Integral closure of rings of integer-valued polynomials on algebras

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Abstract
Let $D$ be an integrally closed domain with quotient field $K$. Let $A$ be a torsion-free $D$-algebra that is finitely generated as a $D$-module. For every $a$ in $A$ we consider its minimal polynomial $\mu_a(X) \in D[X]$, i.e. the monic polynomial of least degree such that $\mu_a(a) = 0$. The ring $\text{Int}_K(A)$ consists of polynomials in $K[X]$ that send elements of $A$ back to $A$ under evaluation. If $D$ has finite residue rings, we show that the integral closure of $\text{Int}_K(A)$ is the ring of polynomials in $K[X]$ which map the roots in an algebraic closure of $K$ of all the $\mu_a(X)$, $a \in A$, into elements that are integral over $D$. The result is obtained by identifying $A$ with a $D$-subalgebra of the matrix algebra $M_n(K)$ for some $n$ and then considering polynomials which map a matrix to a matrix integral over $D$. We also obtain information about polynomially dense subsets of these rings of polynomials.

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1 Introduction
Let $D$ be a (commutative) integral domain with quotient field $K$. The ring $\text{Int}(D)$ of integer-valued polynomials on $D$ consists of polynomials in $K[X]$ that map elements of $D$ back to $D$. More generally, if $E \subseteq K$, then one may define the ring $\text{Int}(E, D)$ of polynomials that map elements of $E$ into $D$.

One focus of recent research ([4], [5], [9], [10], [11]) has been to generalize the notion of integer-valued polynomial to $D$-algebras. When $A \subseteq D$ is a torsion-free module finite $D$-algebra we define $\text{Int}_K(A) := \{ f \in K[X] \mid f(A) \subseteq A \}$. The set $\text{Int}_K(A)$ forms a commutative ring. If we assume that $K \cap A = D$, then $\text{Int}_K(A)$ is contained in $\text{Int}(D)$ (these two facts are indeed equivalent), and often $\text{Int}_K(A)$ shares properties similar to those of $\text{Int}(D)$ (see the references above, especially [5]).

When $A = M_n(D)$, the ring of $n \times n$ matrices with entries in $D$, $\text{Int}_K(A)$ has proven to be particularly amenable to investigation. For instance, [9] Thm. 4.6 shows that the integral

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closure of $\text{Int}_Q(M_n(\mathbb{Z}))$ is $\text{Int}_Q(\mathcal{A}_n)$, where $\mathcal{A}_n$ is the set of algebraic integers of degree at most $n$, and $\text{Int}_Q(\mathcal{A}_n) = \{f \in \mathbb{Q}[X] \mid f(\mathcal{A}_n) \subseteq \mathcal{A}_n\}$. In this paper, we will generalize this theorem and describe the integral closure of $\text{Int}_K(\mathcal{A})$ when $D$ is an integrally closed domain with finite residue rings. Our description (Theorem 3.3) may be considered an extension of both [9, Thm. 4.6] and [2, Thm. IV.4.7] (the latter originally proved in [7, Prop. 2.2]), which states that if $D$ is Noetherian and $D'$ is its integral closure in $K$, then the integral closure of $\text{Int}(D)$ equals $\text{Int}(D, D') = \{f \in K[X] \mid f(D) \subseteq D'\}$.

Key to our work will be rings of polynomials that we dub integral-valued polynomials, and which act on certain subsets of $M_n(K)$. Let $\overline{K}$ be an algebraic closure of $K$. We will establish a close connection between integral-valued polynomials and polynomials that act on elements of $\overline{K}$ that are integral over $D$. We will also investigate polynomially dense subsets of rings of integral-valued polynomials.

In Section 2, we define what we mean by integral-valued polynomials, discuss when sets of such polynomials form a ring, and connect them to the integral elements of $\overline{K}$. In Section 3, we apply the results of Section 2 to $\text{Int}_K(\mathcal{A})$ and prove the aforementioned theorem about its integral closure. Section 4 covers polynomially dense subsets of rings of integral-valued polynomials. We close by posing several problems for further research.

# 2 Integral-valued polynomials

Throughout, we assume that $D$ is an integrally closed domain with quotient field $K$. We denote by $\overline{K}$ a fixed algebraic closure of $K$. When working in $M_n(K)$, we associate $K$ with the scalar matrices, so that we may consider $K$ (and $D$) to be subsets of $M_n(K)$.

For each matrix $M \in M_n(K)$, we let $\mu_M(X) \in K[X]$ denote the minimal polynomial of $M$, which is the monic generator of $N_{K[X]}(M) = \{f \in K[X] \mid f(M) = 0\}$, called the null ideal of $M$. We define $\Omega_M$ to be the set of eigenvalues of $M$ considered as a matrix in $M_n(\overline{K})$, which are the roots of $\mu_M$ in $\overline{K}$. For a subset $S \subseteq M_n(K)$, we define $\Omega_S := \bigcup_{M \in S} \Omega_M$. Note that a matrix in $M_n(K)$ may have minimal polynomial in $D[X]$ even though the matrix itself is not in $M_n(D)$. A simple example is given by $\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \in M_2(K)$, where $q \in K \setminus D$.

**Definition 2.1.** We say that $M \in M_n(K)$ is integral over $D$ (or just integral, or is an integral matrix) if $M$ solves a monic polynomial in $D[X]$. A subset $S$ of $M_n(K)$ is said to be integral if each $M \in S$ is integral over $D$.

Our first lemma gives equivalent definitions for a matrix to be integral.

**Lemma 2.2.** Let $M \in M_n(K)$. The following are equivalent:

(i) $M$ is integral over $D$

(ii) $\mu_M \in D[X]$

(iii) each $\alpha \in \Omega_M$ is integral over $D$

**Proof.** (i) $\Rightarrow$ (iii) Suppose $M$ solves a monic polynomial $f(X)$ with coefficients in $D$. As $\mu_M(X)$ divides $f(X)$, its roots are then also roots of $f(X)$. Hence, the elements of $\Omega_M$ are integral over $D$.

(iii) $\Rightarrow$ (ii) The coefficients of $\mu_M \in K[X]$ are the elementary symmetric functions of its roots. Assuming (iii) holds, these roots are integral over $D$, hence the coefficients of $\mu_M$ are integral over $D$. Since $D$ is integrally closed, we must have $\mu_M \in D[X]$.  

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(ii) ⇒ (i) Obvious.

For the rest of this section, we will study polynomials in \( K[X] \) that take values on sets of integral matrices. These are the integral-valued polynomials mentioned in the introduction.

**Definition 2.3.** Let \( S \subseteq M_n(K) \). Let \( K[S] \) denote the \( K \)-subalgebra of \( M_n(K) \) generated by \( K \) and the elements of \( S \). Define \( S' := \{ M \in K[S] \mid M \text{ is integral} \} \) and \( \text{Int}_K(S, S') := \{ f \in K[X] \mid f(S) \subseteq S' \} \). We call \( \text{Int}_K(S, S') \) a set of integral-valued polynomials.

**Remark 2.4.** In the next lemma, we will prove that forming the set \( S' \) is a closure operation in the sense that \((S')' = S'\). We point out that this construction differs from the usual notion of integral closure in several ways. First, if \( S \) itself is not integral, then \( S \not\subseteq S' \). Second, \( S' \) need not have a ring structure. Indeed, if \( D = \mathbb{Z} \) and \( S = M_2(\mathbb{Z}) \), then both \((1/2, 1/2)\) and \((1/2, 1/2)\) are in \( S' \), but neither their sum nor their product is integral. Lastly, even if \( S \) is a commutative ring then \( S' \) need not be the same as the integral closure of \( S \) in \( K[S] \), because we insist that the elements of \( S \) satisfy a monic polynomial in \( D[X] \) rather than \( S[X] \).

However, if \( S \) is a commutative \( D \)-algebra and it is an integral subset of \( M_n(K) \) then \( S' \) is equal to the integral closure of \( S \) in \( K[S] \) (see Corollary 1 to Proposition 2 and Proposition 6 of [1] Chapt. V).

**Lemma 2.5.** Let \( S \subseteq M_n(K) \). Then, \((S')' = S'\).

**Proof.** We just need to show that \( K[S'] = K[S] \). By definition, \( S' \subseteq K[S] \), so \( K[S'] \subseteq K[S] \). For the other containment, let \( M \in K[S] \). Let \( d \in D \) be a common multiple for all the denominators of the entries in \( M \). Then, \( dM \in S' \subseteq K[S'] \). Since \( 1/d \in K \), we get \( M \in K[S'] \).

An integral subset of \( M_n(K) \) need not be closed under addition or multiplication, so at first glance it may not be clear that \( \text{Int}_K(S, S') \) is closed under these operations. As we now show, \( \text{Int}_K(S, S') \) is in fact a ring.

**Proposition 2.6.** Let \( S \subseteq M_n(K) \). Then, \( \text{Int}_K(S, S') \) is a ring, and if \( D \subseteq S \), then \( \text{Int}_K(S, S') \subseteq \text{Int}(D) \).

**Proof.** Let \( M \in S \) and \( f, g \in \text{Int}_K(S, S') \). Then, \( f(M), g(M) \) are integral over \( D \). By Corollary 2 after Proposition 4 of [1] Chapt. V], the \( D \)-algebra generated by \( f(M) \) and \( g(M) \) is integral over \( D \). So, \( f(M) + g(M) \) and \( f(M)g(M) \) are both integral over \( D \) and are both in \( K[S] \). Thus, \( f(M) + g(M), f(M)g(M) \in S' \) and \( f + g, fg \in \text{Int}_K(S, S') \). Assuming \( D \subseteq S \), let \( f \in \text{Int}_K(S, S') \) and \( d \in D \). Then, \( f(d) \) is an integral element of \( K \). Since \( D \) is integrally closed, \( f(d) \in D \). Thus, \( \text{Int}_K(S, S') \subseteq \text{Int}(D) \).

We now begin to connect our rings of integral-valued polynomials to rings of polynomials that act on elements of \( K \) that are integral over \( D \). For each \( n > 0 \), let

\[ \Lambda_n := \{ \alpha \in K \mid \alpha \text{ solves a monic polynomial in } D[X] \text{ of degree } n \} \]

In the special case \( D = \mathbb{Z} \), we let \( \Lambda_n := \Lambda_n = \{ \text{algebraic integers of degree at most } n \} \subset \mathbb{Q} \).

For any subset \( \mathcal{E} \) of \( \Lambda_n \), define

\[ \text{Int}_K(\mathcal{E}, \Lambda_n) := \{ f \in K[X] \mid f(\mathcal{E}) \subseteq \Lambda_n \} \]

to be the set of polynomials in \( K[X] \) mapping elements of \( \mathcal{E} \) into \( \Lambda_n \). If \( \mathcal{E} = \Lambda_n \), then we write simply \( \text{Int}_K(\Lambda_n) \). As with \( \text{Int}_K(S, S') \), \( \text{Int}_K(\mathcal{E}, \Lambda_n) \) is a ring despite the fact that \( \Lambda_n \) is not.
Proposition 2.7. For any \( E \subseteq \Lambda_n \), \( \text{Int}_K(E, \Lambda_n) \) is a ring, and is integrally closed.

Proof. Let \( \Lambda_{\infty} \) be the integral closure of \( D \) in \( K \). We set \( \text{Int}_K(E, \Lambda_{\infty}) = \{ f \in K[X] \mid f(E) \subseteq \Lambda_{\infty} \} \). Then, \( \text{Int}_K(E, \Lambda_{\infty}) \) is a ring, and by [2] Prop. IV.4.1 it is integrally closed.

Let \( \text{Int}_K(E, \Lambda_{\infty}) = \{ f \in K[X] \mid f(E) \subseteq \Lambda_{\infty} \} \). Clearly, \( \text{Int}_K(E, \Lambda_n) \subseteq \text{Int}_K(E, \Lambda_{\infty}) \). However, if \( \alpha \in E \) and \( f \in K[X] \), then \( [K(f(\alpha)) : K] \leq [K(\alpha) : K] \leq n \), so in fact \( \text{Int}_K(E, \Lambda_n) = \text{Int}_K(E, \Lambda_{\infty}) \).

Finally, since \( \text{Int}_K(E, \Lambda_{\infty}) = \text{Int}_K(E, \Lambda_{\infty}) \cap K[X] \) is the contraction of \( \text{Int}_K(E, \Lambda_{\infty}) \) to \( K[X] \), it is an integrally closed ring, proving the proposition.

Theorem 4.6 in [9] shows that the integral closure of \( \text{Int}_Q(M_n(\mathbb{Z})) \) equals the ring \( \text{Int}_Q(A_n) \). As we shall see (Theorem 2.9), this is evidence of a broader connection between the rings of integral-valued polynomials \( \text{Int}_K(S, S') \) and rings of polynomials that act on elements of \( \Lambda_n \). The key to this connection is the observation contained in Lemma 2.2 that the eigenvalues of an integral matrix in \( M_n(K) \) lie in \( \Lambda_n \) and also the well known fact that if \( M \in M_n(K) \) and \( f \in K[X] \), then the eigenvalues of \( f(M) \) are exactly \( f(\alpha) \), where \( \alpha \) is an eigenvalue of \( M \). More precisely, if \( \chi_M(X) = \prod_{i=1}^n(X - \alpha_i) \) is the characteristic polynomial of \( M \) (the roots \( \alpha_i \) are in \( K \) and there may be repetitions), then the characteristic polynomial of \( f(M) \) is \( \chi_{f(M)}(X) = \prod_{i=1}^n(X - f(\alpha_i)) \). Phrased in terms of our \( \Omega \)-notation, we have:

\[
\text{if } M \in M_n(K) \text{ and } f \in K[X], \text{ then } \Omega_f(M) = f(\Omega_M) = \{ f(\alpha) \mid \alpha \in \Omega_M \}. \tag{2.8}
\]

Using this fact and our previous work, we can equate \( \text{Int}_K(S, S') \) with a ring of the form \( \text{Int}_K(E, \Lambda_n) \).

Theorem 2.9. Let \( S \subseteq M_n(K) \). Then, \( \text{Int}_K(S, S') = \text{Int}_K(\Omega_S, \Lambda_n) \), and in particular \( \text{Int}_K(S, S') \) is integrally closed.

Proof. We first prove this for \( S = \{ M \} \). Using Lemma 2.2 and (2.8), for each \( f \in K[X] \) we have:

\[
f(M) \in S' \iff f(M) \text{ is integral } \iff \Omega_f(M) \subseteq \Lambda_n \iff f(\Omega_M) \subseteq \Lambda_n.
\]

This proves that \( \text{Int}_K(\{ M \}, \{ M \}') = \text{Int}_K(\Omega_M, \Lambda_n) \). For a general subset \( S \) of \( M_n(K) \), we have

\[
\text{Int}_K(S, S') = \bigcap_{M \in S} \text{Int}_K(\{ M \}, S') = \bigcap_{M \in S} \text{Int}_K(\Omega_M, \Lambda_n) = \text{Int}_K(\Omega_S, \Lambda_n).
\]

The above proof shows that if a polynomial is integral-valued on a matrix, then it is also integral-valued on any other matrix with the same set of eigenvalues. Note that for a single integral matrix \( M \) we have these inclusions:

\[
D \subseteq D[M] \subseteq \{ M \}' \subseteq K[M].
\]

Moreover, \( \{ M \}' \) is equal to the integral closure of \( D[M] \) in \( K[M] \) (because \( D[M] \) is a commutative algebra).
3 The case of a $D$-algebra

We now use the results from Section 2 to gain information about $\text{Int}_K(A)$, where $A$ is a $D$-algebra. In Theorem 3.1 below, we shall obtain a description of the integral closure of $\text{Int}_K(A)$.

As mentioned in the introduction, we assume that $A$ is a torsion-free $D$-algebra that is finitely generated as a $D$-module. Let $B = A \otimes_D K$ be the extension of $A$ to a $K$-algebra. Since $A$ is a faithful $D$-module, $B$ contains copies of $D$, $A$, and $K$. Furthermore, $K$ is contained in the center of $B$, so we can evaluate polynomials in $K[X]$ at elements of $B$ and define

$$\text{Int}_K(A) := \{ f \in K[X] \mid f(A) \subseteq A \}.$$ 

Letting $n$ be the vector space dimension of $B$ over $K$, we also have an embedding $B \hookrightarrow M_n(K)$, $b \mapsto M_b$. More precisely, we may embed $B$ into the ring of $K$-linear endomorphisms of $B$ (which is isomorphic to $M_n(K)$) via the map $B \hookrightarrow \text{End}_K(B)$ sending $b \in B$ to the endomorphism $x \mapsto b \cdot x$. Consequently, starting with just $D$ and $A$, we obtain a representation of $A$ as a $D$-subalgebra of $M_n(K)$. Note that $n$ may be less than the minimum number of generators of $A$ as a $D$-module.

In light of the aforementioned matrix representation of $B$, several of the definitions and notations we defined in Section 2 will carry over to $B$. Since the concepts of minimal polynomial and eigenvalue are independent of the representation $B \hookrightarrow M_n(K)$, the following are well-defined:

- for all $b \in B$, $\mu_b(X) \in K[X]$ is the minimal polynomial of $b$. So, $\mu_b(X)$ is the monic polynomial of minimal degree in $K[X]$ that kills $b$. Equivalently, $\mu_b$ is the monic generator of the null ideal $N_{K[X]}(b)$ of $b$. This is the same as the minimal polynomial of $M_b \in M_n(K)$, since for all $f \in K[X]$ we have $f(M_b) = M_{f(b)}$. To ease the notation, from now on we will identify $b$ with $M_b$.

- by the Cayley-Hamilton Theorem, $\deg(\mu_b) \leq n$, for all $b \in B$.

- for all $b \in B$, $\Omega_b = \{\text{roots of } \mu_b \text{ in } \overline{K}\}$. The elements of $\Omega_b$ are nothing else than the eigenvalues of $b$ under any matrix representation $B \hookrightarrow M_n(K)$. If $S \subseteq B$, then $\Omega_S = \bigcup_{b \in S} \Omega_b$.

- $b \in B$ is integral over $D$ (or just integral) if $b$ solves a monic polynomial in $D[X]$.

- $B = K[A]$, since $B$ is formed by extension of scalars from $D$ to $K$.

- $A' = \{ b \in B \mid b \text{ is integral} \}$. By [1, Theorem 1, Chapt. V] $A \subseteq A'$. In particular, this implies $A \cap K = D$ (because $D$ is integrally closed), so that $\text{Int}_K(A) \subseteq \text{Int}(D)$ (see the remarks in the introduction).

- $\text{Int}_K(A, A') = \{ f \in K[X] \mid f(A) \subseteq A' \}$.

Working exactly as in Proposition 2.3, we find that $\text{Int}_K(A, A')$ is another ring of integral-valued polynomials. Additionally, Lemma 2.2 and (2.8) hold for elements of $B$. Consequently, we have

**Theorem 3.1.** $\text{Int}_K(A, A')$ is an integrally closed ring and is equal to $\text{Int}_K(\Omega_A, \Lambda_n)$.

By generalizing results from [3], we will show that if $D$ has finite residue rings, then $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$. This establishes the analogue of [2, Thm. IV.4.7] (originally proved in [2] Prop. 2.2) mentioned in the introduction.
We will actually prove a slightly stronger statement and give a description of \( \text{Int}_K(A, A') \) as the integral closure of an intersection of pullbacks. Notice that
\[
\bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \subseteq \text{Int}_K(A)
\]
because if \( f \in D[X] + \mu_a(X) \cdot K[X] \), then \( f(a) \in D[a] \subseteq A \). We thus have a chain of inclusions
\[
\bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \subseteq \text{Int}_K(A) \subseteq \text{Int}_K(A, A') \tag{3.2}
\]
and our work below will show that this is actually a chain of integral ring extensions.

**Lemma 3.3.** Let \( f \in \text{Int}_K(A, A') \), and write \( f(X) = g(X)/d \) for some \( g \in D[X] \) and some nonzero \( d \in D \). Then, for each \( h \in D[X] \), \( d^{n-1}h(f(X)) \in \bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \).

**Proof.** Let \( a \in A \). Since \( f \in \text{Int}_K(A, A') \), \( m := \mu_f(a) \in D[X] \), and \( \deg(m) \leq n \).

Now, \( m \) is monic, so we can divide \( h \) by \( m \) to get \( h(X) = q(X)m(X) + r(X) \), where \( q, r \in D[X] \), and either \( r = 0 \) or \( \deg(r) < n \). Then,
\[
d^{n-1}h(f(X)) = d^{n-1}q(f(X))m(f(X)) + d^{n-1}r(f(X))
\]
The polynomial \( d^{n-1}q(f(X))m(f(X)) \in K[X] \) is divisible by \( \mu_a(X) \) because \( m(f(a)) = 0 \). Since \( \deg(r) < n \), \( d^{n-1}r(f(X)) \in D[X] \). Thus, \( d^{n-1}h(f(X)) \in D[X] + \mu_a(X) \cdot K[X] \), and since \( a \) was arbitrary the lemma is true.

For the next result, we need an additional assumption. Recall that a ring \( D \) has finite residue rings if for all proper nonzero ideal \( I \subset D \), the residue ring \( D/I \) is finite. Clearly, this condition is equivalent to asking that for all nonzero \( d \in D \), the residue ring \( D/dD \) is finite.

**Theorem 3.4.** Assume that \( D \) has finite residue rings. Then, \( \text{Int}_K(A, A') = \text{Int}_K(\Omega_A, \Lambda_n) \) is the integral closure of both \( \bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \) and \( \text{Int}_K(A) \).

**Proof.** Let \( R = \bigcap_{a \in A} (D[X] + \mu_a(X) \cdot K[X]) \). By Lemma 3.2, it suffices to prove that \( \text{Int}_K(A, A') \) is the integral closure of \( R \). Let \( f(X) = g(X)/d \in \text{Int}_K(A, A') \). By Theorem 3.1, \( \text{Int}_K(A, A') \) is integrally closed, so it is enough to find a monic polynomial \( \phi \in D[X] \) such that \( \phi(f(X)) \in R \).

Let \( \mathcal{P} \subseteq D[X] \) be a set of monic residue representatives for \( \{\mu_f(a)\} \) \( a \in A \) modulo \( (d^{n-1})^2 \).

Since \( D \) has finite residue rings, \( \mathcal{P} \) is finite. Let \( \phi(X) \) be the product of all the polynomials in \( \mathcal{P} \). Then, \( \phi \) is monic and is in \( D[X] \).

Fix \( a \in A \) and let \( m = \mu_f(a) \). There exists \( p(X) \in \mathcal{P} \) such that \( p(X) \) is equivalent to \( m \) mod \( (d^{n-1})^2 \), so \( p(X) = m(X) + (d^{n-1})^2r(X) \) for some \( r \in D[X] \). Furthermore, \( p(X) \) divides \( \phi(X) \), so there exists \( q(X) \in D[X] \) such that \( \phi(X) = p(X)q(X) \). Thus,
\[
\phi(f(X)) = p(f(X))q(f(X)) = m(f(X))q(f(X)) + (d^{n-1})^2r(f(X))q(f(X)) = m(f(X))q(f(X)) + d^{n-1}r(f(X)) \cdot d^{n-1}q(f(X))
\]

As in Lemma 3.3, \( m(f(X))q(f(X)) \in \mu_a(X) \cdot K[X] \) because \( m(f(a)) = 0 \). By Lemma 3.3, \( d^{n-1}r(f(X)) \) and \( d^{n-1}q(f(X)) \) are in \( D[X] + \mu_a(X) \cdot K[X] \). Hence, \( \phi(f(X)) \in D[X] + \mu_a(X) \cdot K[X] \), and since \( a \) was arbitrary, \( \phi(f(X)) \in R \).
Theorem 3.4 says that the integral closure of $\text{Int}_K(A)$ is equal to the ring of polynomials in $K[X]$ which map the eigenvalues of all the elements $a \in A$ to integral elements over $D$.

**Remark 3.5.** By following essentially the same steps as in Lemma 3.3 and Theorem 3.4, one may prove that $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$ without the use of the pullbacks $D[X] + \mu_a(X) \cdot K[X]$. However, employing the pullbacks gives a slightly stronger theorem without any additional difficulty.

In the case $A = M_n(D)$, $\text{Int}_K(M_n(D))$ is equal to the intersection of the pullbacks $D[X] + \mu_M(X) \cdot K[X]$, for $M \in M_n(D)$. Indeed, let $f \in \text{Int}_K(M_n(D))$ and $M \in M_n(D)$. By Remark 2.1 & (3), $\text{Int}_K(M_n(D))$ is equal to the intersection of the pullbacks $D[X] + \chi_M(X)K[X]$, for $M \in M_n(D)$, where $\chi_M(X)$ is the characteristic polynomial of $M$. By the Cayley-Hamilton Theorem, $\mu_M(X)$ divides $\chi_M(X)$ so that $f \in D[X] + \chi_M(X)K[X] \subseteq D[X] + \mu_M(X)K[X]$ and we are done.

**Remark 3.6.** The assumption that $D$ has finite residue rings implies that $D$ is Noetherian. Given that [2 Thm. IV.4.7] (or [7 Prop. 2.2]) requires only the assumption that $D$ is Noetherian, it is fair to ask if Theorem 3.4 holds under the weaker condition that $D$ is Noetherian.

Note that $\Omega_{M_n(D)} = \Lambda_n$ (and in particular, $\Omega_{M_n(2)} = \mathcal{A}_n$). Hence, we obtain the following (which generalizes [9 Thm. 4.6]):

**Corollary 3.7.** If $D$ has finite residue rings, then the integral closure of $\text{Int}_K(M_n(D))$ is $\text{Int}_K(\Lambda_n)$.

The algebra of upper triangular matrices yields another interesting example.

**Corollary 3.8.** Assume that $D$ has finite residue rings. For each $n > 0$, let $T_n(D)$ be the ring of $n \times n$ upper triangular matrices with entries in $D$. Then, the integral closure of $\text{Int}_K(T_n(D))$ equals $\text{Int}(D)$.

**Proof.** For each $a \in T_n(D)$, $\mu_a$ splits completely and has roots in $D$, so $\Omega_{T_n(D)} = D$. Hence, the integral closure of $\text{Int}_K(T_n(D))$ is $\text{Int}_K(\Omega_{T_n(D)}, \Lambda_n) = \text{Int}_K(D, \Lambda_n)$. But, polynomials in $K[X]$ that move $D$ into $\Lambda_n$ actually move $D$ into $\Lambda_n \cap K = D$. Thus, $\text{Int}_K(D, \Lambda_n) = \text{Int}(D)$.

Since $\text{Int}_K(T_n(D)) \subseteq \text{Int}_K(T_{n-1}(D))$ for all $n > 0$, the previous proposition proves that

$$
\cdots \subseteq \text{Int}_K(T_n(D)) \subseteq \text{Int}_K(T_{n-1}(D)) \subseteq \cdots \subseteq \text{Int}(D)
$$

is a chain of integral ring extensions.

## 4 Matrix rings and polynomially dense subsets

For any $D$-algebra $A$, we have $(A')' = A'$, so $\text{Int}_K(A')$ is integrally closed by Theorem 3.4. Furthermore, $\text{Int}_K(A')$ is always contained in $\text{Int}_K(A, A')$. One may then ask: when does $\text{Int}_K(A')$ equal $\text{Int}_K(A, A')$? In this section, we investigate this question and attempt to identify polynomially dense subsets of rings of integral-valued polynomials. The theory presented here is far from complete, so we raise several related questions worthy of future research.

**Definition 4.1.** Let $S \subseteq T \subseteq M_n(K)$. Define $\text{Int}_K(S, T) := \{f \in K[X] \mid f(S) \subseteq T\}$ and $\text{Int}_K(T) := \text{Int}_K(T, T)$. To say that $S$ is polynomially dense in $T$ means that $\text{Int}_K(S, T) = \text{Int}_K(T)$. 

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Thus, the question posed at the start of this section can be phrased as: is \( A \) polynomially dense in \( A' \)?

In general, it is not clear how to produce polynomially dense subsets of \( A' \), but we can describe some polynomially dense subsets of \( M_n(D)' \).

**Proposition 4.2.** For each \( \Omega \subset \Lambda_n \) of cardinality at most \( n \), choose \( M \in M_n(D)' \) such that \( \Omega_M = \Omega \). Let \( S \) be the set formed by such matrices. Then, \( S \) is polynomially dense in \( M_n(D)' \). In particular, the set of companion matrices in \( M_n(D) \) is polynomially dense in \( M_n(D)' \).

**Proof.** We know that \( \text{Int}_K(M_n(D)') \subseteq \text{Int}_K(S, M_n(D)'), \) so we must show that the other containment holds. Let \( f \in \text{Int}_K(S, M_n(D)') \) and \( N \in M_n(D)' \). Let \( M \in S \) such that \( \Omega_M = \Omega_N \). Then, \( f(M) \) is integral, so by Lemma 2.2 and 2.3, \( f(N) \) is also integral.

The proposition holds for the set of companion matrices because for any \( \Omega \subset \Lambda_n \), we can find a companion matrix in \( M_n(D) \) whose elements are the elements of \( \Omega \).

By the proposition, any subset of \( M_n(D) \) containing the set of companion matrices is polynomially dense in \( M_n(D)' \). In particular, \( M_n(D) \) is polynomially dense in \( M_n(D)' \).

When \( D = \mathbb{Z} \), we can say more. In [10] it is shown that \( \text{Int}_Q(A_n) = \text{Int}_Q(A_n, A_n) \), where \( A_n \) is the set of algebraic integers of degree equal to \( n \). Letting \( \mathcal{I} \) be the set of companion matrices in \( M_n(\mathbb{Z}) \) of irreducible polynomials, we have \( \Omega_M = A_n \). Hence, by Corollary 3.7 and Theorem 2.9 \( \mathcal{I} \) is polynomially dense in \( M_n(\mathbb{Z})' \).

 Returning to the case of a general \( D \)-algebra \( A \), the following diagram summarizes the relationships among the various polynomial rings we have considered:

\[
\begin{align*}
\text{Int}_K(A) & \subseteq \text{Int}_K(A, A') = \text{Int}_K(\Omega_A, \Lambda_n) \\
\text{Int}_K(A') & \subseteq \text{Int}_K(\Omega_{A'}, \Lambda_n) \\
\text{Int}_K(M_n(D)') & = \text{Int}_K(\Lambda_n)
\end{align*}
\]

From this diagram, we deduce that \( A \) is polynomially dense in \( A' \) if and only if \( \Omega_A \) is polynomially dense in \( \Omega_{A'} \).

It is fair to ask what other relationships hold among these rings. We present several examples and a proposition concerning possible equalities in the diagram. Again, we point out that such equalities can be phrased in terms of polynomially dense subsets. First, we show that \( \text{Int}_K(A) \) need not equal \( \text{Int}_K(A, A') \) (that is, \( A \) need not be polynomially dense in \( A' \)).

**Example 4.4.** Take \( D = \mathbb{Z} \) and \( A = \mathbb{Z}[\sqrt{-3}] \). Then, \( A' = \mathbb{Z}[\theta] \), where \( \theta = \frac{1 + \sqrt{-3}}{2} \). The ring \( \text{Int}_K(A, A') \) contains both \( \text{Int}_K(A) \) and \( \text{Int}_K(A') \).

If \( \text{Int}_K(A, A') \) equaled \( \text{Int}_K(A') \), then we would have \( \text{Int}_K(A) \subseteq \text{Int}_K(A') \). However, this is not the case. Indeed, working mod 2, we see that for all \( \alpha = a + b\sqrt{-3} \in A \), \( \alpha^2 \equiv a^2 - 3b^2 \equiv a^2 + b^2 \). So, \( \alpha^2(a^2 + 1) \) is always divisible by 2, and hence \( \frac{x^2(x^2+1)}{2} \in \text{Int}_K(A) \). On the other hand, \( \frac{a^2(a^2+1)}{2} = -\frac{1}{2} \), so \( \frac{x^2(x^2+1)}{2} \not\in \text{Int}_K(A') \). Thus, we conclude that \( \text{Int}_K(A') \not\subseteq \text{Int}_K(A, A') \).

The work in the previous example suggests the following proposition.

**Proposition 4.5.** Assume that \( D \) has finite residue rings. Then, \( A \) is polynomially dense in \( A' \) if and only if \( \text{Int}_K(A) \subseteq \text{Int}_K(A') \).
Proof. This is similar to \cite[Thm. IV.4.9]{2}. If $A$ is polynomially dense in $A'$, then $\text{Int}_K(A') = \text{Int}_K(A, A')$, and we are done because $\text{Int}_K(A, A')$ always contains $\text{Int}_K(A)$. Conversely, assume that $\text{Int}_K(A) \subseteq \text{Int}_K(A')$. Then, $\text{Int}_K(A) \subseteq \text{Int}_K(A') \subseteq \text{Int}_K(A, A')$. Since $\text{Int}_K(A')$ is integrally closed by Theorem 3.1 and $\text{Int}_K(A, A')$ is integral over $\text{Int}_K(A)$ by Theorem 3.3 we must have $\text{Int}_K(A') = \text{Int}_K(A, A')$. 

By Proposition 4.2, $\text{Int}_K(M_n(D)) = \text{Int}_K(M_n(D), M_n(D') \cup \{0\}$). There exist algebras other than matrix rings for which $\text{Int}_K(A') = \text{Int}_K(A, A')$. We now present two such examples.

Example 4.6. Let $A = T_n(D)$, the ring of $n \times n$ upper triangular matrices with entries in $D$. Define $T_n(K)$ similarly. Then, $A'$ consists of the integral matrices in $T_n(K)$, and since $D$ is integrally closed, such matrices must have diagonal entries in $D$. Thus, $\Omega_{A'} = D = \Omega_A$. It follows that $\text{Int}_K(T_n(D), T_n(D')) = \text{Int}_K(T_n(D'))$.

Example 4.7. Let $i, j,$ and $k$ be the standard quaternion units satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$ (see e.g. \cite[Ex. 1.1, 1.13]{2} or \cite{3} for basic material on quaternions).

Let $A$ be the $\mathbb{Z}$-algebra consisting of Hurwitz quaternions:

$$A = \{a_0 + a_1i + a_2j + a_3k \mid a_{\ell} \in \mathbb{Z} \text{ for all } \ell \text{ or } a_{\ell} \in \mathbb{Z} + \frac{1}{2} \text{ for all } \ell\}$$

Then, for $B$ we have

$$B = \{q_0 + q_1i + q_2j + q_3k \mid q_{\ell} \in \mathbb{Q}\}$$

It is well known that the minimal polynomial of the element $q = q_0 + q_1i + q_2j + q_3k \in B \setminus \mathbb{Q}$ is $\mu_q(x) = x^2 - 2q_0x + (q_0^2 + q_1^2 + q_2^2 + q_3^2)$, so $A'$ is the set

$$A' = \{q_0 + q_1i + q_2j + q_3k \in B \mid 2q_0, q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{Z}\}$$

As with the previous example, by \cite[4.3]{1}, it is enough to prove that $\Omega_{A'} = \Omega_A$.

Let $q = q_0 + q_1i + q_2j + q_3k \in A'$ and $N = q_0^2 + q_1^2 + q_2^2 + q_3^2 \in \mathbb{Z}$. Then, $2q_0 \in \mathbb{Z}$, so $q_0$ is either an integer or a half-integer. If $q_0 \in \mathbb{Z}$, then $q_0^2 + q_2^2 + q_3^2 = N - q_0^2 \in \mathbb{Z}$. It is known (see for instance \cite[Lem. B p. 46]{12}) that an integer which is the sum of three rational squares is a sum of three integer squares. Thus, there exist $a_1, a_2, a_3 \in \mathbb{Z}$ such that $a_1^2 + a_2^2 + a_3^2 = N - q_0^2$. Then, $q' = q_0 + a_1i + a_2j + a_3k$ is an element of $A$ such that $\Omega_{A'} = \Omega_q$.

If $q_0$ is a half-integer, then $q_0 = \frac{t}{2}$ for some odd $t \in \mathbb{Z}$. In this case, $q_1^2 + q_2^2 + q_3^2 = \frac{4N - t^2}{4} = \frac{u}{4}$, where $u \equiv 3 \mod 4$. Clearing denominators, we get $(2q_1)^2 + (2q_2)^2 + (2q_3)^2 = u$. As before, there exist integers $a_1, a_2$, and $a_3$ such that $a_1^2 + a_2^2 + a_3^2 = u$. But since $u \equiv 3 \mod 4$, each of the $a_i$ must be odd. So, $q' = (t + a_1i + a_2j + a_3k)/2 \in A$ is such that $\Omega_{A'} = \Omega_q$.

It follows that $\Omega_{A'} = \Omega_A$ and thus that $\text{Int}_K(A, A') = \text{Int}_K(A')$.

Example 4.8. In contrast to the last example, the Lipschitz quaternions $A_1 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ (where we only allow $a_{\ell} \in \mathbb{Z}$) are not polynomially dense in $A_1'$. With $A_1$ as in Example 4.4, we have $A_1 \subseteq A$, and both rings have the same $B$, so $A_1' = A'$. Our proof is identical to Example 4.4.

Working mod 2, the only possible minimal polynomials for elements of $A_1 \setminus \mathbb{Z}$ are $X^2$ and $X^2 + 1$. It follows that $f(X) = \frac{x^2(x^2 + 1)}{2} \in \text{Int}_K(A_1)$. Let $\alpha = \frac{1 + i + j + k}{2} \in A'$. Then, the minimal polynomial of $\alpha$ is $X^2 - X + 1$ (note that this minimal polynomial is shared by $\theta = \frac{1 + i + jk}{2}$ in Example 4.4).

Just as in Example 4.4, $f(\alpha) = -\frac{1}{2}$, which is not in $A'$. Thus, $\text{Int}_K(A_1) \not\subseteq \text{Int}(A')$, so $A_1$ is not polynomially dense in $A_1' = A'$ by Proposition 4.6.
5 Further questions

Here, we list more questions for further investigation.

**Question 5.1.** Under what conditions do we have equalities in (4.3)? In particular, what are necessary and sufficient conditions on $A$ for $A$ to be polynomially dense in $A'$? In Examples 4.6 and 4.7, we exploited the fact that if $\Omega_A = \Omega_A'$, then $\text{Int}_K(A, A') = \text{Int}_K(A')$. It is natural to ask whether the converse holds. If $\text{Int}_K(A, A') = \text{Int}_K(A')$, does it follow that $\Omega_A = \Omega_A'$? In other words, if $A$ is polynomially dense in $A'$, then is $\Omega_A$ equal to $\Omega_A'$?

**Question 5.2.** By [2, Proposition IV.4.1] it follows that $\text{Int}(D)$ is integrally closed if and only if $D$ is integrally closed. By Theorem 5.4, we know that if $A = A'$, then $\text{Int}_K(A)$ is integrally closed. Do we have a converse? Namely, if $\text{Int}_K(A)$ is integrally closed, can we deduce that $A = A'$?

**Question 5.3.** In our proof (Theorem 3.4) that $\text{Int}_K(A, A')$ is the integral closure of $\text{Int}_K(A)$, we needed the assumption that $D$ has finite residue rings. Is the theorem true without this assumption? In particular, is it true whenever $D$ is Noetherian?

**Question 5.4.** When is $\text{Int}_K(A, A') = \text{Int}_K(\Omega_A, \Lambda_A)$ a Prüfer domain? When $D = \mathbb{Z}$, $\text{Int}_Q(A, A')$ is always Prüfer by [9, Cor. 4.7]. On the other hand, even when $A = D$ is a Prüfer domain, $\text{Int}(D)$ need not be Prüfer (see [2, Sec. IV.4]).

**Question 5.5.** In Remark 3.5, we proved that $\text{Int}_K(M_n(D))$ equals an intersection of pullbacks: $$\bigcap_{M \in M_n(D)} (D[X] + \mu_M(X) \cdot K[X]) = \text{Int}_K(M_n(D))$$

Does such an equality hold for other algebras?

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**References**

[1] Bourbaki, N.: Commutative Algebra. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass. (1972).

[2] Cahen, J.-P., Chabert, J.-L.: Integer-valued Polynomials. Amer. Math. Soc. Surveys and Monographs, 48, Providence (1997).

[3] Dickson, L. E.: Algebras and Their Arithmetics. Dover Publications, New York (1960).

[4] Evrard, S, Fares, Y., Johnson, K.: Integer valued polynomials on lower triangular integer matrices. Monats. für Math. 170, no. 2, 147-160 (2013).

[5] Frisch, S.: Integer-valued polynomials on algebras. J. Algebra 373, 414-425 (2013).
[6] Frisch, S.: Corrigendum to “Integer-valued polynomials on algebras”. J. Algebra, http://dx.doi.org/10.1016/j.jalgebra.2013.06.003

[7] Gilmer, R., Heinzer W., Lantz, D.: The Noetherian property in rings of integer-valued polynomials. Trans. Amer. Math. Soc. 338, no. 1, 187-199 (1993).

[8] Lam, T. Y.: A First Course in Noncommutative Rings. Springer, New York (1991).

[9] Loper, K. A., Werner, N. J.: Generalized rings of integer-valued polynomials. J. Number Theory 132, no. 11, 2481-2490 (2012).

[10] Peruginelli, G.: Integral-valued polynomials over sets of algebraic integers of bounded degree. accepted for publication by Journal of Number Theory, http://dx.doi.org/10.1016/j.jnt.2013.11.007

[11] Peruginelli, G.: Integer-valued polynomials over matrices and divided differences. published in Monatshefte für Mathematik, http://dx.doi.org/10.1007/s00605-013-0519-9

[12] Serre, J. P.: A Course in Arithmetic. Springer, New York (1996).