NEARBY CYCLE SHEAVES FOR STABLE POLAR REPRESENTATIONS

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Abstract. We extend the results of [GVX] to the setting of a stable polar representation $G|V$ ($G$ connected, reductive over $\mathbb{C}$), satisfying some mild additional hypotheses. Given a $G$-equivariant rank one local system $\mathcal{L}_\chi$ on the general fiber of the quotient map $f: V \to V//G$, we compute the Fourier transform of the corresponding nearby cycle sheaf $P_\chi$ on the zero-fiber $X_0 = f^{-1}(0)$. Our main intended application is to the theory of character sheaves for graded Lie algebras over $\mathbb{C}$.

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1. **Introduction**

We present a calculation of a nearby cycle sheaf with twisted coefficients in the setting of a stable polar representation $G|V$ of a connected group $G$, satisfying some mild additional hypotheses. This provides a simultaneous partial generalization of the results of [Gr1] and [GVX].

1.1. **The main result.** We recall some of the background. The class of polar representations $G|V$, for $G$ connected, reductive over $\mathbb{C}$, was introduced by Dadok and Kac in [DK], as a class of linear actions whose invariant theory works by analogy with the class of adjoint representations. In fact, Dadok and Kac defined polar representations for general reductive $G$, but the polarity condition is most natural when $G$ is connected. For every polar $G|V$
with $G$ connected, there exists a linear subspace $C \subset V$, which is defined uniquely up to the action of $G$, with the property that:

\[(1.1) \quad V \cong C/W,\]

where $W = N_G(C)/Z_G(C)$. The subspace $C$ is called a Cartan subspace of $V$, and the group $W$ is called the Weyl group of $G|V$. The group $W$ is a finite complex reflection group acting on $C$. By a general result on complex reflection groups (see [Br, Theorem 4.1]), the quotient $C/W$ is isomorphic to an affine space of the same dimension as $C$. Polar representations include many of the classical invariant problems of linear algebra, as well as all representations arising from graded Lie algebras, studied by Vinberg in [Vi] (see Section 1.2 below).

In this paper, we consider a polar representation $G|V$ of a connected reductive group $G$, which is stable in the sense of [DK]. This stability condition means that:

\[(1.2) \quad V = C \oplus g \cdot C,\]

where $g = \text{Lie}(G)$. We are interested in the quotient map:

\[f : V \to Q := C/W,\]

arising from the identification $(1.1)$. We assume that:

\[(1.3) \quad \text{rank}(G|V) := \dim C - \dim C^G \geq 1.\]

Let:

\[C^{reg} = \{c \in C \mid Z_W(c) = \{1\}\} \quad \text{and} \quad Q^{reg} = f(C^{reg}).\]

Pick a basepoint $c_0 \in C^{reg}$ and define:

\[(1.4) \quad \bar{c}_0 = f(c_0), \quad X_0 = f^{-1}(0), \quad X_{c_0} = f^{-1}(\bar{c}_0),\]

where $0 \in C/W$ is the image of $0 \in C$. The stability condition $(1.2)$ implies that $X_{c_0} = G \cdot c_0$.

We consider a nearby cycle functor:

\[\psi_f = \psi_f[\bar{c}_0] : \text{Perv}_G(X_{\bar{c}_0}) \to \text{Perv}_G(X_0)_{C^*\text{-conic}},\]

from (shifted) $G$-equivariant local systems on $X_{\bar{c}_0}$ to $G$-equivariant $C^*$-conic perverse sheaves on $X_0$; both considered with coefficients in $\mathbb{C}$. This functor is defined as a specialization to the asymptotic cone, as in [Gr2]. However, in order to justify the notation $\psi_f$, we wish to ensure that the functor $\psi_f[\bar{c}_0]$ “varies in a local system” over the regular locus $Q^{reg} \ni \bar{c}_0$. To accomplish this, we impose the following further condition on the representation $G|V$. We assume that either $G|V$ is visible, meaning that the zero-fiber $X_0$ consists of finitely many $G$-orbits, or we have rank$(G|V) = 1$ (see equation $(1.3)$). Among other things, this ensures that the pure braid group:

\[PB_W := \pi_1(C^{reg}, c_0),\]

acts on the functor $\psi_f[\bar{c}_0]$. 
Our main result, Theorem 2.23, computes the functor $\psi_f$ for local systems of rank one, subject to two additional technical assumptions (see (2.5) and (2.28)). Specifically, given a character:

(1.5) $\chi : I := \pi_1^G(X_{c_0}, c_0) \to \mathbb{G}_m$,

we write $\mathcal{L}_\chi$ for the corresponding rank one $G$-equivariant local system on $X_{c_0}$, and we let:

$$P_\chi = \psi_f(\mathcal{L}_\chi[-]),$$

where $[-]$ denotes the appropriate shift. The statement of Theorem 2.23 takes the following form:

(1.6) $F P_\chi \sim IC((V^*)^{rs}, \mathcal{M}_\chi)$.

Here, $\mathcal{F} : \text{Perv}_G(V)^{c*-\text{conic}} \to \text{Perv}_G(V^*)^{c*-\text{conic}}$ is the topological Fourier transform functor, $(V^*)^{rs}$ is the union of all closed $G$-orbits of maximal dimension in $V^*$, and the RHS is the IC-extension of a certain local system $\mathcal{M}_\chi$ on $(V^*)^{rs}$, with $\text{rank}(\mathcal{M}_\chi) = |W|$. The stability condition (1.2) implies that $(V^*)^{rs} \subset V^*$ is an open subset, and the local system $\mathcal{M}_\chi$ is specified by equation (2.71).

The fact that the sheaf $P_\chi$ admits a description of the form (1.6) follows readily from the results of [Gr2] and [Gr1]. Thus, the main content of Theorem 2.23 is in describing the local system $\mathcal{M}_\chi$. This description proceeds by reduction to the case of rank one. We now give a partial sketch of the construction of $\mathcal{M}_\chi$.

Let $V^{rs} = f^{-1}(Q^{reg})$. By the stability condition (1.2) and the additional assumption (2.5), the subset $V^{rs} \subset V$ is the union of all closed $G$-orbits of maximal dimension. We use a suitable Hermitian inner product on $V$ to pick a basepoint $l_0 \in (V^*)^{rs}$, corresponding to the basepoint $c_0 \in V^{rs}$, and to identify the equivariant fundamental group $\pi_1^G((V^*)^{rs}, l_0)$ with $\tilde{B}_W := \pi_1^G((V^*)^{rs}, c_0)$. Thus, we can view $\mathcal{M}_\chi$ as a representation of the group $\tilde{B}_W$. This group fits into an exact sequence:

(1.7) $1 \to I \to \tilde{B}_W \xrightarrow{\tilde{q}} B_W \to 1$,

where $B_W := \pi_1(Q^{reg}, \bar{c}_0)$ is the braid group of $W$.

Next, we define a subgroup $W^0_\chi \subset W$ as follows. The Weyl group $W$ acts naturally on the characters $\tilde{I} := \text{Hom}(I, \mathbb{G}_m)$. Let $W_\chi = \text{Stab}_W(\chi)$. Let $\{C_\alpha\}_{\alpha \in A}$ be the set of all reflection hyperplanes for $W$. For each $\alpha \in A$, let $W_\alpha = \text{Stab}_W(C_\alpha)$ and let $W_{\alpha, \chi} = W_\alpha \cap W_\chi$. Then $W^0_\chi \subset W$ is the subgroup generated by all the $\{W_{\alpha, \chi}\}_{\alpha \in A}$. In other words, $W^0_\chi$ is the largest complex reflection group contained in $W_\chi$.

Let $p : B_W \to W$ be the natural map, and let:

(1.8) $B^0_W = p^{-1}(W^0_\chi), \quad \tilde{B}^0_W = \tilde{q}^{-1}(B^0_W)$. 
The $\tilde{B}_W$-representation $M_\chi$ is obtained by inducing from the subgroup $\tilde{B}^{\chi,0}_W \subset \tilde{B}_W$, i.e., we have:

$$(1.9) \quad M_\chi = \mathbb{C}[\tilde{B}_W] \otimes_{\mathbb{C}[\tilde{B}^{\chi,0}_W]} M^0_\chi,$$

for some $\tilde{B}^{\chi,0}_W$-representation $M^0_\chi$, with $\dim M^0_\chi = |W^{\chi}|$.

To describe the representation $M^0_\chi$, let $A^0_\chi \subset A$ be the set of all reflection hyperplanes for $W^{\chi}$, i.e., the set of all $\alpha \in A$ with $W_{\alpha,\chi} \neq \{1\}$. To each $\alpha \in A$ we associate a monic polynomial $R_{\chi,\alpha} \in \mathbb{C}[z]$ of degree $|W_{\alpha,\chi}|$. Let $B_{W^{\chi}}$ be the braid group associated to the complex reflection group $W^{\chi}$. Using the polynomials $R_{\chi,\alpha}$ for $\alpha \in A^0_\chi$, we define a Hecke algebra $H_{W^{\chi}}$ as a quotient of the group algebra $\mathbb{C}[B_{W^{\chi}}]$. The algebra $H_{W^{\chi}}$ is naturally a $\tilde{B}^{\chi,0}_W$-representation, and we have:

$$(1.10) \quad M^0_\chi \cong H_{W^{\chi}} \otimes_{\tilde{B}^{\chi,0}_W} \mathbb{C} \cdot \tilde{\rho} \cdot \tau,$$

where $\tilde{\chi} \cdot \tilde{\rho} \cdot \tau : \tilde{B}^{\chi,0}_W \to \mathbb{G}_m$ is a certain character, obtained as a product of three ingredients, introduced in \((2.69), (2.70),\) and \((2.54),\) respectively. In particular, the character $\tilde{\rho}$ encodes the polynomials $R_{\chi,\alpha}$ for $\alpha \in A - A^0_\chi$.

In conclusion, we briefly indicate the origin on the polynomials $R_{\chi,\alpha}$. For each $\alpha \in A$, there is an associated stable polar representation $G_\alpha|V_\alpha$ of rank one. Namely, $G_\alpha \subset G$ is the connected subgroup corresponding to the stabilizer $g_\alpha := Z_g(C_\alpha)$, and $V_\alpha = C \oplus g_\alpha \cdot C \subset V$. Let $f_\alpha : V_\alpha \to Q_\alpha := V_\alpha//G_\alpha$ be the quotient map, and let:

$$X_{0,\alpha} = f^{-1}_\alpha(0), \quad \hat{c}_0 = f_\alpha(c_0), \quad X_{\hat{c}_0,\alpha} = f^{-1}_\alpha(\hat{c}_0), \quad \hat{I}_\alpha = \pi^G_1(X_{\hat{c}_0,\alpha}, c_0).$$

The character $\chi \in \hat{I}$ restricts to a character $\chi^\alpha \in \hat{I}_\alpha$. We can thus apply the construction of the nearby cycle sheaf $P_\chi$ to the data $(G_\alpha|V_\alpha, \chi^\alpha)$, to obtain a sheaf $P_\chi \in \text{Perv}_{G_\alpha}(X_{\hat{0},\alpha})$. The analog of the isomorphism \((1.6)\) for the sheaf $P_\chi$ defines a local system $M_\chi$ on $(V^*_\alpha)^{rs} \subset V^*_\alpha$. The basepoint $l_0 \in (V^*_\alpha)^{rs}$ restricts to a basepoint $l_{0,\alpha} \in (V^*_\alpha)^{rs}$, and the polynomial $R_{\chi,\alpha}$ is defined as the minimal polynomial of the holonomy of $M_\chi$ along a suitable element of $\pi^G_1((V^*_\alpha)^{rs}, l_{0,\alpha})$.

Thus, we see that Theorem \ref{theo:2.23} stops short of giving an explicit description of the local system $M_\chi$. Rather, it describes $M_\chi$ in terms of the much simpler local systems $M_{\chi,\alpha}$ for $\alpha \in A^0_\chi$. However, in many examples, the geometry of the rank one representations $G_\alpha|V_\alpha$ is tractable, leading to an explicit description of $M_\chi$. See, for example, the discussion of \cite{VX2} and \cite{VX3} in Section \ref{subsec:1.2} below.

### 1.2. Motivation and relationship to prior work.

The motivating example for this work is the class of representations arising from graded Lie algebras, studied in \cite{Vi}, and the motivating problem is to understand character sheaves in this setting. More precisely, let $\hat{G}$
be a reductive group over $\mathbb{C}$, and let $\theta$ be an automorphism:

$$\theta : \tilde{G} \to \tilde{G},$$

of finite order $m > 1$. We have an eigenspace decomposition:

$$\tilde{\mathfrak{g}} := \text{Lie}(\tilde{G}) = \bigoplus_{i=0}^{m-1} \tilde{\mathfrak{g}}_i,$$

where $\theta|_{\tilde{\mathfrak{g}}_i} = \exp(2\pi i i/m)$ and $i = \sqrt{-1}$. Decomposition (1.12) defines a grading on the Lie algebra $\tilde{\mathfrak{g}}$. Let $\tilde{G}^0 \subset \tilde{G}$ be the group of fixed points of $\theta$, and let $\tilde{G}^{0,0}$ be the identity component of $\tilde{G}^0$. Note that we have $\text{Lie}(\tilde{G}^0) = \tilde{\mathfrak{g}}_0$. The adjoint action of $\tilde{G}^0$ restricts to an action of $\tilde{G}^{0,0}$ on each of the $\tilde{\mathfrak{g}}_i$. The case of a general $i \in \{1, \ldots, m-1\}$ is readily reduced to the case $i = 1$. We will focus on the representation $G|V$, with $G = \tilde{G}^{0,0}$ and $V = \tilde{\mathfrak{g}}_1$. This representation is polar and visible, but not necessarily stable in the sense of (1.2). However, in many important cases, the representation $G|V$ is stable, and satisfies the hypotheses of Theorem 2.23. Our results can be extended to some nonstable situations (see the discussion of [VX3] below), but we do not explore this direction in the present paper.

By a character sheaf on $V^*$ we mean a simple perverse sheaf $P \in \text{Perv}_{G}(V^*)_{\mathbb{C}^*\text{-conic}}$ with nilpotent singular support:

$$SS(P) \subset V^* \times X_0 \subset V^* \times V,$$

where $X_0 \subset V$ is the null-cone, defined as in (1.4). In situations where the representation $G|V$ is stable, equation (1.6) is a rich source of character sheaves with full support on $V^*$. Indeed, for every character $\chi$ and every simple subquotient $K$ of $M_\chi$, the IC-extension $\text{IC}((V^*)^{rs}, K)$ is a character sheaf. We should note that, in all of the examples of this form that we have studied, the fundamental group $I$ of equation (1.5) is abelian (see Remark 2.2). This partly explains why, in this paper, we restrict attention to local systems $L_\chi$ of rank one.

In [GVX], we considered the special case of the above situation arising from an involutive automorphism $\theta$ as in (1.1), i.e., the case $m = 2$. In fact, in that paper, we studied the sheaf $P_\chi$ equivariantly with respect to the larger group $\tilde{G}^0 \supset \tilde{G}^{0,0}$. The main result of [GVX] ([GVX] Theorem 3.6]) is essentially equivalent to the corresponding claim Theorem 2.23 (see Remark 2.24). The papers [CVX] and [VX1] use the results of [GVX] to provide a classification of character sheaves for classical symmetric pairs, while the papers [VX2] and [VX3] use the results of the present paper to begin a study of character sheaves arising from higher order automorphisms.

The paper [VX2] studies character sheaves arising from the so-called GIT stable gradings. A grading is GIT stable if the corresponding representation space $V$ contains a $G$-stable vector in the sense of Geometric Invariant theory, that is, a vector $v \in V$ such that the orbit $G \cdot v$ is closed and the centralizer $G_v$ is finite. GIT stability implies the stability of $G|V$ in the sense of (1.2), but not vice versa. In this setting, subject to a finite number of exceptions, the geometry of each rank one representation $G_{\alpha}|V_\alpha$ is captured by a normal
crossings stratification of $V_\alpha$, and the polynomials $\bar{R}_{\chi,\alpha}$ can be computed using the well-known quiver description of perverse sheaves on this stratification.

The paper \cite{VX3} studies character sheaves arising from a class of gradings that, conjecturally, afford cuspidal character sheaves. This class includes all GIT stable gradings, as well as some gradings for which the representation $G|V$ is not stable in the sense of (1.2), but possesses certain rather special properties. With some significant additional argument, the methods of the present paper can be extended to these nonstable gradings. In this setting, the polynomials $\bar{R}_{\chi,\alpha}$ and their nonstable analogs can be computed using $\mathcal{D}$-module techniques, by being related to known b-function calculations for certain prehomogeneous vector spaces.

The paper \cite{Gr1} considers the case of nearby cycles with constant coefficients for an arbitrary (not necessarily stable) polar representation $G|V$, which is visible or of rank one. In the stable case, the main result of \cite{Gr1} (\cite{Gr1, Theorem 3.1}) is quite close, but not identical, to the $\chi = 1$ case of Theorem 2.23 (see Remark 2.25). In the present paper, we freely borrow the results of \cite{Gr1}, so we can mostly focus on the aspects of the problem related to the twisting of the local system $\mathcal{L}_\chi$. We should note that the proof of \cite{Gr1, Theorem 3.1} contained a gap, which was fixed in \cite{Gr3}, and the argument presented here depends on this fix (see \cite{Gr3, Section 3} and Proposition 4.5 below).

We now comment on the main difference between the arguments in \cite{GVX} and in the present paper. In both cases, the proof proceeds by analyzing the Picard-Lefschetz theory of the restriction $l_0|_{X_{c_0}}$, for a suitable choice of $l_0 \in (V^*)^{rs}$. The critical points of this restriction are contained in the Cartan subspace $C \subset V$, and are indexed by the Weyl group $W$. In the case considered in \cite{GVX}, the group $W$ is a Coxeter group. This enabled us to use a real structure on $C$, to place all the critical values of $l_0|_{X_{c_0}}$ on the real line $\mathbb{R} \subset \mathbb{C}$. This, in turn, enabled us to choose a special system of “cuts” (lying in the upper half-plane), and to define a distinguished Picard-Lefschetz basis up to sign for the stalk $(\mathfrak{F}P_{\chi/l_0})$. The proof of \cite{GVX, Theorem 3.6} proceeded as a calculation in terms of this distinguished basis. In the present paper, we consider a situation where $W$ need not be a Coxeter group. For a general complex reflection group, we do not know how to arrange the critical values of $l_0|_{X_{c_0}}$ in any special way, to produce a distinguished basis. Thus, to prove Theorem 2.23, we had to find a “more invariant” argument. In particular, the argument presented here differs substantially from the argument in \cite{GVX}, even in the case considered in that paper.

We also comment on the decision to write this paper in the generality of a stable polar representation, as opposed to focusing on the corresponding class of graded Lie algebras of \cite{Vi}. We feel that the class of polar representations neatly abstracts the geometric features of the quotient map $f : V \to Q$ which are needed to carry out our arguments. We hope this makes the arguments clearer than if we worked specifically with graded Lie algebras. In addition, the class of stable polar representations includes many other examples of potential future interest. The downside of working with polar representations generally is that they are...
rather less well known or well understood than graded Lie algebras. As a result, the statement of Theorem 2.23 contains some features which are trivial in all of the examples studied by the authors. See Remarks 2.1 and 2.21. It is possible that these features are “phantoms”, which can be ruled out through further study of polar representations. Regardless, we hope that they do not complicate the application of our results in specific examples.

1.3. Contents of this paper. The paper is organized as follows. Section 2 introduces the ingredients of our main Theorem 2.23, culminating in the statement of the theorem in Section 2.10. At this stage, our definition of the polynomial $\bar{R}_{\chi,\alpha}$, for $\alpha \in A$, remains contingent on some assertions about the representation $G_{\alpha}|V_{\alpha}$, whose proofs are deferred to Section 7. See Propositions 2.13 and 2.19. Our definition of $\bar{R}_{\chi,\alpha}$ does not lead to a general explicit formula. However, Proposition 2.18, which is also proved in Section 7, gives an explicit formula for the constant term $\bar{R}_{\chi,\alpha}(0)$. This formula leads to a number of other explicit formulas later in the paper, such as equation (2.67) for the character $\rho$, underlying the ingredient $\tilde{\rho}$ of equation (1.10).

Section 3 furnishes the proofs of several preliminary results, needed to formulate Theorem 2.23 and providing context for this theorem. In particular, in Section 3.2, we discuss the Hessian $H[v,l]$ of the restriction $l|_{G \cdot v}$, for $v \in C$ and $l \in (g \cdot C)^{\perp} \subset V^\ast$. Such Hessians play a key role in the proof of Theorem 2.23, which is given in Sections 4-9.

In Section 4, we invoke the results of [Gr1] and [Gr2] to show that the Fourier transform $\mathfrak{F}P_{\chi}$ is an IC-sheaf with full support on $V^\ast$. More precisely, we establish isomorphism (1.6) with $M_{\chi} = M(P_{\chi})$, the Morse local system of $P_{\chi}$ at the origin (see Proposition 4.1). We then begin the study of the local system $M(P_{\chi})$ on $(V^\ast)^{rs}$, using the methods of Picard-Lefschetz theory. Fixing a basepoint $l_0 \in (V^\ast)^{rs}$, we interpret the stalk $M_{l_0}(P_{\chi})$ as a relative homology group of the general fiber $X_{c_0}$ with coefficients in $L_{\chi}$, and recall a standard construction of Picard-Lefschetz classes in this group. Using this interpretation, we obtain our first three results about the local system $M(P_{\chi})$. First, we show that rank $M(P_{\chi}) = |W|$. Second, we invoke the results of [Gr1] and [Gr3] to show that there exists a cyclic vector $u_0 \in M_{l_0}(P_{\chi})$ for the microlocal monodromy (i.e., the holonomy) action of $\tilde{B}_{W} \cong \pi^1((V^\ast)^{rs}, l_0)$ (see Proposition 4.5). This existence of a cyclic vector is a key step in identifying the microlocal monodromy as an induced representation, as in (1.9). Finally, we compute the microlocal monodromy action of the subgroup $I \subset \tilde{B}_{W}$ (see exact sequence (1.7)) on Picard-Lefschetz classes in $M_{l_0}(P_{\chi})$ (see Proposition 4.6).

In Section 5, we introduce the monodromy in the family representation:

(1.13) $\mu : B^\chi W := p^{-1}(W_{\chi}) \rightarrow \text{Aut}(P_{\chi})$.

This structure arises out of the dependence of the sheaf $P_{\chi}$ on the choice of the basepoint $c_0 \in C^{reg}$, and it plays a key role in the rest of the argument. In Section 6, we investigate the monodromy action of a braid generator in $B^\chi W \subset B^\chi W$ (see (1.8)). More precisely, for $\alpha \in A$, we let $e_{\alpha} = |W_{\alpha}|/|W_{\alpha,\chi}|$. If $\sigma_{\alpha} \in B_{W}$ is a braid generator corresponding to the hyperplane
C_\alpha \subset C \ (\text{as defined in Section 2.3}), \text{then we have } \sigma^{e_\alpha}_\alpha \in B^0_W, \text{ and we consider the operator } 
abla(\sigma^{e_\alpha}_\alpha) \in \text{Aut}(P_{\chi}). \text{ We interpret the minimal polynomial } \bar{R}^\alpha_{\chi,\alpha} \text{ of this operator in terms of the rank one representation } G_\alpha|V_\alpha \ (\text{see Proposition 6.3}), \text{ and we use the geometry of the full representation } G|V \text{ to establish the degree bound:}

\begin{equation}
\deg \bar{R}^\alpha_{\chi,\alpha} \leq |W_{\alpha,\chi}|,
\end{equation}

(see Proposition 6.6).

In Section 7, we make a detailed study of the rank one representation \(G_\alpha|V_\alpha, \alpha \in A\). Picard-Lefschetz theory in this setting can be understood quite explicitly, in terms of the so-called carousel technique of singularity theory (see Remark 7.2). This enables us to relate the microlocal monodromy and the monodromy in the family actions on the Morse group \(M_{l_0,\alpha}(P_{\alpha})\), where \(l_{0,\alpha} = l_0|_{V_\alpha} \in (V_\alpha^*)^{rs}\) (see Proposition 7.1). We then use this relationship, together with the degree bound (1.14), to complete the definition of the polynomial \(\bar{R}^\alpha_{\chi,\alpha}\), and to relate it to the polynomial \(\bar{R}^\alpha_{\chi,\alpha}\) (see equation (7.37) and Remark 7.9).

In Section 8, we partially extend the relationship between the microlocal monodromy and the monodromy in the family, established in Section 7, to the setting of the full representation \(G|V\) (see Proposition 8.1). A key input here is the discussion of the Hessians in Section 3.2. In Section 9, we assemble the geometric inputs of Sections 4-8 to complete the proof of Theorem 2.23. The argument here is based on the interplay between two commuting actions on the Morse group \(M_{l_0}(P_{\chi})\): the microlocal monodromy action of \(\tilde{B}_W\) and the monodromy in the family action of \(B^0_W\). Finally, in Section 10, we formulate a conjecture regarding the vanishing of certain intersection numbers, which provides some context for the statement and the proof of Theorem 2.23, form the point of view of the classical Picard-Lefschetz formula (see Conjecture 10.2).

A number of arguments in this paper are closely parallel, in whole or in part, to corresponding arguments in [GVX]. In those cases, we refer to [GVX] and indicate the necessary adaptations, rather than repeating the entire argument. See, for example, the proofs of Propositions 2.6, 5.1, and 6.3.

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2. Statement of the main theorem

2.1. The polar representation \(G|V\). Let \(G|V\) be a polar representation of a connected reductive algebraic group over \(\mathbb{C}\) (see [DK]). Let \(C \subset V\) be a Cartan subspace, and let \(W = N_G(C)/Z_G(C)\) be the associated Weyl group; it is a finite complex reflection group acting on \(C\). We assume that:

\[ \text{rank}(G|V) := \dim C - \dim C^G \geq 1. \]
Let \( Q = C/W = V//G \), and let \( f : V \to Q \) be the quotient map. The main properties of \( G|V \) related to the geometry \( f \) are summarized in [Gr1, Section 2.3].

The representation \( G|V \) is called visible if the zero-fiber \( X_0 = f^{-1}(0) \) consists of finitely many \( G \)-orbits. We assume that:

\[
(2.1) \quad \text{either } G|V \text{ is visible or } \text{rank}(G|V) = 1.
\]

We also assume that \( G|V \) is stable in the sense of [DK, p. 506], i.e., that for every \( c \in C \) with \( Z_W(c) = \{1\} \) and every \( v \in V \), we have \( \dim G \cdot c \geq \dim G \cdot v \). Equivalently, this condition can be expressed by equation (1.2) (see [DK, Corollary 2.5]).

We now introduce some notation related to the complex reflection group \( W \). Let \( \{C_\alpha\}_{\alpha \in A} \) be the set of all reflection hyperplanes for \( W \) (where \( A \) is a finite index set). We write:

\[
(2.2) \quad C^{\text{sing}} = \bigcup_{\alpha \in A} C_\alpha.
\]

Let \( g = \text{Lie}(G) \). For every \( \alpha \in A \), let \( g_\alpha = Z_g(C_\alpha) \), let \( G_\alpha \subset G \) be the connected subgroup corresponding to \( g_\alpha \subset g \), and let:

\[
(2.3) \quad V_\alpha = C \oplus g_\alpha \cdot C \subset V.
\]

The subspace \( V_\alpha \) is preserved by the action of \( G_\alpha \), and the representation \( G_\alpha|V_\alpha \) is also polar and stable, with Cartan subspace \( C \) (see [DK, Theorem 2.12]). Let:

\[
(2.4) \quad W_\alpha = N_{G_\alpha}(C)/Z_{G_\alpha}(C),
\]

be the Weyl group associated to \( G_\alpha|V_\alpha \). Note that we have \( \text{rank}(G_\alpha|V_\alpha) \leq 1 \), and therefore, the group \( W_\alpha \) is cyclic, with \( W_\alpha = \{1\} \) iff \( \text{rank}(G_\alpha|V_\alpha) = 0 \) (in which case, we have \( g_\alpha = Z_g(C) \) and \( V_\alpha = C \)). We make the following further “locality” assumption:

\[
(2.5) \quad W_\alpha = Z_W(C_\alpha) \text{ for every } \alpha \in A.
\]

This assumption guarantees that:

\[
(2.6) \quad \text{rank}(G_\alpha|V_\alpha) = 1,
\]

for every \( \alpha \in A \). It also ensures that our definition of \( C^{\text{sing}} \) in equation (2.2) is consistent with the definition of \( c^{\text{sing}} \subset c \) in [DK] (see top of [DK, p. 515]).

**Remark 2.1.** Assumption (2.5) holds in all examples known to the authors. Moreover, in [DK, p. 521, Conjecture 2], Dadok and Kac conjecture that, for every polar representation \( G|V \) of a connected \( G \), the Weyl group \( W \) is generated by the subgroups \( \{W_\alpha\}_{\alpha \in A} \), defined as in (2.4). This property of \( G|V \) is equivalent to our assumption (2.5), as can be seen using [BCM, Corollary 2.36].
2.2. The fundamental group \( \pi_1^G(V^{rs}, c_0) \). Let:

\[
C^{reg} = C - C^{sing}, \quad Q^{reg} = f(C^{reg}) = C^{reg}/W, \quad V^{rs} = f^{-1}(Q^{reg}).
\]

By a theorem of Steinberg ([St, Theorem 1.5]; see also [Leh]), we have:

\[
C^{reg} = \{ c \in C \mid Z_W(c) = \{1\} \}.
\]

The stability assumption (1.2) implies that:

\[
(2.8) \quad V^{rs} = G \cdot C^{reg}.
\]

Assumption (2.5) further implies that \( V^{rs} \) is the union of all closed \( G \)-orbits of maximal dimension in \( V \). We can therefore refer to \( V^{rs} \subset V \) as the regular semisimple locus, and be consistent with the terminology of [DK]. In this subsection, we describe the \( G \)-equivariant fundamental group of \( V^{rs} \).

Pick a basepoint \( c_0 \in C^{reg} \), and let \( \bar{c}_0 = f(c_0) \in Q^{reg} \). We define:

\[
P B_W := \pi_1(C^{reg}, c_0), \quad B_W := \pi_1(Q^{reg}, \bar{c}_0), \quad \tilde{B}_W := \pi_1^G(V^{rs}, c_0).
\]

Here, \( B_W \) is the braid group associated to the complex reflection group \( W \), \( PB_W \subset B_W \) is the subgroup of pure braids, and we use the same conventions on equivariant fundamental groups as in [GVX, Section 2.3]. Namely, given an algebraic group \( G \), we write \( G^0 \subset G \) for the identity component, and we have \( \pi_1^G(pt) = G/G^0 \). Further, if \( G \) acts on a connected variety \( X \) with a basepoint \( x_0 \in X \), we identify \( G \)-equivariant local systems on \( X \) with left representations of \( \pi_1^G(X, x_0) \). In other words, we multiply loops in \( \pi_1^G(X, x_0) \) by tracing out the second loop first. Let \( X_{\bar{c}_0} = f^{-1}(\bar{c}_0) \) and define:

\[
(2.9) \quad I := \pi_1^G(X_{\bar{c}_0}, c_0) = Z_G(C)/Z_G(C)^0,
\]

(where \( Z_G(C)^0 \subset Z_G(C) \) is the identity component). Note that \( I \) is a finite group.

**Remark 2.2.** In all of the applications of the results of this paper currently envisioned by the authors, the group \( I \) happens to be abelian. However, we do not know how generally this is true. The proof of Theorem 2.23 would not be simplified significantly if we assumed that \( I \) is abelian.

Define:

\[
(2.10) \quad \tilde{W} := N_G(C)/Z_G(C)^0.
\]

As in [GVX, Section 2.3], we have a commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & I & \longrightarrow & \tilde{B}_W & \overset{\tilde{q}}{\longrightarrow} & B_W & \longrightarrow & 1 \\
\| & & \| & & \| & \downarrow \tilde{p} & & \| & & \|
\end{array}
\]

\[
\begin{array}{cccccc}
1 & \longrightarrow & I & \longrightarrow & \tilde{W} & \overset{q}{\longrightarrow} & W & \longrightarrow & 1 \\
\end{array}
\]

the right square of which is Cartesian, i.e., we have:

\[
(2.12) \quad \tilde{B}_W \cong \{ (\tilde{w}, b) \in \tilde{W} \times B_W \mid q(\tilde{w}) = p(b) \}.
\]
To define the maps in this diagram, note that we have:
\[
B_W = \pi_1(C^{reg}/W, \bar{c}_0) \cong \pi_1^W(C^{reg}, c_0),
\]
\[
\tilde{B}_W = \pi_1^G(V^{rs}, c_0) \cong \pi_1^{N_G(C)}(C^{reg}, c_0) \cong \pi_1^W(C^{reg}, c_0).
\]
The maps \( p \) and \( \tilde{p} \) are given by mapping \( C^{reg} \) to a point, the map \( q \) is the natural quotient map, and the map \( \tilde{q} \) is induced by \( q \). Note that the maps \( p \) and \( \tilde{p} \) are surjective, and we have:
\[
\ker(p) = \ker(\tilde{p}) = PB_W.
\]

It is helpful to be able to think about the map \( p \) and the isomorphism \((2.12)\) geometrically, in terms of loops. Let \( \gamma: [0, 1] \to Q^{reg} \) be a continuous path with \( \gamma(0) = \gamma(1) = \bar{c}_0 \). Then \( \gamma \) represents an element of \( B_W \). Let \( \tilde{\gamma}: [0, 1] \to C^{reg} \) be the unique lift of \( \gamma \) with \( \tilde{\gamma}(0) = c_0 \). Then the image \( p(\gamma) \in W \) is defined by the equation:
\[
\tilde{\gamma}(1) = p(\gamma)^{-1} c_0.
\]
See [Gr3, Section 2.2] for a detailed discussion of the appearance of the inverse in the RHS of equation \((2.14)\) (in the notation of [Gr3], the map \( p \) would be called \( \eta_c^0 \)).

Suppose now \( \tilde{w} \in \tilde{W} \) is an element with \( q(\tilde{w}) = p(\gamma) \). Pick a representative \( g_1 \in N_G(C) \) of \( \tilde{w} \) and a path \( g: [0, 1] \to G \) with \( g(0) = 1 \) and \( g(1) = g_1 \). The pair \( (g, \gamma) \) defines a path \( \Gamma: [0, 1] \to V^{rs} \) as follows:
\[
\Gamma(t) = \begin{cases} 
\tilde{\gamma}(2t) \in C^{reg} & \text{for } t \in [0, 1/2] \\
g(2t - 1) \tilde{\gamma}(1) \in X_{c_0} & \text{for } t \in [1/2, 1].
\end{cases}
\]

By equation \((2.14)\), we have \( \Gamma(1) = q(\tilde{w})p(\gamma)^{-1} c_0 = p(\gamma)p(\gamma)^{-1} c_0 = c_0 = \Gamma(0) \). Thus, the path \( \Gamma \) is a closed loop, representing an element of \( \tilde{B}_W \). It is not hard to check that \( \tilde{p}(\Gamma) = \tilde{w} \) and \( \tilde{q}(\Gamma) = \gamma \). Thus, the element \( \Gamma \in \tilde{B}_W \) corresponds to the pair \( (\tilde{w}, \gamma) \in \tilde{W} \times B_W \) under the isomorphism \((2.12)\).

Let \( \hat{I} := \text{Hom}(I, \mathbb{G}_m) \) be the set of characters of \( I \). It follows from diagram \((2.11)\) that the conjugation action of \( \tilde{B}_W \) on itself gives rise to an action of \( B_W \) on \( \hat{I} \), and that this action factors through the map \( p: B_W \to W \). We will denote these actions of \( B_W \) and \( W \) on \( \hat{I} \) by “\( \cdot \)”, so that we have:
\[
b \cdot \chi = p(b) \cdot \chi \in \hat{I} \text{ for all } b \in B_W \text{ and } \chi \in \hat{I}.
\]

In the event the group \( I \) is abelian (see Remark \((2.2)\)), we also obtain actions of \( B_W \) and \( W \) on \( I \), which are compatible with the actions on \( \hat{I} \).

2.3. Braid generators. We will need the following construction of elements of \( B_W \). Pick an \( \alpha \in A \). Let:
\[
A_{\alpha} = A - \{\alpha\}, \quad C_{\alpha} = \bigcup_{\beta \in A_{\alpha}} C_{\beta}, \quad C_{\alpha}^{reg} = C_{\alpha} - C_{\alpha}.
\]
Let \( s_\alpha \in W_\alpha \) be the counter-clockwise primitive generator, and let:
\[
(2.18) \quad n_\alpha = |W_\alpha|.
\]
Pick a point \( c_\alpha \in C^\text{reg}_\alpha \) and a nearby point \( c_{\alpha,1} \in C^\text{reg} \), such that:
\[
(2.19) \quad c_\alpha = \frac{1}{n_\alpha} \sum_{w \in W_\alpha} w c_{\alpha,1}.
\]
Let \( \Gamma_\alpha[c_{\alpha,1}] : [0, 1] \to C^\text{reg} \) be the continuous path given by:
\[
(2.20) \quad \Gamma_\alpha[c_{\alpha,1}](t) = c_\alpha + \exp(2\pi i t/n_\alpha)(c_{\alpha,1} - c_\alpha).
\]
Note that \( \Gamma_\alpha[c_{\alpha,1}](0) = c_{\alpha,1} \) and \( \Gamma_\alpha[c_{\alpha,1}](1) = s_\alpha c_{\alpha,1} \). Thus, the composition \( f \circ \Gamma_\alpha[c_{\alpha,1}] : [0, 1] \to Q^\text{reg} \) is a closed loop. Pick a continuous path \( \Gamma : [0, 1] \to C^\text{reg} \) with \( \Gamma(0) = c_0 \) and \( \Gamma(1) = c_{\alpha,1} \). Define:
\[
(2.21) \quad \sigma_\alpha[\Gamma] = (f \circ \Gamma^{-1}) \ast (f \circ \Gamma_\alpha[c_{\alpha,1}]) \ast (f \circ \Gamma) \in B_W,
\]
where “\( \ast \)” denotes the composition of paths. By equation (2.14), we have:
\[
(2.22) \quad p(\sigma_\alpha[\Gamma]) = s_\alpha^{-1}.
\]
We will refer to the element \( \sigma_\alpha[\Gamma] \in B_W \) as a counter-clockwise braid generator for \( \alpha \), and we will write \( B_W[\alpha] \subset B_W \) for the set of all such counter-clockwise braid generators for \( \alpha \).
Note that any two elements \( \sigma_1, \sigma_2 \in B_W[\alpha] \) are conjugate to each other by an element of \( PB_W \), i.e.:
\[
(2.23) \quad \forall \sigma_1, \sigma_2 \in B_W[\alpha], \ \exists b \in PB_W : \sigma_2 = b \sigma_1 b^{-1}.
\]
Note also that, for every \( \alpha_1 \in A, \sigma_1 \in B_W[\alpha_1], \) and \( b \in B_W, \) we have:
\[
(2.24) \quad b \sigma_1 b^{-1} \in B_W[\alpha_2],
\]
where \( \alpha_2 = p(b) \alpha_1 \in A \).

### 2.4. The concept of a regular splitting.

To state our main result, we will need another piece of structure related to diagram (2.11). Namely, a splitting of the top row of (2.11) satisfying a certain “locality” condition with respect to the braid generators in \( B_W \).

Recall the subgroup \( G_\alpha \subset G, \alpha \in A \), introduced following (2.2). For each \( \alpha \in A \), define:
\[
(2.25) \quad I_\alpha := Z_{G_\alpha}(C)/Z_{G_\alpha}(C)^0, \quad \tilde{W}_\alpha := N_{G_\alpha}(C)/Z_{G_\alpha}(C)^0,
\]
(cf. (2.9) and (2.10)). We have a natural projection \( q_\alpha : \tilde{W}_\alpha \to W_\alpha \), and a short exact sequence:
\[
(2.26) \quad 1 \to I_\alpha \to \tilde{W}_\alpha \xrightarrow{q_\alpha} W_\alpha \to 1.
\]
Note that we have \( Z_{G_\alpha}(C)^0 = Z_G(C)^0 \). Therefore, we have inclusions \( \tilde{W}_\alpha \hookrightarrow \tilde{W} \) and \( I_\alpha \hookrightarrow I \).
We will use these inclusions to view \( \tilde{W}_\alpha \) and \( I_\alpha \) as subgroups of \( \tilde{W} \) and \( I \), respectively. Note that, by construction:
\[
(2.27) \quad \text{for every } x \in I \text{ and } \alpha \in A, \text{ we have } x \tilde{W}_\alpha x^{-1} = \tilde{W}_\alpha \text{ and } x I_\alpha x^{-1} = I_\alpha.
\]
Remark 2.3. In the special case where $G|V$ comes from an involutive automorphism $\theta$, as in Section 1.2, it is known that the group $I$ is generated by the subgroups $\{I_\alpha\}_{\alpha \in A}$. See [K, Theorem 7.55] and [GVX, Remark 2.4]. We do not know whether this is the case for a general $G|V$ as in this paper.

Definition 2.4. Let $\tilde{r}: B_W \to \tilde{B}_W$ be a homomorphism which splits the top row of diagram (2.11), i.e., we have $\tilde{q} \circ \tilde{r} = \text{Id}_{B_W}$. We say that $\tilde{r}$ is a regular splitting if the composition $r = \tilde{p} \circ \tilde{r}: B_W \to \tilde{W}$ satisfies:

$$r(\sigma_\alpha) \in \tilde{W}_\alpha \subset \tilde{W},$$

for every $\alpha \in A$ and every $\sigma_\alpha \in B_W[\alpha]$.

Remark 2.5. Let $\tilde{r}: B_W \to \tilde{B}_W$ be a homomorphism which splits the top row of (2.11), and let $\alpha \in A$. Then, by (2.23) and (2.27), we have:

$$r(\sigma_\alpha) \in \tilde{W}_\alpha \text{ for some } \sigma_\alpha \in B_W[\alpha] \iff r(\sigma_\alpha) \in \tilde{W}_\alpha \text{ for every } \sigma_\alpha \in B_W[\alpha].$$

Proposition 2.6. If there exists a nilpotent $x \in X_0$ which is regular for $f$, i.e., we have $\text{rank} \, dxf = \dim Q$, then there exists a regular splitting $\tilde{r}: B_W \to \tilde{B}_W$.

A proof of Proposition 2.6 will be given in Section 3.4. It uses a normal slice to $X_0$ through the smooth point $x \in X_0$ to lift loops in $Q^{reg}$ to loops in $V^{rs}$.

Remark 2.7. A regular nilpotent does not always exist for a stable polar representation $G|V$ of a connected $G$. For example, if $G = \mathbb{G}_m$ acts on $V = \mathbb{C}^2$ by $t: (x, y) \mapsto (t^2 x, t^{-3} y)$, then there is no regular nilpotent for $G|V$. However, regular nilpotents exist in all of the applications envisioned by the authors.

For the rest of this paper, we assume that:

(2.28) a regular splitting homomorphism $\tilde{r}: B_W \to \tilde{B}_W$ exists and has been fixed, and we write $r = \tilde{p} \circ \tilde{r}: B_W \to \tilde{W}$.

Remark 2.8. The splitting homomorphism $\tilde{r}$ will not typically respect the identification (2.13), and the map $r$ will not typically factor through $p$.

2.5. The nearby cycle sheaf $P_\chi$. Throughout this paper, we work with local systems and sheaves of $\mathbb{C}$-vector spaces. Fix a character:

$$\chi \in \hat{I} = \text{Hom}(I, \mathbb{G}_m).$$

Note that, since $I$ is a finite group, all values of $\chi$ are roots of unity in $\mathbb{G}_m = \mathbb{C}^*$. The character $\chi$ gives rise to a rank one $G$-equivariant local system $\mathcal{L}_\chi$ on $X_{\tilde{c}_0}$, with $(\mathcal{L}_\chi)_{c_0} = \mathbb{C}$. As in [GVX, Section 3.1], we associate to the pair $(X_{\tilde{c}_0}, \mathcal{L}_\chi)$ a nearby cycle sheaf:

$$P_\chi \in \text{Perv}_G(X_0)_{\mathbb{C}^*\text{-conic}}.$$
The construction is as follows. We apply base change to the quotient map \( f : V \to Q \), to form a family:

\[
Z_{c_0} = \{(x,k) \in V \times \mathbb{C} \mid f(x) = k c_0\} \to \mathbb{C},
\]

where the multiplicative action of \( \mathbb{C} \) on \( Q = C/W \) is induced by the action on \( C \), so that \( f(kc) = kf(c) \) for all \( k \in \mathbb{C} \) and \( c \in C \). Let \( Z_{c_0}^{rs} = \{(x,k) \in Z_{c_0} \mid k \neq 0\} \), and let \( p_{c_0} : Z_{c_0}^{rs} \to X_{c_0} \) be the map \( p_{c_0} : (x,k) \mapsto k^{-1} x \). By construction, the family (2.29) is \( G \)-equivariant. Let us write:

\[
\psi_{c_0} : \text{Perv}_G(Z_{c_0}^{rs}) \to \text{Perv}_G(X_0),
\]

for the nearby cycle functor with respect to this family. We define:

\[
(2.30) \quad \psi_f[c_0] = \psi_{c_0} \circ p_{c_0}^* : \text{Perv}_G(X_{c_0}) \to \text{Perv}_G(X_0) \quad \text{and} \quad P_\chi = \psi_f[c_0](\mathcal{L}_\chi[-]).
\]

Here and henceforth, \([-\] denotes an appropriate cohomological shift, so that the resulting sheaf is perverse.

By construction, the sheaf \( P_\chi \) is \( \mathbb{C}^* \)-conic, and our main result, Theorem 2.23, describes the topological Fourier transform \( \mathfrak{F} P_\chi \) as an intersection homology sheaf on the dual space \( V^* \). To fix the conventions, we will be using [KS, Definition 3.7.8] to define the equivalence:

\[
\mathfrak{F} : \text{Perv}_G(V)_{\mathbb{C}^* \text{-conic}} \to \text{Perv}_G(V^*)_{\mathbb{C}^* \text{-conic}}
\]

up to a shift. In Section 5, we will introduce a monodromy in the family structure, which reflects the dependence of the sheaf \( P_\chi \) on the basepoint \( c_0 \in C^{reg} \).

2.6. The dual representation \( G|V^* \). The dual representation \( G|V^* \) is also polar and stable, with Cartan subspace:

\[
(2.31) \quad C^* := (g \cdot C)^{\perp} \subset V^*,
\]

and the same Weyl group \( W \) (see [Gr1, Proposition 2.13 (i)]). We define \( (C^*)^{reg} \subset C^* \) by analogy with equation (2.7), and we let \( (V^*)^{rs} = G \cdot (C^*)^{reg} \subset V^* \) (cf. equation (2.8)). One can check that assumption (2.3) implies the analogous assertion about the representation \( G|V^* \), and therefore, the set \( (V^*)^{rs} \) is the union of all closed \( G \)-orbits of maximal dimension in \( V^* \) (cf. discussion following (2.8)).

Proposition 2.9.

(i) There exists a \( G \)-invariant \( \mathbb{C}^* \)-conic (algebraic, Whitney) stratification \( S_0 \) of \( X_0 \), such that Thom’s \( A_f \) condition holds for the pair \( (V^{rs}, S) \), for every \( S \in S_0 \).

(ii) For every \( S_0 \) as in part (i) and every \( l \in (V^*)^{rs} \), the pair \( (0, l) \in (V^*)^{rs} \) is a generic covector at the origin for \( S_0 \), in the sense of stratified Morse theory.

(iii) For every \( S_0 \) as in part (i), we have:

\[
P_\chi \in \text{Perv}_G(X_0, S_0),
\]

i.e., the sheaf \( P_\chi \) is constructible with respect to \( S_0 \).
The reader is referred to \cite{Mat, Section 11} for a definition of the $A_f$ condition (see also \cite{GVX, Section 5.5}). Proposition 2.9 will be proved in Section 3.1. If $G|V$ is visible, we can take $S_0$ in part (i) to be the $G$-orbit stratification. For the rest of this paper, we fix a stratification $S_0$ as in Proposition 2.9 (i), and for the sake of concreteness, we assume that, if $G|V$ is visible, then $S_0$ is the $G$-orbit stratification. Proposition 2.9 implies that the Fourier transform $\mathfrak{F}P_\chi$ restricts to a (shifted) local system on $(V^*)^{rs}$.

Thus, to state our main result, we will need to specify a $G$-equivariant local system on $(V^*)^{rs}$. For this, it will be convenient to identify the $G$-equivariant fundamental groups of $V^{rs}$ and $(V^*)^{rs}$ as follows. Pick a compact form $K \subset G$ and a $K$-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $V$. We follow the convention that $\langle \cdot, \cdot \rangle$ is linear in the first argument, and antilinear in the second. An element $v \in V$ is called semisimple if the orbit $G \cdot v \subset V$ is closed. We say that a semisimple $v \in V$ is of minimal length if it is of minimal length in its $G$-orbit, with respect to $\langle \cdot, \cdot \rangle$. Write $V^{ms} \subset V$ for the set of all semisimple elements of minimal length, and let $V^{mrs} = V^{ms} \cap V^{rs}$. By \cite{KN} (see also \cite{DK, Theorem 1.1}), the set $V^{ms}$ can be characterized as follows:

$$\tag{2.32} V^{ms} = \{v \in V \mid \langle g \cdot v, v \rangle = 0\}.$$ 

By the proof of \cite[Lemma 2.1]{DK}, for every $v \in V^{mrs}$, we have $C_v \subset V^{ms}$, where $C_v$ is the (unique) Cartan subspace containing $v$. Therefore, without loss of generality, we can assume that:

$$\tag{2.33} C \subset V^{ms},$$

and we proceed with this assumption. By \cite{KN} and \cite{DK} Proposition 2.2, we then have:

$$\tag{2.34} V^{ms} = K \cdot C \text{ and } V^{mrs} = K \cdot C^{reg}.$$ 

Also, by (2.32) and the definition of a Cartan subspace (see \cite[p. 504]{DK}), we have:

$$\tag{2.35} C \perp g \cdot C \text{ with respect to } \langle \cdot, \cdot \rangle,$$

(cf. equation (1.2)).

**Proposition 2.10.**

(i) The inclusion map $j_V : V^{mrs} \hookrightarrow V^{rs}$ is a homotopy equivalence.

(ii) The map $j_V$ induces an isomorphism:

$$(j_V)_* : \pi_1^K(V^{mrs}, c_0) \to \pi_1^G(V^{rs}, c_0).$$

(iii) The inclusion $K \hookrightarrow G$ induces an isomorphism $N_K(C)/Z_K(C)^0 \cong \tilde{W}.$

In fact, the set $V^{mrs}$ is a strong deformation retract of $V^{rs}$, but we will not be using this fact. Proposition 2.10 will be proved in Section 3.1. Let $\nu = \nu_{\langle , \rangle} : V \to V^*$ be the antilinear map given by:

$$\tag{2.36} \nu(v_1) : v_2 \mapsto \langle v_2, v_1 \rangle, \quad v_1, v_2 \in V,$$
and let:
\[(2.37) \quad l_0 = \nu(c_0) \in V^*.
\]
By \((2.35)\), we have \(l_0 \in C^*\), and by Proposition \[2.10\] (iii), we have:
\[(2.38) \quad l_0 \in (C^*)^{reg} \subset (V^*)^{rs}.
\]
We use the inner product \(\langle \ , \rangle\) to define subsets \((V^*)^{mrs} \subset (V^*)^{ms} \subset V^*\) by analogy with \(V^{mrs} \subset V^{ms} \subset V\), and we note that assumption \((2.33)\) implies that \(C^* \subset (V^*)^{ms}\). By analogy with Proposition \[2.10\] we have a homotopy equivalence:
\[
\tilde{j}_{V^*} : (V^*)^{mrs} \to (V^*)^{rs},
\]
and an isomorphism:
\[
(j_{V^*})_* : \pi_1^K((V^*)^{mrs}, l_0) \to \pi_1^G((V^*)^{rs}, l_0).
\]
Using equations \((2.32)-(2.34)\) (see also \[Gr1\] Proposition 2.13 (i)), one can check that the map \(\nu\) restricts to a \(K\)-equivariant bijection:
\[
\nu^{mrs} : V^{mrs} \to (V^*)^{mrs}.
\]
Thus, we obtain an isomorphism:
\[(2.40) \quad \nu^{mrs}_* : \tilde{B}_W = \pi_1^G(V^{rs}, c_0) \cong \pi_1^K(V^{mrs}, c_0) \to \pi_1^K((V^*)^{mrs}, l_0) \cong \pi_1^G((V^*)^{rs}, l_0).
\]
We will use this isomorphism to identify the fundamental group \(\pi_1^G((V^*)^{rs}, l_0)\) with \(\tilde{B}_W\).

**Remark 2.11.** As a point of comparison, in \[GVX\] we used a \(G\)-invariant symmetric bilinear form on \(V\) to relate the geometry of \(V^*\) to that of \(V\), but such a form need not exist in the present context. Note that, by the antilinear property of the map \(\nu : V \to V^*\), the isomorphism \(\nu^{mrs}_*\) reverses the direction of braid generators. More precisely, let \(\alpha \in A\) and let \(\tilde{\sigma}_\alpha \in \tilde{B}_W\) be an element satisfying \(\tilde{q}(\tilde{\sigma}_\alpha) \in B_W[\alpha]\). Write \(\tilde{q}^* : \pi_1^G((V^*)^{rs}, l_0) \to \pi_1^K((C^*)^{reg}, l_0)\) for the analog of the map \(\tilde{q}\) of diagram \((2.11)\). Then the image \(\tilde{q}^* \circ \nu^{mrs}_*(\tilde{\sigma}_\alpha)\) is the inverse of a counter-clockwise braid generator (and can be called a clockwise braid generator).

### 2.7. The minimal polynomials \(R_{\chi,\alpha}\).

The essential content of our main result is that the Fourier transform \(\mathcal{F}P_\chi\) can be described in terms of data derived from the rank one representations \(\{G_\alpha[V_\alpha]_{\alpha \in A}\}\) introduced in Section 2.1 (see equation \((2.6)\)). A key such piece of data is a monic polynomial \(R_{\chi,\alpha} \in \mathbb{C}[z]\) of degree \(n_\alpha = |W_\alpha|\), associated to each \(\alpha \in A\), which we now proceed to define.

Fix an \(\alpha \in A\) and a \(\sigma_\alpha \in B_W[\alpha]\). All of the structures described above for the representation \(G|V\) descend to corresponding structures for the rank one representation \(G_\alpha[V_\alpha]\). Note that assumptions \((2.7)\) and \((2.5)\) hold automatically for \(G_\alpha[V_\alpha]\). The character \(\chi\) restricts to a character \(\chi_\alpha\) of the subgroup \(I_\alpha \subset I\) (see \((2.25)\) and the discussion following \((2.26)\)). Let:
\[
f_\alpha : V_\alpha \to Q_\alpha := C/W_\alpha = V_\alpha//G_\alpha,
\]
be the quotient map, and let:
\[(2.41) \quad X_{0,\alpha} = f_\alpha^{-1}(0), \quad Q_\alpha^{reg} = (C - C_\alpha)/W_\alpha, \quad \tilde{c}_0 = f_\alpha(c_0) \in Q_\alpha^{reg}, \quad X_{\tilde{c}_0,\alpha} = f_\alpha^{-1}(\tilde{c}_0).
\]
As in Section 2.5, we have a rank one $G_\alpha$-equivariant local system $L_\chi_\alpha$ on $X_{\tilde{c}_0, \alpha}$, with $(L_\chi_\alpha)_{c_0} = \mathbb{C}$, and a nearby cycle sheaf $P_{\chi_\alpha} \in \text{Perv}_{G_\alpha}(X_{0, \alpha})$.

Next, we define:

\[ V_{rs}^\alpha := f_{\alpha}^{-1}(Q_\alpha^{reg}) = G_\alpha \cdot (C - C_\alpha), \]
\[ B_{W_\alpha} := \pi_1(Q_\alpha^{reg}, \tilde{c}_0) \cong \mathbb{Z}, \quad \tilde{B}_{W_\alpha} := \pi_{1, \alpha}^{G_\alpha}(V_{rs}^\alpha, c_0). \]

Diagram (2.11) has a direct counterpart for the representation $G_\alpha|_{V_\alpha}$:

\[
\begin{array}{cccccc}
1 & \rightarrow & I_\alpha & \rightarrow & \tilde{B}_{W_\alpha} & \overset{\tilde{q}_\alpha}{\rightarrow} & B_{W_\alpha} & \rightarrow & 1 \\
\downarrow & & \downarrow \tilde{p}_\alpha & & \downarrow p_\alpha & & \downarrow & & \\
1 & \rightarrow & I_\alpha & \rightarrow & \tilde{W}_\alpha & \overset{q_\alpha}{\rightarrow} & W_\alpha & \rightarrow & 1 ,
\end{array}
\]

where $p_\alpha, q_\alpha, \tilde{p}_\alpha, \tilde{q}_\alpha$ are the analogs of the maps $p, q, \tilde{p}, \tilde{q}$. Just as for (2.11), the right square of (2.42) is Cartesian (cf. equation (2.12)). Note that the bottom row of (2.42) naturally injects into the bottom row of (2.11), but the top row does not.

Let:

\[ \sigma \in B_{W_\alpha} \cong \mathbb{Z}, \]
be the counter-clockwise generator. Note that we have:

\[ p_\alpha(\sigma) = s_\alpha^{-1} \in W_\alpha , \]
(see equation (2.22)). The regular splitting $\tilde{r} : B_W \rightarrow \tilde{B}_W$, which was fixed in (2.28), defines a splitting homomorphism:

\[ \tilde{r}[\sigma_\alpha] : B_{W_\alpha} \rightarrow \tilde{B}_{W_\alpha} , \]

such that the composition $r[\sigma_\alpha] := \tilde{p}_\alpha \circ \tilde{r}[\sigma_\alpha]$ is given by:

\[ r[\sigma_\alpha](\sigma) = r(\sigma_\alpha) \in \tilde{W}_\alpha \subset \tilde{W} . \]

Note that the splitting homomorphism $\tilde{r}[\sigma_\alpha]$ depends on the choice of the braid generator $\sigma_\alpha \in B_W[\alpha]$. Note also that $\tilde{r}[\sigma_\alpha]$ is automatically a regular splitting in the sense of Definition 2.4. Thus, we can say that, given a choice of $\sigma_\alpha \in B_W[\alpha]$, all of the assumptions (2.1), (2.5), (2.28) hold for the representation $G_\alpha|_{V_\alpha}$.

Next, we consider the dual representation $G_\alpha|_{V_\alpha^*}$. Let $C_\alpha^* = \nu(C_\alpha) = (C^*)^{W_\alpha} \subset C^*$ (see equation (2.36) and Proposition 2.10 (iii)), let $(V_\alpha^*)^{rs} = G_\alpha \cdot (C^* - C_\alpha^*)$, let:

\[ l_{0, \alpha} = l_0|_{V_\alpha} \in (V_\alpha^*)^{rs} , \]
(see equation (2.37)), and let $K_\alpha = K \cap G_\alpha$. Using assumption (2.33) and arguing as in the proof of [DK, Proposition 1.3], one can show that:

\[ K_\alpha \text{ is a compact form of } G_\alpha . \]
As in Section 2.6 (see equation (2.40)), the compact form $K_\alpha \subset G_\alpha$ and the inner product $\langle \cdot, \cdot \rangle$ give rise to an identification:

\[(2.47)\]

\[\pi_{G_\alpha}(\langle V_\alpha^* \rangle^{rs}, l_{0,\alpha}) \simeq \tilde{B}_W.\]

Applying Proposition 2.9 to the representation $G_\alpha|_{V_\alpha}$, we obtain a Morse local system $M_{l_\alpha}(P_\chi_\alpha)$ on $(\langle V_\alpha^* \rangle^{rs},$ of the sheaf $P_\chi_\alpha$ at the origin. In other words, for each $l_\alpha \in (\langle V_\alpha^* \rangle^{rs},$ we have:

\[(2.48)\]

\[M_{l_\alpha}(P_\chi_\alpha) = H^0((\phi - l_\alpha(P_\chi_\alpha))_0),\]

where the LHS is the stalk of $M(P_\chi_\alpha)$ at $l_\alpha$, and the RHS is the stalk cohomology at the origin of the vanishing cycles of $-l_\alpha$ applied to $P_\chi_\alpha$. We use isomorphism (2.47) to write:

\[(2.49)\]

\[\lambda_{l_0,\alpha} : \tilde{B}_W \to \text{Aut}(M_{l_\alpha}(P_\chi_\alpha)),\]

for the holonomy of this local system, and we refer to $\lambda_{l_0,\alpha}$ as the microlocal monodromy for the sheaf $P_\chi_\alpha$.

Let:

\[(2.50)\]

\[R = \{ R \in \mathbb{C}[z] \mid \deg R \geq 1, \ R(0) \neq 0, \ R \text{ is monic} \}.\]

We think of $R$ as the space of all possible minimal polynomials for an invertible element $a \in A$ of an associative $\mathbb{C}$-algebra $A$ with unit. Let:

\[(2.51)\]

\[W_\chi := \text{Stab}_W(\chi),\]

(see equation (2.16)).

**Proposition-Definition 2.12.** For each $\alpha \in A$, we define $R_{\chi,\alpha} \in R$ to be the minimal polynomial of the holonomy operator:

\[(2.52)\]

\[\lambda_{l_0,\alpha} \circ \bar{r}[\sigma_\alpha](\sigma) \in \text{End}(M_{l_\alpha}(P_\chi_\alpha)),\]

for some $\sigma_\alpha \in B_W[\alpha]$. We claim that:

(i) The polynomial $R_{\chi,\alpha}$ is independent of the choice of $\sigma_\alpha \in B_W[\alpha]$.

(ii) For every $\alpha_1, \alpha_2 \in A$ and $w \in W_\chi$, with $\alpha_2 = w\alpha_1$, we have $R_{\chi,\alpha_1} = R_{\chi,\alpha_2}$.

A proof of Proposition-Definition 2.12 will be given in Section 3.3.

**Proposition 2.13.** For each $\alpha \in A$, we have:

\[\deg R_{\chi,\alpha} = \dim M_{l_\alpha}(P_\chi_\alpha) = n_\alpha.\]

A proof of Proposition 2.13 will be given in Section 7.
2.8. **The determinant character** \( \tau \in \hat{I} \). Let \( \det : G \to \mathbb{G}_m \) be the determinant character of the representation \( G|V \).

**Proposition 2.14.** For every \( g \in Z_G(\mathbb{C}) \), we have \( \det(g) = \pm 1 \).

Proposition 2.14 will be proved in Section 3.1. It implies that \( \det : G \to \mathbb{G}_m \) descends to a character:

\[(2.53) \quad \tau : I \to \{ \pm 1 \},\]

(see equation (2.29)). Moreover, the character \( \tau \in \hat{I} \) is fixed by the action of \( W \) on \( \hat{I} \). Therefore, we can use the splitting homomorphism \( \tilde{r} : B_W \to \tilde{B}_W \) (see assumption (2.28)) to extend \( \tau \) to a character:

\[(2.54) \quad \tau : \tilde{B}_W \to \{ \pm 1 \},\]

such that \( \tau \circ \tilde{r}(b) = 1 \) for every \( b \in B_W \).

**Example 2.15.** The following example illustrates that the character \( \tau \) of equation (2.53) can be non-trivial. Take \( V = \text{Hom}(\mathbb{C}^2, \mathbb{C}) \oplus \text{Hom}(\mathbb{C}, \mathbb{C}^3) \), and let \( G = SO(2) \times GL(1) \times SO(3) \) act on \( V \) by:

\[(g_1, a, g_2)(x_1, x_2) = (a x_1 g_1^{-1}, g_2 x_2 a^{-1}).\]

The representation \( G|V \) is polar and stable, of rank one. For this representation, we have: \( W \cong \mathbb{Z}/4, I \cong \mathbb{Z}/2, \tilde{W} \cong W \times I, \) and \( \tau : I \to \{ \pm 1 \} \) is the non-trivial character. Note that the extension \( \tau : \tilde{B}_W \to \{ \pm 1 \} \) of equation (2.54), in this example, will depend on the choice of the splitting homomorphism \( \tilde{r} \).

**Remark 2.16.** The character \( \tau \) of equation (2.53) is analogous to the character \( \tau : I \to \{ \pm 1 \} \) of [GVX, Section 3.4]. However, in the situation (and the notation) of [GVX], the character \( \tau \) is only non-trivial because we consider the action of the full fixed point group \( K = G^\theta \), rather than the identity component \( K^0 \subset K \) (cf. Remark 2.24).

For each \( \alpha \in A \), we can repeat the construction of the character \( \tau \in \hat{I} \) for the representation \( G|V_{\alpha} \), to obtain a character:

\[\tau_\alpha : I_\alpha \to \{ \pm 1 \}.\]

Recall that we have \( I_\alpha \subset I \), as discussed in Section 2.4 following (2.26).

**Proposition 2.17.** For each \( \alpha \in A \), we have \( \tau_\alpha = \tau|_{I_\alpha} \).

A proof of Proposition 2.17 will be given in Section 3.2.

The character \( \tau_\alpha \in \hat{I}_\alpha \) turns out to be related to the value of the minimal polynomial \( R_{\chi,\alpha} \) at zero. Let:

\[(2.55) \quad d_\alpha = \dim X_{\zeta_\alpha,\alpha} ,\]

(see (2.41)). Recall that we write \( n_\alpha = |W_\alpha| \) (see (2.18)), and recall the map \( r = \tilde{p} \circ \tilde{r} : B_W \to \tilde{W} \) defined following assumption (2.28).
Proposition 2.18. Let \( \alpha \in A \), \( \sigma_\alpha \in B_W[\alpha] \). Let \( x = r(\sigma^n_\alpha) \in I_0 \subset I \subset \tilde{W} \) (see diagram (2.11) and Definition 2.4). We then have:

\[
R_{\chi,\alpha}(0) = (-1)^{d_\alpha + 1} \cdot \chi(x) \cdot \tau_\alpha(x).
\]

A proof of Proposition 2.18 will be given in Section 7.

2.9. The group \( W_0^{\chi} \) and the Hecke algebra \( \mathcal{H}_{W_0^{\chi}} \). For each \( \alpha \in A \), let:

\[
W_{\alpha,\chi} = W_\alpha \cap W_\chi \quad \text{and} \quad e_\alpha = n_\alpha/|W_{\alpha,\chi}| \in \mathbb{Z}, \quad \text{so that} \quad W_{\alpha,\chi} = \langle s_\alpha^{e_\alpha} \rangle.
\]

Proposition 2.19. For each \( \alpha \in A \), there exists a polynomial \( \bar{R}_{\chi,\alpha} \in \mathbb{R} \), with \( \deg \bar{R}_{\chi,\alpha} = n_\alpha/e_\alpha \), such that:

\[
R_{\chi,\alpha}(z) = \bar{R}_{\chi,\alpha}(z^{e_\alpha}).
\]

A proof of Proposition 2.19 will be given in Section 7. Clearly, the polynomial \( \bar{R}_{\chi,\alpha} \in \mathbb{R} \) is unique; we can think of it as the minimal polynomial of the holonomy operator:

\[
\lambda_{i_0,\alpha} \circ \bar{r}[\sigma_\alpha^{e_\alpha}] (\sigma_\alpha^{e_\alpha}) \in \text{End}(M_{i_0,\alpha}(P_{\chi_0})),
\]

for some \( \sigma_\alpha \in B_W[\alpha] \). Note that:

\[
\text{the assignment } \alpha \mapsto e_\alpha \text{ is invariant under the action of } W_\chi \text{ on } A.
\]

Therefore, by Proposition-Definition 2.12 (ii), we have:

\[
\text{for every } \alpha_1, \alpha_2 \in A \text{ and } w \in W_\chi, \text{ with } \alpha_2 = w \alpha_1, \text{ we have } \bar{R}_{\chi,\alpha_1} = \bar{R}_{\chi,\alpha_2}.
\]

Remark 2.20. The assignment \( (G|V, \chi) \mapsto \{W_0^{\chi} \subset W_\chi \subset W\} \) can be applied to the pair \( (G_\alpha|V_\alpha, \chi_\alpha) \), to obtain subgroups \( W_0^{\chi_\alpha} \subset W_{\alpha,\chi_\alpha} \subset W_\alpha \). By construction, we then have:

\[
W_{\alpha,\chi} \subset W_{0,\alpha,\chi} = W_{\alpha,\chi_\alpha}.
\]

Let:

\[
A_0^{\chi} = \{ \alpha \in A \mid e_\alpha < n_\alpha \}, \quad A_1^{\chi} = \{ \alpha \in A \mid e_\alpha = n_\alpha \},
\]

so that \( A = A_0^{\chi} \cup A_1^{\chi} \). Let:

\[
C^{\text{reg}}_\chi = C - \bigcup_{\alpha \in A_0^{\chi}} C_\alpha,
\]

and define:

\[
B_{W_\chi}^{0} := \pi_1(C^{\text{reg}}_\chi/W_0^{\chi}, c_0), \quad B_{\chi}^{0} := \pi_1(C^{\text{reg}}/W_0^{\chi}, c_0) = p^{-1}(W_0^{\chi}) \subset B_W,
\]
where the basepoint $c_0 \in C^{reg}$ naturally determines basepoints in $C^{reg}/W^0_\chi$ and $C^{reg}/W^0_\chi$. Note that $B_{W_\chi}$ is the braid group associated to the complex reflection group $W^0_\chi$.

Let:

$$\varphi : B_\chi^0 \rightarrow B_{W_\chi},$$

be the map induced by the inclusion $C^{reg} \rightarrow C^{reg}$. Note that, for every $\alpha \in A^0_\chi$ and $\sigma_\alpha \in B_\chi[\alpha]$, we have $\sigma_\alpha^{e_\alpha} \in B_\chi^0$. Moreover, by (2.60), the image $\varphi(\sigma_\alpha^{e_\alpha})$ is a braid generator for the group $B_{W_\chi}$. We define a Hecke algebra $\mathcal{H}_{W_\chi}$ as the quotient of the group algebra $\mathbb{C}[B_{W_\chi}]$ by all relations of the form:

$$R_{\chi,\alpha}(\varphi(\sigma_\alpha^{e_\alpha})) = 0,$$

for $\alpha \in A^0_\chi$ and $\sigma_\alpha \in B_\chi[\alpha]$, and we write:

$$\eta_\chi : \mathbb{C}[B_{W_\chi}] \rightarrow \mathcal{H}_{W_\chi},$$

for the quotient map. By assertion (2.59) and [Et, Theorem 1.3], we have:

$$\dim \mathcal{H}_{W_\chi} = |W^0_\chi|.$$

The Hecke algebra $\mathcal{H}_{W_\chi}$ encodes the polynomials $R_{\chi,\alpha}$ for $\alpha \in A^0_\chi$. To encode the polynomials $\tilde{R}_{\chi,\alpha}$ for $\alpha \in A^1_\chi$, we define a character:

$$\rho = \rho_\chi : B_\chi^0 \rightarrow \mathbb{G}_m,$$

by requiring that:

$$\rho(\sigma_\alpha^{e_\alpha}) = 1 \quad \text{for all} \quad \alpha \in A^0_\chi \quad \text{and} \quad \sigma_\alpha \in B_\chi[\alpha],$$

(2.66)

$$\tilde{R}_{\chi,\alpha}(\rho(\sigma_\alpha^{n_\alpha})) = 0 \quad \text{for all} \quad \alpha \in A^1_\chi \quad \text{and} \quad \sigma_\alpha \in B_\chi[\alpha].$$

Note that, for $\alpha \in A^1_\chi$, we have $e_\alpha = n_\alpha$ and $\deg \tilde{R}_{\chi,\alpha} = 1$. The character $\rho$ is well-defined by assertion (2.59). Moreover, by Proposition 2.18, we can rewrite condition (2.66) as follows:

(2.67)

$$\rho(\sigma_\alpha^{n_\alpha}) = (-1)^{d_\alpha} \cdot \chi(x) \cdot \tau_\alpha(x),$$

for all $\alpha \in A^1_\chi$ and $\sigma_\alpha \in B_\chi[\alpha]$, and for $x = r(\sigma_\alpha^{n_\alpha}) \in I_\alpha$.

**Remark 2.21.** In all of the examples computed by the authors, the character $\rho$ is trivial. But we can not rule out the possibility that it is non-trivial in some examples.

### 2.10. Statement of the theorem

We begin by constructing a $\mathbb{C}[\tilde{B}_W]$-module $M_\chi$ associated to the character $\chi \in \hat{I}$. Let:

$$B_\chi^x := \text{Stab}_{B_W}(\chi) = p^{-1}(W_\chi), \quad \tilde{B}_W^x := \tilde{q}^{-1}(B_\chi^x), \quad \tilde{B}_W^x,0 := \tilde{q}^{-1}(B_\chi^0),$$

(see equation (2.10)). Note that we have $\tilde{B}_W^x \subset \tilde{B}_W \subset \tilde{B}_W$. The character $\chi$ extends uniquely to a character:

$$\hat{\chi} : \tilde{B}_W^x \rightarrow \mathbb{G}_m,$$
such that \( \hat{\chi} \circ \bar{r} (b) = 1 \) for every \( b \in B_W^\chi \). Let \( C_\chi \) be a copy of \( \mathbb{C} \), viewed as an \( I \)-module via the character \( \chi \). Simultaneously, we view \( C_\chi \) as a \( \tilde{B}_W^\chi \)-module (and therefore a \( \tilde{B}_W^{\chi,0} \)-module) via the character \( \hat{\chi} \) of equation (2.69). Similarly, let \( C_\tau \) be a copy of \( \mathbb{C} \), viewed as a \( \tilde{B}_W \)-module via the character \( \tau \) of equation (2.54). Also, let \( C_\rho \) be a copy of \( \mathbb{C} \), viewed as a \( \tilde{B}_W^{\chi,0} \)-module via the composition:

\[
\tilde{\rho} := \rho \circ \bar{q} : \tilde{B}_W^{\chi,0} \to \mathbb{G}_m.
\]

Recall the group homomorphism \( \varphi : B_W^{\chi,0} \to B_W^\chi \) of equation (2.62), and let:

\[
\tilde{\varphi} := \varphi \circ \bar{q} : \tilde{B}_W^{\chi,0} \to B_W^\chi.
\]

We can view the Hecke algebra \( \mathcal{H}_W^\chi \) as a module over itself via the left multiplication. Simultaneously, we can view \( \mathcal{H}_W^\chi \) as \( \mathbb{C}[B_W^\chi] \)-module via the map \( \eta_\chi \) of equation (2.63), as a \( \mathbb{C}[B_W^{\chi,0}] \)-module via the composition:

\[
\eta_\chi \circ \varphi : \mathbb{C}[B_W^{\chi,0}] \to \mathcal{H}_W^\chi,
\]

and as a \( \mathbb{C}[\tilde{B}_W^{\chi,0}] \)-module via the composition:

\[
\eta_\chi \circ \tilde{\varphi} : \mathbb{C}[\tilde{B}_W^{\chi,0}] \to \mathcal{H}_W^\chi.
\]

Here, and in the rest of the paper, we use the following notational convention.

**Convention 2.22.** Given a map of discrete groups \( \Phi : B_1 \to B_2 \), we use the same symbol for the corresponding map of group algebras \( \Phi : \mathbb{C}[B_1] \to \mathbb{C}[B_2] \). Similarly, given a representation of a discrete group \( \Lambda : B \to \text{Aut}(M) \) on a complex vector space \( M \), we use the same symbol for the corresponding representation of the group algebra \( \Lambda : \mathbb{C}[B] \to \text{End}(M) \).

Consider the tensor product \( \mathbb{C}_\chi \otimes \mathbb{C}_\rho \otimes \mathcal{H}_W^\chi \), taken over \( \mathbb{C} \), and view it as a \( \mathbb{C}[\tilde{B}_W^{\chi,0}] \)-module by combining the \( \mathbb{C}[\tilde{B}_W^{\chi,0}] \)-module structures on \( \mathbb{C}_\chi \), \( \mathbb{C}_\rho \), and \( \mathcal{H}_W^\chi \). We now induce this module to \( \mathbb{C}[\tilde{B}_W] \), and define:

\[
\mathcal{M}_\chi = \left( \mathbb{C}[\tilde{B}_W] \otimes_{\mathbb{C}[\tilde{B}_W^{\chi,0}]} \left( \mathbb{C}_\chi \otimes \mathbb{C}_\rho \otimes \mathcal{H}_W^\chi \right) \right) \otimes \mathbb{C}_\tau.
\]

Note that, by equation (2.64), we have \( \dim \mathcal{M}_\chi = |W| \). We interpret the \( \mathbb{C}[\tilde{B}_W] \)-module \( \mathcal{M}_\chi \) as a \( G \)-equivariant local system on \((V^*)^rs\), whose fiber over the basepoint \( l_0 \in (V^*)^rs \) is equal to \( \mathcal{M}_\chi \), and whose holonomy is given by the \( \mathbb{C}[\tilde{B}_W] \)-module structure, via the identification (2.40).
Theorem 2.23. Let \( G|V \) be as above: a stable polar representation, which is visible or of rank one, satisfies the locality assumption \((2.3)\), and admits a regular splitting as in \((2.28)\). Consider the nearby cycle sheaf \( P_\chi \) defined in \((2.30)\). Its Fourier transform is given by:

\[
\mathcal{F} P_\chi \cong IC((V^*)^{rs}, M_\chi),
\]

where the RHS is the IC-extension of the local system \( M_\chi \) defined in \((2.71)\) above.

Remark 2.24. The statement of Theorem 2.23 is formally very similar to the statement of [GVX] Theorem 3.6], the main difference being that the former applies much more broadly. However, [GVX] Theorem 3.6] is not technically a special case of Theorem 2.23. There are four reasons for this. First, the paper [GVX] works with coefficients in a general integral domain \( k \), while in this paper we work with coefficients in \( \mathbb{C} \). This distinction is not material, given [GVX] Remark 7.2] and the freeness of the \( k \)-module \( M_1(P_\chi) \) in [GVX] Proposition 7.3]. Second, [GVX] Theorem 3.6] computes the polynomials \( R_{\chi,\alpha} \) explicitly. Third, the paper [GVX] deals with a possibly disconnected group \( K \) acting on a symmetric space \( \mathfrak{p} \). And fourth, the group \( W_{a,\chi} \) defined in [GVX] is potentially smaller than the group \( W_{\chi} \) defined for the corresponding situation in this paper, even if \( K \) is connected. Having said that, it is not difficult to verify that the claim of [GVX] Theorem 3.6] for a symmetric pair \((G, K)\), giving rise to a symmetric space \( \mathfrak{p} \), is equivalent to the claim of Theorem 2.23 for the polar representation \( K^0 \mathfrak{p} \) (cf. Remark 2.26 below).

Remark 2.25. The claim of Theorem 2.23 for a polar representation \( G|V \) and the trivial character \( \chi = 1 \) is close, but not equivalent, to the claims of [Gr1] Theorems 3.1 & 5.2] for \( G|V \). More precisely, the latter provide a simultaneous description of the Morse local system \( M(P_1) \) on \((V^*)^{rs}, \) of the sheaf \( P_1 \) at the origin, and of the monodromy in the family action of \( B_\mathfrak{p} \) on \( M(P_1) \). However, the description of \( M(P_1) \) in [Gr1] is not complete. Theorem 2.23, on the other hand, gives a complete description of \( M(P_1) \), but does not discuss the monodromy in the family; see however Remark 2.27 below.

Remark 2.26. In some situations, the factorization of the polynomial \( R_{\chi,\alpha} \) in Proposition 2.19 holds with \( z^{e_\alpha} \) replaced by \( z^{E_\alpha} \), where \( E_\alpha \geq e_\alpha \) for all \( \alpha \in A, E_\alpha > e_\alpha \) for some \( \alpha \in A, \) and the assignment \( \alpha \mapsto E_\alpha \) is invariant under \( W_\chi \). When this is the case, we can define \( W_\chi^1 \subset W_\chi \) to be the proper subgroup generated by the powers \( \{s^{E_\alpha}_{\alpha} \}_{\alpha \in A,} \) and the group \( W_\chi^1 \) can be used in place of \( W_\chi \) in the statement of Theorem 2.23. In the context of GIT stably graded Lie algebras, a natural candidate \( W_\chi^{en} \) for \( W_\chi^1 \) arises from the endoscopic point of view, as explained in [VX2] Section 5].

Remark 2.27. A natural question that is not answered by Theorem 2.23 is the computation of the endomorphism ring \( \text{End}(P_\chi) \). We expect that this ring can be described as follows. Write \( \mathcal{H}_\chi \) for the opposite of the Hecke algebra \( \mathcal{H}_{W_\chi} \). There is a unique homomorphism \( \kappa: \mathcal{C}[B_\chi^{0}] \to \mathcal{H}_\chi \), such that:

\[
\kappa(\sigma^{e_\alpha}_{\alpha}) = k_{\alpha} \cdot \eta_{\chi} \circ (\sigma^{-e_\alpha}_{\alpha}) \quad \text{for all } \alpha \in A_\chi^{0} \text{ and } \sigma_\alpha \in B_\chi[\alpha],
\]

\[
\kappa(\sigma^{n_\alpha}_{\alpha}) = (-1)^{d_{\alpha}} \cdot \chi(\sigma^{-n_\alpha}_{\alpha}) \quad \text{for all } \alpha \in A_\chi^{1} \text{ and } \sigma_\alpha \in B_\chi[\alpha],
\]
where \( \{k_\alpha \in \{\pm 1\}\}_{\alpha \in A} \) are the signs given by Proposition 7.1. The definition of \( \kappa \) is designed to capture the action of \( B_{W,0}^\chi \subset B_{W}^\chi \) on \( M_\chi^0 \subset M_\chi \) (see equation (1.9)), via the monodromy in the family (1.13) (cf. Proposition 8.1 and equation (9.3)). The existence and uniqueness of \( \kappa \) follow from the proof of Theorem 2.23 in Section 9. Use \( \kappa \) to view \( H_\chi^\circ \) as a \( \mathbb{C}[B_{W}^\chi] \)-module, and define:

\[
E := \mathbb{C}[B_{W}^\chi] \otimes_{\mathbb{C}[B_{W}^\chi]} H_\chi^\circ.
\]

Then the \( \mathbb{C}[B_{W}^\chi] \)-module structure on \( E \) descends to a \( \mathbb{C} \)-algebra structure, and the monodromy action (1.13) gives rise to an isomorphism:

\[
\text{End}(P_\chi) \cong E.
\]

In particular, the ring \( \text{End}(P_\chi) \) is generated by the monodromy action (1.13), and we have \( \dim \text{End}(P_\chi) = |W_\chi| \). We expect that isomorphism (2.73) can be established using the methods of this paper, but we have not carried out a proof. We note, however, that in the special case where \( W_\chi^0 = W_\chi \), definition (2.72) reduces to \( E = H_\chi^\circ \), and isomorphism (2.73) follows readily from the statement and the proof of Theorem 2.23.

3. Preliminary results

3.1. Geometric preliminaries. In this subsection, we prove Propositions 2.9, 2.10, and 2.14.

**Proof of Proposition 2.9.** A proof of part (i) is essentially contained in the first paragraph of [Gr1, Section 3]. To wit, if \( G|V \) is visible, we can take \( S_0 \) to be the \( G \)-orbit stratification. If, instead (see assumption (2.1)), we have \( \text{rank}(G|V) = 1 \), we can readily reduce to the case \( \dim Q = 1 \), then use a general result on the existence of \( A_f \) stratifications for functions; see [Hi, p. 248, Corollary 1].

Our proof of part (ii) is similar to the proof of [GVX, Lemma 6.1]. Since the stratification \( S_0 \) is \( C^* \)-conic, it suffices to show that \( 0 \in X_0 \) is the only stratified critical point of the restriction \( l|_{X_0} \), with respect to \( S_0 \). By [Gr1, Proposition 2.13 (i)], the representation \( G|V^* \) is polar with Cartan subspace \( C^* = (g \cdot C)^\perp \subset V^* \) (see (2.31)). Therefore, we can assume, without loss of generality, that \( l \in (C^*)^{reg} = C^* \cap (V^*)^{rs} \). By [Gr1, Proposition 2.13 (ii)], we have:

\[
\{v \in V \mid l|_{g \cdot v} = 0\} = C.
\]

Since the stratification \( S_0 \) is \( G \)-invariant, it remains to observe that \( X_0 \cap C = \{0\} \).

Part (iii) follows from [Gr, Theorem 5.5] and the remark that follows that theorem (cf. proof of [GVX, Corollary 3.2]).
Proof of Proposition 2.10. Let $\mathfrak{K} = \text{Lie}(K)$. By [DK, Proposition 1.3], for every $v \in C^{reg}$, we have:

\begin{equation}
Z_G(v) = Z_G(C) = Z_K(C) \cdot \exp(i Z_K(C)).
\end{equation}

Part (i) follows by considering $j_V : V^{mrs} \rightarrow V^{rs}$ as a map of fiber bundles over $Q^{reg}$. Part (ii) follows from part (i) and the definition of the equivariant $\pi_1$. Part (iii) follows from equation (3.1) and [DK, Lemma 2.7]. □

For every $v \in V$ and $l \in (g \cdot v)^\perp \subset V^*$, the restriction $l|_{G \cdot v}$ has a critical point at $v$, and we write:

\begin{equation}
\mathcal{H}[v, l] \in S\text{ym}^2((g \cdot v)^*),
\end{equation}

for the Hessian of $l|_{G \cdot v}$ at $v$. By [Gr1, Corollary 2.16 (ii)], if $v \in V^{rs}$ and $l \in (g \cdot v)^\perp \cap (V^*)^{rs}$, then $v$ is a Morse critical point of $l|_{G \cdot v}$. In other words, we have:

\begin{equation}
\forall v \in V^{rs}, l \in (g \cdot v)^\perp \cap (V^*)^{rs}, \ \mathcal{H}[v, l] \text{ is a non-degenerate quadratic form on } g \cdot v.
\end{equation}

Proof of Proposition 2.14. Each $g \in Z_G(C)$ preserves the direct sum decomposition $V = C \oplus g \cdot C$ of equation (1.2). Pick a $v \in C^{reg}$ and an $l \in (C^*)^{reg}$. By the definition of a Cartan subspace (see [DK, p. 504]), we have $g \cdot C = g \cdot v$. The element $g$ acts on $g \cdot v$ preserving the Hessian $\mathcal{H}[v, l]$. By (3.3), this implies that $\det(g|_{g \cdot v}) = \pm 1$. By assumption, we have $\det(g|_C) = 1$. The proposition follows. □

3.2. Hessians and the root space decomposition. In Section 3.1 we introduced the Hessian $\mathcal{H}[v, l]$ for every $v \in V$ and $l \in (g \cdot v)^\perp \subset V^*$ (see equation (3.2)). In this subsection, we make a more detailed study of $\mathcal{H}[v, l]$ for $v \in C$ and $l \in C^*$, leading up to a proof of Proposition 2.17. We will need the analog of the root space decomposition for the representation $G[V$, i.e., the following digest of [DK, Theorem 2.12]. Let $m = Z_g(C)$, and recall that we write $g_\alpha = Z_g(C_\alpha), \alpha \in A$.

Theorem 3.1.

(i) For all $\alpha \neq \beta, \alpha, \beta \in A$, we have:

$$g_\alpha \cap g_\beta = m.$$ 

(ii) 

$$g = \bigoplus_{\alpha \in A} g_\alpha \quad \text{and} \quad g/m = \bigoplus_{\alpha \in A} g_\alpha/m.$$ 

(iii) 

$$V = C \bigoplus_{\alpha \in A} g_\alpha \cdot C.$$ 

(iv) For every $v \in C$, we have:

$$g \cdot v = \bigoplus_{\alpha \in A; v \notin C_\alpha} g_\alpha \cdot C.$$
(v) For every $v \in C$, we have:

$$Z_g(v) = \sum_{\alpha \in A : v \in C_{\alpha}} g_{\alpha} \cdot$$

Proof. The only assertion of the theorem not contained in [DK, Theorem 2.12] is the second equation of part (ii). It follows from the first equation of part (ii) and the direct sum decomposition of part (iii).

The following elementary lemma applies generally to representations of algebraic groups over $\mathbb{C}$.

Lemma 3.2. Let $G|V$ be a representation of an algebraic group over $\mathbb{C}$, let $v \in V$, let $l \in (g \cdot v)^\perp \subset V^*$, and let $\mathcal{H}[v, l]$ be the Hessian of $l|_{G \cdot v}$ at $v$. Let $x \in g$ and $w \in g \cdot v \subset V$. Then we have:

$$\mathcal{H}[v, l](x \cdot v, w) = l(x \cdot w).$$

Proof. Let $\Gamma_x : G_a \to G$ be the homomorphism of complex analytic groups generated by $x \in g$. The assignment $t \mapsto (\Gamma_x(t) \cdot v, \Gamma_x(t) \cdot w), t \in G_a$, defines an analytic curve $G_a \to TG \cdot v$. We have:

$$\frac{\partial \Gamma_x(t) \cdot v}{\partial t} \bigg|_{t=0} = x \cdot v \quad \text{and} \quad \frac{\partial \Gamma_x(t) \cdot w}{\partial t} \bigg|_{t=0} = x \cdot w.$$

The lemma follows by the definition of a Hessian.

Recall the inner product $\langle \ , \rangle$ which was fixed in Section 2.6 and is subject to assumption (2.33).

Proposition 3.3. For each $v \in C$ and $l \in C^*$, the direct sum decomposition:

$$g \cdot v = \bigoplus_{\alpha \in A : v \notin C_{\alpha}} g_{\alpha} \cdot C,$$

of Theorem 3.1 (iv) is orthogonal with respect to both the Hessian $\mathcal{H}[v, l]$ and the inner product $\langle \ , \rangle$.

Proof. Begin with the assertion regarding the Hessian $\mathcal{H}[v, l]$. Pick a pair of distinct elements $\alpha, \beta \in A$ with $v \notin C_{\alpha} \cup C_{\beta}$. Let $x_1 \in g_{\alpha}$ and $x_2 \in g_{\beta}$. We need to show that:

$$\mathcal{H}[v, l](x_1 \cdot v, x_2 \cdot v) = 0.$$

Note that:

$$\text{(3.4) for every } v_1 \in C - C_{\beta}, \text{ we have } g_{\beta} \cdot v_1 = g_{\beta} \cdot v.$$
Using (3.4), we can choose elements \( v_1 \in C_\alpha - C_\beta \) and \( x_3 \in g_\beta \), such that \( x_3 \cdot v_1 = x_2 \cdot v \). We then have:

\[
\mathcal{H}[v, l] (x_1 \cdot v, x_2 \cdot v) = \mathcal{H}[v, l] (x_1 \cdot v, x_3 \cdot v_1) = l(x_1 \cdot (x_3 \cdot v_1)) = l([x_1, x_3] \cdot v_1) + l(x_3 \cdot (x_1 \cdot v_1)),
\]

where the second equality follows from Lemma 3.2. It remains to note that each of the summands on the second line of (3.5) vanishes. Indeed, we have:

\[
l([x_1, x_3] \cdot v_1) = 0,
\]

because \( l \in C^* = (g \cdot C)^\perp \subset V^* \), and we have:

\[
l(x_3 \cdot (x_1 \cdot v_1)) = 0,
\]

because \( x_1 \in g_\alpha, v_1 \in C_\alpha \), and so \( x_1 \cdot v_1 = 0 \). This proves the orthogonality with respect to \( \mathcal{H}[v, l] \).

We now proceed to the assertion regarding the inner product \( \langle \ , \rangle \). For each \( \alpha \in A \), let \( R_\alpha = g_\alpha \cap R \). By [DK] Proposition 1.3 and Theorem 3.3 (v), we have:

\[
g_\alpha = R_\alpha \otimes_R \mathbb{C}, \quad \alpha \in A.
\]

(3.6)

Pick a pair of distinct elements \( \alpha, \beta \in A \) with \( v \notin C_\alpha \cup C_\beta \). Let \( x_1 \in R_\alpha, x_2 \in R_\beta \), and \( v_1 \in C_\alpha - C_\beta \). In view of (3.4) and (3.6), it suffices to show that \( \langle x_1 \cdot v, x_2 \cdot v_1 \rangle = 0 \). Using the \( K \)-invariance of the inner product \( \langle \ , \rangle \), we compute:

\[
\langle x_1 \cdot v, x_2 \cdot v_1 \rangle = -\langle v, x_1 \cdot (x_2 \cdot v_1) \rangle = \langle v, x_2 \cdot (x_1 \cdot v_1) \rangle - \langle v, [x_1, x_2] \cdot v_1 \rangle = 0.
\]

Here, \( x_1 \cdot v_1 = 0 \) by construction, and \( [x_1, x_2] \cdot v_1 \in g \cdot C \) is orthogonal to \( v \), by (2.35). \( \Box \)

For each \( \alpha \in A, v \in C - C_\alpha \), and \( l \in C^* \), we note that \( g_\alpha \cdot v = g_\alpha \cdot C \), and we write:

\[
\mathcal{H}_\alpha[v, l] := \mathcal{H}[v, l]|_{g_\alpha \cdot v} \subset \text{Sym}^2((g_\alpha \cdot v)^*).
\]

Assertion (3.3) and Proposition 3.3 imply that the partial Hessian \( \mathcal{H}_\alpha[v, l] \) is a non-degenerate quadratic form on \( g_\alpha \cdot v \), whenever \( v \in C_{\text{reg}}^\alpha \) and \( l \in (C^*)_{\text{reg}}^\alpha \). The following proposition shows that, in fact, this non-degeneracy of the partial Hessian holds more generally. Recall that we write \( C_\alpha^* \subset C^* \) for the reflection hyperplane corresponding to \( \alpha \).

**Proposition 3.4.** The partial Hessian \( \mathcal{H}_\alpha[v, l] \) is a non-degenerate quadratic form on \( g_\alpha \cdot v \) for every \( \alpha \in A, v \in C - C_\alpha \), and \( l \in C^* - C_\alpha^* \).

*Proof.* Apply assertion (3.3) to the rank one representation \( G_\alpha | V_\alpha \). \( \Box \)

*Proof of Proposition 2.17* Consider the \( G_\alpha \)-invariant direct sum decomposition:

\[
V = V_\alpha \oplus g \cdot C_\alpha,
\]

(3.8)
where \( V_\alpha = C \oplus \mathfrak{g}_\alpha \cdot C \) and \( \mathfrak{g} \cdot C_\alpha = \bigoplus_{\beta \in A_{\alpha}} \mathfrak{g}_\beta \cdot C \) (see (2.3), (2.17) and Theorem 3.1 (iii)).

For every \( g \in G_\alpha \), we will write:

\[
\det_\alpha(g) = \det(g|_{V_\alpha}), \quad \det_{\mathfrak{t}_\alpha}(g) = \det(g|_{\mathfrak{g} \cdot C_\alpha}).
\]

We then have:

\[
\det(g) = \det_\alpha(g) \cdot \det_{\mathfrak{t}_\alpha}(g), \quad g \in G_\alpha.
\]

We will show that:

\[
\det_{\mathfrak{t}_\alpha}(g) = 1 \quad \text{for every} \quad g \in Z_{G_\alpha}(C),
\]

and the proposition will follow. As in the proof of Proposition 2.14, we will use considerations related to the Hessian \( \mathcal{H}[v, l] \), for suitable \( v \in C \) and \( l \in C^* \).

Recall the notation of equation (2.17), and define:

\[
C^*_\alpha = \bigcup_{\beta \in A_{\alpha}} C^*_\beta, \quad (C^*_\alpha)^{reg} = C^*_\alpha - C^*_{\mathfrak{t}_\alpha}.
\]

Pick some \( v \in C^{reg} \) and \( l \in (C^*_\alpha)^{reg} \), note that \( \mathfrak{g} \cdot v = \mathfrak{g} \cdot C_\alpha \), and consider the Hessian \( \mathcal{H}[v, l] \), as in (3.2). By Theorem 3.1 (iv) and Propositions 3.3, 3.4, the Hessian \( \mathcal{H}[v, l] \) is a non-degenerate quadratic form on \( \mathfrak{g} \cdot C_\alpha \). By the choice of \( v \in V \) and \( l \in V^* \), the group \( G_\alpha \) acts on \( \mathfrak{g} \cdot C_\alpha \) preserving the Hessian \( \mathcal{H}[v, l] \). Equation (3.10) follows, since \( G_\alpha \) is connected. □

### 3.3. Full equivariance in rank one.

In this subsection, we prove Proposition-Definition 2.12. Instead of giving the shortest proof possible, we begin with a preliminary discussion which will be useful later (see, for example, the proof of Proposition 6.1).

Fix an \( \alpha \in A \). Let \( G^f_\alpha = Z_G(C_\alpha) \subset G \). Note that we have \( G_\alpha = (G^f_\alpha)^0 \). Let \( K^f_\alpha = K \cap G^f_\alpha \).

By analogy with (2.46), one can show that:

\( K^f_\alpha \) is a compact form of \( G^f_\alpha \).

By (2.35) and Proposition 3.3, decomposition (3.8) is orthogonal with respect to \( \langle \cdot, \cdot \rangle \). Since the compact form \( K^f_\alpha \subset G^f_\alpha \) preserves both the subspace \( \mathfrak{g} \cdot C_\alpha \subset V \) and the inner product \( \langle \cdot, \cdot \rangle \), we can conclude that:

\[
\text{the group } G^f_\alpha \text{ acts on } V \text{ preserving the decomposition (3.8)}.
\]

By the locality assumption (2.3), we have:

\[
V_\alpha/\!\!/G^f_\alpha = C/W_\alpha = Q_\alpha.
\]

It follows that both the local system \( \mathcal{L}_{\chi_\alpha} \) on \( X_{\mathfrak{c}_0,} \) and the nearby cycles sheaf \( P_{\chi_\alpha} \) can be viewed \( G^f_\alpha \)-equivariantly (see Section 2.7). Let:

\[
\tilde{B}^f_{W_\alpha} := \pi_{1}^{G^f_\alpha}(V_\alpha, l_{0, \alpha}), \quad \tilde{W}^f_{\alpha} := N_{G^f_\alpha}(C)/Z_{G^f_\alpha}(C)^0.
\]
Evidently, we have \( Z_{Gf}(C) = Z_G(C) \). Therefore, we can think of \( \tilde{W}_f^\alpha \) as a subgroup of \( \tilde{W} \), and we have \( I \subset \tilde{W}_f^\alpha \). Moreover, by (2.5), we have:

\[
\tilde{W}_f^\alpha = q^{-1}(W_{\alpha}) \subset \tilde{W}.
\]

By construction, we have:

\[
\tilde{W}_\alpha \subset \tilde{W}_f^\alpha, \quad \tilde{B}_{W_\alpha} \subset \tilde{B}_{W_f^\alpha}.
\]

By analogy with diagrams (2.11) and (2.42), we have a diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & I & \longrightarrow & \tilde{B}_{W_f^\alpha} & \longrightarrow & \tilde{B}_{W_\alpha} & \longrightarrow & 1 \\
& & | & | & \downarrow p_f & & \downarrow p & & 1 \\
1 & \longrightarrow & I & \longrightarrow & \tilde{W}_f^\alpha & \longrightarrow & W_\alpha & \longrightarrow & 1,
\end{array}
\]

the right square of which is Cartesian. Moreover, the maps \( q_f, q_\alpha, p_f, p_\alpha \) of diagram (2.42) are the restrictions of the maps \( q_f, q_\alpha, p_f, p_\alpha \) of diagram (3.14). Thus, the entire diagram (2.42) embeds into diagram (3.14). This enables us to formulate the following lemma.

**Lemma 3.5.** Let \( \sigma_1, \sigma_2 \in B_W[\alpha] \). Then the elements \( \tilde{r}[\sigma_1](\sigma), \tilde{r}[\sigma_2](\sigma) \in \tilde{B}_{W_\alpha} \) (cf. equation (2.52)) are conjugate in \( \tilde{B}_{W_f^\alpha} \) by an element of \( I \subset \tilde{B}_{W_f^\alpha} \).

**Proof.** By (2.23), there exists an element \( b \in PB_W \), such that \( \sigma_2 = b \sigma_1 b^{-1} \). Let \( x = r(b) \in I \). We then have \( r(\sigma_2) = x r(\sigma_1) x^{-1} \). By equation (2.45) and the Cartesian property of diagram (3.14), we can conclude that \( r[\sigma_2](\sigma) = x r[\sigma_1](\sigma) x^{-1} \), as required. \( \square \)

Isomorphism (2.47) naturally extends to an isomorphism:

\[
\pi_1^{G_f}(V_\alpha^r, l_{0,\alpha}) \cong \tilde{B}_{W_\alpha}^f.
\]

Using this isomorphism, the \( G_f^\alpha \)-equivariant structure on the sheaf \( P_{\chi_\alpha} \) produces a microlocal monodromy action:

\[
\lambda_{l_{0,\alpha}} : \tilde{B}_{W_\alpha}^f \rightarrow \text{Aut}(M_{l_{0,\alpha}}(P_{\chi_\alpha})),
\]

which extends the action of equation (2.49).

**Proof of Proposition-Definition 2.12.** Part (i) follows readily from Lemma 3.5. Indeed, let \( \sigma_1, \sigma_2 \in B_W[\alpha] \) be a pair of braid generators. Using Lemma 3.5 and the action (3.15), we can conclude that the linear transformations:

\[
\lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_1](\sigma), \lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_2](\sigma) \in \text{End}(M_{l_{0,\alpha}}(P_{\chi_\alpha})),
\]

are conjugate to each other by an element of \( \text{Aut}(M_{l_{0,\alpha}}(P_{\chi_\alpha})) \). Therefore, these linear transformations have the same minimal polynomial, as required.
2.6. We begin by describing a homomorphism \( \tilde{r} \).

The restriction \( f|_Y : Y \to Q \) is a local analytic isomorphism near \( x \). Recall the Hermitian inner product \( \langle \cdot , \cdot \rangle \) on \( V \). Pick a small \( \epsilon > 0 \), and let \( C_\epsilon \subset C \) be the open \( \epsilon \)-ball around 0, with respect to \( \langle \cdot , \cdot \rangle \). Let \( Q_\epsilon = f(C_\epsilon) \). We assume that \( \epsilon \) is sufficiently small, so that there exists an open neighborhood \( Y_\epsilon \subset Y \) of \( x \), such that \( f(Y_\epsilon) = Q_\epsilon \) and:

\[
T_x X_0 \oplus T_x Y = T_x V \cong V.
\]

The restriction \( f|_Y : Y \to Q \) is a local analytic isomorphism near \( x \). Recall the Hermitian inner product \( \langle \cdot , \cdot \rangle \) on \( V \). Pick a small \( \epsilon > 0 \), and let \( C_\epsilon \subset C \) be the open \( \epsilon \)-ball around 0, with respect to \( \langle \cdot , \cdot \rangle \). Let \( Q_\epsilon = f(C_\epsilon) \). We assume that \( \epsilon \) is sufficiently small, so that there exists an open neighborhood \( Y_\epsilon \subset Y \) of \( x \), such that \( f(Y_\epsilon) = Q_\epsilon \) and:

\[
f_{\epsilon} := f|_{Y_\epsilon} : Y_\epsilon \to Q_\epsilon,
\]

is a complex analytic isomorphism. Let \( Q_{\epsilon}^{reg} = Q_\epsilon \cap Q^{reg} \), and note that the inclusion \( Q_{\epsilon}^{reg} \to Q^{reg} \) is a homotopy equivalence. Pick a \( k \in (0,1] \) such that:

\[
c_1 := k c_0 \in C_\epsilon,
\]

and let \( \bar{c}_1 = f(c_1) \). We then have:

\[
B_W = \pi_1(Q_{\epsilon}^{reg}, \bar{c}_0) = \pi_1(Q^{reg}, \mathbb{R}_+, \bar{c}_0) = \pi_1(Q^{reg}, \bar{c}_1) \cong \pi_1(Q_{\epsilon}^{reg}, \bar{c}_1).
\]

\[
\tilde{B}_W = \pi_1^G(V^{rs}, c_0) = \pi_1^G(V^{rs}, \mathbb{R}_+, c_0) = \pi_1^G(V^{rs}, c_1).
\]

Let \( X_{\epsilon_1} = f_{\epsilon}^{-1}(\bar{c}_1) \) and \( y_1 = f_{\epsilon}^{-1}(\bar{c}_1) \), so that \( X_{\epsilon_1} \cap Y_\epsilon = \{y_1\} \). Pick a continuous path \( \Gamma_1 : [0,1] \to X_{\epsilon_1} \), with \( \Gamma_1(0) = c_1 \) and \( \Gamma_1(1) = y_1 \). We are now prepared to define the
homomorphism \( \tilde{r} : B_W \to \widetilde{B}_W \). Let \( b \in B_W \) be an element represented by a path \( \Gamma_b : [0,1] \to Q^\text{reg}_\epsilon \), with \( \Gamma_b(0) = \Gamma_b(1) = \bar{c}_1 \), using the identification (3.17). We set:
\[
\tilde{r}(b) = \Gamma_1^{-1} \star (f^{-1}_\epsilon \circ \Gamma_g) \star \Gamma_1 \in \widetilde{B}_W ,
\]
where " \( \star \) " denotes the composition of paths, and we use the identification (3.18). It is clear from the definition that \( \tilde{r} \) is a group homomorphism which splits the top row of diagram (2.11).

**Proposition 3.7.** The homomorphism \( \tilde{r} \) is a regular splitting in the sense of Definition 2.4.

Our proof of Proposition 3.7 will rely on the following lemma.

**Lemma 3.8.** Let \( \alpha \in A \), and let \( \Gamma : [0,1] \to V^r \cap V_\alpha \) be a path with \( \Gamma(0) = \Gamma(1) = c_0 \), representing an element \( b \in \widetilde{B}_W \). Then we have \( \tilde{p}(b) \in \widetilde{W}_\alpha \).

**Proof.** For every \( v \in V^r \cap V_\alpha \), let \( C_v \subset V \) be the Cartan subspace containing \( v \). By the definition of \( V_\alpha \subset V \), we have:
\[
\forall v \in V^r \cap V_\alpha \ \exists g \in G_\alpha : \ C_v = g \cdot C \subset V_\alpha .
\]
The path \( t \mapsto C_{\Gamma(t)} \) can be lifted to a continuous path \( t \mapsto g_t \in G_\alpha \), such that \( g_0 = 1 \) and \( C_{\Gamma(t)} = g_t \cdot C \) for every \( t \in [0,1] \). The element \( \tilde{p}(b) \in \widetilde{W} \) is then represented by \( g_1 \in G_\alpha \) (cf. equation (2.15)). The lemma follows.

**Proof of Proposition 3.7.** This proof is a direct adaptation of the proof of [GVX, Proposition 2.6], and we focus on the aspects specific to the present setting. Pick an \( \alpha \in A \) and a braid generator \( \sigma_\alpha = \sigma_\alpha[\Gamma] \in B_W[\alpha] \), as in equation (2.21). Recall that \( \Gamma : [0,1] \to C^\text{reg} \) is a continuous path, with \( \Gamma(0) = c_0 \) and \( \Gamma(1) = c_{\alpha,1} \), where the point \( c_{\alpha,1} \in C^\text{reg}_\alpha \) lies near a point \( c_\alpha \in C^\text{reg}_\alpha \), which is the projection of \( c_{\alpha,1} \) onto \( C_\alpha \) (see equation (2.19)). We need to show that \( r(\sigma_\alpha) \in \widehat{W}_\alpha \subset \widehat{W} \).

Without loss of generality, we can assume that \( c_1 = c_0 \) (see equation (3.16)), and that the path of equation (2.21), representing \( \sigma_\alpha \in B_W \), is contained entirely within \( Q^\text{reg}_\epsilon \subset Q^\text{reg} \). It follows that \( c_\alpha \in C_\epsilon \). Let:
\[
(3.19) \quad \bar{c}_\alpha = f(c_\alpha), \quad X_{\bar{c}_\alpha} = f^{-1}(\bar{c}_\alpha), \quad y_\alpha = f^{-1}_\epsilon(c_\alpha).
\]
The main idea of the proof is to observe that, in order to specify the element \( r(\sigma_\alpha) \in \widetilde{W} \) up to conjugation by elements of \( I \subset \widehat{W} \), we do not need the entirety of the normal slice \( Y_\epsilon \), but only the germ of it at \( y_\alpha = Y_\epsilon \cap X_{\bar{c}_\alpha} \).

More precisely, we will make use of the following construction. Let \( V^\text{reg} \subset V \) be the set of regular points for \( f \), and let \( X^\text{reg}_{\bar{c}_\alpha} = X_{\bar{c}_\alpha} \cap V^\text{reg} \). Pick a point \( z_\alpha \in X^\text{reg}_{\bar{c}_\alpha} \) and an affine normal
slice $N \subset V$ through $z_\alpha$ to $X^{reg}_{c_\alpha} \subset V^{reg}$. For $\delta > 0$, let $C[c_\alpha, \delta] \subset C$ be the open $\delta$-ball around $c_\alpha$. We will say that $\delta > 0$ is sufficiently small for the normal slice $N$ if we have:

$$C[c_\alpha, 2\delta] \subset C,$$

and there exits a neighborhood $N_{2\delta} \subset N$ of $z_\alpha$, such that:

$$f(N_{2\delta}) = Q[\bar{c}_\alpha, 2\delta] := f(C[c_\alpha, 2\delta]),$$

and the restriction:

$$f_{N,2\delta} := f|_{N_{2\delta}} : N_{2\delta} \to Q[\bar{c}_\alpha, 2\delta],$$

is a complex analytic isomorphism.

Let $\delta_1 = \text{dist}(c_\alpha, c_{\alpha,1})$, and fix a $\delta \in (0, \delta_1]$ which is sufficiently small for $N$. Define:

$$(3.20) \quad c_{\alpha,2} = c_\alpha + (\delta/\delta_1) \cdot (c_{\alpha,1} - c_\alpha) \in C^{reg}, \quad \bar{c}_{\alpha,2} = f(c_{\alpha,2}) \in Q^{reg},$$

$$(3.21) \quad X_{\bar{c}_{\alpha,2}} = f^{-1}(\bar{c}_{\alpha,2}) \subset V^{rs}, \quad z_2 = f^{-1}(\bar{c}_{\alpha,2}) = X_{\bar{c}_{\alpha,2}} \cap N_{2\delta}.$$

Pick a continuous path $\Gamma_2 : [0, 1] \to X_{\bar{c}_{\alpha,2}}$, with $\Gamma_2(0) = c_{\alpha,2}$ and $\Gamma_2(1) = z_2$. The quadruple $[z_\alpha, N, \delta, \Gamma_2]$ defines an element $\tilde{b} = \tilde{b}[z_\alpha, N, \delta, \Gamma_2] \in \tilde{B}_W$ as follows:

$$(3.22) \quad \tilde{b} = \Gamma^{-1} \ast [c_{\alpha,2}, c_{\alpha,1}] \ast \Gamma_2^{-1} \ast (f_{N,2\delta}^{-1} \circ f \circ \Gamma_2[c_{\alpha,2}]) \ast \Gamma_2 \ast [c_{\alpha,1}, c_{\alpha,2}] \ast \Gamma \in \tilde{B}_W,$$

where $[c_{\alpha,1}, c_{\alpha,2}]$ and $[c_{\alpha,2}, c_{\alpha,1}]$ are the straight line paths, and $\Gamma_2[c_{\alpha,2}]$ is defined as in (2.20). In other words, the element $\tilde{b}$ is represented by a small loop inside of $N$ which links the hypersurface $f^{-1}(f(C_\alpha)) \subset V$ in the counter-clockwise direction and is connected to the basepoint $c_0 \in V^{rs}$ via the paths $\Gamma$, $[c_{\alpha,1}, c_{\alpha,2}]$, and $\Gamma_2$.

We now make the following observations regarding the above construction. First, for suitable $\delta$ and $\Gamma_2$, we have:

$$(3.23) \quad \tilde{r}(\sigma_\alpha) = \tilde{b}[y_\alpha, Y, \delta, \Gamma_2].$$

Second, write $\tilde{B}_W/I$ for the set of $I$-orbits in $\tilde{B}_W \supset I$, under the conjugation action (see diagram (2.11)). Then the image of $\tilde{b}[z_\alpha, N, \delta, \Gamma_2]$ in $\tilde{B}_W/I$ is independent of the path $\Gamma_2$. We denote this image by:

$$\tilde{b}[z_\alpha, N, \delta] \in \tilde{B}_W/I.$$ 

Third, by continuity, the element $\tilde{b}[z_\alpha, N, \delta] \in \tilde{B}_W/I$ is independent of the normal slice $N$ and the number $\delta \in (0, \delta_1]$. Thus, we can omit $N$ and $\delta$ from the notation, writing:

$$\tilde{b}[z_\alpha, \delta] = \tilde{b}[z_\alpha] \in \tilde{B}_W/I.$$ 

And fourth, again by continuity, we have:

$$(3.24) \quad \tilde{b}[z_\alpha] \in \tilde{B}_W/I$$

is determined by the connected component of $z_\alpha \in X^{reg}_{\bar{c}_\alpha}$.

Write $\tilde{W}/I$ for the set of $I$-orbits in $\tilde{W} \supset I$, under the conjugation action, and let:

$$\omega : \tilde{W} \to \tilde{W}/I,$$
be the quotient map. The map \( \tilde{p} : \tilde{B}_W \to \tilde{W} \) of diagram (2.11) induces a map:
\[
\tilde{p} : \tilde{B}_W/I \to \tilde{W}/I.
\]
Recall that the conjugation action of \( I \) on \( \tilde{W} \) preserves the subgroup \( \tilde{W}_\alpha \subset \tilde{W} \) (see equation (2.27)). Thus, in view of (3.23), in order to prove the proposition, it suffices to show that:
\[
(3.25) \quad \tilde{p}(\tilde{b}[y_\alpha]) \in \tilde{W}_\alpha/I = \omega(\tilde{W}_\alpha).
\]
We will establish (3.25) by showing that the construction of equation (3.22) can be performed entirely within the subspace \( V_\alpha \subset V \); then applying Lemma 3.8. More precisely, consider the intersection \( X_{\tilde{c}_\alpha} \cap V_\alpha \). Let:
\[
(3.26) \quad \tilde{c}_\alpha = f_\alpha(c_\alpha), \quad X_{\tilde{c}_\alpha,\alpha} = f_\alpha^{-1}(\tilde{c}_\alpha).
\]
Then \( X_{\tilde{c}_\alpha,\alpha} \) is the connected component of \( X_{\tilde{c}_\alpha} \cap V_\alpha \) containing \( c_\alpha \). By Theorem 3.1 (iii)-(iv) (see also equation (3.8)), we have:
\[
(3.27) \quad \text{the subspace } V_\alpha \subset V \text{ is a normal slice to the closed orbit } G \cdot c_\alpha \text{ through } c_\alpha.
\]
Since \( G \cdot c_\alpha \) is the unique closed \( G \)-orbit in \( X_{\tilde{c}_\alpha} \), we can conclude that every \( G \)-orbit in \( X_{\tilde{c}_\alpha} \) meets the connected component \( X_{\tilde{c}_\alpha,\alpha} \). Pick a point \( z_\alpha \in G \cdot y_\alpha \cap X_{\tilde{c}_\alpha,\alpha} \), and note that:
\[
(3.28) \quad z_\alpha \in X_{\tilde{c}_\alpha,\alpha}^{\text{reg}}.
\]
Since \( G \) is connected, and in view of (3.24), we have:
\[
(3.29) \quad \tilde{b}[y_\alpha] = \tilde{b}[z_\alpha] \in \tilde{B}_W/I.
\]
Consider the orbits \( G_\alpha \cdot z_\alpha \subset G \cdot z_\alpha \subset X_{\tilde{c}_\alpha,\alpha}^{\text{reg}} \). Note that \( \{c_\alpha\} \) is the unique closed \( G_\alpha \)-orbit in \( X_{\tilde{c}_\alpha,\alpha} \). Therefore, the point \( c_\alpha \in V_\alpha \) is contained in the closure of \( G_\alpha \cdot z_\alpha \). In view of (3.27), we can conclude that:
\[
(3.30) \quad \text{the intersection } G_\alpha \cdot z_\alpha \cap V_\alpha \text{ is transverse along } G_\alpha \cdot z_\alpha.
\]
Let \( V_\alpha^{\text{reg}} \subset V_\alpha \) be the set of regular points for \( f_\alpha \), and let \( X_{\tilde{c}_\alpha,\alpha}^{\text{reg}} = X_{\tilde{c}_\alpha,\alpha} \cap V_\alpha^{\text{reg}} \). Assertions (3.28) and (3.30) imply that:
\[
(3.31) \quad z_\alpha \text{ is a regular point for the restriction map } f|_{V_\alpha} : V_\alpha \to Q.
\]
Let \( h_\alpha : Q_\alpha \to Q \) be the quotient map, and consider the factorization:
\[
(3.32) \quad f|_{V_\alpha} = h_\alpha \circ f_\alpha : V_\alpha \to Q.
\]
Assertion (3.31) and equation (3.32) imply that \( z_\alpha \in X_{\tilde{c}_\alpha,\alpha}^{\text{reg}} \) and that:
\[
(3.33) \quad \tilde{c}_\alpha \in Q_\alpha \text{ is a regular point of } h_\alpha.
\]
Note that assertion (3.33) can also be inferred directly from the locality assumption (2.5). By (3.32) and (3.33), we have:
\[
(3.34) \quad X_{\tilde{c}_\alpha,\alpha}^{\text{reg}} \subset X_{\tilde{c}_\alpha,\alpha}^{\text{reg}}.
\]
Let $N \subset V_\alpha$ be an affine normal slice through $z_\alpha$ to $X^{reg}_{\alpha,\alpha} \subset V^{reg}_\alpha$. By (3.30) and (3.34), the subspace $N \subset V_\alpha \subset V$ is also an affine normal slice through $z_\alpha$ to $X^{reg}_{\alpha,\alpha} \subset V^{reg}_\alpha$.

We can now use the normal slice $N$ to identify the element $\tilde{b}[z_\alpha]$ of equation (3.29). Pick a $\delta \in (0, \delta_1]$ which is sufficiently small for $N$, and define $c_{\alpha,2} \in C^{reg}$ and $z_2 \in N_{2\delta}$ as in equations (3.20)-(3.21). Let:

$$\tilde{c}_{\alpha,2} = f_\alpha(c_{\alpha,2}), \quad X_{\tilde{c}_{\alpha,2},\alpha} = f^{-1}_\alpha(\tilde{c}_{\alpha,2}).$$

Using the factorization (3.32), it is not hard to check that $z_2 \in X_{\tilde{c}_{\alpha,2},\alpha} \subset X_{\tilde{c}_{\alpha,2}}$. Pick a continuous path $\Gamma_2 : [0,1] \to X_{\tilde{c}_{\alpha,2},\alpha}$, with $\Gamma_2(0) = c_{\alpha,2}$ and $\Gamma_2(1) = z_2$, and consider the construction of the element $\tilde{b} = \tilde{b}[z_\alpha, N, \delta, \Gamma_2] \in \tilde{B}_W$ of equation (3.22). We see that $\tilde{b} \in \tilde{B}_W$ is represented by a closed loop in $V^{rs} \cap V_\alpha$. By Lemma 3.8 it follows that $\tilde{p}(\tilde{b}) \in \tilde{W}_\alpha$. By (3.29), this proves the containment (3.25).

4. Fourier transform of the nearby cycles

As in Section 2.7 (see also [GVX] Section 7.1), we use Proposition 2.9 to obtain a Morse local system $M(P_\chi)$ on $V^{rs}$, of the sheaf $P_\chi$ at the origin. For every $l \in (V^*)^{rs}$ we write:

$$M_l(P_\chi) = M(0,l)(P_\chi),$$

for the corresponding Morse group (cf. equation (2.48)).

**Proposition 4.1.** We have:

$$\mathfrak{F}P_\chi \cong IC((V^*)^{rs}, M(P_\chi)).$$

**Proof.** This is similar to the proof of [GVX] Proposition 7.1. By [KS] Proposition 3.7.12 (ii) (see also the proof of [GVX] Lemma 6.2), we have:

$$\mathfrak{F}P_\chi|_{V^{rs}} \cong M(P_\chi)[\dim V].$$

By [Gr1] Proposition 2.17, the general fiber $X_{\xi_0} \subset V$ of the family $Z_{\xi_0} \to \mathbb{C}$ of equation (2.29) is transverse to infinity in the sense of [Gr2]. The proposition follows from [Gr2] Theorem 1.1 & Remark 1.4. □

Let $l \in (V^*)^{rs}$, and let $C_l = (\mathfrak{g} \cdot l)^\perp \subset V$ be the Cartan subspace corresponding to $l$. Write $Z_l \subset X_{\xi_0}$ for the critical locus of the restriction $l|_{X_{\xi_0}}$. We then have:

$$Z_l = X_{\xi_0} \cap C_l, \quad \text{and therefore } |Z_l| = |W|.$$ 

Write $\xi_l = \text{Re}(l) : V \to \mathbb{R}$ and $d = \dim X_{\xi_0}$. We have the following analog of [GVX] Lemma 7.4.
Lemma 4.2. The Morse group $M_l(P_\chi)$ can be identified as follows:

$$M_l(P_\chi) \cong H_d(X_{\xi_0}, \{x \in X_{\xi_0} \mid \xi_l(x) \geq \xi_0\}; \mathcal{L}_\chi),$$

where $\xi_0$ is any real number with $\xi_0 > \xi_l(c)$ for every $c \in Z_l$.

Proof. This is a consequence of Proposition 2.9. The proof is similar to the proofs of [GVX, Lemmas 6.6, 6.9, 7.4]. □

Corollary 4.3. We have:

$$\dim M_l(P_\chi) = |W|. $$

Proof. This follows form Lemma 4.2, equation (4.1), assertion (3.3), and the generic property of $l$, as expressed by Proposition 2.9 (cf. [GVX, Lemma 6.10]). □

Recall from Section 2.6 that we identify the fundamental group $\pi_1^G((V^*)^{\text{rs}}, l_0)$ with $\bar{B}_W$ (see isomorphism (2.40)). Using this identification, we will write:

(4.2) $\lambda_{l_0} : \bar{B}_W \to \text{Aut}(M_{l_0}(P_\chi))$,

for the microlocal monodromy action arising from the structure of $P_\chi$ as a $G$-equivariant perverse sheaf, i.e., for the holonomy of the local system $M(P_\chi)$.

We will analyze the action $\lambda_{l_0}$ by means of Picard-Lefschetz theory, and we will use the same notation for Picard-Lefschetz classes in the Morse groups $\{M_l(P_\chi)\}$ as in [GVX]. More precisely, let $l \in (V^*)^{\text{rs}}$ and let $\xi_0 > 0$ be as in Lemma 4.2. Let $c \in Z_l$, and let $\gamma : [0, 1] \to \mathbb{C}$ be a smooth path such that:

(P1) $\gamma(0) = l(c)$;

(P2) $\gamma'(t) \neq 0$ for all $t \in [0, 1]$;

(P3) $\gamma(1) = \xi_0$;

(P4) $\gamma(t) \notin l(Z_l)$ for all $t \in (0, 1)$;

(P5) $\gamma(t_1) \neq \gamma(t_2)$ for all $t_1, t_2 \in [0, 1]$ with $t_1 \neq t_2$.

Recall the Hessian $\mathcal{H}(c, l)$ of $l|_{X_{\xi_0}}$ at $c$ (see equation (3.2)). Let:

(4.3) $T_+ [c, \gamma] \subset \mathfrak{g} \cdot c$,

be the positive eigenspace of $\gamma'(0)^{-1} \cdot \mathcal{H}(c, l)$, relative to the inner product $\langle \cdot, \cdot \rangle$, i.e., the direct sum of all the eigenspaces of the real part $\text{Re}(\gamma'(0)^{-1} \cdot \mathcal{H}(c, l))$, corresponding to positive eigenvalues. By assertion (3.3), we have:

$$\dim_\mathbb{R} T_+ [c, \gamma] = d.$$ 

Pick an orientation $o$ of $T_+ [c, \gamma]$ and an element $a \in (\mathcal{L}_\chi)_c$. The quadruple $[c, \gamma, o, a]$ defines a Picard-Lefschetz class:

(4.4) $\text{PL}[c, \gamma, o, a] \in M_l(P_\chi)$,
as in [GVX] Sections 6.2, 7.2.

We now focus on the Morse group \( M_{l_0}(P_X) \). Note that \( l_0(c_0) > 0 \), and we have:

\[
\xi_{l_0}(c) < l_0(c_0) \text{ for every } c \in Z_{l_0} - \{c_0\},
\]

(see equation (2.37)). Fix a \( \xi_0 > l_0(c_0) \), and let \( \gamma_0 : [0, 1] \to \mathbb{C} \) be the straight line path from \( l_0(c_0) \) to \( \xi_0 \). Pick an orientation \( o_0 \) of \( T_+[c_0, \gamma_0] \) and a generator \( 0 \neq a_0 \in (\mathcal{L}_\chi)_{c_0} \). We define:

\[
u = \Pi L[c_0, \gamma_0, o_0, a_0] \in M_{l_0}(P_X).
\]

**Remark 4.4.** Our choice of the basepoint \( l_0 \in (V^*)^{rs} \), which is coordinated with the choice of \( c_0 \in C^{reg} \subset V \) via (2.37), will give us a number of advantages. In particular, it facilitates the definition of the class \( \nu \) in equation (4.5), and this class will play a central role in our argument. However, we point out that the pair \( (c_0, l_0) \) is not generic in the following sense. The critical values \( l_0(Z_{l_0}) \subset \mathbb{C} \) will not, in general, be distinct. For example, if \( W = S_3 \) is the symmetric group on three letters, generated by simple reflections \((s_1, s_2)\), then we will always have \( l_0(s_1 s_2 c_0) = l_0(s_2 s_1 c_0) \).

**Proposition 4.5.** The vector \( \nu \in M_{l_0}(P_X) \) is cyclic for the microlocal monodromy action \( \lambda_{l_0} \) of equation (4.2). More precisely, we have:

\[
\lambda_{l_0}((\mathbb{C}[\tilde{B}_W]) \cdot \nu = M_{l_0}(P_X).
\]

**Proof.** In the case \( \chi = 1 \), the claim of the proposition is equivalent to the claim of [Gr3] Lemma 3.2. Moreover, the proof of [Gr3] Lemma 3.2 goes through without any changes in the presence of the local system \( \mathcal{L}_\chi \).

It is not difficult to describe the microlocal monodromy action (4.2) of the subgroup \( I \subset \tilde{B}_W \) on Picard-Lefschetz classes (see diagram (2.11)).

**Proposition 4.6.** Let \( u = \Pi L[c, \gamma, o, a] \in M_{l_0}(P_X) \) be a Picard-Lefschetz class, with \( c = w c_0 \in Z_{l_0}, w \in W \), and let \( x \in I \). We have:

\[
\lambda_{l_0}(x) \cdot u = (w \cdot \chi)(x) \cdot \tau(x) \cdot u,
\]

where \( \tau \) is the character of equation (2.33).

**Proof.** This is similar to the proofs of [GVX] Lemma 6.19 & equation (7.13)]. More precisely, let \( g \in Z_G(C) \) be an element representing \( x \in I \). Note that we have \( g c = c \) and \( gl_0 = l_0 \). It follows that \( g \) preserves the positive eigenspace \( T_+[c, \gamma] \subset g \cdot c = g \cdot C \). In view of the direct sum decomposition (1.2), the effect of \( g \) on the orientation of \( T_+[c, \gamma] \) is given by \( \tau(x) \). By chasing the definitions of the \( G \)-equivariant local system \( \mathcal{L}_\chi \) and of the \( W \)-action on the characters \( I \), one can check that \( g a = (w \cdot \chi)(x) \cdot a \). The proposition follows by the linearity of the Picard-Lefschetz class \( \Pi L[c, \gamma, o, a] \) in the last two arguments. \( \square \)
5. Construction of the Monodromy in the Family

Recall the groups $W_\chi = \text{Stab}_W(\chi)$ and $B^\chi_W = \text{Stab}_{B^W}(\chi)$ of equations (2.51) and (2.68). As in [GVX, Section 3.2], the sheaf $P_\chi$ naturally carries a representation:

$$\mu : B^\chi_W \to \text{Aut}(P_\chi),$$

which we call the monodromy in the family, and which is constructed as follows. Consider the projection $C/W_\chi \to C/W$, and form a Cartesian commutative diagram as follows:

$$\begin{array}{ccc}
\tilde{V}_\chi & \xrightarrow{g} & V \\
\downarrow f_x & & \downarrow f \\
C/W_\chi & \longrightarrow & C/W.
\end{array}$$

Let $\tilde{V}^{rs}_\chi = f_x^{-1}(C^{reg}/W_\chi) \subset \tilde{V}_\chi$. Let $\tilde{c}_0 \in C^{reg}/W_\chi$ be the image of $c_0 \in C^{reg}$, and let:

$$\tilde{c}_0 = (\hat{c}_0, c_0) \in \tilde{V}^{rs}_\chi.$$

Then the restriction $g : \tilde{V}^{rs}_\chi \to V^{rs}$ is a $G$-equivariant covering map, and we have:

$$\pi_1^G(\tilde{V}^{rs}_\chi, \tilde{c}_0) = \tilde{B}^\chi_W = \tilde{q}^{-1}(B^\chi_W) \subset \tilde{B}_W,$$

(see equation (2.68)).

Recall that the splitting homomorphism $\tilde{r}$ determines a canonical extension $\hat{\chi} : \tilde{B}^\chi_W \to \mathbb{G}_m$ of the character $\chi : I \to \mathbb{G}_m$ (see equation (2.69)). The character $\hat{\chi}$ gives rise to a rank one $G$-equivariant local system $\hat{L}_\chi$ on $\tilde{V}^{rs}_\chi$, with $(\hat{L}_\chi)_{c_0} = \mathbb{C}$. Note that $\hat{c}_0 = f_x(\tilde{c}_0) \in C^{reg}/W_\chi$, and let $X_{\hat{c}_0} = f_x^{-1}(\hat{c}_0)$. We have natural identifications:

$$X_{\hat{c}_0} \cong X_{c_0} \quad \text{and} \quad \hat{L}_\chi|_{X_{\hat{c}_0}} \cong L_\chi.$$

Next, we consider a parametrized version of the family $\mathcal{Z}_{c_0} \to \mathbb{C}$ of equation (2.29):

$$\mathcal{Z}_\chi = \{(\tilde{v}, \hat{c}, k) \in \tilde{V}_\chi \times (C^{reg}/W_\chi) \times \mathbb{C} \mid f_x(\tilde{v}) = k \hat{c}\} \xrightarrow{F} C^{reg}/W_\chi \times \mathbb{C}$$

where $F(\tilde{v}, \hat{c}, k) = (\hat{c}, k)$. Let $F_2 : \mathcal{Z}_\chi \to \mathbb{C}$ be the second component of $F$, and let $\mathcal{Z}^{rs}_\chi = F_2^{-1}(\mathbb{C}^*)$. By abuse of notation, we denote the pull-back of $\hat{L}_\chi$ from $\tilde{V}^{rs}_\chi$ to $\mathcal{Z}^{rs}_\chi$ by the same symbol $\hat{L}_\chi$. Let $\mathcal{Z}_{\chi,0} = F_2^{-1}(0) = X_0 \times (C^{reg}/W_\chi)$, and consider the nearby cycle sheaf:

$$\mathcal{P}_\chi = \psi_{F_2}(\hat{L}_\chi[-]) \in \text{Perv}_G(\mathcal{Z}_{\chi,0}).$$
The stratification $S_0$ of $X_0$, which was fixed in Section 2.6, following Proposition 2.9, induces a stratification $S_{\chi,0}$ of $Z_{\chi,0}$, given by:

$$Z_{\chi,0} = \bigcup_{S \in S_0} S \times (C^{reg}/W_\chi).$$

**Proposition 5.1.** For every $S \in S_0$, the pair of manifolds:

$$(Z^{rs}_S, S \times (C^{reg}/W_\chi)),$$

satisfies Thom’s $A_F^2$ condition.

**Proof.** This is similar to the proof of [GVX, Proposition 3.1], which can be readily adapted to the present setting. □

Let $\text{Perv}_G(Z_{\chi,0}, S_{\chi,0})$ denote the category of $G$-equivariant perverse sheaves on $Z_{\chi,0}$, constructible with respect to $S_{\chi,0}$. Proposition 5.1 and [Gi, Theorem 5.5] imply that:

$$P_\chi \in \text{Perv}_G(Z_{\chi,0}, S_{\chi,0}),$$

(this is similar to the proof of [GVX, Corollary 3.2]). For every $\hat{c} \in C^{reg}/W_\chi$, let $j_{\hat{c}} : X_0 \to Z_{\chi,0}$ be the inclusion $x \mapsto (x, \hat{c}, 0)$. In view of (5.5), this inclusion gives rise to a perverse (i.e., properly shifted) restriction functor:

$$j_{\hat{c}}^* : \text{Perv}_G(Z_{\chi,0}, S_{\chi,0}) \to \text{Perv}_G(X_0, S_0).$$

For each $\hat{c} \in C^{reg}/W_\chi$, we will write:

$$P_{\hat{c}} = j_{\hat{c}}^*(P) \in \text{Perv}_G(X_0, S_0).$$

In view of equation (5.4) and Proposition 5.1, we have:

$$P_{\alpha} \cong P_\chi.$$

The sheaves $\{P_{\hat{c}} \in \text{Perv}_G(X_0, S_0)\}$ form a local system over $C^{reg}/W_\chi \ni \hat{c}$, giving rise to the monodromy representation $\mu$ of equation (5.1). Note that the construction of $\mu$ depends on the choice of the regular splitting $\tilde{r} : B_W \to \tilde{B}_W$ in (2.28).

### 6. Monodromy in the Family for a Braid Generator

Fix an $\alpha \in A$ and a $\sigma_\alpha \in B_W[\alpha]$. In this section, we discuss the monodromy in the family operator $\mu(\sigma_\alpha^\alpha) \in \text{Aut}(P_\chi)$ (see equations (2.50) and (5.1)), and relate it to the corresponding construction for the rank one representation $G_\alpha|V_\alpha$.

Recall the splitting homomorphism $\tilde{r}[\sigma_\alpha] : B_{W_\alpha} \to \tilde{B}_{W_\alpha}$ of equation (2.44), for the rank one representation $G_\alpha|V_\alpha$. Recall that $\chi_\alpha = \chi|I_\alpha \in \tilde{I}_\alpha$, and $p_\alpha$, $q_\alpha$, $\tilde{p}_\alpha$, $\tilde{q}_\alpha$ are the maps of diagram (2.42). Let:

$$W_{\alpha,\chi_\alpha} := \text{Stab}_{W_\alpha}(\chi_\alpha), \quad B_{W_\alpha}^{\chi_\alpha} := p_\alpha^{-1}(W_{\alpha,\chi_\alpha}) \subset B_{W_\alpha}, \quad \tilde{B}_{W_\alpha}^{\chi_\alpha} := \tilde{q}_\alpha^{-1}(B_{W_\alpha}^{\chi_\alpha}) \subset \tilde{B}_{W_\alpha},$$
(cf. (2.51) and (2.68)). By analogy with (5.1), the data \((G_a|V_a, \chi_a, r[s\sigma_a])\) gives rise to a monodromy in the family representation:

\[
\mu[s\sigma_a] : B_{W_a}^{\chi_a} \to \text{Aut}(P_{\chi_a}).
\]

Here, we include \(\sigma_a\) in the notation, because the construction of \(\mu[s\sigma_a]\) uses the splitting homomorphism \(r[s\sigma_a]\). However, we have the following proposition. Recall the subgroup \(W_{a,\chi} = W_a \cap W_\chi \subset W_{a,\chi_a}\) (see equation (2.56) and Remark 2.20), and let:

\[
B_{W_a}^\chi := p_a^{-1}(W_{a,\chi}) \subset B_{W_a}^{\chi_a}, \quad \tilde{B}_{W_a}^\chi := \tilde{q}_a^{-1}(B_{W_a}^\chi) \subset \tilde{B}_{W_a}^{\chi_a}.
\]

**Proposition 6.1.** Write:

\[
\mu_a : B_{W_a}^\chi \to \text{Aut}(P_{\chi_a}),
\]

for the restriction of the homomorphism \(\mu[s\sigma_a]\) to the subgroup \(B_{W_a}^\chi \subset B_{W_a}^{\chi_a}\). The homomorphism \(\mu_a\) is independent of the choice of \(\sigma_a \in B_W[\alpha]\).

**Proof.** Consider the following Cartesian commutative diagram:

\[
\begin{array}{ccc}
\tilde{V}_{a,\chi} & \longrightarrow & \tilde{V}_{a,\chi_a} & \longrightarrow & V_a \\
\downarrow f_{a,\chi} & & \downarrow f_{a,\chi_a} & & \downarrow f_a \\
C/W_{a,\chi} & \longrightarrow & C/W_{a,\chi_a} & \longrightarrow & C/W_a.
\end{array}
\]

Here, the right square is the analog of diagram (5.2) for the data \((G_a|V_a, \chi_a)\), and the left square is obtained by further pulling back the family \(f_{a,\chi_a}\) to \(C/W_{a,\chi}\). Let:

\[
\tilde{V}_{a,\chi_a}^{rs} = f_{a,\chi_a}^{-1}((C - C_a)/W_{a,\chi_a}), \quad \tilde{V}_{a,\chi}^{rs} = (f_{a,\chi})^{-1}((C - C_a)/W_{a,\chi}).
\]

We have:

\[
\pi_1^G(\tilde{V}_{a,\chi_a}^{rs}, c_0) = \tilde{B}_{W_a}^{\chi_a}, \quad \pi_1^G(\tilde{V}_{a,\chi}^{rs}, c_0) = \tilde{B}_{W_a}^\chi \subset \tilde{B}_{W_a}^{\chi_a},
\]

where the basepoint \(c_0 \in C^{reg}\) naturally determines basepoints in \(\tilde{V}_{a,\chi_a}^{rs}\) and \(\tilde{V}_{a,\chi}^{rs}\), as in (5.3).

By analogy with the character \(\tilde{\chi}\) of equation (2.69), we obtain a character \(\tilde{\chi}_a : \tilde{B}_{W_a}^{\chi_a} \to \mathbb{G}_m\). More precisely, the character \(\tilde{\chi}_a\) is determined by requiring that \(\tilde{\chi}_a|I_a = \chi_a\) and:

\[
\tilde{\chi}_a \circ r[s\sigma_a] : B_{W_a}^{\chi_a} \to \mathbb{G}_m \quad \text{is the trivial character.}
\]

Recall the group \(\tilde{B}_{W_a}^\chi \supset \tilde{B}_{W_a}\) of equations (3.12)-(3.13) and diagram (3.14). Define:

\[
\tilde{B}_{W_a}^{\chi_a,f} := (q_a^f)^{-1}(B_{W_a}^{\chi_a}) \subset \tilde{B}_{W_a}^\chi, \quad \tilde{B}_{W_a}^{\chi,f} := (q_a^f)^{-1}(B_{W_a}^{\chi}) \subset \tilde{B}_{W_a}^{\chi_a,f}.
\]

The groups (6.3) and (6.5) can be organized into a diagram of inclusions as follows:

\[
\begin{array}{ccc}
\tilde{B}_{W_a}^{\chi} & \longrightarrow & \tilde{B}_{W_a}^{\chi_a} \\
\downarrow & & \downarrow \\
\tilde{B}_{W_a}^{\chi,f} & \longrightarrow & \tilde{B}_{W_a}^{\chi_a,f}.
\end{array}
\]
The character $\hat{\chi}_\alpha : \tilde{B}_{W_\alpha}^{\chi_\alpha} \to \mathbb{G}_m$ does not, in general, extend to a character of $\tilde{B}_{W_\alpha}^{\chi_\alpha} \supset \tilde{B}_{W_\alpha}^{\chi_\alpha}$. However, there is a unique character $\hat{\chi}_\alpha^f : \tilde{B}_{W_\alpha}^{\chi_\alpha} \to \mathbb{G}_m$, such that $\hat{\chi}_\alpha^f|_I = \chi$ and:

$$\hat{\chi}_\alpha^f \circ \tilde{r}[\sigma] : B_{W_\alpha}^{\chi} \to \mathbb{G}_m$$

is the trivial character.

By Lemma 3.5, condition 6.7 is independent of the choice of $\sigma_\alpha \in B_{W_\alpha}$. Therefore, the character $\hat{\chi}_\alpha$ is independent of $\sigma_\alpha$. By construction, the characters $\hat{\chi}_\alpha$ and $\hat{\chi}_\alpha^f$ match on the upper left corner of diagram (6.6), i.e., we have:

$$\hat{\chi}_\alpha|_{\tilde{B}_{W_\alpha}^{\chi}} = \hat{\chi}_\alpha^f|_{\tilde{B}_{W_\alpha}^{\chi}}.$$

Therefore, the LHS of (6.8) is independent of $\sigma_\alpha$. It remains to note that the monodromy action $\mu_\alpha$ can be defined using the family $f_{\alpha,\chi}$ and the character $\hat{\chi}_\alpha|_{\tilde{B}_{W_\alpha}^{\chi}}$, in place of the family $f_{\alpha,\chi_\alpha}$ and the full character $\hat{\chi}_\alpha$ (see diagram (6.2) and equation (6.3)).

The following proposition relates the monodromy actions $\mu$ and $\mu_\alpha$ of equations (5.1) and (6.1). Recall that $s_\alpha \in W_\alpha \cong \mathbb{Z}/n_\alpha$ and $\sigma \in B_{W_\alpha} \cong \mathbb{Z}$ are the counter-clockwise generators (see Sections 2.3, 2.7). Recall also that $W_{\alpha,\chi} = \langle s_\alpha^{e_\alpha} \rangle$ (see (2.56)) and therefore $B_{W_\alpha}^{\chi} = \langle \sigma^{e_\alpha} \rangle \subset B_{W_\alpha} = \langle \sigma \rangle$.

**Definition 6.2.** Let $\tilde{R}_{\chi,\alpha}^\mu \in \mathcal{R}$ be the minimal polynomial of $\mu_\alpha(\sigma^{e_\alpha}) \in \text{End}(P_{\chi_\alpha})$.

**Proposition 6.3.** The minimal polynomial of $\mu(\sigma^{e_\alpha}) \in \text{End}(P_{\chi})$ is equal to $\tilde{R}_{\chi,\alpha}^\mu$.

**Proof.** This is similar to the proofs of [GVX, Propositions 6.14 & 7.7] and [Gr1, Theorem 5.2]. We briefly outline the argument. Let $\sigma_\alpha = \sigma_\alpha[\Gamma]$, with $\Gamma(0) = c_0$ and $\Gamma(1) = c_{\alpha,1}$, as in equation (2.21). Without loss of generality, we can assume that $c_0 = c_{\alpha,1}$ and the path $\Gamma$ is trivial. Recall the point $c_\alpha \in C_\alpha^{reg}$ defined by equation (2.19). The first main idea of the proof is to decompose the process of specializing the regular value $c_0 \in C^{reg}/W_\chi$ of $f_\chi$ to $0 \in C/W_\chi$ into two steps as follows:

$$\tilde{c}_0 \rightarrow 0 \iff \tilde{c}_0 \rightarrow \tilde{c}_\alpha \rightarrow 0,$$

where $\tilde{c}_\alpha$ is the image of $c_\alpha$ in $C/W_\chi$.

Equation (6.9) is meant to be schematic. To make the idea of this equation precise, let $D_\gamma = \{z \in \mathbb{C} \mid |z| < 2\}$, and let $\gamma : D_\gamma \to C$ be the analytic arc defined by:

$$\gamma(z) = c_\alpha + z \cdot (c_0 - c_\alpha).$$

Next, let $D_\gamma = \{z \in \mathbb{C} \mid |z| < 2^{n_\alpha/e_\alpha}\}$, and let $\bar{\gamma} : D_\gamma \rightarrow C/W_\chi$ be the unique analytic arc such that:

$$f_\chi \circ \gamma(z) = \bar{\gamma}(z^{n_\alpha/e_\alpha}) \quad \text{for every } z \in D_\gamma.$$

We base change the family $f_\chi : \tilde{V}_\chi \to C/W_\chi$ to $D_\gamma$, to obtain a family:

$$f_\gamma : \tilde{V}_\chi \times_{C/W_\chi} \tilde{V}_\chi \to D_\gamma.$$
Let $D_\gamma^* = D_\gamma \cap C^*$, and note that $\gamma(D_\gamma^*) \subset C^{reg} / W_\chi$ (this is a consequence of choosing $c_0 = c_{\alpha,1}$ to be near $c_\alpha \in C_\alpha$). Let $\tilde{V}_\gamma^{rs} = f_\gamma^{-1}(D_\gamma^*)$, note that we have a projection $\tilde{V}_\gamma^{rs} \to V_\chi^{rs}$, and let $\hat{L}_\chi, \gamma$ be the pull-back of $\hat{L}_\chi$ from $\tilde{V}_\gamma^{rs}$ to $\tilde{V}_\gamma^{rs}$. Let:

$$\tilde{c}_\alpha = f(c_\alpha), \ X_{\tilde{c}_\alpha} = f^{-1}(\tilde{c}_\alpha) = f_\chi^{-1}(\tilde{c}_\alpha) = f_\gamma^{-1}(0),$$

(cf. equation (3.19)). Form the nearby cycle sheaf:

$$P_\gamma = \psi_\gamma(\hat{L}_{\chi, \gamma} [-]) \in \text{Perv}_G(X_{\tilde{c}_\alpha}),$$

which, as usual, we make perverse by an appropriate shift, and let $\mu_\gamma : P_\gamma \to P_\gamma$ be the associated (counter-clockwise) monodromy. The pair $(P_\gamma, \mu_\gamma)$ encapsulates the first step of the specialization process in the RHS of (6.9).

For the second step, consider the functor:

$$(6.10) \quad \psi_f[\tilde{c}_\alpha] : \text{Perv}_G(X_{\tilde{c}_\alpha}) \to \text{Perv}_G(X_0),$$

developed in the same manner as the functor $\psi_f[\tilde{c}_0]$ of equation (2.30), now using the family:

$$Z_{\tilde{c}_\alpha} = \{(x, k) \in V \times \mathbb{C} \mid f(x) = k \tilde{c}_\alpha \} \to \mathbb{C};$$

in place of the family (2.29). We claim that:

$$(6.11) \quad \psi_f[\tilde{c}_\alpha](P_\gamma) \cong P_\chi \quad \text{and} \quad \psi_f[\tilde{c}_\alpha](\mu_\gamma) = \mu(\sigma^\alpha_{\tilde{c}_\alpha}).$$

A proof of this claim is analogous to the proof of [GVX, Equation (6.4)] (see proof of [GVX, Proposition 6.14]).

Note that the functor $\psi_f[\tilde{c}_\alpha]$ of equation (6.10) is exact and faithful. Therefore, in view of (6.11), it suffices to show that the minimal polynomial of $\mu_\gamma \in \text{End}(P_\gamma)$ is equal to $\tilde{R}_{X, \alpha}^\mu$. The second main idea of the proof is to recall (see (3.27)) that $V_\alpha \subset V$ is a normal slice to the orbit $G \cdot c_\alpha$ through $c_\alpha$, and therefore, the minimal polynomial of $\mu_\gamma$ can be computed by focusing on the intersection $X_{\tilde{c}_\alpha} \cap V_\alpha$ near the point $c_\alpha$.

To make the above precise, we reuse some of the construction of the proof of Proposition 3.7. Recall the fiber $X_{\tilde{c}_\alpha, \alpha} = f_\alpha^{-1}(f_\alpha(c_\alpha))$ of equation (3.26), which is the connected component of $X_{\tilde{c}_\alpha} \cap V_\alpha$ containing $c_\alpha$. By analogy with (3.30), we have:

$$(6.12) \quad \text{for every } x \in X_{\tilde{c}_\alpha, \alpha}, \text{ the intersection } G \cdot x \cap V_\alpha \text{ is transverse at } x.$$ 

Write $j : X_{\tilde{c}_\alpha, \alpha} \to X_{\tilde{c}_\alpha}$ for the inclusion map. By (6.12), there is a well-defined perverse (i.e., properly shifted) restriction functor:

$$j^* : \text{Perv}_G(X_{\tilde{c}_\alpha}) \to \text{Perv}_{G, \alpha}(X_{\tilde{c}_\alpha, \alpha}).$$

Since $G \cdot c_\alpha$ is the unique closed $G$-orbit in $X_{\tilde{c}_\alpha}$, the functor $j^*$ is faithful.

Recall the zero-fiber $X_{0, \alpha} = f_\alpha^{-1}(0)$ and note that parallel translation by $c_\alpha$ takes $X_{0, \alpha}$ into $X_{\tilde{c}_\alpha, \alpha}$, establishing an equivalence of categories:

$$T_{\tilde{c}_\alpha}^* : \text{Perv}_{G, \alpha}(X_{\tilde{c}_\alpha, \alpha}) \cong \text{Perv}_{G, \alpha}(X_{0, \alpha}).$$
By chasing the definitions, and using the transversality assertion (6.12), one can check that:

\[ T^*_c \circ j^*(P_{\gamma}) \cong P_{\chi} \] and

\[ T^*_c \circ j^*(\mu_{\gamma}) = \mu_{\alpha}(\sigma^{e_{\alpha}}). \]

Since the composition \( T^*_c \circ j^* \) is faithful, the monodromy transformations \( \mu_{\gamma} \) and \( \mu_{\alpha}(\sigma^{e_{\alpha}}) \) have the same minimal polynomial.

By analogy with Proposition-Definition 2.12 (ii) and assertion (2.59), we have the following corollary of Proposition 6.3.

**Corollary 6.4.** For every \( \alpha_1, \alpha_2 \in A \) and \( w \in W_\chi \), with \( \alpha_2 = w \cdot \alpha_1 \), we have \( \bar{R}^\mu_{\chi,\alpha_1} = \bar{R}^\mu_{\chi,\alpha_2} \).

**Proof.** Pick a pair of braid generators \( \sigma_1 \in B_W[\alpha_1] \), \( \sigma_2 \in B_W[\alpha_2] \), and let \( e = c_{\alpha_1} = c_{\alpha_2} \) (see assertion (2.58)). By (2.23) and (2.24), the elements \( \sigma_1^e, \sigma_2^e \in B^\chi_W \) are conjugate to each other. The Corollary follows by Proposition 6.3.

Write:

\[(6.13) \quad \mu_{t_0} : B^\chi_W \to \text{Aut}(M_{t_0}(P_\chi)), \]

for the action induced by the monodromy in the family (5.1). Note that:

\[(6.14) \quad \text{the actions } \lambda_{t_0} \text{ and } \mu_{t_0} \text{ of equations (4.2) and (6.13) commute with each other.} \]

Let \( W/W_\chi \) be the set of left cosets, and write \( \bar{w} = w W_\chi \in W/W_\chi \) for \( w \in W \). For every \( \bar{w} \in W/W_\chi \), let:

\[(6.15) \quad M_{t_0}(P_\chi)[\bar{w}] \subset M_{t_0}(P_\chi), \]

be the linear span of all Picard-Lefschetz classes \( u = PL[c, \gamma, o, a] \in M_{t_0}(P_\chi) \), with \( c = w_1 c_0 \in Z_{t_0} \) and \( w_1 \in \bar{w} \).

**Proposition 6.5.** We have a direct sum decomposition:

\[(6.16) \quad M_{t_0}(P_\chi) = \bigoplus_{\bar{w} \in W/W_\chi} M_{t_0}(P_\chi)[\bar{w}], \]

which is invariant under the monodromy action \( \mu_{t_0} \). For every \( \bar{w} \in W/W_\chi \), we have:

\[ \dim M_{t_0}(P_\chi)[\bar{w}] = |W_\chi|. \]

**Proof.** This follows from Corollary 4.3, Proposition 4.6, and assertion (6.14). Decomposition (6.16) is just the decomposition by the characters of \( I \), under the microlocal monodromy action \( \lambda_{t_0} \) of equation (4.2).

**Proposition 6.6.** We have:

\[ \deg \bar{R}_{\chi,\alpha}^\mu \leq n_\alpha/e_{\alpha}. \]
Proof. Using Proposition 6.3, we interpret $\bar{\mu}^{\mu}_{\chi,\alpha}$ as the minimal polynomial of $\mu(\sigma_{\alpha}^{e_{\alpha}}) \in \text{End}(P_{\chi})$. Next, using Proposition 4.1, we interpret $\bar{\mu}^{\mu}_{\chi,\alpha}$ as the minimal polynomial of $\mu_{l_{0}}(\sigma_{\alpha}^{e_{\alpha}}) \in \text{End}(M_{l_{0}}(P_{\chi}))$. By Proposition 4.5 and assertion (6.14), it suffices to show that:

(6.17) \[ \dim \mathbb{C}[\mu_{l_{0}}(\sigma_{\alpha}^{e_{\alpha}})] \cdot u_{0} \leq n_{\alpha}/e_{\alpha}. \]

Let $\sigma_{\alpha} = \sigma_{\alpha}[\Gamma]$, with $\Gamma(0) = c_{0}$ and $\Gamma(1) = c_{\alpha,1}$, as in equation (2.21). By moving the point $c_{0}$ along the path $\Gamma$, while maintaining the relation $l_{0} = \nu(c_{0}) \in (C^{*})^{\text{reg}}$ (see equations (2.37)-(2.38)), we can reduce inequality (6.17) to the case where $c_{0} = c_{\alpha,1}$ and the path $\Gamma$ is trivial. Proceeding with this assumption, write:

(6.18) \[ Z_{\alpha} = W_{\alpha} \cdot c_{0} \subset Z_{l_{0}} = W \cdot c_{0}, \]

(cf. equation (4.1)). By the choice of the basepoints $c_{0} \in C^{\text{reg}}$ and $l_{0} \in (C^{*})^{\text{reg}}$, we have:

(6.19) \[ \xi_{l_{0}}(c_{1}) > \xi_{l_{0}}(c_{2}) \text{ for every } c_{1} \in Z_{\alpha} \text{ and } c_{2} \in Z_{l_{0}} - Z_{\alpha}, \]

where $\xi_{l_{0}} = \text{Re}(l_{0}) : V \to \mathbb{R}$, as in Section 4. Let:

(6.20) \[ M_{l_{0}}(P_{\chi})[\alpha] \subset M_{l_{0}}(P_{\chi}), \]

be the linear span of all Picard-Lefschetz classes $u = PL[c_{1}, \gamma, o, a]$, with $c_{1} \in Z_{\alpha}$ and the path $\gamma$ satisfying:

(6.21) \[ \xi_{l_{0}} \circ \gamma(t) > \xi_{l_{0}}(c_{2}) \text{ for every } t \in [0, 1] \text{ and } c_{2} \in Z_{l_{0}} - Z_{\alpha}. \]

Note that we have:

(6.22) \[ \dim M_{l_{0}}(P_{\chi})[\alpha] = |Z_{\alpha}| = |W_{\alpha}| = n_{\alpha}. \]

Inequality (6.19) admits a parametrized version, as the point $c_{0}$ traces out the path $\Gamma_{\alpha}[c_{\alpha,1}] : [0, 1] \to (C^{*})^{\text{reg}}$ of equation (2.20). Namely, for every $t \in [0, 1]$, let:

\[ Z_{l_{0},t} = W_{\alpha} \cdot \Gamma_{\alpha}[c_{\alpha,1}](t) \quad \text{and} \quad Z_{\alpha,t} = W_{\alpha} \cdot \Gamma_{\alpha}(c_{\alpha,1})(t). \]

Then we have:

(6.23) \[ \xi_{l_{0}}(c_{1}) > \xi_{l_{0}}(c_{2}) \text{ for every } c_{1} \in Z_{\alpha,t} \text{ and } c_{2} \in Z_{l_{0},t} - Z_{\alpha,t}. \]

By a standard Picard-Lefschetz theory analysis, it follows that:

(6.24) \[ \mu_{l_{0}}(\sigma_{\alpha}^{e_{\alpha}})(M_{l_{0}}(P_{\chi})[\alpha]) = M_{l_{0}}(P_{\chi})[\alpha]. \]

For each $i \in \{0, \ldots, e_{\alpha} - 1\}$, let $w_{i} = s_{\alpha}^{i} \in W_{\alpha}$, consider the coset $\overline{w_{i}} = w_{i} W_{\chi} \in W/W_{\chi}$, and form the intersection:

\[ M_{l_{0}}(P_{\chi})[\alpha, i] = M_{l_{0}}(P_{\chi})[\alpha] \cap M_{l_{0}}(P_{\chi})[\overline{w_{i}}]. \]

Note that $W_{\alpha} \cap \overline{w_{i}} = |W_{\alpha} \chi| = n_{\alpha}/e_{\alpha}$. It follows that:

(6.25) \[ \dim M_{l_{0}}(P_{\chi})[\alpha, i] \geq n_{\alpha}/e_{\alpha}. \]

Proposition 6.5, equations (6.22)-(6.23), and inequality (6.24) imply that:

\[ \dim M_{l_{0}}(P_{\chi})[\alpha, i] = n_{\alpha}/e_{\alpha} \quad \text{and} \quad \mu_{l_{0}}(\sigma_{\alpha}^{e_{\alpha}})(M_{l_{0}}(P_{\chi})[\alpha, i]) = M_{l_{0}}(P_{\chi})[\alpha, i], \]
for every $i \in \{0, \ldots, e_{\alpha} - 1\}$. Since we have $u_{0} \in M_{l_{0}}(P_{\chi})[\alpha, 0]$, inequality (6.17) follows. □

In the next section, we will prove the following sharpening of Proposition 6.6.

**Proposition 6.7.** We have:

$$\deg \bar{R}_{\chi, \alpha}^\mu = n_{\alpha}/e_{\alpha}.$$  

7. **Rank one representations and the simple carousel**

In this section, we prove Propositions 2.13, 2.18, 2.19, and 6.7. We also state and prove Proposition 7.1 relating the actions of the microlocal monodromy and the monodromy in the family for the rank one representation $G_{\alpha}|V_{\alpha}$, $\alpha \in A$.

Fix an $\alpha \in A$. Recall the microlocal monodromy action:

$$\lambda_{l_{0}, \alpha} : \tilde{B}_{W_{\alpha}} \to \text{Aut}(M_{l_{0}, \alpha}(P_{\chi_{\alpha}})),$$

of equation (2.49). Write:

$$\mu_{l_{0}, \alpha} : B_{W_{\alpha}}^{\chi} \to \text{Aut}(M_{l_{0}, \alpha}(P_{\chi_{\alpha}})),$$

for the action induced by the monodromy in the family $\mu_{\alpha}$ of Proposition 6.1. Also, recall that a braid generator $\sigma_{\alpha} \in B_{W}[\alpha]$ gives rise to a splitting homomorphism $\tilde{r}[\sigma_{\alpha}] : B_{W_{\alpha}} \to \tilde{B}_{W_{\alpha}}$, as in (2.44)-(2.45).

**Proposition 7.1.** There exists a sign $k_{\alpha} \in \{\pm 1\}$, such that for every $\sigma_{\alpha} \in B_{W}[\alpha]$, we have:

$$\mu_{l_{0}, \alpha}(\sigma^{e_{\alpha}}) = k_{\alpha} \cdot \lambda_{l_{0}, \alpha} \circ \tilde{r}[\sigma_{\alpha}](\sigma^{-e_{\alpha}}) \in \text{Aut}(M_{l_{0}, \alpha}(P_{\chi_{\alpha}})).$$

The appearance of the inverse in the RHS of (7.2) is related to Remark 2.11. The proofs of Propositions 2.13, 2.18, 2.19, 6.7, 7.1 use a variant of the carousel technique of Lê from singularity theory (see Remark 7.2). For all five propositions, we can assume, without loss of generality, that the basepoint $c_{0} \in C^{reg}$ is located near the hyperplane $C_{\alpha} \subset C$. More precisely, we pick points $c_{\alpha} \in C_{\alpha}^{reg}$ and $c_{\alpha, 1} \in C_{\alpha}^{reg}$ as in equation (2.19), and we assume that:

$$c_{0} = c_{\alpha, 1}.$$

Recall that we write $\tilde{c}_{0} = f_{\alpha}(c_{0}) \in Q_{\alpha}^{reg}$ and $X_{l_{0}, \alpha} = f_{-1}(\tilde{c}_{0})$ (see equation (2.41)). Let $Z_{\alpha} = W_{\alpha} \cdot c_{0}$ be the set of critical points of the restriction $l_{0, \alpha}|X_{l_{0, \alpha}}$ (as in equation 6.18). The critical values $l_{0, \alpha}(Z_{\alpha}) \subset \mathbb{C}$ appear in the complex plane as the vertices of a regular $n_{\alpha}$-gon, centered on $l_{0, \alpha}(c_{\alpha}) \in \mathbb{C}$ (see equation 2.19). The term “carousel”, in the present instance, refers to the collective movement of these critical values, as we vary the basepoints $\tilde{c}_{0} \in Q_{\alpha}^{reg}$ or $l_{0, \alpha} \in (V_{\alpha}^{*})^{rs}$.
Remark 7.2. The carousel technique was introduced in \[\text{Lê}\] and further developed in \[\text{T}i\]. See also \[\text{Mas}\] and its references. We point out one distinction between the arguments in \[\text{Lê}\] and in this subsection. In \[\text{Lê}\], the focus is on the homology of the Milnor fiber \(F_{f,0}\) of a polynomial \(f\) at the origin, relative to the intersection of \(F_{f,0}\) with a generic hyperplane passing through the origin. In this subsection, the focus is on the homology of a Milnor fiber at the origin, relative to the intersection with a generic hyperplane passing near, but not through, the origin. Under suitable conditions, these two relative homology groups are isomorphic to each other. However, monodromy in the family acts very differently on the two, and only the latter is functorial in the nearby cycles of \(f\). A special case of the argument of this section can be found in \[\text{BG\ Section 7}\].

Recall that we write \(d_\alpha = \dim X_{\tilde{c}_0,\alpha}\) (see (2.55)), and recall the rank one \(G_\alpha\)-equivariant local system \(\mathcal{L}_{\chi_\alpha}\) on \(X_{\tilde{c}_0,\alpha}\), corresponding to the character \(\chi_\alpha \in \hat{I}_\alpha\) (see Section 2.7). By Lemma 4.2, applied to the representation \(G_\alpha|_{V_\alpha}\), we have:

\[
M_{l_0,\alpha}(P_{\chi_\alpha}) \cong H_{d_\alpha}(X_{\tilde{c}_0,\alpha}; \{x \in X_{\tilde{c}_0,\alpha} \mid \xi_{l_0}(x) \geq \xi_0\}; \mathcal{L}_{\chi_\alpha}),
\]

where \(\xi_0\) is any real number with \(\xi_0 > l_{0,\alpha}(c_0) = \langle c_0, c_0 \rangle\) (see equation (2.37)). Pick a braid generator:

\[
\sigma_\alpha \in B_{W}[\alpha].
\]

The generator \(\sigma_\alpha\) need not be the obvious one, given by the path \(\Gamma_{\alpha}[c_{\alpha,1}]\) of equation (2.20). We will now use isomorphism (7.4) to construct a collection of Picard-Lefschetz classes:

\[
\{u_{j,\alpha}\}_{j \in J} \subset M_{l_0,\alpha}(P_{\chi_\alpha}) \quad \text{for} \quad J = \{0, \ldots, n_\alpha\}.
\]

The collection \(\{u_{j,\alpha}\}\) will depend on the braid generator \(\sigma_\alpha\), but only through the image \(r(\sigma_\alpha) \in \tilde{W}\).

To begin, recall the data \([c_0, \xi_0, \gamma_0, o_0, a_0]\) used to define the class \(u_0 \in M_{l_0}(P_\alpha)\) of equation (4.5). Note that we have \(\gamma_0'(0) > 0\) and \(T_{c_0}X_{\tilde{c}_0,\alpha} = g_\alpha \cdot c_0 \subset V_\alpha\). Define a real subspace:

\[
T_{+\alpha}[c_0, \gamma_0] \subset g_\alpha \cdot c_0,
\]

by analogy with (1.3), and note that it is just the positive eigenspace of the partial Hessian \(\mathcal{H}_\alpha[c_0, l_0]\) of equation (3.7). Pick an orientation \(o_{0,\alpha}\) of \(T_{+\alpha}[c_0, \gamma_0]\). We define:

\[
u_{0,\alpha} = PL[c_0, \gamma_0, o_{0,\alpha}, a_0] \in M_{l_0,\alpha}(P_{\chi_\alpha}),
\]

by analogy with equation (4.5).

Next, for \(j \in J - \{0\}\), define:

\[
c_j = s^j_\alpha c_0 \in Z_\alpha.
\]
Note that $c_{n_\alpha} = c_0$. For each $j \in J - \{0\}$, let $\gamma_j : [0, 1] \to \mathbb{C}$ be a smooth path with:

(Q1) $\gamma_j(0) = l_{0,\alpha}(c_j)$;
(Q2) $\gamma_j(1) = \xi_0$;
(Q3) $\gamma'_j(0) = \gamma_j(0) - l_{0,\alpha}(c_{\alpha}) = l_{0,\alpha}(c_j - c_{\alpha})$;
(Q4) $\text{Re}(\gamma'_j(t)/(\gamma_j(t) - l_{0,\alpha}(c_{\alpha}))) > 0$ for all $t \in (0, 1]$;
(Q5) $\text{Im}(\gamma'_j(t)/(\gamma_j(t) - l_{0,\alpha}(c_{\alpha}))) < 0$ for all $t \in (0, 1]$,
\[\text{i.e., } \gamma_j(t) \text{ moves clockwise around the center } l_{0,\alpha}(c_{\alpha}) \in \mathbb{C};\]
(Q6) $\int_0^1 \text{Im}(\gamma'_j(t)/(\gamma_j(t) - l_{0,\alpha}(c_{\alpha}))) \, dt = -2\pi j/n_\alpha$.

Figure 1 illustrates conditions (Q1)-(Q6) in the case $n_\alpha = 5$. The path $\tilde{\gamma}_{n_\alpha}$ (shown here as $\tilde{\gamma}_5$) will be introduced in the proof of Proposition 2.18. Note that conditions (Q1)-(Q6)
ensure that:

\[ (7.10) \]

the path \( \gamma_j \) satisfies conditions (P1)-(P5) of Section 4 for the representation

\[ G_\alpha \vert V_\alpha, \text{ the covector } l_{0, \alpha} \in (V_\alpha^*)^{rs}, \text{ and the critical point } c_j \in Z_\alpha. \]

Moreover, conditions (Q1)-(Q6) determine the path \( \gamma_j \) uniquely up to homotopy within the class of all smooth paths satisfying conditions (P1)-(P5).

For each \( j \in J \), we consider the tangent space:

\[ T_{c_j}X_{c_0, \alpha} = \mathfrak{g}_\alpha \cdot c_j = \mathfrak{g}_\alpha \cdot C, \]

the partial Hessian:

\[ \mathcal{H}_\alpha[c_j, l_0] \in Sym^2((\mathfrak{g}_\alpha \cdot C)^*), \]

(see equation (3.7)), and the positive eigenspace:

\[ (7.11) \]

\[ T_{+, \alpha}[c_j] = T_+[c_j, \gamma_j] \subset \mathfrak{g}_\alpha \cdot C, \]

of \( \gamma'_j(0)^{-1} \cdot \mathcal{H}_\alpha[c_j, l_0] \) (cf. equation (4.3)). Note that, for \( j = 0 \), the notation of (7.11) is consistent with the notation of (7.7).

**Lemma 7.3.**

(i) For every \( j \in J - \{0\} \), we have:

\[ \mathcal{H}_\alpha[c_j, l_0] = \exp(-2\pi i j/n_\alpha) \cdot \mathcal{H}_\alpha[c_0, l_0] \quad \text{and} \quad T_{+, \alpha}[c_j] = \exp(2\pi i j/n_\alpha) \cdot T_{+, \alpha}[c_0]. \]

(ii) For every \( j \in J \) and every \( g \in N_K(C) \), representing the element \( s_\alpha' \in W \), we have:

\[ g \cdot T_{+, \alpha}[c_0] = T_{+, \alpha}[c_j]. \]

**Proof.** Let \( C^\perp_\alpha \subset C \) be the orthogonal complement to \( C_\alpha \) with respect to \( \langle \ , \, \rangle \); it is the eigenspace of \( s_\alpha \) with eigenvalue \( 2\pi i/n_\alpha \). Let:

\[ \widetilde{V}_\alpha = C^\perp_\alpha \oplus \mathfrak{g}_\alpha \cdot C \subset V_\alpha, \]

and let:

\[ \bar{f}_\alpha : V_\alpha \to \widetilde{V}_\alpha / G_\alpha = C^\perp_\alpha / W_\alpha \cong \mathbb{C}, \]

be the quotient map. Part (i) follows from the fact that the map \( \bar{f}_\alpha \) is given by a homogenous polynomial of degree \( n_\alpha \). Part (ii) follows from the fact that \( gl = \exp(-2\pi i j/n_\alpha) \cdot l \), while the space \( X_{c_0, \alpha} \) and the inner product \( \langle \ , \, \rangle \) are preserved by the action of \( g \).

For every \( j \in J - \{0\} \), let:

\[ (7.12) \]

\[ o_{j, \alpha} = (\exp(2\pi i j/n_\alpha)), o_{0, \alpha}, \]

be the orientation of \( T_{+, \alpha}[c_j] \) obtained as the push-forward of \( o_{0, \alpha} \) via the scalar multiplication by \( \exp(2\pi i j/n_\alpha) \in \mathbb{C} \); see Lemma 7.3 (i). Note that we have \( o_{n_\alpha, \alpha} = o_{0, \alpha} \).

Note that the \( G_\alpha \)-equivariant structure on \( \mathcal{L}_{x_\alpha} \) gives rise to a \( \widetilde{W}_\alpha \)-equivariant structure on the restriction of \( \mathcal{L}_{x_\alpha} \) to the critical set \( Z_\alpha = W_\alpha \cdot c_0 \). Thus, for every \( \bar{w} \in \widetilde{W}_\alpha \) and \( c_j \in Z_\alpha \), we obtain an action map \( (\mathcal{L}_{x_\alpha})_{c_j} \to (\mathcal{L}_{x_\alpha})_{\bar{w}c_j} \), which we denote by \( a \mapsto \bar{w} \cdot a \).
In view of diagram (2.42) and equations (2.43)-(2.45), for each \( j \in J - \{0\} \), we have:

\[
(7.13) \quad r(\sigma^{-j}) c_0 = p_\alpha(\sigma^{-j}) c_0 = s^j_\alpha c_0 = c_j.
\]

Using (7.13), for each \( j \in J - \{0\} \), we define:

\[
(7.14) \quad a_j = r(\sigma^{-j}) \cdot a_0 \in (L_{\chi_\alpha})_{c_j},
\]

\[
(7.15) \quad u_{j,\alpha} = PL[c_j, \gamma_j, \sigma_{j,\alpha}, a_j] \in M_{\theta_\alpha}(P_{\chi_\alpha}),
\]

where the notation of (4.4) is applied to the representation \( G_{\alpha} | V_\alpha \). This completes the construction of the classes (7.6).

**Remark 7.4.** As noted following equation (7.6), the collection \( \{u_{j,\alpha}\}_{j \in J} \) depends on the choice of the braid generator \( \sigma_{\alpha} \in B_W[\alpha] \) in (7.5). However, one can use Lemma 3.5 and equation (6.8) to show that the class \( u_{j,\alpha} \) is independent of \( \sigma_{\alpha} \) for every \( j \in J \) which is divisible by \( e_\alpha \).

By Lemma (7.3) for every \( j \in J \) and every \( g \in N_K(C) \), representing \( s^j_\alpha \in W_\alpha \), we have an \( \mathbb{R} \)-linear self-map:

\[
(7.16) \quad \exp(-2\pi i j/n_\alpha) \circ g : T_{+,\alpha}[c_0] \to T_{+,\alpha}[c_0].
\]

The map (7.16) preserves the inner product \( \langle , \rangle \), and therefore has determinant \( \pm 1 \). We write \( Sgn_\alpha(g) \in \{\pm 1\} \) for this determinant, or equivalently, the effect of the map (7.16) on the orientation of \( T_{+,\alpha}[c_0] \). Using Proposition 2.10 (iii), it is not hard to check that the assignment \( g \mapsto Sgn_\alpha(g) \) descends to a character:

\[
(7.17) \quad Sgn_\alpha : \tilde{W}_\alpha^f \to \{\pm 1\},
\]

of the subgroup \( \tilde{W}_\alpha^f = q^{-1}(W_\alpha) \subset \tilde{W} \) of equations (3.12)-(3.13) and diagram (3.14).

The character \( Sgn_\alpha \) of equation (7.17) can be described more concretely. Recall that the group \( G_\alpha^f = Z_G(C_\alpha) \) acts on the subspace \( V_\alpha \subset V \) (see assertion (3.11)). Write:

\[
\det_\alpha : G_\alpha^f \to \mathbb{G}_m,
\]

for the determinant of this action (cf. (3.9)). By Proposition 2.14 applied to the representation \( G_\alpha | V_\alpha \), the character \( \det_\alpha \) descends to a character:

\[
\hat{\tau}_\alpha : \tilde{W}_\alpha^f \to \mathbb{G}_m,
\]

satisfying \( \hat{\tau}_\alpha | I_\alpha = \tau_\alpha \). Let \( \zeta_\alpha : W_\alpha \to \mathbb{G}_m \) be the character defined by:

\[
\zeta_\alpha(s_\alpha) = \exp(2\pi i/n_\alpha),
\]

and let:

\[
\tilde{\zeta}_\alpha = \zeta_\alpha \circ q_\alpha^f : \tilde{W}_\alpha^f \to \mathbb{G}_m,
\]

(see diagram (3.14)). By reviewing the definition of the character \( Sgn_\alpha \), and observing that \( T_{+,\alpha}[c_0] \subset g_\alpha \cdot C \) is a real form, one can see that:

\[
(7.18) \quad Sgn_\alpha = \hat{\tau}_\alpha \cdot \tilde{\zeta}_\alpha^{-d_\alpha - 1} : \tilde{W}_\alpha^f \to \mathbb{G}_m,
\]
where \( d_\alpha = \dim X_{\alpha,0} = \dim g_\alpha \cdot C \). Note, however, that the description (7.18) does not make it clear that \( Sgn_\alpha \) takes values in \( \{ \pm 1 \} \). By restricting (7.18) to \( I_\alpha \subset \tilde{W}_\alpha^f \), we obtain:

(7.19)

\[
Sgn_\alpha|_{I_\alpha} = \tau_\alpha.
\]

**Remark 7.5.** By an argument similar to the proof of Proposition 2.14, we have \( \hat{\tau}_\alpha(x) \in \{ \pm 1 \} \) for every \( x \in I \). By Theorem 3.1 (iii), we further have:

\[
\tau(x) = \prod_{\alpha \in A} \hat{\tau}_\alpha(x) \text{ for every } x \in I.
\]

By Lemma 3.5 we have:

(7.20)

\[
\forall \sigma_1, \sigma_2 \in B_W[\alpha] : Sgn_\alpha(r(\sigma_1)) = Sgn_\alpha(r(\sigma_2)).
\]

The following lemma encapsulates the carousel technique and forms the basis of all the main arguments in this section. Let \( J^0 = J - \{ n_\alpha \} \).

**Lemma 7.6.**

(i) The elements \( \{ u_{j,\alpha} \}_{j \in J^0} \) form a basis of \( M_{l_0,\alpha}(P_{x_\alpha}) \).

(ii) We have:

\[
\mu_{l_0,\alpha}(\sigma^{e_\alpha}) u_{0,\alpha} = u_{e_\alpha,\alpha}.
\]

(iii) For every \( j \in J^0 \), we have:

\[
\lambda_{l_0,\alpha} \circ \tilde{r}[\sigma_\alpha] (\sigma^{-1}) u_{j,\alpha} = Sgn_\alpha(r(\sigma_\alpha)) \cdot u_{j+1,\alpha}.
\]

**Proof.** Part (i) follows form the fact that the set \( \{ u_{j,\alpha} \}_{j \in J^0} \) contains exactly one Picard-Lefschetz class for every critical point in \( Z_\alpha \), and the paths \( \{ \gamma_j \}_{j \in J^0} \) are mutually disjoint; see Figure 1.

Part (ii) is a standard carousel argument (cf. proof of [BG, Lemma 7.7 (ii)]). For each \( \tau \in [0, 1] \), we can emulate equation (2.20) to define:

(7.21)

\[
c[\tau] = c_\alpha + \exp(2\pi i \tau e_\alpha/n_\alpha)(c_0 - c_\alpha) \in C^{reg},
\]

(see assumption (7.3)). Note that we have \( c[0] = c_0 \), \( c[1] = c_{e_\alpha} \), and the path:

\[
\tau \mapsto f_\alpha(c[\tau]) \in Q^{reg},
\]

represents the element \( \sigma^{e_\alpha} \in B_{W_\alpha} \). Let \( X_{\tau,\alpha} = f_\alpha^{-1}(f_\alpha(c[\tau])) \), and let \( Z_{\tau,\alpha} \subset X_{\tau,\alpha} \) be the critical locus of the restriction \( l_{0,\alpha}/X_{\tau,\alpha} \). We then have \( Z_{\tau,\alpha} = W_\alpha \cdot c[\tau] \). We can extend the path \( \gamma_0 : [0, 1] \to C \) to a smoothly varying family of smooth paths:

\[
\{ \gamma[\tau] : [0, 1] \to C \}_{\tau \in [0, 1]},
\]

such that \( \gamma[0] = \gamma_0 \), \( \gamma[1] = \gamma_{e_\alpha} \), and each \( \gamma[\tau] \) satisfies conditions (Q1)-(Q5), where we replace \( c_j \) by \( c[\tau] \) and \( \gamma_j \) by \( \gamma[\tau] \). It follows that:

\[
\mu_{l_0,\alpha}(\sigma^{e_\alpha}) u_0 = PL[c_{e_\alpha}, \gamma_{e_\alpha}, o, a],
\]
for a certain orientation \( o \) of \( T_{+,a}[c_{e_a}] \) and a certain \( a \in (\mathcal{L}_{\chi_a})_{c_{e_a}} \), which we now proceed to identify. 

To identify the orientation \( o \), we consider the family of positive eigenspaces:

\[
T_{+,a}[\tau] = T_{+,a}[c[\tau], \gamma[\tau]] \subset g_{a} \cdot c[\tau] = g_{a} \cdot C, \quad \tau \in [0, 1],
\]

defined by analogy with the subspaces \( T_{+,a}[c_j] \) of equation (7.11). The orientation \( o \) is then the parallel translate of the orientation \( o_{0,a} \) of \( T_{+,a}[0] = T_{+,a}[c_0] \) (see equation (7.8)) to an orientation of \( T_{+,a}[1] = T_{+,a}[c_{e_a}] \), in the family (7.22). By analogy with Lemma 7.3 (i), we have:

\[
T_{+,a}[\tau] = \exp(2\pi i \tau e_{a}/n_{a}) \cdot T_{+,a}[0],
\]

and therefore, we have \( o = o_{e_a,a} \) (see equation (7.12)).

To identify the element \( a \in (\mathcal{L}_{\chi_a})_{c_{e_a}} \), we refer to diagram (6.2) from the proof of Proposition 6.1. As in that proof, we can use the family \( f_{a,\chi} \) to define the monodromy action \( \mu_{a,\chi} \). We write \( \hat{c}_{0,a} \in C/W_{a,\chi} \) for the image of \( c_0 \). By analogy with (5.4), we identify the fibers:

\[
(f_{a,\chi})^{-1}(\hat{c}_{0,a}) \cong X_{\hat{c}_{0,a}},
\]

and we use (7.23) to regard \( X_{\hat{c}_{0,a}} \) as a subset of \( V_{a,\chi}^{rs} \). The path:

\[
\tau \mapsto c[\tau], \quad [0, 1] \to C^{reg} \subset V_{a,\chi}^{rs},
\]

determines a path:

\[
\tau \mapsto \hat{c}[\tau], \quad [0, 1] \to \hat{V}_{a,\chi}^{rs},
\]

in the obvious way. Note that \( \hat{c}[0] = c_0 \) and \( \hat{c}[1] = c_{e_a} \). Recall that the choice of the braid generator \( \sigma_{\alpha} \in B_{W_{a}}[\alpha] \) in (7.5) defines an extension \( \hat{\chi}_{a} : \tilde{B}_{W_{a}}^{\chi_{a}} \to G_{m} \) of the character \( \chi_{a} \), as in (6.4). Recall also that the restriction \( \hat{\chi}_{a}\big|_{\tilde{B}_{W_{a}}^{\chi_{a}}} \) is independent of the choice of \( \sigma_{\alpha} \), as noted following (6.8). The character \( \hat{\chi}_{a}\big|_{\tilde{B}_{W_{a}}^{\chi_{a}}} \) gives rise to a rank one \( G \)-equivariant local system \( \hat{\mathcal{L}}_{a,\chi} \) on \( \hat{V}_{a,\chi}^{rs} \), with:

\[
\hat{\mathcal{L}}_{a,\chi}\big|_{\hat{X}_{\hat{c}_{0,a}}} \cong \mathcal{L}_{\chi_{a}},
\]

as in (5.4). The element \( a \in (\mathcal{L}_{\chi_{a}})_{c_{e_a}} \cong (\hat{\mathcal{L}}_{a,\chi})_{c_{e_a}} \) is obtained by parallel transporting \( a_0 \in (\mathcal{L}_{\chi_{a}})_{c_0} \cong (\hat{\mathcal{L}}_{a,\chi})_{c_0} \) along the path (7.24), where the isomorphisms of stalks are given by (7.25).

To compute the element \( a \in (\mathcal{L}_{\chi_{a}})_{c_{e_a}} \), consider the element:

\[
\bar{b} := \bar{r}[\sigma_{\alpha}] (\sigma_{e_a}^{e_a}) \in \tilde{B}_{W_{a}}^{\chi_{a}} \subset \tilde{B}_{W_{a}},
\]

and let \( h_{b} : (\mathcal{L}_{\chi_{a}})_{c_0} \to (\mathcal{L}_{\chi_{a}})_{c_0} \) be the holonomy of the local system \( \hat{\mathcal{L}}_{a,\chi} \) along \( \bar{b} \). By condition (6.4), we have \( h_{\bar{b}}(a_0) = a_0 \). On the other hand, if we represent \( \bar{b} \) by a path \( \Gamma : [0, 1] \to V_{a}^{rs} \), as in equation (2.15), we can compute:

\[
h_{\bar{b}}(a_0) = r(\sigma_{e_a}^{e_a}) \cdot a.
\]
It follows that $a = a_{e_{\alpha}}$, as defined in (7.14). This completes the proof of part (ii).

For part (iii), we apply the construction of equation (2.15) to the element:

$$\tilde{r}[\sigma_{\alpha}] (\sigma^{-1}) \in \tilde{B}_{W_{\alpha}},$$

(cf. proofs of [GVX Proposition 6.12 (i) & Lemma 7.5]). Namely, recall the compact form $K_{\alpha} \subset G_{\alpha}$ of equation (2.46), and use Proposition 2.10 (iii) to pick a representative $g_{1} \in N_{K_{\alpha}}(C)$ of $r(\sigma_{\alpha}^{-1}) \in \tilde{W}_{\alpha}$. Choose a path $g : [0, 1] \to K_{\alpha}$, with $g(0) = 1$ and $g(1) = g_{1}$. By analogy with equation (2.20) and using assumption (7.3) (see also equation (7.21)), define a path $\Gamma_{1} : [0, 1] \to C^{reg}$ by:

$$\Gamma_{1}(t) = c_{\alpha} + \exp(-2\pi i t n_{\alpha})(c_{0} - c_{\alpha}).$$

Note that $\Gamma_{1}(0) = c_{0}$ and $\Gamma_{1}(1) = c_{n_{\alpha}^{-1}}$. Next, define a path $\Gamma_{2} : [0, 1] \to X_{(c_{0}, \alpha)}$ by:

$$\Gamma_{2}(t) = g(t) c_{n_{\alpha}^{-1}}.$$  

Note that $\Gamma_{2}(0) = c_{n_{\alpha}^{-1}}$ and $\Gamma_{2}(1) = c_{0}$. As in equation (2.15), define a path $\Gamma : [0, 1] \to V_{\alpha}^{rs}$ as a composition:

$$\Gamma = \Gamma_{2} \bullet \Gamma_{1}.$$  

The path $\Gamma$ represents the element $\tilde{r}[\sigma_{\alpha}] (\sigma^{-1}) \in \tilde{B}_{W_{\alpha}}$.

Recall the identification of fundamental groups (2.34), which is analogous to (2.40). Write:

$$V_{\alpha}^{mrs} = K_{\alpha} \cdot (C - C_{\alpha}) \subset V_{\alpha}^{rs}$$

and

$$(V_{\alpha}^{rs})^{mrs} = K_{\alpha} \cdot (C^{*} - C_{\alpha}^{*}) \subset (V_{\alpha}^{rs})^{rs},$$

for the minimal semisimple loci (cf. equation (2.34)). Also, write:

$$\nu_{\alpha} : V_{\alpha} \to V_{\alpha}^{rs}, \quad \nu_{\alpha}^{mrs} : V_{\alpha}^{mrs} \to (V_{\alpha}^{rs})^{mrs},$$

for the analogs of the maps $\nu, \nu^{mrs}$ of equations (2.36), (2.39). Note that we have $\Gamma([0, 1]) \subset V_{\alpha}^{mrs}$. Therefore, we can use holonomy along the path:

$$\Gamma^{*} = \nu_{\alpha}^{mrs} \circ \Gamma : [0, 1] \to (V_{\alpha}^{rs})^{mrs},$$

to define the operator $\lambda_{0, \alpha} \circ \tilde{r}[\sigma_{\alpha}] (\sigma^{-1}) \in \text{Aut}(M_{0, \alpha}(P_{\chi_{\alpha}}))$. Let:

$$\Gamma_{1}^{*} = \nu_{\alpha}^{mrs} \circ \Gamma_{1}, \quad \Gamma_{2}^{*} = \nu_{\alpha}^{mrs} \circ \Gamma_{2},$$

so that $\Gamma^{*} = \Gamma_{2}^{*} \bullet \Gamma_{1}^{*}$.

Note that we have $l_{0, \alpha} = \nu_{\alpha}^{mrs}(c_{0})$, and define $l_{\alpha, \alpha} = \nu_{\alpha}(c_{\alpha})$, $l_{1, \alpha} = \nu_{\alpha}^{mrs}(c_{n_{\alpha}^{-1}})$. By equation (2.19) and assumption (7.3), we have:

$$l_{\alpha, \alpha} |_{X_{(c_{0}, \alpha)}} \equiv (c_{\alpha}, c_{\alpha}).$$

By equation (7.26) for $t = 1$ and the antilinear property of the map $\nu_{\alpha}$, we have:

$$l_{1, \alpha} = l_{\alpha, \alpha} + \exp(2\pi i t/n_{\alpha})(l_{0, \alpha} - l_{\alpha, \alpha}).$$

Equation (7.27) enables us to write:

$$\lambda_{0, \alpha} \circ \tilde{r}[\sigma_{\alpha}] (\sigma^{-1}) = \lambda_{2} \circ \lambda_{1},$$

where $\lambda_{1} : M_{0, \alpha}(P_{\chi_{\alpha}}) \to M_{1, \alpha}(P_{\chi_{\alpha}})$ and $\lambda_{2} : M_{1, \alpha}(P_{\chi_{\alpha}}) \to M_{0, \alpha}(P_{\chi_{\alpha}})$ are the holonomy operators for the local system $M(P_{\chi_{\alpha}})$, corresponding to the paths $\Gamma_{1}^{*}$ and $\Gamma_{2}^{*}$. 
Consider the case $j = 0$ of the claim of part (iii). By an argument similar to the proof of part (ii), one can verify that:

$$\lambda_1(u_{0,\alpha}) = PL[c_0, \gamma_1, o_{0,\alpha}, a_0] \in M_{l_{1,\alpha}}(P_{0,\alpha}),$$

(cf. equation (7.14)). To see that the Picard-Lefschetz parameters $[c_0, \gamma_1, o_{0,\alpha}, a_0]$ specify an element of $M_{l_{1,\alpha}}(P_{0,\alpha})$, note that:

$$l_{1,\alpha}(c_0) = l_{0,\alpha}(c_1) = \gamma_1(0).$$

Further, let $l_1 = \nu(c_{n_{\alpha} - 1}) \in V^*$, so that $l_{1,\alpha} = l_1|_{V_{\alpha}}$. By equations (7.28)-(7.29), we have:

$$\mathcal{K}_\alpha[c_0, l_1] = \exp(2\pi i l/n_\alpha) \cdot \mathcal{K}_\alpha[c_0, l_0].$$

It follows that the positive eigenspace of $\gamma_1'(0)^{-1}\mathcal{K}_\alpha[c_0, l_1]$ is equal to the subspace $T_{+_{\alpha}}[c_0] = T_{+_{\alpha}}[c_0, \gamma_0] \subset g_\alpha \cdot C$ of equations (7.7) and (7.11), as required.

For the second step of equation (7.30), we observe that:

$$\lambda_2(PL[c_0, \gamma_1, o_{0,\alpha}, a_0]) = g_1 PL[c_0, \gamma_1, o_{0,\alpha}, a_0] = PL[c_1, \gamma_1, (g_1)_*, o_{0,\alpha}, a_1],$$

(cf. equation (7.14)). It remains to note that, by equation (7.12) and the definition of the character $Sgn_\alpha$ of equation (7.17), we have:

$$(g_1)_*, o_{0,\alpha} = Sgn_\alpha(r(\sigma^{-1})) \cdot o_{1,\alpha}.$$ 

Since $Sgn_\alpha(r(\sigma^{-1})) = Sgn_\alpha(r(\sigma_\alpha))$, we obtain:

$$\lambda(l_{0,\alpha}) \circ \tilde{r}[\sigma_\alpha] (\sigma^{-1}) u_0 = Sgn_\alpha(r(\sigma_\alpha)) \cdot u_1,$$

as required.

The case $j > 0$ is completely analogous to the case $j = 0$. We omit the rest of the details. □

**Proof of Proposition 2.13** We use the braid generator $\sigma_\alpha \in B_W[\alpha]$, which was fixed in (7.5) and utilized in the construction of the Picard-Lefschetz classes (7.6), to define the minimal polynomial $R_{\chi_\alpha}$. Parts (i) and (iii) of Lemma 7.6 imply that:

(7.31) $u_{0,\alpha}$ is a cyclic vector for the action $\lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_\alpha] : B_{W_\alpha} \to \text{Aut}(M_{l_{0,\alpha}}(P_{0,\alpha})).$

The proposition follows from Lemma 7.6 (i) and assertion (7.31). □

**Remark 7.7.** Note that assertion (7.31) implies the claim of Proposition 4.5 for the representation $G_\alpha|V_\alpha$.

**Proof of Proposition 7.1.** Note that the group $B_{W_\alpha} \cong \mathbb{Z}$ is abelian, and the action:

$$\lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_\alpha] : B_{W_\alpha} \to \text{Aut}(M_{l_{0,\alpha}}(P_{0,\alpha}));$$

commutes with the action $\mu_{l_{0,\alpha}}$ of equation (7.1), as in (6.14). Therefore, in view of assertion (7.31), it suffices to check that:

(7.32) $\mu_{l_{0,\alpha}}(\sigma^{e_\alpha}) u_{0,\alpha} = k_\alpha \cdot \lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_\alpha](\sigma^{-e_\alpha}) u_{0,\alpha},$
for some \( k_\alpha \in \{ \pm 1 \} \), which is independent of \( \sigma_\alpha \in B_W[\alpha] \). By Lemma 7.6 (ii)-(iii), equation (7.32) holds with:
\[
(7.33) \quad k_\alpha = \text{Sgn}_\alpha (r(\sigma_\alpha))^{e_\alpha}.
\]
The sign \( k_\alpha \) is independent of \( \sigma_\alpha \) by assertion (7.20). \( \square \)

**Remark 7.8.** Equations (7.33) and (7.18) imply the following explicit formula for \( k_\alpha \):
\[
(7.34) \quad k_\alpha = \tau_\alpha (r(\sigma_\alpha^{n_\alpha})), \quad \sigma_\alpha \in B_W[\alpha], \quad \alpha \in A^1_\chi.
\]
Note that, for \( \alpha \in A^1_\chi \) (where \( e_\alpha = n_\alpha \)), equation (7.34) simplifies as follows:
\[
(7.35) \quad k_\alpha = \tau_\alpha (r(\sigma_\alpha^{n_\alpha})), \quad \sigma_\alpha \in B_W[\alpha], \quad \alpha \in A^1_\chi.
\]
Equation (7.34) plus assertion (2.24) imply that the assignment \( \alpha \mapsto k_\alpha \) is invariant under the action of \( W_\chi \) on \( A \). Also by (7.34), the sign \( k_\alpha \) depends on the character \( \chi \in \hat{I} \) only through the integer \( e_\alpha \). Finally, we note that, in some examples, such as Example 2.15 with \( \chi = 1 \), the sign \( k_\alpha \) will depend on the choice of the regular splitting \( \tilde{r} \) in (2.28).

For the proof of Propositions 2.19 and 6.7, recall the space \( \mathcal{R} \) of all possible minimal polynomials introduced in (2.50). This space is equipped with a natural involution:
\[
\vartheta : \mathcal{R} \to \mathcal{R}, \quad \text{defined by} \quad \vartheta R(z) = z^{\deg R} \cdot R(z^{-1})/R(0).
\]
The significance of the involution \( \vartheta \) is that, if \( A \) is an associative \( \mathbb{C} \)-algebra with unit, \( a \in A \) is an invertible element, and \( R \in \mathcal{R} \) is the minimal polynomial of \( a \), then \( \vartheta R \in \mathcal{R} \) is the minimal polynomial of \( a^{-1} \in A \). Note that we have:
\[
(7.36) \quad \deg(\vartheta R) = \deg(R), \quad R \in \mathcal{R}.
\]

**Proof of Propositions 2.19 and 6.7** We use the braid generator \( \sigma_\alpha \in B_W[\alpha] \), which was fixed in (7.5), to define the minimal polynomials \( R_{\chi, \alpha} \) and \( \bar{R}^\mu_{\chi, \alpha} \). Define \( R_{\chi, \alpha} \in \mathcal{R} \) as the minimal polynomial of the holonomy operator \( \lambda_{\alpha_0, \alpha} \circ \tilde{r}[\sigma_\alpha] (\sigma_\alpha) \) of equation (2.57). For Proposition 2.19 it suffices to establish that:
\[
\deg \bar{R}_{\chi, \alpha} = n_\alpha/e_\alpha.
\]
By Proposition 7.1 and equation (7.36), we have:
\[
(7.37) \quad \bar{R}_{\chi, \alpha} (z) = k_\alpha^{\deg \bar{R}_{\chi, \alpha}} \cdot (\vartheta \bar{R}_{\chi, \alpha}) (k_\alpha \cdot z),
\]
where the overall sign ensures that the RHS is a monic polynomial. Combining this with equation (7.36), we obtain:
\[
\deg \bar{R}_{\chi, \alpha} = \deg \bar{R}_{\chi, \alpha}.
\]
By Proposition 2.13 we have \( \deg \bar{R}_{\chi, \alpha} = n_\alpha \), and therefore:
\[
\deg \bar{R}_{\chi, \alpha} \geq n_\alpha/e_\alpha.
\]
By Proposition 6.6 we have:
\[
\deg \bar{R}_{\chi, \alpha} \leq n_\alpha/e_\alpha.
\]
Propositions 2.19 and 6.7 follow. \( \square \)
Remark 7.9. Having established Proposition 2.19 and using equation (7.34), we can rewrite the relationship (7.37) between minimal polynomials as follows:

\[ \tilde{R}^u_{\chi,\alpha}(z) = \tau_{\alpha}(r(\sigma^n_{\alpha})) \cdot (\vartheta \tilde{R}_{\chi,\alpha})(k_{\alpha} \cdot z), \quad \sigma_{\alpha} \in B_{W[\alpha]} \].

Another way to express this relationship is in terms of roots:

if \( \{ z_i \}_{i=1}^{n_{\alpha}/e_{\alpha}} \) are the roots of \( \tilde{R}_{\chi,\alpha} \), then \( \{ k_{\alpha}/z_i \}_{i=1}^{n_{\alpha}/e_{\alpha}} \) are the roots of \( \tilde{R}^u_{\chi,\alpha} \).

Since the group \( I \) is finite, and all our characters take values in finite subgroups of \( \mathbb{G}_m \), general results on the quasi-unipotence of the monodromy in the family apply, and we can conclude that all roots of the polynomials \( \tilde{R}^u_{\chi,\alpha} \), \( \tilde{R}_{\chi,\alpha} \), and \( R_{\chi,\alpha} \) are roots of unity.

Proof of Proposition 2.18 Let \( \tilde{\gamma}_{n_{\alpha}} : [0,1] \rightarrow \mathbb{C} \) be a smooth path, as shown in Figure 1 for \( n_{\alpha} = 5 \). More precisely, the path \( \tilde{\gamma}_{n_{\alpha}} \) satisfies conditions (P1)-(P5) of Section 4 for the representation \( G_{\alpha}|_{\mathbb{V}_a} \), the covector \( l_{0,\alpha} \in (V^*_a)^{r,s} \), and the critical point \( c_{n_{\alpha}} = c_0 \in \mathbb{Z}_\alpha \), as well as the following conditions:

- \( \tilde{\gamma}_{n_{\alpha}}'(0) = \tilde{\gamma}_0'(0) \);
- \( \xi_{0,\alpha}(\tilde{\gamma}_{n_{\alpha}}(t)) > \xi_{0,\alpha}(c_j) \) for all \( t \in [0,1] \) and \( j \in \{1,\ldots,n_{\alpha} - 1\} \);
- the path \( \tilde{\gamma}_{n_{\alpha}} \) makes one clockwise turn around \( l_{0,\alpha}(c_0) \) before reaching \( \xi_0 \).

Write:

\[ z = \lambda_{l_{0,\alpha}} \circ \tilde{r}[\sigma_{\alpha}] (\sigma^{-1}) \in \text{End}(M_{l_{0,\alpha}}(P_{\chi_\alpha})), \]

and let \( R_z \in \mathcal{R} \) be the minimal polynomial of \( z \). By Lemma 7.6 (iii) and equation (7.19), we have:

\[ z^{n_{\alpha}}(u_{0,\alpha}) = Sgn_{\alpha}(r(\sigma^n_{\alpha})) \cdot u_{n_{\alpha},\alpha} = \tau_{\alpha}(x) \cdot u_{n_{\alpha},\alpha}. \]

By the definition of the classes \( \{ u_{j,\alpha} \} \) for \( j \neq 0 \) in equation (7.13), plus a standard Picard-Lefschetz theory argument, we have:

\[ u_{n_{\alpha},\alpha} = PL[c_0, \tilde{\gamma}_{n_{\alpha}}, o_{0,\alpha}, a_{n_{\alpha}}] = PL[c_0, \tilde{\gamma}_{n_{\alpha}}, o_{0,\alpha}, a_{n_{\alpha}}] + L_1(u_{1,\alpha}, \ldots, u_{j-1,\alpha}), \]

where \( L_1(u_{1,\alpha}, \ldots, u_{j-1,\alpha}) \) is some linear combination of the \( u_{1,\alpha}, \ldots, u_{j-1,\alpha} \). By the definition of \( a_{n_{\alpha}} \in (\mathcal{L}_{\chi_\alpha})_{c_0} \) in (7.14), plus Lemma 7.6 (iii), we can rewrite equation (7.39) as follows:

\[ u_{n_{\alpha},\alpha} = \chi(x^{-1}) \cdot PL[c_0, \tilde{\gamma}_{n_{\alpha}}, o_{0,\alpha}, a_0] + L_2(z(u_{0,\alpha}), \ldots, z^{j-1}(u_{0,\alpha})), \]

where \( L_2(z(u_{0,\alpha}), \ldots, z^{j-1}(u_{0,\alpha})) \) is again a linear combination.

By considering the obvious homotopy between the paths \( \tilde{\gamma}_{n_{\alpha}} \) and \( \gamma_0 \), within the class of all smooth paths satisfying (P1)-(P5) (see observation (7.10)), we obtain:

\[ PL[c_0, \tilde{\gamma}_{n_{\alpha}}, o_{0,\alpha}, a_0] = (-1)^{d_{\alpha}} \cdot PL[c_0, \gamma_0, o_{0,\alpha}, a_0] = (-1)^{d_{\alpha}} \cdot u_{0,\alpha}, \]

(see (7.8)). By combining equations (7.38), (7.40), (7.41), we obtain:

\[ z^{n_{\alpha}}(u_{0,\alpha}) = (-1)^{d_{\alpha}} \cdot \chi(x^{-1}) \cdot \tau_{\alpha}(x) \cdot u_{0,\alpha} + \tau_{\alpha}(x) \cdot L_2(z(u_{0,\alpha}), \ldots, z^{j-1}(u_{0,\alpha})). \]
In view of Proposition 2.13 and assertion (7.31), we can conclude that:

\[ R_z(0) = (-1)^{d+1} \cdot \chi(x^{-1}) \cdot \tau_\alpha(x). \]

It remains to note that \( R_{\chi,\alpha} = \partial R_z \), so \( R_{\chi,\alpha}(0) = R_z(0)^{-1}. \)

\[ \square \]

8. Microlocal monodromy for a braid generator

The following proposition is a partial analog of Proposition 7.1 in the context of the full representation \( G|V \).

**Proposition 8.1.** For each \( \alpha \in A \) and \( \sigma_\alpha \in B_{W[\alpha]} \), we have:

\[ \mu_l(\sigma_\alpha^e) u_0 = k_\alpha \cdot \lambda_0 \circ \tilde{r}(\sigma_\alpha^{-e_\alpha}) u_0 \in M_0(P_\chi), \]

\[ \mu_l(\sigma_\alpha^{-e_\alpha}) u_0 = k_\alpha \cdot \lambda_0 \circ \tilde{r}(\sigma_\alpha^{e_\alpha}) u_0 \in M_0(P_\chi), \]

where \( k_\alpha \in \{\pm 1\} \) is the sign provided by Proposition 7.1.

Our proof of Proposition 8.1 will be an adaptation of the proof of Proposition 7.1 in Section 7. We begin with a preliminary discussion geared towards the analysis of the Hessians at the critical points of \( l_0|_{X_{\chi_\alpha}} \).

Fix an \( \alpha \in A \). Recall the notation of equation (2.17), and define:

\[ C^*_\alpha = \bigcup_{\beta \in A^\alpha} C^*_\beta, \quad (C^*_\alpha)^{reg} = C^*_\alpha - C^*_{\alpha^e}. \]

By Theorem 3.1 (iv), we have:

\[ g \cdot v \supset g \cdot C_\alpha = \bigoplus_{\beta \in A_{\alpha}} g_\beta \cdot C \quad \text{for every} \quad v \in C - C_{\alpha^e}. \]

Consider the image:

\[ \nu(V_\alpha) = C^* \oplus g_\alpha \cdot C^* = (g \cdot C_\alpha)^\perp \subset V^*, \]

(see (2.3) and (2.36)). Here, the first equality follows from (2.35) and (3.6), and the second from Proposition 3.3. Note that restriction to \( V_\alpha \) produces an isomorphism: \( \nu(V_\alpha) \cong V^*_\alpha \).

Let \( V^\alpha = V_\alpha \cap V^\alpha \) (see Section 2.6), and consider the set of pairs:

\[ P_\alpha = \{(v,l) \in V^\alpha \times \nu(V_\alpha) \mid l|_{g \cdot v} = 0 \} \subset V \times V^*. \]

Note that \( P_\alpha \subset V \times V^* \) is a real algebraic subset (see (2.32)), and we have:

\[ P_\alpha = K_\alpha \cdot (C \times C^*), \]

(cf. (2.34)). Let:

\[ P_\alpha^o[C] = (C - C_{\alpha^e}) \times (C^* - C^*_{\alpha^e}) \subset C \times C^* \quad \text{and} \quad P_\alpha^o = K_\alpha \cdot P_\alpha^o[C] \subset P_\alpha. \]

Since the group \( K_\alpha \) is compact, we have:

\[ \text{the subset} \ P_\alpha^o \subset P_\alpha \text{ is open in the classical topology}. \]
Using equation (8.3) and the $K_\alpha$-invariance of the subspace $g \cdot C_\alpha \subset V$, we obtain:

$$g \cdot v \supset g \cdot C_\alpha$$

for every $(v, l) \in P_\alpha^\circ$. Let $H_\alpha = \text{Sym}^2((g \cdot C_\alpha)^*)$ and let $H_\alpha^\circ \subset H_\alpha$ be the open subset consisting of all non-degenerate quadratic forms. Note that $H_\alpha$ inherits a Hermitian metric from $\langle \ , \ \rangle$, and $K_\alpha$ acts on $H_\alpha$ by isometries, preserving the subset $H_\alpha^\circ \subset H_\alpha$. We have a map:

$$\mathcal{H}_\alpha : P_\alpha^\circ \to H_\alpha, \ (v, l) \mapsto \mathcal{H}_\alpha[v, l] := \mathcal{H}[v, l]_{g \cdot C_\alpha},$$

(see (3.2)).

**Lemma 8.2.**

(i) The map $\mathcal{H}_\alpha$ of equation (8.3) is $K_\alpha$-equivariant.

(ii) The map $\mathcal{H}_\alpha$ is continuous in the classical topology.

(iii) We have $\mathcal{H}_\alpha[v, l] \in H_\alpha^\circ$ for every $(v, l) \in P_\alpha^\circ$.

**Proof.** Part (i) follows from the definitions. For part (ii), note that:

$$\text{the restriction } \mathcal{H}_\alpha|_{P_\alpha^\circ[C]} \text{ is complex algebraic, and therefore continuous.}$$

The continuity of $\mathcal{H}_\alpha$ follows from (8.6) plus the $K_\alpha$-equivariance of part (i) and the compactness of $K_\alpha$. Part (iii) follows from Propositions 3.3 and 3.4 plus the $K_\alpha$-equivariance of part (i). \qed

For every $(v, l) \in P_\alpha^\circ$, let:

$$T_{+, \alpha}[v, l] \subset g \cdot C_\alpha,$$

be the positive eigenspace of $\mathcal{H}_\alpha[v, l]$, relative to the inner product $\langle \ , \ \rangle$. Here, as usual, by “the positive eigenspace” we mean the direct sum of all the eigenspaces of the real part $\text{Re}(\mathcal{H}_\alpha[v, l])$, corresponding to positive eigenvalues. By Lemma 8.2 the vector spaces $\{T_{+, \alpha}[v, l]\}$ form a $K_\alpha$-equivariant real vector bundle $T_{+, \alpha}[P_\alpha^\circ]$ over $P_\alpha^\circ$, of rank $d - d_\alpha = \dim C_\alpha$. Moreover, each $T_{+, \alpha}[v, l]$ is a real form of $g \cdot C_\alpha$, i.e., we have $g \cdot C_\alpha = T_{+, \alpha}[v, l] \otimes \mathbb{R} \mathbb{C}$. Of particular interest to us is the orientation bundle of $T_{+, \alpha}[P_\alpha^\circ]$, which we denote by $\mathcal{O}[P_\alpha^\circ]$. It is a $K_\alpha$-equivariant $\mathbb{Z}/2$-torsor over $P_\alpha^\circ$.

Pick a $c_\alpha \in C_\alpha^{reg}$ and let $l_\alpha = \nu(c_\alpha) \in (C_\alpha^{reg})^{reg}$. We then have $(c_\alpha, l_\alpha) \in P_\alpha^\circ$. Use (8.4) to pick an $\epsilon > 0$ such that:

$$P_{\alpha, \epsilon} := \{(v, l) \in P_\alpha \mid \text{dist}(c_\alpha, v) < \epsilon, \text{dist}(l_\alpha, l) < \epsilon\} \subset P_\alpha^\circ,$$

where the distances are taken with respect to $\langle \ , \ \rangle$. Note that $P_{\alpha, \epsilon} \subset P_\alpha^\circ$ is a contractible, $K_\alpha$-invariant open subset. Since the group $K_\alpha$ is connected, we have:

$$\text{the restriction } \mathcal{O}[P_{\alpha, \epsilon}] \text{ of } \mathcal{O}[P_\alpha^\circ] \text{ to } P_{\alpha, \epsilon} \text{ is a trivial } K_\alpha\text{-equivariant } \mathbb{Z}/2\text{-torsor.}$$

**Proof of Proposition 8.1.** The proofs of (8.1) and (8.2) are similar, and we will only give the former. Let $\sigma_\alpha = \sigma_\alpha[\Gamma]$, with $\Gamma(0) = c_0$ and $\Gamma(1) = c_{\alpha, 1}$, as in equation (2.21). As in the
proof of Proposition 6.6 we can reduce to the case where \( c_0 = c_{\alpha, 1} \) and the path \( \Gamma \) is trivial. Moreover, we can assume that:

\[
\text{dist}(c_\alpha, c_{\alpha, 1}) < \epsilon, \tag{8.10}
\]

where \( c_\alpha \in C_{\alpha, 0}^e \) is given by equation (2.19), and \( \epsilon > 0 \) is chosen to satisfy the containment \( P_{\alpha, \epsilon} \subseteq P_\alpha^o \) of equation (8.8). Proceeding with these assumptions, the proof of (8.1) is essentially a paraphrase of the proof of equation (7.32) for \( k_\alpha \) as in equation (7.33) (see the proof of Proposition 7.1). The only non-trivial issue in adapting the argument from the setting of the rank one representation \( G^\alpha \mid V_\alpha \) to the setting of the full representation \( G \mid V \) is the treatment of the orientations of the positive eigenspaces of the Hessians. This issue is essentially handled by invoking Proposition 3.3 and assertion (8.9).

Recall the subset \( Z_\alpha = W_\alpha \cdot c_0 \subseteq Z_0 \) and the subspace \( M_{l_0}(P_\chi)[\alpha] \subseteq M_{l_0}(P_\chi) \), introduced in the proof of Proposition 6.6 (see equations (6.18), (6.20)). In the present argument, the subspace \( M_{l_0}(P_\chi)[\alpha] \) will play the same role as the Morse group \( M_{l_\alpha}(P_{\chi_\alpha}) \) did in Section 7. As in that section, we let \( J = \{0, \ldots, n_\alpha\} \), and we construct a collection of Picard-Lefschetz classes:

\[
\{u_j\}_{j \in J} \subseteq M_{l_0}(P_\chi)[\alpha], \tag{8.11}
\]

by analogy with the classes \( \{u_{j, \alpha}\}_{j \in J} \) of equation (7.6). The class \( u_0 \) has already been defined by equation (4.5).

Recall the critical points \( \{c_j\}_{j \in J - \{0\}} \subseteq Z_\alpha \) of equation (7.9). For each \( j \in J - \{0\} \), let \( \gamma_j : [0, 1] \to \mathbb{C} \) be a smooth path, satisfying conditions (Q1)-(Q6) of Section 7 as well as condition (6.21) from the proof of Proposition 6.6 which we paraphrase as follows:

\[
\xi_{l_0} \circ \gamma_j(t) > \xi_{l_0}(c) \text{ for every } t \in [0, 1] \text{ and } c \in Z_0 - Z_\alpha. \tag{8.12}
\]

Note that these conditions ensure that \( \gamma_j \) satisfies conditions (P1)-(P5) of Section 4 and furthermore, determine \( \gamma_j \) uniquely up to homotopy within the class of all paths satisfying (P1)-(P5).

Next, recall the elements \( a_j \in (\mathcal{L}_{\chi_\alpha})_{c_j}, j \in J - \{0\}, \) of equation (7.14). Note that we have \( \mathcal{L}_{\chi_\alpha} \cong \mathcal{L}_\chi|_{x_{\chi_{\alpha}}}. \) Therefore, we can view each \( a_j \) as an element of \( (\mathcal{L}_\chi)_{c_j}. \)

We now discuss orientations. For each \( j \in J \), let:

\[
T_+[c_j] \subseteq \mathfrak{g} \cdot c_j = \mathfrak{g} \cdot C, \tag{8.13}
\]

be the positive eigenspace of \( \gamma_j'(0)^{-1} : \mathcal{K}[c_j, l_0]. \) By Theorem 3.1 (iv), we have:

\[
\mathfrak{g} \cdot C = \mathfrak{g}_\alpha \cdot C \oplus \mathfrak{g} \cdot C_\alpha. \tag{8.14}
\]

By Proposition 3.3 decomposition (8.14) gives rise to a decomposition:

\[
T_+[c_j] = T_{+, \alpha}[c_j] \oplus T_{+, l_\alpha}[c_j], \tag{8.15}
\]

where \( T_{+, \alpha}[c_j] = T_+[c_j] \cap \mathfrak{g}_\alpha \cdot C \) is the positive eigenspace of equation (7.11), and \( T_{+, l_\alpha}[c_j] = T_+[c_j] \cap \mathfrak{g} \cdot C_\alpha. \)
We have already chosen an orientation $o_{j,\alpha}$ of $T_{+,\alpha}[c_j]$ for every $j \in J$ (see equations (7.8), (7.12)). We also have the orientation $o_0$ of $T_+[c_0]$, used to define the class $u_0$ of equation (4.3). Let $o_{0,\alpha}$ be the orientation of $T_{+,\alpha}[c_0]$ such that:

\begin{equation}
(8.16) \quad o_0 = (o_{0,\alpha}, o_{0,\alpha}),
\end{equation}

in terms for equation (8.15) for $j = 0$. We now proceed to define an orientation $o_{j,\alpha}$ of $T_{+,\alpha}[c_j]$ for every $j \in J - \{0\}$.

By assumption (8.10), we have $(c_j, l_0) \in P_{\alpha,\epsilon} \subset V \times V^*$ for every $j \in J$. Recall the trivial orientation torsor $\mathcal{O}[P_{\alpha,\epsilon}]$ on $P_{\alpha,\epsilon}$ of assertion (8.9). Because of the multiple $\gamma_j(0)^{-1}$ appearing the definition of the subspace $T_+[c_j]$ of equation (8.13), in the notation of (8.7), we have:

\begin{equation}
(8.17) \quad T_{+,\alpha}[c_j] = \exp(\pi i j/n_\alpha) \cdot T_{+,\alpha}[c_j, l_0], \quad j \in J,
\end{equation}

(see Figure 1 and condition (Q3) of Section 7). In particular, we have $T_{+,\alpha}[c_0] = T_{+,\alpha}[c_0, l_0]$. Thus, the orientation $o_{0,\alpha}$ of $T_{+,\alpha}[c_0]$ determines a global section $o_{\alpha}[P_{\alpha,\epsilon}]$ of $\mathcal{O}[P_{\alpha,\epsilon}]$. We denote the value of this section at $(v, l) \in P_{\alpha,\epsilon}$ by $o_{\alpha}[v, l]$, so that $o_{\alpha}[c_0, l_0] = o_{0,\alpha}$. For each $j \in J - \{0\}$, we now use (8.17) to define an orientation:

\begin{equation}
(8.18) \quad o_{j,\alpha} = (\exp(\pi i j/n_\alpha)) \cdot o_{\alpha}[c_j, l_0],
\end{equation}

of $T_{+,\alpha}[c_j]$. Note that we have:

\[ c_{n_\alpha} = c_0, \quad T_{+,\alpha}[c_{n_\alpha}] = T_{+,\alpha}[c_0], \quad \text{and} \quad o_{n_\alpha,\alpha} = (-1)^{d - d_\alpha} \cdot o_{0,\alpha}, \]

where $d - d_\alpha = \dim(\mathfrak{g} \cdot C_\alpha)$.

Next, for each $j \in J - \{0\}$, we use decomposition (8.15) to define an orientation:

\[ o_j = (o_{j,\alpha}, o_{j,\alpha}), \]

of $T_+[c_j]$ and a Picard-Lefshetz class:

\[ u_j = PL[c_j, \gamma_j, o_j, a_j] \in M_0(P_\chi)[\alpha], \]

(cf. equation (7.15)). This completes the construction of the classes (8.11). With these definitions, we can proceed as in the proof of Lemma 7.6 to verify that:

\[ \mu_0(\sigma^{e_\alpha}_\alpha) \cdot u_0 = u_{e_\alpha} \quad \text{and} \quad \lambda_{10} \cdot \tilde{r}(\sigma^{-e_\alpha}_\alpha) \cdot u_0 = k_\alpha \cdot u_{e_\alpha}, \]

where $k_\alpha \in \{\pm 1\}$ is given by equation (7.33). There are only two new features of the argument in this setting. First, one needs to check that critical points in $Z_0 - Z_\alpha$ do not interfere with the paths $\{\gamma_j\}_{j \in J}$. This is guaranteed by the choice of $c_0 = c_{\alpha,1}$ to be near the regular part $C^{reg}_\alpha \subset C_\alpha$, and by assumption (8.12) (cf. proof of assertion (6.23)). Second, in view of (8.15), one needs to match the orientations of the positive eigenspace $T_{+,\alpha}[c_\alpha]$. This matching follows form assertion (8.9) and the choice of the orientations $\{o_{j,\alpha}\}_{j \in J}$ in (8.16) and (8.18). We omit the rest of the details. □
9. Proof of the main theorem

In this section, we assemble the results of Sections 4-8 to prove Theorem 2.23. Recall that we have $B_\chi^\chi,0 \subset B_\chi^\chi$, and define:

$$M_0 = \mu_{l_0}(C[B_\chi^\chi,0]) \cdot u_0 \subset \mu_{l_0}(P_\chi),$$

where $u_0$ is the class of equation (4.5) and $\mu_{l_0}$ is the monodromy in the family of equation (6.13). In the argument that follows, we will show that the subspace $M_0 \subset \mu_{l_0}(P_\chi)$ plays the role of the $\tilde{B}_\chi^\chi,0$-representation $M_0^0$ of equation (1.9). We begin with the following corollary of Proposition 4.6.

**Corollary 9.1.** For every $u \in M_0$ and every $x \in I$, we have:

$$\lambda_{l_0}(x) u = \chi(x) \cdot \tau(x) \cdot u.$$

**Proof.** For $u = u_0$, this is a special case of Proposition 4.6, and for general $u$, this follows from assertion (6.14). \qed

**Lemma 9.2.** The subgroup $B_\chi^\chi,0 \subset B_\chi^\chi$ is generated by all elements of the form $\sigma_\alpha^{e_\alpha} \in B_\chi^\chi$ for $\alpha \in \Lambda$ and $\sigma_\alpha \in B_\chi^\chi[\alpha]$.

**Proof.** By definition, we have $B_\chi^\chi,0 = \pi_1(C^{\text{reg}}/W_\chi^0, c_0)$. The space $C^{\text{reg}}/W_\chi^0$ is the complement of the divisor $C^{\text{sing}}/W_\chi^0$ in the affine space $C/W_\chi^0$. By equation (2.60), each element $\sigma_\alpha^{e_\alpha} \in B_\chi^\chi,0$ is represented by a loop which links this divisor once. The lemma follows, for example, from [BMR, Proposition A1, p. 181], by induction on the number of irreducible components of the divisor $C^{\text{sing}}/W_\chi^0$. \qed

**Lemma 9.3.** We have:

$$\dim M_0 = |W_\chi^0|.$$

**Proof.** The last equality follows from Lemma 9.2, Proposition 8.1 and assertion (6.14). The middle equality follows from Corollary 9.1. And the first equality, once again, follows from assertion (6.14). \qed

**Lemma 9.4.** We have:

$$\dim M_0 = |W_\chi^0|.$$

**Proof.** Recall the decomposition $A = A_\chi^0 \cup A_\chi^1$ of Section 2.9 (see equation (2.61)). By Propositions 6.3 and 6.7, we have:

$$\text{(9.2) for every } \alpha \in A_\chi^1 \text{ and } \sigma_\alpha \in B_\chi[\alpha], \text{ the operator } \mu_{l_0}(\sigma_\alpha^{e_\alpha}) \in \text{Aut}(M_0(P_\chi)) \text{ is a scalar.}$$
To capture these scalars, we define a character:

\[ \rho^\mu = \rho_{\chi}^\mu : B_W^{X,0} \to \mathbb{G}_m, \]

by analogy with the character \( \rho \) of equation (2.65). Namely, we require that:

\[ \rho^\mu(\sigma_{e}^\alpha) = 1 \quad \text{for all} \quad \alpha \in A_1^\chi \quad \text{and} \quad \sigma_{\alpha} \in B_W[\alpha], \]

\[ \bar{R}_{\chi,\alpha}^\mu(\rho^\mu(\sigma_{n}^{\alpha})) = 0 \quad \text{for all} \quad \alpha \in A_1^\chi \quad \text{and} \quad \sigma_{\alpha} \in B_W[\alpha]. \]

Recall that, for \( \alpha \in A_1^\chi \), we have \( e_{\alpha} = n_{\alpha} \) and \( \deg \bar{R}_{\chi,\alpha}^{\mu} = 1 \). The character \( \rho^\mu \) is well-defined by Corollary (6.4). Note that the characters \( \rho \) and \( \rho^\mu \) determine each other by Remark 7.9. I.e., for every \( \alpha \in A_1^\chi \) and \( \sigma_{\alpha} \in B_W[\alpha] \), we have:

\[ \rho(\sigma_{n}^{\alpha}) \cdot \rho^\mu(\sigma_{n}^{\alpha}) = k_{\alpha}, \]

where \( k_{\alpha} \in \{ \pm 1 \} \) is provided by Proposition 7.1. Using equations (2.67) and (7.35), this yields an explicit expression:

(9.3)

\[ \rho^\mu(\sigma_{n}^{\alpha}) = (-1)^{d_{\alpha}} \cdot \chi(r(\sigma_{-n}^{\alpha})), \]

for all \( \alpha \in A_1^\chi \) and \( \sigma_{\alpha} \in B_W[\alpha] \).

Consider the diagram:

(9.4)

\[ B_W^{X,0} \xrightarrow{\varphi} B_W^{X,0} \xrightarrow{\rho^\mu} \mathbb{G}_m, \]

(see equation (2.62)). By an argument as in the proof of Lemma 9.2, the kernel \( \ker(\varphi) \subset B_W^{X,0} \) is generated by all elements of the form \( \sigma_{n}^{\alpha} \) for \( \alpha \in A_1^\chi \) and \( \sigma_{\alpha} \in B_W[\alpha] \). It follows that the restriction of the monodromy action \( \mu_{l_0} \) to \( \ker(\varphi) \subset B_W^{X,0} \) is given by the character \( \rho^\mu \). Therefore, we have:

\[ \mu_{l_0}|_{\ker(\varphi) \cap \ker(\rho^\mu)} = 1, \]

and the restriction of \( \mu_{l_0} \) to \( \ker(\rho^\mu) \subset B_W^{X,0} \) factors through the homomorphism \( \varphi \). Since the restriction \( \varphi|_{\ker(\rho^\mu)} \) is surjective, we obtain an action:

\[ \tilde{\mu}_{l_0} : B_W^{X,0} \to \text{Aut}(M_0(P_{\chi})), \]

such that \( \mu_{l_0}|_{\ker(\rho^\mu)} = \tilde{\mu}_{l_0} \circ \varphi \). By Lemma 9.2 and assertion (9.2), we have:

\[ M_0 = \mu_{l_0}(C[B_W^{X,0}]) \cdot u_0 = \tilde{\mu}_{l_0}(C[B_W^{X,0}]) \cdot u_0. \]

The inequality:

\[ \dim M_0 \leq |W^0_{\chi}|, \]

now follows from Proposition 6.7 and [11, Theorem 1.3], while the inequality:

\[ \dim M_0 \geq |W^0_{\chi}|, \]

follows from Corollary 4.3, Proposition 4.5, and Lemma 9.3. \( \square \)
For every $\alpha \in A$, let $\bar{R}_{\chi,\alpha}^0 \in \mathcal{R}$ be the minimal polynomial of the restriction:

$$\lambda_0 \circ \tilde{r} (\sigma_{\alpha}^\alpha)|_{M_0} \in \mathrm{End}(M_0),$$

for some $\sigma_{\alpha} \in B_W[\alpha]$. Note that $\bar{R}_{\chi,\alpha}^0$ is independent of the choice of $\sigma_{\alpha}$ by assertion (2.23).

**Lemma 9.5.** For every $\alpha \in A$, we have: $\bar{R}_{\chi,\alpha}^0 = \bar{R}_{\chi,\alpha}$. 

**Proof.** Pick a $\sigma_{\alpha} \in B_W[\alpha]$, and let $\bar{R}_{\chi,\alpha}^{\mu,0} \in \mathcal{R}$ be the minimal polynomial of $\mu_0(\sigma_{\alpha}^\alpha)|_{M_0}$. By Propositions 4.1, 4.5, 6.3 and assertion (6.14), we have:

$$\bar{R}_{\chi,\alpha}^{\mu,0} (z) = k_{\alpha}^{\deg \bar{R}_{\chi,\alpha}^0} \cdot (\vartheta \bar{R}_{\chi,\alpha}^0)(k_{\alpha} \cdot z),$$

(cf. equation (7.37)). Equation (9.6) uniquely determines the polynomial $\bar{R}_{\chi,\alpha}^0 \in \mathcal{R}$. Therefore, the lemma follows from equations (7.37) and (9.5)-(9.6).

**Proof of Theorem 2.23.** By Lemma 9.3, the action $\lambda_0 \circ \tilde{r} : B_W \to \operatorname{Aut}(M_0(P_\chi))$ restricts to an action:

$$\hat{\lambda}_0 : B_W^\chi \to \operatorname{Aut}(M_0).$$

By analogy with diagram (9.4), consider the diagram:

$$B_W^\chi \xleftarrow{\varphi} B_W^\chi \xrightarrow{\rho} \mathbb{G}_m.$$ 

Arguing as in the proof of Lemma 9.4 and using Lemma 9.5 for $\alpha \in A^1$, we observe that the restriction of $\hat{\lambda}_0$ to $\ker(\rho) \subset B_W^\chi$ factors through the homomorphism $\varphi$. Since the restriction $\varphi|_{\ker(\rho)}$ is surjective, we obtain an action:

$$\hat{\lambda}_0 : B_W^\chi \to \operatorname{Aut}(M_0),$$

such that $\hat{\lambda}_0|_{\ker(\rho)} = \hat{\lambda}_0 \circ \varphi$.

Lemmas 9.3, 9.4, Lemma 9.5 for $\alpha \in A_0^1$, and equation (2.64) enable us to conclude that:

$$M_0 \cong \mathcal{H}_W^{\chi_0}$$

as $\mathbb{C}[B_W^{\chi_0}]$-modules,

where the $\mathbb{C}[B_W^{\chi_0}]$-module structure on $M_0$ is given by the action $\hat{\lambda}_0$. Combining this with Lemma 9.2 and Lemma 9.5 for $\alpha \in A_1^1$, we obtain:

$$M_0 \cong \mathbb{C}_\rho \otimes \mathcal{H}_W^{\chi_0}$$

as $\mathbb{C}[B_W^{\chi_0}]$-modules,

where the $\mathbb{C}[B_W^{\chi_0}]$-module structure on $M_0$ is given by the action $\lambda_0^\rho$. Combining this further with Corollary 9.1, we obtain:

$$M_0 \cong \mathbb{C}_\chi \otimes \mathbb{C}_T \otimes \mathbb{C}_\rho \otimes \mathcal{H}_W^{\chi_0}$$

as $\mathbb{C}[B_W^{\chi_0}]$-modules,
where the $\mathbb{C}[\tilde{B}_W^0]$-module structure on $M_0$ is given by the microlocal monodromy action $\lambda_0$. By Proposition 4.5 plus a dimension count using Corollary 4.3 and Lemma 9.4, we obtain:

$$M_0(P_\chi) \cong \mathbb{C}[\tilde{B}_W] \otimes_{\mathbb{C}[\tilde{B}_W^0]} M_0 \text{ as } \mathbb{C}[\tilde{B}_W]\text{-modules},$$

where the $\mathbb{C}[\tilde{B}_W]$-module structure on $M_0(P_\chi)$ is again given by $\lambda_0$. It remains to note that the factor $\mathbb{C}_\tau$ of equation (9.7) can be taken outside the tensor product of equation (9.8), because the character $\tau$ of equation (2.33) is preserved by the action of $W$ on $\tilde{I}$. 

\[ \square \]

### 10. A VANISHING CONJECTURE FOR INTERSECTION NUMBERS

The proof of Theorem 2.23 in Sections 4-9 is an elaborate application of Picard-Lefschetz theory. However, it avoids any direct use of the central result of this theory: the Picard-Lefschetz formula. In this section, we remind the reader the statement of this formula in the context of our paper, and formulate a related conjecture (Conjecture 10.2) which, if true, would elucidate the structure behind Theorem 2.23.

In order to state Conjecture 10.2 in a suitable generality, pick an arbitrary basepoint:

$$l \in (C^*)^{reg} \subset (V^*)^{rs}.$$ 

We do not wish to restrict to the case $l = l_0$, partly in view of Remark 4.4. Recall the identification of the Morse group $M_l(P_\chi)$ in Lemma 4.2. Note that we have $Z_l = \{w c_0\}_{w \in W}$, independent of $l$. The generic property of $l \in (V^*)^{rs}$, given by Proposition 2.9 ensures that the restriction $l|_{X_{c_0}}$ is a locally trivial fibration over the non-critical locus $\mathbb{C} - l(Z_l)$. It follows that we have:

$$M_l(P_\chi) \cong H_d(X_{c_0}, X_{c_0, l}; L_\chi),$$

where $X_{c_0, l} = \{x \in X_{c_0} \mid l(x) = \xi_0\}$. Note that each Picard-Lefschetz class $PL[c, \gamma, o, a] \in M_l(P_\chi)$, as in equation (4.4), is constructed as an element of the RHS of equation (10.1).

The space $X_{c_0, l}$ is a smooth manifold of real dimension $2d - 2$, which we orient by the complex orientation. Thus, there is a well-defined intersection pairing:

$$H_{d-1}(X_{c_0, l}; L_\chi) \otimes H_{d-1}(X_{c_0, l}; L_\chi^*) \rightarrow \mathbb{C},$$

where $L_\chi^* \cong L_{\chi^{-1}}$ is the dual of the local system $L_\chi$. We denote this pairing by $a \otimes b \mapsto a \cap b$.

Pick a pair of critical points $c_1, c_2 \in Z_l$ and a triple of paths $\gamma_1, \gamma_2, \gamma_3 : [0, 1] \rightarrow C$, such that $\gamma_1(0) = \gamma_3(0) = l(c_1)$, $\gamma_2(0) = l(c_2)$, $\gamma'_1(0) = \gamma'_3(0)$, and all three paths satisfy conditions (P2)-(P5) of Section 4. Assume that the paths $\gamma_1, \gamma_2, \gamma_3$ do not meet each other, except at the end-points. Also, assume that $l(c_2)$ is the unique element of $l(Z_l)$, contained inside the closed curve formed by the paths $\gamma_1$ and $\gamma_3$, and that $l(c) \neq l(c_2)$ for every $c \in Z_l - \{c_2\}$. Pick orientations $o_1$ of $T_+[c_1, \gamma_1]$ and $o_2$ of $T_+[c_2, \gamma_2]$ (see (4.3)). Also, pick generators $a_1 \in (L_\chi)_{c_1}$,
Let \( a_2 \in (\mathcal{L}_\chi)_{c_2} \). Let \( a_2^* \in (\mathcal{L}_\chi^*)_{c_2} \) be the unique element with \( \langle a_2, a_2^* \rangle = 1 \). Define Picard-Lefschetz classes \( u_1, u_2, u_3 \in H_d(X_{c_0}, X_{c_0, c_0}; \mathcal{L}_\chi) \) and \( u_2^* \in H_d(X_{c_0}, X_{c_0, c_0}; \mathcal{L}_\chi^*) \) as follows:

\[
\begin{align*}
 u_1 &= PL[c_1, \gamma_1, o_1, a_1], \\
 u_2 &= PL[c_2, \gamma_2, o_2, a_2], \\
 u_2^* &= PL[c_2, \gamma_2, o_2, a_2^*].
\end{align*}
\]

The Picard-Lefschetz formula, in this setting, takes the following form:

\[
(10.2)
\]

The Picard-Lefschetz formula in this form can be derived from the discussion in [AGV, Chapter 1.3].

The following is an immediate corollary of Proposition 6.5 and its proof.

**Corollary 10.1.** In the situation of equation (10.3), assume that \( c_1 = w_1 c_0, c_2 = w_2 c_0, w_1, w_2 \in W \), and the cosets \( \bar{w}_1 = w_1 W_\chi \) and \( \bar{w}_2 = w_2 W_\chi \) are distinct. Then we have:

\[
\partial u_1 \cap \partial u_2^* = 0.
\]

**Proof.** First, consider the case \( l = l_0 \). Using the notation of (6.15), and with reference to equation (10.3), we have:

\[
\text{LHS} \in M_{l_0}(P_\chi[\bar{w}_1]) \quad \text{while} \quad \text{RHS} \in M_{l_0}(P_\chi[\bar{w}_2]).
\]

Therefore, by Proposition 6.5 we have \( \text{LHS} = \text{RHS} = 0 \). But \( u_2 \neq 0 \), as a Picard-Lefschetz class with \( a_2 \neq 0 \) (see equation (10.2)). Therefore, we must have \( \partial u_1 \cap \partial u_2^* = 0 \).

For general \( l \in (C^*)^{reg} \), we note that the proof of Proposition 6.5 goes through unchanged for the group \( M_l(P_\chi) \) in place of \( M_{l_0}(P_\chi) \). Therefore, the same argument as in the case \( l = l_0 \) applies.

The following conjecture is a slight modification of Corollary 10.1.

**Conjecture 10.2.** In the situation of equation (10.3), assume that \( c_1 = w_1 c_0, c_2 = w_2 c_0, w_1, w_2 \in W \), and the cosets \( w_1 W_\chi^0 \) and \( w_2 W_\chi^0 \) are distinct, as elements of \( W/W_\chi^0 \). Then we have:

\[
\partial u_1 \cap \partial u_2^* = 0.
\]

Conjecture 10.2 plus equation (10.3) readily imply a version of Proposition 6.5 with the subgroup \( W_\chi \subset W \) replaced by \( W_\chi^0 \), yielding a decomposition:

\[
M_{l_0}(P_\chi) = \bigoplus_{\bar{w} \in W/W_\chi^0} M_{l_0}(P_\chi)[\bar{w}].
\]
Such a decomposition, once established, would enable us to define:
\[ M_0 = M_{l_0}(P_\chi)[\overline{T}], \]
where \( \overline{T} \in W/W^0_\chi \) is the coset of the identity in \( W \), thus replacing the equivalent, but rather less direct, definition (9.1). With this simpler definition, it would be essentially immediate that \( \dim M_0 = |W^0_\chi| \) (cf. Lemma 9.4) and that \( M_0 \subset M_{l_0}(P_\chi) \) is preserved by the microlocal monodromy action \( \lambda_{l_0}|_{\tilde{B}_W^0} \) (cf. Lemma 9.3). Moreover, for every \( \tilde{b} \in \tilde{B}_W \), we would have:
\[ \lambda_{l_0}(\tilde{b}) (M_0) = M_{l_0}(P_\chi)[\overline{w}], \]
where \( w = p \circ \tilde{q}(\tilde{b}) \) and \( \overline{w} = w W^0_\chi \in W/W^0_\chi \).

Thus, in our view, Conjecture 10.2 clarifies the nature of the subspace \( M_0 \subset M_{l_0}(P_\chi) \), and therefore, provides some context for the statement of Theorem 2.23. An independent proof of this conjecture would be likely to provide a simplified proof of Theorem 2.23. At present, we are unable to prove Conjecture 10.2 outside of the cases where \( W^0_\chi = W_\chi \).

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