Anomalous magnon Nernst effect of topological magnonic materials

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Abstract

The magnon transport driven by a thermal gradient in a perpendicularly magnetized honeycomb lattice is studied. The system with the nearest-neighbor pseudodipolar interaction and the next-nearest-neighbor Dzyaloshinskii–Moriya interaction has various topologically nontrivial phases. When an in-plane thermal gradient is applied, a transverse in-plane magnon current is generated. This phenomenon is termed as the anomalous magnon Nernst effect that closely resembles the anomalous Nernst effect for an electronic system. The anomalous magnon Nernst coefficient and its sign are determined by the magnon Berry curvature distributions in the momentum space and magnon populations in the magnon bands. We predict a temperature-induced sign reversal in anomalous magnon Nernst effect under certain conditions.

Keywords: magnon, topology, Berry curvature, Nernst effect

(Some figures may appear in colour only in the online journal)

1. Introduction

Spintronics is about generation, detection and manipulation of spin degree of freedom of particles. Most early studies focused on the electron spins [1]. However, an electric current normally accompanies an electron spin current and consumes much energy, leading to a Joule heating. The Joule heating becomes the critical problem in nano electronics and spintronics although many efforts have been made. Recently, magnon spintronics, or magnonics in which magnons are spin carriers, attract much attention because of their fundamental interest [2, 3] and its lower energy consumption in comparison to that of electron spintronics [4–6].

Nernst effect commonly refers to the generation of a transverse voltage/current by a thermal gradient in an electronic system under a perpendicular magnetic field. In a ferromagnetic metal and in the absence of an external magnetic field, a thermal gradient can generate a transverse charge current or voltage proportional to the vector product of the thermal gradient and the magnetization in the linear response region. This is the anomalous Nernst effect, the thermal electric manifestation of the anomalous Hall effect [7]. It is natural to ask whether there is a similar effect for magnons. Moving magnons experience gyroscopic forces because of nonzero Berry curvature of a magnetic system, although magnons are charge neutral quasiparticles that do not have the Lorentz force, as schematically illustrated in figure 1(a). When the transverse components of the currents due to opposite Berry curvatures do not cancel with each other, a net transverse magnon current is generated when magnons are driven by a longitudinal force such as a thermal gradient in the absence of a magnetic field, which is termed as the anomalous magnon Nernst effect (AMNE). In this paper, we focus on a perpendicularly magnetized honeycomb lattice with the nearest-neighbor pseudodipolar interaction and the next-nearest-neighbor Dzyaloshinskii–Moriya interaction (DMI), whose magnon bands can be topologically nontrivial with various topological phases [8–10]. We investigate the magnon transport of this
system in the presence of a thermal gradient using the semi-classical equations of motion of magnons and the Boltzmann equation in linear response regime. We found that the system has topologically nontrivial magnon bands. The system changes from one topologically nontrivial phase to another as the DMI strength varies. The AMNE coefficient depends on temperature nonmonotonically. It starts from 0 at 0 K and goes back to 0 at high temperature limit with a maximum at an intermediate temperature. The nonmonotonic temperature-dependence of AMNE is due to non-trivial Berry curvature distribution of a given band in the momentum space and thermally activated magnon population in the bands. In certain parameter spaces, there is a sign reversal of the AMNE at low temperature because the magnon Berry curvature near the band bottom at \( \Gamma \) point has small non-zero values of the opposite sign as those near band top at K and K' points with a much bigger value. In the presence of staggered anisotropy on A, B sublattices, the system can also be topologically trivial, and the K and K' valleys contribute opposite transverse magnon currents due to the opposite Berry curvatures. However, the total transverse magnon current does not vanish. The boundary that AMNE coefficient changes its sign is also determined numerically.

2. Model and results

We consider classical magnetic moments on a honeycomb lattice in the xy plane as illustrated in figure 1(a), and the Hamiltonian is

\[
\mathcal{H} = -\frac{J}{2} \sum_{\langle ij \rangle} \mathbf{m}_i \cdot \mathbf{m}_j - \frac{F}{2} \sum_{\langle ij \rangle} (\mathbf{m}_i \cdot \mathbf{e}_y)(\mathbf{m}_j \cdot \mathbf{e}_y)
- D \sum_{\langle ij \rangle} \nu_{ij} \mathbf{z} \cdot (\mathbf{m}_j \times \mathbf{m}_i) - \sum_i \frac{K_i}{2} m_i^2,
\]

(1)

where the first term is the nearest-neighbor ferromagnetic Heisenberg exchange interaction \( J > 0 \). The second and third terms arise from the spin-orbit coupling (SOC) [11, 12]. \( \mathbf{e}_y \) is the unit vector pointing from site \( i \) to \( j \). \( F \) is the strength of the nearest-neighbor pseudodipolar interaction, which is the second-order effect of the SOC. (The nearest-neighbor DMI would be the first-order effect of SOC if it exists, but it vanishes because the center of the A–B bond is an inversion center of the honeycomb lattice.) The next-nearest-neighbor DMI measured by \( D \) is in general non-zero. \( \nu_{ij} = \frac{\alpha}{\sqrt{3}} \mathbf{z} \cdot (\mathbf{e}_i \times \mathbf{e}_j) = \pm 1 \), where \( l \) is the nearest neighbor site of \( i \) and \( j \). The last term is the sublattice-dependent anisotropy whose easy-axis is along the \( z \) direction with anisotropy coefficients of \( K_i = K + \Delta \) for \( i \in A \) and \( K - \Delta \) for \( i \in B \). \( \mathbf{m}_i \) is the unit vector of the magnetic moment at site \( i \). Compared to the Heisenberg exchange interaction as well as DMI and pseudodipolar interaction originated from the SOC, the long-range dipole–dipole interaction is much weaker and is ignored in our model. The spin dynamics is governed by the Landau–Lifshitz–Gilbert (LLG) equation [8, 14],

\[
\frac{d\mathbf{m}_i}{dt} = -\gamma \mathbf{m}_i \times \mathbf{H}^{\text{eff}} + \alpha \mathbf{m}_i \times \frac{d\mathbf{m}_i}{dt},
\]

(2)

where \( \gamma \) is the gyromagnetic ratio and \( \alpha \) is the Gilbert damping constant. \( \mathbf{H}^{\text{eff}} = -\frac{\partial U}{\partial \mathbf{m}} \) is the effective field at site \( i \). The lattice constant \( a \) and \( J \) are used as the length unit and the energy unit. The magnetic field and time are in the units of \( \sqrt{\mu_0 a^3} \) and \( \sqrt{\mu_0 / (J a^2)} \), respectively, where \( \mu_0 \) is the vacuum permeability. When the anisotropy is sufficiently large, spins are perpendicularly magnetized [15]. To obtain the spin wave spectrum, we consider a conservative system (\( \alpha = 0 \)) and linearize the LLG equation following the standard procedures [8] and obtain the magnon spectrum (see appendix). The magnon band structures for \( K = 10J \), \( F = 5J \) and various \( D \) and \( \Delta \) are shown in figures 1(b)–(g). The direct gaps at the valleys can close and reopen as \( D \) and \( \Delta \) varies, resulting in various topological phase transitions. Figures 1(b)–(d) are three gapped bands with different parameters. The colors of the Dirac cones indicate the sign of Berry curvature (red for positive and blue for negative). When \( \Delta = \Delta_c \) (\( \Delta_c \) is defined in the appendix), the gap at K (K') closes, as shown in figures 1(e) and (g) for \( D = 0 \). When \( \Delta = \Delta_c = 0 \) that happens when \( D = \frac{\sqrt{3}J_0^2}{10a^2} \), both K and K' gaps close, as shown in
The Berry curvature $\Omega_n$ of $n$th band and the corresponding Chern number $C_n$ can be calculated by [16–18]

$$\Omega_n = i \nabla_k \times (\psi_n^\dagger \sigma \nabla_k \psi_n);$$

(3)

$$C_n = \frac{1}{2\pi} \int \int_{k \in \text{BZ}} \Omega_n \, d^2k,$$

(4)

where the integration is over the Brillouin zone (BZ), $g = \sigma_0 \otimes \sigma_3$ ($\sigma_i$ are the Pauli matrices), and $\Omega_n = \Omega_n \cdot \hat{z}$ is the $z$ component of the Berry curvature that is given by a gauge-invariant formula similar to that in electronic systems [17]

$$\Omega_n = i \text{Tr} \left[ P_n \left( \frac{\partial P_n \partial P_n}{\partial k_x \partial k_y} - \frac{\partial P_n \partial P_n}{\partial k_y \partial k_x} \right) \right] \hat{z},$$

(5)

where $P_n$ is the projection matrix of the $n$th band defined as $P_n = \psi_n \psi_n^\dagger g$.

Figure 2(a) is the full phase diagram in $D/J - \Delta/J - F/J$ space for $K = 10J$. The various topological phases are classified by Chern numbers $C_l$ and $C_u$ of lower and upper magnon bands. $C_l + C_u = 0$ satisfies the ‘zero sum rule’ [17, 19]. The magnon band Chern number change its value when magnon band gap closes and reopens at valley K or $K'$. Thus, the band gap closing at K or $K'_\prime$ defines two phase boundary surfaces of $\omega^2_L = \omega^2_U$ and $\omega^2_L = \omega^2_U$ (see appendix). The two phase boundary surfaces are $\Delta = \pm \Delta_\sigma$, denoted as the orange surfaces. They divide the whole space into four regions. In the region of $\Delta_\sigma > 0$ and $-\Delta_\sigma < \Delta < \Delta_\sigma$, $C_n = -1$. The density plot of $\Omega$ for $F = 5J$, $\Delta = D = 0$ (O$_1$ in figure 2(a)) is shown in the top panel of figure 2(b). Interestingly, the lower band has two contour curves of $\Omega = \Omega_1 = 0$ around $\Gamma$ denoted by black lines. The two contour curves of $\Omega = 0$ divide the first Brillouin zone into three parts. $\Omega$ is slightly positive inside the inner contour curve around $\Gamma$ for the lower band as shown in the top left panel. Between two contour curves, $\Omega$ is slightly negative. $\Omega$ is positive outside the outer contour curve as shown in the top left panel of figure 2(b), but significant non-zero $\Omega$ occurs only around $K$ and $K'$. In the region of $\Delta_\sigma < 0$ and $\Delta_\sigma < \Delta < -\Delta_\sigma$, the upper magnon band has Chern number 1. The bottom panel of figure 2(b) is the density plot of $\Omega$ of lower (left panel) and upper (right panel) bands for a representative point of $F = 5J$, $\Delta = 0$, $D = 0.4J$ (O$_2$ in figure 2(a)) in this topologically nontrivial phase. The lower band has only one contour curve of $\Omega = 0$ (black dash curve) around $\Gamma$ that divides the first Brillouin zone into two parts. Inside the contour curve, $\Omega$ is slightly positive as shown in the bottom left panel of figure 2(b). It is negative outside the contour curve with significant non-zero value around $K$ and $K'$. The system is in topologically trivial phase for both lower and upper bands in the other two regions. $\Omega$ around $K$ and $K'$ valleys have opposite sign so that the Chern numbers are 0 for both bands. We consider $O$ in figure 2(a) ($F = D = 0$, $\Delta = 1.5J$) as a representative point in the phase. The middle panel of figure 2(b) shows the density plot of $\Omega$ at $O$ for the two bands. Indeed, Berry curvatures $\Omega$ at $K$ and $K'$ have opposite value, and Chern numbers are zero. For $\Delta = 0$, the band gap $\Delta_\sigma$ gap closing at $K$ or $K'_\prime$ and $\Delta = 0$. The color bars are shown at the middle. The contour lines of $\Omega = 0$ are shown by black lines. The white hexagons are the first Brillouin zones.
gaps at K and K' close and reopen at the same time and the Chern number of the upper band changes from $-1$ to $+1$ if we tune the DMI crossing the line of $D = \frac{\gamma n}{r \Delta}$ and $\Delta = 0$ (the green line in figure 2(a)). The system changes from one topologically nontrivial phase to another. The features of the phase diagram discussed above preserve as long as the system ground state is the perpendicular ferromagnetic state.

Let us consider the magnon transport in an infinite system. Applying a thermal gradient along $\hat{x}$ direction, the motion of a magnon wavepacket is governed by the semiclassical equations [13, 20],

$$\dot{\mathbf{r}} = \frac{1}{\hbar} \frac{\partial \epsilon}{\partial \mathbf{k}} - \hat{k} \times \Omega; \quad \dot{\mathbf{k}} = -\frac{1}{\hbar} \frac{\partial \epsilon}{\partial \mathbf{r}} + \frac{1}{\hbar} \mathbf{F},$$

where $\epsilon(\mathbf{r}, \mathbf{k}) = \hbar \omega(\mathbf{k}) + U(\mathbf{r})$ is the energy of the magnon and $U(\mathbf{r})$ is the potential energy due to the inhomogeneity such as thermal gradient (see appendix for detailed derivations) and spatial dependence of material parameters. $\mathbf{F}$ is the total force on the magnon. In the presence of a thermal gradient, the Boltzmann equation of the magnon is

$$\dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} = -\frac{f - f_0}{\tau} \equiv -\frac{f_1}{\tau},$$

where $f(\mathbf{r}, \mathbf{k})$ is the magnon distribution function. $f_0 = 1/(e^{\beta \omega - 1})$ is the Bose–Einstein distribution of zero chemical potential at local temperature $T$ [$\beta = (k_B T)^{-1}$]. $\tau$ is magnon relaxation time. $f_1$ is the deviation of the distribution function from its equilibrium values. In the linear response regime where the thermal gradient is small, $f$ in the left-hand side of equation (8) can be replaced by its equilibrium value $f_0$ and equation (8) becomes

$$\dot{\mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{r}} = -\frac{f_1}{\tau}. \quad (9)$$

Using the following identity

$$\dot{\mathbf{r}} \cdot \frac{\partial f_0}{\partial \mathbf{r}} = \left(-\frac{\hbar \omega}{T} \nabla T\right) \cdot \frac{\partial f_0}{\partial \mathbf{k}} \frac{\hbar \omega}{\hbar \omega \mathbf{k}}.$$ \hspace{1cm} (10)

Equation (9) can be rewritten as

$$\left(-\frac{\hbar \omega}{T} \nabla T\right) \cdot \frac{\partial f_0}{\partial \mathbf{k}} = -\frac{f_1}{\tau}. \quad (11)$$

Thus, one can identify a thermal force $\mathbf{F}_T = \left(-\frac{\hbar \omega}{T} \nabla T\right)$ proportional to the magnon frequency and the thermal gradient. This force is rotationless ($\nabla \times \mathbf{F}_T = 0$) so that it can be written as the gradient of a potential $\mathbf{F}_T = \frac{\omega}{T} \nabla T$ \cite{21}. Inserting (7) into (6) with $\mathbf{F} = \mathbf{F}_T$, we obtain

$$\dot{\mathbf{r}} = \frac{\partial \omega}{\partial \mathbf{k}} + \frac{\omega}{T} \nabla T \times \Omega. \quad (12)$$

The magnon current density is given by $\mathbf{j}_m = \sum_n [\hat{r} f(n, \mathbf{k})]$, where the summation is over all magnon states. We keep the terms linear in the thermal gradient and convert the summation to integration to obtain,

$$\mathbf{j}_m = \nabla \left( -k_B \nabla T \right),$$

where the longitudinal heat conductance $k_{\text{xx}}$ and the anomalous Nernst coefficient $\kappa_{\text{xy}}$ are

$$k_{\text{xx}} = \frac{\tau}{(2\pi)^2} \sum_n \int \frac{\beta \omega_n^2}{\partial \mathbf{k}} \rho(\beta \hbar \omega_n) d^2\mathbf{k},$$

$$\kappa_{\text{xy}} = \frac{1}{(2\pi)^2} \sum_n \int \beta \omega_n \Omega \mathbf{f}_0 (\beta \hbar \omega_n) d^2\mathbf{k}, \quad (14)$$

where $\rho(x) = \frac{2x}{(e^{x} - 1)^2}$, $f_0(x) = \frac{1}{e^x - 1}$, and $n = 1, 2$ labels the lower and upper magnon bands. Figure 3(a) shows the temperature dependence of $k_{\text{xx}}$ and $\kappa_{\text{xy}}$ in two different topologically nontrivial phases specified by $O_1$ and $O_2$ in figure 2(a). In order to have a quantitative feeling about the results, we use $\text{Sr}_2\text{IrO}_4$ parameters of $a = 0.55 \text{ nm}$ \cite{22}, $J = 19.6\mu_0\hbar^2/\alpha^2$ \cite{12}, and $\gamma = 2.21 \times 10^5 \text{ rad/s/\text{A/m}}$ in all the following discussions. The longitudinal heat conductance $k_{\text{xx}}$ is always positive as expected from thermodynamic laws that the magnons move from the hot side to the cold side. (15) says that the AMNE coefficient is determined by the Berry curvature distribution in the momentum space and the magnon equilibrium distribution function. Since the magnon number in the lower band is bigger than that in the higher band according to the Bose–Einstein distribution, the sign of AMNE coefficient is always determined by the Berry curvature of the lower magnon band. At very low temperature, only the magnons near the $\Gamma$ point (band bottom of the lower band) are excited. The sign of AMNE coefficient is determined by $\Omega$ around $\Gamma$, and its value is small because Berry curvature $\Omega$ is very close to zero, although not exactly zero, and the magnon number is also small there. At a higher temperature when the magnon number near K and K' points (band top (bottom) of the lower (upper) band) are large enough and dominate the AMNE due to significant non-zero values of the Berry curvature near there. At even higher temperature when equal-partition theorem holds so that $f_0 \approx k_B T/\hbar \omega$, and $\omega f_0 = k_B T$ is independent of $\omega$. From (15), the AMNE coefficient is zero because $\kappa_{\text{xy}}$ is proportional to $(c_0 + c_1) = 0$ \cite{17, 19}, i.e. the contributions from two bands cancel with each other.

To quantitatively see how the AMNE coefficient $\kappa_{\text{xy}}$ depends on temperature, we first choose two representative points in two distinct topologically nontrivial phases of $C_a = -1$ (for $O_1$) and $C_a = 1$ (for $O_2$). For $O_1$, whose Berry curvature distribution is given in the top panel of figure 2(b), $\kappa_{\text{xy}}$ is always positive, a transverse magnon current along $\mathbf{m}_0 \times (-\nabla T)$, because $\Omega$ are positive near both $\Gamma$ and K (K') points. For $O_2$, at very low temperatures when the magnon number around K and K' are negligible and only the magnons near $\Gamma$ point are excited, $\kappa_{\text{xy}}$ increases and becomes more and more positive initially with the increase of temperature because the magnon number near the $\Gamma$ point increases and $\Omega$ is positive there. However, when magnons near K and K' points are excited, $\kappa_{\text{xy}}$ starts to increase with temperature, and becomes negative after an intermediate temperature because $\Omega$ has large negative values near K and K'. Thus, in this phase the sign of the...
AMNE coefficient reverses at the intermediate temperature. The numerical results of $\kappa_{xy}$ and $\kappa_{xy}$ at higher temperatures are shown in the inset of figure 3(a). The longitudinal heat conductivity $\kappa_{xx}$ saturates at high temperature. AMNE coefficient $\kappa_{xy}$ at $O_1$ ($O_2$) increases from 0 to a maximum positive (negative) value as the temperature increases, and then gradually go back to 0 when magnons in the upper band are thermally excited. This indicates that there is an optimal temperature $T_{\text{max}}$ for the maximal AMNE coefficient. If this temperature does not exceed the Curie temperature, it should be used for the measurement of AMNE.

In the topologically trivial phase, the Berry curvatures $\Omega$ of the same band has opposite values near $K$ and $K'$ points. Thus the contributions to AMNE from different valleys partially cancel each other, and the net transverse magnon current can be in either direction, depending on the parameters. To see this, we calculate $\kappa_{xy}$ for different $D/J$ and $\Delta/J$ parameters that cover all the topological phases. Figure 3(b) is the density plots of $\kappa_{xy}$ as a function of $D/J$ and $\Delta/J$ at different temperatures (for $K = 10J$ and $F = 5J$). The contour lines of $\kappa_{xy} = 0$ are emphasized by dashed lines. At low $T$ ($k_B T = 5J$ and $10J$), $\kappa_{xy}$ is mainly determined by the Berry curvature near $\Gamma$ point so that the sign of $\kappa_{xy}$ is not relevant to phase transitions. $\kappa_{xy}$ does not change sign across the topological phase boundary indicated by the black lines. However, at the high temperature ($k_B T = 15J$), $\kappa_{xy}$ is mainly determined by the Berry curvature near $K$ and $K'$ points. Along $\Delta = 0$, there is a topological phase transition directly from one topologically nontrivial phase to another at $D = D_c$, with the signs of Berry curvature near $K$ and $K'$ points reversed. Thus, the Nernst coefficient $\kappa_{xy}$ changes sign simultaneously. So in the figures for $k_B T = 15J$ and $T = T_{\text{max}}$ in figure 3(b), the contour line of $\kappa_{xy} = 0$ passes through the phase transition point $\Delta = 0$, $D = D_c$.

In the above discussions, we studied the magnon Nernst effect, a transverse magnon current generated by a longitudinal thermal gradient. Similar to electronic systems, there are other related effects, such as a transverse magnon current induced by a longitudinal chemical potential gradient (magnon Hall effect and anomalous magnon Hall effect), and a transverse magnon heat current induced by a longitudinal chemical potential gradient (magnon Peltier effect). These effects can be investigated in the same way as what has done here for the same Berry curvature physics. Similar topological phase transitions and sign-reversal of AMNE was also predicted in pyrochlore lattices [23]. In the calculation of thermal transport coefficients, the thermal energy $k_B T$ is allowed to be much higher than $J$. In real materials, the temperature is limited by the Curie temperature that is order of $J/k_B$. For example, $J = 20 \text{ meV (2.5} \times 10^3 \mu_0 \text{J} / \text{m}^2$) and the Curie temperature is about $240 \text{ K}$ [24] for Sr$_2$IrO$_4$. The sign-reversal temperature is $9.5J/k_B$ as shown in figure 3(a). Thus, the temperature is much smaller than the sign-reversal temperature in this case so the AMNE coefficient should be always positive. In the presence of an external magnetic field along the magnetization direction, the magnon band is shifted upward. The magnon modes become ‘harder’ so that less magnons are excited. Thus, all the thermal conductivity coefficients become smaller compared to the results above. Honeycomb magnet model with DMI has been studied in [9] and [10]. However, they did not consider pseudodipolar interaction, and the sign of magnon Nernst coefficient is determined by the DMI only, but not by the temperature [10]. In our model, when the pseudodipolar interaction is non-zero, there is a temperature-induced sign reversal of the magnon Nernst coefficient as shown in figure 3 due to the $F$-dependent Berry curvature distribution near $\Gamma$ point, although exact reason why the Berry curvature near $\Gamma$ point are so sensitive to pseudodipolar interaction is still not clear.

3. Conclusion

In conclusion, we studied the thermal magnon transport of perpendicularly magnetized honeycomb lattice with the nearest-neighbor pseudodipolar interaction and the next-nearest-neighbor DMI. We showed that the system has various
topological nontrivial phases. Due to the nontrivial Berry curvature, a transverse magnon current appears when a thermal gradient is applied, resulting in an anomalous magnon Nernst effect. The sign of the anomalous magnon Nernst effect is reversed by tuning DMI and temperature.

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Appendix A. Spin wave spectrum

We consider a small deviation of \( \mathbf{m}_0 = (\delta m_{ix}, \delta m_{iy}, 1) \) from its uniform ground state \( m_0 = (0, 0, 1) \) so that \( \sqrt{\delta m_{ix}^2 + \delta m_{iy}^2} \ll 1 \). The spin waves of the linearized LLG equation (2) must satisfy the Bloch theorem due to the periodic structure of the lattice, and have the forms of \( \delta m_{ix} = X_{A,B} e^{i k_{A,B} R} \) and \( \delta m_{iy} = Y_{A,B} e^{i k_{A,B} R} \), where A and B are sublattices and \( R \) is the position of the lattice. For convenience, we define \( \psi_{A,B}^\pm = (X_{A,B} \pm Y_{A,B})/\sqrt{2} \), and \( \psi = (\psi_A^+, \psi_A^-, \psi_B^+, \psi_B^-) \). The spin wave spectrum and wavefunctions are obtained by solving the eigenvalue problem \( gH(k)\psi = \omega_{m}(k)\psi \), where \( H(k) \) is a \( 4 \times 4 \) Hermitian matrix obtained from linearized LLG equation,

\[
H = \begin{pmatrix}
M_A^+ & 0 & -\ell(k) & -g_+(k) \\
0 & M_A^- & -g_-(k) & -\ell(k) \\
-\ell^*(k) & g^*(k) & M_B^- & 0 \\
-g^*(k) & \ell^*(k) & 0 & M_B^+
\end{pmatrix},
\]

with \( \ell(k) = (J + \frac{\gamma}{2}) \sum_{j=1,3} \cos \theta_j \), \( g^* \)(\( k \)) = \( \frac{\gamma}{2} \sum_{j=1,3} e^{\pm 2i\theta_j} \cos \theta_j \) (\( \theta \) is the angle between the \( \mathbf{a}_i \) and \( x \) direction), \( M_A^\pm = M + \Delta \pm \ell(k) \) and \( M_B^\pm = M - \Delta \pm \ell(k) \) with \( M = K + 3J \) and \( d(k) = 2D \sum_{j=1,3} \sin \kappa j (k \cdot \mathbf{b}_j). g = \sigma_0 \otimes \sigma_3 \) (with \( \sigma_0 \) being the \( 2 \times 2 \) identity matrix and \( \sigma_3 \) the Pauli matrix). \( \psi_n \) is the \( n \)th eigenvector of eigen-frequency \( \omega_{m} \), satisfying the generalized orthogonality \( \psi_n^\dagger g \psi = \delta_{ij} \). At K point \( \mathbf{k} = \left( \frac{2\pi}{\sqrt{3}}, \frac{2\pi}{\sqrt{3}} \right) \), the frequencies of the two magnon bands are respectively,

\[
\omega^K_1 = M - 3\sqrt{3}D + \Delta, \\
\omega^K_2 = \sqrt{(M + 3\sqrt{3}D)^2 - \frac{9}{4}F^2 + \Delta}.
\]

Similarly at \( K' \) point \( \mathbf{k} = \left( \frac{2\pi}{\sqrt{3}}, -\frac{2\pi}{\sqrt{3}} \right) \),

\[
\omega^{K'}_1 = M - 3\sqrt{3}D - \Delta, \\
\omega^{K'}_2 = \sqrt{(M + 3\sqrt{3}D)^2 - \frac{9}{4}F^2 + \Delta}.
\]

While \( \omega^{K(2)}_1 = \omega^{K'(2)}_2 \), the gap at \( K(K') \) closes and topological phase transition happens, so that we can obtain the topological phase boundaries. For convenience, we define

\[
\Delta_c = \frac{1}{2} \left[ (M - 3\sqrt{3}D) - \sqrt{(M + 3\sqrt{3}D)^2 - \frac{9}{4}F^2} \right],
\]

and two phase boundary surfaces are \( \Delta = \pm \Delta_c \).

Appendix B. Semiclassical equations of motion for magnon wavepackets

We first consider a uniaxial ferromagnet with exchange stiffness \( A \), an easy-axis anisotropy \( K \) and an external potential \( U(r) \). The LLG equation is

\[
\frac{\partial \mathbf{m}}{\partial t} = -\frac{\gamma}{\mu_0 M_r} \mathbf{m} \times \left[ A \nabla^2 \mathbf{m} + (K_m + U) \mathbf{e}_z \right].
\]

Linearizing the above equation yields \[25\]

\[
i \frac{\partial}{\partial t} \begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix} = \gamma \frac{1}{\mu_0 M_r} \sigma_3 \left( A \nabla^2 - K - U \right) \begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix},
\]

where \( \psi_\pm(r) = m_\pm(r) \pm i m_\mp(r) \) are new continuous fields. For the magnet on a honeycomb lattice and without the external field, the lattice vector \( \mathbf{k} \) is a good quantum number and the linearized LLG equation can be written as a matrix equation as illustrated in the previous section,

\[
\begin{pmatrix}
\psi_A^- \\
\psi_A^+ \\
\psi_B^- \\
\psi_B^+
\end{pmatrix}
= \begin{pmatrix}
\sigma_0 & 0 & -\ell(k) & -g_+(k) \\
0 & M_A^- & -g_-(k) & -\ell(k) \\
-\ell^*(k) & g^*(k) & M_B^- & 0 \\
-g^*(k) & \ell^*(k) & 0 & M_B^+
\end{pmatrix}
\begin{pmatrix}
\psi_A^- \\
\psi_A^+ \\
\psi_B^- \\
\psi_B^+
\end{pmatrix}.
\]

In the following, we focus on equation (B.2) that has the same form as the Schrödinger equation. For convenience, we use the bracket notations so that \( \sigma_1 H_0 | \psi_{ak} \rangle = \omega_{kad} | \psi_{ak} \rangle \) for an eigenstate, where \( H_0 = \frac{\gamma}{\mu_0 M_r} \left[ A \nabla^2 - K \right] \). Knowing \( H_0 \) is Hermitian and \( \sigma_1 \sigma_3 = \sigma_0 \), the generalized orthogonality relationships can be obtained from

\[
\omega_{kad} \langle \psi_{kad} | \sigma_3 \psi_{kad} \rangle = \langle \psi_{kad} | (\sigma_3 H_0) \psi_{kad} \rangle = \langle \psi_{kad} | H_0 \sigma_3 \psi_{kad} \rangle = \langle \psi_{kad} | \sigma_3 H_0 \psi_{kad} \rangle.
\]

so that when \( \omega_{kad} \neq 0 \), \( \langle \psi_{kad} | \sigma_3 \psi_{kad} \rangle | \psi_{kad} \rangle = 0 \).

Then we follow \[26\] to derive the equation of motion for a wavepacket of a given band centered at \( r_e \) (thus we omit the band index \( n \))

\[
| \Psi(r,t) \rangle = \int a(k,t) | \psi_{ek}(r,t; r_e) \rangle dk,
\]

in an external potential \( U(r) \), where \( | \psi_{ek}(r,t; r_e) \rangle \) is the Bloch eigenstate of periodic Hamiltonian \( H_e = H_0 + U(r_e)/\hbar \),
which is the leading term of the Taylor expansion of the total (non-periodic) Hamiltonian near $r_c$,
\[
H \approx H_c + \frac{1}{2\hbar} \left[ (r - r_c) \cdot \frac{\partial U}{\partial r_c} + \frac{\partial U}{\partial r_c} \cdot (r - r_c) \right]. \tag{B.6}
\]
For the wavepacket highly concentrated at $r_c$ and $k_c$, i.e. the width of $a(k,t)$ in $k$ space is much smaller than the size of the Brillouin zone, and the wavepacket size is so small that the external potential does not change much within the size, one needs to keep only the linear terms. The wavepacket center at $r_c$ and $k_c$ yields
\[
\langle \Psi | \sigma_3 r | \Psi \rangle = \langle \Psi | r \sigma_3 | \Psi \rangle = r_c, \tag{B.7}
\]
\[
\langle \Psi | \sigma_3 k | \Psi \rangle = \langle \Psi | k \sigma_3 | \Psi \rangle = k_c. \tag{B.8}
\]
The normalization condition $\langle \Psi | \sigma_3 | \Psi \rangle = 1$ yields
\[
\int |a(k,t)|^2 dk = 1. \tag{B.9}
\]
where the subscript $c$ means all terms are evaluated at $k_c$ because of the narrow wavepacket assumption. The Lagrangian of the system is
\[
L = \langle \Psi | i \sigma_3 \frac{d}{dt} - H | \Psi \rangle \tag{B.10}
\]
which can be easily verified by the fact that (B.2) is recovered by taking the variation of $L$. Here $d/dt$ means the time derivatives with respect to the time explicitly and implicitly through $r_c$ and $k_c$. Similar to [26], by using (B.9), the Lagrangian can be deduced to
\[
L = -\omega_c + k_c \cdot \dot{x}_c + \dot{k}_c \cdot i(u | \sigma_3 \frac{\partial u}{\partial k_c})
+ \dot{x}_c \cdot i(u | \sigma_3 \frac{\partial u}{\partial x_c}) + i(u | \sigma_3 \frac{\partial u}{\partial t}). \tag{B.11}
\]
At $r = r_c$, $U(r_c)$ uniformly shifts the whole band. Thus, the eigenstates remain unchanged while the eigenfrequency changes to $\omega_c = \omega + U(r_c)/\hbar$. Also, the eigenstates do not change with time because $U$ does not explicitly depend on time. So the Lagrangian takes the form
\[
L = -\omega(k) - U(r)/\hbar + k \cdot \dot{x} + \dot{k} \cdot i(u | \sigma_3 \frac{\partial u}{\partial k_c}), \tag{B.12}
\]
in which the subscript $c$ is omitted. The equations of motion are obtained by the Euler equations with respect to $r$ and $k$. We define ‘Berry connection’ $A = -i(u | \sigma_3 \frac{\partial u}{\partial k_c})$ and ‘Berry curvature’ $\Omega = \nabla_k \times A$ so that the equations of motion take the form
\[
\dot{r} = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial k} - k \times \Omega, \tag{B.13}
\]
\[
\dot{k} = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial r}. \tag{B.14}
\]

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