Codes in Rosenbloom-Tsfasman metric: A Survey

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Abstract. This paper gives a systematic survey of research carried out in the theory of codes equipped with Rosenbloom-Tsfasman metric. In classical coding theory setting, codes are investigated with respect to the Hamming metric which can efficiently address the communication problems arising from channels in which channel noise generates equiprobable errors. But however, not all the real world channels are of that nature, especially, when the possible errors form patterns of a specific shape. Rosenbloom and Tsfasman introduced a non-Hamming metric, called Rosenbloom-Tsfasman metric (RT-metric, in short) that can address the problem of reliable information transmission over parallel noisy channels. Martin, Stinson and Skriganov independently introduced the same metric in the context of the theory of uniform distributions. As this metric happened to be a generalization of the classical Hamming metric, it has attracted so much attention from the coding theory research community and as a result a lot of work has been done in this line of research over the past 3 decades. In this paper we would like to present the key developments in the field of codes with RT-metric.

Keywords: Rosenbloom-Tsfasman metric, Linear codes, Self-dual codes, MDS codes.

1 Introduction
In the current digital era, long distance communications between two parties over noisy channels is a daily routine. But reliable information transmission over a noisy channel is a fundamental challenge as the channel noise alters the message during transmission by causes occasional errors. Objects known as codes or error-correcting codes are designed to cope with this problem, and these codes are now commonplace in all storage and communication media ranging from compact disk players, hard disk drives, scanners, mobile phones to deep space communication devices.

While information transfer from one person to another is almost as old as mankind, the mathematical theory of fundamental principles governing modern digital communication is not that old. This began in 1948, when Claude E. Shannon’s seminal paper [1] showed that, with the help of proper encoding and decoding techniques, reliable communication, at any rate, can be accomplished within the capacity of the given communication channel even when the channel is noisy. This basic and ground-breaking finding created the twin fields of coding theory and information theory. However, Shannon’s theory is merely existential but not constructive as it only tells us, in a probabilistic sense, what is possible and what is not, but it doesn’t through light on ways to construct codes that can achieve the channel capacity. Meanwhile, his colleague at Bell Labs, R. W. Hamming paved ways to the actual code construction, encoding and decoding of codes by introducing a class of single error correction codes, now popularly known as Hamming codes and assigning a metric structure to those codes known as Hamming metric [2] in 1950. This contribution of Hamming’s has influenced many researchers and as a result revolutionary development happened in the field of coding theory.

The code and the metric defined by Hamming are as follows: Let $Mat_{m 	imes s}(F_q)$ denote the vector space of all $m \times s$ matrices over $F_q$, a finite field with $q$ elements. A nonempty subset $C$ of $Mat_{m 	imes s}(F_q)$ is called a code in $Mat_{m 	imes s}(F_q)$. In addition, $C$ is called a linear code, if it is a
subspace of $Mat_{m \times s}(\mathbb{F}_q)$. Moreover, $(m \times s, K)_q$ or $(m, s, K)_q$ denotes a code in $Mat_{m \times s}(\mathbb{F}_q)$ with cardinality $K$, and $[m \times s, k]_q$ or $[m, k]_q$ denotes a $k$ dimensional linear code. The Hamming weight $w_H(U)$, of $U \in Mat_{m \times s}(\mathbb{F}_q)$, is defined to be the number of nonzero entries in the matrix $U$ (see [3]). Here $w_H(U_1 - U_2)$ defines the Hamming metric on $Mat_{m \times s}(\mathbb{F}_q)$. Further, the Hamming distance $d_H(U, V)$, between $U, V \in Mat_{m \times s}(\mathbb{F}_q)$, is defined to be the number of positions in which the matrices $U$ and $V$ disagree.

Most of the studies in algebraic coding theory deal with the Hamming metric. If the noise from the channel produces equi-probable errors at all possible positions, then the Hamming metric is the most suitable one. However, when possible errors form patterns of specific shape, this metric is not always well suited to the characteristics of such real channels. Variety of metrics such as Lee metric [4] and Rank metric [5] have been introduced to deal with such channels and the properties of codes with respect to those metrics have been studied. In similar lines, to deal with a particular type of information transmission model over parallel noisy channels, Rosenbloom and Tsfasman introduced a new metric [6]. Moreover, some channels were also considered in which errors tend to occur on the left end just after a spike pulse and where the probability of errors at the right end is negligible. M. M. Skriganov [7] introduced an equivalent version of this metric which addresses the above problem far more adequately than the Hamming metric. This metric, later called as Rosenbloom-Tsfasman Metric (in short, RT-metric), has attracted a lot of researchers’ attention considering the fact that it is a generalization of the Hamming metric.

The organization of this paper is as follows. In Section 2, some basic definitions and notations concerning RT-metric are presented. Section 3 discusses the geometry of the RT-metric, particularly the importance of covering radius and the normal q-ary codes in RT-metric. In Section 4, we discuss the RT-weight enumerators and MacWilliams duality and the most fascinating family of self-dual codes in RT-metric and their practical significance. In Section 5, the linear isometries in RT-metric and the automorphism groups of RT-space are discussed. Sections 6 and 7 discuss works on MDS codes and burst error correction, respectively, in the context of RT-metric. Section 8 throws light on various other contributions in the study of RT-metric codes, the generalizations of RT-metrics such as NRT-metric and poset metric, and finally discusses on the probable applications of codes in RT-metric.

2 Preliminaries
The metric structure introduced by Rosenbloom and Tsfasman [6] on the space $Mat_{m \times s}(\mathbb{F}_q)$ is as follows:

The column weight, or simply weight, of

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in Mat_{m \times 1}(\mathbb{F}_q)$$

is given by

$$w_c(a) = \left\{ \begin{array}{ll} m - \max\{j|a_i = 0, \text{for any } i \leq j\} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{array} \right. \quad (1)$$

with the assumption that $\max \phi = 0$. As an extension, the weight of $A \in Mat_{m \times s}(\mathbb{F}_q)$ is determined by

$$w_c(A) = \sum_{k=1}^{s} w_c(A_k)$$

where $A_k$ indicates the $k^{th}$ column of $A$. If $d_c(A, B) = w_c(A - B)$, for all $A, B \in Mat_{m \times s}(\mathbb{F}_q)$, then $w_c$ specifies a metric on $Mat_{m \times s}(\mathbb{F}_q)$, called as column-metric. It is to be noted that it is simply the usual Hamming metric for $m = 1$. Alternatively, Skriganov, in [8] gave an equivalent
definition of the metric on \( \text{Mat}_{m \times s}(\mathbb{F}_q) \) induced by a weight known as \( \rho \)-weight, which focuses on rows of the matrices instead of columns. Its description is as follows:

For \( u = [u_1, u_2, \ldots, u_s] \in \text{Mat}_{1 \times s}(\mathbb{F}_q) = \mathbb{F}_q^s \), define

\[
\rho(u) = \begin{cases} 
\max\{i|u_i \neq 0, i = 1, 2, \ldots, s\} & \text{if } u \neq 0 \\
0 & \text{if } u = 0
\end{cases}
\]  

(2)

The \( \rho \)-distance between \( u, v \in \mathbb{F}_q^s \) can be defined as \( d_\rho(u, v) = \rho(u - v) \). If \( U, V \in \text{Mat}_{m \times s}(\mathbb{F}_q) \) (also, denoted as \( \mathbb{F}_q^{m \times s} \)), then the RT-weight and RT-distance in \( \text{Mat}_{m \times s}(\mathbb{F}_q) \) are given as follows:

\[
w_\rho(U) = \sum_{i=1}^{m} \rho(u_i)
\]  

(3)

where \( u_i \) indicates the \( i^{th} \) row of \( U \),

\[
d_\rho(U, V) = \rho(U - V), \text{ for every } U, V \in \text{Mat}_{m \times s}(\mathbb{F}_q).
\]  

(4)

It can be easily verified that the column-weight of a codeword is equal to the row-weight of a reversed component of the transposed codeword. Therefore, both metrics, \( m \)-metric and RT-metric, give rise to equivalent codes. The space \( \text{Mat}_{m \times s}(\mathbb{F}_q) \) equipped with RT-metric is called as RT-space.

In [9], Kwankyer Lee defined another variant of the metric as follows. For \( U \in \text{Mat}_{m \times s}(\mathbb{F}_q) \), the weight \( w_\rho(U) \) of \( U \) is defined by

\[
w_\rho(U) = \sum_{j=1}^{s} \text{depth}_j(U),
\]  

(5)

where \( \text{depth}_j(U) = \max\{i|u_{ij} \neq 0\} \), for \( 1 \leq j \leq s \) assuming \( \max\phi = 0 \).

For an RT-metric code \( \mathcal{C} \), the minimum weight \( w_\rho \) and minimum distance \( d_\rho \) are defined as follows:

\[
w_\rho = w_\rho(\mathcal{C}) = \min\{w_\rho(U) | U \in \mathcal{C}, U \neq 0\}
\]

\[
d_\rho = d_\rho(\mathcal{C}) = \min\{d_\rho(U, V) | U, V \in \mathcal{C}, U \neq V\}
\]  

(6)

If \( \varphi(U) \) denotes the Hamming weight of \( U \in \text{Mat}_{m \times s}(\mathbb{F}_q) \), it can be easily seen that the weights \( \varphi \) and \( \rho \) are related by

\[
\varphi(U) \leq \rho(U) \leq s * \varphi(U).
\]  

(7)

As it is impossible to improve these inequalities on the ambient space \( \text{Mat}_{m \times s}(\mathbb{F}_q) \), the \( \rho \)-metric is stronger than Hamming metric for large \( s \), with both coinciding when \( s = 1 \).

### 3 Geometry of RT-Space

The geometry of a code plays an important role in deciding the vital parameters of the code which in turn determine how good a code is. In this section, we discuss the works that focused on the geometric aspects of RT-metric codes.

Consider RT-balls(or \( \rho \)-balls) and RT-spheres(or \( \rho \)-spheres) in \( \text{Mat}_{m \times s}(\mathbb{F}_q) \) with respect to the RT-metric,

\[
B^{(m,s)}(U; r) = \{ V \in \text{Mat}_{m \times s}(\mathbb{F}_q) : d_\rho(U, V) \leq r \}
\]  

(8)

\[
S^{(m,s)}(U; r) = \{ V \in \text{Mat}_{m \times s}(\mathbb{F}_q) : d_\rho(U, V) = r \}
\]  

(9)

where \( 0 \leq r \leq ms \) is an integer. Some authors also denote these RT-ball and RT-sphere respectively as \( B_\rho(U, r) \) and \( S_\rho(U, r) \). Since the cardinalities of these structures do not depend
on centers, without taking the center \( U \) into consideration (or considering the center to be the origin or zero matrix(or zero vector)), we can simply take

\[
B^{(m,s)}(r) = \{ U \in Mat_{m \times s}(\mathbb{F}_q) : w_p(U) \leq r \}
\]

(10)

\[
S^{(m,s)}(r) = \{ U \in Mat_{m \times s}(\mathbb{F}_q) : w_p(U) = r \}
\]

(11)

The RT-balls and RT-spheres were also defined in [8] in a different and equivalent fashion where the subject of study is codes as well as point distributions over \( \mathbb{F}_q \).

The maximum \( r \) for which all the \( p \)-balls of radius \( r \) centered at the codewords in \( \mathcal{C} \) are mutually disjoint is called the packing radius of \( \mathcal{C} \). The minimum \( R \) for which the \( p \)-balls of radius \( R \) centered at the codewords in \( \mathcal{C} \) cover the RT space is called the covering radius of \( \mathcal{C} \).

A code in \( Mat_{m \times s}(\mathbb{F}_q) \) with covering radius \( R \) is denoted by \((m,s,K,d;R)_q\). If the covering radius and packing radius of a code happen to be equal, such a code is called a perfect code.

For, \( m = 1 \), we have,

\[
B^{(1,s)}(u;r) = \{ v = (v_1, v_2, ..., v_s) \in Mat_{1 \times s}(\mathbb{F}_q) : d_p(u,v) \leq r \}
\]

(12)

where \( u = (u_1, u_2, ..., u_s) \in Mat_{1 \times s}(\mathbb{F}_q) \).

Consider

\[
B^{(1,s)}(r) = \{ u = (u_1, u_2, ..., u_s) \in Mat_{1 \times s}(\mathbb{F}_q) : u_i = 0 \text{ for } i > r \}.
\]

(13)

If \( r = 0 \), \( B^{(1,s)}(0) = S^{(1,s)}(0) \) and \( B^{(1,s)}(r) \) is a subspace in \( Mat_{1 \times s}(\mathbb{F}_q) \) with dimension \( r \).

Moreover,

\[
S^{(1,s)}(r) = B^{(1,s)}(r) - B^{(1,s)}(r-1), r \geq 1.
\]

(14)

Note: \( Mat_{1 \times s}(\mathbb{F}_q) \) is divided into a union of distinct spheres:

\[
Mat_{1 \times s}(\mathbb{F}_q) = \bigcup_{r=0}^{s} S^{(1,s)}(r).
\]

(15)

The following subset of integer vectors was indicated in an arbitrary \( m \), by \( Q_{m,s} \subset \mathbb{Z}^m \):

\[
Q_{m,s} = \{ R = (r_1, r_2, ..., r_m) : 0 \leq r_j \leq s, 1 \leq j \leq m \}.
\]

(16)

As the space \( Mat_{m \times s}(\mathbb{F}_q) \) can be regarded as a direct product of \( m \) copies of \( Mat_{1 \times s}(\mathbb{F}_q) \):

\[
Mat_{m \times s}(\mathbb{F}_q) = \prod_{j=1}^{m} Mat_{1 \times s}(\mathbb{F}_q),
\]

(17)

for \( R = (r_1, r_2, ..., r_m) \in Q_{m,s} \), one can introduce the subspaces \( V_R \subset Mat_{m \times s}(\mathbb{F}_q) \),

\[
V_R = \prod_{j=1}^{m} B^{(1,s)}(r_j),
\]

(18)

with

\[
dim V_R = r_1 + r_2 + ... + r_m,
\]

(19)

and subsets \( F_R \subset Mat_{m \times s}(\mathbb{F}_q) \) are said to be fragments as defined as follows:

\[
F_R = \prod_{j=1}^{m} S^{(1,s)}(r_j).
\]

(20)
Notice that $\text{Mat}_{m \times s}(\mathbb{F}_q)$ splits into a disjoint union of fragments (20):

$$\text{Mat}_{m \times s}(\mathbb{F}_q) = \bigcup_{R \in \mathbb{Q}_{m,s}} F_R.$$  \hspace{1cm} (21)

Skriganov proved in [7] that each ball $B^{(m,s)}(r)$ is a union of the subspaces $V_R$, which is given as

$$B^{(m,s)}(r) = \bigcup_{r_1, r_2, ..., r_m = r} V_R.$$  \hspace{1cm} (22)

And he also proved that each sphere $S^{(m,s)}(r)$ is a distinct union of the following fragments $F_R$

$$S^{(m,s)}(r) = \bigcup_{r_1, r_2, ..., r_m = r} F_R.$$  \hspace{1cm} (23)

3.1 Covering Radius in Rosenbloom-Tsfasman metric

Covering radius, one of the four fundamental geometric parameters, of a code, is important in a number of ways [10]. Considering the fact that covering radius is a geometric property of a code that characterizes the maximum ability to correct errors in the context of minimum distance decoding, many researchers have been extensively studying it in conventional Hamming metric during the last several years since the Delsarte’s paper in 1973 [10].

Panek et al [11] explored the problem of packing and covering radii and existence of perfect codes in the case of a poset block metric known as Niederreiter-Rosenbloom-Tsfasman (NRT) metric, a generalization of RT-metric. Yildiz et al [12] had first studied the problem on coverings with respect to RT-metric over finite rings. They established the cardinality in relation to the RT-metric of the minimal $R$-covers of finite rings by extending the results in Nakaoka and dos Santos [13] for 0-short coverings of Finite Chain Rings. Further, they explored $R$-short coverings of rings with RT-metric. R. S. Selvaraj and V. Marka [14] attempted the problem of exploring covering radius for codes of length $s$ over $\mathbb{F}_q$ by introducing concepts known as $l$-cell and partition number.

R. S. Selvaraj and V. Marka [15] established a relationship between the minimum $\rho$-distance of the code and its covering radius and provided a necessary and sufficient condition for an RT-metric code to be MDS, using partition number of a code. They also found that all MDS codes are perfect, and vice versa, which is not the case with codes in other metrics such as Hamming metric [2], Rank metric [5] and Lee metric [4].

Over the years, various construction techniques have been proposed to establish better bounds on the covering radius using two or more known codes in the construction of new code over Hamming metric. One of such methods called Amalgamated Direct Sum (ADS) over binary linear codes - an important but simple technique - on which notion of normality was introduced by Graham and Sloane in [16] to improve the bounds on covering radius of codes thus constructed. In [14], the notion of normality of codes in the Hamming metric has been adapted to Rosenbloom-Tsfasman metric by R.S. Selvaraj and V. Marka who showed that the ADS construction is less significant for higher dimensional RT-metric codes.

4 MacWilliams Duality, Weight Enumerators in RT-metric

Weight enumerators are some specific polynomials associated with every linear code, which play a crucial role in determining the minimum distance of a code and hence in the process of encoding and decoding. The MacWilliams identities, one of the most fascinating results in coding theory, establish a connection between the weight enumerators of a code and its dual [3, 17]. Skriganov [8] is the first to address the problem of extending the notion of MacWilliams identities to the $\rho$-metric and finding $\rho$-weight enumerators of a linear code $\mathcal{C}$ defined as follows:

$$W(\mathcal{C}|z) = \sum_{r=0}^{m_s} w_r(\mathcal{C}) z^r = \sum_{U \in \mathcal{C}} z^{w_\rho(U)}.$$  \hspace{1cm} (24)
The dual code
and it is extended to the inner product of $U = (U_1, U_2, ..., U_m)^T$, $V = (V_1, V_2, ..., V_m)^T \in Mat_{m \times s}(\mathbb{F}_q)$ as
\[
\langle U, V \rangle = \sum_{i=1}^{m} (U_i, V_i).
\]
(26)

The dual code $\mathcal{C}^\perp$ is defined as
\[
\mathcal{C}^\perp = \{ V \in Mat_{m \times s}(\mathbb{F}_q) : \langle V, U \rangle = 0 \text{ for all } U \in \mathcal{C} \}.
\]
(27)

It is clear that $\mathcal{C}^\perp$ is a subspace of $Mat_{m \times s}(\mathbb{F}_q)$, and $(\mathcal{C}^\perp)^\perp = \mathcal{C}$. Furthermore, if $\mathcal{C}$ and $\mathcal{C}^\perp$ are with parameters $[ms, k, d_q]$ and $[ms, k^-, d^-_q]$ respectively, then their corresponding dimensions and cardinalities, respectively, are related by
\[
k + k^- = ms, \ |\mathcal{C}|\ |\mathcal{C}^\perp| = q^{ms}, \ |\mathcal{C}| = q^k, \ |\mathcal{C}^\perp| = q^{ms-k}.
\]
(28)

This inner product for RT-metric differs from the usual inner product used for Hamming metric as it leads to the MacWilliams relations given in [18] where the usual inner product can’t connect the weight enumerators of an RT-metric code and its dual. This inner product is more convincing as was proved in [18] that whenever $C$ is a linear RT-metric MDS code the dual $C^\perp$, with respect to this special inner product, is an MDS code as well. This result may not always hold for other inner product choices. Furthermore, it needs to be further explored to determine if there is a better suitable inner product alternative.

4.1 MacWilliams identity for $\rho$-weight enumeration in RT-metric

MacWilliams in [17] first introduced a classical MacWilliams relations for Hamming weight enumerators and defined as follows
\[
W(\mathcal{C}^\perp | z) = \frac{1}{|\mathcal{C}|} \frac{1}{1 + (q-1)z} W(\mathcal{C} | \frac{1}{1 + (q-1)z}).
\]
(29)

This relation satisfies the $\rho$-weight enumerator (24) in the case of $Mat_{m \times 1}(\mathbb{F}_q)$. Skriganov [8] was first to introduce the following MacWilliams relations for $\rho$-weight enumerator (24) in the case of $Mat_{1 \times s}(\mathbb{F}_q)$:
\[
(qz - 1)W(\mathcal{C}^\perp | z) + 1 - z = |\mathcal{C}^\perp| z^{s+1} W(\mathcal{C} | \frac{1}{qz}) + qz - 1.
\]
(30)

The natural question that one would seek answers to is if one can extend the notion of MacWilliams identities to codes in $Mat_{m \times s}(\mathbb{F}_q)$. Skriganov noted that straightforward extensions don’t exist for $\rho$-weight enumerators (24) and can not be linked to the Macwilliams identities in arbitrary case of $m$ and $s$.

In 2002, Dougherty and Skriganov [18] gave the generalized version of MacWilliams relation of RT-weight enumerator for codes in $Mat_{m \times s}(\mathbb{F}_q)$. Following the method proposed by Skriganov in [8], it was demonstrated that the RT-weight enumerators associated with the orbits of a $\rho$-weight preserving linear group satisfy the MacWilliams-type theorems for a linear code and its dual. They also showed that a multi-dimensional generalization of known Krawtchouk polynomials relates to the corresponding weight spectra of dual codes. In addition, they also
briefly analyzed the connections with the results by Godsil [19], Martin and Stinson [20] on MacWilliams-type theorems corresponding to objects such as association schemes and ordered orthogonal arrays. It turns out that the indicated group is transitive on each sphere for $\rho$-metric, only in the two special cases of $s = 1$ or $m = 1$. In fact, this observation provides an idea of why the MacWilliams-type identities for $\rho$-weight enumerators (24) is possible only in the cited cases. This treatment of the problem is similar to a familiar approach to MacWilliams identities for the Hamming metric by Delsarte [10]. Furthermore, Martin and Stinson studied the relation between ordered codes and ordered orthogonal arrays, $(T, M, S)$-nets, and association schemes and shown that RT-metric also has an important role to play here as well [20]. However, the approach in [18] is inspired by the relationship of codes to uniform distributions with the Rosenbloom-Tsfasman metric, and the formulation of these theories emerged independently. Dougherty and Skriganov have pointed out where equivalent results occur and have made considerable contributions to convince that the $\rho$-metric appears to derive great significance compared to the Hamming metric.

4.2 MacWilliams identity for Complete $\rho$-Weight Enumeration in $Mat_{m \times s}(F_q)$

To overcome the problem of finding MacWilliam type identities in order to link the $\rho$-Weight Enumerators of a code and its dual, Siap [21] introduced a complete $\rho$-weight enumerator of linear code (CWEL in short) over $Mat_{m \times s}(F_q)$, and using these he proved the MacWilliams identity endowed with the RT-metric. Further, in [22] he derived the MacWilliams identity for CWEL in $Mat_{m \times s}(K)$, where $K = F_q[v] / (v^s - l)$ with $l \in F_q$. Subsequently, Siap and Özen generalized the MacWilliams identity for CWEL in $Mat_{m \times s}(R)$, where $R$ is a commutative finite ring [23]. The authors in [24–26] elaborated the structure of linear codes in $Mat_{m \times s}(K)$ when $K$ is either $F_q$ or $F_q[v] / (v^s)$ or a Golay ring. Zhu and Xu [27] defined the Lee-CWEL and the exact-CWEL in $Mat_{m \times s}(F_2 + vF_2)$ with $v^2 = 0$ and derived the corresponding MacWilliams identity. Later, Du and Xu [28] generalized their result to the linear codes in $Mat_{m \times s}(K)$, where $K = F_q + vF_2 + \cdots + v^{s-1}F_q$ and $v' = 0$, defined the exact-CWEL in $Mat_{m \times s}(R)$ and also obtained MacWilliams identities for them. Sharma et.al [29] introduced and studied the split $\rho$-weight enumerator of a linear code in $Mat_{m \times s}(R)$, and proved the MacWilliams identity for the same. They also defined the Lee-CWEL in $Mat_{m \times s}(Z_k)$, and derived the MacWilliams identity for it.

4.3 Self-dual codes in Rosenbloom-Tsfasman metric

The class of self-dual codes is one of the most interesting families of codes. This type of codes are of greatest curiosity because of various reasons, one of them is that self-dual codes are the finest known error correction codes. Such codes have close connections to other mathematical structures such as lattices, modular forms, block designs, and sphere packings [30]. Self-dual codes are also relevant in practice; for example, $G_{24}$, an extended binary Golay [24,12] code used for the Voyager Space probes, launched in 1977 to Jupiter and Saturn, is a self-dual code [31].

The theory of self-dual codes has motivated many researchers in the last 50 years and over those years, work has taken a significant step forward in proposing various techniques to construct self-dual codes, to investigate the properties and identify them eventually [30].

A linear code which has dimension half its length and that is equal to its dual is called self-dual code. The obvious characteristic of a self-dual code is that its weight distribution is the same as its dual’s. A linear code over a field is called formally self-dual.

As the inner product considered in RT-metric is different from that in Hamming metric, most codes in which are self-dual in Hamming metric are not so in RT metric. Owing to this, though a lot of research has been carried out on self-dual codes in Hamming metric, the theory of self-dual codes in RT-metric must be thoroughly investigated from existence to other results. Marka et al [32] had established the necessary and sufficient conditions for RT-metric codes over $F_q^s$ to be self-dual. Recently, using polynomial invariant theory, Santos and Muniz [33] studied binary self-dual codes, shape enumerators in NRT-metric and gave some constructions.
of self-dual codes in $\text{Mat}_{m\times s}(\mathbb{F}_q)$ by extending the results in [32].

5 Linear Isometries in RT-metric
In coding theory, the most important characteristic of code is the distance between certain codewords and between codewords and non-codewords in the metric under consideration. For example, in terms of the Hamming metric, the nearest pair of codewords defines the error correction capacity of the code. In addition, it may be possible to map one code to another using a map that preserves the distances with respect to the metric concerned. Clearly one code would be as good as the other in any practical application as far as error-correction is concerned. This classification of codes is important because it gives only those codes that are essentially distinct, that is, non-equivalent codes that are of interest because of their role in making the search for good codes more effective.

Two $(m \times s, k)_q$-codes $\mathcal{C}_1, \mathcal{C}_2 \subseteq \text{Mat}_{m\times s}(\mathbb{F}_q)$ are equivalent if there exists a mapping

$$\xi : \text{Mat}_{m\times s}(\mathbb{F}_q) \to \text{Mat}_{m\times s}(\mathbb{F}_q)$$

with $\xi(\mathcal{C}_1) = \mathcal{C}_2$ that preserves the RT-distance that is,

$$d_\rho(U, V) = d_\rho(\xi(U), \xi(V)), \text{ for all } U, V \in \text{Mat}_{m\times s}(\mathbb{F}_q) \tag{31}$$

Mapping with the property (31) is called an isometry, if the isometry is linear then we call it linear isometry. The study of linear isometries is named as the automorphism group of the space that preserves the metric, for these linear isometries are in fact automorphisms. The definition of equivalent codes or isometric codes can be defined as follows, using this notion of isometry:

Two linear codes $\mathcal{C}_1, \mathcal{C}_2 \subseteq \text{Mat}_{m\times s}(\mathbb{F}_q)$ are said to be isometric or equivalent if there exists an isometry of $\text{Mat}_{m\times s}(\mathbb{F}_q)$ that maps $\mathcal{C}_1$ onto $\mathcal{C}_2$.

Skriyanov [7] has defined some of the natural linear isometries of the RT-space $\text{Mat}_{m\times s}(\mathbb{F}_q)$. Firstly, the symmetric group $S_m$ of all permutations of rows of a matrix $U \in \text{Mat}_{m\times s}(\mathbb{F}_q)$ preserves the weight $w_2(U)$; so it is a group of isometries. Another group of isometries given by him is as follows: Let $T_s$ indicate a group of all non-singular lower triangular with non-zero diagonal elements over $\text{Mat}_{s\times s}(\mathbb{F}_q)$. Then, it is preserve the weight $\rho : w_2(u\Gamma) = w_2(u)$ for $u = (u_1, u_2, ..., u_s) \in \text{Mat}_{1\times s}(\mathbb{F}_q)$ and $\Gamma \in T_s$. Then consider

$$H_{m,s} = S_m \times T_s^m \tag{32}$$

where

$$T_s^m = \prod_{j=1}^{m} T_s \tag{33}$$

indicates the direct product of $m$ copies of $T_s$ and $S_m$ is the symmetric group of permutations on $m$ symbols. It is not so difficult to see that this set $H_{m,s}$ also forms the group of isometries of the space $\text{Mat}_{m\times s}(\mathbb{F}_q)$. Kwankyu Lee [9] proved that the group $H_{m,s}$ actually constitutes the full group of linear RT-space isometries. Furthermore, since these isometries are linear, that is, automorphisms on the $\text{Mat}_{m\times s}(\mathbb{F}_q)$ space, this group can also be called as the full group of RT-space automorphisms.

6 MDS Codes in Rosenbloom-Tsfasman metric
From a coding perspective [6], and from the corresponding notion in uniform distributions [7], one wants the codewords in a RT-metric code to be as far apart from each other as possible. That essentially means, the smallest RT-distance between any two codewords to be as large as possible. As with codes in the classical setting, one also wants the size of the RT-metric code to be as large as possible. These objectives are contradictory. Hence, what is sought is the largest number of codewords one can have with a large minimum distance, or the greatest possible minimum distance one can have for a given number of codewords. Codes realizing these objectives happen to be MDS codes. Now, we state a formal definition of MDS codes via the Singleton bound for the codes in RT-metric that is given in [6].


Singleton Bound: For every code \( C \subset \text{Mat}_{m \times s}(F_q) \), its cardinality \(|C|\) and minimum RT-distance \( d_q(C) \) are related by
\[
|C| \leq q^{ms - d_q(C) + 1}
\]  
(34)
In particular, every \([ms, k]_q\) code satisfies the bound
\[
d_q(C) \leq ms - k + 1
\]  
(35)

In RT-metric, an \([m \times s, k, d_{p,q}]\) code is said to be a Maximum Distance Separable code (or MDS code in short) if it attains the Singleton bound, that is, if \( d = ms - k + 1 \). MDS codes in \( \text{Mat}_{m \times s}(F_q) \) with the \( \rho \)-metric were defined by Rosenbloom and Tsfasman [6] and related to possible information theory applications. Skriganov [7] related these codes to uniform distributions and furthered their theory. He introduced finite point subsets of a special kind in the \( m \)-dimensional unit-cube, called as optimum distributions, that has a rich interior combinatorial structure in the sense that they can be characterized completely as maximum distance separable codes in \( \text{Mat}_{m \times s}(F_q) \) with the \( \rho \)-metric. He gave general results on these distributions, and thus provided a means to identify each of these distributions with a unique \( q \)-ary code. Then he went on to investigate MDS codes in the \( \rho \)-metric. Optimum distributions were characterized completely in terms of their \( q \)-ary codes. He had shown that a given distribution is an optimum distribution if and only if its code is MDS in the \( \rho \)-metric. Explicit formulas for weight spectra were also proved for arbitrary MDS codes and optimum distributions in the \( \rho \)-metric which further implies certain necessary conditions for the existence of such codes and distributions. V. Marka and R. S. Selvaraj [15] has shown that an optimal code in \( F_q^s \) are MDS and vice-versa. While the MDS codes are optimal in case of codes with the classic Hamming metric, the converse need not be true. In the Hamming metric, optimal codes don’t have to be MDS. Take the binary Reed-Muller \([8, 4, 4]\) code, for example, for Hamming Metric it would be optimal, but not MDS. They have shown that all cyclic codes in \( F_q^s \) are MDS, which is not true for codes in Hamming metric. They also prove that for every MDS code in Hamming metric, there is an equivalent code which is MDS in RT-metric, vice versa.

7 Burst Error Correction and RT-metric

Sometimes, communication systems and the tape or disk memory can cause error clusters, i.e., burst errors (called bursts because they occur in many consecutive bits). Fire [34] introduced the concept of burst errors for classical codes in the space of all \( s \)-tuples with entries from a finite field \( F_q \). For example, imagine a sender sending messages that are \( s \) tuples of \( m \) tuples, each of them transmitting \( q \)-ary symbols on \( m \) parallel channels. These channels produce an interfering noise that causes errors in the message being transmitted. In the study of \( \rho \)-metric array code, the errors considered so far are random errors that can occur in the code matrix anywhere. An important and realistic condition is when the errors in the code matrix are not distributed randomly but in cluster form and are limited to a sub-matrix portion of the code matrix. Motivated by this information transmission problem, Sapna Jain [35] has introduced burst errors by adopting the notion of bursts to \( \rho \)-metric array codes in the space \( \text{Mat}_{m \times s}(F_q) \). She had also established certain bounds in codes for detecting and correcting burst errors on \( \rho \)-metric array codes. The definition of a CT-burst given by Sapna Jain is as follows [35]:

Definition: A burst of order \( ab \) (or \( a \times b \)) \((1 \leq a \leq m, 1 \leq b \leq s)\) in the space \( \text{Mat}_{m \times s}(F_q) \) is an \( m \times s \) matrix in which all the nonzero entries are confined to some \( a \times b \) sub-matrix which has non-zero first and last rows as well as non-zero first and last columns.

For \( a = 1 \), the above definition converted to the classical code burst definition [34].

In classical Hamming metric, for codes in \( F_q^s \), Campopiano [36] obtained an upper bound for the burst error correction, Sapna Jain [37] had given some upper bounds analogous to these Campopiano bounds for both cases, with and without weight constraint.

Sapna Jain [38] also extended the concept of a CT-burst, that had been introduced by Chen and Tang [39] for classical codes, to array codes in \( \text{Mat}_{m \times s}(F_q) \) endowed with RT-metric and established a Rieger’s type bound. There were certain limitations in enumerating the CT burst
errors of RT-metric array code as her work involved solving certain linear equations and then carrying out the computations separately for each weight computation. This tedious task of solving equations and performing repeated computations had been overcome by Irfan Siap [40]. In this work, he introduced the concept of generic burst errors and developed a novel generating function-like approach for computing the number of CT bursts. His method obtains the so-called CT burst error weight enumerator which not only gives the number of CT burst errors of particular weight but also the entire weight spectra in a single computation.

8 RT-metric: Other contributions, Generalizations and Applications

8.1 Some other contributions

J. Quistroff [41] had presented the combinatorial foundations of the RT-metric in a slightly more general setting using words over finite sets. Further, apart from the bounds given in [6], some new bounds induced by combinations and modifications of parameters were also obtained. Panek et al. [11] extended the concept of generalized Wei weights [42] for Rosenbloom-Tsfasman metric and showed that all linear codes satisfy the chain condition in $Mat_{m \times s}(F_q)$. Mehmet Ozen and Irfan Siap had provided a simple decoding technique for linear codes with RT-metric [24].

R. Rajkumar [43] had studied the codes and graphs with RT-metric. In this work, certain new classes of codes such as circulant RT-distance codes, fuzzy RT-distance codes were introduced and their properties like chromatic number, components and degrees of vertices of distance graphs on the spaces $Z_q^n$ and $S_n$ with respect to the RT-metric were studied. Chen et al. [44] studied the RT-metric of matrix product codes over finite commutative rings and also discussed the lower bounds of the dual matrix product codes over finite Frobenius commutative rings. In the $Mat_{m \times s}(F_q)$, V. Marka and R. S. Selvaraj [45] adapted and analyzed the basic parameters such as the covering radius and minimum distance to decomposable codes. Further, they also showed that a decomposable code can be neither MDS nor perfect. In addition, they demonstrated that a decomposable code is self-dual only if its constituent codes are self-dual. Further, they discussed the optimality of decomposable codes.

8.2 Variants and Generalizations of RT-metric

Given length $n$ and minimum Hamming distance $d$, the problem of finding a linear code $C$ with maximum dimension is one of the primary goals of coding theory. In [46–48], Niederreiter generalized this problem by expressing it in matrix form or, equivalently, in terms of collection of vectors in $F_q^n$, with the help of parity-check matrices of codes. Later, in [49], Brualdi et al. introduced poset metrics that are determined by partial orders on the set $\{1, 2, ..., n\}$ of coordinate positions of vectors in $F_q^n$, and further shown that, in a certain poset metric defined by disjoint union of chains, the Niederreiter’s problem is analogous to the classical coding theoretic problem stated in the beginning of this section. In the sequel, the block metric was introduced, by Feng et al., in which the set of coordinate positions of $F_q^n$ are partitioned into families of blocks [50]. Both these metrics, the poset metrics and block metric, are generalizations of the classical Hamming metric. Later, these poset and block metric structures were combined by Alves et al. [51] to obtain a further generalization, the poset-block metrics. NRT-metric is one type of poset-block metrics which is a generalization of the RT-metric. Indeed, the poset metric represents the classical Hamming metric if the poset is an anti-chain one. In addition, the RT-metric is known to be one particular case of all these poset metrics in the sense that the disjoint unions of chains of the same cardinality can be viewed as a poset metric [29]. Andre et al. [52] established a connection between covering codes in Hamming and RT spaces. One could refer to further discussions on the relationship of RT-metric to poset metrics in general and NRT-metrics in particular [53, 54]. Generalizing the classical Lee metric (see [4]) and array RT-metric, Sapna Jain [55] introduced a new pseudo-metric called Generalized-Lee-RT-Pseudo-Metric (in short, the GLRP-Metric) on the module space of all $m \times s$ matrices over the finite ring $Z_q$. 
8.3 Applications of codes in RT-metric

As coding theory arose from the need for reliable communication, various codes in the classical Hamming metric were found to have multiple applications in several information transmission problems. It’s also true that even in areas like cryptography and biology, classical coding theory has applications. In this context, it is normal to ask whether the RT-metric codes have any unique applications relevant to the data transfer problem. Some information transmission models were introduced in [6, 7], where this metric can probably be applied.

Rosenbloom and Tsfasman [6] presented a channel model for information transmission for which RT-metric (particularly, the column metric $d_c$) is a natural quality attribute of a code, where the messages are transmitted over as many as $m$ parallel channels in terms of $s$-tuple of $q$-ary symbols. There is an interfering noise in several consecutive channels, starting from the last, which are occupied by a priority user, who lets them free once he needs them no more, in the order inverse to the order of their occupation. The degree of interference is measured by the total number of $q$-ary symbols sent by the priority user over the main message. It is clear that the minimum distance of a code in the sense of RT-metric characterizes its stability under such interference.

One of the possible applications of the codes, in this metric, according to M. M. Skriganov [7] is as follows: Suppose a sender intends to transmit a sequence $c = (c_1, c_2, c_3, \ldots)$ of digits $c_i \in \mathbb{F}_q$ across a noisy channel. If the channel noise causes equiprobable errors in all possible positions, the Hamming metric addresses such transmission problems quite adequately. However, when possible errors form patterns of a specific shape, the Hamming metric turns out to be very crude. Consider, for example, a channel where a periodic spike-wise perturbation of the periods generates errors. We split the sequence of digits $U$ of length $ms$ into $m$ blocks of length $s$. then, $U = (u_1, u_2, \ldots, u_m)$, $u_j \in \text{Mat}_{1 \times s} (\mathbb{F}_q)$, $1 \leq j \leq m$. The errors in every block $u_j = (u_{j1}, u_{j2}, \ldots, u_{js})$ tend to occur in the left after a spike pulse, while the probability of errors in the right end is insignificant (i.e. $(0, \ldots, 0, 1)$). RT-metric clearly depicts much more accurately the actual structure of the problem described above. Code $C \subseteq \text{Mat}_{m \times s} (\mathbb{F}_q)$ with a high minimum distance $d_p$ will, moreover, detect and correct all channel errors $U' \in \text{Mat}_{1 \times s} (\mathbb{F}_q)$ with $2w_p(U') < d_p$, although the minimum distance of Hamming $d_H(C)$ is small.

9 Conclusion

In this paper, we presented most of the research developments happened in the theory of codes with RT-metric in various notions such as covering properties, MDS codes, linear isometries, weight enumerators, MacWilliams identities, duality and burst error correction. Still there is a lot of scope for research on codes with RT-metric in areas such as decoding techniques, optimal codes and several combinatorial bounds.

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