A Graph Theoretical Approach to the Collatz Problem

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Abstract

Andrei et al. have shown in 2000 that the graph $C$ of the Collatz function starting with root $8$ after the initial loop is an infinite binary tree $A(8)$. According to their result they gave a reformulated version of the Collatz conjecture: the vertex set $V(A(8)) = \mathbb{Z}^+$. In this paper an inverse Collatz function $\overleftarrow{C}$ with eliminated initial loop is used as generating function of a Collatz graph $\overleftarrow{C}(\mathbb{N})$. This graph can be considered as the union of one forest that stems from sequences of powers of $2$ with odd start values and a second forest that is based on branch values $y = 6k + 4$ where two Collatz sequences meet. A proof that the graph $\overleftarrow{C}(1)$ is an infinite binary tree $A(\overleftarrow{C}(1))$ with vertex set $V(A(\overleftarrow{C}(1)) = \mathbb{Z}^+$ completes the paper.

Key Words: 3n+1 Problem, Collatz Conjecture, Collatz Graph, Infinite Tree, Infinite Forest.

MSC-Class: 11B83, 05C05, 05C63

1 The Collatz function and conjecture

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}^+$ be the positive integers, then the Collatz problem relates to the Collatz map $\overleftarrow{C}: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$:

$$\overleftarrow{C}(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod{2}, \\ 3n + 1 & \text{if } n \equiv 1 \pmod{2}, \end{cases} \overleftarrow{C}(n) \in \mathbb{Z}^+, \overleftarrow{C}(n) \equiv 4 \pmod{6}. \tag{1}$$

The famous $3n+1$ or Collatz conjecture now states that for any $n \in \mathbb{Z}^+$ there exists a $k \in \mathbb{N}$ such that:

$$\overleftarrow{C}^{(k)}(n) = 1, \quad [\overleftarrow{C}^{(0)}(n) = n \text{ and } \overleftarrow{C}^{k}(n) = \overleftarrow{C} \circ \overleftarrow{C}^{k-1}(n)].$$

The conjecture excludes the existence of other loops than the trivial terminal cycle $(1, 2, 4, 1, \ldots)$ and of any divergent sequences.

2 The Collatz tree and a modified conjecture

Most papers deal with the dynamics of the Collatz function $\overleftarrow{C}$ or modified versions of it while pure graph theoretical aspects have seldom been considered. Some exceptions are Andaloro [1], Andrei et al. [2, 3], Laarhoven and de Weger [6], Lang [7] and Wirsching [8].
Andrei et al. [2,3] examined a graph $C$ of the Collatz function and showed that a subgraph of $C$ with the vertex set $V \subseteq \mathbb{Z}^+ - \{1, 2, 4\}$ and the value 8 as root is an infinite binary tree $A(8)$. Therefore they called it Collatz tree. According to this result they reformulated the Collatz conjecture to be:

The vertex set of the Collatz tree $A(8)$ is $V = \mathbb{Z}^+ - \{1, 2, 4\}$.

Their conclusions also lead to the fact that every $n > 4$ could be the root of a Collatz tree $A(n)$. Then they concentrate on infinite chain subtrees which are characterized by values which are divisible by 3. Graphs without these chain subtrees are called pruned Collatz graphs [8]. This approach leads to infinite sets of start numbers whose sequences converge at 1.

3 The inverse Collatz function

Let the set $\mathbb{Y} = \{ n > 4 \mid n \equiv 4 \mod 6 \} \subset \mathbb{Z}^+$, then the inverse Collatz map $\overrightarrow{C} : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is:

$$\overrightarrow{C}(n) = \begin{cases} 2n & \text{if } n \in \mathbb{Z}^+, \\ (n-1)/3 & \text{if } n \in \mathbb{Y}, \quad \overrightarrow{C}(n) \equiv 0 \mod 2, \\ 1 & \text{if } n \in \mathbb{Y}, \quad \overrightarrow{C}(n) \equiv 1 \mod 2. \end{cases} \quad (2)$$

Although the two operations of the Collatz function $\overrightarrow{C}$ have the above unique inverses in the definition of $\overrightarrow{C}$, the function $\overrightarrow{C}$ itself is not unique. This is because $\mathbb{Y}$ is a proper subset of $\mathbb{Z}^+$. This leads to the fact that every $y \in \mathbb{Y}$ always has two descendants. It is obvious that the operation $2n$ simply continues its current sequence while the operation $(n-1)/3$ results in an odd number and starts a complete new sequence. Therefore we call the numbers $y$ branch values. As 4 is such a branch value we excluded 4 from the set $\mathbb{Y}$ to avoid the otherwise inevitable initial loop $(1, 2, 4, \frac{1}{8}, \ldots)$.

4 The Collatz graph of the inverse Collatz function

In 1977 Lothar Collatz remarks in a paper on the use of graph representations to study iteration problems of functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ but he did not consider the $3n+1$ problem therein [4]. His idea was to picture such dynamical systems by infinite graphs of the following kind:

**Definition 4.1** Let $(n, f(n)) \in \mathbb{Z}^+$, then an infinite Collatz graph is generally defined by:

$$C_f(V_f, E_f) = \begin{cases} V_f = \mathbb{Z}^+ & \text{the set of vertices}, \\ E_f = \{ n, f(n) \mid (n, f(n)) \in V_f \} & \text{the set of directed edges}. \end{cases} \quad (3)$$

There are important differences between normal graphs and Collatz graphs:

1. The vertex set and the results of the generating function $f$ are restricted to the set $\mathbb{Z}^+$.
2. Vertices and their labels are indistinguishable.
3. The map $f$ determines the set of edges and their direction $n \rightarrow f(n)$.
4. The map $f$ enforce the properties of the vertices/labels.

An example for the above point 4 are the numbers $y \in \mathbb{Y}$. The property induced by the maps $\overrightarrow{C}$ and $\overrightarrow{C}$ is that all $y \equiv 4 \mod 6$. But $\overrightarrow{C}$ named now as Collatz backward function ignores that these numbers are branch values. The inverse Collatz function $\overrightarrow{C}$ is therefore much more appropriate as a generating function of a graph and so we use from now on the map $\overrightarrow{C}$ named
as Collatz forward function for the construction of our Collatz graphs exclusively. Since \( \overrightarrow{C} \) is the inverse map of \( \overleftarrow{C} \) we even relax the demand of Definition 4.1 that the edges have to be directed. Thus we define the common graph for both Collatz functions as:

\[
C_{\overrightarrow{C}} = \begin{cases} 
V_{\overrightarrow{C}} = \mathbb{Z}^+ \\
E_{\overrightarrow{C}} = (n, \overrightarrow{C}(n)) \in V_{\overrightarrow{C}} 
\end{cases}
\]

the set of vertices

(4)

and the Collatz conjecture reads now:

The graph \( C_{\overrightarrow{C}}(1) \) is an undirected infinite binary tree \( A_{\overrightarrow{C}}(1) \) with the vertex set \( V = \mathbb{Z}^+ \).

5 The Collatz graph \( C_{\overrightarrow{C}} \) as union of two infinite forests

Diestel defines a forest as: A graph without circles is called a forest. A connected forest is a tree. Thus a forest is a graph whose components are trees [5].

5.1 The forest \( F_h \)

We now show what happens if we repeatedly apply the operation \( n' = 2n \) of \( \overrightarrow{C} \) to all odd start numbers \( o \in \mathbb{O} = \{n > 0 | n \equiv 1 \pmod{2}\} \). The inverse operation is \( n' = n/2 \) of \( \overleftarrow{C} \) applied to any even number \( n \in \mathbb{E} = \{n > 0 | n \equiv 0 \pmod{2}\} \) until \( n' \) is odd.

**Theorem 5.1** Let \( o \in \mathbb{O} \) and \( d \in \mathbb{N} \), then with \( o \to \infty \) and \( d \to \infty \) the Collatz graph \( C_h \) generated by the function \( h(o,d) = o \cdot 2^d \) is an infinite forest \( F_h \) of distinct infinite trees \( A_h(o) \) with the set of vertices \( V(F_h) = \mathbb{Z}^+ \).

**Proof:** For any fixed \( O \in \mathbb{O} \) and \( d \to \infty \) the infinite sequence \( h(O,d) = O \cdot 2^d \) resembles a single infinite tree \( A_h(O) \) without any branches. Thus with \( o \to \infty \) we get a set of unconnected infinite trees: the forest \( F_h \) with the set of edges \( E(F_h) = \{e | e = o \cdot 2^d, o \cdot 2^{d+1}\} \) (Figure 1). For \( d = 0 \) the codomain of \( h(o,0) \) is the set \( \mathbb{O} \) and for \( d > 0 \) the codomain of \( h(o,d) \) is the set \( \mathbb{E} \). The set of vertices of \( F_h \) is \( V(F_h) = \mathbb{O} \cup \mathbb{E} = \mathbb{Z}^+ \).

**Corollary 5.1** Obviously all vertices \( o \in \mathbb{O} \) as roots of the trees \( A_h(o) \) have one incident edge and all nodes \( v \in \mathbb{E} \) have two incident edges.

5.2 The forest \( F_b \)

Now we exclusively apply the operation \( b = (y-1)/3 \) of \( \overleftarrow{C} \) to all branch numbers \( y > 4 \). The inversion is the operation \( y = 3o + 1 \) of \( \overleftarrow{C} \) applied to all numbers \( o > 1 \).

**Theorem 5.2** Let \( o \in \mathbb{O} \), \( y \in \mathbb{Y} \) and the map \( b: \mathbb{Y} \to \mathbb{O}: b(y) = (y-1)/3 \), then with \( y \to \infty \) the Collatz graph \( C_b \) is an infinite forest \( F_b \) of distinct trees \( A_b(y) \).

**Proof:** \( E(C_b) = \{e | e = (y,o)\} \) and \( V(C_b) = \{\mathbb{Y} \cup \mathbb{O}\} \subset \mathbb{Z}^+ \). Since all edges \( e \in E(C_b) \) are different each edge \( e \) represents a single tree \( A_b(y) \). With \( y \to \infty \) we get the forest \( F_b \) as set of infinitely many unconnected trees \( A_b(y) \) (Figure 2).

**Corollary 5.2** Obviously all vertices \( y \in \mathbb{Y} \) and \( o \in \mathbb{O} \) of the trees \( A_b(y) \) have one incident edge.
Figure 1: Grid graph of the Forest $F_h$. The generating function $h(o,d) = o \cdot 2^d$ dictates the colors indicating the properties of the nodes: $v \equiv 1 \pmod{2}$ black, $v \equiv 4 \pmod{6}$ yellow, $v \equiv 2 \pmod{6}$ grey, $v \equiv 0 \pmod{6}$ white.

Figure 2: Grid graph of the Forest $F_b$. The generating function is $b(y) = (y - 1)/3$ and the properties of the vertices are: $v \equiv 1 \pmod{2}$ black, $v \equiv 4 \pmod{6}$ yellow.
5.3 Consequences of the union of $F_h$ and $F_b$

The separate application of operations of the generating functions $C$ and $\overrightarrow{C}$ split the Collatz graph $C_{\overrightarrow{C}}$ into two different forests. The re-union of $F_h$ and $F_b$ changes the sets of edges and the incidences of the nodes of both forests (Figure 3).

![Collatz graph as union of $F_h(o)$ and $F_b(o)$](image)

Figure 3: The grid graph $C_{\overrightarrow{C}}$. The forest $F_h$ rules the vertical and the forest $F_b$ the diagonal edges of this graph. Circles represent nodes $v \equiv 2 \pmod{6}$, $v \equiv 0 \pmod{6}$ and $v = 4$.

Lemma 5.1 $E(F_h) \cap E(F_b) = \{0\}$.

Proof:
Let $d \in \mathbb{N}$, $o \in \mathbb{O}$, $y \in \mathbb{Y}$, $E(F_h) = \{e|e = (o \cdot 2^d, o \cdot 2^{d+1})\}$ and $E(F_b) = \{e|e = (y, o)\}$, then all $e_{h,2} = o \cdot 2^{d+1}$ of $E(F_h)$ are even and all $e_{b,2} = o$ of $E(F_b)$ are odd and therefore all edges of $E(F_h)$ and $E(F_b)$ are different. ■

Theorem 5.3 $C_{\overrightarrow{C}} = F_h \cup F_b$.

Proof:
Because of Lemma 5.1 the union $E(F_h) \cup E(F_b) = E(C_{\overrightarrow{C}})$ introduces no multiple edges. As $V(F_h) = \mathbb{Z}^+$ and $V(F_b) \subset \mathbb{Z}^+$ therefore $V(C_{\overrightarrow{C}}) = V(F_h) \cup V(F_b) = \mathbb{Z}^+$. ■

Theorem 5.4 All nodes $v \in V(C_{\overrightarrow{C}})$ have at most three incident edges.

Proof:
Due to Lemma 5.1 and Theorem 5.3 we can add and count the incident edges of $E(C_{\overrightarrow{C}})$:
1. The root $v = 1$ is no vertex of $F_b$ and so only has one undirected edge $e = (1, 2)$.
2. For all nodes $o > 1$ there exist two undirected edges $(o, y)$, $(o, 2o)$.
3. For all nodes $y \in \mathbb{Y}$ there exist three undirected edges $(y, y/2)$, $(y, 2y)$, $(y, o)$.
4. For all vertices $v \in \mathbb{E} - \mathbb{Y}$ there exist two undirected edges $(v, v/2)$, $(v, 2v)$. ■
6 Proof of the Collatz conjecture

The detour due to splitting the Collatz graph $C_{\overline{C}}$ into separate components leads to important insights into the overall structure of this graph provoked by the generating function.

**Theorem 6.1** The Collatz graph $C_{\overline{C}}(1)$ is an infinite connected graph with vertex set $\mathbb{Z}^+$.  

**Proof:**

1. According to Theorems 5.1 and 5.3 is $V(C_{\overline{C}}(1)) = \mathbb{Z}^+$ guaranteeing the infiniteness too.
2. The inverse operations of $\overline{C}$ and $\overline{C}$ provoke edges that match bijective maps. Thus $C_{\overline{C}}(1)$ is an undirected graph.
3. The graph $C_{\overline{C}}(1)$ is connected. If we assume that it is not connected, there has to be at least one node $v \neq 1$ which has no edge to a predecessor or successor. But this is a contradiction to the fact that the root $v = 1$ is the only vertex which has just one incident edge. All nodes $v \neq 1$ either have two or three definite incident edges according Item 2 and Theorem 5.4. ■

The graph $C_{\overline{C}}(1)$ is connected indeed but in the representation of (Figure 3) it seems to be an utter mess of edges crossing each other in an arbitrary manner. The existence of circuits cannot be excluded. On the contrary a binary tree is a well structured planar graph whose nodes can be arranged in height-oriented levels (Figure 4).

![Figure 4: Height-oriented binary Collatz tree $A_{\overline{C}}(1)$ up to level $h = 13$ with an indicated continuation for $h = 14$.](image)

If we are able to transform the graph $C_{\overline{C}}(1)$ into a level-oriented binary tree then this is a proof that there cannot be any circuits. This could be done by a recursive procedure that evaluates the function $\overline{C}$ and pulls all subtrees nearer to the tree they descent from. But this fails because the recursion of any subtree will never come to an end as there are no leaves that
terminate the recursive descent. However the function $\overrightarrow{C}$ offers an iterative procedure instead that assigns a height and a dedicated level to all nodes of $C_{\overrightarrow{C}}(1)$. 

**Theorem 6.2** The Collatz graph $C_{\overrightarrow{C}}(1)$ can be transformed to an infinite undirected binary tree $A_{\overrightarrow{C}}(1)$.

**Proof:** We construct $C_{\overrightarrow{C}}(1)$ by using the function $\overrightarrow{C}$ and induction.

1. We assume that $C_{\overrightarrow{C}}(1)$ is a binary tree up to the level $h=13$ and as Figure 4 shows this is true for $h=13$.
2. In respect of $\overrightarrow{C}$ all nodes $v \neq 1$ have one incoming edge only. For even nodes $v$ these are the edges $\overrightarrow{e} = \langle v/2, v \rangle$ and $\overrightarrow{e} = \langle y, o \rangle$ for odd vertices $o \neq 1$ (Theorem 5.4).
3. $\overrightarrow{C}$ creates no nodes on level $h$ that can have an outgoing edge $\overrightarrow{e} = \langle v, 2v \rangle$ or $\overrightarrow{e} = \langle y, o \rangle$ to a vertex on the levels from 0 up to and including $h$ itself since all these nodes are already saturated regarding to their indegree viz. the number of incoming edges.
4. Thus all successors of the nodes of the level $h$ have to be arranged on the next higher level $h' = h + 1$.
5. The constraints of the Items 2 to 4 apply to all nodes of every new level $h'$ and so induction applies ad infinitum since $C_{\overrightarrow{C}}(1)$ is connected.

There are no circuits in $C_{\overrightarrow{C}}(1)$ thus it is an infinite binary tree $A_{\overrightarrow{C}}(1)$ with vertex set $\mathbb{Z}^+$ and therefore the Collatz conjecture is true. ■

**7 References**

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