BOUNDARY SHAPE CONTROL OF NAVIER-STOKES EQUATIONS AND GEOMETRICAL DESIGN METHOD FOR BLADE’S SURFACE IN THE IMPELLER*

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Abstract. In this paper A Geometrical Design Method for Blade’s surface Θ in the impeller is provided here. Θ is a solution to a coupling system consisting of the well-known Navier-Stokes equations and a four order elliptic boundary value problem. The coupling system is used to describe the relations between solutions of Navier-Stokes equations and the geometry of the domain occupied by fluids, and also provides new theory and methods for optimal geometric design of the boundary of domain mentioned above. This coupling system is the Euler-Lagrange equations of the optimal control problem which is describing a new principle of the geometric design for the blade’s surface of an impeller. The control variable is the surface of the blade and the state equations are Navier-Stokes equations with mixed boundary conditions in the channel between two blades. The objective functional depending on the geometry shape of blade’s surface describes the dissipation energy of the flow and the power of the impeller. First we prove the existence of a solution of the optimal control problem. Then we use a special coordinate system of the Navier-Stokes equations to derive the objective functional which depends on the surface Θ explicitly. We also show the weakly continuity of the solution of the Navier-Stokes equations with respect to the geometry shape of the blade’s surface.

Key words. blade, boundary shape control, general minimal surface, Navier-Stokes equations, Euler-Lagrange equations.

AMS subject classifications. 65N30, 76U05, 76M05

1. Introduction. Blade’s shape design for the impeller is driven by the need of improving performance and reliability. So far we have not found a geometric design entirely from mathematical point of view. As it is well known that the blade’s surface is a part of the boundary of the flow domain in the impeller. We can use the technique of the boundary geometric control problem for the Navier-Stokes equations to design the blade’s shape. This idea is motivated by the classical minimal surface which is to find a surface spanning on a closed Jordanian curvilinear C such that

\[ J(\Theta) = \text{Aug} \inf_{S \in \mathcal{F}} J(S) \]

where \( J(S) = \int \int_S dS \) is the area of \( S \).

In this paper we try to propose a principle for a fully mathematical design of the surface of the blade in an impeller. This principle models a general minimal surface by minimizing a functional proposed by us. A key point in this modelling process is theoretical rationality and the realization of our design procedure. Using a tensor analysis technique we realize this procedure and obtain the Euler-Lagrange equations for blade’s surface which is a system coupling an elliptic boundary value problem, the Navier-Stokes equations and linearized Navier-Stokes equations, and prove the existence of solution of the system coupling problem.

This paper is organized as follows. In section 2 we give the main results of this paper. In section 3 we derive the rotating Navier-Stokes equations in the channel.

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*Subsidized by NSFC Project 50136030, 50306019, 40375010, 10471110, 10471109.
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in the impeller with mixed boundary condition under a new coordinate system. We give the minimizing functional problem and derive the Euler-Lagrange equations in section 4. In section 5 another model to design the blade of the impeller is given. In the last section we prove the existence of the solution to the optimal control problem, including the existence of the solution of Navier-Stokes Equations with mixed boundary conditions and the weakly continuous dependence relationships of the solution of Navier-Stokes equation with respect to the geometry shape of the blade's surface and so on.

2. Main Results. Suppose \((x^1, x^2) \in D \subset E^2(2D - EuclidianSpace)\). A smooth mapping \(\Theta(x^1, x^2)\) is the image of a surface. On the other hand, suppose that \((r, \theta, z)\) is a polar cylindrical coordinate system rotating with impeller's angular velocity \(\omega\).

\((\vec{e}_r, \vec{e}_\theta, \vec{k})\) are the corresponding base vectors. \(z\)-axis is the rotating axis of the impeller. \(N\) is the number of blade and \(\varepsilon = \pi/N\). The angle between two successive blades is \(\frac{2\pi}{N}\). The flow passage of the impeller is bounded by \(\partial \Omega = \Gamma_\text{in} \cup \Gamma_\text{out} \cup \Gamma_t \cup \Gamma_b \cup S_+ \cup S_-\). The middle surface \(S\) of the blade is defined as the image \(\tilde{R}\) of the closure of a domain \(D \subset R^2\) where \(\tilde{R} : D \rightarrow R^3\) is a smooth injective mapping which can be expressed by that for any point \(\tilde{R}(D) \in S\)

\[
\tilde{R}(x) = x^2 \vec{e}_r + x^2 \Theta(x^1, x^2) \vec{e}_\theta + x^1 \vec{k}, \forall x = (x^1, x^2) \in \tilde{D},
\]

where \(\Theta \in C^2(D, R)\) is a smooth function. \(x = (x^1, x^2)\) is called a Gaussian coordinate system on \(S\). It is easy to prove that there exists a family \(S_\xi\) of surfaces with a single parameter to cover the domain \(\Omega_\varepsilon\) defined by the mapping \(D \rightarrow S_\xi = \{\tilde{R}(x^1, x^2; \xi) : \forall (x^1, x^2) \in D\} : \)

\[
\tilde{R}(x^1, x^2; \xi) = x^2 \vec{e}_r + x^2(\varepsilon \xi + \Theta(x^1, x^2)) \vec{e}_\theta + x^1 \vec{k},
\]

It is clear that the metric tensor \(a_{\alpha \beta}\) of \(S_\xi\) is homogenous and nonsingular independent of \(\xi\), and is given as follows:

\[
a_{\alpha \beta} = \frac{\partial \tilde{R}}{\partial x^\alpha} \frac{\partial \tilde{R}}{\partial x^\beta}^T = \delta_{\alpha \beta} + r^2 \Theta_{\alpha \beta}, \quad a = \det(a_{\alpha \beta}) = 1 + r^2(\Theta_1^2 + \Theta_2^2) > 0,
\]

From this we establish a curvilinear coordinate system \((x^1, x^2, \xi)\) in \(R^3\),

\[
(r, \theta, z) \rightarrow (x^1, x^2, \xi) : \begin{array}{c}
\begin{aligned}
x^1 &= z, \\
x^2 &= r,
\end{aligned}
\end{array} \xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)),
\]

that maps the flow passage domain

\[
\Omega_\varepsilon = \{\tilde{R}(x^1, x^2, \xi) = x^2 \vec{e}_r + x^2(\varepsilon \xi + \Theta(x^1, x^2)) \vec{e}_\theta + x^1 \vec{k}, \forall (x^1, x^2, \xi) \in \Omega_\varepsilon\}
\]

into a fixed domain in \(E^3(3D Euclidian Space)\):

\[
\Omega = \{(x^1, x^2) \in D, -1 \leq \xi \leq 1\} \text{ in } R^3
\]

which is independent of Surface \(S\) of the blade, and Jacobian

\[
J(\frac{\partial (r, \theta, z)}{\partial (x^1, x^2, \xi)}) = \varepsilon,
\]

therefore the transformation is nonsingular.
Assume that \((x^1, x^2, x^3) = (r, \theta, z)\), as well known that corresponding metric tensor of \(\mathbb{R}^3\) is \(g_{ij} = 1, g_{i2} = r^2, g_{2j} = 1, g_{ij} = 0 (\forall i \neq j)\). According to rule of tensor transformation under coordinate transformation we have following calculation formulae

\[
g_{ij} = g_{ij}^{\prime} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^i}.
\]

Substituting (2.3) into above formulae the metric tensor of \(\mathbb{R}\) coordinate system can be obtain

\[
g_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^j} g_{ij} = g_{\alpha\beta},
\]

Through this paper we denote \(\Theta_{\alpha} = \frac{\partial \Theta}{\partial x^\alpha}\). Its contravariant components are given by

\[
g^{\alpha\beta} = \delta^{\alpha\beta}, g^{33} = -\varepsilon^{-1} \Theta_{\beta}, g^{33} = \varepsilon^{-2} r^{-2} (1 + r^2 |\nabla \Theta|^2),
\]

where \(|\nabla \Theta|^2 = \Theta_1^2 + \Theta_2^2\) and \(\Theta_{\alpha} = \frac{\partial \Theta}{\partial x^\alpha}\).

**Theorem 2.1.** Suppose the \(\Theta\) is a blade’s surface defined by (2.1). Then \(\Theta\) is proposed as a solution of following elliptic boundary value problem:

\[
\begin{align*}
\frac{\partial^2}{\partial x^i \partial x^j} (K^{\alpha\beta\lambda\sigma}(w, \Theta) \frac{\partial^2 \Theta}{\partial x^i \partial x^j}) &+ \frac{\partial^2}{\partial x^i \partial x^j} (r \tilde{\Phi}^{\lambda\sigma}(w, \Theta)) \\
\Theta &\equiv \Theta_0, \ \ \frac{\partial \Theta}{\partial n} = \Theta_s, \text{ on } \partial D
\end{align*}
\]

(2.7)

Combining Navier-Stokes equations and linearized Navier-Stokes equations, where \((w, p)\) and \((\tilde{w}, \tilde{p})\) are solutions of compressible or incompressible rotating Navier-Stokes equations (3.1) or (3.3) and linearized Navier-Stokes equations (3.34) respectively and

\[
K^{\alpha\beta\lambda\sigma}(w, \Theta) = 2 \mu r^3 W^{\alpha\sigma}\delta^{\beta\lambda}, \ W^{\alpha\beta} = \int_{-1}^{1} w^\alpha w^\beta d\xi,
\]

\(\tilde{\Phi}^{\lambda\sigma}, \tilde{\Phi}^{\lambda\sigma}\) are defined by (4.11) respectively.

Variational formulation associated with (2.7) is given by

\[
\begin{align*}
\text{Find } \Theta &\in V_T(D) = \{ q | q \in H^2(D), q|_{\Gamma_0} = \Theta_0, \frac{\partial q}{\partial n} |_{\Gamma_0} = 0 \},
\end{align*}
\]

such that, \(\forall \eta \in H_0^2(D)\)

\[
\int_D \left[ (K^{\alpha\beta\lambda\sigma}(w, \Theta) \Theta_{\nu\mu} + r \tilde{\Phi}^{\lambda\sigma}(w, \Theta)) \eta_{\lambda\sigma} \right] dx \\
+ \int_D \left[ r \tilde{\Phi}^{\lambda\sigma}(w, \Theta) \eta_{\lambda\sigma} + r \tilde{\Phi}^{\lambda\sigma}(w, \tilde{w}, \Theta) \eta \right] dx = 0.
\]

(2.9)

**Theorem 2.2.** Suppose the \(\Theta\) is a blade’s surface defined by (2.1). Then \(\Theta\) is proposed as a solution of following elliptic boundary value problem:

\[
\begin{align*}
- \left( \frac{\partial}{\partial x^i} (K_{ij}(w) \tilde{\Theta} + K^{ij}(w, \Theta) \Theta_{ij}) \right) &+ F_{\nu\mu}(w) \Theta_{\nu\mu} \\
\Theta |_{\gamma} &\equiv \Theta_0,
\end{align*}
\]

(2.10)
where \( K_0(w), K^\lambda(w), F^\nu(w), F^\lambda(w), F_0(w, \Theta) \) are defined by (5.15). The variational formulation associated with (5.11) is given by

\[
\begin{align*}
\text{Find } \Theta \in H^1_0(D) &= \{v|v \in H^1(D), v = \Theta^* \text{ on } \gamma = \partial D \}, \\
&\text{such that, } \forall \eta \in H^1_0(D) \\
&\int_D \left\{ (\Psi_0(w, p, \Theta)\eta + \Psi^\lambda(w, p, \Theta)\eta_\lambda - \mu r^2 W' e^{\partial H' x^3} r \omega \epsilon r) \right\} \, dx = 0,
\end{align*}
\]

where

\[
\Psi^\lambda(w, p, \Theta) = \Psi^\lambda_0(w, p) + \Psi^\lambda_0(w, p)\Theta_\nu + \Psi^\lambda_0(w, p)\Theta_\mu,
\]

and \( \Psi_0(w, p, \Theta), \Psi_0^\lambda(w, p), \Psi_0^\lambda(w, p), \Psi_0^\lambda_0(w, p) \) are defined by (5.10)(5.11).

3. Rotation Navier-Stokes Equations With Mixed Boundary Conditions. At first, we consider the three-dimensional rotating Navier-Stokes equations in a frame rotating around the axis of a rotating impeller with an angular velocity \( \omega \):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho w) &= 0, \\
\rho a &= \nabla \sigma + f, \\
\rho c_v(\frac{\partial T}{\partial t} + w \cdot \nabla T) - \text{div}(\kappa \nabla T) + p \text{div} w &= \Phi = h, \\
p &= p(\rho, T),
\end{align*}
\]

where \( \rho \) is the density of the fluid, \( w \) the velocity of the fluid, \( h \) the heat source, \( T \) the temperature, \( k \) the coefficient of heat conductivity, \( c_v \) specific heat at constant volume, and \( \mu \) viscosity. Furthermore, the strain rate tensor, stress tensor, dissipative function and viscous tensor are given by respectively:

\[
\begin{align*}
\varepsilon_{ij}(w) &= \frac{1}{2}(\nabla_i w_j + \nabla_j w_i); \quad i, j = 1, 2, 3, \\
e^{ij}(w) &= \frac{1}{2}(\varepsilon^{ij}(w) + \varepsilon^{ji}(w)), \\
\sigma_{ij}(w, p) &= A^{ijkm}(w)\varepsilon_{km}(w), \\
A^{ijkm} &= \lambda g^{ij}g^{km} + \mu (g^{ik}g^{jm} + g^{im}g^{jk}), \quad \lambda = -\frac{2}{3}\mu,
\end{align*}
\]

where \( g_{ij} \) and \( g^{ij} \) are the covariant and contravariant components of the metric tensor of dimensional three Euclidian space in the curvilinear coordinate \( (x^1, x^2, x^3) \) define by [2, 4] respectively,

\[
\begin{align*}
\nabla_i w^j &= \frac{\partial w^j}{\partial x^i} + \Gamma^j_{ik} \omega^k; \quad \nabla_i w_j = \frac{\partial w_j}{\partial x^i}, \\
\Gamma^j_{ik} &= g^{il}\frac{\partial g_{kl}}{\partial x^i} + g_{ij} \frac{\partial g_{kl}}{\partial x^i}.
\end{align*}
\]

The absolute acceleration of the fluid is given by

\[
\begin{align*}
a^i &= \frac{\partial w^i}{\partial t} + w^j \nabla_j w^i + 2\varepsilon^{ijk} \omega_j w_k - \omega^2 r^i; \\
a &= \frac{\partial w}{\partial t} + (w \nabla)w + 2\ddot{\mathbf{a}} + \ddot{\mathbf{a}} \times (\ddot{\mathbf{a}} \times \ddot{\mathbf{R}}),
\end{align*}
\]

where \( \ddot{\mathbf{a}} = \omega \dddot{\mathbf{k}} \) is the vector of angular velocity, \( \ddot{\mathbf{k}} \) the unite vector along axis, and \( \dddot{\mathbf{R}} \) the radium vector of the fluid particle. The flow domain \( \Omega_e \) occupied by the fluids in
the channel in the impeller. The boundary $\partial \Omega_c$ of flow domain $\Omega_c$ consists of inflow boundary $\Gamma_{in}$, outflow boundary $\Gamma_{out}$, positive blade’s surface $S_+$, negative blade’s surface $S_-$ and top wall $\Gamma_t$ and Bottom wall $\Gamma_b$:

$$\partial \Omega_c = \Gamma = \Gamma_{in} \cup \Gamma_{out} \cup S_- \cup S_+ \cup \Gamma_t \cup \Gamma_b$$ \tag{3.5}

Boundary conditions are given by

$$\begin{cases}
  w|_{S_- \cup S_+} = 0, & w|_{\Gamma_b} = 0, w|_{\Gamma_t} = 0, \\
  \sigma^{ii}(w, p) n_j|_{\Gamma_{in}} = g_{in}, & \sigma^{ii}(w, p) n_j|_{\Gamma_{out}} = g_{out} \text{ (Natural conditions)} \tag{3.6}
\end{cases}
$$

If the fluid is incompressible and flow is stationary then

$$\begin{cases}
  \text{div} w = 0, \\
  \left( w \nabla \right) w + 2 \omega \times \vec{w} + \nabla p - \nu \text{div}(\nu w) = -\vec{\omega} \times (\vec{\omega} \times \vec{R}) + f, \\
  w|_{\Gamma_0} = 0, & \Gamma_0 = S_+ \cup S_- \cup \Gamma_t \cup \Gamma_b, \\
  (-pm + 2\nu w)|_{\Gamma_{in}} = g_{in}, & \Gamma_1 = \Gamma_{in} \cup \Gamma_{0out}, \\
  (-pm + 2\nu w)|_{\Gamma_{out}} = g_{out}, \\
  w|_{t=0} = w_0(x) 
\end{cases} \tag{3.7}
$$

For the polytropic ideal gas and flow is stationary, system \textbf{(3.1)} turns to the conservation form

$$\begin{cases}
  \text{div}(\rho w) = 0, \\
  \text{div}(\rho w \otimes w) + 2\rho \omega \times w + R\nabla (\rho T) = \mu \Delta w + (\lambda + \mu) \nabla \text{div} w - \rho \omega \times (\omega \times \vec{R}), \tag{3.8}
\end{cases}
$$

while for isentropic ideal gases, it turns

$$\begin{cases}
  \text{div}(\rho w) = 0, \\
  \text{div}(\rho w \otimes w) + 2\rho \omega \times w + \alpha \nabla (\rho \gamma) = 2\mu \text{div}(\epsilon) + \lambda \nabla \text{div} w - \rho \omega \times (\omega \times \vec{R}), \tag{3.9}
\end{cases}
$$

where $\gamma > 1$ is the specific heat ratio and $\alpha$ a positive constant.

The rate of work done by the impeller and global dissipative energy are given by

$$I(S, w(S)) = \int \int_{S_- \cup S_+} \sigma \cdot n \cdot e_\theta w \text{d}S, \quad J(S, w(S)) = \int \int_{\Omega_c} \Phi(w) \text{d}V \tag{3.10}
$$

where $e_\theta$ is base vector along the angular direction in a polar cylindrical coordinate system.

Let us employ new coordinate system defined by $\textbf{(2.2)}$. Flow’s domain $\Omega_c$ is mapped into $\Omega = D \times [-1, 1]$, where $D$ is a domain in $(x^1, x^2) \in \mathbb{R}^2$ surround by four are $\widehat{AB}, \widehat{CD}, \widehat{CB}, \widehat{DA}$ such that

$$\partial D = \gamma_0 \cup \gamma_1, \quad \gamma_0 = \widehat{AB} \cup \widehat{CD}, \quad \gamma_1 = \widehat{CB} \cup \widehat{DA},$$
and there exist four positive functions $\gamma_0(z), \tilde{\gamma}_0(z), \gamma_1(z), \tilde{\gamma}_1(z)$ such that

\[
\begin{align*}
    r := x^2 = \gamma_0(x^1) & \quad \text{on } AB, \quad x^2 = \tilde{\gamma}_0(x^1) \quad \text{on } CD, \\
    r := x^2 = \gamma_1(x^1) & \quad \text{on } DA, \quad x^2 = \tilde{\gamma}_1(x^1) \quad \text{on } BC, \\
    r_0 \leq \gamma_0(z) & \leq r_1 \quad \text{on } AB, \quad r_0 \leq \tilde{\gamma}_0(z) \leq r_1 \quad \text{on } CD, \\
    r_0 \leq \gamma_1(z) & \leq r_1 \quad \text{on } DA, \quad r_0 \leq \tilde{\gamma}_1(z) \leq r_1 \quad \text{on } BC.
\end{align*}
\] (3.11)

Let

\[
\begin{align*}
    \bar{\Gamma}_1 &= \bar{\Gamma}_0 \cup \bar{\Gamma}_1, \\
    \gamma_0 = (D \cap \bar{\Gamma}_0) &\cup (D \cap \bar{\Gamma}_1), \quad \gamma_1 = (D \cup \bar{\Gamma}_0) \cup (D \cup \bar{\Gamma}_1),
\end{align*}
\] (3.13)

where $\bar{R}$ is defined by (2.1).

Let denote

\[
V(\Omega) := \{v|, v \in H^1(\Omega), v|_{\Gamma_0} = 0\}, \quad H^1_\Gamma(\Omega) = \{q|, q \in H^1(\Omega), q|_{\Gamma_0} = 0\}.
\] (3.14)

The variational formulations for Navier-Stokes problem (3.7) and (3.9) are respectively given by

\[
\begin{align*}
    \text{Find } (w, p), w \in V(\Omega), p \in L^2(\Omega), \text{ such that } \\
    a(w, v) + 2(\omega \times w, v) + b(w, w, v) + \\
    -\langle p, \text{div} v \rangle = \langle F, v \rangle, \quad \forall v \in V(\Omega), \\
    (q, \text{div} w) = 0, \quad \forall q \in L^2(\Omega),
\end{align*}
\] (3.15)

and

\[
\begin{align*}
    \text{Find } (w, \rho), w \in V(\Omega), \rho \in L^2(\Omega), \text{ such that } \\
    a(w, v) + 2(\omega \times w, v) + b(\rho w, w, v) + \\
    + (-\alpha p + \lambda \text{div} w, \text{div} v) = \langle F, v \rangle, \quad \forall v \in V(\Omega), \\
    (\nabla q, \rho w) = \langle \rho w q, \theta \rangle|_{\Gamma_1}, \quad \forall q \in H^1_\Gamma(\Omega),
\end{align*}
\] (3.16)

where

\[
\begin{align*}
    \langle F, v \rangle &= \langle f, v \rangle + \langle g, v \rangle|_{\Gamma_0} + \langle g, v \rangle|_{\Gamma_1}, \\
    \langle g, v \rangle &= \langle g_{in}, v \rangle|_{\Gamma_0} + \langle g_{out}, v \rangle|_{\Gamma_1}, \\
    a(w, v) &= \int_{\Omega} A^{ij}k_{ij}(w)\epsilon_{km}(v)\sqrt{\gamma}d\gamma d\xi, \\
    b(w, w, v) &= \int_{\Omega} g_{km}w^j\nabla_j w^k \sqrt{\gamma}d\gamma d\xi.
\end{align*}
\] (3.17)

Next we rewrite (3.7) and (3.9) in new coordinate system. Because second kind of Christoffel symbols in new coordinate system are

\[
\begin{align*}
    \Gamma^\alpha_{\beta\gamma} &= -r\delta_{2\alpha}\Theta_\beta\Theta_\gamma, \quad \Gamma^\alpha_{\beta\gamma} = -\varepsilon r\delta_{2\alpha}\Theta_\beta, \\
    \Gamma^3_{\beta\gamma} &= \varepsilon^{-1}r^{-1}(\delta_{2\alpha}\beta + \delta_{2\beta}\alpha)\Theta_\lambda + \varepsilon^{-1}\Theta_{\alpha\beta} + \varepsilon^{-1}r\Theta_2\Theta_\alpha\Theta_\beta, \\
    \Gamma^3_{\alpha\beta} &= \Gamma^3_{\beta\alpha} = r^{-1}\delta_{2\alpha} + r\Theta_2\Theta_\alpha, \quad \Gamma^3_{\alpha\beta} = -\varepsilon^2 r\delta_{2\alpha}, \quad \Gamma^3_{\alpha\beta} = \varepsilon r\Theta_2.
\end{align*}
\] (3.18)

the covariant derivatives of the velocity field $\nabla_i w^j = \frac{\partial w^j}{\partial x^i} + \Gamma^j_{ik} w^k$ can be expressed as

\[
\begin{align*}
\nabla \omega & = \frac{\partial w}{\partial x^\alpha} - r^\alpha_\beta \Theta_\alpha \Pi(w, \Theta), \\
\nabla \omega^3 & = \frac{\partial w^3}{\partial x^\alpha} + \varepsilon^{-1} (x^2)^{-1} w^2 \Theta_\alpha + \varepsilon^{-1} w^3 \Theta_\alpha, \\
\nabla w^3 & = \frac{\partial w^3}{\partial x^\alpha} - x^2 \varepsilon \delta^\alpha_\beta \Pi(w, \Theta), \\
\nabla \omega^3 & = \frac{\partial w^3}{\partial x^\alpha} - x^2 \varepsilon \delta^\alpha_\beta \Pi(w, \Theta), \\
\text{div} w & = \frac{\partial w}{\partial x^\alpha} + \frac{w^2}{x^2} + \frac{\partial w^3}{\partial x^\beta}, \\
\Pi(w, \Theta) & = \varepsilon w^3 + w^3 \Theta_\beta.
\end{align*}
\] (3.19)

and the deformation tensors are given by

\[
\begin{align*}
\varphi_{ij}(w) & = \varphi_{ij}(w) + \psi_{ij}(w, \Theta); \\
\psi_{ij}(w, \Theta) & = \psi_{ij}^\alpha(w) \Theta_\alpha + \psi_{ij}^\beta(w) \Theta_\beta + \psi_{ij}^r(w, \Theta),
\end{align*}
\] (3.20)

where

\[
\begin{align*}
\varphi_{3\alpha}(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^3} + \frac{\partial w^3}{\partial x^\alpha} \right), \\
\varphi_{3\alpha}(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^3} + \frac{\partial w^3}{\partial x^\alpha} \right), \\
\varphi_{33}(w) & = \varepsilon^2 r^2 \left( \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} \right), \\
\varphi_{3\alpha}(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^3} + \frac{\partial w^3}{\partial x^\alpha} \right), \\
\varphi_{3\alpha}(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^3} + \frac{\partial w^3}{\partial x^\alpha} \right), \\
\psi_{3\alpha}^\beta(w) & = \psi_{3\alpha}^\beta(w), \\
\psi_{3\alpha}^\beta(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^\beta} + \frac{w^\beta}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} + \frac{w^2}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} \right),
\end{align*}
\] (3.21)

\[
\begin{align*}
\psi_{3\alpha}^\beta(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^\beta} + \frac{w^\beta}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} + \frac{w^2}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} \right), \\
\psi_{3\alpha}^\beta(w) & = \frac{1}{2} \left( \frac{\partial w^\alpha}{\partial x^\beta} + \frac{w^\beta}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} + \frac{w^2}{r} \delta_{\alpha_\beta} \delta_{\alpha_\beta} \right),
\end{align*}
\] (3.22)

\[
e_{3\alpha}^\beta(w, \Theta) = \frac{1}{2} r^2 w^\sigma (\Theta_\alpha \Theta_\beta + \Theta_\beta \Theta_\alpha), e_{3\alpha}^\sigma(w) = \frac{1}{2} \varepsilon r^2 w^\sigma \Theta_\sigma, e_{33}^\sigma(w) = 0.
\] (3.23)

The proof is omitted here.

By simply tensor calculations in terms of (3.19), (2.5) and (2.6),

\[
A^{ijkl} e_{kl}(w) e_{ij}(v)
= e_{ij}(w) e_{ij}(v) + 2 g_{ij} g_{ij} e_{3\alpha}(w) e_{3\alpha}(v)
+ \varepsilon^{-1} g_{ij} g_{ij} e_{3\alpha}(w) e_{3\alpha}(v)
- 2 \varepsilon^{-1} g_{ij} g_{ij} e_{3\alpha}(w) e_{3\alpha}(v) e_{3\alpha}(w)
+ \varepsilon^{-1} g_{ij} g_{ij} e_{3\alpha}(w) e_{3\alpha}(v) e_{3\alpha}(w)
+ \varepsilon^{-1} g_{ij} g_{ij} e_{3\alpha}(w) e_{3\alpha}(v) e_{3\alpha}(w)
\] (3.24)

and

\[
\int_\Omega A^{ijkl} \delta_{kl}(w) \delta_{ij}(v) r \varepsilon \xi dx = ((w, v)) + \int_\Omega B(w, v, \Theta) d\xi dx,
\]

where

\[
\begin{align*}
((w, v)) & = \int_\Omega 2 \mu \left[ \varphi_{ij}(w) \varphi_{ij}(v) + 2 (r \varepsilon)^{-2} \varphi_{ij}(w) \varphi_{ij}(v) + (r \varepsilon)^{-4} \varphi_{ij}(w) \varphi_{ij}(v) \right] r \varepsilon \xi dx, \\
\|w\|^2 & := ((w, w)), \\
a(w, v) & = ((w, v)) + \int_\Omega 2 \nu \left[ B(w, v, \Theta) + A(w, v, \Theta) \right] r \varepsilon \xi dx,
\end{align*}
\] (3.25)
In particular, angular velocity vector is given by
\[ B(w, v, \Theta) = 2\mu [2\varepsilon^2(\nabla \Theta)\delta_{\alpha\beta} + \Theta_{\alpha}\Theta_{\beta}] \varphi_{3\alpha}(w) \varphi_{3\beta}(v) + \varepsilon^{-1}r_{\alpha\beta}[\varphi_{3\alpha}(w) \varphi_{3\beta}(v) - 2\varepsilon^{-1}g_{\alpha\beta}\Theta_{\alpha}\varphi_{3\beta}(w) + 2\varepsilon^{-1}g_{\alpha\beta}\Theta_{\alpha}\varphi_{3\beta}(v) + \varphi_{3\alpha}(v) \varphi_{3\beta}(w)] \]
\[ A(w, v, \Theta) = A^{ijkl}[\varphi_{kl}(w)\psi_{ij}(v, \Theta) + \varphi_{ij}(v)\psi_{kl}(w, \Theta) + \psi_{ij}(w, \Theta)\psi_{ij}(v, \Theta)] \]

(3.26)

Next we consider trilinear form. Using (3.19), we have
\[ w^\alpha \nabla_j w^\beta = w^\alpha \frac{\partial w^\beta}{\partial x^j} + w^3 \frac{\partial w^\beta}{\partial x^j} - r\delta_{2\beta} \Pi(w, \Theta) \Pi(w, \Theta), \]
\[ w^\alpha \nabla_j w^\beta = w^3 \frac{\partial w^\beta}{\partial x^j} + w^3 \frac{\partial w^\beta}{\partial x^j} + \varepsilon^{-1}w^\alpha w^\beta \Theta_{\alpha\beta} + \varepsilon^{-1}r\Theta_{2\beta} \Pi(w, \Theta) \Pi(w, \Theta) + \varepsilon^{-1}r^2 w^\alpha \Theta_{\alpha\beta} \Pi(w, \Theta), \]

Therefore
\[ b(w, w, v) = \int_D \int_{-1}^1 \frac{k_{\alpha\beta}(w^\alpha \partial w^\beta}{\partial \xi} \sqrt{\nu} \xi d\xi dx = \int_D \int_{-1}^1 \left[\alpha_{\alpha\beta}(w^\alpha \partial w^\beta}{\partial \xi} + w^3 \frac{\partial w^\beta}{\partial \xi} \right] + \varepsilon^{-1}w^\alpha w^\beta \Theta_{\alpha\beta} + \varepsilon^{-1}r\Theta_{2\beta} \Pi(w, \Theta) \Pi(w, \Theta) + \varepsilon^{-1}r^2 w^\alpha \Theta_{\alpha\beta} \Pi(w, \Theta) \Pi(w, \Theta) \right] \]
\[ \left[ \right] \frac{w^3 \partial w^\beta}{\partial \xi} + \varepsilon^{-1}r\Theta_{2\beta} \Pi(w, \Theta) \Pi(w, \Theta) \right] \]

(3.28)

By similar manner, angular velocity vector is given by
\[ \mathbf{\omega} = \omega e^1 - \varepsilon^{-1}r\Theta_1 e_3, \]
\[ 2\mathbf{\omega} \times \mathbf{\omega} = -2r\omega \delta_{\alpha\beta} (\varepsilon w^3 + \Theta_{\beta}) e^\alpha + 2r\omega \varepsilon^{-1}[\Theta_2 (\varepsilon w^3 + \Theta_{\beta})] - r^2 w^2 e^3 \]
\[ -2r\omega \delta_{\alpha\beta} (\varepsilon w^3 + \Theta_{\beta}) e^\alpha + 2r\omega \varepsilon^{-1}[\Theta_2 (\varepsilon w^3 + \Theta_{\beta})] - r^2 w^2 e^3 \]

(3.29)

while Coliali form
\[ C(w, v) = \int_D \int_{-1}^1 2g_{ij} (\mathbf{\omega} \times w)^j v^i \sqrt{\nu} d\xi dx \]
\[ = \int_D \int_{-1}^1 \left[2\alpha_{\alpha\beta}(\mathbf{\omega} \times w)^\alpha v^\beta + 2r\omega \varepsilon^{-1}[\Theta_2 (\mathbf{\omega} \times w)^3 v^\beta] + (\mathbf{\omega} \times w)^3 v^\alpha \right] \]
\[ + \varepsilon^{-1}r^2 (\mathbf{\omega} \times w)^3 v^\beta \sqrt{\nu} d\xi dx \]
\[ = \int_D \int_{-1}^1 2r\omega [(w^2 \Theta_{\beta} - \delta_{2\beta} \Pi(w, \Theta)) v^\beta + \varepsilon w^2 v^3 \sqrt{\nu} d\xi dx \]

It is very easy to verify that
\[ C(w, w) = 0. \]
Throughout this paper, Latin indices and exponents $i,j,k \ldots$ vary in the set \{1, 2, 3\}, while Greek indices and exponents $\alpha, \beta, \gamma \ldots$ vary in the set \{1, 2\}. Furthermore, the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule.

**Lemma 3.2.** The function $\| \cdot \|_\Omega$ defined by (3.25) is a norm in Hilbert space \[ V(\Omega) := \{ v \in H^1(D)^3, v|_{\Gamma_1} = 0, \} \tag{3.32} \]

**Proof.** Indeed it is enough to prove that $\|w\|_\Omega = 0, \forall w \in V(\Omega) \Rightarrow w = 0.$

This means that $\|w\|_\Omega = 0, \text{i.e., } \varphi_{ij}(w) = 0.$

we have to prove $w = 0.$ Firstly, the following identity is held

$$\partial_\gamma (\partial_\alpha w^\beta) = \partial_\gamma \varphi_{\alpha\beta}(w) + \partial_\alpha \varphi_{\gamma\beta}(w) - \partial_\beta \varphi_{\alpha\gamma}(w).$$

This shows that \( \varphi_{\alpha\beta}(w) = 0, \text{ in } D \Rightarrow \partial_\gamma \partial_\alpha w^\beta = 0, \text{ in } D'(D). \)

By a classical result from distribution theory, each function $w$ is therefore a polynomial of degree $\leq 1$(recall that the set $D$ is connect). In other words, there exist constants $c_\alpha$ and $d_{\alpha\beta}$ such that

$$w^\alpha(x) = c_\alpha + d_{\alpha\beta}x^\beta, \forall x = (x^1, x^2) \in D,$$

But $\varphi_{\alpha\beta}(w) = 0$ also implies that $d_{\alpha\beta} = -d_{\beta\alpha};$ hence there exist two vectors $\vec{c}, \vec{d} \in \mathbb{R}^2$ such that

$$w = \vec{c} + \vec{d} \times \vec{x}, \forall x \in D,$$

Since $w|_{\Gamma_1} = 0$ and the set where such a vector field $w^\alpha$ vanishes is always of zero area unless $\vec{c} = \vec{d} = 0,$ it follows that $w^\alpha = 0$ when area $\tilde{\Gamma}_0 > 0.$ On the other hand, in view of boundary condition (3.13)

$$\varphi_{33}(w) = \varepsilon^2 r^2 (\frac{\partial w^3}{\partial \xi} + \frac{w^2}{r}) = 0 \Rightarrow \frac{\partial w^3}{\partial \xi} = 0 \Rightarrow w^3 = 0,$$

The proof is complete. \[ \square \]

**Lemma 3.3.** The norm $\| \cdot \|_\Omega$ and the norm

$$|w|^2_{1,\Omega} = \int \sum_{i=1}^3 \left[ \sum_{\alpha=1}^2 \left( \frac{\partial w^i}{\partial x^\alpha} \right)^2 + \left( \frac{\partial w^i}{\partial \xi} \right)^2 \right] r \varepsilon d\xi dx, \forall w \in V(\Omega),$$

are equivalent in $V(\Omega), \text{i.e. there exist a constant } C_i(\Omega) > 0, i = 1, 2 \text{ depending upon } \Omega \text{ only such that}$

$$C_1(\Omega)|w|_{1,\Omega} \leq \|w\|_\Omega \leq C_2(\Omega)|w|_{1,\Omega}, \forall w \in V(\Omega), \quad (3.33)$$
Proof. Firstly we indicate that in view of (3.11)(3.12) we assert that there exist a constant \( C_1(\Omega) > 0, i = 1, 2 \) depending upon \( \Omega \) only such that

\[
C_1(\Omega)(\sum_{i,j=1}^{3} \| \varphi_{ij}(w) \|_{0,\Omega}^2)^{1/2} \leq \| w \|_{\Omega} \leq C_2(\Omega)(\sum_{i,j=1}^{3} \| \varphi_{ij}(w) \|_{0,\Omega}^2)^{1/2}, \quad \forall w \in V(\Omega),
\]

and \( \varphi_{ij}(w) \) can be looked as strain tensor in Cartesian coordinates in \( \mathbb{R}^3 \) then according to Korn’s inequality (see \([14],[15]\)) the (\( \sum_{i,j=1}^{3} \| \varphi_{ij}(w) \|_{0,\Omega}^2)^{1/2} \) is equivalent to \( \| w \|_{1,\Omega} \), therefor this reach to (3.33). The proof is complete. \( \square \)

Theorem 3.1. Under the new coordinate system, the stationary Navier-Stokes equations can be explicitly expressed as \( \Theta \):

\[
\begin{align*}
\text{div}(w) &= \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{\partial w^3}{\partial x^3} + \frac{w^2}{r} = 0, \\
N^3(w, p, \Theta) &= \mathcal{L}^i(w, p, \Theta) + \mathcal{N}^i(u, w) = f^i, \quad i = 1, 2, 3,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{L}^i(w, p, \Theta) &= -\nu(\Delta w^\alpha - 2\varepsilon^{-1}\Theta_{\beta} \frac{\partial^2 w^\alpha}{\partial x^\beta \partial x^\alpha} + (r\varepsilon)^{-2}\alpha \frac{\partial^2 w^\alpha}{\partial x^\beta \partial x^\beta} + r^{-1} \frac{\partial w^\alpha}{\partial x^3} \\
&- 2\varepsilon \delta_{2\alpha}(\Theta_{2\alpha} \frac{\partial w^3}{\partial x^3} - \varepsilon^{-1}(\delta_{3\alpha}(\Theta_{3\alpha} + 2\partial_{2\alpha}\Theta_{\lambda} + r\delta_{2\alpha} \Delta \Theta) \frac{\partial w^3}{\partial x^3} - 2r^{-1} \delta_{2\alpha} \frac{\partial w^3}{\partial x^3} \\
&- 2\varepsilon \delta_{2\alpha} \alpha \Theta_{2\alpha} w^3 + \delta_{2\alpha} (r^{-2} \delta_{2\alpha} - 2\Theta_{2\alpha} \Theta_{\delta} - \alpha \delta_{2\alpha} |\nabla \Theta|^2 - \Theta_{\delta} \Theta_{\lambda}) w^\sigma) \\
&- 2\varepsilon \delta_{2\alpha} \alpha \Theta_{2\alpha} \Pi(w, \Theta) + \nabla \alpha p - \varepsilon^{-1} \Theta_{\alpha} \frac{\partial \delta_{2\alpha}}{\partial x^3},
\end{align*}
\]

\[
\begin{align*}
\mathcal{N}^3(w, w) &= w^3 \frac{\partial w^3}{\partial x^3} + w^3 \frac{\partial w^3}{\partial x^3} - r \delta_{2\alpha} \Pi(w, \Theta) \Pi(w, \Theta),
\end{align*}
\]

\[
\begin{align*}
\sum_{i,j=1}^{3} \| \varphi_{ij}(w) \|_{0,\Omega}^2)^{1/2} \leq \| w \|_{\Omega} \leq C_2(\Omega)(\sum_{i,j=1}^{3} \| \varphi_{ij}(w) \|_{0,\Omega}^2)^{1/2}, \quad \forall w \in V(\Omega),
\end{align*}
\]

where \( \Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2_{i} \partial x^2_{i}} + \frac{\partial^2 \varphi}{\partial x^2_{3} \partial x^2_{3}} \), \( \varepsilon^{\alpha\beta} = +1, -1, 0 \) depending on \( (\alpha, \beta) = (1, 2), (2, 1) \), or other case. Under the new coordinate system the domain \( \Omega = \{(x^1, x^2) \in D, -1 \leq \xi \leq 1\} \) is independent of geometry of blade’s surface \( \Theta \).

Proof. Substituting (3.19)(2.5)(2.6) and (3.20) into (3.7), tensor calculations show that (3.14) is valid. The details is omitted here. \( \square \)

By using above formulae we claim that there exist of the Gateaux derivative of the solution of Navier-Stokes equations to be satisfy following linearized Navier-Stokes equation

Theorem 3.2. Assume that there exists a solution \( (w(\Theta), p(\Theta)) \) of Navier-Stokes problem (3.7) such that define a mapping \( \Theta \Rightarrow (w(\Theta), p(\Theta)) \) from \( H^1_0(D) \cap H^2(D) \) to \( H^{1,2}(\Omega) \times L^2(\Omega) \). Then there exist the Gâteaux derivatives of \( (w, p) \) at a point \( \Theta \in H^1_0(D) \cap H^2(D) \) with respect to any direction \( \eta \in H^1_0(D) \cap H^2(D) : \tilde{w} = \frac{\partial w}{\partial \Theta} \eta, \tilde{p} = \frac{\partial p}{\partial \Theta} \eta. \)
\[ D_p \eta \] and satisfy following linearized equations:

\[
\begin{cases}
\text{div}(\hat{w}) = 0 \\
L^i(\hat{w}, \hat{p}, \Theta) + \mathcal{N}^i(\hat{w}, \hat{w}) + \mathcal{N}^i(\hat{w}, w) = R^i(w, p) \\
u \big|_{\Gamma} = 0, \quad \sigma^{ij}(\hat{w}, \hat{p})n_j |_{\Gamma_0} = 0.
\end{cases}
\tag{3.37}
\]

where

\[
R^\alpha(w, p, \Theta) = \nu \delta_{2\alpha} \frac{\partial^2}{\partial x^2} (rw^\alpha \Theta_\lambda) + \nu \varepsilon^{-1} \frac{\partial w^\alpha}{\partial \xi} + \nu \frac{\partial}{\partial x} \left[ -2(\varepsilon)^{-2} \Theta_{\beta} \frac{\partial r w^\alpha}{\partial \xi} + 2r \varepsilon \delta_{2\alpha} \frac{\partial r}{\partial \xi} \right] + (r \varepsilon)^{-1} \left( \delta_{\alpha \lambda} \delta_{2\beta} + 2 \delta_{2\alpha} \delta_{\lambda \beta} \right) \frac{\partial w^\alpha}{\partial \xi} + 2 \varepsilon \delta_{2\alpha} w^3 (4(\delta_{2\beta} + 3\delta_2 \Theta_2)) \\
+ \delta_{2\alpha} w^3 (2(\delta_{2\beta} \Theta_2 + 2(\Theta_2 \delta_{2\beta} + 2r^2 (\Theta_2 \delta_{2\beta} + \delta_{2\beta} \Theta_2)) \nabla \Theta)^2 \\
- 2a_2 \Theta \beta - r \Theta_\beta \right] = \frac{r}{\varepsilon} \left[ \frac{\partial}{\partial \xi} \left( r \delta_2 \nabla (w, \Theta) + \omega \right) w^3 + \varepsilon^{-1} \delta_{\alpha \beta} \frac{\partial r}{\partial \xi} \right]
\tag{3.38}
\]

\[
R^3(w, p, \Theta) = \nu \varepsilon^{-1} \left( -\Delta \left( \frac{w^2}{r} \right) + 3 \partial_{2\alpha} (r^{-1} w^\alpha) \right) + \varepsilon^{-1} \partial_{\lambda} (\frac{w^\alpha}{r} w^\lambda) - 2 \varepsilon \frac{\partial}{\partial x} \left( \frac{w^3}{r} \right) \left( \frac{\partial w^3}{\partial \xi} \right) \\
- \nu \varepsilon^{-1} \frac{\partial}{\partial \xi} \left[ 2(r \varepsilon)^{-1} \Theta_{\beta} \frac{\partial w^3}{\partial \xi} + 2r^{-1} \frac{\partial w^3}{\partial \xi} - r^{-1} \delta_{2\beta} \frac{\partial w^3}{\partial \xi} + r^{-2} \delta_{2\beta} w^2 \right] \\
+ \varepsilon^{-1} \frac{\partial}{\partial \xi} \left[ 2w^3 \frac{\partial w^3}{\partial \xi} - 2 \omega r \omega \delta_{2\beta} \Pi (w, \Theta) \right. \\
+ (r (\Pi (w, \Theta) - w) [2 \Theta_\beta w^3 - \delta_{2\beta} \Pi (w, \Theta)]) \right] + \varepsilon^{-1} \Delta \hat{p} + 2 \varepsilon^{-1} \frac{\partial}{\partial \xi} \left( r \Theta_{\beta} \frac{\partial w}{\partial \xi} \right)
\tag{3.39}
\]

Proof. The Navier-Stokes equations (3.24) read

\[
\frac{\partial w^\alpha}{\partial t} + \frac{w^2}{r} \frac{\partial w^\alpha}{\partial x} + \frac{\partial w^\alpha}{\partial \xi} = 0, \\
\mathcal{N}^\alpha(w, p, \Theta) \hat{e}_\alpha + \mathcal{N}^3(w, p, \Theta) \hat{e}_3 = f^\alpha \hat{e}_\alpha + f^3 \hat{e}_3.
\tag{3.40}
\]

Set Gateaux derivative with respect with \( \Theta \) along any director \( \eta \in \mathcal{W} := H^2(D) \cap H_0^1(D) \) denoted by \( \frac{D}{D \Theta} \eta \). Then from (3.40) we obtain

\[
\begin{aligned}
\frac{D}{D \Theta} \mathcal{N}^\alpha & (w, p, \Theta) \hat{e}_\alpha \eta + \frac{D}{D \Theta} \mathcal{N}^3 (w, p, \Theta) \hat{e}_3 \eta + \mathcal{N}^\alpha (w, p, \Theta) \frac{D}{D \Theta} \eta + \mathcal{N}^3 (w, p, \Theta) \frac{D}{D \Theta} \eta \\
&= f^\alpha \frac{D}{D \Theta} \hat{e}_\alpha \eta + f^3 \frac{D}{D \Theta} \hat{e}_3 \eta, \\
\frac{D}{D \Theta} \mathcal{N}^\alpha & (w, p, \Theta) \hat{e}_\alpha + \frac{D}{D \Theta} \mathcal{N}^3 (w, p, \Theta) \hat{e}_3 + [\mathcal{N}^\alpha (w, p, \Theta) - f^\alpha] \frac{D}{D \Theta} \hat{e}_\alpha + [\mathcal{N}^3 (w, p, \Theta) - f^3] \frac{D}{D \Theta} \hat{e}_3 = 0.
\end{aligned}
\]

Hence,

\[
\begin{aligned}
\frac{D}{D \Theta} \mathcal{N}^\alpha (w, p, \Theta) \eta = \mathcal{L}^\alpha (\hat{w}, \hat{p}, \Theta) \eta + \mathcal{N} \mathcal{N}^\alpha (w, \hat{w}) \eta + \mathcal{N} \mathcal{N}^\alpha (\hat{w}, w) \eta + R^\alpha (w, p, \Theta) \eta = 0, \\
\frac{D}{D \Theta} \mathcal{N}^3 (w, p, \Theta) \eta = \mathcal{L}^3 (\hat{w}, \hat{p}, \Theta) \eta + \mathcal{N} \mathcal{N}^3 (w, \hat{w}) \eta + \mathcal{N} \mathcal{N}^3 (\hat{w}, w) \eta + R^3 (w, p, \Theta) \eta = 0,
\end{aligned}
\]

where \( R^i(w, p, \Theta) \) can be obtain from (3.22)(3.23). Indeed, direction calculations from
\[ -R^\eta = -\nu[-2\varepsilon^{-1}\frac{\partial^2 w^\eta}{\partial \xi \partial x^3} + 2(r\varepsilon)^{-2} r^2 \Theta_\beta \frac{\partial^2 w^\eta}{\partial \xi^3} - 2r\varepsilon \delta_{2a} \frac{\partial^2 w^\eta}{\partial \xi^2} \partial_\eta \] \\
\[ -\nu[(-r\varepsilon)^{-1}(\delta_{2a} \delta_{2a} \eta_2 + 2\delta_{2a} \eta_2 + r\delta_{2a} \lambda) \frac{\partial w^\eta}{\partial \xi} - 2\varepsilon \delta_{2a} \alpha \eta_2 \alpha^3 + 4\varepsilon \delta_{2a} r^2 \Theta_\beta \Theta_2 w^3 \eta_\beta + \delta_{2a}(2\eta_2 \Theta_\sigma - 2\Theta_2 \eta_\sigma - 2r^2(\Theta_2 \eta_\sigma + \eta_2 \Theta_\sigma) \nabla \Theta |^2 - 2\varepsilon \delta_{2a} \alpha \eta_\lambda \alpha^3 - r\eta_\lambda \Theta_{\lambda \sigma} + r\Theta_\lambda \eta_{\lambda \sigma}) w^\sigma] \\
\[ -\nu[\delta_{2a}(\Pi(w, \Theta) + 2\varepsilon \delta_{2a} w^3 \Theta_\lambda \eta_{\lambda \sigma} + \varepsilon^{-1} \delta_{\eta_\lambda} \partial w^\eta \partial_\xi + 2(r\varepsilon)^{-2} r^2 \Theta_\beta \frac{\partial^2 w^\eta}{\partial \xi^3} + 2r\varepsilon \delta_{2a} \frac{\partial w^3}{\partial \xi^2} \eta_\beta + \nu[(r\varepsilon)^{-1}(\delta_{2a} \delta_{2a} + 2\delta_{2a} \delta_{\beta \alpha} \delta_{\gamma} \partial w^\eta \partial_\xi + 2\varepsilon \delta_{2a} w^3(\alpha \delta_{2a} + 2r^2 \Theta_\beta \Theta_2)] \\
\[ +\nu\delta_{2a}(2\Theta_\beta \Theta_\sigma + 2\Theta_2 \delta_{2a} \Theta_\beta + 2r^2 \Theta_2 \delta_{2a} \Theta_\sigma) \nabla \Theta |^2 - 2\varepsilon \delta_{2a} \alpha \eta_\lambda \alpha^3 - r\Theta_{\lambda \sigma} w^\sigma \eta_\beta + \nu\delta_{2a}(\Pi(w, \Theta) + \omega) w^\beta + \varepsilon^{-1} \delta_{\eta_\lambda} \partial w^\eta \partial_\xi] \]

Taking into accounts of the homogenous boundary conditions for the $\eta$ and integrating by parts and using Green formula and incompressible condition

\[ \nu\varepsilon^{-1} \frac{\partial w^\eta}{\partial \xi} - 2\nu\varepsilon^{-1} \frac{\partial^3 w^\eta}{\partial x^3 \partial \xi^3} = -\nu\varepsilon^{-1} \frac{\partial w^\eta}{\partial \xi} \]

we claim

\[ R^\eta(w, p, \Theta) = \nu\delta_{2a} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} (r \varepsilon \delta_{\beta \alpha} \Theta_\lambda) + \nu\varepsilon^{-1} \Delta \frac{\partial w^\eta}{\partial \xi} + 2(\varepsilon)^{-1} \frac{\partial^2 w^\eta}{\partial \xi^3} - 2\varepsilon^{-2} \Theta_\beta \frac{\partial^2 w^\eta}{\partial \xi^2} + 2\varepsilon \delta_{2a} \frac{\partial^2 w^3}{\partial \xi^2} \] \\
\[ +\nu\varepsilon^{-1} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \left[(r\varepsilon)^{-1}(\delta_{2a} \delta_{2a} + 2\delta_{2a} \delta_{\beta \alpha} \delta_{\gamma} \partial w^\eta \partial_\xi + 2\varepsilon \delta_{2a} w^3(\alpha \delta_{2a} + 2r^2 \Theta_\beta \Theta_2)] \\
\[ +\nu\varepsilon^{-1} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \left[(2\delta_{2a} \Theta_\lambda + 2\Theta_2 \delta_{2a} \Theta_\beta + 2r^2 \Theta_2 \delta_{2a} \Theta_\beta) \nabla \Theta |^2 - 2\varepsilon \delta_{2a} \alpha \eta_\lambda \alpha^3 - r\Theta_{\lambda \sigma} w^\sigma \eta_\beta \\
\[ -\nu\varepsilon^{-1} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \left[2\varepsilon \delta_{2a} \Pi(w, \Theta) + \omega) w^\beta + \varepsilon^{-1} \delta_{\eta_\lambda} \partial w^\eta \partial_\xi \right] \]

This is (3.25). Next we consider (3.26). Indeed,

\[ -R^\eta = -\nu[(-2\varepsilon^{-1}\eta_\beta \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} + (r\varepsilon)^{-2} r^2 \Theta_\beta \eta_\beta \frac{\partial^2 w^\eta}{\partial \xi^3} + \varepsilon^{-2} \eta_\beta 2r^2 \Theta_\beta \eta_\beta \frac{\partial w^\eta}{\partial \xi} \\
\[ +2(\varepsilon)^{-1} \delta_{2a} \delta_{2a} \eta_2 + r \eta_2 \Theta_\beta \frac{\partial w^\eta}{\partial \xi} + (r\varepsilon)^{-1}(\delta_{2a} \delta_{2a} + 2\delta_{2a} \delta_{\beta \alpha} \delta_{\gamma} \partial w^\eta \partial_\xi + 2\varepsilon \delta_{2a} w^3(\alpha \delta_{2a} + 2r^2 \Theta_\beta \Theta_2)] \\
\[ +\nu\varepsilon^{-1} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \left[(2\delta_{2a} \Theta_\lambda + 2\Theta_2 \delta_{2a} \Theta_\beta + 2r^2 \Theta_2 \delta_{2a} \Theta_\beta) \nabla \Theta |^2 - 2\varepsilon \delta_{2a} \alpha \eta_\lambda \alpha^3 - r\Theta_{\lambda \sigma} w^\sigma \eta_\beta \\
\[ +\nu\varepsilon^{-1} \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \left[2\varepsilon \delta_{2a} \Pi(w, \Theta) + \omega) w^\beta + \varepsilon^{-1} \delta_{\eta_\lambda} \partial w^\eta \partial_\xi \right] \]

By similar manner and

\[ \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} = \frac{\partial^2 w^\eta}{\partial \xi \partial x^3} \]
we assert

\[ R^3(w, p, \Theta) = \nu \varepsilon^{-1} \left[ \Delta \left( \frac{\partial w^\lambda}{\partial x^\lambda} - \frac{\partial w^\alpha}{\partial x^\alpha} \right) + 3 \partial_\alpha (r^{-1} w^\alpha) + \varepsilon^{-1} \partial_\beta \lambda (w^\beta w^\lambda) \right] \\
+ 2 \nu \varepsilon^{-1} \Delta \frac{\partial w^3}{\partial x^3} - 2 \nu \frac{\partial}{\partial x^\alpha} (\Theta \beta \frac{\partial w^3}{\partial x^\alpha}) \\
- \nu \varepsilon^{-1} \frac{\partial}{\partial x^\alpha} \left[ r (\varepsilon^{-1} \Theta \beta \frac{\partial w^3}{\partial x^\alpha} + 2 r^{-1} \frac{\partial w^3}{\partial x^\alpha} - r^{-1} \delta_{\beta \lambda} \frac{\partial w^3}{\partial x^\alpha} + r^{-2} \delta_{\beta \lambda} w^3) \right] \\
+ \varepsilon^{-1} \partial_\alpha (r \Theta \beta \frac{\partial w^3}{\partial x^\alpha}) \\
= \nu \varepsilon^{-1} \left[ \Delta \left( \frac{\partial w^\alpha}{\partial x^\alpha} \right) + 3 \partial_\alpha (r^{-1} w^\alpha) + \varepsilon^{-1} \partial_\beta \lambda (w^\beta w^\lambda) - 2 \nu \frac{\partial}{\partial x^\alpha} (\Theta \beta \frac{\partial w^3}{\partial x^\alpha}) \\
- \nu \varepsilon^{-1} \frac{\partial}{\partial x^\alpha} \left[ r (\varepsilon^{-1} \Theta \beta \frac{\partial w^3}{\partial x^\alpha} + 2 r^{-1} \frac{\partial w^3}{\partial x^\alpha} - r^{-1} \delta_{\beta \lambda} \frac{\partial w^3}{\partial x^\alpha} + r^{-2} \delta_{\beta \lambda} w^3) \right] \\
+ \varepsilon^{-1} \partial_\alpha (r \Theta \beta \frac{\partial w^3}{\partial x^\alpha}) \right] \\
+ \varepsilon^{-1} \Delta p + 2 \varepsilon^{-1} \frac{\partial}{\partial x^\alpha} \left( r \Theta \beta \frac{\partial p}{\partial x^\alpha} \right) \]

This is (3.26). The proof is complete. \[ \square \]

Assocional variational formulation is given by

\[
\begin{align*}
\text{Find } (\tilde{w}, \tilde{p}) &\in H^1_0(\Omega) \times L^2(\Omega) \text{ such that} \\
&\quad a(\tilde{w}, v) + b(w, \tilde{w}, v) + b(\tilde{w}, \tilde{w}, v) + 2(\omega \times \tilde{w}, v) = (\tilde{p}, \text{div} v), \\
&\quad (q, \text{div} \tilde{w}) = 0, \quad \forall q \in L^2(\Omega), \\
&\quad (R(w, p), v) = \int_\Omega g_\alpha \tilde{R}^\alpha w^\beta dV = \int_\Omega [a_{\alpha \beta} R^\alpha (w, p, \Theta)v^\beta \\
&\quad + \varepsilon r^2 (R^\alpha (w, p, \Theta)v^\beta + R^3 (w, p, \Theta)v^\alpha)] + \varepsilon^2 r^2 R^3 (w, p, \Theta)v^3 \epsilon d\xi d\eta, \\
&\quad \text{div} (\tilde{w} \rho + w \tilde{\rho}) = 0 \\
&\quad \text{div} (\rho \tilde{w} w + \rho w \tilde{w} + \rho w w) + 2 \tilde{\rho} (\omega \times \tilde{w}) + 2 \rho (w \omega \times \tilde{w}) + 2 \rho (\omega \omega \times \tilde{w}) + a g j \nabla_j (\gamma p^{-1} \tilde{p}) \\
&\quad - \nabla_j (A^{jk} e_{km} (\tilde{w})) = S^i (w, p) \\
&\quad (S^i (w, p, \Theta) = -\partial_\beta S^{i \beta} (w, p, \Theta) + \partial_\alpha S^{i \alpha \beta} (w, p; \Theta), \quad (3.44) \\
\end{align*}
\]

where

\[
\begin{align*}
S^{i \beta} (w, p, \Theta) &= \{ r \delta_\beta^\gamma \left[ (\delta_\gamma \Theta_\sigma + \delta_\gamma \Theta_\lambda) w^\lambda w^\sigma + 2 \varepsilon w^3 w^\beta \right] + 2 \rho r \delta_\alpha w^\beta - \varepsilon^{-1} \delta_\alpha \beta \frac{\partial (p \epsilon)}{\partial x^\alpha} \\
&\quad + 2 \nu \varepsilon^{-1} \left[ -\Theta_\alpha g^{jm} \nabla_j e_{3m}^\beta (w) \right] \\
&\quad + \varepsilon^{-1} \left[ (\Theta_\alpha \delta_\beta \lambda + \Theta_\lambda \delta_\alpha \beta) \nabla_\lambda e_{33} (w) + \varepsilon^{-1} (\Theta_\alpha \delta_\beta \lambda + 2 \delta_\beta \lambda \Theta_\lambda + \Theta_\lambda \delta_\alpha \beta) \nabla_\lambda e_{33} (w) \\
&\quad + \delta_\alpha \beta \nabla_\lambda e_{33} (w) - \nabla_\beta e_{33} (w) - \nabla_\lambda e_{33} (w) \\
&\quad - \varepsilon^{-1} [ \Theta_\alpha \delta_\beta \lambda + 3 \delta_\alpha \beta \delta_\lambda (\epsilon_{33} (w) - \varepsilon^{-1} \Theta_\lambda e_{33} (w))] \} \\
\end{align*}
\]

(3.45)
4. A Principle of Geometric Design for Blade’s Surface. We consider naturally to choose global dissipative energy to be object functional for geometric design of the blade surface. The global dissipative energy functional is given by

\[ S^{3\beta}(w, p, \Theta) = -\{(r\varepsilon)^{-1}(\delta_{2\lambda}\delta_{\lambda}^\beta + \delta_{2\beta}^\beta \delta_{\lambda}^\beta + r^2(\delta_{2\lambda}\Theta_\lambda + \Theta_\lambda \delta_{2\lambda}^\beta + \Theta_\beta \delta_{\lambda}^\beta))w^\lambda w^\sigma + 2r(\delta_{2\lambda}\Theta_\lambda + \delta_{2\beta}^\beta \Theta_\lambda \lambda^3w^\lambda + r^2(\delta_{2\lambda}^\beta w^3w^3)\}
+ 2w^2(\varepsilon^{-1}w^\lambda \lambda^3w^\lambda + \varepsilon^{-1}w^\beta \lambda^3w^\beta - \varepsilon^{-1}(\nabla \beta \beta + e^{-2}\Theta_\lambda \beta_\beta)\)
+ 2\varepsilon^{-1}(\beta_\lambda \lambda^3w^\lambda + \beta_\beta \beta_\lambda w^\lambda + \beta_\lambda \beta \beta_\beta w^\lambda)\Theta_\lambda
- 2\varepsilon^{-2}(2\beta_\lambda \lambda^3w^\lambda + \beta_\beta \lambda \lambda^3w^\beta) - 2\nu(r)(r^{-1}e_{33}(w))\)
+ e_{33}(w)(\varepsilon^{-1}(\delta_{2\lambda}^\beta \Theta_\lambda + 3\delta_{2\beta}^\beta \Theta_\lambda + \varepsilon^{-2}(\delta_{2\lambda}^\beta \Theta_\lambda + \Theta_\lambda \delta_{2\lambda}^\beta)\beta_{33}(w))
+ e_{33}(w)(\varepsilon^{-3}\delta_{2\beta}^\beta (4 + 3r^2(\nabla \Theta^2)))\}
\]

\[ \Phi(w, v) = \int_{\Omega} A_{ijkl} e_{kl}(w) e_{ij}(v) d\Omega
\]

\[ J(S) = 1/2 \int_{\Omega} \int_{\Omega} \Phi(w(S), w(S)) d\Omega = \int_{\Omega} \int_{\Omega} A_{ijkl} e_{kl}(w) e_{ij}(w) \sqrt{g} d\Omega d\xi,
\]

\[ A_{ijkl} = \lambda g_{ij} g_{kl} + \mu(g_{ik} g_{lj} + g_{il} g_{jk})]

(4.1)

where \( \Omega = D \times [-1, 1] \) and \( \Omega_e \) is flow passage in Turbomachinery bounded by \( \Gamma_t \cup \Gamma_b \cup \Gamma_{in} \cup \Gamma_{out} \cup S_t \cup S_b \). We propose a principle for geometric design of the blade:

\[ \text{Find a surface } S \text{ of the blade such that } J(\Omega) = \inf_{\Omega \in F} J(S), \]

(4.2)

where \( F \) denotes a set of regular surfaces spanning on a given Jordannian curve \( C = R^3 \). The \( S \) is called a ”general minimal surface which achieves a mimimum of object dissipative energy functional. In other words, from mathematical point of view, this minimum problem of geometric sharp of the surface of the blade is a general minimal surface problem.

Note that (4.2) is an optimal control problem with distribute parameters, control variable is the surface of the blade and the Navier-Stokes equations are the state equations of this control problem.

In sequel, we prove that equation (2.7) is the Eular-Lagrange equations of optimal problem and Navier-Stokes equations are its state equations.

In order to investigate optimal control problem (4.2) we should consider the object functional \( J \) in a fixed domain in new coordinate system.

Let \( \delta \eta = \frac{D}{D\Theta}(\Theta)\eta \) denote Gateaux derivative of \( w \) at \( \Theta \) in direction \( \eta \in V(\Omega) \). Then according to (3.20) , we have following lemma

**Lemma 4.1.** Assume that \( (w, p) \) is a solution of Navier-Stokes equations (3.1)/(3.6) associated with \( \Theta \in H^1(D) \) which define a mapping \((w(\Theta), p(\Theta)) \) from \( W(D) = H^1_0(D) \cap H^2(D) \) to \( H^1(\Omega)^3 \times L^2(\Omega), \Theta \rightarrow (w(\Theta), p(\Theta)) \). Then \((w(\Theta), p(\Theta)) \) is Gateaux differential in \( H^1(D) \) with respect to all direction \( \eta \in W(D) \). The Gateaux derivatives \( \delta \eta = \frac{D}{D\Theta}(\Theta)\eta \) are a solution of (3.37). Then the strain rate tensor \( e_{ij}(w) \) defined by (3.2) possesses Gateaux derivatives \( \frac{D}{D\Theta}(\Theta)\eta \) at any point \( \Theta \in H^2(D) \) along every direction \( \eta \in W(D) \), and

\[ \frac{D}{D\Theta} e_{ij}(w) = e_{ij}(\delta) + e_{ij}^\lambda(w) \eta_\lambda + e_{ij}^{\lambda \sigma}(w) \eta_\lambda \eta_\sigma, \]

(4.3)
where
\[
e^{\lambda}_{\alpha \beta}(w) = \psi^{\lambda}_{\alpha \beta}(w) + (\psi^{\lambda \sigma}_{\alpha \beta}(w) + \psi^{\sigma \lambda}(w))\Theta_\sigma + \frac{1}{2}r^2w^\sigma(\delta_{\alpha \lambda}\Theta_{\sigma \beta} + \delta_{\beta \lambda}\Theta_{\alpha \sigma}),
\]
\[
e^{\lambda}_{\alpha 3}(w) = \frac{1}{2}r^2w^\sigma(\Theta_\alpha \delta_\beta \lambda + \Theta_\beta \delta_\alpha \lambda),
\]
\[
e^{\lambda}_{3 \alpha}(w) = \psi^{\lambda}_{3 \alpha}(w) + (\psi^{\lambda \nu}_{3 \alpha}(w) + \psi^{\nu \lambda}_{3 \alpha}(w))\Theta_\nu,
\]
\[
e^{\lambda}_{33}(w) = \psi^{\lambda}_{33}(w), \quad e^{\lambda}_{33}(w) = 0. \tag{4.4}
\]

Proof. From (3.20) we claim
\[
\frac{\partial e_{ij}(w)}{\partial \Theta} \eta = e_{ij}(\bar{w})\eta + \varphi^{\lambda}_{ij}(w)\Theta_\lambda + \varphi^{\lambda \sigma}_{ij}(\eta_\lambda \Theta_\sigma + \Theta_\lambda \eta_\sigma) + R_{ij}(w, \Theta)\eta,
\]
\[
R_{\alpha \beta}(w, \Theta)\eta = \frac{1}{2}r^2w^\sigma(\delta_{\alpha \lambda}\Theta_{\sigma \beta} + \delta_{\beta \lambda}\Theta_{\alpha \sigma})\eta_\lambda + (\delta_{\beta \alpha}\Theta_\lambda + \delta_{\alpha \lambda}\Theta_\beta)\eta_\lambda, \quad R_{3 \alpha}(w, \Theta)\eta = \frac{1}{2}r^2w^\sigma \delta_\alpha \lambda, \quad R_{33}(w, \Theta) = 0.
\]

This leads to (4.3) and (4.4). The proof is completed. \[\square\]

**Lemma 4.2.** Under the assumptions in Lemma 4.1, the dissipative functions $\Phi(w)$ defined by (4.1) is Gateaux differential at $\Theta \in H^2(D)$ along any direction $\eta \in W$, and
\[
\frac{\partial \Phi(w)}{\partial \Theta} \eta = \Phi(\bar{w}, w)\eta + \Phi^\lambda(w, \Theta)\eta_\lambda + \Phi^{\lambda \sigma}(w, \Theta)\eta_\lambda \eta_\sigma, \tag{4.5}
\]

where
\[
\begin{align*}
\Phi(\bar{w}, w) &= 2A^{ijkl} e_{ij}(\bar{w}) e_{kl}(w), \\
\Phi^\lambda(w, \Theta) &= 2A^{ijkl} e_{ij}^\lambda(w) e_{kl}(w) + M^\lambda(w, \Theta), \\
\Phi^{\lambda \sigma}(w, \Theta) &= 2A^{ijkl} e_{ij}^{\lambda \sigma}(w) e_{kl}(w),
\end{align*} \tag{4.6}
\]

\[
M^\lambda(w, \Theta) = 4\varepsilon^{-2}(\Theta_\lambda \delta_\alpha \beta + \Theta_\alpha \delta_\beta \lambda) e_{3 \alpha}(w) e_{\beta \lambda}(w) + 4\varepsilon^{-4}r^{-2}(\Theta_\lambda \delta_\alpha \beta + \Theta_\beta \delta_\alpha \lambda) e_{3 \alpha}(w) e_{3 \lambda}(w) - 4\varepsilon^{-1}e_{3 \alpha}(w) e_{3 \lambda}(w) - 4\varepsilon^{-1}e_{3 \alpha}(w) e_{3 \lambda}(w). \tag{4.7}
\]

Proof. Indeed, taking (3.24) and (4.3) into account,
\[
\frac{\partial A^{ijkl}}{\partial \Theta} \eta e_{ij}(w) e_{kl}(w) = M^\lambda(w, \Theta)\eta_\lambda, \tag{4.8}
\]

where $M^\lambda(w, \Theta)$ is defined by (4.7). Then
\[
\begin{align*}
\frac{\partial \Phi(w)}{\partial \Theta} \eta &= 2A^{ijkl} \frac{\partial e_{ij}(w)}{\partial \Theta} \eta e_{kl}(w) + \frac{\partial A^{ijkl}}{\partial \Theta} \eta e_{ij}(w) e_{kl}(w), \\
&= 2A^{ijkl} e_{ij}(\bar{w}) e_{kl}(w) \eta + (2A^{ijkl} e_{ij}^\lambda(w) + M^\lambda(w, \Theta))\eta_\lambda \\
&\quad + 2A^{ijkl} e_{ij}^{\lambda \sigma}(w) e_{kl}(w) \eta_\lambda \eta_\sigma, \tag{4.9}
\end{align*}
\]

From this it yields (4.5)(4.6). The proof is complete. \[\square\]

**Theorem 4.1.** Under the assumptions in Lemma 4.2, the object functional $J$ defined by (4.1) has a Gateaux derivative $\text{grad} J \equiv \frac{\partial J}{\partial \Theta}$ in every direction $\eta \in W := H^2(D) \cap H_0^1(D)$. Furthermore $\text{grad} J$ is determined by
\[
< \text{grad}_\Theta(J(\Theta)), \eta > = \int_D [\Phi^0(w; \bar{w})\eta + \Phi^\lambda(w, \Theta)\eta_\lambda + \Phi^{\lambda \sigma}(w, \Theta)\eta_\lambda \eta_\sigma \\
+ 2\mu \varepsilon w^{\sigma \tau} \Theta_{\nu \lambda \eta_\sigma} \varepsilon r d\tau], \tag{4.10}
\]
Substituting (4.4) into above leads to

\[
\begin{aligned}
\hat{\Phi}^0(w; \hat{w}) &= \int_{-1}^1 \Phi^0(w; \hat{w}) d\xi, \\
\hat{\Phi}^\lambda(w, \Theta) &= \int_{-1}^1 \Phi^\lambda(w, \Theta) d\xi, \\
\hat{\Phi}^{\lambda\sigma}(w, \Theta) &= 2\nu^2 [\varepsilon \hat{w}^{\lambda\sigma} \Theta_{\lambda\sigma} + \hat{w}^{\lambda\sigma} \Theta_{\lambda\sigma}], \\
W^{\alpha\beta} &= \int_{-1}^1 w^{\alpha\beta} d\xi, \quad W^{\alpha\beta} := \int_{-1}^1 w^{\alpha\beta} \frac{\partial w^{\beta}}{\partial x^\alpha} d\xi.
\end{aligned}
\tag{4.11}
\]

where \( \hat{w} = \frac{Dw}{D\Theta} \) are the Gâteaux of the solution \((w)\) of Navier-Stokes equations at point \( \Theta \), \( \hat{\Phi}^0, \hat{\Phi}^\lambda \) are defined by (4.6).

**Proof.** Indeed, taking into account of (4.5) and (4.6) we assert that

\[
< \text{grad}_\Theta(J(\Theta)), \eta > = \int_D \int_{-1}^1 \frac{D\Phi(w, \Theta)}{D\Theta} \eta \varepsilon r d\xi dx
\]

where we need to show only last two terms in (4.12). In fact, from (4.4) (4.6) and (3.24)

\[
\Phi^{\lambda\sigma}(w, \Theta) = 2A^{ijkl} e^{\lambda\sigma}_{ij}(w) e_{kl}(w) = 4\nu \{ e^{\lambda\sigma}_{\alpha\beta}(w) e_{\alpha\beta}(w) + 2(g^{33} \delta^{\alpha\beta} + \varepsilon^{-2} \Theta_{\alpha\beta} e_{33}(w) e_{\alpha\beta}(w) + g^{33} g^{33} \delta^{\alpha\beta}(w) e_{33}(w) - 2\varepsilon^{-2} \Theta_{\alpha\beta} e_{33}(w) e_{\alpha\beta}(w) + \Theta_{\alpha\beta} e_{33}(w) [\varepsilon^{-2} \Theta_{\alpha\beta} e_{33}(w) + \Theta_{\alpha\beta} e_{33}(w)] \}
\]

In view of \( e_{33}^{\lambda\sigma}(w) = 0 \) therefore, make rearrangement,

\[
\Phi^{\lambda\sigma}(w, \Theta) = 4\nu \{ e^{\lambda\sigma}_{\alpha\beta}(w) (e_{\alpha\beta}(w) - 2\varepsilon^{-1} \Theta_{\alpha\beta} e_{33}(w) \varepsilon^{-2} \Theta_{\alpha\beta} e_{33}(w)) + 2e_{33}^{\lambda\sigma}(w) \}
\]

Substituting (4.4) into above leads to

\[
\Phi^{\lambda\sigma}(w, \Theta) = 2\nu^2 \varepsilon^{\lambda\sigma} [(2\varepsilon^{-2} |\nabla \Theta|^2 - 2g^{33}) \Theta_{\lambda} e_{33}(w) + \Theta_{\lambda} e_{33}(w) + \varepsilon^{-2} \Theta_{\lambda} e_{33}(w) + \nabla \Theta_{\lambda} e_{33}(w)] \varepsilon^{\alpha\beta} \varepsilon^{-2} \Theta_{\alpha\beta} e_{33}(w) - \Theta_{\lambda} e_{33}(w)]
\]

By virtue of (3.20-22) and the index \((\lambda, \sigma)\) in \( \Phi^{\lambda\sigma} \eta_{\lambda\sigma} \) being symmetry,

\[
\int_{-1}^1 \varepsilon^{\lambda\sigma} \partial_{\xi} w^{\lambda\sigma} d\xi = \int_{-1}^1 \frac{1}{2} \partial_{\xi} (w^{\lambda\sigma}) d\xi = \frac{1}{2} \varepsilon^{\lambda\sigma} w^{\lambda\sigma} |_{\xi=1,-1} = 0 \quad \text{(by boundary conditions)}
\]

we have

\[
\Phi^{\lambda\sigma}(w, \Theta) d\xi = 2\nu^2 \varepsilon^{\lambda\sigma} \partial_{\xi} (w^{\lambda\sigma} \Theta_{\lambda\sigma}) d\xi + 2\nu^2 W^{\nu\sigma} \Theta_{\lambda\nu}
\]

The proof is complete. \( \square \)

Taking integration by part of (4.10) and considering homogenous boundary condition for \( \eta \in W(D) \) it implies

\[
< \text{grad}_\Theta(J(\Theta)), \eta > = \int_D \varepsilon [\partial_{\sigma}(2\mu^3 W^{\nu\sigma} \Theta_{\nu\lambda} + r \hat{\Phi}^{\lambda\sigma}(w, \Theta)) \partial_{\lambda} (r \hat{\Phi}^\lambda(w, \Theta)) + r \hat{\Phi}^0(w, \hat{w})] \eta dx.
\]
From above discussion we obtain directly Theorem 2.1 The Euler-Lagrange equation for the extremum $\Theta$ of $J$ is given by:

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} (2\nu r^3 W^{\nu\sigma} \frac{\partial^2}{\partial x^2}) + \frac{\partial^2}{\partial x^2} (r \Phi^\lambda (w, \Theta)) = \frac{\partial}{\partial x} (r \Phi^\lambda (w, \Theta)) + \Phi^0 (w, \tilde{w}) r = 0, \\
\Theta|_\gamma = \Theta_0, \quad \frac{\partial \Theta}{\partial n}|_\gamma = \Theta^*_\gamma.
\end{align*}
\]  

(4.14)

and variational formulation associated with (4.14) reads

\[
\begin{align*}
\text{Find } \Theta \in V(D) = \{q | q \in H^2(D), q|_\gamma = \Theta_0, \frac{\partial q}{\partial n}|_\gamma = \Theta^*_\gamma \} \text{ such that}
\end{align*}
\]

\[
\int_D \left\{ (K^{\alpha\beta\lambda\sigma} \Theta_{\alpha\beta} + \Phi^\lambda (w, \Theta)) \eta_{\lambda\sigma} + \Phi^\lambda (w, \Theta) \eta_{\lambda} + \Phi^0 (w, \tilde{w}) \eta \right\} \varepsilon r dx, \quad \forall \eta \in H^2_0(D),
\]

(4.15)

where

\[
K^{\alpha\beta\lambda\sigma} := 2\nu r^3 W^{\alpha\sigma} \delta_{\beta\lambda}.
\]

(4.16)

5. Second Model. Let us choose the rate of work done by the impeller under certain ratio of inlet and outlet pressure as minimizing functional (3.10)

\[
I(3) = \int \int_{\mathcal{S} \cup \mathcal{S}_\alpha} \sigma \cdot n \cdot e_\theta e_\theta e_\omega dS,
\]

(5.1)

where $n$ is unite normal vector to the surface $\mathcal{S}$ and $(e_r, e_\theta, k)$ are bases vector in rotating cylinder coordinate system, $\sigma$ is total stress tensor for compressible flow. Our purpose is to find a surface $\mathcal{S}$ of blade such that

\[
I(3) = \inf_{S \in \mathcal{F}} I(S),
\]

(5.2)

where $\mathcal{F}$ denotes a set of regular surface spanning on a given Jordanian curve $C \in R^3$.

Under new coordinate system (5.1) can be rewritten by

\[
I(3) = \int_D \left\{ (\xi^2 - p + 2\nu r^3 W^{\sigma} e_\sigma) \right\} \varepsilon r dx,
\]

(5.3)

Next we compute (5.3). To do this, we give the followings without proof:

\[
\begin{align*}
n(\xi) &= n^i(\xi) e_i, \quad n^\lambda (\xi) = -r \Theta_\lambda / \sqrt{\alpha}, \quad n^3(-1) = (r\varepsilon)^{-1} \frac{1 + \varepsilon^2 \theta^2}{\sqrt{\alpha}}, \\
&n^3(+1) = - (\varepsilon r)^{-1} \frac{1 + \varepsilon^2 \theta^2}{\sqrt{\alpha}}, \quad e_\theta = (r\varepsilon)^{-1} e_3, \quad (e_\theta)^3 = (r\varepsilon)^{-1}.
\end{align*}
\]

(5.4)

Thanks to (2.5) and (5.4) we claim that

\[
\begin{align*}
&\left( -p + \lambda d\nu w \right) g_{33} n^i (e_\theta)^3 = \left( -p + \lambda d\nu w \right) (r\varepsilon)^{-1} (g_{33} n^3 + g_{33} n^3), \\
&\left( -p + \lambda d\nu w \right) (r\varepsilon)^{-1} (g_{33} n^3 + g_{33} n^3), \quad \text{for } \xi = \pm 1,
\end{align*}
\]

(5.5)

Substituting (5.4)(5.5) into (5.3) and simple calculations lead to

\[
\begin{align*}
I(3) = \int_D \left\{ (1 + r^2 \Theta_\theta^2) (-2P + 2\lambda d\nu W) - r^2 |\nabla \Theta|^2 (-2P + 2\lambda d\nu W) \\
+ 2\mu (r\varepsilon)^{-1} (1 + r^2 \Theta_\theta^2) e_{33} (W) - r\Theta_\theta e_{33} (W) \right\} r\varepsilon r dx,
\end{align*}
\]

(5.6)

where we use notation

\[
\begin{align*}
w|_{\xi = -1} = w(-1), \quad w|_{\xi = 1} = w(1), \quad W = (w(1) + w(-1))/2, \quad \tilde{W} = (w(-1) - w(1))/2, \\
P = (p(1) + p(-1))/2, \quad \tilde{P} = (p(-1) - p(1))/2, \\
W_\theta := \frac{dW}{d\Theta}, \quad P_\theta := \frac{dP}{d\Theta}, \quad \tilde{W}_\theta := \frac{d\tilde{W}}{d\Theta}, \quad \tilde{P}_\theta := \frac{d\tilde{P}}{d\Theta},
\end{align*}
\]

(5.7)
Hence
\[ \tilde{P} - P = -p(-1), \quad \tilde{W} - W = -w(-1), \quad (5.8) \]
and in view of Lemma 4.2 the gradient of \( I \) along any direction \( \eta \in H^1_0(D) \) is given by

**Theorem 5.1.** Under the assumptions in Lemma 4.2 the object functional \( I \) defined by (5.1) has a Gâteaux derivative \( \text{grad}_I \equiv \frac{D I}{D \Theta} \) in every direction \( \eta \in H^1_0(D) \). Furthermore grad\textunderscore I is determined by

\[ < \text{grad}_I(\tilde{\eta}), \eta > = \int_D \left[ \Psi_0(w, p, \Theta) \eta + \Psi^\lambda(w, p, \Theta) \eta_\lambda + \Psi^{\lambda\sigma}(w, p, \Theta) \eta_{\lambda\sigma} |r w e r \right] dx, \quad (5.9) \]

where

\[
\begin{align*}
\Psi_0(w, p, \Theta) &:= (1 + r^2 \Theta_0^2)(-2 \tilde{P}_\Theta + 2 \lambda \text{div} \tilde{W}_\Theta) - r^2 |\nabla \Theta|^2(-2 \tilde{P}_\Theta + 2 \lambda \text{div} \tilde{W}_\Theta) \\
+2 \mu(r \varepsilon)^{-1}(r \varepsilon)^{-1}(1 + r^2 \Theta_0^2)e_{33}(\tilde{W}_\Theta) - r \Theta_\sigma e_{33}(\tilde{W}_\Theta), \\
\Psi^\lambda_0(w, p, \Theta) &= \Psi^\lambda_0(w, p) + \Psi^\lambda_0(w, p) \Theta_\nu + \Psi^\lambda_\nu_\mu(w, p) \Theta_\sigma + \Psi^\lambda_\nu_\mu_\mu(w, p, \Theta), \\
\Psi^{\lambda\sigma}(w, p, \Theta) &:= -2 \mu \varepsilon^{-1} \Theta_\sigma e_{33}(W) = -\mu r^2 W^{\sigma \lambda}, \\
\Psi^\lambda_\nu_\mu(w, p, \Theta) &= \frac{2 \mu(r \varepsilon)^{-2} \psi^\lambda_\nu_\mu(W) - \varepsilon^{-1} \varphi^\lambda_\nu_\mu(W),} \\
\Psi^\lambda_\nu_\mu_\mu(w, p, \Theta) &= \frac{-2 \mu \varepsilon^{-2} \psi^\lambda_\nu_\mu(W) + \varphi^\lambda_\nu_\mu(W)}{2}, \\
\Psi^\lambda_\nu_\mu_\nu_\mu(w, p, \Theta) &= \frac{-2 \mu \varepsilon^{-2} \psi^\lambda_\nu_\mu(W) + \varphi^\lambda_\nu_\mu(W)}{2}. \\
\end{align*} \quad (5.10) \]

**Proof.** Firstly, by virtue of (3.21) we assert

\[ \frac{D \text{div} W}{D \Theta} = \text{div} \frac{D W}{D \Theta}. \]

In addition, thanks to (4.7) and denote

\[ W_\Theta := \frac{D W}{D \Theta}, \quad \tilde{W}_\Theta := \frac{D \tilde{W}}{D \Theta}, \quad P_\Theta := \frac{D P}{D \Theta}, \quad \tilde{P}_\Theta := \frac{D \tilde{P}}{D \Theta}. \quad (5.6) \]

shows

\[ < \text{grad}_I(\tilde{\eta}), \eta > = \int_D \left[ 2 r^2 \Theta_2 \eta_\eta(-2 \tilde{P} + 2 \lambda \text{div} \tilde{W}) + (1 + \lambda^2 \Theta_2)(-2 \tilde{P}_\Theta + 2 \lambda \text{div} \tilde{W}_\Theta) \eta \\
-2 \lambda \Theta_\sigma \eta_\sigma(-2 P + 2 \lambda \text{div} W) - r^2 |\nabla \Theta|^2(-2 \tilde{P}_\Theta + 2 \lambda \text{div} \tilde{W}_\Theta) \eta \\
+2 \mu(r \varepsilon)^{-1}(r \varepsilon)^{-1} 2 r^2 \Theta_2 \eta_\sigma e_{33}(W) + (r \varepsilon)^{-1}(1 + r^2 \Theta_0^2) \frac{\partial \eta_\sigma}{\partial \Theta_\sigma}(\tilde{W}) \eta - \eta_\sigma e_{33}(W) - r \Theta_\sigma \frac{\partial \eta_\sigma}{\partial \Theta_\sigma}(\tilde{W}) \eta \right] \]

In view of (4.7)(4.8) and rewriting integrated we claim the first of (5.10). Moreover

\[
\begin{align*}
\Psi^\lambda(w, p, \Theta) &= 2 r^2 \Theta_2 \delta_{33}(\tilde{P} + 2 \lambda \text{div} \tilde{W}) - 2 r^2 \Theta_\lambda(-2 \tilde{P} + 2 \lambda \text{div} \tilde{W}) \\
+2 \mu(r \varepsilon)^{-1}(r \varepsilon)^{-1} 2 r^2 \Theta_2 \delta_{33}(W) + 2 r^2 (r \varepsilon)^{-1} \delta_{33}(\tilde{W}) \\
- r \Theta_\sigma e_{33}(W) - r e_{33}(W), \\
\end{align*} \quad (5.12) \]

Therefore

\[ e_{3\lambda}(w) = \psi_{3\lambda}(w) + \psi_{3\lambda}'(w)\Theta_\nu + \psi_{\nu\mu}\Theta_\nu \Theta_\mu + \frac{1}{2} \varepsilon r^2 w^\sigma \Theta_{\lambda\sigma} \]

Take (4.8) and (3.16)(3.17) into account, simple calculation from (5.12) shows

\[ \Psi^\lambda(w, p, \Theta) = 2\mu r^2 \psi_{3\lambda}(W) - 2\mu e^{-1} \varphi_{3\lambda}(W) + \{(2r^2(-P + 2\lambda \text{div} W) + 4\mu e^{-2}\varphi_{3\lambda}(W))\delta_{2\nu}^2\delta_{2\nu} - 2r^2(-P + 2\lambda \text{div} W)\delta_{2\nu} - 2\mu e^{-1}(\psi_{3\lambda}(W) + \psi_{3\lambda}'(W))\} \Theta_\nu + 2\mu e^{-2}(\psi_{3\lambda}''(W) + 2\psi_{3\lambda}''(W))\delta_{2\nu}^2\delta_{2\nu} - 2\mu e^{-1}(\psi_{3\lambda}''(W) + \psi_{3\lambda}''(W))\} \Theta_\nu \Theta_\mu - 2\mu(\varepsilon r)^{-1} r^2 \varepsilon r^2 W^\sigma \Theta_{\lambda\sigma}. \]

This leads to second of (5.10).

Integrating by parts in (5.9) yields

\[ \varepsilon r^2 \omega \Psi_0(w, p, \Theta) - \frac{\partial}{\partial x}(\varepsilon r^2 \omega \Psi(w, p, \Theta)) + \frac{\partial^2}{\partial x^2}(\varepsilon \omega r^2 \Psi^\lambda(w, p, \Theta)) = 0 \]

Hence the stationary point of minimum problem should satisfies following Eular-Lagrange equation

\[ \varepsilon r^2 \omega \Psi_0(w, p, \Theta) - \frac{\partial}{\partial x}(\varepsilon r^2 \omega \Psi(w, p, \Theta)) + \frac{\partial^2}{\partial x^2}(\varepsilon \omega r^2 \Psi^\lambda(w, p, \Theta)) = 0 \]

However,

\[ \frac{\partial^2}{\partial x^2}(\varepsilon \omega r^2 \Psi^\lambda(w, p, \Theta)) = -\frac{\partial^2}{\partial x^2}(\mu \varepsilon r^4 W^\sigma \Theta_{\lambda}) - \frac{\partial^2}{\partial x^2}(\mu \varepsilon r^4 W^\sigma \frac{\partial}{\partial x^2}(\Theta_{\lambda})) \]

In addition

\[ -\frac{\partial}{\partial x}(\Psi^\lambda_\psi(w, p, \Theta)) + \frac{\partial^2}{\partial x^2}(\varepsilon \omega r^2 \Psi^\lambda w, p, \Theta)) = -\frac{\partial^2}{\partial x^2}(\mu \varepsilon r^4 W^\sigma) \Theta_{\lambda} - \frac{\partial}{\partial x}(\mu \varepsilon r^4 W^\sigma) \Theta_{\lambda} \]

Therefore

\[ -\frac{\partial}{\partial x}(\Psi^\lambda_\psi(w, p, \Theta)) + \frac{\partial^2}{\partial x^2}(\varepsilon \omega r^2 \Psi^\lambda w, p, \Theta)) = -\frac{\partial^2}{\partial x^2}(\mu \varepsilon r^4 W^\sigma) \Theta_{\lambda} - \frac{\partial}{\partial x}(\mu \varepsilon r^4 W^\sigma) \Delta \Theta, \]

where

\[ \Delta \Theta = \Theta_{11} + \Theta_{22}. \]

On the other hand,

\[ -\partial_\lambda(\Psi^\lambda_\psi + \Psi^\lambda_\psi \Theta_\nu + \Psi^\lambda_\psi \Theta_\mu \Theta_\nu) = -\partial_\lambda(\Psi^\lambda_\psi + \partial_\lambda(\Theta_\psi) + \partial_\lambda \Psi^\lambda_\psi \Theta_\lambda \Theta_\nu + \partial_\lambda(\Psi^\lambda_\psi + \Psi^\lambda_\psi \Theta_\mu) \Theta_\lambda \Theta_\nu) \]

To sum up and by simply calculation, we obtain Theorem 2.2

\[ \{ -\left( K_0(w) \Delta \Theta + K^\nu(w, \Theta) \Theta_{\lambda\nu} \right) + F^\nu\mu(w) \Theta_\nu \Theta_\mu + F^\lambda(w) \Theta_\lambda + F_0(w, \Theta) = 0, \Theta_{|\gamma} = \Theta_0, \gamma = \partial D, \]

(5.16)
Let \( c \) be a constant. We consider functional defined in a closed convex set in a Sobolev space, weakly lower semi-continuous on \( U \), then \( J \) is bounded from below and \( J \) achieves its minimum. \( \Omega \) and \( \mu \) are defined by (5.10) and (5.11) respectively. Variational formulation associated with (5.16) is given by

\[
\left\{
\begin{array}{l}
\text{Find } \Theta \in V = \{ q | q \in H^1(D), q_{|\gamma} = \Theta_0 \} \text{ such that } \\
\int_D \left( [\Psi_0(w, p, \Theta)\eta + \Psi^\lambda(w, p, \Theta)\eta_\lambda + \Psi^\lambda_{v\mu}(w, p, \Theta)\eta_{v\mu}] \right) r_{w,\varepsilon} \, dx,
\end{array}
\right.
\]

(5.18)

where \( \Psi_0(w, p, \Theta), \Psi^\lambda(w, p, \Theta), \Psi^\lambda_{v\mu}(w, p, \Theta) \) are defined by (5.10). \( \square \)

### 6. Existence of Solution to the Optimal Control Problem

In this section, we discuss existence of the optimal control problem (3.1) for incompressible case. As well known the object functional

\[
J(\Theta) = \frac{1}{2} \int_{\Omega} A^{ijkl}(\Theta)e_{ij}(w(\Theta))e_{kl}(w(\Theta))\sqrt{\gamma} \, dx = \frac{1}{2} a(w(\Theta), w(\Theta)),
\]

(6.1)

is depending upon the existence of solution \( w \) to Navier-Stokes equations. Since the solution \( w \) of Navier-Stokes equation and \( A^{ijkl} \) are the functions of \( \Theta \), therefore \( J \) is a function of \( \Theta \). However \( J \) can be read as a function of \( w : J(\Theta) = J(w(\Theta)) \). As well known that if there exists Gateaux derivative \( \frac{dJ}{d\Theta} \) of \( J(\Theta) \) with respect to \( \Theta \) at \( \Theta^* \), then the minimum point \( \Theta \) of (3.1) is necessary to satisfies

\[
\text{grad}_S J(\Theta) = 0.
\]

and from this, if \( \Theta \) is regular then Eular-Lagrange equation for \( \Theta \) can be obtained.

At first, it is well known that following theorem give a sufficient condition of the existence

**Theorem 6.1. (Generalized Weierstrass Theorem)\textcite{12}** Let \( X \) be a reflexive Banach Space, and \( U \) a bounded and weakly closed subset of \( X \). If the functional \( J \) is weakly lower semi-continuous on \( U \), then \( J \) is bounded from below and \( J \) achieves its infimum in \( U \). We consider functional defined in a closed convex set in a Sobolev space

\[
V(\Omega) := \{ u | u \in H^{1,p}(\Omega), u|_{\Gamma_0} = 0, \partial \Omega = \Gamma_0 \cup \Gamma_1, \text{meas}(\Gamma_0) \neq 0 \}.
\]

Let

\[
\bar{J}(w) = \int_{\Omega} A^{ijkl}(w)e_{ij}(w)e_{kl}(w)\sqrt{\gamma} \, dx.
\]

**Lemma 6.1.** \( \bar{J}(w) \) is weak lower semi-continuous with respect to \( w \) in \( H^1(\Omega)^3 \).

**Proof.** We firstly need to establish the uniform positive definiteness of the three-dimensional tensor \( A^{ijkl} \). "Uniform" means with respect to all points \( x \in \Omega \) and to symmetric matrices of the order three \( \{ t_{ij} \} \). Namely, there exist a constant \( c(\Omega, \Theta, \mu) > 0 \) such that

\[
A^{ijkl}(x)t_{kl}t_{ij} \geq c \sum_{i,j} |t_{ij}|^2.
\]

(6.2)
See the proof of Th1.8-1 in [P.G.Ciarlet 2000]. Note that $\Omega$ is a Lipschitz domain, $g = det(g_{ij}) = \varepsilon^2 r^2 > 0$. Therefore, we claim

$$a(w, w) \geq \nu c(\Theta, \Theta) \sum_{i,j} \|e_{ij}(w)\|^2_{0,\Omega}$$

On the other hand, the Korn inequality with boundary conditions in curvilinear coordinates (see Th1.7-4 in [P.G.Ciarlet 2000]) shows that

$$\sum_{i,j} \||e_{ij}(v)||^2_{0,\Omega} \geq c(\Omega, \Theta) \|v\|^2_{1,\Omega}, \quad \forall v \in V. \quad (6.3)$$

Hence

$$a(w, w) \geq \mu c(\Omega, \Theta) \|w\|^2_{1,\Omega}, \quad (6.4)$$

We assert that $a(\cdot, \cdot)$ is a equivalent norm in $H^1(\Omega)^3$. In other words, $\bar{J}(\cdot)$ is a equivalent norm in $H^1(\Omega)^3$. As well known that the norm in a Hilbert space , as a functional, is weakly lower semi-continuous. we assert $\bar{J}$ is weakly lower semi-continuous with respect to $w$. Proof is complete. \[\square\]

By virtue of Lemma 3.3 we directly to obtain

**Lemma 6.2.** If the function $w(\Theta)$ of Navier-Stokes equations satisfies the following:

Assumption \quad $P : \Theta_n \rightarrow \Theta_0$ (weakly) $\Rightarrow w_n = w(\Theta_n) \rightarrow w_0 = w(\Theta_0)$ (weakly).

Then functional $J(\Theta)$ is weak lower semi-continuous with respect to $\Theta$.

Finally, we have

**Theorem 6.2.** Assume that the solutions $(w, p)$ of Navier-Stokes equations with mixed boundary condition are weakly continuous with respect to $\Theta$. Then there exists a two dimensional surface $S$ defined by a smooth mapping

$$\Theta : D \rightarrow W \equiv H^2(D) \cap (\Theta \in H^1(D), \Theta|_{\partial D} = \Theta^*, \Theta)$$

such that $J(\Theta)$ achieves its infimum at $\{\Theta, w(\Theta)\}$.

Next we consider the existence of the solutions for Navier-Stokes equations. In deed flow’s domain is unbounded domain. In section 3 we make artificial boundary, inflow boundary $\Gamma_{in}$ and outflow boundary $\Gamma_{out}$, and impose natural boundary conditions (3.6). We also can impose the pressures

$$p|_{\Gamma_{in}} = p_{in}, \quad p|_{\Gamma_{out}} = p_{out},$$

or flux

$$\int_{\Gamma_{in}} pw \cdot nd\Gamma = Q, \quad \int_{\Gamma_{out}} pw \cdot nd\Gamma = Q,$$

Let us consider energy inequality. Owing to

$$(2\omega \times w, w) = 0,$$

Hence moment equations (3.11) show

$$a(w, w) + b(w, w) = (f, w),$$
However,
\[ b(w, w, w) = \frac{1}{2} \int_{\Gamma} |w|^2 w \cdot nd\Gamma, \]
where \(w \cdot n = g_{ij}w^i n^j\) and \(n\) is outward normal unite vector to inflow or out flow boundary. Therefore (6.1) shows that
\[ b(w, w, w) = K_{\text{out}}(w) - K_{\text{in}}(w), \]
It is obvious that we cannot make say \(K_{\text{out}}(w) < \infty\).

**Theorem 6.3.** Suppose that the exterior force \(f\) and normal stress \(g\) of inflow and out flow boundaries \(\Gamma_1 = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}\) satisfy
\[ \|F\|_* := \|f\|_{0, \Omega} + \|g_{\text{in}}\|_{-1/2, \Gamma_{\text{in}}} + \|g_{\text{out}}\|_{-1/2, \Gamma_{\text{out}}} \leq \nu^2/(4c_1c_2), \]
and the mapping \(\Theta\) defined in (2.1) is \(C^2(D)\)-function satisfied by
\[ \Theta \in C^2(D, K_0) \equiv \{\theta \in C^2(D), \|\theta\| \leq K_0, \|\Theta\| \leq K_0\}, \]
where \(K_0\) is a constant. Then there exists a smooth solution of the variational problem (3.11) satisfying
\[ \|\nabla w\|_{0, \Omega} \leq \frac{c(\Omega, \Theta)\mu}{2c_1} [1 - \sqrt{1 - \frac{4c_1c_2\|F\|_*}{(c(\Omega, \Theta)\mu)^2}}], \]
where \(c_1, c_2\) are constants depending of \((\Omega, K_0)\) defined by (6.14)/(6.15).

**Proof.** To prove the theorem for a steady Navier-Stokes problem, it is convenient to construct the solution as a limit of Galerkin approximations in terms of the eigenfunctions of the corresponding stead Stokes problem. This use of the Stokes eigenfunctions (see in Heywood and Rannacher and Turek[20] and Glowinski[12]). Galerkin equations are a system of algebraic equations for constant unknowns and the Galerkin approximation solution \(w\) is a solution of the finite dimensional problem
\[ a(w, v) + 2(\omega \times w, v) + b(w, w, v) = \langle F, v \rangle, \forall v \in V_m := \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_m\} \]
where \(\varphi_i, i = 1, 2, \cdots, m\) are eigenfunctions of corresponding Stokes operator. Let \(S_{\rho}\) denote the spheres in \(V_m\) satisfying inequality (6.10). Assume that \(w_\ast \in S_{\rho}\), we seeking \(w\) such that
\[ a(w, v) + 2(\omega \times w, v) + b(w_\ast, w, v) = \langle F, v \rangle, \forall v \in V_m, \]
(6.12) is uniquely solvable because \( w_s \in S_p \) because the \( w = 0 \) is only solution of the corresponding homogeneous equation (\( F = 0 \)). Indeed, if \( w_s \) satisfies (6.10) and \( w \) satisfies (6.12) with the \( F = 0 \), taking (6.4) and

\[
2(\omega \times w, w) = 0 \tag{6.13}
\]

into account, and combing (3.19) and (6.11) we assert

\[
c(\Omega, \Theta)\mu\|\nabla w\|_{0, \Omega}^2 \leq |b(w_s, w, w)| \leq c_1(\Omega, K_0)\|w_s\|_{L^2(\Omega)}\|\nabla w\|_{0, \Omega} \leq c_1(\Omega, K_0)\|w_s\|_{0, \Omega} \|
abla w\|_{2, \Omega}^2 \leq c_{1}(\Omega, K_0)\frac{c(\Omega, \Theta)\mu}{2c_1}\|\nabla w\|_{2, \Omega}^2,
\]

(6.14)

This implies that \( w = 0 \). In order to apply Brouwer’s fixed point theorem, we have to show the mapping \( w_s \Rightarrow w \) take the ball \( S_p \) defined by (6.10) into itself, suppose that \( w_s \) satisfies (6.10). Similarly to (6.13), for nonhomogeneous equation (\( F \neq 0 \)) ,we obtain

\[
c(\Omega, \Theta)\mu\|\nabla w\|_{0, \Omega}^2 \leq |b(w_s, w, w)| + | < F, w > | \leq c_1(\Omega, K_0)\|\nabla w_s\|_{0, \Omega} \|\nabla w\|_{2, \Omega}^2 + c_2(\Omega, K_0)\|F\|_s\|\nabla w\|_0,
\]

\[
c(\Omega, \Theta)\mu\|\nabla w\|_{0, \Omega} \leq c_1(\Omega, K_0)\|\nabla w_s\|_{0, \Omega} \|\nabla w\|_{0, \Omega} + c_2(\Omega, K_0)\|F\|_s,
\]

(6.15)

Here note that

\[
Fw = g_{ij} F^i w^j = (\delta_{i\beta} + \mu^2 \Theta_{ij} \beta) F^\alpha w^\beta + \varepsilon \Theta_{ij} (F^\alpha w^\beta + F^3 w^\alpha) + \varepsilon^2, w^3 F^3 w^3
\]

Therefore

\[
\|\nabla w\|_0 \leq \frac{c_2\|F\|_s}{c(\Omega, \Theta)\mu - c_1\|\nabla w\|_0} \leq \frac{c_2\|F\|_s}{c(\Omega, \Theta)\mu - c_1\|\nabla w\|_0} \left(1 - \sqrt{1 - \frac{4c_1c_2\|F\|_s}{c(\Omega, \Theta)\mu^2}}\right)
\]

Thus Brouwer’s fixed point theorem can be applied and gives the existence of Galerkin approximations satisfying (6.8). Hence by a standard compactness argument there is at least a subsequence of the Galerkin approximation converging to a weak solution \( w \in V(\Omega) \) of the steady problem (3.11):

\[
\left\{ \begin{array}{l}
\text{Find } (w, p), w \in V(\Omega), p \in L^2(\Omega), \text{such that } \\
\begin{array}{l}
a(w, v) + 2(\omega \times w, v) + b(w, w, v) + \\
- (p, \text{div} v) = < F, v >, \forall v \in V(\Omega),
\end{array} \\
(q, \text{div} w) = 0, \forall q \in L^2(\Omega),
\end{array} \right.
\]

Its smoothness is easily proven if one obtain a further estimate from the Galerkin approximations by setting \( v = - Aw \) in (6.9). This gives

\[
\mu \| Aw \|_0^2 = -2(\omega \times w, Aw) - b(w, w, Aw) + < F, Aw >, \tag{6.14}
\]

Because \( Aw \) is solenoidal, one has the rather unusual trace estimate

\[
|2(\omega \times w, Aw)| \leq c_3\|w\|_0\|Aw\|_0, \quad | < F, Aw > | \leq c_3\|F\|_s\|Aw\|_0,
\]

(6.15)

which we combine with (6.14) and Agmon inequality

\[
\|w\|_\infty \leq c_4\|\nabla w\|_{0, \Omega}^{1/2}\|Aw\|_{0, \Omega}^{1/2}, \forall w \in D(A),
\]

(6.16)
to get
\[
\mu \|Aw\|_0^2 \leq c_4 \|
abla w\|_0^{3/2} \|Aw\|_0^{3/2} + c_3 \|w\|_0 \|Aw\|_0 + c_3 \|F\|_\infty \|Aw\|_0.
\] (6.17)

Then, by using Young’s inequality, we obtain
\[
\mu \|Aw\|_0 \leq \frac{2c_4^2}{\mu^2} \|
abla w\|_0^3 + \frac{8c_2^2}{\mu} \|w\|_0^2 + \frac{8c_2^2}{\mu} \|F\|_\infty^2.
\] (6.18)

which is then inherited by the solution. The full classical smoothness of the solution can now be obtained using the $L^2$-regularity theory for the steady Stokes equations. This completes the proof of Theorem 6.3

It is clear that the bound in (6.10) is not uniform with respect to $\Theta$. Next our goal is to prove the the solution $w(\Theta)$ of Navier-Stokes equations (3.11) with mixed boundary condition is uniformly bounded. At first we prove

**Lemma 6.3.** Assume that
\[
\Theta \in \mathcal{F} = \{ \phi \in C^1(\Omega), \sup_D |\nabla \phi| \leq K_1 := \frac{1}{2\sqrt{5r}}, \}
\] (6.19)

then
\[
a(w(\Theta), w(\Theta)) \geq \mu/2(2c_{\alpha\beta}(w)e_{\alpha\beta}(w) + e_\alpha(\lambda)e_\alpha(\lambda) + e_{3\lambda}(\lambda)e_{3\lambda}(\lambda)) \geq 0 \forall w \in V(\Omega)
\] (6.20)

where $a(\cdot, \cdot)$ is defined by (6.1) and $\|\cdot\|_\Omega$ is defined by (3.25).

**Proof.** By virtue of (4.6)(2.1)($\lambda = 0$) we claim
\[
a(w, w) = 2\mu(c_{\alpha\beta}(w)e_{\alpha\beta}(w) + (2\varepsilon^{-2}\Theta(\alpha)\Theta(\sigma) + 2\delta_{\alpha\sigma}g^{33})e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) + g^{33}g^{33}e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) - 4\varepsilon^{-1}\Theta(\alpha)e_{3\lambda}(\lambda)e_{\alpha\lambda}(\lambda) + 2\varepsilon^{-2}\Theta(\alpha)e_{3\lambda}(\lambda)e_{\alpha\lambda}(\lambda) - 4\varepsilon^{-1}\Theta(\alpha)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda)),
\] (6.21)

By Cauchy inequality and Yang inequality we obtain
\[
4\varepsilon^{-1}\Theta(\alpha)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) \\
4\varepsilon^{-1}\Theta(\alpha)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) \\
2\varepsilon^{-2}\Theta(\alpha)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda)
\leq 8\varepsilon^{-2}(\Theta(\alpha)e_{3\lambda}(\lambda))^2 + \frac{1}{4}(g^{33}e_{3\lambda}(\lambda))^2,
\] (6.22)

To sum up and thanks $g^{33} = \varepsilon^{-2}r^{-2} + \varepsilon^{-2}|\nabla \Theta|^2$ we assert that
\[
a(w, w) \geq 2\mu(c_{\alpha\beta}(w)e_{\alpha\beta}(w) + [-6\varepsilon^{-2}\Theta(\alpha)\Theta(\sigma) + 2\varepsilon^{-2}r^{-2}\delta_{\alpha\sigma} - 14\varepsilon^{-2}|\nabla \Theta|^2\delta_{\alpha\sigma}]e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) + (\frac{1}{2}g^{33}g^{33} - 4\varepsilon^{-4}|\nabla \Theta|^4)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda)),
\]

Note
\[
\Theta(\alpha)e_{3\lambda}(\lambda)e_{3\lambda}(\lambda) \leq (\frac{\sum_{\lambda} \Theta(\alpha)^2}{\sqrt{\sum_{\lambda} e_{3\lambda}(\lambda)^2}})^2 \leq |\nabla \Theta|^2 e_{3\lambda}(\lambda)e_{3\lambda}(\lambda)
\]

\[
\frac{1}{2}g^{33}g^{33} - 4\varepsilon^{-4}|\nabla \Theta|^4 = \frac{\varepsilon^{-4}}{2}(r^{-4} + 2r^{-2}|\nabla \Theta|^2 - 7|\nabla|^4)
\]
Therefore
\[ a(w, w) \geq 2\mu \left( \frac{1}{2} e_{\alpha\beta}(w)e_{\alpha\beta}(w) + 2\varepsilon^{-2}r^{-2}[1 - 10r^2|\nabla \Theta|^2]e_{3\lambda}(w)e_{3\lambda}(w) \\
+ \varepsilon^{-4}r^{-4}(1 + 2r^2|\nabla \Theta|^2 - 7r^4|\nabla \Theta|^4) e_{33}(w)e_{33}(w) \right) \]

It is obvious that
\[ 2\varepsilon^{-2}r^{-2}[1 - 10r^2|\nabla \Theta|^2] \geq \frac{1}{2}\varepsilon^{-2}r^{-2}, \quad \text{if} \quad |\nabla \Theta| \leq \frac{1}{2r}; \]
\[ \varepsilon^{-4}r^{-4}(1 + 2r^2|\nabla \Theta|^2 - 7r^4|\nabla \Theta|^4) \geq \frac{1}{2}\varepsilon^{-4}r^{-4}, \quad \text{if} \quad |\nabla \Theta| \leq \frac{1}{2r}, \]

Hence If \( \Theta \in \mathcal{F} \) then
\[ a(w, w) \geq 2\mu \left( \frac{1}{2} e_{\alpha\beta}(w)e_{\alpha\beta}(w) + \frac{1}{4} e_{3\lambda}(w)e_{3\lambda}(w) + \frac{1}{4} e_{33}(w)e_{33}(w) \right), \]

End the proof. \( \blacksquare \)

**Lemma 6.4.** Assumption in Lemma 4.1 is held such that the constant \( K_0 \) satisfies
\[ K_0 \leq \frac{1}{\sqrt{2\beta_0(\Omega)}}, \quad (6.23) \]

where \( \beta_0(\Omega) \) is a constants depending \( \Omega \) only. Then we have following uniform coerciveness for bilinear form \( a(\cdot, \cdot) \)
\[ a(w, w) \geq \frac{\mu}{4} \| w \|_{\Omega}^2 \geq \frac{\mu}{4} C_1(\Omega) \| w \|_{\Omega}^2, \quad \forall w \in H^1(\Omega)^3, \quad (6.24) \]

where \( \alpha_0(\Omega) \) is a constant independent of \( \Theta \) and \( \| w \|_{\Omega}^2 \) is defined by (3.25).

**Proof.** In view of (3.16),
\[ \| e_{\alpha\beta}(w) \|_{0, \Omega}^2 = \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2 + \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 + \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 + \| e_{\alpha\beta}(w) \|_{0, \Omega}^2 + 2(\varphi_{\alpha\beta}(w), \psi_{\alpha\beta}(w)\varphi_\lambda \Theta_\lambda) \\
+ 2(\varphi_{\alpha\beta}(w), \psi_{\alpha\beta}(w)\varphi_\lambda \Theta_\lambda) + 2(\varphi_{\alpha\beta}(w), \varphi_{\alpha\beta}(w)\varphi_\lambda \Theta_\lambda) \]
\[ + 2(\varphi_{\alpha\beta}(w), \varphi_{\alpha\beta}(w)\varphi_\lambda \Theta_\lambda) + 2(\varphi_{\alpha\beta}(w), \varphi_{\alpha\beta}(w)\varphi_\lambda \Theta_\lambda), \quad (6.25) \]

Using (3.18) it is easy to verify
\[ \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \quad \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2 + \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \]
\[ \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \quad \| \psi_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \]
\[ \| e_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) K_0^2 \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \quad \| e_{\alpha\beta}(w) \|_{0, \Omega}^2 \leq \beta_0(\Omega) K_0^2 \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2, \quad (6.26) \]

**Remark 6.1.** the constant \( \beta_0(\Omega) \) represents different meaning at different place.

By means of Schwarz’s inequality, from (6.25)(6.26) we claim
\[ \| e_{\alpha\beta}(w) \|_{0, \Omega}^2 \geq \| \varphi_{\alpha\beta}(w) \|_{0, \Omega}^2 - K_0^{1/2} \beta_0(\Omega) \| w \|_{\Omega}^2, \quad (6.27) \]

By similar manner we imply
\[ \| e_{3\alpha}(w) \|_{0, \Omega}^2 \geq \| \varphi_{3\alpha}(w) \|_{0, \Omega}^2 - K_0^{1/2} \beta_0(\Omega) \| w \|_{\Omega}^2, \quad \| e_{3\alpha}(w) \|_{0, \Omega}^2 \geq \| \varphi_{3\alpha}(w) \|_{0, \Omega}^2 - K_0^{1/2} \beta_0(\Omega) \| w \|_{\Omega}^2, \]

Let return to (6.20), it yields that
\[ a(w, w) \geq \frac{\mu}{2} \| w \|_{0, \Omega}^2 - K_0^{1/2} \beta_0(\Omega) \| w \|_{\Omega}^2. \]
This yields that if (6.23) is satisfied then, by Lemma 3.3, it yields (6.24).

From Theorem 5.1 and Lemma 4.2 we conclude

Theorem 6.4. Assume the the assumptions in Theorem 6.1 are satisfied and (6.23) held. The \( w(\Theta) \) is a solution of rotating Navier-Stokes equations (3.11) associated with \( \Theta \in C^2(\Omega, K_0) \). Then the following estimate held

\[
\| w(\Theta) \|_{\Omega} \leq \frac{\mu}{2c_1} \left[ 1 - \sqrt{\left( 1 - \frac{4c_1c_2\|F\|_*}{(\mu)^2} \right)} \right],
\]  

(6.28)

The solution is sequently weak continuous with respect to \( \Theta \), i.e. if sequence \( \{\Theta_i\} \) is weak convergent in \( H^1(D) \) then there exist convergent subsequent in \( \{w(\Theta_i)\} \) in \( H^1(\Omega) \).

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