The solution of some nonlinear space-time fractional Fokker-Planck equations by using homotopy perturbation method

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Abstract. In this work, the homotopy perturbation method together with the natural transform are applied to solve some nonlinear space-time fractional Fokker-Planck equations. The numerical and exact solutions in examples are shown to support our results.

1. Introduction

The generalized nonlinear Fokker-Planck equation (FPE), a partial differential equation introduced by Adrian Fokker and Max Planck [1] that describes the time evolution of the probability density function \( u(x, t) \) of space \( x \) and time \( t \), can be written as [2]

\[
\frac{\partial}{\partial t} u(x, t) = \left[ -\frac{\partial}{\partial x} A(x, t, u) + \frac{\partial^2}{\partial x^2} B(x, t, u) \right] u(x, t) \tag{1}
\]

where \( A \) and \( B > 0 \) are the drift and diffusion coefficients, respectively.

This equation has been applied in many branches of natural science [3, 4] such as chemical physics, solid-state physics or circuit theory. Several researchers have tried to find its solutions by using different methods such as variational iteration method (VIM), homotopy perturbation method (HPM) or Adomian decomposition method (ADM).

In recent years, many researchers were interested in fractional partial differential equations. They proposed a number of new methods for solving these equations. In 2007, Odibat and Momani [5] found the solution of the space-time fractional FPE by using VIM and ADM. The analytical solution obtained was a convergent series. They concluded that VIM and ADM are very powerful and efficient to find the solution. In 2011, Garg and Manohar [6] proposed a generalized differential transform method (GDTM) for solving the space-time fractional equation. The results showed that the method is a powerful and efficient technique. In 2012, homotopy analysis method (HAM) introduced by Garg and Choudhary [7] was presented to find solutions of fractional FPE. These solutions are given in the form of a series with easily computable components. This research showed that the process requires less computational work than others. Later in 2012, a homotopy perturbation transform method (HPTM) was presented by Liu [8]. The result obtained from this HPTM converged to the exact solution as same as obtained from the ADM and VIM. In 2013 [9], Yan introduced iterative Laplace transform method to solve space-time fractional FPE. This method gave a converge solution in which each term in series is...
easily computed and required no linearization or small perturbation. Recently in 2015, Dubey et al. [10] proposed a homotopy perturbation Sumudu transform method (HPSTM) to find the solution of space-time FPE. The result obtained using the HPSTM converged to the exact solution similar to that of the ADM. In 2016, homotopy perturbation natural transform method (HPNTM) was introduced by Gupta and Kumar [11] which considered to be a combination of the homotopy perturbation method, natural transform and He’s polynomials. This technique was applied to find the solution of space-time fractional FPE. The numerical results showed that this technique is effective and accurate.

In this work, the homotopy perturbation method and natural transform is applied to find the solution of some nonlinear space-time fractional Fokker-Planck equations:

\[ D_α^t u(x, t) = \left[ -D_β^x A(x, t, u) + D_β^2 x B(x, t, u) \right] u(x, t) \]  

with the initial condition

\[ u(x, 0) = f(x), \quad x \in \mathbb{R}, \]  

where \( 0 < \alpha, \beta < 1 \) are parameters denote the order of the fractional time and space derivatives, respectively. If \( \alpha = \beta = 1 \), then (2) reduces to (1).

2. Preliminaries

The fractional partial differential equation will be explained in the Caputo sense and the natural transform will be used in the process of solving the FPE.

2.1. Definition of fractional derivative

The fractional derivative of order \( \alpha \) of a function \( u(x, t) \) in the sense of Caputo is defined by

\[ D_α^t u(x, t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\eta)^{m-\alpha-1} D^m_\eta u(\eta, t) \, d\eta, \quad m-1 < \alpha < m, \quad \alpha \in \mathbb{R}, \quad m \in \mathbb{N} \]  

where \( \Gamma(\cdot) \) is the gamma function.

2.2. Definition of natural transform

Let \( A \) be a set of function \( f(x, t) \):

\[ A = \{ f(x, t) \mid \exists M, \tau_1, \tau_2 > 0 \text{ such that } |f(x, t)| < Me^{\|t\|/\tau_2}, \quad t \in (-1)^i \times [0, \infty) \} \]  

The natural transform of a function \( f(x, t) \) with respect to \( t \) is defined as

\[ F(x, s, v) = N^+[f(x, t)] = \int_0^\infty e^{-st} f(x, vt) \, dt, \quad s > 0, \quad v > 0. \]  

2.2.1. Natural transform of n-th derivative

is

\[ N^+[D^nf(x, t)] = \frac{s^n}{v^n} F(x, s, v) - \sum_{k=0}^{n-1} \left( \frac{s^{n-k-1}}{v^{n-k}} D^k_t f(x, 0) \right), \quad n \in \mathbb{N}. \]  

2.2.2. Natural transform of fractional derivative of order \( \alpha \)

is

\[ N^+[D_\alpha^t f(x, t)] = \frac{s^\alpha}{v^\alpha} F(x, s, v) - \sum_{k=0}^{n-1} \left( \frac{s^{\alpha-k-1}}{v^{\alpha-k}} D^k_t f(x, 0) \right), \quad n-1 < \alpha < n, \quad n \in \mathbb{N}. \]
3. Method and results
In this section, we will find the solution of a space-time fractional Fokker-Planck equation by using the natural homotopy perturbation method which the drift and diffusion coefficients of FPE are defined.

Consider the nonlinear space-time fractional Fokker-Planck equation:

\[ D_t^\alpha u = \left[-D_x^\beta A + D_x^{2\beta} B\right]u, \quad 0 < \alpha, \beta < 1, \]  

(9)

where

\[ A = ax^{-1} + bx \quad \text{and} \quad B = u, \quad a, b \in \mathbb{R}, \]  

(10)

with the initial condition \( u(x, 0) = x^{\alpha/2} \).

From (9) and (10), applying the natural transform with an initial condition, we have

\[ N^+\left[D_t^\alpha u\right] = N^+\left[-D_x^\beta (au^2 x^{-1} + bx) + D_x^{2\beta} u^2\right] \]  

(11)

\[ \frac{s^\alpha}{v^\alpha} N^+[u] - \frac{s^{\alpha-1}}{v^\alpha} x^{\alpha/2} = N^+\left[-D_x^\beta (au^2 x^{-1} + bx) + D_x^{2\beta} u^2\right] \]  

(12)

\[ N^+[u] = \frac{1}{s} x^{\alpha/2} + \frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (au^2 x^{-1} + bx) + D_x^{2\beta} u^2\right]. \]  

(13)

From (13), by using the inverse natural transform, we can get

\[ u = x^{\alpha/2} + N^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (au^2 x^{-1} + bx) + D_x^{2\beta} u^2\right]\right]. \]  

(14)

Applying the homotopy perturbation method [...] to (14), we can obtain a homotopy:

\[ H(U; p) = U - x^{\alpha/2} - pN^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (ax^{-1}(\sum_{i=0}^{\infty} p^i u_i)^2 + bx (\sum_{i=0}^{\infty} p^i u_i)) + D_x^{2\beta}(\sum_{i=0}^{\infty} p^i u_i)^2\right]\right] = 0, \]  

(15)

or

\[ U = x^{\alpha/2} + pN^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (ax^{-1}(\sum_{i=0}^{\infty} p^i u_i)^2 + bx (\sum_{i=0}^{\infty} p^i u_i)) + D_x^{2\beta}(\sum_{i=0}^{\infty} p^i u_i)^2\right]\right], \]  

(16)

where \( U = \sum_{i=0}^{\infty} p^i u_i \) is a solution of \( H(U; p) \) and \( p \) is an embedding parameter. Obviously, if \( p \to 1 \), (16) becomes (14) and \( u = \lim_{p \to 0} U = u_0 + u_1 + u_2 + ... \).

From (16), we have

\[ \sum_{i=0}^{\infty} p^i u_i = x^{\alpha/2} + pN^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (ax^{-1}(\sum_{i=0}^{\infty} p^i u_i)^2 + bx (\sum_{i=0}^{\infty} p^i u_i)) + D_x^{2\beta}(\sum_{i=0}^{\infty} p^i u_i)^2\right]\right]. \]  

(17)

By comparing the equating terms with identical powers of \( p \), we have

\[ p^0 : u_0 = x^{\alpha/2} \]  

(18)

\[ p^1 : u_1 = N^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (ax^{-1}(u_0^2) + bx u_0) + D_x^{2\beta}(u_0^2)\right]\right] \]

\[ = N^-\left[\frac{v^\alpha}{s^\alpha} N^+\left[-D_x^\beta (ax^{-1}(u_0^{\alpha/2}) + bx (x^{\alpha/2})) + D_x^{2\beta}(x^{\alpha/2})\right]\right]. \]
By setting $\beta = 1$, we have
\begin{equation}
    u(x,t) = x^{\alpha/2}E_\alpha(-b(\frac{a}{2} + 1)t^\alpha),
\end{equation}
where $E_\alpha(-b(\frac{a}{2} + 1)t^\alpha) = \sum_{k=0}^{\infty} \frac{(-b(\frac{a}{2} + 1)t^\alpha)^k}{\Gamma(ak + 1)}$ is a Mittag-Leffler function.

By setting $\alpha = \beta = 1$, (22) becomes
\begin{equation}
    u(x,t) = x^{\alpha/2}e^{-b(\frac{a}{2} + 1)t}.
\end{equation}

The following examples will show how effectiveness of these formulas.
3.1. Example 1
From (9) consider the nonlinear fractional FPE where \( A = 3ux^{-1} \) and \( B = u \), we can get the numerical solution

\[
  u = u_0 + u_1 = x^{3/2} + \left[ -3 \frac{\Gamma(3)}{\Gamma(3 - \beta)} x^{2-\beta} + \frac{\Gamma(4)}{\Gamma(4 - 2\beta)} x^{3-2\beta} \right] \frac{t^\alpha}{\Gamma(\alpha + 1)}. \tag{24}
\]

By setting \( \beta = 1 \), the exact solution is

\[
  u(x, t) = x^{3/2}, \tag{25}
\]
and by setting \( \alpha = \beta = 1 \), we have the exact solution

\[
  u(x, t) = x^{3/2}. \tag{26}
\]

3.2. Example 2
From (9) consider the nonlinear fractional FPE where \( A = 2ux^{-1} - \frac{x}{2} \) and \( B = u \), we can get the numerical solution

\[
  u = u_0 + u_1 = x + \left[ -2 \frac{\Gamma(2)}{\Gamma(2 - \beta)} x^{1-\beta} + \frac{1}{2} \frac{\Gamma(3)}{\Gamma(3 - \beta)} x^{2-\beta} + \frac{\Gamma(3)}{\Gamma(3 - 2\beta)} x^{3-2\beta} \right] \frac{t^\alpha}{\Gamma(\alpha + 1)}. \tag{27}
\]

By setting \( \beta = 1 \), the exact solution is

\[
  u(x, t) = xE_\alpha(t^\alpha), \tag{28}
\]
and by setting \( \alpha = \beta = 1 \), we get the exact solution

\[
  u(x, t) = xe^t. \tag{29}
\]

3.3. Example 3
From (9) consider the nonlinear fractional FPE where \( A = 4ux^{-1} - \frac{x}{3} \) and \( B = u \), we can get the numerical solution

\[
  u = u_0 + u_1 = x^2 + \left[ -4 \frac{\Gamma(4)}{\Gamma(4 - \beta)} x^{3-\beta} + \frac{1}{3} \frac{\Gamma(4)}{\Gamma(4 - \beta)} x^{3-\beta} + \frac{\Gamma(5)}{\Gamma(5 - 2\beta)} x^{4-2\beta} \right] \frac{t^\alpha}{\Gamma(\alpha + 1)}. \tag{30}
\]

By setting \( \beta = 1 \), the exact solution is

\[
  u(x, t) = x^2E_\alpha(t^\alpha), \tag{31}
\]
and by setting \( \alpha = \beta = 1 \), we have the exact solution

\[
  u(x, t) = x^2e^t. \tag{32}
\]
3.4. Example 4
From (9) consider the nonlinear fractional FPE where 
\[ A = \frac{1}{2} u x^{-1} + 3 x \]  
\[ B = u, \]
we can get the numerical solution
\[ u = u_0 + u_1 = x^{\frac{1}{4}} + \left[ -\frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \beta\right)} x^{-\frac{1}{2} - \beta} - 3 \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - \beta\right)} x^{\frac{1}{2} - \frac{3}{2} - \beta} + \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2} - 2\beta\right)} x^{\frac{1}{2} - 2\beta} \right] \frac{t^\alpha}{\Gamma(\alpha + 1)}. \] (33)

By setting \( \beta = 1 \), the exact solution is
\[ u(x, t) = x^{\frac{1}{4}} E_{\alpha\beta}\left(-\frac{15}{4} t^\alpha\right), \] (34)

and by setting \( \alpha = \beta = 1 \), we get the exact solution
\[ u(x, t) = x^{\frac{1}{4}} e^{-\frac{15}{4} t}. \] (35)

4. Conclusion
In this work, the homotopy perturbation method and the natural transform were used for solving some nonlinear space-time fractional Fokker-Planck equation. Our examples show that our results (21) - (23) is effective.

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