Elastic spheres can walk on water

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Incited by public fascination and engineering application, water-skipping of rigid stones and spheres has received considerable study. While these objects can be coaxed to ricochet, elastic spheres demonstrate superior water-skipping ability, but little is known about the effect of large material compliance on water impact physics. Here we show that upon water impact, very compliant spheres naturally assume a disk-like geometry and dynamic orientation that are favourable for water-skipping. Experiments and numerical modelling reveal that the initial spherical shape evolves as elastic waves propagate through the material. We find that the skipping dynamics are governed by the wave propagation speed and by the ratio of material shear modulus to hydrodynamic pressure. With these insights, we explain why softer spheres skip more easily than stiffer ones. Our results advance understanding of fluid-elastic body interaction during water impact, which could benefit inflatable craft modelling and, more playfully, design of elastic aquatic toys.
Water-skipping has been studied for centuries with diverse motivations including: the ancient art of stone skipping\textsuperscript{1,2}; naval application\textsuperscript{3–10}; water-surface craft\textsuperscript{11,12}; and biological\textsuperscript{13,14} and biomimetic\textsuperscript{15} water-walking. While water ricochet of rigid objects has been well studied, the physics underlying the water impact of highly deformable elastic solids remains poorly understood\textsuperscript{12,16,17}. Compliant bodies such as inflatable boats\textsuperscript{13} and elastic aquatic toys\textsuperscript{18,19} exhibit behaviour that is not readily explained within the traditional framework for rigid objects. For such elastic bodies, an understanding of the coupling between the material response and hydrodynamic loading is essential in unravelling the overall dynamics.

An object obliquely impacting a water surface with sufficient inertia will carve a cavity on the air-water interface\textsuperscript{20} and experience a pressure-driven hydrodynamic force dependent on object velocity, geometry and orientation\textsuperscript{4,8,21}. Water-skipping occurs when the upward vertical component of this force is large enough to lift the object off the water surface. Studies born from naval applications ranging from cannonball skipping tactics\textsuperscript{5,8} to the dam-busting Wallis bomb\textsuperscript{10} have revealed an upper bound on the impact angle $\beta_f$ (angle between the free-surface and object velocity vector) below which rigid spheres will skip on water\textsuperscript{7,8,10,21}. Disk-shaped stones are more amenable to skipping, particularly if one orients the stone at just the right angle\textsuperscript{2}.

Further research has revealed more details regarding the oblique water impact of these and other canonical rigid body geometries\textsuperscript{23–26}. The referenced ricochet events are dominated by inertia, with negligible contributions from viscous and surface tension forces\textsuperscript{4,20}. In this regime, the physics of water-skipping also generalize to the water-walking ability of basilisk lizards\textsuperscript{13,14} and some birds\textsuperscript{14,27}, and to surface craft slamming\textsuperscript{21}.

In this work, we investigate the skipping of deformable elastic solid spheres on water. We observe that elastic spheres can skip for impact angles nearly three times larger than predicted for rigid spheres. Experiments and numerical modelling show that the spheres deform throughout impact in response to elastic waves propagating in the material. In some cases these elastic waves actually interact with the air-water interface to create nested cavities. We determine how the deformed geometry scales with material properties and initial impact kinematics. Using an analytical model to relate deformation to the hydrodynamic lift force, we identify the mechanisms by which elastic spheres skip so readily on water. Furthermore, we compute the normal and tangential restitution coefficients and find, surprisingly, that they display analogous behaviour to liquid droplets bouncing on inclined liquid films\textsuperscript{28}. Based on our findings about single impact events, we explain how elastic spheres are able to achieve multiple successive skips on water.

**Results**

**Elastic sphere skipping phenomena.** Prior research has reported an upper bound on the impact angle of $\beta_{\text{max}} = 18^\circ / \sqrt{\rho_s / \rho_w}$ below which rigid spheres (density $\rho_s$) will skip on water (density $\rho_w$)\textsuperscript{7,8,10,21}. We have found that elastic spheres skip at much larger values of $\beta_{\text{max}}$, raising the question of how the elastic response enables this enhanced skipping behaviour. To investigate the mechanisms underlying elastic sphere skipping, we film the water impact of custom-made elastomeric spheres with a high-speed camera viewing from the side. Rigid and elastic spheres having nearly the same radius $R$ and density $\rho_s$ are shown experimentally impacting the water in Fig. 1a,b. Each sphere strikes with approximately the same speed $U_o$ and impact angle $\beta_o$, but the elastic sphere has a shear modulus $G$ that is four orders of magnitude smaller than that of the rigid sphere material. Within a few milliseconds after impact, the elastic sphere deforms dramatically and rides along the front of a cavity on the air-water interface before lifting off the surface. By the time the elastic sphere is two diameters above the surface ($t \approx 25$ ms), the rigid sphere has plunged nearly the same distance below it. The elastic sphere evidently experiences a larger upward vertical force from the water.

The extreme sphere deformation is more evident in Fig. 1c, which shows that the water-contacting surface assumes the shape of a disk with a larger radius than that of the undeformed sphere. The disk is oriented at an attack angle $z$ that, unlike for skipping stones\textsuperscript{4}, changes in time throughout the impact. Large amplitude oscillations excited by the impact can persist in the sphere after lifting off the surface (Fig. 1b and Supplementary Movie 1) or even while the sphere is still in contact with the water (Fig. 1d and Supplementary Movie 2). In the latter case, the sphere vibrations form a group of nested cavities, or a so-called matryoshka cavity\textsuperscript{29}, named after Russian nesting dolls (Fig. 1d). This phenomenon is in contrast to rigid sphere skipping, for which the cavity is asymmetric, but smooth (Fig. 1e).

**Sphere deformation modes.** To examine the sphere deformations more thoroughly, we implement a fully coupled numerical finite-element model in Abaqus (ref. 30) (see Methods section). Figure 2a shows the results of a numerical simulation carried out with the same sphere material properties ($R$, $G$ and $\rho_s$) and impact conditions ($U_o$ and $\beta_o$) as the experimental test shown in Fig. 2b. The numerics reveal an elastic wave propagating around the circumference of the sphere in a counter-clockwise direction. We classify this type of wave propagation, depicted in the line drawing of Fig. 2c, as vibration mode 1\textsuperscript{°}.

In some cases this elastic wave impacts the air-water interface at time $t_w$, thus initiating a matryoshka cavity (as seen numerically and experimentally in Fig. 2a,b). While tempting to attribute these kinematics to rigid body rotation, we find that the elastic wave propagation time $t_w$ is typically much smaller than the measured period of rigid rotation (see Methods and Supplementary Movie 2).

In mode 2 (Fig. 2d, Supplementary Movie 1), the sphere assumes an ellipsoidal shape with oscillating major and minor axes. In mode 1\textsuperscript{°} (Fig. 2e) an elastic wave again propagates around the circumference of the sphere, but in the clockwise direction. Finally, we observe that the attack angle $z$ of the deformed water-contacting face evolves as a result of the elastic wave propagation (Fig. 2a,f and Supplementary Movie 2).

**Skip-enhancing mechanisms.** Based on our experimental observations and numerical simulations (Figs 1 and 2), we hypothesize that the elastic response of the sphere enhances skipping through two mechanisms: (1) by taking on the shape of a flat disk with an increased wetted area; and (2) by acquiring a favourable attack angle.

To connect the suggested skip-enhancing mechanisms to the vertical force acting on the sphere, we propose an analytical model of the coupled fluid-structure interaction. The sphere is idealized as an incompressible, neo-Hookean hyperelastic solid\textsuperscript{31} with shear modulus $G$, radius $R$ and density $\rho_s$. During water impact, the sphere deforms into a disk-shaped ellipsoid inclined at attack angle $z(t)$ to the water surface (Fig. 3a). A set of equations can then be written for the motion of the centre of mass (COM) and sphere deformation in terms of general forces and tractions acting on the body (see Methods section).

To couple the sphere response to the fluid loading, we extend an existing hydrodynamic force model for circular disk-shaped
Figure 1 | Elasticity alters sphere skipping dynamics. (a) High-speed images of an oblique impact show a rigid sphere carving an air cavity into the water as it dives below the surface ($U_0 = 24.3 \text{ m s}^{-1}$; $p_0 = 29.6\times 10^5 \text{kPa}$; $R = 25.8 \text{ mm}$; and $\rho' = \rho_d/\rho_w = 0.959$). Scale bar, 40 mm. (b) A highly compliant elastic sphere significantly deforms upon impact and skips off the surface ($U_0 = 22.0 \text{ m s}^{-1}$; $p_0 = 32.0\times 10^5 \text{kPa}$; $G = 12.3 \text{kPa}$; $R = 26.2 \text{ mm}$; and $\rho' = 0.937$). (c) The deformed sphere resembles a disk-shaped stone oriented at a dynamic attack angle $\alpha$. An inertia-dominated hydrodynamic force $F$ acts on the flattened face, which moves with velocity described by $U_B$ and $b_B$. Scale bar, 40 mm. (d) A surprising consequence of the interaction of sphere vibrations with the liquid interface is the formation of nested air cavities (that is, a matryoshka cavity). (e) A rigid sphere can skip if the impact angle does not exceed $\beta_{\text{max}} = 18^\circ/\sqrt{\rho'\rho}$, leaving a smooth, asymmetric cavity on the surface ($\beta_0 = 17.3^\circ$ for sphere shown).

Figure 2 | Fluid-structure coupling through sphere deformation modes. (a) An Abaqus numerical model captures the skipping behaviour and reveals the local relative strain in the sphere (colour contours). (b) Under the same conditions as simulated in (a) an experiment shows the formation of a nested cavity (or so-called matryoshka cavity), as also shown by the model. (a,c) The model reveals an elastic wave that propagates in a counter-clockwise direction around the sphere (classified as deformation mode 1$^-$). At time $t_w$, the elastic wave strikes the air–water interface. Two other deformation modes are observed: (d) mode 2: the sphere assumes an ellipsoidal shape with oscillating major and minor axes (Supplementary Movie 1); (e) mode 1$: an elastic wave propagates in the clockwise direction. (f) The attack angle $\alpha$ evolves in time in response to the elastic wave propagation (deformation mode 1$^-$ is pictured). The purple lines are experimental measurements of $\alpha$ taken at 1.33 ms intervals.
skipping stones \(^4\) by approximating the water-contacting face as a circular disk with radius \(R_{eq}\) (where \(R_{eq}\) approximates the sphere deformation (see Fig. 3a and Methods section). The force is modelled as

\[
F = \frac{1}{2} \rho_w |U_B|^2 S_w \sin(\alpha/\beta_m) m_2,
\]

where \(U_B\) and \(\beta_m\) describe the velocity of the water-contacting face and the wetted area \(S_w\) proportional to \((R_{eq}/R)^2\). The equation for the vertical motion of the COM is then

\[
\ddot{d}_z = \frac{3}{8\pi R^3} \frac{\rho_w}{\rho_s} |U_B|^2 S_w \sin(\alpha/\beta_m) \cos \alpha - g,
\]

where \(d_z\) is the vertical coordinate of the COM and \(g\) is gravity. We can now predict how the hypothesized mechanisms relate to skipping. First, a larger area

\[
\text{impact in not steady-state, we nonetheless expect } \alpha \text{ to be governed by the speed of elastic waves propagating in the sphere through the characteristic distance } R \text{ such that } \alpha(t) \propto \frac{1}{R} \sqrt{\frac{G}{\rho_s}}. \quad \text{Therefore, we predict that a more compliant sphere (smaller } G) \text{ will assume a smaller rate of change of } \alpha, \text{ thereby increasing the upward vertical force that enables skipping.}
\]

Numerical simulations verify the expected dependence of these two mechanisms on \(G\). Measurements from the Abaqus results show that the rate of change of the attack angle scales as \(\dot{\alpha} \propto \frac{1}{R} \sqrt{\frac{G}{\rho_s}}\) for mode 1 \(^-\) deformations (Fig. 3b). Additionally, we find that the maximum value of \(\lambda_{eq}\) achieved during impact, \(\beta_m\), increases with a decrease in the dimensionless term \(G/\rho_s U_0^2\) (Fig. 3c), which is the ratio of material stiffness to hydrodynamic pressure. Therefore, a smaller \(G\) yields a larger stretch and larger wetted area, as well as a smaller rate of change of \(\alpha\), as predicted.

To confirm that these mechanisms indeed enhance skipping, we perform experiments and simulations over a range of impact conditions and sphere properties and measure the minimum impact speed required to skip \(U_{\text{min}}\) as a function of \(G\) (Fig. 4). Above a certain value of \(G\) \((\approx 10^2 - 10^3 \text{ kPa})\), depending on \(\beta_m\), we recover the rigid skipping regime, in which \(U_{\text{min}}\) is independent of shear modulus but is very sensitive to \(\beta_m\). For rigid spheres impacting above \(\beta_{\text{min}} = 18/\sqrt{\rho^s} \), where \(\rho^s = \rho_d\)
after impact and the interval between
G
of the sphere affects skipping. In this elastic regime, the minimum speed required to skip \( U_{\text{min}} \) decreases monotonically with decreasing shear modulus \( G \), as shown by both experimental and numerical results (triangle and star markers, respectively). In the limit of small \( G \) we expect \( U_{\text{min}} \propto G^{1/5} \), which is confirmed by the numerics. As stiffness increases above \( G \approx 10 \) kPa, \( U_{\text{min}} \) deviates from this relation as larger \( G \) augments the rate of change of \( \alpha \), thereby reducing the upward vertical force component. Our analytical model also captures this change in behaviour (solid coloured lines). For shallow impact angles, \( U_{\text{min}} \) becomes insensitive to shear modulus for \( G > \approx 10^3 \) kPa, indicating that rigid sphere skipping behaviour is recovered. The transition between the elastic and rigid skipping regime occurs at larger \( G \) as \( \beta_s \) increases. In the rigid skipping regime, \( U_{\text{min}} \) is very sensitive to \( \beta_s \) (also evident in Supplementary Fig. 1). The lower bound of the rigid skipping regime is inferred from the dark triangle symbols, which occur for experiments with \( \beta_s \sqrt{\rho} < 14.5^\circ \). The coloured triangle markers at \( G = 5.66 \times 10^5 \) kPa result from experiments where the colour gradient on the marker indicates the uncertainty in \( \beta_s \sqrt{\rho} \). The upper bound on the rigid regime corresponds to \( \beta_s \sqrt{\rho} = 18^\circ \); prior literature suggests that for \( \beta_s \sqrt{\rho} > 18^\circ \) spheres may broach (that is, completely immerse before exiting), but not skip\(^5\). (b-d) Numerical simulations show that increasing \( G \) results in a larger rate of change of \( \alpha \) (purple lines) in the elastic skipping regime. Each image shown is 6 ms after impact and the interval between \( \alpha \) measurements is 2 ms. (e) As shear modulus increases into the rigid regime, the sphere deformation is negligible (black outline is undeformed sphere contour). (f) We have observed broaching for rigid spheres impacting with \( \beta_s \sqrt{\rho} > 18^\circ \). Sphere properties for data in (a): rigid sphere experiments, \( R = 25.8 \) mm, \( \rho^* = 0.959 \), \( G = 5.66 \times 10^5 \) kPa; all other data markers, \( R = 26.2 \) mm, \( \rho^* = 0.937 \) for \( G \leq 12.3 \) kPa and \( \rho^* = 1.03 \) for \( G > 12.3 \) kPa. The purple error bars are characteristic for experimental data. The numerical error bars represent \pm 1/2 of the difference in \( U_o \) between the skipping and non-skipping cases used to compute \( U_{\text{min}} \) error bars are offset for clarity.

\( \rho_w \), prior research suggests spheres may broach (that is, become completely submerged before exiting), but not skip\(^8\) (Fig. 4f). For stiffness values below the rigid regime, the elasticity of the sphere becomes important and \( U_{\text{min}} \) decreases monotonically with decreasing shear modulus. Our analytical model accurately predicts the experimental and numerical results in this regime. The minimum speed is also much less sensitive to \( \beta_s \) in the elastic skipping regime and as a result we observe skipping at impact angles nearly three times larger than for rigid spheres (Fig. 4 and Supplementary Fig. 1).

While our results show that reducing shear modulus has the predicted effect on wetted area and attack angle (Fig. 3b,c), it is unclear whether one or both of these mechanisms are responsible for the observed improved skipping performance. To isolate the mechanisms we consider the limiting case of small \( G \), for which

\[
\frac{\dot{\alpha}}{\dot{\alpha}_0} \approx \frac{1}{B} \sqrt{\frac{G}{\rho}} \to 0 \quad \text{and thus } \cos \alpha \approx 1 \quad \text{over typical impact timescales.}
\]

The expected dependence of \( U_{\text{min}} \) on \( G \) in this limit can be rationalized by scaling analysis of equation 2 (see Methods and Supplementary Note 2), which gives

\[
\frac{2R}{l_c^2} \approx \frac{3}{8\pi \rho \rho^*} U_{\text{min}}^2 \frac{l_{c,\text{max}}^2}{l_{c,\text{max}}^2} - g,
\]

where \( l_c \) is the collision time (that is, time in which the sphere is in contact with the water). For threshold skipping cases, we expect the characteristic acceleration 2R/l\(_c^2\) to be small compared with gravity and thus, to first order, 3U\(_{\text{min}}^2\)l\(_{c,\text{max}}^2\) /8\(\pi\)\(\rho\)\(\rho^*\) \approx g. Furthermore, in the small \( G \) limit our numerical modelling shows that \( \lambda_{\text{max}} \propto (G/\rho_w U_{\text{min}}^2)^{-1/3} \) (Fig. 3c). Applying this dependence and solving for \( U_{\text{min}} \), we find \( U_{\text{min}} \propto (\rho^* \rho^*)^{1/30}(G/\rho_w U_{\text{min}}^{1/5}) \).

Figure 4a shows that \( U_{\text{min}} \) approaches the \( G^{1/5} \) relation in the limit of small \( G \), indicating that the only mechanism by which reducing shear modulus enhances skipping in this limit is through the increased wetted area. However, for larger \( G \) (\( G > \approx 10 \) kPa), \( U_{\text{min}} \) deviates from the \( G^{1/5} \) relation as the coupling between shear modulus and \( \alpha \) becomes important (Fig. 4a-d). As stiffness continues to increase, ultimately the amplitude of the deformations (affecting both \( S_c \) and \( \alpha \)) become negligible and the sphere is effectively rigid (Fig. 4a,e). Consequently, we conclude that decreasing shear modulus below this rigid boundary causes an increase in the upward vertical force that promotes skipping through both of the hypothesized mechanisms, save for the limit of small \( G \) where decreasing shear modulus only affects lift by increasing \( S_c \).

Skipping regimes. As the impact events devolve from clear skipping to water entry, we observe a transitional regime characterized by a matryoshka cavity, in which the sphere still skips (Fig. 1d; Supplementary Movie 2). The matryoshka phenomenon occurs when the total contact time of the sphere with water \( t_c \) is longer than the wave time \( t_w \) associated with mode 1 - elastic wave propagation, such that \( t_c/t_w \gg 1 \). We define \( t_w \) as the time from impact until the circumferential elastic wave strikes the air-water interface (Fig. 2a,b). Experiments over a range of sphere properties and impact conditions reveal that \( t_c/t_w \) is governed by the impact angle \( \beta_s \) and the ratio \( G/\rho_w U_o^2 \) (Fig. 5). The dependence on these terms can be rationalized by scaling analysis with \( U_o \) replacing \( U_{\text{min}} \) in equation 3. When \( t_c \approx t_w \), the characteristic sphere acceleration is much greater than \( g \) such that

\[
2R/t_c^2 \approx 3U_o^2/2 \sqrt{\rho \rho^*} /8\pi \rho \rho^* \quad \text{and we find } t_c \propto R \sqrt{\rho \rho^*}/U_o \lambda_{\text{max}}
\]
Experiments show that the timescale ratio depends on $R/P_w U_o^2$ and $\beta_w$ with seemingly minimal dependence on $R$ (marker size indicates $R$; marker shapes indicate same $G$ as for Fig. 3b). The four coloured patches result from calculations using our analytical model with the same material properties as for the experiments. Varying $R$ in the model over the range experimentally tested results in the spread in the patches, which is small for $t_c/t_w < 1$. For shallow impact angles, scaling analysis predicts $t_c/t_w \propto (G/p_w U_o^2)^{1/2}$ (dashed line), while for steep impact angles we expect $t_c/t_w \propto (G/p_w U_o^2)^{11/12}$ (dash-dot line). These limiting trends capture the general evolution shown by the experimental data. Characteristic error bars are shown.

Methods section). Furthermore, based on the speed of mode 1 of elastic waves, we expect $t_w \propto 1/\sqrt{R}/G/p_w$, which is confirmed experimentally (Supplementary Fig. 2). Combining the scaling for each time gives $t_c/t_w \propto (G/p_w U_o^2)^{1/2}/t_{\max}$. We can now examine the evolution of the timescale ratio in the vicinity of the transitional regime in the limit of shallow ($\beta_w \rightarrow 0$) and steep impact angles, making use of the relationship between $t_{\max}$ and $G/p_w U_o^2$ shown in Fig. 3c. Here, we consider shallow impact angles to be $\beta_w \rightarrow \beta_{w,\text{crit}}$, where $\beta_{w,\text{crit}}$ is the maximum impact angle at which we have observed elastic sphere skipping ($t_{\max} \approx 47^\circ$). In the shallow $\beta_w$ limit, $t_c/t_w \approx 1/2$ occurs at values of $G/p_w U_o^2 > 0.1$, for which $t_{\max} \rightarrow 1$ (Fig. 3c); thus, we anticipate $t_c/t_w \propto (G/p_w U_o^2)^{1/2}$ for shallow angles. For steep $\beta_w$, the transitional regime occurs for $G/p_w U_o^2 < 0.1$, for which $t_{\max} \propto (G/p_w U_o^2)^{-3/2}$ (Fig. 3c), and we expect $t_c/t_w \propto (G/p_w U_o^2)^{11/12}$. These limiting relations capture the evolution of $t_c/t_w$ observed experimentally (Fig. 5) and provide insight into the differences observed at different impact angles. We see that for steeper $\beta_w$ the sphere deformation has a larger effect on the collision time, which gets manifested as a higher sensitivity of $t_c/t_w$ to $G/p_w U_o^2$.

Based on our findings regarding the transitional skipping regime, we hypothesize that the same dimensionless parameters (that is, $\beta_w$ and $G/p_w U_o^2$) govern all deformation modes and associated skipping behaviour. An empirical regime diagram indeed shows that these parameters classify all observed skip types (Fig. 6). Mode 1 is promoted by shallow $\beta_w$, large $U_o$ and/or small $G$. As stiffness becomes large relative to hydrodynamic pressure, the vibration type traverses the mode 2 and mode 1 regimes. Our analytical model correctly predicts the boundary between the mode 1 and 2 skipping ($t_c/t_w < 1$) and mode 1 transitional ($t_c/t_w > 1$) regimes (marked by red line on Fig. 6).
limit \( G \to 0 \) (Fig. 4). When the relative magnitude of droplet surface tension becomes small for liquid droplets impacting liquid layers, bouncing does not occur and the droplet completely merges with the liquid layer\(^{28}\). We conjecture about a similar limit for elastic spheres with \( G \to 0 \) and impacting with \( \text{We} = \rho_w U_{2n}^2 R / \sigma_w \approx 1 \), where \( \sigma_w \) is the surface tension of water. In this limit, we expect the surface tension force from the water to act prominently on the sphere\(^{33}\) and to inhibit sphere reformation, thus preventing recovery of translational kinetic energy from deformational potential energy during impact. As a result, we hypothesize that sphere skipping would ultimately cease in this limit. We anticipate these dynamics would become relevant when \( G \approx \sigma_w / R \) (see Supplementary Fig. 3 and Supplementary Note 3). To validate these predictions is beyond the scope of the present work.

### Discussion

Perhaps the most mesmerizing manifestation of elastic sphere water impact is continual skipping across water (Supplementary Movie 3). To confirm that our physical description of a single skip generalizes to multiple skip events, we predict the placement of successive impacts on the regime diagram (Fig. 6). An experimental investigation using isolated water tanks validates the predictions and shows the sphere traversing through the vibration modes with each impact (Fig. 8). As to how repeated skipping is sustained over very long skipping trajectories (Supplementary Movie 3), we gain insight from the behaviour of the restitution coefficients (Fig. 7). First, \( e_t \) is consistently larger than \( e_r \), which causes \( \beta_t \) to decrease and thus become more favourable with each skip. Second, the restitution coefficients actually become larger as \( U_{2n} \) decreases, until the sphere enters the mode 1\(^{\downarrow} \) skipping and transitional regimes. Therefore, one could say it becomes easier to skip with every skip.

While toy elastic balls may bestow upon the casual sportsman the ability to break the world stone skipping record (88 skips by K. Steiner, Guinness World Records), we believe the physics underlying the elastic sphere impact are common to the large deformation hydroelastic response of surface-riding and skipping compliant bodies. Models of these structures, such as inflatable boats, typically ignore extreme elastic deformation even though it is known to affect drag, stability and slamming loads\(^{12}\). The mechanisms of form and force augmentation, as well as the secondary vibration-induced fluid interactions that we have revealed, can be exploited for functional advantage and incorporated into higher-fidelity hydroelastic models.

### Methods

#### Sphere fabrication and material properties.

In order to control material properties, custom elastomeric spheres were fabricated from a high performance platinum-cure silicone rubber called Dragon Skin produced by Smooth-On, Inc., which consists of two liquid constituent parts. Once the two constituents are mixed, the material sets without requiring heat treatment. The shear modulus was varied by adding a silicone thinner to the mixture before setting, which reduces the material shear modulus by decreasing the polymer cross-linking density. Sphere
materials with three different shear moduli were fabricated by adding 0, 1/3 and 1/2 parts thinner by mass ratio. Before setting, the liquid mixture was placed in a vacuum chamber to remove any entrained air. For our experiments, spheres were fabricated by curing the liquid mixture in smooth, machined aluminium moulds to produce spheres with three different radii: 20.1 ± 0.8 mm; 26.2 ± 0.8 mm; and 48.8 ± 0.9 mm. A rigid sphere with \( R = 25.8 ± 1 \) mm was fabricated from Nylon DuraForm PA using selective laser sintering. The uncertainty on each sphere radius represents the 95% confidence interval based on seven independent measurements. A thin Lucry casing was loosely fitted around each sphere in order to prevent undesired particles from adhering to the surface and to reduce the friction between the sphere and the launching mechanism from which it was fired.

The Sucro piston was so compliant that traditional uniaxial ‘dropgon’ setting on our Instron machine was not feasible as the forces generated were too small to be reliably measured. To overcome this, we performed a test in which the spheres were compressed on the Instron to generate a quasi-static force-displacement curve. This test set-up was then numerically modelled in Abaqus with the sphere material described by a neo-Hookean hyperelastic constitutive model, parameterized by the shear modulus \( G \). We then varied \( G \) to find the value that produced the best fit between the numerically simulated and experimentally measured force-displacement curves. The elastomeric spheres used in our experiments had shear moduli of 97.2, 28.5 and 12.3 kPa corresponding to 0, 1/3 and 1/2 parts thinner, respectively. According to the manufacturer of the rigid sphere (3D Systems—Quick Parts, Solutions), the elastic modulus of the selected laser sintering Nylon DuraForm PA material is 1.59 × 10^10 kPa, which—assuming a Poisson’s ratio of 0.4—gives a shear modulus of \( G = 5.66 × 10^8 \) kPa.

**Sphere skipping experiments and data processing.** Spheres were launched at the water surface from a variable-angle, pressure-driven cannon consisting of a pressure chamber for compressed air, a sliding cylindrical piston and interchangeable barrels. Sliding the cylindrical piston allowed air to flow from the pressure chamber into the barrel, thus forcing the sphere to accelerate out of the barrel and strike the water surface with impact speed \( u_0 \) and angle \( \beta \). Impact events were illuminated with diffuse white back lighting and filmed with either NAC GX-3 or Photron SA3 high-speed cameras acquiring at 1,000–2,000 frames per second (fps).

The impact speed \( u_0 \) and angle \( \beta \) were measured from images of the sphere before water impact using a cross-correlation algorithm. The mean uncertainties on \( u_0 \) and \( \beta \) are ±1.09 m s⁻¹ and ±1.75°, respectively (computed at 95% confidence level). This same algorithm was used to measure the exit speed and angle of the sphere after lifting off the surface. Also, the rigid body rotation of the sphere was tracked after skipping by tracking \( V_r \). These coordinates on the exterior of the Lucry casing was found for mode 1—skip type that the rotational kinetic energy was typically < 8% of the translational kinetic energy after water exit. Furthermore, for these impacts the wave time \( t_w \) was typically < 30% of the period of rigid body rotation measured after skipping. The collision time \( t_c \), wave time \( t_w \) and vibration modes classification were all determined from manual inspection of the high-speed images. The minimum skipping speed \( u_{\text{min}} \) was found experimentally by performing successive experiments with identical conditions but with increasing speed until the sphere skipped.

**Abaqus numerical model.** Details of the Abaqus numerical model of the elastic sphere impact are contained in reference [26]; clarifications relevant for the present work are summarized here. The finite-element model uses the built-in coupled euler–lagrange functionality of Abaqus/Explicit, which couples the contact interaction between the Lagrangian (sphere) and Eulerian (fluid) domains using a penalty method. Direct numerical simulation of the compressible Navier–Stokes equations is performed in the Eulerian domain. For the solid, conservation of momentum is solved with an incompressible neo-Hookean constitutive model describing the sphere. For all numerical model results presented herein, the mesh consists of eight-noded Eulerian hexahedral elements with spatial resolution of 3 mm. The sphere radius \( R \), density \( \rho \) and shear modulus \( G \) as well as the impact speed \( u_0 \) and angle \( \beta \) were set to match experimental values. The three-dimensional computational domain consists of a water tank (length = 30 R, depth = 6.3 R) with a symmetric plane coinciding with the plane of motion. For the numerical results presented in Fig. 4a, \( u_{\text{min}} \) is the average of the impact speeds for a skipping case and the non-skipping case with the nearest \( u_0 \). The numerical marker error bars reported in Fig. 4a represent ±1/2 of the speed difference between the two cases.

**Analytical model of elastic sphere skipping.** An approximate analytical approach to modelling the impact between a compliant elastomeric sphere and a fluid surface is outlined (for a complete derivation, see Supplementary Note 1). Here, we derive the governing equations for the motion and deformation of an incompressible, isotropic, inextensible, inviscid, slender generalized coordinates that are governed by a system of ordinary differential equations (ODEs). We begin by defining a fixed Cartesian coordinate system \( (e_1, e_2, e_3) \) (Fig. 3a). We assume the sphere moves only in the \( e_1 - e_2 \) plane and undergoes no rigid body rotation. The sphere deformation is first described by a rigid sphere of radius \( R \) and the motion of the center of mass \( C = (x, y, z) \) as a function of time \( t \). Suppose the coordinates \( \mathbf{u} = (x, y, z) \) give \( x = 1/2 \). Furthermore, \( S_0 \), \( U_0 \), and \( \beta_0 \) can be written in terms of the generalized coordinates describing the sphere deformation (see Supplementary Note 1). Without describing the pressure distribution, we cannot specify the centre of pressure and, thus, cannot define the \( y_1 \) coordinate at which the traction vector acts in equation 8, which governs the \( x \)-direction. The solution to this problem is specified by the coordinate \( z \) and \( x \) (the Cartesian coordinates of the deformed sphere (Fig. 3b). With \( z \) prescribed, the remaining ODEs for \( \delta_1 \), \( \delta_2 \), \( \lambda_1 \) and \( \lambda_2 \) (equations 6.7,9 and 10) can be solved without further simplification. Inserting the
hydrodynamic force yields equation 2 and the remaining ODEs are:

\[
\dot{t}_e = -\frac{3\rho\nu}{8\pi \nu^2} |U_0|^2 S_0 \sin(\pi + \beta_h) \sin z.
\]  

(11)

\[
\frac{4\pi}{15} \rho \nu R^4 \left[ \frac{\dot{z}_1}{1 + \frac{\dot{z}_2}{\dot{z}_1}} \frac{\dot{z}_2}{(\dot{z}_1 \dot{z}_2)^2} \right] + \frac{4\pi}{15} \rho \nu R^4 \left[ \frac{\dot{z}_1}{1 + \frac{\dot{z}_2}{\dot{z}_1}} \frac{\dot{z}_2}{(\dot{z}_1 \dot{z}_2)^2} \right] - 2 \frac{\dot{z}_1 \dot{z}_2 \dot{z}_1 \dot{z}_2}{(\dot{z}_1 \dot{z}_2)^2} = \frac{8\pi}{15} \rho \nu R^4 \left[ \frac{\dot{z}_1}{1 + \frac{\dot{z}_2}{\dot{z}_1}} \frac{\dot{z}_2}{(\dot{z}_1 \dot{z}_2)^2} \right]
\]

(12)

For cases that barely skip, we have observed that \( t \approx (10^{-1}) \), which gives a characteristic acceleration of \( 2R/t_c^2 \approx O(1) \text{ m s}^{-2} \) (with \( R \approx O(10^{-3}) \)). Therefore, in the case of the minimum impact speed, we expect gravity to be an order of magnitude larger than the sphere acceleration such that

\[
\frac{3\pi}{8\pi \nu R^4} U_{\text{min}}^2 \sin (\pi + \beta_h) \cos z \approx g.
\]  

(14)

In the limit of small G, simulations show that \( \sin(z + \beta_h) \cos z \approx \sin(\pi + \beta_h) \approx g \).

(15)

To determine how \( t_e/t_c \) evolves, we consider a scaling analysis of equation 14 for order 1 (see Supplementary Note 2) and \( U_{\text{max}} \equiv (G/\rho_n U_{\text{min}}^2)^{1/3} \) (see Fig. 3c). Thus, equation 15 leads to

\[
U_{\text{max}} \approx (g \rho_n R^{2/3} / (G/\rho_n U_{\text{min}}^2))^{1/3}.
\]  

(16)

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Author contributions

J.B., T.T.T. and R.C.H. designed the research; J.B., R.C.H., T.T.T. and M.A.J performed the experiments; J.B., R.C.H. and T.T.T. analysed the data; M.A.J developed the Abaqus numerical model; A.F.B. and J.B. developed the analytical model; and J.B. wrote the original manuscript and all authors helped revise it.

Additional information

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This Article contains an error in Fig. 5 that was introduced during the production process. The coloured background is scaled incorrectly relative to the axes and foreground figure elements. The correct version of Fig. 5 appears below.

Figure 5