_{\varepsilon}) \, d\mu^{[k]}(x) .
\]

Moreover, \( Z_{k-1} \) is the smallest factor of \( X \) with this property.

iv) Every \( T^{[k-1]} \)-invariant subset of \( X^{[k-1]} \) is measurable with respect to \( Z_{k-1}^{[k-1]} \). Moreover \( Z_{k-1}^{[k-1]} \) is the smallest factor of \( X \) with this property.

Proof.

The meaning of this statement is perhaps not obvious and we begin with some explanation. We have already introduced the factor map \( \pi_{k-1} : X \to Z_{k-1} \). As we identify \( Z_{k-1} \) with the \( \sigma \)-algebra \( J^{[k]} \) on \( X^{[k]} \), we have also a factor map \( p_{k-1} : X^{[k]} \to Z_{k-1} \) with \( J^{[k]} = p_{k-1}^{-1}(Z_{k-1}) \). Relation \((4.1)\) can be written

\[
\pi_{k-1}(x_0) = p_{k-1}(\bar{x}) \text{ for } \mu^{[k]}-\text{almost every } x = (x_0, \bar{x}).
\]

When \( F \) is an integrable function on \( X^{[k]} \), following our standard convention we write \( \mathbb{E}(F \mid Z_{k-1}) \) for the function on \( Z_{k-1} \) given by \( \mathbb{E}(F \mid Z_{k-1}) \circ p_{k-1} = \mathbb{E}(F \mid J^{[k]}) \). Statement \( i) \) means that for every
bounded function $f$ on $X$ and every bounded function $F$ on $X^{[k]}$,
\[(4.4) \quad \int f(x_0) F(\tilde{x}) d\mu^{[k]}(x) = \int \mathbb{E}(f \mid Z_{k-1}) \mathbb{E}(F \mid Z_{k-1}) d\mu_{k-1}.
\]

This relation is similar to formula (3.1) used to define the measure $\mu^{[k+1]}$ in Section 3.2.

Let $f$ and $F$ be as above. For $i = 1, 2, \ldots, k$ the measure $\mu^{[k]}$ and the function $x \mapsto f(x_0)$ on $X^{[k]}$ are invariant under the transformation $T^{[k]}_i = \text{Id} \times T^{[k]}_i$. Therefore the first integral in (4.4) remains unchanged when the function $F$ is replaced by its conditional expectation with respect to $\mathcal{J}^{[k]}$ and this integral is equal to
\[\int f(x_0) \mathbb{E}(F \mid Z_{k-1}) \circ p_{k-1}(\tilde{x}) d\mu^{[k]}(x).\]

By using (4.3) we rewrite this integral as
\[\int f(x_0) \mathbb{E}(F \mid Z_{k-1}) \circ \pi_{k-1}(x_0) d\mu(x_0).\]

In this last integral we can replace the function $f$ by its conditional expectation with respect to $Z_{k-1}$ and equality (4.3) follows.

\[\text{i)}\] Assume that $\mathbb{E}(f \mid Z_{k-1}) = 0$. Using (4.3) with $F(\tilde{x})$ equal to the product of the functions $C^[[c]] f(x_ε)$ for $ε \in \{0, 1\}^k \setminus \{0\}$, we get that $\|f\|_k = 0$.

Assume conversely that $\|f\|_k = 0$. By using Lemma 3.6 and a density argument, we get that the first integral in (4.4) is equal to zero for any choice of the function $F$ on $X^{[k]}$ and thus in particular when $F = \mathbb{E}(f \mid Z_{k-1}) \circ p_{k-1}$. In this case we have $\mathbb{E}(F \mid Z_{k-1}) = \mathbb{E}(f \mid Z_{k-1})$ and the second integral in (4.4) is equal to $\int \mathbb{E}(f \mid Z_{k-1})^2 d\mu_{k-1}$. As it is also equal to zero we have $\mathbb{E}(f \mid Z_{k-1}) = 0$ and thus $\mathbb{E}(f \mid Z_{k-1}) = 0$.

\[\text{ii)}\] By again using the equality (4.4) we get that the first integral in (4.4) remains unchanged when the function $\mathbb{E}(f_0 \mid Z_{k-1})$ is substituted for the function $f_0$. The same properties holds for the other vertices $ε \in \{0, 1\}^k$ because of the symmetry of the measure $\mu^{[k]}$ (see Lemma 3.5) and this gives (4.2).

Let $\mathcal{Y}$ be a factor of $X$ with the same property. For each bounded function on $X$ with $\mathbb{E}(f \mid \mathcal{Y}) = 0$, equality (4.2) with all functions equal to $f$ gives $\|f\|_k = 0$, and thus $\mathbb{E}(f \mid Z_{k-1}) = 0$ by ii). This shows that $\mathcal{Y} \supset Z_{k-1}$ and achieves the proof of ii).

\[\text{iii)}\] Let $A$ be a $T^{[k-1]}$-invariant subset of $X^{[k-1]}$. By construction of the measure $\mu^{[k]}$ we have $1_A(x') = 1_A(x'')$ for $\mu^{[k]}$ almost every $x = (x', x'') \in X^{[k]}$ and thus $\mu^{[k]}(A \times A) = \mu^{[k-1]}(A)$. On the other
hand,
\[
\mu^{[k]}(A \times A) = \int 1_A \otimes 1_A \, d\mu^{[k]} = \int \mathbb{E}(1_A \otimes 1_A \mid Z_{k-1}^{[k]}) \, d\mu^{[k]}
\]
\[
= \int \mathbb{E}(1_A \mid Z_{k-1}^{[k-1]}) \otimes \mathbb{E}(1_A \mid Z_{k-1}^{[k-1]}) \, d\mu^{[k]}
\]
where the second equality holds because the measure \(\mu^{[k]}\) is relatively independent over \(Z_{k-1}^{[k]}\). Moreover, as the \(\sigma\)-algebra \(Z_{k-1}^{[k-1]}\) and the set \(A\) are invariant under \(T^{[k-1]}\), the function \(\mathbb{E}(1_A \mid Z_{k-1}^{[k-1]})(x') = \mathbb{E}(1_A \mid Z_{k-1}^{[k-1]})(x'')\) for \(\mu^{[k]}\) almost every \(x = (x', x'') \in X^{[k]}\). The last integral is therefore equal to \(\int \mathbb{E}(1_A \mid Z_{k-1}^{[k-1]})^2 \, d\mu^{[k-1]}\). We get that the function \(1_A\) and its conditional expectation on \(Z_{k-1}^{[k-1]}\) have the same norm in \(L^2(\mu)\) and it follows that they are equal and that \(A\) is measurable with respect to \(Z_{k-1}^{[k-1]}\).

The announced minimality property of \(Z_{k-1}\) can be proven by a method similar to the one used in the proof of iii) \(\square\)

**Corollary 4.3.** \(Z_0\) is the trivial factor of \(X\) and \(Z_1\) is its Kronecker factor. The sequence of factors \(\{Z_k: k \geq 0\}\) is increasing.

We therefore have a chain of factor maps:

\[
X \to \cdots \to Z_{k+1} \to Z_k \to Z_{k-1} \to \cdots \to Z_1 \to Z_0.
\]

**Proof.** All these properties follow from part iii) of the Proposition by using the formula \(\|f\|_1 = \int |f| \, d\mu\) for the first one, Lemma 3.2 for the second one and the ordering of the seminorms (Subsection 3.2) for the last one. \(\square\)

**4.3. Systems of order \(k\).**

**Definition.** Let \(k \geq 0\) be an integer. A system of order \(k\) is an ergodic system \(X\) with \(Z_k(X) = X\).

It is easy to check that for any ergodic system \(X\) we have \(Z_k(Z_k(X)) = Z_k(X)\) and thus that \(Z_k(X)\) is a system of order \(k\). By Corollary 4.3 there exists a unique system of order zero, the trivial system. The systems of order 1 are those which are equal to their Kronecker factor, that is the ergodic rotations (see Subsection 3.3.2). Every system of order \(k\) is also a system of order \(\ell\) for every \(\ell > k\).
4.3.1. **Reduction to systems of order \( k \).** We explain here how it suffices to prove the convergence theorems for systems of order \( k \) (for some \( k \)). Let \((X, \mu, T)\) be an ergodic system, \( f_1, f_2, \ldots, f_k \) be bounded functions on \( X \) and consider the averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \ldots f_k(T^{kn} x)
\]

as in Theorem 1.2. Fix \( i \in \{1, 2, \ldots, k\} \). By part ii) of Proposition 4.2, \( \| f_i - \mathbb{E}(f_i \mid Z_{k-1}) \|_k = 0 \) and thus by Proposition 3.3 the difference between the averages (1.2) and the same averages with \( \mathbb{E}(f_i \mid Z_{k-1}) \) substituted for \( f_i \) converge to zero in \( L^2(\mu) \). In order to prove Theorem 1.2, we can thus assume without loss that all the functions \( f_i \) are measurable with respect to \( Z_{k-1} \). Therefore we can assume that the functions are defined on the associated factor system \( Z_{k-1}(X) \): we say that this factor is a *characteristic factor* for the convergence of the averages (1.2). As \( Z_{k-1}(X) \) is a system of order \( k - 1 \), it is sufficient to prove the convergence of these averages under the additional hypothesis that \( X \) is a system of order \( k - 1 \).

The same method applies to the polynomial averages of Theorem 1.3: for every polynomial family \( p_1, p_2, \ldots, p_k \) there exists by Proposition 3.4 an integer \( \ell \) such that it suffices to show the convergence under the additional hypothesis that the system is of order \( \ell \).

4.3.2. **Reduction to nilsystems.** In the rest of the paper we establish a relation between the systems of order \( k \) and the \( k \)-step nilsystem described in Section 2. We need a definition.

**Definition.** For each integer \( i \geq 1 \) let \((X_i, \mu_i, T)\) be a factor of the system \((X, \mu, T)\) and assume that this sequence is increasing, meaning that the sequence \( \{\mathcal{X}_i\} \) of associated sub-\( \sigma \)-algebras of \( \mathcal{X} \) is increasing. We say that \( X \) is the *inverse limit* (or the projective limit) of the sequence \( \{\mathcal{X}_i\} \) if \( \mathcal{X} = \bigvee_i \mathcal{X}_i \), that is, if \( \mathcal{X} = \bigcup_i \mathcal{X}_i \) up to null sets.

**Structure Theorem** ([HK2], Theorem 10.1). For every \( k \geq 1 \), every system of order \( k \) is an inverse limit of a sequence of \( k \)-step nilsystems.

The convergence results (Theorem 1.2 and 1.3) follow easily since they hold for nilsystems (Corollaries 2.4 and 2.5) and pass to inverse limits.

The proof of the Structure Theorem is the longest and the most technical part of the proofs of convergence and in the next section we can only give a relatively vague idea of the strategy. To understand why this proof is long it is perhaps interesting to compare the two
notions involved in the theorem. Systems of order $k$ are defined in terms of abstract ergodic theory, without any mention of a topology or a differentiable structure on the space. The unique ingredients of this *poor structure* are a probability space and a measure preserving transformation. On the other hand, nilsystems are defined in terms of Lie Groups and clearly have a rich structure. Proving the Structure Theorem therefore forces us to build this rich structure from scratch.

4.4. Notes on Section 4

4.4.1. Characteristic factors. The notion of a characteristic factor was already used (without the name) in Furstenberg’s original paper [Fu2] and applies to a wide range of questions. Assume for example that we are dealing with the limit behavior of a sequence of averages depending on some functions. We say that a factor $Y$ of $X$ is characteristic if the difference between the given averages and the averages with each function replaced by its conditional expectation on $Y$ converges to zero (for the notion of convergence in question). We are therefore left with studying the limit behavior with $Y$ substituted for the given system and this can be much easier if $Y$ has a “rich” structure.

Clearly characteristic factors are not unique: a factor containing a characteristic one is characteristic too. It can be proven that the factors $Z_k$ defined here are the smallest possible for our convergence problems. Furstenberg used the much larger *maximal distal factor*. The “structure” of this factor is much weaker than that of a nilmanifold but is sufficiently rich to make the proof of Furstenberg’s Theorem possible.

4.4.2. The case of a nilsystem. The factors $Z_k(X)$ can be described explicitly for nilsystems.

Let $(X, \mu, T)$ be an ergodic $\ell$-step nilsystem. We use the notation of Section 2 and assume moreover that the group $G$ is spanned by its connected component of the identity and the element $t$ defining the transformation $T$ (it is always possible to reduce to this case). For every $k \geq 1$, $(\Lambda G_{k+1})/G_{k+1}$ is a discrete and cocompact subgroup of the nilpotent Lie group $G/G_{k+1}$, and $Z_k(X)$ is the nilsystem

$$Z_k(X) = \frac{G/G_{k+1}}{(\Lambda G_{k+1})/G_{k+1}} = \frac{G}{G_{k+1}\Lambda}$$

endowed with translation by the projection of $t$ on $G/G_{k+1}$. This result was already proved by Parry ([P1]) and Leibman ([L2]) for the Kronecker factor corresponding to the case $k = 1$. Using this formula with $k = \ell$ we get that every ergodic $k$-step nilsystem is a system of order $k$. 
5. On the way to the Structure Theorem

5.1. A group associated to an ergodic system. To every ergodic system \((X, \mu, T)\) we associate a group \(\mathcal{G}(X)\) of measure preserving transformations. The strategy consists in showing that for sufficiently many systems of order \(k\) this group is a nilpotent Lie group and acts “transitively”, so that the system can be given the structure of a nil-system. We need some notation.

Let \(g\) be a measure preserving transformation on \(X\) written \(x \mapsto g \cdot x\) and let \(k \geq 1\) be an integer. For each side \(\alpha\) of \(\{0, 1\}^k\) we define the transformation \(g^{[k]}_\alpha\) of \(X^{[k]}\) by

\[
(g^{[k]}_\alpha \cdot x)_\varepsilon = \begin{cases} g \cdot x_\varepsilon & \text{if } \varepsilon \in \alpha; \\ x_\varepsilon & \text{otherwise.} \end{cases}
\]

This definition coincides with that of the side transformation in the case that \(g = T\).

**Definition.** \(\mathcal{G}(X)\) is the group of measure preserving transformations of \(X\) such that for every integer \(k \geq 1\) and every side \(\alpha\) of \(\{0, 1\}^k\), the transformation \(g^{[k]}_\alpha\) leaves the measure \(\mu^{[k]}\) invariant.

The proofs of the following results can be found in Section 5 of [HK2]. We write \(\mathcal{G}\) instead of \(\mathcal{G}(X)\) except when some ambiguity can occur. This group is a Polish group when endowed with the topology of convergence in probability. It contains \(T\) and every measure preserving transformation of \(X\) commuting with \(T\).

**Lemma 5.1.** Let \(g \in \mathcal{G}(X)\). Then for every \(k\) the transformation \(g\) of \(X\) maps the factor \(Z_k\) to itself and thus induces a transformation of \(Z_k\), which belongs to \(G(Z_k)\).

**Lemma 5.2.** If \(X\) is a system of order \(k\) then \(\mathcal{G}(X)\) is a \(k\)-step nilpotent group.

**Proof.** Let \(g_1, g_2, \ldots, g_{k+1} \in \mathcal{G}\) and \(h = [[[g_1, g_2], g_3], \ldots, g_{k+1}]. \)

Let \(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}\) be the \(k + 1\) sides of \(\{0, 1\}^k\) containing 0. The measure \(\mu^{[k+1]}\) is invariant under each transformation \(g^{[k+1]}_{\alpha_i}\) and thus also under their commutator \( [[[g^{[k+1]}_{\alpha_1}, g^{[k+1]}_{\alpha_2}], g^{[k+1]}_{\alpha_3}], \ldots, g^{[k+1]}_{\alpha_{k+1}}] \). But it is easy to check that this transformation is equal to the transformation \(h^{[k+1]}_0\), given by \((h^{[k+1]}_0 \cdot x)_0 = h \cdot x_0\) and \((h^{[k+1]}_0 \cdot x)_\varepsilon = x_\varepsilon\) for \(\varepsilon \neq 0\).

Let \(A\) be a subset of \(X\). As \(X\) is of level \(k\), \(Z_k(X) = X\) and by definition there exists a subset \(B\) of \(X^{[k+1]}\) with \(1_A(x_0) = 1_B(x)\) for \(\mu^{[k+1]}\) almost every \(x\). Applying the transformation \(h^{[k+1]}_0\) we get that
\(1_A(h \cdot x_0) = 1_B(\tilde{x}) (\mu^{[k+1]} \text{ a.e.})\) and thus that \(1_A(x_0) = 1_A(h \cdot x_0) (\mu\text{-a.e.})\). This shows that every subset \(A\) of \(X\) is invariant under \(h\) and that \(h = \text{Id}. \)

5.2. Relations between two consecutive factors.

**Definition.** Let \((Y, \nu, T)\) be a system, \(U\) a compact abelian group endowed with its Haar measure \(m_U\) and \(\rho: Y \to U\) a measurable map. Let \(X = Y \times U\) be endowed with the measure \(\mu = \nu \times m_U\) and with the transformation \(T\) given by \(T(y, u) = (Ty, u + \rho(y))\). Then we say that \(X\) is an extension of \(Y\) by \(U\) and \(\rho\) is called the cocycle defining the extension.

We note that \(Y\) is a factor of \(X\), with the factor map given by \((y, u) \mapsto y\). For each \(v \in U\), the transformation \(R_v: (y, u) \mapsto (y, u + v)\) of \(X\) preserves \(\mu\) and commutes with \(T\); it is called a vertical rotation.

**Proposition 5.3.** Let \((X, \mu, T)\) be a system of level \(k\) and \((Y, \nu, T) = Z_{k-1}(X)\). Then \(X\) is an extension of \(Y\) by a compact abelian group \(U\). Moreover, the cocycle \(\rho: Y \to U\) defining this extension satisfies:

There exists a map \(F: Y^{[k]} \to U\) with

\[
\sum_{\varepsilon \in \{0,1\}^k} (-1)^{|\varepsilon|} \rho(y_\varepsilon) = F(T^{[k]}y) - F(y)
\]

for \(\nu^{[k]}\)-almost every \(y \in Y^{[k]}\).

A cocycle \(\rho\) satisfying the functional equation (5.1) for some map \(F\) is called a cocycle of type \(k\).

**Idea of the proof.** The first step of the proof uses the notion of an isometric extension as defined by Furstenberg (see [Fu3]). Part [1] of Proposition 4.2 implies that the \(T^{[k]}\)-invariant \(\sigma\)-algebra \(\mathcal{I}^{[k]}\) of \(X^{[k]}\) is measurable with respect to \(\mathcal{W}^{[k]}\), where \(W\) is some factor of \(X\) which is an isometric extension of \(Y\). The minimality property [4] of the same Proposition then gives that \(X = W\), that is, \(X\) is an isometric extension of \(Y\).

In particular there exists a compact group \(U\), acting on \(X\) by measure preserving transformations and inducing the trivial transformation on \(Y\). It is then proven that this group of transformations is included in the center of \(G(X)\) and in particular is an abelian group; this shows that \(X\) is an extension of \(Y\) by the compact abelian group \(U\). We identify \(X\) with \(Y \times U\). Let \(\rho: Y \to U\) be the cocycle defining this extension.

Let \(\chi\) be a character of \(U\), that is a continuous group homomorphism from \(U\) to the circle group \(S^1\). Let \(\phi\) be the function on \(X = Y \times U\)
defined by \( \phi(y, u) = \chi(u) \) and \( \Phi \) the function defined on \( X^{[k]} = Y^{[k]} \times U^{[k]} \) by

\[
\Phi(y, u) = \prod_{\varepsilon \in \{0, 1\}^k} C^{[\varepsilon]} \phi(x_{\varepsilon}) = \chi\left( \sum_{\varepsilon \in \{0, 1\}^k} (-1)^{\varepsilon} u_{\varepsilon} \right)
\]

for \( x = (y, u) \) with \( y \in Y^{[k]} \) and \( u \in U^{[k]} \).

As \( X \) is of type \( k \), \( \|\psi\|_{k+1} \neq 0 \) by part ii) of Proposition 4.2. By construction of the seminorms this means that the function \( \Psi = \mathbb{E}(\Phi | \mathcal{I}^{[k]}) \) is not identically zero. This function is invariant under \( T^{[k]} \) and satisfies \( \Psi(R_u x) = \chi(u) \Psi(x) \) for every \( u \in U \) and \( \mu \)-almost every \( x \in X \). By using the ergodicity property (Lemma 3.8) of \( \mu^{[k]} \) it can be showed that there exists a function with the same properties and everywhere nonzero, and thus a function \( F_\chi \) of modulus 1 with the same properties. The invariance of this function gives:

\[
\chi\left( \sum_{\varepsilon \in \{0, 1\}^k} (-1)^{\varepsilon} \rho_k(y_{\varepsilon}) \right) = F_\chi(T^{[k]}y) F_\chi(y)^{-1}.
\]

The existence of a function \( F_\chi \) with this property for every character \( \chi \) of \( U \) implies the existence of a function \( F \) satisfying (5.1) by classical results about cocycles. \( \square \)

5.3. A technical tool. From this point, the proof of the Structure Theorem proceeds by induction on \( k \). We give here a technical tool used in the induction.

For every \( s \) in the Kronecker factor \( Z_1 \) we have defined in Subsection 3.3.3 a measure \( \mu_s \) on \( X \times X \), invariant under \( T \times T \). For almost every \( s \), the system \( (X \times X, \mu_2, T \times T) \) is ergodic and we denote it by \( X_s \).

**Proposition 5.4.** Let the hypotheses and the notation be as in Proposition 5.3. Then, for almost every \( s \in Z_1 \), \( X_s \) is a system of level \( k \), \( Y_s \) a system of level \( k - 1 \) and \( X_s \) is an extension of \( Y_s \) by the compact abelian group \( U \times U \), given by the cocycle \( (y_0, y_1) \mapsto (\rho(y_0), \rho(y_1)) \) which is of type \( k \).

Moreover, \( Z_{k-1}(X_s) \) is an extension of \( Y_s \) by \( U \), given by the cocycle \( (y_0, y_1) \mapsto \rho(y_0) - \rho(y_1) \) which is of type \( k - 1 \).

This proposition plays the role of a stepladder, allowing us to climb from one level to the next one: assume that some properties have been shown for systems of order \( k - 1 \) and let \( X \) be a system of order \( k \). Let \( U \) and \( \rho \) be as in Proposition 5.3. Then we can use these properties for the systems \( Z_{k-1}(X_s) \) and this gives information on the group \( U \) and the cocycle \( \rho \). This method is used in particular to prove:
Lemma 5.5. Let $X,Y,U$ and $\rho$ be as in Proposition 5.3. Then $U$ is connected.

5.4. Toral systems. Recall that a compact abelian group can be given a structure of Lie group if and only if the connected component of the its unit element is a finite dimensional torus. This property is equivalent to saying that its dual group is finitely generated. It follows that every compact (metrizable) abelian group can be represented as an inverse limit of a sequence of compact abelian Lie groups. In particular every compact connected abelian group can be represented as an inverse limit of a sequence of finite dimensional tori. This motivates the next definition.

Definition. Let $k \geq 1$ be an integer and $(X,\mu,T)$ be a system of order $k$. We say that this system is toral if its Kronecker factor $Z_1(X)$ is a compact abelian Lie group and if $Z_j(X)$ is an extension of $Z_{j-1}(X)$ by a finite dimensional torus for $2 \leq j \leq k$.

The structure Theorem can be split into the next two Propositions:

Proposition 5.6. For every integer $k \geq 1$, every system of order $k$ is the inverse limit of a sequence of toral systems of order $k$.

Proposition 5.7. Every toral system of order $k$ is $k$-step nilsystem.

More precisely, $G(X)$ is a Lie group and acts transitively on $X$. $X$ can be identified with $G(X)/\Lambda$ where $G(X)$ is the subgroup of $G(X)$ spanned by the connected component of its identity and $T$. We give an idea of the method of the proof.

Let $X$ be a system of order $k$; we use the notation of Proposition 5.3. As we assume that the result holds for systems of order $k - 1$ it holds in particular for $Y = Z_{k-1}(X)$. Each element $h$ of $G(Y)$ is lifted to an element $\tilde{h}$ of $G(X)$; this means that $h$ is the transformation induced by $\tilde{h}$ on $Y$ as in Lemma 5.1. This is done first for $h$ belonging to $G(Y)_{k-1}$ then for $h \in G(Y)_{k-1}$, and so on, following the lower central series of $G(Y)$ upwards. Each step uses the functional equation 5.1 satisfied by the cocycle $\rho$.

Appendix: Further comments

A.1. The cubic averages. The paper [HK2] also contains the proof of convergence of another type of multiple ergodic average, the cubic averages, already proven by Bergelson for the cubes of dimension 2:
Theorem (Bergelson [B2]). Let \((X, \mu, T)\) be a system and let \(f, g, h\) be three bounded functions on \(X\). Then the averages
\[
\frac{1}{N^2} \sum_{m,n=0}^{N-1} f(T^mx)g(T^nx)h(T^{m+n}x)
\]
converge in \(L^2(\mu)\) as \(N \to +\infty\).

For the general case we use here the notation introduced at the top of Section 3.

Theorem A.8 (Host & Kra [HK2]). Let \((X, \mu, T)\) be a system and let \(f, \varepsilon \in \{0, 1\}^k \setminus \{0\}\) be \(2^k - 1\) bounded functions on \(X\). Then the averages
\[
(A.2) \quad \frac{1}{N^k} \sum_{n_1, n_2, \ldots, n_k=0}^{N-1} \prod_{\varepsilon \in \{0, 1\}^k \setminus \varepsilon \neq 0} f_\varepsilon(T^{n_\varepsilon}x)
\]
converge in \(L^2(\mu)\).

The strategy for the proof of this theorem is the same as for the averages along arithmetic progressions and for the polynomial averages. Clearly it suffices to prove the result for ergodic systems. First, a bound similar to (3.8) holds for the cubic averages, with a similar proof using a multidimensional version of the van der Corput Lemma. Here \((X, \mu, T)\) is an ergodic system.

Proposition. Let \(f, \varepsilon \in \{0, 1\}^k \setminus \{0\}\) be \(2^k - 1\) functions on \(X\) with \(\|f_\varepsilon\|_\infty \leq 1\) for every \(\varepsilon\). The \(\limsup\) of the norm in \(L^2(\mu)\) of the averages \((A.2)\) is bounded by \(\min_{\varepsilon} \|f_\varepsilon\|_k\).

By the same arguments as in Subsections 4.3.1 and 4.3.2 it is therefore possible to restrict to the case that \(X\) is a \((k - 1)\)-step nilsystem. Then Theorem A.8 follows from a generalization of Theorems 2.1 and 2.2 and of Corollary 2.3 to the case of several commuting translations on a nilmanifold.

The cubic averages are directly linked to the measures \(\mu^{[k]}\):

Proposition. Let \(f, \varepsilon \in \{0, 1\}^k\) be \(2^k\) functions on \(X\). Then
\[
\frac{1}{N^k} \sum_{n_1, n_2, \ldots, n_k=0}^{N-1} \int \prod_{\varepsilon \in \{0, 1\}^k} f_\varepsilon(T^{n_\varepsilon}x) \, d\mu(x) \to \int \prod_{\varepsilon \in \{0, 1\}^k} f_\varepsilon(x_\varepsilon) \, d\mu^{[k]}(x).
\]

By using this result with all functions equal to \(1_A\) and the inequality \(\|1_A\|_k \geq \|1_A\|_1 = \mu(A)\) we get:
Corollary. For every subset $A$ of $X$,

$$\lim_{N \to +\infty} \frac{1}{N^k} \sum_{n_1, n_2, \ldots, n_k = 0}^{N-1} \mu\left( \bigcap_{\varepsilon \in \{0,1\}^k} T^{n \cdot \varepsilon} A \right) \geq \mu(A)^{2k}.$$ 

The same results hold when the averages on $[1, N)^k$ are replaced by averages on a sequence of parallelepipeds whose minimal size tend to infinity, or more generally by averages on a Følner sequence. This version of the last Corollary has a combinatorial interpretation in terms of sets of integer of positive upper density but we do not state it here (see [HK2]).

The convergence a.e. of the averages (A.2) has been recently proven by Assani ([As]).

A.2. Gowers norms and their relations to arithmetic progressions.

A.2.1. The definition. When writing our paper [HK2] we became aware of Gowers’ paper [G1] where he introduced some norms very similar to our seminorms. It turns out that they are identical to our seminorms when computed in the particular case that $X = \mathbb{Z}/N\mathbb{Z}$ endowed with the uniform measure and with the transformation $x \mapsto x + 1 \mod N$. These norms were extensively used by Green and Tao ([GrT], [T]). We recall here their definition, with notation modified in order to fit with ours.

Let $N \geq 2$ be an integer and let $G = \mathbb{Z}/N\mathbb{Z}$ be endowed with its normalized Haar measure $m$. Let $C(G)$ denotes the space of complex valued functions on $G$. For $f \in C(G)$, $\|f\|_1$ is defined to be $|\int f(x) \, dm(x)|$. For $t \in G$ let $f_t$ be the function $x \mapsto f(x + t)$ and define by induction

(A.3) $\quad \|f\|_{k+1} = \left( \int \|f_{f_t}\|_{k}^{2^k} \, dt \right)^{1/2^{k+1}}.

\|f\|_k$ can also be defined by a closed formula. For $t = (t_1, t_2, \ldots, t_k) \in G^k$ and $\varepsilon \in \{0,1\}^k$ we write $\varepsilon \cdot t = \varepsilon_1 t_1 + \varepsilon_2 t_2 + \cdots + \varepsilon_k t_k$. For $f \in C(G)$ and $k \geq 1$ we have by induction

(A.4) $\quad \|f\|_k = \left( \int \prod_{\varepsilon \in \{0,1\}^k} C_{\varepsilon}^{|\varepsilon|} f(x + \varepsilon \cdot t) \, dm(x) \, dm(t_1) \, dm(t_2) \cdots \, dm(t_k) \right)^{1/2^k}.$
An inequality similar to the Cauchy-Schwarz Inequality follows: when \( f_\varepsilon, \varepsilon \in \{0, 1\}^k \), are functions on \( G \),
\[
\left| \prod_{\varepsilon \in \{0, 1\}^k} f_\varepsilon(x + \varepsilon \cdot t) \, dm(x) \, dm(t_1) \, dm(t_2) \ldots \, dm(t_k) \right| \leq \prod_{\varepsilon \in \{0, 1\}^k} \| f_\varepsilon \|_k .
\]
This implies that each map \( f \mapsto \| f \|_k \) is subadditive and hence a seminorm. For \( k = 2 \), relation (A.4) gives:
\[
\| f \|_2 = \left( \sum_{j \in G} |\hat{f}(j)|^4 \right)^{1/4}
\]
where \( \hat{f} \) is the Fourier Transform of \( f \), defined on the dual group \( \hat{G} = G \) of \( G \). This expression can only be zero when \( f \) is the zero function and thus \( \| \cdot \|_2 \) is a norm on \( C \). By induction and by using definition (A.3), it can be checked that for every \( f \in C(G) \) we have \( \| f \|_{k+1} \geq \| f \|_k \) for every \( k \) and thus \( \| \cdot \|_k \) is a norm on \( C(G) \) for every \( k \geq 2 \).

A.2.2. Using Gowers norms. These norms are used by Gowers, Green and Tao to control the integral appearing in the analytic form of Szemerédi’s Theorem.

**Proposition A.9.** Let \( N \geq 2 \) be an integer and let \( \mathbb{Z}/N\mathbb{Z} \) be endowed with its normalized Haar measure. Let \( \ell \geq 2 \) be an integer and \( f_0, f_1, f_2, \ldots, f_{\ell-1} \) be functions on \( \mathbb{Z}/N\mathbb{Z} \) with \( |f_i| \leq 1 \) for \( 0 \leq i \leq \ell - 1 \). Then
\[
\left| \int \int f_0(x) f_1(x + y) f_2(x + 2y) \ldots f_{\ell-1}(x + (\ell - 1)y) \, dm(x) \, dm(y) \right| \leq \min_{0 \leq i \leq \ell - 1} \| f_i \|_{\ell-1} .
\]

The starting point of Gowers’ method can be summarized very roughly as follows. Let \( A \) be a subset of \( G \), \( f = 1_A \) and \( g = f - m(A) \). He distinguishes two cases: If \( \| g \|_{\ell-1} \) is small then the integral in \( \| \cdot \|_{\ell-1} \) is close to the same integral with the constant \( m(A) \) substituted for \( f \) and thus is large. If \( \| g \|_{\ell-1} \) is large then he shows that the restriction of \( f \) to some relatively large subset of \( G \) has some strong arithmetic properties and behaves in some respects like a polynomial; he deduces that the integral is large in this case also.

The way Tao and Green and Tao use Gowers’ norms is much closer to ergodic theory and in particular to the way we use the seminorms here. We try here to make this similarity apparent but
the reader must be advised that the contents of the next few lines is nothing more than an oversimplification.

Tao decomposes any function $f$ on $G$ as a sum of a function $g$ with small norm and a “anti-uniform” function $h$. The contribution of $g$ to the integral in (1.1) is small and the function $h$ can be viewed as the conditional expectation of $f$ relatively to some $\sigma$-algebra, comparable to the factor used in [FuKO] except that it is not invariant under translation. The contribution of $h$ to the integral is bounded from below by using van der Waerden’s Theorem.

The theorem of Green and Tao ([GrT]) about the existence of arithmetic progressions in primes seems completely out of the range of ergodic theory because the primes have zero density in the integers and thus the Correspondence Principle does not apply. However the strategy is similar. They show that in the preceding decomposition the uniform norm of the function $h$ can be bounded independently of any uniform bound of the function $f$, assuming only that this function is bounded by a “quasirandom” function. Therefore the analytic form of Szemerédi’s Theorem can be used to show that the function $h$ gives a “large” contribution to the integral in (1.1). Moreover the function $g$ also is bounded by a quasirandom function and its contribution to the integral is small, due to an extension of Proposition A.9 to this case. They therefore get a generalization of the analytic form of Szemerédi’s Theorem under the weaker hypothesis. This result is then used for a function closely related to the indicator function of primes.

The decomposition used in both cases is parallel to the decomposition

$$f = \mathbb{E}(f \mid Z_{k-1}) + (f - \mathbb{E}(f \mid Z_{k-1}))$$

used in subsection 4.3.1 of this paper but the authors do not need a precise description of the “factor” comparable to the Structure Theorem and do not use the machinery of nilpotent groups. It can be conjectured that there exists a hidden link between the combinatorial constructions and the nilpotent groups and we believe that making this link explicit is a very interesting challenge.

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Mathématiques, Université de Marne la Vallée. 5, Bd. Descartes, Champs sur Marne. 77454 Marne la Vallée Cedex, France

E-mail address: host@math.univ-mlv.fr