NEW HOHLOV TYPE INTEGRAL OPERATOR INVOLVING
CLAUSEN’S HYPERGEOMETRIC FUNCTIONS

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Abstract. We consider the integral operator $I_{a,b,c}^{d,e}(f)(z)$ involving Clausen’s Hyper-
geometric Function by means of convolution introduced by Chandrasekran and Prab-
hakaran for investigation. The conditions on the parameters $a, b, c$ are determined using
the integral operator $I_{a,b}^{c,d}(f)(z)$ to study the geometric properties of Clausen’s Hyper-
geometric Function for various subclasses of univalent functions.

1. Introduction and preliminaries

Let $A$ denote the family of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

with $f(0) = 0$ and $f'(0) = 1$ in the open unit disc $D = \{ z : |z| < 1 \}$ of the complex plane.

Let us denote by $S$, the class of all normalised functions that are analytic and univalent
in $D$; that is $S = \{ f(z) \in A : f(z) \text{ is univalent in } D \}$. If $f(z)$ defined by (1.1) is belongs
to $S$, then,

$$|a_n| \leq n, \text{ for } n \geq 2.$$  \hfill (1.2)

Various subclasses of $S$ are characterized by their geometric properties. Some impor-
tant subclasses of $S$ are the class $C$ of all convex functions, the class $S^\ast$ of all functions
starlike with respect to the origin and the class $K$ of all close-to-convex functions. (For
more details refer \cite{4, 5}.

Robertson \cite{12} introduced the class $S^\ast(\alpha)$, of all starlike functions of order $\alpha$ and
the class $C(\alpha)$ of all convex functions of order $\alpha$. A function in $f(z) \in S$ is a star-
like function of order $\alpha$ in $D$ if and only if $\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha$, for all $z \in D$ where
$0 \leq \alpha < 1$. A function in $f(z) \in S$ is a convex function of order $\alpha$ in $D$ if and only if
$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$ for all $z \in D$, where $0 \leq \alpha < 1$. 

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Convex Functions, Uniformly Starlike Functions and Uniformly Convex Functions.
Our main focus is on the following subclasses of starlike and convex functions that are defined as follows. For \( \lambda > 0 \), \( \mathcal{S}_\lambda^* \), is defined as follows:

\[
\mathcal{S}_\lambda^* = \left\{ f(z) \in \mathcal{A} \mid \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda, \ z \in \mathbb{D} \right\}
\]

and a sufficient condition for which the function \( f(z) \) to be in \( \mathcal{S}_\lambda^* \) is

\[
\sum_{n=2}^{\infty} (n + \lambda - 1)|a_n| \leq \lambda.
\] (1.3)

For \( \lambda > 0 \), \( \mathcal{C}_\lambda \) is defined as follows \( \mathcal{C}_\lambda = \{ f(z) \in \mathcal{A} \mid zf'(z) \in \mathcal{S}_\lambda^* \} \) and a sufficient condition for which the function \( f(z) \) to be in \( \mathcal{C}_\lambda \), is as follows:

\[
\sum_{n=2}^{\infty} n (n + \lambda - 1)|a_n| \leq \lambda.
\] (1.4)

Goodman \[6, 7\] introduced the two new class, the class of all uniformly convex functions denoted by \( \mathcal{UCV} \) and the class of all uniformly starlike functions denoted by \( \mathcal{UST} \). He gave an analytic criterion for a function to be uniformly convex and uniformly starlike. Let \( f(z) \) be a function of the form (1.1). Then \( f \) is in \( \mathcal{UCV} \) if and only if

\[
\Re \left( 1 + \frac{(z - \zeta)}{f''(z)} \right) > 0, \text{ for all } z, \ \zeta \in \mathbb{D}.
\]

Then \( f \) is in \( \mathcal{UST} \) if and only if \( \Re \left( \frac{(z - \zeta)}{f'(z)} \right) > 0, \text{ for all } z, \ \zeta \in \mathbb{D} \). Note that, for \( \zeta = 0 \), the class \( \mathcal{UCV} \) coincides with the class \( \mathcal{C} \) and the class \( \mathcal{UST} \) coincides with the class \( \mathcal{S}^* \). Subsequently, Rønning \[13\] and Ma and Minda \[8\] independently gave the one variable analytic characterization of the class \( \mathcal{UCV} \).

A sufficient condition for the function of form (1.1) to belong to \( \mathcal{UCV} \) \[14\], is given by

\[
\sum_{n=2}^{\infty} n (2n - 1)|a_n| \leq 1.
\] (1.5)

The subclass \( \mathcal{S}_p \) of starlike functions introduced by Rønning \[13\] is defined as

\[
\mathcal{S}_p = \{ F \in \mathcal{S}_* \mid F(z) = zf'(z), f(z) \in \mathcal{UCV} \}.
\]

A sufficient condition for a function \( f(z) \) of form (1.1) to belong to \( \mathcal{S}_p \) is given by

\[
\sum_{n=2}^{\infty} (2n - 1)|a_n| \leq 1.
\] (1.6)

The above result was proved for more general case \( \mathcal{S}_p(\alpha) \) in \[15\].

Let for \( \beta < 1 \),

\[
\mathcal{R}(\beta) = \{ f(z) \in \mathcal{A} : \exists \ \phi \in \mathbb{R} / \Re (e^{i\phi}(f'(z) - \beta)) > 0, \ z \in \mathbb{D} \}.
\]

Note that when \( \beta \geq 0 \) we have \( \mathcal{R}(\beta) \subset \mathcal{S} \) and for \( \beta < 0 \), \( \mathcal{R}(\beta) \) contains also non univalent functions. This class has been widely used to study certain integral transforms.
Further \[1, 10\] and the reference therein. Suppose that \( f(z) \) defined by \((1.1)\) is in the class \( R(\beta) \). Then, by \[9\], we have

\[
|a_n| \leq \frac{2(1 - \beta)}{n}, \ n \geq 2.
\]

\( (1.7) \)

The convolution of two functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) (both \( f(z) \) and \( g(z) \) are analytic functions in \( \mathbb{D} \)) is defined by \( f(z) \ast g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \).

For any non-zero complex variable \( a \), the ascending factorial notation (or Pochhammer symbol) is defined as \((a)_0 = 1\), and \((a)_n = a(a+1)\cdots(a+n-1)\), for \( n = 1, 2, 3, \ldots \).

In literature, there are many articles connecting geometric function theory and Gauss hypergeometric function. But there are very few papers on Clausen’s hypergeometric function. This function is of greater interest, since the famous Bieberbach conjecture has proved by Louis De Branges in 1984 using the Askey-Gasper inequalities for Jacobi Polynomials which involves Clausen’s series.

\[ 3 \mathcal{F}_2 \left( \frac{k + \frac{1}{2}, k - n, n + k + 2}{2k + 1, k + \frac{3}{2}}; e^{-t} \right). \]

The Clausen’s series can be obtained by squaring the Gaussian hypergeometric function. i.e., \([2 \mathcal{F}_1(a, b; a + b + 1/2; z)]^2 = 3 \mathcal{F}_2(2a, 2b, a + b; 2a + 2b, a + b + 1/2; z)\).

The Clausen’s hypergeometric function \( 3 \mathcal{F}_2(a, b, c; d, e; z) \) is defined as

\[
3 \mathcal{F}_2(a, b, c; d, e; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n(1)_n} z^n, \quad |z| < 1
\]

where \( a, b, c, d, e \in \mathbb{C} \) with \( d, e \neq 0, -1, -2, -3 \cdots \).

The Clausen’s hypergeometric function \( 3 \mathcal{F}_2(a, b, c; d, e; z) := 4 \mathcal{F}_3(a, b, c, a_4; d, e, a_4; z) \) have been studied by only few authors. In particular \[11\], Ponnusamy and Sabapathy considered the generalized hypergeometric functions and tried to find condition on the parameter so that \( z 3 \mathcal{F}_2(a, b, c; d, e; z) \) has some geometric properties.

In \[2\], Chandrasekran and Prabhakaran have introduced an integral operator and derived the geometric properties for the Clausen’s Hypergeometric series \( z 3 \mathcal{F}_2(a, b, c; b + 1, c + 1; z) \), in which, the numerator and denominator parameters differs by arbitrary negative integers. Also to determine the conditions on the parameters \( a, b, c, \) such that, the hypergeometric function \( z 3 \mathcal{F}_2(a, b, c; 1 + a - b, 1 + a - c; z) \) associated with the Dixon’s summation formula or its equivalence has the admissibility to be the classes like \( S_\lambda^*, C_\lambda, UCV \), and \( S_p \) is still it is an open problem.

In 2006, Driver and Johnston \[3\] derived a summation formula for Clausen’s hypergeometric function in terms of Gaussian hypergeometric function. We recall their summation formula as follows:
provide $\text{Re}(c) > \text{Re}(b) > 0$ and $\text{Re}(c - a - b) > 0$.

In this paper, we consider an integral operator $I_{a,b,c}^{d,e}(f)(z)$ introduced by Chandrasekran and Prabhakaran [2] and to study the various geometric properties of the above Clausen’s series, the integral operator takes the form as follows

\[ I_{a,b,c}^{d,e}(f)(z) = z 3F2 \left( a, b, b+1/2, c+1/2, c, z \right) * f(z) \]

with $A_1 = 1$ and for $n > 1$,

\[ A_n = \frac{(a)_{n-1} \left( \frac{b}{2} \right)_{n-1} \left( \frac{b+1}{2} \right)_{n-1}}{(c)_{n-1} \left( \frac{c+1}{2} \right)_{n-1}} a_n. \]

The following Lemma is useful to prove our main results.

\textbf{Lemma 1.11.} Let $a, b, c > 0$. Then the following is derived:

\begin{enumerate}
  \item For $c > a + b + 1$,
    \[ \sum_{n=0}^{\infty} \frac{(n+1)(a)_{n} \left( \frac{b}{2} \right)_{n} \left( \frac{b+1}{2} \right)_{n}}{(\frac{c}{2})_{n} \left( \frac{c+1}{2} \right)_{n} (1)_{n}} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left[ \frac{(a)_{2}(b)_{2}}{(c-a)(c-a-b-1)} \right] \times 2F1(a+1, b+2; c-a+1; -1) + 2F1(a, b; c-a; -1) \]

  \item For $c > a + b + 2$,
    \[ \sum_{n=0}^{\infty} \frac{(n+1)^2(a)_{n} \left( \frac{b}{2} \right)_{n} \left( \frac{b+1}{2} \right)_{n}}{(\frac{c}{2})_{n} \left( \frac{c+1}{2} \right)_{n} (1)_{n}} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left[ \frac{(a)_{2}(b)_{4}}{(c-a)(c-a-b)(c-a-b-2)} \right] \times 2F1(a+2, b+4; c-a+2; -1) \]
    \[ + 3 \left( \frac{(a)_{2}}{(c-a)(c-a-b-1)} \right) 2F1(a+1, b+2; c-a+1; -1) + 2F1(a, b; c-a; -1) \]

  \item For $c > a + b + 3$,}

\end{enumerate}
\[
\sum_{n=0}^{\infty} \frac{(n+1)^3(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left[ \frac{(a)_3 (b)_6}{(c-a)_3 (c-a-b-3)_3} \right] \\
\times 2F_1(a+3, b+6; c-a+3; -1) \\
+6 \frac{(a)_2 (b)_4}{(c-a)_2 (c-a-b-2)_2} F_1(a+2, b+4; c-a+2; -1) \\
+7 \frac{a (b)_2}{(c-a) (c-a-b-1)} \\
\times 2F_1(a+1, b+2; c-a+1; -1) + 2F_1(a, b; c-a; -1) \].
\]

(4) For \( a \neq 1, b \neq 1, 2 \) and \( c > \max\{a+1, a+b-1\}, \)

\[
\sum_{n=0}^{\infty} \frac{(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_{n+1}} = \left[ \frac{(c-a-1) (c-a-b) \Gamma(c) \Gamma(c-a-b)}{(a-1) (b-1) (b-2) \Gamma(c-b) \Gamma(c-a)} \right] \\
\times 2F_1(a-1, b-2; c-a-1; -1) - \frac{(c-2)_2}{(a-1) (b-2)_2} \right].
\]

Proof. (1) Using Pochhammer symbol, we can formulate

\[
\sum_{n=0}^{\infty} \frac{(n+1)(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n} = \sum_{n=0}^{\infty} \frac{(a)_{n+1} \left( \frac{b}{2} \right)_{n+1} \left( \frac{b+1}{2} \right)_{n+1}}{\left( \frac{c}{2} \right)_{n+1} \left( \frac{c+1}{2} \right)_{n+1} (1)_{n+1}} + \sum_{n=0}^{\infty} \frac{(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]

Using the formula (1.3) and using the fact that \( \Gamma(a+1) = a\Gamma(a) \), the aforementioned equation reduces to

\[
\sum_{n=0}^{\infty} \frac{(n+1)(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-b) \Gamma(c-a)} \\
\times \left[ \frac{(a) (b)_2}{(c-a) (c-a-b-1)} \right] 2F_1(a+1, b+2; c-a+1; -1) \\
+ 2F_1(a, b; c-a; -1) \right].
\]

Hence, (1) is proved.
(2) Using \((n+1)^2 = n(n-1) + 3n + 1\), we can easily obtain that
\[
\sum_{n=0}^{\infty} \frac{(n+1)^2 (a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]
\[
= \sum_{n=2}^{\infty} \frac{(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_{n-2}} + 3 \sum_{n=1}^{\infty} \frac{(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_{n-1}} + \sum_{n=0}^{\infty} \frac{(a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]
Using the formula (1.8) and using the fact that \(\Gamma(a+1) = a\Gamma(a)\), the aforementioned equation reduces to
\[
\sum_{n=0}^{\infty} \frac{(n+1)^2 (a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]
\[
= \left( \frac{(a)_2 (b)_4}{(c-a-b-2)_2 (c-a)_2} \right) \left( \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} \right)
\times _2F_1(a+2,b+4;c-a+2;-1)
+ 3 \left( \frac{(a)_2 (b)_2}{(c-a-b-1) (c-a)} \right) \left( \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} \right)
\times _2F_1(a+1,b+2;c-a+1;-1)
+ \left( \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \right) _2F_1(a,b;c-a;-1).
\]

Hence,
\[
\sum_{n=0}^{\infty} \frac{(n+1)^2 (a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]
\[
= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{(a)_2 (b)_4}{(c-a)_2 (c-a-b-2)_2} \right]
\times _2F_1(a+2,b+4;c-a+2;-1)
+ 3 \left( \frac{a(b)_2}{(c-a)(c-a-b-1)} \right) _2F_1(a+1,b+2;c-a+1;-1)
+ 2F_1(a,b;c-a;-1),
\]

Which completes the proof of (2).

(3) Using \((n+1)^3 = n(n-1)(n-2) + 6n(n-1) + 7n + 1\), we can write
\[
\sum_{n=0}^{\infty} \frac{(n+1)^3 (a)_n \left( \frac{b}{2} \right)_n \left( \frac{b+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n}
\]
\[\sum (a)_{3} (b)_{6} \sum_{n=0}^{\infty} \frac{(a + 3)_{n} \left(\frac{b}{2} + 3\right)_{n} \left(\frac{b+1}{2} + 3\right)_{n}}{(\frac{c}{2} + 3)_{n} \left(\frac{c+1}{2} + 3\right)_{n} (1)_{n}}
\]

+ 6 \left(\frac{a (b)_{2}}{(c)_{2}} \sum_{n=0}^{\infty} \frac{(a + 2)_{n} \left(\frac{b}{2} + 2\right)_{n} \left(\frac{b+1}{2} + 2\right)_{n}}{(\frac{c}{2} + 2)_{n} \left(\frac{c+1}{2} + 2\right)_{n} (1)_{n}}
\right)

+ 7 \left(\frac{a (b)_{2}}{(c)_{2}} \sum_{n=0}^{\infty} \frac{(a + 1)_{n} \left(\frac{b}{2} + 1\right)_{n} \left(\frac{b+1}{2} + 1\right)_{n}}{(\frac{c}{2} + 1)_{n} \left(\frac{c+1}{2} + 1\right)_{n} (1)_{n}}
\right)

Using the formula (1.8) and using the fact that \(\Gamma(a + 1) = a\Gamma(a)\), the aforementioned equation reduces to

\[\sum_{n=0}^{\infty} \frac{(n + 1)^{3} (a)_{n} \left(\frac{b}{2}\right)_{n} \left(\frac{b+1}{2}\right)_{n}}{(\frac{c}{2})_{n} \left(\frac{c+1}{2}\right)_{n} (1)_{n}}
\]

\[= \Gamma(c) \frac{\Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \left[\frac{(a)_{3} (b)_{6}}{(c - a)_{3} (c - a - b - 3)_{3}} \right] \times 2F_{1}(a + 3, b + 6; c - a + 3; -1)
\]

+ 6 \left(\frac{(a)_{2} (b)_{4}}{(c - a)_{2} (c - a - b - 2)_{2}} \right) 2F_{1}(a + 2, b + 4; c - a + 2; -1)

+ 7 \left(\frac{(a)_{2} (b)_{4}}{(c - a)(c - a - b - 1)} \right) 2F_{1}(a + 1, b + 2; c - a + 1; -1)

which completes the proof.

(4) Let \(a \neq 1, b \neq 1, 2\) and \(c > \max\{a + 1, a + b - 1\}\). It is found that

\[\sum_{n=0}^{\infty} \frac{(a)_{n} \left(\frac{b}{2}\right)_{n} \left(\frac{b+1}{2}\right)_{n}}{(\frac{c}{2})_{n} \left(\frac{c+1}{2}\right)_{n} (1)_{n+1}}
\]

\[= \left(\frac{(c - 1) (c - 2)}{(a - 1) (b - 1) (b - 2)} \right) \sum_{n=0}^{\infty} \frac{(a - 1)_{n} \left(\frac{b}{2} - 1\right)_{n} \left(\frac{b+1}{2} - 1\right)_{n}}{(\frac{c}{2} - 1)_{n} \left(\frac{c+1}{2} - 1\right)_{n} (1)_{n}} - 1
\]

\[= \left(\frac{(c - a - 1) (c - a - b) \Gamma(c) \Gamma(c - a - b)}{(a - 1) (b - 1) (b - 2) \Gamma(c - b) \Gamma(c - a)} \right) \times 2F_{1}(a - 1, b - 2; c - a - 1; -1) - \left(\frac{(c - 2) (c - 1)}{(a - 1) (b - 1) (b - 2)} \right).
\]

Hence the desired result follows. □
2. Starlikeness of \( z_3F_2 \left( a, b, c; \frac{b+1}{2}, \frac{c+1}{2}; z \right) \)

**Theorem 2.1.** Let \( a, b \in \mathbb{C} \setminus \{0\}, \ c > 0 \) and \( c > |a| + |b| + 1 \). A sufficient condition for the function \( z_3F_2 \left( a, b, c; \frac{b+1}{2}, \frac{c+1}{2}, z \right) \) to belong to the class \( S_\lambda \), \( 0 < \lambda \leq 1 \) is that

\[
\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{|a| (|b|)_2}{(c - |a|) (c - |a| - |b| - 1)} \right] \\
\times 2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) + \lambda 2F_1(|a|, |b|; c - |a|; -1) \leq 2\lambda.
\]

**Proof.** Let \( f(z) = z_3F_2 \left( a, \frac{b}{2}, \frac{b+1}{2}, \frac{c}{2}, \frac{c+1}{2}; z \right) \), then, by the equation (1.8), it is enough to show that

\[
T = \sum_{n=2}^{\infty} (n + \lambda - 1)|A_n| \leq \lambda.
\]

Using the fact \(|(a)_n| \leq (|a|)_n\), one can get

\[
T \leq \sum_{n=0}^{\infty} \left( (n + 1) \frac{|a|_n \left( \frac{|b|}{2} \right)_n \left( \frac{|b|+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n} \right) + (\lambda - 1) \sum_{n=0}^{\infty} \left( \frac{|a|_n \left( \frac{|b|}{2} \right)_n \left( \frac{|b|+1}{2} \right)_n}{\left( \frac{c}{2} \right)_n \left( \frac{c+1}{2} \right)_n (1)_n} \right) - \lambda.
\]

Using (1.8) and the result (1) of Lemma 1.11 in the aforesaid equation, we get

\[
T \leq \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{|a| (|b|)_2}{(c - |a|) (c - |a| - |b| - 1)} \right] \\
\times 2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) + \lambda 2F_1(|a|, |b|; c - |a|; -1) \leq \lambda.
\]

Because of (2.2), the above expression is bounded above by \( \lambda \), and hence,

\[
T \leq \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{|a| (|b|)_2}{(c - |a|) (c - |a| - |b| - 1)} \right] \\
\times 2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) + \lambda 2F_1(|a|, |b|; c - |a|; -1) \leq \lambda.
\]

Therefore, \( z_3F_2 \left( a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; z \right) \) belongs to the class \( S_\lambda \). \( \square \)
Theorem 2.3. Let $a, b \in \mathbb{C}\{0\}$, $c > 0$, $|a| \neq 1$, $|b| \neq 1, 2$, and $c > \max\{|a| + 1, |a| + |b| - 1\}$. For $0 < \lambda \leq 1$ and $0 \leq \beta < 1$, assume that

$$\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left( \frac{(\lambda - 1)(c - |a| - |b|)(c - |a| - 1)}{(|a| - 1)(|b| - 1)(|b| - 2)} \right)$$

$$\times {}_2F_1(|a| - 1, |b| - 2; c - |a| - 1; -1) + {}_2F_1(|a|, |b|; c - |a|; -1)$$

(2.4)

$$\leq \lambda \left( 1 + \frac{1}{2(1 - \beta)} \right) + \frac{(\lambda - 1)(c - 1)(c - 2)}{(|a| - 1)(|b| - 1)(|b| - 2)}.$$ 

Then, the integral operator $T_{a, b; \frac{c}{2}, \frac{c+1}{2}}^n(f)$ maps $\mathcal{R}(\beta)$ into $\mathcal{S}_\lambda^*$.

Proof. Let $a, b \in \mathbb{C}\{0\}$, $c > 0$, $|a| \neq 1$, $|b| \neq 1, 2$, and $c > |a| + |b| - 1$. For $0 < \lambda \leq 1$ and $0 \leq \beta < 1$.

Consider the integral operator $T_{a, b; \frac{c}{2}, \frac{c+1}{2}}^n(f)(z)$ defined by (1.9). According to (1.3), we need to show that

$$T = \sum_{n=2}^{\infty} (n + \lambda - 1)|A_n| \leq \lambda,$$

where $A_n$ is given by (1.10). Then, we have

$$T = \sum_{n=2}^{\infty} |n + (\lambda - 1)| \frac{(a)_{n-1} (\frac{b}{2})_{n-1} (\frac{b+1}{2})_{n-1}}{(\frac{c}{2})_{n-1} (\frac{c+1}{2})_{n-1} (1)_n} |a_n|$$

Using (1.7) in the aforementioned equation, we have

$$T \leq 2(1 - \beta) \left[ \sum_{n=0}^{\infty} \left( \frac{|a|_n}{(\frac{c}{2})_n} \right)^n \left( \frac{|b|_n}{(\frac{c+1}{2})_n} \right)^n (\frac{1}{n+1}) - 1 \right] + (\lambda - 1) \sum_{n=0}^{\infty} \left( \frac{|a|_n}{(\frac{c}{2})_n} \right)^n \left( \frac{|b|_n}{(\frac{c+1}{2})_n} \right)^n \left( \frac{1}{n+1} \right) - (\lambda - 1) \right] := T_1.$$

Using the formula (1.8) and the results (1) and (4) of Lemma 1.11, we find that

$$T_1 \leq 2(1 - \beta) \left[ \left( \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \left( \frac{(\lambda - 1)(c - |a| - |b|)(c - |a| - 1)}{(|a| - 1)(|b| - 1)(|b| - 2)} \right) \right.$$

$$\times {}_2F_1(|a| - 1, |b| - 2; c - |a| - 1; -1) + {}_2F_1(|a|, |b|; c - |a|; -1)$$

$$- \frac{(\lambda - 1)(c - 1)(c - 2)}{(|a| - 1)(|b| - 1)(|b| - 2) - \lambda}.$$
Thus, we have the inequalities $T \leq T_1 \leq \lambda$, and hence (2.5) hold. Therefore, it is concluded that the operator $\mathcal{I}_{\frac{a}{2}, \frac{b+1}{2}}(\lambda)$ maps $\mathcal{R}(\beta)$ into $S^*_\lambda$, which completes the proof of the theorem.

When, $\lambda = 1$, we get the following result from Theorem 2.3.

**Corollary 2.6.** Let $a, b \in \mathbb{C}\setminus\{0\}$, $c > 0$, and $c > |a| + |b|$. For $0 \leq \beta < 1$. Assume that

$$
\left( \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \binom{\lambda}{2} |a| |b| |c - |a| - |b| - 2| \leq 1 + \frac{1}{2(1 - \beta)}.
$$

Then, the integral operator $\mathcal{I}_{\frac{a}{2}, \frac{b+1}{2}}(f)$ maps $\mathcal{R}(\beta)$ into $S^*_1$.

**Theorem 2.7.** Let $a, b \in \mathbb{C}\setminus\{0\}$, $c > 0$ and $c > |a| + |b| + 2$. For $0 < \lambda \leq 1$. If

$$
\left( \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \left[ \left( \frac{|a|^2 (|b|)^4}{|c - |a| - |b| - 2|^2} \right) \times 2F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) ight.

\left. + \frac{(\lambda + 2) |a| (|b|)^2}{(c - |a|) (c - |a| - |b| - 1)} \right] 2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1)

(2.8)

then the integral operator $\mathcal{I}_{\frac{a}{2}, \frac{b+1}{2}}(f)$ maps $\mathcal{S}$ to $S^*_\lambda$.

**Proof.** Let $a, b \in \mathbb{C}\setminus\{0\}$, $c > 0$ $c > |a| + |b| + 2$ and $0 < \lambda \leq 1$.

Suppose that the integral operator $\mathcal{I}_{\frac{a}{2}, \frac{b+1}{2}}(f)(z)$ is defined by (1.9). In view of (1.3), it is enough to show that

$$
T = \sum_{n=2}^{\infty} (n + \lambda - 1)|A_n| \leq \lambda.
$$
where $A_n$ is given by (1.10). Using the fact $|(a)_n| \leq |(a)|_n$ and the equation (1.2) in the aforementioned equation, it is derived that

$$T \leq \sum_{n=2}^{\infty} n(n + (\lambda - 1)) \left( \frac{|(a)|_{n-1} \left( b \right) \frac{1}{z} \left( c+1 \right)}{(c-z)_{n-1} \left( c+1 \right)} \right).$$

Using (1) and (2) of Lemma 1.11 it is found that

$$T \leq \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{(a)_2 (|b|)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right] \times _2 F_1 (|a| + 2, |b| + 4; c - |a| + 2; -1)
+ \left( \frac{(\lambda + 2) (|a|)_2 (|b|)_2}{(c - |a|)_2 (c - |a| - |b| - 1)_2} \right) _2 F_1 (|a| + 1, |b| + 2; c - |a| + 1; -1)
+ \lambda_2 F_1 (|a|, |b|; c - |a|; -1) - \lambda.$$

By (2.8), the above expression is bounded above by $\lambda$, and hence,

$$\Gamma(c) \Gamma(c - |a| - |b|) \left[ \frac{(a)_2 (|b|)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right] \times _2 F_1 (|a| + 2, |b| + 4; c - |a| + 2; -1)
+ \left( \frac{(\lambda + 2) (|a|)_2 (|b|)_2}{(c - |a|)_2 (c - |a| - |b| - 1)_2} \right) _2 F_1 (|a| + 1, |b| + 2; c - |a| + 1; -1)
+ \lambda_2 F_1 (|a|, |b|; c - |a|; -1) \leq \lambda.$$ 

Under the stated condition, the integral operator $T_{\frac{a}{\gamma}, \frac{b+1}{\gamma}, \frac{c+1}{\gamma}}$ maps $S$ into $S_{\lambda}$. 

3. Convexity of $z_3 F_2 (a, b, \frac{b+1}{\gamma}, \frac{c+1}{\gamma}; z)$

**Theorem 3.1.** Let $a, b \in \mathbb{C} \setminus \{0\}$, $c > 0$, $c > |a| + |b| + 2$ and $0 < \lambda \leq 1$. A sufficient condition for the function $z_3 F_2 (a, b, \frac{b+1}{\gamma}, \frac{c+1}{\gamma}; z)$ to belong to the class $C_{\lambda}$ is that

$$\left( \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \left[ \frac{(a)_2 (|b|)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right] \times _2 F_1 (|a| + 2, |b| + 4; c - |a| + 2; -1)
+ \left( \frac{(\lambda + 2) (|a|)_2 (|b|)_2}{(c - |a|)_2 (c - |a| - |b| - 1)_2} \right) _2 F_1 (|a| + 1, |b| + 2; c - |a| + 1; -1)
+ \lambda_2 F_1 (|a|, |b|; c - |a|; -1) \leq 2\lambda.$$

**Proof.** The proof is similar to Theorem 2.7. So we omit the details. 

□
Theorem 3.2. Let $a, b \in \mathbb{C}\{0\}$, $c > 0$, $c > |a| + |b| + 1$ and $0 < \lambda \leq 1$. For $0 \leq \beta < 1$, it is assumed that
\[
\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \times \left[ \frac{(|a|)(|b|)^2}{(c - |a|)(c - |a| - |b| - 1)} \right] 2F_1(|a| + 1, |b| + 2; -|a| + 1; -1) \\
+ \lambda_2 2F_1(|a|, |b|; -|a|; -1) \leq \lambda \left( \frac{1}{2(1 - \beta)} + 1 \right).
\]

Then, the operator $T^{a, b, \frac{b+1}{2}}_{\frac{R}{c}}(f)$ maps $\mathcal{R}(\beta)$ into $\mathcal{C}_\lambda$.

Proof. The proof is similar to Theorem 2.3. So we omit the details. \quad \Box

Theorem 3.3. Let $a, b \in \mathbb{C}\{0\}$, $c > 0$, $c > |a| + |b| + 3$ and $0 < \lambda \leq 1$. If
\[
\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \times \left[ \frac{(|a|)(|b|)^6}{(c - |a|)^3(c - |a| - |b| - 3)^3} \right] 2F_1(|a| + 3, |b| + 6; -|a| + 3; -1) \\
+ \lambda_2 \left[ \frac{(\lambda + 5)(|a|^2)(|b|)^2}{(c - |a|)^2(c - |a| - |b| - 2)^2} \right] 2F_1(|a| + 2, |b| + 4; -|a| + 2; -1) \\
+ \lambda_2 \left[ \frac{(3\lambda + 4)(|a|)(|b|)^2}{(c - |a|)(c - |a| - |b| - 1)} \right] 2F_1(|a| + 1, |b| + 2; -|a| + 1; -1) \\
\leq 2\lambda,
\]

then, $T^{a, b, \frac{b+1}{2}}_{\frac{R}{c}}(f)$ maps $\mathcal{S}$ into $\mathcal{C}_\lambda$.

Proof. Let $a, b \in \mathbb{C}\{0\}$, $c > 0$, $c > |a| + |b| + 3$ and $0 < \lambda \leq 1$.

Suppose the integral operator $T^{a, b, \frac{b+1}{2}}_{\frac{R}{c}}(f)(z)$ is defined by (1.3). In view of the sufficient condition given in (1.4), it is enough to prove that
\[
T = \sum_{n=2}^{\infty} n(n + \lambda - 1) |A_n| \leq \lambda.
\]
i.e.,
\[
T = \sum_{n=2}^{\infty} n(n + \lambda - 1) \left| \left( \frac{(a)_{n-1} (\frac{b}{2})_{n-1} (\frac{b+1}{2})_{n-1}}{(\frac{b}{2})_{n-1} (\frac{b+1}{2})_{n-1} (1)_{n-1}} \right) a_n \right| \leq \lambda.
\]
Using the fact that $|(a)_n| \leq (|a|)_n$ and (3.2) in the aforementioned equation, it is derived

$$T \leq \sum_{n=0}^{\infty} \frac{(n+1)^3 (|a|)_{n-1} \left(\frac{|b|}{2}\right)_n \left(\frac{|b|+1}{2}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{c+1}{2}\right)_n (1)_n} + (\lambda - 1) \sum_{n=0}^{\infty} \frac{(n+1)(|a|)_n \left(\frac{|b|}{2}\right)_n \left(\frac{|b|+1}{2}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{c+1}{2}\right)_n (1)_n} - \lambda.$$

Using (2) and (3) of Lemma 1.11, we find that

\[
T \leq \left(\frac{\Gamma(c) \Gamma(c-|a|-|b|)}{\Gamma(c-|a|) \Gamma(c-|b|)}\right) \times \left[\frac{(|a|)_3 (|b|)_6}{(c-|a|)_3 (c-|a|-|b|-3)_3} \right] \, _2F_1(|a| + 3, |b| + 6; c - |a| + 3; -1) \\
+ \left(\frac{(\lambda + 5) (|a|)_2 (|b|)_4}{(c-|a|)_2 (c-|a|-|b|-2)_2}\right) \, _2F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) \\
+ \left(\frac{(3\lambda + 4) (|a|)_2 (|b|)_2}{(c-|a|) (c-|a|-|b|-1)}\right) \, _2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) \\
+ \lambda_2 \, _2F_1(|a|, |b|; c - |a|; -1) \right] - \lambda. \\

By the equation (3.3), the above expression is bounded above by $\lambda$, and hence,

\[
\left(\frac{\Gamma(c) \Gamma(c-|a|-|b|)}{\Gamma(c-|a|) \Gamma(c-|b|)}\right) \times \left[\frac{(|a|)_3 (|b|)_6}{(c-|a|)_3 (c-|a|-|b|-3)_3} \right] \, _2F_1(|a| + 3, |b| + 6; c - |a| + 3; -1) \\
+ \left(\frac{(\lambda + 5) (|a|)_2 (|b|)_4}{(c-|a|)_2 (c-|a|-|b|-2)_2}\right) \, _2F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) \\
+ \left(\frac{(3\lambda + 4) (|a|)_2 (|b|)_2}{(c-|a|) (c-|a|-|b|-1)}\right) \, _2F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) \\
+ \lambda_2 \, _2F_1(|a|, |b|; c - |a|; -1) \right] - \lambda \leq \lambda. \\

Hence, the integral operator $I_{\frac{\alpha}{2}, \frac{b+1}{2}} \, \frac{\beta+1}{2} (f)(z)$ maps $S$ into $C_\lambda$ and the proof is complete. $\square$

4. Admissibility condition of $z \, _3F_2 \left(\frac{a}{2}, \frac{b+1}{2}, \frac{c+1}{2}; \frac{\beta+1}{2}; z\right)$ in $UCV$.

Theorem 4.1. Let $a, b \in \mathbb{C}\setminus\{0\}$, $c > 0$ and $c > |a| + |b| + 2$. A sufficient condition for the function $z \, _3F_2 \left(\frac{a}{2}, \frac{b+1}{2}, \frac{c+1}{2}; \frac{\beta+1}{2}; z\right)$ to belong to the class $UCV$ is that
\[
\frac{(\Gamma(c) \Gamma(c - |a| - |b|))}{(\Gamma(c - |a|) \Gamma(c - |b|))} \times \left[ \left( \frac{2 \left( |a| \right)_2 \left( |b| \right)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right) _2 F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) \\
+ 5 \left( \frac{(|a|) \left( |b| \right)_2}{(c - |a|) (c - |a| - |b| - 1)} \right) _2 F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) \right] + 2 F_1(|a|, |b|; c - |a|; -1) \leq 2.
\]

**Proof.** Let \( a, b \in \mathbb{C} \setminus \{0\}, \ c > 0 \) and \( c > |a| + |b| + 2 \).

Let \( f(z) = z^3 F_2 \left( a, \frac{b + 1}{2}; \frac{c + 1}{2}; z \right) \). Then, by (1.5), it is enough to show that

\[
T = \sum_{n=2}^{\infty} n (2n - 1) |A_n| \leq 1.
\]

where \( A_n \) is given by (1.10). Using the fact \( |(a)_n| \leq (|a|)_n \),

\[
T \leq 2 \sum_{n=0}^{\infty} \left( \frac{(n + 1)^2 (|a|)_n \left( \frac{b}{2} \right)_n \left( \frac{|b| + 1}{2} \right)_n}{\left( \frac{\xi}{2} \right)_n \left( \frac{\xi}{2} + 1 \right)_n (1)_n} \right) - \sum_{n=0}^{\infty} \left( \frac{(n + 1) (|a|)_n (|b|)_n (c)_n}{(|b| + 1)_n (c + 1)_n (1)_n} \right) - 1.
\]

Using (1) and (2) of Lemma 1.11 in the aforementioned equation, we find that

\[
T \leq \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \left( \frac{2 \left( |a| \right)_2 \left( |b| \right)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right) _2 F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) \\
+ 5 \left( \frac{(|a|) \left( |b| \right)_2}{(c - |a|) (c - |a| - |b| - 1)} \right) _2 F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) \right] - 1.
\]

Because of (4.2), the above expression is bounded above by 1, and hence,

\[
\frac{(\Gamma(c) \Gamma(c - |a| - |b|))}{(\Gamma(c - |a|) \Gamma(c - |b|))} \times \left[ \left( \frac{2 \left( |a| \right)_2 \left( |b| \right)_4}{(c - |a|)_2 (c - |a| - |b| - 2)_2} \right) _2 F_1(|a| + 2, |b| + 4; c - |a| + 2; -1) \\
+ 5 \left( \frac{(|a|) \left( |b| \right)_2}{(c - |a|) (c - |a| - |b| - 1)} \right) _2 F_1(|a| + 1, |b| + 2; c - |a| + 1; -1) \right] - 1 \leq 1.
\]

Therefore, \( z^3 F_2 \left( a, \frac{b + 1}{2}; \frac{c + 1}{2}; z \right) \) belongs to the class \( UCV \). \( \square \)
Theorem 4.3. Let \( a, b \in \mathbb{C}\setminus\{0\}, \ c > 0, \ c > |a| + |b| + 1 \) and \( 0 \leq \beta < 1 \). Assume that

\[
\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{2 \,(|a|) \,(|b|)_{2}}{(c - |a|) (c - |a| - |b| - 1)} \right] \times {}_{2}F_{1}(|a| + 1, |b| + 2; c - |a| + 1; -1) + {}_{2}F_{1}(|a|, |b|; c - |a|; -1) \]

\[
\frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{2 \,(|a|) \,(|b|)_{2}}{(c - |a|) (c - |a| - |b| - 1)} \right] \times {}_{2}F_{1}(|a| + 1, |b| + 2; c - |a| + 1; -1) + {}_{2}F_{1}(|a|, |b|; c - |a|; -1) \]

\[
\leq \frac{1}{2(1 - \beta)} + 1.
\]

Then, \( I_{n, \beta, \frac{b + 1}{2}}^{a, \frac{b + 1}{2}} (f) \) maps \( \mathcal{R}(\beta) \) into \( \mathcal{UCV} \).

Proof. Let \( a, b \in \mathbb{C}\setminus\{0\}, \ c > 0, \ c > |a| + |b| + 1 \) and \( 0 < \beta \leq 1 \).

Then consider the integral operator \( I_{n, \beta, \frac{b + 1}{2}}^{a, \frac{b + 1}{2}} (f) \) given in (1.9). According to sufficient condition given in (1.5), it is enough to show that

\[
T := \sum_{n=2}^{\infty} n \,(2n - 1) \,|A_{n}| \leq 1,
\]

where \( A_{n} \) is given by (1.10). Using the fact \(|(a)_{n}| \leq (|a|)_{n}\) and (1.7) in the aforementioned equation, it is found that

\[
T \leq 2(1 - \beta) \sum_{n=2}^{\infty} n \,(2n - 1) \left( \frac{(|a|)_{n-1} \,(\frac{|b|}{2})_{n-1} \,(\frac{|b|+1}{2})_{n-1}}{\left( \frac{c}{2} \right)_{n-1} \,\left( \frac{c+1}{2} \right)_{n-1} \,(1)_{n-1} \,n} \right)
\]

Using the formula (1.8) and (1) of Lemma (1.11) it is derived that

\[
T \leq 2(1 - \beta) \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{2 \,(|a|) \,(|b|)_{2}}{(c - |a|) (c - |a| - |b| - 1)} \right] \times {}_{2}F_{1}(|a| + 1, |b| + 2; c - |a| + 1; -1) + {}_{2}F_{1}(|a|, |b|; c - |a|; -1) \] - 1.

By (4.4), the aforementioned expression is bounded above by 1, and hence,

\[
2(1 - \beta) \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ \frac{2 \,(|a|) \,(|b|)_{2}}{(c - |a|) (c - |a| - |b| - 1)} \right] \times {}_{2}F_{1}(|a| + 1, |b| + 2; c - |a| + 1; -1) + {}_{2}F_{1}(|a|, |b|; c - |a|; -1) \] - 1 \leq 1.

Therefore, the operator \( I_{n, \beta, \frac{b + 1}{2}}^{a, \frac{b + 1}{2}} (f) (z) \) maps \( \mathcal{R}(\beta) \) into \( \mathcal{UCV} \), and the result follows. \( \square \)
5. Inclusion Properties of $z_3F_2 \left( a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; z \right)$ in $S_p$-CLASS

**Theorem 5.1.** Let $a, b \in \mathbb{C}\backslash\{0\}, c > 0$ and $c > |a| + |b| + 1$. A sufficient condition for the function $z_3F_2 \left( a, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; z \right)$ to belong to the class $S_p$ is that

\[
\left( \frac{\Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \left[ \frac{2(|a|)(|b|)_2}{(|c| - |a|)(c - |a| - |b| - 1)} \right] \times \binom{\sum_{n=0}^\infty (|a|)_n (|b|)_n (|b| + 1)_n}{(|a| + 1)(|b| + 2)(c - |a| + 1)(-1) + \binom{\sum_{n=0}^\infty (|a|)_n (|b|)_n (|b| + 1)_n}{(|a| + 1)(|b| - 1)(|b| - 2)} \leq 2.
\]

**Proof.** The proof is similar to Theorem 4.3. So we omit the details. \(\square\)

**Theorem 5.2.** Let $a, b \in \mathbb{C}\backslash\{0\}, c > 0, |a| \neq 1, |b| \neq 1, 2, c > \max\{|a| + 1, |a| + |b| - 1\}$ and $0 \leq \beta < 1$. Assume that

\[
\frac{\Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left[ 2 \binom{\sum_{n=0}^\infty (|a|)_n (|b|)_n (|b| + 1)_n}{(|a| + 1)(|b| - 1)(|b| - 2)} \leq \frac{1}{2(1 - \beta)} + 1.
\]

Then, $I_{\frac{a}{2}, \frac{b}{2}}^{\frac{b+1}{2}}(f)$ maps $R(\beta)$ into $S_p$ class.

**Proof.** Let $a, b \in \mathbb{C}\backslash\{0\}, c > 0, |a| \neq 1, |b| \neq 1, 2, c > \max\{|a| + 1, |a| + |b| - 1\}$ and $0 \leq \beta < 1$.

Consider the integral operator $I_{\frac{a}{2}, \frac{b}{2}}^{\frac{b+1}{2}}(f)$ given by (1.9). In the view of (1.8), it is enough to show that

\[
T := \sum_{n=0}^\infty (2n - 1) |A_n| \leq 1,
\]

where $A_n$ is given by (1.10). It is proven that

\[
T \leq \sum_{n=2}^\infty (2n - 1) \left( \frac{(|a|)_{n-1} |b|_{n-1} (|b| + 1)_{n-1}}{(\frac{a}{2})_{n-1} (\frac{b+1}{2})_{n-1}} \right) |a_n| \leq 1.
\]

Using the inequality $|(a)_n| \leq (|a|)_n$ and (1.7) in the aforesaid equation, it is derived that

\[
T \leq 2(1 - \beta) \left[ \sum_{n=0}^\infty (n + 1)(|a|)_n \frac{|b|}{2}_n \frac{|b| + 1}{2}_n \right] - \sum_{n=0}^\infty \frac{|a|}_n \frac{|b|}{2}_n \frac{|b| + 1}{2}_n \frac{1}{16} (1)_{n+1} - 1.
\]
Using the equation (1.8) and (4) of Lemma 1.11 it is found that

\[ T \leq 2(1 - \beta) \left[ \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right]^{2} 2F_{1}(|a|, |b|; c - |a|; -1) \]

\[ - \left( \frac{(c - |a| - 1) (c - |a| - |b|)}{(|a| - 1)(|b| - 1)(|b| - 2)} \right) 2F_{1}(|a| - 1, |b| - 2; c - |a| - 1; -1) \]

\[ + \frac{(c - 1) (c - 2)}{(|a| - 1)(|b| - 1)(|b| - 2) - 1} \]

By the condition (5.3), the aforementioned expression is bounded above by 1, and hence,

\[ 2(1 - \beta) \left[ \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right]^{2} 2F_{1}(|a|, |b|; c - |a|; -1) \]

\[ - \left( \frac{(c - |a| - 1) (c - |a| - |b|)}{(|a| - 1)(|b| - 1)(|b| - 2)} \right) 2F_{1}(|a| - 1, |b| - 2; c - |a| - 1; -1) \]

\[ + \frac{(c - 1) (c - 2)}{(|a| - 1)(|b| - 1)(|b| - 2) - 1} \leq 1. \]

Under the stated condition, the operator \( T^{b, c; d}_{a, c; d} \) maps \( R(\beta) \) into \( S_{p} \) and the proof is complete. \( \square \)

**Theorem 5.4.** Let \( a, b \in \mathbb{C} \setminus \{0\} \), \( c > 0 \) and \( c > |a| + |b| + 2 \). Suppose \( a, b, \) and \( c \) satisfy the condition

\[ \left( \frac{\Gamma(c) \Gamma(c - |a| - |b|)}{\Gamma(c - |a|) \Gamma(c - |b|)} \right) \times \left[ \frac{2(|a|)_{2} (|b|)_{4}}{(c - |a|)(c - |a| - |b| - 2)_{2}} \right] 2F_{1}(|a| + 2, |b| + 4; c - |a| + 2; -1) \]

\[ + 5 \left( \frac{(|a|)_{2} (|b|)_{2}}{(c - |a|)(c - |a| - |b| - 1)_{2}} \right) 2F_{1}(|a| + 1, |b| + 2; c - |a| + 1; -1) \]

\[ \leq 2. \]

Then, \( T^{b, c; d}_{a, c; d} \) maps \( S \) into \( S_{p} \) class.

**Proof.** The proof is similar to Theorem 4.1. So we omit the details. \( \square \)

**References**

1. M. Anbu Durai and R. Parvatham, *Convolutions with Hypergeometric Functions*, Bull. Malaysian Math. Sc. Soc. (Second Series) **23** (2000) 153-161.
2. K. Chandrasekran and D. J. Prabhakaran *Geometric Properties of Clausen’s Hypergeometric Functions*, Preprint.
3. K. A. Driver and S. J. Johnston, *An integral representation of some hypergeometric functions*, Electron. Trans. Numer. Anal. **25** (2006), 115–120.
4. P. L. Duren, *Univalent functions* (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, (1983).
5. A. W. Goodman, *Univalent functions*, Vol.I and Vol.II, Tampa Florida Mariner Publishing Company, (1983).
6. A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), no. 1, 87–92.
7. A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), no. 2, 364–370.
8. W. C. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. **57** (1992), no. 2, 165–175.
9. T. H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537.
10. R. Parvatham and D.J. Prabhakaran, *On the Hohlov convolution operator of the class $S_p$*, Far East J. Math. Sci. Special volume, (2001), 217–228.
11. S. Ponnusamy and S. Sabapathy, Geometric properties of generalized hypergeometric functions, Ramanujan J. **1** (1997), no. 2, 187–210.
12. Robertson, Malcolm I. S. *On the theory of univalent functions*. Ann. of Math. (2) **37** (1936), no. 2, 374–408.
13. F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.
14. K. G. Subramanian et al., *Subclasses of uniformly convex and uniformly starlike functions*, Math. Japon. **42** (1995), no. 3, 517–522.
15. K. G. Subramanian et al., *Classes of uniformly starlike functions*, Pub. Math. Debrecen **53** (1998), no. 3-4, 309–315.

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