Back Reaction and Semiclassical Approximation of cosmological models coupled to matter

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Bianchi -I, -III, and FRW type models minimally coupled to a massive spatially homogeneous scalar field (i.e. a particle) are studied in the framework of semiclassical quantum gravity. In a first step we discuss the solutions of the corresponding equation for a Schrödinger particle propagating on a classical background. The back reaction of the Schrödinger particle on the classical metric is calculated by means of the Wigner function and by means of the expectation value of the energy-momentum-tensor of the field as a source. Both methods in general lead to different results.

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1. Introduction

Applying the well-known Dirac quantization procedure to general relativity coupled to a free massive scalar field leads to the corresponding Wheeler–DeWitt equation and to the diffeomorphism constraints. Expanding the wave functional in powers of the gravitational constant yields a semiclassical approximation \cite{1,2}, which describes the quantized matter field in a classical curved spacetime, i.e. the quantized field propagate on a classical background defined by the gravitational degrees of freedom. It is then natural to ask in which way the background metric is influenced by the quantized field. Unfortunately no general procedure is yet known for calculating this kind of back reaction from quantum gravity.

The aim of this contribution is the following one:

a) A discussion of the semiclassical approximation for different types of minisuperspaces especially for Bianchi -I, -III, and Friedman-Robertson-Walker type models minimally coupled to a massive particle.

b) To compare the results of two different approaches for calculating the back reaction in detail: The back reaction is calculated first with help of Wigner’s function and second by using the expectation value of the energy-momentum-tensor of the particle as a source.

The paper is organized as follows:

In chapter 2 we sketch the main features of semiclassical quantum gravity.

In chapter 3 the solutions of the corresponding Schrödinger equation of the quantized particle are given.

In chapter 4 we calculate and compare the back reaction on the metric by the different methods mentioned above.

2. Semiclassical Quantum Gravity

Starting from the Wheeler–DeWitt equation for minisuperspaces

\[
-\hbar^{-1/2}G^2\hbar^2G_{abcd}\partial_{h_{ab}}\partial_{h_{cd}} - \sqrt{\hbar}(R - 2\Lambda) + GH_{M}\]

\[\Psi(h_{ab}, \phi) = 0, \quad (1)\]

where $G$ is $16\pi$ times the gravitational constant, $h_{ab}$ the homogeneous three-metric, $\hbar$ its determinant, $R$ the three-dimensional Ricci scalar, $\Lambda$ the cosmological constant, $G_{abcd}$ the metric on

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superspace, \( \partial_{h_{ab}} = \frac{\partial}{\partial h_{ab}} \) and \( H_M \) the Matter-Hamiltonian for the massive spatially homogeneous field \( \phi(t) \)
\[
H_M = -\hbar^2 (2\sqrt{\hbar})^{-1} \partial^2_{\phi} + \sqrt{\hbar} m^2 \phi^2 / 2. \tag{2}
\]

We expand the wave function \( \Psi \) in powers of \( G \)
\[
\Psi = \exp \{ i (G^{-1} S_{-1}(h_{ab}) + S_0(h_{ab}, \phi) + \ldots) / \hbar \}, \tag{3}
\]
where \( S_{-1} \) is a solution of the Hamilton–Jacobi equation of pure gravity. Defining the function \( f(h_{ab}, \phi) \) and the semiclassical time \( t(h_{ab}) \) by
\[
f(h_{ab}, \phi) = F(h_{ab}) \exp(i S_0(h_{ab}, \phi) / \hbar), \tag{4}
\]
\[
\partial_t = G_{abcd}(h_{ab} S_{-1}) \partial_{h_{cd}}, \tag{5}
\]
where \( F(h_{ab}) \) is determined by \( S_{-1} \), we obtain the Schrödinger equation on curved space
\[
\hbar \partial_t f(\phi, t) = H_M (\phi, t, \partial_\phi) f(\phi, t). \tag{6}
\]

Since we are dealing with a time dependent Schrödinger equation, we use the Schrödinger inner-product for the wave function \( f(\phi, t) \)
\[
< f | f > = \int_{-\infty}^{+\infty} d\phi f(\phi, t) f^*(\phi, t), \tag{7}
\]
where \( f^* \) denotes the complex conjugated of \( f \).

3. Explicit solvable Models

Our starting point is the Schrödinger equation \( (\ref{eq:schroedinger}) \) for a massive field. A redefinition of the semiclassical time \( t(h_{ab}) \) yields
\[
\hbar \partial_t f = \left[ \hbar^2 \partial^2_{\phi} + \phi^2 V(\tau) \right] f, \quad V(\tau) = h(\tau) \tag{8}
\]
with a time-dependent potential \( V(\tau) \). Inserting the special ansatz
\[
f_0 = \exp \left( -i \phi^2 \partial_\tau \ln |y(\tau)| / 4 \hbar - \ln |y(\tau)| / 2 \right), \tag{9}
\]
yields for the unknown function \( y(\tau) \) a second order ordinary differential eq.
\[
\frac{d^2 y(\tau)}{d\tau^2} = -V(\tau) y(\tau). \tag{10}
\]
Assuming we have found a solution of \( (\ref{eq:2}) \), we insert the unknown function \( \eta(\phi, \tau) \) defined by
\[
f(\phi, \tau) = f_0(\phi, \tau) \eta(\phi, \tau) \quad \text{in eq. (8).} \quad \text{After a}
\]
Fourier-Transformation \( \phi \rightarrow p, \eta \rightarrow \tilde{\eta} \) we conclude
\[
\partial_\tau \tilde{\eta} - p(\partial_\tau \ln(y)) \partial_p \tilde{\eta} - i \hbar p^2 \tilde{\eta} = 0. \tag{11}
\]
Thus solutions of \( (\ref{eq:8}) \) are given by
\[
f = f_0 \int_{-\infty}^{\infty} dy \exp (i p \phi)
\exp \{ i \hbar p^2 y^2 \int_{\tau}^{\tilde{\tau}} d\tilde{\tau} y^{-2}(\tilde{\tau}) \} F(y), \tag{12}
\]
with an arbitrary function \( F(y) \). A complete set of eigenfunctions with respect to the innerproduct \( (\ref{eq:2}) \) is given by
\[
f_n = y^{-1/2} H_n[\phi(4i \hbar q(\tau))^{-1/2}]
\exp \{ -i \phi^2 \partial_\tau \ln(y) / 4 \hbar + n \int_{\tau}^{\tilde{\tau}} d\tilde{\tau} q^{-2}(\tilde{\tau}) / 2 \}, \tag{13}
\]
where \( H_n \) are the Hermite polynomials of order \( n \) and \( q(\tau) \) is defined by
\[
q(\tau) = y^2(\tau) \tag{14}
\int_{\tau}^{\tilde{\tau}} d\tilde{\tau} y^{-2}(\tilde{\tau}).
\]
Thus, the solutions of the Schrödinger eq. \( (\ref{eq:2}) \) can be calculated from the solutions \( y(\tau) \) of eq. \( (\ref{eq:13}) \). The solutions of eq. \( (\ref{eq:13}) \) have to be chosen in such a way that \( \Re(-i \phi \partial_\tau \ln(y)) \) is positive i.e. the functions \( f_n \) \( (\ref{eq:13}) \) are normalizable.

3.1. FRW Models

A line element of the homogeneous and isotropic Friedman–Robertson–Walker models is
\[
ds^2 = N^2(t) dt^2 - a^2(t) (1 - kr)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \tag{15}
\]
with \( k = \pm 1 \) for the closed and for the hyperbolic universes respectively. For vanishing \( k \) we obtain the well-known DeSitter model, which describes a flat three dimensional space. For nonvanishing \( k \) and cosmological constant \( \Lambda \) we get a complicated expression for the semiclassical time parameter \( \tau \)
\[
\tau = \{ k^{-1} a^{-2} \sqrt{\Lambda a^2 - k}
+ \Lambda k^{-3/2} \arctan(\sqrt{\Lambda k^{-1} a^2 + 1}) \} / 4. \tag{16}
\]
Instead of this we introduce another parameter \( x \) by \( x = k a^2 \) and obtain from the eq. \( (\ref{eq:15}) \) for \( y(\tau) \) the well-known hypergeometric differential equation
\[
x(x-1) \partial^2_x y + (5x/2 - 2) \partial_x y + m^2 y / (4a) = 0. \tag{17}
\]
For vanishing $\Lambda$, $k = -1$, we get with $\tau = a^{-2}/4$ Bessel's eq.

$$9\Lambda \tau^2 \partial^2_y + m^2 y = 0. \quad (18)$$

In the DeSitter case with $k = 0$, $\tau = a^{-3}/(6\sqrt{\Lambda})$ and $\Lambda > 0$ eq. (19) can be written as a special case of Bessel's eq. namely Euler's eq.

$$9\Lambda \tau^2 \partial^2_y + m^2 y = 0. \quad (19)$$

Thus for the DeSitter model the solutions of eqs. (19), (8) can be expressed by elementary functions, depending on the value of $\Lambda/m^2$:

$$\Lambda/m^2 < 4/9 : \quad y = \sqrt{\tau} e^{-ika}, \quad (20)$$
$$\Lambda/m^2 > 4/9 : \quad y = \sqrt{\tau} (k_1 \tau^{k_0} + ik_2 \tau^{-k_0}), \quad (21)$$
$$\Lambda/m^2 = 4/9 : \quad y = \sqrt{\tau} (k_1 \ln(\tau) + ik_2), \quad (22)$$

with $k_0 = \sqrt{m^2/(9\Lambda)-1/4}$, $k_1, k_2 \in \mathbb{R}$ and $k_1 k_2 > 0$ because of normalizability.

### 3.2. Anisotropic Minisuperspaces

Contrary to the FRW models discussed above each of the following two models possesses two gravitational variables $z(t), b(t)$.

#### 3.2.1. Bianchi–I with rotational symmetry

The line element of the Bianchi–I model with rotational symmetry is

$$ds^2 = N^2(t) dt^2 - z^2(t) dr^2 - b^2(t) [d\theta^2 + \sin^2(\theta) d\phi^2]. \quad (23)$$

Here we obtain for the semiclassical time parameter $\tau$

$$\tau = -\frac{2c_0}{b(4Ab^2 - 3c_0 z^2)} \ln \left[ \frac{4Ab^2}{3c_0^2 z^2} \right], \quad (24)$$

where $c_0$ is the separation constant of the solution $S_{-1} = c_0 z^2 b + \Lambda b^4/(12c_0)$ (25) of the Hamilton–Jacobi eq.. Inserting $\tau$ of (24) in eq. (11) we obtain

$$\partial^2_{\tau} y + c_1 \sin^{-2}(c_2 \tau) y = 0, \quad (26)$$

with $c_1 = 3c_0^2 m^2 l^2/(32\Lambda)$ and $c_2 = 3c_0 l/4$, where $l = 4Ab^3/(3c_0^2 z^2 b)$ has to be treated as a constant.

Eq. (26) is related to a hypergeometric one, which can be seen by a change of variables $w = 1 + \exp(2c_2 \tau)$. This transforms eq. (26) into

$$w^2 (1-w) \partial^2_{w} y - w^2 \partial_w y + c_1 y/c_2 = 0. \quad (27)$$

Defining the function $\theta$ by $\theta(w) = y(w)w^{-c}$ with $c = 1/2 + \sqrt{1/4 + c_1/c_2}$, $\theta$ fulfills the hypergeometric differential eq.

#### 3.2.2. Bianchi–III with rotational symmetry

Similar to the line element (23) above, we have

$$ds^2 = N^2(t) dt^2 - z^2(t) dr^2 - b^2(t) [d\theta^2 + \sin^2(\theta) d\phi^2]. \quad (28)$$

Introducing the parameter

$$x = (4c_3^2 z^2 + 1)/(4c_3^2 z^2 - 1), \quad (29)$$

where $c_3$ denotes the separation constant of

$$S_{-1} = c_3 z^2 b + b/(4c_3), \quad (30)$$

eq (14) yields

$$\partial_x (x^2 - 1) \partial_x y + c_4 (1 - x) y = 0, \quad (31)$$

with $c_4 = 2c_3^2 m^2 (4c_3^2 z^2 - 1) b$ as a constant.

This eq. (31) is known from the scattering problem for the hydrogen molecule ion $H_2^+$ first solved by Jaffé [3].

The solutions of the Schrödinger equation (13) can be normalized for each of the models above.

### 4. Back Reaction

Two different approaches for calculating the back reaction of the quantized matter field on the classical metric are investigated. First, using the expectation value of the energy–momentum tensor as a source and second, with help of Wigner’s function [4]. We calculate and compare the results for different solutions of the Schrödinger eq. (8) for the DeSitter model.

#### 4.1. Using the energy-momentum tensor

One of the most familiar definitions of back reaction is to replace $H_M$ by $< f | H_M | f >$ in the classical Hamiltonian constraint, where $f$ is the wave function (8). This is equivalent to define

$$P = G^{-1} \partial_\alpha S_{-1} + < f | \partial_\alpha \beta(a, \phi) | f >, \quad (32)$$

for the gravitational variable $a$ of the DeSitter model, where $P$ denotes the classical momentum conjugate to $a$ and $\beta(a, \phi)$ the phase of the wave function $f$.

Inserting the unperturbed classical expression $P = -G^{-1} \partial_\alpha a^2$ ($N = 1$) in eq. (32) we get a first order differential eq. for the perturbed metric $\bar{a}(t)$. Since the perturbation is exact up to
order $G$ only, we set $\dot{a}(t) = a(t) + Ga_1(t)$, where $a = \exp[\sqrt{\Lambda}(t + t_0)/2]$ is the unperturbed DeSitter metric.

Given the wave function $f(\phi, \tau(a))$ we can calculate $a_1(t)$ explicitly. For the "groundstate" $f_0 = y^{-1/2}\exp(-i\phi^2 \partial_r \ln(y)/(4\hbar))$ we obtain

\[
\Lambda/m^2 < 4/9 : a_1 = -\hbar k_0^2 a^{-2}[k_0 + 1/(4k_0)], \quad (33)
\]

\[
\Lambda/m^2 > 4/9 : a_1 = -\hbar a^{-2}[a^{6k_0}k_0^2 + a^{-6k_0}k_0^2], \quad (34)
\]

\[
\Lambda/m^2 = 4/9 : a_1 = -\hbar a^{-2}[k_0^2(\ln(a))^2 + k_0 \ln(a) + k_{10}], \quad (35)
\]

with real constants $k_5, k_6, k_7, k_8, k_9, k_{10}$. In the massless case we have: $a_1 = -\hbar k_1^2 a^{-5}$, $k_{11} \in R$.

For large universes, $a \rightarrow \infty$, the perturbation is negligible $a_1 \ll a, \dot{a} \rightarrow a$.

If $f$ contains the Hermite polynomial $H_n$ we get

\[
a_1 = -\hbar k_2^2 a^{-2}(2n + 1)[k_0 + n!2^n/(4k_0)] \quad (36)
\]

for $\Lambda/m^2 < 4/9$. Thus, the result $(33)$ for the "groundstate" is only slightly modified. All the perturbations $(33, 34, 35, 36)$ vanish in the "classical limit" $\hbar \rightarrow 0$.

To obtain the additional classical part of the back reaction of order $G$, we take the wave function

\[
f = y^{-1/2}\exp(-i\phi^2 \partial_r \ln(y)/(4\hbar) + i\bar{\phi}/(\hbar y))
\]

\[
+ i\bar{\phi}^2 \int d\tau y^{-2}(\dot{\tau})/\hbar, \quad (37)
\]

and get

\[
a_1 \sim -c_1^2 a^{6k_0 - 2} - c_2^2 a^{-2} - c_3^2 a^{-6k_0 - 2} - c_4^2 a^{6k_0 - 2} \quad (38)
\]

with real constants $c_1, c_2, c_3, c_4$. Here the quantum corrections are given for large $a$. Multiplying the wave function $(37)$ by Hermite polynomials changes the coefficient $c_1^2$ of the quantum correction in $a_1$ only.

### 4.2. Wigner’s function

Wigner’s function $F_W$ is a generalization of a classical correlation function on phase space $[1]$. $F_W(a, P, \phi, P_\phi)$ depending on the gravitational, the matter variable and their momenta is defined by

\[
F_W = Gm^{-1}\int_{-\infty}^{\infty} du dv f^*(a - \hbar u/2, \phi - \hbar v/(2m))
\]

\[
e^{-i(\hbar P_\phi v + P_\phi v)/m}f(a + \hbar uG/2, \phi + \hbar v/(2m)). \quad (39)
\]

Expanding $F_W$ in powers of $G$ leads in order $G^0$ to $F_W \sim \delta(P - G^{-1}\partial_a S_{-1})$ from which we obtain the unperturbed classical expression $P = -G^{-1}\partial_a S_{-1}$ [2]. Integrating over $\phi, P_\phi$ we obtain in order $G$

\[
F_G = (\partial_a S_{-1})^{-1}\int_{-\infty}^{\infty} d\phi |f|^2 \delta(P - G^{-1}\partial_a S_{-1} - \partial_a \beta). \quad (40)
\]

where $\beta(a, \phi)$ is the phase of $f$. The peaks of $F_G$ yield the relation between $a$ and $P$ from which we obtain $a_1(t)$ by integration as in the case of the energy-momentum-tensor.

For the "groundstate" we calculate

\[
\Lambda/m^2 < 4/9 : a_1 = -\hbar k_0^2 k_0^{-2} \quad (33)
\]

\[
\Lambda/m^2 > 4/9 : a_1 \sim -\hbar a^{-2}(k_0^2 a^{6k_0} - k_0^{-2} a^{-6k_0}), \quad (34)
\]

\[
\Lambda/m^2 = 4/9 : a_1 \sim -\hbar k_1^2 a^{-2} \quad (35)
\]

The last two expressions are given in leading order $a \gg 1$ only. The case $\Lambda/m^2 < 4/9$ yields the same result as $(33)$. $\Lambda/m^2 > 4/9$ agrees in leading order with $(34)$. Only $\Lambda/m^2 = 4/9$ leads to a different back reaction compared with $(33)$.

The quantum corrections induced by the wave function $(37)$ remain unchanged.

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