Extremely Non-symmetric, Non-multiplicative, Non-commutative Operator Spaces

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Abstract. Motivated by importance of operator spaces contained in the set of all scalar multiples of isometries ($MI$-spaces) in a separable Hilbert space for $C^*$-algebras and E-semigroups we exhibit more properties of such spaces. For example, if an $MI$-space contains an isometry with shift part of finite multiplicity, then it is one-dimensional. We propose a simple model of a unilateral shift of arbitrary multiplicity and show that each separable subspace of a Hilbert space is the range of a shift. Also, we show that $MI$-spaces are non-symmetric, very unfriendly to multiplication, and prove a Commutator Identity which elucidates the extreme non-commutativity of these spaces.

1. Genesis and Justification

$B(H)$ is the algebra of all linear, bounded operators in a separable Hilbert space $H$ over the complex numbers $\mathbb{C}$, with the identity $I$. A subspace of $H$ is always closed. A shift is always unilateral. An operator space is a linear subspace of $B(H)$. For $A_1, ..., A_n \in B(H)$ $\text{span}(A_1, ..., A_n)$ is the set of all linear combinations of $A_1, ..., A_n$.

Operator spaces contained in the set $MI$ of all scalar multiples of all isometries in a Hilbert space are the subject of investigation in this paper. For brevity, an operator space contained in $MI$ will be called an $MI$-space. Another possible name: ”a subspace of $MI$” is misleading because $MI$ is not a linear space. As will be shown, these are strange spaces, indeed. Just for a start, the shift of multiplicity one and its square cannot belong to one $MI$-space, because their sum does not belong to $MI$. This example will be further explained after Proposition 3.2. Even though the set $MI$ is a semigroup with operator multiplication, it turns out - cf. Proposition 3.2.- that on $MI$-spaces this multiplicative structure trivializes.

My interest in $MI$-spaces came from an attempt to extend the following result of H. Radjavi and P. Rosenthal [R-R]: Each linear space of operators contained in the set of all normal operators is commutative. With John B. Conway [Con-Sz] we replaced ”normal” by ”hyponormal” in that theorem and showed that such result is false. Trying to understand
what really went "wrong" for hyponormals, in [Cat-Sz] MI-spaces were introduced (without that name) and proved to be the culprit. Corollary 3.3. of [Cat-Sz] reads: *If C is a class of operators that contains MI then there are A, B ∈ C such that span(A, B) ⊂ MI ⊂ C and A, B do not commute.* Since the class of hyponormal operators contains MI, the hyponormal case followed. Even though the attempt to extend the above [R-R] result failed (so far), MI-spaces appeared. Concerning their commutative properties - well - they are really bad - worse than found in [Cat-Sz] - cf Corollary 3.5. This is the first justification why MI-spaces are worth attention. I call it the *operator theory justification.*

Another justification comes from C*-algebras. This connection was already made in [Cat-Sz]. A *Cuntz algebra* O_n is a universal C*-algebra generated by isometries S_1,...,S_n ∈ B(H) such that S_1S_1* + ... + S_nS_n* = I - cf. [D]. In [Cat-Sz] Corollary 2.6. states that if S_1,...,S_n ∈ B(H) are generators of a Cuntz algebra then span(S_1,...,S_n) is an MI-space. Also a slight generalization of the converse of this result is proved there. I call this the *C*-algebra justification.

One more justification comes from continuous tensor product systems of Hilbert spaces introduced by William Arveson in [A1] as a continuous analogue of Fock spaces. It turns out that such product systems are a basic structure in studying semigroups of endomorphisms of B(H) called E-semigroups - cf [A2]. Proposition 2.1. of [A1] says that if α is a non-zero normal *-endomorphism of B(H) then there are isometries V_1,V_2,... with mutually orthogonal ranges such that α(A) = ∑ V_nAV_n* for each A ∈ B(H). The linear space E= {T ∈ B(H) : α(A)T = TA for each A ∈ B(H)} is norm closed and T*S commutes with B(H) for each T, S ∈ E , therefore T*S is a scalar multiple of I. By Proposition 2.1. in the next section, E is an MI-space. For a concrete product system and a semigroup α_t, t ≥ 0, of normal *-automorphisms of B(H) the operator spaces E_t defined as above for α_t play a fundamental role in E-semigroup theory. I call this justification the *E-semigroup justification.*

In summary, MI-spaces appear naturally in three areas: operator theory, C*-algebras, and E-semigroups.
2. Geometry

In this section geometric aspects of $M I$-spaces will be discussed. In particular, a geometric model of a shift will be presented in Theorem 2.6., which is, perhaps, of interest on its own.

The basic result on $M I$-spaces is

2.1. Theorem ([Cat-Sz, Theorem 2.3]). Suppose $S \subset B(H)$ is a linear space. Then $S$ is an $M I$-space if and only if for each $A, B \in S$ there is $\lambda \in C$ such that $B^*A = \lambda I$.

Consider the mapping $\langle , \rangle_0: B(H) \times B(H) \to B(H)$ defined by $\langle A, B \rangle_0 = B^*A$ for $A, B \in B(H)$. This mapping satisfies all defining conditions of an inner product, except, in general, being scalar-valued. Denote by $CI$ all scalar multiples of $I$. Theorem 2.1 can be restated as

2.2. Theorem ([Cat-Sz, Theorem 2.4]). Suppose $S \subset B(H)$ is a linear space. Then $S$ is an $M I$-space if and only if the restriction of $\langle , \rangle_0$ to $S \times S$ takes values in $CI$.

Therefore on an $M I$-space $S$ we introduce the inner product as follows: given $A, B \in S$ we let $\langle A, B \rangle = \lambda$ such that $B^*A = \lambda I$ from Theorem 2.1., that is,

$\langle A, B \rangle_0 = B^*A = \langle A, B \rangle I$.

The norm defined by this inner product is the same as the operator norm in $B(H)$ because $\|A\|^2 = \|A^*A\| = \langle A, A \rangle$ for $A \in S$.

2.3. Proposition. The norm closure of an $M I$-space is an $M I$-space

Proof. Let $S$ be an $M I$-space. Take a sequence $A_n \in S$ and $A \in B(H)$. Then there are sequences $\lambda_n \in C$ and $V_n \in B(H)$ isometries such that $A_n = \lambda_n V_n$. Suppose $A_n \to A$. Then $A_n^*A_n \to A^*A$ and $A_n^*A_n = |\lambda_n|^2 V_n^*V_n = |\lambda_n|^2 I$. Therefore $|\lambda_n|^2$ converges to a non-negative number $|\lambda|^2$ for some $\lambda \in C$. Hence $A^*A = |\lambda|^2 I$. By [Cat-Sz, Proposition 2.1], $A \in MI$.

Therefore, the closure of an $M I$-space is a Hilbert space. Notice that the last proof works for any subset of $MI$. 
Let \( S \) be an \( MI \)-space. Two elements \( A, B \in S \) are orthogonal if \( B^*A = 0 \), which means that \( A, B \) have orthogonal ranges. Thus orthonormal vectors in \( S \) are isometries with mutually orthogonal ranges.

In the \( C^* \)-algebra justification the isometries \( S_1, \ldots, S_n \) form an orthonormal basis of \( \text{span}(S_1, \ldots, S_n) \). In the E-semigroup justification, it is proved in [A1, Proposition 2.1] that \( V_1, V_2, \ldots \) are the orthonormal basis of \( E \).

According to the celebrated Wold decomposition theorem, for each isometry \( A \in B(H) \) there are unique subspaces \( H_u, H_s \) of \( H \) which reduce \( A \) such that \( H = H_u \oplus H_s \), the part \( A_u \) in \( H_u \) is unitary, the part \( A_s \) in \( H_s \) is a shift, and \( A = A_u \oplus A_s \). Therefore, the range of \( A \) is \( AH = H_u \oplus A_s H_s \), thus, \( H \ominus AH = H_s \ominus A_s H_s \) is the wandering space for the shift part \( A_s \) of \( A \). From this we get immediately the following generalization of Proposition 2.10 in [Cat-Sz]:

**2.4. Proposition.** If an \( MI \)-space \( S \) contains an isometry \( A \) whose shift part has finite multiplicity then \( S = \text{span}(A) \), thus \( \dim S = 1 \).

**Proof.** Take \( B \) in the orthogonal complement of \( A \) in the Hilbert space \( \text{cl} \ S = \) the norm closure of \( S \), that is, \( A^*B = 0 \). This is justified by Proposition 2.3. Then \( BH \subset \ker A^* = H \ominus AH = \) the wandering space of \( A_s \) - cf. remark above. Since \( A_s \) has finite multiplicity, \( \dim H \ominus AH \) is finite. Since, by Proposition 2.3., \( B \in MI \), this is possible only if \( B = 0 \). Thus \( S \subset \text{cl} \ S = \text{span}(A) \). q.e.d.

Therefore, if \( S \) is an \( MI \)-space and \( \dim S > 1 \) then the shift part of each isometry in \( S \) has infinite multiplicity. In particular,

**2.5. Corollary.** Suppose \( S \) is an \( MI \)-space.

a. If \( S \) contains a unitary operator \( A \) then \( S = \text{span}(A) \).

b. If \( S \) contains \( I \) then \( S = CI \).

The remarks before the last proposition show also that what really matters when considering orthonormal systems in \( MI \)-spaces is the shift parts if the isometries involved. Therefore, now we turn to shifts. The next proposition is elementary. It is included here for the sake
of completeness. Let \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

2.6. Proposition. Suppose \( E = \{e_n : n \in \mathbb{N}\} \) is an orthonormal basis of \( H \). If \( A_0 : E \to H \) is a mapping whose range consists of orthonormal vectors then there is a unique isometry \( A \in B(H) \) such that \( A|E = A_0 \).

Proof. Take any \( x \in H \). Since \( E \) is an orthonormal basis of \( H \), there are unique \( \alpha_n \in \mathbb{C} \) such that \( x = \sum \alpha_n e_n \) and \( \sum |\alpha_n|^2 < \infty \). Define \( Ax = \sum \alpha_n A_0 e_n \). The mapping \( A : H \to H \) is well-defined and preserves inner product. Linearity follows by a standard argument. It is plain that \( A|E = A_0 \) and that such \( A \) is unique. q.e.d.

Now a shift with a given wandering space will be constructed. The construction relies on the following remarkable property of countable sets: If \( X \) is a finite or countable set then \( X \times \mathbb{N} \) is countable.

2.7. Theorem. For each subspace \( M \) of \( H \) with infinite dimensional \( H \ominus M \) there is a shift for which \( M \) is the wandering space.

Proof. Suppose \( \dim M = m \) is finite or countable. Choose an orthonormal basis \( e_{10}, e_{20}, \ldots, e_{m0} \) of \( M \). Let \( X = \{1, \ldots, m\} \). Choose an orthonormal basis of \( H \ominus M \) indexed by \( X \times \mathbb{N} \) as follows:

\[
e_{11}, e_{21}, \ldots, e_{m1} \\
e_{12}, e_{22}, \ldots, e_{m2} \\
\ldots................
\]

This is possible because \( X \times \mathbb{N} \) is countable. Then \( E = \{e_{jk} : (j, k) \in X \times (\mathbb{N} \cup \{0\})\} \) is an orthonormal basis of \( H \). Define \( A_0(e_{jk}) = e_{j,k+1} \) for \( (j, k) \in X \times (\mathbb{N} \cup \{0\}) \). By Proposition 2.6., there is a unique isometry \( A \in B(H) \) such that \( Ae_{jk} = A_0 e_{jk} = e_{j,k+1} \) for \( (j, k) \in X \times (\mathbb{N} \cup \{0\}) \). Since \( A^pM = \text{span}(e_{1p}, \ldots, e_{mp}) \) for \( p \in \mathbb{N} \cup \{0\} \), the subspaces \( A^pM \) and \( A^qM \) are mutually orthogonal for \( p, q \in \mathbb{N} \cup \{0\}, p \neq q \). Moreover, \( H \) is the orthogonal sum of all \( A^pM, p \in \mathbb{N} \cup \{0\} \), because \( E \) is an orthonormal basis of \( H \). Hence \( A \) is a shift with wandering space \( M \). q.e.d.

The construction of the shift in the above proof provides us with a very simple, yet useful model.
2.8. **Corollary.** Each infinite dimensional subspace of a Hilbert space is the range of a shift.

**Proof.** Suppose $K$ is an infinite dimensional subspace of $H$. By Theorem 2.7., there is a shift $A$ with wandering space $H \ominus K$. Since $H \ominus AH$ is the wandering space for $A$, we conclude $K = AH$. q.e.d.

In the operator theory and $C^*$-algebra justification $MI$-spaces are finite dimensional. In the E-semigroup justification to avoid trivial cases the $MI$-spaces $E_t$ have to be infinite dimensional.

Corollary 2.8. shows, in particular, how to construct $MI$-spaces with any dimension and prescribed ranges of isometries in their orthonormal bases. To get an $MI$-space $S$ with $\dim S = d$ finite or countable and mutually orthogonal ranges $K_1, \ldots, K_d$ of isometries in the orthonormal basis of $S$ just use Corollary 2.8. to get shifts $A_1, \ldots, A_d$ with desired properties.

Finally, suppose an $MI$-space should have an orthonormal basis consisting of isometries, some of which with non-trivial unitary part. This can be done exactly the same way as described above for shifts, using the following

2.9. **Corollary.** Suppose $K$ is a subspace of $H$, $K_u$ is a subspace of $K$ with infinite dimensional $K \ominus K_u$, and $U \in B(K_u)$ is a unitary operator. Then there is an isometry $A \in B(H)$ with unitary part $U$ and range $K$.

**Proof.** Let $A_s$ be the shift in $H \ominus K_u$ with range $K \ominus K_u$ as constructed in Theorem 2.7. and Corollary 2.8.. Let $A = U \oplus A_s$ on $H = K_u \oplus (H \ominus K_u)$. Since $AH = UK_u \oplus A_s(H \ominus K_u) = K_u \oplus (K \ominus K_u) = K$, $A$ satisfies all requirements. q.e.d.

Therefore, not only the range, but also the unitary part of an isometry can be arbitrarily prescribed. The only restriction is $K_u \subset K$, but Wold decomposition makes it necessary.

3. **Algebra**

Throughout this section $S$ is an $MI$-space. Now we will justify properties of $MI$-spaces in the title of this paper. First, symmetry.
3.1. Proposition.

a) If $A \in MI$ is such that $A^* \in MI$ then $A$ is a scalar multiple of a unitary operator.

b) If $A \in S$ is such that $A \neq 0$ and $A^* \in S$ then $S = \text{span}(A)$.

Proof. a) Suppose $A \neq 0$. Since $A$ and $A^*$ are in $MI$, there are $\lambda, \mu \in \mathbb{C}$ both non-zero, and isometries $V, W \in B(H)$ such that $A = \lambda V$, $A^* = \mu W$. Hence $\overline{\lambda}V^* = \mu W$ and $(\overline{\lambda}/\mu)V^* = W$ is an isometry. Therefore, $\ker V^* = 0$. Thus $VH = H$ and $V$ is unitary.

b) If such $A$ exists then, by a), it is a non-zero scalar multiple of a unitary operator. By Corollary 2.5.a), $S = \text{span}(A)$. q.e.d.

Therefore, if an $MI$-space is more than one dimensional then it cannot contain the adjoint of any of its elements. Now, let us turn to multiplicative properties.

3.2. Proposition.

a) If $A \in S$ then $\{B \in B(H) : AB \in S\} = CI$.

b) Suppose $S \neq CI$ and $A, B \in S$. Then $AB \in S$ if and only if $B = 0$.

c) If $S \neq CI$ and $A, A^k \in S$ for some $k \in \mathbb{N}$, $k > 1$, then $A = 0$.

Proof. Suppose $A \in S$ and $B \in B(H)$ is such that $AB \in S$. By Proposition 2.1., $A^*(AB) = \lambda I$ for some $\lambda \in \mathbb{C}$. But $A^*A = \mu I$ for some $\mu \in \mathbb{C}$. Therefore $B \in CI$. This proves a). Now, if $A, B \in S$ are such that $AB \in S$ then, by a), $B = \lambda I$ for some $\lambda \in \mathbb{C}$. If $S \neq CI$ then, by Corollary 2.5.b), $I \not\in S$. Therefore, $\lambda = 0$ and $B = 0$, which proves b). Part c) follows from b). q.e.d.

Part c) of this proposition explains the example with the shift of multiplicity one and its square given at the beginning of Section 1. Now it is plain that $MI$-spaces are very unfriendly to the operator multiplication. A simple way of thinking about an operator space could be considering an operator algebra and forgetting about multiplication. Not here. As we see, for an $MI$-space there is no non-trivial chance even to contain a power of its element, not to mention the algebra of polynomials in its element.

Finally, we come back to where we started in the operator theory justification, but with a broader perspective.

If $H$ is just any Hilbert space with inner product $\langle , \rangle$ then the number $\|x\|^2\|y\|^2 - |\langle x, y \rangle|^2$ for $x, y \in H$ does not seem to have any particular significance. It is, certainly, non-negative.
due to Schwarz inequality. In $MI$-spaces, however, this number seems to be rather important - it turns out to be the "measure of non-commutativity" for operators in such spaces, as the following proposition shows. For operators $A, B \in B(H)$ the commutator is defined by $[A, B] = AB - BA$.

3.3. Theorem. If $A, B \in S$ then $[A, B] \in MI$ and

$$[A, B]^*[A, B] = 2(\|A\|^2\|B\|^2 - |< A, B >|^2)I$$

Proof. Just compute:

$$[A, B]^*[A, B] = (AB - BA)^*(AB - BA) = B^*A^*AB - B^*A^*BA - A^*B^*AB + A^*B^*BA =$$

$$B^*\|A\|^2B - B^* < B, A > A - A^* < A, B > B + A^*\|B\|^2A =$$

$$(\|A\|^2\|B\|^2 - < B, A > < A, B > - < A, B > < B, A > + \|B\|^2\|A\|^2)I =$$

$$(2\|A\|^2\|B\|^2 - 2|< A, B >|^2)I.$$

Proposition 2.1. of [Cat-Sz] implies that $[A, B] \in MI$. q.e.d.

It seems the formula proved in this theorem deserves a name - I propose to call it the Commutator Identity. How non-commutative are orthonormal vectors in $S$? The Commutator Identity gives the answer:

3.4. Corollary. If $A, B \in S$ are orthonormal then $[A, B]^*[A, B] = 2I$ and $\|[A, B]\| = \sqrt{2}$.

Can we say anything "positive" about commuting in $MI$-spaces? The answer is a definite no. $MI$-spaces are the most noncommutative linear spaces encountered in the world of operator theory.

3.5. Corollary. Suppose $S$ is an $MI$-space.

a) $A, B \in S$ commute if and only if they are linearly dependent.

b) $S$ is commutative if and only if $\dim S = 0$ or $\dim S = 1$.

Proof. That $A, B$ commute means $[A, B] = 0$, which is the same as $[A, B]^*[A, B] = 0$. By the Commutator Identity, this is equivalent to equality in the Schwarz inequality for $A, B$. As every inner product child knows, this is equivalent to linear dependence of $A, B$. 

8
This proves a). If $\mathcal{S}$ is commutative and $\text{dim}\mathcal{S} \neq 0$ choose a non-zero $A \in \mathcal{S}$. Then each $B \in \mathcal{S}$ commutes with $A$. By a), $A, B$ are linearly dependent. Thus $\text{dim}\mathcal{S} = 1$. q.e.d.

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