Deformation of Hyperbolic Cone-Structures: Study of the non-Colapsing case

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Dedicated to my wife Cynthia.

Abstract. This work is devoted to the study of deformations of hyperbolic cone structures under the assumption that the lengths of the singularity remain uniformly bounded over the deformation. Given a sequence \((M_i, p_i)\) of pointed hyperbolic cone-manifolds with topological type \((M, \Sigma)\), where \(M\) is a closed, orientable and irreducible 3-manifold and \(\Sigma\) an embedded link in \(M\). Assuming that the lengths of the singularity remain uniformly bounded, we prove that either the sequence \(M_i\) collapses and \(M\) is Seifert fibered or a Sol manifold, or the sequence \(M_i\) does not collapse and in this case a subsequence of \((M_i, p_i)\) converges to a complete Alexandrov space of dimension 3 endowed with a hyperbolic metric of finite volume on the complement of a finite union of quasi-geodesics. We apply this result to a conjecture of Thurston and to the case where \(\Sigma\) is a small link in \(M\).

1. Introduction

Fixed a closed, orientable and irreducible 3-manifold \(M\), this text focus deformations of hyperbolic cone structures on \(M\) which are singular along a fixed embedded link \(\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_l\) in \(M\). A hyperbolic cone structure with topological type \((M, \Sigma)\) is a complete intrinsic metric on \(M\) (see section 2 for the definition) such that every non-singular point (i.e. every point in \(M - \Sigma\)) has a neighborhood isometric to an open set of \(H^3\), the hyperbolic space of dimension 3, and that every singular point (i.e. every point in \(\Sigma\)) has a neighborhood isometric to an open neighborhood of a singular point of \(H^3(\alpha)\), the space obtained by identifying the sides of a dihedral of angle \(\alpha \in (0, 2\pi]\) in \(H^3\) by a rotation about the axe of the dihedral. The angles \(\alpha\) are called cone angles and they may vary from one connected component of \(\Sigma\) to the other. By convention, the complete structure on \(M - \Sigma\) (see [Koj2]) is considered as a hyperbolic cone structure with topological type \((M, \Sigma)\) and cone angles equal to zero.

Unlike hyperbolic structures, which are rigid after Mostow, the hyperbolic cone structures can be deformed (see [HK2]). The difficulty to understand these deformations lies in the possibility of degenerating the structure. In other words, the

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Hausdorff-Gromov limit of the deformation (see section 2 for the definition) is only an Alexandrov space which may have dimension strictly smaller than 3, although its curvature remains bounded from below by $-1$ (cf. [Koj]). In fact, the works of Kojima, Hodgson-Kerckhoff and Fuji (see [Koj], [HK] et [Fuj]) show that the degeneration of the hyperbolic cone structures occurs if and only if the singular link of these structures intersects itself over the deformation.

A natural way to study deformations of hyperbolic cone structures on $(M, \Sigma)$ is to consider sequences of hyperbolic cone structures with topological type $(M, \Sigma)$ converging (in the Hausdorff-Gromov sense) to the limit Alexandrov space. To study this kind of sequences, we need the important notion of collapse.

**Definition 1.** We say that a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ collapses if, for every sequence of points $p_i \in M - \Sigma$, the sequence $r_{M_i - \Sigma}^{p_i}$ consisting of their riemannian injectivity radii in $M_i - \Sigma$ converges to zero. Otherwise, we say that the sequence $M_i$ does not collapse.

This dichotomy is very natural and illustrates the intuitive fact that the volume of the sequence may or may not go to zero.

We are interested in studying the following question raised by W. Thurston in 80’s:

**Question 1.** Let $M$ be a closed and orientable hyperbolic manifold and suppose the existence of a simple closed geodesic $\Sigma$ in $M$. Can the hyperbolic structure of $M$ be deformed to the complete hyperbolic structure on $M - \Sigma$ through a path $M_\alpha$ of hyperbolic cone structures with topological type $(M, \Sigma)$ and parametrized by the cone angles $\alpha \in [0, 2\pi]$?

We started studying this question in [Bar2]. In that paper we obtained the following result:

**Theorem 1.** Let $M$ be a closed, orientable and irreducible 3-manifold and let $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l$ be an embedded link in $M$. Suppose the existence of a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ and having cone angles $\alpha_{ij} \in (0, 2\pi]$. Denote by $L_{M_i}(\Sigma_j)$ the length of the connected component $\Sigma_j$ of $\Sigma$ in the hyperbolic cone-manifold $M_i$. If

$$
\sup \{ L_{M_i}(\Sigma_j) : i \in \mathbb{N} \text{ and } j \in \{1, \ldots, l\} \} < \infty
$$

and the sequence $M_i$ collapses, then $M$ is Seifert fibered or a Sol manifold.

As a consequence of this theorem, we obtained the following result yielding some information on Thurston’s question (1).

**Corollary 1.** Let $M$ be a closed and orientable hyperbolic 3-manifold and suppose the existence of a finite union of simple closed geodesics $\Sigma$ in $M$. Let $M_\alpha$ be a (angle decreasing) deformation of this structure along a continuous path of hyperbolic cone-structures with topological type $(M, \Sigma)$ and having cone angles $\alpha \in (L, 2\pi] \subset [0, 2\pi]$ (the same for all components of $\Sigma$). If

$$
\sup \{ L_{M_\alpha}(\Sigma_j) : \alpha \in (L, 2\pi] \text{ and } j \in \{1, \ldots, l\} \} < \infty,
$$

then every convergent (in the Hausdorff-Gromov sense) sequence $M_{\alpha_i}$, with $\alpha_i$ converging to $L$, does not collapses.
The additional hypothesis on the length of the singularity is automatically verified when the holonomy representations of the hyperbolic cone structures are convergent. This happens (cf. [CS]), for example, when \( \Sigma \) is a small link in \( M \) (see section 4.1 for the definition). If the deformation proposed by Thurston in (1) exists, it is a consequence of Thurston’s hyperbolic Dehn surgery Theorem that the length of the singular link must converge to zero. In particular, we have that its length remains uniformly bounded over the deformation. This remark makes clear that the additional hypothesis on the length of the singularity is, in fact, a necessary condition for the existence of the desired deformation.

When a convergent sequence of hyperbolic cone-manifolds collapses, most of the geometric information can be lost. This happens because the dimension of the limit Alexandrov space may be strictly smaller than 3 (see [Bar2]). On the non-collapsing case, however, the limit Alexandrov space must have dimension 3. Our goal is to use all geometric information on the three dimensional limit to study deformations of hyperbolic cone-structures that do not collapse.

The principal result of this paper is the following one:

**Theorem 2.** Let \( M \) be a closed, orientable and irreducible 3-manifold and let \( \Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_l \) be an embedded link in \( M \). Suppose the existence of a sequence \( M_i \) of hyperbolic cone-manifolds with topological type \((M, \Sigma)\) and having cone angles \( \alpha_{ij} \in (0,2\pi] \). Denote by \( L_{M_i}(\Sigma_j) \) the length of the connected component \( \Sigma_j \) of \( \Sigma \) in the hyperbolic cone-manifold \( M_i \). If

\[
\sup \{ L_{M_i}(\Sigma_j) ; i \in \mathbb{N} \text{ and } j \in \{1, \ldots, l\} \} < \infty,
\]

then one of the following statements holds:

i. the sequence \( M_i \) collapses and \( M \) is Seifert fibered or a Sol manifold,
ii. the sequence \( M_i \) does not collapse and there exists a sequence of points \( p_{ik} \in M - \Sigma \) such that the sequence \((M_{ik}, p_{ik})\) converges in the Hausdorff-Gromov sense to a three dimensional pointed Alexandrov space \((Z, z_0)\). The Alexandrov space \( Z \) is endowed with a (noncomplete) hyperbolic metric of finite volume on the complement of a finite union \( \Sigma_Z \) of quasi-geodesics. Moreover, \( Z \) is homeomorphic to \( M \) (in particular, \( Z \) is compact) if there exists \( \varepsilon \in (0,2\pi] \) such that the cone angles \( \alpha_{ij} \) belong to \((\varepsilon,2\pi]\). Moreover, the following three statements are equivalent:

- \( Z \) is compact
- \( \inf \left\{ \text{cone-angle}_{M_{ik}}(\Sigma_j) ; k \in \mathbb{N} \text{ and } \Sigma_j \subset \Sigma \right\} > 0 \)
- \( \inf \left\{ L_{M_{ik}}(\Sigma_j) ; k \in \mathbb{N} \right\} > 0 \), for each component \( \Sigma_j \) of \( \Sigma \).

**Remark 1.** Note that the part (i) in the statement of the previous theorem is precisely the theorem [1].

**Remark 2.** A by-product of the above theorem is that the length of a connected component \( \Sigma_j \) of \( \Sigma \) shrinks down to zero if and only if the same arises for its cone angles \( \alpha_{ij} \) (when \( i \) goes to infinity). If the cone angles are supposed to be the same on all of the connected components of \( \Sigma \), it follows from this fact (see Corollary [2]) that the sequence of cone angles converges to zero if and only if the following three statements hold:

i. \( \sup \{ L_{M_i}(\Sigma) ; i \in \mathbb{N} \} < \infty \)
ii. \( \lim_{i \to \infty} \text{diam}(M_i) = \infty \)
iii. the sequence $M_i$ does not collapse.

As an application of Theorem 2, we obtain the following result related to the Thurston’s question (1).

**Corollary 2.** Let $M$ be a closed and orientable hyperbolic 3-manifold and suppose the existence of a finite union of simple closed geodesics $\Sigma$ in $M$. Let $M_\alpha$ be a deformation of this structure along a continuous path of hyperbolic conical structures with topological type $(M, \Sigma)$ and having cone angles $\alpha \in (\theta, 2\pi] \subset [0, 2\pi]$ (the same for all components of $\Sigma$). Then the following statements are equivalent

i. $\theta = 0$ and the path $M_\alpha$ extends continuously to $[0, 2\pi]$, where $M_0$ denotes $M - \Sigma$ with the complete hyperbolic metric

ii. $\lim_{\alpha \to \theta} \mathcal{L}_{M_\alpha}(\Sigma) = \lim_{\alpha \to \theta} \sum_{i=1}^{l} \mathcal{L}_{M_\alpha}(\Sigma_j) = 0$

iii. There exists a sequence $\alpha_i \in (\theta, 2\pi]$ converging to $\theta$ satisfying

$$\sup \{ \mathcal{L}_{M_\alpha}(\Sigma_j) : \alpha \in (\theta, 2\pi] \text{ and } j \in \{1, \ldots, l\} \} < \infty$$

and such that the sequence $\text{diam}(M_\alpha)$ goes to infinity with $i$.

**Remark 3.** Note that the above corollary provides a necessary and sufficient condition for the existence of the deformation proposed by Thurston. Using the notations in the statement of Thurston’s question (1), we have that

$$\theta = 0 \iff \lim_{\alpha \to \theta} \mathcal{L}_{M_\alpha}(\Sigma) = 0.$$

Supposing in addition that $M$ is not Seifert fibered and that $\Sigma$ is a small link in $M$, we have also the following theorem (see Corollaries 5 and 6) providing universal constants for the hyperbolic cone structures with topological type $(M, \Sigma)$.

**Theorem 3.** Let $M$ be a closed, orientable, irreducible and not Seifert fibered 3-manifold and let $\Sigma$ be a small link in $M$. There exists a constant $V = V(M, \Sigma) > 0$ and a constant $K = K(M, \epsilon) > 0$, for each $\epsilon \in (0, 2\pi)$, such that:

i. $\text{Vol}(M) > V$, for every hyperbolic cone-manifold $M$ with topological type $(M, \Sigma)$,

ii. $\text{diam}(M) < K$, for every hyperbolic cone-manifold $M$ with topological type $(M, \Sigma)$ and having cone angles in the interval $(\epsilon, 2\pi]$.

2. **Metric Geometry**

Given a metric space $Z$, the metric on $Z$ will always be denoted by $d_Z(\cdot, \cdot)$. The open ball of radius $r > 0$ about a subset $A$ of $Z$ is going to be denoted by

$$B_Z(A, r) = \bigcup_{a \in A} \{ z \in Z ; d_Z(z, a) < r \}.$$

A metric space $Z$ is called a length space (and its metric is called intrinsic) when the distance between every pair of points in $Z$ is given by the infimum of the lengths of all rectifiable curves connecting them. When a minimizing geodesic between every pair of points exists, we say that $Z$ is complete.

For all $k \in \mathbb{R}$, denote $\mathbb{M}^2_k$ the complete and simply connected two dimensional riemannian manifold of constant sectional curvature equal to $k$. Given a triple of
points \((x ; y, z)\) of \(Z\), a \textit{comparison triangle} for the triple is nothing but a geodesic triangle \(\Delta_k (\bar{x}, \bar{y}, \bar{z})\) in \(M^3_k\) with vertices \(\bar{x}, \bar{y}\) and \(\bar{z}\) such that
\[
d_{M^3_k} (\bar{x}, \bar{y}) = d_Z (x, y) , \quad d_{M^3_k} (\bar{y}, \bar{z}) = d_Z (y, z) \quad \text{and} \quad d_{M^3_k} (\bar{z}, \bar{x}) = d_Z (z, x).
\]
Note that a comparison triangle always exists when \(k \leq 0\). The \(k\)-angle of the triple \((x ; y, z)\) is, by definition, the angle \(\angle_k (x ; y, z)\) of a comparison triangle \(\Delta_k (\bar{x}, \bar{y}, \bar{z})\) at the vertex \(\bar{z}\) (assuming the triangle exists).

**Definition 2.** A finite dimensional (in the Hausdorff sense) length space \(Z\) is called an Alexandrov space of curvature not smaller than \(k \in \mathbb{R}\) if every point has a neighborhood \(U\) such that, for all points \(x, y, z \in U\), the angles \(\angle_k (x ; y, z)\), \(\angle_k (y ; x, z)\) and \(\angle_k (z ; x, y)\) are well defined and satisfy
\[\angle_k (x ; y, z) + \angle_k (y ; x, z) + \angle_k (z ; x, y) \leq 2\pi.\]

We point out that every hyperbolic cone-manifold is an Alexandrov space of curvature not smaller than \(-1\).

Suppose from now on that \(Z\) is a n dimensional Alexandrov space of curvature not smaller than \(k \in \mathbb{R}\). Consider \(z \in Z\) and \(\lambda \in (0, \pi)\). The point \(z\) is said to be \(\lambda\)-strained if there exists a set \(\{(a_i, b_i) \in Z \times Z ; i \in \{1, \ldots, n\}\}\), called a \(\lambda\)-strainer at \(z\), such that \(\angle_k (x ; a_i, b_i) > \pi - \lambda\) and
\[
\max \left\{ \left| \angle_k (x ; a_i, a_j) - \frac{\pi}{2} \right| , \left| \angle_k (x ; b_i, b_j) - \frac{\pi}{2} \right| , \left| \angle_k (x ; a_i, b_j) - \frac{\pi}{2} \right| \right\} < \lambda
\]
for all \(i \neq j \in \{1, \ldots, n\}\). The set \(R_\lambda (Z)\) of \(\lambda\)-strained points of \(Z\) is called the \textit{set of \(\lambda\)-regular points of \(Z\)}. It is a remarkable fact that \(R_\lambda (Z)\) is an open and dense subset of \(Z\).

Recall now, the notion of (pointed) Hausdorff-Gromov convergence (see [BB1]):

**Definition 3.** Let \((Z_i, z_i)\) be a sequence of pointed metric spaces. We say that the sequence \((Z_i, z_i)\) converges in the (pointed) Hausdorff-Gromov sense to a pointed metric space \((Z, z_0)\), if the following holds: For every \(r > \varepsilon > 0\), there exist \(i_0 \in \mathbb{N}\) and a sequence of (may be non continuous) maps \(f_i : B_{Z_i} (z_i, r) \to Z\) (\(i > i_0\)) such that
\[
i. \quad f_i (z_i) = z_0,
ii. \quad \sup \{d_{Z_i} (f_i (z_1), f_i (z_2)) - d_Z (z_1, z_2) ; z_1, z_2 \in Z\} < \varepsilon,
iii. \quad B_Z (z_0, r - \varepsilon) \subset B_{Z_i} (f_i (B_{Z_i} (z_i, r)), \varepsilon),
iv. \quad f_i (B_{Z_i} (z_i, r)) \subset B_Z (z_0, r + \varepsilon).
\]

Its a fundamental fact that the class of Alexandrov spaces of curvature not smaller than \(k \in \mathbb{R}\) is pre-compact with respect to the notion of convergence in the Hausdorff-Gromov sense. In particular, every pointed sequence of hyperbolic cone-manifolds with constant topological type has a subsequence converging (in the Hausdorff-Gromov sense) to a pointed Alexandrov space which may have dimension strictly smaller than 3, although its curvature remains bounded from below by \(-1\).

### 3. Sequences of Hyperbolic cone-manifolds

Recall that \(M\) denotes a closed, orientable and irreducible differential manifold of dimension 3 and that \(\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_r\) denotes an embedded link in \(M\). A sequence of hyperbolic cone-manifolds with topological type \((M, \Sigma)\) will always be denoted by \(M_i\).
Given a sequence $M_i$ as above, fix indices $i \in \mathbb{N}$ and $j \in \{1, \ldots, l\}$. For sufficiently small radius $R > 0$, the metric neighborhood $B_{M_i}(\Sigma_j, R) = \{x \in M_i : d_{M_i}(x, \Sigma_j) < R\}$ of $\Sigma$ is a solid torus embedded in $M_i$. The supremum of the radius $R > 0$ satisfying the above property will be called normal injectivity radius of $\Sigma_j$ in $M_i$ and it is going to be denoted by $R_i(\Sigma_j)$. Analogously we can define $R_i(\Sigma)$, the normal injectivity radius of $\Sigma$. It is a remarkable fact (see [Fuj] and [HK]) that the existence of a uniform lower bound for $R_i(\Sigma)$ ensures the existence of a sequence of points $p_{ik} \in M_i$ such that the sequence $(M_i, p_{ik})$ converges in the Hausdorff-Gromov sense to a pointed hyperbolic cone-manifold $(M_\infty, p_\infty)$ with topological type $(M, \Sigma)$. Moreover, $M_\infty$ must be compact provides that cone angles of $M_{ik}$ are uniformly bounded from below.

Let us also emphasize that the sequence $\text{Vol}(M_i)$ consisting of the riemannian volumes of the hyperbolic manifolds $M_i - \Sigma$ is always uniformly bounded. More precisely, we have (see [Dun] and [Fra])

$$\text{Vol}(M_i) < \text{Vol}(M_0),$$

where $M_0$ denotes the complete hyperbolic manifold that is homeomorphic to $M - \Sigma$.

The purpose of this section is to prove the Theorem (2). It is divided into two parts. The first part contains some preliminary results whereas the remaining part deal with the proof of Theorem (2).

Let us point out that, throughout the rest of the paper, the term ”component” is going to stand for ”connected component”

### 3.1. Preliminary results

Let us recall some definitions and elementary results which will be important for the proof of Theorem (2). We will begin with the classification of two dimensional embedded torus in $M - \Sigma$ (see [Bar2]).

**Lemma 1.** Suppose that $M - \Sigma$ is hyperbolic and let $T$ be a two dimensional torus embedded in $M - \Sigma$. Then $T$ separates $M$. Moreover, one and only one of the following statements holds:

i. $T$ is parallel to a component of $\Sigma$ (hence it bounds a solid torus in $M$),

ii. $T$ is not parallel to a component of $\Sigma$ and it bounds a solid torus in $M - \Sigma$,

iii. $T$ is not parallel to a component of $\Sigma$ and it is contained in a ball $B$ of $M - \Sigma$. Furthermore, $T$ bounds a region in $B$ which is homeomorphic to the exterior of a knot in $S^3$.

Now let us recall the geometric classification of the thin part of a hyperbolic manifold.

**Definition 4.** Consider $\delta > 0$ and let $\mathcal{M}$ be a hyperbolic manifold of dimension 3 (without boundary and perhaps noncomplete). Define $\mathcal{M}_{\text{thin}}(\delta)$, the $\delta$-thin part of $\mathcal{M}$, by

$$\mathcal{M}_{\text{thin}}(\delta) = \{q \in \mathcal{M} : r_{inj}^M(q) < \delta \ \text{et} \ \exp_q \text{ is defined on } B_{T_q \mathcal{M}}(0, 3\delta)\}.$$

The following result concerning the thin part of hyperbolic manifolds will be needed later.

**Proposition 1.** Let $\mathcal{M}$ be a hyperbolic manifold of dimension 3 (without boundary and perhaps noncomplete) of finite volume. If $\delta > 0$ is small enough,
then each component of $\mathcal{M}_{\text{thin}}(\delta)$ contains a maximal region which is isometric to one of the following models:

i. the quotient of a metric neighborhood of a geodesic $\gamma$ in $\mathbb{H}^3$ by a loxodromic element of $PSL_2(\mathbb{C})$ leaving $\gamma$ invariant and whose translation length is not bigger than $\delta$,

ii. a parabolic cusp of rank 2.

This proposition is a consequence of the existence of a Margulis foliation for the thin part of a hyperbolic manifold. A proof for this proposition is given in [BLP, theorem 5.3] where the authors study the thin part of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ and whose cone angles are not bigger than $\pi$. Note that the condition imposed on the cone angles is used only in the description of the singular components of the thin part. We summarize bellow their proof for the proposition above which, indeed, dispenses with the angle condition.

Consider a hyperbolic manifold $M$ and denote by $\pi: \tilde{M} \to M$ the universal cover of $M$. Let $\delta > 0$ be the constant given by the Margulis lemma (see [KM, KM], [BGS] et [BLP]). Then for every component $P$ of $M_{\text{thin}}(\delta)$, the stabilizer of a component of $\pi^{-1}(P) \subset \tilde{M}$ is an elementary subgroup of $PSL_2(\mathbb{C})$ constituted exclusively either by parabolic or by loxodromic elements. Associated to this group we have a canonical foliation of $\mathbb{H}^3$. The pull-back of this foliation by a developing map gives a foliation on $\pi^{-1}(P)$ which is equivariant by the action of $\pi_1M$. The quotient of this foliation is the Margulis foliation on $P$.

To finish the proof, it is sufficient to show that the leaves of this foliation are two-dimensional torus. Because they are flat, it suffices (Gauss-Bonnet) to verify that the leaves are complete. This however, is a consequence of the fact that injectivity radius is constant on the leaves (see [Bar1]).

3.2. Proof of the Theorem 2 The purpose of this section is to study a non-collapsing sequence $M_i$. Without loss of generality, this hypothesis implies the existence of a sequence $p_i \in M - \Sigma$ satisfying

$$r_0 = \inf \{ r_{M_i}^{M_j}(p_i) ; i \in \mathbb{N} \} > 0,$$

and such that the sequence $(M_i, p_i)$ converges in the Hausdorff-Gromov sense to a pointed Alexandrov space $(Z, z_0)$. By definition of the Hausdorff-Gromov convergence, the ball $B_Z(z_0, r_0)$ is isometric to a ball of radius $r_0$ in $\mathbb{H}^3$ and this implies that $Z$ has dimension equal to 3.

We are interested in the case where the length of the singularity remains uniformly bounded, i.e. where

$$\sup \{ L_{M_i}(\Sigma_j) ; i \in \mathbb{N}, j \in \{1, \ldots, l\}\} < \infty .$$

Since $Z$ has dimension 3, this assumption implies by an Ascoli-type argument (passing to a subsequence if necessary) that each component $\Sigma_j$ of $\Sigma$ satisfies one, and only one, of the following statements:

(1) $\sup_{i \in \mathbb{N}} d_{M_i}(p_i, \Sigma_j) < \infty$ and $\Sigma_j$ converges in the Hausdorff-Gromov sense to a quasi-geodesic $\Sigma^Z_j \subset Z$,

(2) $\lim_{i \in \mathbb{N}} d_{M_i}(p_i, \Sigma_j) = \infty$.

This dichotomy allows us to write $\Sigma = \Sigma_0 \cup \Sigma_\infty$, where $\Sigma_0$ contains the components $\Sigma_j$ of $\Sigma$ which satisfy Item (1) and $\Sigma_\infty$ those that satisfy Item (2).
The following lemma shows that the hypothesis of non-collapsing imposes restrictions on the length and on the cone angles of the singular components of $\Sigma$ contained in $\Sigma_0$.

**Lemma 2.** Suppose that the sequence $M_i$ does not collapse and let $p_i \in M - \Sigma$ be a sequence of points such that $r_0 = \inf \left\{ r_{inj}^{M_i}(p_i) : i \in \mathbb{N} \right\} > 0$. If
\[
L = \sup \left\{ L_{M_i}(\Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\} \right\} < \infty ,
\]
then the following inequalities hold:
\begin{enumerate}
  \item $\inf \left\{ L_{M_i}(\Sigma_j) : i \in \mathbb{N}, \Sigma_j \subset \Sigma_0 \right\} > 0,$
  \item $\inf \{ \alpha_{ij} : i \in \mathbb{N}, \Sigma_j \subset \Sigma_0 \} > 0,$
  \item $\sup \{ R_i(\Sigma_j) : i \in \mathbb{N} \text{ and } \Sigma_j \subset \Sigma_0 \} < \infty.$
\end{enumerate}

**Proof.** Consider $\mathcal{R} > \sup \{ d_{M_i}(p_i, \Sigma_j) : i \in \mathbb{N}, \Sigma_j \subset \Sigma_0 \} + r_0$. Note that, by construction, $\mathcal{R} < \infty$ and $B_{M_i}(p_i, r_0) \subset B_{M_i}(\Sigma_j, \mathcal{R})$, for all $i \in \mathbb{N}$ and all component $\Sigma_j$ of $\Sigma_0$.

Fix $i \in \mathbb{N}$ and fix a component $\Sigma_j$ of $\Sigma_0$. Let $\mathcal{A}$ be a region of $\mathbb{H}^3(\alpha_{ij})$ which is bounded by two planes orthogonal to the singular geodesic $\sigma$ of $\mathbb{H}^3(\alpha_{ij})$ and having distance $L_{M_i}(\Sigma_j)$ between them. Using a developing map for $M_i - \Sigma$ and the minimizing geodesics leaving $\Sigma_j$ orthogonally, the manifold $M_i$ can be developed in a compact domain $K \subset \mathcal{A}$ such that $Vol(K) = Vol(M_i)$.

Since $B_{M_i}(p_i, r_0) \subset B_{M_i}(\Sigma_j, \mathcal{R})$, the development of $B_{M_i}(p_i, r_0)$ in $K$ is contained in $B_{\mathbb{H}^3(\alpha_{ij})}(\sigma, \mathcal{R}) \cap \mathcal{A}$. If $V_0$ represents the volume of a ball of radius $r_0$ in $\mathbb{H}^3$, we have
\[
V_0 = Vol(B_{M_i}(p_i, r_0)) \leq Vol(B_{\mathbb{H}^3(\alpha_{ij})}(\sigma, \mathcal{R}) \cap \mathcal{A}) = \frac{\alpha_{ij}}{2} L_{M_i}(\Sigma_j) \sinh^2(\mathcal{R})
\]
and therefore
\[
L_{M_i}(\Sigma_j) \geq \frac{V_0}{\pi \sinh^2(\mathcal{R})} > 0 \quad \text{and} \quad \alpha_{ij} \geq \frac{2V_0}{L \sinh^2(\mathcal{R})} > 0 .
\]

Finally, item (iii) follows from the fact that the sequence $Vol(M_i)$ is uniformly bounded from above (see [3,1]).

With the preceding notations, set
\[
\Sigma_Z = \bigcup_{\Sigma_j \subset \Sigma_0} \Sigma_j^Z \subset Z .
\]

We present now the main result for the non-collapsing:

**Theorem 4 (non-collapsing).** Suppose that there exists a sequence $p_i \in M - \Sigma$ satisfying
\[
r_0 = \inf \left\{ r_{inj}^{M_i}(p_i) : i \in \mathbb{N} \right\} > 0
\]
and such that the sequence $(M_i, p_i)$ converges in the Hausdorff-Gromov sense to a pointed Alexandrov space $(Z, z_0)$ of dimension 3. If
\[
\sup \left\{ L_{M_i}(\Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\} \right\} < \infty ,
\]
then the following assertions hold:
\begin{enumerate}
  \item $Z - \Sigma_Z$ is a hyperbolic manifold of finite volume whose convex and unbounded ends are finite in number and are parabolic cusps of rank 2,
  \item $Z$ is compact (and therefore homeomorphic to $M$) if and only if $\Sigma_\infty = \emptyset$,
\end{enumerate}
iii. if $Z$ is not compact, there is a bijection between the connected components of $\Sigma_\infty$ and the complete ends of $Z - \Sigma_Z$. In fact, each unbounded end $C_j$ of $Z - \Sigma_Z$ is the Hausdorff-Gromov limit of metric neighborhoods (homeomorphic to solid tori) $B_{M_i}(\Sigma_j, r_i)$ of a component $\Sigma_j$ of $\Sigma_\infty$, where $r_i > 0$ is an increasing sequence going off to infinity. In addition, the cone angles $\alpha_{ij}$ and the lengths of these components converge to 0.

**Proof of Item (i).** According to Fuj Lemma 2, every point of $Z - \Sigma_Z$ is the limit of a sequence of points of $M_i - \Sigma$ whose injectivity radius is uniformly bounded from below. This implies that $Z - \Sigma_Z$ is a (without boundary and noncomplete) hyperbolic manifold. Note that the unbounded ends of $Z$ are those of $Z - \Sigma_Z$.

In view of Proposition (1) (see also BLP Theorem 5.3), to prove item (i) it is sufficient to shows the following:

**Claim:** $Vol (Z - \Sigma_Z) < \infty$.

**Proof of Claim:** Suppose for contradiction the statement is false. Let $K_\infty$ be a compact set of $Z - \Sigma_Z$ whose riemannian volume is strictly greater than $Vol(M_{comp})$. Since the convergence is bilipschitz on compact subsets [CHK Theorem 6.20], there exists an index $i_0 \in \mathbb{N}$ and a compact subset $K_{i_0}$ of $M_{i_0} - \Sigma$ (near $K_\infty$) such that $Vol(M_{comp}) < Vol_{M_{i_0}}(K_{i_0}) \leq Vol(M_{i_0})$.

This is however impossible since $Vol(M_{i_0}) < Vol(M_{comp})$ (see 3.1). 

Proof of Items (ii) and (iii). If $Z$ is compact then $\Sigma_\infty = \emptyset$. Suppose now that $Z$ is not compact. By Lemma 2 we can choose $R > 0$ such that

$$BM_i(\Sigma_j, R(\Sigma_j)) \subset BM_i(p_i, R/2)$$

for all connected component $\Sigma_j$ of $\Sigma_0$ and all $i \in N$. Let $K$ be a compact subset of $Z$ which contains the ball $B_{\infty}(z_0, R)$ (and hence $\Sigma_Z$) in its interior and satisfies

$$Z = Z - \text{int}(K) = C_1 \sqcup \ldots \sqcup C_m,$$

where each $C_k \cong T^2 \times [0, \infty)$ is a cuspidal end of $Z$.

Consider a sequence $C_{i_k} = T^2 \times [0, t_i]$ of compact subsets of $C_1$, where $t_i > 0$ is an unbounded and strictly increasing sequence.

Let $\varepsilon_i > 0$ be a sequence converging to zero. Without loss of generality, there exists (according to [CHK] Theorem 6.20) a sequence of $(1 + \varepsilon_i)$-bilipschitz embeddings $f_{i_k} : C_{i_k} \rightarrow M_i - \Sigma$ onto their images. Therefore, the sequence $B_{i_k} = f_{i_k}(C_{i_k})$ converges in the bilipschitz sense to the compact set $C_{i_1}$.

Consider now a sequence of holonomy representations $\zeta_{i_k} : Z \times Z \rightarrow PSL_2(\mathbb{C})$ for the hyperbolic structures on the interior sets $B_{i_k}$. According to [CHK] Theorem 6.22, we can assume that

$$(3.2) \quad \zeta_{i_k} \circ (f_{i_k})_* \rightarrow \varphi_1,$$

where $\varphi_1 : Z \times Z \rightarrow PSL_2(\mathbb{C})$ is a holonomy representation of the hyperbolic structure in the interior of $C_1$ and where $(f_{i_k})_* : Z \times Z \rightarrow \pi_1(M - \Sigma)$ is the canonical homomorphism induced by the map $f_{i_k}$.
Consider the torus $T_{1i} = f_{1i} \left( T^2 \times \{0\} \right)$ embedded in $M - \Sigma$. Since $K$ contains the ball $B(\varepsilon_0, R)$, the torus $T_{1i}$ cannot be parallel to a component $\Sigma_j$ of $\Sigma_0$. For $i$ sufficiently large, the torus $T_{1i}$ cannot be contained in a ball of $M - \Sigma$. To see this, consider a homotopically nontrivial loop $\gamma_1$ on $T^2 \times \{0\} \subset C_{1i}$. Since $C_1$ is a parabolic cusp, $\varphi_1(\gamma_1)$ is a nontrivial parabolic element of $PSL_2(\mathbb{C})$ and therefore the convergence (3.2) implies that $\zeta_{1i} \circ (f_{1i})_*(\gamma_1)$ is not trivial for $i$ very large. The same then holds for the sequence $(f_{1i})_*(\gamma_1)$.

According to Lemma (1), we can suppose that the torus $T_{1i}$ bounds a solid torus $W_{1i}$ in $M$ (with perhaps a singular soul). Note that
\[
\lim_{i \to \infty} diam_{M_i}(W_{1i}) = \infty,
\]
because $f_{1i}(C_{1i}) \subset W_{1i}$, for all $i \in \mathbb{N}$.

We can repeat the same construction for each cusp $C_k$ of $Z$ in order to obtain sequences of embedded tori $T_{ki} \subset M - \Sigma$ ($k \in \{1, \ldots, m\}$ and $i \in \mathbb{N}$), each of then boundy solid torus $W_{ki}$ in $M - \Sigma_0$. Furthermore whose diameters become infinite with $i$. This yields a sequence of 3-manifolds with torus boundary
\[
\mathcal{M}_i = M_i - \bigcup_{k=1}^{m} W_{ki}
\]
such that $M$ can be obtained by Dehn filling on their boundary components. By construction, the sequence $\mathcal{M}_i$ converges in the Hausdorff-Gromov sense to the compact $K$ and then (by Perelman’s stability theorem [Kap]), we can assume that the manifolds $\mathcal{M}_i$ are all homeomorphic to $K$.

For all $i \in \mathbb{N}$ and all $k \in \{1, \ldots, m\}$, fix a homotopically nontrivial loop $\mu_{ki}$ in $T^2 \times \{0\} \subset C_k$ satisfying:
- the loop $f_{ki} \circ \mu_{ki}$ bounds a disc in $W_{ki}$,
- if, for some index $j \in \mathbb{N}$, a loop $\mu_{kj}$ belongs to the same homotopy class of the loop $\mu_{ki}$, then $\mu_{kj} = \mu_{ki}$.

The rest of the proof is going to be divided in two cases depending on whether or not $\Sigma_0$ is empty.

1st case : $\Sigma_0 = \emptyset$.

Since the link $\Sigma$ was supposed to be non empty, it follows that $\Sigma_\infty \neq \emptyset$. Since the distance between $p_i$ and $\Sigma_\infty$ becomes infinite, we can assume that $\Sigma_\infty$ is contained in the complement of $\mathcal{M}_i$. More precisely, we can also assume (cf. Lemma (1)) that each solid torus of $M_i - \mathcal{M}_i$ contains at most one component of $\Sigma_\infty$ and, in the latter case, this component corresponds to the soul of the torus in question.

The singular set $\Sigma_\infty$ has a finite number of elements. Passing to a subsequence if necessary, we obtain an one-to-one map which associates each component $\Sigma_j$ of $\Sigma_\infty$ to a component $C_{kj}$ of $Z$, that is, the component $\Sigma_j$ is contained in the component $W_{kj,i}$ of $M_i - \mathcal{M}_i$, for all $i \in \mathbb{N}$.

Recall that every connected component $\Sigma_j$ of $\Sigma_\infty$ satisfies $\lim_{i \to \infty} d_{M_i}(p_i, \Sigma_j) = \infty$. Since the tori $T_{kj,i}$ remains at a finite distance to the points $p_i$ and they are parallel to the components $\Sigma_j$, we must have $\lim_{i \to \infty} R_i(\Sigma_j) = \infty$.

Since $\Sigma_0 = \emptyset$ and thanks to [Fu] Theorem 1], we have that the cone angles of $\Sigma$ converge to zero and $Z$ has a complete hyperbolic structure whose ends are
associated with components of $\Sigma_\infty$. In other words, the injection defined above between the components of $\Sigma_\infty$ and the components of $Z$ is, in deed, a bijection.

2nd case : $\Sigma_0 \neq \emptyset$.

Denote by $\Lambda$ the subset of $\{1, \ldots, m\}$ containing the indices that are not associated with components of $\Sigma_\infty$. Denote also by $\Omega$ the subset of $\{1, \ldots, m\}$ containing the indices that are associated with components of $\Sigma_\infty$ whose sequence of cone angles does not converge to zero.

**Lemma 3.** There exist $i_0 \in \mathbb{N}$ satisfying: for each $k \in \Lambda \cup \Omega$, the homotopy classes of loops $\mu_{ki}$ ($i > i_0$) are pairwise distinct.

**Proof of Lemma (4) :**
Suppose for a contradiction that the statement of the lemma does not hold. Without loss of generality, there exists $k_0 \in \Lambda \cup \Omega$ such that all loops $\mu_{k_0i}$ ($i \in \mathbb{N}$) belongs to the same homotopy class. By construction, this implies that the loops $\mu_{k_0i}$ ($i \in \mathbb{N}$) are the same loop, say $\mu$.

Suppose first that $k_0 \in \Lambda$. By construction, for all $i \in \mathbb{N}$.

$$\zeta_{ki} \circ (f_{k_0i})_* (\mu) = \zeta_{k_0i} (f_{k_0i} \circ \mu) = 1_{PSL_2(\mathbb{C})},$$

for all $i \in \mathbb{N}$. Because $\varphi_{k_0} ([\mu])$ is a nontrivial parabolic element of $PSL_2 (\mathbb{C})$, we have a contradiction.

Suppose now that $k_0 \in \Omega$. Then $k_0 = k_j$, for some component $\Sigma_j$ of $\Sigma_\infty$ whose sequence of cone angles converges to $\alpha_\infty \neq 0$. Since the maps $f_{k_0i}$ are $(1 + \varepsilon_i)$-bilipschitz embeddings (with $\varepsilon_i$ shrinks down to zero), the loops $f_{k_0i} \circ \mu$ must have bounded lengths.

As noted in the preceding case, the sequence $R_i (\Sigma_j)$ of the normal injectivity radii of the component $\Sigma_j$ goes off to infinity. Since $\alpha_\infty \neq 0$, the sequence $L_{M_i} (f_{k_0i} \circ \mu)$ formed by the lengths of the loops $f_{k_0i} \circ \mu$ cannot be bounded. This is a contradiction with above paragraph.

As a consequence of the above lemma, we will show that the set $\Lambda \cup \Omega$ is empty.

To do this, the following lemma will be needed:

**Lemma 4.** Given $k \in \Lambda$, there exists $i_0 = i_0 (k) \in \mathbb{N}$ such that the solid tori $W_{ki}$ contains a simple closed geodesic $\sigma_{ki}$, for every $i > i_0$.

**Proof of Lemma (4) :** Fix $k \in \Lambda$ and let

$$\delta = \inf \left\{ \frac{r_{\text{inj}}^{Z-S}}{2} (z) : z \in C_{k_1} \right\} > 0.$$

Since the map $f_{ki \inj}$ : $C_{k_1} \rightarrow B_{ki}$ becomes closer and closer to isometries, there exists $i_1 \in \mathbb{N}$ such that

$$r_{\text{inj}}^{M_i} (q) > \delta,$$

for all $i > i_1$ and for all $q \in B_{ki}$ (in particular, for all $q \in T_{ki}$).

**Claim :** There is $i_2 \in \mathbb{N}$ such that, for all $i > i_2$, we can find a loop $\gamma_{ki}$ in $W_{ki}$ which is homotopically nontrivial in the interior $M - \Sigma$ and has length smaller than $\delta$. 
Proof of Claim: Consider the loops constituted by two geodesic segments with same ends and equal lengths which, furthermore, are smaller than $\frac{\delta}{4}$. Note that there are always homotopically nontrivial, otherwise we would obtain, after development, two distinct geodesic arcs with the same ends and equal lengths in $\mathbb{H}^3$, what is not possible.

The fact that $W_{k_i}$ does not admit this type of loop in its interior is equivalent to saying that all points of $W_{k_i}$ have injectivity radius not smaller than $\frac{\delta}{4}$. This is a contradiction because the sequence $Vol(M_i)$ is uniformly bounded from above (see 3.1) and the diameter of components $W_{k_i}$ becomes infinite. 

Consider $i_0 = \max \{i_1, i_2\}$ and fix $i > i_0$. Let $\gamma_{k_i} \subset W_{k_i}$ be a loop as above. According to [Koj] Lemma 1.2.4], the loop $\gamma_{k_i}$ is freely homotopic (in $M - \Sigma$) to a closed geodesic $\sigma_{k_i} \subset M - \Sigma$. Moreover, the length of $\sigma_{k_i}$ is smaller than $\delta$ because the length of loops is strictly decreasing along this homotopy. Because the points of the torus $T_{k_i}$ have injectivity radius bigger than $\delta$, all the loops involved in this homotopy must lie entirely in the interior of $W_{k_i}$. In particular, $\sigma_{k_i} \subset W_{k_i}$.

If $\sigma_{k_i}$ is not simple, then it gives rise to a loop $\gamma'_{k_i}$ constituted by two geodesic segments with same ends and equal lengths which are smaller than $\frac{\delta}{4}$. This implies that the injectivity radius of the ends of $\gamma'_{k_i}$ is smaller than $\frac{\delta}{4}$. We can apply the same construction for the loop $\gamma'_{k_i}$ in order to obtain a new closed geodesic $\sigma_{k_i} \subset W_{k_i}$ whose length is smaller than $\frac{\delta}{4}$. Since the injectivity radius of points of $W_{k_i}$ bounded from below by compactness, this process must end after a finite number of steps and therefore we can suppose that $\sigma_{k_i}$ is simple. This completes the proof of Lemma (4). 

The following lemma shows that $\Sigma_\infty$ is not empty and the cone angles of its components goes to zero. Moreover the map between the components of $\Sigma_\infty$ and the components of $Z$ must be a bijection.

Lemma 5. The set $\Lambda \cup \Omega$ is empty.

Proof of Lemma 5: According to the above lemma, we can suppose the existence of a simple closed geodesic $\sigma_{k_i}$ in the solid torus $W_{k_i}$, for every $i \in \mathbb{N}$ and every $k \in \Lambda$. If the manifolds $M_i$ are regarded as hyperbolic cone-manifolds with topological type $(M, \Sigma')$, where

$$\Sigma' = \Sigma \cup \bigcup_{k \in \Lambda} \sigma_{k_i}$$

and the cone angles on the geodesics $\sigma_{k_i}$ are equal to $2\pi$, it follows from Lemma (1) that the tori $T_{k_i}$ are parallel to the geodesics $\sigma_{k_i}$. In addition, $M - \Sigma'$ admits a complete hyperbolic structure (see [Koj]2) that will be denoted by $M_0$.

For all $i \in \mathbb{N}$ and all $k \in \Lambda$, denote the homotopy class of the loop $\mu_{ik}$ by $(p_{ki}, q_{ki}) \in \mathbb{Z} \times \mathbb{Z} \approx \pi_1 C_k$. Without loss of generality, the Thurston’s hyperbolic Dehn surgery ([CHK] theorem 1.13) gives a sequence of complete hyperbolic manifolds $\mathcal{M}(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im})$ diffeomorphic to $M - \Sigma$ and such that

$$V_i := Vol(\mathcal{M}(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im})) < Vol(M_0),$$

where $(p_{ki}, q_{ki}) = \infty$, for all $i \in \mathbb{N}$ and all $k \in \{1, \ldots, m\} - \Lambda$.

Since, for each $k \in \Lambda$, the pairs $(p_{ki}, q_{ki})_{i \in \mathbb{N}}$ are pairwise distinct (since the homotopy classes of the loops $\mu_{ik}$ are pairwise distinct), a subsequence $\mathcal{M}(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im})$
such that
\[
\lim_{s \to \infty} \|(p_{ki_s}, q_{ki_s})\| = \lim_{s \to \infty} (p_{ki_s})^2 + (q_{ki_s})^2 = \infty, \quad \text{for every } k \in \Lambda
\]
always exists. Thurston’s hyperbolic Dehn surgery then gives
\[
(3.6) \quad \lim_{s \to \infty} V_{i_s} = Vol (M_{comp}).
\]
Recall that the Riemannian volume of a complete hyperbolic manifold with finite volume is a topological invariant (Mostow’s Theorem). Since the manifolds \( M(p_{i1}, q_{i1}, \ldots, p_{im}, q_{im}) \) are diffeomorphic, the sequence \( V_i \) must be constant. This contradicts the statements of Corollary 3 and Corollary 6. Hence \( M_i - M_\alpha \) cannot have nonsingular components. Therefore, \( \Sigma_\infty \neq \emptyset \) and the map between the components of \( \Sigma_\infty \) and the components of \( Z \) is a bijection.

**Corollary 3.** Suppose that the sequence \( M_i \) does not collapse and verifies
\[
\sup \{ L_M, (\Sigma_j) : i \in \mathbb{N}, j \in \{1, \ldots, l\} \} < \infty.
\]
If there is \( \varepsilon \in (0, 2\pi) \) such that the cone angles \( \alpha_{ij} \) belongs to \((\varepsilon, 2\pi]\), then there exists a sequence of points \( p_{i_s} \in M - \Sigma \) such that the sequence \( (M_{i_s}, p_{i_s}) \) converges in the Hausdorff-Gromov sense to a compact and 3-dimensional pointed Alexandrov space \( (Z, z_0) \) (in fact homeomorphic to \( M \)). Moreover, there exists a finite union of quasi-geodesics such that \( Z - \Sigma_Z \) is a noncomplete hyperbolic manifold of finite volume.

**Remark 4.** Suppose that \( \Sigma \) is not connected. If \( (M_i, p_i) \) is a sequence as in the statement of the Theorem, then the inequality
\[
\sup \{ diam_{M_i}, (\Sigma_i) : i \in \mathbb{N} \} < \infty
\]
is a necessary and sufficient condition to ensure that the sequence \( diam (M_i) \) remains bounded.

We have also the following less immediate corollary:

**Corollary 4.** Let \( M \) be a closed, orientable and irreducible 3-manifold and let \( \Sigma \) be an embedded link in \( M \). Assume that there exists a sequence \( M_i \) of hyperbolic cone-manifolds with topological type \( (M, \Sigma) \) and having the same cone angles \( \alpha_i \in (0, 2\pi] \) for all components of \( \Sigma \). Then there is a pointed subsequence \( M_{i_k} \) converging to \( M_0 \) if and only if the following three conditions hold:

i. \( \sup \{ L_{M_i}, (\Sigma_i) : i \in \mathbb{N} \} < \infty \),

ii. \( \sup \{ diam (M_i) : i \in \mathbb{N} \} = \infty \),

iii. the sequence \( M_i \) does not collapse.

**Proof.** By Kojima’s result (see \([Ko]\)), the existence of a subsequence \( M_{i_k} \) converging to \( M_0 \) is equivalent to the convergence of the cone-angles \( \alpha_{i_k} \) to zero.

Suppose that the sequence \( \alpha_i \) converges to zero. Without loss of generality, we can assume that \( \alpha_i \in (0, \pi] \) for every \( i \in \mathbb{N} \). According to \([Ko]\), there exists a continuous path (parametrized by cone angles) of hyperbolic cone structures with topological type \( (M, \Sigma) \) which connects the hyperbolic cone structure of \( M_0 \) to the complete hyperbolic structure on \( M - \Sigma \). Moreover, by uniqueness of the hyperbolic cone structures with cone angles not bigger than \( \pi \) (see \([Ko]\)), this path contains the hyperbolic cone structures of \( M_i \) for every \( i \in \mathbb{N} \). Then for every point \( p \in M \), the sequence \( (M_i, p) \) converges in the Hausdorff-Gromov sense to \( (M - \Sigma, p) \) with the complete hyperbolic structure. This implies the items (ii) and (iii). The item
(i) is a consequence of Thurston’s hyperbolic Dehn surgery theorem which implies that the sequence $\mathcal{L}_{M_i}(\Sigma)$ converges to zero.

Conversely, suppose now that items (i), (ii) and (iii) are true. Then there exists a sequence of points $p_{ik} \in M - \Sigma$ satisfying

$$\inf \left\{ r_{\text{inj}}(p_{ik}) ; \ k \in \mathbb{N} \right\} > 0$$

and such that the sequence $(M_{ik}, p_{ik})$ converges in the Hausdorff-Gromov sense to a noncompact and 3-dimensional pointed Alexandrov space $(Z, z_0)$. Corollary (3) then shows that the sequence $\alpha_i$ must converge to zero. □

4. Applications

4.1. Small links. An embedded link $\Sigma$ in a 3-manifold $M$ is called small (in $M$) if it has an open tubular neighborhood $U$ such that $M - U$ does not contain an embedded essential surface whose boundary is empty or an a union of meridians of $\Sigma$. An important fact due to W.Thurston and A.Hatcher (see [HT] Lemma 3) is that every 3-manifold containing a small link does not admit an embedded essential surface.

Given a 3-manifold $M$, let $\Sigma$ be an embedded link in $M$. Suppose there exists a sequence $M_i$ of hyperbolic cone-manifolds with topological type $(M, \Sigma)$ and consider the sequence $\mathcal{L}_{M_i}(\Sigma)$ formed by the lengths of the singular set $\Sigma$ in $M_i$.

As a consequence of the Culler-Shalen theory (see [CS]) we have the following proposition:

**Proposition 2.** Let $M_i$ be a sequence of hyperbolic cone-manifolds with topological type $(M, \Sigma)$. If $\Sigma$ is a small link in $M$, then

$$\sup \left\{ \mathcal{L}_{M_i}(\Sigma_j) ; \ i \in \mathbb{N} \text{ and } \Sigma_j \text{ component of } \Sigma \right\} < \infty.$$  

When $\Sigma$ is a small link in $M$, Theorem (2) yields the following corollaries:

**Corollary 5.** Suppose that $M$ is a closed, orientable, irreducible and not Seifert fibered 3-manifold and let $\Sigma$ be an embedded small link in $M$. Then there exists a constant $V = V(M, \Sigma) > 0$ such that $\text{Vol}(M) > V$, for every hyperbolic cone-manifold $M$ with topological type $(M, \Sigma)$.

**Proof.** First note that $M$ is not a $Sol$ manifold. In fact every $Sol$ manifold is foliated by essential two dimensional tori and this is not possible since $\Sigma$ is small (see [HT] Lemma 3).

Suppose that the lower bound $V$ does not exist. Since $\Sigma$ is small in $M$, the non-existence of $V$ implies the existence of a sequence of hyperbolic cone-manifolds $\mathcal{M}_i$ with topological type $(M, \Sigma)$ satisfying

- $\sup \{ \mathcal{L}_{\mathcal{M}_i}(\Sigma_j) ; \ i \in \mathbb{N} \text{ and } \Sigma_j \text{ component of } \Sigma \} < \infty$,
- the sequence $\text{Vol}(\mathcal{M}_i - \Sigma)$ formed by the riemannian volumes of the hyperbolic manifolds $\mathcal{M}_i - \Sigma$ shrinks down to to zero (and therefore the sequence $\mathcal{M}_i$ collapses).

According to Therem (2), $M$ must be Seifert fibered and this contradicts our hypothesis. □

**Corollary 6.** Suppose that $M$ is a closed, orientable, irreducible and not Seifert fibered 3-manifold and let $\Sigma$ be an embedded small link in $M$. Given $\varepsilon \in (0, 2\pi)$, there is a constant $K = K(M, \varepsilon) > 0$ such that $\text{diam}(\mathcal{M}) < K$, for every
hyperbolic cone-manifold $M$ with topological type $(M, \Sigma)$ and having cone angles belonging to $(\varepsilon, 2\pi]$.

PROOF. As seen in the previous corollary, $M$ is not a Sol manifold. Fix $\varepsilon \in (0, 2\pi)$ and suppose that the upper bound $K$ does not exist. Since $\Sigma$ is small in $M$, the non-existence of $K$ implies the existence of a sequence of hyperbolic cone-manifolds $M_i$ with topological type $(M, \Sigma)$, having cone angles $\alpha_{ji} \in (\varepsilon, 2\pi]$ and satisfying

i. $\sup \{L_{M_i}(\Sigma_j) : i \in \mathbb{N} \text{ and } \Sigma_j \text{ component of } \Sigma\} < \infty$,

ii. the sequence $\text{diam}(M_i)$ formed by the diameters of the hyperbolic cone-manifolds $M_i$ go to infinity.

Since $M$ is neither Seifert fibered nor a Sol manifold, it follows from item (i) and Theorem 2 that the sequence $M_i$ does not collapse. Moreover, since the cone angles $\alpha_{ji}$ belong to $(\varepsilon, 2\pi]$, it follows that the sequence $\text{diam}(M_i)$ is bounded and this yields a contradiction with item (ii).

4.2. Proof of Corollary 2. First, we would like to recall that the existence of a deformation $M_\alpha$ as in Corollary 2 is a consequence of the Local Deformation Theorem due to Hodgson and Kerckhoff [HK2].

PROOF. The implication $(i \Rightarrow ii)$ is immediate (see [Koj]). Suppose now that the sequence $L_{M_{\alpha}}(\Sigma)$ converges to 0 when $\alpha$ converges to $\theta$. Then

$$\sup \{L_{M_{\alpha_i}}(\Sigma_j) : i \in \mathbb{N} \text{ and } \Sigma_j \text{ component of } \Sigma\} < \infty,$$

for every sequence $\alpha_i \in (\theta, 2\pi]$ converging to $\theta$. Consider such a sequence $\alpha_i$. Since $M$ is hyperbolic (and therefore is neither Seifert fibered nor a Sol manifold), it follows from theorem 2 that the sequence $M_{\alpha_i}$ does not collapse. Moreover, since the sequence $L_{M_{\alpha_i}}(\Sigma)$ converges to zero, we must have $\lim_{i \to \infty} \text{diam}(M_{\alpha_i}) = \infty$. This concludes the proof of the implication $(ii \Rightarrow iii)$.

To prove $(iii \Rightarrow i)$ take a sequence $\alpha_i$ satisfying item $(iii)$. Again by Theorem 2, it follows that the sequence $M_{\alpha_i}$ does not collapse. Moreover, since the sequence $\text{diam}(M_{\alpha_i})$ is not bounded, we must have $\theta = 0$ because all the components of $\Sigma$ have the same cone angle. Then, by Kojima’s work (see [Koj]), it follows that $M_i$ converges (in the Hausdorff-Gromov sense) to $M_0$. 

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