THE STRUCTURE OF $I_4$-FREE AND TRIANGLE-FREE BINARY MATROIDS

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Abstract. A simple binary matroid is called $I_4$-free if none of its rank-4 flats are independent sets. These objects can be equivalently defined as the sets $E$ of points in $PG(n-1,2)$ for which $|E \cap F|$ is not a basis of $F$ for any four-dimensional flat $F$.

We prove a decomposition theorem that exactly determines the structure of all $I_4$-free and triangle-free matroids. In particular, our theorem implies that the $I_4$-free and triangle-free matroids have critical number at most 2.

1. Introduction

This paper proves an exact structure theorem for the simple binary matroids with no four-element independent flat and two-dimensional subgeometry. In this paper we use standard matroid theory terminology, with some minor modifications that we explain below.

A simple binary matroid is a pair $M = (E,G)$ where $G$ is a finite binary projective geometry $PG(n-1,2)$ and $E$, the ground set, is any subset of the points of $G$. Abusing notation, we write $G$ for the set of points of $G$. For brevity, we will simply refer to a simple binary matroid as a matroid in this paper. Two matroids $M_1 = (E_1,G_1)$ and $M_2 = (E_2,G_2)$ are isomorphic if there exists an isomorphism from $G_1$ to $G_2$ which maps $E_1$ to $E_2$. We say that a matroid $N$ is an induced restriction (or induced submatroid) of $M$ if there exists some subgeometry $F$ of $G$ such that $N = (E \cap F,F)$. We use the notation $M|F$ for the matroid $N$. If $M$ has no induced restriction isomorphic to $N$, then we say that $M$ is $N$-free.

As opposed to simple binary matroids in the usual sense, our matroids are equipped with an extrinsic ambient space $G$. The dimension of $M$ is the dimension of $G$ as a geometry. We do not require the

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ground set to span $G$; if $E$ does not span $G$, then we say that $M$ is rank-deficient, and if $E$ spans $G$, then $M$ is full-rank. This definition of matroids allows us to define the complement of a matroid; given a matroid $M = (E, G)$, the complement of $M$ is the matroid $M^c = (G \setminus E, G)$.

We write $I_n$ for the matroid $(B, G)$ where $B$ is a basis of an $n$-dimensional projective geometry $G$. Note that $(E, G)$ is $I_n$-free if and only if $F \cap E$ is not a basis of $F$ for any $n$-dimensional subgeometry $F$ of $G$. A triangle of $G$ is a two-dimensional subgeometry. If $E$ contains no triangle of $G$, then $M = (E, G)$ is triangle-free. The main result for this paper is a structure theorem for $I_4$-free and triangle-free matroids; such matroids fall into one of two types, each of which arises via simple operations from a basic class of $I_4$-free, triangle-free matroids.

Suppose that $M = (E, G)$ is an $n$-dimensional matroid. We say that $M$ is a doubling if there exists $w \notin E$ for which $E + w = E$. Suppose further that there exists a hyperplane $H$ for which $E \subseteq G \setminus H$ (such matroids are called affine). Then $M' = (E', G')$ is a 0-expansion of $M$ if $G$ is a hyperplane of $G'$ and $E = E'$, and $M' = (E', G')$ is a 1-expansion of $M$ if $G$ is a hyperplane of $G'$ and there exists $x \in G' \setminus G$ such that $E' = E \cup (x + H) \cup \{x\}$. Note that a 0-expansion is simply the embedding of the matroid in a larger projective geometry.

We can now state our main result.

**Theorem 1.1.** A full-rank matroid $M = (E, G)$ is $I_4$-free and triangle-free if and only if either

- $M$ can be obtained from a 1-dimensional matroid by a sequence of 0-expansions and 1-expansions, or
- $M$ can be obtained by a sequence of doublings of $AG^\oplus(n - 1, 2)$ for some $n \geq 3$.

We remark that the two outcomes of Theorem 1.1 are mutually exclusive. For an $n$-dimensional matroid $M = (E, G)$, its critical number is $\chi(M) = n - \omega(M^c)$, where $\omega(M)$ is the dimension of a largest subgeometry of $M$ contained in $E$; the critical number of a matroid can be seen as a matroidal analogue of chromatic number for graphs (see [8] p.588 for a discussion). Then the first outcome in Theorem 1.1 results in a class of matroids with critical number 1 (a matroid with critical
number 1 is called *affine*), whereas the second outcome consists of matroids with critical number 2. It is also worth noting that the second outcome is a much more restrictive class of matroids; up to isomorphism, there are exactly \( n - 3 \) such matroids for any given dimension \( n \geq 4 \).

The proof of Theorem 1.1 falls in two parts, based on the existence of an induced \( C_5 \)-restriction. The matroid \( C_5 \) is the full-rank 4-dimensional matroid on 5 elements adding to zero; it will not be difficult to show that an \( I_4 \)-free, triangle-free matroid \( M \) is affine if and only if it does not contain an induced \( C_5 \)-restriction. It turns out that having an induced \( C_5 \)-restriction greatly restricts the structure of an \( I_4 \)-free, triangle-free matroid.

The case when \( M \) is affine is more involved, and in fact we will first consider the class of \( AI_4 \)-free matroids; we say that a matroid \( M = (E, G) \) is \( AI_4 \)-free if for any basis \( \{x_1, x_2, x_3, x_4\} \subseteq E \) of a four-dimensional subgeometry of \( G \), there exists some \( i \in \{1, 2, 3, 4\} \) such that \( \sum_{j \neq i} x_j \in E \). It turns out that if \( M = (E, G) \) is \( AI_4 \)-free, then we can always find a special hyperplane \( H \) of \( G \) such that either \( E \) or \( G \setminus E \) is contained in either \( H \) or \( G \setminus H \). Once we understand this feature of \( AI_4 \)-free matroids, it will be straightforward to derive the outcome corresponding to the affine matroids in Theorem 1.1. Also, we will provide a structural theorem for \( AI_4 \)-free matroids.

Finally, we remark that Theorem 1.1 settles the case \( s = 4 \) in the following conjecture, made in [2].

**Conjecture 1.2** ([2]). For any \( s \geq 4 \), the class of \( I_s \)-free and triangle-free matroids has bounded critical number.

2. Preliminaries

**Flats, cosets and translates.** We say that a subgeometry of \( G \) is a flat of \( G \). Viewing \( G \) as \( \mathbb{F}_2^n \setminus \{0\} \), a set \( F \subseteq G \) is a flat if and only if \( F \cup \{0\} \) is closed under addition. We write \([F]\) for the set \( F \cup \{0\} \). By convention, we call flats of dimension 2 triangles. A triangle of \( G \) is equivalently a triple \( \{x, y, x + y\} \subseteq G \) with \( x \neq y \). A maximal proper flat of \( G \) is a hyperplane.

A coset of a flat \( F \) in a flat \( G \supseteq F \) is any set of the form \( A = x + [F] \) for some \( x \in G \setminus F \). We do not consider the set \( F \) itself to be a coset; when we wish to include the set \( F \) itself, we use the term translate.

**Induced restrictions.** Recall that if \( B \) is a basis of an \( n \)-dimensional projective geometry \( H \), then we write \( I_n \) for the matroid \((B, H)\). In this paper, we will frequently be obtaining a contradiction by finding
an induced $I_4$-restriction. Therefore, to keep our proofs concise, we will abuse notation and say that the set $B$ itself is an induced $I_n$-restriction. Note also that we will often simply claim that $B$ is an induced $I_4$-restriction rather than explicitly writing out calculations to show that these elements do not span any other points of the matroid; instead enough information will be provided prior to such a claim so that it should be easy to check that $M|\text{cl}(B)$ is indeed an induced $I_4$-restriction.

**Critical number.** Recall that the critical number $\chi(M)$ of an $n$-dimensional matroid $M = (E, G)$ is $n - k$ where $k$ is the size of a largest projective geometry restriction of $M_c = (G \setminus E, G)$. If $\chi(M) = 1$, then $M$ is affine. Below we give a standard characterisation of affine matroids in terms of induced odd circuits. For $n$ odd, the circuit of length $n$, denoted $C_n$, is the full-rank $(n - 1)$-dimensional matroid whose ground set consists of $n$ points that add to zero. Note that odd circuits have critical number exactly 2. The following characterisation is the matroidal analogue of the characterisation of bipartite graphs in terms of excluded odd cycles.

**Theorem 2.1.** A matroid is affine if and only if it has no induced odd circuits.

**Proof.** The forward direction follows by the observation that the critical number does no increase under induced restrictions.

Conversely, suppose that $M = (E, G)$ contains no induced odd circuits, and let $B \subseteq E$ be a basis of $\text{cl}(E)$. As $M$ has no induced odd circuits, it follows through an inductive argument that $\text{cl}(B + B) \cap E = \emptyset$. But $H = \text{cl}(B + B)$ is a hyperplane, so $M$ is affine. \hfill \Box

Note that odd circuits of length 5 or more contain an induced $I_4$-restriction. The following easy consequence is used repeatedly throughout the paper, often without explicit reference.

**Corollary 2.2.** If $M$ is $I_4$-free, triangle-free, then $M$ is affine if and only if $M$ does not contain a $C_5$-restriction.

**Doublings.** Recall that a matroid $M = (E, G)$ is a doubling if there exists $w \in G \setminus E$ for $E = w + E$; we sometimes specify such an element $w$ and say that the matroid is a doubling by $w$. Note that the condition is equivalent to the existence of $w \in G \setminus E$ and a hyperplane $H \subseteq G$ not containing $w$ for which $E = [w] + (E \cap H)$. Hence we sometimes specify such a hyperplane $H$ and say that $M$ is the doubling with respect to the matroid $M|H$ by $w$, if this condition holds. The following lemma
shows that doublings preserve critical number and the absence of most fixed induced submatroids ([2]).

**Lemma 2.3** ([2], Lemma 2.2). Let $M$ be the doubling of a matroid $M_0$. Then

- $\chi(M) = \chi(M_0)$, and
- if $N$ is a matroid that is not a doubling of another matroid, and $M_0$ contains no induced $N$-restriction, then neither does $M$.

In particular, the above lemma applies when $N$ is a triangle or $I_n$ for $n \geq 3$. We will sometimes write $D(M)$ to mean the resulting matroid from doubling $M$ and define $D^{k}(M) = D(D^{k-1}(M))$ recursively for $k \geq 2$.

**Expansion operations.** Suppose that $M$ is an affine matroid, so that there exists a hyperplane $H$ for which $E \subseteq G \setminus H$. Then $M' = (E', G')$ is a 0-expansion of $M$ if $G$ is a hyperplane of $G'$ and $E = E'$. Note that it is simply the embedding of $M$ in an $(n + 1)$-dimensional projective geometry. Clearly 0-expansions preserve critical number and $I_s$-freeness for any $s \geq 1$.

Still supposing that $M$ is affine with a hyperplane $H$ for which $E \subseteq G \setminus H$, $M' = (E', G')$ is a 1-expansion of $M$ if $G$ is a hyperplane of $G'$ and there exists $x \in G' \setminus G$ such that $E' = E \cup (x + H) \cup \{x\}$. Note any two 1-expansions of the same matroid are isomorphic. We have the following easy consequence.

**Lemma 2.4.** Let $M$ be an affine matroid, and let $M'$ be the 1-expansion of $M$. Let $s \geq 4$. Then,

- $M'$ is affine, and
- if $M$ is $I_s$-free, then so is $M'$.

*Proof.* Pick any $y \in G \setminus H$, and let $H' = \text{cl}(H \cup \{x + y\})$. Then $H'$ is a hyperplane of $G'$ for which $H' \cap E = \emptyset$, so $M'$ is affine.

Now, suppose towards a contradiction that $M$ is $I_s$-free but $M'$ is not, so that there exist $F = \{v_1, \ldots, v_s\} \subseteq E'$ for which $M'|\text{cl}(F) \cong I_s$. Note that for any distinct three elements $x, y, z \in E \setminus E$, we have that $x + y + z \in E'$, so $|E' \cap F| \leq 2$. Hence $|E \cap F| \geq 2$. Let $x, y \in E \cap F$, and pick any arbitrary $z \in E \setminus E$. But then $x + y + z \in E'$, a contradiction. \qed

**The matroid $AG^\otimes(n - 1, 2)$.** Let $n \geq 3$. Then $AG^\otimes(n - 1, 2)$ is the $(n+1)$-dimensional matroid $M = (E, G)$ with nested hyperplanes $H_0 \subseteq G_0 \subseteq G$ and $x \in G \setminus G_0$, $y \in G_0 \setminus H_0$ for which $E = G_0 \setminus H_0 \cup \{x, y, x+y\}$. In matroid terminology, $\{x, x+y\}$ is a *series pair*; any basis of $E$ must
include either $x$ or $x+y$. Moreover, the matroid $AG^\bullet(n-1,2)$ is a series extension of $AG(n-1,2)$; contracting the element $x$ or $x+y$ gives the matroid $AG(n-1,2)$. Moreover, if $n \geq 4$, then $\{x, x+y\}$ is the unique series pair. Note that $AG^\bullet(n-1,2)$ always contains an induced $C_5$-restriction. It is easy to verify that $AG^\bullet(n-1,2)$ is $I_4$-free, triangle-free, and has critical number exactly 2.

**AI$_4$-freeness.** For a matroid $M = (E, G)$, we say that $M$ is AI$_4$-free if for any basis $\{x_1, x_2, x_3, x_4\} \subseteq E$ of a four-dimensional subgeometry of $G$, there exists some $i \in \{1, 2, 3, 4\}$ such that $\sum_{j \neq i} x_j \notin E$. It is useful to note that AI$_4$-freeness is preserved under complementation, and this fact will be used without reference.

**Lemma 2.5.** A matroid $M = (E, G)$ is AI$_4$-free if and only if $M^c$ is AI$_4$-free.

**Proof.** Let $B = \{x_1, x_2, x_3, x_4\} \subseteq E$ be an independent set for which $\sum_{j \neq i} x_j \notin E$ for all $1 \leq i \leq 4$. Then consider the independent set $B' = \{w_1, w_2, w_3, w_4\}$ of $G \setminus E$ where $w_i = \sum_{j \neq i} x_j$. Then $\sum_{j \neq i} w_j \in E$ for $1 \leq i \leq 4$. □

**Triangle-freeness.** We will use the following two results concerning triangle-freeness.

**Lemma 2.6.** If a matroid $M = (E, G)$ is AI$_4$-free and triangle-free, then $M$ is affine.

**Proof.** As $M$ is AI$_4$-free, it contains no induced odd circuits of length 5 or more. Therefore $M$ contains no induced odd circuits and is affine by Theorem 2.1. □

**Lemma 2.7** ([2], Corollary 5.2). If $M = (E, G)$ is both triangle-free and $I_3$-free, then $(E, cl(E))$ is an affine geometry.

### 3. The non-affine case

In this section, we will consider the non-affine $I_4$-free, triangle-free matroids. The goal will be to prove the following.

**Theorem 3.1.** If $M = (E, G)$ is an $I_4$-free, triangle-free matroid, then either $M$ is affine, or $(E, cl(E)) \cong D^k(AG^\bullet(n-1,2))$ for $n \geq 3$ and $k \geq 0$.

The proof is by induction on $\dim(M)$. We will first prove the following lemma, which describes the case when we can find a hyperplane $H$ such that $M|H$ contains the doubling of an induced $C_5$-restriction.
This condition turns out to be very strong, as it implies that the matroid \( M \) is also a doubling.

**Lemma 3.2.** Let \( M = (E, G) \) be an \( I_4 \)-free, triangle-free matroid with a flat \( F \) for which \( M|F \cong C_5 \). Let \( w \in E \backslash F \). If \( M|\text{cl}(F \cup \{w\}) \) is the doubling of \( M|F \) with respect to \( w \), then \( M \) is also a doubling with respect to \( w \).

**Proof.** If \( \dim(M) = 5 \), then the result follows trivially, so suppose that \( \dim(M) > 5 \).

Suppose for a contradiction that \( M \) is not a doubling with respect to \( w \). This implies that there exists \( z \in E \backslash \text{cl}(F \cup \{w\}) \) such that \( w + z \notin E \). We will now show that the 6-dimensional matroid \( M|\text{cl}(F \cup \{w, z\}) \) contains an induced \( I_4 \)-restriction.

**3.2.1.** Let \( \{x_1, x_2, x_3\} \) be any three distinct elements of \( F \cap E \). Then \( x_1 + x_2 + x_3 + z \notin E \).

**Subproof:** Suppose not, so that \( x_1 + x_2 + x_3 + z \in E \); then the set \( \{x_1 + x_2 + x_3 + z, w + x_1, w + x_2, w + x_3\} \) is an induced \( I_4 \)-restriction.

We now fix a basis \( \{x_1, x_2, x_3, x_4\} \) of \( F \cap E \). Let \( x_5 = x_1 + x_2 + x_3 + x_4 \), so that \( F \cap E = \{x_1, x_2, x_3, x_4, x_5\} \). We may apply the above claim with \( \{x_1, x_2, x_3\}, \{x_1, x_4, x_5\}, \{x_2, x_4, x_5\} \) and \( \{x_3, x_4, x_5\} \) to obtain that \( x_1 + x_2 + x_3 + z, x_2 + x_3 + z, x_1 + x_3 + z, x_1 + x_2 + z \notin E \). Since \( x_1 + x_2 + x_3 \notin E \), this implies that \( \{x_1, x_2, x_3, z\} \) is an induced \( I_4 \)-restriction, a contradiction. This completes the proof.

The next lemma describes the situation in which there is a hyperplane \( H \) of \( G \) such that \( M|H \) is AG\( \times(n - 1, 2) \). This case turns out to be harder and requires a detailed case analysis.

**Lemma 3.3.** Let \( M = (E, G) \) be an \( n \)-dimensional, \( I_4 \)-free, triangle-free matroid, \( n \geq 5 \). If \( G \) has a hyperplane \( H \) so that \( M|H \cong \text{AG}_{\times}(n - 3, 2) \), then either

- \( E \subseteq H \),
- \( M \) is a doubling of \( M|H \), or
- \( M \cong \text{AG}_{\times}(n - 2, 2) \).

**Proof.** Since \( M|H \cong \text{AG}_{\times}(n - 3, 2) \), there exist a hyperplane \( F \) of \( H \), a hyperplane \( F' \) of \( F \), and elements \( W = \{w_0, w_1\} \subseteq (H \cup E) \backslash F \) so that \( M|H = (W \cup (y + F'), H) \) where \( w_0 + w_1 = y \). We consider the following cases.

**Case 1:** There exists \( z \in E \backslash H \) for which \( y + z \in E \).

We will make a series of straightforward observations to help understand the structure of \( M \).
3.3.1. \((z + F) \cap E = \{y + z\}\)

Subproof: If \(x \in F\setminus(F' \cup \{y\})\), then clearly \(x + z \notin E\) as \(\{x, z, x + z\}\) would be a triangle. If there exists \(x \in F'\) such that \(x + z \in E\), then \(\{y + z, x + z, x + y\}\) is a triangle. ■

3.3.2. For any triangle \(T = \{x_0, x_1, x_0 + x_1\} \subseteq F'\) and \(w \in W\), \(|(z + w + \{x_0 + y, x_1 + y, x_0 + x_1\}) \cap E| > 0\).

Subproof: If not, then \(\{w, z, x_0 + y, x_1 + y\}\) is an induced \(I_4\)-restriction. ■

3.3.3. For any distinct \(x, x' \in F', \{|x + z + w_0, x' + z + w_1\} \cap E| < 2\).

Subproof: If not, then \(\{x + z + w_0, x' + z + w_1, x + x' + y\}\) is a triangle. ■

When \(\dim(F') > 2\) we obtain the following observation.

3.3.4. Provided that \(\dim(F') > 2\), if there exists \(w \in W\) for which \(x_0 + w + z \in E\) for some \(x_0 \in F'\), then \(F' + w + z \subseteq E\).

Subproof: Let \(x \in F'\setminus\{x_0\}\). Since \(\dim(F') > 2\), we may select \(x_1, x_2 \in F'\) such that \(x_0 \notin \text{cl}(\{x_1, x_2\})\) and \(x = x_0 + x_1 + x_2\). Then the set \(\{z, x_0 + w + z, x_1 + y, x_2 + y\}\) is an induced \(I_4\)-restriction if \(x + w + z \notin E\). Therefore \(F' + w + z \subseteq E\). ■

At this point, it is helpful to consider the cases when \(\dim(F') > 2\) and \(\dim(F') = 2\) separately.

Case 1.1: \(\dim(F') > 2\).

We claim that \(M\) is a doubling of \(M|H\). By 3.3.2 there exists \(x \in F'\) and \(w \in W\) for which \(x + z + w \in E\), and by 3.3.4 \(w + z + F' \subseteq E\). By 3.3.3 \((F' + (y + w) + z) \cap E = \emptyset\). Hence, along with 3.3.1 we have that \(E \setminus H = \{z, y + z\} \cup (w + z + F')\). Recall that \(E \cap H = \{w, w + y\} \cup (y + F')\). Therefore we have
\[
E = (E \cap H) \cup (E \setminus H)
= \{w, w + y\} \cup (y + F') \cup \{z, y + z\} \cup (w + z + F')
= [y + w + z] + (\{w, w + y\} \cup (y + F')) = [y + w + z] + (E \cap H).
\]
Since \(y + w + z \notin E\), we conclude that \(M\) is a doubling of \(M|H\).

Case 1.2: \(\dim(F') = 2\).

By 3.3.2 there exists \(x \in F'\) such that \(x + z + w \in E\) for some \(w \in W\). Write \(F' = \text{cl}(x, x')\) for \(x' \in F'\). Note that \(x + z + w \in E\) implies by 3.3.3 that \((z + y + w + F' \setminus \{x\}) \cap E = \emptyset\).

If \(z + z + y + w \in E\), then 3.3.3 gives that \((z + w + F' \setminus \{x\}) \cap E = \emptyset\). Hence \(E \setminus H = \{z\} \cup (z + \{y, x + w, x + y + w\})\). Therefore
3.3.5. For any triangle \( T \subseteq F' \), \( T \cap E \neq \emptyset \).

**Subproof:** If not, then \( (y + T) \cup \{z\} \) is an induced \( I_4 \)-restriction. ■

3.3.6. For any \( x \in F' \), \( x + z \notin E \) if and only if \( (x + z + W) \cap E \neq \emptyset \). Moreover, in this case we have \( |(x + z + W) \cap E| = 1 \).

**Subproof:** To show the forward statement, if \( (x + z + W) \cap E = \emptyset \), \( \{x + y, w_0, w_1, z\} \) is an induced \( I_4 \)-restriction.

For the reverse direction, note that if \( x + z \in E \), then \( \{x + z, x + w + z, w\} \) would be a triangle for some \( w \in W \).

Finally, since there exists no \( t \in G \setminus H \) for which \( \{t, y + t\} \subseteq E \), it follows that \( (x + z + W) \cap E \neq \emptyset \) if and only if \( |(x + z + W) \cap E| = 1 \). ■

3.3.7. For any distinct \( x, x' \in F' \), \( |\{x + w_0 + z, x' + w_1 + z\} \cap E| < 2 \).

**Subproof:** If not, then \( \{x + w_0 + z, x' + w_1 + z, x + x' + y\} \) is a triangle. ■

Let us write \( X_0 = \{x \in F' \mid x + w_0 + z \in E\} \), \( X_1 = \{x \in F' \mid x + w_1 + z \in E\} \) and \( X_2 = \{x \in F' \mid x + z \in E\} \). By 3.3.5 \( X_2 \neq \emptyset \), and 3.3.6 implies that \( (X_0, X_1, X_2) \) partitions \( F' \). Moreover, by 3.3.7 \( |X_0| \neq 0 \) if and only if \( |X_1| = 0 \). Moreover, when we restrict to a triangle, we have the following.

3.3.8. If \( T \subseteq F' \) is a triangle, then

1. if \( |T \cap X_2| = 3 \), or
2. if \( |T \cap X_2| = 1 \) and \( |T \cap X_i| = 2 \) for some \( i \in \{0, 1\} \).

**Subproof:** By 3.3.5 \( |T \cap X_2| > 0 \). By 3.3.6 \( (x + z + W) \cap E = \emptyset \).
Suppose \( \text{(1)} \) does not hold, so that there exists \( x' \in T \) such that \( x' + z \notin E \). By \( \text{3.3.6} \) we know that there exists exactly one \( w \in W \) for which \( x' + z + w \in E \).

If \( x + x' + z \in E \) holds, then by \( \text{3.3.6} \) \((x + x' + z + W) \cap E = \emptyset \). But then, it follows that \( \{x + y, x' + y, x + x' + y, x' + w + z\} \) is an induced \( I_4 \)-restriction. Hence \( x + x' + z \notin E \).

By \( \text{3.3.6} \) and \( \text{3.3.7} \) it follows that \( x + x' + z + w \in E \) and \( x + x' + z + y + w \notin E \), which is \( \text{(2)} \).

The above claim will be enough to settle the case where \( \dim(F') = 2 \). To handle the case \( \dim(F') > 2 \), the following observation will be useful.

**3.3.9.** For either \( i \in \{0, 1\} \), there exist no 3-dimensional flat \( F_0 \subseteq F' \) and a triangle \( T \subseteq F_0 \) for which \( T \subseteq X_2 \) and \( F_0 \setminus T \subseteq X_i \).

**Subproof:** Assume for a contradiction that such a triangle \( T = \text{cl}(x_1, x_2) \) and a flat \( F_0 = \text{cl}(x_0, x_1, x_2) \) exist, \( x_0 \notin T \). But then \( \{x_0 + y, x_1 + y, z, x_0 + x_2 + w_i + z\} \) is an induced \( I_4 \)-restriction.

**Case 2.1:** \( \dim(F') = 2 \).

We will apply \( \text{3.3.8} \) to give a case analysis depending on the value of \( |F' \cap X_2| \).

If \( |F' \cap X_2| = 3 \), then we have that

\[
E = W \cup (y + \text{cl}(F' \cup \{y + z\})),
\]

so that \( M \cong \text{AG}_\#(n-2, 2) \).

If \( |F' \cap X_2| = 1 \) and \( |F' \cap X_i| = 2 \) for \( i \in \{0, 1\} \), let \( x' \in F' \setminus \{x\} \), and \( F'' = \text{cl}(\{x, x' + w_i, y + w_i + z\}) \). Then we have

\[
E = \{w_i, x + y\} \cup ((x + y + w_i) + F''),
\]

where \( F'' \cap E = \emptyset \). Hence \( M \cong \text{AG}_\#(n-2, 2) \).

**Case 2.2:** \( \dim(F') > 2 \).

We claim that \( F' \subseteq X_2 \).

We know from \( \text{3.3.8} \) that \( X_2 \neq \emptyset \). Fix \( v \in X_2 \).

Suppose towards a contradiction that there exists \( v' \in F' \setminus X_2 \), so that \( v' \in X_i \) for some \( i \in \{0, 1\} \).

By \( \text{3.3.8} \) it follows that \( v + v' \in X_i \). Since \( \dim(F') > 2 \), there exists \( v'' \notin \text{cl}(\{v, v'\}) \). If \( v'' \in X_i \), then \( \text{3.3.8} \) applied to triangles implies that \( v + v' + v'' \in X_2 \), \( v' + v'' \in X_2 \) and \( v + v'' \in X_i \), but this contradicts \( \text{3.3.9} \). Similarly, if \( v'' \in X_2 \), then applying \( \text{3.3.8} \) again implies that \( v + v' + v'' \in X_i \), \( v' + v'' \in X_i \), and \( v + v'' \in X_2 \). But then this contradicts \( \text{3.3.9} \) This shows that \( F' \subseteq X_2 \).
Since $F' \subseteq X_2$, we have $E \setminus H = (z + [F'])$, so that
\[ E = (E \cap H) \cup (E \setminus H) = W \cup (y + F') \cup (z + [F']) = W \cup (y + F''), \]
where $F'' = \text{cl}(\{y + z\} \cup F')$ and $F'' \cap E = \emptyset$. Therefore $M \cong AG^\times(n-2,2)$.

We can now prove the main theorem of this section, restated below.

**Theorem 3.4.** If $M = (E, G)$ is an $I_4$-free, triangle-free matroid, then either $M$ is affine, or $(E, \text{cl}(E)) \cong D^k(AG^\times(n-1,2))$ where $n \geq 3$ and $k \geq 0$.

**Proof.** We proceed by induction on $\dim(M)$. The cases where $\dim(M) = 1, 2, 3$ are routine to check, so we may assume that $\dim(M) \geq 4$. We may assume without loss of generality that $M$ is full-rank.

Suppose that $M$ is not affine, so that it contains $C_5$ as an induced restriction. If $\dim(M) = 4$, then $M \cong C_5 = AG^\times(2,2)$, so suppose that $\dim(M) > 4$. By extending a basis of such an induced $C_5$-restriction, we can select a hyperplane $H$ of $G$ such that $M|H$ contains $C_5$ as an induced restriction, and $M|H$ is full-rank. By the inductive hypothesis, we have that $M|H \cong D^k(AG^\times(n-1,2))$ for some $n \geq 3$ and $k \geq 0$.

If $k \geq 1$, then $H$ contains the doubling of an induced $C_5$-restriction. By Lemma 3.2, we have that $M$ is the doubling, say by $w$, of $M|H'$ for some hyperplane $H'$. Note that $M|H'$ is not affine, as the doubling of an affine matroid is always affine. Hence we may apply the inductive hypothesis to $M|H'$ to obtain the required result in this case.

If $k = 0$ then Lemma 3.3 gives the required result.

As an immediate corollary, this shows that the $I_4$-free, triangle-free matroids have critical number at most 2.

**Corollary 3.5.** If $M$ is $I_4$-free and triangle-free, then $\chi(M) \leq 2$.

**Proof.** By Theorem 3.4, it follows that $\chi(M) = 1$, or $(E, \text{cl}(E)) \cong D^k(AG^\times(n-1,2))$ where $n \geq 3$ and $k \geq 0$. Note that $AG^\times(n-1,2)$ has critical number 2 for $n \geq 3$. Since doublings preserve critical number, it follows in the latter case that $\chi(M) = 2$.

4. **AI$_4$-freeness**

In order to understand the structure of $I_4$-free, triangle-free matroids with critical number exactly 1, it is helpful to consider the notion of AI$_4$-freeness and our goal of this section is to prove the following.
Lemma 4.1. If $M = (E, G)$ is $A_I$-free, then there exists a hyperplane $H$ of $G$ such that either $E$ or $G \setminus E$ is contained in either $H$ or $G \setminus H$; that is, either
\begin{itemize}
  \item $E \subseteq H$ ($M$ is rank-deficient),
  \item $G \setminus E \subseteq H$ ($M^c$ is rank-deficient),
  \item $E \cap H = \emptyset$, or
  \item $H \subseteq E$.
\end{itemize}

We first state an important lemma used repeatedly in this section. Note that translates of $U$ include $U$ itself; when we do not want to include $U$ itself, we consider cosets instead.

Lemma 4.2. For every matroid $M = (E, G)$, there is a flat $U$ of $G$ for which $U^c = E + E^c$ and $E$ is a union of translates of $U$. Moreover, if $|E| \geq 2$ then $U \subseteq E + E$, and if $|E| \leq |G| - 2$ then $U \subseteq E^c + E^c$.

Proof. For each set $A \subseteq |G|$, let $S_A = \{s \in |G| : s + A = A\}$; this is a subspace of $|G|$, so has cardinality a power of 2, and since $A$ is a disjoint union of translates of $S_A$ the ratio $|A|/|S_A|$ is an integer. Thus, if $|A|$ is odd, the subspace $S_A$ is trivial.

Since the statement of the lemma is identical when $E$ is replaced by $E^c$, and $|E|$ is odd, we may assume that $|E|$ is even; thus $|E^c|$ is odd, so $S_{E^c}$ is trivial. We show that the flat $U = S_{E^c}\{0\}$ satisfies the lemma.

First, let $e + f \in E + E^c$, where $e \in E$ and $f \in E^c$. If $e + f \in [U]$, then $f = (e + f) + e \in (e + f) + E = E$, a contradiction. Therefore $E + E^c \subseteq G\{[U]\} = U^c$.

Now, let $a \in U^c$. If $a \in E$, then since $S_{E^c}$ is trivial, we have $a + E^c \neq E^c$, so there is some $f \in E^c$ such that $a + f \in G \setminus E^c$; since $a \neq f$ we have $a + f \neq 0$ and so $a + f \in E$. Therefore $a \in E + f \subseteq E + E^c$. If $a \in E^c$, then since $a \notin U$, we have $a + E \neq E$ and so there is some $e' \in E$ for which $a + e' \notin E$. Since $a \neq e'$ we have $a + e' \neq 0$ and so $a + e' \in E^c$; this gives $a \in E + E^c$.

The last two arguments give $U^c = E + E^c$ as required. Since $U = S_{E^c}\{0\}$, $E$ is a union of some translates of $U$, which also implies that if $|E| \geq 2$ then $U \subseteq E + E$, and if $|E| \leq |G| - 2$ then $U \subseteq E^c + E^c$. \[\square\]

The proof of Lemma 4.1 will follow from the following lemmas.

Lemma 4.3. Let $M = (E, G)$ be an $A_I$-free matroid of dimension at least 5, and $H$ be a hyperplane of $G$ such that $|E \setminus H| \leq 1$. Then $G$ has a hyperplane $H'$ such that either $E \subseteq H'$, $E \cap H' = \emptyset$, or $H' \subseteq E$.

Proof. We may assume that $M$ is full-rank and that $|E \setminus H| = 1$, as otherwise the result is trivial. Let $\{v\} = E \setminus H$. Note that $M|H$ must be full-rank, as otherwise, $r(E) \leq r(E \cap H) + 1 \leq \dim(G) - 1$, so
\(M\) is not full-rank. Also, we have \(\sum_{x \in J} x \in E\) for each three-element linearly independent subset \(J\) of \(E \cap H\), since otherwise \(J \cup \{v\}\) would violate \(AI_4\)-freeness. In particular, the matroid \(M|H\) is \(I_3\)-free.

Hence, if \(M|H\) is triangle-free, it follows from Lemma 2.7 that \(M|H\) is an affine geometry, so that \(E \cap H = H\setminus K\) where \(K\) is a hyperplane of \(H\). Then for any \(x \in H\setminus K\), we have that \(E \cap H' = \emptyset\) where \(H' = \text{cl}(K \cup \{v + x\})\) is a hyperplane of \(G\), giving us the required result. Therefore, we may assume that \(M|H\) contains a triangle \(T = \{x_1, x_2, x_1 + x_2\}\), but then for any \(x_3 \in (E \cap H)\setminus T\) it follows from the above observation that \(\{x_1 + x_2 + x_3, x_1 + x_3, x_2 + x_3\} \subseteq E\). Since \(M|H\) is full-rank, it follows that \(H \subseteq E\).

\[\square\]

**Lemma 4.4.** Let \(M = (E,G)\) be an \(AI_4\)-free matroid of dimension at least 5, let \(H\) be a hyperplane of \(G\), and let \(F\) be a hyperplane of \(H\) such that \(E \cap H \subseteq F\). Then there exists a hyperplane \(H'\) of \(G\) such that either \(E\) or \(G\setminus E\) is contained in either \(H'\) or \(G\setminus H'\).

**Proof.** Suppose that no such \(H'\) exists. In particular, \(M\) is full-rank (since otherwise any hyperplane \(H'\) containing \(\text{cl}(E)\) satisfies \(E \subseteq H'\)), and \(|E \cap F| \geq 1\) (since otherwise \(E \subseteq G\setminus H\)). We will first prove the lemma in the case where \(|E \cap F| > 1\).

**Case 1:** \(|E \cap F| > 1\)

Let \(A_0 = H\setminus F\), and let \(A_1, A_2\) denote the two remaining cosets of \(F\) in \(G\). By assumption, we have \(A_0 \cap E = \emptyset\).

**4.4.1.** Let \(v_1 \in E \cap A_1\) and \(v_2 \in E \cap A_2\). If \(u, u'\) are distinct elements of \(F\) with \(u' \in E\), then \(|\{v_1 + v_2, u + u'\} \cap E| \neq 1\).

**Subproof:** Note that \(E \cap (F + v_1 + v_2) = E \cap A_0 = \emptyset\). If \(u + u' \in E\) while \(u + v_1, u + v_2 \notin E\), then \(\{v_1, v_2, u, u'\}\) would violate \(AI_4\)-freeness. If \(u + v_1 \in E\) for some \(i \in \{1, 2\}\) while \(u + u', u + v_{3-i} \notin E\), then \(\{u', v_1, v_2, v_1 + u\}\) would violate \(AI_4\)-freeness.

We may assume that \(|A_1 \cap E| > 1\); if not, we can take a hyperplane \(H' = \text{cl}(F \cup A_2)\), so that \(|E \setminus H'| \leq 1\), and Lemma 4.3 gives a contradiction. Similarly, we may assume \(|A_2 \cap E| > 1\). If \(A_1 \cap E \neq A_i\), select \(v_i \in A_i \setminus E\), and otherwise, choose \(v_i \in A_i \cap E = A_i\), for \(i = 1, 2\). Note that at most one of \(v_1 \in E\) and \(v_2 \in E\) holds; otherwise, \(A_1, A_2 \subseteq E\), so that choosing \(H' = F \cup A_0\) gives \(G\setminus E \subseteq H'\).

Let \(X_i = (v_i + (A_i \cap E)) \setminus \{0\}\) for \(i = 1, 2\). Note that \(X_i \subseteq F\) and \(|X_i| > 1\) for each \(i = 1, 2\) by our choice of \(v_i\). Write \(X_0 = F \cap E\). Note that \(|X_0| > 1\).
Now, 4.4.1 implies
\[
(X_0 + X_0) \cap (X_1 + X_1) \cap (X_2 + X_2) = \emptyset
\]
\[
(X_0 + X_0) \cap (X_1 + X_1) \cap (X_2 + X_2) = \emptyset
\]
\[
(X_0 + X_0) \cap (X_1 + X_1) \cap (X_2 + X_2) = \emptyset
\]
We may now apply Lemma 4.2 (with \(F\) being the ambient space), to obtain flats \(U_i\) for which \(X_i + X_i = U_i\), and \(X_i\) is a union of translates of \(U_i\) for each \(i = 0, 1, 2\) provided \(U_i\) is not empty. Moreover, since \(|X_i| > 1\), we have \(U_i \subseteq X_i + X_i\) for \(i = 0, 1, 2\).

Hence, for any distinct \(i, j, k \in \{1, 2, 3\}\), we have
\[
(X_i + X_i) \subseteq U_j \cup U_k
\]
Recall that \(U_i \subseteq X_i + X_i\) for \(i = 0, 1, 2\). Therefore, for any distinct \(i, j, k \in \{1, 2, 3\}\) we obtain
\[
U_i \subseteq U_j \cup U_k
\]
Note that each of \(U_1, U_2, U_3\) is a flat. This implies that at least two of \(U_0, U_1, U_2\) are identical and the third is contained in the other two. Let us write \(U_i \subseteq U_j = U_k\) for some \(i, j, k \in \{0, 1, 2\}\). Let \(U = U_j = U_k\).

Since \(U \subseteq X_j + X_j\) and \(X_j + X_j \subseteq U \cup U = U\), it follows that \(U = X_j + X_j\), and similarly \(U = X_k + X_k\). We also have \(X_i + X_i \subseteq U\).

In particular, since \(|X_j| > 1\), \(U\) is non-empty. Since \(U = X_j + X_j\) and \(X_j\) is a union of translates of \(U\), it follows that \(X_j\), and similarly \(X_k\), equal a translate of \(U\) in \(F\). In a similar vein, since \(X_i + X_i \subseteq U\), \(X_i\) is contained in a translate of \(U\) in \(F\). To summarise, we have the following.

4.4.2. There exists a non-empty flat \(U \subseteq F\) and \(j, k \in \{0, 1, 2\}, j \neq k\), such that
\[
\cdot \ X_j \ \text{and} \ X_k \ \text{equal a translate} \ U.
\]
\[
\cdot \ X_i \ \text{is contained in a translate of} \ U.
\]
where \(i \in \{0, 1, 2\} \setminus \{j, k\}\).

We say that a set \(X\) is full if it equals a translate of \(U\). Hence, at least two of \(X_0, X_1, X_2\) are full.

We now consider two cases depending on whether \(v_1 \in E\) and \(v_2 \in E\) (recall that at most one of the two can hold at the same time). In the case \(v_1 \in E\) or \(v_2 \in E\), we may assume, by symmetry, that \(v_2 \in E\) and \(v_1 \notin E\).

Case 1.1: \(v_1, v_2 \notin E\)
4.4.3. At most two of \(X_0, X_1, X_2\) are contained in \(U\). Moreover, if precisely two of \(X_0, X_1, X_2\) are contained in \(U\), then \(X_0 \subseteq U\).

Subproof: Suppose first for a contradiction that \(X_i \subseteq U\) for each \(i = 0, 1, 2\). If \(X_1, X_2\) are all full, then since \(|X_0| > 1\), we may select two elements \(y_1, y_2 \in X_0\), then \(\{y_1, y_2, v_1 + y_1, y_2, v_2 + y_1 + y_2\}\) would violate \(AI_4\)-freeness by 4.4.1. If \(X_1\) is not full, then we may select \(x_1 \in X_1, y_1 \notin X_1\), so that \(\{x_1, y_1, x_1 + v_1, x_1 + y_1 + v_2\}\) would violate \(AI_4\)-freeness by 4.4.1. Similarly, if \(X_2\) is not full, a symmetrical argument shows that it would violate \(AI_4\)-freeness, giving a contradiction.

Suppose next that precisely two of \(X_0, X_1, X_2\) are contained in \(U\). For a contradiction, suppose that \(X_0\) is not contained in \(U\), so that \(X_1, X_2 \subseteq U\), and \(X_0\) is contained in a coset of \(U\). If \(X_1, X_2\) are full, then select two elements \(y_1, y_2 \in X_0\), then \(\{y_1, y_2, v_1 + y_1, y_2, v_2 + y_1 + y_2\}\) would violate \(AI_4\)-freeness by 4.4.1. If \(X_1\) is not full, then select \(x_1 \in X_1, y_1 \notin X_1\) and \(z_1 \in X_0\). Then \(\{x_1 + v_1, x_1 + y_1 + v_2, z_1, x_1 + y_1 + z_1\}\) would violate \(AI_4\)-freeness by 4.4.1. By symmetry, the case where \(X_2\) is not full follows, giving a contradiction in all cases.

We are now ready to complete the analysis of Case 1.1. In each of the possible outcomes resulting from 4.4.3, we will show that we can select a hyperplane \(H'\) of \(G\) that satisfies the theorem or find an induced \(I_4\)-restriction, giving a contradiction.

Suppose first that none of \(X_0, X_1, X_2\) is contained in \(U\). Let \(X_i \subseteq B_i\) where \(B_i\) is a coset of \(U\) for \(i = 0, 1, 2\). If \(B_0 = B_1 = B_2\), then we may assume that there is no other coset of \(U\), as otherwise \(M\) is rank-deficient. But then, we may select \(H' = \text{cl}(U \cup \{v_1, v_2\})\), and we have \(E \subseteq G \setminus H'\). Therefore we may assume that \(B_0, B_1, B_2\) are not identical, so we may assume without loss of generality that \(B_0 \neq B_1\). But then \(B_1 \cap \text{cl}(E) = \emptyset\) (to see this, note that a general element \(z\) of \(\text{cl}(E)\) has the form \(v_1 + x_1 + x_2 + y\) where \(x_0 \in B_0 \cup \{0\}, x_1 \in B_1 \cup \{v_1\}, x_2 \in B_2 \cup \{v_2\}, y \in U \cup \{0\}\), and hence \(z \notin B_1\) as otherwise it would force \(x_1 = v_1, x_2 = v_2\), giving \(z \in U \cup B_0\)). Therefore \(M\) is rank-deficient.

Suppose next that precisely one of \(X_0, X_1, X_2\) is contained in \(U\). First, suppose that \(X_0 \subseteq U\). Then, it follows in a similar way that \(B_1 \cap \text{cl}(E) = \emptyset\). Therefore, \(M\) is rank-deficient. Hence we may assume without loss of generality that \(X_1 \subseteq U\), and \(X_0 \subseteq B_0\) and \(X_2 \subseteq B_2\) for (possibly identical) cosets \(B_0, B_2\) of \(U\). Note that we may assume that the only cosets are \(B_0, B_2, B_0 + B_2\), as otherwise \(M\) is rank-deficient. Select \(x \in B_0\), and let \(H' = \text{cl}(U \cup (B_0 + B_2) \cup \{v_1 + x, v_2\})\). Then we have that \(E \subseteq G \setminus H'\).
Finally, we consider the case where precisely two of \(X_0, X_1, X_2\) are contained in \(U\). Suppose without loss of generality that \(X_0, X_1 \subseteq U\) and \(X_2 \subseteq B_2\) where \(B_2\) is a coset of \(U\). But then it follows that 
\[B_2 \cap \text{cl}(E) = \emptyset.\] So \(M\) is rank-deficient.

**Case 1.2:** \(v_1 \notin E, v_2 \in E\).

The fact that \(v_2 \in E\) means that \(A_2 \subseteq E\) by our choice of \(v_2\). Hence \(X_2 = F\), and therefore \(U_2 = F\). Hence, either \(U_0 = F\) or \(U_1 = F\). If \(U_0 = F\), then choosing the hyperplane \(H' = \text{cl}(F \cup A_2)\), we have that \(G \setminus E \subseteq G \setminus H'\). Hence we may assume that \(U_1 = F\). We may also assume that \(F \cap E \neq E\), as otherwise \(H' = \text{cl}(F \cup A_2)\) satisfies \(G \setminus E \subseteq G \setminus H'\) again. Let \(w_1 \in F \setminus E, w_2 \in F \cap E\). Then \(\{v_1 + w_1, w_2, v_2, v_2 + w_1 + w_2\}\) would violate \(AI_4\)-freeness by 4.4.1.

**Case 2:** \([E \cap F] = 1\)

Choose \(F''\) to be a hyperplane of \(F\) such that \(F'' \cap E = \emptyset\). Let \(F' = \text{cl}(F'' \cup \{z\})\) for any \(z \in A_0\) so that \(F' \cap E = \emptyset\), and consider the three cosets of \(F'\) in \(G\), denoted \(A_0', A_1', A_2'\) where we take \(A_0'\) so that \(|A_0' \cap E| = 1\). Let \(v \in A_0' \cap E\).

If \(|A_i' \cap E| > 1\) and \(A_i' \cap E\) is not full-rank, then we may select a hyperplane \(H'\) of \(F' \cup A_1'\) such that \(E \cap (F' \cup A_1') \subseteq H'\), and \(|E \cap H'| > 1\). Case 1 then applies, and the same holds with \(A_2'\). So we may assume without loss of generality that \(|A_i' \cap E| = 1\) or \(A_i' \cap E\) is full rank for each \(i = 1, 2\).

If \(|A_i' \cap E| = 1\) for \(i = 1, 2\), then because \(\text{dim}(M) \geq 5\), \(M\) is rank-deficient, so we may assume without loss of generality that \(A_i' \cap E\) is full-rank. Given three linearly independent vectors \(v_1, v_2, v_3 \in A_i' \cap E\), we must have that \(v_1 + v_2 + v_3 \in E\), as otherwise \(\{v, v_1, v_2, v_3\}\) would violate \(AI_4\)-freeness. Therefore, \(A_i' \subseteq E\), and let \(H' = A_i' \cup F''\). It is then easy to check that the conditions are met to apply Case 1 with the matroid \(M^c\) and the hyperplane \(H'\) to give the required result.

\[\square\]

We can now prove Lemma 4.1.

Proof of Lemma 4.1. Let \(M = (E, G)\) be a counterexample of smallest dimension. If \(\text{dim}(M) = 1, 2, 3\), then we obtain a contradiction from a routine check, hence we may assume \(\text{dim}(M) \geq 4\).

4.4.4. \(\text{dim}(M) \geq 5\). 

Proof. This is a tedious check. If \(\text{dim}(M) = 4\), then replacing \(M\) with \(M^c\) if necessary, we may assume that \(|E| \leq 7\). We may also assume that \(M\) is full-rank. Note that \(M\) needs to contain a \(C_4\)-restriction, on a hyperplane \(H\), since \(M\) is \(AI_4\)-free.
Suppose first that it is an induced $C_4$-restriction. Then $M|H \cong C_4$ and write $E \cap H = \{v_1, v_2, v_3, v_1 + v_2 + v_3\}$. Let $v_4 \in E \setminus H$. Then there exists $v \in E \cap H$ such that $v + v_4 \in E$, as otherwise $M'|\cl(\{v_1 + v_2, v_1 + v_3, v_1 + v_2 + v_3 + v_4\})$ is an $F_7$-restriction. Without loss of generality, suppose that $v_1 + v_4 \in E$. Now, we must have that $v_2 + v_3 + v_4 \in E$ or $v_1 + v_2 + v_3 + v_4 \in E$, as otherwise, $\{v_2, v_3, v_4, v_1 + v_4\}$ would violate $AI_4$-freeness. If $v_2 + v_3 + v_4 \in E$, then $\{v_1, v_2, v_1 + v_3 + v_4\}$ would violate $AI_4$-freeness. If $v_1 + v_2 + v_3 + v_4 \in E$, then $\{v_1, v_2, v_4, v_1 + v_2 + v_3 + v_4\}$ would violate $AI_4$-freeness.

Hence we may assume that it has no induced $C_4$-restriction. Suppose $|H \cap E| = 6$. Recall that $M$ is full-rank and $|E| \leq 7$. We may take $v_4 \in E \setminus H$, and $v_1, v_2, v_3 \in E \cap H$ for which $v_1 + v_2 + v_3 \notin E$, and $\{v_1, v_2, v_3, v_4\}$ would violate $AI_4$-freeness. Hence we may assume $|E \cap H| = 5$. Let $E \cap H = \{v_1, v_2, v_3, v_1 + v_2, v_1 + v_2 + v_3\}$, and pick $v_4 \in E \setminus H$. By symmetry and the fact that $|E| \leq 7$, we may assume that $v_1 + v_4 \notin E$. It follows that $v_1 + v_2 + v_3 + v_4 \in E$, as otherwise $\{v_2, v_3, v_1 + v_2, v_4\}$ would violate $AI_4$-freeness. But then $\{v_1, v_2, v_4, v_1 + v_2 + v_3 + v_4\}$ violates $AI_4$-freeness.

Hence we have that $\dim(M) \geq 5$. Let $k = \dim(M)$. By minimality, for every hyperplane $H$ of $G$, $H$ contains a hyperplane that satisfies one of the four outcomes. If, for any hyperplane $H$ of $G$, the conditions of Lemma 4.4 are satisfied, then Lemma 4.3 provides a contradiction. Hence, we may assume that, for every hyperplane $H$ of $G$, either $M|H$ or $M^c|H$ contains a PG($k - 3, 2$)-restriction, the projective geometry of dimension $k - 2$.

Moreover, since $\dim(M) = k \geq 5$, if $M$ contains a PG($k - 3, 2$)-restriction, then $M^c$ cannot contain a PG($k - 3, 2$)-restriction, as otherwise we would have $\dim(M) \geq 2(k - 2)$, which implies $k \leq 4$. By switching to $M^c$ if necessary, we may therefore suppose that $M^c$ contains a PG($k - 3, 2$)-restriction in every hyperplane. Now, observe that $M$ is triangle-free, since otherwise any hyperplane containing such a triangle cannot contain a PG($k - 3, 2$)-restriction in $M^c$. Therefore, $M$ is both $AI_4$-free and triangle-free, so by Lemma 2.6 it follows that $M$ is affine. 

The rest of this section describes a structural theorem for $AI_4$-free matroids, which will not be used in the proof of our main theorem Theorem 4.1. For the main theorem, we will only use Lemma 4.1.

In light of Lemma 4.1, we define the following four operations for a given $n$-dimensional matroid $M = (E, G)$, which we denote by $\alpha_0$, $\alpha_1$, $\beta_0$ and $\beta_1$. 
First note that all such matroids described are indeed embeddings of $M$ in a projective geometry of dimension $n+1$.

- $\alpha_0(M)$ is the 0-expansion of $M$ (recall that the 0-expansion is an embedding of $M$ in a projective geometry)
- $\alpha_1(M) = (E', G')$ is the $(n+1)$-dimensional matroid such that a copy of $G$ is embedded in $G'$, and $E' = E \cup (G' \setminus G)$
- $\beta_0(M) = (E', G')$ is the $(n+1)$-dimensional matroid with a copy of $G$ embedded in $G'$ and $E' = (w + E) \cup \{w\}$ for $w \in G' \setminus G$
- $\beta_1(M) = (E', G')$ is the $(n+1)$-dimensional matroid with a copy of $G$ embedded in $G'$ such that $E' = G \cup (w + E) \cup \{w\}$ for $w \in G' \setminus G$.

We now state some straightforward facts about these four operations, all of which are easy to verify, and hence the proof is omitted.

**Lemma 4.5.** Let $M = (E, G)$ be a matroid. Then the following hold.

1. For $\gamma \in \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$, if $\gamma(M)$ is $AI_4$-free then $M$ is $AI_4$-free.
2. For $\gamma \in \{\alpha_0, \alpha_1\}$, if $M$ is $AI_4$-free then $\gamma(M)$ is $AI_4$-free.
3. For $\gamma \in \{\alpha_0, \alpha_1\}$, $M$ is $I_3$-free if and only if $\gamma(M)$ is $I_3$-free.
4. For $\gamma \in \{\beta_0, \beta_1\}$, if $M$ is $AI_4$-free and $I_3$-free then $\gamma(M)$ is $AI_4$-free.
5. For $\gamma \in \{\beta_0, \beta_1\}$, if $M$ contains an induced $I_3$-restriction, then $\gamma(M)$ contains an independent set $\{x_1, x_2, x_3, x_4\} \subseteq E$ for which $\sum_{j \neq i} x_j \notin E$ for all $i$ (i.e., $\gamma(M)$ is not $AI_4$-free).
6. $\beta_0(M)$ is $I_3$-free.
7. $\beta_0(M)$ contains an induced $I_3$-restriction unless $E$ is a flat.
8. If $E$ is a flat, then $\beta_0(M) = \gamma_k \cdots \gamma_1(N_0)$ where $N_0$ is a 1-dimensional matroid, and $\gamma_i \in \{\alpha_0, \alpha_1\}$ for $1 \leq i \leq k$.

Using this lemma, we can prove the following structure theorem for $AI_4$-free matroids, stated below.

**Theorem 4.6.** The class of $AI_4$-free matroids is the union of the following two classes. $N_0$ denotes the set of 1-dimensional matroids.

1. $\mathcal{M}_0 = \{\gamma_k \cdots \gamma_0(N_0) \mid k \geq 0, \lambda_i \in \{\alpha_0, \alpha_1, \beta_1\}, N_0 \in N_0\}$
2. $\mathcal{M}_1 = \{\gamma_k \cdots \gamma_0\beta_0(M) \mid k \geq 0, \lambda_i \in \{\alpha_0, \alpha_1\}, M \in \mathcal{M}_0\}$

**Proof.** First note that all such matroids described are indeed $AI_4$-free by statements 2, 3, 4, 6 in Lemma 4.5.

Let $M = (E, G)$ be an $AI_4$-free matroid. By applying Lemma 4.1 and statement 1 in Lemma 4.5 iteratively, there is a sequence of operations $\gamma_i$ such that $M = \gamma_k \gamma_{k-1} \cdots \gamma_1(E)$, and $\gamma_i \in \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$ for $i = 1, \ldots, k$. Let us write $M_l = (E_l, G_l) = \gamma_l \gamma_{l-1} \cdots \gamma_1(E)$ and by construction each $M_l$ is $AI_4$-free.

Now, suppose that $\gamma_l$ is the first occurrence of $\beta_0$, if there is any, so that $\gamma_j \in \{\alpha_0, \alpha_1, \beta_1\}$ for $j = 1, 2, \ldots, l - 1$, and $\gamma_l = \beta_0$. We may
Theorem 5.1. For a full-rank matroid $M = (E, G)$, $M$ is $I_4$-free and triangle-free if and only if

- $M$ can be obtained by a sequence of 0-expansions and 1-expansions of a 1-dimensional matroid, or
- $M$ can be obtained by a sequence of doublings of $AG^{\times}(n-1, 2)$, $n \geq 3$.

Proof. The backward direction follows from Lemmas 2.3, 2.4.

We now prove the forward direction. By Theorem 3.1, $M$ can either be obtained by a sequence of doublings of $AG^{\times}(n-1, 2)$ or $M$ is affine. If the former case holds then we are done, so we may suppose that $M$ is affine, so that there exists a hyperplane $H$ of $G$ for which $E \subseteq G \setminus H$.

If $M$ is the empty matroid, then the result is trivially true, so suppose that $E \neq \emptyset$. Pick $z \in E$, and consider the matroid $M_0 = (F, H)$ where $F = \{v \mid v + z \in E\}$. Since $M$ is $I_4$-free, it follows that $M_0$ is $I_3$-free. Moreover, since $M$ is affine, it follows that $M_0$ is $AI_4$-free.

Let $H'$ be the hyperplane of $H$ from the conclusion of Lemma 4.1. We will now go through each conclusion of Lemma 4.1 to see that in each of the cases, we obtain a 0-expansion or a 1-expansion, proving the result.

Case 1: $F \subseteq H'$.

In this case, let $H'' = \text{cl}(H' \cup \{z\})$, so that $H''$ is a hyperplane of $G$. Then $M|H''$ is an affine matroid with $H'$ satisfying $E \cap H'' \subseteq H'' \setminus H'$. Then

$$E = \{z\} \cup (z + F) \subseteq \{z\} \cup (z + H') \subseteq H''.$$ 

Therefore, $M$ is the 0-expansion of the affine matroid $M|H''$. 

Case 2: $H \setminus F \subseteq H'$.
Let $H'' = \text{cl}(H' \cup \{z\})$ as before. Let $w \in H \setminus H'$. Then

$$E = \{z\} \cup (z + F)$$
$$= \{z\} \cup (z + F \cap H') \cup (z + F \setminus H')$$
$$= \{z\} \cup (z + H') \cup (z + H \setminus H')$$
$$= (E \cap H'') \cup \{z + w\} \cup (z + w + H')$$

Therefore, $M$ is the 1-expansion of the affine matroid $M|H''$.

Case 3: $F \cap H' = \emptyset$.
Note that if $M_0$ is rank-deficient, then we are in Case 1, so assume that $M_0$ is full-rank. Observe that $M_0$ is $I_3$-free and triangle-free (since it is affine). Therefore, Lemma 2.7 implies that $M_0$ is a full-rank affine geometry. We are in Case 2.

Case 4: $H' \subseteq F$.
Let $w \in F \setminus H'$ (if no such $w$ exists, then $M_0$ is rank-deficient and we are in Case 1), and let $H'' = \text{cl}(H' \cup \{z + w\})$.

Then $M|H''$ is an affine matroid with $H'$ as its hyperplane such that $H'' \cap E \subseteq H'' \setminus H'$. Then

$$E = \{z\} \cup (z + F)$$
$$= \{z\} \cup (z + F \cap H') \cup (z + F \setminus H')$$
$$= \{z\} \cup \{z + H'\} \cup \{z + w\} \cup (z + w + F \cap H')$$
$$= \{z\} \cup \{z + H'\} \cup (E \cap H'')$$

Therefore, $M$ is the 1-expansion of the affine matroid $M|H''$.

Thus $M$ is either the 0-expansion or 1-expansion of another affine matroid of smaller dimension. The result now follows by induction on $\dim(M)$. \qed

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