BOUNDS FOR THE RELATIVE N–TH NILPOTENCY DEGREE IN COMPACT GROUPS

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Abstract. The line of investigation of the present paper goes back to a classical work of W. H. Gustafson of the 1973, in which it is described the probability that two randomly chosen group elements commute. In the same work, he gave some bounds for this kind of probability, providing information on the group structure. We have recently obtained some generalizations of his results for finite groups. Here we improve them in the context of the compact groups.

1. Introduction

A compact group $G$ admits a unique left Haar measure $\mu_G$ which is normalized and left-invariant (see [11, Sections 18.1, 18.2, Proposition 18.2.1]). This allows us to assume that $G$ has a unique probability measure space with respect to $\mu_G$ (see [11, Sections 18.1, 18.2] or [10, Section 2]). On the product measure space $G \times G$, it is possible to consider the product measure $\mu_G \times \mu_G$ which is a probability measure. If

$$C_2 = \{(x, y) \in G \times G \mid [x, y] = 1\},$$

then $C_2 = f^{-1}(1)$, where

$$f : (x, y) \in G \times G \mapsto f(x, y) = [x, y] \in G.$$ 

Clearly, $f$ is continuous and $C_2$ is a compact measurable subset of $G \times G$. Therefore it is possible to define

$$d(G) = (\mu_G \times \mu_G)(C_2)$$

as the commutativity degree of $G$. In the finite case $d(G)$ is described in [1, 2, 3, 5, 12, 13, 14]. We may extend the notion of $d(G)$ as follows. Suppose that $n \geq 1$, $G^n$ is the product of $n$-copies of $G$ and $\mu^n_G$ that of $n$-copies of $\mu_G$. We define

$$d^{(n)}(G) = \mu^{n+1}_G(C_{n+1})$$

as the $n$-th nilpotency degree of $G$, where

$$C_{n+1} = \{(x_1, \ldots, x_{n+1}) \in G^{n+1} \mid [x_1, x_2, \ldots, x_{n+1}] = 1\}.$$ 

Obviously, if $G$ is finite, then $G$ is a compact group with the discrete topology and so the Haar measure of $G$ is the counting measure. Then, for a finite group $G$, we have

$$d^{(n)}(G) = \mu^{n+1}_G(C_{n+1}) = \frac{|C_{n+1}|}{|G|^{n+1}}.$$ 

See for details [5, 13].

More generally, let $H$ be a closed subgroup of a compact group $G$. We define

$$D_2 = \{(h, g) \in H \times G \mid [h, g] = 1\}$$

and note that $D_2 = \phi^{-1}(1)$, where

$$\phi : (h, g) \in H \times G \mapsto \phi(h, g) = [h, g] \in G.$$
Clearly, $\phi$ is continuous and $D_2$ is a compact measurable subset of $H \times G$. Note that $\phi$ is the restriction of $f$ to $H \times G$ and this shows that $H$ has to be closed subgroup of $G$, if we want to preserve the topological structure. Then we define

$$d(H,G) = (\mu_H \times \mu_G)(D_2)$$

as the relative commutativity degree of $H$ with respect to $G$. Considering

$$D_{n+1} = \{(h_1, \ldots, h_n, g) \in H^n \times G \mid [h_1, h_2, \ldots, h_n, g] = 1\},$$

we define

$$d^{(n)}(H,G) = (\mu^n_H \times \mu_G)(D_{n+1})$$

as the relative $n$-th nilpotency degree of $H$ with respect to $G$. As already noted, $\cite{1, 5, 12, 13, 14}$ give contributions to the knowledge of the $n$-th nilpotency degree in case of finite groups. Recently, the case of infinite groups can be found in $\cite{4, 6, 7, 8, 9, 15, 16}$. We will try to extend the results in $\cite{3}$ Sections 3, 4, 5 looking at the methods in $\cite{4, 6, 7, 8, 9, 15, 16}$.

### 2. Relative Commutativity Degree

The next statement is useful for proving most of our results.

**Lemma 2.1.** Assume that $G$ is a compact group, $H$ is a closed subgroup of $G$ and $C_G([h_1, \ldots, h_n])$ is the centralizer of the commutator $[h_1, \ldots, h_n]$ in $G$ for some elements $h_1, \ldots, h_n$ in $H$. Then

$$d^{(n)}(H,G) = \int_H \cdots \left( \int_H \mu_G(C_G([h_1, \ldots, h_n]))d\mu_H(h_1) \right) \cdots d\mu_H(h_n),$$

where

$$\mu_G(C_G([h_1, \ldots, h_n])) = \int_G \chi_{D_{n+1}}(h_1, \ldots, h_n, g)d\mu_G(g)$$

and $\chi_{D_{n+1}}$ denotes the characteristic map of the set $D_{n+1}$.

**Proof.** Since

$$\mu_G(C_G([h_1, \ldots, h_n])) = \int_G \chi_{D_{n+1}}(h_1, \ldots, h_n, g)d\mu_G(g),$$

we have by Fubini-Tonelli’s Theorem:

$$d^{(n)}(H,G) = (\mu^n_H \times \mu_G)(D_{n+1}) = \int_{H^n \times G} \chi_{D_{n+1}}(d\mu^n_H \times d\mu_G)$$

$$= \int_H \cdots \left( \int_H \chi_{D_{n+1}}(h_1, \ldots, h_n, g)d\mu_G(g) \right) d\mu_H(h_1) \cdots d\mu_H(h_n)$$

$$= \int_H \cdots \left( \int_H \mu_G(C_G([h_1, \ldots, h_n]))d\mu_H(h_1) \right) \cdots d\mu_H(h_n).$$

\hfill \Box

We recall the following elementary fact, which can be found in $\cite{10}$. See also $\cite{6}$ Lemma 3.1.

**Lemma 2.2.** Assume $H$ is a closed subgroup of a compact group $G$. If $|G : H| = n < \infty$, then $\mu_G(H) = \frac{1}{n}$. If $|G : H| = \infty$, then $\mu_G(H) = 0$.

**Proof.** Assume that $|G : H| = n$ is finite. Then $G = \bigcup_{i=1}^n g_iH$. So we have

$$1 = \mu_G(G) = \mu_G(\bigcup_{i=1}^n g_iH) = \sum_{i=1}^n \mu_G(g_iH) = \sum_{i=1}^n \mu_G(H) = n\mu_G(H)$$
and therefore $\mu_G(H) = \frac{1}{n}$. Now assume that $\alpha = |G : H| = \infty$. Of course, $\alpha > 0$, then $t \alpha > 1$ for some positive integer $t$. By assumption, $G = \bigcup_{i \in I} g_i H$, where $I$ is an infinite set. Choose a subset $J$ of $I$ of cardinality $t$. It follows that

$$1 = \mu_G(G) \geq \mu_G\left(\bigcup_{j \in J} g_j H\right) \geq \sum_{j \in J} \mu_G(g_j H) = t\alpha > 0.$$ 

This contradicts $\mu_G(H) = 0$ and the proof of the lemma follows. \qed

Lemma 2.2 will be used in most of our proofs, even if the following form is more suitable.

**Lemma 2.3.** Assume $H$ is a closed subgroup of a compact group $G$. If $|G : H| \geq n$, then $\mu_G(H) \leq \frac{1}{n}$. If $|G : H| \leq n$, then $\mu_G(H) \geq \frac{1}{n}$. In particular, $|G : H| = n$ if and only if $\mu_G(H) = \frac{1}{n}$.

**Proof.** This follows from an argument as in Lemma 2.2. \qed

Lemma 2.3 allows us to reformulate [5, Theorem 3.10] for infinite groups in terms of the following result. The reader may find exactly the same proof in [15]; here we repeat it, just for sake of completeness and because we want to point out the methods and the ideas which are often used in similar circumstances.

**Theorem 2.4.** Let $H$ be a closed subgroup of a compact group $G$.

(i) If $d(H, G) = \frac{3}{4}$, then $H/(Z(G) \cap H)$ is cyclic of order 2.

(ii) If $d(H, G) = \frac{3}{4}$ and $H$ is nonabelian, then $H/(Z(G) \cap H)$ is 2-elementary abelian of rank 2.

**Proof.** (i). Assume that $d(H, G) = \frac{3}{4}$ and let $K = H \cap Z(G)$. If $h$ is an element of $H$ not belonging to $K$, then $|G : C_G(h)| \geq 2$ and so $\mu_G(C_G(h)) \leq \frac{1}{2}$ by Remark 2.3. On the other hand, if $h$ is an element of $K$, then $\mu_G(C_G(h)) = 1$. From these facts and Lemma 2.1, we have

$$\frac{3}{4} = d(H, G) = \int_H \mu_G(C_G(h))d\mu_H(h)$$

$$= \int_K \mu_G(C_G(h))d\mu_H(h) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h)$$

$$\leq \int_K d\mu_H(h) + \frac{1}{2} \int_{H-K} d\mu_H(h) = \mu_H(K) + \frac{1}{2}(1 - \mu_H(K)).$$

Therefore, $\mu_H(K) \geq \frac{1}{2}$. On the other hand, $K$ is a closed subgroup of the abelian group $H$ and $\mu_H(K) \leq \frac{1}{2}$. Then $\mu_H(K) = \frac{1}{2}$ and so $|H : K| = 2$. This means that $H/K$ is cyclic of order 2, as claimed.

(ii). Assume that $d(H, G) = \frac{3}{4}$ and let $K = H \cap Z(G)$. We may argue as in the previous statement (i). On a hand, we have $\frac{3}{4} = d(H, G) \leq \frac{1}{2} + \frac{1}{2}\mu_H(K)$. Therefore, $\mu_H(K) \geq \frac{1}{2}$. On the other hand, $K$ is a closed subgroup of the nonabelian group $H$ and $\mu_H(K) \leq \frac{1}{2}$, still by Lemma 2.3. This gives $\mu_H(K) = \frac{1}{2}$ so that $|H : K| = 4$. This means that $H/K$ has order 4. Since $H$ is nonabelian, $H/K$ cannot be cyclic. From this, $H/K$ is 2-elementary abelian of rank 2, as claimed. \qed

Note that [5, Theorem 3.10] follows from Theorem 2.4 when we consider a finite group with the counting measure on it. Now we extend [5, Lemma 3.2] to the case of infinite groups. The next result overlaps [6, Lemma 3.2].

**Lemma 2.5.** Let $H$ be a closed subgroup of a compact group $G$. Then

$$\mu_G(C_G(x)) \leq \mu_H(CH(x))$$

for all $x \in G$. 
Theorem 2.7. Let $H$ be a closed subgroup of a compact group $G$. Then
\[ d(G) \leq d(H, G) \leq d(H). \]

Proof. From Lemma 2.5, $\mu_H(C_H(x)) \geq \mu_G(C_G(x))$. Integrating over $H$ and keeping in mind Lemma 2.1, we have
\[ d(H) = \int_H \mu_H(C_H(x))d\mu_H(x) \geq \int_H \mu_G(C_G(x))d\mu_H(x) = d(H, G). \]

(ii) By Lemma 2.1 and noting that $\mu_G(C_G(x)) \leq \frac{1}{2}$ for each noncentral element $x$ of $G$, we have
\[ d(G) = \int_G \mu_G(C_G(x))d\mu_G(x) \]
\[ = \int_{Z(G)} \mu_G(C_G(x))d\mu_G(x) + \int_{G-Z(G)} \mu_G(C_G(x))d\mu_G(x) \]
\[ = \mu_G(Z(G)) + \int_{G-Z(G)} \mu_G(C_G(x))d\mu_G(x) \]
\[ \leq \mu_G(Z(G)) + \frac{1}{2}(1 - \mu_G(Z(G))) = \frac{1}{2} + \frac{1}{2}\mu_G(Z(G)). \]

(iii) By Lemma 2.1 and noting that $\mu_G(C_G(h)) \leq \frac{1}{2}$ for each element $h$ of $H - K$,
\[ d(H, G) = \int_H \mu_G(C_G(h))d\mu_H(h) \]
\[ = \int_K \mu_G(C_G(h))d\mu_H(h) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h) \]
\[ = \mu_H(K) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h) \]
\[ \leq \mu_H(K) + \frac{1}{2}(1 - \mu_H(K)) = \frac{1}{2} + \frac{1}{2}\mu_H(K). \]

\[ \square \]

Theorem 2.7. Let $H$ be a closed subgroup of a compact group $G$. Then
\[ d(G) \leq d(H, G) \leq d(H). \]

Proof. Consider the map
\[ f : hC_H(x) \in \{hC_H(x) \mid h \in H\} \rightarrow f(hC_H(x)) = hC_G(x) \in \{gC_G(x) \mid g \in G\}. \]
f is one-to-one and so $|H : C_H(x)| \leq |G : C_G(x)|$. This implies $\mu_G(C_G(x)) \leq \mu_H(C_H(x))$. \[ \square \]

An important dominance condition is the following.

Theorem 2.6. Let $H$ be a closed subgroup of a compact group $G$. Then
\[ d(G) \leq d(H, G) \leq d(H). \]

Proof. From Lemma 2.5, $\mu_H(C_H(x)) \geq \mu_G(C_G(x))$. Integrating over $H$ and keeping in mind Lemma 2.1, we have
\[ d(H) = \int_H \mu_H(C_H(x))d\mu_H(x) \geq \int_H \mu_G(C_G(x))d\mu_H(x) = d(H, G). \]

(ii) By Lemma 2.1 and noting that $\mu_G(C_G(x)) \leq \frac{1}{2}$ for each noncentral element $x$ of $G$, we have
\[ d(G) = \int_G \mu_G(C_G(x))d\mu_G(x) \]
\[ = \int_{Z(G)} \mu_G(C_G(x))d\mu_G(x) + \int_{G-Z(G)} \mu_G(C_G(x))d\mu_G(x) \]
\[ = \mu_G(Z(G)) + \int_{G-Z(G)} \mu_G(C_G(x))d\mu_G(x) \]
\[ \leq \mu_G(Z(G)) + \frac{1}{2}(1 - \mu_G(Z(G))) = \frac{1}{2} + \frac{1}{2}\mu_G(Z(G)). \]

(iii) By Lemma 2.1 and noting that $\mu_G(C_G(h)) \leq \frac{1}{2}$ for each element $h$ of $H - K$,
\[ d(H, G) = \int_H \mu_G(C_G(h))d\mu_H(h) \]
\[ = \int_K \mu_G(C_G(h))d\mu_H(h) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h) \]
\[ = \mu_H(K) + \int_{H-K} \mu_G(C_G(h))d\mu_H(h) \]
\[ \leq \mu_H(K) + \frac{1}{2}(1 - \mu_H(K)) = \frac{1}{2} + \frac{1}{2}\mu_H(K). \]

\[ \square \]

Note that the upper bounds in [5, Theorem 3.5] follow from Theorem 2.7 when we consider a finite group with the counting measure on it. The lower bounds in [5, Theorem 3.5] cannot be true in the infinite case, as the infinite dihedral group shows.

Corollary 2.8. Assume that $H$ is a closed subgroup of a nonabelian compact group $G$.

(i) If $H \leq Z(G)$, then $d(H, G) = 1$.
(ii) If $H \nsubseteq Z(G)$ and $H$ is abelian, then $d(H, G) \leq \frac{3}{4}$.
(iii) If $H \nsubseteq Z(G)$ and $H$ is nonabelian, then $d(H, G) \leq \frac{5}{8}$. 

\[ \square \]
Proof. (i). Obvious.
(ii). Since $H \nsubseteq Z(G)$, $K = H \cap Z(G) \nsubseteq H$. As in the proof of Theorem 2.4 (ii), we have $\mu_H(K) \leq \frac{1}{2}$. Theorem 2.5 (ii) implies $d(H, G) \leq \frac{1}{2} + \frac{1}{2}(\frac{1}{4}) = \frac{3}{8}$.
(iii). We know from Theorem 2.6 and [10] that $d(H, G) \leq d(H) \leq \frac{3}{8}$. \hfill \Box

Note that [5, Theorem 3.6] follows from Corollary 2.8 when we consider a finite group with the counting measure on it.

**Corollary 2.9.** Let $A$ and $B$ be two closed subgroups of a compact group $G$ such that $A \leq B$. Then $d(A, B) \geq d(A, G) = d(B, G)$.

**Proof.** As in the proof of Theorem 2.6, the condition

$$|A : C_A(x)| \leq |B : C_B(x)| \leq |G : C_G(x)|$$

implies the condition $\mu_A(C_A(x)) \geq \mu_B(C_B(x)) \geq \mu_G(C_G(x))$ for every element $x$ of $G$. Integrating and keeping in mind Lemma 2.1, we have

$$d(A, B) = \int_A \mu_B(C_B(x))d\mu_A(x) \geq \int_A \mu_G(C_G(x))d\mu_A(x) = d(A, G).$$

\hfill \Box

Note that [5, Theorem 3.7] follows from Corollary 2.8 when we consider a finite group with the counting measure on it. We recall to convenience of the reader [5, Lemma 3.8].

**Lemma 2.10.** Let $H$ and $N$ be two closed subgroups of $G$ such that $N \leq H$ and $N$ is normal in $G$. Then $C_H(x)N/N \leq C_H/N(xN)$ for every element $x$ of $G$. Moreover, the equality holds if $N \cap [H, G]$ is trivial.

Then we may formulate another interesting dominance condition as follows.

**Theorem 2.11.** Let $H$ and $N$ be two closed subgroups of a compact group $G$ such that $N \leq H$ and $N$ is normal in $G$. Then $d(H, G) \leq d(H/N, G/N)d(N)$. In particular, the equality holds if $N \cap [H, G]$ is trivial.

**Proof.** Consider $S = \{g \in G \mid |H : C_H(g)|$ is finite\}. We have

$$d(H, G) = \int_G \mu_H(H(g))d\mu_G(g) = \int_S \mu_H(C_H(g))d\mu_G(g)$$

$$= \int_S \frac{\mu_H(C_H(g)N)}{|C_H(g)N : C_H(g)|}d\mu_G(g) = \int_S \mu_H(C_H(g)N)\mu_N(C_N(g))d\mu_G(g).$$

In the last equality we have used the argument just before Theorem 2.4 and the fact that $|C_H(g)N : C_H(g)|$ is finite, getting

$$|C_H(g)N : C_H(g)| = |N : C_H(g) \cap N| = \frac{1}{\mu_N(C_N(g))}.$$  

Now we get:

$$d(H, G) \leq \int_G \mu_H(C_H(g)N)\mu_N(C_N(g))d\mu_G(g)$$

$$= \int_S \left( \int_N \mu_H(C_H(gx)N)\mu_N(C_N(gx))d\mu_N(x) \right) d\mu_G/N(gN).$$

By Lemma 2.10, $\mu_H(C_H(gx)N) = \mu_H\left( \frac{C_H(gx)N}{N} \right) \leq \mu_H\left( C_H(gN) \right)$, then

$$d(H, G) \leq \int_S \mu_G/N(C_G/N(gN)) \left( \int_N \mu_N(C_N(gx))d\mu_N(x) \right) d\mu_G/N(gN).$$
On another hand,
\[ C_2 = \{(x, y) \in N \times N \mid gx, y = 1\} = \{(x, y) \in N \times N \mid gx \in C_G(y) \cap gN\}. \]
If \( x_0 \in C_G(y) \cap gN \neq \emptyset \), then either \( gN = g_0N \) or \( g = g_0t \) for some \( t \in N \), whence \( C_G(y) \cap gN = g_0(C_G(y) \cap N) = g_0C_N(y) \) and so
\[ C_2 = \{(x, y) \in N \times N \mid gx \in g_0C_N(y)\} = \{(x, y) \in N \times N \mid x \in tC_N(y)\}. \]
Therefore
\[ \int_N \mu_N(C_N(gx))d\mu_N(x) \leq \int_N \mu_N(tC_N(y))d\mu_N(y) = \int_N \mu_N(C_N(y))d\mu_N(y) = d(N). \]

Hence
\[ d(H, G) \leq d(N) \int_N \mu_N(C_N(gN))d\mu_G/N(gN) = d(N)d(H/N, G/N). \]

In particular, if \( N \cap [H, G] = 1 \), then \( C_H(g) = C_H(g)N \) and so \( \mu_H(C_G(g)) = \mu_H(C_H(g)N) \)
for all \( g \in G \). Furthermore, we have
\[ \mu_H(C_H(gn)N) = \mu_G/N\left( \frac{C_H(gn)N}{N} \right) = \mu_G/N(C_H(N(gn))). \]
Therefore each inequality becomes equality and so \( d(H, G) = d(H/N, G/N)d(N) \).

3. Relative n–th commutativity degree

The present section is devoted to extend some results of Section 2. For instance, the next statement extends the upper bound in Theorem 2.6.

**Theorem 3.1.** If \( H \) is a closed subgroup of a compact group \( G \), then
\[ d^{(n)}(H, G) \leq d^{(n)}(H). \]

**Proof.** We may argue as in the proof of Theorem 2.6 in order to get
\[ \mu_H(C_H([h_1, ..., h_n])) \geq \mu_G(C_G([h_1, ..., h_n])). \]
Integrating over \( H \) and keeping in mind Lemma 2.1, we have
\[ d^{(n)}(H) = \int_H \ldots \left( \int_H \mu_H(C_H([h_1, ..., h_n]))d\mu_H(h_1) \right) \ldots d\mu_H(h_n) \]
\[ \geq \int_H \ldots \left( \int_H \mu_G(C_G([h_1, ..., h_n]))d\mu_H(h_1) \right) \ldots d\mu_H(h_n) = d^{(n)}(H, G). \]

Note that Theorem 3.1 informs us that the sequence \( \{d^{(n)}(H, G)\}_{n \geq 1} \) is increasing for any compact group \( G \) and any closed subgroup \( H \) of \( G \).

The evidences of the finite case and the considerations of many situations in the infinite case can be summarized in the following result.

**Theorem 3.2.** If \( H \) is a closed subgroup of a compact group \( G \) and \( K = H \cap Z(G) \), then
\[ d^{(n+1)}(H, G) \leq \frac{1}{2}\left( 1 + d^{(n)}(H/K) \right). \]

**Proof.** Let \( A = \{[h_1, ..., h_{n+1}] \in H^{n+1} \mid [h_1, ..., h_{n+1}] \in Z(G) \cap H \} \) and \( B = H^{n+1} - A \). Then
\[ d^{(n+1)}(H, G) = \int_{H^{n+1}} \mu_G([h_1, ..., h_{n+1}])d(\mu_H)^{n+1} \]
\[ = \int_A \mu_G([h_1, ..., h_{n+1}])d(\mu_H)^{n+1} + \int_B \mu_G([h_1, ..., h_{n+1}])d(\mu_H)^{n+1} \]
\[ \leq \mu_H^{n+1}(A) + \frac{1}{2}\mu_H^{n+1}(B) \leq \mu_H^{n+1}(A) + \frac{1}{2}(1 - \mu_H^{n+1}(A)) = \frac{1}{2}(1 + \mu_H^{n+1}(A)). \]
On the other hand,
\[ \mu_H^{n+1}(A) = \int_H \cdots \int_H \mu_K(C_{\overline{K}}([\bar{h}_1, \ldots, \bar{h}_n]))d\mu_H(h_1) \cdots d\mu_H(h_n) \]
\[ = \int_H \cdots \left( \int_K \mu_K(C_{\overline{K}}([\bar{h}_1, \ldots, \bar{h}_n]))d\mu_K(k) \right) \cdots d\mu_H(h_n) \]
\[ = \int_H \cdots \left( \int_K \mu_K(C_{\overline{K}}([\bar{h}_1, \ldots, \bar{h}_n]))d\mu_H(h_1) \right) \cdots d\mu_H(h_n) \]
\[ = \int_H \cdots \int_K \mu_K(C_{\overline{K}}([\bar{h}_1, \ldots, \bar{h}_n]))d\mu_K(h_1) \cdots d\mu_K(h_n) = d^{(n)}(H/K). \]
and the result follows. \(\square\)

Note that [5, Theorem 4.3] follows from Theorem 3.2 when we consider a finite group with the counting measure on it. Note that Theorem 3.2 is true also for groups of the form \(A_i \times B_j\), where \(A_i\) is a compact abelian (infinite) group, \(B_j\) is a finite group, \(i \in I\) and \(j \in J\).

**Corollary 3.3.** If \(G\) is a compact group, then \(d^{(n+1)}(G) \leq \frac{1}{2}(1 + d^{(n)}(G/Z(G))).\)

**Proof.** This follows from Theorem 3.2 with \(H = G\). \(\square\)

It is possible to bound \(d^{n+1}(G)\) as follows.

**Theorem 3.4.** If \(G\) is a compact group, then
\[ d^{(n+1)}(G) \leq \frac{1}{2^n}(2^n - 1 + d(G/Z_n(G))). \]

**Proof.** We may repeat the proof of [5, Theorem 4.5], since we do not need that \(G\) is finite. We should only note that \(Z_n(G)/Z(G) = Z_{n-1}(G/Z(G))\) is a closed subgroup of \(G/Z(G)\). \(\square\)

Note that [5, Theorem 4.5] follows from Theorem 3.4 when we consider a finite group with the counting measure on it. Furthermore Theorem 3.4 is true for compact groups of the form \(A \times B\), where \(A\) is a compact abelian (infinite) group and \(B\) is a finite group. In such a case [5, Theorem 4.5] cannot be applied.

**Theorem 3.5.** Let \(H\) and \(N\) be two closed subgroups of \(G\) such that \(N \leq H\) and \(N\) is normal in \(G\). Then \(d^{(n)}(H, G) \leq d^{(n)}(H/N, G/N)\). In particular, the equality holds if \(N \cap [n, H, G]\) is trivial.

**Proof.** Let \(\lambda\), \(\mu\) and \(\nu\) be corresponding Haar measures on \(N\), \(G\) and \(G/N\) respectively. Consider \(S = \{(h_1, \ldots, h_n) \mid |G : C_G([h_1, \ldots, h_n])| \text{ is finite}\}\). Then
\[ d^{(n)}(H, G) = \int_{H^n} \mu_G(C_G([h_1, \ldots, h_n]))d\mu_H^n = \int_S \mu_G(C_G([h_1, \ldots, h_n]))d\mu_H^n \]
\[ = \int_S \frac{\mu_G(C_G([h_1, \ldots, h_n])N)}{|C_G([h_1, \ldots, h_n])N : C_G([h_1, \ldots, h_n])|}d\mu_H^n \]
\[ = \int_S \mu_G(C_G([h_1, \ldots, h_n])N)\mu_N(C_G([h_1, \ldots, h_n]))d\mu_H^n \]
\[ \leq \int_{H^n} \mu_G(C_G([h_1, \ldots, h_n])N)\mu_N(C_G([h_1, \ldots, h_n]))d\mu_H^n \]
\[ = \int_{H^n} \cdots \int_N \mu_G(C_G([h_1 \cdots, h_n a_n])N)\mu_N(C_G([h_1 a_1, \ldots, h_n a_n])) \]
\[ \cdots d\mu_N(a_1) \cdots d\mu_N(a_n) = \mu_G(C_G([h_1 a_1, \ldots, h_n a_n])N) \]
\[ = \mu_G(C_G([h_1 a_1, \ldots, h_n a_n])N). \]
Therefore
\[
\begin{aligned}
d^{(n)}(H,G) &\leq \int_{\mathbb{X}} \ldots \int_{\mathbb{X}} \mu_{G/N}(C_{\mathbb{X}}([h_1N,\ldots,h_nN])) \\
&\quad \times \mu_N(C_G([h_1 a_1,\ldots,h_n a_n])) \mu_N(a_1) \ldots \mu_N(a_n) \mu_G(h_1N) \ldots \mu_G(h_nN) \\
&\leq \int_{\mathbb{X}} \ldots \int_{\mathbb{X}} \mu_{G/N}(C_{\mathbb{X}}([h_1N,\ldots,h_nN])) \mu_{G/Z}(h_1N) \ldots \mu_{G/Z}(h_nN) = d^{(n)}\left(\frac{H}{N}, \frac{G}{N}\right).
\end{aligned}
\]

An easy consequence is the following.

**Corollary 3.6.** If \( N \) is a closed normal subgroup of a compact group \( G \), then \( d^{(n)}(G) \leq d^{(n)}(G/N) \).

**Proof.** This follows from Theorem 3.5 with \( H = G \). 

\[\square\]

4. Weakening some bounds

In the present section we will give some upper and lower bounds for \( d^{(n)}(G) \) and \( d^{(n)}(H,G) \) by means of the results which have been previously found.

**Corollary 4.1.** If \( G \) is a compact group which is not nilpotent of class at most \( n \), then \( d^{(n)}(G) \leq \frac{2^{n+2}-3}{2^{n+1}} \).

**Proof.** \( G/Z_{n-1}(G) \) is a nonabelian group by the assumptions. From [10],

\[
d(G/Z_{n-1}(G)) \leq \frac{5}{8}
\]

Now Theorem 3.4 gives \( d^{(n)}(G) \leq \frac{1}{2^{n-1}}(2^{n-1} - 1 + \frac{5}{8}) = \frac{2^{n+2}-3}{2^{n+1}} \).

\[\square\]

Note that [5, Theorem 5.1] follows from Corollary 4.1 when we consider a finite group with the counting measure on it.

**Corollary 4.2.** If \( G \) is a nontrivial compact group with trivial center, then \( d^{(n)}(G) \leq \frac{2n-1}{2^{n-1}} \).

**Proof.** Of course, \( Z_n(G) \) is trivial for each \( n \geq 1 \). Thus \( G \) is a nonnilpotent group. In particular \( \mu_G(Z(G)) = 0 \) so Theorem 2.7 (i) implies \( d(G) \leq \frac{1}{n} \). Now the result follows by Theorem 3.4 by induction on \( n \).

\[\square\]

Our last result improves Corollary 2.8 and extends [5, Theorem 5.5].

**Theorem 4.3.** Assume that \( H \) is a proper closed subgroup of a nonabelian compact group \( G \) such that \( n \geq 1 \) and \( K = Z(G) \cap H \).

(i) If \( H \leq Z_n(G) \), then \( d^{(n)}(H,G) = 1 \).

(ii) If \( H \nleq Z_n(G) \) and \( H/K \) is a nilpotent group of class at most \( n-1 \), then \( d^{(n)}(H,G) = 1 \).

(iii) If \( H \nleq Z(G) \) and \( H/K \) is a nonnilpotent group of class at most \( n-1 \), then \( d^{(n)}(H,G) \leq \frac{2^{n+2}-3}{2^{n+1}} \).

**Proof.** (i). This is obvious.

(ii). Of course, if \( H/K \) is nilpotent of class at most \( n-1 \), then \( [\bar{x}_1,\ldots,\bar{x}_n] = K \) for some elements \( \bar{x}_1,\ldots,\bar{x}_n \) of \( H/K \). Therefore, \( G = C_G([\bar{x}_1,\ldots,\bar{x}_n]) \) and we may argue as in Theorem 3.2, getting \( d^{(n)}(H,G) = 1 \).

(iii). Using Corollary 4.1 and the fact that \( H/K \) is nonnilpotent of class at most \( n-1 \), we get \( d^{(n-1)}(H/K) \leq \frac{2^{n+1}-3}{2^{n+1}} \). By Theorem 3.2, we get

\[
d^{(n)}(H,G) \leq \frac{1}{2}(1 + d^{(n-1)}(H/K)) \leq \frac{1}{2}(1 + \frac{2^{n+1}-3}{2^{n+1}}) = \frac{2^{n+2}-3}{2^{n+2}}.
\]

\[\square\]
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