Lyapunov Approach to Consensus Problems

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Abstract

This paper investigates the weighted-averaging dynamic for unconstrained and constrained consensus problems. Through the use of a suitably defined adjoint dynamic, quadratic Lyapunov comparison functions are constructed to analyze the behavior of weighted-averaging dynamic. As a result, new convergence rate results are obtained that capture the graph structure in a novel way. In particular, the exponential convergence rate is established for unconstrained consensus with the exponent of the order of $1 - O(1/(m \log_2 m))$. Also, the exponential convergence rate is established for constrained consensus, which extends the existing results limited to the use of doubly stochastic weight matrices.

I. INTRODUCTION

Over the past decade, distributed control has become an active area in control systems society and there has been considerable interest in distributed computation and decision making problems of all types. Among these are consensus and flocking problems [1], distributed averaging [2], multi-agent coverage problems [3], the rendezvous problem [4], localization of sensors in a multi-sensor network [5] and the distributed management of multi-robot formations [6]. These problems have found applications in a wide range of fields including sensor networks, robotic teams, social networks [7] and electric power grids [8]. Compared with traditional centralized control, distributed control is believed more promising for those large-scale complex networks because of its fault tolerance, cost saving and many inevitable physical constraints such as limited sensing, computation and communication capabilities. One of the basic problems arising in decentralized coordination and control is a consensus problem, also known as an agreement problem [9]–[15]. It arises in a number of applications including coordination of UAV’s, flocking and formation control, tracking in network of robots, and parameter estimation [16]–[25]. In a consensus problem, we have a set of agents each of which has some initial variable (a scalar or a vector). The agents are interconnected over an underlying (possibly time-varying) communication network and each agent has a local view of the network, i.e., each agent is aware of its immediate neighbors in the network and communicates with them only. The goal is to design a distributed and local algorithm that the agents can execute to agree on a common value asymptotically. The algorithm

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needs to be local in the sense that each agent performs local computations and communicates only with its immediate neighbors.

In this paper, we present two novel results for consensus problems and averaging dynamics. The first contribution is the establishment of new convergence rate analysis using Lyapunov approach, which allows us to provide an exponential rate in terms of network structure (such as longest shortest path) and the properties of the weight matrices. This rate result allows us to establish that the convergence rate with the ratio of the form $1 - O(1/\log_2 m)$ is achievable on special tree-like regular graphs. The second contribution is the development of the convergence rate result for a constrained consensus, which is more general than that of [26]. In contrast with [26], we do not require the weight matrices to be doubly stochastic. In fact, it is sufficient to have rooted directed spanning trees contained in the graphs and the existence of a specific adjoint dynamic for the linear consensus dynamic. Our analysis makes use of the Lyapunov comparison functions and absolute probability sequence, which have been developed in [27] in the more general setting of random graphs (see also [28], [29]).

The paper is organized as follows. In Section II, we discuss the weighted-averaging algorithm for consensus problem. In Section III, we review some of the recent results for cut-balanced matrices and the related adjoint dynamics for the linear consensus dynamics. Using these results, we construct suitable Lyapunov comparison functions and study convergence properties of the weighted-averaging algorithm in Section IV for standard consensus problem, while in Section V, we study a projection-based weighted-averaging algorithm for constrained consensus. We conclude with some remarks in Section VI.

**Notation:** For an integer $m \geq 1$, we write $[m]$ to denote the index set $\{1, \ldots, m\}$. We view vectors as column vectors. We write $x'$ to denote the transpose of a vector $x$ and, similarly, we use $A'$ for the transpose of a matrix $A$. A vector is stochastic if its entries are nonnegative and sum to 1. A matrix is said to be stochastic if its rows are stochastic vectors. A matrix is doubly stochastic if both $A$ and its transpose $A'$ are stochastic. A matrix $A$ entries will be denoted by $A_{ij}$ and, also, by $|A|_{ij}$ when convenient. We use $I$ for the identity matrix. To differentiate between the scalar and the vector cases, we use $x_i$ to denote a scalar value associated with agent $i$ and $x_i$ for a vector associated with agent $i$. We write $1$ to denote the vector with all entries equal to 1, where the size of the vector is to be understood from the context. Given a set $S$ with finitely many elements, we use $|S|$ to denote the cardinality of $S$. We use $\| \cdot \|$ for the Euclidean norm, while for other $p$-norms we will write $\| \cdot \|_p$. The Euclidean projection of a point $y$ on a convex closed set $Y$ is denoted by $P_Y[y]$, i.e., $P_Y[y] = \operatorname{argmin}_{z \in Y} \|y - z\|$. The distance of a point $y$ to the set $Y$ is denoted by dist$(y,Y)$, i.e., $\text{dist}(y,Y) = \|y - P_Y[y]\|$.  

II. UNCONSTRAINED CONSENSUS

We consider a set of $m$ agents, denoted by $[m] = \{1, \ldots, m\}$. The agents are embedded in a communication network, which is modeled by a directed graph $G_t = ([m], E_t)$, where $E_t \subseteq [m] \times [m]$ is the set of directed links.
A link \((i, j)\) indicates that agent \(i\) sends information to agent \(j\) at time \(t\). We will work with a sequence \(\{G_t\}\) of directed graphs, where each graph \(G_t\) contains a directed spanning tree rooted at one of the agents. We refer to such a graph as \emph{rooted graph}. The self-loops will be only virtually added to the graphs to model the fact that every agent has access to its own state information. We consider the unconstrained consensus problem, formalized as follows.

**[Unconstrained Consensus]** Design a distributed algorithm obeying the communication structure given by graph \(G_t\) at each time \(t\) and ensuring that, for every set of initial values \(x_i(0) \in \mathbb{R}^n, i \in [m]\), the following limiting behavior emerges:

\[
\lim_{t \to \infty} x_i(t) = c \text{ for all } i \in [m] \text{ and some } c \in \mathbb{R}^n.
\]

The algorithms for solving consensus problems have been mainly constructed using the Laplacians of the graphs \(G_t = ([m], E_t)\), e.g. see [11], [12], [30], or weighted-averaging (through the use of stochastic matrices) [11], [13], [16], [29]. In the scalar case, a well studied approach to the problem is for each agent to use a linear iterative update rule of the following form

\[
x_i(t+1) = W(t)x(t),
\]

where the weights \(W(t)\), \(i, j \in [m]\), are non-negative and the positive values satisfy some conditions with respect to the graph \(G_t\) structure, to be specified soon.

The dynamic in (1) is linear, so we focus on the case where the variables \(x_i\) are scalars, denoted by \(x_i\), as all the results for the vector case follow immediately by coordinate-wise analysis. The agents’ variables \(x_i \in \mathbb{R}, i \in [m]\) are stucked to form a vector \(x \in \mathbb{R}^m\). The existing analysis of the weighted-averaging is based on studying the behavior of the left-matrix products. Specifically, as the iterates \(x(t)\) are related over time by the following linear dynamic:

\[
x(t) = A(t)A(t-1)\cdots A(s+1)A(s)x(s) \quad \text{for } t \geq s \geq 0,
\]

the convergence of the iterates generated by the algorithm is related to the convergence of the matrix products \(A(t)A(t-1)\cdots A(1)A(0)\), as \(t \to \infty\). In particular, when the matrices \(A(t)A(t-1)\cdots A(1)A(0)\) converge to a rank
one matrix, the iterates $x(t)$ converge to a consensus. Concretely, some conditions on the graphs $G_t$ and the matrices $A(t)$ that yield such a convergence are given in the following assumption.

**Assumption 1.** Let $\{G_t\}$ be a graph sequence and $\{A(t)\}$ be a sequence of $m \times m$ matrices that satisfy the following conditions:

(a) Each $A(t)$ is a stochastic matrix that is compliant with the graph $G_t$, i.e., $A_{ij}(t) > 0$ when $(j, i) \in E_t$, for all $t$.
(b) (Aperiodicity) The diagonal entries of each $A(t)$ are positive, $A_{ii}(t) > 0$ for all $t$ and $i \in [m]$.
(c) (Uniform Positivity) There is a scalar $\beta > 0$ such that $A_{ij}(t) \geq \beta$ whenever $A_{ij}(t) > 0$.
(d) (Irreducibility) Each $G_t$ is strongly connected.

The convergence properties of the weighted-averaging algorithm have been extensively studied under Assumption 1 (see [9], [11], [16], [32]). Actually, in this case the matrix sequence $\{A(t)\}$ is known to be ergodic in the sense that the limit

$$\lim_{t \to \infty} A(t) \cdots A(k+1)A(k)$$

exists for all $k \geq 0$.

Moreover, it is known that the convergence rate of these products is geometric. The convergence rate question has been studied in [33–37] for deterministic matrix sequences and in [27], [38], [39] for random sequences. In [36], [40], [41], the convergence rate question was addressed for the cases when the matrices $A(t)$ are doubly stochastic; the best polynomial-time bound on the convergence time was given in [36]. Specifically, the following result is well known.

**Theorem 1.** [Lemma 5.2.1 in [9], Lemma 5 in [36]] Under Assumption 1 we have

$$\lim_{t \to \infty} A(t) \cdots A(k+1)A(k) = 1 \phi'(k)$$

for all $k \geq 0$,

where each $\phi(k)$ is stochastic vector. Furthermore, the convergence rate is geometric: for all $t \geq k \geq 0$,

$$\|A(t) \cdots A(k+1)A(k) - 1 \phi'(k)\|^2 \leq C q^{t-k},$$

where the constants $C > 0$ and $q \in (0, 1)$ depend only on $m$ and $\beta$. When the matrices $A(t)$ are doubly stochastic, we have for all $t \geq k \geq 0$,

$$\left\|A(t) \cdots A(k+1)A(k) - \frac{1}{m}11^T\right\|^2 \leq \left(1 - \frac{\beta}{2m^2}\right)^{t-k}.$$
on the results developed in Touri’s thesis [29] (see also [27], [28]). In this approach, an absolute probability sequence of matrices \( A(t) \) play a critical role in the construction of a Lyapunov comparison function and in establishing its rate of decrease along the iterates of the algorithm.

III. Absolute Probability Sequence

We embark on a study of the important features of stochastic matrices for convergence of the weighted-averaging method. The development here makes use of the notion of an absolute probability sequence associated with a sequence \( \{A(t)\} \) of stochastic matrices. This notion was introduced by Kolmogorov [43].

**Definition 1.** [43] Let \( \{A(t)\} \) be a sequence of stochastic matrices. A sequence of stochastic vectors \( \{\pi(t)\} \) is an absolute probability sequence for \( \{A(t)\} \) if

\[
\pi'(t) = \pi'(t + 1)A(t) \quad \text{for all } t \geq 0. \tag{2}
\]

Blackwell [44] has shown that every sequence of stochastic matrices has an absolute probability sequence. As a direct consequence of Blackwell’s result, every ergodic sequence of stochastic matrices has an absolute probability sequence (an earlier result due to Kolmogorov [43]). In particular, for an ergodic sequence \( \{A(t)\} \) of stochastic matrices we have

\[
\lim_{\tau \to \infty} A(t)A(t-1)\cdots A(1)A(t) = \phi'(t), \tag{3}
\]

and \( \{\phi(t)\} \) is an absolute probability sequence for \( \{A(t)\} \). In general, a sequence \( \{A(t)\} \) of stochastic matrices may have more than one absolute probability sequence. The following example has been communicated to us by B. Touri: if each of the matrices \( A(t) \) is invertible and each \( A(t)^{-1} \) is stochastic, then for any stochastic vector \( u \), we can construct an absolute probability sequence for \( \{A(t)\} \) by letting \( \pi'(0) = u' \) and \( \pi'(t+1) = \pi'(t)A(t)^{-1} \) for all \( t \geq 0 \). Thus, \( \{A(t)\} \) has infinitely many absolute probability sequences.

We show that the absolute probability sequence is unique for an ergodic stochastic matrix sequence.

**Lemma 1.** Let \( \{A(t)\} \) be an ergodic sequence of stochastic matrices (cf. [3]). Then, the vector sequence \( \{\phi(t)\} \) is the unique absolute probability sequence for \( \{A(t)\} \).

**Proof:** Assume that \( \{\pi(t)\} \) is another absolute probability sequence for \( \{A(t)\} \). Then, we have

\[
\pi'(t) = \pi'(t + \tau)A(t + \tau - 1)\cdots A(t + 1)A(t)
\]

for all \( \tau \geq 1 \) and \( t \geq 0 \). Thus,

\[
\pi(t) = \pi'(t + \tau)\left( A(t + \tau - 1)\cdots A(t) - \phi'(t) \right)
\]

\[
+ \pi'(t + \tau)\phi'(t)
\]
\[
\pi'(t + \tau) A(t + \tau - 1) \cdots A(t) - \mathbf{1} \phi'(t) + \phi'(t),
\]
where in the second equality we use \( \pi'(t + \tau) \mathbf{1} = 1 \). By letting \( \tau \to \infty \) and using \( \|\pi'(s)\|_1 = 1 \), we obtain
\[
\|\pi'(t) - \phi'(t)\|_1 \leq \limsup_{\tau \to \infty} \left( \|\pi'(t + \tau)\|_1 \|A(t + \tau - 1) \cdots A(t) - \mathbf{1} \phi'(t)\|_\infty \right)

\leq \lim_{\tau \to \infty} \|A(t + \tau - 1) \cdots A(t) - \mathbf{1} \phi'(t)\|_\infty = 0.
\]

In the subsequent development, it will be important that a sequence \( \{A(t)\} \) of stochastic matrices has an absolute probability sequence of vectors \( \pi(t) \) whose entries are uniformly bounded away from zero. This is the case when each matrix \( A(t) \) is doubly stochastic, as we can use \( \pi'(t) = \frac{1}{m} \mathbf{1} \). Another class of matrices that have this property is a subclass of cut-balanced matrices \( [27] \) (see there the class \( \mathcal{P}^{**} \)). (See Hendrickx and Tsitsiklis \( [45] \) for cut-balancedness as studied for continuous-time systems, and Touri \( [27], [28] \) and Bolouki and Malhamé \( [46] \) for discrete-time systems.)

In what follows, we will work under the following assumption, where we view a rooted tree \( T_t \) as a collection of directed edges from \( E_t \).

**Assumption 2.** Let \( \{G_t\} \) be a graph sequence and \( \{A(t)\} \) be a matrix sequence such that:
(a) (Partial Irreducibility) Each graph \( G_t \) is rooted and each \( A(t) \) is a stochastic matrix that is compliant with a rooted directed spanning tree \( T_t \) of \( G_t \), i.e., \( A_{ij}(t) > 0 \) whenever \( (j, i) \in T_t \) for all \( t \geq 0 \).
(b) (Aperiodicity) The diagonal entries of each \( A(t) \) are positive, \( A_{ii}(t) > 0 \) for all \( t \) and \( i \in [m] \).
(c) (Partial Uniform Positivity) There is a scalar \( \beta > 0 \) such that \( A_{ii}(t) \geq \beta \) and \( A_{ij}(t) \geq \beta \) for all \( (j, i) \in T_t \) and for all \( t \geq 0 \).
(d) The matrix sequence \( \{A(t)\} \) has an absolute probability sequence \( \{\pi(t)\} \) that is uniformly bounded away from zero, i.e., there is \( \delta \in (0, 1) \) such that \( \pi_i(t) \geq \delta \) for all \( i \) and \( t \).

One can show that Assumption 1 implies Assumption 2.

**IV. WEIGHTED-AVERAGING ALGORITHM**

We analyze convergence properties of the weighted-averaging algorithm in (1) by using a suitable Lyapunov comparison function.

**A. Lyapunov Comparison Function**

As indicated in \( [27] \), there are many possible constructions of Lyapunov comparison functions by using convex functions and absolute probability sequences, i.e., the adjoint dynamic in \( [2] \). Here, we focus on the
quadratic case, where the function is of the form:

\[ \varphi(x, v) \triangleq \sum_{i=1}^{m} v_i x_i^2 - (v'x)^2 \quad \text{for } x \in \mathbb{R}^m \text{ and } v \in \mathbb{R}^m_+, \]  

(4)

for suitably chosen vectors \( v \) (which will vary with time). The function \( \varphi \) has an equivalent form:

\[ \varphi(x, v) = \sum_{i=1}^{m} v_i (x_i - (v'x))^2 \quad \text{for } x \in \mathbb{R}^m \text{ and } v \in \mathbb{R}^m_+, \]  

(5)

which can be seen by expanding \((x_i - (v'x))^2\). The quadratic function \( s \mapsto s^2 \) has exact second order expansion, which allows us to obtain the exact expression for the difference \( \varphi(Ax, v) - \varphi(x, A'v) \) for a stochastic matrix \( A \), as seen in the following lemma.

**Lemma 2.** Let \( A \) be an \( m \times m \) stochastic matrix. We then have for all \( x \in \mathbb{R}^m \) and all \( v \in \mathbb{R}^m_+ \),

\[ \varphi(Ax, v) = \varphi(x, A'v) - \frac{1}{2} \sum_{i=1}^{m} v_i \sum_{j=1}^{m} A_{ij} A_{i\ell} (x_j - x_\ell)^2. \]

**Proof:** By the definition of \( \varphi \) we have \( \varphi(Ax, v) = \sum_{i=1}^{m} v_i ([Ax])_i^2 - (v'Ax)^2 \), where \([Ax]_i = \sum_{j=1}^{m} A_{ij} x_j\). We fix an arbitrary index \( i \), and we expand \(([Ax])_i^2\) to obtain

\[ ([Ax])_i^2 = \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij} A_{i\ell} x_j x_\ell. \]

Since \( x_j x_\ell = \frac{1}{2} \left( x_j^2 + x_\ell^2 - (x_j - x_\ell)^2 \right) \), it follows that

\[
([Ax])_i^2 = \frac{1}{2} \sum_{j=1}^{m} A_{ij} \left( \sum_{\ell=1}^{m} A_{i\ell} \right) x_j^2 + \frac{1}{2} \sum_{\ell=1}^{m} A_{i\ell} \left( \sum_{j=1}^{m} A_{ij} \right) x_\ell^2
- \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij} A_{i\ell} (x_j - x_\ell)^2.
\]

Note that \( \sum_{\ell=1}^{m} A_{i\ell} = 1 \) since the matrix \( A \) is stochastic, thus implying

\[ ([Ax])_i^2 = \sum_{j=1}^{m} A_{ij} x_j^2 - \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij} A_{i\ell} (x_j - x_\ell)^2. \]

By multiplying the preceding relation with \( v_i \) and by summing over \( i \), we obtain

\[
\varphi(Ax, v) = \sum_{j=1}^{m} \left( \sum_{i=1}^{m} v_i A_{ij} \right) x_j^2
- \frac{1}{2} \sum_{i=1}^{m} v_i \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij} A_{i\ell} (x_j - x_\ell)^2 - (v'Ax)^2.
\]

Observe that \( \sum_{i=1}^{m} v_i A_{ij} = [A'v]_j \). Therefore, by using the definition of the function \( \varphi \) we find

\[ \varphi(Ax, v) = \varphi(x, A'v) - \frac{1}{2} \sum_{i=1}^{m} v_i \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij} A_{i\ell} (x_j - x_\ell)^2. \]

\[ \square \]
Lemma 2 provides one of the fundamental relations in the assessment of the convergence rate of the weighted-averaging algorithm.

B. Convergence Rate Analysis

In this part, we will first show the convergence of the weighted-averaging algorithm for the scalar case, by considering the decrease of \( \varphi(x(t), \pi(t)) \) over time along the iterate sequence \( \{x(t)\} \), where \( \{\pi(t)\} \) is an absolute probability sequence of \( \{A(t)\} \). The decrease of this function in time can be captured exactly, as follows. Since \( x(t+1) = A(t)x(t) \) and the matrices \( A(t) \) are stochastic, by Lemma 2 it follows

\[
\varphi(x(t+1), \pi(t+1)) = \varphi(A(t)x(t), \pi(t+1)) = \varphi(x(t), A'(t)\pi(t+1)) - D(t),
\]

where

\[
D(t) = \frac{1}{2} \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t) \left(x_j(t) - x_{\ell}(t)\right)^2.
\]

By the definition of the adjoint dynamics in (2), we have

\[
\varphi(x(t+1), \pi(t+1)) = \varphi(x(t), \pi(t)) - D(t).
\]

Note that function \( \varphi(\cdot, \nu) \) induces a semi norm on \( \mathbb{R}^m \) when \( \nu \) is a stochastic vector, and it induces a norm when all the entries \( \nu_i \) are positive. Thus, to properly bound the decrease \( D(t) \) (cf. (3)) of the function \( \varphi(x(t), \pi(t)) \), one would like to have \( \varphi_i(t) > \delta \) for all \( i \), for some \( \delta \) and for all sufficiently large \( t \). This property can be ensured (for all \( t \)) by requiring the additional properties on the matrix sequence \( \{A(t)\} \) and the graph sequence \( \{G_t\} \) such as cut-balancedness (see Lemma 9 in (27)). Once all \( \pi_i(t) \) are bounded uniformly away from zero, to further bound \( D(t) \) from below, we would also like that the value of the sum \( \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t) \) does not vanish in time. These properties are ensured by Assumption 2 which we use to establish the key relation for the decrease amount \( D(t) \), as seen in the following lemma.

**Lemma 3.** Let Assumption 2 hold. Consider the decrement \( D(t) \) given by: for \( t \geq 0 \),

\[
D(t) = \frac{1}{2} \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t) \left(x_j(t) - x_{\ell}(t)\right)^2.
\]

Then, the decrement is bounded from below as follows:

\[
D(t) \geq \frac{\delta \beta^2}{4p^*(t)} \max_{j,\ell \in [m]} \left(x_j(t) - x_{\ell}(t)\right)^2 \quad \text{for } t \geq 0,
\]

where \( \beta > 0 \) and \( \delta > 0 \) are from Assumptions 2(c) and 2(d), respectively, while \( p^*(t) \) is the maximum number of links in any of the directed paths in the tree \( T_t \) of Assumption 2(a).
Proof: We let \( t \geq 0 \) be arbitrary but fixed. By Assumption 2(d), it follows that
\[
D(t) \geq \frac{\delta}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}(t)A_{i\ell}(t)\left(x_j(t) - x_\ell(t)\right)^2.
\]

Let us observe that
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}(t)A_{i\ell}(t) = \sum_{j=1}^{m} \sum_{\ell=1}^{m} (A_{j\ell}(t))^2 A_{\ell\ell}(t),
\]
where \( A_{j\ell} \) denotes \( j\)th column vector of a matrix \( A \). From this relation, we further obtain
\[
D(t) \geq \frac{\delta}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} (A_{j\ell}(t))^2 A_{\ell\ell}(t)\left(x_j(t) - x_\ell(t)\right)^2. \tag{8}
\]

Let \( j^* \) and \( \ell^* \) be two agents such that
\[
\max_{j, \ell \in [m]} |x_j(t) - x_\ell(t)| = |x_{j^*}(t) - x_{\ell^*}(t)|. \tag{9}
\]

Note that for any node \( v \) we must have
\[
\max(|x_p(t) - x_{j^*}(t)|, |x_p(t) - x_{\ell^*}(t)|) \geq \frac{1}{2} |x_{j^*}(t) - x_{\ell^*}(t)|, \tag{10}
\]
for otherwise by the triangle inequality for the norm we would have
\[
|x_{j^*}(t) - x_{\ell^*}(t)| \leq |x_p(t) - x_{j^*}(t)| + |x_p(t) - x_{\ell^*}(t)| < |x_{j^*}(t) - x_{\ell^*}(t)|,
\]
which is a contradiction.

According to Assumption 2(a), in the graph \( G_t \) there is a rooted directed spanning tree \( T_t \). Let agent \( v^* \) be the root node of this tree. Then, relation (10) holds for \( v = v^* \). Without loss of generality let us assume that \( j^* \) attains the maximum in (10) when \( v = v^* \), i.e., \( |x_{v^*}(t) - x_{j^*}(t)| \geq |x_{v^*}(t) - x_{\ell^*}(t)| \), so that we have
\[
|x_{v^*}(t) - x_{j^*}(t)| \geq \frac{1}{2} |x_{j^*}(t) - x_{\ell^*}(t)|. \tag{11}
\]

Since \( v^* \) is the root of the directed spanning tree \( T_t \), there must exist a path from \( v^* \) to \( j^* \), i.e., \( v^* = j_0 \rightarrow j_1 \rightarrow j_2 \cdots \rightarrow j_p = j^* \) with links \( (j_k, j_{k+1}) \) in the tree \( T_t \). Then, using (9) we can write
\[
D(t) \geq \delta \sum_{k=0}^{p-1} (A_{j_k}(t))^2 A_{\ell_{k+1}}(t)\left(x_{j_k}(t) - x_{\ell_{k+1}}(t)\right)^2. \tag{12}
\]

We now look at the coefficients \((A_{j_k}(t))^2 A_{\ell_{k+1}}(t)\) in (12) along the path \( v^* = j_0 \rightarrow j_1 \rightarrow j_2 \cdots \rightarrow j_p = j^* \). For
each $\kappa = 0,\ldots,p - 1$, we have
\[
(A_{jk}(t))^jA_{j,k+1}(t) = \sum_{i=1}^{m} A_{ijk}(t)A_{ik,j+1}(t) \\
\geq A_{j,k+1}(t)A_{j,k+1}(t) \geq \beta^2,
\]
where the last inequality follows by Assumption 2(c). From relations (12) and (13) we see that
\[
D_{j} \geq \delta p^2 \sum_{k=0}^{p-1} (x_{jk}(t) - x_{jk+1}(t))^2.
\]
Since the function $s \rightarrow s^2$ is convex, we have
\[
\frac{1}{p} \sum_{k=0}^{p-1} (x_{jk}(t) - x_{jk+1}(t))^2 \geq \left( \frac{1}{p} \sum_{k=0}^{p-1} (x_{jk}(t) - x_{jk+1}(t)) \right)^2 \geq \left( \frac{1}{p} \sum_{k=0}^{p-1} (x_{jk}(t) - x_{jk+1}(t)) \right)^2,
\]
\[
\Rightarrow \sum_{k=0}^{p-1} (x_{jk}(t) - x_{jk+1}(t))^2 \geq \frac{1}{p} \left( x_{j0}(t) - x_{jp}(t) \right)^2.
\]
Therefore, from the preceding relation and (14), by recalling that $j_0 = v^*$ and $j_p = j^*$, we obtain
\[
D(t) \geq \frac{\delta p^2}{p} (x_{j^*}(t) - x_{j^*}(t))^2.
\]
Finally, using inequality (11) in relation (15) we obtain
\[
D(t) \geq \frac{\delta p^2}{4p} (x_{j^*}(t) - x_{j^*}(t))^2.
\]
Recall that $p$ is the number of links in the path from $v^*$ to $j^*$ in the directed spanning tree $T_\tau$ (rooted at $v^*$) of the graph $G_\tau$. Thus, $p$ is bounded from above by the maximal number of links along the path from $v$ to any other node in the graph $G_\tau$, where the paths are taken along the directed spanning tree rooted at $v^*$. We note that $p^*$ depends on time $t$ which was fixed so far, and we have suppressed this dependence on $t$.
Recall, further that $j^*$ and $\ell^*$ are agents with the maximal difference $|x_j(t) - x_{\ell}(t)|$ (see Eq. [9]). Thus, from the relation in (16) we have $D(t) \geq \frac{\delta p^2}{4p^*(t)} \max_{j,\ell \in [m]} (x_j(t) - x_{\ell}(t))^2$.

Before stating our main result, we provide an auxiliary lemma for use in the forthcoming analysis.

**Lemma 4.** For any stochastic vector $v \in \mathbb{R}^m$ and any $x \in \mathbb{R}^m$ it holds that
\[
\sum_{i=1}^{m} v_i(x_i - v^*x)^2 \leq \max_{1 \leq j,\ell \leq m} (x_j - x_\ell)^2.
\]

**Proof:** Since $v$ is stochastic vector, it follows that $\sum_{i=1}^{m} v_i(x_i - v^*x)^2 \leq \max_{1 \leq k \leq m} (x_k - v^*x)^2$. Without loss of
generality, let us assume that the preceding maximum is attained for \( \kappa = 1 \),

\[
(x_1 - v'x)^2 = \max_{1 \leq k \leq m} (x_k - v'x)^2,
\]

and note that, since \( v'1 = 1 \) we can write \( x_1 - v'x = x_1 v'1 - v'x = v'(x_1 1 - x) \). Using the preceding relation, the fact that \( v \) is a stochastic vector, and the convexity of the function \( s \mapsto s^2 \), we obtain

\[
(x_1 - v'x)^2 = (v'(x_1 1 - x))^2 \leq \sum_{i=1}^m v_i (x_1 - x_i)^2 \leq \max_{1 \leq \ell \leq m} (x_1 - x_\ell)^2.
\]

Therefore, we have

\[
\sum_{i=1}^m v_i (x_i - v'x)^2 \leq \max_{1 \leq \ell \leq m} (x_1 - x_\ell)^2 \leq \max_{1 \leq j, \ell \leq m} (x_j - x_\ell)^2.
\]

With Lemma 3 in place, we can now establish a key relation for the quadratic comparison function. The convergence result of the weighted-averaging algorithm, as well as its convergence rate estimates, will follow from this relation.

**Theorem 2.** Under Assumption 2 for the iterates \( \{x(t)\} \) generated by the weighted-averaging algorithm 1 with any initial vector \( x(0) \in \mathbb{R}^m \), we have for any \( t \geq k \geq 0 \),

\[
\sum_{i=1}^m \pi_i(t) \left( x_i(t) - \pi(0)'x(0) \right)^2 \\
\leq \left( 1 - \frac{\delta \beta^2}{4p^*} \right)^{t-k} \sum_{j=1}^m \pi_j(k) \left( x_j(k) - \pi(0)'x(0) \right)^2,
\]

where \( \beta > 0 \) and \( \delta > 0 \) are from Assumptions 2(c) and 2(d), while \( p^* = \max_{s \geq 0} p^*(s) \) where \( p^*(s) \) is the longest shortest path in the tree \( T_s \) of Assumption 2(a).

**Proof:** The stated relation for \( t = k \) can be seen to hold by inspection. Consider now \( t > k \geq 0 \) where \( t \) and \( k \) are arbitrary but fixed. From relations 6–7 and Lemma 3 we obtain for all \( t \geq 0 \),

\[
\varphi(x(t+1), \pi(t+1)) \leq \varphi(x(t), \pi(t)) - \frac{\delta \beta^2}{4p^*(t)} \max_{j, \ell \in [m]} (x_j(t) - x_\ell(t))^2.
\]

From Lemma 4 it follows that

\[
\max_{1 \leq j, \ell \leq m} (x_j(t) - x_\ell(t))^2 \geq \sum_{j=1}^m \pi_j(t) \left( x_j(t) - \pi(t)'x(t) \right)^2,
\]

thus implying that for all \( t \geq 0 \),

\[
\varphi(x(t+1), \pi(t+1)) \leq \left( 1 - \frac{\delta \beta^2}{4p^*(t)} \right) \sum_{j=1}^m \pi_j(t) \left( x_j(t) - \pi(t)'x(t) \right)^2.
\]
Hence, for all \( t \geq 0 \),
\[
\sum_{i=1}^{m} \pi_i(t+1) \left( x_i(t+1) - \pi(t+1)'x(t+1) \right)^2 \\
\leq \left( 1 - \frac{\delta \beta^2}{4p^*(t)} \right) \sum_{j=1}^{m} \pi_j(t) \left( x_j(t) - \pi(t)'x(t) \right)^2.
\]
Furthermore, from the dynamics in (1) and (2) we can see that for all \( t \geq 1 \),
\[
\pi(t)'x(t) = \pi(t)'A(t-1)x(t-1) = \pi(t-1)'x(t-1) = \cdots = \pi(0)'x(0),
\]
which yields for all \( t \geq 0 \),
\[
\sum_{i=1}^{m} \pi_i(t+1) \left( x_i(t+1) - \pi(0)'x(0) \right)^2 \\
\leq \left( 1 - \frac{\delta \beta^2}{4p^*(t)} \right) \sum_{j=1}^{m} \pi_j(t) \left( x_j(t) - \pi(0)'x(0) \right)^2.
\]

The stated relation follows by recursively using the preceding inequality for \( t, t-1, \ldots, k \), and then using \( p^*(s) \leq p^* \) for all \( s \).

Theorem 2 captures the convergence rate in terms of the longest shortest paths in the graph sequence. The quotient \( q = 1 - \frac{\delta \beta^2}{4p^*} \) indicates the rate at which the information is diffused in the graphs \( \{G_t\} \) over time, with a small \( q \) being desirable for a fast diffusion.

Several immediate consequences of Theorem 2 are in place. First, we observe that from Theorem 2 it follows that the agent iterates converge to the consensus value \( \pi(0)'x(0) \), by virtue of the lower boundedness property of the absolute probability sequence (Assumption 2(d)), i.e., \( \lim_{t \to \infty} x_i(t) = \pi(0)'x(0) \) for all \( i \in [m] \). When the agent variables \( x_i \) are vectors, then by applying Theorem 2 to each coordinate of the vectors, we can see that the iterates \( x_i(t) \) generated by the weighted-averaging algorithm are such that for any initial vectors \( x_i(0) \in \mathbb{R}^n \), \( i \in [m] \), for each coordinate index \( \ell \in [n] \), and for all \( t \geq k \geq 0 \), we have
\[
\sum_{i=1}^{m} \pi_i(t) \left( [x_i(t)]_\ell - c_\ell \right)^2 \\
\leq \left( 1 - \frac{\delta \beta^2}{4p^*} \right)^{t-k} \sum_{j=1}^{m} \pi_j(k) \left( [x_j(k)]_\ell - c_\ell \right)^2,
\]
where \( c_\ell = \sum_{i=1}^{m} \pi_i(0)'[x_i(0)]_\ell \) for all \( \ell \in [n] \). By summing these relations over all coordinate indices \( \ell \in [n] \), we obtain the following result.

**Corollary 1.** Consider the vector-valued consensus problem and let Assumption 2 hold. Then, the iterates \( \{x_i(t)\} \),
i \in [m] \text{ generated by the weighted-averaging algorithm are such that for any initial vectors } \mathbf{x}_i(0) \in \mathbb{R}^n,

\sum_{i=1}^{m} \pi_i(t) \| \mathbf{x}_i(t) - \mathbf{c} \|^2 \leq \left(1 - \frac{\delta p^2}{4p} \right)^{t-k} \sum_{j=1}^{m} \pi_j(k) \| \mathbf{x}_j(k) - \mathbf{c} \|^2

for all \ t \geq k \geq 0, \text{ where the vector } \mathbf{c} \in \mathbb{R}^n \text{ has coordinates given by } c_\ell = \sum_{i=1}^{m} \pi_i(0)^\prime [\mathbf{x}_i(0)]_\ell \text{ for all } \ell \in [n].

Some further implications of Theorem 2 are discussed in the following section.

C. Implications of Theorem 2

We present some implications of Theorem 2 regarding the improvement of the best known rate of \(O(m^2)\) and the convergence properties of the matrix products \(A(t) \cdots A(k+1)A(k)\).

Let Assumption 2 hold, and assume also that the weight matrices \(A(t), t \geq 0,\) are doubly stochastic. Then, we have \(\pi(t) = \frac{1}{m} \mathbf{1}\) and the relation of Theorem 2 reduces to (after multiplication by \(m\)):

\[ \| \mathbf{x}(t) - \bar{\mathbf{x}}(0) \|_2^2 \leq \left(1 - \frac{\beta p^2}{4m p^*} \right)^{t-k} \| \mathbf{x}(k) - \bar{\mathbf{x}}(0) \|_2^2, \]  

(17)

with \(\bar{\mathbf{x}}(0) = \frac{1}{m} \mathbf{x}(0)\). Since the maximum path length from the root to any other node cannot exceed \(m-1\), i.e., \(p^*(s) \leq m - 1\), it follows that

\[ \| \mathbf{x}(t) - \bar{\mathbf{x}}(0) \|_2^2 \leq \left(1 - \frac{\beta p^2}{4m(m-1)} \right)^{t-k} \| \mathbf{x}(k) - \bar{\mathbf{x}}(0) \|_2^2. \]

Thus, when \(\beta\) does not depend on \(m\), the convergence rate has dependency of \(O(m^2)\) in terms of the number \(m\) of agents, which is the same as the rate result in [36]; see Theorem 1.

Suppose now that we want to construct the graphs \(G_t\) such that Assumption 2 holds and we want to get the most favorable rate dependency on \(m\). In this case, the following result is valid.

**Theorem 3.** There is a sequence \(\{G_t\}\) of regular undirected graphs such that for all \(x(0) \in \mathbb{R}^m\) and all \(t \geq k \geq 0,\)

\[ \| \mathbf{x}(t) - \bar{\mathbf{x}}(0) \|_2^2 \leq q^{t-k} \| \mathbf{x}(k) - \bar{\mathbf{x}}(0) \|_2^2, \]

with \(q = 1 - \frac{1}{4^d \log^2 m} \) and \(\bar{\mathbf{x}}(0) = \frac{1}{m} \mathbf{x}(0)\).

**Proof:** We will construct an undirected graph sequence \(\{G_t\}\) that satisfies Assumption 2. Let \(m = 2^d\) for some integer \(d \geq 1\). Let \(t\) be arbitrary but fixed time. Select \(2^d - 1\) agents and construct an undirected binary tree with these agents as nodes. Next, add one extra agent as a root with a single child (see Figure 1a). Thus, each agent \(i\) except for the root and the leaf agents has the degree equal to 3. Consider, now connecting all leaf-nodes with undirected edges (see Figure 1b). Now, all leaf-agents have degree equal to 3 except for the far most left and far most right agents, each of which has the degree equal to 2. Connect these two agents to the root node (see Figure 1c). In this way, the far most left and far most right leaf agents, as well as the...
root agent have degree 3. In the resulting regular undirected graph, we let \( A_{ij}(t) = \frac{1}{4} \) for all \( j \in N_i(t) \cup \{i\} \) and \( \beta = \frac{1}{4} \).

The shortest path from the root agent to any other agent in the graph is at most \( \lceil \frac{d}{2} \rceil \) (going down from the root of the tree to the nodes at the depth \( \lceil \frac{d}{2} \rceil \), and going through the leaf nodes to reach those that are the depth larger than \( \lceil \frac{d}{2} \rceil \)).

Using the same construction, for all times \( t \), we have that \( \{A(t)\} \) is a sequence of doubly stochastic matrices, and therefore \( \pi(t) = \frac{1}{m} \mathbf{1} \) for all \( t \). Thus, Assumption 2 is satisfied, and the estimate in (17) reduces to

\[
\| x(t) - \tilde{x}(0) \mathbf{1} \|_2 \leq \left( 1 - \frac{1}{4 m \lceil \frac{d}{2} \rceil} \right)^{t-k} \| x(k) - \tilde{x}(0) \mathbf{1} \|_2.
\]

The result follows by noting that \( d = \log_2 m \).

Theorem 3 shows that the exponential convergence rate with the ratio of the order \( 1 - O(\frac{1}{m \log_2 m}) \) is achievable for consensus on some tree-like regular undirected graphs. This improves the best known bound with the ratio of the order \( 1 - O(\frac{1}{m^2}) \) for undirected graphs and doubly stochastic matrices [36].

We next consider the implication of Theorem 2 for the convergence of matrix products

\[
A(t:k) \triangleq A(t) \cdots A(k+1)A(k) \quad \text{for all} \quad t \geq k \geq 0,
\]

where \( A(t:k) \triangleq A(k) \) whenever \( t = k \).

**Theorem 4.** If Assumption 2 then for all \( t \geq k \geq 0 \),

\[
\| A(t:k) - \mathbf{1} \pi(k) \|_2 \leq \frac{1}{\delta} \left( 1 - \frac{\delta \beta^2}{4 p^*} \right)^{t-k} \| \mathbf{1} - \mathbf{1} \pi(k) \|_2^2.
\]

**Proof:** By Theorem 2 and the fact that \( \pi'(s)x(s) = \pi'(0)x(0) \) for all \( s \), we have that for all \( t \geq k \geq 0 \),

\[
\sum_{i=1}^{m} \pi_i(t) \left( x_i(t) - \pi(k)' x(k) \right)^2 \\
\leq \left( 1 - \frac{\delta \beta^2}{4 p^*} \right)^{t-k} \sum_{j=1}^{m} \pi_j(k) \left( x_j(k) - \pi(k)' x(k) \right)^2.
\]
Since $\pi_i(k) \leq 1$ for all $i$ and $k$, and $\pi_i(t) \geq \delta$ by Assumption [2](d), it follows that for all $t \geq k \geq 0$,
\[
\|x(t) - \pi(k)'x(k)1\| \leq \frac{1}{\delta} \left(1 - \frac{\delta \beta^2}{4p^2}\right)^{t-k} \|x(k) - \pi(k)'x(k)1\|^2.
\]
Noting that $x(t) = A(t : k)x(k)$ and $\pi(k)'x(k)1 = 1\pi(k)'x(k)$, we can write: for all $t \geq k \geq 0$,
\[
\|A(t : k) - 1\pi(k)'|x(k)\|^2 \leq \frac{1}{\delta} \left(1 - \frac{\delta \beta^2}{4p^2}\right)^{t-k} \|1 - 1\pi(k)'|x(k)\|^2.
\] (18)

Since the matrices $A(t)$ do not depend on the state variables $x(s)$, $0 \leq s < t$, the situation is similar to constructing $\{x(t)\}_{t=k}$ by the truncated matrix sequence $\{A(t)\}_{t=k}$, where the dynamic is started at time $k$ in any state $x(k)$. Then, relation (18) can be seen to hold for any $x(k) \in \mathbb{R}^n$. Let $x(k) = x \in \mathbb{R}^n$ and obtain for all $t \geq k \geq 0$,
\[
\sup_{x \neq 0} \frac{\|A(t : k) - 1\pi(k)'|x\|^2}{\|x\|^2} \leq \frac{1}{\delta} \left(1 - \frac{\delta \beta^2}{4p^2}\right)^{t-k} \sup_{x \neq 0} \frac{\|1 - 1\pi(k)'|x\|^2}{\|x\|^2},
\]
which is equivalent to the stated relation.

We have the following immediate consequence of Theorem [4] by letting $t \to \infty$.

**Corollary 2.** **Under Assumption [2]** the sequence $\{A(t)\}$ is ergodic: $\lim_{t\to\infty} A(t) \cdots A(k) = 1\pi(k)'$ for all $k \geq 0$.

V. **Constrained Consensus**

In this section, we consider consensus problems where the agent values are constrained to given sets. Such constraints are inevitable in a number of applications including motion planning and alignment problems, where each agent’s position is limited to a certain region or range [47]. Constrained consensus was first introduced in [26] where a simple discrete-time projected constrained consensus algorithm was proposed. The analysis of the algorithm in [26] relies on convergence properties of doubly stochastic matrices. An alternative analysis developed in [48] gets around this limitation and also takes into account transmission delays, but the proofs are intricate and no convergence rate results are established. In [49], a continuous-time constrained consensus algorithm was proposed using logarithmic barrier functions. In [50] and [51], discrete-time constrained consensus algorithms were presented for a special case in which the variable of each agent is a scalar quantity.

In the sequel, we will follow the algorithm in [26]. Unlike the existing analysis in [26], [48], we here adopt dynamic system point of view and apply a Lyapunov approach, as done in the unconstrained consensus problem. This approach would allow us to provide an elegant proof of convergence and characterize the convergence rate under appropriate assumptions.

**A. Projected Weighted-Averaging Algorithm**

We assume that each agent has a constraint set $X_i \subseteq \mathbb{R}^n$, which is a convex and closed, and the agents need to agree on a common point $c \in \cap_{i=1}^m X_i$. We will work under the following assumption on the sets $X_i$. 

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Assumption 3. The sets $X_i \subseteq \mathbb{R}^n$ are nonempty, closed, and convex, and their intersection is nonempty, i.e., $X \triangleq \bigcap_{i=1}^m X_i \neq \emptyset$.

The constrained consensus problem is as follows.

**[Constrained Consensus]** Assuming that each agent $i$ knows only its set $X_i$, design a distributed algorithm obeying the communication structure given by graph $G_t$ at each time $t$ and ensuring that, for every set of initial values $x_i(0) \in \mathbb{R}^n$, $i \in [m]$, the following limiting behavior emerges: $\lim_{t \to \infty} x_i(t) = c$ for all $i \in [m]$ and some $c \in X$.

To solve the constrained consensus problem, we consider the algorithm proposed in [26], which has the following form. Assuming that each agent starts with some initial vector $x_i(0) \in X_i$ at time $t = 0$, each agent $i$ updates at times $t = 1, 2, \ldots$, as follows:

$$w_i(t+1) = \sum_{j=1}^m A_{ij}(t)x_j(t),$$

$$x_i(t+1) = P_{X_i}[w_i(t+1)],$$

where $P_{X_i}[\cdot]$ is the Euclidean projection on the set $X_i$.

We will show that, under Assumption 2 and Assumption 3, the algorithm converges to a consensus point in the intersection set $X$. However, unlike the results for unconstrained consensus problems, we cannot characterize the consensus point more precisely. We will also prove that, under some further conditions on the sets $X_i$, the convergence rate of the algorithm is linear. The behavior of the algorithm [19] is very similar to that of the basic weighted-averaging algorithm in [1] for the unconstrained consensus. The intuition comes from the following observation: the iterates of the algorithm [19] satisfy $x_i(t+1) = P_{X_i} \left[ \sum_{j=1}^m A_{ij}(t)x_j(t) \right]$. The inner averaging mapping (defined through $A(t)$) possesses some nice contraction properties under Assumption 2 on the graphs and the matrices $A(t)$. This mapping is followed by a projection mapping, which is non-expansive. Thus, one would expect that the resulting composite map is also contractive, with a nearly the same contraction constant as the averaging map.

The non-expansiveness and few other properties of the projection map are summarized below. Given a (nonempty) closed convex set $Y \subseteq \mathbb{R}^n$, the projection mapping $y \mapsto P_Y[y]$ is non-expansive, i.e.,

$$\|P_Y[x] - y\| \leq \|x - y\| \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in Y,$$

which is one of the key properties used in the analysis of projection-based approaches. This and other properties of the projection mapping can be found, for example, in [52], Volume 2, 12.1.13 Lemma, page 1120. Another useful relation for the projection mapping is given by a variational inequality:

$$(P_Y[x] - x)'(y - P_Y[x]) \geq 0$$

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for all \( x \in \mathbb{R}^n \) and \( y \in Y \). The relation in (21) can be obtained by noting that the vector \( P_Y[x] \) is the unique solution of the minimization problem \( \min_{y \in Y} \|y - x\|^2 \) and by using the optimality condition for the solution. The formal proof of relation (21) can be found for example in [53], Proposition 2.2.1(b), page 55.

### B. Quadratic Lyapunov Comparison Function

Our choice of Lyapunov function is similar to the Lyapunov comparison function [4] for the weighted-averaging algorithm in the case of an unconstrained consensus (see Section IV-B). The similarity is in the use of an adjoint sequence \( \{\pi(t)\} \) associated with the matrix sequence \( \{A(t)\} \) (cf. (2)); however, there is a slight difference in the choice of the centering term \( v'x \) in [4], which is replaced by an arbitrary value. Specifically, we consider the function of the following form: for all \( t \geq 0 \) and \( y \in \mathbb{R}^n \),

\[
V(t, y) = \sum_{i=1}^m \pi_i(t) \|x_i(t) - y\|^2.
\]

When the values of \( y \) are constrained so that \( y \in X \), the function \( V \) has an important decrease property. To establish that property we use the following result.

**Lemma 5.** Let \( v \in \mathbb{R}^m \) be a given vector and let \( \phi \in \mathbb{R}^m \) be a given stochastic vector. Then, we have for any \( s \in \mathbb{R} \),

\[
\langle \phi' v - s \rangle^2 = \frac{1}{2} \left[ \sum_{j=1}^m \sum_{\ell=1}^m \phi_j(v_j - s) \phi_\ell(v_\ell - s) \right] - \frac{1}{2} \left[ \sum_{j=1}^m \phi_j(v_j - s) \right] \left[ \sum_{\ell=1}^m \phi_\ell(v_\ell - s) \right].
\]

**Proof:** We note that \( \phi' 1 = 1 \) since \( \phi \) is stochastic vector. Thus, we have \( \phi' v - s = \phi'(v - s 1) = \phi'(v - s) \phi_\ell(v_\ell - s) \).

Therefore, by taking the square we obtain

\[
\langle \phi' v - s \rangle^2 = \sum_{j=1}^m \sum_{\ell=1}^m \phi_j(v_j - s) \phi_\ell(v_\ell - s).
\]

Using the identity \( ab = \frac{1}{2} [a^2 + b^2 - (a - b)^2] \), which is valid for any \( a, b \in \mathbb{R} \), we can further write

\[
\langle \phi' v - s \rangle^2 = \frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m \phi_j \phi_\ell [(v_j - s)^2 + (v_\ell - s)^2 - (v_j - v_\ell)^2]
\]

\[
= \frac{1}{2} \sum_{j=1}^m \phi_j [v_j - s]^2 + \frac{1}{2} \sum_{\ell=1}^m \phi_\ell [v_\ell - s]^2 - \frac{1}{2} \sum_{j=1}^m \phi_j [v_j - v_\ell]^2
\]

\[
= \sum_{j=1}^m \phi_j [(v_j - s)^2 - \frac{1}{2} \sum_{\ell=1}^m \phi_\ell (v_j - v_\ell)^2],
\]

where the last equality is obtained by using \( \phi' 1 = 1 \).

Using Lemma 5, we have the following decrease property for the function \( V(t, y) \) for \( y \in X \).

**Theorem 5.** Let Assumption 2 and Assumption 3 hold. Then, along the sequences \( \{x_i(t)\} \), \( i \in [m] \), produced by
the algorithm [19], we have for any initial vectors \( x_i(0) \in X_i \), for \( t \geq 0 \) and \( y \in X \),

\[
V(t+1, y) \leq V(t, y) - \frac{\delta \beta^2}{4p^*} \max_{j \in V} \| x_j(t) - x_\ell(t) \|^2,
\]

where the constants \( \beta > 0 \) and \( \delta > 0 \) are from Assumptions [2(c) and 2(d)], respectively, while \( p^* = \max_{t \geq 0} p^*(t) \) with \( p^*(t) \) being the maximum number of edges in any of the paths from a root node to any other node in the tree \( T_t \) from Assumption [2(a)].

Proof: From the definition of \( w_i(t+1) \) in [19], using the fact that the matrix \( A(t) \) is stochastic and applying Lemma 5 (where \( \phi' = A_i(t) \)), we see that the following relation is valid for each coordinate index \( \kappa \in [n] \) of the vector \( w_i(t+1) \): for any \( s \in \mathbb{R} \),

\[
(|w_i(t+1)|_\kappa - s)^2 = \sum_{j=1}^{m} A_{ij}(t)(|x_j(t)|_\kappa - s)^2
\]

\[
- \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t)|x_j(t)|_\kappa - |x_\ell(t)|_\kappa|^2.
\]

Let \( c \in \mathbb{R}^n \) be an arbitrary vector. Then, by letting \( s = c_\kappa \) in the preceding relation and by summing over all coordinate indices \( \kappa \in [n] \), we obtain the following relation: for any \( c \in \mathbb{R}^n \), for all \( i \in [m] \) and all \( t \geq 0 \),

\[
\| w_i(t+1) - c \|^2 = \sum_{j=1}^{m} A_{ij}(t)|x_j(t) - c|^2
\]

\[
- \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t)|x_j(t) - x_\ell(t)|^2.
\]

By multiplying with \( \pi_i(t+1) \) and then summing over all \( i \), we have for any \( c \in \mathbb{R}^n \) and all \( t \geq 0 \),

\[
\sum_{i=1}^{m} \pi_i(t+1)\| w_i(t+1) - c \|^2
\]

\[
= \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} A_{ij}(t)|x_j(t) - c|^2 - D(t), \tag{23}
\]

where the decrement \( D(t) \) is given by: for all \( t \geq 0 \),

\[
D(t) = \frac{1}{2} \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} \sum_{\ell=1}^{m} A_{ij}(t)A_{i\ell}(t)|x_j(t) - x_\ell(t)|^2 \tag{24}
\]

Now, we consider the \( x \)-iterates. By the definition of \( x_i(t+1) \) in [19], we have \( x_i(t+1) = P_{X_i}[w_i(t+1)] \). Thus, by the non-expansiveness property of the projection map \( x \rightarrow P_{X_i}[x] \) (see [20]), we obtain for all \( i \), all \( t \geq 0 \), and all \( y \in X \) (note \( X \subseteq X_i \) for all \( i \)): \( \| x_i(t+1) - y \|^2 \leq \| w_i(t+1) - y \|^2 \). Therefore, by multiplying with \( \pi_i(t+1) \) and then summing over all \( i \), and using the definition of \( V \), we see that

\[
V(t+1, y) \leq \sum_{i=1}^{m} \pi_i(t+1)\| w_i(t+1) - y \|^2. \tag{25}
\]
Letting \( c = y \) in \([23]\) and combining the resulting relation with inequality \([25]\), we obtain

\[
V(t+1, y) \leq \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} A_{ij}(t) \| x_j(t) - y \|^2 - D(t).
\]

Exchanging the order of summations yields

\[
V(t+1, y) \leq \sum_{j=1}^{m} \left( \sum_{i=1}^{m} \pi_i(t+1) A_{ij}(t) \right) \| x_j(t) - y \|^2 - D(t)
= \sum_{j=1}^{m} \pi_j(t) \| x_j(t) - y \|^2 - D(t),
\]

where in the last equality we use \( \pi_j(t) = \sum_{i=1}^{m} \pi_i(t+1) A_{ij}(t) \) (see the adjoint dynamic in \([2]\)). Relation \([26]\) and the definition of \( V(t, y) \) imply that

\[
V(t+1, y) \leq V(t, y) - D(t) \quad \text{for all } t \geq 0 \quad \text{and } y \in X.
\]

It remains to bound the decrement \( D(t) \) in \([27]\) from below. We note that the decrement \( D(t) \) defined in \([24]\) is a vector analog of the decrement \( D(t) \) in Lemma \([3]\). In particular, by defining the decrement \( D_k(t) \) for each coordinate sequence of \( x_i(t) \), it can be seen that

\[
D(t) = \sum_{k=1}^{n} D_k(t),
\]

where for each coordinate \( \kappa \in [n] \) and for all \( t \geq 0 \),

\[
D_k(t) = \frac{1}{2} \sum_{i=1}^{m} \pi_i(t+1) \sum_{j=1}^{m} A_{ij}(t) A_{ij}(t) \left( |x_j(t)|_k - |x_j(t)|_k \right)^2.
\]

Observing that the bound of Lemma \([3]\) is valid for each of the decrements \( D_k(t) \), i.e., for all \( \kappa \in [n] \) and \( t \geq 0 \),

\[
D_k(t) \geq \frac{\delta \beta^2}{4 p^*(t)} \max_{j, \ell \in [m]} \left( |x_j(t)|_k - |x_\ell(t)|_k \right)^2.
\]

By using \( p^*(t) \leq p^* \) and by summing the resulting inequalities over \( \kappa \in [n] \), from relations \([28]\) and \([29]\) we obtain

\[
D(t) \geq \frac{\delta \beta^2}{4 p^*} \sum_{k=1}^{n} \max_{j, \ell \in [m]} \left( |x_j(t)|_k - |x_\ell(t)|_k \right)^2 \quad \text{for } t \geq 0.
\]

By noting that

\[
\sum_{k=1}^{n} \max_{j, \ell \in [m]} \left( |x_j(t)|_k - |x_\ell(t)|_k \right)^2 \geq \max_{j, \ell \in [m]} \| x_j(t) - x_\ell(t) \|^2,
\]

we arrive at the following bound

\[
D(t) \geq \frac{\delta \beta^2}{4 p^*} \max_{j, \ell \in [m]} \| x_j(t) - x_\ell(t) \|^2 \quad \text{for } t \geq 0,
\]

which when combined with relation \([27]\) yields the stated relation. 

Theorem \([5]\) provides the key relation that we use to establish the convergence of the projection-based
C. Convergence and Convergence Rate Results

We first show that the algorithm correctly solves the constrained consensus problem. Then, we investigate the rate of convergence of the algorithm in general case and some special instances.

1) Convergence: The following result proves that the iterates of the algorithm converge to a common point in the set $X$.

**Theorem 6.** Let Assumption 2 and Assumption 3 hold. Then, the sequences $\{x_i(t)\}, i \in [m]$, produced by the algorithm 19 are bounded, i.e., there is a scalar $\rho > 0$ such that

$$\|x_i(t)\| \leq \rho \quad \text{for all } i \in [m] \text{ and all } t \geq 0,$$

and they converge to a common point $x^* \in X$:

$$\lim_{t \to \infty} x_i(t) = x^* \quad \text{for some } x^* \in X \text{ and for all } i \in [m].$$

**Proof:** We use Theorem 5 where we let $\tau$ and $T$ be arbitrary times with $T > \tau \geq 0$. By summing the relations given in Theorem 5 over $t = \tau, \ldots, T - 1$, we obtain for all $y \in X$ and all $T > \tau \geq 0$,

$$V(T, y) - V(\tau, y) = -\frac{\delta \beta^2}{4p^*} \sum_{t=\tau}^{T-1} \max_{j, \ell \in [m]} \|x_j(t) - x_\ell(t)\|^2.$$

Based on relation (30), we first show that each sequence $\{x_i(t)\}$ is bounded. By the definition of $V(t, y)$, from (30) it follows that for all $y \in X$ and $T > \tau \geq 0$,

$$\sum_{i=1}^m \pi_i(T)\|x_i(T) - y\|^2 \leq \sum_{j=1}^m \pi_j(\tau)\|x_j(\tau) - y\|^2$$

$$- \frac{\delta \beta^2}{4p^*} \sum_{t=\tau}^{T-1} \max_{j, \ell \in [m]} \|x_j(t) - x_\ell(t)\|^2.$$

(31)

Letting $\tau = 0$ and dropping the non-negative terms in (31), we find that for all $y \in X$ and all $T > 0$,

$$\sum_{i=1}^m \pi_i(T)\|x_i(T) - y\|^2 \leq \sum_{j=1}^m \pi_j(0)\|x_j(0) - y\|^2.$$

By letting $y \in X$ be arbitrary but fixed and using the fact that the adjoint sequence $\{\pi(t)\}$ is uniformly bonded away from zero (cf. Assumption 2(d)), we conclude that each sequence $\{x_i(t)\}$ is bounded, i.e., there is a scalar $\rho > 0$ such that

$$\|x_i(t)\| \leq \rho \quad \text{for all } i \in [m] \text{ and all } t \geq 0,$$

where $\rho$ depends on $\pi(0)$, the initial points $x_i(0), i \in [m]$, the parameter $\delta$ and the chosen point $y \in X$.

Thus, every sequence $\{x_i(t)\}$ has accumulation points. We next show that all the accumulation points of
these sequences coincide, i.e.,
\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{for all } i, j \in [m].
\] (32)

This follows from (31), where by letting \(\tau = 0\) and using non-negativity of \(V(T,y)\) we find that for all \(T > 0\),
\[
\frac{\delta\beta^2}{4p^2} \sum_{t=0}^{T-1} \max_{0 \leq j, \ell \leq |m|} \|x_j(t) - x_\ell(t)\|^2 \leq \sum_{j=1}^{m} \pi_j(0)\|x_j(0) - y\|^2.
\]

Therefore, by letting \(T \to \infty\) we conclude that the sequences \(\{x_j(t)\}\) have the same accumulation points (i.e., (32) is valid). Since each sequence \(\{x_j(t)\}\) lies in the set \(X_i\) and each set \(X_i\) is closed, it follows the accumulation points of each \(\{x_i(t)\}\) lie in the set \(X_i\). Furthermore, since the accumulation points are the same for all of the sequences \(\{x_i(t)\}, \ i \in [m]\), the accumulation points must be in the intersection of the sets \(X_i\), i.e., in the set \(X\).

Finally, we show that the sequences \(\{x_j(t)\}\) can have only one accumulation point, thus showing that they converge to a common point in the set \(X\). To prove this, we argue by contraposition. Suppose that there are two accumulation points for the sequences \(\{x_i(t)\}, \ i \in [m]\). Let \(\{t_s\}\) and \(\{r_s\}\) be the time sequences along which the iterates \(\{x_i(t)\}\) converge, respectively, to two distinct points, say \(\hat{x} \in X\) and \(\check{x} \in X\), with \(\hat{x} \neq \check{x}\),
\[
\lim_{s \to \infty} x_i(t_s) = \hat{x}, \quad \lim_{s \to \infty} x_i(r_s) = \check{x}, \quad \text{for all } i \in V.
\] (33)

Without loss of generality let us assume that \(t_s > r_s\) for all \(s \geq 1\) (for otherwise we can construct such subsequences from \(\{t_s\}\) and \(\{r_s\}\)). In relation (31), we let \(T = t_s\) and \(\tau = r_s\) for any \(s \geq 1\), and thus, obtain (by omitting the non-negative terms) for all \(y \in X\),
\[
\sum_{i=1}^{m} \pi_i(t_s)\|x_i(t_s) - y\|^2 \leq \sum_{j=1}^{m} \pi_j(r_s)\|x_j(r_s) - y\|^2 \quad \text{for all } s \geq 1.
\]

Letting \(y = \hat{x}\) and recalling that the adjoint sequence \(\{\pi(t)\}\) is bounded away from 0, we see that
\[
\delta \sum_{i=1}^{m} \|x_i(t_s) - \hat{x}\|^2 \leq \sum_{j=1}^{m} \pi_j(r_s)\|x_j(r_s) - \hat{x}\|^2 \quad \text{for all } s \geq 1.
\]

Now, letting \(s \to \infty\) we have
\[
\delta \lim_{s \to \infty} \left( \sum_{i=1}^{m} \|x_i(t_s) - \hat{x}\|^2 \right) \leq \lim_{s \to \infty} \left( \sum_{j=1}^{m} \pi_j(r_s)\|x_j(r_s) - \hat{x}\|^2 \right) \leq \sum_{j=1}^{m} \lim_{s \to \infty} \|x_j(r_s) - \hat{x}\|^2,
\]
where in the last inequality we use \(0 \leq \pi_j(t) \leq 1\) for all \(j\) and \(t\). From relation (33) it follows that
\[
\delta \sum_{i=1}^{m} \|\hat{x} - \check{x}\|^2 \leq 0,
\]
thus implying \(\hat{x} = \check{x}\), which is a contradiction. Hence, the sequences \(\{x_i(t)\}, \ i \in [m]\), must be convergent. ■
Theorem 6 shows that Proposition 2 in [26] holds under weaker assumptions on the graphs and the weights. At first, the requirement in [26] that each matrix $A(t)$ is doubly stochastic is relaxed. At second, while here we assume that each of the graphs $G_t$ is rooted, the results easily extend to the case studied in [26] by assuming that the graphs are rooted over at most $B$ units of time and that the absolute probability sequence exists for such unions of the graphs.

2) Convergence Rate: Our convergence rate results are obtained for sets $X_i$ that satisfy a certain regularity condition which relates the distances from a given point to the sets $X_i$ with the distance from the point to the intersection set $X = \cap_{i=1}^m X_i$. One relation that among these distances always holds. In particular, since $X \subseteq X_i$ for all $i$, it follows that

$$\text{dist}(x, X_i) \leq \text{dist}(x, X) \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad i \in [m]. \quad (34)$$

In our analysis, we need an upper bound on $\text{dist}(x, X)$ in terms of the distances $\text{dist}(x, X_i), i \in [m]$. A related generic question is: when the distances of a given point $y$ to a collection of closed convex sets $\{Y_i, i \in \mathcal{I}\}$ can be related to the distance of $y$ from the intersection set $Y = \cap_{i \in \mathcal{I}} Y_i \neq \emptyset$? This question has been studied in the optimization literature within the terminology of error bounds or metric regularity. In this literature, loosely speaking, the question is when the distance $\text{dist}(y, Y)$ is bounded from above by a constant factor of the maximum distance $\max_{i \in \mathcal{I}} \text{dist}(y, Y_i)$. In general, the index set $\mathcal{I}$ can be infinite, but we restrict our attention to finite index sets only.

We will use the following definition of set regularity.

**Definition 2.** Let $Z \subseteq \mathbb{R}^n$ be a nonempty set. We say that a (finite) collection of closed convex sets $\{Y_i, i \in \mathcal{I}\}$ is regular (in Euclidian norm) with respect to the set $Z$, if there is a constant $r \geq 1$ such that

$$\text{dist}(y, Y) \leq r \max_{i \in \mathcal{I}} \{\text{dist}(y, Y_i)\} \quad \text{for all} \quad y \in Z.$$

We refer to the scalar $r$ as a regularity constant. When the preceding relation holds with $Z = \mathbb{R}^n$, we say that the sets $\{Y_i, i \in \mathcal{I}\}$ are uniformly regular.

In view of relation (34) it follows that the regularity constant $r$ must satisfy $r \geq 1$. Note that the regularity constant $r$ in Definition 2 depends on the set $Z$. It also depends on the choice of the metric and the geometry of the sets $\{Y_i, i \in \mathcal{I}\}$. In general, it is hard to compute $r$, but our algorithm does not require the knowledge of such a constant. We just provide a convergence rate result that captures the dependence on $r$.

In view of Theorem 6 the iterate sequences $\{x_i(t)\}, i \in [m]$, are contained a ball $B(0, \rho)$ centered at the origin with a radius $\rho$. We will assume that the sets $X_i$ are regular with respect to the ball $B(0, \rho)$. Later in Section V-C3 we discuss some sufficient conditions for this regularity assumption to hold. Under such a regularity assumption, we show a result that is critical in the subsequent convergence rate analysis.
Lemma 6. Let Assumption 3 hold. Assume further that the sets \( \{X_i, i \in [m]\} \) are regular with respect to a set \( Z \subseteq \mathbb{R}^n \) with a regularity constant \( r \geq 1 \), and assume that \((X_1 \times \cdots \times X_m) \cap (Z \times \cdots \times Z) \neq \emptyset \). Let \( \phi \in \mathbb{R}^m \) be a given stochastic vector. Then, for all \((x_1, \ldots, x_m) \in (X_1 \times \cdots \times X_m) \cap (Z \times \cdots \times Z) \) we have

\[
\max_{j, \ell} \|x_j - x_\ell\| \geq \frac{1}{r + 1} \max_{p \in [m]} \left\| x_p - \mathbb{P}_X \left[ \sum_{i=1}^m \phi_i x_i \right] \right\|
\]

Proof: Let \((x_1, \ldots, x_m) \in (X_1 \times \cdots \times X_m) \cap (Z \times \cdots \times Z) \) be arbitrary, and define \( u = \sum_{i=1}^m \phi_i x_i \). Let \( \ell \in [m] \) be arbitrary. Consider estimating \( \|x_\ell - \mathbb{P}_X[u]\| \) as follows:

\[
\|x_\ell - \mathbb{P}_X[u]\| \leq \|x_\ell - \mathbb{P}_X[x_\ell]\| + \|\mathbb{P}_X[x_\ell] - \mathbb{P}_X[u]\| \\
\leq r \max_{j \in [m]} \{\text{dist}(x_\ell, X_j)\} + \|x_\ell - u\|
\]

where the first inequality uses the triangle inequality for the norm. The second inequality uses the fact \( \|x_\ell - \mathbb{P}_X[x_\ell]\| = \text{dist}(x_\ell, X) \) and the set regularity assumption for the first term (i.e., \( \text{dist}(y, X) \leq r \max_y \text{dist}(y, X_i) \) for all \( y \in Z \) and the fact \( x_\ell \in Z \)), while the second term is estimated by using the non-expansiveness property of the projection map (see (20)). By the definition of the projection, we have

\[
\text{dist}(x_\ell, X_j) = \min_{j \in X_j} \|x_\ell - y\| \leq \|x_\ell - x_j\|
\]

where the inequality follows by \( x_j \in X_j \) for all \( j \). Thus,

\[
\|x_\ell - \mathbb{P}_X[u]\| \leq r \max_{j \in [m]} \|x_\ell - x_j\| + \|x_\ell - u\|. \tag{35}
\]

Consider now the term \( \|x_\ell - u\| \). By the definition of \( u \), this vector is a convex combination of points \( x_i, i \in [m] \), since \( \phi \) is a stochastic vector. Thus, by the convexity of the Euclidean norm, it follows that

\[
\|x_\ell - u\| = \left\| \sum_{i=1}^m \phi_i (x_\ell - x_i) \right\| \leq \sum_{i=1}^m \phi_i \|x_\ell - x_i\| \leq \max_{i \in [m]} \|x_\ell - x_i\|
\]

By substituting the preceding estimate in relation (35), we obtain

\[
\|x_\ell - \mathbb{P}_X[u]\| \leq (r + 1) \max_{j \in [m]} \|x_\ell - x_j\|.
\]

So far the index \( \ell \) was arbitrary, so by taking the maximum over all \( \ell \in [m] \), we find that

\[
\max_{\ell \in [m]} \|x_\ell - \mathbb{P}_X[u]\| \leq (r + 1) \max_{j, \ell \in [m]} \|x_\ell - x_j\|
\]

and the desired relation follows after dividing by \( r + 1 \).

With Lemma 6 in place, we investigate the rate of decrease of the Lyapunov comparison function \( V(t, y) \), as given in (22). We have the following result.

Theorem 7. Let Assumption 2 and Assumption 3 hold. Assume further that the sets \( \{X_i, i \in [m]\} \) are regular,
with a regularity constant \( r \geq 1 \), with respect to a ball \( B(0, \rho) \) which contains all the iterates \( \{x_i(t)\} \) generated by the algorithm [19]. Consider the following vectors

\[
 u(t) = \sum_{i=1}^{m} \pi_i(t)x_i(t), \quad v(t) = P_X[u(t)], \quad \text{for all } t \geq 0. \tag{36}
\]

Then, the Lyapunov comparison function \( V(t, v(t)) \) decreases at a geometric rate: for all \( t \geq 0 \),

\[
 V(t+1, v(t+1)) \leq \left( 1 - \frac{\delta \beta^2}{4p^*(r+1)^2} \right) V(t, v(t)),
\]

where the scalars \( \delta, \beta \in (0, 1) \) and the integer \( p^* \geq 1 \) are the same as in Theorem 5.

**Proof:** In Theorem 5 we let \( y = v(t) \) with \( v(t) \in X \) and we use the definition of \( u(t) \). Then, we have for all \( t \geq 0 \),

\[
 V(t+1, v(t)) \leq V(t, v(t)) - \frac{\delta \beta^2}{4p^*} \max_{j, \ell \in [m]} \| x_j(t) - x_{\ell}(t) \|^2. \tag{37}
\]

Next, we consider the term \( V(t+1, v(t)) \). We have

\[
 V(t+1, v(t)) = \sum_{i=1}^{m} \pi_i(t+1)\| x_i(t+1) - v(t) \|^2
 = \sum_{i=1}^{m} \pi_i(t+1)\| x_i(t+1) - v(t+1) + (v(t+1) - v(t)) \|^2.
\]

By expanding the squared-norm terms, we obtain

\[
 V(t+1, v(t)) \geq \sum_{i=1}^{m} \pi_i(t+1)\| x_i(t+1) - v(t+1) \|^2
 + 2 \left( \sum_{i=1}^{m} \pi_i(t+1)x_i(t+1) - v(t+1) \right)' (v(t+1) - v(t)),
\]

where the inequality is obtained by dropping the term \( \| v(t+1) - v(t) \|^2 \). In view of the definition of the vector \( u(t+1) \) (cf. (36)), it follows that

\[
 V(t+1, v(t)) = \sum_{i=1}^{m} \pi_i(t+1)\| x_i(t+1) - v(t+1) \|^2
 + 2 (u(t+1) - v(t+1))' (v(t+1) - v(t)),
\]

Since \( v(t+1) \) is the projection of \( u(t+1) \) on the set \( X \) and since \( v(t) \in X \), it further follows that

\[
 (u(t+1) - v(t+1))' (v(t+1) - v(t)) \geq 0
\]
By combining the preceding relation with (37) we can conclude that for all \( t \geq 0 \),
\[
\mathcal{V}(t+1, \mathbf{v}(t)) \geq \sum_{i=1}^{m} \pi_i(t+1) \| \mathbf{x}_i(t+1) - \mathbf{v}(t+1) \|^2 = \mathcal{V}(t+1, \mathbf{v}(t+1)).
\]

By substituting the estimate (39) into inequality (38) we obtain the desired relation.

To estimate the term \( \max_{j,\ell \in [m]} \| \mathbf{x}_j(t) - \mathbf{x}_\ell(t) \|^2 \) from below we use Lemma 6 with the following identification:
\[ Z = B(0, \rho), \mathbf{x}_i = \mathbf{x}_i(t), \phi = \pi(t) \text{ and } \mathbf{u} = \mathbf{u}(t), \] we note that \( \mathbf{x}_i(t) \in Z \) for all \( i \) and \( t \). Thus, by Lemma 6 we have
\[
\max_{j,\ell \in [m]} \| \mathbf{x}_j(t) - \mathbf{x}_\ell(t) \| \geq \frac{1}{r+1} \max_{p \in [m]} \| \mathbf{x}_p(t) - \mathbb{P}_X[\mathbf{u}(t)] \|.
\]

In our notation, we have \( \mathbf{v}(t) = \mathbb{P}_X[\mathbf{u}(t)] \) (see (36)), so by using \( \mathbf{v}(t) \) and by taking squares in the preceding relation we obtain
\[
\max_{j,\ell \in [m]} \| \mathbf{x}_j(t) - \mathbf{x}_\ell(t) \|^2 \geq \frac{1}{(r+1)^2} \max_{p \in [m]} \| \mathbf{x}_p(t) - \mathbf{v}(t) \|^2.
\]

Since the vector \( \pi(t) \) is stochastic, we have
\[
\max_{p \in [m]} \| \mathbf{x}_p(t) - \mathbf{v}(t) \|^2 \geq \sum_{i=1}^{m} \pi_i(t) \| \mathbf{x}_i(t) - \mathbf{v}(t) \|^2 = \mathcal{V}(t, \mathbf{v}(t)),
\]
where the equality uses the definition of \( \mathcal{V}(t, y) = \sum_{i=1}^{m} \pi_i(t) \| \mathbf{x}_i(t) - y \|^2 \) (see (22)). Therefore
\[
\max_{j,\ell \in [m]} \| \mathbf{x}_j(t) - \mathbf{x}_\ell(t) \|^2 \geq \frac{1}{(r+1)^2} \mathcal{V}(t, \mathbf{v}(t)). \tag{39}
\]

By substituting the estimate (39) into inequality (38) we obtain the desired relation.

Using the decrease rate result for the Lyapunov comparison function \( \mathcal{V}(t, y) \) of Theorem 7 and the properties of the adjoint dynamics, we can now estimate the rate of convergence of the iterates \( \{ \mathbf{x}_i(t) \} \).

**Theorem 8.** Let Assumption 2 and Assumption 3 hold. Assume further that the sets \( \{ \mathbf{x}_i, i \in [m] \} \) are regular, with a regularity constant \( r \geq 1 \), with respect to a ball \( B(0, \rho) \) which contains all the iterates \( \{ \mathbf{x}_i(t) \} \) generated by the algorithm (19). Then, the sequences \( \{ \mathbf{x}_i(t), i \in [m], \} \), are such that for all \( t \geq 0 \),
\[
\sum_{j=1}^{m} \text{dist}^2(\mathbf{x}_j(t), X) \leq \frac{1}{\delta} \left( 1 - \frac{\delta \beta^2}{44^* (r+1)^2} \right)^t \mathcal{V}(0, \mathbf{v}(0)),
\]
where \( \mathbf{v}(0) = \mathbb{P}_X[\mathbf{u}(0)] \) with \( \mathbf{u}(0) = \sum_{j=1}^{m} \pi_j(0) \mathbf{x}_j(0) \), while the scalars \( \delta, \beta \in (0, 1) \) and the integer \( p^* \geq 1 \) are the same as in Theorem 5.

**Proof:** From Theorem 7 it can be seen that \( \mathcal{V}(t, \mathbf{v}(t)) \leq \left( 1 - \frac{\delta \beta^2}{44^* (r+1)^2} \right)^t \mathcal{V}(0, \mathbf{v}(0)) \) for all \( t \geq 0 \). The result
follows by recalling that $V(t, y) = \sum_{i=1}^{m} \pi_i(t)\|x_i(t) - y\|^2$, recalling the definition of $v(t)$ (see (36)), and using the fact that the vectors $\pi(t)$ have uniformly bounded entries from below by $\delta > 0$ (cf. Assumption 2(d)). □

Theorem 8 extends the convergence rate result obtained originally in [26], where the convergence rate was analyzed for a special case when the matrices $A(t)$ are doubly stochastic, and the graph is static and complete, i.e., $A(t) = \frac{1}{m}11'$ for all $t$.

3) Sufficient Conditions for Set Regularity: We discuss two cases of sufficient conditions for the set regularity property, namely, the case of a polyhedral set $X$, and the case of $X$ with a nonempty interior.

### Polyhedral Set $X$

Let $X \subseteq \mathbb{R}^n$ be a nonempty polyhedral set. We will show that use the description of $X$ in terms of linear inequalities,

$$X = \{x \in \mathbb{R}^n \mid a_i^j x \leq b_i, \quad i \in \mathcal{I}\},$$

where $\mathcal{I}$ is a finite index set, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all $i$. For such a set, Hoffman in [54] had shown that the distance from any point $x \in \mathbb{R}^n$ to the set $X$ is bounded from above by the maximal distance from $x$ to any of the hyperplanes defined by the linear inequalities, i.e., that there exists a constant $r \geq 1$ such that

$$\text{dist}(x, X) \leq r \max_{i \in \mathcal{I}} \text{dist}(x, H_i) \quad \text{for all } x \in \mathbb{R}^n,$$

(40)

where, for every $i$, the set $H_i$ is the hyperplane given by $H_i = \{x \in \mathbb{R}^n \mid a_i^j x \leq b_i\}$, while the constant $r$ depends on the set of normals $\{a_i, i \in \mathcal{I}\}$ that define the hyperplanes $\{H_i, i \in \mathcal{I}\}$. We will refer to this relation as the Hoffman bound. We will use this bound to show that, when each set $X_i$ is polyhedral, the sets $X_i$ are uniformly regular.

**Proposition 1.** Assume that each set $X_j$, $j \in [m]$, is given by $X_j = \{x \in \mathbb{R}^n \mid (a^{(j)}_i)^\ell x \leq b^{(j)}_\ell, \ell \in \mathcal{I}_j\}$. Also, assume that $X = \bigcap_{i=1}^m X_i$ is nonempty. Then, the sets $X_i$ are uniformly regular with the regularity constant equal to the constant $r$ in the Hoffman bound (40), where $\mathcal{I} = \bigcup_{j=1}^m \mathcal{I}_j$, i.e.,

$$\text{dist}(x, X) \leq r \max_{i \in [m]} \text{dist}(x, X_i) \quad \text{for all } x \in \mathbb{R}^n.$$

**Proof:** Note that the set $X$ is the intersection of the hyperplanes that define the sets $X_i$, i.e.,

$$X = \bigcap_{\ell=1}^m \left(\bigcap_{j=1}^m H^{(j)}_{\ell}\right),$$

where $H^{(j)}_{\ell} = \{x \mid (a^{(j)}_i)^\ell x \leq b^{(j)}_\ell\}$. By the Hoffman bound, there is an $r \geq 1$ such that

$$\text{dist}(x, X) \leq r \max_{j \in [m]} \max_{\ell \in \mathcal{I}_j} \left\{\text{dist}(x, H^{(j)}_{\ell}) \right\} \quad \text{for all } x \in \mathbb{R}^n.$$

(41)

For every $j \in [m]$, we have $H^{(j)}_{\ell} \supseteq X_j$ for all $\ell \in \mathcal{I}_j$, thus implying that for every $j \in [m]$,

$$\max_{\ell \in \mathcal{I}_j} \left\{\text{dist}(x, H^{(j)}_{\ell}) \right\} \leq \text{dist}(x, X_j) \quad \text{for all } x \in \mathbb{R}^n.$$
The preceding relation and (41) yield
\[
\text{dist}(x, X) \leq r \max_{j \in [m]} \{\text{dist}(x, X_j)\} \quad \text{for all } x \in \mathbb{R}^n.
\]

Thus, the sets \(X_i, i \in [m]\) are uniformly regular.

Hence, when the sets \(X_i\) are polyhedral, they are uniformly regular and thus, also regular with respect to any ball \(B(0,\rho)\) that contains the sequences \(\{x_i(t)\}\). Consequently, when the sets \(X_i\) are polyhedral, the regularity condition of Theorem 8 holds.

**Set \(X\) with Nonempty Interior.** The regularity condition also holds when the interior of the intersection set \(X\) is nonempty. The proof uses some ideas from \([55]\) (see the proof of Lemma 5 there). However, in this case, the set regularity property is not global.

**Proposition 2.** Let Assumption 3 hold, and assume that the set \(X = \cap_{j \in [m]} X_j\) has a nonempty interior, i.e., there is a vector \(\tilde{x} \in X\) and a scalar \(\theta > 0\) such that \(\{z \in \mathbb{R}^n \mid \|z - \tilde{x}\| \leq \theta\} \subseteq X\). Let \(Y \subseteq \mathbb{R}^n\) be a bounded set. Then, we have
\[
\text{dist}(x, X) \leq r \max_{j \in [m]} \{\text{dist}(x, X_j)\} \quad \text{for all } x \in Y,
\]
with \(r = \frac{1}{\theta} \max_{y \in Y} \|y - \tilde{x}\|\).

**Proof:** Let \(x \in \mathbb{R}^n\) be arbitrary. Define \(\epsilon = \max_{j \in [m]} \{\text{dist}^2(x, X_j)\}\) and consider the vector \(y = \frac{\epsilon}{\epsilon + \theta} \tilde{x} + \frac{\theta}{\epsilon + \theta} x\).

We show that \(y \in X\). To see this note that we can write for each \(j \in [m]\),
\[
y = \frac{\epsilon}{\epsilon + \theta} (\tilde{x} + \frac{\theta}{\epsilon} (x - \mathbb{P}_{X_j}[x])) + \frac{\theta}{\epsilon + \theta} \mathbb{P}_{X_j}[x].
\]

The vector \(z = \tilde{x} + \frac{\theta}{\epsilon} (x - \mathbb{P}_{X_j}[x])\) satisfies
\[
\|z - \tilde{x}\| = \frac{\theta}{\epsilon} \|x - \mathbb{P}_{X_j}[x]\| \leq \frac{\theta}{\epsilon} \max_{j \in [m]} \|x - \mathbb{P}_{X_j}[x]\| = \theta,
\]
where the last equality follows by the definition of \(\epsilon\) and \(\text{dist}(x, X_j) = \|x - \mathbb{P}_{X_j}[x]\|\). Thus, since \(\tilde{x}\) is an interior point of \(X\), it follows that \(z \in X \subseteq X_i\) for all \(i \in [m]\). Since the vector \(y\) is a convex combination of \(z \in X_j\) and \(\mathbb{P}_{X_j}[x] \in X_j\), by the convexity of the set \(X_j\), it follows that \(y \in X_j\).

Therefore, for each \(j\), the vector \(y\) can be written as a convex combination of two points in \(X_j\), implying that \(y \in X_j\) for all \(j \in [m]\). Consequently, we have \(y \in X\), so that \(\text{dist}(x, X) \leq \|x - y\| = \frac{\epsilon}{\epsilon + \theta} \|x - \tilde{x}\| \leq \frac{\theta}{\epsilon} \|x - \tilde{x}\|\).

Using the definition of \(\epsilon\), we obtain \(\text{dist}(x, X) \leq \frac{1}{\theta} \|x - \tilde{x}\| \max_{j \in [m]} \{\text{dist}(x, X_j)\}\), which is valid for any \(x \in \mathbb{R}^n\).

By using \(\|x - \tilde{x}\| \leq \max_{y \in Y} \|x - \tilde{x}\|\), we arrive at
\[
\text{dist}(x, X) \leq \left(\frac{1}{\theta} \max_{y \in Y} \|y - \tilde{x}\|\right) \max_{j \in [m]} \{\text{dist}(x, X_j)\} \quad \text{for all } x \in Y.
\]
VI. Conclusion

We have investigated the properties of the weighted-averaging dynamic for consensus problem using Lyapunov approach. We have established new convergence rate results in terms of the longest shortest path of spanning trees contained in the graph. For constrained consensus, we established exponential convergence rate assuming some regularity conditions on the constraint sets. These results easily extend to the cases where the underlying graphs are not necessarily rooted at every instant, but rather rooted over a period of time.

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