Graph fusion algebras of $\mathcal{WLM}(p, p')$

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Abstract

We consider the $W$-extended logarithmic minimal model $\mathcal{WLM}(p, p')$. As in the rational minimal models, the so-called fundamental fusion algebra of $\mathcal{WLM}(p, p')$ is described by a simple graph fusion algebra. The fusion matrices in the regular representation thereof are mutually commuting, but in general not diagonalizable. Nevertheless, we show that they can be brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of dimension 1, 3, 5 or 9. The directed graphs associated with the two fundamental modules are described in detail. The corresponding adjacency matrices share a complete set of common generalized eigenvectors organized as a web constructed by interlacing the Jordan chains of the two matrices. This web is here called a Jordan web and it consists of connected subwebs with 1, 3, 5 or 9 generalized eigenvectors. The similarity matrix, formed by concatenating these vectors, simultaneously brings the two fundamental adjacency matrices to Jordan canonical form modulo permutation similarity. The ranks of the participating Jordan blocks are 1 or 3, and the corresponding eigenvalues are given by $2\cos\frac{j\pi}{\rho}$ where $j = 0, \ldots, \rho$ and $\rho = p, p'$. For $p > 1$, only some of the modules in the fundamental fusion algebra of $\mathcal{WLM}(p, p')$ are associated with boundary conditions within our lattice approach. The regular representation of the corresponding fusion subalgebra has features similar to the ones in the regular representation of the fundamental fusion algebra, but with dimensions of the upper-triangular blocks and connected Jordan-web components given by 1, 2, 3 or 8. Some of the key results are illustrated for $W$-extended critical percolation $\mathcal{WLM}(2, 3)$. 
1 Introduction

A central question of much current interest is whether an extended symmetry algebra $W^{[1,2]}$ exists for logarithmic conformal field theories [3,4,5,6] like the logarithmic minimal models $\mathcal{LM}(p,p')$ [7,8,9]. These models contain a countably infinite number of inequivalent Virasoro modules which the extended symmetry should reorganize into a finite number of $W$-extended modules closing under fusion. In the case of the logarithmic minimal models $\mathcal{LM}(1,p')$, the existence and properties of such an extended $W$-symmetry, including the associated fusion rules, are by now largely understood [10,11,12,13,14,15,16,17]. The works [18,19] strongly indicate the existence of a $W_{p,p'}$ symmetry algebra for general augmented minimal models, but offer only very limited insight into the associated fusion algebras. Recently, a detailed description of these fusion algebras has been provided in [20,21,22] generalizing the approach of [17]. Extending ideas originating with Cardy [23,24], this approach uses a strip-lattice implementation of fusion to obtain the fusion rules of the entire series of logarithmic minimal models $\mathcal{LM}(p,p')$ in the $W$-extended picture where they are denoted by $W\mathcal{LM}(p,p')$. It is stressed, that the extended picture is described by the same lattice model as the Virasoro picture.

Contrary to the situation in the Virasoro picture, for $p > 1$, there is no identity nor a pair of so-called fundamental modules in the lattice approach to $W\mathcal{LM}(p,p')$. In [22], we found that one can supplement the set of indecomposable modules associated with boundary conditions by a set of reducible yet indecomposable rank-1 modules. This algebraically enlarged set was shown to yield a well-defined fusion algebra called the fundamental fusion algebra. This algebra is so named since it is generated from repeated fusions of the two fundamental modules $(2,1)_W$ and $(1,2)_W$ in addition to the identity $(1,1)_W$ which is now present for all $p$. It was also found that the fusion algebra generated by modules associated with boundary conditions is an ideal of the fundamental fusion algebra. Further algebraic extensions exist. In particular, for $p > 1$, there are additional irreducible modules not associated with boundary conditions. Their fusion properties have been systematically examined only very recently [25,26,27]. Here we restrict ourselves to the modules generating the fundamental fusion algebra.

The fusion matrices of a standard rational conformal field theory are diagonalizable. This is made manifest by the Verlinde formula [28] where the diagonalizing similarity matrix is the modular $S$-matrix of the characters in the theory. In a logarithmic conformal field theory, on the other hand, there are typically more linearly independent representations than linearly independent characters due to the presence of indecomposable modules of rank greater than 1. Consequently, there is no Verlinde formula in the usual sense and the fusion matrices may not all be diagonalizable. This is indeed the situation for the $W$-extended logarithmic minimal models $W\mathcal{LM}(p,p')$ analyzed in the present work.

In the regular representation of a fusion algebra, the fusion matrices are mutually commuting. Viewing the fusion matrices as adjacency matrices of graphs, the fusion rules are succinctly encoded in these fusion graphs. In this context, the regular representation of a fusion algebra is referred to as the graph fusion algebra. Fusion graphs have been the key to the classification of rational conformal field theories on the cylinder [29,30] and on the torus [31,32,33,34]. In the rational $A$-type theories, the Verlinde algebra yields a diagonal modular invariant, while $D$- and $E$-type theories are related to non-diagonal modular invariants. The Ocneanu algebras arise when considering fusion on the torus, with left and right chiral halves of the theory, and involve Ocneanu graphs. We refer to [35,36,37,38] for earlier results on the interrelation between fusion algebras, graphs and modular invariants. It is our hope that the present work will be a step in the direction of extending these fundamental insights to the logarithmic conformal field theories.

As already indicated, the fusion matrices in the regular representation of the fundamental fusion algebra are mutually commuting, but in general not diagonalizable. Nevertheless, we show that they can be brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of...
dimension 1, 3, 5 or 9. The directed graphs associated with the two fundamental modules are described in detail. They consist of a number of connected components of which there are two prototypes. The adjacency matrices of these tadpole graphs and eye-patch graphs are Jordan decomposed explicitly. Combining them, the adjacency matrices $X$ and $Y$ of the two fundamental graphs are found to share a complete set of common generalized eigenvectors organized as a web constructed by interlacing the Jordan chains of $X$ and $Y$. This web is here called a Jordan web and it consists of connected subwebs with 1, 3, 5 or 9 generalized eigenvectors. The similarity matrix, formed by concatenating these vectors, simultaneously brings $X$ and $Y$ to Jordan canonical form modulo permutation similarity. For $p > 1$, it is simply not possible to properly Jordan decompose them simultaneously. The ranks of the participating Jordan blocks are 1 or 3, and the corresponding eigenvalues are given by $2 \cos \frac{j\pi}{p}$ where $j = 0, \ldots, \rho$ and $\rho = p, p'$. 

For $p = 1$, the fundamental fusion graph with adjacency matrix $Y$ is given by a single eye-patch graph and is thus connected. The fundamental fusion matrix $X$ acts as a conjugation on this eye-patch graph. In contrast to the situation for $p > 1$, as demonstrated in [39], these simple properties allow for the existence of a similarity matrix which simultaneously brings all fusion matrices of the fundamental fusion algebra of $WLM(1,p')$ to Jordan form. The two fundamental fusion matrices, in particular, are both brought to Jordan canonical form by this similarity transformation. The present work is an extension of the paper [39] on $WLM(1,p')$ to the general series of $W$-extended logarithmic minimal models $WLM(p,p')$.

For $p > 1$, only some of the modules in the fundamental fusion algebra of $WLM(p,p')$ are associated with boundary conditions within our lattice approach. The fusion matrices in the regular representation of the corresponding fusion subalgebra have features similar to the ones for the larger fundamental fusion algebra. From [22], we know that the modules associated with boundary conditions form an ideal of the fundamental fusion algebra. Their matrix realizations $N_\mu$ therefore follow from the realizations $N_\mu$ of the generators of the fundamental fusion algebra by elimination of the rows and columns corresponding to the modules not associated with boundary conditions. According to [22], every fusion matrix $N_\mu$ can be written as a polynomial in the fundamental fusion matrices $X$ and $Y$. Likewise, every fusion matrix $N_\mu$ can be written as a polynomial in the auxiliary fusion matrices $\hat{X}$ and $\hat{Y}$ obtained from $X$ and $Y$ by the aforementioned elimination procedure. For $p > 1$, the matrix $\hat{Y}$ does not correspond to a module associated with a boundary condition and is, in this sense, auxiliary. For $p > 2$, this applies to both $\hat{X}$ and $\hat{Y}$. Despite their auxiliary status, the matrices $\hat{X}$ and $\hat{Y}$ are very useful in the description of the spectral decomposition of the fusion matrices $N_\mu$. We refer to the corresponding directed graphs as auxiliary fusion graphs. As in the case of the fundamental fusion graphs, the auxiliary fusion graphs consist of a certain number of connected components of which there are two prototypes: cycle graphs and the eye-patch graphs above. We show that the auxiliary adjacency matrices $\hat{X}$ and $\hat{Y}$ share a complete set of common generalized eigenvectors, and that the corresponding Jordan web consists of connected subwebs with 1, 2, 3 or 8 generalized eigenvectors. We subsequently show that the fusion matrices $N_\mu$ can be brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of dimension 1, 2, 3 or 8.

The remaining part of this paper is organized as follows. Section 2 briefly reviews some basics of $WLM(p,p')$ and its fundamental fusion algebra. The associated graph fusion algebras are formally introduced and the fusion rules involving the fundamental modules are summarized. The cycle, tadpole and eye-patch graphs are defined in Section 3, and the spectral decompositions of their adjacency matrices are worked out in detail. These results are conveniently expressed in terms of Chebyshev polynomials. Using the summarized fusion rules just mentioned, in Section 4, we determine the fundamental and auxiliary fusion graphs as well as their adjacency matrices. We recall that the connected
components of these graphs are of the form discussed in Section 3. In Section 5, we work out the spectral decompositions of the fundamental and auxiliary fusion matrices. In both cases, we determine a complete set of common generalized eigenvectors and describe the corresponding Jordan web and its connected components. The Jordan canonical forms of the fundamental and auxiliary fusion matrices follow readily. Arising as the result of the simultaneous similarity transformation of the general fusion matrices, we also present explicit expressions for the block-diagonal forms of these matrices. Section 6 contains some concluding remarks and indications of future work, while Appendix A provides elementary examples demonstrating that two commuting matrices may not share a complete set of common generalized eigenvectors nor necessarily be brought simultaneously to Jordan form. In Appendix B, the Jordan subwebs formed by the common generalized eigenvectors of \(X\) and \(Y\) are collected in table form with respect to the corresponding eigenvalues. At various places in the paper, some of the key results are illustrated for \(W\)-extended critical percolation \(W\mathcal{L}M(2, 3)\).

2 \(W\)-extended logarithmic minimal models

A logarithmic minimal model \(\mathcal{L}M(p, p')\) is defined [7, 9] for every coprime pair of positive integers \(p < p'\). The model has central charge

\[
c = 1 - 6\frac{(p' - p)^2}{pp'}
\]  

and conformal weights

\[
\Delta_{\rho, \sigma} = \frac{(\rho p' - \sigma p)^2 - (p' - p)^2}{4pp'}, \quad \rho, \sigma \in \mathbb{N}
\]  

(2.2)

Its \(W\)-extension \(W\mathcal{L}M(p, p')\) is discussed in [17, 20, 21, 22] and briefly reviewed in the following. Throughout, we are using the following notation and conventions

\[
Z_{n,m} = \mathbb{Z} \cap [n, m], \quad \epsilon(n) = \frac{1 - (-1)^n}{2}, \quad n \cdot m = 1 + \epsilon(n + m), \quad n, m \in \mathbb{Z}
\]  

(2.3)

and

\[
\kappa, \kappa' \in \mathbb{Z}_{1,2}, \quad a \in \mathbb{Z}_{1, p-1}, \quad b \in \mathbb{Z}_{1, p'-1}, \quad r \in \mathbb{Z}_{1, p}, \quad s \in \mathbb{Z}_{1, p'}
\]  

(2.4)

2.1 Modules associated with boundary conditions

The indecomposable modules in \(W\mathcal{L}M(p, p')\), which can be associated with Yang-Baxter integrable boundary conditions on the strip lattice and \(W\)-invariant boundary conditions in the continuum scaling limit, were identified in [20, 21] by extending constructions in [17] pertaining to the case \(p = 1\). The set of these modules is given by

\[
\{W(\Delta_{\kappa p, b}), W(\Delta_{a, \kappa p'}), W(\Delta_{\kappa p, p'}), (\mathcal{R}^{a, 0}_{\kappa p, s})_W, (\mathcal{R}^{0, b}_{\kappa p'})_W, (\mathcal{R}^{a, b}_{\kappa p, p'})_W\}
\]  

(2.5)

and is of cardinality

\[
6pp' - 2p - 2p'
\]  

(2.6)

Here we have adopted the notation of [25] denoting a \(W\)-irreducible module of conformal weight \(\Delta\) by \(W(\Delta)\). Thus, there are \(2p + 2p' - 2\) irreducible (hence indecomposable rank-1) modules

\[
\{W(\Delta_{\kappa p, b}), W(\Delta_{r, \kappa p'})\}
\]  

(2.7)
where the two modules \( \mathcal{W}(\Delta_{1,1}) = \mathcal{W}(\Delta_{p,p'}) \) are listed twice, in addition to \( 4pp' - 2p - 2p' \) indecomposable rank-2 modules
\[
\{(\mathcal{R}_{1,1})_{p,p'}, (\mathcal{R}_{2,1})_{p,p'}\}
\] (2.8)
and \( 2(p - 1)(p' - 1) \) indecomposable rank-3 modules
\[\{\mathcal{R}_{1,1,1,1}\} \text{ subject to } (\mathcal{R}_{p,2p,p'}_{p,p'}) \equiv (\mathcal{R}_{2p,2p',p'}) \equiv (\mathcal{R}_{p,p',p'}) (2.9)\]
The associative and commutative fusion algebra of the modules (2.5) was determined in [21, 22]. There is no algebra unit or identity for \( p > 1 \), while, for \( p = 1 \), the irreducible module \( \mathcal{W}(\Delta_{1,1}) \) is the identity.

### 2.2 Fundamental fusion algebra

In [22], we found that one can supplement the set of indecomposable modules (2.5) by the \( (p - 1)(p' - 1) \) reducible yet indecomposable rank-1 modules
\[\{(a, b)_{W}\}, \quad \Delta((a, b)_{W}) = \Delta_{a,b} (2.10)\]
with conjectured embedding patterns given by
\[
(a, b)_{W} : (\Delta_{2p-a,b})_{W} = (\Delta_{a,2p'-b})_{W} (2.11)\]
Their characters read
\[
\chi[(a, b)_{W}](q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} (k^2 - 1) \left( q^{(ap' + bp + 2kpp')^2/4pp'} - q^{(ap' - bp + 2kpp')^2/4pp'} \right) (2.12)\]
where \( \eta(z) \) is the Dedekind eta function
\[
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) (2.13)\]
The cardinality of the enlarged set of indecomposable modules is readily seen to be given by
\[
7pp' - 3p - 3p' + 1 (2.14)\]
and this set was shown in [22] to yield a well-defined fusion algebra called the fundamental fusion algebra
\[
\text{Fund}[\mathcal{WLM}(p,p')] = \langle (1,1)_{W}, (2,1)_{W}, (1,2)_{W} \rangle (2.15)\]
This algebra is so named since it is generated from repeated fusions of the two fundamental modules \((2,1)_{W} \) and \((1,2)_{W} \) in addition to the identity \((1,1)_{W} \) which is now present for all \( p \). The module \((1,1)_{W} \) is irreducible for \( p = 1 \) in which case \((1,1)_{W} = \mathcal{W}(\Delta_{1,1}) \). The module \((2,1)_{W} \) is irreducible for \( p = 1, 2 \) in which case \((2,1)_{W} = \mathcal{W}(\Delta_{2,1}) \). The module \((1,2)_{W} \) is irreducible for \( p' = 2 \) in which case \((1,2)_{W} = \mathcal{W}(\Delta_{1,2}) \). From [22], we know that the fusion algebra generated by the modules (2.5) is an ideal of the fundamental fusion algebra. To simplify the notation, we sometimes write \((\mathcal{R}_{r,s})_{W} = (r,s)_{W} \), or \((r,s)_{W} = \mathcal{W}(\Delta_{r,s}) \) if \((r,s)_{W} \) happens to be irreducible.

Further algebraic extensions exist. In particular, for \( p > 1 \), there are irreducible modules not associated with boundary conditions as the ones in (2.5). Their fusion properties have been systematically examined only very recently [25, 26, 27]. Here we restrict ourselves to the modules generating the fundamental fusion algebra.
2.3 Fusion products of fundamental modules

Since the associative and commutative fundamental fusion algebra is generated from repeated fusions of the two fundamental modules \((2, 1)_W\) and \((1, 2)_W\), the complete set of fusion rules can be reconstructed from knowledge of the basic fusion products involving these two modules. Here we list all such fusion products. For \(p = 1\), we have

\[
(2, 1)_W \otimes (\kappa, s)_W = (2 \cdot \kappa, s)_W \\
(2, 1)_W \otimes (\mathcal{R}_{1,\kappa p}^{0,b})_W = (\mathcal{R}_{1,(2,\kappa)p'}^{0,b})_W
\]

while for \(p > 1\), we have

\[
(2, 1)_W \otimes (a, b)_W = (1 - \delta_{a,1})\left((a - 1, b)_W \oplus (a + 1, b)_W\right) \\
(2, 1)_W \otimes (\kappa p, s)_W = (\mathcal{R}_{p, s}^{1,0})_W \\
(2, 1)_W \otimes (a, \kappa p')_W = (1 - \delta_{a,1})\left((a - 1, \kappa p')_W \oplus (a + 1, \kappa p')_W\right) \\
(2, 1)_W \otimes (\mathcal{R}_{\kappa p, s}^{0,0})_W = 2\delta_{a,1}(\kappa p, s)_W \oplus 2\delta_{a,p-1}(2 \cdot \kappa p, s)_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{a,1})(\mathcal{R}_{\kappa p, s}^{a,-1,0})_W \oplus (1 - \delta_{a,p-1})(\mathcal{R}_{\kappa p, s}^{a+1,0})_W \\
(2, 1)_W \otimes (\mathcal{R}_{r,\kappa p}^{0,b})_W = \delta_{r,1}(\mathcal{R}_{r,\kappa p}^{0,b})_W \oplus \delta_{r,p}(\mathcal{R}_{r,\kappa p}^{1,b})_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{r,1})(1 - \delta_{r,p})(\mathcal{R}_{r,\kappa p}^{0,b})_W \oplus (\mathcal{R}_{r+1,\kappa p}^{0,b})_W \\
(2, 1)_W \otimes (\mathcal{R}_{\kappa p, p'}^{a,b})_W = 2\delta_{a,1}(\mathcal{R}_{\kappa p, p'}^{0,b})_W \oplus 2\delta_{a,p-1}(\mathcal{R}_{\kappa p, p'}^{0,b})_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{a,1})(\mathcal{R}_{\kappa p, p'}^{a,b})_W \oplus (1 - \delta_{a,p-1})(\mathcal{R}_{\kappa p, p'}^{a,b+1})_W
\]

Since \(p' > p \geq 1\), we simply have

\[
(1, 2)_W \otimes (a, b)_W = (1 - \delta_{b,1})(a, b - 1)_W \oplus (a, b + 1)_W \\
(1, 2)_W \otimes (\kappa p, b)_W = (1 - \delta_{b,1})(\kappa p, b - 1)_W \oplus (\kappa p, b + 1)_W \\
(1, 2)_W \otimes (r, \kappa p')_W = (\mathcal{R}_{r,\kappa p'}^{0,1})_W \\
(1, 2)_W \otimes (\mathcal{R}_{\kappa p, s}^{0,0})_W = \delta_{s,1}(\mathcal{R}_{\kappa p, 2}^{0,0})_W \oplus \delta_{s,p'}(\mathcal{R}_{\kappa p, p'}^{0,1})_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{s,1})(1 - \delta_{s,p'})(\mathcal{R}_{\kappa p, s-1}^{0,0})_W \oplus (\mathcal{R}_{\kappa p, s+1}^{0,0})_W \\
(1, 2)_W \otimes (\mathcal{R}_{r,\kappa p}^{0,b})_W = 2\delta_{b,1}(r, \kappa p')_W \oplus 2\delta_{b,p'-1}(r, (2 \cdot \kappa)p')_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{b,1})(\mathcal{R}_{r,\kappa p}^{0,b-1})_W \oplus (1 - \delta_{b,p'-1})(\mathcal{R}_{r,\kappa p}^{0,b+1})_W \\
(1, 2)_W \otimes (\mathcal{R}_{\kappa p, p'}^{a,b})_W = 2\delta_{b,1}(\mathcal{R}_{\kappa p, p'}^{a,b})_W \oplus 2\delta_{b,p'-1}(\mathcal{R}_{\kappa p, p'}^{a,b})_W \\
\quad \quad \quad \quad \oplus (1 - \delta_{b,1})(\mathcal{R}_{\kappa p, p'}^{a,b-1})_W \oplus (1 - \delta_{b,p'-1})(\mathcal{R}_{\kappa p, p'}^{a,b+1})_W
\]

for all \(p \in \mathbb{N}\).

2.4 Graph fusion algebras

Let \(I_f\) denote the set of indecomposable modules appearing in the fundamental fusion algebra of \(\mathcal{WLM}(p, p')\). In the regular representation

\[
N_\mu N_\nu = \sum_{\lambda \in I_f} N_{\mu, \nu} \lambda N_\lambda, \quad \mu, \nu \in I_f
\]

(2.19)
of this fusion algebra, the fusion matrices $N_\mu$ are mutually commuting, but in general not diagonalizable. Viewing the fusion matrices as adjacency matrices of graphs, the fusion rules are neatly encoded in the graphs. In this context, (2.19) is referred to as the graph fusion algebra of $WL\mathcal{M}(p, p')$, in this case corresponding to the fundamental fusion algebra.

As demonstrated in Section 4.1 the fundamental fusion graphs, the ones associated to the two fundamental modules, have two particular types of connected and directed components. In Section 3 we discuss the spectral decomposition of the adjacency matrices of these subgraphs. The adjacency matrices of the fundamental fusion graphs themselves are given by the matrix realizations of the two fundamental modules. We use

$$X = \left(1 + \delta_{p,1}\right)N_{(2,1)_W}, \quad Y = N_{(1,2)_W}$$

(2.20)

as a convenient shorthand for these matrices. The normalization of $X$ is chosen to ensure universality of notation in the following. In Section 5 we will demonstrate that $X$ and $Y$ can be simultaneously brought to Jordan form, modulo permutation similarity, by a common similarity transformation. It is recalled that two matrices $A$ and $B$ are permutation similar if for some permutation matrix $P$,

$$A = P^{-1}BP \quad (2.21)$$

In [22], we found that the fundamental fusion algebra is isomorphic to the polynomial ring

$$\mathbb{C}[X, Y]/\left(P_p(X), P_{p'}(Y), P_{p,p'}(X, Y)\right)$$

(2.22)

where

$$P_n(x) = 2(T_n\left(\frac{Z}{2}\right) - 1)U_{n-1}\left(\frac{y}{2}\right), \quad P_{n,n'}(x, y) = \left(T_n\left(\frac{x}{2}\right) - T_{n'}\left(\frac{y}{2}\right)\right)U_{n-1}\left(\frac{x}{2}\right)U_{n'-1}\left(\frac{y}{2}\right) \quad (2.23)$$

Here $T_n(z)$ and $U_n(z)$ denote the Chebyshev polynomials of the first and second kind, respectively. The isomorphism is given by

$$(a, b)_W \leftrightarrow U_{a-1}\left(\frac{X}{2}\right)U_{b-1}\left(\frac{Y}{2}\right)$$

$$\mathcal{W}(\Delta_{kp, s}) \leftrightarrow \frac{1}{r}\left(U_{kp-1}\left(\frac{X}{2}\right)U_{s-1}\left(\frac{Y}{2}\right)\right)$$

$$\mathcal{W}(\Delta_{a,kp'}) \leftrightarrow \frac{1}{r}\left(U_{a-1}\left(\frac{X}{2}\right)U_{kp'-1}\left(\frac{Y}{2}\right)\right)$$

$$\mathcal{W}(\mathcal{R}_{a,0, s})_W \leftrightarrow \frac{2}{r}T_k\left(\frac{X}{2}\right)U_{kp-1}\left(\frac{X}{2}\right)U_{s-1}\left(\frac{Y}{2}\right)$$

$$\mathcal{W}(\mathcal{R}_{0,b, s})_W \leftrightarrow \frac{2}{r}T_k\left(\frac{X}{2}\right)T_b\left(\frac{Y}{2}\right)U_{kp'-1}\left(\frac{Y}{2}\right)$$

$$\mathcal{W}(\mathcal{R}_{a,b, s})_W \leftrightarrow \frac{4}{r}T_k\left(\frac{X}{2}\right)U_{kp-1}\left(\frac{X}{2}\right)T_b\left(\frac{Y}{2}\right)U_{p'-1}\left(\frac{Y}{2}\right) \quad (2.24)$$

where it is noted that

$$U_{kp-1}\left(\frac{X}{2}\right)U_{p'-1}\left(\frac{Y}{2}\right) \equiv U_{p-1}\left(\frac{X}{2}\right)U_{kp'-1}\left(\frac{Y}{2}\right) \quad (\text{mod } P_{p,p'}(X, Y)) \quad (2.25)$$

Identifying the formal entities $X$ and $Y$ appearing in (2.22) with the two fundamental matrices (2.20) of matching notation, we obtain the regular representation (2.19) of the fundamental fusion algebra.

Letting $\mathcal{I}_b$ denote the set of indecomposable modules (2.23) associated with boundary conditions, the regular representation of the corresponding fusion algebra is given by

$$\hat{N}_{\mu} \hat{N}_{\nu} = \sum_{\lambda \in \mathcal{I}_b} \hat{N}_{\mu, \nu} \hat{N}_{\lambda}, \quad \mu, \nu \in \mathcal{I}_b$$

(2.26)
Since this fusion algebra is an ideal of the fundamental fusion algebra, \( \hat{N}_\mu \), for every \( \mu \in \mathcal{I}_b \), is obtained from \( N_\mu \) by elimination of the rows and columns corresponding to the \((p - 1)(p' - 1)\) modules \((2.10)\) not associated with boundary conditions. Indeed, ordering the elements of \( \mathcal{I}_f \) according to \( \mathcal{I}_f = (\mathcal{I}_f \setminus \mathcal{I}_b) \cup \mathcal{I}_b \) yields

\[
N_\mu = \begin{pmatrix}
* & * \\
0 & *
\end{pmatrix}, \quad \mu \in \mathcal{I}_f \setminus \mathcal{I}_b; \\
N_\mu = \begin{pmatrix}
0 & * \\
0 & \hat{N}_\mu
\end{pmatrix}, \quad \mu \in \mathcal{I}_b
\] (2.27)

Utilizing this block-triangular structure for \( X \) and \( Y \)

\[
X = \begin{pmatrix}
* & * \\
0 & \hat{X}
\end{pmatrix}, \quad Y = \begin{pmatrix}
* & * \\
0 & \hat{Y}
\end{pmatrix}
\] (2.28)

we have

\[
N_\mu = \text{pol}_\mu(X, Y) = \begin{pmatrix}
* & * \\
0 & \text{pol}_\mu(X, Y)
\end{pmatrix}
\] (2.29)

where \( \text{pol}_\mu(X, Y) \) is the polynomial appearing in \((2.24)\) for \( \mu \in \mathcal{I}_f \). It follows that we can express the fusion matrices \( \hat{N}_\mu \) in terms of the matrices \( \hat{X} \) and \( \hat{Y} \)

\[
\hat{N}_\mu = \text{pol}_\mu(X, Y), \quad \mu \in \mathcal{I}_b
\] (2.30)

using the same polynomial as in the description of \( N_\mu \) in terms of \( X \) and \( Y \) \((2.29)\). For \( p > 2 \), we have \((2,1)_W, (1,2)_W \in \mathcal{I}_f \setminus \mathcal{I}_b \), in which case the matrices \( \hat{X} \) and \( \hat{Y} \) should be thought of as auxiliary matrices. Similarly, for \( p = 2 \), we have \((1,2)_W \in \mathcal{I}_f \setminus \mathcal{I}_b \). Despite their auxiliary status, the matrices \( \hat{X} \) and \( \hat{Y} \) are very useful in the description of the spectral decomposition of the fusion matrices \( \hat{N}_\mu \). We refer to the corresponding directed graphs as auxiliary fusion graphs. As in the case of the fundamental fusion graphs, the auxiliary ones consist of two particular types of connected and directed components, one of which also appears as subgraphs of the fundamental fusion graphs.

3 Spectral decomposition of adjacency matrices

3.1 Cycle, tadpole and eye-patch graphs

As already mentioned, a fundamental (or auxiliary) fusion graph consists of a number of connected components. There are three prototypes: cycle graphs, tadpole graphs and eye-patch graphs, and they are the topic of the present section. The connected subgraphs of the fundamental fusion graphs are all tadpole or eye-patch graphs, while the connected subgraphs of the auxiliary fusion graphs are all cycle or eye-patch graphs. All of these connected graphs depend on a single integer order parameter \( \rho \geq 2 \).

We refer to a connected and directed graph of the type

\[
\begin{align*}
U_1 & \rightarrow U_2 \rightarrow \cdots \rightarrow U_{\rho-1} \\
L & \rightarrow U_{\rho-1} \\
D_{\rho-1} & \rightarrow \cdots \rightarrow L \\
D_1 & \rightarrow \cdots \\
R & \rightarrow D_1
\end{align*}
\] (3.1)
as a *cycle graph* with order parameter $\rho$. Its order is $2\rho$ and the labeling of the $2\rho$ vertices has been chosen to reflect their position in the graph. The cycle graph with order parameter $\rho = 2$ is given by

$$
\begin{array}{c}
\bigcirc \\
L \quad R
\end{array}
$$

(3.2)

In the ordered basis

$$\{L, U_1, \ldots, U_{\rho-1}, R, D_1, \ldots, D_{\rho-1}\}$$

(3.3)

the adjacency matrix associated to the cycle graph (3.1) is given by

$$
C_\rho =
\begin{pmatrix}
0 & 1 & & & & & \\
2 & 0 & 1 & & & & \\
1 & 0 & & & & & \\
\vdots & & & & & & \\
0 & 1 & & & & & \\
1 & 0 & 2 & & & & \\
0 & 0 & 1 & & & & \\
2 & 0 & 1 & & & & \\
1 & 0 & & & & & \\
\vdots & & & & & & \\
0 & 1 & & & & & \\
1 & 0 & & & & & \\
\end{pmatrix}
$$

(3.4)

The first and $(\rho + 1)$'th rows and columns (corresponding to $L$ and $R$) are emphasized to signal their special status. For $\rho = 2$, the adjacency matrix is

$$
C_2 =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
2 & 0 & 2 & 0 \\
\end{pmatrix}
$$

(3.5)

We refer to a connected and directed graph of the type

$$
\begin{array}{c}
L_1 \quad \ldots \quad L_{\rho-1} \quad L_\rho \\
\bigcirc \\
D_{\rho-1} \quad D_1
\end{array}
$$

(3.6)

as a *tadpole graph* with order parameter $\rho$. Its order is $3\rho - 1$ and the labeling of the $3\rho - 1$ vertices has been chosen to reflect their position in the graph. The tadpole graph with order parameter $\rho = 2$
The adjacency matrix associated to the tadpole graph (3.6) is given by

\[
T_\rho = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
& 0 & 1 & 0 & 1 \\
& & 0 & 1 & 0 \\
& & & 0 & 1 \\
2 & 0 & 1 & 0 & 2 \\
& 2 & 0 & 1 & 0 \\
& & & & 0 & 1 \\
& & & & 1 & 0 \\
2 & 0 & 1 & 0 & 2 \\
\end{pmatrix}
\] (3.9)

The \(\rho\)'th and \((2\rho)\)'th rows and columns (corresponding to \(L_\rho\) and \(R\)) are emphasized to signal their special status. For \(\rho = 2\), the adjacency matrix is

\[
T_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 2 & 0 \\
\end{pmatrix}
\] (3.10)

We also introduce what we call an *eye-patch graph* with order parameter \(\rho\)
which, for \( \rho = 2 \), reduces to

\[
\begin{align*}
L_1 & \rightarrow L_2 & R_2 & \leftarrow R_1 \\
 & \leftarrow D_1 & & \\
\end{align*}
\]

(3.12)

The order of the graph (3.11) is \( 4\rho - 2 \), and the labeling of the \( 4\rho - 2 \) vertices has been chosen to reflect their position in the graph. In the ordered basis

\[
\{L_1, \ldots, L_{\rho - 1}, L_\rho, U_1, \ldots, U_{\rho - 1}, R_1, \ldots, R_{\rho - 1}, R_\rho, D_1, \ldots, D_{\rho - 1}\}
\]

(3.13)

the adjacency matrix associated to the graph (3.11) is given by

(3.14)

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 1 \\
1 & 0 & 1 & \ldots & 1 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
2 & 0 & 1 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 1 & \ldots & 1 & 1 \\
2 & 0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}
\]

The \( \rho \)'th and \( (3\rho - 1) \)'th rows and columns (corresponding to \( L_\rho \) and \( R_\rho \)) are emphasized to signal their special status. For \( \rho = 2 \), the adjacency matrix (3.14) is meant to reduce to

(3.15)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 2 & 0 \\
\end{pmatrix}
\]
Extending the definition of the order parameter to \( \rho = 1 \), we let a cycle, tadpole or eye-patch graph collapse to the following directed order-2 graph

\[
\begin{array}{c}
L \\
\downarrow
\end{array} \begin{array}{c}
R
\end{array}
\] (3.16)

This type of graph is relevant only when considering the series \( \mathcal{WLM}(1, p') \). The corresponding adjacency matrix is

\[
\mathcal{C}_1 = \mathcal{T}_1 = \mathcal{E}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}
\] (3.17)

### 3.2 Spectral decompositions

In preparation for the spectral decomposition of the adjacency matrices (3.4), (3.9) and (3.14), we recall that canonical Jordan blocks of rank-2 or -3 associated to the eigenvalue \( \lambda \) of a matrix \( A \) are given by

\[
\mathcal{J}_{\lambda,2} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \mathcal{J}_{\lambda,3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}
\] (3.18)

They appear in the Jordan decomposition of \( A \) if the eigenvalue \( \lambda \) gives rise to a Jordan chain of length 2 or 3, where a Jordan chain of length 3, in particular, is given by

\[
(A - \lambda I) v^{(2)} = v^{(1)}, \quad (A - \lambda I) v^{(1)} = v^{(0)}, \quad (A - \lambda I)^{\ell+1} v^{(\ell)} = 0, \quad \ell \in \mathbb{Z}_{0,2} \] (3.20)

indicating that the vectors are generalized eigenvectors. A proper eigenvector is merely a special type of generalized eigenvector. We say that an upper-triangular (square) matrix with identical entries \( \lambda \) on the diagonal is a Jordan block if the geometric multiplicity of (the single eigenvalue) \( \lambda \) is 1. It is a Jordan canonical block (as in (3.18)) if the entries on the super-diagonal are 1 while all entries above the super-diagonal are 0. A block-diagonal matrix is of Jordan (canonical) form if every block is a Jordan (canonical) block.

Following [39], we introduce the \( 2\rho - 1 \) functions \( f_h(x) \), \( h \in \mathbb{Z}_{1,2\rho-1} \), defined by

\[
f_k(x) = U_{k-1}(\frac{x}{2}), \quad f_\rho(x) = U_{\rho-1}(\frac{x}{2}), \quad f_{\rho+k}(x) = 2T_k(\frac{x}{2})U_{\rho-1}(\frac{x}{2}) \] (3.21)

Here, and in the following, we are using the convention

\[
k \in \mathbb{Z}_{1,\rho-1}
\] (3.22)

Certain useful properties of \( f_h(x) \) are listed here, while further details can be found in [39]. For \( \rho > 2 \), the functions satisfy recursive relations allowing us to express \( x f_h(x) \), for \( h \in \mathbb{Z}_{1,2\rho-2} \), as

\[
\begin{align*}
&f_2(x) = xf_1(x) \\
&f_{h-1}(x) + f_{h+1}(x) = xf_h(x), \quad h \in \mathbb{Z}_{2,\rho-1} \\
&f_{\rho+1}(x) = xf_\rho(x) \\
&2f_\rho(x) + f_{\rho+2}(x) = xf_{\rho+1}(x) \\
&f_{h-1}(x) + f_{h+1}(x) = xf_h(x), \quad h \in \mathbb{Z}_{\rho+2,2\rho-2}
\end{align*}
\] (3.23)
It follows that

$$
\begin{align*}
    f'_2(x) &= x f'_1(x) + f_1(x), \\
    f'_h(x) + f'_{h+1}(x) &= x f'_h(x) + f_h(x), \\
    f'_{h+1}(x) &= x f'_h(x) + f_h(x), \\
    f''_h(x) &= f''_h(x) + 2 f'_h(x).
\end{align*}
$$

with the conditions on $h$ adopted from (3.23). It is noted that we have not included any relations involving $x f_{2\rho-1}(x)$ for general $x$. Instead, we focus on evaluations at $x = \lambda_j$, for $j \in \mathbb{Z}_{0,\rho}$, where

$$
\begin{align*}
    2(-1)^i f'_\rho(\lambda_j) + f_{2\rho-2}(\lambda_j) &= \lambda_j f_{2\rho-1}(\lambda_j), & j \in \mathbb{Z}_{1,\rho-1} \text{ or } i = j \in \{0, \rho\} \\
    2(-1)^k f''_\rho(\lambda_k) + f_{2\rho-2}(\lambda_k) &= \lambda_k f''_{2\rho-1}(\lambda_k) + f_{2\rho-1}(\lambda_k) \\
    2(-1)^k f''_\rho(\lambda_k) + f_{2\rho-2}(\lambda_k) &= \lambda_k f''_{2\rho-1}(\lambda_k) + 2 f_{2\rho-1}(\lambda_k)
\end{align*}
$$

(3.25)

We also note that

$$
f_h(\lambda_k) = 0, \quad h \in \mathbb{Z}_{\rho,2\rho-1}
$$

(3.26)

A convenient notation to be used below is

$$
\frac{1}{\rho} \partial^{(\ell)}_h f_h(x) = \frac{1}{\ell!} \partial^{\ell}_x f_h(x), \quad \ell \in \mathbb{Z}_{0,2}, \quad h \in \mathbb{Z}_{1,2\rho-1}
$$

(3.27)

where $0! = 1$.

Since the spectral decompositions of $C_\rho$, $T_\rho$ and $E_\rho$ are worked out in Section 3.2.1, 3.2.2 and 3.2.3 for $\rho \geq 2$, we here discuss the rather trivial spectral decomposition of $C_1 = T_1 = E_1$ (3.17) in the framework employed in those sections. With $\rho = 1$, the eigenvalues are

$$
\lambda_j = 2 \cos \frac{j \pi}{\rho}, \quad j \in \mathbb{Z}_{0,\rho}
$$

(3.28)

with corresponding eigenvectors given by

$$
V_0 = \begin{pmatrix} f_\rho(\lambda_0) \\ (-1)^0 f_\rho(\lambda_0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad V_\rho = \begin{pmatrix} f_\rho(\lambda_0) \\ (-1)^\rho f_\rho(\lambda_0) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}
$$

(3.29)

The minimal and characteristic polynomials of $C_1$ are given by

$$
m(C_1) = (C_1 - \lambda_0 I)(C_1 - \lambda_\rho I) = C_1^2 - 4I, \quad \det(\lambda I - C_1) = (\lambda - \lambda_0)(\lambda - \lambda_\rho) = \lambda^2 - 4
$$

(3.30)

while the similarity matrix constructed by concatenating the two eigenvectors

$$
Q_{C_1} = \begin{pmatrix} V_0 & V_\rho \end{pmatrix}
$$

(3.31)

diagonalizes $C_1$

$$
Q_{C_1}^{-1} C_1 Q_{C_1} = \text{diag}(\lambda_0, \lambda_\rho) = \text{diag}(2, -2)
$$

(3.32)
3.2.1 Cycle graphs

It follows from the explicit construction of generalized eigenvectors below that the eigenvalues of $C = C_\rho$, $\rho > 1$, are given by (3.28), while the minimal and characteristic polynomials of $C$ are given by

$$m(C) = (C - \lambda_0 I)(C - \lambda_\rho I) \prod_{k=1}^{\rho-1} (C - \lambda_k I)^2 = (C^2 - 4I)^2 - (\frac{C}{2})$$

$$\det(\lambda I - C) = (\lambda - \lambda_0)(\lambda - \lambda_\rho) \prod_{k=1}^{\rho-1} (\lambda - \lambda_k)^2 = (\lambda^2 - 4)^2 - (\frac{\lambda}{2})$$

(3.33)

This implies that the Jordan canonical form of $C$ consists of $\rho - 1$ rank-2 blocks associated to the eigenvalues $\lambda_k$, and two rank-1 blocks associated to the eigenvalues $\lambda_0 = 2$, $\lambda_\rho = -2$. The number of linearly independent eigenvectors of $C$ is thus $\rho + 1$. Since the null-space of $C$ is empty for $\rho$ odd but one-dimensional for $\rho$ even, the rank of $C$ is

$$\text{rank}(C) = 2\rho - \epsilon(\rho - 1) = 2\rho - 1 + \epsilon(\rho)$$

(3.34)

To establish these results on $C$, we now discuss the two eigenvectors corresponding to $\lambda_0 = 2$, $\lambda_\rho = -2$, and the $\rho - 1$ Jordan chains of length 2 associated to $\lambda_k$. Using the various properties of the functions $f_k(x)$ discussed above, it is straightforward to verify that

$$C_0 = \begin{pmatrix} f_\rho(\lambda_0) \\ \vdots \\ (-1)^0f_\rho(\lambda_0) \\ (-1)^0f_{2\rho-1}(\lambda_0) \end{pmatrix}, \quad C_\rho = \begin{pmatrix} f_\rho(\lambda_\rho) \\ \vdots \\ (-1)^0f_\rho(\lambda_\rho) \\ (-1)^0f_{2\rho-1}(\lambda_\rho) \end{pmatrix}$$

(3.35)

are eigenvectors of $C$

$$CC_0 = \lambda_0 C_0, \quad CC_\rho = \lambda_\rho C_\rho$$

(3.36)

For every $k \in \mathbb{Z}_{1,\rho-1}$, it is likewise verified that

$$C_k^{(0)} = \begin{pmatrix} f_\rho'(\lambda_k) \\ \vdots \\ (-1)^k f_\rho'(\lambda_k) \\ (-1)^k f_{2\rho-1}'(\lambda_k) \end{pmatrix}, \quad C_k^{(1)} = \begin{pmatrix} \frac{1}{2} f_\rho''(\lambda_k) \\ \vdots \\ \frac{1}{2}(-1)^k f_\rho''(\lambda_k) \\ \frac{1}{2}(-1)^k f_{2\rho-1}''(\lambda_k) \end{pmatrix}$$

(3.37)

form the Jordan chain

$$CC_k^{(0)} = \lambda_k C_k^{(0)}, \quad CC_k^{(1)} = \lambda_k C_k^{(1)} + C_k^{(0)}$$

(3.38)

Finally, the $2\rho$-dimensional matrix $Q_C$ is constructed by concatenating the generalized eigenvectors $C_k^{(0)}$ and $C_k^{(1)}$

$$Q_C = \begin{pmatrix} C_0 & C_1^{(0)} & \cdots & C_{\rho-1}^{(0)} \\ C_1^{(1)} & C_2^{(1)} & \cdots & C_{\rho-1}^{(1)} \end{pmatrix}$$

(3.39)

By a similarity transformation, this matrix converts $C$ into its Jordan canonical form

$$J_C = Q_C^{-1} C Q_C = \text{diag}(\lambda_0, J_{\lambda_1,2}, \ldots, J_{\lambda_{\rho-1},2}, \lambda_\rho)$$

(3.40)
3.2.2 Tadpole graphs

It follows from the explicit construction of generalized eigenvectors below that the eigenvalues of $\mathcal{T} = \mathcal{T}_\rho$, $\rho > 1$, are given by (3.28), while the minimal and characteristic polynomials of $\mathcal{T}$ are given by

$$m(\mathcal{T}) = (\mathcal{T} - \lambda_0 I)(\mathcal{T} - \lambda_\rho I) \prod_{k=1}^{\rho-1} (\mathcal{T} - \lambda_k I)^3 = (\mathcal{T}^2 - 4I)U_{\rho-1}^3(\frac{1}{2})$$

$$\det(\lambda I - \mathcal{T}) = (\lambda - \lambda_0)(\lambda - \lambda_\rho) \prod_{k=1}^{\rho-1} (\lambda - \lambda_k)^3 = (\lambda^2 - 4)U_{\rho-1}^3(\frac{1}{2})$$  \hspace{1cm} (3.41)

This implies that the Jordan canonical form of $\mathcal{T}$ consists of $\rho - 1$ rank-3 blocks associated to the eigenvalues $\lambda_k$, and two rank-1 blocks associated to the eigenvalues $\lambda_0 = 2$, $\lambda_\rho = -2$. The number of linearly independent eigenvectors of $\mathcal{T}$ is thus $\rho + 1$. Since the null-space of $\mathcal{T}$ is empty for $\rho$ odd but one-dimensional for $\rho$ even, the rank of $\mathcal{T}$ is

$$\text{rank}(\mathcal{T}) = 3\rho - 1 - \epsilon(\rho - 1) = 3\rho - 2 + \epsilon(\rho)$$  \hspace{1cm} (3.42)

To establish these results on $\mathcal{T}$, we now discuss the two eigenvectors corresponding to $\lambda_0 = 2$, $\lambda_\rho = -2$, and the $\rho - 1$ Jordan chains of length 3 associated to $\lambda_k$. Using the various properties of the functions $f_k(x)$ discussed above, it is straightforward to verify that

$$T_j = \begin{pmatrix} f_1(\lambda_j) \\ \vdots \\ f_{\rho-1}(\lambda_j) \\ f_\rho(\lambda_j) \\ \vdots \\ f_{2\rho-1}(\lambda_j) \\ (-1)^jf_\rho(\lambda_j) \\ \vdots \\ (-1)^jf_{2\rho-1}(\lambda_j) \end{pmatrix}, \quad j = 0, \rho; \quad T_k^{(0)} = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \\ f_\rho(\lambda_k) \\ \vdots \\ f_{2\rho-1}(\lambda_k) \\ (-1)^kf_\rho(\lambda_k) \\ \vdots \\ (-1)^kf_{2\rho-1}(\lambda_k) \end{pmatrix} = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \end{pmatrix} \hspace{1cm} (3.43)$$

are eigenvectors of $\mathcal{T}$

$$\mathcal{T}\mathcal{T}_0 = \lambda_0\mathcal{T}_0, \quad \mathcal{T}\mathcal{T}_k^{(0)} = \lambda_k\mathcal{T}_k^{(0)}, \quad \mathcal{T}\mathcal{T}_\rho = \lambda_\rho\mathcal{T}_\rho$$  \hspace{1cm} (3.44)

For every $k \in \mathbb{Z}_{1,\rho-1}$, it is likewise verified that $T_k^{(0)}$ together with

$$T_k^{(1)} = \begin{pmatrix} f_1'(\lambda_k) \\ \vdots \\ f_{\rho-1}'(\lambda_k) \\ f_\rho'(\lambda_k) \\ \vdots \\ f_{2\rho-1}'(\lambda_k) \\ (-1)^kf_\rho'(\lambda_k) \\ \vdots \\ (-1)^kf_{2\rho-1}'(\lambda_k) \end{pmatrix}, \quad T_k^{(2)} = \begin{pmatrix} \frac{1}{2}f_1''(\lambda_k) \\ \vdots \\ \frac{1}{2}f_{\rho-1}''(\lambda_k) \\ \frac{1}{2}f_\rho''(\lambda_k) \\ \vdots \\ \frac{1}{2}f_{2\rho-1}''(\lambda_k) \end{pmatrix}$$  \hspace{1cm} (3.45)
form the Jordan chain
\[ T T_k^{(0)} = \lambda_k T_k^{(0)}, \quad T T_k^{(1)} = \lambda_k T_k^{(1)} + T_k^{(0)}, \quad T T_k^{(2)} = \lambda_k T_k^{(2)} + T_k^{(1)} \]  
(3.46)

Finally, the \((3\rho - 1)\)-dimensional matrix \(Q_T\) is constructed by concatenating the generalized eigenvectors \((3.43)\) and \((3.45)\)
\[ Q_T = \begin{pmatrix} T_0 & T_1^{(0)} & T_1^{(1)} & \ldots & T_1^{(0)} & T_1^{(1)} & T_1^{(2)} & \ldots & T_1^{(0)} & T_1^{(1)} & T_1^{(2)} & \ldots & T_{\rho-1}^{(0)} & T_{\rho-1}^{(1)} & T_{\rho-1}^{(2)} & \ldots & T_\rho \end{pmatrix} \]  
(3.47)

By a similarity transformation, this matrix converts \(T\) into its Jordan canonical form
\[ J_T = Q_T^{-1} T Q_T = \text{diag}(\lambda_0, J_{\lambda_1,3}, \ldots, J_{\lambda_{\rho-1},3}, \lambda_\rho) \]  
(3.48)

3.2.3 Eye-patch graphs

It follows from the explicit construction of generalized eigenvectors below that the eigenvalues of \(E = \mathcal{E}_\rho\), \(\rho > 1\), are given by \((3.28)\), while the minimal and characteristic polynomials of \(E\) are given by
\[ m(E) = (E - \lambda_0 I)(E - \lambda_\rho I) \prod_{k=1}^{\rho-1} (E - \lambda_k I)^3 = (E^2 - 4I) U_\rho^3 (\frac{E}{2}) \]
\[ \text{det}(\lambda I - E) = (\lambda - \lambda_0)(\lambda - \lambda_\rho) \prod_{k=1}^{\rho-1} (\lambda - \lambda_k)^4 = (\lambda^2 - 4I) U_\rho^4 (\frac{\lambda}{2}) \]  
(3.49)

This implies that the Jordan canonical form of \(E\) consists of \(\rho - 1\) rank-3 blocks associated to the eigenvalues \(\lambda_k\), and \(\rho + 1\) rank-1 blocks associated to the eigenvalues \(\lambda_j\). The number of linearly independent eigenvectors of \(E\) is thus \(2\rho\). Since the null-space of \(E\) is empty for \(\rho\) odd but two-dimensional for \(\rho\) even, the rank of \(E\) is
\[ \text{rank}(E) = 4\rho - 2 - 2\epsilon(\rho - 1) = 4(\rho - 1) + 2\epsilon(\rho) \]  
(3.50)

To establish these results on \(E\), we now discuss the \(\rho - 1\) Jordan chains of length 3 associated to \(\lambda_k\), and the additional \(\rho + 1\) eigenvectors corresponding to \(\lambda_j\). Using the various properties of the functions \(f_k(x)\) discussed above, it is straightforward to verify that
\[
E_j = \begin{pmatrix} f_1(\lambda_j) \\ \vdots \\ f_{\rho-1}(\lambda_j) \\ f_\rho(\lambda_j) \\ \vdots \\ f_{2\rho-1}(\lambda_j) \\ (-1)^j f_1(\lambda_j) \\ \vdots \\ (-1)^j f_{\rho-1}(\lambda_j) \\ (-1)^j f_\rho(\lambda_j) \\ \vdots \\ (-1)^j f_{2\rho-1}(\lambda_j) \end{pmatrix}, \quad j = 0, \rho; \quad E_k = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \\ f_\rho(\lambda_k) \\ \vdots \\ f_{2\rho-1}(\lambda_k) \\ (-1)^{k-1} f_1(\lambda_k) \\ \vdots \\ (-1)^{k-1} f_{\rho-1}(\lambda_k) \\ (-1)^{k-1} f_\rho(\lambda_k) \\ \vdots \\ (-1)^{k-1} f_{2\rho-1}(\lambda_k) \end{pmatrix} = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \\ 0 \\ \vdots \\ 0 \\ (-1)^{k-1} f_1(\lambda_k) \\ \vdots \\ (-1)^{k-1} f_{\rho-1}(\lambda_k) \end{pmatrix} \]  
(3.51)

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The vectors are eigenvectors of $E$.

$$E_k^{(0)} = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \\ f_\rho(\lambda_k) \\ \vdots \\ f_{2\rho-1}(\lambda_k) \\ (-1)^k f_1(\lambda_k) \\ \vdots \\ (-1)^k f_{\rho-1}(\lambda_k) \\ (-1)^k f_\rho(\lambda_k) \\ \vdots \\ (-1)^k f_{2\rho-1}(\lambda_k) \end{pmatrix} = \begin{pmatrix} f_1(\lambda_k) \\ \vdots \\ f_{\rho-1}(\lambda_k) \end{pmatrix}$$  \hspace{1cm} (3.52)

are eigenvectors of $E$

$$E E_0 = \lambda_0 E_0, \quad E E_k = \lambda_k E_k, \quad E E_k^{(0)} = \lambda_k E_k^{(0)}, \quad E E_\rho = \lambda_\rho E_\rho$$  \hspace{1cm} (3.53)

The vectors $E_k$ and $E_k^{(0)}$ are readily seen to be linearly independent. For every $k \in \mathbb{Z}_{1,\rho-1}$, it is likewise verified that $E_k^{(0)}$ together with

$$E_k^{(1)} = \begin{pmatrix} f'_1(\lambda_k) \\ \vdots \\ f'_{\rho-1}(\lambda_k) \\ f''_\rho(\lambda_k) \\ \vdots \\ f''_{2\rho-1}(\lambda_k) \\ (-1)^k f'_1(\lambda_k) \\ \vdots \\ (-1)^k f'_{\rho-1}(\lambda_k) \\ (-1)^k f''_\rho(\lambda_k) \\ \vdots \\ (-1)^k f''_{2\rho-1}(\lambda_k) \end{pmatrix}, \quad E_k^{(2)} = \begin{pmatrix} \frac{1}{2} f''_1(\lambda_k) \\ \vdots \\ \frac{1}{2} f''_{\rho-1}(\lambda_k) \\ \frac{1}{2} (-1)^k f''_\rho(\lambda_k) \\ \vdots \\ \frac{1}{2} (-1)^k f''_{2\rho-1}(\lambda_k) \end{pmatrix}$$  \hspace{1cm} (3.54)

form the Jordan chain

$$E E_k^{(0)} = \lambda_k E_k^{(0)}, \quad E E_k^{(1)} = \lambda_k E_k^{(1)} + E_k^{(0)}, \quad E E_k^{(2)} = \lambda_k E_k^{(2)} + E_k^{(1)}$$  \hspace{1cm} (3.55)

Finally, the $(4\rho-2)$-dimensional matrix $Q_\mathcal{E}$ is constructed by concatenating the generalized eigenvectors (3.51), (3.52) and (3.54)

$$Q_\mathcal{E} = \begin{pmatrix} E_0 & E_1 & E_1^{(0)} & E_1^{(1)} & \cdots & E_{\rho-1} & E_{\rho-1}^{(0)} & E_{\rho-1}^{(1)} & E_{\rho-1}^{(2)} & E_\rho \end{pmatrix}$$  \hspace{1cm} (3.56)
By a similarity transformation, this matrix converts $E$ into its Jordan canonical form

$$J_E = Q_E^{-1}E Q_E = \text{diag}(\lambda_0; \lambda_1, J_{\lambda_1,3}; \ldots; \lambda_{p-1}, J_{\lambda_{p-1},3}; \lambda_p)$$  \hfill (3.57)

## 4 Fundamental and auxiliary fusion graphs

### 4.1 Fundamental fusion graphs

The graphs associated to the two fundamental modules are called fundamental fusion graphs and consist of certain connected components. Every such subgraph is a tadpole graph or an eye-patch graph.

The fundamental fusion graph, whose adjacency matrix is given by $X$, consists of $p' - 1$ tadpole graphs and $p'$ eye-patch graphs, all with order parameter $\rho = p$. For every $b \in \mathbb{Z}_{1,p'-1}$, the $3p - 1$ vertices of the tadpole graphs are given by

$$T_p : \quad (L_r, U_a, R, D_a) = ((r, b)_W, (R_{a,b}^0, 0)_W, (2p, b)_W, (R_{2p,b}^a)_W), \quad r \in \mathbb{Z}_{1,p}, \quad a \in \mathbb{Z}_{1,p-1}$$ \hfill (4.1)

while for every $\beta \in \mathbb{Z}_{0,p'-1}$, the $4p - 2$ vertices of the eye-patch graphs are given by

$$E_p : \quad (L_r, U_a, R_r, D_a) = ((R_{a,b}^0, 0)_W, (R_{p,b}^a)_W, (R_{p,2p}^0, 0)_W, (R_{p,2p,b}^a)_W), \quad r \in \mathbb{Z}_{1,p}, \quad a \in \mathbb{Z}_{1,p-1}$$ \hfill (4.2)

Likewise, the fundamental fusion graph, whose adjacency matrix is given by $Y$, consists of $p - 1$ tadpole graphs and $p$ eye-patch graphs, all with order parameter $\rho = p'$. For every $a \in \mathbb{Z}_{1,p-1}$, the $3p' - 1$ vertices of the tadpole graphs are given by

$$T_{p'} : \quad (L_s, U_b, R, D_b) = ((a, s)_W, (R_{a,b}^0, 0)_W, (a, 2p')_W, (R_{a,2p'}^0, 0)_W), \quad s \in \mathbb{Z}_{1,p'}, \quad b \in \mathbb{Z}_{1,p'-1}$$ \hfill (4.3)

while for every $\alpha \in \mathbb{Z}_{0,p-1}$, the $4p' - 2$ vertices of the eye-patch graphs are given by

$$E_{p'} : \quad (L_s, U_b, R_s, D_b) = ((R_{a,b}^0, 0)_W, (R_{a,b}^0, 0)_W, (R_{2p,b}^0, 0)_W, (R_{2p,2p}^0, 0)_W), \quad s \in \mathbb{Z}_{1,p'}, \quad b \in \mathbb{Z}_{1,p'-1}$$ \hfill (4.4)

In accord with the results obtained in [39] on $\mathcal{WLM}(1,p')$, the graph corresponding to $X$ consists of $2p' - 1$ order-2 graphs (4.11)

$$(L, R) \in \left\{ \left( (1, b)_W, (2, b)_W \right), \left( W(\Delta_{1,p'}), W(\Delta_{2,p'}) \right), \left( (R_{1,1}^0, b)_W, (R_{1,2}^0, b)_W \right), \quad b \in \mathbb{Z}_{1,p'-1} \right\}$$ \hfill (4.5)

where $\Delta_{2,p'} = \Delta_{1,2p'}$, while there is a single eye-patch graph associated to $Y$

$$(L_s, U_b, R_s, D_b) = \left( W(\Delta_{1,s}), (R_{1,1}^0, b)_W, W(\Delta_{2,s}), (R_{1,2}^0, b)_W \right), \quad s \in \mathbb{Z}_{1,p'}, \quad b \in \mathbb{Z}_{1,p'-1}$$ \hfill (4.6)

### 4.1.1 $\mathcal{W}$-extended critical percolation $\mathcal{WLM}(2,3)$

$\mathcal{W}$-extended critical percolation is described by $\mathcal{WLM}(2,3)$ where $p = 2$ and $p' = 3$. In this case, the fundamental fusion graph, whose adjacency matrix is given by $X$, consists of five connected components, all with order parameter 2. The order of the graph is 28. The connected components are the two tadpole graphs

\[
\begin{align*}
& (1,1)_W \quad \text{and} \quad (2,1)_W \\
& (1,2)_W \quad \text{and} \quad (2,2)_W \\
& (4,1)_W \quad \text{and} \quad (4,2)_W
\end{align*}
\]
and the three eye-patch graphs

\[
(1, 3)_W \quad \xrightarrow{\mathcal{R}_{2,3}^{1,0}} \quad (2, 3)_W
\]

\[
(2, 6)_W \quad \xleftarrow{\mathcal{R}_{4,3}^{1,0}} \quad (1, 6)_W
\]

\[
(1, 3)_W \quad \xrightarrow{\mathcal{R}_{2,3}^{1,0}} \quad (2, 3)_W
\]

\[
(2, 6)_W \quad \xleftarrow{\mathcal{R}_{4,3}^{1,0}} \quad (1, 6)_W
\]

\[
(4.8)
\]

and

\[
(\mathcal{R}_{1,3}^{0,1}_W \xrightarrow{\mathcal{R}_{2,3}^{0,1}} \mathcal{R}_{2,6}^{0,1}_W \xleftarrow{\mathcal{R}_{1,6}^{0,1}} \mathcal{R}_{4,3}^{1,1}_W)
\]

\[
(4.9)
\]

and

\[
(\mathcal{R}_{1,3}^{0,2}_W \xrightarrow{\mathcal{R}_{2,3}^{0,2}} \mathcal{R}_{2,6}^{0,2}_W \xleftarrow{\mathcal{R}_{1,6}^{0,2}} \mathcal{R}_{4,3}^{1,2}_W)
\]

\[
(4.10)
\]

Likewise, the fundamental fusion graph, whose adjacency matrix is given by \( Y \), consists of three connected components, all with order parameter 3. The order of the graph is 28. The connected components are the single tadpole graph

\[
(1, 1)_W \quad \xrightarrow{\mathcal{R}_{1,3}^{0,1}_W} \quad (1, 2)_W \quad (1, 3)_W
\]

\[
(1, 6)_W \quad \xleftarrow{\mathcal{R}_{1,6}^{0,1}_W} \quad (1, 3)_W
\]

\[
(4.11)
\]

and the two eye-patch graphs

\[
(2, 1)_W \quad \xrightarrow{\mathcal{R}_{2,3}^{0,1}_W} \quad (2, 2)_W \quad (2, 3)_W
\]

\[
(4, 3)_W \quad \xleftarrow{\mathcal{R}_{2,6}^{0,2}_W} \quad (4, 2)_W \quad (4, 1)_W
\]

\[
(4.12)
\]

and

\[
(\mathcal{R}_{2,3}^{1,1}_W \xrightarrow{\mathcal{R}_{2,3}^{1,2}} \mathcal{R}_{2,3}^{1,0}_W)
\]

\[
(\mathcal{R}_{2,2}^{1,0}_W \xrightarrow{\mathcal{R}_{2,2}^{1,0}} \mathcal{R}_{2,3}^{1,0}_W)
\]

\[
(\mathcal{R}_{2,6}^{1,2}_W \xrightarrow{\mathcal{R}_{2,6}^{1,1}} \mathcal{R}_{2,6}^{1,1}_W)
\]

\[
(4.13)
\]
We recall that
\[(4, 3)_W = \mathcal{W}(\Delta_{4,3}) = \mathcal{W}(\Delta_{2,6}) = (2, 6)_W, \quad (\mathcal{R}_{4,3}^{1,1})_W = (\mathcal{R}_{2,6}^{1,1})_W, \quad (\mathcal{R}_{4,3}^{1,2})_W = (\mathcal{R}_{2,6}^{1,2})_W \quad (4.14)\]

### 4.2 Fundamental fusion matrices

We choose to work with the basis
\[
\begin{align*}
\{ & (1, 1)_W, \ldots, (1, p')_W, (\mathcal{R}_{1, p'}^{0,1})_W, \ldots, (\mathcal{R}_{1, p'}^{0, p'-1})_W, \\
& (1, 2p')_W, (\mathcal{R}_{1, 2p'}^{0,1})_W, \ldots, (\mathcal{R}_{1, 2p'}^{0, p'-1})_W, \\
& \vdots \\
& (p - 1, 1)_W, \ldots, (p - 1, p')_W, (\mathcal{R}_{p-1, p'}^{0,1})_W, \ldots, (\mathcal{R}_{p-1, p'}^{0, p'-1})_W, \\
& (p - 1, 2p')_W, (\mathcal{R}_{p-1, 2p'}^{0,1})_W, \ldots, (\mathcal{R}_{p-1, 2p'}^{0, p'-1})_W, \\
& (p, 1)_W, \ldots, (p, p')_W, (\mathcal{R}_{p, p'}^{0,1})_W, \ldots, (\mathcal{R}_{p, p'}^{0, p'-1})_W, \\
& (2p, 1)_W, \ldots, (2p, p')_W, (\mathcal{R}_{2p, p'}^{0,1})_W, \ldots, (\mathcal{R}_{2p, p'}^{0, p'-1})_W, \\
& (\mathcal{R}_{p, 1})_W, \ldots, (\mathcal{R}_{p, p'})_W, (\mathcal{R}_{p, 2p'})_W, \ldots, (\mathcal{R}_{p, 2p'})_W, \ldots, (\mathcal{R}_{p, 2p'})_W, \\
& \vdots \\
& (\mathcal{R}_{p, 1})_W, \ldots, (\mathcal{R}_{p, p'})_W, (\mathcal{R}_{p, 2p'})_W, \ldots, (\mathcal{R}_{p, 2p'})_W, \ldots, (\mathcal{R}_{p, 2p'})_W \} \quad (4.15)
\end{align*}
\]

in which \(Y\) has the simple form
\[
Y = \text{diag}\left(\begin{array}{c}
\mathcal{T}_{p'}, \ldots, \mathcal{T}_{p'}, \mathcal{E}_{p'}, \ldots, \mathcal{E}_{p'}
\end{array}\right)_{p-1} \quad (4.16)
\]
while \(X\) is the matrix
\[
X = \mathcal{P}^{-1}\text{diag}\left(\mathcal{T}_{p'}, \ldots, \mathcal{T}_{p'}, \mathcal{E}_{p'}, \ldots, \mathcal{E}_{p'}\right)\mathcal{P} = \begin{pmatrix}
0 & I & \ddots & \\
I & 0 & & 0 \\
& \ddots & \ddots & \\
& & I & 0
\end{pmatrix}_{p' \times p'} \quad (4.17)
\]

Here \(\mathcal{P}\) is a permutation matrix, while the \((4p' - 2) \times (4p' - 2)\)-dimensional matrix \(C\) and the \((3p' - 1) \times (4p' - 2)\)-dimensional matrix \(\tilde{I}\) are given by
\[
C = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}, \quad \tilde{I} = \begin{pmatrix}
I_{(2p-1)\times (2p'-1)} & 0_{(2p'-1)\times (p'-1)} & 0_{(2p'-1)\times (p')}
0_{(p')\times (2p'-1)} & I_{(p')\times (p')} & 0_{(p')\times (p')}
\end{pmatrix}_{(2p' - 1)\times (p' - 1)} \quad (4.18)
\]
In (4.17), $X$ is written as a $(2p - 1) \times (2p - 1)$-dimensional matrix whose entries are blocks. Every block (indicated by 0 or $I$) to the left of the leftmost vertical delimiter has $3p' - 1$ columns, while every block (indicated by 0, $I$, $\tilde{I}$, $2I$ or $2C'$) to the right of this delimiter has $4p' - 2$ columns. Likewise, every block (indicated by 0, $I$ or $\tilde{I}$) above the upper vertical delimiter has $3p' - 1$ rows, while every block (indicated by 0, $I$, $2I$ or $2C'$) below this delimiter has $4p' - 2$ rows. For small values of $p$, the matrix $X$ in (4.17) is meant to reduce to

$$X|_{p=2} = \begin{pmatrix} 0 & \tilde{I} \\ 0 & 0 & I \\ 0 & 2I + 2C' & 0 \end{pmatrix}, \quad X|_{p=1} = 2C$$

(4.19)

with $p'$-dependent dimensions of the blocks given as above.

For later convenience, it is noted that

$$\tilde{I}E_0 = T_0, \quad \tilde{I}E_k = T_k^{(0)}, \quad \tilde{I}E_k^{(t)} = T_k^{(t)}, \quad \tilde{I}E_{p'} = T_{p'}$$

(4.20)

and

$$CE_0 = E_0, \quad CE_k = (-1)^{k-1}C_k, \quad CE_k^{(t)} = (-1)^{k}E_k^{(t)}, \quad CE_{p'} = (-1)^{p'}E_{p'}$$

(4.21)

where $k \in \mathbb{Z}_{1,p' - 1}$ and $\ell \in \mathbb{Z}_{0,2}$, while the order parameter appearing in the entries of the vectors is $\rho = p'$. We also note that the basis used in [39] on $\mathcal{WLM}(1,p')$ is different from (4.15) for $p = 1$

$$(p,1)_W, \ldots, (p,p')_W, (\mathcal{R}_{p,p'}^{0,1})_W, \ldots, (\mathcal{R}_{p,p'}^{0,1})_W, \ldots, (2p,1)_W, \ldots, (2p,p')_W, (\mathcal{R}_{p,2p'}^{0,1})_W, \ldots, (\mathcal{R}_{p,2p'}^{0,1})_W$$

(4.22)

The two bases are related by a permutation.

### 4.3 Auxiliary fusion graphs

The two auxiliary fusion graphs consist of certain connected components. Every such subgraph is a cycle graph or an eye-patch graph.

The auxiliary fusion graph, whose adjacency matrix is given by $\hat{X}$, consists of $p' - 1$ cycle graphs and $p'$ eye-patch graphs, all with order parameter $\rho = p$. For every $b \in \mathbb{Z}_{1,p' - 1}$, the $2p$ vertices of the cycle graphs are given by

$$C_p : \quad (L, U_a, R, D_a) = ((p,b)_W, (\mathcal{R}_{p,b}^{a,0})_W, (2p,b)_W, (\mathcal{R}_{2p,b}^{a,0})_W), \quad a \in \mathbb{Z}_{1,p - 1}$$

(4.23)

while for every $\beta \in \mathbb{Z}_{0,p' - 1}$, the $4p - 2$ vertices of the eye-patch graphs are given by

$$E_p : \quad (L_r, U_a, R_r, D_a) = ((\mathcal{R}_{r,p'}^{a,\beta})_W, (\mathcal{R}_{p,2p'}^{a,\beta})_W, (\mathcal{R}_{r,2p'}^{a,\beta})_W, (\mathcal{R}_{2p,2p'}^{a,\beta})_W), \quad r \in \mathbb{Z}_{1,p}, \quad a \in \mathbb{Z}_{1,p - 1}$$

(4.24)

Likewise, the auxiliary fusion graph, whose adjacency matrix is given by $\hat{Y}$, consists of $p - 1$ cycle graphs and $p$ eye-patch graphs, all with order parameter $\rho = p'$. For every $a \in \mathbb{Z}_{1,p - 1}$, the $2p'$ vertices of the cycle graphs are given by

$$C_{p'} : \quad (L, U_b, R, D_b) = ((a,p')_W, (\mathcal{R}_{a,b}^{0,0})_W, (a,2p')_W, (\mathcal{R}_{a,2p'}^{0,0})_W), \quad b \in \mathbb{Z}_{1,p' - 1}$$

(4.25)

while for every $\alpha \in \mathbb{Z}_{0,p' - 1}$, the $4p' - 2$ vertices of the eye-patch graphs are given by

$$E_{p'} : \quad (L_s, U_b, R_s, D_b) = ((\mathcal{R}_{p,s}^{0,0})_W, (\mathcal{R}_{p,b}^{a,0})_W, (\mathcal{R}_{2p,s}^{a,0})_W, (\mathcal{R}_{p,2p'}^{a,0})_W), \quad s \in \mathbb{Z}_{1,p'}, \quad b \in \mathbb{Z}_{1,p' - 1}$$

(4.26)
4.3.1 $W$-extended critical percolation $\mathcal{WLM}(2,3)$

In the case of $W$-extended critical percolation $\mathcal{WLM}(2,3)$, there are exactly two indecomposable modules \((2.10)\) in the fundamental fusion algebra not associated with boundary conditions, namely the identity $\mathbf{1}_W$ and the fundamental module $(1,2)_W$. The auxiliary fusion graph, whose adjacency matrix is given by $\hat{X}$, consists of five connected components, all with order parameter 2. The order of the graph is 26. The connected components are the two cycle graphs

\[
\begin{align*}
(2,1)_W & \quad (4,1)_W \\
(2,2)_W & \quad (4,2)_W \\
(4,1)_W & \quad (2,2)_W
\end{align*}
\]

and the three eye-patch graphs \((4.8), (4.9)\) and \((4.10)\). Likewise, the auxiliary fusion graph, whose adjacency matrix is given by $\hat{Y}$, consists of three connected components, all with order parameter 3. The order of the graph is 26. The connected components are the single cycle graph

\[
\begin{align*}
(1,3)_W & \quad (1,6)_W \\
(1,3)_W & \quad (1,6)_W \\
(1,6)_W & \quad (1,3)_W
\end{align*}
\]

and the two eye-patch graphs \((4.12)\) and \((4.13)\).

4.4 Auxiliary fusion matrices

We choose to work with the basis

\[
\left\{ (1,p')_W, (\mathcal{R}_{p-1,p'}^{1,0})_W, \ldots, (\mathcal{R}_{p-1,p'}^{0,0})_W, (1,2p')_W, (\mathcal{R}_{1,2p'}^{0,1})_W, \ldots, (\mathcal{R}_{1,2p'}^{0,0})_W, \ldots \\
(1,p')_W, (\mathcal{R}_{p-1,p'}^{1,1})_W, \ldots, (\mathcal{R}_{p-1,p'}^{1,1})_W, (1,2p')_W, (\mathcal{R}_{1,2p'}^{0,1})_W, \ldots, (\mathcal{R}_{1,2p'}^{0,0})_W, \ldots \\
\vdots \\
(1,p')_W, (\mathcal{R}_{p-1,p'}^{p-1,0})_W, \ldots, (\mathcal{R}_{p-1,p'}^{p-1,0})_W, (1,2p')_W, (\mathcal{R}_{1,2p'}^{p-1,1})_W, \ldots, (\mathcal{R}_{1,2p'}^{p-1,0})_W, \ldots \\
\vdots \\
(1,p')_W, (\mathcal{R}_{p-1,p'}^{p-1,1})_W, \ldots, (\mathcal{R}_{p-1,p'}^{p-1,1})_W, (1,2p')_W, (\mathcal{R}_{1,2p'}^{p-1,1})_W, \ldots, (\mathcal{R}_{1,2p'}^{p-1,0})_W, \ldots \right\}
\]

(4.29)
in which \( \hat{Y} \) has the simple form

\[
\hat{Y} = \text{diag}(C_{p'}, \ldots, C_{p'}, E_{p'}, \ldots, E_{p'})
\]  

(4.30)

while \( \hat{X} \) is the matrix

\[
\hat{X} = \hat{P}^{-1} \text{diag}(C_{p'}, \ldots, C_{p'}, E_{p'}, \ldots, E_{p'}) \hat{P} = \\
\begin{pmatrix}
0 & I \\
I & 0 \\
& \ddots \\
& & 0 & I \\
& & I & 0 \\
& & \hat{I} & 0 \\
& & \hat{I} & \hat{I} \\
0 & 0 & I & \hat{I} \\
0 & 0 & \hat{I} & \hat{I} \\
\end{pmatrix}
\]  

(4.31)

Here \( \hat{P} \) is a permutation matrix, while the \((4p' - 2) \times (4p' - 2)\)-dimensional matrix \( C \) and the \((2p') \times (4p' - 2)\)-dimensional matrix \( \hat{I} \) are given by

\[
C = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} 0_{(p') \times (p'-1)} & I_{(p') \times (p')} \\ 0_{(p') \times (p'-1)} & 0_{(p') \times (p')} \\ 0_{(p') \times (p'-1)} & 0_{(p') \times (p')} \\ 0_{(p') \times (p')} & I_{(p') \times (p')} \end{pmatrix}
\]  

(4.32)

In (4.31), \( \hat{X} \) is written as a \((2p - 1) \times (2p - 1)\)-dimensional matrix whose entries are blocks. Every block (indicated by 0 or \( I \)) to the left of the leftmost vertical delimiter has \( 2p' \) columns, while every block (indicated by 0, \( I \), \( \hat{I} \), 2\( I \) or 2\( C \)) to the right of this delimiter has \( 4p' - 2 \) columns. Likewise, every block (indicated by 0, \( I \) or \( \hat{I} \)) above the upper vertical delimiter has \( 2p' \) rows, while every block (indicated by 0, \( I \), 2\( I \) or 2\( C \)) below this delimiter has \( 4p' - 2 \) rows. For small values of \( p \), the matrix \( \hat{X} \) in (4.31) is meant to reduce to

\[
\hat{X} \mid_{p=2} = \begin{pmatrix} 0 & \hat{I} \\ 0 & 0 \\ \hat{I} & 0 \\ 0 & 2I + 2\hat{C} \end{pmatrix}, \quad \hat{X} \mid_{p=1} = 2\hat{C}
\]  

(4.33)

with \( p' \)-dependent dimensions of the blocks given as above. For later convenience, it is noted that

\[
\hat{I}E_0 = C_0, \quad \hat{I}E_k = \hat{I}E_k^{(0)} = 0, \quad \hat{I}E_k^{(1)} = C_k^{(0)}, \quad \hat{I}E_k^{(2)} = C_k^{(1)}, \quad \hat{I}E_{p'} = C_{p'}
\]  

(4.34)

where \( k \in \mathbb{Z}_{1,p'-1} \), while the order parameter appearing in the entries of the vectors is \( \rho = p' \).

5 Spectral decomposition of fusion matrices

The objective here is to examine to what extent the fusion matrices \( N_\mu \) (or \( \tilde{N}_\mu \)) can be simultaneously brought to Jordan form. Our first goal is thus to devise a similarity transformation in the form of a matrix \( Q \) (\( \tilde{Q} \)) which simultaneously brings the fundamental (auxiliary) fusion matrices \( X \) and \( Y \) (\( \hat{X} \) and \( \hat{Y} \))...
and \( \hat{Y} \) to Jordan form. For \( p > 1 \), this is only possible modulo permutation similarity. With the Jordan decompositions of \( Y \) and \( \hat{Y} \) implemented, the best we can do is therefore

\[
Q^{-1}XQ = P^{-1}JXP, \quad Q^{-1}YQ = J_Y
\]

and

\[
\hat{Q}^{-1}\hat{X}Q = \hat{P}^{-1}J_X\hat{P}, \quad \hat{Q}^{-1}\hat{Y}\hat{Q} = J_Y
\]

where \( J_X \) and \( J_Y \) \((J_X \) and \( J_Y \)) are Jordan canonical forms of \( X \) and \( Y \) \((\hat{X} \) and \( \hat{Y} \)), while \( P \) \((\hat{P} \) is a permutation matrix. For every fusion matrix \( N_\mu \) in \((2.26)\), it then follows that

\[
Q^{-1}N_\mu Q = Q^{-1}\text{pol}_\mu(X,Y)Q = \text{pol}_\mu(Q^{-1}XQ, Q^{-1}YQ) = \text{pol}_\mu(P^{-1}JXP, J_Y)
\]

\[
= P^{-1}\text{pol}_\mu(x)(J_X)P \text{pol}_\mu(y)(J_Y)
\]

where we have used that the polynomials \( \text{pol}_\mu(X,Y) \) \((2.24)\) factorize and thus can be written as

\[
\text{pol}_\mu(X,Y) = \text{pol}_\mu(x)(X)\text{pol}_\mu(y)(Y)
\]

As we will demonstrate, \( Q^{-1}N_\mu Q \) is a block-diagonal matrix whose blocks are upper-triangular matrices of dimension 1, 3, 5 or 9, while \( P \) is a symmetric permutation matrix. Likewise, for every \( \hat{N}_\mu \) in \((2.26)\), it follows that

\[
\hat{Q}^{-1}\hat{N}_\mu \hat{Q} = \hat{Q}^{-1}\text{pol}_\mu(\hat{X}, \hat{Y})\hat{Q} = \text{pol}_\mu(\hat{Q}^{-1}\hat{X}\hat{Q}, \hat{Q}^{-1}\hat{Y}\hat{Q}) = \text{pol}_\mu(\hat{P}^{-1}J_X\hat{P}, J_Y)
\]

\[
= \hat{P}^{-1}\text{pol}_\mu(x)(J_X)\hat{P} \text{pol}_\mu(y)(J_Y)
\]

where \( \hat{Q}^{-1}\hat{N}_\mu \hat{Q} \) turns out to be a block-diagonal matrix whose blocks are upper-triangular matrices of dimension 1, 2, 3 or 8, while \( \hat{P} \) is a symmetric permutation matrix. By reversing the conjugation in \((5.3)\) or \((5.5)\), one obtains an explicit expression for the given fusion matrix. We will describe the relations \((5.3)\) and \((5.5)\) in detail in Section 5.2.2 and Section 5.3.2.

### 5.1 Jordan webs

As discussed in Appendix A, two commuting matrices need not share a complete set of common generalized eigenvectors. However, we will demonstrate that the two fundamental (or the two auxiliary) adjacency matrices \( X \) and \( Y \) \((\hat{X} \) and \( \hat{Y} \)) do have a common complete set of generalized eigenvectors. These generalized eigenvectors are organized as a web constructed by interlacing the Jordan chains of the two matrices. We refer to such a web as a Jordan web. It consists of a number of connected components or subwebs which we will characterize in the following.

#### 5.1.1 Fundamental fusion matrices

With respect to \( X \) or \( Y \) separately, we only encounter Jordan chains of length 1 or 3. As we will demonstrate in Section 5.2.1, five different types of connected Jordan webs arise in the description of the common generalized eigenvectors \( G_{\lambda,\lambda'}^{(\ell,\ell')} \) of \( X \) and \( Y \)

\[
XG_{\lambda,\lambda'}^{(0,\ell)} = \lambda G_{\lambda,\lambda'}^{(0,\ell)} , \quad XG_{\lambda,\lambda'}^{(1,\ell)} = \lambda G_{\lambda,\lambda'}^{(1,\ell)} + G_{\lambda,\lambda'}^{(0,\ell)} , \quad XG_{\lambda,\lambda'}^{(2,\ell)} = \lambda G_{\lambda,\lambda'}^{(2,\ell)} + G_{\lambda,\lambda'}^{(1,\ell)}
\]

\[
YG_{\lambda,\lambda'}^{(\ell,0)} = \lambda' G_{\lambda,\lambda'}^{(\ell,0)} , \quad YG_{\lambda,\lambda'}^{(\ell,1)} = \lambda' G_{\lambda,\lambda'}^{(\ell,1)} + G_{\lambda,\lambda'}^{(\ell,0)} , \quad YG_{\lambda,\lambda'}^{(\ell,2)} = \lambda' G_{\lambda,\lambda'}^{(\ell,2)} + G_{\lambda,\lambda'}^{(\ell,1)}
\]
These Jordan webs are

\[
\begin{align*}
&W^{(1,1)}_{\lambda,\lambda'} : G^{(0,0)}_{\lambda,\lambda'} & \quad & W^{(1,3)}_{\lambda,\lambda'} : G^{(0,1)}_{\lambda,\lambda'} & \quad & W^{(3,1)}_{\lambda,\lambda'} : G^{(0,0)}_{\lambda,\lambda'} & \quad & W^{(1,0)}_{\lambda,\lambda'} & \quad & W^{(2,0)}_{\lambda,\lambda'} \\
&G^{(0,2)}_{\lambda,\lambda'} & \quad & G^{(0,1)}_{\lambda,\lambda'} & \quad & G^{(0,0)}_{\lambda,\lambda'} & \quad & G^{(1,0)}_{\lambda,\lambda'} & \quad & G^{(2,0)}_{\lambda,\lambda'}
\end{align*}
\]

and

\[
\begin{align*}
&W^{(3,3)^\dagger}_{\lambda,\lambda'} : G^{(0,0)}_{\lambda,\lambda'} & \quad & W^{(3,3)}_{\lambda,\lambda'} : G^{(0,1)}_{\lambda,\lambda'} & \quad & W^{(3,3)}_{\lambda,\lambda'} : G^{(0,0)}_{\lambda,\lambda'} & \quad & W^{(1,0)}_{\lambda,\lambda'} & \quad & W^{(2,0)}_{\lambda,\lambda'} \\
&G^{(0,2)}_{\lambda,\lambda'} & \quad & G^{(0,1)}_{\lambda,\lambda'} & \quad & G^{(0,0)}_{\lambda,\lambda'} & \quad & G^{(1,0)}_{\lambda,\lambda'} & \quad & G^{(2,0)}_{\lambda,\lambda'}
\end{align*}
\]

where a horizontal arrow from \( G_{\lambda,\lambda'} \) to \( G'_{\lambda,\lambda'} \) indicates that \( XG_{\lambda,\lambda'} = \lambda G_{\lambda,\lambda'} + G'_{\lambda,\lambda'} \), while a vertical arrow from \( G_{\lambda,\lambda'} \) to \( G'_{\lambda,\lambda'} \) indicates that \( YG_{\lambda,\lambda'} = \lambda G_{\lambda,\lambda'} + G'_{\lambda,\lambda'} \). We note that these five connected Jordan webs are all subwebs of \( W^{(3,3)}_{\lambda,\lambda'} \). A web of the type \( W^{(1,1)}_{\lambda,\lambda'} \) is merely a common eigenvector of \( X \) and \( Y \) not appearing in any non-trivial Jordan chain.

To describe the matrix realizations of the restrictions of \( X \) and \( Y \) to these connected Jordan webs, we introduce an ordered basis \( B^{(\ell,\ell')}_{\lambda,\lambda'} \) of generalized eigenvectors associated to \( W^{(\ell,\ell')}_{\lambda,\lambda'} \). As always, we favour \( Y \) and thus introduce

\[
B^{(1,1)}_{\lambda,\lambda'} = \{ G^{(0,0)}_{\lambda,\lambda'} \}, \quad B^{(1,3)}_{\lambda,\lambda'} = \{ G^{(0,0)}_{\lambda,\lambda'}, G^{(0,1)}_{\lambda,\lambda'}, G^{(0,2)}_{\lambda,\lambda'} \}, \quad B^{(3,1)}_{\lambda,\lambda'} = \{ G^{(0,0)}_{\lambda,\lambda'}, G^{(1,0)}_{\lambda,\lambda'}, G^{(2,0)}_{\lambda,\lambda'} \}
\]

\[
B^{(3,3)}_{\lambda,\lambda'} = \{ G^{(0,0)}_{\lambda,\lambda'}, G^{(0,1)}_{\lambda,\lambda'}, G^{(0,2)}_{\lambda,\lambda'}, G^{(0,2)}_{\lambda,\lambda'} \}
\]

The corresponding matrix realizations are denoted by \( X^{(\ell,\ell')}_{\lambda,\lambda'} \) and \( Y^{(\ell,\ell')}_{\lambda,\lambda'} \) and are given by

\[
X^{(1,1)}_{\lambda,\lambda'} = \lambda I_{1\times 1}, \quad X^{(1,3)}_{\lambda,\lambda'} = \lambda I_{3\times 3}, \quad X^{(3,1)}_{\lambda,\lambda'} = J_{3,3}
\]

\[
X^{(3,3)^\dagger}_{\lambda,\lambda'} = \begin{pmatrix}
\lambda & 0 & 0 & 1 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 1
\end{pmatrix}, \quad X^{(3,3)}_{\lambda,\lambda'} = \begin{pmatrix}
\lambda & I & 0 \\
0 & \lambda & I \\
0 & 0 & \lambda I
\end{pmatrix}
\]

and

\[
Y^{(1,1)}_{\lambda,\lambda'} = \lambda' I_{1\times 1}, \quad Y^{(1,3)}_{\lambda,\lambda'} = J_{3,3}, \quad Y^{(3,1)}_{\lambda,\lambda'} = \lambda' I_{3\times 3}
\]

\[
Y^{(3,3)^\dagger}_{\lambda,\lambda'} = \text{diag}(J_{3,3}, \lambda', \lambda'), \quad Y^{(3,3)}_{\lambda,\lambda'} = \text{diag}(J_{3,3}, J_{3,3}, J_{3,3})
\]

In (5.10), the nine-dimensional matrix \( X^{(3,3)}_{\lambda,\lambda'} \) is written as a three-dimensional matrix whose entries are three-dimensional matrices.
5.1.2 Auxiliary fusion matrices

With respect to $\hat{X}$ or $\hat{Y}$ separately, we only encounter Jordan chains of length 1, 2 or 3. As we will demonstrate in Section [5.3.1], six different types of connected Jordan webs arise in the description of the common generalized eigenvectors of $\hat{X}$ and $\hat{Y}$.

\[
\hat{X} \hat{G}_{\lambda,\lambda'}^{(0,\ell')} = \lambda \hat{G}_{\lambda,\lambda'}^{(0,\ell')}, \quad \hat{X} \hat{G}_{\lambda,\lambda'}^{(1,\ell')} = \lambda \hat{G}_{\lambda,\lambda'}^{(1,\ell')} + \hat{G}_{\lambda,\lambda'}^{(0,\ell')}, \quad \hat{X} \hat{G}_{\lambda,\lambda'}^{(2,\ell')} = \lambda \hat{G}_{\lambda,\lambda'}^{(2,\ell')} + \hat{G}_{\lambda,\lambda'}^{(1,\ell')}
\]
\[
\hat{Y} \hat{G}_{\lambda,\lambda'}^{(0,\ell')} = \lambda' \hat{G}_{\lambda,\lambda'}^{(0,\ell')}, \quad \hat{Y} \hat{G}_{\lambda,\lambda'}^{(1,\ell')} = \lambda' \hat{G}_{\lambda,\lambda'}^{(1,\ell')} + \hat{G}_{\lambda,\lambda'}^{(0,\ell')}, \quad \hat{Y} \hat{G}_{\lambda,\lambda'}^{(2,\ell')} = \lambda' \hat{G}_{\lambda,\lambda'}^{(2,\ell')} + \hat{G}_{\lambda,\lambda'}^{(1,\ell')}
\]
(5.12)

Three of these Jordan webs are inherited from $X$ and $Y$ as $W_{\lambda,\lambda'}^{(1,1)} \to \hat{W}_{\lambda,\lambda'}^{(1,1)}$, $W_{\lambda,\lambda'}^{(3,1)} \to \hat{W}_{\lambda,\lambda'}^{(3,1)}$ and $W_{\lambda,\lambda'}^{(1,3)} \to \hat{W}_{\lambda,\lambda'}^{(1,3)}$, that is,

\[
\hat{W}_{\lambda,\lambda'}^{(1,1)} : \hat{G}_{\lambda,\lambda'}^{(0,0)} \quad \hat{W}_{\lambda,\lambda'}^{(1,3)} : \hat{G}_{\lambda,\lambda'}^{(0,1)} \quad \hat{W}_{\lambda,\lambda'}^{(3,1)} : \hat{G}_{\lambda,\lambda'}^{(0,0)} \quad \hat{G}_{\lambda,\lambda'}^{(1,0)} \quad \hat{G}_{\lambda,\lambda'}^{(2,0)}
\]
(5.13)

The Jordan web $W_{\lambda,\lambda'}^{(3,3)^{\dagger}}$, on the other hand, breaks down as only the quotient $W_{\lambda,\lambda'}^{(3,3)^{\dagger}}/G_{\lambda,\lambda'}^{(0,0)}$ survives the reduction to $\hat{X}$ and $\hat{Y}$ in the sense that $W_{\lambda,\lambda'}^{(3,3)^{\dagger}} \to \hat{W}_{\lambda,\lambda'}^{(2,1)} \cup \hat{W}_{\lambda,\lambda'}^{(1,2)}$, where

\[
\hat{W}_{\lambda,\lambda'}^{(1,2)} : \quad \hat{W}_{\lambda,\lambda'}^{(2,1)} : \quad \hat{W}_{\lambda,\lambda'}^{(0,0)} \quad \hat{G}_{\lambda,\lambda'}^{(1,0)} \quad \hat{G}_{\lambda,\lambda'}^{(0,0)}
\]
(5.14)

Finally, only the quotient $W_{\lambda,\lambda'}^{(3,3)^{\dagger}}/G_{\lambda,\lambda'}^{(0,0)}$ survives the reduction of the Jordan web $W_{\lambda,\lambda'}^{(3,3)}$ to $\hat{X}$ and $\hat{Y}$. This eight-dimensional connected Jordan web $\hat{W}_{\lambda,\lambda'}^{(3,3)}$ is given by

\[
\hat{W}_{\lambda,\lambda'}^{(3,3)} : \hat{G}_{\lambda,\lambda'}^{(0,0)} \quad \hat{G}_{\lambda,\lambda'}^{(1,0)} \quad \hat{G}_{\lambda,\lambda'}^{(2,0)}
\]
(5.15)

where, with reference to (5.12), $\hat{G}_{\lambda,\lambda'}^{(0,0)} \equiv 0$.

To describe the matrix realizations of the restrictions of $\hat{X}$ and $\hat{Y}$ to these connected Jordan webs, we introduce the $\hat{Y}$-favouring ordered bases $\hat{B}_{\lambda,\lambda'}^{(\ell,\ell')}$ of generalized eigenvectors associated to $\hat{W}_{\lambda,\lambda'}^{(\ell,\ell')}$ by

\[
\hat{B}_{\lambda,\lambda'}^{(1,1)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,0)} \}, \quad \hat{B}_{\lambda,\lambda'}^{(1,3)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,0)}, \hat{G}_{\lambda,\lambda'}^{(0,1)}, \hat{G}_{\lambda,\lambda'}^{(0,2)} \}, \quad \hat{B}_{\lambda,\lambda'}^{(3,1)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,0)}, \hat{G}_{\lambda,\lambda'}^{(1,0)}, \hat{G}_{\lambda,\lambda'}^{(2,0)} \}
\]
\[
\hat{B}_{\lambda,\lambda'}^{(1,2)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,0)}, \hat{G}_{\lambda,\lambda'}^{(0,1)} \}, \quad \hat{B}_{\lambda,\lambda'}^{(2,1)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,0)}, \hat{G}_{\lambda,\lambda'}^{(1,0)} \}
\]
\[
\hat{B}_{\lambda,\lambda'}^{(3,3)} = \{ \hat{G}_{\lambda,\lambda'}^{(0,1)}, \hat{G}_{\lambda,\lambda'}^{(0,2)}, \hat{G}_{\lambda,\lambda'}^{(1,1)}, \hat{G}_{\lambda,\lambda'}^{(1,2)}, \hat{G}_{\lambda,\lambda'}^{(2,0)}, \hat{G}_{\lambda,\lambda'}^{(2,1)}, \hat{G}_{\lambda,\lambda'}^{(2,2)} \}
\]
(5.16)
The corresponding matrix realizations are denoted by $\hat{X}^{(\ell,\ell)}_{λ,λ'}$ and $\hat{Y}^{(\ell,\ell)}_{λ,λ'}$ and are given by

$$
\hat{X}^{(1,1)}_{λ,λ'} = \lambda I_{1 \times 1}, \quad \hat{X}^{(1,3)}_{λ,λ'} = \lambda I_{3 \times 3}, \quad \hat{X}^{(3,1)}_{λ,λ'} = J_{λ,3}
$$

$$
\hat{X}^{(1,2)}_{λ,λ'} = \lambda I_{2 \times 2}, \quad \hat{X}^{(2,1)}_{λ,λ'} = J_{λ,2}, \quad \hat{X}^{(3,3)}_{λ,λ'} = 
\begin{pmatrix}
\lambda & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 1 & 0 & 0 \\
\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 1
\end{pmatrix}
$$

(5.17)

and

$$
\hat{Y}^{(1,1)}_{λ,λ'} = \lambda' I_{1 \times 1}, \quad \hat{Y}^{(1,3)}_{λ,λ'} = J_{λ',3}, \quad \hat{Y}^{(3,1)}_{λ,λ'} = \lambda' I_{3 \times 3}
$$

$$
\hat{Y}^{(1,2)}_{λ,λ'} = J_{λ',2}, \quad \hat{Y}^{(2,1)}_{λ,λ'} = \lambda' I_{2 \times 2}, \quad \hat{Y}^{(3,3)}_{λ,λ'} = \text{diag}(J_{λ',2}, J_{λ',3}, J_{λ',3})
$$

(5.18)

The eight-dimensional matrix $\hat{X}^{(3,3)}_{λ,λ'}$ in (5.17) is obtained from the nine-dimensional matrix $X^{(3,3)}_{λ,λ'}$ in (5.10) by elimination of the first row and column. By similar eliminations, the eight-dimensional matrix $\hat{Y}^{(3,3)}_{λ,λ'}$ in (5.18) is obtained from the nine-dimensional matrix $Y^{(3,3)}_{λ,λ'}$ in (5.11), while the four-dimensional matrices $\text{diag}(\hat{X}^{(1,2)}_{λ,λ'}, \hat{X}^{(2,1)}_{λ,λ'})$ and $\text{diag}(\hat{Y}^{(1,2)}_{λ,λ'}, \hat{Y}^{(2,1)}_{λ,λ'})$ follow from the five-dimensional matrices $X^{(3,3)}_{λ,λ'}$ and $Y^{(3,3)}_{λ,λ'}$, respectively.

5.2 Fundamental fusion algebra

In the following, we write

$$
\beta_i = \frac{2 \cos i \pi}{p}, \quad i \in \mathbb{Z}_{0,p}, \quad \beta'_j = \frac{2 \cos j \pi}{p'}, \quad j \in \mathbb{Z}_{0,p'}
$$

(5.19)

We also recall our label conventions $a \in \mathbb{Z}_{1,p-1}$ and $b \in \mathbb{Z}_{1,p'-1}$ introduced in (2.4).

5.2.1 Fundamental fusion matrices

Due to the block-diagonal structure (4.16) of the fundamental fusion matrix $Y$, its spectral decomposition follows readily from the spectral decompositions of $T_{p'}$ and $E_{p'}$ discussed in Section 3.2.2 and Section 3.2.3 respectively. The Jordan canonical form of $Y$ thus consists of $2p - 1$ rank-1 blocks of eigenvalue $\beta'_j$ for every $j \in \{0, p'\}$, $p$ rank-1 blocks of eigenvalue $\beta'_0$ for every $b \in \mathbb{Z}_{1,p'-1}$, and $2p - 1$ rank-3 blocks of eigenvalue $\beta'_b$ for every $b \in \mathbb{Z}_{1,p'-1}$. Likewise, the Jordan canonical form of $X$ consists of $2p' - 1$ rank-1 blocks of eigenvalue $\beta$ for every $i \in \{0, p\}$, $p'$ rank-1 blocks of eigenvalue $\beta_a$ for every $a \in \mathbb{Z}_{1,p-1}$, and $2p' - 1$ rank-3 blocks of eigenvalue $\beta_a$ for every $a \in \mathbb{Z}_{1,p-1}$.

To characterize the connected components of the Jordan web of the complete set of common generalized eigenvectors of $X$ and $Y$, we choose to work in the $Y$-favouring basis (4.15). A generalized vector $G^{(\ell,\ell)}_{λ,λ'}$ can thus be written as a $(2p - 1)$-dimensional vector whose $p - 1$ upper entries are $(3p' - 1)$-dimensional vectors of the type $T$ appearing in Section 3.2.2, while the $p$ lower entries are $(4p' - 2)$-dimensional vectors of the type $E$ appearing in Section 3.2.3.
The connected subwebs of the type $W^{(1,1)}_{\lambda, \lambda'}$ are given by the following eigenvectors

$$W^{(1,1)}_{\beta_i, \beta_j'} : \quad G^{(0,0)}_{\beta_i, \beta_j'} = \begin{pmatrix} f_1(\beta_i)T_j \\ \vdots \\ f_{p-1}(\beta_i)T_j \\ f_p(\beta_i)E_j \\ f_{2p-1}(\beta_i)E_j \end{pmatrix}, \quad i + j \text{ even}, \quad i \in \{0, p\}, \quad j \in \{0, p'\} \quad (5.20)$$

or

$$W^{(1,1)}_{\beta_i, \beta_j} : \quad G^{(0,0)}_{\beta_i, \beta_j} = \begin{pmatrix} f_1(\beta_i)T_j \\ \vdots \\ f_{p-1}(\beta_i)T_j \\ f_p(\beta_i)E_j \\ f_{2p-1}(\beta_i)E_j \end{pmatrix} = \begin{pmatrix} f_1(\beta_i)T_j \\ \vdots \\ f_{p-1}(\beta_i)T_j \\ 0 \\ 0 \end{pmatrix}, \quad a + j \text{ odd}, \quad j \in \{0, p'\} \quad (5.21)$$

or

$$W^{(1,1)}_{\beta_i, \beta'_b} : \quad G^{(0,0)}_{\beta_i, \beta'_b} = \begin{pmatrix} f_1(\beta_i)T^{(0)}_b \\ \vdots \\ f_{p-1}(\beta_i)T^{(0)}_b \\ f_p(\beta_i)E_b \\ f_{2p-1}(\beta_i)E_b \end{pmatrix}, \quad i + b \text{ odd}, \quad i \in \{0, p\} \quad (5.22)$$

The connected subwebs of the type $W^{(1,3)}_{\lambda, \lambda'}$ consist of the following generalized eigenvectors

$$W^{(1,3)}_{\beta_i, \beta'_b} : \quad G^{(0,\ell')}_{\beta_i, \beta'_b} = \begin{pmatrix} f_1(\beta_i)T^{(\ell')}_b \\ \vdots \\ f_{p-1}(\beta_i)T^{(\ell')}_b \\ f_p(\beta_i)E^{(\ell')}_b \\ f_{2p-1}(\beta_i)E^{(\ell')}_b \end{pmatrix}, \quad i + b \text{ even}, \quad i \in \{0, p\}, \quad \ell' \in \mathbb{Z}_{0,2} \quad (5.23)$$
The connected subwebs of the type \( W^{(3,1)}_{\lambda, \lambda'} \) consist of the following generalized eigenvectors

\[
W^{(3,1)}_{\beta_a, \beta_j'} : \quad G^{(\ell, 0)}_{\beta_a, \beta_j'} = \begin{pmatrix}
\frac{1}{\pi} f_1^{(\ell)}(\beta_a) T_j \\
\vdots \\
\frac{1}{\pi} f_{p-1}^{(\ell)}(\beta_a) T_j \\
\frac{1}{\pi} f_p^{(\ell)}(\beta_a) E_j \\
\vdots \\
\frac{1}{\pi} f_{2p-1}^{(\ell)}(\beta_a) E_j
\end{pmatrix}, \quad a + j \text{ even}, \quad j \in \{0, p'\}, \quad \ell \in \mathbb{Z}_{0, 2}
\] (5.24)

The connected subwebs of the type \( W^{(3,3)^\dagger}_{\lambda, \lambda'} \) consist of the following generalized eigenvectors

\[
W^{(3,3)^\dagger}_{\beta_a, \beta_b'} : \quad G^{(\ell, 0)}_{\beta_a, \beta_b'} = \begin{pmatrix}
\frac{1}{\pi} f_1^{(\ell)}(\beta_a) T_b^{(0)} \\
\vdots \\
\frac{1}{\pi} f_{p-1}^{(\ell)}(\beta_a) T_b^{(0)} \\
\frac{1}{\pi} f_p^{(\ell)}(\beta_a) E_b \\
\vdots \\
\frac{1}{\pi} f_{2p-1}^{(\ell)}(\beta_a) E_b
\end{pmatrix}, \quad G^{(0, \ell')}_{\beta_a, \beta_b'} = \begin{pmatrix}
f_1(\beta_a) T_b^{(\ell')}
\vdots 
 f_{p-1}(\beta_a) T_b^{(\ell')}
 f_p(\beta_a) E_b^{(\ell')}
\vdots 
 f_{2p-1}(\beta_a) E_b^{(\ell')}
\end{pmatrix}, \quad a + b \text{ odd}, \quad \ell, \ell' \in \mathbb{Z}_{0, 2}
\] (5.25)

where (3.26) ensures consistency of the two expressions for \( G^{(0,0)}_{\beta_a, \beta_b'} \). Finally, the connected subwebs of the type \( W^{(3,3)}_{\lambda, \lambda'} \) consist of the following generalized eigenvectors

\[
W^{(3,3)}_{\beta_a, \beta_b'} : \quad G^{(\ell, \ell')}_{\beta_a, \beta_b'} = \begin{pmatrix}
\frac{1}{\pi} f_1^{(\ell)}(\beta_a) T_b^{(\ell')}
\vdots 
 f_{p-1}^{(\ell)}(\beta_a) T_b^{(\ell')}
 f_p^{(\ell)}(\beta_a) E_b^{(\ell')}
\vdots 
 f_{2p-1}^{(\ell)}(\beta_a) E_b^{(\ell')}
\end{pmatrix}, \quad a + b \text{ even}, \quad \ell, \ell' \in \mathbb{Z}_{0, 2}
\] (5.26)

Using properties of the \( T \) and \( E \) vectors as generalized eigenvectors of \( T_{\rho'} \) and \( E_{\rho'} \), together with (4.20) and (4.21), in particular, it is straightforward to prove that the vectors given in (5.20) through (5.26) indeed correspond to the Jordan webs (5.7) and (5.8) consistent with (5.6). We also note that the number \( \mathcal{N}^{(\ell, \ell')} \) of connected Jordan webs of the type \( W^{(\ell, \ell')} \) is given by

\[
\mathcal{N}^{(1, 1)} = \rho + \rho', \quad \mathcal{N}^{(1, 3)} = \rho' - 1, \quad \mathcal{N}^{(3, 1)} = \rho - 1, \quad \mathcal{N}^{(3, 3)^\dagger} = \mathcal{N}^{(3, 3)} = \frac{1}{2}(\rho - 1)(\rho' - 1)
\] (5.27)

consistent with the total number \( 2p + 2p' \) of generalized eigenvectors. In Appendix B we list the connected Jordan subwebs \( W^{(\ell, \ell')}_{\beta_i, \beta_j} \) with respect to the labeling \( i, j \) of the corresponding eigenvalues.

The similarity matrix \( Q \) appearing in (5.11) is constructed by concatenating the common generalized eigenvectors of \( X \) and \( Y \) according to any ordering of the ordered bases (5.9). The permutation matrix
Here we determine the upper-triangular block-diagonal matrix $\{B_{\lambda,\lambda'}^{(1,1)}\} \cup \{B_{\lambda,\lambda'}^{(1,3)}\} \cup \{B_{\lambda,\lambda'}^{(3,1)}\} \cup \{B_{\lambda,\lambda'}^{(3,3)}\}$

\begin{equation}
\{B_{\lambda,\lambda'}^{(1,1)}\} \cup \{B_{\lambda,\lambda'}^{(1,3)}\} \cup \{B_{\lambda,\lambda'}^{(3,1)}\} \cup \{B_{\lambda,\lambda'}^{(3,3)}\} \quad (5.28)
\end{equation}

such that every set (of generalized eigenvectors) of the type $B_{\lambda,\lambda'}^{(1,1)}$ comes before every set (of generalized eigenvectors) of the type $B_{\lambda,\lambda'}^{(1,3)}$, and so on. The Jordan canonical forms $J_X$ and $J_Y$ in (5.1) are then of the form

\begin{equation}
J_X = \text{diag}\left(\lambda, \ldots, \lambda, \ldots, J_{\lambda,3}, \ldots, \text{diag}(J_{\lambda,3}, \lambda, \lambda), \ldots, \text{diag}(J_{\lambda,3}, J_{\lambda,3}, J_{\lambda,3}), \ldots\right)
\end{equation}

\begin{equation}
J_Y = \text{diag}\left(\lambda', \ldots, J_{\lambda',3}, \ldots, \lambda', \ldots, \text{diag}(J_{\lambda',3}, \lambda', \lambda'), \ldots, \text{diag}(J_{\lambda',3}, J_{\lambda',3}, J_{\lambda',3}), \ldots\right)
\end{equation}

We stress that the eigenvalues $\lambda$ and $\lambda'$ vary in these expressions but are always of the form (5.19). The corresponding permutation matrix $P$ is a block-diagonal matrix whose blocks are of dimension 1, 3, 5 or 9, corresponding to the dimensions of the sets $B_{\lambda,\lambda'}^{(\ell,\ell')}$. By a similarity transformation (5.1), these $P$-blocks convert the blocks in $J_X$ into the corresponding upper-triangular matrices $X_{\lambda,\lambda'}^{(\ell,\ell')}$ in (5.10). The $P$-blocks of dimension 1 or 3 are identity matrices, while the $P$-blocks of dimension 5 or 9 are the symmetric permutation matrices

\begin{equation}
P_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \\
P_9 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{equation}

(5.31)

where it is recalled that a symmetric permutation matrix equals its inverse. The symmetric permutation matrix $P$ is thus given by

\begin{equation}
P = \text{diag}\left(1, \ldots, 1, P_5, \ldots, P_5, P_9, \ldots, P_9\right)
\end{equation}

(5.32)

As actions on the connected Jordan webs $W_{\lambda,\lambda'}^{(3,3)\dagger}$ and $W_{\lambda,\lambda'}^{(3,3)}$, these permutations reflect the vertices (generalized eigenvectors) with respect to the line from south-west to north-east through $G_{\lambda,\lambda'}^{(0,0)}$. $P_5$ and $P_9$ thus have one and three fix-points, respectively, in accord with the numbers of units on their diagonals.

### 5.2.2 General fusion matrices

Here we determine the upper-triangular block-diagonal matrix $Q^{-1}N_\mu Q$ obtained from the general fusion matrix $N_\mu$ by a similarity transformation with respect to $Q$ defined according to (5.28). From
and Section 5.2.1, we have that

\[
Q^{-1}N_\mu Q = \text{diag} \left( g(X_{\lambda,\lambda'}^{(1,1)})h(Y_{\lambda,\lambda'}^{(1,1)}), \ldots, g(X_{\lambda,\lambda'}^{(1,3)})h(Y_{\lambda,\lambda'}^{(1,3)}), \ldots, g(X_{\lambda,\lambda'}^{(3,1)})h(Y_{\lambda,\lambda'}^{(3,1)}), \ldots \right)
\]

\[
\frac{1}{2^{(p-1)(p'-1)}} g(X_{\lambda,\lambda'}^{(3,3)\dagger})h(Y_{\lambda,\lambda'}^{(3,3)\dagger}), \ldots, g(X_{\lambda,\lambda'}^{(3,3)})h(Y_{\lambda,\lambda'}^{(3,3)}), \ldots \right)
\]

(5.33)

where, for simplicity, \( g(z) = \text{pol}_{\mu}^{(z)}(z) \) and \( h(z) = \text{pol}_{\mu}^{(y)}(z) \). In a given block \( g(X_{\lambda,\lambda'}^{(\ell,\ell')}h(Y_{\lambda,\lambda'}^{(\ell,\ell')}) \), the pairs of labels \( \lambda, \lambda' \) (eigenvalues of \( X \) and \( Y \)) of \( X_{\lambda,\lambda'}^{(\ell,\ell')} \) and \( Y_{\lambda,\lambda'}^{(\ell,\ell')} \) are the same, while they generally vary from block to block. For the five types of blocks, we have

\[
g(X_{\lambda,\lambda'}^{(1,1)})h(Y_{\lambda,\lambda'}^{(1,1)}) = g(\lambda)h(\lambda') \]

\[
g(X_{\lambda,\lambda'}^{(1,3)})h(Y_{\lambda,\lambda'}^{(1,3)}) = g(\lambda)h(J_{\lambda,3}), \quad g(X_{\lambda,\lambda'}^{(3,1)})h(Y_{\lambda,\lambda'}^{(3,1)}) = g(J_{\lambda,3})h(\lambda') \]

\[
g(X_{\lambda,\lambda'}^{(3,3)\dagger})h(Y_{\lambda,\lambda'}^{(3,3)\dagger}) = \begin{pmatrix}
0 & g(\lambda)h(\lambda') & \frac{1}{2}g(\lambda)h''(\lambda') \\
0 & g(\lambda)h(\lambda'') & \frac{1}{2}g(\lambda)h''(\lambda') \\
0 & 0 & g(\lambda)h(\lambda') \end{pmatrix}
\]

\[
g(X_{\lambda,\lambda'}^{(3,3)})h(Y_{\lambda,\lambda'}^{(3,3)}) = g(J_{\lambda,3}) \times h(J_{\lambda,3})
\]

(5.34)

where \( g(J_{\lambda,3}) \times h(J_{\lambda,3}) \) denotes the nine-dimensional Kronecker product of the two three-dimensional matrices \( g(J_{\lambda,3}) \) and \( h(J_{\lambda,3}) \). It is recalled that, for a function \( f \) expandable as a power series in its argument, we have

\[
f(J_{\lambda,2}) = \begin{pmatrix}
f(\lambda) & f'(\lambda) \\
0 & f(\lambda) \end{pmatrix}, \quad f(J_{\lambda,3}) = \begin{pmatrix}
f(\lambda) & f'(\lambda) & \frac{1}{2}f''(\lambda) \\
0 & f(\lambda) & f'(\lambda) \\
0 & 0 & f(\lambda) \end{pmatrix}
\]

(5.35)

whose ranks depend on \( f'(\lambda) \) and \( f''(\lambda) \). The first of these matrix expressions will be relevant in (5.57) below. This completes the description of the upper-triangular block-diagonal matrix \( Q^{-1}N_\mu Q \).

### 5.2.3 \( W \)-extended critical percolation \( \mathcal{WLM}(2,3) \)

In the case of \( \mathcal{WLM}(2,3) \), the eigenvalues of \( X \) and \( Y \) are

\[
\beta_i = 2 \cos \frac{i\pi}{2}, \quad i \in \{0, 1, 2\}, \quad \beta_j' = 2 \cos \frac{j\pi}{3}, \quad j \in \{0, 1, 2, 3\}
\]

(5.36)

respectively. As displayed in Figure 11, the connected components \( \mathcal{W}_{\beta_i,\beta_j'}^{(\ell,\ell')} \) of the Jordan web associated to the fundamental fusion algebra are neatly organized with respect to the labels \( i \) and \( j \). An example of an ordering of the common generalized eigenvectors of \( X \) and \( Y \) respecting (5.28) is

\[
G_{2,2}^{(0,0)}; G_{0,2}^{(0,0)}; G_{2,-2}^{(0,0)}; G_{2,1}^{(0,0)}; G_{-2,1}^{(0,0)}; G_{2,-1}^{(0,0)}; G_{0,-1}^{(0,1)}; G_{-2,-1}^{(0,1)}; G_{2,-1}^{(0,2)}; G_{0,-2}^{(0,0)}; G_{-2,1}^{(0,2)}; G_{0,1}^{(0,0)}; G_{0,1}^{(1,0)}; G_{0,1}^{(1,1)}; G_{0,1}^{(1,2)}; G_{0,1}^{(2,0)}; G_{0,1}^{(2,1)}; G_{0,1}^{(2,2)}
\]

(5.37)
where we have introduced the abbreviations

\[ W^{(l,e)} = W^{(\beta_3, \beta_4)} \]

of the Jordan web associated to the fundamental fusion algebra of \( W \)-extended critical percolation \( \mathcal{WLM}(2,3) \). The two \( \mathcal{O} \)'s indicate that there are no common generalized eigenvectors corresponding to the pair \( (\beta_0, \beta_3^*) = (2, -2) \) or to the pair \( (\beta_2, \beta_3^*) = (-2, -2) \) of eigenvalues of the fundamental fusion matrices \( X \) and \( Y \).

We define the similarity matrix \( Q \) by concatenating these vectors in the order given. Modulo a similarity transformation, \( Q \) converts \( X \) and \( Y \) into the Jordan canonical forms

\[
J_X = P^{-1}Q^{-1}XQP = \text{diag}(2, 0, -2, 2, -2, 2, 2, 2, -2, -2, -2, -2, J_{0,3}, J_{0,3}, 0, 0, J_{0,3}, J_{0,3}, J_{0,3})
\]

\[
J_Y = Q^{-1}YQ = \text{diag}(2, 2, 2, 1, 1, J_{-1,3}, J_{-1,3}, -2, -2, -2, J_{-1,3}, -1, -1, J_{1,3}, J_{1,3}, J_{1,3})
\]

(5.38)

where \( P \) is the symmetric permutation matrix

\[
P = \text{diag}(I_{14} \times 14, P_5, P_5)
\]

(5.39)

The fusion matrix \( N_\mu \) associated to the general module \( \mu \in \mathcal{I}_f \) is polynomial in \( X \) and \( Y \)

\[
N_{1,1} = I, \quad N_{1,2} = Y, \quad N_{1,3} = Y^2 - I, \quad N_{2,1} = X, \quad N_{2,2} = XY, \quad N_{2,3} = X(Y^2 - I)
\]

\[
N_{1,6} = \frac{1}{2}Y(Y^2 - I)(Y^2 - 3I), \quad N_{2,6} = \frac{1}{2}XY(Y^2 - I)(Y^2 - 3I)
\]

\[
N_{1,4} = \frac{1}{2}X(X^2 - 2I), \quad N_{4,2} = \frac{1}{2}X(X^2 - 2I)Y
\]

\[
N_{1,3} = Y(Y^2 - I), \quad N_{2,3} = (Y^2 - I)(Y^2 - 2I)
\]

\[
N_{1,6}^{0,1} = \frac{1}{2}Y^2(Y^2 - I)(Y^2 - 3I), \quad N_{1,6}^{0,2} = \frac{1}{2}Y(Y^2 - I)(Y^2 - 2I)(Y^2 - 3I)
\]

\[
N_{2,3}^{0,1} = XY(Y^2 - I), \quad N_{2,3}^{0,2} = X(Y^2 - I)(Y^2 - 2I)
\]

\[
N_{2,6}^{0,1} = \frac{1}{2}XY^2(Y^2 - I)(Y^2 - 3I), \quad N_{2,6}^{0,2} = \frac{1}{2}XY(Y^2 - I)(Y^2 - 2I)(Y^2 - 3I)
\]

\[
N_{2,1}^{1,0} = X^2, \quad N_{2,2}^{1,0} = XY, \quad N_{2,3}^{1,0} = X^2(Y^2 - I)
\]

\[
N_{4,1}^{1,0} = \frac{1}{2}X^2(X^2 - 2I), \quad N_{4,2}^{1,0} = \frac{1}{2}XY^2(X^2 - 2I), \quad N_{4,3}^{1,0} = \frac{1}{2}X^2(Y^2 - I)(Y^2 - 2I)
\]

\[
N_{4,1}^{1,1} = \frac{1}{2}X^2(X^2 - 2I)Y(Y^2 - I), \quad N_{4,2}^{1,1} = \frac{1}{2}XY^2(X^2 - 2I)(Y^2 - I)
\]

(5.40)

where we have introduced the abbreviations \( N_{r,s} = N_{(r,s)}^{\mu} \), \( N_{r,s} = N_{W(\Delta_{r,s})} \) and \( N_{r,s}^{\alpha, \beta} = N_{(K_{r,s}^{\alpha, \beta})}^{\mu} \).

The similarity transformation of \( N_\mu \) is the block-diagonal matrix \( Q^{-1}N_\mu Q \) whose blocks are upper-

| \( i \backslash j \) | 0   | 1   | 2   | 3   |
|---------------|-----|-----|-----|-----|
| 0  | \( W_{2,2}^{(1,1)} \) | \( W_{2,1}^{(1,1)} \) | \( W_{2,-1}^{(1,3)} \) | \( \mathcal{O} \) |
| 1  | \( W_{0,2}^{(1,1)} \) | \( W_{0,1}^{(3,3)} \) | \( W_{0,-1}^{(3,3)} \) | \( W_{0,-2}^{(3,1)} \) |
| 2  | \( W_{-2,2}^{(1,1)} \) | \( W_{-2,1}^{(1,1)} \) | \( W_{-2,-1}^{(1,3)} \) | \( \mathcal{O} \) |
triangular matrices. As illustrations of such block-diagonal matrices, we here consider

\[
Q^{-1}N_{4,2}Q = \text{diag}\begin{pmatrix}
-2 & 2 & 0 \\
0 & -2 & 2 \\
0 & 0 & -2
\end{pmatrix}, \begin{pmatrix}
2 & -2 & 0 \\
0 & 2 & -2 \\
0 & 0 & 2
\end{pmatrix}, \begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
Q^{-1}N_{2,3}^{1,1}Q = \text{diag}\begin{pmatrix}
0 & 8 & -12 \\
0 & 0 & 8 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 8 & -12 \\
0 & 0 & 8 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & -6 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
0, 0, 0, 0, 0, \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \times \begin{pmatrix}
0 & 2 & 3 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{pmatrix}
\]

(5.41)

5.3 Fusion algebra associated with boundary conditions

5.3.1 Auxiliary fusion matrices

Due to the the block-diagonal structure \([4.30]\) of the auxiliary fusion matrix \(\hat{Y}\), its spectral decomposition follows readily from the spectral decompositions of \(C_{\beta_i}\) and \(E_{\lambda_i}\) discussed in Section 3.2.1 and \([3.2.3]\) respectively. The Jordan canonical form of \(\hat{Y}\) thus consists of \(2p - 1\) rank-1 blocks of eigenvalue \(\beta_j\) for every \(j \in \{0, p'\}\), \(p\) rank-1 blocks of eigenvalue \(\beta_b\) for every \(b \in \mathbb{Z}_{1,p'-1}\), \(p - 1\) rank-2 blocks of eigenvalue \(\beta_b\) for every \(b \in \mathbb{Z}_{1,p'-1}\), and \(p\) rank-3 blocks of eigenvalue \(\beta_b\) for every \(b \in \mathbb{Z}_{1,p'-1}\). Likewise, the Jordan canonical form of \(\hat{X}\) consists of \(2p' - 1\) rank-1 blocks of eigenvalue \(\beta_i\) for every \(i \in \{0, p\}\), \(p'\) rank-1 blocks of eigenvalue \(\beta_a\) for every \(a \in \mathbb{Z}_{1,p-1}\), \(p' - 1\) rank-2 blocks of eigenvalue \(\beta_a\) for every \(a \in \mathbb{Z}_{1,p-1}\), and \(p'\) rank-3 blocks of eigenvalue \(\beta_a\) for every \(a \in \mathbb{Z}_{1,p-1}\).

To characterize the connected components of the Jordan web of the complete set of common generalized eigenvectors of \(\hat{X}\) and \(\hat{Y}\), we choose to work in the \(\hat{Y}\)-favouring basis \([1.29]\). A generalized vector \(\hat{G}_{(\ell,\ell')}^{(\ell,\ell')}\) can thus be written as a \((2p - 1)\)-dimensional vector whose \(p - 1\) upper entries are \(2p'\)-dimensional vectors of the type \(\hat{C}\) appearing in Section 3.2.1 while the \(p\) lower entries are \((4p' - 2)\)-dimensional vectors of the type \(\hat{E}\) appearing in Section 3.2.3.

The connected subwebs of the type \(\hat{W}_{\lambda,\lambda'}^{(1,1)}\) are given by the following eigenvectors

\[
\hat{W}_{\beta_i,\beta_j}^{(1,1)} : \hat{G}_{\beta_i,\beta_j}^{(0,0)} = \begin{pmatrix}
f_1(\beta_i)C_j \\
\vdots \\
f_{p-1}(\beta_i)C_j \\
f_p(\beta_i)E_j \\
\vdots \\
f_{2p-1}(\beta_i)E_j
\end{pmatrix}, \quad i + j \text{ even, } i \in \{0, p\}, \quad j \in \{0, p'\}
\]

(5.42)
or

\[
\hat{W}_{\beta_a, \beta_j}^{(1,1)} : \hat{G}^{(0,0)}_{\beta_a, \beta_j} \begin{pmatrix}
  f_1(\beta_a)C_j \\
  \vdots \\
  f_p(\beta_a)E_j \\
  f_{p-1}(\beta_a)E_j \\
  \end{pmatrix} = \begin{pmatrix}
  f_1(\beta_a)C_j \\
  \vdots \\
  f_{p-1}(\beta_a)C_j \\
  0 \\
  \end{pmatrix}, \quad a + j \text{ odd}, \ j \in \{0, p'\} \quad (5.43)
\]

or

\[
\hat{W}_{\beta_i, \beta_b}^{(1,1)} : \hat{G}^{(0,0)}_{\beta_i, \beta_b} \begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
  f_p(\beta_i)E_b \\
  f_{p-1}(\beta_i)E_b \\
  \end{pmatrix}, \quad i + b \text{ odd}, \ i \in \{0, p\} \quad (5.44)
\]

The connected subwebs of the type \(\hat{W}^{(1,3)}_{\lambda, \lambda'}\) consist of the following generalized eigenvectors

\[
\hat{W}^{(1,3)}_{\beta_i, \beta_b} : \hat{G}^{(0,\ell')}_{\beta_i, \beta_b} \begin{pmatrix}
  f_1(\beta_i)C_{b}^{(\ell'-1)} \\
  \vdots \\
  f_p(\beta_i)C_{b}^{(\ell'-1)} \\
  f_{p-1}(\beta_i)E_{b}^{(\ell')} \\
  f_{p-2}(\beta_i)E_{b}^{(\ell')} \\
  \end{pmatrix}, \quad i + b \text{ even}, \ i \in \{0, p\}, \ \ell' \in \mathbb{Z}_{0,2} \quad (5.45)
\]

where \(C_{b}^{(-1)} \equiv 0\). The connected subwebs of the type \(\hat{W}^{(3,1)}_{\lambda, \lambda'}\) consist of the following generalized eigenvectors

\[
\hat{W}^{(3,1)}_{\beta_a, \beta_j} : \hat{G}^{(\ell,0)}_{\beta_a, \beta_j} \begin{pmatrix}
  \frac{1}{\pi} f_1^{(\ell)}(\beta_a)C_j \\
  \vdots \\
  \frac{1}{\pi} f_{p-1}^{(\ell)}(\beta_a)C_j \\
  \frac{1}{\pi} f_1^{(\ell)}(\beta_a)E_j \\
  \vdots \\
  \frac{1}{\pi} f_{2p-1}^{(\ell)}(\beta_a)E_j \\
  \end{pmatrix}, \quad a + j \text{ even}, \ j \in \{0, p'\}, \ \ell \in \mathbb{Z}_{0,2} \quad (5.46)
\]
The connected subwebs of the type $\hat{W}^{(1,2)}_{\lambda,\lambda'}$ consist of the following generalized eigenvectors

$$
\hat{W}^{(1,2)}_{\beta_a,\beta'_b} : \hat{G}_{\beta_a,\beta'_b}^{(0,\ell')} = \left( \begin{array}{c}
 f_1 (\beta_a) C_b^{(\ell')} \\
 \vdots \\
 f_{p-1} (\beta_a) C_b^{(\ell')} \\
 0 \\
 \vdots \\
 0
 \end{array} \right), \quad a + b \text{ odd}, \quad \ell' \in \mathbb{Z}_{0,1} \quad (5.47)
$$

The connected subwebs of the type $\hat{W}^{(2,1)}_{\lambda,\lambda'}$ consist of the following generalized eigenvectors

$$
\hat{W}^{(2,1)}_{\beta_a,\beta'_b} : \hat{G}_{\beta_a,\beta'_b}^{(\ell,0)} = \left( \begin{array}{c}
 0 \\
 \vdots \\
 0 \\
 \frac{1}{(\ell+1)!} f_{p}(\beta_a) E_b^{(\ell+1)} \\
 \vdots \\
 \frac{1}{(\ell+1)!} f_{2p-1}(\beta_a) E_b^{(\ell+1)}
 \end{array} \right), \quad a + b \text{ odd}, \quad \ell \in \mathbb{Z}_{0,1} \quad (5.48)
$$

Finally, the connected subwebs of the type $\hat{W}^{(3,3)}_{\lambda,\lambda'}$ consist of the following generalized eigenvectors

$$
\hat{W}^{(3,3)}_{\beta_a,\beta'_b} : \hat{G}_{\beta_a,\beta'_b}^{(\ell,\ell')} = \left( \begin{array}{c}
 \frac{1}{\pi} f_1 (\beta_a) C_b^{(\ell'-1)} \\
 \vdots \\
 \frac{1}{\pi} f_{p-1} (\beta_a) C_b^{(\ell'-1)} \\
 \frac{1}{\pi} f_{p}(\beta_a) E_b^{(\ell')} \\
 \vdots \\
 \frac{1}{\pi} f_{2p-1}(\beta_a) E_b^{(\ell')}
 \end{array} \right), \quad a + b \text{ even}, \quad \ell, \ell' \in \mathbb{Z}_{0,2}, \quad (\ell, \ell') \neq (0,0) \quad (5.49)
$$

where $C_b^{(-1)} \equiv 0$ as above. Using properties of the $C$ and $E$ vectors as generalized eigenvectors of $C_{\ell'}$ and $E_{\ell'}$, together with (5.3) and (5.21), in particular, it is straightforward to prove that the vectors given in (5.42) through (5.49) indeed correspond to the Jordan webs (5.13), (5.14) and (5.15) consistent with (5.12). We also note that the number $\hat{N}(\ell,\ell')$ of connected Jordan webs of the type $\hat{W}(\ell,\ell')$ is given by

$$
\hat{N}^{(1,1)} = p + p', \quad \hat{N}^{(3,1)} = p - 1, \quad \hat{N}^{(1,3)} = p' - 1, \quad \hat{N}^{(2,1)} = \hat{N}^{(1,2)} = \hat{N}^{(3,3)} = \frac{1}{2} (p - 1) (p' - 1) \quad (5.50)
$$

consistent with the total number (2.4) of generalized eigenvectors.

The similarity matrix $Q$ appearing in (7.2) is constructed by concatenating the common generalized eigenvectors of $X$ and $Y$ according to any ordering of the ordered bases (5.16). The permutation matrix $P$ depends on this choice of ordering, and the degree of convenience of such a choice depends on the intended application. Here we consider a general ordering reflecting the partitioning

$$
\{ B_{\lambda,\lambda'}^{(1,1)} \} \cup \{ B_{\lambda,\lambda'}^{(1,3)} \} \cup \{ B_{\lambda,\lambda'}^{(2,1)} \} \cup \left( \{ B_{\lambda,\lambda'}^{(1,2)} \} \cup \{ B_{\lambda,\lambda'}^{(2,1)} \} \right) \cup \{ B_{\lambda,\lambda'}^{(3,3)} \} \quad (5.51)
$$
such that every set (of generalized eigenvectors) of the type \( \hat{B}_{\lambda,\lambda}^{(1,1)} \) comes before every set (of generalized eigenvectors) of the type \( \hat{B}_{\lambda,\lambda}^{(1,3)} \), and so on. In addition, for every pair \( \lambda, \lambda' \) in \( \{ \hat{B}_{\lambda,\lambda}^{(1,2)} \} \) (or equivalently in \( \{ \hat{B}_{\lambda,\lambda'}^{(2,1)} \} \)), the two vectors \( \hat{G}_{\lambda,\lambda'}^{(0,0)} \) and \( \hat{G}_{\lambda,\lambda'}^{(0,1)} \) in \( \hat{B}_{\lambda,\lambda}^{(1,2)} \) are followed immediately by the two vectors \( \hat{G}_{\lambda,\lambda'}^{(0,0)} \) and \( \hat{G}_{\lambda,\lambda'}^{(1,0)} \) in \( \hat{B}_{\lambda,\lambda'}^{(2,1)} \). The Jordan canonical forms \( J_X \) and \( J_Y \) in (5.2) are then of the form

\[
J_X = \operatorname{diag} \left( \lambda, \ldots, \lambda, J_{\lambda,3}, \ldots, \text{diag} \left( J_{\lambda,2}, \lambda, \lambda, \ldots, \text{diag} \left( J_{\lambda,2}, J_{\lambda,3}, J_{\lambda,3} \right) \right) \right) \tag{5.52}
\]

\[
J_Y = \operatorname{diag} \left( \lambda', \ldots, \lambda, J_{\lambda',3}, \ldots, \text{diag} \left( J_{\lambda',2}, \lambda', \lambda', \ldots, \text{diag} \left( J_{\lambda',2}, J_{\lambda',3}, J_{\lambda',3} \right) \right) \right) \tag{5.53}
\]

We stress that the eigenvalues \( \lambda \) and \( \lambda' \) vary in these expressions but are always of the form (5.19). The corresponding permutation matrix \( \hat{P} \) is a block-diagonal matrix whose blocks are of dimension 1, 3, 4 or 8, corresponding to the dimensions of the sets \( \hat{B}_{\lambda,\lambda'}^{(1,1)} \), \( \hat{B}_{\lambda,\lambda'}^{(1,3)} \), \( \hat{B}_{\lambda,\lambda'}^{(3,1)} \), \( \hat{B}_{\lambda,\lambda'}^{(1,2)} \cup \hat{B}_{\lambda,\lambda'}^{(2,1)} \) and \( \hat{B}_{\lambda,\lambda'}^{(3,3)} \), respectively. By a similarity transformation (5.2), these \( \hat{P} \)-blocks convert the blocks in \( J_X \) into the corresponding upper-triangular matrices \( \hat{X}_{\lambda,\lambda'}^{(1,2)} \) in (5.17) (where \( \hat{X}_{\lambda,\lambda'}^{(1,2)} \) and \( \hat{X}_{\lambda,\lambda'}^{(2,1)} \) are viewed as the single four-dimensional matrix \( \text{diag}(\lambda, \lambda, J_{\lambda,2}) \)). The \( \hat{P} \)-blocks of dimension 1 or 3 are identity matrices, while the \( \hat{P} \)-blocks of dimension 4 or 8 are the symmetric permutation matrices

\[
\hat{P}_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \hat{P}_8 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.54}
\]

The symmetric permutation matrix \( \hat{P} \) is thus given by

\[
\hat{P} = \operatorname{diag} \left( 1, \ldots, 1, \hat{P}_4, \hat{P}_4, \hat{P}_4, \ldots, \hat{P}_8, \ldots, \hat{P}_8 \right) \tag{5.55}
\]

Acting on the non-connected Jordan web \( \hat{W}_{\lambda,\lambda'}^{(1,2)} \cup \hat{W}_{\lambda,\lambda'}^{(2,1)} \), the permutation matrix \( \hat{P}_4 \) interchanges the two connected components. As an action on the connected Jordan webs \( \hat{W}_{\lambda,\lambda'}^{(3,3)} \), \( \hat{P}_8 \) reflects the vertices (generalized eigenvectors) with respect to the line from south-west to north-east through \( \hat{G}_{\lambda,\lambda'}^{(1,1)} \) and \( \hat{G}_{\lambda,\lambda'}^{(2,2)} \). \( \hat{P}_8 \) thus has two fix-points in accord with the two units on the diagonal.

By eliminating the first row and column of the permutation matrices \( P_5 \) and \( P_9 \) in (5.31), one obtains the permutation matrices \( \hat{P}_4 \) and \( \hat{P}_8 \), respectively. Likewise, the Jordan canonical forms \( J_X \) and \( J_Y \) follow from the Jordan canonical forms \( J_X \) and \( J_Y \) by elimination of the corresponding rows and columns. Instead of preserving this elimination property, \( \hat{P}_4 \) could have been chosen as the four-dimensional identity matrix in which case the blocks \( \text{diag}(J_{\lambda,2}, \lambda, \lambda) \) in (5.52) are replaced by \( \text{diag}(\lambda, \lambda, J_{\lambda,2}) \).

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5.3.2 General fusion matrices

Here we determine the upper-triangular block-diagonal matrix $\hat{Q}^{-1}\tilde{N}_\mu \hat{Q}$ obtained from the general fusion matrix $\tilde{N}_\mu$ by a similarity transformation with respect to $\hat{Q}$ defined according to (5.51). From [5.5] and Section [5.3.1] we have that

$$\hat{Q}^{-1}\tilde{N}_\mu \hat{Q} = \text{diag} \left( g(\hat{X}_{\lambda,\lambda'}^{(1,1)}) h(\hat{Y}_{\lambda,\lambda'}^{(1,1)}), \ldots, g(\hat{X}_{\lambda,\lambda'}^{(1,3)}) h(\hat{Y}_{\lambda,\lambda'}^{(1,3)}), \ldots, g(\hat{X}_{\lambda,\lambda'}^{(3,1)}) h(\hat{Y}_{\lambda,\lambda'}^{(3,1)}), \ldots \right)$$

$$\frac{1}{2}(p-1)(p'-1)$$

where, as before, $g(z) = \text{pol}_{\mu(z)}(z)$ and $h(z) = \text{pol}_{\mu}(z)$, and where

$$g(\hat{X}_{\lambda,\lambda'}^{(1,1)}) h(\hat{Y}_{\lambda,\lambda'}^{(1,1)}) = g(\lambda) h(\lambda')$$

$$g(\hat{X}_{\lambda,\lambda'}^{(1,3)}) h(\hat{Y}_{\lambda,\lambda'}^{(1,3)}) = g(\lambda) h(J_{\lambda,3}) = g(\lambda) h(\lambda')$$

$$g(\hat{X}_{\lambda,\lambda'}^{(2,1)}) h(\hat{Y}_{\lambda,\lambda'}^{(2,1)}) = g(\lambda) h(J_{\lambda,2}) = g(\lambda) h(\lambda')$$

$$g(\hat{X}_{\lambda,\lambda'}^{(3,3)}) h(\hat{Y}_{\lambda,\lambda'}^{(3,3)}) = \left( \begin{array}{ccccccccc} gh & gh' & 0 & g'h & g'h' & 0 & \frac{1}{2}g''h & \frac{1}{2}g''h' \\ 0 & gh & 0 & 0 & g'h & 0 & 0 & \frac{1}{2}g''h \\ 0 & 0 & gh & gh' & g'h & g'h' & \frac{1}{2}g''h & \frac{1}{2}g''h' \\ 0 & 0 & 0 & gh & gh' & 0 & \frac{1}{2}g''h & \frac{1}{2}g''h' \\ 0 & 0 & 0 & 0 & gh & 0 & g'h & g'h' \\ 0 & 0 & 0 & 0 & 0 & gh & gh' & g'h' \\ 0 & 0 & 0 & 0 & 0 & 0 & gh & g'h' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & gh \end{array} \right)$$

To simplify the notation, we have used the abbreviations $g = g(\lambda)$ and $h = h(\lambda')$. The eight-dimensional matrix $g(\hat{X}_{\lambda,\lambda'}^{(3,3)}) h(\hat{Y}_{\lambda,\lambda'}^{(3,3)})$ in (5.57) is obtained from the nine-dimensional matrix $g(\hat{X}_{\lambda,\lambda'}^{(3,3)}) h(\hat{Y}_{\lambda,\lambda'}^{(3,3)})$ given in (5.51) by elimination of the first row and column. This completes the description of the upper-triangular block-diagonal matrix $\hat{Q}^{-1}\tilde{N}_\mu \hat{Q}$.

5.3.3 $\mathcal{W}$-extended critical percolation $\mathcal{WLM}(2,3)$

As for $X$ and $Y$, the eigenvalues of $\tilde{X}$ and $\tilde{Y}$ are given in (5.36) in the case of $\mathcal{WLM}(2,3)$. As displayed in Figure 2, the connected components $\tilde{W}_{\beta_i,\beta_j}^{(i,j)}$ of the Jordan web associated to the fusion algebra of modules associated with boundary conditions are neatly organized with respect to the labels $i$ and $j$. An example of an ordering of the common generalized eigenvectors of $\tilde{X}$ and $\tilde{Y}$ respecting (5.51) is

$$\hat{G}_{2,2}^{(0,0)}, \hat{G}_{2,0}^{(0,0)}, \hat{G}_{2,2}^{(0,0)}, \hat{G}_{2,1}^{(0,0)}, \hat{G}_{2,-1}^{(0,0)}, \hat{G}_{2,-1}^{(0,0)}, \hat{G}_{2,-1}^{(0,0)}, \hat{G}_{0,-2}^{(0,0)}, \hat{G}_{0,-2}^{(0,0)}, \hat{G}_{0,-2}^{(0,0)}, \hat{G}_{0,-2}^{(0,0)};$$

$$\hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}, \hat{G}_{0,-1}^{(0,0)}$$

We define the similarity matrix $\hat{Q}$ by concatenating these vectors in the order given. Modulo a similarity transformation, $\hat{Q}$ converts $\tilde{X}$ and $\tilde{Y}$ into the Jordan canonical forms

$$J_{\tilde{X}} = \hat{P}^{-1}\tilde{X}\hat{Q}\hat{P} = \text{diag}(2, 0, -2, 2, -2, 2, 2, 2, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$J_{\tilde{Y}} = \hat{Q}^{-1}\tilde{Y}\hat{Q} = \text{diag}(2, 2, 2, 1, 1, 1, 1, 1, -2, -2, 2, 2, -2, -2, 1, 1, 1, 1, 1, 1, 1, 1)$$

(5.59)
The similarity transformation of $\hat{P}$ is the symmetric permutation matrix

$$\hat{P} = \text{diag}(I_{14 \times 14}, \hat{P}_4, \hat{P}_8)$$

The fusion matrix $\hat{N}_\mu$ associated to the general module $\mu \in \mathcal{I}_b$ is polynomial in $\hat{X}$ and $\hat{Y}$. It is given by the same polynomial as in (5.40) but as a function of $\hat{X}, \hat{Y}$ instead of $X, Y$. We recall that the only two modules in the fundamental fusion algebra not associated with boundary conditions are $(1, 1)_W$ and $(1, 2)_W$, that is,

$$\mathcal{I}_f \setminus \mathcal{I}_b = \{(1, 1)_W, (1, 2)_W\}$$

The similarity transformation of $\hat{N}_\mu$ is the block-diagonal matrix $\hat{Q}^{-1} \hat{N}_\mu \hat{Q}$ whose blocks are upper-triangular matrices. As illustrations of such block-diagonal matrices, we here consider

$$\hat{Q}^{-1} \hat{N}_{4,2} \hat{Q} = \text{diag}(4, 0, -4, 2, -2, \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}),$$

$$\hat{Q}^{-1} \hat{N}_{2,3} \hat{Q} = \text{diag}(24, 0, 24, 0, 0, \begin{pmatrix} 0 & 8 & -12 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 8 & -12 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}),$$

where $\hat{P}$ is the symmetric permutation matrix

$$\hat{P} = \text{diag}(I_{14 \times 14}, \hat{P}_4, \hat{P}_8)$$

and $\hat{Q}$ is the symmetric permutation matrix

$$\hat{Q} = \text{diag}(I_{14 \times 14}, \hat{Q}_4, \hat{Q}_8).$$
6 Conclusion

We have extended the work [39] on \( \mathcal{WLM}(1,p') \) by considering the spectral decompositions of the regular representations of the graph fusion algebras of the general \( \mathcal{W} \)-extended logarithmic minimal model \( \mathcal{WLM}(p,p') \). In preparation therefore, we first defined and examined three types of directed and connected graphs, here called cycle, tadpole and eye-patch graphs. As in the rational minimal models, the fundamental fusion algebra of \( \mathcal{WLM}(p,p') \) is described by a simple graph fusion algebra. The graphs associated with the two fundamental modules consist of a number of tadpole and eye-patch graphs. The corresponding adjacency matrices share a complete set of common generalized eigenvectors organized as a web. This Jordan web is constructed by interlacing the Jordan chains of the two matrices and consists of connected subwebs with 1, 3, 5 or 9 generalized eigenvectors. The similarity matrix, formed by concatenating these vectors, simultaneously brings the two fundamental adjacency matrices to Jordan canonical form modulo permutation similarity. By the same similarity transformation, the general fusion matrices are brought simultaneously to block-diagonal forms whose blocks are upper-triangular matrices of dimension 1, 3, 5 or 9. For \( p > 1 \), only some of the modules in the fundamental fusion algebra of \( \mathcal{WLM}(p,p') \) are associated with boundary conditions within our lattice approach. The regular representation of the corresponding fusion subalgebra has features similar to the ones in the regular representation of the fundamental fusion algebra, but with dimensions of the connected Jordan-web components and upper-triangular blocks given by 1, 2, 3 or 8. In addition to eye-patch graphs, cycle graphs appear as connected components of the two auxiliary fusion matrices obtained from the fundamental fusion matrices by elimination of certain rows and columns. The general fusion matrices associated with boundary conditions are conveniently described in terms of the two auxiliary fusion matrices. Some of the key results have been illustrated for \( \mathcal{W} \)-extended critical percolation \( \mathcal{WLM}(2,3) \).

There are several natural continuations of this work, all of which we hope to discuss elsewhere. The first one concerns an algebraic extension of the fundamental fusion algebra of \( \mathcal{WLM}(p,p') \) for \( p > 1 \). It amounts to including all modules arising from fusions of the complete set of irreducible modules in the model as discussed in [25, 26, 27].

The second continuation concerns the derivation of a generalized Verlinde formula from the spectral decomposition of the various fusion matrices of \( \mathcal{WLM}(p,p') \). This problem was solved in [39] for \( p = 1 \). Other approaches to a Verlinde-like formula for \( \mathcal{WLM}(1,p') \) have been proposed in [11, 40, 14, 16, 41]. In the case of the so-called \emph{projective modules} in \( \mathcal{WLM}(p,p') \)

\[
\mathcal{M} = \{ (\kappa p, p')_{X}, (R_{\kappa p,p'})_{X}, (R_{\kappa p,p'})_{Y} ; \; \kappa \in \mathbb{Z}_{1,2}, \; a \in \mathbb{Z}_{1,p-1}, \; b \in \mathbb{Z}_{1,p'-1} \} \tag{6.1}
\]

of which there are \( 2pp' \) [21], the structure of the corresponding Verlinde-like formula [42] resembles the ordinary Verlinde formulas. This is intimately related to the observation that the auxiliary fusion graphs underlying the restrictions of the fundamental matrices \( X \) and \( Y \) to the projective modules are simply given by \( p' \) cycle graphs \( C_{p'} \) and \( p \) cycle graphs \( C_{p} \), respectively. Their spectral decompositions are much simpler than the ones considered here as they only involve rank-1 and rank-2 blocks. The two matrices share a complete set of \( (2pp') \) common generalized eigenvectors with the numbers of connected Jordan webs given by

\[
\mathcal{N}_{\text{proj}}^{(1,1)} = 2, \quad \mathcal{N}_{\text{proj}}^{(1,2)} = p' - 1, \quad \mathcal{N}_{\text{proj}}^{(2,1)} = p - 1, \quad \mathcal{N}_{\text{proj}}^{(2,2)} = \frac{1}{2}(p - 1)(p' - 1) \tag{6.2}
\]

As in the case of the connected Jordan webs associated with the auxiliary (boundary) fusion matrices \( \hat{X} \) and \( \hat{Y} \), cf. Section [5.1.2], the connected Jordan webs \( \hat{X} \) associated with the auxiliary (projective) fusion matrices can be viewed as quotients of the connected Jordan webs associated with the fundamental fusion matrices \( X \) and \( Y \).
The third continuation concerns the spectral decomposition of the (matrix) generators of the Grothendieck ring associated to $\mathcal{WLM}(p,p')$. For $p = 1$, this ring is obtained by elevating the various character identities to equivalence relations between the corresponding generators (modules) of the fusion algebra. For $p > 1$, on the other hand, the situation is more complicated as also pointed out in [25, 27]. Partition functions only concern characters, not the full-fledged fusion algebra. It thus suffices to consider the Grothendieck ring instead of the fusion algebra when discussing partition functions. In such circumstances, one is simply not concerned with the reducible yet indecomposable module structures, only in their characters. Based on spectral decompositions of the regular representation of the Grothendieck ring of $\mathcal{WLM}(1,p')$, a Verlinde-like formula was derived in [41]. In [39], a general framework is outlined within which it makes sense to discuss rings of equivalence classes of fusion-algebra generators. Together with the insight we have just gained by studying the graph fusion algebras and fusion graphs, this may provide the means to classify Grothendieck-like rings associated to $\mathcal{WLM}(p,p')$.

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A Commuting matrices and Jordan forms

Given two commuting $n$-dimensional matrices, there exists a complete chain of subspaces $0 = M_0 \subset M_1 \subset \ldots \subset M_n = \mathbb{C}^n$, $\dim(M_j) = j$, such that $M_j$ is invariant with respect to both matrices for all $j \in \mathbb{Z}_{0,n}$. This fundamental result on commuting matrices readily extends to all finite sets of commuting $n$-dimensional matrices, see [43], for example. It does not, however, imply that the two matrices share a complete set of common generalized eigenvectors. Nor does it imply that the two matrices can be simultaneously brought to Jordan form. It does, on the other hand, imply that the two matrices can be simultaneously brought to upper-block-triangular form.

To illustrate that two commuting matrices $A$ and $B$ do not necessarily share a complete set of generalized eigenvectors, even if they can be simultaneously brought to Jordan form, we consider

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 2 \\ 0 & \lambda \end{pmatrix} \quad (A.1)$$

These matrices are already in Jordan form, albeit $B$ not in Jordan canonical form. The most general Jordan chain associated to $A$ is

$$A \begin{pmatrix} a \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a \neq 0 \quad (A.2)$$

However, since $a \neq 0$, the evaluation

$$B \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} b \\ a \end{pmatrix} + 2 \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (A.3)$$

demonstrates that $A$ and $B$ do not share a complete set of generalized eigenvectors.
To illustrate that it is not always possible to bring a pair of commuting matrices $A$ and $B$ simultaneously to Jordan form, even if they share a complete set of generalized eigenvectors, we consider

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & c & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & d \\ 0 & 0 & \lambda \end{pmatrix}$$  \hspace{1cm} (A.4)

Here we have also defined matrices $C$ and $D$, which, for $c, d \neq 0$, represent general three-dimensional (upper) Jordan forms consisting of a rank-1 and a rank-2 Jordan block with respect to the single eigenvalue $\lambda$. The three vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (A.5)

form a complete set of generalized eigenvectors of both $A$ and $B$. The most general similarity matrices $S$ and $S$ bringing $A$ to the Jordan form $C$ or $D$

$$S^{-1}AS = C, \quad S^{-1}AS = D$$  \hspace{1cm} (A.6)

are given by

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & cS_{1,1} & 0 \\ 0 & S_{3,2} & S_{3,3} \end{pmatrix}, \quad S_{1,1}, S_{3,3} \neq 0; \quad S = \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & dS_{1,2} \\ S_{3,1} & 0 & S_{3,3} \end{pmatrix}, \quad S_{1,2}, S_{3,1} \neq 0$$  \hspace{1cm} (A.7)

However, it is readily verified that neither $S^{-1}BS$ nor $S^{-1}BS$ is in (upper) Jordan form.

## B Jordan webs

In the tables in Figure 3, 4 and 5 we collect the connected Jordan subwebs $W^{(\ell,\ell')}$ formed by the common generalized eigenvectors of the two fundamental fusion matrices $X$ and $Y$. There is a table for each of the three possible parity combinations of $p$ and $p'$. In Figure 3, $p$ and $p'$ are both odd; in Figure 4, $p$ is odd and $p'$ is even; while in Figure 5, $p$ is even and $p'$ is odd. An $\bigcirc$ in position $i, j$ indicates that there are no common generalized eigenvectors corresponding to the pair $\beta_i, \beta_j$ of eigenvalues of $X$ and $Y$. For every parity combination, there are exactly two $\bigcirc$’s in the corresponding table. This reflects the rather trivial observation

$$\mathcal{N}^{(1,1)} + \mathcal{N}^{(1,3)} + \mathcal{N}^{(3,1)} + \mathcal{N}^{(3,3)} + \mathcal{N}^{(3,3)^\dagger} = (p + 1)(p' + 1) - 2$$  \hspace{1cm} (B.1)

Similar tables for the connected Jordan subwebs $\tilde{W}^{(\ell,\ell')}$ associated to the auxiliary fusion matrices $\tilde{X}$ and $\tilde{Y}$ are obtained from the tables in Figure 3, 4 and 5 by the replacements

$$W^{(1,1)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(1,1)}_{\beta_i,\beta_j}, \quad W^{(1,3)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(1,3)}_{\beta_i,\beta_j}, \quad W^{(3,1)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(3,1)}_{\beta_i,\beta_j}$$

$$W^{(3,3)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(3,3)}_{\beta_i,\beta_j}, \quad W^{(1,3)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(1,2)}_{\beta_i,\beta_j} \cup \tilde{W}^{(2,1)}_{\beta_i,\beta_j}, \quad W^{(3,3)}_{\beta_i,\beta_j} \rightarrow \tilde{W}^{(3,3)}_{\beta_i,\beta_j}$$  \hspace{1cm} (B.2)
Figure 3: The connected components $W_{\beta_i,\beta'_{j'}}^{(\ell,\ell')}$ of the Jordan web associated to the fundamental fusion algebra of $WL\mathcal{M}(p,p')$ for $p$ and $p'$ both odd. Since the eigenvalues $\beta_i, \beta'_{j'}$ of $W_{\beta_i,\beta'_{j'}}^{(\ell,\ell')}$ are given by the location in the table, it suffices to indicate the component by the ranks $(\ell, \ell')$. The two $\emptyset$’s reflect that there are no common generalized eigenvectors corresponding to the pairs $\beta_p, \beta'_0$ and $\beta_0, \beta'_p$.
| $i \setminus j$ | 0     | 1     | 2     | 3     | $p' - 3$ | $p' - 2$ | $p' - 1$ | $p'$ |
|---|---|---|---|---|---|---|---|---|
| 0 | (1,1) | (1,1) | (1,3) | (1,1) | (1,1) | (1,3) | (1,1) | (1,1) |
| 1 | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (1,1) |
| 2 | (3,1) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,1) |
| 3 | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (1,1) |
| $p - 3$ | (3,1) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,1) |
| $p - 2$ | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (1,1) |
| $p - 1$ | (3,1) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,3) | (3,1) |
| $p$ | $\varnothing$ | (1,3) | (1,1) | (1,3) | (1,3) | (1,1) | (1,3) | $\varnothing$ | $\varnothing$ |

Figure 4: The connected components $W^{(\ell,\ell')}_{\beta_i,\beta'_j}$ of the Jordan web associated to the fundamental fusion algebra of $\mathcal{WLM}(p, p')$ for $p$ odd and $p'$ even. Since the eigenvalues $\beta_i, \beta'_j$ of $W^{(\ell,\ell')}_{\beta_i,\beta'_j}$ are given by the location in the table, it suffices to indicate the component by the ranks $(\ell, \ell')$. The two $\varnothing$’s reflect that there are no common generalized eigenvectors corresponding to the pairs $\beta_p, \beta'_0$ and $\beta_p, \beta'_{p'}$.
| $i \setminus j$ | 0  | 1  | 2  | 3  | $p' - 3$ | $p' - 2$ | $p' - 1$ | $p'$ |
|---------|-----|-----|-----|-----|---------|---------|---------|-----|
| 0       | (1,1) | (1,1) | (1,3) | (1,1) | ... | (1,3) | (1,1) | (1,3) | Ø |
| 1       | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | ... | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,1) |
| 2       | (3,1) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | ... | (3,3) | (3,3)$\dagger$ | (3,3) | (1,1) |
| 3       | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | ... | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,1) |
| ...     | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $p - 3$ | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | ... | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,1) |
| $p - 2$ | (3,1) | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | ... | (3,3) | (3,3)$\dagger$ | (3,3) | (1,1) |
| $p - 1$ | (1,1) | (3,3) | (3,3)$\dagger$ | (3,3) | ... | (3,3)$\dagger$ | (3,3) | (3,3)$\dagger$ | (3,1) |
| $p$     | (1,1) | (1,1) | (1,3) | (1,1) | ... | (1,3) | (1,1) | (1,3) | Ø |

Figure 5: The connected components $W^{(\ell,\ell')}_{\beta_i,\beta_j'}$ of the Jordan web associated to the fundamental fusion algebra of $\mathcal{WL}M(p,p')$ for $p$ even and $p'$ odd. Since the eigenvalues $\beta_i, \beta_j'$ of $W^{(\ell,\ell')}_{\beta_i,\beta_j'}$ are given by the location in the table, it suffices to indicate the component by the ranks $(\ell, \ell')$. The two Ø’s reflect that there are no common generalized eigenvectors corresponding to the pairs $\beta_0, \beta_{p'}$ and $\beta_p, \beta_{p'}$. 

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