Arbitrary l-state solutions of the Feynman propagator for the Rosen-Morse Potential with a centrifugal term approximation

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Abstract. We present analytic solutions of the Feynman propagator with the Rosen-Morse potential. To do that, an approximation of the centrifugal potential is used and nonlinear space-time transformations are applied. A relation between the original path integral and the Green function of a new quantum soluble system is derived. Explicit expressions of the bound state energy spectra and the eigenfunctions are obtained and compared to those of Schrödinger formalism.

1. Introduction

It is well known that only a limited number of potentials admit exact solutions of the Schrödinger equation [1, 2], especially in the three-dimensional space when the angular momentum \(l \neq 0\). This is due to the presence of the centrifugal term \((1/r^2)\). Motivated by the success in obtaining approximately the bound state solutions of the Feynman propagator with the Manning-Rosen potential [5, 6], we attempt here to solve the arbitrary \(l\)-wave propagator with the Rosen-Morse potential. An improved approximation scheme for the centrifugal term is used as in [7, 8, 9] and non linear space-time transformations are applied.

The Rosen-Morse (RM) potential is given by

\[
V^{RM}(r) = -\frac{V_1}{\cosh^2(r/a)} + V_2 \tanh(r/a).
\]

(1)

Here \(V_1\) and \(V_2\) are two strength parameters; the screening parameter \(a\) has a dimension of length. This potential has been one of the most useful and convenient models to study the energy eigenvalues of diatomic molecules. It is known that for this potential, the Schrödinger equation can be solved exactly when \(l = 0\) either directly [10] or by using the Feynman path integral formalism [11]. Unfortunately, for arbitrary \(l\)-state, the radial Schrödinger equation does not admit exact solutions. In this case, some authors have used the approximation scheme proposed by Lu [7] to study analytically arbitrary \(l\)-wave bound states of the Schrödinger equation and the relativistic wave equation for the Rosen-Morse potential [8, 9].

The purpose of this paper is to study \(l\)-states solutions of the Rosen-Morse potential within the Feynman path integrals formalism as done in the recent works [8, 9] in which the
Nikiforov-Uvarov method is used. The method we propose consists in using the centrifugal term approximation proposed by Lu [7], \( \frac{1}{r^2} \approx \frac{1}{r_{0}^2} \left( C_{0} + \frac{C_{1}}{r_{0} + 1} + \frac{C_{2}}{(r_{0} + 1)^2} \right) \), where the parameters \((C_{l})\) can be expressed in terms of the specific potential parameters, and \( r_{0} = -\tanh^{-1}(\frac{V_{0}}{V}) \) corresponds to the minimum of the potential.

The rest of the paper is organized as follows. In section 2, we present the centrifugal term approximation and apply it to the Rosen-Morse propagator. Then, using the Dru-Kleinert method we determine the \( l \)-wave eigensolutions for the Rosen-Morse potential within the path integral formalism. Finally, in Section 3, we give concluding remarks.

2. Path integral for the the Rosen-Morse propagator

In spherical coordinates, the propagator related to the Rosen-Morse potential, between two time-space points \((\vec{r}', t')\) and \((\vec{r}'' , t'')\), is written [12]:

\[
K(\vec{r}'', t''; \vec{r}', t') = \frac{1}{4\pi r'' r'} \sum_{l=0}^{\infty} (2l + 1) K_{l}(r'', r'; t', t') P_{l}(\cos \theta). \tag{2}
\]

\( P_{l}(\cos \theta) \) is the Legendre polynomial with \( \theta = (\vec{r}'', \vec{r}') \) and

\[
K_{l}(r'', r'; t', t') = \lim_{N \to \infty} \int \frac{i}{\hbar} S_{j} \prod_{j=1}^{N} \left[ \frac{m}{2\pi \hbar \varepsilon} \right]^{\frac{1}{2}} \prod_{j=1}^{N-1} dr_{j}. \tag{3}
\]

Here \( S_{j} = \frac{m}{2\pi} (\Delta r_{j})^{2} - \varepsilon V_{s}(r_{j}) \),

\[
V_{s}(r_{j}) = \frac{\hbar}{2m} \frac{l(l + 1)}{r_{j} r_{j-1}} - \frac{V_{1}}{\cosh^{2}(r_{j}/a)} + V_{2} \tanh(r_{j}/a) \tag{4}
\]

and \( \Delta r_{j} = r_{j} - r_{j-1} \), \( \varepsilon = t_{j} - t_{j-1} \), \( t' = t_{0} \) and \( t'' = t_{N} \). Moreover, setting \( T = t'' - t' \), we can write:

\[
K_{l}(r'', r'; T) := K_{l}(r'' , r'; t', t'). \tag{5}
\]

To calculate the propagator \( K_{l} \) given by equation (3) for \( l \neq 0 \) states, we apply the following approximate scheme to the centrifugal term \([7]\) :

\[
\frac{l(l + 1)}{r_{j}^{2}} \approx \frac{l(l + 1)}{r_{0}^{2}} \left[ C_{0} + C_{1} \frac{e^{-2r_{j}}}{1 + e^{-2r_{j}a}} + C_{2} \left( \frac{e^{-2r_{j}a}}{1 + e^{-2r_{j}a}} \right)^{2} \right], \tag{6}
\]

where

\[
C_{0} = 1 - \left( 1 + e^{-2(r_{0}/a)} \right) \left( 1 + e^{-2(r_{0}/a)} \right) - 3 + 2(r_{0}/a), \quad C_{1} = -2 \left( 1 + e^{2(r_{0}/a)} \right) \times \left[ 3 \left( 1 + e^{-2(r_{0}/a)/2(r_{0}/a)} \right) - (3 + 2(r_{0}/a)) \right], \quad C_{2} = \left( 1 + e^{2(r_{0}/a)} \right) ^{2} \left( \frac{1 + e^{-2(r_{0}/a)/2(r_{0}/a)} \right) \times \left( 3 + 2(r_{0}/a) - \frac{4(r_{0}/a)}{1 + e^{-2(r_{0}/a)}} \right). \]

By substituting the centrifugal term in (4) by its approximation (6), we get

\[
V_{s}(r_{j}) = A \tanh(r_{j}/a) - \frac{B}{\cosh^{2}(r_{j}/a)} + C, \tag{7}
\]

with \( A = (C_{1} - C_{2}) \frac{l(l+1)}{4m_{0}^{2}} + V_{2}, \quad B = C_{2} \frac{l(l+1)}{2m_{0}^{2}} + V_{1} \) and \( C = (C_{2} - C_{1}) \frac{l(l+1)}{4m_{0}^{2}} + \frac{l(l+1)}{2m_{0}^{2}} C_{0} \).
2.1. Space-time transformations

In the following, we study the potential (7) with the Duru-Kleinert method [3]. The path integral for the potential (7) is

\[ K_i(r'', r'; T) = \int \mathcal{D}r(t) \exp \left[ \int_{t'}^{t''} \left( \frac{m}{2} \dot{r}^2 - V_{\text{eff}}(r) \right) dt \right]. \]  

By achieving the point canonical transformation and the time transformation [4],

\[ \begin{cases} r = f(q), \\ dt = |f'(q)|^2 ds, \end{cases} \]

with

\[ f(q) = a \cdot \text{arctanh} \left( 2(\tanh q)^2 - 1 \right). \]

The propagator expression becomes [6]:

\[ \tilde{K}_i(q'', q'; s'') = \exp \left[ \frac{i}{\hbar} s'' \left( E + C - A \right) a^2 \right] \times \]

\[ \int \mathcal{D}q(s) \exp \left[ \frac{i}{\hbar} \int_0^{s''} \frac{m}{2} \dot{q}^2 - \frac{\hbar^2}{2m} \left[ \frac{\eta(n - 1)}{\sinh^2 q} - \frac{\nu(v - 1)}{\cosh^2 q} \right] ds \right] \]

\[ = \exp \left[ \frac{i}{\hbar} s'' \left( E + C - A \right) a^2 \right] \tilde{K}_i^{\text{MPT}}(q'', q'; s'') \]  

with \( \eta = \frac{1}{2} \pm \sqrt{-2ma^2(E + C + A)/\hbar^2}, \nu = \frac{1}{2} \pm \sqrt{1 + 8ma^2/\hbar^2}, \) and \( \tilde{K}_i^{\text{MPT}} \) is the path integral of the modified Pöschl-Teller potential, which is a known solved problem [13, 14, 15] for which the bound states wave functions are explicitly given by [15]:

\[ \chi_{i,n}^{(k_1,k_2)}(q) = N_n^{(k_1,k_2)} \left( \sinh q \right)^{2k_2 - 1/2} \left( \cosh q \right)^{-2k_1 + 3/2} \times \]

\[ F_1(-k_1 + k_2 + k, -k_1 + k_2 - k + 1; 2k_2; - \sinh^2 q), \]

where \( k = k_1 - k_2 - n, \) and

\[ N_n^{(k_1,k_2)} = \frac{1}{\Gamma(2k_2)} \left( \frac{(2k - 1) \Gamma(k_1 + k_2 - k) \Gamma(k_1 + k_2 + k - 1)}{\Gamma(k_1 - k_2 + k) \Gamma(k_1 - k_2 - k + 1)} \right)^{1/2}, \]

The eigenvalues are expressed by

\[ E_n^{\text{MPT}} = -\frac{\hbar^2}{2m} \left[ 2(k_1 - k_2 - n) - 1 \right]^2. \]

2.2. Energy spectrum and wave functions

Calculating \( \tilde{K}_i(q'', q'; s'') \) allows us to obtain the Green function \( G. \) Once this is known, the whole energy spectrum is obtained from the poles. We obtain the corresponding wave functions from the residues at the poles.

Substituting (12) in (11), we get

\[ \tilde{K}_i(q'', q'; s'') = \sum_{n=0}^{N_n} \exp \left\{ \frac{i}{\hbar} s'' \left[ (E + C - A)(a)^2 - E_n^{\text{MPT}} \right] \right\} \chi_{i,n}^{(k_1,k_2)}(q'') \chi_{i,n}^{(k_1,k_2)}(q) \]

\[ + \int_0^\infty dK \exp \left\{ \frac{i}{\hbar} s'' \left[ (E + C - A)(a)^2 - \frac{\hbar^2 K^2}{2m} \right] \right\} \chi_{K}^{(k_1,k_2)}(q'') \chi_{K}^{(k_1,k_2)}(q'). \]
Then, by integrating this latter over the pseudo-time parameters $s''$, the Green function $G$ given in [6] becomes

$$G_l(r'', r'; E) = \sum_{n=0}^{N_m} \frac{\chi_{l,n}^{RM}(k_1,k_2)(r'')}{E_{n,l}^{RM} - E} + \int_0^\infty dK \frac{\chi_{l,K}^{RM}(r'')}{(\hbar^2 K^2/2ma^2) - A - C - E}. \quad (16)$$

As in [4, 11], satisfying the boundary conditions for $r \to 0$ and $r \to \infty$ gives:

$$k_1 = \frac{1}{2} \left(1 + \sqrt{1 + 8mBa^2/\hbar^2}\right) \equiv \frac{1}{2} (1 + s), \quad k_2 = \frac{1}{2} \left(1 + \frac{1}{2} (s - 2n - 1) - 2mA^2/(s - 2n - 1)\right).$$

By substituting these last values in (12) and (14), and considering $u = \frac{1}{2} \left[1 + \tanh \frac{x}{a}\right]$, we get

$$\chi_{l,n}^{RM}(k_1,k_2)(r) = \frac{N_{n}^{(k_1,k_2)}}{\sqrt{a}} k_2^{-(1/2)} (1 - u)^{s/2 - k_2 - n} \times F_1(-n, s - n; 2k_2; u), \quad (17)$$

where

$$E_{n,l}^{RM} = -\frac{\hbar^2 (s - 2n - 1)^2}{8ma^2} - \frac{2mA^2 a^2}{\hbar^2 (s - 2n - 1)^2} + C. \quad (18)$$

The constants entering in this expression have been defined in Section 2.

If we set $\delta = -\frac{1}{4} (1 + s)$, the energy $E_{n,l}^{RM}$ turns to

$$E_{n,l}^{RM} = \frac{l(l+1)C_0}{2mr_0^2} + V_2 - \frac{\hbar^2}{2ma^2} \times \left[\frac{(n+1)^2 + \delta (2n + 1) + (\delta^2 + \delta) + mA^2/\hbar^2}{n + \delta + 1}\right]^2. \quad (19)$$

This expression is perfectly identical to that given in [8, 9], obtained with the framework of the Schrödinger formalism.

### 3. Concluding remarks

In this paper, we have calculated the energy levels of the Feynman kernel with the Rosen-Morse potential. Use is made of an improved approximation scheme for the centrifugal term and a nonlinear space-time transformations in the radial path integral. Explicit expressions of the wave functions are obtained in terms of Jacobi polynomials and the energy eigenvalues are expressed in function of the potential parameters. We have shown that our results are identical with those obtained with the Schrödinger formalism. In future works, we plan to apply this technique to other exponential-type potentials.

### References

[1] L. D. Landau and E. M. Lifshitz (1977) Quantum Mechanics, Non-Relativistic Theory, Pergamon, Oxford.
[2] L. I. Schiff (1955) Quantum Mechanics 3rd edition, McGraw-Hill, New York.
[3] H. Kleinert, Path integrals in Quantum Mechanics, Statistics and Polymer Physics, World scientific, Singapore, 2009.
[4] C. Groshe, F. Steiner, Handbook of Feynman Path Integrals, Springer, Berlin, 1998.
[5] A. Diaf, A. Chouchaoui and R.L. Lombard, Ann. Phys. 317, (2005) 354.
[6] A Diaf and A Chouchaoui, Phys. Scr. 84, (2011) 015004.
[7] Jun Lu, Phys. Scr. 72, (2005) 349.
[8] F. Taskin, Int. J. Theor. Phys. 48, (2009) 26922697.
[9] S.M. Ikhdair, J. Math. Phys. 51, (2010) 023525.
[10] M. M. Nieto, Phys. Rev. A 17, (1978) 1273.
[11] H. Kleinert, I. Mustapic, J. Math. Phys. 31, (1990) 634-662.
[12] D.C. Khandekar, S.V. Lawande, K.V. Bhagwat, Path Integral Methods and their Applications, World Scientific, Singapore, 1986.
[13] Böhm M and Junker G J , Math. Phys. A 117, (1986) 375.
[14] Inomata, A., Kuratsuji, H., Gerry, C.C.: Path Integrals and Coherent States SU(2) and SU(1,1), World Scientific, Singapore, 1992.
[15] C. Groshe J. Phys. A. Math. Gen. 22, (1989) 5073.