AN INVESTIGATION OF
THE CHUNG-FELLER THEOREM

ELI A. WOLFHAGEN

Abstract. In this paper, we shall prove the Chung-Feller Theorem in several ways. We provide an inductive proof, bijective proof, and proofs using generating functions, and the Cycle Lemma of Dvoretzky and Motzkin [2].

1. Introduction

The main focus of this paper is to prove the following result of Chung and Feller [1]:

The Chung Feller Theorem. The number of paths from (0, 0) to (n, n), with steps (0, 1) and (1, 0) and exactly 2k steps above the line x = y is independent of k, for every k such that 0 ≤ k ≤ n. In fact, it is equal to the nth Catalan number.

2. Preliminary Definitions and an Inductive Proof

A Dyck path is a path with steps (0, 1) and (1, 0) that starts at the origin and ends at (n, n) for some positive integer n. We can study the number of paths indicated in the above Chung-Feller theorem in a simpler form if we generalize Dyck paths as seen in [3]. A k-negative path of length 2n is a path from (0, 0) to (2n, 0), with steps (1, 1) and (1, −1), such that exactly 2k of these steps are below the horizontal axis.

A prime Dyck path is a Dyck path that returns to the x-axis only once at the end of the path. A negative prime Dyck path is a a prime Dyck path reflected about the x-axis. In general, any negative Dyck path is the reflection about the x-axis of a Dyck path.

An Inductive Proof of the Chung-Feller Theorem. As a base case, for n = 0, there is only one path with 0 steps. So clearly there is only \(C_0 = 1\) path with 0 steps below the x-axis.

Now, let \(n > 0\) and assume that for all \(i\) such that \(0 ≤ i < n\), the number of \(l\)-negative paths of length \(2i\) is equal to \(C_i\) for all \(l ≤ i\). Therefore, choose some \(k ≤ n\).

Any nonempty \(k\)-negative path either starts out with either a prime Dyck path or a negative prime Dyck path. In the first case, the \(k\)-negative path starts with an up-step followed by a Dyck path of some length \(2p − 2\) which is then followed by a down-step and a \(k\)-negative path of length \(2n − 2p\), for some \(p ≤ n − k\). On the other hand, the second case deals with paths that start out with a down-step, followed by a \((q − 1)\)-negative path of length \(2q − 2\) which is followed by an up-step and a \((k − q)\)-negative path of length \(2n − 2q\), for some \(q ≤ k\).

Let \(N_{\text{up}}\) be the number of \(k\)-negative paths of length \(2n\) that start with an up-step. Let \(N_{\text{down}}\) be the number of \(k\)-negative paths of length \(2n\) that start with a down-step. Let \(N\) be the total number of \(k\)-negative paths of length \(2n\).

If a path starts out with an up-step and a Dyck path of length \(2p − 2\), then the number of such paths is the total number of Dyck paths of length \(2p − 2\) multiplied by the number
of $k$-negative paths of length $2n - 2p < 2n$. By the inductive hypothesis, this means that the number $N_p^+$ of such paths is $C_{p-1}C_{n-p}$. For the total number of $k$-negative paths that start off with an up-step we need to take the sum of $N_p^+$, for all $1 \leq p \leq n-k$. Therefore, we have that the total number of $k$-negative paths of length $2n$ that start with an up-step is

$$N_{\text{up}} = \sum_{p=1}^{n-k} N_p^+ = \sum_{p=1}^{n-k} C_{p-1}C_{n-p}.$$ 

Similarly, we define $N_q^-$ as the number of $k$-negative paths that start out with a down-step and a negative Dyck path of length $2q - 2$. So $N_q^-$ is equal to the number of negative Dyck paths of length $2q - 2$ multiplied by the number of $(k - q)$-negative paths of length $2n - 2q < 2n$. Obviously, since the negative Dyck paths are reflections of (positive) Dyck paths, the number of negative Dyck paths of length $2q - 2$ is equal to the number of Dyck paths of length $2q - 2$. Therefore, we again have the sum of products of two Catalan numbers, and the total number of $k$-negative paths of length $2n$ that start with a down-step is

$$N_{\text{down}} = \sum_{q=1}^{k} C_{q-1}C_{n-q} = \sum_{q=1}^{k} C_{n-q}C_{q-1}.$$ 

Therefore the total number of $k$-negative paths of length $2n$ is equal to

$$\mathcal{N} = N_{\text{up}} + N_{\text{down}} = \sum_{p=1}^{n-k} C_{p-1}C_{n-p} + \sum_{q=1}^{k} C_{n-q}C_{q-1};$$

that is,

$$\mathcal{N} = (C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-k}C_{k}) + (C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_{n-k}C_{k-1}).$$

Reversing the order of the summands in the second parentheses clearly gives

$$\mathcal{N} = (C_0C_{n-1} + \cdots + C_{n-k}C_{k}) + (C_{n-k}C_{k-1} \cdots + C_{n-1}C_0)$$

$$= \sum_{i=0}^{n-1} C_{i}C_{n-i-1}.$$ 

Now, for any Dyck path of length $2n$ we can look at the first nonempty prime path. It will have length $2i + 2$ for some $i \geq 0$, giving a total of $C_i$ such prime paths. Since the total length of the entire path is $2n$, we must have $i \leq n - 1$. Therefore, since the rest of the path is an arbitrary Dyck path of length $2n - 2i - 2$ there are $C_{n-i-1}$ possible paths that can follow the initial prime path of length $2i + 2$. Therefore, the total number of Dyck paths of length $2n$ that start with a prime path of length $2i + 2$ is given by $C_{i}C_{n-i-1}$. Therefore if we sum over $0 \leq i \leq n - 1$ we will get all possible Dyck paths of length $2n$, which is given by $C_n$. Therefore

$$\mathcal{N} = \sum_{i=0}^{n-1} C_{i}C_{n-i-1} = C_n,$$

and so the total number of $k$-negative paths of length $2n$ is equal to $C_n$. $\square$
Let $S_k$ denote the set of all $k$-negative paths of length $2n$. Choose some $k$, such that $0 \leq k < n$.

For $s \in S_k$, find the last positive prime Dyck path $q$. Next factor $s$ into $s = pqr$, where $p$ is the path up to the last positive prime Dyck path and $r$ is the rest of the path after $q$. Since $q$ is the last positive prime Dyck path, the remainder of the path $r$ must be a negative Dyck path. This factorization is unique.

Like any other prime Dyck path, $q$ is composed of an up-step followed by an arbitrary Dyck path $Q$ and a down-step. So $q$ can be rewritten as $q = uQd$, where $u$ denotes an up-step and $d$ denotes a down-step. Define the function $\varphi_+: S_k \to S_{k+1}$ by

$$
\varphi_+ (s) = \varphi_+ (puQdr) = pdruQ,
$$

for any $s \in S_k$.

Given an $s \in S_k$, $\varphi_+ (s) = pdruQ$ by (2). Since $p$ begins and ends on the $x$-axis, the step $d$ is below the $x$-axis. As noted earlier $r$ is an arbitrary negative Dyck path, so in the path $\varphi_+ (s)$, $r$ starts and ends at height $-1$, without going above it. Therefore the step $u$ is negative, starting at height $-1$ and ends on the $x$-axis. Since $Q$ is a Dyck path it contains no negative steps, so altogether $\varphi_+ (s)$ has 2 steps below the $x$-axis in addition to the number of negative steps in path segments $p$ and $r$. Since the only negative steps in the path $s \in S_k$ occur in $p$ and $r$, the number of negative steps in path segments $p$ and $r$ is equal to $2k$. Thus $\varphi_+ (s)$ has $2k + 2$ steps below the $x$-axis and so is a member of $S_{k+1}$.

Now for any $\sigma \in S_{k+1}$ we can find the last negative prime path $\omega$. Thus we can write $\sigma = \pi \omega \varrho$, where $\pi$ is the path up to the last negative prime, and $\varrho$ is the rest of the path which is by construction positive. So since $\omega$ is a negative prime path it can be written as $d\Omega u$, where $\Omega$ is an arbitrary negative Dyck path. Therefore any $\sigma \in S_{k+1}$ can be uniquely written as $\sigma = \pi d\Omega u \varrho$, where $\Omega$ is a negative path and $\varrho$ is a positive path. Therefore, if $s_1 = p_1 u Q_1 d r_1 \neq s_2 = p_2 u Q_2 d r_2 \in S_k$ then either $p_1 \neq p_2$, $Q_1 \neq Q_2$ or $r_1 \neq r_2$, so clearly since $\varphi_+ (s_1) = p_1 d r_1 u Q_1, \varphi_+ (s_2) = p_2 d r_2 u Q_2$, since the factorization in $S_{k+1}$ is unique, $\varphi_+ (s_1) \neq \varphi_+ (s_2)$. Therefore, $\varphi_+ \text{ is injective.}$

Using the factorization on $S_{k+1}$ we can define the injection $\varphi_- : S_{k+1} \to S_k$ which sends $\sigma = \pi d \Omega u \varrho \mapsto \pi u \varrho d \Omega$. Since there are now two fewer negative steps in $\varphi_- (\sigma)$ than in $\sigma$ itself, due to the fact that the up-step now comes before the down-step, $\varphi_- (\sigma) \in S_k$. Additionally, since there is an unique decomposition in $S_k$, $\varphi_-$ is injective. Thus, by the Skroder-Bernstein theorem $|S_k| = |S_{k+1}|$. In fact, $\varphi_- = \varphi_+^\dagger$, since $\varphi_- (pdruQ) = puQdr$ for all $s = puQdr \in S_k$, so $\varphi_+$ is in fact a bijection.

Therefore, we can now bijectively prove the Chung-Feller theorem.

**Bijective Proof.** The cardinality of $S_k$ and $S_{k+1}$ are equal, for all nonnegative $k \leq n - 1$. Since there are $n + 1$ equal sets of paths of length $2n$ each with the same cardinality, and the total number of paths of length $2n$ is $\binom{2n}{n}$, we have the equation

$$(n + 1)K = \binom{2n}{n},$$

where $K$ is the number of $k$-negative paths of length $2n$ for any $k$ such that $0 \leq k \leq n$.

Therefore $K = \frac{1}{n+1} \binom{2n}{n} = C_n$. \(\square\)
4. A Generating Function Approach to the Theorem

In general a generating function is a very useful tool used in enumerative combinatorics. The generating function of an infinite sequence \( \{a_n\} \) can be thought of as the function for which the coefficient of \( x^n \) in the power series expansion about \( x = 0 \), is \( a_n \). That is, if \( A(x) \) is the generating function for the sequence \( \{a_n\} \), then \( A(x) = \sum_{n=0}^{\infty} a_n x^n \).

The generating function can also be obtained more formally by weighting certain combinatorial objects by a variable \( x \). For instance, if \( \{a_n\} \) represents the number of bloops with \( n \) glurps, then \( A(x) \) is obtained by weighting each glurp by \( x \) and summing over all bloops. We will use both formulations in this section to give a straight-forward proof of the Chung-Feller theorem.

The generating function for the Catalan numbers is given by

\[
(3) \quad c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x},
\]

which is obtained by weighting every step in a Dyck path by \( \sqrt{x} \) and summing over all possible Dyck paths.

Let us construct a generating function for arbitrary paths that end on the \( x \)-axis by weighting each step by \( \sqrt{x} \) and each step below the \( x \)-axis by \( \sqrt{tx} \), such that a path with \( 2n \) total steps, \( 2k \) of which are below the \( x \)-axis is given the weight \( t^k x^n \). As described in section 2, the primes of a \( k \)-negative path are either positive prime Dyck paths or negative prime Dyck paths. Let \( P_n \) denote the number of (positive) prime Dyck paths of length \( 2n \).

Such a path of length \( 2n \) consists of an arbitrary Dyck path of length \( 2n - 2 \) sandwiched between an up-step and down-step. So the number of prime Dyck paths of length \( 2n \) is given by the \((n-1)\)th Catalan number, that is \( P_n = C_{n-1} \). Therefore the generating function for the prime Dyck paths is given by

\[
(4) \quad p_+(x) = \sum_{n=1}^{\infty} P_n x^n = \sum_{n=1}^{\infty} C_{n-1} x^n = x \sum_{n=0}^{\infty} C_n x^n = xc(x).
\]

Similarly, for negative prime Dyck paths the generating function, now weighted by \( \sqrt{tx} \) because each step in a negative prime Dyck path is below the \( x \)-axis is given by

\[
(5) \quad p_-(x, t) = \sum_{n=1}^{\infty} P_n (tx)^n = ttx \sum_{n=0}^{\infty} C_n (tx)^n = txc(tx).
\]

So since arbitrary paths can be factored into \( l \) primes (either positive or negative) for some \( l \geq 0 \), the generating function for such paths is

\[
N(x, t) = \sum_{l=0}^{\infty} (p_- + p_+)^l = \frac{1}{1 - (p_- + p_+)}.\]

In this generating function the coefficient of \( t^i x^j \) is the number paths of total length \( 2j \) and with \( 2i \) steps below the \( x \)-axis.
By the definition of \(c(x)\) and equations (4) and (5) we get that
\[
N(t, x) = \frac{1}{1 - xc(x) - txc(tx)}
= \frac{1}{1 - \left(\frac{1 - \sqrt{1 - 4x}}{2x} + tx \frac{1 - \sqrt{1 - 4tx}}{2tx}\right)}
= \frac{2}{\sqrt{1 - 4x} + \sqrt{1 - 4tx}}.
\]
Rationalizing the denominator we see that
\[
N(t, x) = \frac{\sqrt{1 - 4x} - \sqrt{1 - 4tx}}{2x(t - 1)}
= \frac{1 - \sqrt{1 - 4x} - 1 + \sqrt{1 - 4xt}}{2x(1 - t)}
= \frac{1}{1 - t} \left(1 - \frac{\sqrt{1 - 4x}}{2x} - \frac{1 - \sqrt{1 - 4tx}}{2x}\right)
= \frac{1}{1 - t} (c(x) - tc(tx))
= \frac{1}{1 - t} \left(\sum_{n=0}^{\infty} C_n x^n - \sum_{n=0}^{\infty} C_n t^{n+1} x^n\right)
= \sum_{n=0}^{\infty} \frac{1 - t^{n+1}}{1 - t} C_n x^n.
\]

Using \(N(t, x)\) and a simple algebraic identity we can prove the Chung-Feller Theorem.

**Proof.** Because
\[
\frac{1 - t^{n+1}}{1 - t} = 1 + t + t^2 + \cdots + t^n,
\]
equation (6) can be rewritten as
\[
(7) \quad N(t, x) = \sum_{n=0}^{\infty} C_n \left(\sum_{k=0}^{n} t^k\right) x^n.
\]
Thus the coefficient of \(t^k x^n\) in \(N(t, x)\) is equal to \(C_n\) for \(0 \leq k \leq n\). Therefore the number of \(k\)-negative paths of length \(2n\) is equal to \(C_n\) for \(0 \leq k \leq n\). \(\Box\)

5. A **Proof by Reordering**

The Cycle Lemma of Dvoretzky and Motzkin [2] is intricately linked with our main theorem. In [3], the author provides the following proof of the Cycle Lemma using an ordering which naturally reveals the equidistribution of the Chung-Feller Theorem.

**Lemma 1** (Cycle Lemma). *Given a sequence \(\pi = a_1 a_2 a_3 \cdots a_n\) of integers with \(a_i \leq 1\) such that \(\sum_{i=1}^{n} a_i = k > 0\), there are exactly \(k\) values of \(i\) such that all partial sums of the cyclic permutation \(\pi_i = a_{i+1} a_{i+2} \cdots a_{2n+k} a_1 \cdots a_i\) are positive.*
This is a stronger claim than we need, so let us just restrict ourselves to $a_i \in \{-1, 1\}$. For ease of computation for any $p \leq n$, let $s(p) = \sum_{j=1}^{p} a_i$. Given a sequence $\pi$ as above, let us define a new order relation $\preceq$ on \{0, 1, \ldots, n\}. For any $p, q \in \{0, 1, \ldots, n\}$,

$$p \preceq q,$$

if $s(p) < s(q)$ or if $s(p) = s(q)$ and $p > q$.

Additionally, let us define $m_i$ for $i = 0, 1, \ldots, n$ such that $m_i \in \{0, 1, \ldots, n\}$ and there are exactly $i$ elements $m \in \{0, 1, \ldots, n\}$ such that $m < m_i$.

Now, let if we look at the $j$th cyclic shift of $\pi$, denoted $\pi_j = a_{j+1}a_{j+2}\cdots a_{m}a_{1}\cdots a_{j}$, then the partial sums of $\pi_j$, $s^j(p)$, is given by $s^j(p) = \sum_{i=1}^{p-j} a_{i+j} = a_{i+1} + a_{i+2} + \cdots + a_{p}$, where indices are considered modulo $n$. This leads to the equation

$$(8) \quad s^j(p) = \begin{cases} s(p) - s(j), & \text{if } j \leq p \leq n; \\ s(p) - s(j) + k, & \text{if } 0 \leq p < j. \end{cases}$$

**Proposition.** For any sequence $\pi$ with steps $a_i \in \{-1, 1\}$ and sum $k = 1$, the $m_i$th cyclic shift of $\pi$ has exactly $i + 1$ values of $p$ such that $s^{m_i}(p) \leq 0$.

*Proof.* Clearly, $s^{m_i}(m_i) = 0$, so there is at least one such value of $p$. If $l < i$, then let us check $s^{m_i}(m_l)$. If $s(m_l) = s(m_i)$ then $m_l > m_i$, so we know that $s^{m_i}(m_l) = s(m_l) - s(m_i) = 0$. If, however, $s(m_l) < s(m_i)$ then $s^{m_i}(m_l) \leq s(m_l) + 1 - s(m_i) < 1$, so $s^{m_i}(m_l) \leq 0$.

If $l > i$, then either $s(m_l) > s(m_i)$ or $m_l < m_i$ and $s(m_i) = s(m_i)$. In the first case since $s^{m_i}(m_l) \geq s(m_l) - s(m_i)$, clearly $s^{m_i}(m_l) > 0$. Otherwise, if $m_l < m_i$ then $s^{m_i}(m_l) = 1 + s(m_l) - s(m_i)$ so since $s(m_l) = s(m_i)$, $s^{m_i}(m_l) = 1$.

Thus there are exactly $i + 1$ values of $p$, namely $m_0, m_1, \ldots, m_i$, such that $s^{m_i}(p) \leq 0$. □

The proof of the Cycle Lemma naturally follows from this proposition.

*Proof of Cycle Lemma.* Use the order relation $\preceq$ to calculate $m_i$ for each $i = 0, 1, \ldots, n$ and let $\phi = \pi_{m_0}$. By definition, only $s^{m_0}(m_0) = 0$. Since $s^{m_0}(p) = s_{\phi}(p - m_0)$, only $s_{\phi}(0) \leq 0$, so all partial sums of $\phi$ for $p \geq 0$ are positive. □

Let $\pi$ represent a Dyck path from (0, 0) to (2n + 1, 1), by taking the path (1, $a_i$). Let us restrict the ordering $\preceq$ to just the cyclic shifts of $\pi$ that begin with an up-step. Since the order structure of \{0, 1, \ldots, 2n + 1\} is maintained we can order the up-steps $j_k = m_i$ is the up-step with exactly $k$ up-steps $j$ such that $j \preceq j_k$ for $k = 0, 1, 2, \ldots, n$. Therefore, though there will be a total of $i$ vertices on or below the $x$-axis in the $n_k$th cyclic shift of $\pi$, there will be precisely $k$ up-steps that start on or below the $x$-axis.

Since there are $n + 1$ total up-steps, there are $n + 1$ cyclic shifts of $\pi$ that start with an up-step. Now since there is precisely one cyclic shift for every path, then we have that the total number of paths with steps $a_i \in \{-1, 1\}$ and sum 1 that start with an up-step and have exactly $k$ up-steps that start on or below the $x$-axis is given by the fraction $\frac{1}{n+1}$ of the total number of paths from (0, 0) to (2n + 1, 1) that start with an up-step. Thus it is the same as the number of paths from (1, 1) to (2n + 1, 1) which is precisely the number of paths from (0, 0) to (2n, 0); that is, $\binom{2n}{n}$.

Therefore the number of paths from (0, 0) to (2n + 1, 1) that start with an up-step and have exactly $k$ up-steps that start on or below the $x$-axis, for $k = 0, 1, \ldots, n$, is given by
\[ \frac{1}{n+1} \binom{2n}{n} = C_n. \]

If we drop the initial up-step then we are left with a path from \((0, 0)\) to \((2n, 0)\) with exactly \(k\) up-steps below the \(x\)-axis. So there are \(C_n\) paths from \((0, 0)\) to \((2n, 0)\) with exactly \(k\) up-steps (and thus a total of \(2k\) steps) below the \(x\)-axis for each \(k\).

**References**

[1] K.L. Chung, W. Feller, *Fluctuations in coin-tossing*, Proc. Natl. Acad. Sci. USA 35 (1949), 605–608.

[2] A. Dvoretzky, T. Motzkin, *A problem of arrangements*, Duke Math J. 14 (1947), 305–313.

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd ed. New York: John Wiley & Sons, Inc. ©1960, 72–73.

[4] E. Wolhagen, *The cycle lemma and combinatorial interpretations of familiar numbers*, (2004) in preparation.